Joel Rogelio Portada Coacalle

L^2 estimates for the operators $\bar{\partial}$ and $\bar{\partial}_b$

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Supervisor: Gustavo Hoepfner

Co-supervisor: Andrew Seth Raich

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Assinaturas dos membros da comissão examinadora que avaliou e aprovou a Defesa de Tese de Doutorado do candidato Joel Rogelio Portada Coacalle, realizada em 02/08/2019:

Prof. Dr. Gustavo Hoepfner
UFSCar

Prof. Dr. Andrew Seth Raich
UARK

Prof. Dr. Jorge Guillermo Hounie
UFSCar

Prof. Dr. Paulo Domingos Cordaro
USP

Prof. Dr. Gerardo Alvaro Manuel Mendoza Jacobsen
Temple

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"If I have seen further, it is by standing upon the shoulders of giants."
(Sir Isaac Newton.)

Abstract

The purpose of this work is to establish sufficient conditions for closed range estimates on $(0, q)$-forms, for some *fixed* q, $1 \leq q \leq n-1$, for $\bar{\partial}_b$ in both L^2 and L^2-Sobolev spaces in embedded, not necessarily pseudoconvex CR manifolds of hypersurface type. The condition, named weak $Y(q)$, is both more general than previously established sufficient conditions and easier to check. Applications of our estimates include estimates for the Szegö projection as well as an argument that the harmonic forms have the same regularity as the complex Green operator. We use a microlocal argument and carefully construct a norm that is well-suited for a microlocal decomposition of form. We do not require that the CR manifold is the boundary of a domain. Finally, we provide an example that demonstrates that weak $Y(q)$ is an easier condition to verify than earlier, less general conditions.

Keywords Cauchy Riemann operator. Tangential Cauchy Riemann operator. CR manifolds. Weak Y(q) condition. Closed range estimates.

Resumo

O objetivo de este trabalho é estabelecer condições suficientes para estimativas de imagem fechada sob $(0, q)$-formas, com q fixo e $1 \leq q \leq n-1$, para $\bar{\partial}_b$ nos espaços L^2 e L^2 Sobolev sob variedades CR do tipo hipersuperfície. A condição, chamada $Y(q)$ fraca, é mais geral do que as condições suficientes estabelecidas anteriormente e é mais fácil de verificar. As aplicações de nossas estimativas incluem estimativas para a projeção Szegö, bem como um argumento de que as formas harmônicas têm a mesma regularidade que o operador Green complexo. Utilizamos um argumento microlocal e construímos cuidadosamente uma norma que é adequada para uma decomposição microlocal das formas. Não exigimos que a variedade CR seja a fronteira de um domínio. Finalmente, fornecemos um exemplo que demonstra que a condição $Y(q)$ fraca é uma condição mais fácil de verificar que as versões anteriores menos gerais.

Keywords: Operador de Cauchy Riemann. Operador tangencial de Cauchy Riemann. Variedade CR. Condição Y(q) fraca. Estimativas de imagem fechada.

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CHAPTER 1

INTRODUCTION

One of the principal operators that appears in the study of theory of functions of several complex variables, is the Cauchy Riemann operator $\bar{\partial}$ which is deeply related to analytic functions, domains of holomorphy, extension problems of Hartog's type, the Levi problem; and the existence and regularity of solutions for the well known non-homogeneous equation

$$\bar{\partial}u = f \tag{1.1}$$

where f a (p,q)-form defined on a bounded open set $\Omega \subset \mathbb{C}^n$, $n \geq 2$, with smooth boundary $b\Omega$, with the necessary condition that f is $\bar{\partial}$-closed. It is in the last topic where the present work is focused, and its equivalent non homogeneous equation associated to its analogue operator, the tangential Cauchy Riemann operator $\bar{\partial}_b$. Many advances for the existence of solutions for the equation (1.1) were done in decades of 50's and 60's, by Garabedian and Spencer [7], Morrey [23], Kohn [17], Hormander [14]. Two things to remark in these last three works, is the use of Hilbert space tools to solve (1.1), once they considered the (p,q) forms on a more general set, $L^2_{p,q}(\Omega)$, and look for solutions on distributional sense; and the other one is the conditions imposed on the geometry on $b\Omega$. The use of Hilbert space tools allowed then to relate the existence of solutions with the property of closed range of the operator $\bar{\partial}$, once it was defined as a unbounded closed densely defined operator. One of the first geometrical conditions imposed in the boundary of Ω was strongly pseudoconvexity by Kohn in [17], who proved that strongly pseudoconvexity implied not just the closed range property for $\bar{\partial}$ but 1/2 Sobolev estimates (as it is also reiterated by Kohn and Niremberg in [21]). On the other hand Hormander in [14] introduced weighted spaces getting advantage of existence of well behaved functions (plurisubharmonic functions) defined on the environment $\mathbb{C}^n$, and he could imply the property of closed range for the operator $\bar{\partial}$ assuming just pseudoconvexity in $b\Omega$. The Chapter 2 in this work is focused in the study of this operator $\bar{\partial}$ (in fact just in closed range property in L^2), written in the spirit of the Chapter 4 in [4]. And we consider add the study concerning to the study of the operator $\bar{\partial}$, because gives an overview of the philosophy behind technique and machinery used to map the steps to follow in the analysis of tangential Cauchy Riemann operator, which is our principal topic. In Chapter 3 we give a sketch about oldness results and approaches obtained through time about operator $\bar{\partial}_b$, and its origins, and thus give a motivation about the principal topic.

The principal topic of this work is made in Chapters 4 - 6, where we consider the study of operator $\bar{\partial}_b$ on CR manifolds of hypersurface type. A CR manifold of hypersurface type is an odd real dimensional manifold whose tangent bundle splits into a complex subbundle and another line bundle. An appropriate restriction of the $\bar{\partial}$ operator to the complex subbundle yields the tangential Cauchy Riemann operator $\bar{\partial}_b$.

The $\bar{\partial}_b$ operator was introduced by Kohn and Rossi [22] to study the boundary values of holomorphic functions on domains in $\mathbb{C}^n$, and it was soon realized that the $\bar{\partial}_b$-complex was deeply intertwined with the geometry and potential theory of such domains and their boundaries. The story of the L^2-theory of the $\bar{\partial}_b$ operator begins with Shaw [28] and Boas and Shaw [2] (in the top degree) on boundaries of pseudoconvex domains in $\mathbb{C}^n$ and with Kohn [20] on the boundaries of pseudoconvex domains in Stein manifolds. Nicoara [24] established closed range for $\bar{\partial}_b$ (at all form levels) on smooth, embedded, compact, orientable CR manifolds of hypersurface dimension in the case that $n \geq 3$ and Baracco [1] established the $n = 2$ case. Thus, from the point of view closed range, the pseudoconvex case is completely understood.

Harrington and Raich [10] began an investigation of the $\bar{\partial}_b$-problem on non-pseudoconvex CR manifolds of hypersurface type. Specifically, they fixed a level q, $1 \leq q \leq n-2$, and sought a general condition that sufficed to prove closed range of $\bar{\partial}_b$ on $(0,q)$-forms (and in L^2-Sobolev spaces in suitably weighted spaces). They worked on CR manifolds of hypersurface type, and our results generalize theirs by showing that the conclusions they draw are still true with a weaker hypothesis, namely, the *weak* $Y(q)$ condition from [11]. The analysis in [11] is loosely based on the ideas of Shaw and does not use a microlocal argument, but rather $\bar{\partial}$-methods. This requires the CR manifold to be the boundary of a domain, a hypothesis that we relax. The name weak $Y(q)$ stems from the fact that it is a weakening of the classical $Y(q)$ condition, a geometric condition that is equivalent to the complex Green operator (inverse to the Kohn Laplacian) satisfying 1/2-estimates on $(0,q)$-forms.

In this work, we show that the tangential Cauchy-Riemann operator has closed range on $(0,q)$-forms, for a *fixed* q, $1 \leq q \leq n-1$, in L^2 and L^2-Sobolev spaces on a general class of embedded CR manifolds of hypersurface type that satisfy the general geometric condition weak $Y(q)$. We work on a smooth CR submanifold $M \subset \mathbb{C}^n$ that may be neither pseudoconvex nor the boundary of a domain. The weak $Y(q)$ condition, first written down by Harrington and Raich [11] and applied to boundaries of domains in Stein manifolds, is the most general known condition that ensures closed range of the tangential Cauchy-Riemann operator on $(0,q)$-forms. We also provide an example that shows that the generality provided by the definition makes it easier to verify than previous and more restrictive conditions. Additionally, we show that for any Sobolev level, there is a weight such that the (weighted) complex Green operator (inverse to the weighted Kohn Laplacian) is continuous and the harmonic forms in this weighted space are elements of the prescribed Sobolev space.

Also, we generalize both [10] and [11] in the following ways. We do not require our CR manifold to be the boundary of a domain. In effect, we translate the $\bar{\partial}$-techniques of [11] to the microlocal setting. In [10], they prove results akin to our main results, but the "weak $Y(q)$" condition they define is more restrictive than the weak $Y(q)$ condition here. Additionally, we use a reengineered elliptic regularization argument to show that (weighted) harmonic $(0,q)$-forms are smooth, a fact not mentioned in [10, 11]. Additionally, we are careful to monitor the regularized operators and the fact that they preserve orthogonality with the space of (weighted) harmonic forms, a fact that has not been observed before (in part because we prove smoothness of harmonic forms early in regularization process).

Our methods to analyze the tangential Cauchy Riemann Operator involve a microlocal argument in the spirit of [24, 25, 10] and a recently reengineered elliptic regularization that not only allows for a weighted complex Green operator to solve the $\bar{\partial}_b$-problem in a given L^2-Sobolev space, but also shows that the weighted L^2-harmonic forms reside in that Sobolev space [16, 9]. This last fact is not clear from the elliptic regularization methods used in [24, 10]. For a discussion of the weak $Y(q)$ condition and its related, non-symmetrized version, weak $Z(q)$, please see [10, 11, 8, 12, 13] and for discussion on the elliptic regularization method, [9, 16].

The outline of the argument is as follows: we start by proving a basic identity that is well suited to the geometry of M. The problem with basic identities for $\bar{\partial}_b$ is that the Levi form appears with in a term that also contains the derivative in the totally real direction T. The microlocal argument is used to control this term – specifically, we construct a norm based on a microlocal decomposition of our form which allows us to use a version of the sharp Gårding's inequality and eliminate the T from the inner product term. This allows us to prove a basic estimate (Proposition 4.6.1) from the basic identity and the main results are due to careful applications of the basic estimate.

We conclude this chapter giving an outline of the work. As we said before, the Chapter 2 is focused on as well general ideas and procedures to study the $\bar{\partial}$ operator. In Chapter 3 we relate the origins in the study of $\bar{\partial}_b$ operator. Into Chapter 4, we define our notations, give some computations in local coordinates and the microlocal decomposition, to prove the basic estimate, Proposition 4.6.1. In Chapter 5, we prove the Theorem 5.0.2, and we outline how to pass from Theorem 5.0.2 to Theorem 5.0.1. We conclude the work in Chapter 6 with two examples comparing the older version and new version of weak $Z(q)$ condition.

CHAPTER 2

L^2 ESTIMATES FOR THE $\overline{\partial}$ OPERATOR

In this chapter we give some details about the approach given to study of operator $\bar{\partial}$. We follow the process given in [4, Chapter 4]. In the first section we give some definitions and important functional analysis results in the study of operators we are interested. In second section we present the L^2 theory about operators $\bar{\partial}$ and $\Box$ on L^2 as well as some computations related to them. The third and fourth section have L^2 existence theorems about the equation associated to operators $\bar{\partial}$ and $\Box$. The last section has an application of these results to solve the Levi problem.

2.1 Unbounded operators in Hilbert spaces

Let $H_1,\ H_2$ be Hilbert spaces, and $T : \mathrm{Dom}(T) \subset H_1 \to H_2$ be an unbounded linear, closed and densely defined operator, that is, the graph of T, $Graph(T)$, is closed in $H_1 \times H_2$ and $\mathrm{Dom}(T)$ is a dense linear subspace of H_1. If T is not a bounded operator, by the Closed Graph Theorem, $\mathrm{Dom}(T)$ is going to be a proper subset of H_1.

Let $\|.\|_1$ and $\|.\|_2$ denote the norms of H_1 and H_2 respectively. If $v \in H_2^*$ and there is $C \geq 0$ such that $|\,(v, T(u))\,| \leq C\|u\|_1$ for all $u \in \mathrm{Dom}(T)$, by the analytic form of Hahn-Banach Theorem there is a $f_v \in H_1^*$ that extended the linear function $u \in \mathrm{Dom}(T) \mapsto (v, T(u))$ to all H_1 and $|\,(f_v, u)\,| \leq C\|u\|_1$. Moreover, f_v is unique because $\overline{\mathrm{Dom}(T)} = H_1$. We define the adjoint of T by T^* and so $T^* : \mathrm{Dom}(T^*) \subset H_2^* \to H_1^*$ where

$$\mathrm{Dom}(T^*) = \{v \in H_2^* : \exists C \geq 0 \text{ such that } |\,(v, T(u))\,| \leq C\|u\|_1,\ \forall u \in \mathrm{Dom}(T)\}$$

and $T^*(v) := f_v$. If $v,\, w \in \mathrm{Dom}(T^*)$ then $(v + w, T(u)) = (f_v + f_w, u)$ for all $u \in \mathrm{Dom}(T)$. By the uniqueness of f_{v+w} we have $T^*(v + w) = T^*(v) + T^*(w)$. Then T^* is linear. Also, T^* is closed. Indeed, if (v_n) in $\mathrm{Dom}(T^*)$ such that $v_n \to v$ in H_2^* and $T^*(v_n) \to f$ in H_1^*, since

$$(v_n, T(u)) = (T^*(v_n), u)$$

for all $u \in \mathrm{Dom}(T)$ we have $(v, T(u)) = (f, u)$. Then $v \in \mathrm{Dom}(T^*)$ because $|\,(v, T(u))\,| \leq \|f\|\|u\|_1$ for all $u \in \mathrm{Dom}(T)$. Also, $T^*(v) = f$ from the uniqueness of $T^*(v)$.

If we consider the isomorphism $I : F^* \times E^* \to E^* \times F^*$ defined by $I(v, f) = (-f, v)$ we have that $I(G(A^*)) = G(A)^{\perp}$.

Theorem 2.1.1 *Let E and F reflexive Banach spaces. Let $T : \mathrm{Dom}(T) \subset E \to F$ be a unbounded operator that is densely defined and closed. Then $\mathrm{Dom}(T^*)$ is dense in F^*. Thus $T^{**} : \mathrm{Dom}(T^{**}) \subset E^{**} \to F^{**}$ is well defined and it may also be viewed as an unbounded operator from E into F. Then we have $T^{**} = T$.*

Proof. First let's show that $\mathrm{Dom}(T^*)$ is dense in F^*. Let $\varphi \in F^{**}$ such that $\varphi\left(\mathrm{Dom}(T^*)\right) = \{0\}$. Since F is reflexive $\varphi(v) = (v, \xi)$ for all $v \in F^*$ and some $\xi \in F$. Then is sufficient show that $\xi = 0$. If $\xi \neq 0$ then $(0, \xi) \notin Graph(T)$. By the geometric form of Hahn-Banach theorem, there is a $(w, z) \in E^* \times F^* \cong (E \times F)^*$ (with the isomorphism $(f, g) \in E^* \times F^* \mapsto f \times g$ where $f \times g(t, r) = f(t) + g(r)$) such that $(w, z)(u, Tu) < \alpha < (w, z)(0, \xi)$ for all $u \in \mathrm{Dom}(T)$, and for some $\alpha \in \mathbb{R}$. Then

$$(w, u) + (z, Tu) = (w, z)(u, Tu) = 0 \quad \forall u \in \mathrm{Dom}(T).$$

Then $z \in \mathrm{Dom}(T^*)$ and $0 < \alpha < (z, \xi)$. This is a contradiction. Thus $\xi = 0$, and then $\mathrm{Dom}(T^*)$ is dense in F^*.

With this, T^* is considered as an unbounded operator that is densely defined and closed. Then we can define the operator (again closed and densely defined) $T^{**} : \mathrm{Dom}(T^{**}) \subset E^{**} \to F^{**}$.

Since E, F are reflexive spaces, T^{**} is an operator from E into F. As we saw

$$I(G(T^*)) = G(T)^{\perp} \quad I(G(T^{**})) = G(T^*)^{\perp}.$$

Let the isomorphism $S : E^* \times F^* \to F^* \times E^*$ given by $S(f, g) = (g, f)$ for $(f, g) \in E^* \times F^*$. Then $I(G(T^*))^{\perp} = T^{-1}I^{-1}(G(T^*)^{\perp})$, also $I(G(T^{**}))^{\perp} = T^{-1}I^{-1}(G(T^{**})^{\perp})$. Thus

$$\begin{aligned} G(T) = G(T)^{\perp\perp} &= I(I(G(T^{**}))^{\perp})^{\perp} = T^{-1}I^{-1}T^{-1}I^{-1}(G(T^{**})^{\perp\perp}) \\ &= G(T^{**})^{\perp\perp} = G(T^{**}) \end{aligned}$$

because $G(T)$, $G(T^*)$ and $G(T^{**})$ are closed. Then $T = T^{**}$. ∎

Now, we rewrite two important theorems used by Hörmander in [14] to study the closed range property of unbounded operators. Assuming H_1 and H_2 two Hilbert spaces and let $T : H_1 \to H_2$ be linear, closed, and densely defined operator. By the results above $T^* : H_2 \to H_1$ has the same properties, and $T^{**} = T$. By definition of the adjoint operator, the orthogonal complement of the range $\mathrm{Ran}(T)$ of T is the kernel $\mathrm{Ker}(T^*)$ of T^*. Which implies that the orthogonal complement of $\mathrm{Ker}(T^*)$ is the closure $\overline{\mathrm{Ran}(T)}$ of $\mathrm{Ran}(T)$. This means

$$H_1 = \mathrm{Ker}(T) \oplus \overline{\mathrm{Ran}(T^*)}, \tag{2.1}$$

Similarly, we have that $H_2 = \mathrm{Ker}(T^*) \oplus \overline{\mathrm{Ran}(T)}$. The next two theorems appears in [14].

Theorem 2.1.2 *Let H_1 and H_2 be two Hilbert spaces and let $T : H_1 \to H_2$ be linear, closed, and densely defined operator. The following conditions on T are equivalent:*

1. $\mathrm{Ran}(T)$ *is closed.*
2. *There is a constant C such that*

$$\|f\|_1 \leq C\|Tf\|_2, \qquad f \in \mathrm{Dom}(T) \cap \overline{\mathrm{Ran}(T^*)}. \tag{2.2}$$

3. $\mathrm{Ran}(T^*)$ *is closed.*
4. *There is a constant C such that*

$$\|g\|_2 \leq C\|T^*g\|_1, \qquad g \in \mathrm{Dom}(T^*) \cap \overline{\mathrm{Ran}(T)}. \tag{2.3}$$

The best constants in (2.2) and in (2.3) are the same.

Proof. Assume that (1) holds. Since $\overline{\mathrm{Ran}(T^*)} = \mathrm{Ker}(T)$, the restriction of T to $\mathrm{Dom}(T) \cap \overline{\mathrm{Ran}(T^*)}$ is a closed, one to one, linear mapping onto the closed subspace $\mathrm{Ran}(T)$ of H_2. Hence the inverse is continuous by the closed graph theorem, which proves (2). Conversely, (2) obviously implies (1). In a similar way, (3) will be equivalent to (4). Now we prove that (2) implies (4). Notice that

$$|(g, Tf)_2| = |(T^*g, f)_1| \leq C\|T^*g\|_1\|Tf\|_2$$

for $g \in \mathrm{Dom}(T^*)$ and $f \in \mathrm{Dom}(T) \cap \overline{\mathrm{Ran}(T^*)}$. Thus

$$|(g, h)_2| \leq C\|T^*g\|_1\|h\|_2$$

for $g \in \mathrm{Dom}(T^*)$ and $h \in \mathrm{Ran}(T)$. So $|(g, h)_2| \leq C\|T^*g\|_1\|h\|_2$ for $g \in \mathrm{Dom}(T^*)$ and $h \in \overline{\mathrm{Ran}(T)}$, which implies (4) (take a sequence in $\mathrm{Ran}(T)$ converging to g and use this last inequality). Similarly, (4) implies (2). ∎

Let H_3 be another Hilbert space and $S : H_2 \to H_3$ be another unbounded linear, closed densely defined operator, such that $ST = 0$, that is, $\mathrm{Ran}(T) \subset \mathrm{Ker}(S)$.

Theorem 2.1.3 *A necessary and a sufficient condition for* $\mathrm{Ran}(T)$ *and* $\mathrm{Ran}(S)$ *both to be closed is that*

$$\|g\|_2^2 \leq C^2\left(\|T^*g\|_1^2 + \|Sg\|_3^2\right); \qquad g \in \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S), g \perp K := \mathrm{Ker}(T^*) \cap \mathrm{Ker}(S). \tag{2.4}$$

Proof. First note that

$$H_2 = \overline{\mathrm{Ran}(T)} \oplus K \oplus \overline{\mathrm{Ran}(S^*)} \tag{2.5}$$

In fact, since $T^*S^* = 0$ then $\mathrm{Ran}(T)$ and $\mathrm{Ran}(S^*)$ are orthogonal, and the intersection of the orthogonal complements of this spaces, $\mathrm{Ker}(T^*)$ and $\mathrm{Ker}(S)$ respectively, is K. Now, if (2.4) holds, then (4) is valid because S vanishes in $\overline{\mathrm{Ran}(T)}$, and, changing T by S, (2) is valid because T^* vanishes in $\overline{\mathrm{Ran}(S^*)}$. So (2.4) implies that $\mathrm{Ran}(T)$ and $\mathrm{Ran}(S)$ are closed. Reciprocally, if $\mathrm{Ran}(T)$ and $\mathrm{Ran}(S)$ are closed, by the decomposition (2.5), if $g \in \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S)$ with $g \perp K := \mathrm{Ker}(T^*) \cap \mathrm{Ker}(S)$ there exist $\alpha \in \mathrm{Dom}(T^*)$ such that $T^*\alpha_2 \in \mathrm{Dom}(T)$ and $\beta \in \mathrm{Dom}(S)$ such that $S\beta \in \mathrm{Dom}(S^*)$, with $g = TT^*\alpha + S^*S\beta$. Then inequality (2.4) follows by (2.2) and (2.3) and since $ST = 0$, we have $T^*S^* = 0$. ■

2.2 L^2 theory for $\bar\partial$ operator

Let Ω is a open subset in $\mathbb{C}^n$, with $n \geq 2$, $C^\infty_{(p,q)}(\Omega)$ the set of smooth (p,q)-forms in Ω, $C^\infty_{(p,q)}(\overline{\Omega})$ the set of smooth (p,q)-forms in $\mathbb{C}^n$ restricted to Ω. $D_{(p,q)}(\Omega)$ the set of smooth (p,q)-forms with compact support in Ω. Then any (p,q)-form $f \in C^\infty_{(p,q)}(\Omega)$ can be expressed as

$$f = \sum_{I,J}{}' f_{I,J} dz^I \wedge d\bar z^J \tag{2.6}$$

where $I = (i_1, i_2, ..., i_p)$ and $J = (j_1, j_2, ..., j_q)$ are multiindices, $dz^I = dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p}$, $d\bar z^J = d\bar z_{j_1} \wedge d\bar z_{j_2} \wedge ... \wedge d\bar z_{j_q}$, $\sum{}'$ means summation over strictly increasing multiindices, and $f_{I,J}$ are smooth functions defined in Ω for arbitrary I and J so that they are antisymmetric.

Writing

$$dz_j = dx_j + idy_j \ , \qquad d\bar z_j = dx_j - idy_j,$$

and

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \ , \qquad \frac{\partial}{\partial \bar z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

if $f \in C^\infty(\Omega)$ is a complex-valued function then

we define the operators $\bar\partial$ and ∂ for functions by

$$\partial f = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j \quad \text{and} \quad \bar\partial f = \sum_{j=1}^n \frac{\partial f}{\partial \bar z_j} d\bar z_j.$$

Then we will have $df = \partial f + \bar\partial f$. Once we have defined the $\bar\partial$ and ∂ operator on complex valued functions, we define the $\bar\partial$ and ∂ for (p,q)-forms as (2.6) by

$$\partial f = \sum_{I,J}{}' \partial f_{I,J} \wedge dz^I \wedge d\bar z^J,$$

$$\bar{\partial} f = \sum_{I,J}{}' \bar{\partial} f_{I,J} \wedge dz^I \wedge d\bar{z}^J, \tag{2.7}$$

and also we will have $df = \partial f + \bar{\partial} f$.

Since $0 = d^2 = \partial^2 + (\bar{\partial}\partial + \partial\bar{\partial}) + \bar{\partial}^2$ and ∂^2, $\bar{\partial}\partial + \partial\bar{\partial}$, $\bar{\partial}^2$ are of bidegree $(p+2, q)$, $(p+1, q+1)$ and $(p, q+2)$ respectively, we have

$$\partial^2 = 0, \quad \bar{\partial}\partial + \partial\bar{\partial} = 0, \quad \bar{\partial}^2 = 0.$$

For $0 \le q \le n$ and $0 \le q \le n-1$, define $\bar{\partial} = \bar{\partial}_{(p,q)} : C^\infty_{(p,q)}(\Omega) \to C^\infty_{(p,q+1)}(\Omega)$ as in (2.7) and then we have $\mathrm{Ran}(\bar{\partial}_{(p,q)}) \subset \mathrm{Ker}(\bar{\partial}_{(p,q+1)})$. If p is fixed, we can to define the short sequence

$$0 \to C^\infty_{(p,0)}(\Omega) \overset{\bar{\partial}_{(p,0)}}{\longrightarrow} C^\infty_{p,1}(\Omega) \overset{\bar{\partial}_{(p,1)}}{\longrightarrow} \cdots \overset{\bar{\partial}_{(p,n-1)}}{\longrightarrow} C^\infty_{(p,n)}(\Omega) \overset{\bar{\partial}_{(p,n)}}{\longrightarrow} 0$$

and the natural question is, is this a exact sequence?. To answer this question, we need to solve the equation

$$\bar{\partial} u = f, \tag{2.8}$$

with the necessary condition $\bar{\partial} f = 0$, called the compatibility condition.

Example If $n > 1$, $f = \sum_{i=1}^n f_j d\bar{z}_j$ with $f_j \in C^k(\mathbb{C}^n)$ and $k > 0$, such that $\frac{\partial f_i}{\partial \bar{z}_j} = \frac{\partial f_j}{\partial \bar{z}_i}$ (or $\bar{\partial} f = 0$) and $\operatorname{supp} f := \cup_{j=1}^n \operatorname{supp} f_j$ is a compact subset in $\mathbb{C}^n$; the complex-valued function

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_i(\zeta, z_2, ..., z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta}$$

is in $C^k_0(\mathbb{C}^n)$ such that $\bar{\partial} u = f$ (see Theorem 2.3.1 in [15], and observe its implication on the analytic extension of Hartog's theorem; together the clarification made on $n > 1$).

We are interested in solving the equation (2.8), using tools of Hilbert spaces regarding results on unbounded operators and find solutions in a more general sense. For this purpose, we need to define the operator $\bar{\partial}$ appropriately.

From now on, we consider D to be a bounded open subset in $\mathbb{C}^n$. Let $L^2(D)$ denote the space of square integrable functions in D with respect to the Lebesgue measure. Here we are using the volume element $dV = (i/2)^n dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} ... \wedge dz_n \wedge d\overline{z_n}$. Let $L^2_{(p,q)}(D)$ denote the space of (p,q)-forms with coefficients in $L^2(D)$. We define

$$\|f\|^2 := \sum_{I,J}{}' \int_D |f_{I,J}|^2 dV \qquad \forall f = \sum_{I,J}{}' f_{I,J} \in L^2_{(p,q)}(D).$$

This turns $L^2_{p,q}(D)$ into a Hilbert space. And, if we use $(\ ,\ )_D$ to denote the inner product in $L^2_{(p,q)}(D)$, we have that

$$\left(\sum_{I,J}{}' f_{I,J} dz^I \wedge d\bar{z}^J, \sum_{I,J}{}' g_{I,J} dz^I \wedge d\bar{z}^J \right)_D = \sum_{I,J}{}' (f_{I,J}, g_{I,J})$$

where $(\ ,\ )$ is a given inner product in $L^2(D)$. Note that, when $p=q=0$, $(\ ,\ )_D = (\ ,\ )$. Sometimes, when there is no danger confusion, we drop the subscript D in the notation. Also, if $\phi \in C(D)$, we denote $L^2(D,\phi)$ the space of complex-valued functions ψ such that $\psi e^{-\phi/2} \in L^2(D)$, similarly $L^2_{(p,q)}(D,\phi)$ the space of (p,q)-forms with coefficients in $L^2(D,\phi)$. We define

$$\|f\|^2_\phi = {\sum_{I,J}}' \int_D |f_{I,J}|^2 e^{-\phi} dV \qquad \forall f = {\sum_{I,J}}' f_{I,J} \in L^2_{(p,q)}(D,\phi)$$

and $(L^2_{(p,q)}(D,\phi), \|\ \|_\phi)$ is also a Hilbert space. We use $(\ ,\ )_\phi$ to denote the inner product in $L^2_{(p,q)}(D,\phi)$. Note that if $\phi \in C(\overline{D})$ then $L^2_{(p,q)}(D,\phi) = L^2_{(p,q)}(D)$, because in this case we will have

$$\min_{\overline{D}} |e^{-\phi}| \int_D |f|^2 dV \le \int_D |f|^2 e^{-\phi} dV \le \max_{\overline{D}} |e^{-\phi}| \int_D |f|^2 dV \tag{2.9}$$

for all $f \in L^2(D,\phi)$, and the norms $\|\ \|$ and $\|\ \|_\phi$ are equivalents. Similarly, we can to define $L^2_{(p,q)}(D,loc)$ the space of (p,q)-forms with coefficients in $L^2_{loc}(D)$, where $L^2_{loc}(D)$ is a space of complex-valued functions in $L^2(K)$ for all $K \subset D$ compact.
For $0 \le q \le n-1$, we define the formal adjoint operator $\vartheta = \vartheta_{(p,q+1)} : C^\infty_{p,q+1}(D) \to C^\infty_{(p,q)}(D)$ of $\bar\partial = \bar\partial_{(p,q)} : C^\infty_{p,q}(D) \to C^\infty_{p,q+1}(D)$ by requiring

$$(\vartheta f, g) = \left(f, \bar\partial g\right)$$

for all $g \in D_{(p,q)}(D)$. Then if $f \in C^\infty_{(p,q+1)}(D)$ is expressed by (2.6), then

$$\vartheta(f) = (-1)^{p+1} {\sum_{I,K}}' \sum_{l=1}^n \frac{\partial f_{I,lK}}{\partial z_l} dz^I \wedge d\bar z^K \tag{2.10}$$

where $f_{I,lK} = sgn\binom{J}{lK} f_{I,J}$. In fact, if $g = \sum_{I,K}{}' g_{I,K} dz^I \wedge d\bar z^K \in D_{(p,q)}(D)$ then

$$\begin{aligned}
\left(f, \bar\partial g\right) &= \left({\sum_{I,J}}' f_{I,J} dz^I \wedge d\bar z^J, {\sum_{I,K}}' \sum_{l=1}^n \frac{\partial g_{I,K}}{\partial \bar z_l} d\bar z_l \wedge dz^I \wedge d\bar z^K \right) \\
&= (-1)^p \sum_{l=1}^n \left({\sum_{I,J}}' f_{I,J} dz^I \wedge d\bar z^J, {\sum_{I,K}}' \frac{\partial g_{I,K}}{\partial \bar z_l} dz^I \wedge d\bar z_l \wedge d\bar z^K \right) \\
&= (-1)^p \sum_{l=1}^n {\sum_{I,K}}' \left(f_{I,lK}, \frac{\partial g_{I,K}}{\partial \bar z_l} \right)
\end{aligned} \tag{2.11}$$

but, since $g \in D_{(p,q)}(D)$, integrating by parts we have

$$\left(f, \bar\partial g\right) = (-1)^{p+1} \sum_{l=1}^n {\sum_{I,K}}' \left(\frac{\partial f_{I,lK}}{\partial z_l}, g_{I,K} \right). \tag{2.12}$$

Now (2.10) follows from (2.12).

Observe that $\vartheta^2 = 0$, thus, similarly as before, we define $\vartheta_\phi : C^\infty_{p,q+1}(D) \to C^\infty_{p,q}(D)$ the $L^2(D,\phi)$ adjoint of $\bar{\partial} = \bar{\partial}_{(p,q)} : C^\infty_{p,q}(D) \to C^\infty_{p,q+1}(D)$ by requiring

$$(\vartheta_\phi f, g)_\phi = \left(f, \bar{\partial} g\right)_\phi, \quad \forall g \in D_{p,q}(D). \tag{2.13}$$

So, we have the relation between ϑ and ϑ_ϕ given by the following

$$(\vartheta_\phi f, \psi)_\phi = \left(f, \bar{\partial}\psi\right)_\phi = \left(fe^{-\phi}, \bar{\partial}\psi\right) = \left(\vartheta(fe^{-\phi}), \psi\right) = \left(e^\phi \vartheta(fe^{-\phi}), \psi\right)_\phi$$

for all $\psi \in D^\infty_{(p,q)}(D)$, that is $\vartheta_\phi(f) = e^\phi \vartheta(fe^{-\phi})$ for all $f \in C^\infty_{p,q+1}(D)$.

Now, if $\phi \in C^1(\overline{D})$, we extend the definition of the unbounded densely defined operator $\bar{\partial}_{(p,q)} : C^\infty_{p,q}(D) \subset L^2_{(p,q)}(D,\phi) \to L^2_{(p,q+1)}(D,\phi)$, and we will still denote it by $\bar{\partial}_{p,q}$. For distributions we define as follows: an element $u \in L^2_{(p,q)}(D,\phi)$ is in the domain of $\bar{\partial}$, denoted by $\mathrm{Dom}(\bar{\partial})$, if $\bar{\partial}u$, defined in the distribution sense, belong to $L^2_{(p,q)}(D,\phi)$. Since the convergence in $L^2(D)$ implies in convergence in distributions and the differentiation is a continuous operation in distributions, we have that the operator $\bar{\partial}$ is closed. We denote $\bar{\partial}^*_\phi$ to be the adjoint operator of $\bar{\partial}$ and $\mathrm{Dom}(\bar{\partial}^*_\phi)$ the domain of $\bar{\partial}^*_\phi$. When $\phi = 0$ we denote $\bar{\partial}^*$ to be the adjoint operator of $\bar{\partial}$ and $\mathrm{Dom}(\bar{\partial}^*)$ the domain of $\bar{\partial}^*$. By the Theorem 2.1.1 $\bar{\partial}^*$ and $\bar{\partial}^*_\phi$ are unbounded closed densely defined operators.

By the definition of $\mathrm{Dom}(\bar{\partial}^*_\phi)$, an element f belongs to $\mathrm{Dom}(\bar{\partial}^*_\phi)$ if there is a $g \in L^2_{(p,q)}(D,\phi)$ such that for every $\psi \in \mathrm{Dom}(\bar{\partial})$, we have $\left(f, \bar{\partial}\psi\right)_\phi = (g,\psi)_\phi$. We define $\bar{\partial}^*_\phi f = g$. Defining ϑ_ϕ in distributional sense (this means $(\vartheta_\phi f, h)_\phi := \left(f, \bar{\partial}h\right)_\phi$ for any $h \in D_{p,q}(D)$), and if $f \in \mathrm{Dom}(\bar{\partial}^*_\phi)$ then $\vartheta_\phi f = \bar{\partial}^*_\phi f$ in L^2 sense. We make here a small parenthesis to talk about definition functions, then we will continue talking about the domain of $\mathrm{Dom}(\bar{\partial}^*_\phi)$.

Definition 2.2.1 *A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is said to have C^k ($1 \leq k \leq \infty$) boundary at the boundary point p if there exists a real-valued C^k function ρ defined in a some open neighborhood U of p such that*

$$D \cap U = \{x \subset U : \rho(x) < 0\}, \quad bD \cap U = \{x \in U : \rho(x) = 0\}$$

and $d\rho(x) \neq 0$ on $bD \cap U$. The function ρ is called a C^k local defining function for D near ρ. If U is an open neighborhood of $\bar{D}$ then ρ is called a global defining function for D, or simply a defining function for D.

The next lemma give us the relationship between two defining functions.

Lemma 2.2.2 *Let ρ_1 and ρ_2 be two local defining functions for D of class C^k $(1 \le k \le \infty)$ in a neighborhood U of $p \in bD$. Then there exist a positive C^{k-1} function h on U such that*
(1) $\rho_1 = h\rho_2$ on U
(2) $d\rho_1(x) = h(x)d\rho_2(x)$ for all $x \in bD \cap U$

Proof. Suppose first that $p = 0$ and $\rho_2(x) = x_n$ and $bD \cap U = \{x \in U : x_n = 0\}$. Denote $x = (x', x_n)$ where $x' = (x_1, x_2, ..., x_{n-1})$. Then by the Fundamental Theorem of Calculus we have that

$$\rho_1(x) = \rho_1(x', x_n) - \rho_1(p) = \int_0^{x_n} \frac{\partial \rho_1}{\partial x_n}(x', s)ds = x_n \int_0^1 \frac{\partial \rho_1}{\partial x_n}(x', tx_n)dt \tag{2.14}$$

and the function $h(x', x_n) = \int_0^1 \frac{\partial \rho_1}{\partial x_n}(x', tx_n)dt \in C^{k-1}$ and by (2.14) is a positive function in U.

In general case, for each $x \in bD \cap U$, since $d\rho_2(x) \neq 0$, without loss of generality, we may assume that $\frac{\partial \rho_2(x)}{\partial x_n} \neq 0$, thus there are a neighborhood $U_x \subset U$ of x and a C^k diffeomorphism $\Phi : U_x \to \Phi(U_x)$ such that $0 \in \Phi(U_x)$, $\Phi(0) = x$ and $\rho_2 \circ \Phi^{-1}(y_1, ..., y_n) = y_n$. Then $\Phi(bD \cap U_x) = \{y : y_n = 0\}$ and by the first part there is a positive C^{k-1} function H_x defined in $\Phi(U_x)$ such that $\rho_1 \circ \Phi^{-1}(y) = H_x(y)\rho_2 \circ \Phi^{-1}(y)$ for all $y \in \Phi(U_x)$. With this, there is a positive C^{k-1} function h_x defined in U_x such that $\rho_1(z) = h_x(z)\rho_2(z)$ for all $z \in U_x$. Varying x in $bD \cap U$, using the continuity of the function h_x and since $h_x = \rho_1/\rho_2$ in $U_x \setminus bD$, we may define a function h defined in the open set $\cup_{x \in bD} U_x$ such that $\rho_1(z) = h(z)\rho_2(z)$ for all $z \in \cup_{x \in bD} U_x$. In the same way we can extend the definition of h for all $z \in U$ to define $h(z) = \rho_1(z)/\rho_2(z)$ for $z \in U \setminus \cup_{x \in bD} U_x$.

If $k \ge 2$, (2) follows from the product rule. If $k = 1$ and $x \in dD \cap U$, then from the definition of differentiation of ρ_1 in 0 we have $\rho_1(x + v) = d\rho_1(x)v + R(v)$ for some function R such that $\lim_{v \to 0} R(v)/|v| = 0$. By (1), we obtain $d\rho_2(x) = d\rho_1(x)/h(x)$ for all $x \in bD \cap U$. ■

Continuing the discussion about the domain of $\mathrm{Dom}(\bar{\partial}^*_\phi)$, we can see that $C^1_{(p,q)}(\overline{D}) \subset \mathrm{Dom}(\bar{\partial})$, however not every element in $C^\infty_{(p,q)}(\overline{D})$ is in $\mathrm{Dom}(\bar{\partial}^*_\phi)$. The next result is given in [4, Lemma 4.2.1].

Lemma 2.2.3 *Let D be a bounded domain with C^1 boundary bD and ρ be a C^1 defining function for D. For any $f \in \mathrm{Dom}(\bar{\partial}^*_\phi) \cap C^1_{(p,q)}(\bar{D})$, where $\phi \in C(\bar{D})$, f must satisfy the boundary condition*

$$\sigma(\vartheta, d\rho)f(z) = 0, \qquad z \in bD, \tag{2.15}$$

where $\sigma(\vartheta, d\rho)f(z) = \vartheta(\rho f)(z)$ denotes the symbol of ϑ in the $d\rho$ direction evaluated at z. More explicitly, if f is expressed as in (2.6), then f must satisfy

$$\sum_j f_{I,jK}\frac{\partial\rho}{\partial z_j} = 0 \qquad \text{on } bD \text{ for all } I, K, \tag{2.16}$$

where $|I| = p$ and $|K| = q-1$.

Proof. We first assume that $\phi = 0$. By (2.10) we have

$$\begin{aligned}\vartheta(\rho f) &= (-1)^{p+1}\sum_{I,K}{}'\sum_{j=1}^{n}\frac{\partial}{\partial z_j}(\rho f_{I,jK})\,dz^I\wedge d\bar z^K\\ &= (-1)^{p+1}\sum_{I,K}{}'\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}f_{I,jK}dz^I\wedge d\bar z^K + (-1)^{p+1}\sum_{I,K}{}'\sum_{j=1}^{n}\rho\frac{\partial f_{I,jK}}{\partial z_j}dz^I\wedge d\bar z^K\ ,\end{aligned}$$

and (2.15) and (2.16) are equivalent. Now, we can see that the condition given does not depend to the choice of ρ as follows. If ρ_1 is other defining function for D, by Lemma 2.2.2, then there is a continuous positive function h defined in a open neighborhood of D such that $\rho_1 = h\rho$ in D and $d\rho_1 = hd\rho$ on bD. Following the above computations

$$\begin{aligned}\vartheta(\rho_1 f) = \vartheta(\rho_1 f) &= (-1)^{p+1}\sum_{I,K}{}'\sum_{j=1}^{n}\frac{\partial}{\partial z_j}(\rho_1 f_{I,jK})\,dz^I\wedge d\bar z^J\\ &= (-1)^{p+1}\sum_{I,K}{}'\sum_{j=1}^{n}h\frac{\partial}{\partial z_j}(\rho f_{I,jK})\,dz^I\wedge d\bar z^J\\ &= (-1)^{p+1}\sum_{I,K}{}'\sum_{j=1}^{n}h\frac{\partial\rho}{\partial z_j}f_{I,jK}dz^I\wedge d\bar z^J + (-1)^{p+1}\sum_{I,K}{}'\sum_{j=1}^{n}h\rho\frac{\partial f_{I,jK}}{\partial z_j}dz^I\wedge d\bar z^J\\ &= h\vartheta(\rho f),\end{aligned}$$

then (2.15) and (2.16) are independent of the defining function. With this, we can assume that ρ is such that $|d\rho| = 1$ on bD.

First, assume f be $(0,1)$-form and $f = \sum_{j=1}^n f_j d\bar z_j$. Using integration by parts and (2.10), we have, for any $\psi \in C^\infty(\bar D) \subset \mathrm{Dom}(\bar\partial)$, that

$$\begin{aligned}(\vartheta f, \psi) &= \sum_{j=1}^{n}\left(-\frac{\partial f_j}{\partial z_j}, \psi\right)\\ &= \sum_{j=1}^{n}\left(f_j, \frac{\partial\psi}{\partial\bar z_j}\right) - \sum_{j=1}^{n}\int_{bD} f_j\frac{\partial\rho}{\partial\bar z_j}, \bar\psi dS\\ &= \left(f, \bar\partial\psi\right) + \int_{bD}(\sigma(\vartheta, d\rho)f, \psi)\,dS,\end{aligned}$$

where dS is the surface measure of bD (that is). Similarly, for a (p,q)-form f and $\psi \in C^\infty_{(p,q-1)}(\bar D) \subset \mathrm{Dom}(\bar\partial)$, using integration by parts, we obtain

$$(\vartheta f, \psi) = \left(f, \bar\partial\psi\right) + \int_{bD}(\sigma(\vartheta, d\rho)f, \psi)\,dS. \tag{2.17}$$

Since $f \in \mathrm{Dom}(\bar\partial^*)$, $D_{(p,q-1)}(D)$ is dense in $L^2_{p,q-1}(D)$ and $\bar\partial^* f = \vartheta f$ in $D_{(p,q-1)}(D)$, we must have $(\vartheta f, \psi) = \left(\bar\partial^* f, \psi\right) = \left(f, \bar\partial\psi\right)$ for any $\psi \in C^\infty_{p,q-1}(\overline{D})$ and so

$$\int_{bD} (\sigma(\vartheta, d\rho) f, \psi)\, dS = 0 \qquad \text{for all } \psi \in C^\infty_{(p,q-1)}(\bar D).$$

This implies that $\sigma(\vartheta, d\rho) f(z) = 0$ for all $z \in bD$.

The case $\phi \neq 0$ can be proved similarly, using instead of (2.17), the following

$$(\vartheta_\phi f, \psi)_\phi = \left(f, \bar\partial\psi\right)_\phi + \int_{bD} (\sigma(\vartheta, d\rho) f, \psi)\, e^{-\phi} dS. \quad \blacksquare$$

Remark. Reciprocally, if $f \in C^1_{(p,q)}(\overline{D})$ and satisfies one of the equivalent conditions (2.15) or (2.16), by (2.17) (and assuming Ω with at least C^2 boundary), then $f \in \mathrm{Dom}(\bar\partial^*_\phi)$ by the density Lemma 2.3.2 below. With this, a function $f \in C^1_{(p,q)}(\bar D)$ is in $\mathrm{Dom}(\bar\partial^*_\phi)$ if and only if f satisfies one of the equivalent conditions (2.15) or (2.16). Also, observe that the arguments in the proof of this lemma works for elements in $C^\ell_{p,q}(\bar D)$ with $\ell \geq 1$.

Another way to express condition (2.15) or (2.16) is as follows. Let $\vee$ the interior product defined as the dual of the wedge product with $\bar\partial\rho$. For an (p,q)-form f, $\bar\partial\rho \vee f$ is defined as the $(p, q-1)$-form satisfying

$$\left(g \wedge \bar\partial\rho, f\right) = \left(g, \bar\partial\rho \vee f\right), \quad g \in C^\infty_{(p,q-1)}(\mathbb{C}^n).$$

Using this notation, condition (2.15) or (2.16) can be expressed as

$$\bar\partial\rho \vee f = 0 \text{ on } bD. \tag{2.18}$$

Then $f \in C^1_{(p,q)}(\bar D) \cap \mathrm{Dom}(\bar\partial^*_\phi)$ if and only if f satisfies one of the three equivalent conditions (2.15), (2.16) or (2.18).

Now, for fixed $0 \leq p \leq n$, $1 \leq q \leq n$, we define the Laplacian of the $\bar\partial$ complex.

Definition 2.2.4 *Let $\square_{(p,q)} = \bar\partial_{(p,q-1)}\bar\partial^*_{(p,q)} + \bar\partial^*_{(p,q)}\bar\partial_{(p,q-1)}$ be the operator from $L^2_{(p,q)}(D)$ to $L^2_{(p,q)}(D)$ such that*

$$\begin{aligned}\mathrm{Dom}(\square_{(p,q)}) = \{f \in L^2_{(p,q)}(D) :\ & f \in \mathrm{Dom}(\bar\partial_{(p,q)}) \cap \mathrm{Dom}(\bar\partial^*_{(p,q)}),\\ & \bar\partial_{(p,q)} f \in \mathrm{Dom}(\bar\partial^*_{(p,q+1)}),\ \bar\partial^*_{(p,q)} f \in \mathrm{Dom}(\bar\partial_{(p,q-1)})\}.\end{aligned}$$

Proposition 2.2.5 *$\square_{(p,q)}$ is a linear, closed, densely defined self-adjoint and positive operator.*

Proof. Clearly $\Box_{(p,q)}$ is linear. $\Box_{(p,q)}$ is densely defined because $\mathrm{Dom}(\Box_{(p,q)})$ contains $D_{(p,q)}(D)$.

Now we prove that $\Box_{(p,q)}$ is closed. Let $(f_n) \in \mathrm{Dom}(\Box_{(p,q)})$, $f \in L^2_{(p,q)}(D)$ such that $f_n \to f$ in $L^2_{(p,q)}(D)$ and $\Box_{(p,q)} f_n$ converge in $L^2_{(p,q)}(D)$. Since $f_n \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\left(\Box_{(p,q)} f_n, f_n\right) &= \left(\bar\partial\bar\partial^* f_n, f_n\right) + \left(\bar\partial^*\bar\partial f_n, f_n\right) \left(\bar\partial^* f_n, \bar\partial^* f_n\right) + \left(\bar\partial f_n, \bar\partial f_n\right) \\ &= \left\|\bar\partial^* f_n\right\|^2 + \left\|\bar\partial f_n\right\|^2 \geq 0.\end{aligned}$$

Since $\Box_{(p,q)} f_n$ and f_n converge in $L^2_{(p,q)}(D)$ we have that $\bar\partial^* f_n$ and $\bar\partial f_n$ converge in $L^2_{p,q-1}(D)$ and $L^2_{p,q+1}(D)$ respectively. Now, $f \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$ and

$$\bar\partial f_n \to \bar\partial f, \quad \bar\partial^* f_n \to \bar\partial^* f \tag{2.19}$$

in $L^2_{p,q}(D)$, because $\bar\partial$ and $\bar\partial^*$ are closed operators.

Also for $h \in \mathrm{Dom}(\Box_{(p,q)})$

$$\begin{aligned}\left\|\Box_{(p,q)} h\right\|^2 &= \left(\bar\partial\bar\partial^* h + \bar\partial^*\bar\partial h, \bar\partial\bar\partial^* h + \bar\partial^*\bar\partial h\right) \\ &= \left\|\bar\partial\bar\partial^* h\right\|^2 + \left\|\bar\partial^*\bar\partial h\right\|^2 + \\ &\quad + \left(\bar\partial\bar\partial^* h, \bar\partial^*\bar\partial h\right) + \left(\bar\partial^*\bar\partial h, \bar\partial\bar\partial^* h\right) \\ &= \left\|\bar\partial\bar\partial^* h\right\|^2 + \left\|\bar\partial^*\bar\partial h\right\|^2,\end{aligned}$$

we have that $\bar\partial\bar\partial^* f_n$ and $\bar\partial^*\bar\partial f_n$ are Cauchy sequences, then, they converge. Again, since $\bar\partial$ and $\bar\partial^*$ are closed, using (2.19) we have that $\bar\partial f \in \mathrm{Dom}(\bar\partial^*)$, $\bar\partial^* f \in \mathrm{Dom}(\bar\partial)$ and

$$\bar\partial\bar\partial^* f_n \to \bar\partial\bar\partial^* f \quad \text{and} \quad \bar\partial^*\bar\partial f_n \to \bar\partial^*\bar\partial f \text{ in } L^2_{(p,q)}(D). \tag{2.20}$$

Then $f \in \mathrm{Dom}(\Box_{(p,q)})$ and by (2.20), $\Box_{(p,q)} f_n \to \Box_{(p,q)} f$. With this $\Box_{(p,q)}$ is a closed operator.

Now we prove that $\Box_{(p,q)}$ is a self-adjoint operator. Let $\Box^*_{(p,q)}$ be the adjoint operator of $\Box_{(p,q)}$. It is easy to see that $\Box^*_{(p,q)} = \Box_{(p,q)}$ on $\mathrm{Dom}(\Box_{(p,q)}) \cap \mathrm{Dom}(\Box^*_{(p,q)})$, because if $f \in \mathrm{Dom}(\Box_{(p,q)}) \cap \mathrm{Dom}(\Box^*_{(p,q)})$, and $u \in \mathrm{Dom}(\Box_{(p,q)})$

$$\begin{aligned}\left(\Box^*_{(p,q)} f, u\right) &= \left(f, \Box_{(p,q)} u\right) = \left(f, \bar\partial\bar\partial^* u\right) + \left(f, \bar\partial^*\bar\partial u\right) \\ &= \left(\bar\partial^* f, \bar\partial^* u\right) + \left(\bar\partial f, \bar\partial u\right) \\ &= \left(\bar\partial\bar\partial^* f, u\right) + \left(\bar\partial^*\bar\partial f, u\right) = \left(\Box_{(p,q)} f, u\right).\end{aligned}$$

In order to prove that $\mathrm{Dom}(\square_{(p,q)}) = \mathrm{Dom}(\square^*_{(p,q)})$ we consider the operator

$$L_1 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} + I = \square_{(p,q)} + I$$

defined on $\mathrm{Dom}(\square_{(p,q)})$, and we will prove that L_1 is self-adjoint.

Since $\bar{\partial}$ and $\bar{\partial}^*$ are closed and densely defined operators, by a theorem of Von Neumann (see Section 118 in [26]), the operators

$$(I + \bar{\partial}\bar{\partial}^*)^{-1} \quad \text{and} \quad (I + \bar{\partial}^*\bar{\partial})^{-1}$$

are defined everywhere, bounded and self-adjoint.

We define $Q_1 = (I + \bar{\partial}\bar{\partial}^*)^{-1} + (I + \bar{\partial}^*\bar{\partial})^{-1} - I$, then Q_1 is defined everywhere, self-adjoint bounded operator. Note that

$$\begin{aligned}(I + \bar{\partial}\bar{\partial}^*)^{-1} - I &= (I - (I + \bar{\partial}\bar{\partial}^*))(I + \bar{\partial}\bar{\partial}^*)^{-1}\\ &= -\bar{\partial}\bar{\partial}^*(I + \bar{\partial}\bar{\partial}^*)^{-1}\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}(I + \bar{\partial}^*\bar{\partial})^{-1} - I &= (I - (I + \bar{\partial}^*\bar{\partial}))(I + \bar{\partial}^*\bar{\partial})^{-1}\\ &= -\bar{\partial}^*\bar{\partial}(I + \bar{\partial}^*\bar{\partial})^{-1},\end{aligned} \tag{2.22}$$

and (2.21), (2.22) are true everywhere, then $\mathrm{Ran}(I + \bar{\partial}\bar{\partial}^*)^{-1} \subset \mathrm{Dom}(\bar{\partial}\bar{\partial}^*)$ and $\mathrm{Ran}(I + \bar{\partial}^*\bar{\partial})^{-1} \subset \mathrm{Dom}(\bar{\partial}^*\bar{\partial})$. Besides, we have

$$Q_1 = (I + \bar{\partial}^*\bar{\partial})^{-1} - \bar{\partial}\bar{\partial}^*(I + \bar{\partial}\bar{\partial}^*)^{-1}, \tag{2.23}$$

$$Q_1 = (I + \bar{\partial}\bar{\partial}^*)^{-1} - \bar{\partial}^*\bar{\partial}(I + \bar{\partial}^*\bar{\partial})^{-1}. \tag{2.24}$$

By (2.23) and since $\bar{\partial}^2 = 0$, we see that $\mathrm{Ran}(\bar{\partial}\bar{\partial}^*(I + \bar{\partial}\bar{\partial}^*)^{-1}) \subset \mathrm{Dom}(\bar{\partial}^*\bar{\partial})$, so $\mathrm{Ran}(Q_1) \subset \mathrm{Dom}(\bar{\partial}^*\bar{\partial})$ and

$$\bar{\partial}^*\bar{\partial}Q_1 = \bar{\partial}^*\bar{\partial}(I + \bar{\partial}^*\bar{\partial})^{-1}. \tag{2.25}$$

In the same way, by (2.24) since $(\bar{\partial}^*)^2 = 0$, we see that $\mathrm{Ran}(\bar{\partial}^*\bar{\partial}(I + \bar{\partial}^*\bar{\partial})^{-1}) \subset \mathrm{Dom}(\bar{\partial}\bar{\partial}^*)$, so $\mathrm{Ran}(Q_1) \subset \mathrm{Dom}(\bar{\partial}\bar{\partial}^*)$, so

$$\bar{\partial}\bar{\partial}^*Q_1 = \bar{\partial}\bar{\partial}^*(I + \bar{\partial}\bar{\partial}^*)^{-1}. \tag{2.26}$$

Then $\mathrm{Ran}(Q_1) \subset \mathrm{Dom}(L_1)$ and by (2.25), (2.26) and (2.23) we have

$$\begin{aligned}L_1Q_1 &= \bar{\partial}\bar{\partial}^*(I + \bar{\partial}\bar{\partial}^*)^{-1} + \bar{\partial}^*\bar{\partial}(I + \bar{\partial}^*\bar{\partial})^{-1} + Q_1\\ &= \bar{\partial}^*\bar{\partial}(I + \bar{\partial}^*\bar{\partial})^{-1} + (I + \bar{\partial}^*\bar{\partial})^{-1}\\ &= (\bar{\partial}^*\bar{\partial} + I)(I + \bar{\partial}^*\bar{\partial})^{-1}\end{aligned}$$

$$= I. \tag{2.27}$$

Moreover L_1 is injective, because, if $u \in \mathrm{Dom}(L_1)$ and $L_1(u) = 0$, then the computations

$$\begin{aligned} 0 = (L_1 u, u) &= \left(\bar\partial\bar\partial^* u + \bar\partial^*\bar\partial u + u, u\right) \\ &= \left(\bar\partial\bar\partial^* u, u\right) + \left(\bar\partial^*\bar\partial u, u\right) + \|u\|^2 \\ &= \left(\bar\partial^* u, \bar\partial^* u\right) + \left(\bar\partial u, \bar\partial u\right) + \|u\|^2 \\ &= \left\|\bar\partial^* u\right\|^2 + \left\|\bar\partial u\right\|^2 + \|u\|^2 \end{aligned}$$

shows that $u = 0$. With this and (2.27) we have $Q_1 = L_1^{-1}$. Then L_1 is selfadjoint. Then $L_1 - I = \Box_{(p,q)}$ is self-adjoint. As we wished to prove. ∎

The next proposition give us two necessary boundary conditions for (p,q)-forms belong to $\mathrm{Dom}(\Box_{(p,q)})$, they are namely, the $\bar\partial$-Neumann boundary conditions.

Proposition 2.2.6 *Let D be a bounded domain with C^1 boundary and ρ be a C^1 defining function. If $f \in C^2_{(p,q)}(\bar D)$ then $f \in \mathrm{Dom}(\Box_{(p,q)})$ if and only if*

$$\sigma(\vartheta, d\rho) f = 0 \quad \text{and} \quad \sigma(\vartheta, d\rho)\bar\partial f = 0 \quad \text{on } bD. \tag{2.28}$$

If $f = \sum_{I,J}{}' f_{I,J} dz^I d\bar z^J \in C^2_{(p,q)}(\bar D) \cap \mathrm{Dom}(\Box_{(p,q)})$, we have

$$\Box_{(p,q)} f = -\frac{1}{4} \sum_{I,J}{}' \Delta f_{I,J} dz^I \wedge d\bar z^J, \tag{2.29}$$

where $\Delta = 4\sum_{k=1}^n \partial^2/\partial z_k \partial\bar z_k = \sum_{k=1}^n (\partial^2/\partial x_k^2 + \partial^2/\partial y_k^2)$ is the usual Laplacian on functions.

Proof. If $f \in \mathrm{Dom}(\Box_{(p,q)})$ then $f \in \mathrm{Dom}(\bar\partial^*)$ and $\bar\partial f \in \mathrm{Dom}(\bar\partial^*)$ and by Lemma 2.2.3 we obtain (2.28). Conversely, if $f \in C^2_{(p,q)}(\bar D)$ and satisfies (2.28), by observation after Lemma 2.2.3 we have $f \in \mathrm{Dom}(\bar\partial^*)$ and $\bar\partial f \in \mathrm{Dom}(\bar\partial^*)$. Since $f \in C^2_{(p,q)}(\bar D)$ and $f \in \mathrm{Dom}(\bar\partial^*)$, $\bar\partial^* f = \vartheta f \in C^1_{(p,q)}(\bar D)$, so $\bar\partial^* f \in \mathrm{Dom}(\bar\partial)$. And obviously $f \in \mathrm{Dom}(\bar\partial)$. Then $f \in \mathrm{Dom}(\Box_{(p,q)})$.

Let's prove (2.29) for $p = 0$ (the proof for $p \neq 0$ is similar). In this proof, J and R are the multiindices of length q, K and S are multiindices of length $q-1$ and $q+1$ respectively. If $f \in \mathrm{Dom}(\Box_{(p,q)})$ we have $\bar\partial^* f = \vartheta f$ and $\bar\partial^*\bar\partial f = \vartheta\bar\partial f$, because $f \in \mathrm{Dom}(\bar\partial^*)$ and $\bar\partial f \in \mathrm{Dom}(\bar\partial^*)$. With this, if f is written as in this proposition, we have

$$\bar\partial\vartheta f = -\sum_K{}' \sum_j \sum_l \frac{\partial^2 f_{lK}}{\partial\bar z_j \partial z_l} d\bar z_j \wedge d\bar z^K$$

$$
\begin{aligned}
&= -{\sum_K}' \sum_j \frac{\partial^2 f_{jK}}{\partial \bar z_j \partial z_j} d\bar z_j \wedge d\bar z^K - {\sum_K}' \sum_j \left\{ \sum_{l\neq j} \frac{\partial^2 f_{lK}}{\partial \bar z_j \partial z_l} \right\} d\bar z_j \wedge d\bar z^K \\
&= -{\sum_J}' \left\{ \sum_{j\in J} \frac{\partial^2 f_J}{\partial \bar z_j \partial z_j} \right\} d\bar z^J - {\sum_K}' \sum_j \left\{ \sum_{l\neq j} \frac{\partial^2 f_{lK}}{\partial \bar z_j \partial z_l} \right\} d\bar z_j \wedge d\bar z^K \\
&= -{\sum_J}' \left\{ \sum_{j\in J} \frac{\partial^2 f_J}{\partial \bar z_j \partial z_j} \right\} d\bar z^J - {\sum_R}' \left\{ {\sum_{\substack{K \\ K\cup\{j\}=R}}}' \sum_j \left\{ \epsilon^R_{jK} \sum_{l\neq j} \frac{\partial^2 f_{lK}}{\partial \bar z_j \partial z_l} \right\} \right\} d\bar z^R,
\end{aligned} \tag{2.30}
$$

where ϵ^R_{jK} is 0 if $\{j\}\cup K \neq R$ as sets and is the sign of the permutation that reorders jK as R. Let $\{j,J\}'$ denote the multiindex in increasing order with elements in $\{j,J\}$, for some $j\notin J$. Before to proceed in the calculation of $\vartheta\bar\partial f$, note that for any smooth function g and multiindex S

$$
\vartheta(g d\bar z^S) = -{\sum_R}' \left\{ \sum_l \epsilon^S_{lR} \frac{\partial g}{\partial z_l} \right\} d\bar z^R .
$$

Then

$$
\begin{aligned}
\vartheta\bar\partial f &= {\sum_J}' \sum_j \vartheta\{ \frac{\partial f_J}{d\bar z_j} d\bar z_j \wedge d\bar z^J \} = {\sum_J}' \sum_j \epsilon^{\{j,J\}'}_{jJ} \vartheta\{ \frac{\partial f_J}{d\bar z_j} d\bar z^{\{j,J\}'} \} \\
&= -{\sum_J}' \sum_j \epsilon^{\{j,J\}'}_{jJ} \left\{ {\sum_R}' \sum_l \epsilon^{\{j,J\}'}_{lR} \frac{\partial^2 f_J}{\partial z_l \partial \bar z_j} d\bar z^R \right\} \\
&= -{\sum_R}' \left\{ {\sum_J}' \sum_j \sum_l \epsilon^{\{j,J\}'}_{jJ} \epsilon^{\{j,J\}'}_{lR} \frac{\partial^2 f_J}{\partial z_l \partial \bar z_j} \right\} d\bar z^R \\
&= -{\sum_R}' \left\{ \sum_{j\notin R} \frac{\partial f_R}{\partial z_j \partial \bar z_j} \right\} d\bar z^R - {\sum_R}' \left\{ {\sum_J}' \sum_{\substack{j \\ j\neq l}} \sum_l \epsilon^{\{j,J\}'}_{jJ} \epsilon^{\{j,J\}'}_{lR} \frac{\partial^2 f_J}{\partial z_l \partial \bar z_j} \right\} d\bar z^R .
\end{aligned} \tag{2.31}
$$

The first term on the right in this last equality arises from collecting terms where $j=l$, and also from the fact the sum on index j is for $j\notin J$.

Now we claim that the second term in (2.30) is equal to minus the second term in (2.31). To show this, it is enough to inspect the coefficients of $d\bar z^R$. For a fixed R, we proceed to compute the coefficient of the second term in (2.31). Notice first that if $l\neq j$ then $\epsilon^{\{j,J\}'}_{lR}=0$ when $j\notin R$. Fixed $j_0\in R$; for $l\neq j_0$, $\epsilon^{\{j_0,J\}'}_{lR}\neq 0$ just for l and J such that $l\in J$ and $R\setminus\{j_0\} = J\setminus\{l\}$. Denote K_0 the unique multiindex in increasing order such that $\{K_0\} = R\setminus\{j_0\}$. Then

$$
{\sum_J}' \sum_{\substack{l \\ l\neq j_0}} \epsilon^{\{j_0,J\}'}_{j_0 J} \epsilon^{\{j_0,J\}'}_{lR} \frac{\partial^2 f_J}{\partial z_l \partial \bar z_{j_0}} = -\sum_{\substack{l \\ l\neq j_0}} \epsilon^R_{j_0 K_0} \frac{\partial^2 f_{lK_0}}{\partial z_l \partial \bar z_{j_0}} . \tag{2.32}
$$

because the sum in the first member of this last equality is reduced for multiindices J containing K_0, and also $\epsilon_{j_0J}^{\{j_0,J\}'}\epsilon_{lR}^{\{j_0,J\}'} = -\epsilon_{j_0K_0}^{R}\epsilon_{lK_0}^{J}$. The second member in (2.32) is equal to the sum of terms inside of

$$-\sum_{\substack{K\\ K\cup\{j\}=R}}{}' \sum_{j} \left\{ \epsilon_{jK}^{R} \sum_{l\neq j} \frac{\partial^2 f_{lK}}{\partial \bar z_j \partial z_l} \right\}$$

when $j = j_0$ and $K = K_0$. With this we proved our claim. Adding (2.30) and (2.31), we obtain

$$\square_{(0,q)} f = -\sum_{J}{}' \sum_{j=1}^{n} \frac{\partial^2 f_J}{\partial z_j \partial \bar z_j} d\bar z^J = -\frac{1}{4}\sum_{J}{}' \Delta f_J d\bar z^J. \qquad \blacksquare$$

Example Let D be a smooth bounded domain in $\mathbb{C}^n$ such that $0 \in bD$. We assume that for some neighborhood U of 0, $D \cap U = \{Im\ z_n = y_n < 0\} \cap U$. Let $f = \sum_k f_k d\bar z_k \in C^2_{(0,1)}(\overline{D})$ and the support of f lies in $U \cap \bar D$. Then f is in $\mathrm{Dom}(\square_{(0,1)})$ if, and only if satisfies

(a) $f_n = 0$ on $bD \cap U$,

(b) $\frac{\partial f_j}{\partial \bar z_n} = 0$ on $bD \cap U$, $j = 1, ..., n-1$.

If we consider the defining function ρ of D such that $\rho(z) = Im\ z_n = y_n$ in $D \cap U$, since $\partial/\partial z_j$ for $1 \le j \le n-1$ are tangential to $bD \cap U$, that is, $\partial\rho/\partial z_j = 0$ for $j = 1, ..., n-1$, and $\frac{\partial \rho}{\partial z_n}(z) = -i$ on $bD \cap U$, we have (a) from the Lemma 2.2.3. In the same way, since

$$\begin{aligned} \bar\partial f &= \sum_{k=1}^{n} \sum_{j=1,\ j\neq k}^{n} \frac{\partial f_k}{\partial \bar z_j} d\bar z_j \wedge d\bar z_k \\ &= \sum_{1\le j<k\le n} \left\{ \frac{\partial f_k}{\partial \bar z_j} - \frac{\partial f_j}{\partial \bar z_k} \right\} d\bar z_j \wedge d\bar z_k \end{aligned}$$

then

$$\frac{\partial f_k}{\partial \bar z_n} - \frac{\partial f_n}{\partial \bar z_k} = 0 \qquad \forall k = 1, ..., n-1 \text{ on } bD \cap U.$$

By (a), and since $\partial/\partial z_j$ for $1 \le j \le n-1$ are tangential to $bD \cap U$ we have $\partial f_n/\partial \bar z_j(z) = 0$ for $z \in bD \cap U$ for $j = 1, ..., n-1$. Then we obtain (b).

2.3 L^2 existence theorems for $\bar\partial$ in pseudoconvex domains

Let D be a domain with C^2 boundary bD. Let ρ be a C^2 defining function in a neighborhood of D such that $D = \{z|\rho(z) < 0\}$ and $|d\rho| = 1$ on bD. For each $l \in \mathbb{N}$, we set

$$\mathfrak{D}^l_{(p,q)} = \mathrm{Dom}(\bar\partial^*) \cap C^l_{(p,q)}(\bar D)$$

and

$$\mathfrak{D}_{(p,q)} = \mathrm{Dom}(\bar\partial^*) \cap C^\infty_{(p,q)}(\bar D).$$

Let $\phi \in C^2(\bar{D})$ be a fixed function and let

$$\mathfrak{D}^{\phi}_{(p,q)} = \mathrm{Dom}(\bar{\partial}^*_{\phi}) \cap C^{\infty}_{(p,q)}(\bar{D}).$$

By Lemma 2.2.3 (and the remark done after) we have that, $f \in \mathfrak{D}^{\phi}_{(p,q)}$ if and only if $\sigma(\vartheta, d\rho)f(z) = 0$ for any $z \in bD$, a condition independent of ϕ. So $\mathfrak{D}^{\phi}_{(p,q)} = \mathfrak{D}_{(p,q)}$, which is also independent of ϕ. Similarly we have that $\mathrm{Dom}(\bar{\partial}^*_{\phi}) \cap C^{l}_{(p,q)}(\bar{D}) = \mathfrak{D}^{l}_{(p,q)}$.

Let Q^{ϕ} be the form on $\mathfrak{D}_{(p,q)}$ defined by

$$Q^{\phi}(f,f) = \|\bar{\partial} f\|^2_{\phi} + \|\bar{\partial}^*_{\phi} f\|^2_{\phi}.$$

Proposition 2.3.1 (Morrey-Kohn-Hörmander's Identity) *Let $D \subset \mathbb{C}^n$ be a domain with C^2 boundary bD and ρ be a C^2 defining function for D such that $|d\rho| = 1$ on bD. Let $\phi \in C^2(\bar{D})$. For any $f = \sum_{|I|=p,|J|=q}{}' f_{I,J} dz^I d\bar{z}^J \in \mathfrak{D}^{l}_{(p,q)}$,*

$$\begin{aligned}
Q^{\phi}(f,f) &:= \|\bar{\partial} f\|^2_{\phi} + \|\vartheta_{\phi} f\|^2_{\phi} \\
&= \sum_{|I|=p,|K|=q-1}{}' \sum_{i,j} \int_D \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} f_{I,iK} \bar{f}_{I,jK} e^{-\phi} dV \\
&\quad + \sum_{|I|=p,|J|=q}{}' \sum_{k} \int_D \left| \frac{\partial f_{I,J}}{\partial \bar{z}_k} \right|^2 e^{-\phi} dV \\
&\quad + \sum_{|I|=p,|K|=q-1}{}' \sum_{i,j} \int_{bD} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} f_{I,iK} \bar{f}_{I,jK} e^{-\phi} dS. \qquad (2.33)
\end{aligned}$$

Proof. Let $\delta^{\phi}_j u = e^{\phi} \frac{\partial}{\partial z_j}(e^{-\phi} u)$ and $\bar{L}_j = \partial/\partial \bar{z}_j$. Then

$$\|\bar{\partial} f\|^2_{\phi} + \|\vartheta_{\phi} f\|^2_{\phi} = \sum_{I,J,L}{}' \sum_{j,l} \epsilon^{jJ}_{lL} \left(\bar{L}_j(f_{I,J}), \bar{L}_l(f_{I,L})\right)_{\phi} + \sum_{I,K}{}' \sum_{j,k} \left(\delta^{\phi}_j f_{I,jK}, \delta^{\phi}_k f_{I,kK}\right)_{\phi}, \quad (2.34)$$

where $\epsilon^{jJ}_{lL} = 0$, unless $j \notin J$, $l \notin L$ and $\{j\} \cup J = \{l\} \cup L$, in which case ϵ^{jJ}_{lL} is the sign of permutation $\binom{jJ}{lL}$. Rearranging the terms in (2.34) gives

$$\begin{aligned}
\|\bar{\partial} f\|^2_{\phi} + \|\vartheta_{\phi} f\|^2_{\phi} = &\sum_{I,J}{}' \sum_{j} \|\bar{L}_j f_{I,J}\|^2_{\phi} - \sum_{I,K}{}' \sum_{j,k} \left(\bar{L}_k f_{I,jK}, \bar{L}_j f_{I,kK}\right)_{\phi} \\
&+ \sum_{I,K}{}' \sum_{j,k} \left(\delta^{\phi}_j f_{I,jK}, \delta^{\phi}_k f_{I,kK}\right)_{\phi}. \qquad (2.35)
\end{aligned}$$

Note that, if $u,\ v \in C^2(\bar{D})$ applying integration by parts we have

$$\left(u, \delta^{\phi}_j v\right)_{\phi} = -\left(\bar{L}_j u, v\right)_{\phi} + \int_{bD} \frac{\partial \rho}{\partial \bar{z}_j} u \bar{v} e^{-\phi} dS$$

and

$$\left[\delta^{\phi}_j, \bar{L}_k\right] u = \delta^{\phi}_j \bar{L}_k u - \bar{L}_k \delta^{\phi}_j u = u \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}.$$

So,

$$
\begin{aligned}
\left(\delta_j^\phi u, \delta_k^\phi v\right)_\phi &= \left(-\bar{L}_k \delta_j^\phi u, v\right)_\phi + \int_{bD} \frac{\partial \rho}{\partial \bar{z}_k} (\delta_j^\phi u) \bar{v} e^{-\phi} dS \\
&= \left(-\delta_j^\phi \bar{L}_k u, v\right)_\phi + \left(\left[\delta_j^\phi, \bar{L}_k\right] u, v\right)_\phi + \int_{bD} \frac{\partial \rho}{\partial \bar{z}_k} (\delta_j^\phi u) \bar{v} e^{-\phi} dS \\
&= -\overline{\left(v, \delta_j^\phi \bar{L}_k u\right)_\phi} + \left(u \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}, v\right)_\phi + \int_{bD} \frac{\partial \rho}{\partial \bar{z}_k} (\delta_j^\phi u) \bar{v} e^{-\phi} dS \\
&= \left(\bar{L}_k u, \bar{L}_j v\right)_\phi + \left(u \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}, v\right)_\phi \\
&\quad + \int_{bD} \frac{\partial \rho}{\partial \bar{z}_k} (\delta_j^\phi u) \bar{v} e^{-\phi} dS - \int_{bD} \frac{\partial \rho}{\partial z_j} (\bar{L}_k u) \bar{v} e^{-\phi} dS.
\end{aligned} \tag{2.36}
$$

When u, v are in $C^1(\bar{D})$, (2.36) also holds by approximation (convolution with a sequence of mollifiers) since $C^2(\bar{D})$ is a dense subset in $C^1(\bar{D})$ (still the regularization of u and v convergence uniform in compacts to u and v, and so also their derivatives). Using (2.36) for each I, K, it follows that

$$
\begin{aligned}
\sum_{j,k} \left(\delta_j^\phi f_{I,jK}, \delta_k^\phi f_{I,kK}\right)_\phi = \sum_{j,k=1}^n \left(\bar{L}_k f_{I,jK}, \bar{L}_j f_{I,kK}\right)_\phi &+ \sum_{j,k=1}^n \left(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} f_{I,jK}, f_{I,kK}\right)_\phi \\
&+ \sum_{j,k=1}^n \int_{bD} \frac{\partial \rho}{\partial \bar{z}_k} (\delta_j^\phi f_{I,jK}) \bar{f}_{I,kK} e^{-\phi} dS \\
&- \sum_{j,k=1}^n \int_{bD} \frac{\partial \rho}{\partial z_j} (\bar{L}_k f_{I,jK}) \bar{f}_{I,kK} e^{-\phi} dS.
\end{aligned} \tag{2.37}
$$

Since $f \in \mathfrak{D}^1_{(p,q)}$, by (2.16) in Lemma 2.2.3 we have

$$
\sum_{k=1}^n \frac{\partial \rho}{\partial \bar{z}_k} \bar{f}_{I,kK} = 0 \quad \text{on } bD \tag{2.38}
$$

for each I, K. Thus $\sum_{k=1}^n \bar{f}_{I,kK} \frac{\partial}{\partial \bar{z}_k}$ is tangential to bD, then

$$
\sum_{k=1}^n \bar{f}_{I,kK} \frac{\partial}{\partial \bar{z}_k} \left(\sum_{j=1}^n f_{I,jK} \frac{\partial \rho}{\partial z_j}\right) = 0
$$

on bD for each I, K. This implies

$$
\sum_{k=1}^n \sum_{j=1}^n \bar{f}_{I,kK} \frac{\partial \rho}{\partial z_j} \frac{\partial f_{I,jK}}{\partial \bar{z}_k} + \sum_{k=1}^n \sum_{j=1}^n f_{I,jK} \bar{f}_{I,kK} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} = 0 \tag{2.39}
$$

on bD. Combining (2.35)-(2.39), we proved (2.33). ∎

We invoke the next density lemma, whose proof could be find it on [4] as Lemma 4.3.2, or on [30] as Proposition 2.3.

Lemma 2.3.2 *Let D be a domain with C^{l+1} boundary bD, $l \geq 1$ and $\phi \in C^2(\bar{D})$. Then $\mathfrak{D}^l_{(p,q)}$ is dense in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*_\phi)$ in the graph norm*

$$f \to \|f\|_\phi + \|\bar{\partial} f\|_\phi + \|\bar{\partial}^*_\phi f\|_\phi.$$

This density result allows us to work more comfortably on forms with smooth regularity on the boundary with tangential conditions expressed on the Lemma 2.2.3, and get estimates on $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*_\phi)$, as we will see below. Now we will use a concept called pseudoconvexity, and for this reason we make the next definition.

Definition 2.3.3 *Let D be a bounded domain, with C^2 boundary in $\mathbb{C}^n$, $n \geq 2$, and let r be a C^2 defining function for D. The Levi form of the function r at the point $p \in bD$ denoted by $\mathcal{L}_p(r; t)$, is defined by the Hermitian form*

$$\mathcal{L}_p(r; t) := \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) t_j \overline{t_k} \tag{2.40}$$

for all t in

$$T^{1,0}_p(bD) := \left\{ t = (t_1, ..., t_n) \in \mathbb{C}^n : \sum_{j,k=1}^{n} t_j \left(\partial r / \partial z_j\right)(p) = 0 \right\}.$$

Definition 2.3.4 *Let D be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$, and let r be a C^2 defining function for D. D is called pseudoconvex, or Levi pseudoconvex, at $p \in bD$, if the Levi form $\mathcal{L}_p(r; t)$ is nonnegative for any t in $T^{1,0}_p(bD)$. The domain D is said to be strictly (or strongly) pseudoconvex at p, if the Levi form is positive for all such $t \neq 0$. D is called a (Levi) pseudoconvex domain if D is (Levi) pseudoconvex at every boundary point of D. D is called a strictly (or strongly) pseudoconvex domain if D is strictly (or strongly) pseudoconvex at every boundary point of D.*

We can see that this definition is clearly independent of the choice of the defining function r, because, if ρ is another C^2 defining function, then $\rho = hr$ for some C^1 function h with $h > 0$ on some open neighborhood of bD, and so for any $p \in bD$ and $t \in T^{1,0}_p(bD)$ we have

$$\begin{aligned} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) t_j \overline{t_k} &= \sum_{j,k=1}^{n} \frac{\partial r}{\partial z_j}(p) \frac{\partial h}{\partial \bar{z}_k}(p) t_j \overline{t_k} + \sum_{j,k=1}^{n} \frac{\partial h}{\partial z_j}(p) \frac{\partial r}{\partial \bar{z}_k}(p) t_j \overline{t_k} \\ &\quad + h(p) \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) t_j \overline{t_k} \\ &= h(p) \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) t_j \overline{t_k} \end{aligned}$$

that is $\mathcal{L}_p(\rho; t) = h \mathcal{L}_p(r; t)$.

We say that a C^2 real valued function φ on D is (strictly) plurisubharmonic if $\mathcal{L}_z(\varphi, t)$ is (positive) non negative for all $t = (t_1, ..., t_n) \in \mathbb{C}^n$, and all $z \in D$. When we do not have a C^2 boundary for D we define the pseudoconvexity as follows:

Definition 2.3.5 *An open domain D in $\mathbb{C}^n$, $n \geq 2$, is called pseudoconvex if there exists a smooth strictly plurisubharmonic function φ on D, such that, for any $c \in \mathbb{R}$ the set $D_c := \{x \in D : \ \varphi(x) < c\}$ is relatively compact in D. The function φ, if exists, is called exhaustion function for D.*

Proposition 2.3.6 *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary and $\phi \in C^2(\overline{D})$. For every $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*_\phi)$, we have*

$$\sum_{I,K}{}' \sum_{j,k} \int_D \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} f_{I,jK} \bar{f}_{I,kK} e^{-\phi} dV \leq \|\bar{\partial} f\|^2_\phi + \|\bar{\partial}^*_\phi f\|^2_\phi. \tag{2.41}$$

Proof. By the density lemma, Lemma 2.3.2, is sufficient to prove (2.41) for $f \in \mathfrak{D}^1_{(p,q)} = \mathrm{Dom}(\bar{\partial}^*) \cap C^1_{(p,q)}(\bar{D})$. For $f \in \mathfrak{D}^1_{(p,q)}$, f satisfies (2.16) so by the pseudoconvexity of the domain D we have

$$\int_{b\Omega} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} f_{I,iK} \overline{f_{I,jK}} e^{-\phi} dS \geq 0.$$

Then by the equality of Morrey-Kohn-Hörmander, Proposition 2.3.1, we have (2.41). ∎

The next theorem is in [14], as Theorem 2.2.3, and it shows an existence of solution for the equation (2.8) in bounded pseudoconvex domains in $\mathbb{C}^n$. Note how the closure of the range of the operators $\bar{\partial}$ and $\bar{\partial}^*_\phi$ implies on the existence of solutions for this equation.

Theorem 2.3.7 *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. For every $f \in L^2_{(p,q)}(D)$, where $0 \leq p \leq n$, $1 \leq q \leq n$ with $\bar{\partial} f = 0$, one can find $u \in L^2_{(p,q-1)}(D)$ such that $\bar{\partial} u = f$ and*

$$q \int_D |u|^2 \, dV \leq e\delta^2 \int_D |f|^2 \, dV,$$

where $\delta = \sup_{z,w \in D} |z - w|$ is the diameter of D.

Proof. First, we consider D with C^2 boundary. Without loss of generality, we may assume that $0 \in D$. On (2.41) we choice the weight function $\phi = t\,|z|^2 \in C^2(\overline{D})$. So we have

$$\|g\|_\phi = tq \int_D |g|^2 \, e^{-t|z|^2} dV \leq \|\bar{\partial} g\|^2_\phi + \|\bar{\partial}^*_\phi g\|^2_\phi.$$

Thus, if $g \in \mathrm{Dom}(\bar{\partial}^*_\phi) \cap \mathrm{Ker}(\bar{\partial}) \subset \mathrm{Dom}(\bar{\partial}^*_\phi) \cap \mathrm{Dom}(\bar{\partial})$, then

$$tq\|g\|^2_\phi \leq \|\bar{\partial}^*_\phi g\|^2_\phi. \tag{2.42}$$

By the Lemma 2.1.2 we have $\operatorname{Ran}(\bar{\partial})$ is closed in $L^2_{(p,q)}(D,\phi)$ and $\operatorname{Ran}(\bar{\partial}^*_\phi)$ is closed in $L^2_{(p,q-1)}(D,\phi)$.

We claim that: for $f \in \operatorname{Ker}(\bar{\partial})$, there exist a constant $C > 0$ such that

$$\left|(f,g)_\phi\right| \leq C\|\bar{\partial}^*_\phi g\|_\phi \quad \text{for all } g \in \operatorname{Dom}(\bar{\partial}^*\phi).$$

In fact, since $\operatorname{Ran}(\bar{\partial}^*_\phi)$ is closed we have $\operatorname{Ker}(\bar{\partial})^\perp = \operatorname{Ran}(\bar{\partial}^*_\phi)$ and

$$L^2_{p,q}(D,\phi) = \operatorname{Ker}(\bar{\partial}) \oplus \operatorname{Ker}(\bar{\partial})^\perp = \operatorname{Ker}(\bar{\partial}) \oplus \operatorname{Ran}(\bar{\partial}^*_\phi)$$

thereby if $g = g_1 + g_2 \in \operatorname{Dom}(\bar{\partial}^*_\phi)$ with $g_1 \in \operatorname{Ker}(\bar{\partial})$ and $g_2 \in \operatorname{Ker}(\bar{\partial})^\perp = \operatorname{Ran}(\bar{\partial}^*_\phi) \subset \operatorname{Ker}(\bar{\partial}^*_\phi) \subset \operatorname{Dom}(\bar{\partial}^*_\phi)$ (because $\bar{\partial}^2 = 0$) then $g_1 = g - g_2 \in \operatorname{Dom}(\bar{\partial}^*_\phi)$ and by (2.42) we have

$$\left|(f,g_1)_\phi\right| \leq \|f\|_\phi \|g_1\|_\phi \leq \frac{1}{\sqrt{tq}}\|f\|_\phi \|\bar{\partial}^*_\phi g_1\|_\phi$$

and $(f,g_2)_\phi = 0$. Then

$$\left|(f,g)_\phi\right| = \left|(f,g_1)_\phi\right| \leq \frac{1}{\sqrt{tq}}\|f\|_\phi \|\bar{\partial}^*_\phi g_1\|_\phi \leq \frac{1}{\sqrt{tq}}\|f\|_\phi \|\bar{\partial}^*_\phi g\|_\phi.$$

Using the Hahn-Banach theorem and the Riesz representation applied to $\bar{\partial}^*_\phi g \mapsto (f,g)_\phi$, there exist $u \in L^2_{(p,q)}(D,\phi)$ such that $(f, \bar{\partial}^*_\phi g) = (u,g)$ for all $g \in \operatorname{Dom}(\bar{\partial}^*_\phi)$ and

$$\|u\|_\phi \leq \frac{1}{\sqrt{tq}}\|f\|_\phi.$$

This implies that $u = \bar{\partial} f$ in the distribution sense and u satisfies (since, we assumed $0 \in D$, then $|z| \leq \delta^2$ for all $z \in D$)

$$\begin{aligned} q\int_D |u|^2\, dV &\leq q e^{t\delta^2} \int_D |u|^2\, e^{-t|z|^2} dV \\ &\leq \frac{1}{t} e^{t\delta^2} \int_D |f|^2\, e^{-t|z|^2} dV \\ &\leq \frac{1}{t} e^{t\delta^2} \int_D |f|^2\, dV. \end{aligned}$$

Since the function $\frac{1}{t}e^{t\delta^2}$ achieves its minimum when $t = \delta^{-2}$, we have

$$q\int_D |u|^2\, dV \leq e\delta^2 \int_D |f|^2\, dV.$$

This proves the theorem when the boundary bD is C^2.

For a general pseudoconvex domain, we will use the exhaustion of the domain D by a sequence of pseudoconvex domains with C^∞ boundary D_ν. We write

$$D = \cup_{\nu=1}^\infty D_\nu,$$

where each D_ν is a bounded pseudoconvex domain with C^∞ boundary and C^∞ boundary and $D_\nu \subset D_{\nu+1} \subset D$ for each ν. Let δ_ν the diameter for D_ν. Because of the above, on each D_ν, there exists a $u_\nu \in L^2_{(p,q)}(D_\nu)$ such that $\bar{\partial} u_\nu = f$ in D_ν and

$$q \int_{D_\nu} |u_\nu|^2 \, dV \leq e\delta_\nu^2 \int_{D_\nu} |f|^2 \, dV \leq e\delta^2 \int_D |f|^2 \, dV.$$

By Banach-Alaoglu theorem, we can choose a subsequence of u_ν, still denoted by u_ν, such that $u_\nu \rightharpoonup u$ weakly in $L^2_{(p,q-1)}(D)$. Furthermore, u satisfies the estimate

$$q \int_D |u|^2 \, dV \leq \liminf e\delta_\nu^2 \int_{D_\nu} |f|^2 \, dV \leq e\delta^2 \int_D |f|^2 \, dV,$$

and $\bar{\partial} u = f$ in D in the distribution sense. So the theorem is proved. ■

The result above could be obtained on (p, q)-forms locally square integrable functions defined on pseudoconvex domains not necessary bounded, as it is shown on [4] in Theorem 4.3.5. Note that, we have just proved that $\mathrm{Ran}(\bar{\partial}_{(p,q-1)})$ is closed and is equal to $\mathrm{Ker}(\bar{\partial}_{(p,q)})$. Obviously, uniqueness is not guaranteed, for example if $f \in \mathrm{Ker}(\bar{\partial}_{0,1})$ the $u + h$ will be another solution for any holomorphic function h, but we could have uniqueness of solution on a particular subspace as we will see on Corollary 2.4.2.

Even the domain D is not bounded, we could have existence of solutions, as it is established in the next theorem proved by Hörmander as Theorem 2.2.4 in [14].

Theorem 2.3.8 *Let D be a pseudoconvex domain in $\mathbb{C}^n$. For every $f \in L^2_{(p,q)}(D, loc)$, where $0 \leq p \leq n$, $1 \leq q \leq n$ with $\bar{\partial} f = 0$, one can find $u \in L^2_{(p,q-1)}(D, loc)$ such that $\bar{\partial} u = f$.*

Proof. Since D is pseudoconvex domain, there exist a C^∞ strictly plurisubharmonic exhaustion function σ for D. For any $f \in L^2_{(p,q)}(D, loc)$, we can choose a rapidly increasing convex function $\eta(t)$, $t \in \mathbb{R}$ such that $\eta(t) = 0$ when $t \leq 0$ and $f \in L^2_{(p,q)}(D, \eta(\sigma))$. Let $D_\nu = \{z \in D : \sigma(z) < \nu\}$, then $D = \cup_{\nu=1}^\infty D_\nu$, where each D_ν is a bounded pseudoconvex domain with C^∞ boundary and $D_\nu \subset D_{\nu+1} \subset D$ for each ν. Since $\eta(\sigma)$ is plurisubharmonic, the function $\phi = \eta(\sigma) + |z|^2$ is strictly plurisubharmonic with

$$\sum_{j,k=1}^n \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(z) a_j \bar{a}_k \geq |a|^2$$

for all $(a_1, ..., a_n) \in \mathbb{C}^n$ and all $z \in D$. Applying Proposition 2.3.6 to each D_ν we have for any $g \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}_\phi^*)$,

$$q\|g\|^2_{\phi(D_\nu)} \leq \int_{D_\nu} \sum_{I,K}{}' \sum_{j,k} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} g_{I,jK} \bar{g}_{I,kK} e^{-\phi} dV$$

$$\leq \|\bar{\partial} g\|^2_{\phi(D_\nu)} + \|\bar{\partial}^*_\phi g\|^2_{\phi(D_\nu)}.$$

With the same argument in the proof of the Theorem 2.3.7, there exists a $u_\nu \in L^2_{(p,q-1)}(D_\nu, \phi)$ such that $\bar{\partial} u_\nu = f$ in D_ν and

$$q \int_{D_\nu} |u_\nu|^2 e^{-\phi} dV \leq \int_{D_\nu} |f|^2 e^{-\phi} \leq \int_D |f|^2 e^{-\phi} dV < \infty,$$

Due to $L^2_{(p,q)}(D, \phi)$ is a Hilbert space and $\{u_\nu \chi_\nu\}_\nu$ is a bounded sequence in $L^2_{(p,q)}(D, \phi)$, where χ_ν equal to 1 in D_ν and 0 in $D \setminus D_\nu$, there exists a subsequence $\left\{u_{\nu_j} \chi_{\nu_j}\right\}_j$ converging weakly to some $u \in L^2_{(p,q)}(D, \phi)$. Also we have $\bar{\partial} u = f$ in D and

$$q \int_D |u|^2 e^{-\phi} dV \leq \int_D |f|^2 e^{-\phi} dV.$$

The theorem is proved. ■

2.4 L^2 existence theorems for the operator $\square_{(p,q)}$

Assuming that our bounded domain is pseudoconvex, we show the existence of the $\bar{\partial}$-Neumann operator N. This operator will be presented as an inverse operator of the $\square_{(p,q)}$ operator defined before for $1 \leq q \leq n$. Before establishing this result we recall some properties about the $\square_{(p,q)}$ operator, its range $\operatorname{Ran}(\square_{(p,q)})$, and kernel $\operatorname{Ker}(\square_{(p,q)})$. By Proposition 2.2.5, $\operatorname{Ker}(\square_{(p,q)})$ is closed, and by (2.1)

$$L^2_{(p,q)} = \overline{\operatorname{Ran}(\square_{(p,q)})} \oplus \operatorname{Ker}(\square_{(p,q)}).$$

We claim that

$$\operatorname{Ker}(\square_{(p,q)}) = \operatorname{Ker}(\bar{\partial}) \cap \operatorname{Ker}(\bar{\partial}^*) = 0 \quad \text{for } q \geq 1. \tag{2.43}$$

In fact, the first equality follows from the fact, for any $\alpha \in \operatorname{Ker}(\square_{p,q})$ we have

$$\|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2 = \left(\square_{(p,q)}\alpha, \alpha\right) = 0$$

thus $\operatorname{Ker}(\square_{(p,q)}) \subset \operatorname{Ker}(\bar{\partial}) \cap \operatorname{Ker}(\bar{\partial}^*)$. To see the other inclusion is sufficient to note that $\operatorname{Ker}(\bar{\partial}) \cap \operatorname{Ker}(\bar{\partial}^*) \subset \operatorname{Dom}(\square_{(p,q)})$. The second equality in (2.43), we use Theorem 2.3.7 to see that, if $\alpha \in \operatorname{Ker}(\bar{\partial}) \cap \operatorname{Ker}(\bar{\partial}^*)$ then there exists a $(p, q-1)$-form β such that $\bar{\partial}\beta = \alpha$, and since $0 = \bar{\partial}^*\alpha$ we will have

$$\|\alpha\|^2 = \left(\bar{\partial}\beta, \bar{\partial}\beta\right) = \left(\bar{\partial}^*\bar{\partial}\beta, \beta\right) = 0,$$

then, the equality follows.

Now we prove that $\operatorname{Ran}(\square_{(p,q)})$ is closed for $q \geq 1$, and so the existence of an operator N which inverts the operator $\square_{(p,q)}$, as it is established in [4] by Theorem 4.4.1.

Theorem 2.4.1 *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. For each $0 \leq p \leq n$, $1 \leq q \leq n$, there exists a bounded operator $N_{(p,q)} : L^2_{(p,q)}(D) \to L^2_{(p,q)}(D)$ such that*

(a) $\mathrm{Ran}(N_{(p,q)}) \subset \mathrm{Dom}(\square_{(p,q)})$, $N_{(p,q)}\square_{(p,q)} = \square_{(p,q)}N_{(p,q)} = I$ *on* $\mathrm{Dom}(\square_{(p,q)})$.

(b) For any $f \in L^2_{(p,q)}(D)$, $f = \bar\partial\bar\partial^* N_{(p,q)} f \oplus \bar\partial^*\bar\partial N_{(p,q)} f$.

(c) $\bar\partial N_{(p,q)} = N_{(p,q+1)}\bar\partial$ *on* $\mathrm{Dom}(\bar\partial)$, $1 \leq q \leq n-1$.

(d) $\bar\partial^* N_{(p,q)} = N_{(p,q-1)}\bar\partial^*$ *on* $\mathrm{Dom}(\bar\partial^*)$, $2 \leq q \leq n$.

(e) Let δ be the diameter of D. The following estimates hold for any $f \in L^2_{(p,q)}(D)$:

$$\|N_{(p,q)}f\| \leq \frac{e\delta^2}{q}\|f\|,$$

$$\|\bar\partial N_{(p,q)f}\| \leq \sqrt{\frac{e\delta^2}{q}}\|f\|, \tag{2.44}$$

$$\|\bar\partial^* N_{(p,q)f}\| \leq \sqrt{\frac{e\delta^2}{q}}\|f\|. \tag{2.45}$$

Proof. By Theorem 2.3.7 we have $\mathrm{Ran}(\bar\partial_{(p,q-1)}) = \mathrm{Ker}(\bar\partial_{(p,q)})$, and by (2.1) we may write

$$L^2_{(p,q)}(D) = \mathrm{Ran}(\bar\partial_{(p,q-1)}) \oplus \mathrm{Ran}(\bar\partial^*_{(p,q+1)}), \qquad \text{for } q \geq 1. \tag{2.46}$$

Then, if $f \in \mathrm{Dom}(\bar\partial_{(p,q)}) \cap \mathrm{Dom}(\bar\partial^*_{(p,q)})$ and $f = f_1 + f_2$ where $f_1 \in \mathrm{Ran}(\bar\partial_{(p,q-1)})$, $f_2 \in \mathrm{Ran}(\bar\partial^*_{(p,q+1)})$, again by Theorem 2.3.7, and Theorem 2.1.2 (applied on $\bar\partial_{(p,q-1)}$ and $\bar\partial_{(p,q)}$) we have

$$\|f_1\|^2 \leq c_q\|\bar\partial^*_{(p,q)}f_1\|^2,$$
$$\|f_2\|^2 \leq c_{q+1}\|\bar\partial_{(p,q)}f_2\|^2$$

where $c_q = e\delta^2/q$, because in this case $f_1, f_2 \in \mathrm{Dom}(\bar\partial_{(p,q)}) \cap \mathrm{Dom}(\bar\partial^*_{(p,q)})$. Since the decomposition in (2.46) is orthogonal, we obtain

$$\|f\|^2 = \|f_1\|^2 + \|f_2\|^2 \leq c_q\left(\|\bar\partial_{(p,q)}f\|^2 + \|\bar\partial^*_{(p,q)}f\|^2\right)$$

for every $f \in \mathrm{Dom}(\bar\partial_{(p,q)}) \cap \mathrm{Dom}(\bar\partial^*_{(p,q)})$.

Now, if $f \in \mathrm{Dom}(\square_{(p,q)})$, by the last inequality, we will have

$$\begin{aligned}\|f\|^2 &\leq c_q\left(\left(\bar\partial f, \bar\partial f\right) + \left(\bar\partial^* f, \bar\partial^* f\right)\right)\\ &= c_q\left(\square_{(p,q)}f, f\right)\\ &\leq c_q\|\square_{(p,q)}f\|\|f\|.\end{aligned}$$

Hence, for any $f \in \mathrm{Dom}(\Box_{(p,q)})$,

$$\|f\| \leq c_q \|\Box_{(p,q)} f\|. \tag{2.47}$$

Then by Theorem 2.1.2, $\Box_{(p,q)}$ has closed range. Also, by (2.1) and (2.43), $L^2_{(p,q)}(D) = \mathrm{Ran}(\Box_{(p,q)})$. By (2.47), $\Box_{(p,q)}$ is injective. Then there exist a unique inverse $N_{(p,q)} : L^2_{(p,q)}(D) \to \mathrm{Dom}(\Box_{p,q})$ such that $\Box_{(p,q)} N_{(p,q)} = Id$, that is

$$f = \bar\partial\bar\partial^* N_{(p,q)} f + \bar\partial^*\bar\partial N_{(p,q)} f \qquad \text{for any } f \in L^2_{(p,q)}(D).$$

and $N_{(p,q)}\Box_{(p,q)} = Id$ on $\mathrm{Dom}(\Box_{(p,q)})$. The assertions (a) and (b) have been established.

By (b), if $f \in \mathrm{Dom}(\bar\partial)$, we have

$$N_{(p,q+1)}\bar\partial f = N_{(p,q+1)}\bar\partial\, \Box_{(p,q)} N_{(p,q)} f = N_{(p,q+1)}\bar\partial\bar\partial^*\bar\partial N_{(p,q)} f = N_{(p,q+1)}\Box_{(p,q+1)}\bar\partial N_{(p,q)} f = \bar\partial N_{(p,q)} f$$

then (c) follows. (d) follows on a similar way.

The first inequality in (e) follows by (2.47) and (a). To obtain the other inequalities, observe that

$$\begin{aligned} \|\bar\partial N_{(p,q)} f\|^2 + \|\bar\partial^* N_{(p,q)} f\|^2 &= \left(\Box_{(p,q)} N_{(p,q)} f, N_{(p,q)} f\right) = \left(f, N_{(p,q)} f\right) \\ &\leq \|f\| \|N_{(p,q)} f\|, \end{aligned}$$

and (2.44), (2.45) follow by the first inequality. ∎

Corollary 2.4.2 *Let D and $N_{(p,q)}$ be the same as in Theorem 2.4.1, where $0 \leq p \leq n$, $1 \leq q \leq n$. For any $\alpha \in L^2_{(p,q)}(D)$ such that $\bar\partial\alpha = 0$, the $(p, q-1)$-form*

$$u = \bar\partial^* N_{(p,q)} \alpha \tag{2.48}$$

satisfies the equation $\bar\partial u = \alpha$ and the estimate

$$\|u\|^2 \leq \frac{e\delta^2}{q}\|\alpha\|^2. \tag{2.49}$$

The solution u is called the canonical solution to the equation $\bar\partial u = \alpha$ with compatibility condition on α, and it is the unique solution which is orthogonal to $\mathrm{Ker}(\bar\partial_{(p,q-1)})$.

Proof. If u is defined by (2.48) then u is orthogonal to $\mathrm{Ker}(\bar\partial)$. Moreover

$$\bar\partial u = \bar\partial\bar\partial^* N_{(p,q)}\alpha = \Box_{(p,q)} N_{(p,q)}\alpha - \bar\partial^*\bar\partial N_{(p,q)}\alpha = \alpha - \bar\partial^*\bar\partial N_{(p,q)}\alpha$$

then by (c) in Theorem 2.4.1 and the compatibility condition on α we have $\bar\partial^*\bar\partial N_{(p,q)}\alpha = 0$, and so $\bar\partial u = \alpha$. The inequality (2.49) follows by (e) on Theorem 2.4.1. If v is another solution orthogonal to $\mathrm{Ker}(\bar\partial)$, we will have $u - v \in \mathrm{Ker}(\bar\partial)$, and also by the orthogonality to $\mathrm{Ker}(\bar\partial_{(p,q-1)})$, $u - v \in {}^{\perp}\mathrm{Ker}(\bar\partial_{(p,q-1)})$. Then $u - v = 0$. The corollary is proved. ∎

It is possible to define the Neumann operator on the level $(p, 0)$ as it is made in [4] Section 4.4, and with similar properties as given in Theorem 2.4.1.

2.5 Pseudoconvexity and the Levi problem

As an application we did before, we will solve the well-known Levi problem. But before, we have to give a definition of an object involving this problem.

Definition 2.5.1 *A domain D in $\mathbb{C}^n$ is called a domain of holomorphy, if we cannot find two nonempty open sets D_1 and D_2 in $\mathbb{C}^n$ with the following properties:*

(1) D_1 is connected, $D_1 \nsubseteq D$ and $D_2 \subset D_1 \cap D$.

(2) For every holomorphic function f in D there is a holomorphic function $\tilde{f}$ in D_1 satisfying $f = \tilde{f}$ in D_2.

The Levi problem consist fo showing whether pseudoconvex domains are domains of holomorphy. To prove this result, for any $p \in bD$ one must find a holomorphic function f which cannot be continued holomorphically near p.

In the case of D is a strongly pseudoconvex domain with C^∞ boundary bD and $p \in bD$ one can to construct a function f in an open neighborhood U of p such that f is holomorphic in $U \cap D$, $f \in C(\left\{\overline{D} \cap U\right\} \setminus \{p\})$ and $|f| \to +\infty$ when $z \in D$ approaches p. In fact f can be obtained as follows: Let r be a strictly plurisubharmonic defining function for D and assume that $p = 0$. Let

$$F(z) = -2\sum_{i=1}^{n} \frac{\partial r(0)}{\partial z_i} z_i - \sum_{i,j=1}^{n} \frac{\partial^2 r(0)}{\partial z_j \partial z_i} z_j z_i$$

$F(z)$ is holomorphic in $\mathbb{C}^n$, and it is called the Levi polynomial of r at 0. Using Taylor's expansion of r at 0, by the strictly plurisubharmonicity of r, there exists a sufficiently small neighborhood U of 0 and $C > 0$ such that for any $z \in \overline{D} \cap U$,

$$\mathrm{Re}(F) = -r(z) + \sum_{i,j=1}^{n} \frac{\partial^2 r(0)}{\partial z_i \partial \bar{z}_j} z_i \bar{z}_j + O(|z|^3) \geq C|z|^2.$$

Thus $F(z) \neq 0$ when $z \in \left\{\bar{D} \cap U\right\} \setminus \{0\}$. Setting $f = 1/F$, it is easily seen that f is holomorphic in $D \cap U$ which cannot be extended holomorphically across 0.

The general case is proved using the next result.

Theorem 2.5.2 *Let D be a pseudoconvex domain in $\mathbb{C}^n$. For every $f \in C^\infty_{(p,q)}(D)$, where $0 \leq p \leq n$, and $1 \leq q \leq n$, with $\bar{\partial} f = 0$, one can find $u \in C^\infty_{(p,q-1)}(D)$ such that $\bar{\partial} u = f$.*

Proof. Let $f \in C^\infty_{(p,q)}(D)$ with $0 \leq p \leq n$ and $1 \leq q \leq n$, then $f \in L^2_{(p,q)}(D, loc)$, and by the proof of Theorem 2.3.8 there exists a function $\phi \in C^2(D)$ (strictly plurisubharmonic) such that, $f \in L^2_{(p,q)}(D,\phi)$, and there exists $u \in L^2_{(p,q-1)}(D, loc)$ with $\bar{\partial} u = f$, and

$$\|u\|_\phi \leq \|f\|_\phi. \tag{2.50}$$

Then using the inequality (2.50) and repeating the same arguments as in Section 2.4 the L^2 existence theorem for the $\bar{\partial}$-Neumann operator, there exists a weighted $\bar{\partial}$-Neumann operator N_ϕ such that for any $g \in L^2_{(p,q)}(D,\phi)$, we have

$$g = \bar{\partial}\bar{\partial}^*_\phi N_\phi g + \bar{\partial}^*_\phi \bar{\partial} N_\phi g.$$

Since $\bar{\partial} f = 0$ we have $f = \bar{\partial}\bar{\partial}^*_\phi N_\phi f$. Let $u = \bar{\partial}^*_\phi N_\phi f$. It is sufficient to prove that $u \in C^\infty_{(p,q-1)}(D)$. Due to $\bar{\partial}^*_\phi u = 0$, we have $\vartheta u = -A_0 u \in L^2_{(p,q)}(D, loc)$ where A_0 is some zero order operator.

Note that if α is a (p,q)-form in $C^\infty_{(p,q)}(\bar{D})$ with compact support in D, we have

$$\begin{aligned} 4\left(\|\bar{\partial}\alpha\|^2 + \|\vartheta\alpha\|^2\right) &= 4\left(\left(\bar{\partial}\alpha, \bar{\partial}\alpha\right) + (\vartheta\alpha, \vartheta\alpha)\right) \\ &= 4\left(\left(\bar{\partial}^*\bar{\partial}\alpha, \alpha\right) + \left(\bar{\partial}\bar{\partial}^*\alpha, \alpha\right)\right) \\ &= 4\left(\Box\alpha, \alpha\right) = (-\Delta\alpha, \alpha) = \|\nabla\alpha\|. \end{aligned} \tag{2.51}$$

Where Δ is the real Laplacian and ∇ is the gradient, both acting on α componentwise. When $q = 0$ (2.51) also holds since $\Box = \vartheta\bar{\partial} = \bar{\partial}^*\bar{\partial}$ is equal to $-\Delta/4$.

Then the Sobolev 1-norm

$$\|\alpha\|^2_{1(\Omega)} := \|\alpha\|^2 + \|\nabla\alpha\|^2 \leq C\left(\|\alpha\| + \left\|\bar{\partial}\alpha\right\| + \|\vartheta\alpha\|\right) \tag{2.52}$$

with $C > 0$ is a constant.

Let $\tilde{u} = \zeta u$ where $\zeta \in C^\infty_0(D)$ and define $u_\varepsilon = \tilde{u} * \chi_\varepsilon$ where χ is a nonnegative function such that $\chi \in C^\infty_0(\mathbb{C}^n)$, $\int \chi = 1$, $\operatorname{supp}\chi \subset \overline{B(0,1)}$ and $\chi_\varepsilon(z) = \frac{1}{\varepsilon^n}\chi(\frac{z}{\varepsilon})$. It follows, from Young's inequality that $\|u_\varepsilon\| \leq \|\tilde{u}\|$, $\bar{\partial} u_\varepsilon = \bar{\partial}\tilde{u} * \chi_\varepsilon$ and $\vartheta u_\varepsilon = \vartheta\tilde{u} * \chi_\varepsilon$. Taking $\alpha = u_\varepsilon$ in (2.52) we have

$$\begin{aligned} \|u_\varepsilon\|_{1(D)} &\leq C\left(\|u_\varepsilon\| + \left\|\bar{\partial} u_\varepsilon\right\| + \|\vartheta u_\varepsilon\|\right) \\ &\leq C\left(\|\tilde{u}\| + \left\|\bar{\partial}\tilde{u}\right\| + \|\vartheta\tilde{u}\|\right) \end{aligned}$$

thus, there exists a subsequence (u_{ε_j}) such that converges weak in $W^1_{(p,q-1)}(D)$ (Because $W^1(D)$ is a Hilbert space). Since $u_\varepsilon \to \tilde{u}$ in $L^2_{(p,q-1)}(D)$ we have $\tilde{u} \in W^1_{(p,q-1)}(D)$. So $\nabla\tilde{u} \in L^2_{(p,q-1)}(D)$. Due to $\nabla u_\varepsilon = \nabla\tilde{u} * \chi_\varepsilon$, we have $\nabla u_\varepsilon \to \nabla\tilde{u}$ in $L^2_{(p,q-1)}(D)$, then $u_\varepsilon \to \tilde{u}$ in $W^1_{(p,q-1)}(D)$. Thereby, we have $u \in W^1_{(p,q-1)}(D, loc)$.

Applying the process above to ∇u we obtain that $u \in W^2_{(p,q-1)}(D, loc)$. We conclude by induction that $u \in W^{k+1}_{(p,q-1)}(D, loc)$ for any $k \in \mathbb{N}$. By Sobolev's embedding theorem, $u \in C^\infty_{(p,q-1)}(D)$. ■

The following result solves the Levi Problem through the existence theorem for the Cauchy-Riemann equations given by Theorem 2.5.2.

Theorem 2.5.3 *Let D be a domain in $\mathbb{C}^n$, $n \geq 1$. If the next property,*

$$\begin{aligned}&\textit{For every } f \in C^\infty_{p,q}(D) \textit{ where } 0 \leq p \leq n,\ 1 \leq q \leq n \textit{ with } \bar\partial f = 0,\ \textit{one can find}\\ &u \in C^\infty_{(p,q-1)}(D) \textit{ such that } \bar\partial u = f,\end{aligned} \tag{2.53}$$

is satisfied, then D is a domain of holomorphy.

Proof. We will use an induction argument in n to prove the theorem. For $n = 1$ this is obvious because all open set in $\mathbb{C}$ are domains of holomorphy.

Assume the affirmation is true for $n-1$, i.e., if Ω is a domain in $\mathbb{C}^{n-1}$ satisfying the property (2.53) in Ω, then Ω is a domain of holomorphy.

Let $D \subset \mathbb{C}^n$ and (2.53) is fulfilled in D. To prove D is domain of holomorphy, for each $z_0 \in D$ (or maybe, for each z_0 in a dense subset of bD) we need construct a holomorphic function in D which cannot be extended holomorphically across any neighborhood of z_0.

First of all, if for every $z \in D$ and $r_z > 0$ is such that $r_z = dist(z, \mathbb{C}^n \subset D)$, we claim the set $F = \cup_{z\in D}\left(\overline{B(z, r_z)} \cap bD\right)$ is dense in bD. In fact, if $p_0 \in bD$, there exists a sequence $\{z_j\}$ in D such that $|z_j - p_0| \to 0$ when $j \to +\infty$. Taking $\zeta_j \in \mathbb{C}^n$ such that $\zeta_j \in \overline{B(z_j, r_{z_j})} \cap bD$. Then $r_{z_j} \leq |z_j - p_0|$. So we have $|\zeta_j - p_0| \leq |\zeta_j - z_j| + |z_j - p_0| \leq 2\,|z_j - p_0| \to 0$ when $j \to +\infty$. The claim is proved.

Let $z_0 \in F$, and $z_0 \in \overline{B(\zeta_0, r_{\zeta_0})} \cap bD$. Take Σ_0 being the complex $(n-1)$-dimensional hyperplane passing through ζ_0 and z_0. Note that $z_0 \in \Sigma_0 \cap bD$. By a linear transformation we may assume that $z_0 = 0$ and $\Sigma_0 = D \cap \{z_n = 0\}$. Let $A \subset \mathbb{C}^{n-1}$ such that $\Sigma_0 = A \times \{0\}$; $\pi : D \to \mathbb{C}^{n-1}$ with $\pi(z', z_n) = z'$, $D_0 = D \setminus \pi^{-1}(A)$.

To construct the required function, we will need the next claim:

$$\left\{\begin{array}{l}\text{For all } g \in C^\infty_{(p,q)}(A) \text{ with } 0 \leq p \leq n-1,\ 0 \leq q \leq n-1\\ \text{ and } \bar\partial g = 0,\ \text{there exist } G \in C^\infty_{(p,q)}(D) \text{ such that}\\ G|_{A\times\{0\}} = g,\ \bar\partial G = 0\end{array}\right. \tag{2.54}$$

Since Σ_0 and D_0 are relatively closed disjoint subsets of D, using Urysohn's lemma, there exist a smooth function η in D, such that $\eta \equiv 0$ in a neighborhood of Σ_0 and $\eta \equiv 1$ in a neighborhood of D_0.

Let $\tilde g(z \in D) = \eta(z)(\pi^* g)(z)$ where $\pi^* g$ is the pull-back of the form g by π^*, that is, if $g = \sum g_{I,J} dz^I \wedge d\bar z^J$ (note that $n \notin J$) then $\pi^* g = \sum_{I,J,n\notin J} g_{I,J} dz^I \wedge d\bar z^J$. $\bar\partial \tilde g = 0$ in a neighborhood of D_0, and $\bar\partial \tilde g \equiv 0$ in a neighborhood of Σ_0, because $\tilde g = \pi^* g$ in a neighborhood of Σ_0 and $\bar\partial\tilde g = \bar\partial(\pi^* g) = \pi^*\bar\partial g = 0$ (this because π is a holomorphic map). So the $(p, q+1)$-form $\frac{\bar\partial\tilde g}{z_n}$ is well defined in D, and is in $C^\infty_{(p,q+1)}(D)$, and

$$\begin{aligned}\bar\partial\frac{\bar\partial\tilde g}{z_n} &= \bar\partial\left(\frac{1}{z_n}\bar\partial\left[\eta(\pi^* g)\right]\right) = \bar\partial\left(\frac{1}{z_n}\left\{\bar\partial\eta\wedge\pi^* g\right\}\right)\\ &= \frac{1}{z_n}\left(\bar\partial\bar\partial\eta\wedge\pi^* g - \bar\partial\eta\wedge\bar\partial(\pi^* g)\right) = 0.\end{aligned}$$

Since property (2.53) is satisfied in D, there exist $u \in C^\infty_{(p,q)}(D)$ such that $\bar\partial u = \bar\partial\tilde g/z_n$. Let $G(z) = \tilde g(z) - z_n u(z)$. Then $G \in C^\infty_{(p,q)}(D)$,

$$\bar\partial G = \bar\partial\tilde g - \bar\partial(z_n u) = \bar\partial\tilde g - z_n\bar\partial u = \bar\partial\tilde g - \bar\partial\tilde g = 0$$

when $z_n \neq 0$ and also $\bar\partial G = 0$ when $z_n = 0$ because $\bar\partial\tilde g = 0$ on a neighborhood of Σ_0.

If $(z', 0) \in A \times \{0\}$, then $G(z', 0) = \tilde g(z', 0) = g(z')$. This proves affirmation made in (2.54).

We next claim that A is a domain of holomorphy. In fact, if $f \in C^\infty_{p,q}(A)$, $0 \le p \le n-1$, $1 \le q \le n-1$ such that $\bar\partial f = 0$, by (2.54) there exists $F \in C^\infty_{(p,q)}(D)$ such that $\bar\partial F = 0$ and $F|_{A\times\{0\}} = f$. Since property (2.53) is fulfilled in D there exists $U \in C^\infty_{(p,q-1)}(D)$ such that $\bar\partial U = F$; if we define $u(z') = U(z', 0)$; we have $\bar\partial u = f$ and u is in $C^\infty_{(p,q)}(A)$. So by the induction hypothesis, A is domain of holomorphy.

Finally, since A is domain of holomorphy, there exists a holomorphic function $h(z') = h(z_1, ..., z_{n-1})$ in A such that it cannot be extended holomorphically across 0. By (2.54) there exist a function $H \in C^\infty(D)$ such that $\bar\partial H = 0$ in D and $H(z', 0) = h(z')$ for all $z \in A$. So, H is holomorphic function in D which cannot be extended holomorphically across 0, i.e. D is a domain of holomorphy. The theorem is proved. ∎

CHAPTER 3

APPROACH TO THE OPERATOR $\bar{\partial}_b$

In this chapter we want to show some results that appeared over the years in the study of this operator, in order to explain the motivation and the approaches given to it, as well as to report the conditions, methods, papers and some researchers involved in the development of the theory about the operator $\bar{\partial}_b$.

One of the first works where the operator $\bar{\partial}_b$ appeared was in Kohn and Rossi's paper [22]. This paper was focused on extension problems inspirited on Hartog's theorem. As it is known the Hartog's Theorem guarantees the property of holomorphic extension in a hole bounded domain $\Omega \subset \mathbb{C}^n$ for functions whose are holomorphic in connected open subsets of the type $\Omega \setminus K$, with K being a compact set in Ω. The discussion made by Kohn and Rossi starts by asking about the sufficient conditions to imply the same affirmation for functions defined just in $b\Omega$. Even more, the problem was proposed for (p, q)-forms f being restrictions in $b\Omega$ of forms in $\mathbb{C}^n$, looking for (p, q)-forms $\tilde{f}$ such that $\tilde{f}\,\big|_{b\Omega} = f$ and $\bar{\partial}$-closed in Ω, that is $\bar{\partial}\tilde{f} = 0$ in Ω. One of the conditions imposed by Kohn and Rossi was that f had to be $\bar{\partial}_b$-*closed* in $b\Omega$, that is $\bar{\partial}_b f = 0$ in $b\Omega$. The operator $\bar{\partial}_b$ was defined extrinsically in [22] as follows.

Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary $b\Omega$. Denote for the moment $M = b\Omega$. We define $\Lambda^{p,q}T^*(\mathbb{C}^n)|_M$ to be the restriction of the bundle of (p, q)-forms on $\mathbb{C}^n$, $\Lambda^{p,q}T^*(\mathbb{C}^n)$, to M. Define $I^{p,q}$ as the ideal in $\Lambda^{p,q}T^*(\mathbb{C}^n)$ which is generated by ρ and $\bar{\partial}\rho$ where ρ is any smooth function that vanishes in M and $|d\rho| \neq 0$ (that is, ρ is a defining function). Any element in $I^{p,q}$ is of the form $\rho\Phi_1 + \bar{\partial}\rho \wedge \Phi_2$ where $\Phi_1 \in \Lambda^{p,q}T^*(\mathbb{C}^n)$ and $\Phi_2 \in \Lambda^{p,q-1}T^*(\mathbb{C}^n)$, and observe that $I^{p,q}$ is independent of the choice of ρ. Let $I^{p,q}|_M$ denote the restriction of $I^{p,q}$ to M. $I^{p,q}|_M$ is the ideal locally generated by $\bar{\partial}\rho$. Define $\Lambda^{p,q}T^*(M)$ the orthogonal complement of $I^{p,q}|_M$ in $\Lambda^{p,q}T^*(\mathbb{C}^n)|_M$. The space $\Lambda^{p,q}T^*(M)$ is not intrinsic to M, i.e., it is not a sub space of the exterior algebra generated by the complexified cotangent bundle of M, this is due $\bar{\partial}\rho$ in not orthogonal to cotangent bundle of M. $\Lambda^{p,q}T^*(M) = 0$ if either $p > n$ or $q > n - 1$. For an open set $U \subset M$ the space of smooth sections of $\Lambda^{p,q}T^*(M)$ over U will be denoted $\mathcal{E}^{p,q}_M(U)$, and $D^{p,q}_M(U)$ will denote the space of compactly supported elements in $\mathcal{E}^{p,q}_M(U)$. Define $t_M : \Lambda^{p,q}T^*(\mathbb{C}^n)|_M \to \Lambda^{p,q}T^*(M)$ to be the orthogonal projection map on $\Lambda^{p,q}T^*(M)$. We will denote $t_M(f)$ by f_{t_M}. f_{t_M} is called the *tangential part* of f.

Definition 3.0.1 *For an open set $U \subset M$, the tangential Cauchy-Riemann complex $\bar{\partial}_b : \mathcal{E}_M^{p,q}(U) \to \mathcal{E}_M^{p,q+1}(U)$ is defined as follows. For $f \in \mathcal{E}_M^{p,q}(U)$, let $\tilde{U}$ be an open set in $\mathbb{C}^n$ with $\tilde{U} \cap M = U$ and let $\tilde{f} \in \mathcal{E}^{p,q}(\tilde{U})$ with $\tilde{f}_{t_M} = f$ on $\tilde{U} \cap M = U$ then*

$$\bar{\partial}_b f = (\bar{\partial}\tilde{f})_{t_M}.$$

Note that if $\alpha \in \mathcal{E}^{p,q}(\tilde{U})$, $\beta \in \mathcal{E}^{p,q-1}(\tilde{U})$, and if $\rho : \tilde{U} \to \mathbb{R}$ vanishes on $M \cap \tilde{U}$, then $\bar{\partial}(\alpha\rho + \beta \wedge \bar{\partial}\rho) = \rho\bar{\partial}\alpha + (\alpha + \bar{\partial}\beta) \wedge \bar{\partial}\rho$. So $\bar{\partial}_b$ maps smooth sections of $I^{p,q}$ to $I^{p,q+1}$. Then if $\tilde{f}_1, \tilde{f}_2 \in \mathcal{E}^{p,q}(\tilde{U})$ with $(\tilde{f}_1)_{t_M} = (\tilde{f}_2)_{t_M}$ on $M \cap \tilde{U}$ we have $\tilde{f}_1 - \tilde{f}_2 \in I^{p,q}$ so $\bar{\partial}\tilde{f}_1 - \bar{\partial}\tilde{f}_2 \in I^{p,q+1}$, then $(\bar{\partial}\tilde{f}_1)_{t_M} - (\bar{\partial}\tilde{f}_2)_{t_M} = (\bar{\partial}\tilde{f}_1 - \bar{\partial}\tilde{f}_2)_{t_M} = 0$. With this $\bar{\partial}_b$ is well defined. Also, we have

(a) $\bar{\partial}_b(f \wedge g) = \bar{\partial}_b f \wedge g + (-1)^{p+q} f \wedge \bar{\partial}_b g$ for $f \in \mathcal{E}_M^{p,q}$ and $g \in \mathcal{E}^{r,s}$.

(b) $\bar{\partial}_b^2 = \bar{\partial}_b \circ \bar{\partial}_b = 0$.

Note that in case f is a function, the condition $\bar{\partial}_b f = 0$ in $b\Omega$ is equivalent to f satisfying the *tangential Cauchy Riemann equations*, that is

$$\sum_{j=1}^{n} \alpha_j \frac{\partial f}{\partial \bar{z}_j} = 0 \quad \text{on } b\Omega$$

for any $\alpha_1, ..., \alpha_n$ satisfying $\sum_{j=1}^{n} \alpha_j \frac{\partial \rho}{\partial \bar{z}_j} = 0$ in $b\Omega$. These functions f are called *CR functions*. Kohn and Rossi's ideas required of result about existence and regularity up to the boundary of solutions for the operator $\bar{\partial}$, so they impose convexity conditions on the boundary with this objective. To precise, the condition was the $Z(q)$ *condition* (the Levi form, at every point in $b\Omega$, has at least $n - q$ positive eigenvalues or at least $q + 1$ negative eigenvalues), because it was proved by Hormander in [14] that this property offered regularity up to boundary in addition to only solutions.

In [18], Kohn presented an approach to the existence and regularity of solutions for the equation

$$\bar{\partial}_b u = f \tag{3.1}$$

defined just not only for manifolds being the boundary of domains in n dimensional complex manifolds, but in more general structures, now known as CR manifolds. These objects were compact C^∞ manifolds $2n-1$ real dimensional M, endowed with a subbundle S of the complexified tangent bundle $\mathbb{C}TM$ of M which satisfies the next conditions: Every fiber of S has complex dimension $n - 1$, $S \cap \overline{S} = 0$ (so we can define a Hermitian metric such that $S \perp \overline{S}$), and the *integrability condition*, that is, S is preserved by Lie bracket ($[L_1, L_2] \in S$ for any $L_1, L_2 \in S$). This last condition allows us to imply on the compatibility conditions for the equation (3.1), that is $\bar{\partial}_b^2 = 0$. Having in mind this structure, it is possible to present the operator $\bar{\partial}_b$ as a differential operator, satisfying the properties (a) and (b) above, as we prove in Chapter 4 (Section 4.1.1). It is worth clarifying that the approaches of the operator $\bar{\partial}_b$ given above and the given in Chapter 4 are different, because they are extrinsic and intrinsic approaches respectively, but the complexes that these operators provide are isomorphic. Let $\mathcal{E}^{p,q}$ denote the space of smooth sections of $\Lambda^p(S^*) \otimes \Lambda^q(\overline{S}^*)$. One of Kohn's goals was to obtain local estimates (it was in fact subelliptic estimates, once appropriate geometrical conditions were assumed), whose permit him imply just not closed range of the operator $\bar{\partial}_b$, but regularity of solutions. The process implemented was as follows. Let U be an open set in M, $L_1, ..., L_{n-1}$ are vectors field of the type $(1,0)$, that is $L_j \in S$ for $j = 1, \ldots, n-1$, T a purely imaginary vector field in $\mathbb{C}TM$ such that $\{L_1, ..., L_{n-1}, \bar{L}_1, ..., \bar{L}_{n-1}, T\}$ is a basis for $\mathbb{C}TM$ in U. Let c_{jk} be smooth functions in U such that

$$[L_j, \bar{L}_k] = c_{jk}T \mod S \oplus \overline{S}.$$

The Hermitian form defined by c_{jk} is called the Levi form. Define $\vartheta_b : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q-1}$ the formal adjoint of $\bar{\partial}_b$ (throughout an inner product $(\cdot,\cdot)$ defined in the standard way), and denote

$$Q_b(\varphi, \psi) = (\bar{\partial}_b\varphi, \bar{\partial}_b\psi) + (\vartheta_b\varphi, \vartheta_b\psi) + (\varphi, \psi),$$

for any $\varphi, \psi \in \mathcal{E}^{p,q}$. The main estimate established by Kohn in [18] (Theorem 5.3), is stated as follow.

Theorem 3.0.2 *If $x_0 \in M$ and if the Levi form at x_0 has $max(n-q, q+1)$ non-zero eigenvalues of the same sign, then there exist a neighborhood U os x_0 and a constant C such that*

$$\sum_{j=1}^{n-1} \sum_{J \in \mathcal{I}_q} \|L_j f_J\|^2 + \sum_{j=1}^{n-1} \sum_{J \in \mathcal{I}_q} \|\bar{L}_j f_J\|^2 + \sum_{J \in \mathcal{I}_q} |\mathrm{Re}\,(T f_J, f_J)| \leq C Q_b(f,f). \tag{3.2}$$

for any (p,q)-form f whose support lies in U.

The condition imposed in the Levi form in this theorem is now known as $Y(q)$ *condition*. And to obtain local 1/2-estimates, he stated the next (Proposition 6.8).

Theorem 3.0.3 *If U is a coordinate neighborhood in M and if there exist a constant C_0 such that*

$$\sum_{j=1}^{n-1} \sum_{J \in \mathcal{I}_q} \|L_j f_J\|^2 + \sum_{j=1}^{n-1} \sum_{J \in \mathcal{I}_q} \|\bar{L}_j f_J\|^2 + \sum_{J \in \mathcal{I}_q} |\mathrm{Re}\,(T f_J, f_J)| \leq C_0 Q_b(f,f).$$

for any (p,q)-form f whose support lies in U, then for any open set V with $\overline{V} \subset U$ there exists a constant C such that

$$\|f\|_{1/2} \leq C Q_b(f,f) \tag{3.3}$$

for all $f \in \mathcal{E}^{p,q}$ whose support lies in V.

And over orientability conditions in M; that is M can be covered by neighborhoods in which the dual forms of L_j, $1 \leq j \leq n-1$, T have been chosen so that in the intersection of two neighborhoods the duals of T are positive multiples to each others; he was allowed to pass from local estimates to global estimates after an argument of partition of unity, that is, (3.3) is going to be true for any $f \in \mathcal{E}^{p,q}$ (Theorem 6.14 in [18]). Once this good estimates (1/2-estimates) were reached Kohn applied an argument, named by Niremberg and Kohn as “elliptic regularization" in [21], to obtain smooth solutions. This method, still used today, consists in adding ε times an elliptic operator so that the resulting equation becomes elliptic and coercive for $\varepsilon > 0$. This new equation, being coercive elliptic, has a smooth solution u_ε in M and, the method of obtaining a priori estimates applies as well to the new equation as to the original one, and yields estimates for derivatives of u_ε which are independent of ε. Doing $\varepsilon \to 0$ through a sequence ε_j, it follows that a subsequence of u_{ε_j}, together with derivatives, converges to a smooth solution of the original problem. As an example, we explain here how the elliptic regularization works in this case. Define the Kohn Laplacian $\Box_b := \bar{\partial}_b\vartheta_b + \vartheta_b\bar{\partial}_b$. Once it is obtained (3.3) for $f \in \mathcal{E}^{p,q}$, by using an induction process and a small constant/large constant argument, obtain the next estimate (the *a priori* estimates)

$$\|f\|_{s+1/2} \leq C_s\|\Box_b f + f\|_{s-1/2}$$

for $f \in \mathcal{E}^{p,q}$, and for any non negative integer s. Define for $\varepsilon > 0$

$$Q_b^\varepsilon(\varphi, \psi) = Q_b(\varphi, \psi) + \varepsilon K(\varphi, \psi), \quad \text{for } \varphi, \psi \in \mathcal{E}^{p,q}$$

where $K(.,.)$ is the elliptic term (chosen in a suitable way). The ellipticity of $Q_b^\varepsilon(.,.)$ will imply that for any $\alpha \in \mathcal{E}^{p,q}$ (a smooth section) there will exist a unique $\varphi_\varepsilon \in \mathcal{E}^{p,q}$ such that $Q_b^\varepsilon(\varphi_\varepsilon, \psi) = (\alpha, \psi)$ for all $\psi \in \mathcal{E}^{p,q}$, and also (by using the same method to get the a priori estimates) for each integer $s \geq 0$ there exists $C_s > 0$, C_s being *independent* of ε, such that

$$\|\varphi_\varepsilon\|_{s+1/2} \leq C_s\|\alpha\|_{s-1/2}.$$

Then using Rellich's lemma and the diagonal process, there will exist a subsequence φ_{ε_j} converging in $\| \ \|_s$ for every $s > 0$, hence the limit φ is in $\mathcal{E}^{p,q}$ (a smooth section) and satisfies the equation $\Box_b\varphi + \varphi = \alpha$. In this way, Kohn could imply in the existence of smooth solutions for equation (3.1) on CR manifolds of hypersurface type assuming $Y(q)$ condition.

Shawn, in [29], reaffirmed the result obtained by Rosay in [27] but using a more direct method, and also proved the existence of smooth solutions as Kohn did in [18], on manifolds being the boundary of a bounded weakly pseudoconvex domain Ω in $\mathbb{C}^n$ ($n \geq 2$). Part of this result is established as follows: the necessary and sufficient conditions for the solvability and regularity of the solutions for the equation (3.1), where f is a smooth (p, q)-form on $b\Omega$, and $q < n-1$, is $\bar{\partial}_b f = 0$. This result was also valid for forms in top level $q = n-1$, after appropriate orthogonality conditions on f. The approach was done using $\bar{\partial}$-closed extension as done in [22], and using the existence of smooth solutions up to the boundary for the operator $\bar{\partial}$ on weakly pseudoconvex domains obtained by Kohn in [19]. Although this process could be useful to guarantee existence of smooth solutions, the argument could not be applied to establish L^2 closed range estimates for operator $\bar{\partial}_b$. So Shaw gave in [28], a method to obtain results about global solvability of the $\bar{\partial}_b$ complex on the boundary of a bounded weakly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ $(n > 3)$, in L^2 as well as in Sobolev spaces, and exposed their respective estimates, in levels $0 < q \leq n-2$. This procedure includes the construction of *two-sided* $\bar{\partial}$-closed extensions with good estimates of any given $\bar{\partial}_b$-closed form on $b\Omega$, using results already known at that time about existence and regularity for $\bar{\partial}$-Neumann operator, as well on pseudoconvex domains as on strongly pseudoconvex domains. In [2], Boas and Shaw contour the result, offering same estimates obtained by Shaw in [28] in top degree $q = n-1$ (so it allows them work in the boundary of $\Omega \subset \mathbb{C}^2$). It means in particular, for any bounded domain $\Omega \subset \mathbb{C}^2$ with a smooth weakly pseudoconvex boundary $b\Omega$ (with induced CR structure), the operator $\bar{\partial}_b$ has closed range in L^2 as well as W^s for any nonegative integer s (Corollary in [2]).

On the other hand, Kohn in [20] gave an approach to study the operator $\bar{\partial}_b$ on manifolds which were not just boundaries of domains in $\mathbb{C}^n$, as it was studied in [28] and [2]. As Kohn described there, he was interested to know whether $\bar{\partial}_b$ on an embedded, compact CR manifold $M \subset \mathbb{C}^n$, (of higher co-dimension) has closed range. He introduced a microlocal method suited to the study of $\bar{\partial}_b$ on CR manifolds, and proved that if M is compact, pseudoconvex, and the boundary of a smooth complex manifold which admits a strictly plurisubharmonic function defined in a neighborhood of M, then $\bar{\partial}_b$ has closed range. In this work we are focused in study the question stated here by Kohn, and also in obtaining results about closed range estimates as well Nicoara obtained in [24], but imposing weaker hypothesis than pseudoconvexity. We close this chapter here because the results that come after the scopes given so far were mentioned in the Introduction of this work.

CHAPTER 4

L^2 CLOSED RANGE ESTIMATES FOR $\bar{\partial}_b$ AND THE WEAK $Y(q)$-CONDITION

The objective of this chapter is prove $L^2_{0,q}(M)$ estimates (indeed, the estimate (4.15)) in order to prove the closure of the range of the operators $\Box_b$ and $\bar{\partial}_b$. We understand this by dominating the L^2-norm with the energy $Q_b(\cdot,\cdot)$ generated by $\bar{\partial}_b$ and its adjoint as it was done for the operator $\bar{\partial}$. On the first two sections we give the definitions of the key terms including CR manifolds of hypersurface type, the Levi form, the operator $\bar{\partial}_b$, and the sufficient geometrical condition used here to imply the L^2 closed range estimates, named weak $Y(q)$ condition. The weak $Y(q)$ condition is obtained throughout weak $Z(q)$ condition (see definition 4.1.3). A first version of this property, weak $Z(q)$ condition, was given in [10], which required the local existence of a vector field of type $(1,1)$, whose coefficients of the diagonal terms are zeros or one and the other terms are taken to be zeros, such that the sum of any q eigenvalues of the Levi matrix minus the Levi form applied to this (1,1) vector field is not negative. The difference with the new version used here (given by Harrington and Raich in [11]) is basically that we can take this (1,1) vector field with more liberty, taking care that its coefficients make a positive semidefinite hermitian matrix with eigenvalues not bigger than one.

The approach for operator $\bar{\partial}_b$ is similar to what we gave for the operator $\bar{\partial}$, with suitable differences. For example since we work on manifolds with no boundary, boundary terms will not appear on, instead the Levi form will appear with terms with the totally real part of the tangent bundle, or more commonly known “bad direction". To get control of this term we will use microlocal analysis. The microlocal analysis is developed in Section 4.3, and we also prove some technical results in Sections 4.4 and 4.5 below. All this machinery together with the weak $Y(q)$ is used in Section 4.6 below. Finally, on the Subsection 4.6.1 we show the process to handle terms appearing on our main estimate (4.16), and so imply in our objective estimate (4.15).

This technique was developed by Nicoara [24] and refined by Harrington and Raich [10], and in fact many of results used in these works will be used here. Nicoara proved in [24] the closure of the range of $\bar{\partial}_b$ using a weak pseudoconvexity (the Levi form is positive semidefinite) on CR manifolds of hypersurface type with real dimension at least 5. Her result (when $\dim_{\mathbb{R}} \cdot \geq 5$) extends to those obtained in [2, 21, 28] who worked on boundaries of a pseudoconvex domain. Harrington and Raich proved in [10], closure of the range of $\bar{\partial}_b$, at level q, on CR manifolds of hypersurface type assuming a first version of the weak $Y(q)$ condition given in [10].

4.1 Definitions

Let $\mathbb{C}T(\mathbb{C}^n) := T(\mathbb{C}^n) \otimes \mathbb{C}$ denote the complexified tangent bundle of $T(\mathbb{C}^n)$. Let $\{z_1, ...z_n\}$ with $z_j = x_j + iy_j$ be holomorphic coordinates to $\mathbb{C}^n$. Define the vector fields

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \text{ and } \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

and for $z \in \mathbb{C}^n$ define $T_z^{1,0}(\mathbb{C}^n)$ and $T_z^{0,1}(\mathbb{C}^n)$ the complex vector spaces generated by $\{\partial/\partial z_1, ..., \partial/\partial z_n\}$ and $\{\partial/\partial \bar{z}_1, ..., \partial/\partial \bar{z}_n\}$ respectively. Then, is easy to see $T_z^{1,0}(\mathbb{C}^n) \cap T_z^{0,1}(\mathbb{C}^n) = \{0\}$ for any $z \in \mathbb{C}^n$, and $\mathbb{C}T_z(\mathbb{C}^n) = T_z^{1,0}(\mathbb{C}^n) \oplus T_z^{0,1}(\mathbb{C}^n)$ (with the usual Hermitian product). For $0 \le p, q \le n$ and a point $z \in \mathbb{C}^n$, define the space $\Lambda_z^{p,q}T^*(M) = \Lambda^p \left\{T_z^{*1,0}(\mathbb{C}^n)\right\} \hat{\otimes} \Lambda^q \left\{T_z^{*0,1}(\mathbb{C}^n)\right\}$, where $\hat{\otimes}$ denotes the antisymmetric tensor product, $T_z^{*1,0}$ denotes the space generated by $\{dz_1, ..., dz_n\}$ with $dz_j := dx_j + idy_j$, $T_z^{*0,1} = \overline{T_z^{*1,0}}$, and $\Lambda^p \left\{T_z^{*1,0}(\mathbb{C}^n)\right\}$ ($\Lambda^q \left\{T_z^{*0,1}(\mathbb{C}^n)\right\}$) denotes the p-th (q-th) exterior power of $T_z^{*1,0}(\mathbb{C}^n)$ ($T_z^{*0,1}(\mathbb{C}^n)$). And define the bundle of (p,q)-*forms* $\Lambda^{p,q}T^*(M)$ by $\bigcup_{z \in \mathbb{C}^n} \Lambda_z^{p,q}T^*(M)$.

Let M a real manifold of real dimension $2n-1$ with $n \ge 2$, and denote $T(M)$ the tangent bundle of M, and $\mathbb{C}T(M) := T(M) \otimes \mathbb{C}$ the complexified tangent vector bundle over M. A *CR structure manifold of hypersurface type* is defined as follows:

Definition 4.1.1 *Let M a smooth manifold of real dimensional $2n-1$. M is called a CR manifold of hypersurface type if M is equipped with a subbundle of the complexified tangent bundle $\mathbb{C}T(M)$ denoted by $T^{1,0}(M)$ satisfying:*

i $dim_{\mathbb{C}} T_x^{1,0}(M) = n-1$ where $T_x^{1,0}(M)$ is the fiber at each $x \in M$.

ii $T_x^{1,0}(M) \cap T_x^{0,1}(M) = \{0\}$ where $T_x^{0,1}(M)$ is the complex conjugate os $T_x^{1,0}(M)$.

iii If $L, L' \in T^{1,0}(M)$ then $[L, L'] := LL' - L'L$ is in $T^{1,0}(M)$.

$T^{1,0}(M)$ is called the CR structure of M.

If M is a submanifold of $\mathbb{C}^N$ of real dimension $2n-1$, for some $N \ge n$, such that the complex dimension of $T_z^{1,0}(M) := T_z^{1,0}(\mathbb{C}^N) \cap \{T_z(M) \otimes \mathbb{C}\}$ (under the natural inclusions) has complex dimension $n-1$ for all $z \in M$, we can let $T^{1,0}(M) = \bigcup_{z \in M} T_z^{1,0}(M)$, and this will define a CR structure on M of hypersurface type as follows: Obviously, (i) is satisfied. $T_z^{0,1}(M) := \overline{T_z^{1,0}(M)} = T_z^{0,1}(\mathbb{C}^N) \cap \{T_z(M) \otimes \mathbb{C}\}$, so $T_z^{1,0}(M) \cap T_z^{0,1}(M) = \{0\}$, then (ii) is satisfied. Since $T^{1,0}(M) = \left\{T^{1,0}(\mathbb{C}^N)|_M\right\} \cap \mathbb{C}T(M)$, and the bundle $T^{1,0}(\mathbb{C}^N)$ is involutive because the Lie bracket of any two vector fields spanned by $\{\partial/\partial z_1, ..., \partial/\partial z_n\}$ is again spanned by $\{\partial/\partial z_1, ..., \partial/\partial z_n\}$, and the bundle $\mathbb{C}T(M)$ is involutive because the tangent bundle of any manifold is involutive, we obtain (iii). We will call $T^{1,0}(M) = U_{z\in M}T_z^{1,0}(M)$ by induced CR structure on M.

In what follows, we will assume M being a smooth, orientable CR manifold of real dimension $2n-1$ of hypersurface type embedded in $\mathbb{C}^N$. Next we proceed to give the definition of our operator $\bar{\partial}_b$, and here we prefer to give a intrinsic approach of the for our purpose. To see the extrinsic approach, and a equivalence between these approaches, we invite the reader to see the Chapter 8 in [3].

4.1.1 The $\bar{\partial}_b$ operator

Here we give the definition of the operator $\bar{\partial}_b$ on a intrinsic way inspired in [3, Section 1], slightly different for our purposes. This approach could be done for any abstract CR manifold $(M, T^{1,0}(M))$, with a Hermitian metric defined on the complexified tangent bundle $\mathbb{C}T(M)$ such that $T^{1,0}(M)$ is orthogonal to $T^{0,1}(M) = \overline{T^{1,0}(M)}$. For each $p \in M$ we let X_p be the orthogonal complement of $T_p^{1,0}(M) \oplus T_p^{0,1}(M)$ in $\mathbb{C}T(M)$. The space $\{X_p, p \in M\}$ fit together smoothly (since $\mathbb{L}_p \oplus \overline{\mathbb{L}_p}$ does), and so the space $X(M) = \bigcup_{p\in M} X_p$ forms a subbundle of $\mathbb{C}T(M)$. Denote by $T^{*0,1}(M)$, $T^{*1,0}(M)$ and $X^*(M)$ the dual spaces of $T^{0,1}(M)$, $T^{1,0}(M)$ and $X(M)$ respectively. Define the bundles $\Lambda^{p,q}T^*(M) = \Lambda^p(T^{*1,0}(M))\hat{\otimes}\Lambda^q(T^{*0,1}(M))$ and $\Lambda_X^{p,q}T^*(M) = \Lambda^p(T^{*1,0}(M))\hat{\otimes}\Lambda^q(T^{*0,1}(M))\hat{\otimes}X^*(M)$, where $\hat{\otimes}$ denotes the antisymmetric tensor product. Since $dim_{\mathbb{C}}X(M) = 1$ we have $\Lambda^{p,q}T^*(M) = \{0\} = \Lambda_X^{p,q}T^*(M)$ when $p > n-1$ or when $q > n-1$. The pointwise metric on $\mathbb{C}T(M)$ induces a pointwise dual metric on $\mathbb{C}T^*(M)$ in the usual way. The metric for $\mathbb{C}T^*(M)$ extends to a metric on $\Lambda^r(\mathbb{C}T^*(M))$. So we have the following orthogonal decomposition

$$\Lambda^r(\mathbb{C}T^*(M)) = \left(\oplus_{p+q=r}\Lambda^{p,q}T^*(M)\right) \oplus \left(\oplus_{p+q=r-1}\Lambda_X^{p,q}T^*(M)\right).$$

Define $\Pi^{p,q} : \Lambda^r\mathbb{C}T^*(M) \to \Lambda^{p,q}T^*(M)$ and $\Pi_X^{p,q} : \Lambda^r\mathbb{C}T^*(M) \to \Lambda_X^{p,q}T^*(M)$ like the natural projection maps. For a open set $U \subset M$, denote by $\mathcal{E}^r(U)$ the space of r-forms on an open set $U \subset M$, $\mathcal{E}^{p,q}(U)$, $\mathcal{E}_X^{p,q}(U)$ the spaces of smooth section of $\Lambda^{p,q}T^*(M)$ and $\Lambda_X^{p,q}T^*(M)$ over U respectively, and $D_M^{p,q}(U)$ the space of compactly supported elements on U of $\mathcal{E}^{p,q}(U)$. We will omit U in these notations when $U = M$. Let $d_M : \mathcal{E}^r \to \mathcal{E}^{r+1}$ be the exterior derivative.

Definition 4.1.2 *The tangential Cauchy Riemann operator $\bar{\partial}_b : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}$ is defined by $\bar{\partial}_b := \Pi^{p,q+1} \circ d_M$.*

Now if $\phi \in \mathcal{E}^{1,0}$ and $\theta \in \mathcal{E}_X^{0,0}$ we have for $\bar{L}_1, \bar{L}_2 \in \overline{\mathbb{L}}$ then by the Cartan-Frobenius identity, and since $\mathbb{L}$ satisfies the condition of integrability (iii) on the definition of CR manifolds, we will have

$$\left(d_M\phi, \bar{L}_1 \wedge \bar{L}_2\right) = \bar{L}_1\left\{\left(\phi, \bar{L}_2\right)\right\} - \bar{L}_2\left\{\left(\phi, \bar{L}_1\right)\right\} - \left\{\phi, \left[\bar{L}_1, \bar{L}_2\right]\right\} = 0$$

$$\left(d_M\theta, \bar{L}_1 \wedge \bar{L}_2\right) = \bar{L}_1\left\{\left(\theta, \bar{L}_2\right)\right\} - \bar{L}_2\left\{\left(\theta, \bar{L}_1\right)\right\} - \left\{\theta, \left[\bar{L}_1, \bar{L}_2\right]\right\} = 0$$
$$(d_M\theta, L_1 \wedge L_2) = L_1\{(\theta, L_2)\} - L_2\{(\theta, L_1)\} - \{\theta, [L_1, L_2]\} = 0$$

Then $d_M(\mathcal{E}^{1,0}) \subset \mathcal{E}^{2,0} \oplus \mathcal{E}^{1,1} \oplus (\oplus_{r+s=1}\mathcal{E}_X^{r,s})$, and $d_M(\mathcal{E}_X^{0,0}) \subset \oplus_{r+s=1}\mathcal{E}_X^{r,s}$. And also we have $d_M(\mathcal{E}^{0,1}) \subset \mathcal{E}^{2,0} \oplus \mathcal{E}^{1,1} \oplus \mathcal{E}^{0,2} \oplus (\oplus_{r+s=1}\mathcal{E}_X^{r,s})$. In general case, by the product rule of d_M and the cases above, we have

$$d_M(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+2,q-1} \oplus \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1} \oplus (\oplus_{r+s=p+q}\mathcal{E}_X^{r,s}), \qquad d_M(\mathcal{E}_X^{p,q}) \subset \oplus_{r+s=p+q+1}\mathcal{E}_X^{r,s},$$

With this, if ϕ is a smooth (p,q)-form, that is ϕ is an element of $\mathcal{E}^{p,q}$, we have

$$\bar{\partial}_b\phi = d_M\phi - \left(\Pi^{p+2,q-1}d_M\phi + \Pi^{p+1,q}d_M\phi + \sum_{r+s=p+q}\Pi_X^{r,s}d_M\phi\right), \tag{4.1}$$

and

$$\begin{aligned}\bar{\partial}_b \circ \bar{\partial}_b\phi &= \Pi^{p,q+2}(d_M\bar{\partial}_b\phi)\\ &= -\Pi^{p,q+2}\left(d_M\Pi^{p+2,q-1}d_M\phi + d_M\Pi^{p+1,q}d_M\phi + \sum_{r+s=p+q}d_M\Pi_X^{r,s}d_M\phi\right)\\ &= 0\end{aligned}$$

So we will have the complex

$$0 \xrightarrow{\bar{\partial}_b} \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}_b} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}_b} .... \xrightarrow{\bar{\partial}_b} \mathcal{E}^{p,n-1} \xrightarrow{\bar{\partial}_b} 0.$$

Now if f is a smooth (p,q)-form and if g is a (r,s)-form, we will have from the product rule for the exterior derivative

$$\bar{\partial}_b(f\wedge g) = \Pi^{p+r,q+s+1}d_M(f\wedge g) = \Pi^{p+r,q+s+1}((d_Mf)\wedge g) + (-1)^{p+q}\Pi^{p+r,q+s+1}(f\wedge(d_Mg)),$$

By (4.1) we will have

$$\begin{aligned}\Pi^{p+r,q+s+1}((d_Mf)\wedge g) &= \Pi^{p+r,q+s+1}\Bigg(\bar{\partial}_bf\wedge g + (\Pi^{p+2,q-1}d_Mf)\wedge g + (\Pi^{p+1,q}d_Mf)\wedge g\\ &\qquad + \left(\sum_{r+s=p+q}\Pi_X^{r,s}d_Mf\right)\wedge g\Bigg)\\ &= (\bar{\partial}_bf)\wedge g\end{aligned}$$

and also in the same way we have $\Pi^{p+r,q+s+1}(f\wedge(d_Mg)) = f\wedge\bar{\partial}_bg$. Then we will have the product rule for $\bar{\partial}_b$

$$\bar{\partial}_b(f\wedge g) = (\bar{\partial}_bf)\wedge g + (-1)^{p+q}f\wedge\bar{\partial}_bg.$$

On analogous way, the operator $\partial_b : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}$ is defined by $\partial_b := \Pi^{p+1,q} \circ d_M$. We are going to consider just smooth, orientable CR manifolds of hypersurface type embedded in a complex space $\mathbb{C}^N$ with the induced CR structure. It is therefore only natural to choose as a metric the restriction on $\mathbb{C}T(M)$ of the natural Hermitian inner product on $\mathbb{C}^N$. And this metric will be compatible with the induced CR structure, i.e., the vector spaces $T_z^{1,0}(M)$ and $T_z^{0,1}(M)$ will be orthogonal spaces under this inner product. Let ω the real 2-form of type (1,1) associated to this Hermitian metric. So if $L, L' \in T^{1,0}(M)$ then the inner product $\langle L, L' \rangle$ between L and L' is given by

$$\langle L, L' \rangle = \omega(i\overline{L'} \wedge L).$$

We can define a Hermitian inner product on $\mathcal{E}^{0,q}(M)$ by

$$(f, g)_0 = \int \langle f, g \rangle_\omega \, dV$$

where dV is the volume element on M and $\langle f, g \rangle_\omega$ is the induced inner product on $\Lambda^{0,q}(M)$ by ω. The Hermitian inner product above gives rise to an L^2-norm $\|.\|_0$, and we also denote the closure of $\bar{\partial}_b$ in this norm by $\bar{\partial}_b$ (by an abuse of notation). In this way, $\bar{\partial}_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ is a well-defined, closed, densely defined operator, and we define $\bar{\partial}_b^* : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ to be the L^2 adjoint of $\bar{\partial}_b$. The Kohn Laplacian $\Box_b : L^2_{0,q}(M) \to L^2_{0,q}(M)$ is defined as

$$\Box_b := \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*,$$

which is also an unbounded, closed, densely defined operator.

4.1.2 The Levi form

Next, we will assume M orientable and we give a definition of the Levi form as follows. Let γ be a purely imaginary global 1-form on M, that is $\bar{\gamma} = -\gamma$ such that

(a) annihilates $T^{1,0}(M) \bigoplus T^{0,1}(M)$

(b) If $L_1, ..., L_{n-1}$ is a basis of the (1,0)-vector fields in a neighborhood U of one point in M, T is a vector field taken purely imaginary ($\bar{T} = -T$) on U such that $L_1, ..., L_{n-1}, \bar{L}_1, ..., \bar{L}_{n-1}, T$ generate $T(U)$, then $\langle \gamma, T \rangle = -1$ (Here $\langle \cdot, \cdot \rangle$ is related by the metric ω).

The Levi form at a point $x \in M$ is the Hermitian form given by $\left(d\gamma_x, L \wedge \bar{L}' \right)$ where L and L' are two vectors fields in $T_x^{1,0}(U)$, and U is a neighborhood of $x \in M$. For $L, L' \in T^{1,0}(M)$ and by Cartan's Formula and by (a), we have

$$\langle d\gamma, L \wedge \bar{L}' \rangle = -\langle \gamma, \left[L, \bar{L}' \right] \rangle. \tag{4.2}$$

And if c_{jk}^L are such that

$$\left[L_j, \overline{L}_k \right] = c_{jk}^L T \; mod \; T^{1,0}(U) \oplus T^{0,1}(U) \;\; \forall 1 \leq j, k \leq n-1$$

we have $\langle d\gamma, L_j \wedge \overline{L}_k\rangle = c^L_{jk}$. We will call $\left[c^L_{jk}\right]_{1\leq j,k\leq n-1}$ the Levi matrix respect to $L_1, ..., L_{n-1}$. Now, if $S_1, ..., S_{n-1}$ is another basis for $T^{1,0}(M)$ and the change of basis matrix is given by a non singular matrix B then the Levi matrix $[c^S_{jk}]$ respect to the basis $S_1, ..., S_{n-1}$ will be equal to $B^*[c^L_{jk}]B$, where B^* denotes the Hermitian transpose of the matrix B. So the *inertia*, that is, the number of positive, negative and zero eigenvalues (all counting multiplicity), of the the Levi matrix is preserved. Even more, if we assume $L_1, ..., L_{n-1}$ and $S_1, ..., S_{n-1}$ as orthonormal basis, then the eigenvalues of the Levi forms are preserved. When there is no danger of confusion, we drop the superscript L in the notation of the Levi matrix. The CR structure is called (strictly) *pseudoconvex* in some point $p \in M$ if the matrix $[c_{jk}(p)]$, is positive (definite) semidefinite. If the CR structure is (strictly) pseudoconvex in every point, then it is called (strictly) pseudoconvex. And if the matrix $[c_{jk}]$ vanishes completely on a open set $U \subset M$, M is called Levi flat. We say that the CR structure has the $Z(q)$ *property* in some point $p \in M$ if the Levi matrix in the point p has at least $n - q$ positive eigenvalues or at least $q + 1$ negative eigenvalues. And we say that the CR structure has the $Z(q)$ property if it has this property at every point.

4.1.3 The weak $Z(q)$ condition

Now, we introduce the main geometrical hypothesis, given by Harrington and Raich on [11].

Definition 4.1.3 *Let M be a smooth, compact, oriented CR manifold of hypersurface type of real dimension $2n-1$. For $1 \leq q \leq n-1$ we say M satisfies the weak $Z(q)$ condition if there exist a real $\Upsilon \in T^{1,1}(U)$ satisfying*

(A) $|\theta|^2 \geq (i\theta \wedge \overline{\theta})(\Upsilon) \geq 0$ *for all* $\theta \in \Lambda^{1,0}(M)$

(B) $\mu_1 + \mu_2 + ... + \mu_q - i\,\langle d\gamma, \Upsilon\rangle \geq 0$ *on U, where $\mu_1, ..., \mu_{n-1}$ are the eigenvalues of Levi matrix $[c_{jk}]$ (respect to an orthonormal basis) in increasing order.*

(C) $\omega(\Upsilon) \neq q$.

We say M satisfies the weak $Y(q)$ condition if the weak $Z(q)$ and weak $Z(n-1-q)$ conditions are satisfied.

Remark Assume the weak $Z(q)$ condition is satisfied. In a sufficiently small open set $U \subset M$, we can write $\Upsilon = i\sum_{j,k=1}^{n-1} b_{jk}\bar{L}_k \wedge L_j$, in some orthonormal basis $L_1, ..., L_{n-1}$ of $T^{1,0}(U)$, where $[b_{jk}]$ is a hermitian matrix. If we choose a local orthonormal basis $L_1, ..., L_{n-1}$ of $T^{1,0}(U)$ such that the Levi matrix $[c_{jk}(x)]$, at a point $x \in U$, is a diagonal matrix with diagonal entries $\lambda_1, ..., \lambda_{n-1}$ in increasing order, by (A), the diagonal entries of the matrix $[b_{jk}]$ are in $[0,1]$, and so we will have

$$\begin{aligned}
0 &\leq \lambda_1 + ... + \lambda_q - \lambda_1 b_{11} - \lambda_2 b_{22} - ... - \lambda_{n-1} b_{(n-1)(n-1)} \\
&\leq \lambda_1(1-b_{11}) + \lambda_2(1-b_{22}) + ... + \lambda_q(1-b_{qq}) - \lambda_{q+1} b_{(q+1)(q+1)} - ... - \lambda_{n-1} b_{(n-1)(n-1)} \\
&\leq \lambda_q(q - b_{11} - ... - b_{qq}) - \lambda_q(b_{(q+1)(q+1)} - ... - b_{(n-1)(n-1)}) \\
&= \lambda_q(q - \omega(\Upsilon)), \qquad (4.3)
\end{aligned}$$

and also

$$\begin{aligned}
0 &\leq \lambda_1 + ... + \lambda_q - \lambda_1 b_{11} - \lambda_2 b_{22} - ... - \lambda_{n-1} b_{(n-1)(n-1)} \\
&\leq \lambda_1(1-b_{11}) + \lambda_2(1-b_{22}) + ... + \lambda_q(1-b_{qq}) - \lambda_{q+1} b_{(q+1)(q+1)} - ... - \lambda_{n-1} b_{(n-1)(n-1)} \\
&\leq \lambda_{q+1}(q - b_{11} - ... - b_{qq}) - \lambda_{q+1}(b_{(q+1)(q+1)} - ... - b_{(n-1)(n-1)}) \\
&= \lambda_{q+1}(q - \omega(\Upsilon)). \qquad (4.4)
\end{aligned}$$

So, a necessary condition of the weak $Z(q)$ condition appears as follows: if $\omega(\Upsilon) < q$, by (4.3) we will have $\lambda_q \geq 0$ and so, the Levi matrix will have at least $n-q$ nonnegative eigenvalues in x. Now, if $\omega(\Upsilon) > q$, by (4.4) the Levi matrix will have at least $q+1$ nonpositive eigenvalues in x.

Note that the Definition 4.1.3 requires global existence of a real $\Upsilon \in T^{1,1}(M)$ satisfying conditions (A), (B) and (C). Nevertheless conditions (A), (B) are local properties and the third one is local modulo bounded connected components, as it was noted in [11].

Lemma 4.1.4 *For $1 \leq q \leq n-1$, let $\sigma : M \to \{-1, 1\}$ continuous, and suppose for every $p \in M$ there exist an open set U_p such that $U_p \cap M$ is connected and a real $\Upsilon_p \in T^{1,1}(U_p)$ satisfying*

(a) $|\theta|^2 \geq (i\theta \wedge \overline{\theta})(\Upsilon) \geq 0$ *for all* $\theta \in \Lambda^{1,0}(U_p)$

(b) $\mu_1 + \mu_2 + ... + \mu_q - i\langle d\gamma, \Upsilon\rangle \geq 0$ *on U_p, where $\mu_1, ..., \mu_{n-1}$ are the eigenvalues of Levi matrix $[c_{jk}]$ in increasing order.*

(c) $\sigma(p)(\omega(\Upsilon) - q) > 0$ *on* U_p.

Then M satisfies the weak $Z(q)$ condition.

Proof. Since M is compact, there exist a finite cover $\cup_j U_{p_j}$ for M. Let $\{\chi_j\}_j$ a partition of unity subordinate to $\cup_j U_{p_j}$. If we take $\Upsilon = \sum_j \chi_j \Upsilon_{p_j}$, the conditions (A), (B) are satisfied by linearity. Now, if $x \in M$, let $j_1, ..., j_s$ such that $\chi_{j_r}(x) \neq 0$, and $\sum_{r=1}^s \chi_{j_r}(x) = 1$. Since $\cup_{r=1}^s U_{p_{j_r}}$ is connected (because x is there) we will have $\sigma(p_{j_r}) = \sigma(x)$ for $r = 1, ..., s$, it follows that

$$(\omega(\Upsilon) - q\ )|_x = \sum_{r=1}^s \chi_{j_r}(x)\Big(\omega(\Upsilon_{p_{j_r}}) - q\Big)\Big|_x = \sum_{r=1}^s \chi_{j_r}(x)\sigma(x)\sigma(p_{j_r})\Big(\omega(\Upsilon_{p_{j_r}}) - q\Big)\Big|_x$$

$$= \sigma(x)\sum_{r=1}^{s}\chi_{j_r}(x)\sigma(p_{j_r})\Big(\omega(\Upsilon_{p_{jr}})-q\Big)\Big|_x \neq 0,$$

because, in the last equality the sum is a positive number. ■

Examples of weak $Z(q)$ include:

- If the CR structure M is pseudoconvex then it is sufficient consider $\Upsilon = 0$ to imply in the weak $Z(q)$ condition, for any $1 \leq q \leq n-1$.

- $\Upsilon = 0$ also works if the sum of any q eigenvalues of the Levi matrix is nonnegative.

- If $Z(q)$ is satisfied, choose a local orthonormal coordinates $L_1, ..., L_{n-1}$ of $T^{1,0}(U)$ for some neighborhood U in M, where the Levi form is a diagonal matrix at x, that is, $c_{jk}|_x = \delta_{jk}\mu_j$. If there exist at least $n-q$ positive eigenvalues, then $\mu_q > 0$, so we could take $\Upsilon_x = i\sum_{j=1}^{q-1} L_j \wedge \bar{L}_j$ which satisfies (A),

$$0 < \mu_q = \mu_1 + ... + \mu_q - i\left\langle d\gamma, \Upsilon_x\right\rangle,$$

and also $1 = q - \omega(\Upsilon_x) > 0$, and by continuity, this last inequalities will be satisfied on a neighborhood of x. Note that, if for x there exists another Υ'_x satisfying (A) and (B) and (C) in some neighborhood of x then by (4.3) we will have

$$0 \leq \mu_q(q - \omega(\Upsilon'_x)),$$

so, $\omega(\Upsilon'_x) < q$. Thus, the inequality $\omega(\Upsilon_y) < q$ will be satisfied for any y in the connected component of x in M. In the same way, if there exist at least $q+1$ negative eigenvalues at x we choose $\Upsilon_x = i\sum_{j=1}^{q+1} L_j \wedge \bar{L}_j$, which satisfies condition (A),

$$0 < -\mu_{q+1} = \mu_1 + ... + \mu_q - i\left\langle d\gamma, \Upsilon_x\right\rangle,$$

on a neighborhood of x and the inequality $\omega(\Upsilon_y) > q$ will be satisfied for any y in the connected component of x in M. Thus, by Lemma 4.1.4, the weak $Z(q)$ condition will be satisfied on M.

- The argument of the previous item, could be used also when the Levi form has a local diagonalization with increasing entries along the diagonal, and the Levi matrix on each connected components, has at least $n-q$ nonnegative eigenvalues (so we take $\Upsilon_x = i\sum_{j=1}^{q-1} L_j \wedge \bar{L}_j$) or it has at least $q+1$ nonpositive eigenvalues (we take $\Upsilon_x = i\sum_{j=1}^{q+1} L_j \wedge \bar{L}_j$).

The condition weak Z(q) appeared first in [10] where the matrix $[b_{jk}]$ was considered a diagonal matrix with entries like been zero or one. Below, we will refer the weak $Z(q)$, and so the weak $Y(q)$, condition by the concept given in this present work.

If M is a CR manifold satisfying the $Y(q)$ condition, then the Υ corresponding to weak-$Z(q)$ does not need to have relation with the other one corresponding to weak-$Z(n-q-1)$. We will use the notation Υ_q to refer the weak-$Z(q)$ and Υ_{n-q-1} to the weak-$Z(n-q-1)$.

4.2 Special functions

In this section, we want to compare the Hermitian forms $\frac{1}{2}\left(\partial_b\bar{\partial}_b - \bar{\partial}_b\partial_b\right)\phi$ and $\frac{1}{2}\left(\partial\bar{\partial} - \bar{\partial}\partial\right)\phi$, restricted to elements in $T^{1,0}(M)$, for smooth functions ϕ defined on a neighborhood of $M \subset \mathbb{C}^N$. We recall that M is a smooth, orientable CR manifold of real dimension $2n-1$ of hypersurface type embedded in $\mathbb{C}^N$.

Let $U = \tilde{U} \cap \mathbb{C}^N$, with $\tilde{U}$ an open set in $\mathbb{C}^N$, be a local path of M, $\{L_1, ..., L_{n-1}\}$ a local basis of $T^{1,0}(U)$, T a tangential vector totally imaginary such that $\left\{L_1, ..., L_{n-1}, \bar{L}_1, ..., \bar{L}_{n-1}, T\right\}$ is a local basis for $\mathbb{C}T(U)$. Let ω_j, $1 \le j \le n-1$, and γ be the dual elements of L_j, $1 \le j \le n-1$, and T respectively. Define c^i_{jk} as the L_i-component of $\left[L_j, \bar{L}_k\right]$. By the definition of $\bar{\partial}_b$, ∂_b, and Cartan's formula, we have

$$c^i_{jk} = \omega_i(\left[L_j, \bar{L}_k\right]) = -\bar{\partial}_b\omega_i(L_j \wedge \bar{L}_k).$$

Then

$$\bar{\partial}_b\omega_i = -\sum_{j,k=1}^{n-1} c^i_{jk}\omega_j \wedge \bar{\omega}_k, \qquad \partial_b\bar{\omega}_i = \sum_{j,k=1}^{n-1} \overline{c^i_{kj}}\omega_j \wedge \bar{\omega}_k,$$

and

$$\left[L_j, \bar{L}_k\right] = c_{jk}T + \sum_{i=1}^{n-1} c^i_{jk}L_i - \sum_{i=1}^{n-1} \overline{c^i_{kj}}\bar{L}_i. \tag{4.5}$$

where c_{jk} is the Levi matrix associated to the basis $\left\{L_1, ..., L_{n-1}, \bar{L}_1, ..., \bar{L}_{n-1}, T\right\}$. Using the product rule for $\bar{\partial}_b$, for a smooth function ϕ on $\tilde{U}$, we have

$$\partial_b\bar{\partial}_b\phi = \sum_{j,k=1}^{n-1}\left(L_j\bar{L}_k(\phi) + \sum_{i=1}^{n-1}\overline{c^i_{kj}}\bar{L}_i(\phi)\right)\omega_j \wedge \bar{\omega}_k.$$

Similarly we have

$$\bar{\partial}_b\partial_b\phi = \sum_{j,k=1}^{n-1}\left(-\bar{L}_kL_j(\phi) + \sum_{i=1}^{n-1} c^i_{jk}L_i(\phi)\right)\omega_j \wedge \bar{\omega}_k.$$

Then for $1 \le j, k \le n-1$

$$\frac{1}{2}\left(\partial_b\bar{\partial}_b\phi - \bar{\partial}_b\partial_b\phi, L_j \wedge \bar{L}_k\right) = \frac{1}{2}\left(L_j\bar{L}_k + \bar{L}_kL_j + \sum_{i=1}^{n-1}\left(\overline{c^i_{kj}}\bar{L}_i(\phi) + c^i_{jk}L_i(\phi)\right)\right).$$

On the other hand, let $L_n, ..., L_N$ be vector fields in $T^{1,0}(\tilde{U})$ such that $\{L_1, .., L_{n-1}, L_n, ..., L_N\}$ forms a basis for $T^{1,0}(\tilde{U})$. Denote ω_j, for $n \le j \le N$, the dual forms of L_j, for $n \le j \le N$. Let θ^i_{jk} smooth functions such that $\bar{\partial}\omega_i = \sum_{j.k=1}^N \theta^i_{jk}\bar{\omega}_j \wedge \omega_k$. Then by the definition of $\bar{\partial}$ and using Cartan's formula we have

$$\theta^i_{jk} = \bar{\partial}\omega_i(\bar{L}_j \wedge L_k) = -d\omega_i(\big[L_k \wedge \bar{L}_j\big]) = \omega_i(\big[L_k, \bar{L}_j\big]). \tag{4.6}$$

$\theta^i_{jk} = c^i_{kj}$ for $1 \le j, k, i \le n-1$. Also, for $1 \le j, k \le n-1$ we have

$$\frac{1}{2}\left(\partial\bar{\partial}\phi - \bar{\partial}\partial\phi, L_j \wedge \bar{L}_k\right) = \frac{1}{2}\left(L_j\bar{L}_k + \bar{L}_k L_j + \sum_{i=1}^N \left(\overline{\theta^i_{jk}}\bar{L}_i(\phi) + \theta^i_{kj}L_i(\phi)\right)\right).$$

Then if $L = \sum_{j=1}^{n-1} \xi_j L_j$ is an element in $T^{1,0}(U)$

$$\frac{1}{2}\left((\partial\bar{\partial}\phi - \bar{\partial}\partial\phi) - (\partial_b\bar{\partial}_b\phi - \bar{\partial}_b\partial_b\phi), L \wedge \bar{L}\right) = \frac{1}{2}\sum_{j,k=1}^{n-1} \xi_j\bar{\xi}_k \left(\sum_{i=n}^N(\overline{\theta^i_{jk}}\bar{L}_i\phi + \theta^i_{kj}L_i\phi)\right). \tag{4.7}$$

Since $T \in \mathbb{C}T(\mathbb{C}^N)$ (under the natural inclusions) we have that

$$T = \sum_{i=1}^{n-1} \alpha_i L_i + \sum_{i=1}^{n-1} \beta_i \bar{L}_i + \sum_{i=n}^N r_i L_i + \sum_{i=n}^N s_i \bar{L}_i.$$

for smooth functions α_i, β_i, r_i, s_i. By (4.5) and (4.6) we have $r_i c_{jk} = \theta^i_{kj}$ and $s_i c_{jk} = -\overline{\theta^i_{jk}}$ for $n \le i \le N$. Then by (4.7) we have

$$\frac{1}{2}\left((\partial\bar{\partial}\phi - \bar{\partial}\partial\phi) - (\partial_b\bar{\partial}_b\phi - \bar{\partial}_b\partial_b\phi), L \wedge \bar{L}\right) = \frac{1}{2}\sum_{j,k=1}^{n-1} \xi_j\bar{\xi}_k c_{jk} \left(\sum_{i=n}^N(r_i L_i\phi - s_i\bar{L}_i\phi)\right).$$

Observe here that $\nu := \sum_{i=n}^N(r_i L_i - s_i\bar{L}_i)$ is a totally real vector field in $\mathbb{C}T(\tilde{U})$ $(\bar{\nu} = \nu)$, because $\overline{T} = -T$ implies $\overline{r_i} = -s_i$. Now, if we choose $L_n, ..., L_N$ such that $\{L_1, ..., L_N\}$ forms an orthonormal basis for $T^{1,0}(\tilde{U})$, ν is a real vector field orthogonal to $\mathbb{C}T(U)$ ($\langle \nu, L_j\rangle = 0$, $1 \le j \le n-1$ and $\langle \nu, T\rangle = |r_i|^2 - |s_i|^2 = 0$). As a conclusion we made above, we establish the next proposition.

Proposition 4.2.1 *Let M be a smooth, orientable CR manifold of real dimension $2n-1$ of hypersurface type embedded in $\mathbb{C}^N$. If ϕ is smooth function defined on a neighborhood of M, and $L \in T^{1,0}(M)$ then*

$$\frac{1}{2}\left\langle(\partial\bar{\partial}\phi - \bar{\partial}\partial\phi) - (\partial_b\bar{\partial}_b\phi - \bar{\partial}_b\partial_b\phi), L \wedge \bar{L}\right\rangle = \frac{1}{2}\nu(\phi)\left\langle d\gamma, L \wedge \bar{L}\right\rangle.$$

where ν is a smooth real vector field in $\mathbb{C}T(\mathbb{C}^N)$, $\bar{\nu} = \nu$, and orthogonal to $\mathbb{C}TM$.

Observe that since we are working on compact smooth manifold M, $\nu(\phi)$ will be a bounded quantity.

As usual, when it is trying to get L^2 estimates to the operators $\bar{\partial}$, $\bar{\partial}_b$, we need to consider well behaved global functions like weight functions. Let λ be a smooth function defined near to $M \subset \mathbb{C}^n$. We define the next 2-form

$$\Theta^\lambda := \frac{1}{2}(\partial_b\bar{\partial}_b\lambda - \bar{\partial}_b\partial_b\lambda) + \frac{1}{2}\nu(\lambda)d\gamma.$$

Also, we will consider the $(n-1)\times(n-1)$ matrix $\left[\Theta^\lambda_{jk}\right]$ with entries $\Theta^\lambda_{jk} := \langle \Theta^\lambda, L_j \wedge \bar{L}_k\rangle$.

The importance of this 2-form will be seen in computations to control terms whose allow us imply in our basic estimate (4.16), in Section 4.6. By Proposition 4.2.1, we can see if $\lambda = |z|^2$ then we will have $\Theta^{|z|^2} = \partial\bar{\partial}(|z|^2) = -i\omega$. So, if M satisfies the weak $Z(q)$ condition then in local coordinates

- $b_1 + \cdots + b_q - i\left\langle \Theta^\lambda, \Upsilon_q\right\rangle = q - \sum_{l=1}^{n-1} b_{ll} > 0$ if $\omega(\Upsilon_q) = \sum_{l=1}^{n-1} b_{ll} < q$.
- $b_{n-1} + \cdots + b_{n-q} - i\left\langle \Theta^\lambda, \Upsilon_q\right\rangle = q - \sum_{l=1}^{n-1} b_{ll} < 0$ if $\omega(\Upsilon_q) = \sum_{l=1}^{n-1} b_{ll} > q$.

where $\{b_1, ..., b_{n-1}\}$ denote the eigenvalues of $\Theta^{|z|^2}_{jk} = \langle \Theta^{|z|^2}, L_j \wedge \bar{L}_k\rangle = \omega(i\bar{L}_k \wedge L_j)$ in increasing order. Since the inequality is strictly, by compactness of M, there will exist a positive constant B_q such that $\min_M |q - \omega(\Upsilon)| > B_q$.

4.3 Pseudodifferential operators

We will follow the setup for the microlocal analysis in [25]. By the compactness of M, there exists a finite cover $\{U_\nu\}_\nu$, so each U_ν has a special boundary system and can be parameterized by a hypersurface in $\mathbb{C}^n$ (U_ν may be shrunk as necessary).

Let $\xi = (\xi_1, ..., \xi_{2n-2}, \xi_{2n-1}) = (\xi', \xi_{2n-1})$ be the coordinates in Fourier space so that ξ' is the dual variable to the part of $T(M)$ in the maximal complex subspace $(T^{1,0}(M) \oplus T^{0,1}(M))$ and ξ_{2n-1} is dual to the totally real part of $T(M)$, i.e., the“bad" direction T. Define

$$\mathcal{C}^+ = \left\{\xi : \xi_{2n-1} \geq \frac{1}{2}|\xi'| \text{ and } |\xi| \geq 1\right\}; \quad \mathcal{C}^- = \left\{\xi : -\xi \in \mathcal{C}^+\right\};$$
$$\mathcal{C}^0 = \left\{\xi : -\frac{3}{4}|\xi'| \leq \xi_{2n-1} \geq \frac{3}{4}|\xi'|\right\} \cup \{\xi : |\xi| \leq 1\}.$$

$\mathcal{C}^+$ and $\mathcal{C}^-$ are disjoint, but both intersect $\mathcal{C}^0$ nontrivially. Next, we define smooth functions ψ^+, ψ^- and ψ^0, on $\{|\xi| : |\xi|^2 = 1\}$. Let

$$\psi^+(\xi) = 1 \text{ when } \xi_{2n-1} \geq \frac{3}{4}|\xi'| \text{ and } supp\ \psi^+ \subset \left\{\xi : \xi_{2n-1} \geq \frac{1}{2}|\xi'|\right\};$$

$$\psi^-(\xi) = \psi^+(-\xi); \quad \psi^0(\xi) \text{ satisfies } \psi^0(\xi)^2 = 1 - \psi^+(\xi)^2 - \psi^-(\xi)^2.$$

Extend ψ^+, ψ^-, and ψ^0 homogeneously outside of the unit ball, i.e., if $|\xi| \geq 1$, then

$$\psi^+(\xi) = \psi^+(\xi/|\xi|), \quad \psi^-(\xi) = \psi^-(\xi/|\xi|), \quad \text{and} \quad \psi^0(\xi) = \psi^0(\xi/|\xi|).$$

Also, extend ψ^+, ψ^- and ψ^0 smoothly inside the unit ball so that $(\psi^+)^2 + (\psi^-)^2 + (\psi^0)^2 = 1$. Finally, for a fixed constant $A > 0$ to be chosen later, define for any $t > 0$.

$$\psi_t^+(\xi) = \psi^+(\xi/(tA)), \psi_t^-(\xi) = \psi^-(\xi/(tA)), \text{ and } \psi^0(\xi) = \psi^0(\xi/(tA)).$$

Next, let Ψ_t^+, Ψ_t^-, and Ψ_t^0 be the pseudodifferential operators of order zero with symbols ψ_t^+, ψ_t^-, and ψ_t^0, respectively. The equality $(\psi_t^+)^2 + (\psi_t^-)^2 + (\psi_t^0)^2 = 1$ implies that

$$(\Psi_t^+)^*\Psi_t^+ + (\Psi_t^-)^*\Psi_t^- + (\Psi_t^0)^*\Psi_t^0 = Id.$$

We will use pseudodifferential operators that"dominate" a given pseudodifferential operator. Let ψ be a cut-off function and $\tilde\psi$ be another cut-off function so that $\tilde\psi|_{supp\psi} \equiv 1$. If Ψ and $\tilde\Psi$ are pseudodifferential operators with symbols ψ and $\tilde\psi$, respectively, then we say that $\tilde\Psi$ *dominates* Ψ.

For each ν, we can define Ψ_t^+, Ψ_t^- and Ψ_t^0 to act on functions or forms supported in U_ν, so let $\Psi_{\nu,t}^+, \Psi_{\nu,t}^-$ and $\Psi_{\nu,t}^0$ be the pseudodifferential operators of order zero defined on U_ν, and $\mathcal{C}_\nu^+, \mathcal{C}_\nu^-$ and $\mathcal{C}_\nu^0$ be the regions of $\xi-$space dual to U_ν on which the symbol of each of those pseudodifferential operators is supported. Then it follows that

$$(\Psi_{\nu,t}^+)^*\Psi_{\nu,t}^+ + (\Psi_{\nu,t}^-)^*\Psi_{\nu,t}^- + (\Psi_{\nu,t}^0)^*\Psi_{\nu,t}^0 = Id. \tag{4.8}$$

Let $\tilde\Psi_{\mu,t}^+$ and $\tilde\Psi_{\mu,t}^-$ be pseudodifferential operators that dominate $\Psi_{\mu,t}^+$ and $\Psi_{\mu,t}^-$ respectively (where $\Psi_{\mu,t}^+$ and $\Psi_{\mu,t}^-$ are defined on some U_μ). If $\tilde{\mathcal{C}}_\mu^+$ and $\tilde{\mathcal{C}}_\mu^-$ are the supports of the symbols of $\tilde\Psi_{\mu,t}^+$ and $\tilde\Psi_{\mu,t}^-$, respectively, then we can choose $\{U_\mu\}$, $\tilde\psi_{\mu,t}^+$, and $\tilde\psi_{\mu,t}^-$ so that the following result holds.

Lemma 4.3.1 *Let M be a compact, orientable, embedded CR manifold. There is a finite open covering $\{U_\mu\}_\mu$ of M so that if $U_\mu, U_\nu \in \{U_\mu\}$ have nonempty intersection, then there exits a diffeomorphism ϑ between U_ν and U_μ with Jacobian $\mathcal{J}_\vartheta$ such that* *(i)* ${}^t\mathcal{J}_\vartheta(\mathcal{C}^+_\mu) \cap \mathcal{C}^-_\nu = \emptyset$ *and* $\mathcal{C}^+_\nu \cap {}^t\mathcal{J}_\vartheta(\mathcal{C}^-_\nu) = \emptyset$ *where* ${}^t\mathcal{J}_\theta$ *is the inverse of the transpose of the Jacobian of* ϑ*;*

(ii) let ${}^\vartheta\Psi^+_{t,\mu}$, ${}^\vartheta\Psi^-_{t,\mu}$ *and* ${}^\vartheta\Psi^0_{t,\mu}$ *be the transfer of* $\Psi^+_{t,\mu}$, $\Psi^-_{t,\mu}$ *and* $\Psi^0_{t,\mu}$*, respectively via* ϑ*, then on* $\left\{\xi : \xi_{2n-1} \geq \frac{4}{5}|\xi'| \text{ and } |\xi| \geq (1+\varepsilon)tA\right\}$*, the principal symbol of* ${}^\vartheta\Psi^+_{t,\mu}$ *is identically equal to 1, on* $\left\{\xi : \xi_{2n-1} \leq -\frac{4}{5}|\xi'| \text{ and } |\xi| \geq (1+\varepsilon)tA\right\}$*, the principal symbol of* ${}^\vartheta\Psi^-_{t,\mu}$ *is identically equal to 1, and on* $\left\{\xi : -\frac{1}{3}|\xi'| \leq \xi_{2n-1} \leq \frac{1}{3}|\xi'| \text{ and } |\xi| \geq (1+\varepsilon)tA\right\}$*, the principal symbol of* ${}^\vartheta\Psi^0_{t,\mu}$ *is identically equal to 1, where* $\varepsilon > 0$ *and can be very small.*

(iii) Let ${}^\vartheta\tilde{\Psi}^+_{t,\mu}$, ${}^\vartheta\tilde{\Psi}^-_{t,\mu}$ *be the transfer via* ϑ *of* $\tilde{\Psi}^+_{t,\mu}$, $\tilde{\Psi}^-_{t,\mu}$ *respectively. Then the principal symbol of* ${}^\vartheta\tilde{\Psi}^+_{t,\mu}$ *is identically 1 on* $\mathcal{C}^+_\nu$ *and the principal symbol of* ${}^\vartheta\tilde{\Psi}^-_{t,\mu}$ *is identically 1 on* $\mathcal{C}^-_\nu$*;*

(iv) $\tilde{\mathcal{C}}^+_\nu \cap \tilde{\mathcal{C}}^-_\nu = \emptyset$.

We will suppress the left superscript ϑ as it should be clear from the context which pseudodifferential operator must be transferred. The proof of this lemma is contained in Lemma 4.3 and its subsequent discussion in [24]. If P is any of the operators $\Psi^+_{t,\mu}$, $\Psi^-_{t,\mu}$ or $\Psi^0_{t,\mu}$ then it is immediate that

$$D^\alpha_\xi \sigma(P) = \frac{1}{|t|^\alpha} q_\alpha(x,\xi)$$

for $|\alpha| \geq 0$, where $q_\alpha(x,\xi)$ is bounded independently of t.

4.4 Norms

Considering the inner product $(\cdot,\cdot)$ defined above, if ϕ is a real function defined on M, we define the weighted inner product for $(0,q)$-forms f and g, denoted by $(f,g)_\phi$, by $(f,g)_\phi = \left(e^{-\phi}f, g\right)$. For example, if $f = \sum_{J\in\mathcal{I}_q} f_J\bar{\omega}^J$ is a $(0,q)$-form supported on neighborhood U, where $\mathcal{I}_q = \{J = \{j_1,...,j_q\} : 1 \leq j_1 < ... < j_q \leq n-1\}$, and $\omega^J = \omega_{j_1} \wedge \cdots \wedge \omega_{j_q}$, with $\{\omega_1,\ldots,\omega_{n-1}\}$ a local orthonormal basis for the $(1,0)$-forms, we have $\|f\|_\phi = \sum_{J\in\mathcal{I}_q}\|f_J\|_\phi$ where $\|f_J\|_\phi = \int_M |f_J|^2 e^{-\phi}dV$, and we denote the corresponding weighted L^2 space by $L^2_{0,q}(M,e^{-\phi})$.

We now construct a norm that is well adapted to the microlocal analysis. Let $\{U_\mu\}_\mu$ be a covering of M that admits the families of pseudodifferential operators $\left\{\Psi^+_{\mu,t}, \Psi^-_{\mu,t}, \Psi^0_{\mu,t}\right\}$ and a partition of unity $\{\zeta_\mu\}_\mu$ subordinate to the cover satisfying $\sum_\mu \zeta^2_\mu = 1$. For each μ

let $\tilde{\zeta}_\mu$ be a cutoff function that dominates ζ_μ such that supp $\zeta_\mu \in U_\mu$, and ϕ^+, ϕ^- smooth functions defined on M. We define the global inner product and norm as follows:

$$(f,g)_\pm := \sum_\mu \Big[\Big(\tilde{\zeta}_\mu \Psi^+_{\mu,t}\zeta_\mu f^\mu, \tilde{\zeta}_\mu \Psi^+_{\mu,t}\zeta_\mu g^\mu \Big)_{\phi^+} + \Big(\tilde{\zeta}_\mu \Psi^0_{\mu,t}\zeta_\mu f^\mu, \tilde{\zeta}_\mu \Psi^0_{\mu,t}\zeta_\mu g^\mu \Big)_0 + \Big(\tilde{\zeta}_\mu \Psi^-_{\mu,t}\zeta_\mu f^\mu, \tilde{\zeta}_\mu \Psi^-_{\mu,t}\zeta_\mu g^\mu \Big)_{\phi^-} \Big]$$

and

$$|||f|||^2_\pm := \sum_\mu \Big[\|\tilde{\zeta}_\mu \Psi^+_{\mu,t}\zeta_\mu f^\mu\|^2_{\phi^+} + \|\tilde{\zeta}_\mu \Psi^0_{\mu,t}\zeta_\mu f^\mu\|^2_0 + \|\tilde{\zeta}_\mu \Psi^-_{\mu,t}\zeta_\mu f^\mu\|^2_{\phi^-} \Big]$$

where f^μ and g^μ are the forms f and g, respectively, expressed in the local coordinates on U_μ. The superscript μ will often omitted. In the case that $\phi^+(z) = t\,|z|^2$ or $-t\,|z|^2$ and $\phi^-(z) = -t\,|z|^2$ or $t\,|z|^2$, we denote the norm by $|||\ |||_t$ and in general replace the subscript with t (e.g., we write c_t for c_{ϕ^+,ϕ^-}).

For a form f on M, the Sobolev norm of order s is given by the following:

$$\|f\|^2_s = \sum_\mu \|\tilde{\zeta}_\mu \Lambda^s \zeta_\mu f^\mu\|^2_0$$

where Λ is defined to be the pseudodifferential operator with symbol $(1+|\xi|^2)^{1/2}$. The proof of the next is in [25],

Theorem 4.4.1 *There exist constant $c_\pm$ and $C_\pm$ so that*

$$c_\pm \|\varphi\|^2_0 \le |||\varphi|||^2_{\phi^+,\phi^-} \le C_\pm \|\varphi\|^2_0 \tag{4.9}$$

where $c_\pm$ and $C_\pm$ depend on $\max_M \{|\phi^+| + |\phi^-|\}$ *(assuming $tA \ge 1$).*

Proof. It is sufficient to prove the result when φ is a function. Let $r_\pm = \max_M \{|\lambda^+|, |\lambda^-|\}$, then

$$|||\varphi|||^2_\pm \le e^{r_\pm} \sum_\nu \Big(\|\tilde{\zeta}\Psi^+_{\nu,t}\zeta_\nu \varphi^\nu\|^2_0 + \|\tilde{\zeta}\Psi^0_{\nu,t}\zeta_\nu \varphi^\nu\|^2_0 + \|\tilde{\zeta}\Psi^-_{\nu,t}\zeta_\nu \varphi^\nu\|^2_0 \Big)$$

We can express $\tilde{\zeta}\Psi^+_{\nu,t}\zeta_\nu\varphi^\nu = \Psi^+_{\nu,t}\zeta_\nu\varphi^\nu - (1-\tilde{\zeta})\Psi^+_{\nu,t}\zeta_\nu\varphi^\nu$. Then

$$\begin{aligned}(1-\tilde{\zeta}_\nu(x))\Psi^+_{\nu,t}\zeta_\nu\varphi^\nu(x) &= (1-\tilde{\zeta}_\nu(x)) \int e^{2\pi i x\xi}\psi^+_{\nu,t}(\xi)\widehat{\zeta\varphi}(\xi)d\xi \\ &= \int \varphi(y)(1-\tilde{\zeta}_\nu(x))\zeta(y) \int e^{2\pi i \xi(x-y)}\psi^+_{\nu,t}(\xi)d\xi dy.\end{aligned}$$

Define

$$K(x,y) := (1-\tilde{\zeta}_\nu(x))\zeta(y)\int e^{2\pi i\xi(x-y)}\psi^+_{\nu,t}(\xi)d\xi$$

Since $\operatorname{supp}(\zeta) \cap \operatorname{supp}(1-\tilde{\zeta}_\nu)$ is empty, there exists $\beta_\nu > 0$ such that $|x-y| > \beta_\nu$ for any $x \in \operatorname{supp}(1-\tilde{\zeta}_\nu)$ and $y \in \operatorname{supp}\zeta_\nu$. Then $1+|x-y| = \frac{\beta+|x-y|}{\beta} \le |x-y|(1+1/\beta)$, so $|x-y|^{-|\alpha|} \le (1+1/\beta)^{|\alpha|}(1+|x-y|)^{-|\alpha|}$ for any multiindex α. Then, by integration by parts, for $tA \geq 1$, for any $x \in \operatorname{supp}(1-\tilde{\zeta}_\nu)$ and $y \in \operatorname{supp}\zeta_\nu$ and for $|\alpha|$ sufficiently large (e.g., $2n-1 < |\alpha|$), there exists a C_α such that

$$\begin{aligned}
|K(x,y)| &= \left|(1-\tilde{\zeta}_\nu(x))\zeta(y)\frac{1}{(2\pi i)^{|\alpha|}(x-y)^\alpha}\int e^{2\pi\xi(x-y)}D_\xi^\alpha\psi_{\nu,t}^+(\xi)d\xi\right| \\
&\leq \frac{\left|1-\tilde{\zeta}(x)\right||\zeta(y)|}{(2\pi)^{|\alpha|}|x-y|^{|\alpha|}}\left|\int e^{2\pi i\xi(x-y)}\left(\frac{1}{(tA)^{|\alpha|}}\right)D^\alpha\psi_\nu^+\left(\frac{\xi}{tA}\right)d\xi\right| \\
&\leq \frac{\left|1-\tilde{\zeta}(x)\right||\zeta(y)|}{(2\pi)^{|\alpha|}|x-y|^{|\alpha|}(tA)^{|\alpha|}}\int\left(1+\left|\frac{\xi}{tA}\right|\right)^{-|\alpha|}\left(1+\left|\frac{\xi}{tA}\right|\right)^{|\alpha|}\left|D^\alpha\psi_\nu^+\left(\frac{\xi}{tA}\right)\right|d\xi \\
&\leq \frac{\left|1-\tilde{\zeta}(x)\right||\zeta(y)|}{(2\pi)^{|\alpha|}|x-y|^{|\alpha|}(tA)^{|\alpha|}}C_\alpha\int\frac{(tA)^{2n-1}}{(1+|w|)^{|\alpha|}}dw \leq \frac{\left|1-\tilde{\zeta}(x)\right||\zeta(y)|}{(2\pi)^{|\alpha|}|x-y|^{|\alpha|}(tA)^{|\alpha|-2n+1}}C_\alpha \\
&\leq \frac{\left|1-\tilde{\zeta}(x)\right||\zeta(y)|}{(1+|x-y|)^{|\alpha|}}C_\alpha C_{\beta_\nu}.
\end{aligned}$$

Then the operator $(1-\tilde{\zeta}_\nu)\Psi_{\nu,t}^+\zeta_\nu$ is bounded linear operator on L^2 and

$$\|(1-\tilde{\zeta}_\nu)\Psi_{\nu,t}^+\zeta_\nu\varphi\|_0 \leq C\|f\|_0$$

where C is just depending of α and β_ν, but not of A (e.g. see Theorem 6.18 in [5]). On the other hand, although the range of $\Psi_{\nu,t}^+\zeta_\nu$ is not $L^2(U_\nu)$ but $L^2(\mathbb{R}^{2n-1})$, this operator is a smoothing operator outside the $Dom(\zeta_\nu) \subset U_\nu$, and by the definition of $\Psi_{t,\nu}^+$ it is bounded on $L^2(\mathbb{R}^n)$, so

$$\|\tilde{\zeta}_\nu\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu\|_0^2 \leq 2\|\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu\|_0^2 + 2\|(1-\tilde{\zeta}_\nu)\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu\|_0^2 \leq C\|\zeta_\nu\varphi^\nu\|_0^2$$

with C is a constant independent of tA. A similar bound will also hold for $\Psi_{\nu,t}^0$ and $\Psi_{\nu,t}^-$. So it follows the upper bound of the lemma, because the sum on ν is finite.

To get the lower bound we proceed as follows. Since $\sum_\nu \zeta_\nu^2 = 1 = \sum_\nu \tilde{\zeta}_\nu\zeta_\nu^2$, by (4.8), we can write

$$\begin{aligned}
\|\varphi\|_0^2 &= \sum_\nu \|\zeta_\nu\varphi^\nu\|_0^2 \\
&= \sum_\nu \left(((\Psi_{\nu,t}^+)^*\Psi_{\nu,t}^+ + (\Psi_{\nu,t}^0)^*\Psi_{\nu,t}^0 + (\Psi_{\nu,t}^-)^*\Psi_{\nu,t}^-)\zeta_\nu\varphi^\nu, \zeta_\nu\varphi^\nu\right)_0 \\
&= \sum_\nu \left(\|(\tilde{\zeta}_\nu + (1-\tilde{\zeta}_\nu))\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu\|_0^2 + \|(\tilde{\zeta}_\nu + (1-\tilde{\zeta}_\nu))\Psi_{\nu,t}^0\zeta_\nu\varphi^\nu\|_0^2\right. \\
&\quad \left.+ \|(\tilde{\zeta}_\nu + (1-\tilde{\zeta}_\nu))\Psi_{\nu,t}^-\zeta_\nu\varphi^\nu\|_0^2\right).
\end{aligned}$$

Now, $\|(\tilde{\zeta}_\nu + (1-\tilde{\zeta}_\nu))\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu\|_0^2 \leq 2\|\tilde{\zeta}_\nu\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu\|_0^2 + 2\|(1-\tilde{\zeta}_\nu)\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu\|_0^2$. Since $\Psi_{\nu,t}^+\zeta_\nu$ is pseudolocal (indeed, $(1-\tilde{\zeta}_\nu)\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu$ is infinitely smoothing), $\|\tilde{\zeta}_\nu\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu\|_0^2$ controls $\|(1-\tilde{\zeta}_\nu)\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu\|_0^2$ and similarly for $\Psi_{\nu,t}^-$ and $\Psi_{\nu,t}^0$. As a result,

$$\|\varphi\|_0^2 \leq C\sum_\nu\left(\|\tilde{\zeta}_\nu\Psi_{\nu,t}^+\zeta_\nu\varphi^\nu\|_0^2 + \|\tilde{\zeta}_\nu\Psi_{\nu,t}^0\zeta_\nu\varphi^\nu\|_0^2 + \|\tilde{\zeta}_\nu\Psi_{\nu,t}^-\zeta_\nu\varphi^\nu\|_0^2\right)$$

$$\leq C_\pm \sum_\nu \left(\|\tilde\zeta_\nu \Psi^+_{\nu,t}\zeta_\nu\varphi^\nu\|^2_{\phi^+} + \|\tilde\zeta_\nu \Psi^0_{\nu,t}\zeta_\nu\varphi^\nu\|^2_0 + \|\tilde\zeta_\nu \Psi^-_{\nu,t}\zeta_\nu\varphi^\nu\|^2_{\phi^-} \right)$$

Since we are assuming M as compact manifold, ϕ^+ and ϕ^- are bounded. ■

Like a result of the equivalence between the norms $|||\cdot|||_\pm$ we imply the next result, proved in [24] as Corollary 4.6.

Corollary 4.4.2 *There exist a self adjoint operator $E_\pm$ such that*

$$(\varphi,\phi)_0 = (\varphi, E_\pm\phi)_\pm$$

for any two $(0,q)$-forms φ, ϕ in $L^2_{(0,q)}(M)$. $E_\pm$ is the inverse of

$$\sum_\nu \left(\zeta_\nu(\Psi^+_{\nu,t})^*\tilde\zeta_\nu e^{-\phi^+}\tilde\zeta_\nu\Psi^+_{\nu,t}\zeta_\nu + \zeta_\nu(\Psi^0_{\nu,t})^*\tilde\zeta^2_\nu\Psi^0_{\nu,t}\zeta_\nu + \zeta_\nu(\Psi^-_{\nu,t})^*\tilde\zeta_\nu e^{-\phi^-}\tilde\zeta_\nu\Psi^-_{\nu,t}\zeta_\nu \right).$$

Proof. By the equivalence between the norms $|||\ |||_\pm$ and $\|\ \|_0$ and the Riesz Representation theorem there must be bounded operators $E_\pm$ and $F_\pm$, inverse of each other, on $L^2_{(0,q)}(M)$ such that $(f,g)_0 = (f,E_\pm g)_\pm$ and $(f,F_\pm g)_0 = (f,g)_\pm$ for any $f,g \in L^2_{(0,q)}(M)$. Also $E_\pm$ and $F_\pm$ are injective operators and $E^*_\pm = E_\pm = E^{*,\pm}_\pm$, $F^*_\pm = F_\pm = F^{*,\pm}_\pm$. On the other hand, by the definition of $(\cdot,\cdot)_\pm$ we have

$$\begin{aligned}
(f,g)_\pm &= \sum_\nu \Big(\left(\tilde\zeta_\nu\Psi^+_{\nu,t}\zeta_\nu f, e^{-\phi^+}\tilde\zeta_\nu\Psi^+_{\nu,t}\zeta_\nu g\right)_0 + \left(\tilde\zeta_\nu\Psi^0_{\nu,t}\zeta_\nu f, \tilde\zeta_\nu\Psi^0_{\nu,t}\zeta_\nu g\right)_0 \\
&\quad + \left(\tilde\zeta_\nu\Psi^-_{\nu,t}\zeta_\nu f, e^{-\phi^-}\tilde\zeta_\nu\Psi^-_{\nu,t}\zeta_\nu g\right)_0 \Big) \\
&= \sum_\nu \Big(\left(\zeta_\nu f, (\Psi^+_{\nu,t})^*\tilde\zeta_\nu e^{-\phi^+}\tilde\zeta_\nu\Psi^+_{\nu,t}\zeta_\nu g\right)_0 + \left(\zeta_\nu f, (\Psi^0_{\nu,t})^*\tilde\zeta^2_\nu\Psi^0_{\nu,t}\zeta_\nu g\right)_0 \\
&\quad + \left(\zeta_\nu f, (\Psi^-_{\nu,t})^*\tilde\zeta_\nu e^{-\phi^-}\tilde\zeta_\nu\Psi^-_{\nu,t}\zeta_\nu g\right)_0 \Big) \\
&= \left(f, \sum_\nu \left(\zeta_\nu(\Psi^+_{\nu,t})^*\tilde\zeta_\nu e^{-\phi^+}\tilde\zeta_\nu\Psi^+_{\nu,t}\zeta_\nu + \zeta_\nu(\Psi^0_{\nu,t})^*\tilde\zeta^2_\nu\Psi^0_{\nu,t}\zeta_\nu + \zeta_\nu(\Psi^-_{\nu,t})^*\tilde\zeta_\nu e^{-\phi^-}\tilde\zeta_\nu\Psi^-_{\nu,t}\zeta_\nu \right) g \right)
\end{aligned}$$

for any f and g in $L^2_{(0,q)}(M)$. Then

$$F_\pm := \sum_\nu \left(\zeta_\nu(\Psi^+_{\nu,t})^*\tilde\zeta_\nu e^{-\phi^+}\tilde\zeta_\nu\Psi^+_{\nu,t}\zeta_\nu + \zeta_\nu(\Psi^0_{\nu,t})^*\tilde\zeta^2_\nu\Psi^0_{\nu,t}\zeta_\nu + \zeta_\nu(\Psi^-_{\nu,t})^*\tilde\zeta_\nu e^{-\phi^-}\tilde\zeta_\nu\Psi^-_{\nu,t}\zeta_\nu \right).$$ ■

4.5 $\bar\partial_b$ and its adjoints

Working locally in a small open set $U \subset M$, let $L_1,\ldots,L_{n-1}$ denote a basis for $T^{1,0}(U)$, and $\omega_1,\ldots,\omega_{n-1}$ the dual basis for $L_1,\ldots,L_{n-1}$; if f is a function on M, locally we have $\bar\partial_b f = \sum_{j=1}^{n-1}\bar L_j f\bar\omega_j$, and if $f = \sum_{J\in\mathcal{I}_q} f_J\bar\omega^J$ is a $(0,q)$-form then, there exist functions m^J_K such that

$$\bar\partial_b f = \sum_{J\in\mathcal{I}_q, K\in\mathcal{I}_{q+1}} \sum_{j=1}^{n-1} \epsilon^{jJ}_K \bar L_j f_J\bar\omega_K + \sum_{J\in\mathcal{I}_q, K\in\mathcal{I}_{q+1}} f_J m^J_K\bar\omega_K$$

where ϵ^{jJ}_K is equal to 0 if $\{K\} \neq \{j\} \cup J$ and is the sign of the permutation that reorders jJ to K otherwise. We also define

$$f_{jI} = \sum_{J\in\mathcal{I}_q} \epsilon^{jI}_J f_J \tag{4.10}$$

(in this case, $I \in \mathcal{I}_{q-1}$). Let $\bar L_j^*$ be the adjoint of $\bar L_j$ in $(\ ,\ )_0$, $\bar L_j^{*,\phi}$ be the adjoint of $\bar L_j$ in $(\ ,\ )_\phi$. Then on a small neighborhood U we will have $\bar L_j^* = -L_j + \sigma_j$ and $\bar L_j^{*,\phi} = -L_j + L_j\phi + \sigma_j$ where σ_j is smooth function on U. Because we will need it later, we observe that there are smooth functions d^ℓ_{sr} so that $\left[L_j, \bar L_k\right] = c_{jk}T + \sum_{\ell=1}^{n-1}(d^\ell_{jk}L_\ell - \bar d^\ell_{kj}\bar L_\ell)$. Then

$$\begin{aligned}\left[\bar L_r, \bar L_s^{*,\phi}\right] &= \left[\bar L_r, -L_s\right] + \left[\bar L_r, L_s\phi\right] + \left[\bar L_r, \sigma_s\right]\\ &= c_{sr}T + \sum_{\ell=1}^{n-1}(d^\ell_{sr}L_\ell - \bar d^\ell_{rs}\bar L_\ell) + \bar L_r L_s\phi + \bar L_r\sigma_s.\end{aligned} \tag{4.11}$$

We denote the L^2 adjoint of $\bar\partial_b$ in the $L^2_{0,q}(M, e^{-\phi})$ by $\bar\partial_b^{*,\phi}$. For the remainder of this work, ϕ stands for ϕ^+ or ϕ^- and

$$\left|\phi^+(z)\right| = \left|\phi^-(z)\right| = |t|\,|z|^2,$$

though virtually all of our calculations hold for general ϕ (up to the point when our calculation require an analysis of the eigenvalues of the Levi form).

To keep track of the terms that arise in our integration by parts, we use the following shorthand for forms f supported in a neighborhood U_μ (recognizing that these operators depend on our choice of neighborhoods $\{U_\mu\}$):

$$\|\overline{\nabla}_\Upsilon f\|^2_\phi := \sum_{j,k=1}^{n-1}\left(b^{\bar k j}\bar L_k f, \bar L_j f\right)_\phi = \sum_{J\in\mathcal{I}_q}\sum_{j,k=1}^{n-1}\left(b^{\bar k j}\bar L_k f_J, \bar L_j f_J\right)_\phi; \tag{4.12}$$

$$\|\nabla_\Upsilon f\|^2_\phi := \sum_{j,k=1}^{n-1}\left(b^{\bar k j}\bar L_j^{*,\phi} f, \bar L_k^{*,\phi} f\right)_\phi = \sum_{J\in\mathcal{I}_q}\sum_{j,k=1}^{n-1}\left(b^{\bar k j}\bar L_j^{*,\phi} f_J, \bar L_k^{*,\phi} f_J\right)_\phi; \tag{4.13}$$

$$\|\nabla_{\bar L^{*,\phi}} f\|^2_\phi := \sum_{j=1}^{n-1}\|\bar L_j^{*,\phi} f\|^2_\phi = \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j^{*,\phi} f_J\|^2_\phi;$$

$$\|\nabla_{\bar L} f\|^2_\phi := \sum_{j=1}^{n-1}\|\bar L_j f\|^2_\phi = \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j f_J\,\bar\omega^J\|^2_\phi,$$

where $\Upsilon = i\sum_{j,k=1}^{n-1} b^{\bar k j}\bar L_k \wedge L_j$ is a real $(1,1)$ vector defined on U initially satisfying (A) in Definition 4.1.3 . Again, if $f = \sum_{J\in\mathcal{I}_q} f_J\,\bar\omega^J$ is defined locally, then

$$\begin{aligned}\bar\partial_b^* f &= \sum_{I\in\mathcal{I}_{q-1}, J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\epsilon^{jI}_J \bar L_j^* f_J\,\bar\omega^I + \sum_{I\in\mathcal{I}_{q-1}, J\in\mathcal{I}_q} f_J \overline{m}^I_J\,\bar\omega^I\\ &= \sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^{n-1}\bar L_j^* f_{jI}\,\bar\omega^I + \sum_{I\in\mathcal{I}_{q-1}, J\in\mathcal{I}_q} f_J \overline{m}^I_J\,\bar\omega^I\end{aligned}$$

and

$$\bar\partial_b^{*,\phi} f = \sum_{I\in\mathcal{I}_{q-1}} \sum_{j=1}^{n-1} \bar L_j^{*,\phi} f_{jI}\,\bar\omega^I + \sum_{I\in\mathcal{I}_{q-1},J\in\mathcal{I}_q} f_J \overline{m_J^I}\,\bar\omega^I.$$

Note that a consequence of the compactness of M and the boundedness of ϕ, the domains of $\bar\partial_b^*$ and $\bar\partial_b^{*,\phi}$ are equal. Also we have $\bar\partial_b^{*,\phi} = \bar\partial_b^* - \left[\bar\partial_b^*, \phi\right]$. Let $\bar\partial_{b,t}^*$ be the adjoint of $\bar\partial_b$ with respect to the inner product $(\cdot,\cdot)_t$. We also define the weighted Kohn Laplacian $\Box_b$ by $\Box_{b,t} := \bar\partial_b\bar\partial_{b,t}^* + \bar\partial_{b,t}^*\bar\partial_b$ where

$$\mathrm{Dom}(\Box_{b,t}) := \left\{\phi \in L^2_{0,q}(M) : \phi \in \mathrm{Dom}(\bar\partial_b)\cap \mathrm{Dom}(\bar\partial_{b,t}^*),\ \bar\partial_b\phi \in \mathrm{Dom}(\bar\partial_{b,t}^*),\ \text{and } \bar\partial_{b,t}^*\phi \in \mathrm{Dom}(\bar\partial_b)\right\}.$$

The computations proving Lemmas 4.8 and 4.9 and equation (4.4) in [24] can be applied here with only a change of notation, so we have the following two results, recorded here as Lemmas 4.5.1 and 4.5.2. The meaning of the results is that $\bar\partial_{b,t}^*$ acts like $\bar\partial_b^{*,\phi^+}$ (denoted just by $\bar\partial_b^{*,+}$) for forms whose support is basically $\mathcal{C}^+$ and $\bar\partial_b^{*,\phi^-}$ (denoted just by $\bar\partial_b^{*,-}$) on forms whose support is basically $\mathcal{C}^-$.

Lemma 4.5.1 *On smooth $(0,q)$-forms,*

$$\begin{aligned}\bar\partial_{b,\pm}^* = \bar\partial_b^* &- \sum_\mu \zeta_\mu^2\tilde\Psi_{\mu,t}^+\left[\bar\partial_b^*,\phi^+\right] + \sum_\mu \zeta_\mu^2\tilde\Psi_{\mu,t}^-\left[\bar\partial_b^*,\phi^-\right]\\ &+ \sum_\mu\Big(\tilde\zeta_\mu\left[\tilde\zeta_\mu\Psi_{\mu,t}^+\zeta_\mu,\bar\partial_b\right]^*\tilde\zeta_\mu\Psi_{\mu,t}^+\zeta_\mu + \zeta_\mu(\Psi_{\mu,t}^+)^*\tilde\zeta_\mu\left[\bar\partial_b^{*,+},\tilde\zeta_\mu\Psi_{\mu,t}^+\zeta_\mu\right]\tilde\zeta_\mu\\ &+ \tilde\zeta_\mu\left[\tilde\zeta_\mu\Psi_{\mu,t}^-\zeta_\mu,\bar\partial_b\right]^*\tilde\zeta_\mu\Psi_{\mu,t}^-\zeta_\mu + \zeta_\mu(\Psi_{\mu,t}^-)^*\tilde\zeta_\mu\left[\bar\partial_b^{*,-},\tilde\zeta_\mu\Psi_{\mu,t}^-\zeta_\mu\right]\tilde\zeta_\mu + E_A\Big)\end{aligned}$$

where the error term E_A is a sum of order zero terms and "lower order" terms. Also, the symbol of E_A is supported in $\mathcal{C}_\mu^0$ for each μ.

We use the following energy forms in our calculations:

$$\begin{aligned}Q_{b,\pm}(f,g) &:= \left(\bar\partial_b f,\bar\partial_b g\right)_\pm + \left(\bar\partial_{b,\pm} f,\bar\partial_{b,\pm} g\right)_\pm,\\ Q_{b,+}(f,g) &:= \left(\bar\partial_b f,\bar\partial_b g\right)_{\phi^+} + \left(\bar\partial_b^{*,+} f,\bar\partial_b^{*,+} g\right)_{\phi^+},\\ Q_{b,0}(f,g) &:= \left(\bar\partial_b f,\bar\partial_b g\right)_0 + \left(\bar\partial_b^* f,\bar\partial_b^* g\right)_0,\\ Q_{b,-}(f,g) &:= \left(\bar\partial_b f,\bar\partial_b g\right)_{\phi^-} + \left(\bar\partial_b^{*,-} f,\bar\partial_b^{*,-} g\right)_{\phi^-}.\end{aligned}$$

The space of weighted harmonic forms $\mathcal{H}_t^q$ is defined by

$$\begin{aligned}\mathcal{H}_t^q &:= \left\{f\in\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*) : \bar\partial_b f = 0, \bar\partial_{b,t}^* f = 0\right\}\\ &= \left\{f\in\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*) : Q_{b,t}(f,f) = 0\right\}.\end{aligned}$$

We have the following relationship between these energies. It says us that, up to well behaved terms, to estimate the energy $Q_{b,\pm}(.,.)$ is sufficient to estimate the energies $Q_{b,+}(.,.)$ and $Q_{b,-}(.,.)$ applied on parts, up to smooth terms, whose Fourier transform are supported in $\mathcal{C}^+$ and $\mathcal{C}^-$ respectively. See [10, Lemma 3.4] or [24, Lemma 4.9].

Lemma 4.5.2 *If f is a smooth $(0,q)$-form on M, then there exist constant $K, K_\pm$ and K' with $K \geq 1$ so that*

$$\begin{aligned} KQ_{b,\pm}(f,f) + K_\pm \sum_\nu \|\tilde{\zeta}_\nu \tilde{\Psi}^0_{\nu,t}\zeta_\nu f^\nu\|_0^2 + K' \left\|\left| f \right|\right\|_\pm^2 + O_t(\|f\|_{-1}^2) \\ \geq \sum_\nu \Big[Q_{b,+}(\tilde{\zeta}_\nu \Psi^+_{\nu,t}\zeta_\nu f^\nu, \tilde{\zeta}_\nu \Psi^+_{\nu,t}\zeta_\nu f^\nu) \\ + Q_{b,0}(\tilde{\zeta}_\nu \Psi^0_{\nu,t}\zeta_\nu f^\nu, \tilde{\zeta}_\nu \Psi^0_{\nu,t}\zeta_\nu f^\nu) + Q_{b,-}(\tilde{\zeta}_\nu \Psi^-_{\nu,t}\zeta_\nu f^\nu, \tilde{\zeta}_\nu \Psi^-_{\nu,t}\zeta_\nu f^\nu) \Big] \end{aligned}$$

K and K' do not depend on t, ϕ^- or ϕ^+.

4.6 The main estimate

In this section, we compile the technical pieces that will allows us to establish a basic estimate the ground level L^2 estimates for the closure of our operator $\bar{\partial}_b$ and $\Box_{b,t}$.

Proposition 4.6.1 *Let M^{2n-1} be a smooth, compact, orientable CR manifold of hypersurface type embedded in $\mathbb{C}^N$, that satisfies weak $Y(q)$ for some fixed $1 \leq q \leq n-2$. Set*

$$\phi^+(z) = \begin{cases} t|z|^2 & \text{if } \omega(\Upsilon_q) < q \\ -t|z|^2 & \text{if } \omega(\Upsilon_q) > q \end{cases} \quad \text{and} \quad \phi^-(z) = \begin{cases} -t|z|^2 & \text{if } \omega(\Upsilon_{n-1-q}) < n-1-q \\ t|z|^2 & \text{if } \omega(\Upsilon_{n-1-q}) > n-1-q. \end{cases} \tag{4.14}$$

There exist constants K and K_t where K does not depend on t so that

$$t\left\|\left| f \right|\right\|_t^2 \leq KQ_{b,t}(f,f) + K_t\|f\|_{-1}^2, \tag{4.15}$$

for t sufficiently large.

Note that, functions ϕ^+ and ϕ^- are well defined, since the signs of $\omega(\Upsilon_q) - q$ and $\omega(\Upsilon_{n-1-q}) - n - 1 - q$ are constants modulo connected components.

The main work in establishing (4.15) is to prove the following:

$$t\left\|\left| f \right|\right\|_t^2 \leq KQ_{b,t}(f,f) + K\left\|\left| f \right|\right\|_t^2 + K_t \sum_\mu \sum_{J\in\mathcal{I}_q} \|\tilde{\zeta}_\mu \tilde{\Psi}^0_{\mu,t}\zeta_\mu f^\mu_J\|_0^2 + K'_t\|f\|_{-1}^2. \tag{4.16}$$

In order to prove (4.16), we estimate a $(0,q)$-form f with support in neighborhood U for a generic energy form $Q_{b,\phi}(f,g) := (\bar{\partial}_b f, \bar{\partial}_b g)_\phi + (\bar{\partial}_b^{*,\phi} f, \bar{\partial}_b^{*,\phi} g)_\phi$. Throughout the estimate, we will make use of three terms, $E_0(f)$, $\tilde{E}_1(f)$, and $\tilde{E}_2(f)$ to collect the error terms that we will bound later. We want $E_0(f) = O(\|f\|_\phi^2)$ and

$$\tilde{E}_1(f) = \sum_{J,J'\in\mathcal{I}_q} \sum_{j=1}^{n-1} \left(\bar{L}_j f_J, a_{JJ'} f_{J'}\right)_\phi \quad \text{and} \quad \tilde{E}_2(f) = \sum_{J,J'\in\mathcal{I}_q} \sum_{j=1}^{n-1} \left(\bar{L}_j^{*,\phi} f_J, \tilde{a}_{JJ'} f_{J'}\right)_\phi$$

for some collection of smooth functions $a_{JJ'}$ and $\tilde{a}_{JJ'}$ that may change from line to line.

Similarly to the computations done in [25, Lemma 4.2], we have

$$\begin{aligned} Q_{b,\phi}(f,f) = \|\nabla_{\bar{L}} f\|_\phi^2 &+ \sum_{J,J'\in\mathcal{I}_q} \sum_{\substack{j,k=1\\ j\neq k}}^{n-1} \epsilon^{kJ}_{jJ'} \left(\left[\bar{L}_j^{*,\phi}, \bar{L}_k \right] f_J, f_{J'} \right)_\phi \\ &+ \sum_{J\in\mathcal{I}_q} \sum_{j\in J} \left(\left[\bar{L}_j, \bar{L}_j^{*,\phi} \right] f_J, f_J \right)_\phi \\ &+ \tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f). \end{aligned}$$

Using (4.11), $\epsilon^{kJ}_{jJ'} = -\sum_{I\in\mathcal{I}_{q-1}} \epsilon^{jI}_J \epsilon^{kI}_{J'}$ for $j\neq k$, and by (4.10) we have

$$\begin{aligned} Q_{b,\phi}(f,f) &= \|\nabla_{\bar{L}} f\|_\phi^2 + \sum_{J\in\mathcal{I}_q} \sum_{j\in J} (c_{jj} T f_J, f_J)_\phi - \sum_{J,J'\in\mathcal{I}_q} \sum_{j\neq k, j,k=1}^{n-1} \epsilon^{kJ}_{jJ'} (c_{jk} T f_J, f_{J'})_\phi \\ &\quad - \sum_{J,J'\in\mathcal{I}_q} \sum_{j\neq k, j,k=1}^{n-1} \epsilon^{kJ}_{jJ'} \left[\left(\bar{L}_k L_j \phi f_J, f_{J'} \right)_\phi + \left(\sum_{\ell=1}^{n-1} (d^\ell_{jk} L_\ell - \bar{d}^\ell_{kj} \bar{L}_\ell) f_J, f_{J'} \right)_\phi \right] \\ &\quad + \sum_{J\in\mathcal{I}_q} \sum_{j\in J} \left[\left(\bar{L}_j L_j \phi f_J, f_J \right)_\phi + \left(\sum_{\ell=1}^{n-1} (d^\ell_{jj} L_\ell - \bar{d}^\ell_{jj} \bar{L}_\ell) f_J, f_J \right)_\phi \right] \\ &\quad + \tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f) \\ &= \|\nabla_{\bar{L}} f\|_\phi^2 + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} (c_{jk} T f_{jI}, f_{kI})_\phi \\ &\quad + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left[\left(\bar{L}_k L_j \phi f_{jI}, f_{kI} \right)_\phi + \left(\sum_{\ell=1}^{n-1} (d^\ell_{jk} L_\ell - \bar{d}^\ell_{kj} \bar{L}_\ell) f_{jI}, f_{kI} \right)_\phi \right] \\ &\quad + \tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f) \\ &= \|\nabla_{\bar{L}} f\|_\phi^2 + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} (c_{jk} T f_{jI}, f_{kI})_\phi \\ &\quad + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left[\left(\bar{L}_k L_j \phi f_{jI}, f_{kI} \right)_\phi + \left(\sum_{\ell=1}^{n-1} d^\ell_{jk} L_\ell f_{jI}, f_{kI} \right)_\phi \right] \\ &\quad + \tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f) \end{aligned} \tag{4.17}$$

where $\tilde{E}_1$ now includes the term

$$\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\sum_{l=1}^{n-1} \bar{d}^l_{kj} \bar{L}_l f_{jI}, f_{kI} \right)_\phi .$$

Now, since $L_j = -\bar{L}_j^{*,\phi} + L_j\phi + \sigma_j$, by (4.17) we have

$$Q_{b,\phi}(f,f) = \|\nabla_{\bar{L}} f\|_\phi^2 + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \mathrm{Re}\,(c_{jk} T f_{jI}, f_{kI})_\phi$$

$$
\begin{aligned}
&+ \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \mathrm{Re}\left[\left(\bar{L}_k L_j\phi f_{jI}, f_{kI}\right)_\phi + \left(\sum_{\ell=1}^{n-1} d_{jk}^{\ell} L_\ell \phi f_{jI}, f_{kI}\right)_\phi\right]\\
&+ \mathrm{Re}\left(\tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f)\right)
\end{aligned}
$$

and $\tilde{E}_2$ now includes the term

$$
\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\sum_{\ell=1}^{n-1} d_{jk}^{\ell} \bar{L}_\ell^{*,\phi} f_{jI}, f_{kI}\right)_\phi .
$$

Note that

$$
\begin{aligned}
\mathrm{Re} \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\bar{L}_k L_j\phi f_{jI}, f_{kI}\right)_\phi &= \frac{1}{2}\left\{\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\bar{L}_k L_j\phi f_{jI}, f_{kI}\right)_\phi + \overline{\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\bar{L}_k L_j\phi f_{jI}, f_{kI}\right)_\phi}\right\}\\
&= \frac{1}{2}\left\{\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\bar{L}_k L_j\phi f_{jI}, f_{kI}\right)_\phi + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(f_{kI}, \bar{L}_k L_j\phi f_{jI}\right)_\phi\right\}\\
&= \frac{1}{2}\left\{\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\bar{L}_k L_j\phi f_{jI}, f_{kI}\right)_\phi + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(f_{jI}, \bar{L}_j L_k\phi f_{kI}\right)_\phi\right\}\\
&= \frac{1}{2}\left\{\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\bar{L}_k L_j\phi f_{jI}, f_{kI}\right)_\phi + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(L_j \bar{L}_k\phi f_{jI}, f_{kI}\right)_\phi\right\}\\
&= \frac{1}{2}\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left((\bar{L}_k L_j\phi + L_j\bar{L}_k\phi) f_{jI}, f_{kI}\right)_\phi
\end{aligned}
$$

$$
\mathrm{Re} \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\sum_{\ell=1}^{n-1} d_{jk}^{\ell} L_\ell\phi f_{jI}, f_{kI}\right)_\phi = \frac{1}{2}\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\sum_{\ell=1}^{n-1}(d_{jk}^{\ell} L_\ell\phi + \bar{d}_{kj}^{\ell}\bar{L}_\ell\phi) f_{jI}, f_{kI}\right)_\phi
$$

and also

$$
\frac{1}{2}\left(\bar{L}_k L_j\phi + L_j\bar{L}_k\phi\right) + \frac{1}{2}\sum_{\ell=1}^{n-1}(d_{jk}^{\ell} L_\ell\phi + \bar{d}_{kj}^{\ell}\bar{L}_\ell\phi) = \Theta_{jk}^{\phi} - \frac{1}{2}\nu(\phi) c_{jk}.
$$

It follows that

$$
\begin{aligned}
Q_{b,\phi}(f,f) = \|\nabla_{\bar{L}} f\|_\phi^2 &+ \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \mathrm{Re}\,(c_{jk} T f_{jI}, f_{kI})_\phi\\
&+ \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left((\Theta_{jk}^{\phi} - \frac{1}{2}\nu(\phi) c_{jk}) f_{jI}, f_{kI}\right)_\phi\\
&+ \mathrm{Re}\left(\tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f)\right).
\end{aligned} \tag{4.18}
$$

On the other hand, using notation given in (4.12) we have

$$
\begin{aligned}
\|\overline{\nabla}_\Upsilon f\|_\phi^2 &= \sum_{j,k=1}^{n-1} \left(b^{\bar{k}j}\bar{L}_k f, \bar{L}_j f\right)_\phi = \sum_{j,k=1}^{n-1} \left(b^{\bar{k}j}\bar{L}_j^{*,\phi}\bar{L}_k f, f\right)_\phi + \sum_{j,k=1}^{n-1} \left(\bar{L}_j^{*,\phi}(b^{\bar{k}j})\bar{L}_k f, f\right)_\phi\\
&= \sum_{j,k=1}^{n-1} \left[\left(\bar{L}_k \bar{L}_j^{*,\phi} f, b^{\bar{j}k} f\right)_\phi + \left(\left[\bar{L}_j^{*,\phi}, \bar{L}_k\right] f, b^{\bar{j}k} f\right)_\phi + \left(\bar{L}_j^{*,\phi}(b^{\bar{k}j})\bar{L}_k f, f\right)_\phi\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j,k=1}^{n-1}\left[\left(b^{\bar kj}\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}f\right)_\phi + \left(\left[\bar L_j^{*,\phi},\bar L_k\right]f,b^{\bar jk}f\right)_\phi + \left(\bar L_j^{*,\phi}(b^{\bar kj})\bar L_kf,f\right)_\phi\right] \\
&\quad + \sum_{j,k=1}^{n-1}\left(\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}(b^{\bar jk})f\right)_\phi \\
&= \sum_{j,k=1}^{n-1}\left[\left(b^{\bar kj}\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}f\right)_\phi + \left((-c_{jk}T-\sum_{\ell=1}^{n-1}(d_{jk}^\ell L_\ell-\bar d_{jk}^\ell\bar L_\ell)-\bar L_kL_j\phi-\bar L_k\sigma_j)f,b^{\bar jk}f\right)_\phi\right] \\
&\quad + \sum_{j,k=1}^{n-1}\left[\left(\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}(b^{\bar jk})f\right)_\phi + \left(\bar L_j^{*,\phi}(b^{\bar kj})\bar L_kf,f\right)_\phi\right] \\
&= \sum_{j,k=1}^{n-1}\left[\left(b^{\bar kj}\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}f\right)_\phi - \left(b^{\bar kj}c_{jk}Tf,f\right) - \left(\bar L_kL_j\phi f,b^{\bar jk}f\right)_\phi - \sum_{\ell=1}^{n-1}\left(d_{jk}^\ell L_\ell f,b^{\bar jk}f\right)_\phi\right] \\
&\quad + \sum_{j,k=1}^{n-1}\left[\left(\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}(b^{\bar jk})f\right)_\phi + \left(\bar L_j^{*,\phi}(b^{\bar kj})\bar L_kf,f\right)_\phi - \sum_{\ell=1}^{n-1}\left(\bar d_{jk}^\ell\bar L_\ell f,b^{\bar jk}f\right)_\phi\right] + E_0(f) \\
&= \sum_{j,k=1}^{n-1}\left[\left(b^{\bar kj}\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}f\right)_\phi - \left(b^{\bar kj}c_{jk}Tf,f\right)_\phi - \left(\bar L_kL_j\phi f,b^{\bar jk}f\right)_\phi - \sum_{\ell=1}^{n-1}\left(d_{jk}^\ell L_\ell\phi f,b^{\bar jk}f\right)_\phi\right] \\
&\quad + \sum_{j,k=1}^{n-1}\left[\left(\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}(b^{\bar jk})f\right)_\phi + \left(\bar L_j^{*,\phi}(b^{\bar kj})\bar L_kf,f\right)_\phi - \sum_{\ell=1}^{n-1}\left(\bar d_{jk}^\ell\bar L_\ell f,b^{\bar jk}f\right)_\phi\right] \\
&\quad + \sum_{j,k,\ell=1}^{n-1}\left(d_{jk}^\ell\bar L_\ell^{*,\phi}f,b^{\bar jk}f\right)_\phi + E_0(f) \\
&= \sum_{j,k=1}^{n-1}\left[\left(b^{\bar kj}\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}f\right)_\phi - \left(b^{\bar kj}c_{jk}Tf,f\right)_\phi - \left(\frac12(\bar L_kL_j\phi+L_j\bar L_k\phi)f,b^{\bar jk}f\right)_\phi\right] \\
&\quad - \sum_{j,k=1}^{n-1}\left(b^{\bar kj}\frac12\sum_{\ell=1}^{n-1}(d_{jk}^\ell L_\ell\phi+\bar d_{jk}^\ell\bar L_\ell\phi)f,f\right)_\phi + \tilde E_2(f)+\tilde E_1(f)+E_0(f) \\
&= \sum_{j,k=1}^{n-1}\left[\left(b^{\bar kj}\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}f\right)_\phi - \left(b^{\bar kj}c_{jk}Tf,f\right)_\phi - \left(b^{\bar kj}(\Theta_{jk}^\phi-\frac12\nu(\phi)c_{jk})f,f\right)_\phi\right] \\
&\quad + \tilde E_2(f)+\tilde E_1(f)+E_0(f).
\end{aligned} \tag{4.19}
$$

where $\tilde E_1(f)$ includes terms

$$
\sum_{j,k,\ell=1}^{n-1}\left(\bar d_{kj}^\ell\bar L_\ell f,b^{\bar jk}f\right)_\phi, \qquad \sum_{j,k=1}^{n-1}\left(\bar L_j^{*,\phi}(b^{\bar kj})\bar L_kf,f\right)_\phi
$$

and $\tilde E_2(f)$ includes

$$
\sum_{j,k,\ell=1}^{n-1}\left(d_{jk}^\ell\bar L_\ell^{*,\phi}f,b^{\bar jk}f\right)_\phi, \qquad \sum_{j,k=1}^{n-1}\left(\bar L_j^{*,\phi}f,\bar L_k^{*,\phi}(b^{\bar jk})f\right)_\phi.
$$

Motivated by [11, p.1725], we write $\|\nabla_{\bar L}f\|_\phi^2=\left(\|\nabla_{\bar L}f\|_\phi^2-\|\overline\nabla_\Upsilon f\|_\phi^2\right)+\|\overline\nabla_\Upsilon f\|_\phi^2$. Using (4.19) and (4.13) we obtain

$$
Q_{b,\phi}(f,f)=\left(\|\nabla_{\bar L}f\|_\phi^2-\|\overline\nabla_\Upsilon f\|_\phi^2\right)+\|\nabla_\Upsilon f\|_\phi^2+\sum_{I\in\mathcal I_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\left(c_{jk}Tf_{jI},f_{kI}\right)_\phi
$$

$$
\begin{aligned}
&- \sum_{j,k=1}^{n-1} \left[\left(b^{\bar{k}j} c_{jk} Tf, f \right)_\phi + \left(b^{\bar{k}j} (\Theta^\phi_{jk} - \frac{1}{2}\nu(\phi) c_{jk}) f, f \right)_\phi \right] \\
&+ \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left((\Theta^\phi_{jk} - \frac{1}{2}\nu(\phi) c_{jk}) f_{jI}, f_{kI} \right)_\phi + \tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f) \\
= &\left(\|\nabla_{\bar{L}} f\|^2_\phi - \|\overline{\nabla}_\Upsilon f\|^2_\phi \right) + \|\nabla_\Upsilon f\|^2_\phi + \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \operatorname{Re} \left(c_{jk} T f_{jI}, f_{kI} \right)_\phi \\
&- (i \langle d\gamma, \Upsilon \rangle Tf, f)_\phi + \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left((\Theta^\phi_{jk} - \frac{1}{2}\nu(\phi) c_{jk}) f_{jI}, f_{kI} \right)_\phi \\
&- \left(i \langle \Theta^\phi, \Upsilon \rangle f, f \right)_\phi + \left(\frac{1}{2} \nu(\phi) i \langle d\gamma, \Upsilon \rangle f, f \right)_\phi + \tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f).
\end{aligned}
$$

Since

$$
\sum_{J \in \mathcal{I}_q} (a f_J, f_J)_\phi = \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\frac{a \delta^k_j}{q} f_{jI}, f_{kI} \right)_\phi
$$

where (δ^k_j) is the identity matrix I_{n-1}, we have

$$
\begin{aligned}
Q_{b,\phi}(f,f) = &\left(\|\nabla_{\bar{L}} f\|^2_\phi - \|\overline{\nabla}_\Upsilon f\|^2_\phi \right) + \|\nabla_\Upsilon f\|^2_\phi \\
&+ \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \operatorname{Re} \left(\left(c_{jk} - \frac{i \langle d\gamma, \Upsilon \rangle \delta^k_j}{q} \right) T f_{jI}, f_{kI} \right)_\phi \\
&+ \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\left(\Theta^\phi_{jk} - \frac{i \langle \Theta^\phi, \Upsilon \rangle \delta^k_j}{q} \right) f_{jI}, f_{kI} \right)_\phi \\
&- \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\frac{1}{2} \nu(\phi) \left(c_{jk} - \frac{i \langle d\gamma, \Upsilon \rangle \delta^k_j}{q} \right) f_{jI}, f_{kI} \right)_\phi \\
&+ \tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f).
\end{aligned}
$$

Bounding the error terms $\tilde{E}_1(f)$ and $\tilde{E}_2(f)$ uses the same argument, and we demonstrate the bound for $\tilde{E}_1(f)$. Terms of the form $\sum_{j=1}^{n-1} \left(a_j \bar{L}_j g, h \right)_\phi$ comprise $\tilde{E}_1$ for various functions g and h, and we compute

$$
\sum_{j=1}^{n-1} \left(a_j \bar{L}_j g, h \right)_\phi = \sum_{j,k=1}^{n-1} \left((\delta_{jk} - b^{\bar{j}k}) \bar{L}_j g, \bar{a}_k h \right)_\phi + \sum_{j,k=1}^{n-1} \left(b^{\bar{j}k} \bar{L}_j g, \bar{a}_k h \right)_\phi . \tag{4.20}
$$

To estimate the first terms, observe that for $\varepsilon > 0$, a small constant/large constant argument shows that

$$
\begin{aligned}
\left| \sum_{j,k=1}^{n-1} \left((\delta_{jk} - b^{\bar{j}k}) \bar{L}_j g, \bar{a}_k h \right)_\phi \right| &= \left| \sum_{k=1}^{n-1} \left(\sum_{j=1}^{n-1} (\delta_{jk} - b^{\bar{j}k}) \bar{L}_j g, \bar{a}_k h \right)_\phi \right| \\
&\leq \sum_{k=1}^{n-1} \left\| \sum_{j=1}^{n-1} (\delta_{jk} - b^{\bar{j}k}) \bar{L}_j g \right\|_\phi \|a_k h\|_\phi \\
&\leq \varepsilon \sum_{k=1}^{n-1} \left\| \sum_{j=1}^{n-1} (\delta_{jk} - b^{\bar{j}k}) \bar{L}_j g \right\|^2_\phi + O_{\frac{1}{\varepsilon}}(\|h\|^2_\phi).
\end{aligned}
$$

Stepping away from the integration (momentarily), suppose that at some point in U, A is a unitary matrix that diagonalizes the hermitian matrix $\bar{B} = (b^{\bar{j}k})$ of Υ such that $\bar{B} = A^*\Lambda A$, where $\Lambda = diag\{\lambda_1, \ldots, \lambda_{n-1}\}$ and $\lambda_1, \cdots, \lambda_{n-1}$ are the eigenvalues of $\bar{B}$. Consider $[\bar{L}_j g]$ as a column vector with components $[\bar{L}_j g]_k$. Then since $(1-\lambda_j)^2 \leq (1-\lambda_j)$ for all j,

$$\begin{aligned}
\sum_{k=1}^{n-1}\left|\sum_{j=1}^{n-1}(\delta_{jk} - b^{\bar{j}k})(\bar{L}_j g)\right|^2 &= \left|[Id - B]\left[\bar{L}_j g\right]\right|^2 = \left|\left[Id - \bar{B}\right]\left[\overline{\bar{L}_j g}\right]_j\right|^2 \\
&= \left|A^*\left[Id - \Lambda\right]A\left[\overline{\bar{L}_j g}\right]_j\right|^2 = \left|\left[Id - \Lambda\right]A\left[\overline{\bar{L}_j g}\right]_j\right|^2 \\
&= \sum_{k=1}^{n-1}(1-\lambda_k)^2\left|\left[A\left[\overline{\bar{L}_j g}\right]\right]_k\right|^2 \leq \sum_{k=1}^{n-1}(1-\lambda_k)\left|\left[A\left[\overline{\bar{L}_j g}\right]\right]_k\right|^2 \\
&= \left(\sum_{j=1}^{n-1}\left|\bar{L}_j g\right|^2\right) - \left[\overline{\bar{L}_j g}\right]^*\bar{B}\left[\overline{\bar{L}_j g}\right] \\
&= \sum_{j=1}^{n-1}\left|\bar{L}_j g\right|^2 - \sum_{j,k=1}^{n-1} b^{\bar{k}j}\,\overline{\bar{L}_j g}\,\bar{L}_k g.
\end{aligned}$$

Returning to the integration, we now observe,

$$\sum_{k=1}^{n-1}\left\|\sum_{j=1}^{n-1}(\delta_{jk} - b^{\bar{j}k})\bar{L}_j g\right\|_\phi^2 \leq \|\nabla_{\bar{L}} g\|_\phi^2 - \|\overline{\nabla}_\Upsilon g\|_\phi^2.$$

For the second term in (4.20), in a similar way, a small constant/large constant argument shows

$$\sum_{j,k=1}^{n-1}\left(b^{\bar{j}k}\bar{L}_j g, \bar{a}_k h\right)_\phi = \sum_{j,k=1}^{n-1}\left(a_k g, b^{\bar{k}j}\bar{L}_j^{*,\phi} h\right)_\phi + O(\|g\|_\phi \|h\|_\phi)$$

$$\left|\sum_{j,k=1}^{n-1}(a_k g, b^{\bar{k}j}\bar{L}_j^{*,\phi} h)_\phi\right| \leq O_{\frac{1}{\varepsilon}}(\|g\|_\phi^2) + \varepsilon \sum_{k=1}^{n-1}\left\|\sum_{j=1}^{n-1} b^{\bar{k}j}\bar{L}_j^{*,\phi} h\right\|_\phi^2,$$

and linear algebra (as above) helps to establish

$$\begin{aligned}
\sum_{k=1}^{n-1}\left|\sum_{j=1}^{n-1} b^{\bar{k}j}\bar{L}_j^{*,\phi} h\right|^2 &= \left|\bar{B}\left[\bar{L}_j^{*,\phi} h\right]_j\right|^2 = \left|\Lambda A^*\left[\bar{L}_j^{*,\phi} h\right]\right|^2 = \sum_{k=1}^{n-1}\lambda_k^2\left|\left[A^*\left[\bar{L}_j^{*,\phi} h\right]\right]_k\right|^2 \\
&\leq \sum_{k=1}^{n-1}\lambda_k\left|\left[A^*\left[\bar{L}_j^{*,\phi} h\right]\right]_k\right|^2 = \left[\bar{L}_j^{*,\phi} h\right]_j^*\bar{B}\left[\bar{L}_j^{*,\phi} h\right]_j \\
&= \sum_{j,k=1}^{n-1} b^{\bar{k}j}\bar{L}_j^{*,\phi} h\,\overline{\bar{L}_k^{*,\phi} h}.
\end{aligned}$$

Then

$$\sum_{k=1}^{n-1}\left\|\sum_{j=1}^{n-1} b^{\bar{k}j}\bar{L}_j^{*,\phi} h\right\|_\phi^2 \leq \sum_{j,k=1}^{n-1}\left(b^{\bar{k}j}\bar{L}_j^{*,\phi} h, \bar{L}_k^{*,\phi} h\right)_\phi = \|\nabla_\Upsilon h\|_\phi^2$$

Summarizing the above calculations, for ε sufficiently small and f supported in a small neighborhood, we have

$$\begin{aligned} Q_{b,\phi}(f,f) \geq & \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1} \operatorname{Re}\left(\left(c_{jk} - \frac{i\langle d\gamma,\Upsilon\rangle\delta_j^k}{q}\right) T f_{jI}, f_{kI}\right)_\phi \\ & + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\left(\Theta^\phi_{jk} - \frac{i\langle\Theta^\phi,\Upsilon\rangle\delta_j^k}{q}\right) f_{jI}, f_{kI}\right)_\phi \\ & - \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\frac{1}{2}\nu(\phi)\left(\left(c_{jk} - \frac{i\langle d\gamma,\Upsilon\rangle\delta_j^k}{q}\right)\right) f_{jI}, f_{kI}\right)_\phi + O(\|f\|_\phi^2) \end{aligned} \tag{4.21}$$

To handle the T terms, we recall the following results. The first is a well-known multilinear algebra result that appears (among other places) in Straube [30]

Lemma 4.6.2 *Let $B = (b_{jk})_{1\leq j,k\leq n-1}$ be a Hermitian matrix and $1 \leq q \leq n-1$. The following are equivalent:*

i. If $u \in \Lambda^{0,q}$, then $\sum_{K\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1} b_{jk}u_{jK}\overline{u_{kK}} \geq M\,|u|^2$.

ii. The sum of any q eigenvalues of B is at least M.

iii. $\sum_{s=1}^{q}\sum_{j,k=1}^{n-1} b_{jk}t_j^s\overline{t_k^s} \geq M$ for any orthonormal vectors $\{t^s\}_{1\leq s\leq q} \subset \mathbb{C}^{n-1}$.

The next two results are consequences of the sharp Gårding Inequality and appear as [25, Lemma 4.6, Lemma 4.7]

Lemma 4.6.3 *Let f a $(0,q)$-form supported on U so that up to a smooth term $\hat{f}$ is supported in $\mathcal{C}^+$, and let $[h_{jk}]$ a Hermitian matrix such that the sum of any q eigenvalues is ≥ 0. Then*

$$\operatorname{Re}\left\{\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}Tf_{jI}, f_{kI})_\phi\right\} \geq tA\operatorname{Re}\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}f_{jI}, f_{kI})_\phi - O(\|f\|_\phi^2) - O_t(\|\tilde\zeta\tilde\Psi_t^0 f\|_0^2).$$

Lemma 4.6.4 *Let f a $(0,q)$-form supported on U so that up to a smooth term $\hat{f}$ is supported in $\mathcal{C}^-$, and let $[h_{jk}]$ a Hermitian matrix such that the sum of any $n-1-q$ eigenvalues is ≥ 0. Then*

$$\begin{aligned} &\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}(h_{jj}(-T)f_J, f_J)_\phi - \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}(-T)f_{jI}, f_{kI})_\phi\right\} \\ &\geq tA\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}(h_{jj}f_J, f_J)_\phi - \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}f_{jI}, f_{kI})_\phi\right\} - O(\|f\|_\phi^2) - O_t(\|\tilde\zeta\tilde\Psi_t^0 f\|_0^2). \end{aligned}$$

We are now ready to control derivatives in the bad direction T appearing in energies $Q_{b,+}(\cdot,\cdot)$ and $Q_{b,-}(\cdot,\cdot)$.

Lemma 4.6.5 *Let f a $(0,q)$-form supported on U so that up to a smooth term $\hat{f}$ is supported in $\mathcal{C}^+$, and let $[h_{jk}]$ a Hermitian matrix such that the sum of any q eigenvalues is ≥ 0. Then*

$$\mathrm{Re}\left\{\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}Tf_{jI},f_{kI})_\phi\right\}\geq tA\mathrm{Re}\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}f_{jI},f_{kI})_\phi - O(\|f\|_\phi^2)-O_t(\|\tilde{\zeta}_\nu\widetilde{\Psi}_t^0 f\|_0^2)$$

Proof. Let $\widetilde{\Psi}_t^+$ be the pseudodifferential operator of order zero whose symbol dominates $\hat{f}$ (up to smooth error) and is supported in $\widetilde{\mathcal{C}}^+$. By the support condition of f and $\hat{f}$, we will have

$$\begin{aligned}\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}Tf_{jI},f_{kI})_\phi &= \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(h_{jk}Tf_{jI},\left((\widetilde{\Psi}_t^+)^*\widetilde{\Psi}_t^+ + \left(Id-(\widetilde{\Psi}_t^+)^*\widetilde{\Psi}_t^+\right)\right)f_{kI}\right)_\phi\\ &= \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(h_{jk}Tf_{jI},(\widetilde{\Psi}_t^+)^*\widetilde{\Psi}_t^+ f_{kI}\right)_\phi + \textit{smoother terms}\\ &= \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\tilde{\zeta}e^{-\phi}h_{jk}\widetilde{\Psi}_t^+ Tf_{jI},\tilde{\zeta}\widetilde{\Psi}_t^+ f_{kI}\right)_0 + \textit{smoother terms}\\ &= \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\tilde{\zeta}(\widetilde{\Psi}_t^+)^*\tilde{\zeta}^2e^{-\phi}h_{jk}\widetilde{\Psi}_t^+ Tf_{jI},f_{kI}\right)_0 + \textit{smoother terms}\end{aligned}$$

where *smoother terms* are $O(\|f\|_{-1}^2)$. We look for the symbol of the operator $(\widetilde{\Psi}_t^+)^*\tilde{\zeta}^2e^{-\phi}h_{jk}\widetilde{\Psi}_t^+T$. Let $\widetilde{\psi}_t^+(x,\xi)$ be the symbol of $\widetilde{\Psi}_t^+$. Note that $\sigma(T)=\xi_{2n-1}$. Then the symbol of the composition $T\widetilde{\Psi}_t^+$ is given by

$$\sum_\beta\frac{1}{\beta!}\partial_\xi^\beta(\xi_{2n-1})D_x^\beta\widetilde{\psi}_t^+(x,\xi)=\xi_{2n-1}\widetilde{\psi}_t^+(x,\xi)+D_x^{(0,\dots,0,1)}\widetilde{\psi}_t^+(x,\xi).$$

Since $\hat{\phi}$ is supported in $\mathcal{C}^+$ (up to smooth term) and $\widetilde{\psi}_t^+\equiv 1$ on $\mathcal{C}^+$, any of its derivatives will be zero, so $\sigma(\widetilde{\Psi}_t^+T)=\sigma(T\widetilde{\Psi}_t^+)=\xi_{2n-1}\widetilde{\psi}_t^+(x,\xi)$ up to smooth terms when applied to f. Now since $\sigma(\widetilde{\Psi}_t^+)\equiv 1$ on $\mathcal{C}^+$, it follows that $\sigma((\widetilde{\Psi}_t^+)^*)\equiv 1$ on $\mathcal{C}^+$ as well. Then $\sigma((\widetilde{\Psi}_t^+)^*)=\widetilde{\psi}_t^+(x,\xi)$ up to terms supported in $\mathcal{C}^0\setminus\mathcal{C}^+$. This implies

$$\sigma\left((\widetilde{\Psi}_t^+)^*\tilde{\zeta}^2e^{-\phi}h_{jk}\right)=\sum_\beta\frac{1}{\beta!}\partial_\xi^\beta\widetilde{\psi}_t^+(x,\xi)D_x^\beta(\tilde{\zeta}^2e^{-\phi}h_{jk})=\widetilde{\psi}_t^+(x,\xi)\tilde{\zeta}^2e^{-\phi}h_{jk},$$

up to errors on $\mathcal{C}^0\setminus\mathcal{C}^+$. So, on $\mathcal{C}^+$ we will have

$$\sigma\left((\widetilde{\Psi}_t^+)^*\tilde{\zeta}^2e^{-\phi}h_{jk}\widetilde{\Psi}_t^+T\right)=\sum_\beta\frac{1}{\beta!}\partial_\xi^\beta\sigma((\widetilde{\Psi}_t^+)^*\tilde{\zeta}^2e^{-\phi}h_{jk})D_x^\beta\sigma(T\widetilde{\Psi}_t^+)$$

$$
\begin{aligned}
&= \sum_\beta \frac{1}{\beta!}\partial_\xi^\beta(\widetilde{\psi}_t^+(x,\xi)\tilde{\zeta}^2 e^{-\phi}h_{jk})D_x^\beta(\xi_{2n-1}\widetilde{\psi}_t^+(x,\xi)) \\
&= \tilde{\zeta}^2 e^{-\phi}h_{jk}\xi_{2n-1}.
\end{aligned}
$$

Since $\xi_{2n-1} \geq tA$ on $\mathcal{C}^+$ and $\tilde{\zeta}e^{-\phi}h_{jk}$ is such that the sum of any of its q eigenvalues is nonnegative, by the Proposition 4.6.3 there exist a constant C independent of t such that

$$
\begin{aligned}
\operatorname{Re}\left\{\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}Tf_{jI},f_{kI})_\phi\right\} &\geq tA\operatorname{Re}\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\tilde{\zeta}^2e^{-\phi}h_{jk}f_{jI},f_{kI}\right)_0 \\
&\quad - C\|f\|_\phi^2 + O(\|f\|_{-1}^2) + O_t\left(\|\tilde{\zeta}\widetilde{\Psi}_t^0 f\|_0^2\right) \\
&= tA\operatorname{Re}\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}f_{jI},f_{kI})_\phi \\
&\quad - O(\|f\|_\phi^2) - O_t\left(\|\tilde{\zeta}_\nu\widetilde{\Psi}_t^0 f\|_0^2\right),
\end{aligned}
$$

where all the error whose Fourier transforms are supported in $\mathcal{C}^0\setminus\mathcal{C}^+$ have included in $O_t(\|\tilde{\zeta}_\nu\widetilde{\Psi}_t^0 f\|_0^2)$. ∎

By a similar argument, we can prove the following:

Lemma 4.6.6 *Let f a $(0,q)$-form supported on U so that up to a smooth term $\hat{f}$ is supported in $\mathcal{C}^-$, and let $[h_{jk}]$ a Hermitian matrix such that the sum of any n-1-q eigenvalues is ≥ 0. Then*

$$
\begin{aligned}
&\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}(h_{jj}(-T)u_J,u_J)_\phi - \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}(-T)u_{jI},u_{kI})_\phi\right\} \\
&\geq tA\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}(h_{jj}u_J,u_J)_\phi - \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(h_{jk}u_{jI},u_{kI})_\phi\right\} - O(\|u\|_\phi^2) - O_t(\|\tilde{\zeta}_\nu\tilde{\Psi}_t^0u\|_0^2).
\end{aligned}
$$

Now, we are ready to estimate the energies $Q_{b,+}(.,.)$ and $Q_{b,-}(.,.)$. We start with the energy $Q_{b,+}(.,.)$ as follows:

Proposition 4.6.7 *Let $f \in Dom\bar{\partial}_b \cap Dom\bar{\partial}_b^*$ be a $(0,q)$-form supported in U and let ϕ^+ be as in (4.14). Then there exists a constant C so that*

$$
Q_{b,+}\left(\tilde{\zeta}\Psi_t^+f,\tilde{\zeta}\Psi_t^+f\right) + C\|\tilde{\zeta}\Psi_t^+f\|_{\phi^+} + O_t(\|\tilde{\zeta}\tilde{\Psi}_t^0f\|_0^2) \geq tB_q\|\tilde{\zeta}\Psi_t^+f\|_{\phi^+}^2 \tag{4.22}
$$

Proof. By (4.21), the fact that the Fourier transform of $\tilde{\zeta}\Psi_t^+f$ is supported in $\mathcal{C}^+$ up to smooth term, and Proposition 4.6.5, we have

$$
Q_{b,+}(\tilde{\zeta}\Psi_t^+f,\tilde{\zeta}\Psi_t^+f) \geq tA\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\operatorname{Re}\left(\left(c_{jk}-\frac{i\langle d\gamma,\Upsilon_q\rangle\delta_j^k}{q}\right)\tilde{\zeta}\Psi_t^+f_{jI},\tilde{\zeta}\Psi_t^+f_{kI}\right)_{\phi^+}
$$

$$
\begin{aligned}
&+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\left(\Theta^{\phi+}_{jk}-\frac{i\langle\Theta^{\phi^+},\Upsilon_q\rangle\delta^k_j}{q}\right)\tilde\zeta\Psi^+_t f_{jI},\tilde\zeta\Psi^+_t f_{kI}\right)_{\phi^+}\\
&-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\frac{1}{2}\nu(\phi^+)\left(\left(c_{jk}-\frac{i\langle d\gamma,\Upsilon_q\rangle\delta^k_j}{q}\right)\right)\tilde\zeta\Psi^+_t f_{jI},\tilde\zeta\Psi^+_t f_{kI}\right)_{\phi^+}\\
&-O(\|\tilde\zeta\Psi^+_t f\|^2_{\phi^+})-O_t(\|\tilde\zeta\tilde\Psi^0_t f\|^2_0)
\end{aligned}
$$

Now choosing $A\geq\frac{1}{2}\left|\nu(|z|^2)\right|$ (we can because M is compact), $tA-\frac{1}{2}\nu(\phi^+)\geq 0$, and by Lemma 4.6.2 and the last inequality we have

$$
\begin{aligned}
&Q_{b,+}(\tilde\zeta\Psi^+_t f,\tilde\zeta\Psi^+_t f)+C\|\tilde\zeta\Psi^+_t f\|^2_{\phi^+}+O_t(\|\tilde\zeta\tilde\Psi^0_t f\|^2_0)\\
&\qquad\geq\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\left(\Theta^{\phi+}_{jk}-\frac{i\langle\Theta^{\phi^+},\Upsilon_q\rangle\delta^k_j}{q}\right)\tilde\zeta\Psi^+_t f_{jI},\tilde\zeta\Psi^+_t f_{kI}\right)_{\phi^+}. \qquad (4.23)
\end{aligned}
$$

Since $|q-\omega(\Upsilon_q)|>B_q$, if $\omega(\Upsilon_q)>q$ then $\phi^+=-t\,|z|^2$. Then $\Theta^{\phi^+}=ti\omega$, and so $i\langle\Theta^{\phi^+},\Upsilon_q\rangle=-t\omega(\Upsilon_q)$. Thus the sum of any q eigenvalues of $\Theta^{\phi^+}_{jk}-i\langle\Theta^{\phi^+},\Upsilon_q\rangle\delta^k_j/q=-t\delta^k_j+t\omega(\Upsilon_q)\delta^k_j/q$ equals to $t(\omega(\Upsilon_q)-q)>tB_q$. In the same way, if $\omega(\Upsilon_q)<q$ then $\phi^+=t\,|z|^2$ and $\Theta^{\phi^+}=-ti\omega$. So $i\langle\Theta^{\phi^+},\Upsilon_q\rangle=t\omega(\Upsilon_q)$, and the sum of any q eigenvalues of $\Theta^{\phi^+}_{jk}-i\langle\Theta^{\phi^+},\Upsilon_q\rangle\delta^k_j/q=t\delta^k_j-t\omega(\Upsilon_q)\delta^k_j/q$ equals to $t(q-\omega(\Upsilon))>tB_q$. Then using Lemma 4.6.2 in (4.23) we have (4.22). ■

In order to estimate the terms $Q_{b,-}(\tilde\zeta\Psi^-_t f,\tilde\zeta\Psi^-_t f)$ we have to modify the analysis slightly from the $Q_{b,+}$ case. Similarly to (4.18), we have

$$
\begin{aligned}
Q_{b,\phi}(f,f)\geq{}&\|\nabla_{\bar L^*,\phi}f\|^2_\phi+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\,(c_{jk}Tf_{jI},f_{kI})_\phi-\sum_j\mathrm{Re}\,(c_{jj}Tf,f)_\phi\\
&+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left((\Theta^\phi_{jk}-\frac{1}{2}\nu(\phi)c_{jk})f_{jI},f_{kI}\right)_\phi\\
&-\sum_{j=1}^{n-1}\left((\Theta^\phi_{jj}-\frac{1}{2}\nu(\phi)c_{jj})f,f\right)_\phi\\
&-O_\epsilon(\|\nabla_{\bar L^*,\phi}f\|^2_\phi-\|\nabla_\Upsilon f\|^2_\phi)-O_\epsilon(\|\overline\nabla_\Upsilon f\|^2_\phi)-O_{\frac{1}{\epsilon}}(\|f\|^2_\phi)-O(\|f\|^2_\phi). \quad (4.24)
\end{aligned}
$$

Analogously to (4.19), we have

$$
\begin{aligned}
\|\nabla_\Upsilon f\|^2_\phi\geq{}&\sum_{j,k=1}^{n-1}\left[\left(b^{\bar kj}\bar L_k f,\bar L_j f\right)_\phi+\left(b^{\bar kj}c_{jk}Tf,f\right)_\phi+\left(b^{\bar kj}(\Theta^\phi_{jk}-\frac{1}{2}\nu(\phi)c_{jk})f,f\right)_\phi\right]\\
&-O_\epsilon(\|\nabla_{\bar L^*,\phi}f\|^2_\phi-\|\nabla_\Upsilon f\|^2_\phi)-O_\epsilon(\|\overline\nabla_\Upsilon f\|^2_\phi)-O_{\frac{1}{\epsilon}}(\|f\|^2_\phi)-O(\|f\|^2_\phi). \quad (4.25)
\end{aligned}
$$

It now follows from (4.24) and (4.25) that

$$
Q_{b,\phi}(f,f)\geq\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\,(c_{jk}Tf_{jI},f_{kI})_\phi-\sum_{j=1}^{n-1}\mathrm{Re}\,(c_{jj}Tf,f)_\phi
$$

$$
\begin{aligned}
&+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\left(\left(\frac{i\langle d\gamma,\Upsilon\rangle\delta_j^k}{q}\right)Tf_{jI},f_{kI}\right)_\phi\\
&+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\left(\Theta_{jk}^\phi-\frac{1}{2}\nu(\phi)c_{jk}\right)f_{jI},f_{kI}\right)_\phi\\
&+\sum_{j,k=1}^{n-1}\left(b^{\bar{k}j}\left(\Theta_{jk}^\phi-\frac{1}{2}\nu(\phi)c_{jk}\right)f,f\right)_\phi\\
&-\sum_{j=1}^{n-1}\left(\left(\Theta_{jj}^\phi-\frac{1}{2}\nu(\phi)c_{jj}\right)f,f\right)_\phi-O(\|f\|_\phi^2)\\
=&\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\left(c_{jk}Tf_{jI},f_{kI}\right)_\phi-\mathrm{Re}\left(\sum_{j=1}^{n-1}c_{jj}Tf,f\right)_\phi\\
&+\mathrm{Re}\left(i\langle d\gamma,\Upsilon\rangle Tf,f\right)_\phi\\
&+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\Theta_{jk}^\phi f_{jI},f_{kI}\right)_\phi-\left(\sum_{j=1}^{n-1}\Theta_{jj}^\phi f,f\right)\\
&+\left(i\langle\Theta^\phi,\Upsilon\rangle f,f\right)\\
&-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\left(\frac{1}{2}\nu(\phi)c_{jk}f_{jI},f_{kI}\right)_\phi+\left(\frac{1}{2}\nu(\phi)\sum_{j=1}^{n-1}c_{jj}f,f\right)\\
&-\left(\frac{1}{2}\nu(\phi)i\langle d\gamma,\Upsilon\rangle f,f\right)-O(\|f\|_\phi^2).
\end{aligned}\tag{4.26}
$$

If we set

$$
h_{jk}^-=c_{jk}-\delta_{jk}\frac{i\langle d\gamma,\Upsilon\rangle}{n-1-q},\qquad\text{and}\qquad h_{jk}^\Theta=\Theta_{jk}^\phi-\delta_{jk}\frac{i\langle\Theta^\phi,\Upsilon\rangle}{n-1-q}
$$

then we can rewrite (4.26) by

$$
\begin{aligned}
Q_{b,\phi}(f,f)\geq&-\left(\sum_{j=1}^{n-1}h_{jj}^-Tf,f\right)+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(h_{jk}^-Tf_{jI},f_{kI}\right)\\
&-\left(\sum_{j=1}^{n-1}h_{jj}^\Theta f,f\right)+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(h_{jk}^\Theta f_{jI},f_{kI}\right)\\
&+\left(\frac{1}{2}\nu(\phi)\sum_{j=1}^{n-1}h_{jj}^-f,f\right)-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\frac{1}{2}\nu(\phi)h_{jk}^-f_{jI},f_{kI}\right)\\
&-O(\|f\|_\phi^2).
\end{aligned}
$$

Since the sum of q eigenvalues of the matrix $\frac{Tr(H)}{q}Id-H$ is equal to sum of $(n-1-q)$ eigenvalues of the matrix H, we may now proceed as in the proof of Proposition 4.6.7 (with $\Upsilon=\Upsilon_{n-1-q}$) to obtain the following proposition.

Proposition 4.6.8 *Let $f\in\mathrm{Dom}(\bar{\partial}_b)\cap\mathrm{Dom}(\bar{\partial}_b^*)$ be a $(0,q)$-form supported in U and let ϕ^- be as in (4.14). Then there exists a constant C so that*

$$
Q_{b,-}\left(\tilde{\zeta}\Psi_t^-f,\tilde{\zeta}\Psi_t^-f\right)+C\|\tilde{\zeta}\Psi_t^-f\|_{\phi^-}+O_t(\|\tilde{\zeta}\tilde{\Psi}_t^0f\|_0^2)\geq tB_{n-1-q}\|\tilde{\zeta}\Psi_t^-f\|_{\phi^-}^2.
$$

Proof of the estimate (4.16) Increasing the size of K, K' and $K_\pm$ in Lemma 4.5.2, and by Propositions 4.6.7 and 4.6.8 and the definition of $|||.|||_t$, we obtain (4.16). ■

4.6.1 Ellipticity on $\tilde{\mathcal{C}}_0$

In order to handle the term $O_t\left(\|\tilde{\zeta}\widetilde{\Psi}_t^0\|_0^2\right)$ appearing in our estimate (4.16), the objective of this section is to prove the Proposition 4.6.11 below, that is statemented in [25] as the Proposition 4.11. In contrast with the estimates in Propositions (4.6.7) and (4.6.8) for forms supported on $\mathcal{C}^+$ and $\mathcal{C}^-$ up to smooth terms, we have better estimates for forms supported on $\mathcal{C}^0$ up to smooth terms, as we show below.

Lemma 4.6.9 *Let φ be a $(0,q)$-form supported on a path U', such that, up to a smooth term, $\hat{\varphi}$ is supported in $\mathcal{C}^0$. Then exists positive constants C and C' independent of t for which*

$$CQ_{b,0}(\varphi,\varphi) + C'\|\varphi\|_0^2 \geq \|\varphi\|_1^2. \tag{4.27}$$

Proof. On a similar way as in beginning of the last section, using bc/sc argument, for a small $\varepsilon > 0$ we have

$$Q_{b,0}(\varphi,\varphi) \geq (1-\varepsilon)\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar{L}_j\varphi_J\|^2 + \mathrm{Re}\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}(c_{jk}T\varphi_{jI},\varphi_{kI}) + O(\|\varphi\|^2). \tag{4.28}$$

On the other hand, note that $\|T\cdot\|^2 + \|\nabla_{\bar{L}}\cdot\|^2 + \|\nabla_{\bar{L}^*}\cdot\|^2$ dominates $\|.\|_1^2$, and since $\hat{\varphi}$ is supported in $\mathcal{C}^0$ we have

$$\begin{aligned}
\sum_{J\in\mathcal{I}_q}\|T\varphi_J\|^2 = \sum_{J\in\mathcal{I}_q}(T^*T\varphi_J,\varphi_J) &= \sum_{J\in\mathcal{I}_q}(|\xi_{2n-1}|^2\hat{\varphi}_J,\hat{\varphi}_J)\\
&\leq C\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{2n-2}(|\xi_j|^2\hat{\varphi}_J,\hat{\varphi}_J)\\
&\leq C\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{2n-2}(\|\bar{L}_j\varphi_J\|^2 + \|\bar{L}_j^*\varphi_J\|^2) + O(\|\varphi\|^2)
\end{aligned} \tag{4.29}$$

for some constant C. Then to show (4.27), it is sufficient control the derivatives $\bar{L}_j$ and $\bar{L}_j^*$ by $Q_{b,0}(.,.)$ as we will see below. We claim

$$Q_{b,0}(\varphi,\varphi) \geq (1-\varepsilon)\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar{L}_j\varphi_J\|^2 + O(\|\varphi\|^2) \tag{4.30}$$

$$Q_{b,0}(\varphi,\varphi) \geq (1-\varepsilon)\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar{L}_j^*\varphi_J\|^2 + O(\|\varphi\|^2) \tag{4.31}$$

In fact. Proving (4.30) and (4.31) are similar. We show (4.31). First note that by (4.29) and using small constant/large constant we have

$$-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}(c_{jk}T\varphi_{jI},\varphi_{kI}) \geq -\varepsilon C\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}(\|\bar{L}_j\varphi_J\|^2 + \|\bar{L}_j^*\varphi_J\|^2) + O(\|\varphi\|^2). \tag{4.32}$$

For a smooth function α and by small constant/large constant argument, we have $|(\alpha L_j\varphi_J,\varphi_J)|\le\varepsilon\|\bar L_j^*\varphi_J\|^2+O(\|\varphi\|^2)$ and $|(\alpha\bar L_j\varphi_J,\varphi_J)|\le\varepsilon\|\bar L_j^*\varphi_J\|^2+O(\|\varphi\|^2)$. Then

$$\begin{aligned}\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j\varphi_J\|^2 &= \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\Big\{(\big[\bar L_j^*,\bar L_j\big]\varphi_J,\varphi_J)+(\bar L_j^*\varphi_J,\bar L_j^*\varphi_J)\Big\}\\ &= \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\Big\{\|\bar L_j^*\varphi_J\|^2+(-\big[L_j,\bar L_j\big]\varphi_J,\varphi_J)\Big\}+O(\|\varphi\|^2)\\ &= \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\Big\{\|\bar L_j^*\varphi_J\|^2+(-c_{jj}T\varphi_J,\varphi_J)\Big\}+\\ &\quad+\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\sum_{\ell=1}^{n-1}\Big((d_{jj}^\ell L_\ell-\bar d_{jj}^\ell\bar L_\ell)\varphi_J,\varphi_J\Big)+O(\|\varphi\|^2) \qquad (4.33)\\ &\le(1+2\varepsilon)\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j^*\varphi_J\|^2+\mathrm{Re}\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}(-c_{jj}T\varphi_J,\varphi_J)\\ &\quad+O(\|\varphi\|^2). \qquad (4.34)\end{aligned}$$

On the other hand, using a small constant/large constant argument and (4.29)

$$\begin{aligned}-\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\mathrm{Re}(c_{jj}T\varphi_J,\varphi_J) &\le\varepsilon\sum_{J\in\mathcal{I}_q}\|T\varphi_J\|^2+O(\|\varphi\|^2)\\ &\le\varepsilon C\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}(\|\bar L_j\varphi_J\|^2+\|\bar L_j^*\varphi_J\|^2)+O(\|\varphi\|^2)\\ &\le\varepsilon C'\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j^*\varphi_J\|^2+\varepsilon(1+2\varepsilon)\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j^*\varphi_J\|^2\\ &\quad-\varepsilon C'\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\mathrm{Re}(c_{jj}T\varphi_J,\varphi_J)+O(\|\varphi\|^2). \qquad (4.35)\end{aligned}$$

Absorbing terms in this last inequality, we have

$$-\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\mathrm{Re}(c_{jj}T\varphi_J,\varphi_J)\le\frac{2\varepsilon(1+\varepsilon)}{1-\varepsilon C'}\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j^*\varphi_J\|^2+O(\|\varphi\|^2). \qquad (4.36)$$

Thus, by using (4.36) in (4.34) we have

$$\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j\varphi_J\|^2\le(1+2\varepsilon+\frac{2\varepsilon(1+\varepsilon)}{1-\varepsilon C'})\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j^*\varphi_J\|^2+O(\|\varphi\|^2). \qquad (4.37)$$

Also by (4.33)

$$\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j\varphi_J\|^2\ge(1-2\varepsilon)\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j^*\varphi_J\|^2-\mathrm{Re}\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}(c_{jj}T\varphi_J,\varphi_J)+O(\|\varphi\|^2) \quad (4.38)$$

and using a small constant/large constant argument, by (4.29) and (4.37)

$$-\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\mathrm{Re}(c_{jj}T\varphi_J,\varphi_J)\ge-\varepsilon\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j^*\varphi_J\|^2-\varepsilon(1+2\varepsilon+\frac{2\varepsilon(1+\varepsilon)}{1-\varepsilon C'})\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\bar L_j^*\varphi_J\|^2+O(\|\varphi\|^2).$$

Thus, using this last inequality in (4.38), for some $\varepsilon' > 0$ sufficiently small we have

$$\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\left\|\bar{L}_j\varphi_J\right\|^2 \geq (1-\varepsilon')\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\left\|\bar{L}_j^*\varphi_J\right\|^2 + O(\|\varphi\|^2). \tag{4.39}$$

Now, using (4.39), (4.32) in (4.28) we obtain (4.31) after choosing ε', ε sufficiently small. This proves (4.27). ■

Observe the next relation between $\bar{\partial}_b^*$ and $\bar{\partial}_{b,t}^* = \bar{\partial}_{b,\pm}^*$ below. Using the operators $F_\pm$ and $E_\pm$ defined before, we have

$$\begin{aligned}\left(\varphi,\bar{\partial}_b\phi\right)_\pm &= \left(F_\pm\varphi,\bar{\partial}_b\phi\right)_0 = \left(F_\pm\bar{\partial}_b^*\varphi,\phi\right)_0 + \left(\left[\bar{\partial}_b^*,F_\pm\right]\varphi,\phi\right)_0\\ &= \left(\bar{\partial}_b^*\varphi,\phi\right)_\pm + \left(E_\pm\left[\bar{\partial}_b^*,F_\pm\right]\varphi,\phi\right)_\pm.\end{aligned}$$

Then,

$$\bar{\partial}_{b,\pm}^* = \bar{\partial}_b^* + E_\pm\left[\bar{\partial}_b^*,F_\pm\right]. \tag{4.40}$$

Lemma 4.6.10 *Let φ be a $(0,q)$-form supported in U_μ for some μ such that up to smooth term, $\hat{\varphi}$ is supported in $\tilde{\mathcal{C}}_\mu^0$. There exists a positive constants $C>1$ independent of t and another positive constant $C_\pm$ for which:*

$$C\mathrm{Re}\ Q_{b,\pm}(\varphi,E_\pm\varphi) + C_\pm\|\varphi\|_0^2 \geq \|\varphi\|_1. \tag{4.41}$$

Proof. Keeping in mind that $F_\pm$ and $E_\pm$ are self-adjoint and inverse of each other and that $\sum_\mu\zeta_\mu^2 = 1$, we compute $Q_{b,\pm}(.,.)$, in terms of $Q_{b,0}(.,.)$. By (4.40)

$$\begin{aligned}Q_{b,\pm}(\varphi,E_\pm\varphi) &= \left(\bar{\partial}_b\varphi,\bar{\partial}_bE_\pm\varphi\right)_\pm + \left(\bar{\partial}_{b,\pm}^*\varphi,\bar{\partial}_{b,\pm}^*E_\pm\varphi\right)_\pm\\ &= \left(\bar{\partial}_b\varphi,F_\pm\bar{\partial}_bE_\pm\varphi\right)_0\\ &\quad+\sum_\mu\left(\zeta_\mu(\bar{\partial}_b^*+E_\pm\left[\bar{\partial}_b^*,F_\pm\right])\varphi,\zeta_\mu F_\pm(\bar{\partial}_b^*+E_\pm\left[\bar{\partial}_b^*,F_\pm\right])E_\pm\varphi\right)_0\\ &= \left(\bar{\partial}_b\varphi,\bar{\partial}_b\varphi\right)_0 + \left(\bar{\partial}_b\varphi,\left[F_\pm,\bar{\partial}_b\right]E_\pm\varphi\right)_0 + \left(\bar{\partial}_b^*\varphi,\bar{\partial}_b^*\varphi\right)_0 + \left(\bar{\partial}_b^*\varphi,\left[F_\pm,\bar{\partial}_b^*\right]E_\pm\varphi\right)_0\\ &\quad+\sum_\mu\left(\zeta_\mu E_\pm\left[\bar{\partial}_b^*,F_\pm\right]\varphi,\zeta_\mu F_\pm\bar{\partial}_b^*E_\pm\varphi\right)_0\\ &\quad+\sum_\mu\left(\zeta_\mu E_\pm\left[\bar{\partial}_b^*,F_\pm\right]\varphi,\zeta_\mu\left[\bar{\partial}_b^*,F_\pm\right]E_\pm\varphi\right)_0 + \left(\bar{\partial}_b^*\varphi,\left[\bar{\partial}_b^*,F_\pm\right]E_\pm\varphi\right)_0\\ &= Q_{b,0}(\varphi,\varphi) + \left(\bar{\partial}_b\varphi,\left[F_\pm,\bar{\partial}_b\right]E_\pm\varphi\right)_0 + \left(\bar{\partial}_b^*\varphi,\left[F_\pm,\bar{\partial}_b^*\right]E_\pm\varphi\right)_0\\ &\quad+\sum_\mu\left(\zeta_\mu E_\pm\left[\bar{\partial}_b^*,F_\pm\right]\varphi,\zeta_\mu F_\pm\bar{\partial}_b^*E_\pm\varphi\right)_0\\ &\quad+\sum_\mu\left(\zeta_\mu E_\pm\left[\bar{\partial}_b^*,F_\pm\right]\varphi,\zeta_\mu\left[\bar{\partial}_b^*,F_\pm\right]E_\pm\varphi\right)_0 + \left(\bar{\partial}_b^*\varphi,\left[\bar{\partial}_b^*,F_\pm\right]E_\pm\varphi\right)_0.\end{aligned} \tag{4.42}$$

Since $F_\pm$ and $E_\pm$ are of order zero

$$\left|\left(\bar\partial_b\varphi,\left[F_\pm,\bar\partial_b\right]E_\pm\varphi\right)_0\right|\le\varepsilon\|\bar\partial_b\varphi\|_0^2+C_{\pm,1/\varepsilon}\|\varphi\|_0^2,$$

$$\left|\left(\bar\partial_b^*\varphi,\left[F_\pm,\bar\partial_b^*\right]E_\pm\varphi\right)_0\right|\le\varepsilon\|\bar\partial_b^*\varphi\|_0^2+C_{\pm,1/\varepsilon}\|\varphi\|_0^2,$$

$$\left|\left(\bar\partial_b^*\varphi,\left[\bar\partial_b^*,F_\pm\right]E_\pm\varphi\right)_0\right|\le\varepsilon\|\bar\partial_b^*\varphi\|_0^2+C_{\pm,1/\varepsilon}\|\varphi\|_0^2,$$

$$\left|\sum_\mu\left(\zeta_\mu E_\pm\left[\bar\partial_b^*,F_\pm\right]\varphi,\zeta_\mu\left[\bar\partial_b^*,F_\pm\right]E_\pm\varphi\right)_0\right|\le C_\pm\|\varphi\|_0^2,$$

$$\begin{aligned}\left|\sum_\mu\left(\zeta_\mu E_\pm\left[\bar\partial_b^*,F_\pm\right]\varphi,\zeta_\mu F_\pm\bar\partial_b^*E_\pm\varphi\right)_0\right|&\le\left|\sum_\mu\left(\zeta_\mu E_\pm\left[\bar\partial_b^*,F_\pm\right]\varphi,\zeta_\mu F_\pm\left[\bar\partial_b^*,E_\pm\right]\varphi\right)_0\right|\\&\quad+\left|\sum_\mu\left(\zeta_n uE_\pm\left[\bar\partial_b^*,F_\pm\right]\varphi,\zeta_\mu\bar\partial_b^*\varphi\right)_0\right|\\&\le\varepsilon\|\bar\partial_b^*\varphi\|_0^2+C_{\pm,1+1/\varepsilon}\|\varphi\|_0^2,\end{aligned}$$

for any small constant $\varepsilon>0$ and some positives constants $C_{\pm,1/\varepsilon}$ and $C_{\pm,1+1/\varepsilon}$. Taking a constant ε sufficient small, by (4.42) and by these six last inequalities above, we can prove that there exists a positive constant $C'>1$ independent of t and another positive constant $C_\pm$ such that

$$C'\mathrm{Re}\ Q_{b,\pm}(\varphi,E_\pm\varphi)+C_\pm\|\varphi\|_0^2\ge Q_{b,0}(\varphi,\varphi).$$

Now we can apply Lemma 4.6.9 with U_μ as U' and $\tilde{\mathcal{C}}_\mu^0$ as $\mathcal{C}^0$ to see that

$$cQ_{b,0}(\varphi,\varphi)+c'\|\varphi\|_0^2\ge\|\varphi\|_1^2$$

with c and c' independent of t. Putting this last two inequalities together and enlarging $C'>1$ (that does not depend on t), it follows the inequality (4.41). ■

Proposition 4.6.11 *For any $\varepsilon>0$, there exists $C_{\varepsilon,\pm}>0$ so that*

$$\|\tilde\zeta_\mu\Psi_{\mu,t}^0\zeta_\mu\varphi^\mu\|_0^2\le\varepsilon Q_{b,\pm}(\varphi^\mu,\varphi^\mu)+C_{\varepsilon,\pm}\|\varphi^\mu\|_{-1}^2.$$

Proof. Note that $\|\tilde\zeta_\mu\Psi_{\mu,t}^0\zeta_\mu\varphi^\mu\|_0=\|\Lambda^{-1}\tilde\zeta_\mu\Psi_{\mu,t}^0\zeta_\mu\varphi^\mu\|_1$. The Fourier transform of the $(0,q)$-form $\tilde\zeta_\mu\Psi_{\mu,t}^0\zeta_\mu\varphi^\mu$, up to smooth terms, is supported in $\mathcal{C}^0$, so we can apply the Lemma 4.6.10 with $\varphi=\tilde\zeta_\mu\Psi_{\mu,t}^0\zeta_\mu\varphi^\mu$. Although the range of Λ^{-1} is outside U_μ, we can write $\Lambda^{-1}\tilde\zeta_\mu=\zeta_\mu'\Lambda^{-1}\tilde\zeta_\mu+(1-\zeta_\mu')\Lambda^{-1}\tilde\zeta_\mu$ where $\zeta_\mu{}'$ is a smooth bump function that is identically one on the support of $\tilde\zeta_\mu$. Then $(1-\zeta_\mu')\Lambda^{-1}\tilde\zeta_\mu$ is infinitely smoothing and hence can be absorbed in the $\|\varphi\|_{-1}^2$ term. By the Lemma 4.6.10 we have

$$\begin{aligned}\|\zeta_\mu'\Lambda^{-1}\tilde\zeta_\mu\Psi_{\mu,t}^0\zeta_\mu\varphi^\mu\|_1^2&\le C_\pm\|\zeta_\mu'\Lambda^{-1}\tilde\zeta_\mu\Psi_{\mu,t}^0\zeta_\mu\varphi^\mu\|_0^2\\&\quad+C'\mathrm{Re}\ Q_{b,\pm}(\zeta_\mu'\Lambda^{-1}\tilde\zeta_\mu\Psi_{\mu,t}^0\zeta_\mu\varphi^\mu,E_\pm\zeta_\mu'\Lambda^{-1}\tilde\zeta_\mu\Psi_{\mu,t}^0\zeta_\mu\varphi^\mu)\end{aligned}$$

$$\leq C_\pm \|\varphi^\mu\|_{-1} + C' \text{Re } Q_{b,\pm}(\zeta'_\mu \Lambda^{-1} \tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu, E_\pm \zeta'_\mu \Lambda^{-1} \tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu).$$

Let $P = \zeta'_\mu \Lambda^{-1}$ and $P^{*,\pm}$ be the adjoint of P.

$$\left(\zeta_\mu{}' \Lambda^{-1} u, v\right)_t = \left(u, \Lambda^{-1} \zeta_\mu{}' F_\pm v\right)_0 = \left(u, E_\pm \Lambda^{-1} \zeta_\mu{}' F_\pm v\right)_t = \left(u, \Lambda^{-1} \zeta_\mu{}' + \left[E_\pm, \Lambda^{-1} \zeta_\mu{}'\right] F_\pm v\right)_t,$$

That is, $P^{*,\pm} = \Lambda^{-1}\zeta_\mu{}' + [E_\pm, \Lambda^{-1}\zeta_\mu{}'] F_\pm$, then $P^{*,\pm} - P = [\Lambda^{-1}, \zeta_\mu] + [E_\pm, \Lambda^{-1}\zeta_\mu{}'] F_\pm$ is a pseudodifferential operators of order -2. In the same way $(E_\pm P)^{*,\pm} - E_\pm P$ is a pseudodifferential operators of order -2. By Lemma 2.4.2 in [6] we have

$$\begin{aligned}
\text{Re } Q_{b,\pm}(P\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu, E_\pm P\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu) &\leq \frac{1}{2} Q_{b,\pm}(P\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu, P\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu) \\
&\quad + \frac{1}{2} Q_{b,\pm}(E_\pm P\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu, E_\pm P\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu) \\
&= \frac{1}{2} \text{Re } Q_{b,\pm}(\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu, P^{*,\pm} P\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu) \\
&\quad + \frac{1}{2} \text{Re } Q_{b,\pm}(\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu, (E_\pm P)^{*,\pm} E_\pm P\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu) \\
&\quad + O_\pm \left(\|\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu \varphi^\mu\|^2_{-1}\right) \\
&\leq \varepsilon Q_{b,\pm}(\varphi^\mu, \varphi^\mu) + C_{\varepsilon,\pm} \|\varphi^\mu\|^2_{-1}. \qquad \blacksquare
\end{aligned}$$

Proof of Proposition 4.6.1 By Proposition 4.6.11, choosing sufficiently large constants K and t in (4.16) we obtain (4.15). $\blacksquare$

CHAPTER 5

CLOSURE AND REGULARITY THEOREMS FOR THE OPERATOR $\bar{\partial}_b$ AND $\Box_b$

Now that we have the tools of last Chapter, we can prove strong closed range estimates using many of the arguments of [10]. We do, however, use a substantially different elliptic regularization to pay particular attention to the regularity of the weighted harmonic forms, the relationship of the harmonic forms with the regularized operators, and an especially detailed look at the induction base case.

Specifically, the objective of this chapter is to prove the next result about closure and regularity of the $\bar{\partial}_b$ and the $\Box_b$ operators.

Theorem 5.0.1 *Let M^{2n-1} be an embedded C^∞ compact, orientable CR manifold of hypersurface type that satisfies weak $Y(q)$ for some fixed q, $1 \leq q \leq n-2$. Then the following hold:*

1. *The operators $\bar{\partial}_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ and $\bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ have closed range;*
2. *The operators $\bar{\partial}^*_b : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ and $\bar{\partial}^*_b : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ have closed range;*
3. *The Kohn Laplacian $\Box_b := \bar{\partial}_b\bar{\partial}^*_b + \bar{\partial}^*_b\bar{\partial}_b$ has closed range on $L^2_{0,q}(M)$;*
4. *The complex Green operator G_q exists and is continuous on $L^2_{0,q}(M)$;*
5. *The canonical solution operators, $\bar{\partial}^*_b G_q : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ and $G_q\bar{\partial}^*_b : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ are continuous;*
6. *The canonical solution operators, $\bar{\partial}_b G_q : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ and $G_q\bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ are continuous;*
7. *The space of the harmonic forms $\mathcal{H}_{0,q}(M)$, defined to be the $(0,q)$-forms annihilated by $\bar{\partial}_b$ and $\bar{\partial}^*_b$, is finite dimensional;*
8. *If $\alpha \in L^2_{0,q}(M) \cap {}^\perp\mathcal{H}_{0,q}(M)$ is $\bar{\partial}_b$-closed, then there exists $u \in L^2_{0,q-1}(M)$ so that*

$$\bar{\partial}_b u = \alpha$$

and $\|u\|_0 \leq C\|\alpha\|_0$ for some constant C independent of α;

9. *The Szegö projection* $S_q = I - \bar{\partial}_b^* \bar{\partial}_b G_q$ *is continuous on* $L^2_{0,q}(M)$.

We start by proving the result by considering weighted spaces but then in the last section we show the results in the unweighted spaces follows from the equivalence of the norms. From now on, the functions ϕ^+ and ϕ^- are going to be as (4.14). The dependence of t of this functions will be considered. We denote now the norm $|||\cdot|||_\pm$ by $|||\cdot|||_t$, and in general we replace the subscript $\pm$ with t (e.g. we write c_t for $\phi_\pm$).

Theorem 5.0.2 *Let* M^{2n-1} *be a* C^∞ *compact, orientable, weakly* $Y(q)$ *CR manifold of hypersurface type embedded in* $\mathbb{C}^N$, $N \geq n$, *and* $1 \leq q \leq n-2$. *For each* $s \geq 0$ *there exists* $T_s \geq 0$ *so that the following hold:*

i. *The operators* $\bar{\partial}_b : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q+1}(M, |||\cdot|||_t)$ *and* $\bar{\partial}_b : L^2_{0,q-1}(M, |||\cdot|||_t) \to L^2_{0,q}(M, |||\cdot|||_t)$ *have closed range. Additionally, for any* $s > 0$ *if* $t \geq T_s$, *then* $\bar{\partial}_b : H^s_{0,q}(M, |||\cdot|||_t) \to H^s_{0,q+1}(M, |||\cdot|||_t)$ *and* $\bar{\partial}_b : H^s_{0,q-1}(M, |||\cdot|||_t) \to H^s_q(M, |||\cdot|||_t)$ *have closed range.*

ii. *The operators* $\bar{\partial}^*_{b,t} : L^2_{0,q+1}(M, |||\cdot|||_t) \to L^2_{0,q}(M, |||\cdot|||_t)$ *and* $\bar{\partial}^*_{b,t} : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q-1}(M, |||\cdot|||_t)$ *have closed range; Additionally, if* $t \geq T_s$, *then* $\bar{\partial}^*_{b,\pm} : H^s_{0,q+1}(M, |||\cdot|||_t) \to H^s_{0,q}(M, |||\cdot|||_t)$ *and* $\bar{\partial}^*_{b,\pm} : H^s_{0,q}(M, |||\cdot|||_t) \to H^s_{0,q-1}(M, |||\cdot|||_t)$ *have closed range.*

iii. *The Kohn Laplacian* $\Box_{b,t} := \bar{\partial}_b \bar{\partial}^*_{b,t} + \bar{\partial}^*_{b,t} \bar{\partial}_b$ *has closed range on* $L^2_{0,q}(M, |||\cdot|||_t)$, *and if* $t \geq T_s$, $\Box_{b,t}$ *also has closed range on* $H^s_{0,q}(M, |||\cdot|||_t)$.

iv. *The space of (weighted) harmonic forms* $\mathcal{H}^q_t(M)$, *defined to be the* $(0,q)$*-forms annihilated by* $\bar{\partial}_b$ *and* $\bar{\partial}^*_{b,t}$, *is finite dimensional.*

v. *The complex Green operator* $G_{q,t}$ *exists and is continuous on* $L^2_{0,q}(M, |||\cdot|||_t)$ *and also on* $H^s_{0,q}(M, |||\cdot|||_t)$ *if* $t \geq T_s$.

vi. *The canonical solution operator for* $\bar{\partial}_b$, $\bar{\partial}^*_{b,t} G_{q,t} : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q-1}(M, |||\cdot|||_t)$ *is continuous. Additionally,* $\bar{\partial}^*_{b,t} G_{q,t} : H^s_{0,q}(M, |||\cdot|||_t) \to H^s_{0,q-1}(M, |||\cdot|||_t)$ *is continuous if* $t \geq T_s$.

vii. *The canonical solution operator for* $\bar{\partial}^*_{b,t}$, $\bar{\partial}_b G_{q,t} : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q+1}(M, |||\cdot|||_t)$ *is continuous. Additionally,* $\bar{\partial}_b G_{q,t} : H^s_{0,q}(M, |||\cdot|||_t) \to H^s_{0,q+1}(M, |||\cdot|||_t)$ *is continuous if* $t \geq T_s$.

viii. *The Szegö projection* $S_{q,t} = I - \bar{\partial}^*_{b,t} \bar{\partial}_b G_{q,t}$ *is continuous on* $L^2_{0,q}(M, |||\cdot|||_t)$ *. Additionally, if* $t \geq T_s$ *then* $S_{q,t}$ *is continuous on* $H^s_{0,q}(M, |||\cdot|||_t)$ *.*

We start doing some observations about the space of weighted harmonic $(0,q)$-forms denoted by $\mathcal{H}^q_t(M)$ and defined by

$$\begin{aligned}\mathcal{H}^q_t(M) &:= \left\{f \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b) : \bar{\partial}_b f = 0, \bar{\partial}^*_{b.\pm} f = 0\right\} \\ &= \left\{f \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b) : Q_{b,\pm}(f,f) = 0\right\}.\end{aligned}$$

Lemma 5.0.3 *Let M be a smooth, embedded CR manifold of hypersurface type that satisfies $Y(q)$ weakly. If $t > 0$ is suitably large and the functions ϕ^+, ϕ^- are as in (4.14), then*

(i) $\mathcal{H}^q_t$ is finite dimensional;

(ii) There exists C that does not depend on ϕ^+ and ϕ^- so that for all $(0,q)$-forms $u \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b)$ satisfying $u \perp \mathcal{H}^q_t$ (with respect to $(\cdot,\cdot)_t$) we have

$$|||u|||^2_t \le C Q_{b,t}(u,u). \tag{5.1}$$

Proof. If $u \in \mathcal{H}^q_t(M)$ by (4.15) we have $|||u|||_t \le C_t \|u\|_{-1}$. Then the identity map from $H^{-1}_{0,q}(M) \cap \mathcal{H}^q_t(M)$ to $L^2_{0,q}(M) \cap \mathcal{H}^q_t(M)$ is bounded. By Rellich's theorem, $L^2_{0,q}(M) \cap \mathcal{H}^q_t(M)$ is compactly embedded in $H^{-1}_{0,q}(M) \cap \mathcal{H}^q_t(M)$ via the identity map. The composition of a bounded operator with a compact operator is a compact operator, which implies the identity map from $L^2_{0,q}(M) \cap \mathcal{H}^q_t(M)$ to $L^2_{0,q}(M) \cap \mathcal{H}^q_t(M)$ is compact. It follows that the unit sphere on $\mathcal{H}^q_t(M)$ is compact. Then $\mathcal{H}^q_t(M)$ is finite dimensional.

We prove (5.1) by contradiction. Assume (5.1) is not true, then there exists a sequence $\{u_k\} \subset {}^\perp\mathcal{H}^q_t(M)$ with $|||u_k|||_t = 1$, so that

$$1 = |||u_k|||^2_t \ge k Q_{b,t}(u_k, u_k). \tag{5.2}$$

Again, by the Rellich's theorem $L^2_{0,q}(M) \cap {}^\perp\mathcal{H}^q_t(M)$ is compact in $H^{-1}_{0,q}(M) \cap {}^\perp\mathcal{H}^q_t(M)$, then there exists a subsequence u_{k_j} that converges in $H^{-1}_{0,q}(M)$. Now using (4.15), we obtain

$$|||u_{k_r} - u_{k_s}|||^2_t \le C_t \left(Q_{b,t}(u_{k_r}, u_{k_r}) + Q_{b,t}(u_{k_s}, u_{k_s}) + \|u_{k_r} - u_{k_s}\|^2_{-1}\right)$$

Then, by (5.2) and since u_{k_j} converges in $H^{-1}_{0,q}(M)$, u_{k_j} is a Cauchy sequence in $L^2_{0,q}(M)$, so it converges in $L^2_{0,q}(M)$. Also (5.2) implies that u_{k_j} converges in the $Q_{b,t}(\cdot,\cdot)^{1/2}$ norm, so in the $\left(Q_{b,t}(\cdot,\cdot) + |||\cdot|||^2_t\right)^{1/2}$ norm. If u is the limit of u_{k_j} in $L^2_{0,q}(M)$ then $u \in {}^\perp\mathcal{H}^q_t(M)$ and $|||u|||_t = 1$, and by the closure of operator $\bar{\partial}_b$ and $\bar{\partial}^*_{b,t}$, $u \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b)$. However, a consequence of (5.2) is that $u \in \mathcal{H}^q_t(M)$. Then $u = 0$ and this is a contradiction. ∎

Now by the Theorem 2.1.3 we have the closure of the range of the operators $\bar{\partial}_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ and $\bar{\partial}^*_{b,t} : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ in the norm $|||.|||_t$. Then by the Theorem 2.1.2 we have the closure of the range of the operators $\bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ and $\bar{\partial}^*_{b,t} : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ in the norm $|||.|||_t$.

5.1 Existence of the Green operator $G_{q,t}$

Here we prove the existence of the Green operator, which is defined as the inverse of the Kohn Laplacian $\Box_{b,t} := \bar{\partial}_b\bar{\partial}^*_{b,t} + \bar{\partial}^*_{b,t}\bar{\partial}_b$ where

$$\mathrm{Dom}(\Box_{b,t}) := \left\{\phi \in L^2_{0,q}(M) : \phi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_{b,t}),\ \bar{\partial}_b\phi \in \mathrm{Dom}(\bar{\partial}^*_{b,t}),\ \text{and}\ \bar{\partial}^*_{b,t}\phi \in \mathrm{Dom}(\bar{\partial}_b)\right\}.$$

For this purpose, we will use the next theorem established in [24] in Lemma 5.4.

Lemma 5.1.1 *Let H be a Hilbert space equipped with the inner product $(.,.)$ and corresponding norm $\|.\|$ and Q a positive definite Hermitian form defined on a dense $D \subset H$ satisfying*

$$\|\varphi\|^2 \leq CQ(\varphi,\varphi) \tag{5.3}$$

for all $\varphi \in D$. D and Q are such that D is a Hilbert space under the inner product $Q(.,.)$. Then there exists a unique self-adjoint operator injective F with $\mathrm{Dom}(F) \subset D$ satisfying

$$Q(\varphi,\phi) = (F\varphi,\phi)$$

for all $\varphi \in \mathrm{Dom}(F)$ and $\phi \in D$. F is called the Friedrich's representative.

Proof. For each $\alpha \in H$, we define the antilinear functional Ω_α on $(D, \|.\|)$ given by $\Omega_\alpha : D \ni \phi \to (\alpha,\phi)$. By (5.3)

$$|\Omega_\alpha(\phi)| \leq \|\alpha\|\|\phi\| \leq C\|\alpha\|Q(\phi,\phi)^{1/2}$$

so by the Riesz representation theorem there exists some $\varphi_\alpha \in D$ such that $Q(\varphi_\alpha,\phi) = (\alpha,\phi)$ for every $\phi \in D$. Let S be the operator associated to $\alpha \in H$ with $\varphi_\alpha \in D$, that is, $S : H \ni \alpha \to \varphi_\alpha \in D$. Again by (5.3) we have

$$\|S\alpha\|^2 \leq CQ(S\alpha,S\alpha) = C(\alpha,S\alpha) \leq C\|\alpha\|\|S\alpha\|,$$

so S is a bounded operator and $\|S\alpha\| \leq \|\alpha\|$. Now, if $S\alpha = 0$ then $(\alpha,\phi) = Q(S\alpha,\phi) = 0$ for all $\phi \in D$. Since D is dense in H, $(\alpha,\phi) = 0$ for all $\phi \in H$ and $\alpha = 0$. This implies that S is injective. Also

$$(S\alpha,\beta) = \overline{(\beta,S\alpha)} = \overline{Q(S\beta,S\alpha)} = Q(S\alpha,S\beta) = (\alpha,S\beta),$$

so S is self-adjoint. Define $F := S^{-1} : \mathrm{Ran}(S) \to H$. F is self-adjoint, $\mathrm{Dom}(F) = \mathrm{Ran}(S) \subset D$, F is onto and

$$Q(\varphi,\phi) = (F\varphi,\phi)$$

for all $\varphi \in \mathrm{Dom}(F)$ and $\phi \in D$. ■

In order to use Lemma 5.1.1, we need the following lemma.

Lemma 5.1.2 *$\left(\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)\cap{}^\perp\mathcal{H}_t^q(M), Q_{b,t}(.,.)^{1/2}\right)$ is a Hilbert space and $\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)\cap{}^\perp\mathcal{H}_t^q(M)$ is dense in ${}^\perp\mathcal{H}_t^q(M)$.*

Proof. Suppose $\{u_\ell\}\subset\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)\cap{}^\perp\mathcal{H}_t^q(M)$ is a Cauchy sequence with respect to the norm $Q_{b,t}(\cdot,\cdot)^{1/2}$. Then $\bar\partial_b u_\ell$ and $\bar\partial_{b,t}^* u_\ell$ are Cauchy sequences in $L^2_{0,q+1}(M,\left|\left|\left|\cdot\right|\right|\right|_t)$ and $L^2_{0,q-1}(M,\left|\left|\left|\cdot\right|\right|\right|_t)$, respectively, so they converge to $v_1\in L^2_{0,q+1}(M,\left|\left|\left|\cdot\right|\right|\right|_t)$ and $v_2\in L^2_{0,q-1}(M,\left|\left|\left|\cdot\right|\right|\right|_t)$ respectively. By (5.1), this means $\{u_\ell\}$ is a Cauchy sequence in $L^2_{0,q}(M,\left|\left|\left|\cdot\right|\right|\right|_t)$, hence converges to some $u\in L^2_{0,q}(M,\left|\left|\left|\cdot\right|\right|\right|_t)$. Thus $u\in\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)$, $\bar\partial_b u=v_1$, and $\bar\partial_{b,t}^* u=v_2$ since $\bar\partial_b$ and $\bar\partial_{b,t}^*$ are closed operators. Since $0=(u_\ell,w)_t$ for all $w\in\mathcal{H}_t^q$ and $\left|\left|\left|u_\ell-u\right|\right|\right|_t\to 0$, $u\in{}^\perp\mathcal{H}_t^q(M)$. Thus $u\in\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)\cap{}^\perp\mathcal{H}_t^q$.

Next, suppose $u\in{}^\perp\mathcal{H}_t^q(M)$ is nonzero and $u_\ell\in\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)$ satisfies $u_\ell\to u$ on $L^2_{0,q}(M,\left|\left|\left|\cdot\right|\right|\right|_t)$. Let $v_\ell=(I-H_t^q)u_\ell$, with H_t^q the orthogonal projection onto $\mathcal{H}_t^q$. The forms $v_\ell\in{}^\perp\mathcal{H}_t^q(M)\cap\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)$. Since $u\neq 0$, it cannot be the case that $v_\ell=0$ for every ℓ. Since $\left|\left|\left|u_\ell\right|\right|\right|_t^2=\left|\left|\left|H_t^q u_\ell\right|\right|\right|_t^2+\left|\left|\left|v_\ell\right|\right|\right|_t^2$, and the forms $H_t^q u_\ell$ and v_ℓ are orthogonal, $H_t^q u_\ell$ and v_ℓ both converge in $L^2_{0,q}(M,\left|\left|\left|\cdot\right|\right|\right|_t)$. Let $\alpha=\lim_{\ell\to\infty}H_t^q u_\ell$, $v=\lim_{\ell\to\infty}v_\ell$, and since $H_t^q u_\ell=u_\ell-v_\ell$, $\alpha=u-v\in{}^\perp\mathcal{H}_t^q(M)$. However, $\alpha\in\mathcal{H}_t^q$ since $\mathcal{H}_t^q$ is closed, forcing $\alpha=0$. Thus, $\left|\left|\left|u-v_\ell\right|\right|\right|_t\le\left|\left|\left|u-u_\ell\right|\right|\right|_t+\left|\left|\left|H_t^q u_\ell\right|\right|\right|_t\to 0$. Consequently $\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)\cap{}^\perp\mathcal{H}_t^q(M)$ is dense in ${}^\perp\mathcal{H}_t^q(M)$. ■

We now can establish the existence and L^2-continuity of the complex Green operator $G_{q,t}$ using the following well-known result (we adapt the presentation and argument in [24, Corollary 5.5]).

Corollary 5.1.3 *Let M be a smooth compact, orientable embedded CR manifold of hypersurface type that satisfies weak $Y(q)$. If $t>0$ is suitable large, ϕ^+,ϕ^- are as in (4.14), and $\alpha\in{}^\perp\mathcal{H}_t^q$, then there exists a unique $\varphi_t\in{}^\perp\mathcal{H}_t^q\cap\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)$ such that*

$$Q_{b,t}(\varphi_t,\phi)=(\alpha,\phi)_t\,,\qquad\forall\phi\in\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*).$$

We define the Green operator $G_{q,t}$ to be the operator that maps α into φ_t. $G_{q,t}$ is a bounded operator, and if additionally α is closed, then $u_t=\bar\partial_{b,t}^*G_{q,t}\alpha$ satisfies $\bar\partial_b u_t=\alpha$. We define $G_{q,t}$ to be identically 0 on $\mathcal{H}_t^q(M)$.

Proof. We apply Lemma 5.1.1 with $H={}^\perp\mathcal{H}_t^q(M)$, $\left|\left|\left|\cdot\right|\right|\right|_t$ as $\|\cdot\|$, $Q_{q,t}(\cdot,\cdot)$ as Q and $\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)\cap{}^\perp\mathcal{H}_t^q(M)$ as D. By the Lemma 5.1.2, for t suitably large and (5.1), we have both the existence of the Friedrichs representative F, (and using the notation on Lemma 5.1.1), the existence of its inverse self-adjoint $S:{}^\perp\mathcal{H}_t^q(M)\to\mathrm{Ran}(S)=\mathrm{Dom}(F)\subset\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)\cap{}^\perp\mathcal{H}_t^q(M)$. We affirm $\mathrm{Dom}(\Box_{b,t})\cap{}^\perp\mathcal{H}_t^q(M)\subset\mathrm{Dom}(F)$. In fact, let $\alpha\in\mathrm{Dom}(\Box_{b,t})\cap{}^\perp\mathcal{H}_t^q(M)$ and $\beta=\Box_{b,t}\alpha$. Then $\beta\in{}^\perp\mathcal{H}_t^q(M)$ and $Q_{b,t}(S\beta,\phi)=$

$(\beta, \phi)_t = Q_{b,t}(\alpha, \phi)$ for any $\phi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*) \cap {}^{\perp}\mathcal{H}_t^q(M)$. Then $\alpha = S\beta$ and $\alpha \in \mathrm{Ran}(S) = \mathrm{Dom}(F)$. Moreover $\mathrm{Dom}(\Box_{b,t}^*) \cap {}^{\perp}\mathcal{H}_t^q(M) \supset \mathrm{Dom}(F^*)$. Since $\Box_{b,t}$ and F are self-adjoint, $\mathrm{Dom}(\Box_{b,t}) \cap {}^{\perp}\mathcal{H}_t^q(M) = \mathrm{Dom}(F)$. Then $F = \Box_{b,t}$ on $\mathrm{Dom}(\Box_{b,t}) \cap {}^{\perp}\mathcal{H}_t^q(M)$. So $G_{q,t} := S : {}^{\perp}\mathcal{H}_t^q(M) \to \mathrm{Dom}(\Box_{b,t}) \cap {}^{\perp}\mathcal{H}_t^q(M)$, and

$$Q_{b,t}(G_{q,t}\alpha, \phi) = (\alpha, \phi)_t, \qquad \forall \phi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*) \cap {}^{\perp}\mathcal{H}_t^q(M)$$

with $G_{b,t}\alpha$ unique on $\mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*) \cap {}^{\perp}\mathcal{H}_t^q(M)$, and $|||G_{q,t}\alpha|||_t \leq C|||\alpha|||_t$. Since $\alpha \perp \mathcal{H}_t^q(M)$, also we have

$$Q_{b,t}(G_{q,t}\alpha, \phi) = (\alpha, \phi)_t, \qquad \forall \phi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*) \cap \mathcal{H}_t^q(M). \tag{5.4}$$

So (5.4) is satisfied for all $\phi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*)$. ∎

5.2 Smoothness of harmonic forms

Here we will prove that $\mathcal{H}_t^q(M) \subset H_{0,q}^s(M, |||\cdot|||_t)$ for t sufficiently large, assuming (4.15). We adapt the arguments of [16, 9]. See also [24, 19].

Fix $s \geq 1$. For forms $f, g \in H_{0,q}^1(M, |||\cdot|||_t)$, set

$$Q_{b,t}^{\delta,\nu}(f,g) = Q_{b,t}(f,g) + \delta Q_{d_b}(f,g) + \nu\,(f,g)_t$$

where $Q_{d_b}(\cdot,\cdot)$ is the hermitian inner product associated to the de Rham exterior derivative d_b, i.e., $Q_{d_b}(u,v) = (d_b u, d_b v)_t + \left(d_{b,t}^* u, d_{b,t}^* v\right)_t$, and $\delta, \nu \geq 0$. Also note that $Q_{b,t}^{0,\nu}(f,g) = Q_{b,t}(f,g) + \nu\,(f,g)_t$ for $f, g \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*)$. Then

$$|||\varphi|||_t^2 \leq \frac{1}{\nu} Q_{b,t}^{\delta,\nu}(\varphi,\varphi).$$

for all $\varphi \in H_{0,q}^1(M, |||\cdot|||_t)$ if $\delta > 0$ and all $\varphi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*)$ if $\delta = 0$. By Lemma 5.1.1, there exist self-adjoint operators (for $0 \leq \delta \leq 1$ and $0 < \nu \leq 1$) $\Box_{b,t}^{\delta,\nu} : \mathrm{Dom}(\Box_{b,t}^{\delta,\nu}) \to L_{0,q}^2(M, |||\cdot|||_t)$, with inverses $G_{q,t}^{\delta,\nu} : L_{0,q}^2(M, |||\cdot|||_t) \to \mathrm{Dom}(\Box_{b,t}^{\delta,\nu})$ satisfying

$$|||G_{q,t}^{\delta,\nu}\varphi|||_t^2 \leq \frac{1}{\nu}|||\varphi|||_t^2 \tag{5.5}$$

for all $\varphi \in L_{0,q}^2(M, |||\cdot|||_t)$ and all $\delta \in [0,1]$.

Our goal is to prove

$$\|G_{q,t}^{0,\nu}\varphi\|_{H^s} \leq K_t\|\varphi\|_{H^s} + C_{t,s}\|G_{q,t}^{0,\nu}\varphi\|_0. \tag{5.6}$$

In fact, (5.6) is the main tool that we need to prove that $\mathcal{H}_t^q(M) \subset H_{0,q}^s(M, |||\cdot|||_t)$, for t sufficiently large.

We now prove (5.6). The operator $\Box_{b,t}^{\delta,\nu}$ is elliptic when $\delta > 0$ which means that $G_{q,t}^{\delta,\nu} : H_{0,q}^s(M, |||\cdot|||_t) \to H_{0,q}^{s+2}(M, |||\cdot|||_t)$.

If $\varphi \in H^s_{0,q}(M, |||\cdot|||_t)$, then

$$\|G^{\delta,\nu}_{q,t}\varphi\|^2_{H^s} = \|\Lambda^s G^{\delta,\nu}_{q,t}\varphi\|^2_0 \le C_t |||\Lambda^s G^{\delta,\nu}_{q,t}\varphi|||^2_t.$$

Since $G^{\delta,\nu}_{q,t}\varphi \in H^{s+2}_{0,q}(M, |||\cdot|||_t)$, the basic estimate yields

$$\begin{aligned} |||\Lambda^s G^{\delta,\nu}_{q,t}\varphi|||^2_t &\le \frac{K}{t} Q_{b,t}(\Lambda^s G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi) + C_{t,s}\|G^{\delta,\nu}_{q,t}\varphi\|^2_{H^{s-1}} \\ &\le \frac{K}{t} Q^{\delta,\nu}_{b,t}(\Lambda^s G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi) + C_{t,s}\|G^{\delta,\nu}_{q,t}\varphi\|^2_{H^{s-1}}. \end{aligned} \tag{5.7}$$

On the other hand, if $(\Lambda^s)^{*,t}$ is the adjoint of Λ^s under the inner product $(\cdot,\cdot)_t$, then

$$(\Lambda^s u, v)_t = (u, \Lambda^s F_t v)_0 = (u, E_t\Lambda^s F_t v)_t = (u, (\Lambda^s + [E_t, \Lambda^s]F_t)v)_t$$

implies that $(\Lambda^s)^{*,t} = \Lambda^s + P^{s-1}_t$, where P^{s-1}_t is a pseudodifferential operator of order $s-1$ depending on t. A careful integration by parts shows that

$$\begin{aligned} &|||\bar\partial_b \Lambda^s G^{\delta,\nu}_{q,t}\varphi|||^2_t \\ &\quad= \left(\bar\partial^*_{b,t}\Lambda^s\bar\partial_b G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t + \left([\bar\partial_b, \Lambda^s]G^{\delta,\nu}_{q,t}\varphi, \bar\partial_b\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\quad= \left(\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t + \left([\bar\partial_b, \Lambda^s]G^{\delta,\nu}_{q,t}\varphi, \bar\partial_b\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t + \left([\bar\partial^*_{b,t}, \Lambda^s]\bar\partial_b G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\quad= \left(\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t + \left([\bar\partial_b, \Lambda^s]G^{\delta,\nu}_{q,t}\varphi, \bar\partial_b\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\qquad+ \left(\Lambda^s\bar\partial_b G^{\delta,\nu}_{q,t}\varphi, (\Lambda^{-s})^{*,t}\left[(\Lambda^s)^{*,t}, \bar\partial_b\right]\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\quad= \left(\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t + \left([\bar\partial_b, \Lambda^s]G^{\delta,\nu}_{q,t}\varphi, \bar\partial_b\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\qquad+ \left(\Lambda^s\bar\partial_b G^{\delta,\nu}_{q,t}\varphi, \left(\left[(\Lambda^s)^{*,t}, \bar\partial_b\right] + \Lambda^{-s}\left[\left[(\Lambda^s)^{*,t}, \bar\partial_b\right], \Lambda^s\right] + P^{-s-1}_t\left[(\Lambda^s)^{*,t}, \bar\partial_b\right]\Lambda^s\right) G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\quad= \left(\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t + \left([\bar\partial_b, \Lambda^s]G^{\delta,\nu}_{q,t}\varphi, \bar\partial_b\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\qquad+ \left(\bar\partial_b\Lambda^s G^{\delta,\nu}_{q,t}\varphi, \left(\left[(\Lambda^s)^{*,t}, \bar\partial_b\right] + \Lambda^{-s}\left[\left[(\Lambda^s)^{*,t}, \bar\partial_b\right], \Lambda^s\right] + P^{-s-1}_t\left[(\Lambda^s)^{*,t}, \bar\partial_b\right]\Lambda^s\right) G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\qquad+ \left([\Lambda^s, \bar\partial_b]G^{\delta,\nu}_{q,t}\varphi, \left(\left[(\Lambda^s)^{*,t}, \bar\partial_b\right] + \Lambda^{-s}\left[\left[(\Lambda^s)^{*,t}, \bar\partial_b\right], \Lambda^s\right] + P^{-s-1}_t\left[(\Lambda^s)^{*,t}, \bar\partial_b\right]\Lambda^s\right) G^{\delta,\nu}_{q,t}\varphi\right)_t. \end{aligned}$$

Applying the same sequence of integration by parts and commutators we have

$$\begin{aligned} &|||\bar\partial^*_{b,t} \Lambda^s G^{\delta,\nu}_{q,t}\varphi|||^2_t \\ &\quad= \left(\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t + \left([\bar\partial^*_{b,t}, \Lambda^s]G^{\delta,\nu}_{q,t}\varphi, \bar\partial^*_{b,t}\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\qquad+ \left(\bar\partial^*_{b,t}\Lambda^s G^{\delta,\nu}_{q,t}\varphi, \left(\left[(\Lambda^s)^{*,t}, \bar\partial^*_{b,t}\right] + \Lambda^{-s}\left[\left[(\Lambda^s)^{*,t}, \bar\partial^*_{b,t}\right], \Lambda^s\right] + P^{-s-1}_t\left[(\Lambda^s)^{*,t}, \bar\partial^*_{b,t}\right]\Lambda^s\right) G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\qquad+ \left([\Lambda^s, \bar\partial^*_{b,t}]G^{\delta,\nu}_{q,t}\varphi, \left(\left[(\Lambda^s)^{*,t}, \bar\partial^*_{b,t}\right] + \Lambda^{-s}\left[\left[(\Lambda^s)^{*,t}, \bar\partial^*_{b,t}\right], \Lambda^s\right] + P^{-s-1}_t\left[(\Lambda^s)^{*,t}, \bar\partial^*_{b,t}\right]\Lambda^s\right) G^{\delta,\nu}_{q,t}\varphi\right)_t; \end{aligned}$$

$$\begin{aligned} &|||d_b \Lambda^s G^{\delta,\nu}_{q,t}\varphi|||^2_t \\ &\quad= \left(\Lambda^s d^*_{b,t} d_b G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t + \left([d_b, \Lambda^s]G^{\delta,\nu}_{q,t}\varphi, d_b\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t \\ &\qquad+ \left(d_b\Lambda^s G^{\delta,\nu}_{q,t}\varphi, \left(\left[(\Lambda^s)^{*,t}, d_b\right] + \Lambda^{-s}\left[\left[(\Lambda^s)^{*,t}, d_b\right], \Lambda^s\right] + P^{-s-1}_t\left[(\Lambda^s)^{*,t}, d_b\right]\Lambda^s\right) G^{\delta,\nu}_{q,t}\varphi\right)_t \end{aligned}$$

$$+\left([\Lambda^s, d_b]G^{\delta,\nu}_{q,t}\varphi, \left(\left[(\Lambda^s)^{*,t}, d_b\right] + \Lambda^{-s}\left[\left[(\Lambda^s)^{*,t}, d_b\right], \Lambda^s\right] + P_t^{-s-1}\left[(\Lambda^s)^{*,t}, d_b\right]\Lambda^s\right)G^{\delta,\nu}_{q,t}\varphi\right)_t;$$

$$\begin{aligned}
&\left\|\left|d^*_{b,t}\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right\|\right|_t^2 \\
&\quad= \left(\Lambda^s d_b d^*_{b,t}G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t + \left([d^*_{b,t}, \Lambda^s]G^{\delta,\nu}_{q,t}\varphi, d^*_{b,t}\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right)_t \\
&\qquad+ \left(d^*_{b,t}\Lambda^s G^{\delta,\nu}_{q,t}\varphi, \left(\left[(\Lambda^s)^{*,t}, d^*_{b,t}\right] + \Lambda^{-s}\left[\left[(\Lambda^s)^{*,t}, d^*_{b,t}\right], \Lambda^s\right] + P_t^{-s-1}\left[(\Lambda^s)^{*,t}, d^*_{b,t}\right]\Lambda^s\right)G^{\delta,\nu}_{q,t}\varphi\right)_t \\
&\qquad+ \left([\Lambda^s, d^*_{b,t}]G^{\delta,\nu}_{q,t}\varphi, \left(\left[(\Lambda^s)^{*,t}, d^*_{b,t}\right] + \Lambda^{-s}\left[\left[(\Lambda^s)^{*,t}, d^*_{b,t}\right], \Lambda^s\right] + P_t^{-s-1}\left[(\Lambda^s)^{*,t}, d^*_{b,t}\right]\Lambda^s\right)G^{\delta,\nu}_{q,t}\varphi\right)_t.
\end{aligned}$$

Since $\bar\partial^*_{b,t} = \bar\partial^*_b + P^0_t$, $d^*_{b,t} = d^*_b + P^0_t$, using, small constant/large constant argument, we can absorb terms to obtain

$$Q^{\delta,\nu}_{b,t}(\Lambda^s G^{\delta,\nu}_{q,t}\varphi, \Lambda^s G^{\delta,\nu}_{q,t}\varphi) \le C\left\|\left|\Lambda^s\varphi\right\|\right|_t^2 + C_s\left\|\left|\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right\|\right|_t^2 + C_{t,s}\|G^{\delta,\nu}_{q,t}\varphi\|^2_{H^{s-1}} \tag{5.8}$$

where C does not depend t, s, δ, or ν, and C_s does not depend on t, δ, or ν. By (5.7), for t sufficiently large

$$\|G^{\delta,\nu}_{q,t}\varphi\|^2_{H^s} \le K_t\|\varphi\|^2_{H^s} + C_{t,s}\|G^{\delta,\nu}_{q,t}\varphi\|^2_{H^{s-1}}.$$

By induction, we can reduce the H^{s-1}-norm to an L^2-norm, and by (5.5), we observe

$$\|G^{\delta,\nu}_{q,t}\varphi\|^2_{H^s} \le K_t\|\varphi\|^2_{H^s} + C_{t,s,\nu}\|\varphi\|^2_0,$$

uniformly in $\delta > 0$. Then there exists a sequence $\{G^{\delta_k,\nu}_{q,t}\varphi\}_k$ converging weakly to an element u_ν in $H^s_{0,q}(M, |||\cdot|||_t)$ when $\delta_k \to 0$, and satisfying both

$$\|u_\nu\|_{H^s} \le K_t\|\varphi\|_{H^s} + C_{t,s,\nu}\|\varphi\|_0 \quad \text{and} \quad \|u_\nu\|_{H^s} \le K_t\|\varphi\|_{H^s} + C_{t,s}\|u_\nu\|_0. \tag{5.9}$$

Since $H^s_{0,q}(M, |||\cdot|||_t)$ embeds compactly in $H^{s'}_{0,q}(M, |||\cdot|||_t)$ for $s' < s$, it follows that $G^{\delta_k,\nu}_{q,t}\varphi \to u_\nu$ strongly in $H^{s'}_{0,q}(M, |||\cdot|||_t)$. Also, observe that the next conclusion is not automatic in the $s = 1$ case.

$$\begin{aligned}
\left\|\left|\bar\partial_b G^{\delta,\nu}_{q,t}\varphi\right\|\right|_t^2 + \left\|\left|\bar\partial^*_{b,t}G^{\delta,\nu}_{q,t}\varphi\right\|\right|_t^2 &\le Q^{\delta,\nu}_{q,t}(G^{\delta,\nu}_{q,t}\varphi, G^{\delta,\nu}_{q,t}\varphi) \\
&= \left(\varphi, G^{\delta,\nu}_{q,t}\varphi\right)_t \le |||\varphi|||_t\,|||G^{\delta,\nu}_{q,t}\varphi|||_t \le C_\nu|||\varphi|||_t^2,
\end{aligned} \tag{5.10}$$

and, moreover, $\bar\partial_b G^{\delta_k,\nu}_{q,t}\varphi$ and $\bar\partial^*_{b,t}G^{\delta_k,\nu}_{q,t}\varphi$ are Cauchy sequences in L^2. Indeed, assuming $\delta_k \le \delta_j$ we have

$$\begin{aligned}
&\left\|\left|\bar\partial_b G^{\delta_k,\nu}_{q,t}\varphi - \bar\partial_b G^{\delta_j,\nu}_{q,t}\varphi\right\|\right|_t^2 + \left\|\left|\bar\partial^*_{b,t}G^{\delta_k,\nu}_{q,t}\varphi - \bar\partial^*_{b,t}G^{\delta_j,\nu}_{q,t}\varphi\right\|\right|_t^2 \\
&\qquad\le Q^{\delta_k,\nu}_{b,t}(G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi, G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi) \\
&\qquad= \left(\varphi, G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi\right)_t - Q^{\delta_k,\nu}_{q,t}(G^{\delta_j,\nu}_{q,t}\varphi, G^{\delta_k,\nu}_{q,t}\varphi) + Q^{\delta_k,\nu}_{q,t}(G^{\delta_j,\nu}_{q,t}\varphi, G^{\delta_j,\nu}_{q,t}\varphi) \\
&\qquad\le \left(\varphi, G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi\right)_t - Q^{\delta_k,\nu}_{q,t}(G^{\delta_j,\nu}_{q,t}\varphi, G^{\delta_k,\nu}_{q,t}\varphi) + Q^{\delta_j,\nu}_{q,t}(G^{\delta_j,\nu}_{q,t}\varphi, G^{\delta_j,\nu}_{q,t}\varphi) \\
&\qquad= \left(\varphi, G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi\right)_t - \left(G^{\delta_j,\nu}_{q,t}\varphi, \varphi\right)_t + \left(\varphi, G^{\delta_j,\nu}_{q,t}\varphi\right)_t \\
&\qquad\le |||\varphi|||_t\,|||G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi|||_t.
\end{aligned}$$

Since $\bar\partial_b$ and $\bar\partial^*_{b,t}$ are closed operators it follows that $u_\nu \in \mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial^*_b)$, $\bar\partial_b G^{\delta_k,\nu}_{q,t}\varphi \to \bar\partial_b u_\nu$ and $\bar\partial^*_{b,t}G^{\delta_k,\nu}_{q,t}\varphi \to \bar\partial^*_{b,t}u_\nu$ in L^2. This means $G^{\delta_k,\nu}_{q,t}\varphi$ converges strongly to u_ν in the $Q^{0,\nu}_{b,t}(\cdot,\cdot)^{1/2}$-norm. Thus, we will have, for any $v \in H^2_{0,q}(M, |||\cdot|||_t)$, by (5.5),

$$\begin{aligned}\left|Q^{0,\nu}_{b,t}(G^{\delta_k,\nu}_{q,t}\varphi - G^{0,\nu}_{q,t}\varphi, v)\right| &= \Big|Q^{\delta_k,\nu}_{b,t}(G^{\delta_k,\nu}_{q,t}\varphi, v) - \delta_k\left(d_bG^{\delta_k,\nu}_{q,t}\varphi, d_bv\right)_t \\ &\quad -\delta_k\left(d^*_{b,t}G^{\delta_k,\nu}_{q,t}\varphi, d^*_{b,t}v\right)_t - (\varphi,v)_t\Big| \\ &= \delta_k\left|\left(G^{\delta_k,\nu}_{q,t}\varphi, (d^*_{b,t}d_b + d_bd^*_{b,t})v\right)_t\right| \le \delta_k C_{\nu,t}|||\varphi|||_t\|v\|_2.\end{aligned}$$

It now follows that $G^{0,\nu}_{q,t}\varphi = u_\nu$ and

$$\|G^{0,\nu}_{q,t}\varphi\|_{H^s} \le K_t\|\varphi\|_{H^s} + C_{t,s}\|G^{0,\nu}_{q,t}\varphi\|_0. \tag{5.11}$$

With this last inequality, we can now prove that $\mathcal{H}^q_t(M) \subset H^s_{0,q}(M)$, for t sufficiently large. We proved that $\mathcal{H}^q_t(M)$ is finite dimensional, so assume $dim\mathcal{H}^q_t(M) = N$. If $N = 0$ there is nothing to prove. Otherwise assuming $\theta_0 = 0$ and let $\theta_j \in H^s_{0,q}(M)$ for $0 \le j \le l < N$, where $\{\theta_1, ..., \theta_N\}$ is a basis for $\mathcal{H}^q_t(M)$. We will construct some $\theta \in H^s \cap \mathcal{H}^q_t(M)$ such that $\|\theta\|_0 = 1$ and $(\theta,\theta_j)_t = 0$ for all $j \le l$. Let $\alpha \in H^s_{0,q}(M)$ such that $\alpha \perp \theta_j$ for all $j \le l$ and α is not orthogonal to θ_{l+1} (α exists because otherwise it would imply $\theta_{l+1} = 0$). By (5.11) we have

$$\|G^{0,\nu}_{q,t}\alpha\|_{H^s} \le C_{t,s}(\|\alpha\|_{H^s} + \|G^{0,\nu}_{q,t}\alpha\|_0),$$

for all $\nu \in (0,1)$. We claim $\left\{\|G^{0,\nu}_{q,t}\alpha\|_0, 0<\nu<1\right\}$ is unbounded. If it were bounded, then by the last inequality, there would be a sequence $G^{0,\nu_k}_{q,t}\alpha$ converging weakly to $u \in H^s_{0,q}(M)$. Additionally, by a previous argument for $G^{\delta,\nu}_{q,t}$ (done in (5.10)), $u \in \mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial^*_b)$, $G^{0,\nu_k}_{q,t}\alpha \to u$ strongly in the $Q_{b,t}(.,.)^{1/2}$-norm and so

$$\begin{aligned}Q_{b,t}(u,\phi) &= \lim_{k\to+\infty} Q_{b,t}(G^{0,\nu_k}_{q,t}\alpha,\phi) = \lim_{k\to+\infty} Q^{0,\nu_k}_{b,t}(G^{0,\nu_k}_{q,t}\alpha,\phi) - \nu_k\left(G^{0,\nu}_{q,t}\alpha,\phi\right)_t \\ &= (\alpha,\phi) - \lim_{k\to+\infty}\nu_k\left(G^{0,\nu_k}_{q,t}\alpha,\phi\right)_t = (\alpha,\phi)_t\,, \qquad \forall\phi \in \mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial^*_b)\end{aligned}$$

(because we assumed $|||G^{0,\nu}_{q,t}\alpha|||_t$ is bounded) for every $\phi \in \mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial^*_b)$. Setting $\phi = \theta_j$ the left-hand side is zero for all j, whereas the right-hand side is non zero for $j = l+1$. The claim is therefore established. It now follows that there exists some subsequence $\left\{G^{0,\nu_k}_{q,t}\alpha\right\}$ such that $\lim_{k\to\infty}\|G^{0,\nu_k}_{q,t}\alpha\|_0 = \infty$. We set $\gamma_k = \frac{G^{0,\nu_k}_{q,t}\alpha}{\|G^{0,\nu_k}_{q,t}\alpha\|_0}$ and note that

$$Q^{0,\nu_k}_{b,t}(\gamma_k,\phi) = \frac{(\alpha,\phi)_t}{\|G^{0,\nu_k}_{q,t}\alpha\|_0}. \tag{5.12}$$

By (5.11) there exists a subsequence γ_{k_j} which converges weakly in $H^s_{0,q}(M)$ to some $\theta \in H^s_{0,q}(M)$, so it converges strongly in $L^2_{0,q}(M)$. Since $\|\gamma_{k_j}\|_0 = 1$, we have $\|\theta\|_0 = 1$ and

$$|||\bar\partial_b\gamma_{k_j}|||^2_t + |||\bar\partial^*_{b,t}\gamma_{k_j}|||^2_t \le Q^{0,\nu_{k_j}}_{b,t}(\gamma_{k_j},\gamma_{k_j}) = \frac{1}{\|G^{0,\nu_{k_j}}_{q,t}\alpha\|_0}\left(\alpha,\gamma_{k_j}\right)_t$$

$$\leq \frac{C_t |||\alpha|||_t}{\|G_{q,t}^{0,\nu_{k_j}}\|_0}$$

Then $\bar{\partial}_b \gamma_{k_j} \to 0$ and $\bar{\partial}^*_{b,t}\gamma_{k_j} \to 0$ in $L^2_{0,q}(M)$. Since $\bar{\partial}_b$ and $\bar{\partial}^*_{b,t}$ are closed operators, $\theta \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b)$, $\bar{\partial}_b\theta = 0$, and $\bar{\partial}^*_{b,t}\theta = 0$. Thus $\theta \in \mathcal{H}^q_t$. Finally, by (5.12)

$$\nu_{k_j}\left(\gamma_{k_j}, \theta_r\right) = Q_{q,t}^{0,\nu_{k_j}}(\gamma_{k_j}, \theta_r) = \frac{(\alpha, \theta_r)_t}{\|G_{q,t}^{0,\nu_{k_j}}\|_0} = 0,$$

this means γ_{k_j} is orthogonal to θ_r for $r = 1, ..., l$. Since γ_{k_j} converges strongly in $L^2_{0,q}(M)$ to θ; θ will be orthogonal to θ_r for $r = 1, ..., l$. So with this, we have proved $\mathcal{H}^q_t(M) \subset H^s_{0,q}(M)$ for t sufficiently large.

5.3 Regularity of the Green operator and the canonical solutions.

In this section we assume t is sufficiently large and the weighted harmonic $(0,q)$-forms, if they exist, are elements of $H^1_{0,q}(M)$, so ${}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M) \neq \{0\}$. We use an elliptic regularization argument. The operator $G_{q,t} : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q}(M, |||\cdot|||_t) \cap {}^\perp\mathcal{H}^q_t(M)$. Consequently, the regularity result for $G_{q,t}$ must be on ${}^\perp\mathcal{H}^q_t(M) \cap H^s_{0,q}(M)$ for $s \geq 0$. Continuity on all of $H^s_{0,q}(M)$ then follows because we already established that harmonic forms are elements of $H^s_{0,q}(M)$.

The quadratic form $Q^\delta_{q,t}(\cdot,\cdot) := Q^{\delta,0}_{q,t}(\cdot,\cdot)$ is an inner product on $H^1_{0,q}(M)$. By (5.1)

$$|||u|||_t^2 \leq C Q_{b,t}(u,u) \leq C Q^\delta_{b,t}(u,u) \tag{5.13}$$

for all $u \in H^1_{0,q}(M) \cap {}^\perp\mathcal{H}^q_t(M)$. If $f \in L^2_{0,q}(M)$ then

$$|(f,g)_t| \leq |||f|||_t |||g|||_t \leq |||f|||_t C^{1/2} Q^\delta_{b,t}(g,g)$$

for all $g \in {}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M)$. This means the mapping $g \mapsto (f,g)_t$ is a bounded conjugate linear functional on ${}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M)$. By the Riesz Representation Theorem, there exists an element $G^\delta_{q,t}f \in {}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M)$ such that $(f,g)_t = Q^\delta_{b,t}(G^\delta_{q,t}f, g)$ for all $g \in {}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M)$.

Moreover, by (5.13)

$$C^{-1} |||G^\delta_{q,t}f|||_t^2 \leq Q^\delta_{b,t}(G^\delta_{q,t}f, G^\delta_{q,t}f) = \left(f, G^\delta_{q,t}f\right)_t \leq |||f|||_t |||G^\delta_{q,t}f|||_t,$$

where C is independent of δ. Consequently,

$$|||G^\delta_{q,t}f|||_t \leq C|||f|||_t. \tag{5.14}$$

Since $Q^\delta_{b,t}(\cdot,\cdot)$ satisfies $Q^\delta_{b,t}(f,f) \geq \delta |||\Lambda^1 f|||_t$ for every $f \in H^1_{0,q}(M)$, the bilinear form $Q^\delta_{b,t}(\cdot\cdot)$ is elliptic on $H^1_{0,q}(M)$. This means that $\varphi \in H^s_{0,q}(M)$ implies $G^\delta_{q,t}\varphi \in H^{s+2}_{0,q}(M)$

(before, we only knew that $G^\delta_{q,t}\varphi \in {}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M)$).

Let $\varphi \in H^s_{0,q}(M)$, then

$$\|G^\delta_{q,t}\varphi\|^2_{H^s} = \|\Lambda^s G^\delta_{q,t}\varphi\|^2_0 \le C_t \left|\left|\left| \Lambda^s G^\delta_{q,t}\varphi \right|\right|\right|^2_t. \tag{5.15}$$

We apply the basic estimate to $G^\delta_{q,t}\varphi \in H^{s+2}_{0,q}(M)$ and observe

$$\left|\left|\left| \Lambda^s G^\delta_{q,t}\varphi \right|\right|\right|^2_t \le \frac{K}{t} Q_{b,t}(\Lambda^s G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi) + C_{t,s}\|G^\delta_{q,t}\varphi\|^2_{H^{s-1}}. \tag{5.16}$$

Using the argument of (5.8), we can establish

$$\begin{aligned} Q_{b,t}(\Lambda^s G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi) &\le Q^\delta_{b,t}(\Lambda^s G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi) \\ &\le C \left|\left|\left| \Lambda^s \varphi \right|\right|\right|^2_t + C_s \left|\left|\left| \Lambda^s G^\delta_{q,t}\varphi \right|\right|\right|^2_t + C_{t,s}\|G^\delta_{q,t}\varphi\|^2_{H^{s-1}} \end{aligned} \tag{5.17}$$

where C is independent of t, s, δ, and ν and C_s is independent of t, δ, and ν.

Plugging (5.17) into (5.16) we have

$$\left|\left|\left| \Lambda^s G^\delta_{q,t}\varphi \right|\right|\right|^2_t \le \frac{K}{t}\left(C_t\|\varphi\|^2_s + C_s \left|\left|\left| \Lambda^s G^\delta_{q,t}\varphi \right|\right|\right|^2_t \right) + C_{t,s}\|G^\delta_{q,t}\varphi\|^2_{H^{s-1}},$$

and choosing t sufficiently large to absorb terms, it follows that

$$\left|\left|\left| \Lambda^s G^\delta_{q,t}\varphi \right|\right|\right|^2_t \le K_t\|\varphi\|^2_{H^s} + C_{t,s}\|G^\delta_{q,t}\varphi\|^2_{H^{s-1}}, \tag{5.18}$$

since $\left|\left|\left| \Lambda^s G^\delta_{q,t}\varphi \right|\right|\right|_t < \infty$. Plugging (5.18) into (5.15)

$$\|G^\delta_{q,t}\varphi\|^2_{H^s} \le K_t\|\varphi\|^2_{H^s} + C_{t,s}\|G^\delta_{q,t}\varphi\|^2_{H^{s-1}}.$$

Using (5.14) and induction, for the last inequality above we have

$$\|G^\delta_{q,t}\varphi\|^2_{H^s} \le K_t\|\varphi\|^2_{H^s} + C_{t,s}\|\varphi\|^2_0. \tag{5.19}$$

With (5.19) in hand, we now turn to sending $\delta \to 0$, in a similar manner to [10]. If $\varphi \in H^s_{0,q}(M)$ then $\left\{G^\delta_{q,t}\varphi : 0 < \delta < 1\right\}$ is bounded in $H^s_{0,q}(M)$, so there exists $\delta_k \to 0$ and $\tilde{u} \in H^s_{0,q}(M)$ so that $G^{\delta_k}_{q,t}\varphi \to \tilde{u}$ weakly in $H^s_{0,q}(M)$. Since the inclusion of $H^s_{0,q}(M)$ in $L^2_{0,q}(M)$ is compact, we have $G^{\delta_k}_{q,t}\varphi \to \tilde{u}$ strongly in $L^2_{0,q}(M)$ and $\tilde{u} \in {}^\perp\mathcal{H}^q_t(M)$. Also we have

$$\|\tilde{u}\|^2_{H^s} \le K_t\|\varphi\|^2_{H^s} + C_{t,s}\|\varphi\|^2_0. \tag{5.20}$$

Also,

$$\left|\left|\left| \bar\partial_b G^\delta_{q,t}\varphi \right|\right|\right|^2_t + \left|\left|\left| \bar\partial^*_{b,t} G^\delta_{q,t}\varphi \right|\right|\right|^2_t \le Q^\delta_{b,t}(G^\delta_{q,t}\varphi, G^\delta_{q,t}\varphi) = \left(\varphi, G^\delta_{q,t}\varphi\right)_t \le \left|\left|\left| \varphi \right|\right|\right|_t \left|\left|\left| G^\delta_{q,t}\varphi \right|\right|\right|_t \le C_t \left|\left|\left| \varphi \right|\right|\right|^2_t,$$

and, as in the previous section, we can prove $\bar\partial_b G^\delta_{q,t}\varphi$ and $\bar\partial^*_{b,t} G^\delta_{q,t}\varphi$ are Cauchy sequences in $L^2_{0,q+1}(M)$ and $L^2_{0,q-1}(M)$ respectively. Since $\bar\partial_b$ and $\bar\partial^*_{b,t}$ are closed operators we will have

$\tilde u \in \mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)$, $\bar\partial_b G^\delta_{q,t}\varphi \to \bar\partial_b\tilde u$ and $\bar\partial^*_{b,t}G^\delta_{q,t}\varphi \to \bar\partial^*_{b,t}\tilde u$ in $L^2_{0,q+1}(M)$ and $L^2_{0,q-1}(M)$ respectively, and

$$\left|\left|\left|\bar\partial_b\tilde u\right|\right|\right|_t^2 + \left|\left|\left|\bar\partial^*_{b,t}\tilde u\right|\right|\right|_t^2 \le C_t\left|\left|\left|\varphi\right|\right|\right|_t^2. \tag{5.21}$$

Consequently if $v \in H^{s+2}_{0,q}(M)$, then $\lim Q^{\delta_k}_{b,t}(G^{\delta_k}_{q,t}\varphi, v) = Q_{b,t}(\tilde u, v)$. However, $Q^{\delta_k}_{b,t}(G^{\delta_k}_{q,t}\varphi, v) = (\varphi, v)_t = Q_{b,t}(G_{q,t}\varphi, v)$. So by uniqueness $G_{q,t}\varphi = \tilde u$ and by (5.20) we have

$$\|G_{q,t}\varphi\|^2_{H^s} \le K_t\left|\left|\left|\varphi\right|\right|\right|^2_{H^s} + C_{t,s}\|\varphi\|_0^2, \tag{5.22}$$

and by (5.21)

$$\left|\left|\left|\bar\partial_b G_{q,t}\varphi\right|\right|\right|_t^2 + \left|\left|\left|\bar\partial^*_{b,t}G_{q,t}\varphi\right|\right|\right|_t^2 \le C_t\left|\left|\left|\varphi\right|\right|\right|_t^2.$$

These two last equations prove the continuity of $G_{q,t}$ on $H^s_{0,q}(M)$, and as well as $\bar\partial_b G_{q,t}$ and $\bar\partial^*_{b,t}G_{q,t}$ in $L^2_{0,q}(M)$.

Now, we show some estimates to prove the regularity of canonical solutions. Let

$$(\kappa) := \left|\left|\left|\Lambda^s\bar\partial_b G^\delta_{q,t}\varphi\right|\right|\right|_t^2 + \left|\left|\left|\Lambda^s\bar\partial^*_{b,t}G^\delta_{q,t}\varphi\right|\right|\right|_t^2 + \delta(\left|\left|\left|\Lambda^s d_b G^\delta_{q,t}\varphi\right|\right|\right|_t^2 + \left|\left|\left|\Lambda^s d_b^* G^\delta_{q,t}\varphi\right|\right|\right|_t^2).$$

By integration by parts we have

$$\begin{aligned}
\left|\left|\left|\Lambda^s\bar\partial_b G^\delta_{q,t}\varphi\right|\right|\right|_t^2 &= \left(\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi\right)_t + \left(\left[\bar\partial^*_{b,t}, \Lambda^s\right]\bar\partial_b G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi\right)_t \\
&\quad + \left(\Lambda^s\bar\partial_b G^\delta_{q,t}\varphi, \left[\Lambda^s, \bar\partial_b\right] G^\delta_{q,t}\varphi\right)_t, \\
\left|\left|\left|\Lambda^s\bar\partial^*_{b,t} G^\delta_{q,t}\varphi\right|\right|\right|_t^2 &= \left(\Lambda^s\bar\partial_b\bar\partial^*_{b,t} G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi\right)_t + \left(\left[\bar\partial_b, \Lambda^s\right]\bar\partial^*_{b,t} G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi\right)_t \\
&\quad + \left(\Lambda^s\bar\partial^*_{b,t} G^\delta_{q,t}\varphi, \left[\Lambda^s, \bar\partial^*_{b,t}\right] G^\delta_{q,t}\varphi\right)_t, \\
\left|\left|\left|\Lambda^s d_b G^\delta_{q,t}\varphi\right|\right|\right|_t^2 &= \left(\Lambda^s d_b^* d_b G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi\right)_t + \left([d_b^*, \Lambda^s]\, d_b G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi\right)_t \\
&\quad + \left(\Lambda^s d_b G^\delta_{q,t}\varphi, [\Lambda^s, d_b]\, G^\delta_{q,t}\varphi\right)_t, \\
\left|\left|\left|\Lambda^s d_b^* G^\delta_{q,t}\varphi\right|\right|\right|_t^2 &= \left(\Lambda^s d_b d_b^* G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi\right)_t + \left([d_b, \Lambda^s]\, d_b^* G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi\right)_t \\
&\quad + \left(\Lambda^s d_b^* G^\delta_{q,t}\varphi, [\Lambda^s, d_b^*]\, G^\delta_{q,t}\varphi\right)_t.
\end{aligned}$$

Then using a small constant/large constant argument and the absorbing of error terms by (κ) we have

$$(\kappa) \le C\left|\left|\left|\Lambda^s\varphi\right|\right|\right|_t^2 + C_{t,s}\|G^\delta_{q,t}\varphi\|^2_{H^s}.$$

where C does not depend on t, s, ν, δ. So by (5.19), for t sufficiently large

$$\|\bar\partial_b G^\delta_{q,t}\varphi\|_{H^s} + \|\bar\partial^*_{b,t}G^\delta_{q,t}\varphi\|_{H^s} \le C_{t,s}(\|\varphi\|_{H^s} + \|\varphi\|_0). \tag{5.23}$$

In particular

$$\|\bar\partial_b G^{\delta_k}_{q,t}\varphi\|_{H^s} + \|\bar\partial^*_{b,t}G^{\delta_k}_{q,t}\varphi\|_{H^s} \le C_{t,s}(\|\varphi\|_{H^s} + \|\varphi\|_0), \tag{5.24}$$

where $G_{q,t}^{\delta_k}\varphi$ is the same subsequence that we took to prove (5.22).We proved that $G_{q,t}^{\delta_k}\varphi \to G_{q,t}\varphi$ strongly in $L^2_{0,q}(M)$. Now, by (5.24), there exist subsequences $\bar{\partial}_b G_{q,t}^{\delta_{k_j}}\varphi$ and $\bar{\partial}_{b,t}^* G_{q,t}^{\delta_{k_j}}\varphi$ that converges weakly to $v_1 \in H^s_{q+1}(M)$ and $v_2 \in H^s_{q-1}(M)$ respectively, and

$$\|v_1\|_{H^s} + \|v_2\|_{H^s} \leq C_{t,s}(\|\varphi\|_{H^s} + \|\varphi\|_0).$$

Since $\bar{\partial}_b G_{q,t}^{\delta_{k_j}}\varphi \to v_1$, $\bar{\partial}_{b,t}^* G_{q,t}^{\delta_{k_j}}\varphi \to v_2$ strongly in $L^2_{0,q+1}(M)$ and $L^2_{0,q-1}(M)$ respectively and $G_{q,t}^{\delta_k}\varphi \to G_{q,t}\varphi$ strongly in $L^2_{0,q}(M)$, then $v_1 = \bar{\partial}_b G_{q,t}\varphi$ and $v_2 = \bar{\partial}_{b,t}^* G_{q,t}\varphi$. So

$$\|\bar{\partial}_b G_{q,t}\varphi\|_{H^s} + \|\bar{\partial}_{b,t}^* G_{q,t}\varphi\|_{H^s} \leq C_{t,s}(\|\varphi\|_{H^s} + \|\varphi\|_0).$$

This proves the continuity of the canonical solution operators in $H^s_{0,q}(M)$.

5.3.1 Some facts about the canonical solutions

Here we make computations in order to find weak solutions to the $\bar{\partial}_b$ and $\bar{\partial}_{b,t}^*$ problems. All of this sections will be done for elements on ${}^\perp\mathcal{H}_t^q(M)$. Remember that here, in ${}^\perp\mathcal{H}_t^q(M)$, we will have $\Box_{b,t}G_{q,t} = Id$ and $G_{q,t}\Box_{b,t} = Id$ on $\mathrm{Dom}(\Box_{b,t})$.

Claim 1. If α is a $(0,q)$-form in $\mathrm{Ker}(\bar{\partial}_b)$, then $\bar{\partial}_{b,t}^*\bar{\partial}_b G_{q,t}\alpha = 0$, and if it is in $\mathrm{Ker}(\bar{\partial}_{b,t}^*)$ then $\bar{\partial}_b\bar{\partial}_{b,t}^* G_{q,t}\alpha = 0$. In fact, since $0 = \bar{\partial}_b\alpha = \bar{\partial}_b\bar{\partial}_{b,t}^*\bar{\partial}_b G_{q,t}\alpha + \bar{\partial}_b\bar{\partial}_b\bar{\partial}_{b,t}^* G_{q,t}\alpha = \bar{\partial}_b\bar{\partial}_{b,t}^*\bar{\partial}_b G_{q,t}\alpha$ we have $0 = \left(\bar{\partial}_b\bar{\partial}_{b,t}^*\bar{\partial}_b G_{q,t}\alpha, \bar{\partial}_b G_{q,t}\alpha\right)_t = \|\bar{\partial}_{b,t}^*\bar{\partial}_b G_{q,t}\alpha\|_t^2$. Then $\bar{\partial}_{b,t}^*\bar{\partial}_b G_{q,t}\alpha = 0$. The other equality is proved in a similar way.

With this claim and the fact $\Box_{b,t}G_{q,t} = Id$, we have that $\bar{\partial}_{b,t}^* G_{q,t}\alpha$ satisfies weakly the equation $\bar{\partial}_b u = \alpha$. We will call $\bar{\partial}_{b,t}^* G_{q,t}$ the canonical solution operator to $\bar{\partial}_b$. Similarly, we can prove that the canonical solution operator to $\bar{\partial}_{b,t}^*$ is given by $\bar{\partial}_b G_{q,t}$.

Also, we can see that these canonical solutions, given by the operator $\bar{\partial}_{b,t}^* G_{q,t}$ and $\bar{\partial}_b G_{q,t}$, are the unique solutions orthogonal to $\mathrm{Ker}(\bar{\partial}_b)$ and $\mathrm{Ker}(\bar{\partial}_{b,t}^*)$ respectively. For example if u is in ${}^\perp\mathrm{Ker}(\bar{\partial}_b)$ and $\bar{\partial}_b u = \alpha = \bar{\partial}_b\bar{\partial}_{b,t}^* G_{q,t}\alpha$ we will have $u - \bar{\partial}_{b,t}^* G_{q,t}\alpha \in \mathrm{Ker}(\bar{\partial}_b)$, and so $u = \bar{\partial}_{b,t}^* G_{q,t}\alpha$.

Claim 2 $G_{q,t}$ commutes with the operators $\bar{\partial}_b\bar{\partial}_{b,t}^*$ and $\bar{\partial}_{b,t}^*\bar{\partial}_b$ on $\mathrm{Dom}(\Box_{b,t}) \cap \mathrm{Ker}(\bar{\partial}_b)$ and $\mathrm{Dom}(\Box_{b,t}) \cap \mathrm{Ker}(\bar{\partial}_{b,t}^*)$ respectively. In fact, by above if $\bar{\partial}_b\alpha = 0$ then $\bar{\partial}_{b,t}^*\bar{\partial}_b G_{q,t}\alpha = 0$. Then since $\Box_{b,t}G_{q,t} = G_{q,t}\Box_{b,t}$ on $\mathrm{Dom}(\Box_{b,t})$, we have $\bar{\partial}_b\bar{\partial}_{b,t}^* G_{q,t}\alpha = \Box_{b,t}G_{q,t}\alpha = G_{q,t}\Box_{b,t}\alpha = G_{q,t}\bar{\partial}_b\bar{\partial}_{b,t}^*\alpha$. The other equality is proved similarly.

Claim 3 $G_{q,t}(\mathrm{Ker}(\bar{\partial}_b) \cap {}^\perp\mathcal{H}_t^q(M)) \subset \mathrm{Ker}(\bar{\partial}_b)$ and $G_{q,t}(\mathrm{Ker}(\bar{\partial}_{b,t}^*) \cap {}^\perp\mathcal{H}_t^q(M)) \subset \mathrm{Ker}(\bar{\partial}_{b,t}^*)$. In fact, if $\bar{\partial}_b\alpha = 0$, by the claim 1 $\bar{\partial}_{b,t}^*\bar{\partial}_b G_{q,t}\alpha = 0$, then $0 = \left(\bar{\partial}_{b,t}^*\bar{\partial}_b G_{q,t}\alpha, G_{q,t}\alpha\right)_t =$

$\left|\left|\left|\bar\partial_b G_{q,t}\alpha\right|\right|\right|_t^2$. So $\bar\partial_b G_{q,t}\alpha = 0$. The other equality is proved similarly.

5.4 The Szego projection $S_{q,t}$

The Szego projection $S_{q,t}$ is the projection of $L^2_{0,q}(M)$ onto $\mathrm{Ker}(\bar\partial_b)$. We claim that

$$S_{q,t} = I - \bar\partial^*_{b,t}\bar\partial_b G_{q,t} \tag{5.25}$$

and by the claim 2 above, $S_{q,t} = I - G_{q,t}\bar\partial^*_{b,t}\bar\partial_b$. In fact, obviously (5.25) is true in $\mathcal{H}^q_t(M)$. Now if $\alpha \in \mathrm{Ker}(\bar\partial_b) \cap {}^\perp\mathcal{H}^q_t(M)$, by the claim 1 we will have $\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\alpha = 0$; so (5.25) is checked. If $\alpha \in {}^\perp\mathrm{Ker}(\bar\partial_b) \cap {}^\perp\mathcal{H}^q_t(M)$, since $\alpha - \bar\partial^*_{b,t}\bar\partial_b G_{q,t}\alpha = \bar\partial_b\bar\partial^*_{b,t}G_{q,t}\alpha$ and $\bar\partial^*_{b,t}\bar\partial_b G_{q,t} \perp \mathrm{Ker}(\bar\partial_b)$, we have $\bar\partial_b\bar\partial^*_{b,t}G_{q,t}\alpha \perp \mathrm{Ker}(\bar\partial_b)$. But $\bar\partial_b\bar\partial^*_{b,t}G_{q,t}\alpha \in \mathrm{Ker}(\bar\partial_b)$, then $\bar\partial_b\bar\partial^*_{b,t}G_{q,t}\alpha = 0$; so (5.25) is satisfied, because in this case $\alpha - \bar\partial^*_{b,t}\bar\partial_b G_{q,t}\alpha = 0$.

In the same way we can prove that the projection of $L^2_{0,q}(M)$ on $\mathrm{Ker}(\bar\partial^*_{b,t})$ is given by $I - \bar\partial_b\bar\partial^*_{b,t}G_{q,t}$.

We know that $\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi \in L^2_{0,q}(M)$, because we proved that $\mathrm{Ran}(G_{q,t}) = \mathrm{Dom}(\Box_{b,t}) \cap {}^\perp\mathcal{H}^q_t(M)$, but we can have a quantitative bound for it as follows:

$$\begin{aligned}\left|\left|\left|\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi\right|\right|\right|_t^2 &= \left(\bar\partial_b\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi, \bar\partial_b G_{q,t}\varphi\right)_t = \left(\bar\partial_b\varphi, \bar\partial_b G_{q,t}\varphi\right)_t \\ &= \left(\varphi, \bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi\right)_t \le \left|\left|\left|\varphi\right|\right|\right|_t \left|\left|\left|\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi\right|\right|\right|_t\end{aligned} \tag{5.26}$$

Then, since $\left|\left|\left|\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi\right|\right|\right|_t < \infty$, we will have

$$\left|\left|\left|\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi\right|\right|\right|_t \le \left|\left|\left|\varphi\right|\right|\right|_t. \tag{5.27}$$

And this proves that the operator $S_{q,t}$ is continuous in $L^2_{0,q}(M)$.

Note that in the same way; it is true that $\left|\left|\left|\bar\partial_b\bar\partial^*_{b,t}G_{q,t}\varphi\right|\right|\right|_t \le \left|\left|\left|\varphi\right|\right|\right|_t$. In fact we will have that, if $\varphi \in {}^\perp\mathcal{H}^q_t(M)$, the decomposition $\varphi = \bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi + \bar\partial_b\bar\partial^*_{b,t}G_{q,t}\varphi$ is orthogonal and so $\left|\left|\left|\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi\right|\right|\right|_t^2 + \left|\left|\left|\bar\partial_b\bar\partial^*_{b,t}G_{q,t}\varphi\right|\right|\right|_t^2 = \left|\left|\left|\varphi\right|\right|\right|_t^2$.

Now let $s > 0$. In order to prove the continuity of the operator $S_{q,t}$ in $H^s_{0,q}(M)$, it is suffices to show that the operator $\bar\partial^*_{b,t}\bar\partial_b G_{q,t}$ is continuous in $H^s_{0,q}(M)$, this is, it suffices to prove

$$\left|\left|\left|\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi\right|\right|\right|_t^2 \le C_{t,s}\left|\left|\left|\Lambda^s\varphi\right|\right|\right|_t^2,$$

for some constant $C_{t,s} > 0$. In this case, we can not do just integration by parts like above in (5.26) to get (5.27), because we do not know if $\left|\left|\left|\Lambda^s \bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi\right|\right|\right|_t$ is finite. Instead we will use a density argument.

Let t sufficiently large such that $G_{q,t}$ is continuous in $H^{s+3}_{0,q}(M)$ and $H^s_{0,q}(M)$, $\bar{\partial}_b G_{q,t}$ continuous in $H^s_{0,q}(M)$. Let $\varphi \in H^s_{0,q}(M)$ and $\varphi_j \in H^{s+3}_{0,q}(M)$, such that $\varphi_j \to \varphi$ in $H^s_{0,q}(M)$. Then for any j we have

$$\begin{aligned}
\left|\left|\left|\Lambda^s \bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j\right|\right|\right|_t^2 &= \left(\Lambda^s \bar{\partial}_b\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j, \Lambda^s\bar{\partial}_b G_{q,t}\varphi_j\right)_t + \left(\left[\bar{\partial}_b, \Lambda^s\right]\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j, \Lambda^s\bar{\partial}_b G_{q,t}\varphi_j\right)_t \\
&\quad + \left(\Lambda^s\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j, \left[\Lambda^s, \bar{\partial}^*_{b,t}\right]\bar{\partial}_b G_{q,t}\varphi_j\right)_t \\
&= \left(\Lambda^s\bar{\partial}_b\varphi_j, \Lambda^s\bar{\partial}_b G_{q,t}\varphi_j\right)_t + \left(\left[\bar{\partial}_b, \Lambda^s\right]\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j, \Lambda^s\bar{\partial}_b G_{q,t}\varphi_j\right)_t \\
&\quad + \left(\Lambda^s\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j, \left[\Lambda^s, \bar{\partial}^*_{b,t}\right]\bar{\partial}_b G_{q,t}\varphi_j\right)_t \\
&= \left(\left[\Lambda^s, \bar{\partial}_b\right]\varphi_j, \Lambda^s\bar{\partial}_b G_{q,t}\varphi_j\right)_t + \left(\Lambda^s\varphi_j, \left[\bar{\partial}^*_{b,t}, \Lambda^s\right]\bar{\partial}_b G_{q,t}\varphi_j\right)_t \\
&\quad + \left(\Lambda^s\varphi_j, \Lambda^s\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j\right)_t + \left(\left[\bar{\partial}_b, \Lambda^s\right]\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j, \Lambda^s\bar{\partial}_b G_{q,t}\varphi_j\right)_t \\
&\quad + \left(\Lambda^s\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j, \left[\Lambda^s, \bar{\partial}^*_{b,t}\right]\bar{\partial}_b G_{q,t}\varphi_j\right)_t \\
&\le C_t\left|\left|\left|\Lambda^s\varphi_j\right|\right|\right|_t\left|\left|\left|\Lambda^s\bar{\partial}_b G_{q,t}\varphi_j\right|\right|\right|_t + C_t\left|\left|\left|\Lambda^s\varphi_j\right|\right|\right|_t\left|\left|\left|\Lambda^s\bar{\partial}_b G_{q,t}\varphi_j\right|\right|\right|_t \\
&\quad + \left|\left|\left|\Lambda^s\varphi_j\right|\right|\right|_t\left|\left|\left|\Lambda^s\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j\right|\right|\right|_t + C_t\left|\left|\left|\Lambda^s\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j\right|\right|\right|_t\left|\left|\left|\Lambda^s\bar{\partial}_b G_{q,t}\varphi_j\right|\right|\right|_t \\
&\quad + C_t\left|\left|\left|\Lambda^s\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j\right|\right|\right|_t\left|\left|\left|\Lambda^s\bar{\partial}_b G_{q,t}\varphi_j\right|\right|\right|_t.
\end{aligned}$$

Then using a large constant/small constant argument we have

$$\left|\left|\left|\Lambda^s\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j\right|\right|\right|_t \le C_t\left|\left|\left|\Lambda^s\varphi_j\right|\right|\right|_t.$$

Then $\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_j$ is bounded in $H^s_{0,q}(M)$. Then there exists a subsequence $\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi_{j_k}$ converging weakly to an element $\beta \in H^s_{0,q}(M)$ (in particular in $L^2_{0,q}(M)$), and

$$\|\beta\|_{H^s} \le C_t\|\varphi\|_{H^s}. \tag{5.28}$$

Since $\bar{\partial}_b G_{q,t}$ is continuous in $H^s_{0,q}(M)$, $\bar{\partial}_b G_{q,t}\varphi_{j_k}$ converges to $\bar{\partial}_b G_{q,t}\varphi$ in $H^s_{0,q}(M)$, in particular in $L^2_{0,q}(M)$. Since $\bar{\partial}^*_{b,t}$ is a closed operator, $\beta = \bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi$, and (5.28) is satisfied with $\bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi$ instead of β. This proves the continuity of the Szego projection in $H^s_{0,q}(M)$ for t sufficiently large.

5.5 Proof of the Theorem 5.0.1

Since $\bar{\partial}_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ has closed range in the norm $\left|\left|\left|.\right|\right|\right|_t$, by Theorem 2.1.2 and by (4.9), we have $\bar{\partial}_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ and $\bar{\partial}^*_b : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$

have closed range in the norm $\|.\|_0$. In the same way, by Theorem 2.1.2 and by (4.9) we will have the closure of the range of the operators $\bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ and $\bar{\partial}^*_b : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ in the norm $\|.\|_0$. Now by Theorem 2.1.3 we will have the estimate

$$\|f\|_0 \leq C\left(\|\bar{\partial}_b f\|_0 + \|\bar{\partial}^*_b f\|_0\right)$$

for every $f \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b)$ and $f \in {}^{\perp}\mathcal{H}^q(M)$, where $\mathcal{H}^q(M)$ is the space of the $(0, q)$ harmonic forms, that is,

$$\mathcal{H}^q(M) := \left\{f \in f \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b) : \bar{\partial}_b f = 0, \text{ and } \bar{\partial}^*_b f = 0\right\}.$$

And so, again by the Theorem 2.1.3, the operator $\Box_b := \bar{\partial}_b\bar{\partial}^*_b + \bar{\partial}^*_b\bar{\partial}_b$ has closed range. In a similar manner to our argument for the existence of the weighted Green operator $G_{q,t}$, we can prove the existence of the Green operator G_q, defined to be the inverse of the operator $\Box_b$ on ${}^{\perp}\mathcal{H}^q(M)$ and $\equiv 0$ in $\mathcal{H}^q(M)$. Also the operator G_q is continuous on $L^2_{0,q}(M)$. And by Hodge decomposition for every $f \in {}^{\perp}\mathcal{H}^q(M)$, $f = \bar{\partial}_b\bar{\partial}^*_b G_q f \oplus \bar{\partial}^*_b\bar{\partial}_b G_q f$ we have

$$\|\bar{\partial}_b\bar{\partial}^*_b G_q f\|_0 + \|\bar{\partial}^*_b\bar{\partial}_b G_q f\|_0 = \|f\|_0 \tag{5.29}$$

and

$$\|\bar{\partial}^*_b G_q f\|^2_0 + \|\bar{\partial}_b G_q f\|^2_0 = \left(\bar{\partial}_b\bar{\partial}^*_b G_q f, G_q f\right)_0 + \left(\bar{\partial}^*_b\bar{\partial}_b G_q f, G_q f\right)_0 = \|f\|_0\|G_q f\|_0$$

Thus, the canonical solutions $\bar{\partial}^*_b G_q f : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ and $\bar{\partial}_b G_q f : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ are continuous ($G_q \equiv 0$ on $\mathcal{H}^q(M)$).

Now, if t is sufficient large (so $\mathcal{H}^q_t(M)$ has finite dimension and there exists the Green operator $G_{q,t}$) observe that if f is a $(0, q)$-form such that $f \in \mathrm{Ker}(\bar{\partial}_b)$ and $f \perp \mathrm{Ran}(\bar{\partial}_b)$ ($\perp$ respect to $(.,.)_t$), we will have $f \perp \bar{\partial}_b\bar{\partial}^*_{b,t}G_{q,t}f$, and since $\mathrm{Ran}(\bar{\partial}^*_{b,t}) \perp \mathrm{Ker}(\bar{\partial}_b)$, $f \perp \bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}f$. Then by Hodge decomposition $f = \bar{\partial}_b\bar{\partial}^*_{b,t}G_{q,t}f + \bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}f + H^q_t f$, we have $f = H^q_t f \in \mathcal{H}^q_t(M)$. This means $\mathrm{Ker}(\bar{\partial}_b) = \mathrm{Ran}(\bar{\partial}_b) \oplus \mathcal{H}^q_t(M)$ ($\oplus$ with respect to $(.,.)_t$) for any t sufficiently large. In the same way, we can show $\mathrm{Ker}(\bar{\partial}_b) = \mathrm{Ran}(\bar{\partial}_b) \oplus \mathcal{H}^q_t(M)$ ($\oplus$ with respect to $(.,.)_0$). But since $\mathrm{Ker}(\bar{\partial}_b)$ and $\mathrm{Ran}(\bar{\partial}_b)$ do not depend on the weights ϕ^+ and ϕ^-, it follows that the dimensions of $\mathcal{H}^q_t(M)$ and $\mathcal{H}^q(M)$ are equal. So $\mathcal{H}^q(M)$ is finite dimensional, with dimension equal to $\dim \mathcal{H}^q_t(M)$.

As we proved above, the Szego projection S_q, defined as the projection (with respect to $(.,.)_0$) is given by $I - \bar{\partial}^*_b\bar{\partial}_b G_q$, and it is continuous by (5.29).

CHAPTER 6

EXAMPLES

In this Chapter, we present two examples involving the two versions of weak $Z(q)$. The first example compares the two version showing whose satisfies the new version of weak $Z(q)$ condition but not the older one given by Harrington and Raich in [10]. The second example, inspired in the first one, shows that the new version of weak $Y(q)$ is satisfied easier than the older weak-$Y(q)$ condition (OW-$Z(q)$ condition).

6.1 Example 1

In order to compare the two versions of weak $Z(q)$ condition we remember here the older weak $Z(q)$ condition (OW-$Z(q)$).

Definition 6.1.1 (OW-$Z(q)$) *Let M be a smooth, compact, oriented CR manifold of hypersurface type of real dimension $2n-1$. We say M satisfies OW-$Z(q)$ condition is satisfied at p if there exists*

(i) a neighborhood $U \subset M$ containing p;

(ii) an integer $m = m(U) \neq q$;

(iii) an orthonormal basis $L_1, ..., L_{n-1}$ of $T^{1,0}(U)$ so that $\mu_1 + ... + \mu_q - (c_{11} + ...c_{mm}) \geq 0$ on U, where $\mu_1, ..., \mu_{n-1}$ are the eigenvalues of the Levi form in increasing order.

We say M satisfies the OW-$Z(q)$ condition if M satisfies OW-$Z(q)$ at every point in M, and the condition $m > q$ or $m < q$ is independent of p. As above, M satisfies OW-$Y(q)$ condition at p if M satisfies OW-$Z(q)$ and OW-$Z(n-1-q)$.

The next example was given by Harrington and Raich in [11]. OW-$Z(q)$ is not satisfied but weak $Z(q)$ is.

In $\mathbb{C}^3$ consider the hypersurface $M = \{z : \rho(z) = 0\}$ with $\rho(z) = -Imz_3 + P(z_1, z_2)$ where $P(z_1, z_2) = 2x\,|z_2|^2 - xy^4$ where we denote $z_1 = x + yi$. Then $L_j = \partial/\partial z_j + 2i\partial P/\partial z_j \partial/\partial z_3 \in T^{1,0}(M)$, for $j = 1, 2$. The Levi matrix C^L in this basis is given by

$$C^L = \begin{bmatrix} -3xy^2 & z_2 \\ \bar{z}_2 & 2x \end{bmatrix}.$$

We claim that the Levi matrix C^L does not satisfy OW-$Z(2)$ condition at the origin. In fact, let a_{jk}, for $1 \leq j,k \leq 2$, be smooth functions on a neighborhood of the origin such that the matrix $A := [a_{jk}]$ be a 2×2 nonsingular matrix on a neighborhood of the origin. Let u_1 and u_2 be vectors of type $(1,0)$ with $u_j = \sum_{k=1}^{2} a_{jk}L_k$. The Levi matrix C^u in the basis $\{u_1, u_2\}$ is given by $C^u = AC^LA^*$.

Now since A is not a singular matrix, the inertia of C^u (the number of positive, negative and zero eigenvalues, all counting multiplicity) is equal to the inertia of the matrix C^L. Then C^u has a negative eigenvalue and the other eigenvalue is zero when $y = 0 = z_2$ and $x < 0$. Then second condition in 6.1.1, and so OW-$Z(2)$, can not be satisfied with $m = 0$.

Now, assume OW-$Z(2)$ is satisfied with $m = 1$ in a neighborhood of the origin. Then by (iii) in Definition 6.1.1, the trace of the matrix $C^u - \frac{1}{2}Tr(C^u diag\,\{1,0\})Id = C^u - (c^u_{11}/2)Id$ is nonnegative. A direct calculation shows

$$\begin{aligned} Tr(C^u - (c^u_{11}/2)Id) &= c^L_{11}\left|a_{21}\right|^2 + c^L_{22}\left|a_{22}\right|^2 + 2Re(c^L_{12}a_{21}\overline{a_{22}}) \\ &= -3xy^2\left|a_{21}\right|^2 + 2x\left|a_{22}\right|^2 - 2Re(z_2a_{21}\overline{a_{22}}). \end{aligned}$$

Observe here that $Tr(C^u - (c^u_{11}/2)Id)$ admits a minimum value when $x = z_2 = 0$, so any first derivatives on x or z_2 applied on $x = z_2 = 0$ must be equal to zero. Then

$$\begin{aligned} -3y^2\left|a_{21}(0,y,0)\right|^2 + 2\left|a_{22}(0,y,0)\right|^2 &= 0, \\ a_{21}(0,y,0)\overline{a_{22}}(0,y,0) &= 0. \end{aligned}$$

This implies $a_{21}(0,y,0) = 0 = a_{22}(0,y,0)$ for any $y \neq 0$. But this is contradiction, because A was supposed to be a non-singular matrix on a neighborhood of the origin. Thus OW-$Z(2)$ is not satisfied with $m = 1$. So, OW-$Z(2)$ condition is not satisfied at the origin (in any Hermitian metric considered in M, because of the arbitrariness of matrix A).

Now consider the next matrix R given by

$$R = \begin{bmatrix} 2 & 0 \\ 0 & 3y^2 \end{bmatrix}.$$

Let a_{jk}, for $1 \leq j,k \leq 2$, be smooth functions on a neighborhood of the origin such that the matrix $A := [a_{jk}]$ be a 2×2 nonsingular matrix on a neighborhood of the origin, and a matrix D such that $DA = Id$. Let C^u the Levi matrix given by the basis $u_j = \sum_{k=1}^{2} a_{jk}L_k$, for $j = 1,2$. Define for $t > 0$ the matrix $B = [b_{jk}]$ by $B := Id - tD^*RD$. A direct calculation gives

$$Tr(C^u - \frac{1}{2}(Tr(C^uB))Id) = Tr(C^u) - Tr(C^uB) = tTr(AC^LA^*D^*RD) = tTr(C^LR) = 0.$$

Since D is nonsingular matrix and R has nonnegative eigenvalues, for t sufficiently small the matrix B will have nonnegative eigenvalues in $[0,1]$. On the other hand, if A is such that the basis $\{u_1, u_2\}$ is an orthonormal basis, defining the (1,1)-vector Υ defined by $\Upsilon = i\sum_{j,k=1}^{2} b_{jk}u_1 \wedge \overline{u_2}$, we have $\omega(\Upsilon) = Tr(B) = 2 - tTr(D^*BD) > 0$ for sufficiently small t, because in this case D^*BD has a positive eigenvalue and the other is nonnegative (so $Tr(D^*BD) > 0$). Thus weak $Z(2)$ is satisfied at the origin.

6.2 Example 2

Inspired by this last example, we now show a hypersurface in $\mathbb{C}^5$ where weak $Y(2)$ is satisfied and OW-$Y(2)$ is practically impossible to check.

In $\mathbb{C}^5$, let $M = \{z : \rho(z) = 0\}$ with $\rho = -Imz_5 + P(z_1, z_2, z_3, z_4)$ where $P(z_1, z_2, z_3, z_4) = 2x|z_2|^2 - xy^4 + |z_3|^2 + |z_4|^2$, and $z_1 = x + iy$. Let $L_j := \partial/\partial z_j - 2i\partial P/\partial z_j \partial/\partial z_5$, so $L_j \in T^{1,0}(M := \{\rho = 0\})$. Then the Levi matrix C^L associated to the basis $\{L_j\}_{1\leq j\leq 4}$ is given by

$$C^L = \begin{bmatrix} -3xy^2 & z_2 & 0 & 0 \\ \overline{z_2} & 2x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that, since C^L has three positive eigenvalues whenever either $z_2 \neq 0$ or both $x \neq 0$ and $y \neq 0$, it follows that $Z(2)$ is satisfied on a dense subset of $M \cap U$, with U a neighborhood of the origin.

Let a_{jk}, for $1 \leq j, k \leq 4$, be smooth functions on a neighborhood of the origin such that the matrix $A := [a_{jk}]$ be a 4×4 non-singular matrix on a neighborhood of the origin. Let $u_j = \sum_{k=1}^{4} a_{jk}L_k$ for $1 \leq j \leq 4$. Then the Levi matrix C^u in the basis $\{u_j\}$ for $1 \leq j \leq 4$ is given by $C^u = AC^LA^*$. Then a direct calculation gives us

$$c_{jj}^u = 2Re(z_2 a_{j1}\overline{a_{j2}}) - 3xy^2|a_{j1}| + 2x|a_{j2}|^2 + |a_{j3}|^2 + |a_{j4}|^2.$$

Since A is not a singular matrix C^u will have the same inertia of C^L. So C^u will have one negative eigenvalue, one zero eigenvalue, and two positive eigenvalues when $y = 0 = z_2$ and $x < 0$. The OW-$Z(2)$ condition cannot be satisfied with $m = 0$, because in this case the sum of the two minor eigenvalues will be negative.

In the same way, when $x = z_2 = 0$, C^u has two zero eigenvalues and two positive eigenvalues. So OW-$Z(2)$ condition cannot be satisfied with $m = 4$, because in this case the trace of C^u will be positive.

Now, assume that the OW-$Z(2)$ condition is satisfied with $m = 3$. When $x = z_2 = 0$, the sum of two minor eigenvalues of C^u is zero. Then OW-$Z(2)$ condition implies $(c^u_{11} + c^u_{22} + c^u_{33})|_{x=z_2=0} \leq 0$. So $a_{13} = a_{14} = a_{23} = a_{24} = a_{33} = a_{34} = 0$ when $x = z_2 = 0$. But this will imply that $detA\,|_{x=z_2=0} = 0$. But this is contradiction because A was supposed to be a nonsingular matrix on a neighborhood of origin. Then OW-$Z(2)$ condition can not be satisfied with $m = 3$.

Now, in order to prove the OW-$Z(2)$ condition is satisfied, the unique possibility for m is 1. So we have to show that the sum of two minor eigenvalues (sometimes being zero, and sometimes being negative) is greater than or equal to c^u_{11}, for the second condition in Definition 6.1.1 be satisfied. But this appears not to be easy (or true).

Now, we will prove that new version of weak $Z(2)$ condition is satisfied on $U \cap M$, with U a neighborhood of the origin, and considering the Euclidian metric. Let $A = [a_{jk}]$ a 4×4 matrix defined by

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}$$

with A_j a 2×2 matrix of smooth functions for $j = 1, 2, 3$, such that the set of vector $\{u_1, ..., u_4\}$ with $u_j = \sum_{k=1}^{4} a_{jk} L_k$ for $1 \leq j \leq 4$, is an orthonormal basis for $T^{1,0}(M)$. Then A is a nonsingular matrix, and in particular A_1 is also a nonsingular matrix. The Euclidean metric gives the next condition on A,

$$Id = A(Id - F)A^*$$

where F is a matrix such that $F|_{z=0} = 0$. This implies that

$$(Id - A_1 A_1^*)|_{z=0} = 0. \tag{6.1}$$

Now, the Levi matrix C^u written in the basis $\{u_j\}$ is equal to

$$AC^L A^* = \begin{bmatrix} A_1 C A_1^* & A_1 C A_2^* \\ A_2 C A_1^* & A_2 C A_2^* + A_3 A_3^* \end{bmatrix}$$

where

$$C = \begin{bmatrix} -3xy^2 & z_2 \\ \overline{z_2} & 2x \end{bmatrix}.$$

Since u_1 and u_2 are in the span of L_1 and L_2, the sum of the smallest eigenvalues of the Levi form is given by

$$\mu_1 + \mu_2 = \mathcal{L}(i\bar{u}_1 \wedge u_1 + i\bar{u}_2 \wedge u_2) = Tr(C^u diag\,\{1, 1, 0, 0\}) = Tr(A_1 C A_1^*).$$

For $t > 0$, define $\Upsilon = i \sum_{j,k=11}^{4} b_{kj} \bar{u}_j \wedge u_k$ with

$$[b_{jk}] = diag\{1,1,0,0\} - t\ diag\left\{D_1^*\ diag\left\{2, 3y^3\right\} D_1, 0, 0\right\},$$

where D_1 is a matrix such that $D_1 A_1 = Id$. Then

$$\begin{aligned}\mu_1 + \mu_2 - \mathcal{L}(\Upsilon) &= Tr(A_1 C A_1^*) - Tr(C^u[b_{jk}]) \\ &= Tr(A_1 C A_1^*) - Tr(A_1 C A_1^*) + tTr\left(A_1 C A_1^* D_1^*\ diag\left\{2, 3y^2\right\}\ D_1\right) \\ &= tTr(C\ diag\left\{2, 3y^2\right\}) \\ &= 0.\end{aligned}$$

By (6.1), for t sufficiently small, the matrix $[b_{jk}]$ will have eigenvalues in $[0, 1]$. Also, for t sufficiently small $\omega(\Upsilon) = Tr([b_{jk}]) = 2 - tTr(D_1\ diag\{2, y^2\}\ D_1^*) < 2$ because D_1 is a nonsingular matrix and $diag\{2, 3y^2\}$ has one positive eigenvalues and the other eigenvalue is nonnegative. Then the example is satisfying the new version of the weak Z(2) condition at the origin.

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