Applications of topological degree theory to Generalized ODEs

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São Carlos, December 4, 2021.

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Profa. Dra. Karina Schiabel

"Now to him who is able to do far more abundantly than all that we ask for or think, according to the power at work within us, to him be glory in the church and in Christ Jesus throughout all generations, forever and ever. Amen." (Ephesians 3.20,21)

Acknowledgment

First of all, I thank my Lord and Saviour, Jesus Christ, for the gift of life and my salvation, for his valuable blood poured out for me in the cross of calvary, for his presence in my daily routine and for his Holy Spirit that gives me the peace which surpasses all understanding. "Yours, O LORD, is the greatness and the power and the glory and the victory and the majesty, for all that is in the heavens and in earth is yours. Yours is the kingdom, O LORD, and you are exalted as head above all." (1 Chronicles 29.11). In this sense, this thesis is a manner to honour his name.

One of the tools that Jesus used to make me know him was Ana Carolina Bergantim. Before becoming an official Ph.D. student, we met and she showed me God's love and word and we narrowed our relation becoming sisters and friends. Then, I started to pray constantly asking for my Ph.D. Although God did gave me a place to do my Pd.D., directed my studies and bestowed upon me with enough grades to conclude the course, he gave me far more abundantly than all that I asked for, like taking me to the Czech Republic. So, he called me out of the darkness of idolatry that I lived during 29 years into his marvellous light.

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Resumo

Neste trabalho, apresentamos resultados originais sobre a teoria de Equações Diferenciais Ordinárias Generalizadas (escrevemos EDOs generalizadas), através do uso de ferramentas da teoria do Grau Topológico. Em particular, provamos resultados sobre

- Existência de pontos de bifurcação e aplicamos os resultados às equações diferenciais em medida;

- Diferenciabilidade do operador solução de EDOs generalizadas, incluindo, também, um teorema do tipo Alternativa de Fredholm; as aplicações foram direcionadas às equações diferenciais em medida;

- Existência de soluções periódicas de EDOs generalizadas lineares em que utilizamos não somente resultados da teoria do Grau Topológico, mas também da teoria de Operadores de Fredholm;

- Existência de soluções "*afim periódicas*" de EDOs generalizadas.

Vale mencionar que 3 artigos originais são provenientes desta tese, a saber, [11, 12, 13]. Tais artigos encontram-se em fase final de preparação e serão submetidos à publicação em breve.

Além do que mencionamos, também generalizamos os resultados da minha dissertação de mestrado contidos em um artigo já submetido, em coautoria com J. Mawhin e M. Federson (veja [14]). Enquanto que em tal artigo tratamos da existência de soluções periódicas de EDOs generalizadas envolvendo funções de variação limitada, na presente tese consideramos o caso em que tais funções estão no espaço das funções regradas. Estes novos resultados fazem parte de um capítulo do livro intitulado "*Generalized ODEs in Abstract Spaces and Applications*", organizado pelos editores M. Federson, E. Bonotto e J. Mesquita. O livro será publicado pela Wiley em 2020 (veja [10]).

Abstract

In this work, we present original results concerning the theory of Generalized Ordinary Differential Equations (we write generalized ODEs for short) using tools from the Topological Degree theory. In particular, we proved results on

- Existence of bifurcation points and we applied the results to measure differential equations;

- Differentiability of the solution operator of generalized ODEs, including a Fredholm Alternative-type theorem, and we applied the results to measure differential equations;

- Existence of periodic solutions of linear generalized ODEs to which we applied not only results from the topological degree theory, but also from the Fredholm operator theory;

- Existence of affine-periodic solutions of generalized ODEs.

It is worth mentioning that the present work generated 3 original articles (see [11, 12, 13]) which are in their final stages of preparation and will be submitted for publication soon.

In addition to the above, we also generalized the results from my Master Thesis which are contained in a submitted article, coauthored by J. Mawhin and M. Federson (see [14]). While in such article we deal with the existence of periodic solutions of generalized ODEs involving bounded variation functions, in the present work we consider the regulated functions. Such new results are part of a chapter in the book entitled "*Generalized ODEs in Abstract Spaces and Applications*" and organized by the editors M. Federson, E. Bonotto and J. Mesquita. The book will be published by Wiley in 2020 (see [10]).

Contents

Introduction

The aim of this work is employ tools of the theory of Topological Degree to obtain results in the framework of generalized ordinary differential equations (we write generalized ODEs for short).

It well-known that generalized ODEs, introduced by the Jaroslav Kurzweil in 1957 (see [22, 23]), are within a useful theory which handles well jumps and highly oscillatory behavior. This nice property if due to the fact that generalized ODE are in fact integral equations whose integral is in the sense of J. Kurzweil.

In order to describe, at a glance, the potential of the theory of generalized ODEs, it is worth mentioning that they were born within the theory of non-absolute integration created and developed by Jaroslav Kurzweil and, independently, by Ralph Henstock. The Kurzweil-Henstock integral is able to deal not only with many discontinuities, but also with functions of unbounded variation. As a matter of fact, the Kurzweil-Henstock integral of functions taking values in finite dimensional spaces encompass the integrals of Riemann, Lebesgue and Newton.

As we mentioned in the abstract, we proved results concerning the following properties of generalized ODEs:

- Existence of periodic solutions of generalized ODEs;

- Existence of bifurcation points;

- Differentiability of the solution operator;

- A Fredholm Alternative-type theorem;

- Existence of periodic solutions of linear generalized ODEs;

- Existence of affine-periodic solutions of generalized ODEs.

All such properties were obtained through the application of the Topological Degree theory. Throughout this work, we show, in detail, the nuances of the tools we employed.

Concerning the applications, we have to say that, in this work, we choose to apply the new results to measure differential equations (we write MDEs, for short) and, in some

cases to impulsive differential equations. Our choice is justified by two facts. The first
fact concerns the power of MDEs. It is well-known that MDEs allow one to treat more
general situations than those handled by difference differential equations, ordinary differ-
ential equations and impulsive differential equations. The beginning of the investigations
in MDEs is considered to date from the late 60's and early 70's, with the works by W.
Schmaedeke [34], and P. Das and R. Sharma [3, 4]. Since then, several authors have been
studying this class of equations. We can mention, for instance, [5, 7, 6, 20, 28, 29, 35],
where qualitative properties of solutions were investigated. Moreover, MDEs encompass
not only impulsive differential equation (see [16, Theorem 3.1]), but also dynamic equa-
tions on time scales (see [17, Theorem 4.3]). Thus, MDEs is a very important class of
differential equations. The second motive for applying our results to MDEs is that they
can be regarded as generalized ODEs (see [17, Theorems 3.8 and 3.9]). This means that
MDEs are special cases of generalized ODEs.

Now, we specify some details concerning organization of the present work as well as
the articles generated from it.

The two initial chapters are devoted to basic theories. In Chapter 1, we collect those
results from the Topological Degree theory which we will use throughout this dissertation.
In Chapter 2, we did the same concerning the theory of generalized ODEs. The following
chapter, namely, chapters 3 to 7, are related to the articles we produced during my
doctor course in Mathematics at the Federal University of São Carlos and my stage at the
Institute of Mathematics of the Academy of Sciences of the Czech Republic. We describe
them as follows.

Chapter 3 contains the results of our paper entitled "*Existence and bifurcation of peri-
odic solutions for generalized ordinary differential equations*", coauthored by M. Federson
and J. Mawhin (see [14]) and it is a generalization of the results from my Master Thesis.
We showed an existence result of a periodic solution of generalized ODEs in the frame-
work of regulated functions instead of functions of bounded variation. In order to do
this, we had to replace the Helly's Choice Theorem by an Ascoli-Arzelà-type theorem for
regulated functions. We also employed the Topological Degree theory to get the results.
We applied the main result to impulsive ODEs. A more general result of Chapter 3 is part
of a chapter in the book entitled "*Generalized ODEs in Abstract Spaces and Applications*"
and organized by the editors M. Federson, E. Bonotto and J. Mesquita.

Chapters 4 and 5 concern the results contained in the paper entitled "*Bifurcation
theory and differentiability for generalized ODEs*", coauthored by M. Federson and K.
Schiabel (see [11]), where we established a result on the existence of a bifurcation point
with respect to an arbitrary solution of a generalized ODE. In order to do this, we defined
an operator Φ, which is characterized by the solution of a generalized ODE, and we used
such operator to obtain the main result. The theory of the topological degree was crucial

to our result. We also established a formula for the derivative of the operator Φ, which is linear and, in turn, is a solution of a linear generalized differential equation. This fact allowed us to state and prove a Fredholm Alternative for linear equations which involve the derivatives of such operator Φ. Finally, we applied our results to MDEs.

Chapter 6 concerns the results contained in the paper entitled "*Periodic solutions of linear generalized ODEs and applications*", coauthored by M. Federson and K. Schiabel (see [13]), we established a result on the existence of periodic solutions of linear generalized ODE. In order to do that, we defined operators L and N from the space of regulated functions to itself and L is a Fredholm operator. We proved that there exists a correspondence between the solutions of $L(x) = N(x)$ and the periodic solutions of linear generalized ODEs. Then, it was possible to prove our main result on the existence of a solution of $L(x) = N(x)$. The theories of Topological Degree and of Fredholm Linear Operators were also employed to obtain the main result.

Finally, in the paper entitled "*Affine-Periodic solutions for generalized ODEs*", coauthored by M. Federson and R. Grau (see [12]), we considered an $n \times n$ matrix Q with entries in $\mathbb{R}$ and $T > 0$ and we proved a result on the existence of what we call a (Q, T)-*affine-periodic solution* of a generalized ordinary differential equation using tools from Functional Analysis. In Chapter 7, we proved the same result by means of the Topological Degree theory inspired on the papers [39] and [40]. As a matter of fact, the definition of (Q, T) is more general than the usual definition of periodic solutions stated in the literature. Indeed, when we consider $Q = I$, where I is the identity matrix, we obtain the classic notion of periodicity and, when $Q = -I$, we are in the classic case of anti-periodicity. Thus, our definition encompasses the classic definitions.

Chapter 1

Preliminaries: Degree Theory

The aim of this initial chapter is to compile some of the basic results of the topological degree theory which will be used throughout our work. In particular, we recall results from both the finite dimension and the infinite dimension cases, which are, respectively, those results concerning the Brouwer degree and those concerning the Leray-Schauder degree. We also include useful results from the Fredholm theory for linear operators.

The main references for this section are [30], [2] and [8].

1.1 The Brower degree

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $C^k(\overline{\Omega}, \mathbb{R}^n)$ be the space of all functions, from the closure $\overline{\Omega}$ of Ω to the n dimensional Euclidean space $\mathbb{R}^n$, which are k-times differentiable on Ω whose the derivatives can be extended continuously to $\overline{\Omega}$ and it is endowed with the following norm

$$||\varphi||_k = \max_{0 \le j \le k} \sup_{x \in \Omega} ||D^{(j)} \varphi(x)||.$$

where $D^{(j)} \varphi(x)$ denotes the j derivative of φ at the point x.

Suppose $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ and $S = \{x \in \Omega, \ J(\varphi)(x) = 0\}$, where $J(\varphi)(x)$ denotes the Jacobian matrix of φ in x. Let $b \in \mathbb{R}^n$ be such that $b \notin \varphi(\partial\Omega) \cup \varphi(S)$, where $\partial\Omega$ denotes the boundary of Ω. If $x \in \varphi^{-1}(\{b\})$, then $J(\varphi)(x) \neq 0$. Therefore, by the Inverse Application Theorem, there exist neighborhoods U of x and V of b such that $\varphi|_U : U \to \varphi(U) = V$ is a diffeomorphism.

We assert that $\varphi^{-1}(\{b\})$ is finite. Indeed. Since $\varphi^{-1}(\{b\})$ is closed in $\overline{\Omega}$, $\varphi^{-1}(\{b\})$ is closed and bounded in $\mathbb{R}^n$. Thus, $\varphi^{-1}(\{b\})$ is a compact set. Also, for every $x_j \in \varphi^{-1}(\{b\})$, $j \in \mathbb{N}$, there exists an open ball in U, centered at x_j and with radius $r_j > 0$, denoted by

$B_{r_j}(x_j)$, such that

$$\varphi^{-1}(\{b\}) \subset \cup_{x_j \in \varphi^{-1}(\{b\})} B_{r_j}(x_j), \quad j \in \mathbb{N}.$$

Since $\{B_{r_j}(x_j)\}_{j \in \mathbb{N}}$ is an open cover of $\varphi^{-1}(\{b\})$, we can find a finite subcover satisfying

$$\varphi^{-1}(\{b\}) \subset \cup_{j=1}^{k} B_{r_j}(x_j),$$

which shows that $\varphi^{-1}(\{b\})$ is finite, that is,

$$\varphi^{-1}(\{b\}) = \{\xi_1, \xi_2, \xi_3, \ldots, \xi_k\},$$

where $J(\varphi)(\xi_i) \neq 0$, for all $i \in \{1, 2, 3, \ldots, k\}$.

In the next lines, we recall the definition of the Brouwer degree and we give an example borrowed from [1], on how to calculate it.

Definition 1.1.1. Suppose $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ and $b \notin \varphi(\partial\Omega) \cup \varphi(S)$. We define the Brouwer degree with respect to the triple (φ, Ω, b), by the integer number

$$d(\varphi, \Omega, b) = \sum_{\xi_i \in \varphi^{-1}(\{b\})} \operatorname{sgn}(J(\varphi)(\xi_i))$$

where sgn is the function given by

$$\operatorname{sgn}(t) = \begin{cases} 1, & \text{if } t > 0, \\ -1, & \text{if } t < 0. \end{cases}$$

Example 1.1.2. Consider the function $\varphi : \Omega \to \mathbb{R}$, defined by $\varphi(x) = \sin x$ with $\Omega = \left(0, \frac{5\pi}{2}\right)$ and $b = \frac{\pi}{4}$. We want to calculate $d(\varphi, \Omega, b)$. In order to do this, we need to verify that $b \notin \varphi(\partial\Omega) \cup \varphi(S)$. This means means that $d\left(\sin x, \left(0, \frac{5\pi}{2}\right), \frac{\pi}{4}\right)$ is well defined. Notice that

$$\partial\Omega = \left\{0, \frac{5\pi}{2}\right\}, \quad S = \left\{x \in \left(0, \frac{5\pi}{2}\right); \cos x = 0\right\} = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}, \quad \varphi(\partial\Omega) = \{0, 1\},$$

$$\varphi(S) = \{-1, 1\}, \quad \text{then } \varphi(\partial\Omega) \cup \varphi(S) = \{-1, 0, 1\}.$$

Therefore, $\frac{\pi}{4} \notin \{-1, 0, 1\}$. Thus, $\varphi^{-1}\left(\frac{\pi}{4}\right) = \{\xi_1, \xi_2, \xi_3\}$. By the definition of the Brower degree (Definition 1.1.1),

$$d\left(\sin x, \left(0, \frac{5\pi}{2}\right), \frac{\pi}{4}\right) = \sum_{\xi_i \in \varphi^{-1}(\frac{\pi}{4})} \operatorname{sgn}(J(\varphi)(\xi_i)).$$

Then,

$$
\begin{aligned}
d\left(\sin x, \left(0, \frac{5\pi}{2}\right), \frac{\pi}{4}\right) &= \operatorname{sgn}(\varphi'(\xi_1)) + \operatorname{sgn}(\varphi'(\xi_2)) + \operatorname{sgn}(\varphi'(\xi_3)), \\
&= 1 + (-1) + 1 = 1.
\end{aligned}
$$

It is important to mention that the definition of the Brouwer degree holds for continuous functions. The following result is presented in [2], Theorem 21.5 and Corollary 2.16. It says that there exists the function, which defines the Brouwer degree, with respect to continuous functions.

Theorem 1.1.3. *Let E be a finite dimensional Banach space. Then, for every open and bounded subset Ω of E and every $z \in E$, there exists a function*

$$
deg[\cdot, \Omega, z] : D_z(\Omega, E) \to \mathbb{Z}
$$

called Brouwer degree, where $D_z(\Omega, E) = \{f \in C(\overline{\Omega}, E); \; z \notin f(\partial\Omega)\}$, which satisfies the following properties:

(i) (Normalization): If $z \in \Omega$, then $deg(I, \Omega, z) = 1$.

(ii) (Homotopy invariance): Let $J \subseteq \mathbb{R}$ be a nonempty compact interval. Assume that $h \in C(\overline{\Omega} \times J, E)$ and $y \in C(J, E)$ satisfy

$$
y(\lambda) \notin h(\partial\Omega \times \{\lambda\}) \;\; \text{for each} \;\; \lambda \in J.
$$

Then,
$$
deg[h(\cdot, \lambda), \Omega, y(\lambda)]
$$
is well-defined and is independent of $\lambda \in J$.

Corollary 1.1.4. *Let Ω be open and bounded in a Banach space E of finite dimension n, and for $f \in C(\overline{\Omega}, E)$ assume that z does not belong to $f(\partial\Omega)$. Then*

$$
deg[-f, \Omega, z] = (-1)^n deg[f, \Omega, z].
$$

1.2   The Leray-Schauder degree

In order to understand the concept of degree for functions whose domain is a subset of a general Banach space, we recall some elements of the Leray-Schauder degree theory. For more details see [8, 18, 30].

Let E be a Banach space, $\Omega \subset E$ be open and bounded and $T \in C(\overline{\Omega}, E)$ be an operator such that $T(\overline{\Omega})$ is contained in a finite subset of E. The operator $\Phi = I - T$

is called a perturbation of finite dimension of the identity I. Let A, B be subsets of E. Then the distance between A and B is given by

$$\rho(A, B) = \inf \{ \, \|a - b\|, \quad a \in A, b \in B\}.$$

Definition 1.2.1. Let E be a Banach space and $z \in E$ be such that $z \notin \Phi(\partial\Omega)$. Suppose F is a finite dimensional subspace of E which contains $T(\overline{\Omega})$ and z. Then we define the Leray- Schauder degree of Φ with respect to Ω at a point z by the integer number

$$deg_{LS}(\Phi, \Omega, z) = d(\Phi|_{\overline{\Omega} \cap F}, \Omega \cap F, z)$$

where the right-hand side of the equality is the Brouwer degree defined according to Theorem 1.1.3.

Definition 1.2.2. Let E be a Banach space and $\Omega \subset E$, An operator $T : \Omega \to E$ is said to be compact, whenever T is continuous and $\overline{T(B)}$ is a compact set in E, for every bounded set $B \subset \Omega$.

In the next lines, we present the definition of the Leray- Schauder degree for compact operators in Banach spaces. The construction of the Leray- Schauder degree for compact operators can be found in [18], Chapter 7.

Definition 1.2.3. Let E be a Banach space and $\Omega \subset E$ be an open bounded set. Let $T : \overline{\Omega} \to E$ be a compact operator and let $z \notin (I - T)(\partial\Omega)$. The Leray-Schauder degree is defined by

$$deg_{LS}[I - T, \Omega, z] = deg_{LS}[I - \widehat{T}, \Omega \cap V, z]$$

where $\widehat{T} : \overline{\Omega} \to E$ is continuous such that $\widehat{T}(\overline{\Omega})$ is of finite dimension and

$$\|\widehat{T}(x) - T(x)\| < \rho(z, (I - T)(\partial\Omega))$$

and V is any linear space of finite dimension containing z and $\widehat{T}(\overline{\Omega})$.

Moreover, the Leray-Schauder degree satisfies the following properties:

(i) $deg_{LS}[I, \Omega, z] = 1$, for $z \in \Omega$.

(ii) If $deg_{LS}[I - T, \Omega, z] \neq 0$, then $z \in (I - T)(\Omega)$.

(iii) $deg_{LS}[I - H(t, \cdot), \Omega, y(t)]$ is independent of $t \in [0, 1]$, whenever $H : [0, 1] \times \overline{\Omega} \to E$ is compact, $y : [0, 1] \to E$ is continuous and $y(t)$ does not belong to $(I - H(t, \cdot))(\partial\Omega)$, for every $t \in [0, 1]$.

In what follows, let us introduce the definition of homotopy of compact transformations.

Definition 1.2.4. Let E be a Banach space and $M \subset E$. Let $H : [0,1] \times M \longrightarrow E$. We say that H is a homotopy of compact transformations on M if

(a) For each $\lambda \in [0,1]$ fixed, $H(\lambda, x)$ is compact on M.

(b) For every $\varepsilon > 0$ and for every bounded $L \subset M$, there is $\delta > 0$, such that

$$\|H(\lambda_1, x) - H(\lambda_2, x)\| \leq \varepsilon \tag{1.2.1}$$

whenever $x \in L$ and $|\lambda_1 - \lambda_2| < \delta$.

The next result is presented in [18] and it assures an important property of the Leray-Schauder degree, which it will be useful in the following chapters.

Theorem 1.2.5. *[Invariance under Homotopy] Let E be a Banach space and $\Omega \subset E$ be an open bounded set. Assume that $H : [0,1] \times \overline{\Omega} \longrightarrow E$ is a homotopy of compact transformations on $\overline{\Omega}$. Set*

$$\phi_\lambda = I - H(\lambda, \cdot)$$

for $\lambda \in [0,1]$ and assume that $z \notin \phi_\lambda(\partial\Omega)$, for every $\lambda \in [0,1]$. Then, $deg_{LS}[I - H(\lambda, \cdot), \Omega, z]$ is independent of λ.

Now, we recall the concept of a local Leray-Schauder degree which we refer to as Leray-Schauder index. This definition is presented in [25].

Definition 1.2.6. Let $T : \overline{\Omega} \to E$ satisfy the conditions of Definition 1.2.3. Denote by $B_R(x_0)$ the open ball in Ω with center in x_0 and radius R. Suppose that x_0 is an isolated fixed point of T, that is, there exists $R > 0$ such that $T(x) \neq x$, for every $x \in B_R(x_0) \setminus \{x_0\}$. Then $deg_{LS}[I - T, B_R(x_0), 0]$ is defined and does not depend on R, for sufficiently small $R > 0$. Its value is called the *Leray-Schauder index* of $I - T$ at x_0 and it is denoted simply by $ind_{LS}[I - T, x_0]$.

1.3 Fredholm Theory

Let X, Y be Banach spaces and $\mathcal{L}(X, Y)$ be the space of linear bounded operators from X to Y. The kernel and range of the operator A are denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$ respectively.

Definition 1.3.1. Let X, Y be Banach spaces. An operator $A \in \mathcal{L}(X, Y)$ is a Fredholm operator, if it satisfies the following conditions:

(i) The dimension of $\mathcal{N}(A)$ is finite (we write $\dim \mathcal{N}(A) < \infty$).

(ii) $\mathcal{R}(A)$ is closed in Y

(iii) The codimension of $\mathcal{R}(A)$ is finite (we write codim $\mathcal{R}(A) < \infty$).

Definition 1.3.2. The index of a Fredholm operator is given by

$$i(A) = \dim \mathcal{N}(A) - \text{codim } \mathcal{R}(A). \tag{1.3.1}$$

The next result is known as the Fredholm Alternative for operators in Banach spaces. For more details, see [33], Theorem 4.12.

Theorem 1.3.3. *Let X be a Banach space and let $K : X \longrightarrow X$ be a compact operator. Set $A = I - K$. Then,*

(i) *dim $\mathcal{N}(A) = $ codim $\mathcal{R}(A)$ is finite;*

(ii) *$\mathcal{R}(A)$ is closed in Y.*

In particular, either $\mathcal{R}(A) = X$ and $N(A) = 0$, or $\mathcal{R}(A) \neq X$ and $\mathcal{N}(A) \neq 0$.

The next result can be found in [33] and it assures, under some conditions, the existence of a projetion function.

Proposition 1.3.4. *Let X_1 be a closed subspace of a normed vector space X and let M be a finite dimensional subspace of X such that $M \cap X_1 = \{0\}$. Then*

$$X_2 = X_1 \oplus M$$

is a closed subspace of X. Moreover, the operator P defined by

$$P(x) = \begin{cases} x, & x \in M \\ 0, & x \in X_1 \end{cases}$$

is in $\mathcal{L}(X_2)$.

If L is a Fredholm operator of index 0, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that

$$\mathcal{R}(P) = \mathcal{N}(L)$$

and

$$\mathcal{R}(L) = \mathcal{N}(Q)$$

It follows that $L|_{\mathcal{D}(L) \cap \mathcal{N}(P)} : (I - P)X \to \mathcal{R}(L)$ is invertible. We denote the inverse of $L|_{\mathcal{D}(L) \cap \mathcal{N}(P)}$ by K_p, If Ω is an open bounded subset of X, the mapping N will be called L-compact on Ω, whenever $QN(\overline{\Omega})$ is bounded and $Kp(I - Q)N : \Omega \to X$ is compact. Since $\mathcal{R}(Q)$ is isomorphic to $\mathcal{N}(L)$, there exists an isomorphism $J : \mathcal{R}(Q) \to \mathcal{N}(L)$.

The next result, known as Mawhin's Continuation Theorem, can be found in [19], page 40. It will be essential to prove the main result of Chapter 6.

Theorem 1.3.5. *[Mawhin's Continuation Theorem] Let $\Delta \subset G$ be open and bounded and X, Y, Z be Banach spaces. Suppose $L : \mathcal{D}(L) \subset X \to Z$ is a Fredholm operator of index 0 and $N : Y \to Z$ is L-compact on $\overline{\Delta}$. Assume that the following conditions are satisfied:*

(i) If $x \in \mathcal{D}(L) \cap \partial\Omega$, then $Lx \neq \lambda Nx$, for every $\lambda \in (0,1)$;

(ii) If $x \in \mathcal{N}(L) \cap \partial\Delta$, then $QNx \neq 0$;

(iii) $deg(JQN, \Delta \cap \mathcal{N}(L), 0) \neq 0$, where $J : \mathcal{R}(Q) \longrightarrow \mathcal{N}(L)$.

Then, the equation

$$Lx = Nx$$

has at least one solution $x \in \overline{\Delta} \cap \mathcal{D}(L)$.

Chapter 2

Generalized ODEs

In this chapter, we recall a few basic properties of the Kurzweil integration theory. For more details, see [31, 22]. Throughout this chapter, $T \in (0, \infty)$ is fixed and $\| \, . \, \|$ stands for the usual norm in $\mathbb{R}^n$.

2.1 Generalized ODEs

By a tagged partition of a compact interval $[0, T] \subset \mathbb{R}$, we mean a finite collection of point-interval-pairs $(\tau_j, [t_{j-1}, t_j])_{i=j}^n$, where $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$ and $\tau_j \in [t_{j-1}, t_j]$, for $j = 1, ..., n$. Given a function $\delta : [0, T] \to (0, \infty)$, we say that the tagged partition $(\tau_i, [t_{j-1}, t_j])_{i=1}^n$ is δ-fine, if

$$[t_{j-1}, t_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)), \quad \text{for every } j = 1, \ldots, n.$$

Definition 2.1.1. A function $U : [0, T] \times [0, T] \to \mathbb{R}^n$ is called *Kurzweil integrable* in $[0, T]$, if there is an $I \in \mathbb{R}^n$ such that for a given $\varepsilon > 0$, there is a $\delta : [0, T] \to (0, \infty)$ such that for every δ-fine tagged partition $(\tau_j, [t_{j-1}, t_j])_{j=1}^n$ of $[0, T]$, we have

$$\left\| \sum_{j=1}^n [U(\tau_j, t_j) - U(\tau_j, t_{j-1})] - I \right\| \leq \varepsilon.$$

In such a case, we write $I = \int_0^T DU(\tau, t)$.

Remark 2.1.2. If the integral $\int_a^b DU(\tau, t)$ exists, then we define

$$\int_b^a DU(\tau, t) = -\int_a^b DU(\tau, t), \quad \text{if } a < b$$

and

$$\int_a^b DU(\tau, t) = 0, \ \ \text{if } a = b,$$

where $a, b \in [0, T]$.

Definition 2.1.3. Consider an open set $B \subset \mathbb{R}^n$ and a function $F : B \times [0, T] \to \mathbb{R}^n$. The function $x : [0, T] \to \mathbb{R}^n$ is called a *solution* of the *generalized ordinary differential equation* (we write generalized ODE for short)

$$\frac{dx}{d\tau} = DF(x, t) \tag{2.1.1}$$

whenever $x(s) \in B$, for every $s \in [0, T]$ and

$$x(s) - x(0) = \int_0^s DF(x(\tau), t), \ \ \ \text{for all } s \in [0, T]. \tag{2.1.2}$$

Definition 2.1.4. Let $B \subset \mathbb{R}^n$ be open and $\Omega = B \times [0, T]$. Assume that $h : [0, T] \to \mathbb{R}$ is a nondecreasing left continuous function and $\omega : [0, +\infty) \to \mathbb{R}$ is an increasing and continuous function such that $\omega(0) = 0$. We say that a function $F : \Omega \to \mathbb{R}^n$ belongs to the class $\mathcal{F}(\Omega, h, \omega)$, if it satisfies

$$\|F(z, t_2) - F(z, t_1)\| \le |h(t_2) - h(t_1)|, \tag{2.1.3}$$

$$\|F(z, t_2) - F(z, t_1) - F(y, t_2) + F(y, t_1)\| \le \omega(\|z - y\|)|h(t_2) - h(t_1)|, \tag{2.1.4}$$

for all $z, y \in B$ and $t_1, t_2 \in [0, T]$.

A function $x : [0, T] \to \mathbb{R}^n$ is called regulated, if the lateral limits $x(t^-) = \lim_{\tau \to t^-} x(\tau)$ and $x(s^+) = \lim_{\tau \to s^+} x(\tau)$ exist, for all $t \in (0, T]$ and all $s \in [0, T)$. We denote by G the space of regulated functions $x : [0, T] \to \mathbb{R}^n$, endowed with the usual supremum norm $\|x\|_\infty = \sup_{t \in [0, T]} \|x(t)\|$. The fact that G is a Banach space is well-known (see [37]).

The following lemma combines two statements from [31] (see Lemma 3.9 and Corollary 3.15).

Lemma 2.1.5. *Assume that $F : \Omega \to \mathbb{R}^n$ belongs to the class $\mathcal{F}(\Omega, h, \omega)$. If $x : [0, T] \to B$ is a regulated function, then the integral $\int_0^T DF(x(\tau), t)$ exists and*

$$\left\| \int_0^T DF(x(\tau), t) \right\| \le h(T) - h(0).$$

Moreover, the function $s \longmapsto \int_0^s DF(x(\tau), t)$ is of bounded variation in $[0, T]$ and hence, it is also regulated.

We also need the following lemma, which can be found in [31], (see Lemma 3.12).

Lemma 2.1.6. *Assume that $F : \Omega \to \mathbb{R}^n$ belongs to the class $\mathcal{F}(\Omega, h, \omega)$. Then, every solution x of*

$$\frac{dx}{d\tau} = DF(x, t)$$

is regulated on $[0, T]$.

The next estimate follows directly from the definition of the Kurzweil integral and it can be found in [36].

Lemma 2.1.7. *Let $U : [0, T]^2 \to \mathbb{R}^n$ be Kurzweil integrable. Assume there exist functions $f : [0, T] \to \mathbb{R}$ and $g : [0, T] \to \mathbb{R}$ such that f is regulated, g is nondecreasing, and*

$$\|U(\tau, t) - U(\tau, s)\| \leq f(\tau)|g(s) - g(t)| \quad \text{for all } t, s, \tau \in [0, T].$$

Then

$$\left\| \int_0^T DU(x(\tau), t) \right\| \leq \int_0^T f(\tau) dg(\tau).$$

Remark 2.1.8. Note that, for $F : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ and a constant $a \in \mathbb{R}^n$, $\sum_{j=1}^{|d|}[F(a, t_j) - F(a, t_{j-1})] = F(a, T) - F(a, 0)$, for any tagged division $d = (\tau_j, [t_{j-1}, t_j])_{j=1}^n$ of $[0, T]$, we have

$$\int_0^T DF(a, t) = F(a, T) - F(a, 0). \tag{2.1.5}$$

As the last result of this quick overview of the basis of the GODEs theory, we recall an important property of the class $F \in \mathcal{F}(\Omega, h_R, \omega_R)$ which will be useful in the next section. A proof of it can be found in [26, Lemma 5].

Lemma 2.1.9. *Let $B \subset \mathbb{R}^n$ be open and assume that $F : B \times [0, T] \to \mathbb{R}^n$ belongs to the class $F \in \mathcal{F}(\Omega, h, \omega)$. If $x, y : [0, T] \to B$ are regulated functions, then*

$$\left\| \int_0^T D[F(x(\tau), t) - F(y(\tau), t)] \right\| \leq \int_0^T \omega(\|x(t) - y(t)\|) dh(t).$$

The next proposition presents a characterization of relatively compact subsets of the space G of regulated functions from $[0, T]$ to $\mathbb{R}^n$. For the proof of such fact, we refer to [37, Corollary 4.3.8].

Proposition 2.1.10. *Let $\mathcal{A} \subset G$. Assume that the set $\{x(0), \ x \in \mathcal{A}\}$ is bounded and there exists a nondecreasing function $h : [0, T] \to \mathbb{R}^n$ such that*

$$\|x(t) - x(s)\| \leq |h(t) - h(s)|, \ \text{for every} \ \ t, s \in [0, T] \ \text{and} \ x \in \mathcal{A}.$$

Then $\mathcal{A}$ is relatively compact in G.

The next result is taken from [31] and it is an analogous result to Gronwall's inequality for the Kurzweil Stieltjes integral.

Theorem 2.1.11. *Let $k : [0, T] \to [0, \infty)$ be a nondecreasing left-continuous function, $K > 0$, $\gamma \geq 0$. Suppose that $\psi : [0, T] \longrightarrow [0, \infty)$ is bounded and satisfies*

$$\psi(\xi) \leq K + \gamma \int_0^\xi \psi(\tau) dk(\tau), \quad \text{for every } \xi \in [0, T].$$

Then $\psi(\xi) \leq K \ \exp[\gamma(k(\xi) - k(0))]$, for every $\xi \in [0, T]$.

The following result is a Substitution Theorem for the Kurzweil integral. It can be found in [31, Theorem 1.18].

Lemma 2.1.12. *Assume that $\phi : [c, d] \to \mathbb{R}$ is a continuous strictly monotone function on $[c, d]$. Let $U : [\phi(c), \phi(d)] \times [\phi(c), \phi(d)] \to \mathbb{R}^n$ be given. If one of the integrals*

$$\int_{\phi(c)}^{\phi(d)} DU(\tau, t), \qquad \int_c^d DU(\phi(\tau), \phi(t))$$

exists, then the other also exists and we have

$$\int_{\phi(c)}^{\phi(d)} DU(\tau, t) = \int_c^d DU(\phi(\tau), \phi(t)).$$

The next result can be found in [15, Corollary 3.14], for the case $\Omega = X \times [t_0, +\infty)$, where X is a Banach space. Here, we state the case when $X = \mathbb{R}^n$.

Theorem 2.1.13. *Let $\Omega = \mathbb{R}^n \times [t_0, +\infty)$ and $F \in \mathcal{F}(\Omega, h)$, where the function h is nondecreasing and left–continuous. Then for every $(x_0, s_0) \in \Omega$, there exists a unique maximal solution of (2.1.1), defined in $[s_0, +\infty)$, with $x(s_0) = x_0$.*

2.2 Linear generalized ODEs

In this section, our goal is to present the basic concepts and properties of linear generalized ODEs. For more details, see [31].

Definition 2.2.1. Assume that functions $A : [0, T] \longrightarrow \mathcal{L}(\mathbb{R}^n)$ and $x : [0, T] \longrightarrow \mathbb{R}^n$ are given. We say that the Perron-Stieltjes integral $\int_0^T d[A(s)]x(s)$ exists if there is an element $J \in \mathbb{R}^n$ such that, for every $\varepsilon > 0$, there is a gauge δ on $[0, T]$ satisfying

$$\left\| \sum_{j=1}^n [A(t_j) - A(t_{j-1})]x(\tau_j) - J \right\| < \varepsilon$$

for every δ-fine tagged partition $(\tau_j, [t_{j-1}, t_j])_{j=1}^n$ of $[0, T]$. We write $J = \int_0^T d[A(s)]x(s)$.

Remark 2.2.2. For the case $a \in [0, T]$, it is convenient to set $\int_a^a d[A(s)]x(s) = 0$ and, $\int_0^T d[A(s)]x(s) = -\int_T^0 d[A(s)]x(s)$.

Definition 2.2.3. The function $x : [0, T] \to \mathbb{R}^n$ is called a *solution* of the *generalized linear ordinary differential equation*

$$\frac{dx}{d\tau} = D[A(t)x] \tag{2.2.1}$$

if it satisfies

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} [A(s)]x(s)$$

for every $t_1, t_2 \in [0, T]$.

An operator $A : [0, T] \longrightarrow \mathcal{L}(\mathbb{R}^n)$ is of bounded variation if it satisfies

$$\underset{[0,T]}{var}(A) = \sup \left\{ \sum_{j=1}^n \|A(t_j) - A(t_{j-1})\|_{\mathcal{L}(\mathbb{R}^n)} \right\} < \infty$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < .... < t_n = T$ of $[0, T]$. The set of all the bounded variation operators is denoted by $BV([0, T], \mathcal{L}(\mathbb{R}^n))$. The following results are taken from [32].

Proposition 2.2.4. *Assume that* $A \in BV([0, T], \mathcal{L}(\mathbb{R}^n))$. *Let* $x : [0, T] \longrightarrow \mathbb{R}^n$ *be a regulated function. Then the integral* $\int_0^T d[A(s)]x(s)$ *exists and satisfies*

$$\left\| \int_0^T d[A(s)]x(s) \right\| \leqslant \underset{[0,T]}{var}(A) \underset{s \in [0,T]}{\sup} \|x(s)\|.$$

Proposition 2.2.5. *Assume that* $A \in BV([0, T], \mathcal{L}(\mathbb{R}^n))$. *Let* $x_i : [0, T] \longrightarrow \mathbb{R}^n$ *be regulated functions, for every* $i = \{1, 2\}$. *Then, for every* $c_1, c_2 \in \mathbb{R}$ *the integral* $\int_0^T d[A(s)](c_1 x_1(s) + c_2 x_2(s))$ *exists and*

$$\int_0^T d[A(s)](c_1 x_1(s) + c_2 x_2(s)) = c_1 \int_0^T d[A(s)]x_1(s) + c_2 \int_0^T d[A(s)]x_2(s).$$

The next result is taken from [32] and it ensures that the uniform convergence theorem holds for Perron-Stieltjes integrals.

Theorem 2.2.6. *Assume that* $A \in BV([0, T], \mathcal{L}(\mathbb{R}^n))$. *Let* $x, x_n : [0, T] \longrightarrow \mathbb{R}^n$ *be regulated functions. Suppose that the sequence* x_n *converges on* $[0, T]$ *uniformly to* x, *that*

is,

$$\lim_{n \to \infty} \|x_n(s) - x(s)\| = 0$$

uniformly on $[0, T]$, *then the integral* $\int_0^T d[A(s)]x(s)$ *exists and*

$$\int_0^T d[A(s)]x(s) = \lim_{n \to \infty} \int_0^T d[A(s)]x_n(s).$$

Chapter 3

Periodic solutions of nonautonomous Generalized ODEs

In this chapter, our goal is to introduce the concept of periodic solutions for generalized ODEs and to establish an existence result. The results presented in this chapter are contained in [14] and a more general existence theorem on (θ, T)-periodic solutions for generalized ODEs, inspired by [14], is presented in [10], where $T > 0$ and $\theta \in \mathbb{R}$.

3.1 Introduction

We now are interested in proving a result which ensures the existence of at least one T-periodic solution of a generalized ODE of the type

$$\frac{dx}{d\tau} = DF(x,t), \tag{3.1.1}$$

where $F : B_R \times [0,T] \to \mathbb{R}^n$ and $B_R \subset \mathbb{R}^n$ denotes the open ball with center in 0 and radius $R > 0$. To do it, we will use the Leray-Schauder degree theory.

In this chapter, we assume:

(A1) $F : B_R \times [0,T] \to \mathbb{R}^n$ satisifies Definition 2.1.4, that is, $F \in \mathcal{F}(B_R \times [0,T], h_R, \omega_R)$, for all $R > 0$.

Definition 3.1.1. Let $T > 0$ be fixed. We say that a function $x : [0,T] \to \mathbb{R}^n$ is a *T-periodic solution* of the generalized ODE

$$\frac{dx}{d\tau} = DF(x,t) \tag{3.1.2}$$

if it is a solution of (3.1.2) and, moreover, $x(0) = x(T)$.

Recall that G denotes the space of all regulated funtions $x : [0, T] \to \mathbb{R}^n$ endowed with the supremum norm $\| \cdot \|_\infty$.

In what follows, let the operator

$$\mathcal{M} : G \longrightarrow G$$

$$(\lambda, x) \longmapsto \mathcal{M}(\lambda, x),$$

be defined, for each $s \in [0, T]$, by

$$\mathcal{M}(\lambda, x)(s) = x(0) + \int_0^T DF(x(\tau), t) + \int_0^s DF(x(\tau), t). \qquad (3.1.3)$$

By Lemma 2.1.5, it is clear that the operator $\mathcal{M}$ is well-defined.

The next theorem describes a one-to-one correspondence between T-periodic solutions of (3.1.2) and the fixed points of the operator $\mathcal{M}$, given by (3.1.3).

Theorem 3.1.2. *Assume that $(A1)$ are valid. A function $x : [0, T] \to \mathbb{R}^n$ is a T-periodic solution of (3.1.2), if and only if, x is a fixed point of operator $\mathcal{M} : G \to G$, given by (3.1.3).*

Proof. Suppose x is a T-periodic solution of (3.1.2). Therefore x is regulated (by Lemma 2.1.6) and for every $s \in [0, T]$,

$$x(s) = x(0) + \int_0^s DF(x(\tau), t).$$

In particular,

$$x(T) = x(0) + \int_0^T DF(x(\tau), t).$$

Since x is a T-periodic solution of equation (3.1.2), we have $x(T) = x(0)$, and hence, $\int_0^T DF(x(\tau), t) = 0$. Thus, for every $s \in [0, T]$,

$$x(s) = x(0) + \int_0^T DF(x(\tau), t) + \int_0^s DF(x(\tau), t) \qquad (3.1.4)$$

which implies

$$\mathcal{M}(x)(s) = x(s), \quad s \in [0, T].$$

Conversely, let $x \in G$ such that

$$x(s) = x(0) + \int_0^T DF(x(\tau), t) + \int_0^s DF(x(\tau), t) \qquad (3.1.5)$$

for every $s \in [0, T]$. Taking $s = 0$ in (3.1.5), we obtain

$$\int_0^T DF(x(\tau), t) = 0. \tag{3.1.6}$$

Taking $s = T$ in (3.1.5) and using (3.1.6), we have $x(T) = x(0)$.

Inserting (3.1.6) in (3.1.5), we obtain,

$$x(s) - x(0) = \int_0^s DF(x(\tau), t)$$

for every $s \in [0, T]$, which implies that x is a T-periodic solution of (3.1.2). $\qquad \square$

In order to use the Leray-Schauder degree, we need to prove that the operator $\mathcal{M}$, given by (3.1.3), is compact on G, that is, $\mathcal{M}$ is continuous on G and $\mathcal{M}$ takes bounded set of G into relatively compact sets of G. The following propositions prove these statements.

Proposition 3.1.3. *Assume that (A1) are satisfied. Then, the operator $\mathcal{M} : G \to G$ defined in (3.1.3) is continuous.*

Proof. Let $x, y \in G$ such that $x(s), y(s) \in B_R$, for some $R > 0$ and for all $s \in [0, T]$. Hence,

$$\|\mathcal{M}(\lambda, y) - \mathcal{M}(\lambda, x)\|_\infty = \sup_{s \in [0,T]} \|\mathcal{M}(\lambda, y)(s) - \mathcal{M}(\lambda, x)(s)\|$$

$$\leq \|y(0) - x(0)\| + \left\| \int_0^T [DF(y(\tau), t) - DF(x(\tau), t)] \right\|$$

$$+ \sup_{s \in [0,T]} \left\{ \left\| \int_0^s [DF(y(\tau), t) - DF(x(\tau), t)] \right\| \right\}. \tag{3.1.7}$$

By Lemma 2.1.9,

$$\left\| \int_0^s D[F(y(\tau), t) - F(x(\tau), t)] \right\| \leq \int_0^s \omega_R(\|y(t) - x(t)\|) dh_R(t). \tag{3.1.8}$$

for every $s \in [0, T]$. Then,

$$\|\mathcal{M}(\lambda, y) - \mathcal{M}(\lambda, x)\|_\infty \leq \|y - x\|_\infty + 2\omega_R(\|y - x\|_\infty) \int_0^T dh_R(t)$$

and the proof is complete. $\qquad \square$

The next result ensures that the operator $\mathcal{M} : BV \to BV$ defined in (3.1.3) maps bounded set of G into relatively compact sets of G.

Proposition 3.1.4. *Assume that* (A1) *are satisfied. Then, the set*

$$\mathcal{A} = \{\mathcal{M}(\lambda, x), \ x \in M\}$$

is relatively compact in G for every $M \subset G$ bounded set.

Proof. We have

$$\|\mathcal{M}(\lambda, x)(0)\| = \left\|x(0) + \int_0^T DF(x(\tau), t)\right\|,$$

for every $x \in M$.

By Lemma 2.1.5, we have

$$\|\mathcal{M}(\lambda, x)(s') - \mathcal{M}(\lambda, x)(s)\| = \left\|\int_s^{s'} DF(\lambda, x(\tau), t)\right\| \leq |h_R(s') - h_R(s)|,$$

for every $s, s' \in [0, T]$ and every $x \in M$. The proof follows by Proposition 2.1.10. $\qquad \square$

3.2   An existence result

Keeping the notations and terminology of the previous section, the next result ensures that the generalized ODE (3.1.1) has at least one T-periodic solution $x \in \Delta \subset G$ open and bounded.

Theorem 3.2.1. *Assume that* (A1) *are valid. Suppose there exists an open and bounded subset $\Delta \subset G$ such that the following statements are valid:*

(i) *For every $\lambda \in (0, 1]$, the equation*

$$\frac{dx}{d\tau} = \lambda DF(x, t) \qquad (3.2.1)$$

   does not admit a T-periodic solution x on G such that $x \in \partial\Delta$.

(ii) *The equation*

$$\psi(a) := F(a, T) - F(a, 0) = 0$$

   does not admit a solution $a \in \partial\Delta \cap \mathbb{R}^n$ (where $\mathbb{R}^n$ is viewed as the set of constant functions in G).

(iii) *$deg(\psi, \Delta \cap \mathbb{R}^n, 0) \neq 0$.*

Then the generalized ODE (3.1.1) has at least one T-periodic solution $x \in \Delta$.

Proof. Let us define the operator $\mathcal{H} : \overline{\Delta} \times [0, 1] \to G$ by

$$\mathcal{H}(x, \lambda)(s) = x(0) + \int_0^T DF(x(\tau), t) + \lambda \int_0^s DF(x(\tau), t) \qquad (3.2.2)$$

for every $\lambda \in [0, 1]$ and $s \in [0, T]$. Take first $\lambda = 1$. By Theorem 3.1.2, the fixed points of the operator $\mathcal{H}$ are the T-periodic solutions of (3.1.1). Thus, by hypothesis (i), $\mathcal{H}(x, 1) \neq x$ for every $x \in \partial \Delta$.

For $\lambda \in (0, 1)$, if x is a fixed point of the operator $\mathcal{H}$, then

$$x(s) = x(0) + \int_0^T DF(x(\tau), t) + \lambda \int_0^s DF(x(\tau), t) \qquad (3.2.3)$$

Taking $s = 0$ in (3.2.3), we obtain

$$\int_0^T DF(x(\tau), t) = 0 \qquad (3.2.4)$$

and taking $s = T$ in (3.2.3) and using (3.2.4), we obtain $x(T) = x(0)$.

On the other hand, by (3.2.4), we have

$$x(T) = x(0) + \lambda \int_0^T DF(x(\tau), t).$$

Therefore x is a T-periodic solution of (3.2.1). Hence, for every $\lambda \in (0, 1)$, the fixed points of $\mathcal{H}(\cdot, \lambda)$ are T-periodic solutions of (3.2.1). Thus, by hypothesis (i), $\mathcal{H}(x, \lambda) \neq x$ for $\lambda \in (0, 1)$, and $x \in \partial \Delta$.

Now, consider the case $\lambda = 0$. If x is a fixed point of $\mathcal{H}$, then

$$x(s) = x(0) + \int_0^T DF(x(\tau), t), \quad s \in [0, T], \qquad (3.2.5)$$

which implies that x is constant, that is, $x(s) = a$ in $[0, T]$. Thus, from (3.2.5), we obtain

$$\int_0^T DF(a, t) = 0,$$

which, by (2.1.5) is equivalent to $F(a, T) - F(a, 0) = 0$.

By hypothesis (ii), it is clear that $\mathcal{H}(x, 0) \neq x$ for every $x \in \partial \Delta$.

Then, combining all the cases above, we conclude that

$$\mathcal{H}(u, \lambda) \neq u, \quad \text{for every pair } (u, \lambda) \in \partial \Delta \times [0, 1]$$

and hence, we have

$$0 \notin I - \mathcal{H}(\cdot, \lambda)(\partial \Delta), \ \lambda \in [0,1].$$

By Propositions 3.1.3 and 3.1.4, it is easy to conclude that the operator $\mathcal{H}$ is a homotopy of compact transformations on $\overline{\Delta}$. Therefore, by Theorem 1.2.5, we conclude

$$deg_{LS}[I - \mathcal{H}(\cdot, 1), \Delta, 0] = deg_{LS}[I - \mathcal{H}(\cdot, 0), \Delta, 0].$$

Then, clearly, $\mathcal{H}(x, \{0\}) \subset \mathbb{R}^n$, for every $x \in \partial \Delta$ and using Definition 1.2.1 and Corollary 1.1.4, we have

$$
\begin{aligned}
deg_{LS}[I - \mathcal{H}(\cdot, 0), \Delta, 0] &= deg((I - \mathcal{H}(\cdot, 0)|_{\mathbb{R}^n}, \Delta \cap \mathbb{R}^n, 0) \\
&= deg[-\psi, \Delta \cap \mathbb{R}^n, 0] \\
&= (-1)^n deg[\psi, \Delta \cap \mathbb{R}^n, 0] \neq 0. \quad (3.2.6)
\end{aligned}
$$

The fact that the last degree in equation (3.2.6) is different of zero follows from hypothesis (iii). Thus, $d_{LS}[I - \mathcal{H}(\cdot, 1), \Delta] \neq 0$. By (ii) of Definition 1.2.3, there exists $x \in \Delta$ such that $\mathcal{H}(x, 1) = x$ and, hence, $x \in \Delta$ is a fixed point of $\mathcal{H}(\cdot, 1)$ and, consequently, by Theorem 3.1.2, x is a T-periodic solution of (3.1.1). $\qquad \square$

3.3   Applications to Impulsive Differential Equations

In this section, our goal is to apply the results from the previous sections to impulsive differential equations (we write IDEs for short). In order to do this, use the correspondence between the solutions of IDEs and the solutions of generalized ODEs. Such correspondences can be found in [31], Chapter 5.

Consider the following impulsive differential equation

$$x'(t) = f(x(t), t), \quad t \neq t_i, \ \Delta x(t_i) := I_i(x(t_i)), \quad i = 1, \ldots, m \quad (3.3.1)$$

where $0 < t_1 < \ldots < t_m \leq T$ are pre-assigned moments of impulse and the functions $I_i : B_R \to \mathbb{R}^n$ are continuous, for $i = 1, \ldots, m$ and

$$\Delta x(t_i) := x(t_i+) - x(t_i-) = x(t_i+) - x(t_i)$$

that is, we assume that x is left continuous at $t = t_i$ and the lateral limit $x(t_i+)$ exists, for $i = 1, \ldots, m$.

The impulsive system (3.3.1) is equivalent to the integral equation

$$x(t) = x(0) + \int_0^t f(x(s), s)ds + \sum_{0 < t_i \le t} I_i(x(t_i)), \quad t \in [0, T] \tag{3.3.2}$$

where the integral exists in the Lebesgue sense.

For $d \in [0, T)$, we define the left continuous Heaviside funtion as follows:

$$H_d(t) = \begin{cases} 0, & t \le d \\ 1, & t > d \end{cases}$$

Then,

$$\sum_{0 < t_i \le t} I_i(x(t_i)) = \sum_{i=1}^m I_i(x(t_i))H_{t_i}(t), \quad t \in [0, T]. \tag{3.3.3}$$

Therefore, equation (3.3.2) is equivalent to

$$x(t) = x(0) + \int_0^t f(x(s), s)ds + \sum_{i=1}^m I_i(x(t_i))H_{t_i}(t), \quad t \in [0, T].$$

Now, we denote by $L^1([0, T], \mathbb{R}^n)$ the space of all Lebesgue integral functions $x : [0, T] \to \mathbb{R}^n$ with finite integral. Let $f : B_R \times [0, T] \to \mathbb{R}^n$ be a function such that the following conditions are satisfied:

(A2) for any $z \in B_R$, $f(z, \cdot) \in L^1([0, T], \mathbb{R}^n)$

(A3) there exists $M_1 \in L^1([0, T], \mathbb{R})$ such that

$$\left\| \int_{t_1}^{t_2} f(z, s) \, ds \right\| \le \int_{t_1}^{t_2} M_1(s) \, ds$$

for all $t_1, t_2 \in [0, T]$ and all $z \in B_R$.

(A4) there exists $N_1 \in L^1([0, T], \mathbb{R})$ such that

$$\left\| \int_{t_1}^{t_2} [f(z, s) - f(y, s)] \, ds \right\| \le \|z - y\| \int_{t_1}^{t_2} N_1(s) \, ds$$

for all $t_1, t_2 \in [0, T]$ and all $z, y \in B_R$.

In what follows, we also assume the following hypotheses:

(A5) There exists $K_1 > 0$ such that $\|I_i(z)\| \le K_1$ for all $z \in B_R$ and $i = 1, \ldots, m$.

(A6) There exists $K_2 > 0$ such that $\|I_i(z) - I_i(y)\| \le K_2\|z - y\|$ for all $z, y \in B_R$ and $i = 1, \ldots, m$.

For each pair (z,t) in $B_R \times [0,T]$, we define

$$F(z,t) := \int_0^t f(z,s)\,ds + \sum_{i=1}^m I_i(z)H_{t_i}(t). \tag{3.3.4}$$

Then, defining $h : [0,T] \to \mathbb{R}$ by

$$h(t) := \int_0^t [M_1(s) + N_1(s)]\,ds + \max\{K_1, K_2\}\sum_{i=1}^m H_{t_i}(t), \quad t \in [0,T]. \tag{3.3.5}$$

Note that h is a nondecreasing and left continuous function. As the calculations in [31, Chapter 5], one can conclude that:

(a) h is nondecreasing and left continuous

(b) $F \in \mathcal{F}(B_R \times [0,T], h)$

(c) $\displaystyle\int_0^t DF(x(\tau),s) = \int_0^t f(x(s),s)\,ds + \sum_{0 < t_i \le t} I_i(x(t_i)), \quad t \in [0,T].$

Under all conditions above, $x : [0,T] \to \mathbb{R}^n$ is a solution of the impulsive differential equation (3.3.1) if and only if it is a solution of the generalized ODE (3.1.1), where F is given by (3.3.4) (for more details, see Theorem 5.20 in [31]).

The next result states that Theorem 3.2.1 of the previous section is satisfied for IDEs.

Theorem 3.3.1. *Suppose that* $(A2) - (A6)$ *are satisfied. Assume there exists an open bounded set* $\Delta \subset G$ *such that the following conditions hold:*

1. For any $\lambda \in (0,1]$, *the impulsive equation*

$$x'(t) = \lambda f(x(t),t), \ t \ne t_i, \quad \Delta x(t_i) = \lambda I_i(x(t_i)) \quad (i = 1,\dots,m) \tag{3.3.6}$$

has no T-periodic solution $x \in G \cap \partial\Delta$.

2. The equation

$$\phi(a) := \int_0^T f(a,s)\,ds + \sum_{0 < t_i \le T} I_i(a) = 0 \tag{3.3.7}$$

has no solution on $\partial\Delta \cap \mathbb{R}^n$.

3. $deg[\phi, \Delta \cap \mathbb{R}^n, 0]$ is different from zero.

Then the IDE (3.3.1) has at least one T-periodic solution in Δ.

Chapter 4

Bifurcation theory for Generalized ODEs

In this chapter we will introduce the concept of a bifurcation point with respect to an arbitrary solution of the generalized ODE

$$\frac{dx}{d\tau} = DF(\lambda, x, t) \tag{4.0.1}$$

where $F : \Lambda_0 \times B \times [0, T] \to \mathbb{R}^n$, and $\Lambda_0 \subset \mathbb{R}$, $B \subset \mathbb{R}^n$ are open sets. We will prove an existence result. We borrow some ideas from [14]. All the results present in this chapter are new and are contained in [11].

4.1 Existence of a bifurcation point

Throughout this chapter $B \subset \mathbb{R}^n$ and $\Lambda_0 \subset \mathbb{R}$ are open sets. Recall that G denotes the space of all regulated funtions $x : [0, T] \to \mathbb{R}^n$ endowed with the supremum norm $\| \cdot \|_\infty$.

Consider the following assumptions:

(B1) For each $\lambda \in \Lambda_0$, the function $F(\lambda, \cdot, \cdot)$ satisfies Definition 2.1.4, that is, $F(\lambda, \cdot, \cdot) \in \mathcal{F}(B \times [0, T], h, \omega)$, for each $\lambda \in \Lambda_0$.

(B2) $x_0 \in G$ is a solution to (4.0.1), for each $\lambda \in \Lambda_0$.

In order to define the concept of a bifurcation point with respect to the solution x_0 of generalized ODE (4.0.1), we need to reformulate equation (4.0.1). To do that, let us assume the following condition:

(B3) There is $\eta > 0$ such that if $x \in G$ and $\|x - x_0\|_\infty < \eta$, then $x(t) \in B$ for all $t \in [0, T]$.

In what follows, we denote by $B(x_0, \eta) \subset G$ the open ball centered in x_0 with radius $\eta > 0$. Now, under the assumptions $(B1) - (B3)$, we can define the operator

$$\Phi : \Lambda_0 \times B(x_0, \eta) \longrightarrow G$$

such that, for each $s \in [0, T]$,

$$\Phi(\lambda, x)(s) = x(0) + \int_0^s DF(\lambda, x(\tau), t) \ \text{ for all } \ \lambda \in \Lambda_0, \ x \in B(x_0, \eta). \tag{4.1.1}$$

By virtue of Lemmas 2.1.5 and 2.1.6, it is easy to see that for any $\lambda \in \Lambda_0$, $\Phi(\lambda, \cdot)$ maps $B(x_0, \eta)$ into G and equation (4.0.1) is equivalent to finding a fixed point of $\Phi(\lambda, \cdot)$, for a given $\lambda \in \Lambda_0$.

Now, we are able to introduce the definition of a bifurcation point with respect to the solution x_0 of equation $\Phi(\lambda, x) = x$, where Φ is given by (4.1.1).

Definition 4.1.1. The couple $(\lambda_0, x_0) \in \Lambda_0 \times B(x_0, \eta)$ is said to be a bifurcation point of the equation $\Phi(\lambda, x) = x$, if every neighborhood of (λ_0, x_0) in $\Lambda_0 \times B(x_0, \eta)$ contains a solution (λ, x) of the equation $\Phi(\lambda, x) = x$ such that $x \neq x_0$.

Our intent is to find conditions ensuring the existence of a bifurcation point with respect to $x_0 \in G$ of the equation $\Phi(\lambda, x) = x$, where Φ is given by (4.1.1). In order to do this, we will use the Leray-Schauder degree theory presented in Chapter 1.

At first, we will prove that $\Phi : \Lambda_0 \times B(x_0, \eta) \to G$ given by (4.1.1) is compact in $\Lambda_0 \times B(x_0, \eta)$, meaning that $\Phi : \Lambda_0 \times B(x_0, \eta) \to G$ is continuous with respect to the pair (λ, x) and $\Phi : \Lambda_0 \times B(x_0, \eta) \to G$ takes bounded sets of $\Lambda_0 \times B(x_0, \eta)$ into relatively compact sets of G.

In what follows, we need the continuity with respect to the pair (λ, x) of the operator $\Phi : \Lambda_0 \times B(x_0, \eta) \to G$, given by (4.1.1). Conditions ensuring this are provided by the next proposition.

Proposition 4.1.2. *Assume that $(B1) - (B3)$ are satisfied. Let Φ be given by (4.1.1). Moreover, assume that*

(B4) *There is a function $g : [0, T] \to \mathbb{R}$ nondecreasing and left continuous and such that for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\|F(\lambda_1, w, t) - F(\lambda_2, v, t) - F(\lambda_1, w, s) + F(\lambda_2, v, s)\| < \varepsilon \, |g(t) - g(s)|$$

for all $w, v \in B$, $t, s \in [0, T]$ and $\lambda_1, \lambda_2 \in \Lambda_0$ fulfilling $|\lambda_1 - \lambda_2| + \|w - v\| < \delta$.

Then the operator Φ is continuous on $\Lambda_0 \times B(x_0, \eta)$.

Proof. Let $\varepsilon > 0$ be given and let $\delta \in (0, \varepsilon)$ be such that assertion (B4) is true. Then, using assumption (B4) and Lemma 2.1.7, we get

$$\|\Phi(\lambda_1, x) - \Phi(\lambda_2, y)\|_\infty \;\; \leq \|x(0) - y(0)\| + \int_0^T \varepsilon \, dg$$
$$< \varepsilon \left[1 + g(T) - g(0) \right]$$

for $x, y \in B(x_0, \eta)$ and $\lambda_1, \lambda_2 \in \Lambda_0$ such that

$$|\lambda_1 - \lambda_2| + \|x - y\|_\infty < \delta,$$

wherefrom our statement follows. $\qquad \square$

In the next lines, we will prove that the operator $\Phi : \Lambda_0 \times B(x_0, \eta) \to G$ is compact in $\Lambda_0 \times B(x_0, \eta)$.

Proposition 4.1.3. *Assume that* $(B1) - (B4)$ *are satisfied and let the operator* Φ *be given by* (4.1.1). *Then* $\Phi(\lambda, \cdot) : B(x_0, \eta) \to G$ *is compact for each* $\lambda \in \Lambda_0$. *Moreover* $\Phi : \Lambda_0 \times B(x_0, \eta) \to G$ *is compact, as well.*

Proof. Using Lemma 2.1.5 and taking into account that the function h in assumption (B1) does not depend on $\lambda \in \Lambda_0$, we get that the estimate

$$\|\Phi(\lambda, x)(s') - \Phi(\lambda, x)(s)\| = \left\| \int_s^{s'} DF(\lambda, x(\tau), t) \right\| \leq |h(s') - h(s)| \qquad (4.1.2)$$

is true for every $s, s' \in [0, T]$, $x \in B(x_0, \eta)$ and every $\lambda \in \Lambda_0$. By Proposition 2.1.10 this means that the set $\{\Phi(\lambda, x); \lambda \in \Lambda_0, x \in M\}$ is relatively compact for every subset M of $B(x_0, \eta)$ in G. Using this fact and Proposition 4.1.2, the proof is complete. $\qquad \square$

The next result follows the ideas of Theorem 26.5 from [2]. Here it is presented in the framework of generalized ODEs. However, while in [2] the author employs the Brower degree theory, here we employ the corresponding theory for infinite dimensional spaces which is the Leray-Schauder degree theory. In this manner, we obtain the desired results for generalized ODE whose solutions lie in the space of regulated functions G.

Theorem 4.1.4. *Assume that* $(B1)-(B4)$ *are satisfied. Furthermore, let there be* $\lambda_0 \in \Lambda_0$ *and* $\gamma > 0$ *such that* $[\lambda_0 - \gamma, \lambda_0 + \gamma] \subset \Lambda_0$ *and the following statements are true*

(B5) *For every* $\lambda \in [\lambda_0 - \gamma, \lambda_0 + \gamma] \setminus \{\lambda_0\}$ *there are neighborhoods* $J_\lambda \subset \Lambda_0$ *of* λ *in* $\mathbb{R}$ *and* $U_\lambda \subset B(x_0, \eta)$ *of* x_0 *in* G *such that, if* $\lambda \in J_\lambda$, *then* x_0 *is the only solution of* (4.0.1) *in* U_λ.

Finally, let the operator Φ *is given by* (4.1.1) *and let*

$$i_0(I - \Phi(\lambda_0 - \gamma, \cdot), x_0) \neq i_0(I - \Phi(\lambda_0 + \gamma, \cdot), x_0). \qquad (4.1.3)$$

Then (λ_0, x_0) *is a bifurcation point of equation* (4.0.1).

Proof. Suppose that (λ_0, x_0) is not a bifurcation point of (4.0.1). Then, obviously, (λ_0, x_0) is not a bifurcation point of the equation $\Phi(\lambda, x) = x$, as well. Thus, combining this fact together with hypothesis (B5), we deduce that for each $\lambda \in \overline{J} = [\lambda_0 - \gamma, \lambda_0 + \gamma]$ there are a neighborhood $J_\lambda \subset \Lambda_0$ of λ and a neighborhood $U_\lambda \subset B(x_0, \eta)$ of x_0 in G such that

$$\Phi(\lambda, x) \neq x \quad \text{for every} \ \ (\lambda, x) \in J_\lambda \times (\overline{U}_\lambda \setminus \{x_0\}). \qquad (4.1.4)$$

Since $\overline{J}$ is compact, we can find a finite number of points λ_i in $\overline{J}$ such that $\bigcup_i J_{\lambda_i} = \overline{J}$. Set

$$U = \bigcap_{j=1}^{m} U_{\lambda_j}.$$

Then $\Phi(\lambda, x) \neq x$ for all $(\lambda, x) \in \overline{J} \times (\overline{U} \setminus \{x_0\})$, which implies that x_0 is an isolated fixed point of $\Phi(\lambda, \cdot)$ for each $\lambda \in \overline{J}$ and, in particular,

$$\Phi(\lambda, x) \neq x \quad \text{for every pair} \ (\lambda, x) \in \overline{J} \times \partial U,$$

By Proposition 4.1.3, Φ is compact with respect to (λ, x) in $\Lambda_0 \times B(x_0, \eta)$. Consequently, by the homotopy property of the Leray-Schauder degree (see (iii) in Definition 1.2.3) and Definition 1.2.6, we have

$$i_0(I - \Phi(\lambda_0 - \gamma, \cdot), x_0) = \deg_{LS}(I - \Phi(\lambda_0 - \gamma, \cdot), U) = \deg_{LS}(I - \Phi(\lambda_0 + \gamma, \cdot), U)$$
$$= i_0(I - \Phi(\lambda_0 + \gamma, \cdot), x_0),$$

which contradicts assumption (4.1.3) and this completes the proof. $\square$ $\hfill\square$

4.2   Applications to Measure Differential Equations

In this section, our goal is to apply the results from the previous sections to measure differential equations (we write MDEs for short). In order to do this, we will use the correspondence between the solutions of a measure differential equations and the solutions of a generalized ODE. Such result can be found in [31].

Keeping the notations from previous section, consider functions $f : B \times [0, T] \to \mathbb{R}^n$ and $u : [0, T] \to \mathbb{R}$. It is known from the literature ([31], Chapter 5) that under some

assumptions, the MDE

$$Dx = f(x,t)Du$$

where Dx and Du are distributional derivatives of the functions x and u in the sense of distributions of L. Schwartz, is equivalent to the integral form

$$x(t) = x(0) + \int_0^t f(x,s)du(s), \quad t \in [0,T]$$

and it can be regarded as a generalized ODE whose right-hand side is given by

$$F(x,t) = \int_0^t f(x,s)du(s).$$

We are interested in applying our main results from the previous sections to MDEs whose integral form is

$$x(t) = x(0) + \int_0^t f(\lambda,x,s)du(s), \quad t \in [0,T] \tag{4.2.1}$$

where $u : [0,T] \to \mathbb{R}$ is a nondecreasing and left continuous function and instead of considering the function f defined on $B \times [0,T]$, we consider f defined on $\Lambda_0 \times B \times [0,T]$.

Let $f : \Lambda_0 \times B \times [0,T] \to \mathbb{R}^n$ be a function such that the following conditions are satisfied:

(B6) for every $z \in B$ and $\lambda \in \Lambda_0$, the integral $\int_0^T f(\lambda,z,s)du(s)$ exists.

(B7) There exists a function $M_1 : [0,T] \to \mathbb{R}$ such that $\int_0^T M_1(s)ds < \infty$ and, moreover,

$$\left\| \int_{t_1}^{t_2} f(\lambda,z,s)du(s) \right\| \leq \int_{t_1}^{t_2} M_1(s)\,du(s)$$

for all $\lambda \in \Lambda_0$, $t_1, t_2 \in [0,T]$ and all $z \in B$.

(B8) There exists a function $L_1 : [0,T] \to \mathbb{R}$ such that $\int_0^T L_1(s)ds < \infty$ and, moreover,

$$\left\| \int_{t_1}^{t_2} [f(\lambda,z,s) - f(\lambda,w,s)]\,du(s) \right\| \leq \|z - w\| \int_{t_1}^{t_2} L_1(s)\,du(s)$$

for all $\lambda \in \Lambda_0$, $t_1, t_2 \in [0,T]$ and all $z, w \in B$.

For each $(\lambda,z,t) \in \Lambda_0 \times B \times [0,T]$, we define

$$F(\lambda,z,t) = \int_0^t f(\lambda,z,s)du(s). \tag{4.2.2}$$

It is easy to check that $F : \Lambda_0 \times B \times [0, T] \to \mathbb{R}^n$ satisfies Definition 2.1.4. Indeed, consider the function

$$h(t) = \int_0^t M_1(s) du(s) + \int_0^t L_1(s) du(s), \quad t \in [0, T].$$

Then, for each $\lambda \in \Lambda_0$, we obtain $F(\lambda, \cdot, \cdot) \in \mathcal{F}(B \times [0, T], h)$. Moreover, according to Proposition 5.12 in [31], we have

$$\int_0^t DF(\lambda, x(\tau), s) = \int_0^t f(\lambda, x(s), s) du(s), \quad \text{for all } t \in [0, T] \text{ and } x \in G.$$

Under the assumptions above, we also assume that

(B9) $x_0 \in G$ is a solution of equation (4.2.1), for each $\lambda \in \Lambda_0$.

Let us assume hypothesis $(B3)$. Now, we are able to define the operator

$$\phi(\lambda, x)(t) = x(0) + \int_0^t f(\lambda, x(s), s) du(s) \tag{4.2.3}$$

for every $\lambda \in \Lambda_0$, $x \in B(x_0, \eta)$ and $t \in [0, T]$.

The next result ensures, under some conditions, the existence of a bifurcation point of equation (4.2.1). This is equivalent to finding the existence of a bifurcation point of $\phi(\lambda, x) = x$, where the operator ϕ is given by (4.2.3).

Theorem 4.2.1. *Assume that $(B3)$, $(B6) - (B9)$ are satisfied and ϕ is given by (4.2.3). Furthermore, let there be $\lambda_0 \in \Lambda_0$ and $\gamma > 0$ such that $[\lambda_0 - \gamma, \lambda_0 + \gamma] \subset \Lambda_0$ and the following statements hold*

  (i) *For every $\lambda \in [\lambda_0 - \gamma, \lambda_0 + \gamma] \setminus \{\lambda_0\}$ there are neighborhoods $J_\lambda \subset \Lambda_0$ of λ in $\mathbb{R}$ and $U_\lambda \subset B(x_0, \eta)$ of x_0 in G such that, if $\lambda \in J_\lambda$, then x_0 is the only solution of (4.2.1) in U_λ.*

  (ii) *For every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\left\| \int_s^t [f(\lambda_1, w, r) - f(\lambda_2, v, r)] du(r) \right\| \leq \varepsilon |u(t) - u(s)|$$

  for all $w, v \in B$, $t, s \in [0, T]$ and $\lambda_1, \lambda_2 \in \Lambda_0$ fulfilling $|\lambda_1 - \lambda_2| + \|w - v\| < \delta$.

Finally, let

$$i_0(I - \phi(\lambda_0 - \gamma, \cdot), 0) \neq i_0(I - \phi(\lambda_0 + \gamma, \cdot), 0).$$

Then $(\lambda_0, 0)$ is a bifurcation point of equation (4.2.1).

Proof. It is sufficient to prove that assumption (B4) of Proposition 4.1.2 is satisfied, for F given by (4.2.2). By (ii), given $\varepsilon > 0$, there is $\delta \in (0, \varepsilon)$ such that

$$\|F(\lambda_1, w, t) - F(\lambda_2, v, t) - F(\lambda_1, w, s) + F(\lambda_2, v, s)\| =$$

$$= \left\| \int_s^t [f(\lambda_1, w, r) - f(\lambda_2, v, r)] du(r) \right\| \leq \varepsilon |u(t) - u(s)|$$

wherever $|\lambda_1 - \lambda_2| + \|w - v\| < \delta$. Then, applying Theorem 4.1.4, the proof is complete. $\square$

Chapter 5

Differentiability for Generalized ODEs

The results present in this chapter are new and are contained in [11].

5.1 Necessary conditions

As in the previous chapters, G denotes the set of all regulated functions from $[0, T]$ to $\mathbb{R}^n$. Recall also that as in Chapter 4, $B(x_0, \eta) \subset G$ denotes the open ball with center in x_0 and radius $\eta > 0$ and $\Lambda_0 \subset \mathbb{R}$ is an open set. Now, the set $B \subset \mathbb{R}^n$ is open and convex.

Throughout this chapter, we assume that assumptions $(B1) - (B3)$ from Chapter 4 are satisfied. We are interested in the derivative with respect to the second variable of the operator

$$\Phi(\lambda, x)(s) = x(0) + \int_0^s DF(\lambda, x(\tau), t) \tag{5.1.1}$$

for all $\lambda \in \Lambda_0$, $x \in B(x_0, \eta)$, $s \in [0, T]$ and $F : \Lambda_0 \times B \times [0, T] \to \mathbb{R}^n$.

For convenience, we will assume that the following conditions are satisifed:

(C1) For every fixed pair $(\lambda, t) \in \Lambda_0 \times [0, T]$, the function $y \longmapsto F(\lambda, y, t)$ is differentiable on B.

(C2) There is a nondecreasing left continuous function $\widetilde{h} : [0, T] \to \mathbb{R}$ such that

$$\|F_y'(\lambda, y, t) - F_y'(\lambda, y, s)\| \leq |\widetilde{h}(t) - \widetilde{h}(s)| \quad \text{for all} \ \ t, s \in [0, T] \ \ \text{and} \ \ \lambda \in \Lambda_0$$

where $F_y'(\lambda, y, t)$ denotes the derivative of F with respect to its second variable.

(C3) There is a continuous increasing function $\widetilde{\omega} : [0, +\infty) \to \mathbb{R}$ such that $\widetilde{\omega}(0) = 0$ and

$$\|F_y'(\lambda, y, s) - F_y'(\lambda, y, t) - F_y'(\lambda, w, s) + F_y'(\lambda, w, t)\| \leq \widetilde{\omega}(\|y - w\|) \, |\widetilde{h}(t) - \widetilde{h}(s)|$$

for all $y, w \in B$, $t, s \in [0, T]$ and $\lambda \in \Lambda_0$.

Proposition 5.1.1. *Assume that the conditions* $(B1) - (B3)$, $(C1) - (C3)$ *are satisfied. Then, for each* $\lambda_0 \in \Lambda_0$ *and each* $x \in B(x_0, \eta)$ *the derivative* $\Phi'_x(\lambda_0, x)$ *of* $\Phi(\lambda_0, \cdot)$ *at* x *is given by*

$$\begin{cases} \left(\Phi'_x(\lambda_0, x) \, z \right)(r) = z(0) + \int_0^r D[F'_x(\lambda_0, x(\tau), t) \, z(\tau)] \\ \\ \hspace{4cm} \textit{for } z \in G \ \ \textit{and} \ \ r \in [0, T]. \end{cases} \tag{5.1.2}$$

Proof. Let $x \in B(x_0, \eta)$ and $\lambda_0 \in \Lambda_0$ be given and let

$$\left(\Psi(\lambda_0, x) \, z \right)(r) = z(0) + \int_0^r D[F'_x(\lambda_0, x(\tau), t) \, z(\tau)] \ \ \text{for } z \in G \text{ and } r \in [0, T].$$

Obviously, the operator $\Psi(\lambda_0, x) : G \to G$ is linear and bounded. In fact, for every $z \in G$, we have

$$\|\Psi(\lambda, x) \, z\|_\infty = \sup_{r \in [0,T]} \|\Psi(\lambda, x) z(r)\| = \sup_{r \in [0,T]} \left\| z(0) + \int_0^r D[F'_x(\lambda, x(\tau), t) \, z(\tau)] \right\|$$

$$\leq \|z(0)\| + \sup_{r \in [0,T]} \int_0^r \|z(\tau)\| \, d\widetilde{h}(\tau) \leq [1 + (\widetilde{h}(T) - \widetilde{h}(0))] \, \|z\|_\infty,$$

where the last inequality follows from Lemma 2.1.7 and (C2).

It remains to show that the relation

$$\lim_{\|z\| \to 0} \frac{\|\Phi(\lambda_0, x + z) - \Phi(\lambda_0, x) - \Psi(\lambda_0, x) \, z\|_\infty}{\|z\|_\infty} = 0$$

is true, as well. Thus, let $z \in B(x_0, \eta)$ be given. Notice that for every $r \in [0, T]$ we have

$$\frac{\Phi(\lambda_0, x + z)(r) - \Phi(\lambda_0, x)(r) - (\Psi(\lambda_0, x) \, z)(r)}{\|z\|_\infty}$$

$$= \frac{1}{\|z\|_\infty} \int_0^r D[F(\lambda_0, x(\tau) + z(\tau), t) - F(\lambda_0, x(\tau), t) - F'_x(\lambda_0, x(\tau), t) \, z(\tau)],$$

i.e.

$$\begin{cases} \dfrac{\Phi(\lambda_0, x+z)(r) - \Phi(\lambda_0, x)(r) - (\Psi(\lambda_0, x) \, z)(r)}{\|z\|_\infty} \\ \\ \hspace{2cm} = \int_0^r DU(\tau, t), \end{cases}$$

where

$$U(\tau, t) = \frac{F(\lambda_0, x(\tau) + z(\tau), t) - F(\lambda_0, x(\tau), t) - F'_x(\lambda_0, x(\tau), t) \, z(\tau)}{\|z\|_\infty}$$

for $\tau, t \in [0, T]$. Furthermore, using the fact that B is convex, we can apply the Mean

Value Theorem for vector-valued functions (see e.g. [21], Lemma 8.11) to rearrange the difference

$$
\left\{
\begin{aligned}
&U(\tau,t) - U(\tau,s) \\
&= \left(\int_0^1 F_x'(\lambda_0, \theta\,(x(\tau)+z(\tau)) + (1-\theta)\,x(\tau), t)\, d\theta \right. \\
&\quad - \int_0^1 F_x'(\lambda_0, \theta\,(x(\tau)+z(\tau)) + (1-\theta)\,x(\tau), s)\, d\theta \\
&\quad \left. - \int_0^1 F_x'(\lambda_0, x(\tau), t)\, d\theta - \int_0^1 F_x'(\lambda_0, x(\tau), s)\, d\theta \right) \frac{z(\tau)}{\|z\|_\infty} \\
&\text{for}\ \ t, s, \tau \in [0, T].
\end{aligned}
\right.
\tag{5.1.3}
$$

Using assumption (C3) we obtain

$$
\left\{
\begin{aligned}
&\big\| F_x'(\lambda_0, \theta\,(x(\tau)+z(\tau)) + (1-\theta)\,x(\tau), t) \\
&\qquad - F_x'(\lambda_0, \theta\,(x(\tau)+z(\tau)) + (1-\theta)\,x(\tau), s) \\
&\qquad - F_x'(\lambda_0, x(\tau), t) + F_x'(\lambda_0, x(\tau), s) \big\| \\
&\quad \le \omega(\|z\|_\infty)\, |\widetilde{h}(t) - \widetilde{h}(s)| \\
&\text{for}\ \ \theta \in [0,1]\ \ \text{and}\ t, s, \tau \in [0, T].
\end{aligned}
\right.
\tag{5.1.4}
$$

and, inserting (5.1.4) into (5.1.3), we conclude that the inequality

$$
\|U(\tau,t) - U(\tau,s)\| \le \omega(\|z\|_\infty)\, |\widetilde{h}(t) - \widetilde{h}(s)|
$$

is true for all $t, s, \tau \in [0, t]$. Now, we can use Lemma 2.1.7 to deduce the inequality

$$
\sup_{r \in [0,T]} \left\| \int_0^r DU(\tau,t) \right\| \le \int_0^T \omega(\|z\|_\infty)\, d\widetilde{h}
$$

$$
= \omega(\|z\|_\infty)\, [\widetilde{h}(T) - \widetilde{h}(0)],
$$

wherefrom, letting $\|z\|_\infty \to 0$, our statement follows immediately. $\qquad\square$

5.2   Applications to Bifurcation Theory

In order to prove the main result of this section, we will need the continuity of the operator $\Phi_x' : \Lambda_0 \times B(x_0, \eta) \to \mathcal{L}(G)$ with respect to the couple (λ, x), where Φ_x' is given by (5.1.2).

Proposition 5.2.1. *Assume that the conditions* $(B1) - (B3)$, $(C1) - (C3)$ *are satisfied. Moreover, assume:*

(C4) *there exists a nondecreasing, left-continuous function $\widetilde{g} : [0, T] \to \mathbb{R}$ such that for any $\varepsilon > 0$ we can find $\delta > 0$ such that*

$$
\begin{cases}
\|F'_x(\lambda_1, w, t) - F'_x(\lambda_2, v, t) - F'_x(\lambda_1, w, s) + F'_x(\lambda_2, v, s)\| < \varepsilon \left|\widetilde{g}(t) - \widetilde{g}(s)\right| \\[2mm]
\text{for all } w, v \in B, \ t, s \in [0, T] \ \text{ and } \ \lambda_1, \lambda_2 \in \Lambda_0 \\[2mm]
\text{fulfilling } |\lambda_1 - \lambda_2| + \|w - v\| < \delta.
\end{cases} \tag{5.2.1}
$$

Then the mapping $\Phi'_x : \Lambda_0 \times B(x_0, \eta) \to \mathcal{L}(G)$ is continuous.

Proof. Let $\varepsilon > 0$ be given, there is $\delta \in (0, \varepsilon)$ such that (5.2.1) is true. Then, using (5.2.1) and Lemma 2.1.7, we get

$$
\|\Phi'_x(\lambda_1, x) - \Phi'_x(\lambda_2, y)\|_{\mathcal{L}(G)} \leq \int_0^T \varepsilon d\widetilde{g}(s) < \varepsilon \left[\widetilde{g}(T) - \widetilde{g}(0)\right]
$$

for $x, y \in B(x_0, \eta)$ and $\lambda_1, \lambda_2 \in \Lambda_0$ such that

$$
|\lambda_1 - \lambda_2| + \|x - y\|_\infty < \delta,
$$

wherefrom our statement follows. $\qquad\square$

The next theorem is the main result of this section. Its proof follows the ideas similar to those from the proof of Proposition 26.3 em [2].

Theorem 5.2.2. *Suppose that $(B1) - (B4)$ and $(C1) - (C4)$ are satisfied and let $\lambda_0 \in \Lambda_0$ and $x_0 \in B(x_0, \eta)$ be given. Let the operator Φ be defined by (5.1.1) and let $I - \Phi'_x(\lambda_0, x_0)$ be an isomorphism of G onto G. Then (λ_0, x_0) is not a bifurcation point of the equation $\Phi(\lambda, x) = x$.*

Proof. Suppose that $I - \Phi'_x(\lambda_0, x_0)$ is an isomorphism of G onto G, First, notice that, due to (B2), we have

$$
\Phi(\lambda, x_0) = x_0 \quad \text{for all } \ \lambda \in \Lambda_0. \tag{$*$}
$$

Furthermore, by propositions 4.1.2 and 5.2.1, we have $\Phi \in C^1(\Lambda_0 \times B(x_0, \eta))$. Hence by the Implicit Function Theorem (see [9], Theorem 4.2.1) there exist neighborhoods $\mathcal{V} \subset \Lambda_0$ of λ_0, $\mathcal{W} \subset B(x_0, \eta)$ of x_0 and a unique regulated function $z \in \mathcal{W}$ such that

$$
z - \Phi(\lambda, z) = 0 \quad \text{for every } \lambda \in \mathcal{V}.
$$

On the other hand, due to ($*$), we have $\Phi(\lambda, x_0) = x_0$ for every $\lambda \in \mathcal{V}$. Therefore, $z = x_0$ is the unique solution of equation

$$
z = \Phi(\lambda, z) \quad \text{for } \ \lambda \in \mathcal{V}.
$$

This means that (λ_0, x_0) can not be a bifurcation point of the equation $\Phi(\lambda, x) = x$. $\square$

5.3   A Fredholm Alternative

The next result is inspired in [31, Proposition 6.3]. However, in the present section, our approach is through linear generalized ODEs involving the derivative with respect to the second variable of the operator Φ, given by (5.1.2).

Theorem 5.3.1. *Suppose that* $(B1), (B3), (C1)-(C3)$ *are satisfied and* $x \in B(x_0, \eta) \subset G$. *Then,*

(i) *either the equation*

$$z(r) - z(0) - \int_0^r D[F_x'(\lambda_0, x(\tau), t)z(\tau)] = p(r), \quad r \in [0, T] \qquad (5.3.1)$$

has a unique solution in G, *for every* $p \in G$.

(ii) *or the equation*

$$z(r) - z(0) - \int_0^r D[F_x'(\lambda_0, x(\tau), t)z(\tau)] = 0, \quad r \in [0, T] \qquad (5.3.2)$$

has at least one nontrivial solution in G.

Proof. Given $\lambda_0 \in \Lambda_0$ and $x \in B(x_0, \eta)$. Let us recall equation (5.1.2)

$$\left(\Phi_x'(\lambda_0, x)\, z\right)(r) = z(0) + \int_0^r D[F_x'(\lambda_0, x(\tau), t)\, z(\tau)]$$

for every $z \in G$ and $r \in [0, T]$.

We assert that $\Phi_x'(\lambda_0, x)$ is compact on G. In the proof of Theorem 5.1.1, we obtain $\Phi_x'(\lambda_0, x) \in \mathcal{L}(G)$. It remains to show that $\Phi_x'(\lambda_0, x)$ takes bounded sets of G into relatively compact sets of G.

Let $M \subset G$ be bounded. If $z \in M$, then there is $c > 0$ such that $\|z\|_\infty \leq c$. By $(C2)$ and Lemma 2.1.7, we obtain

$$\|\Phi_x'(\lambda_0, x)(z)(r') - \Phi_x'(\lambda_0, x)(z)(r)\| = \left\|\int_r^{r'} D[F_x'(\lambda_0, x(\tau), t)\, z(\tau)]\right\|$$

$$\leq \int_r^{r'} \|z(\tau)\|\, d\widetilde{h}(\tau) \leq c\,|\widetilde{h}(r') - \widetilde{h}(r)|$$

for every $r, r' \in [0, T]$ and $z \in M$. By Proposition 2.1.10, we have $\{\Phi'_x(\lambda_0, x)(z), \ z \in M\}$ is relatively compact, for every bounded $M \subset G$. This completes our statement.

Using the Fredholm Alternative for Banach spaces (see [33], Theorem 4.12), we have either $\mathcal{R}(I - \Phi'_x(\lambda_0, x)) = G$ and $\mathcal{K}(I - \Phi'_x(\lambda_0, x)) = \{0\}$ or $\mathcal{R}(I - \Phi'_x(\lambda_0, x)) \neq G$ and $\mathcal{K}(I - \Phi'_x(\lambda_0, x)) \neq \{0\}$. These facts imply our assertion and the proof is complete. $\square$

Remark 5.3.2. If (i) occurs in Theorem 5.3.1, it means that the generalized linear differential equation

$$z(r) = z(0) + \int_0^r D[F'_x(\lambda_0, x(\tau), t) \, z(\tau)] \quad \text{for} \ \ r \in [0, T] \tag{5.3.3}$$

has only the trivial solution on G. Therefore, $(\lambda_0, 0)$ is not a bifurcation point of equation (5.3.3).

5.4   Applications to Measure Differential Equations

Keeping the notations from previous section, consider functions $f : B \times [0, T] \to \mathbb{R}^n$ and $u : [0, T] \to \mathbb{R}$. It is known from the literature ([31], Chapter 5) that under some assumptions, the MDE

$$Dx = f(x, t) Du$$

where Dx and Du are distributional derivatives of the functions x and u in the sense of distributions of L. Schwartz, is equivalent to the integral form

$$x(t) = x(0) + \int_0^t f(x, s) du(s), \quad t \in [0, T]$$

and it can be regarded as a generalized ODE whose right-hand side is given by

$$F(x, t) = \int_0^t f(x, s) du(s).$$

In what follows, we want to prove that Theorem 5.1.1 and Theorem 5.3.1 of the previous section holds for MDEs with integral form

$$x(t) = x(0) + \int_0^t f(\lambda, x, s) du(s), \quad t \in [0, T]. \tag{5.4.1}$$

where $f : \Lambda_0 \times B \times [0, T] \to \mathbb{R}^n$ and $u : [0, T] \to \mathbb{R}$ is a nondecreasing and left continuous function.

We need to assume that:

(C5) For each fixed pair $(\lambda, t) \in \Lambda_0 \times [0, T]$, the function $y \mapsto f(\lambda, y, t)$ is differentiable

on B.

(C6) The derivative of $f : \Lambda_0 \times B \times [0, T] \to \mathbb{R}^n$ with respect to $y \in B$, denoted by f_y, is continuous on B.

In the sequel, we calculate the derivative of F with respect to its second variable, where F is given by

$$F(\lambda, y, t) = \int_0^t f(\lambda, y, s) du(s).$$

for every $(\lambda, y, t) \in \Lambda_0 \times B \times [0, T]$.

Therefore, we obtain

$$F'_y(\lambda, y, t) = \int_0^t f_y(\lambda, y, s) du(s), \tag{5.4.2}$$

where $f_y(\lambda, y, s) = \dfrac{\partial f}{\partial y}(\lambda, y, s)$ denotes the derivative of function f with respect to its second variable.

In what follows, we need to ensure that F'_y, given by (5.4.2), satisfies Definition 2.1.4. In order to obtain that, we will assume that the following conditions are satisfied:

(C7) There is a function $\widetilde{M_1} : [0, T] \to \mathbb{R}$ such that $\displaystyle\int_0^T \widetilde{M_1}(s) du(s) < \infty$ and

$$\left\| \int_{t_1}^{t_2} f_y(\lambda, y, s) du(s) \right\| \leq \int_{t_1}^{t_2} \widetilde{M_1}(s) \, du(s)$$

holds, for all $t_1, t_2 \in [0, T]$, $y \in B$ and $\lambda \in \Lambda_0$.

(C8) There is a function $\widetilde{L_1} : [0, T] \to \mathbb{R}$ such that $\displaystyle\int_0^T \widetilde{L_1}(s) du(s) < \infty$ and

$$\left\| \int_{t_1}^{t_2} [f_y(\lambda, y, s) - f_y(\lambda, w, s)] \, du(s) \right\| \leq \|y - w\| \int_{t_1}^{t_2} \widetilde{L_1}(s) \, du(s)$$

holds, for all $t_1, t_2 \in [0, T]$, $y, w \in B$ and $\lambda \in \Lambda_0$.

Analogously, we consider

$$\widetilde{h_1}(t) = \int_0^t \widetilde{M_1}(s) du(s) + \int_0^t \widetilde{L_1}(s) du(s), \quad t \in [0, T].$$

Therefore, $F'_y(\lambda, \cdot, \cdot) \in \mathcal{F}(B \times [0, T], \widetilde{h_1})$, for each $\lambda \in \Lambda_0$. Moreover, by Lemma 5.1 (item

2) in [36], we have

$$\int_0^t D[F_x'(\lambda, x(\tau), s)z(\tau)] = \int_0^t f_x(\lambda, x(\tau), \tau)z(\tau)du(\tau),$$

for all $\lambda \in \Lambda_0$, $t \in [0, T]$ and $x, z \in G$.

The next result characterizes the derivative of the operator ϕ, given by (4.2.3), with respect to its second variable.

Proposition 5.4.1. *Suppose that conditions* $(B3)$, $(B6)-(B8)$, $(C5)-(C8)$ *are satisfied. Then, for each* $\lambda_0 \in \Lambda_0$ *and for each* $x \in B(x_0, \eta)$, *the derivative* $\phi_x'(\lambda_0, x)$ *of* $\phi(\lambda, \cdot)$ *at* x *is given by*

$$\phi_x'(\lambda_0, x)z(t) = z(0) + \int_0^t f_x(\lambda_0, x(\tau), \tau)z(\tau)du(\tau), \ \text{for all } z \in G.$$

Theorem 5.4.2. *Suppose that conditions* $(B3)$, $(B6) - (B8)$, $(C5) - (C8)$ *are satisfied. Let* $x \in B(x_0, \eta)$. *Then, either*

(i) *the equation*

$$z(t) - z(0) - \int_0^t f_x(\lambda_0, x(\tau), \tau)z(\tau)du(\tau) = p(t), \quad t \in [0, T]$$

has a unique solution in G, *for every* $p \in G$.
or

(ii) *the equation*

$$z(t) - z(0) - \int_0^t f_x(\lambda_0, x(\tau), \tau)z(\tau)du(\tau) = 0, \quad t \in [0, T]$$

has at least one nontrivial solution in G.

Remark 5.4.3. Owing to the fact that measure differential equations encompass differential equation with impulses (see [16, Theorem 3.1]) and also dynamic equations on time scales (see [17, Theorem 4.3]), our results apply to these types of equations as well.

Chapter 6

Periodic solutions of linear generalized ODEs

In this chapter, we present a result that ensures the existence of a periodic solution of linear generalized ODEs. It is well-known from the literature that there exists a correspondence between MDE and generalized ODEs (for more details, see [31]). Then we can apply our results to MDEs. These results are contained in [13].

6.1 An existence theorem

Throughout this section, let us consider $\mathbb{R}^n$ equiped with the usual norm $\| \cdot \|$. Recall that G is the space of regulated functions from $[0, T]$ to $\mathbb{R}^n$, with the supremum norm.

Denote by $\mathcal{L}(\mathbb{R}^n)$ the space of linear bounded operators $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, endowed with the norm

$$\|S\|_{\mathcal{L}(\mathbb{R}^n)} = \sup_{\|x\| \leq 1} \|Sx\|.$$

In this chapter, we want to prove a result that ensures the existence of a T-periodic solution of a linear generalized ODE given by

$$\frac{dx}{d\tau} = D[A(t)x] \tag{6.1.1}$$

where $A \in BV([0, T], \mathcal{L}(\mathbb{R}^n))$ and $x : [0, T] \longrightarrow \mathbb{R}^n$ is regulated. Notice that Equation (6.1.1) is equivalent to the integral form

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} D[A(s)]x(s)$$

for every $t_1, t_2 \in [0, T]$.

Definition 6.1.1. Let $T > 0$ be fixed. A function $x : [0, \infty) \to \mathbb{R}^n$ is said to be a

T-periodic solution of (6.1.1), if x is a solution of (6.1.1) and, moreover,

$$x(t) = x(t + T), \quad \text{for all } t \in [0, +\infty).$$

In the next lines, assume that there is a constant $d > 0$ fulfilling the following conditions:

(D1) For every $\varphi \in G$ satisfying $\|\varphi(0)\| > d$, we have

$$[A(T) - A(0)]\varphi(0) \neq 0.$$

(D2) If $z \in \mathbb{R}^n$ such that $A(t + T)z = A(t)z$, for all $t \in [0, +\infty)$, then $\|z\| \leq d$.

Lemma 6.1.2. *Assume that (D2) is valid. The existence of a T–periodic solution $x :$ $[0, +\infty) \to \mathbb{R}^n$ of (6.1.1) is equivalent to the existence of a solution of the boundary value problem*

$$\begin{cases} \dfrac{dx}{d\tau} = D[A(t)x] \\ \quad x(0) = x(T). \end{cases} \tag{6.1.2}$$

Proof. Assume that $x : [0, +\infty) \to \mathbb{R}^n$ is a T-periodic solutions of (6.1.1). Then

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} d[A(s)](x(s), \qquad t_2, t_1 \in [0, +\infty) \tag{6.1.3}$$

and

$$x(t + T) = x(t), \qquad t \in [0, +\infty). \tag{6.1.4}$$

Define the function $z : [0, T] \to \mathbb{R}^n$ given by

$$z(t) := x(t + T), \qquad t \in [0, T]. \tag{6.1.5}$$

We assert that z is a solution of the boundary value problem (6.1.2). Indeed, given $t_2, t_1 \in [0, T]$, we have

$$z(t_2) - z(t_1) = x(t_2 + T) - x(t_1 + T) = \int_{t_1+T}^{t_2+T} d[A(s)]x(s).$$

$$= \int_{\phi(t_1)}^{\phi(t_2)} d[A(s)]x(s) = \int_{t_1}^{t_2} d[A(s+T)]x(s+T) = \int_{t_1}^{t_2} d[A(s)]x(s+T) = \int_{t_1}^{t_2} d[A(s)]z(s)$$

where the last inequality follows from Lemma 2.1.12, taking $\phi(\xi) := \xi + T$ and by (D2).

Also, by (6.1.4) and (6.1.5), we have

$$z(T) = x(T + T) = x(T) = z(0).$$

Thus, z is a solution of the boundary value problem (6.1.2).

On the other hand, assume that there exists a solution $u : [0, T] \to \mathbb{R}^n$ of the boundary value problem (6.1.2). Then, u is a solution of (6.1.1) and $u(T) = u(0)$.

By Theorem 2.1.13, there exists a unique (maximal) solution $y : [0, +\infty) \to \mathbb{R}^n$ of (6.1.1) with $x(0) = u(T)$. Then, by the uniqueness $y|_{[0,T]} = u$, that is, y is a extension of u.

We assert that $y(t + T) = y(t)$, for all $t \in [0, +\infty)$.

Indeed, using the same arguments as above, we can prove that the function $\phi(t) := y(t + T)$ is a solution of (6.1.1) with $x(0) = u(T) = u(0)$.

Now, since $\phi(t)$ and $y(t)$ are solutions of (6.1.1) with $x(0) = u(T) = u(0)$, the uniqueness of a solution yields

$$\phi(t) = y(t), \text{ that is, } y(t + T) = y(t)$$

for all $t \in [0, +\infty)$ and this completes the proof. $\square$

In what follows, due to Lemma 6.1.2, it is possible to change our problem to find a solution of the boundary value problem (6.1.2). Now, let us introduce the operators:

$$L : G \longrightarrow G$$

$$x \longmapsto L(x)$$

given by
$$L(x)(t) = x(t) - x(0), \quad \text{for every } t \in [0, T]. \tag{6.1.6}$$

Consider also the operator
$$N : G \longrightarrow G$$

$$x \longmapsto N(x)$$

given by

$$N(x)(t) = \int_0^T D[A(s)]x(s) + \int_0^t D[A(s)]x(s), \quad \text{for every } t \in [0, T]. \tag{6.1.7}$$

We want to prove that there exists at least one solution $x \in G$ of the linear equation

$$L(x) = N(x) \tag{6.1.8}$$

that is,

$$x(t) = x(0) + \int_0^T D[A(s)]x(s) + \int_0^t D[A(s)]x(s) \tag{6.1.9}$$

for every $t \in [0, T]$. In order to do this, let us recall some results. The next result can be found in [13].

Proposition 6.1.3. *Suppose x is a solution of equation* (6.1.8). *Then x is a solution of the boundary value problem* (6.1.2).

Proof. Let $x : [0, T] \longrightarrow \mathbb{R}^n$ be a solution of (6.1.8). Then

$$x(t) = x(0) + \int_0^T d[A(s)]x(s) + \int_0^t d[A(s)]x(s) \tag{6.1.10}$$

for every $t \in [0, T]$. Taking $t = 0$ in (6.1.10), we have

$$\int_0^T d[A(s)]x(s) = 0. \tag{6.1.11}$$

Inserting (6.1.11) in (6.1.10), we obtain

$$x(t) = x(0) + \int_0^t d[A(s)]x(s), \text{ for all } t \in [0, T]. \tag{6.1.12}$$

Taking $t = T$ in (6.1.10) and using (6.1.11), we conclude that $x(T) = x(0)$.

Therefore x is a solution of the boundary value problem (6.1.2). $\qquad\square$

In what follows, we need to define the range and the kernel of the operator L, given by (6.1.6). The kernel of the operator L is given by

$$\mathcal{N}(L) = \left\{ x \in G; \ x(s) = x(0), \text{ for every } t \in [0, T] \right\} \tag{6.1.13}$$

and the range of the operator L is given by

$$\mathcal{R}(L) = \left\{ y \in G; \ y(s) = x(s) - x(0), \text{ for every } t \in [0, T] \right\}. \tag{6.1.14}$$

We define the set $\mathcal{A} = \{ y \in G; \ y(0) = 0\}$ and we assert that

$$\mathcal{R}(L) = \mathcal{A}. \tag{6.1.15}$$

Proof. Clearly, $\mathcal{R}(L) \subset \mathcal{A}$. On the other hand, let $y \in \mathcal{A}$. Then $y \in G$ and $y(0) = 0$. Thus, there exists $x = y \in G$ satisfying

$$L(x)(t) = L(y)(t) = y(t) - y(0) = y(t), \quad \text{for all } t \in [0, T]$$

Therefore $y \in \mathcal{R}(L)$, which implies $\mathcal{A} \subset \mathcal{R}(L)$ and our statement follows. $\qquad\square$

The next result is new and can be found in [13]. It assures that L is a Fredholm operator of index 0. Recall that we introduced the concept of the Fredholm operators in Definition 1.3.1.

Proposition 6.1.4. *The operator L, given by* (6.1.6), *is a Fredholm operator of index 0.*

Proof. We can rewrite the operator L, given by (6.1.6), in the form

$$L(x)(s) = x(s) - K(x)(s) \tag{6.1.16}$$

where $K : G \longrightarrow G$ is given by

$$K(x)(s) = x(0), \quad \text{for every } s \in [0, T].$$

Clearly, K is a compact operator in G. By Theorem 1.3.3, we obtain $\dim \mathcal{N}(L) =$ codim $\mathcal{R}(L)$ is finite and $\mathcal{R}(L)$ is closed in G. Therefore L is a Fredholm operator and its index is given by

$$i(L) = \dim \mathcal{N}(L) - \text{codim } \mathcal{R}(L) = 0,$$

from where our statement follows. $\qquad\square$

In order to prove the main theorem of this chapter, we need to find the projections P and Q as we mentioned in Chapter 1. This is the role of the next proposition.

Proposition 6.1.5. *Let L be a Fredholm operator of index zero, where L is given by* (6.1.6). *Then there exist continuous projections $P : G \longrightarrow G$ and $Q : G \longrightarrow G$ such that*

$$\mathcal{R}(L) = \mathcal{N}(Q)$$

and

$$\mathcal{N}(L) = \mathcal{R}(P).$$

Proof. Let us consider the operator

$$Q : G \longrightarrow G \qquad x \longmapsto Q(x)$$

given by

$$Q(x)(t) = x(0), \quad \text{for every } t \in [0, T]. \tag{6.1.17}$$

Notice that Q is idempotent, that is, $Q^2 = Q$ and Q is a bounded linear operator. Moreover, the kernel of the operator Q is given by

$$\mathcal{N}(Q) = \left\{ x \in G; \ Q(x)(t) = 0, \text{ for every } t \in [0, T] \right\}$$

$$= \left\{ x \in G; \ x(0) = 0 \right\}.$$

Thus, by (6.1.15), we conclude that $\mathcal{N}(Q) = \mathcal{R}(L)$.

Since L is a Fredholm operator, $\mathcal{R}(L)$ is closed in G and $\mathcal{N}(L)$ is finite. Notice also that

$$\mathcal{R}(L) \cap \mathcal{N}(L) = \{0\}. \tag{6.1.18}$$

Indeed. Let $z \in \mathcal{R}(L) \cap \mathcal{N}(L)$. Then $z \in \mathcal{R}(L)$, hence $z(0) = 0$ and $L(z)(t) = 0$, for all $t \in [0, T]$. Then

$$0 = L(z)(t) = z(t) - z(0) = z(t), \text{ for all } t \in [0, T],$$

Therefore $z(t) = 0$, for all $t \in [0, T]$ and (6.1.18) is satisfied. Then,

$$\mathcal{R}(L) \oplus \mathcal{N}(L) = G.$$

By Proposition 1.3.4, there exists an operator

$$P : G \longrightarrow G$$

$$x \longmapsto P(x)$$

defined by

$$P(x) = \begin{cases} x, & x \in \mathcal{N}(L) \\ 0, & x \in \mathcal{R}(L) \end{cases} \tag{6.1.19}$$

and $P \in \mathcal{L}(G)$. Clearly P is idempotent and $\mathcal{N}(P) = \mathcal{R}(L)$. Therefore our proof is complete. $\qquad\square$

In what follows, let us assume the following conditions:

(D3) There exists a nondecreasing left-continuous function $h : [0, T] \longrightarrow \mathbb{R}$ satisfying

$$\|A(t) - A(s)\| \leq |h(t) - h(s)|, \text{ for every } t, s \in [0, T]$$

(D4) If x is a solution of equation $L(x) = \lambda N(x)$, for all $\lambda \in (0,1)$ then there exists $D > 0$, satisfying $\|x(t)\| \leq D$, for every $t \in [0,T]$.

Now, let us fix $M > 0$ fulfilling $M > \max\{d, D\}$ and define the set

$$\Delta = \{\ x \in G;\ \ \|x\|_\infty < M\}. \tag{6.1.20}$$

We want to prove that the operator N, given by (6.1.7), is L-compact on $\overline{\Delta}$. This means that we need to ensure the following conditions

- $QN(\overline{\Delta})$ is bounded.

- Let Q, P be the operators defined in (6.1.17) and (6.1.19) respectively. Then

$$(L_p)^{-1}(I - Q)N : \overline{\Delta} \longrightarrow G$$

  is compact, where L_p is the operator L restrict to $\mathcal{N}(P)$, that is, $L_p : \mathcal{N}(P) \longrightarrow L(\mathcal{N}(P))$.

Thus, we need the following two propositions.

Proposition 6.1.6. *Let Q and N be operators given by (6.1.17) and (6.1.7) respectively and $A \in BV([0,T], \mathcal{L}(\mathbb{R}^n))$. Then the set*

$$\mathcal{B} = \{\ QN(x),\ x \in \overline{\Delta}\ \}$$

is bounded in G.

Proof. Suppose $x \in \overline{\Delta}$. Then $\|x\|_\infty \leq M$. Furthermore

$$\|QN(x)\|_\infty = \sup_{t \in [0,T]} \|QN(x)(t)\|$$

$$\leq \sup_{t \in [0,T]} \left\| Q\left(\int_0^T d[A(s)]x(s) + \int_0^t d[A(s)]x(s))\right)\right\|$$

$$= \left\|\int_0^T d[A(s)]x(s)\right\| \leq \underset{[0,T]}{var}(A)\sup_{s \in [0,T]} \|x(s)\| \leq \left[\underset{[0,T]}{var}(A)\right] M,$$

where the last inequality follows from Proposition 2.2.4. Therefore the proof is complete. $\qquad\square$

Proposition 6.1.7. *Let Q, P be the operators defined in (6.1.17) and (6.1.19) respectively. Assume also that (D3) holds. Then*

$$(L_p)^{-1}(I - Q)N : \overline{\Delta} \longrightarrow G$$

is compact, where L_p is the operator L restrict to $\mathcal{N}(P)$, that is, $L_p : \mathcal{N}(P) \longrightarrow \mathcal{R}(L)$.

Proof. Let $x \in \mathcal{N}(P)$ be an arbitrary element. By the definition of P in (6.1.19) and from (6.1.15), we have $x \in G$ and $x(0) = 0$. Thus,

$$L(x)(s) = x(s) - x(0) = x(s), \quad \text{for every } s \in [0, T].$$

Therefore, $L_p \equiv I$, where I is the identity operator in $\mathcal{N}(P)$. We also conclude that $(L_p)^{-1} = I$.

Now, it is sufficient to show that the operator

$$(I - Q)N : \overline{\Delta} \longrightarrow G$$

is compact. Given $x \in \overline{\Delta}$ and $t \in [0, T]$, we have

$$(I - Q)N(x)(t) = (I - Q)\left(\int_0^T d[A(s)]x(s) + \int_0^t d[A(s)]x(s)\right)$$

$$= \left(\int_0^T d[A(s)]x(s) + \int_0^t d[A(s)]x(s)\right) - Q\left(\int_0^T d[A(s)]x(s) + \int_0^t d[A(s)]x(s)\right)$$

$$= \left(\int_0^T d[A(s)]x(s) + \int_0^t d[A(s)]x(s)\right) - \int_0^T d[A(s)]x(s)$$

$$= \int_0^t d[A(s)]x(s).$$

Thus,

$$(I - Q)N(x)(t) = \int_0^t d[A(s)]x(s), \quad \text{for all } t \in [0, T] \text{ and } x \in \overline{\Delta}.$$

It is sufficient to prove that $\{(I - Q)N(x); \quad x \in \overline{\Delta}\}$ is relatively compact. Moreover, by Assumption (D3), we conclude that

$$\|(I - Q)N(x)(t_2) - (I - Q)N(x)(t_1)\| = \left\|\int_{t_1}^{t_2} d[A(s)]x(s)\right\|$$

$$\leq |h(t_2) - h(t_1)|\|x\|_\infty \leq |h(t_2) - h(t_1)|M$$

for every $t_1, t_2 \in [0, T]$ and $x \in \overline{\Delta}$. The result follows from Proposition 2.1.10. $\qquad\square$

Lemma 6.1.8. *Let $\Delta = \{ x \in G; \quad \|x\|_\infty < M\}$. Consider the operators L, N, Q presented in (6.1.6), (6.1.7) and (6.1.17) respectively. Then, the following conditions are satisfied:*

(i) *If $x \in \partial\Delta$, then $Lx \neq \lambda Nx$, for every $\lambda \in (0, 1)$.*

(ii) *If $x \in \mathcal{N}(L) \cap \partial\Delta$, then $QNx \neq 0$.*

(iii) $deg(JQN, \Delta \cap \mathcal{N}(L), 0) \neq 0$, where $JQN : \mathcal{N}(L) \longrightarrow \mathcal{N}(L)$.

Proof. The condition (i) is satisfied, since (D4) holds.

Now, we will prove item (ii). If $x \in \mathcal{N}(L) \cap \partial\Delta$,

$$x(t) = x(0), \quad \text{for all } t \in [0, T]$$

and $\|x\|_\infty = M > 0$. In particular, $\|x(0)\| = M$.

By the definition of the operators Q and N, we have

$$QN(x)(t) = Q\left(\int_0^T d[A(s)]x(s) + \int_0^t d[A(s)]x(s)\right)$$

$$= \int_0^T d[A(s)]x(s),$$

for every $x \in G$ and $t \in [0, T]$. In particular, if $x \in \mathcal{N}(L) \cap \partial\Delta$, we obtain

$$QN(x)(t) = \int_0^T d[A(s)]x(0) = [A(T) - A(0)]x(0) \neq 0, \quad \text{for all } t \in [0, T]. \tag{6.1.21}$$

since (D1) holds and $\|x(0)\| = M$.

Finally, it remains to show that condition (iii) is satisfied.

By the definition of the operator Q, its range is given by

$$\mathcal{R}(Q) = \{y \in G; \quad y(t) = x(0), \text{ for all } t \in [0, T]\}$$

and $\mathcal{R}(Q) = \mathcal{N}(L)$, by (6.1.13). Furthermore $J : \mathcal{R}(Q) \longrightarrow \mathcal{N}(L)$ is an isomorfism, the spaces $\mathcal{R}(Q)$ and $\mathcal{N}(L)$ are finite dimensional and their dimensions are n. Then it is possible to identify them with $\mathbb{R}^n$. Therefore, we can take J as the identity operator from $\mathcal{R}(Q)$ to $\mathcal{N}(L)$.

Consider the operator JQN restrict to $\mathcal{N}(L)$, which denote by JQN:

$$JQN : \mathcal{N}(L) \longrightarrow \mathcal{N}(L)$$

$$x \longmapsto JQN(x)$$

given by

$$JQN(x)(t) = JQN(x)(t) = Q\left(\int_0^T d[A(s)]x(s) + \int_0^t d[A(s)]x(s)\right) = \int_0^T d[A(s)]x(s)$$

for every $t \in [0, T]$.

We need to calculate $deg(JQN, \Delta \cap \mathcal{N}(L), 0)$. Notice that $\mathcal{N}(L)$ is finite dimensional, hence it is sufficient to use the Brower degree theory. Note that $deg(JQN, \Delta \cap \mathcal{N}(L), 0)$ is well defined, that is,

$$0 \notin JQN(\partial(\Delta \cap \mathcal{N}(L))).$$

holds, by the same arguments used in the proof of item (ii).

Let us consider the operator $H : [0,1] \times \left(\overline{\Delta \cap \mathcal{N}(L)}\right) \longrightarrow \mathbb{R}^n$ given by

$$H(\lambda, z(0)) = (1 - \lambda)z(0) + \lambda \int_0^T DA(s)z(0), \quad \text{for all } z \in \overline{\Delta \cap \mathcal{N}(L)}. \tag{6.1.22}$$

In what folllows, we need to show that the following assertion

$$0 \notin H(\lambda, \cdot)(\partial(\Delta \cap \mathcal{N}(L))), \quad \text{for every } \lambda \in [0,1]$$

is true. In fact, consider the following cases:

Case 1: If $\lambda = 0$ in (6.1.22), then

$$H(0, z(0)) = z(0), \quad \text{for all } z \in \partial(\Delta \cap \mathcal{N}(L))$$

and $\|H(\lambda, z(0))\| = \|z(0)\| = M \neq 0$.

Case 2: If $\lambda = 1$ in (6.1.22), then

$$H(1, z(0)) = \int_0^T DA(s)z(0), \quad \text{for all } z \in \partial(\Delta \cap \mathcal{N}(L))$$

and $H(1, z(0)) = JQN(z(0))$, thus the proof follows analogously to item (ii).

Case 3: If $\lambda \in (0,1)$ in (6.1.22), then

$$\|H(\lambda, z(0))\| \leq |1 - \lambda| \|z(0)\| + \left\| \lambda \int_0^T DA(s)z(0) \right\|$$

$$< \|z(0)\| + \|A(T) - A(0)\| \|z(0)\| \neq 0$$

since $(H1)$ holds. Therefore, by item (ii) and (i) of Theorem 1.1.3, we obtain

$$deg(JQN, \Delta \cap \mathcal{N}(L), 0) = deg(H(0, \cdot), \Delta \cap \mathcal{N}(L), 0) = deg(Id, \Delta \cap \mathcal{N}(L), 0) = 1 \neq 0$$

from where the proof is complete.   □

Finally, we present the main result of this chapter. It ensures, under some hypotheses, the existence of a T-periodic solution of equation (6.1.1). This is a new result in the literature and it is contained in [13]. We will prove it using Mawhin's Continuation

Theorem (see Theorem 1.3.5).

Theorem 6.1.9. *Let* $\Delta = \left\{ \, x \in G; \quad \|x\|_\infty < M \right\}$ *and* $A \in BV([0, T], \mathcal{L}(\mathbb{R}^n))$. *Suppose valid the conditions* $(D1) - (D4)$. *Then, the linear generalized ODE* $(6.1.1)$ *has at least one* T-periodic solution $x \in \overline{\Delta}$.

Proof. By Lemma 6.1.8, the conditions of Theorem 1.3.5 are satisfied, then there exists a solution $x \in \overline{\Delta}$ of equation (6.1.8). By Proposition 6.1.3, we conclude that x is a solution of the boundary value problem (6.1.2).

By Lemma 6.1.2, there is a T-periodic solution $\hat{x} : [0, \infty) \to \mathbb{R}^n$ of (6.1.1), which is an extension of $x : [0, T] \to \mathbb{R}^n$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6.2   Applications to Measure Differential Equations

In this section, we present a correspondence between the solutions of a linear generalized ODE and the solutions of a linear measure differential equation. Our goal is to apply the main result of the previous section in the framework of linear measure differential equations.

In the previous section, we consider the linear generalized ODE of the form

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} d[A(s)]x(s) \tag{6.2.1}$$

for every $t_1, t_2 \in [0, T]$, where $A \in BV([0, T], \mathcal{L}(\mathbb{R}^n))$ and $x : [0, T] \longrightarrow \mathbb{R}^n$ is a regulated function.

Let us consider the linear measure differential equation

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} b(s)x(s)dg(s), \quad \text{for all } t_1, t_2 \in [0, T], \tag{6.2.2}$$

where, for each $t \in [0, T]$, the operator $b(t) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is linear and $g : [0, T] \longrightarrow \mathbb{R}$ is a nondecreasing function.

We will show, under some hypotheses, that the linear measure equation (6.2.2) is equivalent to linear generalized ODE (6.2.1), where A is defined as follows: for each $t \in [0, T]$, the operator $A(t) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

$$A(t)z = \int_0^t b(s)z \; dg(s). \tag{6.2.3}$$

Obviously, $A(t) \in \mathcal{L}(\mathbb{R}^n)$. In what follows, let us assume the following conditions:

(D5) The integral $\displaystyle\int_0^T b(s)z \; dg(s)$ exists, for every $z \in \mathbb{R}^n$.

(D6) There exists a Kurzweil-Stieljes integrable function $S : [0, T] \longrightarrow \mathbb{R}^+$ with respect to g such that

$$\left\| \int_{t_1}^{t_2} b(s)z\, ds \right\| \leq \|z\| \int_{t_1}^{t_2} S(s)\, dg(s)$$

for all $t_1, t_2 \in [0, T]$, $t_1 \leq t_2$ and all $z \in \mathbb{R}^n$.

Note that the operator A has bounded variation. Indeed, for every $z \in \mathbb{R}^n$, $0 \leq t_1 \leq t_2 \leq T$, and by definition of A in (6.2.3) and (D6), we have

$$\|[A(t_2) - A(t_1)]z\| = \left\| \int_{t_1}^{t_2} b(s)z\, dg(s) \right\| \leq \|z\| \int_{t_1}^{t_2} S(s)\, dg(s)$$

Then,

$$\|[A(t_2) - A(t_1)]\|_{\mathcal{L}(\mathbb{R}^n)} \leq \int_{t_1}^{t_2} S(s)\, dg(s)$$

for every $t_1, t_2 \in [0, T]$, and $t_1 \leq t_2$. This implies that A is of bounded variation on $[0, T]$.

The following result ensures a relation between linear measure functional differential equations and linear generalized ordinary differential equations. A similar theorem for more general equations is presentend in [27], see Theorems 4.4 and 4.5. Therefore, it is sufficient to check that the hypotheses of Theorems 4.4 and 4.5 in [27] are satisfied for our particular case.

Theorem 6.2.1. *Assume that* $g : [0, T] \longrightarrow \mathbb{R}$ *is a nondecreasing function, for each* $t \in [0, T]$, *the operator* $b(t) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is linear and the conditions (D5)-(D6) are satisfied. A function* $x : [0, T] \longrightarrow \mathbb{R}^n$ *is a solution of*

$$x(t) = x(0) + \int_0^t b(s)x(s)dg(s), \quad \text{for all } t \in [0, T],$$

if and only if, $x : [0, T] \longrightarrow \mathbb{R}^n$ *is a solution of*

$$x(t) = x(0) + \int_0^t d[A(s)]x(s) \quad \text{for all } t \in [0, T],$$

where A *is given by* (6.2.3).

Proof. Let us consider $F(z, t) = A(t)z$, where A is given by (6.2.3). Let $B_1(0) = \{z \in \mathbb{R}^n, \|z\| \leq 1\}$ and $\Omega = B_1(0) \times [0, T]$. We want to prove that F satisfies Definition 2.1.4. Indeed,

$$\|F(z, t_2) - F(z, t_1)\| = \|A(t_2)z - A(t_1)z\| = \left\| \int_{t_1}^{t_2} b(s)z\, dg(s) \right\|$$

$$\leq \|z\| \int_{t_1}^{t_2} S(s)\, dg(s) \leq \int_{t_1}^{t_2} S(s)\, dg(s) \leq |h(t_2) - h(t_1)|$$

for every $(z, t_2), (z, t_1) \in \Omega$, $t_1 \leq t_2$ and $h(t) = \int_0^t S(s)\, dg(s)$ is a nondecreasing function, since M is positive on $[0, T]$ and g is nondecreasing. We have,

$$\|F(z, t_2) - F(z, t_1) - F(y, t_2) + F(y, t_1)\| = \|A(t_2)z - A(t_1)z - A(t_2)y + A(t_1)y\| =$$

$$\left\| \int_{t_1}^{t_2} b(s)(z - y)\, dg(s) \right\| \leq \|z - y\| \int_{t_1}^{t_2} S(s)\, dg(s) \leq \|z - y\| |h(t_2) - h(t_1)|,$$

for every $(z, t_2), (z, t_1), (y, t_2), (y, t_1) \in \Omega$, and $t_1 \leq t_2$. Therefore, $F \in \mathcal{F}(\Omega, h)$. □

Now, we are able to introduce the main result of Section 6.1 in the framework of linear measure differential equations. In the sequel, assume that there is a constant $\widehat{d} > 0$ fulfilling the following conditions:

(D7) For every $\varphi \in G$ satisfying $\|\varphi(0)\| > \widehat{d}$, we have

$$[b(T) - b(0)]\varphi(0) \neq 0.$$

(D8) If $z \in \mathbb{R}^n$ satisfies $\int_t^{t+T} b(s)z\, dg(s) = 0$, for all $t \in [0, +\infty)$, then $\|z\| \leq \widehat{d}$.

Is is easy to check that hypotheses (D7) and (D8) imply hypotheses (D1) and (D2) respectively.

Notice that assumption (D3) holds. Indeed, taking h as in the proof of Theorem 6.2.1, we have

$$\|A(t) - A(s)\| \leq |h(t) - h(s)| \tag{6.2.4}$$

for every $t, s \in [0, T]$.

Now, consider the operators

$$l : G \longrightarrow G, \qquad x \longmapsto l(x)$$

given by

$$l(x)(t) = x(t) - x(0), \quad \text{for every } t \in [0, T] \tag{6.2.5}$$

and

$$n : G \longrightarrow G \qquad x \longmapsto n(x)$$

given by

$$n(x)(t) = \int_0^T b(s)x(s)dg(s) + \int_0^t b(s)x(s)dg(s), \quad \text{for every } t \in [0, T]. \tag{6.2.6}$$

By Theorem 6.2.1, we have,

$$\int_0^t d[A(s)]x(s)dg(s) = \int_0^t b(s)x(s)dg(s), \quad \text{for all } t \in [0, T].$$

Now, we also assume the condition:

(D9) if x is a solution of equation $l(x) = \lambda n(x)$, for all $\lambda \in (0, 1)$ then there exists $D > 0$, satisfying $\|x(t)\| \le D$, for every $t \in [0, T]$.

Thus, we just need to consider $m > \max\{D, \widehat{d}\}$ and define $\Delta = \{x \in G, \ \|x\|_\infty < m\}$. Therefore, we conclude that n is l-compact on $\overline{\Delta}$ and we obtain the following result:

Theorem 6.2.2. *Suppose that the conditions* $(D5) - (D9)$ *are satisfied. Then, the linear measure diferential equation* (6.2.2) *has at least one solution* $x \in \Delta = \big\{ \ x \in G; \ \ \|x\|_\infty < m \big\}$.

Chapter 7

Affine-periodic solutions of Generalized ODEs

The goal of this chapter is to introduce the concept of affine-periodic solutions in the framework of generalized ODEs and to prove an existence result. The concept of an affine-periodic solution was first introduced by Yong Li et al. in [38].

This chapter is inspired on articles [39], [40]. The results presented in Section 7.1 are new and are contained in [12].

7.1   General considerations

Let $GL_n(\mathbb{R})$ denote the general linear group over $\mathbb{R}$ is the group of $n \times n$ invertible matrices of real numbers. Let $\Omega = \mathbb{R}^n \times [0, +\infty)$ and consider the generalized ODE given by

$$\frac{dx}{d\tau} = DF(x, t), \tag{7.1.1}$$

where

(F1) $F \in \mathcal{F}(\Omega, h)$ as in Definition 2.1.4.

(F2) There are $Q \in GL_n(\mathbb{R})$ and $T > 0$ such that $F(x, t + T) = QF(Q^{-1}x, t)$, for all $t \geq 0$.

In the sequel, we introduce the concept of affine-periodic solution in the framework of generalized ODEs. This concept was first introduced in [12].

Definition 7.1.1. Consider the generalized ODE given by (7.1.1) and assume (F1) and (F2) hold. A function $x : [0, +\infty) \to \mathbb{R}^n$ is said to be a (Q, T)-*affine-periodic solution* of

the generalized ODE (7.1.1), if x is a solution of (7.1.1) and, moreover,

$$x(t + T) = Qx(t), \qquad t \in [0, +\infty).$$

The next lemma is new and is contained in [12].

Lemma 7.1.2. *The existence of a (Q, T)-affine–periodic solution $x : [0, +\infty) \to \mathbb{R}^n$ of the generalized ODE (7.1.1) is equivalent to the existence of a solution of the boundary value problem (7.1.1) with $x(T) = Qx(0)$.*

Proof. Assume that $x : [0, +\infty) \to \mathbb{R}^n$ is a (Q, T)–affine-periodic solutions of the generalized ODE (7.1.1). Then

$$x(\tau_2) - x(\tau_1) = \int_{\tau_1}^{\tau_2} DF(x(\tau), t), \qquad \tau_2, \tau_1 \in [0, +\infty)$$

and

$$x(t + T) = Qx(t), \qquad t \in [0, +\infty). \tag{7.1.2}$$

Define the function $z : [0, T] \to \mathbb{R}^n$ given by

$$z(t) := Q^{-1}x(t + T), \qquad t \in [0, T]. \tag{7.1.3}$$

We assert that z is a solution of the boundary value problem (7.1.1) with $x(T) = Qx(0)$. Indeed, given $s_2, s_1 \in [0, T]$, we have

$$
\begin{aligned}
z(s_2) - z(s_1) \quad &= \quad Q^{-1}x(s_2 + T) - Q^{-1}x(s_1 + T) \\
&= \quad Q^{-1}\left(x(s_2 + T) - x(s_1 + T)\right) \\
&= \quad Q^{-1}\left(\int_{s_1+T}^{s_2+T} DF(x(\tau), t)\right) \\
&= \quad Q^{-1}\left(\int_{\phi(s_1)}^{\phi(s_2)} DF(x(\tau), t)\right), \quad \phi(\xi) := \xi + T \\
&\overset{\underset{Lemma\ 2.1.12}{\downarrow}}{=} \quad Q^{-1}\left(\int_{s_1}^{s_2} DF(x(\tau + T), t + T)\right) \\
&\overset{\underset{(F2)}{\downarrow}}{=} \quad Q^{-1}\left(\int_{s_1}^{s_2} D\left[QF(Q^{-1}x(\tau + T), t)\right]\right) \\
&= \quad \left(\int_{s_1}^{s_2} D\left[F(Q^{-1}x(\tau + T), t)\right]\right) \\
&= \quad \int_{s_1}^{s_2} DF(z(\tau), t),
\end{aligned}
$$

that is, z is a solution of (7.1.1). Also, by (7.1.2) and (7.1.3), we have

$$z(T){=}Q^{-1}x(T+T){=}Q^{-1}Qx(T) = QQ^{-1}x(T){=}Qz(0).$$

Reciprocally, assume that there exists a solution $u : [0,T] \to \mathbb{R}^n$ of the boundary value problem (7.1.1) with $x(T) = Qx(0)$. Then u is solution of the generalized ODE (7.1.1) on $[0,T]$ and $u(T) = Qu(0)$.

By (F1) and Theorem 2.1.13, there exists a unique (maximal) solution $y : [0, +\infty) \to \mathbb{R}^n$ of the generalized ODE (7.1.1) with $x(0) = Q^{-1}u(T)$. Then, by the uniqueness $y|_{[0,T]} = u$, that is, y is a extension of u.

We will show that $y(t + T) = Qy(t)$ for all $t \in [0, +\infty)$.

Indeed, using the same arguments as above, one can prove that the function $\phi(t) := Q^{-1}y(t+T)$ is a solution of the initial value problem (7.1.1) with $x(0) = Q^{-1}u(T) = u(0)$.

Now, since $\phi(t)$ and $y(t)$ are solutions of initial value problem (7.1.1) with $x(0) = Q^{-1}u(T) = u(0)$, the uniqueness of a solution yields

$$\phi(t) = y(t), \text{ that is, } y(t + T) = Qy(t)$$

for all $t \in [0, +\infty)$ and this completes the proof. $\square$

7.2 An existence theorem

The next result ensures that the generalized ODE (7.1.1) has at least one (Q,T)-affine periodic solution. It is important to mention that the calculations used to prove this result coming from the following papers [39] and [40], but here we are dealing with the Kurzweil integral, which encompasses many types of integrals such as Riemann and Lesbegue.

A new version of the following theorem using different tools can be found in [12].

Theorem 7.2.1. *Let $D \subset \mathbb{R}^n$ be an open bounded set. Suppose the following conditions are satisfied for generalized ODE (7.1.1):*

(i) For each $\lambda \in [0,1]$, every affine-periodic solution $x(t)$ of generalized ODE (7.1.1) satisfies the following property: if $x(t) \in \overline{D}$, then

$$x(t) \notin \partial D, \text{ for every } t \in [0,T].$$

(ii) If $Ker(I - Q) \neq \{0\}$, then

$$deg(g, D \cap B, 0) \neq 0,$$

where B has finite dimension and the funtion g is given by

$$g(a) := \frac{1}{T} \int_0^T D\left[PF(a,t)\right] \tag{7.2.1}$$

and $P : \mathbb{R}^n \longrightarrow Ker(I - Q)$ is the orthogonal projection. Then the generalized ODE (7.1.1) has at least one (Q,T)-affine periodic solution x_* in G, satisfying $x_*(t) \in \overline{D}$, for every $t \in [0, T]$.

Proof. Consider the boundary value problem

$$\frac{dx}{d\tau} = D\left[\lambda F(x,t)\right] \tag{7.2.2}$$

$$x(T) = Qx(0), \tag{7.2.3}$$

where $\lambda \in [0, 1]$. Let $x(t)$ be any solution of (7.2.2)-(7.2.3). Then

$$x(T) = x(0) + \lambda \int_0^T DF(x(\tau), t).$$

By (7.2.3), we have

$$Qx(0) = x(0) + \lambda \int_0^T DF(x(\tau), t).$$

Denote $x(0)$ by x_0, then

$$(I - Q)x_0 = -\lambda \int_0^T DF(x(\tau), t)$$

We need to consider two cases with respect to $Ker(I - Q)$:

Case 1. $Ker(I - Q) \neq \{0\}$
Case 2. $Ker(I - Q) = \{0\}$

The proof of Case 1 is extensive, therefore we will first list the main steps.

Consider the set

$$X = \{x \in G : x(t) \in D\}.$$

and define the operator

$$S : [0, 1] \times X \to G$$

$$(\lambda, x) \longmapsto S(\lambda, x)$$

given by

$$S(\lambda, x)(s) := x^0_{ker} + \frac{1}{T} \int_0^T D\left[PF(x(\tau), t)\right] - \lambda \mathcal{T}^{-1}(I-P) \int_0^T DF(x(\tau), t) + \lambda \int_0^s DF(x(\tau), t)$$

for every $s \in [0, T]$, where $\mathcal{T} := (I - Q)\big|_{Im(I-Q)}$. We will prove that every fixed point x of $S(\lambda, \cdot)$ is a solution of (7.2.2)-(7.2.3). Then after some calculations it will follow that x is a solution of (7.2.2)-(7.2.3). The next step of Case 1 will be to prove the existence of fixed points of the operator $S(1, \cdot)$. In order to do this, we use results from Chapter 1, concerning the topological degree theory. We wil prove that a certain operator $H(\lambda, x)$, defined in terms of the orthogonal projection $P : \mathbb{R}^n \longrightarrow Ker(I - Q)$, is compact and this requires some work. Following this, we will prove that the degree of $I - H$ is well-defined. This also requires a few lines. At this point, we will consider, still in Case 1, $\lambda = 0$ and $\lambda \in (0, 1]$. We will prove that the degree of $I - H$ does not vanish, so that condition (ii) of Definition 1.2.3 is fulfilled and this will imply the existence of a fixed point of the operator $S(1, \cdot)$.

Finally we go to Case 2 which follows easier by some facts which can be derived similarly as in the proof of Case 1. So let us begin the proof of Case 1.

Case 1. $Ker(I - Q) \neq \{0\}$

In this case, $(I - Q)^{-1}$ does not exist. Thus, using a coordinate transformation, we can define

$$Q = \begin{pmatrix} I & 0 \\ 0 & Q_1 \end{pmatrix},$$

where $(I - Q_1)^{-1}$ exists.

Consider the orthogonal projection $P : \mathbb{R}^n \longrightarrow Ker(I - Q)$. Then

$$\begin{aligned}
(I - Q)x_0 &= (I - Q)(x^0_{Ker} + x^0_\perp) \\
&= -\lambda \int_0^T DF(x(\tau), t) \\
&= -\lambda \int_0^T D\left[PF(x(\tau), t)\right] - \lambda \int_0^T D\left[(I - P)F(x(\tau), t)\right] \quad (7.2.4)
\end{aligned}$$

where $x^0_{Ker} \in Ker(I - Q), x^0_\perp \in Im(I - Q)$ and $x_0 = x^0_{Ker} + x^0_\perp$.

Define the operator $\mathcal{T} := (I - Q)\big|_{Im(I-Q)}$. It is clear that $\mathcal{T}^{-1}$ exists. Therefore, (7.2.4) is equivalent to

$$(I - Q)x^0_{Ker} = -\int_0^T PDF(x(\tau), t) = 0$$

since $x_{Ker}^0 \in Ker(I - Q)$ and

$$(I - Q)x_\perp^0 = -\lambda \int_0^T (I - P)DF(x(\tau), t). \tag{7.2.5}$$

Also, since $\mathcal{T}^{-1}$ exists, we can rewrite (7.2.5) as

$$x_\perp^0 = -\lambda \mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t). \tag{7.2.6}$$

Now, let us consider the set

$$X = \{x \in G : x(t) \in D\}.$$

and define the operator

$$S : [0, 1] \times X \to G$$

$$(\lambda, x) \longmapsto S(\lambda, x)$$

given by

$$S(\lambda, x)(s) := x_{ker}^0 + \frac{1}{T}\int_0^T D\left[PF(x(\tau), t)\right] - \lambda \mathcal{T}^{-1}(I-P)\int_0^T DF(x(\tau), t) + \lambda \int_0^s DF(x(\tau), t)$$

for every $s \in [0, T]$.

We want to prove that every fixed point x of $S(\lambda, \cdot)$ is a solution of (7.2.2)-(7.2.3).

Assume that x is a fixed point of $S(\lambda, \cdot)$. Then

$$x(s) = x_{Ker}^0 + \frac{1}{T}\int_0^T D\left[PF(x(\tau), t)\right] - \lambda \mathcal{T}^{-1}(I - P)\int_0^T DF(x(\tau), t) + \lambda \int_0^s DF(x(\tau), t) \tag{7.2.7}$$

Taking $s = 0$ in (7.2.7), we have

$$x_0 = x(0) = x_{Ker}^0 + \frac{1}{T}\int_0^T D\left[PF(x(\tau), t)\right] - \lambda \mathcal{T}^{-1}(I - P)\int_0^T DF(x(\tau), t).$$

Then, by (7.2.6), we have

$$x_0 = x_{Ker}^0 + x_0^\perp + \frac{1}{T}\int_0^T D\left[PF(x(\tau), t)\right] = x_0 + \frac{1}{T}\int_0^T D\left[PF(x(\tau), t)\right].$$

Therefore

$$\frac{1}{T}\int_0^T D\left[PF(x(\tau), t)\right] = 0. \tag{7.2.8}$$

Since $x_0 = x_{Ker}^0 + x_\perp^0$, by (7.2.6), we have

$$x_0 = x_{Ker}^0 - \lambda \mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t). \tag{7.2.9}$$

Applying Q in (7.2.9) and remembering that $x_{Ker}^0 \in Ker(I - Q)$, we have

$$Qx_0 = Qx_{Ker}^0 - \lambda Q\mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t)$$
$$= x_{Ker}^0 - \lambda Q\mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t). \tag{7.2.10}$$

On the other hand, notice that

$$\lambda(I - Q)\mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t) = \lambda(I - P) \int_0^T DF(x(\tau), t)$$
$$= \lambda(I - P) \int_0^T DF(x(\tau), t) + \lambda \int D\left[PF(x(\tau), t)\right]$$
$$= \lambda \int_0^T DF(x(\tau), t). \tag{7.2.11}$$

Using (7.2.11), we obtain

$$\lambda Q\mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t) = \lambda \left(I - (I - Q)\right) \mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t)$$
$$= \lambda \mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t) - \lambda \int_0^T DF(x(\tau), t). \tag{7.2.12}$$

Then, by (7.2.8), (7.2.10) and (7.2.12), we conclude that

$$Qx_0 = x_{Ker}^0 - \lambda \mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t) + \lambda \int_0^T DF(x(\tau), t)$$

$$= x_{Ker}^0 + \frac{1}{T} \int_0^T PDF(x(\tau), t) - \lambda \mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t) + \lambda \int_0^T DF(x(\tau), t)$$

$$= x(T).$$

Thus, $Qx_0 = x(T)$.

By (7.2.8), we have

$$x(s) = x_{Ker}^0 + \frac{1}{T} \int_0^T PDF(x(\tau), t) - \lambda \mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t) + \lambda \int_0^s DF(x(\tau), t)$$

$$= x^0_{Ker} + x^0_\perp + \lambda \int_0^s DF(x(\tau), t)$$

$$= x_0 + \lambda \int_0^s DF(x(\tau), t)$$

since (7.2.9) holds. Therefore, x is a solution of (7.2.2)-(7.2.3).

Now, we need to prove the existence of fixed points of the operator $S(1, \cdot)$. In order to do that, let $M > 0$ be a constant such that

$$\sup_{\substack{s \in [0,T] \\ x \in X}} \left\| \int_0^s DF(x(\tau), t) \right\| < M.$$

and

$$G_\lambda = \left\{ x \in G : \left| \frac{x(t) - x(s)}{t - s} \right| \le \lambda M; \ \forall \, t \ne s \right\}.$$

Let $\alpha_\lambda : G \longrightarrow G_\lambda$, that is, for all $x \in G$, $\alpha_\lambda(x) \in G_\lambda$ and $\alpha_\lambda(x) = x$, whenever $x \in G_\lambda$. We will use the topological degree theory from the previous section to prove the existence of a fixed point of $S(\lambda, \cdot)$.

Let us consider the set

$$\widetilde{D} = \{ x \in X; x(t) \in \overline{D}, \forall t \in [0, 1] \}.$$

and define the operator

$$H : [0, 1] \times \widetilde{D} \to G$$

$$(\lambda, x) \longmapsto H(\lambda, x)$$

given by

$$H(\lambda, x)(s) = x^0_{ker} + \frac{1}{T} \int_0^T D\left[PF(\alpha_\lambda \circ x(\tau), t)\right]$$

$$- \lambda \mathcal{T}^{-1}(I - P) \int_0^T DF(\alpha_\lambda \circ x(\tau), t) + \lambda \int_0^s DF(\alpha_\lambda \circ x(\tau), t),$$

for every $s \in [0, T]$. It is clear that $H(\lambda, x) \in G$.

We need to prove that H is a homotopy of compact transformations on $\widetilde{D}$. By Definition 1.2.4, it is sufficient to prove:

(a) for each $x \in \widetilde{D}$, $H(\lambda, x)$ is continuous in $\lambda \in [0, 1]$.

(b) $H(\lambda, x)$ is continuous in $x \in \widetilde{D}$ and for each $\lambda \in [0, 1]$ fixed.

(c) $H(\lambda, x)$ is relatively compact in $\widetilde{D}$.

We will prove (a). Given $\lambda_1, \lambda_2 \in [0,1]$ and $s \in [0,T]$, we have

$$\|H(\lambda_1, x)(s) - H(\lambda_2, x)(s)\|$$

$$\leq \left\| \frac{1}{T} \int_0^T D\left[PF(\alpha_{\lambda_1} \circ x(\tau), t)\right] - \frac{1}{T} \int_0^T D\left[PF(\alpha_{\lambda_2} \circ x(\tau), t)\right] \right\|$$

$$+ |\lambda_2 - \lambda_1| \|\mathcal{T}^{-1}(I-P)\| \left\| \int_0^T DF(\alpha_{\lambda_2} \circ x(\tau), t) - \int_0^T DF(\alpha_{\lambda_1} \circ x(\tau), t) \right\|$$

$$+ |\lambda_1 - \lambda_2| \left\| \int_0^s DF(\alpha_{\lambda_1} \circ x(\tau), t) - \int_0^s DF(\alpha_{\lambda_2} \circ x(\tau), t) \right\|$$

$$\leq \frac{1}{T} \int_0^T \|\alpha_{\lambda_1} \circ x(t) - \alpha_{\lambda_2} \circ x(t)\| dh(t) + |\lambda_2 - \lambda_1| \|\mathcal{T}^{-1}(I-P)\| 2 \int_0^T dh(t) + |\lambda_1 - \lambda_2| 2 \int_0^s dh(t)$$

$$\leq \frac{1}{T} \int_0^T \|(\alpha_{\lambda_1} - \alpha_{\lambda_2})x\|_\infty dh(t) + |\lambda_2 - \lambda_1| \|\mathcal{T}^{-1}(I-P)\| 2 \int_0^T dh(t) + |\lambda_1 - \lambda_2| 2 \int_0^s dh(t)$$

$$\leq \frac{1}{T} |\lambda_2 - \lambda_1| \|x\|_\infty [h(T) - h(0)] + |\lambda_2 - \lambda_1| \|\mathcal{T}^{-1}(I-P)\| 2[h(T) - h(0)]$$

$$+ |\lambda_1 - \lambda_2| 2[h(T) - h(0)]$$

where we used the fact that F satisfies the conditions of Definition 2.1.4 and the fact that α_λ is continuous on $\lambda \in [0,1]$. Therefore, $H(\lambda, x)$ is continuous in $\lambda \in [0,1]$.

Now, we will prove (b). Let $\varepsilon > 0$, $0 < \delta < \epsilon$, $x, y \in \widetilde{D}$ such that $\|x - y\| < \delta$ and $s \in [0,T]$. Then

$$\|H(\lambda, x)(s) - H(\lambda, y)(s)\| =$$

$$\leq \|x_{ker}^0 - y_{ker}^0\| + \left\| \frac{1}{T} \int_0^T D\left[PF(\alpha_\lambda \circ x(\tau), t)\right] - \frac{1}{T} \int_0^T D\left[PF(\alpha_\lambda \circ y(\tau), t)\right] \right\|$$

$$+ |\lambda| \|\mathcal{T}^{-1}(I-P)\| \left\| \int_0^T DF(\alpha_\lambda \circ y(\tau), t) - \int_0^T DF(\alpha_\lambda \circ x(\tau), t) \right\|$$

$$+ |\lambda| \left\| \int_0^s DF(\alpha_\lambda \circ y(\tau), t) - \int_0^s DF(\alpha_\lambda \circ x(\tau), t) \right\|$$

$$\leq \|x_{ker}^0 - y_{ker}^0\| + \frac{1}{T} \int_0^T \|\alpha_\lambda \circ x(t) - \alpha_\lambda \circ y(t)\| dh(t)$$

$$+ |\lambda| \|\mathcal{T}^{-1}(I-P)\| \int_0^T \|\alpha_\lambda \circ x(t) - \alpha_\lambda \circ y(t)\| dh(t)$$

$$+|\lambda| \int_0^s \|\alpha_\lambda \circ x(t) - \alpha_\lambda \circ y(t)\| dh(t)$$

$$\leq \varepsilon + \frac{1}{T} \int_0^T \|\alpha_\lambda(x-y)\|_\infty dh(t)$$

$$\leq |\lambda| \|\mathcal{T}^{-1}(I-P)\| \int_0^T \|\alpha_\lambda(x-y)\|_\infty dh(t) + |\lambda| \int_0^s \|\alpha_\lambda(x-y)\|_\infty dh(t)$$

where the inequality follows from Lemma 2.1.5 and by the fact that α_λ is continuous on G, for each $\lambda \in [0,1]$.

We claim, for each $\lambda \in [0,1]$ fixed, $H(\lambda, x)$ is relatively compact in $\widetilde{D}$, that is, (c) holds.

In fact, let $M \subset \widetilde{D}$ be bounded and consider the set

$$\mathcal{A} = \{H(\lambda, x), \ x \in M\}.$$

We need to prove that $\mathcal{A}$ is relatively compact in G. Notice that

$$\|H(\lambda, x)(0)\| = \left\| -\lambda \mathcal{T}^{-1}(I-P) \int_0^T DF(\alpha_\lambda \circ x(\tau), t) \right\|$$

$$\leq \left\| \mathcal{T}^{-1}(I-P) \right\| [h(T) - h(0)]$$

for every $x \in M$, where the last inequality follows from Lemma 2.1.5. In fact, the operator $\mathcal{T}^{-1}(I-P) : \mathbb{R}^n \to \mathbb{R}^n$ is linear, thus it is bounded. Therefore, $\|H(\lambda, x)(0)\|$ is bounded.

By Lemma 2.1.5, we have

$$\|H(\lambda, x)(s') - H(\lambda, x)(s)\| = \left\| \lambda \int_{s'}^s DF(\alpha_\lambda \circ x(\tau), t) \right\| \leq |h(s) - h(s')|,$$

for every $s, s' \in [0,T]$ and every $x \in M$. Then, by Theorem 2.1.10, $\mathcal{A}$ is relatively compact and this completes the proof.

In what follows, we will prove that

$$0 \notin (I - H(\lambda, \cdot))(\partial \widetilde{D}), \quad \text{for every } \lambda \in [0,1]$$

where I is the identity operator on $\widetilde{D}$.

Let us assume, by contradiction, that

$$0 \in (I - H(\lambda, \cdot))(\partial \widetilde{D}), \quad \text{for every } \lambda \in [0,1].$$

Then, there exists $\widehat{x} \in \partial \widetilde{D}$ such that

$$I(\widehat{x}) = H(\lambda, \widehat{x}), \quad \text{for every } \lambda \in [0, 1] \tag{7.2.13}$$

We need to consider two cases: $\lambda = 0$ and $\lambda \in (0, 1]$.

Let $\lambda = 0$. Then

$$G_0 = \{x \in G : \left| \frac{x(t) - x(s)}{t - s} \right| \leq 0; \forall t \neq s \}.$$

Thus $\alpha_0 \circ \widehat{x}(t) = p$, for every $t \in [0, T]$. By (7.2.13), we obtain

$$\widehat{x}(s) = \widehat{x}^0_{Ker} + \frac{1}{T} \int_0^T D\left[PF(\alpha_0 \circ \widehat{x}(\tau), t)\right] = H(0, \widehat{x})(s)$$

which implies

$$\widehat{x}(s) = \widehat{x}^0_{Ker} + \frac{1}{T} \int_0^T D\left[PF(p, t)\right] = H(0, \widehat{x})(s)$$

This means that $\hat{x}(s) = p$, for every $s \in [0, T]$. Thus

$$\frac{1}{T} \int_0^T D\left[PF(p, t)\right] = 0.$$

By the definition of $g(a)$ in (7.2.1), we have $g(p) = 0$.

On the other hand, $\widehat{x}(t) \in \partial \widetilde{D}$. Then there exists $t_0 \in [0, T]$ such that $\widehat{x}(t_0) \in \partial \, D$. Since $\widehat{x}(t) = p$, for every $t \in [0, T]$, we have $p \in \partial D$. Thus, $p \in \partial D$ and $g(p) = 0$, which contradicts (ii), because we assumed $deg(g, D, 0) \neq 0$.

Now, consider $\lambda \in (0, 1]$. By (7.2.13), we obtain

$$\widehat{x}(s) = \widehat{x}^0_{ker} + \frac{1}{T} \int_0^T PDF(\alpha_\lambda \circ \widehat{x}(\tau), t)$$

$$-\lambda \mathcal{T}^{-1}(I - P) \int_0^T DF(\alpha_\lambda \circ \widehat{x}(\tau), t) + \lambda \int_0^s DF(\alpha_\lambda \circ \widehat{x}(\tau), t) = H(\lambda, \widehat{x})(s).$$

Note that

$$\left| \frac{\widehat{x}(t) - \widehat{x}(s)}{t - s} \right| = \frac{\lambda}{|t - s|} \left\| \int_s^t DF(\alpha_{\widehat{\lambda}} \circ x(\tau), t) \right\| \leqslant \frac{\lambda}{|t - s|} |t - s| M = \lambda M.$$

Therefore, we conclude $\widehat{x} \in G_{\widehat{\lambda}}$. It means that $\alpha_{\widehat{\lambda}} \circ \widehat{x} = \widehat{x}$. Then,

$$\widehat{x}(s) = \widehat{x}^0_{ker} + \frac{1}{T} \int_0^T D\left[PF(\widehat{x}(\tau), t)\right] - \widehat{\lambda} \mathcal{T}^{-1}(I - P) \int_0^T DF(\widehat{x}(\tau), t) + \widehat{\lambda} \int_0^s DF(\widehat{x}(\tau), t)$$

As before, one can prove that $\widehat{x}$ is a solution of (7.2.2)-(7.2.3). However, $\widehat{x} \in \partial \widetilde{D}$, which contradicts the hypothesis (i), since $\widehat{x} \notin \partial \widetilde{D}$.

Notice that the set $H(0, \{x\})$ is contained in the space of the constant functions on G, for every $x \in \partial \widetilde{D}$, denoted by B. Then, using Theorem 1.2.5, we obtain

$$deg_{LS}(I - H(1, \cdot), \widetilde{D}, 0) = deg_{LS}(Id - H(0, \cdot), \widetilde{D}, 0)$$

$$deg(I - H(0, \cdot)|_{\widetilde{D} \cap B}, \widetilde{D} \cap B, 0)$$

$$= deg(-g, \widetilde{D} \cap B, 0) = (-1)^n deg(g, \widetilde{D} \cap B, 0) \neq 0$$

where this equality follows from Definition 1.2.1, Corollary 1.1.4 and (ii). Therefore, there exists $\widehat{x}_* \in \widetilde{D}$, such that

$$H(1, \widehat{x}_*) = \widehat{x}_*$$

Consequently,

$$\widehat{x}_* = H(1, \widehat{x}_*) = S(1, \widehat{x}_*)$$

Thus, $\widehat{x}_*$ is a fixed point of $S(1, \cdot)$ in X. Then, $\widehat{x}_*$ is a solution of (7.2.2)-(7.2.3).

Case 2. $Ker(I - Q) = \{0\}$

If $Ker(I - Q) = \{0\}$, then $(I - Q)^{-1}$ exists. Thus,

$$x_0 = -\lambda \mathcal{T}^{-1}(I - P) \int_0^T DF(\alpha_\lambda \circ x(\tau), t) \tag{7.2.14}$$

Define the operator

$$H : [0, 1] \times \widetilde{D} \to G([0, T], \mathbb{R}^n)$$

$$(\lambda, x) \longmapsto H(\lambda, x)$$

given by

$$H(\lambda, x)(s) = -\lambda \mathcal{T}^{-1}(I - P) \int_0^T DF(x(\tau), t) + \lambda \int_0^s DF(\alpha_\lambda \circ x(\tau), t)$$

for every $s \in [0, T]$.

In order to use the topological degree theory, we need to prove that $H(\lambda, x)$ is a homotopy of compact transformations in $\widetilde{D}$. But this fact follows analogously to what we did in Case 1. Moreover, as in Case 1, we can conclude that

$$0 \notin (I - H)(\partial \widetilde{D} \times [0, 1]).$$

By Theorem 1.2.5 and (i) from Definition 1.2.3, we obtain

$$deg_{LS}(I - H\ (1, \cdot), \widetilde{D}) =$$

$$= deg_{LS}(I - H\ (0, \cdot), \widetilde{D})$$

$$= deg_{LS}(I, \widetilde{D}, 0) = 1 \neq 0.$$

By (ii) from Definition 1.2.3, there exists $\widehat{x}_* \in \widetilde{D}$ such that

$$I(\widehat{x}_*) = H\ (1, \widehat{x}_*),$$

that is,

$$\widehat{x}_*(s) = \widehat{x}_*(0) + \int_0^s DF(\widehat{x}_*(\tau), t)$$

since (7.2.14) holds. Therefore, $\widehat{x}_*$ is a solution of (7.2.2)-(7.2.3).

We conclude, by Lemma 7.1.2, that the generalized ODE (7.1.1) has a (Q,T)-affine periodic solution $x_* : [0, \infty) \to \mathbb{R}^n$, which is an extension of $\widehat{x}_* : [0, T] \to \mathbb{R}^n$. $\qquad\qquad\square$

Bibliography

[1] O. Almeida, Teoria do Grau e Aplicações, Universidade Federal de Campina Grande Centro de Ciências e Tecnologia, 2006.

[2] H. Amann, *Ordinary Differential Equations: An introduction to nonlinear analysis*, Gruyter Studies in Mathematics, 13. Walter de Gruyter Co., Berlin, 1990.

[3] P. C. Das; R. R. Sharma, On optimal controls for measure delay-differential equations, *J. SIAM Control* 9, (1971), 43-61.

[4] P. C. Das; R. R. Sharma, Existence and stability of measure differential equations, *Czech. Math. J.* 22(97), (1972), 145-158.

[5] B. C. Dhage, On system of abstract measure integro-differential inequalities and applications, *Bull. Inst. Math. Acad. Sinica* 18, (1989), 65-75.

[6] B. C. Dhage; J. R. Graef, On stability of abstract measure delay integro-differential equations, *Dynam. Systems Appl.* 19, (2010), 323-334.

[7] B. C. Dhage, S. S. Bellale, Abstract measure integro-differential equations, *Global J. Math. Anal.* 1, (2007), 91-108.

[8] K. Deimling, *Nonlinear Functional Analysis,* Springer-Verlag, Berlin, 1985.

[9] P. Drábek; J. Milota, *Methods of Nonlinear Analysis. Applications to Differential Equations*, Birkhäuser Advanced Texts/Basler Lehrbücher, Birkhäuser (2007), 568 pp.

[10] M. Federson, E. M. Bonotto, J. G. Mesquita, *Generalized Ordinary Differential Equations in Abstract Spaces and Applications*, Wiley, 2021.

[11] M. Federson; R. Grau, C. Mesquita; K. Schiabel, Differentiability of solutions for generalized ODEs, preprint.

[12] M. Federson; R. Grau; C. Mesquita, Affine-Periodic solutions for generalized ODEs, preprint.

[13] R. Collegari, M. Federson; C. Mesquita; K. Schiabel, Periodic boundary value problems for generalized ODEs and applications, preprint.

[14] M. Federson; J. Mawhin; C. Mesquita, Existence and bifurcation of periodic solutions and bifurcation points for generalized ordinary differential equations, *Bulletin des Sciences Mathématiques* 169 (2021): 102991.

[15] M. Federson; R. Grau; J. G. Mesquita, Prolongation of solutions of measure differential equations and dynamical equations on time scales, *Math. Nach.* 292(1), (2019), 22-55.

[16] M. Federson; J.G. Mesquita; A. Slavík, Basic results for functional differential and dynamic equations involving impulses. *Math. Nachr.* 286(2-3) (2013), 181-204.

[17] M. Federson; J. Mesquita; A. Slavík, Measure functional equations and functional dynamic equations on time scales, *J. Differential Equations* 252 (2012) 3816-3847.

[18] I. Fonseca, W. Gangbo. Deimling, *Degree Theory in Analysis and Applications,* Oxford, 1995.

[19] R. E. Gaines; J. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lectures Notes in Math, 568, Springer-Verlag, 1977.

[20] S. R. Joshi, S. G. Deo, On abstract measure delay differential equations, *An. Stiint. Univ. Al I. Cuza Iasi* XXVI, (1980), 327-335.

[21] W. G. Kelley; A. C. Peterson, *The theory of differential equations.* Classical and qualitative. Second edition. Universitext. Springer, New York, 2010.

[22] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, *Czechoslovak Math. J.* 7(82) (1957) 418-448.

[23] J. Kurzweil, *Generalized Ordinary Differential Equations. Not Absolutely Continuous Solutions*, Series in Real Analysis, 11. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[24] L. H. Loomis, S. Sternberg, *Advanced Calculus*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1968.

[25] J. Mawhin, Leray-Schauder degree - A half century of extensions and applications, *Topological Methods in Nonlinear Analysis*, 14 (1999), 195-228.

[26] J. G. Mesquita; A. Slavík, Periodic averaging theorems for various types of equations, *J. Math. Anal. Appl.*, 387(2) (2012), 863-877.

[27] G. A. Monteiro, A. Slavık, Linear measure functional differential equations with infinite delay, *Mathematische Nachrichten 287, no. 11-12,* 1363–1382 (2014).

[28] S. G. Pandit, S. G. Deo, *Differential systems involving impulses.* Lecture Notes in Mathematics, 954, Springer-Verlag, Berlin-New York, 1982.

[29] J. Persson, Fundamental theorems for linear measure differential equations, *Math. Scand.* 62(1), (1988), 19-43.

[30] E. H. Rothe *Introduction to various aspects of degree theory in Banach spaces.* Mathematical Surveys and Monographs, 23. American Mathematical Society, Providence, RI, 1986.

[31] Š. Schwabik, *Generalized Ordinary Differential Equations,* Series in Real Analysis, 5. World Scientific Publishing Co., Inc., River Edge, NJ, 1992.

[32] Š. Schwabik, Linear Stieltjes integral equations in Banach spaces, *Math. Bohem.* 124(4), (1999), 433-457.

[33] M. Schechter, *Principles of Functional Analysis*, Second edition. Graduate Studies in Mathematics, 36. American Mathematical Society, Providence, RI, 2002.

[34] W. W. Schmaedeke, Optimal control theory for nonlinear vector differential equations containing measures, *J. SIAM Control* 3, (1965), 231-280.

[35] R. R. Sharma, An abstract measure differential equations, *Proc. Am. Math. Soc.* 32, (1972), 503-510.

[36] A. Slavík, Generalized differential equations: differentiability of solutions with respect to initial conditions and parameters, *J. Math. Anal. Appl.* 402(1) (2013), 261-274.

[37] M. Tvrdý; A. Slavík; G. Monteiro, *Kurzweil-Stieltjes Integral Theory and Applications,* Series in Real Analysis, Vol 14, World Scientific Publishing Co., Inc., River Edge, NJ, 2018.

[38] Y. Zhang, X. Yang; Y. Li, *Affine-periodic solutions for dissipative systems.* Abstract and Applied Analysis. Vol. 2013. Hindawi, 2013.

[39] C. Wang, X. Yang; Y. Li, *Affine-periodic solutions for nonlinear differential equations,* Rocky Mountain J. Math. 46(5): 1717-1737 (2016).

[40] C. Wang., Y. Li. , *Affine-periodic solutions for nonlinear dynamic equations on time scales,* Advances in Difference Equations, v. 2015, n. 1, p. 1-16, 2015.