Universidade Federal de São Carlos
Centro de Ciências Exatas e Tecnologia
Departamento de Matemática

Ordinary and Twisted K-Theory

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"Was aus Liebe getan wird,

geschieht immer jenseits von Gut und Böse."

Friedrich Wilhelm Nietzsche,

Jenseits von Gut und Böse - Vorspiel einer Philosophie der Zukunft.

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Resumo

O tópico principal desta dissertação é a K-Teoria Ordinária e Torcida. Começamos descrevendo teorias cohomológicas generalizadas através dos Axiomas de Eilenberg-Steenrod a fim de estabelecer a K-Teoria Ordinária nesses termos. Isto nos permite deduzir suas propriedades estruturais do arcabouço da cohomologia generalizada. Então, expomos as noções elementares da Geometria de Spin para relacioná-la com a K-Teoria Ordinária através do Teorema de Atiyah-Bott-Shapiro. Este resultado nos permite definir o isomorfismo de Thom bem como o mapa de integração, que é conhecido como mapa de Gysin. Depois disso, refraseamos a K-Teoria Ordinária por meio da aplicação do Índice, que nos fornece uma interpretação da K-Teoria através de classes de homotopia de funções contínuas. Em seguida, lidamos com a K-Teoria Torcida. Primeiro, introduzimos o grupo de Grothendieck dos fibrados vetoriais torcidos como um modelo para a K-Teoria Torcida de ordem finita. Então, descrevemos o modelo de dimensão infinita, através de fibrados apropriados de operadores de Fredholm, que lida com classes de torção de qualquer ordem. Finalmente, comparamos estes dois modelos no contexto de ordem finita.

Palavras-chave. K-Teoria Topológica; Teorias cohomológicas generalizadas; K-Teoria Ordinária; Geometria de Spin; Isomorfismo de Thom; Mapa de Gysin; Operadores de Fredholm; Aplicação do Índice; K-Teoria Torcida.

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Abstract

The main topic of this thesis consists in Ordinary and Twisted Topological K-Theory. We begin by describing generalized cohomology theories through the Eilenberg-Steenrod Axioms, in order to set Ordinary K-Theory in these terms. This allows us to deduce its structural properties from the framework of generalized cohomology. Then, we expose the elementary notions of Spin Geometry to relate it to Ordinary K-Theory through the Atiyah-Bott-Shapiro Theorem. This result enables us to construct the Thom isomorphism as well as the integration map, which is known as Gysin map. After that, we rephrase Ordinary K-Theory by means of the Index map, which provides an interpretation of K-Theory through homotopy classes of continuous functions. Afterwards, we deal with Twisted K-Theory. First, we introduce the Grothendieck group of twisted vector bundles as a model for finite-order Twisted K-Theory. Then, we describe the infinite-dimensional model, through suitable bundles of Fredholm operators, that holds for twisting classes of any order. Finally, we compare these two models in the finite-order setting.

Keywords. Topological K-Theory; Generalized cohomology theories; Ordinary K-Theory; Spin Geometry; Thom isomorphism; Gysin map; Fredholm operators; Index map; Twisted K-Theory.

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Ordinary and Twisted K-Theory

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Contents

Introduction

This work is placed in the area of mathematics that is called Algebraic Topology. More precisely, it is inserted in the setting of cohomology theories, especially dealing with K-Theory. This specialization has a transparent historical justification behind it, since K-Theory was the first generalized cohomology theory to appear in the literature. In fact, Algebraic K-theory has its origins in the late 1950s due to a generalization by Alexander Grothendieck (1928-2014) of the famous Riemann-Roch Theorem. Roughly speaking, Grothendieck associated a group $K(X)$ to each X in some family of algebraic spaces. In this framework, he recovered the classical Riemann-Roch Theorem as a special case of a general result involving K-groups. In particular, Grothendieck told us that

> The way I first visualized a K-group was as a group of “classes of objects” of an abelian (or more generally, additive) category, such as coherent sheaves on an algebraic variety, or vector bundles, etc. I would presumably have called this group $C(X)$ (X being a variety or any other kind of “space”), C the initial letter of “class”, but my past in Functional Analysis may have prevented this, as $C(X)$ designates also the space of continuous functions on X (when X is a topological space). Thus, I reverted to K instead of C, since my mother tongue is German, Class = Klasse (in German), and the sounds corresponding to C and K are the same. [11, p. 2]

Afterwards, Friedrich Hirzebruch (1927-2012) and Michael Atiyah (1929-2019) realized that these ideas could be exported to the world of Algebraic Topology. The resulting K-theory of topological spaces, which we refer to as Ordinary K-Theory, turned out to be quite powerful. In fact, for example, some of its early conquests are the determination of the maximum number of linearly independent vector fields on spheres, a classification theorem for real division algebras and the Atiyah-Singer Index Theorem. Years later, in the late 1960s, Max Karoubi (1938 -) introduced Twisted

K-Theory, also known as K-Theory with local coefficients, in his doctoral dissertation. These versions of Topological K-Theory, Ordinary and Twisted, are the main subjects of this thesis. In a nutshell, we begin by considering generalized cohomology theories from an axiomatic viewpoint à la Eilenberg-Steenrod. After that, we discuss the main models of Ordinary K-Theory, including tools from Spin Geometry in order to construct the Thom isomorphism and the Gysin map. Finally, we deal with Twisted K-Theory.

For completeness, we describe below the content of each chapter. We also provide an overview of the references that we reviewed in this work. We emphasize that more information about them will be given in convenient parts of the main text.

In Chapter 1, we provide a presentation of generalized cohomology theories. Thanks to this starting point, in the next chapters we will be able to set the properties of K-Theory that descend directly from the Eilenberg-Steenrod axioms. In particular, we establish the exact sequence of a triple and the Mayer-Vietoris sequences using [13, pp. 3 - 53]. We also discuss the additivity axiom, originally introduced in [28], and multiplicative structures on cohomology theories, following [21, pp. 38-40]. Our presentation is thoroughly realized through the language and the notations of [13], so that it is quite unitary. Since [13] mainly deals with homology theories, we adapted it to the cohomological setting. Other meaningful references for this chapter are the following ones: [37] for historical notes; [26] for the language of category theory, widely used in this thesis; [17] for some algebraic notions; [12], [25] and [29] for the basic concepts of general topology.

In Chapter 2, we expose the main notions of Ordinary K-Theory as a generalized cohomology theory, taking advantage of the results proved in the previous chapter. The main references are [2, pp. 43-94] and [19, pp. 52-111]. However, some applications of K-Theory could not be written without [15, pp. 38-72]. We also used [1], [3], [23, p. 65, pp. 70-76] and [33].

In Chapter 3, we present the basic notions on Spin Geometry, that are essential to construct the Thom isomorphism and the Gysin map, the latter being the integration map in K-Theory. In particular, we carefully analyze the notions of spin and spin^c structure on vector bundles. The exposition of these topics in based on

[2, pp. 102-116], [6], [9, pp. 37-47], [23, pp. 7-40, 58-70, 77-85] and [34].

In Chapter 4, we present another relevant model of Ordinary K-Theory. The latter is realized through homotopy classes of functions taking values in the space of Fredholm operators on an infinite-dimensional separable Hilbert space. This model will be particularly useful in Chapter 5. The main references for this part of the text are [2, pp. 153 - 162], [8, pp. 7-18, 33-43], [22] and [32, pp. 1-23, 55-67, 119-125, 175-183].

In Chapter 5, we develop two relevant models of Twisted K-Theory. We begin by introducing the Grothendieck group of twisted vector bundles as a model of finite-order Twisted K-Theory. Afterwards, we describe the infinite-dimensional model, through suitable bundles of Fredholm operators, that holds for twisting classes of any order. Finally, we compare these two models in the finite-order setting. We also consider a suitable version of the Thom isomorphism in this framework. We used [4], [6, pp. 5-8, 30-36, 43-45, 53-54], [7, pp. 42-43] and [20].

We conclude the main part of the thesis with "Further Perspectives". Here we indicate some topics that can be studied in a near future thanks to the subjects treated in this thesis.

Afterwards, we present six appendixes in which the reader can find many elementary concepts used in the thesis. We provide concise expositions, which hopefully turn the text more readable. The biggest part of the subjects treated in the appendixes is present in the references, but in a way that we did not manage to cite directly without loss of clearness.

In Appendix A, we provide an outline of direct limits of abelian groups. This algebraic notion is essential to define the compactly-supported generalized cohomology groups, used in Chapter 1. It also appears in the definition of compactly-supported Twisted K-Theory in Chapter 5. For the categorical approach to this topic, we used [26, pp. 105-112].

In Appendix B, we set a basic group-theoretical tool, that is essential to define K-Theory, namely, the Grothendieck group of an abelian semigroup. The idea behind such concept consists in finding the minimal extension of an abelian semigroup to an abelian group, although it turns out not to be an extension in general. We followed

[2, pp. 42-43], that presents such construction for generic abelian semigroups, without assuming the existence of a unit. The notions presented here are mainly used in Chapters 2 and 5, but they appear throughout the whole work.

In Appendix C, we consider the fundamental notion underlying Ordinary K-Theory, that is, the one of vector bundle. Since the corresponding theory is extensive, we only selected some of its initial concepts and the results that play essential roles in the main text. We used [2, pp. 1 - 41], [15, pp. 4 - 37], [16, pp. 85 - 109], [18, pp. 24 - 39], [19, pp. 1 - 51], [24, pp. 249 - 271] and [31]. The notions presented here are mainly used in Chapter 2.

In Appendix D, we describe classical constructions with topological spaces: wedge sum, smashed product, cones and suspensions. We restrict them to compact Hausdorff spaces, since they are the spaces we used to construct K-Theory in Chapter 2. We followed [14, pp. 8-10].

In Appendix E, we explain the elementary concepts on real division algebras. Moreover, we provide some historical notes on the main real division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. This is done because we think that it is a way to understand the importance of the Bott-Milnor-Kervaire Theorem presented in Chapter 2, which was one of the first achievements of K-Theory. We conclude our presentation with two classical results about these algebras, which explain why they are relevant and, in a certain sense, unique. Our exposition was based on [5], [10], [27], [36] and [38].

Finally, in Appendix F, we review the initial concepts on principal bundles and the results that play an essential role in our exposition. Moreover, since this theory is, under a certain viewpoint, equivalent to the one of vector bundles, we introduce some notions that show this equivalence. We mainly used [30, pp. 28-35] and [35, pp. 111-118].

Chapter 1

Generalized Cohomology Theories

In this chapter, we describe some of the structural properties of generalized cohomology theories. This technical work is worth doing because, as we shall see later in Chapter 2, many results are then immediate once we prove that the mathematical framework under consideration is a generalized cohomology theory. In order to write this part of the text, we used as main reference [13, pp. 3 - 53]. Nevertheless, Section 1.9 could not be completed without [28] as well as Section 1.12 could not be written without [21, pp. 38-40]. In addition, [37] was used as a reference for some historical facts involving Homological Algebra.

1.1 Admissible categories of topological spaces

The notion of admissible category of topological spaces, which is described here, will be used when we set the axioms for generalized cohomology theories. We begin with the following definition.

Definition 1.1 (The category of ordered pairs of topological spaces)**.** *We define the* ***category of ordered pairs of topological spaces****, and denote it by* Top_2*, to be the one whose:*

- *objects are ordered pairs* (X, A) *in which* X *is a topological space and* $A \subseteq X$ *is equipped with the induced topology; and*

- *morphisms are continuous functions $f : X \to Y$ such that $f(A) \subseteq B$, usually denoted by $f : (X, A) \to (Y, B)$.* $\diamondsuit$

Historical experience shows that the natural environment to set a generalized cohomology theory is a *convenient subcategory* of Top_2. In fact, in some concrete generalized cohomology theories, important theoretical results do not hold if we consider the whole Top_2. We will enlighten shortly the precise meaning of the word "convenient". First, let us show that we can select subcategories of Top_2 in a myriad of ways. For example, we can consider:

(1) TopO_2 to be the non-full subcategory whose objects are ordered pairs of topological spaces and whose morphisms are open continuous maps;

(2) TopHd_2 to be the full subcategory whose objects are ordered pairs of Hausdorff spaces;

(3) TopD_2 to be the full subcategory whose objects are ordered pairs of topological spaces endowed with the discrete topology;

(4) $\mathrm{TopHdCpt}_2$ to be the full subcategory whose objects are ordered pairs (X, A) in which X is compact Hausdorff;

(5) $\mathrm{TopHdCCpt}_2$ to be the full subcategory whose objects are ordered pairs (X, A) in which X is compact Hausdorff and A is a closed subspace of X;

(6) $\mathrm{TopHdLocCptP}_2$ to be the non-full subcategory whose objects are ordered pairs (X, A) in which X is locally compact Hausdorff, and whose morphisms are proper continuous maps; *and*

(7) $\mathrm{TopHdLocCCptP}_2$ to be the non-full subcategory whose objects are ordered pairs (X, A) in which X is locally compact Hausdorff and A is a closed subspace of X, and whose morphisms are proper continuous maps.

As we can see from the preceding examples, there are basically two procedures to set a subcategory of Top_2 (note that, in some cases, these procedures are applied at the same time):

- the first one is to restrict pairs of topological spaces (as in Examples (2), (3), (4), (5), (6) and (7)); *and*
- the second one is to restrict morphisms between pairs of spaces (as in Examples (1), (6) and (7)).

Therefore, the idea behind the word "convenient" is that, in order to develop a generalized cohomology theory in a subcategory of Top_2, we need to be careful about not restricting too much the pairs of spaces and their morphisms. This elementary idea is formalized by the following definition.

Definition 1.2 (Admissible category of topological spaces)**.** *A subcategory $\mathscr{C}$ of* Top_2 *is an **admissible category of topological spaces** if it satisfies all of the four conditions listed below. In this situation, the pairs and the maps that belong to $\mathscr{C}$ are said to be **admissible**.*

(1) If $(X, A) \in \mathscr{C}$, then all pairs and inclusions maps of the following lattice of (X, A) are in $\mathscr{C}$, where 0 denotes the empty set.

$$\begin{tikzcd} & & (X,0) \arrow[dr] & & \\ (0,0) \arrow[r] & (A,0) \arrow[ur] \arrow[dr] & & (X,A) \arrow[r] & (X,X) \\ & & (A,A) \arrow[ur] & & \end{tikzcd}$$

(2) If $f : (X, A) \to (Y, B)$ is in $\mathscr{C}$, then (X, A) and (Y, B) are in $\mathscr{C}$ together with all maps that f defines from members of the lattice of (X, A) into corresponding members of the lattice of (Y, B).

(3) If $\mathbb{I} := [0, 1]$ and $(X, A) \in \mathscr{C}$, then

$$(X, A) \times \mathbb{I} \;:=\; (X \times \mathbb{I},\, A \times \mathbb{I})$$

is in $\mathscr{C}$ together with the maps $\iota_0, \iota_1 : (X, A) \to (X, A) \times \mathbb{I}$ given by $\iota_0(x) = (x, 0)$ and $\iota_1(x) = (x, 1)$ for all $x \in X$.

(4) There exists an ordered pair $(\Omega, 0)$ in $\mathscr{C}$ where Ω consists of a single point. In addition, if (X, A) and $(Y, 0)$ are in $\mathscr{C}$, Y is a single point and $f : (Y, 0) \to (X, A)$ is in Top_2*, then f is also in $\mathscr{C}$.* $\diamondsuit$

Notation 1.3 (Identity maps)**.** *In an admissible category of topological spaces, the identity maps of admissible pairs are always admissible because $\mathscr{C}$ is defined as a subcategory of* Top_2*. Hereafter, $\mathrm{id}_{(X,A)} : (X, A) \to (X, A)$ is our notation for the **identity map** on the admissible pair (X, A).* $\diamondsuit$

Notation 1.4 (Ordered pairs with empty second components)**.** *In an admissible category of topological spaces, we abbreviate an admissible pair $(X, 0)$ simply by X. In particular, we shall say that X is an **admissible space** if its corresponding pair $(X, 0)$ is admissible.* $\diamondsuit$

The reader can prove that Top_2 and Examples (2), (5) and (7) are admissible categories of topological spaces. On the other hand, Examples (1), (3), (4) and (6) are non-admissible categories of topological spaces. We prove this latter statement in the sequence. In fact:

- TopO_2 is non-admissible because a continuous map from a point into a space is not always open. For example, the inclusion of the origin in any non-trivial Euclidean space is not an open map. In other words, TopO_2 does not verify Condition (4);

- TopD_2 is non-admissible because the product of a nonempty discrete space with the unit interval is not a discrete space. In other words, TopD_2 does not verify Condition (3);

- $\mathrm{TopHdCpt}_2$ is non-admissible because, if $(X, A) \in \mathrm{TopHdCpt}_2$ is such that A is not a closed subspace of X, then $(A, 0)$ does not belong to $\mathrm{TopHdCpt}_2$. This happens because every compact subspace of a Hausdorff space is necessarily closed. In other words, $\mathrm{TopHdCpt}_2$ does not verify Condition (1); *and*

- $\mathrm{TopHdLocCptP}_2$ is non-admissible because, if $(X, A) \in \mathrm{TopHdLocCptP}_2$ is such that X is a compact Hausdorff space and A is not a closed subspace of X, then the inclusion $\iota : A \to X$ is not a proper map. Indeed, we have that $\iota^{-1}(X) = A$ is not compact. In other words, we have that $\mathrm{TopHdLocCptP}_2$ does not verify Condition (1).

We leave to the reader the search for subcategories of Top_2 that do not verify Condition (2) of Definition 1.2. The interesting problem is to find subcategories of Top_2 that satisfies all of the conditions of Definition 1.2 but this one. To close this section, we present the following important definitions which are useful to define the generalized cohomology theories.

Definition 1.5 (Homotopy and homotopic maps)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces. Let $\iota_0, \iota_1 : (X, A) \to (X, A) \times \mathbb{I}$ be the maps presented in the third condition of Definition 1.2. In addition, let $f, g : (X, A) \to (Y, B)$ be admissible maps. A* ***homotopy*** *between f and g is an admissible map $\Theta : (X, A) \times \mathbb{I} \to (Y, B)$ such that the diagrams*

$$\begin{tikzcd}
(X,A) \arrow[r, "\iota_0"'] \arrow[rr, bend left, "f"] & (X,A)\times\mathbb{I} \arrow[r, "\Theta"'] & (Y,B)
\end{tikzcd}$$

$$\begin{tikzcd}
(X,A) \arrow[r, "\iota_1"'] \arrow[rr, bend left, "g"] & (X,A)\times\mathbb{I} \arrow[r, "\Theta"'] & (Y,B)
\end{tikzcd}$$

are commutative. If there exists an admissible homotopy between f and g, then these maps are said to be ***homotopic****.* $\diamondsuit$

Remark 1.6 (Homotopy of maps is a compatible equivalence relation on the class of morphisms of an admissible category)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces. The relation of homotopy of maps on the class of morphisms of $\mathscr{C}$ is defined as follows: two admissible maps are related if and only if there exists a homotopy between them. The reader can readily prove that this is an equivalence relation. Furthermore, this relation is compatible with the composition in $\mathscr{C}$. This means that $r \circ f$ is homotopic to $s \circ g$ whenever f is homotopic to g and r is homotopic to s. These facts allow us to set the following definition.* $\diamondsuit$

Definition 1.7 (The quotient category by the relation of homotopy of maps)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces. We define the* ***quotient category of $\mathscr{C}$ by the relation of homotopy of maps****, and denote it by $[\mathscr{C}]$, to be the one whose:*

- *objects are the same objects of $\mathscr{C}$; and*
- *morphisms are the equivalence classes of morphisms of $\mathscr{C}$ under the relation of homotopy of maps.* $\diamondsuit$

Remark 1.8 (The quotient category is non-admissible)**.** *An admissible category of topological spaces $\mathscr{C}$ is a suitable subcategory of* Top_2. *Since the morphisms of $[\mathscr{C}]$ are not morphisms of* Top_2, *$[\mathscr{C}]$ cannot be a subcategory of* Top_2. *A fortiori, $[\mathscr{C}]$ cannot be an admissible category of topological spaces.* $\diamondsuit$

1.2 Axioms for generalized cohomology theories

Cohomology Theory was turned into an axiomatic theory by **Samuel Eilenberg** (1913-1998) and **Norman Steenrod** (1910-1971) in the last century. These men set the postulates that are known today as the **Eilenberg-Steenrod Axioms**. The study of generalized cohomology theories starts removing one of Eilenberg-Steenrod Axioms: the **Dimension Axiom**[(1)]. In this section, we use the notion of admissible category of topological spaces to state the following definition which contains the axioms for a generalized cohomology theory.

Definition 1.9 (Generalized cohomology theory)**.** *Consider:*

- *$\mathscr{C}$ to be an admissible category of topological spaces;*
- *$(h^n)_{n\in\mathbb{Z}}$ to be a sequence of contravariant functors from $\mathscr{C}$ into the category of abelian groups $\mathscr{G}_{ab}$. We call $h^n(X,A)$ the* ***nth generalized relative cohomology group*** *of the admissible pair (X,A). Especially, we call $h^n(X) = h^n(X,0)$ the* ***nth generalized absolute cohomology group*** *of the*

[(1)]The Dimension Axiom states that the cohomology groups of a one-point space are trivial in all degrees with the possible exception of degree zero.

admissible space X. Being $f : (X, A) \to (Y, B)$ an admissible map of pairs, we say that $h^n(f) : h^n(Y, B) \to h^n(X, A)$ is its ***nth generalized induced homomorphism****; and*

- *$(\delta^n)_{n\in\mathbb{Z}}$ to be a sequence of functions that assign to each admissible pair (X, A) a homomorphism $\delta^n_{(X,A)} : h^{n-1}(A) \to h^n(X, A)$. We call $\delta^n_{(X,A)}$ the* ***nth generalized coboundary operator*** *of (X, A).*

These three pieces of data are said to be a ***generalized cohomology theory*** *if the following four axioms are satisfied.*

(1) ***Commutativity Axiom****. For every admissible map $f : (X, A) \to (Y, B)$ and every $n \in \mathbb{Z}$, the following diagram is commutative.*

$$\begin{array}{ccc}
h^{n-1}(B) & \xrightarrow{h^{n-1}(f|_A)} & h^{n-1}(A) \\
\Big\downarrow{\scriptstyle \delta^n_{(Y,B)}} & & \Big\downarrow{\scriptstyle \delta^n_{(X,A)}} \\
h^n(Y, B) & \xrightarrow[h^n(f)]{} & h^n(X, A)
\end{array}$$

In other words, if $f : (X, A) \to (Y, B)$ is admissible and $f\,|_A\colon A \to B$ is the map defined by f, then the two ways of mapping $h^{n-1}(B)$ into $h^n(X, A)$ presented in the previous diagram have to coincide.

(2) ***Exactness Axiom****. If $i : A \to X$ and $j : X \to (X, A)$ are the natural inclusion maps, then the following sequence composed of groups and of group homomorphisms is exact.*

$$\cdots \longrightarrow h^{n-1}(A) \xrightarrow{\delta^n_{(X,A)}} h^n(X, A) \xrightarrow{h^n(j)} h^n(X) \xrightarrow{h^n(i)} h^n(A) \longrightarrow \cdots$$

In other words, we require $\operatorname{Im}\delta^n_{(X,A)} = \operatorname{Ker} h^n(j)$, $\operatorname{Im} h^n(j) = \operatorname{Ker} h^n(i)$ and $\operatorname{Im} h^n(i) = \operatorname{Ker}\delta^{n+1}_{(X,A)}$ for all $n \in \mathbb{Z}$. This exact sequence is called the ***generalized cohomology sequence*** *of the admissible pair (X, A).*

(3) ***Homotopy Axiom***. *Admissible homotopic maps have the same generalized induced homomorphisms in all degrees. More explicitly, if* $f, g : (X, A) \to (Y, B)$ *are admissible homotopic maps, then*

$$h^n(f) = h^n(g) : h^n(Y, B) \to h^n(X, A)$$

for every $n \in \mathbb{Z}$.

(4) ***Excision Axiom***. *If* U *is open in* X *and its closure is contained in the interior of* A*, then the inclusion map* $(X-U, A-U) \to (X, A)$*, from now on called an* ***excision map*** *or just an* ***excision****, if admissible, induces isomorphisms from* $h^n(X, A)$ *onto* $h^n(X - U, A - U)$ *for all* $n \in \mathbb{Z}$. ♢

Remark 1.10 (On an equivalent formulation of the Excision Axiom)**.** *In a generalized cohomology theory, we have that the following statement is equivalent to the Excision Axiom.*

> *Let* X_1 *and* X_2 *be subsets of an admissible space* X *such that* X_1 *is closed and* X *is the union of interiors of* X_1 *and* X_2. *If*
>
> $$i : \ (X_1, X_1 \cap X_2) \ \to \ (X_1 \cup X_2, X_2)$$
>
> *is admissible, then* $h^n(i) : h^n(X_1 \cup X_2, X_2) \to h^n(X_1, X_1 \cap X_2)$ *is an isomorphism for all* $n \in \mathbb{Z}$.

Indeed, assuming the Excision Axiom as in Definition 1.9, the preceding assertion follows since $i : (X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$ *is the excision map obtained from the pair* $(X_1 \cup X_2, X_2)$ *relatively to* $U = X - X_1$. *Conversely, the Excision Axiom follows from the preceding assertion by taking* $A = X_2$ *and* $U = X - X_1$. *This proves our claim.* ♢

The beauty of an axiomatic treatment lies in the simplification obtained in some proofs of theorems. As a matter of fact, proofs based directly on the axioms are usually simple and conceptual. Furthermore, no one is faced at the end of a proof by the question: Does the proof still hold if another generalized cohomology theory replaces the one used? To close this section, we present our first illustrative examples of these ideas for generalized cohomology theories.

Theorem 1.11 (Trivial generalized cohomology groups)**.** *In a generalized cohomology theory, if (X, A) is an admissible pair and the inclusion map $i : A \to X$ is such that $h^n(i) : h^n(X) \to h^n(A)$ is an isomorphism for all $n \in \mathbb{Z}$, then $h^n(X, A)$ is trivial for all $n \in \mathbb{Z}$.*

Proof. Let n be an integer number and $j : X \to (X, A)$ be an inclusion map. The following section of the generalized cohomology sequence of (X, A) is exact by the Exactness Axiom.

$$h^{n-1}(X) \xrightarrow{h^{n-1}(i)} h^{n-1}(A) \xrightarrow{\delta^n_{(X,A)}} h^n(X, A) \xrightarrow{h^n(j)} h^n(X) \xrightarrow{h^n(i)} h^n(A)$$

Therefore, since $h^n(i)$ is an isomorphism, $\operatorname{Ker} h^n(i)$ is trivial. Thence, once $\operatorname{Im} h^n(j) = \operatorname{Ker} h^n(i)$, we have

$$\operatorname{Ker} h^n(j) \ = \ h^n(X, A).$$

Correspondingly, since $h^{n-1}(i)$ is an isomorphism, it follows that

$$\operatorname{Im} h^{n-1}(i) \ = \ h^{n-1}(A).$$

Thus, once $\operatorname{Ker} \delta^n_{(X,A)} = \operatorname{Im} h^{n-1}(i)$, we have that $\operatorname{Im}\delta^n_{(X,A)}$ is trivial. Hence, $h^n(X, A)$ is trivial because $\operatorname{Im} \delta^n_{(X,A)} = \operatorname{Ker} h^n(j)$, as we wished. □

Corollary 1.12 (The generalized cohomology groups of a pair with equal components)**.** *In a generalized cohomology theory, if (X, X) is an admissible pair, then $h^n(X, X)$ is trivial for all $n \in \mathbb{Z}$.*

Proof. This result follows from Theorem 1.11 since the identity map $\mathrm{id}_X : X \to X$ induces $\mathrm{id}_{h^n(X)} : h^n(X) \to h^n(X)$ for all $n \in \mathbb{Z}$. □

Corollary 1.13 (The generalized cohomology groups of the pair composed of empty components)**.** *In a generalized cohomology theory, the generalized absolute cohomology group $h^n(0)$ is trivial for all $n \in \mathbb{Z}$.*

Proof. The assertion follows from Corollary 1.12 since the pair composed of empty components has equal components. □

1.3 Homomorphisms between generalized cohomology sequences

The main result of this section is that an admissible map of pairs induces a homomorphism of exact sequences between the generalized cohomology sequences of its domain and codomain. After that, we set a classical result from Homological Algebra to establish a condition under which generalized cohomology sequences are isomorphic. Finally, we discuss some changeable features in the framework of generalized cohomology theories with respect to Definition 1.9. We begin with the following remark that must be kept in mind.

Remark 1.14 (Maps defined by a map of pairs). *Let $\mathscr{C}$ be an admissible category of topological spaces. Every admissible map of pairs $f : (X, A) \to (Y, B)$ defines the maps*

$$f_1 : X \to Y \qquad \textit{and} \qquad f_2 : A \to B.$$

The reader can readily prove that f_1 and f_2 are admissible maps by writing them as compositions between f and convenient admissible inclusions. An important fact is that, although f_1 and f_2 are defined by the same formula than f, these three maps are different maps of pairs in general. ♢

Definition 1.15 (The generalized induced homomorphism between generalized cohomology sequences). *In a generalized cohomology theory, if $f : (X, A) \to (Y, B)$ is an admissible map of pairs, then the sequence of group homomorphisms*

$$h(f) := (\cdots,\ h^{n-1}(f_1),\ h^{n-1}(f_2),\ h^n(f),\ h^n(f_1),\ h^n(f_2),\ \cdots)$$

*is the **generalized induced homomorphism** of f between the generalized cohomology sequences of (Y, B) and (X, A).* ♢

Theorem 1.16 (The generalized induced homomorphism between generalized cohomology sequences is a homomorphism of exact sequences). *In a generalized cohomology theory, if $f : (X, A) \to (Y, B)$ is an admissible map of pairs, then $h(f)$*

is a homomorphism of exact sequences between the generalized cohomology sequences of (Y, B) and (X, A).

Proof. Let $i : A \to X$, $j : X \to (X, A)$, $i' : B \to Y$ and $j' : Y \to (Y, B)$ be inclusion maps. To verify the statement of the theorem we have to prove that the following diagram is commutative.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & h^n(Y,B) & \xrightarrow{h^n(j')} & h^n(Y) & \xrightarrow{h^n(i')} & h^n(B) & \xrightarrow{\delta^{n+1}_{(Y,B)}} & h^{n+1}(Y,B) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle h^n(f)} & & \Big\downarrow{\scriptstyle h^n(f_1)} & & \Big\downarrow{\scriptstyle h^n(f_2)} & & \Big\downarrow{\scriptstyle h^{n+1}(f)} & & \\
\cdots & \longrightarrow & h^n(X,A) & \xrightarrow[h^n(j)]{} & h^n(X) & \xrightarrow[h^n(i)]{} & h^n(A) & \xrightarrow[\delta^{n+1}_{(X,A)}]{} & h^{n+1}(X,A) & \longrightarrow & \cdots
\end{array}$$

In fact, we have $h^n(j) \circ h^n(f) = h^n(f_1) \circ h^n(j')$ and $h^n(i) \circ h^n(f_1) = h^n(f_2) \circ h^n(i')$ because of the functoriality of h^n since $f \circ j = j' \circ f_1$ and $f_1 \circ i = i' \circ f_2$. In turn, $\delta^{n+1}_{(X,A)} \circ h^n(f_2) = h^{n+1}(f) \circ \delta^{n+1}_{(Y,B)}$ because it is just a restatement of the Commutativity Axiom. $\square$

Lemma 1.17 (The Five Lemma)**.** *The following commutative diagram of abelian groups and homomorphisms has exact rows.*

$$\begin{array}{ccccccccc}
C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & C_4 & \longrightarrow & C_5 \\
\Big\downarrow{\scriptstyle \varphi_1} & & \Big\downarrow{\scriptstyle \varphi_2} & & \Big\downarrow{\scriptstyle \varphi_3} & & \Big\downarrow{\scriptstyle \varphi_4} & & \Big\downarrow{\scriptstyle \varphi_5} \\
D_1 & \longrightarrow & D_2 & \longrightarrow & D_3 & \longrightarrow & D_4 & \longrightarrow & D_5
\end{array}$$

If φ_1, φ_2, φ_4 and φ_5 are isomorphisms, then φ_3 is also an isomorphism.

Proof. The reader can find a proof of this result in [13, p. 16], which is where the lemma in question first appeared according to [37, p. 17]. This last reference is a good one to acquire some knowledge on the history and on the main problems of Homological Algebra. $\square$

Corollary 1.18 (Isomorphism of generalized cohomology sequences)**.** *In a generalized cohomology theory, if $f : (X, A) \to (Y, B)$ is an admissible map of pairs such that*

$h^n(f_1) : h^n(Y) \to h^n(X)$ and $h^n(f_2) : h^n(B) \to h^n(A)$ are isomorphisms for all $n \in \mathbb{Z}$, then

$$h^n(f) : h^n(Y, B) \to h^n(X, A)$$

is also an isomorphism for all $n \in \mathbb{Z}$. In particular, we have that $h(f)$ is an isomorphism of exact sequences between the generalized cohomology sequences of (Y, B) and (X, A).

Proof. Let n be an integer number and $i : A \to X$, $j : X \to (X, A)$, $i' : B \to Y$ and $j' : Y \to (Y, B)$ be inclusion maps. The following diagram is commutative due to Theorem 1.16.

$$\begin{tikzcd}
h^{n-1}(Y) \arrow[r, "h^{n-1}(i')"] \arrow[d, "h^{n-1}(f_1)"] & h^{n-1}(B) \arrow[r, "\delta^n_{(Y,B)}"] \arrow[d, "h^{n-1}(f_2)"] & h^n(Y, B) \arrow[r, "h^n(j')"] \arrow[d, "h^n(f)"] & h^n(Y) \arrow[r, "h^n(i')"] \arrow[d, "h^n(f_1)"] & h^n(B) \arrow[d, "h^n(f_2)"] \\
h^{n-1}(X) \arrow[r, "h^{n-1}(i)"'] & h^{n-1}(A) \arrow[r, "\delta^n_{(X,A)}"'] & h^n(X, A) \arrow[r, "h^n(j)"'] & h^n(X) \arrow[r, "h^n(i)"'] & h^n(A)
\end{tikzcd}$$

Moreover, the Exactness Axiom says that the preceding commutative diagram has exact rows. Therefore, since $h^{n-1}(f_1)$, $h^{n-1}(f_2)$, $h^n(f_1)$ and $h^n(f_2)$ are isomorphisms, it follows from the Five Lemma that $h^n(f) : h^n(Y, B) \to h^n(X, A)$ is also an isomorphism, as we wished. □

To close this section, as we said at the beginning, we discuss some changeable features in the framework of generalized cohomology theories with respect to Definition 1.9. This discussion may help the reader to perfect his or her understanding of the data involved in generalized cohomology theories. We begin with the following definition.

Definition 1.19 (The category of exact sequences of abelian groups)**.** *We define the* ***category of exact sequences of abelian groups****, and denote it by* SeqExactAb*, to be the one whose:*

- *objects are infinite exact sequences of abelian groups. That is, exact sequences of the form*

$$\cdots \longrightarrow C_{n-1} \xrightarrow{\varphi_n} C_n \xrightarrow{\varphi_{n+1}} C_{n+1} \longrightarrow \cdots,$$

where C_n is an abelian group and $\varphi_n : C_{n-1} \to C_n$ is a morphism of abelian groups for all $n \in \mathbb{Z}$; and

- *morphisms are homomorphisms of exact sequences of abelian groups. That is, sequences of morphisms of abelian groups $(\xi_n : C_n \to D_n)_{n\in\mathbb{Z}}$ such that the following diagram is commutative.*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n-1} \arrow[r] \arrow[d, "\xi_{n-1}"] & C_n \arrow[r] \arrow[d, "\xi_n"] & C_{n+1} \arrow[r] \arrow[d, "\xi_{n+1}"] & \cdots \\
\cdots \arrow[r] & D_{n-1} \arrow[r] & D_n \arrow[r] & D_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Note that the rows of the preceding diagram are tacitly assumed to be exact sequences of abelian groups. ◊

Remark 1.20 (The generalized cohomology functors and their domain and codomain categories). *To set a generalized cohomology theory, we considered a sequence $(h^n)_{n\in\mathbb{Z}}$ of contravariant functors from an admissible category $\mathscr{C}$ into the category of abelian groups $\mathscr{G}_{ab}$. Obviously, h^n sends:*

- *a pair (X, A) into the nth generalized cohomology group $h^n(X, A)$; and*
- *a map $f : (X, A) \to (Y, B)$ into the nth generalized induced homomorphism $h^n(f) : h^n(Y, B) \to h^n(X, A)$.*

Moreover, the Homotopy Axiom says that homotopic admissible maps have the same image through these functors. This allows a refinement of each h^n through the homotopy equivalence of maps. The refinement is the contravariant functor $[h]^n$ from $[\mathscr{C}]$ into $\mathscr{G}_{ab}$ that sends:

- *a pair (X, A) into the nth generalized cohomology group $h^n(X, A)$; and*
- *a class $[f : (X, A) \to (Y, B)]$ into the nth generalized induced homomorphism $h^n(f) : h^n(Y, B) \to h^n(X, A)$.*

Consequently, we could have defined a generalized cohomology theory requiring a sequence of contravariant functors from $[\mathscr{C}]$ *into* $\mathscr{G}_{ab}$ *instead of a sequence of contravariant functors from* $\mathscr{C}$ *into* $\mathscr{G}_{ab}$*. In this situation, the Homotopy Axiom would have been an obvious consequence of the mathematical structure in question. Furthermore, we could have considered a contravariant functor from* $[\mathscr{C}]$ *into the category of exact sequences of abelian groups* SeqExactAb. *More explicitly, this functor would send:*

- *a pair* (X, A) *into its generalized cohomology sequence; and*
- *a class* $[f : (X, A) \to (Y, B)]$ *into the generalized induced homomorphism of* f *between the generalized cohomology sequences of* (Y, B) *and* (X, A)*.*

In this situation, the Commutativity Axiom, the Exactness Axiom and the Homotopy Axiom would have been obvious consequences of the mathematical structures in question. The reader can easily combine these constructions to produce other ways of establishing the data and the axioms of Definition 1.9. All of these approaches are equivalent and then a matter of choice. $\diamondsuit$

1.4 Homeomorphic pairs and generalized cohomology groups and sequences

In this section, we prove that, in a generalized cohomology theory, the generalized cohomology groups and sequences are intrinsically the same for admissible homeomorphic pairs. The results that are established here can be seen as immediate consequences of the results involving homotopy equivalences that we shall present later. However, the importance of homeomorphisms is a sufficient reason to set this section independently. We begin with the following definition.

Definition 1.21 (Homeomorphism and homeomorphic pairs of topological spaces)**.** *In an admissible category of topological spaces, two admissible pairs* (X, A) *and* (Y, B) *are* ***homeomorphic*** *if there exist admissible maps* $f : (X, A) \to (Y, B)$ *and*

$g : (Y, B) \to (X, A)$ such that $g \circ f = \mathrm{id}_{(X,A)}$ and $f \circ g = \mathrm{id}_{(Y,B)}$. In this situation, we say that f and g are inverse ***homeomorphisms****.* $\diamondsuit$

Remark 1.22 (Homeomorphism of pairs is an equivalence relation on the class of objects of an admissible category)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces. The relation of homeomorphism of pairs on the class of objects of $\mathscr{C}$ is defined in the following manner: two admissible pairs of topological spaces are related if and only if there exists an admissible homeomorphism between them. The reader can readily prove that this is an equivalence relation.* $\diamondsuit$

Theorem 1.23 (Invariance of the cohomology groups under homeomorphisms of pairs)**.** *In a generalized cohomology theory, if (X, A) and (Y, B) are admissible homeomorphic pairs, then $h^n(X, A)$ is isomorphic to $h^n(Y, B)$ for all $n \in \mathbb{Z}$. In other words, admissible homeomorphic pairs have isomorphic generalized cohomology groups.*

Proof. Let n be an integer number and $f : (X, A) \to (Y, B)$ be a homeomorphism of pairs. The functorial properties of h^n imply

$$\begin{aligned} h^n(f) \circ h^n(f^{-1}) &= h^n(f^{-1} \circ f) = h^n \mathrm{id}_{(X,A)} = \mathrm{id}_{h^n(X,A)} \quad \text{and} \\ h^n(f^{-1}) \circ h^n(f) &= h^n(f \circ f^{-1}) = h^n \mathrm{id}_{(Y,B)} = \mathrm{id}_{h^n(Y,B)}. \end{aligned}$$

Hence, $h^n(f^{-1})$ is a group isomorphism from $h^n(X, A)$ onto $h^n(Y, B)$. Then, $h^n(X, A)$ is isomorphic to $h^n(Y, B)$, as we wished. $\square$

Corollary 1.24 (Isomorphism of the generalized cohomology sequences induced from an admissible homeomorphism of pairs)**.** *In a generalized cohomology theory, if (X, A) and (Y, B) are admissible homeomorphic pairs, then the generalized cohomology sequences of (X, A) and (Y, B) are isomorphic. In other words, admissible homeomorphic pairs have isomorphic generalized cohomology sequences.*

Proof. Let $f : (X, A) \to (Y, B)$ be a homeomorphism of pairs. It follows from the preceding result that $h(f)$ is an isomorphism between the generalized cohomology sequences of (Y, B) and (X, A), as we wished. $\square$

1.5 The reduced generalized cohomology groups and sequences

In this section, we present the reduced generalized cohomology groups and sequences. In addition, we study their relation to the generalized cohomology groups and sequences defined in the previous sections. These new mathematical objects are important tools to simplify various calculus in cohomology theory. We begin with the following definition.

Definition 1.25 (Collapsible topological spaces). *Let $\mathscr{C}$ be an admissible category of topological spaces and Ω be an admissible single point. Let X be an admissible topological space. If the only possible map $p_X : X \to \Omega$ is admissible, then X is said to be a* ***collapsible space****.* $\diamondsuit$

Remark 1.26 (Collapsibility of an admissible space is independent of the choice of the admissible single point). *Let $\mathscr{C}$ be an admissible category of topological spaces and X be an admissible topological space. If Ω and Γ are admissible single points, then X is collapsible with respect to Ω if and only if it is collapsible with respect to Γ. In fact, if X is collapsible with respect to Ω, then $p_X : X \to \Omega$ is admissible. Let $f : \Omega \to \Gamma$ be the only possible admissible map. It follows that $q_X = f \circ p_X : X \to \Gamma$ is admissible, and then that X is collapsible with respect to Γ. The converse can be proved in the exactly same way considering $f^{-1} : \Gamma \to \Omega$ instead of $f : \Omega \to \Gamma$. Therefore, we are allowed to say that an admissible topological space is collapsible without worrying about any specific choice of the admissible single point.* $\diamondsuit$

Remark 1.27 (A space can be collapsible in an admissible category of topological spaces but non-collapsible in another one). *In the first section of this chapter, we have seen that* Top_2*,* TopHd_2*,* $\mathrm{TopHdCCpt}_2$ *and* $\mathrm{TopHdLocCCptP}_2$ *are admissible categories of topological spaces. It is evident that every admissible space in* Top_2*, in* TopHd_2 *and in* $\mathrm{TopHdCCpt}_2$ *is collapsible. On the other hand, the collapsible spaces in* $\mathrm{TopHdLocCCptP}_2$ *are the compact ones since the preimage of a point by a proper map must be compact. Hence, for example, the Euclidean spaces are all collapsible in* Top_2 *and in* TopHd_2 *but* *non-collapsible in* TopHdLocCCptP$_2$. *It is to be noted that every space in* TopHdCCpt$_2$ *is collapsible in* TopHdLocCCptP$_2$. ♢

Theorem 1.28 (Categorical consequences of maps with collapsible codomains). *Let $\mathscr{C}$ be an admissible category of topological spaces and Ω be an admissible single point. Let X and Y be admissible spaces and $f : X \to Y$ be an admissible map. If Y is collapsible, then X is collapsible. Furthermore, if (X, A) is an admissible pair, and X is collapsible, then A is collapsible and $p_{(X,A)} : (X, A) \to (\Omega, \Omega)$ is admissible.*

Proof. The first statement is a trivial consequence of the third property of admissible categories. If (X, A) is admissible, the inclusion map $A \to X$ is admissible. If, in addition, X is collapsible, the composite map $A \to X \to \Omega$ is admissible, and then A is collapsible. Since $X \to \Omega$ is admissible, so is $(X, X) \to (\Omega, \Omega)$. Therefore, $(X, A) \to (X, X) \to (\Omega, \Omega)$ is also admissible. □

Definition 1.29 (The reduced generalized cohomology groups of an admissible collapsible space). *In a generalized cohomology theory, let Ω be an admissible single point and X be an admissible collapsible space. In this situation, for all $n \in \mathbb{Z}$:*

- *the homomorphic image of $h^n(\Omega)$ under $h^n(p_X) : h^n(\Omega) \to h^n(X)$ is denoted by $h^n(\Omega)_X$; and*
- *the quotient group of $h^n(X)$ by $h^n(\Omega)_X$ is said to be the* ***nth reduced generalized cohomology group*** *of X, and is denoted by $\widetilde{h}^n(X)$. In other words, $\widetilde{h}^n(X)$ is defined as the cokernel of $h^n(p_X) : h^n(\Omega) \to h^n(X)$.* ♢

Theorem 1.30 (The reduced generalized cohomology groups of a point). *Let $\mathscr{C}$ be an admissible category of topological spaces and Ω be an admissible single point. If Γ is another admissible single point, then $h^n(\Gamma) = h^n(\Omega)_\Gamma$ for all $n \in \mathbb{Z}$. In particular, $\widetilde{h}^n(\Gamma)$ is the trivial group for all $n \in \mathbb{Z}$.*

Proof. Let n be an integer number. The first claim is immediate since $p_\Gamma : \Gamma \to \Omega$ is a homeomorphism. Therefore, since $\widetilde{h}^n(\Gamma)$ is the quotient group of $h^n(\Gamma)$ by $h^n(\Omega)_\Gamma$ and $h^n(\Gamma) = h^n(\Omega)_\Gamma$, $\widetilde{h}^n(\Gamma)$ is the trivial group. □

Theorem 1.31 (A cohomological consequence of maps with collapsible codomains)**.** *In a generalized cohomology theory, if Ω is an admissible single point and $f : X \to Y$ is an admissible map such that Y is collapsible, then $h^n(f) : h^n(Y) \to h^n(X)$ maps $h^n(\Omega)_Y$ isomorphically onto $h^n(\Omega)_X$ for all $n \in \mathbb{Z}$.*

Proof. Since Y is a collapsible space, we have that $p_Y : Y \to \Omega$ is an admissible map; moreover, since $f : X \to Y$ is an admissible map, we have that X is also collapsible. Then, let n be an integer number and $i : \Omega \to Y$ be an admissible map. It is clear that $p_Y \circ i = \mathrm{id}_\Omega$. Consequently, we have that $h^n(p_Y) : h^n(\Omega) \to h^n(Y)$ is a monomorphism because

$$\mathrm{id}_{h^n(\Omega)} \;=\; h^n(p_Y \circ i) \;=\; h^n(i) \circ h^n(p_Y).$$

Similarly, $h^n(f) \circ h^n(p_Y) = h^n(p_Y \circ f) : h^n(X) \to h^n(\Omega)$ is a monomorphism because $p_Y \circ f \circ i = \mathrm{id}_\Omega$ implies

$$\mathrm{id}_{h^n(\Omega)} \;=\; h^n(p_Y \circ f \circ i) \;=\; h^n(i) \circ h^n(p_Y \circ f).$$

These two facts prove that $h^n(f) : h^n(Y) \to h^n(X)$ is an isomorphism from $\operatorname{Im} h^n(p_Y) = h^n(\Omega)_Y$ onto $\operatorname{Im} h^n(p_Y \circ f) = h^n(\Omega)_X$, as we wished. $\square$

Definition 1.32 (The reduced generalized induced homomorphisms of an admissible map with collapsible codomain)**.** *In a generalized cohomology theory, if $f : X \to Y$ is an admissible map such that Y is a collapsible space, then the homomorphism $\widetilde{h}^n(f) : \widetilde{h}^n(Y) \to \widetilde{h}^n(X)$ generated by $h^n(f)$ is called the* ***nth reduced generalized induced homomorphism****.* $\diamondsuit$

Corollary 1.33 (The kernel of the generalized induced homomorphism is isomorphic to the kernel of the reduced generalized induced homomorphism)**.** *In a generalized cohomology theory, if Ω is an admissible single point and $f : X \to Y$ is an admissible map such that Y is a collapsible space, then* $\operatorname{Ker} h^n(f)$ *is isomorphic to* $\operatorname{Ker} \widetilde{h}^n(f)$ *for all $n \in \mathbb{Z}$.*

Proof. Let n be an integer number. Since Y is a collapsible space, $p_Y : Y \to \Omega$ is admissible; moreover, since $f : X \to Y$ is an admissible map, we have that X is also collapsible. Therefore, $p_X : X \to \Omega$ is also admissible. Consequently, the reduced generalized cohomology groups $\widetilde{h}^n(X)$ and $\widetilde{h}^n(Y)$ are well-defined. Then, let $\pi_X^n : h^n(X) \to \widetilde{h}^n(X)$ and $\pi_Y^n : h^n(Y) \to \widetilde{h}^n(Y)$ be the natural quotient maps. The commutativity of the following diagram is a straightforward computation, that we leave to the reader.

$$\begin{tikzcd} h^n(Y) \arrow[r, "h^n(f)"] \arrow[d, "\pi_Y^n"'] & h^n(X) \arrow[d, "\pi_X^n"] \\ \widetilde{h}^n(Y) \arrow[r, "\widetilde{h}^n(f)"'] & \widetilde{h}^n(X) \end{tikzcd}$$

We claim that the restriction map $\pi_Y^n \mid_{\operatorname{Ker} h^n(f)}: \operatorname{Ker} h^n(f) \to \operatorname{Ker} \widetilde{h}^n(f)$ is a well-defined isomorphism. Indeed:

- π_Y^n *maps* $\operatorname{Ker} h^n(f)$ *into* $\operatorname{Ker} \widetilde{h}^n(f)$. This assertion is an immediate consequence of the commutativity of the preceding diagram. In fact, if $u \in \operatorname{Ker} h^n(f)$, then $\widetilde{h}^n(f)(\pi_Y^n(u)) = \pi_X^n(h^n(f)(u)) = \pi_X^n(0) = [0]$. Therefore, we have proved that $\pi_Y^n(u) \in \operatorname{Ker} \widetilde{h}^n(f)$;

- $\pi_Y^n \mid_{\operatorname{Ker} h^n(f)}$ *is injective.* Let $u \in \operatorname{Ker} h^n(f)$ be in such manner that $\pi_Y^n(u) = [0]$. This condition implies $u \in h^n(\Omega)_Y$. Therefore, $u \in \operatorname{Ker} h^n(f) \mid_{h^n(\Omega)_Y}$. Since $h^n(f) \mid_{h^n(\Omega)_Y}: h^n(\Omega)_Y \to h^n(\Omega)_X$ is an isomorphism by Theorem 1.31, we have $u = 0$. This proves the injectivity of $\pi_Y^n \mid_{\operatorname{Ker} h^n(f)}$; *and*

- $\pi_Y^n \mid_{\operatorname{Ker} h^n(f)}$ *is surjective.* Let $[u] \in \operatorname{Ker} \widetilde{h}^n(f)$. This assumption implies $[h^n(f)(u)] = [0]$. Thus, $h^n(f)(u) \in h^n(\Omega)_X$. Consequently, there exists $v \in h^n(\Omega)$ such that $h^n(f)(u) = h^n(p_X)(v)$. Since $p_X = p_Y \circ f$, it follows that $h^n(p_X)(v) = h^n(f)(h^n(p_Y)(v))$. Therefore, $h^n(f)(u) = h^n(f)(h^n(p_Y)(v))$. Then, $h^n(f)(u - h^n(p_Y)(v)) = 0$ implies $u - h^n(p_Y)(v) \in \operatorname{Ker} h^n(f)$. Furthermore, $\pi_Y^n(u - h^n(p_Y)(v)) = [u]$ because $h^n(p_Y)(v) \in h^n(\Omega)_Y$. This proves the surjectivity of $\pi_Y^n \mid_{\operatorname{Ker} h^n(f)}$. $\square$

Theorem 1.34 (The coboundary operator in reduced generalized cohomology)**.** *In a generalized cohomology theory, if Ω is an admissible single point and (X, A) is an* *admissible pair of topological spaces such that X is a collapsible space, then $h^{n-1}(\Omega)_A$ lies in the kernel of $\delta^n_{(X,A)} : h^{n-1}(A) \to h^n(X,A)$ for all $n \in \mathbb{Z}$. Therefore, this map induces the homomorphism $\widetilde{\delta}^n_{(X,A)} : \widetilde{h}^{n-1}(A) \to h^n(X,A)$ which is the* ***nth coboundary operator in the reduced generalized cohomology****.*

Proof. Let n be an integer number. Since X is collapsible, $p_X : X \to \Omega$ is admissible. Therefore, $p_A : A \to \Omega$ and $p_{(X,A)} : (X,A) \to (\Omega,\Omega)$ are admissible maps because of Theorem 1.28. Hence, the following diagram is not only well-defined but also commutative by the Commutativity Axiom.

$$\begin{tikzcd}
h^{n-1}(\Omega) \arrow[r, "h^{n-1}(p_A)"] \arrow[d, "\delta^n_{(\Omega,\Omega)}"'] & h^{n-1}(A) \arrow[d, "\delta^n_{(X,A)}"] \\
h^n(\Omega,\Omega) \arrow[r, "h^n p_{(X,A)}"'] & h^n(X,A)
\end{tikzcd}$$

Therefore, if $u \in h^{n-1}(\Omega)_A$, then there exists $v \in h^{n-1}(\Omega)$ such that $u = h^{n-1}(p_A)(v)$. Thus,

$$\delta^n_{(X,A)}(u) = \delta^n_{(X,A)}(h^{n-1}(p_A)(v)) = h^n p_{(X,A)}(\delta^n_{(\Omega,\Omega)}(v)) = h^n p_{(X,A)}(0) = 0$$

because $\delta^n_{(\Omega,\Omega)}(v) \in h^n(\Omega,\Omega)$ and $h^n(\Omega,\Omega)$ is the trivial group by Corollary 1.12. Hence, $h^{n-1}(\Omega)_A$ really lies in the kernel of $\delta^n_{(X,A)} : h^{n-1}(A) \to h^n(X,A)$. Consequently, it is defined the coboundary operator in the reduced generalized cohomology as the natural map $\widetilde{\delta}^n_{(X,A)} : \widetilde{h}^{n-1}(A) \to h^n(X,A)$ induced by $\delta^n_{(X,A)} : h^{n-1}(A) \to h^n(X,A)$ from $h^{n-1}(A)/h^{n-1}(\Omega)_A = \widetilde{h}^{n-1}(A)$ into $h^n(X,A)$. $\square$

In the next paragraphs, we use the results developed in this section to establish the reduced generalized cohomology sequence of an admissible pair whose first component is a collapsible space.

Definition 1.35 (The reduced generalized cohomology sequence of an admissible pair whose first component is a collapsible space)**.** *In a generalized cohomology theory, if (X,A) is an admissible pair such that X is a collapsible space, then the* ***reduced generalized cohomology sequence*** *of (X,A) is*

$$\cdots \longrightarrow \widetilde{h}^{n-1}(A) \xrightarrow{\widetilde{\delta}^{n}_{(X,A)}} h^n(X,A) \xrightarrow{\widetilde{h}^n(j)} \widetilde{h}^n(X) \xrightarrow{\widetilde{h}^n(i)} \widetilde{h}^n(A) \longrightarrow \cdots,$$

where $i : A \to X$ and $j : X \to (X,A)$ are inclusions, $\widetilde{\delta}^n_{(X,A)} : \widetilde{h}^{n-1}(A) \to h^n(X,A)$ is the nth coboundary operator in the reduced generalized cohomology, and the maps $\widetilde{h}^n(i) : \widetilde{h}^n(X) \to \widetilde{h}^n(A)$ and $\widetilde{h}^n(j) : h^n(X,A) \to \widetilde{h}^n(X)$ are the nth reduced generalized induced homomorphisms generated by, respectively, $h^n(i) : h^n(X) \to h^n(A)$ and $h^n(j) : h^n(X,A) \to h^n(X)$. ♢

Theorem 1.36 (Exactness of the reduced generalized cohomology sequence)**.** *In a generalized cohomology theory, if Ω is an admissible single point, (X,A) is an admissible pair such that X is a collapsible space and $i : (\Omega,\Omega) \to (X,A)$ is any admissible map, then the generalized cohomology sequence of (X,A) decomposes into the direct sum of two exact subsequences:*

(1) the kernel of $h(i)$; and

(2) the isomorphic image of the generalized cohomology sequence of (Ω,Ω) under $h\,p_{(X,A)}$.

Furthermore, the first subsequence is isomorphic to the reduced generalized cohomology sequence of (X,A) under factorization of the generalized cohomology sequence of (X,A) by the second subsequence. In particular, the reduced generalized cohomology sequence of (X,A) is exact.

Proof. This result follows from purely algebraic arguments. The reader can find a proof of it in [13, pp. 21-22]. □

Definition 1.37 (The reduced generalized induced homomorphism between reduced generalized cohomology sequences)**.** *In a generalized cohomology theory, if $f : (X,A) \to (Y,B)$ is an admissible map of pairs such that X and Y are collapsible spaces, then we say that the sequence of group homomorphisms*

$$\widetilde{h}(f) := (\cdots, \widetilde{h}^{n-1}(f_1), \widetilde{h}^{n-1}(f_2), h^n(f), \widetilde{h}^n(f_1), \widetilde{h}^n(f_2), \cdots)$$

*is the **reduced generalized induced homomorphism** of f between the reduced generalized cohomology sequences of (Y, B) and (X, A).* $\diamondsuit$

Theorem 1.38 (The reduced generalized induced homomorphism between reduced generalized cohomology sequences is a homomorphism of exact sequences)**.** *In a generalized cohomology theory, if $f : (X, A) \to (Y, B)$ is an admissible map of pairs such that X and Y are collapsible spaces, then $\widetilde{h}(f)$ is a homomorphism of exact sequences between the reduced generalized cohomology sequences of (Y, B) and (X, A).*

Proof. This result is an immediate consequence of Theorem 1.16. $\square$

To close this section, we establish another version of the reduced generalized cohomology groups, which will be useful later. In addition, we prove that the new version of the reduced generalized cohomology groups is isomorphic to the first one, although the isomorphism is not usually canonical. The absence of canonicity is what makes the new theory interesting. We leave to the reader the construction of the new reduced generalized cohomology sequence of an admissible pair because the details are analogous to the ones we have just seen.

Definition 1.39 (The pointed reduced generalized cohomology groups)**.** *In a generalized cohomology theory, let Ω be an admissible single point, X be an admissible space and $x \in X$ be such that the map $i_x : \Omega \to X$ defined by $i_x(\Omega) = x$ is an admissible map. Henceforth, for all $n \in \mathbb{Z}$, the kernel of $h^n(i_x) : h^n(X) \to h^n(\Omega)$ is called the **nth pointed reduced generalized cohomology group** of X, and is denoted by $\widetilde{h}^n_x(X)$. Furthermore, we shall write $u^n(x)$ instead of $h^n(i_x)(u) \in h^n(\Omega)$ for all $n \in \mathbb{Z}$ and for all $u \in h^n(X)$.* $\diamondsuit$

Theorem 1.40 (The pointed reduced generalized induced homomorphism)**.** *In a generalized cohomology theory, let Ω be an admissible single point and $f : X \to Y$ be an admissible map. Therefore, for all $n \in \mathbb{Z}$, if $x \in X$, $y = f(x)$ and $u \in h^n(Y)$, then $h^n(f \circ i_x)(u) = u^n(y)$. Thus, $h^n(f) : h^n(Y) \to h^n(X)$ maps $\widetilde{h}^n_y(Y)$ into $\widetilde{h}^n_x(X)$ and $\widetilde{h}^n_y(Y)$ contains* $\operatorname{Ker} h^n(f)$*. Hence, it is defined $\widetilde{h}^n_x(f) : \widetilde{h}^n_y(Y) \to \widetilde{h}^n_x(X)$ which is the **nth pointed reduced generalized induced homomorphism**.*

Proof. Let n be an integer number. The first claim follows from $f \circ i_x = i_y$. Moreover, this yields the commutativity of the following diagram since $h^n(f \circ i_x) = h^n(i_x) \circ h^n(f)$ and $u^n(y) = h^n(i_y)(u)$ for all $u \in h^n(Y)$.

$$\begin{tikzcd} h^n(Y) \arrow[r, "h^n(i_y)"] \arrow[d, "h^n(f)"'] & h^n(\Omega) \arrow[d, equal] \\ h^n(X) \arrow[r, "h^n(i_x)"'] & h^n(\Omega) \end{tikzcd}$$

Therefore:

- given $u \in \widetilde{h}^n_y(Y) = \operatorname{Ker} h^n(i_y)$, we have $h^n(i_x)(h^n(f)(u)) = h^n(i_y)(u) = 0$. Thus, $h^n(f)$ maps $\widetilde{h}^n_y(Y)$ into $\operatorname{Ker} h^n(i_x) = \widetilde{h}^n_x(X)$; *and*

- given $u \in \operatorname{Ker} h^n(f)$, we have $h^n(i_y)(u) = h^n(i_x)(h^n(f)(u)) = h^n(i_x)(0) = 0$. Thus, $\operatorname{Ker} h^n(f)$ is contained in $\operatorname{Ker} h^n(i_y) = \widetilde{h}^n_y(Y)$.

Consequently, it is indeed defined the pointed reduced generalized induced homomorphism $\widetilde{h}^n_x(f) : \widetilde{h}^n_y(Y) \to \widetilde{h}^n_x(X)$ as the restriction $h^n(f) \mid_{\widetilde{h}^n_y(Y)} : \widetilde{h}^n_y(Y) \to \widetilde{h}^n_x(X)$, as we wished. □

Theorem 1.41 (The connection between the generalized cohomology groups and the pointed reduced generalized cohomology groups of an admissible space)**.** *In a generalized cohomology theory, if Ω is an admissible single point and X is a collapsible space, then the unique admissible map $p_X : X \to \Omega$ induces an isomorphism from $h^n(\Omega)$ onto $h^n(\Omega)_X$ for all $n \in \mathbb{Z}$. Moreover, $h^n(X)$ decomposes as the direct sum $\widetilde{h}^n_x(X) \oplus h^n(\Omega)_X$ for all $n \in \mathbb{Z}$ and for all $x \in X$ such that $i_x : \Omega \to X$ is an admissible map.*

Proof. Let n be an integer number and $x \in X$ be such that the map $i_x : \Omega \to X$ given by $i_x(\Omega) = x$ is admissible. Since the composition $p_X \circ i_x$ is the identity map on Ω, the composition $h^n(i_x) \circ h^n(p_X)$ is the identity map on $h^n(\Omega)$. Therefore, $h^n(p_X)$ is a monomorphism. This proves that $p_X : X \to \Omega$ induces an isomorphism from $h^n(\Omega)$ onto $\operatorname{Im} h^n(p_X) = h^n(\Omega)_X$. On the other hand, $h^n(X)$ decomposes as $\widetilde{h}^n_x(X) \oplus h^n(\Omega)_X$ because:

- $\widetilde{h}^n_x(X) = \operatorname{Ker} h^n(i_x)$ and $h^n(\Omega)_X = \operatorname{Im} h^n(p_X)$ have only the zero element in their intersection. In fact, if $u \in \operatorname{Ker} h^n(i_x) \cap \operatorname{Im} h^n(p_X)$, there exists $v \in h^n(\Omega)$ such that $h^n(p_X)(v) = u$. Then,

$$v \;=\; \mathrm{id}_{h^n(\Omega)}(v) \;=\; (h^n(i_x) \circ h^n(p_X))(v) \;=\; h^n(i_x)(u) \;=\; 0$$

implies $u = h^n(p_X)(v) = h^n(p_X)(0) = 0$; *and*

- every $u \in h^n(X)$ is the sum of an element from $\widetilde{h}^n_x(X)$ with an element from $h^n(\Omega)_X$. In fact, let $v = u - w$ where $w = (h^n(p_X) \circ h^n(i_x))(u)$. It is evident that $w \in h^n(\Omega)_X = \operatorname{Im} h^n(p_X)$. In turn,

$$h^n(i_x)(w) \;=\; (h^n(i_x) \circ h^n(p_X) \circ h^n(i_x))(u) \;=\; h^n(i_x)(u)$$

implies $h^n(i_x)(v) = h^n(i_x)(u - w) = h^n(i_x)(u) - h^n(i_x)(w) = 0$. Therefore, we have $v \in \operatorname{Ker} h^n(i_x) = \widetilde{h}^n_x(X)$. The statement is proved because $u = v + w$, as we wished. □

Corollary 1.42 (The pointed reduced generalized cohomology groups and the reduced generalized cohomology groups are always isomorphic)**.** *In a generalized cohomology theory, if Ω is an admissible single point, X is an admissible collapsible space and $i : \Omega \to X$ is any fixed admissible map, then $\widetilde{h}^n(X)$ is isomorphic to $\widetilde{h}^n_{i(\Omega)}(X)$ for all $n \in \mathbb{Z}$.*

Proof. Let n be an integer number and $p_X : X \to \Omega$ be the only possible map that is admissible because X is supposed to be a collapsible space. The natural quotient map $\pi^n_X : h^n(X) \to \widetilde{h}^n(X)$ is also defined. We know that the following sequence is a short exact sequence since $h^n(p_x)$ is injective, $\operatorname{Im} h^n(p_X) = h^n(\Omega)_X = \operatorname{Ker} \pi^n_X$, and π^n_X is trivially surjective.

$$0 \longrightarrow h^n(\Omega) \xrightarrow{h^n(p_X)} h^n(X) \xrightarrow{\pi^n_X} \widetilde{h}^n(X) \longrightarrow 0$$

To show that $h^n(p_x)$ is injective, we use the admissible map $i : \Omega \to X$. Indeed, once $p_X \circ i = \mathrm{id}_\Omega$ implies $\mathrm{id}_{h^n(\Omega)} = h^n(p_X \circ i) = h^n(i) \circ h^n(p_X)$, the assertion is proved. However, this equation also proves that the preceding short exact sequence is a split short exact sequence. As a consequence, there exists an isomorphism between $h^n(X)$ and

$h^n(\Omega) \oplus \widetilde{h}^n(X)$. In turn, this direct sum is isomorphic to $h^n(\Omega)_X \oplus \widetilde{h}^n(X)$ since $h^n(p_X)$ is an isomorphism from $h^n(\Omega)$ onto $h^n(\Omega)_X$. Thus, $h^n(\Omega)_X \oplus \widetilde{h}^n(X)$ is isomorphic to $h^n(\Omega)_X \oplus \widetilde{h}^n_{i(\Omega)}(X)$ by Theorem 1.41. Actually, the isomorphisms considered here can be chosen to produce an isomorphism

$$\varphi : h^n(\Omega)_X \oplus \widetilde{h}^n(X) \to h^n(\Omega)_X \oplus \widetilde{h}^n_{i(\Omega)}(X)$$

such that $\varphi(u, 0) = (u, 0)$ for all $u \in h^n(\Omega)_X$. This fact implies that $\widetilde{h}^n(X)$ is isomorphic to $\widetilde{h}^n_{i(\Omega)}(X)$. Indeed, let $\pi_{\widetilde{h}^n_{i(\Omega)}(X)} : h^n(\Omega)_X \oplus \widetilde{h}^n_{i(\Omega)}(X) \to \widetilde{h}^n_{i(\Omega)}(X)$ be the projection onto the second variable. The composition $\pi_{\widetilde{h}^n_{i(\Omega)}(X)} \circ \varphi : h^n(\Omega)_X \oplus \widetilde{h}^n(X) \to \widetilde{h}^n_{i(\Omega)}(X)$ is such that $\operatorname{Ker}(\pi_{\widetilde{h}^n_{i(\Omega)}(X)} \circ \varphi) = h^n(\Omega)_X$. Therefore, the *First Isomorphism Theorem* shows that $\widetilde{h}^n_{i(\Omega)}(X)$ is isomorphic to the quotient group of $h^n(\Omega)_X \oplus \widetilde{h}^n(X)$ by $h^n(\Omega)_X$. Since this very same theorem also implies that this quotient group is isomorphic to $\widetilde{h}^n(X)$, we are done here. □

Remark 1.43 (The isomorphisms between the pointed reduced generalized cohomology groups and the reduced generalized cohomology groups)**.** *In a generalized cohomology theory, the existence of an isomorphism between $\widetilde{h}^n(X)$ and $\widetilde{h}^n_{i(\Omega)}(X)$ for each fixed admissible map $i : \Omega \to X$ allows us to see the reduced generalized cohomology of X as the kernel of $h^n(i) : h^n(X) \to h^n(\Omega)$. This characterization is usually more tractable than the first one because kernels are much more concrete objects than cokernels. Another important fact is that the absence of canonicity for an isomorphism between $\widetilde{h}^n(X)$ and $\widetilde{h}^n_{i(\Omega)}(X)$ is related to the choice of $i(\Omega) \in X$. Indeed, once we started considering any admissible map $i : \Omega \to X$, there are many choices for the image $i(\Omega)$; for each admissible choice, there is an isomorphism between the groups under consideration.* ♢

1.6 Homotopy and generalized cohomology groups and sequences

In this section, we prove that, in a generalized cohomology theory, the generalized cohomology groups and sequences are intrinsically the same for admissible homotopic pairs. We also prove here an important fact involving contractible topological spaces, which in many senses are seen to be the simplest topological spaces. We begin with the following definition.

Definition 1.44 (Homotopy equivalence of pairs)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces. Admissible pairs (X, A) and (Y, B) are said to be* ***homotopically equivalent*** *if there exist two maps $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ such that the composition $g \circ f$ is homotopic to the identity map on (X, A) and the composition $f \circ g$ is homotopic to the identity map on (Y, B). The maps f and g are said to be a* ***homotopy equivalence****. Frequently, each of the maps f and g will be referred to as a homotopy equivalence.* $\diamondsuit$

Remark 1.45 (Homotopy of pairs is an equivalence relation on the class of objects of an admissible category)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces. The relation of homotopy of pairs on the class of objects of $\mathscr{C}$ is defined in the following manner: two admissible pairs of topological spaces are related if and only if there exists a homotopy equivalence between them. Once again, the reader can readily prove that this is an equivalence relation.* $\diamondsuit$

Remark 1.46 (Homeomorphisms and homotopy equivalences)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces. Any two homeomorphic pairs are homotopically equivalent. This happens because the identity map on any admissible pair is homotopic to itself. On the contrary, homotopically equivalent pairs are not necessarily homeomorphic pairs. In fact, for example, non-trivial Euclidean spaces are homotopically equivalent to a one-point space in* Top_2. *However, there is no homeomorphism between these spaces since there is no bijection between their sets of points. This shows that the notion of homotopy equivalence generalizes the one of homeomorphism.* $\diamondsuit$

Theorem 1.47 (Invariance of the generalized cohomology groups under homotopy equivalence of pairs)**.** *In a generalized cohomology theory, if (X, A) and (Y, B) are admissible homotopically equivalent pairs, then $h^n(X, A)$ is isomorphic to $h^n(Y, B)$ for all $n \in \mathbb{Z}$. In other words, admissible homotopically equivalent pairs have isomorphic generalized cohomology groups.*

Proof. Let n be an integer number and $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ be an admissible homotopy equivalence. Since $g \circ f$ is homotopic to $\mathrm{id}_{(X,A)}$, it follows from the Homotopy Axiom that

$$h^n(f) \circ h^n(g) \;=\; h^n(g \circ f) \;=\; h^n\mathrm{id}_{(X,A)} \;=\; \mathrm{id}_{h^n(X,A)}.$$

Similarly, once $f \circ g$ is homotopic to $\mathrm{id}_{(Y,B)}$, it follows $h^n(g) \circ h^n(f) = \mathrm{id}_{h^n(Y,B)}$. Therefore, $h^n(f)$ and $h^n(g)$ are inverse isomorphisms between $h^n(X, A)$ and $h^n(Y, B)$, proving what we wished. □

Corollary 1.48 (An isomorphism of generalized cohomology groups induced by almost homotopically equivalent admissible pairs)**.** *In a generalized cohomology theory, if $f : (X, A) \to (Y, B)$ is an admissible map of pairs in such manner that $f_1 : X \to Y$ and $f_2 : A \to B$ are homotopy equivalences, then $h^n(X, A)$ is isomorphic to $h^n(Y, B)$ for all $n \in \mathbb{Z}$.*

Proof. It follows from the proof of Theorem 1.47 that $h^n(f_1) : h^n(Y) \to h^n(X)$ and $h^n(f_2) : h^n(B) \to h^n(A)$ are isomorphisms for all $n \in \mathbb{Z}$. It is then conspicuous from Theorem 1.18 that $h^n(f) : h^n(Y, B) \to h^n(X, A)$ is an isomorphism for all $n \in \mathbb{Z}$, proving what we wished. □

Corollary 1.49 (Isomorphism of the generalized cohomology sequences induced by an admissible homotopy equivalence)**.** *In a generalized cohomology theory, if (X, A) and (Y, B) are admissible homotopically equivalent pairs, then their generalized cohomology sequences are isomorphic. In other words, admissible homotopically equivalent pairs have isomorphic generalized cohomology sequences. The same claim holds for reduced generalized cohomology sequences.*

Proof. Let $f : (X, A) \to (Y, B)$ be a homotopy equivalence. It is a consequence of Theorem 1.16 and a consequence of the proof of Theorem 1.47 that $h(f)$ is an isomorphism of exact sequences between the generalized cohomology sequences of (Y, B) and (X, A). We leave to the reader the proof of the statement for reduced generalized cohomology sequences. □

To close this section, we present the notion of contractible topological spaces and an immediate consequence of this idea from the viewpoint of generalized cohomology groups. Usually, some authors take this consequence to be the definition of contractible spaces.

Definition 1.50 (Contractible spaces). *Let $\mathscr{C}$ be an admissible category of topological spaces. An admissible space is a* ***contractible space*** *if there is a homotopy between its identity map and a constant map.* ♢

Theorem 1.51 (Contractibility and homotopy equivalences). *In an admissible category of topological spaces, if X is a contractible space, then X is homotopically equivalent to any of its points. Therefore, in a generalized cohomology theory, Theorem 1.47 implies that $h^n(X)$ is isomorphic to $h^n(x)$ for all $x \in X$.*

Proof. Since X is a contractible space, we know that there exist $x_0 \in X$ and an admissible homotopy $\Theta : X \times \mathbb{I} \to X$ in such manner that $\Theta(x, 0) = \mathrm{id}_X(x)$ and $\Theta(x, 1) = x_0$ for all $x \in X$. Then, let $f : \{x_0\} \to X$ be the admissible map defined by $f(x_0) = x_0$. We have that:

- $\Theta(\cdot, 1) \circ f$ is the identity map on $\{x_0\}$; *and*
- Θ is a homotopy connecting $f \circ \Theta(\cdot, 1)$ to the identity map on X.

Thus, we have that $\Theta(\cdot, 1)$ and f form a homotopy equivalence. Hence, it follows that X is homotopically equivalent to $\{x_0\}$. Consequently, since any two points of X are homotopically equivalent (in fact, they are homeomorphic to each other) and homotopy of pairs is an equivalence relation, this yields that X is homotopically equivalent to any of its points, as we wished. □

1.7 The generalized cohomology sequence of a triple

In this section, we set another helpful tool in the calculus of generalized cohomology groups, namely, the generalized cohomology sequence of a triple. The main theorem here, whose proof is a long technical computation that we provide to the reader in detail, establishes the exactness of this sequence. We begin with the following definition.

Definition 1.52 (Admissible triples and their maps)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces. We say that:*

- *(X, A, B) is an **admissible triple of spaces**, where X, A and B are admissible spaces such that $B \subseteq A \subseteq X$ and A and B are equipped with the induced topology, if the inclusion maps $i^{(A,B)}_{(X,B)} : (A, B) \to (X, B)$ and $j^{(X,B)}_{(X,A)} : (X, B) \to (X, A)$ are admissible; and*
- *$f : (X, A, B) \to (Y, C, D)$ is an **admissible map of triples**, where (X, A, B) and (Y, C, D) are admissible triples, if it is a continuous map $f : X \to Y$ such that $f(A) \subseteq C$, $f(B) \subseteq D$ and the maps $f_1 : (X, A) \to (Y, C)$, $f_2 : (X, B) \to (Y, D)$ and $f_3 : (A, B) \to (C, D)$ are admissible.* $\diamondsuit$

Notation 1.53 (The inclusion maps associated with an admissible triple of spaces)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces and (X, A, B) be an admissible triple of spaces. The inclusion maps associated with the pairs (X, A), (X, B) and (A, B) are denoted by $i^A_X : A \to X$, $i^B_X : B \to X$, $i^B_A : B \to A$, $j^X_{(X,A)} : X \to (X, A)$, $j^X_{(X,B)} : X \to (X, B)$ and $j^A_{(A,B)} : A \to (A, B)$. Moreover, we will not put these maps between parentheses when we take their induced homomorphisms to avoid overloading the notation.* $\diamondsuit$

Definition 1.54 (The generalized cohomology sequence of a triple)**.** *In a generalized cohomology theory, we define the **generalized cohomology sequence of the triple** (X, A, B) as the sequence*

$$\cdots \longrightarrow h^{n-1}(A, B) \xrightarrow{\delta^n_{(X,A,B)}} h^n(X, A) \xrightarrow{h^n j^{(X,B)}_{(X,A)}} h^n(X, B) \xrightarrow{h^n i^{(A,B)}_{(X,B)}} h^n(A, B) \longrightarrow \cdots,$$

*where $\delta^n_{(X,A,B)} : h^{n-1}(A, B) \to h^n(X, A)$, which is called the **nth generalized coboundary operator of the triple** (X, A, B), is the composition between $h^{n-1} j^A_{(A,B)} : h^{n-1}(A, B) \to h^{n-1}(A)$ and $\delta^n_{(X,A)} : h^{n-1}(A) \to h^n(X, A)$.* $\diamondsuit$

Theorem 1.55 (The generalized cohomology sequence of a triple is exact)**.** *In a generalized cohomology theory, if (X, A, B) is an admissible triple of spaces, then its generalized cohomology sequence is exact.*

Proof. Let n be an integer number. We have that each square in the following diagram is commutative.

$$\begin{tikzcd}
h^{n-1}(X,B) \arrow[r, "h^{n-1}i^{(A,B)}_{(X,B)}"] \arrow[d, "h^{n-1}j^X_{(X,B)}"'] & h^{n-1}(A,B) \arrow[d, "h^{n-1}j^A_{(A,B)}"] & & \\
h^{n-1}(X) \arrow[r, "h^{n-1}i^A_X"'] & h^{n-1}(A) \arrow[r, "h^{n-1}i^B_A"] \arrow[d, "\delta^n_{(X,A)}"'] & h^{n-1}(B) \arrow[d, "\delta^n_{(X,B)}"] & \\
& h^n(X,A) \arrow[r, "h^n j^{(X,B)}_{(X,A)}"'] & h^n(X,B) \arrow[r, "h^n i^{(A,B)}_{(X,B)}"] \arrow[d, "h^n j^X_{(X,B)}"'] & h^n(A,B) \arrow[d, "h^n j^A_{(A,B)}"] \\
& & h^n(X) \arrow[r, "h^n i^A_X"'] & h^n(A)
\end{tikzcd}$$

Indeed:

- $h^{n-1}i^A_X \circ h^{n-1}j^X_{(X,B)} = h^{n-1}j^A_{(A,B)} \circ h^{n-1}i^{(A,B)}_{(X,B)}$ because we have the equality of inclusions $j^X_{(X,B)} \circ i^A_X = i^{(A,B)}_{(X,B)} \circ j^A_{(A,B)}$;
- $h^n j^{(X,B)}_{(X,A)} \circ \delta^n_{(X,A)} = \delta^n_{(X,B)} \circ h^{n-1}i^B_A$ because it is a restatement of the Commutativity Axiom since i^B_A is the restriction of $j^{(X,B)}_{(X,A)}$ to B; *and*
- $h^n i^A_X \circ h^n j^X_{(X,B)} = h^n j^A_{(A,B)} \circ h^n i^{(A,B)}_{(X,B)}$ because we have the equality of inclusions $j^X_{(X,B)} \circ i^A_X = i^{(A,B)}_{(X,B)} \circ j^A_{(A,B)}$.

Moreover:

(a). $h^n j^X_{(X,A)} = h^n j^X_{(X,B)} \circ h^n j^{(X,B)}_{(X,A)}$ because we have the equality of inclusions $j^X_{(X,A)} = j^{(X,B)}_{(X,A)} \circ j^X_{(X,B)}$;

(b). $h^n i_X^B = h^n i_A^B \circ h^n i_X^A$ because we have the equality of inclusion maps $i_X^B = i_X^A \circ i_A^B$;

(c). $\delta^n_{(A,B)} = h^n i^{(A,B)}_{(X,B)} \circ \delta^n_{(X,B)}$ because it is a restatement of the Commutativity Axiom since it is equivalent to $\delta^n_{(A,B)} \circ \mathrm{id}_{h^{n-1}(B)} = h^n i^{(A,B)}_{(X,B)} \circ \delta^n_{(X,B)}$ and id_B is the restriction of $i^{(A,B)}_{(X,B)}$ to B; *and*

(d). $\delta^n_{(X,A,B)} = \delta^n_{(X,A)} \circ h^{n-1} j^A_{(A,B)}$ because of the definition of $\delta^n_{(X,A,B)}$.

The preceding relations are used to prove the following six assertions, which complete the proof of this theorem.

(1) $h^n i^{(A,B)}_{(X,B)} \circ h^n j^{(X,B)}_{(X,A)} : h^n(X,A) \to h^n(A,B)$ *is the trivial homomorphism. Thus,* $\operatorname{Im} h^n j^{(X,B)}_{(X,A)} \subseteq \operatorname{Ker} h^n i^{(A,B)}_{(X,B)}$.

Note that $j^{(X,B)}_{(X,A)} \circ i^{(A,B)}_{(X,B)} : (A,B) \to (X,A)$ can be expressed as the composition of the inclusion maps $k : (A,B) \to (A,A)$ and $l : (A,A) \to (X,A)$. Since $h^n(A,A) = 0$ by Corollary 1.12, $h^n(l) : h^n(X,A) \to h^n(A,A)$ is the trivial homomorphism. Therefore,

$$h^n i^{(A,B)}_{(X,B)} \circ h^n j^{(X,B)}_{(X,A)} = h^n(k) \circ h^n(l) = 0,$$

as we wished.

(2) *If* $u \in h^n(X,B)$ *and* $h^n i^{(A,B)}_{(X,B)}(u) = 0$*, which is the same as* $u \in \operatorname{Ker} h^n i^{(A,B)}_{(X,B)}$*, then there exists* $u' \in h^n(X,A)$ *such that* $h^n j^{(X,B)}_{(X,A)}(u') = u$*, which is the same as* $u \in \operatorname{Im} h^n j^{(X,B)}_{(X,A)}$*. Thus,* $\operatorname{Ker} h^n i^{(A,B)}_{(X,B)} \subseteq \operatorname{Im} h^n j^{(X,B)}_{(X,A)}$.

Since $h^n i^A_X h^n j^X_{(X,B)}(u) = h^n j^A_{(A,B)} h^n i^{(A,B)}_{(X,B)}(u) = h^n j^A_{(A,B)}(0) = 0$ because of the upper commutative square in the preceding diagram, $h^n j^X_{(X,B)}(u) \in \operatorname{Ker} h^n i^A_X$. Then, the exactness of the generalized cohomology sequence of (X,A) implies that there exists $\alpha \in h^n(X,A)$ such that $h^n j^X_{(X,A)}(\alpha) = h^n j^X_{(X,B)}(u)$. Therefore, Item (a) yields

$$\begin{aligned} h^n j^X_{(X,B)}\left(u - h^n j^{(X,B)}_{(X,A)}(\alpha)\right) &= h^n j^X_{(X,B)}(u) - h^n j^X_{(X,B)} h^n j^{(X,B)}_{(X,A)}(\alpha) \\ &= h^n j^X_{(X,B)}(u) - h^n j^X_{(X,A)}(\alpha) \\ &= 0. \end{aligned}$$

In other words, we have $u - h^n j_{(X,A)}^{(X,B)}(\alpha) \in \operatorname{Ker} h^n j_{(X,B)}^{X}$. Consequently, we know that there exists $\beta \in h^{n-1}(B)$ such that $\delta_{(X,B)}^n(\beta) = u - h^n j_{(X,A)}^{(X,B)}(\alpha)$ because of the exactness of the generalized cohomology sequence of (X, B). Thus, Item (c) and the fact that $h^n i_{(X,B)}^{(A,B)} \circ h^n j_{(X,A)}^{(X,B)} : h^n(X, A) \to h^n(A, B)$ is the trivial homomorphism imply

$$\begin{aligned}
\delta_{(A,B)}^n(\beta) &= h^n i_{(X,B)}^{(A,B)} \delta_{(X,B)}^n(\beta) \\
&= h^n i_{(X,B)}^{(A,B)} \left(u - h^n j_{(X,A)}^{(X,B)}(\alpha)\right) \\
&= h^n i_{(X,B)}^{(A,B)}(u) - h^n i_{(X,B)}^{(A,B)} h^n j_{(X,A)}^{(X,B)}(\alpha) \\
&= 0.
\end{aligned}$$

Otherwise stated, we have $\beta \in \operatorname{Ker} \delta_{(A,B)}^n$. For this reason and the exactness of the generalized cohomology sequence of (A, B), there exists $\gamma \in h^{n-1}(A)$ such that $h^{n-1} i_A^B(\gamma) = \beta$. Then, it is a consequence of the middle commutative square in the preceding diagram that

$$\begin{aligned}
h^n j_{(X,A)}^{(X,B)}\left(\alpha + \delta_{(X,A)}^n(\gamma)\right) &= h^n j_{(X,A)}^{(X,B)}(\alpha) + h^n j_{(X,A)}^{(X,B)} \delta_{(X,A)}^n(\gamma) \\
&= h^n j_{(X,A)}^{(X,B)}(\alpha) + \delta_{(X,B)}^n h^{n-1} i_A^B(\gamma) \\
&= h^n j_{(X,A)}^{(X,B)}(\alpha) + \delta_{(X,B)}^n(\beta) \\
&= h^n j_{(X,A)}^{(X,B)}(\alpha) + u - h^n j_{(X,A)}^{(X,B)}(\alpha) \\
&= u.
\end{aligned}$$

Thereby, it is proved that there exists $u' \in h^n(X, A)$ such that $h^n j_{(X,A)}^{(X,B)}(u') = u$, as we wished.

(3) $\delta_{(X,A,B)}^n \circ h^{n-1} i_{(X,B)}^{(A,B)} : h^{n-1}(X, B) \to h^n(X, A)$ *is the trivial homomorphism. Thus,* $\operatorname{Im} h^{n-1} i_{(X,B)}^{(A,B)} \subseteq \operatorname{Ker} \delta_{(X,A,B)}^n$.

Because of Item (d) and the upper commutative square in the preceding diagram,

$$\delta_{(X,A,B)}^n \circ h^{n-1} i_{(X,B)}^{(A,B)} = \delta_{(X,A)}^n \circ h^{n-1} j_{(A,B)}^{A} \circ h^{n-1} i_{(X,B)}^{(A,B)} = \delta_{(X,A)}^n \circ h^{n-1} i_X^A \circ h^{n-1} j_{(X,B)}^{X}.$$

Since the generalized cohomology sequence of (X, A) is exact, $\delta_{(X,A)}^n \circ h^{n-1} i_X^A = 0$. Then, $\delta_{(X,A,B)}^n \circ h^{n-1} i_{(X,B)}^{(A,B)} = 0$, as we wished.

(4) *If* $v \in h^{n-1}(A,B)$ *and* $\delta^n_{(X,A,B)}(v) = 0$*, which is the same as* $v \in \operatorname{Ker}\delta^n_{(X,A,B)}$*, then there exists* $v' \in h^{n-1}(X,B)$ *such that* $h^{n-1}i^{(A,B)}_{(X,B)}(v') = v$*, which is the same as* $v \in \operatorname{Im} h^{n-1}i^{(A,B)}_{(X,B)}$*. Thus,* $\operatorname{Ker}\delta^n_{(X,A,B)} \subseteq \operatorname{Im} h^{n-1}i^{(A,B)}_{(X,B)}$.

Since $\delta^n_{(X,A)}h^{n-1}j^A_{(A,B)}(v) = \delta^n_{(X,A,B)}(v) = 0$ because of Item (d), it follows $h^{n-1}j^A_{(A,B)}(v) \in \operatorname{Ker}\delta^n_{(X,A)}$. Then, $h^{n-1}i^A_X(\alpha) = h^{n-1}j^A_{(A,B)}(v)$ for some $\alpha \in h^{n-1}(X)$ because of the exactness of the generalized cohomology sequence of (X,A). Thus, since the exactness of the generalized cohomology sequence of (A,B) implies $h^{n-1}i^B_A h^{n-1}j^A_{(A,B)} = 0$, Item (b) yields

$$h^{n-1}i^B_X(\alpha) = h^{n-1}i^B_A h^{n-1}i^A_X(\alpha) = h^{n-1}i^B_A h^{n-1}j^A_{(A,B)}(v) = 0.$$

In other words, we have $\alpha \in \operatorname{Ker} h^{n-1}i^B_X$. Then, the exactness of the generalized cohomology sequence of (X,B) implies that there exists $\beta \in h^{n-1}(X,B)$ such that $h^{n-1}j^X_{(X,B)}(\beta) = \alpha$. As a consequence of the lower commutative square in the preceding diagram,

$$\begin{aligned} h^{n-1}j^A_{(A,B)}\left(v - h^{n-1}i^{(A,B)}_{(X,B)}(\beta)\right) &= h^{n-1}j^A_{(A,B)}(v) - h^{n-1}j^A_{(A,B)}h^{n-1}i^{(A,B)}_{(X,B)}(\beta) \\ &= h^{n-1}j^A_{(A,B)}(v) - h^{n-1}i^A_X h^{n-1}j^X_{(X,B)}(\beta) \\ &= h^{n-1}j^A_{(A,B)}(v) - h^{n-1}i^A_X(\alpha) \\ &= 0. \end{aligned}$$

Stated differently, we have $v - h^{n-1}i^{(A,B)}_{(X,B)}(\beta) \in \operatorname{Ker} h^{n-1}j^A_{(A,B)}$. Hence, the exactness of the generalized cohomology sequence of (A,B) yields the existence of $\gamma \in h^{n-2}(B)$ such that $\delta^{n-1}_{(A,B)}(\gamma) = v - h^{n-1}i^{(A,B)}_{(X,B)}(\beta)$. Then, it is a consequence of Item (c) that

$$\begin{aligned} h^{n-1}i^{(A,B)}_{(X,B)}\left(\beta + \delta^{n-1}_{(X,B)}(\gamma)\right) &= h^{n-1}i^{(A,B)}_{(X,B)}(\beta) + h^{n-1}i^{(A,B)}_{(X,B)}\delta^{n-1}_{(X,B)}(\gamma) \\ &= h^{n-1}i^{(A,B)}_{(X,B)}(\beta) + \delta^{n-1}_{(A,B)}(\gamma) \\ &= h^{n-1}i^{(A,B)}_{(X,B)}(\beta) + v - h^{n-1}i^{(A,B)}_{(X,B)}(\beta) \\ &= v. \end{aligned}$$

Thereby, it is proved that there exists $v' \in h^{n-1}(X,B)$ such that $h^{n-1}i^{(A,B)}_{(X,B)}(v') = v$, as we wished.

(5) $h^n j_{(X,A)}^{(X,B)} \circ \delta_{(X,A,B)}^n : h^{n-1}(A,B) \to h^n(X,B)$ *is the trivial homomorphism. Thus,* $\operatorname{Im} \delta_{(X,A,B)}^n \subseteq \operatorname{Ker} h^n j_{(X,A)}^{(X,B)}$.

Because of Item (d) and the middle commutative square in the preceding diagram,

$$h^n j_{(X,A)}^{(X,B)} \circ \delta_{(X,A,B)}^n = h^n j_{(X,A)}^{(X,B)} \circ \delta_{(X,A)}^n \circ h^{n-1} j_{(A,B)}^A = \delta_{(X,B)}^n \circ h^{n-1} i_A^B \circ h^{n-1} j_{(A,B)}^A.$$

Since the generalized cohomology sequence of (A,B) is exact, $h^n i_A^B \circ h^n j_{(A,B)}^A = 0$. Then, $h^n j_{(X,A)}^{(X,B)} \circ \delta_{(X,A,B)}^n = 0$, as we wished.

(6) *If* $w \in h^n(X,A)$ *and* $h^n j_{(X,A)}^{(X,B)}(w) = 0$*, which is the same as* $w \in \operatorname{Ker} h^n j_{(X,A)}^{(X,B)}$*, then there exists* $w' \in h^{n-1}(A,B)$ *such that* $\delta_{(X,A,B)}^n(w') = w$*, which is the same as* $w \in \operatorname{Im} \delta_{(X,A,B)}^n$*. Thus,* $\operatorname{Ker} h^n j_{(X,A)}^{(X,B)} \subseteq \operatorname{Im} \delta_{(X,A,B)}^n$.

Since $h^n j_{(X,A)}^X(w) = h^n j_{(X,B)}^X h^n j_{(X,A)}^{(X,B)}(w) = h^n j_{(X,B)}^X(0) = 0$ because of Item (a), it follows $w \in \operatorname{Ker} h^n j_{(X,A)}^X$. Then, the exactness of the generalized cohomology sequence of (X,A) implies that there exists $\alpha \in h^{n-1}(A)$ in such manner that $\delta_{(X,A)}^n(\alpha) = w$. Consequently, the middle commutative square in the preceding diagram yields

$$\delta_{(X,B)}^n h^{n-1} i_A^B(\alpha) = h^n j_{(X,A)}^{(X,B)} \delta_{(X,A)}^n(\alpha) = h^n j_{(X,A)}^{(X,B)}(w) = 0.$$

In other words, we have $h^{n-1} i_A^B(\alpha) \in \operatorname{Ker} \delta_{(X,B)}^n$. As a consequence of the exactness of the generalized cohomology sequence of (X,B), we know that there exists $\beta \in h^{n-1}(X)$ such that $h^{n-1} i_X^B(\beta) = h^{n-1} i_A^B(\alpha)$. Then, it follows from Item (b) that

$$\begin{aligned} h^{n-1} i_A^B \left(\alpha - h^{n-1} i_X^A(\beta)\right) &= h^{n-1} i_A^B(\alpha) - h^{n-1} i_A^B h^{n-1} i_X^A(\beta) \\ &= h^{n-1} i_A^B(\alpha) - h^{n-1} i_X^B(\beta) \\ &= 0. \end{aligned}$$

Said differently, we have $\alpha - h^{n-1} i_X^A(\beta) \in \operatorname{Ker} h^{n-1} i_A^B$. Therefore, we know that the exactness of the generalized cohomology sequence of (A,B) implies that there exists $\gamma \in h^{n-1}(A,B)$ such that $h^{n-1} j_{(A,B)}^A(\gamma) = \alpha - h^{n-1} i_X^A(\beta)$. Thus, once the exactness of the generalized cohomology sequence of (X,A) implies $\delta_{(X,A)}^n \circ h^{n-1} i_X^A = 0$, Item (d) yields

$$\begin{aligned}\delta^n_{(X,A,B)}(\gamma) &= \delta^n_{(X,A)}h^{n-1}j^A_{(A,B)}(\gamma) \;=\; \delta^n_{(X,A)}\left(\alpha - h^{n-1}i^A_X(\beta)\right)\\ &= \delta^n_{(X,A)}(\alpha) - \delta^n_{(X,A)}h^{n-1}i^A_X(\beta)\\ &= \delta^n_{(X,A)}(\alpha)\\ &= w.\end{aligned}$$

Thereby, it is proved that there exists $w' \in h^{n-1}(A,B)$ such that $\delta^n_{(X,A,B)}(w') = w$, as we wished. □

Definition 1.56 (The generalized induced homomorphism between generalized cohomology sequences of triples)**.** *In a generalized cohomology theory, if $f : (X,A,B) \to (Y,C,D)$ is an admissible map of triples, then we say that the sequence of group homomorphisms*

$$h(f) \;:=\; (\cdots,\ h^{n-1}(f_2),\ h^{n-1}(f_3),\ h^n(f_1),\ h^n(f_2),\ h^n(f_3),\ \cdots)$$

*is the **generalized induced homomorphism** of f between the generalized cohomology sequences of the triples (Y,C,D) and (X,A,B).* ♢

Theorem 1.57 (The generalized induced homomorphism between generalized cohomology sequences of triples is a homomorphism of exact sequences)**.** *In a generalized cohomology theory, if $f : (X,A,B) \to (Y,C,D)$ is an admissible map of triples, then $h(f)$ is a homomorphism of exact sequences between the generalized cohomology sequences of the triples (Y,C,D) and (X,A,B).*

Proof. To verify the statement of this theorem we have to prove that the following diagram is commutative.

$$\begin{tikzcd}
\cdots \arrow[r] & h^{n-1}(C,D) \arrow[r, "\delta^n_{(Y,C,D)}"] \arrow[d, "h^{n-1}(f_3)"] & h^n(Y,C) \arrow[r, "h^n j^{(Y,D)}_{(Y,C)}"] \arrow[d, "h^n(f_1)"] & h^n(Y,D) \arrow[r, "h^n i^{(C,D)}_{(Y,D)}"] \arrow[d, "h^n(f_2)"] & h^n(C,D) \arrow[r] \arrow[d, "h^n(f_3)"] & \cdots \\
\cdots \arrow[r] & h^{n-1}(A,B) \arrow[r, "\delta^n_{(X,A,B)}"'] & h^n(X,A) \arrow[r, "h^n j^{(X,B)}_{(X,A)}"'] & h^n(X,B) \arrow[r, "h^n i^{(A,B)}_{(X,B)}"'] & h^n(A,B) \arrow[r] & \cdots
\end{tikzcd}$$

In other words, we have to prove the following three relations for all $n \in \mathbb{Z}$. In fact, if n is an integer number, then:

- $\delta^n_{(X,A,B)} \circ h^{n-1}(f_3) = h^n(f_1) \circ \delta^n_{(Y,C,D)}$. To prove that this relation holds we consider the following diagram where the admissible map $f \mid_A: A \to C$ is the restriction of f to A.

$$\begin{tikzcd}
h^{n-1}(C,D) \arrow[r, "h^{n-1}j^C_{(C,D)}"] \arrow[d, "h^{n-1}(f_3)"] \arrow[rr, bend left, "\delta^n_{(Y,C,D)}"] & h^{n-1}(C) \arrow[r, "\delta^n_{(Y,C)}"] \arrow[d, "h^{n-1}(f|_A)"] & h^n(Y,C) \arrow[d, "h^n(f_1)"] \\
h^{n-1}(A,B) \arrow[r, "h^{n-1}j^A_{(A,B)}"'] \arrow[rr, bend right, "\delta^n_{(X,A,B)}"'] & h^{n-1}(A) \arrow[r, "\delta^n_{(X,A)}"'] & h^n(X,A)
\end{tikzcd}$$

  Since $f_3 \circ j^A_{(A,B)} = j^C_{(C,D)} \circ f \mid_A$, the square on the left-hand side is commutative. In turn, the square on the right-hand side is commutative because it is a restatement of the Commutativity Axiom. Therefore, the whole diagram is commutative, which ensures the relation in question;

- $h^n j^{(X,B)}_{(X,A)} \circ h^n(f_1) = h^n(f_2) \circ h^n j^{(Y,D)}_{(Y,C)}$. This relation is an obvious consequence of the equality $f_1 \circ j^{(X,B)}_{(X,A)} = j^{(Y,D)}_{(Y,C)} \circ f_2$; *and*

- $h^n i^{(A,B)}_{(X,B)} \circ h^n(f_2) = h^n(f_3) \circ h^n i^{(C,D)}_{(Y,D)}$. This relation is an obvious consequence of the equality $f_2 \circ i^{(A,B)}_{(X,B)} = i^{(C,D)}_{(Y,D)} \circ f_3$. □

Theorem 1.58 (Isomorphism of generalized cohomology groups whose pairs of spaces come from an admissible triple of spaces)**.** *In a generalized cohomology theory, let (X, A, B) be an admissible triple. In this situation:*

(1) if $i^B_A : B \to A$ induces isomorphisms from $h^n(A)$ onto $h^n(B)$ for all $n \in \mathbb{Z}$, then $j^{(X,B)}_{(X,A)} : (X, B) \to (X, A)$ induces isomorphisms from $h^n(X, A)$ onto $h^n(X, B)$ for all $n \in \mathbb{Z}$; and

(2) if $i^A_X : A \to X$ induces isomorphisms from $h^n(X)$ onto $h^n(A)$ for all $n \in \mathbb{Z}$, then $i^{(A,B)}_{(X,B)} : (A, B) \to (X, B)$ induces isomorphisms from $h^n(X, B)$ onto $h^n(A, B)$ for all $n \in \mathbb{Z}$.

Proof.

(1) Since the inclusion map $i_A^B : B \to A$ induces isomorphisms from $h^n(A)$ onto $h^n(B)$ for all $n \in \mathbb{Z}$, Theorem 1.11 says that $h^n(A, B)$ is the trivial group for all $n \in \mathbb{Z}$. Therefore, the generalized cohomology sequence of the triple (X, A, B) is the following one.

$$\cdots \longrightarrow 0 \longrightarrow h^n(X, A) \xrightarrow{h^n j_{(X,A)}^{(X,B)}} h^n(X, B) \longrightarrow 0 \longrightarrow \cdots$$

The preceding sequence is exact by Theorem 1.55. Consequently, we have that $h^n j_{(X,A)}^{(X,B)} : h^n(X, A) \to h^n(X, B)$ is an isomorphism for all $n \in \mathbb{Z}$. Then, $j_{(X,A)}^{(X,B)} : (X, B) \to (X, A)$ induces isomorphisms from $h^n(X, A)$ onto $h^n(X, B)$ for all $n \in \mathbb{Z}$, as we wished.

(2) The proof of this assertion is analogous to the proof of the first part of this theorem. We leave the details to the reader. □

1.8 Deformation retracts and the Excision Axiom

In this section, we present the important notion of deformation retracts. Roughly speaking, a deformation retract is a homotopy of admissible pairs in which the homotopy equivalence is composed of an inclusion map. We begin with the following definition.

Definition 1.59 (Retract, deformation retract and strong deformation retract)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces and (X, A) be an admissible pair. An admissible pair $(Y, B) \subseteq (X, A)$ is called a:*

- ***retract** of (X, A) if there exists an admissible map $r : (X, A) \to (Y, B)$ such that $r(y) = y$ for all $y \in Y$. We say that such a map $r : (X, A) \to (Y, B)$ is a **retraction** of (X, A) into (Y, B);*
- ***deformation retract** of (X, A) if there exists a retraction $r : (X, A) \to (Y, B)$ such that its composition with the inclusion map $(Y, B) \to (X, A)$ is homotopic to*

the identity map on (X, A). *We say that such a map* $r : (X, A) \to (Y, B)$ *is a* ***deformation retraction*** *of* (X, A) *into* (Y, B)*; and*

- ***strong deformation retract*** *of* (X, A) *if there exists a deformation retraction* $r : (X, A) \to (Y, B)$ *in which the homotopy* $\Theta : X \times \mathbb{I} \to X$ *between its composition with the inclusion map* $(Y, B) \to (X, A)$ *and the identity map on* (X, A) *can be chosen in such manner that* $\Theta(y, t) = y$ *for all* $y \in Y$ *and all* $t \in \mathbb{I}$. *We say that such a map* $r : (X, A) \to (Y, B)$ *is a* ***strong deformation retraction*** *of* (X, A) *into* (Y, B). $\diamondsuit$

Remark 1.60 (On retracts, deformation retracts and strong deformation retracts)**.** *Let* $\mathscr{C}$ *be an admissible category of topological spaces and* (X, A) *be an admissible pair. If* (Y, B) *is an admissible pair contained in* (X, A) *and the inclusion map* $i : (Y, B) \to (X, A)$ *is admissible, then:*

- (Y, B) *is a retract of* (X, A) *if and only if there exists an admissible map* $r : (X, A) \to (Y, B)$ *such that* $r \circ i$ *is the identity map on* (Y, B). *Thus, the equality*

$$\mathrm{id}_{h^n(Y,B)} \;=\; h^n(r \circ i) \;=\; h^n(i) \circ h^n(r)$$

implies that $h^n(i)$ *is an epimorphism and that* $h^n(r)$ *is a monomorphism for all* $n \in \mathbb{Z}$*; and*

- *if* (Y, B) *is a strong deformation retract of* (X, A)*, then* (Y, B) *is also a deformation retract of* (X, A). *In fact, a strong deformation retraction of* (X, A) *into* (Y, B) *is a deformation retraction of* (X, A) *into* (Y, B). *On the other hand,* (Y, B) *being a deformation retract of* (X, A) *does not imply that* (Y, B) *is also a strong deformation retract of* (X, A). *The reader can find an example for this claim in [25, p. 215]. The discrepancy between these two notions will not play an important role in this work since the majority of its concrete examples is composed of strong deformation retracts.* $\diamondsuit$

Theorem 1.61 (Split exact sequence induced by a special retraction)**.** *In a generalized cohomology theory, if an admissible pair* (X, A) *is such that the inclusion* $i : A \to X$ *is a retraction, then the sequence*

$$0 \longrightarrow h^n(X,A) \xrightarrow{h^n(j)} h^n(X) \xrightarrow{h^n(i)} h^n(A) \longrightarrow 0$$

is split exact for all $n \in \mathbb{Z}$*, where* $j : X \to (X, A)$ *is the natural inclusion. Therefore, in particular, we have that* $h^n(X)$ *is isomorphic to the direct sum* $h^n(X, A) \oplus h^n(A)$ *for all* $n \in \mathbb{Z}$*. This implies that, if* (X, Ω) *is an admissible pair where* Ω *is a one-point space, then the relative group* $h^n(X, \Omega)$ *is isomorphic to the pointed reduced group* $\widetilde{h}^n_\Omega(X)$ *for all* $n \in \mathbb{Z}$.

Proof. Since $i : A \to X$ is a retraction, there exists an admissible map $r : X \to A$ such that $r \circ i = \mathrm{id}_A$. Consequently, if the preceding sequence is exact, then it clearly splits because

$$h^n(i) \circ h^n(r) \;=\; \mathrm{id}_{h^n(A)}.$$

Thus, we only have to prove the exactness of the sequence in question. We claim that this follows from the exactness of the generalized cohomology sequence of the admissible pair (X, A). Indeed, since $h^n(i) : h^n(X) \to h^n(A)$ is an epimorphism for all $n \in \mathbb{Z}$, we have

$$\operatorname{Ker} \delta^{n+1}_{(X,A)} \;=\; \operatorname{Im} h^n(i) \;=\; h^n(A).$$

Thus, $\delta^n_{(X,A)}$ is trivial for all $n \in \mathbb{Z}$. This allows us to change its domain and codomain as in the preceding sequence without losing exactness, which finishes the proof of the theorem. □

Theorem 1.62 (Homotopy equivalence induced from a deformation retract)**.** *Let* $\mathscr{C}$ *be an admissible category of topological spaces and* (X, A) *be an admissible pair. If* (Y, B) *is a deformation retract of* (X, A)*, then the inclusion map* $i : (Y, B) \to (X, A)$ *and any deformation retraction of* (X, A) *into* (Y, B) *form a homotopy equivalence. In particular, in a generalized cohomology theory, the proof of Theorem 1.47 implies that* $h^n(X, A)$ *is isomorphic to* $h^n(Y, B)$ *for all* $n \in \mathbb{Z}$.

Proof. Let $r : (X, A) \to (Y, B)$ be a deformation retraction of (X, A) into (Y, B). Then, $r \circ i$ is the identity map on (Y, B) and $i \circ r$ is homotopic to the identity map on (X, A). Once the identity map on (Y, B) is homotopic to itself, we have that r and i form a homotopy equivalence. □

Corollary 1.63 (Isomorphism of generalized cohomology sequences induced by an admissible deformation retract). *In a generalized cohomology theory, if (Y, B) is a deformation retract of an admissible pair (X, A), then the inclusion $i : (Y, B) \to (X, A)$ induces an isomorphism of exact sequences between the generalized cohomology sequences of (X, A) and (Y, B). The same claim holds considering reduced generalized cohomology sequences.*

Proof. This is a consequence of Corollary 1.49 since $i : (Y, B) \to (X, A)$ is a homotopy equivalence between (Y, B) and (X, A). $\square$

To close this section, we present two extensions of the Excision Axiom. We remind the reader that this axiom asserts that, if (X, A) is an admissible pair and U is an open subset of X whose closure is contained in the interior of A, then the excision map $(X - U, A - U) \to (X, A)$, if admissible, induces isomorphisms of the generalized cohomology groups in all dimensions. We prove below that the condition of the closure of U to be contained in the interior of A can be relaxed in two special cases to U just contained in A. However, this relaxation of the hypothesis in question is not generally reasonable.

Theorem 1.64 (The first extension of the Excision Axiom). *In a generalized cohomology theory, let (X, A) be an admissible pair and U and V be open subsets of X such that the closure of V is contained in U which is contained in A. If the inclusion maps $i : (X - U, A - U) \to (X - V, A - V)$ and $j : (X - V, A - V) \to (X, A)$ are admissible and $(X - U, A - U)$ is a deformation retract of $(X - V, A - V)$, then the inclusion map $j \circ i : (X - U, A - U) \to (X, A)$ induces isomorphisms of the generalized cohomology groups in all dimensions.*

Proof. Let n be an integer number. Since $j : (X - V, A - V) \to (X, A)$ is an excision map, the Excision Axiom implies that $h^n(j) : h^n(X, A) \to h^n(X - V, A - V)$ is an isomorphism. Furthermore, $h^n(i) : h^n(X - V, A - V) \to h^n(X - U, A - U)$ is an isomorphism because of Theorem 1.62. Thus, we have that $h^n(j \circ i) = h^n(i) \circ h^n(j)$ is an isomorphism from $h^n(X, A)$ onto $h^n(X - U, A - U)$. This proves that the inclusion $j \circ i : (X - U, A - U) \to (X, A)$ induces isomorphisms of the generalized cohomology groups in all dimensions, as we wished. $\square$

Theorem 1.65 (The second extension of the Excision Axiom)**.** *In a generalized cohomology theory, let (X, A) be an admissible pair and U be an open subset of X contained in A. Moreover, assume that there exists a subset B of X containing A in such manner that:*

(a). the inclusion maps $i : (X - U, A - U) \to (X, A)$, $j : (X, A) \to (X, B)$, $k : (X - U, B - U) \to (X, B)$ and $l : (X - U, A - U) \to (X - U, B - U)$ are admissible;

(b). the closure of U is contained in the interior of B;

(c). A is a deformation retract of B; and

(d). $A - U$ is a deformation retract of $B - U$.

Then, $i : (X - U, A - U) \to (X, A)$ induces isomorphisms of the generalized cohomology groups in all dimensions.

Proof. Let n be an integer number. We tacitly use Item (a) whenever appears an induced homomorphism. It is clear that

$$h^n(k) : h^n(X, B) \to h^n(X - U, B - U)$$

is an isomorphism from Item (b) and the Excision Axiom. Furthermore, $h^n(A)$ is isomorphic to $h^n(B)$ and $h^n(A - U)$ is isomorphic to $h^n(B - U)$ because of Item (c), Item (d) and Corollary 1.63. Hence,

$$h^n(j) : h^n(X, B) \to h^n(X, A) \qquad \text{and} \qquad h^n(l) : h^n(X - U, B - U) \to h^n(X - U, A - U)$$

are isomorphisms because of Theorem 1.58. Therefore, since the equality $j \circ i = k \circ l$ yields $h^n(i) \circ h^n(j) = h^n(l) \circ h^n(k)$, it follows that

$$h^n(i) \;=\; h^n(l) \circ h^n(k) \circ h^n(j)^{-1}$$

is an isomorphism from $h^n(X, A)$ onto $h^n(X - U, A - U)$. This proves that the inclusion map $i : (X - U, A - U) \to (X, A)$ induces isomorphisms of the generalized cohomology groups in all dimensions, as we wished. $\square$

1.9 The Direct Sum Theorem and Milnor's Additivity Axiom

In this section, we prove an important theorem connecting the generalized cohomology groups of an admissible pair to the generalized cohomology groups of the components of a suitable decomposition of the spaces that belong to the pair in question. After that, we set an axiom suggested by John Milnor (1931 -) in [28, p. 337] which treats the question of a possible extension of this theorem. We begin with the following definition.

Definition 1.66 (Projective direct product representation of a group). *Let C be a group and $\left(C_\alpha\right)_{\alpha\in\Lambda}$ be a family of groups indexed by an indexing set Λ. For each family of group homomorphisms $\Phi = \left(\varphi_\alpha : C \to C_\alpha\right)_{\alpha\in\Lambda}$, we know that it is determined the group homomorphism*

$$\begin{aligned} \prod_{\alpha\in\Lambda}\varphi_\alpha : C &\to \prod_{\alpha\in\Lambda} C_\alpha, \\ c &\mapsto \left(\varphi_\alpha(c)\right)_{\alpha\in\Lambda}. \end{aligned}$$

If $\prod_{\alpha\in\Lambda}\varphi_\alpha$ is an isomorphism of C onto $\prod_{\alpha\in\Lambda} C_\alpha$, then Φ is called a ***projective direct product representation*** *of C, and each component φ_α is called a* ***projection****. In this situation, for each sequence $(c_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} C_\alpha$, there is a unique $c \in C$ such that $\varphi_\alpha(c) = c_\alpha$ for all $\alpha \in \Lambda$.* ♢

Remark 1.67 (Projective direct sum representation of a group). *Let $\left(C_\alpha\right)_{\alpha\in\Lambda}$ be a family of groups indexed by an indexing set Λ. We remind the reader that the direct product $\prod_{\alpha\in\Lambda} C_\alpha$ and the direct sum $\bigoplus_{\alpha\in\Lambda} C_\alpha$ are different objects. Roughly speaking, direct products are formed from a collection of groups taking all possible combinations of elements of these groups. In turn, direct sums are formed from a collection of groups taking only the combinations of elements of these groups which has a finite number of elements different from the identity elements. Indeed, we can think about direct products and direct sums in Group Theory as we think about box and product topologies in General Topology, respectively. However, there is an important* *case to consider. When the indexing set Λ is finite, it is clear that the direct product coincides with the direct sum. Hence, when we have a finite projective direct product representation of a group, we say that it is a* ***projective direct sum representation*** *of the group in question, and we use the notation associated with direct sums instead of the one associated with direct products.* ♢

Lemma 1.68 (Sufficient conditions for existence of a projective direct sum representation of a group)**.** *The following diagram of groups and homomorphisms has each of its triangles commutative.*

$$\begin{tikzcd}
C_1 & & C_2 \\
& C \arrow[ul, "\psi_1"'] \arrow[ur, "\psi_2"] & \\
C_2' \arrow[uu, "\eta_1"] \arrow[ur, "\varphi_2"] & & C_1' \arrow[uu, "\eta_2"'] \arrow[ul, "\varphi_1"']
\end{tikzcd}$$

If $\mathrm{Im}(\varphi_1) \subseteq \mathrm{Ker}(\psi_1)$, $\mathrm{Im}(\varphi_2) = \mathrm{Ker}(\psi_2)$ *and* η_1 *and* η_2 *are isomorphisms, then* (ψ_1, ψ_2) *is a projective direct sum representation of* C*.*

Proof. The reader can find a proof of this result in [13, p. 32]. □

Theorem 1.69 (The Direct Sum Theorem)**.** *In a generalized cohomology theory, let:*

- (X, A) *be an admissible pair;*
- $X = \bigcup_{\alpha=1}^{m} X_\alpha$ *be a union of disjoint sets each of which are closed and open in* X*;*
- $A_\alpha \subseteq X_\alpha$ *be such that* $A = \bigcup_{\alpha=1}^{m} A_\alpha$*;*
- *all pairs formed of the sets* X_α *and* A_α *and all their unions are admissible and all inclusion maps of such pairs are admissible; and*
- $i_\alpha : (X_\alpha, A_\alpha) \to (X, A)$ *be an inclusion map for each* α *between* 1 *and* m*, both included.*

Then, the family $\big(h^n(i_\alpha) : h^n(X, A) \to h^n(X_\alpha, A_\alpha)\big)_{\alpha=1}^{m}$ *yields a projective direct sum representation of* $h^n(X, A)$ *for all* $n \in \mathbb{Z}$. *In particular,* $h^n(X, A)$ *is isomorphic to* $\bigoplus_{\alpha=1}^{m} h^n(X_\alpha, A_\alpha)$ *for all* $n \in \mathbb{Z}$.

Proof. We prove the result using the Finite Induction Principle.

- *Induction basis.* The theorem is obvious for $m = 1$.

- *Induction hypothesis.* Suppose the theorem holds for the admissible pair (Y, B), where $Y = \bigcup_{\alpha=1}^{m-1} X_\alpha$ and $B = \bigcup_{\alpha=1}^{m-1} A_\alpha$. In other words, assume that the maps $h^n(i'_\alpha) : h^n(Y, B) \to h^n(X_\alpha, A_\alpha)$ yield a projective direct sum representation of $h^n(Y, B)$ for all $n \in \mathbb{Z}$, where $i'_\alpha : (X_\alpha, A_\alpha) \to (Y, B)$ is an inclusion map for each α between 1 and $(m - 1)$, both included.

- *Induction step.* Let n be an integer number and consider the following diagram of inclusion maps.

$$\begin{tikzcd}
(X, X_m \cup A) & & & & (X, Y \cup A_m) \\
& & (X, A) \arrow[ull, "j_m"'] \arrow[urr, "j'"] & & \\
(Y \cup A_m, A) \arrow[uu, "k'"] \arrow[urr, "l'"] & & & & (X_m \cup A, A) \arrow[uu, "k_m"'] \arrow[ull, "l_m"'] \\
(Y, B) \arrow[u, "h'"] \arrow[uurr, "i'"'] & & & & (X_m, A_m) \arrow[u, "h_m"'] \arrow[uull, "i_m"]
\end{tikzcd}$$

It is clear that h_m, $k_m \circ h_m$, h' and $k' \circ h'$ are excision maps. Then, $h^n(h_m)$, $h^n(k_m)$, $h^n(h')$ and $h^n(k')$ are isomorphisms by the Excision Axiom. Further, the exactness of the generalized cohomology sequences of the triples $(X, Y \cup A_m, A)$ and $(X, X_m \cup A, A)$ implies $\operatorname{Im} h^n(j') = \operatorname{Ker} h^n(l')$ and $\operatorname{Im} h^n(j_m) = \operatorname{Ker} h^n(l_m)$.

Thus, the hypotheses of Lemma 1.68 are fulfilled and the induced homomorphisms $h^n(l')$ and $h^n(l_m)$ form a projective direct sum representation of the generalized cohomology group $h^n(X, A)$. In a similar way, we have that $h^n(i')$ and $h^n(i_m)$ yield a projective direct sum representation of $h^n(X, A)$. These facts can be seen more easily in the following diagram.

$$
\begin{tikzcd}
h^n(X, X_m \cup A) \arrow[dd, "h^n(k')"'] \arrow[dr, "h^n(j_m)"] & & h^n(X, Y \cup A_m) \arrow[dl, "h^n(j')"'] \arrow[dd, "h^n(k_m)"] \\
& h^n(X, A) \arrow[dl, "h^n(l')"'] \arrow[dr, "h^n(l_m)"] \arrow[ddl, "h^n(i')"] \arrow[ddr, "h^n(i_m)"'] & \\
h^n(Y \cup A_m, A) \arrow[d, "h^n(h')"'] & & h^n(X_m \cup A, A) \arrow[d, "h^n(h_m)"] \\
h^n(Y, B) & & h^n(X_m, A_m)
\end{tikzcd}
$$

Additionally, $h^n(i_\alpha) = h^n(i'_\alpha) \circ h^n(i')$ for all α between 1 and $(m - 1)$, both included. Consequently, the homomorphisms $h^n(i_\alpha) : h^n(X, A) \to h^n(X_\alpha, A_\alpha)$ form a projective direct sum representation of $h^n(X, A)$, as we wished. $\square$

A natural question that the reader may be asking himself or herself now is about the existence of "The Direct Product Theorem". More explicitly, the reader may be asking if it is also true that, in all generalized cohomology theories, under the hypotheses of *The Direct Sum Theorem* adapted to a decomposition of the admissible pair (X, A) in an arbitrary number of components, we have that the natural admissible inclusion maps induce a projective direct product representation of $h^n(X, A)$ for all $n \in \mathbb{Z}$. The answer for this question is negative, and references in which this claim is proven can be found in [28, p. 337]. This fact together with the trueness of the desired assertion in Singular Cohomology, as well as in other relevant generalized cohomology theories, led to the following axiom.

Definition 1.70 (Milnor's Additivity Axiom and additive generalized cohomology theories)**.** *In a generalized cohomology theory, let:*

- (X, A) *be an admissible pair;*
- Λ *be an indexing set;*
- $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ *be a union of disjoint sets each of which are closed and open in* X*;*
- $A_\alpha \subseteq X_\alpha$ *be such that* $A = \bigcup_{\alpha\in\Lambda} A_\alpha$*;*
- *all pairs formed of the sets* X_α *and* A_α *and all their unions are admissible and all inclusion maps of such pairs are admissible; and*
- $i_\alpha : (X_\alpha, A_\alpha) \to (X, A)$ *be an inclusion map for each* $\alpha \in \Lambda$*.*

We say that ***Milnor's Additivity Axiom*** *is the assertion that the family of generalized induced homomorphisms* $\big(h^n(i_\alpha) : h^n(X, A) \to h^n(X_\alpha, A_\alpha)\big)_{\alpha\in\Lambda}$ *produces a projective direct product representation of* $h^n(X, A)$ *for all* $n \in \mathbb{Z}$*. In particular, this axiom implies that* $h^n(X, A)$ *is isomorphic to* $\prod_{\alpha\in\Lambda} h^n(X_\alpha, A_\alpha)$ *for all* $n \in \mathbb{Z}$*. Moreover, if a generalized cohomology theory verifies Milnor's Additivity Axiom, we say that it is an* ***additive generalized cohomology theory****.* $\diamondsuit$

Remark 1.71 (On Milnor's Additivity Axiom)**.** *Some authors say that Milnor's Additivity Axiom is "Milnor's Multiplicativity Axiom". Moreover, they add Milnor's Additivity Axiom among Eilenberg-Steenrod Axioms to define generalized cohomology theories. In this work, we will not follow these conventions. In particular, we will say "additive generalized cohomology theory" when we mean a generalized cohomology theory that verifies Milnor's Additivity Axiom. We think that this is more respectful with the history behind these concepts.* $\diamondsuit$

1.10 Triads and proper triads

In this section, we generalize the notions of admissible triple and generalized cohomology sequence of an admissible triple, presenting the concepts of proper triad and generalized cohomology sequence of a proper triad. These new mathematical objects will play an important role when we discuss the generalized Mayer-Vietoris sequences, which are fundamental tools in the calculus of generalized cohomology groups. We begin with the following definition.

Definition 1.72 (Triads and proper triads)**.** *In a generalized cohomology theory, let X be a topological space and X_1 and X_2 be subspaces of X. We say that $(X; X_1, X_2)$ is a:*

- ***triad*** *if X, X_1, X_2, $X_1 \cup X_2$, $X_1 \cap X_2$ and all pairs formed from these spaces are admissible, and all their inclusion maps are admissible; and*
- ***proper triad*** *if it is a triad and the inclusions*

$$\begin{aligned} k_1 : (X_2, X_1 \cap X_2) &\to (X_1 \cup X_2, X_1) \quad \textit{and} \\ k_2 : (X_1, X_1 \cap X_2) &\to (X_1 \cup X_2, X_2) \end{aligned}$$

induce isomorphisms of the generalized cohomology groups in all dimensions. ♢

Remark 1.73 (Some related triads and proper triads)**.** *In a generalized cohomology theory, if $(X; X_1, X_2)$ is a triad, then:*

- *$(X; X_2, X_1)$ is also a triad, which is distinct from $(X; X_1, X_2)$ unless $X_1 = X_2$. Moreover, $(X; X_1, X_2)$ is a proper triad if and only if $(X; X_2, X_1)$ is a proper triad;*
- *$(X_1 \cup X_2; X_1, X_2)$ is also a triad, which is distinct from $(X; X_1, X_2)$ unless $X = X_1 \cup X_2$. Moreover, $(X; X_1, X_2)$ is a proper triad if and only if $(X_1 \cup X_2; X_1, X_2)$ is a proper triad.* ♢

Remark 1.74 (Some kinds of triads that are proper in all generalized cohomology theories)**.** *It is not hard to find examples of triads that are proper in a generalized cohomology theory but non-proper in another one. However, some kinds of triads are proper in all generalized cohomology theories. For example:*

- *if $(X; X_1, X_2)$ is a triad such that $X_2 \subseteq X_1$, then it is a proper triad. In fact, it follows that*

$$\begin{aligned} k_1 : (X_2, X_2) &\rightarrow (X_1, X_1) \quad \textit{and} \\ k_2 : (X_1, X_2) &\rightarrow (X_1, X_2) \end{aligned}$$

are the inclusion maps that we have to consider. Hence, since $h^n(X_1, X_1)$ and $h^n(X_2, X_2)$ are trivial groups for all $n \in \mathbb{Z}$ by Corollary 1.12, we have that the group homomorphism

$$h^n(k_1) : h^n(X_1, X_1) \rightarrow h^n(X_2, X_2)$$

is the only possible one for all $n \in \mathbb{Z}$. This is clearly a group isomorphism for all $n \in \mathbb{Z}$. Moreover, since $k_2 : (X_1, X_2) \rightarrow (X_1, X_2)$ is the identity map, we have that the group homomorphism

$$h^n(k_2) : h^n(X_1, X_2) \rightarrow h^n(X_1, X_2)$$

is the identity for all $n \in \mathbb{Z}$, which is also a group isomorphism for all $n \in \mathbb{Z}$. Consequently, $(X; X_1, X_2)$ is a proper triad; and

- *if $(X_1 \cup X_2; X_1, X_2)$ is a triad such that X_1 and X_2 are closed in $X_1 \cup X_2$, and the closure of $X_1 - (X_1 \cap X_2)$ in $X_1 \cup X_2$ is disjoint from the closure of $X_2 - (X_1 \cap X_2)$ in $X_1 \cup X_2$, then it is also a proper triad. Indeed, these hypotheses imply that the inclusions*

$$\begin{aligned} k_1 : (X_2, X_1 \cap X_2) &\rightarrow (X_1 \cup X_2, X_1) \quad \textit{and} \\ k_2 : (X_1, X_1 \cap X_2) &\rightarrow (X_1 \cup X_2, X_2) \end{aligned}$$

are excision maps. Therefore, the triad $(X_1 \cup X_2; X_1, X_2)$ is proper because of the Excision Axiom. The reader can produce more examples of similar kinds of triads that are proper in all generalized cohomology theories considering Theorem 1.64 and Theorem 1.65. $\diamondsuit$

Theorem 1.75 (Necessary and sufficient condition for a triad to be proper)**.** *In a generalized cohomology theory, a triad $(X; X_1, X_2)$ is proper if and only if the inclusions*

$$\begin{aligned} i_1 : (X_1, X_1 \cap X_2) &\to (X_1 \cup X_2, X_1 \cap X_2) \quad \textit{and} \\ i_2 : (X_2, X_1 \cap X_2) &\to (X_1 \cup X_2, X_1 \cap X_2) \end{aligned}$$

induce a projective direct sum representation of $h^n(X_1 \cup X_2, X_1 \cap X_2)$ *for all* $n \in \mathbb{Z}$. *In other words, we have that a triad* $(X; X_1, X_2)$ *is proper if and only if, for each* $n \in \mathbb{Z}$ *and for each* $(u_1, u_2) \in h^n(X_1, X_1 \cap X_2) \oplus h^n(X_2, X_1 \cap X_2)$, *there exists a unique* $u \in h^n(X_1 \cup X_2, X_1 \cap X_2)$ *in such manner that* $h^n(i_1)(u) = u_1$ *and* $h^n(i_2)(u) = u_2$.

Proof. Let the following diagram be composed of admissible pairs and inclusion maps from the triad in question.

$$\begin{tikzcd}
(X_1 \cup X_2, X_1) & & (X_1 \cup X_2, X_2) \\
& (X_1 \cup X_2, X_1 \cap X_2) \arrow[ul, "j_1"'] \arrow[ur, "j_2"] & \\
(X_2, X_1 \cap X_2) \arrow[uu, "k_1"] \arrow[ur, "i_2"'] & & (X_1, X_1 \cap X_2) \arrow[uu, "k_2"'] \arrow[ul, "i_1"]
\end{tikzcd}$$

For all $n \in \mathbb{Z}$, the following diagram has commutative triangles because the preceding diagram of inclusions verifies this property. Furthermore, $\operatorname{Ker} h^n(j_1) = \operatorname{Im} h^n(i_1)$ and $\operatorname{Ker} h^n(j_2) = \operatorname{Im} h^n(i_2)$ for all $n \in \mathbb{Z}$ since the generalized cohomology sequences of the triples $(X_1 \cup X_2, X_1, X_1 \cap X_2)$ and $(X_1 \cup X_2, X_2, X_1 \cap X_2)$ are exact by Theorem 1.55. Thus, if $(X; X_1, X_2)$ is a proper triad, then $h^n(k_1)$ and $h^n(k_2)$ are isomorphisms for all $n \in \mathbb{Z}$. Therefore, Lemma 1.68 implies that $h^n(i_1)$ and $h^n(i_2)$ yield a projective direct sum representation of the generalized cohomology group $h^n(X_1 \cup X_2, X_1 \cap X_2)$ for all $n \in \mathbb{Z}$, as we wished.

$$\begin{tikzcd}
h^n(X_1 \cup X_2, X_1) \arrow[dr, "h^n(j_1)"] \arrow[dd, "h^n(k_1)"'] & & h^n(X_1 \cup X_2, X_2) \arrow[dl, "h^n(j_2)"'] \arrow[dd, "h^n(k_2)"] \\
& h^n(X_1 \cup X_2, X_1 \cap X_2) \arrow[dl, "h^n(i_2)"] \arrow[dr, "h^n(i_1)"'] & \\
h^n(X_2, X_1 \cap X_2) & & h^n(X_1, X_1 \cap X_2)
\end{tikzcd}$$

Conversely, if the inclusion maps i_1 and i_2 induce a projective direct sum representation of the generalized cohomology group $h^n(X_1 \cup X_2, X_1 \cap X_2)$ for all $n \in \mathbb{Z}$, then the group homomorphism

$$\begin{aligned} h^n(i_1) \oplus h^n(i_2) : h^n(X_1 \cup X_2, X_1 \cap X_2) &\to h^n(X_1, X_1 \cap X_2) \oplus h^n(X_2, X_1 \cap X_2), \\ u &\mapsto \big(h^n(i_1)(u),\, h^n(i_2)(u)\big), \end{aligned}$$

is an isomorphism for all $n \in \mathbb{Z}$. Then, we first claim that $h^n(j_1)$ is a monomorphism for all $n \in \mathbb{Z}$. Indeed, since $h^{n-1}(i_1) \oplus h^{n-1}(i_2)$ is an epimorphism, $h^{n-1}(i_1)$ is also an epimorphism. Hence, by the exactness of the generalized cohomology sequence of the triple $(X_1 \cup X_2, X_1, X_1 \cap X_2)$, we have that

$$\operatorname{Ker} \delta^n_{(X_1 \cup X_2, X_1, X_1 \cap X_2)} = \operatorname{Im} h^{n-1}(i_1) = h^{n-1}(X_1, X_1 \cap X_2)$$

implies $\operatorname{Im} \delta^n_{(X_1 \cup X_2, X_1, X_1 \cap X_2)} = 0$. This proves that $\operatorname{Ker} h^n(j_1) = \operatorname{Im} \delta^n_{(X_1 \cup X_2, X_1, X_1 \cap X_2)} = 0$ for all $n \in \mathbb{Z}$. It is also true that $h^n(j_2)$ is a monomorphism for all $n \in \mathbb{Z}$, but since the proof of this assertion is analogous to the one we have just done we leave the details to the reader. Now, we claim that $h^n(k_1)$ is an isomorphism for all $n \in \mathbb{Z}$. In fact, if n is an integer number, then:

- $h^n(k_1)$ *is a monomorphism.* Let $u \in h^n(X_1 \cup X_2, X_1)$ be such that $h^n(k_1)(u) = 0$. We have to show that $u = 0$ to ensure that $h^n(k_1)$ is a monomorphism. Since we have:

$$[h^n(i_1) \oplus h^n(i_2)](h^n(j_1)(u)) = \big(h^n(i_1)h^n(j_1)(u), h^n(i_2)h^n(j_1)(u)\big) = (0, 0)$$

  because $h^n(i_2)h^n(j_1)(u) = h^n(k_1)(u) = 0$ and because $h^n(i_1) \circ h^n(j_1)$ is the trivial homomorphism by the exactness of the generalized cohomology sequence of the triple $(X_1 \cup X_2, X_1, X_1 \cap X_2)$, it follows $h^n(j_1)(u) = 0$ once $h^n(i_1) \oplus h^n(i_2)$ is a monomorphism. Therefore, it follows $u = 0$ since $h^n(j_1)$ was shown to be a monomorphism; *and*

- $h^n(k_1)$ *is an epimorphism.* Let $u \in h^n(X_2, X_1 \cap X_2)$. To ensure that $h^n(k_1)$ is an epimorphism, we have to prove that there exists $v \in h^n(X_1 \cup X_2, X_1)$ such that

$h^n(k_1)(v) = u$. Because $h^n(i_1) \oplus h^n(i_2)$ is an epimorphism, we know that there exists $w \in h^n(X_1 \cup X_2, X_1 \cap X_2)$ such that

$$[h^n(i_1) \oplus h^n(i_2)](w) = (0, u).$$

Moreover, since

$$[h^n(i_1) \oplus h^n(i_2)](w) = (h^n(i_1)(w), h^n(i_2)(w)),$$

we have that

$$h^n(i_1)(w) = 0 \qquad \text{and} \qquad h^n(i_2)(w) = u.$$

Hence, once $w \in \operatorname{Ker} h^n(i_1)$ and $\operatorname{Ker} h^n(i_1) = \operatorname{Im} h^n(j_1)$ because of the exactness of the generalized cohomology sequence of the triple $(X_1 \cup X_2, X_1, X_1 \cap X_2)$, there exists $v \in h^n(X_1 \cup X_2, X_1)$ such that

$$h^n(j_1)(v) = w.$$

In this situation, it follows that

$$h^n(k_1)(v) = h^n(i_2)h^n(j_1)(v) = h^n(i_2)(w) = u.$$

The proof that $h^n(k_2)$ is an isomorphism is analogous to the proof that $h^n(k_1)$ is an isomorphism. Then, we also leave these details to the reader. Hence, $(X; X_1, X_2)$ is a proper triad, as we wished. $\square$

Definition 1.76 (The generalized cohomology sequence of a proper triad)**.** *In a generalized cohomology sequence, we define the* ***generalized cohomology sequence of the proper triad*** $(X; X_1, X_2)$ *as the sequence*

$$\begin{tikzcd}
& h^n(X, X_1 \cup X_2) \arrow[r, "h^n j^{(X,X_2)}_{(X,X_1 \cup X_2)}"] & h^n(X, X_2) \arrow[d, "h^n i^{(X_1,X_1 \cap X_2)}_{(X,X_2)}"] & \\
\cdots \arrow[r] & h^{n-1}(X_1, X_1 \cap X_2) \arrow[u, "\delta^n_{(X;X_1,X_2)}"] & h^n(X_1, X_1 \cap X_2) \arrow[r] & \cdots,
\end{tikzcd}$$

where $i^{(X_1,X_1\cap X_2)}_{(X,X_2)} : (X_1, X_1 \cap X_2) \to (X, X_2)$ *and* $j^{(X,X_2)}_{(X,X_1\cup X_2)} : (X, X_2) \to (X, X_1 \cup X_2)$ *are inclusion maps, and* $\delta^n_{(X;X_1,X_2)} : h^{n-1}(X_1, X_1 \cap X_2) \to h^n(X, X_1 \cup X_2)$*, named the* ***nth generalized coboundary operator of the proper triad*** $(X; X_1, X_2)$*, consists of the composition between* $h^{n-1}(k_2)^{-1} : h^{n-1}(X_1, X_1 \cap X_2) \to h^{n-1}(X_1 \cup X_2, X_2)$ *and the nth generalized coboundary operator of the triple* $(X, X_1 \cup X_2, X_2)$*, which is the map* $\delta^n_{(X,X_1\cup X_2,X_2)} : h^{n-1}(X_1 \cup X_2, X_2) \to h^n(X, X_1 \cup X_2)$*.* ♢

Remark 1.77 (The relation between the generalized cohomology sequences of triples and triads)**.** *Let* $(X; X_1, X_2)$ *be a triad such that* X_2 *is a subset of* X_1*. Then, it is immediate that* (X, X_1, X_2) *is an admissible triple. Moreover, it was shown in Remark 1.74 that* $(X; X_1, X_2)$ *is a proper triad. Now, we claim that the generalized cohomology sequence of the proper triad* $(X; X_1, X_2)$ *coincides with the generalized cohomology sequence of the triple* (X, X_1, X_2)*. Indeed, this happens because*

$$\delta^n_{(X;X_1,X_2)} = \delta^n_{(X,X_1,X_2)}$$

for all $n \in \mathbb{Z}$ *since the inverse of the generalized induced homomorphisms generated by the inclusion map*

$$k_2 : (X_1, X_2) \to (X_1, X_2)$$

is the identity homomorphism for all $n \in \mathbb{Z}$*. In particular, this assertion shows that the generalized cohomology sequence of a proper triad generalizes the generalized cohomology sequence of a triple, which is widely expected since proper triads generalize admissible triples.* ♢

Theorem 1.78 (The generalized cohomology sequence of a proper triad is exact)**.** *In a generalized cohomology theory, if* $(X; X_1, X_2)$ *is a proper triad, then its generalized cohomology sequence is exact.*

Proof. We prove this result showing that the generalized cohomology sequence of the proper triad $(X; X_1, X_2)$ is isomorphic to the generalized cohomology sequence of the admissible triple $(X, X_1 \cup X_2, X_2)$. Note the sufficiency of this argument since exactness is invariant under isomorphism of sequences, and since the latter sequence is exact by Theorem 1.55.

$$\begin{tikzcd}
\cdots \arrow[r] & h^{n-1}(X,X_2) \arrow[d, "h^{n-1}i^{(X_1\cup X_2,X_2)}_{(X,X_2)}"] \arrow[ddd, bend right=60, "\mathrm{id}_{h^{n-1}(X,X_2)}"'] & h^n(X,X_2) \arrow[r] \arrow[ddd, bend left=60, "\mathrm{id}_{h^n(X,X_2)}"] & \cdots \\
& h^{n-1}(X_1\cup X_2,X_2) \arrow[r, "\delta^n_{(X,X_1\cup X_2,X_2)}"] \arrow[d, "h^{n-1}(k_2)"] & h^n(X,X_1\cup X_2) \arrow[u, "h^n j^{(X,X_2)}_{(X,X_1\cup X_2)}"'] \arrow[d, "\mathrm{id}_{h^n(X,X_1\cup X_2)}"] & \\
& h^{n-1}(X_1,X_1\cap X_2) \arrow[r, "\delta^n_{(X;X_1,X_2)}"] & h^n(X,X_1\cup X_2) \arrow[d, "h^n j^{(X,X_2)}_{(X,X_1\cup X_2)}"] & \\
\cdots \arrow[r] & h^{n-1}(X,X_2) \arrow[u, "h^{n-1}i^{(X_1,X_1\cap X_2)}_{(X,X_2)}"'] & h^n(X,X_2) \arrow[r] & \cdots
\end{tikzcd}$$

Indeed, as shown by the preceding diagram, the generalized cohomology sequence of the proper triad $(X;X_1,X_2)$ is obtained from the generalized cohomology sequence of the triple $(X,X_1\cup X_2,X_2)$ by replacing the group $h^{n-1}(X_1\cup X_2,X_2)$ by the isomorphic group $h^{n-1}(X_1,X_1\cap X_2)$ under $h^{n-1}(k_2):h^{n-1}(X_1\cup X_2,X_2)\to h^{n-1}(X_1,X_1\cap X_2)$, and defining $\delta^n_{(X;X_1,X_2)}$ so that $\delta^n_{(X;X_1,X_2)}\circ h^{n-1}(k_2)=\delta^n_{(X,X_1\cup X_2,X_2)}$. Thus, the generalized cohomology sequence of $(X;X_1,X_2)$ is isomorphic to the generalized cohomology sequence of the triple $(X,X_1\cup X_2,X_2)$, as we wished. $\square$

Definition 1.79 (Maps of triads and of proper triads)**.** *In a generalized cohomology theory, let $(X;X_1,X_2)$ and $(Y;Y_1,Y_2)$ be (proper) triads. We say that $f:(X;X_1,X_2)\to(Y;Y_1,Y_2)$ is an **admissible map of (proper) triads** if $f:X\to Y$ is a continuous map such that:*

- *$f(X_1)\subseteq Y_1$;*
- *$f(X_2)\subseteq Y_2$; and*

- *the maps of pairs* $f_1 : (X, X_1 \cup X_2) \to (Y, Y_1 \cup Y_2)$, $f_2 : (X, X_1) \to (Y, Y_1)$, $f_3 : (X, X_2) \to (Y, Y_2)$, $f_4 : (X_1, X_1 \cap X_2) \to (Y_1, Y_1 \cap Y_2)$, $f_5 : (X_1 \cup X_2, X_2) \to (Y_1 \cup Y_2, Y_2)$ *and* $f_6 : (X, X_1 \cap X_2) \to (Y, Y_1 \cap Y_2)$ *are all admissible.* $\diamondsuit$

Definition 1.80 (The generalized induced homomorphism between generalized cohomology sequences of proper triads)**.** *In a generalized cohomology theory, if* $f : (X; X_1, X_2) \to (Y; Y_1, Y_2)$ *is an admissible map of triads, then we say that the sequence of group homomorphisms*

$$h(f) = (\cdots,\ h^{n-1}(f_3),\ h^{n-1}(f_4),\ h^n(f_1),\ h^n(f_3),\ h^n(f_4),\ \cdots)$$

is the ***generalized induced homomorphism*** *of* f *between the generalized cohomology sequences of the proper triads* $(Y; Y_1, Y_2)$ *and* $(X; X_1, X_2)$. $\diamondsuit$

Theorem 1.81 (Homomorphism of generalized cohomology sequences of proper triads induced by an admissible map of proper triads)**.** *In a generalized cohomology theory, if* $f : (X; X_1, X_2) \to (Y; Y_1, Y_2)$ *is an admissible map of proper triads, then* $h(f)$ *is a homomorphism of exact sequences between the generalized cohomology sequences of the triads* $(Y; Y_1, Y_2)$ *and* $(X; X_1, X_2)$.

Proof. We have to prove the following three relations for all $n \in \mathbb{Z}$. In fact, if n is an integer number, then:

- $h^{n-1} i_{(X,X_2)}^{(X_1, X_1 \cap X_2)} \circ h^{n-1}(f_3) = h^{n-1}(f_4) \circ h^{n-1} i_{(Y,Y_2)}^{(Y_1, Y_1 \cap Y_2)}$. This relation is an obvious consequence of the equality $f_3 \circ i_{(X,X_2)}^{(X_1, X_1 \cap X_2)} = i_{(Y,Y_2)}^{(Y_1, Y_1 \cap Y_2)} \circ f_4$;

- $\delta^n_{(X;X_1,X_2)} \circ h^{n-1}(f_4) = h^n(f_1) \circ \delta^n_{(Y;Y_1,Y_2)}$. To prove this relation we consider the following diagram.

$$\begin{tikzcd}
h^{n-1}(Y_1, Y_1 \cap Y_2) \arrow[r, "h^{n-1}(k_2')^{-1}"] \arrow[d, "h^{n-1}(f_4)"] \arrow[rr, bend left, "\delta^n_{(Y;Y_1,Y_2)}"] & h^{n-1}(Y_1 \cup Y_2, Y_2) \arrow[r, "\delta^n_{(Y,Y_1 \cup Y_2,Y_2)}"] \arrow[d, "h^{n-1}(f_5)"] & h^n(Y, Y_1 \cup Y_2) \arrow[d, "h^n(f_1)"] \\
h^{n-1}(X_1, X_1 \cap X_2) \arrow[r, "h^{n-1}(k_2)^{-1}"'] \arrow[rr, bend right, "\delta^n_{(X;X_1,X_2)}"'] & h^{n-1}(X_1 \cup X_2, X_2) \arrow[r, "\delta^n_{(X,X_1 \cup X_2,X_2)}"'] & h^n(X, X_1 \cup X_2)
\end{tikzcd}$$

We have $h^{n-1}(f_4) \circ h^{n-1}(k_2') = h^{n-1}(k_2) \circ h^{n-1}(f_5)$ since $k_2' \circ f_4 = f_5 \circ k_2$. Then, it follows $h^{n-1}(k_2)^{-1} \circ h^{n-1}(f_4) = h^{n-1}(f_5) \circ h^{n-1}(k_2')^{-1}$, proving that the square on the left-hand side is commutative. In turn, the square on the right-hand side is commutative because of Theorem 1.57; *and*

- $h^n(f_3) \circ h^n j^{(Y,Y_2)}_{(Y,Y_1 \cup Y_2)} = h^n j^{(X,X_2)}_{(X,X_1 \cup X_2)} \circ h^n(f_1)$. This relation is an obvious consequence of the equality $j^{(Y,Y_2)}_{(Y,Y_1 \cup Y_2)} \circ f_3 = f_1 \circ j^{(X,X_2)}_{(X,X_1 \cup X_2)}$. This completes the proof of the commutativity of the following diagram.

$$
\begin{tikzcd}
\cdots \arrow[r] & h^{n-1}(Y,Y_2) \arrow[d, "h^{n-1} i^{(Y_1,Y_1\cap Y_2)}_{(Y,Y_2)}"] \arrow[ddd, bend right=60, "h^{n-1}(f_3)"'] & h^n(Y,Y_2) \arrow[r] \arrow[ddd, bend left=60, "h^n(f_3)"] & \cdots \\
& h^{n-1}(Y_1, Y_1 \cap Y_2) \arrow[r, "\delta^n_{(Y;Y_1,Y_2)}"] \arrow[d, "h^{n-1}(f_4)"] & h^n(Y, Y_1 \cup Y_2) \arrow[u, "h^n j^{(Y,Y_2)}_{(Y,Y_1\cup Y_2)}"'] \arrow[d, "h^n(f_1)"] & \\
& h^{n-1}(X_1, X_1 \cap X_2) \arrow[r, "\delta^n_{(X;X_1,X_2)}"] & h^n(X, X_1 \cup X_2) \arrow[d, "h^n j^{(X,X_2)}_{(X,X_1\cup X_2)}"] & \\
\cdots \arrow[r] & h^{n-1}(X,X_2) \arrow[u, "h^{n-1} i^{(X_1,X_1\cap X_2)}_{(X,X_2)}"'] & h^n(X,X_2) \arrow[r] & \cdots
\end{tikzcd}
$$

This finishes the proof of the theorem. $\square$

1.11 The generalized Mayer-Vietoris sequences

In this section, we set the last notorious helpful tools in various calculus of the generalized cohomology groups, namely, the generalized Mayer-Vietoris cohomology sequences of proper triads. The main theorems here, whose proofs are just technical computations that we leave to the reader, set the exactness of these sequences. We begin with the following algebraic lemma.

Lemma 1.82 (Hexagonal Lemma)**.** *The following diagram of groups and homomorphisms has each of its triangles commutative.*

$$
\begin{tikzcd}
 & C_0 \arrow[dl, "\xi_1"'] \arrow[dr, "\xi_2"] \arrow[dd, "\varphi_0"] & \\
C_1' & & C_2' \\
 & C \arrow[ul, "\psi_1"] \arrow[ur, "\psi_2"'] \arrow[dd, "\psi_0"] & \\
C_2 \arrow[uu, "\eta_1"] \arrow[ur, "\varphi_2"] \arrow[dr, "\zeta_1"'] & & C_1 \arrow[uu, "\eta_2"'] \arrow[ul, "\varphi_1"'] \arrow[dl, "\zeta_2"] \\
 & C_0' &
\end{tikzcd}
$$

If $\mathrm{Im}(\varphi_1) \subseteq \mathrm{Ker}(\psi_1)$*,* $\mathrm{Im}(\varphi_2) = \mathrm{Ker}(\psi_2)$*,* $\mathrm{Im}(\varphi_0) \subseteq \mathrm{Ker}(\psi_0)$ *and* η_1 *and* η_2 *are isomorphisms, then* $\zeta_1 \circ \eta_1^{-1} \circ \xi_1 = -\zeta_2 \circ \eta_2^{-1} \circ \xi_2$*.*

Proof. This result is an immediate consequence of Lemma 1.68. The reader can find its proof in [13, p. 38]. $\square$

Theorem 1.83 (A consequence of the Hexagonal Lemma)**.** *In a generalized cohomology theory, let the following commutative diagram be composed of admissible pairs and inclusion maps which come from a proper triad* $(X_1 \cup X_2; X_1, X_2)$*.*

$$
\begin{tikzcd}
(X_2, X_1 \cap X_2) \arrow[dr, "i_2"] \arrow[dd, "k_1"'] & & (X_1, X_1 \cap X_2) \arrow[dl, "i_1"'] \arrow[dd, "k_2"] \\
 & (X_1 \cup X_2, X_1 \cap X_2) \arrow[dl, "j_1"'] \arrow[dr, "j_2"] & \\
(X_1 \cup X_2, X_1) & & (X_1 \cup X_2, X_2) \\
 & X_1 \cup X_2 \arrow[uu, "j"] \arrow[ul, "l_1"] \arrow[ur, "l_2"'] & \\
X_2 \arrow[ur, "m_2"] & & X_1 \arrow[ul, "m_1"'] \\
 & X_1 \cap X_2 \arrow[uu, "i"] \arrow[ul, "h_2"] \arrow[ur, "h_1"'] &
\end{tikzcd}
$$

The preceding diagram induces the following one, which is also commutative and in such manner that

$$h^n(l_1)\circ h^n(k_1)^{-1}\circ \delta^n_{(X_2,X_1\cap X_2)} \;=\; -h^n(l_2)\circ h^n(k_2)^{-1}\circ \delta^n_{(X_1,X_1\cap X_2)}$$

for all $n\in\mathbb{Z}$.

$$\begin{tikzcd}
& h^{n-1}(X_1\cap X_2) \arrow[dl, "\delta^n_{(X_2,X_1\cap X_2)}"'] \arrow[dr, "\delta^n_{(X_1,X_1\cap X_2)}"] \arrow[dd, "\delta^n_{(X_1\cup X_2,X_1\cap X_2)}"] & \\
h^n(X_2,X_1\cap X_2) & & h^n(X_1,X_1\cap X_2) \\
& h^n(X_1\cup X_2,X_1\cap X_2) \arrow[ul, "h^n(i_2)"'] \arrow[ur, "h^n(i_1)"] \arrow[dd, "h^n(j)"] & \\
h^n(X_1\cup X_2,X_1) \arrow[uu, "h^n(k_1)"] \arrow[ur, "h^n(j_1)"] \arrow[dr, "h^n(l_1)"'] & & h^n(X_1\cup X_2,X_2) \arrow[uu, "h^n(k_2)"'] \arrow[ul, "h^n(j_2)"'] \arrow[dl, "h^n(l_2)"] \\
& h^n(X_1\cup X_2) \arrow[dl, "h^n(m_2)"'] \arrow[dr, "h^n(m_1)"] \arrow[dd, "h^n(i)"] & \\
h^n(X_2) \arrow[dr, "h^n(h_2)"'] & & h^n(X_1) \arrow[dl, "h^n(h_1)"] \\
& h^n(X_1\cap X_2) &
\end{tikzcd}$$

Proof. The diagram in question is commutative because the one of inclusion maps is commutative, and because the Commutativity Axiom yields the equalities $h^n(i_2)\circ\delta^n_{(X_1\cup X_2,X_1\cap X_2)} = \delta^n(X_2,X_1\cap X_2)$ and $h^n(i_1)\circ\delta^n_{(X_1\cup X_2,X_1\cap X_2)} = \delta^n(X_1,X_1\cap X_2)$ for all $n\in\mathbb{Z}$. Moreover, for all $n\in\mathbb{Z}$:

- $\operatorname{Ker} h^n(j_1) = \operatorname{Im} h^n(i_1)$ and $\operatorname{Ker} h^n(j_2) = \operatorname{Im} h^n(i_2)$ because the generalized cohomology sequences of the triples $(X_1\cup X_2, X_1, X_1\cap X_2)$ and $(X_1\cup X_2, X_2, X_1\cap X_2)$ are exact by Theorem 1.55;

- $\operatorname{Im} \delta^n_{(X_1 \cup X_2, X_1 \cap X_2)} = \operatorname{Ker} h^n(j)$ because the generalized cohomology sequence of the pair $(X_1 \cup X_2, X_1 \cap X_2)$ is exact by the Exactness Axiom; *and*

- $h^n(k_1)$ and $h^n(k_2)$ are isomorphisms because $(X_1 \cup X_2; X_1, X_2)$ is supposed to be a proper triad.

Therefore, since we have seen that all hypotheses of Lemma 1.82 hold, it follows $h^n(l_1) \circ h^n(k_1)^{-1} \circ \delta^n_{(X_2, X_1 \cap X_2)} = -h^n(l_2) \circ h^n(k_2)^{-1} \circ \delta^n_{(X_1, X_1 \cap X_2)}$ for all $n \in \mathbb{Z}$, as we wished. $\square$

In the next paragraphs, we use the notations and the diagrams of Theorem 1.83 to define and study the generalized Mayer-Vietoris cohomology sequence of a proper triad in which the main space is the union of its subspaces. This convention will be undone when we reach Theorem 1.89, where new notations and diagrams need to be considered to define and study the generalized relative Mayer-Vietoris cohomology sequence of a generic proper triad.

Definition 1.84 (The generalized Mayer-Vietoris cohomology sequence of a proper triad)**.** *In a generalized cohomology theory, we define the* ***generalized Mayer-Vietoris cohomology sequence*** *of the proper triad $(X_1 \cup X_2; X_1, X_2)$ as the sequence:*

$$\begin{tikzcd}
 & h^n(X_1 \cup X_2) \arrow[rr, "\Phi_n"] & & h^n(X_1) \oplus h^n(X_2) \arrow[d, "\Psi_n"] & \\
\cdots \arrow[r] & h^{n-1}(X_1 \cap X_2) \arrow[u, "\Delta^n_{(X_1 \cup X_2; X_1, X_2)}"] & & h^n(X_1 \cap X_2) \arrow[r] & \cdots ,
\end{tikzcd}$$

where:

- $\Psi_n : h^n(X_1) \oplus h^n(X_2) \to h^n(X_1 \cap X_2)$ *maps each* $(u_1, u_2) \in h^n(X_1) \oplus h^n(X_2)$ *into* $h^n(h_1)(u_1) - h^n(h_2)(u_2) \in h^n(X_1 \cap X_2)$*;*

- $\Phi_n : h^n(X_1 \cup X_2) \to h^n(X_1) \oplus h^n(X_2)$ *maps each* $v \in h^n(X_1 \cup X_2)$ *into* $\big(h^n(m_1)(v), h^n(m_2)(v)\big) \in h^n(X_1) \oplus h^n(X_2)$*; and*

- $\Delta^n_{(X_1 \cup X_2; X_1, X_2)} : h^{n-1}(X_1 \cap X_2) \to h^n(X_1 \cup X_2)$, *named the* ***nth generalized Mayer-Vietoris coboundary operator*** *of* $(X_1 \cup X_2; X_1, X_2)$, *is the composition* $-h^n(l_1) \circ h^n(k_1)^{-1} \circ \delta^n_{(X_2, X_1 \cap X_2)}$ *which coincides with the composition* $h^n(l_2) \circ h^n(k_2)^{-1} \circ \delta^n_{(X_1, X_1 \cap X_2)}$ *by Theorem 1.83.* ♢

Theorem 1.85 (Relation between the generalized coboundary operator of a proper triad and the generalized Mayer-Vietoris coboundary operator of an associated proper triad)**.** *In a generalized cohomology theory, if* $(X; X_1, X_2)$ *is a proper triad, then the following diagram is commutative for all* $n \in \mathbb{Z}$.

$$\begin{tikzcd}
h^{n-1}(X_1 \cap X_2) \arrow[r, "\Delta^n_{(X_1 \cup X_2; X_1, X_2)}"] \arrow[d, "\delta^n_{(X_1, X_1 \cap X_2)}"'] & h^n(X_1 \cup X_2) \arrow[d, "\delta^{n+1}_{(X, X_1 \cup X_2)}"] \\
h^n(X_1, X_1 \cap X_2) \arrow[r, "\delta^{n+1}_{(X; X_1, X_2)}"'] & h^{n+1}(X, X_1 \cup X_2)
\end{tikzcd}$$

Proof. For all $n \in \mathbb{Z}$, we have $\delta^n_{(X, X_1 \cup X_2, X_2)} = \delta^n_{(X, X_1 \cup X_2)} \circ h^{n-1}(l_2)$ by Definition 1.54, $\delta^n_{(X; X_1, X_2)} = \delta^n_{(X, X_1 \cup X_2, X_2)} \circ h^{n-1}(k_2)^{-1}$ by Definition 1.76, and $\Delta^n_{(X_1 \cup X_2; X_1, X_2)} = h^n(l_2) \circ h^n(k_2)^{-1} \circ \delta^n_{(X_1, X_1 \cap X_2)}$ by Definition 1.84. Therefore, it follows that

$$\begin{aligned}
\delta^{n+1}_{(X; X_1, X_2)} \circ \delta^n_{(X_1, X_1 \cap X_2)} &= \delta^{n+1}_{(X, X_1 \cup X_2, X_2)} \circ h^n(k_2)^{-1} \circ \delta^n_{(X_1, X_1 \cap X_2)} \\
&= \delta^{n+1}_{(X, X_1 \cup X_2)} \circ h^n(l_2) \circ h^n(k_2)^{-1} \circ \delta^n_{(X_1, X_1 \cap X_2)} \\
&= \delta^{n+1}_{(X, X_1 \cup X_2)} \circ \Delta^n_{(X_1 \cup X_2; X_1, X_2)}
\end{aligned}$$

for all $n \in \mathbb{Z}$, as we wished. □

Theorem 1.86 (The generalized Mayer-Vietoris cohomology sequence is exact)**.** *In a generalized cohomology theory, the generalized Mayer-Vietoris cohomology sequence of a proper triad* $(X_1 \cup X_2; X_1, X_2)$ *is exact.*

Proof. Let n be an integer number. The following six assertions complete the proof of this theorem.

(1) $\Psi_n \circ \Phi_n : h^n(X_1 \cup X_2) \to h^n(X_1 \cap X_2)$ *is the trivial homomorphism. Thus,* $\mathrm{Im}(\Phi_n) \subseteq \mathrm{Ker}(\Psi_n)$.

(2) *If* $(u_1, u_2) \in h^n(X_1) \oplus h^n(X_2)$ *and* $\Psi_n(u_1, u_2) = 0$, *which is the same as* $(u_1, u_2) \in \mathrm{Ker}(\Psi_n)$, *then there exists* $u' \in h^n(X_1 \cup X_2)$ *such that* $\Phi_n(u') = (u_1, u_2)$, *which is the same as* $(u_1, u_2) \in \mathrm{Im}(\Phi_n)$. *Thus,* $\mathrm{Ker}(\Psi_n) \subseteq \mathrm{Im}(\Phi_n)$.

(3) $\Phi_n \circ \Delta^n_{(X_1 \cup X_2; X_1, X_2)} : h^{n-1}(X_1 \cap X_2) \to h^n(X_1) \oplus h^n(X_2)$ *is the trivial homomorphism. Thus,* $\mathrm{Im}\, \Delta^n_{(X_1 \cup X_2; X_1, X_2)} \subseteq \mathrm{Ker}(\Phi_n)$.

(4) *If* $v \in h^n(X_1 \cup X_2)$ *and* $\Phi_n(v) = (0,0)$, *which is the same as* $v \in \mathrm{Ker}(\Phi_n)$, *then there exists* $v' \in h^{n-1}(X_1 \cap X_2)$ *such that* $\Delta^n_{(X_1 \cup X_2; X_1, X_2)}(v') = v$, *which is the same as* $v \in \mathrm{Im}\, \Delta^n_{(X_1 \cup X_2; X_1, X_2)}$. *Thus,* $\mathrm{Ker}(\Phi_n) \subseteq \mathrm{Im}\, \Delta^n_{(X_1 \cup X_2; X_1, X_2)}$.

(5) $\Delta^n_{(X_1 \cup X_2; X_1, X_2)} \circ \Psi_{n-1} : h^{n-1}(X_1) \oplus h^{n-1}(X_2) \to h^n(X_1 \cup X_2)$ *is the trivial homomorphism. Thus,* $\mathrm{Im}(\Psi_{n-1}) \subseteq \mathrm{Ker}\, \Delta^n_{(X_1 \cup X_2; X_1, X_2)}$.

(6) *If* $w \in h^{n-1}(X_1 \cap X_2)$ *and* $\Delta^n_{(X_1 \cup X_2; X_1, X_2)}(w) = 0$, *which is the same as* $w \in \mathrm{Ker}\Delta^n_{(X_1 \cup X_2; X_1, X_2)}$, *then there exists* $(w_1, w_2) \in h^{n-1}(X_1) \oplus h^{n-1}(X_2)$ *such that* $\Psi_{n-1}(w_1, w_2) = w$, *which is the same as* $w \in \mathrm{Im}(\Psi_{n-1})$. *Thus,* $\mathrm{Ker}\, \Delta^n_{(X_1 \cup X_2; X_1, X_2)} \subseteq \mathrm{Im}(\Psi_{n-1})$.

We leave these instructive details to the reader. In order to complete them, we recommend a closer look at the proof of Theorem 1.55. $\square$

Definition 1.87 (The generalized induced homomorphism between generalized Mayer-Vietoris cohomology sequences of proper triads)**.** *In a generalized cohomology theory, if* $f : (X_1 \cup X_2; X_1, X_2) \to (Y_1 \cup Y_2; Y_1, Y_2)$ *is an admissible map of proper triads, then we say that the sequence of group homomorphisms*

$$h(f) := (\cdots,\ h^{n-1}(f_4 \mid_{X_1 \cap X_2}),\ h^n(f_1 \mid_{X_1 \cup X_2}),\ h^n(f_2 \mid_{X_1}) \oplus h^n(f_3 \mid_{X_2}),\ h^n(f_4 \mid_{X_1 \cap X_2}),\ \cdots)$$

is the ***generalized induced homomorphism*** *of* f *between the generalized Mayer-Vietoris cohomology sequences of the proper triads* $(Y_1 \cup Y_2; Y_1, Y_2)$ *and* $(X_1 \cup X_2; X_1, X_2)$. $\diamondsuit$

Theorem 1.88 (Homomorphism of generalized Mayer-Vietoris cohomology sequences induced by a map of triads)**.** *In a generalized cohomology theory, if $f : (X_1 \cup X_2; X_1, X_2) \to (Y_1 \cup Y_2; Y_1, Y_2)$ is an admissible map of proper triads, then $h(f)$ is a homomorphism of exact sequences between the generalized Mayer-Vietoris cohomology sequences of the triads $(Y_1 \cup Y_2; Y_1, Y_2)$ and $(X_1 \cup X_2; X_1, X_2)$.*

Proof. To verify the statement of this theorem we have to prove that the following diagram is commutative.

$$\begin{tikzcd}
\cdots \arrow[r] & h^{n-1}(Y_1 \cap Y_2) \arrow[d, "\Delta^n_{(Y_1 \cup Y_2; Y_1, Y_2)}"] \arrow[ddd, bend right=60, "h^{n-1}(f_4|_{X_1 \cap X_2})"'] & h^n(Y_1 \cap Y_2) \arrow[r] \arrow[ddd, bend left=60, "h^n(f_4|_{X_1 \cap X_2})"] & \cdots \\
 & h^n(Y_1 \cup Y_2) \arrow[r, "\Phi'_n"] \arrow[d, "h^n(f_1|_{X_1 \cup X_2})"] & h^n(Y_1) \oplus h^n(Y_2) \arrow[u, "\Psi'_n"] \arrow[d, "h^n(f_2|_{X_1}) \oplus h^n(f_3|_{X_2})"] & \\
 & h^n(X_1 \cup X_2) \arrow[r, "\Phi_n"] & h^n(X_1) \oplus h^n(X_2) \arrow[d, "\Psi_n"] & \\
\cdots \arrow[r] & h^{n-1}(X_1 \cap X_2) \arrow[u, "\Delta^n_{(X_1 \cup X_2; X_1, X_2)}"] & h^n(X_1 \cap X_2) \arrow[r] & \cdots
\end{tikzcd}$$

We leave the details to the reader. □

To close this section, we present the generalized relative Mayer-Vietoris cohomology sequence of a generic proper triad. This sequence is not a generalization of the one that we have just studied in this section. In fact, we will see later when we apply the new sequence to a proper triad in which the main space is the union of its subspaces that it yields a conclusion which the first sequence does not yield. Since the proofs here are essentially the same as the preceding ones, we will leave them to the reader. We begin with the following theorem, whose notations and diagrams will be considered until the end of this section.

Theorem 1.89 (Another consequence of the Hexagonal Lemma)**.** *In a generalized cohomology theory, let the following (disconnected) commutative diagram be composed of admissible pairs of topological spaces and of inclusion maps which come from a proper triad* $(X; X_1, X_2)$*.*

$$\begin{tikzcd}
 & (X, X_1 \cap X_2) & \\
(X_2, X_1 \cap X_2) \arrow[ur, "p_1"] \arrow[dr, "i_2"] \arrow[dd, "k_1"'] & & (X_1, X_1 \cap X_2) \arrow[ul, "p_2"'] \arrow[dl, "i_1"'] \arrow[dd, "k_2"] \\
 & (X_1 \cup X_2, X_1 \cap X_2) \arrow[uu, "j"] \arrow[dl, "q_1"] \arrow[dr, "q_2"'] & \\
(X_1 \cup X_2, X_1) & & (X_1 \cup X_2, X_2)
\end{tikzcd}$$

$$\begin{tikzcd}
 & (X, X_1 \cup X_2) & \\
(X, X_2) \arrow[ur, "m_2"] & & (X, X_1) \arrow[ul, "m_1"'] \\
 & (X, X_1 \cap X_2) \arrow[uu, "i"] \arrow[ul, "h_2"] \arrow[ur, "h_1"'] &
\end{tikzcd}$$

The preceding diagrams induce the following one, which is also commutative and in such manner that

$$\delta^n_{(X, X_1 \cup X_2, X_1)} \circ h^{n-1}(k_1)^{-1} \circ h^{n-1}(p_1) \;=\; -\delta^n_{(X, X_1 \cup X_2, X_2)} \circ h^{n-1}(k_2)^{-1} \circ h^{n-1}(p_2)$$

for all $n \in \mathbb{Z}$*.*

$$\begin{tikzcd}
 & h^{n-1}(X, X_1 \cap X_2) \arrow[dl, "h^{n-1}(p_1)"'] \arrow[dr, "h^{n-1}(p_2)"] \arrow[dd, "h^{n-1}(j)"] & \\
h^{n-1}(X_2, X_1 \cap X_2) & & h^{n-1}(X_1, X_1 \cap X_2) \\
 & h^{n-1}(X_1 \cup X_2, X_1 \cap X_2) \arrow[ul, "h^{n-1}(i_2)"] \arrow[ur, "h^{n-1}(i_1)"'] \arrow[dd, "\delta^n_{(X, X_1 \cup X_2, X_1 \cap X_2)}"] & \\
h^{n-1}(X_1 \cup X_2, X_1) \arrow[uu, "h^{n-1}(k_1)"] \arrow[ur, "h^{n-1}(q_1)"] \arrow[dr, "\delta^n_{(X, X_1 \cup X_2, X_1)}"'] & & h^{n-1}(X_1 \cup X_2, X_2) \arrow[uu, "h^{n-1}(k_2)"'] \arrow[ul, "h^{n-1}(q_2)"'] \arrow[dl, "\delta^n_{(X, X_1 \cup X_2, X_2)}"] \\
 & h^n(X, X_1 \cup X_2) \arrow[dl, "h^n(m_2)"'] \arrow[dr, "h^n(m_1)"] \arrow[dd, "h^n(i)"] & \\
h^n(X, X_2) \arrow[dr, "h^n(h_2)"'] & & h^n(X, X_1) \arrow[dl, "h^n(h_1)"] \\
 & h^n(X, X_1 \cap X_2) &
\end{tikzcd}$$

Proof. The proof of this result is analogous to the proof of Theorem 1.83. Then, we leave the details to the reader. □

Definition 1.90 (The generalized relative Mayer-Vietoris cohomology sequence of a proper triad)**.** *In a generalized cohomology theory, we define the* ***generalized relative Mayer-Vietoris cohomology sequence*** *of the proper triad* $(X; X_1, X_2)$ *as the sequence:*

$$\begin{tikzcd}
 & h^n(X, X_1 \cup X_2) \arrow[rr, "\Phi_n"] & & h^n(X, X_1) \oplus h^n(X, X_2) \arrow[d, "\Psi_n"] & \\
\cdots \arrow[r] & h^{n-1}(X, X_1 \cap X_2) \arrow[u, "\Delta^n_{(X; X_1, X_2)}"] & & h^n(X, X_1 \cap X_2) \arrow[r] & \cdots ,
\end{tikzcd}$$

where:

- $\Psi_n : h^n(X, X_1) \oplus h^n(X, X_2) \to h^n(X, X_1 \cap X_2)$ *maps each pair of elements* $(u_1, u_2) \in h^n(X, X_1) \oplus h^n(X, X_2)$ *into* $h^n(h_1)(u_1) - h^n(h_2)(u_2) \in h^n(X, X_1 \cap X_2)$*;*

- $\Phi_n : h^n(X, X_1 \cup X_2) \to h^n(X, X_1) \oplus h^n(X, X_2)$ *maps each* $v \in h^n(X, X_1 \cup X_2)$ *into* $\big(h^n(m_1)(v), h^n(m_2)(v)\big) \in h^n(X, X_1) \oplus h^n(X, X_2)$*; and*

- $\Delta^n_{(X;X_1,X_2)} : h^{n-1}(X, X_1 \cap X_2) \to h^n(X, X_1 \cup X_2)$*, named the* ***nth generalized relative Mayer-Vietoris coboundary operator*** *of* $(X; X_1, X_2)$*, is the composition* $-\delta^n_{(X,X_1 \cup X_2,X_1)} \circ h^{n-1}(k_1)^{-1} \circ h^{n-1}(p_1)$ *which coincides with the composition* $\delta^n_{(X,X_1 \cup X_2,X_2)} \circ h^{n-1}(k_2)^{-1} \circ h^{n-1}(p_2)$ *by Theorem 1.89.* ♢

Theorem 1.91 (The generalized relative Mayer-Vietoris cohomology sequence is exact)**.** *In a generalized cohomology theory, the generalized relative Mayer-Vietoris cohomology sequence of a proper triad* $(X; X_1, X_2)$ *is exact.*

Proof. The proof of this result is analogous to the proof of Theorem 1.86. Then, we leave the details to the reader. □

Remark 1.92 (The generalized relative Mayer-Vietoris cohomology sequence of a proper triad in which the main space is the union of its subspaces)**.** *In a generalized cohomology theory, let* $(X_1 \cup X_2; X_1, X_2)$ *be a proper triad. The generalized Mayer-Vietoris cohomology sequence of* $(X_1 \cup X_2; X_1, X_2)$ *is different from the generalized relative Mayer-Vietoris cohomology sequence of* $(X_1 \cup X_2; X_1, X_2)$*. In particular, since* $h^n(X_1 \cup X_2, X_1 \cup X_2)$ *is the trivial group for all* $n \in \mathbb{Z}$ *by Corollary 1.12, the last sequence is*

$$\begin{tikzcd}
 & h^n(X_1 \cup X_2, X_1) \oplus h^n(X_1 \cup X_2, X_2) \arrow[r, "\Psi_n"] & h^n(X_1 \cup X_2, X_1 \cap X_2) \arrow[d] & \\
\cdots \arrow[r] & 0 \arrow[u] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Therefore, since the preceding sequence is exact because of Theorem 1.91, we have that Ψ_n *is an isomorphism from* $h^n(X_1 \cup X_2, X_1) \oplus h^n(X_1 \cup X_2, X_2)$ *onto* $h^n(X_1 \cup X_2, X_1 \cap X_2)$ *for* *all $n \in \mathbb{Z}$. In other words, for each $n \in \mathbb{Z}$ and each element $u \in h^n(X_1 \cup X_2, X_1 \cap X_2)$, there exists a unique $(u_1, u_2) \in h^n(X_1 \cup X_2, X_1) \oplus h^n(X_1 \cup X_2, X_2)$ in such manner that $u = h^n(h_1)(u_1) - h^n(h_2)(u_2)$.* $\diamondsuit$

Definition 1.93 (The generalized induced homomorphism between generalized relative Mayer-Vietoris cohomology sequences of proper triads)**.** *In a generalized cohomology theory, if $f : (X; X_1, X_2) \to (Y; Y_1, Y_2)$ is an admissible map of proper triads, then we say that the sequence of group homomorphisms*

$$h(f) := (\cdots,\ h^{n-1}(f_6),\ h^n(f_1),\ h^n(f_2) \oplus h^n(f_3),\ h^n(f_6),\ \cdots)$$

is the ***generalized induced homomorphism*** *of f between the generalized relative Mayer-Vietoris cohomology sequences of the proper triads $(Y; Y_1, Y_2)$ and $(X; X_1, X_2)$.* $\diamondsuit$

Theorem 1.94 (Homomorphism of generalized relative Mayer-Vietoris cohomology sequences induced by a map of triads)**.** *In a generalized cohomology theory, if $f : (X; X_1, X_2) \to (Y; Y_1, Y_2)$ is an admissible map of proper triads, then $h(f)$ is a homomorphism of exact sequences between the generalized relative Mayer-Vietoris cohomology sequences of the triads $(Y; Y_1, Y_2)$ and $(X; X_1, X_2)$.*

Proof. The proof of this result is analogous to the proof of Theorem 1.88. We leave the details to the reader. $\square$

1.12 Multiplicative generalized cohomology theories

In this section, we present axioms for multiplicative structures in generalized cohomology theories. In general, the fact that cohomology theories can be enriched with multiplicative structures turns them into stronger sources of information when compared to homology theories. It is to be noted that the exposition below is not common in the references, which prefer to treat multiplicative structures of particular cohomology theories. We begin with the following definition in which we select between all of the admissible categories of topological spaces the ones that can support multiplicative structures.

Definition 1.95 (Multiplicative category of topological spaces). *An admissible category of topological spaces $\mathscr{C}$ is a **multiplicative category of topological spaces** if it satisfies the following two conditions.*

(1) If (X, A) and (Y, B) are admissible pairs, then the pair

$$(X, A) \times (Y, B) \; := \; (X \times Y,\, Z),$$

where $Z := (A \times Y) \cup (X \times B)$, is also admissible.

(2) If $f : (X, A) \to (X', A')$ and $g : (Y, B) \to (Y', B')$ are admissible maps of pairs, then the natural map $f \times g : (X, A) \times (Y, B) \to (X', A') \times (Y', B')$ is also admissible. ♢

Definition 1.96 (Multiplicative generalized cohomology theories). *In a generalized cohomology theory based on a multiplicative category of topological spaces $\mathscr{C}$, a **multiplicative structure** is a map μ that sends $m, n \in \mathbb{Z}$ and $(X, A), (Y, B) \in \mathscr{C}$ into a group homomorphism*

$$\mu^{m,n}_{(X,A),(Y,B)} : h^m(X, A) \otimes h^n(Y, B) \to h^{m+n}(X \times Y, Z)$$

that satisfies the following five axioms. Here $\otimes$ denotes the usual tensor product of abelian groups. For convenience, we will denote $\mu^{m,n}_{(X,A),(Y,B)}$ simply by $\mu_{m,n}$ (although it is an abuse of notation).

*(1) **Naturality Axiom**. For every integer numbers $m, n \in \mathbb{Z}$ and every admissible maps $f : (X, A) \to (X', A')$ and $g : (Y, B) \to (Y', B')$, the following diagram is commutative.*

$$\begin{tikzcd}
h^m(X,\, A) \otimes h^n(Y,\, B) \arrow[r, "\mu_{m,n}"] & h^{m+n}(X \times Y,\, Z) \\
h^m(X',\, A') \otimes h^n(Y',\, B') \arrow[r, "\mu_{m,n}"'] \arrow[u, "h^m(f) \otimes h^n(g)"] & h^{m+n}(X' \times Y',\, Z') \arrow[u, "h^{m+n}(f \times g)"']
\end{tikzcd}$$

(2) ***Excision-compatibility Axiom***. *For every integer numbers $m, n \in \mathbb{Z}$ and every admissible pairs (X, A) and (Y, B), if the excision maps*

$$\begin{aligned} \eta :\ (A \times Y,\, A \times B) &\ \rightarrow\ (Z,\, X \times B) \quad \text{and} \\ \theta :\ (X \times B,\, A \times B) &\ \rightarrow\ (Z,\, A \times Y) \end{aligned}$$

are admissible and induce isomorphisms in all degrees, then the following two diagrams are commutative. Note that η is the excision of $(Z,\, X \times B)$ with respect to $(X - A) \times B$ as well as θ is the excision of $(Z,\, A \times Y)$ with respect to $A \times (Y - B)$.

$$\begin{tikzcd}
h^m(A) \otimes h^n(Y,\, B) \arrow[rr, "\delta^m_{(X,A)} \otimes \mathrm{id}_{h^n(Y,B)}"] \arrow[d, "\mu_{m,n}"'] & & h^{m+1}(X,\, A) \otimes h^n(Y,\, B) \arrow[dd, "\mu_{m+1,n}"] \\
h^{m+n}(A \times Y,\, A \times B) \arrow[d, "h^{m+n}(\eta)^{-1}"'] & & \\
h^{m+n}(Z,\, X \times B) \arrow[rr, "\delta^{m+n}_{(X \times Y,\, Z,\, X \times B)}"'] & & h^{m+n+1}(X \times Y,\, Z)
\end{tikzcd}$$

$$\begin{tikzcd}
h^m(X,\, A) \otimes h^n(B) \arrow[rr, "(-1)^m \mathrm{id}_{h^m(X,A)} \otimes \delta^n_{(Y,B)}"] \arrow[d, "\mu_{m,n}"'] & & h^m(X,\, A) \otimes h^{n+1}(Y,\, B) \arrow[dd, "\mu_{m,n+1}"] \\
h^{m+n}(X \times B,\, A \times B) \arrow[d, "h^{m+n}(\theta)^{-1}"'] & & \\
h^{m+n}(Z,\, A \times Y) \arrow[rr, "\delta^{m+n}_{(X \times Y,\, Z,\, A \times Y)}"'] & & h^{m+n+1}(X \times Y,\, Z)
\end{tikzcd}$$

(3) ***Associativity Axiom***. *For every integer numbers $k, m, n \in \mathbb{Z}$ and every admissible pairs (X, A), (Y, B) and (W, C), the following diagram is commutative, where*

$$\begin{aligned} U &:= (C \times X) \cup (W \times A) \quad \textit{and} \\ V &:= (C \times X \times Y) \cup (W \times Z). \end{aligned}$$

$$\begin{tikzcd} h^k(W,\, C) \otimes h^m(X,\, A) \otimes h^n(Y,\, B) \arrow[r, "\mathrm{id}_{h^k(W,C)} \otimes \mu_{m,n}"] \arrow[d, "\mu_{k,m} \otimes \mathrm{id}_{h^n(Y,B)}"'] & h^k(W,\, C) \otimes h^{m+n}(X \times Y,\, Z) \arrow[d, "\mu_{k,m+n}"] \\ h^{k+m}(W \times X,\, U) \otimes h^n(Y,\, B) \arrow[r, "\mu_{k+m,n}"'] & h^{k+m+n}(W \times X \times Y,\, V) \end{tikzcd}$$

(4) ***Unit Axiom***. *Being Ω a one-point space in $\mathscr{C}$, there exists an element $1 \in h^0(\Omega)$ such that*

$$\begin{aligned} (h^m(i_1) \circ \mu_{0,m})(1 \otimes u) &= u \quad \textit{and} \\ (h^m(i_2) \circ \mu_{m,0})(u \otimes 1) &= u \end{aligned}$$

for all $m \in \mathbb{Z}$ and all $u \in h^n(X, A)$, where $i_1 : (X, A) \to (\Omega \times X, \Omega \times A)$ and $i_2 : (X, A) \to (X \times \Omega, A \times \Omega)$ are the natural inclusions. It is to be noted that these inclusions are homeomorphisms since the natural projections are their inverses.

(5) ***Commutativity Axiom***. *For every integer numbers $m, n \in \mathbb{Z}$ and every admissible pairs (X, A) and (Y, B), if*

$$Z^{-1} := (B \times X) \cup (Y \times A),$$

then the following diagram is commutative, where

$$\nu_{n,m}(u \otimes v) := (-1)^{mn} v \otimes u$$

for all $u \otimes v \in h^m(X, A) \otimes h^n(Y, B)$, and

$$\begin{aligned} \alpha : (X \times Y,\, Z) &\to (Y \times X,\, Z^{-1}), \\ (x,y) &\mapsto (y,x). \end{aligned}$$

$$\begin{tikzcd} h^m(X,\,A) \otimes h^n(Y,\,B) \arrow[r, "\nu_{n,m}"] \arrow[d, "\mu_{m,n}"'] & h^n(Y,\,B) \otimes h^m(X,\,A) \arrow[d, "\mu_{n,m}"] \\ h^{n+m}(X \times Y,\, Z) & h^{n+m}(Y \times X,\, Z^{-1}) \arrow[l, "h^{n+m}(\alpha)"] \end{tikzcd}$$

A generalized cohomology theory equipped with a multiplicative structure is called a ***multiplicative generalized cohomology theory****. A multiplicative structure μ is also said to be an* ***external multiplication****. Moreover, if $u \in h^m(X, A)$ and $v \in h^n(Y, B)$, then $\mu_{m,n}(u \otimes v)$ is denoted by $u \times v$ and is called the* ***cross product*** *of these elements.* ♢

Remark 1.97 (On the Excision-compatibility Axiom)**.** *In a multiplicative generalized cohomology theory, let (X, A) and (Y, B) be admissible pairs of topological spaces. Let the excision maps*

$$\begin{aligned} \eta :\ (A \times Y,\, A \times B) &\to (Z,\, X \times B) \quad \text{and} \\ \theta :\ (X \times B,\, A \times B) &\to (Z,\, A \times Y) \end{aligned}$$

be as in Condition (2) of Definition 1.96. These maps induce isomorphisms in all degrees if $(X - A) \times B$ and $A \times (Y - B)$ are open subsets of Z such that their closures are contained in the interiors of $X \times B$ and $A \times Y$, respectively. In particular, these conditions are satisfied if A and B are both open and closed subsets of X and Y, respectively. ♢

Theorem 1.98 (Internal and external multiplications)**.** *In a multiplicative generalized cohomology theory, let (X, A) and (X, B) be admissible pairs of topological spaces. If $\Delta : X \to X \times X$ is the diagonal map, then the composition $\varphi_{m,n}$ defined by the commutative diagram*

$$\begin{tikzcd}
h^m(X,\, A) \otimes h^n(X,\, B) \arrow[r, "\mu_{m,n}"'] \arrow[rr, bend left, "\varphi_{m,n}"] & h^{m+n}(X \times X, Z) \arrow[r, "h^{m+n}(\Delta)"'] & h^{m+n}(X, A \cup B)
\end{tikzcd}$$

is a natural homomorphism, which is called the ***internal multiplication*** *or the* ***cup product****. Moreover, the following diagram is commutative for all integer numbers* $m, n \in \mathbb{Z}$ *and all admissible pairs* (X, A) *and* (Y, B)*, where* $\pi_1 : (X \times Y, A \times Y) \to (X, A)$ *and* $\pi_2 : (X \times Y, X \times B) \to (Y, B)$ *are the natural projections onto the first and the second factors, respectively.*

$$\begin{tikzcd}
h^m(X \times Y, A \times Y) \otimes h^n(X \times Y, X \times B) \arrow[r, "\varphi_{m,n}"] & h^{m+n}(X \times Y, Z) \\
h^m(X,\, A) \otimes h^n(Y,\, B) \arrow[u, "h^m(\pi_1) \otimes h^n(\pi_2)"] \arrow[ur, "\mu_{m,n}"'] &
\end{tikzcd}$$

This shows that we can recover the external multiplication from the internal multiplication in a canonical way.

Proof. For every integer numbers $m, n \in \mathbb{Z}$ and every admissible map $f : X \to X'$ for which $f(A) \subseteq A'$ and $f(B) \subseteq B'$, the naturality of $\varphi_{m,n}$ is ensured by the following commutative diagram.

$$\begin{tikzcd}
h^m(X, A) \otimes h^n(X, B) \arrow[r, "\mu_{m,n}"] \arrow[rr, bend left, "\varphi_{m,n}"] & h^{m+n}(X \times X, Z) \arrow[r, "h^{m+n}(\Delta)"] & h^{m+n}(X, A \cup B) \\
h^m(X', A') \otimes h^n(X', B') \arrow[u, "h^m(f) \otimes h^n(f)"] \arrow[r, "\mu_{m,n}"'] \arrow[rr, bend right, "\varphi_{m,n}"'] & h^{m+n}(X' \times X', Z') \arrow[u, "h^{m+n}(f \times f)"] \arrow[r, "h^{m+n}(\Delta')"'] & h^{m+n}(X', A' \cup B'). \arrow[u, "h^{m+n}(f)"]
\end{tikzcd}$$

Note that this diagram is commutative because its right-hand square is commutative by the functoriality of the contravariant functor h^{m+n} since $(f \times f) \circ \Delta = \Delta' \circ f$, and because its left-hand square is commutative by the Naturality Axiom. Finally, note that the last assertion of the statement follows from the fact that the following diagram is commutative, where

$$U := (A \times Y \times X \times Y) \cup (X \times Y \times X \times B).$$

$$\begin{tikzcd}
h^m(X \times Y, A \times Y) \otimes h^n(X \times Y, X \times B) \arrow[r, "\mu_{m,n}"] & h^{m+n}(X \times Y \times X \times Y, U) \arrow[d, "h^{m+n}(\Delta)"] \\
h^m(X, A) \otimes h^n(Y, B) \arrow[u, "h^m(\pi_1) \otimes h^n(\pi_2)"] \arrow[r, "\mu_{m,n}"'] & h^{m+n}(X \times Y, Z)
\end{tikzcd}$$

In turn, this diagram is commutative because of the Naturality Axiom since we have $h^{m+n}(\Delta) = h^{m+n}(\pi_1 \times \pi_2)^{-1}$. $\square$

Remark 1.99 (Graded commutative ring and graded module of a multiplicative generalized cohomology theory)**.** *In a multiplicative generalized cohomology theory, let Ω be an admissible single point. In this situation, the reader can prove that the following assertions are true.*

- *If X is an admissible collapsible space and*

$$h(X) := \bigoplus_{n \in \mathbb{Z}} h^n(X),$$

then $h(X)$ is a graded commutative ring with unit under the internal multiplication. In fact, its unit is the obvious one formed from $h^0(p_X)(1)$, where $p_X : X \to \Omega$ is the only possible map and $1 \in h^0(\Omega)$ is the element whose existence is ensured by the Unit Axiom.

- *If (X, A) is an admissible pair such that X is a collapsible space and*

$$h(X, A) := \bigoplus_{n \in \mathbb{Z}} h^n(X, A),$$

then $h(X, A)$ is a graded module over the graded commutative ring with unit $h(X)$ defined above. ♢

1.13 Compactly-supported cohomology

In this section, we describe an important construction from the generalized cohomology groups, namely, the compactly-supported generalized cohomology groups. This new object is especially interesting when one desires to enlarge the scope of a particular generalized cohomology theory. This last phrase will become clearer when we define the compactly-supported K-Theory groups in Section 2.10. We begin with the following definition.

Definition 1.100 (Compatible family of compact subspaces of an admissible space)**.** *Let $\mathscr{C}$ be an admissible category of topological spaces. Being X an admissible space, we say that the* ***compatible family of compact subspaces of X*** *is the collection $\mathfrak{K}_{\mathscr{C}}(X)$ whose elements are the compact subspaces of X that satisfy the following three conditions.*

(1) If $K, L \in \mathfrak{K}_{\mathscr{C}}(X)$, then $K \cup L \in \mathfrak{K}_{\mathscr{C}}(X)$.

(2) If $K \in \mathfrak{K}_{\mathscr{C}}(X)$, then $(X, X - K)$ is an admissible pair in $\mathscr{C}$.

(3) If $K, L \in \mathfrak{K}_{\mathscr{C}}(X)$ with $K \subseteq L$, then the inclusion $i^X_{KL} : (X, X - L) \to (X, X - K)$ is an admissible map of pairs in $\mathscr{C}$. ♢

Definition 1.101 (Generalized compactly-supported cohomology groups)**.** *In a generalized cohomology theory based on an admissible category $\mathscr{C}$, let X be an admissible space. Being n an integer number, we say that the* ***nth direct system of generalized cohomology groups of X relative to compact subspaces*** *is the triple*

$$\mathfrak{A}_X^n := \big(\mathfrak{K}_{\mathscr{C}}(X),\ (h^n(X, X-K))_{K\in\mathfrak{K}_{\mathscr{C}}(X)},\ (h^n i_{KL}^X : h^n(X, X-K) \to h^n(X, X-L))_{K,L\in\mathfrak{K}_{\mathscr{C}}(X)}\big),$$

where $h^n i_{KL}^X : h^n(X, X-K) \to h^n(X, X-L)$ *is the nth induced homomorphism of the inclusion map* $i_{KL}^X : (X, X-L) \to (X, X-K)$ *if* K *is contained in* L*, and is the trivial homomorphism otherwise. Furthermore, we define the* ***nth compactly-supported generalized cohomology group of*** X*, and denote it by* $h_c^n(X)$*, to be the direct limit of abelian groups*

$$h_c^n(X) := \lim_{\longrightarrow K} h^n(X, X-K),$$

which is equipped with the family $(\iota_K^n : h^n(X, X-K) \to h_c^n(X))_{K\in\mathfrak{K}_{\mathscr{C}}(X)}$ *of morphisms of abelian groups.* ♢

Remark 1.102 (On the notions presented above)**.** *We have the following facts about the notions presented above.*

- *We have that Condition (1) of Definition 1.100 ensures that* $\mathfrak{K}_{\mathscr{C}}(X)$ *is a direct set with respect to the partial order given by the inclusion of compact subspaces. Indeed, for any* $K, L \in \mathfrak{K}_{\mathscr{C}}(X)$*, we have* $K \subseteq K\cup L$*,* $L \subseteq K \cup L$ *and* $K\cup L \in \mathfrak{K}_{\mathscr{C}}(X)$*.*

- *We have that Condition (2) of Definition 1.100 ensures that, for every* $K \in \mathfrak{K}_{\mathscr{C}}(X)$*, it makes sense taking the generalized relative cohomology group* $h^n(X, X-K)$ *for all* $n \in \mathbb{Z}$*.*

- *We have that Condition (3) of Definition 1.100 ensures that, for every* $K, L \in \mathfrak{K}_{\mathscr{C}}(X)$ *such that* $K \subseteq L$*, it makes sense taking the generalized induced homomorphism* $h^n i_{KL}^X : h^n(X, X-K) \to h^n(X, X-L)$ *for all* $n \in \mathbb{Z}$*.*

- *The three items above ensure that* $\mathfrak{A}_X^n$ *is well-defined for all* $n \in \mathbb{Z}$*. However, in order to prove that* $\mathfrak{A}_X^n$ *is a direct system of abelian groups, we still have to show that* $h^n i_{KK}^X = \mathrm{id}_{h^n(X,X-K)}$ *for all* $K \in \mathfrak{K}_{\mathscr{C}}(X)$ *and that, for every* $K, L, M \in \mathfrak{K}_{\mathscr{C}}(X)$ *that verify* $K \subseteq L \subseteq M$*, we have* $h^n i_{KM}^X = h^n i_{LM}^X \circ h^n i_{KL}^X$*. These equations are immediate consequences of the functoriality of the generalized cohomology theory under consideration.*

- *The four items above ensure that the direct limit* $h_c^n(X) = \varinjlim_K h^n(X, X - K)$ *is well-defined. Moreover, because of Theorem A.7, we know that two classes* $[u], [v] \in h_c^n(X)$ *are equal, where* $u \in h^n(X, X-K)$ *and* $v \in h^n(X, X-L)$ *with* $K, L \in \mathfrak{K}_{\mathscr{C}}(X)$, *if and only if there exists* $M \in \mathfrak{K}_{\mathscr{C}}(X)$ *for which* $K \subseteq M$, $L \subseteq M$ *and* $h^n i_{KM}^X(u) = h^n i_{LM}^X(v)$.

- *If* X *is a compact space, then* $h_c^n(X)$ *is isomorphic to* $h^n(X)$ *for all* $n \in \mathbb{Z}$. *This happens since* $\mathfrak{K}_{\mathscr{C}}(X)$ *contains a unique maximal compact subspace of* X, *which is* X *itself.* ◇

Remark 1.103 (On the Excision Axiom in the literature). *The Excision Axiom is not always stated as in Definition 1.9. In fact, its other common version is equal to ours but not requiring the openness condition on* U. *The motivation for this version (which evidently restricts the range of generalized cohomology theories, being then a stronger axiom) is that it holds in Singular Cohomology and, as we shall see later, in K-Theory. This stronger version is the one used until the end of this section.* ◇

Theorem 1.104 (Isomorphism with the one-point Alexandroff compactification of a locally compact Hausdorff space). *In a generalized cohomology theory based on an admissible category of topological spaces* $\mathscr{C}$, *let* X *be an admissible locally compact Hausdorff space. Suppose that:*

(a). (X^+, ∞) *is admissible, where* $X^+ = X \sqcup \{\infty\}$ *is the one-point Alexandroff compactification of* X. *In particular, note that this implies that* X^+ *is an admissible space;*

(b). $\mathfrak{K}_{\mathscr{C}}(X) \subseteq \mathfrak{K}_{\mathscr{C}}(X^+)$;

(c). the excision map $i_K : (X, X-K) \to (X^+, X^+ - K)$ *is admissible for every* $K \in \mathfrak{K}_{\mathscr{C}}(X)$;

(d). the inclusion map $j_K : (X^+, \infty) \to (X^+, X^+ - K)$ *is admissible for every* $K \in \mathfrak{K}_{\mathscr{C}}(X)$; *and*

(e). there exists a fundamental set of open neighborhoods $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ *of* ∞ *in* X^+*, where*

$$U_\alpha \;=\; X^+ - K_\alpha$$

with $K_\alpha \in \mathfrak{K}_{\mathscr{C}}(X)$ *for all* $\alpha \in \Lambda$*, such that* U_α *is contractible for each* $\alpha \in \Lambda$[2]*. See Figure 1.1.*

Under these conditions, we have that the compactly-supported generalized cohomology group $h^n_c(X)$ *is isomorphic to the pointed reduced generalized cohomology group* $\widetilde{h}^n_\infty(X^+)$ *for all* $n \in \mathbb{Z}$*.*

Proof. We have the following facts.

- The diagram below is commutative because its corresponding diagram of inclusions is easily seen to be commutative.

$$\begin{tikzcd}
h^n(X,\, X-K) \arrow[r, "h^n(i_K)^{-1}"] \arrow[d, "h^n i^X_{KL}"'] & h^n(X^+,\, X^+-K) \arrow[d, "h^n i^{X^+}_{KL}"] \\
h^n(X,\, X-L) \arrow[r, "h^n(i_L)^{-1}"'] & h^n(X^+,\, X^+-L)
\end{tikzcd}$$

  Note that $h^n(i_K)^{-1}$ is well-defined for all $K \in \mathfrak{K}_{\mathscr{C}}(X)$ because of the Excision Axiom (see Remark 1.103). Therefore, by taking the direct limit on both sides, we obtain the map

$$\Phi_n :\; h^n_c(X) \;\to \varinjlim_{K\in\mathfrak{K}_{\mathscr{C}}(X)} h^n(X^+, X^+-K),$$

  which the reader can readily prove to be an isomorphism since each of its components is an isomorphism.

- The diagram below is commutative because its corresponding diagram of inclusions is easily seen to be commutative.

---

[2] The reader can prove that the existence of such a fundamental set of open neighborhoods is ensured if X^+ is locally contractible in ∞. However, as one could expect, we cannot ensure *a priori* that the elements of such a collection of open subspaces of X^+ are formed from compact subspaces in $\mathfrak{K}_{\mathscr{C}}(X)$.

$$\begin{tikzcd}
h^n(X^+,\, X^+ - K) \arrow[r, "h^n(j_K)"] \arrow[d, "h^n i^{X^+}_{KL}"'] & h^n(X^+,\, \infty) \arrow[d, equal] \\
h^n(X^+,\, X^+ - L) \arrow[r, "h^n(j_L)"'] & h^n(X^+,\, \infty)
\end{tikzcd}$$

Therefore, by taking the direct limit, we obtain the map

$$\Psi_n : \varinjlim_{K \in \mathfrak{K}_{\mathscr{C}}(X)} h^n(X^+, X^+ - K) \;\to\; h^n(X^+,\, \infty),$$

which we now prove to be an isomorphism. Indeed, for each $K \in \mathfrak{K}_{\mathscr{C}}(X)$, since $\mathfrak{U}$ is a fundamental set of open neighborhoods of ∞ in X^+, there exists $\alpha \in \Lambda$ such that

$$U_\alpha \;=\; X^+ - K_\alpha \;\subseteq\; X^+ - K.$$

Hence, we have that

$$h^n(X^+,\, X^+ - K) \qquad \text{and} \qquad h^n(X^+,\, U_\alpha)$$

are identified in the direct limit. Moreover, by hypothesis, $U_\alpha = X^+ - K_\alpha$ is contractible. Thus, it follows that

$$h^n(X^+,\, U_\alpha) \qquad \text{and} \qquad h^n(X^+,\, \infty)$$

are isomorphic because of Theorem 1.58. The reader may convince himself or herself that this ensures our claim.

Consequently, since $\widetilde{h}^n_\infty(X^+)$ is isomorphic to $h^n(X^+, \infty)$ for all $n \in \mathbb{Z}$, the theorem is proved because

$$\Psi_n \circ \Phi_n :\; h^n_c(X) \;\to\; h^n(X^+,\, \infty)$$

is an isomorphism between the compactly-supported group $h^n_c(X)$ and $h^n(X^+, \infty)$ for all $n \in \mathbb{Z}$. $\square$

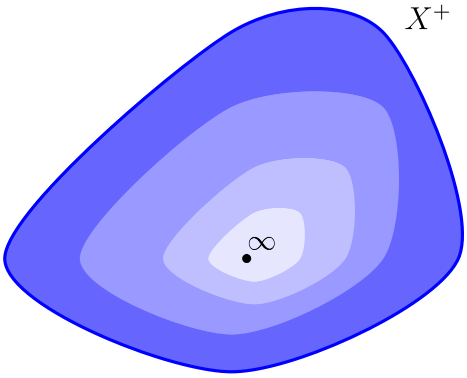

Figure 1.1: This picture represents the one-point Alexandroff compactification X^+ of a space X that admits a fundamental set of open neighborhoods of ∞ as above. Note that, by definition, an open set in this collection is the complement in X^+ of a compact subspace in $\mathfrak{K}_{\mathscr{C}}(X)$.

Remark 1.105 (On the hypotheses of the preceding result)**.** *The reader can prove that we can weaken Items (c) and (d) of Theorem 1.104 by just requiring the existence of a cofinal family of compact subspaces in $\mathfrak{K}_{\mathscr{C}}(X)$ for which the properties stated in these items are verified.* $\diamondsuit$

Example 1.106 (The thesis of the preceding result is not always true)**.** *Let X be the surface of countable-infinite genus obtained by connected summing two-dimensional tori. In Singular Cohomology, we have*

$$\begin{aligned} \widetilde{H}^1(X^+) &\simeq \prod_{n\in\mathbb{N}} \mathbb{Z} \quad \textit{and} \\ H^1_c(X) &\simeq \bigoplus_{n\in\mathbb{N}} \mathbb{Z}. \end{aligned}$$

Therefore, in Singular Cohomology, $H^n_c(X)$ is not isomorphic to $\widetilde{H}^n(X^+)$ for all $n \in \mathbb{Z}$. Thus, in a generalized cohomology theory, the conclusion of the preceding result is not always true. $\diamondsuit$

Finally, in order to finish this section, we show that the multiplicative structures studied before are well-behaved with respect to the compactly-supported cohomology groups. This is done in the following theorem, whose proof we leave as an exercise to the reader.

Theorem 1.107 (Multiplicative structures in compactly-supported groups)**.** *In a multiplicative generalized cohomology theory,*

$$\begin{aligned} [\varphi_{m,n}] : h_c^m(X) \otimes h_c^n(X) &\to h_c^{n+m}(X), \\ [u] \otimes [v] &\mapsto [\varphi_{m,n}(u \otimes v)], \end{aligned}$$

is well-defined, as well as

$$\begin{aligned} [\mu_{m,n}] : h_c^m(X) \otimes h_c^n(Y) &\to h_c^{n+m}(X \times Y), \\ [u] \otimes [v] &\mapsto [\mu_{m,n}(u \otimes v)]. \end{aligned}$$

Proof. We leave the details of this proof to the reader since they just consist in proving that

$$[\varphi_{m,n}(u \otimes v)] \qquad \text{and} \qquad [\mu_{m,n}(u \otimes v)]$$

do not depend on the representing elements u and v of the compactly-supported classes $[u]$ and $[v]$, respectively. □

Chapter 2

Ordinary K-Theory as a Generalized Cohomology Theory

In this chapter, we expose the main notions on Ordinary K-Theory as a generalized cohomology theory, taking advantage of the results proved in Chapter 1. In order to write this part of the text, we used as main references [2, pp. 43-94] and [19, pp. 52-111]. However, Sections 2.6 and 2.8 could not be written without [15, pp. 38-72] as well as Sections 2.9 and 2.11 could not be completed without [1], [3], [23, p. 65, pp. 70-76] and [33].

2.1 Absolute K-Theory

In this section, we start the study of Ordinary K-Theory defining its most elementary notions, namely, the absolute K-Theory group and the induced group homomorphisms. We begin with the following definition.

Definition 2.1 (The category of compact Hausdorff topological spaces). *We define the* ***category of compact Hausdorff topological spaces****, and denote it by* TopHdCpt*, to be the one whose:*

- *objects are compact Hausdorff spaces; and*
- *morphisms are continuous functions $f : X \to Y$ where X and Y are compact Hausdorff spaces.* $\diamondsuit$

Remark 2.2 (On some results from Appendixes B and C)**.** *We have the following facts from the appendixes.*

- *Let S be an abelian semigroup. Theorems B.3 and B.4 say that there exists a unique, up to a unique isomorphism, Grothendieck group $K(S)$ of S. Moreover, Theorem B.4 and Remark B.7 imply that:*
  - *the generic element of $K(S)$ is a formal difference of classes $[a] - [b] \in K(S)$; and*
  - *two classes $[a]$ and $[b]$ in $K(S)$ coincide if and only if there exists $s \in S$ for which $a + u = b + u$.*
- *Given a topological space X, the set* Vect_X *of isomorphism classes of complex vector bundles on X is an abelian semigroup*(1) *when equipped with the direct sum operation*

$$\begin{aligned} \oplus : \mathrm{Vect}_X \times \mathrm{Vect}_X &\to \mathrm{Vect}_X, \\ ([E], [F]) &\mapsto [E \oplus F], \end{aligned}$$

  by Theorem C.38.

These two pieces of information are the ones that allow us to set the following definition. ♢

Definition 2.3 (The absolute K-Theory group of a compact Hausdorff space)**.** *Let X be an object in* TopHdCpt *and* Vect_X *be the semigroup of isomorphism classes of complex vector bundles on X with respect to the induced direct sum. The* ***absolute K-Theory group*** *of X, hereafter denoted by $K(X)$, is the Grothendieck group associated to* Vect_X. ♢

Remark 2.4 (On the elements of the absolute K-Theory group of a compact Hausdorff space)**.** *Let X be an object in* TopHdCpt. *It follows from Remark 2.2 and from Definition 2.3 that:*

(1)In fact, the induced direct sum operation turns Vect_X into an abelian monoid. This happens because the isomorphism class of the product vector bundle with trivial typical fiber is its identity element.

- *the generic element of* $K(X)$ *is a formal difference of classes* $[[E]] - [[F]] \in K(X)$*; and*

- *two classes* $[[E]]$ *and* $[[F]]$ *in* $K(X)$ *coincide if and only if there exists a complex vector bundle* G *for which* $[E] \oplus [G] = [F] \oplus [G]$*. That is,* $[[E]] = [[F]]$ *in* $K(X)$ *if and only if there exists a complex vector bundle* G *for which* $E \oplus G$ *and* $F \oplus G$ *are isomorphic over* X*.* $\diamondsuit$

Remark 2.5 (The reason for restricting the framework of absolute K-Theory groups to TopHdCpt)**.** *The reader may be asking himself or herself why we are restricting to compact Hausdorff spaces in Definition 2.3 if the conclusions of Remark 2.4, which were immediately obtained from Remark 2.2, are still true for all classes of topological spaces. We now present answers for this question. However, these answers will only become clear in next sections, where they turn out to be essential properties of K-Theory groups of compact Hausdorff spaces. Indeed, let* X *be an object in* TopHdCpt*. Thus:*

- *if* $[[E]]$ *and* $[[F]]$ *are coincident classes in* $K(X)$*, then there exists a complex vector bundle* G *such that*
$$[E] \oplus [G] \;=\; [F] \oplus [G].$$
Therefore, because of Theorem C.51, we know that there exists a complex vector bundle H *such that* $G \oplus H$ *is isomorphic to some trivial vector bundle of rank* $n \in \mathbb{N}$*, which we will also denote by* n*. Hence,* $[[E]] = [[F]]$ *in* $K(X)$ *if and only if there exists a trivial vector bundle* n *for which* $E \oplus n$ *and* $F \oplus n$ *are isomorphic over* X*; and*

- *given a class* $[[E]] - [[F]] \in K(X)$*, Theorem C.51 ensures the existence of a complex vector bundle* G *such that* $F \oplus G$ *is isomorphic to some trivial vector bundle* n*. Therefore,*
$$\begin{aligned} [[E]] - [[F]] &= [[E]] + [[G]] - [[F]] - [[G]] \\ &= [[E \oplus G]] - [[F \oplus G]] \\ &= [[E \oplus G]] - [[n]]. \end{aligned}$$

In other words, we have that any K-Theory class of a compact Hausdorff space can be represented as a formal difference between a generic vector bundle and a trivial vector bundle. ♢

Notation 2.6 (On the rank of K-Theory classes)**.** *Let X be an object in* TopHdCpt *and $x \in X$. Given a class $\alpha = [[E]] - [[F]] \in K(X)$, we will use the notation* $\mathrm{rk}_x(\alpha)$ *to indicate* $\mathrm{rk}_x(E) - \mathrm{rk}_x(F)$*.* ♢

Definition 2.7 (Pullback in absolute K-Theory)**.** *Let $f : X \to Y$ be a morphism in* TopHdCpt*. We say that the* ***pullback of f in absolute K-Theory*** *is the morphism of abelian groups*

$$\begin{aligned} K(f) : K(Y) &\to K(X), \\ [[E]] - [[F]] &\mapsto [[f^*E]] - [[f^*F]], \end{aligned}$$

where f^*E and f^*F are the pullbacks of the vector bundles E and F through f, respectively. Note that $K(f)$ is well-defined because the pullbacks of isomorphic vector bundles are also isomorphic. ♢

Remark 2.8 (Categorical interpretation of the absolute K-Theory data presented above)**.** *Being $\mathscr{G}_{ab}$ the standard category of abelian groups, we have the contravariant functor*

$$\begin{aligned} K : \mathrm{TopHdCpt} &\to \mathscr{G}_{ab}, \\ X &\mapsto K(X), \\ f : X \to Y &\mapsto K(f) : K(Y) \to K(X). \end{aligned}$$

Indeed, if $f : X \to Y$ and $g : Y \to Z$ are morphisms in TopHdCpt*, then*

$$\begin{aligned} K(\mathrm{id}_X) &= \mathrm{id}_{K(X)} \quad \textit{and} \\ K(g \circ f) &= K(f) \circ K(g) \end{aligned}$$

by Theorem C.55. Furthermore, since Theorem C.57 imply that the pullbacks of vector bundles through homotopic continuous maps are isomorphic over X, the contravariant functor

$$\begin{aligned}[K] : [\mathrm{TopHdCpt}] &\rightarrow \mathscr{G}_{ab}, \\ X &\mapsto K(X), \\ [f : X \rightarrow Y] &\mapsto K(f) : K(Y) \rightarrow K(X),\end{aligned}$$

is well-defined, where [TopHdCpt] *is the quotient of* TopHdCpt *by the relation of homotopy of maps, which is an equivalence relation that is compatible with the composition in* TopHdCpt. ♢

Example 2.9 (Absolute K-Theory groups of contractible compact Hausdorff spaces)**.** *Let X be a contractible space in* TopHdCpt. *Since every vector bundle on X is trivial by Corollary C.58,* Vect_X *is composed of an isomorphism class for each possible dimension of the typical fiber of a trivial vector bundle on X. Thus,* Vect_X *is isomorphic to the additive monoid $\mathbb{N}$. Consequently, $K(X)$ is the additive group $\mathbb{Z}$. In particular, if Ω is a one-point space, since it is a contractible compact Hausdorff space, $K(\Omega)$ is the additive group $\mathbb{Z}$. In addition:*

- *every element in $K(\Omega)$ is a difference between two classes of trivial vector bundles $[[n]] - [[m]] \in K(\Omega)$, which we hereafter identify with the integer number $n - m \in \mathbb{Z}$; and*
- *given a continuous map $f : \Omega \rightarrow X$, where X is any compact Hausdorff space, if $[[E]] - [[F]] \in K(X)$, then the pullback $K(f)([[E]] - [[F]])$ is the integer number* $\mathrm{rk}_{f(\Omega)}(E) - \mathrm{rk}_{f(\Omega)}(F)$. ♢

Example 2.10 (Absolute K-Theory group of the circle)**.** *Let $\mathbb{S}^1$ be the unit circle, canonically embedded in the Euclidean plane. We claim that every complex vector bundle on $\mathbb{S}^1$ is trivial. Indeed, let $\mathbb{S}^1_+$ and $\mathbb{S}^1_-$ be the superior and inferior semicircles, respectively. Every complex vector bundle on $\mathbb{S}^1$ with rank $n \in \mathbb{N}$ is isomorphic to a vector bundle obtained from the disjoint union*

$$(\mathbb{S}^1_+ \times \mathbb{C}^n) \sqcup (\mathbb{S}^1_- \times \mathbb{C}^n)$$

by a quotient by an equivalence relation which, given a continuous function $f : \mathbb{S}^0 \rightarrow \mathrm{GL}(n, \mathbb{C})$, identifies (z, w) with $(z, f(w))$ for all $z \in \mathbb{S}^0$ and all $w \in \mathbb{C}^n$.

Thus, since the isomorphism class of this kind of vector bundle only depends on the homotopy class of f, our initial assertion is proved. In fact, since $\mathbb{S}^0$ is discrete and $\mathrm{GL}(n, \mathbb{C})$ *is connected, every continuous map f is homotopic to the unit constant function. Therefore, $K(\mathbb{S}^1)$ is the group of the integer numbers equipped with the ordinary sum.* ♢

Remark 2.11 (On the reasoning used to obtain the absolute K-Theory groups of the spaces considered in Examples 2.9 and 2.10)**.** *In the preceding examples, we deduced the absolute K-Theory groups of the spaces under consideration by successively applying these three steps:*

(1) we set the compact Hausdorff space X;

(2) we found the semigroup of isomorphism classes of vector bundles Vect_X*; and*

(3) we calculated the Grothendieck group $K(X)$ of Vect_X.

The repeated use of this process may mislead the reader, suggesting that this is the natural way to find the K-Theory groups. In fact, finding Vect_X *for each given compact Hausdorff space X is an extremely hard and unsolved problem. Then, the strategy that we will follow in the next sections is to set K-Theory as a generalized cohomology theory, which will allow us to use all the calculation tools developed in Chapter 1. Therefore, it must be clear that, since we still do not have the tools to calculate the K-Theory groups, the reasoning applied to Examples 2.9 and 2.10 is the best we could do right now. More than that, the theory that we will develop in the next sections is a way to avoid the problem of explicitly calculating* Vect_X*. In summary, if we could achieve the semigroups of isomorphism classes of vector bundles, then there would be no use for K-Theory. However, since the first approach is intractable, K-Theory is the way we have to understand these objects.* ♢

Remark 2.12 (The absolute K-Theory groups given by a differentiable manifold)**.** *Let r be a natural number or ∞. In addition, let $\mathcal{M}$ be a real $\mathcal{C}^r$-manifold. Because of Corollary C.71, we have that the Grothendieck group of $\mathcal{M}$ as a topological manifold $K(\mathcal{M})$ is isomorphic to the Grothendieck group of $\mathcal{M}$ as a real $\mathcal{C}^r$-manifold*

$K(\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^r})$. *More explicitly, this isomorphism tells us that, if we have a differential manifold, then it suffices to consider its semigroup of differentiable isomorphism classes of vector bundles to obtain its absolute K-Theory information. Moreover, because of Remark C.72:*

- *if $\mathcal{M}$ is a real analytic manifold, then $K(\mathcal{M})$ is always isomorphic to $K(\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^\omega})$, where $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^\omega}$ is the semigroup of analytic isomorphism classes of vector bundles on $\mathcal{M}$; and*
- *if $\mathcal{M}$ is a complex manifold, then $K(\mathcal{M})$ is not always isomorphic to $K(\mathrm{Vect}_{\mathcal{M}}^{\mathcal{H}})$, where $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{H}}$ is the semigroup of holomorphic isomorphism classes of vector bundles on $\mathcal{M}$.* ♢

2.2 Reduced K-Theory

In this section, we define the reduced K-Theory groups and the induced group homomorphisms. We also show that these objects are isomorphic to each other, although the isomorphism is not usually canonical. In addition, we prove that there is an explicit relation between them and the absolute K-Theory group. We begin with the following definition.

Definition 2.13 (The category of compact Hausdorff pointed topological spaces)**.** *We define the* ***category of compact Hausdorff pointed topological spaces****, and denote it by* $\mathrm{TopHdCpt}_+$*, to be the one whose:*

- *objects are ordered pairs (X, x_0) in which X is a compact Hausdorff space and $x_0 \in X$; and*
- *morphisms are continuous functions $f : X \to Y$ such that $f(x_0) = y_0$, usually denoted by $f : (X, x_0) \to (Y, y_0)$.* ♢

Definition 2.14 (The reduced K-Theory group of a compact Hausdorff pointed space)**.** *Let (X, x_0) be an object in* $\mathrm{TopHdCpt}_+$ *and $i : \{x_0\} \to X$ be the inclusion map. We define*

$$\widetilde{K}(X, x_0) \;:=\; \operatorname{Ker} K(i),$$

where $K(i) : K(X) \to K(x_0)$ *is the pullback of* i *in K-Theory. This new group is said to be the* ***reduced K-Theory group*** *of the pointed space* (X, x_0)*. More explicitly,* $\widetilde{K}(X, x_0)$ *is formed by the K-Theory classes* $[[E]] - [[F]] \in K(X)$ *in such manner that* $\mathrm{rk}_{x_0}(E) = \mathrm{rk}_{x_0}(F)$*.* ♢

Remark 2.15 (On the reduced K-Theory group of a compact Hausdorff pointed space)**.** *Let* (X, x_0) *be an object in* $\mathrm{TopHdCpt}_+$*. Note that:*

- *every class in* $\widetilde{K}(X, x_0)$ *can be represented in the form* $[[E]] - [[\mathrm{rk}_{x_0}(E)]]$*. This implies that* $\widetilde{K}(X, x_0)$ *only depends on the connected component of* x_0 *in* X*. Hence, if* X *is connected, then the condition for which* $[[E]] - [[F]] \in \widetilde{K}(X, x_0)$ *simply becomes* $\mathrm{rk}(E) = \mathrm{rk}(F)$*. Therefore, in this situation, each class of reduced K-Theory can represented as* $[[E]] - [[\mathrm{rk}(E)]]$*. Therefore,* $\widetilde{K}(X, x_0)$ *does not depend on* $x_0 \in X$*; and*
- *we have the canonical isomorphism*

$$\begin{aligned} \Phi_{(X,x_0)} : K(X) &\to \widetilde{K}(X, x_0) \oplus \mathbb{Z}, \\ \alpha &\mapsto (\alpha - [[\mathrm{rk}_{x_0}(\alpha)]]),\ [[\mathrm{rk}_{x_0}(\alpha)]]). \end{aligned}$$

  This shows the relation between the absolute and the reduced K-Theory groups, which is $K(X)$ *being isomorphic to the direct sum of* $\widetilde{K}(X, x_0)$ *with one copy of* $\mathbb{Z}$*. In particular, this* $\mathbb{Z}$ *factor corresponds to the subgroup of* $K(X)$ *generated by the trivial vector bundles.* ♢

Remark 2.16 (On the image of relative K-Theory groups by pullbacks in absolute K-Theory)**.** *Let* $f : (X, x_0) \to (Y, y_0)$ *be a morphism in* $\mathrm{TopHdCpt}_+$*. If* $\alpha \in \widetilde{K}(Y, y_0)$*, then the pullback* $K(f) : K(Y) \to K(X)$ *is such that* $K(f)(\alpha) \in \widetilde{K}(X, x_0)$*. Indeed, let* $i : \{x_0\} \to X$ *and* $j : \{y_0\} \to Y$ *be the inclusion maps. In addition, let* $\eta : \{x_0\} \to \{y_0\}$ *be the only possible map. The reader can readily prove that the following diagram is commutative.*

$$\begin{tikzcd}
\{x_0\} \arrow[r, "\eta"] \arrow[d, "i"'] & \{y_0\} \arrow[d, "j"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Therefore, if $\alpha \in \tilde{K}(Y, y_0)$*, then*

$$K(i)K(f)(\alpha) \;=\; K(\eta)K(j)(\alpha) \;=\; K(\eta)0 \;=\; 0.$$

This proves that $K(f)(\alpha) \in \tilde{K}(X, x_0)$*, as we wished. Consequently, we are allowed to set the following definition.* ♢

Definition 2.17 (Pullback in pointed reduced K-Theory)**.** *Let* $f : (X, x_0) \to (Y, y_0)$ *be a morphism in* TopHdCpt$_+$*. We say that the **pullback of f in pointed reduced K-Theory** is the morphism of abelian groups*

$$\widetilde{K}(f) \;:=\; K(f)\,|_{\widetilde{K}(Y,y_0)}\colon \widetilde{K}(Y, y_0) \to \widetilde{K}(X, x_0). \qquad \diamondsuit$$

Remark 2.18 (Categorical interpretation of the reduced K-Theory data presented above)**.** *We have the contravariant functor*

$$\begin{aligned}
\widetilde{K} : \mathrm{TopHdCpt}_+ \;&\to\; \mathscr{G}_{ab}, \\
(X, x_0) \;&\mapsto\; \widetilde{K}(X, x_0), \\
f : (X, x_0) \to (Y, y_0) \;&\mapsto\; \widetilde{K}(f) : \widetilde{K}(Y, y_0) \to \widetilde{K}(X, x_0).
\end{aligned}$$

Let $f, g : (X, x_0) \to (Y, y_0)$ *be morphisms in* TopHdCpt$_+$*. We say that a homotopy in* TopHdCpt$_+$ *between f and g is a homotopy* $H : X \times \mathbb{I} \to Y$ *in* TopHdCpt *between these two maps such that*

$$H(x_0, t) \;=\; y_0$$

for all $t \in \mathbb{I}$*. Then, considering* [TopHdCpt$_+$]*, which is the quotient of* TopHdCpt$_+$ *by the compatible equivalence relation of homotopy of pointed maps, we have the contravariant functor*

$$\begin{array}{rcl}[\widetilde{K}] : [\mathrm{TopHdCpt}_+] & \to & \mathscr{G}_{ab}, \\ (X, x_0) & \mapsto & \widetilde{K}(X, x_0), \\ [f : (X, x_0) \to (Y, y_0)] & \mapsto & \widetilde{K}(f) : \widetilde{K}(Y, y_0) \to \widetilde{K}(X, x_0). \end{array}$$

Clearly, $[f]$ *is defined up to homotopy in* $\mathrm{TopHdCpt}_+$. *Nevertheless, given morphisms* $f, g : (X, x_0) \to (Y, y_0)$ *in* $\mathrm{TopHdCpt}_+$, *it is possible that* f *and* g *are homotopic as morphisms in* TopHdCpt *but not as morphisms in* $\mathrm{TopHdCpt}_+$. *Even in this case, we have* $\widetilde{K}(f) = \widetilde{K}(g)$ *in reduced K-Theory. In fact, we have* $K(f) = K(g)$ *in absolute K-Theory and, moreover, the pullback in absolute K-Theory sends* $\widetilde{K}(Y, y_0)$ *into* $\widetilde{K}(X, x_0)$. *Therefore,*

$$\widetilde{K}(f) \ = \ K(f) \mid_{\widetilde{K}(Y,y_0)} = \ K(g) \mid_{\widetilde{K}(Y,y_0)} = \ \widetilde{K}(g).$$

Hence, since a homotopy in $\mathrm{TopHdCpt}_+$ *is also a homotopy in* TopHdCpt, *we could define the contravariant functor*

$$[\widetilde{K}] : \ [\mathrm{TopHdCpt}_+] \to \mathscr{G}_{ab},$$

considering $[\mathrm{TopHdCpt}_+]$ *to be the quotient of* $\mathrm{TopHdCpt}_+$ *by the compatible equivalence relation of homotopy of maps. However, this quotient is unnatural in* $\mathrm{TopHdCpt}_+$. *In any case, the pullback in reduced K-Theory is invariant.* $\diamondsuit$

The preceding remark finishes the definitions and the construction of the reduced K-Theory for compact Hausdorff pointed spaces. In the next and last paragraphs of this section, we define an equivalent version of reduced K-Theory. This new approach defines the ideas in question for compact Hausdorff spaces which do not have a special marked point *a priori*.

Definition 2.19 (The reduced K-Theory group of a compact Hausdorff space)**.** *Let* Ω *be a one-point space and* X *be an object in* TopHdCpt. *In addition, let* $p_X : X \to \Omega$ *be the only possible map and* $K(p_X) : K(\Omega) \to K(X)$ *be its pullback in absolute K-Theory. We say that the quotient of* $K(X)$ *by* $\mathrm{Im}\, K(p_X)$, *which we denote by* $\widetilde{K}(X)$, *is the* ***reduced K-Theory group*** *of* X. *More directly,* $\widetilde{K}(X)$ *is the cokernel of* $K(p_X) : K(\Omega) \to K(X)$. $\diamondsuit$

Remark 2.20 (On the relative K-Theory groups and the pullback in absolute K-Theory)**.** *Let* Ω *be a one-point space and* $f : X \to Y$ *be a morphism in* TopHdCpt*. In addition, let* $p_X : X \to \Omega$ *and* $p_Y : Y \to \Omega$ *be the only possible maps. The reader can readily prove that the diagram*

$$\begin{tikzcd} X \arrow[r, "p_X"'] \arrow[rr, bend left, "f"] & \Omega & Y \arrow[l, "p_Y"] \end{tikzcd}$$

is commutative. Therefore, for every $\alpha \in K(\Omega)$*, we have*

$$K(f)K(p_Y)(\alpha) \;=\; K(p_X)(\alpha).$$

Consequently, it is well-defined the map induced by $K(f)$ *from the quotient of* $K(Y)$ *by* $\operatorname{Im} K(p_Y)$ *into the quotient of* $K(X)$ *by* $\operatorname{Im} K(p_X)$*. This allows us to set the following definition.* ♢

Definition 2.21 (Pullback in reduced K-Theory)**.** *Let* $f : X \to Y$ *be a morphism in* TopHdCpt*. We say that the* ***pullback of f in reduced K-Theory*** *is the morphism of abelian groups*

$$\begin{aligned} \widetilde{K}(f) : \widetilde{K}(Y) &\to \widetilde{K}(X), \\ [\alpha] &\mapsto [K(f)(\alpha)]. \end{aligned}$$ ♢

Remark 2.22 (Categorical interpretation of the reduced K-Theory data presented above)**.** *We have the contravariant functor*

$$\begin{aligned} \widetilde{K} : \mathrm{TopHdCpt} &\to \mathscr{G}_{ab}, \\ X &\mapsto \widetilde{K}(X), \\ f : X \to Y &\mapsto \widetilde{K}(f) : \widetilde{K}(Y) \to \widetilde{K}(X). \end{aligned}$$

The reader can readily prove that the pullback in K-Theory of compact Hausdorff spaces is still homotopy invariant. Therefore, we have that it is well-defined the contravariant functor

$$\begin{aligned}
[\widetilde{K}] : [\mathrm{TopHdCpt}] &\to \mathscr{G}_{ab}, \\
X &\mapsto \widetilde{K}(X), \\
[f : X \to Y] &\mapsto \widetilde{K}(f) : \widetilde{K}(Y) \to \widetilde{K}(X),
\end{aligned}$$

where [TopHdCpt] *is the quotient of* TopHdCpt *by the relation of homotopy of maps, which is an equivalence relation that is compatible with the composition in* TopHdCpt*.* ♢

Remark 2.23 (Relation between the two versions of reduced K-Theory)**.** *Let Ω be a one-point space and (X, x_0) be an object in* $\mathrm{TopHdCpt}_+$*. By definition, $\widetilde{K}(X)$ is the quotient of $K(X)$ by* $\mathrm{Im}\, K(p_X)$*, where $p_x : X \to \Omega$ is the only possible map. Thus, we have the short exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{K(p_X)} K(X) \xrightarrow{\pi_X} \widetilde{K}(X) \longrightarrow 0,$$

where $\pi_X : K(X) \to \widetilde{K}(X)$ is the natural projection that sends $\alpha \in K(X)$ into $[\alpha] \in \widetilde{K}(X)$. The fact that $K(X)$ is isomorphic to $\widetilde{K}(X) \oplus \mathbb{Z}$ is still true. However, the isomorphism is canonical only if X is connected. Indeed, in order to find an isomorphism between $K(X)$ and $\widetilde{K}(X) \oplus \mathbb{Z}$, we have to choose a right inverse $i : \Omega \to X$ for $p_X : X \to \Omega$, which is equivalent to fix a point in X. Then, $K(i) : K(X) \to \mathbb{Z}$ splits the preceding short exact sequence. Therefore, we have the isomorphism

$$\begin{aligned}
\Phi^{\pi_X}_{K(i)} : K(X) &\to \widetilde{K}(X) \oplus \mathbb{Z}, \\
\alpha &\mapsto (\pi_X(\alpha),\, K(i)(\alpha)).
\end{aligned}$$

The group $\widetilde{K}(X, x_0)$ is a subgroup of $K(X)$ while $\widetilde{K}(X)$ is a quotient of $K(X)$. In particular, in order to embed $\widetilde{K}(X)$ in $K(X)$, we have to choose a point $x_0 \in X$, which is generally a non-canonical procedure in a space without a marked point. Nevertheless, when X is connected, $\widetilde{K}(X)$ and $\widetilde{K}(X, x_0)$ are canonically isomorphic because $\widetilde{K}(X, x_0)$ does not depend on $x_0 \in X$. ♢

2.3 Relative K-Theory

In this section, we define the last fundamental building blocks of Ordinary K-Theory, namely, the relative K-Theory group and the induced group homomorphisms. We begin with the following definition in which stands the admissible category of topological spaces that we will use as grounding for K-Theory as a generalized cohomology theory.

Definition 2.24 (The category of pairs of compact Hausdorff topological spaces)**.** *We define the* ***category of pairs of compact Hausdorff topological spaces****, and denote it by* $\mathrm{TopHdCCpt}_2$*, to be the one whose:*

- *objects are ordered pairs* (X, A) *in which* X *is a compact Hausdorff space and* $A \subseteq X$ *is a closed subspace; and*
- *morphisms are continuous functions* $f : X \to Y$ *such that* $f(A) \subseteq B$*, usually denoted by* $f : (X, A) \to (Y, B)$*.* ♢

Definition 2.25 (The relative K-Theory group of a pair of compact Hausdorff spaces)**.** *Let* (X, A) *be an object in* $\mathrm{TopHdCCpt}_2$*. We define*

$$K(X, A) := \widetilde{K}(X/A,\, A/A).$$

This new group is said to be the ***relative K-Theory group*** *of the pair of spaces* (X, A)*.* ♢

Remark 2.26 (On the relative K-Theory group of a pair of compact Hausdorff spaces)**.** *Let* (X, A) *be an object in* $\mathrm{TopHdCCpt}_2$*. Note that:*

- *since* A *is a closed subspace of the compact Hausdorff* X*, the quotient* X/A *is also a compact Hausdorff space. Therefore,* $(X/A, A/A)$ *is really an element of* $\mathrm{TopHdCpt}_+$*. This allows us to set the relative K-Theory group as in Definition 2.25; and*

- *in Singular Cohomology, the relative cohomology of a pair* (X, A) *coincides up to isomorphism with the reduced cohomology of the pointed space* $(X/A, A/A)$ *if* (X, A) *is a cofibration or a good pair. In general, this does not happen. In K-Theory, the relative cohomology of a compact Hausdorff pair* (X, A) *is equal to the reduced cohomology of the compact Hausdorff pointed space* $(X/A, A/A)$ *by definition.* $\diamondsuit$

Notation 2.27 (On the pointed space induced by a pair of compact Hausdorff spaces)**.** *Let* (X, A) *be an object in* $\mathrm{TopHdCCpt}_2$. *Once the quotient of* X *by* A *has a natural marked point, which is the quotient of* A *by* A, *we will say that the former belongs to* $\mathrm{TopHdCpt}_+$ *without mentioning its marked point. In particular, we will use this convention in pointed reduced K-Theory, which should cause no confusion with the reduced K-Theory of Definition 2.19.* $\diamondsuit$

Remark 2.28 (Morphism of pointed spaces induced by a morphism of pairs of spaces)**.** *Let* $f : (X, A) \to (Y, B)$ *be a morphism in* $\mathrm{TopHdCCpt}_2$. *Then, we have the morphism*

$$\begin{aligned} \bar{f} : X/A &\to Y/B, \\ [x] &\mapsto [f(x)], \end{aligned}$$

in $\mathrm{TopHdCpt}_+$. *The reader can readily prove that this morphism is well-defined using that* $f(A) \subseteq B$. $\diamondsuit$

Definition 2.29 (Pullback in relative K-Theory)**.** *Let* $f : (X, A) \to (Y, B)$ *be a morphism in* $\mathrm{TopHdCCpt}_2$. *We say that the* ***pullback of*** $\boldsymbol{f}$ ***in relative K-Theory*** *is the morphism of abelian groups*

$$K(f) := \widetilde{K}(\bar{f}) : \widetilde{K}(Y/B) \to \widetilde{K}(X/A),$$

where $\bar{f} : X/A \to Y/B$ *is the natural map of compact Hausdorff pointed spaces defined in Remark 2.28. Finally, we will write* $K(f) : K(Y, B) \to K(X, A)$ *instead of* $K(f) : \widetilde{K}(Y/B) \to \widetilde{K}(X/A)$. $\diamondsuit$

Remark 2.30 (Categorical interpretation of the relative K-Theory data presented above)**.** *We have the contravariant functor*

$$\begin{aligned} K : \mathrm{TopHdCCpt}_2 &\rightarrow \mathscr{G}_{ab}, \\ (X,A) &\mapsto K(X,A), \\ f:(X,A)\rightarrow (Y,B) &\mapsto K(f): K(Y,B)\rightarrow K(X,A). \end{aligned}$$

Let $f, g : (X,A) \rightarrow (Y,B)$ *be morphisms in* $\mathrm{TopHdCCpt}_2$*. We say that a homotopy in* $\mathrm{TopHdCCpt}_2$ *between* f *and* g *is a homotopy* $H : X \times \mathbb{I} \rightarrow Y$ *in* $\mathrm{TopHdCpt}$ *between these two maps such that* $H(a,t) \in B$ *for all* $a \in A$ *and all* $t \in \mathbb{I}$*. Then, if there exists a homotopy of pairs* H *between* $f, g : (X,A) \rightarrow (Y,B)$*, it is well-defined the continuous function*

$$\bar{H} : (X/A) \times \mathbb{I} \rightarrow Y/B$$

according to Remark 2.28. It can be readily proved that this map is a homotopy of pointed maps between $\bar{f}, \bar{g} : X/A \rightarrow Y/B$*. Thus, we have*

$$K(f) \;=\; \widetilde{K}(\bar{f}) \;=\; \widetilde{K}(\bar{g}) \;=\; K(g).$$

Therefore, we have the contravariant functor

$$\begin{aligned} [K] : [\mathrm{TopHdCCpt}_2] &\rightarrow \mathscr{G}_{ab}, \\ (X,A) &\mapsto K(X,A), \\ [f]:(X,A)\rightarrow (Y,B) &\mapsto K(f): K(Y,B)\rightarrow K(X,A), \end{aligned}$$

where $[\mathrm{TopHdCCpt}_2]$ *is the quotient of* $\mathrm{TopHdCCpt}_2$ *by the compatible equivalence relation of homotopy of maps of pairs.* $\diamondsuit$

2.4 First relations

In this section, we establish the first relations between the absolute, reduced and relative K-Theory groups through exact sequences. It is to be noted that the results exposed here have technical proofs, which can be skipped if the reader prefers. However, the statements of these results cannot be ignored since they are essential in Sections 2.5 and 2.6 to set the K-Theory cohomology sequences and their exactness. We begin with the following theorem.

Theorem 2.31 (Exact sequence involving absolute and relative K-Theory groups). *Let (X, A) be an object in* TopHdCCpt$_2$. *In addition, let $i : A \to X$ and $\pi : X \to X/A$ be the natural inclusion and projection, respectively. The pullback in absolute K-Theory $K(\pi) : K(X/A) \to K(X)$ can be restricted to*

$$K(\pi)\mid_{K(X,A)}: K(X, A) \to K(X),$$

since $K(X, A)$ is a subgroup of $K(X/A)$. Moreover, we will continue to denote this restriction by $K(\pi) : K(X, A) \to K(X)$. Therefore, we have that the sequence

$$K(X, A) \xrightarrow{\quad K(\pi) \quad} K(X) \xrightarrow{\quad K(i) \quad} K(A).$$

is exact. This means that the image of $K(\pi)$ is the set of classes of $K(X)$ whose restriction to A is zero.

Proof. Let us first prove that $\operatorname{Im} K(\pi)$ is a subset of $\operatorname{Ker} K(i)$. Indeed, consider $[[E']] - [[F']] \in K(X, A)$. The vector bundles E' and F' on X/A have the same rank on the marked point A/A. This happens because of Definition 2.14 since $K(X, A) = \widetilde{K}(X/A)$. In addition, it follows from Corollary C.62 that the restrictions to A of $E := \pi^*E'$ and $F := \pi^*F'$ are trivial. Therefore,

$$\begin{aligned} K(i)K(\pi)([[E']] - [[F']]) &= K(i)([[\pi^*E]] - [[\pi^*F]]) \\ &= K(i)([[E]] - [[F]]) \\ &= [[E\mid_A]] - [[F\mid_A]] \\ &= 0, \end{aligned}$$

as we wished. Finally, we prove that $\operatorname{Ker} K(i)$ is a subset of $\operatorname{Im} K(\pi)$. In fact, let $\alpha \in \operatorname{Ker} K(i)$. We can represent α in the form $\alpha = [[E]] - [[n]]$ where E and n are a generic and a trivial vector bundles on X, respectively, because of Remark 2.5. By definition,

$$K(i)(\alpha) \;=\; K(i)[[E]] - K(i)[[n]] \;=\; [[E\mid_A]] - [[n\mid_A]] \;=\; 0.$$

Therefore, it also follows from Remark 2.5 that there exists a trivial vector bundle m on X for which $(E \oplus m) \mid_A$ is isomorphic to $(n \oplus m) \mid_A$. Consequently, because of Corollary C.62, the vector bundle $E \oplus m$ is the pullback of a vector bundle on X/A. In particular, let $\alpha : (E \oplus m) \mid_A \to A \times \mathbb{C}^{n+m}$ be a global trivialization of $(E \oplus m) \mid_A$. Then, it is defined the quotient E' of $E \oplus m$ by the α-equivalence relation, which is such that $[\pi^*(E')] = [E \oplus m]$. Thus,

$$\begin{aligned} K(\pi)([[E']] - [[n \oplus m]]) &= K(\pi)[[E']] - K(\pi)[[n \oplus m]] \\ &= [[\pi^* E']] - [[n \oplus m]] \\ &= [[E \oplus m]] - [[n \oplus m]] \\ &= [[E]] - [[n]] \\ &= \alpha. \end{aligned}$$

□

Definition 2.32 (The category of pointed pairs of compact Hausdorff topological spaces)**.** *We define the* ***category of pointed pairs of compact Hausdorff pointed topological spaces****, and denote it by* $\mathrm{TopHdCCpt}_{2+}$*, to be the category whose:*

- *objects are ordered triples (X, A, a_0) in which (X, A) belongs to* $\mathrm{TopHdCCpt}_2$ *and $a_0 \in A$; and*
- *morphisms are continuous functions $f : X \to Y$ such that $f(A) \subseteq B$ and $f(a_0) = b_0$, usually denoted by $f : (X, A, a_0) \to (Y, B, b_0)$.* ♢

Corollary 2.33 (Exact sequence involving pointed reduced and relative K-Theory groups)**.** *Let (X, A, a_0) be an object in* $\mathrm{TopHdCCpt}_{2+}$*. Moreover, let $i : (A, a_0) \to (X, a_0)$ and $\pi : (X, a_0) \to X/A$ be the natural inclusion and projection, respectively. Then, the sequence*

$$K(X, A) \xrightarrow{\widetilde{K}(\pi)} \widetilde{K}(X, a_0) \xrightarrow{\widetilde{K}(i)} \widetilde{K}(A, a_0)$$

is exact.

Proof. It follows from Remark 2.15 and from Theorem 2.31 that the sequence

$$K(X,A) \longrightarrow \widetilde{K}(X,a_0)\oplus\mathbb{Z} \longrightarrow \widetilde{K}(A,a_0)\oplus\mathbb{Z}$$

is exact. Now, let $i_{a_0}:\{a_0\}\to A$ and $j_{a_0}:\{a_0\}\to X$ be the inclusion maps. We claim that, for each $\alpha\in K(X)$ such that $K(i)(\alpha)=0$, we have $K(j_{a_0})(\alpha)=0$. Indeed, since $j_{a_0}=i\circ i_{a_0}$, we obtain

$$K(j_{a_0})(\alpha) \;=\; (K(i_{a_0})\circ K(i))(\alpha) \;=\; K(i_{a_0})0 \;=\; 0.$$

Thus, $\alpha\in\widetilde{K}(X,a_0)$. Therefore, the image of $K(\pi)$ is a subset of $\widetilde{K}(X,a_0)$. This finishes the proof of the theorem. □

Corollary 2.34 (Exact sequence involving reduced and relative K-Theory groups)**.** *Let (X,A) be an object in* $\mathrm{TopHdCCpt}_2$*. In addition, let $i:A\to X$ and $\pi:X\to X/A$ be the natural inclusion and projection, respectively. In this situation, we define the homomorphism*

$$\widetilde{K}(\pi):\,K(X,A)\,\to\,\widetilde{K}(X)$$

to be the composition between $K(\pi):K(X,A)\to K(X)$ of Theorem 2.31 with the projection $K(X)\to\widetilde{K}(X)=K(X)/\mathbb{Z}$, where the factor $\mathbb{Z}$ is the one in the isomorphism of Remark 2.15. Moreover, the homomorphism $K(i):K(X)\to K(A)$ of Theorem 2.31 defines

$$\widetilde{K}(i):\,\widetilde{K}(X)=K(X)/\mathbb{Z}\,\to\,\widetilde{K}(A)=K(A)/\mathbb{Z}$$

since its image of any trivial vector bundle is also trivial. Therefore, we have that the sequence

$$K(X,A) \xrightarrow{\;\widetilde{K}(\pi)\;} \widetilde{K}(X) \xrightarrow{\;\widetilde{K}(i)\;} \widetilde{K}(A)$$

is exact.

Proof. This assertion follows from Theorem 2.31 since the projection $K(X)\to\widetilde{K}(X)$ restricts to isomorphisms $\operatorname{Im}K(\pi)\to\operatorname{Im}\widetilde{K}(\pi)$ and $\operatorname{Ker}K(i)\to\operatorname{Ker}\widetilde{K}(i)$. Indeed, given a class $[\alpha]\in\operatorname{Ker}\widetilde{K}(i)$ or a class $[\alpha]\in\operatorname{Im}\widetilde{K}(\pi)$, there exists a unique $\alpha\in\operatorname{Ker}K(i)=\operatorname{Im}K(\pi)$ because, if we add a trivial vector bundle to α, then the resulting image is not zero in $K(A)$. This means that $[\alpha] \in \operatorname{Ker} \widetilde{K}(i)$ if and only if there exists a unique $\alpha \in \operatorname{Ker} K(i) = \operatorname{Im} K(\pi)$. In turn, this happens if and only if $[\alpha] \in \operatorname{Im} \widetilde{K}(\pi)$. $\square$

Remark 2.35 (On the exact sequences of the preceding corollaries)**.** *Because of Corollary C.64, if A is contractible, then the homomorphisms induced by π in the exact sequences of Corollaries 2.33 and 2.34 are bijections. Indeed, since $[\pi^*]$ is an isomorphism between* $\mathrm{Vect}(X/A)$ *and* $\mathrm{Vect}(X)$*, it is extended as an isomorphism between the Grothendieck groups $K(X/A)$ and $K(X)$. In turn, this isomorphism restricts to isomorphisms between the reduced K-Theory groups. It is to be noted that this claim holds independently of the embedding of A into X being a cofibration, which is needed in Singular Cohomology.* $\diamondsuit$

2.5 K-Theory of negative degree

In this section, we extend the absolute, reduced and relative K-Theory groups to other degrees, giving then the first explicit step towards the construction of the data presented in Definition 1.9. In order to do this, we use the constructions and the notations presented in Appendix D. However, we are not yet capable of constructing K-Theory groups in all degrees. In fact, here we restrict ourselves to an extension of the K-Theory groups to negative degrees. We begin with the following remark that justifies this restriction.

Remark 2.36 (The suspension isomorphism in Singular Cohomology)**.** *In Singular Cohomology, being (X, x_0) an object in* Top_2 *and ΣX its reduced suspension, we have a canonical isomorphism between $\widetilde{H}^n(X, x_0)$ and $\widetilde{H}^{n+1}(\Sigma X)$ for all $n \in \mathbb{Z}$, which is known as the* ***suspension isomorphism****. Iteratively, we obtain from this a canonical isomorphism*

$$\widetilde{H}^{k-n}(X, x_0) \simeq \widetilde{H}^k(\Sigma^n X)$$

for all $k, n \in \mathbb{Z}$. Additionally, since the absolute group $H^n(X)$ is canonically isomorphic to the reduced group $\widetilde{H}^n(X_+)$ where $X_+ = X \sqcup \{\infty\}$, we also have a canonical isomorphism

$$H^{k-n}(X) \simeq \widetilde{H}^k(\Sigma^n X_+).$$

In this context, the case in which $k = 0$ is not interesting because we only obtain trivial groups, with a possible exception when $n = 0$ (which is simple to be handled). Nonetheless, in the K-Theory framework, we define the negative degree groups through this case, as below. $\diamondsuit$

Definition 2.37 (K-Theory groups and homomorphisms of negative degree)**.** *Being n a natural number, we give the following definitions.*

- *The* ***nth negative degree absolute K-Theory group*** *of a compact Hausdorff space X, which is hereafter denoted by $K^{-n}(X)$, is the pointed reduced K-Theory group*

$$\widetilde{K}(\Sigma^n X_+).$$

  In addition, being $f : X \to Y$ a morphism of compact Hausdorff spaces, we define the ***nth negative degree pullback of f in absolute K-Theory****, and denote it by $K^{-n}(f) : K^{-n}(Y) \to K^{-n}(X)$, to be the pullback in pointed reduced K-Theory*

$$\widetilde{K}(\Sigma^n f_+) : \widetilde{K}(\Sigma^n Y_+) \to \widetilde{K}(\Sigma^n X_+).$$

- *The* ***nth negative degree pointed reduced K-Theory group*** *of an object $(X, x_0) \in \mathrm{TopHdCpt}_+$, which is hereafter denoted by $\widetilde{K}^{-n}(X, x_0)$, is the pointed reduced K-Theory group*

$$\widetilde{K}(\Sigma^n X).$$

  In addition, being $f : (X, x_0) \to (Y, y_0)$ a morphism of pointed compact Hausdorff spaces, we define the ***nth negative degree pullback of f in pointed reduced K-Theory****, and denote it by $\widetilde{K}^{-n}(f) : \widetilde{K}^{-n}(Y, y_0) \to \widetilde{K}^{-n}(X, x_0)$, to be the pullback in pointed reduced K-Theory*

$$\widetilde{K}(\Sigma^n f) : \widetilde{K}(\Sigma^n Y) \to \widetilde{K}(\Sigma^n X).$$

- *The* ***nth negative degree relative K-Theory group*** *of an object $(X, A) \in \mathrm{TopHdCCpt}_2$, which is hereafter denoted by $K^{-n}(X, A)$, is the pointed reduced K-Theory group*

$$\widetilde{K}^{-n}(X/A).$$

In addition, being $f : (X, A) \to (Y, B)$ a morphism of pairs of compact Hausdorff spaces, we define the ***nth negative degree pullback of f in relative K-Theory****, and denote it by $K^{-n}(f) : K^{-n}(Y, B) \to K^{-n}(X, A)$, to be the pullback in pointed reduced K-Theory*

$$\widetilde{K}^{-n}(\bar{f}) : \widetilde{K}^{-n}(Y/B) \to \widetilde{K}^{-n}(X/A).$$

It is to be noted that

$$\begin{aligned} K^0(X) &= K(X), \\ \widetilde{K}^0(X, x_0) &= \widetilde{K}(X, x_0) \quad \textit{and} \\ K^0(X, A) &= K(X, A). \end{aligned}$$

Evidently, these equations hold up to isomorphism, but such isomorphisms will not be carried any further. ♢

Remark 2.38 (The categorical structure of the negative degree K-Theory)**.** *Being n a natural number, the groups and the homomorphisms defined above induce the following contravariant functors.*

- *Consider the covariant functor*

$$\begin{aligned} {}_+ : \mathrm{TopHdCpt} &\to \mathrm{TopHdCpt}_+, \\ X &\mapsto X_+, \\ f : X \to Y &\mapsto f_+ : X_+ \to Y_+. \end{aligned}$$

  The composition of functors

$$\begin{aligned} K^{-n} := \widetilde{K} \circ \Sigma^n \circ {}_+ : \mathrm{TopHdCpt} &\to \mathscr{G}_{ab}, \\ X &\mapsto K^{-n}(X), \\ f : X \to Y &\mapsto K^{-n}(f) : K^{-n}(Y) \to K^{-n}(X), \end{aligned}$$

  where $\widetilde{K}$ and Σ^n are the contravariant and covariant functors defined in Remark 2.18 and in Definition D.4, respectively, represents the negative degree absolute K-Theory data presented above.

- *The composition of functors*

$$\begin{aligned}\widetilde{K}^{-n} \;:=\; \widetilde{K}\circ\Sigma^n :\; \mathrm{TopHdCpt}_+ &\;\to\; \mathscr{G}_{ab},\\ (X,x_0) &\;\mapsto\; \widetilde{K}^{-n}(X,x_0),\\ f:(X,x_0)\to(Y,y_0) &\;\mapsto\; \widetilde{K}^{-n}(f):\widetilde{K}^{-n}(Y,y_0)\to\widetilde{K}^{-n}(X,x_0),\end{aligned}$$

where $\widetilde{K}$ and Σ^n are as in the preceding item, represents the negative degree pointed reduced K-Theory data presented above.

- *Consider the covariant functor*

$$\begin{aligned}/:\; \mathrm{TopHdCCpt}_2 &\;\to\; \mathrm{TopHdCpt}_+,\\ (X,A) &\;\mapsto\; X/A,\\ f:(X,A)\to(Y,B) &\;\mapsto\; \bar{f}:X/A\to Y/B.\end{aligned}$$

The composition of functors

$$\begin{aligned}K^{-n} \;:=\; \widetilde{K}^{-n}\circ / :\; \mathrm{TopHdCCpt}_2 &\;\to\; \mathscr{G}_{ab},\\ (X,A) &\;\mapsto\; K^{-n}(X,A),\\ f:(X,A)\to(Y,B) &\;\mapsto\; K^{-n}(f):K^{-n}(Y,B)\to K^{-n}(X,A),\end{aligned}$$

represents the negative degree relative K-Theory data presented above. $\diamondsuit$

Remark 2.39 (An overview of the results that will be proven here and in the next sections)**.** *Being n a natural number and (X,x_0) an object in* $\mathrm{TopHdCpt}_+$*, we prove below that*

$$\widetilde{K}^{-n}(X,x_0) \qquad \text{and} \qquad \operatorname{Ker} K^{-n}(i)$$

are isomorphic, where $i:\{x_0\}\to X$ is the inclusion map. This establishes a natural correspondence with Definition 2.14, which could then be adapted to define the negative degree pointed reduced K-Theory groups. In particular, it suggests defining the ***nth negative degree reduced K-Theory group*** *of a compact Hausdorff space X, which is hereafter denoted by $\widetilde{K}^{-n}(X)$, to be*

$$\operatorname{Coker} K^{-n}(p_X),$$

where Ω *is a one-point space and* $p_X : X \to \Omega$ *is the only possible map. Thence, as one could expect, we prove the existence of an isomorphism between* $\widetilde{K}^{-n}(X, x_0)$ *and* $\widetilde{K}^{-n}(X)$*, which is non-canonical if* X *is not connected. In addition, we prove an isomorphism*

$$K^{-n}(X) \;\simeq\; \widetilde{K}^{-n}(X, x_0) \oplus K^{-n}(x_0).$$

Therefore, being Ω *a one-point space, we also obtain a direct sum decomposition*

$$K^{-n}(X) \;\simeq\; \widetilde{K}^{-n}(X) \oplus K^{-n}(\Omega).$$

Finally, since we will prove

$$K^{-2n}(\Omega) \;\simeq\; \mathbb{Z} \qquad \textit{and} \qquad K^{-2n-1}(\Omega) \;\simeq\; 0,$$

we obtain the isomorphisms

$$K^{-2n}(X) \;\simeq\; \widetilde{K}^{-2n}(X) \oplus \mathbb{Z} \qquad \textit{and} \qquad K^{-2n-1}(X) \;\simeq\; \widetilde{K}^{-2n-1}(X).$$

These isomorphisms are canonical if we consider the pointed reduced K-Theory or if X *is connected.* ♢

Now, before proving some of the results shown in Remark 2.39, we prove that the negative degree K-Theory groups induce a left long exact sequence, which will be completed to a long exact sequence when we define the positive degree K-Theory groups. First, however, let us consider the following lemma that contains some natural ´ isomorphisms.

Lemma 2.40 (Natural isomorphisms in K-Theory)**.** *Let* (X, A, a_0) *be an object in* $\mathrm{TopHdCCpt}_{2+}$*. There are isomorphisms*

$$\begin{aligned} K(X, A) \;&\simeq\; \widetilde{K}(C(X, A)) \quad \textit{and} \\ \widetilde{K}^{-1}(A, a_0) \;&\simeq\; \widetilde{K}(C'X \cup CA). \end{aligned}$$

Proof. Since X/A is homeomorphic to $C(X,A)/CA$, we have that

$$K(X,A) = \widetilde{K}(X/A) \qquad \text{and} \qquad \widetilde{K}(C(X,A)/CA)$$

are isomorphic. Further, once $C(A)$ is a contractible subspace of $C(X,A)$, we also have that

$$\widetilde{K}(C(X,A)/CA) \qquad \text{and} \qquad \widetilde{K}(C(X,A))$$

are isomorphic. Hence, if $\varphi : C(X,A)/CA \to X/A$ is the natural homeomorphism and $\pi_1 : C(X,A) \to C(X,A)/CA$ is the natural projection, then we obtain the isomorphism between $K(X,A)$ and $\widetilde{K}(C(X,A))$ given by the composition of the maps in the sequence

$$K(X,A) = \widetilde{K}(X/A) \xrightarrow{\widetilde{K}(\varphi)} \widetilde{K}(C(X,A)/CA) \xrightarrow{\widetilde{K}(\pi_1)} \widetilde{K}(C(X,A)).$$

Choosing the natural marked point of $C(X,A)$, which is mapped by π_1 into the natural marked point of the quotient $C(X,A)/C(A)$, it is well-defined $\widetilde{K}(\pi_1)$ above. Additionally, since the suspension SA is homeomorphic to $C(X,A)/X$, we have that

$$\widetilde{K}(SA) \qquad \text{and} \qquad \widetilde{K}(C(X,A)/X)$$

are isomorphic. Furthermore, since $C(X,A)/X$ is homeomorphic to the quotient $(C'X \cup CA)/C'X$, and $C'X$ is a contractible subspace of $C'X \cup CA$, we have that

$$\widetilde{K}(SA), \qquad \widetilde{K}(C(X,A)/X), \qquad \tilde{K}(C'X \cup CA/C'X) \qquad \text{and} \qquad \tilde{K}(C'X \cup CA)$$

are isomorphic. Moreover, by restricting the isomorphism $K(\pi) : K(\Sigma A) \to K(SA)$ of Definition D.4 to the pointed reduced K-Theory groups, we also have that the abelian groups

$$\widetilde{K}^{-1}(A, a_0) = \widetilde{K}(\Sigma A) \qquad \text{and} \qquad \widetilde{K}(SA)$$

are isomorphic for any marked point of SA belonging to $\{a_0\} \times \mathbb{I}$. Let us call by v the inferior vertex of SA and by w the vertex of $C'X$. Thence, we obtain the isomorphism between $\widetilde{K}^{-1}(A, a_0)$ and $\widetilde{K}(C'X \cup CA, w)$ given by the composition of the maps in the sequence

$$
\begin{tikzcd}
\widetilde{K}^{-1}(A,a_0)=\widetilde{K}(\Sigma A) \arrow[d, "\widetilde{K}(\pi)"'] & & \\
\widetilde{K}(SA,v) \arrow[r, "\widetilde{K}(\psi)"'] & \widetilde{K}((C'X\cup CA)/C'X) \arrow[r, "\widetilde{K}(\pi_2)"'] & \widetilde{K}(C'X\cup CA,w),
\end{tikzcd}
$$

where $\psi : (C'X \cup CA)/C'X \to SA$ denotes the natural homeomorphism and $\pi_2 : C'X \cup CA \to (C'X \cup CA)/C'X$ denotes the natural projection. □

Theorem 2.41 (Left long exact sequences in Ordinary K-Theory)**.** *Let (X, A, a_0) be an object in* TopHdCCpt$_{2+}$*. We have the left long exact sequence in pointed reduced K-Theory*

$$
\begin{tikzcd}
 & \cdots \arrow[dll, dashed, to path={}] & \\
K^{-n-1}(X,A) \arrow[r, "\widetilde{K}^{-n-1}(\pi)"] & \widetilde{K}^{-n-1}(X,a_0) \arrow[r, "\widetilde{K}^{-n-1}(i)"] & \widetilde{K}^{-n-1}(A,a_0) \arrow[dll, "\delta^{-n}_{(X,A,a_0)}"'] \\
K^{-n}(X,A) \arrow[r, "\widetilde{K}^{-n}(\pi)"'] & \widetilde{K}^{-n}(X,a_0) \arrow[r, "\widetilde{K}^{-n}(i)"'] & \widetilde{K}^{-n}(A,a_0) \arrow[dll, dashed] \\
K^{-1}(X,A) \arrow[r, "\widetilde{K}^{-1}(\pi)"] & \widetilde{K}^{-1}(X,a_0) \arrow[r, "\widetilde{K}^{-1}(i)"] & \widetilde{K}^{-1}(A,a_0) \arrow[dll, "\delta_{(X,A,a_0)}"'] \\
K(X,A) \arrow[r, "\widetilde{K}(\pi)"'] & \widetilde{K}(X,a_0) \arrow[r, "\widetilde{K}(i)"'] & \widetilde{K}(A,a_0).
\end{tikzcd}
$$

Moreover, if we substitute X and A by X_+ and A_+, respectively, then we obtain the left long exact sequence in absolute K-Theory

$$
\begin{tikzcd}
 & \cdots \arrow[dl, dashed] & \\
K^{-n-1}(X,A) \arrow[r, "K^{-n-1}(\pi)"] & K^{-n-1}(X) \arrow[r, "K^{-n-1}(i)"] & K^{-n-1}(A) \arrow[dll, "\delta^{-n}_{(X,A)}"] \\
K^{-n}(X,A) \arrow[r, "K^{-n}(\pi)"'] & K^{-n}(X) \arrow[r, "K^{-n}(i)"'] & K^{-n}(A) \arrow[dll, dashed] \\
K^{-1}(X,A) \arrow[r, "K^{-1}(\pi)"] & K^{-1}(X) \arrow[r, "K^{-1}(i)"] & K^{-1}(A) \arrow[dll, "\delta_{(X,A)}"] \\
K(X,A) \arrow[r, "K(\pi)"'] & K(X) \arrow[r, "K(i)"'] & K(A).
\end{tikzcd}
$$

Finally, if we quotient this last sequence by the appropriate subgroups generated by the trivial vector bundles, then we obtain the left long exact sequence in reduced K-Theory.

$$
\begin{tikzcd}
 & \cdots \arrow[dl, dashed] & \\
K^{-n-1}(X,A) \arrow[r, "\widetilde{K}^{-n-1}(\pi)"] & \widetilde{K}^{-n-1}(X) \arrow[r, "\widetilde{K}^{-n-1}(i)"] & \widetilde{K}^{-n-1}(A) \arrow[dll, "\widetilde{\delta}^{-n}_{(X,A)}"] \\
K^{-n}(X,A) \arrow[r, "\widetilde{K}^{-n}(\pi)"'] & \widetilde{K}^{-n}(X) \arrow[r, "\widetilde{K}^{-n}(i)"'] & \widetilde{K}^{-n}(A) \arrow[dll, dashed] \\
K^{-1}(X,A) \arrow[r, "\widetilde{K}^{-1}(\pi)"] & \widetilde{K}^{-1}(X) \arrow[r, "\widetilde{K}^{-1}(i)"] & \widetilde{K}^{-1}(A) \arrow[dll, "\widetilde{\delta}_{(X,A)}"] \\
K(X,A) \arrow[r, "\widetilde{K}(\pi)"'] & \widetilde{K}(X) \arrow[r, "\widetilde{K}(i)"'] & \widetilde{K}(A).
\end{tikzcd}
$$

Proof. It is sufficient to prove the exactness of the first left long sequence. Hence, because of the canonical homeomorphisms between the quotient $C(X,A)/CX$ and SA, and between the quotient $(C'X \cup CA)/C(X,A)$ and SX, we have the exact sequences in $\mathrm{TopHdCpt}_+$[2]

$$A \xrightarrow{\quad i \quad} X \xrightarrow{\quad \pi \quad} X/A,$$

$$X \xrightarrow{\quad i' \quad} C(X,A) \xrightarrow{\quad \pi' \quad} SA \quad \text{and}$$

$$C(X,A) \xrightarrow{\quad i'' \quad} C'X \cup CA \xrightarrow{\quad \pi'' \quad} SX.$$

We now prove that these exact sequences in $\mathrm{TopHdCpt}_+$ induce the exact sequences in K-Theory

$$K(X,A) \xrightarrow{\widetilde{K}(\pi)} \widetilde{K}(X,a_0) \xrightarrow{\widetilde{K}(i)} \widetilde{K}(A,a_0),$$

$$\widetilde{K}^{-1}(A,a_0) \xrightarrow{\delta_{(X,A,a_0)}} K(X,A) \xrightarrow{\widetilde{K}(\pi)} \widetilde{K}(X,a_0) \quad \text{and}$$

$$\widetilde{K}^{-1}(X,a_0) \xrightarrow{\widetilde{K}(i)} \widetilde{K}^{-1}(A,a_0) \xrightarrow{\delta_{(X,A,a_0)}} K(X,A).$$

In fact, the first sequence coincides with the one in Corollary 2.33. The second sequence is the one that turns the diagram

$$\begin{tikzcd}
\widetilde{K}(SA,v) \arrow[r, "\widetilde{K}(\pi')"] & \widetilde{K}(C(X,A)) \arrow[r, "\widetilde{K}(i')"] & \widetilde{K}(X,a_0) \\
\widetilde{K}^{-1}(A,a_0) \arrow[u, "\widetilde{K}(\pi)"] \arrow[r, "\delta_{(X,A,a_0)}"'] & K(X,A) \arrow[u, "\widetilde{K}(\pi_1)\circ\widetilde{K}(\varphi)"] \arrow[r, "\widetilde{K}(\pi)"'] & \widetilde{K}(X,a_0). \arrow[u, equal]
\end{tikzcd}$$

---

[2] Let $f : (X,x_0) \to (Y,y_0)$ be a morphism in $\mathrm{TopHdCpt}_+$. We say that the **kernel** of f is the preimage $f^{-1}(y_0) \subseteq X$. Therefore, it is clear that the notion of exactness is also defined in the category $\mathrm{TopHdCpt}_+$.

into a commutative one, where $\widetilde{K}(\pi)$ is the restriction of the isomorphism given in Definition D.4 and $\widetilde{K}(\pi_1) \circ \widetilde{K}(\varphi)$ is the obvious composition from Lemma 2.40. In particular, note that $\delta_{(X,A,a_0)}$ is defined by the preceding diagram as the composition $\widetilde{K}(\varphi)^{-1} \circ \widetilde{K}(\pi_1)^{-1} \circ \widetilde{K}(\pi') \circ \widetilde{K}(\pi)$. In a similar manner, the third sequence is the one that turns the diagram

$$\begin{tikzcd}
\widetilde{K}(SX,u) \arrow[r, "\widetilde{K}(\pi'')"] & \widetilde{K}(C'X \cup CA, w) \arrow[r, "\widetilde{K}(i'')"] & \widetilde{K}(C(X,A)) \\
\widetilde{K}^{-1}(X,a_0) \arrow[r, "\widetilde{K}(i)"'] \arrow[u, "\widetilde{K}(\pi)"] & \widetilde{K}^{-1}(A,a_0) \arrow[r, "\delta_{(X,A,a_0)}"'] \arrow[u, "\widetilde{K}(\pi_2) \circ \widetilde{K}(\psi) \circ \widetilde{K}(\pi)"] & K(X,A). \arrow[u, "\widetilde{K}(\pi_1) \circ \widetilde{K}(\varphi)"]
\end{tikzcd}$$

into a commutative one, where $\widetilde{K}(\pi)$ is the restriction of the isomorphism given in Definition D.4 and $\widetilde{K}(\pi_2) \circ \widetilde{K}(\psi) \circ \widetilde{K}(\pi)$ is the obvious composition from Lemma 2.40. Thence, gluing these three exact sequences together, we obtain the exact sequence with five terms

$$\begin{tikzcd}
\widetilde{K}^{-1}(A,a_0) \arrow[r, "\delta_{(X,A,a_0)}"] & K(X,A) \arrow[r, "\widetilde{K}(\pi)"] & \widetilde{K}(X,a_0) \arrow[d, "\widetilde{K}(i)"] \\
\widetilde{K}^{-1}(X,a_0) \arrow[u, "\widetilde{K}^{-1}(i)"] & & \widetilde{K}(A,a_0).
\end{tikzcd}$$

Since the quotient $\Sigma X/\Sigma A$ is homeomorphic to $\Sigma(X/A)$, one can prove (using induction) that $\Sigma^n X/\Sigma^n A$ is homeomorphic to $\Sigma^n(X/A)$ for all $n \in \mathbb{N}$. Consequently, substituting X and A by $\Sigma^n X$ and $\Sigma^n A$ in the preceding sequence, respectively, we obtain the exact sequence

$$\begin{tikzcd}
\widetilde{K}^{-n-1}(A,a_0) \arrow[r, "\delta^{-n}_{(X,A,a_0)}"] & K^{-n}(X,A) \arrow[r, "\widetilde{K}^{-n}(\pi)"] & \widetilde{K}^{-n}(X,a_0) \arrow[d, "\widetilde{K}^{-n}(i)"] \\
\widetilde{K}^{-n-1}(X,a_0) \arrow[u, "\widetilde{K}^{-n-1}(i)"] & & \widetilde{K}^{-n}(A,a_0).
\end{tikzcd}$$

Therefore, gluing these last exact sequences together, we obtain the desired left long exact sequence. □

Theorem 2.42 (Split exact sequence induced by a special retraction)**.** *Let* (X, A) *be an object in* $\mathrm{TopHdCCpt}_2$ *such that the inclusion* $i : A \to X$ *is a retraction. For all* $n \in \mathbb{N}$*, we have that*

$$0 \longrightarrow K^{-n}(X, A) \xrightarrow{K^{-n}(\pi)} K^{-n}(X) \xrightarrow{K^{-n}(i)} K^{-n}(A) \longrightarrow 0$$

is a split short exact sequence. Therefore,

$$\begin{aligned} \Phi_n : K^{-n}(X) &\to K^{-n}(X, A) \oplus K^{-n}(A), \\ \alpha &\mapsto (\alpha - K^{-n}(i \circ r)(\alpha), K^{-n}(i)(\alpha)), \end{aligned}$$

where $r : X \to A$ *is a left inverse for the inclusion* $i : A \to X$*, is an isomorphism between* $K^{-n}(X)$ *and* $K^{-n}(X, A) \oplus K^{-n}(A)$*. This proves that* $K^{-n}(X)$ *is isomorphic to* $\widetilde{K}^{-n}(X, x_0) \oplus K^{-n}(x_0)$*, as claimed in Remark 2.39. Analogously, if* (X, A, a_0) *is an object in* $\mathrm{TopHdCCpt}_{2+}$*, then*

$$0 \longrightarrow K^{-n}(X, A) \xrightarrow{\widetilde{K}^{-n}(\pi)} \widetilde{K}^{-n}(X, a_0) \xrightarrow{\widetilde{K}^{-n}(i)} \widetilde{K}^{-n}(A, a_0) \longrightarrow 0$$

is a split short exact sequence. The same result holds true considering the other reduced version of K-Theory.

Proof. The reader can readily adapt the proof of Theorem 1.61 in order to prove this result. □

Corollary 2.43 (A direct decomposition of the pointed reduced K-Theory groups of a product of pointed compact Hausdorff spaces)**.** *Let* (X, x_0) *and* (Y, y_0) *be objects in* $\mathrm{TopHdCpt}_+$*. In addition, let* $\pi_1 \colon X \times Y \to X$*,* $\pi_2 \colon X \times Y \to Y$*,* $i_1 \colon X \to X \times Y$ *and* $i_2 \colon Y \to X \times Y$ *be the canonical projections and inclusions. Thence, we have that the map*

$$\Psi_n : \widetilde{K}^{-n}(X \times Y, (x_0, y_0)) \to \widetilde{K}^{-n}(X \wedge Y) \oplus \widetilde{K}^{-n}(Y, y_0) \oplus \widetilde{K}^{-n}(X, x_0)$$

defined by

$$\Psi_n(\alpha) := (\alpha - \widetilde{K}^{-n}(i_1 \circ \pi_1)(\alpha) - \widetilde{K}^{-n}(i_2 \circ \pi_2)(\alpha),\ \widetilde{K}^{-n}(i_2 \circ \pi_2)(\alpha),\ \widetilde{K}^{-n}(i_1 \circ \pi_1)(\alpha))$$

is a isomorphism.

Proof. Since $i_1 : X \to X \times Y$ and $[i_2] : Y \to (X \times Y)/X$ are retractions, Theorem 2.42 yields

$$\begin{aligned}
\widetilde{K}^{-n}(X \times Y, (x_0, y_0)) &\simeq K^{-n}(X \times Y, X) \oplus \widetilde{K}^{-n}(X, x_0) \\
&\simeq \widetilde{K}^{-n}((X \times Y)/X) \oplus \widetilde{K}^{-n}(X, x_0) \\
&\simeq K^{-n}((X \times Y)/X,\ Y) \oplus \widetilde{K}^{-n}(Y, y_0) \oplus \widetilde{K}^{-n}(X, x_0) \\
&\simeq \widetilde{K}^{-n}(X \wedge Y) \oplus \widetilde{K}^{-n}(Y, y_0) \oplus \widetilde{K}^{-n}(X, x_0).
\end{aligned}$$

This finishes the proof of this result once the reader can explicitly write the isomorphisms indicated above in order to show that their composition coincides with the map Ψ_n set in the statement. □

Remark 2.44 (Ensuring some of the results stated before)**.** *Let (X, x_0) be an object in* $\mathrm{TopHdCpt}_+$*. We have that the relative group $K^{-n}(X, x_0)$ is canonically isomorphic to the pointed reduced group $\widetilde{K}^{-n}(X, x_0)$ for all $n \in \mathbb{N}$. Therefore, considering the first sequence in Theorem 2.42 with $A = \{x_0\}$, we have that the group $K^{-n}(X, A)$ becomes $\widetilde{K}^{-n}(X, x_0)$, which is isomorphic to the image of $K^{-n}(\pi)$ since this map is injective. In turn, this image coincides with the kernel of $K^{-n}(i)$. This proves the first part of Remark 2.39, allowing us to reproduce the discussion made in Remark 2.23 for negative degree reduced K-Theory groups.* ♢

2.6 The Bott Periodicity Theorem and K-Theory of positive degree

In this section, we present a powerful result in Ordinary K-Theory, namely, the Bott Periodicity Theorem. We do not prove this theorem here because of its extent and because it is crystal clear in the literature. However, it allows us to finish the process of extending the K-Theory groups to other degrees, which was started in the preceding section. We begin with the following remark in which stands the basic tool behind the theorem in question, which are the natural multiplicative structures in K-Theory.

Remark 2.45 (Internal and external multiplicative structures in Ordinary K-Theory)**.** *Let X be a compact Hausdorff space. Because of Remark C.42 and Theorem C.43, we have the product*

$$\begin{aligned} \otimes: \ K(X) \otimes K(X) &\to K(X), \\ [[E]] \otimes [[F]] &\mapsto [[E \otimes F]], \end{aligned}$$

which turns $K(X)$ into a commutative ring with unit. Additionally, if Y is another compact Hausdorff space, then we have the external multiplication (see Definition 1.96) given by

$$\begin{aligned} \boxtimes: \ K(X) \otimes K(Y) &\to K(X \times Y), \\ [[E]] \otimes [[F]] &\mapsto [[E \boxtimes F]], \end{aligned}$$

where

$$E \boxtimes F \ := \ \pi_1^* E \otimes \pi_2^* F$$

with $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ being the natural projections onto the first and the second factors, respectively. It is to be noted that the fiber of $E \boxtimes F$ over $(x, y) \in X \times Y$ coincides with $E_x \otimes F_y$. Moreover, if $X = Y$ and $\Delta : X \to X \times X$ is the diagonal map, then

$$E \otimes F \ = \ \Delta^*(E \boxtimes F).$$

This shows that the product on $K(X)$ is an internal multiplication in the sense of Theorem 1.98. Furthermore, if we choose marked points $x_0 \in X$ and $y_0 \in Y$, then we obtain by restriction

$$\boxtimes: \ \widetilde{K}(X, x_0) \otimes \widetilde{K}(Y, y_0) \ \to \ \widetilde{K}(X \wedge Y),$$

where the marked point of $X \wedge Y$ is the natural one. Indeed, let $\alpha \in \widetilde{K}(X, x_0)$ and $\beta \in \widetilde{K}(Y, y_0)$. If $i_1 : X \to X \times Y$ and $i_2 : Y \to X \times Y$ are the canonical inclusions, then

$$i_1^*(\alpha \boxtimes \beta) \;=\; i_1^*(\pi_1^*\alpha \otimes \pi_2^*\beta) \;=\; i_1^*\pi_1^*\alpha \otimes i_1^*\pi_2^*\beta \;=\; \alpha \otimes i_1^*\pi_2^*\beta \;=\; \alpha \otimes (\pi_2 \circ i_1)^*\beta.$$

Since $\pi_2 \circ i_1 : X \to Y$ *is a constant function, we have* $(\pi_2 \circ i_1)^*\beta = 0$. *Thus, it follows that* $i_1^*(\alpha \boxtimes \beta) = 0$. *Analogously, one can readily prove that* $i_2^*(\alpha \boxtimes \beta) = 0$. *Therefore, we obtain* $\alpha \boxtimes \beta \in \widetilde{K}(X \wedge Y)$. *Therefore, substituting* X *and* Y *by* $\Sigma^n X$ *and* $\Sigma^m Y$, *respectively, we obtain the product*

$$\boxtimes : \; \widetilde{K}^{-n}(X, x_0) \,\otimes\, \widetilde{K}^{-m}(Y, y_0) \;\to\; \widetilde{K}^{-n-m}(X \wedge Y).$$

In a similar manner, if we substitute X *and* Y *by* $\Sigma^n X_+$ *and* $\Sigma^m Y_+$, *respectively, then we obtain the product*

$$\boxtimes : \; K^{-n}(X) \,\otimes\, K^{-m}(Y) \;\to\; K^{-n-m}(X \times Y).$$

once $X_+ \wedge Y_+$ *is canonically homeomorphic to* $(X \times Y)_+$. *In both cases, if* $X = Y$, *then the pullback of these maps through the diagonal map is the internal multiplication in K-Theory.* ♢

Notation 2.46 (Spheres and discs in Euclidean spaces)**.** *Let* n *be a natural number. Hereafter, we denote the* ***n-dimensional sphere*** *by* $\mathbb{S}^n$ *and the* ***n-dimensional disc*** *by* $\mathbb{D}^n$. ♢

Definition 2.47 (Canonical line bundle on the two-dimensional sphere)**.** *We say that the* ***canonical line bundle on the two-dimensional sphere*** $\mathbb{S}^2$ *is the quotient* η *of the disjoint union*

$$(\mathbb{D}^2 \times \mathbb{C}) \,\sqcup\, (\mathbb{D}^2 \times \mathbb{C})$$

by the relation that identifies $(z, w)_1$ *with* $(z, zw)_2$ *for all* $z \in \mathbb{S}^1$ *and all* $w \in \mathbb{C}$, *where* $(z, w)_1$ *indicates* (z, w) *in the first copy of* $\mathbb{D}^2 \times \mathbb{C}$ *and* $(z, w)_2$ *indicates* (z, w) *in the second copy of* $\mathbb{D}^2 \times \mathbb{C}$. ♢

Theorem 2.48 (Bott Periodicity Theorem)**.** *Let* (X, x_0) *be an object in* $\mathrm{TopHdCpt}_+$. *In addition, let* η *be the canonical line bundle on the two-dimensional sphere* $\mathbb{S}^2$. *Then, we have the map*

$$\begin{aligned} B : \; \widetilde{K}(X, x_0) \;&\to\; \widetilde{K}^{-2}(X, x_0), \\ \alpha \;&\mapsto\; [[\eta - 1]] \boxtimes \alpha, \end{aligned}$$

since $\widetilde{K}^{-2}(X,x_0)$ *is canonically isomorphic to* $\widetilde{K}(\mathbb{S}^2 \wedge X)$*. Furthermore, this map is an isomorphism of rings.*

Proof. The reader can find proofs of this result in [2, pp. 44-64] and in [15, pp. 41-55]. The treatment given by the first reference is more technical and general than the one given by the second reference. This may help the reader in choosing which one of them to follow. □

Remark 2.49 (Extending the Bott Periodicity Theorem to the other K-Theory groups)**.** *The Bott Periodicity Theorem ensures that (complex) Ordinary K-Theory is* 2*-periodic. This happens because, in Theorem 2.48, if we substitute:*

- X *by* $\Sigma^n X$*, then we obtain the isomorphism of rings*

$$B^n_{(X,x_0)} : \widetilde{K}^{-n}(X,x_0) \to \widetilde{K}^{-n-2}(X,x_0);$$

- X *by* $\Sigma^n X_+$*, then we obtain the isomorphism of rings*

$$B^n_X : K^{-n}(X) \to K^{-n-2}(X); \text{ and}$$

- X *by* $\Sigma^n(X/A)$*, then we obtain the isomorphism of rings*

$$B^n_{(X,A)} : K^{-n}(X,A) \to K^{-n-2}(X,A).$$

Therefore, when considering a K-Theory group, the only important information is the parity of its degree. In other words, the only significant K-Theory groups are the ones of degree 0 *and* -1*.* ♢

Definition 2.50 (K-Theory groups and homomorphisms of positive degree)**.** *Let* n *be a natural number. Because of Remark 2.49, we extend the K-Theory groups to positive degrees as follows.*

- *The* ***nth positive degree absolute K-Theory group*** *of a compact Hausdorff space* X*, which is hereafter denoted by* $K^n(X)$*, is defined as the negative K-Theory group*

$$K^{-n}(X).$$

*In addition, being $f : X \to Y$ a morphism of compact Hausdorff spaces, we define the **nth positive degree pullback of f in absolute K-Theory**, and denote it by $K^n(f) : K^n(Y) \to K^n(X)$, to be the nth negative degree pullback*

$$K^{-n}(f) : K^{-n}(Y) \to K^{-n}(X).$$

- *The **nth positive degree pointed reduced K-Theory group** of an object $(X, x_0) \in \mathrm{TopHdCpt}_+$, which is hereafter denoted by $\widetilde{K}^n(X, x_0)$, is defined as the negative K-Theory group*

  $$\widetilde{K}^{-n}(X, x_0).$$

  *In addition, being $f : (X, x_0) \to (Y, y_0)$ a morphism of pointed compact Hausdorff spaces, we define the **nth positive degree pullback of f in pointed reduced K-Theory**, and denote it by $\widetilde{K}^n(f) : \widetilde{K}^n(Y, y_0) \to \widetilde{K}^n(X, x_0)$, to be the nth negative degree pullback*

  $$\widetilde{K}^{-n}(f) : \widetilde{K}^{-n}(Y, y_0) \to \widetilde{K}^{-n}(X, x_0).$$

- *The **nth positive degree relative K-Theory group** of an object $(X, A) \in \mathrm{TopHdCCpt}_2$, which is hereafter denoted by $K^n(X, A)$, is defined as the negative degree K-Theory group*

  $$K^{-n}(X, A).$$

  *In addition, being $f : (X, A) \to (Y, B)$ a morphism of pairs of compact Hausdorff spaces, we define the **nth positive degree pullback of f in relative K-Theory**, and denote it by $K^n(f) : K^n(Y, B) \to K^n(X, A)$, to be the nth negative degree pullback*

  $$K^{-n}(f) : K^{-n}(Y, B) \to K^{-n}(X, A).$$

The reader can readily set the categorical structure of the positive degree K-Theory using Remark 2.38. $\diamondsuit$

Corollary 2.51 (Long exact sequence in Ordinary K-Theory)**.** *Let (X, A) be an object in* $\mathrm{TopHdCCpt}_2$*. In addition, let $i : A \to X$ and $\pi : X \to X/A$ be the canonical inclusion and projection, respectively. In this situation, we have the long exact sequence in K-Theory*

$$\cdots \longrightarrow K^{n-1}(A) \xrightarrow{\delta^n_{(X,A)}} K^n(X, A) \xrightarrow{K^n(\pi)} K^n(X) \xrightarrow{K^n(i)} K^n(A) \longrightarrow \cdots,$$

where $\delta^n_{(X,A)} : K^{n-1}(A) \to K^n(X, A)$ is naturally defined through the following commutative diagram.

$$\begin{tikzcd}
K^{n-1}(A) = K^{-n+1}(A) \arrow[r, "B^{n-1}_A"'] \arrow[rr, bend left, "\delta^n_{(X,A)}"] & K^{-n-1}(A) \arrow[r, "\delta^{-n}_{(X,A)}"'] & K^{-n}(X, A) = K^n(X, A).
\end{tikzcd}$$

In fact, since (complex) K-Theory is 2-periodic, this sequence reduces to the exact rectangle with six significant groups

$$\begin{tikzcd}
K(X, A) \arrow[r, "K(\pi)"] & K(X) \arrow[r, "K(i)"] & K(A) \arrow[d, "\delta^{-1}_{(X,A)}"] \\
K^{-1}(A) \arrow[u, "\delta_{(X,A)}"] & K^{-1}(X) \arrow[l, "K^{-1}(i)"] & K^{-1}(X, A). \arrow[l, "K^{-1}(\pi)"]
\end{tikzcd}$$

Proof. This result is a consequence of Theorem 2.41 and of the ideas presented in this section. $\square$

2.7 Ordinary K-Theory as a generalized cohomology theory

In this section, we finally prove that the data defined above establish a generalized cohomology theory. After this is done, as we mentioned before, all of the results proven in Chapter 1 hold true. In particular, we immediately obtain the

K-Theory exact sequence of a triple, the K-Theory exact sequence of a proper triad, the K-Theory Mayer-Vietoris absolute and relative exact sequences, *et reliqua*. We begin with the following theorem.

Theorem 2.52 (K-Theory as a generalized cohomology theory)**.** *Let:*

- $\mathrm{TopHdCCpt}_2$ *be the category of pairs of compact Hausdorff spaces;*
- $(K^n : \mathrm{TopHdCCpt}_2 \to \mathscr{G}_{ab})_{n\in\mathbb{Z}}$ *be the sequence of contravariant relative functors in K-Theory; and*
- $(\delta^n)_{n\in\mathbb{Z}}$ *be the sequence of maps that send a pair* $(X, A) \in \mathrm{TopHdCCpt}_2$ *into the homomorphism* $\delta^n_{(X,A)} : K^{n-1}(A) \to K^n(X, A)$.

We have that the three pieces of data above are a generalized cohomology theory (see Definition 1.9).

Proof. We still only have to verify the Excision Axiom, which we now prove to hold in the stronger version presented in Remark 1.103. Indeed, if $(X, A) \in \mathrm{TopHdCCpt}_2$ and U is a subset of X whose closure is contained in the interior of A, then we claim that the inclusion $i : (X - U, A - U) \to (X, A)$ induces isomorphisms

$$K^n(i) : \; K^n(X, A) \;\to\; K^n(X - U, A - U)$$

for all $n \in \mathbb{Z}$. This happens because one can prove that the preceding inclusion induces the canonical homeomorphism $\bar{i} : (X - U)/(A - U) \to X/A$, which ensures the desired isomorphisms. $\square$

Remark 2.53 (Ordinary K-Theory is an additive generalized cohomology theory)**.** *Since the category* $\mathrm{TopHdCCpt}_2$ *is the building block for K-Theory, we have that Ordinary K-Theory is an additive generalized cohomology theory. In fact, any decomposition of a pair* $(X, A) \in \mathrm{TopHdCCpt}_2$ *as in Definition 1.70 has to be a finite one (because a compact Hausdorff space must have a finite number of connected components). Then, the claim is obvious because Theorem 1.69 holds in K-Theory once it is a generalized cohomology theory.* $\diamondsuit$

Example 2.54 (K-Theory groups of spheres). *We have the following facts.*

- *In Example 2.9, we have proved that, if Ω is a one-point space, then $K(\Omega)$ is isomorphic to the integer numbers equipped with the usual sum. Consequently, because of the isomorphism in Remark 2.15 and because $\Sigma\Omega$ is homeomorphic to Ω, we have that $K^{-1}(\Omega)$ is trivial. Therefore, it follows from the Bott Periodicity Theorem that*
$$K^{-2n}(\Omega) \simeq \mathbb{Z} \qquad \text{and} \qquad K^{-2n-1}(\Omega) \simeq 0,$$
as claimed in Remark 2.39.

- *In Example 2.10, we have proved that, if $\mathbb{S}^1$ is the one-dimensional sphere, then $K(\mathbb{S}^1)$ is isomorphic to the integer numbers equipped with the usual sum.*

These results, together with the fact that
$$\widetilde{K}^n(\mathbb{S}^k) \simeq \widetilde{K}^n(\Sigma^k\mathbb{S}^0) \simeq \widetilde{K}^{n-k}(\mathbb{S}^0) = K^{-n-k}(\Omega)$$
for all $k \in \mathbb{N}$ and all $n \in \mathbb{Z}$, allow us to set Table 2.1, which contains all of the K-theory groups of the spheres. $\diamondsuit$

n	k	$\widetilde{K}^n(\mathbb{S}^k)$	$K^n(\mathbb{S}^k)$
even	even	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$
even	odd	0	$\mathbb{Z}$
odd	even	0	0
odd	odd	$\mathbb{Z}$	$\mathbb{Z}$

Table 2.1: K-Theory groups of spheres.

2.8 An application

In this brief section, we show the relevance of the theory developed in this chapter by exhibiting one of its great achievements in the last century, which is a classification theorem for real division algebras known as the Bott-Milnor-Kervaire Theorem. Moreover, we expose a solution to the problem of the tangent bundles of spheres being trivial. We do not prove these results here because of their extent and because they are crystal clear in the literature. However, we provide sketches of their proofs in order to explain why K-Theory solved them. The reader who desires to understand a bit more about the problem of the real division algebras is invited to read Appendix E, where we not only establish the elementary facts on real (division) algebras but also present a historical perspective to them. We begin with the following definition.

Definition 2.55 (Topological and differential notions applied to spheres)**.** *Let n be a natural number. We say that the n-dimensional sphere $\mathbb{S}^n$ is:*

- *an **H-space** if there exists a continuous binary operation $\cdot : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n$ having a two-sided identity element*[(3)]*; and*

- ***parallelizable** if its tangent bundle $\mathrm{T}\,\mathbb{S}^n$ is trivial. We remind the reader that it is equivalent to the existence of n linearly independent vector fields on $\mathbb{S}^n$. This equivalence follows from Theorem C.20 since vector fields on $\mathbb{S}^n$ are global sections in $\Gamma(\mathrm{T}\,\mathbb{S}^n)$.* ♢

Lemma 2.56 (Relation between real division algebras, parallelizable spheres and H-spaces)**.** *Let n be a non-zero natural number. We have that the following assertions hold true.*

(1) If $\mathbb{R}^n$ is a real division algebra, then $\mathbb{S}^{n-1}$ is an H-space.

(2) If $\mathbb{S}^{n-1}$ is parallelizable, then $\mathbb{S}^{n-1}$ is an H-space.

Proof.

[(3)]It is to be noted that a topological space being an H-space is weaker than it being a topological group. This happens because the first notion does not require associativity and inverses for the binary operation, while the second one does require these properties. Indeed, for example, $\mathbb{S}^1$ and $\mathbb{S}^3$ are topological groups with the multiplications being the ones restricted from the complex numbers $\mathbb{C}$ and from the quaternions $\mathbb{H}$, respectively. In turn, $\mathbb{S}^7$ is an H-space with the multiplication being the one restricted from the octonions $\mathbb{O}$. However, it is not a topological group since this multiplication lacks associativity (see Example E.7).

(1) If $\mathbb{R}^n$ is a real division algebra (see Definition E.1), then

$$\begin{array}{rcl} \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} & \to & \mathbb{S}^{n-1}, \\ (x,\, y) & \mapsto & \dfrac{xy}{|xy|}, \end{array}$$

is a well-defined continuous binary operation having a two-sided identity element, where the norm in question is the Euclidean one. This proves that $\mathbb{S}^{n-1}$ is an H-space.

(2) If $\mathbb{S}^{n-1}$ is parallelizable, then let $v_1, \cdots, v_{n-1}$ be linearly independent vector fields on $\mathbb{S}^{n-1}$. Because of the *Gram-Schmidt Orthonormalization Process*, we may assume that the vectors $x, v_1(x), \cdots, v_{n-1}(x)$ are orthonormal for all $x \in \mathbb{S}^{n-1}$. In addition, we may also assume that

$$v_1(e_1) \;=\; e_2,\; \cdots,\; v_{n-1}(e_1) \;=\; e_n,$$

where $e_1, \cdots, e_n$ is the standard basis on $\mathbb{R}^n$. This is possible because, if it is not the case, then we can change the sign of v_{n-1} to correct the orientations and thence deform the vector fields in question near $e_1 \in \mathbb{R}^n$. Now, let $\alpha_x : \mathbb{R}^n \to \mathbb{R}^n$ be the linear isometry that sends the standard basis into $x, v_1(x), \cdots, v_{n-1}(x)$ for all $x \in \mathbb{S}^{n-1}$. Therefore,

$$\begin{array}{rcl} \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} & \to & \mathbb{S}^{n-1}, \\ (x,\, y) & \mapsto & \alpha_x(y), \end{array}$$

defines an H-space structure on $\mathbb{S}^{n-1}$ with the two-sided identity element being e_1 since $\alpha_{e_1} = \mathrm{id}_{\mathbb{R}^n}$ and $\alpha_x(e_1) = x$ for all $x \in \mathbb{S}^{n-1}$. This finishes the proof of the lemma. □

Theorem 2.57 (Main results of this section)**.** *The following assertions are true only if* $n = 1, 2, 4$ *or* 8.

(1) There exists a real division algebra structure for an n-dimensional vector space $\mathscr{A}$.

(2) The sphere $\mathbb{S}^{n-1}$ *is parallelizable.*

The first assertion is called the ***Bott-Milnor-Kervaire Theorem****.*

Proof. First, we prove that we can restrict the problem of real division algebras to the one when the underlying n-dimensional vector space is the Euclidean space $\mathbb{R}^n$. Indeed, if $\mathscr{A}$ is an n-dimensional vector space, then let $\alpha : \mathscr{A} \to \mathbb{R}^n$ be a linear isomorphism. The diagram

$$\begin{tikzcd}
\mathscr{A} \times \mathscr{A} \arrow[r, "m_{\mathscr{A}}"] \arrow[d, "\alpha\times\alpha"'] & \mathscr{A} \arrow[d, "\alpha"] \\
\mathbb{R}^n \times \mathbb{R}^n \arrow[r, "m_{\mathbb{R}^n}"'] & \mathbb{R}^n
\end{tikzcd}$$

proves our assertion. In fact:

- if $m_{\mathscr{A}} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is a real division algebra structure on $\mathscr{A}$, then

$$m_{\mathbb{R}^n} := \alpha \circ m_{\mathscr{A}} \circ (\alpha \times \alpha)^{-1} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

  is a real division algebra structure on $\mathbb{R}^n$; *and*

- if $m_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a real division algebra structure on $\mathbb{R}^n$, then

$$m_{\mathscr{A}} := \alpha^{-1} \circ m_{\mathbb{R}^n} \circ (\alpha \times \alpha) : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$$

  is a real division algebra structure on $\mathscr{A}$.

Therefore, in order to prove the assertions of the statement, it suffices to show that $\mathbb{S}^{n-1}$ is an H-space only if $n = 1, 2, 4$ or 8. This is a consequence of Lemma 2.56. In turn, in order to prove this last assertion, we use the following K-Theory arguments. The *Bott Periodicity Theorem* ensures that:

- the reduced K-Theory group $\widetilde{K}(\mathbb{S}^n)$ is the group of integer numbers for n even and trivial for n odd (see Table 2.1). This comes from repeated application of the periodicity isomorphism

$$\begin{aligned} B : \widetilde{K}(X, x_0) &\to \widetilde{K}(\mathbb{S}^2 \wedge X), \\ \alpha &\mapsto [[\eta - 1]] \boxtimes \alpha, \end{aligned}$$

where η is the canonical line bundle on $\mathbb{S}^2$ (see Definition 2.47). In particular, we immediately see that the generator of the ring $\widetilde{K}(\mathbb{S}^{2k})$ is the k-fold external product

$$[[\eta - 1]] \boxtimes \cdots \boxtimes [[\eta - 1]].$$

Moreover, we have that the multiplication in $\widetilde{K}(\mathbb{S}^{2k})$ is trivial since this ring is the k-fold tensor product of the ring $\widetilde{K}(\mathbb{S}^2)$, which one can prove that has trivial multiplication;

- the external product

$$\boxtimes : \ \widetilde{K}(\mathbb{S}^{2k}) \otimes \widetilde{K}(X) \ \to \ \widetilde{K}(\mathbb{S}^{2k} \wedge X)$$

is an isomorphism since it is an iterate of the periodicity isomorphism of the preceding item; *and*

- the external product

$$\boxtimes : \ K(\mathbb{S}^{2k}) \otimes K(X) \ \to \ K(\mathbb{S}^{2k} \times X)$$

is an isomorphism. Since external product is a ring homomorphism, the isomorphism between $\widetilde{K}(\mathbb{S}^{2k} \wedge X)$ and $\widetilde{K}(\mathbb{S}^{2k}) \otimes \widetilde{K}(X)$ is a ring isomorphism. For example, since $K(\mathbb{S}^{2k})$ can be described as the quotient ring $\mathbb{Z}[\gamma]/(\gamma^2)$, we can deduce that $K(\mathbb{S}^{2k} \times \mathbb{S}^{2l})$ is $\mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ where α and β are the pullbacks of the generators of $\widetilde{K}(\mathbb{S}^{2k})$ and $\widetilde{K}(\mathbb{S}^{2l})$ under the natural projections of $\mathbb{S}^{2k} \times \mathbb{S}^{2l}$ onto its factors. Thus, we have that an additive basis for $K(\mathbb{S}^{2k} \times \mathbb{S}^{2l})$ is $\{1, \alpha, \beta, \alpha\beta\}$.

Thence, the proof splits in the following two cases.

(1) *If k is a non-zero natural number, then $\mathbb{S}^{2k}$ is not an H-space.* This claim follows from the last item above. Indeed, suppose that $\mu : \mathbb{S}^{2k} \times \mathbb{S}^{2k} \to \mathbb{S}^{2k}$ is an H-space multiplication. Hence, we have that the induced homomorphism of K-rings has the form

$$K(\mu) : \ \mathbb{Z}[\gamma]/(\gamma^2) \ \to \ \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2).$$

We claim that

$$K(\mu)(\gamma) \;=\; \alpha + \beta + m\alpha\beta$$

for some $m \in \mathbb{Z}$. In fact, we have that the composition

$$\mathbb{S}^{2k} \xrightarrow{\quad i \quad} \mathbb{S}^{2k} \times \mathbb{S}^{2k} \xrightarrow{\quad \mu \quad} \mathbb{S}^{2k}$$

is the identity, where i is the inclusion into either of the subspaces $\mathbb{S}^{2k} \times \{1\}$ or $\{1\} \times \mathbb{S}^{2k}$ with 1 being the identity element of the H-space structure μ. Thus, $K(i)$ for i the inclusion onto the first factor sends α to γ and β to 0. Consequently, the coefficient of α in $K(\mu)(\gamma)$ must be 1. In a similar manner, the coefficient of β in $K(\mu)(\gamma)$ must also be 1. However, this leads to a contradiction since it implies

$$K(\mu)(\gamma^2) \;=\; (\alpha + \beta + m\alpha\beta)^2 \;=\; 2\alpha\beta,$$

which is impossible since $\gamma^2 = 0$.

(2) *If k is a natural number different from* 1, 2 *and* 4, *then $\mathbb{S}^{2k-1}$ is not an H-space.* This is the hard part of the proof. The main idea is to associate to a map $f : \mathbb{S}^{2k-1} \times \mathbb{S}^{2k-1} \to \mathbb{S}^{2k-1}$ a map $\widehat{f} : \mathbb{S}^{4k-1} \to \mathbb{S}^{2k}$, and then show that the *Hopf invariant* of $\widehat{f}$ is equal to plus or minus the unit if f is an H-space multiplication. Consequently, the problem is solved proving that a map $g : \mathbb{S}^{4k-1} \to \mathbb{S}^{2k}$ has Hopf invariant equal to plus or minus the unit only when $n = 1, 2$ or 4. For this, one has to prove the existence of a special kind of ring homomorphism in the K-Theory framework, which is known as *Adams Operations*. The reader can find all the details in [15, pp. 59-72]. □

Remark 2.58 (Complementing the preceding result)**.** *We have the following facts.*

- *There is a real division algebra structure for an n-dimensional vector space $\mathscr{A}$ when $n = 1, 2, 4$ or 8. Once and again, we can restrict ourselves to the case when $\mathscr{A}$ coincides with $\mathbb{R}^n$. Therefore, the problem is solved since the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$ are real division algebras of dimensions* 1, 2, 4 *and* 8, *respectively.*

- *The spheres* $\mathbb{S}^0$*,* $\mathbb{S}^1$*,* $\mathbb{S}^3$ *and* $\mathbb{S}^7$ *are parallelizable. In order to prove this claim, we use the notations of Examples E.3, E.4 and E.5. Indeed, we can explicitly construct a sufficient number of linearly independent vector fields on these spheres that ensure their parallelizability. In fact:*

  - $\mathbb{S}^0$ *is parallelizable because its tangent bundle has rank zero;*
  - $\mathbb{S}^1$ *is parallelizable because* $v : \mathbb{S}^1 \to \mathrm{T}\,\mathbb{S}^1$*,* $z \mapsto (z, e_1 z)$*, is a global vector field on* $\mathbb{S}^1$*;*
  - $\mathbb{S}^3$ *is parallelizable because*

$$\begin{aligned} v_1 : \mathbb{S}^3 &\to \mathrm{T}\,\mathbb{S}^3,\ z \mapsto (z, e_1 z),\\ v_2 : \mathbb{S}^3 &\to \mathrm{T}\,\mathbb{S}^3,\ z \mapsto (z, e_2 z),\\ v_3 : \mathbb{S}^3 &\to \mathrm{T}\,\mathbb{S}^3,\ z \mapsto (z, e_3 z), \end{aligned}$$

  are three linearly independent vector fields on $\mathbb{S}^3$*; and*
  - $\mathbb{S}^7$ *is parallelizable because*

$$\begin{aligned} v_1 : \mathbb{S}^7 &\to \mathrm{T}\,\mathbb{S}^7,\ z \mapsto (z, e_1 z),\\ v_2 : \mathbb{S}^7 &\to \mathrm{T}\,\mathbb{S}^7,\ z \mapsto (z, e_2 z),\\ v_3 : \mathbb{S}^7 &\to \mathrm{T}\,\mathbb{S}^7,\ z \mapsto (z, e_3 z),\\ v_4 : \mathbb{S}^7 &\to \mathrm{T}\,\mathbb{S}^7,\ z \mapsto (z, e_4 z),\\ v_5 : \mathbb{S}^7 &\to \mathrm{T}\,\mathbb{S}^7,\ z \mapsto (z, e_5 z),\\ v_6 : \mathbb{S}^7 &\to \mathrm{T}\,\mathbb{S}^7,\ z \mapsto (z, e_6 z),\\ v_7 : \mathbb{S}^7 &\to \mathrm{T}\,\mathbb{S}^7,\ z \mapsto (z, e_7 z), \end{aligned}$$

  are seven linearly independent vector fields on $\mathbb{S}^7$*.* ♢

2.9 Euler characteristic

In this section, we will return to the relative K-Theory groups. Indeed, until now, we have a discrepancy between the treatment of this idea and the form we introduced absolute and reduced K-Theories. In fact, these latter concepts were introduced in a concrete and geometrical way. The absolute K-Theory classes were presented as *virtual* vector bundles, which admit minus signs before them. In turn, the reduced K-Theory classes were presented as absolute K-Theory classes whose rank is equal to zero. On the other hand, the motivation for Definition 2.25 was shown in Remark 2.26, but it said nothing about the geometric structure of the relative K-Theory classes. Among other things, we will fix this omission here. We begin with the following definition.

Definition 2.59 (The category of exact sequences of vector bundles). *Let* (X, A) *be an object in* TopHdCCpt$_2$. *We define the* ***category of exact sequences of vector bundles on*** $\boldsymbol{(X, A)}$, *and denote it by* $\mathcal{C}_1(X, A)$, *to be the category whose:*

- *objects are triples* $E = (E_1, E_0, \alpha)$ *where* E_1 *and* E_0 *are vector bundles on* X *and* $\alpha : E_1 \mid_A \to E_0 \mid_A$ *is an isomorphism over* A. *This can be equivalently stated saying that the sequence*

$$0 \longrightarrow E_1 \mid_A \xrightarrow{\alpha} E_0 \mid_A \longrightarrow 0$$

  is exact; and

- *morphisms* $\varphi : E \to F$ *between* $E = (E_1, E_0, \alpha)$ *and* $F = (F_1, F_0, \beta)$ *are pairs of morphisms of vector bundles* $(\varphi_1 : E_1 \to F_1,\ \varphi_0 : E_0 \to F_0)$ *in such manner that the diagram*

$$\begin{tikzcd} E_1 \mid_A \arrow[r, "\alpha"] \arrow[d, "\varphi_1|_A"'] & E_0 \mid_A \arrow[d, "\varphi_0|_A"] \\ F_1 \mid_A \arrow[r, "\beta"'] & F_0 \mid_A \end{tikzcd}$$

  is commutative. An ***isomorphism*** *in* $\mathcal{C}_1(X, A)$ *is a morphism whose components are isomorphisms over* X. ♢

Definition 2.60 (Elementary sequences and an equivalence relation induced by them). *Let* (X, A) *be an object in* TopHdCCpt$_2$. *We say that an* ***elementary sequence*** *in* $\mathcal{C}_1(X, A)$ *is an object of the form*

$$E \;=\; (E,\, E,\, \mathrm{id}_{E|_A}).$$

In addition, given $E = (E_1, E_0, \alpha)$ *and* $F = (F_1, F_0, \beta)$ *in* $\mathcal{C}_1(X, A)$*, we say that they are equivalent if and only if there exist elementary sequences* Q *and* P *in* $\mathcal{C}_1(X, A)$ *for which*

$$E \oplus Q \qquad \text{and} \qquad F \oplus P$$

are isomorphic[(4)]*. This definition naturally gives rise to an equivalence relation on the class of objects of* $\mathcal{C}_1(X, A)$*. The set of such equivalence classes is hereafter denoted by* $\mathcal{L}_1(X, A)$*.* ♢

Notation 2.61 (The canonical commuting isomorphism)**.** *Let* X *be a topological space and* E *and* F *be vector bundles on* X*. Hereafter, we will denote by* $\eta_{E,F}$ *the canonical isomorphism*

$$\begin{aligned} \eta_{E,F} : F \oplus E \;&\to\; E \oplus F, \\ (a, b) \;&\mapsto\; (b, a). \end{aligned}$$ ♢

Theorem 2.62 (Natural structure of abelian group)**.** *Let* (X, A) *be an object in* $\mathrm{TopHdCCpt}_2$*. The binary operation*

$$\begin{aligned} \oplus :\; \mathcal{L}_1(X, A) \times \mathcal{L}_1(X, A) \;&\to\; \mathcal{L}_1(X, A), \\ ([E_1,\, E_0,\, \alpha],\, [F_1,\, F_0,\, \beta]) \;&\mapsto\; [E_1 \oplus F_1,\, E_0 \oplus F_0,\, \alpha \oplus \beta], \end{aligned}$$

is well-defined. More than that, it turns the set of equivalences classes $\mathcal{L}_1(X, A)$ *into an abelian group.*

Proof. Let $E = (E_1, E_0, \alpha)$ and $E' = (E_1', E_0', \alpha')$ represent the same class in $\mathcal{L}_1(X, A)$. Analogously, let $F = (F_1, F_0, \beta)$ and $F' = (F_1', F_0', \beta')$ represent the same class in $\mathcal{L}_1(X, A)$. We claim that

$$(E_1 \oplus F_1,\, E_0 \oplus F_0,\, \alpha \oplus \beta) \qquad \text{and} \qquad (E_1' \oplus F_1',\, E_0' \oplus F_0',\, \alpha' \oplus \beta')$$

[(4)] When A is empty, since an object in $\mathcal{C}_1(X)$ is just a pair $E = (E_1, E_0) \in \mathrm{VectBdl}_X^2$, we have that $E = (E_1, E_0)$ is related to $F = (F_1, F_0)$ if and only if there exist vector bundles Q and P for which $E_1 \oplus Q$ is isomorphic to $F_1 \oplus P$, and $E_0 \oplus Q$ is isomorphic to $F_0 \oplus P$. This is an enlightening situation for this equivalence.

represent the same class in $\mathcal{L}_1(X, A)$, which proves that the binary operation defined in the statement is well-defined. Indeed, since $[E] = [E']$ in $\mathcal{L}_1(X, A)$, there exist elementary sequences Q and P in $\mathcal{C}_1(X, A)$ for which there exists an isomorphism of vector bundles $(\gamma_1, \gamma_0) : E \oplus Q \to E' \oplus P$. Similarly, there exit elementary sequences Q' and P' in $\mathcal{C}_1(X, A)$ for which there exists and isomorphism of vector bundles $(\gamma_1', \gamma_0') : F \oplus Q' \to F' \oplus P'$. Thus,

$$(E_1 \oplus F_1 \oplus Q \oplus Q',\ E_0 \oplus F_1 \oplus Q \oplus Q',\ \alpha \oplus \beta \oplus \mathrm{id}_{Q|_A} \oplus \mathrm{id}_{Q'|_A})$$

and

$$(E_1' \oplus F_1' \oplus P \oplus P',\ E_0' \oplus F_1' \oplus P \oplus P',\ \alpha' \oplus \beta' \oplus \mathrm{id}_{P|_A} \oplus \mathrm{id}_{P'|_A})$$

are isomorphic. Indeed, one can readily see that this isomorphism is ensured by the commutative diagram

$$\begin{tikzcd}
(E_1 \oplus F_1 \oplus Q \oplus Q')\,|_A \arrow[r, "\alpha \oplus \beta \oplus \mathrm{id}_{Q|_A} \oplus \mathrm{id}_{Q'|_A}"] \arrow[d, "\mathrm{id}_{E_1|_A} \oplus \eta_{Q,F_1}|_A \oplus \mathrm{id}_{Q'|_A}"] & (E_0 \oplus F_0 \oplus Q \oplus Q')\,|_A \arrow[d, "\mathrm{id}_{E_0|_A} \oplus \eta_{Q,F_0}|_A \oplus \mathrm{id}_{Q'|_A}"] \\
(E_1 \oplus Q \oplus F_1 \oplus Q')\,|_A \arrow[d, "\gamma_1|_A \oplus \gamma_1'|_A"] & (E_0 \oplus Q \oplus F_0 \oplus Q')\,|_A \arrow[d, "\gamma_0|_A \oplus \gamma_0'|_A"] \\
(E_1' \oplus P \oplus F_1' \oplus P')\,|_A \arrow[d, "\mathrm{id}_{E_1'|_A} \oplus \eta_{F_1',P}|_A \oplus \mathrm{id}_{P'|_A}"] & (E_0' \oplus P \oplus F_0' \oplus P')\,|_A \arrow[d, "\mathrm{id}_{E_0'|_A} \oplus \eta_{F_0',P}|_A \oplus \mathrm{id}_{P'|_A}"] \\
(E_1' \oplus F_1' \oplus P \oplus P')\,|_A \arrow[r, "\alpha' \oplus \beta' \oplus \mathrm{id}_{P|_A} \oplus \mathrm{id}_{P'|_A}"'] & (E_0' \oplus F_0' \oplus P \oplus P')\,|_A\,.
\end{tikzcd}$$

Furthermore, this binary operation evidently turns $\mathcal{L}_1(X, A)$ into an abelian monoid. Thus, we only have to prove the existence of inverses. In fact, let $[E, F, \alpha] \in \mathcal{L}_1(X, A)$. We claim that

$$-[E_1,\, E_0,\, \alpha] \;=\; [E_0,\, E_1,\, \alpha^{-1}].$$

To prove this assertion, it suffices to show that $(E_1 \oplus E_0, E_0 \oplus E_1, \alpha \oplus \alpha^{-1})$ is isomorphic to an elementary sequence when summed up with another one. Indeed, let P be a trivializing addendum for E_0 (see Theorem C.51). This means that there exist a trivial vector bundle T and an isomorphism of vector bundles $\beta : E_0 \oplus P \to T$ over X. Hence, we have that

$$(E_1 \oplus E_0 \oplus P,\, E_0 \oplus E_1 \oplus P,\, \alpha \oplus \alpha^{-1} \oplus \mathrm{id}_{P|_A}) \quad \text{and} \quad (E_1 \oplus T,\, E_1 \oplus T,\, \mathrm{id}_{(E \oplus T)|_A})$$

are isomorphic. Indeed, one can readily see that this isomorphism is ensured by the commutative diagram

$$\begin{tikzcd}
(E_1 \oplus E_0 \oplus P)\,|_A \arrow[r, "\alpha \oplus \alpha^{-1} \oplus \mathrm{id}_{P|_A}"] \arrow[dd, "\mathrm{id}_{E_1|_A} \oplus \beta|_A"'] & (E_0 \oplus E_1 \oplus P)\,|_A \arrow[d, "\eta_{E_1,E_0}|_A \oplus \mathrm{id}_{P|_A}"] \\
 & (E_1 \oplus E_0 \oplus P)\,|_A \arrow[d, "\mathrm{id}_{E_1|_A} \oplus \beta|_A"] \\
(E_1 \oplus T)\,|_A \arrow[r, "\mathrm{id}_{(E_1 \oplus T)|_A}"'] & (E_1 \oplus T)\,|_A\,.
\end{tikzcd}$$

□

Theorem 2.63 (An interpretation for absolute K-Theory)**.** *Let X be an object in* TopHdCCpt$_2$*. We have that*

$$\begin{aligned} \varphi_X : \mathcal{L}_1(X) &\to K(X), \\ [E_1, E_0] &\mapsto [[E_0]] - [[E_1]], \end{aligned}$$

is a group isomorphism. Moreover, this isomorphism is natural in the sense that diagram

$$\begin{tikzcd}
\mathcal{L}_1(X) \arrow[r, "\varphi_X"] & K(X) \\
\mathcal{L}_1(Y) \arrow[u, "\mathcal{L}_1(f)"] \arrow[r, "\varphi_Y"'] & K(Y) \arrow[u, "K(f)"']
\end{tikzcd}$$

is commutative for all $f : X \to Y$ *in* $\mathrm{TopHdCCpt}_2$*, where* $\mathcal{L}_1(f)[E_1, E_0] = [f^*E_1, f^*E_0]$ *for all* $[E_1, E_0] \in \mathcal{L}_1(Y)$*.*

Proof. Let us first prove that φ_X is well-defined. Indeed, let $E = (E_1, E_0)$ and $E' = (E_1', E_0')$ represent the same class in $\mathcal{L}_1(X)$. Then, there exist vector bundle Q and P such that $E_1 \oplus Q$ is isomorphic to $E_1' \oplus P$, and $E_0 \oplus Q$ is isomorphic to $E_0' \oplus P$. Consequently,

$$\begin{aligned}
\varphi_X[E_1, E_0] &= [[E_0]] - [[E_1]] \\
&= [[E_0]] + [[Q]] - [[E_1]] - [[Q]] \\
&= [[E_0 \oplus Q]] - [[E_1 \oplus Q]] \\
&= [[E_0' \oplus P]] - [[E_1' \oplus P]] \\
&= [[E_0']] + [[P]] - [[E_1']] - [[P]] \\
&= [[E_0']] - [[E_1']] \\
&= \varphi_X[E_1', E_0'].
\end{aligned}$$

Now, let us prove that φ_X is a group homomorphism. In fact, if $[E_1, E_0], [F_1, F_0] \in \mathcal{L}_1(X)$, then

$$\begin{aligned}
\varphi_X([E_1, E_0] \oplus [F_1, F_0]) &= \varphi_X[E_1 \oplus F_1,\ E_0 \oplus F_0] \\
&= [[E_0 \oplus F_0]] - [[E_1 \oplus F_1]] \\
&= [[E_0]] + [[F_0]] - [[E_1]] - [[F_1]] \\
&= ([[E_0]] - [[E_1]]) + ([[F_0]] - [[F_1]]) \\
&= \varphi_X[E_1, E_0] + \varphi_X[F_1, F_0].
\end{aligned}$$

Moreover, since any class in absolute K-Theory can be represented as a formal difference $[[E]] - [[F]]$, $\varphi_X[F, E] = [[E]] - [[F]]$. This proves surjectivity. Finally, if $[E_1, E_0], [F_1, F_0] \in \mathcal{L}_1(X)$ are such that

$$[[E_0]] - [[E_1]] = \varphi_X[E_1, E_0] = \varphi_X[F_1, F_0] = [[F_0]] - [[F_1]],$$

then

$$[[E_0 \oplus F_1]] \;=\; [[E_0]] + [[F_1]] \;=\; [[F_0]] + [[E_1]] \;=\; [[F_0 \oplus E_1]].$$

Thus, there exists $G \in \mathrm{VectBdl}_X$ such that $E_0 \oplus F_1 \oplus G$ is isomorphic to $F_0 \oplus E_1 \oplus G$. We set $Q := F_0 \oplus G$ and $P := E_0 \oplus G$. Consequently, we obtain that $E_1 \oplus Q$ is isomorphic to $F_1 \oplus P$, and that $E_0 \oplus Q$ is isomorphic to $F_0 \oplus P$. Hence, $[E_1, E_0] = [F_1, F_0]$. This ensures injectivity. The last claim of the statement is a straightforward computation that we leave to the reader. □

Definition 2.64 (An Euler characteristic in K-Theory)**.** *Consider the contravariant functor*

$$\begin{aligned} \mathcal{L}_1 : \mathrm{TopHdCCpt}_2 \;&\to\; \mathscr{G}_{ab}, \\ (X, A) \;&\mapsto\; \mathcal{L}_1(X, A), \\ f : (X, A) \to (Y, B) \;&\mapsto\; \mathcal{L}_1(f) : \mathcal{L}_1(Y, B) \to \mathcal{L}_1(X, A), \end{aligned}$$

where

$$\mathcal{L}_1(f)[E_1,\, E_0,\, \alpha] \;:=\; [f^*E_1,\, f^*E_0,\, f^*\alpha]$$

for all $[E_1,\, E_0\, \alpha] \in \mathcal{L}_1(Y, B)$*. Moreover, consider* $K : \mathrm{TopHdCCpt}_2 \to \mathscr{G}_{ab}$ *to be the contravariant functor defined in Remark 2.30. An* ***Euler characteristic for*** $\boldsymbol{\mathcal{L}_1}$ *is a natural transformation*

$$\chi_1 \;=\; \{\chi_1(X, A) : \mathcal{L}_1(X, A) \to K(X, A)\}_{(X,A) \in \mathrm{TopHdCCpt}_2}$$

between the functors $\mathcal{L}_1$ *and* K *such that*

$$\chi_1(X)[E_1,\, E_0] \;=\; [[E_0]] - [[E_1]]$$

for all $X = (X, 0) \in \mathrm{TopHdCCpt}_2$*. We remind the reader that* $\chi_1 : \mathcal{L}_1 \to K$ *being a natural transformation means that the following diagram is commutative for all* $f : (X, A) \to (Y, B)$ *in* $\mathrm{TopHdCCpt}_2$*.*

$$\begin{tikzcd} \mathcal{L}_1(X, A) \arrow[r, "\chi_1(X,A)"] & K(X, A) \\ \mathcal{L}_1(Y, B) \arrow[u, "\mathcal{L}_1(f)"] \arrow[r, "\chi_1(Y,B)"'] & K(Y, B) \arrow[u, "K(f)"'] \end{tikzcd}$$

◊

Theorem 2.65 (Existence of an Euler characteristic for $\mathcal{L}_1$)**.** *There exists an Euler characteristic χ_1 for $\mathcal{L}_1$.*

Proof. Let $[E_1, E_0, \alpha] \in \mathcal{L}_1(X, A)$ where (X, A) is an object in TopHdCCpt_2. Subsequently, we define an element $\chi_1(X, A)[E_1, E_0, \alpha] \in K(X, A)$ in such manner that the map

$$\begin{aligned} \chi_1(X,A):\ \mathcal{L}_1(X,A) &\to K(X,A), \\ [E_1,\, E_0,\, \alpha] &\mapsto \chi_1(X,A)[E_1,E_0,\alpha], \end{aligned}$$

is naturally defined with respect to (X, A) and that

$$\chi(X) \;=\; [[E_0]] - [[E_1]]$$

for all $X = (X, 0) \in \text{TopHdCCpt}_2$. Indeed, let

$$\begin{aligned} X_0 &:= X \times \{0\} \quad \text{and} \\ X_1 &:= X \times \{1\}. \end{aligned}$$

We set $\mathfrak{X}$ to be the identification space obtained as the quotient of the disjoint union $X_0 \sqcup X_1$ by the equivalence relation that identifies $(a, 0)$ with $(a, 1)$ for all $a \in A$. The natural sequence

$$0 \longrightarrow K(\mathfrak{X},\, X_0) \xrightarrow{K(j)} K(\mathfrak{X}) \xrightarrow{K(i)} K(X_0) \longrightarrow 0$$

is an split exact sequence since $i : X_0 \to \mathfrak{X}$ is a retraction (see Theorem 1.61). In particular, we can consider its obvious right inverse $\rho : \mathfrak{X} \to X_0$. Furthermore, we have the isomorphism

$$\varphi:\ K(\mathfrak{X},\, X_0) \to K(X,\, A)$$

which is induced by the inclusion $(X, A) \to (\mathfrak{X}, X_0)$ that identifies X with X_1 (see Theorem 1.62). Thence, from $[E_1, E_0, \alpha] \in \mathcal{L}_1(X, A)$, we define the vector bundle F on $\mathfrak{X}$ by setting $F\mid_{X_0} = E_0$, $F\mid_{X_1} = E_1$ and identifying these restrictions over A via the isomorphism $\alpha : E_1\mid_A \to E_0\mid_A$. The reader can prove that this vector bundle is well-defined up to isomorphism and that

$$[[F]] - [[\rho^*(E_0)]] \in \operatorname{Ker} K(i).$$

Therefore, there exists a unique element $\chi_1(X,A)[E_1,E_0,\alpha] \in K(X,A)$ for which

$$(K(j) \circ \varphi^{-1})\chi_1(X,A)[E_1,E_0,\alpha] \;=\; [[F]] - [[\rho^*(E_0)]].$$

This defines the homomorphism $\chi_1(X,A)$ that verifies the conditions presented at the beginning. $\square$

Among other important things, we now show that an Euler characteristic $\chi_1 : \mathcal{L}_1 \to K$ is a natural isomorphism between the functors in question, which clearly extends Theorem 2.63. In order to do this, we begin with the following technical result that gives reasonable conditions under which one can monomorphically (respectively, isomorphically) extend monomorphisms (respectively, isomorphisms) of vector bundles.

Lemma 2.66 (Extension of monomorphisms and of isomorphisms of vector bundles)**.** *Let (X,A) be an object in* $\mathrm{TopHdCCpt}_2$*. In addition, let E and F be vector bundles on X. If $\alpha : E\mid_A \to F\mid_A$ and $\beta : E \to F$ are monomorphisms such that $\beta\mid_A$ is homotopic to α, then α can be extended as a monomorphism of vector bundles to the whole X. The same claim holds for isomorphisms.*

Proof. The reader can find a proof of this result in [2, pp. 89-90]. $\square$

Lemma 2.67 (Euler characteristic of a pair of compact Hausdorff spaces whose second component is a one-point space)**.** *Let (X,A) be an object in* $\mathrm{TopHdCCpt}_2$*. In addition, let $i : A \to X$ and $j : X \to (X,A)$ be the inclusion maps. If A is a one-point space, then*

$$0 \longrightarrow \mathcal{L}_1(X,A) \xrightarrow{\mathcal{L}_1(j)} \mathcal{L}_1(X) \xrightarrow{\mathcal{L}_1(i)} \mathcal{L}_1(A)$$

is an exact sequence. Consequently, in this situation, we have that, if $\chi_1 : \mathcal{L}_1 \to K$ is an Euler characteristic for $\mathcal{L}_1$, then $\chi_1(X,A) : \mathcal{L}_1(X,A) \to K(X,A)$ is an isomorphism.

Proof. Let $\chi_1 : \mathcal{L}_1 \to K$ be an Euler characteristic for $\mathcal{L}_1$ (see Theorem 2.65). The fact that

$$\operatorname{Im}\mathcal{L}_1(j) \subseteq \operatorname{Ker}\mathcal{L}_1(i)$$

follows from the commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{L}_1(X,A) \arrow[r, "\mathcal{L}_1(j)"] \arrow[d, "\chi_1(X,A)"'] & \mathcal{L}_1(X) \arrow[r, "\mathcal{L}_1(i)"] \arrow[d, "\chi_1(X)"] & \mathcal{L}_1(A) \arrow[d, "\chi_1(A)"] \\
K(X,A) \arrow[r, "K(j)"'] & K(X) \arrow[r, "K(i)"'] & K(A)
\end{tikzcd}$$

In fact, since $\chi_1(A)$ is an isomorphism because of Theorem 2.63, we have

$$\mathcal{L}_1(i) \circ \mathcal{L}_1(j) \;=\; \chi_1(A)^{-1} \circ K(i) \circ K(j) \circ \chi_1(X,A).$$

Once $K(i) \circ K(j)$ is the trivial homomorphism, our assertion is proved. More than that, we have

$$\operatorname{Ker}\mathcal{L}_1(i) \subseteq \operatorname{Im}\mathcal{L}_1(j).$$

Indeed, if (E_1, E_0) represents an element of $\mathcal{L}_1(X)$ whose image in $\mathcal{L}_1(A)$ is zero, then E_1 and E_0 have the same dimension over A. Consequently, there exists an isomorphism $\alpha : E_1 \mid_A \to E_0 \mid_A$ because A is a one-point space. Thus, $\mathcal{L}_1(j)[E_1, E_0, \alpha] = [E_1, E_0]$ proves our second assertion. Therefore, we have just concluded the exactness of the sequence

$$\mathcal{L}_1(X,A) \xrightarrow{\mathcal{L}_1(j)} \mathcal{L}_1(X) \xrightarrow{\mathcal{L}_1(i)} \mathcal{L}_1(A).$$

Now, we have to show that $\mathcal{L}_1(j)$ is injective. Note that this is equivalent to prove that the trivial class in $\mathcal{L}_1(X,A)$ is the only one that is mapped by $\mathcal{L}_1(j)$ into the trivial class of $\mathcal{L}_1(X)$. In fact, let $[E_1, E_0, \alpha] \in \mathcal{L}_1(X,A)$ have image zero in $\mathcal{L}_1(X)$. Then, there exists a vector bundle P and an isomorphism $\beta : E_1 \oplus P \to E_0 \oplus P$. Then, $\beta \mid_A \circ\, (\alpha \oplus \mathrm{id}_{P|_A})^{-1}$ is an automorphism of $(E_0 \oplus P) \mid_A$. Since A is a one-point space, any such automorphism must be homotopic to the identity. Hence, by Lemma 2.66, $\beta \mid_A \circ\, (\alpha \oplus \mathrm{id}_{P|_A})^{-1}$ extends to $\gamma : E_0 \oplus P \to E_0 \oplus P$. Thus, we have the following commutative diagram.

$$\begin{tikzcd}
(E_1 \oplus P)\mid_A \arrow[r, "\alpha \oplus \mathrm{id}_{P|_A}"] \arrow[d, "\beta|_A"'] & (E_0 \oplus P)\mid_A \arrow[d, "\gamma|_A"] \\
(E_0 \oplus P)\mid_A \arrow[r, "\mathrm{id}_{(E_0 \oplus P)|_A}"'] & (E_0 \oplus P)\mid_A
\end{tikzcd}$$

Hence, (E_1, E_0, α) represents the trivial class in $\mathcal{L}_1(X, A)$, as we wished. Thus, $\mathcal{L}_1(j) : \mathcal{L}_1(X, A) \to \mathcal{L}_1(X)$ is an injection. The last claim of the statement is proved as follows. First, note that the surjectivity of $\chi_1(X, A)$ is obvious. Thence, note that the injectivity of $\chi_1(X, A)$ is ensured by the following commutative diagram and by Theorem 2.63.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_1(X, A) \arrow[r, "\mathcal{L}_1(j)"] \arrow[d, "\chi_1(X,A)"'] & \mathcal{L}_1(X) \arrow[d, "\chi_1(X)"] \\
0 \arrow[r] & K(X, A) \arrow[r, "K(j)"'] & K(X)
\end{tikzcd}$$

Indeed, since $\chi_1(X)$ is an isomorphism by Theorem 2.63, it is an injection. Thus, the composition $\chi_1(X) \circ \mathcal{L}_1(j)$ is injective. Consequently, $K(j) \circ \chi_1(X, A)$ is also injective. Therefore, by set-theoretic arguments, we have that $\chi_1(X, A)$ is an injection, as we wished. □

Theorem 2.68 (Euler characteristic of a pair of compact Hausdorff spaces)**.** *Let* (X, A) *be an object in* $\mathrm{TopHdCCpt}_2$*. If* $\pi : (X, A) \to (X/A, A/A)$ *is the canonical projection, then*

$$\mathcal{L}_1(\pi) : \mathcal{L}_1(X/A, A/A) \to \mathcal{L}_1(X, A)$$

is an isomorphism. Consequently, if $\chi_1 : \mathcal{L}_1 \to K$ *is an Euler characteristic for* $\mathcal{L}_1$*, then* $\chi_1(X, A) : \mathcal{L}_1(X, A) \to K(X, A)$ *is an isomorphism.*

Proof. Let $\chi_1 : \mathcal{L}_1 \to K$ be an Euler characteristic for $\mathcal{L}_1$. We have that

$$\chi_1(X/A, A/A) : \mathcal{L}_1(X/A, A/A) \to K(X/A, A/A)$$

is an isomorphism by Lemma 2.67. Moreover, since the map

$$\begin{aligned} \bar{\pi} : X/A &\to (X/A)/(A/A), \\ [x] &\mapsto [\pi(x)], \end{aligned}$$

is a homeomorphism, we also have that

$$K(\pi) \;=\; \widetilde{K}(\bar{\pi}) : \; K(X/A, A/A) \to K(X, A)$$

is an isomorphism. Thus,

$$K(\pi) \circ \chi_1(X/A, A/A) : \mathcal{L}_1(X/A, A/A) \to K(X, A)$$

is an isomorphism. In particular, this map is injective. Therefore, since the following diagram is commutative because $\chi_1 : \mathcal{L}_1 \to K$ is an Euler characteristic, it follows that $\chi_1(X, A) \circ \mathcal{L}_1(\pi)$ is injective. Hence, we conclude that $\mathcal{L}_1(\pi)$ is injective by set-theoretic arguments.

$$\begin{tikzcd} \mathcal{L}_1(X, A) \arrow[r, "\chi_1(X,A)"] & K(X, A) \\ \mathcal{L}_1(X/A, A/A) \arrow[u, "\mathcal{L}_1(\pi)"] \arrow[r, "\chi_1(X/A,A/A)"'] & K(X/A, A/A) \arrow[u, "K(\pi)"'] \end{tikzcd}$$

Now let $(E_1, E_0, \alpha) \in \mathcal{C}_1(X, A)$. In addition, let P be a trivializing addendum for E_1 (see Theorem C.51). That is, P is a vector bundle on X for which there exist an isomorphism $\beta : E_1 \oplus P \to X \times \mathscr{V}$ over X, where $X \times \mathscr{V}$ is the product bundle with typical fiber $\mathscr{V}$. We first claim that

$$[E_1,\, E_0,\, \alpha] \;=\; [X \times \mathscr{V},\, E_0 \oplus P,\, \gamma],$$

where

$$\gamma \;:=\; (\alpha \oplus \mathrm{id}_{P|_A}) \circ \beta \,|_{(F|_A)}^{-1} \colon \; A \times \mathscr{V} \;\to (E_0 \oplus P)\,|_A\,.$$

In fact, considering the elementary sequence $(P, P, \mathrm{id}_{P|_A}) \in \mathcal{C}_1(X, A)$, the following commutative diagram shows that the sum $(E_1, E_0, \alpha) \oplus (P, P, \mathrm{id}_{P|_A})$ is isomorphic to $(X \times \mathscr{V}, E_0 \oplus P, \gamma)$, as we wished.

$$\begin{tikzcd}
(E_1 \oplus P)\,|_A \arrow[r, "\alpha \oplus \mathrm{id}_{P|_A}"] \arrow[d, "\beta|_A"'] & (E_0 \oplus P)_A \arrow[d, "\mathrm{id}_{(E_0 \oplus P)|_A}"] \\
A \times \mathscr{V} \arrow[r, "\gamma"'] & (E_0 \oplus P)\,|_A
\end{tikzcd}$$

Furthermore,

$$\mathcal{L}_1(\pi)[X \times \mathscr{V},\, (E_0 \oplus P)/\gamma,\, \gamma/\gamma] \;=\; [X \times \mathscr{V},\, E_0 \oplus P,\, \gamma],$$

where $(E_0 \oplus P)/\gamma$ is defined in Section C.10. This equality is straightforward from the proof of Corollary C.62, although it is not immediate to be geometrically visualized. Thus, we conclude that $\mathcal{L}_1(\pi)$ is surjective. Hence, we have that $\mathcal{L}_1(\pi)$ is an isomorphism, as desired. Therefore,

$$\chi_1(X, A) \;=\; K(\pi) \,\circ\, \chi_1(X/A, A/A) \,\circ\, \mathcal{L}_1(\pi)^{-1}.$$

This proves the last part of the statement since $\chi_1(X, A)$ is the composition of three isomorphisms. $\square$

Corollary 2.69 (Uniqueness of Euler characteristics for $\mathcal{L}_1$)**.** *If χ_1 and χ_1' are Euler characteristics for $\mathcal{L}_1$, then $\chi_1 = \chi_1'$.*

Proof. Because of Theorem 2.68, for each $(X, A) \in \mathrm{TopHdCCpt}_2$, we have that the map $\chi_1(X, A) : \mathcal{L}_1(X, A) \to K(X, A)$ is invertible. Therefore, it is defined the natural transformation

$$\chi_1^{-1} \;:=\; \{\chi_1(X, A)^{-1} : K(X, A) \to \mathcal{L}_1(X, A)\}_{(X,A) \in \mathrm{TopHdCCpt}_2}$$

between K and $\mathcal{L}_1$. In addition, once the composition of natural transformations is also a natural transformation, we have the following natural transformation between K and itself

$$\chi_1' \circ \chi_1^{-1} \;=\; \{\chi_1'(X,A) \circ \chi_1^{-1}(X,A) : K(X,A) \to K(X,A)\}_{(X,A) \in \mathrm{TopHdCCpt}_2}.$$

We claim that $\chi_1'(X) \circ \chi_1^{-1}(X) = \mathrm{id}_{K(X)}$ for all $X = (X, 0) \in \mathrm{TopHdCCpt}_2$. Indeed, according to Definition 2.64,

$$\chi_1'(X)[E_1,\, E_0] \;=\; [[E_0]] - [[E_1]] \;=\; \chi_1(X)[E_1,\, E_0]$$

for all $[E_1, E_0] \in \mathcal{L}_1(X)$. Thus, $\chi_1'(X) = \chi_1(X)$ for all $X = (X, 0) \in \mathrm{TopHdCCpt}_2$, which proves our assertion. Consequently, since the reader can readily prove that the equality

$$\chi_1'(X,A) \circ \chi_1(X,A)^{-1} \;=\; (\chi_1'(X/A) \circ \chi_1(X/A)^{-1}) \mid_{K(X,A)}$$

holds for all $(X, A) \in \mathrm{TopHdCCpt}_2$, the theorem is proved because

$$\chi_1'(X/A) \circ \chi_1(X/A)^{-1} \;=\; \mathrm{id}_{K(X/A)},$$

as we showed before. □

Corollary 2.70 (Dependence on the homotopy class)**.** *Let* (X, A) *be an object in* $\mathrm{TopHdCCpt}_2$*. We have that the class of* (E_1, E_0, α) *in* $\mathcal{L}_1(X, A)$ *only depends on the homotopy class of* $\alpha : E_1 \mid_A \to E_0 \mid_A$.

Proof. First, we set $(Y, B) := (X, A) \times \mathbb{I}$. Thence, if $\beta : E_1 \mid_A \to E_0 \mid_A$ is an isomorphism of vector bundles which is homotopic to α, then we consider a homotopy $\Theta : E_1 \mid_A \times \mathbb{I} \to E_0 \mid_A$ between them. Thus, being $\pi : (Y, B) \to (X, A)$ the natural projection onto the first factor, we have a natural isomorphism of vector bundles

$$\begin{aligned} \gamma_\Theta :\; \pi^* E_1 \mid_B \;&\to\; \pi^* E_0 \mid_B, \\ (e,\, x,\, t) \;&\mapsto\; (\Theta(e,t),\, x,\, t). \end{aligned}$$

Now, let $i_0, i_1 : (X, A) \to (Y, B)$ be given by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$ for all $x \in X$. Since these maps are homotopic, being the identity on (Y, B) a homotopy between them, we have

$$K(i_0) \;=\; K(i_1) :\; K(Y, B) \;\to\; K(X, A).$$

Moreover, we have

$$\begin{aligned}\mathcal{L}_1(i_0)[\pi^*E_1,\,\pi^*E_0,\,\gamma_\Theta] &= [E_1,\,E_0,\,\alpha] \quad \text{and}\\ \mathcal{L}_1(i_1)[\pi^*E_1,\,\pi^*E_0,\,\gamma_\Theta] &= [E_1,\,E_0,\,\beta].\end{aligned}$$

This happens because

$$\pi\circ i_0 = \mathrm{id}_{(X,A)} = \pi\circ i_1$$

and because

$$i_0^*\gamma_\Theta = \alpha \qquad \text{and} \qquad i_1^*\gamma_\Theta = \beta.$$

Finally, if $\chi_1 : \mathcal{L}_1 \to K$ is the Euler characteristic for $\mathcal{L}_1$, then the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{L}_1(X,A) \arrow[r, "\chi_1(X,A)"] & K(X,A) \\
\mathcal{L}_1(Y,B) \arrow[u, "\mathcal{L}_1(i_0)"] \arrow[r, "\chi_1(Y,B)"] \arrow[d, "\mathcal{L}_1(i_1)"'] & K(Y,B) \arrow[u, "K(i_0)"'] \arrow[d, "K(i_1)"] \\
\mathcal{L}_1(X,A) \arrow[r, "\chi_1(X,A)"'] & K(X,A)
\end{tikzcd}$$

Thus, once every map in this diagram is an isomorphism[(5)], we have $\mathcal{L}_1(i_0) = \mathcal{L}_1(i_1)$. Therefore,

$$\begin{aligned}[E_1, E_0, \alpha] &= \mathcal{L}_1(i_0)[\pi^*E_1, \pi^*E_0, \gamma_\Theta]\\ &= \mathcal{L}_1(i_1)[\pi^*E_1, \pi^*E_0, \gamma_\Theta]\\ &= [E_1, E_0, \beta],\end{aligned}$$

as we wished. □

[(5)] This happens because i_0 and i_1 are homotopy equivalences between (X, A) and (Y, B). Indeed, the reader can easily prove that the equality $\pi \circ i_j = \mathrm{id}_{(X,A)}$ holds. Moreover, one can readily show that the map

$$\begin{aligned}\Gamma : (Y,B)\times\mathbb{I} &\to (Y,B),\\ (x,\,t,\,s) &\mapsto (x,\,t\cdot s),\end{aligned}$$

is a homotopy between $i_j\circ\pi$ and $\mathrm{id}_{(Y,B)}$. This proves our assertion. Note that this technical arguments just express the idea that a deformation retraction of (Y,B) into (X,A) is obtained by crushing the cylinder $X\times\mathbb{I}$ on one of its bases, which is an operation that obviously crushes the cylinder $A\times\mathbb{I}$ on its corresponding base.

Remark 2.71 (Some necessary complements for this section). *At this point, the reader may be asking himself or herself why we wrote, for instance, $\mathcal{C}_1(X, A)$ and $\mathcal{L}_1(X, A)$ instead of $\mathcal{C}(X, A)$ and $\mathcal{L}(X, A)$, respectively. This is a righteous question because, apparently, we have overloaded the notation without need. However, this is only partially true. In fact, from a strict viewpoint, we overloaded the notion since most of what we wrote could lose their subindexes without producing confusion. Nevertheless, from a broader viewpoint, for each non-zero natural number n, we can define the category $\mathcal{C}_n(X, A)$ whose:*

- *objects are $(2n+1)$-tuples $(E_n, \cdots, E_0, \alpha_n, \cdots, \alpha_1)$ where E_i is a vector bundle on X for each i between 0 and n, both included, and $\alpha_i : E_i \mid_A \to E_{i-1} \mid_A$ is a morphism of vector bundles for i between 1 and n, both included, in such manner that the sequence*

$$0 \longrightarrow E_n \mid_A \xrightarrow{\alpha_n} E_{n-1} \mid_A \longrightarrow \cdots \longrightarrow E_1 \mid_A \xrightarrow{\alpha_1} E_0 \mid_A \longrightarrow 0$$

is exact. For convenience, we will usually denote an object in $\mathcal{C}_n(X, A)$ by $E = (E_i, \alpha_i)$; and

- *morphisms $\varphi : E \to F$ between $E = (E_i, \alpha_i)$ and $F = (F_i, \beta_i)$ are collections of morphisms of vector bundles $\varphi_i : E_i \to F_i$ for i between 0 and n, both included, such that the diagram*

$$\begin{tikzcd}
0 \arrow[r] & E_n \mid_A \arrow[r, "\alpha_n"] \arrow[d, "\varphi_n|_A"] & E_{n-1} \mid_A \arrow[r] \arrow[d, "\varphi_{n-1}|_A"] & \cdots \arrow[r] & E_1 \mid_A \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1|_A"] & E_0 \mid_A \arrow[r] \arrow[d, "\varphi_0|_A"] & 0 \\
0 \arrow[r] & F_n \mid_A \arrow[r, "\beta_n"'] & F_{n-1} \mid_A \arrow[r] & \cdots \arrow[r] & F_1 \mid_A \arrow[r, "\beta_1"'] & F_0 \mid_A \arrow[r] & 0
\end{tikzcd}$$

is commutative. A morphism $\varphi : E \to F$ in $\mathcal{C}_n$ between $E = (E_i, \alpha_i)$ and $F = (F_i, \beta_i)$ is an ***isomorphism*** *if $\varphi_i : E_i \to F_i$ is an isomorphism for i between 0 and n, both included.*

Moreover, we can define an elementary sequence in $\mathcal{C}_n(X, A)$ to be an object of the form

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow E_i \mid_A \xrightarrow{\alpha_i} E_{i-1} \mid_A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0,$$

where $E_i = E_{i-1}$ *and* $\alpha_i = \mathrm{id}_{E_i|_A}$. *Thence, given* E *and* F *in* $\mathcal{C}_n(X,A)$, *we can say that they are equivalent if and only if there exist elementary sequences* $Q_1, \cdots, Q_r$ *and* $P_1, \cdots, P_s$ *for which*

$$E \oplus \bigoplus_{i=1}^{r} Q_i \qquad \textit{and} \qquad F \oplus \bigoplus_{i=1}^{s} P_i$$

are isomorphic. This definition naturally gives rise to an equivalence relation on the class of objects of $\mathcal{C}_n(X,A)$. *The set of such equivalence classes, which we hereafter denote by* $\mathcal{L}_n(X,A)$, *has a natural abelian group structure. This allows us to define the contravariant functor*

$$\begin{aligned}
\mathcal{L}_n : \mathrm{TopHdCCpt}_2 &\to \mathscr{G}_{ab}, \\
(X,A) &\mapsto \mathcal{L}_n(X,A), \\
f:(X,A)\to(Y,B) &\mapsto \mathcal{L}_n(f):\mathcal{L}_n(Y,B)\to\mathcal{L}_n(X,A),
\end{aligned}$$

where

$$\mathcal{L}_n(f)[E_i,\, \alpha_i] \;:=\; [f^*E_i,\, f^*\alpha_i]$$

for all $[E_i, \alpha_i] \in \mathcal{L}_n(Y,B)$. *In addition, being* $K : \mathrm{TopHdCCpt}_2 \to \mathscr{G}_{ab}$ *the contravariant functor defined in Remark 2.30, we can define an* ***Euler characteristic for*** $\boldsymbol{\mathcal{L}_n}$ *as a natural transformation*

$$\chi_n \;=\; \{\chi_n(X,A) : \mathcal{L}_n(X,A) \to K(X,A)\}_{(X,A)\in\mathrm{TopHdCCpt}_2}$$

between the functors $\mathcal{L}_n$ *and* K *such that*

$$\chi_n(X)[E_n,\, \cdots,\, E_0] \;=\; \sum_{i=1}^{n}(-1)^i[[E_i]]$$

for all $X = (X,0) \in \mathrm{TopHdCCpt}_2$. *As before, there exists a unique Euler characteristic for* $\mathcal{L}_n$ *for all non-zero natural number* n. *In particular,* $\mathcal{L}_n(X,A)$ *is always isomorphic to* $K(X,A)$. *We will not prove this result here, but we will give a brief sketch for its existence part. The reader will find the complete proof of this result in the references indicated below. Indeed, let the canonical inclusion of* $\mathcal{C}_n(X,A)$ *into* $\mathcal{C}_{n+1}(X,A)$ *be the faithful covariant functor*

$$\begin{aligned} I_n : \mathcal{C}_n(X,A) &\to \mathcal{C}_{n+1}(X,A), \\ (E_n, \cdots, E_0, \alpha_n, \cdots, \alpha_1) &\mapsto (0, E_n, \cdots, E_0, 0 \to E_n \mid_A, \alpha_n, \cdots, \alpha_1), \\ (\varphi_n : E_n \to F_n, \cdots, \varphi_0 : E_0 \to F_0) &\mapsto (0 \to 0,\ \varphi_n : E_n \to F_n,\ \cdots,\ \varphi_0 : E_0 \to F_0). \end{aligned}$$

The map between objects of I_n induces a homomorphism

$$\iota_n : \mathcal{L}_n(X,A) \to \mathcal{L}_{n+1}(X,A).$$

Thence, since one can prove that ι_n is always an isomorphism, we can define the Euler characteristic for $\mathcal{L}_n$

$$\chi_n := \chi_1 \circ \iota_{1,n}^{-1}$$

between the contravariant functors $\mathcal{L}_n$ and K, where $\iota_{1,n}$ is the isomorphism indicated in the following diagram.

$$\begin{tikzcd} \mathcal{L}_1(X,A) \arrow[r, "\iota_1"'] \arrow[rrrr, bend left, "\iota_{1,n}"] & \mathcal{L}_2(X,A) \arrow[r] & \cdots \arrow[r] & \mathcal{L}_{n-1}(X,A) \arrow[r, "\iota_{n-1}"'] & \mathcal{L}_n(X,A) \end{tikzcd}$$

We can also define $\mathcal{L}_\infty(X,A)$ to be the direct limit of the direct system

$$(\mathbb{N},\ (\mathcal{L}_n(X,A))_{n\in\mathbb{N}},\ (\iota_{n,m} : \mathcal{L}_n(X,A) \to \mathcal{L}_m(X,A))_{n,m\in\mathbb{N}}),$$

where $\iota_{n,m}$ is the trivial homomorphism if $m < n$, and is the obvious composition if $n \leq m$. Thus, we obtain a family of isomorphisms $\iota_{n,\infty} : \mathcal{L}_n(X,A) \to \mathcal{L}_\infty(X,A)$ indexed by the non-zero natural numbers. Any one of the isomorphisms of this family proves the existence of an isomorphism $\chi : \mathcal{L}_\infty(X,A) \to K(X,A)$. This is done using the same reasoning that proved the existence of an Euler characteristic for $\mathcal{L}_n$. Thus, the overloading of the notation mentioned at the beginning of this remark is explained by the ideas that we have just exposed here. The reader who fells the urge to deepen his or her knowledge on this interesting topic will find in [2, pp. 87-94], [3] and [23, pp. 64-65] good references. Finally, note that the results that we established in this section can be trivially extended to the relative K-Theory group $K^m(X,A)$ for all $m \in \mathbb{Z}$. $\diamondsuit$

2.10 Compactly-supported K-Theory

In this section, we explicitly set the compactly-supported K-Theory groups. This is mainly done because these groups are essential in Section 3.8 in order to define the Thom isomorphisms in K-Theory, which is a fundamental result that enables us to discuss integration in K-Theory through the Gysin map. We begin with the following definition.

Definition 2.72 (The category of locally compact Hausdorff spaces). *We define the* ***category of locally compact Hausdorff spaces****, and denote it by* TopHdLocCptP*, to be the category whose:*

- *objects are locally compact Hausdorff spaces; and*
- *morphisms are proper continuous functions. We remind the reader that a function is proper if its preimage of any compact subspace of the codomain is compact in the domain.* ♢

Definition 2.73 (The one-point Alexandroff compactification covariant functor). *We define the covariant functor*

$$\begin{aligned} {}^{+}: \text{TopHdLocCptP} &\to \text{TopHdCpt}, \\ X &\mapsto X^{+}, \\ f: X \to Y &\mapsto f^{+}: X^{+} \to Y^{+}. \end{aligned}$$

This functor is known as the ***one-point Alexandroff compactification covariant functor****.* ♢

Definition 2.74 (Compactly-supported K-Theory). *Consider the composition of functors*

$$\begin{aligned} K_c^n := \widetilde{K}^n \circ {}^{+}: \text{TopHdLocCptP} &\to \mathscr{G}_{ab}, \\ X &\mapsto \widetilde{K}^n(X^{+}), \\ f: X \to Y &\mapsto \widetilde{K}^n(f^{+}): \widetilde{K}^n(Y^{+}) \to \widetilde{K}^n(X^{+}), \end{aligned}$$

for each $n \in \mathbb{Z}$. We say that:

- *the* ***nth compactly-supported K-Theory group*** *of a locally compact Hausdorff space* X*, which is denoted by* $K_c^n(X)$*, is the nth pointed reduced K-Theory group* $\widetilde{K}^n(X^+)$*; and*

- *the* ***nth compactly-supported induced homomorphism in K-Theory*** *of a proper continuous map* $f : X \to Y$ *between locally compact Hausdorff spaces, which is denoted by* $K_c^n(f) : K_c^n(Y) \to K_c^n(X)$*, is the nth induced homomorphism* $\widetilde{K}^n(f^+) : \widetilde{K}^n(Y^+) \to \widetilde{K}^n(X^+)$*.* $\diamondsuit$

Remark 2.75 (On compactly-supported K-Theory groups)**.** *We have the following facts.*

- *If* X *is a compact Hausdorff space, then* $K_c^n(X)$ *is canonically isomorphic to* $K^n(X)$*.*

- *If* X *and* Y *are locally compact Hausdorff spaces, then, considering the product between pointed reduced K-Theory groups in Remark 2.45, we obtain the external multiplication*

$$\boxtimes : \; K_c^n(X) \otimes K_c^m(Y) \;\to\; K_c^{n+m}(X \times Y),$$

  since the reader can readily prove that $X^+ \wedge Y^+$ *is canonically homeomorphic to* $(X \times Y)^+$*.*

- *The compactly-supported version of K-Theory is not homotopic invariant. Indeed, for example, the real line* $\mathbb{R}$ *has the same homotopy type as a one-point space* Ω*, but*

$$\begin{aligned} K_c^1(\Omega) \;&\simeq\; K^1(\Omega) \;=\; 0 \quad \textit{and} \\ K_c^1(\mathbb{R}) \;&=\; \widetilde{K}^1(\mathbb{R}^+) \;\simeq\; \widetilde{K}^1(\mathbb{S}^1) \;\simeq\; \mathbb{Z}. \end{aligned}$$

- *We can also define the relative version of compactly-supported K-Theory. In fact, if* (X, A) *is a pair of locally compact Hausdorff spaces for which the inclusion* $A \to X$ *is a proper map, then we define*

$$K_c^n(X, A) \;:=\; K_c^n(X/A) \;=\; \widetilde{K}^n((X/A)^+)$$

  for all $n \in \mathbb{Z}$*. Moreover, once we have a canonical homeomorphism between* X^+/A^+ *and* $(X/A)^+$*, we have*

$$K_c^n(X, A) \simeq \widetilde{K}^n(X^+/A^+).$$

for all $n \in \mathbb{Z}$. ♢

2.11 Real K-Theory

In this final section, we recapitulate the main notions of this chapter to set real K-Theory. Indeed, until now, we have only considered K-Theory based on complex vector bundles. However, there is an obvious analog to K-Theory based on real vector bundles. Here we pinpoint the main differences between these two versions. It is to be noted that there are other versions of K-theory, as the one that the reader can find in [23, pp. 70-76], that we do not address in this work. We begin with the following definition.

Definition 2.76 (The absolute real K-Theory group of a compact Hausdorff space)**.** *Let X be an object in* TopHdCpt *and* Vect_X *be the semigroup of isomorphism classes of real vector bundles on X with respect to the induced direct sum. The* ***absolute real K-Theory group*** *of X, hereafter denoted by $K(X)$, is the Grothendieck group associated to* Vect_X*.* ♢

Definition 2.77 (Pullback in real absolute K-Theory)**.** *Let $f : X \to Y$ be a morphism in* TopHdCpt*. We say that the* ***pullback of f in real absolute K-Theory*** *is the morphism of abelian groups*

$$\begin{aligned} K(f) : K(Y) &\to K(X), \\ [[E]] - [[F]] &\mapsto [[f^*E]] - [[f^*F]], \end{aligned}$$

where f^*E and f^*F are the pullbacks of the vector bundles E and F through f, respectively. Note that $K(f)$ is well-defined because the pullbacks of isomorphic vector bundles are also isomorphic. ♢

Remark 2.78 (Categorical interpretation of the real absolute K-Theory data presented above)**.** *Being $\mathscr{G}_{ab}$ the standard category of abelian groups, we have the contravariant functor*

$$\begin{aligned} K : \mathrm{TopHdCpt} &\to \mathscr{G}_{ab}, \\ X &\mapsto K(X), \\ f : X \to Y &\mapsto K(f) : K(Y) \to K(X). \end{aligned}$$

Furthermore, since Theorem C.57 imply that the pullbacks of vector bundles through homotopic continuous maps $f, g : X \to Y$, where X and Y are compact Hausdorff spaces, are isomorphic over X, the contravariant functor

$$\begin{aligned} [K] : [\mathrm{TopHdCpt}] &\to \mathscr{G}_{ab}, \\ X &\mapsto K(X), \\ [f : X \to Y] &\mapsto K(f) : K(Y) \to K(X), \end{aligned}$$

is well-defined, where [TopHdCpt] *is the quotient of* TopHdCpt *by the relation of homotopy of maps, which is an equivalence relation that is compatible with the composition in* TopHdCpt*.* ♢

From the functors of the preceding remark, we construct the real versions of (pointed) reduced and relative K-Theory groups and homomorphisms in the exactly same manner as was done in the complex case. Moreover, all of the results of this chapter hold true for real K-Theory, adapting them to the Real Bott Periodicity Theorem set below.

Theorem 2.79 (Real Bott Periodicity Theorem)**.** *Let (X, x_0) be an object in* $\mathrm{TopHdCpt}_+$*. There exists an isomorphism of rings*

$$\widetilde{K}(X, x_0) \to \widetilde{K}^{-8}(X, x_0).$$

Proof. The reader can find more details and further developments of this result in [23, p. 63]. □

In particular, we have that the real K-Theory is also an additive generalized cohomology theory. Differing from the complex case, which is 2-periodic, we have that real K-Theory is 8-periodic. We finish this section, and then the chapter, showing in the following remark the relevance of this version of K-theory by exhibiting one of its great achievements in the last century, which is a theorem that properly answers the question of what is the maximal number of linearly independent vector fields on a sphere.

Remark 2.80 (An application of real K-Theory)**.** *Using real K-Theory, one can prove that, if n is a non-zero natural number that we uniquely decompose in the form*

$$n \;=\; (2\alpha - 1)\, 2^{4\beta+\gamma},$$

then there exist at most $8\beta + 2^{\gamma} - 1$ linearly independent vector fields on $\mathbb{S}^{n-1}$. The reader can find proofs of this result in [1] and in [33]. The treatment given by the first reference is more technical than the one given by the second reference. This may help the reader in choosing which one of them to follow. $\diamondsuit$

Chapter 3

Spin Geometry and Ordinary K-Theory

In this chapter, we expose the necessary concepts of Spin Geometry in order to set the Thom isomorphisms and the Gysin map in Ordinary K-Theory. In particular, we study the Clifford algebras and their classification, given through Sylvester's Law of Inertia. Furthermore, we work with the representation theory of Clifford algebras, which leads us directly to the Atiyah-Bott-Shapiro Theorem. Afterwards, we deal with Pin and Spin groups, in order to introduce the notion of spin and spin^c structures on vector bundles. In order to write this part of the text, we used as main references [2, pp. 102-116], [6] and k [23, pp. 7-40, 58-70, 77-85]. However, Sections 3.6 and 3.7 could not be written without [9, pp. 37-47] as well as Section 3.8 could not be completed without the presence of [34].

3.1 Clifford algebras

In this section, we develop the fundamental notion that will be used throughout this chapter, namely, the Clifford algebras. This concept will be used later to deepen our present comprehension of Ordinary K-Theory. In particular, these special algebras are the basic mathematical structure used in the Atiyah-Bott-Shapiro Theorem, which will allow us to prove the existence of Thom classes in K-Theory. We begin by fixing the following notation.

Notation 3.1 (On real and complex numbers). *When we do not desire to distinguish between the field of real numbers and the field of complex numbers, we shall write* $\mathbb{K}$ *to symbolize any of them. In particular, the vector spaces considered below are always real or complex.* ♢

Definition 3.2 (Category of vector spaces and symmetric bilinear forms). *We define the* ***category of vector spaces and symmetric bilinear forms****, and denote it by* VectSymBF*, to be the category whose:*

- *objects are ordered pairs* $(\mathscr{V}, s)$ *where* $\mathscr{V}$ *is a finite-dimensional vector space and* $s : \mathscr{V} \times \mathscr{V} \to \mathbb{K}$ *is a symmetric bilinear form; and*
- *morphisms are linear maps* $\varphi : \mathscr{V} \to \mathscr{W}$ *such that* $s_{\mathscr{W}} \circ \varphi = s_{\mathscr{V}}$*, usually denoted by* $\varphi : (\mathscr{V}, s_{\mathscr{V}}) \to (\mathscr{W}, s_{\mathscr{W}})$*. In other words, morphisms are linear maps that preserve the symmetric bilinear forms.* ♢

Definition 3.3 (Category of vector spaces and quadratic forms). *We define the* ***category of vector spaces and quadratic forms****, and denote it by* VectQF*, to be the category whose:*

- *objects are ordered pairs* $(\mathscr{V}, q)$ *where* $\mathscr{V}$ *is a finite-dimensional vector space and* $q : \mathscr{V} \to \mathbb{K}$ *is a quadratic form; and*
- *morphisms are linear maps* $\varphi : \mathscr{V} \to \mathscr{W}$ *such that* $q_{\mathscr{W}} \circ \varphi = q_{\mathscr{V}}$*, usually denoted by* $\varphi : (\mathscr{V}, q_{\mathscr{V}}) \to (\mathscr{W}, q_{\mathscr{W}})$*. In other words, morphisms are linear maps that preserve the quadratic forms.* ♢

Definition 3.4 (Clifford algebras). *We give the following definitions.*

- *Let* $(\mathscr{V}, s)$ *be an object in* VectSymBF*. The* ***Clifford algebra*** $\mathrm{Cl}(\mathscr{V}, s)$ *of* $(\mathscr{V}, s)$ *is said to be the associative free algebra with unit generated by* $\mathscr{V}$ *and submitted to the relations*
$$vw + wv \;=\; -2s(v, w) \cdot 1 \tag{3.1}$$
for all $v, w \in \mathscr{V}$.

- *Let* $(\mathscr{V}, q)$ *be an object in* VectQF*. The* ***Clifford algebra*** $\mathrm{Cl}(\mathscr{V}, q)$ *of* $(\mathscr{V}, q)$ *is said to be the associative free algebra with unit generated by* $\mathscr{V}$ *and submitted to the relations*
$$v^2 = -q(v) \cdot 1 \tag{3.2}$$
for all $v \in \mathscr{V}$*.* ♢

Remark 3.5 (The Clifford algebras seen through the isomorphic categories of vector spaces equipped with symmetric bilinear forms and quadratic forms)**.** *We have the following facts.*

- *Let* $(\mathscr{V}, s)$ *be an object in* VectSymBF*. We define*
$$\begin{aligned} q_s : \mathscr{V} &\mapsto \mathbb{K}, \\ v &\mapsto s(v,\, v). \end{aligned}$$
We have $(\mathscr{V}, q_s) \in$ VectQF*. In addition, if* $\varphi : (\mathscr{V}, s_{\mathscr{V}}) \to (\mathscr{W}, s_{\mathscr{W}})$ *is a morphism in* VectSymBF*, then it follows that* $\varphi : (\mathscr{V}, q_{s_{\mathscr{V}}}) \to (\mathscr{W}, q_{s_{\mathscr{W}}})$ *is a morphism in* VectQF*.*

- *Let* $(\mathscr{V}, q)$ *be an object in* VectQF*. We define*
$$\begin{aligned} s_q : \mathscr{V} \times \mathscr{V} &\to \mathbb{K}, \\ (v,\, w) &\mapsto \frac{1}{2}\,(q(v+w) - q(v) - q(w)). \end{aligned}$$
We have $(\mathscr{V}, s_q) \in$ VectSymBF*. In addition, if* $\varphi : (\mathscr{V}, q_{\mathscr{V}}) \to (\mathscr{W}, q_{\mathscr{W}})$ *is a morphism in* VectQF*, then it follows that* $\varphi : (\mathscr{V}, s_{q_{\mathscr{V}}}) \to (\mathscr{W}, s_{q_{\mathscr{W}}})$ *is a morphism in* VectSymBF*.*

Since $q_{s_q} = q$ *and* $s_{q_s} = s$*, we have that*

$$\begin{aligned} Q : \mathrm{VectSymBF} &\to \mathrm{VectQF}, \\ (\mathscr{V},\, s) &\mapsto (\mathscr{V},\, q_s), \\ \varphi : (\mathscr{V},\, s_{\mathscr{V}}) \to (\mathscr{W},\, s_{\mathscr{W}}) &\mapsto \varphi : (\mathscr{V},\, q_{s_{\mathscr{V}}}) \to (\mathscr{W},\, q_{s_{\mathscr{W}}}), \end{aligned}$$

is an isomorphism between VectSymBF *and* VectQF*. Indeed, its inverse is the covariant functor*

$$\begin{aligned} S:\ \mathrm{VectQF} &\ \to\ \mathrm{VectSymBF}, \\ (\mathscr{V},\, q) &\ \mapsto\ (\mathscr{V},\, s_q), \\ \varphi : (\mathscr{V},\, q_{\mathscr{V}}) \to (\mathscr{W},\, q_{\mathscr{W}}) &\ \mapsto\ \varphi : (\mathscr{V},\, s_{q_{\mathscr{V}}}) \to (\mathscr{W},\, s_{q_{\mathscr{W}}}). \end{aligned}$$

Consequently, symmetric bilinear forms and quadratic forms are indistinguishable from a categorical viewpoint. Thus, we have that the following facts on Clifford algebras hold true.

- *Let* $(\mathscr{V}, s)$ *be an object in* VectSymBF*. The relations presented in Equation* (3.1) *are equivalent to*
$$v^2 \ =\ -q_s(v) \cdot 1$$
for all $v \in \mathscr{V}$*. Therefore, we have that* $\mathrm{Cl}(\mathscr{V}, s)$ *is canonically isomorphic to* $\mathrm{Cl}(\mathscr{V}, q_s)$*.*

- *Let* $(\mathscr{V}, q)$ *be an object in* VectQF*. The relations presented in Equation* (3.2) *are equivalent to*
$$vw + wv \ =\ -2s_q(v, w) \cdot 1$$
for all $v, w \in \mathscr{V}$*. Therefore, we have that* $\mathrm{Cl}(\mathscr{V}, q)$ *is canonically isomorphic to* $\mathrm{Cl}(\mathscr{V}, s_q)$*.*

In this text, we use these representations of the Clifford algebras interchangeably. Moreover, when the symmetric bilinear form or the quadratic form are understood, we write $\mathrm{Cl}(\mathscr{V})$*.* $\diamondsuit$

Theorem 3.6 (An alternative presentation to Clifford algebras)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF*. In addition, for all* $m \in \mathbb{N}$*, let* $\mathscr{V}^{\otimes m}$ *be the* m*-times tensor product of* $\mathscr{V}$ *with itself. We define*

$$\mathscr{T}(\mathscr{V}) \ :=\ \bigoplus_{i \in \mathbb{N}} \mathscr{V}^{\otimes i}.$$

Moreover, we define $\mathscr{I}(\mathscr{V}, q)$ *to be the ideal in* $\mathscr{T}(\mathscr{V})$ *formed by the elements of the form* $v \otimes v + q(v) \cdot 1$ *for* $v \in \mathscr{V}$*. We have that* $\mathrm{Cl}(\mathscr{V}, q)$ *is isomorphic to the* *quotient of* $\mathscr{T}(\mathscr{V})$ *by* $\mathscr{I}(\mathscr{V}, q)$. *Consequently, since* $\mathscr{V}$ *is a subset of* $\mathscr{T}(\mathscr{V})$ *up to a canonical isomorphism, there exists a map*

$$\iota_{\mathscr{V}} : \mathscr{V} \to \mathrm{Cl}(\mathscr{V}, q)$$

whose image generates the whole Clifford algebra.

Proof. The assertion is obvious from the definitions of the objects in question. We leave the details to the reader. □

Theorem 3.7 (Extending linear maps to Clifford algebras)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF. *In addition, let* $\mathcal{A}$ *be an associative algebra and* $\varphi : \mathscr{V} \to \mathcal{A}$ *be a linear map such that*

$$\varphi(v)^2 = -q(v) \cdot 1 \tag{3.3}$$

for all $v \in \mathscr{V}$. *Under these conditions, we have that* φ *extends uniquely to an algebra homomorphism*

$$\Phi : \mathrm{Cl}(\mathscr{V}, q) \to \mathcal{A}.$$

Furthermore, $\mathrm{Cl}(\mathscr{V}, q)$ *is the unique associative algebra with this property. This is the universal property of the Clifford algebras.*

Proof. The reader can readily prove that $\varphi : \mathscr{V} \to \mathcal{A}$ extends to a unique algebra homomorphism

$$\mathscr{T}(\mathscr{V}) \to \mathcal{A}.$$

Because of Equation (3.3), this homomorphism is trivial on $\mathscr{I}(\mathscr{V}, q)$. Thus, it descends to unique algebra homomorphism

$$\Phi : \mathrm{Cl}(\mathscr{V}, q) \to \mathcal{A},$$

as we wished. Now, let $\mathcal{B}$ be an associative algebra with unit over $\mathbb{K}$ equipped with an embedding $i : \mathscr{V} \to \mathcal{B}$ with the property that any linear map $\varphi : \mathscr{V} \to \mathcal{A}$ as above extends uniquely to an algebra homomorphism $\Phi : \mathcal{B} \to \mathcal{A}$. Thence, the isomorphism between $\mathscr{V} \subseteq \mathrm{Cl}(\mathscr{V}, q)$ and $i(\mathscr{V}) \subseteq \mathcal{B}$ induces an algebra isomorphism between $\mathrm{Cl}(\mathscr{V}, q)$ and $\mathcal{B}$. □

Remark 3.8 (Functorial behavior of the Clifford algebras implied by the preceding result)**.** *Let* $\varphi : (\mathscr{V}, q_{\mathscr{V}}) \to (\mathscr{W}, q_{\mathscr{W}})$ *be a morphism in* VectQF. *Additionally, let* $\iota_{\mathscr{W}} : \mathscr{W} \to \mathrm{Cl}(\mathscr{W}, q_{\mathscr{W}})$ *be the map defined in Theorem 3.6. Because of Theorem 3.7, we have that there exists a unique algebra homomorphism* $\Phi : \mathrm{Cl}(\mathscr{V}, q_{\mathscr{V}}) \to \mathrm{Cl}(\mathscr{W}, q_{\mathscr{W}})$ *that extends* $\iota_{\mathscr{W}} \circ \varphi : \mathscr{V} \to \mathrm{Cl}(\mathscr{W}, q_{\mathscr{W}})$. *In particular, because of uniqueness, the reader can prove that the induced map of a composition coincides with the composition of the induced maps. Consequently, it follows that the induced map of an isomorphism is also an isomorphism.* ♢

Lemma 3.9 (Two linear maps)**.** *Let* $(\mathscr{V}, s)$ *be an object in* VectSymBF. *For any* $m \in \mathbb{N}$*, let* $\Lambda^m(\mathscr{V})$ *denote the vector space of* m*-forms on* $\mathscr{V}$*. For any* $m \in \mathbb{N}$ *and* $v \in \mathscr{V}$*, we define the linear maps*

$$\begin{aligned} l_v^m : \Lambda^m(\mathscr{V}) &\to \Lambda^{m+1}(\mathscr{V}), \\ \alpha &\mapsto v \wedge \alpha, \end{aligned}$$

and

$$\begin{aligned} \delta_v^m : \Lambda^m(\mathscr{V}) &\to \Lambda^{m-1}(\mathscr{V}), \\ v_1 \wedge \cdots \wedge v_m &\mapsto \sum_{i=1}^m (-1)^{i-1}\, s(v, v_i)\, v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_m, \end{aligned}$$

which are such that the compositions

$$l_v^{m+1} \circ l_v^m : \Lambda^m(\mathscr{V}) \to \Lambda^{m+2}(\mathscr{V}) \qquad \text{and} \qquad \delta_v^{m-1} \circ \delta_v^m : \Lambda^m(\mathscr{V}) \to \Lambda^{m-2}(\mathscr{V})$$

are trivial homomorphisms. Moreover,

$$l_v^{m-1} \circ \delta_v^m + \delta_v^{m+1} \circ l_v^m = q(v) \cdot \mathrm{id}_{\Lambda^m(\mathscr{V})}.$$

It is to be note that, in order to define δ_v^m*, we tacitly supposed* m *strictly greater than zero. Furthermore, as usual, the hat in* $\widehat{v_i}$ *indicates the removal of the element* v_i *from the expression.*

Proof. The triviality of the composition $l_v^{m+1} \circ l_v^m$ is obvious since, for all $\alpha \in \Lambda^m(\mathscr{V})$, we have

$$(l_v^{m+1} \circ l_v^m)(\alpha) \;=\; l_v^{m+1}(v \wedge \alpha) \;=\; v \wedge v \wedge \alpha \;=\; 0.$$

In turn, we have that the triviality of the composition $\delta_v^{m-1} \circ \delta_v^m$ is proved as follows. Indeed, if

$$\alpha \;=\; v_1 \wedge \cdots \wedge v_m \in \Lambda^m(\mathscr{V}),$$

then

$$\begin{aligned}
(\delta_v^{m-1} \circ \delta_v^m)(\alpha) \;&=\; \delta_v^{m-1}\left(\sum_{i=1}^{m}(-1)^{i-1}\, s(v,v_i)\, v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_m\right)\\
&=\; \sum_{i=2}^{m}\sum_{j=1}^{i-1}(-1)^{i+j}\, s(v,v_i)\, s(v,v_j)\, v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_m\\
&\quad\; + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(-1)^{i+j-1} s(v,v_i) s(v,v_j) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_m\\
&=\; \sum_{i=2}^{m}\sum_{j=1}^{i-1}(-1)^{i+j}\, s(v,v_i)\, s(v,v_j)\, v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_m\\
&\quad\; - \sum_{j=2}^{m}\sum_{i=1}^{j-1}(-1)^{i+j}\, s(v,v_j)\, s(v,v_i)\, v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_m\\
&=\; 0.
\end{aligned}$$

Finally, using the notation presented above, the last equality of the statement is proved as follows. In fact,

$$(l_v^{m-1} \circ \delta_v^m)(\alpha) \;=\; \sum_{i=1}^{m}(-1)^{i-1}\, s(v,v_i)\, v \wedge v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_m$$

and

$$\begin{aligned}
(\delta_v^{m+1} \circ l_v^m)(\alpha) \;&=\; \delta_v^{m+1}(v \wedge \alpha)\\
&=\; q(v) \cdot \alpha + \sum_{i=1}^{m}(-1)^{i}\, s(v,v_i)\, v \wedge v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_m\\
&=\; q(v) \cdot \alpha - \sum_{i=1}^{m}(-1)^{i-1}\, s(v,v_i)\, v \wedge v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_m.
\end{aligned}$$

Consequently,

$$(l_v^{m-1} \circ \delta_v^m + \delta_v^{m+1} \circ l_v^m)(\alpha) \;=\; q(v) \cdot \alpha,$$

as we wished. □

Theorem 3.10 (Clifford algebras of non-trivial spaces are also non-trivial)**.** *We have that the map*

$$\iota_{\mathscr{V}} : \mathscr{V} \to \mathrm{Cl}(\mathscr{V})$$

defined in Theorem 3.6 is injective. This allows us to identify $\iota_{\mathscr{V}}(v) \in \mathrm{Cl}(\mathscr{V})$ with $v \in \mathscr{V}$ when there can be no confusion. In particular, if $\mathscr{V}$ is a non-trivial vector space, then $\mathrm{Cl}(\mathscr{V})$ is a non-trivial algebra.

Proof. To prove the desired injectivity, we construct a representation of $\mathrm{Cl}(\mathscr{V})$ over the exterior algebra

$$\Lambda(\mathscr{V}) \;:=\; \bigoplus_{i\in\mathbb{N}} \Lambda^i(\mathscr{V})$$

as follows. First, we consider

$$\begin{aligned} \eta :\; \mathscr{T}(\mathscr{V}) \;&\to\; \operatorname{End}\Lambda(\mathscr{V}), \\ v \;&\mapsto\; l_v - \delta_v, \end{aligned}$$

where

$$l_v \;:=\; \bigoplus_{i\in\mathbb{N}} l_v^i \qquad \text{and} \qquad \delta_v \;:=\; \bigoplus_{i\in\mathbb{N}} \delta_v^i$$

being l_v^i and δ_v^i the special linear operators defined in Lemma 3.9. Therefore, because of this very same lemma, and because Equation (3.2) holds true, we have that η projects to the map

$$\begin{aligned} \bar\eta :\; \mathrm{Cl}(\mathscr{V}) \;&\to\; \operatorname{End}\Lambda(\mathscr{V}), \\ \iota_{\mathscr{V}}(v) \;&\mapsto\; l_v - \delta_v. \end{aligned}$$

Using the definitions of the maps l_v and δ_v presented above, the reader can readily prove the equality

$$\eta(v)(1) \;=\; (l_v - \delta_v)(1) \;=\; v.$$

This implies

$$\bar\eta(\iota_{\mathscr{V}}(v))(1) \;=\; v.$$

Consequently, if v is non-zero, then $\iota_{\mathscr{V}}(v)$ is also non-zero. This finishes the proof of the theorem. $\square$

Remark 3.11 (Natural filtration on a Clifford algebra)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF. *We have*

$$\mathscr{T}(\mathscr{V}) = \bigoplus_{i \in \mathbb{N}} \mathscr{V}^{\otimes i}.$$

Consequently, if we define

$$\mathscr{V}_{\otimes m} := \bigoplus_{i=0}^{m} \mathscr{V}^{\otimes i}$$

for all $m \in \mathbb{N}$, *then we obtain the filtration*

$$\mathscr{V}_{\otimes 0} \subseteq \mathscr{V}_{\otimes 1} \subseteq \cdots \subseteq \mathscr{V}_{\otimes m} \subseteq \cdots \subseteq \mathscr{T}(\mathscr{V}).$$

In particular, the reader can readily prove the existence of a natural isomorphism between $\mathscr{V}_{\otimes m}/\mathscr{V}_{\otimes(m-1)}$ *and* $\mathscr{V}^{\otimes m}$. *Furthermore, if* $\pi : \mathscr{T}(\mathscr{V}) \to \mathrm{Cl}(\mathscr{V}, q)$ *is the natural projection and*

$$\mathscr{F}^m(\mathscr{V}) := \pi(\mathscr{V}_{\otimes m})$$

for all $m \in \mathbb{N}$, *then we obtain the filtration*

$$\mathscr{F}^0(\mathscr{V}) \subseteq \mathscr{F}^1(\mathscr{V}) \subseteq \cdots \subseteq \mathscr{F}^m(\mathscr{V}) \subseteq \cdots \subseteq \mathrm{Cl}(\mathscr{V}, q).$$

It is to be noted that this last filtration is compatible with the Clifford product, which means that

$$\mathscr{F}^m(\mathscr{V}) \cdot \mathscr{F}^n(\mathscr{V}) \subseteq \mathscr{F}^{m+n}(\mathscr{V})$$

for all $m, n \in \mathbb{N}$. *Moreover, it is to be noted that* $\mathscr{F}^m(\mathscr{V})$ *is generated by the products of at most* m *vectors of* $\mathscr{V}$. *As a consequence, we have that there can be products of* m *vectors contained in* $\mathscr{F}^n(\mathscr{V})$ *with* $n < m$. *For instance, being* $v, w \in \mathscr{V}$, *we have* $vww \in \mathscr{F}^1(\mathscr{V})$, *although it is the product of three vectors, since the equality* $vww = -q(w) \cdot v$ *holds.* $\diamondsuit$

Lemma 3.12 (Understanding the projected map defined in the proof of the preceding theorem)**.** *If*

$$\bar{\eta} : \mathrm{Cl}(\mathscr{V}) \to \mathrm{End}\,\Lambda(\mathscr{V})$$

is the map defined in Theorem 3.10, then, for all non-zero natural number m *and all* $v_1, \cdots, v_{m+1} \in \mathscr{V}$, *we have*

$$\bar{\eta}(v_1 \cdots v_{m+1})(1) \;=\; v_1 \wedge \cdots \wedge v_{m+1} \;+\; \alpha_m,$$

where

$$\alpha_m \;\in\; \Lambda_m(\mathscr{V}) \;:=\; \bigoplus_{i=0}^{m} \Lambda^i(\mathscr{V}).$$

Proof. We prove the result using the Finite Induction Principle.

- *Induction basis.* If $m = 0$, then we have seen that $\bar{\eta}(v)(1) = v$ for all $v \in \mathscr{V}$. Therefore, in this situation, we have $\alpha_0 = 0$. Consequently, the statement is true for $m = 0$.

- *Induction hypothesis.* We suppose that the lemma holds for some $m \in \mathbb{N}$ and all $v_1, \cdots, v_{m+1} \in \mathscr{V}$.

- *Induction step.* Being $v_1, \cdots, v_{m+2} \in \mathscr{V}$, it follows from the induction hypothesis that

$$\begin{aligned}
\bar{\eta}(v_1 \cdots v_{m+2})(1) \;&=\; \bar{\eta}(v_1)\bar{\eta}(v_2 \cdots v_{m+2})(1) \\
&=\; \bar{\eta}(v_1)(v_2 \wedge \cdots \wedge v_{m+2} + \alpha_m) \\
&=\; v_1 \wedge \cdots \wedge v_{m+2} + v_1 \wedge \alpha_m - \delta_{v_1}(v_2 \wedge \cdots \wedge v_{m+2} + \alpha_m).
\end{aligned}$$

  Since δ_{v_1} decreases degrees, we have

$$\alpha_{m+1} \;:=\; (v_1 \wedge \alpha_m) - \delta_{v_1}(v_2 \wedge \ldots \wedge v_{m+2} + \alpha_m) \;\in\; \Lambda_{m+1}(\mathscr{V}),$$

  as desired. □

Theorem 3.13 (The Clifford and exterior algebras)**.** *Let* $(\mathscr{V}, s)$ *be an object in* VectSymBF. *The map*

$$\begin{aligned}
\Phi_{\mathscr{V}} :\; \Lambda(\mathscr{V}) \;&\to\; \mathrm{Cl}(\mathscr{V}, s), \\
v_1 \wedge \cdots \wedge v_m \;&\mapsto\; [v_1 \cdots v_m],
\end{aligned}$$

is a canonical isomorphism of vector spaces, but not of algebras. In particular, we have $\dim \mathrm{Cl}(\mathscr{V}, s) = 2^{\dim(\mathscr{V})}$.

Proof. According to the proof of Theorem 3.10, we have

$$\Lambda(\mathscr{V}) = \bigoplus_{i\in\mathbb{N}} \Lambda^i(\mathscr{V}).$$

Nevertheless, since $\Lambda^m(\mathscr{V})$ is trivial when $\dim(\mathscr{V}) < m$, we have a canonical vector space isomorphism

$$\Lambda(\mathscr{V}) \simeq \bigoplus_{i=0}^{\dim(\mathscr{V})} \Lambda^i(\mathscr{V}). \tag{3.4}$$

Moreover, because of what we have shown in Remark 3.11, we have a canonical vector space isomorphism

$$\mathrm{Cl}(\mathscr{V}, s) \simeq \bigoplus_{i\in\mathbb{N}} \mathscr{F}^i(\mathscr{V})/\mathscr{F}^{i-1}(\mathscr{V}). \tag{3.5}$$

Now, let $\Phi^m_{\mathscr{V}} : \Lambda^m(\mathscr{V}) \to \mathscr{F}^m(\mathscr{V})/\mathscr{F}^{m-1}(\mathscr{V})$ be the restriction of $\Phi_{\mathscr{V}}$. Because of the Isomorphisms (3.4) and (3.5), it is clear that, if we prove that $\Phi^m_{\mathscr{V}}$ is an isomorphism for all $m \in \mathbb{N}$, then the theorem follows. First, however, we have to prove that $\Phi^m_{\mathscr{V}}$ is well-defined. This have to be done since the elements $v_1 \wedge \ldots \wedge v_m$ do not form a basis for $\Lambda^m(\mathscr{V})$. Let

$$\begin{aligned} \Psi^m_{\mathscr{V}} : \mathscr{V}^{\otimes m} &\to \mathscr{F}^m(\mathscr{V})/\mathscr{F}^{m-1}(\mathscr{V}), \\ v_1 \otimes \cdots \otimes v_m &\mapsto [v_1 \cdots v_m]. \end{aligned}$$

This map is well-defined since it is the composition of $\pi : \mathscr{T}(\mathscr{V}) \to \mathrm{Cl}(\mathscr{V}, s)$ restricted to $\mathscr{V}^{\otimes m}$ with the projection $\mathscr{F}^m(\mathscr{V}) \to \mathscr{F}^m(\mathscr{V})/\mathscr{F}^{m-1}(\mathscr{V})$. We shall prove that $\Phi^m_{\mathscr{V}}$ is the restriction of $\Psi^m_{\mathscr{V}}$ to $\Lambda^m(\mathscr{V})$. For this, let Σ_m denote the permutation group on m letters. Thence, if $(i\ i+1) \in \Sigma_m$ is a transposition of consecutive elements, then we obtain

$$\begin{aligned} [v_1 \cdots v_{i+1} v_i \cdots v_m] &= [v_1 \cdots v_{i-1} \left(-v_i v_{i+1} - 2s(v_i, v_{i+1}) \cdot 1\right) v_{i+2} \cdots v_m] \\ &= -[v_1 \cdots v_i v_{i+1} \cdots v_m] - [2s(v_i, v_{i+1}) v_1 \cdots v_{i-1} v_{i+2} \cdots v_m] \\ &= -[v_1 \cdots v_i v_{i+1} \cdots v_m], \end{aligned}$$

where this last equality holds because

$$2s(v_i, v_{i+1}) v_1 \cdots v_{i-1} v_{i+2} \cdots v_m \in \mathscr{F}^{m-2}(\mathscr{V}) \subseteq \mathscr{F}^{m-1}(\mathscr{V}).$$

Therefore, since the collection of all transpositions of consecutive elements generates Σ_m, we have

$$\Psi^m_{\mathscr{V}}(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}) \;=\; (-1)^\sigma \Psi^m_{\mathscr{V}}(v_1 \otimes \cdots \otimes v_i)$$

for all $\sigma \in \Sigma_m$. Consequently, it follows that the restriction of $\Psi^m_{\mathscr{V}}$ to $\Lambda^m(\mathscr{V})$ satisfies the equality

$$\Psi^m_{\mathscr{V}}(v_1 \wedge \cdots \wedge v_m) \;=\; \Psi^m_{\mathscr{V}}(v_1 \otimes \cdots \otimes v_m).$$

This proves that the function

$$v_1 \wedge \cdots \wedge v_m \;\mapsto\; \Psi^m_{\mathscr{V}}(v_1 \otimes \cdots \otimes v_m),$$

which coincides with $\Phi^m_{\mathscr{V}}$, is well-defined. Furthermore,

- $\Phi^m_{\mathscr{V}}$ *is surjective.* Indeed, if it is given a class $[v_1 \cdots v_m] \in \mathscr{F}^m(\mathscr{V})/\mathscr{F}^{m-1}(\mathscr{V})$, then we have

  $$\Phi^m_{\mathscr{V}}(v_1 \wedge \cdots \wedge v_m) \;=\; [v_1 \cdots v_m].$$

  In particular, this proves that $\mathscr{F}^m(\mathscr{V}) = \mathscr{F}^{\dim(\mathscr{V})}(\mathscr{V})$ for all m greater than or equal to $\dim(\mathscr{V})$; *and*

- $\Phi^m_{\mathscr{V}}$ *is injective.* In fact, we explicitly show its inverse. Let $p_m : \Lambda(\mathscr{V}) \to \Lambda^m(\mathscr{V})$ be the natural projection. We define

  $$\begin{aligned} \Xi^m_{\mathscr{V}} : \; \mathscr{F}^m(\mathscr{V})/\mathscr{F}^{m-1}(\mathscr{V}) \;&\to\; \Lambda^m(\mathscr{V}), \\ [v_1 \cdots v_m] \;&\mapsto\; p_m(\bar{\eta}(v_1 \cdots v_m)(1)), \end{aligned}$$

  where $\bar{\eta}$ is the map defined in the proof of Theorem 3.10. Because of Lemma 3.12, we have

  $$\Xi^m_{\mathscr{V}}[v_1 \cdots v_m] \;=\; v_1 \wedge \cdots \wedge v_m.$$

  Therefore, we obtain $\Xi^m_{\mathscr{V}} = (\Phi^m_{\mathscr{V}})^{-1}$, as desired. This finishes the proof of the theorem. $\square$

Remark 3.14 (Improving our comprehension on the Clifford algebras)**.** *Let* $(\mathscr{V}, s)$ *be an object in* VectSymBF. *If* $\mathscr{B}_1 = \{e_1, \cdots, e_{\dim(\mathscr{V})}\}$ *is a basis for* $\mathscr{V}$*, then collection of* m*-forms*

$$\mathscr{B}_m := \{e_{k_1} \wedge \cdots \wedge e_{k_m}\}_{1 \leq k_1 < \cdots < k_m \leq \dim(\mathscr{V})}$$

is a basis for $\Lambda^m(\mathscr{V})$ *for all* m *between* 1 *and* $\dim(\mathscr{V})$, *both included. Consequently, the collection*

$$\mathscr{B} := \bigcup_{i=0}^{\dim(\mathscr{V})} \mathscr{B}_i$$

is a basis for $\Lambda(\mathscr{V})$, *where* $\mathscr{B}_0$ *is defined to be the singleton containing* $1 \in \mathbb{K}$. *Because of Theorem 3.13,*

$$\Phi_{\mathscr{V}}(\mathscr{B}) := \bigcup_{i=0}^{\dim(\mathscr{V})} \Phi_{\mathscr{V}}(\mathscr{B}_i)$$

is a basis for the vector space $\mathrm{Cl}(\mathscr{V}, s)$. *Therefore, we may think about an element in* $\mathrm{Cl}(\mathscr{V}, s)$ *as an* m*-form for some* m *between* 1 *and* $\dim(\mathscr{V})$, *both included. Nevertheless, the wedge product is substituted by the Clifford product, which is not even isomorphic to the former. In fact, leaving the isomorphism of* $\mathrm{Cl}(\mathscr{V}, s)$ *with* $\Lambda(\mathscr{V})$ *implicit, it can be proved that*

$$v \cdot \eta \simeq v \wedge \eta - \delta_v(\eta)$$

for all $v \in \mathscr{V}$ *and all* $\eta \in \mathrm{Cl}(\mathscr{V})$. ♢

Remark 3.15 (Important decomposition of Clifford algebras)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF. *We have*

$$\mathrm{Cl}(\mathscr{V}, q) \simeq \mathrm{Cl}^0(\mathscr{V}, q) \oplus \mathrm{Cl}^1(\mathscr{V}, q), \tag{3.6}$$

where:

- $\mathrm{Cl}^0(\mathscr{V}, q)$ *is the subalgebra of* $\mathrm{Cl}(\mathscr{V}, q)$ *generated by the products of an even number of vectors of* $\mathscr{V}$*; and*
- $\mathrm{Cl}^1(\mathscr{V}, q)$ *is the subalgebra of* $\mathrm{Cl}(\mathscr{V}, q)$ *generated by the products of an odd number of vectors of* $\mathscr{V}$.

This claim can be proved by using two different viewpoints. Indeed:

- *consider the split*

$$\mathscr{T}(\mathscr{V}) = \mathscr{T}^0(\mathscr{V}) \oplus \mathscr{T}^1(\mathscr{V}),$$

obtained by sorting out the homogeneous generators of even and odd degree in $\mathscr{T}(\mathscr{V})$. *Correspondingly,*

$$\mathscr{I}(\mathscr{V},q) \;=\; \mathscr{I}^0(\mathscr{V},q) \,\oplus\, \mathscr{I}^1(\mathscr{V},q).$$

This split of $\mathscr{I}(\mathscr{V},q)$ *is possible because its generators* $v\otimes v + q(v)\cdot 1$ *are sums of two terms with even degrees. In fact, since the generic element of* $\mathscr{I}(\mathscr{V},q)$ *has the form*

$$\sum_{i=1}^{k} \gamma_i \otimes (v_i \otimes v_i - q(v_i)\cdot 1) \otimes \theta_i,$$

it splits in the sum of the terms such that $\delta_i := \deg(\gamma_i) + \deg(\theta_i)$ *is even with the terms such that* δ_i *is odd. As a consequence of these splittings, we have the isomorphism*

$$\mathrm{Cl}(\mathscr{V},q) \;\simeq\; \mathscr{T}^0(\mathscr{V})/\mathscr{I}^0(\mathscr{V},q) \,\oplus\, \mathscr{T}^1(\mathscr{V})/\mathscr{I}^1(\mathscr{V},q) \;=\; \mathrm{Cl}^0(\mathscr{V},q)\oplus \mathrm{Cl}^1(\mathscr{V},q),$$

as we wished; and

- *consider the linear operator*

$$\begin{aligned} \alpha:\, \mathscr{V} \;&\to\; \mathscr{V},\\ v \;&\mapsto\; -v. \end{aligned}$$

Let $\tau : \mathrm{Cl}(\mathscr{V},q) \to \mathrm{Cl}(\mathscr{V},q)$ *be its induced homomorphism from Remark 3.8. We have that* $\mathrm{Cl}^0(\mathscr{V},q)$ *and* $\mathrm{Cl}^1(\mathscr{V},q)$ *are the eigenspaces relatively to the eigenvalues* 1 *and* -1 *of* τ. *Thus, we can prove the desired result by showing that any element in* $\mathrm{Cl}(\mathscr{V},q)$ *can be written as a sum of an element in* $\mathrm{Cl}^0(\mathscr{V},q)$ *with an element in* $\mathrm{Cl}^1(\mathscr{V},q)$. *This happens because eigenspaces associated to different eigenvalues are necessarily disjoint. In fact, for all* $\eta\in\mathrm{Cl}(\mathscr{V},q)$, *we have the decomposition*

$$\eta \;=\; \frac{1}{2}(\eta+\tau(\eta)) \,+\, \frac{1}{2}(\eta-\tau(\eta)),$$

where $\eta+\tau(\eta)\in\mathrm{Cl}^0(\mathscr{V},q)$ *and* $\eta-\tau(\eta)\in\mathrm{Cl}^1(\mathscr{V},q)$. *As a consequence, we are done here*

Furthermore, we have

$$\mathrm{Cl}^i(\mathscr{V},q)\cdot\mathrm{Cl}^j(\mathscr{V},q) \;\subseteq\; \mathrm{Cl}^{i+j}(\mathscr{V},q), \tag{3.7}$$

where $i, j \in \mathbb{Z}_2$. The decomposition in (3.6) *together with the property in* (3.7) *turn* $\mathrm{Cl}(\mathscr{V}, q)$ *into a* ***$\mathbb{Z}_2$-graded algebra****. The structure of $\mathbb{Z}_2$-graded algebra provides a modification on the tensor product of algebras, which we denote by $\widehat{\otimes}$, that relates the Clifford algebra of a direct sum with the Clifford algebras of its summands. In fact, if $\mathscr{V} = \mathscr{V}_1 \oplus \mathscr{V}_2$ is such that*

$$q(v_1 + v_2) \;=\; q(v_1) + q(v_2)$$

for all $v_1 \in \mathscr{V}_1$ and all $v_2 \in \mathscr{V}_2$, then there exists a natural isomorphism

$$\mathrm{Cl}(\mathscr{V}, q) \;\simeq\; \mathrm{Cl}(\mathscr{V}_1) \,\widehat{\otimes}\, \mathrm{Cl}(\mathscr{V}_2),$$

where the quadratic forms on $\mathscr{V}_1$ and $\mathscr{V}_2$ are the restrictions of the quadratic form on $\mathscr{V}$. This fact will be important in the next section. The reader who desires to understand a bit more about this construction may find in [23, pp. 11-12] an interesting reference. $\diamondsuit$

3.2 Classification of Clifford algebras

In this section, we present the complete classification of the real and complex Clifford algebras. This interesting topic complements the formal study developed in the preceding section. In fact, it supplies us with many concrete examples. The essential result that enables such classification is Sylvester's Law of Inertia, which is a famous theorem from Linear Algebra that will be remarked in time. We begin by fixing the following notation.

Notation 3.16 (Usual matrix algebras)**.** *Let n be a natural number. We shall denote by $\mathbb{R}(n)$, $\mathbb{C}(n)$ and $\mathbb{H}(n)$ the algebra of square matrices of order n over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, respectively.* $\diamondsuit$

Definition 3.17 (Clifford algebra induced by a real canonical form)**.** *Let n be a natural number. For each natural numbers a and b such that $a + b$ is at most n, we define the* ***canonical real quadratic form***

$$\begin{aligned} q^n_{a,b} : \mathbb{R}^n &\to \mathbb{R}, \\ (x_1, \cdots, x_n) &\mapsto \sum_{i=1}^{a} x_i^2 - \sum_{i=1}^{b} x_{i+a}^2. \end{aligned}$$

The representative matrix of the quadratic form $q^n_{a,b}$ *with respect to the canonical basis is given by*

$$\begin{bmatrix} I_a & 0 & 0 \\ 0 & -I_b & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}(n). \tag{3.8}$$

We say that $\mathrm{Cl}_n(a,b)$ *is the Clifford algebra of* $\mathbb{R}^n$ *with respect to the real quadratic form* $q^n_{a,b} : \mathbb{R}^n \to \mathbb{R}$. *In addition, if* $q^n_{a,b}$ *is non-degenerate, that is, if* $n = a + b$, *then* $\mathrm{Cl}_n(a,b)$ *is denoted by* $\mathrm{Cl}(a,b)$. *Finally, we agree on denoting* $\mathrm{Cl}(n,0)$ *simply by* $\mathrm{Cl}(n)$. ♢

Remark 3.18 (On real Clifford algebras)**.** *Let* n *be a natural number. The real version of* ***Sylvester's Law of Inertia*** *says that any real quadratic form* q *on an* n*-dimensional real vector space* $\mathscr{V}$ *admits a basis under which its representative matrix coincides with* (3.8) *for some appropriate* $a, b \in \mathbb{N}$. *Thus, there exists an isomorphism* $\varphi : \mathscr{V} \to \mathbb{R}^n$ *such that*

$$q^n_{a,b} \circ \varphi = q.$$

As a consequence of this reasoning, it follows from Remark 3.8 that there exists an algebra isomorphism

$$\Phi : \mathrm{Cl}(\mathscr{V}, q) \to \mathrm{Cl}_n(a,b).$$

This proves that any real Clifford algebra is isomorphic to some $\mathrm{Cl}_n(a,b)$. *Clearly, if the quadratic form is non-degenerate, then its Clifford algebra is isomorphic to some* $\mathrm{Cl}(a,b)$. *Therefore, we can rephrase the problem of classifying all of the real Clifford algebras to the problem of classifying all of the real Clifford algebras* $\mathrm{Cl}_n(a,b)$. *In fact, it suffices to classify the real Clifford algebras* $\mathrm{Cl}(a,b)$. *Indeed, consider the decomposition* $\mathbb{R}^n = \mathbb{R}^{a+b} \oplus \mathbb{R}^{n-a-b}$ *where* $q^n_{a,s} \mid_{\mathbb{R}^{a+b}}$ *is non-degenerate and* $q^n_{a,b} \mid_{\mathbb{R}^{n-a-b}}$ *is trivial. Because of Remark 3.15, we have*

$$\mathrm{Cl}_n(a,b) \;\simeq\; \mathrm{Cl}(a,b)\,\widehat{\otimes}\,\mathrm{Cl}_{n-a-b}(0,0). \tag{3.9}$$

Thence, since $\mathrm{Cl}_{n-a-b}(0,0)$ *is canonically isomorphic to the exterior algebra* $\Lambda(\mathbb{R}^{n-a-b})$*, it is clear from Equation* (3.9) *that we only have to classify the Clifford algebras* $\mathrm{Cl}(a,b)$*, as we claimed.* ♢

Definition 3.19 (Clifford algebra induced by a complex canonical form)**.** *Let n be a natural number. For each natural number a between* 0 *and n, both included, we define the* ***canonical complex quadratic form***

$$\begin{aligned} q_a^n : \mathbb{C}^n &\to \mathbb{C}, \\ (z_1,\cdots,z_n) &\mapsto \sum_{i=1}^{a} z_i^2. \end{aligned}$$

The representative matrix of the quadratic form q_a^n with respect to the canonical basis is given by

$$\begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}(n). \tag{3.10}$$

We say that $\mathbb{C}l_n(a)$ is the Clifford algebra of $\mathbb{C}^n$ with respect to the complex quadratic form $q_a^n : \mathbb{C}^n \to \mathbb{C}$. In addition, if q_a^n is non-degenerate, that is, if $n = a$, then $\mathbb{C}l_n(n)$ is denoted by $\mathbb{C}l(n)$. ♢

Remark 3.20 (On complex Clifford algebras)**.** *Let n be a natural number. The complex version of Sylvester's Law of Inertia says that any complex quadratic form q on an n-dimensional complex vector space $\mathscr{V}$ admits a basis under which its representative matrix coincides with* (3.10) *for some appropriate $a \in \mathbb{N}$. Thus, we have that there exists an isomorphism $\varphi : \mathscr{V} \to \mathbb{C}^n$ such that*

$$q_a^n \circ \varphi \;=\; q.$$

As a consequence of this reasoning, it follows from Remark 3.8 that there exists an algebra isomorphism

$$\Phi :\; \mathrm{Cl}(\mathscr{V},q) \;\to\; \mathbb{C}l_n(a).$$

This proves that any complex Clifford algebra is isomorphic to some $\mathbb{C}l_n(a)$. *Clearly, if the quadratic form is non-degenerate, then its Clifford algebra is isomorphic to some* $\mathbb{C}l(n)$. *Therefore, we can rephrase the problem of classifying all of the complex Clifford algebras to the problem of classifying all of the complex Clifford algebras* $\mathbb{C}l_n(a)$. *In fact, it suffices to classify the complex Clifford algebras* $\mathbb{C}l(n)$. *Indeed, consider the decomposition* $\mathbb{C}^n = \mathbb{C}^a \oplus \mathbb{C}^{n-a}$ *where* $q_a^n\mid_{\mathbb{C}^a}$ *is non-degenerate and* $q_a^n\mid_{\mathbb{C}^{n-a}}$ *is trivial. Because of Remark 3.15, we have*

$$\mathbb{C}l_n(a) \simeq \mathbb{C}l(a) \mathbin{\widehat{\otimes}} \mathbb{C}l_{n-a}(0). \tag{3.11}$$

Therefore, as before, it follows from Equation (3.11) *that we only have to classify the Clifford algebras* $\mathbb{C}l(n)$, *as we claimed.* $\diamondsuit$

Remark 3.21 (A relation between real and complex Clifford algebras)**.** *We have the following facts.*

- *If* $\mathscr{V}$ *is a real vector space, then we can define its complexification* $\mathscr{V}_{\mathbb{C}} := \mathscr{V} \otimes_{\mathbb{R}} \mathbb{C}$. *Moreover, any real symmetric bilinear form on* $\mathscr{V}$ *can be extended by* $\mathbb{C}$*-linearity to a complex symmetric bilinear form on* $\mathscr{V}_{\mathbb{C}}$. *With respect to this data, we have that the Clifford algebra* $\mathrm{Cl}(\mathscr{V}_{\mathbb{C}})$ *is canonically isomorphic to the complexification* $\mathrm{Cl}(\mathscr{V}) \otimes_{\mathbb{R}} \mathbb{C}$. *Indeed, it suffices to identify* $v \otimes z \in \mathrm{Cl}(\mathscr{V}) \otimes_{\mathbb{R}} \mathbb{C}$ *with* $v \otimes z \in \mathscr{V}_{\mathbb{C}}$. *In particular, the complexification of* $\mathrm{Cl}_n(a,b)$ *is isomorphic to* $\mathbb{C}l_n(a+b)$. *Further, we have that* $\mathrm{Cl}^0(\mathscr{V}_{\mathbb{C}})$ *and* $\mathrm{Cl}^1(\mathscr{V})$ *are canonically isomorphic to* $\mathrm{Cl}^0(\mathscr{V}) \otimes_{\mathbb{R}} \mathbb{C}$ *and* $\mathrm{Cl}^1(\mathscr{V}) \otimes_{\mathbb{R}} \mathbb{C}$, *respectively.*

- *If* $\mathscr{V}$ *is a complex vector space, then we can define its realification* $\mathscr{V}_{\mathbb{R}}$ *by restricting its scalar product to real numbers. Moreover, any complex symmetric form on* $\mathscr{V}$ *can be restricted to a real symmetric bilinear form on* $\mathscr{V}_{\mathbb{R}}$. *With respect to this data, we have that the Clifford algebra* $\mathrm{Cl}(\mathscr{V}_{\mathbb{R}})$ *is canonically isomorphic to the realification of* $\mathrm{Cl}(\mathscr{V})$, *where an isomorphism is given by the identity map. Furthermore, we have that* $\mathrm{Cl}^0(\mathscr{V}_{\mathbb{R}})$ *and* $\mathrm{Cl}^1(\mathscr{V}_{\mathbb{R}})$ *are canonically isomorphic to the realifications of* $\mathrm{Cl}^0(\mathscr{V})$ *and* $\mathrm{Cl}^1(\mathscr{V})$, *respectively.* $\diamondsuit$

Theorem 3.22 (The first concrete examples of real and complex Clifford algebras)**.** *We have the isomorphisms*

$$\begin{aligned} \mathrm{Cl}(1) &\simeq \mathbb{C} \\ \mathrm{Cl}(0,1) &\simeq \mathbb{R}\oplus\mathbb{R} \\ \mathrm{Cl}(1,1) &\simeq \mathbb{R}(2) \\ \mathrm{Cl}(2) &\simeq \mathbb{H} \\ \mathrm{Cl}(0,2) &\simeq \mathbb{R}(2). \end{aligned}$$

As immediate consequences of the first and forth of these isomorphisms, we obtain the isomorphisms

$$\mathbb{C}l(1) \simeq \mathbb{C}\oplus\mathbb{C} \qquad \textit{and} \qquad \mathbb{C}l(2) \simeq \mathbb{C}(2).$$

Proof. Indeed, we have that:

- $\mathrm{Cl}(1)$ is the vector space generated by 1 and e_1, with the only interesting multiplication being $e_1^2 = -1$. Therefore, we obtain the desired isomorphism by identifying $i \in \mathbb{C}$ with $e_1 \in \mathrm{Cl}(1)$. Moreover, because of Remark 3.21, we have the isomorphism

$$\begin{aligned} \Phi:\ \mathbb{C}l(1) &\to \mathbb{C}\oplus\mathbb{C}, \\ \frac{1}{2}(1+ie_1) &\mapsto (1,0),\ \frac{1}{2}(1-ie_1) \ \mapsto\ (0,1). \end{aligned}$$

- $\mathrm{Cl}(0,1)$ is the vector space generated by 1 and e_1, with the only interesting multiplication being $e_1^2 = 1$. Therefore, we obtain the desired isomorphism through the linear map

$$\begin{aligned} \Phi:\ \mathrm{Cl}(0,1) &\to \mathbb{R}\oplus\mathbb{R}, \\ \frac{1}{2}(1+e_1) &\mapsto (1,0),\ \frac{1}{2}(1-e_1) \ \mapsto\ (0,1). \end{aligned}$$

- $\mathrm{Cl}(1,1)$ is the vector space generated by 1, e_1 and e_2, with the only interesting multiplications being $e_1^2 = -1$, $e_2^2 = 1$ and $e_1e_2 = -e_2e_1$. Thus, we obtain the desired isomorphism through the linear map

$$\begin{aligned}\Phi:\ \mathrm{Cl}(1,1) &\to \mathbb{R}(2),\\ e_1 &\mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ e_2 \mapsto \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.\end{aligned}$$

- $\mathrm{Cl}(2)$ is the vector space generated by 1, e_1 and e_2, with the only interesting multiplications being $e_1^2 = e_2^2 = -1$ and $e_1e_2 = -e_2e_1$. Thus, we obtain the desired isomorphism by making the obvious identifications (see Example E.4). Moreover, we have the isomorphism

$$\begin{aligned}\Phi:\ \mathbb{C}l(2) &\to \mathbb{C}(2),\\ e_1 &\mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\ e_2 \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.\end{aligned}$$

- $\mathrm{Cl}(0,2)$ is the vector space generated by 1, e_1 and e_2, with the only interesting multiplications being $e_1^2 = e_2^2 = 1$ and $e_1e_2 = -e_2e_1$. Thus, we obtain the desired isomorphism through the linear map

$$\begin{aligned}\Phi:\ \mathrm{Cl}(0,2) &\to \mathbb{R}(2),\\ e_1 &\mapsto \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ e_2 \mapsto \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.\end{aligned}$$

This finishes the proof of the theorem. $\square$

We have seen hitherto that the problem of classifying (real or complex) Clifford algebras can be rephrased to the problem of classifying the (real or complex) Clifford algebras of $\mathbb{K}^n$ obtained from the non-degenerate canonical quadratic forms. We now classify these algebras. We start with the complex case, which is much simpler than the real case.

Theorem 3.23 (Periodicity of complex Clifford algebras)**.** *Let n be a natural number. The linear map*

$$\begin{array}{rcl}\Phi_n : \ \mathbb{C}l(n+2) & \to & \mathbb{C}l(n) \otimes_{\mathbb{C}} \mathbb{C}l(2), \\ 1 & \mapsto & 1, \\ \gamma_1 & \mapsto & \beta_1 \otimes i\alpha_1\alpha_2, \\ & \vdots & \\ \gamma_n & \mapsto & \beta_n \otimes i\alpha_1\alpha_2, \\ \gamma_{n+1} & \mapsto & 1 \otimes \alpha_1, \\ \gamma_{n+2} & \mapsto & 1 \otimes \alpha_2, \end{array}$$

is an algebra isomorphism, where $\{\alpha_1, \alpha_2\}$*,* $\{\beta_1, \cdots, \beta_n\}$ *and* $\{\gamma_1, \cdots, \gamma_{n+2}\}$ *are the canonical bases of* $\mathbb{R}^2$*,* $\mathbb{R}^n$ *and* $\mathbb{R}^{n+2}$*, respectively.*

Proof. We start by showing that Φ_n preserves products. In fact, it suffices to show that this map preserves the relations in (3.1). This task consists in a pile of straightforward computations. To keep clearness, we exemplify one of them. Indeed, for j and k between 1 and n, both included,

$$\begin{array}{rcl}\gamma_j\gamma_k + \gamma_k\gamma_j & = & (-\delta_{jk}) \otimes 1 \\ & = & (\beta_j\beta_k + \beta_k\beta_j) \otimes 1 \\ & = & (\beta_j\beta_k) \otimes 1 + (\beta_k\beta_j) \otimes 1 \\ & = & (\beta_j \otimes i\alpha_1\alpha_2)(\beta_k \otimes i\alpha_1\alpha_2) + (\beta_k \otimes i\alpha_1\alpha_2)(\beta_j \otimes i\alpha_1\alpha_2)\end{array}$$

because

$$(i\alpha_1\alpha_2)^2 \ = \ -\alpha_1\alpha_2\alpha_1\alpha_2 \ = \ (\alpha_1)^2(\alpha_2)^2 \ = \ (-1)(-1) = 1$$

once $\alpha_1\alpha_2 = -\alpha_2\alpha_1$. Here we have that δ_{jk} denotes the usual Kronecker delta. That is, we have

$$\delta_{jk} \ = \ \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, since Φ_n is a linear map by definition, it is an algebra homomorphism. Now, we prove that Φ_n is bijective. As a matter of fact, we have that Φ_n is surjective since the generators $\beta_j \otimes 1$ and $1 \otimes \alpha_k$ of its codomain belong to its image. This happens because

$$\beta_j \otimes 1 \;=\; \beta_j \otimes -i^2 \;=\; i(\beta_j \otimes i\alpha_1\alpha_2\alpha_1\alpha_2) \;=\; i(\beta_j \otimes i\alpha_1\alpha_2)(1 \otimes \alpha_1)(1 \otimes \alpha_2).$$

Thence, the injectivity of Φ_n is obvious from the Rank-Nullity Theorem since both $\mathbb{C}l(n+2)$ and $\mathbb{C}l(n) \otimes_{\mathbb{C}} \mathbb{C}l(2)$ have dimension 2^{n+2}. This finishes the proof of the theorem. $\square$

Corollary 3.24 (Classification of complex Clifford algebras)**.** *We have the isomorphisms*

$$\mathbb{C}l(2n) \;\simeq\; \mathbb{C}(2^n) \qquad \textit{and} \qquad \mathbb{C}l(2n+1) \;\simeq\; \mathbb{C}(2^n) \oplus \mathbb{C}(2^n)$$

for all $n \in \mathbb{N}$.

Proof. This result follows from Theorems 3.22 and 3.23. We leave the immediate details to the reader. $\square$

Finally, we handle the classification of the real Clifford algebras obtained from the non-degenerate canonical quadratic forms. The complete characterization of these algebras, which immediately follows from the following three results, is shown in Table 3.1.

Theorem 3.25 (Reducing the classification of real Clifford algebras to a smaller class)**.** *Let a and b be natural numbers. The linear map*

$$\begin{aligned}
\Phi_{a,b} :\; \mathrm{Cl}(a+1, b+1) &\;\to\; \mathrm{Cl}(a,b) \otimes_{\mathbb{R}} \mathrm{Cl}(1,1), \\
1 &\;\mapsto\; 1, \\
\gamma_1 &\;\mapsto\; \beta_1 \otimes \alpha_1\alpha_2, \\
&\;\;\vdots \\
\gamma_a &\;\mapsto\; \beta_a \otimes \alpha_1\alpha_2, \\
\gamma_{a+1} &\;\mapsto\; 1 \otimes \alpha_1, \\
\gamma_{a+2} &\;\mapsto\; \beta_{a+1} \otimes \alpha_1\alpha_2, \\
&\;\;\vdots \\
\gamma_{a+b+1} &\;\mapsto\; \beta_{a+b} \otimes \alpha_1\alpha_2, \\
\gamma_{a+b+2} &\;\mapsto\; 1 \otimes \alpha_2,
\end{aligned}$$

is an algebra isomorphism, where we have that $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \cdots, \beta_a, \beta_{a+1}, \cdots, \beta_{a+b}\}$ *and* $\{\gamma_1, \cdots, \gamma_a, \gamma_{a+1}, \gamma_{a+2}, \cdots, \gamma_{a+b+2}\}$ *are the canonical bases of* $\mathbb{R}^2$, $\mathbb{R}^{a+b}$ *and* $\mathbb{R}^{a+b+2}$, *respectively. Consequently:*

- *if* $a < b$, *then*
$$\mathrm{Cl}(a, b) \simeq \mathrm{Cl}(0, b-a) \otimes_{\mathbb{R}} \mathbb{R}(2^a);$$
- *if* $a = b$, *then*
$$\mathrm{Cl}(a, b) \simeq \mathbb{R}(2^a);$$
- *if* $b < a$, *then*
$$\mathrm{Cl}(a, b) \simeq \mathrm{Cl}(a-b) \otimes_{\mathbb{R}} \mathbb{R}(2^b).$$

These last three isomorphisms show that we can rephrase the problem of classifying all of the Clifford algebras $\mathrm{Cl}(a, b)$ *to the problem of classifying only the ones of the forms* $\mathrm{Cl}(a)$ *and* $\mathrm{Cl}(0, b)$.

Proof. The proof of this result is analogous to the proof of Theorem 3.23. We leave the details to the reader. □

Lemma 3.26 (Two more isomorphisms in the framework of Clifford algebras)**.** *Let* n *be a natural number. We have the isomorphisms*

$$\mathrm{Cl}(n+2) \simeq \mathrm{Cl}(0, n) \otimes_{\mathbb{R}} \mathrm{Cl}(2) \qquad \text{and} \qquad \mathrm{Cl}(0, n+2) \simeq \mathrm{Cl}(n) \otimes_{\mathbb{R}} \mathrm{Cl}(0, 2),$$

which are define by

$$\begin{aligned} 1 &\mapsto 1, \\ \gamma_1 &\mapsto \beta_1 \otimes \alpha_1\alpha_2, \\ &\vdots \\ \gamma_n &\mapsto \beta_n \otimes \alpha_1\alpha_2, \\ \gamma_{n+1} &\mapsto 1 \otimes \alpha_1, \\ \gamma_{n+2} &\mapsto 1 \otimes \alpha_2, \end{aligned}$$

where $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \cdots, \beta_n\}$ *and* $\{\gamma_1, \cdots, \gamma_{n+2}\}$ *are the canonical bases of* $\mathbb{R}^2$, $\mathbb{R}^n$ *and* $\mathbb{R}^{n+2}$, *respectively.*

Proof. Once and again, the proof of this result is analogous to the proof of Theorem 3.23. We leave the details to the reader. □

Theorem 3.27 (Periodicity of real Clifford algebras)**.** *For all $n \in \mathbb{N}$, we have the isomorphisms*

$$\mathrm{Cl}(n+8) \simeq \mathrm{Cl}(n) \otimes_{\mathbb{R}} \mathrm{Cl}(8) \qquad \textit{and} \qquad \mathrm{Cl}(0,n+8) \simeq \mathrm{Cl}(0,n) \otimes_{\mathbb{R}} \mathrm{Cl}(0,8),$$

where

$$\mathrm{Cl}(8) \simeq \mathbb{R}(16) \simeq \mathrm{Cl}(0,8).$$

Proof. Due to Lemma 3.26, we have

$$\begin{aligned} \mathrm{Cl}(n+8) &\simeq \mathrm{Cl}(n) \otimes_{\mathbb{R}} \mathrm{Cl}(0,2) \otimes_{\mathbb{R}} \mathrm{Cl}(2) \otimes_{\mathbb{R}} \mathrm{Cl}(0,2) \otimes_{\mathbb{R}} \mathrm{Cl}(2) \\ &\simeq \mathrm{Cl}(n) \otimes_{\mathbb{R}} \mathrm{Cl}(8), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Cl}(0,n+8) &\simeq \mathrm{Cl}(0,n) \otimes_{\mathbb{R}} \mathrm{Cl}(2) \otimes_{\mathbb{R}} \mathrm{Cl}(0,2) \otimes_{\mathbb{R}} \mathrm{Cl}(2) \otimes_{\mathbb{R}} \mathrm{Cl}(0,2) \\ &\simeq \mathrm{Cl}(0,n) \otimes_{\mathbb{R}} \mathrm{Cl}(0,8). \end{aligned}$$

Additionally,

$$\begin{aligned} \mathrm{Cl}(8) &\simeq \mathrm{Cl}(0,2) \otimes_{\mathbb{R}} \mathrm{Cl}(2) \otimes_{\mathbb{R}} \mathrm{Cl}(0,2) \otimes_{\mathbb{R}} \mathrm{Cl}(2) \\ &\simeq \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{H} \\ &\simeq \mathbb{R}(4) \otimes_{\mathbb{R}} \mathbb{R}(4) \\ &\simeq \mathbb{R}(16) \\ &\simeq \mathbb{R}(4) \otimes_{\mathbb{R}} \mathbb{R}(4) \\ &\simeq \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}(2) \\ &\simeq \mathrm{Cl}(2) \otimes_{\mathbb{R}} \mathrm{Cl}(0,2) \otimes_{\mathbb{R}} \mathrm{Cl}(2) \otimes_{\mathbb{R}} \mathrm{Cl}(0,2) \\ &\simeq \mathrm{Cl}(0,8). \end{aligned}$$

This finishes the proof of the theorem. □

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\oplus\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32)\oplus\mathbb{H}(32)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64)\oplus\mathbb{R}(64)$	$\mathbb{R}(218)$	$\mathbb{C}(128)$
8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128)\oplus\mathbb{R}(128)$	$\mathbb{R}(256)$

Table 3.1: This table describes all of the Clifford algebras $\mathrm{Cl}(a, b)$ for a and b between 0 and 8, both included. It is to be noted that, as usual, a varies in columns while b varies in rows.

3.3 Representations of Clifford algebras

In this section, we expose some facts on the representation theory for Clifford algebras. Interestingly, we have that most of these facts are direct consequences of the classification theorems that we have set in the preceding section. In particular, we establish the important notion of Clifford multiplication. We begin with the following definition.

Definition 3.28 (Representation of a Clifford algebra and Clifford multiplication)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF*. A* ***representation*** *of the Clifford algebra* $\mathrm{Cl}(\mathscr{V}, q)$ *is an algebra homomorphism*

$$\rho : \ \mathrm{Cl}(\mathscr{V}, q) \ \rightarrow \ \mathrm{End}(\mathscr{W}),$$

where $\mathrm{End}(\mathscr{W})$ *denotes the algebra of linear transformations of a finite-dimensional vector space* $\mathscr{W}$*. The space* $\mathscr{W}$ *is then a* $\mathrm{Cl}(\mathscr{V}, q)$*-module. We simplify notation by writing*

$$\eta \cdot w \ := \ \rho(\eta)(w) \tag{3.12}$$

for all $w \in \mathscr{W}$ *and all* $\eta \in \mathrm{Cl}(\mathscr{V}, q)$*. The product defined in Equation* (3.12) *is referred to as* ***Clifford multiplication****.* ♢

Definition 3.29 (Reducible and irreducible representations of a Clifford algebra)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF *and* $\rho : \mathrm{Cl}(\mathscr{V}, q) \rightarrow \mathrm{End}(\mathscr{W})$ *be a representation of the Clifford algebra* $\mathrm{Cl}(\mathscr{V}, q)$*. We say that* ρ *is* ***reducible*** *if* $\mathscr{W}$ *can be written as a nontrivial direct sum*

$$\mathscr{W} \ = \ \mathscr{W}_1 \oplus \mathscr{W}_2$$

such that

$$\rho(\eta)(\mathscr{W}_1) \ \subseteq \ \mathscr{W}_1 \qquad \text{and} \qquad \rho(\eta)(\mathscr{W}_2) \ \subseteq \ \mathscr{W}_2$$

for all $\eta \in \mathrm{Cl}(\mathscr{V}, q)$*. In this situation, we have a decomposition of the representation given by*

$$\rho \ = \ \rho\,|_{\mathscr{W}_1} \oplus \rho\,|_{\mathscr{W}_2},$$

where

$$\rho\,|_{\mathscr{W}_i}(\eta) \ := \ \rho(\eta)\,|_{\mathscr{W}_i}$$

for all $\eta \in \mathrm{Cl}(\mathscr{V}, q)$ *and all* $i \in \{1, 2\}$*. In turn, we say that* ρ *is* ***irreducible*** *if it is not reducible. In other words,* ρ *is said to be irreducible if it does not admit proper invariant subspaces.* ♢

Theorem 3.30 (Irreducible representations play a fundamental role)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF*. Every representation* $\rho : \mathrm{Cl}(\mathscr{V}, q) \to \mathrm{End}(\mathscr{W})$ *of* $\mathrm{Cl}(\mathscr{V}, q)$ *can be decomposed into a direct sum*

$$\rho \;=\; \bigoplus_{i=1}^{m} \rho_i : \mathrm{Cl}(\mathscr{V}, q) \to \mathrm{End}(\mathscr{W}_i)$$

where ρ_i *is an irreducible representation of* $\mathrm{Cl}(\mathscr{V}, q)$ *for each* i *between* 1 *and* m*, both included.*

Proof. We only have to prove the statement if ρ is reducible. In this situation, as we have seen in Definition 3.29, ρ can be decomposed into a direct sum $\rho = \rho_1 \oplus \rho_2$. If either ρ_1 or ρ_2 are reducible, then ρ can be further decomposed. The essential fact is that this process must stop because $\mathscr{W}$ is a finite-dimensional $\mathrm{Cl}(\mathscr{V}, q)$-module. This finishes the proof of the theorem. □

Definition 3.31 (Equivalence of representations of a Clifford algebra)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF*. Two representations*

$$\rho_1 : \mathrm{Cl}(\mathscr{V}, q) \to \mathrm{End}(\mathscr{W}_1) \qquad \textit{and} \qquad \rho_2 : \mathrm{Cl}(\mathscr{V}, q) \to \mathrm{End}(\mathscr{W}_2)$$

of $\mathrm{Cl}(\mathscr{V}, q)$ *are said to be* ***equivalent*** *if there exists a linear isomorphism* $F : \mathscr{W}_1 \to \mathscr{W}_2$ *such that the diagram*

$$\begin{tikzcd} \mathscr{W}_1 \arrow[r, "\rho_1(\eta)"] \arrow[d, "F"'] & \mathscr{W}_1 \arrow[d, "F"] \\ \mathscr{W}_2 \arrow[r, "\rho_2(\eta)"'] & \mathscr{W}_2 \end{tikzcd}$$

is commutative for all $\eta \in \mathrm{Cl}(\mathscr{V}, q)$*. This defines an equivalence relation on the set of representations of* $\mathrm{Cl}(\mathscr{V}, q)$*.* ♢

Remark 3.32 (Irreducible representations of the Clifford algebras). *From Section 3.2, we have that every Clifford algebra* $\mathrm{Cl}(a,b)$ *is of the form* $\mathbb{K}(2^m)$ *or* $\mathbb{K}(2^m) \oplus \mathbb{K}(2^m)$ *for* $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$. *The representation theory of such algebras is particularly simple. Indeed, one can prove that:*

- *the natural representation*

$$\rho_{2^m} : \mathbb{K}(2^m) \to \mathrm{End}(\mathbb{K}^{2^m})$$

given by

$$\rho_{2^m}(A)(v) := Av$$

for all $v \in \mathbb{K}^{2^m}$, *is the only irreducible representation of the matrix algebra* $\mathbb{K}(2^m)$ *up to equivalence; and*

- *the algebra* $\mathbb{K}(2^m) \oplus \mathbb{K}(2^m)$ *has two inequivalent irreducible representations up to equivalence, which are*

$$\begin{aligned} \rho^1_{2^m} : \mathbb{K}(2^m) \oplus \mathbb{K}(2^m) &\to \mathrm{End}(\mathbb{K}^{2^m}), \\ (A,\, B) &\mapsto \rho_{2^m}(A), \end{aligned}$$

and

$$\begin{aligned} \rho^2_{2^m} : \mathbb{K}(2^m) \oplus \mathbb{K}(2^m) &\to \mathrm{End}(\mathbb{K}^{2^m}), \\ (A,\, B) &\mapsto \rho_{2^m}(B). \end{aligned}$$

$\diamondsuit$

Theorem 3.33 (Number of inequivalent irreducible representations of a Clifford algebra). *We have the following facts.*

- *Let* a *and* b *be natural numbers. In addition, let* $\nu_{a,b}$ *be the number of inequivalent irreducible representations of the Clifford algebra* $\mathrm{Cl}(a,b)$. *Whenever* b *is equal to zero, for convenience, we shall write* ν_a *instead of* $\nu_{a,0}$. *Under these conditions, we have*

$$\nu_{a,b} = \begin{cases} 2 & \text{if } a - b \equiv 3 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

- *Let n be a natural number. In addition, let $\nu_n^{\mathbb{C}}$ be the number of inequivalent irreducible representations of the Clifford algebra $\mathbb{C}l(n)$. Under these circumstances, we have*

$$\nu_n^{\mathbb{C}} = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. This result is an immediate consequence of the classification of Clifford algebras shown in Section 3.2. □

Theorem 3.34 (More information on some Clifford algebras)**.** *Let n be a natural number. In addition, let:*

- *d_n be the dimension of any irreducible* $\mathrm{Cl}(n)$*-module;*
- *$d_n^{\mathbb{C}}$ be the dimension of any irreducible* $\mathbb{C}l(n)$*-module;*
- *$\mathfrak{M}_n$ be the Grothendieck group of equivalence classes of irreducible representations of the Clifford algebra* $\mathrm{Cl}(n)$*; and*
- *$\mathfrak{M}_n^{\mathbb{C}}$ be the Grothendieck group of equivalence classes of irreducible representations of the Clifford algebra* $\mathbb{C}l(n)$*.*

For n between 1 and 8, both included, the elements ν_n, $\nu_n^{\mathbb{C}}$, d_n, $d_n^{\mathbb{C}}$, $\mathfrak{M}_n$ and $\mathfrak{M}_n^{\mathbb{C}}$ are as in Table 3.2.

n	$\mathrm{Cl}(n)$	ν_n	d_n	$\mathfrak{M}_n$	$\mathbb{C}l(n)$	$\nu_n^{\mathbb{C}}$	$d_n^{\mathbb{C}}$	$\mathfrak{M}_n^{\mathbb{C}}$
1	$\mathbb{C}$	1	2	$\mathbb{Z}$	$\mathbb{C}\oplus\mathbb{C}$	2	1	$\mathbb{Z}\oplus\mathbb{Z}$
2	$\mathbb{H}$	1	4	$\mathbb{Z}$	$\mathbb{C}(2)$	1	2	$\mathbb{Z}$
3	$\mathbb{H}\oplus\mathbb{H}$	2	4	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	2	2	$\mathbb{Z}\oplus\mathbb{Z}$
4	$\mathbb{H}(2)$	1	8	$\mathbb{Z}$	$\mathbb{C}(4)$	1	4	$\mathbb{Z}$
5	$\mathbb{C}(4)$	1	8	$\mathbb{Z}$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	2	4	$\mathbb{Z}\oplus\mathbb{Z}$
6	$\mathbb{R}(8)$	1	8	$\mathbb{Z}$	$\mathbb{C}(8)$	1	8	$\mathbb{Z}$
7	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	2	8	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	2	8	$\mathbb{Z}\oplus\mathbb{Z}$
8	$\mathbb{R}(16)$	1	16	$\mathbb{Z}$	$\mathbb{C}(16)$	1	16	$\mathbb{Z}$

Table 3.2: This table contains the Clifford algebras $\mathrm{Cl}(n)$ and $\mathbb{C}l(n)$ as well as the elements ν_n, $\nu_n^{\mathbb{C}}$, d_n, $d_n^{\mathbb{C}}$, $\mathfrak{M}_n$ and $\mathfrak{M}_n^{\mathbb{C}}$ for n between 1 and 8, both included.

Furthermore, for n greater than 8, these elements can be easily computed from

$$\begin{aligned}
\nu_{m+8k} &= \nu_m \\
\nu^{\mathbb{C}}_{m+2k} &= \nu^{\mathbb{C}}_m \\
d_{m+8k} &= 2^{4k} d_m \\
d^{\mathbb{C}}_{m+2k} &= 2^k d^{\mathbb{C}}_m \\
\mathfrak{M}_{m+8k} &\simeq \mathfrak{M}_m \\
\mathfrak{M}^{\mathbb{C}}_{m+2k} &\simeq \mathfrak{M}^{\mathbb{C}}_m,
\end{aligned}$$

where m and k are nonzero natural numbers.

Proof. This result is an immediate consequence of Corollary 3.24 and Theorem 3.27. We leave the details to the reader. □

Remark 3.35 (On an object defined in the preceding theorem)**.** *Let n be a natural number and $\mathscr{M}_n$ be the set of equivalence classes of irreducible representations of* $\mathrm{Cl}(n)$*. In the preceding theorem, we defined $\mathfrak{M}_n$ to be the Grothendieck group of $\mathscr{M}_n$. We have the following facts.*

- *The binary operation on $\mathscr{M}_n$ is the direct sum. More explicitly, if $[\rho], [\sigma] \in \mathscr{M}_n$, then*

$$[\rho] \oplus [\sigma] := [\rho \oplus \sigma].$$

  The reader can prove that this definition makes sense by showing that it does not depend on any representing element.

- *Two elements $[[\rho]]$ and $[[\sigma]]$ of $\mathfrak{M}_n$ coincide if and only if there exists $[\theta] \in \mathscr{M}_n$ for which*

$$[\rho \oplus \theta] = [\rho] \oplus [\theta] = [\sigma] \oplus [\theta] = [\sigma \oplus \theta].$$

- *Since any representation can be decomposed into a direct sum of irreducible ones (Theorem 3.30), we have that it naturally corresponds to an element in $\mathfrak{M}_n$ (with positive coefficients).*

It is to be noted that, mutatis mutandis, the same observations hold true in the complex case. ♢

Remark 3.36 (Tensor product of irreducible representations). *Let n be a natural number. We have the following facts.*

- *The tensor product of irreducible representations of $\mathrm{Cl}(n)$ and $\mathrm{Cl}(8)$ gives an irreducible representation of $\mathrm{Cl}(n+8) \simeq \mathrm{Cl}(n) \otimes \mathrm{Cl}(8)$.*
- *The tensor product of irreducible representations of $\mathbb{C}l(n)$ and $\mathbb{C}l(2)$ gives an irreducible representation of $\mathbb{C}l(n+2) \simeq \mathbb{C}l(n) \otimes \mathbb{C}l(2)$.*

Nevertheless, in general, $\mathrm{Cl}(n) \otimes \mathrm{Cl}(m)$ and $\mathbb{C}l(n) \otimes \mathbb{C}l(m)$ are not Clifford algebras. Hence, to find a multiplicative structure in the representations of Clifford algebras, we consider $\mathbb{Z}_2$-graded modules. $\diamondsuit$

Definition 3.37 (The category of $\mathbb{Z}_2$-graded modules for a Clifford algebra). *Let n be a natural number. We define the **category of $\mathbb{Z}_2$-graded $\mathrm{Cl}(n)$-modules**, and denote it by $\mathbb{Z}_2\mathrm{Mod}_n$, to be the category whose:*

- *objects are $\mathbb{Z}_2$-graded $\mathrm{Cl}(n)$-modules $(\mathscr{W}, \mathscr{W}^0, \mathscr{W}^1)$. More explicitly, we have that an object is a $\mathrm{Cl}(n)$-module $\mathscr{W}$ equipped with a decomposition $\mathscr{W} = \mathscr{W}^0 \oplus \mathscr{W}^1$ such that*
$$\mathrm{Cl}^i(n) \cdot \mathscr{W}^j \subseteq \mathscr{W}^{i+j}$$
for $i, j \in \mathbb{Z}_2$; and
- *morphisms are $\mathrm{Cl}(n)$-module homomorphisms $\varphi : \mathscr{W}_1 \to \mathscr{W}_2$ for which $\varphi(\mathscr{W}_1^0) \subseteq \mathscr{W}_2^0$ and $\varphi(\mathscr{W}_1^1) \subseteq \mathscr{W}_2^1$, which are usually denoted by $\varphi : (\mathscr{W}_1, \mathscr{W}_1^0, \mathscr{W}_1^1) \to (\mathscr{W}_2, \mathscr{W}_2^0, \mathscr{W}_2^1)$, as expected.*

Note that, mutatis mutandis, we can define the category of $\mathbb{Z}_2$-graded $\mathbb{C}l(n)$-modules $\mathbb{Z}_2\mathrm{Mod}_n^{\mathbb{C}}$. $\diamondsuit$

Theorem 3.38 (An equivalence of categories involving $\mathbb{Z}_2\mathrm{Mod}_n$). *Let n be a natural number and Mod_{n-1} be the category of $\mathrm{Cl}(n-1)$-modules. There exists an equivalence between $\mathbb{Z}_2\mathrm{Mod}_n$ and Mod_{n-1}. The same is true considering the categories $\mathbb{Z}_2\mathrm{Mod}_n^{\mathbb{C}}$ and $\mathrm{Mod}_{n-1}^{\mathbb{C}}$.*

Proof. Let $(\mathscr{W}, \mathscr{W}^0, \mathscr{W}^1)$ be an object in $\mathbb{Z}_2\mathrm{Mod}_n$. It is immediate to see that $\mathscr{W}^0$ is a $\mathrm{Cl}^0(n)$-module. Therefore, since one can prove that $\mathrm{Cl}^0(n)$ is canonically isomorphic to $\mathrm{Cl}(n-1)$, it follows that $\mathscr{W}^0$ is a $\mathrm{Cl}(n-1)$-module. Thence, it makes all sense to define

$$\begin{aligned} \Phi : \mathbb{Z}_2\mathrm{Mod}_n &\to \mathrm{Mod}_{n-1}, \\ (\mathscr{W}, \mathscr{W}^0, \mathscr{W}^1) &\mapsto \mathscr{W}^0, \\ \varphi : (\mathscr{W}_1, \mathscr{W}_1^0, \mathscr{W}_1^1) \to (\mathscr{W}_2, \mathscr{W}_2^0, \mathscr{W}_2^1) &\mapsto \varphi\,|_{\mathscr{W}_1^0} : \mathscr{W}_1^0 \to \mathscr{W}_2^0. \end{aligned}$$

Moreover, we define

$$\begin{aligned} \Psi : \mathrm{Mod}_{n-1} &\to \mathbb{Z}_2\mathrm{Mod}_n, \\ \mathscr{W} &\mapsto (\mathrm{Cl}(n) \otimes_{\mathrm{Cl}^0(n)} \mathscr{W},\, \mathrm{Cl}^0(n) \otimes_{\mathrm{Cl}^0(n)} \mathscr{W},\, \mathrm{Cl}^1(n) \otimes_{\mathrm{Cl}^0(n)} \mathscr{W}), \\ \varphi &\mapsto \mathrm{id}_{\mathrm{Cl}(n)} \otimes_{\mathrm{Cl}^0(n)} \varphi. \end{aligned}$$

The reader can readily prove that Φ and Ψ are equivalences of categories inverse to each other. $\square$

Remark 3.39 (The $\mathbb{Z}_2$-graded tensor product)**.** *Let n and m be natural numbers. In addition, let $(\mathscr{W}_1, \mathscr{W}_1^0, \mathscr{W}_1^1)$ be an object in $\mathbb{Z}_2\mathrm{Mod}_n$ and $(\mathscr{W}_2, \mathscr{W}_2^0, \mathscr{W}_2^1)$ be an object in $\mathbb{Z}_2\mathrm{Mod}_m$. We set*

$$\mathscr{W}_1 \mathbin{\widehat{\otimes}} \mathscr{W}_2 := (\mathscr{W}_1 \otimes \mathscr{W}_2,\, \mathscr{W}_1^0 \otimes \mathscr{W}_2^0 + \mathscr{W}_1^1 \otimes \mathscr{W}_2^1,\, \mathscr{W}_1^0 \otimes \mathscr{W}_2^1 + \mathscr{W}_1^1 \otimes \mathscr{W}_2^0),$$

which is an object in $\mathbb{Z}_2\mathrm{Mod}_{n+m}$ with respect to the action of $\mathrm{Cl}(n) \mathbin{\widehat{\otimes}} \mathrm{Cl}(m)$ on $\mathscr{W}_1 \otimes \mathscr{W}_2$ given by

$$(w_1 \otimes w_2) \cdot (w_3 \otimes w_4) := (-1)^{\deg(w_2)\deg(w_3)} (w_1 w_3) \otimes (w_2 w_4).$$

Here the degree of an element is the obvious one induced by the decompositions of $\mathscr{W}_1$ and $\mathscr{W}_2$. We left implicit the canonical isomorphism between $\mathrm{Cl}(n) \mathbin{\widehat{\otimes}} \mathrm{Cl}(m)$ and $\mathrm{Cl}(n+m)$. Once and again, all of these notions are still true considering the complex framework. $\diamondsuit$

Theorem 3.40 (Another Grothendieck group in the context of Clifford algebras)**.** *Let n and m be natural numbers. In analogy with the groups $\mathfrak{M}_n$ and $\mathfrak{M}_n^{\mathbb{C}}$ defined in* *the statement of Theorem 3.34, we define* $\widehat{\mathfrak{M}}_n$ *and* $\widehat{\mathfrak{M}}_n^{\mathbb{C}}$ *to be the Grothendieck groups of* $\mathbb{Z}_2$*-graded* $\mathrm{Cl}(n)$*-modules and* $\mathbb{C}l(n)$*-modules, respectively. Because of Theorem 3.38, we have*

$$\widehat{\mathfrak{M}}_n \simeq \mathfrak{M}_{n-1} \qquad \textit{and} \qquad \widehat{\mathfrak{M}}_n^{\mathbb{C}} \simeq \mathfrak{M}_{n-1}^{\mathbb{C}}. \tag{3.13}$$

Moreover, we have natural pairings

$$\begin{aligned} \widehat{\mathfrak{M}}_n \otimes_{\mathbb{Z}} \widehat{\mathfrak{M}}_m &\to \widehat{\mathfrak{M}}_{n+m} \\ \widehat{\mathfrak{M}}_n^{\mathbb{C}} \otimes_{\mathbb{Z}} \widehat{\mathfrak{M}}_m^{\mathbb{C}} &\to \widehat{\mathfrak{M}}_{n+m}^{\mathbb{C}} \end{aligned}$$

induced by the $\mathbb{Z}_2$*-graded tensor product. We have that these pairings are associative. Thus,*

$$\widehat{\mathfrak{M}} := \bigoplus_{i\in\mathbb{N}} \widehat{\mathfrak{M}}_i \qquad \textit{and} \qquad \widehat{\mathfrak{M}}^{\mathbb{C}} := \bigoplus_{i\in\mathbb{N}} \widehat{\mathfrak{M}}_i^{\mathbb{C}}$$

have the structure of graded rings.

Proof. This result is an immediate consequence of Theorem 3.38 and Remark 3.39. We leave the details to the reader. □

3.4 The Atiyah-Bott-Shapiro Theorem

In this section, we present the Atiyah-Bott-Shapiro Theorem. This result was originally proved in [3]. In our text, it will be mainly considered when we study the Thom isomorphisms in K-Theory. Roughly speaking, the theorem in question will ensure us the existence of Thom classes. We begin with the theorem itself, although its statement does not make sense at a first glance, and then we explain the ideas that are behind it. For a complete proof, we recommend to the reader the original reference mentioned above.

Theorem 3.41 (The Atiyah-Bott-Shapiro Theorem). *Let* Ω *be a one-point space. We define*

$$K^{-*}(\Omega) := \bigoplus_{i\in\mathbb{N}} K(\mathbb{D}^i, \mathbb{S}^{i-1})$$

where $\mathbb{D}^i$ *and* $\mathbb{S}^{i-1}$ *are the closed unit disc and sphere in* $\mathbb{R}^i$*, respectively. There exists a graded ring homomorphism*

$$\varphi:\ \widehat{\mathfrak{M}}^{\mathbb{C}}/\tau(\widehat{\mathfrak{M}}^{\mathbb{C}})\ \to\ K^{-*}(\Omega).$$

Note that, mutatis mutandis, we obtain a graded ring homomorphism involving the real K-Theory.

Proof. Let n be a natural number. Here we use the notations of Section 2.9. For each $(\mathscr{W},\mathscr{W}^0,\mathscr{W}^1)\in\mathbb{Z}_2\mathrm{Mod}_n^{\mathbb{C}}$, we define

$$\varphi_n(\mathscr{W},\mathscr{W}^0,\mathscr{W}^1)\ :=\ [\mathbb{D}^n\times\mathscr{W}^0,\,\mathbb{D}^n\times\mathscr{W}^1,\mu_n]\ \in\ K(\mathbb{D}^n,\,\mathbb{S}^{n-1}),$$

where

$$\begin{aligned}\mu_n:\ \mathbb{S}^n\times\mathscr{W}^0\ &\to\ \mathbb{S}^n\times\mathscr{W}^1,\\ (x,\,w)\ &\mapsto\ (x,\,x\cdot w).\end{aligned}$$

We have that the map

$$\begin{aligned}\mathbb{Z}_2\mathrm{Mod}_n^{\mathbb{C}}\ &\to\ K(\mathbb{D}^n,\,\mathbb{S}^{n-1}),\\ (\mathscr{W},\mathscr{W}^0,\mathscr{W}^1)\ &\mapsto\ \varphi_n(\mathscr{W},\mathscr{W}^0,\mathscr{W}^1),\end{aligned}$$

is an additive homomorphism. Moreover, we have that $\varphi_n(\mathscr{W},\mathscr{W}^0,\mathscr{W}^1)$ depends only on the isomorphism class of the $\mathbb{Z}_2$-graded $\mathbb{C}l(n)$-module $(\mathscr{W},\mathscr{W}^0,\mathscr{W}^1)$. Therefore, we obtain a homomorphism

$$\begin{aligned}\widehat{\mathfrak{M}}_n^{\mathbb{C}}\ &\to\ K(\mathbb{D}^n,\,\mathbb{S}^{n-1}),\\ [\mathscr{W},\mathscr{W}^0,\mathscr{W}^1]\ &\mapsto\ \varphi_n(\mathscr{W},\mathscr{W}^0,\mathscr{W}^1).\end{aligned}$$

Now, we consider

$$\begin{aligned}i_n:\ \mathbb{R}^n\ &\to\ \mathbb{R}^{n+1},\\ (x_1,\cdots,x_n)\ &\mapsto\ (x_1,\cdots,x_n,0),\end{aligned}$$

to be the natural inclusion. By Remark 3.8, this map induces a homomorphism of algebras $\mathbb{C}l(n)\to\mathbb{C}l(n+1)$. Restricting the action from $\mathbb{C}l(n+1)$ to $\mathbb{C}l(n)$ thereby induces a homomorphism $\tau_n:\widehat{\mathfrak{M}}_{n+1}^{\mathbb{C}}\to\widehat{\mathfrak{M}}_n^{\mathbb{C}}$. Thence, suppose that $(\mathscr{W},\mathscr{W}^0,\mathscr{W}^1)$ is a $\mathbb{Z}_2$-graded $\mathbb{C}l(n)$-module which can be obtained from a $\mathbb{C}l(n+1)$-module in the above fashion. This means that the Clifford multiplications of $\mathbb{R}^n$ on $\mathscr{W}$ extends to all of $\mathbb{R}^{n+1}$. As a consequence, if we set

$$\begin{aligned} \alpha : \mathbb{D}^n &\to [0,1], \\ x &\mapsto \sqrt{1-|x|^2}, \end{aligned}$$

then we may extend the isomorphism μ_n to all of $\mathbb{D}^n$ by setting

$$\begin{aligned} \bar{\mu}_n : \mathbb{D}^n \times \mathscr{W}^0 &\to \mathbb{D}^n \times \mathscr{W}^1, \\ (x, w) &\mapsto (x, x \cdot w + \alpha(x) e_{n+1} \cdot w), \end{aligned}$$

where e_{n+1} is a unit vector orthogonal to $\mathbb{R}^n$. Since $\bar{\mu}_n$ is an isomorphism, $\varphi_n(\mathscr{W}, \mathscr{W}^0, \mathscr{W}^1)$ must be zero. Therefore, the map $\widehat{\mathfrak{M}}_n^{\mathbb{C}} \to K(\mathbb{D}^n, \mathbb{S}^{n-1})$ defined above descends to a homomorphism

$$\varphi_n : \widehat{\mathfrak{M}}_n^{\mathbb{C}} / \tau_n(\widehat{\mathfrak{M}}_{n+1}^{\mathbb{C}}) \to K(\mathbb{D}^n, \mathbb{S}^{n-1}).$$

The second isomorphism in (3.13) ensures that

$$\widehat{\mathfrak{M}}_n^{\mathbb{C}} / \tau_n(\widehat{\mathfrak{M}}_{n+1}^{\mathbb{C}}) \simeq \mathfrak{M}_{n-1}^{\mathbb{C}} / \tau_n(\mathfrak{M}_n^{\mathbb{C}}).$$

Further, algebraic arguments show that

$$\mathfrak{M}_{n-1}^{\mathbb{C}} / \tau_n(\mathfrak{M}_n^{\mathbb{C}}) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ \mathbb{Z} & \text{if } n \text{ is even}. \end{cases}$$

Finally, we clarify the graded ring structures mentioned in the statement. Because of Theorem 3.40,

$$\widehat{\mathfrak{M}}^{\mathbb{C}} / \tau(\widehat{\mathfrak{M}}^{\mathbb{C}}) := \bigoplus_{i \in \mathbb{N}} \widehat{\mathfrak{M}}_i^{\mathbb{C}} / \tau_i(\widehat{\mathfrak{M}}_{i+1}^{\mathbb{C}})$$

is a graded ring. Moreover, since

$$K(\mathbb{D}^n, \mathbb{S}^{n-1}) \simeq \widetilde{K}(\mathbb{S}^n) \simeq K^{-n}(\Omega),$$

$K^{-*}(\Omega)$ is also a graded ring. Finally, note that the map $\varphi : \widehat{\mathfrak{M}}^{\mathbb{C}} / \tau(\widehat{\mathfrak{M}}^{\mathbb{C}}) \to K^{-*}(\Omega)$ in defined by

$$\bigoplus_{i \in \mathbb{N}} \varphi_i : \widehat{\mathfrak{M}}_i^{\mathbb{C}} / \tau_i(\widehat{\mathfrak{M}}_{i+1}^{\mathbb{C}}) \to K(\mathbb{D}^i, \mathbb{S}^{i-1}) \simeq K^{-i}(\Omega).$$

This finishes our exposition. $\square$

3.5 Pin and Spin groups

In this section, we present the notion that obliged us to develop the preceding study on Clifford algebras, namely, the Pin and Spin groups. These mathematical objects are of fundamental importance in this chapter. In fact, they are not only used in the study of spin and spin^c structures, but they are also applied in the study of Thom isomorphisms in K-Theory to ensure the existence of Thom classes. We begin with the following definition.

Definition 3.42 (Group of units of a Clifford algebra)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF. *We define*

$$\mathrm{Cl}^\times(\mathscr{V}, q) \;:=\; \{\eta \in \mathrm{Cl}(\mathscr{V}, q) \,:\, \eta\eta^{-1} = \eta^{-1}\eta = 1 \textit{ for some } \eta^{-1} \in \mathrm{Cl}(\mathscr{V}, q)\}.$$

This group is said to be the ***multiplicative group of units*** *in the Clifford algebra* $\mathrm{Cl}(\mathscr{V}, q)$. ♢

Remark 3.43 (Examples of elements in the multiplicative group of units of a Clifford algebra)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF. *We define*

$$\mathscr{V}_q^\times \;:=\; \{v \in \mathscr{V} \,:\, q(v) \textit{ is not zero}\}.$$

We have that $\mathscr{V}_q^\times$ *is contained in the multiplicative group of units* $\mathrm{Cl}^\times(\mathscr{V}, q)$. *Indeed, if* $v \in \mathscr{V}_q^\times$, *then*

$$v^{-1} \;=\; -\frac{v}{q(v)}.$$

In particular, if the symmetric bilinear form is an inner product, then every non-zero vector in $\mathscr{V}$ *is an element in* $\mathrm{Cl}^\times(\mathscr{V}, q)$. ♢

Theorem 3.44 (The adjoint representation)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF. *In addition, let* $\mathrm{GLCl}(\mathscr{V}, q)$ *be the group of automorphisms of* $\mathrm{Cl}(\mathscr{V}, q)$. *The* ***adjoint representation*** *of* $\mathrm{Cl}(\mathscr{V}, q)$ *is*

$$\begin{aligned} \mathrm{Ad} : \; \mathrm{Cl}^\times(\mathscr{V}, q) &\;\to\; \mathrm{GLCl}(\mathscr{V}, q), \\ \eta &\;\mapsto\; \mathrm{Ad}_\eta, \end{aligned}$$

where

$$\mathrm{Ad}_\eta(x) \;:=\; \eta x \eta^{-1}$$

for all $x \in \mathrm{Cl}(\mathscr{V}, q)$. *We have*

$$\mathrm{Ad}_v(\mathscr{V}) \;=\; \mathscr{V}$$

for all $v \in \mathscr{V}_q^{\times}$. *In fact, the reader can readily prove that this equality follows from the fact that*

$$-\mathrm{Ad}_v(w) \;=\; w - 2\,\frac{s_q(v,\, w)}{q(v)}\, v \tag{3.14}$$

for all $w \in \mathscr{V}$.

Proof. Being $v \in \mathscr{V}_q^{\times}$, we only have to prove that Equation (3.14) holds for all $w \in \mathscr{V}$. Indeed, since

$$v^{-1} \;=\; -\frac{v}{q(v)},$$

it follows from Equation (3.1) that

$$q(v)\,\mathrm{Ad}_v(w) \;=\; q(v) v w v^{-1} \;=\; v^2 w + 2\, s_q(v,\, w)\, v \;=\; -q(v) w + 2\, s_q(v,\, w)\, v,$$

as we wished. □

Remark 3.45 (Consequences of the preceding result)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF. *We are lead by Theorem 3.44 to consider the subgroup of elements* $\eta \in \mathrm{Cl}^{\times}(\mathscr{V}, q)$ *for which*

$$\mathrm{Ad}_\eta(\mathscr{V}) \;=\; \mathscr{V}.$$

We have that $\mathscr{V}_q^{\times}$ *is contained in such a subgroup because of Theorem 3.44. Moreover, if* $v \in \mathscr{V}_q^{\times}$, *then it follows from Equation* (3.14) *that*

$$(q \circ \mathrm{Ad}_v)(w) \;=\; q(w)$$

for all $w \in \mathscr{V}$. *Under these circumstances, we define* $\mathrm{P}(\mathscr{V}, q)$ *to be the subgroup of the multiplicative group of units* $\mathrm{Cl}^{\times}(\mathscr{V}, q)$ *generated by the elements of* $\mathscr{V}_q^{\times}$. *More explicitly, we have that an element of* $\mathrm{P}(\mathscr{V}, q)$ *is a product* $v_1 \cdots v_r \in \mathrm{Cl}(\mathscr{V}, q)$ *such that* $v_1, \cdots, v_r \in \mathscr{V}_q^{\times}$. *Further,*

$$\mathrm{Ad}: \mathrm{P}(\mathscr{V}, q) \rightarrow \mathrm{O}(\mathscr{V}, q)$$

is a representation, where

$$\mathrm{O}(\mathscr{V}, q) := \{\varphi \in \mathrm{GL}(\mathscr{V}) : q \circ \varphi = q\}$$

is the ***orthogonal group*** *of* $(\mathscr{V}, q)$*. The following definition presents the most important subgroups of* $\mathrm{P}(\mathscr{V}, q)$*.* ♢

Definition 3.46 (The Pin and Spin groups)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF*. We give the following definitions.*

- *The* Pin ***group*** *of* $(\mathscr{V}, q)$ *is the subgroup* $\mathrm{Pin}(\mathscr{V}, q)$ *of* $\mathrm{P}(\mathscr{V}, q)$ *generated by the elements* $v \in \mathscr{V}$ *for which*
$$q(v) = 1 \qquad \text{or} \qquad q(v) = -1.$$
- *The* Spin ***group*** *of* $(\mathscr{V}, q)$ *is the subgroup* $\mathrm{Spin}(\mathscr{V}, q)$ *of* $\mathrm{Pin}(\mathscr{V}, q)$ *defined as the group intersection*
$$\mathrm{Spin}(\mathscr{V}, q) := \mathrm{Pin}(\mathscr{V}, q) \cap \mathrm{Cl}^0(\mathscr{V}, q).$$

More explicitly, we have that an element of $\mathrm{Pin}(\mathscr{V}, q)$ *is a product* $v_1 \cdots v_r \in \mathrm{P}(\mathscr{V}, q)$ *in such manner that* $q(v_i) = 1$ *or* $q(v_i) = -1$ *for all* i *between* 1 *and* r*, both included. In addition, an element of* $\mathrm{Spin}(\mathscr{V}, q)$ *is a product* $v_1 \cdots v_r \in \mathrm{Pin}(\mathscr{V}, q)$ *such that* r *is an even number.* ♢

Remark 3.47 (The twisted adjoint representation)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF*. The right-hand side of Equation* (3.14) *coincides with the map* $\rho_v : \mathscr{V} \rightarrow \mathscr{V}$ *given by the reflection across the hyperplane*

$$v^{\perp} := \{w \in \mathscr{V} : s_q(v, w) = 0\}.$$

We have that ρ_v *fixes* $v^{\perp}$ *and maps* v *into* $-v$*. Unfortunately, there is a minus sign on the left-hand side of Equation* (3.14)*. This defect can be removed considering the* ***twisted adjoint representation***

$$\begin{aligned}\widetilde{\mathrm{Ad}}:\ \mathrm{Cl}^\times(\mathscr{V},q) &\to \mathrm{GLCl}(\mathscr{V},q),\\ \eta &\mapsto \widetilde{\mathrm{Ad}}_\eta,\end{aligned}$$

where

$$\widetilde{\mathrm{Ad}}_\eta(x) := \tau(\eta)x\eta^{-1}$$

for all $x \in \mathrm{Cl}(\mathscr{V},q)$. *In the preceding formula,* $\tau : \mathrm{Cl}(\mathscr{V},q) \to \mathrm{Cl}(\mathscr{V},q)$ *is the unique extension of the linear map* $\alpha : \mathscr{V} \to \mathscr{V}$ *defined in Remark 3.15. The reader can readily prove that*

- *If* $\eta, \theta \in \mathrm{Cl}^\times(\mathscr{V},q)$, *then*
$$\widetilde{\mathrm{Ad}}_{\eta\theta} = \widetilde{\mathrm{Ad}}_\eta \circ \widetilde{\mathrm{Ad}}_\theta.$$
- *if* $\eta \in \mathrm{Cl}^\times(\mathscr{V},q) \cap \mathrm{Cl}^0(\mathscr{V},q)$, *then*
$$\widetilde{\mathrm{Ad}}_\eta = \mathrm{Ad}_\eta.$$

Furthermore, if $v \in \mathscr{V}_q^{\times}$, *then*

$$\widetilde{\mathrm{Ad}}_v(w) = w - 2\,\frac{s_q(v,w)}{q(v)}\,v \tag{3.15}$$

for all $w \in \mathscr{V}$. *This is an immediate consequence of Equation* (3.14) *since, for all* $v \in \mathscr{V}$, *we have*

$$\tau(v) = \alpha(v) = -v.$$

Under these conditions, we define the subgroup of $\mathrm{Cl}^\times(\mathscr{V},q)$

$$\widetilde{\mathrm{P}}(\mathscr{V},q) := \{\eta \in \mathrm{Cl}^\times(\mathscr{V},q) : \widetilde{\mathrm{Ad}}_\eta(\mathscr{V}) = \mathscr{V}\}^{(1)}.$$

Note that $\mathrm{P}(\mathscr{V},q) \subseteq \widetilde{\mathrm{P}}(\mathscr{V},q) \subseteq \mathrm{Cl}^\times(\mathscr{V},q)$ *because of Theorem 3.44. In [23, p. 16], it is proved that* $\widetilde{\mathrm{Ad}}_\eta : \mathscr{V} \to \mathscr{V}$ *preserves the quadratic form* q *for every* $\eta \in \widetilde{\mathrm{P}}(\mathscr{V},q)$. *As a consequence, we have a homomorphism*

$$\widetilde{\mathrm{Ad}}:\ \widetilde{\mathrm{P}}(\mathscr{V},q) \to \mathrm{O}(\mathscr{V},q).$$

---

(1) One can prove that $\widetilde{\mathrm{P}}(\mathscr{V},q)$ is not so different from $\mathrm{P}(\mathscr{V},q)$. In fact, either $\widetilde{\mathrm{P}}(\mathscr{V},q) = \mathrm{P}(\mathscr{V},q)$ or the quotient of $\widetilde{\mathrm{P}}(\mathscr{V},q)$ by $\mathrm{P}(\mathscr{V},q)$ is isomorphic to $\mathbb{Z}_2$.

In particular, we obtain a homomorphism

$$\widetilde{\mathrm{Ad}} : \mathrm{P}(\mathscr{V}, q) \to \mathrm{O}(\mathscr{V}, q)$$

such that

$$\widetilde{\mathrm{Ad}}_{v_1 \cdots v_r} = \rho_{v_1} \circ \cdots \circ \rho_{v_r}, \tag{3.16}$$

where ρ_{v_i} *is the reflection across* $v_i^{\perp}$ *for all* i *between* 1 *and* r*, both included. Thus, the image of* $\mathrm{P}(\mathscr{V}, q)$ *under* $\widetilde{\mathrm{Ad}}$ *is the subgroup of* $\mathrm{O}(\mathscr{V}, q)$ *generated by the reflections. Hence, because of the Cartan-Dieudonné Theorem, the image of* $\mathrm{P}(\mathscr{V}, q)$ *under* $\widetilde{\mathrm{Ad}}$ *is the whole* $\mathrm{O}(\mathscr{V}, q)$. ♢

Theorem 3.48 (The special orthogonal group)**.** *Let* $(\mathscr{V}, q)$ *be an object in* VectQF*. We define*

$$\mathrm{SP}(\mathscr{V}, q) := \mathrm{P}(\mathscr{V}, q) \cap \mathrm{Cl}^0(\mathscr{V}, q).$$

More explicitly, an element of $\mathrm{SP}(\mathscr{V}, q)$ *is a product* $v_1 \cdots v_r \in \mathrm{P}(\mathscr{V}, q)$ *such that* r *is an even number. Moreover, since* $\mathscr{V}$ *is finite-dimensional, we define the* ***special orthogonal group*** *of* $(\mathscr{V}, q)$

$$\mathrm{SO}(\mathscr{V}, q) := \{\varphi \in \mathrm{O}(\mathscr{V}, q) : \det(\varphi) = 1\}.$$

We have that

$$\widetilde{\mathrm{Ad}} : \mathrm{SP}(\mathscr{V}, q) \to \mathrm{SO}(\mathscr{V}, q)$$

is surjective.

Proof. Initially, note that

$$\det(\rho_v) = -1$$

for all $v \in \mathscr{V} - \{0\}$. In order to prove this claim, let $\{v_1, \cdots, v_{\dim(\mathscr{V})}\}$ be a basis for $\mathscr{V}$ in such manner that $v_1 = v$ and $s_q(v, v_i) = 0$ for i between 2 and $\dim(\mathscr{V})$, both included[2]. Therefore, by definition of the reflection ρ_v, we have $\rho_v(v_1) = -v_1$

[2] The existence of such a basis is easily proved. Indeed, since v is a nonzero vector in $\mathscr{V}$, we can find $\dim(\mathscr{V}) - 1$ vectors in $\mathscr{V}$ that, together with v, form a basis for $\mathscr{V}$. Then, applying the usual *Gram-Schmidt Process*, we can turn these $\dim(\mathscr{V})$ vectors into new vectors that are q-orthogonal, keeping v intact, as desired.

and $\rho(v_i) = v_i$ for all i between 2 and $\dim(\mathscr{V})$, both included. As a consequence, we have $\det(\rho_v) = -1$, as claimed. Hence, because of the *Cartan-Dieudonné Theorem*, we have

$$\mathrm{SO}(\mathscr{V}, q) = \{\rho_{v_1} \circ \cdots \circ \rho_{v_r} : v_i \in \mathscr{V}_q^{\times} \text{ and } r \text{ is even}\}.$$

Thence, $\widetilde{\mathrm{Ad}} : \mathrm{SP}(\mathscr{V}, q) \to \mathrm{SO}(\mathscr{V}, q)$ is surjective because of Equation (3.16). This finishes the proof of the theorem. □

Remark 3.49 (On the preceding result)**.** *Let $(\mathscr{V}, q)$ be an object in* VectQF*. In light of Remark 3.47 and Theorem 3.48, it is natural to ask whether the homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{P}(\mathscr{V}, q) \to \mathrm{O}(\mathscr{V}, q)$ restricted to* $\mathrm{Pin}(\mathscr{V}, q)$ *and* $\mathrm{Spin}(\mathscr{V}, q)$ *maps onto* $\mathrm{O}(\mathscr{V}, q)$ *and* $\mathrm{SO}(\mathscr{V}, q)$*, respectively. In fact, it seems likely since, at a first glance, we have the equality*

$$\rho_{tv} = \rho_v$$

for all $t \in \mathbb{K} - \{0\}$. Therefore, we have that one should be able to normalize any $v \in \mathscr{V}_q^{\times}$ to have q-length equal to 1 *or* -1*. Of course, since q is quadratic, we have the equation*

$$q(tv) = t^2\, q(v).$$

Clearly, at least one of the equations

$$t^2 = \frac{1}{q(v)} \qquad \textit{and} \qquad t^2 = -\frac{1}{q(v)}$$

are solvable in $\mathbb{K}$[(3)]*. This is the main property used in the proof of the following theorem, which is the principal result of this section.* ♢

Theorem 3.50 (The twisted adjoint representation restricted to the Pin and Spin groups)**.** *Let $(\mathscr{V}, q)$ be an object in* VectQF*. If q is non-degenerate, then there exist short exact sequences*

$$0 \longrightarrow \Omega \longrightarrow \mathrm{Pin}(\mathscr{V}, q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(\mathscr{V}, q) \longrightarrow 0$$

(3) Note that both equations are well-defined since $q(v)$ is not zero because $v \in \mathscr{V}_q^{\times}$. Thence, if $\mathbb{K}$ is the field of complex numbers, then both equations are solvable. In turn, if $\mathbb{K}$ is the field of real numbers, then only one of them is solvable.

and

$$0 \longrightarrow \Omega \longrightarrow \mathrm{Spin}(\mathscr{V}, q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(\mathscr{V}, q) \longrightarrow 0$$

where

$$\Omega = \begin{cases} \mathbb{Z}_2 = \{1, -1\} & \text{if} \quad \mathbb{K} = \mathbb{R}, \\ \mathbb{Z}_4 = \{1, -1, i, -i\} & \text{if} \quad \mathbb{K} = \mathbb{C}. \end{cases}$$

Proof. The first two maps of the preceding sequences are inclusions. Thus, in order to prove exactness, we just have to show that:

- Ω *is the kernel of* $\widetilde{\mathrm{Ad}}$. Indeed, if $\eta \in \mathrm{Pin}(\mathscr{V}, q)$ is in the kernel of $\widetilde{\mathrm{Ad}}$, then it is proved in [23, p. 19] that $\eta^2 = 1$ or $\eta^2 = -1$. This establishes the kernel of $\widetilde{\mathrm{Ad}}$ in both cases, as above.

- $\widetilde{\mathrm{Ad}}$ *is surjective.* This claim follows immediately from Remark 3.49 because of the *Cartan-Dieudonné Theorem.*

This finishes the proof of the theorem. $\square$

In order to close this section, we examine the real case of Pin and Spin groups in more detail. This is done because the information that is needed in the next sections are mainly obtained from the real Spin groups. We begin with the following definition.

Definition 3.51 (The groups of the Euclidean space equipped with the canonical real quadratic forms)**.** *Let n be a natural number. For each natural numbers a and b such that $n = a + b$, we define*

$$\begin{aligned} \mathrm{O}(a, b) &:= \mathrm{O}(\mathbb{R}^n,\, q^n_{a,b}) \\ \mathrm{SO}(a, b) &:= \mathrm{SO}(\mathbb{R}^n,\, q^n_{a,b}) \\ \mathrm{Pin}(a, b) &:= \mathrm{Pin}(\mathbb{R}^n,\, q^n_{a,b}) \\ \mathrm{Spin}(a, b) &:= \mathrm{Spin}(\mathbb{R}^n,\, q^n_{a,b}) \\ \mathrm{P}(a, b) &:= \mathrm{P}(\mathbb{R}^n,\, q^n_{a,b}) \\ \widetilde{\mathrm{P}}(a, b) &:= \widetilde{\mathrm{P}}(\mathbb{R}^n,\, q^n_{a,b}). \end{aligned}$$

Here $q^n_{a,b} : \mathbb{R}^n \to \mathbb{R}$ *is the canonical real quadratic form presented in Definition 3.17. For convenience, we denote* $\mathrm{O}(n,0) \simeq \mathrm{O}(0,n)$ *simply by* $\mathrm{O}(n)$*, and* $\mathrm{SO}(n,0) \simeq \mathrm{SO}(0,n)$ *simply by* $\mathrm{SO}(n)$*. Finally, it is to be noted that, since we have the equality* $q^n_{a,b} = -q^n_{b,a}$*,* $\mathrm{P}(a,b) = \widetilde{\mathrm{P}}(a,b)$. $\diamondsuit$

Lemma 3.52 (The fundamental group of the special orthogonal groups)**.** *We have the following facts.*

(1) The fundamental group of $\mathrm{SO}(1)$ *is trivial.*

(2) The fundamental group of $\mathrm{SO}(2)$ *is isomorphic to* $\mathbb{Z}$*.*

(3) The fundamental group of $\mathrm{SO}(n)$ *is isomorphic to* $\mathbb{Z}_2$ *for all* $n \in \mathbb{N} - \{0,1,2\}$*.*

Proof. Indeed:

(1) we have that $\mathrm{SO}(1)$ is the trivial group. Thus, it is clear that its fundamental group is also trivial.

(2) we have that $\mathrm{SO}(2)$ is homeomorphic to the unit circle $\mathbb{S}^1$. Thus, it is clear that its fundamental group is isomorphic to $\mathbb{Z}$.

(3) we consider the fibration $\pi : \mathrm{SO}(n+1) \to \mathbb{S}^n$ with fiber $\mathrm{SO}(n)$ defined as follows. Let us fix a point in $\mathbb{S}^n$, for example, $e_{n+1} \in \mathbb{R}^{n+1}$. For each $A \in \mathrm{SO}(n+1)$, we define

$$\pi(A) := A\, e_{n+1} \in \mathbb{S}^n.$$

The map π is surjective since $\mathrm{SO}(n+1)$ acts transitively on $\mathbb{S}^n$[(4)]. Moreover, let $u \in \mathbb{S}^n$ and $A \in \mathrm{SO}(n+1)$ be such that $Ae_{n+1} = u$. We have that the elements of $\pi^{-1}(u)$ are of the form $A' = RA$, where R is a rotation that fixes e_{n+1}. Hence, $R \in \mathrm{SO}(n)$. Evidently, this proves that the fiber over u is

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[(4)] In order to prove that $\mathrm{SO}(n+1)$ acts transitively on $\mathbb{S}^n$, it suffices to show that, for each $u, v \in \mathbb{S}^n$, there exists $A \in \mathrm{SO}(n+1)$ such that $Au = v$. This is a simple task to be done. Indeed, we can choose n vectors in $\mathbb{R}^{n+1}$ that, together with u, form a basis $\mathcal{A}$ for $\mathbb{R}^{n+1}$. Analogously, we can find n vectors in $\mathbb{R}^{n+1}$ that, together with v, form a basis $\mathcal{B}$ for $\mathbb{R}^{n+1}$. These bases can be taken orthonormal because, if it is not the case, then we can apply the *Gram-Schmidt Process* in order to ensure the property in question. Moreover, we have that $\mathcal{A}$ and $\mathcal{B}$ can be taken positively-oriented with their first elements being u and v, respectively. Therefore, the automorphism A of $\mathbb{R}^{n+1}$ that sends $\mathcal{A}$ into $\mathcal{B}$ is the desired element of $\mathrm{SO}(n+1)$.

diffeomorphic to $\mathrm{SO}(n)$ for all $u \in \mathbb{S}^n$. Finally, note that π is locally trivial. Indeed, if U is an open neighborhood of u, then there exists a smooth function $A : U \to \mathrm{SO}(n+1)$ such that

$$A(x)(e_{n+1}) \;=\; x$$

for all $x \in U$. Thence, an element of $\pi^{-1}(x)$ has the form $A'(x) = RA(x)$, where $R \in \mathrm{SO}(n)$. This produces the local chart

$$\begin{aligned} \varphi :\ \pi^{-1}(U) \;&\to\; U \times \mathrm{SO}(n), \\ RA(x) \;&\mapsto\; (x,\, R). \end{aligned}$$

As a consequence, we can consider the long exact sequence in homotopy that is associated to the fibration $\pi : \mathrm{SO}(n+1) \to \mathbb{S}^n$. This sequence contains the exact sequence

$$0 \;=\; \pi_2(\mathbb{S}^n) \longrightarrow \pi_1\mathrm{SO}(n) \longrightarrow \pi_1\mathrm{SO}(n+1) \longrightarrow \pi_1(\mathbb{S}^n) \;=\; 0.$$

This proves that

$$\pi_1\mathrm{SO}(n) \;\simeq\; \pi_1\mathrm{SO}(n+1).$$

Thus, we only have to calculate the fundamental group of $\mathrm{SO}(3)$. In fact, we have $\pi_1\mathrm{SO}(3) \simeq \mathbb{Z}_2$. In order to prove this, let $R_{u,\theta}$ denote the rotation around the axis $u \in \mathbb{R}^3$ of angle $\theta \in [0, 2\pi]$, according to the right-hand rule. In addition, let $\mathbb{D}^3$ be the closed unit disc in $\mathbb{R}^3$. Thence, being $|\cdot| : \mathbb{R}^3 \to [0, \infty)$ the usual Euclidean norm, we define

$$\begin{aligned} \Phi :\ \mathbb{D}^3 \;&\to\; \mathrm{SO}(3), \\ v \;&\mapsto\; \begin{cases} \mathrm{id}_{\mathbb{R}^3} & \text{if} \quad v = 0, \\ R_{v,\,\pi|v|} & \text{otherwise.} \end{cases} \end{aligned}$$

The reader can readily prove that Φ is continuous. Moreover, it is surjective since any rotation can be achieved by fixing an axis and a rotation angle. Further, we have that $\Phi\,|_{\mathbb{D}^3 - \mathbb{S}^2} \colon \mathbb{D}^3 - \mathbb{S}^2 \to \mathrm{SO}(3)$ is injective. Indeed, if $u, v \in \mathbb{D}^3 - \mathbb{S}^2$, then:

- if u and v form a linearly independent family in $\mathbb{R}^3$, then $\Phi(u)$ is different from $\Phi(v)$ since these rotations have different axes; *and*
- if $u = \lambda v$, then $\Phi(u)$ is different from $\Phi(v)$ since these rotations have different rotation angles. In fact, the rotation angles are the same only if $\lambda = 1$, which implies $u = v$.

In turn, if $v \in \mathbb{S}^2$, then

$$\Phi(v) = \Phi(-v).$$

More precisely, given distinct vectors $u, v \in \mathbb{D}^3$, we have that $\Phi(u) = \Phi(v)$ if and only if $u, v \in \mathbb{S}^2$ and $u = -v$. Because of that, Φ projects to a homeomorphism between the real projective space $\mathbb{RP}^3$ and $\mathrm{SO}(3)$. As a consequence, it follows that $\pi_1\mathrm{SO}(3)$ is isomorphic to $\pi_1(\mathbb{RP}^3)$. Luckily, it is a well-know fact that $\pi_1(\mathbb{RP}^3)$ is isomorphic to $\mathbb{Z}_2$.

This finishes the proof of the lemma. $\square$

Remark 3.53 (Important consequences of the preceding lemma)**.** *In Lemma 3.52, we have seen that $\pi_1\mathrm{SO}(2) \simeq \mathbb{Z}$ and $\pi_1\mathrm{SO}(n) \simeq \mathbb{Z}_2$ for all natural number n greater than 2. Therefore, there exists a unique nontrivial two-sheeted covering of $\mathrm{SO}(n)$ for all $n \in \mathbb{N} - \{0, 1\}$. Furthermore, this two-sheeted covering is the universal covering for all n greater than 2. The map $z \mapsto z^2$ is the nontrivial two-sheeted covering of $\mathrm{SO}(2)$. The other cases are treated by the following result, which is the main theorem in the real case of* Pin *and* Spin *groups.* $\diamondsuit$

Theorem 3.54 (The Pin and Spin groups and two-sheeted coverings of the orthogonal group and of the special orthogonal group)**.** *Let n be a natural number. For each natural numbers a and b in such manner that $n = a + b$, we have that there exist short exact sequences*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(a, b) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(a, b) \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(a, b) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(a, b) \longrightarrow 0$$

Furthermore, if a and b are not both equal to 1, then these two-sheeted coverings are nontrivial over each connected component of $\mathrm{O}(a,b)$*. In particular, in the special case*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(n) \longrightarrow 0,$$

we have that the map $\widetilde{\mathrm{Ad}}$ *represents the universal covering homomorphism of* $\mathrm{SO}(n)$ *for all* $n \in \mathbb{N} - \{0,1,2\}$.

Proof. The short exact sequences in the statement are consequences of Theorem 3.50. In turn, in order to prove that the two-sheeted coverings are nontrivial, it suffices to join 1 and -1 by a continuous path in $\mathrm{Spin}(a,b)$. In fact, since a and b are not both equal to 1, there exist orthogonal vectors $u, v \in \mathbb{R}^n$ such that $q^n_{a,b}(u) = q^n_{a,b}(v) = 1$ or $q^n_{a,b}(u) = q^n_{a,b}(v) = -1$. We set

$$\begin{aligned} \gamma :\ [0,1] &\ \rightarrow\ \mathrm{Spin}(a,b), \\ t &\ \mapsto\ \left[u \cos\left(\frac{\pi t}{2}\right) + v \sin\left(\frac{\pi t}{2}\right)\right]\left[v \sin\left(\frac{\pi t}{2}\right) - u \cos\left(\frac{\pi t}{2}\right)\right]. \end{aligned}$$

As a consequence:

- if $q^n_{a,b}(u) = q^n_{a,b}(v) = 1$, then

  $$\gamma(0) = -u^2 = q^n_{a,b}(u) \cdot 1 = 1$$

  and

  $$\gamma(1) = v^2 = -q^n_{a,b}(v) \cdot 1 = -1;$$

- if $q^n_{a,b}(u) = q^n_{a,b}(v) = -1$, then

  $$\gamma(0) = -u^2 = q^n_{a,b}(u) \cdot 1 = -1$$

  and

  $$\gamma(1) = v^2 = -q^n_{a,b}(v) \cdot 1 = 1.$$

This finishes the proof of the theorem. $\square$

3.6 Spin structures

In this section, we start the process of using the algebraic notions presented in this chapter to study the geometry of vector bundles. In fact, we establish a fundamental notion to study the Thom isomorphisms, namely, the spin structures of oriented Euclidean vector bundles. We begin by remembering some ideas with the following remark.

Remark 3.55 (On real vector bundles). *Let $\pi : E \to X$ be a real vector bundle with typical fiber $\mathscr{V}$. We remind the reader of the following notions.*

- *We say that E is an Euclidean vector bundle if it is equipped with an inner product*

$$\langle\, , \rangle :\ E \oplus E \to \mathbb{R},$$

  which is a continuous function that restricts in each fiber to an inner product. Because of Theorem C.49, it is always possible to turn a real vector bundle into an Euclidean vector bundle.

- *We say that E is orientable if it can be equipped with an orientation. If it is equipped with an orientation, then we say that E is oriented. In turn, in order to define an orientation of E, let $\mathscr{O}_x$ be an orientation of $E_x = \pi^{-1}(x)$ for each $x \in X$. Thence,*

$$\mathscr{O} := \{\mathscr{O}_x\}_{x\in X}$$

  is said to be an orientation of E provided that, for a fixed orientation of $\mathscr{V}$ and for each $x \in X$, there exists a local chart (U_x, φ_x) of E in x such that the linear isomorphism $\varphi_x \mid_{E_y}: E_y \to \{y\} \times \mathscr{V}$ is orientation preserving for all $y \in U_x$. Finally, we have that E is orientable if and only if it admits an oriented atlas. An oriented atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$ of E is an atlas for which the transition functions $\varphi_{ij} : U_{ij} \to \mathrm{GL}(\mathscr{V})$ are such that $\det \varphi_{ij}(x)$ is positive for all $x \in U_{ij}$ and all $i, j \in I$. It is not always possible to equip E with an orientation. For example, a line bundle is orientable if and only if it is trivial. The following result establishes a necessary and sufficient condition for the orientability of a vector bundle. ♢

Notation 3.56 (On Čech cohomology). *Hereafter, we use the notation and the ideas of Čech cohomology that are established in [9, pp. 37-47]. Since the subject in question is widely known and [9] gives a fairly complete and didactic approach for it, we do not elaborate our own details about this topic. Instead, we focus on its applications to our context.* ♢

Lemma 3.57 (Orientability of vector bundles). *Let $\pi : E \to X$ be a real vector bundle with typical fiber $\mathscr{V}$. We assume that there exists an atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$ such that U_{ij} is connected for all $i, j \in I$. Being $\{\varphi_{ij} : U_{ij} \to \mathrm{GL}(\mathscr{V})\}_{i,j\in I}$ the set of transition functions of $\Phi_{\mathfrak{U}}$, we define*

$$\epsilon_{ij} \;:=\; \mathrm{sgn}(\det\varphi_{ij}) \;=\; \begin{cases} 1 & \textit{if} \quad \det\varphi_{ij} \;>\; 0, \\ -1 & \textit{if} \quad \det\varphi_{ij} \;<\; 0. \end{cases}$$

We set

$$\omega_1(E) \;:=\; [\{\epsilon_{ij}\}_{i,j\in I}] \;\in\; \check{H}^1(X; \mathbb{Z}_2).$$

This is the ***first Stiefel-Whitney class*** *of E. We have that E is orientable if and only if $\omega_1(E)$ is trivial. In other words, the first Stiefel-Whitney class measures the obstruction to orientability.*

Proof. We remind the reader that $\mathrm{GL}(\mathscr{V})$ has exactly two connected components, which are

$$\begin{aligned} \mathrm{GL}^+(\mathscr{V}) \;&:=\; \{\varphi \in \mathrm{GL}(\mathscr{V}) \,:\, \det(\varphi) > 0\} \quad \text{and} \\ \mathrm{GL}^-(\mathscr{V}) \;&:=\; \{\psi \in \mathrm{GL}(\mathscr{V}) \,:\, \det(\psi) < 0\}. \end{aligned}$$

This ensures that ϵ_{ij} is well-defined for all $i, j \in I$. Indeed, since U_{ij} is connected and φ_{ij} is continuous, $\varphi_{ij}(U_{ij}) \subseteq \mathrm{GL}(\mathscr{V})$ is connected. Therefore,

$$\varphi_{ij}(U_{ij}) \;\subseteq\; \mathrm{GL}^+(\mathscr{V}) \qquad \text{or} \qquad \varphi_{ij}(U_{ij}) \;\subseteq\; \mathrm{GL}^-(\mathscr{V}).$$

In both cases, we have that the sign of the determinant of φ_{ij} is constant, as we wished. Furthermore, we have that $\omega_1(E)$ is well-defined in the Čech cohomology group $\check{H}^1(X; \mathbb{Z}_2)$ because:

- $\{\epsilon_{ij}\}_{i,j\in I}$ *is a cocycle.* In fact,

$$\begin{aligned}\epsilon_{ij}\epsilon_{jk}\epsilon_{ki} &= \operatorname{sgn}(\det\varphi_{ij})\operatorname{sgn}(\det\varphi_{jk})\operatorname{sgn}(\det\varphi_{ki})\\ &= \operatorname{sgn}(\det\varphi_{ij}\varphi_{jk}\varphi_{ki})\\ &= \operatorname{sgn}(\det\mathrm{id}_{\mathscr{V}})\\ &= 1\end{aligned}$$

  for all $i,j,k\in I$. As a consequence, we obtain $\check{\delta}^1\{\epsilon_{ij}\}_{i,j\in I} = 1$, which proves our assertion.

- $\omega_1(E)$ *only depends on* E. Indeed, let $\Psi_{\mathfrak{U}} = \{(U_i,\psi_i)\}_{i\in I}$ be another atlas of E based on $\mathfrak{U}$. Being $\{\psi_{ij} : U_{ij}\to \mathrm{GL}(\mathscr{V})\}_{i,j\in I}$ the set of transition functions of $\Psi_{\mathfrak{U}}$, we set $\epsilon'_{ij} := \operatorname{sgn}(\det\psi_{ij})$. We claim that

$$[\{\epsilon_{ij}\}_{i,j\in I}] \;=\; [\{\epsilon'_{ij}\}_{i,j\in I}] \in \check{H}^1(X;\mathbb{Z}_2).$$

  This happens because, since $\Phi_{\mathfrak{U}}$ and $\Psi_{\mathfrak{U}}$ are atlases of E based on the same open cover $\mathfrak{U}$ of X, we know that there exists a family $\{\eta_i : U_i\to \mathrm{GL}(\mathscr{V})\}_{i\in I}$ in such manner that

$$(\psi_{ij})_x \;=\; (\eta_j)_x\circ(\varphi_{ij})_x\circ(\eta_i)_x^{-1}$$

  for all $x\in U_{ij}$ and all $i,j\in I$. Hence,

$$\det(\psi_{ij})_x \;=\; \det(\eta_j)_x\,\det(\varphi_{ij})_x\,\det(\eta_i)_x^{-1}$$

  for all $x\in U_{ij}$ and all $i,j\in I$. Consequently,

$$\operatorname{sgn}(\det\psi_{ij}) \;=\; \operatorname{sgn}(\det\eta_j)\operatorname{sgn}(\det\varphi_{ij})\operatorname{sgn}(\det\eta_i)^{-1}$$

  for all $i,j\in I$. In other words, if we set $\nu_i := \operatorname{sgn}(\det\eta_i)$ for all $i\in I$, then we obtain $\epsilon'_{ij} = \nu_j\,\epsilon_{ij}\,\nu_i^{-1}$ for all $i,j\in I$. Since $\{\nu_i\}_{i\in I}\in \check{C}^0(\mathfrak{U};\mathbb{Z}_2)$, we are done here.

Finally, let us prove the last part of the statement. If E is orientable, then we can choose an oriented atlas $\Phi_{\mathfrak{U}} = \{(U_i,\varphi_i)\}_{i\in I}$ with U_{ij} being connected for all $i,j\in I$. Thence, since

$$\epsilon_{ij} \;=\; \operatorname{sgn}(\det \varphi_{ij}) \;=\; 1$$

for all $i, j \in I$, it follows that $\omega_1(E)$ is trivial. Conversely, we assume that $\omega_1(E)$ is trivial. Let $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i \in I}$ be any atlas of E and $\varphi : \mathscr{V} \to \mathscr{V}$ be any orientation reversing automorphism of $\mathscr{V}$. Since $\omega_1(E)$ is trivial, we have that there exists a family $\{\nu_i\}_{i \in I} \in \check{C}^0(\mathfrak{U}; \mathbb{Z}_2)$ for which $\epsilon_{ij} = \nu_j \nu_i^{-1}$ for all $i, j \in I$. Under these circumstances, we set

$$\varphi_i' \;:=\; \begin{cases} \varphi_i & \text{if} \quad \nu_i \;=\; 1, \\ \varphi_i \circ \varphi & \text{if} \quad \nu_i \;=\; -1, \end{cases}$$

for each $i \in I$. The reader can prove that $\Phi'_{\mathfrak{U}} := \{(U_i, \varphi_i')\}_{i \in I}$ is an oriented atlas. This finishes the proof of the lemma. □

Remark 3.58 (A class of orientable vector bundles)**.** *Let X be a simply connected paracompact Hausdorff space. Every real vector bundle on X is orientable. This happens because*

$$H^1(X, \mathbb{Z}_2) \;\simeq\; \operatorname{Hom}(H_1(X), \mathbb{Z}_2)$$

is trivial. In fact, since $H_1(X)$ is the abelianization of the fundamental group of X, which is trivial once X is simply connected, $\operatorname{Hom}(H_1(X), \mathbb{Z}_2)$ *is trivial. Moreover, since X is paracompact Hausdorff,*

$$\check{H}^1(X; \mathbb{Z}_2) \;\simeq\; H^1(X, \mathbb{Z}_2).$$

Therefore, the first Stiefel-Whitney class of any real vector bundle can only be trivial. Evidently, we are tacitly restricting the real vector bundles to the ones which admit an atlas as in Lemma 3.57. ◇

Definition 3.59 (Spin structure)**.** *Let $\pi : E \to X$ be an n-dimensional oriented Euclidean vector bundle. In addition, let $\pi_{\mathrm{SO}} : \mathrm{SO}(E) \to X$ be the $\mathrm{SO}(n)$-principal bundle of oriented orthonormal frames of E. We say that a **spin structure** on E is a $\mathrm{Spin}(n)$-principal bundle $\pi_{\mathrm{Spin}} : \mathrm{Spin}(E) \to X$ equipped with a two-sheeted covering $\xi : \mathrm{Spin}(E) \to \mathrm{SO}(E)$ such that, if $p_n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the projection, then (ξ, p_n) is a morphism of principal bundles over X. In this situation, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Spin}(E)\times\mathrm{Spin}(n) \arrow[rr, "\xi\times p_n"] \arrow[d, "\cdot"'] & & \mathrm{SO}(E)\times\mathrm{SO}(n) \arrow[d, "\cdot"] \\
\mathrm{Spin}(E) \arrow[rr, "\xi"] \arrow[dr, "\pi_{\mathrm{Spin}}"'] & & \mathrm{SO}(E) \arrow[dl, "\pi_{\mathrm{SO}}"] \\
& X &
\end{tikzcd}$$

♢

Remark 3.60 (On principal bundles)**.** *The notions of principal bundles that we have invoked in the preceding definition can be found in Appendix F. The only one that demands further commentaries is the* $\mathrm{SO}(n)$*-principal bundle of oriented orthonormal frames of* E*. Indeed, assuming* $\pi : E \to X$ *to be an* n*-dimensional oriented Euclidean vector bundle,* $\mathrm{SO}(E)$ *can be described as follows. Let*

$$\{\varphi_{ij} : U_{ij} \to \mathrm{SO}(n)\}_{i,j\in I}$$

be the set of transition functions of an atlas $\Phi_{\mathfrak{U}}$ *of* E*. In addition, consider the disjoint union*

$$D^{\mathfrak{U}}_{\mathrm{SO}(n)} := \bigsqcup_{i\in I} U_i \times \mathrm{SO}(n).$$

If $x \in U_{ij}$ *and* $\varphi \in \mathrm{SO}(n)$*, then we denote by* $(x,\varphi)_i$ *the pair* $(x,\varphi) \in U_i \times \mathrm{SO}(n)$ *and by* $(x,\varphi)_j$ *the pair* $(x,\varphi) \in U_j \times \mathrm{SO}(n)$*. We define* $\mathrm{SO}(E)$ *as the quotient of* $D^{\mathfrak{U}}_{\mathrm{SO}(n)}$ *by the equivalence relation that identifies* $(x,\varphi)_i$ *with* $(x, (\varphi_{ij})_x \circ \varphi)_j$ *for all* $(x,\varphi) \in U_{ij} \times \mathrm{SO}(n)$ *and all* $i,j \in I$*. We have*

$$\begin{aligned}
\pi_{\mathrm{SO}} :\ \mathrm{SO}(E) &\to X, \\
[(x,\,\varphi)_i] &\mapsto x,
\end{aligned}$$

as well as the $\mathrm{SO}(n)$*-right action*

$$\begin{aligned}
\cdot :\ \mathrm{SO}(E)\times\mathrm{SO}(n) &\to \mathrm{SO}(E), \\
([(x,\,\varphi)_i],\ \psi) &\mapsto [(x,\,\varphi\circ\psi)_i].
\end{aligned}$$

Apparently, $\mathrm{SO}(n)$ *depends on the atlas* $\Phi_{\mathfrak{U}}$ *of* E*. However, this is not the case since* $\mathrm{SO}(E)$ *can be defined in an equivalent manner that does not depend on* $\Phi_{\mathfrak{U}}$*. In fact, we can also define* $\mathrm{SO}(E)$ *to be the* $\mathrm{SO}(n)$*-principal bundle whose fibers are the sets of orientation-preserving orthogonal maps. The reader may deduce the details by studying the analogous ones that were developed about the frame bundle of a vector bundle in Section F.7.* ♢

Definition 3.61 (Equivalence of spin structures)**.** *Let* $\pi : E \to X$ *be an* n*-dimensional oriented Euclidean vector bundle. Two spin structures* $\xi : \mathrm{Spin}(E) \to \mathrm{SO}(E)$ *and* $\xi' : \mathrm{Spin}'(E) \to \mathrm{SO}(E)$ *on* E *are said to be* ***equivalent*** *if there exists an isomorphism of* $\mathrm{Spin}(n)$*-principal bundles* $\varphi : \mathrm{Spin}(E) \to \mathrm{Spin}'(E)$ *such that the following diagram is commutative.*

$$\begin{tikzcd} \mathrm{Spin}(E) \arrow[r, "\xi"'] \arrow[rr, bend left, "\varphi"] & \mathrm{SO}(E) & \mathrm{Spin}'(E) \arrow[l, "\xi'"] \end{tikzcd}$$ ♢

Theorem 3.62 (Existence of spin structures)**.** *Let* $\pi : E \to X$ *be an* n*-dimensional oriented Euclidean vector bundle. We have that there exists a spin structure on* E *if and only if there exist:*

(1) an atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$ *where* $\{\varphi_{ij} : U_{ij} \to \mathrm{SO}(n)\}_{i,j\in I}$ *is its set of transition functions; and*

(2) a set of liftings of the transition functions $\{s_{ij} : U_{ij} \to \mathrm{Spin}(n)\}_{i,j\in I}$ *in such manner that*

$$s_{ki}|_{U_{ijk}} \cdot s_{jk}|_{U_{ijk}} \cdot s_{ij}|_{U_{ijk}} = 1 \tag{3.17}$$

for all $i, j, k \in I$*. We remind the reader that* s_{ij} *being a lifting of* φ_{ij} *means that the following diagram is commutative.*

$$\begin{tikzcd} U_{ij} \arrow[r, "\varphi_{ij}"'] \arrow[rr, bend left, "s_{ij}"] & \mathrm{SO}(n) & \mathrm{Spin}(n) \arrow[l, "p_n"] \end{tikzcd} \tag{3.18}$$

Proof. If there exists a spin structure $\xi : \mathrm{Spin}(E) \to \mathrm{SO}(E)$ on E, then we consider an atlas $\Psi_{\mathfrak{U}} = \{(U_i, s_i)\}_{i\in I}$ of $\pi_{\mathrm{Spin}} : \mathrm{Spin}(E) \to X$ whose set of transition functions is $\{s_{ij} : U_{ij} \to \mathrm{Spin}(n)\}_{i,j\in I}$. Evidently, Equation (3.17) is verified since it involves transition functions of a principal bundle. Moreover, we have that there exists an atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$ of $\mathrm{SO}(E)$ whose transition functions $\varphi_{ij} : U_{ij} \to \mathrm{SO}(E)$ verify the commutativity of Diagram (3.18)[(5)]. Conversely, we assume that Conditions (1) and (2) of the statement are verified. We know that $\Phi_{\mathfrak{U}}$ determines an isomorphism between $\mathrm{SO}(E)$ and the quotient of the disjoint union

$$\bigsqcup_{i\in I} U_i \times \mathrm{SO}(n)$$

by the equivalence relation that identifies $(x, \varphi)_i$ with $(x,\, \varphi_{ij}(x) \circ \varphi)_j$ for all $x \in U_{ij}$ and all $i, j \in I$. This suggests how to construct a spin structure. Indeed, let $\mathrm{Spin}(E)$ be the quotient of the disjoint union

$$\bigsqcup_{i\in I} U_i \times \mathrm{Spin}(n)$$

by the equivalence relation that identifies $(x, s)_i$ with $(x,\, s_{ij}(x) \cdot s)_j$ for all $x \in U_{ij}$ and all $i, j \in I$. We have that

$$\begin{aligned} \xi :\ \mathrm{Spin}(E) &\to \mathrm{SO}(E), \\ [(x,\, s)] &\mapsto [(x,\, p_n(s))], \end{aligned}$$

is a spin structure on E. The reader can prove this claim since Equation (3.17) ensures that $\mathrm{Spin}(E)$ is a $\mathrm{Spin}(n)$-principal bundle. □

[(5)]Let $\pi_P : P \to X$ and $\pi_Q : Q \to X$ be principal bundles with structure groups G and H, respectively. In addition, let $(\varphi : P \to Q, \rho : G \to H)$ be a morphism of principal bundles over X and $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$ be an atlas of P defined by the local sections in $\{s_i : U_i \to P\}_{i\in I}$. The local sections in

$$\{r_i := \varphi \circ s_i : U_i \to Q\}_{i\in I}$$

define an atlas $\Psi_{\mathfrak{U}} = \{(U_i, \psi_i)\}_{i\in I}$ of Q. Moreover, if $\{\varphi_{ij} : U_{ij} \to G\}_{i\in I}$ is the set of transition functions of $\Phi_{\mathfrak{U}}$, then

$$\{\psi_{ij} := \rho \circ \varphi_{ij} : U_{ij} \to H\}_{i\in I}$$

is the set of transition functions of $\Psi_{\mathfrak{U}}$. The reader can prove this result by defining $\psi_i : \pi_Q^{-1}(U_i) \to U_i \times H$, $\varphi \circ s_i(x) \cdot h \mapsto (x, h)$. Furthermore, the reader can readily deduce the claim of the main text from this theorem.

Remark 3.63 (Local behavior of spin structures). *Let X be a paracompact Hausdorff space and $\pi : E \to X$ be an n-dimensional oriented Euclidean vector bundle. In addition, let $\{\varphi_{ij} : U_{ij} \to \mathrm{SO}(n)\}_{i,j\in I}$ be the set of transition functions of an atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$ of E. Up to a refinement of $\Phi_{\mathfrak{U}}$, we can find a set of liftings for the transition functions*

$$\{s_{ij} : U_{ij} \to \mathrm{Spin}(n)\}_{i,j\in I}$$

such that $p_n \circ s_{ij} = \varphi_{ij}$. Indeed, since X is paracompact Hausdorff, we can find a locally finite refinement of $\Phi_{\mathfrak{U}}$. Thus, being $x \in X$, we consider $U_{i_1}, \cdots, U_{i_m}$ to be the elements of $\mathfrak{U}$ containing x. For each k and h between 1 and m, both included, we choose a neighborhood V_{kh} of $\varphi_{i_k i_h}(x)$ in $\mathrm{SO}(n)$ such that $p_n^{-1}(V_{kh})$ is the disjoint union of two open sets homeomorphic to V_{kh}. We set

$$U_x := \bigcap_{k,h=1}^{m} \varphi_{i_k i_h}^{-1}(V_{kh}).$$

Moreover, for all $x \in X$, we choose a function φ_{i_k} for k between 1 and m, both included. We set

$$\varphi_x := \varphi_{i_k}|_{U_x}.$$

This gives us an atlas $\{(U_x, \varphi_x)\}_{x\in X}$ of E for which every transition function admits a lifting to $\mathrm{Spin}(n)$*. Furthermore, any spin structure on U_x is trivial because the only spin structure on $U_x \times \mathrm{SO}(n)$, up to equivalence, is $U_x \times \mathrm{Spin}(n)$. In other words,* $\mathrm{Spin}(E)|_{U_x}$ *is equivalent to $U_x \times \mathrm{Spin}(n)$. This information characterizes the local behavior of the spin structures.* ♢

Remark 3.64 (The second Stiefel-Whitney class in the framework of vector bundles). *Let X be a paracompact Hausdorff space and $\pi : E \to X$ be an n-dimensional oriented Euclidean vector bundle. In addition, being $\Phi_{\mathfrak{U}}$ an atlas of E, let $\{s_{ij} : U_{ij} \to \mathrm{Spin}(n)\}_{i,j\in I}$ be a set of liftings for its transition functions $\{\varphi_{ij} : U_{ij} \to \mathrm{SO}(n)\}_{i,j\in I}$ such that we have $p_n \circ s_{ij} = \varphi_{ij}$ for all $i, j \in I$. Since*

$$\left(\varphi_{ki}|_{U_{ijk}}\right)_x \circ \left(\varphi_{jk}|_{U_{ijk}}\right)_x \circ \left(\varphi_{ij}|_{U_{ijk}}\right)_x = 1$$

for all $x \in X$ an all $i, j, k \in I$, we have

$$s_{ki} \cdot s_{jk} \cdot s_{ij} \;=\; \epsilon_{ijk} \cdot 1,$$

where $\epsilon_{ijk} \in \mathbb{Z}_2 = \{1, -1\}$ *for all* $i, j, k \in I$. *Thus, we obtain* $\{\epsilon_{ijk}\}_{i,j,k \in I} \in \check{C}^2(\mathfrak{U}, \mathbb{Z}_2)$. *With some abuse of notation, we write*

$$\{\epsilon_{ijk}\}_{i,j,k \in I} \;=\; \check{\delta}^1 \{s_{ij}\}_{i,j \in I}.$$

Because of that, we have $\check{\delta}^2 \{\epsilon_{ijk}\}_{i,j,k \in I} \;=\; \check{\delta}^2 \check{\delta}^1 \{s_{ij}\}_{i,j \in I} \;=\; 0$. *Hence, it is well-defined* $[\{\epsilon_{ijk}\}_{i,j,k \in I}] \in H^2(\mathfrak{U}, \mathbb{Z}_2)$. *Considering the direct limit on the open coverings of* X, *we obtain*

$$[\{\epsilon_{ijk}\}_{i,j,k \in I}] \in H^2(X, \mathbb{Z}_2).$$

Now we claim that this cohomology class only depends on E. *Indeed, if we consider different liftings* $r_{ij} : U_{ij} \to \mathrm{Spin}(n)$ *with* $r_{ki} \cdot r_{jk} \cdot r_{ij} = \rho_{ijk} \cdot 1$ *for all* $i, j, k \in I$, *then it follows that*

$$r_{ij} \;=\; s_{ij}\, \epsilon_{ij}$$

where $\epsilon_{ij} \in \mathbb{Z}_2$ *for all* $i, j \in I$. *As a consequence,*

$$\rho_{ijk} \;=\; \epsilon_{ijk} \cdot \check{\delta}^1 \{\epsilon_{ij}\}_{i,j \in I}.$$

In other words,

$$[\{\rho_{ijk}\}_{i,j,k \in I}] \;=\; [\{\epsilon_{ijk}\}_{i,j,k \in I}]. \tag{3.19}$$

Moreover, if we choose an atlas of $\mathrm{SO}(E)$ *which produces transition functions* $\{\psi_{ij} : U_{ij} \to \mathrm{SO}(n)\}_{i,j \in I}$, *then*

$$\psi_{ij} \;=\; \eta_i \circ \varphi_{ij} \circ \eta_j^{-1}$$

for all $i, j \in I$. *Thus, choosing the liftings* $r_{ij} = \eta_j \circ s_{ij} \circ \eta_i^{-1}$, *we obtain*

$$\rho_{ijk} \cdot 1 \;=\; (\eta_i \circ s_{ki} \circ \eta_k^{-1}) \cdot (\eta_k \circ s_{jk} \circ \eta_j^{-1}) \cdot (\eta_j \circ s_{ij} \circ \eta_i^{-1}) \;=\; \epsilon_{ijk} \cdot 1$$

for all $i, j, k \in I$. *Therefore, we recover Equation* (3.19). *Finally, before the next definition, we remind the reader that, since* X *is paracompact Hausdorff,* $\check{H}^2(X; \mathbb{Z}_2)$ *is isomorphic to* $H^2(X, \mathbb{Z}_2)$. $\diamondsuit$

Definition 3.65 (The second Stiefel-Whitney class). *Let X be a paracompact Hausdorff space and $\pi : E \to X$ be an n-dimensional oriented Euclidean vector bundle. Using the notation of Remark 3.64, we set*

$$\omega_2(E) \;:=\; [\{\epsilon_{ijk}\}_{i,j,k\in I}] \in \check{H}^2(X;\mathbb{Z}_2) \;\simeq\; H^2(X,\mathbb{Z}_2).$$

This is the ***second Stiefel-Whitney class*** *of E. If X is a smooth manifold, then we define its second Stiefel-Whitney class $\omega_2(X)$ to be the second Stiefel-Whitney class of its tangent bundle.* ◇

Corollary 3.66 (Existence of spin structures through the second Stiefel-Whitney class). *Let X be a paracompact Hausdorff space and $\pi : E \to X$ be an n-dimensional oriented Euclidean vector bundle. We have that there exists a spin structure on E if and only if $\omega_2(E)$ is trivial.*

Proof. Here we use the notation of Remark 3.64. If there exists a spin structure on E, then it follows from Theorem 3.62 that we can choose $\epsilon_{ijk} = 1$ for all $i,j,k \in I$. As a consequence, it follows that $\omega_2(E)$ is trivial. Conversely, if $\omega_2(E)$ is trivial, then

$$s_{ki}\cdot s_{jk}\cdot s_{ij} \;=\; \epsilon_{ki}\,\epsilon_{jk}\,\epsilon_{ij}\cdot 1$$

for all $i,j,k \in I$. We set

$$r_{ij} \;:=\; \epsilon_{ij}\, s_{ij}$$

or all $i,j,k \in I$. Consequently, we have just obtained liftings $r_{ij} : U_{ij} \to \mathrm{Spin}(n)$ for $\varphi_{ij} : U_{ij} \to \mathrm{SO}(n)$ such that

$$r_{ki}\cdot r_{jk}\cdot r_{ij} \;=\; 1$$

for all $i,j,k \in I$. Thence, Theorem 3.62 ensures the existence of a spin structure on E, as desired. □

Remark 3.67 (On the preceding corollary). *Let X be a paracompact Hausdorff space and $\pi : E \to X$ be an n-dimensional oriented Euclidean vector bundle. The preceding result shows that the second Stiefel-Whitney class $\omega_2(E)$ measures the obstruction to the existence of a spin structure on E. In fact, Corollary 3.66 says that, if $\omega_2(E)$*

is trivial, then it is possible to find local liftings of the transition functions of $\mathrm{SO}(E)$ *to* $\mathrm{Spin}(n)$*, and multiply some of them by* -1*, in such manner that the cocycle condition holds. At the same time, this procedure cannot be done if* $\omega_2(E)$ *is not trivial. Putting it together with Remark 3.55 and Lemma 3.57, the following definition arises naturally.* $\diamondsuit$

Definition 3.68 (Spin bundle and spin manifold)**.** *Let* X *be a paracompact Hausdorff space and* $\pi : E \to X$ *be an oriented Euclidean vector bundle. We say that* E *is a* ***spin bundle*** *provided that* $\omega_1(E)$ *and* $\omega_2(E)$ *are both trivial. Furthermore, if* X *is a smooth manifold, then we say that it is a* ***spin manifold*** *if its tangent bundle is a spin bundle.* $\diamondsuit$

Theorem 3.69 (The number of inequivalent spin structures of a spin bundle)**.** *Let* $\pi : E \to X$ *be an* n*-dimensional spin bundle. The number of inequivalent spin structures on* E *is the order of* $H^1(X, \mathbb{Z}_2)$*.*

Proof. Let $\Phi_{\mathfrak{U}}$ be an atlas of $\mathrm{SO}(E)$ and $\{s_{ij} : U_{ij} \to \mathrm{Spin}(n)\}_{i\in I}$ be a set of liftings for its transition functions satisfying (3.17). The total space of the corresponding spin structure is the quotient of the disjoint union

$$\bigsqcup_{i\in I} U_i \times \mathrm{Spin}(n) \tag{3.20}$$

by the equivalence relation that identifies $(x, s)_i$ with $(x,\, s_{ij}(x) \cdot s)_j$ for all $x \in U_{ij}$ and all $i, j \in I$. Any other spin structure on E descends from a set of liftings $\{s_{ij}\epsilon_{ij} : U_{ij} \to \mathrm{Spin}(n)\}_{i,j\in I}$ where $\epsilon_{ij} \in \mathbb{Z}_2$ for all $i, j \in I$. Since the cocycle condition must be verified, we necessarily have $\check{\delta}^1\{\epsilon_{ij}\}_{i,j\in I} = 1$. Consequently, it is well-defined

$$[\{\epsilon_{ij}\}_{i,j\in I}] \in \check{H}^1(X, \mathbb{Z}_2). \tag{3.21}$$

We have that one of these spin structures is equivalent to the first if and only if (3.21) is trivial, which means that there exists $\{\epsilon_i\}_{i\in I} \in \check{C}^0(\mathfrak{U}, \mathbb{Z}_2)$ for which $\check{\delta}^0\{\epsilon_i\}_{i\in I} = \{\epsilon_{ij}\}_{i,j\in I}$. Indeed:

- if there exists an equivalence of spin structures ξ between the spin structures given by the quotient of the disjoint union in (3.20):

- by the equivalence relation that identifies $(x, s)_i$ with $(x,\, s_{ij}(x)\,\cdot\, s)_j$ for all $x \in U_{ij}$ and all $i, j \in I$; *and*
- by the equivalence relation that identifies $(x, s)_i$ with $(x,\, \epsilon_{ij}\, s_{ij}(x)\,\cdot\, s)_j$ for all $x \in U_{ij}$ and all $i, j \in I$,

then we have

$$\xi(x,\, s)_i \;=\; (x,\, \epsilon_i s)_i \tag{3.22}$$

where $\epsilon_i \in \mathbb{Z}_2$ for all $i \in I$. This happens because ξ commutes with the projections onto $\mathrm{SO}(E)$. Therefore,

$$(x,\, \epsilon_j s_{ij}(x)s)_j \;=\; \xi(x,\, s_{ij}(x)s)_j \;=\; \xi(x,\, s)_i \;=\; (x,\, \epsilon_i s)_i \;=\; (x,\, \epsilon_i \epsilon_{ij} s_{ij}(x)s)_j.$$

Consequently,

$$\epsilon_j \;=\; \epsilon_i\, \epsilon_{ij}$$

for all $i, j \in I$. Equivalently, $\epsilon_{ij} = \epsilon_j \epsilon_i^{-1}$ for all $i, j \in I$. This proves the triviality of (3.21), as desired; *and*

- if (3.21) is trivial, then the reader can readily prove that (3.22) defines an equivalence of spin structures, where $\{\epsilon_i\}_{i\in I}$ is any family in $\check{C}^0(\mathfrak{U},\, \mathbb{Z}_2)$ for which $\check{\delta}^0\{\epsilon_i\}_{i\in I} = \{\epsilon_{ij}\}_{i,j\in I}$.

Summarizing, let us fix a spin structure $\mathrm{Spin}(E)$ on E. Given $\alpha \in \check{H}^1(X,\, \mathbb{Z}_2)$, we have just proved that the spin structure $\mathrm{Spin}(E)\cdot\alpha$ is equivalent to $\mathrm{Spin}(E)$ if and only if $\alpha = 1$. As a consequence, we have that $\mathrm{Spin}(E)\cdot\alpha$ is equivalent to $\mathrm{Spin}(E)\cdot\beta$ if and only if $\alpha^{-1}\beta = 1$. In other words, we have that $\mathrm{Spin}(E)\cdot\alpha$ is equivalent to $\mathrm{Spin}(E)\cdot\beta$ if and only if $\alpha = \beta$. This finishes the proof of the theorem. □

Remark 3.70 (Spin structures and inner products)**.** *Let E be an n-dimensional oriented Euclidean bundle. The existence of a spin structure on E does not depend on its inner product. This is a non-trivial consequence of the fact that* $\mathrm{SO}(n)$ *is a deformation retract of* $\mathrm{GL}^+(n)$. ♢

3.7 Spin^c structures

In this section, we continue the process of using the algebraic notions that were presented in this chapter to study the geometry of vector bundles. In fact, we establish another fundamental notion to study the Thom isomorphisms, namely, the spin^c structures of oriented Euclidean vector bundles. We begin with the following definition.

Definition 3.71 (The spin^c groups)**.** *Let n be a natural number. We define the **spin^c group***

$$\mathrm{Spin}^c(n) \ := \ \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$$

where:

- $\mathrm{U}(1)$ *is the subgroup of the complex numbers composed of the unit elements; and*
- *the product $\times_{\mathbb{Z}_2}$ denotes the identification of the elements $(s, -\lambda)$ and $(-s, \lambda)$ for all $s \in \mathrm{Spin}(n)$ and all $\lambda \in \mathrm{U}(1)$. This justifies an element of $\mathrm{Spin}^c(n)$ being denoted by $[s, \lambda]$.* ♢

Remark 3.72 (The spin^c groups and two-sheeted coverings)**.** *Let n be a natural number. We define*

$$\begin{aligned} p_n^c : \ \mathrm{Spin}^c(n) &\ \rightarrow \ \mathrm{SO}(n) \times \mathrm{U}(1), \\ [s,\, \lambda] &\ \mapsto \ (p_n(s),\, \lambda^2), \end{aligned}$$

where $p_n : \mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$ is the usual two-sheeted covering of $\mathrm{SO}(n)$. This map not only is well-defined but it also is a two-sheeted covering of $\mathrm{SO}(n) \times \mathrm{U}(1)$. In particular, we have the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{p_n^c} \mathrm{SO}(n) \times \mathrm{U}(1) \longrightarrow 0.$$

Here $\mathbb{Z}_2$ is the subgroup of $\mathrm{Spin}^c(n)$ generated by $[1, -1] \in \mathrm{Spin}^c(n)$. Furthermore, $\mathrm{Spin}^c(n) \subseteq \mathbb{C}l(n)$. Indeed, since $\mathbb{C}l(n)$ is canonically isomorphic to $\mathrm{Cl}(n) \otimes \mathbb{C}$, it follows from the fact that $\mathrm{Spin}^c(n)$ is obtained by $\mathbb{Z}_2$-tensoring $\mathrm{Spin}(n)$ with the unit complex numbers. ♢

Definition 3.73 (Spin^c structure)**.** *Let $\pi : E \to X$ be an n-dimensional oriented Euclidean vector bundle. In addition, let:*

- $\pi_{\mathrm{SO}} : \mathrm{SO}(E) \to X$ *be the* $\mathrm{SO}(n)$*-principal bundle of oriented orthonormal frames of E defined in Remark 3.60; and*
- $\pi_{\mathrm{U}} : \mathrm{U}(L) \to X$ *be the* $\mathrm{U}(1)$*-principal bundle of unitary frames of a given Hermitian line bundle L, which the reader can readily define by inspiring himself or herself in Remark 3.60.*

*We say that a **spin^c structure** on E is a* $\mathrm{Spin}^c(n)$*-principal bundle*

$$\pi_{\mathrm{Spin}^c} : \mathrm{Spin}^c(E) \to X$$

equipped with a two-sheeted covering

$$\xi^c : \mathrm{Spin}^c(E) \to \mathrm{SO}(E) \times_{\pi_{\mathrm{SO}},\pi_{\mathrm{U}}} \mathrm{U}(L)$$

(see Remark C.52) such that, if $p_n^c : \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times \mathrm{U}(1)$ is the projection, then (ξ^c, p_n^c) is a morphism of principal bundles over X. In this situation, the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Spin}^c(E) \times \mathrm{Spin}^c(n) \arrow[rr, "\xi^c \times p_n^c"] \arrow[d, "\cdot"'] & & (\mathrm{SO}(E) \times_{\pi_{\mathrm{SO}},\pi_{\mathrm{U}}} \mathrm{U}(L)) \times (\mathrm{SO}(n) \times \mathrm{U}(1)) \arrow[d, "\cdot"] \\
\mathrm{Spin}^c(E) \arrow[rr, "\xi^c"] \arrow[dr, "\pi_{\mathrm{Spin}^c}"'] & & \mathrm{SO}(E) \times_{\pi_{\mathrm{SO}},\pi_{\mathrm{U}}} \mathrm{U}(L) \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

In this diagram, $\pi : \mathrm{SO}(E) \times_{\pi_{\mathrm{SO}},\pi_{\mathrm{U}}} \mathrm{U}(L) \to X$ is given by $\pi(s,u) := \pi_{\mathrm{SO}}(s) = \pi_{\mathrm{U}}(u)$ for all $(s,u) \in \mathrm{SO}(E) \times_{\pi_{\mathrm{SO}},\pi_{\mathrm{U}}} \mathrm{U}(L)$. $\diamondsuit$

Definition 3.74 (Equivalence of spin^c structures)**.** *Let $\pi : E \to X$ be an n-dimensional oriented Euclidean vector bundle. Two spin^c structures*

$$\begin{aligned} \xi^c : \mathrm{Spin}^c(E) &\to \mathrm{SO}(E) \times_{\pi_{\mathrm{SO}},\,\pi_{\mathrm{U}}} \mathrm{U}(L) \quad \textit{and} \\ \xi^{c\,\prime} : \mathrm{Spin}^{c\,\prime}(E) &\to \mathrm{SO}(E) \times_{\pi_{\mathrm{SO}},\,\pi_{\mathrm{U}}} \mathrm{U}(M) \end{aligned}$$

are said to be ***equivalent*** *provided that there exists an isomorphism of* $\mathrm{Spin}^c(n)$*-principal bundles $\varphi^c : \mathrm{Spin}^c(E) \to \mathrm{Spin}^{c\prime}(E)$ such that the following diagram is commutative.*

$$\begin{tikzcd} \mathrm{Spin}^c(E) \arrow[r, "\pi_1 \circ \xi^c"'] \arrow[rr, bend left, "\varphi^c"] & \mathrm{SO}(E) & \mathrm{Spin}^{c'}(E) \arrow[l, "\pi_1 \circ \xi^{c\,\prime}"] \end{tikzcd}$$

In this diagram, $\pi_1 : \mathrm{SO}(E) \times_{\pi_{\mathrm{SO}},\,\pi_{\mathrm{U}}} \mathrm{U}(\cdot) \to \mathrm{SO}(E)$ is given by $\pi_1(s,u) = s$ for all $(s,u) \in \mathrm{SO}(E) \times_{\pi_{\mathrm{SO}},\pi_{\mathrm{U}}} \mathrm{U}(\cdot)$. ♢

Definition 3.75 (The third integral Stiefel-Whitney class)**.** *Let X be a paracompact Hausdorff space and $\pi : E \to X$ be an n-dimensional oriented Euclidean vector bundle. In addition, let the following short exact sequence be the usual one.*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

Being

$$\Phi : \; H^2(X, \mathbb{Z}_2) \to H^3(X, \mathbb{Z})$$

the Bockstein homomorphism in Singular Cohomology with respect to the preceding short exact sequence, we define

$$W_3(E) \;:=\; \Phi(\omega_2(E))$$

where $\omega_2(E) \in H^2(X, \mathbb{Z}_2)$ is the second Stiefel-Whitney class of E. This is the ***third integral Stiefel-Whitney class*** *of E. If X is a smooth manifold, then we define its third integral Stiefel-Whitney class $W_3(X)$ to be the third integral Stiefel-Whitney class of its tangent bundle.* ♢

Theorem 3.76 (Existence of spin^c structures through the third integral Stiefel-Whitney class)**.** *Let X be a paracompact Hausdorff space and $\pi : E \to X$ be an n-dimensional oriented Euclidean vector bundle. We have that there exists a spin^c structure on E if and only if $W_3(E)$ is trivial.*

Proof. In order to construct a (local) spin^c lifting of E, we fix a complex Hermitian line bundle $\pi_L : L \to X$ as a part of the initial data. We then obtain the corresponding unitary frame bundle $\pi_{\mathrm{U}} : \mathrm{U}(L) \to X$. For each open set U_i of an appropriate open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X, we choose a spin lifting $\pi^i_{\mathrm{Spin}} : \mathrm{Spin}(E_i) \to U_i$ with a two-sheeted covering $\xi_i : \mathrm{Spin}(E_i) \to \mathrm{SO}(E_i)$. Furthermore, we choose a lifting $\pi^i_L : \mathrm{U}(L_i) \to U_i$ where $L_i := L \mid_{U_i}$ for all $i \in I$, equipped with a two-sheeted covering $\eta_i : \mathrm{U}(L_i) \to \mathrm{U}(L_i)$ compatible with

$$\begin{aligned} \rho : \mathrm{U}(1) &\to \mathrm{U}(1), \\ \lambda &\mapsto \lambda^2. \end{aligned}$$

Under these circumstances, we obtain the spin^c lifting

$$\pi^i_{\mathrm{Spin}^c} : \mathrm{Spin}^c(E_i) \to U_i$$

where

$$\mathrm{Spin}^c(E_i) := \mathrm{Spin}(E_i) \times_{\mathbb{Z}_2, \pi^i_{\mathrm{Spin}}, \pi^i_L} \mathrm{U}(L_i),$$

with the two-sheeted covering

$$\xi^c_i := \xi_i \times \eta_i : \mathrm{Spin}^c(E_i) \to \mathrm{SO}(E_i) \times_{\pi^i_{\mathrm{SO}}, \pi^i_L} \mathrm{U}(L_i).$$

Now we fix principal bundle isomorphisms

$$\varphi'_{ij} : \mathrm{Spin}(E_i) \mid_{U_{ij}} \to \mathrm{Spin}(E_j) \mid_{U_{ij}},$$

lifting the identity $\mathrm{SO}(E_i) \mid_{U_{ij}} = \mathrm{SO}(E_j) \mid_{U_{ij}}$. It follows that

$$\varphi'_{ki} \circ \varphi'_{jk} \circ \varphi'_{ij} = \epsilon_{ijk} \cdot I$$

for all $i, j, k \in I$. We also fix principal bundle isomorphisms

$$\psi'_{ij} : \mathrm{U}(L_i)\mid_{U_{ij}} \to \mathrm{U}(L_j)\mid_{U_{ij}},$$

lifting the identity $\mathrm{U}(L_i)\mid_{U_{ij}} = \mathrm{U}(L_j)\mid_{U_{ij}}$. It follows that

$$\psi'_{ki} \circ \psi'_{jk} \circ \psi'_{ij} \;=\; \theta_{ijk} \;=\; \pm 1$$

for all $i, j, k \in I$. Thence, we obtain the principal bundle isomorphisms

$$\varphi^c_{ij} \;:=\; \varphi'_{ij} \times \psi'_{ij} : \mathrm{Spin}^c(E_i)\mid_{U_{ij}} \to \mathrm{Spin}^c(E_j)\mid_{U_{ij}},$$

lifting the identity $(\mathrm{SO}(E_i) \times \mathrm{U}(L_i))\mid_{U_{ij}} = (\mathrm{SO}(E_j) \times \mathrm{U}(L_j))\mid_{U_{ij}}$. It follows that

$$\varphi^c_{ki} \circ \varphi^c_{jk} \circ \varphi^c_{ij} \;=\; \epsilon_{ijk}\theta_{ijk}$$

for all $i, j, k \in I$. We can construct a global bundle $\mathrm{Spin}^c(E)$ if and only if it is possible to choose these data in such a way that

$$\epsilon_{ijk}\theta_{ijk} \;=\; 1$$

for all $i, j, k \in I$. This is equivalent to $\theta_{ijk} = \epsilon_{ijk}$ for all $i, j, k \in I$. Fixing a set of local unitary sections $z_i : U_i \to L\mid_{U_i}$, we obtain the set of transition functions $h_{ij} \colon U_{ij} \to \mathrm{U}(1)$ for which we have $z_i = h_{ij} z_j$ for all $i, j \in I$. Consecutively, we lift the sections z_i to $z'_i : U_i \to \mathrm{U}(L_i)$ so that we get transition functions $h'_{ij} : U_{ij} \to \mathrm{U}(1)$ such that $\psi'_{ij} z'_i = h'_{ij} z'_j$ for all $i, j \in I$. It follows that

$$h'_{ki}\, h'_{jk}\, h'_{ij} \;=\; \theta_{ijk}$$

for all $i, j, k \in I$. Hence, there exists a global spin^c lifting of E if and only if it is possible to find a cochain $\{h'_{ij}\}_{i,j \in I} \in \check{C}^2(\mathfrak{U}; \underline{\mathrm{U}}(1))$ such that

$$h'_{ki}\, h'_{jk}\, h'_{ij} \;=\; \epsilon_{ijk} \cdot I$$

for all $i, j, k \in I$. This is equivalent to the triviality of $[\{\epsilon_{ijk}\}_{i,j,k \in I}]$ as a $\underline{\mathrm{U}}(1)$-cocycle. In turn, this is equivalent to $W_3(E)$ being trivial since it can be proved that $\check{H}^2(X, \underline{\mathrm{U}}(1))$ is isomorphic to $H^3(X; \mathbb{Z})$. This last fact is true because, in the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \underline{\mathbb{R}} \xrightarrow{\exp} \underline{\mathrm{U}}(1) \longrightarrow 0,$$

the sheaf of real functions is acyclic. Therefore, we have that the associated long exact sequence consists of isomorphisms between $\check{H}^n(X, \underline{\mathrm{U}}(1))$ and $H^{n+1}(X;\mathbb{Z})$ for all $n \in \mathbb{N}$. $\square$

Definition 3.77 (Spin^c bundle and spin^c manifold)**.** *Let X be a paracompact Hausdorff space and $\pi : E \to X$ be an oriented Euclidean vector bundle. We say that E is a **spin^c bundle** provided that $\omega_1(E)$ and $W_3(E)$ are both trivial. Furthermore, if X is a smooth manifold, then we say that it is a **spin^c manifold** if its tangent bundle is a spin^c bundle.* $\diamondsuit$

Remark 3.78 (Spin and spin^c structures)**.** *We have that the following facts hold true.*

- *Any vector bundle that admits a spin structure carries a correspondingly canonically determined spin^c structure. Indeed, if $\pi : E \to X$ admits a spin structure* $\mathrm{Spin}(E)$*, then*

$$\mathrm{Spin}^c(E) := \mathrm{Spin}(E) \times_{\mathbb{Z}_2} \mathrm{U}(1)$$

  is a spin^c structure of E, where $\mathbb{Z}_2$ acts diagonally by $(-1,-1)$ and where $\mathrm{U}(1)$ *is the trivial circle bundle. In addition, there are examples in the literature of spin^c bundles that admit no spin structures. Therefore, spin^c bundles are more common than spin bundles. Consequently, spin^c manifolds are more common than spin manifolds.*

- *Any complex vector bundle carries a canonically determined spin^c structure. The reader can find this construction in [23, pp. 392-393]. Therefore, we have that every complex manifold (in fact, every almost complex manifold) is canonically a spin^c manifold.* $\diamondsuit$

Although spin^c structures are more common than spin structures, they do not always exist. For example, the reader can find in [23, pp. 393-394] a non-spin^c manifold that is contained, up to open embeddings, in every non-spin^c manifold belonging to a special class of manifolds.

Theorem 3.79 (The number of inequivalent spin^c structures of a spin^c bundle)**.** *Let $\pi : E \to X$ be an n-dimensional spin^c bundle. The number of inequivalent spin^c structures on E is the order of $H^2(X, \mathbb{Z})$.*

Proof. The reader can adapt the arguments given in the proof of Theorem 3.69 to prove this result. □

3.8 Thom isomorphisms

In this section, we present the Thom isomorphisms in K-Theory. This is the furthest achievement of this thesis on the subject of Ordinary K-Theory. This result is a non-trivial consequence of the tools from Spin Geometry that we have presented until now. Here we will be able to present only a brief sketch of proof for it. We begin with the following remarks.

Remark 3.80 (An important module structure involving K-Theory)**.** *Let X be an object in* TopHdCpt *and $\pi : E \to X$ be a real vector bundle. We have that E is locally compact Hausdorff. The reader can readily prove that this follows from X being locally compact Hausdorff because of the existence of the local trivializations for E. As as consequence, we set*

$$K_c(E) := \bigoplus_{i \in \mathbb{Z}} K_c^i(E).$$

We have that $K_c(E)$ has a natural $K(X)$-module structure. Indeed, using the product in Remark 2.75, we define

$$\begin{aligned} \cdot : \; K^m(X) \otimes K_c^n(E) &\to K_c^{m+n}(E), \\ \alpha \otimes \beta &\mapsto K^{m+n}(i^+)(\alpha \boxtimes \beta). \end{aligned}$$

where the map $i^+ : E^+ \to (X \times E)^+$ is the only continuous extension of the proper map defined by

$$\begin{aligned} i : \; E &\to X \times E, \\ e &\mapsto (\pi(e), e). \end{aligned}$$

The reader can readily prove that this defines a $K(X)$-module structure on $K_c(E)$. In fact, one can prove that $K_c(E)$ is a unitary module with respect to this $K(X)$-module structure. ♢

Remark 3.81 (Some of the ideas involved in Thom isomorphisms)**.** *Let n be a natural number. In addition, let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous function. In Singular Cohomology, we consider the pullback*

$$H^n(f) : \; H^n(\mathbb{S}^n) \; \to \; H^n(\mathbb{S}^n).$$

Up to isomorphism, $H^n(f) : \mathbb{Z} \to \mathbb{Z}$ since $H^n(\mathbb{S}^n)$ is isomorphic to $\mathbb{Z}$. The degree of $f : \mathbb{S}^n \to \mathbb{S}^n$ is defined as

$$\deg(f) \; := \; H^n(f)(1) \; \in \; \mathbb{Z}.$$

If $g : \mathbb{S}^n \to \mathbb{S}^n$ is another continuous map, then one can prove that

$$\deg(g \circ f) \; = \; \deg(g) \cdot \deg(f).$$

As a consequence, since $\deg(\mathrm{id}_{\mathbb{S}^n}) = 1$, we have that, if f is a homeomorphism, then $\deg(f) = 1$ or $\deg(f) = -1$. Indeed, once there exists $f^{-1} : \mathbb{S}^n \to \mathbb{S}^n$ in such manner that $f^{-1} \circ f = \mathrm{id}_{\mathbb{S}^n}$, we have

$$1 \; = \; \deg(\mathrm{id}_{\mathbb{S}^n}) \; = \; \deg(f^{-1} \circ f) \; = \; \deg(f^{-1}) \cdot \deg(f),$$

which proves our assertion. Furthermore, it can be proved that, if f is a homeomorphism, then

$$\deg(f) \; = \; \begin{cases} 1 & \textit{if } f \textit{ preserves the orientation on } \mathbb{S}^n, \\ -1 & \textit{otherwise.} \end{cases}$$

In particular, let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism. The one-point Alexandroff compactification

$$\varphi^+ : \; (\mathbb{R}^n)^+ \simeq \mathbb{S}^n \; \to \; (\mathbb{R}^n)^+ \simeq \mathbb{S}^n$$

is a homeomorphism. Therefore,

$$\deg(\varphi^+) \; = \; \begin{cases} 1 & \textit{if } \varphi^+ \textit{ preserves the orientation on } \mathbb{S}^n, \\ -1 & \textit{otherwise.} \end{cases}$$

It can be proved that φ^+ *preserves the orientation on* $\mathbb{S}^n$ *if and only if* $\det(\varphi)$ *is positive. Now let* Ω *be a one-point space. We have*

$$K(\Omega) \simeq K_c^n(\mathbb{R}^n). \tag{3.23}$$

Indeed,

$$K_c^n(\mathbb{R}^n) \simeq \widetilde{K}^n(\mathbb{S}^n) \simeq \mathbb{Z} \simeq K(\Omega).$$

Since $\mathbb{R}^n$ *is a real vector bundle on* Ω, *the idea of Thom isomorphisms is to generalize* (3.23) *to any real vector bundle. Nevertheless, in order to do this, we have to establish an adequate hypothesis, which is orientability. In fact, let* E *be an oriented real vector bundle (see Remark 3.55). In addition, let* $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$ *be an oriented atlas of* E. *The transition functions of* $\Phi_{\mathfrak{U}}$ *can be compactified in each point* $x \in X$, *defining maps* $\varphi_x^+ : \mathbb{S}^n \to \mathbb{S}^n$ *for which*

$$\deg(\varphi_x^+) = 1$$

for all $x \in X$. *In particular, for a local chart* $\varphi_i \colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$, *we consider the element*

$$H^n(\varphi_i)_x^+(1) \in H^n(E_x^+, \mathbb{Z})$$

for each $x \in U_i$. *We have that* $H^n(\varphi_i)_x^+(1)$ *generates* $H^n(E_x^+, \mathbb{Z})$ *for all* $x \in U_i$. *Moreover, these elements do not depend on the chosen chart and continuously vary with* $x \in X$. *In fact,* E *is orientable if and only if it is possible to continuously choose a generator of* $H^n(E_x^+, \mathbb{Z})$ *for all* $x \in X$. *This idea produces the following definition in the K-Theory framework.* ♢

Definition 3.82 (Weak orientation in K-Theory). *Let* $\pi : E \to X$ *be an* n*-dimensional real vector bundle. A **weak orientation in K-Theory** is a continuous choice of generator for* $\widetilde{K}^n(E_x^+) \simeq \mathbb{Z}$ *where* $x \in X$. *We say that* E *is **weak orientable** if it admits a weak orientation in K-Theory.* ♢

The preceding definition is not convenient for our purposes. Indeed, one can prove that the existence of a weak orientation in K-Theory is equivalent to the existence of an orientation in the sense of Remark 3.55. In other words, orientation and weak orientation in K-Theory are equivalent notions. Nevertheless, with respect to

Singular Cohomology, the notion of weak orientability provides interesting results such as the Poincaré Duality Theorem for smooth manifolds. This does not happen in the K-Theory framework. This fact demands a better definition of orientability in K-Theory. In fact, **René Thom** (1923-2002) extended the notion of orientability by proving the following result.

Theorem 3.83 (Thom isomorphisms in Singular Cohomology)**.** *Let $\pi : E \to X$ be an n-dimensional oriented real vector bundle. If we continuously choose a generator for $\widetilde{H}^n(E_x^+) \simeq H_c^n(E_x)$ for each $x \in X$, then there exists a unique element $\alpha \in H_c^n(E)$, which is called the **Thom class** of E, whose restriction to E_x is the chosen generator of $H_c^n(E_x)$ for all $x \in X$. Furthermore, considering the usual module structure given by the cup product*

$$H^m(X;\mathbb{Z}) \otimes H_c^n(E;\mathbb{Z}) \xrightarrow{\smile} H_c^{m+n}(E;\mathbb{Z}),$$

we have that

$$\begin{aligned} T_m : H^m(X;\mathbb{Z}) &\to H_c^{m+n}(E;\mathbb{Z}), \\ u &\mapsto u \smile \alpha, \end{aligned}$$

*is a group isomorphism for all $m \in \mathbb{Z}$. These group isomorphisms, which do not form a ring isomorphism in general, are said to be the **Thom isomorphisms** of the vector bundle in question.*

Proof. The reader can find a complete proof of this result in [34]. □

Remark 3.84 (Thom classes and orientability)**.** *Let $\pi : E \to X$ be an oriented real vector bundle. It follows from Theorem 3.83 that there exist exactly two Thom classes for E, each of which induces one of the two orientations of E. In the smooth manifold setting, a Thom class consists in a differential form with compact support whose restriction to each fiber is a unitary volume form of its one-point Alexandroff compactification. In Singular Cohomology, the orientability of E is equivalent to the existence of:*

- *a continuous choice of generator for the* $\mathrm{rk}(E)$*th integral cohomology of each compactified fiber; and*

- *a global integral class with compact support for E that restricts to a generator in each fiber.*

For an arbitrary cohomology theory, these two properties are inequivalent, being the second one stronger than the first one. More concretely, in K-Theory, a continuous choice of generator for the integral cohomology of each fiber does not imply the existence of a global class that restricts to the chosen generators. Because of that, the existence of a global class turns out to be the best definition for orientability. This justifies the following definition. $\diamondsuit$

Definition 3.85 (Thom class in K-Theory)**.** *Let $\pi : E \to X$ be an n-dimensional real vector bundle. In K-Theory, a* ***Thom class*** *of E consists of an element $u \in K_c^n(E)$ for which the restriction to each fiber $u_x \in K_c^n(E_x) \simeq K_c^n(\mathbb{R}^n) \simeq \mathbb{Z}$ is a generator. We say that E is* ***K-orientable*** *provided that there exists a Thom class of E in K-Theory. In this situation, a choice of a Thom class of E is a* ***K-orientation*** *of the vector bundle in question.* $\diamondsuit$

Definition 3.86 (Spinor bundles)**.** *Let $\pi_P : P \to X$ be a G-principal bundle and $\rho : G \to \mathrm{GL}(\mathscr{V})$ be a topological representation of G where $\mathscr{V}$ is a finite-dimensional vector space. In Definition F.38, we defined the ρ-associated bundle of P, which is hereafter denoted by $\mathcal{E}_\rho(P)$. Thence, being $\pi_E : E \to X$ an n-dimensional oriented Euclidean vector bundle:*

- *if E is a spin bundle, then we say that a* ***complex spinor bundle*** *of E consists of a μ-associated bundle*

$$S_{\mathbb{C}}(E) := \mathcal{E}_\mu \mathrm{Spin}(E)$$

of a spin structure $\xi : \mathrm{Spin}(E) \to \mathrm{SO}(E)$ of E where $\mathscr{V}$ is a left module for $\mathbb{C}l(n)$ and where $\mu : \mathrm{Spin}(n) \to \mathrm{SO}(\mathscr{V}) \subset \mathrm{GL}(\mathscr{V})$ is the representation given by left multiplication by elements of $\mathrm{Spin}(n)$; and

- *if E is just a spin^c bundle, then we say that a* ***complex spinor bundle*** *of E consists of a Δ-associated bundle*

$$S^c_{\mathbb{C}}(E) := \mathcal{E}_\Delta \mathrm{Spin}^c(E)$$

of a spin^c structure $\xi : \mathrm{Spin}^c(E) \to \mathrm{SO}(E) \times U(L)$ of E where $\mathscr{V}$ is a left module for $\mathrm{Cl}(n)$ and where $\Delta : \mathrm{Spin}^c(n) \to \mathrm{GL}(\mathscr{V})$ is given by restriction of the $\mathrm{Cl}(n)$-representation to $\mathrm{Spin}^c(n)$. $\diamondsuit$

Remark 3.87 (Final concepts for establishing Thom isomorphisms)**.** *Let $\pi : E \to X$ be an $2n$-dimensional spin bundle. In addition, let $S_{\mathbb{C}}(E)$ be the irreducible complex spinor bundle of E. We can split $S_{\mathbb{C}}(E)$ into a direct sum*

$$S_{\mathbb{C}}(E) \;=\; S_{\mathbb{C}}^{+}(E) \oplus S_{\mathbb{C}}^{-}(E)$$

of $\mathrm{Cl}^0(E)$-modules. Here $\mathrm{Cl}(E)$ denotes the bundle of Clifford algebras generated by E which is such that $\mathrm{Cl}(E)_x = \mathrm{Cl}(E_x)$ for all $x \in X$. Moreover, $\mathrm{Cl}^0(E)$ and $\mathrm{Cl}^1(E)$ are defined analogously. Now consider the global section of $\mathrm{Cl}(E) \otimes \mathbb{C}$ which at $x \in X$ is given by

$$\omega_{\mathbb{C}} \;=\; i^n \, e_1 \cdots e_{2n}$$

for any positively oriented orthonormal basis $\{e_1, \cdots, e_n\}$ of the fiber E_x. Thence, we have

$$\omega_{\mathbb{C}}^2 \;=\; 1$$

and

$$e \cdot \omega_{\mathbb{C}} \;=\; -\omega_{\mathbb{C}} \cdot e$$

for all $e \in \mathrm{Cl}^1(E) \otimes \mathbb{C}$. We define $S_{\mathbb{C}}^{+}(E)$ and $S_{\mathbb{C}}^{-}(E)$ to be the 1 and -1 eigenbundles for Clifford multiplication by $\omega_{\mathbb{C}}$, respectively. If E is just a spin^c bundle, then we can analogously consider the irreducible complex spinor bundle $S_{\mathbb{C}}^{c}(E)$ of E and split it into a direct sum

$$S_{\mathbb{C}}^{c}(E) \;=\; S_{\mathbb{C}}^{c+}(E) \oplus S_{\mathbb{C}}^{c-}(E).$$

Now let $\mathbb{D}(E)$ and $\mathbb{S}(E)$ be the unit disc bundle and unit sphere bundle, respectively. The pullbacks of $S_{\mathbb{C}}^{+}(E)$ and $S_{\mathbb{C}}^{-}(E)$ through $\pi : E \to X$ over $\mathbb{D}(E)$ are canonically isomorphic on $\mathbb{S}(E)$ by the map

$$\mu : \; \pi^* S_{\mathbb{C}}^{+}(E) \;\to\; \pi^* S_{\mathbb{C}}^{-}(E)$$

given at $e \in \mathbb{S}(E)$ by

$$\mu_e(\sigma) \;=\; e\cdot\sigma$$

for all $\sigma \in \pi^* S^+_{\mathbb{C}}(E)_e$. These objects determine

$$[\pi^* S^+_{\mathbb{C}}(E),\, \pi^* S^-_{\mathbb{C}}(E),\, \mu] \,\in\, K^{2n}(\mathbb{D}(E), \mathbb{S}(E)) \,\simeq\, K^{2n}_c(E)^{(6)}.$$

We can then set the following result. ♢

Theorem 3.88 (Thom isomorphisms in K-Theory)**.** *Let X be a compact Hausdorff space and $\pi : E \to X$ be a $2n$-dimensional Euclidean bundle. If E is a spin bundle, then the class*

$$s(E) \;:=\; [\pi^* S^+_{\mathbb{C}}(E),\, \pi^* S^-_{\mathbb{C}}(E),\, \mu] \,\in\, K^{2n}_c(E)$$

is a K-orientation of E. This remains true if E is just a spin^c bundle. For this, we have to change $S_{\mathbb{C}}(E) = S^+_{\mathbb{C}}(E) \oplus S^-_{\mathbb{C}}(E)$ by $S^c_{\mathbb{C}}(E) = S^{c+}_{\mathbb{C}}(E) \oplus S^{c-}_{\mathbb{C}}(E)$. Furthermore, we have that

$$\begin{aligned} T_m :\; K^m(X) \;&\to\; K^{m+2n}_c(E),\\ \alpha \;&\mapsto\; \alpha\cdot s(E), \end{aligned}$$

*is an isomorphism for all $m \in \mathbb{Z}$. The multiplication here is the one of the module structure defined in Remark 3.80. These group isomorphisms, which do not form a ring isomorphism in general, are said to be the **Thom isomorphisms** of the vector bundle in question.*

Proof. The complete proof of this result can be found in [23, pp. 384-388]. Here we provide a short sketch of it. Indeed, we begin by saying that a compactly-supported K-Theory class of E has the **Bott periodicity property** provided that it determines a K-orientation in any local trivialization of E over a closed subset of X. Thence, it can be proved that any such class having the Bott periodicity property is a global

(6) In fact, according the construction of Section 2.9, we have $[\pi^* S^+_{\mathbb{C}}(E), \pi^* S^-_{\mathbb{C}}(E), \mu] \in K(\mathbb{D}(E), \mathbb{S}(E))$. Nervertheless, since $K(\mathbb{D}(E), \mathbb{S}(E))$ is isomorphic to $K^{2n}(\mathbb{D}(E), \mathbb{S}(E))$ because of the Bott Periodicity Theorem, we can consider $[\pi^* S^+_{\mathbb{C}}(E), \pi^* S^-_{\mathbb{C}}(E), \mu] \in K^{2n}(\mathbb{D}(E), \mathbb{S}(E))$. Furthermore, we have that $K^{2n}(\mathbb{D}(E), \mathbb{S}(E))$ is isomorphic to $K^{2n}_c(E)$ because, when X is a compact Hausdorff space, there is a canonical homeomorphism between the one-point Alexandroff compactification E^+ and the quotient of $\mathbb{D}(E)$ by $\mathbb{S}(E)$. In general, the quotient space of $\mathbb{D}(E)$ by $\mathbb{S}(E)$ is referred to in the literature as the **Thom space** of E.

K-orientation of E. As a consequence, the statement follows by just proving that $s(E)$ has the Bott periodicity property. In fact, if C is any closed subset of X for which we have a local trivialization $\varphi_C : E\mid_C \to X \times \mathbb{R}^{2n}$, then the class $s(E)$ becomes the element

$$s(E\mid_C) \in K_c^{2n}(C \times \mathbb{R}^{2n}),$$

which is the pullback to the product of the canonical generator of $K_c^{2n}(\mathbb{R}^{2n}) \simeq \mathbb{Z}$ given by the Atiyah-Bott-Shapiro Theorem (Theorem 3.41). This implies that $s(E)$ has the Bott periodicity property. $\square$

Corollary 3.89 (Thom isomorphisms in K-Theory for all vector bundles)**.** *Let X be a compact Hausdorff space and $\pi : E \to X$ be a $(2n-1)$-dimensional Euclidean bundle. If E is a spin (spin^c) bundle, then it is K-orientable. Furthermore, there are natural isomorphisms*

$$T_m : K^m(X) \to K_c^{m+2n-1}(E)$$

for all $m \in \mathbb{Z}$. These are also said to be the ***Thom isomorphisms*** *of the vector bundle in question.*

Proof. We have that the direct sum $E \oplus \mathbb{R}$ of E with the trivial bundle $X \times \mathbb{R} \to X$ is a spin (spin^c) bundle. We then obtain a K-orientation and the Thom isomorphisms for E by properly restricting the ones of $E \oplus \mathbb{R}$ from Theorem 3.88. This finishes the proof of the corollary. $\square$

Remark 3.90 (Extending the Thom isomorphisms)**.** *The preceding results are still true considering vector bundles on locally compact Hausdorff spaces. This claim can be found in [23, p. 389].* $\diamondsuit$

3.9 Gysin map

In this final section, we present an object that follows from the Thom isomorphisms in K-Theory, which is the Gysin map. This is the integration map in K-Theory. We only present here its definition and first properties since the applications would demand much more time to be developed. We begin with the following definition that treats an important particular case.

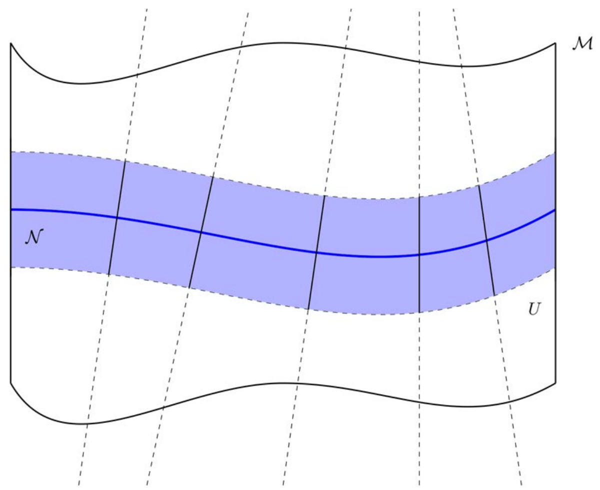

Figure 3.1: This picture gives a geometric visualization for the Gysin map described in Definition 3.91. Indeed, it says that, using the Thom isomorphism, we can associate to a cohomology class of $\mathcal{N}$ a class of $N_{\mathcal{N}}\mathcal{M}$. Thence, using the induced homomorphism of φ_U^+, we map the resulting class of $N_{\mathcal{N}}\mathcal{M}$ to a cohomology class of U. Finally, applying the induced homomorphism of η_U, we map this last cohomology class of U to a class of $\mathcal{M}$. This shows that the Gysin map extends a cohomology class of $\mathcal{N}$ to a class of $\mathcal{M}$. The price paid in this process is a translation in the degree of the initial cohomology class by the rank of the normal bundle $N_{\mathcal{N}}\mathcal{M}$, which is due to the use of the Thom isomorphism.

Definition 3.91 (Gysin map of an embedding)**.** *Let $\mathcal{M}$ be a smooth manifold. In addition, let $\mathcal{N}$ be an embedded compact submanifold of $\mathcal{M}$ for which the normal bundle $N_{\mathcal{N}}\mathcal{M} \to \mathcal{N}$ is K-orientable. We remind the reader that $N_{\mathcal{N}}\mathcal{M}$ is the quotient of the tangent bundle of $\mathcal{M}$ restricted to $\mathcal{N}$ by the tangent bundle of $\mathcal{N}$. Since $\mathcal{N}$ is compact, there exists a tubular neighborhood U of $\mathcal{N}$ in $\mathcal{M}$. In other words, there exists an open subset U of $\mathcal{M}$ containing $\mathcal{N}$ for which we have a homeomorphism $\varphi_U : U \to N_{\mathcal{N}}\mathcal{M}$. Being $i : \mathcal{N} \to \mathcal{M}$ the natural embedding, we define a group homomorphism called **Gysin map***

$$i_!^m : K^m(\mathcal{N}) \to K_c^{m+\dim(\mathcal{M})-\dim(\mathcal{N})}(\mathcal{M})$$

for each $m \in \mathbb{Z}$. In general, we have that these group homomorphisms do not form a ring homomorphism. Furthermore, as one could naturally expect, if $\mathcal{M}$ is also compact, then we obtain

$$i_!^m : K^m(\mathcal{N}) \to K^{m+\dim(\mathcal{M})-\dim(\mathcal{N})}(\mathcal{M}).$$

Being

$$\begin{aligned} \eta_U :\ \mathcal{M}^+ &\to U^+, \\ x &\mapsto \begin{cases} x & \text{if } x \in U, \\ \infty & \text{if } x \in \mathcal{M}^+ - U, \end{cases} \end{aligned}$$

we define

$$i_!^m \ := \ \widetilde{K}^{m+\dim(\mathcal{M})-\dim(\mathcal{N})}(\eta_U) \circ \widetilde{K}^{m+\dim(\mathcal{M})-\dim(\mathcal{N})}(\varphi_U^+) \circ T_m$$

for all $m \in \mathbb{Z}$. ♢

Definition 3.92 (Gysin map of a continuous function)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be compact smooth manifolds. In addition, let $f : \mathcal{N} \to \mathcal{M}$ be any continuous map. Under these circumstances:*

- *let $j : \mathcal{N} \to \mathbb{R}^N$ be an embedding. Thence, consider the embedding*

$$(f,\, j) :\ \mathcal{N} \ \to \ \mathcal{M} \times \mathbb{R}^N.$$

  It is to be noted that this construction is possible because of the Whitney Embedding Theorem; and

- *since $(\mathcal{M} \times \mathbb{R}^N)^+$ is homeomorphic to $\mathcal{M}^+ \wedge (\mathbb{R}^N)^+$, which is homeomorphic to $\mathcal{M}^+ \wedge \mathbb{S}^N$, we have that $(\mathcal{M} \times \mathbb{R}^N)^+$ is homeomorphic to $\Sigma^N \mathcal{M}_+$. Therefore, for all $m \in \mathbb{Z}$, we have the suspension isomorphism*

$$s_m^N :\ K_c^m(\mathcal{M} \times \mathbb{R}^N) \ \to \ K^{m-N}(\mathcal{M}).$$

We define the ***Gysin map*** *of f*

$$f_!^m :\ K^m(\mathcal{N}) \ \to \ K^{m+\dim(\mathcal{M})-\dim(\mathcal{N})}(\mathcal{M})$$

as the group homomorphism

$$f_!^m \ := \ s^N_{m+N+\dim(\mathcal{M})-\dim(\mathcal{N})} \ \circ \ (f,j)_!^m$$

for all $m \in \mathbb{Z}$. It can be proved that this map does not depend on $j : \mathcal{N} \to \mathbb{R}^N$. This is a consequence of the fact that any two embeddings of a smooth manifold in a sufficiently *large Euclidean space are isotopic. That is, they are homotopic through a homotopy composed only of embeddings.* ♢

Theorem 3.93 (Some of the properties of the Gysin map)**.** *Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{S}$ be compact smooth manifolds. In addition, let $f : \mathcal{N} \to \mathcal{M}$ and $g : \mathcal{M} \to \mathcal{S}$ be continuous maps. The following facts are true.*

(1) The Gysin map

$$f_!^m : K^m(\mathcal{N}) \to K^{m+\dim(\mathcal{M})-\dim(\mathcal{N})}(\mathcal{M})$$

only depends on the homotopy class of $f : \mathcal{N} \to \mathcal{M}$ for each $m \in \mathbb{Z}$. In particular, it does not depend on any embedding.

(2) For each $m, n \in \mathbb{Z}$,

$$f_!^{m+n}(\alpha \cdot K^n(f)(\beta)) = f_!^m(\alpha) \cdot \beta$$

where $\alpha \in K^m(\mathcal{N})$ and $\beta \in K^n(\mathcal{M})$.

(3) We have

$$(g \circ f)_!^m = g_!^{m+\dim(\mathcal{M})-\dim(\mathcal{N})} \circ f_!^m$$

for all $m \in \mathbb{Z}$.

Proof. The reader can find a complete proof of this result in [19, p. 233]. □

Remark 3.94 (Extending the preceding definitions)**.** *Here we use the notations of Definitions 3.91 and 3.92. If $\mathcal{N}$ is locally compact, then, by means of an immersion $i : \mathcal{N} \to \mathcal{M}$, we can define the Gysin map exactly as before. This happens because, under these circumstances, there exists a tubular neighborhood of $\mathcal{N}$ in $\mathcal{M}$ as well. Thence, we can also define the Gysin map of a proper map $f : \mathcal{N} \to \mathcal{M}$. More than that, we can define the Gysin map of any continuous map $f : \mathcal{N} \to \mathcal{M}$ that factors out by a proper embedding.* ♢

Chapter 4

Rephrasing Ordinary K-Theory

In this chapter, we conclude the study of Ordinary K-Theory by presenting a different but equivalent viewpoint for this theory. Indeed, here we use the language and the initial results of Functional Analysis to set an interpretation of K-Theory through homotopy classes of functions whose codomain is the space of Fredholm operators, which are continuous linear operators defined on separable Hilbert spaces for which the kernel and the cokernel are finite-dimensional. This model for K-Theory will be particularly interesting in Chapter 5. In order to write this part of the text, we used as main references [2, pp. 153 - 162], [8, pp. 7-18, 33-43], [22] and [32, pp. 1-23, 55-67, 119-125, 175-183].

4.1 Fredholm operators

In this section, we present the basic language of Functional Analysis that is needed to develop the model of Fredholm operators for K-Theory. In particular, we recover the notion of separable Hilbert space in order to define and study some properties of Fredholm operators. The reader who is familiar with these notions may skip this section and return to it later if it is necessary. We begin with the following remark.

Remark 4.1 (Elementary notions on Functional Analysis). *Let H be a complex Hilbert space. That is, H is a complex vector space which is equipped with an Hermitian product*

$$\langle\,,\,\rangle :\ H \times H \to \mathbb{C}$$

such that the induced norm

$$\begin{aligned} |\cdot| :\ H &\to [0,\infty), \\ u &\mapsto \sqrt{\langle u,\, u\rangle}, \end{aligned}$$

turns H into a Banach space. Moreover, suppose that H is separable. In other words, suppose that there exists a dense countable subset of H. By Zorn's Lemma, this is equivalent to the existence of a countable orthonormal basis for H. Therefore, one can prove that every infinite-dimensional separable Hilbert space is isometrically isomorphic to the famous space ℓ^2 of infinite sequences of complex numbers $z = (z_n)_{n\in\mathbb{N}}$ for which the series

$$\sum_{n=0}^{\infty} |z_n|^2$$

converges, equipped with the Hermitian product given by

$$\langle z,\, w\rangle := \sum_{n=0}^{\infty} z_n\,\overline{w}_n$$

for all $z, w \in \ell^2$. This may help the inexperienced reader to have a more concrete picture of infinite-dimensional separable Hilbert spaces. Under these hypotheses, we consider $\mathscr{H}$ to be the algebra of all bounded operators on H. We equip $\mathscr{H}$ with the norm topology, where

$$\begin{aligned} |\cdot| :\ \mathscr{H} &\to [0,\infty), \\ T : H \to H &\mapsto \sup_{u\in H;\ |u|=1} |T(u)|. \end{aligned}$$

This makes $\mathscr{H}$ into a Banach space. In particular, the group of units $\mathscr{H}^*$ of $\mathscr{H}$ forms an open set. Furthermore, by the Closed Graph Theorem, any $T \in \mathscr{H}$ which is an algebraic automorphism is also a topological automorphism. This means that, if T^{-1} exists in $\mathscr{H}$, then $T \in \mathscr{H}^*$. $\diamondsuit$

Definition 4.2 (Fredholm operator)**.** *An operator $T \in \mathscr{H}$ is a* ***Fredholm operator*** *if* $\operatorname{Ker} T$ *and* $\operatorname{Coker} T$ *are finite-dimensional. We denote the collection of all Fredholm operators on H by $\mathscr{F}_H$.* $\diamondsuit$

Theorem 4.3 (The image of a Fredholm operator is always closed)**.** *If $T \in \mathscr{F}_H$, then $T(H)$ is closed in H.*

Proof. We have that the restriction

$$T\,|_{(\operatorname{Ker} T)^{\perp}}\colon\ (\operatorname{Ker} T)^{\perp} \to T(H)$$

is a bijection. Then, we define an extension of this map

$$\hat{T}:\ (\operatorname{Ker} T)^{\perp} \oplus \mathbb{C}^{\dim(\operatorname{Coker} T)} \to T(H) \oplus \operatorname{Coker} T,$$

by sending a basis of $\mathbb{C}^{\dim(\operatorname{Coker} T)}$ into a basis of $\operatorname{Coker} T$. This map is a continuous bijection. Therefore, by the Open Mapping Theorem, we conclude that it is a homeomorphism. Thus,

$$T(H) = \hat{T}((\operatorname{Ker} T)^{\perp})$$

is closed. This happens because $(\operatorname{Ker} T)^{\perp}$ is easily seen to be closed since $\operatorname{Ker} T$ is closed. □

Corollary 4.4 (The adjoint of a Fredholm operator is also Fredholm)**.** *We remind the reader that the* ***adjoint*** *of an operator $T \in \mathscr{H}$ is the unique linear operator $T^* \in \mathscr{H}$ for which*

$$\langle Tu,\, v\rangle = \langle u,\, T^* v\rangle$$

for all $u, v \in H$. We have that $T \in \mathscr{F}_H$ if and only if $\operatorname{Ker} T$ *and* $\operatorname{Ker} T^*$ *are both finite-dimensional. In particular, an operator is Fredholm if and only if its adjoint is Fredholm.*

Proof. Since $\operatorname{Ker} T^* = T(H)^{\perp}$, $H = T(H) \oplus T(H)^{\perp}$ and $T(H)$ is closed (Theorem 4.3), we have

$$\operatorname{Ker} T^* = T(H)^{\perp} \simeq H/T(H) = \operatorname{Coker} T.$$

Thus, $\operatorname{Ker} T^*$ is finite-dimensional if and only if $\operatorname{Coker} T$ is finite-dimensional. Therefore, T is Fredholm if and only if T^* is Fredholm. □

Theorem 4.5 (Relation between Fredholm operators and the Calkin Algebra)**.** *Let $\mathscr{K}_H$ be the ideal of $\mathscr{H}$ formed by the compact operators*[1]*. The* ***Calkin Algebra*** *of H is the quotient $\mathscr{H}/\mathscr{K}_H$. Let*

$$\pi : \mathscr{H} \to \mathscr{H}/\mathscr{K}_H$$

be the natural projection. Under these circumstances, we have that $T \in \mathscr{H}$ is a Fredholm operator if and only if $\pi(T)$ is invertible in the Calkin Algebra of H. In particular, we have that an operator $T \in \mathscr{H}$ is Fredholm if and only if there exists $S \in \mathscr{H}$ in such manner that

$$S \circ T = \mathrm{id}_H + K \qquad \textit{and} \qquad T \circ S = \mathrm{id}_H + L$$

where $K, L \in \mathscr{K}_H$.

Proof. The reader can find a proof of this result in [8, p. 14]. □

Corollary 4.6 (The open subspace of Fredholm operators)**.** *The collection of Fredholm operators on H is open in $\mathscr{H}$.*

Proof. The set $\mathscr{A}_H$ of invertible elements in the Calkin Algebra is open. Hence, since the natural projection $\pi : \mathscr{H} \to \mathscr{H}/\mathscr{K}_H$ is continuous, $\mathscr{F}_H = \pi^{-1}(\mathscr{A}_H)$ is open in $\mathscr{H}$. It is to be noted that this last equality is an immediate consequence of Theorem 4.5. □

Corollary 4.7 (The subalgebra of Fredholm operators)**.** *The collection of Fredholm operators on H is a subalgebra of $\mathscr{H}$.*

Proof. We first have to verify that $\mathscr{F}_H$ is a vector subspace of $\mathscr{H}$. We leave this straightforward computation to the reader. Afterwards, we have to prove that, if T and S are Fredholm operators, then the composition $T \circ S$ is also a Fredholm operator. In fact, if T and S are Fredholm operators, then there exist $P, Q \in \mathscr{H}$ such that

---

[1] We remind the reader that $T \in \mathscr{H}$ is a compact operator provided that $T(A)$ is precompact in H whenever A is a bounded subset of H. In addition, we have that $T(A)$ is precompact in H if its closure is compact in this space.

$$P \circ T = \mathrm{id}_H + K, \qquad T \circ P = \mathrm{id}_H + L,$$
$$Q \circ S = \mathrm{id}_H + M \quad \text{and} \quad S \circ Q = \mathrm{id}_H + N,$$

where $K, L, M, N \in \mathscr{K}_H$ (Theorem 4.5). Therefore,

$$\begin{aligned}(T \circ S) \circ (Q \circ P) &= T \circ (S \circ Q) \circ P \\ &= T \circ (\mathrm{id}_H + N) \circ P \\ &= (T + T \circ N) \circ P \\ &= T \circ P + T \circ N \circ P \\ &= \mathrm{id}_H + L + T \circ N \circ P.\end{aligned}$$

Analogously, we obtain

$$(Q \circ P) \circ (T \circ S) = \mathrm{id}_H + M + Q \circ K \circ S.$$

Therefore, since $L + T \circ N \circ P$ and $M + Q \circ K \circ S$ are compact operators on H, we have that $T \circ S$ is a Fredholm operator (Theorem 4.5). This finishes the proof of the theorem. □

4.2 Index of Fredholm operators

In this section, we define and study the notion of Fredholm index. This concept produces a particular case of the main theorem of this chapter. However, the naturality and the simplicity of the arguments used to establish this particular case are sufficient reasons to set it as a motivation for the desired general result. In fact, it is the logical path applied to this section. We begin with the following definition.

Definition 4.8 (Index of a Fredholm operator). *Let $T \in \mathscr{F}_H$. We say that the integer number*

$$\mathrm{index}_T := \dim(\mathrm{Ker}\, T) - \dim(\mathrm{Coker}\, T)$$

is the ***index*** *of T.* ♢

Example 4.9 (Fredholm operators on a finite-dimensional complex Hilbert space)**.** *If H is a finite-dimensional complex Hilbert space, then it is separable. Indeed, let $\mathcal{A}$ be a basis for H, which is necessarily finite since H is finite-dimensional. Thence, the collection of all linear combinations of the elements of $\mathcal{A}$ with rational coefficients is a dense countable subset of H. Moreover, we have that every $T \in \mathscr{H}$ is a Fredholm operator. Finally, since* $\operatorname{Coker} T = H/T(H)$, $\dim(\operatorname{Coker} T) = \dim H - \dim T(H)$. *Therefore,*

$$\begin{aligned}
\operatorname{index}_T &= \dim(\operatorname{Ker} T) - \dim(\operatorname{Coker} T) \\
&= \dim(\operatorname{Ker} T) + \dim T(H) - \dim H \\
&= \dim H - \dim H \\
&= 0
\end{aligned}$$

by the Rank-Nullity Theorem. ♢

Example 4.10 (Index of the adjoint of a Fredholm operator)**.** *Let $T \in \mathscr{F}_H$. Because of Corollary 4.4, $T^* \in \mathscr{F}_H$. In addition, this result allows us to explicitly calculate the index of T^*. Indeed,*

$$\begin{aligned}
\operatorname{index}_{T^*} &= \dim(\operatorname{Ker} T^*) - \dim(\operatorname{Coker} T^*) \\
&= \dim(\operatorname{Ker} T^*) - \dim(\operatorname{Ker} T^{**}) \\
&= \dim(\operatorname{Ker} T^*) - \dim(\operatorname{Ker} T) \\
&= -\operatorname{index}_T.
\end{aligned}$$

♢

Theorem 4.11 (The index of Fredholm operators defined on an infinite-dimensional separable Hilbert space is surjective)**.** *If H is an infinite-dimensional separable Hilbert space, then*

$$\begin{aligned}
\operatorname{index} : \mathscr{F}_H &\to \mathbb{Z}, \\
T &\mapsto \operatorname{index}_T,
\end{aligned}$$

is surjective. Note that this claim is not true for finite-dimensional Hilbert spaces (see Example 4.9).

Proof. Let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis for H (see Remark 4.1). For each $n \in \mathbb{N}$, we define:

- the **right shift**

$$\begin{array}{rcl} S_{-n} : H & \to & H, \\ \sum_{i=0}^{\infty} z_i e_i & \mapsto & \sum_{i=0}^{\infty} z_i e_{i+n}, \end{array}$$

  which is a Fredholm operator. Indeed:

  - S_{-n} is clearly injective. Thus, we have that $\operatorname{Ker} S_{-n}$ is trivial, being then finite-dimensional; and
  - $S_{-n}(H)$ is the complement of the subspace generated by $e_0, \cdots, e_{n-1} \in H$. Thus, we have that $\operatorname{Coker} S_{-n}$ is isomorphic to this vector subspace, being then finite-dimensional.

  Therefore, we have

$$\operatorname{index}_{S_{-n}} \;=\; \dim(\operatorname{Ker} S_{-n}) - \dim(\operatorname{Coker} S_{-n}) \;=\; 0 - n \;=\; -n; \;\; \textit{and}$$

- the **left shift**

$$\begin{array}{rcl} S_{n} : H & \to & H, \\ \sum_{i=0}^{\infty} z_i e_i & \mapsto & \sum_{i=0}^{\infty} z_{i+n} e_{i}, \end{array}$$

  which is a Fredholm operator. Indeed:

  - S_n has as its kernel the subspace generated by $e_0, \cdots, e_{n-1} \in H$, being then finite-dimensional; and
  - S_n is clearly surjective. Thus, we have that $\operatorname{Coker} S_n$ is trivial, being then finite-dimensional.

  Therefore, we have

$$\operatorname{index}_{S_{n}} \;=\; \dim(\operatorname{Ker} S_{n}) - \dim(\operatorname{Coker} S_{n}) \;=\; n - 0 \;=\; n.$$

This finishes the proof of the theorem. □

Remark 4.12 (Index and injectivity). *If $T \in \mathscr{F}_H$ is invertible, then $T^{-1} \in \mathscr{F}_H$ and* $\operatorname{index}_T = 0$. *Indeed, if T is invertible, then* $\operatorname{Ker} T$ *and* $\operatorname{Coker} T$ *are both trivial. Consequently,*

$$\dim(\operatorname{Ker} T) \;=\; \dim(\operatorname{Coker} T) \;=\; 0,$$

which clearly implies our assertions. On the contrary, T is not necessarily invertible if $\operatorname{index}_T = 0$. *In fact, this follows from Example 4.9. Nevertheless, being* $\operatorname{index}_T = 0$, *we have that:*

- *if T is injective, then it is also surjective. This happens because, being injective, T has trivial kernel. Hence, since* $\dim(\operatorname{Ker} T) = 0$, *we obtain* $\dim(\operatorname{Coker} T) = 0$. *Thus, T has trivial cokernel; and*

- *if T is surjective, then it is also injective. This happens because, being surjective, T has trivial cokernel. Hence, since* $\dim(\operatorname{Coker} T) = 0$. *we obtain* $\dim(\operatorname{Ker} T) = 0$. *Thus, T has trivial kernel.*

In particular, the map of Theorem 4.11 is not injective since all invertible operators have zero index. ♢

Lemma 4.13 (Exact sequences of finite-dimensional vector spaces). *Let the following sequence of finite-dimensional vector spaces and linear maps be exact.*

$$0 \longrightarrow \mathscr{V}_1 \longrightarrow \mathscr{V}_2 \longrightarrow \cdots \longrightarrow \mathscr{V}_{n-1} \longrightarrow \mathscr{V}_n \longrightarrow 0$$

In this situation,

$$\sum_{i=1}^{n} (-1)^i \dim(\mathscr{V}_i) \;=\; 0.$$

Proof. The reader can find a proof of this result in [8, pp. 16-17]. □

Theorem 4.14 (The index of a composition of Fredholm operators). *Let $T, S \in \mathscr{F}_H$. Because of Corollary 4.7, it makes sense asking about the index of $T \circ S \in \mathscr{F}_H$. Indeed, we have*

$$\mathrm{index}_{T\circ S} \;=\; \mathrm{index}_T + \mathrm{index}_S.$$

Thus, the map of Theorem 4.11 is a group homomorphism. In particular, if $T \in \mathscr{F}_H$ *is invertible, then* $\mathrm{index}_{T^{-1}} = -\mathrm{index}_T$.

Proof. The reader can readily prove that the sequence of finite-dimensional vector spaces and linear maps

$$\begin{tikzcd}
 & \mathrm{Ker}(T\circ S) \arrow[r, "S"] & \mathrm{Ker}\,T \arrow[r, "\pi"] & \mathrm{Coker}\,S \arrow[r, "T"] & \mathrm{Coker}(T\circ S) \arrow[d, "\beta"] & \\
0 \arrow[r] & \mathrm{Ker}\,S \arrow[u, "\alpha"] & & & \mathrm{Coker}\,T \arrow[r] & 0
\end{tikzcd}$$

is exact, where α is the inclusion, π is the projection and β is the map that sends an equivalence class modulo $(T\circ S)(H)$ into an equivalence class modulo $T(H)$. Thus, by Lemma 4.13, we obtain that the alternate sum of the dimensions of the vector spaces in this sequence is zero. This implies our assertion since such alternate sum can be written as

$$\mathrm{index}_{T\circ S} - \mathrm{index}_T - \mathrm{index}_S \;=\; 0.$$

This finishes the proof of the theorem. $\square$

Theorem 4.15 (The continuity of the index map of Fredholm operators)**.** *The group homomorphism* $\mathrm{index} : \mathscr{F}_H \to \mathbb{Z}$ *defined in Theorem 4.11 is locally constant. Therefore, it is continuous.*

Proof. Let $T \in \mathscr{F}_H$. In addition, let $P : (\mathrm{Ker}\,T)^{\perp} \to H$ be the inclusion and $Q : H \to T(H)$ be the orthogonal projection of H onto $T(H)$. Since P has trivial kernel and

$$\mathrm{Coker}\,P \;=\; H/(\mathrm{Ker}\,T)^{\perp} \;\simeq\; \mathrm{Ker}\,T,$$

P is Fredholm and

$$\mathrm{index}_P \;=\; \dim(\mathrm{Ker}\,P) - \dim(\mathrm{Coker}\,P) \;=\; -\dim(\mathrm{Ker}\,T).$$

Similarly, once

$$\operatorname{Ker} Q \;=\; T(H)^{\perp} \;\simeq\; \operatorname{Coker} T$$

and $\operatorname{Coker} Q$ is trivial, Q is Fredholm and

$$\operatorname{index}_Q \;=\; \dim(\operatorname{Ker} Q) - \dim(\operatorname{Coker} Q) \;=\; \dim(\operatorname{Coker} T).$$

Consequently,

$$\operatorname{index}_T + \operatorname{index}_P + \operatorname{index}_Q \;=\; 0. \tag{4.1}$$

Since $Q \circ T \circ P : (\operatorname{Ker} T)^{\perp} \to T(H)$ is invertible,

$$\varepsilon \;:=\; |(Q \circ T \circ J)^{-1}|$$

is positive. Thus, if $S \in \mathscr{F}_H$ is such that

$$|T - S| \;<\; \frac{\varepsilon^{-1}}{|Q|\,|P|},$$

then we obtain

$$\begin{aligned} |Q \circ T \circ P - Q \circ S \circ P| \;&=\; |Q \circ (T - S) \circ P| \\ &\leq\; |Q|\,|T - S|\,|P| \\ &<\; \varepsilon^{-1}. \end{aligned}$$

This proves that $Q \circ S \circ P$ is invertible. Thence, $\operatorname{index}_{Q \circ S \circ P} = 0$ by Remark 4.12. Hence, Theorem 4.14 yields

$$\operatorname{index}_S + \operatorname{index}_P + \operatorname{index}_Q \;=\; 0. \tag{4.2}$$

From Equations (4.1) and (4.2), we obtain $\operatorname{index}_T = \operatorname{index}_S$. This finishes the proof of the theorem. □

Corollary 4.16 (Index of translations by compact operators)**.** *Let $T \in \mathscr{F}_H$ and $K \in \mathscr{K}_H$. Then, $T + K \in \mathscr{F}_H$ and*

$$\operatorname{index}_{T+K} \;=\; \operatorname{index}_T.$$

Proof. Since $\pi(T+K) = \pi(T)$, where $\pi : \mathscr{H} \to \mathscr{H}/\mathscr{K}_H$ is the natural projection, $T+K$ is a Fredholm operator if and only if T is Fredholm (Theorem 4.5). Now, consider the continuous path

$$\begin{aligned} \alpha : [0,1] &\to \mathscr{F}_H, \\ t &\mapsto T + tK. \end{aligned}$$

Once the index is locally constant because of Theorem 4.15, this path ensures that $\operatorname{index}_{T+tK} = \operatorname{index}_T$ for all $t \in [0,1]$. In particular, we obtain $\operatorname{index}_{T+K} = \operatorname{index}_T$, as we wished. □

Lemma 4.17 (Path-connectedness of the space of invertible operators)**.** *The subalgebra of invertible operators $\mathscr{H}^*$ of $\mathscr{H}$ is path-connected.*

Proof. The reader can find a proof of this result in [8, pp. 18-21]. □

Theorem 4.18 (Bijection induced by the index between path-connected components of the space of Fredholm operators and the integer numbers)**.** *Let $\pi_0\mathscr{F}_H$ denote the set of path-connected components of $\mathscr{F}_H$. The Fredholm index defined in this section induces a bijection $\pi_0\mathscr{F}_H \to \mathbb{Z}$.*

Proof. We only have to prove injectivity since surjectivity was shown in Theorem 4.11. We define

$$\mathscr{F}_H^n := \{T \in \mathscr{F}_H : \operatorname{index}_T = n\}$$

for each $n \in \mathbb{Z}$. Since the Fredholm index is locally constant (Theorem 4.15), we have that injectivity follows provided we prove that $\mathscr{F}_H^n$ is path-connected. This is what is done now. Indeed:

- if $n = 0$, then let $T \in \mathscr{F}_H^0$. Because of Lemma 4.17, it suffices to prove that T can be connected to an invertible operator by a path. In fact, since $\operatorname{index}_T = 0$, we have $\dim(\operatorname{Ker} T) = \dim(\operatorname{Ker} T^*)$. Thus, let $\{v_i\}_{i=1}^{\dim(\operatorname{Ker} T)}$ and $\{w_i\}_{i=1}^{\dim(\operatorname{Ker} T)}$ be bases for $\operatorname{Ker} T$ and $\operatorname{Ker} T^*$, respectively. Therefore, under these circumstances, if we have

$$u = u_0 + \sum_{i=1}^{\dim(\operatorname{Ker} T)} \lambda_i v_i \in H$$

with $u_0 \in (\operatorname{Ker} T)^\perp$, then we define

$$\varphi(u) \;:=\; \sum_{i=1}^{\dim(\operatorname{Ker} T)} \lambda_i w_i.$$

This correspondence sets an operator $\varphi \in \mathscr{H}$ for which $\operatorname{Ker}\varphi = (\operatorname{Ker} T)^\perp$ and $\varphi(H) = \operatorname{Ker} T^*$. We claim that $T+\varphi$ is invertible. In fact, it is clearly surjective. Moreover, it is injective because, if $u \in \operatorname{Ker}(T+\varphi)$, then $T(u) = -\varphi(u)$, which implies $u = 0$. This very same reasoning proves that $T + t\varphi$ is invertible for all $t \in (0,\infty)$. Thus,

$$\begin{aligned} \gamma :\ [0,1] \ &\to\ \mathscr{F}^0_H, \\ t\ &\mapsto\ T + t\varphi, \end{aligned}$$

is a path contained in the space $\mathscr{F}^0_H$ that connects T with an invertible operator, as desired;

- if $n > 0$, then let $T \in \mathscr{F}^n_H$. In addition, let S_{-n} and S_n be the right and left shifts defined in the proof of Theorem 4.11, respectively. Because of Corollary 4.7, we have $T \circ S_{-n} \in \mathscr{F}_H$. Furthermore, because of Theorem 4.14, $\operatorname{index}_{T\circ S_{-n}} = 0$ since $\operatorname{index}_{S_{-n}} = -n$. Thus, we have $T \circ S_{-n} \in \mathscr{F}^0_H$. Once $S_{-n} \circ S_n = \mathrm{id}_H$,

$$T \;=\; (T \circ S_{-n}) \circ S_n \;\in\; \mathscr{F}^0_H \circ S_n.$$

  Hence, $\mathscr{F}^n_H \subseteq \mathscr{F}^0_H \circ S_n$. Additionally, $\mathscr{F}^0_H \circ S_n \subseteq \mathscr{F}^n_H$ because of Theorem 4.14. Consequently, it follows that $\mathscr{F}^n_H = \mathscr{F}^0_H \circ S_n$ is path-connected because of the preceding item; *and*

- if $n < 0$, then the path-connectedness of $\mathscr{F}^n_H$ is immediate from the preceding item since

$$\mathscr{F}^n_H \;=\; (\mathscr{F}^{-n}_H)^*$$

  because of Example 4.10. □

Remark 4.19 (Interpreting the preceding result through homotopy classes and K-Theory)**.** *Let Ω be a one-point space. The reader can readily prove that there exists a natural bijection*

$$\alpha : [\Omega, \mathscr{F}_H] \to \pi_0\mathscr{F}_H,$$

where $[\Omega, \mathscr{F}_H]$ *is the set of homotopy classes of continuous maps from* Ω *into* $\mathscr{F}_H$. *Indeed, Fredholm operators* T *and* S *are in the same path-connected component of* $\mathscr{F}_H$ *if and only if the functions* $\Omega \mapsto T$ *and* $\Omega \mapsto S$ *are homotopic. Moreover, we have* $K(\Omega) = \mathbb{Z}$ *by Example 2.9. Consequently, Theorem 4.18 can be restated saying that there exists a group isomorphism*

$$\text{index} : [\Omega, \mathscr{F}_H] \to K(\Omega).$$

The goal of the next section is to generalize this result establishing a group isomorphism when we change Ω *by any compact Hausdorff space. This assertion is known as the* ***Atiyah-Jänich Theorem****.* ♢

4.3 Atiyah-Jänich Theorem

In this section, we fulfill the program of this chapter stating and proving the Atiyah-Jänich Theorem, which was mentioned in the last remark of the preceding section. This result is the one that gives us the interpretation of K-Theory through homotopy classes of functions whose codomain is the space of Fredholm operators. We begin with the result itself.

Theorem 4.20 (Atiyah-Jänich Theorem)**.** *For any compact Hausdorff space* X*, we have a natural group isomorphism*

$$\text{index} : [X, \mathscr{F}_H] \to K(X).$$

In other words, the space of Fredholm operators $\mathscr{F}_H$ *is a* ***classifying*** *or* ***representing space*** *for K-Theory.*

Proof. The proof of this result will be completed at the end of this section by a series of lemmas and theorems. □

Remark 4.21 (An intuitive idea to prove the Atiyah-Jänich Theorem that unhappily does not work)**.** *Let* X *be a compact Hausdorff space and* $T : X \to \mathscr{F}_H$ *be a continuous function. We define*

$$\operatorname{Ker} T \;:=\; \bigsqcup_{x \in X} \operatorname{Ker} T_x,$$

where $T_x := T(x) \in \mathscr{F}_H$. *The intuitive idea to define the generalized index map of the Atiyah-Jänich Theorem is setting*

$$\operatorname{index}[T] \;:=\; [\operatorname{Ker} T] - [\operatorname{Coker} T]$$

for all $[T] \in [X, \mathscr{F}_H]$. *Nevertheless, this is not generally possible since* $\operatorname{Ker} T$ *and* $\operatorname{Coker} T$ *usually do not have vector bundle structures. In fact, for example, consider the continuous function*

$$\begin{aligned} T : \mathbb{R} \;&\to\; \mathscr{F}_{\mathbb{C}}, \\ x \;&\mapsto\; T_x, \end{aligned}$$

where

$$T_x(z) \;=\; xz$$

for all $z \in \mathbb{C}$. *We have* $\dim(\operatorname{Ker} T_0) = 0$ *and* $\dim(\operatorname{Ker} T_x) = 1$ *for all non-zero* x. *Since the real line is connected, any vector bundle on it must have constant rank. Thus,* $\operatorname{Ker} T$ *cannot be a vector bundle. The reader will note that the technical lemmas presented below to construct the generalized index map are the way we have to bypass this difficulty.* ♢

Lemma 4.22 (Existence of well-behaved neighborhoods in the space of Fredholm operators)**.** *Let* $T \in \mathscr{F}_H$ *and let* V *be a closed finite-codimensional subspace of* H *such that* $V \cap \operatorname{Ker} T$ *is trivial. Then, there exists an open neighborhood* U *of* T *in* $\mathscr{H}$ *such that:*

(1) $V \cap \operatorname{Ker} S$ *is trivial for all* $S \in U$*; and*

(2) the disjoint union

$$\bigsqcup_{S \in U} H/S(V),$$

topologized as a quotient space of the product space $U \times H$*, is a trivial vector bundle on* U*.*

Proof. For each $S \in \mathscr{H}$, we define the linear map

$$\begin{aligned} \varphi_S : V \oplus T(V)^\perp &\to H, \\ (u, v) &\mapsto S(u) + v. \end{aligned}$$

Then, we have the continuous linear map

$$\begin{aligned} \varphi : \mathscr{H} &\to \mathscr{L}(V \oplus T(V)^\perp, H), \\ S &\mapsto \varphi_S, \end{aligned}$$

where $\mathscr{L}$ stands for the space of all continuous linear maps with the norm topology. We have that φ_T is an isomorphism. In fact, φ_T is injective because $T \mid_V : V \to H$ is injective once $V \cap \operatorname{Ker} T$ is trivial. Moreover, we have that φ_T is surjective because

$$\varphi_T(V \oplus T(V)^\perp) = T(V) \oplus T(V)^\perp = H.$$

Since the isomorphisms form an open set in $\mathscr{L}$, there exists a neighborhood U of T in $\mathscr{H}$ in such manner that φ_S is an isomorphism for all $S \in U$. This implies the claims of the statement. Indeed:

- since φ_S is injective for all $S \in U$, it follows that $\operatorname{Ker} S$ is trivial for all $S \in U$. Thus, $V \cap \operatorname{Ker} S$ is trivial for all $S \in U$; *and*

- since $S \in U$ is an isomorphism,

  $$H/S(V) \simeq S^{-1}(H)/V \simeq T(V)^\perp.$$

  Consequently,

  $$\bigsqcup_{S \in U} H/S(V) \simeq \bigsqcup_{S \in U} \{S\} \times T(V)^\perp = U \times T(V)^\perp.$$

  This proves that the disjoint union in question is a trivial vector bundle since the reader can prove that $T(V)^\perp$ is finite-dimensional once $T \in \mathscr{F}_H$ and V is finite-codimensional. $\square$

Theorem 4.23 (Vector bundles induced by a continuous map with codomain in the space of Fredholm operators)**.** *Let X be a compact Hausdorff space and $T : X \to \mathscr{F}_H$ be a continuous map. Then, there exists a closed finite-codimensional subspace V of H such that:*

(1) $V \cap \operatorname{Ker} T_x$ is trivial for all $x \in X$; and

(2) $H/T(V) := \bigsqcup_{x\in X} H/T_x(V)$, topologized as a quotient space of $X \times H$, is a vector bundle on X.

Proof. For each $x \in X$, let $V_x := (\operatorname{Ker} T_x)^{\perp}$ and let U_x be an open neighborhood of T_x in $\mathscr{H}$ as in Lemma 4.22. We define

$$W_x := T^{-1}(U_x)$$

for all $x \in X$. The collection $\{W_x\}_{x\in X}$ is an open cover of X. As a consequence, since X is a compact space, we can take a finite subcover $\{W_{x_i}\}_{i=1}^{m}$ of X. Thence, we define

$$V := \bigcap_{i=1}^{m} W_{x_i}.$$

We have that V is a closed finite-codimensional subspace of H[2]. Moreover, V satisfies the first condition of this theorem because of Condition (1) of Lemma 4.22. Further, V satisfies the second condition of this theorem because we can apply Condition (2) of Lemma 4.22 to T_x for each $x \in X$. This ensures that $\bigsqcup_y H/T_y(V)$ is locally trivial near x for all $x \in X$. Therefore, $H/T(V)$ is locally trivial, being then a vector bundle, as desired. □

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[2] The fact that V is closed follows from it being the intersection of closed subspaces of H. To prove that it is finite-codimensional, we show that any finite intersection of finite-codimensional subspaces of H is also finite-codimensional. Indeed, let E and F be finite-codimensional subspaces of H. We define the linear map

$$\begin{aligned} \Phi : H &\to (H/E) \oplus (H/F), \\ v &\mapsto (v + E, V + F). \end{aligned}$$

The kernel of this map is $E \cap F$. Therefore, Φ induces the injective linear map

$$H/(E \cap F) \to (H/E) \oplus (H/F).$$

The injectivity of this induced map not only ensures that $E \cap F$ is finite-codimensional, but also yields the inequality $\operatorname{codim}(E \cap F) \leq \operatorname{codim}(E) + \operatorname{codim}(F)$. The reader can prove the general case using induction.

Remark 4.24 (Splitting a map involving the vector bundle of the preceding result). *We can split the natural map*

$$\begin{aligned} \rho:\ X \times H &\rightarrow H/T(V), \\ (x, v) &\mapsto [v]_x, \end{aligned}$$

where $[v]_x$ denotes the equivalence class of v in $H/T_x(V)$. More precisely, we can find a continuous map

$$\varphi : H/T(V) \rightarrow X \times H$$

commuting with the projections on X and such that

$$\rho \circ \varphi \ =\ \mathrm{id}_{H/T(V)}.$$

Indeed, by definition, ρ splits locally. Thus, being $\mathfrak{U} = \{U_i\}_{i=1}^m$ a finite open cover of X, we can choose splittings φ_i over U_i for each i between 1 and m, both included. Thence, we have

$$\theta_{ij} \ :=\ \varphi_i - \varphi_j :\ H/T(V)\,|_{U_{ij}} \rightarrow U_{ij} \times V,$$

where $U_{ij} := U_i \cap U_j$ for all i and j between 1 and m, both included. Therefore, if $\Sigma = \{\sigma_i\}_{i=1}^m$ is a partition of the unity subordinated to the open cover $\mathfrak{U}$, we define the map

$$\theta_i \ :=\ \sum_{j=1}^{m} \sigma_j \theta_{ij}.$$

Consequently, not only θ_i is defined on U_i, but also $\varphi = \varphi_i - \theta_i$ is well-defined and gives the required splitting. ♢

Definition 4.25 (The generalized index of the Atiyah-Jänich Theorem). *Let X be a compact Hausdorff space and $T : X \rightarrow \mathscr{F}_H$ be a continuous map. In this situation, we define:*

- *a* ***choice*** *for T to be a closed finite-codimensional subspace V of H that satisfies the conditions of Theorem 4.23; and*
- *being V a choice for T,*

$$\text{index}_T \;:=\; [[X \times H/V]] - [[H/T(V)]] \;\in\; K(X),$$

This rule defines the index map in Theorem 4.20. ♢

Remark 4.26 (The generalized index is well-defined)**.** *Let X be a compact Hausdorff space and $T : X \to \mathscr{F}_H$ be a continuous map. First, we prove that the generalized index of T defined above does not depend on the choice V for T. Indeed, let W be another choice for T. Evidently, $V \cap W$ is also a choice for T. Therefore, we may assume that W is a subspace of V. In this situation, we have the exact sequences of vector bundles*

$$0 \longrightarrow X \times V/W \longrightarrow X \times H/W \longrightarrow X \times H/V \longrightarrow 0,$$

$$0 \longrightarrow X \times V/W \longrightarrow H/T(W) \longrightarrow H/T(V) \longrightarrow 0.$$

Consequently,

$$[[X \times H/V]] - [[X \times H/W]] \;=\; [[X \times V/W]] \;=\; [[H/T(V)]] - [[H/T(W)]].$$

Thence,

$$\begin{aligned} \text{index}_T &= [[X \times H/V]] - [[H/T(V)]] \\ &= [[X \times V/W]] \\ &= [[X \times H/W]] - [[H/T(W)]], \end{aligned}$$

as claimed. Further, the generalized index is clearly functorial. Thus, if $f : Y \to X$ is a continuous map, then

$$\text{index}_{T \circ f} \;=\; K(f)(\text{index}_T).$$

This follows from the fact that a choice V for T is also a choice for $T \circ f$. Moreover, let $S : X \to \mathscr{F}_H$ be a continuous map homotopic to T. Under these circumstances, there exists a homotopy $\mathfrak{T} : X \times \mathbb{I} \to \mathscr{F}_H$ between T and S. Therefore, we have that $\text{index}_{\mathfrak{T}} \in K(X \times \mathbb{I})$ *restricts to* $\text{index}_T \in K(X \times \{0\})$ *and to* $\text{index}_S \in K(X \times \{1\})$*. Since*

$$\begin{aligned} K(X \times \mathbb{I}) &\to K(X \times \{0\}) \simeq K(X) \\ K(X \times \mathbb{I}) &\to K(X \times \{1\}) \simeq K(X) \end{aligned}$$

are isomorphisms,

$$\mathrm{index}(T) = \mathrm{index}(S).$$

As a consequence,

$$\mathrm{index}: [X, \mathscr{F}_H] \to K(X)$$

is well-defined. ♢

Lemma 4.27 (The generalized index presented in Definition 4.25 is also a group homomorphism)**.** *Let X be a compact Hausdorff space and $T, S : X \to \mathscr{F}_H$ be continuous maps. Under these circumstances, the composition of T and S is defined to be the continuous map*

$$\begin{aligned} T \circ S: X &\to \mathscr{F}_H, \\ x &\mapsto T_x \circ S_x. \end{aligned}$$

We have

$$\mathrm{index}_{T \circ S} = \mathrm{index}_T + \mathrm{index}_S.$$

In particular, if $T : X \to \mathscr{F}_H$ is such that $T_x \in \mathscr{F}_H$ is invertible for all $x \in X$, then $\mathrm{index}_{T^{-1}} = -\mathrm{index}_T$*, where*

$$\begin{aligned} T^{-1}: X &\to \mathscr{F}_H, \\ x &\mapsto T_x^{-1}. \end{aligned}$$

Proof. Let W be a choice for T. We may assume $S(H) \subseteq W$. Indeed, if it is not the case, then we can replace S by the homotopic map $\pi_W \circ S$, where $\pi_W : H \to H$ is the projection onto W, without changing the index. In addition, let V be a choice for S. The reader can prove that V is also a choice for $T \circ S$. Therefore, we have the exact sequence of vector bundles

$$0 \longrightarrow W/S(V) \xrightarrow{\ T\ } H/(T \circ S)(V) \longrightarrow H/T(W) \longrightarrow 0.$$

Consequently,

$$[[H/(T\circ S)(V)]] \;=\; [[W/S(V)]] + [[H/T(W)]].$$

Furthermore, we have

$$[[W/S(V)]] \;=\; [[H/S(V)]] - [[X\times H/W]].$$

Hence,

$$\begin{aligned}
\operatorname{index}_{T\circ S} &= [[X\times H/V]] - [[H/(T\circ S)(V)]],\\
&= [[X\times H/V]] - [[W/S(V)]] - [[H/T(W)]]\\
&= [[X\times H/V]] - [[H/S(V)]] + [[X\times H/W]] - [[H/T(W)]]\\
&= \operatorname{index}_S + \operatorname{index}_T,
\end{aligned}$$

as required. □

Theorem 4.28 (An special exact sequence involving the generalized index map defined above)**.** *Let X be a compact Hausdorff space. We have an exact sequence of groups and group homomorphisms*

$$[X,\mathscr{H}^*] \longrightarrow [X,\mathscr{F}_H] \xrightarrow{\text{index}} K(X) \longrightarrow 0.$$

Proof. Let $T : X \to \mathscr{F}_H$ be a continuous map of index zero. This means that, for any choice V for T, we have

$$[[X\times H/V]] \;=\; [[H/T(V)]].$$

Therefore, we have that there exists a trivial vector bundle P on X for which $(X\times H/V)\oplus P$ and $H/T(V)\oplus P$ are isomorphic. Equivalently, if we replace V by one of its closed subspaces W such that $\dim(V/W) = \dim P$[3], then we obtain as a consequence

$$X\times H/W \;\simeq\; H/T(W).$$

[3] This is possible because, once P has finite dimension, we are allowed to choose $\dim P$ linearly independent vectors in V. Thence, we can consider the subspace U of V generated by these vectors. Consequently, it suffices to set $W := U^{\perp}$. Indeed, W is closed, once it is an orthogonal complement, and the quotient space V/W is isomorphic to U. Therefore, this subspace of V has the same dimension of P by construction.

Indeed,

$$\begin{aligned} X \times H/W &\simeq (X \times H/V) \oplus (X \times V/W) \\ &\simeq (X \times H/V) \oplus P \\ &\simeq H/T(V) \oplus P \\ &\simeq H/T(V) \oplus (X \times V/W) \\ &\simeq H/T(V) \oplus (X \times T(V)/T(W)) \\ &\simeq H/T(W). \end{aligned}$$

Consequently, splitting

$$\rho : \; X \times H \;\to\; H/T(W) \;\simeq\; X \times H/W$$

as in Remark 4.24, we obtain a continuous map

$$\varphi : \; X \times H/W \;\to\; X \times H$$

that is linear on the fibers and commutes with the projections onto X. Then, we have the continuous map

$$\begin{aligned} \Phi : \; X &\;\to\; \mathscr{L}(H/W,\, H), \\ x &\;\mapsto\; \Phi_x, \end{aligned}$$

where

$$\Phi_x[v] \;=\; (\pi \circ \varphi)(x,\, [v])$$

for all $[v] \in H/W$, being $\pi : X \times H \to H$ the natural projection onto the second factor. As a consequence of this, it follows from the construction of the splitting in question that the map

$$\begin{aligned} T + \Phi : \; X &\;\to\; \mathscr{H}^*, \\ x &\;\mapsto\; T_x + \Phi_x, \end{aligned}$$

is continuous. Hence,

$$\begin{aligned} \mathfrak{T}:\ X\times\mathbb{I} &\to \mathscr{F}_H, \\ (x,\,t) &\mapsto T+t\,\Phi, \end{aligned}$$

is a homotopy of maps connecting T with $T+\Phi$. This proves exactness in the middle of the sequence in the statement. Thence, it only remains to show that the index is surjective. With this purpose in mind, let E be a vector bundle on X. In addition, let $\pi_x : V \to V$ denote the projection onto the subspace corresponding to E_x for all $x \in X$. We define

$$\begin{aligned} S:\ X &\to \mathscr{F}_{H\otimes V} \simeq \mathscr{F}_H, \\ x &\mapsto S_{-1}\otimes\pi_x+\mathrm{id}_H\otimes(1-\pi_x), \end{aligned}$$

where S_{-1} is the shift defined in the proof of Theorem 4.11. The reader can prove that S_x is injective for all $x \in X$ and

$$H\otimes V/S(H\otimes V) \simeq E.$$

These facts imply

$$\mathrm{index}_S = -[[E]].$$

Moreover, the constant map

$$\begin{aligned} k:\ X &\to \mathscr{F}_H, \\ x &\mapsto S_k, \end{aligned}$$

where S_k is also the shift defined in the proof of Theorem 4.11, has index $[[k]] \in K(X)$. Consequently,

$$\mathrm{index}_{k\circ S} = [[k]]-[[E]]$$

because of Lemma 4.27. Therefore, since we have that every element of $K(X)$ is of the form $[[k]]-[[E]]$, this shows that the index is surjective. The proof of the theorem is then completed. □

Ultimately, the Atiyah-Jänich Theorem (Theorem 4.20) follows from Theorems 4.28 and 4.29. This last fundamental result is named after **Nicolaas Kuiper**

(1920-1994). In fact, Theorem 4.29 ensures that the exact sequence of Theorem 4.28 is a short exact sequence, which is equivalent to the generalized index map being an isomorphism.

Theorem 4.29 (Kuiper's Theorem)**.** *The topological group $\mathscr{H}^*$ is contractible. As a consequence, if X is a compact Hausdorff space, then we have that $[X, \mathscr{H}^*]$ is the trivial group.*

Proof. The reader can find a complete proof of this result in [22, pp. 27-28]. This proof is quite technical, involving notions such as partitions of the unity, nerve of coverings, CW-complexes, *et reliqua*. The reader may skip it since its details are not needed in this thesis. □

Remark 4.30 (Hilbert bundles and its analogous K-Theory)**.** *Changing in Definition C.1 the finite-dimensional vector space $\mathscr{V}$ by a complex Hilbert space H, we obtain the notion of* ***Hilbert bundle****. Using Kuiper's Theorem, one can prove that, if X is a compact Hausdorff space, then every Hilbert bundle on X is trivial. In particular, if we take the Grothendieck group of the monoid of isomorphism classes of Hilbert bundles on X, then we obtain the trivial group. Therefore, we have that it is not interesting to consider an infinite-dimensional version of K-Theory through the obvious adaptation of the finite-dimensional case treated in Chapter 2. More details will be given in Chapter 5.* ♢

Chapter 5

Twisted K-Theory

In this chapter, we close the text by putting together all of the theory studied before in order to develop models of Twisted K-Theory. We begin by introducing the Grothendieck group of twisted vector bundles as a model for finite-order Twisted K-Theory. Afterwards, we describe the infinite-dimensional model, through suitable bundles of Fredholm operators, that holds for twisting classes of any order. Finally, we compare these two models in the finite-order setting. We also consider versions of the Thom isomorphisms in this framework. We used [4], [6, pp. 5-8, 30-36, 43-45, 53-54], [7, pp. 42-43] and [20].

5.1 Twisted vector bundles

In this section, we present the fundamental notion that one must know in order to understand the finite-dimensional model of Twisted K-Theory, which is the one of twisted vector bundles. This concept has an obvious parallel with the one presented in Appendix C. This parallel will become even more evident in the next sections. We begin with the following notation that must be kept in mind until the end of the chapter.

Notation 5.1 (Good covers and Čech cohomology). *In this chapter, X always denotes a paracompact Hausdorff space that admits a good cover $\mathfrak{U} = \{U_i\}_{i\in I}$. We remind the reader that $\mathfrak{U}$ being a good cover means that it is an open cover for which every finite intersection*

$$U_{i_0 \cdots i_n} := \bigcap_{j=0}^{n} U_{i_j}$$

is contractible[(1)]. *We also extend to this chapter the conventions on Čech cohomology of Notation 3.56. In particular:*

- *we denote by* $\underline{\mathrm{U}}(n)$ *the sheaf of* $\mathrm{U}(n)$*-valued continuous functions on* X*; and*
- *when* $n = 1$*, we denote by* $\check{C}^m(\mathfrak{U}, \underline{\mathrm{U}}(1))$*,* $\check{Z}^m(\mathfrak{U}, \underline{\mathrm{U}}(1))$ *and* $\check{H}^m(\mathfrak{U}, \underline{\mathrm{U}}(1))$ *the corresponding* m*-cochains,* m*-cocycles and* m*-cohomology classes, with respect to the good cover* $\mathfrak{U}$. ♢

Definition 5.2 (Twisted vector bundle)**.** *Consider a* 2*-cochain*

$$\zeta := \{\zeta_{ijk}\}_{i,j,k \in I} \in \check{C}^2(\mathfrak{U}, \underline{\mathrm{U}}(1)).$$

We say that an n*-dimensional* $\boldsymbol{\zeta}$***-twisted vector bundle*** E *on* X *is a collection of* n*-dimensional trivial Hermitian vector bundles* $\{\pi_i : E_i \to U_i\}_{i \in I}$ *(see Definition C.48) and of unitary vector bundle isomorphisms* $\{\varphi_{ij} : E_i \mid_{U_{ij}} \to E_j \mid_{U_{ij}}\}_{i,j \in I}$ *such that the equality*

$$\varphi_{ki} \mid_{E_k|_{U_{ijk}}} \circ \varphi_{jk} \mid_{E_j|_{U_{ijk}}} \circ \varphi_{ij} \mid_{E_i|_{U_{ijk}}} = \zeta_{ijk} \cdot \mathrm{id}_{E_i|_{U_{ijk}}} \tag{5.1}$$

holds for all $i, j, k \in I$*. The notations used in this definition will be applied in the whole chapter.* ♢

Remark 5.3 (Ordinary vector bundles are twisted vector bundles up to identification)**.** *The following facts hold true.*

- *If* E *is an ordinary vector bundle, then we can equip it with an Hermitian product by Theorem C.49. We choose a good cover* $\mathfrak{U} = \{U_i\}_{i \in I}$ *of* X *such that* $E \mid_{U_i}$ *is trivial for all* $i \in I$*. Thence,* E *is a* ζ*-twisted vector bundle where* $\zeta_{ijk} = 1$ *for all* $i, j, k \in I$*. Indeed, we set*

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(1) There exist paracompact Hausdorff spaces that do not admit good covers. For example, every space that is not locally contractible, such as the Hawaiian earring, does not admit a good cover. Nevertheless, it is proved in [7, pp. 42-43] that every smooth manifold admits a good cover. Moreover, it is proved in [7, p. 43] that the set of good covers of a smooth manifold is cofinal in the set of all open covers. This means that every open cover admits a refinement that is a good cover. We shall admit that this property holds for X when necessary.

$$E_i \;:=\; E\,|_{U_i}$$

for all $i \in I$ *and*

$$\varphi_{ij} \;:=\; \mathrm{id}_{E_i|_{U_{ij}}} \,:\, E_i\,|_{U_{ij}} \to E_j\,|_{U_{ij}}$$

for all $i, j \in I$.

- *If* E *is a* ζ*-twisted vector bundle on* X *such that* $\zeta_{ijk} = 1$ *for all* $i, j, k \in I$, *then the quotient of the disjoint union* $\bigsqcup_{i\in I} E_i$ *by the equivalence relation that identifies* v *with* $\varphi_{ij}(v)$ *for all* $v \in E_i\,|_{U_{ij}}$ *and all* $i, j \in I$ *is an ordinary vector bundle.*

These facts show the relation between ordinary and twisted vector bundles. In fact, they show that twisted vector bundles generalize ordinary vector bundles, as one would naturally expect. ♢

Lemma 5.4 (On the 2-cochain of a ζ-twisted vector bundle)**.** *In a* ζ*-twisted vector bundle* E, *we have that* $\zeta = \{\zeta_{ijk}\}_{i,j,k\in I}$ *is necessarily a* 2*-cocycle. As a consequence, the cohomology class*

$$[\zeta] \,\in\, \check{H}^2(\mathfrak{U},\, \underline{\mathrm{U}}(1)) \,\simeq\, H^3(X,\, \mathbb{Z})$$

is well-defined.

Proof. Leaving some restrictions implicit, we have

$$\begin{aligned}
(\check{\delta}^2\zeta)_{ijkl}\cdot \mathrm{id}_{E_i|_{U_{ijkl}}} &= (\zeta_{jkl}\,\zeta_{ikl}^{-1}\,\zeta_{ijl}\,\zeta_{ijk}^{-1})\cdot \mathrm{id}_{E_i|_{U_{ijkl}}}\\
&= (\zeta_{jkl}\,\zeta_{ikl}^{-1}\,\zeta_{ijl})\cdot(\zeta_{ijk}^{-1}\cdot \mathrm{id}_{E_i|_{U_{ijkl}}})\\
&= (\zeta_{jkl}\,\zeta_{ikl}^{-1}\,\zeta_{ijl})\cdot(\varphi_{ji}\circ\varphi_{kj}\circ\varphi_{ik})\\
&= (\zeta_{jkl}\,\zeta_{ikl}^{-1})\cdot(\zeta_{ijl}\cdot \mathrm{id}_{E_i|_{U_{ijkl}}})\circ(\varphi_{ji}\circ\varphi_{kj}\circ\varphi_{ik})\\
&= (\zeta_{jkl}\,\zeta_{ikl}^{-1})\cdot(\varphi_{li}\circ\varphi_{jl}\circ\varphi_{ij})\circ(\varphi_{ji}\circ\varphi_{kj}\circ\varphi_{ik})\\
&= \zeta_{jkl}\cdot(\zeta_{ikl}^{-1}\cdot \mathrm{id}_{E_i|_{U_{ijkl}}})\circ(\varphi_{li}\circ\varphi_{jl}\circ\varphi_{ij})\circ(\varphi_{ji}\circ\varphi_{kj}\circ\varphi_{ik})\\
&= \zeta_{jkl}\cdot(\varphi_{ki}\circ\varphi_{lk}\circ\varphi_{il})\circ(\varphi_{li}\circ\varphi_{jl}\circ\varphi_{ij})\circ(\varphi_{ji}\circ\varphi_{kj}\circ\varphi_{ik})\\
&= \varphi_{ki}\circ\varphi_{lk}\circ\varphi_{jl}\circ(\zeta_{jkl}\cdot \mathrm{id}_{E_j|_{U_{ijkl}}})\circ\varphi_{kj}\circ\varphi_{ik}\\
&= \varphi_{ki}\circ\varphi_{lk}\circ\varphi_{jl}\circ(\varphi_{lj}\circ\varphi_{kl}\circ\varphi_{jk})\circ\varphi_{kj}\circ\varphi_{ik}\\
&= 1\cdot \mathrm{id}_{E_i|_{U_{ijkl}}}
\end{aligned}$$

for all $i, j, k, l \in I$. It follows from the definitions of Čech cohomology that $[\zeta]$ is well-defined. Additionally, the reader may recall that $\check{H}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$ is isomorphic to $H^3(X, \mathbb{Z})$ from the end of the proof of Theorem 3.76. This finishes the proof of the lemma. $\square$

Definition 5.5 (Morphisms of twisted vector bundles)**.** *Let*

$$\begin{aligned} E &= (\{\pi_{E_i} : E_i \to U_i\}_{i \in I}, \{\varphi_{ij} : E_i\,|_{U_{ij}} \to E_j\,|_{U_{ij}}\}_{i,j \in I}) \quad \textit{and} \\ F &= (\{\pi_{F_i} : F_i \to U_i\}_{i \in I}, \{\psi_{ij} : F_i\,|_{U_{ij}} \to F_j\,|_{U_{ij}}\}_{i,j \in I}) \end{aligned}$$

*be ζ-twisted vector bundles. A **morphism of ζ-twisted vector bundles** from E to F is a collection of vector bundle morphisms*

$$f = \{f_i : E_i \to F_i\}_{i \in I}$$

such that the diagram

$$\begin{tikzcd} E_i\,|_{U_{ij}} \arrow[r, "f_i|_{E_i|_{U_{ij}}}"] \arrow[d, "\varphi_{ij}"'] & F_i\,|_{U_{ij}} \arrow[d, "\psi_{ij}"] \\ E_j\,|_{U_{ij}} \arrow[r, "f_j|_{E_j|_{U_{ij}}}"'] & F_j\,|_{U_{ij}} \end{tikzcd}$$

*is commutative for all $i, j \in I$. We say that $f : E \to F$ is a **unitary morphism** if $f_i : E_i \to F_i$ is unitary for each $i \in I$. In addition, an **isomorphism** is an invertible morphism, which is equivalent to $f_i : E_i \to F_i$ being a vector bundle isomorphism for all $i \in I$. We denote by $\mathrm{VB}_\zeta(X)$ the **set of isomorphism classes** of ζ-twisted vector bundles on X. Finally, the **category of ζ-twisted vector bundles** $\mathrm{TVectBdl}_X^\zeta$ is established as in Definition C.6.* $\diamondsuit$

Now we approach twisted vector bundles as we approached ordinary vector bundles through nonabelian Čech cohomology in Appendix C. Indeed, the following definition generalizes to the nonabelian setting the basic tools of Čech cohomology in low degree.

Definition 5.6 (Twisted first degree nonabelian Čech cohomology)**.** *We have that a Čech* 0*-cochain of the sheaf* $\underline{\mathrm{U}}(n)$ *is a collection of continuous functions*

$$\{g_i : U_i \to \mathrm{U}(n)\}_{i \in I}.$$

Similarly, we have that a Čech 1*-cochain of the sheaf* $\underline{\mathrm{U}}(n)$ *is a collection of continuous functions*

$$\{g_{ij} : U_{ij} \to \mathrm{U}(n)\}_{i,j \in I}.$$

We denote the sets of 0*-cochains and* 1*-cochains of* $\underline{\mathrm{U}}(n)$ *by* $\check{C}^0(\mathfrak{U}, \underline{\mathrm{U}}(n))$ *and* $\check{C}^1(\mathfrak{U}, \underline{\mathrm{U}}(n))$*, respectively. For* $\zeta = \{\zeta_{ijk}\}_{i,j,k \in I} \in \check{C}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$*, a* 1*-cochain* $\{g_{ij}\}_{i,j \in I} \in \check{C}^1(\mathfrak{U}, \underline{\mathrm{U}}(n))$ *is a* ***ζ-cocycle*** *provided that*

$$g_{ki}\, g_{jk}\, g_{ij} = \zeta_{ijk} \cdot I_n$$

for all $i, j, k \in I$*. We denote the set of* ζ*-cocycles by* $\check{Z}^1_\zeta(\mathfrak{U}, \underline{\mathrm{U}}(n))$*. We have an action of* 0*-cochains on* 1*-cochains defined by*

$$\begin{aligned} \cdot : \check{C}^0(\mathfrak{U}, \underline{\mathrm{U}}(n)) \times \check{C}^1(\mathfrak{U}, \underline{\mathrm{U}}(n)) &\to \check{C}^1(\mathfrak{U}, \underline{\mathrm{U}}(n)), \\ (\{g_i\}_{i \in I}, \{g_{ij}\}_{i,j \in I}) &\mapsto \{g_i\, g_{ij}\, g_j^{-1}\}_{i,j \in I}. \end{aligned}$$

The reader can readily prove that this action determines an equivalence relation on $\check{C}^1(\mathfrak{U}, \underline{\mathrm{U}}(n))$*. It can also be proved that this relation restricts to an equivalence relation on* $\check{Z}^1_\zeta(\mathfrak{U}, \underline{\mathrm{U}}(n))$*. The quotient of* $\check{Z}^1_\zeta(\mathfrak{U}, \underline{\mathrm{U}}(n))$ *by the action of* 0*-cochains, which we hereafter denote by*

$$\check{H}^1_\zeta(\mathfrak{U}, \underline{\mathrm{U}}(n)),$$

is the ***ζ-twisted cohomology set*** *of degree* 1 *and rank* n*. This finishes the construction of twisted nonabelian Čech cohomology.* ♢

Remark 5.7 (On the twisted first degree nonabelian Čech cohomology)**.** *Because of Remark 5.3, when* $\zeta_{ijk} = 1$ *for all* $i, j, k \in I$*, Definition 5.6 becomes the ordinary first degree nonabelian Čech cohomology of Definition C.27. It classifies the isomorphism classes of* n*-dimensional ordinary vector bundles on* X *(Remark C.29). We can prove that the twisted first degree nonabelian Čech cohomology plays the same role for twisted vector bundles. In fact, let*

$$\zeta \;=\; \{\zeta_{ijk}\}_{i,j,k\in I} \;\in\; \check{C}^2(\mathfrak{U},\, \underline{\mathrm{U}}(1)).$$

In addition, consider

$$E \;=\; (\{\pi_i :\, E_i \to U_i\}_{i\in I},\, \{\varphi_{ij} :\, E_i\,|_{U_{ij}} \to E_j\,|_{U_{ij}}\}_{i,j\in I})$$

to be an n-dimensional ζ-twisted vector bundle. For each $i \in I$, we choose n pointwise linearly independent local sections $s_{1,i}, \cdots, s_{n,i} : U_i \to E_i$ of unit norm, determining vector bundle isomorphisms

$$\begin{aligned} \xi_i :\, E_i \;&\to\; U_i \times \mathbb{C}^n, \\ \sum_{j=1}^{n} \lambda_j s_{j,i}(x) \;&\mapsto\; (x,\, (\lambda_1, \cdots, \lambda_n)). \end{aligned}$$

We have that the isomorphisms $\varphi_{ij} : E_i\,|_{U_{ij}} \to E_j\,|_{U_{ij}}$ determine local transition functions $g_{ij} : U_{ij} \to \mathrm{U}(n)$ such that

$$\varphi_{ij}(\xi_i^{-1}(x,\, \lambda)) \;=\; \xi_j^{-1}(x,\, g_{ij}(x)\cdot\lambda)$$

for all $x \in U_{ij}$ and all $\lambda \in \mathbb{C}^n$. It follows from this equation that Equation (5.1) *is equivalent to*

$$g_{ki}\, g_{jk}\, g_{ij} \;=\; \zeta_{ijk}\cdot I_n$$

for all $i, j, k \in I$. As a consequence, we have $\{g_{ij}\}_{i,j\in I} \in \check{Z}^1_\zeta(\mathfrak{U},\, \underline{\mathrm{U}}(n))$. Finally, it is straightforward to verify, as for ordinary vector bundles, that the cohomology class $[\{g_{ij}\}_{i,j\in I}] \in \check{H}^1_\zeta(\mathfrak{U},\, \underline{\mathrm{U}}(n))$ only depends on the isomorphism class of E. Consequently, we have that

$$\begin{aligned} \mathrm{VB}_\zeta(X) \;&\to\; \check{H}^1_\zeta(\mathfrak{U},\, \underline{\mathrm{U}}(n)), \\ [E] \;&\mapsto\; [\{g_{ij}\}_{i,j\in I}], \end{aligned}$$

is an isomorphism, as we wished. $\diamondsuit$

Theorem 5.8 (On the cohomology class of Lemma 5.4)**.** *In an n-dimensional ζ-twisted vector bundle, it is a torsion class*

$$[\zeta] \;\in\; \check{H}^2(\mathfrak{U},\, \underline{\mathrm{U}}(1)).$$

Proof. Here we use the notation of Remark 5.7. In fact, computing the determinants, we obtain

$$\det(g_{ki})\,\det(g_{jk})\,\det(g_{ij}) \;=\; \det(\zeta_{ijk}\cdot I_n) \;=\; \zeta_{ijk}^n$$

for all $i, j, k \in I$. Therefore, since $\det(g_{ij})$ is a $\mathrm{U}(1)$-valued function, it follows that

$$[\zeta]^n \;=\; [\{\zeta_{ijk}^n\}_{i,j,k\in I}] \,\in\, \check{H}^2(\mathfrak{U},\, \underline{\mathrm{U}}(1))$$

is a trivial cocycle. In particular, note that the order of $[\zeta]$ divides n. This finishes the proof of the theorem. $\square$

Theorem 5.9 (Dependence on the cocycle)**.** *Suppose that* $\zeta, \xi \in \check{Z}^2(\mathfrak{U},\, \underline{\mathrm{U}}(1))$ *are cohomologous cocycles. Then, let* $\eta = \{\eta_{ij}\}_{i,j\in I}$ *be such that* $\xi = \zeta\cdot\check{\delta}^1\eta$. *We have that the map*

$$\begin{aligned} \Phi_\eta :\; \check{H}^1_\zeta(\mathfrak{U},\, \underline{\mathrm{U}}(n)) \;&\to\; \check{H}^1_\xi(\mathfrak{U},\, \underline{\mathrm{U}}(n)),\\ [\{g_{ij}\}_{i,j\in I}] \;&\mapsto\; [\{g_{ij}\cdot\eta_{ij}\}_{i,j\in I}], \end{aligned}$$

is an isomorphism.

Proof. The reader can readily prove that Ψ_η is the two-sided inverse of Φ_η, where

$$\begin{aligned} \Psi_\eta :\; \check{H}^1_\xi(\mathfrak{U},\, \underline{\mathrm{U}}(n)) \;&\to\; \check{H}^1_\zeta(\mathfrak{U},\, \underline{\mathrm{U}}(n)),\\ [\{h_{ij}\}_{i,j\in I}] \;&\mapsto\; [\{h_{ij}\cdot\eta_{ij}^{-1}\}_{i,j\in I}]. \end{aligned}$$

This finishes the proof of the theorem. $\square$

Remark 5.10 (On the preceding result)**.** *Because of Remark 5.7, Theorem 5.9 can be equivalently stated as*

$$\begin{aligned} \Phi_\eta :\; \mathrm{VB}_\zeta(X) \;&\to\; \mathrm{VB}_\xi(X),\\ [\{E_i\}_{i\in I},\, \{\varphi_{ij}\}_{i,j\in I}] \;&\mapsto\; [\{E_i\}_{i\in I},\, \{\varphi_{ij}\cdot\eta_{ij}\}_{i,j\in I}], \end{aligned}$$

being an isomorphism. This approach shows that the set $\mathrm{VB}_\zeta(X)$ *only depends on* $[\zeta] \in \check{H}^2(\mathfrak{U},\, \underline{\mathrm{U}}(1))$. *Nevertheless, this dependence is non-canonical since* Φ_η *depends* *on η by construction. More precisely, the choice of η is unique up to a cocycle and the equality $\Phi_\eta = \Phi_\chi$ holds if and only if $\eta^{-1}\chi$ is a coboundary. Therefore, the set of isomorphisms of the form Φ_η is a $\check{H}^1(\mathfrak{U}, \underline{\mathrm{U}}(1))$-torsor. In fact, since $\check{H}^1(\mathfrak{U}, \underline{\mathrm{U}}(1))$ is isomorphic to $H^2(X, \mathbb{Z})$, the first Chern class implies that the set of isomorphisms of the form Φ_η is a Pic_X-torsor, the latter corresponding to the group of ordinary line bundles on X (see Corollary C.45). In particular, if $H^2(X, \mathbb{Z})$ is trivial, then we can define $\mathrm{VB}_{[\zeta]}(X)$ canonically. In general, $\mathrm{VB}_\zeta(X)$ depends on the cocycle ζ up to the tensor product by a line bundle.* ♢

Definition 5.11 (Pullback of a cocycle)**.** *Let $\mathfrak{V} = \{V_\alpha\}_{\alpha\in J}$ be a good cover that refines $\mathfrak{U} = \{U_i\}_{i\in I}$. By definition, there exists a function $\phi : J \to I$ such that $V_\alpha \subseteq U_{\phi(\alpha)}$ for every $\alpha \in J$. Given a cocycle $\zeta = \{\zeta_{ijk}\}_{i,j,k\in I}$ based on $\mathfrak{U}$, we define its* ***pullback*** *through ϕ to be the cocycle based on $\mathfrak{V}$*

$$\phi^*\zeta \ = \ \{\tilde{\zeta}_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma\in J}$$

where

$$\tilde{\zeta}_{\alpha\beta\gamma} \ := \ \zeta_{\phi(\alpha)\phi(\beta)\phi(\gamma)}\,|_{V_{\alpha\beta\gamma}}$$

for all $\alpha, \beta, \gamma \in J$. ♢

Remark 5.12 (On the pullback of cocycles)**.** *Here we use the notation of Definition 5.11. We have that*

$$\begin{aligned} \Phi_\phi :\ \mathrm{VB}_\zeta(X) \ &\to\ \mathrm{VB}_{\phi^*\zeta}(X), \\ [\{E_i\}_{i\in I},\, \{\varphi_{ij}\}_{i,j\in I}] \ &\mapsto\ [\{\tilde{E}_\alpha\}_{\alpha\in J},\, \{\tilde{\varphi}_{\alpha\beta}\}_{\alpha,\beta\in J}], \end{aligned}$$

where

$$\begin{aligned} \tilde{E}_\alpha \ &:=\ E_{\phi(\alpha)}|_{V_\alpha} \quad \text{and} \\ \tilde{\varphi}_{\alpha\beta} \ &:=\ \varphi_{\phi(\alpha)\phi(\beta)}|_{V_{\alpha\beta}} \end{aligned}$$

for all $\alpha, \beta \in J$, is an isomorphism. Furthermore, in an analogous way, one can define an isomorphism

$$\Phi_\phi :\ \check{H}^1_\zeta(\mathfrak{U}, \underline{\mathrm{U}}(n)) \ \to\ \check{H}^1_{\phi^*\zeta}(\mathfrak{V}, \underline{\mathrm{U}}(n)).$$

Such isomorphisms depend on the function ϕ, which shows that they are non-canonical in general. Nevertheless, since the cohomology class represented by $\phi^*\zeta$ does not depend on ϕ, the isomorphism Φ_ϕ is canonical when $H^2(X,\,\mathbb{Z})$ is trivial. In this case, the sets $\check{H}^1_{[\zeta]}(X,\,\underline{\mathrm{U}}(n))$ are well-defined and

$$\mathrm{VB}_{[\zeta]}(X) \;\simeq\; \bigsqcup_{n\in\mathbb{N}} \check{H}^1_{[\zeta]}(X,\,\underline{\mathrm{U}}(n))$$

canonically. ♢

Definition 5.13 (Non-integral twisted vector bundles)**.** *Let ζ be a constant cocycle, that is, let $\zeta \in \check{Z}^2(\mathfrak{U},\,\mathrm{U}(1))$. We say that a ζ-twisted vector bundle is a* ***non-integral vector bundle****. We denote the set of non-integral vector bundles with twisting class ζ by* $\mathrm{NIVB}_\zeta(X)$. ♢

Remark 5.14 (On non-integral vector bundles)**.** *Let $\zeta \in \check{Z}^2(\mathfrak{U},\,\mathrm{U}(1))$. We have that the image of $[\zeta]$ in the cohomology of $\underline{\mathrm{U}}(1)$ is always a torsion class. However, the image of $[\zeta]$ in the cohomology of $\mathrm{U}(1)$ is not necessarily torsion. In fact, it is torsion if the transition functions can be chosen constant, which easily follows by adapting the proof of Theorem 5.8. In this case, we have isomorphisms analogous to the ones of Theorem 5.9 and Remark 5.10, but with respect to a $\mathrm{U}(1)$-cochain η. Furthermore, it follows that the set of isomorphisms of the form Φ_η is a torsor over the image of the natural map*

$$\check{H}^1(\mathfrak{U},\,\mathrm{U}(1)) \,\to\, \check{H}^1(\mathfrak{U},\,\underline{\mathrm{U}}(1)),$$

where

$$\check{H}^1(\mathfrak{U},\,\mathrm{U}(1)) \;\simeq\; H^1(X,\,\mathbb{R}/\mathbb{Z}) \qquad \textit{and} \qquad \check{H}^1(\mathfrak{U},\,\underline{\mathrm{U}}(1)) \;\simeq\; H^2(X,\,\mathbb{Z}).$$

One can prove that this image is canonically isomorphic to $\mathrm{Tor}\,H^2(X,\,\mathbb{Z})$. Therefore, if $\mathrm{Tor}\,H^2(X,\,\mathbb{Z})$ is trivial, then $\mathrm{NIVB}_{[\zeta]}(X)$ is canonically defined. In particular, note that this conclusion follows if X is simply connected. In fact, if X is simply connected, then $H_1(X,\,\mathbb{Z})$ is trivial. Hence, since $H^1(X,\,\mathbb{R}/\mathbb{Z})$ is isomorphic to the group of homomorphisms from $H_1(X,\mathbb{Z})$ into $\mathbb{R}/\mathbb{Z}$ by the Universal Coefficient Theorem, we are done here. ♢

Notation 5.15 (The groups of roots of unity). *Let r be a non-zero natural number. We denote by Γ_r the subgroup of* $\mathrm{U}(1)$ *formed by rth roots of unity. We have that Γ_r is the image of the group embedding*

$$\begin{aligned} \mathbb{Z}_r &\to \mathrm{U}(1), \\ a &\mapsto e^{2\pi i \frac{a}{r}}. \end{aligned}$$

In addition, we set

$$\Gamma_\infty := \bigcup_{r \in \mathbb{N}-\{0\}} \Gamma_r \qquad \text{and} \qquad \mathbb{Z}_\infty := \mathbb{Q}.$$

We have that Γ_∞ is the image of the group embedding

$$\begin{aligned} \mathbb{Z}_\infty &\to \mathrm{U}(1), \\ q &\mapsto e^{2\pi i q}. \end{aligned}$$

$\diamondsuit$

Definition 5.16 (More non-integral twisted vector bundles). *Let r be a non-zero natural number or ∞. We say that a* $\mathbb{Z}_r$***-non-integral vector bundle*** *is a ζ-twisted vector bundle where $\zeta \in \check{Z}^2(\mathfrak{U}, \Gamma_r)$. We denote the set of $\mathbb{Z}_r$-non-integral vector bundles with twisting class ζ by* $\mathrm{NIVB}^r_\zeta(X)$. $\diamondsuit$

Remark 5.17 (On the preceding non-integral vector bundles). *Let r be a non-zero natural number or ∞. The set of isomorphisms of the form Φ_η is a torsor over the image of the natural map*

$$\check{H}^1(\mathfrak{U}, \mathbb{Z}_r) \to \check{H}^1(\mathfrak{U}, \underline{\mathrm{U}}(1)),$$

where

$$\check{H}^1(\mathfrak{U}, \mathbb{Z}_r) \simeq \mathrm{Hom}(H_1(X,\mathbb{Z}), \mathbb{Z}_r) \qquad \text{and} \qquad \check{H}^1(\mathfrak{U}, \underline{\mathrm{U}}(1)) \simeq H^2(X, \mathbb{Z}).$$

This image is canonically isomorphic to the subgroup of $\mathrm{Tor}\, H^2(X, \mathbb{Z})$ *formed by classes of order r, that we denote by* $\mathrm{Tor}_r H^2(X, \mathbb{Z})$. *Therefore, if* $\mathrm{Tor}_r H^2(X, \mathbb{Z})$ *is trivial, then* $\mathrm{NIVB}^r_{[\zeta]}(X)$ *is canonically defined. As before, this conclusion holds if X is simply connected.* $\diamondsuit$

5.2 Absolute Twisted K-Theory

In this section, we define the most elementary notions of Twisted K-Theory, namely, the absolute Twisted K-Theory group and the induced group homomorphisms. In fact, any general definition of Twisted K-theory involves some infinite-dimensional geometric objects, like projective Hilbert bundles. Nevertheless, this is not necessary when the twisting class has finite order, as we summarize below. We begin with the following definition.

Definition 5.18 (The absolute Twisted K-Theory group)**.** *Let X be a paracompact Hausdorff space as in Notation 5.1. In addition, let $E = (\{E_i\}_{i\in I},\, \{\varphi_{ij}\}_{i,j\in I})$ and $F = (\{F_i\}_{i\in I},\, \{\psi_{ij}\}_{i,j\in I})$ be ζ-twisted vector bundles on X, with $\zeta \in \check{Z}^2(\mathfrak{U},\, \underline{\mathrm{U}}(1))$. We then define the* ***direct sum*** *of E and F as being the ζ-twisted vector bundle on X given by*

$$E \oplus F \;:=\; (\{E_i \oplus F_i\}_{i\in I},\, \{\varphi_{ij} \oplus \psi_{ij}\}_{i,j\in I}).$$

This direct sum induces

$$\begin{aligned} \oplus :\; \mathrm{VB}_\zeta(X) \times \mathrm{VB}_\zeta(X) &\;\to\; \mathrm{VB}_\zeta(X), \\ ([E],\, [F]) &\;\mapsto\; [E \oplus F]. \end{aligned}$$

We have that the set $\mathrm{VB}_\zeta(X)$, endowed with this direct sum operation, is an abelian semigroup. We then define its corresponding Grothendieck group, which we hereafter call the ***ζ-twisted absolute K-theory group*** *of X. We shall denote this group simply by $K_\zeta(X)$.* $\diamondsuit$

Definition 5.19 (Pullback in absolute Twisted K-Theory)**.** *Let $f : X \to Y$ be a continuous map between paracompact Hausdorff spaces. We suppose that there exists a good cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of Y for which*

$$f^*\mathfrak{U} \;:=\; \{f^{-1}(U_i)\}_{i\in I}$$

is a good cover of X. In this situation, being a 2-cocycle $\zeta = \{\zeta_{ijk}\}_{i\in I} \in \check{Z}^2(\mathfrak{U},\, \underline{\mathrm{U}}(1))$, we define

$$f^*\zeta \;:=\; \{\zeta_{ijk} \circ f\}_{i\in I} \;\in\; \check{Z}^2(f^*\mathfrak{U},\, \underline{\mathrm{U}}(1)).$$

Moreover, being $E = (\{E_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ *a* ζ*-twisted vector bundle, we define the* $f^*\zeta$*-twisted vector bundle*

$$f^*_\zeta E := (\{f^*E_i\}_{i\in I},\ \{f^*\varphi_{ij}\}_{i,j\in I}),$$

where f^*E_i *and* $f^*\varphi_{ij}$ *are the usual pullbacks of* E_i *and* φ_{ij} *through* f*, respectively. Analogously, one can define the pullback of a morphism of twisted vector bundles. Finally, we define*

$$\begin{aligned} K_\zeta(f) :\ K_\zeta(Y) &\rightarrow K_{f^*\zeta}(X), \\ [[E]] - [[F]] &\mapsto [[f^*_\zeta E]] - [[f^*_\zeta F]], \end{aligned}$$

which is hereafter called the ***pullback of*** $\boldsymbol{f}$ ***in absolute Twisted K-Theory****, as one could expect.* ♢

Remark 5.20 (On the absolute Twisted K-Theory data presented above)**.** *The following facts hold true for* $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$*.*

- *Being* X *a paracompact Hausdorff space and* $\mathfrak{U} = \{U_i\}_{i\in I}$ *any good cover of* X*, we have*

$$K_\zeta(\mathrm{id}_X) = \mathrm{id}_{K_\zeta(X)}.$$

- *Being* $f : X \rightarrow Y$ *and* $g : Y \rightarrow Z$ *continuous functions between paracompact Hausdorff spaces for which there exists a good cover* $\mathfrak{U} = \{U_i\}_{i\in I}$ *of* Z *such that* $g^*\mathfrak{U}$ *and* $f^*g^*\mathfrak{U}$ *are good covers of* Y *and* X*, respectively. Under these conditions, we have*

$$K_\zeta(g\circ f) = K_{g^*\zeta}(f) \circ K_\zeta(g).$$

Furthermore, one can prove that the pullback presented in the preceding definition is homotopy invariant. ♢

Remark 5.21 (An interesting fact in Ordinary K-Theory)**.** *Let* n *be a non-zero natural number and*

$$\mathbb{T}^n := \prod_{j=1}^{n} \mathbb{S}^1$$

be the n-dimensional torus. In addition, we consider the natural embeddings defined supposing $1 \in \mathbb{S}^1$ as a marked point

$$i_j : \mathbb{T}^{n-1} \times X \to \mathbb{T}^n \times X$$

for j between 1 and n. The reader can prove, using induction, that we have a canonical isomorphism

$$K^{-n}(X) \simeq \bigcap_{j=1}^{n} \operatorname{Ker} K(i_j).$$

This intersection is a subgroup of $K(\mathbb{T}^n \times X)$. The following definition is enlightened by these facts. $\diamondsuit$

Definition 5.22 (Absolute Twisted K-Theory groups of negative degree)**.** *Here we use the notations of Remark 5.21. Fixing a good cover of $\mathbb{S}^1$ such as the one in Figure 5.1, we easily obtain a good cover of $\mathbb{T}^n \times X$ through Cartesian product. We also consider the natural projection $\pi_n : \mathbb{T}^n \times X \to X$. Under these circumstances, for $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$, we define*

$$K_\zeta^{-n}(X) := \bigcap_{j=1}^{n} \operatorname{Ker} K_{\pi_n^*\zeta}(i_j),$$

which is the subgroup of $K_{\pi_n^*\zeta}(\mathbb{T}^n \times X)$ that we call ***nth negative degree absolute Twisted K-Theory group*** *of X.* $\diamondsuit$

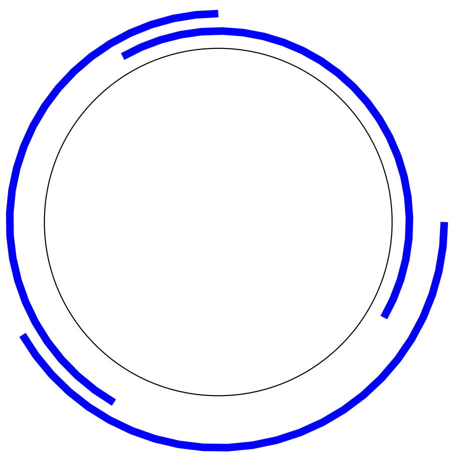

Figure 5.1: The open arcs of circle drawn in blue projects radially to the internal circle. The projected sets form a good cover of $\mathbb{S}^1$. There exists no good cover of $\mathbb{S}^1$ with only two open subsets.

Definition 5.23 (Negative degree pullback in absolute Twisted K-Theory). *Let n be a natural number and $f : X \to Y$ be a continuous map between paracompact Hausdorff spaces as in Definition 5.19. In addition, let $\pi_n : \mathbb{T}^n \times Y \to Y$ be the natural projection. We define*

$$K_\zeta^{-n}(f) : \; K_\zeta^{-n}(Y) \;\to\; K_{f^*\zeta}^{-n}(X)$$

to be the obvious restriction of

$$K_{\pi_n^*\zeta}(\mathrm{id}_{\mathbb{T}^n} \times f) : \; K_{\pi_n^*\zeta}(\mathbb{T}^n \times Y) \;\to\; K_{(\mathrm{id}_{\mathbb{T}^n} \times f)^* \pi_n^* \zeta}(\mathbb{T}^n \times X).$$

This new homomorphism is the ***nth negative degree pullback of f in absolute Twisted K-Theory****.* ♢

Remark 5.24 (Bott Periodicity Theorem in Twisted K-Theory). *Before extending to positive degrees the Twisted K-Theory data presented above, we have to introduce the Bott Periodicity Theorem. For this, we need a multiplicative structure in Twisted K-Theory. Being $E = (\{E_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I})$ a ζ-twisted vector bundle on X and $F = (\{F_i\}_{i \in I}, \{\psi_{ij}\}_{i,j \in I})$ a ξ-twisted vector bundle on X, with $\zeta, \xi \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$, we define the* ***tensor product*** *of E and F as being the $\zeta\xi$-twisted vector bundle on X given by*

$$E \otimes F \;:=\; (\{E_i \otimes F_i\}_{i \in I},\, \{\varphi_{ij} \otimes \psi_{ij}\}_{i,j \in I}).$$

This tensor product induces

$$\begin{aligned} \boxtimes : \; \mathrm{VB}_\zeta(X) \otimes \mathrm{VB}_\xi(Y) \;&\to\; \mathrm{VB}_{(\pi_X^*\zeta)(\pi_Y^*\xi)}(X \times Y), \\ [E] \otimes [F] \;&\mapsto\; [(\pi_X)_\zeta^* E \,\otimes\, (\pi_Y)_\xi^* F], \end{aligned}$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the natural projections. Considering the corresponding K-Theory classes, we obtain

$$K_\zeta(X) \,\otimes\, K_\xi(Y) \;\to\; K_{(\pi_X^*\zeta)(\pi_Y^*\xi)}(X \times Y).$$

From Definition 5.22 and the natural homeomorphism between $(\mathbb{T}^n \times X) \times (\mathbb{T}^m \times Y)$ and $\mathbb{T}^{n+m} \times X \times Y$, we obtain

$$K_\zeta^{-n}(X) \otimes K_\xi^{-m}(Y) \to K^{-n-m}_{(\pi_X^*\zeta)(\pi_Y^*\xi)}(X \times Y).$$

Composing this last map with the pullback though the diagonal map $\Delta : X \times X \to X$, *we obtain*

$$K_\zeta^{-n}(X) \otimes K_\xi^{-m}(X) \to K_{\zeta\xi}^{-n-m}(X).$$

Now we establish the Bott Periodicity Theorem. For this, we consider the dual of the tautological line bundle of $\mathbb{P}^1(\mathbb{C})$, *whose pullback through the map* $p : \mathbb{T}^2 \to \mathbb{S}^2$ *pictured in Figure 5.2 is a line bundle* η *on the torus. The result in question ensures that*

$$\begin{aligned} B_n :\ K_\zeta^{-n}(X) &\to K_\zeta^{-n-2}(X), \\ \alpha &\mapsto (\eta - 1)\,\alpha, \end{aligned}$$

is a group isomorphism for all $n \in \mathbb{N}$. *This fact is the one that justifies the following definition.* ♢

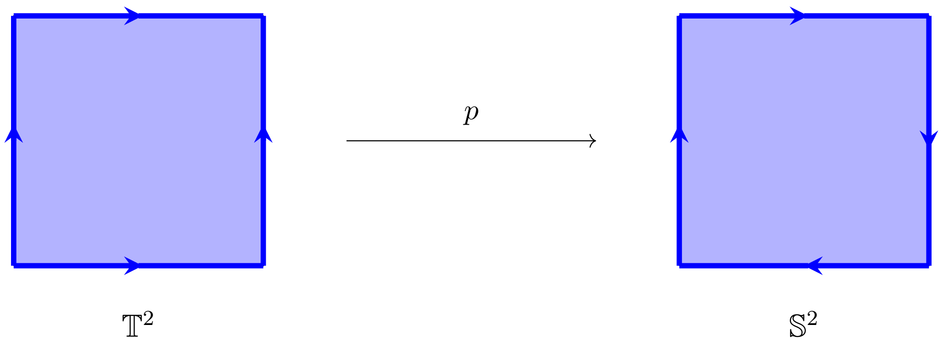

Figure 5.2: The map $p : \mathbb{T}^2 \to \mathbb{S}^2$ acts as the identity on the identification squares of $\mathbb{T}^2$ and $\mathbb{S}^2$.

Definition 5.25 (The absolute Twisted K-Theory groups and pullbacks of positive degree)**.** *Let* n *be a natural number and* $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$. *The* ***nth positive degree absolute Twisted K-Theory group*** X, *which is hereafter denoted by* $K_\zeta^n(X)$, *is defined as the negative K-Theory group* $K_\zeta^{-n}(X)$. *In addition, being* $f : X \to Y$ *a continuous map between paracompact Hausdorff spaces as in Definition 5.19, we define the* ***nth positive degree pullback of f in absolute Twisted K-Theory****, and denote it by* $K_\zeta^n(f) : K_\zeta^n(Y) \to K_{f^*\zeta}^n(X)$, *to be the nth negative degree pullback* $K_\zeta^{-n}(f) : K_\zeta^{-n}(Y) \to K_{f^*\zeta}^{-n}(X)$. ♢

Remark 5.26 (Dependence on the cocycle). *Suppose that $\zeta, \xi \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$ are cohomologous cocycles. Then, let $\eta = \{\eta_{ij}\}_{i,j\in I}$ be such that $\xi = \zeta \cdot \check{\delta}^1\eta$. We have that the isomorphism in Remark 5.10 extends to the corresponding Grothendieck groups, defining the isomorphism*

$$\Phi_\eta : K_\zeta(X) \to K_\xi(X).$$

This shows that the isomorphism class of $K_\zeta(X)$ only depends on $[\zeta]$. As before, we have that this dependence is non-canonical. In fact, the set of isomorphisms of the form Φ_η is a torsor over $\check{H}^1(\mathfrak{U}, \underline{\mathrm{U}}(1)) \simeq H^2(X, \mathbb{Z})$. In particular, if $H^2(X,\mathbb{Z})$ is trivial, then $K_{[\zeta]}(X)$ is canonically defined and does not depend on the cover. Finally, note that we are free to choose $\zeta \in \check{Z}^2(\mathfrak{U}, \mathrm{U}(1))$ or $\zeta \in \check{Z}^2(\mathfrak{U}, \Gamma_r)$. Mutatis mutandis, all of the previous considerations keep on holding. ♢

5.3 Relative and Reduced Twisted K-Theory

In this section, we define the last fundamental tools of Twisted K-Theory, namely, the reduced and relative Twisted K-Theory groups and homomorphisms. The ideas presented here descend directly from Section 2.9. We begin with the following definition.

Definition 5.27 (Reduced and relative Twisted K-Theory groups). *Let A be a subspace of a paracompact Hausdorff space X for which there exists a good cover $\mathfrak{U} = \{U_i\}_{i\in I}$ such that*

$$\mathfrak{U}\,|_A := \{U_i \cap A\}_{i\in I}$$

is a good cover of A. In addition, let $i : A \to X$ be the inclusion and $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$. We define

$$i^*\zeta \in \check{Z}^2(\mathfrak{U}\,|_A, \underline{\mathrm{U}}(1)).$$

*We denote by $\mathcal{L}_\zeta(X,A)$ the set of triples (E_1, E_0, α), where E_1 and E_0 are ζ-twisted vector bundles on X and $\alpha : E_1\,|_A \to E_0\,|_A$ is an isomorphism of $i^*\zeta$-twisted vector bundles. Two triples (E_1, E_0, α) and (F_1, F_0, β) are said to be **isomorphic** if there exist isomorphisms of ζ-twisted vector bundles $f_1 : E_1 \to F_1$ and $f_0 : E_0 \to F_0$ such that the diagram*

$$\begin{tikzcd}
E_1 \mid_A \arrow[r, "\alpha"] \arrow[d, "f_1|_A"'] & E_0 \mid_A \arrow[d, "f_0|_A"] \\
F_1 \mid_A \arrow[r, "\beta"'] & F_0 \mid_A
\end{tikzcd}$$

is commutative. In this case, we write

$$(E_1, E_0, \alpha) \;\simeq\; (F_1, F_0, \beta).$$

Furthermore, we define

$$\begin{aligned}
\oplus :\; \mathcal{L}_\zeta(X,\, A) \times \mathcal{L}_\zeta(X,\, A) \;&\to\; \mathcal{L}_\zeta(X,\, A), \\
((E_1, E_0, \alpha),\, (F_1, F_0, \beta)) \;&\mapsto\; (E_1 \oplus F_1, E_0 \oplus F_0, \alpha \oplus \beta).
\end{aligned}$$

A triple of the form $(E, E, \mathrm{id}_{E|_A})$ *is said to be an* ***elementary triple****. Furthermore, we say that two triples* (E_1, E_0, α) *and* (F_1, F_0, β) *are equivalent if and only if there exist elementary triples*

$$(G,\, G,\, \mathrm{id}_{G|_A}) \qquad \textit{and} \qquad (H,\, H,\, \mathrm{id}_{H|_A})$$

such that

$$(E_1,\, E_0,\, \alpha) \;\oplus\; (G,\, G,\, \mathrm{id}_{G|_A}) \;\simeq\; (F_1,\, F_0,\, \beta) \;\oplus\; (H,\, H,\, \mathrm{id}_{H|_A}).$$

This is an equivalence relation on $\mathcal{L}_\zeta(X, A)$*. The* ***relative Twisted K-Theory group*** *of the pair* (X, A)*, which we denote by* $K_\zeta(X, A)$*, is the quotient of* $\mathcal{L}_\zeta(X, A)$ *by this equivalence relation. Moreover, when* A *contains a single point* $x_0 \in X$*, we define the* ***reduced Twisted K-Theory group*** *of* (X, x_0) *as*

$$\widetilde{K}_\zeta(X, x_0) \;:=\; K_\zeta(X, A)$$

Since the reduced groups are special cases of the relative ones, we concentrate on the relative setting henceforth. ♢

Remark 5.28 (On the preceding definition)**.** *Here we use the notation of Definition 5.27. We have that $K_\zeta(X, A)$ is an abelian group since its neutral elemental is the class of any elementary triple and*

$$-[E_1, E_0, \alpha] \;=\; [E_0, E_1, \alpha^{-1}].$$

These facts can be easily proven by adapting the proof of Theorem 2.62. Additionally, when A is empty, we recover the usual group $K_\zeta(X)$ by identifying $[E_1, E_0]$ with $[[E_0]] - [[E_1]]$. Once and again, the reader can prove this claim by adapting the proof of Theorem 2.63. ♢

Definition 5.29 (Pullback in relative Twisted K-Theory)**.** *Let $f : (X, A) \to (Y, B)$ be a map of pairs between paracompact Hausdorff spaces for which there exists a good cover $\mathfrak{U}$ of Y such that $\mathfrak{U}\mid_B$ is a good cover of B, $f^*\mathfrak{U}$ is a good cover of X and $f^*\mathfrak{U}\mid_A$ is a good cover of A. Under these circumstances, with $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$, we define the group homomorphism*

$$\begin{aligned} K_\zeta(f) : K_\zeta(Y,\, B) \;&\to\; K_{f^*\zeta}(X,\, A), \\ [E_1, E_0, \alpha] \;&\mapsto\; [f_\zeta^* E_1,\, f_\zeta^* E_0,\, f_\zeta^* \alpha]. \end{aligned}$$

This new map is hereafter called the ***pullback of f in relative Twisted K-Theory****, as one could expect.* ♢

Definition 5.30 (Relative Twisted K-Theory groups and homomorphisms of all degrees)**.** *The extension of relative groups and homomorphisms to all degrees is analogous to the one of Section 5.2. In particular:*

- *considering the natural embeddings*

  $$i_j :\; (\mathbb{T}^{n-1} \times X,\, \mathbb{T}^{n-1} \times A) \;\to\; (\mathbb{T}^n \times X,\, \mathbb{T}^n \times A)$$

  for j between 1 and b, both included, we set

  $$K_\zeta^{-n}(X, A) \;:=\; \bigcap_{j=1}^{n} \operatorname{Ker} K_{\pi_n^*\zeta}(i_j),$$

  which is the subgroup of $K_{\pi_n^*\zeta}(\mathbb{T}^n \times X,\, \mathbb{T}^n \times A)$, where

$$\pi_n : (\mathbb{T}^n \times X, \mathbb{T}^n \times A) \to (X, A)$$

is the natural projection, that we hereafter call ***nth negative degree relative Twisted K-Theory group*** *of X; and*

- *we have a natural product*

$$\begin{aligned} K^n_\zeta(X) \times K^m_\xi(X, A) &\to K^{n+m}_{\zeta\xi}(X, A), \\ [[E]] \otimes [F_1, F_0, \alpha] &\mapsto [E \otimes F_1, E \otimes F_0, \mathrm{id}_{E|_A} \otimes \alpha]. \end{aligned}$$

Moreover, the Bott periodicity morphism

$$\begin{aligned} B_n : K^{-n}_\zeta(X, A) &\to K^{-n-2}_\zeta(X, A), \\ \alpha &\mapsto (\eta - 1)\, \alpha, \end{aligned}$$

is well-defined. Thus, we define the ***nth positive degree relative Twisted K-Theory group*** *X, which is hereafter denoted by $K^n_\zeta(X, A)$, as the negative K-Theory group $K^{-n}_\zeta(X, A)$.* ♢

Remark 5.31 (Dependence on the cocycle)**.** *We have that the isomorphism presented in Remark 5.10 extends to the relative setting of Twisted K-Theory. In fact, it suffices to apply the isomorphism in question to both E_1 and E_0 in the triple (E_1, E_0, α). Hence, the isomorphism class of $K^n_\zeta(X, A)$ only depends on $[\zeta]$. In particular, if $H^2(X, \mathbb{Z})$ is trivial, then $K^n_{[\zeta]}(X, Y)$ is canonically defined. The reader can extend this reasoning for $\zeta \in \check{Z}^2(\mathfrak{U}, \mathrm{U}(1))$ or $\zeta \in \check{Z}^2(\mathfrak{U}, \Gamma_r)$.* ♢

5.4 Compactly-supported Twisted K-Theory

In this section, we establish the compactly-supported Twisted K-Theory groups. In addition, we set induced homomorphisms of open embeddings in this framework. This is mainly done because these compactly-supported groups are essential in the next section to define the Thom isomorphisms in Twisted K-Theory. We begin with the following notation.

Notation 5.32 (A special difference of subspaces)**.** *In this section, X is always a locally compact space*(2)*. In this situation, given a compact subset K of X, we define $X \setminus\setminus K$ as the closure of $X - K$ in X. Equivalently, $X \setminus\setminus K$ is the complement of the interior of K in X.* ♢

Definition 5.33 (Compactly-supported Twisted K-Theory groups)**.** *Let n be an integer number and $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$. We give the following definitions.*

- *A compact subspace K of X is **$\mathfrak{U}$-compact** if $\mathfrak{U}\,|_{X\setminus\setminus K}$ is a good cover of $X \setminus\setminus K$. The set formed by the $\mathfrak{U}$-compact subspaces of X is denoted by $\mathfrak{K}_{\mathfrak{U}}(X)$. This set is easily proved to be a directed set with respect to the partial order given by inclusion. Moreover, we shall assume that the good cover $\mathfrak{U}$ of X is refined enough so that*

$$X \;=\; \bigcup_{K\,\in\,\mathfrak{K}_{\mathfrak{U}}(X)} K.$$

- *The **nth compactly-supported Twisted K-Theory group of X**, which is denoted by $K^n_{\zeta,c}(X)$, is the direct limit*

$$K^n_{\zeta,c}(X) \;:=\; \lim_{\longrightarrow K} K^n_\zeta(X, X \setminus\setminus K)$$

of the direct system $\mathfrak{A}^n_{X,\mathfrak{U},\zeta}$ formed by $\mathfrak{K}_{\mathfrak{U}}(X)$, $(K^n_\zeta(X, X \setminus\setminus K))_{K\in\mathfrak{K}_{\mathfrak{U}}(X)}$ and $(K^n_\zeta i^X_{KL} : K^n_\zeta(X, X \setminus\setminus K) \to K^n_\zeta(X, X \setminus\setminus L))_{K,L\in\mathfrak{K}_{\mathfrak{U}}(X)}$. Here $K^n_\zeta i^X_{KL}$ coincides with the induced homomorphism of the inclusion $i^X_{KL} : (X, X \setminus\setminus L) \to (X, X \setminus\setminus K)$ if K is contained in L, and coincides with the trivial homomorphism otherwise. This direct limit is equipped with the family of morphisms of abelian groups $(\iota^n_K : K^n_\zeta(X, X \setminus\setminus K) \to K^n_{\zeta,c}(X))_{K\in\mathfrak{K}_{\mathfrak{U}}(X)}$.

---

(2)In this chapter, X is always a paracompact Hausdorff space. Therefore, when we require X to be locally compact, we are restricting X to a smaller class of topological spaces. Indeed, we remind the reader that:

- there are examples of locally compact Hausdorff spaces that are not paracompact. In fact, there are examples of locally compact Hausdorff spaces that are not even normal. One of them is the deleted Tychonoff plank; *and*
- there are examples of paracompact Hausdorff spaces that are not locally compact. One of them is the Sorgenfrey line. Nevertheless, every second countable Hausdorff space that is locally compact is also paracompact.

An element of $K^n_{\zeta,c}(X)$ is an equivalence class $[\alpha]$ where $\alpha \in K^n_\zeta(X, X \setminus K)$ for some $K \in \mathfrak{K}_{\mathfrak{U}}(X)$. Moreover, $[\alpha], [\beta] \in K^n_{\zeta,c}(X)$ are equal, where $\alpha \in K^n_\zeta(X, X \setminus K)$ and $\beta \in K^n_\zeta(X, X \setminus L)$ with $K, L \in \mathfrak{K}_{\mathfrak{U}}(X)$, if and only if there exists $M \in \mathfrak{K}_{\mathfrak{U}}(X)$ for which $K \subseteq M$, $L \subseteq M$ and $K^n_\zeta i^X_{KM}(\alpha) = K^n_\zeta i^X_{LM}(\beta)$. $\diamondsuit$

Definition 5.34 (Compactly-supported Twisted K-Theory homomorphisms)**.** *Let n be an integer number and $f : X \to Y$ be an open embedding for which there exists a good cover $\mathfrak{U}$ of Y such that $f^*\mathfrak{U}$ is a good cover of X. For any $f^*\mathfrak{U}$-compact subset K of X, from the embedding of pairs $f_K : (X, X \setminus K) \to (Y, Y \setminus f(K))$, we obtain the induced morphism*

$$K^n_\zeta(f_K) :\ K^n_\zeta(Y, Y \setminus f(K)) \to K^n_\zeta(X, X \setminus K).$$

This map is an excision isomorphism. Furthermore, if $K \subseteq L$ with $K, L \in \mathfrak{K}_{f^*\mathfrak{U}}(X)$, then the diagram

$$\begin{tikzcd}
K^n_\zeta(X,\, X \setminus K) \arrow[r, "K^n_\zeta(f_K)^{-1}"] \arrow[d, "K^n_\zeta i^X_{KL}"'] & K^n_\zeta(Y,\, Y \setminus f(K)) \arrow[d, "K^n_\zeta i^Y_{f(K)f(L)}"] \\
K^n_\zeta(X,\, X \setminus L) \arrow[r, "K^n_\zeta(f_L)^{-1}"'] & K^n_\zeta(Y,\, Y \setminus f(L))
\end{tikzcd}$$

*is commutative. Therefore, we obtain an induced morphism between the direct limits, which we denote by $K^n_{\zeta,c}(f) : K^n_{\zeta,c}(X) \to K^n_{\zeta,c}(Y)$. We call $K^n_{\zeta,c}(f)$ the **nth compactly-supported induced homomorphism in Twisted K-Theory**. This construction turns $K^n_{\zeta,c}$ into a covariant functor.* $\diamondsuit$

Remark 5.35 (Dependence on the cocycle)**.** *The isomorphism of Remark 5.10 extended to relative Twisted K-Theory induces an isomorphism between the compactly-supported groups. Thence, the isomorphism class of $K^n_{\zeta,c}(X)$ only depends on $[\zeta]$. In particular, if $H^2(X, \mathbb{Z})$ is trivial, then $K^n_{[\zeta],c}(X)$ is canonically defined. The reader can extend this reasoning for $\zeta \in \check{Z}^2(\mathfrak{U}, \mathrm{U}(1))$ or $\zeta \in \check{Z}^2(\mathfrak{U}, \Gamma_r)$.* $\diamondsuit$

Remark 5.36 (The natural multiplicative structures on the framework of compactly-supported Twisted K-Theory)**.** *Let m and n be integer numbers and $\zeta, \xi \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$. We have the product*

$$\begin{aligned} K^m_\zeta(X) \otimes K^n_{\xi,c}(X) &\rightarrow K^{m+n}_{\zeta\xi,c}(X), \\ \alpha \otimes [\beta] &\mapsto [\alpha \cdot \beta], \end{aligned}$$

where $\alpha \cdot \beta$ is an instance of $K^m_\zeta(X) \otimes K^n_\xi(X, X \setminus\setminus K) \rightarrow K^{m+n}_{\zeta\xi}(X, X \setminus\setminus K)$. Moreover, we have the product

$$\begin{aligned} K^m_{\zeta,c}(X) \otimes K^n_{\xi,c}(X) &\rightarrow K^{m+n}_{\zeta\xi,c}(X), \\ [\alpha] \otimes [\beta] &\mapsto [\alpha \cdot \beta], \end{aligned}$$

where $\alpha \cdot \beta$ is an instance of $K^m_\zeta(X, X \setminus\setminus K) \otimes K^n_\xi(X, X \setminus\setminus K) \rightarrow K^{m+n}_{\zeta\xi}(X, X \setminus\setminus K)$. These are natural products in the framework of compactly-supported Twisted K-Theory, as desired. $\diamondsuit$

Theorem 5.37 (Real integration)**.** *Let n be an integer number. We have the canonical isomorphism*

$$\int_{\mathbb{R}} : K^n_{\zeta,c}(\mathbb{R} \times X) \rightarrow K^{n-1}_{\zeta,c}(X),$$

which is the induced homomorphism of the open embedding $i\colon \mathbb{R} \times X \rightarrow \mathbb{S}^1 \times X$ defined by the natural map $\mathbb{R} \rightarrow \mathbb{R}^+ \approx \mathbb{S}^1$.

Proof. We have

$$K^n_{\zeta,c}(\mathbb{R} \times X) = \varinjlim_{m\in\mathbb{N},\, K\in\mathfrak{K}_{\mathfrak{U}}(X)} K^n_\zeta((\mathbb{R} \times X), (\mathbb{R} \times X) \setminus\setminus i([-m, m] \times K))$$

because all the elements $[-m, m] \times K$, where $m \in \mathbb{N}$ and $K \in \mathfrak{K}_{\mathfrak{U}}(X)$, form a cofinal subset of $\mathfrak{K}_{i^*\mathfrak{U}}(\mathbb{R} \times X)$. We have that the right-hand side of the preceding equation is the group of compactly-supported classes in $\mathbb{S}^1 \times X$ relative to $\{\infty\} \times X$(3). In turn, such a group is the kernel of $K^n_{\zeta,c}(i_\infty) : K^n_{\zeta,c}(\mathbb{S}^1 \times X) \rightarrow K^n_{\zeta,c}(X)$, which is exactly $K^{n-1}_{\zeta,c}(X)$, where $i_\infty : X \rightarrow X \times \mathbb{S}^1$, $x \mapsto (x, \infty)$. This finishes the proof of the theorem. $\square$

(3)Indeed, the compact support in $\mathbb{S}^1 \times X$ only concerns X. Thus, we can define relative classes with respect to the subspace $\{\infty\}$ of $\mathbb{S}^1$ by considering pairs of the form $(\mathbb{S}^1 \times K, \{\infty\} \times K)$ and applying the direct limit.

Remark 5.38 ($\mathbb{S}^1$ - integration). *Let n be an integer number. We can also define the integration map*

$$\int_{\mathbb{S}^1} : K^n_\zeta(\mathbb{S}^1 \times X) \to K^{n-1}_\zeta(X),$$

calling ζ both the twisting cocycle on X and its pullback on $\mathbb{S}^1 \times X$. Let us consider the embedding $i_1 : X \to \mathbb{S}^1 \times X$, defined through a marked point of $\mathbb{S}^1$, and the projection $\pi_1 : \mathbb{S}^1 \times X \to X$. We set

$$\begin{aligned} \int_{\mathbb{S}^1} : K_\zeta(\mathbb{S}^1 \times X) &\to K^{-1}_\zeta(X), \\ \alpha &\mapsto \alpha - K_\zeta(\pi_1) K_\zeta(i_1)(\alpha). \end{aligned}$$

The reader can readily prove that $\alpha - K_\zeta(\pi_1)K_\zeta(i_1)(\alpha) \in \operatorname{Ker} K_\zeta(i_1) = K^{-1}_\zeta(X)$ since $\pi_1 \circ i_1 = \mathrm{id}_X$. Now, since $K^{-1}_\zeta(\mathbb{S}^1 \times X) \subset K_\zeta(\mathbb{S}^1 \times \mathbb{S}^1 \times X)$ and $K^{-2}_\zeta(X) \subset K^{-1}_\zeta(\mathbb{S}^1 \times X)$, we define

$$\int_{\mathbb{S}^1} : K^{-1}_\zeta(\mathbb{S}^1 \times X) \to K^{-2}_\zeta(X)$$

as the restriction of $\int_{\mathbb{S}^1} : K_\zeta(\mathbb{S}^1 \times \mathbb{S}^1 \times X) \to K^{-1}_\zeta(\mathbb{S}^1 \times X)$. Finally, we have that this construction can be iterated and it can be extended to positive degrees by the Bott Periodicity Theorem. $\diamondsuit$

5.5 Thom isomorphisms in Twisted K-Theory

In this section, we present the Thom isomorphisms in Twisted K-Theory. This is the furthest achievement of this thesis on the subject of finite-order Twisted K-Theory. This result also needs the tools from Spin Geometry presented in Chapter 3. We begin with the following remark.

Remark 5.39 (Rephrasing some notions from Spin Geometry). *Here we recall some facts on Spin Geometry in order to fix the notation within the framework of twisted bundles. For this, let $\pi : E \to X$ be a $2r$-dimensional Euclidean oriented vector bundle. We have that the good cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X induces the good cover of E given by*

$$\pi^* \mathfrak{U} = \{E_i := \pi^{-1}(U_i)\}_{i \in I}.$$

Now we consider the orthonormal frame bundle $\pi_{\mathrm{SO}} : \mathrm{SO}(E) \to X$ *and the corresponding restrictions* $\pi^i_{\mathrm{SO}} : \mathrm{SO}(E_i) \to U_i$ *for each* $i \in I$*. Since* U_i *is contractible, we can choose a spin lift* $\pi^i_{\mathrm{Spin}} : \mathrm{Spin}(E_i) \to U_i$ *for each* $i \in I$*. Moreover, we can fix principal bundle isomorphisms*

$$\varphi_{ij} : \mathrm{Spin}(E_i) \mid_{U_{ij}} \to \mathrm{Spin}(E_j) \mid_{U_{ij}},$$

lifting the identity $\mathrm{SO}(E_i) \mid_{U_{ij}} = \mathrm{SO}(E_j) \mid_{U_{ij}}$*. Under these circumstances, we have that*

$$\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} \;=\; \epsilon_{ijk} \cdot 1$$

for all $i, j, k \in I$*. Thence, we obtain*

$$w_2(E) \;=\; [\{\epsilon_{ijk}\}_{i,j,k \in I}] \;\in\; H^2(X, \mathbb{Z}_2),$$

that vanishes if and only if there exists a global spin lift. Furthermore, we call

$$\rho : \mathrm{Spin}(2r) \;\to\; \mathrm{U}(2^r) \;\subseteq\; \mathrm{GL}(\mathbb{C}^{2^r})$$

the natural unitary representation of $\mathrm{Spin}(2r)$*, acting on* $S := \mathbb{C}^{2^r}$*, that splits in the two irreducible chirality representations* $S = S_+ \oplus S_-$*. From each local spin lift* $\mathrm{Spin}(E_i)$*, we obtain the associated vector bundle*

$$S(E_i) \;:=\; \mathcal{E}_\rho \mathrm{Spin}(E_i)$$

of rank 2^r *(the bundle of spinors), with the chirality splitting* $S(E_i) = S_+(E_i) \oplus S_-(E_i)$*. Being* $\epsilon := \{\epsilon_{ijk}\}_{i,j,k \in I}$*, we get the* ϵ*-twisted vector bundle*

$$S(E) \;:=\; (\{S(E_i)\}_{i \in I}, \{\varphi'_{ij}\}_{i,j \in I}),$$

where

$$\varphi'_{ij} \;:=\; \mathcal{E}_\rho(\varphi_{ij}) : \; \mathcal{E}_\rho \mathrm{Spin}(E_i) \;\to\; \mathcal{E}_\rho \mathrm{Spin}(E_j)$$

for all $i, j \in I$*. We also have the natural global splitting of* ϵ*-twisted vector bundles* $S(E) = S_+(E) \oplus S_-(E)$*.* $\diamondsuit$

Theorem 5.40 (Thom isomorphisms in Twisted K-Theory)**.** *Here we use the notations of Remark 5.39. Choosing a refinement map $\phi\colon J \to I$ from $\pi^*\mathfrak{U}$ to a convenient good cover $\mathfrak{V}$ of E, and using the product established in Remark 5.36, we obtain the group isomorphism*

$$\begin{aligned} T_m : \; K^m_\zeta(X) \;&\to\; K^{m+n}_{\phi^*(\zeta\epsilon),\, c}(E), \\ \alpha \;&\mapsto\; K^m_\zeta(\pi)(\alpha) \cdot u, \end{aligned}$$

for all $m \in \mathbb{Z}$. These group isomorphisms, which do not form a ring isomorphism in general, are said to be the ***Thom isomorphisms in Twisted K-Theory*** *of the vector bundle in question. Of course, we identified ζ and ϵ in X with $\pi^*\zeta$ and $\pi^*\epsilon$ in E as twisting cocycles.*

Proof. Now we explain some important details that are not clear in the statement of the theorem. Let us begin by considering the projection $\pi : E \to X$ and the pullback on E

$$\pi^* S(E) \;=\; \pi^* S_+(E) \,\oplus\, \pi^* S_-(E).$$

We define the morphism of twisted vector bundles

$$\mu : \; \pi^* S_+(E) \;\to\; \pi^* S_-(E)$$

as follows. For any fixed point $e \in E_x$, with $x \in U_i$, the morphism μ acts between the fibers $S_+(E_i)_x$ and $S_-(E_i)_x$ as the Clifford multiplication by $e \in \mathbb{C}l(E_x) = \mathbb{C}l(E_i)_x$. It demands a straightforward computation to prove that μ is actually a morphism of twisted bundles and that it is an isomorphism on the closure of the complement of the disk bundle $\mathbb{D}(E)$ of E. Thus, refining $\pi^*\mathfrak{U}$ on E in a suitable way, we obtain a good cover $\mathfrak{V} = \{V_j\}_{j\in J}$ of E such that $\mathbb{D}(E)$ is $\mathfrak{V}$-compact and the union of the $\mathfrak{V}$-compact sets is the whole E(4). In this way, we obtain a class

$$\tilde{u} \;:=\; [\pi^* S_+(E),\, \pi^* S_-(E),\, \mu] \;\in\; K^n_\epsilon(E,\, E \setminus\!\!\setminus \mathbb{D}(E)),$$

representing a compactly-supported class

(4) For example, if we fix a trivialization $E_i \to U_i \times \mathbb{R}^n$, then we have the cover formed by the sets $U_i \times B$ where B is an open ball in $\mathbb{R}^n$.

$$u \in K^n_{\epsilon,c}(E),$$

the latter being a (twisted) Thom class. In all of these constructions, we tacitly assumed n to be even. However, if n is odd, then we can consider a Thom class in $E \oplus \mathbb{R} \to X$, which is the direct sum of E with the trivial line bundle $X \times \mathbb{R} \to X$. Therefore, we obtain

$$u \in K^{n+1}_{\epsilon,c}(E \oplus \mathbb{R}) = K^{n+1}_{\epsilon,c}(E \times \mathbb{R}) \simeq K^n_{\epsilon,c}(E),$$

the last isomorphism being the one of Theorem 5.37. This completes the construction of the Thom isomorphisms. □

Remark 5.41 (Dependence on the cocycle)**.** *In general, the Thom isomorphisms defined in Theorem 5.40 depend on ζ, ϵ and ϕ. Nevertheless, they become canonical when $H^2(X,\mathbb{Z})$ is trivial. In fact, in this case, $H^2(E,\mathbb{Z})$ is trivial as well, since E retracts by deformation on X. It follows that both $K^m_\zeta(X)$ and $K^{m+n}_{\zeta\epsilon,c}(E)$ only depend on the cohomology class of their twisting cocycle. Hence, the isomorphism in question can be written intrinsically as*

$$\begin{aligned} T_m : K^m_{[\zeta]}(X) &\to K^{m+n}_{[\zeta]+W_3(E),c}(E), \\ \alpha &\mapsto K^m_\zeta(\pi)(\alpha)\cdot u. \end{aligned}$$

Since ϵ is constant, if ζ is also constant, then we obtain a canonical isomorphism similar to the preceding one on any manifold such that $\operatorname{Tor} H^2(X;\mathbb{Z})$ *is trivial. For this, we have to replace $W_3(E)$ by $w_2(E)$.* ♢

Remark 5.42 (Thom isomorphism and spin^c structures)**.** *Let us suppose that $W_3(E)$ is trivial. Under this hypothesis, we will show how to recover the Thom isomorphisms in Ordinary K-theory from the Thom isomorphisms in Theorem 5.40. Choosing $\zeta = \epsilon$, we get the isomorphism*

$$T_m : K^m_\epsilon(X) \to K^{m+n}_c(E).$$

Since $W_3(E)$ is trivial and since $W_3(E)$ is the twisting (integral) class represented by ϵ, we have that $K^m_\epsilon(X)$ is isomorphic to $K^m(X)$ in a non-canonical way. In order to find an isomorphism as in Theorem 5.9, we must fix a trivialization of ϵ in $\underline{\mathrm{U}}(1)$. If $h = \{h_{ij}\}_{i,j\in I}$ is such a trivialization, we obtain

$$h_{ki}\, h_{jk}\, h_{ij} \;=\; \epsilon_{ijk} \cdot I$$

for all $i, j, k \in I$. This means that the choice of a spin^c structure is equivalent to the choice of an isomorphism

$$\Phi_h :\; K^m_\epsilon(X) \;\to\; K^m(X)$$

(see Remark 5.10). The composition between Φ_h^{-1} and the Thom isomorphism of Theorem 5.40 is the ordinary Thom isomorphism, with respect to the Thom class induced by the chosen spin^c-structure[(5)]. One can prove that the choice of ϵ as a representative of $w_2(E)$ is immaterial. $\diamondsuit$

5.6 Twisted Hilbert bundles

In this section, we present the fundamental notions that one must know in order to understand the infinite-dimensional model of Twisted K-Theory, namely, the twisted Hilbert bundles and the projective Hilbert bundles. We begin by fixing the following notation.

Notation 5.43 (On Hilbert spaces)**.** *In this chapter, H is always a separable complex Hilbert space. We remind the reader that we compiled some information on this kind of Hilbert space in Remark 4.1. Furthermore, we shall denote the space of Fredholm operators on H by $\mathscr{F}_H$, and the space of continuous functions from $\mathscr{F}_H$ into itself by $\mathcal{C}(\mathscr{F}_H)$. The following definition generalizes the notion of ordinary Hilbert bundle (see Remark 4.30) as well as twisted vector bundles generalized the notion of ordinary vector bundles.* $\diamondsuit$

Definition 5.44 (Twisted Hilbert bundle)**.** *Consider a 2-cocycle*

$$\zeta \;:=\; \{\zeta_{ijk}\}_{i,j,k\in I} \;\in\; \check{Z}^2(\mathfrak{U},\, \underline{\mathrm{U}}(1)).$$

[(5)]In fact, if we consider the associated bundles $M_i := \mathcal{E}_1 \mathrm{U}(L_i)$, where 1 denotes the fundamental representation of U(1), and the isomorphisms $\psi'_{ij} := \mathcal{E}_1 \psi_{ij} : M_i \mid_{U_{ij}} \to M_j \mid_{U_{ij}}$, then we obtain the ϵ-twisted line bundle

$$\sqrt{L} \;:=\; (\{M_i\}_{i\in I},\, \{\psi'_{ij}\}_{i,j\in I}),$$

such that $\sqrt{L} \otimes \sqrt{L} = L$. With this language, the isomorphism Φ_h can also be written in the form $\Phi_{\sqrt{L}} \colon K^m_\epsilon(X) \to K^m(X)$, $\alpha \mapsto \alpha \otimes \sqrt{L}$.

We say that a ***ζ-twisted Hilbert bundle*** *with fiber H on X is a collection of trivial Hilbert bundles $\{\pi_i : E_i \to U_i\}_{i\in I}$ with fiber H and of Hilbert bundle isomorphisms $\{\varphi_{ij} : E_i\mid_{U_{ij}} \to E_j\mid_{U_{ij}}\}_{i,j\in I}$ such that*

$$\varphi_{ki}\mid_{E_k|_{U_{ijk}}} \circ\ \varphi_{jk}\mid_{E_j|_{U_{ijk}}} \circ\ \varphi_{ij}\mid_{E_i|_{U_{ijk}}} = \zeta_{ijk}\cdot \mathrm{id}_{E_i|_{U_{ijk}}}$$

for all $i,j,k \in I$. We denote the set of ζ-twisted Hilbert bundles on X by $\widetilde{\mathrm{VB}}_\zeta(X)$. Moreover, the notations used in this definition will be applied hereafter in this whole chapter. ♢

Remark 5.45 (On the preceding definition)**.** *We have the following facts.*

- *Morphisms and isomorphisms of twisted Hilbert bundles are defined, mutatis mutandis, as in Definition 5.5.*

- *Because of Theorem 5.8, if there exists a ζ-twisted vector bundle, then $[\zeta]$ is torsion. In addition, it can be proved that, for every torsion class in $\check{H}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$, there exists a corresponding twisted vector bundle. In turn, for every class in $\check{H}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$, not necessarily of finite order, there corresponds a twisted Hilbert bundle. The reader can find a proof of this claim in [4]. The main difference with respect to the finite-order setting is that any two ζ-twisted Hilbert bundles, for a fixed ζ, are isomorphic. The reader can find a proof of this claim in [20]. In particular, this last assertion ensures that every ordinary Hilbert bundle is trivial (see Remark 4.30).* ♢

Definition 5.46 (Projective bundle and projective Hilbert bundle)**.** *We give the following definitions.*

- *A* ***projective bundle*** *with typical fiber $\mathbb{P}(\mathscr{V})$[(6)] is a fiber bundle that admits an atlas whose transitions functions are projective transformations induced from automorphisms of $\mathscr{V}$ at each point of X.*

[(6)]The **associated projective space** $\mathbb{P}(\mathscr{V})$ of a complex vector space $\mathscr{V}$ is the quotient of $\mathscr{V} - \{0\}$ by the equivalence relations that identifies $v, w \in \mathscr{V} - \{0\}$ if and only if there exists $\lambda \in \mathbb{C}$ for which $v = \lambda w$.

- *The* ***associated projective bundle*** $\mathbb{P}(E)$ *of a twisted Hilbert bundle E is the one obtained by projecting each fiber to the corresponding projective space. In particular, $\mathbb{P}(E)_x = \mathbb{P}(E_x)$ for all $x \in X$.* $\diamondsuit$

Lemma 5.47 (Equivalence between projective bundles and projective Hilbert bundles)**.** *Every projective bundle is a projective Hilbert bundle up to isomorphism.*

Proof. Let $\pi : P \to X$ be a projective bundle with typical fiber $\mathbb{P}(\mathscr{V})$. Using local triviality, we can define a projective Hilbert bundle E whose associated projective bundle $\mathbb{P}(E)$ is isomorphic to P. For this, let

$$\Phi_{\mathfrak{U}} = \{(U_i,\, \varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{P}(\mathscr{V}))\}_{i\in I}$$

be an atlas of P whose transitions functions $\{\varphi_{ij} : U_{ij} \to \operatorname{Aut}\mathbb{P}(\mathscr{V})\}_{i,j\in I}$ are projective transformations induced from automorphisms of $\mathscr{V}$ at each point of X. Under these circumstances, we set

$$E := (\{U_i \times \mathscr{V}\}_{i\in I},\, \{\tilde{\varphi}_{ij}\}_{i,j\in I}),$$

where

$$\begin{aligned} \tilde{\varphi}_{ij} : U_{ij} \times \mathscr{V} &\to U_{ij} \times \mathscr{V}, \\ (x,\, v) &\mapsto (x,\, (\tilde{\varphi}_{ij})_x(v)), \end{aligned}$$

being $(\tilde{\varphi}_{ij})_x$ an automorphism of $\mathscr{V}$ that induces $(\varphi_{ij})_x$. Now the reader can fulfill the details to prove the assertion. $\square$

Remark 5.48 (On projective bundles)**.** *Let $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$. Because of Lemma 5.47, there exists a surjective map from isomorphism classes of twisted Hilbert bundles to isomorphism classes of projective bundles. In the finite-dimensional case, such a map is not injective for a fixed ζ. This happens because, for example, every line bundle projects to the trivial one. On the contrary, in the infinite-dimensional case, the unique isomorphism class of ζ-twisted Hilbert bundles induces a unique isomorphism class of projective bundles. Moreover, if ζ and ξ are cohomologous with $\xi = \zeta \cdot \check{\delta}^1\eta$, then we have the bijection*

$$\begin{aligned} \Phi_\eta : \widetilde{\mathrm{VB}}_\zeta(X) &\to \widetilde{\mathrm{VB}}_\xi(X) \\ (\{E_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I}) &\mapsto (\{E_i\}_{i\in I}, \{\varphi_{ij}\cdot\eta_{ij}\}_{i,j\in I}). \end{aligned}$$

As a consequence of $\mathbb{P}(E) = \mathbb{P}(\Phi_\eta(E))$, the isomorphism class of $\mathbb{P}(E)$ only depends on the class $[\zeta] \in \check{H}^2(\mathfrak{U}, \underline{\mathrm{U}}(1)) \simeq H^3(X, \mathbb{Z})$ (see [4]). In particular, $H^3(X, \mathbb{Z})$ classifies projective Hilbert bundles on X. Further, if $\check{\delta}^1\eta = 1$, then, since any two ζ-twisted bundles are necessarily isomorphic, there exists an isomorphism

$$f = \{f_i\}_{i\in I} : E \to \Phi_\eta(E).$$

This means $f_i : E_i \to E_i$ and $(\varphi_{ij}\cdot\eta_{ij}) \circ f_i = f_j \circ \varphi_{ij}$ for all $i, j \in I$. Hence, f induces an automorphism $\bar{f} : \mathbb{P}(E) \to \mathbb{P}(E)$. We claim that any automorphism $\bar{f}$ can be realized in this way from suitable η and f. In fact, by local triviality, we can lift $\bar{f}$ to $f_i \colon E_i \to E_i$ for each $i \in I$. Since the family $\{f_i\}_{i\in I}$ glues to $\bar{f}$, we have that there exists η_{ij} such that

$$f_j \circ \varphi_{ij} = (\varphi_{ij} \circ f_i)\cdot\eta_{ij} = (\varphi_{ij}\cdot\eta_{ij}) \circ f_i$$

for all $i, j \in I$. The latter condition implies

$$\check{\delta}^1\eta = 1.$$

Indeed, we have

$$\begin{aligned} f_i \circ \varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} &= \eta_{ki}\cdot\varphi_{ki} \circ f_k \circ \varphi_{jk} \circ \varphi_{ij} \\ &= \eta_{ki}\eta_{jk}\cdot\varphi_{ki} \circ \varphi_{jk} \circ f_j \circ \varphi_{ij} \\ &= \eta_{ki}\eta_{jk}\eta_{ij}\cdot\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} \circ f_i. \end{aligned}$$

Canceling the extra terms on both sides, we obtain the thesis. Moreover, the only freedom in constructing the cocycle η was the choice of the liftings. Any other choice is of the form $f_i\xi_i$, that replaces η by $\eta\cdot\check{\delta}^0\xi$. Therefore,

$$\begin{aligned} \Phi : \mathrm{Aut}\,\mathbb{P}(E) &\to \check{H}^1(\mathfrak{U}, \underline{\mathrm{U}}(1)) \simeq H^2(X, \mathbb{Z}), \\ \bar{f} &\mapsto [\{\eta_{ij}\}_{i,j\in I}], \end{aligned}$$

is well-defined. One can prove that it is a group homomorphism. Furthermore, it follows from the previous construction that $\bar{f} \in \operatorname{Aut} \mathbb{P}(E)$ *lifts to an automorphism of* E *if and only if* $\Phi(\bar{f})$ *is zero. Thus,* $\Phi(\bar{f})$ *can be thought of as the obstruction to the existence of such a lifting. Note that these observations prove the following result.* ♢

Theorem 5.49 (On the automorphisms of a projective Hilbert bundle)**.** *We have that* $\Phi : \operatorname{Aut} \mathbb{P}(E) \to H^2(X, \mathbb{Z})$ *is surjective. Furthermore, we have that its kernel coincides with the connected component of the identity of* $\operatorname{Aut} \mathbb{P}(E)$*. Therefore, it follows that* Φ *induces a canonical bijection between the connected components of* $\operatorname{Aut} \mathbb{P}(E)$ *and* $H^2(X, \mathbb{Z})$*.*

Proof. The surjectivity is proved above. We leave the proof of the other assertions to the reader. □

5.7 Infinite-dimensional Twisted K-Theory

In this section, we use the language of twisted Hilbert bundles to establish the infinite-dimensional version of Twisted K-Theory. As we pointed out before, the advantage of this version is that it holds for twisting classes of any order. We begin with the following definition.

Definition 5.50 (Twisted K-Theory)**.** *Let* $E = (\{E_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ *be any* ζ*-twisted Hilbert bundle where* $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$*. We shall denote by* $P_{\mathbb{P}(E)}$ *the bundle of projective reference frames of* $\mathbb{P}(E)$[(7)]*. We have a natural adjoint action of*

$$\mathrm{PU}(H) := \mathrm{U}(H)/\mathrm{U}(1)$$

on $\mathscr{F}_H$ *by conjugation, that we denote by*

$$\rho : \mathrm{PU}(H) \to \mathcal{C}(\mathscr{F}_H).$$

[(7)] This object is defined as the one presented in Definition F.30. In particular, we substitute the linear isomorphisms by projective transformations, which are bijections between projective spaces induced by linear isomorphisms of the corresponding vector spaces. Note that we followed a similar pattern in Remark 3.60.

Hence, we construct the associated $\mathscr{F}_H$-bundle

$$F_{\mathbb{P}(E)} \;:=\; \mathcal{E}_\rho P_{\mathbb{P}(E)}.$$

The set of global sections of $F_{\mathbb{P}(E)}$ is denoted by $\Gamma(F_{\mathbb{P}(E)})$. Additionally, we denote by $\bar{\Gamma}(F_{\mathbb{P}(E)})$ the corresponding quotient of $\Gamma(F_{\mathbb{P}(E)})$ by the equivalence relation of homotopy of sections. The latter carries a natural abelian group structure, induced by composition of Fredholm operators. The ***Twisted K-theory group*** *$K^\infty_\zeta(X)$ is defined as the abelian group $\bar{\Gamma}(F_{\mathbb{P}(E)})$.* ♢

Remark 5.51 (On pointwise invertible sections)**.** *Since the space of bounded invertible operators in H is contractible, we have that any pointwise invertible section of $F_{\mathbb{P}(E)}$ is homotopic to the identity. Therefore, if a section is pointwise invertible in a subset of X, then we consider it trivial on such a subset. This fact justifies the following definition.* ♢

Definition 5.52 (Compactly-supported Twisted K-Theory)**.** *Let $E = (\{E_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ be any ζ-twisted Hilbert bundle where $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$. A section of $F_{\mathbb{P}(E)}$ is said to be* ***compactly-supported*** *if it is pointwise invertible in the complement of a compact subset of X. We denote by $\Gamma_c(F_{\mathbb{P}(E)})$ and $\bar{\Gamma}_c(F_{\mathbb{P}(E)})$ the space of compactly-supported global sections of $F_{\mathbb{P}(E)}$ and its quotient up to compactly-supported homotopy, respectively. We define the* ***compactly-supported Twisted K-theory group*** *$K^\infty_{\zeta,c}(X)$ as the abelian group $\bar{\Gamma}_c(F_{\mathbb{P}(E)})$*[8]. ♢

Remark 5.53 (Dependence on the cocycle)**.** *Apparently, Definitions 5.50 and 5.52 depend on E, not only on ζ. Nevertheless, fixing two ζ-twisted Hilbert bundles E and F, we have that an isomorphism $f : E \to F$ is unique up to an automorphism of E. It follows from Lemma 5.49 that the induced isomorphism $\mathbb{P}(f) : \mathbb{P}(E) \to \mathbb{P}(F)$ is unique up to an automorphism of $\mathbb{P}(E)$ connected to the identity, the latter inducing the identity on $\bar{\Gamma}(F_{\mathbb{P}(E)})$ and $\bar{\Gamma}_c(F_{\mathbb{P}(E)})$. Hence, $K^\infty_\zeta(X)$ and $K^\infty_{\zeta,c}(X)$ are canonically defined. On the*

[8] When X is a compact space, Definitions 5.50 and 5.52 are equivalent. Actually, we will only apply Definition 5.50 when X is compact. Hence, it would be sufficient to state Definition 5.52 for every (locally compact) space.

contrary, the definition is not canonical if we only fix the cohomology class $[\zeta]$. *In fact, let us consider a* ζ*-twisted Hilbert bundle* E *and a* ξ*-twisted Hilbert bundle* F*, such that* $\xi = \zeta \cdot \check{\delta}^1\eta$. *We have the isomorphism*

$$\Phi_\eta : K_\zeta^\infty(X) \to K_\xi^\infty(X), \tag{5.2}$$

analogous to the one in the finite-order setting, defined as follows. First, we fix an isomorphism $\bar{f} : \mathbb{P}(E) \to \mathbb{P}(F)$*, belonging to the inverse image of* $[\eta]$ *through* $\Phi : \mathrm{Aut}\,\mathbb{P}(E) \to H^2(X, \mathbb{Z})$ *(see Remark 5.48). Then, we apply the induced one between the corresponding Twisted K-theory groups. This is equivalent to inducing the identity between* $\bar{\Gamma}(F_{\mathbb{P}(E)})$ *and* $\bar{\Gamma}(F_{\mathbb{P}(\Phi_\eta(E))})$*, that represents* $K_\zeta^\infty(X)$ *and* $K_\xi^\infty(X)$*, respectively. The isomorphism in* (5.2) *depends on* η *up to coboundaries. Equivalently, the set of isomorphisms of the form* (5.2) *is a torsor over* $H^2(X, \mathbb{Z})$. *Hence, if* $\zeta = \xi$*, then we obtain an action of* $H^2(X, \mathbb{Z})$ *on* $K_\zeta(X)$. *Only the quotient up to such an action is well-defined. Of course, if* $H^2(X, \mathbb{Z})$ *is trivial, then we have the canonical group* $K_{[\zeta]}^\infty(X)$*, as in the finite-order setting. Analogous considerations hold about compactly-supported K-theory.* ♢

5.8 Comparison isomorphism

In this section, we prove that the finite-order and the infinite-dimensional Twisted K-Theory coincide, up to isomorphism, in the finite-order setting. We begin with the following remark.

Remark 5.54 (Technical facts on pullbacks)**.** *Let* $\mathfrak{V} = \{V_\alpha\}_{\alpha \in J}$ *be a good cover of* X *that is a refinement of* $\mathfrak{U} = \{U_i\}_{i \in I}$ *through the function* $\phi : J \to I$. *This means that* $V_\alpha \subseteq U_{\phi(\alpha)}$ *for every* $\alpha \in J$. *We set* $\hat{\zeta} := \phi^*\zeta$. *Under these circumstances, we obtain the function*

$$\begin{aligned} \Phi_\phi : \widetilde{\mathrm{VB}}_\zeta(X) &\to \widetilde{\mathrm{VB}}_{\hat{\zeta}}(X), \\ (\{E_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I}) &\mapsto (\{F_\alpha\}_{\alpha \in J}, \{\psi_{\alpha\beta}\}_{\alpha,\beta \in J}), \end{aligned}$$

where $F_\alpha := E_{\phi(\alpha)}|_{V_\alpha}$ *and* $\psi_{\alpha\beta} := \varphi_{\phi(\alpha)\phi(\beta)}|_{V_{\alpha\beta}}$ *for all* $\alpha, \beta \in J$. *Moreover, for every* $E \in \widetilde{\mathrm{VB}}_\zeta(X)$*, we have the isomorphism*

$$\bar{\Phi}_{\phi,E} : \mathbb{P}(E) \to \mathbb{P}(\Phi_\phi(E)), \tag{5.3}$$

whose inverse identifies the projectivized fiber $\mathbb{P}(F_\alpha)_x$ *with* $\mathbb{P}(E_{\phi(\alpha)})_x$ *for all* $x \in X$. *Therefore, we obtain the isomorphism*

$$\bar{\Phi}_\phi : K_\zeta^\infty(X) \to K_{\hat{\zeta}}^\infty(X), \tag{5.4}$$

where $\bar{\Gamma}(F_{\mathbb{P}(E)})$ *represents Twisted K-Theory in the domain and* $\bar{\Gamma}(F_{\mathbb{P}(\Phi_\phi(E))})$ *represents Twisted K-Theory in the codomain, for any* $E \in \widetilde{\mathrm{VB}}_\zeta(X)$. *This latter isomorphism is well-defined because, fixing an isomorphism of* ζ*-twisted Hilbert bundles* $f : E \to F$, *we obtain the induced isomorphism of* $\hat{\zeta}$*-twisted bundles* $\phi^* f : \Phi_\phi(E) \to \Phi_\phi(F)$, *where* $\phi^* f = \{f_{\phi(\alpha)} |_{V_\alpha}\}_{\alpha \in J}$, *which is such that diagram*

$$\begin{tikzcd}
\mathbb{P}(E) \arrow[r, "\overline{f}"] \arrow[d, "\overline{\Phi}_{\phi,E}"'] & \mathbb{P}(F) \arrow[d, "\overline{\Phi}_{\phi,E'}"] \\
\mathbb{P}(\Phi_\phi(E)) \arrow[r, "\overline{\phi^* f}"'] & \mathbb{P}(\Phi_\phi(F))
\end{tikzcd}$$

is commutative. In particular, it follows that (5.4) *does not depend on the chosen twisted Hilbert bundle* $E \in \widetilde{\mathrm{VB}}_\zeta(X)$. ◇

Theorem 5.55 (The comparison isomorphism). *Let* $\mathfrak{U} = \{U_i\}_{i \in I}$ *be a good cover of a compact Hausdorff space* X. *In addition, let* $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$ *be a cocycle that represents a finite-order cohomology class. We set* $\hat{\zeta} := \phi^*\zeta$. *Then, there exists an isomorphism*

$$\Theta : K_\zeta^\infty(X) \to K_{\hat{\zeta}}(X),$$

where $K_{\hat{\zeta}}(X)$ *is the group presented in Definition 5.18 while* $K_\zeta^\infty(X)$ *is the one presented in Definition 5.50.*

Proof. We choose a good finite refinement $\mathfrak{V} = \{V_k\}_{k=1}^m$ of $\mathfrak{U}$, through a refinement function $\phi : \{1, \dots, m\} \to I$, such that $\overline{V}_k \subseteq U_{\phi(k)}$ for every k between 1 and m, both included(9). Furthermore, we fix an N-dimensional ζ-twisted vector bundle $\tilde{E}$, for any suitable $N \in \mathbb{N}$. Up to isomorphism, it is straightforward to prove that we can represent $\tilde{E}$ in the form

$$\tilde{E} \;:=\; (\{U_i \times \mathbb{C}^N\}_{i \in I},\, \{g_{ij}\}_{i,j \in I}),$$

where $g_{ij} : U_{ij} \to \mathrm{U}(N)$ for all $i, j \in I$. Then, we consider the ζ-twisted Hilbert bundle given by

$$E \;:=\; \tilde{E} \otimes H,$$

where H denotes the trivial bundle $X \times H$. Equivalently,

$$E \;:=\; (\{U_i \times (\mathbb{C}^N \otimes H)\}_{i \in I},\, \{g_{ij} \otimes 1\}_{i,j \in I}).$$

Since $\mathbb{C}^N \otimes H$ is canonically isomorphic to H, we have that the bundle E satisfies Definition 5.44. Hence, applying the first map presented in Remark (5.54), we obtain the $\hat{\zeta}$-twisted Hilbert bundle

$$\hat{E} \;:=\; \Phi_\phi(E) \;=\; (\{V_k \times (\mathbb{C}^N \otimes H)\}_{k=1}^m,\, \{\hat{g}_{kl} \otimes 1\}_{k,l=1}^m),$$

where $\hat{g}_{kl} = g_{\phi(k)\phi(l)} \,|_{V_{kl}}$ for all k and l between 1 and m, both included. Now let us consider a section $s \in \Gamma(F_{\mathbb{P}(E)})$, projecting to $[s] \in \bar{\Gamma}(F_{\mathbb{P}(E)})$, the latter representing $K^\infty_\zeta(X)$ up to canonical identification. Since the local bundles $U_i \times (\mathbb{C}^N \otimes H)$ are already trivialized, we have that the section s corresponds to a family of sections $s_i : U_i \to \mathscr{F}_{\mathbb{C}^N \otimes H}$ such that

$$s_i \;=\; (g_{ij} \otimes 1) \cdot s_j \cdot (g_{ij}^{-1} \otimes 1)$$

for all $i, j \in I$. Thus, applying the isomorphism in (5.3), we obtain

$$t \;:=\; \phi^* s \in \Gamma(F_{\mathbb{P}(\Phi_\phi(E))}),$$

(9)Under our hypotheses, it is always possible to find such a refinement $\mathfrak{V}$ of $\mathfrak{U}$. In fact, since X is (para)compact, there exists a refinement $\mathfrak{W} = \{W_i\}_{i \in I}$ of $\mathfrak{U}$ such that $\overline{W}_i \subseteq U_i$ for every $i \in I$ (see [29, p. 258]). Since good covers form a cofinal subset of the set of open covers of X, we are allowed to choose a good refinement $\mathfrak{V}' = \{V'_\alpha\}_{\alpha \in J}$ of $\mathfrak{W}$. Then, since X is compact, we can extract a finite (necessarily good) subcover $\mathfrak{V} = \{V_k\}_{k=1}^m$ of $\mathfrak{V}'$. We have $V_k = V'_{\alpha(k)} \subseteq W_{\phi(k)}$ for every k between 1 and m, both included, with respect to a suitable function $\phi : \{1, \dots m\} \to I$. Hence, $\overline{V}_k \subseteq \overline{W}_{\phi(k)} \subseteq U_{\phi(k)}$ for all k between 1 and m, both included.

represented by the family

$$t_k := s_{\phi(k)} |_{V_k} \colon V_k \to \mathscr{F}_{\mathbb{C}^N \otimes H}$$

for all k between 1 and m, both included. Additionally, we have the natural identification

$$\mathbb{C}^N \otimes H \simeq H^{\oplus N}.$$

We call $\pi_j : H^{\oplus N} \to H$ the jth canonical projection for all j between 1 and N, both included. By construction, the functions t_k can be extended to $\overline{V}_k$ (the extension being $s_{\phi(k)} |_{\overline{V}_k}$). Hence, for each $x \in \overline{V}_k$ and for every k between 1 and m, both included, we consider

$$\mathfrak{V}_{x,k} := \bigcap_{j=1}^{N} \pi_j(\operatorname{Ker} t_k(x))^{\perp} \subseteq H.$$

Such a space is closed and finite-codimensional. In fact, $\operatorname{Ker} t_k(x)$ is finite-dimensional, since $t_k(x)$ is Fredholm. Hence, each projection $\pi_j(\operatorname{Ker} t_k(x))$ is also finite-dimensional. It follows that the orthogonal complement is closed and finite-codimensional. Thus, the same holds about the finite intersection $\mathfrak{V}_{x,k}$. Hence, $\mathfrak{V}_{x,k}^{\oplus N}$ is closed and finite-codimensional in $H^{\oplus N}$. Moreover,

$$(\mathfrak{V}_{x,k}^{\oplus N}) \cap \operatorname{Ker} t_k(x)$$

is trivial. This is immediate from the fact that

$$\mathfrak{V}_{x,k}^{\oplus N} \subseteq \operatorname{Ker} t_k(x)^{\perp}.$$

In fact, if $v = (v_1, \dots, v_N) \in \mathfrak{V}_{x,k}^{\oplus N}$ and $w = (w_1, \dots, w_N) \in \operatorname{Ker} t_k(x)$, then, for every j between 1 and N, both included, we have

$$v_j \in \pi_j(\operatorname{Ker} t_k(x))^{\perp} \qquad \text{and} \qquad w_j \in \pi_j(\operatorname{Ker} t_k(x)).$$

Thence, we have $\langle v_j, w_j \rangle = 0$, which implies $\langle v, w \rangle = 0$. Following the proof of Theorem 4.23, for each $x \in \overline{V}_k$ there exists a neighborhood $W_{x,k} \subseteq \overline{V}_k$ such that $\mathfrak{V}_{x,k}^{\oplus N} \cap \operatorname{Ker} t_k(y)$ is trivial for every $y \in W_{x,k}$. The family $\{W_{x,k}\}_{x \in \overline{V}_k}$ is an open cover of the compact space $\overline{V}_k$. Therefore, we extract a finite subcover, that we denote by $\{W_{x_h,k}\}_{h=1}^{n_k}$. We set

$$\mathfrak{V}_k := \bigcap_{h=1}^{n_k} \mathfrak{V}_{x_h,k} \qquad \text{and} \qquad \mathfrak{V} := \bigcap_{k=1}^{m} \mathfrak{V}_k.$$

It follows that $\mathfrak{V}^{\oplus N}$ is closed and finite-codimensional in $H^{\oplus N}$ and that $\mathfrak{V}^{\oplus N} \cap \operatorname{Ker} t_k(x)$ is trivial for every $x \in \overline{V}_k$ and every k between 1 and m, both included. Moreover, we have

$$(\hat{g}_{kl})_x(\mathfrak{V}^{\oplus N}) = \mathfrak{V}^{\oplus N}$$

for every $x \in \overline{V}_{kl}$ and every k and l between 1 and m, both included, since the transition functions act as complex matrices of order N on $H^{\oplus N}$. Projecting to the quotient, we obtain the pointwise isomorphism

$$(\overline{g}_{kl})_x : H^{\oplus N}/\mathfrak{V}^{\oplus N} \to H^{\oplus N}/\mathfrak{V}^{\oplus N}.$$

Since $(\overline{g}_{kl})_x$ is defined for every $x \in \overline{V}_{kl}$, it is particularly defined for every $x \in V_{kl}$. Hence, obtain the $\hat{\zeta}$-twisted finite-dimensional vector bundle on X

$$F_s := (\{V_k \times (H^{\oplus N}/\mathfrak{V}^{\oplus N})\}_{k=1}^m, \{\overline{g}_{kl}\}_{k,l=1}^m).$$

We set

$$H^{\oplus N}/s_k(\mathfrak{V}^{\oplus N}) := \bigsqcup_{x\in V_k} H^{\oplus N}/(s_k)_x(\mathfrak{V}^{\oplus N}),$$

as a quotient space of $V_k \times H^{\oplus N} \simeq \bigsqcup_{x\in V_k} H^{\oplus N}$. By Lemma 4.22, the space $H^{\oplus N}/s_k(\mathfrak{V}^{\oplus N})$ is a vector bundle on V_k. Thus, since V_k is contractible, it is a trivial vector bundle. Moreover, we obtain a well-defined isomorphism

$$\overline{\overline{g}}_{kl} : H^{\oplus N}/s_k(\mathfrak{V}^{\oplus N}) \to H^{\oplus N}/s_l(\mathfrak{V}^{\oplus N})$$

since

$$(\hat{g}_{kl})_x((s_k)_x(\mathfrak{V}^{\oplus N})) = (s_l)_x((\hat{g}_{kl})_x(\mathfrak{V}^{\oplus N})) = (s_l)_x(\mathfrak{V}^{\oplus N}).$$

Therefore, we have the $\hat{\zeta}$-twisted finite-dimensional vector bundle on X

$$G_s := (\{H^{\oplus N}/s_k(\mathfrak{V}^{\oplus N})\}_{k=1}^m, \{\overline{\overline{g}}_{kl}\}_{k,l=1}^m).$$

With respect to the data above, we have the isomorphism

$$\begin{aligned} \hat{\Theta}:\ K_{\hat{\zeta}}^{\infty}(X) &\to K_{\hat{\zeta}}(X), \\ [s] &\mapsto [[F_s]] - [[G_s]]. \end{aligned}$$

We set

$$\Theta := \Phi_{\phi}^{-1} \circ \hat{\Theta} \circ \Phi_{\phi} :\ K_{\zeta}^{\infty}(X) \to K_{\zeta}(X). \tag{5.5}$$

One can prove that it is actually an isomorphism by following the same line of the appendix of [2], adapted to the twisted framework. Moreover, one can prove that, when $H^2(X, \mathbb{Z})$ is trivial, (5.5) does not depend on the representative ζ of the class $[\zeta] \in \check{H}^2(X, \underline{\mathrm{U}}(1)) \simeq H^3(X, \mathbb{Z})$. This finishes the proof of the theorem and the main part of the thesis. $\square$

Further Perspectives

Here we list some topics that can be studied in a near future thanks to the subjects treated in this thesis.

- The first topic is the Atiyah-Singer Index Theorem and its numerous and remarkable applications.

- The second topic consists in studying the following references and the notions surrounding them.

  - FREED, D. S. and HOPKINS, M. J.; TELEMAN, C. **Loop groups and twisted K-theory I**, Journal of Topology, Volume 4, Issue 4, December 2011, pp. 737-798.

  - FREED, D. S. and HOPKINS, M. J.; TELEMAN, C. **Loop groups and twisted K-theory II**, Journal of the American Mathematical Society, Volume 26, Number 3, July 2013, pp. 595-644.

  - FREED, D. S. and HOPKINS, M. J.; TELEMAN, C. **Loop groups and twisted K-theory III**, Annals of Mathematics 174 (2011), pp. 947-1007.

  These papers contain non-trivial applications of Twisted K-Theory that show its significance to Mathematics.

- The third and final topic is the differential extension of the cohomology theories studied in this thesis.

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Appendix A

Direct Limits of Abelian Groups

In this appendix, we set an essential tool to define the compactly-supported generalized cohomology groups, namely, the direct limit of abelian groups. The categorical approach given to this topic is based on [26, pp. 105-112]. The reader can easily generalize the ideas presented here to several categories, among which is the category of sets. Finally, it is to be noted that the notions presented here are essentially used in Chapter 1.

A.1 Direct systems of abelian groups

Definition A.1 (Direct system of abelian groups)**.** *We say that a triple*

$$\mathfrak{A} = \big(\Lambda,\ (C_\alpha)_{\alpha\in\Lambda},\ (\iota_{\alpha\beta} : C_\alpha \to C_\beta)_{\alpha,\beta\in\Lambda}\big)$$

is a ***direct system of abelian groups*** *provided that:*

- Λ *is a direct set. This means that* Λ *is a partially ordered set equipped with a partial order* $\prec$ *such that, for all* $\alpha, \beta \in \Lambda$*, there exists* $\gamma \in \Lambda$ *for which* $\alpha \prec \gamma$ *and* $\beta \prec \gamma$*;*
- $(C_\alpha)_{\alpha\in\Lambda}$ *is a family of abelian groups indexed by* Λ*; and*
- $(\iota_{\alpha\beta} : C_\alpha \to C_\beta)_{\alpha,\beta\in\Lambda}$ *is a collection of homomorphisms of abelian groups such that* $\iota_{\alpha\alpha} = \mathrm{id}_{C_\alpha}$ *for all* $\alpha \in \Lambda$ *and, for all* $\alpha, \beta, \gamma \in \Lambda$ *that verify* $\alpha \prec \beta \prec \gamma$*,* $\iota_{\alpha\gamma} = \iota_{\beta\gamma} \circ \iota_{\alpha\beta}$*.* $\diamondsuit$

Notation A.2 (On direct systems of abelian groups). *Henceforth, the notation of Definition A.1 will be used without explicit mention. In particular, we will denote a direct system of abelian groups simply by* $\mathfrak{A}$. $\diamondsuit$

Example A.3 (Direct system of subgroups of a fixed abelian group). *Let C be an abelian group and $\mathfrak{C}$ be its family of subgroups. Clearly, $\mathfrak{C}$ is a direct set with respect to the partial order given by the inclusion of subgroups. Thus, being $A, B \in \mathfrak{C}$, if we define $\iota_{A,B} : A \to B$ to be the inclusion map if A is contained in B, and the trivial homomorphism otherwise, then*

$$\mathfrak{A}_C \ := \ \big(\mathfrak{C},\ \mathfrak{C},\ (\iota_{A,B} : A \to B)_{A,B\in\mathfrak{C}}\big)$$

is a direct system of abelian groups, which we call the ***direct system of subgroups of the abelian group C****.* $\diamondsuit$

A.2 Direct limits of direct systems of abelian groups

Definition A.4 (Direct limit of a direct system of abelian groups). *We say that an abelian group $\varinjlim_{\alpha} C_\alpha$ is a* ***direct limit*** *of $\mathfrak{A}$ if there exists a family $(\iota_\alpha : C_\alpha \to \varinjlim_{\alpha} C_\alpha)_{\alpha\in\Lambda}$ of homomorphisms of abelian groups in such manner that the following two conditions are satisfied.*

(1) For all α, $\beta \in \Lambda$ that verify $\alpha \prec \beta$, the following diagram of morphisms of abelian groups is commutative.

$$\begin{tikzcd} C_\alpha \arrow[r, "\iota_{\alpha\beta}"'] \arrow[rr, bend left, "\iota_\alpha"] & C_\beta \arrow[r, "\iota_\beta"'] & \varinjlim_{\alpha} C_\alpha \end{tikzcd} \tag{A.1}$$

(2) Let C be any abelian group and $(\varphi_\alpha : C_\alpha \to C)_{\alpha\in\Lambda}$ be any family of homomorphisms of abelian groups for which the following diagram is commutative for all $\alpha, \beta \in \Lambda$ that verify $\alpha \prec \beta$.

$$\begin{tikzcd} C_\alpha \arrow[r, "\iota_{\alpha\beta}"'] \arrow[rr, bend left, "\varphi_\alpha"] & C_\beta \arrow[r, "\varphi_\beta"'] & C \end{tikzcd} \tag{A.2}$$

There exists a unique morphism of abelian groups $\varphi : \varinjlim_\alpha C_\alpha \to C$ *such that the following diagram is commutative.*

$$\begin{tikzcd} C_\alpha \arrow[r, "\iota_{\alpha\beta}"] \arrow[d, "\iota_\alpha"'] & C_\beta \arrow[d, "\varphi_\beta"] \\ \varinjlim_\alpha C_\alpha \arrow[r, "\varphi"'] & C \end{tikzcd} \tag{A.3}$$

♢

Example A.5 (The direct limit of the direct system of Example A.3)**.** *Let C be an abelian group. The direct system $\mathfrak{A}_C$ of Example A.3 has as its direct limit C with the family of homomorphisms of abelian groups being given by the inclusions of the subgroups of C in C itself.* ♢

A.3 Existence and uniqueness up to isomorphism

Theorem A.6 (Uniqueness of the direct limit up to a unique isomorphism)**.** *If* $\varinjlim_\alpha C_\alpha$ *and* $\varinjlim_\alpha{}' C_\alpha$ *are direct limits of* $\mathfrak{A}$ *with respect to the families of morphisms of abelian groups* $\Phi = (\iota_\alpha : C_\alpha \to \varinjlim_\alpha C_\alpha)_{\alpha \in \Lambda}$ *and* $\Phi' = (\iota'_\alpha : C_\alpha \to \varinjlim_\alpha{}' C_\alpha)_{\alpha \in \Lambda}$*, respectively, then there necessarily exists a unique isomorphism of abelian groups* $\varphi : \varinjlim_\alpha C_\alpha \to \varinjlim_\alpha{}' C_\alpha$ *in such manner that the following diagram is commutative for every* $\alpha \in \Lambda$*.*

$$\begin{tikzcd} \varinjlim_\alpha C_\alpha \arrow[rr, bend left, "\varphi"] & C_\alpha \arrow[l, "\iota_\alpha"] \arrow[r, "\iota'_\alpha"'] & \varinjlim_\alpha{}' C_\alpha \end{tikzcd} \tag{A.4}$$

Proof. The diagrams presented in the statement and in the proof of this theorem are reformulations of Diagram (A.3) when α coincides with β. Moreover, we tacitly use the commutativity of Diagrams (A.1) and (A.2) with respect to the families Φ and

Φ'. Indeed, there exist unique morphisms of abelian groups $\varphi : \varinjlim_{\alpha} C_\alpha \to \varinjlim'_{\alpha} C_\alpha$ and $\varphi' : \varinjlim'_{\alpha} C_\alpha \to \varinjlim_{\alpha} C_\alpha$ such that Diagrams (A.4) and (A.5) are commutative for every $\alpha \in \Lambda$.

$$\begin{tikzcd} \varinjlim'_{\alpha} C_\alpha \arrow[rr, bend left, "\varphi'"] & C_\alpha \arrow[l, "\iota'_\alpha"'] \arrow[r, "\iota_\alpha"'] & \varinjlim_{\alpha} C_\alpha \end{tikzcd} \tag{A.5}$$

Then, the following diagrams are also commutative.

$$\begin{tikzcd} \varinjlim_{\alpha} C_\alpha \arrow[rr, bend left, "\varphi'\circ\varphi"] & C_\alpha \arrow[l, "\iota_\alpha"'] \arrow[r, "\iota_\alpha"'] & \varinjlim_{\alpha} C_\alpha \end{tikzcd} \qquad \begin{tikzcd} \varinjlim'_{\alpha} C_\alpha \arrow[rr, bend left, "\varphi\circ\varphi'"] & C_\alpha \arrow[l, "\iota'_\alpha"'] \arrow[r, "\iota'_\alpha"'] & \varinjlim'_{\alpha} C_\alpha \end{tikzcd}$$

The homomorphisms represented in the upper arrows of the preceding diagrams are unique. Therefore, since the identity maps also turn these diagrams commutative, we have

$$\varphi' \circ \varphi \;=\; \mathrm{id}_{\varinjlim_{\alpha} C_\alpha} \qquad \text{and} \qquad \varphi \circ \varphi' \;=\; \mathrm{id}_{\varinjlim'_{\alpha} C_\alpha}.$$

Hence, φ is the unique group isomorphism for which Diagram (A.4) is commutative for every $\alpha \in \Lambda$. $\square$

Theorem A.7 (Existence of the direct limit of a direct system of abelian groups)**.** *We define the disjoint union*

$$D_{\mathfrak{A}} \;:=\; \bigsqcup_{\alpha\in\Lambda} C_\alpha.$$

Moreover, we define a relation on $D_{\mathfrak{A}}$ as follows. Being $\alpha, \beta \in \Lambda$, we say that $x \in C_\alpha$ is related to $y \in C_\beta$ if and only if there exists $\gamma \in \Lambda$ for which $\alpha \prec \gamma$, $\beta \prec \gamma$ and $\iota_{\alpha\gamma}(x) = \iota_{\beta\gamma}(y)$. This is an equivalence relation on $D_{\mathfrak{A}}$. Furthermore, the quotient of $D_{\mathfrak{A}}$ by this equivalence relation, which we hereafter denote by $\mathfrak{D}_{\mathfrak{A}}$, has a natural abelian group structure that turns it into the direct limit of the direct system $\mathfrak{A}$.

Proof. We start by showing that the relation defined in the statement of this theorem is an equivalence relation on $D_{\mathfrak{A}}$. Indeed, let $\alpha, \beta, \eta \in \Lambda$. Then, we have that this relation is:

- *reflexive.* In fact, $x \in C_\alpha$ is related to itself since $\alpha \prec \alpha$ and $\iota_{\alpha\alpha}(x) = x = \iota_{\alpha\alpha}(x)$;

- *symmetric.* In fact, if $x \in C_\alpha$ is related to $y \in C_\beta$, then there exists $\gamma \in \Lambda$ for which $\alpha \prec \gamma$, $\beta \prec \gamma$ and $\iota_{\alpha\gamma}(x) = \iota_{\beta\gamma}(y)$. Thus, tautologically, there exists $\gamma \in \Lambda$ such that $\beta \prec \gamma$, $\alpha \prec \gamma$ and $\iota_{\beta\gamma}(y) = \iota_{\alpha\gamma}(x)$. This proves that y is also related to x; *and*

- *transitive.* In fact, assume that $x \in C_\alpha$ is related to $y \in C_\beta$. Then, there exists $\gamma_{x,y} \in \Lambda$ for which $\alpha \prec \gamma_{x,y}$, $\beta \prec \gamma_{x,y}$ and $\iota_{\alpha\gamma_{x,y}}(x) = \iota_{\beta\gamma_{x,y}}(y)$. Moreover, assume that $y \in C_\beta$ is related to $z \in C_\eta$. Thus, there exists $\gamma_{y,z} \in \Lambda$ such that $\beta \prec \gamma_{y,z}$, $\eta \prec \gamma_{y,z}$ and $\iota_{\beta\gamma_{y,z}}(y) = \iota_{\eta\gamma_{y,z}}(z)$. Since Λ is a direct set, there exists $\gamma \in \Lambda$ such that $\gamma_{x,y} \prec \gamma$ and $\gamma_{y,z} \prec \gamma$. Hence, we claim that $\alpha \prec \gamma$, $\eta \prec \gamma$ and $\iota_{\alpha\gamma}(x) = \iota_{\eta\gamma}(z)$. These relations prove that x is also related to z. Indeed, we have $\alpha \prec \gamma_{x,y} \prec \gamma$, $\eta \prec \gamma_{y,z} \prec \gamma$ and

$$\begin{aligned}
\iota_{\alpha\gamma}(x) &= (\iota_{\gamma_{x,y}\gamma} \circ \iota_{\alpha\gamma_{x,y}})(x) \\
&= (\iota_{\gamma_{x,y}\gamma} \circ \iota_{\beta\gamma_{x,y}})(y) \\
&= \iota_{\beta\gamma}(y) \\
&= (\iota_{\gamma_{y,z}\gamma} \circ \iota_{\beta\gamma_{y,z}})(y) \\
&= (\iota_{\gamma_{y,z}\gamma} \circ \iota_{\eta\gamma_{y,z}})(z) \\
&= \iota_{\eta\gamma}(z),
\end{aligned}$$

  since we also have the comparisons $\beta \prec \gamma_{x,y} \prec \gamma$ and $\beta \prec \gamma_{y,z} \prec \gamma$ in the direct set under consideration.

Consequently, the quotient $\mathfrak{D}_{\mathfrak{A}}$ of $D_{\mathfrak{A}}$ by this equivalence relation is well-defined. Now, let $x \in C_\alpha$ and $y \in C_\beta$ where $\alpha, \beta \in \Lambda$. Being $\gamma \in \Lambda$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$, we define

$$[x] + [y] \;:=\; [\iota_{\alpha\gamma}(x) + \iota_{\beta\gamma}(y)].$$

The reader can readily prove that this binary operation on $\mathfrak{D}_{\mathfrak{A}}$ is well-defined. Hence, $\mathfrak{D}_{\mathfrak{A}}$ is an abelian group, which is the direct limit of the direct system of abelian groups $\mathfrak{A}$. More explicitly, we claim that $\mathfrak{D}_{\mathfrak{A}} = \varinjlim_{\alpha} C_\alpha$. In fact, for each $\alpha \in \Lambda$, we define

$$\begin{aligned}\iota_\alpha : C_\alpha &\to \mathfrak{D}_{\mathfrak{A}},\\ x &\mapsto [x].\end{aligned}$$

Thence:

- let $\alpha, \beta \in \Lambda$ be such that $\alpha \prec \beta$. Then, we claim that Diagram (A.1) is commutative, clearly substituting $\varinjlim_\alpha C_\alpha$ by $\mathfrak{D}_{\mathfrak{A}}$. As a matter of fact, note that each $x \in C_\alpha$ is related to $\iota_{\alpha\beta}(x) \in C_\beta$. Indeed, since Λ is a direct set, there exists $\gamma \in \Lambda$ such that $\alpha \prec \beta \prec \gamma$. Thus,

$$\iota_{\alpha\gamma}(x) \;=\; (\iota_{\beta\gamma} \circ \iota_{\alpha\beta})(x) \;=\; \iota_{\beta\gamma}(\iota_{\alpha\beta}(x)).$$

  Therefore, for every $x \in C_\alpha$, we have

$$\iota_\alpha(x) \;=\; [x] \;=\; [\iota_{\alpha\beta}(x)] \;=\; (\iota_\beta \circ \iota_{\alpha\beta})(x),$$

  which proves our claim; *and*

- let C be an abelian group and $(\varphi_\alpha : C_\alpha \to C)_{\alpha\in\Lambda}$ be a family of homomorphisms of abelian groups such that Diagram (A.2) is commutative for all $\alpha, \beta \in \Lambda$ that verify $\alpha \prec \beta$. Then, we claim that there exists a unique morphism of abelian groups $\varphi : \mathfrak{D}_{\mathfrak{A}} \to C$ such that Diagram (A.3) is commutative, clearly substituting $\varinjlim_\alpha C_\alpha$ by $\mathfrak{D}_{\mathfrak{A}}$. In fact, if $x \in C_\eta$ for some $\eta \in \Lambda$, then we define

$$\varphi[x] \;:=\; \varphi_\eta(x) \in C.$$

  This map is well-defined because, if x is related to y, where $y \in C_\delta$ for some $\delta \in \Lambda$, then $\varphi_\delta(y) = \varphi_\eta(x)$. Indeed, since x is related to y, there exists $\gamma \in \Lambda$ such that $\eta \prec \gamma$, $\delta \prec \gamma$ and $\iota_{\eta\gamma}(x) = \iota_{\delta\gamma}(y)$. Thus,

$$\varphi_\delta(y) \;=\; (\varphi_\gamma \circ \iota_{\delta\gamma})(y) \;=\; (\varphi_\gamma \circ \iota_{\eta\gamma})(x) \;=\; \varphi_\eta(x).$$

  Moreover, it is evident that this map turns the preceding diagram commutative since $\varphi_\alpha(x) = (\iota_{\alpha\beta} \circ \varphi_\beta)(x)$ for all $x \in C_\alpha$. Finally, the reader can prove the uniqueness part of the assertion. $\square$

Remark A.8 (On the proof of the preceding result)**.** *Using the notation of the proof of Theorem A.7, the reader can readily prove that the map* $\varphi : \varinjlim_{\alpha} C_\alpha \to C$ *is bijective if and only if*

$$C = \bigcup_{\alpha \in \Lambda} \varphi_\alpha(C_\alpha).$$

Furthermore, $\varphi_\alpha(x) = \varphi_\beta(y)$ *if and only if there exists* $\gamma \in \Lambda$ *for which* $\alpha \prec \gamma$*,* $\beta \prec \gamma$ *and* $\iota_{\alpha\gamma}(x) = \iota_{\beta\gamma}(y)$*.* ♢

Remark A.9 (On a generalization of the preceding constructions)**.** *The reader may have noted that, in all of the technical constructions above, we did not use the neutral element or the inverses of the elements of the abelian groups. In fact, these elements were only used to prove that the direct limit of abelian groups is also an abelian group. Hence, one can readily prove that this appendix can be restated for abelian semigroups. More than that, its definitions and its results are, mutatis mutandis, exactly the same for abelian semigroups. We leave to the reader the completion of the immediate details.* ♢

Appendix B

Grothendieck Groups

In this appendix, we set a basic algebraic tool from Group Theory that is essential to define the K-Theory groups, namely, the Grothendieck group of an abelian semigroup. The idea behind such concept is to find the minimal extension of an abelian semigroup into an abelian group, although this is not always the case. We follow [2, pp. 42-43] that presents such construction for abelian semigroups, differing from the majority of the references that restrict themselves to abelian monoids. Finally, it is to be noted that the notions presented here are essentially used in Chapters 2 and 5.

B.1 Definition

Definition B.1 (Grothendieck group)**.** *A **Grothendieck group** of an abelian semigroup S is a pair $(K(S), \iota_S)$ such that:*

- *$K(S)$ is an abelian group;*
- *$\iota_S : S \to K(S)$ is a semigroup homomorphism; and*
- *if C is any abelian group and $\varphi : S \to C$ is any semigroup homomorphism, then there exists a unique group homomorphism $\xi : K(S) \to C$ such that the following diagram is commutative.*

$$\begin{tikzcd} K(S) \arrow[rr, bend left, "\xi"] & S \arrow[l, "\iota_S"'] \arrow[r, "\varphi"'] & C \end{tikzcd}$$ ◇

Example B.2 (Grothendieck groups of the natural numbers)**.** *If we consider the abelian monoid of natural numbers with the usual sum, then its Grothendieck group is the abelian group of the integer numbers with the usual sum. However, if we consider the abelian monoid of natural numbers with the usual product, then its Grothendieck group is the trivial group. Moreover, if we consider the abelian monoid of non-zero natural numbers with the usual product, then its Grothendieck group is the abelian group of positive rational numbers with the usual product.* ♢

B.2 Existence and uniqueness up to isomorphism

Theorem B.3 (Uniqueness of the Grothendieck group up to a unique isomorphism)**.** *If $(K(S),\, \iota_S)$ and $(K'(S),\, \iota'_S)$ are Grothendieck groups of an abelian semigroup S, then there exists a unique group isomorphism $\xi : K(S) \to K'(S)$ in such manner that the following diagram is commutative.*

$$\begin{tikzcd} K(S) \arrow[rr, bend left, "\xi"] & S \arrow[l, "\iota_S"'] \arrow[r, "\iota'_S"'] & K'(S) \end{tikzcd} \tag{B.1}$$

Proof. There exist unique morphisms of abelian groups $\xi : K(S) \to K'(S)$ and $\xi' : K'(S) \to K(S)$ such that Diagrams (B.1) and (B.2) are commutative.

$$\begin{tikzcd} K'(S) \arrow[rr, bend left, "\xi'"] & S \arrow[l, "\iota'_S"'] \arrow[r, "\iota_S"'] & K(S) \end{tikzcd} \tag{B.2}$$

Then, the following diagrams are also commutative.

$$\begin{tikzcd} K(S) \arrow[rr, bend left, "\xi' \circ \xi"] & S \arrow[l, "\iota_S"'] \arrow[r, "\iota_S"'] & K(S) \end{tikzcd} \qquad \begin{tikzcd} K'(S) \arrow[rr, bend left, "\xi \circ \xi'"] & S \arrow[l, "\iota'_S"'] \arrow[r, "\iota'_S"'] & K'(S) \end{tikzcd}$$

The homomorphisms represented in the upper arrows of the preceding diagrams are unique. Therefore, since the identity maps also turn these diagrams commutative, we have

$$\xi' \circ \xi \;=\; \mathrm{id}_{K(S)} \qquad \text{and} \qquad \xi \circ \xi' \;=\; \mathrm{id}_{K'(S)}.$$

Hence, ξ is the unique group isomorphism for which Diagram (B.1) is commutative, as we wished. □

Theorem B.4 (Existence of a Grothendieck group). *Let S be an abelian semigroup. We define:*

- *$K(S)$ as being the abelian group given by the following free presentation:*
  - *the generators of $K(S)$ are the elements of S. We denote an element $a \in S$ by $[a] \in K(s)$ when it is thought as a generator of $K(S)$;*
  - *the relations in $K(S)$ are the expressions $[a + b] - [a] - [b]$ for all $a, b \in S$. Here $[a + b]$ denotes the class of the sum of a and b in the abelian semigroup S; and*
- *$\iota_S : S \to K(S)$ as being the morphism of abelian semigroups that is given by $\iota_S(a) = [a]$ for all $a \in S$.*

Every element of $K(S)$ is a difference between two classes of generators. Moreover, $(K(S),\ \iota_S)$ is a Grothendieck group of S.

Proof. The first claim is obvious since the general element of $K(S)$ is given by

$$\sum_{j=1}^{n} [a_j] \;-\; \sum_{j=1}^{m} [b_j],$$

which coincides with the difference of classes of generators

$$\left[\sum_{j=1}^{n} a_j\right] \;-\; \left[\sum_{j=1}^{m} b_j\right].$$

For the second claim, let C be an abelian group and $\varphi : S \to C$ be a morphism of abelian semigroups. We define $\xi : K(S) \to C$ as the morphism of abelian groups that sends $[a] \in K(S)$ into $\varphi(a) \in C$, which is tacitly linearly extended to the whole group $K(S)$. It is clear that the image under ξ of every relation $[a + b] - [a] - [b]$ is zero for all $a, b \in S$. Thus, ξ is well-defined. Furthermore, $\varphi = \xi \circ \iota_S$ by construction.

Finally, we have to prove that ξ is the only morphism of abelian groups that verifies this last property. Indeed, if $\eta : K(S) \to C$ is a morphism of abelian groups such that $\varphi = \eta \circ \iota_S$, then, for all $[a] \in K(S)$, we have $\eta[a] = \eta(\iota_S(a)) = \varphi(a) = \xi[a]$. Consequently, since η and ξ take the same values over the generators of $K(S)$, we have $\eta = \xi$, as we wished. □

Remark B.5 (Covariant functor defined by the Grothendieck group of abelian semigroups)**.** *Let the semigroup homomorphism $\iota_S : S \to K(S)$ be implicit, saying that $K(S)$ is a Grothendieck group of an abelian semigroup S. Hence, if $\mathscr{S}_{ab}$ and $\mathscr{G}_{ab}$ are the categories of abelian semigroups and groups, respectively, then we have the covariant functor*

$$\begin{aligned} K : \mathscr{S}_{ab} &\to \mathscr{G}_{ab}, \\ S &\mapsto K(S), \\ \varphi : S \to R &\mapsto K(\varphi) : K(S) \to K(R), \end{aligned}$$

where $K(\varphi)$ is the only morphism of abelian groups that turns the following diagram commutative.

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\iota_S"'] & R \arrow[d, "\iota_R"] \\ K(S) \arrow[r, "K(\varphi)"'] & K(R) \end{tikzcd}$$

More explicitly, $K(\varphi) : K(S) \to K(R)$ is the group homomorphism that sends each generator $[s] \in K(S)$ into $[\varphi(s)] \in K(R)$, which is tacitly linearly extended to the whole group $K(S)$. ♢

B.3 Understanding the structure

Theorem B.6 (Another existence result for Grothendieck groups)**.** *Let S be an abelian semigroup. We define:*

- *$K'(S)$ as being:*

  - *as a set, the quotient of $S \times S$ by the equivalence relation that identifies (a, b) with (c, d) in $S \times S$ if and only if there exists $u \in S$ in such manner that $a + d + u = b + c + u$; and*
  - *as an abelian group, equipped with the commutative binary operation $[a, b] + [c, d] = [a + c, b + d]$. Note that $[a, a]$ is the identity in $K'(S)$ and that $[b, a]$ is the inverse of $[a, b]$ in $K'(S)$; and*

- *$\iota'_S : S \to K'(S)$ as being the morphism of abelian semigroups that is given by $\iota'_S(a) = [a + a, a]$ for all $a \in S$.*

We have that $(K'(S),\ \iota'_S)$ is a Grothendieck group of S.

Proof. Let C be an abelian group and $\varphi : S \to C$ be a morphism of abelian semigroups. We define $\xi' : K'(S) \to C$ as being the morphism of abelian groups that sends $[a, b] \in K'(S)$ into $\varphi(a) - \varphi(b) \in C$. We have that ξ' is well-defined. Indeed, if (a, b) is related to (c, d), then there exists $u \in S$ in such manner that $a + d + u = b + c + u$. Consequently,

$$\varphi(a) + \varphi(d) + \varphi(u) \ = \ \varphi(b) + \varphi(c) + \varphi(u).$$

Therefore,

$$\xi'[a, b] \ = \ \varphi(a) - \varphi(b) \ = \ \varphi(c) - \varphi(d) \ = \ \xi'[c, d].$$

Moreover, we have $\varphi = \xi' \circ \iota'_S$ by construction. Finally, we have to prove that ξ' is the only morphism of abelian groups that verifies this last property. In fact, if $\eta' : K'(S) \to C$ is a morphism of abelian groups such that $\varphi = \eta' \circ \iota'_S$, then, for all $[a, b] \in K'(S)$, we have

$$\eta'[a, b] \ = \ \eta'[a + a, a] - \eta'[b + b, b] \ = \ \eta'(\iota'_S(a)) - \eta'(\iota'_S(b)) \ = \ \varphi(a) - \varphi(b) \ = \ \xi'[a, b].$$

This finishes the proof of the theorem. □

Remark B.7 (Understanding the structure of Grothendieck groups by means of Theorems B.4 and B.6)**.** *We know from Theorem B.3 that there exists a unique group isomorphism between the Grothendieck groups defined in Theorems B.4 and B.6. More* *properly, there exists a unique group isomorphism* $\xi' : K'(S) \to K(S)$ *such that* $\iota_S = \xi' \circ \iota'_S$. *Therefore, for all* $a \in S$, *we have*

$$[a] \;=\; \iota_S(a) \;=\; \xi' \circ \iota'_S(a) \;=\; \xi'[a+a, a].$$

Consequently,

$$\xi'[a,b] \;=\; \xi'[a+a,a] - \xi'[b+b,b] \;=\; [a]-[b].$$

This shows the explicit definition of ξ'. *However, this also shows when* $[a] = [b]$ *in* $K(S)$. *Indeed, since* ξ' *is an isomorphism, we have* $[a] = [b]$ *in* $K(S)$ *if and only if*

$$[a+a,a] \;=\; (\xi')^{-1}[a] \;=\; (\xi')^{-1}[b] \;=\; [b+b,b]$$

in $K'(S)$. *This condition is the same as requiring the existence of* $u \in S$ *in such manner that*

$$a+u \;=\; b+u.$$

In fact, we have $[a+a,a] = [b+b,b]$ *if and only if there exists* $v \in S$ *such that*

$$a+(a+b+v) \;=\; 2a+b+v \;=\; a+2b+v \;=\; b+(a+b+v),$$

which is the same as the existence of $u \in S$ *such that* $a+u = b+u$ *because we can take* $u = a+b+v$. $\diamondsuit$

Appendix C

Ordinary Vector Bundles

In this appendix, we set the fundamental notion that one must know in order to understand K-Theory, which is the one of vector bundles. However, since the theory of vector bundles is extensive, we only expose here its initial concepts and the results that play an essential role in the main text. Hence, it must be clear that we do not intend to give a complete exposition of this subject in any sense. Indeed, we think that the reader who fells the urge to deepen his or her knowledge in this interesting topic may find in [2, pp. 1 - 41], [15, pp. 4 - 37], [16, pp. 85 - 109], [18, pp. 24 - 39], [19, pp. 1 - 51], [24, pp. 249 - 271] and [31] good references. It is to be noted that the notions presented here are mainly used in Chapter 2, although some of them appear generalized in Chapter 5.

C.1 First definitions

Definition C.1 (Vector bundle). *Let X be a connected topological space and $\mathscr{V}$ be a finite-dimensional real or complex vector space. A* ***vector bundle*** *on X with* ***typical fiber*** *$\mathscr{V}$ is defined by the following data:*

- *a topological space E;*
- *a surjective continuous function $\pi : E \to X$; and*
- *a vector space structure on $\pi^{-1}(x)$ for every $x \in X$,*

such that the following two conditions are satisfied.

(1) *For every $x \in X$, there exist an open neighborhood U of x in X and a homeomorphism*

$$\varphi : \pi^{-1}(U) \to U \times \mathscr{V}$$

verifying the commutativity of the following diagram, assuming that $\mathscr{V}$ is endowed with the topology that is naturally induced by its finite-dimensional vector space structure.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\pi"'] \arrow[rr, bend left, "\varphi"] & U & U \times \mathscr{V} \arrow[l, "\pi_U"]
\end{tikzcd}$$

(2) *The function $\varphi \mid_{\pi^{-1}(y)}: \pi^{-1}(y) \to \{y\} \times \mathscr{V}$ is linear for every $y \in U$. Therefore, being bijective, it is a vector space isomorphism for every $y \in U$.*

If X is not connected, then a vector bundle on X is defined by a vector bundle on each connected component of X. In this situation, the typical fiber depends on each connected component of X. $\diamondsuit$

Notation C.2 (On vector bundles)**.** *Henceforth, the notation of Definition C.1 will be used without explicit mention. In particular, we will denote a vector bundle with typical fiber $\mathscr{V}$ by $\pi : E \to X$. Moreover, we will often denote the whole bundle by E, for convenience.* $\diamondsuit$

Definition C.3 (Standard nomenclature in the framework of vector bundles)**.** *Let $\pi : E \to X$ be a vector bundle. We say that:*

- *for every $x \in X$, the vector space $\pi^{-1}(x)$ is the **fiber** of E in x, which is hereafter denoted by E_x;*
- *E and X are, respectively, the **total space** and the **base space** of the vector bundle $\pi : E \to X$;*
- *a **local chart** or **local trivialization** of E is a pair (U, φ_U) where:*
  - *$U \subseteq X$ is open; and*
  - *$\varphi_U : \pi^{-1}(U) \to U \times \mathscr{V}$ is a homeomorphism satisfying Conditions (1) and (2) of Definition C.1.*

Moreover, if $x \in U$, then the local chart (U, φ_U) is also said to be a ***local chart in x****;*

- *an* ***atlas*** *of E is a family $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i \in I}$ where:*
  - *$\mathfrak{U} = \{U_i\}_{i \in I}$ is an open cover of X; and*
  - *(U_i, φ_i) is a local chart of E for all $i \in I$.*

  Note that the existence of an atlas of E follows from Conditions (1) and (2) of Definition C.1; and

- *given a point $x_0 \in X$, the dimension of the vector space E_{x_0} is the* ***rank*** *of E in x_0, which is hereafter denoted by $\mathrm{rk}_{x_0}(E)$. Since the rank only depends on the connected component of x_0 and coincides with the dimension of the typical fiber, if X is connected, then we denote it by $\mathrm{rk}(E)$. Furthermore, if the rank of E is one, then E is said to be a* ***line bundle****.* $\diamondsuit$

Remark C.4 (On the topology of the fibers of a vector bundle)**.** *In a vector bundle $\pi : E \to X$, we have that the topology of each fiber E_x of E, induced as a topological subspace of the total space E, coincides with the topology induced by its finite-dimensional vector space structure. Indeed, fixing any local chart (U, φ_U) of E, because of Condition (1) of Definition C.1, we have that the map $\varphi_U |_{E_y} : E_y \to \{y\} \times \mathscr{V}$ is a homeomorphism for every $y \in U$.* $\diamondsuit$

C.2 Morphisms and categories of vector bundles

Definition C.5 (Vector bundle morphisms)**.** *Let $\pi_E : E \to X$ and $\pi_F : F \to Y$ be vector bundles. We give the following definitions.*

- *A* ***vector bundle morphism*** *from E into F is a continuous function $f : E \to F$ such that:*
  - *there exists a (unique) continuous function $g : X \to Y$ in such manner that $\pi_F \circ f = g \circ \pi_E$; and*
  - *$f |_{E_x} : E_x \to F_{g(x)}$ is linear for every $x \in X$.*

This means that the following diagram is commutative with f being linear in each fiber.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

- *If $X = Y$, then we say that a* ***vector bundle morphism over X*** *from E into F is a vector bundle morphism $f : E \to F$ in such manner that the induced function $g : X \to X$ is the identity map.*

Moreover, we say that an invertible vector bundle morphism (over X) is a ***vector bundle isomorphism*** *(**over X**).* $\diamondsuit$

Definition C.6 (Categories of vector bundles)**.** *We say that:*

- VectBdl *is the* ***category of vector bundles*** *whose objects are vector bundles and whose morphisms are vector bundle morphisms;*

- VectBdl($\mathscr{V}$) *is the* ***category of vector bundles with fixed typical fiber*** $\mathscr{V}$ *whose objects are vector bundles with typical fiber $\mathscr{V}$ and whose morphisms are vector bundle morphisms;*

- VectBdl$_X$ *is the* ***category of vector bundles on X*** *whose objects are vector bundles on X and whose morphisms are vector bundle morphisms over X; and*

- VectBdl$_X(\mathscr{V})$ *is the* ***category of vector bundles on X with fixed typical fiber*** $\mathscr{V}$ *whose objects are vector bundles on X with typical fiber $\mathscr{V}$ and whose morphisms are vector bundle morphisms over X.* $\diamondsuit$

Remark C.7 (On the categories of vector bundles)**.** *We have the following diagram of categories indicating the inclusion relations between* VectBdl*,* VectBdl($\mathscr{V}$)*,* VectBdl$_X$ *and* VectBdl$_X(\mathscr{V})$*.*

$$\begin{tikzcd} \mathrm{VectBdl}(\mathscr{V}) \arrow[r, two heads] & \mathrm{VectBdl} \\ \mathrm{VectBdl}_X(\mathscr{V}) \arrow[r, two heads] \arrow[u] & \mathrm{VectBdl}_X \arrow[u] \end{tikzcd}$$

Indeed, $\mathrm{VectBdl}(\mathscr{V})$, $\mathrm{VectBdl}_X$ *and* $\mathrm{VectBdl}_X(\mathscr{V})$ *are subcategories of* $\mathrm{VectBdl}$*, being* $\mathrm{VectBdl}(\mathscr{V})$ *its only full subcategory. Moreover,* $\mathrm{VectBdl}_X(\mathscr{V})$ *is at the same time a subcategory of* $\mathrm{VectBdl}(\mathscr{V})$ *and of* $\mathrm{VectBdl}_X$*. However,* $\mathrm{VectBdl}_X(\mathscr{V})$ *is only full as a subcategory of* $\mathrm{VectBdl}_X$*.* ♢

Definition C.8 (Sets of equivalence classes of vector bundles)**.** *We say that:*

- Vect *is the quotient of the class of objects of* $\mathrm{VectBdl}$ *by its equivalence relation of isomorphism of vector bundles. In other words,* Vect *is the* ***set of isomorphism classes of vector bundles****; and*
- Vect_X *is the quotient of the class of objects of* $\mathrm{VectBdl}_X$ *by its equivalence relation of isomorphism of vector bundles on X. In other words,* Vect_X *is the* ***set of isomorphism classes of vector bundles on X****.*

The sets of isomorphism classes of vector bundles $\mathrm{Vect}(\mathscr{V})$ *and* $\mathrm{Vect}_X(\mathscr{V})$ *are defined in a similar manner.* ♢

C.3 Trivial bundles and restrictions

Definition C.9 (Product and trivial vector bundles)**.** *Let X be a connected topological space. We say that:*

- *the* ***product vector bundle with typical fiber*** $\mathscr{V}$ *is the projection onto the first factor $\pi : X \times \mathscr{V} \to X$ with the natural vector space structure induced by $\mathscr{V}$ on each fiber; and*
- *a vector bundle $\pi : E \to X$ with typical fiber $\mathscr{V}$ is* ***trivial*** *if it is isomorphic over X to the product bundle $X \times \mathscr{V}$. In this situation, an isomorphism from E onto the product bundle is called a* ***trivialization*** *of E.* ♢

Definition C.10 (Restriction of a vector bundle). *Let $\pi : E \to X$ be a vector bundle. Given a topological subspace $Y \subseteq X$, the **restriction** of E to Y, which is hereafter denoted by $E\,|_Y$, is the vector bundle $\pi\,|_{\pi^{-1}(Y)}\colon \pi^{-1}(Y) \to Y$ with the induced vector space structure on each fiber on Y.* $\diamondsuit$

Remark C.11 (On the restriction of vector bundles). *Let $\pi : E \to X$ be a vector bundle and Y be a topological subspace of X. Then:*

- *the restriction $E\,|_Y$ is a vector bundle because we can verify Conditions (1) and (2) of Definition C.1 by restricting a local chart (U, φ_U) of E to the local chart $(U \cap Y,\ \varphi_U\,|_{\pi^{-1}(U\cap Y)})$ of $E\,|_Y$; and*
- *if (U, φ_U) is a local chart of E, then $\varphi_U : \pi^{-1}(U) \to U \times \mathscr{V}$ is a vector bundle isomorphism over U between $E\,|_U$ and the product bundle. Therefore, a vector bundle is **locally trivial** by definition.* $\diamondsuit$

Remark C.12 (Covariant functor defined by the restriction of vector bundles). *Let X be a topological space and Y be a subspace of X. Then, we have the following covariant functor*

$$\begin{aligned} |_Y\colon \mathrm{VectBdl}_X &\to \mathrm{VectBdl}_Y, \\ E &\mapsto E\,|_Y, \\ f : E \to F &\mapsto f\,|_Y\colon E\,|_Y \to F\,|_Y, \end{aligned}$$

where $f\,|_Y$ is the natural map that sends $a \in E\,|_Y$ into $f(a) \in F\,|_Y$. In fact, since $f : E \to Y$ is a vector bundle morphism over X, we know that $\pi_F \circ f = \pi_E$. Therefore, we have

$$f(\pi_E^{-1}(Y)) \subseteq \pi_F^{-1}(Y).$$

Once $E\,|_Y = \pi_E^{-1}(Y)$ and $F\,|_Y = \pi_F^{-1}(Y)$, it follows that $f\,|_Y$ is well-defined. In turn, the fact

$$\pi_F\,|_{\pi_F^{-1}(Y)} \circ f\,|_Y = \pi_E\,|_{\pi_E^{-1}(Y)},$$

which proves that $f\,|_Y$ is a vector bundle morphism over Y since it is obviously continuous and linear in each fiber, is immediate from the fact that f is a vector bundle morphism over X. $\diamondsuit$

Definition C.13 (Common trivializing open cover for a family of vector bundles on the same base space). *Let X be a topological space and $\Pi = \{\pi_\alpha : E_\alpha \to X\}_{\alpha\in\Lambda}$ be a family of vector bundles on X. A **common trivializing open cover** of X for Π is an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X in such manner that $E_\alpha |_{U_i}$ is trivial for all $\alpha \in \Lambda$ and all $i \in I$.* ♢

Remark C.14 (Existence of common trivializing open covers for finite families of vector bundles). *Let $\pi_E : E \to X$ and $\pi_F : F \to X$ be vector bundles. There exists a common trivializing open cover of X for E and F. Indeed, there exist open covers:*

- *$\mathfrak{U} = \{U_i\}_{i\in I}$ of X such that $E |_{U_i}$ is trivial for all $i \in I$; and*
- *$\mathfrak{V} = \{V_j\}_{j\in J}$ of X such that $F |_{V_j}$ is trivial for all $j \in J$.*

Therefore, for each $x \in X$, there exist $i_x \in I$ and $j_x \in J$ such that $x \in U_{i_x}$ and $x \in V_{j_x}$. We define $W_x := U_{i_x} \cap V_{j_x}$ for every $x \in X$. The reader can readily prove that $\mathfrak{W} := \{W_x\}_{x\in X}$ ensures our assertion. More than that, using induction, the reader can prove that there exists a common trivializing open cover for any finite number of vector bundles. ♢

C.4 Sections of vector bundles

Notation C.15 (On real and complex numbers). *When we do not desire to distinguish between the field of real numbers and the field of complex numbers, we shall write $\mathbb{K}$ to symbolize any of them.* ♢

Definition C.16 (Global and local sections of a vector bundle). *Let $\pi : E \to X$ be a vector bundle. We say that:*

- *a **(global) section** of E is a continuous function $s : X \to E$ in such manner that $\pi \circ s = \mathrm{id}_X$. The **set of sections** of E, which has a natural real or complex vector space structure and a natural module structure over the ring of continuous* *functions $\mathcal{C}^0(X, \mathbb{K})$, endowed with the pointwise sum and exterior product, is hereafter denoted by $\Gamma(E)$; and*

- *if $U \subseteq X$ is open, then a global section $s : U \to E\,|_U$ of the restriction $E\,|_U$ is said to be a **local section** of E. Moreover, if $x \in U$, then s is also called a **local section in x**.* ♢

Theorem C.17 (Local charts induce bijections between the set of local sections and the ring of continuous functions)**.** *Let $\pi : E \to X$ be a vector bundle. If (U, φ_U) is a local chart of E, then it induces a bijection between $\Gamma(E\,|_U)$ and $\mathcal{C}^0(U, \mathscr{V})$. Moreover, this bijection is a $\mathcal{C}^0(U, \mathbb{K})$-module isomorphism.*

Proof. Let $s : U \to E\,|_U$ be a local section of E. If $x \in U$, then, since $\pi_U \circ \varphi_U = \pi$ and $\pi \circ s = \mathrm{id}_U$, we have

$$\varphi_U(s(x)) \;=\; (x,\, v(x))$$

with $v(x) \in \mathscr{V}$. Therefore, we obtain the function $v : U \to \mathscr{V}$, which is obviously continuous once it is the composition between $\varphi_U \circ s$ and the projection onto the second factor $\pi_{\mathscr{V}} : U \times \mathscr{V} \to \mathscr{V}$. Conversely, given a continuous function $v : U \to \mathscr{V}$, we obtain the local section of E

$$\begin{aligned} s : U \;&\to\; E\,|_U, \\ x \;&\mapsto\; \varphi_U^{-1}(x,\, v(x)). \end{aligned}$$

The reader can readily prove the last claim of the theorem, which states that this bijection is a $\mathcal{C}^0(U, \mathbb{K})$-module isomorphism. □

Remark C.18 (Pushforward of sections induced by a morphism of vector bundles which covers a homeomorphism)**.** *Let $\pi_E : E \to X$ and $\pi_F : F \to Y$ be vector bundles. If $f : E \to F$ is a morphism of vector bundles covering a homeomorphism $g : X \to Y$, then, for any open subset $U \subseteq Y$, we obtain the following $\mathcal{C}^0$-module isomorphism, called **f-pushforward of sections***

$$\begin{aligned} f_* : \Gamma(E\,|_{g^{-1}(U)}) \;&\to\; \Gamma(F\,|_U), \\ s \;&\mapsto\; f \circ s \circ g^{-1}. \end{aligned}$$

In addition, if f is an isomorphism, then f_* is also an isomorphism, whose inverse is the f^{-1}-pushforward of sections, which is usually said to be the ***f-pullback of sections****. Moreover, Theorem C.17 is a particular case of this construction applied to the isomorphism $\varphi_U : E\,|_U \to U \times \mathscr{V}$ because $\Gamma(U \times \mathscr{V})$ is isomorphic to $\mathcal{C}^0(U, \mathscr{V})$. Finally, note that, if f is a morphism over X, then we obtain $f_* : \Gamma(E\,|_U) \to \Gamma(F\,|_U)$, $s \mapsto f \circ s$.* $\diamondsuit$

Remark C.19 (On sections of vector bundles)**.** *Let $\pi : E \to X$ be a vector bundle. It is to be noted that:*

- *there always exists the* ***vanishing global section*** $0 : X \to E$ *of E, which is defined by* $0(x) := 0_x$ *for all $x \in X$, where 0_x denotes the origin of the vector space E_x; and*
- *for any $x \in X$ such that* $\mathrm{rk}_x(E)$ *is positive, there exists a local section of E in x which does not vanish in any point. Indeed, it suffices to fix a local chart (U, φ_U) of E in x and a non-zero vector $v \in \mathscr{V}$ in order to define $s(y) := \varphi^{-1}(y, v)$ for every $y \in U$. Note that this is equivalent to consider the section s induced by the constant function v as in Theorem C.17.* $\diamondsuit$

The following result gives a characterization of trivial vector bundles through their global sections. Indeed, it says that a vector bundle is trivial if and only if there exist its rank of pointwise independent global sections. In particular, this result complements Remark C.19, proving that there exist vector bundles such that all of their global sections are zero somewhere. In fact, as examples, it suffices to consider non-trivial vector bundles of rank one.

Theorem C.20 (Equivalence between triviality of a vector bundle and the existence of pointwise independent global sections)**.** *Let $\pi : E \to X$ be a vector bundle. Then, E is trivial if and only if there exist* $\mathrm{rk}(E)$ *pointwise independent global sections $s_1, \cdots, s_{\mathrm{rk}(E)} \in \Gamma(E)$. Moreover, a basis of $\mathscr{V}$ induces a bijection between the set of trivializations of E and the set of families $\{s_1, \cdots, s_{\mathrm{rk}(E)}\}$ of* $\mathrm{rk}(E)$ *pointwise independent global sections of E.*

Proof. Let $\{a_1, \cdots, a_{\mathrm{rk}(E)}\}$ be a basis of $\mathscr{V}$. If E is a trivial vector bundle, then let $\varphi_X : E \to X \times \mathscr{V}$ be one of its trivializations. For each n between 1 and $\mathrm{rk}(E)$, both included, it suffices to set

$$\begin{aligned} s_n : X &\to E, \\ x &\mapsto \varphi_X^{-1}(x, a_n), \end{aligned}$$

to obtain a family of $\mathrm{rk}(E)$ pointwise independent global sections of E. Conversely, given a family $s_1, \cdots, s_{\mathrm{rk}(E)} \in \Gamma(E)$ of $\mathrm{rk}(E)$ pointwise independent global sections of E, we obtain the trivialization

$$\begin{aligned} \varphi_X : E &\to X \times \mathscr{V}, \\ \sum_{n=1}^{\mathrm{rk}(E)} \lambda_n s_n(x) &\mapsto \left(x, \sum_{n=1}^{\mathrm{rk}(E)} \lambda_n a_n \right). \end{aligned}$$

The reader can readily prove that these assignments are inverse to each other, completing the last claim of the theorem. □

Remark C.21 (Another interpretation of local triviality of vector bundles)**.** *We have seen in Remark C.11 that, given a generic vector bundle E, a choice of a local chart (U, φ_U) is equivalent to a choice of a trivialization of $E\mid_U$. Hence, because of Theorem C.20, it is equivalent to a choice of* $\mathrm{rk}(E)$ *pointwise independent local sections $s_1, \cdots, s_{\mathrm{rk}(E)} : U \to E\mid_U$, which are obviously pointwise independent global sections of the restriction $E\mid_U$.* ♢

C.5 Subbundles of vector bundles

Definition C.22 (Subbundle of a vector bundle)**.** *Let $\pi : E \to X$ be a vector bundle. We say that a* ***vector subbundle*** *F of E is a vector bundle of the form $\pi\mid_F : F \to X$ where $F \subseteq E$ is a topological subspace, and $F_x \subseteq E_x$ is a vector subspace for every $x \in X$.* ♢

Remark C.23 (On subbundles of vector bundles)**.** *Let $\pi : E \to X$ be a vector bundle. Note that:*

- *we put no constraints on the typical fiber of a subbundle F of E, but it is necessarily a vector subspace of the typical fiber of E up to isomorphism; and*

- *if F is a subbundle of E, then the inclusion map $i : F \to E$ is an injective vector bundle morphism over X. Indeed, it is continuous, since it is an embedding of a topological subspace, and it is linear in each fiber, since it is an inclusion of a vector subspace.*

We also observe that, when we restrict a vector bundle to Y, we are only considering the fibers over the points of Y, but we take the whole fiber in each point. On the other hand, considering a subbundle of a vector bundle, we restrict each fiber to a vector subspace, but in the whole X. Evidently, we can apply both operations at the same time, considering the restriction of a subbundle. ♢

The next result of this section enlightens subbundles of vector bundles. Indeed, it shows a correspondence between subbundles and pointwise independent local sections of the main vector bundle.

Theorem C.24 (Subbundles and local sections of a vector bundle). *Let $\pi : E \to X$ be a vector bundle. If $F \subseteq E$ is such that $F_x \subseteq E_x$ is a vector subspace for every $x \in X$, then $\pi\ |_F \colon F \to X$, where F is endowed with the induced topology and each F_x is endowed with the induced vector space structure, is a vector subbundle of E if and only if, for every $x \in X$, there exist an open neighborhood U of x in X and pointwise independent local sections $s_1, \cdots, s_n \in \Gamma(E\ |_U)$, where n depends on x, such that $\{s_1(y), \cdots, s_n(y)\}$ is a basis for F_y for every $y \in U$. In particular, this implies that the dimension of the vector space F_x is locally constant in x.*

Proof. ($\Rightarrow$). Since F is a vector bundle, there exists a local chart of F in every $x \in X$. Fixing a basis of the typical fiber $\mathscr{V}$, such a chart is equivalent to a choice of pointwise independent local sections $s_1, \cdots, s_{\mathrm{rk}_x(F)} \in \Gamma(F\ |_U)$, which form a basis of F_y for every $y \in U$. Once F is endowed with the induced topology, $s_1, \cdots, s_{\mathrm{rk}_x(F)} : U \to E\ |_U$ are continuous. Moreover, since the projection $\pi\ |_F \colon F \to X$ is the restriction of $\pi : E \to X$, we have $\pi \circ s_i = \mathrm{id}_U$ for every i between 1 and $\mathrm{rk}_x(F)$, both included. Therefore, $s_1, \cdots, s_{\mathrm{rk}_x(F)} \in \Gamma(E\ |_U)$. ($\Leftarrow$). Since the sections $s_1, \cdots, s_n \in \Gamma(E\ |_U)$ are pointwise independent and span F_y for every $y \in U$, they define a local chart $\varphi_U : (\pi\ |_F)^{-1}(U) \to U \times \mathbb{K}^n$ for every $x \in U$. Hence, F is a vector bundle with typical fiber $\mathbb{K}^n$. □

C.6 Transition functions of vector bundles

Remark C.25 (Relation between the local charts of a vector bundle endowed with an atlas). *Let* $\pi : E \to X$ *be a vector bundle endowed with an atlas* $\Phi_{\mathfrak{U}} = \{(U_i,\, \varphi_i)\}_{i\in I}$*. Let* (U_i, φ_i) *and* (U_j, φ_j) *be any local charts of* $\Phi_{\mathfrak{U}}$ *such that* $U_{ij} := U_i \cap U_j$ *is nonempty. Then, considering* $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathscr{V}$ *and* $\varphi_j : \pi^{-1}(U_j) \to U_j \times \mathscr{V}$*, we obtain the composition*

$$\widetilde{\varphi}_{ij} \;:=\; \varphi_j\,|_{\pi^{-1}(U_{ij})} \circ \big(\varphi_i\,|_{\pi^{-1}(U_{ij})}\big)^{-1} : U_{ij} \times \mathscr{V} \to U_{ij} \times \mathscr{V}.$$

For every $x \in U_{ij}$*, we have the automorphism* $(\widetilde{\varphi}_{ij})_x \in \mathrm{GL}(\mathscr{V})$ *such that*

$$\widetilde{\varphi}_{ij}(x,\, v) \;=\; (x,\, (\widetilde{\varphi}_{ij})_x(v)).$$

Consequently, given $a \in E_x$ *such that* $\varphi_i(a) = (x, v)$*, it follows that the corresponding representation in* U_j *is* $\varphi_j(a) = (x,\, (\widetilde{\varphi}_{ij})_x(v))$*. These facts allow us to set the following definition.* $\diamondsuit$

Definition C.26 (Transition functions of a vector bundle). *Let* $\pi : E \to X$ *be a vector bundle endowed with an atlas* $\Phi_{\mathfrak{U}} = \{(U_i,\, \varphi_i)\}_{i\in I}$*. If* $U_{ij} := U_i \cap U_j$ *is nonempty, then the* ***transition function*** *of* E *from* U_i *to* U_j *is defined by*

$$\begin{aligned} \varphi_{ij} : U_{ij} &\;\to\; \mathrm{GL}(\mathscr{V}), \\ x &\;\mapsto\; (\widetilde{\varphi}_{ij})_x. \end{aligned}$$

We will denote $(\widetilde{\varphi}_{ij})_x$*, that is,* $\varphi_{ij}(x)$*, equivalently by* $(\varphi_{ij})_x$*. Moreover, it is immediate to verify that the transition functions satisfy the following condition, called the* ***cocycle condition****:*

$$\big(\varphi_{jk}\,|_{U_{ijk}}\big)_x \circ \;\big(\varphi_{ij}\,|_{U_{ijk}}\big)_x \;=\; \big(\varphi_{ik}\,|_{U_{ijk}}\big)_x$$

for all $x \in U_{ijk} := U_i \cap U_j \cap U_k$*. In particular,* $(\varphi_{ii})_x = \mathrm{id}_{\mathrm{GL}(\mathscr{V})}$ *for all* $x \in U_i$ *and* $(\varphi_{ij})_x = (\varphi_{ji})_x^{-1}$ *for all* $x \in U_{ij}$*. We will frequently omit the subindexes* x *in the preceding formulas, admitting that whenever appears a composition it is happening in the topological group* $\mathrm{GL}(\mathscr{V})$*.* $\diamondsuit$

Definition C.27 (First degree nonabelian Čech cohomology of $\underline{\mathrm{GL}}(\mathscr{V})$). *Let X be a topological space and $\mathfrak{U} = \{U_i\}_{i\in I}$ be one of its open covers. Being $\mathscr{V}$ a finite dimensional vector space, we set*

$$\check{Z}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V})) \;:=\; \Big\{\{\varphi_{ij} : U_{ij} \to \mathrm{GL}(\mathscr{V})\}_{i,j\in I} \,:\, \varphi_{jk}\mid_{U_{ijk}} \circ \varphi_{ij}\mid_{U_{ijk}} = \varphi_{ik}\mid_{U_{ijk}}\Big\}.$$

We introduce in $\check{Z}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V}))$ the relation defined as follows: two of its families $\{\varphi_{ij}\}_{i,j\in I}$ and $\{\psi_{ij}\}_{i,j\in I}$ are related if and only if there exists a family $\{\eta_i : U_i \to \mathrm{GL}(\mathscr{V})\}_{i\in I}$ such that

$$(\psi_{ij})_x \;=\; (\eta_j)_x \circ (\varphi_{ij})_x \circ (\eta_i)_x^{-1}$$

for all $x \in U_{ij}$ and all $i, j \in I$. The reader can readily prove that this is an equivalence relation on $\check{Z}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V}))$. We set $\check{H}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V}))$ as the quotient of $\check{Z}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V}))$ by this equivalence relation. $\diamondsuit$

Remark C.28 (On the first degree nonabelian Čech cohomology of $\underline{\mathrm{GL}}(\mathscr{V})$). *Let $\pi : E \to X$ be a vector bundle endowed with an atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$. Being $\{\varphi_{ij}\}_{i,j\in I}$ the set of transition functions of E, Definition C.26 ensures that the equivalence class*

$$[\{\varphi_{ij}\}_{i,j\in I}] \in \check{H}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V}))$$

is well-defined. Furthermore, the reader can readily prove that it does not depend on the homeomorphisms of $\Phi_{\mathfrak{U}}$. Therefore, the class $[\{\varphi_{ij}\}_{i,j\in I}]$ only depends on the isomorphism class of E among the vector bundles that are trivial on each element of the open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X. More than that, one can prove that an equivalence class of transition functions in $\check{H}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V}))$ determines a unique up to isomorphism vector bundle with typical fiber $\mathscr{V}$ that is trivial on each element of the open cover in question. $\diamondsuit$

Remark C.29 (Dependence on the open cover of the first degree nonabelian Čech cohomology of $\underline{\mathrm{GL}}(\mathscr{V})$). *Let X be a topological space. When an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ is fixed, we only consider vector bundles E such that $E\mid_{U_i}$ is trivial for every $i \in I$. In general, this condition does not hold. However, for every vector bundle E, there* *exists a refinement* $\mathfrak{V} = \{V_j\}_{j\in J}$ *of* $\mathfrak{U}^{(1)}$ *such that* $E\mid_{V_j}$ *is trivial for every* $j \in J^{(2)}$. *Then, let us consider the partial order* $\prec$ *on the set of open covers of* X *which says that* $\mathfrak{U} \prec \mathfrak{V}$ *if and only if* $\mathfrak{V}$ *if a refinement of* $\mathfrak{U}$. *If* $\mathfrak{U} = \{U_i\}_{i\in I}$ *and* $\mathfrak{V} = \{V_\alpha\}_{\alpha\in\Lambda}$ *are such that* $\mathfrak{U} \prec \mathfrak{V}$, *then there exists a function* $\rho : \Lambda \to I$ *such that* $V_\alpha \subseteq U_{\rho(\alpha)}$. *Therefore, we obtain the function*

$$\begin{aligned} \widehat{\rho} : \check{Z}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V})) &\to \check{Z}^1(\mathfrak{V}, \underline{\mathrm{GL}}(\mathscr{V})), \\ \{\varphi_{ij}\}_{i,j\in I} &\mapsto \{\varphi_{\rho(\alpha)\rho(\beta)} \mid_{V_{\rho(\alpha)\rho(\beta)}}\}_{\alpha,\beta\in J}. \end{aligned}$$

Furthermore, it is straightforward to verify that $\widehat{\rho}$ *projects to the function*

$$\begin{aligned} \widehat{\rho}^{\,*} : \check{H}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V})) &\to \check{H}^1(\mathfrak{V}, \underline{\mathrm{GL}}(\mathscr{V})), \\ [\{\varphi_{ij}\}_{i,j\in I}] &\mapsto [\{\varphi_{\rho(\alpha)\rho(\beta)} \mid_{V_{\rho(\alpha)\rho(\beta)}}\}_{\alpha,\beta\in J}]. \end{aligned}$$

More than that, we have that $\widehat{\rho}^{\,*}$ *does not depend on* ρ. *Indeed, if* $\rho, \nu : \Lambda \to I$ *are two functions such that* $V_\alpha \subseteq U_{\rho(\alpha)}$ *and* $V_\alpha \subseteq U_{\nu(\alpha)}$, *then* $V_\alpha \subseteq U_{\rho(\alpha)\nu(\alpha)}$ *and the reader can prove that the family* $\{\varphi_{\rho(\alpha)\nu(\alpha)} \mid_{V_\alpha}\}_{\alpha\in\Lambda}$ *realizes the equivalence between* $\widehat{\rho}(\{\varphi_{ij}\}_{i,j\in I})$ *and* $\widehat{\nu}(\{\varphi_{ij}\}_{i,j\in I})$. *Consequently, the assignment that sends each open cover* $\mathfrak{U}$ *of* X *into the set* $\check{H}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V}))$ *is a direct system of sets. Therefore, we obtain the direct limit of sets*[3]

$$\check{H}^1(X, \underline{\mathrm{GL}}(\mathscr{V})) := \varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V})).$$

The interesting fact is that one can prove that there exists a bijection between $\mathrm{Vect}_X(\mathscr{V})$ *and* $\check{H}^1(X, \underline{\mathrm{GL}}(\mathscr{V}))$. $\diamondsuit$

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(1) Let $\mathfrak{U} = \{U_i\}_{i\in I}$ and $\mathfrak{V} = \{V_j\}_{j\in J}$ be two open covers of a topological space X. We say that $\mathfrak{V}$ is a **refinement** of $\mathfrak{U}$ if there exists a function $\rho : J \to I$ such that $V_j \subseteq U_{\rho(j)}$ for all $j \in J$. In the set of open covers of X, we denote by $\mathfrak{U} \prec \mathfrak{V}$ the fact that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. The reader can readily prove that this is a partial order relation on the set of open covers of X. Moreover, in the set of atlases of a vector bundle, there exists a partial order relation as well, defined as follows. Given $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$ and $\Psi_{\mathfrak{V}} = \{(V_j, \psi_j)\}_{j\in J}$ atlases of the same vector bundle, we set $\Phi_{\mathfrak{U}} \prec \Psi_{\mathfrak{V}}$ if and only if $\mathfrak{U} \prec \mathfrak{V}$ through a function $\rho : J \to I$ such that $\psi_j = \varphi_{\rho(j)} \mid_{V_j}$ for every $j \in J$. In this situation, we also say that $\Psi_{\mathfrak{V}}$ is a **refinement** of $\Phi_{\mathfrak{U}}$.

(2) In fact, for every $x \in X$, we fix $i \in I$ such that $x \in U_i$. Then, we consider a local chart (U_x, φ_{U_x}) of E in x and set $V_x := U_x \cap U_i$. Clearly, $\mathfrak{V} := \{V_x\}_{x\in X}$ is a refinement of $\mathfrak{U}$ such that $E\mid_{V_x}$ is trivial for every $x \in X$.

(3) The reader can easily adapt Appendix A in order to define and explicitly characterize the direct limit of sets.

Remark C.30 (On the geometric interpretation of the first degree nonabelian Čech cohomology of $\underline{\mathrm{GL}}(\mathscr{V})$). *Let $\pi_E : E \to X$ be a vector bundle endowed with an atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$. Given a vector bundle $\pi_F : F \to X$ isomorphic to E, if $f : F \to E$ is an isomorphism of vector bundles over X, then $\Phi^f_{\mathfrak{U}} := \{(U_i, \varphi_i \circ f\mid_{\pi_F^{-1}(U_i)})\}_{i\in I}$ is an atlas of F inducing the same transition functions of $\Phi_{\mathfrak{U}}$. Fixing a basis $\mathcal{A}$ of $\mathscr{V}$, this equivalently means that the families of local sections $\{s_{i,k}\}_{i\in I;\, 0<k\leq \mathrm{rk}(E)}$ and $\{f^* s_{i,k}\}_{i\in I;\, 0<k\leq \mathrm{rk}(E)}$ induce the same transition functions, where f^* is the f-pullback of sections defined in Remark C.18. Conversely, if F is a vector bundle endowed with a family of local sections $\{t_{i,k}\}_{i\in I;\, 0<k\leq \mathrm{rk}(E)}$ that induces the same transition functions of E with respect to the family $\{s_{i,k}\}_{i\in I;\, 0<k\leq \mathrm{rk}(E)}$, then there exists an isomorphism $f : F \to E$ such that $t_{i,k} = f^* s_{i,k}$ for all $i \in I$ and all k between 1 and $\mathrm{rk}(E)$, both included. In fact, it suffices to set*

$$f\left(\sum_{k=1}^{\mathrm{rk}(E)} \lambda_k t_{i,k}(x)\right) := \sum_{k=1}^{\mathrm{rk}(E)} \lambda_k s_{i,k}(x).$$

Therefore, we can conclude that:

- *fixing an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X and a basis $\mathcal{A}$ of $\mathscr{V}$, a family $\{\varphi_{ij}\}_{i,j\in I} \in \check{Z}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V}))$ corresponds geometrically to a vector bundle $\pi : E \to X$ endowed with a family $\{s_{i,k}\}_{i\in I;\, 0<k\leq \mathrm{rk}(E)}$ of pointwise independent local sections, up to isomorphism respecting the local sections through pullback (or pushforward);*

- *fixing an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X, a class $[\{\varphi_{ij}\}_{i,j\in I}] \in \check{H}^1(\mathfrak{U}, \underline{\mathrm{GL}}(\mathscr{V}))$ corresponds geometrically to a vector bundle $\pi : E \to X$ such that $E\mid_{U_i}$ is trivial for every $i \in I$, up to isomorphism; and*

- *a class $[[\{\varphi_{ij}\}_{i,j\in I}]] \in \check{H}^1(X, \underline{\mathrm{GL}}(\mathscr{V}))$ corresponds geometrically to a vector bundle $\pi : E \to X$, up to isomorphism.* ♢

C.7 Operations on vector bundles

In this section, we will see that natural operations on vector spaces, specially direct sum and tensor product, can be extended to the framework of vector bundles. The only problem that we will face is the question of how one should topologize the resulting spaces. The beauty of the answer that will be presented here is the fact that it gives a general method for extending operations from vector spaces to vector bundles, handling all the situations uniformly. We begin with the following definition.

Definition C.31 (The category of n-tuples of finite-dimensional vector spaces)**.** *Let* FDVectSps *be the category of finite-dimensional (real or complex) vector spaces. For each non-zero natural number n, we define* ***the category of n-tuples of finite-dimensional vector spaces****, and denote it by* $\mathrm{FDVectSps}^n$*, to be the n-times product category of* FDVectSps*. More explicitly, we have that* $\mathrm{FDVectSps}^n$ *is the category whose:*

- *objects are ordered n-tuples $(\mathscr{V}_1, \cdots, \mathscr{V}_n)$, where $\mathscr{V}_i$ is a finite-dimensional vector space for each i between 1 and n, both included; and*
- *morphisms are sequences of linear maps $(\Phi_1 : \mathscr{V}_1 \to \mathscr{W}_1, \cdots, \Phi_n : \mathscr{V}_n \to \mathscr{W}_n)$, usually denoted by $\Phi : (\mathscr{V}_1, \cdots, \mathscr{V}_n) \to (\mathscr{W}_1, \cdots, \mathscr{W}_n)$.* $\diamondsuit$

Remark C.32 (Desired relation between operations on vector spaces and on vector bundles)**.** *We say that:*

- *an* ***operation on vector spaces*** *is a functor $T : \mathrm{FDVectSps}^n \to \mathrm{FDVectSps}$; and*
- *an* ***operation on vector bundles*** *is a functor $\Theta : \mathrm{VectBdl}_X^n \to \mathrm{VectBdl}_X$, where $\mathrm{VectBdl}_X^n$ is analogously the n-times product category of $\mathrm{VectBdl}_X$.*

The main idea of this section is to extend[(4)] *an operation on vector spaces T to an operation on vector bundles Θ_T in such manner that the action of this last one on each fiber coincides with the action of the former. For this, however, we need to require an additional property of the operation on vector spaces. Indeed, since a vector bundle is a continuous (locally trivial) family of vector spaces, it is natural to ask T to obey some continuity hypothesis. This is done in order to ensure that the action of Θ_T is well-behaved*

(4) The word "extension" is appropriate when used in this context because, if X is a one-point space, then $\mathrm{VectBdl}_X$ is canonically isomorphic to FDVectSps. Consequently, $\mathrm{VectBdl}_X^n$ is canonically isomorphic to $\mathrm{FDVectSps}^n$.

when we go from a fiber to the ones near it. This idea is formalized by the following definition. ♢

Definition C.33 (Continuous operation on vector spaces)**.** *We say that a functor* T : FDVectSps^n $\to$ FDVectSps *is a* ***continuous operation on vector spaces*** *if its action on morphisms*

$$\begin{aligned} T_{\mathscr{V},\mathscr{W}} : \mathrm{Hom}_{\mathrm{FDVectSps}^n}(\mathscr{V},\mathscr{W}) &\to \mathrm{Hom}_{\mathrm{FDVectSps}}(T(\mathscr{V}),T(\mathscr{W})), \\ \Phi : \mathscr{V} \to \mathscr{W} &\mapsto T_{\mathscr{V},\mathscr{W}}(\Phi) : T(\mathscr{V}) \to T(\mathscr{W}), \end{aligned}$$

is continuous for all objects $\mathscr{V} = (\mathscr{V}_1, \cdots, \mathscr{V}_n)$ *and* $\mathscr{W} = (\mathscr{W}_1, \cdots, \mathscr{W}_n)$ *of* FDVectSps^n*. The topologies of the domain and of the codomain are the ones induced by their finite-dimensional vector space structures. We will often omit the subindexes above and admit that* T *is covariant in all of its factors. The reader can deal with the cases where* T *is a contravariant or a mixed functor.* ♢

Theorem C.34 (Operation on vector bundles induced by a continuous operation on vector spaces)**.** *Let* T : FDVectSps^n $\to$ FDVectSps *be a continuous operation on vector spaces. For each topological space* X*, there exists an induced operation on vector bundles*

$$\Theta_T : \mathrm{VectBdl}_X^n \to \mathrm{VectBdl}_X,$$

whose action on each fiber coincides with the action of the initial operation on vector spaces.

Proof. We define $\Theta_T : \mathrm{VectBdl}_X^n \to \mathrm{VectBdl}_X$ to be the functor whose actions on objects and on morphisms are given as follows.

- *Action on objects.* Let $E_1, \cdots, E_n$ be vector bundles on X. Then, as a set, the total space of the vector bundle $\Theta_T(E_1, \cdots, E_n)$ is given by

  $$\Theta_T(E_1, \cdots, E_n) := \bigsqcup_{x \in X} T((E_1)_x, \cdots, (E_n)_x).$$

  This precisely means that the action of Θ_T on each fiber coincides with the one of T. The projection $\Theta_T(E_1, \cdots, E_n) \to X$ is the obvious one given by the disjoint union structure of $\Theta_T(E_1, \cdots, E_n)$. However, we still have to define the topology on this total space. This will be done after we define the action of Θ_T on morphisms.

- *Action on morphisms.* Let $\Phi_1 : E_1 \to F_1, \cdots, \Phi_n : E_n \to F_n$ be morphisms of vector bundles over X. We define

$$\Theta_T(\Phi_1, \cdots, \Phi_n) : \ \Theta_T(E_1, \cdots, E_n) \ \to \ \Theta_T(F_1, \cdots, F_n)$$

on each fiber as $T((\Phi_1)_x, \cdots, (\Phi_n)_x) : T((E_1)_x, \cdots, (E_n)_x) \to T((F_1)_x, \cdots, (F_n)_x)$. We still have to prove the continuity of $\Theta_T(\Phi_1, \cdots, \Phi_n)$ with respect to the topology of its domain and codomain. This is done now, together with the definition of these topologies.

Indeed, the topology on $\Theta_T(E_1, \cdots, E_n)$ and the continuity of $\Theta_T(\Phi_1, \cdots, \Phi_n)$ are handled in the following three steps.

- Let $E_1, \cdots, E_n$ be product vector bundles on X. That is, $E_i = X \times \mathscr{V}_i$ for every i between 1 and n, both included. In this case, there exists a canonical bijection between

$$\Theta_T(X \times \mathscr{V}_1, \cdots, X \times \mathscr{V}_n) \qquad \text{and} \qquad X \times T(\mathscr{V}_1, \cdots, \mathscr{V}_n).$$

Therefore, we define the topology on the set $\Theta_T(X \times \mathscr{V}_1, \cdots, X \times \mathscr{V}_n)$ to be the one induced by this canonical bijection from the product topology of $X \times T(\mathscr{V}_1, \cdots, \mathscr{V}_n)$. Moreover, let $\Phi_i : X \times \mathscr{V}_i \to X \times \mathscr{W}_i$ be a morphism of vector bundles over X for each i between 1 and n, both included. Equivalently, $\Phi_i : X \to \mathrm{Hom}_{\mathrm{FDVectSps}}(\mathscr{V}_i, \mathscr{W}_i)$ for each i between 1 and n, both included. Thus, we define the natural map

$$\Phi : \ X \ \to \ \mathrm{Hom}_{\mathrm{FDVectSps}^n}((\mathscr{V}_1, \cdots, \mathscr{V}_n), (\mathscr{W}_1, \cdots, \mathscr{W}_n)).$$

Since T is a continuous operation on vector bundles, we have that the composition $T \circ \Phi : X \to \mathrm{Hom}_{\mathrm{FDVectSps}}(T(\mathscr{V}_1, \cdots, \mathscr{V}_n), T(\mathscr{V}_1, \cdots, \mathscr{V}_n))$ is continuous. Hence, $\Theta_T(\Phi_1, \cdots, \Phi_n) : X \times T(\mathscr{V}_1, \cdots, \mathscr{V}_n) \to X \times T(\mathscr{W}_1, \cdots, \mathscr{W}_n)$

is also continuous. Furthermore, if $\Phi_1, \cdots, \Phi_n$ are isomorphisms of vector bundles over X, then necessarily $\Theta_T(\Phi_1, \cdots, \Phi_n)$ is an isomorphism of vector bundles over X.

- Now, let $E_1, \cdots, E_n$ be trivial vector bundles on X. Then, we can choose a global trivialization $\alpha_i : E_i \to X \times \mathscr{V}_i$ for each i between 1 and n, both included. Consequently, we have

$$\Theta_T(\alpha_1, \cdots, \alpha_n) : \ \Theta_T(E_1, \cdots, E_n) \ \to \ \Theta_T(X \times \mathscr{V}_1, \cdots, X \times \mathscr{V}_n),$$

which is a bijection because it is bijective on each fiber. Once the topology on $\Theta_T(X \times \mathscr{V}_1, \cdots, X \times \mathscr{V}_n)$ is known from the preceding item, we give to $\Theta_T(E_1, \cdots, E_n)$ the least topology that turns $\Theta_T(\alpha_1, \cdots, \alpha_n)$ into a homeomorphism[(5)]. Moreover, let $\Phi_i : E_i \to F_i$ be a morphism of trivial vector bundles over X for each i between 1 and n, both included. If $\alpha_i : E_i \to X \times \mathscr{V}_i$ and $\beta_i : F_i \to X \times \mathscr{W}_i$ are, respectively, global trivializations of E_i and F_i for each i between 1 and n, both included, then

$$\Theta_T(\beta_1, \cdots, \beta_n) \circ \Theta_T(\Phi_1, \cdots, \Phi_n) \circ \Theta_T(\alpha_1^{-1}, \cdots, \alpha_n^{-1}),$$

which sends $\Theta_T(X \times \mathscr{V}_1, \cdots, X \times \mathscr{V}_n)$ into $\Theta_T(X \times \mathscr{W}_1, \cdots, X \times \mathscr{W}_n)$, is continuous because of the preceding item. Therefore, $\Theta_T(\Phi_1, \cdots, \Phi_n)$ is also continuous. In addition, it follows from this construction that, if $Y \subseteq X$, then the induced topology on $\Theta_T(E_1, \cdots, E_n) \mid_Y$ coincides with the topology on $\Theta_T(E_1 \mid_Y, \cdots, E_n \mid_Y)$, as expected.

- Finally, let $E_1, \cdots, E_n$ be generic vector bundles on X. There exists a common trivializing open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X for this finite family of vector bundles because of Remark C.14. We define the topology on $\Theta_T(E_1, \cdots, E_n)$ as follows. We declare a subset $V \subseteq \Theta_T(E_1, \cdots, E_n)$ as open if and only if its

(5) The choice of the trivializations plays no role in this construction. Indeed, let $\beta_i : E_i \to X \times \mathscr{V}_i$ be another global trivialization of E_i for each i between 1 and n, both included. Then, it follows from the preceding item that $\Theta(\beta_1^{-1} \circ \alpha_1, \cdots, \beta_n^{-1} \circ \alpha_n) : \Theta_T(E_1, \cdots, E_n) \to \Theta_T(E_1, \cdots, E_n)$ is an isomorphism of vector bundles over X. Thus, we have that $\Theta(\beta_1^{-1} \circ \alpha_1, \cdots, \beta_n^{-1} \circ \alpha_n)$ is a homeomorphism between the topologies of $\Theta_T(E_1, \cdots, E_n)$ induced by the families of trivializations $\{\alpha_i\}$ and $\{\beta_i\}$.

intersection with $\Theta_T(E_1 \mid_{U_i}, \cdots, E_n \mid_{U_i})$ is open for all $i \in I^{(6)}$. Moreover, let $\Phi_i : E_i \to F_i$ be a morphism of vector bundles over X for each i between 1 and n, both included. We have that

$$\Theta_T(\Phi_1, \cdots, \Phi_n) : \Theta_T(E_1, \cdots, E_n) \to \Theta_T(F_1, \cdots, F_n)$$

is continuous on each element of the open cover $\mathfrak{U}$ because of the preceding item. Thus, it is continuous. In addition, the reader can readily prove that, if $Y \subseteq X$, then the induced topology on $\Theta_T(E_1, \cdots, E_n) \mid_Y$ coincides with the topology on $\Theta_T(E_1 \mid_Y, \cdots, E_n \mid_Y)$, as expected.

This finishes the proof of the theorem. □

Remark C.35 (A characterization of the operations on vector bundles through transition functions)**.** *For each topological space X and each continuous operation on vector spaces T :* FDVectSps$^n \to$ FDVectSps*, we have that Theorem C.34 produces an operation on vector bundles Θ_T :* VectBdl$^n \to$ VectBdl*. Furthermore, being $\mathscr{V}_1, \cdots, \mathscr{V}_n$ finite-dimensional vector spaces, we have the function on isomorphism classes of vector bundles*

$$[\Theta_T] : \prod_{i=1}^{n} \mathrm{Vect}_X(\mathscr{V}_i) \to \mathrm{Vect}_X(T(\mathscr{V}_1, \cdots, \mathscr{V}_n)).$$

Therefore, because of the conclusion of Remark C.29, we have the characterization of this map by means of transition functions

$$[\Theta_T]^* : \prod_{i=1}^{n} \check{H}^1(X, \underline{\mathrm{GL}}(\mathscr{V}_i)) \to \check{H}^1(X, \underline{\mathrm{GL}}(T(\mathscr{V}_1, \cdots, \mathscr{V}_n))).$$

In particular, let $E_1, \cdots, E_n$ be vector bundles on X whose typical fibers are $\mathscr{V}_1, \cdots, \mathscr{V}_n$, respectively. There exists a common trivializing open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X for this

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(6) We have that:

- this topology does not depend on the chosen common trivializing open cover of X. Indeed, if $\mathfrak{U}'$ is another one, then $\mathfrak{U}$ and $\mathfrak{U}'$ define the same topology as any common refinement $\mathfrak{U}''$ of them; *and*
- in general, given a vector bundle $E \to X$ and an open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X, $V \subseteq E$ is open if and only if its intersection with $E \mid_{U_i}$ is open for each $i \in I$. This happens because $\{E \mid_{U_i}\}_{i \in I}$ is an open cover of E. Therefore, if T is the identity on FDVectSps, then Θ_T is the identity on VectBdl$_X$, as expected.

finite family of vector bundles because of Remark C.14. Thus, let $\{\varphi_{ij}^k\}_{i,j\in I}$ *be a set of transition functions that represents* E_k *with respect to* $\mathfrak{U}$ *for each* k *between* 1 *and* n*, both included. The reader can prove that, if* $\rho : T(\mathscr{V}, \cdots, \mathscr{V}_n) \to \mathscr{W}$ *is a finite-dimensional vector space isomorphism, then the transition functions* $\{\varphi_{ij}\}_{i,j\in I}$ *of* $\Theta_T(E_1, \cdots, E_n)$ *are given by*

$$\varphi_{ij}(x) \;=\; \rho \circ T(\varphi_{ij}^1(x), \cdots, \varphi_{ij}^n(x)) \circ \rho^{-1}.$$

Note that, when T *is a contravariant or a mixed functor, we have to change* $\varphi_{ij}^k(x)$ *by its inverse in the preceding equality. However, this change must be done only in the contravariant components of its domain.* ♢

C.7.1 Direct sum of vector bundles

Definition C.36 (Direct sum of vector bundles)**.** *The following functor is the* ***direct sum functor****:*

$$\begin{aligned} \oplus : \mathrm{FDVectSps}^2 &\;\to\; \mathrm{FDVectSps}, \\ (\mathscr{V}_1, \mathscr{V}_2) &\;\mapsto\; \mathscr{V}_1 \oplus \mathscr{V}_2, \\ (\Phi_1 : \mathscr{V}_1 \to \mathscr{W}_1,\, \Phi_2 : \mathscr{V}_2 \to \mathscr{W}_2) &\;\mapsto\; \Phi_1 \oplus \Phi_2 :\, \mathscr{V}_1 \oplus \mathscr{V}_2 \;\to\; \mathscr{W}_1 \oplus \mathscr{W}_2, \end{aligned}$$

where

$$(\Phi_1 \oplus \Phi_2)(v_1, v_2) \;:=\; (\Phi_1(v_1), \Phi_2(v_2))$$

for all $(v_1 v_2) \in \mathscr{V}_1 \oplus \mathscr{V}_2$*. This is a continuous operation on vector spaces. For each topological space* X*, we say that the corresponding vector bundle operation* $\Theta_\oplus : \mathrm{VectBdl}_X^2 \to \mathrm{VectBdl}_X$ *induced by Theorem C.34 is the* ***direct sum of vector bundles*** *on* X*. For convenience, if* E *and* F *are vector bundles on* X*, then we will write* $E \oplus F$ *instead of* $\Theta_\oplus(E, F)$*.* ♢

Remark C.37 (On the direct sum of vector bundles)**.** *Let* X *be a topological space. We have the following facts about the direct sum of vector bundles on* X*.*

- *Let* E*,* F *and* G *be vector bundles on* X*. Then,* $(E \oplus F) \oplus G$ *is isomorphic to* $E \oplus (F \oplus G)$ *over* X*. Consequently, being* $E_1, \cdots, E_n$ *vector bundles on* X*, it is defined the direct sum*

$$\bigoplus_{i=1}^{n} E_i \;=\; E_1 \oplus \cdots \oplus E_n$$

up to isomorphism.

- *Let* $\rho : \mathbb{K}^n \oplus \mathbb{K}^m \to \mathbb{K}^{n+m}$ *be the canonical isomorphism. For any linear maps* $A : \mathbb{K}^n \to \mathbb{K}^n$ *and* $B : \mathbb{K}^m \to \mathbb{K}^m$*, represented by the matrices* A *and* B *with respect to the canonical bases of* $\mathbb{K}^n$ *and* $\mathbb{K}^m$*, respectively, we have that the natural linear map* $A \oplus B : \mathbb{K}^{n+m} \to \mathbb{K}^{n+m}$ *is represented in the canonical basis of* $\mathbb{K}^{n+m}$ *by the matrix*

$$A \oplus B \;=\; \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Consequently, let E *and* F *be vector bundles on* X *with typical fibers* $\mathbb{K}^{\mathrm{rk}(E)}$ *and* $\mathbb{K}^{\mathrm{rk}(F)}$*, respectively*(7)*. Moreover, let* $\{\varphi_{ij}\}_{i,j\in I}$ *and* $\{\psi_{ij}\}_{i,j\in I}$ *be representing sets of transition functions for* E *and* F*, respectively. Then, the corresponding transition functions of* $E \oplus F$ *are given by* $\{\varphi_{ij} \oplus \psi_{ij}\}_{i,j\in I}$. $\diamondsuit$

Theorem C.38 (Direct sum up to isomorphism)**.** *Let* X *be a topological space. Then, the direct sum of vector bundles on* X *induces the commutative and associative binary operation in* Vect_X

$$\begin{aligned} \oplus : \mathrm{Vect}_X \times \mathrm{Vect}_X \;&\to\; \mathrm{Vect}_X, \\ ([E],\, [F]) \;&\mapsto\; [E \oplus F]. \end{aligned}$$

In other words, we have that this induced binary operation turns Vect_X *into an abelian semigroup.*

Proof. The reader can readily prove this result. More than that, one can prove that this induced operation turns Vect_X into an abelian monoid. This happens because the isomorphism class of the product vector bundle with trivial typical fiber is its identity element. Nevertheless, $Vect_X$ is not a group with this operation because of its lack of

(7) This is no restriction on the vector bundles. Indeed, if $\pi : E \to X$ is a vector bundle with typical fiber $\mathscr{V}$, then fix an isomorphism $\eta : \mathscr{V} \to \mathbb{K}^{\mathrm{rk}(E)}$. For each local chart $\varphi_U : \pi^{-1}(U) \to U \times \mathscr{V}$, we obtain a local chart $(\mathrm{id}_U \times \eta) \circ \varphi_U : \pi^{-1}(U) \to U \times \mathbb{K}^{\mathrm{rk}(E)}$. Thus, E is also a vector bundle on X with typical fiber $\mathbb{K}^{\mathrm{rk}(E)}$. Once and again, here, we tacitly assumed X to be connected. The reader can adapt this construction for the general case.

inverses. Indeed, once the direct sum does not decrease the rank of the vector bundles, any isomorphism class with positive rank cannot have an inverse in Vect_X with respect to this operation. $\square$

C.7.2 Dual vector bundle

Definition C.39 (Dual of a vector bundle)**.** *The following contravariant functor is the* ***dual functor****:*

$$\begin{aligned} * : \mathrm{FDVectSps} &\to \mathrm{FDVectSps}, \\ \mathscr{V} &\mapsto \mathscr{V}^*, \\ \Phi : \mathscr{V} \to \mathscr{W} &\mapsto \Phi^T : \mathscr{W}^* \to \mathscr{V}^*, \end{aligned}$$

where

$$\Phi^T(\varphi) := \varphi \circ \Phi$$

for all $\varphi \in \mathscr{W}^*$. This is a continuous operation on vector spaces. Therefore, for each topological space X, we say that the corresponding vector bundle operation $\Theta_* : \mathrm{VectBdl}_X \to \mathrm{VectBdl}_X$ induced by Theorem C.34 is the ***dual of vector bundles*** *on X. For convenience, if E is a vector bundle on X, then we will write E^* instead of $\Theta_*(E)$.* $\diamondsuit$

Remark C.40 (On the dual of a vector bundle)**.** *Let X be a topological space. We have the following facts about the dual of a vector bundle on X.*

- *Since $\mathscr{V}$ is canonically isomorphic to $(\mathscr{V}^*)^*$ for all $\mathscr{V} \in \mathrm{FDVectSps}$, it follows that, if E is a vector bundle on X, then E is isomorphic to $(E^*)^*$ over X.*

- *Let $\rho : (\mathbb{K}^n)^* \to \mathbb{K}^n$ be the canonical isomorphism. For any linear map $A : \mathbb{K}^n \to \mathbb{K}^n$, represented by the matrix A with respect to the canonical basis of $\mathbb{K}^n$, we have that $A^* : (\mathbb{K}^n)^* \to (\mathbb{K}^n)^*$ is represented by the matrix A^T with respect to the canonical basis of $(\mathbb{K}^n)^*$. Consequently, let E be a vector bundle with typical fiber $\mathbb{K}^{\mathrm{rk}(E)}$. Moreover, let $\{\varphi_{ij}\}_{i,j\in I}$ be a representing set of transition functions for E. Then, the corresponding transition functions of E^* are given by the set $\{(\varphi_{ij}^{-1})^T\}_{i,j\in I}$.* $\diamondsuit$

C.7.3 Tensor product of vector bundles

Definition C.41 (Tensor product of vector bundles). *The following functor is the* ***tensor product functor****:*

$$\begin{aligned} \otimes : \mathrm{FDVectSps}^2 &\to \mathrm{FDVectSps}, \\ (\mathscr{V}_1, \mathscr{V}_2) &\mapsto \mathscr{V}_1 \otimes \mathscr{V}_2, \\ (\Phi_1 : \mathscr{V}_1 \to \mathscr{W}_1,\, \Phi_2 : \mathscr{V}_2 \to \mathscr{W}_2) &\mapsto \Phi_1 \otimes \Phi_2 :\ \mathscr{V}_1 \otimes \mathscr{V}_2 \ \to\ \mathscr{W}_1 \otimes \mathscr{W}_2, \end{aligned}$$

where

$$(\Phi_1 \otimes \Phi_2)(v_1 \otimes v_2) \ := \ \Phi_1(v_1) \otimes \Phi_2(v_2)$$

for all $v_1 \otimes v_2 \in \mathscr{V}_1 \otimes \mathscr{V}_2$, being tacitly linearly extended to the whole vector space $\mathscr{V}_1 \otimes \mathscr{V}_2$. This is a continuous operation on vector spaces. For each topological space X, we say that the corresponding vector bundle operation $\Theta_\otimes : \mathrm{VectBdl}_X^2 \to \mathrm{VectBdl}_X$ induced by Theorem C.34 is the ***tensor product of vector bundles*** *on X. For convenience, if E and F are vector bundles on X, then we will write $E \otimes F$ instead of $\Theta_\otimes(E, F)$.* ♢

Remark C.42 (On the tensor product of vector bundles). *Let X be a topological space. We have the following facts about the tensor product of vector bundles on X.*

- *Let E, F and G be vector bundles on X. Then, $(E \otimes F) \otimes G$ is isomorphic to $E \otimes (F \otimes G)$ over X. Moreover, $E \otimes (F \oplus G)$ is isomorphic to $(E \otimes F) \oplus (E \otimes G)$ over X. Consequently, being $E, E_1, \cdots, E_n$ vector bundles on X, it is defined the tensor product*

  $$\bigotimes_{i=1}^{n} E_i \ = \ E_1 \otimes \cdots \otimes E_n$$

  up to isomorphism, and the vector bundles

  $$E \otimes \left(\bigoplus_{i=1}^{n} E_i \right) \qquad \textit{and} \qquad \bigoplus_{i=1}^{n} (E \otimes E_i)$$

  are isomorphic over X. In other words, we have that the tensor product of vector bundles is distributive with respect to the direct sum of vector bundles up to isomorphism.

- *Let $\rho : \mathbb{K}^n \otimes \mathbb{K}^m \to \mathbb{K}^{nm}$ be the canonical isomorphism. For any linear maps $A : \mathbb{K}^n \to \mathbb{K}^n$ and $B : \mathbb{K}^m \to \mathbb{K}^m$, represented by the matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ with respect to the canonical bases of $\mathbb{K}^n$ and $\mathbb{K}^m$, respectively, we have that the natural linear map $A \otimes B : \mathbb{K}^{nm} \to \mathbb{K}^{nm}$ is represented in the canonical basis of $\mathbb{K}^{nm}$ by the matrix $A \otimes B$ whose element in line ij and column lk is given by the product $a_{il}b_{jk}$. More directly, we have that $A \otimes B$ is the matrix obtained multiplying each element of A by the whole matrix B. Consequently, let E and F be vector bundles on X with typical fibers $\mathbb{K}^{\mathrm{rk}(E)}$ and $\mathbb{K}^{\mathrm{rk}(F)}$, respectively. Moreover, let $\{\varphi_{ij}\}_{i,j\in I}$ and $\{\psi_{ij}\}_{i,j\in I}$ be representing sets of transition functions for E and F, respectively. Then, the corresponding transition functions of $E \otimes F$ are given by the set $\{\varphi_{ij} \otimes \psi_{ij}\}_{i,j\in I}$.* ♢

Theorem C.43 (Tensor product up to isomorphism)**.** *Let X be a topological space. Then, the tensor product of vector bundles on X induces the commutative and associative binary operation in* Vect_X

$$\begin{aligned} \otimes : \mathrm{Vect}_X \times \mathrm{Vect}_X &\to \mathrm{Vect}_X, \\ ([E], [F]) &\mapsto [E \otimes F]. \end{aligned}$$

In other words, we have that this induced binary operation turns Vect_X *into an abelian semigroup.*

Proof. The reader can readily prove this result. More than that, one can prove that this induced operation turns Vect_X into an abelian monoid. This happens because the isomorphism class of the product vector bundle with one-dimensional typical fiber is its identity element. Nevertheless, Vect_X is not a group with this operation because of its lack of inverses. □

Definition C.44 (Set of isomorphism classes of line bundles)**.** *Let X be a topological space. We say that* Pic_X *is the* ***set of isomorphism classes of line bundles*** *on X. In other words,* Pic_X *is the subset of* Vect_X *composed by the isomorphism classes of rank-one vector bundles.* ♢

Corollary C.45 (The Picard group)**.** *Let X be a topological space. Then, the tensor product of vector bundles on X induces the commutative and associative binary operation in* Pic_X

$$\begin{aligned} \otimes : \mathrm{Pic}_X \times \mathrm{Pic}_X &\to \mathrm{Pic}_X, \\ ([L], [M]) &\mapsto [L \otimes M]. \end{aligned}$$

Furthermore, we have that this induced binary operation turns Pic_X *into an abelian group. This group is the* ***Picard group*** *of line bundles on X.*

Proof. The first part of the statement is an immediate consequence of Theorem C.43. On the other hand, the reader can prove that any $[L] \in \mathrm{Pic}_X$ has as its inverse $[L^*] \in \mathrm{Pic}_X$. This follows from the fact that, if E is a vector bundle on X, then $E^* \otimes E$ is isomorphic to $\mathrm{End}(E)$, as will be shown in the following subsection. An interesting fact is that there exists an isomorphism $c_1 : \mathrm{Pic}_X \to H^2(X, \mathbb{Z})$ which is called the **first Chern class**. This is a complete invariant in the sense that any isomorphism class $[L] \in \mathrm{Pic}_X$ is completely determined by $c_1[L] \in H^2(X, \mathbb{Z})$. □

C.7.4 Hom and End of vector bundles

Definition C.46 (Hom and End of vector bundles)**.** *The following functor, which is contravariant in the first variable and covariant in the second one, is the* ***morphism functor****:*

$$\begin{aligned} \mathrm{Hom} : \mathrm{FDVectSps}^2 &\to \mathrm{FDVectSps}, \\ (\mathscr{V}_1, \mathscr{V}_2) &\mapsto \mathrm{Hom}(\mathscr{V}_1, \mathscr{V}_2), \\ (\Phi_1 : \mathscr{V}_1 \to \mathscr{W}_1,\, \Phi_2 : \mathscr{V}_2 \to \mathscr{W}_2) &\mapsto \mathrm{Hom}(\Phi_1, \Phi_2) : \mathrm{Hom}(\mathscr{W}_1, \mathscr{V}_2) \to \mathrm{Hom}(\mathscr{V}_1, \mathscr{W}_2), \end{aligned}$$

where

$$\mathrm{Hom}(\Phi_1, \Phi_2)(\varphi) := \Phi_2 \circ \varphi \circ \Phi_1$$

for all $\varphi \in \mathrm{Hom}(\mathscr{W}_1, \mathscr{V}_2)$. This is a continuous operation on vector spaces. For each topological space X, we say that the corresponding vector bundle operation $\Theta_{\mathrm{Hom}} : \mathrm{VectBdl}_X^2 \to \mathrm{VectBdl}_X$ *induced by Theorem C.34 is the* **Hom** ***of vector***

bundles *on X. For convenience, if E and F are vector bundles on X, then we will write* $\textsc{Hom}(E,F)$ *instead of* $\Theta_{\mathrm{Hom}}(E,F)$*. Furthermore, we consider the diagonal functor*

$$\begin{aligned}\Delta : \mathrm{VectBdl}_X &\to \mathrm{VectBdl}_X^2,\\ E &\mapsto (E,E),\\ \Phi : E \to F &\mapsto (\Phi : E \to F, \Phi : E \to F).\end{aligned}$$

We say that the composition $\Theta_{\mathrm{Hom}} \circ \Delta : \mathrm{VectBdl} \to \mathrm{VectBdl}$ *is the* $\textbf{E}\textsc{\textbf{nd}}$ ***of vector bundles*** *on X. For convenience, if E is a vector bundle on X, then we will write* $\textsc{End}(E)$ *instead of* $(\Theta_{\mathrm{Hom}} \circ \Delta)(E)$. ♢

Remark C.47 (On the $\textsc{Hom}$ and on the $\textsc{End}$ of vector bundles)**.** *Let X be a topological space. We have the following facts about the* $\textsc{Hom}$ *and the* $\textsc{End}$ *of vector bundles on X.*

- *Consider the diagonal functor*

  $$\begin{aligned}\Delta' : \mathrm{FDVectSps} &\to \mathrm{FDVectSps}^2,\\ \mathscr{V} &\mapsto (\mathscr{V},\mathscr{V}),\\ \Phi : \mathscr{V} \to \mathscr{W} &\mapsto (\Phi : \mathscr{V} \to \mathscr{W}, \Phi : \mathscr{V} \to \mathscr{W}).\end{aligned}$$

  We have that $\mathrm{Hom} \circ \Delta' : \mathrm{FDVectSps} \to \mathrm{FDVectSps}$ *is a continuous operation on vector spaces. Then, let* $\Theta_{\mathrm{Hom}\,\circ\,\Delta'} : \mathrm{VectBdl} \to \mathrm{VectBdl}$ *be its corresponding vector bundle operation induced by Theorem C.34. The reader can readily prove that, if E is a vector bundle on X, then* $\textsc{End}(E) = (\Theta_{\mathrm{Hom}} \circ \Delta)(E)$ *is isomorphic to* $\Theta_{\mathrm{Hom}\,\circ\,\Delta'}(E)$*.*

- *For all* $\mathscr{V},\mathscr{W} \in \mathrm{FDVectSps}$*, we have that* $\mathrm{Hom}(\mathscr{V},\mathscr{W})$ *is canonically isomorphic to $\mathscr{V}^* \otimes \mathscr{W}$. Thus, if E and F are a vector bundles on X, then* $\textsc{Hom}(E,F)$ *is isomorphic to $E^* \otimes F$ over X. In particular, we have that* $\textsc{End}(E)$ *is isomorphic to $E^* \otimes E$ over X.*

- *Let E and F be vector bundles on X with typical fibers $\mathscr{V}$ and $\mathscr{W}$, respectively. Moreover, let $\{\varphi_{ij}\}_{i,j\in I}$ and $\{\psi_{ij}\}_{i,j\in I}$ be representing sets of transition functions* *for E and F, respectively. The preceding item allows us to see the corresponding transition functions of* $\mathrm{Hom}(E,F)$ *and* $\mathrm{End}(E)$ *as the ones of* $E^* \otimes F$ *and* $E^* \otimes E$ *described in Remark C.42, respectively. Nevertheless, it is interesting to note the following approach. Indeed, the corresponding transition functions of* $\mathrm{Hom}(E,F)$ *are given by the set* $\{\mathrm{Hom}(\varphi_{ij},\psi_{ij}) : U_{ij} \to \mathrm{Hom}(\mathrm{GL}(\mathscr{V}),\mathrm{GL}(\mathscr{W}))\}_{i,j\in I}$ *where* $\mathrm{Hom}(\varphi_{ij},\psi_{ij})_x(\varphi) = (\psi_{ij})_x \circ \varphi \circ (\varphi_{ij})_x^{-1}$ *for all* $x \in U_{ij}$ *and all* $\varphi \in \mathrm{GL}(\mathscr{V})$*. In particular, the corresponding transition functions of* $\mathrm{End}(E)$ *are given by the set* $\{\mathrm{End}(\varphi_{ij}) : U_{ij} \to \mathrm{End}(\mathrm{GL}(\mathscr{V}))\}_{i,j\in I}$ *where* $\mathrm{End}(\varphi_{ij})_x(\varphi) = (\varphi_{ij})_x \circ \varphi \circ (\varphi_{ij})_x^{-1}$ *for all* $x \in U_{ij}$ *and all* $\varphi \in \mathrm{GL}(\mathscr{V})$. ♢

C.8 Inner and Hermitian products on vector bundles

Definition C.48 (Inner and Hermitian products on vector bundles). *We give the following definitions.*

- *An **inner product** on a real vector bundle* $\pi : E \to X$ *is a continuous function* $\langle\ ,\ \rangle : E \oplus E \to \mathbb{R}$ *that restricts in each fiber to an inner product (that is, a positive definite symmetric bilinear form). Moreover, if a real vector bundle is equipped with an inner product, then it is called an **Euclidean vector bundle**.*

- *An **Hermitian product** on a complex vector bundle* $\pi : E \to X$ *is a continuous function* $\langle\ ,\ \rangle : E \oplus E \to \mathbb{C}$ *that restricts in each fiber to an Hermitian product (that is, a positive definite antisymmetric sesquilinear form). Moreover, if a complex vector bundle is equipped with an Hermitian product, then it is called an **Hermitian vector bundle**.* ♢

The next result answers the natural question that the reader may be asking himself or herself now, which consists in finding conditions for the existence of inner and Hermitian products in real and complex vector bundles, respectively. The interesting fact is that a topological condition on the base space of vector bundles is sufficient to answer this question.

Theorem C.49 (Existence of inner and Hermitian products on vector bundles on paracompact Hausdorff spaces). *Let $\pi : E \to X$ be a real (respectively, complex) vector bundle. If X is a paracompact Hausdorff space, then there exists an inner (respectively, Hermitian) product on E.*

Proof. The reader can readily prove this result following the next two steps.

(1) Use local charts
$$\varphi_{U_i} : \pi^{-1}(U_i) \to U_i \times \mathscr{V}$$
of E to define local inner (respectively, Hermitian) products on E by pullbacking the natural inner (respectively, Hermitian) product on the product bundle $U_i \times \mathscr{V}$ induced by an inner (respectively, Hermitian) product of $\mathscr{V}$.

(2) Use the fact that X is a paracompact Hausdorff space to choose a partition of the unit subordinated to the open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X, which is produced by the choices of local charts of E that were made in the preceding item, in order to carefully glue together all the local inner (respectively, Hermitian) products on E, which were also obtained in the preceding item, into a global inner (respectively, Hermitian) product on E. □

The following result generalizes a well-known theorem in Linear Algebra. The theorem which we are referring to says that in a finite-dimensional vector space there always exists an orthogonal complement for each of its vector subspaces. In fact, the subsequent result ensures a corresponding theorem in the framework of vector bundles on paracompact Hausdorff spaces.

Theorem C.50 (Existence of complements of subbundles in vector bundles on paracompact Hausdorff spaces). *Let $\pi : E \to X$ be a vector bundle. If X is a paracompact Hausdorff space, then for every vector subbundle F of E there exists another vector subbundle $F^{\perp}$ of E such that the direct sum $F \oplus F^{\perp}$ is isomorphic to E over X.*

Proof. The reader can find a proof of this result in [15, p. 12]. □

The next and last result of this section is one of the most important theorems in this appendix. It concerns the existence of trivializing addenda for vector bundles on compact Hausdorff spaces. In other words, it says that vector bundles on compact Hausdorff spaces are trivial up to summing them with other vector bundles. In this situation, a trivializing summand is called a **trivializing addendum** for the vector bundle under consideration.

Theorem C.51 (Existence of trivializing addenda for vector bundles on compact Hausdorff spaces)**.** *Let $\pi_E : E \to X$ be a vector bundle. If X is a compact Hausdorff space, then there exists a vector bundle $\pi_F : F \to X$ such that the direct sum $E \oplus F$ is a trivial vector bundle.*

Proof. The reader can find a proof of this result in [15, p. 13]. □

C.9 Pullback of vector bundles

Remark C.52 (Fiber product of topological spaces)**.** *Let $f : X \to Z$ and $g : Y \to Z$ be continuous functions. The fiber product between X and Y with respect to f and g is the topological subspace of $X \times Y$*

$$X \times_{f,g} Y \;:=\; \big\{(x,y) \in X \times Y \;:\; f(x) = g(y)\big\}.$$

We define the projections

$$\begin{aligned} \pi_1 : X \times_{f,g} Y &\to X, \\ (x,y) &\mapsto x, \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \pi_2 : X \times_{f,g} Y &\to Y, \\ (x,y) &\mapsto y. \end{aligned}$$

We have that the fiber product verifies the following universal property. Given a pair of continuous functions $h : W \to X$ and $k : W \to Y$, there exists a unique continuous function from W into $X \times_{f,g} Y$, usually denoted by $h \times_{f,g} k$, such that the following diagram is commutative.

$$
\begin{tikzcd}
W \arrow[dr, "h\times_{f,g}k" description] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\
& X \times_{f,g} Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}
$$

In particular, the fiber product of spaces with respect to continuous functions with the same codomain is unique up to a unique homeomorphism. We use fiber products to set the following definition. ♢

Definition C.53 (Pullback of a vector bundle)**.** *Let $f : X \to Y$ be a continuous map and $\pi : E \to Y$ be a vector bundle. We say that the* ***pullback of E through f*** *is the vector bundle $\pi^* : f^*E \to X$ where $f^*E := E \times_{\pi,f} X$ and $\pi^*(a,x) := x$ for all $(a,x) \in f^*E$.* ♢

Remark C.54 (On the pullback of a vector bundle being a vector bundle)**.** *Let $f : X \to Y$ be a continuous map and $\pi : E \to Y$ be a vector bundle. We can verify that f^*E is a vector bundle with the same typical fiber of E up to canonical isomorphism. In fact, for every $x \in X$, we have $(f^*E)_x = E_{f(x)} \times \{x\}$, which is canonically isomorphic to $E_{f(x)}$. Hence, the fiber of f^*E in x is canonically isomorphic to the one of E in $f(x)$. More precisely, the fiber of f^*E in x is canonically homeomorphic to the one of E in $f(x)$ and we endow it with the induced vector space structure, making the fibers in question isomorphic as vector spaces. Moreover, fixing $x \in X$, let (U, φ_U) be a local chart of E in $f(x)$. Setting $V := f^{-1}(U)$, we obtain the local chart (V, ψ_V) of f^*E in x where $\psi_V : (\pi^*)^{-1}(V) \to V \times \mathscr{V}$ is given by $\psi_V(a) = (\pi^*(a), \pi_{\mathscr{V}} \circ \varphi_U(a))$, where $\pi_{\mathscr{V}} : U \times \mathscr{V} \to \mathscr{V}$ is the projection onto the second factor. This finishes the proof of our last claim.* ♢

Theorem C.55 (Important properties of the pullbacks of vector bundles)**.** *We have the following properties of the pullbacks of vector bundles.*

(1) If $f : X \to Y$ and $g : Y \to Z$ are continuous maps and E is a vector bundle on Z, then $(g \circ f)^*E$ is isomorphic to f^*g^*E over X.

(2) If X is a topological space and E is a vector bundle on X, then id_X^*E is isomorphic to E over X.

(3) If $f : X \to Y$ is a continuous map and E and F are vector bundles on Y, then $f^*(E \oplus F)$ is isomorphic to $f^*E \oplus f^*F$ over X. In other words, the pullback of vector bundles commutes with the direct sum up to isomorphism.

(4) If $f : X \to Y$ is a continuous map and E and F are vector bundles on Y, then $f^*(E \otimes F)$ is isomorphic to $f^*E \otimes f^*F$ over X. In other words, the pullback of vector bundles commutes with the tensor product up to isomorphism.

Proof. These properties are straightforwardly implied by the fact that the pullback of a vector bundle is unique up to isomorphism. The reader who wants more details may find in [31, pp. 5 - 7] a good reference. □

Remark C.56 (Covariant functor defined by the pullback of vector bundles)**.** *If $f : X \to Y$ is a continuous function, then we have the covariant functor*

$$\begin{aligned} f^* : \mathrm{VectBdl}_Y(\mathscr{V}) &\to \mathrm{VectBdl}_X(\mathscr{V}), \\ E &\mapsto f^*E, \\ \eta : E \to F &\mapsto f^*\eta : f^*E \to f^*F, \end{aligned}$$

where $f^*\eta : f^*E \to f^*F$ is given by $f^*\eta(a, x) = (\eta(a), x)$ for all $(a, x) \in f^*E$. Furthermore, if $g : Y \to Z$ is also a continuous functions, then Item (1) of Theorem C.55 says that the covariant functor $(g \circ f)^* : \mathrm{VectBdl}_Z \to \mathrm{VectBdl}_X$ is isomorphic to the composition of the covariant functors f^* and g^* given by $f^* \circ g^* : \mathrm{VectBdl}_Z \to \mathrm{VectBdl}_X$. Also, Item (2) of Theorem C.55 says that $\mathrm{id}_X^* : \mathrm{VectBdl}_X \to \mathrm{VectBdl}_X$ is isomorphic to the identity functor. ♢

Theorem C.57 (Invariance of the pullbacks of a vector bundle through homotopic maps with paracompact Hausdorff domains)**.** *Let $\pi : E \to Y$ be a vector bundle and $f, g : X \to Y$ be continuous homotopic maps. If X is a paracompact Hausdorff space, then the pullbacks f^*E and g^*E are isomorphic over X.*

Proof. The reader can find a proof of this result in the case in which X is a compact Hausdorff space in [2, p. 17]. In turn, a proof of the general case can be seen in [15, pp. 20 - 21] and in [31, pp. 7 - 8]. □

Corollary C.58 (Bijection of isomorphism classes of vector bundles induced by a homotopy equivalence between paracompact Hausdorff spaces)**.** *Let X and Y be paracompact Hausdorff spaces in such manner that there exists a homotopy equivalence $f : X \to Y$ between them. Then,*

$$\begin{aligned} [f^*] : \mathrm{Vect}_Y(\mathscr{V}) &\to \mathrm{Vect}_X(\mathscr{V}), \\ [E] &\mapsto [f^*E], \end{aligned}$$

which is the quotient of the function between objects of the covariant functor from Remark C.56 by the isomorphism equivalence relation, is a bijection between $\mathrm{Vect}_Y(\mathscr{V})$ *and* $\mathrm{Vect}_X(\mathscr{V})$*. In particular, we have that every vector bundle on a contractible paracompact Hausdorff space is trivial.*

Proof. Since $f : X \to Y$ is a homotopy equivalence, there exists a continuous function $g : Y \to X$ such that $g \circ f$ is homotopic to $\mathrm{id}_X : X \to X$ and $f \circ g$ is homotopic to $\mathrm{id}_Y : Y \to Y$. Therefore, for all vector bundles E on X and F on Y, Item (2) of Theorem C.55 and Theorem C.57 imply

$$\begin{aligned} (g \circ f)^*E &\simeq \mathrm{id}_X^*E \simeq E \quad \text{and} \\ (f \circ g)^*F &\simeq \mathrm{id}_Y^*F \simeq F. \end{aligned}$$

Thus, we have $[(g \circ f)^*] = \mathrm{id}_{\mathrm{Vect}_X(\mathscr{V})}$ and $[(f \circ g)^*] = \mathrm{id}_{\mathrm{Vect}_Y(\mathscr{V})}$. Furthermore, Item (1) of Theorem C.55 implies

$$\begin{aligned} [f^*] \circ [g^*] &= [(g \circ f)^*] \quad \text{and} \\ [g^*] \circ [f^*] &= [(f \circ g)^*]. \end{aligned}$$

Consequently, it follows that $[g^*]$ is the inverse function of $[f^*]$. Hence, we have the desired bijection between $\mathrm{Vect}_Y(\mathscr{V})$ and $\mathrm{Vect}_X(\mathscr{V})$. Finally, the last claim of the theorem is obvious since a contractible paracompact Hausdorff space is homotopically equivalent to a point space, which is evidently a paracompact Hausdorff space, and since every vector bundle on a point space is trivial. □

Corollary C.59 (Vector bundles on cylinders are no new information)**.** *Let X be a paracompact Hausdorff space and E be a vector bundle on $X \times \mathbb{I}$, where $\mathbb{I}$ is the usual real unit interval. For any fixed $j \in \mathbb{I}$, we have that E and $\pi_j^*(E\ |_{X\times\{j\}})$ are isomorphic, where*

$$\begin{aligned} \pi_j : X \times \mathbb{I} &\to X \times \mathbb{I}, \\ (x, t) &\mapsto (x, j). \end{aligned}$$

Proof. Since the product of paracompact Hausdorff spaces is paracompact Hausdorff, $X \times \mathbb{I}$ is a paracompact Hausdorff space. Thus, once $\pi_j, \mathrm{id}_{X\times\mathbb{I}} : X \times \mathbb{I} \to X \times \mathbb{I}$ are homotopic maps, it follows from Theorem C.57 that $\pi_j^*(E\ |_{X\times\{j\}})$ is isomorphic to $\mathrm{id}_{X\times\mathbb{I}}^*(E\ |_{X\times\{j\}})$. Moreover, Item (2) of Theorem C.55 implies that $\mathrm{id}_{X\times\mathbb{I}}^*(E\ |_{X\times\{j\}})$ is isomorphic to $E\ |_{X\times\{j\}}$. Therefore, the result is proved because $E\ |_{X\times\{j\}}$ is clearly isomorphic to E. □

C.10 Collapsing vector bundles

Lemma C.60 (Existence of a local extension for an isomorphism of vector bundles defined on a closed subspace of a compact Hausdorff space)**.** *Let A be a closed subspace of a compact Hausdorff space X. In addition, let E and F be vector bundles on X. If $f : E\ |_A \to F\ |_A$ is an isomorphism over A, then there exist an open subspace U of X containing A and an extension $F : E\ |_U \to F\ |_U$ which is an isomorphism over U.*

Proof. The reader can find a proof of this result in [2, p. 17]. □

Theorem C.61 (Collapsing a vector bundle defined on a compact Hausdorff space)**.** *Let A be a closed subspace of a compact Hausdorff space X. In addition, let E be a vector bundle on X for which there exists an isomorphism $\alpha : E\,|_A \to A \times \mathscr{V}$, which will be referred to as a trivialization of E over A. Being $\pi : A \times \mathscr{V} \to \mathscr{V}$ the canonical projection onto the second factor, we define an equivalence relation on $E\,|_A$ as follows: $a, b \in E\,|_A$ are related if and only if*

$$(\pi \circ \alpha)(a) \;=\; (\pi \circ \alpha)(b).$$

Trivially, we extend this relation on $E\,|_{X-A}$ as the identity. Thus, if E/α is the quotient space of E by this (extended) equivalence relation, then it is a vector bundle on X/A. Moreover, its isomorphism class depends only on the homotopy class of the trivialization in question.

Proof. The quotient space E/α has a natural structure of a family of vector spaces on X/A. This happens because the process described just identified the fibers of E over A through α, leaving the other ones intact. More than that, this reasoning shows that, to prove that E/α is a vector bundle, we only have to verify the local triviality of E/α at the base point A/A of X/A. Because of Lemma C.60, we can extend α to an isomorphism $\widetilde{\alpha} : E\,|_U \to U \times \mathscr{V}$ for some open subspace U of X containing A. Then, $\widetilde{\alpha}$ induces an isomorphism between

$$(E\,|_U)/\alpha \qquad \text{and} \qquad (U/A) \times \mathscr{V},$$

which establishes the local triviality of E/α at A/A. For the last claim of the statement, suppose that α and β are homotopic trivializations of E over A. This means that we have a trivialization γ of $E \times \mathbb{I}$ over $A \times \mathbb{I} \subseteq X \times \mathbb{I}$ inducing α and β at the two end points of $\mathbb{I}$. Then, consider the natural map

$$\begin{aligned} \pi : \ (X/A) \times \mathbb{I} \ &\to \ (X \times \mathbb{I})/(A \times \mathbb{I}), \\ ([x],\, t) \ &\mapsto \ [x,\, t]. \end{aligned}$$

We have that the vector bundle $\pi^*((E \times \mathbb{I})/\gamma)$ on $(X/A) \times \mathbb{I}$ is such that its restriction to $(X/A) \times \{0\}$ coincides with E/α, and that its restriction to $(X/A) \times \{1\}$ coincides with E/β. Consequently, it follows from Theorem C.57 that E/α is isomorphic to E/β, as desired. $\square$

Corollary C.62 (Pullback of vector bundles of the natural projection). *Let A be a closed subspace of a compact Hausdorff space X. The projection $\pi : X \to X/A$ induces the pullback*

$$\begin{aligned} [\pi^*] : \mathrm{Vect}_{X/A} &\to \mathrm{Vect}_X, \\ [E] &\mapsto [\pi^* E], \end{aligned}$$

which is the quotient of the function between objects of the covariant functor $\pi^* : \mathrm{VectBdl}_{X/A} \to \mathrm{VectBdl}_X$ from Remark C.56 by the isomorphism equivalence relation of vector bundles. Then,

$$\mathrm{Im}\,[\pi^*] \;=\; \{[E] \in \mathrm{Vect}_X : E\,|_A \;\textit{ is a trivial vector bundle}\}.$$

Proof. Once A/A is a one-point space, the fact that $(\pi^* E)\,|_A$ is trivial follows from the equality

$$(\pi^* E)\,|_A = \;(\pi\,|_A)^*(E|_{A/A}).$$

On the other hand, let E be a vector bundle on X such that $E\,|_A$ is trivial. Then, let $\alpha : E\,|_A \to A \times \mathscr{V}$ be a trivialization. We have that the collapsed vector bundle E/α on X/A is such that

$$[\pi^*(E/\alpha)] \;=\; [E].$$

This equality is straightforward, although it is not immediate to be visualized. This finishes the proof of the theorem. □

Remark C.63 (On the proof of Corollary C.62). *In general, E/α obtained in the proof of the preceding result is not unique (not even up to isomorphism). Thus, $[\pi^*]$ is not injective. Nevertheless, its isomorphism class only depends on the homotopy class of the trivialization α of $E\,|_A$. Hence, if we choose non-homotopic trivializations, then the resulting vector bundles may not be isomorphic. Indeed, let $X = \mathbb{I}$, $A = \partial\,\mathbb{I}$ and E be the trivial vector bundle $\mathbb{I} \times \mathbb{R}$. Therefore:*

- *if we choose the identity trivialization, then we obtain the trivial vector bundle $\mathbb{S}^1 \times \mathbb{R}$; and*
- *if we choose the trivialization given by $\alpha(0,t) = (0,t)$ and $\alpha(1,t) = (1,-t)$, then we obtain the Möbius bundle, which is a non-trivial vector bundle.*

In the literature, the reader can find similar examples in the framework of complex vector bundles. However, there is an exception in this reasoning, which is proved in the following corollary. ♢

Corollary C.64 (Bijection induced by the natural projection)**.** *Let A be a closed contractible subspace of a compact Hausdorff space X. Then, the map $[\pi^*]$ defined in Corollary C.62 is a bijection.*

Proof. Let E be a vector bundle on X. Because of Corollary C.58, it follows that $E\mid_A$ is trivial. Then, let $\alpha : E\mid_A \to A \times \mathscr{V}$ be a trivialization of E over A. Moreover, we have that two such trivializations differ by an automorphism of $A \times \mathscr{V}$[8]. That is, by a map $A \to \mathrm{GL}(\mathscr{V})$. However, since $\mathrm{GL}(\mathscr{V})$ is connected (because it is homeomorphic to $\mathrm{GL}(n)$ for some $n \in \mathbb{N}$) is connected and $\mathscr{V}$ is contractible, α is unique up to homotopy. Thus, the isomorphism class of E/α is uniquely determined by that of E. Hence, we have constructed a map $\mathrm{Vect}_X \to \mathrm{Vect}_{X/A}$ which is clearly a two-sided inverse for $[\pi^*]$. Therefore, $[\pi^*]$ is a bijection. □

C.11 Smooth vector bundles

Definition C.65 ($\mathcal{C}^r$-vector bundles)**.** *Let $\mathcal{M}$ be a real $\mathcal{C}^r$-manifold where r is a natural number or ∞. A* ***$\mathcal{C}^r$-vector bundle*** *on $\mathcal{M}$ is a vector bundle $\pi : E \to \mathcal{M}$ such that:*

- *E is a real $\mathcal{C}^r$-manifold;*
- *π is a $\mathcal{C}^r$-function; and*
- *each homeomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathscr{V}$ of Item (1) of Definition C.1 is a $\mathcal{C}^r$-diffeomorphism.*

---

[8]Indeed, if $\beta : E\mid_A \to A \times \mathscr{V}$ is another trivialization of E over A, then

$$\alpha = (\alpha \circ \beta^{-1}) \circ \beta,$$

where $\alpha \circ \beta^{-1} : A \times \mathscr{V} \to A \times \mathscr{V}$ is the automorphism of $A \times \mathscr{V}$ under which the trivializations α and β of E over A differ.

*Moreover, we usually say that a $\mathcal{C}^0$-vector bundle is a **(continuous) vector bundle**, and that a $\mathcal{C}^\infty$-vector bundle is a **smooth vector bundle**.* ♢

Definition C.66 ($\mathcal{C}^p$-vector bundle morphisms)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be, respectively, a real $\mathcal{C}^r$-manifold and a real $\mathcal{C}^s$ manifold. In addition, let $\pi : E \to \mathcal{M}$ and $\pi' : F \to \mathcal{N}$ be, respectively, a $\mathcal{C}^r$-vector bundle and a $\mathcal{C}^s$-vector bundle. For any p between* 0 *and* $\min\{r, s\}$*, both included:*

- *a **$\mathcal{C}^p$-vector bundle morphism** from E into F is a vector bundle morphism $f : E \to F$ that is a $\mathcal{C}^p$-function which covers a $\mathcal{C}^p$-function $g : \mathcal{M} \to \mathcal{N}$; and*
- *if $\mathcal{M} = \mathcal{N}$, then a **$\mathcal{C}^p$-vector bundle morphism over $\mathcal{M}$** from E into F is a $\mathcal{C}^p$-vector bundle morphism $f : E \to F$ such that the induced function $g : \mathcal{M} \to \mathcal{M}$ is the identity map.*

*Furthermore, we say that an invertible $\mathcal{C}^p$-vector bundle morphism (over $\mathcal{M}$) is a **$\mathcal{C}^p$-vector bundle isomorphism (over $\mathcal{M}$)**. Finally, when $p = 0$ or $p = \infty$, we will use a nomenclature for $\mathcal{C}^p$-vector bundle morphisms analogous to the one we set in Definition C.65.* ♢

Remark C.67 (On $\mathcal{C}^r$-vector bundles)**.** *All the notions and results that we have discussed up to now about vector bundles keeps on holding for $\mathcal{C}^r$-vector bundles, provided we require that all of the topological spaces involved are real $\mathcal{C}^r$-manifolds, all of the topological subspaces are real $\mathcal{C}^r$-submanifolds (embedded or immersed) and that all of the continuous functions involved are $\mathcal{C}^r$-functions (in particular, each homeomorphism must be a $\mathcal{C}^r$-diffeomorphism). Especially, we have:*

- *the categories of $\mathcal{C}^r$-vector bundles* $\mathrm{VectBdl}^{\mathcal{C}^r}$, $\mathrm{VectBdl}^{\mathcal{C}^r}(\mathscr{V})$, $\mathrm{VectBdl}^{\mathcal{C}^r}_{\mathcal{M}}$ *and* $\mathrm{VectBdl}^{\mathcal{C}^r}_{\mathcal{M}}(\mathscr{V})$ *together with the following diagram indicating their subcategory relationships;*

$$\begin{tikzcd}
\mathrm{VectBdl}^{\mathcal{C}^r}(\mathscr{V}) \arrow[r, two heads] & \mathrm{VectBdl}^{\mathcal{C}^r} \\
\mathrm{VectBdl}^{\mathcal{C}^r}_{\mathcal{M}}(\mathscr{V}) \arrow[u] \arrow[r, two heads] & \mathrm{VectBdl}^{\mathcal{C}^r}_{\mathcal{M}} \arrow[u]
\end{tikzcd}$$

- *the sets of* $\mathcal{C}^r$*-isomorphism classes of* $\mathcal{C}^r$*-vector bundles* $\mathrm{Vect}^{\mathcal{C}^r}$, $\mathrm{Vect}^{\mathcal{C}^r}(\mathscr{V})$, $\mathrm{Vect}^{\mathcal{C}^r}_{\mathcal{M}}$ *and* $\mathrm{Vect}^{\mathcal{C}^r}_{\mathcal{M}}(\mathscr{V})$*. Note that* $\mathrm{Vect}^{\mathcal{C}^r}_{\mathcal{M}}$ *is an abelian semigroup with the binary operation induced by the direct sum;*

- *the set* $\Gamma_r(E)$ *of* $\mathcal{C}^r$*-global sections of a* $\mathcal{C}^r$*-vector bundle* $\pi : E \to \mathcal{M}$*;*

- *the transition functions of a* $\mathcal{C}^r$*-vector bundle being* $\mathcal{C}^r$*-diffeomorphisms;*

- *the direct sum and the tensor product of* $\mathcal{C}^r$*-vector bundles being a* $\mathcal{C}^r$*-vector bundle; and*

- *the invariance of the pullbacks of a* $\mathcal{C}^r$*-vector bundle through* $\mathcal{C}^r$*-homotopic maps with a real* $\mathcal{C}^r$*-manifold as domain. This happens because manifolds are paracompact Hausdorff by definition.* ♢

Theorem C.68 ($\mathcal{C}^r$-vector bundles on $\mathcal{C}^\infty$-manifolds)**.** *Let* $\mathcal{M}$ *be a real* $\mathcal{C}^\infty$*-manifold. Every* $\mathcal{C}^r$*-vector bundle* $\pi : E \to \mathcal{M}$ *has a compatible smooth vector bundle structure. Moreover, such a structure is unique up to* $\mathcal{C}^\infty$*-isomorphism over* $\mathcal{M}$*.*

Proof. The reader can find a proof of this result in [16, p. 101]. This proof uses the notion of classifying maps of $\mathcal{C}^r$-vector bundles, which we will not explain here. Indeed, the idea behind the proof of the existence of a smooth vector bundle structure is to approximate a $\mathcal{C}^r$-classifying map for the $\mathcal{C}^r$-vector bundle $\pi : E \to \mathcal{M}$ by a homotopic $\mathcal{C}^\infty$-map, and then apply Theorem C.57. The uniqueness up to isomorphism of such a smooth structure is handled similarly. □

Remark C.69 (On the compatible smooth vector bundle structure of a $\mathcal{C}^r$-vector bundle)**.** *Let* $\mathcal{M}$ *be a real* $\mathcal{C}^\infty$*-manifold. We have just seen that a* $\mathcal{C}^r$*-vector bundle* $\pi : E \to \mathcal{M}$ *has a compatible smooth vector bundle structure. This means that the* $\mathcal{C}^r$*-manifold* E *admits a real* $\mathcal{C}^\infty$*-manifold structure, hereafter denoted by* $\widetilde{E}$*, such that* $\widetilde{\pi} : \widetilde{E} \to \mathcal{M}$*,* $a \mapsto \pi(a)$*, is a smooth vector bundle. In particular, note that* π *and* $\widetilde{\pi}$ *are equal as functions.* ♢

Corollary C.70 (More about $\mathcal{C}^r$-vector bundles on $\mathcal{C}^\infty$-manifolds)**.** *Let* $\mathcal{M}$ *be a real* $\mathcal{C}^\infty$*-manifold. Every* $\mathcal{C}^r$*-vector bundle* $\pi : E \to \mathcal{M}$ *is* $\mathcal{C}^r$*-isomorphic over* $\mathcal{M}$ *to a smooth vector bundle.*

Proof. We can apply Theorem C.68 to endow $\pi : E \to \mathcal{M}$ with a compatible smooth vector bundle structure. Therefore, the identity map $\mathrm{id}_E : E \to \widetilde{E}$ is a $\mathcal{C}^r$-isomorphism over $\mathcal{M}$, which proves what we wished. $\square$

Corollary C.71 (Relation between $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^r}$ and $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^\infty}$)**.** *Let $\mathcal{M}$ be a real $\mathcal{C}^\infty$-manifold. The abelian semigroups $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^r}$ and $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^\infty}$ are isomorphic. In particular, $\mathrm{Vect}_{\mathcal{M}}$ is isomorphic to $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^\infty}$.*

Proof. Let E and F be $\mathcal{C}^r$-vector bundles on $\mathcal{M}$. Since the compatible smooth vector bundle structure of a $\mathcal{C}^r$-vector bundle is unique up to $\mathcal{C}^\infty$-isomorphism by Theorem C.65, we have that $\widetilde{E} \oplus \widetilde{F}$ and $\widetilde{E \oplus F}$ are $\mathcal{C}^\infty$-isomorphic. This proves that the map from $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^r}$ into $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^\infty}$ that sends $[E]_{\mathcal{C}^r}$ into $[\widetilde{E}]_{\mathcal{C}^\infty}$ is a semigroup homomorphism. In addition, the reader can readily prove that this semigroup homomorphism is invertible exhibiting its inverse. Therefore, $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^r}$ is isomorphic to $\mathrm{Vect}_{\mathcal{M}}^{\mathcal{C}^\infty}$, as we wished. $\square$

Remark C.72 (Holomorphic vector bundles on complex manifolds)**.** *In this section, we considered real and complex vector bundles based on real $\mathcal{C}^\infty$-manifolds. However, what about the trueness of its results for holomorphic vector bundles based on complex manifolds? First of all, we observe that the proof of Theorem C.68 cannot be adapted to this framework since there is no "Holomorphic Approximation Theorem" for classifying maps of $\mathcal{C}^\infty$-vector bundles on complex manifolds. More than that, it is known that Theorem C.68 is false in this context because there exist examples in the literature of holomorphic vector bundles which are smoothly isomorphic but not holomorphically isomorphic. Consequently, the abelian semigroup of isomorphism classes of $\mathcal{C}^\infty$-vector bundles on a complex manifold is not always isomorphic to the abelian semigroup of isomorphism classes of holomorphic vector bundles on the same complex manifold. Nevertheless, if we consider real $\mathcal{C}^\omega$-manifolds instead of real $\mathcal{C}^\infty$-manifolds in Theorem C.68, then the result also holds true. The reader can prove this claim following the comments of [16, p. 101].* $\diamondsuit$

Appendix D

Constructions with compact Hausdorff spaces

In this appendix, we describe classical constructions with topological spaces: wedge sum, smashed product, cones and suspensions. We restrict them to the framework of compact Hausdorff spaces since they are mainly used in Chapter 2 to study K-Theory. However, even the reader who is unfamiliar with these constructions will note that they can be extended to all kinds of topological spaces. The notations of Chapter 2 are used here to establish these mathematical objects. We follow [14, pp. 8-10] in this presentation.

D.1 Wedge sum

Definition D.1 (Wedge sum of pointed compact Hausdorff spaces)**.** *Let* (X, x_0) *and* (Y, y_0) *be objects in* $\mathrm{TopHdCpt}_+$*. We define the* ***wedge sum of*** $(\boldsymbol{X}, \boldsymbol{x_0})$ ***and*** $(\boldsymbol{Y}, \boldsymbol{y_0})$*, and denote it by* $X \vee Y$*, to be the union*

$$X \vee Y \;:=\; (X \times \{y_0\}) \cup (\{x_0\} \times Y),$$

which is naturally a pointed compact Hausdorff space. In fact, $X \vee Y$ *is compact Hausdorff because it is a finite union of products of compact Hausdorff spaces, and* $(x_0, y_0) \in X \vee Y$ *is its natural marked point, which is hereafter omitted. Furthermore, we have the following functor with two covariant variables, which is called the* ***wedge sum functor****:*

$$
\begin{aligned}
\vee : \ \mathrm{TopHdCpt}_+ \times \mathrm{TopHdCpt}_+ &\to \mathrm{TopHdCpt}_+, \\
\big((X, x_0),\, (Y, y_0)\big) &\mapsto X \vee Y, \\
\big(f : (X, x_0) \to (Y, y_0),\, g : (Z, z_0) \to (W, w_0)\big) &\mapsto f \vee g : X \vee Z \to Y \vee W,
\end{aligned}
$$

where

$$
(f \vee g)(x,\, z) \ := \ (f(x),\, g(z))
$$

for all $(x, z) \in X \vee Z$. *The reader will find a simple but helpful visualization of this construction in Figure D.1.*

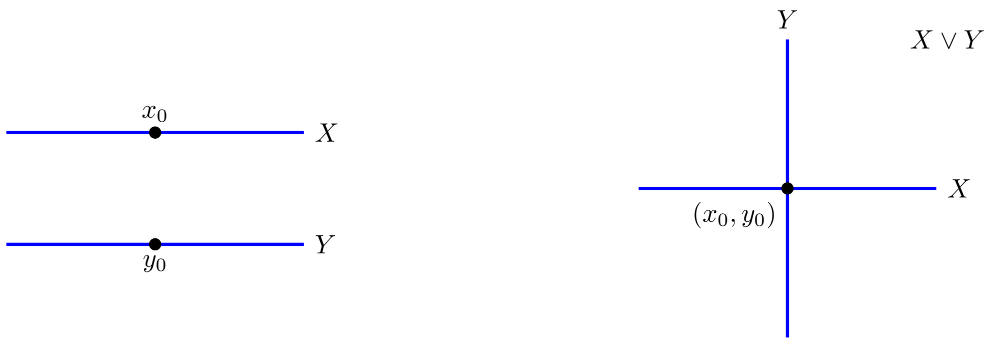

Figure D.1: In the images above, (X, x_0) and (Y, y_0) are the line segments with marked points on the left. We have that the wedge sum $X \vee Y$ is the cross on the right, whose intersection point $(x_0, y_0) \in X \vee Y$ is its natural marked point. Note that X and Y are embedded in $X \vee Y$. ♢

D.2 Smashed product

Definition D.2 (Smashed product of pointed compact Hausdorff spaces)**.** *Let* (X, x_0) *and* (Y, y_0) *be objects in* $\mathrm{TopHdCpt}_+$. *We define the* ***smashed product of*** $\boldsymbol{(X, x_0)}$ ***and*** $\boldsymbol{(Y, y_0)}$, *and denote it by* $X \wedge Y$, *to be the identification space*

$$
X \wedge Y \ := \ \frac{X \times Y}{X \vee Y},
$$

which is naturally a pointed compact Hausdorff space. In fact, $X \wedge Y$ *is compact Hausdorff because it is the quotient of a compact Hausdorff space by one of its closed subspaces, and*

$$
\frac{X \vee Y}{X \vee Y} \in X \wedge Y
$$

is its natural marked point, which is hereafter omitted. Moreover, we have the following functor with two covariant variables, which is called the ***smashed product functor****:*

$$\begin{aligned} \wedge : \mathrm{TopHdCpt}_+ \times \mathrm{TopHdCpt}_+ &\to \mathrm{TopHdCpt}_+, \\ \big((X, x_0),\, (Y, y_0)\big) &\mapsto X \wedge Y, \\ \big(f : (X, x_0) \to (Y, y_0),\, g : (Z, z_0) \to (W, w_0)\big) &\mapsto f \wedge g : X \wedge Z \to Y \wedge W, \end{aligned}$$

where

$$(f \wedge g)[x,\, z] \;:=\; [f(x),\, g(z)]$$

for all $[x,\, z] \in X \wedge Z$. Furthermore, the reader can readily prove that, being n a non-zero natural number, the smashed product of n copies of $\mathbb{S}^1$ is homeomorphic to $\mathbb{S}^n$. The reader will find a simple but helpful visualization of this construction in Figure D.2.

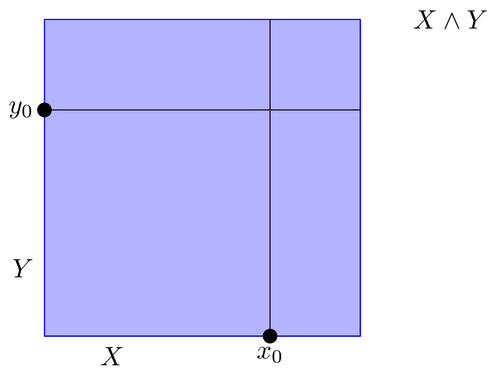

Figure D.2: In the image above, (X, x_0) and (Y, y_0) are the line segments with marked points at the bottom and on the left of the square. We have that the smashed product $X \wedge Y$ is obtained by collapsing the wedge sum $X \vee Y \subseteq X \wedge Y$ to its marked point $(x_0, y_0) \in X \vee Y$. ♢

D.3 Cones

Definition D.3 (Absolute and relative cones of compact Hausdorff spaces)**.** *We define $\mathbb{I} := [0, 1]$ and $-\mathbb{I} := [-1, 0]$, and we give the following definitions.*

- *Let X be an object in* TopHdCpt*. We define the* ***cone of X****, and denote it by CX, to be the identification space*

$$CX := \frac{X \times \mathbb{I}}{X \times \{1\}},$$

which is naturally a pointed compact Hausdorff space. In fact, CX is compact Hausdorff since it is the quotient of a compact Hausdorff space by one of its closed subspaces, and

$$\frac{X \times \{1\}}{X \times \{1\}} \in CX,$$

is its natural marked point, which is hereafter omitted. Moreover, we have the following covariant functor, which is called the ***cone functor****:*

$$\begin{aligned} C : \mathrm{TopHdCpt} &\to \mathrm{TopHdCpt}_+, \\ X &\mapsto CX, \\ f : X \to Y &\mapsto Cf : CX \to CY, \end{aligned}$$

where

$$Cf([x,\, t]) := [f(x),\, t]$$

for all $[x,\, t] \in CX$. Analogously, we define the ***negative cone of X****, and denote it by $C'X$, to be the identification space:*

$$C'X := \frac{X \times -\mathbb{I}}{X \times \{-1\}},$$

which is naturally a pointed compact Hausdorff space. Similarly, we can define the ***negative cone functor****. In particular, note that the intersection of CX and $C'X$ is the base X.*

- *Let (X, A) be an object in $\mathrm{TopHdCCpt}_2$. We define the* ***cone of (X, A)****, and denote it by $C(X, A)$, to be the identification space obtained from the disjoint union $X \sqcup CA$ by collapsing every $a \in A$ to $(a, 0) \in CA$. This is also a pointed compact Hausdorff space with the natural marked point of the cone of A. In particular, it is to be noted that the cone of X coincides with the cone of (X, X). Furthermore, we have the following covariant functor, which is called the* ***relative cone functor****:*

$$\begin{aligned} C: \mathrm{TopHdCCpt}_2 &\to \mathrm{TopHdCpt}_+, \\ (X, A) &\mapsto C(X, A), \\ f : (X, A) \to (Y, B) &\mapsto Cf : C(X, A) \to C(Y, B), \end{aligned}$$

where

$$Cf([x, t]) := [f(x), t]$$

for all $[x, t] \in C(X, A)$.

Finally, the reader will find a simple but helpful visualization of these constructions in Figure D.3.

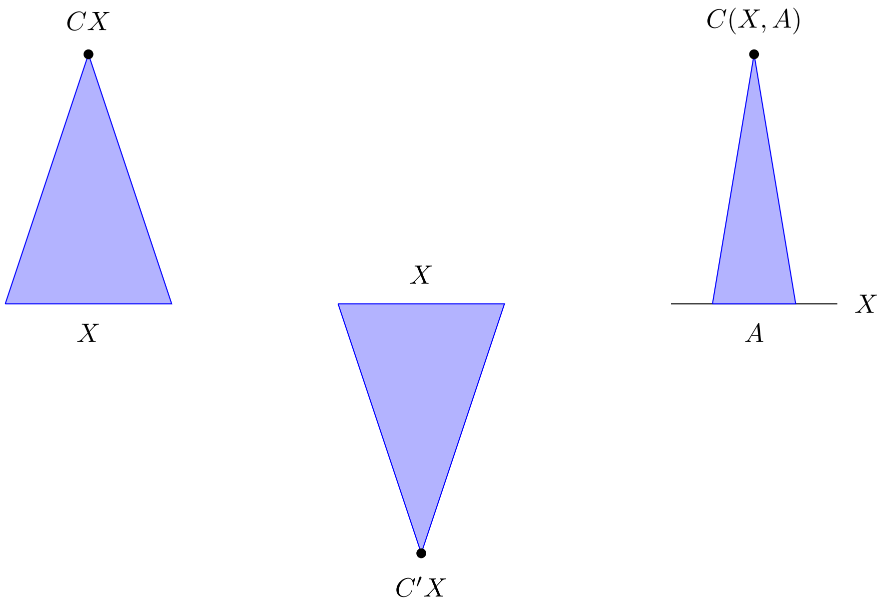

Figure D.3: In the images above, X is the whole line segment at the bottom of the triangles. On the left, we have the cone of X. In the middle, we have the negative cone of X. Finally, on the right, being A the central part of the line segment X, we have the cone of (X, A), which is obtained collapsing the subspace A of X with the base of the cone of A. ♢

D.4 Suspensions

Definition D.4 (Absolute and reduced suspensions of compact Hausdorff spaces)**.** *We define* $J := [-1, 1]$ *and we give the following definitions.*

- *Let X be an object in* TopHdCpt*. We define the* ***suspension of X****, and denote it by SX, to be the identification space*

$$SX := \frac{X \times J}{X \times \{1\},\, X \times \{-1\}},$$

where the comma in the denominator means that each of the spaces considered are being collapsed to a different point. The suspension of X is compact Hausdorff since it is the quotient of a compact Hausdorff space by one of its closed subspaces. However, differing from the preceding cases, it does not have a natural marked point once we are divided between

$$\frac{X \times \{1\}}{X \times \{1\}} \in SX \qquad \text{and} \qquad \frac{X \times \{-1\}}{X \times \{-1\}} \in SX.$$

This is coherent with SX being homeomorphic to the union of the cones CX and $C'X$, which cannot have a natural marked point since each of its components has one. Moreover, we have the following covariant functor, which is called the ***suspension functor****:*

$$\begin{aligned} S : \mathrm{TopHdCpt} &\to \mathrm{TopHdCpt}, \\ X &\mapsto SX, \\ f : X \to Y &\mapsto Sf : SX \to SY, \end{aligned}$$

where

$$Sf([x,\, t]) := [f(x),\, t]$$

for all $[x, t] \in SX$.

- *Let (X, x_0) be an object in* $\mathrm{TopHdCpt}_+$*. We define the* ***suspension of (X, x_0)****, and denote it by ΣX, to be the identification space*

$$\Sigma X := \frac{X \times J}{(X \times \{1\}) \cup (X \times \{-1\}) \cup (\{x_0\} \times J)},$$

which is naturally a pointed compact Hausdorff space. In fact, ΣX is compact Hausdorff since it is the quotient of a compact Hausdorff space by one of its closed subspaces, and

$$\frac{(X \times \{1\}) \cup (X \times \{-1\}) \cup (\{x_0\} \times J)}{(X \times \{1\}) \cup (X \times \{-1\}) \cup (\{x_0\} \times J)} \in \Sigma X$$

is its natural marked point, which is hereafter omitted. Moreover, ΣX *is homeomorphic to the spaces*

$$\mathbb{S}^1 \wedge X \qquad \text{and} \qquad \frac{SX}{\{x_0\} \times J}$$

This last presentation of ΣX *shows that the natural projection* $\pi : SX \to \Sigma X$ *induces the isomorphism in absolute K-Theory* $K(\pi) : K(\Sigma X) \to K(SX)$ *since* $\{x_0\} \times J$ *is contractible. Furthermore, we have the following covariant functor, which is called the* ***reduced suspension functor****:*

$$\begin{aligned} \Sigma : \ \mathrm{TopHdCpt}_+ &\to \mathrm{TopHdCpt}_+, \\ (X, x_0) &\mapsto \Sigma X, \\ f : (X, x_0) \to (Y, y_0) &\mapsto \Sigma f : \Sigma X \to \Sigma Y, \end{aligned}$$

where

$$\Sigma f([x,\, t]) \ := \ [f(x),\, t]$$

for all $[x, t] \in \Sigma X$*. Being* n *a non-zero natural number, we have that the reduced suspension functor can be iterated* n *times, producing the* ***n-reduced suspension functor****:*

$$\begin{aligned} \Sigma^n : \ \mathrm{TopHdCpt}_+ &\to \mathrm{TopHdCpt}_+, \\ (X, x_0) &\mapsto \Sigma^n X, \\ f : (X, x_0) \to (Y, y_0) &\mapsto \Sigma^n f : \Sigma^n X \to \Sigma^n Y. \end{aligned}$$

Since $\Sigma^n X$ *is homeomorphic to* $\mathbb{S}^n \wedge X$ *for all non-zero natural number* n*, and* $\mathbb{S}^0 \wedge X$ *is homeomorphic to* X*, we define the* ***0-reduced suspension functor*** *as the identity on* $\mathrm{TopHdCpt}_+$*. In addition, if* X *is an object in* $\mathrm{TopHdCpt}$ *and* $X_+ := X \sqcup \{\infty\}$ *where* ∞ *is an independent connected component in* X_+ *and is its marked point, then* ΣX_+ *is homeomorphic to the identification space*

$$\frac{X \times J}{(X \times \{1\}) \cup (X \times \{-1\})}.$$

Finally, the reader will find a simple but helpful visualization of these constructions in Figure D.4.

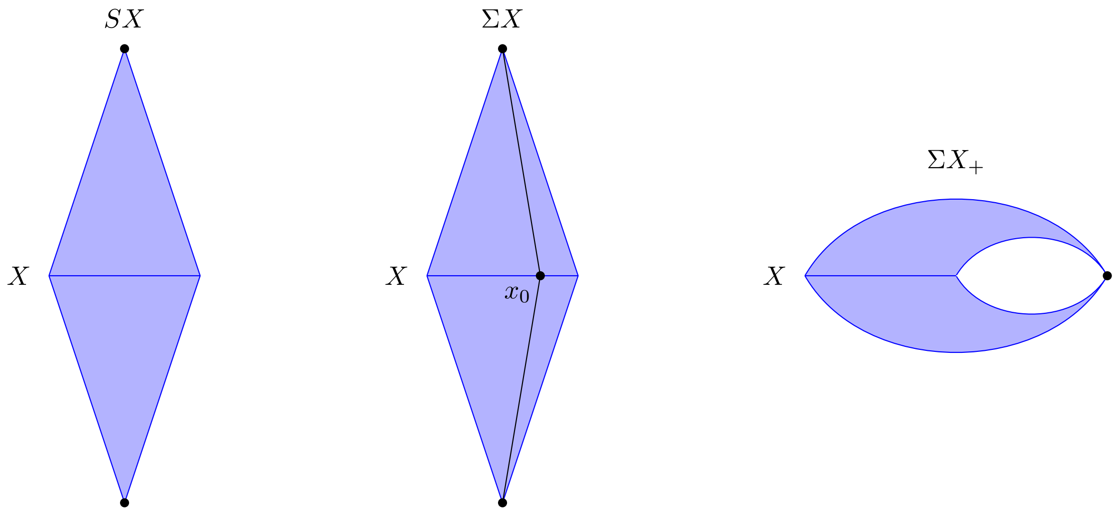

Figure D.4: In the images above, X is the interior blue line segment. On the left, we have the suspension of X, which coincides with the union of the cones CX and $C'X$. In the middle, considering $x_0 \in X$ to be a given marked point in X, the reduced suspension of (X, x_0) is obtained by collapsing the interior black line segment to the marked point $x_0 \in X$. Finally, on the right, we have the reduced suspension of $X_+ = X \sqcup \{\infty\}$ where ∞ is an independent connected component in X_+ and is its marked point. $\diamondsuit$

Appendix E

Real Division Algebras

In this appendix, we explain the elementary concepts of real division algebras. Moreover, we provide some historical notes on the main real division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. This is a way to understand the importance of the Bott-Milnor-Kervaire Theorem presented in Chapter 2, which was one of the first achievements of K-Theory. We finish our presentation with two classical results about these real division algebras which explain why they are relevant and, in a certain sense, unique. Our exposition is based on [5], [10], [27], [36] and [38].

E.1 First definitions and historical examples

Definition E.1 (Real algebra and real division algebra)**.** *Let $\mathscr{A}$ be a finite-dimensional real vector space and*

$$m : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$$

be a bilinear map, which we will hereafter call a ***multiplication*** *in $\mathscr{A}$. The pair $(\mathscr{A}, m)$ is said to be:*

- *a* ***real algebra*** *provided that there exists a non-zero element $1 \in \mathscr{A}$ such that $m(1, a) = m(a, 1) = a$ for all $a \in \mathscr{A}$; and*
- *a* ***real division algebra*** *provided that it is a real algebra in which there are no zero divisors. This means that, if $a, b \in \mathscr{A}$ are such that $m(a, b) = 0$, then necessarily either $a = 0$ or $b = 0$.*

We will say that $\mathscr{A}$ is a real (division) algebra, omitting its multiplication, and we will write ab instead of $m(a, b)$ for all $a, b \in \mathscr{A}$. $\diamondsuit$

Example E.2 (The real division algebra $\mathbb{R}$)**.** *Here we define the* ***real division algebra of the real numbers****, denoted by $\mathbb{R}$, and give some ideas about its historical development. Let 1 be the only vector of the canonical basis of the real Euclidean one-dimensional space. Then:*

- *as a vector space, $\mathbb{R}$ is the real Euclidean one-dimensional space, whose elements are real multiples of 1; and*
- *as a real division algebra, $\mathbb{R}$ has the multiplication coincident with its vector space scalar product.*

Historically, it is not an easy task to choose since when the real numbers are part of Mathematics. That is, when should we start telling the history of the real numbers? Is it appropriate to start:

- *in Prehistory with the cavemen and the counting of hunts and provisions?*
- *in the Ancient Egypt with the practical problems surrounding the plantings on the Nile margins?*
- *in the discovery of the irrational numbers by the Pythagoreans or even with Eudoxus and his work on incommensurability of quantities?*
- *in the European Middle Ages with the construction of a meaning for negative numbers as independent entities?*
- *in somewhere else in the history of eastern civilizations?*

When is it appropriate to start? That is not a simple question to be answered. In particular, note that trying to see the real numbers as the historical evolution of the naturals, integers, rationals and irrationals is not coherent with the timeline presented in the items before. In fact, for instance, the irrationals appear before the negative numbers. Thus, the classical pedagogical presentation of the numerical sets play no role in this discussion.

*Maybe, considering the nowadays stage of development of Mathematics, a plausible and direct answer to that question is the first formalization of the real numbers. Nevertheless, this is another problem: What is this first one? In [38] the reader can find a compilation of various formalization of the real numbers, which curiously does not begin by **Dedekind's construction** of 1872. Furthermore, this paper may help the reader to realize the idea behind a formalization of the real numbers, which is to extract an intuitive property of them and then set it as an axiom in order to derive their familiar properties.* ♢

Example E.3 (The real division algebra $\mathbb{C}$)**.** *Here we define the **real division algebra of the complex numbers**, denoted by $\mathbb{C}$, and give some ideas about its historical development. Let 1 and e_1 be the vectors of the canonical basis of the real Euclidean two-dimensional space. Then:*

- *as a vector space, $\mathbb{C}$ is the real Euclidean two-dimensional space, whose elements are linear combinations of the vectors of its canonical basis. Hence, for each $z \in \mathbb{C}$, there exist unique $\alpha, \alpha_1 \in \mathbb{R}$ such that*

$$z = \alpha + \alpha_1 e_1; \qquad \textit{and}$$

- *as a real division algebra, $\mathbb{C}$ has the multiplication bilinearly induced by the vector relation $e_1^2 = -1$.*

*Historically, the complex numbers appeared in the surroundings of the problem of explicitly solving a third degree polynomial equation. The mathematicians that are nowadays associated to this kind of equation are **Girolamo Cardano** (1501 - 1576) and **Niccolò Fontana** (1500 - 1557). This last one is usually known as **Tartaglia**, which means "stammerer" in Italian. This nickname is due to serious wounds in his jaw and palate, acquired during a French invasion against Venice, which left him with a speech impediment.*

*Nonetheless, the first man who solved the cubic equation was **Scipione del Ferro** (1465-1526), who was a professor at the Bologna University. After accomplishing his solution, he trusted the formula to a student of his called **Antonio Maria del Fiore***

(XVI-XVII). After some time, Fiore challenged Tartaglia to a mathematical contest, for which Tartaglia rediscovered del Ferro's formula. More than that, Tartaglia won the competition answering all the problems proposed by del Fiore, while this one could solve none of the problems suggested by Tartaglia. In turn, Tartaglia told his formula, without his proof, to Cardano, who then swore to secrecy. Having the formula, Cardano deduced its proof. After that, he found out that del Ferro had discovered the formula before Tartaglia. Then, he published it in his book Ars Magna (1545). It is important to note that Cardano mentioned del Ferro as first author and Tartaglia as an independent solver.

Probably, Cardano introduced the complex numbers in his book Ars Magna. Nevertheless, it is known that ***Rafael Bombelli*** *(1526 - 1572) was responsible for the current notation* $\sqrt{-1}$*, which he named "più di meno" at the time, while he was studying the application of Cardano-Tartaglia Formula to the equation* $x^3 = 15x + 4$*. Other men whose names appear in the history of complex numbers are* ***Leonhard Euler*** *(1707-1783),* ***Jean-Robert Argand*** *(1768-1822),* ***Carl Friedrich Gauss*** *(1777-1855) and* ***William Rowan Hamilton*** *(1805-1865). The interested reader can find more details in [27].* $\diamondsuit$

Example E.4 (The real division algebra $\mathbb{H}$)**.** *Here we define the* ***real division algebra of the quaternions****, denoted by* $\mathbb{H}$*, and give some ideas about its historical development. Let* 1*,* e_1*,* e_2 *and* e_3 *be the vectors of the canonical basis of the real Euclidean four-dimensional space. Then:*

- *as a vector space,* $\mathbb{H}$ *is the real Euclidean four-dimensional space, whose elements are linear combinations of the vectors of its canonical basis. Hence, for each* $q \in \mathbb{H}$*, there exist unique* $\alpha, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ *such that*

$$q = \alpha + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3; \qquad \textit{and}$$

- *as a real division algebra,* $\mathbb{H}$ *has the multiplication bilinearly induced by Table E.1, which can be easily deduced from the mnemonic diagram presented in Figure E.1.*

Historically, ***William Rowan Hamilton*** *(1805-1865) was responsible for the introduction of the quaternions in Mathematics. Interestingly, before developing the quaternions, he was involved with the complex numbers. In 1833, he completed his* ***Pair Theory****, which was understood at the time as a new algebraic representation for the complex numbers. Nowadays, Hamilton's formulation of the complex numbers is their definition in any first course. In fact, in his Pair Theory, Hamilton represented a complex number as an ordered pair* (a,b)*, where* a *and* b *are real numbers, and defined the sum operation*

$$(a,b)+(c,d) \;:=\; (a+c, b+d),$$

and the multiplication operation

$$(a,b)(c,d) \;:=\; (ac-bd, ad+bc).$$

As a natural step, Hamilton tried to extend the complex numbers to a new algebraic structure in which each element would be composed of one real part and two distinct imaginary parts. This idea would be known as his ***Triplets Theory****. Inspired by the way one represents rotations in the plane using complex numbers, Hamilton was carried into this search for his desire to represent rotations in the three-dimensional space in a similar manner. Indeed, much of his work after finding out the quaternions was to publicize them through the idea that they were intrinsically linked with Geometry and Physics.*

Nevertheless, Hamilton had failed to create a new algebra for more than ten years, until he found an answer on October 16th, 1843, while he walked with his wife, across the Royal Canal in Dublin, going to a meeting of the Royal Irish Academy. In that moment, he realized that he would need three imaginary parts instead of two. In fact, he noted that the three distinct imaginary parts, which he named i*,* j *and* k*, should verify the conditions*

$$i^2 \;=\; j^2 \;=\; k^2 \;=\; ijk \;=\; -1.$$

Then, he wrote his results on the stone of the Brougham Bridge, which we unfortunately cannot find today because of the action of time. The reader can find more details in

[10]. In turn, [36] contains interesting ideas involving differentiability of quaternionic functions, being a classical reference which complements some notions and questions that the reader will find in [10]. $\diamondsuit$

$\cdot$	$\mathbf{1}$	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$
$\mathbf{1}$	1	e_1	e_2	e_3
$\boldsymbol{e_1}$	e_1	-1	e_3	$-e_2$
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1
$\boldsymbol{e_3}$	e_3	e_2	$-e_1$	-1

Table E.1: This table describes the quaternionic multiplication of the vectors of the canonical basis. In fact, it describes the result of multiplying the bold element in its ith row by the bold element in its jth column.

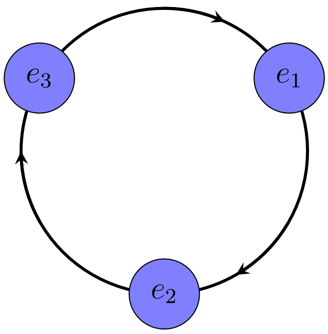

Figure E.1: The arrows in this circular diagram indicate the positive sign to obtain the third element from the product of the other ones. For example, $e_3e_1 = e_2$ and $e_1e_2 = e_3$. If we multiply two elements linked by an arrow in the opposite direction, then we have to put a minus sign in front of the third element. For instance, $e_3e_2 = -e_1$ and $e_2e_1 = -e_3$. Moreover, we have to remember that $e_1^2 = e_2^2 = e_3^2 = -1$ by definition. This allows us to deduce the equation $e_1e_2e_3 = -1$, which is also an important relation in the framework of the quaternions.

$\cdot$	$\mathbf{1}$	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	$\boldsymbol{e_4}$	$\boldsymbol{e_5}$	$\boldsymbol{e_6}$	$\boldsymbol{e_7}$
$\mathbf{1}$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\boldsymbol{e_1}$	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
$\boldsymbol{e_2}$	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
$\boldsymbol{e_3}$	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
$\boldsymbol{e_4}$	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
$\boldsymbol{e_5}$	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
$\boldsymbol{e_6}$	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
$\boldsymbol{e_7}$	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Table E.2: This table describes the octonionic multiplication of the vectors of the canonical basis. In fact, it describes the result of multiplying the bold element in its ith row by the bold element in its jth column.

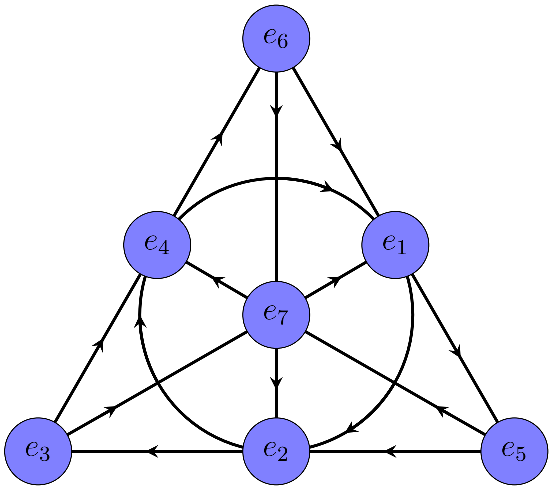

Figure E.2: The mathematical object that allowed the construction of this diagram is known as the **Fano Plane**, which was developed by **Gino Fano** (1871 - 1952). This is the **finite projective plane** with the least number of points and lines. Indeed, it has seven points and seven lines, with three points on each line and three lines through each point. We use the arrows in this diagram to indicate the positive sign to obtain the third element of each line from the product of the other ones. For example, $e_4e_6 = e_3$ and $e_7e_2 = e_6$. If we multiply two elements linked by an arrow in the opposite direction, then we have to put a minus sign in front of the third element. For instance, $e_1e_4 = -e_2$ and $e_1e_7 = -e_3$. Furthermore, we have to remember that $e_1^2 = e_2^2 = e_3^2 = e_4^2 = e_5^2 = e_6^2 = e_7^2 = -1$ by definition.

Example E.5 (The real division algebra $\mathbb{O}$)**.** *Here we define the* ***real division algebra of the octonions****, denoted by $\mathbb{O}$, and give some ideas about its historical development. Let 1, e_1, e_2, e_3, e_4, e_5, e_6 and e_7 be the vectors of the canonical basis of the real Euclidean eight-dimensional space. Then:*

- *as a vector space, $\mathbb{O}$ is the real Euclidean eight-dimensional space, whose elements are linear combinations of the vectors of its canonical basis. Hence, for each $r \in \mathbb{O}$, there exist unique $\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \in \mathbb{R}$ such that*

$$r = \alpha + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7; \qquad \textit{and}$$

- *as a real division algebra, $\mathbb{O}$ has the multiplication bilinearly induced by Table E.2, which can be easily deduced from the mnemonic diagram presented in Figure E.2.*

Historically, the octonions were first described by ***John Thomas Graves*** *(1806 - 1870), who was a Hamilton's friend since both attended together the Trinity*

College in Dublin. In fact, Graves' interest in algebra was particularly responsible for Hamilton's enterprise on the complex numbers and on the triplets. At the same day of his decisive walk across the Royal Canal, Hamilton sent a letter to Graves describing the quaternions. Graves answered greeting him by the boldness of his idea, adding that:

> *"There is still something in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties."*

Moreover, Graves asked "If with your alchemy you can make three pounds of gold, why should you stop there?"

On December 26th, 1843, Graves wrote to Hamilton a description of a new normed division algebra of eight dimensions, which he called ***octaves****. On January, 1844, Graves sent three letters to Hamilton expanding his discoveries. He even considered the idea of a* ***General Theory of*** $\mathbf{2^m}$***-ions*** *and tried to construct a normed division algebra of sixteen dimensions. On July, 1844, Hamilton answered Graves pinpointing that the octonions were non-associative. Indeed, Hamilton invented the term* ***associative*** *at that moment. Therefore, one can say that the octonions were essential to enlighten the notion of associativity in Algebra. Then, Hamilton offered himself to publicize Graves' discovery. However, since he was always engaged with the quaternions he had just created, Hamilton kept postponing such offering.*

In the meantime, the young ***Arthur Cayley*** *(1821 - 1895) was thinking on the quaternions since Hamilton announced their existence. On March, 1845, he published an article on the* ***Philosophical Magazine*** *entitled "On Jacobi's Elliptic Functions, in Reply to the Rev. B. Bronwin; and on Quaternions". In a significant part of this article, Cayley tried to refute another paper, in which the author pointed out errors in his work on elliptic functions. Apparently, Cayley gave a brief description of the octonions in this work. In fact, Cayley's article was so full of errors that it was omitted from his collected works, with the exception of the part in which he treated the octonions.*

Annoyed with being beaten to publication, Graves attached a postscript in one of his articles who would appear on the next edition of the Philosophical Magazine saying that he knew about the octonions since the Christmas of 1843. On June 14th, 1847, Hamilton *wrote a small note to the* ***Transactions of The Royal Irish Academy*** *alleging Graves' pioneerism. Nonetheless, it was too late, the octonions had already entered in history as* ***Cayley's numbers****. The reader can find more details in [5], which is also the main reference for the purposes of this appendix because it contains much information about the real (division) algebras.* ♢

Definition E.6 (Special kinds of real algebras)**.** *Let $\mathscr{A}$ be a real algebra. We say that it:*

- *is* ***commutative*** *if $ab = ba$ for all $a, b \in \mathscr{A}$;*
- *is* ***associative*** *if $(ab)c = a(bc)$ for all $a, b, c \in \mathscr{A}$;*
- *is* ***alternative***(1) *if $a(bb) = (ab)b$ and $(aa)b = a(ab)$ for all $a, b \in \mathscr{A}$. This is equivalent to the fact that every subalgebra of $\mathscr{A}$ generated by two elements is associative;*
- *is* ***normed*** *if it has a norm $|\cdot| : \mathscr{A} \to [0, \infty)$ in such manner that $|a|\,|b| = |ab|$ for all $a, b \in \mathscr{A}$; and*
- *has* ***multiplicative inverses*** *if, for every non-zero $a \in \mathscr{A}$, there exits $a^{-1} \in \mathscr{A}$ such that $aa^{-1} = a^{-1}a = 1$.* ♢

Example E.7 (On the algebras presented before)**.** *We have the following facts about the real algebras that we presented before.*

- *$\mathbb{R}$ is an associative and commutative real division algebra. The proofs of these assertions are consequences of the formalization which one chooses for the real numbers.*
- *$\mathbb{C}$ is an associative and commutative real division algebra. The reader can readily prove these claims.*

(1)This nomenclature follows from the fact that the *associator* $[\cdot,\cdot,\cdot] : \mathscr{A} \times \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ given by $[a, b, c] = (ab)c - a(bc)$ "alternates" in an alternative algebra. That is, the associator changes sign under an odd permutation of the letters a, b and c, but remains unchanged under an even permutation. At this point, the reader may have noted the parallel between the associator and the *commutator* $[\cdot,\cdot] : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ given by $[a, b] = ab - ba$, which is identically zero in a commutative algebra.

- $\mathbb{H}$ *is an associative and non-commutative real division algebra. In fact, it is non-commutative since* $e_1e_2 = e_3$ *and* $e_2e_1 = -e_3$. *We leave to the reader the straightforward computations which prove the associativity of the quaternions.*

- $\mathbb{O}$ *is an alternative, non-associative and non-commutative real division algebra. In fact, it is non-commutative because* $e_1e_2 = e_4$ *and* $e_2e_1 = -e_4$. *Moreover, it is non-associative because*

$$\begin{aligned} (e_1e_2)e_3 &= e_4e_3 = -e_6 \quad \textit{and} \\ e_1(e_2e_3) &= e_1e_5 = e_6. \end{aligned}$$

  Thus, note that the expression $e_1e_2e_3e_4e_5e_6e_7$ *in* $\mathbb{O}$, *which is analogous to the expression* $e_1e_2e_3 = -1$ *in* $\mathbb{H}$, *has no meaning. We leave to the reader the straightforward computations which prove the alternance of the octonions.*

All these four real division algebras have multiplicative inverses. Indeed, with the exception of the real numbers in which we have to prove the existence of multiplicative inverses by means of a formalization, all these proofs are again straightforward computations. Note that Tables E.1 and E.2 may help with the quaternions and the octonions, respectively. Furthermore, all these four real division algebras are normed with respect to the canonical Euclidean norm. ♢

Remark E.8 (On real algebras)**.** *We have the following instructive facts about generic real algebras.*

- *In a real algebra, the absence of zero divisors for the multiplication of* $\mathscr{A}$ *is equivalent to the operations of left and right multiplication by non-zero elements being invertible. Indeed, since* $\mathscr{A}$ *is a finite-dimensional vector space and these operations are linear maps, the Rank-Nullity Theorem says that we only have to prove their injectivities. We leave these proofs to reader.*

- *Every associative real algebra is an alternative real algebra. However, the converse is false as the octonions show in Example E.7.*

- *In a normed division algebra, we have* $|1| = 1$. *In fact, the result is obvious since we have the equality* $|1|^2 = |1|\,|1| = |1|$.

- *A normed real algebra $\mathscr{A}$ is necessarily a division algebra. Indeed, if $\mathscr{A}$ has zero divisors, then it cannot be a normed algebra. This happens because, if $a, b \in \mathscr{A}$ are zero divisors, then the norm of the product ab is zero despite the product of the norms of a and b being non-zero.*

- *An alternative real algebra with multiplicative inverses is a real division algebra. Moreover, an alternative and commutative real algebra has multiplicative inverses if and only if it is a real division algebra. On the other hand, there exist alternative and non-commutative real division algebras without multiplicative inverses. For example, if we only change Table E.1 declaring $e_1^2 = e_2 - 1$, then e_1 has no multiplicative inverse. Indeed, in this situation, we have that $e_3 - e_1$ and $-(e_1 + e_3)$ are left and right inverses for e_1, respectively. Thus, since this new quaternionic algebra is associative, we are done here.* $\diamondsuit$

E.2 Morphisms of real algebras

Definition E.9 (Homomorphisms of real algebras and of real division algebras)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be real (division) algebras. A linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is a **homomorphism of real (division) algebras** if $\varphi(1) = 1$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathscr{A}$. Furthermore:*

- *a **monomorphism of real (division) algebras** is an injective homomorphism of real (division) algebras;*

- *an **epimorphism of real (division) algebras** is a surjective homomorphism of real (division) algebras; and*

- *an **isomorphism of real (division) algebras** is an invertible homomorphism of real (division) algebras.* $\diamondsuit$

Remark E.10 (On homomorphisms of real algebras)**.** *We have the following facts about homomorphisms of real algebras.*

- *Let $\mathscr{A}$ be a real (division) algebra. The real division algebra of the real numbers* *is considered a subalgebra of* $\mathscr{A}$ *by means of the monomorphism of real (division) algebras*

$$\begin{aligned} \iota : \mathbb{R} &\to \mathscr{A}, \\ \alpha &\mapsto \alpha 1. \end{aligned}$$

- *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be real (division) algebras and* $\varphi : \mathscr{A} \to \mathscr{B}$ *be a linear map such that:*

  $$\varphi(ab) = \varphi(a)\varphi(b) \text{ for all } a, b \in \mathscr{A}.$$

  If φ *is surjective, then it is a homomorphism of real (division) algebras. Indeed, by hypothesis, for all* $b \in \mathscr{B}$*, there exists* $a \in \mathscr{A}$ *in such manner that* $\varphi(a) = b$*. Therefore,*

  $$\varphi(1)\, b = \varphi(1)\varphi(a) = \varphi(1a) = \varphi(a) = b.$$

  Analogously, $b\,\varphi(1) = b$*. From the uniqueness of the multiplicative identity,* $\varphi(1) = 1$*. This proves our claim.* ♢

Definition E.11 (Categories of real algebras and of real division algebras)**.** *We say that:*

- AlgR *is the* ***category of real algebras*** *whose objects are real algebras and whose morphisms are homomorphisms of real algebras; and*
- AlgDR *is the* ***category of real division algebras*** *whose objects are real division algebras and whose morphisms are homomorphisms of real division algebras.*

It is to be noted that AlgDR *is a full subcategory of* AlgR *since every homomorphism of real algebras between real division algebras is a homomorphism of real division algebras.* ♢

Definition E.12 (Anti-homomorphisms of real algebras and of real division algebras)**.** *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be real (division) algebras. A linear map* $\varphi : \mathscr{A} \to \mathscr{B}$ *is an* ***anti-homomorphism of real (division) algebras*** *if* $\varphi(1) = 1$ *and* $\varphi(ab) = \varphi(b)\varphi(a)$ *for all* $a, b \in \mathscr{A}$*. Furthermore:*

- *an* ***anti-monomorphism of real (division) algebras*** *is an injective anti-homomorphism of real (division) algebras;*

- *an* ***anti-epimorphism of real (division) algebras*** *is a surjective anti-homomorphism of real (division) algebras; and*

- *an* ***anti-isomorphism of real (division) algebras*** *is an invertible anti-homomorphism of real (division) algebras.* ♢

Remark E.13 (On anti-homomorphisms of real algebras)**.** *We have the following facts about anti-homomorphisms of real algebras.*

- *Let $\mathscr{A}$ and $\mathscr{B}$ be real (division) algebras and $\varphi : \mathscr{A} \to \mathscr{B}$ be a linear map. Then:*

  - *if φ is surjective and $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in \mathscr{A}$, then it is an anti-homomorphism of real (division) algebras. We leave the details to the reader, recommending a closer look into the arguments which we used in Remark E.10; and*

  - *if $\mathscr{B}$ is a commutative real (division) algebra, then $\varphi : \mathscr{A} \to \mathscr{B}$ is an anti-homomorphism of real (division) algebras if and only if it is a homomorphism of real (division) algebras.*

- *It is not possible to define a category of real (division) algebras whose morphisms are anti-homomorphisms of real (division) algebras. This happens because the composition of two anti-homomorphisms of real (division) algebras is a homomorphism of real (division) algebras. Nonetheless, we can define a category of real (division) algebras whose morphisms are anti-homomorphisms of real (division) algebras and homomorphisms of real (division) algebras, but this is not standard.* ♢

E.3 Cayley-Dickson algebras

Definition E.14 (Anti-involution and real star-algebra)**.** *We say that a* ***real (division) star-algebra*** *is a pair $(\mathscr{A}, *)$ in which:*

- *$\mathscr{A}$ is a real (division) algebra; and*

- $* : \mathscr{A} \to \mathscr{A}$ *is an* ***anti-involution of the real (division) algebra*** $\mathscr{A}$*, that is, an anti-isomorphism of the real (division) algebra* $\mathscr{A}$ *whose inverse coincides with itself.*

We will say that $\mathscr{A}$ *is a real (division) star-algebra, omitting its anti-involution, and we will write* a^* *instead of* $*(a)$ *for all* $a \in \mathscr{A}$*. Moreover, we will say that the real star-algebra* $\mathscr{A}$ *is* ***nicely normed*** *if the sum* $a + a^*$ *is a real multiple of* $1 \in \mathscr{A}$ *and the products* aa^* *and* a^*a*, which have two coincide, are a positive real multiple of* $1 \in \mathscr{A}$ *for all non-zero element* $a \in \mathscr{A}$*.* $\diamondsuit$

Example E.15 (The real division star-algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$)**.** *We have that:*

- $\mathbb{R}$ *is a real division star-algebra with respect to the anti-involution* $* : \mathbb{R} \to \mathbb{R}$ *given by* $\alpha^* = \alpha$*;*
- $\mathbb{C}$ *is a real division star-algebra with respect to the anti-involution* $* : \mathbb{C} \to \mathbb{C}$ *given by* $(\alpha + \alpha_1 e_1)^* = \alpha - \alpha_1 e_1$*;*
- $\mathbb{H}$ *is a real division star-algebra with respect to the anti-involution* $* : \mathbb{H} \to \mathbb{H}$ *given by* $(\alpha + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3)^* = \alpha - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3$*;*
- $\mathbb{O}$ *is a real division star-algebra with respect to the anti-involution* $* : \mathbb{O} \to \mathbb{O}$ *given by* $(\alpha + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7)^* = \alpha - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3 - \alpha_4 e_4 - \alpha_5 e_5 - \alpha_6 e_6 - \alpha_7 e_7$.

The reader can readily prove with straightforward computations that all these four real star-algebras are nicely normed. $\diamondsuit$

Remark E.16 (On nicely normed real star-algebras)**.** *We have the following facts about nicely normed real star-algebras.*

- *If* $\mathscr{A}$ *is a nicely normed real star-algebra, then it has multiplicative inverses. Indeed, it suffices to see that, for every non-zero element* $a \in \mathscr{A}$*, the inverse* a^{-1} *of* a *is given by*

$$a^{-1} = \frac{1}{aa^*}\, a^*.$$

- *If $\mathscr{A}$ is nicely normed and alternative, then it is a normed real algebra. In fact, we define the norm*

$$\begin{aligned} |\cdot| : \mathscr{A} &\rightarrow [0, \infty), \\ a &\mapsto \sqrt{aa^*}. \end{aligned}$$

We claim that $|a|\,|b| = |ab|$ for all $a, b \in \mathscr{A}$. Indeed, since $\mathscr{A}$ is alternative, we have

$$|ab|^2 \;=\; (ab)(ab)^* \;=\; ab(b^*a^*) \;=\; a(bb^*)a^* \;=\; aa^*\,|b|^2 \;=\; |a|^2\,|b|^2$$

for all $a, b \in \mathscr{A}$, which proves the assertion. $\diamondsuit$

Definition E.17 (Cayley-Dickson algebra of a real star-algebra)**.** *The* ***Cayley-Dickson algebra of a real star-algebra*** $(\mathscr{A}, *)$ *is said to be the real star-algebra* $\mathrm{CD}(\mathscr{A})$ *in such manner that:*

- *as a vector space,* $\mathrm{CD}(\mathscr{A})$ *is the direct sum $\mathscr{A} \oplus \mathscr{A}$;*
- *as a real algebra,* $\mathrm{CD}(\mathscr{A})$ *has the multiplication* $\mathrm{CD}(\mathscr{A}) \times \mathrm{CD}(\mathscr{A}) \rightarrow \mathrm{CD}(\mathscr{A})$ *given by $(a, b)(c, d) = (ac - db^*, a^*d + cb)$; and*
- *as a real star-algebra,* $\mathrm{CD}(\mathscr{A})$ *has the anti-involution* $* : \mathrm{CD}(\mathscr{A}) \rightarrow \mathrm{CD}(\mathscr{A})$ *given by $(a, b)^* = (a^*, -b)$.* $\diamondsuit$

Theorem E.18 (Relations between a real star-algebra and its Cayley-Dickson algebra)**.** *We have the following facts about a real star-algebra $\mathscr{A}$ and its Cayley-Dickson algebra* $\mathrm{CD}(\mathscr{A})$.

(1) $\mathscr{A}$ is nicely normed if and only if $\mathrm{CD}(\mathscr{A})$ *is nicely normed.*

(2) $\mathscr{A}$ is associative and nicely normed if and only if $\mathrm{CD}(\mathscr{A})$ *is alternative and nicely normed.*

Proof. These facts are proved by straightforward computations that we leave to the reader fulfill. $\square$

Example E.19 (The real algebra $\mathbb{S}$). *The* ***real algebra of the sedenions*** *is the Cayley-Dickson algebra* $\mathbb{S} := \mathrm{CD}(\mathbb{O})$. *More explicitly,* $\mathbb{S}$ *is the real algebra given by the sixteen-dimensional real Euclidean space equipped with the multiplication bilinearly induced by Table E.3. The sedenions are our first example of a real algebra with zero divisors. Indeed, for instance,*

$$\begin{aligned}(e_3 + e_{10})(e_6 - e_{15}) &= e_3e_6 - e_3e_{15} + e_{10}e_6 - e_{10}e_{15} \\ &= e_5 - e_{12} + e_{12} - e_5 \\ &= 0.\end{aligned}$$

The reader can find more examples of zero divisors in $\mathbb{S}$. *Therefore, not only the sedenions cannot be normed, but also Theorem E.18 implies that* $\mathbb{S}$ *is not an alternative algebra. In fact, since* $\mathbb{O}$ *and* $\mathbb{S} = \mathrm{CD}(\mathbb{O})$ *are nicely normed, we have that* $\mathbb{S}$ *is alternative if and only if* $\mathbb{O}$ *is associative. Thus, since* $\mathbb{O}$ *is non-associative, it follows that* $\mathbb{S}$ *is non-alternative.* $\diamondsuit$

Remark E.20 (The real division star-algebra of the real numbers generates an infinite family of real star-algebras through the Cayley-Dickson algebra construction). *We have*

$$\mathrm{CD}(\mathbb{R}) = \mathbb{C}, \qquad \mathrm{CD}(\mathbb{C}) = \mathbb{H} \qquad \textit{and} \qquad \mathrm{CD}(\mathbb{H}) = \mathbb{O}.$$

More than that, iteratively applying the Cayley-Dickson algebra from the real numbers, we obtain an infinite family of nicely normed real star-algebras, each of which has dimension equal to a power of two. An important fact is that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ *and* $\mathbb{O}$ *are the only normed alternative real division algebras of this family. Indeed, all the other algebras of this family, from the sedenions, have zero divisors and are non-alternative because they contain copies of this sixteen-dimensional algebra. In particular, having zero divisors, these algebras cannot be normed.* $\diamondsuit$

$\cdot$	$\mathbf{1}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$	$\mathbf{e_4}$	$\mathbf{e_5}$	$\mathbf{e_6}$	$\mathbf{e_7}$	$\mathbf{e_8}$	$\mathbf{e_9}$	$\mathbf{e_{10}}$	$\mathbf{e_{11}}$	$\mathbf{e_{12}}$	$\mathbf{e_{13}}$	$\mathbf{e_{14}}$	$\mathbf{e_{15}}$
$\mathbf{1}$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
$\mathbf{e_1}$	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
$\mathbf{e_2}$	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
$\mathbf{e_3}$	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
$\mathbf{e_4}$	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
$\mathbf{e_5}$	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
$\mathbf{e_6}$	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
$\mathbf{e_7}$	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
$\mathbf{e_8}$	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\mathbf{e_9}$	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
$\mathbf{e_{10}}$	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
$\mathbf{e_{11}}$	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
$\mathbf{e_{12}}$	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
$\mathbf{e_{13}}$	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
$\mathbf{e_{14}}$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
$\mathbf{e_{15}}$	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

Table E.3: This table describes the sedenionic multiplication of the vectors of the canonical basis. In fact, it describes the result of multiplying the bold element in its ith row by the bold element in its jth column.

E.4 Classical theorems

In Section 2.8, we exposed the following result due to **Raoul Bott** (1923-2005), **John Milnor** (1931-) and **Michel Kervaire** (1927-2007), whose proof uses Ordinary K-Theory.

Theorem E.21 (Bott-Milnor-Kervaire Theorem). *Every real division algebra has dimension* 1, 2, 4 *or* 8. □

This result was independently proved by Bott-Milnor and by Kervaire in 1958, according to [5, p. 150]. Moreover, as we mentioned before, the reader can find a detailed proof of it in [15, pp. 59-72]. When we look to the preceding section of this appendix, an interesting consequence of Bott-Milnor-Kervaire Theorem is that there is no way of changing the multiplication of the sedenions induced by Table E.3 to turn it into a division algebra. More generally, it is not possible to change the multiplication of the Cayley-Dickson algebras, starting from the sedenions, to turn them into division algebras. On the other hand, we have proved that there exist real division algebras in dimensions 1, 2, 4 and 8. Indeed, we have $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ in Examples E.2, E.3, E.4 and E.5, respectively. Nevertheless, these algebras are not the only real division algebras in these dimensions up to isomorphism (with the obvious exception of the real numbers). Indeed:

- in dimension 2, one can consider the hyperbolic complex numbers $\mathbb{C}_{\mathcal{H}}$ that are defined exactly as the complex numbers, but declaring $e_1^2 = 1$. The reader can promptly prove that there can be no isomorphism of real division algebras between $\mathbb{C}$ and $\mathbb{C}_{\mathcal{H}}$;
- in dimension 4, one can consider the quaternionic algebra defined in the last item of Remark E.8. This algebra cannot be isomorphic to $\mathbb{H}$ since its element e_1 has no inverse. In fact, an isomorphism of real division algebras has to map inverses into inverses; *and*
- in dimension 8, one can consider the Cayley-Dickson algebra of the quaternionic algebra of the preceding item. This cannot be isomorphic to $\mathbb{O}$ by the same reasoning presented above.

Therefore, we could ask if the historical division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are also special from a strictly mathematical viewpoint. Subsequently, we present two positive answers for this question.

Theorem E.22 (Zorn's Theorem)**.** *The only alternative real division algebras are* $\mathbb{R}$*,* $\mathbb{C}$*,* $\mathbb{H}$ *and* $\mathbb{O}$*.* $\square$

Theorem E.23 (Hurwitz's Theorem)**.** *The only normed real division algebras are* $\mathbb{R}$*,* $\mathbb{C}$*,* $\mathbb{H}$ *and* $\mathbb{O}$*.* $\square$

The first theorem was proved by **Max Zorn** (1906-1993) in a paper of 1930 that was correlated to his doctoral thesis. The reader can find a mention to Zorn's original work in [5, p. 150]. Moreover, the reader can find an interesting sketch of proof of Hurwitz's Theorem in [5, pp. 156-159]. It is to be noted that, although this result was first proved by **Adolf Hurwitz** (1859-1919) in a paper of 1898, the sketch presented in this reference is the one a modern proof that uses the ideas of Clifford algebras developed in Chapter 3. This gives us one more reason to study these objects, which are intrinsically linked to K-Theory (see Section 3.4).

Appendix F

Principal Bundles

In this appendix, we set the fundamental notion that one must know in order to understand the spin and spin^c structures that we deal with in the main text, which is the one of principal bundles. We only expose here the initial concepts and the results that play an essential role in our exposition. However, since the theory of principal bundles is, under a certain viewpoint, equivalent to the one of vector bundles, we introduce some notions that show this equivalence. The reader who fells the urge to deepen his or her knowledge in this interesting topic may find in [30, pp. 28-35] and [35, pp. 111-118] good references. Finally, it is to be noted that the notions presented here are mainly used in Chapter 3.

F.1 First definitions

Definition F.1 (Principal bundle)**.** *Let X be a connected topological space and G be a topological group. A **principal bundle** on X with **structure group** G is defined by the following data:*

- *a topological space P;*
- *a surjective continuous function $\pi : P \to X$; and*
- *a continuous right action of G on P such that $\pi(x \cdot g) = \pi(x)$ for all $x \in X$ and all $g \in G$,*

such that the following two conditions are satisfied.

(1) For every $x \in X$, there exists an open neighborhood U of x in X and a homeomorphism

$$\varphi : \pi^{-1}(U) \to U \times G$$

verifying the commutativity of the following diagram with $\varphi(\pi^{-1}(y)) = \{y\} \times G$ for every $y \in U$.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\pi"'] \arrow[rr, bend left, "\varphi"] & U & U \times G \arrow[l, "\pi_U"] \end{tikzcd}$$

(2) For every $y \in U$ and every $g, h \in G$, we have the compatibility condition $\varphi^{-1}(y, h) \cdot g = \varphi^{-1}(y, hg)$.

If X is not connected, then a principal bundle on X is defined by a principal bundle on each connected component of X. In this situation, the structure group depends on each connected component of X. ♢

Notation F.2 (On principal bundles)**.** *Henceforth, the notation of Definition F.1 will be used without explicit mention. In particular, we will denote a principal bundle with structure group G by $\pi : P \to X$. Moreover, we will often denote the whole bundle by P, for convenience.* ♢

Definition F.3 (Standard nomenclature in the framework of principal bundles)**.** *Let $\pi : P \to X$ be a principal bundle. We say that:*

- *for every $x \in X$, the topological space $\pi^{-1}(x)$ is the **fiber** of P in x, which is hereafter denoted by P_x;*
- *P and X are, respectively, the **total space** and the **base space** of the principal bundle $\pi : P \to X$;*
- *a **local chart** or **local trivialization** of P is a pair (U, φ_U) where:*
  - *$U \subseteq X$ is open; and*

- $\varphi_U : \pi^{-1}(U) \to U \times G$ *is a homeomorphism satisfying Conditions (1) and (2) of Definition F.1.*

Moreover, if $x \in U$*, then the local chart* (U, φ_U) *is also said to be a* ***local chart in x****; and*

- *an* ***atlas*** *of* P *is a family* $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$ *where:*
  - $\mathfrak{U} = \{U_i\}_{i\in I}$ *is an open cover of* X*; and*
  - (U_i, φ_i) *is a local chart of* P *for all* $i \in I$.

Note that the existence of an atlas of P *follows from Conditions (1) and (2) of Definition F.1.* ♢

Remark F.4 (On principal bundles and their right actions)**.** *Let* $\pi : P \to X$ *be a principal bundle. Because of Conditions (1) and (2) of Definition F.1, the* G*-orbit of* $p \in P_x$ *is the whole* P_x*. Moreover, we have that* P_x *is a* G*-torsor*[(1)]*. Hence, the projection* π *induces a homeomorphism between* P/G *and* X*. In particular, the* G*-action is free in the whole* P*.* ♢

Remark F.5 (Initial comparison between vector bundles and principal bundles)**.** *In a vector bundle, by definition, each fiber is a vector space, which is required to be isomorphic to the typical fiber. On the other hand, in a principal bundle, by definition, the fibers are not groups, but only torsors with respect to the structure group. In particular, we have the canonical embedding of the base of a vector bundle into its total space, which is given by the vanishing global section, but no embedding of the base of a principal bundle into its total space. This shows an asymmetry between these two notions. Nevertheless, vector bundles and principal bundles turn out to be symmetric from another viewpoint, which we will briefly describe in the end of this appendix.* ♢

[(1)] A G-torsor is, roughly speaking, a group that has forgotten its identity element. In fact, given any (non-empty) torsor with respect to a group G, we recover a group isomorphic to G by making what is known as a trivialization of the G-torsor, which roughly corresponds to choosing an identity element. More precisely, we say that a G-torsor is a non-empty set A together with a right action $\alpha : A \times G \to A$ of G such that the map $\pi_A \times \alpha : A \times G \to A \times A$ is an isomorphism, where $\pi_A : A \times G \to A$ is the natural projection onto the first factor. In addition, a trivialization of a G-torsor A is a bijection between A and the underlying set of G.

F.2 Morphisms and categories of principal bundles

Definition F.6 (Principal bundle morphisms)**.** *Let $\pi_P : P \to X$ and $\pi_Q : Q \to Y$ be principal bundles with structure groups G and H, respectively. We give the following definitions.*

- *A* ***principal bundle morphism*** *from P into Q is a pair (f, ρ), where $f : P \to Q$ is a continuous function and $\rho : G \to H$ is a topological group homomorphism, such that:*
  - *there exists a (unique) continuous function $g : X \to Y$ in such manner that $\pi_Q \circ f = g \circ \pi_P$; and*
  - *$f(p \cdot h) = f(p) \cdot \rho(h)$ for all $p \in P$ and all $h \in G$.*

  This means that the following diagram is commutative.

$$\begin{tikzcd}
P \times G \arrow[r, "f \times \rho"] \arrow[d, "\cdot"'] & Q \times H \arrow[d, "\cdot"] \\
P \arrow[r, "f"] \arrow[d, "\pi_P"'] & Q \arrow[d, "\pi_Q"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

- *If $X = Y$, then we say that a* ***principal bundle morphism over X*** *from P into Q is a principal bundle morphism $f : P \to Q$ in such manner that the induced function $g : X \to X$ is the identity map.*

- *If $G = H$, a* ***G-principal bundle morphism*** *is a morphism of the form (f, id_G). Moreover, if $X = Y$ and f induces $g = \mathrm{id}_X$, then we call it a* ***G-principal bundle morphism over X****.*

In all of these cases, we say that an invertible principal bundle morphism (over X) is a ***principal bundle isomorphism (over X)****.* ♢

Theorem F.7 (When principal bundle morphisms are principal bundle isomorphisms). *Let $\pi_P : P \to X$ and $\pi_Q : Q \to Y$ be principal bundles with structure groups G and H, respectively. Then:*

(1) if $(f : P \to Q,\ \rho : G \to H)$ is a principal bundle morphism, then it is a principal bundle isomorphism if and only if g is a homeomorphism and ρ is a group isomorphism;

(2) if $X = Y$ and $(f : P \to Q,\ \rho : G \to H)$ is a principal bundle morphism over X, then it is a principal bundle isomorphism if and only if ρ is a group isomorphism; and

(3) if $X = Y$, $G = H$ and $(f : P \to Q,\ \mathrm{id}_G : G \to G)$ is a G-principal bundle morphism over X, then it necessarily is a G-principal bundle isomorphism over X.

Proof. Evidently, we only have to prove that the first statement holds true. Indeed, note that, if (f, ρ) is a principal bundle isomorphism, then clearly g is a homeomorphism and ρ is a group isomorphism. Conversely, if g is a homeomorphism and ρ is a group isomorphism, then:

- *f is injective.* Let $p, q \in P$ be such that $f(p) = f(q)$. Since $g \circ \pi_P = \pi_Q \circ f$, it follows that $(g \circ \pi_P)(p) = (g \circ \pi_P)(q)$. Hence, once g is a homeomorphism, $\pi_P(p) = \pi_P(q)$. This last equation implies the existence of $h \in G$ for which $q = p \cdot h$. Moreover,

$$f(p) = f(q) = f(p \cdot h) = f(p) \cdot \rho(h)$$

  implies $\rho(h) = 1_H$. Therefore, since ρ is an isomorphism, $h = 1_G$. Consequently, we have $p = q$. This proves that f is injective.

- *f is surjective.* Let $q \in Q$. Since π_Q is surjective, there exists $y \in Y$ for which $q \in Q_y$. Moreover, since g is a homeomorphism, there exists a unique $x \in X$ such that $g(x) = y$. Thus, we have $q \in Q_{g(x)}$. Hence, for any $p \in P_x$, we have $f(p) = q \cdot h$ for a suitable $h \in H$. Consequently, once $\rho^{-1}(h^{-1}) \in G$ is well-defined because ρ is an isomorphism, we have

$$f(p \cdot \rho^{-1}(h^{-1})) \;=\; f(p) \cdot \rho(\rho^{-1}(h^{-1})) \;=\; f(p) \cdot h^{-1} \;=\; (q \cdot h) \cdot h^{-1} \;=\; q.$$

This proves that f is surjective.

- (f^{-1}, ρ^{-1}) *is a principal bundle morphism.* By hypothesis, $\rho^{-1} : H \to G$ is a topological group homomorphism. Moreover, the reader can readily prove that $f^{-1}(q \cdot h) = f^{-1}(q) \cdot \rho^{-1}(h)$ for all $q \in Q$ and all $h \in H$. Thus, it only remains to show that $f^{-1} : Q \to P$ is continuous. Choosing local charts (U, φ) and $(g(U), \psi)$ of P and Q, respectively, we define the function $\alpha : U \to H$ by means of the equality
$$f(\varphi^{-1}(x, 1_G)) \;=\; \psi^{-1}(g(x), \alpha(x)).$$
Obviously, α is continuous because it is the composition of $\psi \circ f \circ \varphi^{-1}(\cdot, 1_G)$ with the natural projection onto the second factor. For all $h \in G$, it follows from the preceding equation that
$$f(\varphi^{-1}(x, h)) \;=\; \psi^{-1}(g(x), \alpha(x)\rho(h)).$$
Therefore, for all $h \in H$, we have
$$f^{-1}(\psi^{-1}(g(x), h)) \;=\; \varphi^{-1}(x, \rho^{-1}(\alpha(x)^{-1}h)).$$
This immediately implies that the composition $f^{-1} \circ \psi^{-1} \circ (g \times 1_H)$ is continuous. Consequently, we have that f^{-1} is also continuous because ψ^{-1} and $g \times 1_H$ are homeomorphisms.

This finishes the proof of the theorem. □

Definition F.8 (Categories of principal bundles)**.** *We say that:*

- PrincBdl *is the* ***category of principal bundles*** *whose objects are principal bundles and whose morphisms are principal bundle morphisms;*
- PrincBdl(G) *is the* ***category of principal bundles with fixed structure group G*** *whose objects are principal bundles with structure group G and whose morphisms are G-principal bundle morphisms;*

- $\mathrm{PrincBdl}_X$ *is the* ***category of principal bundles on X*** *whose objects are principal bundles on X and whose morphisms are principal bundle morphisms over X; and*

- $\mathrm{PrincBdl}_X(G)$ *is the* ***category of principal bundles on X with fixed structure group G*** *whose objects are principal bundles on X with structure group G and whose morphisms are G-principal bundle morphisms over X. Note that this category if a* ***groupoid*** *since its morphisms are always isomorphisms by Theorem F.7.* ♢

Remark F.9 (On the categories of principal bundles)**.** *We have the following diagram of categories indicating the inclusion relations between* PrincBdl*,* PrincBdl(G)*,* $\mathrm{PrincBdl}_X$ *and* $\mathrm{PrincBdl}_X(G)$*.*

$$\begin{tikzcd}
\mathrm{PrincBdl}(G) \arrow[r] & \mathrm{PrincBdl} \\
\mathrm{PrincBdl}_X(G) \arrow[r] \arrow[u] & \mathrm{PrincBdl}_X \arrow[u]
\end{tikzcd}$$

Differently from the diagram in Remark C.7, the horizontal arrows of this diagram are not full. This happens because, fixing the structure group G, the topological group homomorphism $\rho : G \to G$ of any principal bundle morphism (f, ρ) is obliged to be the identity map. ♢

Definition F.10 (Sets of equivalence classes of principal bundles)**.** *We say that:*

- Princ *is the quotient of the class of objects of* PrincBdl *by its equivalence relation of isomorphism of principal bundles. In other words,* Princ *is the* ***set of isomorphism classes of principal bundles****; and*

- Princ_X *is the quotient of the class of objects of* $\mathrm{PrincBdl}_X$ *by its equivalence relation of isomorphism of principal bundles on X. In other words,* Princ_X *is the* ***set of isomorphism classes of principal bundles on X****.*

The sets of isomorphism classes of principal bundles $\mathrm{Princ}(G)$ *and* $\mathrm{Princ}_X(G)$ *are defined in a similar manner.* $\diamondsuit$

F.3 Trivial bundles and restrictions

Definition F.11 (Product and trivial principal bundles)**.** *Let* X *be a connected topological space. We say that:*

- *the* ***product principal bundle with structure group G*** *is the projection onto the first factor* $\pi : X \times G \to X$ *equipped with the trivial action*

$$(x, h) \cdot k \;:=\; (x, hk)$$

  for all $(x, h) \in X \times G$ *and all* $k \in G$*; and*

- *a principal bundle* $\pi : P \to X$ *with structure group* G *is* ***trivial*** *if it is isomorphic over* X *and as a* G*-principal bundle to the product bundle* $X \times G$*. In this situation, an isomorphism from* P *onto the product bundle is said to be a* ***trivialization*** *of* P*.* $\diamondsuit$

Definition F.12 (Restriction of a principal bundle)**.** *Let* $\pi : P \to X$ *be a principal bundle. Given a topological subspace* $Y \subseteq X$*, the* ***restriction*** *of* P *to* Y*, which is hereafter denoted by* $P\,|_Y$*, is the principal bundle* $\pi\,|_{\pi^{-1}(Y)}\colon \pi^{-1}(Y) \to Y$ *with the induced* G*-action on each fiber on* Y*.* $\diamondsuit$

Remark F.13 (On the restriction of principal bundles)**.** *Let* $\pi : P \to X$ *be a principal bundle and* Y *be a topological subspace of* X*. Then:*

- *the restriction* $P\,|_Y$ *is a principal bundle because we can verify Conditions (1) and (2) of Definition F.1 by restricting a local chart* (U, φ_U) *of* P *to the local chart* $(U \cap Y,\ \varphi_U\,|_{\pi^{-1}(U \cap Y)})$ *of* $P\,|_Y$*; and*

- *if* (U, φ_U) *is a local chart of* P*, then* $\varphi_U : \pi^{-1}(U) \to U \times G$ *is a principal bundle isomorphism over* U *between* $P\,|_U$ *and the product bundle. Therefore, a principal bundle is* ***locally trivial*** *by definition.* $\diamondsuit$

Remark F.14 (Covariant functor defined by the restriction of principal bundles). *Let X be a topological space and Y be a subspace of X. Then, we have the following covariant functor*

$$\begin{aligned} |_Y\colon \operatorname{PrincBdl}_X &\to \operatorname{PrincBdl}_Y, \\ P &\mapsto P\,|_Y, \\ (f : P \to Q,\ \rho : G \to H) &\mapsto (f\,|_Y\colon P\,|_Y \to Q\,|_Y,\ \rho : G \to H), \end{aligned}$$

where $f\,|_Y$ is the natural map that sends $a \in P\,|_Y$ into $f(a) \in Q\,|_Y$. The reader can readily prove that this map is not only well-defined but also is a principal bundle morphism over Y. To complete these details, we recommend a closer look at the arguments used in Remark C.12. ♢

Definition F.15 (Common trivializing open cover for a family of principal bundles on the same base space). *Let X be a topological space and $\Pi = \{\pi_\alpha : P_\alpha \to X\}_{\alpha\in\Lambda}$ be a family of principal bundles on X. A* ***common trivializing open cover*** *of X for Π is an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X in such manner that $P_\alpha\,|_{U_i}$ is trivial for all $\alpha \in \Lambda$ and all $i \in I$.* ♢

Remark F.16 (Existence of common trivializing open covers for finite families of principal bundles). *Let X be a topological space. Using induction, the reader can readily prove that there exists a common trivializing open cover of X for any finite number of principal bundles. To complete the details, we recommend a closer look at the arguments used in Remark C.14.* ♢

F.4 Sections of principal bundles

Notation F.17 (On real and complex numbers). *When we do not desire to distinguish between the field of real numbers and the field of complex numbers, we shall write $\mathbb{K}$ to symbolize them.* ♢

Definition F.18 (Global and local sections of a principal bundle). *Let $\pi : P \to X$ be a principal bundle. We say that:*

- *a* ***(global) section*** *of P is a continuous function $s : X \to P$ in such manner that $\pi \circ s = \mathrm{id}_X$. The* ***set of sections*** *of P, which is naturally a torsor with respect to the group of continuous functions $\mathcal{C}^0(X, G)$, is hereafter denoted by $\Gamma(P)$; and*

- *if $U \subseteq X$ is open, then a global section $s : U \to P|_U$ of the restriction $P|_U$ is said to be a* ***local section*** *of P. Moreover, if $x \in U$, then s is also called a* ***local section in x****.* ♢

Theorem F.19 (Local charts induce bijections between the set of local sections and the group of continuous functions)**.** *Let $\pi : P \to X$ be a principal bundle. If (U, φ_U) is a local chart of P, then it induces a bijection between $\Gamma(P|_U)$ and $\mathcal{C}^0(U, G)$. Moreover, this bijection is a $\mathcal{C}^0(U, G)$-torsor isomorphism.*

Proof. The proof of this result is similar to the one of Theorem C.17. The reader can fulfill the details. □

The following result gives a characterization of trivial principal bundles through their global sections. Indeed, it says that a principal bundle is trivial if and only if it admits a global section. This shows one more discrepancy between vector bundles and principal bundles. In fact, as one can readily see comparing Theorems C.20 and F.20, the behavior of global and local sections in these frameworks are radically different.

Theorem F.20 (Equivalence between triviality of a principal bundle and the existence of a global section)**.** *Let $\pi : P \to X$ be a principal bundle with structure group G. Then, P is trivial if and only if there exists a global section of P. Furthermore, there exists a canonical bijection between the set of trivializations of P and the set of its global sections.*

Proof. If $f : P \to X \times G$ is a trivialization of P, then

$$\begin{aligned} s : X &\to P, \\ x &\mapsto f^{-1}(x, 1_G), \end{aligned}$$

is a global section of P. Moreover, if $s : X \to P$ is a global section of P, then we obtain the trivialization

$$\begin{aligned} f : P &\to X \times G, \\ s(x) \cdot g &\mapsto (x, g). \end{aligned}$$

The reader can readily prove that these assignments are inverse to each other, being canonical bijections between the set of trivializations of P and the set of its global sections. □

Remark F.21 (Another interpretation of local triviality of principal bundles)**.** *We have seen in Remark F.13 that, given a principal bundle P, a choice of a local chart (U, φ_U) is equivalent to a choice of a trivialization of $P\mid_U$. Hence, because of Theorem F.20, it is equivalent to a choice of a local section $s : U \to P$, which is obviously a global section of $P\mid_U$.* ♢

F.5 Subbundles of principal bundles

Definition F.22 (Subbundle of a principal bundle)**.** *Let $\pi : P \to X$ be a principal bundle and H be a subgroup of the structure group G. We say that a **principal subbundle** Q of P with structure group H is an H-principal bundle of the form $\pi\mid_Q : Q \to X$, where Q is a topological subspace of P and the action of H on Q is the restriction of the action of H on P.*

Remark F.23 (On subbundles of principal bundles)**.** *Let $\pi : P \to X$ be a principal bundle and Q be a subbundle of P. Note that:*

- *if $h \in H$ and we consider its G-action on P, then $Q \cdot h \subseteq Q$. Therefore, since Q is obviously a subset of $Q \cdot h$, we have $Q = Q \cdot h$; and*

- *the inclusion $(i, j) : (Q, H) \to (P, G)$ is a morphism of principal bundles over X. In fact, i is continuous since it is the inclusion of a topological subspace, j is the inclusion of a topological subgroup by definition and $i(q \cdot h) = i(q) \cdot j(h)$ for all $q \in Q$ and all $h \in H$.*

We also observe that, when we restrict a principal bundle to Y, we are only considering the fibers over the points of Y, but we take the whole fiber in each point. On the other hand, considering a subbundle of a principal bundle, we restrict each fiber to a topological subspace with the group action being the one induced by the restriction, but in the whole X. Evidently, we can apply both operations at the same time, considering the restriction of a subbundle. ♢

The next result of this section enlightens subbundles of principal bundles. Indeed, it shows a correspondence between subbundles and local sections of the main principal bundle.

Theorem F.24 (Subbundles and local sections of principal bundles). *Let $\pi : P \to X$ be a principal bundle and H be a subgroup of the structure group G. If Q is a topological subspace of P such that $Q \cdot h \subseteq Q$ for all $h \in H$, then $\pi \mid_Q : Q \to X$, where Q is endowed with the induced topology and Q_x is endowed with the induced action of G for all $x \in X$, is a principal subbundle of P if and only if, for every $x \in X$, there exists an open neighborhood U of x in X and a local section $s \in \Gamma(P \mid_U)$ such that $s(y) \in Q$ for every $y \in U$.*

Proof. ($\Rightarrow$). Since Q is an H-principal bundle, it admits a local chart in every $x \in X$. Note that such a chart is equivalent to a local section $s \in \Gamma(Q \mid_U)$. Thus, once Q is endowed with the induced topology, it follows that $s : U \to P$ is continuous. Moreover, since the projection $\pi \mid_Q : Q \to X$ is the restriction of $\pi : P \to X$, it follows that $\pi \circ s = \mathrm{id}_U$. Hence, $s \in \Gamma(P \mid_U)$ is such that $s(y) \in Q$ for every $y \in U$. ($\Leftarrow$). For every $x \in X$, by hypothesis, there exists an open neighborhood U of x in X and a local section $s \in \Gamma(P \mid_U)$ such that $s(y) \in Q$ for every $y \in U$. This section defines a local chart $\varphi : (\pi \mid_Q)^{-1}(U) \to U \times H$ in $x \in X$. Therefore, Q is a principal subbundle of P, as we wished. □

F.6 Transition functions of principal bundles

Remark F.25 (Relation between the local charts of a principal bundle endowed with an atlas). *Let $\pi : P \to X$ be a principal bundle endowed with an atlas*

$\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$. *In addition, let* (U_i, φ_i) *and* (U_j, φ_j) *be any local charts of* $\Phi_{\mathfrak{U}}$ *such that* $U_{ij} := U_i \cap U_j$ *is nonempty. Then, consider* $\varphi_i : \pi^{-1}(U_i) \to U_i \times G$ *and* $\varphi_j : \pi^{-1}(U_j) \to U_j \times G$. *Fixing* $x \in U_{ij}$, *if* $p \in P_x$ *is such that* $\varphi_i(p) = (x, 1_G)$, *then its corresponding representation in* U_j *is of the form* $\varphi_j(p) = (x, g_{ij}(x))$ *with* $g_{ij}(x) \in G$. *Moreover, given any other point* $q \in P_x$, *there exists a unique* $g \in G$ *such that* $q = p \cdot g$. *Therefore,*

$$\varphi_i(q) \;=\; \varphi_i(p) \cdot g \;=\; (x, 1_G) \cdot g \;=\; (x, g).$$

Analogously, we have $\varphi_j(q) = (x, g_{ij}(x) \cdot g)$. *This means that the transition function is given by left multiplication by a fixed* $g_{ij}(x) \in G$ *for every* $x \in U_{ij}$. *This fact allows us to set the following definition.* ♢

Definition F.26 (Transition functions of a principal bundle)**.** *Let* $\pi : P \to X$ *be a principal bundle endowed with an atlas* $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$. *If* $U_{ij} := U_i \cap U_j$ *is nonempty, then the* ***transition function*** *of* P *from* U_i *to* U_j *is given by*

$$\begin{aligned} g_{ij} : U_{ij} &\;\to\; G, \\ x &\;\mapsto\; \pi_G(\varphi_j \circ \varphi_i^{-1}(x, 1_G)), \end{aligned}$$

where $\pi_G : U_{ij} \times G \to G$ *is the natural projection onto the second factor. Moreover, it is immediate to verify that the transition functions satisfy the following condition, called the* ***cocycle condition****:*

$$g_{jk}\,|_{U_{ijk}}(x) \cdot g_{ij}\,|_{U_{ijk}}(x) \;=\; g_{ik}\,|_{U_{ijk}}(x),$$

for all $x \in U_{ijk} := U_i \cap U_j \cap U_k$. *In particular,* $g_{ii}(x) = 1_G$ *for all* $x \in U_i$ *and* $g_{ij}(x) = g_{ji}(x)^{-1}$ *for all* $x \in U_{ij}$. *We will frequently omit the point* x *in the preceding formulas, admitting that whenever appears a product it is happening in the topological group* G. ♢

Definition F.27 (First degree nonabelian Čech cohomology of $\underline{G}$)**.** *Let* X *be a topological space and* $\mathfrak{U} = \{U_i\}_{i\in I}$ *be one of its open covers. Being* G *a topological group, we set*

$$\check{Z}^1(\mathfrak{U}, \underline{G}) \;:=\; \big\{\{g_{ij} : U_{ij} \to G\}_{i,j\in I} \;:\; g_{jk}\,|_{U_{ijk}} \cdot \;\; g_{ij}\,|_{U_{ijk}} = \; g_{ik}\,|_{U_{ijk}}\;\big\}.$$

We introduce in $\check{Z}^1(\mathfrak{U}, \underline{G})$ *the relation defined as follows: two of its families* $\{g_{ij}\}_{i,j\in I}$ *and* $\{h_{ij}\}_{i,j\in I}$ *are related if and only if there exists a family* $\{\eta_i : U_i \to G\}_{i\in I}$ *in such manner that*

$$h_{ij}(x) \;=\; \eta_j(x) \,\cdot\, g_{ij}(x) \,\cdot\, \eta_i(x)^{-1}$$

for all $x \in U_{ij}$ *and all* $i, j \in I$*. The reader can readily prove that this is an equivalence relation on* $\check{Z}^1(\mathfrak{U}, \underline{G})$*. We set* $\check{H}^1(\mathfrak{U}, \underline{G})$ *as the quotient of* $\check{Z}^1(\mathfrak{U}, \underline{G})$ *by this equivalence relation.* ♢

Remark F.28 (On the first degree nonabelian Čech cohomology of $\underline{G}$)**.** *Let* $\pi : P \to X$ *be a principal bundle endowed with an atlas* $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$*. Being* $\{g_{ij}\}_{i,j\in I}$ *the set of transition functions of* P*, Definition F.26 ensures that the equivalence class*

$$[\{g_{ij}\}_{i,j\in I}] \;\in\; \check{H}^1(\mathfrak{U}, \underline{G})$$

is well-defined. Furthermore, the reader can readily prove that it does not depend on the homeomorphisms of $\Phi_{\mathfrak{U}}$*. Therefore, the class* $[\{g_{ij}\}_{i,j\in I}]$ *only depends on the isomorphism class of* P *among the principal bundles that are trivial on each element of the open cover* $\mathfrak{U} = \{U_i\}_{i\in I}$ *of* X*. More than that, one can prove that an equivalence class of transition functions in* $\check{H}^1(\mathfrak{U}, \underline{G})$ *determines a unique up to isomorphism principal bundle with structure group* G *that is trivial on each element of the open cover in question. Furthermore, we obtain the direct limit*

$$\check{H}^1(X, \underline{G}) \;:=\; \varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \underline{G}).$$

The interesting fact is that one can prove that there exists a bijection between $\mathrm{Princ}_X(G)$ *and* $\check{H}^1(X, \underline{G})$*.* ♢

Remark F.29 (On the geometric interpretation of the first degree nonabelian Čech cohomology of $\underline{G}$)**.** *Repeating the same reasoning developed in Remark C.30, we conclude that:*

- *fixing an open cover* $\mathfrak{U} = \{U_i\}_{i\in I}$ *of* X*, a family* $\{g_{ij}\}_{i,j\in I} \in \check{Z}^1(\mathfrak{U}, \underline{G})$

corresponds geometrically to a G-principal bundle $\pi : P \to X$ *endowed with a family* $\{s_i\}_{i \in I}$*, up to G-isomorphism respecting the local sections through pullback (or pushforward);*

- *fixing an open cover* $\mathfrak{U} = \{U_i\}_{i\in I}$ *of* X*, a class* $[\{g_{ij}\}_{i,j\in I}] \in \check{H}^1(\mathfrak{U}, \underline{G})$ *corresponds geometrically to a G-principal bundle* $\pi : P \to X$ *such that* $P\,|_{U_i}$ *is trivial for every* $i \in I$*, up to G-isomorphism; and*
- *a class* $[[\{g_{ij}\}_{i,j\in I}]] \in \check{H}^1(X, \underline{G})$ *corresponds geometrically to a G-principal bundle* $\pi : P \to X$*, up to G-isomorphism.* ♢

F.7 Frame bundle

In this section, we show that the transition functions of a vector bundle with typical fiber $\mathscr{V}$ can also be thought of as the ones of a principal bundle with structure group $\mathrm{GL}(\mathscr{V})$. This idea is largely expected since the product coincides with the composition in $\mathrm{GL}(\mathscr{V})$. We start the formalization of this reasoning in the following definition, where we first consider vector bundles equipped with atlases. This approach enlightens the structure of the desired principal bundle from the one of the initial vector bundle.

Definition F.30 (Frame bundle of a vector bundle equipped with an atlas)**.** *Let* X *be a topological space and* E *be a vector bundle on* X *with typical fiber* $\mathscr{V}$ *equipped with an atlas* $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$*. In addition, let* $\{\varphi_{ij} : U_{ij} \to \mathrm{GL}(\mathscr{V})\}_{i,j\in I}$ *be the set of transition functions of* E *with respect to* $\Phi_{\mathfrak{U}}$*. Furthermore, consider the disjoint union*

$$D^{\mathfrak{U}}_{\mathrm{GL}(\mathscr{V})} := \bigsqcup_{i\in I} U_i \times \mathrm{GL}(\mathscr{V}).$$

If $x \in U_{ij}$ *and* $\varphi \in \mathrm{GL}(\mathscr{V})$*, then we denote by* $(x,\varphi)_i$ *the pair* $(x,\varphi) \in U_i \times \mathrm{GL}(\mathscr{V})$ *and by* $(x,\varphi)_j$ *the pair* $(x,\varphi) \in U_j \times \mathrm{GL}(\mathscr{V})$*. We define* $\mathrm{GL}(E,\Phi_{\mathfrak{U}})$ *as the quotient of* $D^{\mathfrak{U}}_{\mathrm{GL}(\mathscr{V})}$ *by the equivalence relation that identifies* $(x,\varphi)_i$ *with* $(x, (\varphi_{ij})_x \circ \varphi)_j$ *for all* $(x,\varphi) \in U_{ij} \times \mathrm{GL}(\mathscr{V})$ *and all* $i,j \in I$*. The* ***frame bundle of*** $\boldsymbol{E}$ ***relative to*** $\boldsymbol{\Phi_{\mathfrak{U}}}$ *is the GL*$(\mathscr{V})$*-principal bundle*

$$\begin{aligned}\pi : \mathrm{GL}(E, \Phi_{\mathfrak{U}}) &\to X, \\ [(x, \varphi)_i] &\mapsto x,\end{aligned}$$

whose continuous $\mathrm{GL}(\mathscr{V})$*-right action on* $\mathrm{GL}(E, \Phi_{\mathfrak{U}})$ *is given by*

$$\begin{aligned}\cdot : \mathrm{GL}(E, \Phi_{\mathfrak{U}}) \times \mathrm{GL}(\mathscr{V}) &\to \mathrm{GL}(E, \Phi_{\mathfrak{U}}), \\ ([(x, \varphi)_i],\ \psi) &\mapsto [(x, \varphi \circ \psi)_i].\end{aligned}$$ ♢

With respect to the preceding reasoning, the only inconvenient of the principal bundle obtained in the definition above is that it apparently depends on the atlas that comes together with the vector bundle. Nevertheless, there is no such dependence since we give below an equivalent intrinsic definition of the frame bundle, independent of any atlas.

Remark F.31 (Vector bundle of linear isomorphisms)**.** *Let X be a topological space. It is immediate from Definition C.46 that, given vector bundles E and F on X with the same typical fiber $\mathscr{V}$, it is defined their morphism bundle* $\textsc{Hom}(E, F)$ *with typical fiber* $\mathrm{End}(\mathscr{V})$*. Moreover,* $\textsc{Hom}(E, F)_x$ *is the vector space of linear maps* $\mathrm{Hom}(E_x, F_x)$ *for all $x \in X$. Now, let us consider* $\textsc{Iso}(E, F)$ *to be topological subspace of* $\textsc{Hom}(E, F)$ *defined by*

$$\textsc{Iso}(E, F)_x \ := \ \mathrm{Iso}(E_x, F_x)$$

for all $x \in X$. In general, this is not a vector subbundle of $\textsc{Hom}(E, F)$*, being only a fiber subbundle*(2)*. More than that,* $\textsc{Iso}(E, F)$ *has a natural* $\mathrm{GL}(\mathscr{V})$*-right action if and only if $E = X \times \mathscr{V}$. This can be seen using the approach of transition functions presented in Remark C.47. Indeed, this shows that the transition functions of* $\textsc{Hom}(E, F)$ *and* $\textsc{Iso}(E, F)$ *are given by*

$$(\sigma_{ij})_x(\varphi) \ = \ (\psi_{ij})_x \circ \varphi \circ (\varphi_{ij})_x^{-1}, \tag{F.1}$$

(2)In fact, vector bundles and principal bundles are enriched cases of *fiber bundles*. In this work, fiber bundles will not play an important role by themselves. However, as the reader will see shortly in the same paragraph of this footnote, it is important to know that all the elementary notions considered here are also defined to this broader concept. Therefore, for instance, we have categories of fiber bundles, morphisms of fiber bundles, sections of fiber bundles, transition functions of fiber bundles, *et reliqua*. The reader who fells the urge to deepen his or her knowledge in this interesting topic may find in [18, pp. 11-23, 61-66] a good reference.

which are not coordinate changes of a principal bundle unless $(\varphi_{ij})_x = \mathrm{id}_{\mathscr{V}}$*. In this case, we have the right-action* $f_x \cdot \varphi := f_x \circ \varphi$ *for all* $x \in X$*,* $\varphi \in \mathrm{GL}(\mathscr{V})$ *and all* $f_x \in \mathrm{ISO}(X \times \mathscr{V}, F)_x$*.* ♢

Definition F.32 (Frame bundle of a vector bundle)**.** *Let* X *be a topological space and* E *be a vector bundle with typical fiber* $\mathscr{V}$*. We say that the* ***frame bundle of E*** *is the* $\mathrm{GL}(\mathscr{V})$*-principal bundle* $\pi_{\mathrm{GL}(E)} : \mathrm{GL}(E) \to X$ *where*

$$\mathrm{GL}(E) := \mathrm{ISO}(X \times \mathscr{V}, E)$$

and the $\mathrm{GL}(\mathscr{V})$*-right action on* $\mathrm{GL}(E)$ *is given by* $f_x \cdot \varphi := f_x \circ \varphi$ *for all* $x \in X$*,* $\varphi \in \mathrm{GL}(\mathscr{V})$ *and all* $f_x \in \mathrm{GL}(E)_x$*.* ♢

Theorem F.33 (Equivalence between the frame bundles from Definitions F.30 and F.32)**.** *Let* X *be a topological space and* E *be a vector bundle on* X *with typical fiber* $\mathscr{V}$ *equipped with an atlas* $\Phi_{\mathfrak{U}} = \{(U_i, \psi_i)\}_{i \in I}$*. There exists a canonical isomorphism over* X *between the frame bundle* $\mathrm{GL}(E, \Phi_{\mathfrak{U}})$ *of Definition F.30 and the frame bundle* $\mathrm{GL}(E)$ *of Definition F.32.*

Proof. For every $x \in X$, let $i \in I$ be such that $x \in U_i$ and let $f_x \in \mathrm{GL}(E)_x$. Then, we define the isomorphism

$$\begin{aligned} \Phi : \mathrm{GL}(E) &\to \mathrm{GL}(E, \Phi_{\mathfrak{U}}), \\ f_x &\mapsto [(x, (\psi_i)_x \circ f_x)_i]. \end{aligned}$$

It suffices to prove that Φ is well-defined since, in this situation, it is clearly invertible because $(\psi_i)_x$ is invertible for all $x \in U_i$. Indeed, if we choose $j \in I$ for which $x \in U_{ij}$, then, using Equation (F.1) with $\varphi_{ij} = \mathrm{id}_{\mathscr{V}}$, we have

$$\begin{aligned} [(x, (\psi_j)_x \circ f_x)_j] &= [(x, (\sigma_{ij})_x((\psi_i)_x \circ f_x))_j] \\ &= [(x, (\psi_{ij})_x \circ (\psi_i)_x \circ f_x)_j] \\ &= [(x, (\psi_i)_x \circ f_x)_i]. \end{aligned}$$

This finishes the proof of the theorem. □

F.8 Associated bundle

In this section, we invert the reasoning of the preceding one. Indeed, we show that the transition functions of a principal bundle with structure group $\mathrm{GL}(\mathscr{V})$ can also be thought of as the ones of a vector bundle with typical fiber $\mathscr{V}$. Furthermore, we show that, fixing a topological representation $\rho : G \to \mathrm{GL}(\mathscr{V})$ where $\mathscr{V}$ is finite-dimensional vector space, the transition functions of a principal bundle with structure group G can also be thought of as the ones of a vector bundle with typical fiber $\mathscr{V}$. We begin the formalization of this reasoning in the following definition, where we first consider principal bundles equipped with atlases. This approach enlightens the structure of the desired vector bundle from the one of the initial principal bundle.

Definition F.34 (Associated bundle of a principal bundle equipped with an atlas)**.** *Let X be a topological space and P be a principal bundle on X with structure group $\mathrm{GL}(\mathscr{V})$ equipped with an atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$. In addition, let $\{\varphi_{ij} : U_{ij} \to \mathrm{GL}(\mathscr{V})\}_{i,j\in I}$ be the set of transition functions of P with respect to $\Phi_{\mathfrak{U}}$. Furthermore, consider the disjoint union*

$$D^{\mathfrak{U}}_{\mathscr{V}} := \bigsqcup_{i\in I} U_i \times \mathscr{V}.$$

If $x \in U_{ij}$ and $v \in \mathscr{V}$, then we denote by $(x, v)_i$ the pair $(x, v) \in U_i \times \mathscr{V}$ and by $(x, v)_j$ the pair $(x, v) \in U_j \times \mathscr{V}$. We define $\mathcal{E}(P, \Phi_{\mathfrak{U}})$ as the quotient of $D^{\mathfrak{U}}_{\mathscr{V}}$ by the equivalence relation that identifies $(x, v)_i$ with $(x, (\varphi_{ij})_x(v))_j$ for all $(x, v) \in U_{ij} \times \mathscr{V}$ and all $i, j \in I$. The ***associated bundle of P relative to $\Phi_{\mathfrak{U}}$*** *is the vector bundle with typical fiber $\mathscr{V}$*

$$\begin{aligned} \pi : \mathcal{E}(P, \Phi_{\mathfrak{U}}) &\to X, \\ [(x,\, v)_i] &\mapsto x, \end{aligned}$$

whose natural finite-dimensional vector space structure induced by $\mathscr{V}$ in each fiber is given by

$$\begin{aligned} [(x,\, v)_i] + [(x,\, w)_i] &:= [(x,\, v + w)_i] \quad \textit{and} \\ \lambda[(x,\, v)_i] &:= [(x,\, \lambda v)_i] \end{aligned}$$

$\diamondsuit$

Once and again, with respect to the preceding reasoning, the only inconvenient of the vector bundle obtained in the definition above is that it apparently depends on the atlas that comes together with the principal bundle. Nevertheless, there is no such dependence since we give below an equivalent intrinsic definition of the associated bundle, independent of any atlas.

Remark F.35 (On trivializations of the fibers of principal bundles)**.** *Let X be a topological space and P be a principal bundle on X with structure group $\mathrm{GL}(\mathscr{V})$. Fixing $x \in X$, consider the fiber P_x of P. According to Remark F.4, fixing a point $p_0 \in P_x$ is equivalent to fix a trivialization $\varphi_x : P_x \to \mathrm{GL}(\mathscr{V})$, $p_0 \cdot g \mapsto g$, of the fiber P_x. Thus, $p_0 \in P_x$ induces the function*

$$\begin{array}{rcl} \xi_x : \ P_x \times \mathscr{V} & \to & \mathscr{V}, \\ (p_0 \cdot g,\, v) & \mapsto & g(v). \end{array}$$

This function is clearly surjective since

$$\xi_x(p_0, v) \;=\; \xi_x(p_0 \cdot \mathrm{id}_{\mathscr{V}},\, v) \;=\; \mathrm{id}_{\mathscr{V}}(v) \;=\; v.$$

Nevertheless, in general, it is not injective. In fact, let us verify when $\xi_x(p, v) = \xi_x(q, w)$. We have $q = p \cdot g$ and $p = p_0 \cdot g_0$ for unique $g, g_0 \in \mathrm{GL}(\mathscr{V})$. Hence, we have to check when

$$g_0(v) \;=\; \xi_x(p_0 \cdot g_0,\, v) \;=\; \xi_x(p_0 \cdot g_0 g,\, w) \;=\; g_0 g(w).$$

Since $g, g_0 \in \mathrm{GL}(\mathscr{V})$, this last equation is equivalent to $w = g^{-1}(v)$. Therefore, we have $\xi_x(p, v) = \xi_x(p \cdot g, g^{-1}(v))$. For this reason, we introduce in $P_x \times \mathscr{V}$ the equivalence relation that identifies (p, v) with $(p \cdot g, g^{-1}(v))$ for all $p \in P_x$ and all $g \in \mathrm{GL}(\mathscr{V})$. Then, considering $\mathcal{E}(P)_x$ to be the quotient of $P_x \times \mathscr{V}$ by this equivalence relation, $p_0 \in P_x$ induces the homeomorphism

$$\begin{array}{rcl} \eta_x : \ \mathcal{E}(P)_x & \to & \mathscr{V}, \\ [(p_0,\, v)] & \mapsto & v. \end{array}$$

This homeomorphism depends on $p_0 \in P_x$. Nonetheless, once the equivalence relation on $\mathcal{E}(P)_x$ does not have this dependence on $p_0 \in P_x$, we have that $\mathcal{E}(P)_x$ is well-defined *starting from P. Moreover, $\mathcal{E}(P)_x$ admits a natural finite-dimensional vector space structure given by*

$$\begin{aligned} [(p,\, v)] + [(p,\, w)] &:= [(p,\, v+w)] \quad \textit{and} \\ \lambda[(p,\, v)] &:= [(p,\, \lambda v)]. \end{aligned}$$

These facts allow us to set the following definition. ◇

Definition F.36 (Associated bundle of a $\mathrm{GL}(\mathscr{V})$-principal bundle)**.** *Let X be a topological space and $\pi : P \to X$ be a principal bundle with structure group $\mathrm{GL}(\mathscr{V})$. We define $\mathcal{E}(P)$ as the quotient of $P \times \mathscr{V}$ by the equivalence relation that identifies (p, v) with $(p \cdot g, g^{-1}(v))$ for all $p \in P$ and all $g \in \mathrm{GL}(\mathscr{V})$. We say that the* ***associated bundle of P*** *is the vector bundle with typical fiber $\mathscr{V}$*

$$\begin{aligned} \pi_{\mathcal{E}(P)} :\ \mathcal{E}(P) &\to X, \\ [(p,\, v)] &\mapsto \pi(p), \end{aligned}$$

whose natural finite-dimensional vector space structure induced by $\mathscr{V}$ in each fiber is given by

$$\begin{aligned} [(p,\, v)] + [(p,\, w)] &:= [(p,\, v+w)] \quad \textit{and} \\ \lambda[(p,\, v)] &:= [(p,\, \lambda v)]. \end{aligned}$$

◇

Theorem F.37 (Equivalence between the frame bundles from Definitions F.34 and F.36)**.** *Let X be a topological space and P be a principal bundle on X with structure group $\mathrm{GL}(\mathscr{V})$ equipped with an atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i \in I}$. There exists a canonical isomorphism over X between the associated bundle $\mathcal{E}(P, \Phi_{\mathfrak{U}})$ of Definition F.34 and the associated bundle $\mathcal{E}(P)$ of Definition F.36.*

Proof. For every $x \in X$, let $i \in I$ be such that $x \in U_i$ and let $[(p, v)] \in \mathcal{E}(P)_x$. We have $[(p, v)] = [(\varphi^{-1}(x, 1_G), v_i)]$ for a unique $v_i \in \mathscr{V}$. Then, we define the isomorphism

$$\begin{aligned} \Phi :\ \mathcal{E}(P) &\to \mathcal{E}(P, \Phi_{\mathfrak{U}}), \\ [(p,\, v)] &\mapsto [(x,\, v_i)_i]. \end{aligned}$$

It suffices to prove that Φ is well-defined since, in this situation, it is clearly invertible because the one-to-one correspondece between v and v_i. Indeed, if we choose $j \in I$ for which $x \in U_{ij}$, then

$$\begin{aligned}[(p,\, v)] &= [(\varphi_i^{-1}(x,\, 1_G),\, v_i)]\\ &= [(\varphi_j^{-1}(x,\, g_{ij}(x)),\, v_i)]\\ &= [(\varphi_j^{-1}(x,\, 1_G),\, g_{ij}(x)(v_i))].\end{aligned}$$

Consequently, we have $v_j = g_{ij}(x)(v_i)$. Thus, it follows $[(x, v_j)_j] = [(x, v_i)_i]$. This finishes the proof of the theorem. □

Definition F.38 (Associated bundle of a principal bundle)**.** *Let X be a topological space, $\pi : P \to X$ be a G-principal bundle and $\rho : G \to \mathrm{GL}(\mathscr{V})$ be a topological representation of G where $\mathscr{V}$ is a finite-dimensional vector space. We define $\mathcal{E}_\rho(P)$ as the quotient of $P \times \mathscr{V}$ by the equivalence relation that identifies (p, v) with $(p \cdot g,\, \rho(g)^{-1}(v))$ for all $p \in P$ and all $g \in \mathrm{GL}(\mathscr{V})$. We say that the* ***$\rho$-associated bundle of P*** *is the vector bundle with typical fiber $\mathscr{V}$*

$$\begin{aligned}\pi_{\mathcal{E}_\rho(P)} : \mathcal{E}_\rho(P) &\to X,\\ [(p,\, v)] &\mapsto \pi(p),\end{aligned}$$

whose natural finite-dimensional vector space structure induced by $\mathscr{V}$ in each fiber is given by

$$\begin{aligned}[(p,\, v)] + [(p,\, w)] &:= [(p,\, v + w)] \quad \text{and}\\ \lambda[(p,\, v)] &:= [(p,\, \lambda v)].\end{aligned}$$

◊

Remark F.39 (On the associated bundle of Definition F.38)**.** *Let X be a topological space, $\pi : P \to X$ be a G-principal bundle and $\rho : G \to \mathrm{GL}(\mathscr{V})$ be a topological representation of G where $\mathscr{V}$ is a finite-dimensional vector space. We have the following facts about the associated bundle of the preceding definition.*

- *We have that, if $G = \mathrm{GL}(\mathscr{V})$ and $\rho = \mathrm{id}_{\mathrm{GL}(\mathscr{V})}$, then $\mathcal{E}_{\mathrm{id}_{\mathrm{GL}(\mathscr{V})}}(P)$ is equal to $\mathcal{E}(P)$. This proves that Definition F.38 is a generalization of Definition F.36, as was naturally expected.*

- *One can prove that $\mathcal{E}_\rho(P)$ only depends on the homotopy class of the representation $\rho : G \to \mathrm{GL}(\mathscr{V})$. This means that, if $\rho, \sigma : G \to \mathrm{GL}(\mathscr{V})$ are homotopic maps, then $\mathcal{E}_\rho(P)$ is isomorphic to $\mathcal{E}_\sigma(P)$ over X. Therefore, in particular, if G is contractible, then there exists only one $\mathcal{E}_\rho(P)$ up to isomorphism.* $\diamondsuit$

F.9 Equivalence with vector bundles

In this section, we establish an equivalence between vector and principal bundles. This will show that, although these two concepts have their discrepancies, as we have seen along this appendix, the ample similarities that the reader may have noted in their definitions and in the commentaries above are not pure chance. In fact, we put together the frame and the associated bundles defined before to show the equivalence between vector and principal bundles. Roughly speaking, we will show that these constructions produces natural equivalences between the category of principal bundles and the category of vector bundles. However, we will have to consider a restriction on the morphisms of vector bundles in order to this reasoning work. We begin with the following remark.

Remark F.40 (The associated bundle of the frame bundle of a vector bundle)**.** *Let X be a topological space and E be a vector bundle on X with typical fiber $\mathscr{V}$ equipped with an atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i\in I}$. The frame bundle* $\mathrm{GL}(E, \Phi_{\mathfrak{U}})$ *is defined from the transition functions of E induced by $\Phi_{\mathfrak{U}}$, as in Definition F.30. This* $\mathrm{GL}(\mathscr{V})$*-principal bundle is endowed with an atlas $\Psi_{\mathfrak{U}}$ with the same transition functions as the ones of $\Phi_{\mathfrak{U}}$. Similarly, $\mathcal{E}(\mathrm{GL}(E, \Phi_{\mathfrak{U}}), \Psi_{\mathfrak{U}})$ is defined by the transition functions of* $\mathrm{GL}(E, \Phi_{\mathfrak{U}})$ *induced by $\Psi_{\mathfrak{U}}$, as in Definition F.34. This vector bundle with typical fiber $\mathscr{V}$ is endowed with an atlas $\Sigma_{\mathfrak{U}}$ with the same transition functions as the ones of $\Psi_{\mathfrak{U}}$, which coincide with the ones of $\Phi_{\mathfrak{U}}$. Thus, E and $\mathcal{E}(\mathrm{GL}(E, \Phi_{\mathfrak{U}}), \Psi_{\mathfrak{U}})$ are canonically isomorphic over X, being one isomorphism*

$$\begin{aligned} \Theta_E : E &\to \mathcal{E}(\mathrm{GL}(E, \Phi_{\mathfrak{U}}), \Psi_{\mathfrak{U}}), \\ e_x &\mapsto [(x,\, (\varphi_i)_x(e_x))_i], \end{aligned}$$

where $i \in I$ is such that $x \in U_i$. We can also set a similar correspondence considering the intrinsic Definitions F.32 and F.36. Indeed, since $\mathcal{E}(\mathrm{GL}(E))_x$ is the quotient of $\mathrm{Iso}(\mathscr{V}, E_x) \times \mathscr{V}$ *by the equivalence relation that identifies (φ_x, v) with $(\varphi_x \circ \psi, \psi^{-1}(v))$ for every $\psi \in \mathrm{GL}(\mathscr{V})$, we have the canonical isomorphism*

$$\begin{aligned} \Theta_E : \mathcal{E}(\mathrm{GL}(E)) &\to E, \\ [(\varphi_x, v)] &\mapsto \varphi_x(v). \end{aligned} \qquad \diamondsuit$$

Remark F.41 (The frame bundle of the associated bundle of a principal bundle)**.** *Let X be a topological space. As in Remark F.40, given a $\mathrm{GL}(\mathscr{V})$-principal bundle P on X equipped with an atlas $\Phi_{\mathfrak{U}} = \{(U_i, \varphi_i)\}_{i \in I}$, we have that $\mathcal{E}(P, \Phi_{\mathfrak{U}})$ is naturally endowed with an atlas $\Psi_{\mathfrak{U}}$ with the same transition functions as the ones of $\Phi_{\mathfrak{U}}$. Thus, we have the canonical isomorphism*

$$\begin{aligned} \Xi_P : P &\to \mathrm{GL}(\mathcal{E}(P, \Phi_{\mathfrak{U}}), \Psi_{\mathfrak{U}}), \\ p_x &\mapsto [(x, (\varphi_i)_x(p_x))_i], \end{aligned}$$

where $i \in I$ is such that $x \in U_i$. Analogously, we can set a similar correspondence considering the intrinsic definitions. Indeed, since $\mathrm{GL}(\mathcal{E}(P))_x$ coincides with $\mathrm{Iso}(\mathscr{V}, \mathcal{E}(P)_x)$*, where $\mathcal{E}(P)_x$ is the quotient of $P_x \times \mathscr{V}$ by the equivalence relation that identifies (p_x, v) with $(p \cdot g, g^{-1}(v))$ for all $g \in \mathrm{GL}(\mathscr{V})$, we have the canonical isomorphism*

$$\begin{aligned} \Xi_P : P &\to \mathrm{GL}(\mathcal{E}(P)), \\ p_x &\mapsto (v \mapsto [(p_x, v)]). \end{aligned} \qquad \diamondsuit$$

The preceding remarks suggest that there exists an equivalence between the category of vector bundles on X with typical fiber $\mathscr{V}$ and the category of principal bundles on X with structure group $\mathrm{GL}(\mathscr{V})$. This is not the case since a morphism of $\mathrm{GL}(\mathscr{V})$-principal bundles over X is necessarily an isomorphism (see Theorem F.7), while there exist non-invertible morphisms between vector bundles on X with typical fiber $\mathscr{V}$. Categorically, this means that $\mathrm{PrincBdl}_X(\mathrm{GL}(\mathscr{V}))$ is a groupoid while $\mathrm{VectBdl}_X(\mathscr{V})$ is not. Nevertheless, we can consider the non-full subcategory $\mathrm{VectBdlIso}_X(\mathscr{V})$ of $\mathrm{VectBdl}_X(\mathscr{V})$ whose objects are all vector bundles, and whose morphisms are only the isomorphisms. By construction, $\mathrm{VectBdlIso}_X(\mathscr{V})$ is a groupoid. Moreover, we have the following result.

Theorem F.42 (Equivalence between vector and principal bundles). *Consider the following covariant functor, which is called the* ***frame functor****:*

$$\begin{aligned} \mathrm{GL} : \mathrm{VectBdlIso}_X(\mathscr{V}) &\to \mathrm{PrincBdl}_X(\mathrm{GL}(\mathscr{V})), \\ E &\mapsto \mathrm{GL}(E), \\ \varphi : E \to F &\mapsto \mathrm{GL}(\varphi) : \mathrm{GL}(E) \to \mathrm{GL}(F), \end{aligned}$$

where

$$\mathrm{GL}(\varphi)(\psi_x) := \varphi_x \circ \psi_x$$

for all $\psi_x \in \mathrm{GL}(E)_x$ and all $x \in X$. Furthermore, consider the following covariant functor, which is called the ***associated functor****:*

$$\begin{aligned} \mathcal{E} : \mathrm{PrincBdl}_X(\mathrm{GL}(\mathscr{V})) &\to \mathrm{VectBdlIso}_X(\mathscr{V}), \\ P &\mapsto \mathcal{E}(P), \\ \varphi : P \to Q &\mapsto \mathcal{E}(\varphi) : \mathcal{E}(P) \to \mathcal{E}(Q), \end{aligned}$$

where

$$\mathcal{E}(\varphi)[(p_x, v)] := [(\varphi_x(p_x), v)]$$

for all $[(p_x, v)] \in \mathcal{E}(P)_x$ and all $x \in X$. These functors are equivalences of groupoids inverse to each other.

Proof. We have to prove that $\mathcal{E} \circ \mathrm{GL} : \mathrm{VectBdlIso}_x(\mathscr{V}) \to \mathrm{VectBdlIso}_X(\mathscr{V})$ is naturally isomorphic to the covariant identity functor $\mathrm{Id}_{\mathrm{VectBdlIso}_X(\mathscr{V})}$, and that $\mathrm{GL} \circ \mathcal{E} : \mathrm{PrincBdl}_X(\mathrm{GL}(\mathscr{V})) \to \mathrm{PrincBdl}_X(\mathrm{GL}(\mathscr{V}))$ is naturally isomorphic to the covariant identity functor $\mathrm{Id}_{\mathrm{PrincBdl}_X(\mathrm{GL}(\mathscr{V}))}$. This means that we have to exhibit families of natural isomorphisms $\Theta = \{\Theta_E : (\mathcal{E} \circ \mathrm{GL})(E) \to E\}_{E \in \mathrm{VectBdlIso}_X(\mathscr{V})}$ and $\Xi = \{\Xi_P : P \to (\mathrm{GL} \circ \mathcal{E})(P)\}_{P \in \mathrm{PrincBdl}_X(\mathrm{GL}(\mathscr{V}))}$ in such manner that the following square diagrams are commutative for all $\varphi \in \mathrm{Iso}(E, F)$ and all $\psi \in \mathrm{Hom}(P, Q)$.

$$
\begin{tikzcd}
(\mathcal{E}\circ \mathrm{GL})(E) \arrow[r, "\Theta_E"] \arrow[d, "(\mathcal{E}\circ\mathrm{GL})(\varphi)"'] & E \arrow[d, "\varphi"] \\
(\mathcal{E}\circ \mathrm{GL})(F) \arrow[r, "\Theta_F"'] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "\Xi_P"] \arrow[d, "\psi"'] & (\mathrm{GL}\circ \mathcal{E})(P) \arrow[d, "(\mathrm{GL}\circ\mathcal{E})(\psi)"] \\
Q \arrow[r, "\Xi_Q"'] & (\mathrm{GL}\circ \mathcal{E})(Q)
\end{tikzcd}
$$

We claim that the isomorphisms Θ_E and Ξ_P defined at the end of Remarks F.40 and F.41, respectively, are such that the preceding square diagrams are commutative. Indeed:

- with respect to the first diagram, we have

$$
\begin{aligned}
(\mathcal{E}\circ \mathrm{GL})(\varphi) : (\mathcal{E}\circ \mathrm{GL})(E) &\to (\mathcal{E}\circ \mathrm{GL})(F), \\
[(\psi_x,\, v)] &\mapsto [(\varphi_x \circ \psi_x,\, v)],
\end{aligned}
$$

  where $\psi_x : \mathscr{V} \to E_x$. Therefore,

$$
(\Theta_F \circ (\mathcal{E}\circ \mathrm{GL})(\varphi))[(\psi_x,\, v)] \;=\; \Theta_F[(\varphi_x \circ \psi_x,\, v)] \;=\; (\varphi_x \circ \psi_x)(v).
$$

  Moreover,

$$
(\varphi \circ \Theta_E)[(\psi_x,\, v)] \;=\; \varphi(\psi_x(v)) \;=\; (\varphi_x \circ \psi_x)(v).
$$

  Hence, the diagram commutes.

- with respect to the second diagram, $(\mathrm{GL}\circ\mathcal{E})(\psi) : (\mathrm{GL}\circ\mathcal{E})(P) \to (\mathrm{GL}\circ\mathcal{E})(Q)$ is given as follows. Let us fix $\varphi_x \in (\mathrm{GL}\circ\mathcal{E})(P)$. This means that we fix $\varphi_x : \mathscr{V} \to \mathcal{E}(P)_x$. Thus,

$$
(\mathrm{GL}\circ\mathcal{E})(\psi)(\varphi_x) \;=\; \mathcal{E}(\psi)_x \circ \varphi_x.
$$

  Therefore, if $\varphi_x(v) = [(q_x, w)]$, then

$$
(\mathcal{E}(\psi)_x \circ \varphi_x)(v) \;=\; \mathcal{E}(\psi)_x[(q_x,\, w)] \;=\; [(\psi_x(q_x),\, w)].
$$

Consequently, let $p_x \in P$. Applying Ξ_P, we obtain the morphism φ_x that sends p_x into the map $v \mapsto [(p_x, v)]$. Then, applying $(\mathrm{GL} \circ \mathcal{E})(\psi)$, we obtain the morphism $v \mapsto [(\psi_x(p_x), v)]$. This coincides with $(\Xi_Q \circ \psi)(p_x)$. Hence, the diagram commutes.

This finishes the proof of the theorem. $\square$

Remark F.43 (Generalizations of Theorem F.42)**.** *In the preceding result, we proved that the covariant functors in Diagram* (F.2) *are equivalences of groupoids inverse to each other.*

$$\begin{tikzcd} \mathrm{VectBdlIso}_X(\mathscr{V}) \arrow[rr, bend left, "\mathrm{GL}"] & & \mathrm{PrincBdl}_X(\mathrm{GL}(\mathscr{V})) \arrow[ll, bend left, "\mathcal{E}"] \end{tikzcd} \tag{F.2}$$

This idea can be generalized removing the fixed base space, allowing it to vary, as in Diagram (F.3)*. In this case,* VectBdlMaps($\mathscr{V}$) *is the non-full subcategory of* VectBdl($\mathscr{V}$) *whose:*

- *objects are all vector bundles; and*
- *morphisms are* ***vector-bundle maps****. We say that a vector-bundle map between $\pi_E : E \to X$ and $\pi_F : F \to Y$ is a vector bundle morphism $f : E \to F$, which covers a continuous function $g : X \to Y$, such that $f_x : E_x \to F_{g(x)}$ is an isomorphism for all $x \in X$.*

$$\begin{tikzcd} \mathrm{VectBdlMaps}(\mathscr{V}) \arrow[rr, bend left, "\mathrm{GL}"] & & \mathrm{PrincBdl}(\mathrm{GL}(\mathscr{V})) \arrow[ll, bend left, "\mathcal{E}"] \end{tikzcd} \tag{F.3}$$

It is to be noted that neither VectBdlMaps($\mathscr{V}$) *nor* PrincBdl(GL($\mathscr{V}$)) *are groupoids. Moreover, the covariant functors* GL *and* $\mathcal{E}$ *in Diagram* (F.3) *are defined as in*

Theorem F.42. However, we remark that, since $\mathrm{GL}(E) = \textsc{Iso}(X \times \mathscr{V}, E)$ *and* $\mathrm{GL}(F) = \textsc{Iso}(Y \times \mathscr{V}, F)$*, we have* $\varphi_x \circ \psi_x \in \mathrm{GL}(F)_{\varphi'(x)}$ *where* $\varphi' : X \to Y$ *is the functions covered by* φ*. Similarly, we have to keep in mind that in this new context we have* $\varphi_x(p_x) \in Q_{\varphi'(x)}$*.*

$$\begin{tikzcd} \mathrm{VectBdlMapsPairs} \arrow[r, bend left, "\mathrm{GL}"] & \mathrm{PrincBdlGL} \arrow[l, bend left, "\mathcal{E}"] \end{tikzcd} \tag{F.4}$$

Finally, the most general equivalence between vector and principal bundles is the one sketched in Diagram (F.4)*, where the typical fibers of the vector bundles are also allowed to vary. In this diagram, the categories* VectBdlMapsPairs *and* PrincBdlGL *are defined as follows.*

- VectBdlMapsPairs *is the category whose:*
  - *objects are all vector bundles; and*
  - *morphisms are pairs* (f, ξ) *where* $f : E \to F$ *is a vector-bundle map and* $\xi : \mathscr{V} \to \mathscr{W}$ *is a vector-space isomorphism between the typical fibers of* E *and* F*.*
- PrincBdlGL *is the category whose:*
  - *objects are* $\mathrm{GL}(\mathscr{V})$*-principal bundles where* $\mathscr{V}$ *is any finite-dimensional vector space; and*
  - *morphisms are principal bundle morphisms of the form* (f, Ψ_ξ) *where* $\xi : \mathscr{V} \to \mathscr{W}$ *is a vector-space isomorphism and* $\Psi_\xi : \mathrm{GL}(\mathscr{V}) \to \mathrm{GL}(\mathscr{W})$ *is its induced morphism of groups.*

Furthermore, it is to be noted that the covariant functors GL *and* $\mathcal{E}$ *in Diagram* (F.4) *are defined exactly as before, but considering the composition with* $\xi : \mathscr{V} \to \mathscr{W}$ *in the natural way.* ♢

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