

Universidade Federal de São Carlos
Programa de Pós-Graduação em Matemática

Symmetric and Non-Symmetric Jack Functions

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São Carlos 2020

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Dissertação apresentada ao Programa de Pós-Graduação em Matemática da Universidade Federal de São Carlos como parte dos requisitos para obtenção do título de Mestre em Matemática.

São Carlos 2020



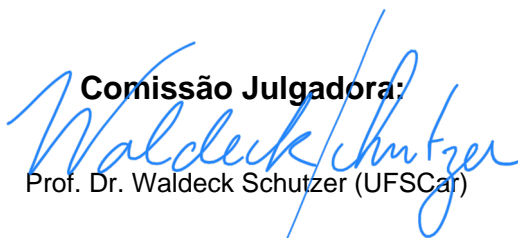
UNIVERSIDADE FEDERAL DE SÃO CARLOS

Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Matemática

Folha de Aprovação

Defesa de Dissertação de Mestrado do candidato Felipe de Mattos Chafik Hindi, realizada em 29/05/2020.

Comissão Julgadora:



Prof. Dr. Waldeck Schutzer (UFSCar)

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Prof. Dr. Adriano Adrega de Moura (UNICAMP)

O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Código de Financiamento 001.

O Relatório de Defesa assinado pelos membros da Comissão Julgadora encontra-se arquivado junto ao Programa de Pós-Graduação em Matemática.

Acknowledgements

I am tremendously thankful

to Waldeck Schützer, my advisor, for his patient mentorship and reassuring guidance;

to my mother and my advisor, for their attentive revisions to my dissertation;

to my brother, my parents and grandmother, for their monumental support;

to the Brazilian working class, whose collective surplus funds public academic and research institutions, making scientific and mathematical progress possible in my country;

to the international working class and every person or institution diligently pursuing the democratization of scientific literature access throughout the world;

to CAPES (Coordenação de Aperfeiçoamento Pessoal de Nível Superior) and MEC (Ministério da Educação) for the scholarship I was granted;

to UFSCar (Universidade Federal de São Carlos) and its department of mathematics for providing me with the necessary infrastructure to accomplish my academic pursuits.

Abstract

The goal of this dissertation is to present the theory of Jack functions¹ from the standpoint of algebraic combinatorics. The presentation of symmetric functions is largely centred on the first chapter of MacDonal’s “Symmetric Functions and Hall’s Polynomials” [7]. Symmetric and non-symmetric Jack functions are characterized by Sahi-Knop’s [4] combinatorial formulas. Moreover, Stanley’s Pieri-type rule [15] for symmetric Jack functions and Schultzer’s [13] Pieri-type rule for non-symmetric functions are thoroughly described.

Resumo

O objetivo dessa dissertação é apresentar a teoria de funções simétricas e de Jack (simétricas ou não), pela perspectiva da combinatória algébrica. A exposição das funções simétricas revolve extensamente sobre o primeiro capítulo do livro “Symmetric Functions and Hall’s Polynomials” [7] de MacDonal. Funções de Jack simétricas e não simétricas são caracterizadas conforme as fórmulas combinatoriais de Sahi e Knop [4]. Ademais, regras do tipo Pieri para funções de Jack simétricas, devida a Stanley [15] e não simétricas, devida a Schultzer [13], são minuciosamente descritas.

¹Symmetric and non-symmetric.

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1 Introduction

The study of symmetric polynomials dates from a long time. In 1882, they were studied by Kostka [5], who was the first to calculate numbers of semi-standard tableaux for determined shapes and weights, after whom they are named. In 1901 Schur showed in his doctorate thesis [12] that characters of irreducible representations of the $GL(n)$ and S_n are given by the family of symmetric polynomials which was later named after him. Later, Pieri provided a formula in the context of Schubert Calculus for the linear decomposition between Schur and complete and elementary² polynomials. Littlewood, Richardson (1934) and Robinson (1938) [6, 10] generalized Pieri's rule for the product of arbitrary Schur polynomials. More recently, in 2009, Pieri's formula was generalized for the product of a skew Schur polynomial and complete (or elementary) polynomials [1].

A parameter α is introduced to algebras of symmetric polynomials by setting the field of coefficients to be the field of rational polynomials $\mathbb{Q}(\alpha)$. Jack polynomials $J_\lambda(\alpha)$ are a family of polynomials which are a base (as a $\mathbb{Q}(\alpha)$ -module) for this new polynomial algebra with a parameter. They are particularly noteworthy for generalizing — up to normalization — known families of symmetric polynomials for different values of α . Namely: Schur (for $\alpha = 1$); conjugate elementary (for $\alpha = 0$); monomial (for $\alpha \rightarrow \infty$) and the two types of zonal polynomials (for $\alpha = 2$ and $\frac{1}{2}$).

Jack polynomials were first introduced by Henry Jack in 1970, after whom they were named, in the context of developing tools to carry out an integration problem [3]. They were later separately proved by Sekiguchi and Debiard [14, 2] to be equivalently defined as simultaneous eigenfunctions of linear operators which were then named after them. MacDonalld furthered the theory of Jack functions, providing, among other results, an inner-product definition of Jack functions [7]. He also contributed to Stanley's work [15], who proved the Pieri rule analogous for Jack functions and introduced skew Jack

²Which are Schur polynomials indexed by single row or single column partitions.

functions, among other results. In 1996, Sahi and Knop provided a purely combinatorial definition of Jack polynomials [4].

Non-symmetric analogues of Jack polynomials were introduced by Opdam in 1995, defined as the family of simultaneous eigenfunctions of Cherednik operators [8]. In the following year, two equivalent definitions were provided by Sahi and Knop in the form of a combinatorial and a recursive formula [4]. The first step towards a Pieri-like rule for non-symmetric polynomials was proved by Waldeck Schützer [13].

In the study of symmetric polynomials, it is noticeable that the (finite) specific number of variables of a polynomial algebra is not particularly relevant for most results concerning them. With the purpose of establishing a more general approach, the theory of symmetric polynomials is recontextualized in the framework of dimension-independent objects called symmetric functions, which can be thought of as polynomial-like objects with countably infinitely many variables³. Although no longer polynomials, symmetric functions are defined so as to make sense of sum and multiplication just like polynomials, effectively defining an algebra. This algebra (and so symmetric functions themselves) can be projected onto finite dimension polynomial algebras by zeroing all variables from some point on.

The first chapter is dedicated to an introduction to the theory of symmetric functions and unabashedly draws greatly from the first chapter of MacDonal’s book “Symmetric Functions and Hall Polynomials” [7], howbeit reformulating some of its arguments. Fundamental results concerning the six main families of symmetric functions (monomial; elementary; complete; forgotten; power and Schur) are proven, such as how they relate to each other through transition matrices; the Pieri Rule and, of course, Littlewood-Richardson-Robinson’s theorem. The established framework of symmetric functions is maintained throughout the text.

³Despite being defined over infinitely many variables, symmetric functions over a commutative ring with identity A are indeed functions over the domain of almost null sequences of A .

The second chapter introduces the algebra of symmetric functions over the rational polynomial field $\mathbb{Q}(\alpha)$ and the families of Jack functions $(J_\lambda(\alpha))$ and skew Jack functions $(J_{\lambda/\mu}(\alpha))$. Sahi-Knop's combinatorial formula is strongly utilized both as a definition of Jack functions and as means to prove formulas for special cases of both Jack and skew Jack functions. The Pieri-like rule for Jack functions given by Stanley [15] is also explained and special cases are presented.

In the last chapter, an algebra of finitely non-symmetric functions is defined while maintaining the same function framework of the symmetric case. Similarly to the symmetric case, the algebra of non-symmetric functions (hence non-symmetric functions themselves) can be projected onto finite dimension polynomial algebras by zeroing all variables from some point on. Non-symmetric Jack functions are a base for this algebra which become Jack functions upon symmetrization. Because of this, besides their intrinsic importance, non-symmetric Jack functions may facilitate the pursuit of results regarding their symmetrical counterparts. The main result in this chapter is Schützer's theorem, which outlines how to combinatorially linearly decompose the product between an arbitrary non-symmetric function and a non-symmetric function indexed by a composition with weight 1. No further generalization is yet known.

2 Symmetric Functions

2.1 Compositions, Partitions and Tableaux

Compositions, partitions and tableaux play fundamental roles in the theory of symmetric functions. This section is dedicated to a brief clarification of their properties and the notation adopted throughout the text.

2.1.1 Preliminary Definitions

Compositions are sequences of non-negative integers with finitely many positive terms and *partitions* are compositions whose terms are non-increasingly ordered. They are usually denoted as

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots) \text{ or } (\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \dots)$$

where in the second notation, each λ_i is necessarily different than its neighbours λ_{i-1} and λ_{i+1} , and m_i is the multiplicity of each λ_i . In the case of partitions, since (positive) λ_i are disposed in strictly decreasing order, the multiplicity m_j of each λ_j equals the total amount of λ_j occurrences in λ . Let λ be a composition or partition. The *length* of λ , denoted by $\ell(\lambda)$, is the index of its last positive term. The *weight* of λ , denoted by $|\lambda|$ is the sum of its entries. Equivalently, if λ has weight n , it is also said to be a partition or composition of n , which is denoted by $\lambda \vdash n$.

Examples:

1. $(5, 0^2, 5^2, 3) = (5, 0, 0, 5, 5, 3)$ is a composition with length 6 and weight 18;
2. $(5, 4^3, 2, 0) = (5, 4, 4, 4, 2, 0)$ is a partition with length 5 and weight 19.

The set of all compositions will be denoted by \mathbb{N}^∞ and its subsets of compositions of n by \mathbb{N}_n^∞ . The set of all partitions will be denoted by \mathcal{P} and its subsets of partitions of n by \mathcal{P}_n .

The non-increasing rearrangement of a composition λ is necessarily a partition and is denoted by λ^+ . This defines a surjection $\lambda \mapsto \lambda^+$ from \mathbb{N}_n^∞ onto \mathcal{P}_n , hence from \mathbb{N}^∞ onto \mathcal{P} . \mathbb{N}^∞ and \mathcal{P} can also be bijectively related by

$$\begin{array}{ccc} \mathbb{N}^\infty & \rightarrow & \mathcal{P} \\ (\lambda_1, \lambda_2, \lambda_3, \dots) & \mapsto & \left(\sum_{i \geq 1} \lambda_i, \sum_{i \geq 2} \lambda_i, \sum_{i \geq 3} \lambda_i, \dots \right) \end{array}$$

whose inverse is

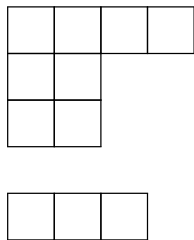
$$\begin{array}{ccc} \mathcal{P} & \rightarrow & \mathbb{N}^\infty \\ (\lambda_1, \lambda_2, \lambda_3, \dots) & \mapsto & (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - \lambda_4, \dots) \end{array}$$

Partitions and compositions are also commonly represented by *diagrams*, defined either by the set

$\{(i, j) \in \mathbb{N}^2 \mid 1 \leq j \leq \lambda_i\}$ in the Anglophone convention or

$\{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq \lambda_j\}$ in the Francophone convention.

Diagrams may be pictorially represented as sets of squares displayed according to their coordinates where the first coordinate (increasing downwards in the Anglophone convention and upwards in the Francophone convention) indicates the row and the second coordinate (increasing rightwards in both conventions) indicates the column of each square. For example, the composition $(4, 2, 2, 0, 3)$ may also be represented by



in the Anglophone convention, which will be the one adopted throughout the text.

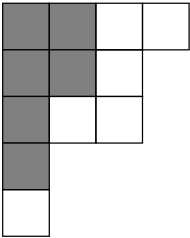
For most purposes, compositions and their diagrams may be conflated without ambiguity. For example, a composition μ is said to be contained in

another composition λ (denoted by $\mu \subseteq \lambda$) if the same is true about their diagrams. When diagrams must be explicitly distinguished, they are denoted by $\text{diag}(\lambda)$.

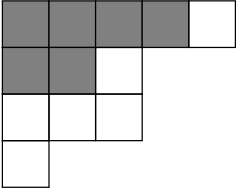
Skew diagrams are the set difference between diagrams of compositions such that one contains the other. That is, $\forall \mu \subseteq \lambda$,

$$\lambda/\mu := \text{diag}(\lambda) \setminus \text{diag}(\mu)$$

They can also be represented pictorially. For example, $(4, 3, 3, 1, 1)/(2, 2, 1, 1)$ can be represented by the clear part of the diagram:

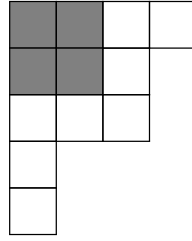


The *conjugate* of a given diagram (be it skew or not) is the diagram obtained by its reflection along the main diagonal and the conjugate of a partition λ , denoted by λ' is the partition for which their diagrams are conjugated. Since the conjugate diagram of a composition is the diagram of another composition if, and only if, they are both partitions, conjugation of compositions which are not partitions is not defined. The conjugate of a skew diagram λ/μ is λ'/μ' . Diagrams or partitions equal to their conjugates are said to be *self-conjugated*. The conjugate diagram of the example above is:



By looking at a partition's diagram, it can be directly seen that if $\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3, \dots)$ is the conjugate of $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, each λ_i is given by $\lambda'_i = \#\{j : \lambda_j \geq i\}$.

A *path* in a diagram is a sequence of its elements x_1, x_2, x_3, \dots such that $|x_{k+1} - x_k| = 1$ for each k . A subset of a diagram is said to be *connected* if for each pair of its points, it contains a path connecting them. *Connected components* of a diagram are its maximal connected subsets. Skew diagrams are said to be *border strips* when they are connected and do not contain 2×2 blocks of points. The *border* of a partition λ is the maximal border strip within it whose shape is λ/μ for some partition μ . For example, the border of $(4, 3, 3, 1, 1)$ is the clear part of



A skew diagram λ/μ is a *m-horizontal strip* if $m = |\lambda| - |\mu|$ and its connected components belong to single rows, or equivalently, $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all i . *m-vertical strips* are defined similarly to their conjugate counterparts, but with connected components belonging to single columns instead.

2.1.2 Operations in \mathcal{P}

Let

$$\begin{aligned}\lambda &= (\lambda_1, \lambda_2, \lambda_3, \dots) = (1^{m_1^\lambda}, 2^{m_2^\lambda}, 3^{m_3^\lambda}, \dots)^+ \\ \mu &= (\mu_1, \mu_2, \mu_3, \dots) = (1^{m_1^\mu}, 2^{m_2^\mu}, 3^{m_3^\mu}, \dots)^+\end{aligned}$$

be arbitrary partitions.

The following commutative operations are defined on \mathcal{P} :

- $\lambda \cup \mu = (\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \mu_3, \dots)^+ = (1^{m_1^\lambda + m_1^\mu}, 2^{m_2^\lambda + m_2^\mu}, 3^{m_3^\lambda + m_3^\mu}, \dots)^+$
- $\lambda \times \mu = (1^{m_1(\lambda, \mu)}, 2^{m_2(\lambda, \mu)}, 3^{m_3(\lambda, \mu)}, \dots)^+$

being $m_k(\lambda, \mu) = \sum_{i, j > k} (m_k^\lambda m_j^\mu + m_i^\lambda m_k^\mu)$ for each k . This definition is the same as the partition obtained by the non-decreasing rearrangement of all numbers given by $\min_{i, j \geq 1} \{\lambda_i, \mu_j\}$.

The following commutative operations are defined on both \mathbb{N}^∞ and \mathcal{P} :

- $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \lambda_3 + \mu_3, \dots)$
- $\lambda \cdot \mu = (\lambda_1\mu_1, \lambda_2\mu_2, \lambda_3\mu_3, \dots)$

Example:

$$(4, 3^2, 1^2) + (3, 2, 1^2) = (7, 5, 4, 2, 1)$$

$$(4, 3^2, 1^2) \cup (3, 2, 1^2) = (4, 3^3, 2, 1^4)$$

$$(4, 3^2, 1^2) \cdot (3, 2, 1^2) = (12, 6, 3, 1)$$

$$(4, 3^2, 1^2) \times (3, 2, 1^2) = (3^3, 2^3, 1^{14})$$

These operations are related to each other by the following propositions:

Proposition 1. *For all $\mu, \lambda \in \mathcal{P}$,*

$$(\lambda \cup \mu)' = \lambda' + \mu'$$

$$(\lambda \times \mu)' = \lambda' \cdot \mu'$$

Proof:

$$\begin{aligned} (\lambda \cup \mu)'_i &= \#\{j : (\lambda \cup \mu)_j \geq i\} \\ &= \#\{j : \lambda_j \geq i\} + \#\{j : \mu_j \geq i\} \\ &= \lambda'_i + \mu'_i \end{aligned}$$

$$\begin{aligned} (\lambda \times \mu)'_i &= \#\{j : (\lambda \times \mu)_j \geq i\} \\ &= \#\{(j, k) : \lambda_j, \mu_k \geq i\} \\ &= \lambda'_i \cdot \mu'_i \end{aligned}$$

□

Proposition 2. *For all $\mu, \lambda \in \mathcal{P}$,*

$$\nu \cdot (\lambda + \mu) = (\nu \cdot \lambda) + (\nu \cdot \mu)$$

$$\nu \times (\lambda \cup \mu) = (\nu \times \lambda) \cup (\nu \times \mu)$$

Proof: The first distributive property is evident from the definitions and, together with the previous proposition, implies the second, for

$$\nu \times (\lambda \cup \mu) = (\nu' \cdot (\lambda' + \mu'))' = (\nu' \cdot \lambda' + \nu' \cdot \mu') = (\nu \times \lambda) \cup (\nu \times \mu).$$

□

It can be concluded that $(\mathcal{P}_n, +, \cdot)$ and $(\mathcal{P}_n, \cup, \times)$ are unitary commutative rings with additive neutral element (0) and respective units (1^n) and (n) .

2.1.3 Orderings on \mathcal{P} and \mathbb{N}^∞

Partitions and compositions may be partially or totally ordered in different ways, such as:

- Natural Ordering

$$\lambda \geq \mu \Leftrightarrow \sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i \quad \forall k \geq 1$$

This is a total ordering on \mathcal{P}_n for $n \leq 4$ and a partial ordering on \mathcal{P}_n for $n > 5$, hence also a partial ordering on \mathcal{P} and \mathbb{N}^∞ . Since this is the most commonly used ordering, it will be denoted by \geq with no subscript.

- Inclusion (\mathcal{I}) and Conjugate Inclusion (\mathcal{I}') orderings

$$\lambda \geq_{\mathcal{I}} \mu \Leftrightarrow \lambda = \mu \cup \nu \quad \text{for some partition } \nu$$

$$\lambda \geq_{\mathcal{I}'} \mu \Leftrightarrow \lambda' = \mu' \cup \nu \quad \text{for some partition } \nu$$

These are both partial orderings on \mathcal{P} .

- Reverse Lexicographical (\mathcal{L}) and Conjugate Reverse Lexicographical (\mathcal{L}') orderings

$$\lambda \geq_{\mathcal{L}} \mu \Leftrightarrow \begin{cases} \lambda = \mu & \text{or} \\ \lambda_j > \mu_j & \text{for } j = \min\{i : \lambda_i \neq \mu_i\} \end{cases}$$

$$\lambda \geq_{\mathcal{L}'} \mu \Leftrightarrow \begin{cases} \lambda = \mu & \text{or} \\ \mu_j > \lambda_j & \text{for } j = \max\{i : \lambda_i \neq \mu_i\} \end{cases}$$

These are total orderings on \mathbb{N}^∞ , hence also on \mathcal{P} .

- “Regrouping” Ordering (\mathcal{R})

Let $S_{\ell, \tilde{\ell}}$ denote the set of surjections from $\{1, 2, \dots, \ell\}$ to $\{1, 2, \dots, \tilde{\ell}\}$:

$$\lambda \geq_{\mathcal{R}} \mu \Leftrightarrow \exists g \in S_{\ell(\lambda), \ell(\mu)} \text{ such that } \sum_{j \in g^{-1}(i)} \mu_j = \lambda_i \quad \forall i \in \{1, 2, \dots, \tilde{\ell}\}$$

This means that $\lambda \geq_{\mathcal{R}} \mu$ iff there is a way to rearrange and sum μ components (without repetition) to form λ . This is a total ordering on \mathcal{P}_n for $n \leq 3$ and a partial ordering for $n > 3$, hence also a partial ordering on \mathcal{P} and \mathbb{N}^∞

Observations:

- For every $\lambda, \mu \in \mathcal{P}$, if $\lambda \geq_{\mathcal{I}} \mu$, then μ can be obtained from λ by removing some rows. Similarly, if $\lambda \geq_{\mathcal{I}'} \mu$, then μ can be obtained from λ by removing some columns. It follows that $\lambda \geq_{\mathcal{I}} \mu \Leftrightarrow \lambda' \geq_{\mathcal{I}'} \mu'$.
- If $\geq_{\mathcal{A}}^{\mathcal{P}}$ is an ordering on \mathcal{P} and $\Psi : \mathbb{N}^\infty \rightarrow \mathcal{P}$ is a function, the condition

$$\lambda \geq_{\mathcal{A}}^{\mathbb{N}^\infty} \mu \Leftrightarrow \Psi(\lambda) \geq_{\mathcal{A}}^{\mathcal{P}} \Psi(\mu), \quad \forall \lambda, \mu \in \mathbb{N}^\infty$$

defines the $\geq_{\mathcal{A}}^{\mathbb{N}^\infty}$ ordering on \mathbb{N}^∞ . Moreover, if Ψ is injective and $\geq_{\mathcal{A}}^{\mathcal{P}}$ is a total ordering, then $\geq_{\mathcal{A}}^{\mathbb{N}^\infty}$ is a total ordering as well.

Proposition 3. For every $\lambda, \mu \in \mathbb{N}^\infty$,

$$\lambda \geq_{\mathcal{L}} \mu \Leftrightarrow \lambda' \geq_{\mathcal{L}'} \mu'$$

Proof: Only one of the directions needs to be proved, as the other direction is analogous. Suppose $\lambda \neq \mu$, then

$$\begin{aligned} \lambda \geq_{\mathcal{L}} \mu &\Rightarrow \exists i \text{ such that } \begin{cases} \lambda_j = \mu_j & \forall j < i \\ \lambda_i > \mu_i \end{cases} \\ &\Rightarrow \exists i \text{ such that } \begin{cases} \lambda'_j = \mu'_j & \forall j > i \\ \lambda'_i > \mu'_i \end{cases} \\ &\Rightarrow \mu' \geq_{\mathcal{L}'} \lambda' \end{aligned}$$

□

Proposition 4. For every $n \in \mathbb{N}$ and every $\lambda, \mu \in \mathcal{P}_n$,

$$\lambda \geq \mu \Leftrightarrow \mu' \geq \lambda'$$

Proof: Only one direction needs to be proved.

$$\begin{aligned} \lambda \not\geq \mu &\Rightarrow \exists i = \min \left\{ k : \sum_{j=1}^k \lambda_j < \sum_{j=1}^k \mu_j \right\} \\ &\Rightarrow \lambda_i < \mu_i \text{ and } \sum_{j>i} \lambda_j > \sum_{j>i} \mu_j \\ &\Rightarrow \sum_{j=1}^{\mu_i} (\lambda'_j - i) \leq \sum_{j=1}^{\lambda_i} (\lambda'_j - i) < \sum_{j=1}^{\mu_i} (\mu'_j - i) \\ &\Rightarrow \sum_{j=1}^{\mu_i} \lambda'_j < \sum_{j=1}^{\mu_i} \mu'_j \\ &\Rightarrow \mu \not\geq \lambda \end{aligned}$$

□

Proposition 5. For all $n \in \mathbb{N}$ and all $\lambda, \mu \in \mathcal{P}_n$:

$$\lambda \geq \mu \Rightarrow \begin{cases} \lambda \geq_{\mathcal{L}} \mu \\ \lambda \geq_{\mathcal{L}'} \mu \end{cases}$$

Proof: By contradiction,

- $(\lambda \geq \mu \Rightarrow \lambda \geq_{\mathcal{L}} \mu)$

Since $\geq_{\mathcal{L}}$ is a total order on \mathcal{P} , and in particular on \mathcal{P}_n ,

$$\begin{aligned} \lambda \not\geq_{\mathcal{L}} \mu &\Rightarrow \lambda \neq \mu \text{ and } \mu \geq_{\mathcal{L}} \lambda \\ &\Rightarrow \exists i \text{ such that } \begin{cases} \mu_j = \lambda_j \quad \forall j < i \\ \mu_i > \lambda_i \end{cases} \\ &\Rightarrow \lambda \not\geq \mu \end{aligned}$$

- $(\lambda \geq \mu \Rightarrow \lambda \geq_{\mathcal{L}'} \mu)$

Since $\geq_{\mathcal{L}'}$ is a total order on \mathcal{P} , and in particular on \mathcal{P}_n ,

$$\begin{aligned}
\lambda \not\geq_{\mathcal{L}'} \mu &\Rightarrow \lambda \neq \mu \text{ and } \mu \geq_{\mathcal{L}'} \lambda \\
&\Rightarrow \exists i \text{ such that } \begin{cases} \mu_j = \lambda_j & \forall j > i \\ \lambda_i > \mu_i \end{cases} \\
&\Rightarrow \sum_{j \geq i} \lambda_j > \sum_{j \geq i} \mu_j \\
&\Rightarrow \sum_{j=1}^{i-1} \lambda_j = n - \sum_{j \geq i} \lambda_j < n - \sum_{j \geq i} \mu_j = \sum_{j=1}^{i-1} \mu_j \\
&\Rightarrow \lambda \not\geq \mu
\end{aligned}$$

□

Observations:

- The converse is not generally true. For example,

$$\begin{cases} (6, 3, 3) \geq_{\mathcal{L}} (5, 5, 1, 1) \\ (6, 3, 3) \geq_{\mathcal{L}'} (5, 5, 1, 1) \end{cases} \text{ but } (6, 3, 3) \not\geq (5, 5, 1, 1)$$

- The condition that λ and μ have the same weight is only required for the second part of the proof, meaning that the natural ordering implies the reverse lexicographical ordering for all partitions, regardless of weight.

Proposition 6. For all $n \in \mathbb{N}$ and all $\lambda, \mu \in \mathcal{P}_n$,

$$\lambda \geq_{\mathcal{R}} \mu \Rightarrow \lambda \geq \mu$$

Proof: Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ and denote $\mu^{(i)} = (\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots)$.

The inequality $\lambda \geq_{\mathcal{R}} \mu$ means that there exists some surjection $g \in S_{\ell(\mu), \ell(\lambda)}$ such that $\lambda = \bigcup_{i \geq 1} \left(\sum_{j \in g^{-1}(i)} \mu_j \right)$. Now since $(a + b) \geq (a, b)$ for all $a, b \geq 0$,

$$\mu^{(i)(j)} \cup (\lambda_i + \lambda_j) \geq \mu^{(i)(j)} \cup (\mu_i, \mu_j) = \mu.$$

Applying this procedure finitely many times reveals that $\lambda \geq \mu$.

□

Observation: The converse is not generally true. For example, $(3, 1) \geq (2, 2)$, but $(3, 1) \not\geq_{\mathcal{R}} (2, 2)$.

2.1.4 Special Quantities

Some quantities regarding compositions and diagrams are commonly used and merit their own definition. For an arbitrary composition λ , the following compositions and integer quantities are defined:

$$\begin{aligned} \bar{\lambda} &:= (\max\{0, \lambda_1 - 1\}, \max\{0, \lambda_2 - 1\}, \max\{0, \lambda_3 - 1\}, \dots) \\ \lambda^* &:= (\lambda_{\ell(\lambda)} - 1, \lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)-1}) \\ \varepsilon(\lambda) &:= (-1)^{|\lambda| - \ell(\lambda)} = (-1)^{|\bar{\lambda}|} \\ z_\lambda &:= \prod_{i \geq 1} \lambda_i^{m_i} m_i! \quad \text{where } \lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \dots) \\ \lambda! &:= \prod_{i \geq 1} \lambda_i! \\ \binom{n}{\lambda} &:= \frac{n!}{\lambda!} \end{aligned}$$

Example:

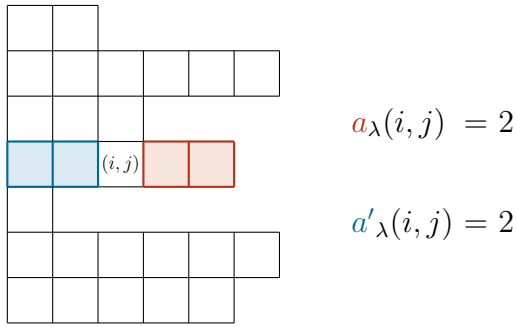
| | $(4, 1^2, 0^2, 2)$ | $(5, 2^3, 1^2)$ |
|------------------------|--------------------|------------------|
| $\bar{\lambda}$ | $(3, 0^4, 1)$ | $(4, 1^3)$ |
| λ^* | $(1, 3, 1^2)$ | $(0, 5, 2^3, 1)$ |
| $\varepsilon(\lambda)$ | 1 | -1 |
| z_λ | 0 | 480 |
| $\lambda!$ | 48 | 960 |

Also define, for each element $(i, j) \in \text{diag}(\lambda)$,

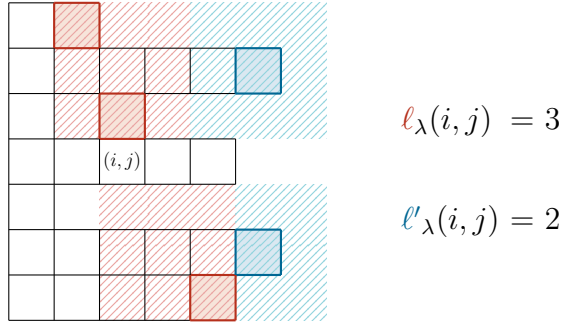
- Arm-length: $a_\lambda(i, j) = \lambda_i - j$
- Coarm-length: $a'_\lambda(i, j) = j - 1$
- Leg-length: $\ell_\lambda(i, j) = \#\{k < i \mid j \leq \lambda_k + 1 \leq \lambda_i\} + \#\{k > i \mid j \leq \lambda_k \leq \lambda_i\}$
- Coleg-length: $\ell'_\lambda(i, j) = \#\{k < i \mid \lambda_k \geq \lambda_i\} + \#\{k > i \mid \lambda_k > \lambda_i\}$
- Hook-length: $h_\lambda(i, j) = a_\lambda(i, j) + \ell_\lambda(i, j) + 1$
- Cohook-length: $h'_\lambda(i, j) = a'_\lambda(i, j) + \ell'_\lambda(i, j) + 1$

Example: Let $\lambda = (2, 6, 3, 5, 1, 6, 5)$ and $(i, j) = (4, 3) \in \text{diag}(\lambda)$.

The arm-length $a_\lambda(i, j)$ is numerically equivalent to the number of squares to the right of (i, j) in $\text{diag}(\lambda)$, indicated by red in the following diagram. Conversely, the coarm-length $a'_\lambda(i, j)$ is numerically equivalent to the number of squares to the left of (i, j) , indicated by blue in the following diagram.



The leg-length $\ell_\lambda(i, j)$ is numerically equivalent to the number of λ rows whose rightmost square lies within the red hatched region in the following diagram. Conversely, the coleg-length $\ell'_\lambda(i, j)$ is numerically equivalent to the number of λ rows whose rightmost square lies within the blue hatched region in the following diagram.



Finally, hook and cohook-lengths for this example are given by:

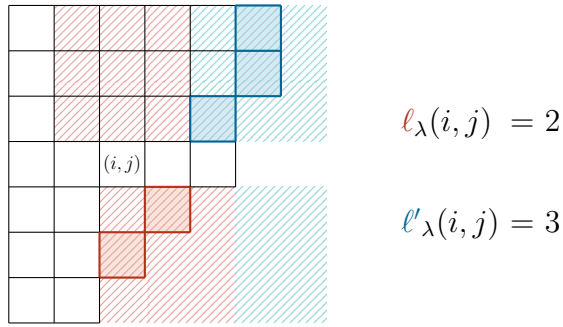
$$h_\lambda(i, j) = a_\lambda(i, j) + \ell_\lambda(i, j) + 1 = 6$$

$$h'_\lambda(i, j) = a'_\lambda(i, j) + \ell'_\lambda(i, j) + 1 = 5$$

As a direct consequence of the definition, when λ is a partition, leg-lengths and coleg-lengths simplify to

- Leg-length: $\ell_\lambda(i, j) = \lambda'_j - i$
- Coleg-length: $\ell'_\lambda(i, j) = i - 1$

Example: Let $\lambda = (6, 6, 5, 5, 4, 3, 2)$ and $(i, j) = (4, 3) \in \text{diag}(\lambda)$.

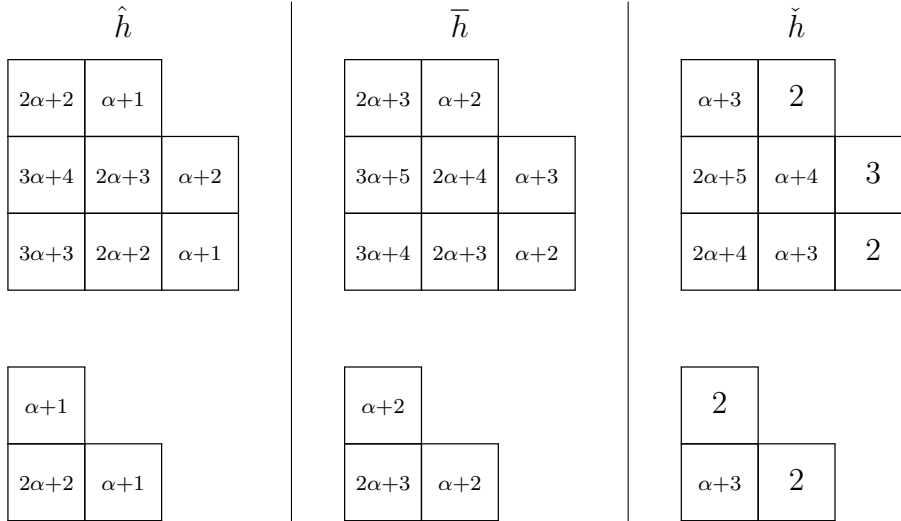


For an arbitrary composition λ and each $(i, j) \in \text{diag}(\lambda)$, we also define the following polynomials in α :

- $\check{h}_\lambda(i, j)(\alpha) := a_\lambda(i, j)\alpha + \ell_\lambda(i, j) + 1$
- $\hat{h}_\lambda(i, j)(\alpha) := a_\lambda(i, j)\alpha + \ell_\lambda(i, j) + \alpha$

- $\bar{h}_\lambda(i, j)(\alpha) := a_\lambda(i, j)\alpha + \ell_\lambda(i, j) + \alpha + 1$
- $\varphi_\lambda(\alpha) := \prod_{i \geq 1} \prod_{j=0}^{\lambda_i - 1} (j\alpha + 1)$

Example: Let $\lambda = (2, 3, 3, 0, 1, 2)$. The correspondent $\hat{h}(s)$, $\bar{h}(s)$ and $\check{h}(s)$ are assigned to each box $s \in \text{diag}(\lambda)$ in the diagrams below.



For the same composition $\lambda = (2, 3, 3, 0, 1, 2)$,

$$\varphi_\lambda(\alpha) = (\alpha + 1) \cdot ((2\alpha + 1)(\alpha + 1))^2 \cdot 1 \cdot 1 \cdot (\alpha + 1) = (2\alpha + 1)^2(\alpha + 1)^4$$

2.1.5 Tableaux

Let λ be an arbitrary composition or skew partition. A *tableau* T of *shape* λ , denoted by $sh(T) = \lambda$, is a labelling of boxes in $\text{diag}(\lambda)$ by positive integers. Its *weight*, denoted by $|T|$, is the composition whose each i -th component is given by the number of T boxes labelled i .

Tableaux whose labelling is non-decreasing along rows and strictly increasing down columns, whose shape and weight are both partitions⁴ are said to be *semi-standard* or *column-strict*. *Row-strict* tableaux can be similarly defined switching the non-decreasing and strictly increasing conditions, so they

⁴Possibly a skew partition in the case of the shape.

are strictly increasing along rows and non-decreasing down columns. *Standard* tableaux are labellings in which numbers $1, 2, \dots, |sh(T)|$ appear exactly once and in strictly increasing order over both rows and columns, hence are simultaneously column and row-strict. Since by definition there is exactly one occurrence of each of the numbers $1, 2, \dots, |sh(T)|$, standard tableaux must necessarily have weight $(1^{|sh(T)|})$. When a tableau T does not necessarily satisfy column or row-strict conditions it is often said to be a *generalized* tableau.

Examples:

- | | | |
|---|---|---|
| | 2 | 3 |
| 1 | | |

 has shape $(3, 1)/(1)$, weight (1^3) and is standard;
- | | |
|---|---|
| 1 | |
| 2 | 3 |

 has shape $(1, 2)$, weight (1^3) and is not standard;
- | | |
|---|---|
| 1 | 1 |
| 2 | 3 |

 has shape $(2, 2)$, weight $(2, 1^2)$ and is semi-standard;
- | | |
|---|---|
| 1 | 2 |
| 3 | 3 |

 has shape $(2, 2)$, weight $(1^2, 2)$ and is not semi-standard.

Notation: Let Tab denote the set of all tableaux. The following notation will be adopted:

- $\widehat{\text{Tab}}(\lambda) = \{T \in \text{Tab} : sh(T) = \lambda\}$;
- $\widehat{\text{Tab}}(\lambda, \nu) = \{T \in \text{Tab} : sh(T) = \lambda \text{ and } |T| = \nu\}$;
- $\text{Tab}(\lambda) = \{T \in \text{Tab} : sh(T) = \lambda \text{ and } T \text{ is semi-standard}\}$;
- $\text{Tab}(\lambda, \nu) = \{T \in \text{Tab} : sh(T) = \lambda, |T| = \nu \text{ and } T \text{ is semi-standard}\}$.

2.2 Algebra of Symmetric Functions

Let A be an arbitrary commutative ring with identity. The symmetric group S_n bijectively acts over any polynomial ring $A[x_1, x_2, \dots, x_n]$ in a natural fashion: each permutation $\sigma \in S_n$ accordingly rearranging variables x_1, \dots, x_n to $x_{\sigma(1)}, \dots, x_{\sigma(n)}$. That is, for $p \in A[x_1, x_2, \dots, x_n]$:

$$\sigma \cdot p(x_1, x_2, \dots, x_n) = p(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

Let for $\alpha \in \mathbb{N}^n$. Denoting $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ and $p(x_1, x_2, \dots, x_n) = \sum_{\alpha} c_{\alpha} x^{\alpha}$, the action becomes

$$\begin{aligned} p(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) &= \sum_{\alpha} c_{\alpha} x_{\sigma(1)}^{\alpha_1} x_{\sigma(2)}^{\alpha_2} \dots x_{\sigma(n)}^{\alpha_n} \\ &= \sum_{\alpha} c_{\alpha} x_1^{\alpha_{\sigma^{-1}(1)}} x_2^{\alpha_{\sigma^{-1}(2)}} \dots x_n^{\alpha_{\sigma^{-1}(n)}} \\ &= \sum_{\alpha} c_{\alpha} x^{\sigma^{-1}(\alpha)} \\ &= \sum_{\alpha} c_{\sigma(\alpha)} x^{\alpha} \end{aligned} \tag{1}$$

Examples:

1. $(1\ 2)(4\ 5) \cdot (x_1^4 x_2^2 x_4 + x_2^4 x_4^2 x_5) = x_1^4 x_2^2 x_4 + x_2^4 x_4^2 x_5$
2. $(1\ 4\ 5)(2\ 3) \cdot (x_1^4 x_2^2 x_4 + x_2^4 x_4^2 x_5) = x_4^4 x_3^2 x_5 + x_3^4 x_5^2 x_1$

Polynomials invariant under this action are called *symmetric polynomials* and the subset of such polynomials forms the sub-algebra of symmetric polynomials over A , which will be denoted by $\Lambda_A[x_1, x_2, \dots, x_n]$. The submodule of symmetric polynomials over A which are homogeneous with degree k (also called k -homogeneous), together with the zero polynomial, will be denoted by $\Lambda_A^k[x_1, x_2, \dots, x_n]$.

Let $p(x_1, x_2, \dots, x_n) = \sum_{\alpha} c_{\alpha} x^{\alpha}$ be the monomial decomposition of a symmetric polynomial. Because of (1), whenever indices $\alpha_{(1)}, \alpha_{(2)} \in \mathbb{N}^n$ are such that $\alpha_{(1)}^+ = \alpha_{(2)}^+ = \lambda$ for some partition λ , their correspondent coefficients must be equal to each other, that is $c_{\alpha_{(1)}} = c_{\alpha_{(2)}} = c_{\lambda}$. It follows that

$p(x_1, x_2, \dots, x_n)$ can be decomposed in terms of sums of distinct monomials x^α such that $\alpha^+ = \lambda$ for some partition λ .

$$p(x_1, x_2, \dots, x_n) = \sum_{\alpha} c_{\alpha} x^{\alpha} = \sum_{\lambda} c_{\lambda} \sum_{\alpha^+ = \lambda} x^{\alpha}$$

These sums are symmetric polynomials themselves, appropriately called monomial symmetric polynomials.

Definition 2.2.1 (Monomial Symmetric Polynomials). *For all $n \geq 1$ and $\lambda \in \mathcal{P}_n$,*

$$m_{\lambda}(x) := \sum_{\substack{\alpha^+ = \lambda \\ \ell(\alpha) \leq n}} x^{\alpha}$$

If $\lambda = (0)$, $m_{(0)} := 1$.

Let $\lambda = (\lambda_1^{\gamma_1}, \lambda_2^{\gamma_2}, \dots, \lambda_{\ell}^{\gamma_{\ell}}) \in \mathcal{P}_n$ and consider the quotient ring $\frac{S_n}{\text{stab}(\lambda)}$, where $\text{stab}(\lambda) = \{\sigma \in S_n : \sigma(\lambda) = \lambda\} \cong S_{\gamma_1} \times S_{\gamma_2} \times \dots \times S_{\gamma_{\ell}}$. Note that two permutations $\sigma', \sigma'' \in S_n$ act alike on λ if, and only if they are within the same coset, that is, $[\sigma'] = [\sigma''] \in \frac{S_n}{\text{stab}(\lambda)}$. Hence the action $[\sigma] \cdot \lambda := \sigma(\lambda)$ for some $\sigma \in [\sigma]$ is well-defined. Now for each $\alpha \in \mathbb{N}^n$ such that $\alpha^+ = \lambda$ there is exactly one $[\sigma] \in \frac{S_n}{\text{stab}(\lambda)}$ such that $[\sigma] \cdot \lambda = \alpha$. This fact yields an alternative expression for monomial symmetric polynomials:

$$m_{\lambda}(x) = \sum_{[\sigma] \in \frac{S_n}{\text{stab}(\lambda)}} x^{[\sigma] \cdot \lambda}$$

And since $\#\text{stab}(\lambda) = \#S_{\gamma_1} \times S_{\gamma_2} \times \dots \times S_{\gamma_{\ell}} = \prod_i \gamma_i!$, it also follows that

$$m_{\lambda}(x) = \frac{1}{\prod_i \gamma_i!} \sum_{\sigma \in S_n} x^{\sigma(\lambda)}.$$

Since every symmetric polynomial in $\Lambda_A[x_1, x_2, \dots, x_n]$ may be decomposed in terms of monomial symmetric polynomials, and these are clearly linearly independent, the family of monomial symmetric polynomials over n variables $\{m_{\lambda}(x_1, x_2, \dots, x_n)\}_{\ell(\lambda) \leq n}$ is an A -base for $\Lambda_A[x_1, x_2, \dots, x_n]$ as an

A -module. For a similar reason, so is the restriction $\{m_\lambda(x_1, x_2, \dots, x_n)\}_{\substack{\ell(\lambda) \leq n \\ \lambda \vdash k}}$ an A -base as an A -module for $\Lambda_A^k[x_1, x_2, \dots, x_n]$.

Examples:

1. $m_{(3,1,1)}(x_1, x_2, x_3) = x_1^3 x_2 x_3 + x_2^3 x_1 x_3 + x_3^3 x_1 x_2$
2. $m_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2$

It is often the case that results regarding symmetric polynomials do not directly depend on their specific number of variables. It is therefore useful to approach the theory of symmetric polynomials from a perspective which generalizes these results for polynomial algebras over arbitrarily many finite variables. To that end, dimension independent mathematical objects which generalize symmetric polynomials, called symmetric functions, are defined.

In order to construct them, firstly consider the homomorphisms of A -modules for each $n, m \in \mathbb{N}$:

$$\begin{aligned} \rho_{m,n} : \Lambda_A[x_1, x_2, \dots, x_n] &\rightarrow \Lambda_A[x_1, x_2, \dots, x_m] \\ m_\lambda(x_1, x_2, \dots, x_n) &\mapsto \begin{cases} 0 & \text{if } m < \ell(\lambda) \\ m_\lambda(x_1, x_2, \dots, x_m) & \text{if } m \geq \ell(\lambda) \end{cases} \end{aligned}$$

and their restriction $\rho_{m,n}^k : \Lambda_A^k[x_1, x_2, \dots, x_n] \rightarrow \Lambda_A^k[x_1, x_2, \dots, x_m]$.

Observations:

- When $m \leq n$, $\rho_{m,n} \circ \rho_{n,m} = \text{id}_n$, so in particular, $\rho_{n,m}$ is a monomorphism and $\rho_{m,n}$ is an epimorphism;
- When $l \leq m \leq n$, $\rho_{l,m} \circ \rho_{m,n} = \rho_{l,n}$ and $\rho_{n,m} \circ \rho_{m,l} = \rho_{n,l}$.

Now let $\prod_{m \in \mathbb{N}} \Lambda_A[x_1, x_2, \dots, x_m]$ be the A -algebra of sequences of symmetric polynomials in increasing numbers of variables with element-wise sum and multiplication, that is, if $(p_m)_{m \in \mathbb{N}}, (q_m)_{m \in \mathbb{N}} \in \prod_{m \in \mathbb{N}} \Lambda_A[x_1, x_2, \dots, x_m]$,

$(p_m)_{m \in \mathbb{N}} + (q_m)_{m \in \mathbb{N}} = (p_m + q_m)_{m \in \mathbb{N}}$ and $(p_m)_{m \in \mathbb{N}} \cdot (q_m)_{m \in \mathbb{N}} = (p_m \cdot q_m)_{m \in \mathbb{N}}$.
 Since each $\rho_{m,n}$ is an A -module homomorphism, so are ρ_n defined by:

$$\begin{aligned} \rho_n : \Lambda_A[x_1, x_2, \dots, x_n] &\rightarrow \prod_{m \in \mathbb{N}} \Lambda_A[x_1, x_2, \dots, x_m] \\ p &\mapsto (\rho_{m,n}(p))_{m \in \mathbb{N}} \end{aligned}$$

Moreover, $\pi'_n \circ \rho_n = \text{id}_n$ where π'_n denotes the projection homomorphism $\pi'_n : (p_m)_{m \in \mathbb{N}} \mapsto p_n$. In particular, each ρ_n is a monomorphism.

Proposition 7. $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n)$ is an A -sub-algebra of $\prod_{n \in \mathbb{N}} \Lambda_A[x_1, x_2, \dots, x_n]$.

Proof:

Firstly notice that $m \leq n \Rightarrow \text{Im}(\rho_{k,m}) \subseteq \text{Im}(\rho_{k,n})$, $\forall k \in \mathbb{N}$, for:

If $m \leq n \leq k$, $\rho_{k,n} \circ \rho_{n,m} = \rho_{k,m}$, which implies $\text{Im}(\rho_{k,m}) \subseteq \text{Im}(\rho_{k,n})$;

If $k \leq n$, $\rho_{k,n}$ is surjective, so $\text{Im}(\rho_{k,m}) \subseteq \text{Im}(\rho_{k,n})$, $\forall m \in \mathbb{N}$.

This in turn implies that $\text{Im}(\rho_m) \subseteq \text{Im}(\rho_n)$ for all $m \leq n$ and therefore $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n)$ is closed under all algebra operations. Finally, since each $\text{Im}(\rho_n)$ is itself an A -sub-algebra of $\prod_{n \in \mathbb{N}} \Lambda_A[x_1, x_2, \dots, x_n]$, so must $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n)$ be.

Observation: The A -algebra $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n)$ implicitly depends on the particular order in which variables x_1, x_2, \dots are introduced by each $\Lambda_A[x_1, x_2, \dots, x_n]$ in the A -algebra of sequences $\prod_{n \in \mathbb{N}} \Lambda_A[x_1, x_2, \dots, x_n]$. For each sequence of variables $X = (x_{i_1}, x_{i_2}, x_{i_3}, \dots)$, this dependency can be made explicit in $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n)$ by denoting $\left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) \right)_X$.

For any set $x = \{x_i : i \in \mathbb{N}\}$ or list $x = (x_i)_{i \in \mathbb{N}}$, denote the set of sequences obtained by some ‘‘rearrangement’’ of x by $\text{seq}(x)$, that is, $\text{seq}(x) := \{(x_{b(i)})_{i \in \mathbb{N}} \mid b : \mathbb{N} \rightarrow \mathbb{N} \text{ is a bijection}\}$, and let $\overset{\text{seq}}{\sim}$ be the equivalence relation in $\bigcup_{X \in \text{seq}(x)} \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) \right)_X$ defined by $p(X) \overset{\text{seq}}{\sim} p'(X') \Leftrightarrow p(X) = p'(X)$, for all $X, X' \in \text{seq}(x)$, $p(X) \in \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) \right)_X$ and $p'(X') \in \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) \right)_{X'}$. With this equivalence relation, the algebra of symmetric functions can be finally defined.

Definition 2.2.2. Let $x = \{x_1, x_2, \dots\}$ be a countably infinite set of variables. The *A-Algebra of Symmetric Functions* over x , denoted by $\Lambda_A[x]$, is defined by the set

$$\Lambda_A[x] := \frac{\bigcup_{X \in \text{seq}(x)} \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) \right)_X}{\underset{\sim}{\text{seq}}}$$

with the operations

- $a[p_1] := [ap_1]$;
- $[p_1] + [p_2] := [p_1 + p_2]$;
- $[p_1][p_2] := [p_1p_2]$;

defined for all $a \in A$, and $p_1, p_2 \in \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) \right)_X$ for some $X \in \text{seq}(x)$.

Similarly, the *A-Module of Homogeneous Symmetric Functions of Degree k* is defined by

$$\Lambda_A^k[x] := \frac{\bigcup_{y \in \text{seq}(x)} \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^k) \right)_y}{\underset{\sim}{\text{seq}}}$$

with operations

- $a[p_1] := [ap_1]$;
- $[p_1] + [p_2] := [p_1 + p_2]$;

again defined for all $a \in A$, and $p_1, p_2 \in \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) \right)_X$ for some $X \in \text{seq}(x)$.

Elements of either $\Lambda_A[x]$ or $\Lambda_A^k[x]$ are called symmetric functions.

Observations:

- These operations are well-defined since cosets $[ap_1(X)]$, $[p_1(X) + p_2(X)]$ and $[p_1(X)p_2(X)]$ do not depend on the choice of $X \in \text{seq}(x)$;
- For any $X \in \text{seq}(x)$, $\Lambda_A[x] \cong \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) \right)_X$ and $\Lambda_A^k[x] \cong \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^k) \right)_X$. This is the case because by the definition of $\Lambda_A[x]$, $f \mapsto [f]$ is an A -algebra monomorphism between $\left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) \right)_X$ and $\Lambda_A[x]$. Since all $[f] \in \Lambda_A[x]$ contain some $f \in \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) \right)_X$ it is also surjective. The proof for A -modules $\Lambda_A^k[x]$ is completely analogous.

The algebra of symmetric functions and the modules of homogeneous symmetric functions are related by the proposition:

Proposition 8. *Let $x = \{x_1, x_2, \dots\}$ be a countably infinite set of variables. Then*

$$\Lambda_A[x] = \bigoplus_{k \in \mathbb{N}} \Lambda_A^k[x]$$

as a graded algebra.

Proof: Due to the previous observation, it need only be proved that $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) = \bigoplus_{k \in \mathbb{N}} \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^k) \right)$ as a graded algebra. Firstly, it will be proved that $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n)$ is the direct sum of the A -modules $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^k)$ and then that $fg \in \bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^{k_1+k_2})$ for all $k_1, k_2 \in \mathbb{N}$, $f \in \bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^{k_1})$ and $g \in \bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^{k_2})$.

Any polynomial algebra is the direct sum of modules of homogeneous polynomials as a graded algebra, so in particular,

$$\Lambda_A[x_1, x_2, \dots, x_n] = \bigoplus_{k \in \mathbb{N}} \Lambda_A^k[x_1, x_2, \dots, x_n], \quad \forall n \in \mathbb{N}.$$

Moreover, letting $f = \sum_{k=0}^J f^{(k)}$ be the unique decomposition of a symmetric polynomial $f \in \Lambda_A[x_1, x_2, \dots, x_n]$ in its symmetric homogeneous components,

$$\rho_{m,n}(f) = \sum_{k=0}^J \rho_{m,n}(f^{(k)}) = \sum_{k=0}^J \rho_{m,n}^k(f^{(k)}), \quad \forall m \in \mathbb{N}.$$

Therefore

$$\rho_n(f) = \sum_{k=0}^J \rho_n(f^{(k)}) = \sum_{k=0}^J \rho_n^k(f^{(k)}).$$

Finally, since uniqueness of the decomposition is preserved and $\rho_n^k(f^{(k)})$ for all $k \in \mathbb{N}$, $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n) = \bigoplus_{k \in \mathbb{N}} \left(\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^k) \right)$.

Now for arbitrary $k_1, k_2 \in \mathbb{N}$, let $f \in \bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^{k_1})$ and $g \in \bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^{k_2})$, that is, there are $N_1, N_2 \in \mathbb{N}$ and $f_{N_1} \in \Lambda_A^{k_1}[x_1, x_2, \dots, x_{N_1}]$, $g_{N_2} \in \Lambda_A^{k_2}[x_1, x_2, \dots, x_{N_2}]$ such that $f = \rho_{N_1}^{k_1}(f_{N_1})$ and $g = \rho_{N_2}^{k_2}(g_{N_2})$. Since $m \leq n \Rightarrow \text{Im}(\rho_m^k) \subseteq \text{Im}(\rho_n^k)$ for each $k \in \mathbb{N}$, $f = \rho_N^{k_1}(f_N)$ and $g = \rho_N^{k_2}(g_N)$ for $N = \max\{N_1, N_2\}$. Therefore

$$\begin{aligned} fg &= \rho_n^{k_1}(f_N) \rho_n^{k_2}(g_N) \\ &= (\rho_{m,N}^{k_1}(f_N))_{m \in \mathbb{N}} (\rho_{m,N}^{k_2}(g_N))_{m \in \mathbb{N}} \\ &= (\rho_{m,N}^{k_1}(f_N) \rho_{m,N}^{k_2}(g_N))_{m \in \mathbb{N}} \\ &= (\rho_{m,N}(f_N) \rho_{m,N}(g_N))_{m \in \mathbb{N}} \\ &= (\rho_{m,N}(f_N g_N))_{m \in \mathbb{N}} \\ &= (\rho_{m,N}^{k_1+k_2}(f_N g_N))_{m \in \mathbb{N}} \\ &= \rho_n^{k_1+k_2}(f_N g_N) \in \bigcup_{n \in \mathbb{N}} \text{Im}(\rho_n^{k_1+k_2}). \end{aligned}$$

□

Symmetric functions can be projected onto symmetric polynomials in a natural fashion. Let $x = \{x_1, x_2, \dots\}$ be a countably infinite set of variables and $s = \{x_{i_1}, x_{i_2}, \dots, x_{i_n}\}$ a subset of x . The projection-like A -homomorphism

$$\begin{aligned} \pi_n : \Lambda_A[x] &\rightarrow \Lambda_A[s] \\ [(f_m)_{m \in \mathbb{N}}] &\mapsto f_n(x_{i_1}, x_{i_2}, \dots, x_{i_n}) \end{aligned}$$

is well-defined, for it is independent of the choice of $f = (f_m)_{m \in \mathbb{N}}$ in $[f]$. Moreover, $f = \rho_N(f_N)$ for some $N \in \mathbb{N}$ and $f_N \in \Lambda_A[x_1, x_2, \dots, x_N]$, so $\pi_m(f) = \rho_{m,N}(f_N)$, $\forall m \in \mathbb{N}$ and

$$\pi_m = \rho_{m,n} \circ \pi_n, \quad \forall n \leq m \in \mathbb{N}.$$

Despite not being actually polynomials, symmetric functions are in fact functions in the sense that they can be evaluated for the appropriate domain. Namely, the set of almost-null sequences of A , denoted by A^∞ .

Proposition 9. *For each $a = (a_1, a_2, \dots) \in A^\infty$, there is a unique homomorphism $ev_a : \Lambda_A[x] \rightarrow A$ which maps $[p] \mapsto \pi_n(p)(a_1, a_2, \dots, a_n)$ for some $n \geq \max\{i : a_i \neq 0\}$.*

Proof: Firstly, notice that this homomorphism is indeed a well-defined, since for all $n \geq \ell = \max\{i : a_i \neq 0\}$ and all $[p] \in \Lambda_A[x]$,

$$\pi_n([p])(a_1, \dots, a_\ell, \underbrace{0, \dots, 0}_{n-\ell}) = \rho_{n,\ell} \circ \pi_\ell([p])(a_1, \dots, a_\ell) = \pi_\ell([p])(a_1, \dots, a_\ell).$$

Now $ev_a = ev_{(a_1, a_2, \dots, a_\ell)} \circ \pi_\ell$, where $ev_{(a_1, a_2, \dots, a_\ell)}$ is the evaluation homomorphism

$$\begin{aligned} ev_{(a_1, a_2, \dots, a_\ell)} : \Lambda_A[x_1, x_2, \dots, x_n] &\rightarrow A \\ p(x_1, x_2, \dots, x_n) &\mapsto p(a_1, a_2, \dots, a_n), \end{aligned}$$

which uniquely maps $p(x_1, x_2, \dots, x_n)$ onto $p(a_1, a_2, \dots, a_n)$. Therefore ev_a must also be the unique homomorphism $\Lambda_A[x] \rightarrow A$ which maps $[p]$ onto $\pi_\ell(p)(a_1, a_2, \dots, a_n)$.

□

Symmetric functions can also be “partially evaluated” in the sense that only finitely many variables are replaced by elements of the commutative ring A . This is done by the evaluation-like homomorphism:

Proposition 10. *Let $s = \{x_{i_1}, x_{i_2}, \dots, x_{i_n}\}$ be a finite subset of the countably infinite set of variables $x = \{x_1, x_2, \dots\}$. For each $a = (a_1, a_2, \dots, a_n) \in A^n$, there is a unique homomorphism $\bar{ev}_a : \Lambda_A[x] \rightarrow \Lambda_A[x \setminus s]$ which maps*

$$[f] \mapsto [\rho_{N-n}(\pi_N(f)(x_1, x_2, \dots, x_N)|_{x_{i_j}=a_j, \forall x_{i_j} \in s})]$$

for all $N \geq i_n + \ell$, where $f = \rho_\ell(f_\ell)$ for some $f_\ell \in \Lambda_A[x_1, x_2, \dots, x_\ell]$.

Proof: Without loss of generality, let $s = \{x_1, x_2, \dots, x_n\}$. Then $\bar{e}v_a$ is well defined and unique if the map

$$\bar{e}v_a : [f] \mapsto [\rho_{N-n}(\pi_N(f)(a_1, \dots, a_n, x_{n+1}, \dots, x_N))]$$

does not depend on the choice of $N \geq n + \ell$.

If $f = \rho_\ell(f_\ell)$, then $f_\ell \in \Lambda_A[x_1, x_2, \dots, x_\ell]$ has a linear decomposition in terms of monomial polynomials indexed by partitions with length no greater than ℓ ,

$$f_\ell = \sum_{\ell(\lambda) \leq \ell} c_\lambda m_\lambda(x_1, x_2, \dots, x_\ell).$$

Then

$$\begin{aligned} & \pi_N(f)(a_1, \dots, a_n, x_{n+1}, \dots, x_N) \\ &= \rho_{N,\ell}(f_\ell)(a_1, \dots, a_n, x_{n+1}, \dots, x_N) \\ &= \sum_{\ell(\lambda) \leq \ell} c_\lambda m_\lambda(a_1, \dots, a_n, x_{n+1}, \dots, x_N) \\ &= \sum_{\ell(\lambda) \leq \ell} c_\lambda \sum_{\alpha^+ = \lambda} a_1^{\alpha_1} \cdots a_n^{\alpha_n} x_{n+1}^{\alpha_{n+1}} \cdots x_N^{\alpha_N} \\ &= \sum_{\ell(\lambda) \leq \ell} c'_\lambda m_\lambda(x_{n+1}, \dots, x_N) \text{ for some } c'_\lambda. \end{aligned}$$

The condition $N \geq n + \ell$ guarantees that $m_\lambda(x_{n+1}, \dots, x_N)$ is well-defined for all λ such that $\ell(\lambda) \leq \ell$. Furthermore,

$$\rho_{N-n}(m_\lambda(x_{n+1}, \dots, x_N)) = \sum_{\alpha^+ = \lambda} (x_{n+1}^{\alpha_1} x_{n+2}^{\alpha_2} x_{n+3}^{\alpha_3} \cdots)$$

does not depend on $N \geq n + \ell$, so neither does $[\rho_{N-n}(\pi_N(f)(a_1, \dots, a_n, x_{n+1}, \dots, x_N))]$.

□

Observation: “Evaluation” homomorphisms ev_a and $\bar{e}v_a$ motivate an intuitive representation of symmetric functions as infinite sums — over infinitely many variables — of polynomials. These are formal sums which behave like polynomials in the sense that they follow similar summation and

multiplication rules, as well as the previously mentioned variable evaluation rules.

Example: let

$$\begin{cases} f(x_1, x_2) = x_1x_2^2 + x_1^2x_2 - 2x_1 - 2x_2 = m_{(2,1)}(x_1, x_2) - 2m_{(1)}(x_1, x_2) \\ g(x_1, x_2) = 3x_1x_2 - x_1^2 - x_2^2 + 2 = 3m_{(1,1)}(x_1, x_2) - m_{(2)}(x_1, x_2) + 2m_{(0)}(x_1, x_2) \end{cases}$$

so $[f] = [\rho_2(f(x_1, x_2))]$ and $[g] = [\rho_2(g(x_1, x_2))]$ may be written as

$$\begin{aligned} [f] &= \sum_{i \neq j} x_i^2 x_j - 2 \sum_i x_i \\ [g] &= 3 \sum_{i < j} x_i x_j - \sum_i x_i^2 + 2. \end{aligned}$$

With this notation, $[f] + [g]$ may be expressed by

$$\sum_{i \neq j} x_i^2 x_j + 3 \sum_{i < j} x_i x_j - \sum_i x_i^2 - 2 \sum_i x_i + 2,$$

and $[f][g]$ by

$$\begin{aligned} & \left(\sum_{i \neq j} x_i^2 x_j - 2 \sum_i x_i \right) \left(3 \sum_{i < j} x_i x_j - \sum_i x_i^2 + 2 \right) \\ &= 3 \sum_{\substack{i \neq j \\ k < t}} x_i^2 x_j x_k x_t - \sum_{\substack{i \neq j \\ k}} x_i^2 x_j x_k^2 + 2 \sum_{i \neq j} x_i^2 x_j - 6 \sum_{\substack{i \\ k < t}} x_i x_k x_t + 2 \sum_{i, k} x_i x_k^2 - 4 \sum_i x_i \\ &= 3 \sum_{\substack{i \neq j, k, t \\ j < k < t}} x_i^2 x_j x_k x_t - 6 \sum_{\substack{i \neq j, k \\ j < k}} x_i^2 x_j x_k - 2 \sum_{\substack{i < j \\ k \neq i, j}} x_i^2 x_j^2 x_k + 3 \sum_{i \neq j \neq k \neq i} x_i^3 x_j^2 x_k \\ &+ 4 \sum_{i \neq j} x_i^2 x_j - \sum_{i \neq j} x_i^3 x_j^2 - \sum_{i \neq j} x_i^4 x_j + 2 \sum_i x_i^3 - 4 \sum_i x_i. \end{aligned}$$

Similarly, the evaluation of $[f]$ and $[g]$ for $x_1 = 2$ and $x_2 = -1$ can be

expressed by

$$\begin{aligned}
[f] \Big|_{\substack{x_1=2 \\ x_2=-1}} &= \left(\sum_{i \neq j} x_i^2 x_j \right) \Big|_{\substack{x_1=2 \\ x_2=-1}} - 2 \left(\sum_i x_i \right) \Big|_{\substack{x_1=2 \\ x_2=-1}} \\
&= \left(2^2(-1) + 2(-1)^2 + ((-1)^2 + 2^2) \sum_{3 \leq i} x_i + ((-1) + 2) \sum_{3 \leq i} x_i^2 + \sum_{\substack{3 \leq i, j \\ i \neq j}} x_i^2 x_j \right) \\
&\quad - 2 \left(2 + (-1) + \sum_{3 \leq i} x_i \right) \\
&= \sum_{\substack{3 \leq i, j \\ i \neq j}} x_i^2 x_j + \sum_{3 \leq i} x_i^2 + 3 \sum_{3 \leq i} x_i - 4 \\
[g] \Big|_{\substack{x_1=2 \\ x_2=-1}} &= 3 \left(\sum_{i < j} x_i x_j \right) \Big|_{\substack{x_1=2 \\ x_2=-1}} - \left(\sum_i x_i^2 \right) \Big|_{\substack{x_1=2 \\ x_2=-1}} + 2 \\
&= 3 \left(2(-1) + (2 + (-1)) \sum_{3 \leq i} x_i + \sum_{3 \leq i < j} x_i x_j \right) - \left(2^2 + (-1)^2 + \sum_{3 \leq i} x_i^2 \right) + 2 \\
&= 3 \sum_{3 \leq i < j} x_i x_j - \sum_{3 \leq i} x_i^2 + 3 \sum_{3 \leq i} x_i - 9.
\end{aligned}$$

From this point on, a simpler notation shall be adopted.

Notation:

- Unless it must be made explicit, the A -algebra $\Lambda_A[x]$ and A -modules $\Lambda_A^k[x]$ of symmetric functions will be denoted without mention to the set of variables by Λ_A and Λ_A^k , respectively;
- Similarly, unless it must be made explicit, symmetric functions will be denoted without brackets nor mention to the set of variables;
- When $A = \mathbb{Z}$ the algebra and modules of symmetric functions will be denoted without mention to the coefficient ring by Λ and Λ^k , respectively;
- Let $a = (a_1, a_2, \dots, a_n) \in A^n$ and $f \in \Lambda_A$, The evaluation $\bar{e}v_a(f)$ will be denoted by $f(a_1, a_2, \dots, a_n, x_1, x_2, \dots)$;
- If x and y are non-intersecting sets of variables, $\Lambda_A[x, y] := \Lambda_A[x \cup y]$ and $f(x, y) := f(x \cup y) \in \Lambda_A[x \cup y]$;
- If $x = \{x_1, x_2, \dots\}$ and \mathcal{A} is a non-empty subset of positive integer indices, denote $x_{\mathcal{A}} = \{x_i : i \in \mathcal{A}\}$, so $f(x_{\mathcal{A}}) \in \Lambda_A[x_{\mathcal{A}}]$.

2.3 Λ bases

There are six families of symmetric functions, which are indexed by partitions $\lambda \in \mathcal{P}$, of particular importance:

- Monomial Functions (m_λ)
- Elementary Functions (e_λ)
- Complete Functions (h_λ)
- Forgotten Functions (f_λ)
- Power Functions (p_λ)
- Schur Functions (s_λ)

With the exception of power functions, which are a \mathbb{Q} -base for $\Lambda_{\mathbb{Q}}$ as a \mathbb{Q} -module, these functions are \mathbb{Z} -bases for Λ as \mathbb{Z} -modules.

Furthermore, power functions indexed by partitions with a single row ($\{p_{(r)}\}$) are a \mathbb{Q} -base for $\Lambda_{\mathbb{Q}}$ as a \mathbb{Q} -algebra and elementary and complete functions ($\{e_{(r)}\}$ and $\{h_{(r)}\}$) are \mathbb{Z} -bases for Λ as a \mathbb{Z} -algebra.

Another important family of symmetric functions, though not a base for Λ , are the Skew Schur functions. As their name suggest, Skew Schur functions are indexed by skew diagrams instead of partitions ($s_{\lambda/\mu}$) and generalize “regular” Schur functions ($s_\lambda = s_{\lambda/(0)}$). In addition to their intrinsic relevance, they are useful assets for proving results regarding “regular” Schur functions.

Examples of these new functions in terms of monomial functions are given in the appendix in the form of transition matrices.

2.3.1 Monomial Functions

Monomial functions are defined from monomial polynomials by

$$m_\lambda := \rho_{\ell(\lambda)}(m_\lambda(x_1, \dots, x_{\ell(\lambda)})) \quad (2)$$

and can also be expressed by the formal sum

$$m_\lambda = \sum_{\alpha^+ = \lambda} x^\alpha. \quad (3)$$

Examples:

- $m_{(3,2,1)} = \sum_{i \neq j \neq k \neq i} x_i^3 x_j^2 x_k$
- $m_{(2,2,1)} = \sum_{\substack{i < j \\ k \neq i, j}} x_i^2 x_j^2 x_k$
- $m_{(r)} = \sum_i x_i^r$
- $m_{(a^r)} = \sum_{i_1 < \dots < i_r} x_{i_1}^a \cdots x_{i_r}^a$

Proposition 11. *Monomial functions are a \mathbb{Z} -base for Λ as a \mathbb{Z} -module.*

Proof: For every $f \in \Lambda$ there is some $\ell \in \mathbb{N}$ and $f_\ell \in \Lambda[x_1, x_2, \dots, x_\ell]$ such that $f = \rho_\ell(f_\ell)$. Since f_ℓ has a unique linear decomposition in terms of monomial polynomials $f_\ell = \sum_{\ell(\lambda) \leq \ell} c_\lambda m_\lambda(x_1, x_2, \dots, x_\ell)$,

$$f = \rho_\ell(f_\ell) = \sum_{\ell(\lambda) \leq \ell} c_\lambda \rho_\ell(m_\lambda(x_1, x_2, \dots, x_\ell)) = \sum_{\ell(\lambda) \leq \ell} c_\lambda m_\lambda$$

so f also has a unique decomposition in terms of monomial functions.

□

Proposition 12. *For two sets of variables $x = \{x_1, x_2, \dots\}$ and $y = \{y_1, y_2, \dots\}$, monomial functions satisfy the relation*

$$m_\lambda(x, y) = \sum_{\mu \cup \nu = \lambda} m_\mu(x) m_\nu(y).$$

Proof: This is proven by induction on the length of λ .

The statement is trivially true for $\ell(\lambda) = 0$, in which case $\lambda = (0)$ and $m_{(0)}(x, y) = 1 = m_{(0)}(x)m_{(0)}(y)$.

Suppose the statement is valid for all partitions with length L . Let $\lambda' = (\lambda_1, \lambda_2, \dots, \lambda_L)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_L, \lambda_{L+1})$. Denoting $x^{(i)} = \{x\} \setminus \{x_i\}$ and $y^{(i)} = \{y\} \setminus \{y_i\}$,

$$\begin{aligned}
m_\lambda(x, y) &= \sum_i x_i^{\lambda_{L+1}} m_{\lambda'}(x^{(i)}, y) + \sum_i y_i^{\lambda_{L+1}} m_{\lambda'}(x, y^{(i)}) \\
&= \sum_{\mu \cup \nu = \lambda'} \sum_i x_i^{\lambda_{L+1}} m_\mu(x^{(i)}) m_\nu(y) + \sum_{\mu \cup \nu = \lambda'} y_i^{\lambda_{L+1}} m_\mu(x) m_\nu(y^{(i)}) \\
&= \sum_{\mu \cup \nu = \lambda'} m_{\mu \cup (\lambda_{L+1})}(x) m_\nu(y) + \sum_{\mu \cup \nu = \lambda'} m_\mu(x) m_{\nu \cup (\lambda_{L+1})}(y) \\
&= \sum_{\mu \cup \nu = \lambda} m_\mu(x) m_\nu(y).
\end{aligned}$$

□

Corollary: For some partition $\lambda \neq (0)$ and some positive integer N , consider the lists of variables (where now each $x_i^{(j)}$ denotes a variable belonging to the list $x^{(j)}$)

$$\begin{aligned}
x^{(1)} &= (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots) \\
x^{(2)} &= (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots) \\
&\vdots \\
x^{(N)} &= (x_1^{(N)}, x_2^{(N)}, x_3^{(N)}, \dots)
\end{aligned}$$

Then

$$m_\lambda(x^{(1)}, x^{(2)}, \dots, x^{(N)}) = \sum_{\bigcup_{i=1}^N \mu^{(i)} = \lambda} m_{\mu^{(1)}}(x^{(1)}) \cdot m_{\mu^{(2)}}(x^{(2)}) \cdots m_{\mu^{(N)}}(x^{(N)})$$

Where each $\mu^{(i)}$ is a partition and $\bigcup_{i=1}^N \mu^{(i)} = \mu^{(1)} \cup \mu^{(2)} \cup \dots \cup \mu^{(N)}$.

Proof: This is proven by induction on N .

By the previous proposition, the statement is true for $N = 1$.

Suppose the statement is holds for N and let $y = (x^{(1)}, x^{(2)}, \dots, x^{(N)})$.

$$\begin{aligned}
& m_\lambda(x^{(1)}, x^{(2)}, \dots, x^{(N)}, x^{(N+1)}) \\
&= m_\lambda(y, x^{(N+1)}) \\
&= \sum_{\mu \cup \nu = \lambda} m_\mu(y) m_\nu(x^{(N+1)}) \\
&= \sum_{\mu \cup \nu = \lambda} \sum_{\bigcup_{i=1}^N \mu^{(i)} = \mu} m_{\mu^{(1)}}(x^{(1)}) m_{\mu^{(2)}}(x^{(2)}) \cdots m_{\mu^{(N)}}(x^{(N)}) m_\nu(x^{(N+1)}) \\
&= \sum_{\bigcup_{i=1}^{N+1} \mu^{(i)}} m_{\mu^{(1)}}(x^{(1)}) m_{\mu^{(2)}}(x^{(2)}) \cdots m_{\mu^{(N+1)}}(x^{(N+1)}).
\end{aligned}$$

□

2.3.2 Elementary Functions

Elementary functions e_λ are defined in terms of $m_{(1^r)}$ as

$$\begin{aligned}
e_r &= m_{(1^r)} \\
e_\lambda &= \prod_i e_{\lambda_i},
\end{aligned}$$

and the generating function for $(e_r)_{r \in \mathbb{N}}$ is the formal sum

$$E(t) := \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t), \quad (4)$$

meaning that each e_r may also be defined as the coefficient of t^r in $\prod_{i \geq 1} (1 + x_i t)$.

Proposition 13.

- a) $(e_\lambda)_{\lambda \in \mathcal{P}}$ is a \mathbb{Z} -base for Λ as a \mathbb{Z} -module;
- b) $(e_r)_{r \in \mathbb{N}}$ is a \mathbb{Z} -base for Λ as a \mathbb{Z} -algebra.

Proof: Since each e_λ is uniquely obtained (apart from order of multiplication) from a product of e_r 's, these statements are equivalent, so only one needs to be proved.

So as to prove the first statement, it will be shown that the linear decomposition of elementary functions in terms of monomial functions

$$e_\lambda = \sum_{\mu} c_{\lambda\mu} m_\mu$$

is invertible.

By definition, each elementary function e_{λ_i} can be expressed by the formal sum

$$e_{\lambda_i} = \sum_{k_1 < \dots < k_{\lambda_i}} x_{k_1} \cdots x_{k_{\lambda_i}},$$

so $e_\lambda = \prod_i e_{\lambda_i}$ itself is given by

$$\prod_{i=1}^{\ell} \left(\sum_{k_1^{(i)} < \dots < k_{\lambda_i}^{(i)}} x_{k_1^{(i)}} x_{k_2^{(i)}} \cdots x_{k_{\lambda_i}^{(i)}} \right) = \sum_{k_1^{(1)} < \dots < k_{\lambda_1}^{(1)}} \cdots \sum_{k_1^{(\ell)} < \dots < k_{\lambda_\ell}^{(\ell)}} \left(\prod_{i=1}^{\lambda_i} x_{k_1^{(i)}} x_{k_2^{(i)}} \cdots x_{k_{\lambda_i}^{(i)}} \right).$$

where $\ell = \ell(\lambda)$. Therefore each monomial term x^α that appears in e_λ must have the form

$$\left(x_{k_1^{(1)}} x_{k_2^{(1)}} \cdots x_{k_{\lambda_1}^{(1)}} \right) \left(x_{k_1^{(2)}} x_{k_2^{(2)}} \cdots x_{k_{\lambda_2}^{(2)}} \right) \cdots \left(x_{k_1^{(\ell)}} x_{k_2^{(\ell)}} \cdots x_{k_{\lambda_\ell}^{(\ell)}} \right) \quad (5)$$

where $k_j^{(i)} < k_{j+1}^{(i)}$ for all $(i, j) \in \text{diag}(\lambda)$.

Since e_λ is symmetric, all x^α such that $\alpha^+ = \mu$ must have the same coefficient $c_{\lambda\mu}$, thence the number of occurrences of x^μ in e_λ .

Each product (5) yielding x^μ can be univocally associated with a labelling with weight μ of the diagram of λ by $(i, j) \mapsto k_j^{(i)}$. By construction, these labellings must be strictly increasing along rows, therefore $\mu \leq \lambda'$. Moreover, for $\mu = \lambda'$, there is only one possible such labelling, so $c_{\lambda\lambda'} = 1$.

It follows that

$$e_\lambda = \sum_{\mu \leq \lambda'} c_{\lambda\mu} m_\mu = m_{\lambda'} + \sum_{\mu > \lambda} c_{\lambda\mu'} m_{\mu'}. \quad (6)$$

Let $\lambda \vdash n$ and index partitions of n according to the (total) reverse lexicographical ordering

$$(1^n) = \lambda^{(1)} <_{\mathcal{L}} \lambda^{(2)} <_{\mathcal{L}} \cdots <_{\mathcal{L}} \lambda^{(P-1)} <_{\mathcal{L}} \lambda^{(P)} = (n)$$

where $P = \#\mathcal{P}_n$. This ordering is generalized by the (partial) natural ordering, so $\lambda^{(i_1)} < \lambda^{(i_2)} \Rightarrow \lambda^{(i_1)} <_{\mathcal{L}} \lambda^{(i_2)} \Rightarrow i_1 < i_2$. Moreover, partition conjugation inverts the order, that is, $\mu < \lambda \Leftrightarrow \lambda' < \mu'$, so for this indexation: $(\lambda^{(k)})' = \lambda^{(P+1-k)}$, $\forall k \in \{1, \dots, P\}$. With these remarks, (6) implies

$$e_{\lambda^{(i)}} = m_{\lambda^{(P+1-i)}} + \sum_{j>i} c_{\lambda^{(i)}\lambda^{(P+1-j)}} m_{\lambda^{(P+1-j)}}.$$

This new formula omits the fact that $c_{\lambda^{(i)}\lambda^{(P+1-j)}} = 0$ whenever $\lambda^{(i)} \not\preceq (\lambda^{(j)})'$ (in the natural ordering), but highlights the triangular relation between elementary and monomial functions. With this notation, (6) is expressed in matrix form like so

$$\begin{bmatrix} e_{\lambda^{(1)}} \\ e_{\lambda^{(2)}} \\ e_{\lambda^{(3)}} \\ \vdots \\ e_{\lambda^{(p-1)}} \\ e_{\lambda^{(p)}} \end{bmatrix} = \begin{bmatrix} 1 & c_{\lambda^{(1)}\lambda^{(p-1)}} & c_{\lambda^{(1)}\lambda^{(p-2)}} & \dots & c_{\lambda^{(1)}\lambda^{(1)}} \\ & 1 & c_{\lambda^{(2)}\lambda^{(p-2)}} & \dots & c_{\lambda^{(2)}\lambda^{(1)}} \\ & & 1 & & c_{\lambda^{(3)}\lambda^{(1)}} \\ & & & \ddots & \vdots \\ & & & & c_{\lambda^{(p-1)}\lambda^{(1)}} \\ & & & & 1 \end{bmatrix} \begin{bmatrix} m_{\lambda^{(p)}} \\ m_{\lambda^{(p-1)}} \\ m_{\lambda^{(p-2)}} \\ \vdots \\ m_{\lambda^{(2)}} \\ m_{\lambda^{(1)}} \end{bmatrix} \quad (7)$$

Where

$$c_{\lambda\mu} = \#\{T \in \widehat{\text{Tab}}(\lambda, \mu) : T \text{ strictly increasing along rows}\}. \quad (8)$$

Elementary and monomial functions indexed by partitions with same weight are therefore related by an upper-triangular matrix with entries in \mathbb{Z} and with only 1's in its diagonal. Since 1 is invertible in \mathbb{Z} and \mathbb{Z} is an integral domain, the matrix itself is invertible. This shows that every monomial symmetric function can be finitely expressed as a linear combination of elementary functions, and completes the proof.

□

Proposition 14. For all $r \in \mathbb{N}$,

$$e_{(1^r)} = \sum_{\lambda \vdash r} \binom{r}{\lambda} \frac{1}{\prod_i \gamma_i!} m_\lambda,$$

where $\lambda = (\lambda_1^{\gamma_1}, \lambda_2^{\gamma_2}, \dots)$.

Proof: By the same argument in (5), monomial terms that appear in $e_{(1^r)}$ have the form $x_{i_1} \cdots x_{i_r}$ with no relation between indices. Therefore the coefficient of each m_λ in the monomial decomposition of elementary functions equals the number of ways x^λ can be obtained by multiplying variables in $\{x_1, x_2, \dots\}$. This is the number of permutations of variables $x_1, x_2, \dots, x_{\ell(\lambda)}$, where each x_i occurs exactly λ_i times, accounted for repetitions of repeated terms in λ , which is exactly $\binom{r}{\lambda} \frac{1}{\prod_i \gamma_i!}$.

□

Proposition 15. For all $a \geq b$,

$$e_{(a,b)} = \sum_{k=0}^{\lfloor \frac{a+b}{2} \rfloor} \binom{a+b-2k}{a-k} m_{(2^k, 1^{a+b-2k})}.$$

Proof: From the definition, $e_{(a,b)} = e_a e_b = \sum_{\substack{i_1 < \dots < i_a \\ i'_1 < \dots < i'_b}} (x_{i_1} \cdots x_{i_a})(x_{i'_1} \cdots x_{i'_b})$,

so monomial terms that appear in $e_{(a,b)}$ are of the form $x_{j_1}^2 \cdots x_{j_k}^2 x_{j'_1} \cdots x_{j'_{k'}}$ with $j_1 < \dots < j_k$, $j'_1 < \dots < j'_{k'}$ and $2k + k' = a + b$. The coefficient of each of these terms is the total number of products $(x_{i_1} \cdots x_{i_a})(x_{i'_1} \cdots x_{i'_b})$ with $i_1 < \dots < i_a$ and $i'_1 < \dots < i'_b$ which yield $x_{j_1}^2 \cdots x_{j_k}^2 x_{j'_1} \cdots x_{j'_{k'}}$. This is equivalent to the number of sets $\{i_1, \dots, i_a\}$ and $\{i'_1, \dots, i'_b\}$ such that

$$\begin{aligned} \{j_1, \dots, j_k\} &= \{i_1, \dots, i_a\} \cap \{i'_1, \dots, i'_b\} \\ \{j'_1, \dots, j'_{k'}\} &= \{i_1, \dots, i_a\} \cup \{i'_1, \dots, i'_b\} \setminus \{i_1, \dots, i_a\} \cap \{i'_1, \dots, i'_b\} \end{aligned}$$

which is $\binom{a+b-2k}{a-k}$. Finally, since k ranges from 0 to $\lfloor \frac{a+b}{2} \rfloor$,

$$\begin{aligned} e_{(a,b)} &= \sum_{k=0}^{\lfloor \frac{a+b}{2} \rfloor} \sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_{a+b-2k}}} \binom{a+b-2k}{a-k} x_{i_1}^2 \cdots x_{i_k}^2 x_{j_1} \cdots x_{j_{a+b-2k}} \\ &= \sum_{k=0}^{\lfloor \frac{a+b}{2} \rfloor} \binom{a+b-2k}{a-k} m_{(2^k, 1^{a+b-2k})}. \end{aligned}$$

□

Examples of elementary functions in terms of monomial functions are given in appendix A.1.

2.3.3 Complete Functions

Complete functions are defined in terms of m_λ as

$$\begin{aligned} h_r &:= \sum_{\lambda \vdash r} m_\lambda \\ h_\lambda &:= \prod_i h_{\lambda_i}. \end{aligned}$$

The generating function for $(h_r)_{r \in \mathbb{N}}$ is the formal sum

$$H(t) := \sum_{r \geq 0} h_r t^r \tag{9}$$

which can also be expressed as

$$\sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} \left(\sum_{k \geq 0} x_i^k t^k \right) = \prod_{i \geq 1} (1 - x_i t)^{-1}, \tag{10}$$

so each h_r can alternatively be defined as the coefficient of t^r in the generalized Taylor series expansion of $\prod_{i \geq 1} (1 - x_i t)^{-1}$. Comparing $H(t)$ with $E(t)$, it is clear that:

$$H(t)E(-t) = 1 = H(-t)E(t)$$

Equivalently:

$$\sum_{r=0}^n (-1)^r e_r h_{n-r} = 0 \quad \forall n > 0. \quad (11)$$

Proposition 16.

a) (h_λ) is a \mathbb{Z} -base for Λ as a \mathbb{Z} -module;

b) (h_r) is a \mathbb{Z} -base for Λ as a \mathbb{Z} -algebra.

Proof: As for the case for elementary functions, each h_λ is uniquely obtained (apart from order of multiplication) from a product of h_r 's, so these statements are equivalent, and only one of them needs to be proved.

The first statement is a consequence of (4) and (10), as they imply:

$$H(-t)E(t) = 1$$

From which it follows that

$$e_r = - \sum_{k=0}^{r-1} (-1)^{r-k} e_k h_{r-k} \quad \forall r \geq 1. \quad (12)$$

So e_r 's are algebraically generated by h_r 's. Since h_r 's are linearly independent and (e_r) is a base for Λ as a \mathbb{Z} -algebra, so must (h_r) be.

□

Observation: since $h_1 = e_1$,

$$h_{(1^r)} = e_{(1^r)} = \sum_{\lambda \vdash r} \binom{r}{\lambda} \frac{1}{\prod_i \gamma_i!} m_\lambda,$$

where $\lambda = (\lambda_1^{\gamma_1}, \lambda_2^{\gamma_2}, \dots)$.

Further examples of complete functions in terms of monomial functions are given in appendix A.2.

2.3.4 Forgotten Functions

Since both (e_r) and (h_r) are \mathbb{Z} -bases for Λ as a \mathbb{Z} -algebra, ω as defined below is an algebra isomorphism

$$\begin{aligned}\omega : \Lambda &\rightarrow \Lambda \\ e_r &\mapsto h_r\end{aligned}$$

Note that it is equivalent to define ω as an automorphism of \mathbb{Z} -modules by $\omega : e_\lambda \mapsto h_\lambda$, since $e_\lambda = \prod_i e_{\lambda_i}$ and $h_\lambda = \prod_i h_{\lambda_i}$.

Applying ω and some manipulation to (12), one sees that

$$\sum_{k=0}^r (-1)^k h_{r-k} (e_k - \omega(h_k)) = 0 \quad \forall r \geq 1.$$

and since $(h_r)_{r \in \mathbb{N}}$ forms a \mathbb{Z} -base for Λ as a \mathbb{Z} -algebra, it follows that $\omega(h_r) = e_r$ for all $r \in \mathbb{N}$. Therefore ω is an involution in Λ .

Forgotten functions are called so because they are best described by the involution ω rather than directly. They are defined as

$$f_\lambda := \omega(m_\lambda).$$

From their definition, forgotten functions are clearly a \mathbb{Z} -base for Λ as a \mathbb{Z} -module and $\omega(f_\lambda) = m_\lambda$.

Examples of forgotten functions in terms of monomial functions are given in appendix A.3.

2.3.5 Power Functions

Power Functions are defined in terms of m_λ as

$$\begin{aligned}p_r &:= m_{(r)} \\ p_\lambda &:= \prod_i p_{\lambda_i}\end{aligned}$$

and the generating function for (p_r) is the formal sum

$$P(t) := \sum_{r \geq 0} p_{r+1} t^r,$$

which can be expressed in terms of generating functions for elementary and complete functions.

$$\begin{aligned} P(t) &= \sum_{i \geq 1} \sum_{r \geq 0} x_i^{r+1} t^r \\ &= \sum_{i \geq 1} x_i (1 - x_i t)^{-1} \\ &= \sum_{i \geq 1} \frac{d}{dt} \log((1 - x_i t)^{-1}) \\ &= \frac{d}{dt} \log \left(\prod_{i \geq 1} (1 - x_i t)^{-1} \right) \\ &= \begin{cases} \frac{H'(t)}{H(t)} \\ \frac{E'(-t)}{E(-t)} \end{cases} \end{aligned} \tag{13}$$

Meaning power sums p_r are the coefficients of t^{r-1} in $\frac{H'(t)}{H(t)}$ or $\frac{E'(-t)}{E(-t)}$.

Now since $P(t)H(t) = H'(t)$ and $P(t)E(-t) = E'(-t)$,

$$\begin{aligned} \sum_{r \geq 0} \left(\sum_{k=0}^r p_{k+1} h_{r-k} \right) t^r &= \sum_{r \geq 0} (r+1) h_{r+1} t^r \\ \sum_{r \geq 0} \left(\sum_{k=0}^r p_{k+1} (-1)^{r-k} e_{r-k} \right) t^r &= \sum_{r \geq 0} (r+1) (-1)^r e_{r+1} t^r \end{aligned}$$

Comparing coefficients for each t^r :

$$r h_r = \sum_{k=1}^r p_k h_{r-k} \tag{14}$$

$$r e_r = \sum_{k=1}^r p_k (-1)^{k-1} e_{r-k} \tag{15}$$

Also from (13),

$$\begin{aligned}
H(t) &= \exp\left(\sum_{r \geq 0} \frac{p_{r+1}}{r+1} t^r\right) \\
&= \prod_{r \geq 1} \exp\left(\frac{p_r}{r} t^r\right) \\
&= \prod_{r \geq 1} \sum_{m_r=0}^{\infty} \left(\frac{(p_r t^r)^{m_r}}{r^{m_r} m_r!}\right) \\
&= \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda} t^{|\lambda|}
\end{aligned} \tag{16}$$

where, recall that for $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \dots)$, $z_{\lambda} = \prod_{r \geq 1} m_r! \lambda_r^{m_r}$.

Proposition 17.

- a) (p_{λ}) is a \mathbb{Q} -base for $\Lambda_{\mathbb{Q}}$ as a \mathbb{Q} -module;
- b) (p_r) is a \mathbb{Q} -base for $\Lambda_{\mathbb{Q}}$ as a \mathbb{Q} -algebra.

Proof: Since each p_{λ} is uniquely obtained (apart from order of multiplication) from a product of p_r 's, these statements are equivalent. So as to prove the first statement and demonstrate whence the necessity of altering which ring Λ should be a module over comes from, a similar approach to that for elementary functions will be undertaken. It will be shown that the linear decomposition of power functions in terms of monomial functions

$$p_{\lambda} = \sum_{\mu} L_{\lambda\mu} m_{\mu}$$

is triangular and not singular, hence invertible over \mathbb{Q} .

Recall that $S_{\ell, \tilde{\ell}}$ ($\ell \geq \tilde{\ell}$) denotes the set of surjections from $\{1, 2, \dots, \ell\}$ onto $\{1, 2, \dots, \tilde{\ell}\}$ and for $g \in S_{\ell, \tilde{\ell}}$, define

$$\lambda^{(g)} = \left(\sum_{j \in g^{-1}(1)} \lambda_j, \sum_{j \in g^{-1}(2)} \lambda_j, \dots, \sum_{j \in g^{-1}(\tilde{\ell})} \lambda_j \right)^+.$$

Then

$$\begin{aligned}
p_\lambda &= \prod_i p_{\lambda_i} \\
&= \sum_{k_1, \dots, k_\ell} x_{k_1}^{\lambda_1} x_{k_2}^{\lambda_2} \cdots x_{k_\ell}^{\lambda_\ell} \\
&= \sum_{\sigma \in S_\ell} \sum_{k_1 \leq \dots \leq k_\ell(\lambda)} x_{k_1}^{\lambda_{\sigma(1)}} x_{k_2}^{\lambda_{\sigma(2)}} \cdots x_{k_\ell}^{\lambda_{\sigma(\ell)}} \\
&= \sum_{\tilde{\ell}=1}^{\ell} \sum_{g \in S_{\ell, \tilde{\ell}}} \sum_{k_1 < \dots < k_{\tilde{\ell}}} x_{k_1}^{\lambda_1^{(g)}} x_{k_2}^{\lambda_2^{(g)}} \cdots x_{k_{\tilde{\ell}}}^{\lambda_{\tilde{\ell}}^{(g)}} \\
&= \sum_{\tilde{\ell}=1}^{\ell} \sum_{g \in S_{\ell, \tilde{\ell}}} m_{\lambda^{(g)}}.
\end{aligned}$$

Grouping monomial functions with same index,

$$p_\lambda = \sum_{\lambda \leq_{\mathcal{R}} \mu} L_{\lambda\mu} m_\mu \quad (17)$$

where

$$L_{\lambda\mu} = \#\{g \in S_{\ell, \ell(\mu)} : \mu = \lambda^{(g)}\}. \quad (18)$$

Now since $\lambda \leq_{\mathcal{R}} \mu \Rightarrow \lambda \leq \mu$, equation (17) — similarly to the case for elementary functions — implies that for every fixed integer n and $\lambda \vdash n$, the ordered sets of power functions (p_λ) and monomial functions (m_λ) are related by a triangular operator. Moreover, (18) implies that for all λ :

$$L_{\lambda\lambda} = \#\{g \in S_\ell : \lambda^{(g)} = \lambda\} = \prod_{i \geq 1} \#\{j : \lambda_j = i\}! = \prod_{i \geq 1} m_i^{(\lambda)}!$$

where $\lambda = (\lambda_1^{m_1^{(\lambda)}}, \lambda_2^{m_2^{(\lambda)}}, \lambda_3^{m_3^{(\lambda)}}, \dots)$. So

$$\det L = \prod_{\lambda \vdash n} \prod_{i \geq 1} m_i^{(\lambda)}!$$

Therefore for each n , the matrix $L = [L_{\lambda\mu}]_{\lambda, \mu \vdash n}$ is invertible in $\mathbb{Z}\left[\frac{1}{p_1}, \frac{1}{p_2}, \dots, \frac{1}{p_k}\right]$, where $p_1, \dots, p_k \leq n$ are the first prime numbers lesser than or equal to n . Finally, for the whole algebra of symmetric functions, power functions (p_λ) are a \mathbb{Q} -base for $\Lambda_{\mathbb{Q}}$ as a \mathbb{Q} -module.

□

As for what happens when the involution ω is applied to p_λ , equations (14) and (15) yield

$$\sum_{k=1}^r p_k (-1)^{k-1} e_{r-k} = r e_r = r \omega(h_r) = \sum_{k=1}^r \omega(p_k) \omega(h_{r-k}) = \sum_{k=1}^r \omega(p_k) e_{r-k}$$

So $\omega(p_k) = (-1)^{k-1} p_k$, which implies

$$\omega(p_\lambda) = \varepsilon(\lambda) p_\lambda. \quad (19)$$

Observation: The monomial decomposition of power functions indexed by partitions p_λ can be obtained by repeatedly expanding each product $p_{\lambda_i} m_\mu$ in terms of monomial functions. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ and $r \in \mathbb{N}$, then

$$\begin{aligned} p_r m_\lambda &= \left(\sum_i x_i^r \right) \left(\sum_{\alpha^+ = \lambda} x^\alpha \right) \\ &= \sum_i \sum_{\alpha^+ = \lambda} x^{\alpha + r \varepsilon_i} \\ &= \sum_i \sum_{\alpha^+ = (\lambda + r \varepsilon_i)^+} x^\alpha \\ &= m_{\lambda \cup (r)} + \sum_{i=1}^{\ell} m_{(\lambda_1, \dots, \lambda_{i-1}, \lambda_i + r, \lambda_{i+1}, \dots, \lambda_\ell)^+}. \end{aligned}$$

Or denoting $\lambda = (\lambda_1^{\gamma_1}, \lambda_2^{\gamma_2}, \dots, \lambda_\ell^{\gamma_\ell})$,

$$p_r m_\lambda = m_{\lambda \cup (r)} + \sum_{i=1}^{\ell} \gamma_i m_{(\lambda_i + r, \lambda_1^{\gamma_1}, \dots, \lambda_i^{\gamma_i - 1}, \dots, \lambda_\ell^{\gamma_\ell})}.$$

Proposition 18. *For every positive integer n and every partition $\lambda = (\lambda_1, \lambda_2, \dots)$, let $n\lambda$ denote the partition $(n\lambda_1, n\lambda_2, \dots)$. Then*

$$p_\lambda = \sum_{\mu} c_\mu m_\mu \quad \Rightarrow \quad p_{n\lambda} = \sum_{\mu} c_\mu m_{n\mu}.$$

Proof: This follows directly from the identities

$$p_{n\lambda}(x_1, x_2, \dots) = \prod_{i=1}^{\ell(\lambda)} \sum_j x_j^{n\lambda_i} = \prod_{i=1}^{\ell(\lambda)} \sum_j (x_j^n)^{\lambda_i} = p_\lambda(x_1^n, x_2^n, \dots)$$

$$m_{n\mu}(x_1, x_2, \dots) = \sum_{\alpha^+ = n\mu} x^\alpha = \sum_{\alpha^+ = \mu} x^{n\alpha} = m_\mu(x_1^n, x_2^n, \dots).$$

□

This proposition together with the previous observation, facilitates the computation of the monomial decomposition of power functions.

Examples: For all cases listed below, assume $a > b > c > 0$.

1. $p_{(a^r)} = \sum_{\lambda \vdash r} \binom{r}{\lambda} \frac{1}{\prod_i \gamma_i!} m_{a\lambda}$, where $\lambda = (\lambda_1^{\gamma_1}, \lambda_2^{\gamma_2}, \dots)$;
2. $p_{(a,b)} = m_{(a,b)} + m_{(a+b)}$;
3. $p_{(a,b,c)} = m_{(a,b,c)} + m_{(a+b,c)} + m_{(a+c,b)} + m_{(a,b+c)} + m_{(a+b+c)}$ where $a \neq b+c$;
4. $p_{(a+b,a,b)} = m_{(a+b,a,b)} + m_{(a+2b,a)} + m_{(2a+b,b)} + 2m_{(a+b,a+b)} + m_{(2a+2b)}$.

Further examples of monomial expansions of power functions are listed in appendix A.4.

2.3.6 Schur and Skew Schur Functions

Skew Schur functions are the first family of functions seen in this section which are neither a base for Λ , nor indexed by partitions, but skew partitions instead. They do, however, generalize regular Schur functions, which are, in turn, \mathbb{Z} -bases for Λ indexed by partitions.

Unlike previously mentioned bases, there is no most appropriate or generally preferred way to define Schur functions. The definition chosen in this text helps illustrate the close connection between Schur and Skew Schur functions, and strongly relies on the following lemma.

Lemma 1. Let μ, λ be two partitions, and p, q positive integers such that

$$\begin{aligned}\max\{\ell(\lambda), \ell(\mu)\} &\leq p \\ \max\{\ell(\lambda'), \ell(\mu')\} &\leq q\end{aligned}$$

then

$$\det \left[h_{\lambda_i - \mu_j - i + j} \right]_{1 \leq i, j \leq p} = \det \left[e_{\lambda'_i - \mu'_j - i + j} \right]_{1 \leq i, j \leq q}$$

setting $h_r = e_r = 0$ for $r < 0$.

Proof:

Let $N = p + q$ and consider the $N \times N$ matrices

$$H = [h_{i-j}]_{0 \leq i, j < N} \quad \text{and} \quad E = [(-1)^{i-j} e_{i-j}]_{0 \leq i, j < N}.$$

Since $h_0 = e_0 = 1$ and $h_r = e_r = 0$ for $r < 0$, H and E are lower triangular matrices with 1's in their diagonals, hence their determinants are both 1. Because of (11), their product yields

$$HE = EH = \left[\sum_{k=0}^{i-j} (-1)^k e_k h_{i-j-k} \right]_{0 \leq i, j < N} = [e_0 h_0 \delta_{ij}]_{0 \leq i, j < N} = I_{N \times N}$$

So E and H are each other's inverses.

In order to proceed, it is indispensable to make use of the following relation for minors. Let A be a non-singular $N \times N$ matrix. Let J_r and J_c be strictly increasing non-empty subsequences of $1, \dots, N$, both of which with $p < N$ elements, and J'_l and J'_c their respective increasing complementary sequences, both of which with $N - p < N$ elements. Then the minors A_{J_r, J_c} of A and $(A^{-1})_{J'_l, J'_c}$ of A^{-1} are related by the equation

$$\det(A_{J_r, J_c}) = (-1)^{\sum_{i=1}^p (J_r(i) + J_c(i))} \det A \cdot \det((A^{-1})_{J'_l, J'_c}) \quad (20)$$

The identity above can be directly applied to the H minor with row indices (J_r) given by $\lambda_i + p - i$ and column indices (J_c) given by $\mu_j + p - j$, and the correspondent E complementary minor with row indices (J'_r) given

by $p - 1 + i - \lambda'_i$ and column indices (J'_c) given by $p - 1 + j - \mu'_j$. With these substitutions, (20) becomes

$$\begin{aligned} \det (H_{\lambda_i+p-i, \mu_j+p-j}) &= (-1)^{\sum_i^p (\lambda_i+p-i) + \sum_j^p (\mu_j+p-j)} \det(H) \det \left(E_{p-1+i-\lambda'_i, p-1+j-\mu'_j} \right) \\ &= (-1)^{|\lambda|+|\mu|} \det \left(E_{p-1+i-\lambda'_i, p-1+j-\mu'_j} \right) \end{aligned}$$

which implies the desired equality, for:

$$\begin{aligned} &\det [h_{\lambda_i-\mu_j-i+j}]_{1 \leq i, j \leq p} \\ &= \det (H_{\lambda_i+p-i, \mu_j+p-j}) \\ &= (-1)^{|\lambda|+|\mu|} \det \left(E_{p-1+i-\lambda'_i, p-1+j-\mu'_j} \right) \\ &= (-1)^{|\lambda|+|\mu|} \det \left[(-1)^{\lambda'_i-\mu'_j-i+j} e_{\lambda'_i-\mu'_j-i+j} \right]_{1 \leq i, j \leq q} \\ &= (-1)^{|\lambda|+|\mu|} (-1)^{(|\lambda|-\frac{q(q-1)}{2}) + (-|\mu|+\frac{q(q-1)}{2})} \det \left[e_{\lambda'_i-\mu'_j-i+j} \right]_{1 \leq i, j \leq q} \\ &= \det \left[e_{\lambda'_i-\mu'_j-i+j} \right]_{1 \leq i, j \leq q}. \end{aligned}$$

□

Corollary: Letting $\mu = 0$ in the previous lemma implies

$$\det [h_{\lambda_i-i+j}]_{1 \leq i, j \leq p} = \det [e_{\lambda'_i-i+j}]_{1 \leq i, j \leq q}.$$

It should be noted that as long as p and q are sufficiently large, that is, larger than the largest non-zero index of λ, μ and λ', μ' respectively, their actual values are not relevant for the computation of the determinant. This is so because $\lambda_i = \mu_j = 0$ for $i, j > \max\{\ell(\lambda), \ell(\mu)\}$ and $\lambda'_i = \mu'_j = 0$ for $i, j > \max\{\ell(\lambda'), \ell(\mu')\}$. The resulting bottom-left square corners of both matrices $[h_{\lambda_i-\mu_j-i+j}]$ and $[e_{\lambda'_i-\mu'_j-i+j}]$ are upper triangular with 1's in their diagonals, leaving the determinants unchanged.

Proposition 19. *Let $p \geq \max\{\ell(\lambda), \ell(\mu)\}$.*

$$\mu \not\subseteq \lambda \Rightarrow \det [h_{\lambda_i-\mu_j-i+j}]_{1 \leq i, j \leq p} = 0$$

Proof: If there exists r such that $\mu_r > \lambda_r$, then:

$$\lambda_i - \mu_j - i + j < 0, \quad \forall (i, j) \in \{r, \dots, p\} \times \{1, \dots, r\}$$

Consequently $[h_{\lambda_i - \mu_j - i + j}]_{1 \leq i, j \leq p}$ has a $(p - r) \times r$ block of zeroes in its bottom left-hand corner, so its determinant vanishes. □

For all partitions μ and λ with $\mu \subseteq \lambda$, “regular” and skew Schur functions are respectively defined as follows.

Definition 2.3.7. *Schur* and *Skew Schur functions* for partitions λ and $\mu \subset \lambda$ are defined as the determinants

$$\begin{aligned} s_\lambda &:= \det \left[h_{\lambda_i - i + j} \right] = \det \left[e_{\lambda'_i - i + j} \right] \\ s_{\lambda/\mu} &:= \det \left[h_{\lambda_i - \mu_j - i + j} \right] = \det \left[e_{\lambda'_i - \mu'_j - i + j} \right] \end{aligned}$$

It is very clear how skew Schur functions generalize Schur functions and the notation is not ambiguous for $[h_{\lambda_i - 0 - i + j}] = [h_{\lambda_i - i + j}]$, so $s_{\lambda/0} = s_\lambda$. Another immediate consequence of the definition is

$$\begin{aligned} s_{(1^n)} &= e_n = m_{(1^n)} \\ s_{(n)} &= h_n. \end{aligned}$$

Examples:

1. $s_{(2,1)}$

$$s_{(2,1)} = \begin{vmatrix} h_{2-1+1} & h_{2-1+2} \\ h_{1-2+1} & h_{1-2+2} \end{vmatrix} = \begin{vmatrix} h_2 & h_3 \\ h_0 & h_1 \end{vmatrix} = h_{(2,1)} - h_3 = 2m_{(1^3)} + m_{(2,1)}$$

2. $s_{(3,3,1)/(2,1)}$

$$\begin{aligned} s_{(3,3,1)/(2,1)} &= \begin{vmatrix} h_{3-2-1+1} & h_{3-1-1+2} & h_{3-0-1+3} \\ h_{3-2-2+1} & h_{3-1-2+2} & h_{3-0-2+3} \\ h_{1-2-3+1} & h_{1-1-3+2} & h_{1-0-3+3} \end{vmatrix} = \begin{vmatrix} h_1 & h_3 & h_5 \\ h_0 & h_2 & h_4 \\ 0 & 0 & h_1 \end{vmatrix} \\ &= s_{(2,1)} h_1 = 8m_{(1^4)} + 4m_{(2,1^2)} + 2m_{(2^2)} + m_{(3,1)} \end{aligned}$$

Since $\omega(h_{\lambda_i - \mu_j - i + j}) = e_{\lambda_i - \mu_j - i + j}$, another direct consequence of Lemma 1 is the corollary:

Corollary: For all partitions $\mu \subset \lambda$,

$$\begin{aligned}\omega(s_{\lambda/\mu}) &= s_{\lambda'/\mu'} \\ \omega(s_\lambda) &= s_{\lambda'}\end{aligned}$$

Theorem 1. *Schur functions of degree n form a base for Λ^n . Consequentially, Schur functions form a base for Λ .*

Proof: A similar approach to that for proposition 13 will be undertaken. It will be shown that the linear decomposition of Schur functions in terms of some other \mathbb{Z} -base of Λ_n is invertible. Consider the Schur function for arbitrary $\lambda \vdash n$ and $\ell(\lambda) = \ell$,

$$s_\lambda = \det \left[h_{\lambda_i - i + j} \right]_{1 \leq i, j \leq \ell} = \sum_{\nu \vdash n} c_{\lambda\nu} h_\nu.$$

For each $\sigma \in S_\ell$, let $(\lambda_1 - 1 + \sigma(1), \lambda_2 - 2 + \sigma(2), \dots, \lambda_\ell - \ell + \sigma(\ell))$ be denoted by $\lambda(\sigma)$. Each ν that shows up in the sum is equal to $\lambda(\sigma)^+$ for some $\sigma \in S_\ell$, and its respective coefficient $c_{\lambda\nu}$ is obtained by collecting all of ν occurrences in $\det [h_{\lambda_i - i + j}]_{1 \leq i, j \leq \ell}$, accounting for signs:

$$c_{\lambda\nu} = \sum_{\substack{\sigma \in S_\ell \\ \lambda(\sigma)^+ = \nu}} \text{sgn}(\sigma) \tag{21}$$

Each index r for which $\lambda(\sigma)_r < \lambda_r$, or equivalently, $\sigma(r) < r$, can be uniquely associated with the smaller index $\sigma(r)$ for which $\sigma^2(r) > \sigma(r)$, and consequentially, $\lambda(\sigma)_{\sigma(r)} > \lambda_{\sigma(r)}$. Therefore

$$\lambda(\sigma) > \lambda, \quad \forall \sigma \in S_\ell / \{\text{id}_\ell\}.$$

In particular, $\lambda(\sigma)^+ \geq \lambda(\sigma) > \lambda$ for all permutations $\sigma \in S_\ell / \{\text{id}_\ell\}$ because $\lambda(\sigma)^+$ is a partition. Therefore $\sigma = \text{id}_\ell$ is the only permutation for which $\lambda(\sigma)^+ = \lambda$, so $c_{\lambda\lambda} = 1$. This means that

$$s_\lambda = h_\lambda + \sum_{\nu > \lambda} c_{\lambda\nu} h_\nu \tag{22}$$

Since $\lambda \vdash n$ is arbitrary, one such equation holds for each $\lambda \vdash n$. Let $P = \#\mathcal{P}_n$ and once again index partitions in \mathcal{P}_n according to the reverse lexicographical ordering

$$(1^n) = \lambda^{(1)} <_{\mathcal{L}} \lambda^{(2)} <_{\mathcal{L}} \cdots <_{\mathcal{L}} \lambda^{(P-1)} <_{\mathcal{L}} \lambda^{(P)} = (n)$$

Expressing equations (22) in matrix form yields:

$$\begin{bmatrix} s_{\lambda^{(1)}} \\ s_{\lambda^{(2)}} \\ s_{\lambda^{(3)}} \\ \vdots \\ s_{\lambda^{(p-1)}} \\ s_{\lambda^{(p)}} \end{bmatrix} = \begin{bmatrix} 1 & c_{\lambda^{(1)}\lambda^{(2)}} & c_{\lambda^{(1)}\lambda^{(3)}} & \cdots & c_{\lambda^{(1)}\lambda^{(p-1)}} & c_{\lambda^{(1)}\lambda^{(p)}} \\ & 1 & c_{\lambda^{(2)}\lambda^{(3)}} & \cdots & c_{\lambda^{(2)}\lambda^{(p-1)}} & c_{\lambda^{(2)}\lambda^{(p)}} \\ & & 1 & \cdots & c_{\lambda^{(3)}\lambda^{(p-1)}} & c_{\lambda^{(3)}\lambda^{(p)}} \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & c_{\lambda^{(p-1)}\lambda^{(p)}} \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} h_{\lambda^{(1)}} \\ h_{\lambda^{(2)}} \\ h_{\lambda^{(3)}} \\ \vdots \\ h_{\lambda^{(p-1)}} \\ h_{\lambda^{(p)}} \end{bmatrix} \quad (23)$$

The matrix which expresses the linear decomposition of Schur functions in terms of complete functions is upper triangular with 1's in its diagonal, so n -homogeneous Schur functions $(s_{\lambda})_{\lambda \vdash n}$ are indeed a \mathbb{Z} -base for Λ^n for any n . This is equivalent to state that Schur functions are a base for Λ as a module.

□

Schur polynomials are simply the restriction of Schur functions to finitely many variables in the sense that for every n ,

$$\pi_n s_{\lambda}(x_1, x_2, \dots) = s_{\lambda}(x_1, x_2, \dots, x_n).$$

However, this consideration allows for an alternative definition which involves determinants whose components are variables x_1, x_2, \dots, x_n themselves, instead of symmetric functions.

Firstly, let $\alpha \in \mathbb{N}^n$, with $n \geq |\alpha|$ be an arbitrary composition. Consider the determinant ratio

$$\frac{\det \left[x_i^{\alpha_j + n - j} \right]_{n \times n}}{\det \left[x_i^{n - j} \right]_{n \times n}} \quad (24)$$

Both determinants would have repeated rows and therefore vanish if $x_{k_1} = x_{k_2}$ for any pair $k_1 < k_2$. So for all pairs $k_1 < k_2$, $(x_{k_1} - x_{k_2})$ must be a common factor for both of them. Now the denominator determinant is actually the Vandermond polynomial, which can be expressed precisely as the product $\prod_{i < j} (x_i - x_j)$. This means that

$$\det [x_i^{n-j}] = \prod_{i < j} (x_j - x_i) \left| \det [x_i^{\alpha_j + n - j}] \right|,$$

so the expression in (24) is indeed a polynomial on variables x_1, x_2, \dots, x_n . It is also symmetric because

$$\sigma \cdot \left(\frac{\det [x_i^{\alpha_j + n - j}]}{\det [x_i^{n-j}]} \right) = \frac{\det [x_{\sigma(i)}^{\alpha_j + n - j}]}{\det [x_{\sigma(i)}^{n-j}]} = \frac{\text{sgn}(\sigma) \det [x_i^{\alpha_j + n - j}]}{\text{sgn}(\sigma) \det [x_i^{n-j}]}.$$

With these remarks, the alternate definition for symmetric Schur polynomials can finally be stated.

Theorem 2. *Let λ be any partition and $n \geq \ell(\lambda)$. Then*

$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{\det [x_i^{\lambda_j + n - j}]_{n \times n}}{\det [x_i^{n-j}]_{n \times n}} \quad (25)$$

Proof: Schur functions, as were defined, are determinants involving either complete or elementary functions. Schur polynomials follow the same definition but with complete and elementary symmetric polynomials instead. Along this proof, in order to avoid notation cluttering, complete and elementary symmetric polynomials will be denoted without reference to their variables, that is, exactly like complete and elementary functions (as in h_λ and e_λ). This should cause no confusion as symmetric functions do not appear in the proof. There will be cases in which some variable x_k will be zeroed (or, equivalently, removed) from a symmetric elementary polynomial. In these cases, also for simplicity of notation, these polynomials will be denoted by

$$e_\lambda^{(k)} = e_\lambda(x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n).$$

Identities (4) and (10) imply that

$$\begin{aligned}
\sum_{r \geq 0} x_k^r t^r &= (1 - x_k t)^{-1} = \left(\prod_{i \geq 1} (1 - x_i t)^{-1} \right) \left(\prod_{\substack{i \geq 1 \\ i \neq k}} (1 - x_i t) \right) \\
&= \left(\sum_{r \geq 0} h_r t^r \right) \left(\sum_{r \geq 0} (-1)^r e_r^{(k)} t^r \right) \\
&= \sum_{r \geq 0} \left(\sum_{q=0}^r h_q (-1)^{r-q} e_{r-q}^{(k)} \right) t^r
\end{aligned}$$

So, for $r < n$, and considering $h_j = 0$ for $j < 0$,

$$x_k^r = \sum_{q=0}^r h_{r-q} (-1)^q e_q^{(k)} = \sum_{j=1}^n h_{r-n+j} (-1)^{n-j} e_{n-j}^{(k)}$$

When expressed in matrix form, these equalities become

$$\begin{bmatrix} x_j^{\alpha_i} \end{bmatrix}_{n \times n} = \begin{bmatrix} h_{\alpha_i - n + j} \end{bmatrix}_{n \times n} \begin{bmatrix} (-1)^{n-i} e_{n-i}^{(j)} \end{bmatrix}_{n \times n}$$

from which determinants can finally be calculated. For $\alpha_i = n - i$, the middle matrix becomes simply $[h_{-i+j}]_{n \times n}$, and its determinant is already known to be 1. The resulting identity yields

$$\det \begin{bmatrix} x_j^{n-i} \end{bmatrix}_{n \times n} = \det \begin{bmatrix} (-1)^{n-i} e_{n-i}^{(j)} \end{bmatrix}_{n \times n}$$

Applying this back to the general case of the determinants of the matrix identity, and substituting $\alpha_i = \lambda_i + n - i$:

$$\det \begin{bmatrix} h_{\lambda_i - i + j} \end{bmatrix} = \frac{\det \begin{bmatrix} x_j^{\lambda_i + n - i} \end{bmatrix}}{\det \begin{bmatrix} x_j^{n-i} \end{bmatrix}}$$

which completes the proof. □

Corollary: It follows immediately from (25) that

$$s_{(\lambda_1, \dots, \lambda_{\ell-1}, \lambda_{\ell})}(x_1, \dots, x_{\ell}) = \left(\prod_{i=1}^{\ell} x_i^{\lambda_{\ell}} \right) s_{(\lambda_1 - \lambda_{\ell}, \dots, \lambda_{\ell-1} - \lambda_{\ell}, 0)}(x_1, \dots, x_{\ell}).$$

Examples:

1. $s_{(3,1,1)}(x_1, x_2, x_3)$

$$s_{(3,1,1)}(x_1, x_2, x_3) = \frac{\begin{vmatrix} x_1^5 & x_1^2 & x_1 \\ x_2^5 & x_2^2 & x_2 \\ x_3^5 & x_3^2 & x_3 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}} = x_1 x_2 x_3 (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$$

2. $s_{(b,a)}(x_1, x_2)$

$$s_{(b,a)}(x_1, x_2) = \frac{\begin{vmatrix} x_1^{b+1} & x_1^a \\ x_2^{b+1} & x_2^a \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}} = \sum_{k=a}^b x_1^k x_2^{b+a-k}$$

3. $s_{(a^n)}(x_1, \dots, x_n)$

$$s_{(a^n)}(x_1, \dots, x_n) = \frac{|x_i^{a+n-j}|}{|x_i^{n-j}|} = \left(\prod_{i=1}^n x_i^a \right) \frac{|x_i^{n-j}|}{|x_i^{n-j}|} = \prod_{i=1}^n x_i^a$$

4. $s_{(na, (n-1)a, \dots, a)}(x_1, \dots, x_n)$

$$\begin{aligned} s_{(na, (n-1)a, \dots, a)}(x_1, \dots, x_n) &= \frac{|x_i^{(n-j+1)a+n-j}|}{|x_i^{n-j}|} \\ &= \left(\prod_i x_i^a \right) \frac{|(x_i^{a+1})^{n-j}|}{|x_i^{n-j}|} \\ &= \left(\prod_{i=1}^n x_i^a \right) \frac{\prod_{i < j} (x_i^{a+1} - x_j^{a+1})}{\prod_{i < j} (x_i - x_j)} \\ &= \left(\prod_{i=1}^n x_i^a \right) \prod_{i < j} \left(\sum_{k=0}^a x_i^k x_j^{a-k} \right) \end{aligned}$$

Further examples of Schur functions in terms of monomial functions are given in appendix A.5.

2.4 Relations between Λ bases

2.4.1 Bilinear Form

For every pair of A -bases of Λ_A as an A -module indexed by partitions, u and v , a bilinear form can be defined by

$$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}. \quad (26)$$

Coefficients which appear in the expansion of an arbitrary symmetric function over bases u and v can be expressed in terms of this bilinear form

$$\begin{cases} q = \sum_{\mu} c_{\mu}^{(u)} u_{\mu} \Rightarrow c_{\mu}^{(u)} = \langle q, v_{\mu} \rangle \\ q = \sum_{\mu} c_{\mu}^{(v)} v_{\mu} \Rightarrow c_{\mu}^{(v)} = \langle u_{\mu}, q \rangle \end{cases} \quad (27)$$

Multiple bilinear forms can be defined through (26), most of which are not particularly interesting. So as to motivate a specific definition of the bilinear form, it will first be shown:

Theorem 3. *For two countably infinite sets of variables: $x = \{x_1, x_2, \dots\}$ and $y = \{y_1, y_2, \dots\}$,*

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \begin{cases} \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(x) p_{\lambda}(y) & \text{(I.)} \\ \sum_{\lambda} h_{\lambda}(x) m_{\lambda}(y) = \sum_{\lambda} h_{\lambda}(y) m_{\lambda}(x) & \text{(II.)} \\ \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y) & \text{(III.)} \end{cases}$$

where λ runs over the set of all partitions \mathcal{P} .

Proof:

Part I. Applying (16) to the set of variables $s = \{x_i y_j : i, j \geq 1\}$,

$$\begin{aligned}
\prod_{i,j} (1 - x_i y_j)^{-1} &= \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(s) \\
&= \sum_{\lambda} z_{\lambda}^{-1} \prod_{k=1}^{\ell(\lambda)} \left(\sum_{i,j} (x_i y_j)^{\lambda_k} \right) \\
&= \sum_{\lambda} z_{\lambda}^{-1} \prod_{k=1}^{\ell(\lambda)} \left(\sum_i x_i^{\lambda_k} \right) \left(\sum_j y_j^{\lambda_k} \right) \\
&= \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(x) p_{\lambda}(y).
\end{aligned}$$

Part II. Applying (10) to variables x ,

$$\begin{aligned}
\prod_{i,j} (1 - x_i y_j)^{-1} &= \prod_j \left(\sum_r h_r(x) y_j^r \right) \\
&= \sum_{\alpha \in \mathbb{N}^{\infty}} h_{\alpha}(x) y^{\alpha} \\
&= \sum_{\lambda} h_{\lambda}(x) m_{\lambda}(y)
\end{aligned}$$

Part III. Let $\eta = (n-1, n-2, \dots, 0)$ and consider now finitely many variables $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$.

By Part II:

$$\det \left[x_i^{n-j} \right] \det \left[y_i^{n-j} \right] \prod_{i,j=1}^n (1 - x_i y_j)^{-1} = \sum_{\lambda} \left(h_{\lambda}(x) \det \left[x_i^{n-j} \right] m_{\lambda}(y) \det \left[y_i^{n-j} \right] \right)$$

Since $h_{\alpha} = h_{\alpha+}$,

$$\sum_{\lambda} \left(h_{\lambda}(x) \det \left[x_i^{n-j} \right] m_{\lambda}(y) \det \left[y_i^{n-j} \right] \right) = \sum_{\alpha} \sum_{\sigma} \left(h_{\alpha}(x) \det \left[x_i^{n-j} \right] \operatorname{sgn}(\sigma) y^{\alpha+\sigma(\eta)} \right)$$

Substituting $\beta = \alpha + \sigma(\eta)$

$$\begin{aligned} \sum_{\alpha} \sum_{\sigma} \left(h_{\alpha}(x) \det \left[x_i^{n-j} \right] \operatorname{sgn}(\sigma) y^{\alpha + \sigma(\eta)} \right) &= \sum_{\beta} \sum_{\sigma} \left(h_{\beta - \sigma(\eta)}(x) \det \left[x_i^{n-j} \right] \operatorname{sgn}(\sigma) y^{\beta} \right) \\ &= \sum_{\beta} \left(\det \left[x_i^{\beta_j + n - j} \right] y^{\beta} \right) \\ &= \sum_{\lambda} \left(\det \left[x_i^{\lambda_j + n - j} \right] \det \left[y_i^{\lambda_j + n - j} \right] \right) \end{aligned}$$

Finally, dividing by $\det \left[x_i^{n-j} \right] \det \left[y_i^{n-j} \right]$ on both sides,

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y)$$

□

Lemma 2. Consider two sets of variables $x = \{x_1, x_2, \dots\}$ and $y = \{y_1, y_2, \dots\}$. Let $u = \{u_{\lambda}\}_{\lambda \in \mathcal{P}}$ and $v = \{v_{\lambda}\}_{\lambda \in \mathcal{P}}$ be orthogonal Λ bases with respect to $\langle \cdot, \cdot \rangle$, meaning $\langle u_{\lambda}, v_{\mu} \rangle = \delta_{\lambda\mu}$, $\forall \mu, \lambda \in \mathcal{P}$. For every other pair of Λ bases $u' = \{u'_{\lambda}\}_{\lambda \in \mathcal{P}}$ and $v' = \{v'_{\lambda}\}_{\lambda \in \mathcal{P}}$, the following conditions are equivalent:

a) $\langle u'_{\lambda}, v'_{\mu} \rangle = \delta_{\lambda\mu}$, $\forall \mu, \lambda \in \mathcal{P}$

b) $\sum_{\lambda} u'_{\lambda}(x) v'_{\mu}(y) = \sum_{\lambda} u_{\lambda}(x) v_{\mu}(y)$

Proof: Each base element of u' and v' can be decomposed in terms of u or v like so

$$u'_{\lambda} = \sum_{\rho} a_{\lambda\rho} u_{\rho} \quad \text{and} \quad v'_{\mu} = \sum_{\sigma} b_{\mu\sigma} v_{\sigma}.$$

Now $\langle u'_{\lambda}, v'_{\mu} \rangle = \delta_{\lambda\mu}$ if, and only if

$$\sum_{\rho} a_{\lambda\rho} b_{\mu\rho} = \left\langle \sum_{\rho} a_{\lambda\rho} u_{\rho}, \sum_{\sigma} b_{\mu\sigma} v_{\sigma} \right\rangle = \delta_{\lambda\mu}.$$

Moreover,

$$\sum_{\lambda} u'_{\lambda}(x) v'_{\mu}(y) = \sum_{\lambda} \sum_{\rho, \sigma} a_{\lambda\rho} u_{\rho}(x) b_{\lambda\sigma} v_{\sigma}(y) = \sum_{\rho, \sigma} u_{\rho}(x) v_{\sigma}(y) \sum_{\lambda} a_{\lambda\rho} b_{\lambda\sigma},$$

so $\sum_{\lambda} u'_{\lambda}(x)v'_{\mu}(y) = \sum_{\lambda} u_{\lambda}(x)v_{\mu}(y)$ if, and only if $\sum_{\lambda} a_{\lambda\rho}b_{\lambda\sigma} = \delta_{\rho\sigma}$. Finally,

$$\sum_{\rho} a_{\lambda\rho}b_{\mu\rho} = \delta_{\lambda\mu} \Leftrightarrow \sum_{\lambda} a_{\lambda\rho}b_{\lambda\sigma} = \delta_{\rho\sigma}$$

so the conditions are equivalent. □

This result, together with theorem 3, motivates a particularly convenient specification of the bilinear form. Without ambiguity, the bilinear form is equivalently defined by either one of the expressions below:

(a) $\langle h_{\lambda}, m_{\mu} \rangle = \delta_{\lambda\mu};$

(b) $\langle p_{\lambda}, p_{\mu} \rangle = \delta_{\lambda\mu}z_{\lambda};$

(c) $\langle s_{\lambda}, s_{\mu} \rangle = \delta_{\lambda\mu}.$

Furthermore, from (19) and $\langle p_{\lambda}, p_{\mu} \rangle = \delta_{\lambda\mu}z_{\lambda}$ (and the fact that (p_{λ}) is a \mathbb{Q} -base for $\Lambda_{\mathbb{Q}}$), the involution ω is an isometry for this bilinear form, that is,

$$\langle \omega(u), \omega(v) \rangle = \langle u, v \rangle.$$

Therefore applying the involution ω to (a) yields another equivalent expression which can be added to the list,

(d) $\langle f_{\lambda}, e_{\mu} \rangle = \delta_{\lambda\mu}.$

2.4.2 Products between Schur Functions

Skew Schur functions' linear decompositions over "regular" Schur functions

$$s_{\lambda/\mu} = \sum_{\nu} \langle s_{\lambda/\mu}, s_{\nu} \rangle s_{\nu}$$

are closely related to linear decompositions of products between Schur functions themselves

$$s_\mu s_\nu = \sum_{\lambda} \langle s_\mu s_\nu, s_\lambda \rangle s_\lambda$$

in the following way:

Theorem 4. *For all partitions ν and skew partitions λ/μ ,*

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\mu s_\nu, s_\lambda \rangle \quad (28)$$

Proof: Let

$$s_{\lambda/\mu} = \sum_{\nu} c_{\nu\mu}^\lambda s_\nu \quad (29)$$

be the linear decomposition of $s_{\lambda/\mu}$ in terms of Schur functions and denote $(n-1, n-2, \dots, 1, 0)$ by η . Then:

$$\begin{aligned} \sum_{\lambda} s_{\lambda/\mu}(x) s_\lambda(y) &= \frac{1}{\det [y_i^{n-j}]} \sum_{\lambda} \det [h_{\lambda_i - \mu_j - i + j}(x)] \det [y_i^{\lambda_j + n - j}] \\ &= \frac{1}{\det [y_i^{n-j}]} \sum_{\lambda} \sum_{\sigma} \operatorname{sgn}(\sigma) h_{\lambda + \eta - \sigma(\mu + \eta)} \det [y_i^{\lambda_j + n - j}] \\ &= \frac{1}{\det [y_i^{n-j}]} \sum_{\alpha} \sum_{\sigma} h_{\alpha}(x) \operatorname{sgn}(\sigma) \det [y_i^{\alpha_{\sigma(j)} + \mu_j + n - j}] \\ &= \frac{1}{\det [y_i^{n-j}]} \sum_{\nu} h_{\nu}(x) m_{\nu}(y) \det [y_i^{\mu_j + n - j}] \\ &= s_{\mu}(y) \prod_{i,j} (1 - x_i y_j)^{-1} \\ &= \sum_{\nu} s_{\nu}(x) s_{\nu}(y) s_{\mu}(y) \end{aligned}$$

Finally, applying the Schur function linear decomposition of $s_{\lambda/\mu}$ and comparing coefficients with respect to variables x

$$s_{\mu} s_{\nu} = \sum_{\lambda} c_{\nu\mu}^{\lambda} s_{\lambda}$$

□

Lemma 3. Let $x = \{x_1, x_2, \dots\}$ and $y = \{y_1, y_2, \dots\}$. If $s_\mu s_\nu = \sum_\lambda c_{\nu\mu}^\lambda s_\lambda$, then

$$s_\lambda(x, y) = \sum_{\mu, \nu \subset \lambda} c_{\nu\mu}^\lambda s_\nu(x) s_\mu(y).$$

Proof: Consider another set of variables $z = \{z_1, z_2, z_3, \dots\}$.

$$\begin{aligned} \sum_\mu \sum_\lambda s_{\lambda/\mu}(x) s_\lambda(z) s_\mu(y) &= \sum_\mu s_\mu(y) s_\mu(z) \prod_{i,k} (1 - x_i z_k)^{-1} \\ &= \prod_{j,k} (1 - y_j z_k)^{-1} \prod_{i,k} (1 - x_i z_k)^{-1} \\ &= \sum_\lambda s_\lambda(x, y) s_\lambda(z) \end{aligned}$$

Comparing coefficients with respect to variables z :

$$s_\lambda(x, y) = \sum_\mu s_{\lambda/\mu}(x) s_\mu(y) = \sum_{\mu, \nu} c_{\nu\mu}^\lambda s_\nu(x) s_\mu(y)$$

The condition $\mu, \nu \subset \lambda$ naturally arises because $c_{\nu\mu}^\lambda$ is otherwise zero.

□

Lemma 4. Let $x = \{x_1, x_2, x_3, \dots\}$ and $y = \{y_1, y_2, y_3, \dots\}$. Then:

$$s_{\lambda/\mu}(x, y) = \sum_{\mu \subset \nu \subset \lambda} s_{\lambda/\nu}(x) s_{\nu/\mu}(y)$$

Proof: The same trick from the previous lemma can be applied.

$$\begin{aligned} \sum_\lambda s_{\lambda/\mu}(x, y) s_\mu(z) &= s_\lambda(x, y, z) \\ &= \sum_\nu s_{\lambda/\nu}(x) s_\nu(y, z) \\ &= \sum_{\nu, \mu} s_{\lambda/\nu}(x) s_{\nu/\mu}(y) s_\mu(z) \end{aligned} \tag{30}$$

The result again follows from coefficient comparison with respect to variables z .

□

Theorem 5.

$$s_{\lambda/\mu} = \sum_{T \in \text{Tab}(\lambda/\mu)} x^{|T|} \quad (31)$$

Proof: The conclusion from the previous lemma can be easily extended for sets of variables over countably arbitrarily many alphabets $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \dots)$ as follows.

$$\begin{aligned} s_{\lambda/\mu}(x^{(1)}, x^{(2)}, \dots) &= \sum_{(\nu^{(0)}, \dots, \nu^{(n)})} s_{\nu^{(1)}/\mu}(x^{(1)}) \cdots s_{\nu^{(i)}/\nu^{(i+1)}}(x^{(i)}) \cdots s_{\lambda/\nu^{(n-1)}}(x^{(n)}) \\ &= \sum_{(\nu^{(0)}, \dots, \nu^{(n)})} \left(\prod_{i=1}^n s_{\nu^{(i)}/\nu^{(i-1)}}(x^{(i)}) \right) \end{aligned} \quad (32)$$

where the sum is done over all partition sequences $(\nu^{(0)}, \dots, \nu^{(n)})$ such that

$$\mu = \nu^{(0)} \subset \nu^{(1)} \subset \dots \subset \nu^{(n-1)} \subset \nu^{(n)} = \lambda.$$

By labelling all squares in each section $\nu^{(i)}/\nu^{(i-1)}$ with i , each sequence $(\nu^{(0)}, \dots, \nu^{(n)})$ can be uniquely associated with a tableau of shape $sh(T) = \lambda/\mu$ and weight

$$|T| = (|\nu^{(1)}/\nu^{(0)}|, |\nu^{(2)}/\nu^{(1)}|, \dots, |\nu^{(n)}/\nu^{(n-1)}|).$$

Furthermore, if now each of the arbitrarily many alphabets $x^{(i)}$ consists of a single word, say x_i , then for each sequence $(\nu^{(0)}, \dots, \nu^{(n)})$,

$$\prod_{i=1}^n s_{\nu^{(i)}/\nu^{(i-1)}}(x^{(i)}) = \prod_{i=1}^n x_i^{|\nu^{(i)}/\nu^{(i-1)}|} = x^{|T|}$$

So equation (32) becomes

$$s_{\lambda/\mu} = \sum_{sh(T)=\lambda/\mu} x^{|T|}.$$

By grouping all tableaux of same weight, apart from rearrangements, this equation can be further simplified to

$$\begin{aligned} s_{\lambda/\mu} &= \sum_{T \in \text{Tab}(\lambda/\mu)} m_{|T|^+} \\ &= \sum_{\nu} K_{\lambda/\mu, \nu} m_{\nu} \end{aligned} \quad (33)$$

where $K_{\lambda/\mu, \nu} = \#\text{Tab}(\lambda/\mu, \nu)$. Numbers $K_{\lambda/\mu, \nu}$ are named after Carl Kostka, who first defined them in his early work on symmetric polynomials in 1882 [5]. When $\mu = 0$, (33) simplifies to

$$s_{\lambda} = \sum_{\nu} K_{\lambda, \nu} m_{\nu} \quad (34)$$

The maximal weight with respect to the natural ordering a semi-standard tableau of shape $\lambda = \lambda/(0)$ can have is obtained by labelling each i -th λ row by i . Any more than that, and it wouldn't be strictly increasing along columns. Consequentially, $K_{\lambda, \nu} = 0$ for all $\nu > \lambda$ and $K_{\lambda, \lambda} = 1$. It follows that, for all positive integers n , Kostka number matrices $[K_{\lambda, \nu}]_{\lambda, \nu \vdash n}$ are triangular with 1's along their diagonals.

Since $[K_{\lambda, \nu}]_{\lambda, \nu \vdash n}$ is invertible (in the multiplicative monoid of integer matrices) for all $n \geq 1$, this is an alternative proof that Schur functions are a \mathbb{Z} -base for Λ .

Applying the bilinear form to (33)

$$\langle s_{\lambda/\mu}, h_{\nu} \rangle = \langle s_{\lambda}, s_{\mu} h_{\nu} \rangle = K_{\lambda/\mu, \nu} \quad (35)$$

Specializing $\nu = (r)$, h_{ν} becomes $s_{(r)}$, and

$$K_{\lambda/\mu, (r)} = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is a horizontal } r\text{-strip} \\ 0 & \text{otherwise} \end{cases}$$

so (35) implies the following theorem, known as Pieri's Rule after its author who proved it in 1893.

Theorem 6. [9]

$$s_\lambda s_{(r)} = \sum_{\nu/\lambda \text{ } r\text{-horizontal strip}} s_\nu$$

$$s_\lambda s_{(1^r)} = \sum_{\nu/\lambda \text{ } r\text{-vertical strip}} s_\nu$$

The second statement is a consequence of the involution ω applied to the first. Recently, a generalized version of the Pieri Rule for skew Schur functions was proven.

Theorem 7. [1, Theorem 3.2]

$$s_{\lambda/\mu} s_{(r)} = \sum_{k=0}^r (-1)^k \sum_{\substack{\nu/\lambda \text{ } (r-k)\text{-horizontal strip} \\ \eta/\mu \text{ } k\text{-vertical strip}}} s_{\nu/\eta}$$

$$s_{\lambda/\mu} s_{(1^r)} = \sum_{k=0}^r (-1)^k \sum_{\substack{\nu/\lambda \text{ } (r-k)\text{-vertical strip} \\ \eta/\mu \text{ } k\text{-horizontal strip}}} s_{\nu/\eta}$$

2.4.3 Transition Matrices

Let $u = \{u_\lambda\}_{\lambda \in \mathcal{P}}$ and $v = \{v_\lambda\}_{\lambda \in \mathcal{P}}$ be bases for Λ as a module such that $\{u_\lambda\}_{\lambda \in \mathcal{P}_k}$ and $\{v_\lambda\}_{\lambda \in \mathcal{P}_k}$ are also bases for each Λ^k . The $\Lambda \rightarrow \Lambda$ operator which effects the base transformation from u to v is represented by the *transition matrix* from u to v , which is denoted by $M(u, v)$.

Transition matrices are block diagonal infinite matrices whose diagonal blocks are the invertible square matrices of order $|\mathcal{P}_k|$ which effect the base transformation from $\{v_\lambda\}_{\lambda \in \mathcal{P}_k}$ to $\{u_\lambda\}_{\lambda \in \mathcal{P}_k}$ for each k . By construction, for any two given transition matrices, the intersection between the support of any row of one and the support of any column of the other is either 0 or $|\mathcal{P}_k|$ for some $k \in \mathbb{N}$. Therefore multiplication between transition matrices is well-defined and associativity holds. Since each of their diagonal blocks are invertible matrices, transition matrices themselves are also invertible.

Up to this point the endeavour of transforming between Λ bases has been already achieved for:

- Elementary and monomial functions. According to (7) and (8),

$$M(m, e) = [c_{\lambda\mu}]_{\lambda, \mu \in \mathcal{P}}$$

where $c_{\lambda\mu} = \#\{T \in \widehat{\text{Tab}}(\lambda, \mu) : T \text{ strictly increasing along rows}\}$.

- Power and monomial functions. According to (17) and (18),

$$M(m, p) = [L_{\lambda\mu}]_{\lambda, \mu \in \mathcal{P}} = L$$

where $L_{\lambda\mu} = \#\{g \in S_{\ell, \ell(\mu)} : \mu = \lambda^g\}$.

- Schur and complete functions. According to (23) and (21),

$$M(s, h) = \left[\sum_{\substack{\sigma \text{ such that} \\ \lambda(\sigma)^+ = \mu}} \text{sgn}(\sigma) \right]_{\lambda, \mu \in \mathcal{P}}$$

where $\lambda(\sigma) = (\lambda_1 - 1 + \sigma(1), \lambda_2 - 2 + \sigma(2), \dots, \lambda_\ell - \ell + \sigma(\ell))$.

- Finally, for Schur and monomial functions. According to (33),

$$M(s, m) = [K_{\lambda, \mu}]_{\lambda, \mu \in \mathcal{P}}$$

where $K_{\lambda, \mu} = \#\text{Tab}(\lambda, \mu)$.

Because $\omega(s_\lambda) = s_{\lambda'}$, the matrix form of the involution operator ω under the Schur base is given by $J = [\delta_{\lambda\mu'}]_{\lambda, \mu \in \mathcal{P}}$. Furthermore, like for finite matrices, $M(u, v)M(v, w) = M(u, w)$ for all bases u, v and w . Finally, $M(u, v) = M(\omega(u), \omega(v))$ for all bases u and v .

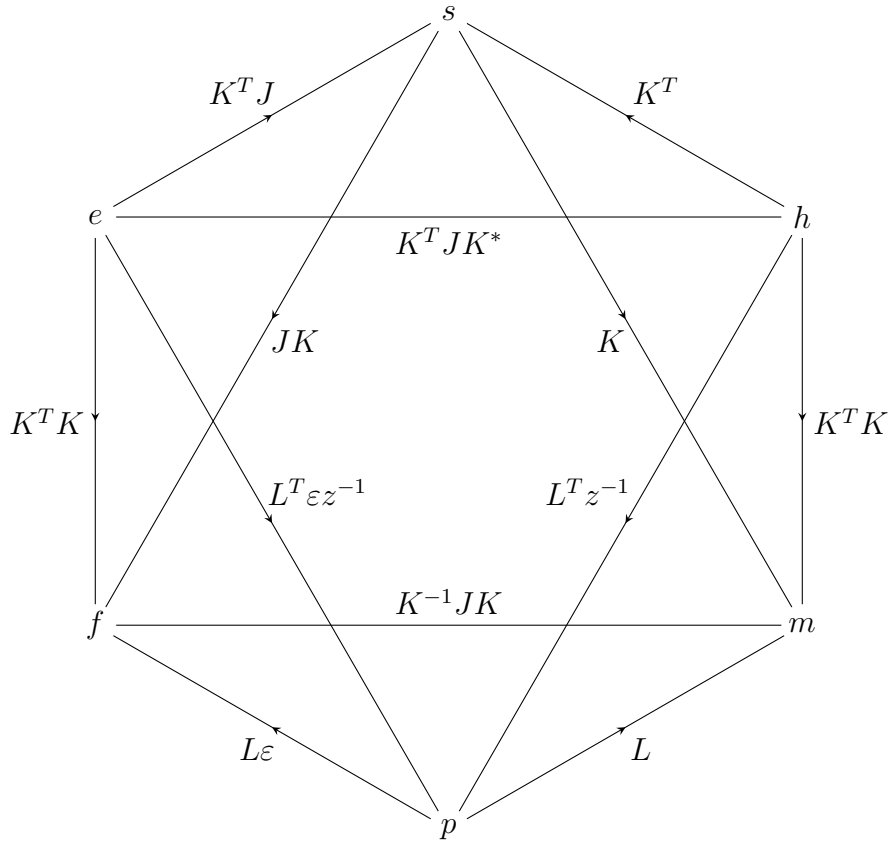
This is (more than) enough for all remaining transition matrices to be calculated in terms of just K, L and J . Denoting $A^* = (A^T)^{-1} = (A^{-1})^T$ and

$$\begin{aligned} \varepsilon &= [\varepsilon_\lambda \delta_{\lambda\mu}]_{\lambda, \mu \in \mathcal{P}} \\ z &= [z_\lambda \delta_{\lambda\mu}]_{\lambda, \mu \in \mathcal{P}}, \end{aligned}$$

the following transition matrix table is obtained.

| | m | f | e | h | s | p |
|-----|------------|-----------------|--------------------|--------------|-----------|-------------------------|
| m | I | $K^{-1}JK$ | $K^{-1}JK^*$ | $K^{-1}K^*$ | K^{-1} | L^{-1} |
| f | $K^{-1}JK$ | I | $K^{-1}K^*$ | $K^{-1}JK^*$ | $K^{-1}J$ | $L^{-1}\varepsilon$ |
| e | K^TJK | K^TK | I | K^TJK^* | K^TJ | $L^T\varepsilon z^{-1}$ |
| h | K^TK | K^TJK | K^TJK^* | I | K^T | L^Tz^{-1} |
| s | K | JK | JK^* | K^* | I | KL^{-1} |
| p | L | εL | $z\varepsilon L^*$ | zL^* | LK^{-1} | I |

These relations may also be expressed in the format of a graph where each of the six bases is a vertex and each oriented edge from base u to v is labelled by the correspondent transition matrix $M(u, v)$. Edges connecting bases for which $M(u, v) = M(v, u)$ should have no orientation. In the following image, edges connecting bases on opposite sides of the graph have been omitted for the sake of clarity.



2.5 Littlewood-Richardson-Robinson Rule

Each product between Schur functions has a linear decomposition in terms of Schur functions themselves

$$s_\mu s_\lambda = \sum_{\nu} c'_{\mu\lambda} s_\nu. \quad (36)$$

While it is completely trivial to describe products of two power functions, elementary functions or complete functions as a linear decomposition of functions on their own bases, the same is not true for the remaining bases. These products can still be calculated algebraically with the tools developed thus far, for example, setting $M(s, e) = [a_{\lambda\mu}]$ and $M(e, s) = [a'_{\lambda\mu}]$,

$$s_\lambda s_\mu = \left(\sum_{\nu} a_{\lambda\nu} e_\nu \right) \left(\sum_{\eta} a_{\mu\eta} e_\eta \right) = \sum_{\nu, \eta} a_{\lambda\nu} a_{\mu\eta} e_{\nu \cup \eta},$$

and these terms may be regrouped and converted back to Schur basis

$$s_\lambda s_\mu = \sum_{\gamma} \left(\sum_{\nu \cup \eta = \gamma} a_{\lambda\nu} a_{\mu\eta} \right) e_\gamma = \sum_{\theta} \left(\sum_{\nu \cup \eta = \gamma} a_{\lambda\nu} a_{\mu\eta} x \right) a'_{\gamma\theta} s_\theta.$$

But this is not a very satisfactory solution because calculating $\sum_{\nu \cup \eta = \gamma} a_{\lambda\nu} a_{\mu\eta}$ can be exceedingly complicated. The goal is to deduce a purely combinatorial formula for $c'_{\mu\lambda}$. Before stating the theorem itself, it is necessary to provide some definitions.

For every tableau T , a *word* $w(T)$ is the sequence of symbols obtained by the reading of T from right to left and up to bottom, as in the example

$$T = \left. \begin{array}{cccc} & & 1 & 1 & 2 \\ & & 2 & 2 & 3 & 3 \\ 1 & 3 & 4 & & & \\ 2 & 4 & & & & \end{array} \right\} \begin{array}{l} \leftarrow 211 \\ \leftarrow 3322 \\ \leftarrow 431 \\ \leftarrow 42 \end{array} \Rightarrow w(T) = 211332243142$$

If a word $w = w(T)$ can be obtained from a tableau $T \in \text{Tab}(\lambda/\mu, \nu)$, it is said to have *weight* ν , denoted by $|w|$ and to be *compatible* with λ/μ . Multiple words can be compatible with the same shape and every word is always compatible with multiple shapes.

Let $w = a_1 a_2 \dots a_N$ be a word in the symbols $1, 2, \dots$. The difference between the number of occurrences of the symbol r and the number of occurrences of the symbol $r - 1$ in the truncated word $a_1 a_2 \dots a_P$ is the *r-index of P in w* and is denoted by $\text{ind}_P^r(w)$:

$$\text{ind}_P^r(w) := \#\{i \leq P | a_i = r\} - \#\{i \leq P | a_i = r - 1\}.$$

The *r-index of w* is defined as the maximal *r-index of P in w* and is denoted by $\text{ind}^r(w)$:

$$\text{ind}^r(w) := \max_P \{\text{ind}_P^r(w)\}.$$

The word $w = a_1 a_2 \dots a_N$ is a *lattice permutation* if all of its *r-indices* are non-positive. That is, for all P and r ,

$$\#\{j \leq P | a_j = r\} \leq \#\{j \leq P | a_j = r - 1\}.$$

The subset of $\text{Tab}(\lambda/\mu, \nu)$ of semi-standard tableaux of shape λ/μ and weight ν whose words are lattice permutations is denoted by $\text{Tab}^0(\lambda/\mu, \nu)$.

With these definitions in hand, Littlewood-Richardson-Robinson's Theorem can be finally stated:

Theorem 8. *Littlewood-Richardson-Robinson*

$$\langle s_\mu s_\lambda, s_\nu \rangle = \#\text{Tab}^0(\nu/\mu, \lambda) \tag{37}$$

Before starting off the proof, it is necessary to lay out some more definitions.

Definition 2.5.1. Functions $S_{a,r}$, where $1 \leq a < r$ are functions which map words with positive r -index onto words with lower (but possibly still positive) r -index. Let P be the position in w at which $\text{ind}^r(w)$ is attained for the first time. The word $S_{r-1,r}(w)$ is obtained by replacing the symbol r at position P by the symbol $r - 1$. For each $1 \leq a < r$, the function $S_{a,r}$ is given by the composition

$$S_{a,r} := S_{a,a+1} \circ S_{a+1,a+2} \circ \cdots \circ S_{r-1,r},$$

which is well-defined so long as $\text{ind}^t(S_{t,r}(w)) > 0$ for all $a < t < r$. Unless stated otherwise, a will be chosen to be the least integer for which this composition is well-defined, that is $a = \max \{t < r : \text{ind}^t(S_{t,r}(w)) \leq 0\}$.

Examples:

1. $S_{2,3} \circ S_{2,3}(133233) = S_{2,3}(133232) = 132232$
2. $S_{1,4}(114223) = S_{1,3}(113223) = S_{1,2}(112223) = 112213$

Observations:

1. Once again let P be the position at which $\text{ind}^r(w)$ is attained for the first time. The effect of the application of $S_{r-1,r}$ over w upon t -indices of w , for $t \leq r$, is

$$\begin{aligned} \text{ind}_\rho^t(S_{r-1,r}(w)) &= \text{ind}_\rho^t(w), & \forall \rho, \forall t < r - 1 \\ \text{ind}_\rho^{r-1}(S_{r-1,r}(w)) &= \begin{cases} \text{ind}_\rho^{r-1}(w) & \text{if } \rho < P \\ \text{ind}_\rho^{r-1}(w) + 1 & \text{if } \rho \geq P \end{cases} \end{aligned} \quad (38)$$

$$\text{ind}_\rho^r(S_{r-1,r}(w)) = \begin{cases} \text{ind}_\rho^r(w) & \text{if } \rho < P \\ \text{ind}_\rho^r(w) - 2 & \text{if } \rho \geq P \end{cases} \quad (39)$$

In particular, $\text{ind}^{r-1}(S_{r-1,r}(w))$ can increase by 1 with respect to $\text{ind}^{r-1}(w)$ and $\text{ind}^r(S_{r-1,r}(w))$ will decrease at least by 1 with respect to $\text{ind}^r(w)$.

2. As utilized throughout the proof, functions $S_{a,r}$ will be applied over words not only with positive r -index, but also non-positive t -indices for all $t < r$. Due to the previous observation, this can only be the case if $\text{ind}^t(w) = 0$ and $\text{ind}^t(S_{t,r}(w)) = 1$ for all $a < t < r$. Furthermore, if a is chosen as $\max \{t < r : \text{ind}^t(S_{t,r}(w)) \leq 0\}$, the resulting word $S_{a,r}(w)$ will also have non-positive t -indices for all $t < r$.

Definition 2.5.2. Functions $R_{a,r}$, where $1 \leq a < r$ are functions which map labelled diagrams onto labelled diagrams. For any labelled diagram M whose r -th row is non-empty, $R_{a,r}(M)$ is obtained by taking the rightmost block of the r -th row of M and placing it at the right of the rightmost block of the a -th row of M .

Example:

$$R_{1,3} \left(\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & & \\ \hline 3 & 3 & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & & & \\ \hline 3 & & & \rightarrow \\ \hline \end{array}$$

Observations:

1. Just like for functions $S_{a,r}$, $R_{b,a} \circ R_{a,r} = R_{b,r}$ for all $b < a < r$;
2. These functions are weight-invariant, that is $|M| = |R_{a,r}(M)|$ for all labelled diagrams M ;
3. The effect of functions $R_{a,r}$ on the shape of a diagram M is the same as the effect of $S_{a,r}$ on the weight of a word w , meaning

$$|w| = sh(M) \Rightarrow |S_{a,r}(w)| = sh(R_{a,r}(M));$$

4. The inverse of $R_{a,r}$ is the function $R_{r,a}$ which takes the rightmost block of the a -th row of a labelled diagram and places it at the right of the rightmost of its r -th row;

Proof of Theorem 37: The proof of the Littlewood-Richardson-Robinson's Theorem will follow the following strategy.

1. Firstly it will be shown that functions $S_{a,r}$ have the following properties

(a) They are injective;

(b) They preserve shape compatibility, that is, for every word w

$$w \text{ is compatible with } \lambda/\mu \Leftrightarrow S_{a,r}(w) \text{ is compatible with } \lambda/\mu.$$

(c) If the words w , $S_{a,r}(w)$ and $S_{b,r} \circ S_{a,r}(w)$ are lattice permutations with respect to $1, 2, \dots, r-1$, that is

$$\text{ind}^t(w), \text{ind}^t(S_{a,r}(w)), \text{ind}^t(S_{b,r} \circ S_{a,r}(w)) \leq 0, \quad \forall t < r,$$

then $b \leq a$.

2. The next step is to devise an algorithm which

(a) Associates each semi-standard tableau T with a pair of tableaux in $\text{Tab}^0(\text{sh}(T), \nu) \times \text{Tab}(\nu, |T|)$ for some partition $\nu \geq |T|$;

(b) Through this association, provides a bijection between $\text{Tab}(\lambda/\mu, \pi)$ and $\bigcup_{\nu} \text{Tab}^0(\lambda/\mu, \nu) \times \text{Tab}(\nu, \pi)$, implying

$$\#\text{Tab}(\lambda/\mu, \pi) = \sum_{\nu} \#\text{Tab}^0(\lambda/\mu, \nu) \langle s_{\nu}, h_{\pi} \rangle.$$

3. The final step is to employ the equality above to prove the theorem.

Now that there is a laid out strategy, what is left is to prove the lemmas in-between.

Lemma 5. $S_{a,r}$ is injective.

Proof: In order to prove this, it is enough to show that $S_{r-1,r}$ is injective.

Take any word w with a positive r -index and apply $S_{r-1,r}$. If a substitution occurs in the P -th position of the word, by (39),

$$\text{ind}_\rho^r(S_{r-1,r}(w)) = \begin{cases} \text{ind}_\rho^r(w) & \text{if } \rho < P \\ \text{ind}_\rho^r(w) - 2 & \text{if } \rho \geq P \end{cases}$$

If P is not the first position at which a symbol r occurs in w , there must be some $P' < P$ for which $\text{ind}_{P'}^r(w)$ — and therefore also $\text{ind}_{P'}^r(S_{r-1,r}(w))$ — is exactly $\text{ind}_{P'}^r(w) - 1$. Therefore, if $\text{ind}^r(S_{r-1,r}(w)) \geq 0$, its maximal r -index must be attained before the position P . Now let Q be the last position at which $\text{ind}_Q^r(S_{r-1,r}(w)) = \text{ind}^r(S_{r-1,r}(w))$. The first $r - 1$ symbol after Q in $S_{r-1,r}(w)$ is precisely the r symbol in the P -th position of w which underwent replacement by $S_{r-1,r}$.

If P is the first position at which a symbol r occurs in w , then no $r - 1$ symbols appear before it and $\text{ind}^r(w) = 1$. Clearly $S_{r-1,r}(w)$ has a negative r -index. In this case, it is easy to see that the first $r - 1$ symbol in $S_{r-1,r}(w)$ is precisely the symbol at position P which underwent replacement by $S_{r-1,r}$.

It follows that it is possible to devise an algorithm for computing $S_{r-1,r}^{-1}(w)$ as follows: if $\text{ind}^r(w) \geq 0$, let Q be the last position at which $\text{ind}_Q^r(w) = \text{ind}^r(w)$ and replace the first $r - 1$ symbol after Q by r . In case $\text{ind}^r(w) < 0$, simply replace the first $r - 1$ symbol in w by r . Since $S_{r-1,r}$ is always reversible, it is injective.

□

Examples:

$$1. S_{2,4}^{-1}(124423) = S_{3,4}^{-1} \circ S_{2,3}^{-1}(1\underline{2}4423) = S_{3,4}^{-1}(13442\underline{3}) = 134424$$

$$2. S_{1,2}^{-1} \circ S_{1,2}^{-1}(2\underline{1}121) = S_{1,2}^{-1}(221\underline{2}1) = 22122$$

Lemma 6. *For any skew partition λ/μ ,*

$$w \text{ is compatible with } \lambda/\mu \Leftrightarrow S_{a,r}(w) \text{ is compatible with } \lambda/\mu.$$

Proof: It is sufficient to prove the result for $S_{r-1,r}$.

Let T_1 and T_2 be (generalized) tableaux with shape λ/μ for which $w(T_2) = S_{r-1,r}(w(T_1))$. Then T_2 must differ from T_1 in exactly one entry. More specifically, there is a single coordinate $(i, j) \in \lambda/\mu$ and an integer P for which $T_1(i, j) = r = w(T_1)_P$ and $T_2(i, j) = r - 1 = w(T_2)_P$.

By contradiction, if one of them is a semi-standard tableau whose word satisfy the lattice permutation, but the other isn't, there are exactly four possibilities:

(\Rightarrow) (T_2 is not semi-standard)

(a) $T_2(i, j) = r - 1$ and $T_2(i, j - 1) = r$

$$\Rightarrow T_1(i, j) = r \text{ and } T_1(i, j - 1) = r$$

(b) $T_2(i, j) = r - 1$ and $T_2(i - 1, j) = r - 1$

$$\Rightarrow T_1(i, j) = r \text{ and } T_1(i - 1, j) = r - 1$$

(\Leftarrow) (T_1 is not semi-standard)

(c) $T_1(i, j) = r$ and $T_1(i, j + 1) = r - 1$

$$\Rightarrow T_2(i, j) = r - 1 \text{ and } T_2(i, j + 1) = r - 1$$

(d) $T_1(i, j) = r$ and $T_1(i + 1, j) = r$

$$\Rightarrow T_2(i, j) = r - 1 \text{ and } T_2(i + 1, j) = r$$

Therefore cases (a) and (c) are equivalent to the existence of a labelling of a semi-standard tableau T for which $w(T)$ is a lattice permutation with a horizontal pair of boxes labelled with the same symbol — as in $\boxed{a} \boxed{a}$ — for which the rightmost of the two is the one for which $\text{ind}^a(w(T))$ is first attained. This is contradictory because the leftmost of the two symbols a

would come after the rightmost in $w(T)$, and should therefore have higher a -index.

(b) is the case when there is a labelling of a semi-standard tableau T for which $w(T)$ is a lattice permutation with a vertical pair of boxes labelled as in $\begin{array}{|c|} \hline a-1 \\ \hline a \\ \hline \end{array}$, and the downward square is labelled by the first a symbol for which $\text{ind}^a(w(T))$ is attained. When this happens, that downward square is labeled by the leftmost a symbol of a string of, say, s squares labeled by a . Immediately above this string, there must be a string of s squares labeled by $a - 1$. The content of each square to the right of the a^s string must be greater than a , and the content of each square to the left of the $(a - 1)^s$ string must be smaller than $a - 1$. It follows that $w(T)$ contains a segment of the form

$$(a - 1)^s \dots a^s$$

with no a , nor $a - 1$ symbol in between these strings. Notice, however, that the a -index of the position of the last a in the segment above is the same as that of the first element preceding the string of $(a - 1)^s$. This contradicts the assumption that that square was labelled by the first a symbol for which $\text{ind}^a(w(T))$ was attained.

Case (d) is the case when there is a labelling of a semi-standard tableau T for which $w(T)$ is a lattice permutation with a vertical pair of squares labelled as in $\begin{array}{|c|} \hline a-1 \\ \hline a \\ \hline \end{array}$ and the downward square is labelled by the last a symbol for which $\text{ind}^a(w)$ is first attained. Once again, when this happens, that downward square is labelled by the leftmost a symbol of a a^s string immediately below a $(a - 1)^s$ string. This is the same situation proven impossible in (b).

Lemma 7. *Let $a, b < r$. If $\text{ind}^t(w), \text{ind}^t(S_{a,r}(w)), \text{ind}^t(S_{b,r} \circ S_{a,r}(w)) \leq 0$ for all $t < r$, then $b \leq a$.*

Proof: Recall that a is the least integer $t < r$ for which $S_{t,r}(w)$ is well-defined and b is the least integer $t < r$ for which $S_{t,r}(S_{a,r}(w))$ is well-defined, so that

$$\begin{aligned} \text{ind}^t(w) = 0 & \quad \text{and} \quad \text{ind}^t(S_{t,r}(w)) = 1 & \quad \forall a < t < r \\ \text{ind}^t(S_{a,r}(w)) = 0 & \quad \text{and} \quad \text{ind}^t(S_{t,r}(S_{a,r}(w))) = 1 & \quad \forall b < t < r \end{aligned}$$

and $\text{ind}^a(S_{a,r}(w)), \text{ind}^b(S_{b,r}(S_{a,r}(w))) \leq 0$. According to (38),

$$\text{ind}_\rho^{s-1}(S_{s-1,s}(w)) = \begin{cases} \text{ind}_\rho^{s-1}(w) & \text{if } \rho < P \\ \text{ind}_\rho^{s-1}(w) + 1 & \text{if } \rho \geq P \end{cases}$$

meaning that for each s , the position of the symbol replacement due to $S_{s-1,s}$ cannot occur to the left of the position of the symbol replacement due to the previous function $S_{s-2,s-1}$, and so on. Therefore if each $S_{s-1,s}$ dictates a replacement at the position P_{r-s} in w or Q_{r-s} in $S_{a,r}(w)$,

$$\begin{aligned} P_0 &\leq P_1 \leq P_2 \leq \cdots \leq P_{r-a-1} \\ Q_0 &\leq Q_1 \leq Q_2 \leq \cdots \leq Q_{r-b-1}. \end{aligned}$$

Since indices $\text{ind}_\rho^s(w)$ increase and decrease in steps of 1 with respect to the position ρ , all possible indices from 0 to $\text{ind}_\rho^s(w)$ are attained at positions between 1 and ρ . In particular, if P is the first position at which $\text{ind}^s(w)$ is attained and $\text{ind}^s(w) > 0$, there is at least one position $Q < P$ — which can be chosen to be the first — at which $\text{ind}_Q^s(w) = \text{ind}^s(w) - 1$. This implies that if P is a position in w at which a symbol replacement occurs due to some function $S_{s-1,s}$, then $Q < P$ is the position in $S_{s-1,s}(w)$ at which a symbol is replaced by another application of $S_{s-1,s}$. Therefore

$$Q_0 < P_0 \leq Q_1 < P_1 \leq \cdots$$

Moreover, by the same argument, since $\text{ind}^t(S_{t,r}(w)) = 1$ for $a < t < r$,

$$\text{ind}^t(S_{t-1,r}(w)) = \text{ind}^t(S_{t-1,t} \circ S_{t,r}(w)) = \text{ind}^t(S_{t,r}(w)) - 1 = 0,$$

and as $S_{a,t-1}$ does not affect t -indices, $\text{ind}^t(S_{a,r}(w)) = 0$ for all $a < t < r$.

Because of this, for each $a < t < r$,

$$\begin{aligned}
\text{ind}^t(S_{t,r} \circ S_{a,r}(w)) &= \text{ind}_{Q_t}^t(S_{t,r} \circ S_{a,r}(w)) && \text{by definition of } Q_t \\
&= \text{ind}_{P_t}^t(S_{t+1,r} \circ S_{a,r}(w)) + 1 && \text{because } Q_t < P_t \\
&= \text{ind}_{P_t}^t(S_{a,r}(w)) + 1 \\
&= \text{ind}^t(S_{a,r}(w)) + 1 && \text{by definition of } P_t \\
&= 1
\end{aligned}$$

Therefore $b = \max \{t < r : \text{ind}^t(S_{t,r} \circ S_{a,r}(w)) \leq 0\} \leq a$.

□

The algorithm is devised in the following manner:

Description of the Algorithm

1. For each tableau T in $\text{Tab}(\lambda/\mu, \pi)$, let M_1 be the only tableau with both shape and weight π and $w_1 = w(T)$. The idea is to successively apply functions $S_{a,r} \times R_{a,r}$ to (w_1, M_1) until the resulting pair (w_ℓ, M_ℓ) is such that w_ℓ is a lattice permutation. Notice that since functions $S_{a,r}$ preserve shape compatibility and functions $R_{a,r}$ preserve weight, w_ℓ is compatible with λ/μ and $|M_\ell| = \pi$. Moreover, since $|w_1| = \pi = sh(M_1)$, and the effect of functions $S_{a,r}$ over weights of words is the same as the effect of functions $R_{a,r}$ over shapes of diagrams, $|w_\ell| = sh(M_\ell)$.
2. The first step of the algorithm is to take the least r_1 for which $\text{ind}^{r_1}(w_1) > 0$ and apply $S_{a',r_1} \times R_{a',r_1}$ to (w_1, M_1) , where a' is the largest $t < r_1$ for which $\text{ind}^t(S_{t,r_1}(w)) \leq 0$. If $\text{ind}^{r_1}(S_{a',r_1}(w_1)) > 0$, apply $S_{a'',r_1} \times R_{a'',r_1}$ to the resulting pair $(S_{a',r_1}(w_1), R_{a',r_1}(M_1))$, where a'' is the largest $t < r_1$ for which $\text{ind}^t(S_{t,r_1} \circ S_{a',r_1}(w)) \leq 0$. Keep applying functions $S_{a,r_1} \times R_{a,r_1}$ until the resulting word has non-positive r_1 -index. Call the resulting pair (w_2, M_2) .

3. The second step is to take the least r_2 for which $\text{ind}^{r_2}(w_2) > 0$ and repeat the previous procedure until there are no more positive r_2 -indices. Keep repeating this procedure until there are no more positive r -indices, in which case the resulting word will by construction be a lattice permutation.
4. Finally, since the resulting word w_ℓ is compatible with λ/μ , it can be uniquely associated with the tableau $T_\ell \in \text{Tab}^0(\lambda/\nu)$ for which $w(T_\ell) = w_\ell$. The output of the algorithm is the pair (T_ℓ, M_ℓ) .

It is elucidating to view a worked out example of the algorithm.

Example: Let T be the following semi-standard tableau with shape $(8, 8, 5, 3, 2)/(5, 3, 2, 1)$ and weight $|T| = (5, 5, 5)$

$$T = \begin{array}{cccccc} & & & & 1 & 1 & 2 \\ & & & 2 & 2 & 3 & 3 & 3 \\ & & 1 & 3 & 3 & & & \\ & 1 & 2 & & & & & \\ 1 & 2 & & & & & & \end{array}$$

The first step of the algorithm is to construct the pair (w_1, M_1) , where $w_1 = w(T)$ and M_1 is the only tableau with both shape and weight $|T|$:

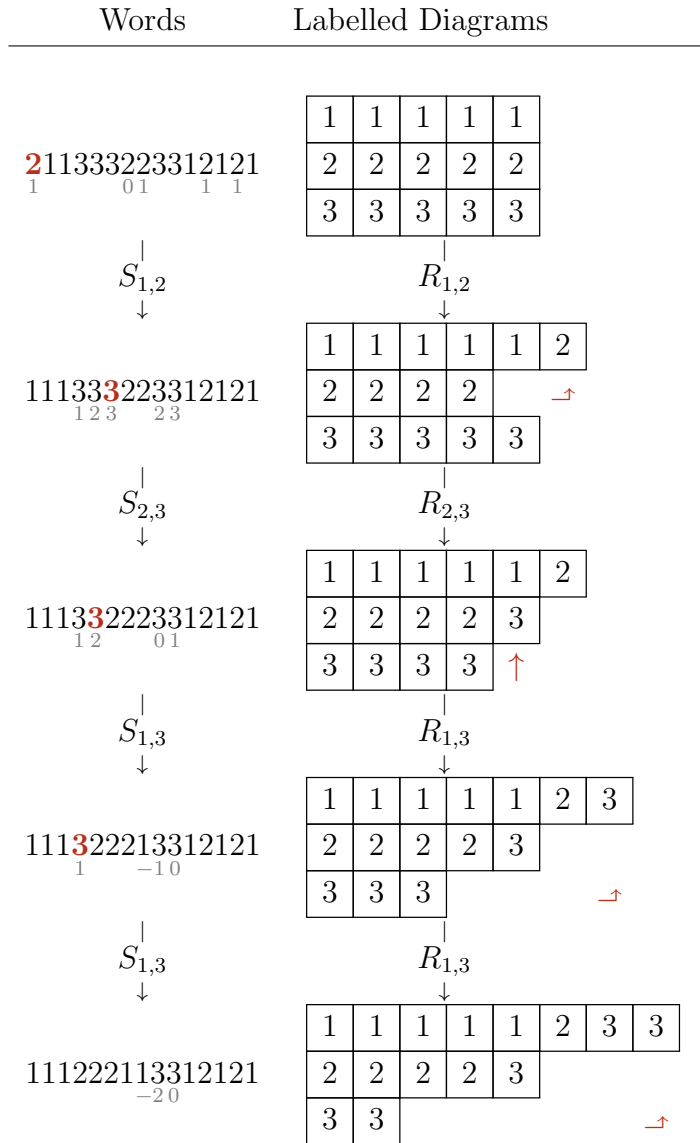
$$w_1 = 211333223312121$$

$$M_1 = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{array}$$

Next, w_1 's 2-indices are evaluated and if at least one of them is positive, $(S_{1,2}(w_1), R_{1,2}(M_1))$ is calculated for the next step. The procedure repeats until there are no more positive 2-indices, and then repeats again for 3-indices.

In the following table, each line indicates a step in the algorithm. Pertinent r -indices are evaluated and displayed in gray below each r symbol. The first maximal occurrence of a positive r -index is displayed in red. Each word

and labelled diagram pair (w, M) is simultaneously operated by an appropriate $S_{a,r} \times R_{a,r}$ pair, indicated by downward arrows along the algorithm's steps.



$$\text{Therefore } \left\{ \begin{array}{l} w' = S_{1,3}^2 \circ S_{2,3} \circ S_{1,2}(w_1) = 111222113312121 \\ M' = R_{1,3}^2 \circ R_{2,3} \circ R_{1,2}(M_1) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 & 3 & & & \\ \hline 3 & 3 & & & & & & \\ \hline \end{array} \end{array} \right.$$

There is a unique tableau T' with shape $sh(T') = sh(T) = (8, 8, 5, 3, 2)/(5, 3, 2, 1)$ whose correspondent word is $w(T') = w' = 111222113312121$. Hence from this result the pair of tableaux

$$(T', M') = \left(\begin{array}{cccccc} & & & & 1 & 1 & 1 \\ & & & 1 & 1 & 2 & 2 & 2 \\ & & 1 & 3 & 3 & & & \\ & 1 & 2 & & & & & \\ 1 & 2 & & & & & & \end{array} , \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 3 & & & \\ 3 & 3 & & & & & & \end{array} \right)$$

is obtained, concluding the algorithm. Notice that $|M'| = |T| = (5^3)$ and $|T'| = (8, 5, 3) = sh(M')$, so the resulting pair (T', M') belongs to $\bigcup_{\nu} \text{Tab}^0(sh(T), \nu) \times \text{Tab}(\nu, |T|)$ as expected.

Lemma 8. *The algorithm is bijective*

Proof: It suffices to show that the steps of the algorithm can be reversed, effectively constructing a new algorithm which functions as the inverse of the former.

1. For each pair of tableaux $(T, M) \in \bigcup_{\nu} \text{Tab}^0(\lambda/\mu, \nu) \times \text{Tab}(\nu, \pi)$, take the pair $(w'_1, M'_1) = (w(T), M)$. The idea is to apply functions $S_{a,r}^{-1} \times R_{a,r}^{-1}$ to (w'_1, M'_1) in the opposite order of that of the application of $S_{a,r} \times R_{a,r}$ in the former algorithm until the resulting tableau on the second coordinate has shape π .
2. Notice that since $M'_1 = M$ is semi-standard and $sh(M)$ is a partition, for any symbol r , there are no r -labeled boxes below the r -th row, nor t -labeled boxes where $t < r$ to the right of any r -labeled box.

Take the largest symbol r_1 for which there are r_1 -labeled boxes above the r_1 -th row in M'_1 , and apply functions $S_{a,r_1}^{-1} \times R_{a,r_1}^{-1}$ (starting from the lowest a all the way to the largest $a < r_1$) to the pair (w'_1, M'_1) until the resulting tableau on the second coordinate has no more r_1 -labeled boxes outside the r_1 -th row. Call the resulting pair (w'_2, M'_2) .

3. The next step is to take the largest symbol r_2 for which there are r_2 -labeled boxes above the r_2 -th row in M'_2 and repeat the same procedure as before, applying functions $S_{a,r_2}^{-1} \times R_{a,r_2}^{-1}$ starting from the lowest possible a all the way to the largest $a < r_2$ to (w'_2, M'_2) until the resulting tableau on the second coordinate has no more r_2 -labeled boxes outside the r_2 -th row. Call the resulting pair (w'_3, M'_3) . Keep repeating these steps until all boxes are labeled by the symbol corresponding to their row, meaning the shape of the resulting tableau will be the same as its weight, π . Call the resulting pair (w', M') .
4. Notice that since $|w'_1| = |T| = sh(M'_1)$ and the effect of functions $S_{a,r}^{-1}$ over weights of words is the same as the effect of functions $R_{a,r}^{-1}$ over shapes of diagrams, the resulting word w' has weight $|w'| = sh(M') = \pi$. Moreover, since w'_1 is compatible with λ/μ , so is w' .

The word w' can therefore be uniquely associated with the tableau $T' \in \text{Tab}(\lambda/\mu, \pi)$ for which $w(T') = w'$, concluding the algorithm.

The reversed algorithm can be illustrated by following the steps of the former algorithm backwards in the previous example.

□

It is already known from (35) that $\langle s_\mu s_\nu, s_\lambda \rangle = \langle s_{\lambda/\mu}, h_\nu \rangle = K_{\lambda/\mu, \nu}$ and the previous lemma shows that that

$$\#\text{Tab}(\lambda/\mu, \pi) = \# \bigcup_{\nu} \text{Tab}^0(\lambda/\mu, \nu) \times \text{Tab}(\nu, \pi)$$

$$\text{so } \langle s_{\lambda/\mu}, h_\pi \rangle = \sum_{\nu} \#\text{Tab}^0(\lambda/\mu, \nu) \langle s_\nu, h_\pi \rangle.$$

But since complete functions (h_π) are a base for Λ ,

$$\langle s_{\lambda/\mu}, s_\pi \rangle = \#\text{Tab}^0(\lambda/\mu, \pi)$$

therefore $s_{\lambda/\mu} = \sum_{\nu} \#\text{Tab}^0(\lambda/\mu, \nu) s_\nu$, concluding the proof.

■

3 Jack Functions

The coefficient ring of integers or rational numbers over which “regular” symmetric functions are defined can be generalized by the introduction of a parameter α by setting the field of coefficients to be the field of rational polynomials $\mathbb{Q}(\alpha)$. In doing so, a new algebra of symmetric functions $\Lambda_{\mathbb{Q}(\alpha)} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}(\alpha)$ is defined. All previously established results pertaining to \mathbb{Z} or \mathbb{Q} -bases for Λ remain valid as they become $\mathbb{Q}(\alpha)$ -bases for $\Lambda_{\mathbb{Q}(\alpha)}$.

Jack Functions $J_\lambda(\alpha)$ (or simply J_λ) are a family of symmetric functions indexed by partitions with coefficients in $\mathbb{Q}(\alpha)$ which are a $\mathbb{Q}(\alpha)$ -base for $\Lambda_{\mathbb{Q}(\alpha)}$ as a $\mathbb{Q}(\alpha)$ -module. They are very closely related to known families of symmetric functions, as up to normalization, different specializations of the parameter α yield Schur ($\alpha = 1$), conjugate elementary ($\alpha = 0$), monomial ($\alpha = \infty$) and both zonal ($\alpha = 2$ or $\frac{1}{2}$) functions.

Since first introduced by Jack [3], multiple equivalent definitions have been provided:

1. Simultaneous eigenfunctions of Sekiguchi-Debiard operators [14, 2];

$$\alpha^2 \sum_{i \geq 1} x_i^2 \frac{\partial^2}{\partial x_i^2} + 2\alpha \sum_{i \neq j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i}$$

2. Unique functions of the form

$$J_\lambda(\alpha) = \sum_{\mu \leq \lambda} v_\mu(\alpha) m_\mu$$

which are orthogonal with respect to the bilinear form

$$\langle p_\lambda, p_\mu \rangle_\alpha = \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda\mu} \tag{40}$$

3. Functions obtained by Sahi-Knop’s combinatorial formula.

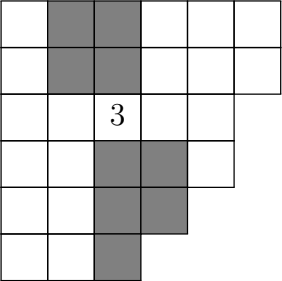
The latter will be described in detail shortly. Notice that the first two definitions determine Jack Functions only up to normalization. This is not a problem as they are defined over the field $\mathbb{Q}(\alpha)$, though conventionally some normalization must be specified. Sahi-Knop's combinatorial formula is particularly convenient because, as it will be seen, it automatically generates normalized functions.

3.1 Combinatorial Formula

Definition 3.1.1. Let T be a (generalized) tableau of shape λ . T is an *admissible tableau* if, for all $(i, j) \in \text{diag}(\lambda)$:

- (a) $T(i, j) \neq T(i', j - 1)$ for $i' < i$ (and $\lambda_i \geq j > 1$)
- (b) $T(i, j) \neq T(i', j)$ for $i' \neq i$
- (c) $T(i, j) \neq T(i', j + 1)$ for $i' > i$ (and $\lambda_i > j \geq 1$)

Items (a) and (c) are redundant, but help visualize how the labelling is displayed in admissible tableaux. In the following example, if $T(3, 2) = 3$, none of the gray boxes can be labeled 3.



The set of admissible tableaux with shape λ and weight μ is denoted by $\text{Tab}^{\text{ad}}(\lambda, \mu)$ and the set of admissible tableaux with shape λ , by $\text{Tab}^{\text{ad}}(\lambda)$.

Definition 3.1.2. A point (i, j) of a tableau T is said to be *critical* if $T(i, j) = T(i, j - 1)$.

Example: in the following tableau, critical points are indicated by a red circle.

| | | | |
|---|---|---|---|
| 1 | 3 | 1 | ① |
| 3 | 4 | ④ | |
| 5 | ⑤ | 2 | |
| 4 | 2 | | |

For a given labelling T , define the following polynomial in α

Definition 3.1.3. $d_T(\alpha) := \prod_{s \text{ critical}} \bar{h}_\lambda(s)$

Recall that the weight of a tableaux T is the composition $|T| = \nu$ defined by $\nu_i = \#\{s \in sh(T) \mid T(s) = i\}$.

Although originally defined as simultaneous eigenfunctions of a family of differential operators, Jack Functions have an alternative, purely combinatorial definition, proven equivalent to the former by Sahi and Knop.

Definition 3.1.4. [4, Theorem 5.1]

$$J_\lambda(\alpha) = \sum_{T \in Tab^{ad}(\lambda)} d_T(\alpha) x^{|T|} \tag{41}$$

Throughout the rest of the text, Jack functions will be denoted without reference to the variable α , as simply J_λ . In the interest of expressing this result from a symmetric function perspective, terms $x^{|T|}$ can be collected into monomial functions and all such functions which may happen to be counted repeatedly for different admissible labellings can be collected again.

If T is an admissible tableau and $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ is an injection, $T' = \sigma \circ T$ is also admissible and there exists another injection $\bar{\sigma} : \mathbb{N} \rightarrow \mathbb{N}$ such that $T = \bar{\sigma} \circ T'$. Since the composition of injective functions is itself injective, this condition defines the equivalence relation (\sim) in the set of admissible tableaux:

$$T \sim T' \Leftrightarrow \exists \sigma : \mathbb{N} \rightarrow \mathbb{N} \text{ injective such that } T' = \sigma \circ T$$

Temporarily adopting the notation $[T] = \{T' \mid T' \sim T\} = \{\sigma \circ T \mid \sigma : \mathbb{N} \rightarrow \mathbb{N} \text{ injective}\}$ for equivalence classes, it is well-defined to consider $[T]$ admissible if, and only if, some (and therefore every) $T \in [T]$ is admissible. Furthermore, because criticality does not change under relabelling, $d_T(\alpha) = d_{T'}(\alpha)$ for all $T \sim T'$, hence $d_{[T]}(\alpha) := d_T(\alpha)$ for some $T \in [T]$ is also well-defined.

For each admissible tableau T , there are $\prod_i (|T|_i)!$ different admissible tableaux $T' \sim T$ such that $|T'| = |T|$. However, it is always the case that $|T'|^+ = |T|^+$ for $T' \sim T$, so $|[T]| := |T|^+$ for some $T \in [T]$ is well-defined. Finally, (41) can be rewritten in terms of monomial functions

$$\begin{aligned} J_\lambda &= \sum_{[T] \text{ admissible}} d_{[T]}(\alpha) \left(\prod_i |[T]|_i! \right) m_{|[T]|} \\ &= \sum_{\mu \leq \lambda} \left(\sum_{T \in \text{Tab}^{\text{ad}}(\lambda, \mu)} d_T(\alpha) \right) \left(\prod_i M_i^{\mu_i}! \right) m_\mu \end{aligned} \quad (42)$$

where $\mu = (\mu_1^{M_1^\mu}, \mu_2^{M_2^\mu}, m_3^{M_3^\mu}, \dots)$.

This formula might seem somewhat abstract at the moment, but an example shall help clarify how exceedingly helpful it can be for calculating Jack Functions.

Example: Formula (42) will be utilized to obtain $J_{(3,2)}$.

In the following table weights μ are listed on the leftmost column and their correspondent $\prod_i M_i^{\mu_i}!$ terms are calculated on the middle column. Tableaux equivalence classes are listed according to their respective weights on the rightmost column. Each tableau's critical points are indicated with a red circle, and its correspondent $d_{[T]}(\alpha)$ polynomial is written immediately below.

| μ | $\prod_i M_i^{\mu_i}!$ | $T \in \text{Tab}^{\text{ad}}((3, 2), \mu)$ |
|------------|------------------------|--|
| $(3, 2)$ | 1 | $(2\alpha+2)(\alpha+2)$ $(2\alpha+2)$ $(\alpha+1)$ $(\alpha+1)$ |
| $(3, 1^2)$ | 2 | $(2\alpha+1)(\alpha+1)$ $(\alpha+1)$ $(\alpha+1)$ |
| $(2^2, 1)$ | 2 | $(2\alpha+2)$ $(2\alpha+2)$ $(2\alpha+2)$ 1 $(\alpha+1)$ $(\alpha+1)$ $(\alpha+1)$ $(\alpha+1)$ |
| $(2, 1^3)$ | 6 | $(2\alpha+2)$ 1 $(\alpha+1)$ 1 1 1 1 $(\alpha+1)$ |
| (1^5) | 120 | 1 |

Collecting all terms according to (42),

$$\begin{aligned}
J_{(3,2)} &= m_{(3,2)}(1) \left(2(\alpha+1)^2(\alpha+2) + 2(\alpha+1)^2 \right) + \\
&\quad m_{(3,1^2)}(2) \left((2\alpha+1)(\alpha+1) + (\alpha+1) + (\alpha+1) \right) + \\
&\quad m_{(2^2,1)}(2) \left(2(\alpha+1)^2 + 2(\alpha+1) + 2(\alpha+1) + 1 + (\alpha+1) + (\alpha+1)^2 + 1 \right) + \\
&\quad m_{(2,1^3)}(6) \left(2(\alpha+1) + 1 + (\alpha+1) + 1 + 1 + 1 + (\alpha+1) \right) + \\
&\quad m_{(1^5)}(120) (1) \\
&= 2(\alpha+1)^2(\alpha+3)m_{(3,2)} + 2(\alpha+1)(2\alpha+3)m_{(3,1^2)} + 2(3\alpha+5)(\alpha+2)m_{(2^2,1)} + \\
&\quad 24(\alpha+2)m_{(2,1^3)} + 120m_{(1^5)}
\end{aligned}$$

Observation: Formula (42) implies the coefficient of each m_μ of any J_λ must be a multiple of $\prod_i M_i^{\mu_i}!$.

3.2 Bilinear Form

Bilinear forms $\langle \cdot, \cdot \rangle_\alpha$ and $\langle \cdot, \cdot \rangle$ are very closely related. The involution ω is still an isometry for $\langle \cdot, \cdot \rangle_\alpha$, for

$$\begin{aligned} \langle \omega(p_\lambda), \omega(p_\mu) \rangle_\alpha &= \langle \varepsilon(\lambda)p_\lambda, \varepsilon(\mu)p_\mu \rangle_\alpha \\ &= \varepsilon(\lambda)\varepsilon(\mu)\langle p_\lambda, p_\mu \rangle_\alpha \\ &= \varepsilon(\lambda)\varepsilon(\mu)z_\lambda\alpha^{\ell(\lambda)}\delta_{\lambda\mu} \\ &= \langle p_\lambda, p_\mu \rangle_\alpha \end{aligned}$$

Now consider α to be evaluated by some positive integer. Denote $\alpha \cdot (\nu_1, \nu_2, \dots) = (\alpha\nu_1, \alpha\nu_2, \dots)$. Then for any partitions $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \dots)$ and μ ,

$$\begin{aligned} \langle p_\lambda, p_\mu \rangle_\alpha &= \alpha^{\ell(\lambda)}z_\lambda\delta_{\lambda\mu} \\ &= (\alpha^{m_1+m_2+\dots}) \left(\prod_{i \geq 1} \lambda_i^{m_i} m_i! \right) \delta_{\lambda\mu} \\ &= \left(\prod_{i \geq 1} (\alpha\lambda_i)^{m_i} m_i! \right) \delta_{\lambda,\mu} \\ &= z_{\alpha \cdot \lambda} \delta_{\alpha \cdot \lambda, \alpha \cdot \mu} \\ &= \langle p_{\alpha \cdot \lambda}, p_{\alpha \cdot \mu} \rangle \end{aligned}$$

Although the bilinear form $\langle \cdot, \cdot \rangle_\alpha$ is originally defined in terms of power functions, it may also be defined in terms of any $\mathbb{Q}(\alpha)$ -base of $\Lambda_{\mathbb{Q}(\alpha)}$, including Jack functions. Since they are orthogonal with respect to $\langle \cdot, \cdot \rangle_\alpha$, all that is left for this bilinear form to be thoroughly characterized in terms of Jack functions is the following case, given by Stanley.

Theorem 9. [15, Proposition 3.6]

$$\langle J_\lambda, J_\lambda \rangle_\alpha = \prod_{s \in \lambda} \hat{h}_\lambda(s) \check{h}_\lambda(s) \quad (43)$$

Example: For $\lambda = (3, 3, 2, 1)$, each box $s \in \text{diag}(\lambda)$ in both tableaux below has been assigned its correspondent $\hat{h}(s)$ and $\check{h}(s)$ value. $\langle J_\lambda, J_\lambda \rangle_\alpha$ is given by the product of their entries.

$$\langle J_\lambda, J_\lambda \rangle_\alpha = \left(\begin{array}{|c|c|c|} \hline 3\alpha+3 & 2\alpha+2 & \alpha+1 \\ \hline 3\alpha+2 & 2\alpha+1 & \alpha \\ \hline 2\alpha+1 & \alpha & \\ \hline \alpha & & \\ \hline \end{array} \right) \cdot \left(\begin{array}{|c|c|c|} \hline 2\alpha+4 & \alpha+3 & 2 \\ \hline 2\alpha+3 & \alpha+2 & 1 \\ \hline \alpha+2 & 1 & \\ \hline 1 & & \\ \hline \end{array} \right)$$

$$= 24\alpha^3(\alpha+1)^3(\alpha+2)^3(\alpha+3)(2\alpha+1)^2(2\alpha+3)(3\alpha+2)$$

3.3 Special Cases

Some special cases of coefficients $v_{\lambda\mu}(\alpha)$ that appear in the linear decomposition $J_\lambda = \sum_{\mu} v_{\lambda\mu}(\alpha)m_\mu$ for arbitrary λ can be directly calculated. Likewise, for some special cases of Jack functions, their entire decomposition can be directly calculated.

3.3.1 Special Cases of Coefficients

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_\ell) \vdash r$. It is immediate from Sahi-Knop's combinatorial formula that

$$v_{\lambda(1^r)} = r!$$

Despite its triviality, this result is nonetheless important because it determines the normalization of functions obtained through said formula. With a little more work, it can also be calculated

$$v_{\lambda(2,1^{r-2})} = \frac{\left(r + \sum_i \lambda_i^2 \right) \alpha + r^2 - \sum_i \lambda_i^2}{2} (r-2)!$$

$$= (\alpha - 1)n(\lambda)(r-2)! + \frac{r!}{2}$$

where $n(\lambda) = \sum_{i \geq 1} \binom{\lambda'_i}{2}$.

The cases $\mu = (1^r)$ and $\mu = (2, 1^{r-2})$ are easy to calculate by Sahi-Knop combinatorial formula, but the more general case where first components of μ coincide with those of λ , and the remaining ones are 1 was calculated by Stanley before such formula was known.

Proposition 20. [15, Proposition 7.1] If $\mu = (\lambda_1, \lambda_2, \dots, \lambda_k, 1^{\lambda_{k+1} + \dots + \lambda_\ell})$,

$$v_{\lambda\mu} = (\lambda_{k+1} + \dots + \lambda_\ell)! \left(\prod_{\substack{(i,j)=s \in \lambda \\ i \leq k}} \check{h}_\lambda(s) \right) \quad (44)$$

Corollary: $v_{\lambda\lambda} = \prod_{s \in \lambda} \check{h}_\lambda(s)$

3.3.2 Special Cases of Jack Functions

Jack functions indexed by single row partitions are given by:

Proposition 21. [15, Proposition 2.2 a]

$$J_{(n)} = \sum_{\mu \vdash n} \binom{n}{\mu} \varphi_\mu(\alpha) m_\mu \quad (45)$$

where $\varphi_\mu(\alpha) = \prod_{i \geq 1} \prod_{j=0}^{\mu_i-1} (j\alpha + 1)$.

Jack functions indexed by $(2^a, 1^b)$ can only have coefficients indexed by $(2^{a'}, 1^{b'})$ with $a' \leq a$ (and $b' = b + 2a - 2a'$). It follows as a corollary of (44) that

Proposition 22. [15, Proposition 7.2]

$$J_{(2^a, 1^b)} = \sum_{r=0}^a \frac{a!(2a - 2r + b)!}{(a - r)!} \left(\prod_{i=1}^r (\alpha + a + b - r + i) \right) m_{(2^r, 1^{2a-2r+b})}$$

Proof: Since

$$\begin{aligned}\check{h}_{(2^a, 1^b)}(i, 1) &= \alpha + a + b - i + 1 \\ \check{h}_{(2^a, 1^b)}(i, 2) &= a - i + 1\end{aligned}\quad \forall i \leq a$$

and according to (44), if $\eta = (2^r, 1^{2a-2r+b})$,

$$\begin{aligned}v_{(2^a, 1^b)\eta} &= (2a - 2r + b)! \prod_{\substack{(i,j) \in (2^a, 1^b) \\ i \leq r}} \check{h}_{(2^a, 1^b)}(i, j) \\ &= (2a - 2r + b)! \left(\prod_{i=1}^r (a - i + 1) \right) \left(\prod_{i=1}^r (\alpha + a + b - i + 1) \right)\end{aligned}$$

□

In order to prove the next special case, it is first necessary to establish the following proposition.

Proposition 23. *For any fixed $n \in \mathbb{N}$ and $\mu \vdash n$,*

$$\sum_{\substack{T \in \text{Tab}^{\text{ad}}((n), \mu) \\ T(1,1)=i}} d_T(\alpha) = \binom{n-1}{\mu - \varepsilon_i} \varphi_\mu(\alpha) \quad (46)$$

Proof: All row-shaped tableaux are admissible. Let T be a tableau such that $sh(T) = (n)$ and $T(1,1) = i$. Either $T(1,2) = i$, in which case $(1,2)$ is critical, or $T(1,2) \neq i$, in which case it is not. Applying Sahi-Knop's combinatorial formula,

$$\begin{aligned}\sum_{\substack{T \in \text{Tab}^{\text{ad}}((n), \mu) \\ T(1,1)=i}} d_T(\alpha) &= \left[((n-1)\alpha + 1) \sum_{\substack{T \in \text{Tab}^{\text{ad}}((n), \mu) \\ T(1,2)=T(1,1)=i}} d_T(\alpha) + \sum_{\substack{T \in \text{Tab}^{\text{ad}}((n), \mu) \\ T(1,2) \neq T(1,1)=i}} d_T(\alpha) \right] \\ &= (n-1)\alpha \sum_{\substack{T \in \text{Tab}^{\text{ad}}((n-1), \mu - \varepsilon_i) \\ T(1,1)=i}} d_T(\alpha) + \sum_{T \in \text{Tab}^{\text{ad}}((n-1), \mu - \varepsilon_i)} d_T(\alpha) \\ &= \sum_{k=1}^{\mu_i} \frac{(n-1)!}{(n-k)!} \alpha^{k-1} \sum_{T \in \text{Tab}^{\text{ad}}((n-k), \mu - k\varepsilon_i)} d_T(\alpha) \quad (\text{by induction}) \\ &= \binom{n}{\mu} \frac{\mu_i}{n} \frac{\varphi_\mu(\alpha)}{\varphi_{(\mu_i)}(\alpha)} \sum_{k=1}^{\mu_i} \frac{(\mu_i - 1)!}{(\mu_i - k)!} \alpha^{k-1} \varphi_{(\mu_i - k)}(\alpha) \quad (\text{by 45})\end{aligned}$$

But

$$\sum_{k=1}^{\mu_i} \frac{(\mu_i - 1)!}{(\mu_i - k)!} \alpha^{k-1} \varphi_{(\mu_i - k)}(\alpha) = \sum_{k=1}^{\mu_i} \left(\prod_{j=\mu_i - k}^{\mu_i - 2} \alpha(j+1) \right) \left(\prod_{j=1}^{\mu_i - k - 1} (\alpha j + 1) \right) = \varphi_{(\mu_i)}(\alpha)$$

Plugging this back in the previous equation,

$$\sum_{k=1}^{\mu_i} \frac{(\mu_i - 1)!}{(\mu_i - k)!} \alpha^{k-1} \varphi_{(\mu_i - k)}(\alpha) = \binom{n}{\mu} \frac{\mu_i}{n} \frac{\varphi_{\mu}(\alpha)}{\varphi_{(\mu_i)}(\alpha)} \varphi_{(\mu_i)}(\alpha) = \binom{n}{\mu} \frac{\mu_i}{n} \varphi_{\mu}(\alpha)$$

□

Hook-shaped Jack functions can be thoroughly described in terms of monomial functions.

Proposition 24. *Let $J_{(1+b, 1^a)} = \sum_{\mu} v_{(1+b, 1^a)\mu}(\alpha) m_{\mu}$. Then $v_{(1+b, 1^a)\mu}(\alpha)$ is given by*

$$\frac{a!b!}{\mu!} \varphi_{\bar{\mu}}(\alpha) \cdot \left[\begin{array}{l} \sum_{k=0}^{\ell(\mu) - a - 1} \sum_{r=0}^{a+1} \alpha^k \binom{k+r}{r} \binom{\ell(\mu) - k - r}{a+1-r} (a+1) e_{r+k}(\bar{\mu}) \\ + \sum_{k=1}^{\ell(\mu) - a} \sum_{r=0}^a \alpha^k \binom{k+r}{r+1} \binom{\ell(\mu) - k - r}{a-r} (r+1) e_{r+k}(\bar{\mu}) \\ + \sum_{k=1}^{\ell(\mu) - a} \sum_{r=0}^a \alpha^k \binom{k+r-1}{r} \binom{\ell(\mu) + 2 - k - r}{a-r} m_{(2, 1^{k+r-1})}(\bar{\mu}) \end{array} \right]$$

Proof: Let μ be a partition such that $\mu \leq (b+1, 1^a)$ and $\ell = \ell(\mu)$ (so $\ell > a$). Fix $\mathcal{I}, \mathcal{J} \subset \{1, 2, \dots, \ell\}$ complementary subsets such that $\#\mathcal{I} = a+1$ and $\#\mathcal{J} = \ell - a - 1$.

Then

$$\begin{aligned}
\sum_{\substack{T \in \text{Tab}^{\text{ad}}((b+1, 1^a), \mu) \\ T(\{1, \dots, a+1\} \times \{1\}) = \mathcal{I}}} d_T(\alpha) &= \sum_{i \in \mathcal{I}} \left(\sum_{\substack{T \in \text{Tab}^{\text{ad}}((b+1, 1^a), \mu) \\ T(\{1, \dots, a+1\} \times \{1\}) = \mathcal{I} \\ T(1, 1) = i}} d_T(\alpha) \right) \\
&= \sum_{i \in \mathcal{I}} \left(a! \sum_{\substack{T \in \text{Tab}^{\text{ad}}((b+1), \mu - \varepsilon_{\mathcal{I}} + \varepsilon_i) \\ T(1, 1) = i}} d_T(\alpha) \right) \\
&= a! \sum_{i \in \mathcal{I}} \binom{b}{\mu - \varepsilon_{\mathcal{I}}} \varphi_{\mu - \varepsilon_{\mathcal{I}} + \varepsilon_i}(\alpha) \\
&= a! \binom{b}{\mu - \varepsilon_{\mathcal{I}}} \varphi_{\mu - \varepsilon_{\mathcal{I}}}(\alpha) \sum_{i \in \mathcal{I}} (\bar{\mu}_i \alpha + 1) \\
&= \frac{a! b!}{\bar{\mu}!} \varphi_{\bar{\mu}}(\alpha) \left(\prod_{i \in \mathcal{I}} (\bar{\mu}_i + 1) \right) \left(\prod_{j \in \mathcal{J}} (\bar{\mu}_j \alpha + 1) \right) \left(\sum_{i \in \mathcal{I}} (\bar{\mu}_i \alpha + 1) \right) \\
&= \frac{a! b!}{\bar{\mu}!} \varphi_{\bar{\mu}}(\alpha) \left(\sum_{r=0}^{a+1} e_r(\bar{\mu}_{\mathcal{I}}) \right) \left(\sum_{k=0}^{\ell-a-1} \alpha^k e_k(\bar{\mu}_{\mathcal{J}}) \right) \left(\alpha e_1(\bar{\mu}_{\mathcal{I}}) + (a+1) \right) \\
&= \frac{a! b!}{\bar{\mu}!} \varphi_{\bar{\mu}}(\alpha) \left(\sum_{r=0}^{a+1} (\alpha m_{(2, 1^{r-1})}(\bar{\mu}_{\mathcal{I}}) + \alpha(r+1) e_{r+1}(\bar{\mu}_{\mathcal{I}}) + (a+1) e_r(\bar{\mu}_{\mathcal{I}})) \right) \\
&\quad \left(\sum_{k=0}^{\ell-a-1} \alpha^k e_k(\bar{\mu}_{\mathcal{J}}) \right)
\end{aligned}$$

Now, summing over all subsets \mathcal{I} of $\{1, \dots, \ell\}$ such that $\#\mathcal{I} = a+1$,

$$\begin{aligned}
\sum_{\substack{\mathcal{I} \subset \{1, \dots, \ell\} \\ \#\mathcal{I} = a+1}} e_r(\bar{\mu}_{\mathcal{J}}) e_k(\bar{\mu}_{\mathcal{I}}) &= \binom{k+r}{k} \binom{\ell-k-r}{a+1-r} e_{r+k}(\bar{\mu}) \\
\sum_{\substack{\mathcal{I} \subset \{1, \dots, \ell\} \\ \#\mathcal{I} = a+1}} m_{(2, 1^r)}(\bar{\mu}_{\mathcal{J}}) e_k(\bar{\mu}_{\mathcal{I}}) &= \binom{k+r}{k} \binom{\ell-k-r+1}{a-r} m_{(2, 1^{r+k})}(\bar{\mu}).
\end{aligned}$$

Applying Sahi-Knop's formula,

$$v_{(1+b, 1^a)\mu}(\alpha) = \sum_{T \in \text{Tab}^{\text{ad}}((b+1, 1^a), \mu)} d_T(\alpha) = \sum_{\substack{\mathcal{I} \subset \{1, \dots, \ell\} \\ \#\mathcal{I} = a+1}} \left(\sum_{\substack{T \in \text{Tab}^{\text{ad}}((b+1, 1^a), \mu) \\ T(\{1, \dots, a+1\} \times \{1\}) = \mathcal{I}}} d_T(\alpha) \right)$$

So finally

$$v_{(1+b,1^a)\mu}(\alpha) = \frac{a!b!}{\bar{\mu}!} \varphi_{\bar{\mu}}(\alpha) \left[\begin{aligned} & \sum_{r=0}^a \sum_{k=0}^{\ell-a-1} \alpha^{k+1} \binom{r+k}{k} \binom{\ell-r-k+1}{\ell-a-k+1} m_{(2,1^{r+k})}(\bar{\mu}) \\ & + \sum_{r=1}^{a+1} \sum_{k=0}^{\ell-a-1} \alpha^{k+1} \binom{k+r}{k} \binom{\ell-k-r}{a+1-r} r e_{r+k}(\bar{\mu}) \\ & + \sum_{r=0}^{a+1} \sum_{k=0}^{\ell-a-1} \alpha^k \binom{k+r}{k} \binom{\ell-k-r}{a+1-r} (a+1) e_{r+k}(\bar{\mu}) \end{aligned} \right]$$

which is equivalent to the stated formula.

□

3.4 Pieri Rule

Completely analogously to the case of symmetric functions over integer or rational coefficients, the product of two Jack functions can be linearly decomposed in terms of Jack functions themselves.

$$J_\nu J_\mu = \sum_{\lambda} c_{\mu\nu}^\lambda(\alpha) J_\lambda \quad (47)$$

Since Jack functions are a base for $\Lambda_{\mathbb{Q}(\alpha)}$ as a $\mathbb{Q}(\alpha)$ -module, $\mathbb{Q}(\alpha)$ is a field and there are finitely many partitions with any given fixed weight, the coefficients $c_{\lambda\mu}^\nu(\alpha)$ can always be found using tools from linear algebra. Ideally, these coefficients could be given by a combinatorial rule, much like Littlewood-Richardson-Robinson's, without the need for lengthy computations. However, no such rule is yet known, except for a few special cases.

The analogous Pieri Rule for symmetric functions, that is, the special case for formula (47) when one of the partitions λ or μ is either a single column or a single row, was proven by Stanley in 1989. The rule is stated in terms of the inner product $\langle \cdot, \cdot \rangle_\alpha$.

Theorem 10. [15, Theorem 6.1] Let $n \in \mathbb{N}$ and $\mu, \lambda \in \mathcal{P}$. Then $\langle J_\mu J_{(n)}, J_\lambda \rangle_\alpha = 0$ unless $\mu \subseteq \lambda$ and λ/μ is a horizontal n -strip, in which case

$$\langle J_\mu J_{(n)}, J_\lambda \rangle_\alpha = \left(\prod_{s \in \mu} A_{\lambda/\mu}^\mu(s) \right) \left(\prod_{s \in (n)} \hat{h}_{(n)}(s) \right) \left(\prod_{s \in \lambda} B_{\lambda/\mu}^\lambda(s) \right) \quad (48)$$

where

$$A_{\lambda/\mu}^\eta(s) := \begin{cases} \hat{h}_\eta(s) & \text{if } \lambda/\mu \text{ contains a square in the same column as } s, \\ \check{h}_\eta(s) & \text{otherwise} \end{cases}$$

$$B_{\lambda/\mu}^\eta(s) := \begin{cases} \check{h}_\eta(s) & \text{if } \lambda/\mu \text{ contains a square in the same column as } s, \\ \hat{h}_\eta(s) & \text{otherwise} \end{cases}$$

Notice that $\prod_{s \in (n)} \hat{h}_{(n)}(s) = \alpha^n n!$

Applying the inner product to (47), coefficients $c_{\nu\mu}^\lambda(\alpha)$ can be isolated

$$c_{\nu\mu}^\lambda(\alpha) = \frac{\langle J_\nu J_\mu, J_\lambda \rangle_\alpha}{\langle J_\lambda, J_\lambda \rangle_\alpha}$$

and since $\langle J_\lambda, J_\lambda \rangle_\alpha = \prod_{s \in \lambda} \hat{h}_\lambda(s) \check{h}_\lambda(s)$, Stanley's theorem can be restated so as to directly calculate these coefficients:

$$c_{\mu,(n)}^\lambda(\alpha) = (\alpha^n n!) \frac{\prod_{s \in \mu} A_{\lambda/\mu}^\mu(s)}{\prod_{s \in \lambda} A_{\lambda/\mu}^\lambda(s)} \quad (49)$$

Examples:

1. $J_{(2,1)} J_{(2)}$

There are four partitions ν such that $(2,1) \subseteq \nu$ and $\nu/(2,1)$ is a 2-horizontal strip: $(4,1)$, $(3,2)$, $(3,1^2)$ and $(2^2,1)$. Equations (48) and (49) have pictorial interpretations which we can take advantage of. In the following calculations, each box of the tableaux has been assigned its respective $A_{\nu/(2^2)}^{(2^2)}(s)$ or $A_{\nu/(2^2)}^\nu(s)$ value, which will then be multiplied in order to obtain $\prod_{s \in (2^2)} A_{\nu/(2^2)}^{(2^2)}(s)$ and $\prod_{s \in \nu} A_{\nu/(2^2)}^\nu(s)$ respectively. Boxes in ν which are not in (2^2)

have been colour-coded dark blue (■) and boxes in either (2^2) or each ν sitting on the same column of a dark blue box have been colour-coded light blue (■). To both dark and light blue boxes, their $\hat{h}(s)$ value is assigned, whereas to white boxes their $\check{h}(s)$ value is assigned.

$$c_{(2,1)(2)}^{(4,1)} = 2\alpha^2 \frac{\begin{array}{|c|c|} \hline \alpha+2 & 1 \\ \hline 1 & \\ \hline \hline \hline 3\alpha+2 & 2\alpha+1 & \color{darkblue}2\alpha & \color{darkblue}\alpha \\ \hline 1 & & & \\ \hline \end{array}}{(3\alpha+2)(2\alpha+1)} = \frac{(\alpha+2)}{(3\alpha+2)(2\alpha+1)}$$

$$c_{(2,1)(2)}^{(3,2)} = 2\alpha^2 \frac{\begin{array}{|c|c|} \hline \alpha+2 & \color{lightblue}\alpha \\ \hline 1 & \\ \hline \hline \hline 2\alpha+2 & \color{lightblue}2\alpha+1 & \color{darkblue}\alpha \\ \hline \alpha+1 & \color{darkblue}\alpha & \\ \hline \end{array}}{(2\alpha+1)(\alpha+1)^2} = \frac{(\alpha+2)\alpha}{(2\alpha+1)(\alpha+1)^2}$$

$$c_{(2,1)(2)}^{(3,1^2)} = 2\alpha^2 \frac{\begin{array}{|c|c|} \hline \color{lightblue}2\alpha+1 & 1 \\ \hline \color{lightblue}\alpha & \\ \hline \hline \hline 3\alpha+2 & \alpha+1 & \color{darkblue}\alpha \\ \hline \color{lightblue}\alpha+1 & & \\ \hline \color{darkblue}\alpha & & \\ \hline \end{array}}{(3\alpha+2)(\alpha+1)^2} = \frac{2\alpha(2\alpha+1)}{(3\alpha+2)(\alpha+1)^2}$$

$$c_{(2,1)(2)}^{(2^2,1)} = 2\alpha^2 \frac{\begin{array}{|c|c|} \hline 2\alpha+1 & \alpha \\ \hline \alpha & \\ \hline 2\alpha+2 & \alpha+1 \\ \hline 2\alpha+1 & \alpha \\ \hline \alpha & \\ \hline \end{array}}{\alpha^2} = \frac{\alpha^2}{(\alpha+1)^2}$$

So finally

$$J_{(2^2)}J_{(2)} = \frac{(\alpha+2)}{(3\alpha+2)(2\alpha+1)}J_{(4,1)} + \frac{(\alpha+2)\alpha}{(2\alpha+1)(\alpha+1)^2}J_{(3,2)} + \frac{2\alpha(2\alpha+1)}{(3\alpha+2)(\alpha+1)^2}J_{(3,1^2)} + \frac{\alpha^2}{(\alpha+1)^2}J_{(2^2,1)}$$

2. a) $J_{(1^m)}J_{(n)}$

The same method can be utilized to calculate the more general product $J_{(1^m)}J_{(n)}$. There are two partitions ν such that $\nu/(1^m)$ is a n -horizontal strip: $(n+1, 1^{m-1})$ and $(n, 1^m)$.

$$c_{(n+1,1^{m-1})} = (\alpha^n n!) \frac{\begin{array}{|c|} \hline m \\ \hline \vdots \\ \hline 1 \\ \hline n\alpha+m & n\alpha & \cdots & \alpha \\ \hline \vdots \\ \hline 1 \\ \hline \end{array}}{\alpha^n n!} = \frac{m}{n\alpha+m}$$

$$c_{(n,1^m)} = (\alpha^n n!) \frac{\begin{array}{c} \alpha+ \\ m-1 \\ \vdots \\ \alpha \\ n\alpha+m \quad \cdots \quad \alpha \\ \alpha+ \\ m-1 \\ \vdots \\ \alpha \end{array}}{n\alpha + m} = \frac{n\alpha}{n\alpha + m}$$

Finally,

$$J_{(1^m)} J_{(n)} = \frac{n\alpha J_{(n,1^m)} + m J_{(n+1,1^{m-1})}}{n\alpha + m}$$

2. b) $J_{(a^m)} J_{(n)}$

More generally, there are $1 + \min\{a, n\}$ partitions ν such that $\nu/(a^m)$ is a horizontal n -strip, all of the form $(a+n-r, a^{m-1}, r)$ with $0 \leq r \leq \min\{a, n\}$. Therefore

$$J_{(a^m)} J_{(n)} = \sum_{r=0}^{\min\{a,n\}} \frac{((a+n-2r)\alpha + m) \prod_{k=0}^{a-r-1} (k\alpha + m)}{\prod_{k=0}^a ((n-r+k)\alpha + m)} \binom{a}{r} \binom{n}{r} r! \alpha^r J_{(a+n-r, a^{m-1}, r)}$$

3. $J_\lambda J_{(1)}$

Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \dots, \lambda_\ell^{m_\ell})$, denote $m_0 = 0 = \lambda_{\ell+1}$ and $M_k = \sum_{i=0}^{k-1} m_i$. The only ν for which $\lambda \subseteq \nu$ and $|\nu/\lambda| = 1$ are those obtained by adding a box to a “corner” of the diagram of λ , that is, in each coordinate $(1 + M_k, 1 + \lambda_k)$ for $k \in \{1, \dots, \ell + 1\}$.

Fix a k and consider ν obtained by adding a box to the $(1 + M_k, 1 + \lambda_k)$ coordinate of the diagram of λ . Consider the product $\prod_{s \in \lambda} A_{\nu/\lambda}^\lambda(s)$. The only

boxes' values that must be recalculated upon adhering this box (so as to obtain $\prod_{s \in \lambda} A_{\nu/\lambda}^\nu(s)$) are those on the line $1 + M_k$ and those on the column $1 + \lambda_k$. Furthermore, the value assigned to the box $(1 + M_k, 1 + \lambda_k)$ itself is necessarily α , so

$$\begin{aligned}
c_{\lambda,(1)}^\nu &= \alpha \frac{\prod_{s \in \lambda} A_{\nu/\lambda}^\lambda(s)}{\prod_{s \in \lambda} A_{\nu/\lambda}^\nu(s)} \\
&= \alpha \frac{\left(\prod_{j=1}^{\lambda_k} A_{\nu/\lambda}^\lambda(1 + M_k, j) \right) \left(\prod_{i=1}^{M_k} A_{\nu/\lambda}^\lambda(i, 1 + \lambda_k) \right)}{\left(\prod_{j=1}^{\lambda_k} A_{\nu/\lambda}^\nu(1 + M_k, j) \right) \alpha \left(\prod_{i=1}^{M_k} A_{\nu/\lambda}^\nu(i, 1 + \lambda_k) \right)} \\
&= \left(\prod_{j=1}^{\lambda_k} \frac{\check{h}_\lambda(1 + M_k, j)}{\check{h}_\nu(1 + M_k, j)} \right) \left(\prod_{i=1}^{M_k} \frac{\hat{h}_\lambda(i, 1 + \lambda_k)}{\hat{h}_\nu(i, 1 + \lambda_k)} \right) \\
&= \left(\prod_{q=k}^{\ell} \frac{\check{h}_\lambda(1 + M_k, \lambda_q - 1)}{\check{h}_\nu(1 + M_k, \lambda_{q+1} + 1)} \right) \left(\prod_{q=0}^{k-1} \frac{\hat{h}_\lambda(M_{q+1} - 1, 1 + \lambda_k)}{\hat{h}_\nu(M_q + 1, 1 + \lambda_k)} \right) \\
&= \left(\prod_{q=k}^{\ell} \frac{(\lambda_k - \lambda_q)\alpha + M_q - M_{k-1}}{(\lambda_k - \lambda_{q+1})\alpha + M_q - M_{k-1}} \right) \left(\prod_{q=1}^{k-1} \frac{(\lambda_q - \lambda_k)\alpha + M_{k-1} - M_q}{(\lambda_q - \lambda_k)\alpha + M_{k-1} - M_{q-1}} \right)
\end{aligned} \tag{50}$$

3.5 Skew Jack Functions

Similarly to Schur functions, Jack functions can be generalized to a broader family, now defined over skew diagrams instead of just partitions. Let $\mu \subseteq \lambda$. Skew Jack functions $J_{\lambda/\mu}$ are defined to satisfy the familiar inner product equation $\langle J_{\lambda/\mu}, J_\nu \rangle_\alpha = \langle J_\mu J_\nu, J_\lambda \rangle_\alpha$ for all J_ν . Equivalently,

$$J_{\lambda/\mu} = \sum_{\nu} \frac{\langle J_\mu J_\nu, J_\lambda \rangle_\alpha}{\langle J_\nu, J_\nu \rangle_\alpha} J_\nu \tag{51}$$

As is the case for Skew Schur functions, the inner product condition

whereby skew Jack functions are defined implies that the linear decomposition of skew Jack functions (51) carries the same information as the linear decomposition of Jack function products (47). In other words, the problem of decomposing products of Jack functions can be translated to the problem of decomposing skew Jack functions. The converse is also true and since a Pieri type formula for Jack functions is known, so can an analogous formula be derived for skew Jack functions.

Let

$$J_{\lambda/(n)} = \sum_{\nu} c_{\nu}^{\lambda/(n)}(\alpha) J_{\nu}$$

Plugging (48) and (43) in (51), it follows that

$$c_{\nu}^{\lambda/(n)}(\alpha) = (\alpha^n n!) \frac{\prod_{s \in \lambda} B_{\lambda/\nu}^{\lambda}(s)}{\prod_{s \in \nu} B_{\lambda/\nu}^{\nu}(s)} \quad (52)$$

and each ν must be such that $\nu \subseteq \lambda$ and λ/ν is a n -horizontal strip.

We now explore some special cases of non-symmetric Jack functions which can be deduced from (52).

Examples:

1. $J_{\lambda/(\lambda_1)}$, where $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \dots)$

There is only one partition ν such that λ/ν is a λ_1 -horizontal strip: $\nu = (\lambda_1^{m_1-1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \dots)$. For this case, every block of both $\text{diag}(\nu)$ and $\text{diag}(\lambda)$ share a column with a block in $\text{diag}(\lambda/\nu)$, so $B_{\lambda/\nu}(s) = \check{h}(s)$.

Hence

$$c_{\nu}^{\lambda/(\lambda_1)} = (\alpha^{\lambda_1} \lambda_1!) \frac{\prod_{s \in \lambda} \check{h}_{\lambda}(s)}{\prod_{s \in \nu} \check{h}_{\nu}(s)} = (\alpha^{\lambda_1} \lambda_1!) \prod_{j=1}^{\lambda_1} \check{h}_{\lambda}(1, j)$$

Which implies

$$J_{\lambda/(\lambda_1)} = (\alpha^{\lambda_1} \lambda_1!) \left(\prod_{j=1}^{\lambda_1} \check{h}_{\lambda}(1, j) \right) J_{(\lambda_1^{m_1-1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \dots)}$$

2. $J_{\lambda/(1)}$

Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \dots, \lambda_\ell^{m_\ell})$ and once again denote $M_k = \sum_{i=0}^{k-1} m_i$. The only partitions $\nu \subseteq \lambda$ for which $|\lambda/\nu| = 1$ are those obtained by removing a box from a ‘‘corner’’ of the diagram of λ , that is, from each coordinate (M_k, λ_k) . Let ν be the partition for which $\lambda/\nu = (M_k, \lambda_k)$. Repeating a similar procedure to that for calculating (50),

$$\begin{aligned}
c_\nu^{\lambda/(1)} &= \alpha \frac{\prod_{s \in \lambda} B_{\lambda/\nu}^\lambda(s)}{\prod_{s \in \nu} B_{\lambda/\nu}^\nu(s)} \\
&= \alpha \frac{\left(\prod_{j=1}^{\lambda_k-1} B_{\lambda/\nu}^\lambda(M_k, j) \right) (1) \left(\prod_{i=1}^{M_k-1} B_{\lambda/\nu}^\lambda(i, \lambda_k) \right)}{\left(\prod_{j=1}^{\lambda_k-1} B_{\lambda/\nu}^\nu(M_k, j) \right) \left(\prod_{i=1}^{M_k-1} B_{\lambda/\nu}^\nu(i, \lambda_k) \right)} \\
&= \alpha \left(\prod_{j=1}^{\lambda_k-1} \frac{\hat{h}_\lambda(M_k, j)}{\hat{h}_\nu(M_k, j)} \right) \left(\prod_{i=1}^{M_k-1} \frac{\check{h}_\lambda(i, \lambda_k)}{\check{h}_\nu(i, \lambda_k)} \right) \\
&= \alpha \left(\frac{\hat{h}_\lambda(M_k, \lambda_{k+1} + 1)}{\hat{h}_\nu(M_k, \lambda_k - 1)} \left(\prod_{q=k+1}^{\ell} \frac{\hat{h}_\lambda(M_k, \lambda_{q+1} + 1)}{\hat{h}_\nu(M_k, \lambda_q)} \right) \right) \\
&\quad \left(\frac{\check{h}_\lambda(M_{k-1}, \lambda_k)}{\check{h}_\nu(M_k - 1, \lambda_k)} \left(\prod_{q=1}^{k-1} \frac{\check{h}(M_{q-1} + 1, \lambda_k)}{\check{h}_\nu(M_q, \lambda_k)} \right) \right)
\end{aligned}$$

But $\hat{h}_\nu(M_k, \lambda_k - 1) = \alpha$ and $\hat{h}_\lambda(M_k, \lambda_{k+1} + 1) = \alpha(\lambda_k - \lambda_{k+1})$, so
 $\check{h}_\nu(M_k - 1, \lambda_k) = 1$ and $\check{h}_\lambda(M_{k-1}, \lambda_k) = m_k$,

$$\begin{aligned}
c_\nu^{\lambda/(1)} &= \alpha(\lambda_k - \lambda_{k+1})m_k \cdot \left(\prod_{q=k+1}^{\ell} \frac{\alpha(\lambda_k - \lambda_{q+1}) + M_q - M_k}{\alpha(\lambda_k - \lambda_q) + M_q - M_k} \right) \\
&\quad \left(\prod_{q=1}^{k-1} \frac{\alpha(\lambda_q - \lambda_k) + M_k - M_{q-1}}{\alpha(\lambda_q - \lambda_k) + M_k - M_q} \right)
\end{aligned}$$

4 Non-symmetric Jack Functions

It has been emphasized that the A -algebra of symmetric functions Λ_A is comprised not of symmetric polynomials, but of dimension independent objects which generalize the notion of symmetric polynomials. Nevertheless, $\Lambda_{\mathbb{Q}(\alpha)}$ is generated as a $\mathbb{Q}(\alpha)$ -algebra by families of symmetric functions indexed by non-negative integers such as e_r , h_r and p_r and as such is isomorphic to a polynomial algebra over infinitely many variables, say, $p = (p_0, p_1, p_2, \dots)$.

$$\Lambda_{\mathbb{Q}(\alpha)} \cong \mathbb{Q}(\alpha)[p_0, p_1, p_2, \dots] = \mathbb{Q}(\alpha)[p]$$

In the interest of defining an algebra which generalizes *all* homogeneous polynomials over arbitrarily many finite dimensions, a new algebra can be defined — still drawing from the same framework of symmetric functions — by tweaking $\Lambda_{\mathbb{Q}(\alpha)}$'s definition so as to account for finitely many non-symmetric variables. This can be done by reintroducing independent variables $x = (x_1, x_2, x_3, \dots)$ to $\Lambda_{\mathbb{Q}(\alpha)}$ as means to create an algebra isomorphic to $\mathbb{Q}(\alpha)[p, x]$, that is, whose elements can be viewed as polynomials over both p and x . The goal of the following section is to properly formalize this new algebra.

4.1 Algebra of Non-Symmetric Functions

The construction of the algebra of non-symmetric functions is similar to that of symmetric functions, except that in this case, the order of variables is important, so henceforth x will solely refer to the countable sequence of variables $x = (x_1, x_2, \dots)$ as opposed to the countable set of variables $\{x_1, x_2, \dots\}$. Let A be an arbitrary commutative ring with identity. For non-negative integers m, N , with $m \geq N$, the following notation will be adopted.

Notation:

- $\Lambda_A[x_1, \dots, x_N | x_{N+1}, \dots, x_m] := A[x_1, \dots, x_N] \Lambda_A[x_{N+1}, \dots, x_m]$ denotes the algebra generated by products of polynomials in $A[x_1, x_2, \dots, x_N]$ and $\Lambda_A[x_{N+1}, x_{N+2}, \dots, x_m]$, that is, the A -sub-algebra of $A[x_1, x_2, \dots, x_m]$ of polynomials which are symmetric on the variables $x_{N+1}, x_{N+2}, \dots, x_m$.
- $\Lambda_A^k[x_1, \dots, x_N | x_{N+1}, \dots, x_m] := \bigoplus_{k_1+k_2=k} A^{k_1}[x_1, \dots, x_N] \Lambda_A^{k_2}[x_{N+1}, \dots, x_m]$ denotes the A -sub-module of $A^k[x_1, x_2, \dots, x_m]$ of k -homogeneous polynomials which are symmetrical on the variables $x_{N+1}, x_{N+2}, \dots, x_m$.

Observations:

- Each $p(x_1, \dots, x_m) \in \Lambda_A[x_1, \dots, x_N | x_{N+1}, \dots, x_m]$ can be uniquely decomposed as a sum

$$p(x_1, \dots, x_m) = \sum_{\mu} q_{\mu}(x_1, x_2, \dots, x_N) m_{\mu}(x_{N+1}, x_{N+2}, \dots, x_m).$$

- $\Lambda_A[x_1, \dots, x_N | x_{N+1}, \dots, x_m]$ is the direct sum of its k -homogeneous sub-modules

$$\Lambda_A[x_1, \dots, x_N | x_{N+1}, \dots, x_m] = \bigoplus_k \Lambda_A^k[x_1, \dots, x_N | x_{N+1}, \dots, x_m]$$

as a graded algebra.

Consider the A -algebra homomorphisms for all $n, m \geq N$:

$$\rho_{N|m,n} : \Lambda_A[x_1, \dots, x_N | x_{N+1}, \dots, x_n] \rightarrow \begin{cases} \Lambda_A[x_1, \dots, x_N | x_{N+1}, \dots, x_m] \\ \sum_{\mu} q_{\mu} m_{\mu}(x_{N+1}, \dots, x_m, 0, \dots, 0) & \text{if } n \geq m \\ \sum_{\mu} q_{\mu} m_{\mu}(x_{N+1}, \dots, x_n, x_{n+1}, \dots, x_m) & \text{if } n < m \end{cases}$$

and their natural restriction for k -homogeneous polynomial modules

$$\rho_{N|m,n}^k : A^k[x_1, \dots, x_N | x_{N+1}, \dots, x_n] \rightarrow A^k[x_1, \dots, x_N | x_{N+1}, \dots, x_m].$$

Analogously to functions $\rho_{m,n}$, if $m \leq n$, $\rho_{N|m,n} \circ \rho_{N|n,m} = \text{id}_m$, so $\rho_{N|n,m}$ is a monomorphism and $\rho_{N|m,n}$ is an epimorphism and the same holds for $\rho_{N|n,m}^k$ and $\rho_{N|m,n}^k$, for each $k \in \mathbb{N}$. Also denote, without loss of generality, $\rho_{N|m,n} = \rho_{m,n}|_{\Lambda_A[x_1, \dots, x_N|x_{N+1}, \dots, x_n]}$ and $\rho_{N|m,n}^k = \rho_{m,n}^k|_{\Lambda_A^k[x_1, \dots, x_N|x_{N+1}, \dots, x_n]}$ for all $m < N$. Since for every $n \geq N$, functions $\rho_{N|m,n}$ are A -algebra homomorphisms, so are the functions

$$\begin{aligned} \rho_{N|n} : \Lambda_A[x_1, \dots, x_N|x_{N+1}, \dots, x_n] &\rightarrow \prod_{m \in \mathbb{N}} A[x_1, \dots, x_m] \\ p &\mapsto (\rho_{N|m,n}(p))_{m \in \mathbb{N}} \end{aligned}$$

where once again, $\prod_{m \in \mathbb{N}} A[x_1, \dots, x_m]$ is the A -algebra of sequences of polynomials in increasing numbers of variables with element-wise sum and multiplication. Similarly, for all $n \geq N$ and $k \in \mathbb{N}$, functions

$$\begin{aligned} \rho_{N|n}^k : \Lambda_A^k[x_1, \dots, x_N|x_{N+1}, \dots, x_n] &\rightarrow \prod_{m \in \mathbb{N}} A^k[x_1, \dots, x_m] \\ p &\mapsto (\rho_{N|m,n}^k(p))_{m \in \mathbb{N}} \end{aligned}$$

are A -module homomorphisms.

Also analogously to functions $\rho_{m,n}$ and $\rho_{m,n}^k$,

$$m \leq n \Rightarrow \begin{cases} \text{Im}(\rho_{N|m}) \subseteq \text{Im}(\rho_{N|n}) \\ \text{Im}(\rho_{N|m}^k) \subseteq \text{Im}(\rho_{N|n}^k), \forall k \in \mathbb{N} \end{cases}$$

so for each $N \in \mathbb{N}$, $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_{N|n})$ is an A -sub-algebra of $\prod_{m \in \mathbb{N}} A[x_1, \dots, x_m]$ and for every $k \in \mathbb{N}$, $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_{N|n}^k)$ is an A -sub-module of $\prod_{m \in \mathbb{N}} A^k[x_1, \dots, x_m]$.

Proposition 25. $\bigcup_{n, N \in \mathbb{N}} \text{Im}(\rho_{N|n})$ is an A -sub-algebra of $\prod_{m \in \mathbb{N}} A[x_1, \dots, x_m]$

Proof: Firstly let $m', N \in \mathbb{N}$ and $n, m \geq N$. Then analogously to functions $\rho_{m,n}$,

If $m \leq n \leq m'$, $\rho_{N|m',m} = \rho_{N|m',n} \circ \rho_{N|n,m}$, which implies $\text{Im}(\rho_{N|m',m}) \subseteq \text{Im}(\rho_{N|m',n})$

If $m' \leq n$, $\rho_{N|m',n}$ is surjective, so $\text{Im}(\rho_{N|m',m}) \subseteq \text{Im}(\rho_{N|m',n})$.

Hence $\text{Im}(\rho_{N|m}) \subseteq \text{Im}(\rho_{N|n})$ for all $N \in \mathbb{N}$ and $n \geq m \geq N$. Therefore for each $N \in \mathbb{N}$, $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_{N|n})$ is closed under all algebra operations. Since each $\text{Im}(\rho_{N|n})$ is itself an A -sub-algebra of $\prod_{m \in \mathbb{N}} A[x_1, \dots, x_m]$, so must $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_{N|n})$ be.

Now for all $N, M \in \mathbb{N}$, $N \leq M$ implies $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_{M|n}) \subseteq \bigcup_{n \in \mathbb{N}} \text{Im}(\rho_{N|n})$ because $\Lambda_A[x_1, \dots, x_M | x_{M+1}, \dots, x_n] \subseteq \Lambda_A[x_1, \dots, x_N | x_{N+1}, \dots, x_n]$. Therefore $\bigcup_{n, N \in \mathbb{N}} \text{Im}(\rho_{N|n})$ is closed under all A -algebra operations. Since each $\bigcup_{n \in \mathbb{N}} \text{Im}(\rho_{N|n})$ is itself an A -sub-algebra of $\prod_{m \in \mathbb{N}} A[x_1, \dots, x_m]$, so is $\bigcup_{n, N \in \mathbb{N}} \text{Im}(\rho_{N|n})$.

□

Observation: By the same arguments $\bigcup_{n, N \in \mathbb{N}} \text{Im}(\rho_{N|n}^k)$ is an A -sub-module of $\prod_{m \in \mathbb{N}} A^k[x_1, \dots, x_m]$ for every $k \in \mathbb{N}$.

Definition 4.1.1. Let $x = (x_1, x_2, \dots)$ be a countably infinite set of variables. Then

$\widehat{\Lambda}_A[x] := \bigcup_{N, n \in \mathbb{N}} \text{Im}(\rho_{N|n})$ is the A -algebra of non-symmetric functions and

$\widehat{\Lambda}_A^k[x] := \bigcup_{N, n \in \mathbb{N}} \text{Im}(\rho_{N|n}^k)$ are the k -homogeneous A -modules of non-symmetric functions over x .

These definitions are related by the following proposition.

Proposition 26. Let $x = (x_1, x_2, \dots)$ be a countably infinite set of variables. Then

$$\widehat{\Lambda}_A[x] = \bigoplus_{k \in \mathbb{N}} \widehat{\Lambda}_A^k[x]$$

as a graded algebra.

Proof: Since $\Lambda_A[x_1, \dots, x_N | x_{N+1}, \dots, x_n] = \bigoplus_k \Lambda_A^k[x_1, \dots, x_N | x_{N+1}, \dots, x_n]$ for all $N \leq n$, each polynomial $f \in \Lambda_A[x_1, \dots, x_N | x_{N+1}, \dots, x_n]$ may be uniquely decomposed in its k -homogeneous components. Let $f = \sum_{k=0}^J f^{(k)}$ be such decomposition, where $f^{(k)} \in \Lambda_A^k[x_1, \dots, x_N | x_{N+1}, \dots, x_n]$ for each k . Then:

$$\rho_{N|m,n}(f) = \sum_{k=0}^J \rho_{N|m,n}(f^{(k)}) = \sum_{k=0}^J \rho_{N|m,n}^k(f^{(k)}), \quad \forall m \in \mathbb{N}.$$

Therefore

$$\rho_{N|n}(f) = \sum_{k=0}^J \rho_{N|n}(f^{(k)}) = \sum_{k=0}^J \rho_{N|n}^k(f^{(k)}).$$

is the unique decomposition of f in components of $\widehat{\Lambda}_A^k[x]$.

Now for arbitrary $k_1, k_2 \in \mathbb{N}$, let $f \in \bigcup_{n, N \in \mathbb{N}} \text{Im}(\rho_{N|n}^{k_1})$ and $g \in \bigcup_{n, N \in \mathbb{N}} \text{Im}(\rho_{N|n}^{k_2})$, that is, there are $N_1, N_2, m_1, m_2 \in \mathbb{N}$ and $f' \in \Lambda_A^{k_1}[x_1, \dots, x_{N_1} | x_{N_1+1}, \dots, x_{m_1}]$, $g' \in \Lambda_A^{k_2}[x_1, \dots, x_{N_2} | x_{N_2+1}, \dots, x_{m_2}]$ such that $f = \rho_{N_1|m_1}^{k_1}(f')$ and $g = \rho_{N_2|m_2}^{k_2}(g')$. Now since for all $k \in \mathbb{N}$,

$$\begin{aligned} m \leq n &\Rightarrow \text{Im}(\rho_{N|m,n}^k) \subseteq \text{Im}(\rho_{N|m,n}^k), \quad \forall N \in \mathbb{N} \\ M \leq N &\Rightarrow \text{Im}(\rho_{M|m,n}^k) \subseteq \text{Im}(\rho_{N|m,n}^k), \quad \forall n, m \in \mathbb{N} \end{aligned}$$

take $m = \max\{m_1, m_2\}$ and $N = \max\{N_1, N_2\}$, so that $f = \rho_{N|m}^{k_1}(f')$ and $g = \rho_{N|m}^{k_2}(g')$. Therefore

$$\begin{aligned} fg &= \rho_{N|m}^{k_1}(f') \rho_{N|m}^{k_2}(g') \\ &= (\rho_{N|m',m}^{k_1}(f'))_{m' \in \mathbb{N}} (\rho_{N|m,m'}^{k_2}(g'))_{m' \in \mathbb{N}} \\ &= (\rho_{N|m',m}^{k_1}(f') \rho_{N|m',m}^{k_2}(g'))_{m' \in \mathbb{N}} \\ &= (\rho_{N|m',m}(f') \rho_{N|m',m}(g'))_{m' \in \mathbb{N}} \\ &= (\rho_{N|m',m}(f'g'))_{m' \in \mathbb{N}} \\ &= (\rho_{N|m',m}^{k_1+k_2}(f'g'))_{m' \in \mathbb{N}} \\ &= \rho_n^{k_1+k_2}(f'g') \in \bigcup_{N, n \in \mathbb{N}} \text{Im}(\rho_{N|n}^k). \end{aligned}$$

□

Observations:

- Both the algebra of non-symmetric functions and its sub-modules of k -homogeneous non-symmetric functions will be denoted without reference to their variables by $\widehat{\Lambda}_A$ and $\widehat{\Lambda}_A^k$, respectively.
- Non-symmetric functions will often be expressed as objects with a non-symmetric polynomial components (in finitely many variables) and symmetric components (in countably infinitely many variables).

The algebra of non-symmetric functions $\widehat{\Lambda}_A$ is intimately related to Λ_A . In fact, there is a natural epimorphism

$$\begin{array}{ccc} \widehat{\Lambda}_A & \rightarrow & \Lambda_A \\ f(x_1, x_2, x_3, \dots) & \mapsto & f(\underbrace{0, \dots, 0}_{\substack{\text{quantity of} \\ \text{non-symmetric} \\ \text{variables of } f}}, x_1, x_2, x_3, \dots) \end{array}$$

which effects the symmetrization of non-symmetric functions.

Every (non-null) symmetric function has infinitely many inverse images under this epimorphism and the image of a base for $\widehat{\Lambda}_{\mathbb{Q}(\alpha)}$ is necessarily a base for $\Lambda_{\mathbb{Q}(\alpha)}$. In particular, the appropriately called family of Non-Symmetric Jack Functions is a base for $\widehat{\Lambda}_{\mathbb{Q}(\alpha)}$ — indexed by compositions instead of partitions — which becomes the family of Jack functions upon symmetrization. Non-symmetric Jack polynomials are denoted by $F_\lambda(\alpha)$ (or simply F_λ) and can be defined in multiple ways.

- Simultaneous eigenfunction polynomials of Cherednik operators;

$$\xi_i := \alpha x_i \frac{\partial}{\partial x_i} + \sum_{k < i} \frac{x_i}{x_i - x_k} (1 - s_{ik}) + \sum_{k > i} \frac{x_k}{x_i - x_k} (1 - s_{ik}) + 1 - i$$

- Family of polynomials orthogonal with respect to the inner product [11];

$$\langle f, g \rangle_\alpha := \text{constant term of } f(x)g(x^{-1}) \prod_{i \neq j} (1 - x_i x_j^{-1})^{\frac{1}{\alpha}}$$

- Sahi-Knop's Combinatorial Formula;
- Sahi-Knop's Recursive Formula.

The latter two of which are described in the following sections.

4.2 Combinatorial Formula

Sahi-Knop's combinatorial formula for non-symmetric Jack functions is very similar to that for Jack functions, with concepts of tableaux admissibility and critical points being just slightly altered.

Definition 4.2.1. Let T be a (generalized) tableau of shape λ (where now $\lambda \in \mathbb{N}^\infty$ instead of \mathcal{P}), that is, a labelling T with numbers $1, 2, 3, \dots$ of the boxes (i, j) in the Ferrers diagram of λ . Then T is *0-admissible* if it would be admissible upon the juxtaposition of a 0-th column with the numbers $1, 2, 3, \dots$ in increasing order.

Example:

| | | | | | | | | | | | | | |
|---|---|---|---|--|--|-------------------------|---|---|---|---|---|--|-----------|
| 1 | 1 | 2 | | | | is 0-admissible because | 1 | 1 | 2 | | | | is admis- |
| 5 | 3 | 5 | 5 | | | | 2 | 5 | 3 | 5 | 5 | | |
| 3 | 4 | 4 | | | | | 3 | 3 | 4 | 4 | | | |
| 4 | 2 | | | | | | 4 | 4 | 2 | | | | |

sible.

However, albeit admissible,

| | | | | | | | |
|---|---|--|-----------------------------|---|---|---|--------------------|
| 2 | | | is not 0-admissible because | 1 | 2 | | is not admissible. |
| 1 | 1 | | | 2 | 1 | 1 | |

The set of all 0-admissible tableaux of shape λ and weight μ (where λ and μ are compositions) is denoted by $\text{Tab}^{0\text{-ad}}(\lambda, \mu)$ and the set of 0-admissible tableaux of shape λ is denoted by $\text{Tab}^{0\text{-ad}}(\lambda)$.

Definition 4.2.2. A point (i, j) is said to be *0-critical* if it is critical for the diagram formed by the juxtaposition of a 0-th column with the numbers $1, 2, 3, \dots$ in increasing order.

Example: 0-critical points are those indicated by a red circle in the diagram because those are the critical points in

| | | | |
|---|---|---|---|
| ① | ① | 2 | |
| 5 | 3 | 5 | ⑤ |
| ③ | 4 | ④ | |
| ④ | 2 | | |

| | | | |
|---|---|---|---|
| ① | ① | 2 | |
| 5 | 3 | 5 | ⑤ |
| ③ | 4 | ④ | |
| ④ | 2 | | |

Now for any given labelling $T \in \text{Tab}^{\text{ad}}(\lambda)$, define the polynomial in α .

Definition 4.2.3.

$$d_T^0(\alpha) := \prod_{s \text{ 0-critical}} \bar{h}_\lambda(s)$$

Then for each composition $\lambda \in \mathbb{N}^\infty$, non-symmetric Jack Functions may be defined by the following combinatorial formula, proven equivalent to former definitions by Sahi and Knop.

Theorem 11. [4, Theorem 5.1]

$$F_\lambda(\alpha) := \sum_{T \in \text{Tab}^{0\text{-ad}}(\lambda)} d_T^0(\alpha) x^{|T|} \tag{53}$$

Non-symmetric Jack Functions will henceforth be denoted without reference to the variable α , as simply F_λ .

Examples:

1. $F_{(2,1)}$

Firstly write all 0-admissible types of tableaux which must be considered⁵, which are 20 for $F_{(2,1)}$. For following tableaux, $i \neq j \neq k \neq i$ ($i, j, k > 2$) and below each tableau is its correspondent $d_T(\alpha)x^{|T|}$ term. Once again, 0-critical points are indicated by a red circle.

⁵For this example, tableaux are of the same “type” if there is a bijection between them which acts as the identity for boxes labelled 1 and 2 (since $\ell(2,1) = 2$).

| | | | | |
|---|---|---|---|---|
| $\begin{array}{ c c } \hline \textcircled{1} & \textcircled{1} \\ \hline \textcircled{2} & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline \textcircled{1} & 2 \\ \hline \textcircled{2} & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline \textcircled{1} & \textcircled{1} \\ \hline i & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline i & 2 \\ \hline \textcircled{2} & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline 2 & \textcircled{2} \\ \hline i & \\ \hline \end{array}$ |
| $2(\alpha + 1)^3 x_1^2 x_2$ | $2(\alpha + 1)^2 x_1 x_2^2$ | $2(\alpha + 1)^2 x_1^2 x_i$ | $(\alpha + 1) x_2^2 x_i$ | $(\alpha + 1) x_2^2 x_i$ |
| $\begin{array}{ c c } \hline \textcircled{1} & i \\ \hline \textcircled{2} & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline 2 & 1 \\ \hline i & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline i & 1 \\ \hline \textcircled{2} & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline \textcircled{1} & 2 \\ \hline i & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline j & i \\ \hline k & \\ \hline \end{array}$ |
| $2(\alpha + 1)^2 x_1 x_2 x_i$ | $x_1 x_2 x_i$ | $(\alpha + 1) x_1 x_2 x_i$ | $2(\alpha + 1) x_1 x_2 x_i$ | $x_i x_j x_k$ |
| $\begin{array}{ c c } \hline i & \textcircled{i} \\ \hline \textcircled{2} & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline 2 & i \\ \hline i & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline \textcircled{1} & i \\ \hline i & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline i & \textcircled{i} \\ \hline j & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline j & i \\ \hline i & \\ \hline \end{array}$ |
| $(\alpha + 1)^2 x_2 x_i^2$ | $x_2 x_i^2$ | $(\alpha + 1) x_1 x_i^2$ | $(\alpha + 1) x_i^2 x_j$ | $x_i^2 x_j$ |
| $\begin{array}{ c c } \hline i & 1 \\ \hline j & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline \textcircled{1} & i \\ \hline j & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline j & i \\ \hline \textcircled{2} & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline i & 2 \\ \hline j & \\ \hline \end{array}$ | $\begin{array}{ c c } \hline 2 & i \\ \hline j & \\ \hline \end{array}$ |
| $x_1 x_i x_j$ | $2(\alpha + 1) x_1 x_i x_j$ | $(\alpha + 1) x_2 x_i x_j$ | $x_2 x_i x_j$ | $x_2 x_i x_j$ |

Summing these terms for all $i \neq j \neq k \neq i$ and $i, j, k > 2$

$$\begin{aligned}
F_{(2,1)} = & 2(\alpha + 1)^3 x_1^2 x_2 + 2(\alpha + 1)^2 x_1 x_2^2 + \left((\alpha^2 + 2\alpha + 2)x_2 + (\alpha + 1)x_1 \right) \left(\sum_{i>2} x_i^2 \right) \\
& + \left(2(\alpha + 1)^2 x_1^2 + (2\alpha + 3)(\alpha + 2)x_1 x_2 + 2(\alpha + 1)x_2^2 \right) \left(\sum_{i>2} x_i \right) \\
& + \left((2\alpha + 3)x_1 + (\alpha + 3)x_2 \right) \left(\sum_{\substack{i \neq j \\ i, j > 2}} x_i x_j \right) + (\alpha + 2) \left(\sum_{\substack{i \neq j \\ i, j > 2}} x_i^2 x_j \right) \\
& + \left(\sum_{\substack{i \neq j \neq k \neq i \\ i, j, k > 2}} x_i x_j x_k \right)
\end{aligned}$$

Now expressing these formal sums in terms of symmetric functions:

$$\begin{aligned}
\sum_{i>2} x_i &= m_{(1)}(x_3, x_4, \dots) = m_1 - x_1 - x_2 \\
\sum_{i>2} x_i^2 &= m_{(2)}(x_3, x_4, \dots) = m_2 - x_1^2 - x_2^2 \\
\sum_{\substack{i \neq j \\ i, j > 2}} x_i x_j &= 2m_{(1^2)}(x_3, x_4, \dots) = 2(m_{(1^2)} - (x_1 + x_2)m_{(1)} + x_1^2 + x_1 x_2 + x_2^2) \\
\sum_{\substack{i \neq j \\ i, j > 2}} x_i^2 x_j &= m_{(2,1)}(x_3, x_4, \dots) \\
&= m_{(2,1)} - (x_1^2 + x_2^2)m_1 - (x_1 + x_2)m_2 + (2x_1^3 + x_1^2 x_2 + x_1 x_2^2 + 2x_2^3) \\
\sum_{\substack{i \neq j \neq k \neq i \\ i, j, k > 2}} x_i x_j x_k &= 6m_{(1^3)}(x_3, x_4, \dots) \\
&= 6(m_{(1^3)} - (x_1 + x_2)m_{(1^2)} + (x_1^2 + x_1 x_2 + x_2^2)m_1 - (x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3))
\end{aligned}$$

So finally

$$\begin{aligned}
F_{(2,1)} &= 2(\alpha + 1)^3 x_1^2 x_2 + 2(\alpha + 1)^2 x_1 x_2^2 + \left((\alpha^2 + 2\alpha + 2)x_2 + (\alpha + 1)x_1 \right) (m_2 - x_1^2 - x_2^2) \\
&\quad + \left(2(\alpha + 1)^2 x_1^2 + (2\alpha + 3)(\alpha + 2)x_1 x_2 + 2(\alpha + 1)x_2^2 \right) (m_1 - x_1 - x_2) \\
&\quad + 2 \left((2\alpha + 3)x_1 + (\alpha + 3)x_2 \right) (m_{(1^2)} - (x_1 + x_2)m_{(1)} + x_1^2 + x_1 x_2 + x_2^2) \\
&\quad + (\alpha + 2) (m_{(2,1)} - (x_1^2 + x_2^2)m_1 - (x_1 + x_2)m_2 + (2x_1^3 + x_1^2 x_2 + x_1 x_2^2 + 2x_2^3)) \\
&\quad + 6 (m_{(1^3)} - (x_1 + x_2)m_{(1^2)} + (x_1^2 + x_1 x_2 + x_2^2)m_1 - (x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3)) \\
&= (\alpha^2) \left((2\alpha + 1)x_1^2 x_2 - x_2^3 - 2x_1^3 \right) + \left((2\alpha + 1)x_1 x_2 + (2\alpha - 1)x_1^2 - x_2^2 \right) (\alpha)m_{(1)} \\
&\quad + (x_1 + (\alpha + 1)x_2)(\alpha)m_{(2)} + (2\alpha)(2x_1 + x_2)m_{(1^2)} + (\alpha + 2)m_{(2,1)} + 6m_{(1^3)}
\end{aligned}$$

2. $F_{(1,2)}$.

Repeat the same procedure. For $F_{(1,2)}$, there are 13 types of tableaux which must be considered

$$\begin{array}{ccccc}
\begin{array}{|c|} \hline \textcircled{1} \\ \hline \end{array} & \begin{array}{|c|c|} \hline i & \\ \hline \end{array} & \begin{array}{|c|c|} \hline i & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline j & \\ \hline \end{array} \\
\begin{array}{|c|c|} \hline j & i \\ \hline \end{array} & \begin{array}{|c|c|} \hline j & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline j & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline j & i \\ \hline \end{array} & \begin{array}{|c|c|} \hline \textcircled{2} & i \\ \hline \end{array} \\
(\alpha + 2)x_1x_ix_j & x_1x_ix_j & x_2x_ix_j & x_2x_ix_j & (2\alpha + 1)x_2x_ix_j \\
\begin{array}{|c|} \hline \textcircled{1} \\ \hline \end{array} & \begin{array}{|c|} \hline \textcircled{1} \\ \hline \end{array} & \begin{array}{|c|} \hline i \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline i \\ \hline \end{array} \\
\begin{array}{|c|c|} \hline \textcircled{2} & i \\ \hline \end{array} & \begin{array}{|c|c|} \hline i & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline \textcircled{2} & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline i & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline \textcircled{2} & \textcircled{2} \\ \hline \end{array} \\
(2\alpha + 1)(\alpha + 2)x_1x_2x_i & (\alpha + 2)x_1x_2x_i & (2\alpha + 1)x_1x_2x_i & x_1x_2x_i & (2\alpha + 1)(\alpha + 1)x_2^2x_i \\
\begin{array}{|c|} \hline \textcircled{1} \\ \hline \end{array} & \begin{array}{|c|} \hline j \\ \hline \end{array} & \begin{array}{|c|} \hline j \\ \hline \end{array} & & \\
\begin{array}{|c|c|} \hline \textcircled{2} & \textcircled{2} \\ \hline \end{array} & \begin{array}{|c|c|} \hline i & \textcircled{i} \\ \hline \end{array} & \begin{array}{|c|c|} \hline k & i \\ \hline \end{array} & & \\
(2\alpha + 1)(\alpha + 2)(\alpha + 1)x_1x_2^2 & (\alpha + 1)x_i^2x_j & x_ix_jx_k & &
\end{array}$$

Summing these terms for all $i \neq j \neq k \neq i$ and $i, j, k > 2$

$$\begin{aligned}
F_{(1,2)} &= (2\alpha + 1)(\alpha + 2)(\alpha + 1)x_1x_2^2 + \left(2(\alpha + 1)(\alpha + 3)x_1x_2 + (2\alpha + 1)(\alpha + 1)x_2^2\right) \left(\sum_{i>2} x_i\right) \\
&+ \left((\alpha + 3)x_1 + (2\alpha + 3)x_2\right) \left(\sum_{\substack{i \neq j \\ i, j > 2}} x_ix_j\right) + (\alpha + 1) \left(\sum_{\substack{i \neq j \\ i, j > 2}} x_i^2x_j\right) \\
&+ \left(\sum_{\substack{i \neq j \neq k \neq i \\ i, j, k > 2}} x_ix_jx_k\right) \\
&= \left((1 - \alpha)x_1^3 + (2\alpha + 1)(\alpha + 1)x_1x_2^2 - (2\alpha + 1)(x_1^2x_2 + x_2^3)\right) (\alpha) \\
&+ \left((2\alpha + 1)(x_1x_2 + x_2^2) - 3x_1^2\right) (\alpha)m_{(1)} + \alpha(\alpha + 2)(x_1)m_{(2)} \\
&+ 2\alpha(x_1 + 2x_2)m_{(1,1)} + 2(x_2 - x_1)m_{(1,1)} + (\alpha + 2)m_{(2,1)} + 6m_{(1,1,1)}
\end{aligned}$$

Observation: Notice that the symmetric parts of both $F_{(2,1)}$ and $F_{(1,2)}$ are exactly the same, and are equal to

$$(\alpha + 2)m_{2,1} + 6m_{(1,1,1)} = J_{(2,1)}$$

This is no coincidence, but rather an example of the general case given by the following theorem.

Theorem 12. [4, Theorem 4.10] If $\lambda \in \mathbb{N}^\infty$,

$$J_{\lambda^+}(x_1, x_2, x_3, \dots) = F_\lambda(\underbrace{0, \dots, 0}_{\ell(\lambda)}, x_1, x_2, x_3, \dots)$$

Sahi-Knop's combinatorial formula for non-symmetric Jack functions together with (45) and (46) can be utilized to calculate non-symmetric Jack functions indexed by a composition with a single row.

Proposition 27.

$$F_{n\varepsilon_k} = J_{(n)} + \sum_{j=1}^n \frac{(n-1)!}{(n-j)!} \alpha^{j-1} (n\alpha x_k^j - p_j(x_1, \dots, x_{k-1})) J_{(n-j)}$$

Proof: Firstly notice that

- if $i < k$, admissible T are not 0-admissible
- if $i = k$, admissible T are 0-admissible and $d_T^0(\alpha) = (n\alpha + 1)d_T(\alpha)$
- if $i > k$, admissible T are 0-admissible and $d_T^0(\alpha) = d_T(\alpha)$

So

$$\begin{aligned} F_{n\varepsilon_k} &= \sum_{i \geq 1} \left(\sum_{\substack{T \in \text{Tab}^{0\text{-ad}}(n\varepsilon_k) \\ T(k,1)=i}} d_T^0(\alpha) x^{|T|} \right) \\ &= (n\alpha + 1) \sum_{\substack{T \in \text{Tab}^{\text{ad}}(n) \\ T(1,1)=k}} d_T(\alpha) x^{|T|} + \sum_{i > k} \left(\sum_{\substack{T \in \text{Tab}^{\text{ad}}(n) \\ T(1,1)=i}} d_T(\alpha) x^{|T|} \right) \\ &= J_{(n)} + n\alpha \sum_{j=1}^n \frac{(n-1)!}{(n-j)!} \alpha^{j-1} x_k^j J_{(n-j)} - \sum_{i < k} \sum_{j=1}^n \frac{(n-1)!}{(n-j)!} \alpha^{j-1} x_i^j J_{(n-j)} \\ &= J_{(n)} + \sum_{j=1}^n \frac{(n-1)!}{(n-j)!} \alpha^{j-1} (n\alpha x_k^j - p_j(x_1, \dots, x_{k-1})) J_{(n-j)} \end{aligned}$$

□

4.3 Recursion Formula

Let $S_{\mathbb{N}}$ be the group of bijections $\mathbb{N} \rightarrow \mathbb{N}$ which change finitely many elements in \mathbb{N} , that is, for each $\phi \in S_{\mathbb{N}}$, the set $\{i \in \mathbb{N} : \phi(i) \neq i\}$ is finite. For each $n \in \mathbb{N}$, denote $\sigma_n = (1 \ 2 \ 3 \ \cdots \ n) \in S_{\mathbb{N}}$ and consider the natural action of $S_{\mathbb{N}}$ over $\widehat{\Lambda}_{\mathbb{Q}(\alpha)}$

$$\sigma_n f(x_1, x_2, \dots, x_n, x_{n+1}, \dots) = f(x_n, x_1, \dots, x_{n-1}, x_{n+1}, \dots)$$

Recall that $\bar{h}_\lambda(s) = (a_\lambda(s)+1)\alpha + (\ell_\lambda(s)+1)$ and $\lambda^* = (\lambda_L - 1, \lambda_1, \dots, \lambda_{L-1}, 0, \dots)$ and define the $\widehat{\Lambda}_{\mathbb{Q}(\alpha)} \rightarrow \widehat{\Lambda}_{\mathbb{Q}(\alpha)}$ operator

$$Y_\lambda := \bar{h}_\lambda(L, 1)x_L\sigma_L + \sum_{i>L} x_i\sigma_i$$

Then Sahi-Knop recursion formula states that, for each $\lambda \in \mathbb{N}^\infty$,

Theorem 13. [4, Theorem 5.1]

$$F_\lambda = Y_\lambda (F_{\lambda^*}) \tag{54}$$

Now consider the finite sequence $\text{seq}(\lambda) = (\lambda, \lambda^*, \lambda^{**}, \dots, \lambda^{*\dots*})$, where $|\lambda^{*\dots*}| = 1$. Since $F_{(0)} = 1$, the formula provided by the theorem can be restated as

$$F_\lambda = Y_\lambda \circ Y_{\lambda^*} \circ Y_{\lambda^{**}} \circ \cdots \circ Y_{\lambda^{*\dots*}} (1).$$

The recursion formula can be stated as an alternative definition of non-symmetric Jack functions and also used as means to prove results concerning them, like the following proposition.

Proposition 28.

$$F_{(0^M, 1^N)} = \sum_{k=0}^N k! \left(\prod_{j=M+k+1}^{M+N} (\alpha + j) \right) e_{N-k}(x_{M+1}, \dots, x_{M+N}) e_k(x_{M+N+1}, \dots) \tag{55}$$

Proof: This is proven by induction on N , while fixating $\ell(0^M, 1^N) = M + N = L$.

For $N = 1$,

$$\begin{aligned}
F_{(0^M, 1)} &= \bar{h}_{(0^M, 1)} x_{M+1} \sigma_{M+1}(1) + \sum_{i>M+1} x_i \sigma_i(1) \\
&= (\alpha + M + 1) x_{M+1} + \sum_{i>L} x_i \\
&= 0! (\alpha + M + 1) e_1(x_{M+1}) e_0(x_{M+2}, \dots) + 1! e_0(x_{M+1}) e_1(x_{M+2}, \dots)
\end{aligned}$$

and the formula holds. Suppose the result is valid for N ,

$$\begin{aligned}
F_{(0^{M-1}, 1^{N+1})} &= h_{(0^{M-1}, 1^{N+1})}(L, 1) x_L \sigma_L(F_{(0^M, 1^N)}) + \sum_{i>L} x_i \sigma_i(F_{(0^M, 1^N)}) \\
&= (\alpha + M) \sum_{k=0}^N \left(\prod_{j=M+k+1}^L (\alpha + j) \right) k! x_L e_{N-k}(x_M, \dots, x_{L-1}) e_k(x_{L+1}, \dots) + \\
&\quad \sum_{i>L} \sum_{k=0}^N \left(\prod_{j=M+k+1}^L (\alpha + j) \right) k! x_i e_{N-k}(x_M, \dots, x_{L-1}) e_k^{(i)}(x_L, \dots)
\end{aligned}$$

where $e_k^{(i)}(x_q, x_{q+1}, \dots) = e_k(x_q, x_{q+1}, \dots, x_{i-1}, x_{i+1}, \dots)$.

Now since $e_k(x_L, x_{L+1}, \dots) = x_L e_{k-1}(x_{L+1}, \dots) + e_k(x_{L+1}, \dots)$ and

$\sum_{i>L} x_i e_q^{(i)}(x_{L+1}, \dots) = (q+1) e_{q+1}(x_{L+1}, \dots)$, $F_{(0^{M-1}, 1^{N+1})}$ is equal to

$$\begin{aligned}
& (\alpha + M) \sum_{k=0}^N \left(\prod_{j=M+k+1}^L (\alpha + j) \right) k! x_L e_{N-k}(x_M, \dots, x_{L-1}) e_k(x_{L+1}, \dots) + \\
& \sum_{k=0}^N \left(\prod_{j=M+k+1}^L (\alpha + j) \right) k! e_{N-k}(x_M, \dots, x_{L-1}) \left(k x_L e_k(x_{L+1}, \dots) + (k+1) e_{k+1}(x_{L+1}, \dots) \right) \\
= & \sum_{k=0}^N \left(\prod_{j=M+k+1}^L (\alpha + j) \right) (\alpha + M + k) k! x_L e_{N-k}(x_M, \dots, x_{L-1}) e_k(x_{L+1}, \dots) + \\
& \sum_{k=0}^N \left(\prod_{j=M+k+1}^L (\alpha + j) \right) (k+1)! e_{N-k}(x_M, \dots, x_{L-1}) e_{k+1}(x_{L+1}, \dots) \\
= & \left(\prod_{j=M}^L (\alpha + j) \right) x_L e_N(x_M, \dots, x_{L-1}) + (N+1)! e_{N+1}(x_{L+1}, \dots) + \\
& \sum_{k=1}^N \left(\prod_{j=M+k}^L (\alpha + j) \right) k! \left(e_{N-k+1}(x_M, \dots, x_{L-1}) + x_L e_{N-k}(x_M, \dots, x_{L-1}) \right) e_k(x_{L+1}, \dots) + \\
= & \sum_{k=0}^{N+1} \left(\prod_{j=M+k}^L (\alpha + j) \right) k! e_{N+1-k}(x_M, \dots, x_L) e_k(x_{L+1}, \dots)
\end{aligned}$$

□

Examples:

1. $F_{(1,0,2)}$

The recursive formula yields $F_{(1,0,2)} = Y_{(1,0,2)}(F_{(1,1)})$, but $F_{(1,1)}$ is given by (55)

$$F_{(1,1)} = (a+1)(a+2)x_1x_2 + (a+2)(x_1+x_2)m_{(1)}(x_3, x_4, \dots) + 2m_{(1,1)}(x_3, x_4, \dots)$$

so

$$\begin{aligned} F_{(1,0,2)} &= \bar{h}_{(1,0,2)}(3, 1)x_3\sigma_3(F_{(1,1)}) + \sum_{i>3} x_i\sigma_i(F_{(1,1)}) \\ &= (2\alpha + 3)x_3 \left((\alpha + 1)(\alpha + 2)x_3x_1 + (\alpha + 2)(x_3 + x_1)(x_2 + m_{(1)}(x_4, x_5, \dots)) \right. \\ &\quad \left. + 2(x_2m_{(1)}(x_4, x_5, \dots) + m_{(1,1)}(x_4, x_5, \dots)) \right) \\ &\quad + \sum_{i>3} x_i \left((\alpha + 1)(\alpha + 2)x_ix_1 + (\alpha + 2)(x_i + x_1)(x_2 + m_{(1)}(x_4, x_5, \dots)) \right. \\ &\quad \left. + 2(x_2m_{(1)}(x_4, x_5, \dots) + m_{(1,1)}(x_4, x_5, \dots)) \right) \\ &= (\alpha + 2)(2\alpha + 3)(x_1x_2x_3 + (\alpha + 1)x_1x_3^2 + x_2x_3^2) + \\ &\quad (\alpha + 2)(x_1x_2 + 2(\alpha + 2)x_1x_3 + 4x_2x_3 + (2\alpha + 3)x_3^2)m_{(1)}(x_4, x_5, \dots) + \\ &\quad (\alpha + 2)((\alpha + 1)x_1 + x_2 + x_3)m_{(2)}(x_4, x_5, \dots) + \\ &\quad 2((\alpha + 2)x_1 + 2x_2 + (2\alpha + 5)x_3)m_{(1^2)}(x_4, x_5, \dots) + \\ &\quad (\alpha + 2)m_{(2,1)}(x_4, x_5, \dots) + 6m_{(1^3)}(x_4, x_5, \dots) \end{aligned}$$

4.4 Pieri Rule

Upon symmetrization, many results concerning non-symmetric Jack functions extend to symmetric Jack functions. For example, Jack functions J_λ can be calculated by applying the recursion formula for a non-symmetric F_λ and symmetrizing the resulting function. Furthermore, if a Pieri or Littlewood-Richardson-Robinson type rule were known for non-symmetric Jack functions, it would be similarly extendable for symmetric functions. The extra information carried by non-symmetric Jack functions might provide useful strategies

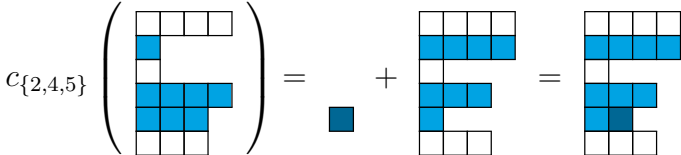
for proving results regarding “regular” Jack functions. The first step towards a Pieri rule for the non-symmetric case is due to Waldeck Schützer, but before it can be stated, some definitions must be provided.

Definition 4.4.1. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ and $L = \{L_1, L_2, \dots, L_J\}$ a non-empty subset of $\{1, 2, \dots, \ell\}$ with $L_1 < L_2 < \dots < L_J$. Furthermore, for $\sigma \in S_{\mathbb{N}}$, $\lambda \in \mathbb{N}^\infty$, let $\sigma \cdot \lambda$ be the action which permutes λ entries. Then

$$c_L(\lambda) := \varepsilon_{L_J} + (L_1 \ L_2 \ \dots \ L_J)^{-1} \cdot \lambda$$

Example: $c_{\{2,4,5\}}(4, 1, 1, 4, 3, 3)$

For illustration purposes, the operation will be performed employing diagrams. Below, cyan rows (■) are those whose indices are in L , and ε_k 's square is coloured blue (■).



so the resulting composition is $(4, 4, 1, 3, 2, 3)$.

Definition 4.4.2. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$. A subset $L = \{L_1, L_2, \dots, L_J\}$ of $\{1, 2, \dots, \ell\}$ is *maximal with respect to* λ if

- $\lambda_{L_1} \neq \lambda_i$ for $i < L_1$;
- $\lambda_{L_j} \neq \lambda_i$ for $L_{j-1} < i < L_j$;
- $\lambda_{L_1} \neq \lambda_i - 1$ for $L_J < i$.

Example: Let $\lambda = (3^{10}, 0^{10}, 1^{10}, 3^{10}, 4^{10}, 2^{10})$. Any subset of $\{1, 11, 21, 31, 41, 51\}$ which contains 51 and either 11 or 21 is maximal in λ , however

- $\{1, 31, 41, 51\}$ is not because $\lambda_{31} = \begin{cases} \lambda_2 \\ \vdots \\ \lambda_{10} \end{cases}$

- $\{1, 11, 21, 41\}$ is not because $\lambda_1 = \begin{cases} \lambda_{42} - 1 \\ \vdots \\ \lambda_{50} - 1 \end{cases}$

Proposition 29. [13, Proposition 3.1.6] Let L be maximal with respect to λ and $c_L(\lambda) = c_M(\lambda)$. Then $M \subseteq L$.

Corollary: Given η and λ , $c_L(\lambda) = \eta$ determines maximal L uniquely.

Let $g_{k\eta}^\lambda$ denote the coefficients of F_η in the linear decomposition of $F_{\varepsilon_k} F_\lambda$:

$$F_{\varepsilon_k} F_\lambda = \sum_{\eta} g_{k\eta}^\lambda F_\eta$$

Coefficients $g_{k\eta}^\lambda$'s which appear in this decomposition are entirely determined by the theorem:

Theorem 14. [13, Proposition 3.1.6] Denote $L = \{L_1, L_2, \dots, L_\ell\} = \{i : \eta_i \neq \lambda_i\}$. Coefficients $g_{k\eta}^\lambda$ are nonzero if, and only if, all of the following conditions are satisfied

1. $\lambda = c_L(\eta)$;
2. $\exists i \geq k$ such that either $\eta_i \neq \lambda_i$ or $\eta_i = \eta_{L_1} + 1$;
3. If $\eta_{L_1} = \eta_1 = \eta_2 = \dots = \eta_k$, then $L_1 \leq k$.

Examples:

1. $\eta = (a^m)$

For any non-empty subset $L = \{L_1, \dots, L_J\} \subseteq \{1, \dots, m\}$, $c_L(a^m) = (a^m) + \varepsilon_{L_J}$

- By condition 1 of the theorem, $L = \{L_J\}$;
- By condition 2 of the theorem, $L_J \geq k$;
- By condition 3 of the theorem, $L_J \leq k$.

therefore for $\eta = (a^m)$, $\lambda = (a^m) + \varepsilon_k$ is the only case of non-zero $g_{k\eta}^\lambda$.

2. $\eta = (a^m, b^n)$ and $k \leq m$

For this case, and a non-empty subset $L \subseteq \{1, \dots, m+n\}$, there are 3 possibilities

(a) $L \subseteq \{1, 2, \dots, m\}$;

And by the same argument of the previous example, if $L \subseteq \{1, \dots, m\}$, $L = \{k\}$ and $\lambda = (a^m, b^n) + \varepsilon_k$.

(b) $L \subseteq \{m+1, m+2, \dots, m+n\}$;

Condition 3 of the theorem no longer applies and by a partial version of the same argument of the previous example, if $L \subseteq \{m+1, \dots, m+n\}$, $L = \{i\}$ for $i > m$ and $\lambda = (a^m, b^n) + \varepsilon_i$

(c) $L \cap \{1, 2, \dots, m\} \neq \emptyset \neq L \cap \{m+1, m+2, \dots, m+n\}$.

- By condition 1 of the theorem, $\lambda = (a^{i-1}, b, a^{m-i}, b^{j-1}, a, b^{n-j}) + \varepsilon_j$ for some $i \in \{1, \dots, m\}, j \in \{m+1, \dots, m+n\}$.
- Condition 2 is already satisfied
- By condition 3 of the theorem, $i \leq k$.

And so the set of compositions λ for which $g_{k\eta}^\lambda$ is non-zero is

$$\{\eta + \varepsilon_k\} \cup \{\eta + \varepsilon_i : i > m\} \cup \{(i \ j) \cdot \eta + \varepsilon_j : i \leq k \text{ and } j > m\}$$

3. $\eta = (a^m, b^n)$ and $k > m$

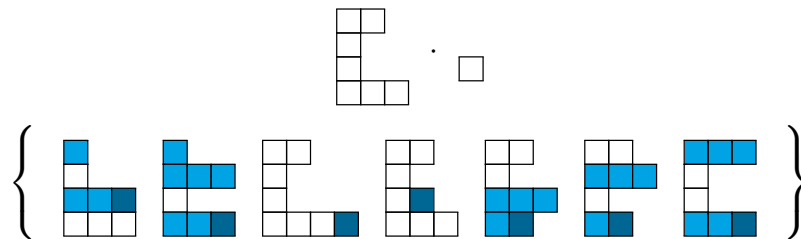
This is very similar to the previous example, except that now $L \not\subseteq \{1, \dots, m\}$.

For this case, the set of compositions λ for which $g_{k\eta}^\lambda$ is non-zero is

$$\{\eta + \varepsilon_i : i \geq m\} \cup \{(i \ j) \cdot \eta + \varepsilon_j : k \leq j \text{ and } j > m\}$$

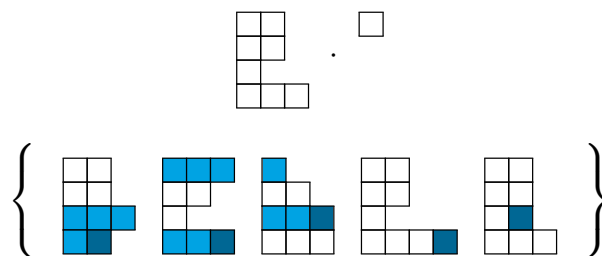
4. $\eta = (2, 1, 1, 3)$ and $k = 3$.

For this and the next example, a more visual approach is taken. The following conjugations are those λ for which $g_{k\eta}^\lambda \neq 0$. Cyan rows denote those indexed by elements in L and dark blue square denotes where a square was added in $c_L(\eta)$.



5. $\eta = (2, 2, 1, 3)$ and $k = 1$.

The set of compositions λ for which $g_{k\eta}^\lambda$ is non-zero is



In order to fully characterize these coefficients $g_{k\eta}^\lambda$, some more definitions must be laid out. Let $L = \{L_1, L_2, \dots, L_J\} = \{i : \eta_i \neq \lambda_i\}$ maximal with respect to η and let $1 \leq k \leq \ell(\eta)$.

Definition 4.4.3.

$$h_\eta^*(i, j) := \begin{cases} \hat{h}_\eta(i, j) & \text{if } i \notin L \\ 1 & \text{if } (i, j) = (L_p, \eta_{L_{p+1}} + 1) \text{ for } p < J \\ 1 & \text{if } (i, j) = (L_J, \eta_{L_1} + 2) \\ \bar{h}_\eta(i, j) & \text{otherwise} \end{cases}$$

$$h_*^\lambda(i, j) := \begin{cases} \bar{h}_\lambda(i, j) & \text{if } i \notin L \\ -1 & \text{if } (i, j) = (L_p, \lambda_{L_{p-1}} + 1) \text{ for } p > 1 \\ -1 & \text{if } (i, j) = (L_1, \eta_{L_J}) \\ \hat{h}_\lambda(i, j) & \text{otherwise} \end{cases}$$

$$h_\eta^{*(k)}(i, j) := \begin{cases} \hat{h}_\eta(i, j) & \text{if } i \notin L \text{ or } i = k \in L \\ 1 & \text{if } (i, j) = (L_p, \eta_{L_{p+1}} + 1) \text{ for } p < J \\ & \text{such that } k \neq L_p \\ 1 & \text{if } (i, j) = (L_J, \eta_{L_1} + 2) \text{ and } k \neq L_J \\ \bar{h}_\eta(i, j) & \text{otherwise} \end{cases}$$

$$h_{*(k)}^\lambda(i, j) := \begin{cases} \bar{h}_\lambda(i, j) & \text{if } i \notin L \text{ or } i = k \in L \\ -1 & \text{if } (i, j) = (L_p, \lambda_{L_{p-1}} + 1) \text{ for } p > 1 \\ & \text{such that } k \neq L_p \\ -1 & \text{if } (i, j) = (L_1, \eta_{L_J}) \text{ and } k \neq L_J \\ \hat{h}_\lambda(i, j) & \text{otherwise} \end{cases}$$

$$b_{\eta\lambda}(\alpha) := \frac{\left(\prod_{s \in \eta} h_\eta^*(s) \right) \left(\prod_{s \in \lambda} h_*^\lambda(s) \right)}{\left(\prod_{s \in \lambda} \check{h}_\lambda(s) \hat{h}_\lambda(s) \right)} \quad b_{\eta\lambda}^{(k)}(\alpha) := \frac{\left(\prod_{s \in \eta} h_\eta^{*(k)}(s) \right) \left(\prod_{s \in \lambda} h_{*(k)}^\lambda(s) \right)}{\left(\prod_{s \in \lambda} \check{h}_\lambda(s) \hat{h}_\lambda(s) \right)}$$

And with that, Waldeck's theorem can finally be stated.

Theorem 15. [13, Theorem 3.3.5]

Let $F_{\varepsilon_k} F_\eta = \sum_\lambda g_{k\eta}^\lambda F_\eta$. If $g_{k\eta}^\lambda \neq 0$,

$$g_{k\eta}^\lambda = \begin{cases} (c_\lambda(k, \lambda_k) - c_\lambda(L_J, \lambda_{L_J}))b_{\eta\lambda} + (\alpha + k)b_{\eta\lambda}^{(k)} & \text{if } k \in L \\ (c_\lambda(L_p, \lambda_{L_p}) - c_\lambda(L_J, \lambda_{L_J}))b_{\eta\lambda} & \text{if } k \notin L, k < L_J \text{ and} \\ & L_p < k < L_{p+1} \text{ for some } p \\ -\alpha b_{\eta\lambda} & \text{if } k < L_1 \end{cases} \quad (56)$$

While these definitions may look cumbersome, they have a very nice pictorial correspondence, which will be explored in the following examples.

Examples:

1. $\eta = (2, 1, 1, 3)$, $k = 3$ and $\lambda = (2, 3, 1, 2)$

The set of λ for which $g_{3(2,1,1,3)}^\lambda \neq 0$ has already been described to be $\{(1, 1, 3, 3); (1, 3, 1, 3); (2, 1, 1, 4); (2, 1, 2, 4); (2, 1, 3, 2); (2, 3, 1, 2); (3, 1, 1, 3)\}$

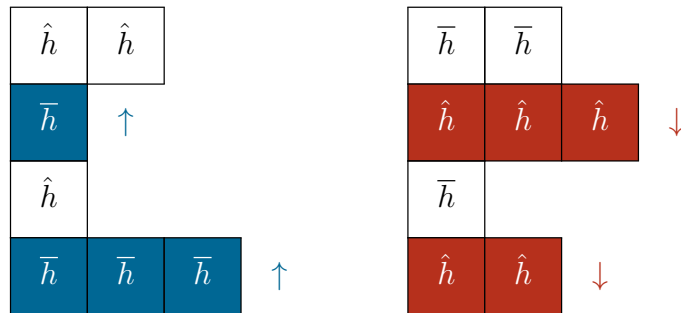
For $\lambda = (2, 3, 1, 2)$, $L = \{2, 4\}$ and it is already maximal with respect to η . Since $L_1 < k = 3 < L_2$, the second case listed in the Theorem must be applied.

$$\begin{aligned} g_{k\eta}^\lambda &= (c_\lambda(L_p, \lambda_{L_p}) - c_\lambda(L_J, \lambda_{L_J}))b_{\eta\lambda} \\ &= (c_{(2,3,1,3)}(2, 3) - c_{(2,1,1,3)}(4, 3)) \frac{\left(\prod_{s \in (2,1,1,3)} h_{(2,1,1,3)}^*(s) \right) \left(\prod_{s \in (2,3,1,3)} h_{(2,3,1,3)}^*(s) \right)}{\left(\prod_{s \in (2,3,1,2)} \check{h}_{(2,3,1,2)}(s) \hat{h}_{(2,3,1,3)}(s) \right)} \end{aligned}$$

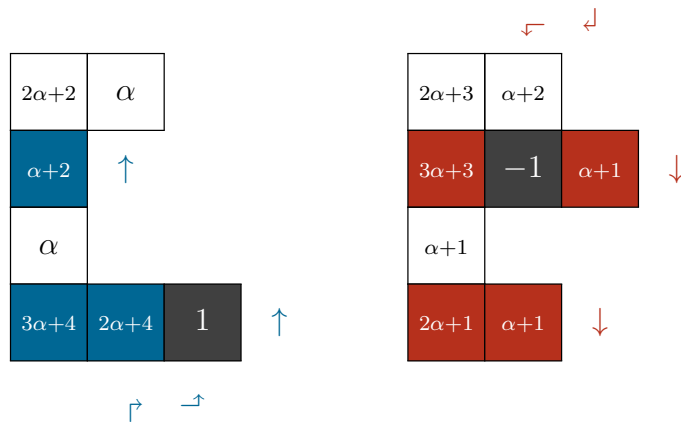
$$(c_{(2,3,1,2)}(2, 3) - c_{(2,3,1,2)}(4, 2)) = (3\alpha - 0) - (2\alpha - 2) = (\alpha + 2)$$

The next step is to calculate $\prod_{s \in \eta} h_\eta^*(s)$ and $\prod_{s \in \lambda} h_\lambda^*(s)$. And for this, we introduce what's been coined a "jeu de flèches".

Draw the diagrams of η and λ and fill their L rows — colour-coded blue (■) for η and red (■) for λ — with their respective correspondent \bar{h} and \hat{h} values, and their remaining rows with their respective correspondent \bar{h} and \hat{h} values, like so



The *jeu de flèches* functions as follows. Project each arrow to the next L row (in the direction it points to), moving arrows in η (\uparrow) one column to the right as they cycle down and arrows in λ (\downarrow) one column to the left as they cycle up. The content of whichever square in L is “hit” by an arrow is replaced by 1 in η and by -1 in λ .



Finally, the coefficient $g_{k\eta}^\lambda$ can be computed

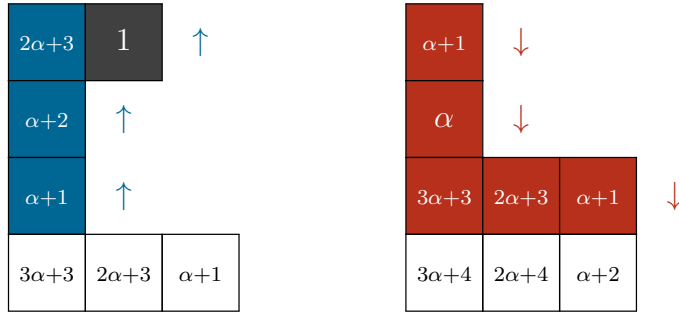
$$g_{k\eta}^\lambda = (\alpha+2) \frac{\left(\begin{array}{|c|c|} \hline 2\alpha+2 & \alpha \\ \hline \alpha+2 & \\ \hline \alpha & \\ \hline 3\alpha+4 & 2\alpha+4 & 1 \\ \hline \end{array} \right) \left(\begin{array}{|c|c|c|} \hline 2\alpha+3 & \alpha+2 & \\ \hline 3\alpha+3 & -1 & \alpha+1 \\ \hline \alpha+1 & & \\ \hline 2\alpha+1 & \alpha+1 & \\ \hline \end{array} \right)}{\left(\begin{array}{|c|c|c|} \hline \alpha+3 & 2 & \\ \hline 2\alpha+4 & \alpha+3 & 2 \\ \hline 1 & & \\ \hline \alpha+2 & 2 & \\ \hline \end{array} \right) \left(\begin{array}{|c|c|c|} \hline 2\alpha+2 & \alpha+1 & \\ \hline 3\alpha+3 & 2\alpha+2 & \alpha+1 \\ \hline \alpha & & \\ \hline 2\alpha+1 & \alpha+1 & \\ \hline \end{array} \right)} = \frac{\alpha(\alpha+2)^2(2\alpha+3)(3\alpha+4)}{16(\alpha+1)(\alpha+3)^2}$$

2. $\eta = (2, 1, 1, 3)$, $k = 3$ and $\lambda = (1, 1, 3, 3)$

For $\lambda = (1, 1, 3, 3)$, $L = \{1, 3\}$, but the maximal L with respect to η is $\{1, 2, 3\}$. Since $k = 3 \in L$, the first case listed in the theorem must be applied. Moreover, since $k = 3 = L_J$, $(c_\lambda(k, \lambda_k) - c_\lambda(L_J, \lambda_{L_J})) = 0$, and so

$$g_{k\eta}^\lambda = (\alpha+k)b_{\eta\lambda}^{(k)} = (\alpha+3) \frac{\left(\prod_{s \in (2, 1^2, 3)} h_{(2, 1^2, 3)}^{*(k)}(s) \right) \left(\prod_{s \in (1^2, 3^2)} h_{*(k)}^{(1^2, 3^2)}(s) \right)}{\left(\prod_{s \in (1^2, 3^2)} \hat{h}_{(1^2, 3^2)}(s) \hat{h}_{(1^2, 3^2)}(s) \right)}$$

Repeat the *jeu de flèches*, but with the following difference with respect to the previous case. Since $k = e \in L$, the third row must be “shielded” from any arrows for the evaluation of $b_{\eta\lambda}^{(k)}$.

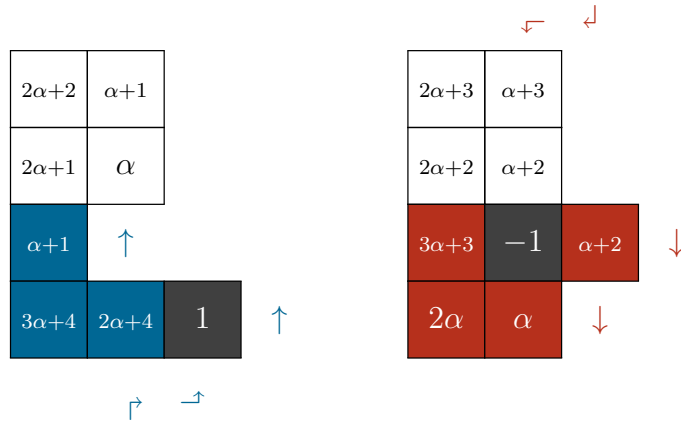


So the coefficient is

$$g_{k\eta}^\lambda = (\alpha+3) \frac{\left(\begin{array}{|c|c|} \hline 2\alpha+3 & 1 \\ \hline \alpha+2 & \\ \hline \alpha+1 & \\ \hline 3\alpha+3 & 2\alpha+3 & \alpha+1 \\ \hline \end{array} \right) \left(\begin{array}{|c|c|c|} \hline \alpha+1 & & \\ \hline \alpha & & \\ \hline 3\alpha+3 & 2\alpha+3 & \alpha+1 \\ \hline 3\alpha+4 & 2\alpha+4 & \alpha+2 \\ \hline \end{array} \right)}{\left(\begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & & \\ \hline 2\alpha+4 & \alpha+4 & 2 \\ \hline 2\alpha+3 & \alpha+3 & 1 \\ \hline \end{array} \right) \left(\begin{array}{|c|c|c|} \hline \alpha+1 & & \\ \hline \alpha & & \\ \hline 3\alpha+3 & 2\alpha+3 & \alpha+1 \\ \hline 3\alpha+2 & 2\alpha+2 & \alpha \\ \hline \end{array} \right)} = \frac{3(2\alpha+3)(\alpha+1)^2(\alpha+2)^2}{8(3\alpha+2)(\alpha+4)\alpha}$$

3. $\eta = (2, 2, 1, 3)$, $k = 1$ and $\lambda = (2, 2, 3, 2)$

$L = \{3, 4\}$ is already maximal with respect to η . Since $k = 1 < L_1 = 3$, the last listed case of the theorem must be applied: $g_{k\eta}^\lambda = -\alpha b_{\eta\lambda}$.



$$g_{k\eta}^\lambda = -\alpha \frac{\left(\begin{array}{|c|c|} \hline 2\alpha+2 & \alpha+1 \\ \hline 2\alpha+1 & \alpha \\ \hline \alpha+1 & \\ \hline 3\alpha+4 & 2\alpha+4 & 1 \\ \hline \end{array} \right) \left(\begin{array}{|c|c|c|} \hline 2\alpha+3 & \alpha+3 & \\ \hline 2\alpha+2 & \alpha+2 & \\ \hline 3\alpha+3 & -1 & \alpha+2 \\ \hline 2\alpha & \alpha & \\ \hline \end{array} \right)}{\left(\begin{array}{|c|c|} \hline \alpha+3 & 3 \\ \hline \alpha+2 & 2 \\ \hline 2\alpha+4 & \alpha+4 & 3 \\ \hline \alpha+1 & 1 \\ \hline \end{array} \right) \left(\begin{array}{|c|c|c|} \hline 2\alpha+2 & \alpha+2 & \\ \hline 2\alpha+1 & \alpha+1 & \\ \hline 3\alpha+3 & 2\alpha+3 & \alpha+2 \\ \hline 2\alpha & \alpha & \\ \hline \end{array} \right)} = \frac{\alpha^2(\alpha+1)(3\alpha+4)}{9(\alpha+4)(\alpha+2)}$$

4.5 Future Pursuits

In spite of how well-established the theory of symmetric functions currently is, there are plenty of open problems and conjectures yet to be resolved. Most notably for the scope of this text, the Littlewood-Richardson-Robinson analogue for symmetric and non-symmetric Jack functions.

Besides its own intrinsic interest, the study of non-symmetric functions is a promising technique for working out problems concerning symmetric functions. The recursive formula given by Sahi and Knop is an indication of their potential. Two natural directions in which the current knowledge of non-symmetric functions could be furthered are the generalizations of Schültzer's rule for the cases

(a) $F_\lambda F_{n\varepsilon_k}$

(b) $F_\lambda F_\mu$ where μ is a single-column composition.

Upon symmetrization, both of these cases reduce to the two cases of the Pieri rule for symmetric Jack functions. We believe these are worthwhile directions to be pursued.

A Transition Matrices

Expressing the transition matrix $M(u, v)$ which effects the base change between v and u is tantamount to expressing v in terms of u . In this appendix complete; elementary; forgotten; power; Schur and Jack functions are expressed in terms of monomial functions through transition matrices for ⁶ $|\lambda| \leq 6$.

In order to express a transition matrix, a total ordering of partitions in \mathcal{P}_n must be specified, which was for this case the reverse lexicographical ordering (\mathcal{L}), meaning

$$\mathcal{P}_2: (2) >_{\mathcal{L}} (1^2)$$

$$\mathcal{P}_3: (3) >_{\mathcal{L}} (2, 1) >_{\mathcal{L}} (1^3)$$

$$\mathcal{P}_4: (4) >_{\mathcal{L}} (3, 1) >_{\mathcal{L}} (2^2) >_{\mathcal{L}} (2, 1^2) >_{\mathcal{L}} (1^4)$$

$$\mathcal{P}_5: (5) >_{\mathcal{L}} (4, 1) >_{\mathcal{L}} (3, 2) >_{\mathcal{L}} (3, 1^2) >_{\mathcal{L}} (2^2, 1) >_{\mathcal{L}} (2, 1^3) >_{\mathcal{L}} (1^5)$$

$$\mathcal{P}_6: (6) >_{\mathcal{L}} (5, 1) >_{\mathcal{L}} (4, 2) >_{\mathcal{L}} (4, 1^2) >_{\mathcal{L}} (3^2) >_{\mathcal{L}} (3, 2, 1) >_{\mathcal{L}} (3, 1^3) \\ >_{\mathcal{L}} (2^3) >_{\mathcal{L}} (2^2, 1^2) >_{\mathcal{L}} (2, 1^4) >_{\mathcal{L}} (1^6)$$

$$\text{So for instance, } M_4(s, m) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ means that}$$

$$\begin{bmatrix} s_{(4)} \\ s_{(3,1)} \\ s_{(2^2)} \\ s_{(2,1^2)} \\ s_{(1^4)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{(4)} \\ m_{(3,1)} \\ m_{(2^2)} \\ m_{(2,1^2)} \\ m_{(1^4)} \end{bmatrix}$$

⁶Case $|\lambda| = 1$ is trivial: $h_1 = e_1 = f_1 = p_1 = s_1 = J_1$

A.1 Complete Functions

$$\begin{aligned}
 M_2(h, m) &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\
 M_3(h, m) &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \\
 M_4(h, m) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 6 \\ 1 & 3 & 4 & 7 & 12 \\ 1 & 4 & 6 & 12 & 24 \end{bmatrix} \\
 M_5(h, m) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 7 & 10 \\ 1 & 3 & 4 & 7 & 8 & 13 & 20 \\ 1 & 3 & 5 & 8 & 11 & 18 & 30 \\ 1 & 4 & 7 & 13 & 18 & 33 & 60 \\ 1 & 5 & 10 & 20 & 30 & 60 & 120 \end{bmatrix} \\
 M_6(h, m) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 2 & 3 & 4 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 3 & 5 & 7 & 6 & 8 & 11 & 15 \\ 1 & 3 & 4 & 7 & 4 & 8 & 13 & 9 & 14 & 21 & 30 \\ 1 & 2 & 3 & 4 & 4 & 6 & 8 & 7 & 10 & 14 & 20 \\ 1 & 3 & 5 & 8 & 6 & 12 & 19 & 15 & 24 & 38 & 60 \\ 1 & 4 & 7 & 13 & 8 & 19 & 34 & 24 & 42 & 72 & 120 \\ 1 & 3 & 6 & 9 & 7 & 15 & 24 & 21 & 33 & 54 & 90 \\ 1 & 4 & 8 & 14 & 10 & 24 & 42 & 33 & 58 & 102 & 180 \\ 1 & 5 & 11 & 21 & 14 & 38 & 72 & 54 & 102 & 192 & 360 \\ 1 & 6 & 15 & 30 & 20 & 60 & 120 & 90 & 180 & 360 & 720 \end{bmatrix}
 \end{aligned}$$

A.2 Elementary Functions

$$M_2(e, m) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$M_3(e, m) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

$$M_4(e, m) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 1 & 2 & 5 & 12 \\ 1 & 4 & 6 & 12 & 24 \end{bmatrix}$$

$$M_5(e, m) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 3 & 10 \\ 0 & 0 & 0 & 1 & 2 & 7 & 20 \\ 0 & 0 & 1 & 2 & 5 & 12 & 30 \\ 0 & 1 & 3 & 7 & 12 & 27 & 60 \\ 1 & 5 & 10 & 20 & 30 & 60 & 120 \end{bmatrix}$$

$$M_6(e, m) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 9 & 30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 6 & 20 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 8 & 22 & 60 \\ 0 & 0 & 0 & 1 & 0 & 3 & 10 & 6 & 18 & 48 & 120 \\ 0 & 0 & 0 & 0 & 1 & 3 & 6 & 6 & 15 & 36 & 90 \\ 0 & 0 & 1 & 2 & 2 & 8 & 18 & 15 & 34 & 78 & 180 \\ 0 & 1 & 4 & 9 & 6 & 22 & 48 & 36 & 78 & 168 & 360 \\ 1 & 6 & 15 & 30 & 20 & 60 & 120 & 90 & 180 & 360 & 720 \end{bmatrix}$$

A.3 Forgotten Functions

$$\begin{aligned}
 M_2(f, m) &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \\
 M_3(f, m) &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
 M_4(f, m) &= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -3 & -2 & -2 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\
 M_5(f, m) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 & 0 \\ -4 & -3 & -3 & -2 & -2 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\
 M_6(f, m) &= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -2 & -2 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 6 & 3 & 4 & 1 & 4 & 2 & 0 & 3 & 1 & 0 & 0 \\ -5 & -4 & -4 & -3 & -4 & -3 & -2 & -3 & -2 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
 \end{aligned}$$

A.4 Power Functions

$$M_2(p, m) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$M_3(p, m) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 6 \end{bmatrix}$$

$$M_4(p, m) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 \\ 1 & 4 & 6 & 12 & 24 \end{bmatrix}$$

$$M_5(p, m) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 2 & 0 & 0 \\ 1 & 3 & 4 & 6 & 6 & 6 & 0 \\ 1 & 5 & 10 & 20 & 30 & 60 & 120 \end{bmatrix}$$

$$M_6(p, m) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 6 & 2 & 3 & 6 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 2 & 4 & 4 & 0 & 6 & 4 & 0 & 0 \\ 1 & 4 & 7 & 12 & 8 & 16 & 24 & 18 & 24 & 24 & 0 \\ 1 & 6 & 15 & 30 & 20 & 60 & 120 & 90 & 180 & 360 & 720 \end{bmatrix}$$

A.5 Schur Functions

$$M_2(s, m) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$M_3(s, m) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_4(s, m) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_5(s, m) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_6(s, m) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 & 2 & 3 & 3 & 4 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 & 1 & 3 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 4 & 8 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A.6 Jack Functions

Matrices with polynomial entries are bigger and less straightforward to read than with numeric entries. For the purpose of clarity, a table format is adopted for displaying Jack function transition matrices. Moreover, since coefficients of m_μ of any given Jack function are multiples of $\prod_i M_i^\mu!$ where $\mu = (\mu_1^{M_1^\mu}, \mu_2^{M_2^\mu}, \mu_3^{M_3^\mu}, \dots)$ (42), they will be expressed in terms of multiples $(\prod_i M_i^\mu!)m_\mu$ of monomial functions instead of m_μ .

| | $m_{(2)}$ | $2m_{(1^2)}$ |
|-------------|----------------|--------------|
| $J_{(2)}$ | $(\alpha + 1)$ | 1 |
| $J_{(1^2)}$ | 0 | 1 |

| | $m_{(3)}$ | $m_{(2,1)}$ | $6m_{(1^3)}$ |
|-------------|-----------------------------|-----------------|--------------|
| $J_{(3)}$ | $(2\alpha + 1)(\alpha + 1)$ | $3(\alpha + 1)$ | 1 |
| $J_{(2,1)}$ | 0 | $(\alpha + 2)$ | 1 |
| $J_{(1^3)}$ | 0 | 0 | 1 |

| | $m_{(4)}$ | $m_{(3,1)}$ | $2m_{(2^2)}$ | $2m_{(2,1^2)}$ | $24m_{(1^4)}$ |
|---------------|---|------------------------------------|----------------------------------|-----------------------------------|---------------|
| $J_{(4)}$ | $(\alpha + 1)(2\alpha + 1)$ $(3\alpha + 1)(4\alpha + 1)$ | $4(\alpha + 1)$ $(2\alpha + 1)$ | $3(\alpha + 1)^2$ | $6(\alpha + 1)$ | 1 |
| $J_{(3,1)}$ | 0 | $2(\alpha + 1)^2$ | $2(\alpha + 1)$ | $(\alpha + 1)$ $(3\alpha + 5)$ | 1 |
| $J_{(2^2)}$ | 0 | 0 | $(\alpha + 1)$ $(\alpha + 2)$ | $2(\alpha + 2)$ | 1 |
| $J_{(2,1^2)}$ | 0 | 0 | 0 | $(\alpha + 3)$ | 1 |
| $J_{(1^4)}$ | 0 | 0 | 0 | 0 | 1 |

| | $m_{(5)}$ | $m_{(4,1)}$ | $m_{(3,2)}$ | $2m_{(3,1^2)}$ |
|---------------|---|---|---------------------------------------|-------------------------------------|
| $J_{(5)}$ | $(\alpha + 1)(2\alpha + 1)$ $(3\alpha + 1)(4\alpha + 1)$ | $5(\alpha + 1)(2\alpha + 1)$ $(3\alpha + 1)$ | $10(\alpha + 1)^2$ $(2\alpha + 1)$ | $10(\alpha + 1)$ $(2\alpha + 1)$ |
| $J_{(4,1)}$ | 0 | $(\alpha + 1)(2\alpha + 1)$ $(3\alpha + 2)$ | $3(\alpha + 1)$ $(3\alpha + 2)$ | $(\alpha + 1)(8\alpha + 7)$ |
| $J_{(3,2)}$ | 0 | 0 | $2(\alpha + 1)^2(\alpha + 2)$ | $2(\alpha + 1)(\alpha + 2)$ |
| $J_{(3,1^2)}$ | 0 | 0 | 0 | $(\alpha + 1)(2\alpha + 3)$ |
| $J_{(2^2,1)}$ | 0 | 0 | 0 | 0 |
| $J_{(2,1^3)}$ | 0 | 0 | 0 | 0 |
| $J_{(1^5)}$ | 0 | 0 | 0 | 0 |

| | $2m_{(2^2,1)}$ | $6m_{(2,1^3)}$ | $120m_{(1^5)}$ |
|---------------|-----------------------------|------------------|----------------|
| $J_{(5)}$ | $15(\alpha + 1)^2$ | $10(\alpha + 1)$ | 1 |
| $J_{(4,1)}$ | $3(\alpha + 1)(\alpha + 4)$ | $3(2\alpha + 3)$ | 1 |
| $J_{(3,2)}$ | $(3\alpha + 5)(\alpha + 2)$ | $4(\alpha + 2)$ | 1 |
| $J_{(3,1^2)}$ | $2(2\alpha + 3)$ | $(3\alpha + 7)$ | 1 |
| $J_{(2^2,1)}$ | $(\alpha + 3)(\alpha + 2)$ | $2(\alpha + 3)$ | 1 |
| $J_{(2,1^3)}$ | 0 | $(\alpha + 4)$ | 1 |
| $J_{(1^5)}$ | 0 | 0 | 1 |

| | $m_{(6)}$ | $m_{(5,1)}$ | $m_{(4,2)}$ | $2m_{(4,1^2)}$ |
|-----------------|--|--|--|--|
| $J_{(6)}$ | $(\alpha + 1)(2\alpha + 1)$ $(3\alpha + 1)(4\alpha + 1)$ $(5\alpha + 1)$ | $6(\alpha + 1)(2\alpha + 1)$ $(3\alpha + 1)(4\alpha + 1)$ | $15(\alpha + 1)^2$ $(2\alpha + 1)(3\alpha + 1)$ | $15(\alpha + 1)$ $(2\alpha + 1)(3\alpha + 1)$ |
| $J_{(5,1)}$ | 0 | $2(\alpha + 1)(2\alpha + 1)^2$ $(3\alpha + 1)$ | $8(2\alpha + 1)^2(\alpha + 1)$ | $3(\alpha + 1)(2\alpha + 1)$ $(5\alpha + 3)$ |
| $J_{(4,2)}$ | 0 | 0 | $2(\alpha + 1)^3(3\alpha + 2)$ | $2(\alpha + 1)^2(3\alpha + 2)$ |
| $J_{(4,1^2)}$ | 0 | 0 | 0 | $3(\alpha + 1)^2(2\alpha + 1)$ |
| $J_{(3^2)}$ | 0 | 0 | 0 | 0 |
| $J_{(3,2,1)}$ | 0 | 0 | 0 | 0 |
| $J_{(3,1^3)}$ | 0 | 0 | 0 | 0 |
| $J_{(2^3)}$ | 0 | 0 | 0 | 0 |
| $J_{(2^2,1^2)}$ | 0 | 0 | 0 | 0 |
| $J_{(2,1^4)}$ | 0 | 0 | 0 | 0 |
| $J_{(1^6)}$ | 0 | 0 | 0 | 0 |

| | $2m_{(3^2)}$ | $m_{(3,2,1)}$ | $6m_{(3,1^3)}$ | $6m_{(2^3)}$ |
|-----------------|--|--|------------------------------------|--|
| $J_{(6)}$ | $10(\alpha + 1)^2(2\alpha + 1)^2$ | $60(\alpha + 1)^2(2\alpha + 1)$ | $20(\alpha + 1)(2\alpha + 1)$ | $15(\alpha + 1)^3$ |
| $J_{(5,1)}$ | $6(\alpha + 1)^2$ $(2\alpha + 1)$ | $4(\alpha + 1)$ $(5\alpha^2 + 20\alpha + 11)$ | $4(\alpha + 1)$ $(5\alpha + 4)$ | $12(\alpha + 1)^2$ |
| $J_{(4,2)}$ | $2(\alpha + 1)^2$ $(3\alpha + 2)$ | $8(\alpha + 1)^2$ $(\alpha + 4)$ | $4(\alpha + 1)$ $(2\alpha + 3)$ | $(\alpha + 1)$ $(3\alpha^2 + 7\alpha + 10)$ |
| $J_{(4,1^2)}$ | 0 | $18(\alpha + 1)^2$ | $2(\alpha + 1)(4\alpha + 5)$ | $6(\alpha + 1)$ |
| $J_{(3^2)}$ | $2(\alpha + 1)^2(\alpha + 2)$ $(2\alpha + 1)$ | $12(\alpha + 1)^2$ $(\alpha + 2)$ | $4(\alpha + 1)$ $(\alpha + 2)$ | $4(\alpha + 1)$ $(\alpha + 2)$ |
| $J_{(3,2,1)}$ | 0 | $(2\alpha + 3)(\alpha + 2)^2$ | $(\alpha + 2)(2\alpha + 3)$ | $(\alpha + 2)(2\alpha + 3)$ |
| $J_{(3,1^3)}$ | 0 | 0 | $2(\alpha + 1)(\alpha + 2)$ | 0 |
| $J_{(2^3)}$ | 0 | 0 | 0 | $(\alpha + 1)$ $(\alpha + 2)(\alpha + 3)$ |
| $J_{(2^2,1^2)}$ | 0 | 0 | 0 | 0 |
| $J_{(2,1^4)}$ | 0 | 0 | 0 | 0 |
| $J_{(1^6)}$ | 0 | 0 | 0 | 0 |

| | $4m_{(2^2,1^2)}$ | $24m_{(2,1^4)}$ | $720m_{(1^6)}$ |
|-----------------|--------------------------------|------------------|----------------|
| $J_{(6)}$ | $45(\alpha + 1)^2$ | $15(\alpha + 1)$ | 1 |
| $J_{(5,1)}$ | $3(\alpha + 1)(5\alpha + 13)$ | $2(5\alpha + 7)$ | 1 |
| $J_{(4,2)}$ | $(9\alpha^2 + 37\alpha + 34)$ | $(7\alpha + 13)$ | 1 |
| $J_{(4,1^2)}$ | $3(\alpha^2 + 8\alpha + 9)$ | $6(\alpha + 2)$ | 1 |
| $J_{(3^2)}$ | $3(3\alpha^2 + 11\alpha + 10)$ | $6(\alpha + 2)$ | 1 |
| $J_{(3,2,1)}$ | $3(\alpha^2 + 6\alpha + 8)$ | $(4\alpha + 11)$ | 1 |
| $J_{(3,1^3)}$ | $6(\alpha + 2)$ | $3(\alpha + 3)$ | 1 |
| $J_{(2^3)}$ | $3(\alpha + 3)(\alpha + 2)$ | $3(\alpha + 3)$ | 1 |
| $J_{(2^2,1^2)}$ | $(\alpha + 4)(\alpha + 3)$ | $2(\alpha + 4)$ | 1 |
| $J_{(2,1^4)}$ | 0 | $(\alpha + 5)$ | 1 |
| $J_{(1^6)}$ | 0 | 0 | 1 |

References

- [1] Assaf, S. H. and McNamara, P. R. W. A Pieri Rule for Skew Shapes. *Journal of Combinatorial Theory, Series A*, 118(1): pages 277–290, 2011.
- [2] Debiard, A. Polynômes de Tchébychev et de Jacobi Dans un Espace Euclidien de Dimension P. *Comptes Rendus de l'Académie des Sciences de Paris*, I. 296: pages 529–532, 1983.
- [3] Jack, H. A class of Symmetric Functions with a Parameter. *Proceedings of the Royal Society of Edinburgh, A*. 69(1): pages 1–18, 1970.
- [4] Knop, F. and Sahi, S. A Recursion and a Combinatorial Formula for Jack Polynomials. *Inventiones Mathematicae*, 128: pages 9–22, 1997.
- [5] Kostka, C. Über den Zusammenhang Zwischen Einigen Formen von Symmetrischen Funktionen. *Crelle*, 93: pages 89–123, 1882.
- [6] Littlewood, D. E. and A R Richardson, A. R. Group characters and Algebra. *Philosophical Transactions of the Royal Society of London, Series A. Containing Papers of a Mathematical or Physical Character*, 233: pages 99–141, 1934.
- [7] Macdonald, I. G. Symmetric Functions and Hall polynomials. *Oxford university press*, 1998.
- [8] Opdam, E. M. et al. Harmonic Analysis for Certain Representations of Graded Hecke Algebras. *Acta Mathematica*, 175(1): pages 75–121, 1995.
- [9] Pieri, M. Sul Problema Degli Spazi Secanti. *Istituto Lombardo di Scienze e Lettere, Serie II.*,26: pages 534–546, 02 1893.
- [10] Robinson. G. de B. On the Representations of the Symmetric Group. *American Journal of Mathematics*, 60(3): pages 745–760, 1938.

- [11] Sahi, S. A New Scalar Product for Nonsymmetric Jack Polynomials. *International Mathematics Research Notices*. volume 1996(20): pages 997–1004, 1996.
- [12] Schur, I. Über Eine Klasse von Matrizen, die Sich Einer Gegebenen Matrix Zuordnen Lassen, 1901.
- [13] Schützer, W. Degree 1 Pieri Formula for Nonsymmetric Jack Polynomials, 06 2016.
- [14] Sekiguchi, J. Zonal Spherical Functions on Some Symmetric Spaces. *Publications of the Research Institute for Mathematical Sciences*, 12(Supplement): pages 455–459, 1977.
- [15] Stanley, R. P. Some combinatorial Properties of Jack Symmetric Functions. *Advances in Mathematics*, 77(1): pages 76–115, 1989.