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Pseudo-parallel immersions in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, and constant anisotropic mean curvature surfaces in $\mathbb{R}^3$

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To the memory of my beloved mother, Denize.

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Abstract

In this Ph.D. thesis, we investigate two topics in Differential Geometry. The first topic refers to the study of pseudo-parallel submanifolds in the ambient spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. We complete the partial classification given by F. Lin and B. Yang. As a consequence, we classify minimal and constant mean curvature pseudo-parallel hypersurfaces. We also prove a characterization of pseudo-parallel surfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, for $n \geq 4$, and the non-existence of pseudo-parallel surfaces with non-vanishing normal curvature, when $n = 3$.

The second part of the thesis is devoted to the study of constant anisotropic mean curvature surfaces in $\mathbb{R}^3$. We obtain a Bernstein-type Theorem for multigraphs with constant anisotropic mean curvature, an anisotropic version of a theorem proved by D. Hoffman, R. Osserman and R. Schoen, in 1982. As a consequence, we prove that complete surfaces with non-zero constant anisotropic mean curvature and whose Gaussian curvature does not change sign are either the Wulff shape or cylinders. We prove uniform height estimates for vertical graphs with non-zero constant anisotropic mean curvature and planar boundary, a generalization of the theorem proved by W. Meeks, in 1988, and we obtain uniform height estimates for compact embedded surfaces with non-zero constant anisotropic mean curvature and planar boundary, as a corollary. We also prove, under certain symmetry hypothesis on the anisotropy function, the non-existence of properly embedded surfaces in $\mathbb{R}^3$ with non-zero constant anisotropic mean curvature and with just one end.

Resumen

En esta tesis doctoral, investigamos dos tópicos en Geometría Diferencial. El primer tópico se refiere al estudio de subvariedades pseudo-paralelas en los espacios producto $\mathbb{S}^n \times \mathbb{R}$ y $\mathbb{H}^n \times \mathbb{R}$. Completamos la clasificación parcial dada por F. Lin y B. Yang. Como consecuencia, clasificamos las hipersuperficies pseudo-paralelas minimales y con curvatura media constante. También probamos una caracterización de las superficies pseudo-paralelas en $\mathbb{S}^n \times \mathbb{R}$ y $\mathbb{H}^n \times \mathbb{R}$, cuando $n \geq 4$, y la no existencia de superficies pseudo-paralelas con curvatura normal que no se anula, cuando $n = 3$.

La segunda parte de la tesis está dedicada al estudio de las superficies de curvatura media anisotrópica constante en $\mathbb{R}^3$. Obtenemos un Teorema tipo-Bernstein para multigrafos con curvatura media anisotrópica constante, una versión anisotrópica de un teorema probado por D. Hoffman, R. Osserman y R. Schoen, en 1982. Como consecuencia, demostramos que las superficies completas con curvatura media anisotrópica constante no nula y cuya curvatura gaussiana no cambia de signo son o bien la forma de Wulff o bien cilindros. Probamos acotaciones uniformes de altura para grafos verticales con curvatura media anisotrópica constante no nula y borde plano, una generalización del teorema probado por W. Meeks, en 1988, y obtenemos acotaciones uniformes de altura para superficies compactas embebidas con curvatura media anisotrópica constante no nula y borde plano, como un corolario. También demostramos, bajo ciertas hipótesis de simetría en la función de anisotropía, que no hay superficies propiamente embebidas en $\mathbb{R}^3$ con curvatura media anisotrópica constante no nula y con solo un final.

Resumo

Nesta tese de doutorado, investigamos dois tópicos em geometria diferencial. O primeiro tópico se refere ao estudo das subvariedades pseudo-paralelas nos espaços ambientes $\mathbb{S}^n \times \mathbb{R}$ e $\mathbb{H}^n \times \mathbb{R}$. Completamos a classificação parcial dada por F. Lin e B. Yang. Como consequência, classificamos as hipersuperfícies pseudo-paralelas mínimas e com curvatura média constante. Também provamos uma caracterização das superfícies pseudo-paralelas em $\mathbb{S}^n \times \mathbb{R}$ e $\mathbb{H}^n \times \mathbb{R}$, quando $n \geq 4$, e a não existência de superfícies pseudo-paralelas com curvatura normal que não se anula, quando $n = 3$.

A segunda parte da tese se dedica ao estudo das superfícies de curvatura média anisotrópica constante em $\mathbb{R}^3$. Obtemos um Teorema do tipo Bernstein para multigráficos com curvatura média anisotrópica constante, uma versão anisotrópica de um teorema provado por D. Hoffman, R. Osserman e R. Schoen, em 1982. Como consequência, provamos que superfícies completas com curvatura média anisotrópica constante não nula e cuja curvatura gaussiana não muda de sinal são ou a forma de Wulff ou os cilindros. Provamos estimativas uniformes de altura para gráficos verticais com curvatura média anisotrópica constante não nula e bordo plano, uma generalização do teorema provado por W. Meeks, em 1988, e obtemos estimativas uniformes de altura para superfícies compactas mergulhadas com curvatura média anisotrópica constante não nula e bordo plano, como um corolário. Também provamos, sob certas hipóteses de simetria na função de anisotropia, a não existência de superfícies propriamente mergulhadas em $\mathbb{R}^3$ com curvatura média anisotrópica constante não nula e com apenas um fim.

Contents

Introduction

This work is divided in two main themes. The first part concerns the study of pseudo-parallel immersions in the product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. It was developed with my advisor, G.A. Lobos, with the colaboration of A.J.Y. Hancco. The second part is devoted on the study of constant anisotropic mean curvature surfaces in $\mathbb{R}^3$. It is the result of a joint work with my other advisor, J.A. Gálvez, and P. Mira, most of the results obtained in a year of investigation in Spain, as a part of the PDSE program, by CAPES.

I.1 Introduction to Part I

Semi-symmetric manifolds are a well-known and natural generalization of locally symmetric manifolds, they were introduced by E. Cartan in [22] and classified by Z.I. Szabó (see [96] and [97]). Investigation of several properties of semi-symmetric manifolds gives rise to their next generalization: the pseudo-symmetric manifolds. For example, *every totally umbilic submanifold of a semi-symmetric manifold, with parallel mean curvature vector, is pseudo-symmetric* (see [2]). The class of pseudo-symmetric manifolds is very large, and many examples of pseudo-symmetric manifolds which are not semi-symmetric have been constructed (see e.g. [34], [37] and references therein). In the last three decades, a big amount of results both intrinsic and extrinsic involving this class of manifolds have been published by several authors. Consequently many particular results are known, see, for example, [30], [34], [35], [36], [37], [38], [39], but a full classification is not available yet.

On the other hand, in the Submanifold Theory, extrinsic conditions analogue to local symmetry, semi-symmetry and pseudo-symmetry have been introduced and studied quite intensively. The notion of locally parallel immersions was introduced by D. Ferus in [45] as an extrinsic analogous to local symmetry and the same author obtained a local classification of such immersions in the Euclidean space and in the spheres of constant sectional curvature (see [44]) while in the hyperbolic spaces two classifications were obtained independently by Backes-Reckziegel (see [11]) and Takeuchi (see [99]). Curiously, we need to mention H.B. Lawson, who classified parallel hypersurfaces in the spheres before the definition of this class of submanifolds had been introduced, as we can see in [69].

Semi-parallel immersions were defined by J. Deprez in [32] as an extrinsic analogous to semi-symmetry. Many results on semi-parallel submanifolds can be found, for example, in [8], [32], [33], [41], [42], [76] and [77], but a classification is not available yet. However, we can find a complete classification of semi-parallel hypersurfaces in [33] for the Euclidean space and in [41] for real space forms.

An extrinsic analogous to pseudo-symmetry was first introduced by A.C. Asperti, G.A. Lobos and F. Mercuri in [9] in space forms: the class of pseudo-parallel submanifolds. Namely, an isometric immersion $f : M^n \to \tilde{M}^m$ is said to be *pseudo-parallel* if its second fundamental form α satisfies the following condition:

$$\tilde{R}(X,Y)\cdot\alpha = \phi(X\wedge Y)\cdot\alpha, \tag{I.1.1}$$

for some smooth real-valued function ϕ on M^n, where $\tilde{R}$ is the curvature tensor corresponding to the Van der Waerden-Bortolotti connection $\tilde{\nabla} = \nabla \oplus \nabla^{\perp}$ of f and for any $X, Y \in T_xM^n$, $x \in M^n$, $X \wedge Y$ denotes the endomorphism defined by

$$(X\wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y, \qquad Z \in T_xM^n.$$

In Equation (I.1.1) the notation means

$$[R(X,Y)\cdot\alpha](Z,W) = R^{\perp}(X,Y)\alpha(Z,W) - \alpha(R(X,Y)Z,W) - \alpha(Z,R(X,Y)W);$$
$$[(X\wedge Y)\cdot\alpha](Z,W) = -\alpha((X\wedge Y)Z,W) - \alpha(Z,(X\wedge Y)W),$$

for $Z, W \in T_xM^n$, where $R^{\perp}$ is the normal curvature tensor of f.

Some notable conclusions on the study of pseudo-parallel immersions in space forms are included in the references [10] and [72], where the authors proved that pseudo-parallel surfaces are either surfaces with flat normal bundle or isotropic surfaces in the sense of B. O'Neill [83] (i.e., surfaces whose ellipse of curvature at any point is a circle). In particular, they proved that pseudo-parallel surfaces of space forms with non-vanishing normal curvature in codimension 2 are superminimal surfaces in the sense of Bryant [14] (i.e., surfaces which are minimal and isotropic). They also obtained a classification of pseudo-parallel hypersurfaces in space forms. Essentially, such hypersurfaces are either *quasi-umbilic* hypersurfaces or *cyclids of Dupin*.

The next step is to study these kind of submanifolds in other ambient spaces. For example, in almost complex manifolds works were carried out by G. Lobos and M. Ortega in [73], where a local classification for pseudo-parallel real hypersurfaces was given. Essentially, pseudo-parallel real hypersurfaces M^{2n-1}, $n \geq 2$, of a complex space form $\tilde{M}^n(4c)$ with constant holomorphic sectional curvature $4c$ are either tubes over a totally geodesic $\mathbb{C}P^{n-1}$ or horospheres in $\mathbb{C}H^{n-1}$ or tubes over a totally geodesic $\mathbb{C}H^{n-1}$. This classification was useful later in the problem of classifying Hopf hypersurfaces with constant principal curvatures in the complex hyperbolic space $\mathbb{C}H^n$, as we can see in the eighth chapter of [24].

Other natural choices for ambient spaces are the remaining conformally flat symmetric spaces, $\mathbb{S}^n(c)\times\mathbb{R}$, $\mathbb{H}^n(c)\times\mathbb{R}$ and $\mathbb{S}^p(c)\times\mathbb{H}^{n+1-p}(-c)$ (see [90]), since they are Riemannian products of space forms and have the simplest curvature tensors apart from space forms. Here $\mathbb{S}^n(c)$ and $\mathbb{H}^n(c)$ denote the n-dimensional sphere and the n-dimensional hyperbolic space, respectively, and c denotes their sectional curvatures. Recently the product spaces $\mathbb{S}^n\times\mathbb{R}$ and $\mathbb{H}^n\times\mathbb{R}$ have attracted the attention of many mathematicians. Some important studies of submanifolds in these ambient spaces include: a generalization of the Hopf's differential for surfaces in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$, due to U. Abresch and H. Rosenberg in [1], permitting authors to prove that topological spheres with constant mean curvature immersed in these spaces are surfaces of revolution; a Fundamental Theorem for hypersurfaces in $\mathbb{S}^n\times\mathbb{R}$ and $\mathbb{H}^n\times\mathbb{R}$, obtained by B. Daniel in [29]; a classification of hypersurfaces with constant sectional curvature of $\mathbb{S}^n\times\mathbb{R}$ and $\mathbb{H}^n\times\mathbb{R}$, given by F. Manfio and R. Tojeiro in [81], for $n \geq 3$, and by J.A. Aledo, J.M. Espinar and J.A. Gálvez in [3] and [4], for $n = 2$.

With respect to the extrinsic notions we are interested in this work, we mention G. Calvaruso, D. Kowalczyk, J. Van der Veken and L. Vrancken, who in [16] and [102] obtained a local classification of umbilical, parallel and semi-parallel hypersurfaces of $\mathbb{H}^n\times\mathbb{R}$ and

$\mathbb{S}^n \times \mathbb{R}$, respectively. One interesting consequence of these classifications is that umbilicity does not imply parallelism (in space forms, as a consequence of Codazzi's Equation, totally geodesic and umbilical hypersurfaces are all parallel). A complete classification of umbilical submanifolds with any codimension in $\mathbb{S}^n \times \mathbb{R}$ was obtained by B. Mendonça and R. Tojeiro in [80] and the same authors gave a complete classification of parallel submanifolds in a product of two space forms in [79].

The study of pseudo-parallel hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ was started by F. Lin and B. Yang in [70] with a classification of pseudo-parallel hypersurfaces. They obtained an algebraic description of the Weingarten operator, which is presented in Lemma 3.1 of [70]. Also, they gave the geometric description of such hypersurfaces, except for a missing case, the class in which the Weingarten operator has three distinct eigenvalues. One of our objectives in this thesis is to complete this classification. A main step was to show that, even in the case in which the shape operator has three distinct eigenvalues, the tangent component T of $\frac{\partial}{\partial t}$ (the unit vector that spans the second factor of the ambient space) is in fact a principal direction. On the other hand, in [101] R. Tojeiro described explicitly this class of hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.

Putting together [10], [70] and [101], we show the following theorem:

Theorem I.1.1. *Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a pseudo-parallel hypersurface with three distinct principal curvatures. Then, $M^n = M^{n-1} \times \mathbb{R}$ and there exists a semi-parallel hypersurface $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ such that either $f(x,s) = (g(x), s)$ or f is given by*

$$f(x,s) = \cos(s)g(x) + \sin(s)N(x) + a(s)\frac{\partial}{\partial t}, \quad \text{if} \quad \epsilon = 1; \tag{I.1.2}$$

$$f(x,s) = \cosh(s)g(x) + \sinh(s)N(x) + a(s)\frac{\partial}{\partial t}, \quad \text{if} \quad \epsilon = -1; \tag{I.1.3}$$

for a linear function $a : \mathbb{R} \to \mathbb{R}$ with nowhere vanishing derivative. Here N denotes the unit normal vector field of g and the elements in equations (I.1.2) *and* (I.1.3) *are seen by their canonical inclusions in $\mathbb{E}^{n+2}$.*

In this way, from Theorem I.1.1 and the results obtained in [70], the classification of pseudo-parallel hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ becomes:

Theorem I.1.2 (Classification Theorem)**.** *Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a pseudo-parallel hypersurface. Then one of the following occurs:*

(i) $n = 2$ and ϕ is the Gaussian curvature;

(ii) f is umbilical;

(iii) f is a rotation hypersurface;

(iv) $f : M^{n-1} \times \mathbb{R} \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is given by $f(x,s) = (g(x), s)$, for a semi-parallel hypersurface $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$;

(v) There exists a semi-parallel hypersurface $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ such that $f : M^{n-1} \times \mathbb{R} \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is given by equations (I.1.2) *and* (I.1.3) *in terms of g and a linear function $a : \mathbb{R} \to \mathbb{R}$ with nowhere vanishing derivative.*

As a consequence of Theorem I.1.2 we obtain a classification of minimal and constant mean curvature pseudo-parallel hypersurfaces.

Corollary I.1.3. *Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ $(n \geq 3)$ be a pseudo-parallel hypersurface with constant mean curvature. Then f is either totally geodesic, a rotation hypersurface with constant mean curvature, or it is given as in item (iv) of Theorem I.1.2, where $g(M^{n-1})$ is an open part of $\mathbb{S}^k(c_1) \times \mathbb{S}^{n-k-1}(c_2)$ (resp. $\mathbb{H}^k(c_1) \times \mathbb{S}^{n-k-1}(c_2)$) if $\epsilon = 1$ (resp. $\epsilon = -1$), for some real numbers c_1, c_2 satisfying $\frac{1}{c_1} + \frac{1}{c_2} = \epsilon$ and some $k \in \{1, \dots, n-2\}$.*

Corollary I.1.4. *Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ $(n \geq 3)$ be a pseudo-parallel hypersurface. If f is minimal, then f is either a totally geodesic hypersurface, a minimal rotation hypersurface, or $\epsilon = 1$ and f is given as in item (iv) of Theorem I.1.2, where $g(M^{n-1})$ is an open part of $\mathbb{S}^k(\frac{n-1}{k}) \times \mathbb{S}^{n-k-1}(\frac{n-1}{n-k-1})$ for some $k \in \{1, \dots, n-2\}$.*

We recall that minimal and constant mean curvature rotation hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ were classified in [40].

In this work we also started the study of pseudo-parallel surfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ (with $\epsilon \neq 0$). The second main result of the first part of this work is a characterization of pseudo-parallel surfaces with non-vanishing normal curvature as isotropic surfaces, generalizing a similar result in space forms given by Asperti-Lobos-Mercuri in [10].

Theorem I.1.5. *Let $f : M^2 \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a ϕ-pseudo-parallel surface with nowhere vanishing normal curvature. Then f is λ-isotropic and*

$$\begin{aligned} &r(x)^2 = K - \phi > 0, \\ &\lambda^2 = 4K - 3\phi + \epsilon(\|T\|^2 - 1) > 0, \\ &\|H\|^2 = 3K - 2\phi + \epsilon(\|T\|^2 - 1) \geq 0, \end{aligned}$$

where K is the Gaussian curvature, λ is a smooth real-valued function on M^2, $r(x)$ is the radius of the ellipse of curvature of f at $x \in M^2$, H is the mean curvature vector field of f and T is the tangent part of $\frac{\partial}{\partial t}$, the canonical unit vector field tangent to the second factor of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. Conversely, if f is λ-isotropic then f is pseudo-parallel.

As a consequence of Theorem I.1.5, we have the following corollary concerning the non-existence of pseudo-parallel surfaces with non-vanishing normal curvature in co-dimension two.

Corollary I.1.6. There is no pseudo-parallel surface $f : M^2 \to \mathbb{Q}^3_\epsilon \times \mathbb{R}$ with nowhere vanishing normal curvature.

We organized the first part of the thesis into two chapters. In the first chapter we introduce the notations we use along the whole work and recall some concepts of the Submanifold Theory. We also recall the space forms and the product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, including for these last ones their Fundamental Equations for surfaces and hypersurfaces, essential tools that we make use in the next chapter. The next section is devoted to present rotation hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and the characterization of this class of hypersurfaces in terms of the second fundamental form. In the following section we recall the geometric description given by R. Tojeiro of hypersurfaces with T as a principal direction.

The second chapter is devoted to our study of pseudo-parallel immersions in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. We begin with precise definitions of the classes of submanifolds we are interested, recalling a characterization of semi-parallel hypersurfaces in space forms in terms of its Weingarten operator, and the classification of semi-parallel hypersurfaces in space forms and in the product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. In the following section we make an improvment of Lemma 3.1, in [70], showing that the second case in item (iii) does not occur and that T is the principal direction with related principal curvature equals to zero. After it, we present in the third section the proof of Theorem I.1.1 and the Classification of pseudo-parallel hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ or $\mathbb{H}^n \times \mathbb{R}$, after we put our result togheter with the partial classification given in [70]. We also give the proofs of Corollaries I.1.3 and I.1.4. In next two sections we study the surface case, observing that any isometric immersion $f : M^2 \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ with flat normal bundle is pseudo-parallel (see Proposition 2.4.2) and proving other auxiliary propositions. Then we prove Theorem I.1.5 and Corollary I.1.6. Although there no exist pseudo-parallel surfaces with non-vanishing normal curvature, as stated in Corollary I.1.6, the class of pseudo-parallel surfaces in $\mathbb{Q}^3_\epsilon \times \mathbb{R}$ is not empty. In Section 2.6 we give examples of semi-parallel surfaces which are not parallel as well as examples of pseudo-parallel surfaces in $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$ which are neither semi-parallel nor pseudo-parallel surfaces in a slice. Finally, we remark that pseudo-parallel surfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with $n \geq 4$ and non-vanishing normal curvature do exist, as shown in Examples 2.6.3, 2.6.5 and 2.6.6 in the Section 2.6.

I.2 Introduction to Part II

Surfaces of constant mean curvature are one of the oldest and most interesting topics in Differential Geometry. The reason for their importance relies on the fact that they appear as local solutions for the problem of minimizing area among the surfaces that enclose a prescribed volume: the famous Isoperimetric Problem.

Along the years, surfaces of constant mean curvature (CMC surfaces, for short) attracted the attention of many mathematicians and many progress have been done. Among the most important problems on CMC surfaces are those that involve global geometric or topological conditions as completeness, compactness, embeddedness, properness and stability. Some important achievements include: the Alexandrov's Theorem (see [6]), which states that the only compact CMC surface embedded in $\mathbb{R}^3$ is the round sphere; the well known Hopf's Theorem (see [60]), which states the only topological sphere with constant mean curvature immersed in the Euclidean space $\mathbb{R}^3$ is the round sphere; the results found in [78], where W. Meeks proved that any properly embedded annulus with constant mean curvature is contained in a solid half-cylinder. He also concluded that there is no proper embedded CMC surface Σ with just one end and if Σ has only two ends, then it is contained in a solid cylinder.

However surfaces of constant mean curvature are also objects of interest for physicists. For example, when two non mixing materials come into contact, the interface between them may often be represented as a surface. According to the law of least action, the equilibrium surface will form in such a way that it attempts to minimize its surface energy subject to constraints and additional forces imposed by the environment. For homogeneous materials, like soapy water, the surface energy (surface tension) is isotropic and proportional to the area of the surface interface. In this case, the process of minimization leads to the formation of minimal surfaces (when no volume constraints are imposed) and constant mean curvature surfaces (when a volume constraint is imposed), the mathematical models for soap films and soap bubbles, respectively. For other types of materials, like some fluids or cooling liquid crystals, a process of crystallization can eventually occur. In this case, their atomic or molecular structures assume a regular repeating pattern, and to model the shape of the interface of the fluid with its environment, we need to take the internal structure of the material into account: the usual isotropic surface energy must be replaced by an anisotropic one, that is, an energy that depends on the direction of the surface at each point.

Anisotropic surface energies were typically given by the functional

$$\mathcal{F}(\psi) = \int_\Sigma F(N(x))d\Sigma, \tag{I.2.1}$$

where $F : \Omega \subset \mathbb{S}^2 \to \mathbb{R}$ is a smooth function over an open subset Ω of $\mathbb{S}^2$ and $\psi : \Sigma \to \mathbb{R}^3$ is an immersion with unit normal field N. In particular, if $F \equiv 1$, then $\mathcal{F}$ becomes the well known area functional.

In this thesis we discuss surfaces that are equilibrium points of the functional $\mathcal{F}$, imposing or not volume constraints into the problem. The Euler–Lagrange equation for these problems characterize the equilibrium surfaces in terms of a certain quantity assigned on each of its points, that we call the anisotropic mean curvature. Namely, if Σ is any surface and $N : U \subset \Sigma \to \mathbb{S}^2$ is a unit normal vector field of Σ defined in an open subset U, the anisotropic mean curvature Λ of Σ with respect to N is given by the relation:

$$\Lambda(x) := -\operatorname{div}_\Sigma((\operatorname{grad}_{\mathbb{S}^2} F)_{N(x)}) + 2F(N(x))H(x), \tag{I.2.2}$$

where $H(x)$ denotes the mean curvature of Σ at x with respect to N and $(\operatorname{grad}_{\mathbb{S}^2} F)_{N(x)}$ is seen as a vector of $T_x\Sigma$, after we identify it with $T_{N(x)}\mathbb{S}^2$.

For each choice of a function $F : \mathbb{S}^2 \to \mathbb{R}$, there is a notable surface that we need to bear in mind: the Wulff shape of F (or the crystal of F). It was discovered by a russian chrystallographer called George Wulff, in the beginning of the twentieth century. Later, with the machinery of the Geometric Measure Theory, it was proven by J. Taylor in [100] that the Wulff shape is in fact the absolute minimizer of the functional $\mathcal{F}$ among all closed surfaces prescribing the same volume, a result known as the Wulff's Theorem.

Although the Wulff shape is well defined even when F is defined only in an open subset of $\mathbb{S}^2$, in this work we will consider only the case where F is a smooth function defined over the whole sphere $\mathbb{S}^2$. We also impose the "convexity condition": $(\operatorname{Hess}_{\mathbb{S}^2} F)_y + F(y)\langle , \rangle : T_y\mathbb{S}^2 \times T_y\mathbb{S}^2 \to \mathbb{R}$ is a positive-definite bilinear form, for any point $y \in \mathbb{S}^2$. The reason for these conditions is purely technical. They imply that the Wulff shape is a strictly convex, compact smooth surface, and from Wulff's Theorem we may infer that the Wulff shape has constant anisotropic mean curvature. From its minimizing and compactness properties, the Wulff shape will play in the theory of CAMC surfaces a similar role as the round sphere does in the theory of CMC surfaces.

Another reason to impose the convexity condition relies on the fact that it is possible to show that the equation for constant anisotropic mean curvature surfaces is absolut elliptic. This means that CAMC surfaces satisfy a Maximum Principle, in the sense of E. Hopf. In other words, if two surfaces have the same constant anisotropic mean curvature, and if the unit normal vectors (corresponding to the anisotropic mean curvature) of both surfaces coincide at a contact point in such a way that one surface lies on one side of the other in a neighborhood of such contact point, then the two surfaces agree in this neighborhood. Since this principle is one of the most important tools used in the study of CMC surfaces and it is also valid for CAMC surfaces, we expect that most of the results on CMC surfaces has extensions to CAMC surfaces.

Many studies on CAMC surfaces were carried up to this date, specially in the last two decades, when many advances were achieved. For example, B. Palmer proved in [84] that up to homotheties, the Wulff shape is the only closed, oriented, stable CAMC surface, an anisotropic extension of the famous theorem of J.L. Barbosa and M.P. do Carmo, in [12]. From the partnership of B. Palmer with M. Koiso arose many other important results. Some of them include: the study of cappilary surfaces; the description of the so called anisotropic Delaunay surfaces (i.e., rotation CAMC surfaces) as well the rolling construction to obtain their profile curves, obtained in [64] and [66], and generalizing the construction of Delaunay in [31]; they also proved an anisotropic version of the Hopf's Theorem in [65]. We also mention the works of H. Li and his collaborators. Among their results we cite an anisotropic version of the Alexandrov Theorem, obtained in [56] and the characterization of the Wulff shape in terms of higher order anisotropic mean curvatures, in [52]. Other interesting results we may cite are the classification of anisotropic isoparametric hypersurfaces due to Ge, in [46], and the study of CAMC helicoids by Kuhns, in [68]. We finally mention [71], where J.H.S. de Lira and M. Melo extend the notion of anisotropic mean curvature for immersed hypersurfaces of arbitrary Riemannian manifolds.

Motivated by the similarities between the theory of CMC and CAMC surfaces, our objective in this thesis will be present the anisotropic version of three important theorems. The first main result is a Bernstein-type Theorem for complete CAMC multigraphs, an anisotropic version of the theorem proved by D. Hoffman, R. Osserman and R. Schoen in [58].

Theorem I.2.1. Let Σ be a vertical multi-graph in $\mathbb{R}^3$, that is, for any $p \in \Sigma$, $T_p\Sigma$ is not a vertical plane. Suppose that Σ is complete and has constant anisotropic mean curvature. Then Σ is a plane.

An immediate consequence of Theorem I.2.1 is the following corollary.

Corollary I.2.2. Let Σ be a complete CAMC surface whose Gauss map image is contained in a closed hemisphere of $\mathbb{S}^2$. Then Σ is either a plane or a CAMC cylinder.

The second main result we obtained in the second part the thesis is also a consequence of Theorem I.2.1, where we generalize a theorem about CMC surfaces whose Gauss curvature does not change sign, a result found in [63], due to T. Klotz and R. Osserman.

Theorem I.2.3. Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed surface of constant anisotropic mean curvature $\Lambda \neq 0$. If the Gaussian curvature of Σ does not change sign then Σ is one of the following surfaces:

(i) a CAMC cylinder;

(ii) the Wulff shape (up to a homothety).

In the third main theorem of the second part of this thesis we develop an uniform upper bound for the maximum height of CAMC graphs with planar boundary, depending only on the anisotropic mean curvature. More presicely, we have

Theorem I.2.4. Let $\Lambda \neq 0$ be a real number and let $v \in \mathbb{S}^2$ be any unit vector. Then there is a constant $C = C(\Lambda)$ such that for any closed (not necessary bounded) domain Ω of the plane $\Pi = \{v\}^\perp$ and smooth function $u : \Omega \to \mathbb{R}$ that vanishes on $\partial\Omega$ and whose graph Σ over Π is a Λ-CAMC surface, the height of any point $p \in \Sigma$ relative to Π is at most C.

The isotropic version of Theorem I.2.4 was first obtained by W. Meeks in [78], and lately generalized by J.A. Aledo, J.M. Espinar and J.A. Gálvez in [5], for more general classes of surfaces that satisfy the Maximum Principle, such as special Weingarten surfaces, i.e., surfaces that satisfies a relation $H = f(H^2 - K)$, for a certain function f, where H and K denotes the constant mean curvature and the Gaussian curvature of the surfaces, respectively. Although Theorem I.2.4 does not provide optimum estimates, we recall that its isotropic version was proved to be a fundamental tool for the study of properly embedded CMC surfaces in $\mathbb{R}^3$. Among the consequences of these height estimates for graphs are the non-existence of properly embedded CMC surfaces with only one end, as we mentioned previously in the second paragraph, and that any two-ended properly embedded CMC surfaces are revolution surfaces.

Our main difficulty in the proof of Theorem I.2.4 relies on the fact that the functional $\mathcal{F}$ is not necessarily invariant under reflections. In other words, if Π is a plane and Σ is a CAMC surface, then the image of Σ under the reflection over Π is not necessarily a CAMC surface, since the values of F in a point an its reflected image could be distinct. This fact prohibits the use of the Alexandrov's Method of Moving Planes, that was used in the proof of the isotropic version of Theorem I.2.4.

As a corollary of Theorem I.2.4 we prove uniform estimates for compact surfaces with non-zero constant anisotropic mean curvature and planar boundary.

Corollary I.2.5. Suppose that the anisotropy function F is invariant under the reflection in $\mathbb{S}^2$ that fixes the geodesic $\mathbb{S}^2 \cap \{v\}^\perp$, for some $v \in \mathbb{S}^2$. Let Σ be any compact Λ-CAMC surface ($\Lambda \neq 0$), that is embedded in $\mathbb{R}^3$ and whose boundary is contained in the plane $\{v\}^\perp$. There exists a positive constant C depending only on Λ such that the height of any point $p \in \Sigma$ relative to $\{v\}^\perp$ is at most C.

Under certain symmetry hypothesis on the anisotropy function and proving an anisotropic version of the famous Meeks' Separation Lemma (see [78] and [67]), we are able to prove the non-existence of properly embedded surfaces in $\mathbb{R}^3$ with non-zero constant anisotropic mean curvature and with just one end.

Theorem I.2.6. Let $\Sigma \subset \mathbb{R}^3$ be a properly embedded Λ-CAMC ($\Lambda \neq 0$) surface with finite topology and at most one end. Consider three linearly independent vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ and suppose in addition that the anisotropy function F is invariant under the reflections in $\mathbb{S}^2$ which fix the geodesics $\mathbb{S}^2 \cap \{v_i\}^\perp$, for $i \in \{1, 2, 3\}$. Then, up to a homothety, Σ is the Wulff shape.

We also organize the second part of this thesis into two chapters. In chapter three we devote attention on the introduction of the basic concepts on the theory of constant anisotropic mean curvature surfaces. In the first section we present the variational problem, whose critical points are our object of interest. To characterize such critical points in terms of the anisotropy function and the mean curvature, we recall the first variation formula, which leads us to the definition of the anisotropic mean curvature of a surface. Next, we present the Wulff shape, relating its geometric construction with its analytic description. This allow us, in the following section, to define the anisotropic analogous of the normal Gauss map and the second fundamental form. In particular, the anisotropic mean curvature of a surface will be given by the trace of the anisotropic second fundamental form. A brief section with examples of CAMC surfaces was included, where a comparison between the isotropic and anisotropic cases can be made by the reader. Namely, we recall the construction of rotation CAMC surfaces, CAMC helicoids and CAMC cylinders. This last one, in special, will play a crucial role in the results obtained in the next chapter.

The proofs of the main results of this thesis, namely, Theorems I.2.1, I.2.3 and I.2.4, are the contents of Chapter four. To overcome the difficulties related to the Method of Moving Planes, we follow the ideas found in [15]. First, we adapt a Compactness Theorem for CAMC surfaces with bounded second fundamental form and whose anisotropic mean curvatures converge to a pre-fixed real number. This result allows us to obtain complete CAMC surfaces as limits of sequences of CAMC surfaces over compact sets, and it will be applied along the whole chapter. Another useful result we adapt from [15] was a priori second fundamental form estimates for CAMC surfaces whose anisotropic mean curvature is bounded by a prefixed positive number and whose Gauss map omits a disk of prefixed area. A key step for its proof was the use of a Bernstein-type Theorem for anisotropic minimal surfaces, due to H.B. Jenkins (see [62]), that we also recall in the text. We also prove horizonal diameter estimates in the sense of [78], i.e., diameter estimates for the connected components of horizontal slices of CAMC graphs defined on closed domains with zero boundary values. These estimates depend only on the anisotropic mean curvature. Its proof is based on the ideas of [5]. Following these auxiliary results, we give a proof of Theorem I.2.1. Although the isotropic version of Theorem I.2.1 is proven using arguments on the harmonicity of the Gauss map N of the surface and properties of the equation $\Delta N + \|dN\|^2 N = 0$ (where Δ denotes the Laplacian operator of the surface), our proof is geometric and part of it is based on the ideas found in [51]. Putting together Theorem I.2.1 and the auxiliary results, we are able to prove Theorem I.2.4 and its consequences, including the study of properly embedded CAMC surfaces in $\mathbb{R}^3$ with finite topology, where we conclude with the proof of Theorem I.2.6.

Finally, for the reader convenience, we add an detailed appendix about the Maximum Principle. We recall the Maximum Principle for linear and quasi-linear second-order elliptic differential operators and we stablish a geometric version of the Maximum Principle, known as the Tangency Principle, for the anisotropic mean curvature operator.

Introducción

Este trabajo está dividido en dos temas principales. La primera parte se refiere al estudio de las inmersiones pseudo-paralelas en los espacios producto $\mathbb{S}^n \times \mathbb{R}$ y $\mathbb{H}^n \times \mathbb{R}$. Esa parte fue desarrollada con mi director de tesis, G.A. Lobos, y con la colaboración de A.J.Y. Hancco. La segunda parte se dedica al estudio de las superficies de curvatura media anisotrópica constante en $\mathbb{R}^3$. Ese estudio es el resultado de un trabajo conjunto con mi otro director, J.A. Gálvez, y P. Mira, la mayoría de los resultados obtenidos durante un año de investigación en España, como parte del programa PDSE, por CAPES.

I.3 Introducción a la Parte I

Las variedades semi-simétricas son una generalización natural y bien conocida de las variedades localmente simétricas, fueron introducidas por E. Cartan en [22] y clasificadas por Z.I. Szabó (ver [96] y [97]). La investigación de varias propiedades de las variedades semi-simétricas da lugar a su próxima generalización: las variedades pseudo-simétricas. Por ejemplo, *cualquier subvariedad totalmente umbilical de una variedad semi-simétrica, con vector de curvatura media paralelo, es pseudo-simétrica* (ver [2]). La clase de variedades pseudo-simétricas es muy grande, y se han construido muchos ejemplos de variedades pseudo-simétricas que no son semi-simétricas (véanse, por ejemplo, [34], [37] y sus referencias). En las últimas tres décadas, varios autores han publicado una gran cantidad de resultados tanto intrínsecos como extrínsecos acerca de esta clase de variedades. Como consecuencia, se conocen muchos resultados particulares, véase, por ejemplo, [30], [34], [35], [36], [37], [38], [39], pero una clasificación completa aún no existe.

Por otro lado, en la Teoría de Subvariedades, condiciones extrínsecas análogas a las de simetría-local, de semi-simetría y de pseudo-simetría han sido introducidas y estudiadas con bastante intensidad. Ferus introdujo la noción de inmersiones localmente paralelas en [45] como un análogo extrínseco a la simetría-local y el mismo autor obtuvo una clasificación local de tales inmersiones en el espacio euclídeo y en las esferas (ver [44]) mientras que en los espacios hiperbólicos se obtuvieron dos clasificaciones independientemente por Backes-Reckziegel (ver [11]) y Takeuchi (ver [99]). Curiosamente, debemos mencionar a H.B. Lawson, que clasificó las hipersuperficies paralelas en las esferas antes de que se introdujera la definición de esta clase de subvariedades, como podemos ver en [69].

Las inmersiones semi-paralelas fueron definidas por J. Deprez en [32] como un análogo extrínseco a la semi-simetría. Se pueden encontrar muchos resultados acerca de subvariedades semi-paralelas, por ejemplo, en [8], [32], [33], [41], [42], [76] y [77], pero una clasificación aún no está disponible. Sin embargo, podemos encontrar una clasificación completa de las hipersuperficies semi-paralelas en [33] para el espacio euclídeo y en [41] para formas espaciales.

Un análogo extrínseco a la pseudo-simetría fue introducido por primera vez por A.C. Asperti, G.A. Lobos y F. Mercuri en [9] en formas espaciales: la clase de subvariedades pseudo-paralelas. A saber, una inmersión isométrica $f : M^n \to \tilde{M}^m$ se dice que es *pseudo-paralela* si su segunda forma fundamental α satisface la siguiente condición:

$$\tilde{R}(X,Y)\cdot\alpha = \phi(X\wedge Y)\cdot\alpha, \tag{I.3.1}$$

para alguna función a valores reales ϕ definida en M^n, donde $\tilde{R}$ es el tensor de curvatura correspondiente a la conexión de Van der Waerden-Bortolotti $\tilde{\nabla} = \nabla \oplus \nabla^{\perp}$ de f y para cualquier $X, Y \in T_xM^n$, $x \in M^n$, $X \wedge Y$ denota el endomorfismo definido por

$$(X\wedge Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y, \qquad Z \in T_xM^n.$$

En la ecuación (I.3.1) la notación significa

$$[R(X,Y)\cdot\alpha](Z,W) = R^{\perp}(X,Y)\alpha(Z,W) - \alpha(R(X,Y)Z,W) - \alpha(Z,R(X,Y)W);$$
$$[(X\wedge Y)\cdot\alpha](Z,W) = -\alpha((X\wedge Y)Z,W) - \alpha(Z,(X\wedge Y)W),$$

para $Z, W \in T_xM^n$, donde $R^{\perp}$ es el tensor de curvatura normal de f.

Algunas conclusiones notables sobre el estudio de las inmersiones pseudo-paralelas en formas espaciales se incluyen en las referencias [10] y [72], donde los autores probaron que las superficies pseudo-paralelas son superficies con fibrado normal plano o superficies isotrópicas en el sentido de B. O'Neill [83] (es decir, superficies cuya elipse de curvatura en cualquier punto es un círculo). En particular, demostraron que las superficies pseudo-paralelas en formas espaciales con curvatura normal que no se anula y codimensión 2 son superficies superminimales en el sentido de Bryant [14] (es decir, superficies que son minimales y isotrópicas). También obtuvieron una clasificación de las hipersuperficies pseudo-paralelas en formas espaciales. Esencialmente, tales hipersuperficies son o bien hipersuperficies *cuasi-umbilicales* o bien *cíclides de Dupin*.

El siguiente paso es estudiar este tipo de subvariedades en otros espacios ambientes. Por ejemplo, en variedades cuasi-complejas trabajos fueron realizados por G. Lobos y M. Ortega en [73], donde una clasificación local para las hipersuperficies pseudo-paralelas reales fue dada. Esencialmente, las hipersuperficies pseudo-paralelas reales M^{2n-1}, $n \geq 2$, de una forma espacial compleja $\tilde{M}^n(4c)$ con curvatura seccional holomorfa constante $4c$ o son tubos sobre un $\mathbb{C}P^{n-1}$ totalmente geodésico o bien horosferas en $\mathbb{C}H^{n-1}$ o tubos sobre un $\mathbb{C}H^{n-1}$ totalmente geodésico. Esta clasificación fue útil más tarde en el problema de clasificar las hipersuperficies de Hopf con curvaturas principales constantes en el espacio hiperbólico complejo $\mathbb{C}H^n$, como podemos ver en el octavo capítulo de [24].

Otras opciones naturales para espacios ambientes son los espacios simétricos conformemente planos restantes, $\mathbb{S}^n(c)\times\mathbb{R}$, $\mathbb{H}^n(c)\times\mathbb{R}$ y $\mathbb{S}^p(c)\times\mathbb{H}^{n+1-p}(-c)$ (ver [90]), ya que son productos riemannianos de formas espaciales y tienen los tensores de curvatura más sencillos, después de las formas espaciales. Aquí $\mathbb{S}^n(c)$ y $\mathbb{H}^n(c)$ denotan la esfera n-dimensional y el espacio hiperbólico n-dimensional, respectivamente, y c denota sus curvaturas seccionales. Recientemente, los espacios producto $\mathbb{S}^n\times\mathbb{R}$ y $\mathbb{H}^n\times\mathbb{R}$ han atraído la atención de muchos matemáticos. Algunos estudios importantes de subvariedades en estos espacios ambientes incluyen: una generalización de la diferencial de Hopf para superficies en $\mathbb{S}^2\times\mathbb{R}$ y $\mathbb{H}^2\times\mathbb{R}$, debido a U. Abresch y H. Rosenberg in [1], lo que permite a los autores demostrar que las esferas topológicas con curvatura media constante inmersas en estos espacios son superficies de revolución; un Teorema Fundamental para hipersuperficies en $\mathbb{S}^n\times\mathbb{R}$ y $\mathbb{H}^n\times\mathbb{R}$, obtenido por B. Daniel en [29]; una clasificación de hipersuperficies con curvatura seccional constante de $\mathbb{S}^n\times\mathbb{R}$ y $\mathbb{H}^n\times\mathbb{R}$, dada por F. Manfio y R. Tojeiro en [81], para $n \geq 3$, y por J.A. Aledo, J.M. Espinar y J.A. Gálvez en [3] y [4], para $n = 2$..

Con respecto a las nociones extrínsecas que estamos interesados en ese trabajo, mencionamos a G. Calvaruso, D. Kowalczyk, J. Van der Veken y L. Vrancken, quienes en [16] y [102] obtuvieron una clasificación local de las superficies umbilicales, paralelas y semi-paralelas de $\mathbb{H}^n \times \mathbb{R}$ y $\mathbb{S}^n \times \mathbb{R}$, respectivamente. Una consecuencia interesante de estas clasificaciones es que la umbilicidad no implica paralelismo (en formas espaciales, como consecuencia de la ecuación de Codazzi, las hipersuperficies totalmente geodésicas y umbilicales son todas paralelas). Posteriormente, B. Mendonça y R. Tojeiro obtuvieron en [80] una clasificación completa de las subvariedades umbilicales con cualquier codimensión en $\mathbb{S}^n \times \mathbb{R}$ y los mismos autores dieron una clasificación completa de subvariedades paralelas en un producto de dos formas espaciales en [79].

El estudio de hipersuperficies pseudo-paralelas en $\mathbb{S}^n \times \mathbb{R}$ y $\mathbb{H}^n \times \mathbb{R}$ fue iniciado por F. Lin y B. Yang, en [70], con una clasificación de hipersuperficies pseudo-paralelas. Obtuvieron una descripción algebraica del operador de Weingarten, que es presentado en el Lema 3.1 de [70]. Además, dieron la descripción geométrica de tales hipersuperficies, a excepción de un caso que falta: la clase en la que el operador de Weingarten tiene tres valores propios distintos. Uno de nuestros objetivos en esta tesis es completar esta clasificación. Un paso clave fue mostrar que, incluso en el caso en que el operador forma tiene tres valores propios distintos, la componente tangente T de $\frac{\partial}{\partial t}$ (el vector unitario tangente al segundo factor del espacio ambiente) es, de hecho, una dirección principal. Por otro lado, en [101] R. Tojeiro describió explícitamente esta clase de hipersuperficies en $\mathbb{S}^n \times \mathbb{R}$ y $\mathbb{H}^n \times \mathbb{R}$.

Al juntar los resultados de [10], [70] y [101], mostramos el siguiente teorema:

Teorema I.3.1. *Sea $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ una hipersuperficie pseudo-paralela con tres curvaturas principales distintas. Entonces, $M^n = M^{n-1} \times \mathbb{R}$ y existe una hipersuperficie semi-paralela $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ tal que $f(x,s) = (g(x), s)$ o f viene dada por*

$$f(x,s) = \cos(s)g(x) + \sin(s)N(x) + a(s)\frac{\partial}{\partial t}, \quad si \quad \epsilon = 1; \tag{I.3.2}$$

$$f(x,s) = \cosh(s)g(x) + \sinh(s)N(x) + a(s)\frac{\partial}{\partial t}, \quad si \quad \epsilon = 1; \tag{I.3.3}$$

en términos de g y una función lineal $a : \mathbb{R} \to \mathbb{R}$ con derivada que nunca se anula. Aquí N denota el campo normal unitario de g y los elementos de las ecuaciones (I.3.2) *y* (I.3.3) *son vistos como sus inclusiones canónicas en $\mathbb{E}^{n+2}$*

En ese sentido, del Teorema I.3.1 y los resultados obtenidos en [70], la clasificación de las hipersuperficies pseudo-paralelas de $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ se torna:

Teorema I.3.2 (Teorema de Clasificación)**.** *Sea $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ una hipersuperficie pseudo-paralela. Entonces ocurre una de las siguientes:*

(i) $n = 2$ *y* ϕ *es la curvatura gaussiana;*

(ii) f *es umbilical;*

(iii) f *es una hipersuperficie de rotación;*

(iv) $f : M^{n-1} \times \mathbb{R} \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ *es dada por* $f(x,s) = (g(x), s)$*, para una hipersuperficie semi-paralela* $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$*;*

(v) Existe una hipersuperficie semi-paralela $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ *tal que* $f : M^{n-1} \times \mathbb{R} \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ *es dada por las ecuaciones* (I.3.2) *y* (I.3.3) *en términos de* g *y de una función lineal* $a : \mathbb{R} \to \mathbb{R}$ *con derivada que nunca se anula.*

Como consecuencia del Teorema I.3.2 obtenemos una clasificación de hipersuperficies pseudo-paralelas minimales y de curvatura media constante.

Corolario I.3.3. *Sea $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ $(n \geq 3)$ una hipersuperficie pseudo-paralela con curvatura media constante. Entonces f es o totalmente geodésica, o una hipersuperficie de rotación con curvatura media constante, o es dada como en el ítem (iv) del Teorema I.3.2, donde $g(M^{n-1})$ es un abierto de $\mathbb{S}^k(c_1) \times \mathbb{S}^{n-k-1}(c_2)$ (resp. $\mathbb{H}^k(c_1) \times \mathbb{S}^{n-k-1}(c_2)$) si $\epsilon = 1$ (resp. $\epsilon = -1$), para algunos números reales c_1, c_2 satisfaciendo $\frac{1}{c_1} + \frac{1}{c_2} = \epsilon$ y algún $k \in \{1, \ldots, n-2\}$.*

Corolario I.3.4. *Sea $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ $(n \geq 3)$ una hipersuperficie pseudo-paralela. Si f es minimal, entonces f es una hipersuperficie totalmente geodésica, o una hipersuperficie de revolución minimal, o $\epsilon = 1$ y f es dada como en el ítem (iv) del Teorema I.3.2, donde $g(M^{n-1})$ es un abierto de $\mathbb{S}^k(\frac{n-1}{k}) \times \mathbb{S}^{n-k-1}(\frac{n-1}{n-k-1})$ para algún $k \in \{1, \ldots, n-2\}$.*

Recordamos que las hipersuperficies de rotación minimales y con curvatura media constante en $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ fueron clasificadas en [40].

En este trabajo también comenzamos el estudio de superficies pseudo-paralelas en $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ (con $\epsilon \neq 0$). El segundo resultado principal de la primera parte de este trabajo es una caracterización de superficies pseudo-paralelas con curvatura normal que no se anula como superficies isotrópicas, generalizando un resultado similar en formas espaciales dadas por Asperti-Lobos-Mercuri en [10].

Teorema I.3.5. *Sea $f : M^2 \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ una superficie ϕ-pseudo-paralela cuya curvatura normal no es idénticamente nula en M^2. Entonces f es λ-isotrópica y*

$$
\begin{aligned}
&r(x)^2 = K - \phi > 0,\\
&\lambda^2 = 4K - 3\phi + \epsilon(\|T\|^2 - 1) > 0,\\
&\|H\|^2 = 3K - 2\phi + \epsilon(\|T\|^2 - 1) \geq 0,
\end{aligned}
$$

donde K es la curvatura gaussiana, λ es una función suave definida en M^2, $r(x)$ es el radio de la elipse de curvatura de f en $x \in M^2$, H es el vector curvatura media de f y T es la parte tangente de $\frac{\partial}{\partial t}$, el campo vectorial unitario tangente al segundo factor de $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. Recíprocamente, si f es λ-isotrópica, entonces f es pseudo-paralela.

Como una consecuencia del Teorema I.3.5, tenemos el siguiente corolario que establece la inexistencia de superficies pseudo-paralelas con curvatura normal que no se anula, en codimensión 2.

Corolario I.3.6. No existe superficie pseudo-paralela $f : M^2 \to \mathbb{Q}^3_\epsilon \times \mathbb{R}$ con curvatura normal que no se anula.

Organizamos la primera parte de la tesis en dos capítulos. En el primer capítulo presentamos las notaciones que usamos a lo largo de todo el trabajo y recordamos algunos conceptos de la Teoría de Subvariedades. También recordamos las formas espaciales y los espacios producto $\mathbb{S}^n \times \mathbb{R}$ y $\mathbb{H}^n \times \mathbb{R}$, incluyendo para estos últimos sus Ecuaciones Fundamentales para superficies e hipersuperficies, herramientas esenciales que usaremos en el próximo capítulo. La siguiente sección está dedicada a presentar las hipersuperficies de rotación de $\mathbb{S}^n \times \mathbb{R}$ y $\mathbb{H}^n \times \mathbb{R}$ y la caracterización de esa clase de hipersuperficies en términos de la segunda forma fundamental. En la siguiente sección recordamos la descripción geométrica dada por R. Tojeiro de las hipersuperficies que tienen a T como una dirección principal.

El segundo capítulo está dedicado a nuestro estudio de inmersiones pseudo-paralelas en $\mathbb{S}^n \times \mathbb{R}$ y $\mathbb{H}^n \times \mathbb{R}$. Comenzamos con definiciones precisas de las clases de subvariedades que nos interesan, recordando una caracterización de las hipersuperficies semi-paralelas en formas espaciales en términos del operador de Weingarten, y la clasificación de las hipersuperficies semi-paralelas en formas espaciales y en los espacios producto $\mathbb{S}^n \times \mathbb{R}$ y $\mathbb{H}^n \times \mathbb{R}$. En la siguiente sección realizamos una mejora del Lemma 3.1, en [70], donde probamos que el segundo caso en el ítem (iii) no ocurre y que T es la dirección principal con curvatura principal relacionada igual a cero. A continuación, presentamos en la tercera sección la prueba del Teorema I.3.1 y la Clasificación de las hipersuperficies pseudo-paralelas de $\mathbb{S}^n \times \mathbb{R}$ y $\mathbb{H}^n \times \mathbb{R}$, después de poner nuestro resultado junto con la clasificación parcial dada en [70]. También proporcionamos las pruebas de los Corolarios I.3.3 y I.3.4. En las siguientes dos secciones estudiamos el caso de superficies, observando que cualquier inmersión isométrica $f : M^2 \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ con fibrado normal plano es pseudo-paralela (ver Proposición 2.4.2) y probamos otras proposiciones auxiliares. Luego demostramos el Teorema I.3.5 y el Corolario I.3.6. Aunque no existen superficies pseudo-paralelas con curvatura normal no idénticamente nula, como se indica en el Corolario I.3.6, la clase de superficies pseudo-paralelas en $\mathbb{Q}^3_\epsilon \times \mathbb{R}$ es no vacía. En la Sección 2.6 damos ejemplos de superficies semi-paralelas que no son paralelas, así como ejemplos de superficies pseudo-paralelas en $\mathbb{S}^3 \times \mathbb{R}$ y $\mathbb{H}^3 \times \mathbb{R}$ que no son superficies semi-paralelas ni pseudo-paralelas en un corte $\mathbb{Q}^3_\epsilon \times \{t\}$. Finalmente, observamos que existen superficies pseudo-paralelas en $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ con $n \geq 4$ y curvatura normal no idénticamente nula, como se muestra en los Ejemplos 2.6.3, 2.6.5 y 2.6.6 en la Sección 2.6.

I.4 Introducción a la Parte II

Las superficies de curvatura media constante son uno de los temas más antiguos e interesantes en Geometría Diferencial. La razón de su importancia reside en el hecho de que surgen como soluciones locales para el problema de minimizar el área entre las superficies que encierran un volumen prescrito: el famoso Problema Isoperimétrico.

A lo largo de los años, las superficies de curvatura media constante (superficies CMC, para abreviar) atrajeron la atención de muchos matemáticos y se han hecho muchos progresos desde entonces. Entre los problemas más importantes en las superficies de CMC están aquellos que envólven condiciones geométricas o topológicas globales como completitud, compacidad, embebimiento, ser propiamente inmersa y estabilidad. Algunos logros importantes incluyen: el Teorema de Alexandrov (ver [6]), que establece que la única superficie CMC compacta y embebida en $\mathbb{R}^3$ es la esfera redonda; el conocido Teorema de Hopf (ver [60]), que establece que la única esfera topológica con curvatura media constante inmersa en el espacio euclídeo $\mathbb{R}^3$ es la esfera redonda; los resultados que se encuentran en [78], donde W. Meeks demostró que cualquier anillo propiamente embebido con curvatura media constante está contenido en un semi-cilindro sólido. También concluyó que no hay una superficie CMC Σ propiamente embebida con solo un final y si Σ tiene solo dos finales, entonces está contenida en un cilindro sólido.

Sin embargo, las superficies de curvatura media constante también son objetos de interés para los físicos. Por ejemplo, cuando dos materiales que no se mezclan entran en contacto, la interfaz entre ellos a menudo se puede representar como una superficie. De acuerdo con la ley de mínima acción, la superficie de equilibrio se formará de tal manera que intente minimizar su energía superficial sujeta a restricciones y fuerzas adicionales impuestas por el medio ambiente. Para materiales homogéneos, como el agua con jabón, la energía superficial (tensión superficial) es isotrópica y proporcional al área de la interfaz de la superficie. En este caso, el proceso de minimización conduce a la formación de superficies minimales (cuando no se imponen restricciones de volumen) y superficies de curvatura media constante (cuando se impone restricción al volumen), los modelos matemáticos para películas de jabón y pompas de jabón, respectivamente. Para otros tipos de materiales, como algunos fluidos o cristales líquidos en enfriamiento, puede ocurrir un proceso de cristalización. En este caso, sus estructuras atómicas o moleculares asumen un patrón regular repetitivo, y para modelar la forma de la interfaz del fluido con su entorno, debemos tener en cuenta la estructura interna del material: la energía superficial isotrópica habitual debe ser reemplazada por una anisotrópica, es decir, una energía que depende de la dirección de la superficie en cada punto.

Las energías superficiales anisotrópicas són típicamente dadas por el funcional

$$\mathcal{F}(\psi) = \int_{\Sigma} F(N(x))d\Sigma, \tag{I.4.1}$$

donde $F : \Omega \subset \mathbb{S}^2 \to \mathbb{R}$ es una función suave sobre un subconjunto abierto Ω de $\mathbb{S}^2$ y $\psi : \Sigma \to \mathbb{R}^3$ es una inmersión con campo normal unitario N. En particular, si $F \equiv 1$, entonces $\mathcal{F}$ se convierte en el bien conocido funcional área.

En esta tesis discutimos las superficies que son puntos de equilibrio del funcional $\mathcal{F}$, imponiendo o no restricciones de volumen en el problema. La ecuación de Euler-Lagrange para estos problemas caracteriza las superficies de equilibrio en términos de una cierta cantidad asignada en cada uno de sus puntos, que llamamos curvatura media anisotrópica. Es decir, si Σ es cualquier superficie y $N : U \subset \Sigma \to \mathbb{S}^2$ es un campo vectorial normal unitario de Σ definido en un subconjunto abierto U, la curvatura media anisotrópica Λ de Σ con respecto a N viene dada por la relación:

$$\Lambda(x) := -\operatorname{div}_\Sigma((\operatorname{grad}_{\mathbb{S}^2} F)_{N(x)}) + 2F(N(x))H(x), \tag{I.4.2}$$

donde $H(x)$ denota la curvatura media de Σ en x con respecto a N y $(\operatorname{grad}_{\mathbb{S}^2} F)_{N(x)}$ se ve como un vector de $T_x\Sigma$, después de identificarlo con $T_{N(x)}\mathbb{S}^2$.

Para cada elección de una función $F : \mathbb{S}^2 \to \mathbb{R}$, hay una superficie notable que debemos tener en cuenta: la forma Wulff de F (o el cristal de F). Fue descubierto por un cristalógrafo ruso llamado George Wulff, a principios del siglo XX. Más tarde, con la maquinaria de la Teoría Geometrica de la Medida, J. Taylor demostró en [100] que la forma de Wulff es, de hecho, el minimizante absoluto del funcional $\mathcal{F}$ entre todas las superficies cerradas prescribiendo el mismo volumen, un resultado conocido como el Teorema de Wulff.

Aunque la forma de Wulff esté bien definida mismo cuando F está definida solamente en un subconjunto abierto de $\mathbb{S}^2$, en este trabajo consideraremos solo el caso donde F es una función suave definida sobre toda la esfera $\mathbb{S}^2$. También imponemos la "condición de convexidad": $(\operatorname{Hess}_{\mathbb{S}^2} F)_y + F(y)\langle , \rangle : T_y\mathbb{S}^2 \times T_y\mathbb{S}^2 \to \mathbb{R}$ es una forma bilineal definida positiva, para cualquier punto $y \in \mathbb{S}^2$. La razón de estas condiciones es puramente técnica. Implican que la forma de Wulff es una superficie suave estrictamente convexa y compacta, y del Teorema de Wulff podemos inferir que la forma de Wulff tiene una curvatura media anisotrópica constante. Desde sus propiedades de minimización y compacidad, la forma de Wulff jugará en la teoría de superficies CMAC un papel similar al de la esfera redonda en la teoría de superficies CMC.

Otra razón para imponer la condición de convexidad reside en el hecho de que es posible demostrar que la ecuación para superficies de curvatura media anisotrópica constante es absolutamente elíptica. Esto significa que las superficies CMAC satisfacen un Principio del Máximo, en el sentido de E. Hopf. En otras palabras, si dos superficies tienen la misma curvatura media anisotrópica constante, y si los vectores unitarios normales (correspondientes a la curvatura media anisotrópica) de ambas superficies coinciden en un punto de contacto de tal manera que una superficie se encuentra en un lado de la otra en un entorno del punto de contacto, entonces las dos superficies coinciden en ese entorno. Dado que este principio es una de las herramientas más importantes utilizadas en el estudio de las superficies CMC y también es válido para las superficies CMAC, esperamos que la mayoría de los resultados en las superficies CMC tengan extensiones a las superficies CMAC.

Muchos estudios sobre superficies CMAC se llevaron a cabo hasta esta fecha, especialmente en las últimas dos décadas, cuando se lograron muchos avances. Por ejemplo, B. Palmer demostró en [84] que salvo homotecias, la forma de Wulff es la única superficie CMAC cerrada, orientada y estable, una extensión anisotrópica del famoso teorema de J.L. Barbosa y M.P. do Carmo, en [12]. De la asociación de B. Palmer con M. Koiso surgieron muchos otros resultados importantes. Algunos de ellos incluyen: el estudio de las superficies capilares; la descripción de las llamadas superficies anisotrópicas de Delaunay (es decir, superficies CMAC de revolución), así como la construcción rodante para obtener sus curvas perfiles, obtenida en [64] y [66], y generalizando la construcción de Delaunay en [31]; también probaron una versión anisotrópica del Teorema de Hopf en [65]. También mencionamos los trabajos de H. Li y sus colaboradores. Entre sus resultados, citamos una versión anisotrópica del Teorema de Alexandrov, obtenida en [56] y la caracterización de la forma de Wulff en términos de curvaturas medias anisotrópicas de alto orden, en [52]. Otros resultados interesantes que podemos citar son la clasificación de las hipersuperficies isoparamétricas anisotrópicas debido a Ge, en [46], y el estudio de los helicoides CMAC por Kuhns, en [68]. Finalmente mencionamos [71], donde J.H.S. de Lira y M. Melo extienden la noción de curvatura media anisotrópica para hipersuperficies inmersas en variedades riemannianas arbitrarias.

Motivado por las similitudes entre la teoría de las superficies CMC y CMAC, nuestro objetivo en esta tesis será presentar la versión anisotrópica de tres teoremas importantes. El primer resultado principal es un Teorema tipo-Bernstein para multigrafos completos de CMAC, una versión anisotrópica del teorema probado por D. Hoffman, R. Osserman y R. Schoen en [58].

Teorema I.4.1. Sea Σ un multigrafo vertical en $\mathbb{R}^3$, es decir, para cualquier $p \in \Sigma$, $T_p\Sigma$ no es un plano vertical. Suponga que Σ es completa y tiene curvatura media anisotrópica constante. Entonces Σ es un plano.

Una consecuencia inmediata del Teorema I.4.1 es el siguiente corolario.

Corolario I.4.2. Sea Σ una superficie CMAC completa cuya imagen por su aplicación de Gauss esté contenida en un hemisferio cerrado de $\mathbb{S}^2$. Entonces Σ es o bien un plano o bien un cilindro CMAC.

El segundo resultado principal de la segunda parte de la tesis es también una consecuencia del Teorema I.4.1, donde generalizamos un teorema sobre superficies CMC cuya curvatura de Gauss no cambia de signo, un resultado encontrado en [63], debido a T. Klotz y R. Osserman.

Teorema I.4.3. Sea $\Sigma \subset \mathbb{R}^3$ una superficie inmersa completa de curvatura media anisotrópica constante $\Lambda \neq 0$. Si la curvatura gaussiana de Σ no cambia de signo, entonces Σ es una de las siguientes superficies:

(i) un cilindro CMAC;

(ii) la forma de Wulff (salvo homotecias).

En el tercer teorema principal de la segunda parte de esta tesis, obtenemos una acotación uniforme para la altura máxima de los gráficos CMAC con borde plano, que depende solo de la curvatura media anisotrópica. Más precisamente, tenemos

Teorema I.4.4. Sea $\Lambda \neq 0$ una constante real y sea $v \in \mathbb{S}^2$ un vector unitario cualquier. Entonces hay una constante $C = C(\Lambda)$ tal que para cualquier dominio cerrado (no necesariamente acotado) Ω del plano $\Pi = \{v\}^\perp$ y una función suave $u : \Omega \to \mathbb{R}$ que se anula en $\partial\Omega$ y cuyo grafo Σ sobre Π es una superficie Λ-CMAC, la altura de cualquier punto $p \in \Sigma$ en relación a Π es a lo más C.

La versión isotrópica del Teorema I.4.4 fue obtenida por primera vez por W. Meeks en [78], y después generalizada por J.A. Aledo, J.M. Espinar y J.A. Gálvez en [5], para clases más generales de superficies que satisfacen el Principio del Máximo, como las superficies especiales de Weingarten, es decir, superficies que satisfacen una relación $H = f(H^2 - K)$, para cierta función f, donde H y K denotan la curvatura media constante y la curvatura gaussiana de las superficies, respectivamente. Aunque el Teorema I.4.4 no proporciona acotaciones óptimas, recordamos que su versión isotrópica ha sido una herramienta fundamental para el estudio de superficies CMC propiamente embebidas. Entre las consecuencias de estas acotaciones de altura para grafos están la inexistencia de superficies CMC propiamente embebidas con un solo final, como mencionamos anteriormente en el segundo párrafo, y que cualquier superficie CMC propiamente embebida de dos finales es una superficie de revolución.

Nuestra principal dificultad en la prueba del Teorema I.4.4 reside en el hecho de que el funcional $\mathcal{F}$ no es necesariamente invariante bajo reflexiones. En otras palabras, si Π es un plano y Σ es una superficie CMAC, entonces la imagen de Σ bajo la reflexión sobre Π no es necesariamente una superficie CMAC, ya que los valores de F en un punto y su imagen reflejada pueden ser distintos. Este hecho prohíbe el uso del Método de los Planos Moviles de Alexandrov, que fue empleado en la prueba de la versión isotrópica del Teorema I.4.4.

Como corolario del Teorema I.4.4, demostramos acotaciones uniformes para superficies compactas con CMAC no nula y borde plano.

Corolario I.4.5. Suponga que la función de anisotropía F es invariante bajo la reflexión en $\mathbb{S}^2$ que fija la geodésica $\mathbb{S}^2 \cap \{v\}^\perp$, para algún $v \in \mathbb{S}^2$. Sea Σ cualquier superficie Λ-CMAC ($\Lambda \neq 0$) compacta, embebida en $\mathbb{R}^3$ y con borde contenido en el plano $\{v\}^\perp$. Entonces existe una constante positiva C que depende solo de Λ, de modo que la altura de cualquier punto $p \in \Sigma$ en relación a $\{v\}^\perp$ es a lo más C.

Bajo cierta hipótesis de simetría sobre la función de anisotropía y probando una versión anisotrópica del famoso Lema de Separación de Meeks (véanse [78] y [67]), podemos demostrar la inexistencia de superficies propiamente embebidas en $\mathbb{R}^3$ con curvatura media anisotrópica constante y con solo un final.

Teorema I.4.6. Sea $\Sigma \subset \mathbb{R}^3$ una superficie Λ-CMAC ($\Lambda \neq 0$) propiamente embebida con topología finita y a lo más un final. Considere tres vectores linealmente independientes $v_1, v_2, v_3 \in \mathbb{R}^3$ y suponga además que la función de anisotropía F es invariante bajo las reflexiones en $\mathbb{S}^2$ que fijan las geodésicas $\mathbb{S}^2 \cap \{v_i\}^\perp$, para $i \in \{1, 2, 3\}$. Entonces, salvo homotecias, Σ es la forma de Wulff.

Organizamos la segunda parte de esta tesis en dos capítulos. En el capítulo tres damos atención a la introducción de los conceptos básicos sobre la teoría de las superficies de curvatura media anisotrópica constante. En la primera sección presentamos el problema variacional, cuyos puntos críticos son nuestro objeto de interés. Para caracterizar estos puntos críticos en términos de la función de anisotropía y la curvatura media, recordamos la primera fórmula de variación, que nos lleva a la definición de la curvatura media anisotrópica de una superficie. A continuación, presentamos la forma de Wulff, relacionando su construcción geométrica con su descripción analítica. Esto nos permite, en la siguiente sección, definir los análogos anisotrópicos de la aplicación normal de Gauss y la segunda forma fundamental. En particular, la curvatura media anisotrópica de una superficie estará dada por la traza de la segunda forma fundamental anisotrópica. Se incluyó una breve sección con ejemplos de superficies CMAC, donde el lector puede hacer una comparación entre los casos isotrópico y anisotrópico. A saber, recordamos la construcción de superficies de rotación CMAC, helicoides CMAC y cilindros CMAC. Este último, en especial, jugará un papel crucial en los resultados obtenidos en el próximo capítulo.

Las pruebas de los principales resultados de esta tesis, a saber, los Teoremas I.4.1, I.4.3 y I.4.4, son los contenidos del Capítulo cuatro. Para superar las dificultades relacionadas con el Método de los Planos Móviles, seguimos las ideas que se encuentran en [15]. Primero, adaptamos un Teorema de Compacidad para superficies CMAC con segunda forma fundamental acotada y cuyas curvaturas medias anisotrópicas convergen a un número real prefijado. Este resultado nos permite obtener superficies CMAC completas como límites de secuencias de superficies CMAC sobre conjuntos compactos, y se aplicará a lo largo de todo el capítulo. Otro resultado útil que adaptamos de [15] fue acotaciones a priori de la segunda forma fundamental para superficies CMAC cuya curvatura media anisotrópica está acotada por una constante prefijada y cuya aplicación de Gauss omite un disco de área prefijada. Un paso clave para su prueba fue el uso de un Teorema tipo-Bernstein para superficies minimales anisotrópicas, debido a H.B. Jenkins (ver [62]), que también recordamos en el texto. También demostramos acotaciones de diámetro horizontal en el sentido de [78], es decir, acotaciones de diámetro para las componentes conexas de cortes horizontales de gráficos CMAC definidos en dominios cerrados y que se anulan en el borde. Estas acotaciones solo dependen de la curvatura media anisotrópica. Su prueba se basa en las ideas de [5]. Siguiendo estos resultados auxiliares, damos una prueba del Teorema I.4.1. Aunque la versión isotrópica del Teorema I.4.1 se prueba utilizando argumentos sobre la armonicidad de la aplicación de Gauss N de la superficie y las propiedades de la ecuación $\Delta N + \|dN\|^2 N = 0$ (donde Δ denota el operador Laplaciano de la superficie), nuestra prueba es geométrica y parte de ella se basa en las ideas encontradas en [51]. Al juntar el Teorema I.4.1 y los resultados auxiliares, podemos probar el Teorema I.4.4 y sus consecuencias, incluyendo el estudio de superficies CMAC propiamente embebidas en $\mathbb{R}^3$ con topología finita, donde concluimos con la demostración del Teorema I.4.6.

Finalmente, para la conveniencia del lector, agregamos un apéndice detallado sobre el Principio del Máximo. Recordamos el Principio Máximo para operadores diferenciales elípticos de segundo orden lineales y cuasi-lineales y establecemos una versión geométrica del Principio del Máximo, conocido como Principio de Tangencia, para el operador de curvatura media anisotrópica.

Part I

PSEUDO-PARALLEL IMMERSIONS IN $\mathbb{S}^n \times \mathbb{R}$ AND $\mathbb{H}^n \times \mathbb{R}$

Chapter 1

Preliminaries and basic notations

The idea of this chapter is to recall basic notions of Submanifold Theory and fix the notation we use along this work. For the reader interested in a detailed introduction to Submanifold Theory we recomend [28].

1.1 Basics of Submanifold Theory

Let M^n and $\tilde{M}^m$ be two smooth manifolds of dimensions n and m, respectively, with $m > n$. We say that a smooth map $f : M^n \to \tilde{M}^m$ is an **immersion** if its differential $df(x) = f_* : T_xM \to T_{f(x)}\tilde{M}$ is injective, for all $x \in M^n$. Two particular cases have special names: when $n = 2$ we say that f is an **immersed surface**; when $n + 1 = m$ we say that f is an **immersed hypersurface**.

An immersion $f : M^n \to \tilde{M}^m$ between two Riemannian manifolds (M^n, g) and $(\tilde{M}^m, \tilde{g})$ is said to be an **isometric immersion** if

$$\tilde{g}(f_*X, f_*Y) = g(X, Y),$$

for all $x \in M^n$ and for all $X, Y \in T_xM$.

If $(\tilde{M}^m, \tilde{g})$ is a Riemannian manifold and $f : M^n \to \tilde{M}^m$ is an immersion, then f induces a Riemannian metric $f^*\tilde{g}$ on M^n by

$$(f^*\tilde{g})_x(X, Y) = \tilde{g}_{f(x)}(f_*X, f_*Y), \quad x \in M^n, \quad X, Y \in T_xM.$$

In this case $f : (M^n, f^*\tilde{g}) \to (\tilde{M}^m, \tilde{g})$ becomes an isometric immersion.

From now until the end of this section we consider $f : (M^n, g) \to (\tilde{M}^m, \tilde{g})$ an isometric immersion, $\{X_1, \ldots, X_n\}$ a local frame of M^n and define $g_{ij} = g(X_i, X_j)$. In terms of this frame, any vector W can be written as $\sum_{i,j=1}^n g^{ij} g(W, X_i)X_j$, where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$.

We denote the tangent bundles of M and $\tilde{M}$ respectively by TM and $T\tilde{M}$, and $f^*T\tilde{M}$ refers to the vector bundle over M^n whose fiber at x is $T_{f(x)}\tilde{M}$. The orthogonal complement of f_*T_xM in $T_{f(x)}\tilde{M}$ is called the normal space of f at x and is denoted by $N_fM(x)$. The normal bundle of f is the vector subbundle of $f^*T\tilde{M}$ whose fiber at a point $x \in M^n$ is $N_fM(x)$. Smooth sections of TM, also called tangent vector fields of M^n, are denoted by $\mathfrak{X}(M)$ while sections of N_fM, also called normal vector fields to f, are denoted by $\Gamma(N_fM)$.

The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{M}^m$ induces a connection $\hat{\nabla}$ on $f^*T\tilde{M}$ by the relation

$$\hat{\nabla}_X Z = \tilde{\nabla}_{f_*X}(Z \circ f),$$

for every $x \in M^n$, $X \in T_xM$ and $Z \in \mathfrak{X}(\tilde{M})$.

We always identify $\hat{\nabla}$ and $\tilde{\nabla}$, and use $\tilde{\nabla}_{f_*X} f_*Y$ instead of $\hat{\nabla}_X f_*Y$. Given vector fields $X, Y \in \mathfrak{X}(M)$, we can decompose $\tilde{\nabla}_{f_*X} f_*Y$ as

$$\tilde{\nabla}_{f_*X} f_*Y = (\tilde{\nabla}_{f_*X} f_*Y)^T + (\tilde{\nabla}_{f_*X} f_*Y)^{\perp}$$

with respect to the orthogonal direct sum

$$f^*T\tilde{M} = f_*TM \oplus N_fM.$$

It is easy to check that $\nabla_X Y = (f_*)^{-1}(\tilde{\nabla}_{f_*X} f_*Y)^T$ defines a compatible torsion-free connection on TM, which must therefore coincide with the Levi-Civita connection of M^n. The application $\alpha : T_xM \times T_xM \to N_fM(x)$ defined by

$$\alpha(X, Y) = (\tilde{\nabla}_{f_*X} f_*Y)^{\perp},$$

is called the **second fundamental form** of f at x. Notice that α is symmetric, since $[f_*X, f_*Y] = f_*[X, Y]$ is tangent, for every $X, Y \in T_xM$. Thus, we have the **Gauss formula**:

$$\tilde{\nabla}_{f_*X} f_*Y = f_*\nabla_X Y + \alpha(X, Y) \tag{1.1.1}$$

The **Weingarten operator** of f at $x \in M^n$ with respect to $\xi \in N_fM(x)$ is defined by

$$g(A_\xi X, Y) = \tilde{g}(\alpha(X, Y), \xi), \quad X, Y \in T_xM.$$

If $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma(N_fM)$, then

$$\tilde{g}(\tilde{\nabla}_{f_*X}\xi, f_*Y) = -\tilde{g}(\xi, \tilde{\nabla}_{f_*X} f_*Y) = -\tilde{g}(\xi, \alpha(X, Y)) = g(A_\xi X, Y),$$

whence we conclude that $-f_*A_\xi X$ is the tangent part of $\tilde{\nabla}_{f_*X}\xi$. On the other hand, the normal component

$$\nabla^{\perp}_X \xi = (\tilde{\nabla}_{f_*X}\xi)^{\perp}, \quad X \in TM, \xi \in N_fM,$$

defines a torsion-free connection $\nabla^{\perp}$ in N_fM compatible with $\tilde{g}$, called the **normal connection** of f. Thus, we have the **Weingarten formula**:

$$\tilde{\nabla}_{f_*X}\xi = -f_*A_\xi X + \nabla^{\perp}_X \xi(X, Y). \tag{1.1.2}$$

The **mean curvature vector** of f at $x \in M^n$ is the normal vector defined by

$$\mathcal{H} = \frac{1}{n}\,\mathrm{trace}\,\alpha = \frac{1}{n}\sum_{i,j=1}^{n} g^{ij}\alpha(X_i, X_j).$$

In terms of the Weingarten operator, we have

$$n\tilde{g}(\mathcal{H}, \xi) = \mathrm{trace}\, A_\xi = \sum_{i,j=1}^{n} g^{ij} g(A_\xi X_i, X_j).$$

When f is a hypersurface, a smooth unit normal vector field $N \in N_fM$ is locally unique, up to sign. In this case we write the Gauss and Weingarten formulas as

$$\tilde{\nabla}_{f_*X} f_*Y = f_*\nabla_X Y + g(A_N X, Y)N,$$

and

$$\tilde{\nabla}_{f_*X} N = -f_*A_N X,$$

respectively, and the mean curvature vector of f can be written as

$$\mathcal{H}(x) = H(x)N(x), \quad x \in M^n,$$

where the quantity $H(x)$ is called the **mean curvature** of f at x, with respect to N.

For a given Riemannian manifold (M^n, g), we denote curvature tensor at $x \in M^n$ by R and adopt the convention:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in T_x M.$$

The sectional curvature of M^n with respect to span$\{X, Y\} \subset T_x M$ is expressed by

$$K(X,Y) = \frac{g(R(X,Y)Y, X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

We denote the curvature tensor of the normal bundle $N_f M$ by $R^\perp$, and it is given by

$$R^\perp(X,Y)\xi = \nabla^\perp_X \nabla^\perp_Y \xi - \nabla^\perp_Y \nabla^\perp_X \xi - \nabla^\perp_{[X,Y]}\xi, \quad X, Y \in \mathfrak{X}(M), \quad \xi \in \Gamma(N_f M).$$

When $R^\perp = 0$ on M^n we say that f has **flat normal bundle**, or f has **vanishing normal curvature**.

If R, $\tilde{R}$ and $R^\perp$ are the curvature tensors of TM, $T\tilde{M}$ and $N_f M$, respectively, from Gauss and Weingarten formulas we can deduce three important equations, called **compatibility equations** of the isometric immersion f.

Gauss Equation:

$$(\tilde{R}(f_*X, f_*Y)f_*Z)^T = f_* R(X,Y)Z + A_{\alpha(X,Z)}Y - A_{\alpha(Y,Z)}X;$$

Codazzi Equation:

$$(\tilde{R}(f_*X, f_*Y)f_*Z)^\perp = (\nabla^\perp_X \alpha)(Y,Z) - (\nabla^\perp_Y \alpha)(X,Z),$$

where

$$(\nabla^\perp_X \alpha)(Y,Z) = \nabla^\perp_X \alpha(Y,Z) - \alpha(\nabla^\perp_X Y, Z) - \alpha(Y, \nabla^\perp_X Z),$$

or, equivalently, Codazzi Equation is:

$$(\tilde{R}(f_*X, f_*Y)\xi)^\perp = \nabla_Y A_\xi X - \nabla_X A_\xi Y - A_\xi [X,Y];$$

Ricci Equation:

$$(\tilde{R}(f_*X, f_*Y)\xi)^\perp = R^\perp(X,Y)\xi + \alpha(A_\xi X, Y) - \alpha(X, A_\xi Y), \qquad (1.1.3)$$

for $X, Y, Z \in T_x M$ and $\xi \in N_f M(x)$.

The sectional curvature of M^n with respect to span$\{X, Y\} \subset T_x M$ can be computed extrinsically through Gauss Equation. It becomes

$$K(X,Y) = \tilde{K}(f_*X, f_*Y) + \frac{\tilde{g}(\alpha(X,X), \alpha(Y,Y)) - \tilde{g}(\alpha(X,Y), \alpha(X,Y))}{g(X,X)g(Y,Y) - g(X,Y)^2}. \qquad (1.1.4)$$

1.2 Space forms and the product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

A space form is a simply connected, complete, Riemannian manifold with constant sectional curvature. It is well known that, up to homotheties there are only three examples of space forms: the n-dimensional Euclidean space $\mathbb{R}^n$, with vanishing sectional curvature everywhere; the n-dimensional sphere $\mathbb{S}^n$ given by

$$\mathbb{S}^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}; \sum_{i=1}^{n+1} x_i^2 = 1\},$$

whose sectional curvature is 1 everywhere; and the n-dimensional hyperbolic space $\mathbb{H}^n$, given by

$$\mathbb{H}^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{L}^{n+1}; -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -1, x_1 > 0\},$$

whose sectional curvature is -1 everywhere. Here $\mathbb{L}^{n+1}$ is the $(n+1)$-dimensional Minkowski space, that is, the $(n+1)$-dimensional euclidean space $\mathbb{R}^{n+1}$ endowed with the inner product

$$\langle (x_1, x_2, \dots, x_{n+1}), (y_1, y_2, \dots, y_{n+1}) \rangle = -x_1 y_1 + \sum_{i=2}^{n+1} x_i y_i.$$

We adopt the following notation to represent a space form: $\mathbb{Q}^n_\epsilon$ refers to either $\mathbb{S}^n$ or $\mathbb{H}^n$ according to whether $\epsilon = 1$ or $\epsilon = -1$, respectively. Some works also consider the case $\epsilon = 0$ when they are referring $\mathbb{Q}^n_0$ as the Euclidean space $\mathbb{R}^n$. In this work we always assume that $\epsilon \in \{-1, 1\}$.

In order to make computations in these manifolds, it is convenient to consider the inclusion $i : \mathbb{Q}^n_\epsilon \to \mathbb{E}^{n+1}$, where $\mathbb{E}^{n+1}$ stands for either Euclidean space $\mathbb{R}^{n+1}$ or Lorentzian space $\mathbb{L}^{n+1}$, according as $\epsilon = 1$ or $\epsilon = -1$, respectively. Identifying $T_x\mathbb{E}^{n+1}$ with $\mathbb{E}^{n+1}$ itself, the outward pointing normal vector of $\mathbb{Q}^n_\epsilon$ at $x = (x_1, \dots, x_n, x_{n+1})$ is the position vector $i(x) = (x_1, \dots, x_n, x_{n+1})$. With respect to the position vector, i is umbilical and its second fundamental form at x is given by

$$\alpha^i(X, Y) = -\epsilon \langle X, Y \rangle i(x), \quad X, Y \in T_x\mathbb{Q}^n_\epsilon.$$

Denoting by D the Levi-Civita connection of $\mathbb{E}^{n+1}$, we can recover the Levi-Civita connection $\overline{\nabla}$ of $\mathbb{Q}^n_\epsilon$ through Gauss formula for the immersion i. Explicitly,

$$i_* \overline{\nabla}_X Y = D_{i_*X} i_* Y - \alpha^i(X, Y) = D_{i_*X} i_* Y + \epsilon \langle X, Y \rangle i,$$

for $X, Y \in \mathfrak{X}(\mathbb{Q}^n_\epsilon)$.

The first part of this thesis is devoted to the study of submanifolds with $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ as the ambient space. Let $\frac{\partial}{\partial t}$ be the canonical unit vector field tangent to the second factor of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, that is, $\frac{\partial}{\partial t}(x, t) = c'(0)$, where $c : \mathbb{R} \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is the vertical line $c(s) = (x, t + s)$. Considering the canonical inclusion $k := i \times Id_{\mathbb{R}} : \mathbb{Q}^n_\epsilon \times \mathbb{R} \to \mathbb{E}^{n+2}$, the outward pointing unit normal vector of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ in $\mathbb{E}^{n+2}$ is given by $\zeta(x, t) = (i(x), 0)$. With respect to ζ, the second fundamental form of k at (x, t) is given by the relations:

$$\alpha^k(X, Y) = -\epsilon \langle X, Y \rangle \zeta, \quad \alpha^k\left(X, \frac{\partial}{\partial t}\right) = 0, \quad \text{and} \quad \alpha^k\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 0,$$

for $X, Y \in \{\frac{\partial}{\partial t}\}^{\perp}$. We can recover the Levi-Civita connection $\tilde{\nabla}$ of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ through the Gauss formula as:

$$k_* \tilde{\nabla}_X Y = D_{k_* X} k_* Y - \alpha^k(X, Y) = D_{k_* X} k_* Y + \epsilon(\langle X, Y\rangle - \langle X, \frac{\partial}{\partial t}\rangle\langle Y, \frac{\partial}{\partial t}\rangle)\zeta,$$

for $X, Y \in \mathfrak{X}(\mathbb{Q}^n_\epsilon \times \mathbb{R})$.

For a given isometric immersion $f : M^m \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$, it is convenient to consider the following decomposition of $\frac{\partial}{\partial t}$ in its tangent and normal parts:

$$\frac{\partial}{\partial t} = f_* T + \cos\theta \eta, \tag{1.2.1}$$

for some smooth function θ defined on M, some vector field $T \in \mathfrak{X}(M)$ and some unit normal vector field $\eta \in N_f M$. We remark this expression is well defined only locally, since M^m could not be orientable. This is not a problem since we are interested in a local study of the submanifolds of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$.

In our study of pseudo-parallel submanifolds of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ we make use of the Compatibility Equations in two special cases: when f is a hypersurface (i.e. $m = n$) and when f is a surface (i.e. $m = 2$). For the first case, denoting by A^f_η its Weingarten operator in the η direction, the Compatibility Equations was derived in [40]. They are:

Gauss Equation:

$$\begin{aligned}\langle R(X, Y)Z, W\rangle = \epsilon(&\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle \\ &- \langle X, T\rangle\langle W, T\rangle\langle Y, Z\rangle + \langle X, Z\rangle\langle Y, T\rangle\langle W, T\rangle \\ &+ \langle X, T\rangle\langle Z, T\rangle\langle Y, W\rangle - \langle Y, T\rangle\langle Z, T\rangle\langle X, W\rangle) \\ &+ \langle A^f_\eta X, W\rangle\langle A^f_\eta Y, Z\rangle - \langle A^f_\eta X, Z\rangle\langle A^f_\eta Y, W\rangle,\end{aligned} \tag{1.2.2}$$

Codazzi Equation:

$$\nabla_X A^f_\eta Y - \nabla_Y A^f_\eta X - A^f_\eta[X, Y] = \epsilon \cos\theta(\langle Y, T\rangle X - \langle X, T\rangle Y), \tag{1.2.3}$$

and more two equations can be derived:

3^{th} Compatibility Equation:

$$\nabla_X T = \cos\theta A^f_\eta X, \tag{1.2.4}$$

4^{th} Compatibility Equation:

$$X(\cos\theta) = -\langle A^f_\eta X, T\rangle. \tag{1.2.5}$$

Consider now a surface $f : M^2 \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ and let $\{e_1, e_2\}$ be an orthonormal local frame for M^2. Set $\alpha_{ij} = \alpha(e_i, e_j)$. By δ_{ij} we mean the Kronecker's Delta. The Compatibility Equations for a submanifold of $f : M^m \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ of any codimension was first derived in [79]. Adapting those equations to the surface case, they become:

Gauss Equation:

$$\begin{aligned}R(e_1, e_2)e_k = \epsilon(&\delta_{2k} e_1 - \delta_{1k} e_2 - \langle e_2, T\rangle\langle e_k, T\rangle e_1 + \delta_{1k}\langle e_2, T\rangle T \\ &- \delta_{2k}\langle e_1, T\rangle T + \langle e_1, T\rangle\langle e_k, T\rangle e_2) + A^f_{\alpha_{2k}} e_1 - A^f_{\alpha_{1k}} e_2.\end{aligned} \tag{1.2.6}$$

Codazzi Equation:

$$(\nabla^\perp_{e_1}\alpha)(e_2, e_k) - (\nabla^\perp_{e_2}\alpha)(e_1, e_k) = \epsilon(\delta_{1k}\langle e_2, T\rangle - \delta_{2k}\langle e_1, T\rangle)\cos\theta\eta. \tag{1.2.7}$$

Ricci Equation:

$$R^\perp(e_1, e_2)\xi = \alpha(e_1, A^f_\xi e_2) - \alpha(A^f_\xi e_1, e_2), \tag{1.2.8}$$

for $\xi \in \Gamma(N_f M)$.

1.3 Rotation hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

Now, let us recall the definition of a rotational hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. Consider a three-dimensional vector subspace P^3 of $\mathbb{E}^{n+2}$ which contains the x_{n+2}-axis and $P^2 \subset P^3$ a vector subspace which also contains the x_{n+2}-axis. Denote by $\mathcal{I}$ the group of isometries of $\mathbb{E}^{n+2}$ which leave $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ invariant and leave P^2 pointwise fixed. Finally, let γ be a curve in $\mathbb{Q}^1_\epsilon \times \mathbb{R} \approx \mathbb{Q}^n_\epsilon \times \mathbb{R} \cap P^3$ which does not intersect P^2. We define the rotation hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with profile curve γ and axis P^2 as the $\mathcal{I}$-orbit of γ.

Up to an isometry of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, we can assume that P^3 is spanned by the canonical vectors $\{e_1, e_{n+1}, e_{n+2}\} \subset \mathbb{E}^{n+2}$.

An interesting characterization of rotation hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ was obtained by F. Dillen, J. Fastenakels and J. Van der Veken in [40] and later generalized for arbitrary codimension by B. Mendonça and R. Tojeiro in [80].

Theorem 1.3.1. (see [40] and [80]) *If $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is a hypersurface whose Weingarten operator is given by:*

$$A^f_\eta = \begin{bmatrix} \lambda & & & \\ & \mu & & \\ & & \ddots & \\ & & & \mu \end{bmatrix}, \quad \textit{with} \quad A^f_\eta T = \lambda T,$$

then, f is a rotation hypersurface.

It is worth to mention that in [80] Theorem 1.3.1 was proved under the extra unnecessary assumption that $\mu = \mu(\lambda)$ that appears in [40].

1.4 Hypersurfaces with a canonical principal direction

In [101], the hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ for which T is everywhere a principal direction and $\cos\theta$ is nowhere vanishing were classified as follows:

Theorem 1.4.1. (See [101]) *Let $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ be a hypersurface and let $g_s : M^{n-1} \to \mathbb{Q}^n_\epsilon$ be the family of its parallel hypersurfaces, that is, g_s is given in terms of the relation*

$$\tilde{g}_s(x) = C_\epsilon(s)\tilde{g}(x) + S_\epsilon(s)\tilde{N}(x), \tag{1.4.1}$$

where N is an unit normal vector field to g,

$$C_\epsilon(s) = \begin{cases} \cos(s), & \textit{if} \quad \epsilon = 1 \\ \cosh(s), & \textit{if} \quad \epsilon = -1 \end{cases} \quad \textit{and} \quad S_\epsilon(s) = \begin{cases} \sin(s), & \textit{if} \quad \epsilon = 1 \\ \sinh(s), & \textit{if} \quad \epsilon = -1, \end{cases}$$

and for the canonical inclusion $i : \mathbb{Q}^n_\epsilon \to \mathbb{E}^{n+1}$ we define $\tilde{g}(x) = (i \circ g(x), 0)$, $\tilde{N}(x) = (N(x), 0)$ and $\tilde{g}_s(x) = (i \circ g_s(x), 0)$. Define

$$f : M^n := M^{n-1} \times \mathbb{R} \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$$

by the relation

$$\tilde{f}(x, s) = \tilde{g}_s(x) + a(s)k_* \frac{\partial}{\partial t}, \tag{1.4.2}$$

for some smooth function $a : \mathbb{R} \to \mathbb{R}$ *with nowhere vanishing derivative, where* $k = (i \times Id_{\mathbb{R}}) : \mathbb{Q}^n_\epsilon \times \mathbb{R} \to \mathbb{E}^{n+2}$ *is the canonical inclusion and* $\tilde{f} = k \circ f$*. Then, the map* f *defines, at regular points, a hypersurface that has* T *as a principal direction. Conversely, any hypersurface* $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ *that has* T *as a principal direction and such that* $\cos\theta$ *is nowhere vanishing is locally given by this way.*

Geometrically, $\tilde{f}(M^n)$ is obtained by parallel transporting the curve

$$s \mapsto C_\epsilon(s)\tilde{g}(x) + S_\epsilon(s)\tilde{N}(x) + a(s)k_*\frac{\partial}{\partial t}$$

of a fixed fiber $\operatorname{span}\{\tilde{g}(x), \tilde{N}(x), k_*\frac{\partial}{\partial t}\}$ of $N_{\tilde{g}}M^{n-1}$ with respect to the normal connection of $\tilde{g}$.

In the conditions of Theorem 1.4.1, the Weingarten operators of f and g_s are related by (see [101], pg.207):

$$A^f_\eta X = -\frac{a'(s)}{\sqrt{1 + a'(s)^2}} A^{g_s}_{N_s} X, \quad \forall X \in TM^{n-1}, \tag{1.4.3}$$

and the principal curvature in the $\frac{\partial}{\partial s}$-direction is $\dfrac{a''(s)}{(1 + a'(s)^2)^{3/2}}$, where

$$N_s = -\epsilon S_\epsilon(s) i \circ g(x) + C_\epsilon(s)N(x)$$

is an unit normal vector field to g_s at x. Moreover, in [101] it was proven that

$$f_*T = \frac{a'(s)}{1 + a'(s)^2} f_* \frac{\partial}{\partial s}.$$

So, we conclude that $\dfrac{a''(s)}{(1 + a'(s)^2)^{3/2}}$ is the principal curvature of f with respect to T.

We recall that the number $\dfrac{a'(s)}{\sqrt{1 + a'(s)^2}}$ coincides with $\sin\theta$, which in turn is equals to $\|T\|$. Putting together these informations we conclude that the matrix of A^f_η has the following form:

$$A^f_\eta = \begin{bmatrix} \dfrac{a''(s)}{(1 + a'(s)^2)^{3/2}} & 0 \\ 0 & -\sin\theta A^{g_s}_{N_s} \end{bmatrix}. \tag{1.4.4}$$

1.5 Classical operators

Let V be an n-dimensional vector space endowed with an inner product $g : V \times V \to \mathbb{R}$ and let $\{X_1, \dots, X_n\}$ be a generic basis of V. Define $g_{ij} = g(X_i, X_j)$. In terms of this basis, any vector $X \in V$ can be written as $\sum_{i,j=1}^n g^{ij} g(X, X_i)X_j$, where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$. Consider $B : V \times V \to \mathbb{R}$ a bilinear form and $\beta : V \to V$ a linear operator. The **trace** of B is the number defined by

$$\operatorname{trace} B = \sum_{i,j=1}^n g^{ij} g(BX_i, X_j),$$

and the **trace** of β as the number defined by

$$\operatorname{trace} \beta = \sum_{i,j=1}^n g^{ij} \beta(X_i, X_j).$$

Let (M^n, g) be a Riemannian manifold. Consider $u : M^n \to \mathbb{R}$ a smooth function. The **gradient** of u is the unique vector field on M^n that satisfies:

$$g(\operatorname{grad}_M u, X) = X(u).$$

At some point $x \in M^n$, we can write the gradient of u in terms of a generic basis $\{X_1, \dots, X_n\}$ of T_xM as:

$$\operatorname{grad}_M u = \sum_{i,j=1}^{n} g^{ij} g(\operatorname{grad}_M u, X_i) X_j = \sum_{i,j=1}^{n} g^{ij} X_i(u) X_j,$$

We also define the symmetric bilinear form $\operatorname{Hess}_M u : T_xM \times T_xM \to \mathbb{R}$, that we call the **Hessian** of u by the following relation:

$$\operatorname{Hess}_M u(X, Y) = X(Y(u)) - \nabla_X Y(u).$$

Given a vector field $X \in \mathfrak{X}(M)$, we define the divergence of X at $x \in M^n$ as

$$\operatorname{div}_M(X) = \operatorname{trace}(Z \mapsto \nabla_Z X) = \sum_{i,j=1}^{n} g^{ij} g(\nabla_{X_i} X, X_j).$$

An useful property of the divergence is the product rule: for a smooth function u defined on M^n and a vector field $X \in \mathfrak{X}(M)$ we have

$$\operatorname{div}_M(uX) = u \operatorname{div}_M(X) + g(X, \operatorname{grad}_M u).$$

We also recall the classical Divergence Theorem.

Proposition 1.5.1. Let (M^n, g) be an oriented Riemannian manifold (with or without boundary). Denote by η the co-normal vector field along ∂M pointing outward and $\tilde{g}$ the induced Riemannian metric on ∂M. Then

$$\int_M \operatorname{div}_M X dV_g = \int_{\partial M} g(X, \eta) dV_{\tilde{g}},$$

where dV_g and $dV_{\tilde{g}}$ are the volume forms of (M, g) and $(\partial M, \tilde{g})$, respectively.

Finally, for a smooth function $u : M^n \to \mathbb{R}$, its **Laplacian** is defined by:

$$\Delta_M u = \operatorname{div}_M(\operatorname{grad}_M u) = \operatorname{trace}(\operatorname{Hess}_M(u)) = \sum_{i,j=1}^{n} g^{ij} g(\nabla_{X_i} \operatorname{grad}_M u, X_j)$$
$$= \sum_{i,j=1}^{n} g^{ij} \{X_i(X_j(u)) - \nabla_{X_i} X_j(u)\}.$$

For the Laplacian operator we have a similar product rule: given smooth functions u, v defined on M^n, it is valid that

$$\Delta_M(uv) = u\Delta_M v + v\Delta_M u + 2g(\operatorname{grad}_M u, \operatorname{grad}_M v).$$

Chapter 2

Pseudo-parallel immersions in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

In this chapter we present our study of pseudo-parallel immersions in the product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. First section beggins with precise definitions of the intrinsic and extrinsic notions that motivates the introduction of the concept of pseudo-parallelism. We also recall some characterization and classification theorems concerning semi-parallel hypersurfaces in space forms and in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, which will be useful in our study of pseudo-parallel hypersurfaces. In the second section we present the characterization of pseudo-parallel hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with at least three distinct principal curvatures, and discuss the relation between pseudo-symmetric and pseudo-parallel hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. Third section is devoted to present the geometric description of pseudo-parallel hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with three distinct principal curvatures, and putting our results together with the partial results obtained in [70], we are able to give the Classification Theorem of pseudo-parallel hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. We also present a classification of pseudo-parallel hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ that are minimal and of constant mean curvature. The next two sections concern the study of the surface case, where we prove a characterization theorem, and in particular the non-existence of pseudo-parallel surfaces in $\mathbb{Q}^3_\epsilon \times \mathbb{R}$ with non-vanishing normal curvature. Finally, we give examples of semi-parallel surfaces which are not parallel as well as examples of pseudo-parallel surfaces in $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$ which are neither semi-parallel nor pseudo-parallel surfaces in a slice, and we exhibit examples of pseudo-parallel surfaces with non-vanishing normal curvature in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, for $n \geq 4$.

2.1 Semi-parallel and pseudo-parallel submanifolds

In this section we recall the notions that appear along this chapter, and in particular the concept of pseudo-parallel immersions, the topic of our interest. We also include some classification results obtained by other authors. We begin by defining some classes of Riemannian manifolds that generalize space forms. These intrinsic notions motivated the classes of submanifolds we are interested in.

Definition 2.1.1. A Riemannian manifold M^n is said to be:

1. *Locally-symmetric* if
$$(\nabla_X R)(Y, Z, W) = 0; \tag{2.1.1}$$

2. *Semi-symmetric* if
$$(R(X, Y) \cdot R)(U, V, W) = 0; \tag{2.1.2}$$

3. *Pseudo-symmetric* if

$$(R(X,Y)\cdot R)(U,V,W) = \phi[(X\wedge Y)\cdot R](U,V,W), \tag{2.1.3}$$

for some smooth real-valued function ϕ on M^n and for any vectors X,Y,Z,U,V and W tangent to M^n.

Here the notation means

$$\begin{aligned}(\nabla_X R)(Y,Z,W) &= \nabla_X R(Y,Z)W - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z)W\\ &\quad - R(Y,Z)\nabla_X W;\\ (R(X,Y)\cdot R)(U,V,W) &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W\\ &\quad - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W;\\ [(X\wedge Y)\cdot R](U,V,W) &= (X\wedge Y)R(U,V)W - R((X\wedge Y)U,V)W\\ &\quad - R(U,(X\wedge Y)V)W - R(U,V)(X\wedge Y)W.\end{aligned}$$

Geometrical meanings of local, semi- and pseudo-symmetry can be found in [48] and [19] (see exercise 8.14). Locally-symmetric manifolds were first studied in the 1920's by E. Cartan, whose investigation has lead to the development of the theory of semi-symmetric manifolds (see [23]). It is clear that locally-symmetric manifolds are semi-symmetric, but the converse is not true. E. Cartan and H. Takagi present examples of semi-symmetric manifolds that are not locally-symmetric (see [98]).

After years of investigation by various mathematicians, semi-symmetric submanifolds were finally classified in the 1980's by Z.I. Szabó (see [96] and [97]). Meanwhile, other mathematicians started to investigate umbilical submanifolds in semi-symmetric ambient-spaces. From these studies originated the concept of pseudo-symmetry, introduced initially by R. Deszcz in [37]. Obvious examples of pseudo-symmetric manifolds are the semi-symmetric ones. However, these classes are not equal, and examples of pseudo-symmetric manifolds that are not semi-symmetric can be found in [37] and in the references therein.

The attempt to obtain an extrinsic theory analogous to the one we just mentioned motivates the definition of extrinsic notions as local-, semi- and pseudo-parallelism, that we present in Definition 2.1.2. In particular, these classes of immersions generalize the concept of total geodesy. From the view point of isometric immersions, the study of these classes of submanifolds permit us to compare different ambient spaces and improve our understanding of their geometry.

Definition 2.1.2. An isometric immersion $f : M^n \to \tilde{M}^m$ is said to be:

1. *Totally geodesic* if

$$\alpha(X,Y) = 0; \tag{2.1.4}$$

2. *Umbilical* if the mean curvature vector field $\mathcal{H}$ of f satisfies

$$\alpha(X,Y) = \langle X,Y\rangle\mathcal{H}; \tag{2.1.5}$$

3. *Locally-parallel* if

$$(\nabla^{\perp}_X\alpha)(Y,Z) = 0; \tag{2.1.6}$$

4. *Semi-parallel* if

$$(\tilde{R}(X,Y)\cdot\alpha)(Z,W) = 0; \tag{2.1.7}$$

5. *Pseudo-parallel* if

$$(\tilde{R}(X,Y)\cdot\alpha)(Z,W)=\phi[(X\wedge Y)\cdot\alpha](Z,W), \tag{2.1.8}$$

for some smooth real-valued function ϕ on M^n and for any vectors X,Y,Z and W tangent to M^n.

Here the notation means

$$\begin{aligned}(\nabla^{\perp}_X\alpha)(Y,Z)&=\nabla^{\perp}_X\alpha(Y,Z)-\alpha(\nabla_XY,Z)-\alpha(Y,\nabla_XZ);\\ (\tilde{R}(X,Y)\cdot\alpha)(Z,W)&=R^{\perp}(X,Y)[\alpha(Z,W)]-\alpha(R(X,Y)Z,W)-\alpha(Z,R(X,Y)W);\\ [(X\wedge Y)\cdot\alpha](Z,W)&=-\alpha((X\wedge Y)Z,W)-\alpha(Z,(X\wedge Y)W).\end{aligned}$$

The concepts of total geodesy and umbilicity are well known in the literature: an isometric immersion $f:M^n\to\tilde{M}^m$ is totally geodesic if and only if the image of any geodesics of M^n by f is also a geodesic of $\tilde{M}^m$. Umbilical submanifolds are the simplest submanifolds apart totally geodesic ones. The image of an umbilical immersion is equally curved in all tangent directions. We can find geometrical meanings of local-, semi- and pseudo-parallelism in [103].

Locally-parallel immersions were introduced by Ferus in [44], in the Euclidean space, which also obtained a classification of such immersions in the Euclidean space and in the spheres while in the hyperbolic spaces two classifications were obtained independently by Backes-Reckziegel (see [11]) and Takeuchi (see [99]). We also mention H.B. Lawson, who classified locally-parallel hypersurfaces in the spheres before the definition of this class of submanifolds had been introduced, as we can see in [69]. In [33] J. Deprez introduced the notion of semi-parallelism. It is clear that locally-parallel immersions are also semi-parallel, but these classes are not equal, since examples of semi-parallel submanifolds that are not locally-parallel can be found in [33], where a classification of semi-parallel hypersurfaces in the Euclidean space was also given. For space forms with non-zero sectional curvature, a classification of semi-parallel hypersurfaces was obtained by F. Dillen in [41]. The notion of pseudo-parallelism was first intruduced by A. Asperti, G.A. Lobos and F. Mercuri in [9]. The obvious examples of pseudo-parallel submanifolds are the semi-parallel ones, but there are pseudo-parallel immersions that are not semi-parallel. Examples can be found in [9] and [10]. Also in [10] we can find a classification of pseudo-parallel hypersurfaces in space-forms: they are either quasi-umbilical hypersurfaces or cyclids of Dupin.

When the ambient space is a space form, locally-parallel immersions are locally-symmetric as well semi-parallel and pseudo-parallel immersions are semi-symmetric and pseudo-symmetric, respectively. In other words, local, semi- and pseudo-parallelism are extrinsic analogue of local, semi- and pseudo-symmetry, respectively. Unfortunatelly, it does not occur when we consider $\mathbb{Q}^n_\epsilon\times\mathbb{R}$ as the ambient space. For example, as we will see in Remark 2.2.5, in these ambient spaces umbilical hypersurfaces are semi-parallel but not necessarily semi-symmetric. We prove that pseudo-parallel hypersurfaces of $\mathbb{Q}^n_\epsilon\times\mathbb{R}$ are pseudo-symmetric (see Corollary 2.2.4).

Since we are interested in the spaces $\mathbb{S}^n\times\mathbb{R}$ and $\mathbb{H}^n\times\mathbb{R}$, where the sectional curvature of the first factor is non-zero, we present here the results of F. Dillen, that we will use later. The first useful result is a characterization of semi-parallel hypersurfaces of space forms in terms of the Weingarten operator.

Proposition 2.1.3. (see [41]) Let $f:M^n\to\mathbb{Q}^{n+1}_\epsilon$ $(\epsilon\neq0)$ be a hypersurface. The following assertions are equivalent:

(1) f is semi-parallel;

(2) the Weingarten operator of f at each point $x \in M^n$ has the following form:

$$A_N = \begin{bmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \mu & & \\ & & & & \ddots & \\ & & & & & \mu \end{bmatrix}, \quad \text{with} \quad \lambda\mu = -\epsilon \quad \text{or} \quad \lambda = \mu. \tag{2.1.9}$$

The following result is the Classification Theorem of semi-parallel hypersurfaces of space forms.

Theorem 2.1.4. (see [41]) Let $f : M^n \to \mathbb{Q}_\epsilon^{n+1}$ be a semi-parallel hypersurface. Then there are three possibilities:

(1) $n = 2$ and M^2 is flat;

(2) f is a parallel immersion. Thus, $f(M^n)$ is an open part of $\mathbb{S}^k(c_1) \times \mathbb{S}^{n-k}(c_2)$ (resp. $\mathbb{H}^k(c_1) \times \mathbb{S}^{n-k}(c_2)$) if $\epsilon = 1$ (resp. $\epsilon = -1$), for some $0 \leq k \leq n$ and some c_1, c_2 satisfying $\frac{1}{c_1} + \frac{1}{c_2} = \epsilon$.

(3) f is a rotation surface (in the sense of [20]) whose profile curve is a helix.

Considering $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ as the ambient space, the first studies of parallel and semi-parallel immersions started with J. Van der Veken, L. Vrancken in [102] and G. Calvaruso, D. Kowalczyk and J. Van der Veken in [16], where a classification of the hypersurface case was obtained. One interesting conclusion of their work was that total umbilicity does not imply parallelism, a fact that do occur in space forms. Indeed, if $f : M^n \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$ is an umbilical hypersurface, the parallelism condition gives

$$0 = (\nabla_X A_N) = \nabla_X A_N Y - A_N \nabla_X Y = X(\lambda) Y,$$

for all $X, Y \in \mathfrak{X}(M)$, whence we conclude that λ is constant on M^n. On the other hand, using Codazzi Equation we have

$$0 = \epsilon \cos\theta(\langle T, T\rangle X - \langle X, T\rangle T) = \epsilon \cos\theta \|T\|^2 X$$

for $X \neq 0$ orthogonal to T. When $\cos\theta = 0$, we conclude that $f(M^n)$ is an open part a vertical cylinder $g(M^{n-1}) \times \mathbb{R}$, for some hypersurface $g : M^{n-1} \to \mathbb{Q}_\epsilon^n$. In particular, f is totally geodesic, since one of its principal curvatures is zero. When $\|T\|^2 = 0$, we conclude that $f(M^n)$ is an open part of $\mathbb{Q}_\epsilon^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. Again, f is totally geodesic.

Semi-parallel hypersurfaces were classified based on the characterization of their Weingarten operators, an idea that we also use in this work. The most interesting case that appeared is when the Weingarten operator has two distinct eigenvalues, being one of them of multiplicity one and T as related principal direction, and with an additional condition on the product of them. In such case, Theorem 1.3.1 was of fundamental importance. The Classification Theorem of semi-parallel hypersufaces of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ is:

Theorem 2.1.5. (see [102] and [16]) Let $f : M^n \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$ be a semi-parallel hypersurface. Then:

(i) $n = 2$ and M^2 is flat;

(ii) f is umbilical,

(iii) $f(M^n)$ is an open part of a rotation surface in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ whose profile curve is either a vertical line or it is parametrized by

$$\gamma(s) = \left(\cos s, 0, \dots, 0, \sin s, \int_{s_0}^{s} \sqrt{C \cos^2 \sigma - 1} d\sigma \right), \quad \text{if} \quad \epsilon = 1 \text{ and}$$

$P^2 = \text{span}\{e_1, e_{n+2}\}$; or, for $\epsilon = -1$,

$$\gamma(s) = \left(\cosh s, 0, \dots, 0, \sinh s, \int_{s_0}^{s} \sqrt{C \cosh^2 \sigma - 1} d\sigma \right), \text{ if } P^2 = \text{span}\{e_1, e_{n+2}\};$$

$$\gamma(s) = \left(\cosh s, 0, \dots, 0, \sinh s, \int_{s_0}^{s} \sqrt{C \sinh^2 \sigma - 1} d\sigma \right), \text{ if } P^2 = \text{span}\{e_{n+1}, e_{n+2}\};$$

$$\gamma(s) = \left(s, 0, \dots, 0, \frac{-1}{2s}, \int_{s_0}^{s} \sqrt{C - \frac{1}{2\sigma^2}} d\sigma \right), \text{ if } P^2 = \text{span}\{\frac{1}{\sqrt{2}}(e_1 + e_{n+1}), e_{n+2}\},$$

where P^2 is the axis of f.

(iv) there exists a semi-parallel hypersurface $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ such that $M^n = M^{n-1} \times \mathbb{R}$ and $f(x,t) = (g(x), t)$, for $x \in M^{n-1}$ and $t \in \mathbb{R}$. In other words, $f(M^n)$ is an open part of a vertical cylinder over a semi-parallel hypersurface of $\mathbb{Q}^n_\epsilon$.

2.2 Pseudo-parallel hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

In this section we present an improvement of the third case of Lemma 3.1 in [70], concerning the Weingarten operator of a pseudo-parallel hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with at least three distinct eigenvalues. We also discuss the relation between pseudo-parallel and pseudo-symmetric hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$.

We begin with a general assumption we will adopt along the section. Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a hypersurface and consider $U = V \cup \text{int}(M^n - V)$, where $V = \{x \in M^n; \theta(x) \neq 0\}$. It is easy to see that U is open and dense on M^n and for each connected component U_γ of U we have either $U_\gamma \subset V$ or $U_\gamma \subset \text{int}(M^n - V)$. If $x \in \text{int}(M^n - V)$ then $\frac{\partial}{\partial t}$ is orthogonal to f on a small neighborhood of x and we conclude that $f(M)$, in such a neighborhood, is an open part of $\mathbb{Q}^n_\epsilon \times \{t_0\}$, for some $t_0 \in \mathbb{R}$, that is, a totally geodesic hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. So, from now on we will assume that θ is nowhere vanishing.

Next, we have the following remark:

Remark 2.2.1. From the definition of pseudo-parallelism we can deduce that any surface $f : M^2 \to \mathbb{Q}^2_\epsilon \times \mathbb{R}$ is pseudo-parallel by taking ϕ as the Gaussian curvature of M^2. Also, it is well known that the curvature tensor of a Riemannian manifold M^n with constant sectional curvature c is given by $R(X,Y)Z = c(X \wedge Y)Z$, and consequently any hypersurface $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is pseudo-parallel with $\phi = c$.

The following lemma is an improvement of Lemma 3.1 in [70].

Lemma 2.2.2. Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a pseudo-parallel hypersurface with at least three eigenvalues. Then the Weingarten operator A^f_η of f has exactly three eigenvalues and it has the following form:

$$A^f_\eta = \begin{bmatrix} 0 & & & & & & \\ & \lambda & & & & & \\ & & \ddots & & & & \\ & & & \lambda & & & \\ & & & & \mu & & \\ & & & & & \ddots & \\ & & & & & & \mu \end{bmatrix}, \tag{2.2.1}$$

with $\phi = \epsilon \cos^2 \theta$, $\lambda\mu = -\epsilon \sin^2 \theta$ and $T = \|T\| e_1$.

Proof. Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a pseudo-parallel hypersurface and fix a point $x \in M^n$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_x M^n$ that satisfies

$$A^f_\eta e_i = \lambda_i e_i, \quad \forall i \in \{1, \dots, n\}.$$

In this basis we write

$$T = \sum_{i=1}^{n} T_i e_i.$$

The pseudo-parallelism condition is equivalent to the following equations

$$(\lambda_i - \lambda_j)(\lambda_i\lambda_j + \epsilon(1 - (T_i)^2 - (T_j)^2)) - \phi) = 0, \tag{2.2.2}$$
$$\epsilon(\lambda_i - \lambda_k) T_j T_k = 0, \quad \forall i \neq j \neq k \neq i. \tag{2.2.3}$$

Recall we are assuming that $T \neq 0$, otherwise we know that $f(M^n)$ is just a totally geodesic slice $\mathbb{Q}^n_\epsilon \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. As a consequence, $\cos^2 \theta \neq 1$.

If $\lambda_i \neq \lambda_j$ are distinct eigenvalues of A^f_η, one of the following situations necessarily occurs:

Situation 1: $T \notin \text{span}\{e_i, e_j\}$. In this case, there exists $k \in \{1, \dots, n\} - \{i, j\}$ such that $T_k \neq 0$. By equation (2.2.3), we conclude that $T_i = T_j = 0$. So, $T \perp \text{span}\{e_i, e_j\}$. In particular, by equation (2.2.2), we have that $\lambda_i\lambda_j = -\epsilon + \phi$.

Situation 2: $T \in \text{span}\{e_i, e_j\}$. Using equation (2.2.2) we get $\lambda_i\lambda_j = -\epsilon \cos^2 \theta + \phi$.

With these conditions, we analyse the number of eigenvalues, their multiplicities and prove that T is a principal direction. Let us suppose that A^f_η has at least three distinct eigenvalues, λ, μ and ν. We define A, B and C the sets of indexes corresponding to ν, λ and μ, respectively. We know from Situations 1 and 2 that $\lambda\mu, \lambda\nu, \mu\nu \in \{-\epsilon \cos^2 \theta + \phi, -\epsilon + \phi\}$. So, at least two of these products are equal. If none of them is zero, and for example $\lambda\nu = \mu\nu$, we conclude that $\lambda = \mu$, a contradiction. Therefore, one of them must be zero. Moreover, we have proven that A^f_η admits at most two eigenvalues which are not zero. So, we can suppose that A^f_η has exactly three distinct eigenvalues, λ, μ and $\nu = 0$.

Because of $\lambda\nu = \mu\nu = 0$, $0 \neq \lambda\mu$ and $0, \lambda\mu \in \{-\epsilon \cos^2 \theta + \phi, -\epsilon + \phi\}$, we have two cases to consider:

Case 1: $0 = -\epsilon \cos^2 \theta + \phi$. In particular $\lambda\mu = -\epsilon + \phi$. Since $\lambda\nu = \mu\nu = 0 = -\epsilon \cos^2 \theta + \phi$, by Situation 2 we have that $T \in \text{span}\{e_i, e_j\}$, $\forall i \in A$, $\forall j \in B \cup C$. But this occurs only if $\mathbf{card}(A) = 1$. So we conclude that 0 has multiplicity one and $T = \|T\| e_1$.

Case 2: $0 = -\epsilon + \phi$. In particular, $\lambda\nu = \mu\nu = 0 = -\epsilon + \phi$. By Situation 1 we have that $T \perp \text{span}\{e_i, e_j\}$, $\forall i \in A$, $\forall j \in B \cup C$. But this implies $T = 0$, which is a contradiction. ☐

Remark 2.2.3. A hypersurface $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is ψ-pseudo-symmetric if, and only if, for every distinct indexes $i, j, k, l \in \{1, \dots, n\}$ we have:

$$T_j T_k(\lambda_i - \lambda_k)(\lambda_j - \lambda_l) = 0, \tag{2.2.4}$$

$$T_j T_k(\epsilon(1 - \|T\|^2) + \lambda_j\lambda_k - \psi) = 0, \tag{2.2.5}$$

$$\begin{aligned}&[\epsilon(1 - T_i^2 - T_j^2) + \lambda_i\lambda_j - \psi](\lambda_i - \lambda_j)\lambda_k \\ &- \epsilon(T_i^2 - T_j^2)[\epsilon(1 - \|T\|^2) + \lambda_i\lambda_j - \psi] = 0.\end{aligned} \tag{2.2.6}$$

Corollary 2.2.4. Any pseudo-parallel hypersurface in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ is pseudo-symmetric.

Proof. Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a ϕ-pseudo-parallel immersion. By Lemma 2.2.2, T is a principal direction and we can suppose $T = \|T\|e_1$. With this simplification, the pseudo-parallelism condition becomes

$$(\epsilon + \lambda_i\lambda_j - \phi)(\lambda_i - \lambda_j) = 0, \tag{2.2.7}$$

$$(\epsilon(1 - \|T\|^2) + \lambda_1\lambda_j - \phi)(\lambda_1 - \lambda_j) = 0, \tag{2.2.8}$$

for $i, j > 1$ mutually distinct. Analogously, the pseudo-symmetry condition with $\psi = \phi$ becomes:

$$(\epsilon + \lambda_i\lambda_j - \phi)(\lambda_i - \lambda_j)\lambda_1 = 0, \tag{2.2.9}$$

$$(\epsilon + \lambda_i\lambda_j - \phi)(\lambda_i - \lambda_j)\lambda_k = 0, \tag{2.2.10}$$

$$(\epsilon(1 - \|T\|^2) + \lambda_1\lambda_j - \phi)(\epsilon\|T\|^2 - (\lambda_1 - \lambda_j)\lambda_k) = 0, \tag{2.2.11}$$

for $i, j, k > 1$ mutually distinct. Putting (2.2.7) and (2.2.8) together with (2.2.9),(2.2.10) and (2.2.11) and assuming that $T \neq 0$ we obtain the following condition,

$$\epsilon\|T\|^2(\epsilon(1 - \|T\|^2) + \lambda_1\lambda_j - \phi) = 0, \qquad \forall j > 1. \tag{2.2.12}$$

If f is not umbilical, item (ii) in Lemma 3.1 of [70] and Lemma 2.2.2 imply that $\lambda_1 \neq \lambda_j$, for all $j \in \{2, \dots, n\}$. Using (2.2.8) we get (2.2.12). If f is umbilical, we know f is ϕ-pseudo-parallel for any function ϕ. In this case we choose $\phi = \epsilon(1 - \|T\|^2) + \lambda^2$ and conclude again that f is ϕ-pseudo-symmetric. □

Remark 2.2.5. Although in space forms semi-parallelism is an extrinsic analogous to semi-symmetry (a fact proved by J. Deprez and F. Dillen, see [33] and [41]), from the proof of Corollary 2.2.4 we see that this is not true for hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, since in these ambient spaces umbilical hypersurfaces are semi-parallel but not semi-symmetric in general.

2.3 Classification of pseudo-parallel hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

In this section we prove the main result of the chapter and exhibit the classification of pseudo-parallel hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$.

Theorem 2.3.1. *Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a pseudo-parallel hypersurface with three distinct principal curvatures. Then, $M^n = M^{n-1} \times \mathbb{R}$ and there exists a semi-parallel hypersurface $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ such that either $f(x, s) = (g(x), s)$ or f is given by equation* (1.4.2) *in terms of g and a linear function $a : \mathbb{R} \to \mathbb{R}$ with nowhere vanishing derivative.*

Proof. By Lemma 2.2.2 we know that T is a principal direction of f.

If $\cos\theta$ vanishes identically, then f is semiparallel and $T = \frac{\partial}{\partial t}$ is tangent everywhere to f. In this case $M^n = M^{n-1} \times \mathbb{R}$ and there exists a semi-parallel hypersurface $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ such that $f(x,s) = (g(x), s)$ (see item (iv) in Theorem 2.1.5).

Assume now that $\cos\theta$ is nowhere vanishing. Theorem 1.4.1 yields that there exists a hypersurface $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ such that f is given by equation (1.4.2) in terms of g and a function $a : \mathbb{R} \to \mathbb{R}$ with non-vanishing derivative.

Since $X(\cos\theta) = -\langle A^f_\eta X, T\rangle = 0$, for all $X \in TM^n$, we have that $\cos\theta$ is constant over M^n. By Corollary 2 in [101] we conclude that $a(s)$ is a linear funcion. By Lemma 2.2.2 the product of the non zero principal curvatures of f is constant and equal to $-\epsilon \sin^2\theta$. Using this information in equation (1.4.3), we conclude that $A^{g_s}_{N_s}$ has two distinct principal curvatures whose product is constant in x and also in s and it is equals to $-\epsilon$, since $a'(s)/\sqrt{1 + a'(s)^2} = \sin\theta$ (see equation 8 in [101]). By Proposition 2.1.3 we conclude that g is a semi-parallel hypersurface of $\mathbb{Q}^n_\epsilon$. □

Remark 2.3.2. An interesting class of hypersurfaces $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ consists of those for which θ is constant on M^n. Such hypersurfaces are called **constant angle hypersurfaces**. It is worth to mention that constant angle hypersurfaces were fully described by R. Tojeiro in [101]. Namely, if $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is a constant angle hypersurface, then $M^n = M^{n-1} \times I$ and there exists a hypersurfaces $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ for which f is given either by $f(x,s) = (g(x), s)$ or by equation (1.4.2), in terms of a linear function $a : I \to \mathbb{R}$. Moreover, the converse is true. In particular, in the statement of Theorem 2.3.1, the pseudo-parallel hypersurface f is a constant angle hypersurface.

With respect to pseudo-parallel hypersurfaces in $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ with exactly two principal curvatures, Lemma 3.1 in [70] provides that one of the principal curvatures has multiplicity one and T is its corresponding principal direction. By Theorem 1.3.1, we conclude that f is a rotation hypersurface. In the next proposition we prove indeed that any rotation hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ is pseudo-parallel.

Proposition 2.3.3. *Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a rotation hypersurface. Then f is pseudo-parallel with $\phi(s) = \lambda(s)\mu(s) + \epsilon\cos^2\theta(s)$, where λ and μ are its principal curvatures.*

Proof. Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a rotation hypersurface. By Theorem 1.3.1, its Weingarten operator is given by $\mathrm{diag}(\lambda, \mu, \dots, \mu)$, and we conclude this proof taking $\phi(s) = \lambda(s)\mu(s) + \epsilon\cos^2\theta(s)$ in (2.2.2). □

Putting together Remark 2.2.1, the results obtained in [70], Theorem 2.3.1 and Proposition 2.3.3, the Classification Theorem of pseudo-parallel hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ can be stated as follows.

Theorem 2.3.4 (Classification Theorem)**.** *Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a pseudo-parallel hypersurface. Then one of the following occurs:*

(i) $n = 2$ and ϕ is the Gaussian curvature;

(ii) f is umbilical;

(iii) f is a rotation hypersurface;

(iv) $f : M^{n-1} \times \mathbb{R} \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is given by $f(x,s) = (g(x), s)$, for a semi-parallel hypersurface $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$;

(v) *There exists a semi-parallel hypersurface* $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ *such that* $f : M^{n-1} \times \mathbb{R} \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ *is given by equation* (1.4.2) *in terms of* g *and a linear function* $a : \mathbb{R} \to \mathbb{R}$ *with nowhere vanishing derivative.*

Remark 2.3.5. An interesting class of hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ consists of those that have constant sectional curvature, and as we observed in Remark 2.2.1, such hypersurfaces are also pseudo-parallel. We could ask how these hypersurfaces fit in the Classification Theorem. We recall that constant sectional curvature hypersurfaces in product spaces were investigated by other authors. The surface case were studied by J. Aledo, J.M. Espinar and J.A. Gálvez, in [3] and [4]. Among their results there are uniform height estimates for compact graphs and a proof of the nonexistence of complete surfaces of constant Gaussian curvature $c \in (-\infty, -1) \cup (0, 1)$ in $\mathbb{S}^2(1) \times \mathbb{R}$ and $c < -1$ in $\mathbb{H}^2(-1) \times \mathbb{R}$. For $n \geq 3$, a full classification of constant sectional curvature hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ was obtained by F. Manfio and R. Tojeiro in [81]. It was proven that for $n \geq 4$, constant sectional curvature hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ must be open subsets of rotation hypersurfaces. Nonrotational examples appear when $n = 3$, but all of them have 0 as a principal curvature in the T-direction, and they can be constructed by means of Theorem 1.4.1 (see section 6 in [81]). The totally geodesic slice $\mathbb{Q}^3_\epsilon \times \{0\}$ has constant sectional curvature ϵ and appears in case (ii), while a flat surface $g : M^2 \to \mathbb{Q}^n_\epsilon$ is semi-parallel and it gives rise (through equation (1.4.2)) to a flat hypersurface in case (iv). We conclude from these works that we have examples of constant sectional curvature hypersurfaces for each case appearing in Theorem 2.3.4.

As a consequence of Theorem 2.3.4, we can classify all pseudo-parallel hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ which have constant mean curvature.

Corollary 2.3.6. *Let* $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ *(*$n \geq 3$*) be a pseudo-parallel hypersurface with constant mean curvature. Then* f *is either totally geodesic, a rotation hypersurface with constant mean curvature, or it is given as in item (iv) of Theorem 2.3.4, where* $g(M^{n-1})$ *is an open part of* $\mathbb{S}^k(c_1) \times \mathbb{S}^{n-k-1}(c_2)$ *(resp.* $\mathbb{H}^k(c_1) \times \mathbb{S}^{n-k-1}(c_2)$*) if* $\epsilon = 1$ *(resp.* $\epsilon = -1$*), for some real constants* c_1, c_2 *satisfying* $\frac{1}{c_1} + \frac{1}{c_2} = \epsilon$ *and some* $k \in \{1, \dots, n-2\}$.

Proof. If $\cos\theta$ vanishes identically, f is given as in item (iv) of Theorem 2.3.4. Then g is semi-parallel and has constant mean curvature. It follows easily that either g is umbilical or isoparametric (i.e. its principal curvatures are constant) with two distinct principal curvatures (see Proposition 2.1.3). In the first case, f is a rotation hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ whose profile curve is a vertical line. In the second case, $g(M^{n-1})$ is an open part of $\mathbb{S}^k(c_1) \times \mathbb{S}^{n-k-1}(c_2)$ (resp. $\mathbb{H}^k(c_1) \times \mathbb{S}^{n-k-1}(c_2)$) if $\epsilon = 1$ (resp. $\epsilon = -1$), for some c_1, c_2 satisfying $\frac{1}{c_1} + \frac{1}{c_2} = \epsilon$ and some $k \in \{1, \dots, n-2\}$.

Thus, we can suppose that $\cos\theta$ is nowhere vanishing.

When f is umbilical, its unique principal curvature is constant over M^n. Using Codazzi's equation for $X \in \{T\}^\perp$, we conclude that:

$$0 = \cos\theta(\langle T, T\rangle X - \langle X, T\rangle T) = \cos\theta \sin^2\theta X.$$

Then, $\sin\theta$ vanishes identically and $f(M)$ is an open part of a totally geodesic slice $\mathbb{Q}^n_\epsilon \times \{t_0\}$, for some real constant t_0.

When f has exactly two distinct principal curvatures, it is a rotation hypersurface with constant mean curvature. An explicit expression of the profile curve is obtained using the formulas of Theorem 3 in [101].

Finally, we suppose that f has exactly three distinct principal curvatures. Then f is given as in item (v) of Theorem 2.3.4. As we observed in the proof of Theorem 2.3.1, the parallels g_s of g are all semi-parallel. Therefore the product of the principal curvatures of g_s is equals to $-\epsilon$. Moreover, since $\cos\theta$ is constant, from equation (1.4.3) we conclude that the mean curvature of g_s is constant on x and s. Using these two informations, we conclude that the principal curvatures of g_s are constant on x and s.

On the other hand, by a direct computation, the principal curvatures $\hat{\lambda}(x)$ and $\hat{\mu}(x)$ of g_s are related with the principal curvatures $\tilde{\lambda}(x)$ and $\tilde{\mu}(x)$ of g at $x \in M^{n-1}$ by the following expression (see [81]):

$$\hat{\lambda}(x) = \frac{\epsilon + \cot_\epsilon(s)\tilde{\lambda}(x)}{\cot_\epsilon(s) - \tilde{\lambda}(x)}, \qquad \hat{\mu}(x) = \frac{\epsilon + \cot_\epsilon(s)\tilde{\mu}(x)}{\cot_\epsilon(s) - \tilde{\mu}(x)},$$

where $\cot_\epsilon(s) := \frac{C_\epsilon(s)}{S_\epsilon(s)}$. Thus, imposing that $\hat{\lambda}(x)$ and $\hat{\mu}(x)$ are constant on x and s, we conclude that $\hat{\lambda}(x)\hat{\mu}(x) = \epsilon$, which is a contradiction. $\square$

In particular, we have also a classification of minimal pseudo-parallel hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$.

Corollary 2.3.7. *Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ ($n \geq 3$) be a pseudo-parallel hypersurface. If f is minimal, then f is either a totally geodesic hypersurface, a minimal rotation hypersurface, or $\epsilon = 1$ and f is given as in item (iv) of Theorem 2.3.4, where $g(M^{n-1})$ is an open part of $\mathbb{S}^k(\frac{n-1}{k}) \times \mathbb{S}^{n-k-1}(\frac{n-1}{n-k-1})$ for some $k \in \{1, \dots, n-2\}$.*

Proof. We observe that when $\cos\theta$ vanishes identically it is known that minimal semi-parallel hypersurfaces of $\mathbb{H}^n(-1)$ are totally geodesic (see Corollary 6.3 in [9]). We also recall that minimal rotation hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ were classified in [40] (see Theorems 6 and 7). The rest of the proof is similar to the proof of Corollary 2.3.6. $\square$

Remark 2.3.8. The converse statement of Corollary 2.2.4 is false. For example, let $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ be a semi-symmetric hypersurface which is not semi-parallel. Such hypersurfaces do exist and for $n \geq 4$ its Weingarten operator has the following form: $\mathrm{diag}(\lambda, 0, \dots, 0)$. By taking $f : M^{n-1} \times \mathbb{R} \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ given by $f(x, s) = (g(x), s)$, according to equations (2.2.4)-(2.2.6) we obtain a semi-symmetric (and in particular pseudo-symmetric) hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. But since g is not semi-parallel, by item (iv) of Theorem 2.3.4 we conclude that f is not pseudo-parallel.

2.4 Pseudo-parallel surfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

In this section we prove some auxiliary results concerning pseudo-parallel surfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, that will be useful later.

Let $f : M^2 \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a pseudo-parallel surface, $\{e_1, e_2\}$ be an orthonormal frame and $\alpha_{ij} := \alpha(e_i, e_j)$, where α is the second fundamental form of f. It follows from Ricci Equation that

$$R^\perp(e_1, e_2)\xi \in \mathrm{span}\{\alpha(X, Y); X, Y \in TM\}, \quad \text{for all} \quad \xi \in N_f M.$$

Thus, equation (1.2.8) is equivalent to the following equation:

$$R^\perp(e_1, e_2)\alpha_{ij} = \langle \alpha_{12}, \alpha_{ij}\rangle(\alpha_{11} - \alpha_{22}) + \langle \alpha_{22} - \alpha_{11}, \alpha_{ij}\rangle\alpha_{12}. \tag{2.4.1}$$

On the other hand, the condition of pseudo-parallelism is equivalent to the following two equations:

$$R^{\perp}(e_1, e_2)\alpha_{ii} = (-1)^i 2(K - \phi)\alpha_{12}, \tag{2.4.2}$$
$$R^{\perp}(e_1, e_2)\alpha_{12} = (K - \phi)(\alpha_{11} - \alpha_{22}), \tag{2.4.3}$$

where

$$K = \epsilon(1 - \|T\|^2) + \langle \alpha_{11}, \alpha_{22} \rangle - \|\alpha_{12}\|^2 \tag{2.4.4}$$

is the Gaussian curvature of M^2. As a consequence, we have the next lemma.

Lemma 2.4.1. *Let $f : M^2 \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a pseudo-parallel surface. Then, the mean curvature vector field H of f satisfies $R^{\perp}(X, Y)H = 0$, for all $X, Y \in TM$.*

Proof. Immediate by equation (2.4.2), since $H = \frac{1}{2}(\alpha_{11} + \alpha_{22})$. □

Proposition 2.4.2. Let $f : M^2 \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a surface with flat normal bundle. Then f is a pseudo-parallel immersion.

Proof. Since f has flat normal bundle, by equations (2.4.2) and (2.4.3) we conclude that f is ϕ-pseudo-parallel by taking $\phi = K$, where K is the Gaussian curvature of M^2. □

In the following, we have two propositions that are useful to construct examples of pseudo-parallel surfaces.

Proposition 2.4.3. Let $f : M^m \to \mathbb{Q}^n_c$ be an isometric immersion and let $j : \mathbb{Q}^n_c \to \mathbb{Q}^n_c \times \mathbb{R}$ be a totally geodesic immersion. If f is ϕ-pseudo-parallel, then $j \circ f$ is ϕ-pseudo-parallel.

Proof. In this proof, we denote the second fundamental form of f and $j \circ f$ respectively by α^f and $\alpha^{j\circ f}$. In the same way, we denote the normal curvature tensors of f and $j \circ f$ respectively by $R^{\perp}_f$ and $R^{\perp}_{j\circ f}$. Since j is a totally geodesic immersion, we have the following relations:

$$\alpha^{j\circ f}(Z, W) = j_* \alpha^f(Z, W),$$
$$R^{\perp}_{j\circ f}(X, Y)\alpha^{j\circ f}(Z, W) = j_* R^{\perp}_f(X, Y)\alpha^f(Z, W),$$

Therefore, applying Definition 2.1.8 we obtain

$$\begin{aligned}
(\tilde{R}(X, Y) \cdot \alpha^{j\circ f})(Z, W) &= R^{\perp}_{j\circ f}(X, Y)\alpha^{j\circ f}(Z, W) - \alpha^{j\circ f}(R(X, Y)Z, W) \\
&- \alpha^{j\circ f}(Z, R(X, Y)W) \\
&= j_* R^{\perp}_f(X, Y)\alpha^f(Z, W) - j_*\alpha^f(R(X, Y)Z, W) \\
&- j_*\alpha^f(Z, R(X, Y)W) \\
&= \phi\{-j_*\alpha^f((X \wedge Y)Z, W) - j_*\alpha^f(Z, (X \wedge Y)W)\} \\
&= \phi\{-\alpha^{j\circ f}((X \wedge Y)Z, W) - \alpha^{j\circ f}(Z, (X \wedge Y)W)\} \\
&= \phi[(X \wedge Y) \cdot \alpha^{j\circ f}](Z, W).
\end{aligned}$$

□

Proposition 2.4.4. Let $f : M^m \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be an isometric immersion and let $j : \mathbb{Q}^n_\epsilon \times \mathbb{R} \to \mathbb{Q}^{n+l}_c \times \mathbb{R}$ be a totally geodesic immersion. If f is ϕ-pseudo-parallel, then $j \circ f$ is ϕ-pseudo-parallel.

Proof. It is analogous to the proof of Proposition 2.4.3. □

2.5 Characterization of pseudo-parallel surfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

In this section we present our main result concerning pseudo-parallel surfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. To do so, we begin by recalling the concepts of ellipse of curvature and isotropic surfaces, in the sense of B. O'Neill (see [83]), and the relation between them.

Let $f : M^2 \to \tilde{M}^m$ $(m \geq 4)$ be any surface, with second fundamental form $\alpha : TM \times TM \to N_f M$. Consider $\{e_1, e_2\}$ an orthonormal frame in M^2 and define $\alpha_{ij} := \alpha(e_i, e_j)$. If $x \in M^2$ and $X \in T_x M$ with $\|X\| = 1$, we can write $X = \cos te_1 + \sin te_2$, for some $t \in [0, 2\pi)$. In this way,

$$\begin{aligned}
\alpha(X,X) &= \alpha(\cos te_1 + \sin te_2, \cos te_1 + \sin te_2) = \cos^2 t\alpha_{11} + 2\cos t \sin t\alpha_{12} + \sin^2 t\alpha_{22} \\
&= (\frac{\alpha_{11}+\alpha_{22}}{2}) + (\cos^2 t - \frac{1}{2})\alpha_{11} + \sin(2t)\alpha_{12} + (\sin^2 t - \frac{1}{2})\alpha_{22} \\
&= H(x) + (\frac{1}{2} - \sin^2 t)\alpha_{11} + \sin(2t)\alpha_{12} + (\sin^2 t - \frac{1}{2})\alpha_{22} \\
&= H(x) + (\sin^2 t - \frac{1}{2})(\alpha_{22} - \alpha_{11}) + \sin(2t)\alpha_{12} \\
&= H(x) + \frac{1}{2}\cos(2t)(\alpha_{11} - \alpha_{22}) + \sin(2t)\alpha_{12},
\end{aligned} \tag{2.5.1}$$

where $H(x)$ denotes the mean curvature vector of f at x.

Thus, if $\{\alpha_{11}-\alpha_{22}, \alpha_{12}\}$ is linearly independent, the map $t \mapsto \alpha(\cos te_1+\sin te_2, \cos te_1+\sin te_2)$ parametrizes an ellipse centered at $H(x)$, in the affine plane spanned by the vectors $\alpha_{11} - \alpha_{22}$ and α_{12}, that passes through $H(x)$. By this reason, we are motivated to call the set $\{\alpha(X,X); X \in T_x M \quad \text{with} \quad \|X\| = 1\}$ as the **ellipse of curvature** of f at x. Notice that in this definition, the ellipse of curvature could also be degenerated, like a line segment or a point. Also, it is worth to mention that the ellipse of curvature of f at $x \in M^2$ is a circle if and only if for some (and hence for any) orthonormal basis $\{e_1, e_2\}$ of $T_x M$ it holds that

$$\langle \alpha_{11} - \alpha_{22}, \alpha_{12} \rangle = 0 \quad \text{and} \quad \|\alpha_{11} - \alpha_{22}\| = 2\|\alpha_{12}\| > 0.$$

Recalling the definition by B. O'Neill (see [83]), a surface $f : M^2 \to \tilde{M}^m$ is called **λ-isotropic** if there exists a non-negative smooth function $\lambda : M^2 \to \mathbb{R}$ such that for each $x \in M^2$ and $X \in T_x M$ with $\|X\| = 1$, we have $\|\alpha(X,X)\| = \lambda(x)$. In the following proposition we discuss briefly the relation between the concepts of isotropy and the ellipse of curvature.

Proposition 2.5.1. Let $f : M^2 \to \tilde{M}^{4+p}$ $(p \geq 0)$ be a surface. If f is λ-isotropic, then either:

i. f is umbilical, in the case where $R^\perp = 0$, or;

ii. if f has non-vanishing normal curvature, its ellipse of curvature at each point $x \in M^2$ is a circle with center $H(x)$, which lies in a 2-dimensional affine subspace of $N_f M(x)$ orthogonal to the vector $H(x)$ and whose radius $r(x) > 0$ satisfies $\lambda(x)^2 = r(x)^2 + \|H(x)\|^2$.

In particular, if $p = 0$ in the last case, we also conclude that f is minimal. The converse is also true.

Proof. First, let us assume that f is λ-isotropic and fix an arbitrary point $x \in M^2$. To simplify the proof we also assume that an orthonormal basis $\{e_1, e_2\}$ of T_xM^2 is chosen in such a way that $\langle \alpha_{11} - \alpha_{22}, \alpha_{12} \rangle = 0$, which is always possible, up to a rotation of $\{e_1, e_2\}$.

We have that

$$2\langle H(x), \alpha_{11} - \alpha_{22} \rangle = \langle \alpha_{11} + \alpha_{22}, \alpha_{11} - \alpha_{22} \rangle = \|\alpha_{11}\|^2 - \|\alpha_{22}\|^2 = \lambda(x)^2 - \lambda(x)^2 = 0.$$

On the other hand, if $X = \cos te_1 + \sin te_2 \in T_xM^2$ denotes any unit tangent vector and $y(t) := \frac{1}{2}\cos(2t)(\alpha_{11} - \alpha_{22}) + \sin(2t)\alpha_{12}$, from equation (2.5.1) we have that $\alpha(X, X) = H(x) + y(t)$ and consequently

$$\lambda(x)^2 = \|H(x) + y(t)\|^2 = \|H(x)\|^2 + 2\langle H(x), y(t) \rangle + \|y(t)\|^2, \quad \text{for all} \quad t \in \mathbb{R} \tag{2.5.2}$$

In particular, applying equation (2.5.2) for $t = \frac{\pi}{4}$ and $t = \frac{-\pi}{4}$, we deduce that $\langle H(x), \alpha_{12} \rangle = 0$. In other words, $H(x)$ is orthogonal to $\{\alpha_{11} - \alpha_{22}, \alpha_{12}\}$. This also implies that $\langle H(x), y(t) \rangle = 0$, for all $t \in \mathbb{R}$, and consequently

$$\lambda(x)^2 = \|H(x)\|^2 + \|y(t)\|^2, \quad \text{for all} \quad t \in \mathbb{R}.$$

Thus the map $t \mapsto \|y(t)\|$ is constant.

By rewritting Ricci Equation (1.1.3) as $R^\perp(e_1, e_2) = (\alpha_{11} - \alpha_{22}) \wedge \alpha_{12}$, we see that f has non-vanishing normal curvature if and only if $\{\alpha_{11} - \alpha_{22}, \alpha_{12}\}$ is linearly independent. Then, if f has flat normal bundle, since we are assuming $\langle \alpha_{11} - \alpha_{22}, \alpha_{12} \rangle = 0$, at least one of these vectors is zero, but from the fact that $\mapsto \|y(t)\|$ is constant, we conclude that both are zero, which means that f is umbilical. On the other hand, if f has non-vanishing normal curvature, in particular $\|y(\frac{\pi}{4})\| = \|\alpha_{12}\| > 0$, whence we conclude that the ellipse of curvature of f at x is a circle centered at $H(x)$, lying in an affine plane of $N_fM(x)$ orthogonal to $H(x)$, whose radius $r(x) > 0$ satisfies $r(x)^2 = \lambda(x)^2 - \|H(x)\|^2$. If, in particular, $p = 0$ in this last case, since $H(x)$ is orthogonal to the plane spanned by $\{\alpha_{11} - \alpha_{22}, \alpha_{12}\}$, we conclude that $H(x) = 0$.

Now we assume that for all $x \in M^2$ the ellipse of curvature of f at x is a circle of radius $r(x) > 0$ centered at $H(x)$, lying in an affine plane of $N_fM(x)$ that is orthogonal to the vector $H(x)$. Then, for all $t \in \mathbb{R}$ we have that $y(t) := \frac{1}{2}\cos(2t)(\alpha_{11} - \alpha_{22}) + \sin(2t)\alpha_{12}$ has norm $r(x)$ and is orthogonal to $H(x)$. Thus,

$$\begin{aligned} \|\alpha(\cos te_1 + \sin te_2, \cos te_1 + \sin te_2)\|^2 &= \|H(x) + y(t)\|^2 \\ &= \|H(x)\|^2 + 2\langle H(x), y(t) \rangle + \|y(t)\|^2 \\ &= \|H(x)\|^2 + r(x)^2, \end{aligned}$$

for all $t \in \mathbb{R}$, whence we deduce that f is λ-isotropic with $\lambda(x)^2 = H(x)^2 + r(x)^2$.

Finally, if f is umbilical, by definition $\alpha(X, X) = H(x)$, for any unit tangent vector, which implies that f is λ-isotropic with $\lambda(x)^2 = \|H(x)\|^2$. □

Now we present our main result, which is a characterization of pseudo-parallel surfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with nowhere vanishing normal curvature.

Theorem 2.5.2. *Let $f : M^2 \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a ϕ-pseudo-parallel surface with nowhere vanishing normal curvature. Then f is λ-isotropic and*

$$r(x)^2 = K(x) - \phi(x) > 0, \quad \forall x \in M^2 \tag{2.5.3}$$

$$\lambda^2 = 4K - 3\phi + \epsilon(\|T\|^2 - 1) > 0, \tag{2.5.4}$$

$$\|H\|^2 = 3K - 2\phi + \epsilon(\|T\|^2 - 1) \geq 0, \tag{2.5.5}$$

where K denotes the Gaussian curvature, λ is a smooth real-valued function on M^2, $r(x)$ is the radius of the ellipse of curvature of f at $x \in M^2$, H is the mean curvature vector field of f and T is the tangent part of $\frac{\partial}{\partial t}$, the canonical unit vector field tangent to the second factor of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$. Conversely, if f is λ-isotropic then f is pseudo-parallel.

Proof. Let us suppose that $f : M^2 \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$ is pseudo-parallel with non-vanishing normal curvature. Combining equations (2.4.1) to (2.4.4) we get

$$\langle \alpha_{12}, \alpha_{ii}\rangle(\alpha_{11} - \alpha_{22}) + \{2(-1)^{i+1}(K - \phi) + \langle \alpha_{ii}, \alpha_{22} - \alpha_{11}\rangle\}\alpha_{12} = 0, \tag{2.5.6}$$
$$\{\|\alpha_{12}\|^2 + (\phi - K)\}(\alpha_{11} - \alpha_{22}) + \langle \alpha_{22} - \alpha_{11}, \alpha_{12}\rangle\alpha_{12} = 0. \tag{2.5.7}$$

From the assumption that f has non-vanishing normal curvature, and rewritting Ricci Equation (1.1.3) as $R^\perp(e_1, e_2) = (\alpha_{11} - \alpha_{22}) \wedge \alpha_{12}$, we conclude that $\{\alpha_{11} - \alpha_{22}, \alpha_{12}\}$ is linearly independent. Using this fact and equations (2.5.6) and (2.5.7) we obtain

$$\langle \alpha_{12}, \alpha_{11}\rangle = \langle \alpha_{12}, \alpha_{22}\rangle = 0, \tag{2.5.8}$$
$$\langle \alpha_{22} - \alpha_{11}, \alpha_{ii}\rangle = (-1)^i 2(K - \phi), \tag{2.5.9}$$
$$\|\alpha_{12}\|^2 = K - \phi > 0. \tag{2.5.10}$$

From equation (2.4.4) we get

$$\langle \alpha_{11}, \alpha_{22}\rangle = 2K - \phi + \epsilon(\|T\|^2 - 1), \tag{2.5.11}$$
$$\|\alpha_{11}\|^2 = \|\alpha_{22}\|^2 = 4K - 3\phi + \epsilon(\|T\|^2 - 1) > 0, \tag{2.5.12}$$
$$\|\alpha_{11} - \alpha_{22}\|^2 = 4(K - \phi) > 0, \tag{2.5.13}$$
$$\|H\|^2 = 3K - 2\phi + \epsilon(\|T\|^2 - 1). \tag{2.5.14}$$

In particular, since $(\alpha_{11} - \alpha_{22}) \perp \alpha_{12}$ and $\|\alpha_{11} - \alpha_{22}\| = 2\|\alpha_{12}\| > 0$, the ellipse of curvature of f at x is a circle. By equations (2.5.8) and (2.5.9) we may also deduce that $H(x)$ is orthogonal to $\{\alpha_{11} - \alpha_{22}, \alpha_{12}\}$. Therefore, from Proposition 2.5.1 we conclude that f is λ-isotropic with $\lambda^2 = 4K - 3\phi + \epsilon(\|T\|^2 - 1)$, and the radius $r(x)$ of the ellipse of curvature of f at x satisfies $r(x)^2 = K(x) - \phi(x)$.

Conversely, let us assume that f is λ-isotropic. Set $X = \cos\theta e_1 + \sin\theta e_2$. Then

$$\begin{aligned}
\lambda^2 &= \|\alpha(X, X)\|^2 \\
&= (\cos^4\theta + \sin^4\theta)\lambda^2 + 2\sin^2\theta\cos^2\theta\langle \alpha_{11}, \alpha_{22}\rangle \\
&\quad + 4\sin^3\theta\cos\theta\langle \alpha_{22}, \alpha_{12}\rangle + 4\sin\theta\cos^3\theta\langle \alpha_{11}, \alpha_{12}\rangle \\
&\quad + 4\sin^2\theta\cos^2\theta\|\alpha_{12}\|^2.
\end{aligned}$$

Since λ does not depend on θ, taking the derivative with respect to θ we get

$$0 = \left.\frac{d\lambda^2}{d\theta}\right|_{\theta=0} = \frac{d}{d\theta}(\|\alpha(X, X)\|^2)|_{\theta=0} = 4\langle \alpha_{11}, \alpha_{12}\rangle,$$
$$0 = \left.\frac{d\lambda^2}{d\theta}\right|_{\theta=\frac{\pi}{2}} = \frac{d}{d\theta}(\|\alpha(X, X)\|^2)|_{\theta=\frac{\pi}{2}} = -4\langle \alpha_{22}, \alpha_{12}\rangle.$$

On the other hand, with $Y = \frac{1}{\sqrt{2}}(e_1 + e_2)$ we get

$$\begin{aligned}
\lambda^2 &= \|\alpha(Y, Y)\|^2 \\
&= \frac{1}{4}\{2\lambda^2 + 4\|\alpha_{12}\|^2 + 2\langle \alpha_{11}, \alpha_{22}\rangle\},
\end{aligned}$$

that is,

$$\lambda^2 = 2\|\alpha_{12}\|^2 + \langle \alpha_{11}, \alpha_{22} \rangle.$$

Using this and Gauss Equation we get

$$\|\alpha_{12}\|^2 = \frac{1}{3}\{\lambda^2 - K + \epsilon(1 - \|T\|^2)\}.$$

From the Ricci Equation $R^\perp(e_1, e_2)\alpha_{ii} = \langle \alpha_{22} - \alpha_{11}, \alpha_{ii} \rangle \alpha_{12}, i = 1, 2$, we obtain

$$\begin{aligned} R^\perp(e_1, e_2)\alpha_{ii} &= \langle \alpha_{22} - \alpha_{11}, \alpha_{ii} \rangle \alpha_{12} \\ &= (-1)^i 2\|\alpha_{12}\|^2 \alpha_{12} \\ &= (-1)^i \frac{2}{3}\{\lambda^2 - K + \epsilon(1 - \|T\|^2)\}\alpha_{12}, \end{aligned}$$

Using the Ricci Equation once more we obtain

$$\begin{aligned} R^\perp(e_1, e_2)\alpha_{12} &= \|\alpha_{12}\|^2(\alpha_{11} - \alpha_{22}) \\ &= \frac{1}{3}\{\lambda^2 - K + \epsilon(1 - \|T\|^2)\}(\alpha_{11} - \alpha_{22}). \end{aligned}$$

Therefore, taking $\phi = \frac{4K - \lambda^2 + \epsilon(\|T\|^2 - 1)}{3}$, we conclude that f is pseudo-parallel according to equations (2.4.2) and (2.4.3). □

As a consequence of Theorem 2.5.2, we have the following non-existence result:

Corollary 2.5.3. There is no pseudo-parallel surface $f : M^2 \to \mathbb{Q}^3_\epsilon \times \mathbb{R}$ with nowhere vanishing normal curvature.

The following result we recall is due to M. Sakaki (see [91]), and it plays a vital role in the proof of Corollary 2.5.3.

Theorem 2.5.4. (see [91]) *Let $f : M^2 \to \mathbb{Q}^3_\epsilon \times \mathbb{R}$ be a minimal surface with $\epsilon \neq 0$. If the ellipse of curvature of f is a circle at any point $x \in M^2$, then f is totally geodesic.*

Now we are ready to prove Corollary 2.5.3.

Proof of Corollary 2.5.3. Suppose that a surface $f : M^2 \to \mathbb{Q}^3_\epsilon \times \mathbb{R}$ as in the statement do exist. Since f has non flat normal bundle, for any $x \in M^2$ we have that $R^\perp(x)(e_1, e_2) : N_f M(x) \to N_f M(x)$ is a non zero anti-symmetric linear operator, defined in a two-dimensional vector space. Thus, by Lemma 2.4.1 we conclude that $H(x) = 0$. But from Theorem 2.5.4, we conclude that f is totally geodesic and in particular, $R^\perp(x) = 0$, which is a contradiction. □

2.6 Some examples

We now introduce the first examples of semi-parallel and pseudo-parallel surfaces of $\mathbb{Q}^3_\epsilon \times \mathbb{R}$ which are not locally-parallel and semi-parallel, respectively, and that are not just inclusions of surfaces of $\mathbb{Q}^3_\epsilon$ into $\mathbb{Q}^3_\epsilon \times \mathbb{R}$.

Example 2.6.1. A general construction of submanifolds of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with flat normal bundle and T as a principal direction can be found in [80], by Mendonça-Tojeiro. For our purpose, based on this work, the construction becomes: let $g : J \to \mathbb{Q}^3_\epsilon$ be a regular curve and $\{\xi_1, \xi_2\}$ an orthonormal set of vector fields normal to g. Put

$$\begin{aligned}
&\tilde{g} = i \circ j \circ g,\\
&\tilde{\xi}_k = i_* j_* \xi_k, \quad \text{for} \quad k \in \{1, 2\},\\
&\tilde{\xi}_0 = \tilde{g}, \quad \tilde{\xi}_3 = i_* \frac{\partial}{\partial t},
\end{aligned}$$

where $j : \mathbb{Q}^3_\epsilon \to \mathbb{Q}^3_\epsilon \times \mathbb{R}$ and $i : \mathbb{Q}^3_\epsilon \times \mathbb{R} \to \mathbb{E}^5$ are the canonical inclusions. If $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) : I \to \mathbb{Q}^2_\epsilon \times \mathbb{R}$ is a smooth regular curve with $\alpha_3'(s) \neq 0, \quad \forall s \in I$, we have the following isometric immersion $f : M^2 = J \times I \to \mathbb{Q}^3_\epsilon \times \mathbb{R}$ given by

$$\tilde{f}(x, s) = (i \circ f)(x, s) = \sum_{k=0}^{3} \alpha_k(s) \tilde{\xi}_k(x). \tag{2.6.1}$$

At regular points, f is an isometric immersion with flat normal bundle and T as a principal direction. Conversely, if $f : M^2 \to \mathbb{Q}^3_\epsilon \times \mathbb{R}$ is an isometric immersion with flat normal bundle and T as a principal direction, then f is given by (2.6.1) for some isometric immersion $g : \mathbb{Q}^3_\epsilon \times \mathbb{R}$ and some smooth regular curve $\alpha : I \to \mathbb{Q}^2_\epsilon \times \mathbb{R}$ whose its last coordinate has non-vanishing derivative.

In particular, when dealing with pseudo-parallel surfaces in $\mathbb{Q}^3_\epsilon \times \mathbb{R}$, at least those that have T as a principal direction are fully described by this method.

We now construct two simple examples. Let us define

$$C_\epsilon(s) = \begin{cases} \cos(s), & \text{if} \quad \epsilon = 1 \\ \cosh(s), & \text{if} \quad \epsilon = -1 \end{cases} \qquad \text{and} \quad S_\epsilon(s) = \begin{cases} \sin(s), & \text{if} \quad \epsilon = 1 \\ \sinh(s), & \text{if} \quad \epsilon = -1. \end{cases}$$

(a) By taking

$$\begin{aligned}
&\tilde{g}(x) = (C_\epsilon(\theta(x)), S_\epsilon(\theta(x)), 0, 0, 0),\\
&\tilde{\xi}_1(x) = (0, 0, 1, 0, 0), \quad \tilde{\xi}_2(x) = (0, 0, 0, 1, 0),\\
&\alpha_0(s) = \sqrt{1 - \epsilon d^2}, \quad \alpha_1(s) = d \cos s, \quad \alpha_2(s) = d \sin s, \quad \alpha_3(s) = s.
\end{aligned}$$

where $0 < d < 1$, if $\epsilon = 1$, or $d > 0$, if $\epsilon = -1$, and $\theta : \mathbb{R}^2 \to \mathbb{R}$ is the smooth function given by

$$\theta(u) = \frac{u}{\sqrt{1 - \epsilon d^2}},$$

we obtain that

$$f(x, s) = (\sqrt{1 - \epsilon d^2} C_\epsilon(\theta(x)), \sqrt{1 - \epsilon d^2} S_\epsilon(\theta(x)), d \cos s, d \sin s, s).$$

Consider $\{e_1, e_2, e_3, e_4\}$ the orthonormal frame adapted to f, given by:

$$\begin{aligned}
e_1(x, s) &= \frac{f_* \partial_x}{\|f_* \partial_x\|} = (-\epsilon S_\epsilon(\theta(x)), C_\epsilon(\theta(x)), 0, 0, 0);\\
e_2(x, s) &= \frac{f_* \partial_s}{\|f_* \partial_s\|} = \frac{1}{\sqrt{1 + d^2}} (0, 0, -d \sin s, d \cos s, 1);\\
e_3(x, s) &= (-\epsilon d C_\epsilon(\theta(x)), -\epsilon d S_\epsilon(\theta(x)), \sqrt{1 - \epsilon d^2} \cos s, \sqrt{1 - \epsilon d^2} \sin s, 0);\\
e_4(x, s) &= \frac{1}{\sqrt{1 + d^2}} (0, 0, -\sin s, \cos s, -d).
\end{aligned}$$

Using this frame we have

$$\begin{aligned} T(x,s) &= \langle \frac{\partial}{\partial t}, e_1(x,s)\rangle e_1(x,s) + \langle \frac{\partial}{\partial t}, e_2(x,s)\rangle e_2(x,s) = \langle \frac{\partial}{\partial t}, e_1(x,s)\rangle e_1(x,s) \\ &= \frac{1}{1+d^2}(0,0,-d\sin s, d\cos s, 1), \end{aligned}$$

which implies that $1 - \|T\|^2 = \frac{d^2}{1+d^2}$. If we consider the decomposition $\alpha_{ij} = h^3_{ij}e_3 + h^4_{ij}e_4$, we have that

$$\begin{aligned} h^3_{11} &= \frac{\epsilon d}{\sqrt{1-\epsilon d^2}} \qquad h^3_{22} = \frac{-d\sqrt{1-\epsilon d^2}}{1+d^2} \\ h^3_{12} &= h^4_{11} = h^4_{12} = h^4_{22} = 0. \end{aligned}$$

Thus, we may compute the Gaussian curvature of f. It becomes

$$\begin{aligned} K &= \epsilon(1-\|T\|^2) + \langle \alpha_{11}, \alpha_{22}\rangle - \|\alpha_{12}\|^2 \\ &= \epsilon(1-\|T\|^2) + h^3_{11}h^3_{22} + h^4_{11}h^4_{22} - (h^3_{12})^2 - (h^4_{12})^2 \\ &= \frac{\epsilon d^2}{1+d^2} + \left(\frac{\epsilon d}{\sqrt{1-\epsilon d^2}}\right)\left(\frac{-d\sqrt{1-\epsilon d^2}}{1+d^2}\right) = 0. \end{aligned}$$

According to Proposition 2.4.2, f is a semi-parallel surface in $\mathbb{Q}^3_\epsilon \times \mathbb{R}$. Let us see that f is not locally-parallel. To do so, consider $\tilde{\nabla}$ the Levi-Civita connection of $\mathbb{E}^5$. Since $e_2 = \frac{f_*\partial_s}{\|f_*\partial_s\|}$, we have that

$$\tilde{\nabla}_{e_2}e_2 = \frac{1}{\sqrt{1+d^2}}\frac{\partial e_2(x,s)}{\partial s} = \frac{1}{1+d^2}(0,0,-d\cos s, -d\sin s, 0),$$

and as a consequence

$$\nabla_{e_2}e_2 = \langle \tilde{\nabla}_{e_2}e_2, e_1\rangle e_1 + \langle \tilde{\nabla}_{e_2}e_2, e_2\rangle e_2 = \langle \tilde{\nabla}_{e_2}e_2, e_1\rangle e_1 = 0.$$

Thus $\nabla_{e_2}e_2 = 0$, which implies that

$$\begin{aligned} (\nabla^\perp_{e_2}\alpha)(e_2,e_2) &= \nabla^\perp_{e_2}\alpha(e_2,e_2) - 2\alpha(\nabla_{e_2}e_2, e_2) = \nabla^\perp_{e_2}\alpha(e_2,e_2) \\ &= \nabla^\perp_{e_2}(h^3_{22}e_3 + h^4_{22}e_4) = \nabla^\perp_{e_2}(h^3_{22}e_3) \\ &= e_2(h^3_{22})e_3 + h^3_{22}\nabla^\perp_{e_2}e_3 = h^3_{22}\nabla^\perp_{e_2}e_3. \end{aligned}$$

But $\tilde{\nabla}_{e_2}e_4 = \frac{1}{\sqrt{1+d^2}}\frac{\partial e_4}{\partial s} = \frac{1}{1+d^2}(0,0,-\cos s, -\sin s, 0)$. Using this information, we obtain

$$\begin{aligned} \nabla^\perp_{e_2}e_3 &= \langle \tilde{\nabla}_{e_2}e_3, e_3\rangle e_3 + \langle \tilde{\nabla}_{e_2}e_3, e_4\rangle e_4 = \langle \tilde{\nabla}_{e_2}e_3, e_4\rangle e_4 \\ &= -\langle e_3, \tilde{\nabla}_{e_2}e_4\rangle e_4 = \frac{\sqrt{1-\epsilon d^2}}{1+d^2}(-\cos^2 s - \sin^2 s)e_4 \\ &= -\frac{\sqrt{1-\epsilon d^2}}{1+d^2}e_4 \neq 0. \end{aligned}$$

Therefore f is not locally parallel.

(b) Another example can be obtained by taking $0 < d < 1$, if $\epsilon = 1$ or $d > 1$, if $\epsilon = -1$, and

$$\begin{aligned}
&\tilde{g}(x) = (0, \cos(x), \sin(x), 0, 0, 0),\\
&\tilde{\xi}_1(x) = (1, 0, 0, 0, 0), \quad \tilde{\xi}_2(x) = (0, 0, 0, 1, 0),\\
&\alpha_0(s) = dS_\epsilon(s), \quad \alpha_1(s) = dC_\epsilon(s), \quad \alpha_2(s) = \sqrt{\epsilon(1-d^2)}, \quad \alpha_3(s) = s,
\end{aligned}$$

In this case

$$f(x,s) = (dC_\epsilon(s), dS_\epsilon(s)\cos(x), dS_\epsilon(s)\sin x, \sqrt{\epsilon(1-d^2)}, s).$$

Now, consider $\{e_1, e_2, e_3, e_4\}$ the orthonormal frame adapted to f given by:

$$\begin{aligned}
e_1(x,s) &= \frac{f_*\partial_s}{\|f_*\partial_s\|} = \frac{1}{\sqrt{1+d^2}}(-\epsilon dS_\epsilon(s), dC_\epsilon(s)\cos x, dC_\epsilon(s)\sin x, 0, 1);\\
e_2(x,s) &= \frac{f_*\partial_x}{\|f_*\partial_x\|} = (0, -\sin x, \cos x, 0, 0);\\
e_3(x,s) &= (\sqrt{\epsilon(1-d^2)}C_\epsilon(s), \sqrt{\epsilon(1-d^2)}S_\epsilon(s)\cos x, \sqrt{\epsilon(1-d^2)}S_\epsilon(s)\sin x, -\epsilon d, 0);\\
e_4(x,s) &= \frac{1}{\sqrt{1+d^2}}(\epsilon S_\epsilon(s), -C_\epsilon(s)\cos x, -C_\epsilon(s)\sin x, 0, d).
\end{aligned}$$

As in the previous example,

$$\begin{aligned}
T(x,s) &= \langle\frac{\partial}{\partial t}, e_1(x,s)\rangle e_1(x,s) + \langle\frac{\partial}{\partial t}, e_2(x,s)\rangle e_2(x,s) = \langle\frac{\partial}{\partial t}, e_1(x,s)\rangle e_1(x,s)\\
&= \frac{1}{1+d^2}(-\epsilon dS_\epsilon(s), dC_\epsilon(s)\cos x, dC_\epsilon(s)\sin x, 0, 1),
\end{aligned}$$

which implies that $1 - \|T\|^2 = \frac{d^2}{1+d^2}$. The coefficients of the second fundamental form in this frame becomes

$$\begin{aligned}
&h^3_{11} = \frac{-d\sqrt{\epsilon(1-d^2)}}{1+d^2} \qquad h^3_{22} = \frac{-\sqrt{\epsilon(1-d^2)}}{d} \qquad h^4_{22} = \frac{C_\epsilon(s)}{dS_\epsilon(s)\sqrt{1+d^2}}\\
&h^3_{12} = h^4_{11} = h^4_{12} = 0.
\end{aligned}$$

and the Gaussian curvature is given by

$$\begin{aligned}
K &= \epsilon(1-\|T\|^2) + h^3_{11}h^3_{22} + h^4_{11}h^4_{22} - (h^3_{12})^2 - (h^4_{12})^2\\
&= \epsilon(1-\|T\|^2) + h^3_{11}h^3_{22}\\
&= \frac{\epsilon d^2}{1+d^2} + \left(\frac{-d\sqrt{\epsilon(1-d^2)}}{1+d^2}\right)\left(\frac{-\sqrt{\epsilon(1-d^2)}}{d}\right) = \frac{\epsilon}{1+d^2} \neq 0.
\end{aligned}$$

According to Proposition 2.4.2 we deduce that the surface f is pseudo-parallel but not semi-parallel.

Question 2.6.2. Are there examples of pseudo-parallel surfaces in $\mathbb{Q}^3_\epsilon \times \mathbb{R}$ ($\epsilon \neq 0$) for which T is not a principal direction?

The next three examples show us that for $n > 3$ there exists pseudo-parallel surfaces with non-vanishing normal curvature.

Example 2.6.3. Let $\mathbb{S}^2_{1/3}$ be the 2-sphere of sectional curvature $\frac{1}{3}$ and consider $f : \mathbb{S}^2_{1/3} \to \mathbb{S}^4$ the classical *Veronese surface*, given by

$$f(x, y, z) = (\frac{1}{\sqrt{3}}xy, \frac{1}{\sqrt{3}}xz, \frac{1}{\sqrt{3}}yz, \frac{1}{2\sqrt{3}}(x^2 - y^2), \frac{1}{6}(x^2 + y^2 - 2z^2)),$$

which is a locally-parallel, minimal and λ-isotropic immersion (as we can see in [21], [61] and [93]) in $\mathbb{S}^4$ with non-vanishing normal curvature. If $i : \mathbb{S}^4 \to \mathbb{S}^4 \times \mathbb{R}$ is the totally geodesic inclusion given by $i(x) = (x, 0)$, then by Proposition 2.4.3 we have that $i \circ f$ is a pseudo-parallel immersion in $\mathbb{S}^4_1 \times \mathbb{R}$ with non-vanishing normal curvature.

Conjecture 2.6.4. Up to isometries, there are no pseudo-parallel surfaces in $\mathbb{Q}^4_\epsilon \times \mathbb{R}$ with non-vanishing normal curvature that are minimal besides the surface in Example 2.6.3.

Example 2.6.5. It's known by Chern in [25] that: "Any minimal immersion of a topological 2-sphere into $\mathbb{S}^4$ is a superminimal immersion". So, by Theorem 2.5.2, we have that any minimal immersion of a topological 2-sphere into a slice $\mathbb{S}^4 \times \mathbb{R}$ with non-vanishing normal curvature is pseudo-parallel with $\phi = \frac{4K-1-\lambda^2}{3}$. Moreover, if the Gaussian curvature is not constant, the immersion is not semi-parallel.

Example 2.6.6. Let $f : \mathbb{R}^2 \to \mathbb{S}^5$ be the surface given by

$$f(x, y) = \frac{2}{\sqrt{6}}(\cos u \cos v, \cos u \sin v, \frac{\sqrt{2}}{2}cos(2u), \sin u \cos v, \sin u \sin v, \frac{\sqrt{2}}{2}\sin(2u)),$$

where $u = \frac{1}{\sqrt{2}}x$, $v = \frac{\sqrt{6}}{2}y$.

This example, that appears in [92], is a minimal λ-isotropic flat torus with $\lambda = \frac{1}{\sqrt{2}}$ and non-vanishing normal curvature. In particular, f is a pseudo-parallel immersion in $\mathbb{S}^5$ with $\phi = \frac{-1}{2}$.

Thus, if $i : \mathbb{S}^5 \to \mathbb{S}^5 \times \mathbb{R}$ is the totally geodesic inclusion given by $i(x) = (x, 0)$, by Proposition 2.4.3 we have that $i \circ f$ is a pseudo-parallel immersion in $\mathbb{S}^5 \times \mathbb{R}$ with non-vanishing normal curvature.

Part II

CONSTANT ANISOTROPIC MEAN CURVATURE SURFACES IN $\mathbb{R}^3$

Chapter 3

Constant anisotropic mean curvature surfaces

In this chapter we present the theory of constant anisotropic mean curvature surfaces. We begin by introducing the variational problem whose critical points are the object of our study. Then we derive the first variation formula for the anisotropic area functional, that gives rise to the concept of anisotropic mean curvature, and we recall the second variation formula and the stability operator. After it, we introduce the most fundamental surface of constant anisotropic mean curvature: the Wulff shape. It is presented through a nice geometric description known as the Wulff's construction, and its algebraic formulation in terms of the function of anisotropy is also presented. In the following section we define the anisotropic Gauss map of a surface, the anisotropic analogous of the usual Gauss map, but whose range now is the Wulff shape instead the unit sphere. We can recover the anisotropic mean curvature of a surface by taking the derivative of its anisotropic Gauss map. We finish this introduction chapter showing some interesting examples. Some pictures of them and their isotropic counterparts are included. Among the examples we present the CAMC cylinders, that will play a crucial role in the next chapter.

3.1 The variational problem

Let $F : \mathbb{S}^n \to \mathbb{R}$ be a smooth-positive function. Consider the following functional:

$$\mathcal{F}(\psi) = \int_\Sigma F(N(x))d\Sigma, \tag{3.1.1}$$

where $\psi : \Sigma \to \mathbb{R}^{n+1}$ is an immersed hypersurface with Gauss map $N : \Sigma \to \mathbb{S}^n$. Notice that when $F \equiv 1$, $\mathcal{F}$ becomes the well-known area functional. We also recall the volume functional, that is given by the following relation:

$$\mathcal{V}(\psi) = \int_\Sigma \langle \psi(x), N(x) \rangle d\Sigma. \tag{3.1.2}$$

If $U \subset \mathbb{R}^{n+1}$ is a bounded domain whose boundary is a finite union of smooth immersed hypersurfaces $\{\psi_i : \Sigma_i \to \mathbb{R}^{n+1}\}_{i=1}^k$, the number $\sum_{i=1}^k \mathcal{V}(\psi_k)$ is the volume of U.

A natural question to ask is:

What are the critical points of the functional $\mathcal{F}$?

To answer this question we need to derive a variation formula for $\mathcal{F}$. A **smooth variation** (or deformation) of an immersed hypersurface $\psi : \Sigma \to \mathbb{R}^n$ is an application of the form $\Psi : (-\epsilon, \epsilon) \times \Sigma \to \mathbb{R}^{n+1}$ given by

$$\Psi(t, x) = \psi(x) + s(t)\xi(x)$$

where $s : (-\epsilon, \epsilon) \to \mathbb{R}$ is a smooth real-valued function with $s(0) = 0$ and $\xi : \Sigma \to T\mathbb{R}^{n+1} \approx \mathbb{R}^{n+1}$ is a smooth vector field along Σ, called **variation vector field**. When $\xi(x) \perp \psi_* T_x\Sigma$ we say that Ψ is a normal variation.

Let $\psi : \Sigma \to \mathbb{R}^{n+1}$ be a smooth, oriented immersed hypersurface. To obtain a first variation formula for the functional $\mathcal{F}$ we consider a normal variation $\Psi : (-\epsilon, \epsilon) \times \Sigma \to \mathbb{R}^{n+1}$ of Σ given by:

$$\psi_t(x) = \Psi(t, x) = \psi(x) + tu(x)N(x).$$

where $u : \Sigma \to \mathbb{R}$ is a smooth function with compact support and N is the Gauss map of ψ.

We recall that the first variation of the volume form $d\Sigma_t$ and the Gauss map N_t of ψ_t are given respectively by $\frac{d}{dt}d\Sigma_t = -nu(x)H(x)$ and $\frac{d}{dt}N_t(x) = -\operatorname{grad}_\Sigma u(x)$. Using these informations together with the Divergence Theorem, the derivative of $t \mapsto \mathcal{F}(\psi_t)$ at $t = 0$ becomes:

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(\Sigma_t) &= \frac{d}{dt}\int_\Sigma F(N_t(x))d\Sigma_t = \int_\Sigma \frac{d}{dt}F(N_t(x))d\Sigma + \int_\Sigma F(N(x))\frac{d}{dt}d\Sigma_t \\
&= \int_\Sigma \langle (\operatorname{grad}_{\mathbb{S}^n} F) \circ N(x), -\operatorname{grad}_\Sigma u(x)\rangle d\Sigma - n\int_\Sigma u(x)H(x)F(N(x))d\Sigma \\
&= -\int_\Sigma \{\operatorname{div}_\Sigma(u \cdot \operatorname{grad}_{\mathbb{S}^n} F) \circ N)(x) - u(x)\operatorname{div}_\Sigma((\operatorname{grad}_{\mathbb{S}^n} F) \circ N)\}d\Sigma \\
&\quad - n\int_\Sigma u(x)H(x)F(N(x))d\Sigma \\
&= -\int_{\partial\Sigma} u(x)\langle(\operatorname{grad}_{\mathbb{S}^n} F) \circ N, \eta\rangle d(\partial\Sigma) + \int_\Sigma u(x)\operatorname{div}_\Sigma((\operatorname{grad}_{\mathbb{S}^n} F) \circ N)(x)d\Sigma \\
&\quad - n\int_\Sigma u(x)H(x)F(N(x))d\Sigma \\
&= -\int_\Sigma u(x)[-\operatorname{div}_\Sigma((\operatorname{grad}_{\mathbb{S}^n} F) \circ N)(x) + nF(N(x))H(x)]d\Sigma,
\end{aligned}$$

that is,

$$\frac{d}{dt}\mathcal{F}(\Sigma_t) = -\int_\Sigma u(x)[-\operatorname{div}_\Sigma((\operatorname{grad}_{\mathbb{S}^n} F) \circ N(x) + nF(N(x))H(x)]d\Sigma. \tag{3.1.3}$$

Definition 3.1.1. The quantity $-\operatorname{div}_\Sigma((\operatorname{grad}_{\mathbb{S}^n} F) \circ N)(x) + nF(N(x))H(x)$ that appears in the integrand in equation (3.1.3) is called the **anisotropic mean curvature** of ψ with respect to F and N at $x \in \Sigma$. We denote this quantity by $\Lambda(x)$.

By equation (3.1.3) we obtain a conclusion: ψ is a critical point of the functional $\mathcal{F}$ if, and only if Λ vanishes identically.

In this work, we are also interested in the study of the critical points of $\mathcal{F}$ among the immersions that preserve volume. For this purpose, we have the following characterization:

Proposition 3.1.2. Let $F : \mathbb{S}^n \to \mathbb{R}$ be a smooth positive function and let $\psi : \Sigma \to \mathbb{R}^{n+1}$ be a smooth, oriented immersed surface. The following three statements are equivalent:

i. the anisotropic mean curvature Λ of ψ with respect to F is constant $\Lambda \equiv \Lambda_0$;

ii. for any compactly supported variation ψ_t of ψ that preserves the volume we have that $\frac{d}{dt}\mathcal{F}(\psi_t) = 0$;

iii. for any compactly supported variation ψ_t of ψ we have that $\frac{d}{dt}[\mathcal{F}(\psi_t)+\Lambda_0\mathcal{V}(\psi_t)] = 0$.

To prove Proposition 3.1.2, we need the following auxiliary result.

Lemma 3.1.3. (see [75]) Let $\psi : \Sigma \to \mathbb{R}^{n+1}$ be an immersed hypersurface and let $u : \Sigma \to \mathbb{R}$ be a piecewise smooth real-valued function with compact support satisfying $\int_\Sigma u(x)d\Sigma = 0$. Then there exists a normal variation ψ_t of ψ that preserves volume and such that $\frac{d}{dt}\psi_t(x) = u(x)N(x)$ at $t = 0$, where $N : \Sigma \to \mathbb{S}^n$ is the Gauss map of ψ.

Proof of Proposition 3.1.2. We first prove that item [iii] implies item [ii]. Let ψ_t be a normal variation of ψ that preserves volume. Then $t \mapsto \mathcal{V}(\psi_t)$ is constant. Therefore $0 = \frac{d}{dt}\left[\mathcal{F}(\psi_t) - \Lambda_0\mathcal{V}(\psi_t)\right] = \frac{d}{dt}\mathcal{F}(\psi_t)$.

Now, let us assume that item [ii] implies item [i]. Let $p \in \Sigma$ a point and choose $D \subset \Sigma$ a relatively compact domain that contains p in its interior. We define

$$\Lambda_0 = \frac{\int_D \Lambda(x)d\Sigma}{\int_D 1d\Sigma}. \tag{3.1.4}$$

Our task is to prove that $\Lambda = \Lambda_0$ on D. Let us suppose that $\Lambda(q) \neq \Lambda_0$, for some $q \in D$. We define

$$D_+ = \{x \in D; \Lambda(x) - \Lambda_0 > 0\}, \quad \text{and} \quad D_- = \{x \in D; \Lambda(x) - \Lambda_0 < 0\}.$$

Since Λ is continuous (because H, F and N are continuous) we conclude that D_+ and D_- are open subsets of D. Without loss of generality we can assume that $q \in D_+$. By the definition of Λ_0 we have

$$\int_D \Lambda(x) - \Lambda_0 d\Sigma = 0,$$

whence we conclude that D_- is also non-empty. It is possible to choose positive smooth functions $v, w : \Sigma \to \mathbb{R}$ such that $p \in \text{supp}(v) \subset D_+$, $\text{supp}(w) \subset D_-$ and

$$\int_D (v(x) + w(x))(\Lambda(x) - \Lambda_0)d\Sigma = 0. \tag{3.1.5}$$

Put $u := (v + w)(\Lambda - \Lambda_0)$. We have that $\text{supp}(u) \subset D_+ \cup D_-$, and in particular $u(x) = 0$ for all $x \in \partial D$. Also, by equation 3.1.5 we have that $\int_\Sigma u(x)d\Sigma = 0$. Using Lemma 3.1.3, there exists a normal variation ψ_t of ψ that preserves volume and satisfies

$$\frac{d}{dt}\psi_t(x) = u(x)N(x).$$

Applying the first variation formula and using the hypothesis, we conclude at $t = 0$ that

$$\begin{aligned} 0 = \frac{d}{dt}\mathcal{F}(\psi_t) &= -\int_\Sigma u(x)\Lambda(x)d\Sigma = -\int_\Sigma u(x)\Lambda(x)d\Sigma + \Lambda_0\int_\Sigma u(x)d\Sigma \\ &= -\int_\Sigma u(x)(\Lambda(x) - \Lambda_0)d\Sigma = -\int_\Sigma (v(x) + w(x))(\Lambda(x) - \Lambda_0)^2 d\Sigma < 0, \end{aligned}$$

leading to a contradiction. Therefore we conclude that Λ is locally constant, and since Σ is connected, $\Lambda \equiv \Lambda_0$, proving that item [i] is valid.

Finally, assume that item [i] is valid. By the formula (3.1.3) and the first variation of the volume we have at $t = 0$ that

$$0 = \frac{d}{dt}\left[\mathcal{F}(\psi_t) + \Lambda_0 \mathcal{V}(\psi_t)\right] = -\int_\Sigma u(x)\Lambda(x)d\Sigma + \Lambda_0 \int_\Sigma u(x)d\Sigma$$
$$= -\int_\Sigma [\Lambda(x) - \Lambda_0]u(x)d\Sigma.$$

Since u is a generic smooth compactly supported function on Σ, we conclude that $\Lambda(x) = \Lambda_0$, for all $x \in \Sigma$, which proves that item [iii] is valid. □

Sometimes we refer a constant anisotropic mean curvature surface using the abbreviation CAMC-surface, or Λ-CAMC surface when we want to specify that it anisotropic mean curvature is Λ.

The second variation of $\mathcal{F}$ is obtained in a similar way as we did for the first variation. The computations and details can be found in [64]. Here we only recall that if $\psi : \Sigma \to \mathbb{R}$ is an immersion with constant anisotropic mean curvature Λ, the second variation formula of the functional $\mathcal{F} + \Lambda\mathcal{V}$ is given by

$$\frac{d^2}{dt^2}\mathcal{F}(\psi_t) = -\int_\Sigma u(x)\operatorname{div}_\Sigma((D^2F \circ N + F \circ N)\operatorname{grad}_\Sigma u)(x)d\Sigma$$
$$-\int_\Sigma \|(D^2F_{N(x)} + F(N(x))) \cdot dN_x\|^2 u(x)^2 d\Sigma, \quad (3.1.6)$$

where D^2F_y is the operator that represents $\operatorname{Hess}_{\mathbb{S}^n} F_y$, that is,

$$\langle D^2F_yX, Y\rangle = \operatorname{Hess}_{\mathbb{S}^n} F_y(X, Y), \quad \forall X, Y \in T_y\mathbb{S}^n.$$

This formula motivates the following definition: denoting by $C_c^\infty(\Sigma)$ the space of compactly supported smooth functions on Σ, the **stability operator** of $\mathcal{F}$ is the linear operator $L : C_c^\infty(\Sigma) \to C_c^\infty(\Sigma)$ given by:

$$Lu(x) := \operatorname{div}_\Sigma((D^2F \circ N + F \circ N)\operatorname{grad}_\Sigma u)(x) + \|(D^2F_{N(x)} + F(N(x))) \cdot dN_x\|^2 u(x).$$

In terms of the stability operator we can write equation (3.1.6) as

$$\frac{d^2}{dt^2}\mathcal{F}(\psi_t) = -\int_\Sigma uLud\Sigma.$$

If $\psi : \Sigma \to \mathbb{R}^{n+1}$ is an isometric immersion with constant anisotropic mean curvature, we say that ψ is **stable** if the stability operator L is negative-definite.

3.2 The Wulff's construction

Let $F : \mathbb{S}^n \to \mathbb{R}$ a smooth positive function. For each $x \in \mathbb{S}^n$, consider $P(x)$ the hyperplane orthogonal to the position vector x that lies at $F(x)x$. Then $P(x)$ defines two closed half-spaces of $\mathbb{R}^{n+1}$, being one of them the half-space where x points outward, that we call $P_-(x)$.

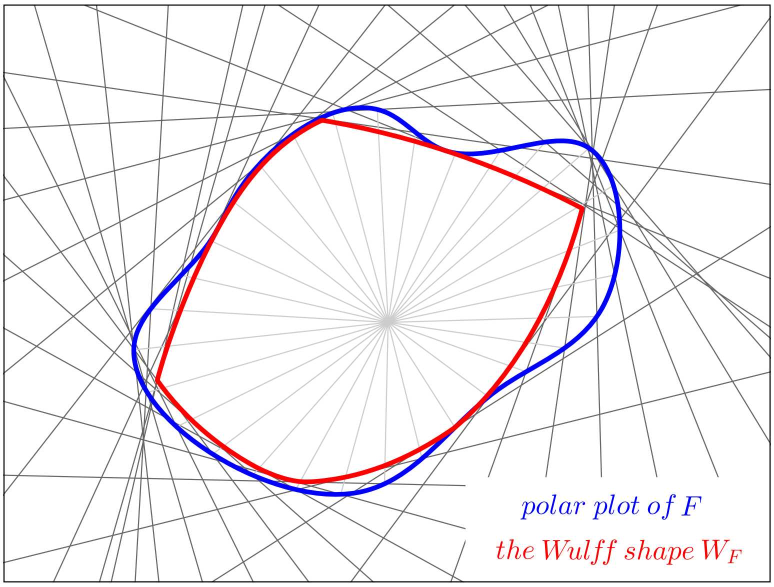

Figure 3.1: The Wulff's construction.

Definition 3.2.1. The **Wulff shape** of F is the defined as the boundary of the intersection of all the half-spaces $P_-(x)$ when x varies in $\mathbb{S}^n$. We denote this set by W_F. Symbolically W_F is given by

$$W_F = \partial\{y \in \mathbb{R}^{n+1}; \langle y, x\rangle \leq F(x), \quad \forall x \in \mathbb{S}^n\}. \tag{3.2.1}$$

Notice that when $F \equiv 1$, W_F becomes the n-dimensional sphere $\mathbb{S}^n$.

Remark 3.2.2. In this work we always impose that

$$(\mathrm{Hess}_{\mathbb{S}^n} F)_y(\cdot,\cdot) + F(y)\langle\cdot,\cdot\rangle : T_x\Sigma \times T_x\Sigma \to \mathbb{R} \tag{3.2.2}$$

is a positive symmetric bilinear form, for all $x \in \Sigma$. We call this assumption on F as the **convexity condition**. The reason of this name will become clear now.

Consider the application $G : \mathbb{S}^n \to \mathbb{R}^{n+1}$ given by:

$$G(y) = i_*(\mathrm{grad}_{\mathbb{S}^n} F)_y + F(y)y, \tag{3.2.3}$$

where $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is the canonical inclusion. For all $X \in T_x\mathbb{S}^n$ we have that

$$\begin{aligned} dG_y(X) &= (D_{i_*X}\,\mathrm{grad}_{\mathbb{S}^n} F)_y + \langle(\mathrm{grad}_{\mathbb{S}^n} F)_y, X\rangle y + F(y)i_*X \\ &= i_*\nabla_X\,\mathrm{grad}_{\mathbb{S}^n} F - \langle\mathrm{grad}_{\mathbb{S}^n} F, X\rangle y + \langle(\mathrm{grad}_{\mathbb{S}^n} F)_y, X\rangle y + F(y)i_*X \\ &= i_*\nabla_X\,\mathrm{grad}_{\mathbb{S}^n} F + F(y)i_*X. \end{aligned}$$

From this equation we conclude that G is an immersion. Indeed, if $dG_y(X) = 0$ for some $X \neq 0$, then $\|\nabla_X\,\mathrm{grad}_{\mathbb{S}^n} F\| = F(y)\|X\|$, but since we are assuming the convexity condition on F, we have

$$\begin{aligned} 0 = \langle dG_y(X), dG_y(X)\rangle &= \|\nabla_X\,\mathrm{grad}_{\mathbb{S}^n} F\| + 2F(y)\langle\nabla_X\,\mathrm{grad}_{\mathbb{S}^n} F, X\rangle + F(y)^2\|X\|^2 \\ &= \|\nabla_X\,\mathrm{grad}_{\mathbb{S}^n} F\| + 2F(y)(\mathrm{Hess}_{\mathbb{S}^n} F)_y(X,X) + F(y)^2\|X\|^2 \\ &> \|\nabla_X\,\mathrm{grad}_{\mathbb{S}^n} F\| - 2F(y)^2\|X\|^2 + F(y)^2\|X\|^2 \\ &= \|\nabla_X\,\mathrm{grad}_{\mathbb{S}^n} F\| - F(y)^2\|X\|^2, \end{aligned}$$

which is a contradiction. We have also that

$$\langle dG_y(X), y\rangle = \langle i_*\nabla_X \operatorname{grad}_{\mathbb{S}^n} F + F(y)i_*X, y\rangle = 0,$$

which means that the outward pointing Gauss map of $G(\mathbb{S}^n)$ is given by $N_W : x \mapsto G^{-1}(x)$. Thus we are able to compute the second fundamental form of G. We have:

$$\begin{aligned}\langle -d(N_W \circ G)_y(X), dG_y(Y)\rangle &= -\langle i_*X, dG_y(Y)\rangle \\ &= -\langle i_*X, i_*\nabla_Y \operatorname{grad}_{\mathbb{S}^n} F + F(y)i_*Y\rangle \\ &= -(\operatorname{Hess}_{\mathbb{S}^n} F)_y(X, Y) - F(y)\langle X, Y\rangle.\end{aligned}$$

The convexity condition implies that $G(\mathbb{S}^n)$ has strictly positive Gaussian curvature, and in particular, it encloses a strictly convex domain of $\mathbb{R}^{n+1}$.

Now, since $i_* \operatorname{grad}_{\mathbb{S}^n} F$ is orthogonal to y we have

$$\langle G(y), y\rangle = F(y), \tag{3.2.4}$$

whence we conclude that $G(y) \in P(y)$. From the fact that $N_W(G(y)) = y$ and $G(y) \in P(y) \cap (G(y) + T_{G(y)}G(\mathbb{S}^n))$, we conclude that $P(y) = G(y) + T_{G(y)}G(\mathbb{S}^n)$. In particular, by the convexity of $G(\mathbb{S}^n)$ we also conclude that $G(\mathbb{S}^n)$ is contained in $P_-(y)$. Therefore $G(\mathbb{S}^n)$ is the Wulff shape W_F.

Remark 3.2.3. It is useful to know that given a compact smooth hypersurface $W \subset \mathbb{R}^{n+1}$ which bounds a strictly convex domain, we can recover the function $F : \mathbb{S}^n \to \mathbb{R}$ for which W is its Wulff Shape. For this purpose, equation (3.2.4) gives a hint: if N_W is the outward pointing Gauss map of W, then $N_W : W \to \mathbb{S}^n$ is a diffeomorphism and we recover F by just defining $F(x) = \langle (N_W)^{-1}x, x\rangle$.

An important feature of the Wulff shape is its minimizing property, which is the content of the next Theorem:

Theorem 3.2.4. (Wulff's Theorem, see Thm. 1.1 in [100]) Let $F : \mathbb{S}^n \to \mathbb{R}$ a smooth positive function and let W_F be its Wulff shape. Then $\mathcal{F}(W_F) \leq \mathcal{F}(\psi)$, among all hypersurfaces $\psi : \Sigma \to \mathbb{R}^{n+1}$ enclosing the same volume as W_F. Moreover, the equality holds if and only if $\psi(\Sigma)$ a rescaling of W_F.

Remark 3.2.5. The Wulff's construction were indeed formulated for integrands $F : \mathbb{S}^n \to \mathbb{R}$ that are not neccessarly smooth. The same conclusion of Theorem 3.2.4 is valid, where Riemannian manifolds give place to integral currents, a kind of more general spaces.

As a consequence Theorem 3.2.4 we conclude that W_F has constant anisotropic mean curvature and that it is stable. Actually, the Wulff shape plays the same role for the anisotropic area functional $\mathcal{F}$ as the unit sphere $\mathbb{S}^n$ for the area functional. In [56], for example, authors proved an anisotropic version of the famous Alexandrov Theorem (see [6]):

Theorem 3.2.6. (see [56]) Let $\psi : \Sigma \to \mathbb{R}^{n+1}$ be a compact hypersurface without boundary embedded in Euclidean space, and let $F : \mathbb{S}^n \to \mathbb{R}$ be a positive, smooth function that satisfies the convexity condition. If the anisotropic mean curvature of ψ with respect to F is constant, then up to translations and homotheties, $\psi(\Sigma)$ is the Wulff shape.

An anisotropic version of the Hopf Theorem was given independently by M. Koiso and B. Palmer in [65] and by N. Ando in [7]. Namely, the Wulff shape W_F is the only topological 2-sphere immersed in $\mathbb{R}^3$ with constant anisotropic mean curvature. And many other results concerning the Wulff shape can be found in the literature. For the interested reader we mention for example [52], [53], [54], [55], [56], [65], [82] and [84].

3.3 The anisotropic Gauss map

In this section we define the anisotropic version of the Weingarten operator, that represents the differential of the anisotropic Gauss map and whose trace is the anisotropic mean curvature. We compare the isotropic and anisotropic Weingarten operators and rewrite the stability operator in terms of the last one.

Let us fix a smooth positive function $F : \mathbb{S}^n \to \mathbb{R}$ satisfying the convexity condition. Consider an oriented immersed hypersurface $\psi : \Sigma \to \mathbb{R}^{n+1}$ with a fixed choice of Gauss map, that we denote by $N : \Sigma \to \mathbb{S}^n$. Since the Wulff shape W_F of F is a compact smooth hypersurface that bounds a convex region of $\mathbb{R}^{n+1}$, its pointing outward Gauss map $N_W : W_F \to \mathbb{S}^n$ is a diffeomorphism and the composition

$$\nu := (N_W)^{-1} \circ N : \Sigma \to W_F \tag{3.3.1}$$

is a well defined smooth application, that we call **anisotropic Gauss map** of ψ with respect to F. For each $p \in \Sigma$, the normal space of W_F at $\nu(p)$ is spanned by $N(p)$, since $N_W(\nu(p)) = N(p)$. Then the differential of ν at p, which is an application from $T_p\Sigma$ to $T_{\nu(p)}W_F$, can be regarded as an application from $T_p\Sigma$ to itself. If A and A_W are the Weingarten operators of $\psi(\Sigma)$ and W_F with respect to N and N_W, respectively, then

$$\langle -d\nu_p \cdot X, Y\rangle = \langle -d(N_W)^{-1}_{N(p)} dN_p \cdot X, Y\rangle = \langle -A_W^{-1} \circ A \cdot X, Y\rangle,$$

for any $X, Y \in T_p\Sigma$, that is, $-d\nu_p$ is represented by $A_F := -A_W^{-1} \circ A : T_p\Sigma \to T_p\Sigma$, that we call the **anisotropic Weingarten operator** of ψ with respect to F (and N).

Proposition 3.3.1. The anisotropic mean curvature satisfies:

$$\Lambda(p) = \mathrm{trace}_\Sigma(-d\nu_p) = \mathrm{trace}_\Sigma(A_F).$$

Proof. First, we notice that

$$\begin{aligned}\mathrm{div}_\Sigma(\mathrm{grad}_{\mathbb{S}^n} F \circ N)_p &= \sum_{i,j=1}^n g^{ij} \langle \nabla_{X_i}(\mathrm{grad}_{\mathbb{S}^n} F \circ N), X_j\rangle \\ &= \sum_{i,j=1}^n g^{ij} \langle \tilde{\nabla}_{X_i}(\mathrm{grad}_{\mathbb{S}^n} F \circ N), i_* X_j\rangle.\end{aligned}$$

Thus, using the parametrization $G : \mathbb{S}^n \to W_F$ given in equation (3.2.3) and recalling that $(N_W)^{-1} = G$, we obtain

$$\begin{aligned}\sum_{i,j=1}^n g^{ij} \langle -d\nu_p \cdot X_i, X_j\rangle &= \sum_{i,j=1}^n -g^{ij} \langle dG_{N(p)} dN_p \cdot X_i, X_j\rangle \\ &= \sum_{i,j=1}^n -g^{ij} \langle i_* \nabla_{dN_p \cdot X_i} \mathrm{grad}_{\mathbb{S}^n} F + F(N(p)) dN_p X_i, i_* X_j\rangle \\ &= -\sum_{i,j=1}^n g^{ij} \langle \nabla_{dN_p \cdot X_i} (\mathrm{grad}_{\mathbb{S}^n} F)_{N(p)}, X_j\rangle \\ &+ F(N(p)) \sum_{i,j=1}^n g^{ij} \langle -dN_p X_i, X_j\rangle \\ &= -\mathrm{div}_\Sigma(\mathrm{grad}_{\mathbb{S}^n} F \circ N)_p + nF(N(p))H(p) = \Lambda(p).\end{aligned}$$

□

It is interesting to notice that the Wulff shape has constant anisotropic mean curvature equals to -2, with respect to the outward pointing unit normal N_W, since $\nu(p) = (N_W)^{-1} \circ N_W(p) = p$ and therefore $-d\nu_p = -Id$.

Another useful observation is the behaviour of the anisotropic mean curvature with respect to homotheties. When $\psi : \Sigma \to \mathbb{R}^{n+1}$ is an isometric immersion with anisotropic mean curvature Λ with respect to choice of a Gauss map $N : \psi(\Sigma) \to \mathbb{S}^n$, then for any $c \neq 0$ the immersion $\tilde{\psi} : \Sigma \to \mathbb{R}^{n+1}$ given by $\tilde{\psi}(x) = c\psi(x)$ has anisotropic mean curvature $\tilde{\Lambda}(x) = \frac{1}{c}\Lambda(x)$, with respect to $\tilde{N} : \tilde{\psi}(\Sigma) \to \mathbb{S}^n$ given by $\tilde{N}(\tilde{\psi}(x)) = N(\psi(x))$, for $x \in \Sigma$. Indeed, if $\{X_1, \ldots, X_n\}$ is a basis of $T_p\Sigma$, then the coefficients of the first fundamental form of ψ and $\tilde{\psi}$ at p are respectively given by $g_{ij} = \langle d\psi_p(X_i), d\psi_p(X_j)\rangle$ and $\tilde{g}_{ij} = \langle d\tilde{\psi}_p(X_i), d\tilde{\psi}_p(X_j)\rangle = c^2 g_{ij}$. Since $\tilde{\nu}(\tilde{\psi}(p)) = N_W^{-1}(\tilde{N}(\tilde{\psi}(p))) = N_W^{-1}(N(\psi(p))) = \nu(\psi(p))$, we have

$$\begin{aligned}
\tilde{\Lambda}(p) := \operatorname{trace}(-d\tilde{\nu}_{\tilde{\psi}(p)}) &= \sum_{i,j=1}^{n} \tilde{g}^{ij} \langle -d\tilde{\nu}_{\tilde{\psi}(p)} d\tilde{\psi}_p X_i, d\tilde{\psi}_p X_j\rangle \\
&= \sum_{i,j=1}^{n} \frac{g^{ij}}{c^2} \langle -d\nu_{\psi(p)} d\psi_p X_i, c d\psi_p X_j\rangle \\
&= \frac{1}{c} \sum_{i,j=1}^{n} g^{ij} \langle -d\nu_{\psi(p)} d\psi_p X_i, d\psi_p X_j\rangle \\
&= \frac{1}{c} \operatorname{trace}(-d\nu_{\psi_p}) = \frac{1}{c}\Lambda(p).
\end{aligned}$$

In particular, the image of the Wulff shape by the antipodal map has constant anisotropic mean curvature equals 2 with respect to the inward point unit normal map.

There are some inequality relations involving the isotropic and the anisotropic Weingarten operators, as we can see in the next proposition.

Proposition 3.3.2. The following assertions are true:

(a) $\|A_F\|^2 = \operatorname{trace}_\Sigma(A_F^t A_F) = \frac{h_{11}^2}{\mu_1^2} + \frac{h_{12}^2}{\mu_1^2} + \frac{h_{12}^2}{\mu_2^2} + \frac{h_{22}^2}{\mu_2^2}$;

(b) $4K_F \leq \Lambda^2 \leq \|A_F\|^2 + 2K_F$;

(c) $\frac{1}{M^2}\|A\|^2 \leq \|A_F\|^2 \leq \frac{1}{m^2}\|A\|^2$;

(d) $A_F^2 - \Lambda A_F + K_F Id = 0$;

(e) $\operatorname{trace}_\Sigma(A_F^2) = \Lambda^2 - 2K_F$;

(f) $\operatorname{trace}_\Sigma(AA_F) = \operatorname{trace}_\Sigma(A_F A) = \frac{h_{11}^2}{\mu_1} + \frac{h_{12}^2}{\mu_1} + \frac{h_{12}^2}{\mu_2} + \frac{h_{22}^2}{\mu_2}$;

(g) $H^2 \leq (h_{11}^2 + h_{22}^2) \leq 2M^2\|A_F\|^2$,

where $m = \inf\{min\{\mu_1(\nu(p)), \mu_2(\nu(p))\}; p \in \Sigma\}$ and $M = \sup\{\max\{\mu_1(\nu(p)), \mu_2(\nu(p))\}; p \in \Sigma\}$ and $K_F := \det(A_F)$.

Proof. Items (a), (e) and (f) are just computations. Items (c) is consequence of item (a). Item (d) is the Cayley-Hamilton Theorem. Thus we prove only items (b) and (g). We have that

$$\begin{aligned}\Lambda^2 &= \left(\frac{h_{11}}{\mu_1} + \frac{h_{22}}{\mu_2}\right)^2 = \frac{h_{11}^2}{\mu_1^2} + 2\frac{h_{11}h_{22}}{\mu_1\mu_2} + \frac{h_{22}^2}{\mu_2^2} \\ &= \frac{h_{11}^2}{\mu_1^2} - 2\frac{h_{11}h_{22}}{\mu_1\mu_2} + \frac{h_{22}^2}{\mu_2^2} + 4\frac{h_{11}h_{22}}{\mu_1\mu_2} \\ &= \left(\frac{h_{11}}{\mu_1} - \frac{h_{22}}{\mu_2}\right)^2 + 4\frac{h_{11}h_{22}}{\mu_1\mu_2} \\ &\geq 4\frac{h_{11}h_{22}}{\mu_1\mu_2} \geq 4\left(\frac{h_{11}h_{22} - h_{12}^2}{\mu_1\mu_2}\right) = 4K_F.\end{aligned}$$

On the other hand, using item (a) we get

$$\begin{aligned}\|A_F\|^2 &= \frac{h_{11}^2}{\mu_1^2} + \frac{h_{12}^2}{\mu_1^2} + \frac{h_{12}^2}{\mu_2^2} + \frac{h_{22}^2}{\mu_2^2} \\ &= \left(\frac{h_{11}}{\mu_1} + \frac{h_{22}}{\mu_2}\right)^2 - 2\frac{h_{11}h_{22}}{\mu_1\mu_2} + h_{12}^2\left(\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}\right) \\ &= \left(\frac{h_{11}}{\mu_1} + \frac{h_{22}}{\mu_2}\right)^2 - 2\frac{h_{11}h_{22}}{\mu_1\mu_2} + h_{12}^2\frac{\mu_1^2 + \mu_2^2}{\mu_1^2\mu_2^2} \\ &\geq \left(\frac{h_{11}}{\mu_1} + \frac{h_{22}}{\mu_2}\right)^2 - 2\frac{h_{11}h_{22}}{\mu_1\mu_2} + 2h_{12}^2\frac{\mu_1\mu_2}{\mu_1^2\mu_2^2} \\ &= \left(\frac{h_{11}}{\mu_1} + \frac{h_{22}}{\mu_2}\right)^2 - 2\frac{h_{11}h_{22} - h_{12}^2}{\mu_1\mu_2} = \Lambda^2 - 2K_F.\end{aligned}$$

Putting them together, we obtain

$$4K_F \leq \Lambda^2 \leq \|A_F\|^2 + 2K_F.$$

This proves item (b). To proof (g), we notice that

$$\begin{aligned}\|A_F\|^2 &= \frac{h_{11}^2}{\mu_1^2} + \frac{h_{12}^2}{\mu_2^2} + \frac{h_{12}^2}{\mu_1^2} + \frac{h_{22}^2}{\mu_2^2} \geq \frac{h_{11}^2}{M^2} + \frac{h_{12}^2}{M^2} + \frac{h_{12}^2}{M^2} + \frac{h_{22}^2}{M^2} \\ &\geq \frac{1}{M^2}(h_{11}^2 + h_{22}^2)\end{aligned}$$

and

$$H^2 = (h_{11} + h_{22})^2 = h_{11}^2 + 2h_{11}h_{22} + h_{22}^2 \leq 2(h_{11}^2 + h_{22}^2)$$

putting them together, we obtain

$$H^2 \leq 2(h_{11}^2 + h_{22}^2) \leq 2M^2\|A_F\|^2.$$

□

Now, we write the anisotropic mean curvature in terms of the isotropic principal curvatures. It will be useful in the next chapter. For this purpose, let $\Sigma \subset \mathbb{R}^3$ be a surface and let $N : \Sigma \to \mathbb{S}^2$ be its Gauss map. Let us fix a point $p \in \Sigma$. Consider $\{E_1, E_2\}$ the principal directions of Σ at p, i.e.,

$$-dN_pE_i = \kappa_i(p)E_i, \qquad i \in 1, 2,$$

where $\kappa_1(p)$ and $\kappa_2(p)$ are the principal curvatures of Σ at p. On the other hand, if $\{e_1, e_2\}$ are the principal directions of $W_F = G(\mathbb{S}^2)$ at $G(N(p))$, and $\mu_1(N(p))$, $\mu_2(N(p))$ are the principal curvatures related to e_1 and e_2, respectively, we already know that

$$dG_{N(p)}e_i = \frac{1}{\mu_i(N(p))}e_i, \qquad i \in 1, 2,$$

Since we can identify $T_p\Sigma$ with $T_{N(p)}\mathbb{S}^2$, we can write $\{E_1, E_2\}$ in terms of the basis $\{e_1, e_2\}$ as:

$$E_1 = a_{11}(p)e_1 + a_{21}(p)e_2,$$
$$E_2 = a_{12}(p)e_1 + a_{22}(p)e_2,$$

with $a_{11}(p)^2 + a_{21}(p)^2 = a_{12}(p)^2 + a_{22}(p)^2 = 1$. In this way, if $\nu = G \circ N$ is the anisotropic Gauss map of Σ, we have:

$$\begin{aligned}
-d\nu_p E_i = -dG_{N(p)}dN_p E_i = \kappa_i(p)dG_{N(p)}E_i = \kappa_i(p)\sum_{j=1}^{2} a_{ji}(p)dG_{N(p)}e_j \\
= \sum_{j=1}^{2} \kappa_i(p)\frac{a_{ji}(p)}{\mu_j(N(p))}e_j.
\end{aligned}$$

Evaluating the anisotropic mean curvature we obtain

$$\begin{aligned}
\Lambda = \operatorname{trace}_\Sigma(-d\nu_p) &= \sum_{i=1}^{2}\langle -d\nu_p E_i, E_i\rangle \\
&= \sum_{i=1}^{2}\left\langle \sum_{j=1}^{2}\kappa_i(p)\frac{a_{ji}(p)}{\mu_j(N(p))}e_j, \sum_{k=1}^{2} a_{ki}(p)e_k \right\rangle \\
&= \sum_{i,j=1}^{2}\kappa_i(p)\frac{a_{ji}(p)^2}{\mu_j(N(p))}.
\end{aligned}$$

Defining $\gamma_1(p) = \frac{a_{11}(p)^2}{\mu_1(N(p))} + \frac{a_{21}(p)^2}{\mu_2(N(p))}$ and $\gamma_2(p) = \frac{a_{12}(p)^2}{\mu_1(N(p))} + \frac{a_{22}(p)^2}{\mu_2(N(p))}$, we can write:

$$\Lambda(p) = \gamma_1(p)\kappa_1(p) + \gamma_2(p)\kappa_2(p). \tag{3.3.2}$$

Thus, formula (3.3.2) relates the anisotropic mean curvature and the isotropic principal curvatures. We remark that if

$$m := \inf\{\mu_1(y), \mu_2(y); y \in W_F\} \quad \text{and} \quad M := \sup\{\mu_1(y), \mu_2(y); y \in W_F\},$$

then we have that $\frac{1}{M} \leq \gamma_1(p), \gamma_2(p) \leq \frac{1}{m}$, for all $p \in \Sigma$.

Before we end this section, we need to do some remarks about the Stability operator, that can be written in terms of the anisotropic Weingarten operator A_F and the Weingarten operator of the Wulff shape A_W. Indeed, let us consider the second order differential operator Δ_F in Σ given by

$$\Delta_F u := \operatorname{div}_\Sigma(-A_W^{-1}\operatorname{grad}_\Sigma u),$$

where A_W is evaluated at the point $\nu(x)$ and $u \in C^2(\Sigma)$. When $F \equiv 1$, we have $A_W = -Id$ and therefore Δ_F is the usual Laplacian operator.

As a consequence of this definition, we have:

$$\begin{aligned} Lu &:= \operatorname{div}_\Sigma(-A_W^{-1}\operatorname{grad}_\Sigma u) + \operatorname{trace}_\Sigma(-A_W^{-1}A^2)u \\ &= \operatorname{div}_\Sigma(-A_W^{-1}\operatorname{grad}_\Sigma u) + \operatorname{trace}_\Sigma(A_F A)u \\ &= \Delta_F u + \operatorname{trace}_\Sigma(A_F A)u \end{aligned}$$

that is,

$$Lu := \Delta_F u + \operatorname{trace}_\Sigma(A_F A)u$$

Remark 3.3.3. Consider a local frame $\{X_1, \dots, X_n\}$ on Σ. In this frame we can write:

$$\Delta_F u = -\sum_{i,j=1}^{n} g^{ij}\{-A_W^{-1}\nabla_{X_i}X_j(u) - X_i(-A_W^{-1}X_j(u))\}.$$

Thus, we conclude that $\Delta_F u$ depends only on the first and second derivatives of u, but not on u. Moreover, it was proven in [71] that the operator Δ_F is elliptic in Σ. Using this information we conclude that the stability operator L is also elliptic, and since $\operatorname{trace}_\Sigma(A_F A)$ is a positive number, in view of the Maximum Principle (see Theorem A.1.3) if $Lu = 0$, then u does not have an interior non-positive local minimum, unless u is constant.

Proposition 3.3.4. (see [26]) If $\psi : \Sigma \to \mathbb{R}^3$ is an isometric immersion with constant anisotropic mean curvature and $N : \Sigma \to \mathbb{S}^2$ is its Gauss map then

$$\Delta_F N = -\operatorname{trace}_\Sigma(A_F A)N.$$

3.4 Some examples

This section is devoted to show some classes of examples of CAMC-surfaces that is already known. Examples are useful in many sense. They provide a way to test conjectures and help us to understand better the theory. They also permit us to compare similar results with different anisotropy functions or with different ambient spaces. In the following subsections we present rotation and helicoidal examples with constant anisotropic mean curvatures and the CAMC-cylinders, that play a vital role in the results of the next chapter.

3.4.1 CAMC revolution surfaces

In the study of constant mean curvature surfaces of $\mathbb{R}^3$, the most important examples are the Delaunay surfaces, i.e., the surfaces of revolution, found by C. Delaunay in [31]. Apart of the round spheres and round cylinders there are three more types of Delaunay surfaces: unduloids, nodoids and catenoids. An interesting property of a Delaunay surface Σ is that their profile curve can be generated by rolling a conic section C along a line without slipping and taking the trace of one of the loci of C (see Fig. 3.2). Namely, when C is a parabola, Σ is a catenoid; when C is an ellipse, Σ is an unduloid; and when C is a hyperbola, Σ is a nodoid.

Besides being nice examples, Delaunay surfaces appear in a fundamental way in the study of more general constant mean curvature surfaces. For example, in [67] it was shown that the ends of any properly immersed surface with nonzero constant mean curvature in $\mathbb{R}^3$ are asymptotic to a Delaunay surface. Also, Delaunay surfaces are useful to compare with other constant mean curvature surfaces through the Maximum Principle.

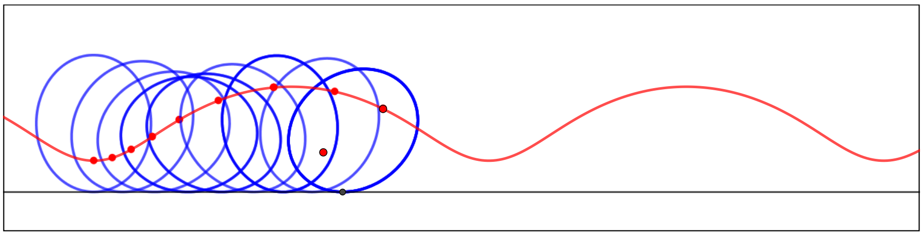

Figure 3.2: The unduloid (in red) as the geometric locus of a focus of a rolling ellipse (in blue).

Revolution surfaces with constant anisotropic mean curvature was first studied by Koiso and Palmer in [64]. We recall that in order to admit a rotationally symmetric surface we need to impose that the function $F : \mathbb{S}^2 \to \mathbb{R}$ is rotationally invariant, that is, for some $v \in \mathbb{S}^2$ (the axis of rotation) and some function $f : [-1, 1] \to \mathbb{R}$ we can write $F(w) = f(\langle v, w\rangle)$. This condition is equivalent to require that the Wulff shape be a rotation surface with axis parallel to v. The examples obtained in [64] are quite similar to those of the isotropic case. If the Wulff shape is a rotation surface whose axis is, for simplicity, parallel to $e_3 = (0, 0, 1)$ and whose profile curve is given by $s \mapsto (u(s), 0, v(s))$, where s is the arc length, then apart of the Wulff shape and the round cylinder with axis parallel to e_3, a Delaunay surface parametrized by $\psi(s,t) = (\alpha(s)\cos t, \alpha(s)\sin t, \beta(s))$ satisfies

I. $\alpha(u) = \frac{c}{2u}$ if it is the *anisotropic catenoid* (the anisotropic minimal surface), for some constant $c > 0$;

II. $\alpha(u) = \frac{u \pm \sqrt{u^2 + \Lambda c}}{-\Lambda}$ if it is the *anisotropic unduloid*, for some constants $\Lambda < 0$ and $c > 0$, where $\alpha(u)$ is defined for $|u| \geq \sqrt{-\Lambda c}$;

III. $\alpha(u) = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}$ if it is the *anisotropic nodoid*, for some constants $\Lambda < 0$ and $c < 0$, where $\alpha(u(s))$ is defined for $-\infty < s < \infty$.

In both items, $\beta(u) = \int^u \alpha'(u)(v \circ u^{-1})'(u)du$. Moreover, it was shown that anisotropic unduloids are embedded periodic surfaces, while the anisotropic nodoids are only immersed and periodic, as occurs in the isotropic case.

Actually, the construction of anisotropic Delaunay surfaces is valid in the case where F is defined only in an open part of $\mathbb{S}^2$ and the convexity condition does not hold everywhere. But in our case where these two conditions hold, an additional information about catenoids is valid: the profile curve is a graph over the whole of the vertical axis.

In [66] authors showed that the profile of anisotropic Delaunay surfaces can be obtained as the trace of a point held in a fixed position relative to a curve that is rolled without slipping along a line, generalizing the same construction of Delaunay in [31]. In the same paper a more general construction of anisotropic Delaunay surfaces is given: instead of the Wulff shape is assumed to be a surface of revolution, they only assume that it has the property that all of its intersections with horizontal planes are mutually homothetic.

3.4.2 CAMC helicoidal surfaces

Another important class of surfaces are constituted by the helicoidal surfaces, that is, those surfaces which are invariant under a helicoidal motion. It is known that helicoidal surfaces with constant mean curvature arise as isometric deformations of Delaunay

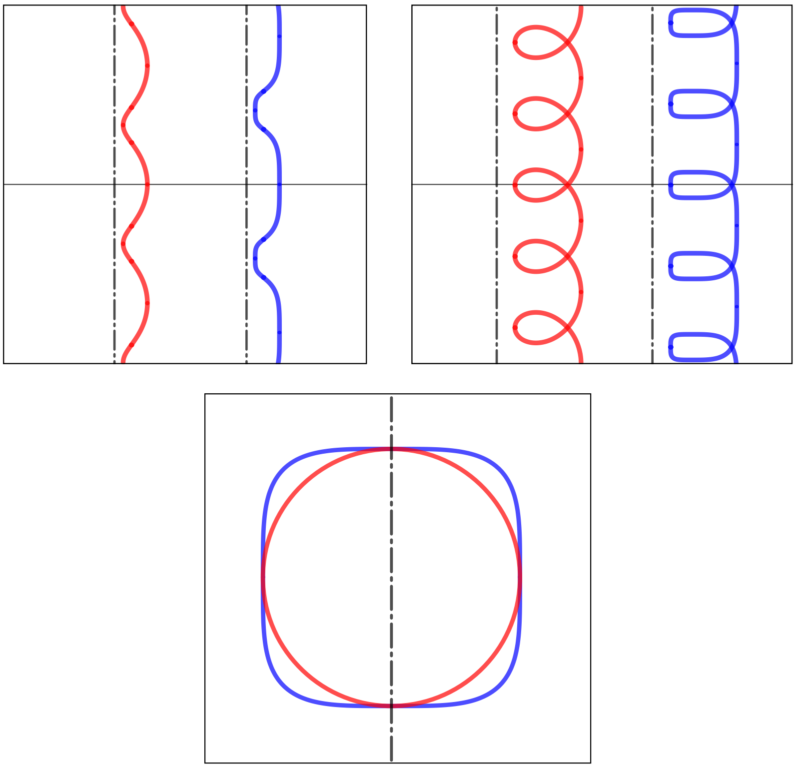

Figure 3.3: Comparison between profile curves of unduloids, nodoids and the Wulff shapes. The isotropic examples are coloured in red while their anisotropic counterpart are coloured in blue. The Wulff shape here is given by $\{(x, y, z) \in \mathbb{R}^3; (x^2 + y^2)^2 + z^4 = 1\}$.

surfaces (see [17]). The best known example is the isometric deformation between the catenoid and the helicoid.

By the nature of helicoidal motions, we need to impose again that the Wulff shape is a rotation surface, and for simplicity, its axis is parallel to $e_3 = (0, 0, 1)$.

Any helicoidal surface can be parametrized by an application $\psi : I \times \mathbb{R} \to \mathbb{R}^3$ whose expression is:

$$\psi(r, \theta) = (r \cos \theta, r \sin \theta, g(r) + \lambda\theta), \tag{3.4.1}$$

where I is an interval of $\mathbb{R}$ and λ is a fixed real constant, the **pitch** of the surface. When g is constant, for example, we have the classical helicoids.

As observed in [68], the classical helicoid has zero anisotropic mean curvature for every rotationally symmetric anisotropic area functional. More generally, authors in [68] derived a expression for the first derivative of g to obtain a helicoidal surface with constant anisotropic curvature Λ through the representation in equation (3.4.1). Namely, $g(r)$ can be obtained by integrating the following equation:

$$[w(r)f(w(r)) - w(r)^2 f'(w(r))]rg'(r) = \frac{\Lambda r^2}{2} + C,$$

where $F(x, y, z) = f(z)$, $w(r) = \frac{r}{\sqrt{r^2+r^2g'(r)^2+\lambda^2}}$ and C is a real constant. When $\lambda = 0$, we can recover the representation of anisotropic Delaunay surfaces that we showed in section 3.4.1.

Helicoidal surfaces can be written in another useful form, called twizzler representation: the parametrization $\psi : I \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\psi(s, t) = (\alpha(s) \cos(\omega t) - \beta(s) \sin(\omega t), \beta(s) \cos(\omega t) + \alpha(s) \sin(\omega t), t + C), \tag{3.4.2}$$

where C and ω are real constants and $s \mapsto (\alpha(s), \beta(s), 0)$ is a curve parametrized by arc length called **generating curve** of ψ. Geometrically the surface is obtained as the orbit of the generating curve under the helical action

$$t \cdot (x, y, z) = ((x\cos(\omega t) - y\sin(\omega t), x\cos(\omega t) + y\sin(\omega t)), z + t).$$

We define the curve

$$\eta_1(s) := -(\alpha'(s)\alpha(2) + \beta'(s)\beta(s)) \quad \text{and} \quad \eta_2(s) := \alpha(s)\beta'(s) - \beta(s)\alpha'(s),$$

called **treadmill sled** of ψ. Geometrically this curve is obtained as the trace of the origin when the curve (α, β) rolls on a treadmill located at the origin which is aligned along the x-axis. For example, the treadmill sled of a circle with center at the origin is just a point, while the treadmill sled of a circle of radius R whose center is at a distance r of the origin, is a circle of radius r with center $(0, R)$.

As showed in Theorem IV.I of [68], a necessary and sufficient condition for a helicoidal surface represented in equation (3.4.2) have constant anisotropic mean curvature Λ is

$$\Lambda(\eta_1^2 + \eta_2^2) + \frac{2\eta_2}{w\left(\frac{\omega\eta_1}{\sqrt{1+\omega^2\eta_1^2}}\right)\sqrt{1+\omega^2\eta_1^2}} + A = 0,$$

where A is a real constant and $w(z) = \frac{1}{f(z) - zf'(z)}$. We can recover the curve $s \mapsto (\alpha(s), \beta(s))$ through the expression (see Prop. IV.I in [68]):

$$(\alpha(s), \beta(s)) = (\eta_2(s)\sin(\omega_0) - \eta_1(s)\cos(\omega_0), -\eta_1(s)\sin(\omega_0) - \eta_2(s)\cos(\omega_0)),$$

where $\omega_0 = \int \frac{1}{\eta_1^2} d\eta_2$.

3.4.3 CAMC cylinders

One of the simplest kind of surfaces are cylinders. In the isotropic case, cylinders with constant mean curvature are included in the family of rotation examples. However we notice that cylinders of constant anisotropic mean curvature is known for any choice of anisotropy function. Indeed, in [46] authors classify the so called anisotropic isoparametric hypersurfaces of the Euclidean space, i.e., hypersurfaces whose anisotropic Weingarten operator is constant. Apart of hyperplanes and homotheties of the Wulff shape, the unique example of anisotropic isoparametric hypersurface in the Euclidean space are the CAMC-cylinders ("Wulff shape cylinder" in the nomenclature of [46]). In this thesis, cylinders with constant anisotropic mean curvature will play a vital role in Theorem 4.4.1. Thus, for completeness present the detailed construction of such cylinders in $\mathbb{R}^3$ in the following proposition.

Proposition 3.4.1. (see [46]) Let $F : \mathbb{S}^2 \to \mathbb{R}$, be a smooth, positive function and let W_F be its Wulff shape. For each $v \in \mathbb{S}^2$, there exists a curve Γ such that the cylinder $\mathcal{C}_v$ parametrized by $\psi(s, t) = \Gamma(s) + tv$ (where s is a parameter for Γ) is a CAMC surface with $\Lambda = -2$. Moreover, up to translations, such a cylinder is unique and if $\pi_v : \mathbb{R}^3 \to \{v\}^\perp$ denotes the orthogonal projection onto the plane $\{v\}^\perp$ and $\Omega_v = \mathrm{int}(\pi_v(W_F))$, then $\pi_v(\Gamma)$ lies inside Ω_v. In particular, up to translations $\mathcal{C}_v$ intersects W_F transversally.

Proof. Let us consider $G : \mathbb{S}^2 \to \mathbb{R}^3$ given by $G(x, y, z) = \mathrm{grad}_{\mathbb{S}^2} F_{(x,y,z)} + F(x, y, z)(x, y, z)$, which is a parametrization of W_F. Given $v \in \mathbb{S}^2$, we consider the great circle $\tilde{\Gamma} = \mathbb{S}^2 \cap \{v\}^\perp$

and we define $\Gamma := G(\tilde{\Gamma})$. Then we define $\psi(s,t) = \Gamma(s) + tv$, whose image is a cylinder that we call $\mathcal{C}_v$.

Let s be an arc length parameter to $\tilde{\Gamma}$, which also works as a parameter for Γ (since G is a diffeomorphism) and denote by N_W and A_W the Gauss map and its Weingarten operator with respect to W_F, respectively. Notice that A_W is negative-definite, since W_F is convex and N_W points outward W_F. Also, we recall that $N_W = G^{-1}$. We claim that $\mathcal{C}_v$ is well defined, that is, $\{\Gamma'(s), v\}$ is linearly independent for all s. Indeed, if $\Gamma'(s_0)$ were parallel to v for some s_0, we would have

$$\begin{aligned}
&\langle N_W(\Gamma(s_0)), v\rangle = \langle \tilde{\Gamma}(s_0), v\rangle = 0\\
\Rightarrow &\langle dN_W(\Gamma'(s_0)), v\rangle = 0\\
\Rightarrow &\langle dN_W(\Gamma'(s_0)), \Gamma'(s_0)\rangle = 0.
\end{aligned}$$

But $\langle dN_W(\Gamma'(s_0)), \Gamma'(s_0)\rangle = -\langle A_W\Gamma'(s_0), \Gamma'(s_0)\rangle > 0$. We have a contradiction.

Now, notice that the outward pointing unit of $\mathcal{C}_v$ at $\psi(s,t)$ coincide with $\tilde{\Gamma}(s)$, the outward unit normal of $\tilde{\Gamma}$ in $\{v\}^{\perp}$. Indeed, we have that

$$\begin{aligned}
\langle \tilde{\Gamma}(s), \Gamma'(s)\rangle &= \frac{d}{ds}\langle \tilde{\Gamma}(s), \Gamma(s)\rangle - \langle \tilde{\Gamma}'(s), \Gamma(s)\rangle\\
&= \frac{1}{2}\frac{d}{ds}\langle \tilde{\Gamma}(s), (\operatorname{grad}_{\mathbb{S}^2} F)_{\tilde{\Gamma}(s)} + F(\tilde{\Gamma}(s))\tilde{\Gamma}(s)\rangle\\
&- \frac{1}{2}\langle \tilde{\Gamma}'(s), (\operatorname{grad}_{\mathbb{S}^2} F)_{\tilde{\Gamma}(s)} + F(\tilde{\Gamma}(s))\tilde{\Gamma}(s)\rangle\\
&= \frac{1}{2}\frac{d}{ds}F(\tilde{\Gamma}(s)) - \langle \tilde{\Gamma}'(s), (\operatorname{grad}_{\mathbb{S}^2} F)_{\tilde{\Gamma}(s)}\rangle = 0.
\end{aligned}$$

Thus, the anisotropic Gauss map of $\mathcal{C}_v$ evaluated at $\psi(s,t)$ is given by $\nu(s,t) = G(\tilde{n}(s))$, whence we conclude that

$$\begin{aligned}
\partial_s\psi(s,t) &= \Gamma'(s)\\
\partial_t\psi(s,t) &= v\\
\partial_s\nu(s,t) &= (G\circ\tilde{n})'(s) = \Gamma'(s) = \partial_s\psi(s,t)\\
\partial_t\nu(s,t) &= 0.
\end{aligned}$$

Therefore $\partial_s\psi(s,t)$ and $\partial_t\psi(s,t)$ are anisotropic principal directions with anisotropic principal curvatures -1 and 0, respectively, whence we conclude that $\Lambda = \operatorname{trace} -d\nu_{(s,t)} = -1$. Also by the fact that the outward unit normal at $\psi(s,t)$ is $\tilde{\Gamma}(s)$, we conclude that $\pi_v(\Gamma)$ coincides with $\partial\Omega_v$. In particular $\pi_v(\Gamma)$ is a planar convex curve.

Finally, we replace Γ by its image under the half scale homothety about the origin. The resulting cylinder has constant anisotropic mean curvature -2 with respect to to its outward pointing Gauss map and up to translations, it intersects the Wulff shape transversally. □

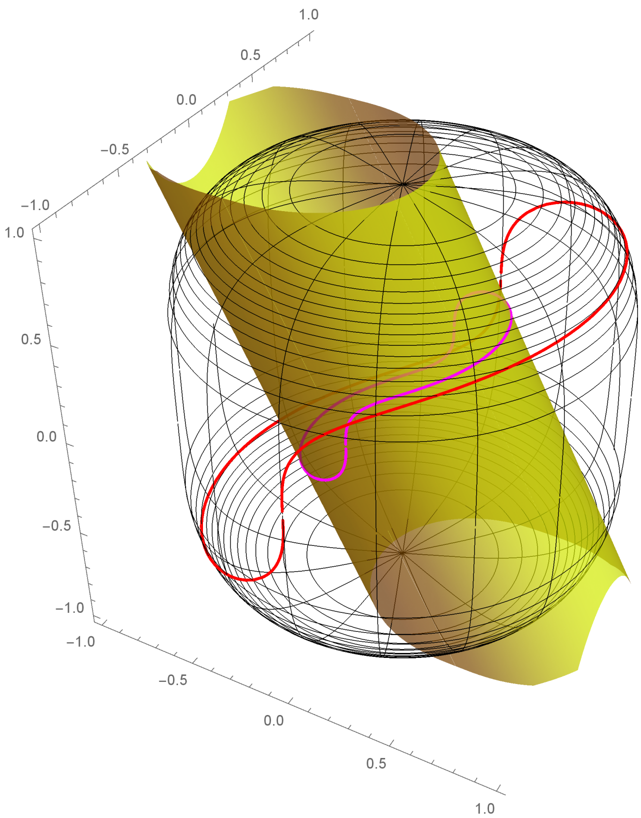

Figure 3.4: Example of a CAMC anisotropic cylinder for $W_F = \{(x, y, z) \in \mathbb{R}^3; (x^2 + y^2)^2 + y^4 = 1\}$.

Chapter 4

Main results

In this chapter, we present the main results of this thesis. The first main result is a Bernstein-type theorem for CAMC multigraphs, which states that the only CAMC complete multigraphs are the planes. For its proof, we use some ideas found in [51]. Although this theorem is important by itself, it also has nice applications: for example, one interesting consequence is our second main result, which states that complete CAMC surfaces whose Gaussian curvature does not change sign are either CAMC cylinders or homotheties of the Wulff shape, generalizing the isotropic version proved by T. Klotz and R. Osserman in [63].

Another consequence of the Bernstein-type Theorem for CAMC multigraphs is the third main result of this chapter: uniform height estimates for CAMC graphs over closed domains with boundary contained in a plane. These estimates depend only on the anisotropic mean curvature. As a corollary, we also obtain height estimates for compact CAMC surfaces with planar boundary, with the additional hypothesis that the Wulff shape is symmetric with respect to the plane which contains the boundary of the surface. Using these estimates, we are able to prove, under certain hypothesis, that any properly embedded CAMC surface with finite topology and at most one end is in fact the Wulff shape, up to a homothety.

Our main difficulty in generalize such theorems for CAMC surfaces relies on the fact that the anisotropic mean curvature is not necessarily invariant under reflections of the ambient space, since F can assume different values in a point and its reflected image. Because of this difficulty, part of this chapter is devoted to prove some auxiliary results: a Compactness Theorem is useful to obtain complete Λ-CAMC surfaces as a limit of compact pieces of a sequence of surfaces whose anisotropic mean curvature converges to Λ; another useful result are uniform estimates for the diameter of horizontal slices of Λ-CAMC graphs over closed domains, whose boundary is contained in a plane; we also obtained a priori second fundamental form estimates for surfaces with bounded anisotropic mean curvature and whose Gauss maps omit a disk. With these three Theorems we are able to prove the Height Estimates for CAMC graphs.

In this chapter, all surfaces will have anisotropic mean curvature computed with respect to a positive, smooth function $F : \mathbb{S}^2 \to \mathbb{R}$ which satisfies the convexity condition 3.2.2.

4.1 Compactness theorem

We begin recalling a well-known result in surface theory that can be found in [88].

Theorem 4.1.1. (see [88], Prop. 2.3) Let Σ be an immersed surface in $\mathbb{R}^3$ whose second fundamental form A satisfies $|A| < \frac{1}{4\delta}$ for some constant $\delta > 0$. Then for any $x \in \Sigma$ with $d_\Sigma(x, \partial\Sigma) > 4\delta$ there is a neighborhood of x in Σ which is a graph of a function u over the Euclidean disk of radius $\sqrt{2}\delta$ centered at x in the tangent plane of Σ at x. Moreover, u satisfies

$$|u| < 2\delta, \quad |\operatorname{grad}_{\mathbb{R}^2} u| < 1 \quad \text{and} \quad |\operatorname{Hess}_{\mathbb{R}^2} u| < \tfrac{1}{\delta}.$$

Our first auxiliary result is the following Compactness Theorem:

Theorem 4.1.2. Let $\{\Sigma_n\}_n$ be a sequence of CAMC surfaces (possibly with boundary) in $\mathbb{R}^3$. For each $n \in \mathbb{N}$, take $p_n \in \Sigma_n$ and assume that the following conditions hold:

(i) There exists a sequence of positive numbers $\{r_n\}_n$ with $r_n \to \infty$ such that the geodesic disks $D_{\Sigma_n}(p_n, r_n)$ centered at p_n of radius r_n are well defined and contained in the interior of Σ_n, i.e., $d_{\Sigma_n}(p_n, \partial\Sigma_n) \geq r_n$;

(ii) $p_n \to p$ for a certain $p \in \mathbb{R}^3$;

(iii) If $|\sigma_n|$ denotes the length of the anisotropic second fundamental form of Σ_n, then there exists $C > 0$ such that $|\sigma_n(x)| \leq C$ for every $n \in \mathbb{N}$ and every $x \in \Sigma_n$;

(iv) $\Lambda_n \to \Lambda \in \mathbb{R}$, where Λ_n denotes the (constant) anisotropic mean curvature of Σ_n.

Then for any $k \geq 2$ there exists a subsequence of $\{\Sigma_n\}_n$ that converges uniformly on compact sets in the C^k-topology to a complete (possibly non-connected) immersed surface Σ without boundary, that passes through p, with bounded curvature and constant anisotropic mean curvature Λ.

Remark 4.1.3. As a consequence of item (c) in Proposition 3.3.2, the condition (iii) in Theorem 4.1.2 can be replaced by the existence of a constant $C' > 0$ such that the norm of the isotropic second fundamental form A_n of Σ_n satisfies $\|A_n\| \leq C'$, for every $n \in \mathbb{N}$ and every $x \in \Sigma_n$.

Proof of Theorem 4.1.2. The proof is based on the ideas found in [15].

1^{st} *Part:* In view of items (i) and (iii), the conditions of Theorem 4.1.1 is verified. Thus there exist constants $\delta, M > 0$ that only depend on C (and not on Λ_n or Σ_n) such that for n sufficiently large:

a. an open neighborhood of $p_n \in \Sigma$ is the graph of a function u_n defined over the Euclidean disk D_δ centered at the origin with radius δ in $T_{p_n}\Sigma_n$.

b. the C^2-norm of u_n in D_δ is not greater than M.

Since each Σ_n has anisotropic mean curvature Λ_n, choosing adequate coordinates (x_n, y_n, z_n) in $\mathbb{R}^3$ so that $T_{p_n}\Sigma_n$ coincides with the plane $\{z_n = 0\}$, each function u_n is a solution in D_δ of the anisotropic mean curvature equation:

$$\Lambda_n = A_0(p,q)\tilde{F}(N_n) + \sum_{i=1}^{4} A_i(p,q)\tilde{F}_{x_i}(N_n) + \sum_{1\leq i\leq j\leq 4} A_{i,j}(p,q)\tilde{F}_{x_i x_j}(N_n) \qquad (4.1.1)$$

where A_0, A_i and $A_{i,j}$ are as in equation (A.3.1). We notice that such equations are quasilinear elliptic PDE's whose coefficients of second order depends smoothly on the derivatives of u_n.

We now use a trick that allow us to use the Schauder estimates for linear second-order PDE's in order to obtain a uniform boundedness of $\{u_n\}_n$ in the $C^{2,\alpha}$-norms over compact domains. For each $n \in \mathbb{N}$, consider the following linear second-order PDE:

$$L_n[u] := \tilde{a}_n(x_n, y_n)\frac{\partial^2 u}{\partial x_n^2} + \tilde{b}_n(x_n, y_n)\frac{\partial^2 u}{\partial x_n y_n} + \tilde{c}_n(x_n, y_n)\frac{\partial^2 u}{\partial y_n^2} = \Lambda_n, \tag{4.1.2}$$

where the coefficients $\tilde{a}_n$, $\tilde{b}_n$ and $\tilde{c}_n$ are the same as the second order coefficients of 4.1.1, but evaluated on the solution $u_n(x_n, y_n)$ of 4.1.1 and its first-order derivatives. By construction, $L_n[u_n] = \Lambda_n$ and $\tilde{a}_n$, $\tilde{b}_n$ and $\tilde{c}_n$ depends smoothly on the derivatives of u_n.

Notice that condition (b) above automatically implies that $\{u_n\}_n$ is uniformly bounded in the $C^{1,\alpha}(D_\delta)$-norm, for all $0 < \alpha < 1$. Thus, the coefficients of the equations (4.1.2) are all uniformly bounded in the $C^{0,\alpha}(D_\delta)$-norm. Then, using the Schauder estimates (see [47], Corollary 6.3, or Theorem A.4.3), for any $0 < \delta' < \delta$ we conclude that there exists a constant C' (again independent of n) such that $\|u_n\| \le C'$ in the $C^{2,\alpha}(D_{\delta'})$-norm. In particular, all coefficients of $L[u_n] = \Lambda_n$ are uniformly bounded in the $C^{1,\alpha}(D_{\delta'})$-norm. Using a bootstrap argument, we eventually obtain

$$\|u_n\|_{C^{k,\alpha}(D_{\delta'})} \le C'', \quad 0 < \alpha < 1,$$

for some constant C'' independent of n.

Thus we may apply the Arzela-Ascoli Theorem. There is a subsequence $\{u_n\}_{n\in N_1}$ of $\{u_n\}_{n\in\mathbb{N}}$ that converges on $D_{\delta'}$ in the C^k-topology to a function $u \in C^k(D_{\delta'})$, which by item (iv) is a solution of

$$\Lambda = A_0(p,q)\tilde{F}(N) + \sum_{i=1}^{4} A_i(p,q)\tilde{F}_{x_i}(N) + \sum_{1\le i\le j\le 4} A_{ij}(p,q)\tilde{F}_{x_i x_j}(N),$$

that is, the graph Σ of u is a Λ-CAMC surface in $\mathbb{R}^3$ with Gauss map N, that passes through p and has second fundamental form bounded by C, by items (ii) and (iii).

2^{nd} *Part:* Consider some $y \in D_{\delta'}$, and let $q \in \Sigma$ be the corresponding point in the graph of u. For each $n \in N_1$ there exist points $q_n \in \mathbb{R}^3$ in the graphs of u_n, all corresponding to y. It is clear that $\{q_n\}_{n\in N_1}$ converges to q. This means that $\{q_n\}_{n\in N_1}$ and q can play the same role as $\{p_n\}_{n\in\mathbb{N}}$ and p did in the first part of the proof. Thus, passing to a subsequence if necessary so that condition (i) in the statement is fulfilled, we can repeat the same process above to obtain a Λ-CAMC surface in $\mathbb{R}^3$ that extends Σ and is well defined around q as a vertical graph over the disk of radius δ' centered at the origin of $T_q\Sigma$.

Applying the argument of the last paragraph with a suitable choice of the points $y \in D_{\delta'}$, it is possible, after a finite number of steps, to extend Σ in a geodesic disk centered at p with radius $\delta_1 := (2 - \frac{1}{2})\delta'$. More generally, replacing $D_{\delta'}$ by D_{δ_1} we can extend Σ in a geodesic disk of radius $\delta_2 = (3 - \frac{1}{2} - \frac{1}{4})\delta'$, and proceeding inductively, for any radius $\delta_l = (l + 1 - \sum_{j=1}^{l} \frac{1}{2^j})\delta'$, with $l \in \mathbb{N}$. By a standard diagonal process Σ can be extended to a complete Λ-CAMC surface (which will also be denoted by Σ), that passes through p, and whose second fundamental form is bounded by C. Moreover, Σ is by construction a limit in the C^k topology on compact sets of the diagonal subsequence of $\{\Sigma_n\}_{n\in\mathbb{N}}$ mentioned above. Finally, it is important to observe that other connected components could also appear in this process. This completes the proof. □

4.2 Second fundamental form and horizontal diameter estimates

The next lemma is based in an analogous result by W. Meeks, in [78]. It gives estimates for the connected components of horizontal slices of a CAMC graph with planar boundary, and it will be useful later. Put in other words, the geometrical meaning of this lemma is that, from a certain height of the planar boundary, the connected components of the graph are contained in a cylinder of uniform radius.

Lemma 4.2.1. Let $\Sigma \subset \mathbb{R}^3$ be a graph $z = u(x, y)$ over a closed (not necessarily bounded) domain $\Omega \subset \mathbb{R}^2$, with zero boundary values. Assume that Σ has constant anisotropic mean curvature $\Lambda \neq 0$. Let d be the diameter of W_F (the supremum of the Euclidean distance between two of its points) and put $R = d\sqrt{3}/|\Lambda|$. Then, for every $t > 2R$, the diameter of each connected component of $\Sigma \cap \{|z| = t\}$ is at most $2R$. In particular, all connected components of $\Sigma \cap \{|z| \geq t\}$ for $t > 2R$ are compact.

Proof. The proof is based on the ideas of Theorem 4 in [5] (see also Theorem 6.2 in [43]), which in turn is a simplification of the original proof due by W. Meeks in [78], Lemma 2.4. Here, for simplicity, we assume $\Lambda = 2$. The general case follows by taking a homothety. Let $P(t)$ be the foliation of $\mathbb{R}^3$ by horizontal planes, being $P(t)$ the plane at height t. Consider W the image of the Wulff shape W_F by the antipodal map. In other words, W has anisotropic mean curvature equals to 2 with respect to the inward pointing unit normal vector. It is possible to show that W is contained in some sphere of radius R. To fix ideas, let $\tilde{o}$ denote the center of such a sphere.

Since Σ is a graph, its Gauss map N for which the anisotropic mean curvature is evaluated satisfies either $\langle N, e_3\rangle < 0$ or $\langle N, e_3\rangle > 0$ on Σ. Without loss of generality, let us suppose that the first case occurs. Then, by the maximum principle we can also suppose that $\Sigma \subset \{z \geq 0\}$, that is, u is non-negative. Indeed, if there exists an interior point $q \in \Omega$ where $u(q) < 0$, considering $\hat{o} \in W$ as the highest point of W relative to e_3, since $\overline{B_{\mathbb{R}^2}(q, R)} \cap \overline{\Omega}$ is compact, for very negative values of s, the image $W(s)$ of W by the translation that takes $\hat{o}$ to $q + se_3$ does not intersect Σ. Thus, increasing s we eventually obtain an interior contact point between $W(s)$ and Σ. In this scenario, the normals of $W(s)$ and Σ coincide at the contact point, and locally, $W(s)$ lies above Σ. This contradicts the Maximum Principle.

Let us suppose that the theorem is not true. Then for some $t > 2R$ there exists at least one such connected component $C(t)$ of $\Sigma(t) := P(t) \cap \Sigma$ whose diameter is greater than $2R$. Without loss of generality we can also assume that $P(t)$ intersects Σ transversally. Thus, the connected components of $\Sigma(t)$ are disjoint regular simple curves which do not touch $\partial\Omega \times \{t\}$. Moreover, by the Jordan-Brower Theorem, these curves are the boundary of connected regions in $P(t)$ whose points lie below the graph of u (see Fig. 4.1).

Denote by $\Omega(t) \subset P(t)$ the region of $P(t)$ bounded by $C(t)$. We can find two points $p, q \in \Omega(t)$ such that $d_{\mathbb{R}^3}(p, q) > 2R$. Let $\beta : [a, b] \to \Omega(t)$ be a simple curve contained in $\Omega(t)$ joining p to q and such that the Euclidean distance from p to q is maximal, among the points of β. Let $\mathcal{R}$ the "rectangle" given by

$$\mathcal{R} = \{\alpha_s(r); s \in [a, b], r \in [0, t]\}$$

where α_s is the geodesic with initial data $\alpha_s(0) = \beta(s)$ and $(\alpha_s)'(r) = -e_3$, being r an arc length parameter for α_s.

Since Σ is a graph and β lies in $\Omega(t)$, we conclude that $\mathcal{R} \subset Q$, where Q is the connected region of $\mathbb{R}^3$ between Σ and the plane $P(0) = \{z = 0\}$. Moreover, if $\hat{\eta}$ is a

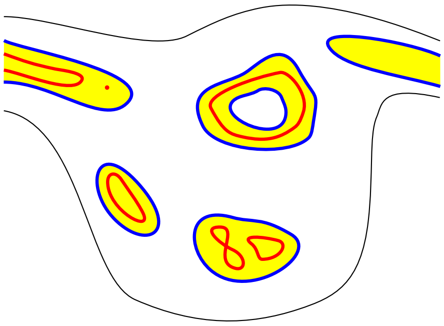

Figure 4.1: In yellow, $u^{-1}([t,+\infty))$; in blue, $u^{-1}(\{t\})$; and in red, $u^{-1}(\{s\})$, for $s>t$.

horizontal unit vector orthogonal to $p-q$, by construction of β, $\mathcal{R}$ divides the solid region

$$S=\{x+l\hat{\eta};x\in\mathcal{R},l\in\mathbb{R}\}$$

in two connected components, C_1 and C_2, with $\partial C_1\cap\partial C_2=\mathcal{R}$.

Let $\tilde{p}\in\mathcal{R}$ be a point whose distance to $\partial\mathcal{R}$ is greater than R. Such a point exists, since the distance between $P(0)$ and $P(t)$ and the distance between α_a and α_b are both greater than $2R$. Define η as the horizontal line passing through $\tilde{p}$, parallel to $\hat{\eta}$. Notice that the distance of any point in η to ∂S is greater than R.

Let r be a parameter for η and consider, for each $r\in\mathbb{R}$, the translation of the Wulff shape from $\tilde{o}$ to $\eta(r)$, which we will denote by $W(r)$. Note that $W(r)$ is contained in the interior of S, for any $r\in\mathbb{R}$. Also, for r sufficiently large, $W(r)$ is completely contained in C_1, since $\mathcal{R}$ is compact. We then decreases r and consider only the piece $\tilde{W}(r):=W(r)\cap C_2$ which has gone through $\mathcal{R}$. As we decrease r more and more, we obtain one of the following two cases: either $\tilde{W}(r)$ could go through $\mathcal{R}$ completely, and end up being entirely contained in Q (Fig. 4.2 (a)(b)(c)) or we obtain a first contact point between $\tilde{W}(r)$ and ∂Q (Fig. 4.2 (d)(e)). In the first case we can move $\tilde{W}(r)$ upwards until it reaches a first contact point with Σ. However, in both cases the contact point between $\tilde{W}(r)$ and Σ occurs in the inner side of Σ, since $\tilde{W}(r)\subset Q$. Moreover, the normal vectors of $\tilde{W}(r)$ and Σ coincide at the contact point and $W(r)$ is locally above Σ, whence we conclude that both surfaces agree, by the Maximum Principle. This finishes the proof.

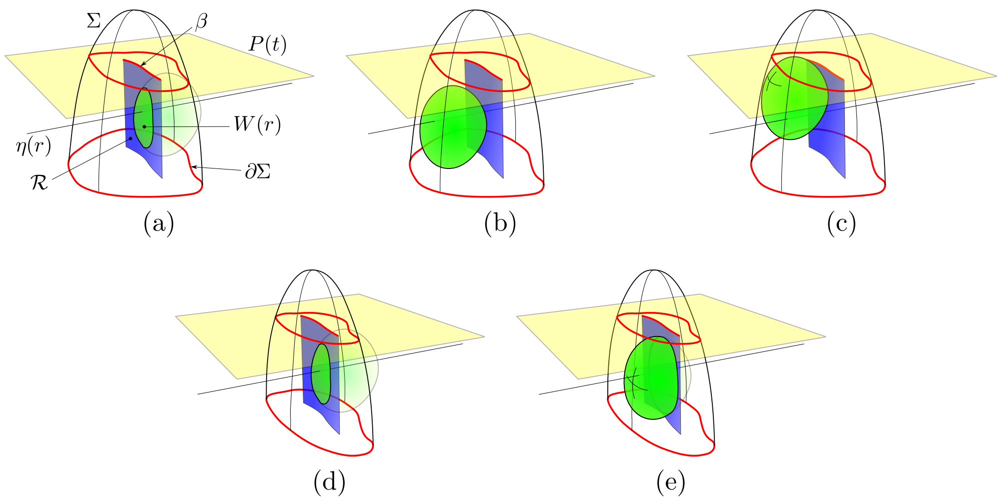

Figure 4.2: Two possible scenarios to obtain an inner contact point.

□

We recall a very important theorem due by Jenkins: a Bernstein's Theorem for anisotropic minimal surfaces.

Theorem 4.2.2. (see [62], Cor. pg. 198) Let Σ be an anisotropic minimal surface. Assume that Σ is oriented and complete. If its Gauss map omits a disk of $\mathbb{S}^2$, then Σ is a plane.

Now we are in conditions to present a Second Fundamental Form Estimates in terms of the anisotropic mean curvature and the Gauss map.

Theorem 4.2.3. Let M, d, ρ be positive constants. Then there exists $C = C(M, d, \rho) > 0$ such that the following assertion is true:

Let Σ be any complete, immersed oriented surface in $\mathbb{R}^3$ with constant anisotropic mean curvature Λ, possibly with non-empty boundary. Denote by N and σ its Gauss map and its second fundamental form, respectively. Assume that

(i) $|\Lambda| \leq M$;

(ii) $N(\Sigma)$ omits a spherical disk of radius ρ.

Then for any $p \in \Sigma_d := \{x \in \Sigma; d_\Sigma(x, \partial\Sigma) \geq d\}$, we have

$$|\sigma|(p) \leq C.$$

Here d_Σ and $|\sigma|$ denotes the intrinsic distance on Σ and the norm of the anisotropic second fundamental form of Σ, respectively.

Proof. The idea of this proof is based on some of the arguments of Theorem 4.2, in [15]. Arguing by contradiction we are able to exhibit an anisotropic minimal graph that is complete, whose second fundamental form is not identically zero. It contradicts Theorem 4.2.2.

Suppose that for each $n \in \mathbb{N}$ there is an isometric immersion $f_n : \Sigma_n \to \mathbb{R}^3$ with constant anisotropic mean curvature Λ_n and Gauss map $N_n : \Sigma_n \to \mathbb{S}^2$ verifying

(a) $|\Lambda_n| \leq M$;

(b) $N_n(\Sigma_n)$ omits a spherical disk of radius ρ;

and a point $p_n \in \Sigma_n$ such that $d_{\Sigma_n}(p_n, \partial\Sigma_n) \geq d$ but $|\sigma_n(p_n)| > n$. Here d_{Σ_n} and σ_n denote the intrinsic distance on Σ_n and the second fundamental form of f_n, respectively. Passing to a subsequence, if necessary, we can suppose that for all $n \in \mathbb{N}$, $N_n(\Sigma_n)$ omits the same open spherical disk of radius $\rho/2$. For convenience, we can change the Euclidean coordinates and assume that such disk is centered in the north pole of $\mathbb{S}^2$.

Consider the compact intrinsic metric disk $D_n = D_{\Sigma_n}(p_n, d/2)$ in Σ_n, whose distance from $\partial\Sigma_n$ is at least $d/2$. Let q_n be the maximum on D_n of the function

$$h_n(x) = |\sigma_n(x)| d_{\Sigma_n}(x, \partial D_n), \quad x \in D_n.$$

Clearly q_n lies in the interior of D_n, as h_n vanishes in ∂D_n. Putting $\lambda_n = |\sigma_n(q_n)|$ and $r_n = d_{\Sigma_n}(q_n, \partial D_n)$, by hypothesis we have

$$\begin{aligned} \lambda_n r_n &= |\sigma_n(q_n)| d_{\Sigma_n}(q_n, \partial D_n) = h(q_n) \geq h(p_n) \\ &= |\sigma_n(p_n)| d_{\Sigma_n}(p_n, \partial D_n) = \frac{d}{2}|\sigma_n(p_n)| > \frac{dn}{2}. \end{aligned} \tag{4.2.1}$$

Since $r_n = d_{\Sigma_n}(q_n, \partial D_n) \leq \frac{d}{2}$, we conclude that $\lambda_n \to \infty$ as $n \to \infty$. Also, notice that for every $z_n \in B_{\Sigma_n}(q_n, r_n/2)$ we have (see Fig. 4.3)

$$d_{\Sigma_n}(z_n, \partial D_n) > \frac{r_n}{2} = \frac{1}{2} d_{\Sigma_n}(q_n, \partial D_n) \tag{4.2.2}$$

Consider now the immersed oriented surfaces $g_n : D_{\Sigma_n}(q_n, r_n/2) \to \mathbb{R}^3$ obtained by

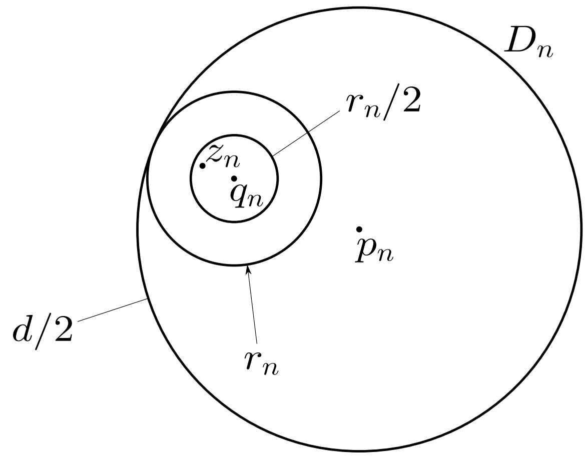

Figure 4.3: Auxiliary illustration exhibiting constructions done in the proof.

applying a rescaling of factor λ_n to the restriction of f_n to $D_{\Sigma_n}(q_n, r_n/2)$, that is, $g_n = \lambda_n f_n$ restricted to $D_{\Sigma_n}(q_n, r_n/2)$. For short, we will sometimes write M_n to denote this immersed surface given by g_n. If we denote the second fundamental form of g_n by $\hat{\sigma}_n$, using equation (4.2.2), for any point $z_n \in D_{\Sigma_n}(q_n, r_n/2)$ we obtain

$$\begin{aligned} |\hat{\sigma}_n(z_n)| = \frac{|\sigma_n(z_n)|}{\lambda_n} &= \frac{h_n(z_n)}{\lambda_n d_{\Sigma_n}(z_n, \partial D_n)} \leq \frac{h_n(q_n)}{\lambda_n d_{\Sigma_n}(z_n, \partial D_n)} \\ &= \frac{d_{\Sigma_n}(q_n, \partial D_n)}{d_{\Sigma_n}(z_n, \partial D_n)} \leq 2. \end{aligned} \tag{4.2.3}$$

In particular the sequence $\{M_n\}_n$ has uniformly bounded second fundamental form. Also, notice that $|\hat{\sigma}(q_n)| = 1$, by construction. We recall that the **radius** of a compact Riemannian surface with boundary is the maximum distance of points in the surface to its boundary. In our case, the radius of M_n is at least $\lambda_n r_n/2$ (if we compare the graph M_n to its domain, in the Euclidean metric), which by equation (4.2.1) diverges to infinity as $n \to \infty$.

Let now $\tilde{M}_n$ denote the translation of M_n that takes the point $g_n(q_n)$ to the origin of $\mathbb{R}^3$, and let $\xi_n \in \mathbb{S}^2$ denote the Gauss map image of M_n at q_n. After passing to a subsequence, we may assume that $\xi_n \to \xi$ as $n \to \infty$, for some $\xi \in \mathbb{S}^2$. Thus $\{\tilde{M}_n\}_n$ is a sequence of surfaces with the norm of the second fundamental forms uniformly bounded by 2, and equal to 1 at the origin.

Using Theorem 4.1.1, we conclude that there exist positive constants δ_0, μ (independent of n) such that for n large we can view a neighborhood of the origin in $\tilde{M}_n$ as a graph of a function u_n over the disk D_n^0 of radius δ_0 of its tangent plane $T_0\tilde{M}_n = \{\xi_n\}^\perp$, and such that $\|u_n\|_{C^2(D_n^0)} \leq \mu$. Since $\xi_n \to \xi$ in $\mathbb{S}^2$, after making δ_0 smaller and μ larger, if necessary, and for n large enough, we have that the same properties hold with respect to the ξ-direction, that is:

(c) an open neighborhood of the origin in $\tilde{M}_n$ is the graph $x_3 = u_n(x_1, x_2)$ of a function u_n over the Euclidean disk $\mathcal{D}_0 = D(0, \delta_0)$ of radius δ_0 in $\{\xi\}^\perp$; here (x_1, x_2, x_3) are orthonormal Euclidean coordinates centered at the origin, with $\frac{\partial}{\partial x_3} = \xi$.

(d) the C^2-norm of u_n in $\mathcal{D}_0$ is at most μ

Let $\tilde{\Lambda}_n$ denote the constant anisotropic mean curvature of $\tilde{M}_n$. By item (a) and the fact that the factors λ_n diverge to $+\infty$ we conclude that $\{\tilde{\Lambda}_n\}_n$ converges to zero. Also notice that, since the graph of u_n has constant anisotropic mean curvature $\tilde{\Lambda}_n$, then u_n is a solution to the linear elliptic PDE for u:

$$\tilde{\Lambda}_n = A_0(p,q)\tilde{F}(N) + \sum_{i=1}^{4} A_i(p,q)\tilde{F}_{x_i}(N) + \sum_{1\leq i\leq j\leq 4} A_{ij}(p,q)\tilde{F}_{x_i x_j}(N), \tag{4.2.4}$$

where the coefficients A_0, A_i and A_{ij} are as in equation (A.3.1), with $p = u_{x_1}$, $q = u_{x_2}$, $r = u_{x_1x_1}$, $s = u_{x_1x_2}$, $t = u_{x_2x_2}$ being the derivatives of u with respect to the variables (x_1, x_2) and $N = \frac{(-u_{x_1}, -u_{x_2}, 1)}{\sqrt{1+u_{x_1}^2+u_{x_1}^2}}$. As, by condition (d) above, the functions u_n are uniformly bounded in the $C^{1,\alpha}$-norm in $\mathcal{D}_0$, we conclude that all coefficients of (4.2.4) are bounded in the $C^{0,\alpha}(\mathcal{D}_0)$-norm. By the Schauder Estimates, for any $0 < \delta < \delta_0$ the $C^{2,\alpha}$-norm of the functions u_n on $D(0,\delta)$ are uniformly bounded.

Once here, we may repeat the last part of the proof in Theorem 4.1.2 using the Arzela-Ascoli Theorem and a diagonal argument, and conclude that a subsequence of the surfaces $\tilde{M}_n$ converges uniformly on compact sets in the C^2-topology to a complete anisotropic minimal surface M^∞ (since by construction $\{\tilde{\Lambda}_n\}_n$ converges to zero) of bounded curvature that passes through the origin. Moreover, the norm of the second fundamental form of M^∞ at the origin is equal to 1. Also, since all the surfaces $\tilde{M}_n$ have been obtained by translations and homotheties in $\mathbb{R}^3$ of the original immersions $f_n : \Sigma_n \to \mathbb{R}^3$, and since all the Gauss map images of the f_n omit an open spherical disk of radius $\rho/2$ of the north pole in $\mathbb{S}^2$, it follows that M^∞ also omits such an open disk. By Theorem 4.2.2, we deduce that M^∞ is a plane. This contradicts the fact that the norm of the second fundamental form of M^∞ at the origin is equal to 1, finishing the proof. □

Remark 4.2.4. In view of Proposition 3.3.2, item (c), the second fundamental form Σ can be replaced by the anisotropic second fundamental form of Σ in the statement of Theorem 4.2.3 without altering the conclusion.

4.3 Bernstein-type theorem for CAMC multigraphs

In this section we present a Bernstein-type Theorem for CAMC multigraphs, i.e., surfaces whose tangent plane at any of its points are not vertical (equivalently, each point on the surface has a neighborhood in which the surfaces can be viewed as a vertical graph). But first we need the following proposition.

Proposition 4.3.1. There is no entire vertical graph with constant mean anisotropic curvature $\Lambda \neq 0$.

Proof. This proof is analogous to the classic CMC case (see [75], Cor. 3.3.3, for example). Let $\Sigma = u(\mathbb{R}^2)$ be an entire vertical Λ-CAMC graph, for some smooth function $u : \mathbb{R}^2 \to \mathbb{R}$. Fix $p = (0, 0, u(0, 0)) \in \Sigma$ and take $R > \frac{\sqrt{3}}{|\Lambda|}d$, where d is the diameter of the Wulff shape W_F. There exists $M > 0$ such that $|u(x)| \leq M$, for all $x \in B_{\mathbb{R}^2}((0,0), R)$. Consider $\tilde{W}$ the image of W_F by the homothety of scaling factor $-\Lambda/2$. Thus $\tilde{W}$ has constant anisotropic mean curvature Λ and it is possible to show that $\tilde{W}$ is contained in a round sphere of radius R centered at some point $\hat{o} \in \mathbb{R}^3$. Then we define $W(r)$ as the image of $\tilde{W}$ by the translation that takes the point $\hat{o}$ to $p+re_3$, where $e_3 = (0,0,1)$. For $r > M+R$, $W(r)$ is completely above Σ. Decreasing r, we eventually obtain a first contact point between Σ and $W(r)$. If the normal vector of both surfaces coincide at the contact point, we apply the Maximum Principle to conclude that Σ is contained in $W(r)$, which is a contradiction. Otherwise, we replace e_3 by $-e_3$ in the previous construction and proceed in the same way. At this time the normals to Σ and $W(r)$ at the contact point will coincide. □

Now we are able to proof the first main result of this thesis:

Theorem 4.3.2. Let Σ be a complete vertical multigraph, that is, for any $p \in \Sigma$, $T_p\Sigma$ is not a vertical plane. Suppose that Σ has constant anisotropic mean curvature Λ. Then Σ is a plane.

Proof. If $\Lambda = 0$, the conclusion follows from Theorem 4.2.2. So, we can suppose, up to a homothety, that $\Lambda = 2$, and by Proposition 4.3.1 we can also suppose that Σ is not an entire graph.

Before proceed, we need to make some considerations and recall important facts. Let N be a unit vector field along Σ. Since Σ is a vertical multigraph, without loss of generality we can assume that Σ is a multigraph with respect to the plane $\{z = 0\}$ and $\langle N, e_3\rangle$ is strictly positive. In particular, the Gauss map of Σ omits a hemisphere of $\mathbb{S}^2$, and by the completeness of Σ we may apply Theorem 4.2.3 to conclude it has bounded second fundamental form. Thus, by Theorem 4.1.1, there is $\delta > 0$ such that for any point $p \in \Sigma$, Σ is a graph over the disk $D(0,\delta) \subset T_p\Sigma$ of radius δ, centered at the origin of $T_p\Sigma$. Such graph will be denoted by $\mathcal{U}_p$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$ be the canonical projection. We will denote by $\mathcal{U}_p^v$ the image of $\mathcal{U}_p$ under the vertical translation that takes p to $(\pi(p), 0)$.

Yet, we recall that given $q \in \mathbb{R}^2$, $v \in \mathbb{S}^2$ and $\delta > 0$, there exists exactly one curve Γ that passes through q such that $\Gamma \times \mathbb{R}$ is a vertical 2-CAMC cylinder whose normal vector at $(q,0)$ is v, as in Proposition 3.4.1. We denote by $\Gamma(q,\delta)$ the piece of Γ of length δ with q as its mid-point. Also, if $p \in \Gamma \times \mathbb{R}$ is one of its points, we denote by $\mathcal{D}(p,\delta)$ the neighborhood of p in $\Gamma \times \mathbb{R}$ that is a graph over $D(0,\delta) \subset T_p(\Gamma \times \mathbb{R})$. It is clear that the projection of $\mathcal{D}(p,\delta)$ by π coincides with $\Gamma(\pi(p),\delta)$.

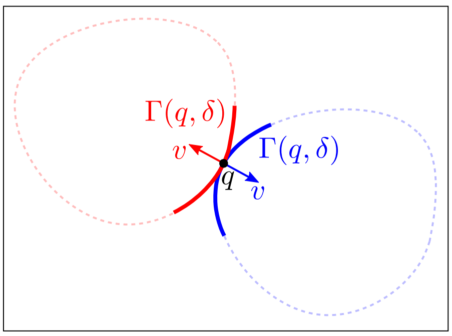

Figure 4.4: Two pieces of curves $\Gamma(q,\delta)$ that passes through q but have opposite unit normals there differ by a translation in $\mathbb{R}^3$.

With these considerations in mind, we begin proving the following assertion:

Claim 1: Let $\{p_n\}_n \subset \Sigma$ be any sequence such that $\pi(p_n) \to (q_0, 0) \in \mathbb{R}^2 \times \{0\}$ and $\langle N(p_n), e_3\rangle \to 0$, that is, the planes $T_{p_n}\Sigma$ are becoming vertical as $n \to \infty$. Then the surfaces $\mathcal{U}_{p_n}^v$ converge to $\mathcal{D}((q_0,0),\delta)$, for some 2-CAMC cylinder $\Gamma \times \mathbb{R}$ that passes through $(q_0,0)$. The convergence is in the C^2-topology.

Proof of Claim 1. By the compactness of $\mathbb{S}^2$ we can suppose, up to a subsequence, that $\{N(p_n)\}_n$ converges to a horizontal vector $v_0 \in \mathbb{S}^2$, since $\langle N(p_n), e_3\rangle \to 0$. In particular, the planes $T_{p_n}\Sigma$ converge to the vertical plane Q orthogonal to v_0 that passes through $(q_0, 0)$. We choose Γ as the unique horizontal curve that passes through q_0 such that $\Gamma\times\mathbb{R}$ is a vertical 2-CAMC cylinder whose normal direction at $(q_0, 0)$ coincides with v_0.

Since $\mathcal{U}^v_{p_n}$ are graphs over $D(0,\delta) \subset T_{p_n}\Sigma$ and the planes $T_{p_n}\Sigma$ converge to $Q = T_{(q_0,0)}(\Gamma\times\mathbb{R})$, given any $0 < \delta' < \delta$, for n sufficiently large, the surfaces $\mathcal{U}^v_{p_n}$ are bounded horizontal graphs over the disk $D(0,\delta') \subset Q$. Applying the same arguments of the proof of Theorem 4.2.3, we deduce that a subsequence of these graphs converges to a 2-CAMC surface S in the C^2-topology. Moreover, S is tangent to Q at $(q_0, 0)$ and is a horizontal graph over $D(0,\delta) \subset Q$.

Now, we show that $S = \mathcal{D}((q_0, 0), \delta)$. If it was not the case, then the intersection of S and $\mathcal{D}((q_0, 0), \delta)$ at $(q_0, 0)$ would be non-transversal, and therefore it would consist of m smooth curves passing through $(q_0, 0)$, $m \geq 2$, meeting transversally at $(q_0, 0)$ (see [60], Corollary 4.6, pg. 159). In a neighborhood of $(q_0, 0)$, these curves separate S into $2m$ components. Adjacent components lie on opposite sides of $\mathcal{D}((q_0, 0), \delta)$. Hence in a neighborhood of $(q_0, 0)$ in S, the unit normal vector field of S alternates from pointing up and down (or vice-versa) with respect to e_3. Since $\mathcal{U}^v_{p_n}$ converges to S in the C^2-topology, for n sufficiently large we have also that the unit normals points up and down, contradicting the fact that $\langle N, e_3\rangle$ is positive.

Finally, it is worth to emphasize that the uniqueness of the limit $\mathcal{D}((q_0, 0), \delta)$ is sufficient to prove that the whole sequence $\{\mathcal{U}^v_n\}_n$ converges $\mathcal{D}((q_0, 0), \delta)$, and not only a subsequence. □

So, let us fix an arbitrary point $p \in \Sigma$ and assume that Σ is, in a neighborhood of p, a vertical graph of a function $u(x_1, x_2)$ defined on $B_{\mathbb{R}^2}(\pi(p), R)$, the open ball of $\mathbb{R}^2 \cong \mathbb{R}^2\times\{0\}$ of radius R centered at $\pi(p)$. Since we are assuming that Σ is not an entire graph, there exists a largest value of R where u is well defined. In particular, there exists a point $q \in \partial B_{\mathbb{R}^2}(\pi(p), R)$ where u cannot be extended on any of its neighborhoods.

Claim 2: For any sequence $\{q_n\}_n \subset B_{\mathbb{R}^2}(\pi(p), R)$ converging to q, the tangent planes $T_{p_n}\Sigma$ (where $p_n := (q_n, u(q_n))$) converges to a vertical plane P. Moreover, such a plane is tangent to $B_{\mathbb{R}^2}(\pi(p), R)$ at q.

Proof of Claim 2. First, we prove that any subsequence of $T_{p_n}\Sigma$ converges to a vertical plane. If it was not the case, since $\mathbb{S}^2$ is compact, there would exists a subsequence $\{q_{n_k}\}_k$ such that $\{N(p_{n_k})\}_k$ converges to some unit vector N_0 satisfying $\langle N_0, e_3\rangle \neq 0$, and since Σ has bounded geometry and δ is the same for all points of Σ, for q_{n_k} close enough to q, the uniform graphs $\mathcal{U}_{p_{n_k}}$ provide an extension of u on a neighborhood of q, which is a contradiction.

Now we prove the second part. Let P be the vertical plane tangent to $\partial B_{\mathbb{R}^2}(\pi(p), R)$ at q. Suppose that for some subsequence $\{q_{n_k}\}_k$ the planes $T_{p_{n_k}}\Sigma$ converge to a vertical plane Q, $Q \neq P$. By Claim 1, $\mathcal{U}^v_{p_{n_k}}$ converges in the C^2-topology to $\mathcal{D}((q, 0), \delta)$, for a certain curve Γ. We know that $\mathcal{D}((q, 0), \delta)$ is tangent to Q at $(q, 0)$, and since $Q \neq P$, this implies that there are points of $\Gamma(q, \delta)$ in the interior of $B_{\mathbb{R}^2}(\pi(p), R)$. Let z be one of these interior points. There exists a sequence $\{z_k\}_k$ with $z_k \in \mathcal{U}^v_{p_{n_k}}$ such that $z_k \to z$. Since the tangent plane of $\mathcal{D}((q, 0), \delta)$ at z is also vertical, the gradient of u running along $\{z_k\}_k$ diverges (to see this, recall the expression of the normal vector field of a graph and verify what occurs with the gradient as the normal tends to be horizontal). But the gradient of u at z is well defined, since z is an interior point of $B_{\mathbb{R}^2}(\pi(p), R)$. We have a contradiction. □

Putting together Claims 1 and 2 we have the following conclusion: there is a curve Γ, profile of a 2-CAMC cylinder, that passes through q, tangent to $B_{\mathbb{R}^2}(\pi(p), R)$ at q and such that for any sequence $\{q_n\}_n \subset B_{\mathbb{R}^2}(\pi(p), R)$ converging to q, $\{\mathcal{U}^v_{(q_n, f(q_n))}\}_n$ converges uniformly in the C^2-topology to the neighborhood $\mathcal{D}((q,0),\delta)$ of $(q,0)$ in $\Gamma \times \mathbb{R}$. The curve Γ is unique, since its unit normal at $(q,0)$ is the limit of the normals of $\mathcal{U}^v_{(q_n, f(q_n))}$ at $(q_n, 0)$. Also, $\Gamma(q,\delta)$ lies outside $B_{\mathbb{R}^2}(\pi(p), R)$, otherwise, by the same arguments as in the proof of Claim 2, we would have an interior point z of $B_{\mathbb{R}^2}(\pi(p), R)$ where $\mathrm{grad}_{\mathbb{R}^2} u(x)$ diverges, as $x \in B_{\mathbb{R}^2}(\pi(p), R)$ approaches z.

Claim 3: Let $\gamma : [0,1) \to \mathbb{R}^2$ be the semi-open line segment parametrized by $\gamma(t) = tq + (1-t)\pi(p)$. The function $z(t) = u(\gamma(t))$ satisfies either $\lim_{t\to 1^-} z(t) = +\infty$ or $\lim_{t\to 1^-} z(t) = -\infty$, according to whether $\gamma(t)$ lies either inside or outside Γ, for values of t close to 1.

Proof of Claim 3. Consider any sequence $\{t_n\}_n \subset [0,1)$ converging to 1. Since $\gamma(t_n) \to q$, Claim 2 implies that $T_{(\gamma(t_n), z(t_n))}\Sigma$ are converging to a vertical plane, and as a consequence, $z'(t_n) \to \pm\infty$. So we can assume that the curve $C := (\gamma, u \circ \gamma)$ is strictly monotonous for values of t close to 1.

Now, if z were bounded above, there would be a sequence $\{t_n\}_n \subset [0,1)$ with $t_n \to 1^-$ and such that $\{(\gamma(t_n), z(t_n))\}_n$ converges to a point $(q,c) \in \mathbb{R}^3$, for some $c \in \mathbb{R}$, that is, (q,c) is a cluster point of C. Moreover, the monotonicity of γ implies that C would have finite length up till (q,c), whence we conclude that $\{(\gamma(t_n), z(t_n))\}_n$ is a Cauchy sequence on Σ. But since Σ is complete, we would have that $(q,c) \in \Sigma$, and in particular, Σ would have a vertical tangent plane at (q,c), contradicting the fact it is a vertical multi-graph.

Finally, since we are supposing that $\langle N, e_3 \rangle > 0$ on Σ, we deduce that if $\lim_{t\to 1^-} z(t) = +\infty$ (resp. $-\infty$) then the (horizontal) limit of the unit normals of Σ along $(\gamma(t), z(t))$ points in the same direction as the vector $(\pi(p) - q, 0)$ (resp. $(q - \pi(p), 0)$), which is the unit vector of the limit cylinder $\Gamma \times \mathbb{R}$ at $(q,0)$. Thus, the last assertion follows from the fact that such a limit cylinder is oriented with respect to the its inner unit normal. □

Let s be an arc length parameter to Γ and denote by $q(s) \in \Gamma$ the point at distance s from $q = q(0)$ along Γ. Consider the following subset of $\mathbb{R}^2$:

$$\mathcal{O}_\epsilon = \{\Gamma(s) + t n_\Gamma(s); s \in [-\delta, \delta], t \in (0, \epsilon)\}, \tag{4.3.1}$$

where $n_\Gamma(s) \in \mathbb{R}^2$ denotes the unit normal vector of Γ at $q(s)$ that points toward $\pi(p)$ for $s = 0$.

For $t_0 \in [0,1)$ we define:

$$\Sigma_{t_0} = \bigcup_{t_0 < t < 1} \mathcal{U}_{(\gamma(t), z(t))}, \tag{4.3.2}$$

which is a connected neighborhood of the curve $\{(\gamma(t), z(t)); t_0 < t < 1\} \subset \Sigma$.

Claim 4: There exists $t_0 > 0$ such that Σ_{t_0} does not intersect $\Gamma \times \mathbb{R}$. Moreover, Σ_{t_0} is a vertical graph over an open domain in $\mathbb{R}^2$ that contains $\mathcal{O}_\epsilon$, for some $\epsilon > 0$. Also, this graph extends u on $\mathcal{O}_\epsilon$ and satisfies $u(x) \to \pm\infty$ when $x \in \mathcal{O}_\epsilon$ converges to a point in Γ.

Proof of Claim 4. Consider $Q(s)$ the vertical plane that is orthogonal to Γ and passes through $q(s)$. Recall that $\mathcal{U}^v_{(\gamma(t), z(t))}$ converges in the C^2-topology to $\mathcal{D}((q,0),\delta)$ as $t \to 1^-$.

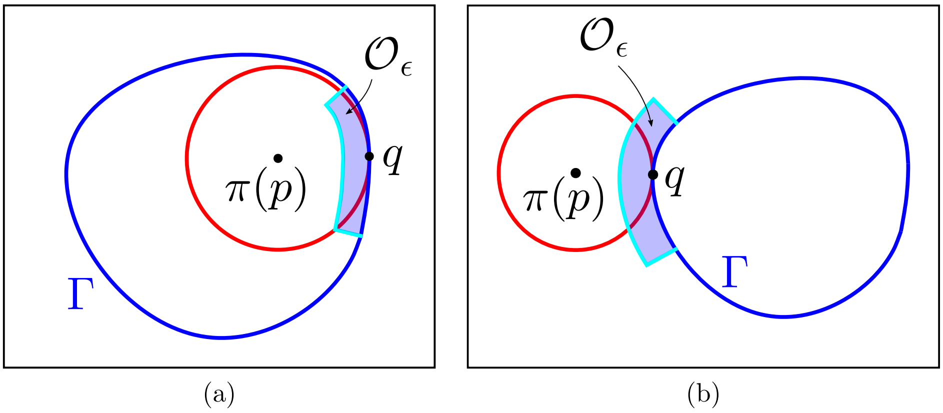

Figure 4.5: The small tubular neighborhood $\mathcal{O}_\epsilon$ of Γ around q, according to two possible scenarios: (a) $n_\Gamma(0) = (\pi(p) - q, 0)$; (b) $n_\Gamma(0) = (q - \pi(p), 0)$.

In particular, from the convergence in C^1-topology and since tranversality is a stable property, there exists $t_0 > 0$ such that $Q(s)$ intersects Σ_{t_0} transversally for all $s \in [-\delta, \delta]$.

Consider the curves $C(s) := \Sigma_{t_0} \cap Q(s)$. By Claim 3, $C(0) = C$ does not intersect $\Gamma \times \mathbb{R}$. Let us see that $C(s)$ does not intersect $\Gamma \times \mathbb{R}$, for all $s \in (0, \delta]$ (the argument is similar for $s \in [-\delta, 0)$).

Suppose that for some $s_0 \in (0, \delta]$, $C(s_0)$ contains either a point in $\Gamma \times \mathbb{R}$ or a point in the opposite side of $\Gamma \times \mathbb{R}$ relative to $C(0)$. In the second case, by continuity, for some intermediate value $0 < s_1 \leq s_0$, $C(s_1)$ has a point on $\Gamma \times \mathbb{R}$. Thus we can assume that $C(s_0)$ contain a point in $\Gamma \times \mathbb{R}$. But the convergence of $\mathcal{U}^v_{(\gamma(t), z(t))}$ to $\mathcal{D}((q, 0), \delta)$ as $t \to 1^-$ implies that $C(s_0)$ converges to $\{q(s_0)\} \times \mathbb{R}$ as the height diverges to infinity. These two facts obliges $C(s_0)$ to have a point where the tangent vector is vertical (see Fig. 4.6 (a)), contradicting the fact that Σ_{t_0} is a vertical multigraph. As a consequence, we conclude that for all $s \in [-\delta, \delta]$, $C(s)$ does not intersect $\Gamma \times \mathbb{R}$, and since these curves are asymptotic to $\Gamma \times \mathbb{R}$, there exists $\epsilon > 0$ such that Σ_{t_0} is a vertical graph over an open domain in $\mathbb{R}^2$ which contains $\mathcal{O}_\epsilon$ with the property that $u(x) \to \pm\infty$ when $x \in \mathcal{O}_\epsilon$ converges to a point in Γ (see Fig. 4.6 (b)). $\square$

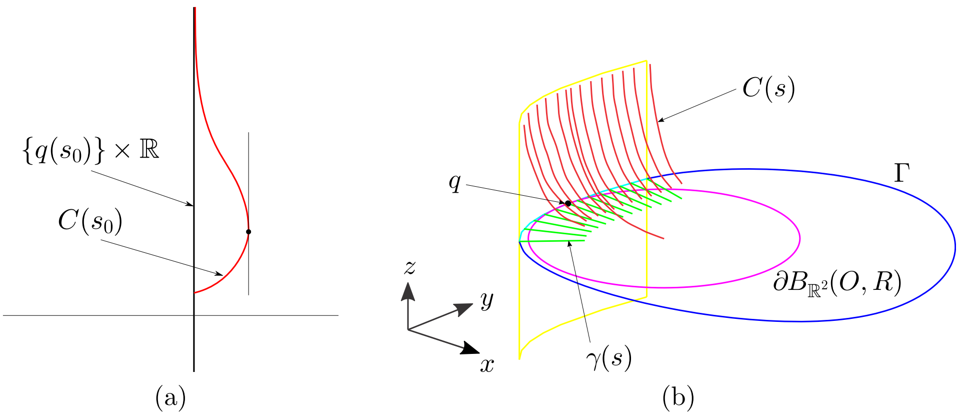

Figure 4.6: (a) A vertical tangent in the opposite side of $\Gamma \times \mathbb{R}$ relative to $B_{\mathbb{R}^2}(O, R)$; (b) The curves $C(s)$ for $s \in [-\delta, \delta]$ extend u in $\mathcal{O}_\epsilon$.

Next, notice that $q(\delta/2)$ plays the same role as q in Claims 2, 3 and 4. Replacing q and C by $q(\delta/2)$ and $C(\delta/2)$, and since δ is uniform for any point of Σ, applying the same arguments of Claim 4, we are able to extend u near Γ in a bigger domain of the form

$$\{\Gamma(s) + tn_\Gamma(s); s \in [-\delta/2, 3\delta/2], t \in (0, \epsilon')\},$$

for some $0 < \epsilon' \leq \epsilon$. More generally, this argument can be applied inductively in both sides of Γ relative to q, and if $\Gamma(s)$ is an injective parametrization in the interval $(a, b]$, for some $a < 0 < b$, then there exists $\tilde{\epsilon} > 0$ for which we can extend u on the open domain

$$V_{\tilde{\epsilon}} = \{\Gamma(s) + tn_\Gamma(s); s \in (a, b), t \in (0, \tilde{\epsilon})\}. \tag{4.3.3}$$

At this point we observe that the conclusions we obtained up to now are also applicable

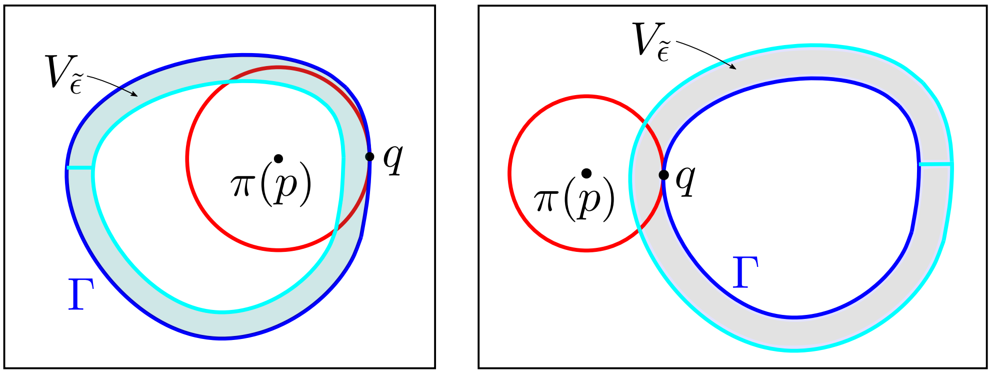

Figure 4.7: The tubular neighborhood of Γ according to equation (4.3.3).

for any point $p \in \Sigma$ and its associated point $q \in \partial B_{\mathbb{R}^2}(\pi(p), R)$, according to with R is the maximum radius where Σ can be written as a vertical graph over the open ball $B_{\mathbb{R}^2}(\pi(p), R)$, and its associated 2-CAMC cylinder $\Gamma \times \mathbb{R}$.

Another important observation here is that, since Σ is a multigraph (not necessarily a graph), we can no longer guarantee that any extension u in a neighborhood Γ as in equation (4.3.3), but for values of s in a bigger interval $(a', b') \supset (a, b)$, will glue together continuously as s reaches the values a and b. Despite this multi-evaluation difficulty, we will prove in the following the existence of a domain big enough where u is a graph and for which we can compare such a graph with the Wulff shape through the Maximum Principle. To do so, we split our arguments into two cases, according to whether γ lies locally either inside or outside the planar region bounded by Γ, for values of t close to 1.

Case 1: Let us suppose that there exists a point $p \in \Sigma$ such that its associated point $q \in \partial B_{\mathbb{R}^2}(\pi(p), R)$ and its associated 2-CAMC cylinder $\Gamma \times \mathbb{R}$ verify that the interior planar region bounded by Γ, that we call Ω, contain points of the line segment joining $\pi(p)$ to q.

In this case, we claim that u can be extended on the whole Ω. Indeed, consider $\tilde{q} \in \Gamma$ where the unit normals of Γ at q and $\tilde{q}$ coincide. Let $v := \tilde{q} - q$ and for $t \in [0, 1]$ define $\Gamma_t = \Gamma + tv$, the image of Γ under the translation by the vector tv. In particular $\Gamma_0 = \Gamma$ and for $0 < t \leq 1$, Γ_t divides Ω into two bounded regions (only one region, if $t = 1$). We are interested only in the region whose boundary contains q, that we call $\Omega(t)$.

By the arguments after Claim 4, we know that for sufficiently small values of t, u can be extended to $\Omega(t)$. Let us suppose that there exists a first value $0 < t_0 < 1$ for which u cannot be extended to $\Omega(t)$, for $t > t_0$. Then, there exists a point $q_0 \in \Gamma_{t_0}$ where u cannot be extended on any of its neighborhoods. Since u can be extended in an interior tubular neighborhood of $\Gamma - \{\tilde{q}\}$, we conclude that $q_0 \in \Omega$. Consider a point p_0 on the graph of u and $R' > 0$ such that the open ball $B_{\mathbb{R}^2}(\pi(p_0), R') \subset \Omega(t_0)$ and $q_0 \in$

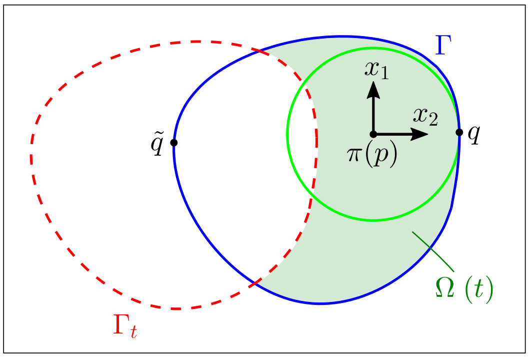

Figure 4.8: Definitions of the curve $\Gamma(t)$ and the open domain $\Omega(t)$.

$\partial B_{\mathbb{R}^2}(\pi(p_0), R')$. Notice that by construction u is well defined on $\overline{B_{\mathbb{R}^2}(\pi(p_0), R')} - \{q_0\}$. Reasoning as in Claims 1, 2 and 3, there exists a curve Γ' which is a translated copy of Γ, that passes through q_0 and have there a tangency point with $B_{\mathbb{R}^2}(\pi(p_0), R')$. Also, as $x \in B_{\mathbb{R}^2}(\pi(p_0), R')$ approaches to Γ', $u(x)$ diverges to either $+\infty$ or $-\infty$ according to whether $B_{\mathbb{R}^2}(\pi(p_0), R')$ is contained in the interior or the exterior open region delimited by Γ', respectively. For the first case, in a small neighborhood $U \subset \Gamma'$ of q_0, we have that $U - \{q_0\}$ is also contained in $\Omega(t_0)$, and in these points we deduce that u diverges to $+\infty$, contradicting the fact that u is well defined in $\Omega(t_0)$ (see Fig. 4.9 (a)). In the second case, note that Γ' intersects Γ. Let $x \in \Gamma \cap \Gamma'$ be one of these points (see Fig. 4.9 (b)). If $\{x_n\} \subset \Omega(t_0)$ is converging to x, by the fact that $x \in \Gamma$ we deduce that $u(x_n) \to +\infty$. On the other hand, since $x \in \Gamma'$ we also deduce that $u(x_n) \to -\infty$, leading to a contradiction.

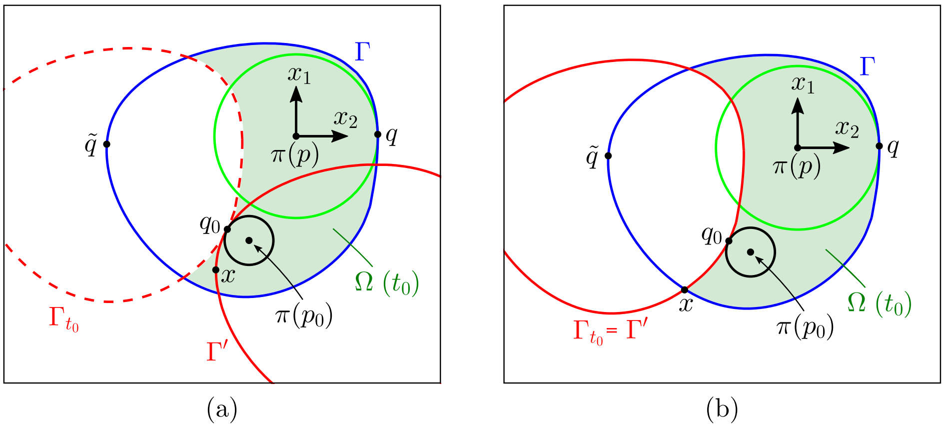

Figure 4.9: Extension of u on $\Omega(t_0)$. Two possible scenarios occur: (a) $\Gamma' \neq \Gamma_{t_0}$; (b) $\Gamma' = \Gamma_{t_0}$.

From these informations, we conclude that Σ contains a vertical graph of a function u defined over the open domain Ω bounded by Γ. Moreover, if $x \in \Omega$ approaches to a point in Γ, then $u(x) \to +\infty$.

Now, consider W the image of the Wulff shape by the antipodal map. With respect to the inner pointing unit normal, W has anisotropic mean curvature 2. Recall that $G : \mathbb{S}^2 \to \mathbb{R}^3$ given by $G(x, y, z) = \operatorname{grad}_{\mathbb{S}^2} F(x, y, z) + F(x, y, z)(x, y, z)$ is a parametrization of the Wulff shape. Thus $-G$ parametrizes W and since it is a diffeomorphism, we define the *south hemisphere* of W as the set $\mathcal{S} = -G(\mathbb{S}^2_+)$, where $\mathbb{S}^2_+ = \{(y_1, y_2, y_3) \in \mathbb{S}^2; y_3 \geq 0\}$. In view of Proposition 3.4.1, consider $W(0)$ a copy of W obtained after a translation in a way that its orthogonal projection over $\mathbb{R}^2 \times \{0\}$ contains $\overline{\Omega}$ in its interior. In other words, $\Gamma \times \mathbb{R}$ is transversal to $W(0)$. Let $W(r)$ be the image of $W(0)$ after a vertical translation by re_3. We define $\mathcal{S}(r)$ analogously. For very negative values of r, $\mathcal{S}(r) \cap \Sigma = \emptyset$. Increasing r we eventually obtain a first interior contact point between $\mathcal{S}(r)$ and Σ. As the normal vectors of $\mathcal{S}(r)$ and Σ at the contact point coincide, we conclude by the Maximum Principle that $\Sigma = W(r)$, leading to a contradiction since Σ is not compact.

Case 2: For any point $p \in \Sigma$ with associated point $q \in \partial B_{\mathbb{R}^2}(\pi(p), R)$ and associated 2-CAMC cylinder $\Gamma \times \mathbb{R}$, $\pi(p)$ lies outside the planar region defined by Γ.

Take any point $p_0 \in \Sigma$ and consider $R > 0$ the maximum radius for which Σ is the vertical graph of a smooth function u defined on $B(\pi(p_0), R)$. Let $q_0 \in \partial B(\pi(p_0), R)$ be a point where u cannot be extended on any of its neighborhoods and let $\Gamma_0 \times \mathbb{R}$ denote the 2-CAMC vertical cylinder tangent to $B(\pi(p_0), R)$ at q_0 on its concave side.

Consider $\tilde{q}_0 \in \Gamma_0 - \{q_0\}$ the unique point for which the normal lines of Γ_0 at q_0 and $\tilde{q}_0$ are parallel. Let $2\Gamma_0$ be the curve obtained as the image of Γ_0 by a double scale homothety followed by a translation in such a way that $2\Gamma_0$ is tangent to Γ_0 at $\tilde{q}_0$ and contains Γ_0 in its interior (see Fig. 4.10).

Let $H : [0, 1] \times \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth, injective homotopy so that for each $t \in [0, 1]$, $H_t := H(t, \cdot) : \mathbb{S}^1 \to \mathbb{R}^2$ parametrizes the image of Γ_0 by a homothety followed by a translation, in such a way that: (a) for $0 \leq t < 1$, $H_t(\mathbb{S}^1)$ is contained in the interior region bounded by $2\Gamma_0$; (b) $H_0(\mathbb{S}^1)$ is contained in $B(\pi(p_0), R) \cup \{q_0\}$ and it is tangent to $\partial B(\pi(p_0), R)$ at q_0; (c) $H_1(\mathbb{S}^1) = 2\Gamma_0$.

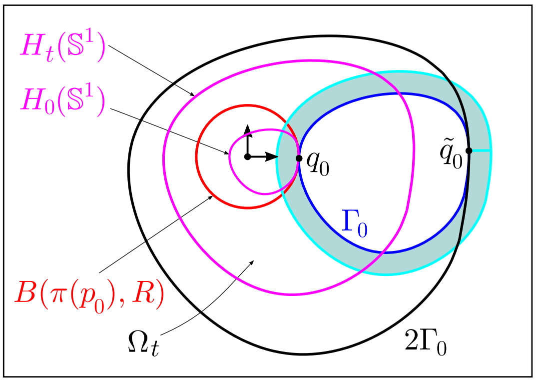

Figure 4.10: Definitions of the convex curve $H_t(\mathbb{S}^1)$ and the compact domain Ω_t.

For each $t \in [0, 1]$, let Ω_t denote the compact convex domain bounded by $H_t(\mathbb{S}^1)$, and consider S_t the connected component of $\Sigma \cap (\Omega_t \times \mathbb{R})$ that contains p_0. If $0 \leq t_1 \leq t_2 \leq 1$, it is clear that $S_{t_1} \subset S_{t_2}$. Our objective is to show that S_1 is a vertical graph, that is, u can be extended on $\pi(S_1)$. To do so, consider $\mathcal{I}$ the set of values $t \in [0, 1]$ for which S_t is a graph of (an extension of) u defined over a domain $\tilde{\Omega}_t \subset \Omega_t$ given as

$$\tilde{\Omega}_t = \Omega_t - (D_0 \cup D_1 \cup \cdots \cup D_{m(t)}), \tag{4.3.4}$$

for some $m(t) \in \mathbb{N}$. Here each D_i is the compact convex domain bounded by some translation of the curve Γ_0, and $D_i \cap D_j \cap \Omega_t = \emptyset$, for $0 \leq i < j \leq m$. Note that in such decompositions $D_i \cap \Omega_t$ could consist of only one point.

First we note that $0 \in \mathcal{I}$, in which case $\tilde{\Omega}_0 = \Omega_0 - D_0$, being D_0 the compact domain bounded by Γ_0. We also observe that if $0 < t_1 < t_2$ and $t_2 \in \mathcal{I}$, then $t_1 \in \mathcal{I}$. Therefore, $\mathcal{I}$ is an interval of the form $[0, a)$ or $[0, a]$ for some $a \leq 1$. Moreover, the same domains D_i appearing in the decomposition of $\tilde{\Omega}_{t_1}$ (according to equation (4.3.4)) will appear in the decomposition of $\tilde{\Omega}_{t_2}$. In particular, the numbers $m(t)$ are non-decreasing with respect to t.

Suppose that $\mathcal{I} = [0, a]$ for some $0 < a < 1$. In particular, there exists $m(a) \in \mathbb{N}$ such that S_a is a graph over

$$\tilde{\Omega}_a = \Omega_a - (D_0 \cup D_1 \cup \cdots \cup D_{m(a)}). \tag{4.3.5}$$

We want to prove that for small values $\epsilon > 0$, $S_{a+\epsilon}$ is a graph over Ω_a minus the same domains $D_0, D_1, \ldots, D_{m(a)}$. Note that for any $i \in \{1, \ldots, m(a)\}$ we have that $\Gamma_i \cap \partial\Omega_a$ either consists of two points or Γ_i is tangent to Ω_a on its concave side. Indeed, if it does not occur, we conclude that $D_i \subset \Omega_a$, and in particular $D_i \subset \Omega_1$, which implies that $D_i \cap D_0 \neq \emptyset$, since two translations of Γ_0 do not fit inside $2\Gamma_0$ without have an intersection point. But again $D_i \subset \Omega_a$ implies that $D_i \cap D_0 \cap \Omega_a \neq \emptyset$, which is a contradiction. Thus, in any case, by the arguments of the first part of this proof u can be extended in a small exterior tubular neighborhood of each curve Γ_i, around their intersection points with $\partial\Omega_a$. Note that there are finitely many such intersection points, since the intersection of each Γ_i with $\partial\Omega_a$ consists of at most two points. Thus, we can choose such exterior tubular neighborhoods so small that they do not overlap each other (see Fig. 4.11). The remaining points of $\partial\Omega_a$ constitute a compact set satisfying that for each of its points there is a small neighborhood where u can be extended. Therefore, for small values $\epsilon > 0$, the domains $D_0, D_1, \ldots, D_{m(a)}$ are mutually disjoint inside $\Omega_{a+\epsilon}$ and $S_{a+\epsilon}$ is a graph over $\Omega_{a+\epsilon} - (D_0 \cup D_1 \cup \cdots \cup D_{m(a)})$.

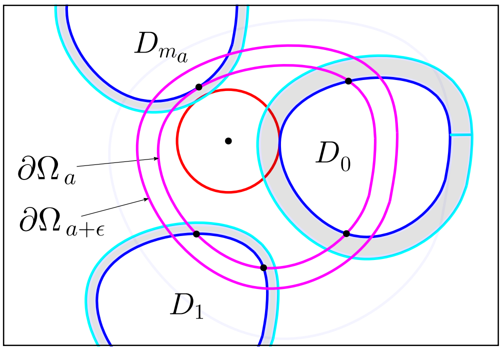

Figure 4.11: Decomposition of the domain $\tilde{\Omega}_{a+\epsilon}$.

Assume next that $\mathcal{I} = [0, a)$, $a \leq 1$. Note that S_a is a graph. To see this, suppose that there would be two points $p_1, p_2 \in S_a$ with $\pi(p_1) = \pi(p_2)$. We assume that $\pi(p_1) = \pi(p_2)$ do not lie in $\partial\Omega_a$, otherwise since S_a is a vertical multigraph and $p_1 \neq p_2$, we could find $\tilde{p}_1$ close to p_1 and $\tilde{p}_2$ close to p_2 in S_a such that $\tilde{p}_1 \neq \tilde{p}_2$ and whose projections under π coincide and lie in the interior of Ω_a. Since S_a is connected, there are two curves $c_1, c_2 : [0, 1] \to S_a$ joining p_1 to p_0 and p_2 to p_0, respectively. If $\pi \circ c_1([0, 1])$ intersects $\partial\Omega_a$, from the fact that $X = \{s \in [0, 1]; \pi \circ c_1(s) \in \partial\Omega_a\}$ is compact and that for every $s \in X$ there is a neighborhood of $c_1(s)$ in S_a that is a vertical graph, we can cover $c_1(X)$ with such neighborhoods and after extracting a finite subcover, it is possible to deform c_1 in order to obtain a new curve $\tilde{c}_1 : [0, 1] \to S_a$ joining p_1 to p_0, whose image under π does not intersect $\partial\Omega_a$. The same argument is valid for c_2. Thus $\pi \circ c_1$ and $\pi \circ c_2$ are compact curves contained in the interior of Ω_a, implying the existence of $b < a$ so that $\pi \circ c_1([0, 1]), \pi \circ c_2([0, 1]) \subset \Omega_b$. In particular we conclude that p_1 and p_2 belongs to S_b. But it contradicts the fact that S_b is a graph. Therefore S_a is a graph.

Now we prove that S_a is a graph over Ω_a, with exception of exactly the same domains D_i which appear in the decomposition of Ω_b, for $b < a$, and domains bounded by translations of Γ_0 that are tangent to $\partial\Omega_a$ on its concave side.

Given any $b < a$, we can write

$$\tilde{\Omega}_b = \Omega_b - (D_0 \cup D_1 \cup \cdots \cup D_{m(b)}), \tag{4.3.6}$$

for some $m(b) \in \mathbb{N}$, where each D_i is the compact convex domain bounded by some translation of the curve Γ_0, and $D_i \cap D_j \cap \Omega_b = \emptyset$, for any $0 \le i < j \le m(b)$. Moreover, since $c \in \mathcal{I}$ for any $b < c < a$, we conclude that $D_i \cap D_j \cap \Omega_c = \emptyset$, for all $0 \le i < j \le m(b)$. Thus, if $D_i \cap D_j \cap \Omega_a \neq \emptyset$ for some $0 \le i < j \le m(b)$, the intersection necessarily occur in $\partial\Omega_a$ (see Fig. 4.12 (a)). Let us see that neither this scenario occur. Without loss of

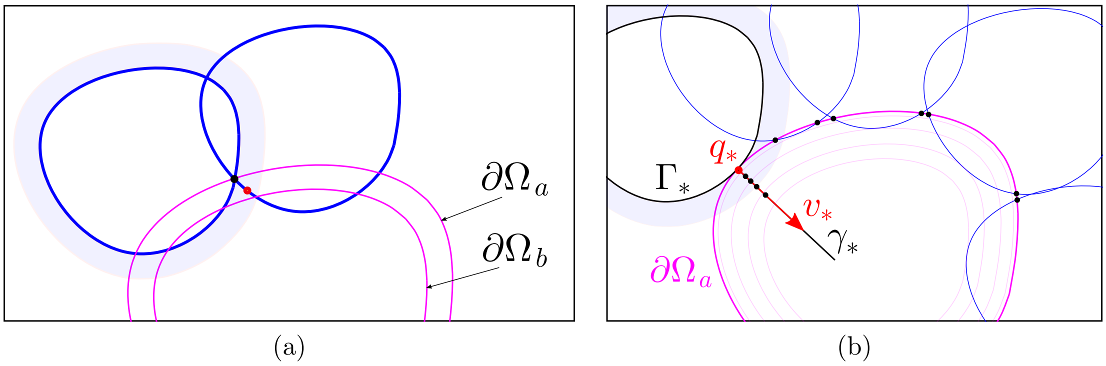

Figure 4.12: (a) The intersection $\Gamma_i \cap \Gamma_j$ occurs necessarily in $\partial\Omega_a$; (b) Auxiliary figure to prove the boundedness of $m(t)$.

generality, we can suppose $j > 0$. We claim that Γ_j has points outside Ω_a. Indeed, if it does not occur, we would have that $\Gamma_j \subset \Omega_a$, implying that $\Gamma_j \cap \Gamma_0 \cap \Omega_a \neq \emptyset$. As $\Gamma_j \neq \Gamma_0$, the intersection $\Gamma_j \cap \Gamma_0$ consists of either one point, if Γ_j and Γ_0 are tangent, or two points, otherwise. In the first case, the tangency point is not $\tilde{q}_0$, otherwise Γ_j would be tangent to $2\Gamma_0$ on its concave side, and in particular Γ_j would lie outside Ω_b. However, we know that u can be extended in a small exterior tubular neighborhood of Γ_0 around this tangency point, and this gives a contradiction. Thus, the intersection $\Gamma_j \cap \Gamma_0$ consists of exactly two points, and at least one of them necessarily belongs to Ω_c, for some $0 < c < a$, since $\Gamma_j \subset \Omega_a$ and Γ_j could have at most one point in common with $\partial\Omega_a$. In other words, we would conclude that $D_j \cap D_0 \cap \Omega_c \neq \emptyset$, which contradicts $c \in \mathcal{I}$. Thus, by the arguments in the first part of the proof we can extend u in small exterior tubular neighborhood of Γ_j around its intersections with $\partial\Omega_a$, and in particular this implies that $D_i \cap D_j \cap \Omega_a = \emptyset$.

On the other hand, since $t \mapsto m(t)$ is non-decreasing, we need to show that $m(t)$ is bounded as $t \to a^-$. If it does not happen, we would have a strictly monotonous sequence $\{t_n\}_n$ with $t_n \to a^-$ and domains $\{D_{m(t_n)}\}_n$ (which rises in the decomposition of $\tilde{\Omega}_{t_n}$), each one bounded by translations Γ_{t_n} of Γ_0, and mutually disjoint inside Ω_a. We also can suppose that Γ_{t_n} are tangent to $\partial\Omega_{t_n}$ on its concave side, and that $\Gamma_{t_n} \cap \partial\Omega_a = \{q_1^n, q_2^n\}$. In particular there exists a cluster point $q_* \in \partial\Omega_a$ of $\{q_1^n, q_2^n\}_n$. Let $v_* \in \mathbb{R}^2$ denote the inner unit normal of $\partial\Omega_a$ at q_* and consider the curve $\gamma_* : (0,1] \to \mathbb{R}^2$ given by $\gamma_*(s) = q_* + sv_*$ (see Fig. 4.12 (b)). Note that u is well defined along γ_*, for small values $s > 0$. Given any sequence $\{s_n\}_n \subset (0,1]$ with $s_n \to 0^+$, we have that the planes $T_{p_n}\Sigma$ ($p_n := (\gamma_*(s_n), u(\gamma_*(s_n)))$) are becoming vertical, otherwise by the same arguments as in the proof of Claim 2 we could extend u in a neighborhood of q_*. Thus Claim 1 implies the uniform graphs $\mathcal{U}_{p_n}^v$ converge to $\mathcal{D}((q_*, 0), \delta)$, for some CAMC cylinder $\Gamma_* \times \mathbb{R}$ that passes through $(q_*, 0)$. Moreover, the unit normal of $\mathcal{D}((q_*, 0), \delta)$ at $(q_*, 0)$ is parallel to

v_*. Indeed, if it is not the case, $\Gamma_*(q_*, \delta)$ would have points in the interior of Ω_b, for all $b < a$ sufficiently close to a. Suppose that for some $b < a$ and some $i \in \{1, \dots, m(b)\}$, the curve $\Gamma_*(q_*, \delta)$ intersects ∂D_i. In this case, since $q_* \notin D_i$, there is a point $x \in \Gamma_*(q_*, \delta)$ that lies outside D_i, but inside the exterior tubular neighborhood of ∂D_i for which u is well defined. Choosing $b < c < a$ sufficiently close to a, we can also suppose that $x \in \tilde{\Omega}_c$. If $\Gamma_*(q_*, \delta)$ does not intersect ∂D_i for any $b < a$ and any $i \in \{1, \dots, m(b)\}$, the existence of such a point $x \in \Gamma_*(q_*, \delta)$ is trivial. In any case, for each $n \in \mathbb{N}$ there exists $b_n = (\pi(b_n), u(\pi(b_n))) \in \mathcal{U}^v_{p_n}$ so that $\{b_n\}_n$ converges to $(x, 0)$. But since the planes $T_{b_n}\Sigma$ are becoming vertical, we conclude that $|\operatorname{grad} u(\pi(b_n))| \to \infty$, which is impossible since $\pi(b_n) \to x$ and u is well defined around x. Thus, the unit normal of $\mathcal{D}((q_*, 0), \delta)$ at $(q_*, 0)$ is parallel to v_*. Moreover, as $\{q_1^n, q_2^n\}_n$ converges to q_*, at points sufficiently close to q_1^n and q_1^n the normals of the graph of u are becoming horizontal and point inward $\partial D_{m(t_n)}$, whence we may also deduce that the unit normal of $\mathcal{D}((q_*, 0), \delta)$ at $(q_*, 0)$ coincides with $-v_*$. In particular, Γ_* is tangent to $\partial\Omega_a$ at q_* on its concave side. From now on, the arguments in the proofs of Claims 3 and 4 are also applicable for $\mathcal{D}((q_*, 0), \delta)$, and we conclude that u can be extended in a small exterior tubular neighborhood of Γ_* around q_*, as in equation (4.3.4). In other words, q_* is an isolated point of $\partial\Omega_a$, which is a contradiction. We denote $m_a := \sup\{m(t); t < a\}$.

Now, if y is any point in $\partial(\Omega_a - \pi(S_a))$ which does not lie in the boundary of the domains D_i for any $b < a$, then there exists a curve Γ_y, translation of Γ_0, which is tangent to $\partial\Omega_a$ at y on its concave side, such that u can be extended in a unique way on an small exterior tubular neighborhood of Γ_y around y. From the same arguments we use previously, we deduce that there are only finitely many such points y in $\partial\Omega_a$. If $y_{m_a+1}, \dots, y_{m_a+k} \in \partial\Omega_a$ denote such points, the domains $D_{m_a+1}, \dots, D_{m_a+k}$ corresponding to the curves $\Gamma_{y_1}, \dots, \Gamma_{y_k}$, respectively, together with the domains $D_0, D_1, \dots, D_{m_a}$ satisfy:

- $D_i \cap D_j \cap \Omega_a = \emptyset, \qquad \forall 0 \le i < j \le m_a + k;$
- $\pi(S_a) = \Omega_a - (D_0 \cup D_1 \cup \dots \cup D_{m_a} \cup D_{m_a+1} \cup \dots \cup D_{m_a+k}),$

and this proves that $a \in \mathcal{I}$.

We conclude from all these arguments that $\mathcal{I} = [0, 1]$ and S_1 is a graph of an extension of u defined over

$$\tilde{\Omega}_1 = \Omega_1 - (D_0 \cup D_1 \cup \dots \cup D_{m(1)}), \tag{4.3.7}$$

for some $m(1) \in \mathbb{N}$, where $\{D_0, D_1, \dots, D_{m(1)}\}$ consists of mutually disjoint compact domains such that ∂D_i is a translation of Γ_0, for each $0 \le i \le m(1)$. Moreover, by Claim 3 $u(x) \to -\infty$ as $x \in \tilde{\Omega}_1$ approaches to ∂D_i, for any $i \in \{0, 1, \dots, m(1)\}$.

As we did in Case 1, consider W the image of the Wulff shape by the antipodal map and $W(0)$ the copy of W obtained after a translation in such a way that its orthogonal projection over $\mathbb{R}^2 \times \{0\}$ coincides with Ω_1. Let $W(r)$ be the image of $W(0)$ after a vertical translation by re_3. For very positive values of r, $W(r) \cap S_1 = \emptyset$. Decreasing r we eventually obtain a first contact point $x \in W(r) \cap S_1$. We observe that x does not belong to ∂S_1, otherwise the unit normal of S_1 at this point would be horizontal, contradicting the fact that Σ is a vertical multigraph. Thus x is an interior contact point, and as the normal vectors of $W(r)$ and S_1 at x coincide, we conclude that $W(r) = S_1$, by the Maximum Principle, leading to a contradiction. Therefore, the complete multigraph Σ with constant anisotropic mean curvature 2 does not exist, finishing the proof. □

An immediate consequence of Theorem 4.3.2 is the following corollary.

Corollary 4.3.3. Let Σ be a complete CAMC surface whose Gauss map image is contained in a closed hemisphere of $\mathbb{S}^2$. Then Σ is either a plane or a CAMC cylinder.

Proof. The hypothesis implies that there exists $a \in \mathbb{S}^2$ such that $\langle N, a\rangle \geq 0$ on Σ. We recall that Proposition 3.3.4 implies

$$\Delta_F \langle N, a\rangle + \mathrm{trace}_\Sigma(AA_F)\langle N, a\rangle = 0.$$

Since this equation is elliptic, by the Maximum Principle, either $\langle N, a\rangle = 0$ or $\langle N, a\rangle > 0$. In the first case we deduce that Σ is either a CAMC cylinder or a plane parallel to a. In the second case, we deduce that Σ is a CAMC multigraph, and Theorem 4.3.2 implies Σ is a plane. □

Another interesting corollary of Theorem 4.3.2 concerns about CAMC surfaces whose Gauss curvature does not change sign.

Theorem 4.3.4. Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed surface of constant anisotropic mean curvature $\Lambda \neq 0$. If the Gaussian curvature of Σ does not change sign then Σ is one of the following surfaces:

(i) a CAMC cylinder;

(ii) the Wulff shape (up to a homothety).

Our proof of Theorem 4.3.4 relies on very different techniques to those used in the proof of its isotropic version, given by T. Klotz and R. Osserman, in [63]. For this reason, we need to recall three results:

Theorem 4.3.5. (Pogorelov's Theorem, see [86] and [87]; Hartman-Nirenberg Theorem, see [50]) Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed surface with vanishing Gaussian curvature everywhere. Then Σ is a generalized cylinder.

Theorem 4.3.6. (Sacksteder's Theorem; see [94] or [18]) Let $\Sigma \subset \mathbb{R}^3$ be a complete and oriented immersed surface with non-negative Gaussian curvature. Suppose that at least for one point on Σ, the Gaussian curvature is positive. Then Σ is embedded in $\mathbb{R}^3$ and it is the boundary of an open convex subset of $\mathbb{R}^3$.

Theorem 4.3.7. (The principal curvature theorem for hypersurfaces; see [95]) Let Σ be a complete immersed orientable hypersurface in $\mathbb{R}^{n+1}$, which is not a hyperplane, and let A denote its Weingarten operator with respect to a global unit normal field. Let $\Omega \subset \mathbb{R}-\{0\}$ be the set of nonzero values assumed by the eigenvalues of A and put $\Omega^+ = \Omega \cap (0, +\infty)$ and $\Omega^- = \Omega \cap (-\infty, 0)$.

(i) If Ω^+ and Ω^- are both nonempty, $\inf \Omega^+ = \sup \Omega^- = 0$;

(ii) If Ω^+ or Ω^- is empty then the closure of Ω is connected.

Proof of Theorem 4.3.4. We begin by observing that up to a homothety, we may assume that $\Lambda > 0$.

Let us suppose that the Gaussian curvature of Σ is non-negative. If it vanishes everywhere, by Theorem 4.3.5 we obtain item (i). So, we can suppose that there exists a point on Σ where the Gaussian curvature is positive. In this case we apply Theorem 4.3.6 to conclude that Σ is embedded in $\mathbb{R}^3$ and that it is the boundary of an open convex subset of $\mathbb{R}^3$. If Σ is compact, the anisotropic version of the Alexandrov's Theorem (see [56]) implies that it is the Wulff shape, up to a homothety. Thus we may assume that Σ is not compact. In particular, since Σ is the boundary of an open convex set, if $N : \Sigma \to \mathbb{S}^2$ denotes its Gauss map (for which Λ is evaluated), we have that $N(\Sigma)$ is contained in a closed hemisphere of $\mathbb{S}^2$. By Corollary 4.3.3 we deduce that Σ is a Λ-CAMC cylinder.

If the Gaussian curvature is non-positive, the principal curvatures of Σ have opposite signs, let us say, $\kappa_1(p) \leq 0$ and $\kappa_2(p) \geq 0$, for all $p \in \Sigma$. Recalling formula 3.3.2 we have

$$\Lambda = \gamma_1(p)\kappa_1(p) + \gamma_2(p)\kappa_2(p),$$

where $\frac{1}{M} \leq \gamma_1(p), \gamma_2(p) \leq \frac{1}{m}$ and m, M are respectively the infimum and supremum of the principal curvatures of the Wulff shape. Therefore,

$$\gamma_2(p)\kappa_2(p) = \Lambda - \gamma_1(p)\kappa_1(p) \geq \Lambda,$$

which implies

$$\kappa_2(p) \geq \frac{\Lambda}{\gamma_2(p)} \geq m\Lambda.$$

So, we have $k_1(p) \leq 0 \leq m\Lambda \leq k_2(p)$, whence we conclude that Σ is a flat surface by applying Theorem 4.3.7. Applying Theorem 4.3.5 again we conclude that Σ is a CAMC cylinder. □

4.4 Height estimates for CAMC graphs

Putting together the results of the previous sections, we are able to prove uniform height estimates for CAMC graphs, that is stated precisely in the next theorem:

Theorem 4.4.1. Let $\Lambda \neq 0$ be a real number. Then there is a constant $C = C(\Lambda) > 0$ such that for any closed (not necessary bounded) domain Ω of the plane $\Pi = \{v\}^\perp$ and smooth function $u : \Omega \to \mathbb{R}$ that vanishes on $\partial\Omega$ and whose graph Σ over Π is a Λ-CAMC surface, the height of any point $p \in \Sigma$ relative to Π is at most C.

Remark 4.4.2. The best constant C in Theorem 4.4.1 satisfies the following property

$$C(\mu\Lambda) = \frac{1}{|\mu|}C(\Lambda), \quad \text{for any} \quad \mu \neq 0. \tag{4.4.1}$$

Proof of Theorem 4.4.1. The proof is by contradiction, and for simplicity we will assume that $\Lambda > 0$. Let us suppose that there is no uniform height estimates for Λ-CAMC graphs with planar boundary. Thus, for each $n \in \mathbb{N}$ there are a unit vector v_n, a Λ-CAMC graph Σ_n over $\Pi_n = \{v_n\}^\perp$ with $\partial\Sigma_n \subset \Pi_n$ and a point $p_n \in \Sigma_n$ such that the height of p_n relative to Π_n is greater than n.

Since $\theta_n := \langle N_n, v_n\rangle$ is a non-vanishing function, for all $n \in \mathbb{N}$, up to a subsequence of $\{\Sigma_n\}_n$ (that we also call it $\{\Sigma_n\}_n$) we can suppose, without loss of generality, that $\langle N_n, v_n\rangle > 0$ on Σ, for all $n \in \mathbb{N}$, and that $v_n \to v_0$, for some unit vector v_0. By the Maximum Principle, as in the proof of Lemma 4.2.1, we conclude that Σ_n is contained in the half-space $\{y \in \mathbb{R}^3; \langle y, v_n\rangle \leq 0\}$, for all $n \in \mathbb{N}$. So, from now on, for simplicity, we adopt a new coordinate system in $\mathbb{R}^3$ where $v_0 = e_3 = (0, 0, 1)$.

Let $\Sigma_n^* := \Sigma_n \cap \{y \in \mathbb{R}^3; \langle y, v_n\rangle \leq 2R\}$, where R is given by Lemma 4.2.1. For n large enough we have $p_n \in \Sigma_n^*$. Let $q_n \in \Sigma_n^0$ be the point of maximum distance relative to Π_n and let Σ_n^1 be the image of Σ_n^0 by the translation that takes q_n to the origin of $\mathbb{R}^3$. Since $\theta_n > 0$, for all $n \in \mathbb{N}$, by Theorem 4.2.3 (with $d = 2R$) there is a constant

M that only depends on Λ (but not on n) such that $\{|\sigma_n|\}_n$ is uniformly bounded by M, where σ_n denotes the second fundamental form of Σ_n^0. Notice that the distance in $\mathbb{R}^3$ of the origin to $\partial\Sigma_n^0$ is diverging to $+\infty$, since $d_{\mathbb{R}^3}(p_n, \Pi_n) > n$. In other words, this means that the geodesic disks $D_{\Sigma_n^0}(0, n)$ are contained in the interior of Σ_n^0, and we may apply Theorem 4.1.2 in order to obtain a subsequence of $\{\Sigma_n^0\}_n$ that converges uniformly on compact sets in the C^2-topology to a complete (possibly non-connected) Λ-CAMC surface Σ_∞ that passes through the origin and whose norm of its second fundamental form σ_∞ is bounded by M. Moreover, Σ_∞ is tangent to the horizontal plane at the origin, since $N_n(q_n) = v_n \to e_3$. From now on, we only consider the connected component of Σ_∞ that passes through the origin, and we still denoting it by Σ_∞.

Let us define $\theta_\infty = \langle N_\infty, e_3\rangle$, where N_∞ is the unit normal vector field of Σ_∞, with $N_\infty((0,0,0)) = e_3$. As the graphs Σ_n^0 converge uniformly on compact sets to Σ_∞ and $\theta_n > 0$, for all $n \in \mathbb{N}$, we conclude that $\theta_\infty \geq 0$ on Σ_∞. In other words, the Gauss map image of Σ_∞ is contained in a closed hemisphere of $\mathbb{S}^2$, and by Corollary 4.3.3 we conclude that Σ_∞ is a Λ-CAMC cylinder. In particular, the image of N_∞ is a great circle passing through $N_\infty((0,0,0)) = e_3$, contradicting that θ_∞ is non-negative. □

A first consequence of Theorem 4.4.1 are Height Estimates for compact CAMC surfaces with planar boundary. Since its proof involves the Alexandrov's Reflection Method, we need to impose an additional assumption: the plane that contains the boundary of the surface is also a symmetry plane for the Wulff shape. For the proof of these Height Estimates for compact CAMC surfaces, we need to do the following consideration

Proposition 4.4.3. let $s : \mathbb{R}^3 \to \mathbb{R}^3$ be a symmetry with respect to some plane $P_0 \subset \mathbb{R}^3$, and suppose that the anisotropy function $F : \mathbb{S}^2 \to \mathbb{R}$ is invariant under s, that is, $F(s(y)) = F(y)$, for all $y \in \mathbb{S}^2$. If $\Sigma \subset \mathbb{R}^3$ is any surface with anisotropic Gauss map $\nu : \Sigma \to W_F$, then

$$\tilde{\nu}(s(p)) = s(\nu(p)), \quad \forall p \in \Sigma.$$

where $\tilde{\nu} : s(\Sigma) \to W_F$ is the anisotropic Gauss map of $s(\Sigma)$, the image of Σ by s.

Proof. Let $N_W : W_F \to \mathbb{S}^2$ denote the outward pointing Gauss map of the Wulff shape W_F and consider $N : \Sigma \to \mathbb{S}^2$ and $\tilde{N} : s(\Sigma) \to \mathbb{S}^2$ the Gauss maps of Σ and $s(\Sigma)$, respectively. First, we notice that

- $s(N_W(z)) = N_W(s(z))$, for all $z \in W_F$;
- $s(N_W^{-1}(y)) = N_W^{-1}(s(y))$, for all $y \in \mathbb{S}^2$.

To see this, consider $\tilde{F} : \mathbb{R}^3 - \{0\} \to \mathbb{R}$ the homogeneous extension of F, that is, $\tilde{F}(x) = |x| F\left(\frac{x}{|x|}\right)$, for $x \in \mathbb{R}^3 - \{0\}$. In this way we have that

$$N_W^{-1}(y) = (\operatorname{grad}_{\mathbb{S}^2} F)_y + 2F(y)y = (\operatorname{grad}_{\mathbb{R}^3} \tilde{F})_y, \quad \forall y \in \mathbb{S}^2.$$

This information together with the fact that s is an orthogonal transformation, we obtain

$$\begin{aligned}\langle(\operatorname{grad}_{\mathbb{R}^3} \tilde{F} \circ s)_y, w\rangle &= d(\tilde{F} \circ s)_y \cdot w = d\tilde{F}_{s(y)} \cdot ds_y \cdot w = \langle(\operatorname{grad}_{\mathbb{R}^3} \tilde{F})_{s(y)}, ds_y \cdot w\rangle \\ &= \langle(\operatorname{grad}_{\mathbb{R}^3} \tilde{F})_{s(y)}, s(w)\rangle = \langle s^{-1}((\operatorname{grad}_{\mathbb{R}^3} \tilde{F})_{s(y)}), w\rangle, \quad \forall w \in \mathbb{R}^3\end{aligned}$$

On the other hand, it is easy to see that $\tilde{F}$ is invariant under s, and this information leads us to

$$\langle(\operatorname{grad}_{\mathbb{R}^3} \tilde{F} \circ s)_y, w\rangle = \langle(\operatorname{grad}_{\mathbb{R}^3} \tilde{F})_y, w\rangle, \quad \forall w \in \mathbb{R}^3,$$

implying that

$$(\mathrm{grad}_{\mathbb{R}^3} \tilde{F})_{s(y)} = s((\mathrm{grad}_{\mathbb{R}^3} \tilde{F})_y)$$

Therefore,

$$N_W^{-1}(s(y)) = (\mathrm{grad}_{\mathbb{R}^3} \tilde{F})_s(y) = s((\mathrm{grad}_{\mathbb{R}^3} \tilde{F})_y) = s(N_W^{-1}(y)), \quad \forall y \in \mathbb{S}^2,$$

that is, $s \circ N_W^{-1} = N_W^{-1} \circ s$. In particular, if $z = N_W^{-1}(y)$, for some $y \in \mathbb{S}^2$, then

$$\begin{aligned} s(N_W(z)) &= s(N_W(N_W^{-1}(y))) = s(y); \qquad \text{and} \\ N_W(s(z)) &= N_W(s(N_W^{-1}(y))) = s(N_W(N_W^{-1}(y))) = s(y), \end{aligned}$$

which also proves that $s \circ N_W = N_W \circ s$.

Finally, the relation between the anisotropic Gauss maps of Σ and $s(\Sigma)$ is given by

$$\tilde{\nu} \circ s := N_W^{-1} \circ \tilde{N} \circ s = N_W^{-1} \circ s \circ N = s \circ N_W^{-1} \circ N := s \circ \nu.$$

Thus,

$$\begin{aligned} -d\tilde{\nu}_{s(p)} \cdot ds_p &= -d(\tilde{\nu} \circ s)_p = -d(s \circ \nu)_p = ds_{\nu(p)} \cdot (-d\nu_p) \\ \Rightarrow -d\tilde{\nu}_{s(p)} &= ds_{\nu(p)} \cdot (-d\nu_p) \cdot (ds_p)^{-1} = s \circ (-d\nu_p) \circ s^{-1}. \end{aligned}$$

In particular, $\mathrm{trace}(-d\tilde{\nu}_{s(p)}) = \mathrm{trace}(-d\nu_p)$ and $\det(-d\tilde{\nu}_{s(p)}) = \det(-d\nu_p)$. □

As a particular consequence of Proposition 4.4.3, if Σ has constant anisotropic mean curvature Λ, then $s(\Sigma)$ also has constant anisotropic mean curvature Λ, and we can apply the Alexandrov's Reflection Method. Thus, we can now state our Height Estimates for compact CAMC surfaces with planar boundary.

Theorem 4.4.4. Suppose that the anisotropy function $F : \mathbb{S}^2 \to \mathbb{R}$ is invariant under the reflection of $\mathbb{S}^2$ that fixes the geodesic $\mathbb{S}^2 \cap \{v\}^\perp$, for some $v \in \mathbb{S}^2$. Let Σ be any compact Λ-CAMC surface ($\Lambda \neq 0$), that is embedded in $\mathbb{R}^3$ and whose boundary is contained in the plane $\{v\}^\perp$. If $C = C(\Lambda)$ is given as in Theorem 4.4.1, then the height of any point $p \in \Sigma$ relative to $\{v\}^\perp$ is at most $2C$.

Proof. The proof is an application of the Alexandrov Reflection Method, based on the classical one due by W. Meeks in [78]. For simplicity, we consider $v = e_3$. Let $\{P(t)\}_{t\in\mathbb{R}}$ be the foliation of $\mathbb{R}^3$ by horizontal planes, being $P(t)$ the image of $\mathbb{R}^2 \times \{0\}$ after a translation by the vector te_3. We define $\Sigma^+(t) = \Sigma \cap \{z \geq t\}$ and $\tilde{\Sigma}(t)$ the image of $\Sigma^+(t)$ under the reflection of $\mathbb{R}^3$ that fixes $P(t)$.

By contradiction, we suppose that there is a point $p \in \Sigma$ such that its height with respect to $P(0)$ is greater than $2C$. Since Σ is compact, for $t > 0$ sufficiently large, $P(t) \cap \Sigma = \emptyset$. There is a first value $t_1 > 2C$ in which $P(t_1)$ meets Σ. For values of t sufficiently close to t_1, we have that

(i) $\Sigma^+(t)$ is a graph over $P(t)$;

(ii) $\tilde{\Sigma}(t)$ lies inside the open region bounded by Σ and $P(0)$.

Now we decrease t until we reach a first value $t_2 < t_1$ when items (i) and (ii) fail to occur simultaneously. By Theorem 4.4.1, item (i) certainly fails when $t < t_1 - C$, whence we conclude that $t_2 \geq t_1 - C > C$ and that $\tilde{\Sigma}(t_2)$ does not intersect $\partial\Sigma$. Thus, $\tilde{\Sigma}(t_2)$ and Σ have an interior contact point or there exists some point on $\Sigma \cap P(t_2)$ whose tangent plane is vertical, which means that $\tilde{\Sigma}(t_2)$ and Σ have a boundary tangent point (see 4.13). In both cases the normal vectors of $\tilde{\Sigma}(t_2)$ and Σ coincide at the contact point, and by the Maximum Principle we conclude that $\Sigma = \Sigma(t_2) \cup \tilde{\Sigma}(t_2)$, that is, Σ is a compact surface with no boundary. □

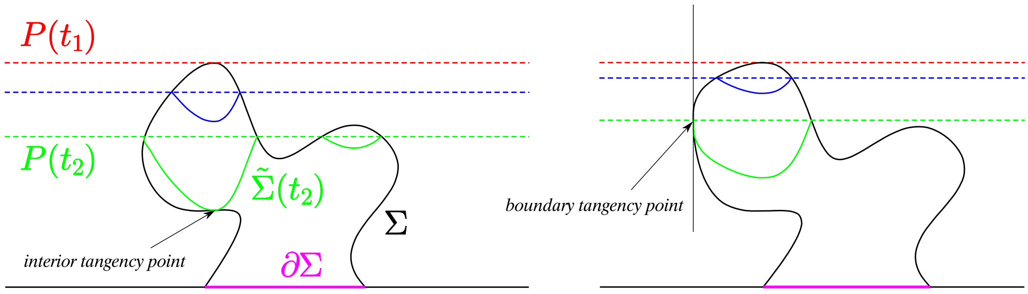

Figure 4.13: Two possible scenarios when $t = t_2$.

Another consequence of the Height Estimates for CAMC graphs is the following theorem.

Theorem 4.4.5. Let Σ be any compact Λ-CAMC ($\Lambda \neq 0$) surface, that is embedded in $\mathbb{R}^3$ and whose boundary is contained in the plane $\{v\}^\perp$, for some $v \in \mathbb{S}^2$. Consider three linearly independent vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ and suppose that the anisotropy function $F : \mathbb{S}^2 \to \mathbb{R}$ is invariant under the reflections in $\mathbb{S}^2$ that fix the geodesics $\mathbb{S}^2 \cap \{v_i\}^\perp$, for $i \in \{1, 2, 3\}$. Let d denote the diameter of $\partial\Sigma$. Then there exists a constant $C = C(\Lambda, d)$ such that the distance of any point $p \in \Sigma$ relative to $\{v\}^\perp$ is at most C.

Proof. For simplicity, we assume that $v = e_3$. Also, we suppose that e_3 is not parallel to v_i, for $i \in \{v_1, v_2, v_3\}$, otherwise the conclusion follows directly from Theorem 4.4.1. Let R be the radius of a circle in $\mathbb{R}^2 \times \{0\}$ centered at the origin O which contains $\partial\Sigma$. For example, after a translation of Σ we can take $R = \frac{\sqrt{3}d}{2}$.

For each $i \in \{1, 2, 3\}$ we consider P_i and Q_i the two supporting planes of $B(O, R)$, both perpendicular to the vector v_i, with P_i lying at v_i. Let P_i^+ and Q_i^+ the half-spaces defined by P_i and Q_i which are disjoint of $B(O, R)$, respectively. By Theorem 4.4.1, there is a constant C', depending only on Λ, such that the heights of $\Sigma \cap P_i^+$ with respect to P_i and $\Sigma \cap Q_i^+$ with respect to Q_i is smaller than C', for $i \in \{1, 2, 3\}$. Notice that these height estimates do not depend on the diameters of $\Sigma \cap P_i$ and $\Sigma \cap Q_i$.

Put $\tilde{P}_i = P_i + C'v_i$ and $\tilde{Q}_i = Q_i - C'v_i$. If S_i denotes the slab between $\tilde{P}_i$ and $\tilde{Q}_i$, for $i = \{1, 2, 3\}$, since $\{v_1, v_2, v_3\}$ is linearly independent, we conclude that $\mathcal{P} = S_1 \cap S_2 \cap S_3$ is a bounded open region of $\mathbb{R}^3$ containing $B(O, R)$. Moreover, $\mathcal{P}$ is contained in a open ball $B(O, C)$, whose radius C depends on R (which in turn depends on d) and C', finishing the proof. $\square$

4.5 Properly embedded CAMC surfaces

We say that a Riemannian surface Σ has **finite topology** if there is a compact surface without boundary K and a finite quantity of points $p_1, \ldots, p_n \in K$ such that Σ is diffeomorphic to $K - \{p_1, \ldots, p_n\}$. In this case, for each $i \in \{1, \ldots, n\}$ there is a neighborhood D_i of p_i in K where D_i has the same topology of the closed disk $\overline{B(0,1)} \subset \mathbb{R}^2$ and such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Identifying Σ with $K - \{p_1, \ldots, p_n\}$, each subset $A_i = D_i - \{p_i\}$ is called an **end** of Σ. Notice that each A_i has the same topology of the annulus $\mathbb{S}^1 \times [0, 1)$.

Remark 4.5.1. If Σ has finite topology and is properly embedded in $\mathbb{R}^3$ then its ends do not intercept and diverge to the infinity.

The following auxiliary result concerns the geometry of the annular ends of a properly embedded surface with constant anisotropic mean curvature. Its isotropic version was firs proved by W. Meeks in [78]. Later, a simpler proof appeared in [67], whose ideas involved there are also applicable in the anisotropic case, with the Wulff shape playing the role of the round sphere. By this reason we omit the proof.

Lemma 4.5.2 (Separation Lemma)**.** Let $A \subset \mathbb{R}^3$ be a properly embedded annulus (i.e., A is diffeomorphic to $\mathbb{S}^1 \times [0,1)$) with constant anisotropic mean curvature $\Lambda \neq 0$ and let P_1 and P_2 two parallel planes in $\mathbb{R}^3$. Denote by P_1^+ (resp. P_2^+) the half-space of $\mathbb{R}^3$ defined by P_1^+ (resp. P_2^+) which does not contain P_2 (resp. P_1). If the distance between P_1 and P_2 is greater than $\frac{2\sqrt{3}d}{|\Lambda|}$, where d denotes the diameter of the Wulff shape, then all the connected components of $\Sigma \cap P_1^+$ or $\Sigma \cap P_2^+$ are compact.

Putting together the height estimates for compact CAMC surfaces given in Corollary 4.4.4 and Lemma 4.5.2 we can infer an important theorem about properly embedded CAMC surfaces with at most one end and an additional hypothesis under the anisotropy function:

Theorem 4.5.3. Let $\Sigma \subset \mathbb{R}^3$ be a properly embedded Λ-CAMC ($\Lambda \neq 0$) surface with finite topology and at most one end. Consider three linearly independent vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ and suppose in addition that the anisotropy function F is invariant under the reflections in $\mathbb{S}^2$ which fix the geodesics $\mathbb{S}^2 \cap \{v_i\}^\perp$, for $i \in \{1,2,3\}$. Then, up to a homothety, Σ is the Wulff shape.

Proof. The idea of this proof is to show that Σ is contained in a slab defined by two parallel planes orthogonal to v_1. This conclusion applied also for v_2 and v_3 permit us to deduce that Σ is contained in a bounded region defined as the intersection of three slabs provided by v_1, v_2 and v_3. Thus, from the hypothesis that Σ is properly immersed, we conclude it is a compact CAMC surface, and since it is also embedded, Theorem 3.2.6 implies that Σ is the Wulff shape, up to a homothety.

Let P_1 and P_2 be two parallel planes orthogonal to v_1, whose distance is greater than $D := \frac{2\sqrt{3}d}{|\Lambda|}$, where d is the diameter of the Wulff shape. We can suppose, without loss of generality that both P_1 and P_2 intersect Σ, otherwise we can translate P_1 and P_2 along v_1 preserving their distance in such a way that either Σ is contained in the slab defined by P_1 and P_2, that is our desired scenario, or both P_1 and P_2 intersect Σ. Consider also a new coordinate system (x,y,z) in $\mathbb{R}^3$ where v_1 spans the z-axis and $P_1 = \{z = D\}$ and $P_2 = \{z = -D\}$. Since Σ is a properly embedded surface with at most one end, we can decompose $\Sigma = \Sigma_0 \cup A$, where Σ_0 is an embedded compact surface with boundary and A is a properly embedded annulus. In particular, A satisfy the hypothesis of Lemma 4.5.2, and we can deduce that all the connected components of $\Sigma \cap \{z \geq D\}$ or $\{z \leq D\}$ are compact. Without loss of generality, we can suppose that $\Sigma \cap \{z \geq D\}$ has this property. Thus, each connected component of $\Sigma \cap \{z \geq D\}$ is a Λ-CAMC surface with planar boundary, whence we deduce that $\Sigma \subset \{z \leq D + C\}$, where C is the uniform height bound given in Corollary 4.4.4.

Now consider the planes $\{z = D - 2C\}$ and $\{z = -D - 2C\}$. By applying Lemma 4.5.2 for these planes, we conclude that all the connected components of $\Sigma \cap \{z \geq D - 2C\}$ or $\Sigma \cap \{z \leq -D - 2C\}$ are compact. If $\Sigma \cap \{z \geq D - 2C\}$ would have this property, the same arguments of the last paragraph permit us to deduce that $\Sigma \subset \{z \geq D - 2C + C = D - C\}$, contradicting our assumption that Σ intersects the plane $\{z = D\}$. Thus we deduce that the connected components of $\Sigma \cap \{z \leq -D - 2C\}$ are compact and using Corollary 4.4.4

again, we conclude that $\Sigma \subset \{z \geq -D - 3C\}$. Therefore, Σ is contained in the slab $\{-D - 3C \leq z \leq D + C\}$, finishing the proof. □

Appendix A

The Maximum Principle

For convenience to the reader, we include this appendix, where we present the details of the main technique we use along the second part of this thesis: the Maximum Principle. The first section is devoted to the interior and boundary versions of the Maximum Principle for second-order elliptic linear operators. In the second section, similar versions of the Maximum Principle for quasi-linear operators are obtained. It is done by reducing to the linear case. Finally, in the third section we present the notions to establish the geometric version of the Maximum Principle, known as the Tangency Principle, that is formulated at the end of this section specifically for anisotropic mean curvatures.

A.1 Linear second-order partial differential operators

Let $\Omega \subset \mathbb{R}^n$ be a domain (i.e. an open and connected subset of $\mathbb{R}^n$). In this section we are concerned on linear second order differential operators $L : C^2(\Omega) \to C^0(\Omega)$ of the form:

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x), \tag{A.1.1}$$

where a_{ij}, b_i and c are, for every $i, j \in \{1, \ldots, n\}$, continuous real-valued functions defined on Ω. Here we also suppose that the second order coefficients are symmetric, that is, $a_{ij} = a_{ji}$, for every $i, j \in \{1, \ldots, n\}$.

Definition A.1.1. The operator L given in equation (A.1.1) is said to be:

i. **elliptic** at $x \in \Omega$ if the matrix $[a_{ij}(x)]$ is positive definite. If this condition is valid for all $x \in \Omega$ we say just that L is elliptic;

ii. **uniformly elliptic** if the eigenvalues of $[a_{ij}(x)]$ are bounded below and above by a positive constant;

iii. **locally uniformly elliptic** if for each $x \in \Omega$ there exists a neighborhood U of x so that L is uniformly elliptic.

More precisely, if $\{\lambda_1(x), \ldots, \lambda_n(x)\}$ are the eigenvalues of $[a_{ij}(x)]$ and

$$\lambda(x) = \min\{\lambda_i(x); 1 \le i \le n\},$$
$$\Lambda(x) = \max\{\lambda_i(x); 1 \le i \le n\},$$

then L is elliptic at $x \in \Omega$ if and only if

$$0 < \lambda(x) \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda(x) \sum_{i=1}^{n} \xi_i^2, \tag{A.1.2}$$

for every $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n - \{0\}$. In the sense of equation (A.1.2), L is uniformly elliptic if and only if $0 < \lambda \leq \lambda(x) \leq \Lambda(x) \leq \Lambda$ for some positive numbers λ and Λ. Thus, L is locally uniformly elliptic if for any $x_0 \in \Omega$ there exist a neighborhood $U \subset \Omega$ of x_0 and positive constants $\lambda(x_0)$ and $\Lambda(x_0)$ such that

$$\lambda(x_0)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda(x_0)|\xi|^2, \quad \forall x \in U, \quad \forall \xi \in \mathbb{R}^n.$$

The simplest linear second-order elliptic operators we must bear in mind is the Laplacian Operator $\Delta : C^2(\Omega) \to C^0(\Omega)$. It is given by:

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \tag{A.1.3}$$

It is easy to check that the Laplacian Operator is uniformly elliptic on $\mathbb{R}^n$. When $n = 2$, from the Theory of holomorphic functions it is known that a sub-solution of the Laplacian does not have a local maximum in the interior of its domain. This phenomenon also happens for any elliptic second-order linear differential operator as we can see in the next theorem.

Theorem A.1.2. (Interior Maximum Principle) Let $\Omega \subset \mathbb{R}^n$ be a domain and let $L : C^2(\Omega) \to C^0(\Omega)$ be an elliptic second-order linear differential operator, as in equation (A.1.1), with $c \leq 0$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu > 0$ on Ω.

i. If $c \equiv 0$, u does not have a maximum in Ω;

ii. If $c \leq 0$, u does not have a non-negative local maximum in Ω.

Proof. Let $x_0 \in \Omega$ be an interior local maximum of u. Then the Hessian matrix of u at x_0, $\operatorname{Hess} u(x_0)$, is negative semi-definite. Consider $A(x_0) = [a_{ij}(x_0)]$ the matrix of the second-order coefficients of L, evaluated at x_0. In particular, $A(x_0)$ is symmetric and positive-definite, since we are assuming that L is elliptic. Thus, there exists an orthogonal matrix P that diagonalizes $A(x_0)$, that is,

$$PA(x_0)P^{-1} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

for some $\lambda_i > 0$, $i \in \{1, \dots, n\}$.

Using that $\partial_i u(x_0) = 0$ for all $i \in \{1, \dots, n\}$, a direct computation gives

$$\operatorname{trace}(A(x_0) \operatorname{Hess} u(x_0)) = \sum_{i,j=1}^{n} a_{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) = Lu(x_0) - c(x_0)u(x_0).$$

Either item [i] or item [ii] being satisfied, in both cases we conclude that $\operatorname{trace}(A(x_0) \operatorname{Hess} u(x_0)) > 0$, since $Lu > 0$ on Ω.

On the other hand, since the trace is invariant for similar matrices, we have

$$\begin{aligned}\operatorname{trace}(A(x_0)\operatorname{Hess}u(x_0)) &= \operatorname{trace}(PA(x_0)\operatorname{Hess}u(x_0)P^{-1})\\ &= \operatorname{trace}(PA(x_0)P^{-1}P\operatorname{Hess}u(x_0)P^{-1})\\ &= \operatorname{trace}(PA(x_0)P^{-1}C) = \sum_{i=1}^{n}\lambda_i c_{ii},\end{aligned}$$

where $C := P\operatorname{Hess}u(x_0)P^{-1} = [c_{ij}]$. Since $\operatorname{Hess}u(x_0)$ is negative semi-definite and P is orthogonal, we conclude that C is also negative-semi definite. Thus, for the canonical basis $\{e_1,\ldots,e_n\}$ of $\mathbb{R}^n$ we have $c_{ii} = \langle Ce_i, e_i\rangle \le 0$, for all $i \in \{1,\ldots,n\}$. Therefore

$$\operatorname{trace}(A(x_0)\operatorname{Hess}u(x_0)) \le 0,$$

which is a contradiction. □

Theorem A.1.3. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $L : C^2(\Omega) \to C^0(\Omega)$ be a locally uniformly elliptic second-order linear differential operator, as in equation (A.1.1). Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu \ge 0$ on Ω.

i. If $c \equiv 0$ and u has a local maximum at $x_0 \in \Omega$, then u is constant in a neighborhood of x_0;

ii. If $c \le 0$ and u has a non-negative local maximum at $x_0 \in \Omega$, then u is constant in a neighborhood of x_0.

Proof. Let x_0 in Ω an interior local maximum of u. Since L is locally uniformly elliptic, there exists $r > 0$ and positive numbers $\lambda(x_0)$ and $\Lambda(x_0)$ such that

$$u(x) \le u(x_0)$$

and

$$\lambda(x_0)|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda(x_0)|\xi|^2,$$

for all $x \in B(x_0, r)$ and for all $\xi \in \mathbb{R}^n$. Here $B(x_0, r)$ denotes the Euclidean ball centered at x_0 of radius r.

Let us suppose, by contradiction, that u is non-constant in any neighborhood of x_0. This means that:

- $\exists y \in B(x_0, r/2)$ such that $u(y) < u(x_0)$, and
- $\exists \delta_0 > 0$ such that $\overline{B(y,\delta_0)} \subset B(x_0, r)$ and $x_0 \in \partial B(y,\delta_0)$.

Set $U(x_0) = \{x \in \Omega; u(x) = u(x_0)\}$ and define

$$\delta := \inf\{\rho > 0; \overline{B(y,\rho)} \subset B(x_0, r) \quad\text{and}\quad \partial B(y,\rho) \cap U(x_0) \ne \emptyset\}.$$

Claim: $0 < \delta \le \delta_0$. Indeed, since $u(y) < u(x_0)$, by continuity there exists $0 < \tilde{\delta} < \delta_0$ such that $u(x) < u(x_0)$, for all $x \in \overline{B(y,\tilde{\delta})}$. Since $B(y,\delta_0) \subset B(x_0,r)$ we can enlarge $\tilde{\delta}$ until $\overline{B(y,\tilde{\delta})}$ reach a first point $x' \in \partial B(y,\tilde{\delta})$ such that $u(x') = u(x_0)$. It necessarily occurs for some $0 < \tilde{\delta} \le \delta_0$ since $x_0 \in B(y,\delta_0)$.

Conclusion: from the definition of δ, $u(x) < u(x_0)$ for all $x \in B(y,\delta)$ and there exists $x' \in \partial B(y,\delta)$ with $u(x') = u(x_0)$.

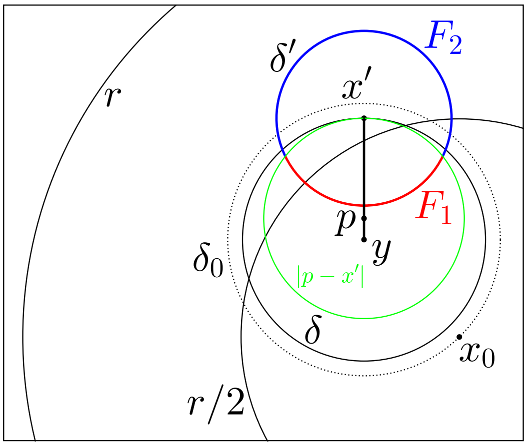

Figure A.14: Auxilliary illustration for the proof of Theorem A.1.3.

Take $0<\delta'<\delta$ so that $\overline{B(x',\delta')}\subset B(x_0,r)$ and a point $p\in[y,x']=\{sy+(1-s)x';s\in[0,1]\}$ so that $|p-x'|>\delta'$ (see Fig. A.14). In particular, $u(x)\leq u(x_0)$, for all $x\in B(x',\delta')$.

Let K be a constant to be determined and define

$$v(x)=e^{-K|x-p|^2}-e^{-K|x'-p|^2},\qquad x\in\overline{B(x',\delta')}. \tag{A.1.4}$$

Then, we compute Lv to obtain

$$\begin{aligned}Lv(x)=e^{-K|x-p|^2}&\left(4K^2\sum_{i,j=1}^n a_{ij}(x)(x_i-p_i)(x_j-p_j)-2K\sum_{i=1}^n a_{ii}(x)\right)\\&-2Ke^{-K|x-p|^2}\sum_{i=1}^n b_i(x)(x_i-p_i)+c(x)v(x)\\&=4K^2e^{-K|x-p|^2}\left(\sum_{i,j=1}^n a_{ij}(x)(x_i-p_i)(x_j-p_j)\right)\\&-2Ke^{-K|x-p|^2}\left(\sum_{i=1}^n a_{ii}(x)+\sum_{i=1}^n b_i(x)(x_i-p_i)\right)\\&+e^{-K|x-p|^2}\left(c(x)-c(x)e^{K(|x-p|^2-|x'-p|^2)}\right)\\&=e^{-K|x-p|^2}(4K^2M(x)-2KN(x)+P(x)),\end{aligned}$$

where

$$\begin{aligned}M(x)&:=\sum_{i,j=1}^n a_{ij}(x)(x_i-p_i)(x_j-p_j);\\N(x)&:=\sum_{i=1}^n a_{ii}(x)+\sum_{i=1}^n b_i(x)(x_i-p_i);\\P(x)&:=c(x)-c(x)e^{K(|x-p|^2-|x'-p|^2)}.\end{aligned}$$

Since $\overline{B(x',\delta')}$ is compact and M,N and P are continuous, we can find constants $\tilde{M},\tilde{N},\tilde{P}\in\mathbb{R}$, independent of K, such that, for every $x\in B(x',\delta')$, $M(x)\geq\tilde{M}$, $N(x)\leq\tilde{N}$ and $P(x)\geq\tilde{P}$, where $\tilde{M}>0$ since L is uniformly elliptic on $B(x',\delta')$. Therefore,

$$Lv(x)\geq e^{-K|x-p|^2}(4K^2\tilde{M}-2K\tilde{N}+\tilde{P}),\quad\forall x\in\overline{B(x',\delta')},$$

Note that $P(x)$ is bounded below by a constant independent of K, given by

$$c_1 := \min\{c(x); x \in \overline{B(x', \delta')}\}.$$

Thus, if $c \equiv 0$ in equation (A.1.1), we can take $\tilde{P} \equiv 0$.

So, since $\tilde{M} > 0$, we can find $K > 0$ such that $4K^2\tilde{M} - 2K\tilde{N} + \tilde{P} > 0$ and thus

$$Lv(x) > 0, \text{ for all } x \in \overline{B(x', \delta')}.$$

Now, for all $t > 0$ we have

$$L(u + tv) = Lu + tLv \geq tLv > 0 \text{ in } B(x', \delta'),$$

since $Lu \geq 0$ on Ω.

Let $\partial B(x', \delta') = F_1 \cap F_2$, where $F_1 = \partial B(x', \delta') \cap \overline{B(p, |x' - p|)}$ and $F_2 = \partial B(x', \delta') - F_1$ (see Fig. A.14). Note that

a. if $x \in F_2$, then $|x - p| > |x' - p|$ and so $v(x) < 0$;

b. if $x \in F_1$, then $u(x) < u(x_0)$.

Putting together items a and b and the fact that F_1 is compact, we can find $t > 0$ such that

$$u(x) + tv(x) < u(x_0), \quad \text{for all} \quad x \in \partial B(x', \delta'). \tag{A.1.5}$$

Let $z \in \overline{B(x', \delta')}$ the maximum of $u + tv$ on $\overline{B(x', \delta')}$, that is

$$u(z) + tv(z) \geq u(x) + tv(x) \text{ for all } x \in \overline{B(x', \delta')}.$$

Since $u(z) + tv(z) \geq u(x') + tv(x') = u(x') = u(x_0)$, we conclude from (A.1.5) that $z \in B(x', \delta')$ and it is a maximum (non-negative maximum, if $c \leq 0$) of $u + tv$ on $\overline{B(x', \delta')}$.

Applying Theorem A.1.2 for $u + tv$ in $B(x', \delta')$, we get a contradiction. Thus, there exists $r > 0$ such that $u(x) = u(x_0)$ for all $x \in B(x_0, r)$. □

Corollary A.1.4. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $L : C^2(\Omega) \to C^0(\Omega)$ be a locally uniformly elliptic second-order linear differential operator, as in equation (A.1.1). Assume that $Lu \geq 0$ and $u \leq 0$ on Ω. Let $x_0 \in \Omega$ be a point such that $u(x_0) = 0$. Then $u \equiv 0$ on $\overline{\Omega}$.

Proof. Consider the following set

$$U = \{x \in \Omega; u(x) = 0\}.$$

By the continuity of u, U is a closed subset of Ω. Moreover, $U \neq \emptyset$, since $x_0 \in U$. If we can proof that U is also a open subset of Ω we conclude that $U = \Omega$, and by continuity, $u \equiv 0$ on $\overline{\Omega}$. For this purpose, consider $L_0 : C^2(\Omega) \to C^0(\Omega)$ given by $L_0 = L + (q - c)Id$, where $q(x) = \min\{c(x), 0\}$. That is,

$$L_0 v(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial v}{\partial x_i}(x) + q(x)v(x).$$

Then L_0 is also a locally uniformly elliptic second-order linear differential operator and since $q - c \leq 0$ on Ω, we have that

$$L_0 u = Lu + (q - c)u \geq 0 \quad \text{on} \quad \Omega$$

Thus, if $y \in U$, then y is a local maximum of u and applying Theorem A.1.3 we conclude that there exists $\epsilon > 0$ such that $u \equiv u(y) = u(x_0)$ on $B(y, \epsilon)$. Therefore U is open. □

Remark A.1.5. Using the arguments of the proof of Corollary A.1.4 we can show that in Theorem A.1.3, the neighborhood of x_0 where u is constant coincides with the connected component of the neighborhood where x_0 is a local maximum of u. This remark will be useful later.

Theorem A.1.6. (Boundary Maximum Principle) Let $\Omega \subset \mathbb{R}^n$ be a domain such that $\partial\Omega$ is smooth and let $L : C^2(\Omega) \to C^0(\Omega)$ be a locally uniformly elliptic second-order linear differential operator, as in equation (A.1.1). Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu \geq 0$ on Ω. Let $x_0 \in \partial\Omega$ so that

- u is C^1 at x_0;
- $u(x_0) \geq u(x)$ for all $x \in \Omega \cap B(x_0, \epsilon)$, for some $\epsilon > 0$;
- $\frac{\partial u}{\partial \eta}(x_0) \geq 0$, where η is the inward normal to $\partial\Omega$.

Then,

i. If $c \equiv 0$, u is constant in a neighborhood of x_0;

ii. If $c \leq 0$ and $u(x_0) \geq 0$, u is constant in a neighborhood of x_0.

Proof. Let us assume, by contradiction, that u is non-constant in any neighborhood of x_0. Since $\partial\Omega$ is smooth, there exists $\rho > 0$ and $x' \in \Omega$ such that $B(x', \rho) \subset \Omega$ and $\partial B(x', \rho) \cap \partial\Omega = \{x_0\}$.

Let $0 < \rho' < \min\{\rho, \epsilon\}$ be such that $u(x) \leq u(x_0)$ for all $x \in B(x_0, \rho') \cap (\Omega \cup \{x_0\})$. Consider the compact set

$$K := \{x \in \Omega; |x - x_0| \leq \rho', |x - x'| \leq \rho\} = \overline{B(x_0, \rho')} \cap \overline{B(x', \rho)}$$

and define

$$v(x) = e^{-\delta|x-x'|^2} - e^{-\delta|x'-x_0|^2}, \qquad x \in K,$$

where δ is a constant to be determined.

In the same way we did in the proof of Theorem A.1.3, we can find $\delta > 0$ such that

$$Lv(x) > 0, \quad \text{for all} \quad x \in K.$$

Take $0 < \tilde{\rho} < \rho'$ so that $B(x_0, \tilde{\rho}) \cap B(x', \tilde{\rho}) = \emptyset$ and consider

$$\tilde{K} = \{x \in \Omega; |x - x_0| \leq \tilde{\rho}, |x - x'| \leq \rho\} = \overline{B(x_0, \tilde{\rho})} \cap \overline{B(x', \rho)}$$

and write $\partial\tilde{K} = F_1 \cup F_2$, where $F_1 = \overline{\partial\tilde{K} \cap B(x', \rho)}$ and $F_2 = \partial\tilde{K} - F_1$ (see Fig. A.15). Therefore, we have

i. If $x \in F_2$, $|x - x'| = \rho = |x_0 - x'|$. Therefore $v(x) = 0$.

ii. If $x \in F_1$, then $u(x) < u(x_0)$. Indeed, let us suppose that $u(x) = u(x_0)$ for some $x \in F_1$. Since $F_1 \subset B(x_0, \epsilon) \cap \Omega$ and, by hypothesis, $u(y) \leq u(x_0)$ for all $y \in B(x_0, \epsilon) \cap \Omega$, we conclude that x would be a maximum for u restricted to $B(x_0, \epsilon) \cap \Omega$. Then, applying Theorem A.1.3 with Remark A.1.5, $u \equiv u(x_0)$ is constant in $B(x_0, \epsilon) \cap \Omega$, which contradicts our assumption that u is non-constant on any neighborhood of x_0.

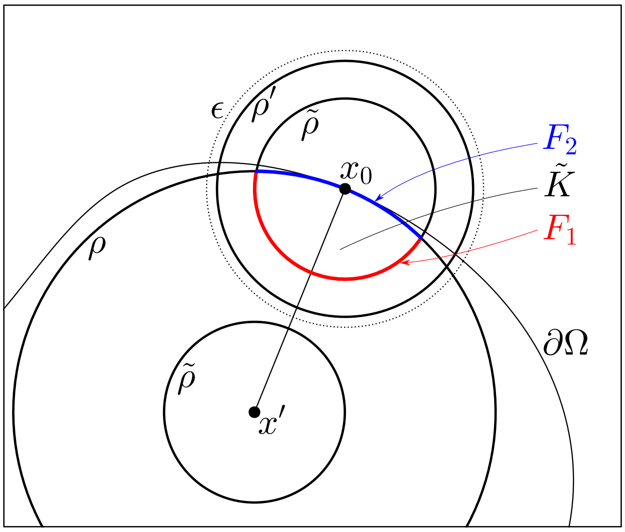

Figure A.15: Auxilliary illustration for the proof of Theorem A.1.6.

Thus, since F_1 is compact, from item ii we can choose $t > 0$ such that

$$u(x) + tv(x) \leq u(x_0), \quad \text{for all} \quad x \in F_1.$$

Putting this information together with item i, we conclude that

$$u(x) + tv(x) \leq u(x_0), \quad \text{for all} \quad x \in \partial\tilde{K}.$$

Define $\xi(x) = u(x) + tv(x) - u(x_0)$, for $x \in \tilde{K}$. We have that

$$L\xi = Lu + tLv - cu(x_0) > 0,$$

since $Lu \geq 0$ on K by hypothesis and we know that $Lv > 0$ on K. We have also that $\xi \leq 0$ on $\partial\tilde{K}$ and $\xi(x_0) = 0$. Thus, applying Theorem A.1.2 we conclude that

$$\xi(x) \leq 0, \quad \text{for all} \quad x \in \tilde{K}.$$

Using this information and the fact that $s \mapsto \xi(x_0 + s\eta)$ has a right derivative at $s = 0$ (because u is C^1 at x_0), we conclude that $\frac{\partial \xi}{\partial \eta}(x_0) \leq 0$. Thus,

$$t\frac{\partial v}{\partial \eta}(x_0) = \frac{\partial \xi}{\partial \eta}(x_0) - \frac{\partial u}{\partial \eta}(x_0) \leq 0,$$

since $\frac{\partial u}{\partial \eta}(x_0) \geq 0$, by hypothesis. On the other hand, computing $t\frac{\partial v}{\partial \eta}(x_0)$ we get:

$$t\frac{\partial v}{\partial \eta}(x_0) = -2t\delta\langle \eta, x_0 - x'\rangle e^{-\delta|x_0 - x'|^2} > 0,$$

since $x_0 - x'$ is orthogonal to $\partial\Omega$ at x_0 pointing outward. Thus we get a contradiction. □

Corollary A.1.7. Let $\Omega \subset \mathbb{R}^n$ be a domain such that $\partial\Omega$ is smooth and let $L : C^2(\Omega) \to C^0(\Omega)$ be a locally uniformly elliptic second-order linear differential operator, as in equation (A.1.1). Assume that $Lu \geq 0$ and $u \leq 0$ on Ω. Let $x_0 \in \partial\Omega$ be a point such that $u(x_0) = 0$. Assume also that u is C^1 at x_0 and $\frac{\partial u}{\partial \eta}(x_0) \geq 0$, where η is the normal vector of $\partial\Omega$ pointing inward. Then $u \equiv 0$ on $\overline{\Omega}$.

Proof. Consider $U = \{x \in \Omega; u(x) = 0\}$. The same arguments of the proof of Corollary A.1.4 shows that U is an open and closed subset of Ω. We need only guarantee that $U \neq \emptyset$. Let L_0 be as in the proof of Corollary A.1.4. Applying Theorem A.1.6 to L_0 we conclude that there exists $\epsilon > 0$ such that u is constant and equals to $u(x_0)$ on $B(x_0, \epsilon) \cap \Omega$. This implies that $U \neq 0$, since $u(x_0) = 0$ and x_0 is a boundary point of Ω. □

A.2 Quasilinear second-order partial differential operators

Now we consider quasilinear second order differential operators, that is, differential operators that are linear on the second order terms. Our objective is to extend the Maximum Principles of the previous section to this kind of differential operator.

Let $\Omega \subset \mathbb{R}^n$ be a domain and consider the quasilinear second order differential operator $Q : C^\infty(\Omega) \to C^\infty(\Omega)$ of the form:

$$Qu := \sum_{i,j=1}^{n} a_{ij}(x, u, \nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + b(x, u, \nabla u), \tag{A.2.1}$$

where a_{ij}, b are, for every $i, j \in \{1, \ldots, n\}$, continuous real-valued functions defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Here ∇u denotes the gradient of u, that is, $\nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})$, and we suppose again that the second order coefficients are symmetric, that is, $a_{ij} = a_{ji}$, for every $i, j \in \{1, \ldots, n\}$.

Definition A.2.1. The operator Q given in A.2.1 is said to be:

i. **elliptic** at $(x, p, q) \in \Omega \times \mathbb{R}^n \times \mathbb{R}$ if the matrix $[a_{ij}(x, p, q)]$ is positive definite. If this condition is valid for all $(x, p, q) \in \Omega \times \mathbb{R}^n \times \mathbb{R}$ we say just that Q is elliptic;

ii. **uniformly elliptic** if the eigenvalues of $[a_{ij}(x, p, q)]$ are bounded below and above by a positive constant;

iii. **locally uniformly elliptic** if for each $(x, p, q) \in \Omega$ there exists a neighborhood U of (x, p, q) so that L is uniformly elliptic.

In other words, if $\{\lambda_1(x, p, q), \ldots, \lambda_n(x, p, q)\}$ are the eigenvalues of $[a_{ij}(x, p, q)]$ and

$$\lambda(x, p, q) = \min \lambda_i(x, p, q); 1 \leq i \leq n,$$
$$\Lambda(x, p, q) = \max \lambda_i(x, p, q); 1 \leq i \leq n,$$

then Q is elliptic if and only if

$$0 < \lambda(x, p, q) \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, p, q)\xi_i\xi_j \leq \Lambda(x, p, q) \sum_{i=1}^{n} \xi_i^2, \tag{A.2.2}$$

for every $(x, p, q) \in \Omega \times \mathbb{R}^n \times \mathbb{R}$ and for every $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n - \{0\}$. In the sense of equation (A.2.2), Q is uniformly elliptic if and only if $0 < \lambda \leq \lambda(x, p, q) \leq \Lambda(x, p, q) \leq \Lambda$ for some positive numbers λ and Λ. Thus, L is locally uniformly elliptic if for any $(x_0, p_0, q_0) \in \Omega \times \mathbb{R}^n \times \mathbb{R}$ there exist a neighborhood U of (x_0, p_0, q_0) and positive constants $\lambda(x_0, p_0, q_0)$ and $\Lambda(x_0, p_0, q_0)$ such that

$$\lambda(x_0, p_0, q_0)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, p, q)\xi_i\xi_j \leq \Lambda(x_0, p_0, q_0)|\xi|^2, \quad \forall (x, p, q) \in U, \quad \forall \xi \in \mathbb{R}^n.$$

Lemma A.2.2. (Hadamard Lemma) Let $U \subset \mathbb{R}^n$ be a convex domain and let $f : U \to \mathbb{R}$ be a C^1 function. Then

$$f(x) - f(y) = \sum_{i=1}^{n} h_i(x, y)(x_i - y_i)$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and

$$h_i(x, y) = \int_0^1 \frac{\partial f}{\partial x_i}(sx + (1-s)y)ds,$$

for all $i \in \{1, \dots, n\}$ and for all $x, y \in U$.

Proof. Given $x, y \in U$, consider the function $g : [0,1] \to U$ given by $g(s) = f(sx+(1-s)y)$. Since U is convex, g is well defined. Using the Chain's Rule and the Fundamental Theorem of Calculus, we have

$$\begin{aligned} f(x) - f(y) = g(1) - g(0) &= \int_0^1 g'(s)ds \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(sx + (1-s)y)(x_i - y_i)ds \\ &= \sum_{i=1}^n h_i(x, y)(x_i - y_i). \end{aligned}$$

□

Lemma A.2.3. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $u^k : C^2(\Omega) \cap C^0(\overline{\Omega})$, $k = 1, 2$ be two functions that satisfies $Qu^2 \geq Qu^1$ on $\overline{\Omega}$, where Q is an second-order quasi-linear differential operator given as in equation (A.2.1).

Then $u := u^2 - u^1$ satisfies an elliptic linear equation $Lu \geq 0$, where L is an elliptic second-order linear differential operator. Moreover, if Q is locally uniformly elliptic, then L is also locally uniformly elliptic

Proof. Define $\phi : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ by

$$\phi(x, z, p, q) = \sum_{i,j=1}^n a_{ij}(x, z, p)q_{ij} + b(x, z, p),$$

where $p = (p_1, \dots, p_n) \in \mathbb{R}^n$, $q = (q_{11}, \dots, q_{1n}, \dots, q_{n1}, \dots, q_{nn}) \in \mathbb{R}^n$ and $x = (x_1, \dots, x_n) \in \Omega$.

Since $Q(u^2) \geq Q(u^1)$, we have

$$\phi(x, u^2(x), p^2(x), q^2(x)) - \phi(x, u^1(x), p^1(x), q^1(x)) \geq 0, \quad \forall x \in \Omega, \tag{A.2.3}$$

where

$$\begin{aligned} &q^k(x) = (\partial^2_{11}u^k(x), \dots, \partial^2_{1n}u^k(x), \dots, \partial^2_{n1}u^k(x), \dots, \partial^2_{nn}u^k(x)), \quad \text{and} \\ &p^k(x) = (\partial_1 u^k(x), \dots, \partial_n u^k(x)), \quad \text{for} \quad k = 1, 2. \end{aligned}$$

Using Hadamard Lemma, we can rewrite inequality (A.2.3) as

$$\sum_{i,j=1}^n A_{ij}(x)(\partial^2_{ij}u^2(x) - \partial^2_{ij}u^1(x)) + \sum_{i=1}^n B_i(x)(\partial_i u^2(x) - \partial_i u^1(x)) \tag{A.2.4}$$

$$+ C(x)(u^2(x) - u^1(x)) \geq 0, \tag{A.2.5}$$

for all $x \in \Omega$, where

$$\begin{aligned}
A_{ij}(x) &= \int_0^1 \frac{\partial \phi}{\partial q_{ij}}(x, su^2(x) + (1-s)u^1(x), s \operatorname{grad} u^2(x) + (1-s) \operatorname{grad} u^1(x))ds; \\
&= \int_0^1 a_{ij}(x, su^2(x) + (1-s)u^1(x), s \operatorname{grad} u^2(x) + (1-s) \operatorname{grad} u^1(x))ds; \\
B_i(x) &= \int_0^1 \frac{\partial \phi}{\partial p_i}(x, su^2(x) + (1-s)u^1(x), s \operatorname{grad} u^2(x) + (1-s) \operatorname{grad} u^1(x))ds; \\
C(x) &= \int_0^1 \frac{\partial \phi}{\partial z}(x, su^2(x) + (1-s)u^1(x), s \operatorname{grad} u^2(x) + (1-s) \operatorname{grad} u^1(x))ds,
\end{aligned}$$

where $\frac{\partial\phi}{\partial q_{ij}}$, $\frac{\partial\phi}{\partial p_i}$ and $\frac{\partial\phi}{\partial z}$ denotes the partial derivatives of $\phi(x, z, p, q)$ in the directions $(0, 0, 0, e_{ij})$, $(0, 0, e_i, 0)$ and $(0, 1, 0, 0)$, respectively.

Defining $L : C^2(\Omega) \to C^0(\Omega)$ by

$$Lu(x) = \sum_{i,j=1}^{n} A_{ij}(x)\partial^2_{ij}u(x) + \sum_{i=1}^{n} B_i(x)\partial_i u(x) + C(x)u(x) = 0,$$

we get a second-order linear differential operator, and by equation (A.2.5) we conclude that $u := u^2 - u^1$ satisfies $Lu \geq 0$.

Denoting by $\tau(x, s) = (x, su^2(x) + (1-s)u^1(x), s \operatorname{grad} u^2(x) + (1-s) \operatorname{grad} u^1(x))$, since Q is elliptical, we have that

$$\lambda(\tau(x, s))|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(\tau(x, s))\xi_i\xi_j \leq \Lambda(\tau(x, s))|\xi|^2.$$

Therefore

$$\int_0^1 \lambda(\tau(x, s))ds|\xi|^2 \leq \sum_{i,j=1}^{n} \int_0^1 a_{ij}(\tau(x, s))ds\xi_i\xi_j = \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j \leq \sum_{i,j=1}^{n} \Lambda(\tau(x, s))ds|\xi|^2,$$

and since $\lambda(\tau(x, s))$ and $\Lambda(\tau(x, s))$, we conclude that $\int_0^1 \lambda(\tau(x, s))ds$ and $\int_0^1 \Lambda(\tau(x, s))ds$ are also positive, whence we deduce that L is elliptic. From these arguments we also conclude that L is locally uniformly elliptic when Q is locally uniformly elliptic. □

Theorem A.2.4. Let $u^1, u^2 : \Omega \to \mathbb{R}$ be two functions of class C^2 defined on an open domain Ω of either $\mathbb{R}^n$ or the half-space $\{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_n \geq 0\}$ such that $0 \in \Omega$. Assume also that $Qu^2 \geq Qu^2$, where Q is a locally uniformly elliptic second-order quasi-linear differential operator given as in equation (A.2.1). Then,

i. if 0 is an interior point at Ω, $u^2(0) = u^1(0)$ and $u^2 \leq u^1$ in Ω, then $u^2 \equiv u^1$ in Ω;

ii. if 0 is a boundary point at Ω, $u^2(0) = u^1(0)$, $\frac{\partial u^2}{\partial x_n}(0) = \frac{\partial u^1}{\partial x_n}(0)$ and $u^2 \leq u^1$ in Ω, then $u^2 \equiv u^1$ in Ω.

Proof. Consider $u := u^2 - u^1$. Applying Lemma A.2.3 we conclude that there exists an elliptic second order differential operator L such that $Lu \geq 0$. The conclusion follows from applying Corollary A.1.4 in item [i] and Corollary A.1.7 in item [ii]. □

A.3 The Tangency Principle for anisotropic mean curvatures

In this section we present a geometric interpretation of the Maximum Principle, known as the Geometric Tangency Principle, that is the main tool used along this thesis. We begin showing an explicit form of the anisotropic mean curvature for graphs. After it, we recall the notions of interior tangency point and boundary tangency point between two surfaces. Finally, we present the Geometric Tangency Principle applied for anisotropic mean curvatures.

Let $F : \mathbb{S}^2 \to \mathbb{R}$ be a smooth, positive function and consider $\tilde{F} : \mathbb{R}^3 - \{0\} \to \mathbb{R}$ its homogeneous extension, that is

$$\tilde{F}(x) = |x| F\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^3 - \{0\},$$

and consider $\psi : (u,v) \in \Omega \mapsto (u, v, h(u,v)) \in \mathbb{R}^3$ the parametrization of the graph of a function h defined over some domain $\Omega \subset \mathbb{R}^2$. We may expand the expression

$$\Lambda(x) = -\operatorname{div}_\Sigma((\operatorname{grad}_{\mathbb{S}^n} F) \circ N)(x) + nF(N(x))H(x)$$

in terms of $\tilde{F}$ and h. Thus ψ has anisotropic mean curvature Λ if and only if h satisfies the following PDE:

$$\begin{aligned}\Lambda = {} & A_0(p,q)\tilde{F}(N) + A_1(p,q)\tilde{F}_x(N) + A_2(p,q)\tilde{F}_y(N) + A_3(p,q)\tilde{F}_z(N) \\ & + A_{11}(p,q)\tilde{F}_{xx}(N) + A_{12}(p,q)\tilde{F}_{xy}(N) + A_{13}(p,q)\tilde{F}_{xz}(N) \\ & + A_{22}(p,q)\tilde{F}_{yy}(N) + A_{23}(p,q)\tilde{F}_{yz}(N) + A_{33}(p,q)\tilde{F}_{zz}(N),\end{aligned} \tag{A.3.1}$$

where N is the Gauss map of ψ, $p = h_u$, $q = h_v$, $r = h_{uu}$, $s = h_{uv}$, $t = h_{vv}$ and

$$\begin{aligned} A_k(p,q) &:= a_k(p,q)r + 2b_k(p,q)s + c_k(p,q)t \\ A_{kl}(p,q) &:= a_{kl}(p,q)r + 2b_{kl}(p,q)s + c_{kl}(p,q)t \end{aligned}$$

with

$$\begin{array}{lll}
a_0(p,q) = \frac{1+q^2}{(1+p^2+q^2)^{\frac{3}{2}}} & b_0(p,q) = \frac{-pq}{(1+p^2+q^2)^{\frac{3}{2}}} & c_0(p,q) = \frac{1+p^2}{(1+p^2+q^2)^{\frac{3}{2}}} \\
a_1(p,q) = \frac{p\left(1+q^2\right)}{(1+p^2+q^2)^2} & b_1(p,q) = \frac{-p^2q}{(1+p^2+q^2)^2} & c_1(p,q) = \frac{p\left(1+p^2\right)}{(1+p^2+q^2)^2} \\
a_2(p,q) = \frac{q\left(1+q^2\right)}{(1+p^2+q^2)^2} & b_2(p,q) = \frac{-pq^2}{(1+p^2+q^2)^2} & c_2(p,q) = \frac{q\left(1+p^2\right)}{(1+p^2+q^2)^2} \\
a_3(p,q) = \frac{-\left(1+q^2\right)}{(1+p^2+q^2)^2} & b_3(p,q) = \frac{pq}{(1+p^2+q^2)^2} & c_3(p,q) = \frac{-\left(1+p^2\right)}{(1+p^2+q^2)^2}
\end{array}$$

and

$$\begin{array}{lll}
a_{11}(p,q) = \frac{\left(1+q^2\right)^2}{(1+p^2+q^2)^{\frac{5}{2}}} & b_{11}(p,q) = \frac{-pq\left(1+q^2\right)}{(1+p^2+q^2)^{\frac{5}{2}}} & c_{11}(p,q) = \frac{p^2q^2}{(1+p^2+q^2)^{\frac{5}{2}}} \\
a_{12}(p,q) = \frac{-2pq\left(1+q^2\right)}{(1+p^2+q^2)^{\frac{5}{2}}} & b_{12}(p,q) = \frac{2p^2q^2+1+p^2+q^2}{(1+p^2+q^2)^{\frac{5}{2}}} & c_{12}(p,q) = \frac{-2pq\left(1+p^2\right)}{(1+p^2+q^2)^{\frac{5}{2}}} \\
a_{13}(p,q) = \frac{2p\left(1+q^2\right)}{(1+p^2+q^2)^{\frac{5}{2}}} & b_{13}(p,q) = \frac{q\left(1-p^2+q^2\right)}{(1+p^2+q^2)^{\frac{5}{2}}} & c_{13}(p,q) = \frac{-2pq^2}{(1+p^2+q^2)^{\frac{5}{2}}} \\
a_{22}(p,q) = \frac{p^2q^2}{(1+p^2+q^2)^{\frac{5}{2}}} & b_{22}(p,q) = \frac{-pq\left(1+p^2\right)}{(1+p^2+q^2)^{\frac{5}{2}}} & c_{22}(p,q) = \frac{\left(1+p^2\right)^2}{(1+p^2+q^2)^{\frac{5}{2}}} \\
a_{23}(p,q) = \frac{-2p^2q}{(1+p^2+q^2)^{\frac{5}{2}}} & b_{23}(p,q) = \frac{p\left(1+p^2-q^2\right)}{(1+p^2+q^2)^{\frac{5}{2}}} & c_{23}(p,q) = \frac{2q\left(1+p^2\right)}{(1+p^2+q^2)^{\frac{5}{2}}} \\
a_{33}(p,q) = \frac{p^2}{(1+p^2+q^2)^{\frac{5}{2}}} & b_{33}(p,q) = \frac{pq}{(1+p^2+q^2)^{\frac{5}{2}}} & c_{33}(p,q) = \frac{q^2}{(1+p^2+q^2)^{\frac{5}{2}}}
\end{array}$$

It is important to notice that equation (A.3.1) is a quasi-linear second-order PDE whose coefficients of first and second order are smooth functions that do not depend on h, but only on its first derivatives. Since we are assuming that $D^2F + F \cdot Id$ is positive definite, it is possible to show that such PDE is also elliptic. Thus, we may apply the results of the previous section. For this purpose, we present now the notions of tangency and boundary interior point between two surfaces.

Definition A.3.1. Let Σ_1 and Σ_2 be two immersed surfaces in $\mathbb{R}^3$. Let $p \in \Sigma_1 \cap \Sigma_2$ be an intersection point such that:

- p is an interior point for both Σ_1 and Σ_2;
- $T_p\Sigma_1 = T_p\Sigma_2$;
- the unit normal vectors of Σ_1 and Σ_2 at p coincide.

In these conditions, we say that p is an **interior tangent point** of Σ_1 and Σ_2.

Definition A.3.2. Let Σ_1 and Σ_2 be two surfaces with boundary immersed in $\mathbb{R}^3$. Let $p \in \Sigma_1 \cap \Sigma_2$ be an intersection point such that:

- p is a boundary point for both Σ_1 and Σ_2, that is, $p \in \partial\Sigma_1 \cap \partial\Sigma_2$;
- $T_p\Sigma_1 = T_p\Sigma_2$;
- the unit normal vectors of Σ_1 and Σ_2 at p coincide.
- the interior co-normal vectors of Σ_1 and Σ_2 at p coincide.

In these conditions, we say that p is an **boundary tangent point** of Σ_1 and Σ_2.

Let Σ be an immersed surface of $\mathbb{R}^3$, with or without boundary, and let $p \in \Sigma$ be one of its points. Denote by $T_p\Sigma$ the tangent plane of Σ at p. In a sufficiently small domain Ω of $T_p\Sigma$ (that contains the origin, if p is an interior point of Σ or such that $p \in \partial\Omega$, if p is a boundary point of Σ) we can write Σ as the graph of a smooth function $h : \Omega \subset T_p\Sigma \to \mathbb{R}$. More precisely, if $N(p)$ is a unit vector orthogonal to Σ at p and $\{E_1, E_2\}$ is an orthonormal basis of $T_p\Sigma$, then in a small neighborhood of p, Σ is parametrized in the following form:

$$(x, y) \in \Omega \mapsto xE_1 + yE_2 + h(x, y)N(p), \tag{A.3.2}$$

for some smooth function $h : \Omega \subset T_p\Sigma \to \mathbb{R}$.

Now, consider two surfaces Σ_1 and Σ_2 (with or without boundary) immersed in $\mathbb{R}^3$. Assume that $p \in \Sigma_1 \cap \Sigma_2$ is an interior or boundary tangent point, according to Definitions A.3.1 and A.3.2. Since the normal vectors and tangent planes of Σ_1 and Σ_2 coincide at p, we can write both Σ_1 and Σ_2 as a graph of smooth functions $h_1, h_2 : \Omega \to \mathbb{R}$ defined in the same domain through equation (A.3.2). Thus, we can compare Σ_1 and Σ_2 through the following definition:

Definition A.3.3. We say that Σ_1 is **above** (resp. **below**) Σ_2 around p if $h_1 \geq h_2$ (resp. $h_1 \leq h_2$) in a neighborhood of $(0, 0) \in \Omega$, and we denote this by $\Sigma_2 \leq \Sigma_1$ (resp. $\Sigma_1 \leq \Sigma_2$).

Theorem A.3.4. Let $F : \mathbb{S}^2 \to \mathbb{R}$ be a smooth, positive function that satisfies the convexity condition: $D^2F(x) + F(x)Id$ is positive definite for all $x \in \mathbb{S}^2$. Let Σ_1 and Σ_2 be two immersed surfaces in $\mathbb{R}^3$ and let $p \in \Sigma_1 \cap \Sigma_2$ be either an interior or a boundary tangency point between Σ_1 and Σ_2. Assume that:

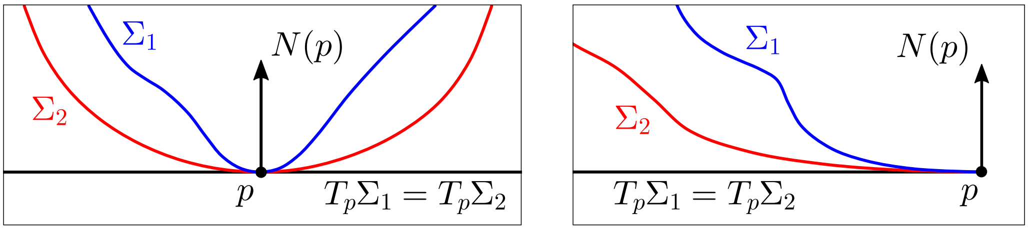

Figure A.16: The interior and boundary versions of the Tangency Principle.

i. $\Sigma_1 \geq \Sigma_2$ around p;

ii. $\Lambda_1 \leq \Lambda_2$ around p.

Then Σ_1 and Σ_2 agree on a neighborhood of p. In particular, $\Lambda_1 = \Lambda_2$ in this neighborhood.

Proof. First of all, we recall that the anisotropic mean curvature operator (that we call Q in this proof) is quasi-linear (see equation (A.3.1)), and since we are assuming that F satisfies the convexity condition, it is also locally uniformly elliptic.

For simplicity we assume that $p = (0, 0, 0)$, the tangent planes of Σ_1 and Σ_2 at p coincide with the plane $\{z = 0\}$, that we identify with $\mathbb{R}^2$, and the normals of Σ_1 and Σ_2 at p coincide with $e_3 = (0, 0, 1)$. For a sufficiently small domain Ω of $\mathbb{R}^2$ that contains the origin if p is an interior tangency point or $0 \in \partial\Omega$ if p is a boundary tangency point, we consider smooth real-valued functions u_1 and u_2 defined on $\overline{\Omega}$ whose graphs coincide around p with Σ_1 and Σ_2, respectively. The hypothesis implies that $u_2 \leq u_1$ on $\overline{\Omega}$ and also that $Qu_1 \leq Qu_2$. Define $u = u_2 - u_1$. Then $u \leq 0$ on $\overline{\Omega}$ and by Theorem A.2.4 we conclude that there exists an uniformly elliptic second-order linear differential operator L, as in equation (A.1.1), such that $Lw \geq 0$. Moreover, since Q does not depend on u, by the arguments in the proof of Lemma A.2.3 we deduce that $c \equiv 0$ in equation (A.1.1).

We need to consider two cases:

Case 1: p is an interior tangency point between Σ_1 and Σ_2. In this case the origin is a local maximum of u and applying Corollary A.1.4, we conclude that $u \equiv 0$ on $\overline{\Omega}$, that is, $\Sigma_1 = \Sigma_2$ around p.

Case 2: p is a boundary tangency point between Σ_1 and Σ_2. By Definition A.3.2 the normals and interior co-normals of Σ_1 and Σ_2 coincide at p, whence we deduce that

$$\frac{\partial u}{\partial \eta}(0) = \frac{\partial u_2}{\partial \eta}(0) - \frac{\partial u_1}{\partial \eta}(0) = 0.$$

where η is the inward co-normal vector of $\overline{\Omega}$ at the origin. Since $u(0) = 0$, applying Corollary A.1.7 we conclude again that $u \equiv 0$ on $\overline{\Omega}$. □

A.4 Other theorems

Theorem A.4.1. (Ascoli–Arzelà) Let K be a compact metric space and let $\mathcal{F}$ be a bounded subset of $C^0(K)$. Assume that $\mathcal{F}$ is uniformly equicontinuous, that is,

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon, \quad \forall f \in \mathcal{F}.$$

Then the closure of $\mathcal{F}$ in $C^0(K)$ is compact.

Definition A.4.2. Let $U \subset \mathbb{R}^n$ be a bounded open domain. We denote by $C^{k;\alpha}(\overline{U})$, $(0 < \alpha \leq 1)$ the space consisting of functions $f \in C^k(\overline{U})$ satisfying $[f]_{k;\alpha;U} < \infty$. This space is indeed a Banach space equipped with the norm

$$\|f\|_{k;\alpha;U} := \|f\|_{k;U} + [f]_{k;\alpha;U}.$$

Theorem A.4.3. (Interior Schauder Estimates for Classical Solutions) For $\alpha \in (0,1)$, let $u \in C^{2;\alpha}(U)$ be a solution of an second-order linear equation

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x),$$

under the additional assumptions

i. Ellipticity of L: form some $0 < \lambda \leq \Lambda$,

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \text{for all} \quad x \in U \quad \text{and} \quad \xi \in \mathbb{R}^n; \tag{A.4.1}$$

ii. $C^{0;\alpha}$-boundedness of the coefficients: $a_{ij}, b_i, c \in C^{0;\alpha}(\overline{U})$, for some $\alpha \in (0,1)$ and

$$\frac{1}{\lambda}\left(\sum_{i,j=1}^{n} \|a_{ij}\|_{0,\alpha,U} + \sum_{i=1}^{n} \|b_i\|_{0,\alpha,U} + \|c\|_{0,\alpha,U}\right) \leq \Lambda_\alpha. \tag{A.4.2}$$

Then for $U' \subset\subset U$, we have

$$\|u\|_{2;\alpha;U'} \leq C\left(\frac{1}{\lambda}\|f\|_{\alpha;U} + \|u\|_{0;U}\right),$$

where C depends only on $n, \alpha, \frac{\Lambda}{\lambda}; \Lambda_\alpha$ and $\text{dist}(U'; \partial U)$.

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