Universidade Federal de São Carlos
Centro de Ciências Exatas e de Tecnologia
Departamento de Matemática

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Autor: *Wagner Carvalho Sgobbi*

Orientador: *Daniel Vendrúscolo*
Coorientador: *Peter Wong*

São Carlos, 4 de março de 2022.

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"No princípio era o Verbo, e o Verbo estava com Deus, e o Verbo era Deus. Ele estava no princípio com Deus. Todas as coisas foram feitas por intermédio dele, e, sem ele, nada do que foi feito se fez." (João 1.1-3).

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To whom would I dedicate this thesis? Who made it possible to me to write this work, after all? Even if it sounds tautological, why did the things that worked out in this research really work out? How did I get to understand everything I did? Who, or what allowed me this? These wonderings lead us to the question below (see [66] and [73] for more details on the subject).

Why can we understand the world around us with science, in particular with mathematics? The rational intelligibility of the universe, that is, the fact that it can be - at least in part - understood by our rational minds is the basis of all science making. The certainty that we can obtain correct information from what is around us with the use of our minds is the foundation for any study we do. When we write either articles, theses or books, we trust that what we are writing is true, for our mind is reliable and capable of understanding how things really work. For Einstein, the fact that we can comprehend the universe is something astonishing, as we see in his two quotes below:

"The most incomprehensible thing about the universe is that it is comprehensible."

"You find it strange that I consider the comprehensibility of the world [...] as a miracle or as an eternal mystery. Well, a priori, one should expect a chaotic world, which cannot be grasped by the mind in any way [...] the kind of order created by Newton's theory of gravitation, for example, is wholly different. Even if man proposes the axioms of the theory, the success of such a project presupposes a high degree of ordering of the objective world, and this could not be expected a priori. That is the 'miracle' which is being constantly reinforced as our knowledge expands."

What is even more interesting is that not only we comprehend the universe, we comprehend it with the help of the mathematical language. As Eugene Wigner told,

"The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious, and there is no rational explanation for it... it is an article of faith".

What is, then, the reason of this strict relation between the mathematical language we use and the objective world? How to explain this philosophical "axiom" in which all of us scientists believe and that we assume, even without noticing? I disagree with Wigner's claim that "there is no rational explanation for it" and I want to show the dear reader my rational (although supernatural) reasons. I agree with the writer, bioethicist and Emeritus Professor of Mathematics at the University of Oxford John Lennox when he answers:

"The reason science works is that the universe out there and the human mind in here (who does the science) are ultimately the product of the same intelligent divine mind. Human beings are made, we are told, in God's image. And that means that science can be done."

There is, therefore, a common source, a common reason between the human being and the universe which allows such precise interaction. This ultimate source can be described by the term "Logos", a greek term with Hellenistic, Judaic, Platonic, Greek and Johannine origins, among others. The Logos, initially understood as "the eternal principle of order in the universe", or the "rational order of the universe", without which "everything would be a chaos, and nature would be unintelligible", can also be translated (in a more dynamic way) as "word", "verb", "reason" or "logic". The Logos is the ultimate reason of the existence and order of all things.

But what, or who, exactly, is the Logos? My point is to claim that the Logos is not only a philosophical idea, but a person.

The "word", or "verb", manifested in the expression "let there be light" in Genesis 1, which gives order to the chaos, is the Logos. The Logos is the one who was in the beginning, the one that was with god and that was God (John 1.1). It is the one "through whom also he [God] created the world" (Hebrews 1.2) and without which "was not any thing made that was made" (John 1.3). He is "the image of the invisible God, the firstborn of all creation. For by him all things were created" (Colossians 1.15,16). The Logos (verb, word) "became flesh and dwelt among us, and we have seen his glory, glory as of the only Son from the Father, full of grace and truth" (John 1.14). His name is Jesus Christ. His name "is", not "was", for he lives and reigns forever.

It happens, therefore, that the Logos, the reason why the universe exists and by which all human science can be done (including my Ph.D. thesis) is also my personal God, the true God, which has died and ressurected for my sins; he is the cause, the means and the purpose of all my existence: "For from him and through him and to him are all things. To him be glory forever. Amen" (Romans 11.36). It is to the Lord Jesus Christ, therefore, that I dedicate this thesis with joy.

Now, I would like to wish my sincere acknowledgements.

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Antes de agradecer, quero dedicar esta tese a alguém especial.

A quem dedicar esta tese? Quem é que me possibilitou escrever este trabalho, no final das contas? Mesmo que soe tautológico, por que é que as coisas que deram certo nesta pesquisa realmente deram certo? Como cheguei a entender tudo que entendi? Quem, ou o que me permitiu isto? Estas perguntas nos levam à questão abaixo (veja [66] e [73] para mais detalhes sobre o assunto).

Por que conseguimos entender o mundo à nossa volta com ciência, em particular com matemática? A inteligibilidade racional do universo, ou seja, o fato de que ele pode ser - pelo menos em parte - compreendido por nossas mentes racionais é a base de todo o fazer científico. A certeza de que podemos obter informações corretas daquilo que nos rodeia com o uso de nossas mentes é o alicerce para qualquer estudo que fazemos. Quando escrevemos um artigo, tese ou livro, estamos confiantes de que o que estamos escrevendo é verdadeiro, pois nossa mente é confiável e é capaz de entender como as coisas funcionam de verdade. Para Einstein, o fato de que conseguimos compreender o universo é algo espantoso, como vemos em suas duas citações:

"A coisa mais incompreensível acerca do universo é que ele é compreensível."

"Você acha estranho que eu considere a compreensibilidade do mundo [...] como um milagre ou como um eterno mistério. Bem, a priori, deveríamos esperar um mundo caótico, que a mente não pudesse captar de modo algum [...] a espécie de ordem criada pela teoria de gravidade de Newton, por exemplo, é totalmente diferente. Mesmo que o homem proponha os axiomas da teoria, o sucesso de um projeto dessa natureza pressupõe um alto grau de ordenamento do mundo objetivo, e isso não poderia ser esperado a priori. Esse é o 'milagre' que está sendo constantemente ratificado à medida que o nosso conhecimento se expande."

O que é mais interessante ainda é que, além de compreendermos o universo, o compreendemos com o auxílio da linguagem matemática. Como disse Eugene Wigner,

"A enorme utilidade da matemática nas ciências naturais é algo que beira o mistério, e não há explicação racional para isso... é um artigo de fé."

Qual é, então, a razão desta estreita ligação entre a linguagem matemática que usamos e o mundo objetivo? Como explicar este "axioma" filosófico no qual todos nós cientistas cremos e que assumimos, mesmo sem perceber? Discordo de Wigner em sua afirmação de que *"não há explicação racional para isso"* e quero apresentar ao querido leitor minhas razões racionais (e sobrenaturais). Estou junto com o escritor, bioeticista e professor emérito de matemática da Universidade de Oxford John Lennox em sua resposta:

"A razão pela qual a ciência funciona é que o universo lá fora e a mente aqui dentro (que faz a ciência) são de forma última o produto da mesma mente inteligente divina. Os seres humanos foram criados, como se diz, à imagem e semelhança de Deus. E isto significa que a ciência pode

ser feita."

Há então uma fonte comum, uma razão comum de ser entre o ser humano e o universo, que permite tão precisa interação. Esta fonte última de razão pode ser descrita pelo termo "Logos", termo grego de origens helenísticas, judaicas, platônicas, gregas e joaninas, entre outras. O Logos, inicialmente entendido como "o princípio eterno de ordem no universo" ou a "ordem racional do universo", sem o qual "tudo seria um caos, e a natureza seria ininteligível", também pode ser traduzido (de forma mais dinâmica) como "palavra", "verbo", "razão" ou "lógica". O Logos é a razão última da existência e da ordem de todas as coisas.

Mas o que, ou quem, exatamente, é o Logos? O meu ponto é afirmar que o Logos não é apenas uma ideia filosófica, mas uma pessoa.

A "palavra" ou "verbo", manifestada na expressão "haja luz" em Gênesis 1.1, que dá ordem ao caos, é o Logos. O Logos é aquele que existe desde o princípio, que estava com Deus e que era Deus (João 1.1). É aquele "por meio do qual [Deus] também fez o universo" (Hebreus 1.2) e sem o qual "nada do que foi feito se fez" (João 1.3). Ele é "a imagem do Deus invisível, o primogênito de toda a criação; pois, nele, foram criadas todas as coisas" (Colossenses 1.15,16). O Logos (verbo, palavra) "se fez carne e habitou entre nós, cheio de graça e de verdade, e vimos a sua glória, glória como do unigênito do Pai. (João 1.14). Seu nome é Jesus Cristo. Seu nome "é", e não "foi", pois ele vive e reina para sempre.

Acontece, então, que o Logos, a razão pela qual o universo existe e pela qual toda ciência humana pode ser feita (inclusive minha tese de doutorado) também é meu Deus pessoal, o Deus verdadeiro, que morreu e ressuscitou pelos meus pecados; ele é a causa, o meio e o propósito de toda minha existência: "Porque dele e por meio dele, e para ele, são todas as coisas; glória, pois, a ele eternamente. Amém" (Romanos 11.36). É ao Senhor Jesus Cristo, portanto, que dedico esta tese com alegria.

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Abstract

In this thesis we study property R_∞ for some classes of finitely generated groups by the use of the BNS invariant Σ^1 and some other geometric tools. In the combinatorial chapters of the work (4, 5, 6, 10 and 11), we compute Σ^1 for the family of Generalized Solvable Baumslag-Solitar groups Γ_n and use it to obtain a new proof of R_∞ for them, by using Gonçalves and Kochloukova's paper [42]. Then, we get nice information on finite index subgroups H of any Γ_n by finding suitable generators and a presentation, and by computing their Σ^1. This gives a new proof of R_∞ for H and for every finite direct product of such groups. We also show that no nilpotent quotients of the groups Γ_n have R_∞. With a help of Cashen and Levitt's paper [19], we give an algorithmic classification of all possible shapes for Σ^1 of GBS and GBS_n groups and show how to use it to obtain some partial twisted-conjugacy information in some specific cases. Furthermore, we show that the existence of certain spherically convex and invariant k-dimensional polytopes in the character sphere of a finitely generated group G can guarantee R_∞ for G. In the geometric chapters (7 through 9), we study property R_∞ for hyperbolic and relatively hyperbolic groups. First, we give a didactic presentation of the (already known) proof of R_∞ for hyperbolic groups given by Levitt and Lustig in [68] (which also uses Paulin's paper [81]). Then, we expand and analyse the sketch of proof of R_∞ for relatively hyperbolic groups given by A. Fel'shtyn on his survey paper [31]: we point out the valid arguments and difficulties of the proof, exhibit what would be a complete proof based on his sketch and show an example where the proof method doesn't work.

Keywords: property R_∞; topology; BNS invariants; combinatorial group theory; geometric group theory.

Resumo

Nesta tese estudamos a propriedade R_∞ para algumas classes de grupos finitamente gerados através do uso do BNS invariante Σ^1 e de algumas outras ferramentas geométricas. Nos capítulos combinatórios do trabalho (4, 5, 6, 10 e 11), computamos Σ^1 para a família dos grupos de Baumslag-Solitar solúveis generalizados Γ_n e o usamos para obter uma nova prova de R_∞ para tais grupos, usando o artigo de Gonçalves e Kochloukova [42]. Então, obtemos boas informações sobre os subgrupos H de índice finito de qualquer Γ_n encontrando geradores adequados, uma presentação e computando seu Σ^1. Com isto, obtemos uma nova prova de R_∞ para H e para qualquer produto direto finito de tais grupos. Também provamos que nenhum quociente nilpotente dos grupos Γ_n tem R_∞. Com a ajuda do artigo de Cashen e Levitt [19], damos uma classificação algorítmica de todos os possíveis formatos do invariante Σ^1 para grupos GBS e GBS_n e mostramos como usá-lo para obter algumas informações parciais sobre classes de conjugação torcida em alguns casos específicos. Além disso, provamos que a existência de certos poliedros esfericamente convexos e invariantes na esfera de caracteres de um grupo finitamente gerado arbitrário G pode garantir R_∞ para G. Nos capítulos geométricos (7 a 9), estudamos a propriedade R_∞ para grupos hiperbólicos e relativamente hiperbólicos. Primeiro, apresentamos de forma didática a prova (já conhecida) de R_∞ para grupos hiperbólicos dada por Levitt e Lustig em [68] (que também usa o artigo [81] de Paulin). Então, expandimos e analisamos o rascunho de prova de R_∞ para grupos relativamente hiperbólicos dado por Fel'shtyn em seu artigo [31]: mostramos os argumentos válidos e as dificuldades da prova, exibimos como seria uma prova completa baseada em seu rascunho e damos um exemplo onde tal método de prova não funciona.

Palavras-chave: propriedade R_∞; topologia; invariantes BNS; teoria combinatória de grupos; teoria geométrica de grupos.

Sumário

Introduction

Hello, dear reader. It is good to have you here. This entire thesis is built around the study of property R_∞ for finitely generated groups. In this introduction, we will catch you up on our motivations and strategies for the work and also give you some historical background. At the end of it, we describe the general structure of the thesis, the main content of each chapter and the main original contributions for the theory.

We start with the R_∞ property. Let G be any group and φ an automorphism of G. Two elements $g, h \in G$ are φ-twisted conjugated (or just twisted conjugated) if there exists $z \in G$ such that $zg\varphi(z)^{-1} = h$. The number of equivalence classes in G given by this relation is denoted by $R(\varphi)$ and called the Reidemeister number of φ. A group G is said to have property R_∞ if $R(\varphi) = \infty$ for every $\varphi \in Aut(G)$, that is, if every automorphism of G has an infinite number of twisted conjugacy classes. The search for groups with this property started mainly in 1994 in the paper [35], where the authors Fel'shtyn and Hill were studying the Reidemeister zeta function with applications to Nielsen Theory. This is an indicator of the topological nature of this property. Indeed, counting the topological Reidemeister number of a self-homeomorphism of a space X in Nielsen theory is the same as counting the algebraic Reidemeister number of the induced automorphism in the fundamental group of X. We show this relation at the end of Section 1.1 (see also [57, 97]). With the use of property R_∞ it has been shown, for example, that for any integer $n \geq 5$, there exists an n-dimensional nilmanifold M such that every self homeomorphism $f : M \to M$ is isotopic to a fixed point free map (see [45]). We can, therefore, see twisted conjugacy classes and property R_∞ as a generalization of topological properties and, ultimately, that's probably the reason why we topologists are interested on this subject. Furthermore, according to [36], twisted conjugacy has connections with Arthur-Selberg theory [3, 89], algebraic geometry [51], Galois cohomology [87], the theory of linear algebraic groups [91] and representation theory [39, 79, 90].

Since 1994 with the paper [35], the task of enlarging the list of groups with property R_∞ is an active research topic in both combinatorial and geometric group theory. The list below contains some of these groups and is based mainly on the list in the paper [36], together with some more recent discoveries. It is not exhaustive and does not follow any particular order.

- Baumslag-Solitar groups $BS(m, n)$, except for $BS(1, 1)$, and some nilpotent quotients of them [32, 40];

- Generalized Baumslag-Solitar groups (or GBS groups), as well as any group which is quasi-isometric to them [67, 93];

- the groups Γ_n, that is, the solvable generalization of the Baumslag-Solitar groups $BS(1, n)$, as well as any group which is quasi-isometric to them [94];

- non-elementary Gromov hyperbolic groups [30, 68];
- a large class of saturated weakly branch groups [37, 49, 53];
- Thompson's groups F and T, generalized Thompson's groups $F_{n,0}$ and their finite direct products [8, 18, 44];
- Houghton's groups [43, 58];
- Symplectic groups $Sp(2n, \mathbb{Z})$, some mapping class groups and the full braid groups $B_n(S)$ with $n \geq 4$ strands, where S is either the disk D^2 or the sphere S^2 [33];
- all pure Artin braid groups P_n for $n \geq 3$ [25];
- some Artin groups of infinite type [60];
- some extensions of $SL(n, \mathbb{Z})$, $PSL(n, \mathbb{Z})$, $GL(n, \mathbb{Z})$, $PGL(n, \mathbb{Z})$, $Sp(2n, \mathbb{Z})$ and $PSp(2n, \mathbb{Z})$ [74];
- $GL(n, K)$ and $SL(n, K)$, over some special integral domains K and with $n > 2$ [77];
- irreducible lattices inside some Lie groups [75];
- some metabelian groups of the form $\mathbb{Q}^n \rtimes \mathbb{Z}$ and $\mathbb{Z}[1/p]^n \rtimes \mathbb{Z}$ [34];
- Lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ if and only if $2|n$ or $3|n$ [46];
- many different classes of free nilpotent groups N_{rc} of rank r and nilpotency class c, as well as some free solvable groups S_{rt} of rank r and class t [24, 45, 85];
- some crystallographic groups [26, 41, 54, 70].

Plenty of different techniques have been used to enlarge the list above, and each paper has its own technical particularities. However, many of them could be classified according to their use of some of the general strategies below. Some papers are listed as examples:

1) short exact sequences, especially the ones containing characteristic subgroups and quotients of the group in question [42, 94];
2) isogredience classes [30, 38, 68];
3) the Σ-invariant of the group [42];
4) nilpotent quotients of the group [22, 23, 24, 40];
5) actions by isometries of the group (or related groups) on trees or hyperbolic spaces [93] (also see the appendix of [33]);
6) representation theory [25, 38];
7) non-abelian cohomology groups [38].

In this thesis we come across the first five items of the list, but our main focus is on items 3), 4) and 5). Items 3) and 4) correspond to the combinatorial part of the work (Part II and Appendix) and item 5) to the geometric one (Part III). Now we will talk about each part.

In the year of 2010, the topologist D. L. Gonçalves, together with the group theorist D. Kochloukova found out that Σ-theory can be used to guarantee property R_∞ in the combinatorial context of finitely generated groups (see [42]). To summarize, they considered the fact that property R_∞ can be deduced by looking to some characteristic quotients of the group (item 1)) and realized that the invariance under automorphisms of Σ^1, in some special cases, can produce some of these quotients which are good enough.

The first of the most known versions of the Σ-invariants was defined in 1987 by R. Bieri, W. Neumann and R. Strebel in [9] for arbitrary finitely generated groups (hence the name "BNS"). Given a finitely generated group G and a finitely generated G-operator group A, they associated to it a subset $\Sigma_A = \Sigma_A(G)$ of the character sphere $S(G)$ defined by the finite (or not) generation of A over a finitely generated submonoid of G. There they showed many general properties of Σ_A, the most known being its openness in $S(G)$ and a characterization of the finitely generated normal subgroups of G containing G'. Since then, the amount of research on this invariant has grown considerably and led to the discovery of many connections with other areas of mathematics. To get a little taste of what we are saying, the BNS invariant $\Sigma_{G'}$ of the fundamental group G of a smooth closed 3-manifold X is characterized by the existence of non-vanishing 1-forms on X which are also non-vanishing on ∂X. Also, let X be a hyperbolic 3-manifold and consider the known Thurston's norm on the second homology group $H_2(X)$. The unitary ball is then a polytope homeomorphic to $S(\pi_1(X))$, and the interior of its faces can be exactly seen as $\Sigma(\pi_1(X))$. One can also find close relations of the BNS-invariant with group actions on $\mathbb{R}$-trees, fibering of manifolds over S^1 and others. Because of this, it is worth to call the collection of theorems about this invariant by "Σ-theory".

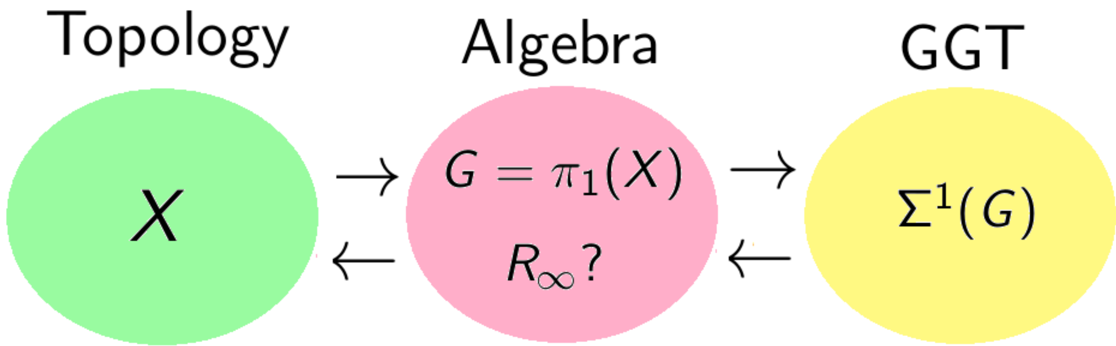
Despite this good amount of nice properties, the Σ-invariant is in general hard to be effectively computed. It would be of precious help to find an equivalent but more simple definition. In fact, the desired definition was somewhat hidden inside the original paper [9], in part (*ii*) of Proposition 3.4. Fortunately, later, Robert Bieri and, independently, Gaël Meigniez realized that fact. They were able to rewrite that property in (*ii*) in terms of some kind of connectivity inside the Cayley graph of G (Definition 3.7, where the BNS-invariant is denoted by $\Sigma^1(G)$). With this graph definition it is easier to derive the basic properties (see our Chapter 3) and, in particular, to see that $\Sigma^1(G)$ (and also its complement $\Sigma^1(G)^c$) is invariant under all automorphisms of G (Theorem 3.18).

So, even without knowing precisely the structure of $Aut(G)$, it becomes possible to relate $\Sigma^1(G)$ with some properties about the automorphisms of G, in particular with property R_∞, as it is done in [42]. The topological nature of this property, together with Gonçalves and Kochloukova's paper [42] and the geometric and graph-theoretical aspect of Σ^1 may have been the main ingredients to bring the attention of the topologist P. Wong. At that time, J. Taback and him had already dealed with GBS groups and the groups Γ_n, both generalizations of Baumslag-Solitar groups (see the three first items of our first list above) and knew they are R_∞-groups. The fact that R_∞ for $BS(1,n)$ can be shown by the use of Σ-theory and the paper [42] naturally arose the question of whether Σ-theory could also be used to deduce R_∞ for Γ_n

and the GBS groups. This was the first goal of the combinatorial part of our project, and we get a positive answer for the groups Γ_n. Furthermore, we show how to "algorithmically" compute Σ^1 for a GBS - and GBS_n - group (more details below).

The second goal is related to the following situation: it is well known that property R_∞ is not invariant under quasi-isometry in general. However, it may be invariant inside some specific families of groups. Taback and Wong show this is true inside the family of the groups Γ_n and of GBS groups ([94] and [93], respectively). In particular, any finite index subgroup H of any Γ_n is an R_∞ group; so, we asked: can Σ-theory be applied to show R_∞ for H? The answer turned out to be positive.

The main strategy of this combinatorial part is, therefore, to compute the geometric invariants of some finitely generated groups in order to guarantee property R_∞ for them, or at least to see in what cases this can be done. From a topologist's (illustrative) point of view, we start with a topological space X and, in order to obtain information about its lifting and Nielsen properties, we investigate the twisted conjugacy classes of its fundamental group G. To do that, we compute its Σ-invariant and see, for example, if it can guarantee property R_∞ for G and, consequently, nice topological properties of X. This is illustrated in the next figure, where GGT means "Geometric Group Theory".



Although the Σ^1 invariant is in the core of the combinatorial part of our thesis, in the last year of research an extra paper called my attention: in [22] (2020), the authors D. L. Gonçalves and K. Dekimpe studied the R_∞ property for nilpotent quotients of the Baumslag-Solitar groups $BS(m,n)$, after having done the same for free groups, free nilpotent groups and free solvable groups in [24] and also for surface groups in [23]. So, in the same way knowing information of the Σ^1 invariant for $BS(1,n)$ was the motivation to investigate $\Sigma^1(\Gamma_n)$ in Chapter 5, knowing which nilpotent quotients of $BS(1,n)$ had R_∞ could help us to understand the same property for Γ_n. Based on this idea, we showed that no nilpotent quotients of the groups Γ_n have R_∞.

After the combinatorial part of the work was complete, the quasi-isometric and geometric likeness of Taback and Wong's papers [94] and [93] turned our attention to two special families of groups: hyperbolic groups and relatively hyperbolic groups, the former being a subfamily of the latter. Hyperbolic groups were first defined by Gromov ([50], 1987) and definitely became one of the most studied families of groups in Geometric Group Theory since then. For the general theory of hyperbolic groups, first connections with other areas of mathematics and some applications, we refer [21], [47] and [50]. The most interesting fact for us is that property R_∞ for (non-elementary) hyperbolic groups was implicitly shown by Levitt-Lustig's paper [68] in 2000 (and explicitly by [30] in 2001). At some point, the proof uses the well-known fact that the family of hyperbolic groups is closed under quasi-isometry (we can also say that hyperbolicity is a "quasi-isometry invariant"). Intuitively, therefore, a generalization of this argument to show R_∞ for the larger class of relatively hyperbolic groups would need to use quasi-isometry invariance for this class, something that was still not known.

Later, in 2006, Drutu [28] published a paper on quasi-isometry invariance for relatively hyperbolic groups, breaking this barrier. So, in 2010, by quoting Drutu's paper, together with Belegradek-Szczepanski [5] and others, Fel'shtyn claimed property R_∞ to be true for (non-elementary) relatively hyperbolic groups in his survey paper (Theorem 3.3 in [31]). However, only a sketch of a proof is given there, based mainly on Levitt and Lustig's proof [68]. Our idea, therefore, was to study that proof sketch in all its details to maybe exhibit a more complete and didactic proof of property R_∞ for relatively hyperbolic groups, something still not present in the literature. However, my general conclusion is that such a proof, based on that sketch, is at least more complicated than it looks like, for it involves extra difficulties that do not exist in the hyperbolic case. To know my specific conclusions, the readers must do what they do best: keep reading.

General structure and chapters

Here we give an overview of the structure of this thesis, together with a general description of each chapter.

Admittedly, this Ph.D. thesis turned out to be quite long. This is because I decided to give the reader the option of reading a didactic presentation of (virtually) all the preliminary background needed for the rest of the work. This is done in Part I (chapters 1 through 4), so that the reader which is not used to the subject and the language can catch up with minimal time. Chapter 1 contains the preliminaries on Combinatorial Group Theory, which are needed in chapters 5, 6, 10 and 11. Similarly, Chapter 2 contains the geometric preliminaries for chapters 7 through 9. In Chapter 3, we present the basic results on Σ-theory we will need in chapters 5,6 and 11, based on the notes [92]. The reader may take a quick look at the table of contents (Sumário) and skip the sections of these three chapters which he already knows. Chapter 4, although counted as preliminary, is original. There, with a help of the results of [42], we show that the existence of some invariant closed convex polytopes in the character sphere $S(G)$ can also guarantee property R_∞ for a finitely generated group G. Since this result has a similar fashion to the ones in Section 3.3, we decided to keep it as a preliminary chapter.

After the preliminaries, we have the combinatorial Part II (chapters 5 and 6), the geometric Part III (chapters 7 through 9) and the combinatorial Appendix (chapters 10 and 11).

Part II is the one who contains the most number of original contributions, where we apply Σ-theory to study property R_∞ for the Generalized Solvable Baumslag-Solitar groups Γ_n. In Chapter 5, we compute the Σ^1 invariant of Γ_n and guarantee property R_∞ for them by using [42]. After this, in Chapter 6, we compute the Σ^1 invariant and guarantee R_∞ for all finite index subgroups H of Γ_n by finding good generators (with a help from Bogopolski's paper [12]) and a good group presentation for them. Finally, we discuss whether such finite index subgroups H are (isomorphic to) Γ_k for some $k \geq 1$ (not all of them are).

Part III can be summarized as a study of property R_∞ for hyperbolic and relatively hyperbolic groups. We had the final purpose of studying the sketch of proof given by Fel'shtyn in

[31] on property R_∞ for (non-elementary) relatively hyperbolic groups. Before doing that, we carefully read Levitt and Lustig's Section 3 of the paper [68] and their (implicit) proof of R_∞ for the particular case of a hyperbolic group. So, Chapter 7 contains an exhibition of a proof for a slight generalization of Levitt and Lustig's result, in order to be also applicable to the relative case. In simple words, their result shows that the existence of some "special" action of G on an $\mathbb{R}$-tree T is sufficient to guarantee $R(\varphi) = \infty$ for a fixed automorphism φ of G. Then, in Chapter 8 we give a more detailed presentation of the R_∞ proof in the hyperbolic case; that is, we exhibit Paulin's proof (Theorem A of [81]) of the fact that hyperbolic groups admit those "special" actions. Finally, in Chapter 9 we exhibit what would be a proof of property R_∞ for non-elementary relatively hyperbolic groups by following Fel'shtyn's sketch, that is, by adapting the proof of the hyperbolic case we give in chapter 8. We show that the proof would be complete if it wasn't for Lemma 9.29, which we believe is not true in general (although nothing prevents it to be true in some particular examples). To convince the reader of this, in Section 9.5 we show an example where Lemma 9.29 does not work. We decided to maintain the incomplete proof in this thesis, anyway, to give the reader an idea of what a proof could look like.

At last, the Appendix. In Chapter 10, we follow [23] to define the R_∞ nilpotency index of a group with property R_∞. Then, by developing some theoretical background similar to the one in [22] and by doing some matrix computations, we calculate the R_∞ nilpotency index for all groups Γ_n, showing it to be infinite. This is equivalent to say that none of the nilpotent quotients of the groups Γ_n are R_∞ groups. In Chapter 11, we use a result from Cashen and Levitt in [19] to algorithmically classify the possible shapes of the Σ-invariant of GBS and GBS_n groups, given the associated finite graph of groups. We then use this to get some partial twisted conjugacy results (not necessarily R_∞ results) on some special cases.

Original contributions

Let us point out some original contributions of this thesis for the literature, in the natural order of the text.

In general, this thesis is useful for any reader who wants a first contact with property R_∞, Σ-theory and geometric group theory. We tried to keep all the text - including the more technical proofs - very readable and enjoyable for anyone with basic math knowledge. I believe this is a didactic contribution. Now let us get more specific.

In Section 1.3, we give a detailed exibition and proof of the Reidemeister-Schreier algorithm (Theorem 1.50) by using group actions and the Cayley graph language of Serre's book [86]. This may be a more intuitive and enjoyable reading for topologists, in comparison with the strictly combinatorial proofs that can be found in the literature. We give examples and it is possible to see the generators for the subgroup H naturally appearing in the drawings. I think this proof may turn Reidemeister-Schreier's Theorem less counter-intuitive (as it was to me before).

We can say all results contained in chapters 4 through 6 are original work. Let us list and comment them briefly.

In Chapter 4, Theorem 4.28 guarantees property R_∞ for any finitely generated group whose character sphere contains certain spherically convex invariant polytopes. Although this property may have been in the minds of a few specialists as folklore, we couldn't find any proof in the literature for it. Our proof is built from scratch and quite detailed. Therefore, I believe it can be considered as original work.

Chapter 5 contains the first known computation of the Σ^1-invariant for the Generalized solvable Baumslag-Solitar groups Γ_n (Theorem 5.2). In fact, it computes the Σ^1-invariant for a slightly bigger family. This theorem is used to guarantee property R_∞ for these groups. Although property R_∞ was already known for the groups Γ_n (see [94]), we have the first proof that uses Σ-theory. The partial generalizations of Theorems 5.5 and 5.7 are also original.

In Chapter 6, let H be a finite index subgroup of Γ_n. We have computed for the first time: a suitable family of generators for H (Theorem 6.6), a presentation for H (Theorem 6.8) and the Σ^1-invariant for H (Theorem 6.10). As in the Γ_n case, it is already known via [94] that H has property R_∞, but our proof is the first one that uses Σ-theory. In the last section, we showed that the family of Generalized Solvable Baumslag-Solitar groups Γ_n do not have the property of being closed under finite index subgroups, which is also original work. This is important, for it distinguishes this family from the family of Solvable Baumslag-Solitar groups $BS(1,n)$, which was shown to have this property by Bogopolski in [12].

Finally, the geometric part. The general content of Chapter 7 is not original, but Theorem 7.4 is a restatement of Levitt and Lustig's Section 3 in [68] in a slightly more general way. It turned out that the sufficient condition they find to guarantee an infinite Reidemeister number $R(\varphi)$ for a fixed automorphism of G is quite general, so we rewrote it in terms of an arbitrary finitely generated group, so that this can be useful for future applications. That being said, Chapter 8 contains an exhibition of Paulin's proof ([81], Theorem A) of the fact that non-elementary hyperbolic groups satisfy that sufficient condition, being only a didactic contribution. Chapter 9 contains the first careful exhibition in the literature of what would be a complete proof of property R_∞ for finitely generated non-elementary relatively hyperbolic groups (Theorem 9.27), based on Fel'shtyn's sketch. Although we showed the proof to be probably incomplete (see Lemma 9.29 and Section 9.5), we believe this may still help the discussion on the veracity of property R_∞ for this family of groups in the future.

Chapter 10 contains, as the main result, the first computation of the R_∞ nilpotency index of the groups Γ_n, shown to be infinite in Theorem 10.14. This also establishes a good distinction between Γ_n and Baumslag-Solitar groups, which have finite R_∞ nilpotency index in most cases (see [22]). But it also contains fresh information about their nilpotent quotients $\Gamma_{n,c}$, such as the first computation of their torsion subgroup (Proposition 10.5), of the terms of their lower central series (Proposition 10.4) and of a presentation for them (Corollary 10.10).

Chapter 11 contains a presentation of some results of Cashen and Levitt's paper [19] on the Σ-invariants of graphs of groups. We decided to put it there so the reader could get aware of the techniques used. So, of course, these results are not original. The original work of Chapter 11 consists of the more "algorithmic" part; that is, if a graph of groups (G,Γ) is explicitly given, we show which calculations must be performed to determine the shape of $\Sigma^1(G)$. For example, in the GBS case, the existence or not of killing circuits determines if $\Sigma^1(G)$ is either empty, or the whole sphere, or two antipodal open hemispheres. The same is done for the GBS_n groups, but the possible cases are more complicated to be described here. The following results are original, together with all the ones preceeding them: Lemmas 11.9 through 11.12; Theorem 11.14 through Corollary 11.18; Theorem 11.23; Corollaries 11.25 through 11.29.

Parte I

Preliminaries

Capítulo 1

Combinatorial preliminaries

In this chapter, we want not only to fix notations and familiarize with them, but also to present the combinatorial background in an organized, intuitive and not so rigorous way (some proofs will be omitted), to clarify the reader's mind in the combinatorial chapters of our work (chapters 5, 6, 10 and 11). Since we don't want to make an extensive text, we need to assume that the reader is at least familiarized with basic set theory, Algebra and basic facts about free groups and group presentations (which can be found in [56] and [71]).

1.1 The R_∞ property

Definition 1.1. Given a group G and an automorphism $\varphi \in Aut(G)$, we say that $g, h \in G$ are φ-twisted conjugated (or just twisted conjugated) and denote $g \sim_\varphi h$ if there exists $z \in G$ such that

$$zg\varphi(z)^{-1} = h.$$

It is straightforward to verify that this is an equivalence relation in G.

Definition 1.2. The equivalence classes of the relation $\sim_\varphi$ in G are called φ-twisted conjugacy classes (or just twisted classes). We denote the set of twisted classes by $\Re(\varphi) = \{[g]_\varphi \mid g \in G\}$ and sometimes we denote a class $[g]_\varphi$ only by $[g]$. The Reidemeister number of φ is $R(\varphi) = \text{card}\,\Re(\varphi)$, i.e., the number of twisted conjugacy classes of φ in G.

Example 1.3. Let $G = \mathbb{Z}^n = \mathbb{Z} \oplus ... \oplus \mathbb{Z}$ be a finitely generated torsion-free abelian group, and let $\varphi \in Aut(G)$. Here we use additive notation. Given $g, h \in G$ we have by definition

$$\begin{aligned} g \sim_\varphi h &\iff z + g - \varphi(z) = h \text{ for some } z \in G \\ &\iff g - h = \varphi(z) - z = (\varphi - Id)(z) \text{ for some } z \in G \\ &\iff g - h \in \text{im}(\varphi - Id) \\ &\iff \overline{g} = \overline{h} \text{ in } \frac{G}{\text{im}(\varphi - Id)}. \end{aligned}$$

Then $R(\varphi)$ is exactly the index of the subgroup $\text{im}(\varphi - Id)$ in $\mathbb{Z}^n$. Since $\varphi - Id \in Hom(\mathbb{Z}^n, \mathbb{Z}^n) \simeq M(n \times n, \mathbb{Z})$, we can associate $\varphi - Id$ to its integer $n \times n$ matrix

$$A = [(\varphi - Id)(e_1)\ ...\ (\varphi - Id)(e_n)]$$

whose i^{th} column is the vector $(\varphi - Id)(e_i)$. If $\det(A) = 0$, then one of the columns are generated by the other $n-1$ ones. This implies $\mathrm{im}(\varphi - Id)$ has rank $r < n$ and therefore it has infinite index in $\mathbb{Z}^n$, by the following Lemma 1.4. Then $R(\varphi) = \infty$. If $\det(A) \neq 0$, it is known that the index of $\mathrm{im}(\varphi - Id)$ in $G = \mathbb{Z}^n$ is exactly $|\det(A)|$, which is also the volume in $\mathbb{R}^n$ of the parallelepiped given by the vectors $(\varphi - Id)(e_1), ..., (\varphi - Id)(e_n)$ (see figure). To summarize, identify $\varphi - Id$ with A. Then

$$R(\varphi) = \begin{cases} |\det(\varphi - Id)|, \text{ if } \det(\varphi - Id) \neq 0, \\ \infty, \text{ if } \det(\varphi - Id) = 0. \end{cases}$$

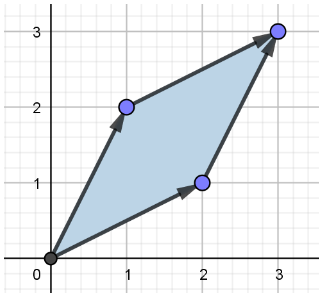


Figura 1.1: The index of $H = \langle (2,1), (1,2) \rangle$ in $\mathbb{Z}^2$ is $2.2 - 1.1 = 3$, the area of the parallelepiped above.

In particular, for the automorphism $\varphi = -Id$ we have $R(\varphi) = |\det(-Id - Id)| = |\det(-2Id)| = 2^n < \infty$, so $G = \mathbb{Z}^n$ has not property R_∞.

Lemma 1.4. *Let $n \geq 1$. If $H \leq \mathbb{Z}^n$ is a finite index subgroup of $\mathbb{Z}^n$, then $rk(H) \geq n$.*

Demonstração. Let us consider $\mathbb{Z}^n \subset \mathbb{R}^n$ as a subset of the real n-dimensional vector space $\mathbb{R}^n$ in the usual way, and let us use additive notation. Suppose by contradiction that $rk(H) = r < n$ and write $H = \langle h_1, ..., h_r \rangle$. Let W be the real subspace of $\mathbb{R}^n$ generated by the vectors $h_1, ..., h_r$. We have the set inclusion $H \subset W$. Since $dim(W) \leq r < n$, W is a proper subspace of $\mathbb{R}^n$ and so at least one of the canonical vectors e_i must be outside W. Fix such e_i. Then $\langle e_i \rangle \cap W = \{0\}$ as subspaces (because if $\lambda e_i \in W$ for some $\lambda \neq 0$ we would have $e_i = \frac{1}{\lambda}(\lambda e_i) \in W$) and in particular $\{je_i \mid j \in \mathbb{Z}\} \cap H = \{0\}$. Then the set $\{je_i \mid j \in \mathbb{Z}\}$ is an infinite set of distinct coset representatives of H in $\mathbb{Z}^n$, because if $j \neq j'$ then $je_i - j'e_i = (j - j')e_i \notin H$. This shows that H has infinite index in $\mathbb{Z}^n$, contradiction. □

Now we define our main object of study:

Definition 1.5. We say that a group G has property R_∞ when $R(\varphi) = \infty$ for all $\varphi \in Aut(G)$. In other words, when every automorphism of G has an infinite number of twisted conjugacy classes.

One of the most basic tools for studying twisted conjugacy classes is the following: consider a commutative diagram of group homomorphisms, where φ and ψ are isomorphisms.

$$\begin{array}{ccc} G & \xrightarrow{\eta} & H \\ \varphi\downarrow & \circlearrowright & \downarrow\psi \\ G & \xrightarrow{\eta} & H \end{array}$$

Then the function

$$\begin{aligned} \hat{\eta} :\Re(\varphi) &\longrightarrow \Re(\psi) \\ [g]_\varphi &\longmapsto [\eta(g)]_\psi \end{aligned}$$

is well defined. Indeed,

$$\begin{aligned} [g]_\varphi = [g']_\varphi \quad &\Rightarrow \quad zg\varphi(z)^{-1} = g',\ z \in G \\ &\Rightarrow \quad \eta(z)\eta(g)\eta\varphi(z)^{-1} = \eta(g'),\ z \in G \\ &\Rightarrow \quad \eta(z)\eta(g)\psi(\eta(z))^{-1} = \eta(g'),\ \eta(z) \in H \\ &\Rightarrow \quad [\eta(g)]_\psi = [\eta(g')]_\psi. \end{aligned}$$

Observation 1.6. It is obvious from the definition that if η is surjective, then $\hat{\eta}$ is surjective as well. Also, if η is an isomorphism, $\hat{\eta}$ is bijective with inverse $\widehat{\eta^{-1}}$ and therefore $R(\varphi) = R(\psi)$. In fact, if we replace the right arrows η in the diagram by left ones with η^{-1} we get a commutative diagram because

$$\varphi\eta^{-1} = \eta^{-1}\eta\varphi\eta^{-1} = \eta^{-1}\psi\eta\eta^{-1} = \eta^{-1}\psi.$$

Then $\widehat{\eta^{-1}}$ is well defined and easily we have $\hat{\eta} \circ \widehat{\eta^{-1}} = Id = \widehat{\eta^{-1}} \circ \hat{\eta}$.

Lemma 1.7. *Consider the following commutative diagram of group homomorphisms*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 1 \\ & & \downarrow\xi & \circlearrowright & \downarrow\varphi & \circlearrowright & \downarrow\psi & & \\ 1 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 1 \end{array}$$

where the horizontal lines are exact sequences, that is, i is injective, p is surjective and $\ker(p) = \operatorname{im}(i)$. *Denote $Fix(\psi) = \{c \in C \mid \psi(c) = c\}$. Then*

1) if $R(\psi) = \infty$, then $R(\varphi) = \infty$;

2) if $R(\xi) = \infty$ and $Fix(\psi)$ is finite, then $R(\varphi) = \infty$.

Demonstração. Item 1) is easy: since p is surjective, $\hat{p} : \Re(\varphi) \to \Re(\psi)$ is surjective. Then $R(\varphi) \geq R(\psi) = \infty$ by hypothesis and we get $R(\varphi) = \infty$. Let us show item 2). The set $Fix(\psi)$ is a subgroup of C. We define an action of $Fix(\psi)$ on the set $\Re(\xi)$ in the following way: let $c \in Fix(\psi)$ and $[a]_\xi \in \Re(\xi)$. Take $b \in B$ such that $p(b) = c$. Since $p \circ i$ is the zero homomorphism and $\psi(c) = c$ we have

$$p(bi(a)\varphi(b)^{-1}) = p(b)pi(a)p\varphi(b)^{-1} = p(b)p\varphi(b)^{-1} = c\psi p(b)^{-1} = c\psi(c)^{-1} = cc^{-1} = 1,$$

then $bi(a)\varphi(b)^{-1} \in \ker(p) = \mathrm{im}(i)$ and there must be $\alpha \in A$ such that $i(\alpha) = bi(a)\varphi(b)^{-1}$ (note that this α is unique because i is injective). We then define $c \cdot [a]_\xi = [\alpha]_\xi$.

Let us show that this is a well defined action. Suppose we had chosen two different $b, b' \in B$ such that $p(b) = c = p(b')$ and also two different class representatives $[a]_\xi = [a']_\xi$. We must show that $p(b) \cdot [a]_\xi = p(b') \cdot [a']_\xi$. Write $p(b) \cdot [a]_\xi = [\alpha]_\xi$ and $p(b') \cdot [a']_\xi = [\alpha']_\xi$, that is, α and α' are the unique elements in A such that $i(\alpha) = bi(a)\varphi(b)^{-1}$ and $i(\alpha') = b'i(a')\varphi(b')^{-1}$, respectively. Now, from $p(b) = p(b')$ we obtain that $b^{-1}b' \in \ker(p) = \mathrm{im}(i)$ and $b^{-1}b' = i(\tilde{a})$, $\tilde{a} \in A$. Also, since $[a]_\xi = [a']_\xi$ we have $xa\xi(x)^{-1} = a'$ for $x \in A$ and then applying i in both sides and using that $\varphi \circ i = i \circ \xi$ we have $i(x)i(a)\varphi(i(x))^{-1} = i(a')$. Since $\mathrm{im}(i) = \ker(p) \lhd B$ we can write $bi(\tilde{a}x)b^{-1} = i(\tilde{\alpha})$ for some $\tilde{\alpha} \in A$. Now, by using all these identities we obtain

$$\begin{aligned}
i(\alpha') &= b'i(a')\varphi(b')^{-1} \\
&= bi(\tilde{a})i(x)i(a)\varphi i(x)^{-1}\varphi(bi(\tilde{a}))^{-1} \\
&= bi(\tilde{a}x)i(a)\varphi i(x)^{-1}\varphi(i(\tilde{a}))^{-1}\varphi(b)^{-1} \\
&= bi(\tilde{a}x)b^{-1}i(\alpha)\varphi(b)\varphi i(x)^{-1}\varphi(i(\tilde{a}))^{-1}\varphi(b)^{-1} \\
&= (bi(\tilde{a}x)b^{-1})i(\alpha)\varphi(bi(x)^{-1}i(\tilde{a})^{-1}b^{-1}) \\
&= (bi(\tilde{a}x)b^{-1})i(\alpha)\varphi(bi(\tilde{a}x)b^{-1})^{-1} \\
&= i(\tilde{\alpha})i(\alpha)\varphi(i(\tilde{\alpha}))^{-1} \\
&= i(\tilde{\alpha})i(\alpha)i(\xi(\tilde{\alpha})^{-1}) \\
&= i(\tilde{\alpha}\alpha\xi(\tilde{\alpha})^{-1})
\end{aligned}$$

then $\alpha' = \tilde{\alpha}\alpha\xi(\tilde{\alpha})^{-1}$ because i is injective and we have by definition that $[\alpha']_\xi = [\alpha]_\xi$, as desired.

Let us show that this is a group action. First, if $c = 1$, choose the preimage $b = 1$. We have by definition $1 \cdot [a]_\xi = [\alpha]_\xi$, where α is the unique element of A such that $i(\alpha) = 1i(a)\varphi(1)^{-1} = i(a)$. Since i is injective we have $\alpha = a$ and $1 \cdot [a]_\xi = [a]_\xi$. Second, let $c, c' \in C$. We will show that $(cc') \cdot [a]_\xi = c \cdot (c' \cdot [a]_\xi)$. Choose preimages b and b' for c and c', respectively. Write $c' \cdot [a]_\xi = [\alpha']$. Now, for the element cc' choose the preimage bb'. So $(cc') \cdot [a]_\xi = [\alpha'']_\xi$, where by definition α'' is the unique element such that

$$i(\alpha'') = (bb')i(a)\varphi(bb')^{-1} = bb'i(a)\varphi(b')^{-1}\varphi(b)^{-1} = bi(\alpha')\varphi(b)^{-1}.$$

Therefore by uniqueness $[\alpha'']_\xi = c \cdot [\alpha']_\xi$ and so

$$(cc') \cdot [a]_\xi = [\alpha'']_\xi = c \cdot [\alpha']_\xi = c \cdot (c' \cdot [a]_\xi),$$

and we have a group action, as desired.

Now we finally show item 2): note that two classes $[a]_\xi, [\alpha]_\xi$ are in the same $Fix(\psi)$-orbit if and only if $[i(a)]_\varphi = [i(\alpha)]_\varphi$. Indeed, $c \cdot [a]_\xi = [\alpha]_\xi$ for $c \in Fix(\varphi)$ implies $i(\alpha) = bi(a)\varphi(b)^{-1}$ for $b \in B$ which implies $[i(a)]_\varphi = [i(\alpha)]_\varphi$. On the other hand, $[i(a)]_\varphi = [i(\alpha)]_\varphi$ implies $i(\alpha) = bi(a)\varphi(b)^{-1}$ for $b \in B$. This identity implies that $p(b) \in Fix(\psi)$, because

$$1 = pi(\alpha) = p(bi(a)\varphi(b)^{-1}) = p(b)pi(a)p\varphi(b)^{-1} = p(b)p\varphi(b)^{-1} = p(b)\psi p(b)^{-1}.$$

Then $p(b) \cdot [a]_\xi = [\alpha]$, by definition. Now, $Fix(\psi)$ is finite, which means that all the orbits are finite. So the infinite set $\Re(\xi)$ is being partitioned in finite orbits, which implies that we must have an infinite number of orbits. By the equivalence shown above, we then have infinite classes $[i(a)]_\varphi$ in B. Then $R(\varphi) = \infty$, as desired. □

Throughout the thesis we will also need the following

Proposition 1.8. *If G is a group and $\varphi \in Aut(G)$, then $R(\varphi) = R(\varphi^{-1})$.*

Demonstração. It is enough to show $R(\varphi) \geq R(\varphi^{-1})$ for every φ, for then applying this to φ^{-1} we also obtain $R(\varphi^{-1}) \geq R((\varphi^{-1})^{-1}) = R(\varphi)$ and therefore $R(\varphi) = R(\varphi^{-1})$, as desired. So, we just have to show that there is a surjective map from $\Re(\varphi)$ to $\Re(\varphi^{-1})$. Define $f : \Re(\varphi) \to \Re(\varphi^{-1})$ by putting $f([x]_\varphi) = [x^{-1}]_{\varphi^{-1}}$ (for $x \in G$). To see that it is well defined, suppose $[x]_\varphi = [y]_\varphi$. Then there is $z \in G$ such that $y = zx\varphi(z)^{-1}$. Therefore

$$y^{-1} = (zx\varphi(z)^{-1})^{-1} = \varphi(z)x^{-1}z^{-1} = \varphi(z)x^{-1}\varphi^{-1}(\varphi(z))^{-1},$$

which implies $f([x]_\varphi) = [x^{-1}]_{\varphi^{-1}} = [y^{-1}]_{\varphi^{-1}} = f([y]_\varphi)$, as desired. To see that f is surjective, just note that for every $[x]_{\varphi^{-1}} \in \Re(\varphi^{-1})$, we have $f([x^{-1}]_\varphi) = [x]_{\varphi^{-1}}$. This completes the proof. □

In a similar fashion of the Reidemeister number, we can define algebrically the number of isogredience classes and what is called "property S_∞":

Definition 1.9. Let G be a group and let $\pi : Aut(G) \to Out(G) = \frac{Aut(G)}{Inn(G)}$ be the natural projection onto the quotient of the automorphism group $Aut(G)$ by its (normal) subgroup of inner automorphisms of the form $\gamma_g : G \to G$, $\gamma_g(h) = ghg^{-1}$ (for any $h \in G$). Let $\Psi = \pi(\alpha) \in Out(G)$ be any element. Given two automorphisms $\gamma_r\alpha, \gamma_s\alpha \in \pi^{-1}(\Psi)$, we say they are isogredient if there is $g \in G$ such that $\gamma_g\gamma_r\alpha\gamma_g^{-1} = \gamma_s\alpha$. We define $S(\Psi)$ to be the cardinality of the set of such isogredience classes given by the relation above, that is, $S(\Psi) = \text{card}(\pi^{-1}(\Psi)/\sim)$. We say that a group G has property S_∞ if $S(\Psi) = \infty$ for every $\Psi \in Out(G)$.

Proposition 1.10. *Let G be a group and $\Psi = \pi(\alpha) \in Out(G)$. If $S(\Psi) = \infty$ then $R(\alpha) = \infty$. In particular, if G has property S_∞ then it also has property R_∞.*

Demonstração. Denote by $Z(G)$ the center of G, let $f : G \to G/Z(G)$ with $f(g) = \overline{g}$ be the natural projection and $\overline{\alpha}$ be the naturally induced automorphism on this quotient. Given two automorphisms of the form $\gamma_r\alpha$ and $\gamma_s\alpha$, we claim that they are isogredient if and only if $\overline{r}$ and

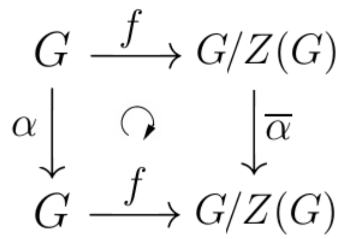
$\overline{s}$ are $\overline{\alpha}$-conjugated in $G/Z(G)$. Indeed, if there is $g \in G$ such that $\gamma_g \gamma_r \alpha \gamma_g^{-1} = \gamma_s \alpha$, then since $\alpha \gamma_g^{-1} = \gamma_{\alpha(g^{-1})} \alpha$ we have $\gamma_g \gamma_r \gamma_{\alpha(g^{-1})} \alpha = \gamma_s \alpha$, so

$$\gamma_{gr\alpha(g^{-1})} = \gamma_g \gamma_r \gamma_{\alpha(g^{-1})} = \gamma_s, \text{ or } \gamma_{gr\alpha(g^{-1})s^{-1}} = Id,$$

and by definition of center we have $gr\alpha(g^{-1})s^{-1} \in Z(G)$, or $\overline{g}.\overline{r}.\overline{\alpha}(\overline{g})^{-1} = \overline{s}$ in $G/Z(G)$, as desired. Suppose now that $\overline{r}$ and $\overline{s}$ are $\overline{\alpha}$-conjugated in $G/Z(G)$ by an element $\overline{g}$. Then, by definition, there is $z \in Z(G)$ such that $gr\alpha(g)^{-1}s^{-1} = z$. Then, since $\gamma_z = Id$ we have

$$\gamma_g \gamma_r \alpha \gamma_g^{-1} = \gamma_g \gamma_r \gamma_{\alpha(h^{-1})} \alpha = \gamma_{gr\alpha(h^{-1})} \alpha = \gamma_{zs} \alpha = \gamma_s \alpha,$$

which shows the claim. Because of this, we have exactly $S(\Psi) = R(\overline{\alpha})$ and so $R(\overline{\alpha}) = \infty$ by hypothesis. Since we have the commuting diagram



and f is surjective, we get $R(\alpha) = \infty$ by using Lemma 1.7. The last assertion is an immediate consequence. □

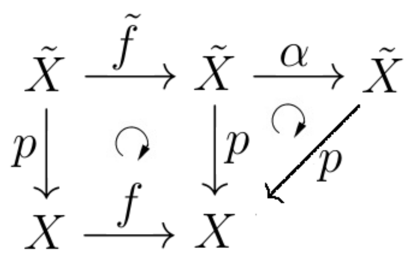
1.2 Topological motivation for twisted conjugacy

In spite of having a purely algebraic definition, twisted conjugacy classes arise from a topological viewpoint. This section will provide us a brief explanation of this fact. The reader just needs to be familiar with basic facts on covering spaces, liftings and the classic fundamental group.

Let X be a topological space with universal covering $p : \tilde{X} \to X$ and fix points $x_0 \in X$ and $\tilde{x_0} \in \tilde{X}$ with $p(\tilde{x_0}) = x_0$. Let $f : X \to X$ be a homeomorphism and fix a lifting $\widetilde{f(x_0)}$ of the point $f(x_0)$. Since $\tilde{X}$ is simply connected, from lifting theory there exists a unique lift $\tilde{f} : \tilde{X} \to \tilde{X}$ of f such that $\tilde{f}(\tilde{x_0}) = \widetilde{f(x_0)}$ and $p\tilde{f} = fp$. Now, consider the covering transformation set,

$$\mathcal{D} = \{\alpha : \tilde{X} \to \tilde{X} \mid \alpha \text{ is continous and } p\alpha = p\}.$$

This is a group with usual composition. Again, from lifting theory, any other lifting of f is of the form $\alpha\tilde{f}$ and this form is unique, that is, if $\alpha\tilde{f} = \alpha'\tilde{f}$ then $\alpha = \alpha'$ in $\mathcal{D}$.



Because of this, we can define a function $\tilde{f}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ in the following way: given $\alpha \in \mathcal{D}$, since $\tilde{f}\alpha$ is another lifting of f (because $p\tilde{f}\alpha = fp\alpha = fp$) we must have $\tilde{f}\alpha = \alpha'\tilde{f}$ for a unique $\alpha' \in \mathcal{D}$ and we set $\tilde{f}_{\mathcal{D}}(\alpha) = \alpha'$. In other words, $\tilde{f}_{\mathcal{D}}$ is characterized by the equation $\tilde{f}\alpha = \tilde{f}_{\mathcal{D}}(\alpha)\tilde{f}$. Then

$$\tilde{f}_{\mathcal{D}}(\alpha)\tilde{f}_{\mathcal{D}}(\beta)\tilde{f} = \tilde{f}_{\mathcal{D}}(\alpha)\tilde{f}\beta = \tilde{f}\alpha\beta = \tilde{f}_{\mathcal{D}}(\alpha\beta)\tilde{f},$$

so by uniqueness we have $\tilde{f}_{\mathcal{D}}(\alpha\beta) = \tilde{f}_{\mathcal{D}}(\alpha)\tilde{f}_{\mathcal{D}}(\beta)$ and $\tilde{f}_{\mathcal{D}}$ is a homomorphism. Since f is a homeomorphism, $\tilde{f}_{\mathcal{D}}$ is an isomorphism.

Let us define now the topological Reidemeister number of f. One can define an equivalence relation in the set $\mathcal{L}(f)$ of all liftings of f in the following way:

$$g \sim g' \Leftrightarrow \exists\, \beta \in \mathcal{D} : \beta g \beta^{-1} = g'.$$

The topological Reidemeister number $R(f)$ is defined by the number of such classes

$$R(f) = \mathrm{card}(\mathcal{L}(f)/\sim).$$

This number arises when we count the fixed points $Fix(f)$ of f (see [57]):

Proposition 1.11. *Under the conditions above, we have* $Fix(f) = \bigcup_{g\in\mathcal{L}(f)} p(Fix(g))$*, and, given two liftings* g *and* g'*, the sets* $p(Fix(g))$ *and* $p(Fix(g'))$ *are the same if* $g \sim g'$ *and disjoint if* $g \nsim g'$*. In other words,*

$$Fix(f) = \bigsqcup_{[g]\in\mathcal{L}(f)/\sim} p(Fix(g)).$$

It is worth to observe that $R(f)$ coincides exactly with the number of fixed point classes $p(Fix(g))$ of f given above. It couldn't have a more topological fashion!

We are ready to prove the main result of this section. It says that this topological (and fixed-point-counting) Reidemeister number coincides with the algebraic Reidemeister number of the induced automorphism in the fundamental group.

Proposition 1.12. *Under the conditions above,* $R(f) = R(f_*)$ *where* f_* *is the induced automorphism in* $\pi_1(X, x_0)$*.*

Demonstração. First we show that $R(f) = R(\tilde{f}_{\mathcal{D}})$. Remember that any lifting of f is of the form $\alpha\tilde{f}$ for a unique $\alpha \in \mathcal{D}$. Then, given two liftings $\alpha\tilde{f}$ and $\alpha'\tilde{f}$ of f, we have

$$\begin{aligned}
\alpha\tilde{f} \sim \alpha'\tilde{f} \quad &\Leftrightarrow \quad \exists\, \beta \in \mathcal{D} : \beta\alpha\tilde{f}\beta^{-1} = \alpha'\tilde{f} \\
&\Leftrightarrow \quad \exists\, \beta \in \mathcal{D} : \beta\alpha\tilde{f}_{\mathcal{D}}(\beta^{-1})\tilde{f} = \alpha'\tilde{f} \\
&\Leftrightarrow \quad \exists\, \beta \in \mathcal{D} : \beta\alpha\tilde{f}_{\mathcal{D}}(\beta^{-1}) = \alpha' \\
&\Leftrightarrow \quad \alpha \sim_{\tilde{f}_{\mathcal{D}}} \alpha' \text{ in } \mathcal{D},
\end{aligned}$$

so the number of fixed point classes $R(f)$ is exactly the number of twisted conjugacy classes $R(\tilde{f}_{\mathcal{D}})$ in $\mathcal{D}$.

Now we show that $R(\tilde{f}_{\mathcal{D}}) = R(f_*)$. From covering space theory (see [76]) there is an isomorphism

$$\begin{aligned} F : \pi_1(X, x_0) &\longrightarrow \mathcal{D} \\ [\gamma] &\longmapsto \alpha \text{ such that } \alpha(\tilde{x_0}) = \tilde{\gamma}(1), \end{aligned}$$

where $\tilde{\gamma}$ is the unique lifting of the path γ starting at $\tilde{x_0}$, and α is the unique covering transformation that sends the point $\tilde{x_0} = \tilde{\gamma}(0)$ to the point $\tilde{\gamma}(1)$.

Consider then the following diagram:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \tilde{f}_{\mathcal{D}}} \\ \pi_1(X, x_0) & \xrightarrow{F} & \mathcal{D} \end{array}$$

If we show that it is a commuting diagram we are done by Observation 1.6, since F is an isomorphism. On one hand, by definition,

$$Ff_*[\gamma] = F[f \circ \gamma] = \alpha' \text{ such that } \alpha'(\widetilde{f(x_0)}) = \widetilde{f \circ \gamma}(1),$$

where $\widetilde{f \circ \gamma}$ is the lifting of $f \circ \gamma$ starting at $\widetilde{f(x_0)}$. On the other hand,

$$\begin{aligned} \tilde{f}_{\mathcal{D}} F[\gamma] &= \tilde{f}_{\mathcal{D}}(\alpha) \quad (\text{where } \alpha(\tilde{x_0}) = \tilde{\gamma}(1)) \\ &= \alpha'' \quad (\text{such that } \alpha'' \tilde{f} = \tilde{f}\alpha). \end{aligned}$$

We are then left to show that $\alpha' = \alpha''$. By uniqueness it suffices to show that these two transformations coincide in the point $\tilde{f}(\tilde{x_0}) = \widetilde{f(x_0)}$. But

$$\alpha''(\tilde{f}(\tilde{x_0})) = \alpha'' \tilde{f}(\tilde{x_0}) = \tilde{f}\alpha(\tilde{x_0}) = \tilde{f} \circ \tilde{\gamma}(1),$$

and

$$\alpha'(\tilde{f}(\tilde{x_0})) = \alpha'(\widetilde{f(x_0)}) = \widetilde{f \circ \gamma}(1).$$

so we just have to see that $\widetilde{f \circ \gamma} = \tilde{f} \circ \tilde{\gamma}$. By uniqueness of liftings, it suffices to see that they are liftings of the same path with the same initial point. But

$$p\tilde{f}\tilde{\gamma} = fp\tilde{\gamma} = f\gamma = p\widetilde{f\gamma},$$

and

$$\tilde{f} \circ \tilde{\gamma}(0) = \tilde{f}(\tilde{x_0}) = \widetilde{f(x_0)} = \widetilde{f \circ \gamma}(0),$$

as desired. Then $R(f) = R(\tilde{f}_{\mathcal{D}}) = R(f_*)$.

□

When we want to count fixed points of a map f, we use Nielsen fixed point theory, in which the Nielsen number $N(f)$ is a lower bound for the minimal possible number $M[f]$ of fixed points of any map in the homotopy class $[f]$. For a large class of spaces (for example, for all compact manifolds of dimension ≥ 3 (see [97]) we do have $N(f) = M[f]$ and then, if $N(f) = 0$, f can be deformed by homotopy to a fixed point free map. Because of this, the Nielsen number is the main object of study in the theory. But for many spaces we also have either $N(f) = 0$ and $R(f) = \infty$, or $N(f) = R(f) < \infty$ (for example, for all Nilmanifolds). Then we can study $R(f)$ to count fixed points instead of $N(f)$. By the above proposition, we just have to count twisted conjugacy classes in the fundamental group. Just to visualize: if we consider a compact Nilmanifold of dimension ≥ 3, for example, then property R_∞ in its fundamental group would imply that every self homeomorphism f is deformable by homotopy to a fixed point free map, for $R(f) = R(f_*) = \infty$ implies $N(f) = 0$, which implies the desired property.

1.3 Graphs, Cayley graphs and basic constructions

Here we follow a notation similar to the one in [86]. All the omitted proofs can be found there. This section is a collection of definitions and constructions which will be used in different parts of the text. The reader which is already familiarized with the language may skip it and come back when needed.

Definition 1.13. A graph Γ is a 5-uple $(V(\Gamma), E(\Gamma), o, t, -)$, where

- $V(\Gamma)$ is called the set of vertices;
- $E(\Gamma)$ is called the set of edges;
- $o : E(\Gamma) \to V(\Gamma)$ with $y \mapsto o(y)$ and $o(y)$ is called the origin of y;
- $t : E(\Gamma) \to V(\Gamma)$ with $y \mapsto t(y)$ and $t(y)$ is called the terminus of y;
- $- : E(\Gamma) \to E(\Gamma)$ with $y \mapsto \overline{y}$ and $\overline{y}$ is called the inverse edge of y,

and such that $\overline{y} \neq y$, $\overline{\overline{y}} = y$ and $t(\overline{y}) = o(y)$ for all edges y.

There is the obvious association of a "simple enough" graph with its drawing. For example, if $V(\Gamma) = \{P, Q\}$, $E(\Gamma) = \{y, \overline{y}\}$, $o(y) = P$ and $t(y) = Q$, we may call Γ a segment and associate to it either one the following figures:

Definition 1.14. A morphism $f : \Gamma \to \Gamma'$ between two graphs Γ and Γ' consists of two maps $f_v : V(\Gamma) \to V(\Gamma')$ and $f_e : E(\Gamma) \to E(\Gamma')$ such that $o(f_e(y)) = f_v(o(y))$, $t(f_e(y)) = f_v(t(y))$ and $f_e(\overline{y}) = \overline{f_e(y)}$ for every edge $y \in E(\Gamma)$. If both f_v and f_e are injective (resp. surjective, bijective) we say that f is injective (resp. surjective, isomorphism). An isomorphism $f : \Gamma \to \Gamma$ is called and automorphism of Γ. The set of automorphisms of Γ with the natural composition operation is a group and is denoted by $Aut(\Gamma)$.

Definition 1.15. An orientation for a graph Γ is a subset $E^+(\Gamma) \subset E(\Gamma)$ such that $E(\Gamma) = E^+(\Gamma) \sqcup \overline{E^+(\Gamma)}$ is the disjoint union of the set $E^+(\Gamma)$ and the set of its inverse edges $\overline{E^+(\Gamma)}$ by the function $-$.

An orientation in Γ corresponds to choosing a direction for each edge in the drawing of Γ, or choosing between y and $\overline{y}$ for each edge y. We sometimes call $y \in E^+(\Gamma)$ an oriented edge, due to this geometric representation. When we are thinking about the drawing of Γ, we call $\{y, \overline{y}\}$ a geometric edge, since both edges represent the same "line drawing". The vertices $o(y), t(y)$ of an edge y are called the extremities of y. Two vertices are said adjacent if they are extremities of an edge.

Definition 1.16. Let Γ, Γ' be graphs and let $E^+(\Gamma)$ be an orientation for Γ. An oriented morphism $f : \Gamma \to \Gamma'$ consists of two maps $f_v : V(\Gamma) \to V(\Gamma')$ and $f_e : E^+(\Gamma) \to E(\Gamma')$ such that $o(f_e(y)) = f_v(o(y))$ and $t(f_e(y)) = f_v(t(y))$ for every $y \in E^+(\Gamma)$.

Observation 1.17. Any oriented morphism $f : \Gamma \to \Gamma'$ induces a graph morphism $f : \Gamma \to \Gamma'$. To define the morphism, we just have to extend the map $f_e : E^+(\Gamma) \to E(\Gamma')$ to a map $f_e : E(\Gamma) \to E(\Gamma')$, so we have to define $f_e(y)$ for $y \in \overline{E^+(\Gamma)}$. But since $\overline{y} \in E^+(\Gamma)$, $f_e(\overline{y})$ is defined. So we define $f_e(y) = \overline{f_e(\overline{y})}$. It is straightforward to verify that f is an authentic graph morphism. Furthermore, suppose $E^+(\Gamma')$ is an orientation for Γ' and that the oriented morphism $f : \Gamma \to \Gamma'$ given initially is a bijection between the vertices $f_v : V(\Gamma) \to V(\Gamma')$ and also between the oriented edges $f_e : E^+(\Gamma) \to E^+(\Gamma')$. Then, since the inversion maps $y \mapsto \overline{y}$ in both graphs are also bijections, it is easy to verify that the extended map $f_e : E(\Gamma) \to E(\Gamma')$ is also a bijection and so the graph morphism f is a graph isomorphism.

Definition 1.18. A path γ of length n in a graph Γ is a finite sequence $\gamma = y_1, ..., y_n$ of edges such that $t(y_i) = o(y_{i+1})$ for $1 \leq i \leq n-1$. A backtracking in γ is a subsequence having the form $..., y_{i-1}, y_i, \overline{y_i}, y_{i+2}, ...$ for some i. We say that γ is injective if the vertices $o(y_1), ..., o(y_n), t(y_n)$ are pairwise distinct. We say that γ is closed if $t(y_n) = o(y_1)$. A loop is a closed path of length 1. We say that γ is a circuit if it is closed and the path $y_1, ..., y_{n-1}$ is injective (the latter condition only needed if $n > 1$). Note, then, that every loop is also a circuit.

Definition 1.19 (Combinatorial trees)**.** We say that two vertices P, Q in a graph Γ are connected by a path $\gamma = y_1, ..., y_n$ in Γ if $o(y_1) = P$ and $t(y_n) = Q$. The graph Γ is said to be connected if every two vertices in Γ are connected by a path in Γ. A (combinatorial) tree is a nonempty connected graph with no circuits. We usually denote a tree by T, instead of Γ. In a tree T, a geodesic is a path with no backtrackings.

Note that, if a path γ has a backtracking, we can remove it by considering the path $\gamma' = y_1, ..., y_{i-1}, y_{i+2}, ..., y_n$. Then, if two vertices can be connected by a path, they can also be connected by a path without backtrackings.

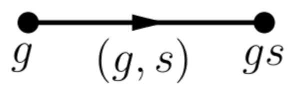
Trees play a fundamental role in graph theory because of the following property (see [86]):

Proposition 1.20. *Every two vertices in a tree can be connected by a unique geodesic, and any geodesic in a tree is an injective path.*

Definition 1.21. Given a group G and a nonempty subset $S \subset G$, the Cayley graph $\Gamma = \Gamma(G, S)$ is given by

- $V(\Gamma) = G$;
- $E(\Gamma) = (G \times S) \sqcup Z$, where Z is a disjoint set from $G \times S$ with a bijection $f : G \times S \to Z$;
- $o : E(\Gamma) \to V(\Gamma)$ is defined by $(g, s) \mapsto g$ and $f(g, s) \mapsto gs$;
- $t : E(\Gamma) \to V(\Gamma)$ is defined by $(g, s) \mapsto gs$ and $f(g, s) \mapsto g$;
- $- : E(\Gamma) \to E(\Gamma)$ is exactly the function f in $G \times S$ and the function f^{-1} in Z.

To be more intuitive, the "drawing" of Γ is given by all the connections of the form



for all $g \in G$ and $s \in S$. If $s \in S$, we denote $(g, s^{-1}) = f(gs^{-1}, s)$. This corresponds to walking through the edge (gs^{-1}, s) in the opposite direction, so (g, s^{-1}) starts at $gs^{-1}s = g$ and finishes at gs^{-1}. A path p in Γ is characterized by its initial vertex $g \in G$ and the (oriented) edges on which it walks. So we will denote a path in Γ by $p = (g, s_1 ... s_n)$ with $s_i \in S^{\pm 1}$. This means that p starts in g, walks through the edge (g, s_1) until gs_1, then walks through the edge (gs_1, s_2) until gs_1s_2, and so on, always by right multiplication. The set of paths is denoted by $P(\Gamma)$. Sometimes we will denote the edges in the picture only by the label s. An orientation for $\Gamma(G, S)$ is given by $E^+ = G \times S$; it will always be its orientation, unless we say otherwise.

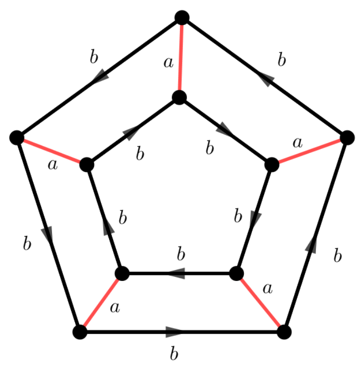


Figura 1.2: Cayley graph of the dihedral group $D_5 = \langle a, b \mid a^2 = 1, b^5 = 1, abab = 1 \rangle$ with $S = \{a, b\}$. Multiplying by a corresponds to crossing the red edges. Multiplying by b corresponds to walking in the black edges in the indicated direction.

The shape of the Cayley graph detects free generation:

Proposition 1.22. *Let G be a group, $S \subset G$ and denote by $\Gamma = \Gamma(G, S)$ its Cayley graph. Then Γ is a tree if and only if G is a free group with basis S.*

Demonstração. Note that connecting a vertex g to 1 in Γ by a path $p = (1, s_1 ... s_n)$ corresponds to writing $g = s_1 ... s_n$ for $s_i \in S^{\pm}$. Then Γ is connected if and only if $G = \langle S \rangle$. Let us show the proposition. If Γ is a tree, then we must have $G = \langle S \rangle$ and also $S \cap S^{-1} = \emptyset$ (otherwise, we would have either $1 \in S$ and $p = (1, 1)$ would be a loop, or $s' = s^{-1} \in S$ for some element

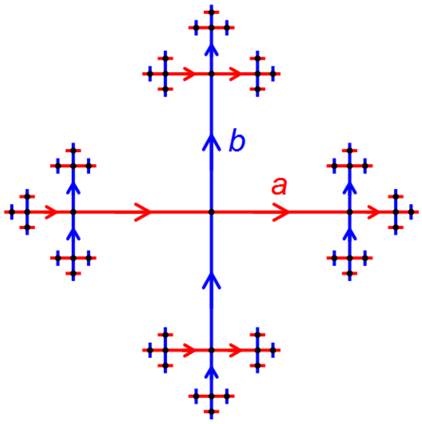


Figura 1.3: a portion of the Cayley graph of the free group $F_2 = \langle a, b\rangle$ with $S = \{a, b\}$. Multiplying by a is walking right and multiplying by b is walking up.

$s \in S$ and then $p = (1, ss')$ would be a circuit in Γ). To show that G is free with basis S, we just need to show that there is no equation of the form $s_1^{\epsilon_1}...s_n^{\epsilon_n} = 1$ in G with $s_i \in S$, $\epsilon_i = \pm 1$ and $s_i^{\epsilon_i}s_{i+1}^{\epsilon_{i+1}} \neq 1$ (see [71], page 4, Proposition 1.9). Suppose by contradiction that there is such an equation, and suppose it has minimal length. Then $n \geq 3$, because if $n = 1$, $p = (1, s_1^{\epsilon_1})$ would be a loop and the case $n = 2$ we just treated above. Note that the vertices $P_0 = 1$ and $P_i = s_1^{\epsilon_1}...s_i^{\epsilon_i}$ for $1 \leq i \leq n-1$ are pairwise distinct, because if $P_i = P_{i+k}$ we would have an equation $s_{i+1}^{\epsilon_{i+1}}...s_{i+k}^{\epsilon_{i+k}} = 1$ in G with length $< n$, contradiction. Since $n \geq 3$ and the vertices are pairwise distinct with $P_n = P_0$, the path $p = (1, s_1^{\epsilon_1}...s_n^{\epsilon_n})$ is a circuit in Γ, contradiction. Then G is free with basis S. Conversely, suppose that G is free with basis S. Then Γ is connected. We just have to show that Γ does not contain any circuit. Suppose by contradiction that $p = (g, s_1...s_n)$ is a circuit in Γ with $s_i \in S^{\pm}$. Since p has no backtrackings, the word $s_1...s_n$ is reduced in G, and since p is closed we have $gs_1...s_n = g$, which implies $s_1...s_n = 1$ in G. Then G cannot be a free group, contradiction. □

Definition 1.23. A subgraph Γ' of $\Gamma = (V(\Gamma), E(\Gamma), o, t, -)$ is given by two subsets $V(\Gamma') \subset V(\Gamma)$ and $E(\Gamma') \subset E(\Gamma)$ such that the restrictions of o, t and $-$ to Γ' are well defined (i.e., we have $o : E(\Gamma') \to V(\Gamma')$, $t : E(\Gamma') \to V(\Gamma')$ and $- : E(\Gamma') \to E(\Gamma')$). In other words, Γ' is a graph with the respective restrictions of o, t and $-$. We denote this relation by $\Gamma' \leq \Gamma$. A subtree of Γ is a subgraph that is a tree. A maximal tree T in Γ is a maximal element in the set of all subtrees of Γ with the partial order given by the subgraph relation "$\leq$". This is the same as saying that T is a subtree of Γ and if T' is a subtree of Γ with $T \leq T'$, then $T = T'$.

By the well known Zorn's Lemma, every nonempty graph Γ has a maximal tree. Also, one can show that every tree inside Γ is contained in a maximal one. By a simple proof by contradiction, one can also prove

Proposition 1.24. *If T is a maximal tree in Γ then $V(T) = V(\Gamma)$, that is, T contais all the vertices of Γ.*

When does the removal of an edge in a connected graph "breaks" it in two connected pieces? The answer will be useful in Chapter 11 and is the following:

Lemma 1.25. *Let y_0 be an edge of a connected graph Γ and let $\Gamma - y_0$ denote the subgraph of Γ obtained by geometrically removing y_0 from Γ, that is, $V(\Gamma - y_0) = V(\Gamma)$ and $E(\Gamma - y_0) = E(\Gamma) - \{y_0, \overline{y_0}\}$. If there is a maximal tree T of Γ not containing y_0 then $\Gamma - y_0$ is connected. If not, then $\Gamma - y_0$ contains exactly two connected components, each one containing one of the extremities of y_0.*

In the latter situation we say y_0 is a *separating* edge of Γ.

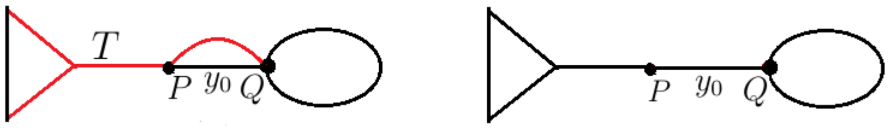


Figura 1.4: y_0 is separating on the right, but not on the left.

Demonstração. In the former case, let T be such a maximal tree and P, Q be any two vertices of $\Gamma - y_0$. By Proposition 1.24, P, Q are vertices of T, so there is a path p connecting them in T (therefore in Γ). Since T does not contain y_0, the path p is inside $\Gamma - y_0$, as desired.

In the latter case, y_0 must be a segment with distinct extremities $P = o(y_0)$ and $Q = t(y_0)$ (see figure). We claim that every path in Γ from P to Q must cross y_0. Indeed, let p be such path. We can extract from p an injective subpath $\tilde{p}$ from P to Q, without backtrackings. Then $\tilde{p}$, as a subgraph, is itself a tree and must be contained in a maximal tree T. By hypothesis, T contains y_0. But then $\tilde{p}$ and the edge y_0 itself are by definition geodesics in T from P to Q. By uniqueness, $\tilde{p}$ is the path y_0 and so p crosses y_0, which shows the claim. Because of this, there is no path from P to Q in $\Gamma - y_0$, which means that this graph contains at least two connected components, say, Γ_P and Γ_Q. Let us show that they are the only ones. Let v be a vertex in $\Gamma - y_0$ which is not in Γ_P and let us show that it is in Γ_Q. Let p be a path in Γ from v to P. Since such a path cannot exist inside $\Gamma - y_0$ (because v is not in Γ_P) it must cross y_0 at least once. Consider the first time p crosses y_0. If it crossed "in the right direction", that is, first over P then over Q, then the restriction of p from v to P would connect them inside $\Gamma - y_0$, contradiction. So, when p crosses the edge y_0 for the first time, it must be "in the left direction", that is, first Q then P. But then the restriction of p from v to Q connects these vertices inside $\Gamma - y_0$, which shows that v is in Γ_Q, as desired. □

Definition 1.26. A closed path in a graph is contractible if, after removing all its backtrackings, we obtain the constant path.

Proposition 1.27. *Let γ be a closed path in a graph Γ. If γ does not contain any circuit, then it is contractible.*

Demonstração. Since γ does not contain any circuit, the edges of γ form a connected subgraph of Γ with no circuits, that is, a tree. Since every tree is contained in a maximal tree, γ must be contained in a maximal tree, say, T. Let P be the origin and terminus of γ. Then, after removing all the backtrackings of γ we obtain a geodesic γ' from P to P in T, by definition. By the uniqueness of geodesics in a tree, we have that γ' is the constant path, so γ is contractible. □

Definition 1.28. A vertex P in a graph Γ is terminal when it is the terminus of exactly one edge of Γ.

The properties about terminal vertices we will need later are all summarized in the following proposition (see [86]):

Proposition 1.29. *Every finite tree must contain at least one terminal vertex. If P is terminal in a graph Γ and y is the edge with $t(y) = P$, let $\Gamma - P$ be the subgraph of Γ defined as follows: $V(\Gamma - P) = V(\Gamma) - \{P\}$, $E(\Gamma - P) = E(T) - \{y, \overline{y}\}$. Then $\Gamma - P$ is a tree if and only if Γ is a tree.*

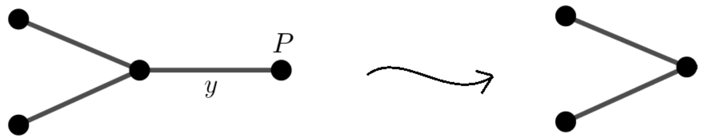


Figura 1.5: the tree T on the left and the tree $T - P$ on the right

Other basic constructions we will need are the following:

Definition 1.30. Let Γ be a graph and $T \leq \Gamma$ a subtree. The contraction of T is how we call the graph denoted by Γ/T, which is defined by:

- $V(\Gamma/T) = V(\Gamma)/\sim$, where $P \sim Q$ iff $P = Q$ or $P, Q \in V(T)$;
- $E(\Gamma/T) = E(\Gamma) - E(T)$;
- $o : E(\Gamma/T) \to V(\Gamma/T)$ with $y \mapsto [o(y)]$ ([.] denotes the class of an element);
- $t : E(\Gamma/T) \to V(\Gamma/T)$ with $y \mapsto [t(y)]$;
- $- : E(\Gamma/T) \to E(\Gamma/T)$ is the restriction of $- : E(\Gamma) \to E(\Gamma)$.

Geometrically one can imagine that the entire tree T is being contracted to one single vertex and all the other edges in Γ are preserved. Obviously, if T is a maximal tree, then all the vertices become just one (thanks to Proposition 1.24) and Γ/T is what we call a "bouquet", as in next figure. More generally, we could define the contraction of a family of disjoint subtrees $\Lambda = \sqcup_\alpha T_\alpha$ by defining $V(\Gamma/\Lambda) = V(\Gamma)/\sim$, where $P \sim Q$ iff $P = Q$ or $P, Q \in V(T_\alpha)$ for some α and $E(\Gamma/\Lambda) = E(\Gamma) - E(\Lambda)$. This means that each tree T_α is being contracted to a single vertex. Intuitively, since contracting a tree do not "kill" any circuits, we have the following important property (see [86], pg. 23, Corollary 2):

Proposition 1.31. *Let Γ be a graph and $\Lambda = \sqcup_\alpha T_\alpha$ the disjoint union of a family of subtrees of Γ. Then Γ is a tree if and only if Γ/Λ is a tree.*

Definition 1.32. Let Γ be a tree and $A \subset V(\Gamma)$. The subtree T generated by A is the smallest (minimal, in the subgraph relation "$\leq$") tree of Γ which contains all the vertices in A.

Observation 1.33. It is straightforward to show that T is the tree consisting of all the edges and vertices of all the geodesics in Γ connecting all the vertices in A to each other. With a similar argument one can also show that T is the tree consisting of all the edges and vertices of all the geodesics in Γ connecting the vertices in A to a fixed vertex P in A.

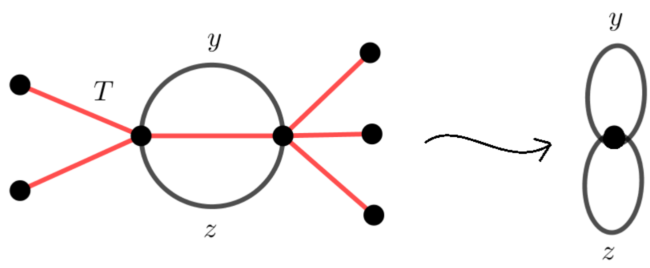


Figura 1.6: the tree T in red is contracted to one vertex

The construction above shall not be confused with the following:

Definition 1.34. Let Γ be any graph and $A \subset V(\Gamma)$. The subgraph $\Gamma_A \leq \Gamma$ induced by A is given by

- $V(\Gamma_A) = A$;
- $E(\Gamma_A) = \{y \in E(\Gamma) \mid o(y), t(y) \in A\}$.

By the above definition, a path γ in Γ is contained in Γ_A if and only if the extremities of all its edges are in A, or, let's say, if γ runs only over vertices in A. Note also that Γ_A need not be connected, even if Γ is.

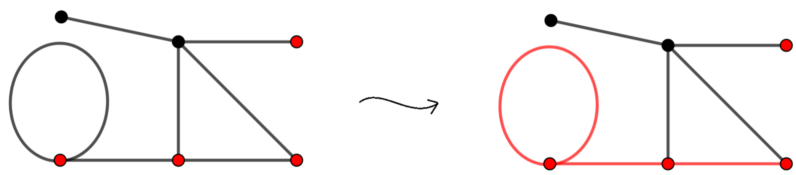


Figura 1.7: The vertices A on the left, not connected Γ_A on the right

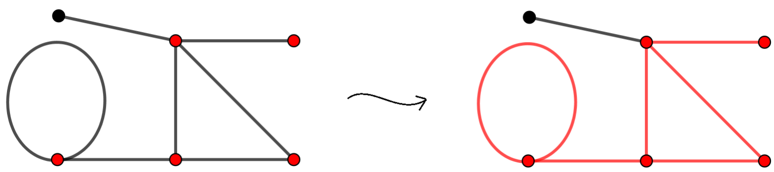


Figura 1.8: The vertices A on the left, connected Γ_A on the right

Definition 1.35. An action of a group G on a graph Γ is a group homomorphism $\varphi : G \to Aut(\Gamma)$. When such an action exists we say that G acts on Γ. Given $g \in G$, the automorphism $\varphi(g)$ then consists of two bijective maps $\varphi(g)_v$ (resp. $\varphi(g)_e$) between the vertices (resp. edges) of Γ, so we abbreviate $\varphi(g)_v(P)$ by $g \cdot P$ for any vertex P and $\varphi(g)_e(y)$ by $g \cdot y$ for any edge y. With this notation, the action satisfies

1) $(gh) \cdot P = g \cdot (h \cdot P)$ and $1 \cdot P = P$ for any vertex P and $g, h \in G$;

2) $(gh) \cdot y = g \cdot (h \cdot y)$ and $1 \cdot y = y$ for any edge y and $g, h \in G$;

3) $o(g \cdot y) = g \cdot o(y)$ and $t(g \cdot y) = g \cdot t(y)$ for any edge y and $g \in G$;

4) $g \cdot \overline{y} = \overline{g \cdot y}$ for any edge y and $g \in G$.

An inversion in such an action is a pair $(g, y) \in G \times E(\Gamma)$ such that $g \cdot y = \overline{y}$. We say that G acts without inversion in Γ when such an inversion does not exists. We say that the action is free and that G acts freely on Γ when G acts without inversion and $g \cdot P = P$ implies $g = 1$.

Example 1.36. It is easy to verify that if G is a group and $S \subset G$, then G acts freely on its Cayley graph $\Gamma(G, S)$ if we define the action as $g \cdot h = gh$ on a vertex h and $g \cdot (h, s) = (gh, s)$ on an edge (h, s).

Definition 1.37. If G acts on Γ without inversion, we define the quotient graph, or the orbit graph G/Γ by:

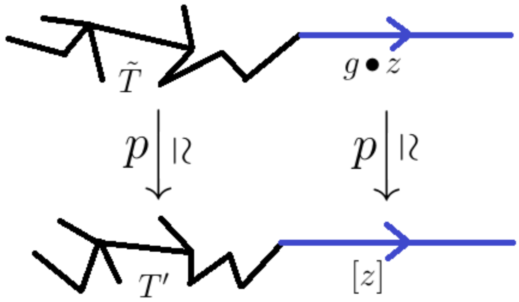
- $V(G/\Gamma) = V(\Gamma)/\sim$, where $P \sim P'$ if $g \cdot P = P'$ for some $g \in G$;
- $E(G/\Gamma) = E(\Gamma)/\sim$, where $y \sim y'$ if $g \cdot y = y'$ for some $g \in G$;
- $o([y]) = [o(y)]$, $t([y]) = [t(y)]$ and $\overline{[y]} = [\overline{y}]$ for any edge y.

It is easy to verify that G/Γ is a graph by using the properties of an action. In particular, the action being without inversion is what guarantees that $\overline{[y]} \neq [y]$ for any edge $[y]$, because $[\overline{y}] = [y]$ would give rise to an inversion $g \cdot y = \overline{y}$, by definition. It is straightforward to see that there is a natural surjective morphism $p : \Gamma \to G/\Gamma$ with $p(P) = [P]$ and $p(y) = [y]$. The vertex and edge classes $[P]$ and $[y]$ are called the orbits of the action.

Definition 1.38. Let a group G act without inversions on a graph Γ and let $p : \Gamma \to G/\Gamma$ be the natural projection defined above. If $\Lambda \leq G/\Gamma$, we say that a subgraph $\tilde{\Lambda} \leq \Gamma$ is a lift of Λ if $p(\tilde{\Lambda}) = \Lambda$ and the restriction morphism $p|_{\tilde{\Lambda}} : \tilde{\Lambda} \to \Lambda$ is an isomorphism. We also say that Λ is lifted to $\tilde{\Lambda}$.

Proposition 1.39. *Let a group G act without inversions on a graph Γ and let $p : \Gamma \to G/\Gamma$ be the projection. Every subtree T of G/Γ can be lifted to a subtree $\tilde{T}$ of Γ.*

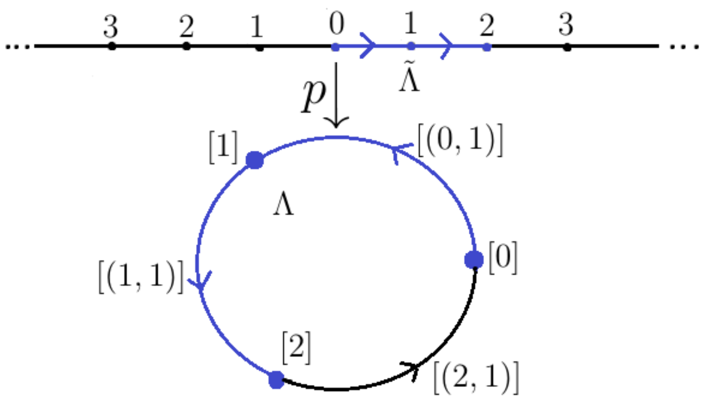
Demonstração. Let $\Omega = \{\tilde{T} \leq \Gamma \mid \tilde{T}$ is a tree, $p(\tilde{T}) \subset T$ and $p|_{\tilde{T}} : \tilde{T} \to T$ is injective$\}$. Obviously, $\Omega \neq \emptyset$: indeed, if $[P]$ is any vertex of T, then the single vertex P, thought as a subtree of Γ, is in Ω. Let us ordinate Ω by the subgraph relation "$\leq$". Since the union $\cup_\alpha \tilde{T}_\alpha$ of any chain $\{\tilde{T}_\alpha\}_\alpha$ in Ω is an upper bound for the chain (homework for the reader), by Zorn's Lemma there is a maximal element in Ω, which we will also denote by $\tilde{T}$. We also have the injection $p|_{\tilde{T}} : \tilde{T} \to T$. Let $T' = p(\tilde{T}) \leq T$, so we have an isomorphism $p|_{\tilde{T}} : \tilde{T} \to T'$ and T' is also a tree. We just have to show that $T' = T$. Suppose by contradiction that there is an edge $[z]$ in T and outside T', and because T is connected we can also suppose without loss of generality that its origin $[o(z)]$ is in T' (see the next figure). Note then that its terminus $[t(z)]$ must not be in T', because if both extremities of the edge $[z]$ were in T' we could connect them inside T' by a geodesic γ, and then the concatenation of γ with the edge $[z]$ would be a circuit in T. So, the union of T' with the edge $[z]$ is also a tree by Proposition 1.29. Now, $[o(z)] \in T' = p(\tilde{T})$ by construction, so write $[o(z)] = [P_0]$ for some vertex P_0 in $\tilde{T}$. We must have $g \cdot o(z) = P_0$ for some $g \in G$. Then the edge $g \cdot z$ is a lift of $[z]$ (because $p(g \cdot z) = [g \cdot z] = [z]$) whose origin is in $\tilde{T}$ (because $o(g \cdot z) = g \cdot o(z) = P_0$). This edge must also be outside $\tilde{T}$, because if $g \cdot z$ was in $\tilde{T}$ its projection $[z]$ would be in $p(\tilde{T}) = T'$, contradiction. Similarly, its terminus $t(g \cdot z)$ is outside $\tilde{T}$, because its projection is $[t(z)]$ which is not in T'. So the union of $\tilde{T}$ and the edge $g \cdot z$ is also a tree by Proposition 1.29 and we have p as an isomorphism (see figure) between these extended trees. Since $[z]$ is in T, it is also an injection into T. Therefore the extended tree in Γ is in Ω and $\tilde{T}$ is not maximal, contradiction. This completes the proof.



□

Definition 1.40. If a group G acts without inversion on a graph Γ, a tree of representatives of Γ mod G is any lifting of a maximal tree of G/Γ.

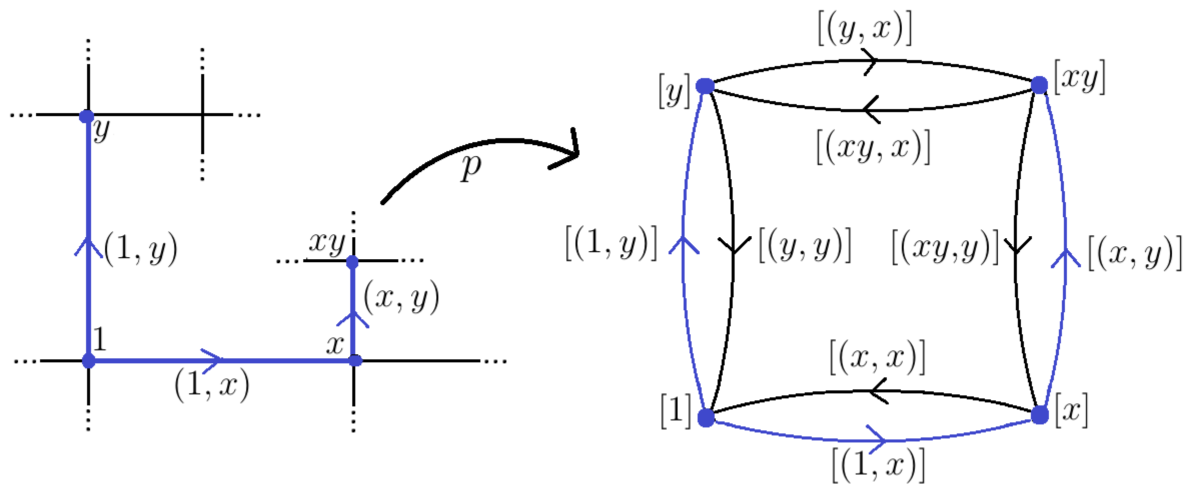
Example 1.41. Define an action of $\mathbb{Z}$ on its Cayley graph $\Gamma = \Gamma(\mathbb{Z}, \{1\})$ by $n \cdot m = 3n + m$ for $n \in \mathbb{Z}$ and a vertex m in Γ, and $n \cdot (m, 1) = (3n + m, 1)$ for an edge $(m, 1)$ in Γ. It has no inversions and since $[3] = [0]$ the orbit graph $\mathbb{Z}/\Gamma$ is isomorphic to a circuit of length 3 (see figure). The blue path $\Lambda = [(0,1)], [(1,1)]$ is a maximal tree in the orbit graph, so the blue path $\tilde{\Lambda} = (0,1), (1,1)$ is a lift for it and a tree of representatives of Γ mod $\mathbb{Z}$, by definition.



Example 1.42. The free group F_2 on two generators x and y acts naturally on its Cayley graph $\Gamma = \Gamma(F_2, \{x, y\})$. So, every subgroup of F_2 also acts on Γ. Let $H \leq F_2$ be the subgroup consisting of the words whose sum of all the x-exponents is even, as well as the sum of all the y-exponents. In other words, $H = \ker(\varphi)$ where

$$\begin{aligned} \varphi : & F_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \\ & w \longmapsto ((w)^x, (w)^y). \end{aligned}$$

Note that φ is surjective, so by the isomorphism theorem we have $F_2/H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and H has index 4 in F_2 with $\{1, x, y, xy\}$ as a collection of coset representatives. Since the action of $h \in H$ on a vertex g of Γ is always given by $h \cdot g = hg$, the orbit $[g]$ of a vertex g is exactly the coset Hg. Since there are only 4 cosets, the orbit graph H/Γ has only 4 vertices $[1], [x], [y]$ and $[xy]$. There are also only 8 geometric edges, as the figure shows.



If the three edges $[(1, y)], [(1, x)]$ and $[(x, y)]$ are chosen as a maximal tree of H/Γ, we can lift them to the three edges $(1, y), (1, x)$ and (x, y) which form the blue tree of representatives of Γ mod H.

Proposition 1.43. *Let G act on a graph Γ. Then there is no inversion if and only if Γ has an orientation E^+ which is preserved by the action (that is, $y \in E^+,\ g \in G \Rightarrow g \cdot y \in E^+$, or $G \cdot E^+ \subset E^+$).*

Demonstração. ($\Leftarrow$) Let E^+ be such orientation and suppose by contradiction that there is an inversion $g \cdot y = \overline{y}$. We have either $y \in E^+$ or $\overline{y} \in E^+$. In the former case, we have $\overline{y} = g \cdot y \in G \cdot E^+ - E^+$, contradiction. In the latter case, $y = \overline{\overline{y}} = \overline{g \cdot y} = g \cdot \overline{y} \in G \cdot E^+ - E^+$, also a contradiction. ($\Rightarrow$) Suppose the action has no inversion. Then we can consider the orbit graph G/Γ. Choose any orientation W in G/Γ and define $E^+ = \{y \in E(\Gamma) \mid [y] \in W\}$. It is straightforward to check that E^+ is an orientation for Γ. To see that $G \cdot E^+ \subset E^+$, let $g \in G$ and $y \in E^+$. We have $[g \cdot y] = [y] \in W$, so by definition $g \cdot y \in E^+$, as desired. □

For example, we know that if $S \subset G$, the group G acts freely (in particular without inversion) on $\Gamma(G, S)$. The orientation $G \times S$ of $\Gamma(G, S)$ is always preserved since $g \cdot (h, s) = (gh, s) \in G \times S$ for every $g \in G$ and $(h, s) \in G \times S$.

1.4 The Reidemeister-Schreier algorithm

Finding a presentation for a subgroup $H \leq G$ in terms of a presentation of G is not an easy task in general. The first results in this direction were obtained by Schreier (1927) and Reidemeister (1932), leading to the names "Reidemeister-Schreier" method, process, theorem or algorithm. There are many versions of the Reidemeister-Schreier Theorem, for example in [72] (section 2.3, Theorem 2.9, page 94) and [71] (Proposition 4.1, Chapter II, page 103). We are going to show a more geometric version of it, based on some results of [86] about group actions and Cayley graphs. This result will be useful especially in Chapter 6.

Theorem 1.44. *Let a group G act freely on a tree X. Let $\tilde{\Lambda}$ be a tree of representatives of X mod G (associated to a maximal tree Λ of G/X) and let $E^+ = E^+(X)$ be an orientation of X which is preserved under the action (see Proposition 1.43). Let*

$$S = \{s \neq 1 \in G \mid \exists\ y \in E^+ \text{ with } o(y) \in \tilde{\Lambda} \text{ and } t(y) \in s\tilde{\Lambda}\}.$$

Then G is a free group with basis S.

Demonstração. The basic idea is to show that $\Gamma(G,S) \simeq X'$, where X' is a quotient of the tree X given by the contraction of some disjoint subtrees of X. Since X is a tree, X' is also a tree by Proposition 1.31 and then so it is $\Gamma(G,S)$. By Proposition 1.22, G must be free with basis S.

Let us define the subtrees of X which we will contract. Denote by $p : X \to G/X$ the orbit projection. Every $g \in G$ induces an automorphism of X, so all the $g\tilde{\Lambda}$, $g \in G$, are also subtrees of X. Furthermore, we claim that they are pairwise disjoint. Indeed, suppose two of them, say, $g\tilde{\Lambda}$ and $h\tilde{\Lambda}$ have a common vertex. So $g \cdot P = h \cdot Q$ for P, Q vertices of $\tilde{\Lambda}$. Then $h^{-1}g \cdot P = Q$ implies $p(P) = p(Q)$. Since $p|_{\tilde{\Lambda}}$ is an isomorphism we have $P = Q$. So $h^{-1}g \cdot P = P$, but since the action is free we must have $h^{-1}g = 1$, or $h = g$, which shows the claim. Denote this family of disjoint subtrees by $G \cdot \tilde{\Lambda}$. There is a bijection $G \cdot \tilde{\Lambda} \to G$ with $g\tilde{\Lambda} \to g$. Define $X' = X/(G \cdot \tilde{\Lambda})$ their contraction in X, which we already know is a tree.

Each subtree $g\tilde{\Lambda}$ in X becomes a single vertex in X' which we denote by $(g\tilde{\Lambda})$. We claim that these are the only vertices of X'. In fact, for every vertex P of X, $[P] = p(P) \in V(G/X) = V(\Lambda) = p(V(\tilde{\Lambda}))$ (using Proposition 1.24), so $[P] = [Q]$ for some vertex Q of $\tilde{\Lambda}$. By definition, $P = g \cdot Q \in g\tilde{\Lambda}$ for some $g \in G$. This shows that every vertex of X is inside some $g\tilde{\Lambda}$, so by definition of X' every vertex of X' must be some $(g\tilde{\Lambda})$. Because the $g\tilde{\Lambda}$ are pairwise disjoint, the vertices $(g\tilde{\Lambda})$ are pairwise distinct, so we have a bijection $V(X') \to G \cdot \tilde{\Lambda}$ with $(g\tilde{\Lambda}) \mapsto g\tilde{\Lambda}$. By putting this bijection together with the one we obtained in the previous paragraph we have a bijection

$$\begin{aligned} \alpha : V(X') &\longrightarrow G = V(\Gamma(G,S)) \\ (g\tilde{\Lambda}) &\longmapsto g. \end{aligned}$$

Then, by Observation 1.17, to create an isomorphism $\alpha : X' \to \Gamma(G,S)$ we just have to put an orientation in X' and define an oriented morphism $\alpha : X' \to \Gamma(G,S)$ which is also a bijection between the oriented edges. Remember that $G \times S$ is the orientation of $\Gamma(G,S)$.

By definition, $E(X') = E(X) - E(G \cdot \tilde{\Lambda})$. Define $E^+(X') = E^+(X) - E(G \cdot \tilde{\Lambda})$. We claim that this is an orientation for X'. Indeed, using that $y \in E(G \cdot \tilde{\Lambda}) \Leftrightarrow \overline{y} \in E(G \cdot \tilde{\Lambda})$ (because $E(G \cdot \tilde{\Lambda})$ is a subgraph of X), we have

$$\begin{aligned} y \in E(X') = E(X) - E(G \cdot \tilde{\Lambda}) \quad &\Rightarrow \quad y \in E(X) = E^+(X) \cup \overline{E^+(X)} \\ &\Rightarrow \quad y \in E^+(X) \text{ or } \overline{y} \in E^+(X) \\ &\Rightarrow \quad y \in E^+(X) - E(G \cdot \tilde{\Lambda}) \text{ or } \overline{y} \in E^+(X) - E(G \cdot \tilde{\Lambda}), \end{aligned}$$

which shows that $E(X') = E^+(X') \cup \overline{E^+(X')}$. Also,

$$\begin{aligned} y \in E^+(X') = E^+(X) - E(G \cdot \tilde{\Lambda}) \quad &\Rightarrow \quad y \in E^+(X) \\ &\Rightarrow \quad \overline{y} \notin E^+(X) \\ &\Rightarrow \quad \overline{y} \notin E^+(X) - E(G \cdot \tilde{\Lambda}) = E^+(X') \\ &\Rightarrow \quad y \notin \overline{E^+(X')}, \end{aligned}$$

and so $E^+(X') \cap \overline{E^+(X')} = \emptyset$, which shows the claim.

Now we define the oriented morphism $\alpha : X' \to \Gamma(X,S)$ by defining the map $\alpha : E^+(X') \to G \times S$. Given $y \in E^+(X')$, we must have $o(y) = (g\tilde{\Lambda})$ and $t(y) = (g'\tilde{\Lambda})$ in X' for $g \neq g' \in G$ (if $g = g'$, y would be a loop in the tree X', contradiction). This implies $s = g^{-1}g' \in S$ by definition of S. Indeed, $s \neq 1$ because $g \neq g'$. Also, the edge $g^{-1} \cdot y$ is in $E^+(X)$ (because y is and the action preserves orientation) and is such that $o(g^{-1} \cdot y) = g^{-1} \cdot o(y) \in g^{-1}g\tilde{\Lambda} = \tilde{\Lambda}$ and $t(g^{-1} \cdot y) = g^{-1} \cdot t(y) \in g^{-1}g'\tilde{\Lambda} = s\tilde{\Lambda}$. Then we can define $\alpha(y) = (g,s)$. To see that α is an oriented morphism, note that

- $\alpha(o(y)) = \alpha((g\tilde{\Lambda})) = g = o(g,s) = o(\alpha(y))$;
- $\alpha(t(y)) = \alpha((g'\tilde{\Lambda})) = g' = gg^{-1}g' = gs = t(g,s) = t(\alpha(y))$.

Finally, let us check that $\alpha : E^+(X') \to G \times S$ is a bijection:

- Let $(g,s) \in G \times S$. Let $y \in E^+(X)$ such that $o(y) \in \tilde{\Lambda}$ and $t(y) \in s\tilde{\Lambda}$. We claim that $y' = g \cdot y$ is an element of $E^+(X')$ such that $\alpha(y') = (g,s)$. Indeed, first note that $y \in E^+(X')$. If that was not the case we would have y an edge of some $h\tilde{\Lambda}$ for some $h \in G$. Then on one hand we would have $o(y) \in h\tilde{\Lambda} \cap \tilde{\Lambda}$ which would imply $h = 1$, and on the other hand we would also have $t(y) \in h\tilde{\Lambda} \cap s\tilde{\Lambda}$ which would imply $h = s$, so $s = 1$, contradiction. Now, $y' = g \cdot y$ is also in $E^+(X)$ because the action preserves the orientation of X and if y' were in $G \cdot \tilde{\Lambda}$ we would have $y = g^{-1} \cdot y' \in g^{-1} \cdot G \cdot \tilde{\Lambda} \subset G \cdot \tilde{\Lambda}$, contradiction. So $y' \in E^+(X')$. Let us compute $\alpha(y')$. Since in X $o(y') = o(g \cdot y) = g \cdot o(y) \in g\tilde{\Lambda}$ and $t(y') = t(g \cdot y) = g \cdot t(y) \in gs\tilde{\Lambda}$, in X' we have $o(y') = (g\tilde{\Lambda})$ and $t(y') = (gs\tilde{\Lambda})$. So by definition $\alpha(y') = (g,s')$ where $s' = g^{-1}(gs) = s$. So $\alpha(y') = (g,s)$ and α is surjective.

- Let $y, \tilde{y} \in E^+(X')$ such that $(g, s) = \alpha(y) = \alpha(\tilde{y}) = (\tilde{g}, \tilde{s})$. By definition of α we have $o(y) = (g\tilde{\Lambda})$, $t(y) = (g'\tilde{\Lambda})$ and $s = g^{-1}g'$. Similarly, $o(\tilde{y}) = (\tilde{g}\tilde{\Lambda})$, $t(\tilde{y}) = (\tilde{g}'\tilde{\Lambda})$ and $\tilde{s} = \tilde{g}^{-1}\tilde{g}'$. Then $g^{-1}g' = s = \tilde{s} = \tilde{g}^{-1}\tilde{g}' = g^{-1}\tilde{g}'$ and so $g' = \tilde{g}'$. Then, both edges y and $\tilde{y}$ in X' start in $(g\tilde{\Lambda})$ and finish in $(g'\tilde{\Lambda})$. Since X' is a tree, $y = \tilde{y}$. So α is injective, which completes the proof.

□

Before walking in the direction of Reidemeister-Schreier Theorem, we point out two direct and beautiful consequences of the theorem above which are important tools in geometric and combinatorial group theory.

Corollary 1.45. *A group G is free if and only if it acts freely on a tree.*

Demonstração. If G is free, say, with a basis Z, then by Proposition 1.22 $\Gamma(G, Z)$ is a tree on which G acts freely. Conversely, if G acts freely on a tree X, by Theorem 1.44 G is free (with the basis S given there). □

Corollary 1.46 (Schreier's Theorem)**.** *Any subgroup of a free group is free.*

Demonstração. Let G be a free group and $H \leq G$. By the previous corollary, let X be a tree on which G acts freely. Then it is easy to see that the restriction of this action to H is also a free action of H in X. Again, by the previous corollary, H is a free group. □

After reading the corollary above, one could ask: if H is free, how to find a basis for it? The answer is given by the next theorem.

Definition 1.47. Let G be a group and $H \leq G$. A Scherier's transversal T of G mod H is a collection of coset representatives of G mod H (that is, $T \subset G$ such that $G = \sqcup_{t \in T} Ht$ (disjoint union)) with $1 \in T$ and with the following property: "if $t = x_1^{\epsilon_1}...x_n^{\epsilon_n} \in T$, $\epsilon_i = \pm 1$ is a reduced word, then every initial track $x_1^{\epsilon_1}...x_i^{\epsilon_i}$ of t is also in T, for $i = 1, ..., n$". Given such a Schreier's transversal and $g \in G$, we denote by $\overline{g}$ the (unique) element of T such that $Hg = H\overline{g}$.

Theorem 1.48 (Explicit Schreier's Theorem)**.** *Let F be a free group with basis X and $H \leq F$ a subgroup. Then*

a) There is a Schreier's transversal T of F mod H;

b) If T is any Schreier's transversal T of F mod H, the set

$$R = \{tx\overline{tx}^{-1} \mid t \in T, x \in X, tx \notin T\}$$

is a basis for the free group H.

Demonstração. a) Let $\Gamma = \Gamma(F, X)$ be the tree (Proposition 1.22) on which F acts freely. The restriction of this action to H is also a free action of H on Γ. Let Λ be a maximal tree of H/Γ and lift it to a tree of representatives $\tilde{\Lambda}$ of Γ mod H. As in the proof of Theorem 1.44, the $h\tilde{\Lambda}$, $h \in H$ form a family of disjoint trees whose vertices partition the vertices of Γ. Since all the $h\tilde{\Lambda}$ are isomorphic to $\tilde{\Lambda}$ and have the same projection $p(h\tilde{\Lambda}) = p(\tilde{\Lambda}) = \Lambda$, all of them are also trees of representatives. So, by exchanging $\tilde{\Lambda}$ by some $h\tilde{\Lambda}$ if necessary, we can assume that $\tilde{\Lambda}$ contains the vertex 1 of Γ.

Let $T = V(\tilde{\Lambda}) \subset V(\Gamma) = F$. We claim that T is a Schreier's transversal of F mod H. First, note that $1 \in T$ by construction. Since $p|_{\tilde{\Lambda}} : \tilde{\Lambda} \to \Lambda$ is an isomorphism, in particular we have a bijection $p : T = V(\tilde{\Lambda}) \to V(\Lambda)$. But by Proposition 1.24, $V(\Lambda) = V(H/\Gamma) = V(\Gamma)/\sim = F/\sim$, where the orbit relation is

$$g \sim g' \Leftrightarrow hg = g' \text{ for some } h \in H \Leftrightarrow g' \in Hg \Leftrightarrow Hg = Hg'.$$

Then the orbits $[g]$ are precisely the cosets Hg and we have the bijection $p : T \to \{Hg \mid g \in F\}$ with $p(t) = Ht$. This shows that $F = \sqcup_{t \in T} Ht$. Finally, let $t = x_1^{\epsilon_1} ... x_n^{\epsilon_n} \in T$, $\epsilon_i = \pm 1$ be a reduced word in T. Then $p = (1, x_1^{\epsilon_1} ... x_n^{\epsilon_n})$ is a path from 1 to t in Γ. Since t is a reduced word, p has no backtrackings and since Γ is a tree, p is a geodesic. Now, 1 and t are also vertices of the tree $\tilde{\Lambda}$. Connect them by a geodesic p' inside $\tilde{\Lambda}$ using Proposition 1.20. Since p' is also a geodesic in the tree Γ, again by Proposition 1.20 we have $p = p'$. So p is a path inside $\tilde{\Lambda}$ and therefore its verices $x_1^{\epsilon_1} ... x_i^{\epsilon_i}$, $i = 1, ..., n$ are vertices of $\tilde{\Lambda}$, that is, $x_1^{\epsilon_1} ... x_i^{\epsilon_i} \in T$.

b) If T is any Schreier's transversal of F mod H, let $\tilde{\Lambda} \leq \Gamma$ be the subtree generated by the elements (vertices) of T (Definition 1.32). By Observation 1.33, $\tilde{\Lambda}$ consists of the edges and vertices of all the geodesics in Γ connecting $t \in T$ to $1 \in T$. We claim that $\tilde{\Lambda}$ is a tree of representatives of Γ mod H. First, let us see that $V(\tilde{\Lambda}) = T$. Every element t of T is the end of a geodesic of Γ connecting 1 to t, so obviously $t \in V(\tilde{\Lambda})$. Conversely, if w is a vertex of some geodesic $p = (1, x_1 ... x_n)$ connecting 1 to some $t \in T$ in Γ, then we have $w = x_1 ... x_i$ for some i and because T is a Schreier's transversal we must have $w = x_1 ... x_i \in T$. Now we show the claim. Let $\Lambda = p(\tilde{\Lambda}) \leq H/\Gamma$. We must show that Λ is a maximal tree of H/Γ and that the restriction $p : \tilde{\Lambda} \to \Lambda$ is an isomorphism. It is obviously surjective. Furthermore, $p : T \to V(H/\Gamma)$ is a bijection, because the vertices of H/Γ are exactly the cosets Hg for $g \in F$ (see item *a*)) and because $F = \sqcup_{t \in T} Ht$. Since $V(\Lambda) = p(V(\tilde{\Lambda})) = p(T) = V(H/\Gamma)$, $p : \tilde{\Lambda} \to \Lambda$ is a bijection on the vertices. But it is straightforward to show the following general property: "if $\varphi : \Gamma \to \Gamma'$ is any graph morphism, Γ is a tree and $\varphi_v : V(\Gamma) \to V(\Gamma')$ is injective, then $\varphi_e : E(\Gamma) \to E(\Gamma')$ is also injective". Then, in our case, $p : E(\tilde{\Lambda}) \to E(\Lambda)$ is also injective and, since it is also surjective, $p : \tilde{\Lambda} \to \Lambda$ is an isomorphism. Then Λ is a tree which contains all the vertices of H/Γ and so it is a maximal tree. This shows the claim.

Now we show what we want. Let us apply Theorem 1.44 to our case. H is a group acting freely on a tree Γ, $\tilde{\Lambda}$ is a tree of representatives of Γ mod H and $E^+ = F \times X$ is an orientation of Γ which is preserved by the action. By Theorem 1.44, then, H is a free group with basis

$$R = \{r \neq 1 \in H \mid \exists\, (g, x) \in F \times X \text{ with } g \in \tilde{\Lambda} \text{ and } gx \in r\tilde{\Lambda}\}.$$

But $g \in \tilde{\Lambda}$ as a vertex means $g \in V(\tilde{\Lambda}) = T$. Then we rewrite

$$R = \{r \neq 1 \in H \mid \exists\, (t, x) \in T \times X \text{ with } tx \in rT\}.$$

The condition $tx \in rT$ is also equivalent to $tx = ru$ for some $u \in T$, or $r = txu^{-1}$ for $u \in T$. Note also that $txu^{-1} = r \in H$ is equivalent to $Htx = Hu$, so by uniqueness $u = \overline{tx}$. Then

$r = txu^{-1} = tx\overline{tx}^{-1}$ and

$$R = \{tx\overline{tx}^{-1} \neq 1 \mid t \in T, x \in X\}.$$

Finally, since $tx \in rT$ and the trees $h\tilde{\Lambda}$ are pairwise disjoint (in particular its vertices hT), we have $tx \in T \Leftrightarrow rT = T \Leftrightarrow r = 1$, so we can replace the expression $tx\overline{tx}^{-1} = r \neq 1$ by $tx \notin T$. Then

$$R = \{tx\overline{tx}^{-1} \mid t \in T, x \in X, tx \notin T\}$$

is a basis for the free group H, as desired. □

Example 1.49. Let us go back to Example 1.42. The tree of representatives we chose there has 4 vertices, which are $T = \{1, x, y, xy\}$. By Theorem 1.48, T is a Schreier's transversal of F_2 mod H. Then, to find a free basis for H we just have to find the elements $t \in T$ and $z \in X = \{x, y\}$ such that $tz \notin T$ and then compute $tz\overline{tz}^{-1}$. The 8 elements of the form tz are $x, y, x^2, xy, yx, y^2, xyx$ and xy^2. The ones outside T are x^2, yx, y^2, xyx and xy^2. For each one of these, we compute $tz\overline{tz}^{-1}$. Since $\overline{x^2} = 1, \overline{yx} = xy, \overline{y^2} = 1, \overline{xyx} = y$ and $\overline{xy^2} = x$, a basis for H is

$$R = \{x^2, yxy^{-1}x^{-1}, y^2, xyxy^{-1}, xy^2x^{-1}\}.$$

There is also a more geometric way to see this. Remember that since the basis for F_2 is $X = \{x, y\}$, the oriented edges in Γ are the ones going up or right. By Theorem 1.44, a basis for H consist of the elements $1 \neq r \in H$ such that there is an oriented edge y in Γ starting at $\tilde{\Lambda}$ and finishing in $r\tilde{\Lambda}$. So we just have to look at the oriented edges which start at Γ (and leave it) and see which trees of representatives they touch. We obtain the same 5 elements after all. See the next figure.

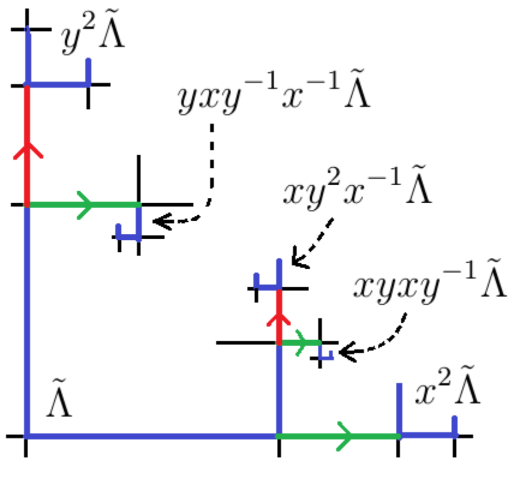


Figura 1.9: there are exactly 5 oriented edges starting $\tilde{\Lambda}$ and leaving it.

Theorem 1.50 (Reidemeister-Schreier). *Let $G = \langle X \mid R \rangle$ be a group and $H \leq G$ a subgroup. Let $\varphi : F_X \to G$ be the projection morphism such that $\ker(\varphi) = \ll R \gg^{F_X}$ (the normal closure of the set R in the free group F_X) and define $\tilde{H} = \varphi^{-1}(H) \leq F_X$. If $T \subset F_X$ is a Schreier's transversal of F_X mod $\tilde{H}$, then*

$$H = \left\langle \{\gamma(t, x) \mid t \in T, x \in X, tx \notin T\} \mid \{\tau(trt^{-1}) \mid t \in T, r \in R\} \right\rangle$$

is a presentation for H, where $\gamma(t,x) = tx\overline{tx}^{-1}$ for $t \in T, x \in X^{\pm}$ and

$$\tau(w) = \gamma(1,x_1)\gamma(\overline{x_1},x_2)\gamma(\overline{x_1x_2},x_3)...\gamma(\overline{x_1...x_{n-1}},x_n)$$

for a word $w = x_1...x_n$ in F_X.

Demonstração. Since T is a Schreier's transversal of F_X mod $\tilde{H}$, by Theorem 1.48 $\tilde{H}$ is a free group with basis $X' = \{\gamma(t,x) \mid t \in T, x \in X, tx \notin T\}$. So we can denote $\tilde{H} = F_{X'}$ as a free group on the basis X'. The restriction of the surjective morphism $\varphi : F_X \to G$ to the subgroup $\tilde{H} = F_{X'}$ gives us the surjective projection morphism

$$\varphi_{\tilde{H}} : \tilde{H} \to \varphi(\tilde{H}) = \varphi(\varphi^{-1}(H)) = H,$$

which we can denote by $\varphi' : F_{X'} \to H$. So, by the definition of group presentations we are just left to show that

$$\ker(\varphi') = \ll \{\tau(trt^{-1}) \mid t \in T, r \in R\} \gg^{F_{X'}}$$

(note that the normal closure on the right is in the subgroup $F_{X'}$, not in the whole group F_X). But we have

$$\ker(\varphi') = \tilde{H} \cap \ker(\varphi) = \varphi^{-1}(H) \cap \varphi^{-1}(\{1\}) = \varphi^{-1}(\{1\}) = \ker(\varphi) = \ll R \gg^{F_X},$$

so we must show that $\ll R \gg^{F_X} = \ll \{\tau(trt^{-1}) \mid t \in T, r \in R\} \gg^{F_{X'}}$. Let us show that. The reader shall remember the definitions of normal closure in a group.

($\supset$) Since $F_{X'} \leq F_X$, we have $\ll \{\tau(trt^{-1}) \mid t \in T, r \in R\} \gg^{F_{X'}} \subset \ll \{\tau(trt^{-1}) \mid t \in T, r \in R\} \gg^{F_X}$. So it is enough to show that $\ll \{\tau(trt^{-1}) \mid t \in T, r \in R\} \gg^{F_X} \subset \ll R \gg^{F_X}$. Since $\ll R \gg^{F_X}$ is a normal subgroup of F_X it is enough to show that $\tau(trt^{-1}) \in \ll R \gg^{F_X}$ for all $t \in T, r \in R$. Now, using that $\overline{\overline{g}g'} = \overline{gg'}$ for all $g, g' \in F_X$ note that, for every word $w = x_1...x_n$ in F_X, $x_i \in X^{\pm}$, we have

$$\begin{aligned}
\tau(w) &= \gamma(1,x_1)\gamma(\overline{x_1},x_2)\gamma(\overline{x_1x_2},x_3)...\gamma(\overline{x_1...x_{n-1}},x_n) \\
&= x_1\overline{x_1}^{-1}\overline{x_1}x_2\overline{\overline{x_1}x_2}^{-1}\overline{x_1x_2}x_3\overline{\overline{x_1x_2}x_3}^{-1}...\overline{x_1...x_{n-1}}x_n\overline{\overline{x_1...x_{n-1}}x_n}^{-1} \\
&= x_1\overline{x_1}^{-1}\overline{x_1}x_2\overline{x_1x_2}^{-1}\overline{x_1x_2}x_3\overline{x_1x_2x_3}^{-1}...\overline{x_1...x_{n-1}}x_n\overline{x_1...x_{n-1}x_n}^{-1} \\
&= x_1...x_n\overline{x_1...x_n}^{-1} = w\overline{w}^{-1},
\end{aligned}$$

so for every $h \in \tilde{H}$ we have $\tau(h) = h\overline{h}^{-1} = h$. Since $r \in R \subset \ll R \gg^{F_X} \lhd F_X$, the conjugate element trt^{-1} is also in $\ll R \gg^{F_X} = \ker(\varphi) \subset \tilde{H}$ and therefore $\tau(trt^{-1}) = trt^{-1} \in \ll R \gg^{F_X}$, as desired.

($\subset$) An arbitrary element of $\ll R \gg^{F_X}$ is a finite product of elements of the form grg^{-1} with $g \in F_X$ and $r \in R$. Since $\ll \{\tau(trt^{-1}) \mid t \in T, r \in R\} \gg^{F_{X'}}$ is a subgroup of F_X, it is enough to show that $grg^{-1} \in \ll \{\tau(trt^{-1}) \mid t \in T, r \in R\} \gg^{F_{X'}}$. But $g \in F_X = \sqcup_{t \in T}\tilde{H}t$, so write $g = ht$ for $h \in \tilde{H}$ and $t \in T$. Then

$$grg^{-1} = htrt^{-1}h^{-1} = h\tau(trt^{-1})h^{-1}$$

and since $h \in \tilde{H} = F_{X'}$, the element above is by definition inside $\ll \{\tau(trt^{-1}) \mid t \in T, r \in$

$R\} \gg^{F_{X'}}$, as desired. This completes the proof. □

Corollary 1.51. *If G is finitely generated and $H \leq G$ is a finite index subgroup, then H is finitely generated.*

Demonstração. By the proof of Proposition 6.2, the subgroup $\tilde{H} \leq F_X$ is of finite index. Then the Schreier's transversal T of F_X mod $\tilde{H}$ (which exists by Theorem 1.48) is also finite. By Theorem 1.50, the set $\{\gamma(t,x) \mid t \in T, x \in X, tx \notin T\}$ generates H. But it is a finite set, since T and X are finite. □

Observation 1.52. The triumph of the Reidemeister-Schreier Theorem is that it is algorithmic. The projection $\varphi : F_X \to G$ can be written as $\varphi(w) = w$, where $w = x_1...x_n, x_i \in X^{\pm}$ is being considered as a word in the domain F_X and as the product of the generators $x_1...x_n$ of G in the codomain G. If a Schreier's transversal T of G mod H is known, then using Proposition 6.2 we see that $\tilde{T} = \{w \in F_X \mid w = \varphi(w) \in T\}$ is a Schreier's transversal of F_X mod $\tilde{H}$ (from now on we will identify $\tilde{T} = T$). We do the following: we just have to find which are the elements $\gamma(t,x)$ such that $tx \notin T$, which will be the generators of H, and then write all the relations $\tau(trt^{-1})$ in terms of these generators. Note that the generators are only the $\gamma(t,x)$ for $x \in X$, not for $x \in X^{-1}$. So if some expression of the form $\gamma(t,x^{-1})$ with $x \in X$ appears in the expression of $\tau(trt^{-1})$, we must figure it out which generator it represents. But

$$\gamma(\overline{tx^{-1}},x)^{-1} = \left(\overline{tx^{-1}}x\overline{\overline{tx^{-1}}x}^{-1}\right)^{-1} = \left(\overline{tx^{-1}}x\overline{tx^{-1}x}^{-1}\right)^{-1} = (\overline{tx^{-1}}xt^{-1})^{-1} = tx^{-1}\overline{tx^{-1}}^{-1} = \gamma(t,x^{-1}),$$

so we replace $\gamma(t,x^{-1})$ by the expression $\gamma(\overline{tx^{-1}},x)^{-1}$, which is the inverse of the generator $\gamma(\overline{tx^{-1}},x)$ of H, as one can see in the next example.

Example 1.53. Let us use Example 1.49 to use the Reidemeister-Schreier algorithm. Let $G = \mathbb{Z} \oplus \mathbb{Z} = \langle x,y \mid xyx^{-1}y^{-1} \rangle$ and $H = 2\mathbb{Z} \oplus 2\mathbb{Z} \leq G$. Let us compute a presentation for H (our intuition tells us to expect some presentation also having the form $\langle a,b \mid aba^{-1}b^{-1} \rangle$). We have $X = \{x,y\}$, $R = \{xyx^{-1}y^{-1}\}$ and the projection $\varphi : F_2 \to G$ can be identified with $\varphi(x) = (1,0)$ and $\varphi(y) = (0,1)$. Then $\tilde{H} = \varphi^{-1}(H) \leq F_2$ is exactly the subgroup of Example 1.49. $T = \{1,x,y,xy\}$ is a Schreier's transversal of F_2 mod $\tilde{H}$ and the generators of H are

$$\begin{aligned} a &= x^2 = \gamma(x,x), \\ b &= y^2 = \gamma(y,y), \\ c &= yxy^{-1}x^{-1} = \gamma(y,x), \\ d &= xyxy^{-1} = \gamma(xy,x), \\ e &= xy^2x^{-1} = \gamma(xy,y). \end{aligned}$$

Now let us compute the 4 relations. Remember that $\gamma(t,z) = 1$ if and only if $tz \in T$:

$$\begin{aligned}\tau(xyx^{-1}y^{-1}) &= \gamma(1,x)\gamma(x,y)\gamma(xy,x^{-1})\gamma(y,y^{-1})\\ &= \gamma(xy,x^{-1})\\ &= \gamma(\overline{xyx^{-1}},x)^{-1}\\ &= \gamma(y,x)^{-1}\\ &= c^{-1},\end{aligned}$$

$$\begin{aligned}\tau(xxyx^{-1}y^{-1}x^{-1}) &= \gamma(1,x)\gamma(x,x)\gamma(1,y)\gamma(y,x^{-1})\gamma(xy,y^{-1})\gamma(x,x^{-1})\\ &= \gamma(x,x)\gamma(y,x^{-1})\\ &= \gamma(x,x)\gamma(\overline{yx^{-1}},x)^{-1}\\ &= \gamma(x,x)\gamma(xy,x)^{-1}\\ &= ad^{-1},\end{aligned}$$

$$\begin{aligned}\tau(yxyx^{-1}y^{-1}y^{-1}) &= \gamma(1,y)\gamma(y,x)\gamma(xy,y)\gamma(x,x^{-1})\gamma(1,y^{-1})\gamma(y,y^{-1})\\ &= \gamma(y,x)\gamma(xy,y)\gamma(1,y^{-1})\\ &= \gamma(y,x)\gamma(xy,y)\gamma(\overline{y^{-1}},y)^{-1}\\ &= \gamma(y,x)\gamma(xy,y)\gamma(y,y)^{-1}\\ &= ceb^{-1},\end{aligned}$$

and

$$\begin{aligned}\tau(xyxyx^{-1}y^{-1}y^{-1}x^{-1}) &= \gamma(1,x)\gamma(x,y)\gamma(xy,x)\gamma(y,y)\gamma(1,x^{-1})\gamma(x,y^{-1})\gamma(xy,y^{-1})\gamma(x,x^{-1})\\ &= \gamma(xy,x)\gamma(y,y)\gamma(1,x^{-1})\gamma(x,y^{-1})\\ &= \gamma(xy,x)\gamma(y,y)\gamma(\overline{x^{-1}},x)^{-1}\gamma(\overline{xy^{-1}},y)^{-1}\\ &= \gamma(xy,x)\gamma(y,y)\gamma(x,x)^{-1}\gamma(xy,y)^{-1}\\ &= dba^{-1}e^{-1}.\end{aligned}$$

Then a presentation for H is

$$\begin{aligned}H &= \left\langle a,b,c,d,e \mid c^{-1},ad^{-1},ceb^{-1},dba^{-1}e^{-1}\right\rangle\\ &= \left\langle a,b,d,e \mid ad^{-1},eb^{-1},dba^{-1}e^{-1}\right\rangle\\ &= \left\langle a,b,e \mid eb^{-1},aba^{-1}e^{-1}\right\rangle\\ &= \left\langle a,b \mid aba^{-1}b^{-1}\right\rangle,\end{aligned}$$

as desired.

Open question: could we also give a geometric approach (like the one above) to Johnson's method? This question is due to a personal communication with professor Vinicius Casteluber Laass (DMAT - UFBa - Brazil) and Johnson's method can be found in Chapter 13 of [59].

1.5 Commutators and lower central series

For the work of Chapter 10, we need to know some basic properties about commutators and lower central series of a group. Our approach here will be minimal.

Definition 1.54. Let G be a group. Given two elements $x, y \in G$, we denote $[x, y] = xyx^{-1}y^{-1}$ and call it the commutator of x and y. We also denote by $x^y = yxy^{-1}$ the conjugate of x by y. If $z \in G$ is another element, we define $[x, y, z]$ as $[x, y, z] = [[x, y], z]$ and recursively define $[x_1, ..., x_k] = [[x_1, ..., x_{k-1}], x_k]$ for $k \geq 4$ and $x_1, ..., x_k \in G$.

Here are some basic identities of commutators, whose proofs are straightforward and will be omitted:

Proposition 1.55. *Let G be a group and $x, y, z \in G$. Then*

a) $xy = [x, y]yx = yx[x^{-1}, y^{-1}]$;

b) $[x, y]^{-1} = [y, x]$;

c) $[x, yz] = [x, y][x, z][z, x, y]$;

d) $[xy, z] = [y, z][z, y, x][x, z]$;

e) $[y^{-1}, x, z]^y[z^{-1}, y, x]^z[x^{-1}, z, y]^x = 1$.

Definition 1.56 (Commutator subgroup)**.** Given two subgroups $H, K \leq G$, we define the subgroup $[H, K] \leq G$ as the subgroup generated by all commutators $[h, k]$ (with $h \in H$ and $k \in K$) and call it the commutator of H and K in G. If $J \leq G$ is another subgroup, we define $[H, K, J]$ as $[H, K, J] = [[H, K], J]$ and recursively define $[H_1, ..., H_k] = [[H_1, ..., H_{k-1}], H_k]$ for $k \geq 4$ and $H_1, ..., H_k \leq G$.

It is easy to see that $[H, K] = [K, H]$ for any subgroups $H, K \leq G$. Also, the group $[H_1, ..., H_k]$ is generated by all elements of the form $[h_1, ..., h_k]$ (with $h_i \in H_i$), which we call k-fold commutators.

Definition 1.57 (Lower central series and nilpotent groups)**.** Given any group G, the lower central series of G is the sequence $\gamma_1(G), \gamma_2(G), \gamma_3(G), ...$ of subgroups of G defined by

- $\gamma_1(G) = G$;
- $\gamma_2(G) = [G, G] = [\gamma_1(G), G]$;
- $\gamma_{k+1}(G) = [\gamma_k(G), G]$ for any $k \geq 1$.

It is not hard to prove by induction that $\gamma_i(G) \supset \gamma_{i+1}(G)$, so it is (setwise) a decreasing sequence. For $c \geq 1$, we say G is nilpotent of class c if $\gamma_{c+1}(G) = 1$ and if c is the smallest positive integer with this property.

One can prove by easy induction that the subgroups $\gamma_i(G)$ are all characteristic in G, that is, $\varphi(\gamma_i(G)) = \gamma_i(G)$ for all automorphisms $\varphi \in Aut(G)$. In particular, they are invariant under conjugation.

Lemma 1.58. *If G is a group and $i, j \geq 1$, then $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$.*

Demonstração. Induction on j. For $j = 1$, we have $[\gamma_i(G), \gamma_j(G)] = [\gamma_i(G), G] = \gamma_{i+1}(G) = \gamma_{i+j}(G)$, as desired. Suppose now that, for some fixed j, we have $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$ for any $i \geq 1$ and let us show that $[\gamma_i(G), \gamma_{j+1}(G)] \leq \gamma_{i+j+1}(G)$ for any $i \geq 1$. Given such i, we have

$$[\gamma_i(G), \gamma_{j+1}(G)] = [\gamma_{j+1}(G), \gamma_i(G)] = [[\gamma_j(G), G], \gamma_i(G)] = [\gamma_j(G), G, \gamma_i(G)],$$

so every generator of $[\gamma_i(G), \gamma_{j+1}(G)]$ is of the form $[x_j^{-1}, z, y_i]$ for $x_j \in \gamma_j(G)$, $z \in G$ and $y_i \in \gamma_i(G)$. If we show this generator is in $\gamma_{i+j+1}(G)$ we are done. By Proposition 1.55, item e) and basic computations we obtain

$$\begin{aligned} [x_j^{-1}, z, y_i] &= (([z^{-1}, y_i, x_j]^z)^{-1}([y_i^{-1}, x_j, z]^{y_j})^{-1})^{x_j^{-1}} \\ &= ([x_j, [z^{-1}, y_i]]^z [z, [y_i^{-1}, x_j]]^{y_j})^{x_j^{-1}} \\ &= [x_j, [z^{-1}, y_i]]^{x_j^{-1}z} [z, [y_i^{-1}, x_j]]^{x_j^{-1}y_j}. \end{aligned}$$

Since $[z^{-1}, y_i] \in [G, \gamma_i(G)] = \gamma_{i+1}(G)$ we have by induction that $[x_j, [z^{-1}, y_i]] \in [\gamma_j(G), \gamma_{i+1}(G)] \leq \gamma_{i+j+1}(G)$, and since this subgroup is characteristic we get $[x_j, [z^{-1}, y_i]]^{x_j^{-1}z} \in \gamma_{i+j+1}(G)$. Similarly, using induction we show that $[z, [y_i^{-1}, x_j]]^{x_j^{-1}y_j} \in \gamma_{i+j+1}(G)$ and so $[x_j^{-1}, z, y_i] \in \gamma_{i+j+1}(G)$, which completes the proof. □

Definition 1.59. Given $k \geq 1$ and two elements $x, y \in G$, we say x and y are congruent modulo $\gamma_k(G)$ and denote $x = y \bmod \gamma_k(G)$ if $x\gamma_k(G) = y\gamma_k(G)$, or, equivalently, if $x^{-1}y \in \gamma_k(G)$. This means x and y project onto the same element in $G/\gamma_k(G)$.

The following propositions are the most important ones for Chapter 10.

Proposition 1.60. *Let $k, m, n \geq 1$ and let $x, y, z \in G$ be elements of a group G such that $x \in \gamma_k(G)$, $y \in \gamma_m(G)$ and $z \in \gamma_n(G)$. Then*

a) $xy = yx$ mod $\gamma_{k+m}(G)$;

b) $[x, yz] = [x, y][x, z]$ mod $\gamma_{k+m+n}(G)$;

c) $[xy, z] = [x, z][y, z]$ mod $\gamma_{k+m+n}(G)$.

Demonstração. For item a), just note that $[x^{-1}, y^{-1}] \in \gamma_{k+m}(G)$ and so $xy\gamma_{k+m}(G) = yx[x^{-1}, y^{-1}]\gamma_{k+m}(G) = yx\gamma_{k+m}(G)$. Now, item b). By Lemma 1.58, we have $[z, x, y] = [[z, x], y] \in [[\gamma_n(G), \gamma_k(G)], \gamma_m(G)] \leq [\gamma_{k+n}(G), \gamma_m(G)] \leq \gamma_{k+n+m}(G)$; therefore,

$$[x, yz]\gamma_{k+m+n}(G) = [x, y][x, z][z, x, y]\gamma_{k+m+n}(G) = [x, y][x, z]\gamma_{k+m+n}(G).$$

Item c) is similar: by using Lemma 1.58, we get that $[z, y, x] \in \gamma_{k+n+m}(G)$, that $[[z, y, x]^{-1}, [x, z]^{-1}] \in \gamma_{k+n+m}(G)$ (actually, this element is in $\gamma_{2k+2n+m}(G) \leq \gamma_{k+n+m}(G)$) and similarly that $[[y,z]^{-1},[x,z]^{-1}] \in \gamma_{k+n+m}(G)$. Now, using Proposition 1.55 we have

$$\begin{aligned} [xy,z]\gamma_{k+n+m}(G) &= [y,z][z,y,x][x,z]\gamma_{k+n+m}(G) \\ &= [y,z][x,z][z,y,x][[z,y,x]^{-1},[x,z]^{-1}]\gamma_{k+n+m}(G) \\ &= [y,z][x,z]\gamma_{k+n+m}(G) \\ &= [x,z][y,z][[y,z]^{-1},[x,z]^{-1}]\gamma_{k+n+m}(G) \\ &= [x,z][y,z]\gamma_{k+n+m}(G), \end{aligned}$$

as we wanted. □

At last, we need a little information about the quotients $\gamma_k(G)/\gamma_{k+1}(G)$:

Proposition 1.61. *If G is finitely generated by elements $x_1,...,x_r$ then, for any $k \geq 1$, $\gamma_k(G)/\gamma_{k+1}(G)$ is abelian and finitely generated by the cosets of the k-fold commutators $[x_{i_1},...,x_{i_k}]$, where $1 \leq i_j \leq r$.*

Demonstração. The proof comes from [72] but we rewrite it here. First, note that $\gamma_k(G)/\gamma_{k+1}(G)$ is abelian because, by Lemma 1.58, $[\gamma_k(G),\gamma_k(G)] \leq \gamma_{2k}(G) \leq \gamma_{k+1}(G)$, so all commutators in the quotient are trivial. Let us show by induction that the cosets of those k-fold commutators generate $\gamma_k(G)/\gamma_{k+1}(G)$. The case $k = 1$ is trivial for we know that G is generated by the elements x_i (or the 1-fold commutators), so their cosets generate $G/\gamma_2(G)$, as desired. Suppose the proposition is true for some $k \geq 1$ and let us show it for $k+1$. We know $\gamma_{k+1}(G) = [\gamma_k(G),G]$ is generated by the elements $[h,g]$, where $h \in \gamma_k(G)$ and $g \in G$. By projecting h in $\gamma_k(G)/\gamma_{k+1}(G)$ and using induction we can write $h = h_1^{\epsilon_1}...h_s^{\epsilon_s}h'$, where $\epsilon_i = \pm 1$, the h_i are the k-fold commutators of the induction and $h' \in \gamma_{k+1}(G)$. By using Proposition 1.60 above, we have

$$[h,g] = [h_1^{\epsilon_1}...h_s^{\epsilon_s}h',g] = [h_1,g]^{\epsilon_1}...[h_s,g]^{\epsilon_s}[h',g] \bmod \gamma_{2k+1}(G),$$

and since $\gamma_{2k+1}(G) \leq \gamma_{k+2}(G)$ this equality is true modulo $\gamma_{k+2}(G)$. Since $[h',g] \in \gamma_{k+2}(G)$, we then have $[h,g] = [h_1,g]^{\epsilon_1}...[h_s,g]^{\epsilon_s} \bmod \gamma_{k+2}(G)$, so we can say $\gamma_{k+1}(G)/\gamma_{k+2}(G)$ is generated by the cosets of the elements $[\tilde{h},g]$, where $\tilde{h}$ is one of those k-fold commutators and $g \in G$. Now write $g = x_{i_1}^{\epsilon_1}...x_{i_s}^{\epsilon_s}$ for some $\epsilon_i = \pm 1$ and $1 \leq i_j \leq r$. Again by using Proposition 1.60 we get

$$[\tilde{h},g] = [\tilde{h},x_{i_1}^{\epsilon_1}...x_{i_s}^{\epsilon_s}] = [\tilde{h},x_{i_1}]^{\epsilon_1}...[\tilde{h},x_{i_s}]^{\epsilon_s} \bmod \gamma_{k+2}(G),$$

so $\gamma_{k+1}(G)/\gamma_{k+2}(G)$ is generated by the cosets of the elements of the form $[\tilde{h},x_{i_j}]$, which are all $(k+1)$-fold commutators of the elements $x_1,...,x_r$. This completes the proof. □

1.6 Graphs of groups and its fundamental groups

We need the following notion to deal with GBS groups, since they will be defined in Chapter 11 as the fundamental groups of some special graphs of groups.

Definition 1.62. A graph of groups (G,Γ) consists of:

- a graph Γ;

- a choice of an arbitrary group G_P for each vertex P of Γ;
- a choice of an arbitrary group G_y for each edge y of Γ such that $G_{\overline{y}} = G_y$ for all y;
- a choice of a monomorphism $f_y : G_y \to G_{t(y)}$ for all edges y.

We can also think that for each edge y there are two monomorphisms $f_y : G_y \to G_{t(y)}$ and $f_{\overline{y}} : G_y \to G_{o(y)}$. We call the G_P the vertex groups and the G_y the edge groups.

We will give below the definition of the fundamental group of a graph of groups using the presentation for it. There are other equivalent ones, for example, using the direct limit of groups (see [86]).

Definition 1.63. Let (G, Γ) be a graph of groups with orientation $E^+ = E^+(\Gamma)$ and T a maximal tree in Γ. Let $G_P = \langle X_P \mid R_P \rangle$ be presentations for the vertex groups. The fundamental group of (G, Γ) (sometimes also denoted by $\pi_1(G, \Gamma, T)$ or just $\pi_1(\Gamma)$) is

$$\begin{aligned} G \;\; = \;\; & \langle \sqcup_P X_P \sqcup \{g_y \mid y \in E^+\} \mid \sqcup_P R_P \sqcup \{g_y f_y(a) {g_y}^{-1} f_{\overline{y}}(a)^{-1} \mid y \in E^+, a \in G_y\} \\ & \sqcup \{g_y \mid y \in E^+ \cap E(T)\}\rangle. \end{aligned}$$

When necessary, we denote it by $\pi_1(G, \Gamma, T)$ or just $\pi_1(\Gamma)$.

Let's get an intuition of the presentation above. For generators, we have all the generators of the G_P and one extra generator g_y (called stable letter) for each oriented edge y outside T. The relations are the ones from all the G_P and the equalities

$$g_y f_y(a) {g_y}^{-1} = f_{\overline{y}}(a)$$

for all $y \in E^+ - E(T), a \in G_y$ and

$$f_y(a) = f_{\overline{y}}(a)$$

for all $y \in E^+ \cap E(T), a \in G_y$. One can show that G is independent (up to isomorphism) from the choices of the tree T, the orientation E^+ and the presentations $G_P = \langle X_P \mid R_P \rangle$ (this last independence is due to the direct limit definition in [86]). The groups G_P and G_y can always be seen as subgroups of G.

Definition 1.64. Given three groups $G = \langle X \mid R \rangle$, $H = \langle Y \mid S \rangle$ and A with two monomorphisms $f : A \hookrightarrow H$ and $g : A \hookrightarrow G$, the amalgamated product $G *_A H$ is the fundamental group of the segment of groups Γ with $V(\Gamma) = \{P, Q\}$, $E(\Gamma) = \{y, \overline{y}\}$, $G_P = G$, $G_Q = H$, $G_y = G_{\overline{y}} = A$ and monomorphisms $f_y = f$ and $f_{\overline{y}} = g$. This means that

$$G = \left\langle X \sqcup Y \mid R \sqcup S \sqcup \{f(a)g(a)^{-1} \mid a \in A\} \right\rangle.$$

Definition 1.65. Let $G = \langle X \mid R \rangle$ and $A \leq G$ with inclusion $l : A \hookrightarrow G$ and another monomorphism $\theta : A \hookrightarrow G$. The HNN extension G' of (A, G, θ) is the fundamental group of the loop of groups Γ with $V(\Gamma) = \{P\}$, $E(\Gamma) = \{y, \overline{y}\}$, $G_P = G$, $G_y = G_{\overline{y}} = A$ and monomorphisms $f_y = l$ and $f_{\overline{y}} = \theta$. This means that

$$G' = \left\langle X, t \mid R, tat^{-1}\, \theta(a)^{-1},\; a \in A \right\rangle.$$

Figura 1.10: segment of groups on the left and loop of groups on the right

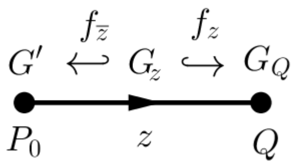
Amalgamated products and HNN extensions are vastly studied classes of groups. Because of this, we won't prove its properties when necessary, but only give the reader references for them. Most of the basic theory and properties can be found in [86] and [71].

The construction of the fundamental group of graph of groups is recursive: intuitively saying, instead of taking the more complicated graph of groups, we can divide it in steps, each step involving a simpler graph of groups, and in the end we obtain the same graph of groups. Let us give two examples:

Example 1.66 (Reconstruction of trees)**.** Let T be a tree with Q a terminal vertex being the terminus of the edge z with $o(z) = P_0$, and consider the subtree $T' = T - Q$. Let $G_P = \langle X_P \mid R_P \rangle$ and $G_y = \langle X_y \mid R_y \rangle$ be the respective presentations for the vertex and edge groups of T. Let us denote by G the fundamental group of T and G' the fundamental group of T'. By definition, we have

$$G' = \left\langle \sqcup_{P \in V(T')} X_P \mid \sqcup_{P \in V(T')} R_P \sqcup \{ f_y(g) = f_{\overline{y}}(g) \mid y \in E(T'), g \in G_y \} \right\rangle .$$

Now, identifying $f_{\overline{z}} : G_z \hookrightarrow G_{P_0}$ with its composition with the inclusion $G_z \hookrightarrow G_{P_0} \hookrightarrow G'$, consider the following segment of groups:



Now, the fundamental group of this graph is

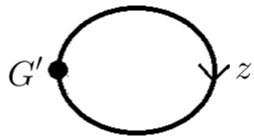
$$\begin{aligned} G' *_{G_z} G_Q \quad &= \quad \langle \sqcup_{P \in V(T')} X_P \sqcup X_Q \mid \sqcup_{P \in V(T')} R_P \sqcup \{ f_y(g) = f_{\overline{y}}(g) \mid y \in E(T') \} \\ &\qquad \sqcup R_Q \sqcup \{ f_z(g) = f_{\overline{z}}(g) \} \rangle \\ &= \quad \langle \sqcup_{P \in V(T)} X_P \mid \sqcup_{P \in V(T)} R_P \sqcup \{ f_y(g) = f_{\overline{y}}(g) \mid y \in E(T), g \in G_y \} \rangle \\ &= G, \end{aligned}$$

and we re-obtained the fundamental group of the entire tree T.

Example 1.67 (Reconstruction of bouquets)**.** Let Γ be a bouquet with vertex $V(\Gamma) = \{P\}$, vertex group $G_P = \langle X \mid R \rangle$, orientation E^+ and at least one edge. Fix one edge z and denote by Γ' the "sub bouquet" obtained by removing z and $\overline{z}$ from Γ. Denote by G and G' the fundamental groups of Γ and Γ', respectively. By definition, taking the one-vertex maximal tree we have

$$G' = \left\langle X \sqcup \{ g_y \;\; y \in E^+ - \{z\} \} \mid R \sqcup \{ g_y f_y(a) {g_y}^{-1} f_{\overline{y}}(a)^{-1} \mid y \in E^+ - \{z\}, a \in G_y \} \right\rangle .$$

Now, identifying the monomorphisms $f_z, f_{\overline{z}} : G_z \hookrightarrow G_P$ with its compositions with the inclusions $G_z \hookrightarrow G_P \hookrightarrow G'$, consider the following loop of groups Z:



$$G' \underset{f_{\overline{z}}}{\overset{f_z}{\leftleftarrows}} G_z$$

The fundamental group of this new graph of groups Z is

$$\begin{aligned} \pi_1(Z) &= \langle X \sqcup \{g_y \mid y \in E^+ - \{z\}\}, g_z \mid R \sqcup \{g_y f_y(a) {g_y}^{-1} f_{\overline{y}}(a)^{-1} \mid y \in E^+ - \{z\}, a \in G_y\}, \\ &\qquad g_z f_z(a) {g_z}^{-1} f_{\overline{z}}(a)^{-1}, a \in G_z\rangle \\ &= \langle X \sqcup \{g_y \mid y \in E^+\} \mid R \sqcup \{g_y f_y(a) {g_y}^{-1} f_{\overline{y}}(a)^{-1} \mid y \in E^+, a \in G_y\}\rangle \\ &= G \end{aligned}$$

and we got again the fundamental group of the whole bouquet. This reconstruction also works if we remove a finite number of edges of the bouquet and start "putting them back" one by one by repeating the argument above many times. We re-obtain the whole fundamental group after all.

Capítulo 2

Geometric preliminaries

As we did in Chapter 1, here we give the reader the notation and background necessary for the geometric part of our work (ch. 7 through 9). Most proofs will be omitted to make the text more compact, but we will give further theory references in the beginning of each section. The reader must be used to the notions of metric spaces and isometries, basic topology and group actions.

2.1 Hyperbolic spaces

We refer [55] for a great survey of hyperbolic spaces and their boundaries. Our approach here will be minimal and with (almost) no proofs.

Definition 2.1. Let (X, d) be a metric space and $x, y \in X$. A geodesic between x and y is a path $\alpha : [0, d(x, y)] \to X$ such that $\alpha(0) = x$, $\alpha(d(x, y)) = y$ and $d(\alpha(t), \alpha(t')) = |t - t'|$ for every $t, t' \in [0, d(x, y)]$. We say that X is a geodesic space (sometimes "length space" in the literature) if for every two points $x, y \in X$, there is a unique geodesic between x and y.

Because of the uniqueness of α the set $[x, y] = \text{im}(\alpha)$ is well defined and is often also called by the geodesic between x and y, or geodesic segment, or geodesic arc. If $x \neq y$ we also call $[x, y]$ a non-degenerate geodesic or arc. Note that, as a subset, $[x, y] = [y, x]$ and that $[x, y]$ is by definition always isometric to a compact interval of $\mathbb{R}$.

A standard and important fact about geodesic spaces is the following (see [14]):

Proposition 2.2. *Let (X, d) be geodesic and complete. Then X is locally compact if and only if every closed ball $\overline{B}(x, r)$ ($x \in X$, $r \geq 0$) is compact. In particular, any closed ball $\overline{B}(x, r)$ in a proper (i.e. complete and locally compact) geodesic space is compact.*

We will need to use the ends of a space, as well as the boundary of it:

Definition 2.3. Given a geodesic space (X, d), a (geodesic) line is a map $c : \mathbb{R} \to X$ such that $d(c(t), c(t')) = |t - t'|$ for every $t, t' \in \mathbb{R}$. Similarly, a (geodesic) ray in X is a map $r : [0, \infty) \to X$ such that $d(r(t), r(t')) = |t - t'|$ for every $t, t' \in [0, \infty)$. Given two such rays r_1, r_2, we say that they have "the same end" if for every compact set $K \subset X$ there is $N \geq 0$ such that $r_1[N, \infty)$ and $r_2[N, \infty)$ are both contained in the same connected component of $X - K$. This is an equivalence relation on the set of geodesic rays of X. The equivalence class of a ray r will be denoted by $end(r)$ and the set of classes $Ends(X)$ will be called the ends of X.

Sometimes we will identify rays or lines with their images in the space X.

Definition 2.4. Given a geodesic space (X, d) and two rays r_1, r_2, we say that they have "the same value at ∞" if there is a number $K \geq 0$ such that $d(r_1(t), r_2(t)) \leq K$ for every $t, t' \in [0, \infty)$. This is also an equivalence relation on the set of geodesic rays of X. The class of a ray r will be denoted by $r(\infty)$ and the set of classes ∂X will be called the boundary of X. If $c : \mathbb{R} \to X$ is a geodesic line, we define $c(\infty) = (c|_{[0,\infty)})(\infty)$ and $c(-\infty) = \tilde{c}(\infty)$, where $\tilde{c} : [0, \infty) \to X$ is the ray $\tilde{c}(t) = c(-t)$.

Now we proceed to define a hyperbolic space.

Definition 2.5. Given a geodesic space (X, d), a subset $A \subset X$ and $r \geq 0$, the r-neighborhood of A in X is $N_r(A) = \{x \in X \mid \exists\, a \in A \text{ such that } d(a, x) \leq r\}$. Given $x, y, z \in X$, the geodesic triangle of x, y and z is denoted by $\Delta(x, y, z) = [x, y] \cup [x, z] \cup [y, z]$. We call the three geodesics involved the edges of $\Delta(x, y, z)$.

Definition 2.6. Let $\delta \geq 0$ and (M, d) a geodesic space. We say that a geodesic triangle Δ in X is δ-slim if every edge of Δ is contained in the δ-neighborhood of the union of the two other edges. We say that a geodesic metric space is δ-hyperbolic if every geodesic triangle in X is δ-slim.

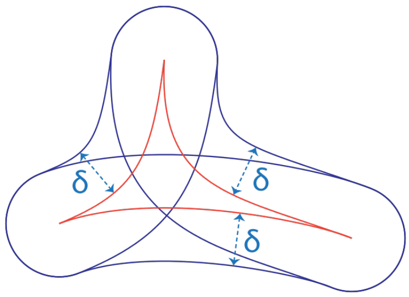


Figura 2.1: source: Wikipedia

The boundaries of hyperbolic spaces have many known interesting properties that will be explored on this thesis. To list some of them, we will use the following lemma (see [55] for a proof):

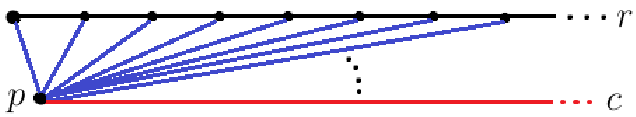
Lemma 2.7 (Arzelà-Ascoli)**.** *Let X, Y be metric spaces, with X separable and Y compact. Then every equicontinuous sequence of maps $f_n : X \to Y$ contains a subsequence converging uniformly (on compacts) to a uniformly continuous map $f : X \to Y$.*

This lemma is incredibly useful in the theory of hyperbolic spaces, for it can be used to produce geodesic rays or lines (therefore, elements on ∂X) from some sequences of other geodesics or geodesic rays. An example of application is

Proposition 2.8. *Let (X, d) be a proper geodesic space, $p \in X$ and $q \in \partial X$. Then there is a geodesic ray $c : [0, \infty) \to X$ such that $c(0) = p$ and $c(\infty) = q$.*

Demonstração. This is only a sketch of the proof for illustration. Let $r : [0, \infty) \to X$ be a geodesic ray with $r(\infty) = q$ (by definition). Of course, if $r(0) = p$ we are done, so assume

$r(0) \neq p$. Define a sequence of maps $c_n : [0, \infty) \to X$ in the following way: define c_n to be the geodesic $[p, r(n)]$ (of course, in the interval $[0, d(p, r(n))]$) and put $c_n(t) = r(n)$ for $t \in [d(p, r(n)), \infty)$. It is an equicontinuous family from the separable space $[0, \infty)$. Note that X is not necessarily compact (but it is complete); so, with some minor adaptations on the proof of Arzelà-Ascoli's Lemma we can indeed guarantee that a subsequence of $(c_n)_n$ converges uniformly (on compacts) to a uniformly continuous map $c : [0, \infty) \to X$ (see next figure).

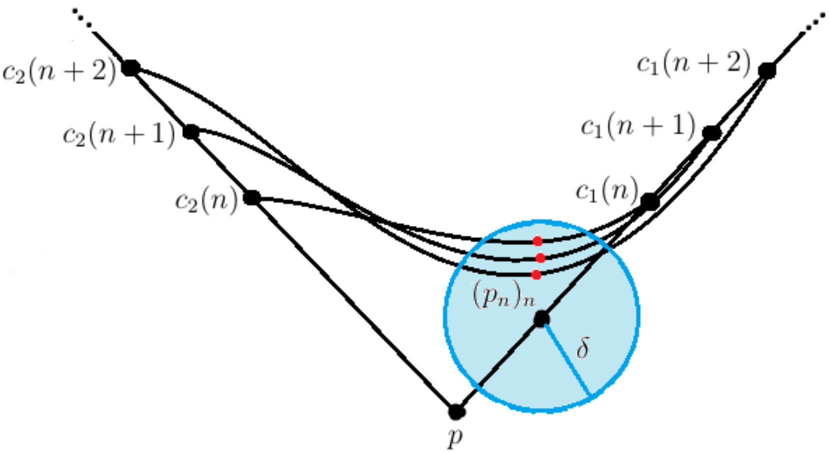


Since it is the limit of geodesics, c can be shown to be a geodesic line starting from p. Finally, by the geometric construction of the c_n we can show $c(\infty) = r(\infty) = q$. □

A similar use of the Arzelà-Ascoli lemma can be seen in the following proposition, in the case of hyperbolic spaces:

Proposition 2.9. *If (X, d) is a proper geodesic and δ-hyperbolic space, then for every pair of distinct points $q_1, q_2 \in \partial X$ there is a geodesic line $c : \mathbb{R} \to X$ such that $c(-\infty) = q_1$ and $c(\infty) = q_2$. Furthermore, c is on the closed δ-neighborhood of the union of geodesic rays representing p and q*

Demonstração. This is again only a sketch. Fix a point $p \in X$. By the previous proposition, let c_1 and c_2 be rays starting at p with $c_i(\infty) = q_i$. Let $k \geq 0$ be such that $d(c_1(k), \mathrm{im}(c_2)) > \delta$ (see next figure). For each $n > k$, we consider the geodesic triangle with vertices $c_1(n)$, $c_2(n)$ and p. Since it is δ-slim, there must be a point $p_n \in [c_1(n), c_2(n)] \cap \overline{B}(c_1(k), \delta)$.



By the compacity of $\overline{B}(c_1(k), \delta)$ (Proposition 2.2) we can assume $(p_n)_n$ to converge. Then, by using the same argument of the previous proposition, a subsequence of the geodesics $[p_n, c_1(n)]$ must converge by the Arzelà-Ascoli's Theorem. Now we look to this subsequence on the other side, that is, the sequence $([c_2(n_k), p_{n_k}])_k$. By the same argument, a subsequence of it must converge, so we can assume the sequence $([c_1(n_k), c_2(n_k)])_k$ of maps converges. The limit can be shown to be a geodesic line c with $c(-\infty) = c_1(\infty) = q_1$ and $c(\infty) = c_2(\infty) = q_2$. The last assertion follows by the definition of δ-hyperbolic space and by the construction of c. □

From now on, we continue with our proper geodesic δ-hyperbolic space X. Next we will give some intuitive ideas of the construction of a topology for the space $\overline{X} = X \cup \partial X$ and some of the properties that will be used later. Most of the details are found in [55].

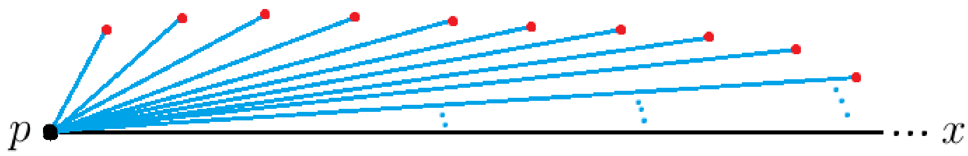
Definition 2.10. We say a map $c : I \to X$ is a generalized ray if either $I = [0, R]$ and c is a geodesic or $I = [0, \infty)$ and c is a geodesic ray. In the former case, we will denote $c(t) = c(R)$ for $t \geq R$ and also denote $c(\infty) = c(R)$.

Because of the definition above, the set $\overline{X} = X \cup \partial X$ can be seen as

$$\overline{X} = X \cup \partial X = \{c(\infty) \mid c \text{ is a generalized ray in } X\}.$$

Definition 2.11. Fix $p \in X$. Let $(x_n)_n$ be a sequence of elements of $\overline{X}$ and $x \in \overline{X}$. We say $x_n \to x$ (as $n \to \infty$) if there are generalized rays c_n with $c_n(0) = p$ and $c_n(\infty) = x_n$ such that every subsequence of $(c_n)_n$ contains a subsequence converging uniformly on compact sets to a generalized ray c with $c(\infty) = x$. Now, we define a topology on $\overline{X}$ by defining its closed sets: a subset $F \subset \overline{X}$ is closed on $\overline{X}$ if for every sequence $(x_n)_n \subset F$ converging to a point $x \in \overline{X}$ we have $x \in F$.

It is possible to show this topology does not depend on the point p chosen. In the case $x_n \in X$ for every n and $x \in \partial X$, the convergence $x_n \to x$ can be intuitively seen as next figure shows.



A natural question is: if x and the x_n are all in X, is this convergence equivalent to the metric convergence in (X, d)? Intuitively we can see this: suppose $x_n \to x$ by the definition above. Then, since uniform convergence implies pointwise convergence, the ends x_n of the finite geodesics c_n converge in X to the end x of the limit geodesic c. On the other hand, if $x_n \to x$ in the usual sense, we can apply Arzelà-Ascoli's Lemma (in a similar way we did above) to the geodesics $[p, x_n]$ to guarantee the convergence $x_n \to x$ according to definition above.

Because of this, since the two convergences in X coincide, the closed sets (which are characterized by convergence) of both topologies on X must coincide and therefore both topologies on X coincide. Therefore

Proposition 2.12. *The inclusion map $X \hookrightarrow \overline{X}$ is a homeomorphism onto its image. In particular, X is open in $\overline{X}$ and therefore ∂X is closed.* □

A sketch for a proof of the following (surprising) result can be found in [55].

Theorem 2.13. *$\overline{X}$ is metrizable, that is, there is a metric on $\overline{X}$ whose induced topology coincides with the one defined above.*

Assuming theorem above, we can show

Proposition 2.14. *$\overline{X}$ and ∂X are compact metric spaces.*

Demonstração. We won't write all details. By theorem above both can be seen as metric spaces. Since ∂X is closed on $\overline{X}$, it is enough to show that $\overline{X}$ is compact, or sequentially compact, since it is a metric space. Let $(x_n)_n$ be a sequence in $\overline{X}$ and let us find a converging subsequence. The set $\{n \in \mathbb{N} \mid x_n \in X\}$ is either infinite or finite, so we may assume without loss of generality that either $x_n \in X$ for all n or $x_n \in \partial X$ for all n. In the latter case, we can suppose by Proposition 2.8 that all rays emerge from a fixed point p. Then an easy adaptation of Arzelà-Ascoli's Lemma give us the desired converging subsequence. In the former case, we have two subcases: if the sequence $(x_n)_n \subset X$ is bounded, then it is contained in a closed ball $\overline{B}(z, R)$. Since such balls are compact we find the convergent subsequence. If $(x_n)_n$ is not bounded fix a point $z \in X$. There must be a subsequence $(x_{n_k})_k$ such that $x_{n_k} \notin \overline{B}(z, k)$. By applying the Arzelà-Ascoli's Lemma for the sequence of geodesics $([z, x_{n_k}])_k$ we find a subsequence of it (say $([z, x_{n_j}])_j$) converging uniformly on compacts to a geodesic ray c, and by definition of the convergence in $\overline{X}$ we have exactly $x_{n_j} \to c(\infty)$ in $\overline{X}$, as desired. □

Let $g : X \to X$ be an isometry, denoted by $x \mapsto gx$. If c is a geodesic ray in X, then the map $gc : [0, \infty) \to X$ with $(gc)(t) = gc(t)$ is a geodesic ray, for $d(gc(t), gc(t')) = d(c(t), c(t')) = |t - t'|$ for every $t, t' \geq 0$. We define a map $g : \partial X \to \partial X$ by putting $g(c(\infty)) = gc(\infty)$. It is clear that $c(\infty) = c'(\infty)$ implies $gc(\infty) = gc'(\infty)$, so g is a well defined map. Furthermore, let us see it is continuous. Since g is an isometry, is straightforward to show that if a sequence $(c_n)_n$ of geodesic rays converge uniformly on compact sets to a geodesic ray c, then the sequence $(gc_n)_n$ of geodesic rays converge uniformly on compact sets to gc. Because of this, it is straightforward to see that if $x_n \to x$ in ∂X, then $gx_n \to gx$, which shows the desired continuity. Since the isometry g^{-1} also induces a continuous map and $gg^{-1} = g^{-1}g = Id_{\partial X}$, we have

Proposition 2.15. *If $g : X \to X$ is an isometry, the induced map $g : \partial X \to \partial X$ is a homeomorphism.* □

It is well known in the literature an equivalent definition of hyperbolicity that involves the Gromov product:

Definition 2.16. Let (X, d) be any metric space and let $w \in X$. The Gromov product of points $x, y \in X$ with respect to w is defined as

$$\langle x, y \rangle_w = \frac{1}{2}[d(x, w) + d(y, w) - d(x, y)].$$

Definition 2.17. Let (X, d) be a metric space and $\delta \geq 0$. We say X is (δ)-hyperbolic (note the parenthesis on δ) if for every $x, y, z, w \in X$,

$$\langle x, z \rangle_w \geq \min\{\langle x, y \rangle_w, \langle y, z \rangle_w\} - \delta.$$

The equivalence is the following. For a proof, see [55].

Proposition 2.18. *If X is a geodesic metric space, then X is hyperbolic (Definition 2.6) if and only if there is $\delta \geq 0$ such that X is (δ)-hyperbolic. Furthermore, X is 0-hyperbolic if and only if it is (0)-hyperbolic.*

2.2 Quasi-isometries

In geometric group theory, quasi-isometries have the same importance homeomorphisms have in classic topology, i.e., they are one of the most important criteria for comparing spaces with respect to the characteristics the theory wants to preserve.

Definition 2.19 (Quasi Isometry and QI-embedding)**.** Let (X, d_X) and (Y, d_Y) be any metric spaces. We say a map $f : X \to Y$ is a quasi-isometric embedding if there are constants $\lambda \geq 1$ and $\epsilon \geq 0$ such that

$$d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + \epsilon \quad \text{and} \quad d_X(x, x') \leq \lambda d_Y(f(x), f(x')) + \epsilon$$

for all $x, x' \in X$. We also say that f is a (λ, ϵ)-QI-embedding. If $\epsilon = 0$, we say f is λ-bi-Lipschitz. We say f is a quasi-isometry, or a (λ, ϵ)-quasi-isometry, if f is a (λ, ϵ)-QI-embedding and, in addition, there is $K \geq 0$ such that

$$\text{For every } y \in Y, \text{ there is } x \in X \text{ such that } d_Y(y, f(x)) \leq K.$$

In this case we denote $X \overset{QI}{\sim} Y$.

It can bee easily seen that the quasi-isometry relation $\overset{QI}{\sim}$ is an equivalence relation. For the basic theory about quasi-isometries, we reference [14] and [55].

A useful example of QI is found in the context of Cayley graphs. Let G be a finitely generated group and $S \subset G$ be a finite generating set. Denote by $\Gamma = \Gamma(G, S)$ the Cayley graph. Given a vertex $g \in G$, it can be written in terms of words in the generators of S. denote by $|g|$ the minimum length of a word in S that represents the element G (which is always attained, since words have only non-negative integer length). With this, we define a metric on the set G by putting $d(g, g') = |g^{-1}g'|$. This distance can be easily interpreted geometrically: it is the minimum length of any combinatorial path between g and g'. If we think of the edges as compact segments of $\mathbb{R}$ with length 1, we can extend naturally the distance to $d : \Gamma(G, S) \times \Gamma(G, S) \to \mathbb{R}$ and the Cayley graph Γ turns out to be a geodesic space. With this in hands, one can show the following

Proposition 2.20 ([14])**.** *Let G be finitely generated and S, S' be two arbitrary finite sets of generators for G. Then $\Gamma(G, S) \overset{QI}{\sim} \Gamma(G, S')$.*

Definition 2.21. We say two finitely generated groups G and H are quasi-isometric, and write $G \overset{QI}{\sim} H$, if $\Gamma(G, S) \overset{QI}{\sim} \Gamma(H, S')$ for some finite sets of generators $S \subset G$ and $S' \subset H$.

By the proposition above, if $\Gamma(G, S) \overset{QI}{\sim} \Gamma(H, S')$ for some finite such sets of generators, it must be true for *any* such sets. Some standard facts about QI of groups are:

Proposition 2.22. *If G is finitely generated and $H \leq G$ is finite index, then $G \overset{QI}{\sim} H$.*

Proposition 2.23 (QI invariance of hyperbolicity)**.** *Let X, Y be geodesic metric spaces. If $X \overset{QI}{\sim} Y$, then X is hyperbolic if and only if Y is hyperbolic. In particular, if G, H are quasi-isometric groups, then G is hyperbolic if and only if H is hyperbolic.*

2.3 ℝ-trees

The trees we dealt with in the previous chapter are also known in the literature as "simplicial" or "combinatorial" trees, because they arise in a more combinatorial fashion, instead of a topological one. They are a special kind of the trees we are going to deal with here: the ℝ-trees. These have a more geometric and topological characterization. This section is mainly based on [2], with a few adaptations.

Definition 2.24. An ℝ-tree is a metric space (T, d) such that:

a) T is a geodesic space;

b) For every $x, y, z \in T$ there exists $w \in T$ such that $[x, y] \cap [x, z] = [x, w]$;

c) If $[x, y] \cap [y, z] = \{y\}$, then $[x, z] = [x, y] \cup [y, z]$.

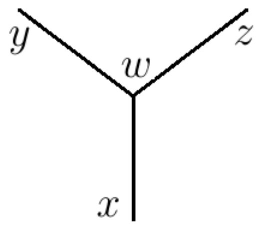
The point w of item *b*) above can be shown to be unique and will be denoted by $w = Y(y, x, z)$. In the case of item *c*), the point y is in the interior of the geodesic $[x, z]$ and we write $[x, z] = [x, y, z]$. Moreover, there is actually a well-defined total order "$\leq$" in every geodesic segment, so we can similarly write $[x_0, x_n] = [x_0, x_1, ..., x_{n-1}, x_n]$ when the points x_i are in the geodesic $[x_0, x_n]$ and $x_0 \leq x_1 \leq ... \leq x_{n-1} \leq x_n$.

Definition 2.25. If T is an ℝ-tree and $T' \subset T$, we say that T' is a subtree if T' is a convex subset of T (that is, $x, y \in T' \Rightarrow [x, y] \subset T'$). This is equivalent to say that T' is an ℝ-tree with the induced metric from T. We say that the subset T' is a closed-subtree of T if every nonempty intersection $T' \cap [x, y]$ of T' with a geodesic segment $[x, y]$ is also a geodesic segment of T.

It is not hard to see that every closed-subtree T' is also a subtree (a convex subset), and that if T' is a subtree and is closed with the induced topology of T, then T' is a closed-subtree of T.

Below we state the main basic properties of ℝ-trees we are going to use. We are based on [2] (p. 271-286), where the reader may find all the proofs.

Proposition 2.26 (The Y proposition)**.** *Let T be an ℝ-tree $x, y, z \in T$ and $w = Y(y, x, z)$ as in Definition 2.24. Then*



- $[y, w] \cap [w, z] = \{w\}$ *(and therefore $[y, z] = [y, w, z] = [y, w] \cup [w, z]$);*
- $d(y, z) = d(y, x) + d(z, x) - 2d(w, x)$*;*
- $Y(x, y, z) = Y(y, x, z) = Y(x, z, y) = Y(y, z, x) = Y(z, x, y) = Y(z, y, x)$.

With this proposition one can show

Proposition 2.27 (The subtree proposition). *Let $T_1, ..., T_n$ be subtrees of an R-tree T.*

- *If $T_i \cap T_{i+1} \neq \emptyset$ for $i = 1, ..., n-1$ then $T_1 \cup ... \cup T_n$ is a subtree;*
- *If $T_i \cap T_j \neq \emptyset$ for every $1 \leq i, j \leq n$ then $T_1 \cap ... \cap T_n \neq \emptyset$ is a nonempty subtree.*

The subtree proposition, by its turn, implies

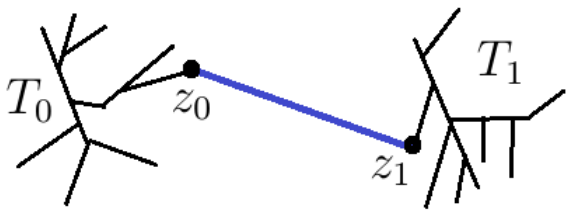
Proposition 2.28 (Piecewise geodesic). *Let T be an $\mathbb{R}$-tree and $x_0, x_1, ..., x_n \in T$. Then the following are true:*

- *$[x_0, x_n] \subset [x_0, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, x_n]$;*
- *If $d(x_0, x_n) = \Sigma_{i=0}^{n-1} d(x_i, x_{i+1})$ then $[x_0, x_n] = [x_0, x_1, ..., x_n]$;*
- *If $[x_{i-1}, x_i] \cap [x_i, x_{i+1}] = \{x_i\}$ for every $1 \leq i \leq n-1$ and if $x_i \neq x_{i+1}$ for every $1 \leq i \leq n-2$, then $d(x_0, x_n) = \Sigma_{i=0}^{n-1} d(x_i, x_{i+1})$ (and so $[x_0, x_n] = [x_0, x_1, ..., x_n]$).*

The less trivial of the propositions is

Proposition 2.29 (The bridge). *If T_0 and T_1 are disjoint closed-subtrees of an $\mathbb{R}$-tree T, there is a unique geodesic segment $[z_0, z_1]$ of T such that*

$$(z_0, z_1) \in T_0 \times T_1, \text{ and for every } (x_0, x_1) \in T_0 \times T_1 \text{ we have } [z_0, z_1] \subset [x_0, x_1].$$



Furthermore, for $i = 1, 2$ we have $[z_0, z_1] \cap T_i = \{z_i\}$ and $d(z_0, z_1) = d(T_0, T_1)$. The geodesic $[z_0, z_1]$ is called the bridge between T_0 and T_1 in T.

Most papers use the following characterization of $\mathbb{R}$-trees:

Proposition 2.30. *A geodesic metric space (T, d) is an $\mathbb{R}$-tree if and only if for every $x, y \in T$ there is a unique topological embedding $\gamma : [0, d(x, y)] \to T$ from x to y, with image being the geodesic $[x, y]$.*

Let us show the important fact that the $\mathbb{R}$-trees are the 0-hyperbolic spaces.

Proposition 2.31. *A geodesic metric space (T, d) is an $\mathbb{R}$-tree if and only if it is 0-hyperbolic.*

Demonstração. First note that, by Definition 2.6, T is 0-hyperbolic if and only if every edge of a geodesic triangle is contained in the union of the other two edges. Suppose first that T is an $\mathbb{R}$-tree and let $\Delta(x, y, z)$ be a geodesic triangle. It is enough to show that $[x, y] \subset [x, z] \cup [z, y]$, the other cases being similar. Let $w = Y(y, x, z)$. Using the Y-proposition, we have

$$[x, y] = [x, w, y] = [x, w] \cup [w, y] \subset [x, z] \cup [z, y],$$

as desired. Suppose now T is 0-hyperbolic and let us check items $a)$, $b)$ and $c)$ of Definition 2.24. T is geodesic by assumption, so we have $a)$. To show $b)$, let $x, y, z \in T$ and let $\alpha : [0, d(x, y)] \to T$ (let $\alpha' : [0, d(x, z)] \to T$) be the geodesic from x to y (from x to z). Let $m = \min\{d(x, y), d(x, z)\}$ and

$$A = \{t \in [0, m] \mid \alpha(t) = \alpha'(t)\}.$$

Of course, $0 \in A \neq \emptyset$ and A is bounded, so let $s = \sup A$. We claim that $t \in A \Rightarrow [0, t] \subset A$. Indeed, if $t \in A$, let $\alpha, \alpha' : [0, t] \to T$ be the restrictions. Their images are geodesics between the points $\alpha(0) = x = \alpha'(0)$ and $\alpha(t) = \alpha'(t)$, so by uniqueness of geodesic α and α' must coincide in $[0, t]$, and so by definition we have $[0, t] \subset A$, which shows the claim. This shows that A is an interval containing 0 and inside $[0, s]$. If we show that $s \in A$ we will then have $A = [0, s]$. If $s = 0$ then s is obviously in A. Suppose $s > 0$. By definition of supremum we have, for n large enough, a sequence $(t_n)_n \subset A$ such that $0 < s - \frac{1}{n} < t_n \leq s$, so $\lim_n t_n = s$. By hypothesis, $\alpha(t_n) = \alpha'(t_n)$ (or $d(\alpha(t_n), \alpha'(t_n)) = 0$) for every n in the sequence. Now, since α, α' and d are continuous we have

$$0 = \lim_n d(\alpha(t_n), \alpha'(t_n)) = d(\lim_n \alpha(t_n), \lim_n \alpha'(t_n)) = d(\alpha(\lim_n t_n), \alpha'(\lim_n t_n)) = d(\alpha(s), \alpha'(s)),$$

so $s \in A$ and $A = [0, s]$. Now define $w = \alpha(s)$. We claim that $[x, y] \cap [x, z] = [x, w]$. Of course $(\supset)$ is true. To see $(\subset)$, suppose we have a point p in $[x, y] \cap [x, z]$, that is, suppose $p = \alpha(t) = \alpha'(t')$ for some t, t'. Because α, α' are geodesics we have

$$t' = |t' - 0| = d(\alpha'(t'), \alpha'(0)) = d(\alpha(t), x) = d(\alpha(t), \alpha(0)) = |t - 0| = t,$$

so $\alpha(t) = \alpha'(t)$ and $t \in A = [0, s]$. Therefore $p = \alpha(t) \in \alpha[0, s] = [x, w]$, as desired. This shows $b)$. Let us show $c)$: let $x, y, z \in T$ such that $[x, y] \cap [y, z] = \{y\}$ and let $\alpha : [0, d(x, y)] \to T$, $\alpha' : [0, d(y, z)] \to T$ and $\alpha'' : [0, d(x, z)] \to T$ be the geodesics representing $[x, y], [y, z]$ and $[x, z]$, respectively. Because of the 0-hyperbolicity we have $[x, z] \subset [x, y] \cup [y, z]$. We have to show that $[x, y] \subset [x, z]$ and $[y, z] \subset [x, z]$. For every $0 \leq t < d(x, y)$, we know $\alpha(t) \in [x, y] \subset [x, z] \cup [y, z]$. But $\alpha(t) \neq y$. So, since $[x, y] \cap [y, z] = \{y\}$ the only possibility is $\alpha(t) \in [x, z]$. So $\alpha[0, d(x, y)) \subset [x, z]$ and since $[x, z]$ is compact (therefore closed) we must have (by taking a sequence) $y = \alpha(d(x, y)) \in [x, z]$ as well. Therefore $[x, y] \subset [x, z]$. Now, $[y, z]$ and $\alpha''[d(x, y), d(x, z)]$ must be both geodesics from y to z. So they coincide and therefore $[y, z] = \alpha''[d(x, y), d(x, z)] \subset [x, z]$, as we wanted. This shows that $[x, z] = [x, y] \cup [y, z]$ and finishes the proposition. □

As a consequence of this and of Proposition 2.18, we get

Corollary 2.32. *A geodesic metric space (X, d) is an ℝ-tree if and only if for every $x, y, z, w \in X$,*

$$\langle x, z \rangle_w \geq \min\{\langle x, y \rangle_w, \langle y, z \rangle_w\}.$$

Because ℝ-trees are very special hyperbolic spaces ($\delta = 0$), the notions of ends and boundary coincide. It is easy to show the following

Proposition 2.33. *If r_1 and r_2 are rays in an ℝ-tree T, the following are equivalent:*

- $\mathrm{im}(r_1) \cap \mathrm{im}(r_2)$ *is not bounded;*

- *end*$(r_1) =$ *end*(r_2) *in Ends*(T);
- $r_1(\infty) = r_2(\infty)$ *in* ∂T.

2.4 Isometries and actions on $\mathbb{R}$-trees

The next few pages are based on [20]. Here we will mostly denote the image of a point x under an isometry g (definition below) by gx instead of $g(x)$ or $g\cdot x$, as it is sometimes in the literature. This will make the notation easier, and we will make sure the reader knows if a letter represents a point or a map.

Definition 2.34. Let T be an $\mathbb{R}$-tree. An isometry of T is a map $g : T \to T$ such that $d(gx, gy) = d(x, y)$ for every $x, y \in T$. For such isometry, the length of g is denoted by $\|g\|$ and defined as

$$\|g\| = \inf_{x\in T} d(x, gx) \geq 0.$$

Before we start talking about the isometries of $\mathbb{R}$-trees, let us note that the length is invariant under conjugation.

Lemma 2.35. *If g, h are isometries of an $\mathbb{R}$-tree T, then $\|g\| = \|hgh^{-1}\|$.*

Demonstração. Given $x \in T$, $d(x, hgh^{-1}x) = d(h^{-1}x, gh^{-1}x) \geq \inf_{y\in T} d(y, gy) = \|g\|$. Since this is true for every x we have $\|hgh^{-1}\| = \inf_{x\in T} d(x, hgh^{-1}x) \geq \|g\|$. Similarly, given $x \in T$ we have

$$d(x, gx) = d(h^{-1}hx, gh^{-1}hx) = d(hx, hgh^{-1}hx) \geq \inf_{y\in T} d(y, hgh^{-1}y) = \|hgh^{-1}\|,$$

from where we get $\|g\| = \inf_{x\in T} d(x, gx) \geq \|hgh^{-1}\|$. Thus, $\|g\| = \|hgh^{-1}\|$. □

Now, we are going to state the main propositions we need to know especially for Chapter 7. The first thing to do is to understand the two main types of isometries on $\mathbb{R}$-trees and its special characteristic sets:

Proposition 2.36 (Classification of isometries)**.** *Let g be an isometry of an $\mathbb{R}$-tree T. Let*

$$C_g = \{x \in T \mid d(x, gx) = \|g\|\}.$$

Then C_g is a nonempty closed-subtree of T which is invariant under g. Furthermore, the following assertions hold:

1) If $\|g\| = 0$, then $C_g = Fix(g)$ is the set of fixed points of g;

2) If $\|g\| > 0$, then C_g is isometric to $\mathbb{R}$ and g acts on C_g as a translation by $\|g\|$;

3) For every $x \in T$, $d(x, gx) = \|g\| + 2d(x, C_g)$;

4) The middle point of any geodesic of the form $[x, gx]$ in T is in C_g.

Definition 2.37. Let g and T as above. If $\|g\| = 0$ we call g an eliptic isometry and C_g turns out to be the fixed point set of g. If $\|g\| > 0$, we call g a hyperbolic isometry and C_g the translation axis of g, where there is a well defined orientation (the direction of the translation).

The proposition above shows that $C_g \neq \emptyset$, which means that the infimum $\|g\| = \inf_{x \in T} d(x, gx)$ is always attained. This implies that if g does not have a fixed point then $\|g\| > 0$. So, the eliptic isometries are the ones with fixed points and the hyperbolic isometries are the ones without them.

Proposition 2.38. *Let g, h be two isometries of an $\mathbb{R}$-tree T. If either*

1) $C_g \cap C_h = \emptyset$, or

2) $C_g \cap C_h$ is a single point and g, h are hyperbolic,

then

$$\|hg\| = \|h^{-1}g\| = \|g\| + \|h\| + 2d(C_g, C_h).$$

In case 1), C_{hg} contains the bridge between C_g and C_h.

Proposition 2.39. *Let g, h be two hyperbolic isometries of an $\mathbb{R}$-tree T. Then*

1) $C_g \cap C_h \neq \emptyset \Leftrightarrow \max\{\|hg\|, \|h^{-1}g\|\} = \|g\| + \|h\|$;

2) $\|hg\| > \|h^{-1}g\| \Leftrightarrow C_g \cap C_h$ contains a non-degenerate segment whose orientations induced by g and h coincide.

Of course, if $h = g$ is hyperbolic, then $C_g \cap C_g$ is nonempty and contains a non-degenerate segment with same orientation, so by items 1) and 2) above we have

$$\|g^2\| = \max\{\|g^2\|, \|g^{-1}g\|\} = \|g\| + \|g\| = 2\|g\|.$$

It follows that g^2 is also hyperbolic and $C_g \subset C_{g^2}$. Also, both axes are isometric to $\mathbb{R}$, so they must be equal. By induction we get

Corollary 2.40. *If g is a hyperbolic isometry of an $\mathbb{R}$-tree T and $n \geq 1$, then g^n is hyperbolic, $\|g^n\| = n\|g\|$ and $C_{g^n} = C_g$.*

Now we turn to actions by isometries on $\mathbb{R}$-trees. The reader should be used with the language of group actions.

Definition 2.41. We say that a group G acts by isometries on an $\mathbb{R}$-tree (T, d) if G acts on T and every $g \in G$ induces an isometry of T (that will also be denoted by g). We use the expression $G \curvearrowright T$ for actions. The translation length function of such action by isometries is denoted by $l : G \to \mathbb{R}$ and defined by $l(g) = \|g\| = \inf_{x \in T} d(x, gx)$. We say the action is non-trivial if $l \neq 0$. If we need to clarify we can denote l by l_T, l_d, $l_{(G,T)}$ or even $l_{(G,T,d)}$.

By using the subtrees proposition we can get

Proposition 2.42. *If G is a finitely generated group acting by isometries on an $\mathbb{R}$-tree T, then the action is trivial if and only if there exists a point in T fixed by all G.*

An action by isometries $G \curvearrowright T$ also induces a well defined action in the set of rays of T by putting $(gr)(t) = g(r(t))$. We also get well-defined actions (not necessarily by isometries) $G \curvearrowright Ends(T)$ and $G \curvearrowright \partial T$ by putting $g(end(r)) = end(gr)$ and $g(r(\infty)) = (gr)(\infty)$.

Since most actions in this work are by isometries, we will sometimes call them only by actions.

Definition 2.43. We say an action by isometries $G \curvearrowright T$ of a group G on an $\mathbb{R}$-tree T is reducible if either

1) Every element g is eliptic (i.e. the action is trivial), or

2) There is a G-invariant line in T, or

3) There is $end(r) \in Ends(T)$ fixed by all G.

An irreducible action is an action which is not reducible. That means none of the items above are satisfied.

Definition 2.44. We say an action by isometries $G \curvearrowright T$ of a group G on an $\mathbb{R}$-tree T is semi-simple if either

1) It has a global fixed point in T, or

2) There is a G-invariant line in T, or

3) It is an irreducible action.

With irreducible actions, a translation axis C_g must always be disjoint of some translation axis C_h. We can see this directly from

Proposition 2.45. *If $G \curvearrowright T$ is an action by isometries and if there exists g hyperbolic such that $C_g \cap C_h \neq \emptyset$ for every hyperbolic element $h \in G$, then the action is reducible.*

Now we will define some specific types of actions and state the characterizations given in [20].

Definition 2.46. Let $G \curvearrowright T$ be an action. Given a geodesisc arc $[x, y]$, the stabilizer of $[x, y]$ is the subgroup $Stab([x, y]) = \{g \in G \mid gz = z \ \forall \ z \in [x, y]\}$ (similarly we define the stabilizer of any subset $S \subset T$). The action is said to be small if every arc stabilizer is virtually cyclic, that is, it contains a finite index cyclic subgroup.

For the next definition, remember that any isometry of $\mathbb{R}$ is either a translation or the composition of a translation with the reflection $x \mapsto -x$. The first type preserves and the second type reverses the orientation of $\mathbb{R}$. Now, if L is any G-invariant line of an action $G \curvearrowright T$, we say G reverses the orientation of L if there is $g \in G$ such that the isometry g induced in $L \simeq \mathbb{R}$ reverses its orientation. Otherwise, we say that G preserves the orientation of L, i.e., every isometry $g \in G$ induced on L is a translation.

Definition 2.47. Let $G \curvearrowright T$ be an action of type 2) in 2.43, that is, with at least one G-invariant line. We say the action is dihedral if G reverses the orientation of every G-invariant line. Otherwise, i.e., if G preserves the orientation of some G-invariant line, we say the action is a shift.

We can classify the non-trivial actions in the following way:

Proposition 2.48. *Any non-trivial action $G \curvearrowright T$ is of one (and only one) of the following types:*

- *With a fixed end;*
- *Dihedral;*
- *Irreducible.*

Demonstração. This is just an observation. If the action has a fixed end, we are done. Suppose it has no fixed ends. Then item 3) of definition 2.43 is false. Since the action is non-trivial, item 1) is also false. So either 2) is false (and the action is irreducible) or it is true and the action is either dihedral or a shift. But every shift fixes 2 ends of T (determined by the G-invariant line whose orientation is preserved by G), so the action must be dihedral. This finishes the proof. □

Below we summarize the main properties of the three types of non-trivial actions above. We highlight the close relationship they have with its translation length functions.

Theorem 2.49 (Fixed end actions). *Let $G \curvearrowright T$ be a non-trivial action by isometries on an $\mathbb{R}$-tree T. The following are equivalent:*

a) There is $end(r) \in Ends(T)$ fixed by all G;

b) $l(g) = |\rho(g)|\ \forall\ g \in G$, where $\rho : G \to \mathbb{R}$ is a homomorphism;

c) $\|ghg^{-1}h^{-1}\| = 0$ for every $g, h \in G$.

Theorem 2.50 (Dihedral actions). *Let $G \curvearrowright T$ be a non-trivial action by isometries on an $\mathbb{R}$-tree T. The following are equivalent:*

a) The action is dihedral;

b) $l(g) = \tilde{l}(f(g))$, where $f : G \to Isom(\mathbb{R})$ is a homomorphism whose image contains a reflection and $\tilde{l}$ is the translation length function of the natural action $Isom(\mathbb{R}) \curvearrowright \mathbb{R}$;

c) $\|ghg^{-1}h^{-1}\| = 0$ for every hyperbolic elements g, h but there are elements $a, b \in G$ such that $\|aba^{-1}b^{-1}\| > 0$.

In the context of irreducible actions, there are two interesting facts:

Proposition 2.51. *Let g, h be hyperbolic isometries of an $\mathbb{R}$-tree T. If either $C_g \cap C_h$ is empty or a geodesic segment of length less than $\min\{\|g\|, \|h\|\}$, then the subgroup $\langle g, h\rangle \leq Isom(T)$ is free of rank 2.*

Theorem 2.52 (Irreducible actions). *Let $G \curvearrowright T$ be a non-trivial action by isometries on an $\mathbb{R}$-tree T. The following are equivalent:*

a) The action is irreducible;

b) There are hyperbolic elements g, h such that $\|ghg^{-1}h^{-1}\| \neq 0$;

c) There are hyperbolic elements g, h such that $C_g \cap C_h$ is a non-degenerate geodesic;

d) G contains a free group of rank 2 acting freely and properly discontinuously on T.

Let us now define minimal actions and prove powerful tools for Chapter 7:

Definition 2.53. We say that an action $G \curvearrowright T$ is minimal if T is a minimal G-invariant tree, that is, there is no G-invariant subtree $T' \subset T$ other than T itself.

A standard fact in the literature is that it is easy to obtain a minimal action:

Proposition 2.54 (Minimal action)**.** *If $G \curvearrowright T$ is a non-trivial action, there is a unique G-invariant minimal subtree $T' \subset T$. The subtree T' is exactly the union of all translation axes of the hyperbolic elements of G. Therefore, a non-trivial action $G \curvearrowright T$ is minimal if and only if $T = T'$. Furthermore, T' is contained in every G-invariant subtree of T.*

Most minimal actions also have a useful uniqueness, up to equivariant isometry:

Theorem 2.55. *Suppose that $G \curvearrowright (T_1, d_1)$ and $G \curvearrowright (T_2, d_2)$ are two minimal semi-simple actions of a group G on any $\mathbb{R}$-trees, with the same translation length function. Then there exists a G-equivariant isometry $h : (T_1, d_1) \to (T_2, d_2)$, that is, a bijection such that $d_2(h(x), h(y)) = d_1(x, y)$ for every $x, y \in T_1$ and $h(g \cdot x) = g \cdot h(x)$ for every $g \in G$, $x \in T_1$. If either action is not a shift then the equivariant isometry is unique.*

Theorem 2.56. *If $G \curvearrowright T$ is a minimal and irreducible action, then every geodesic $[x, x']$ is contained in the translation axis of some hyperbolic isometry $g \in G$.*

Demonstração. If $x = x'$ the result is trivial since the action is minimal. So, suppose $x \neq x'$ and, since T is minimal, let g, g' be hyperbolic elements such that $x \in C_g$ and $x' \in C_{g'}$. If $x \in C_{g'}$ or $x' \in C_g$, we are also done, so suppose neither x nor x' are in the intersection (possibly empty) $C_g \cap C_{g'}$. If $C_g \cap C_{g'} = \emptyset$, let $[z, z']$ be the bridge between C_g and $C_{g'}$. If $C_g \cap C_{g'} \neq \emptyset$, by the Y-proposition 2.26 we get a point $z \in C_g \cap C_{g'}$ such that $[x, x'] = [x, z, x']$, and we define $z' = z$. In any of the two cases we have $[x, x'] = [x, z, z', x']$. Now, since the action is irreducible, by Proposition 2.45 there are translation axes C_h and $C_{h'}$ disjoint from C_g and $C_{g'}$, respectively. So, let $\alpha = [w, y]$ and $\alpha' = [w', y']$ be the bridges from C_h to C_g and from $C_{h'}$ to $C_{g'}$, respectively. By acting replacing α by $g^n\alpha$ for some $n \in \mathbb{Z}$ and replacing C_h by $g^n C_h$, if necessary, we can suppose $x \in [y, z]$. Similarly, suppose $x' \in [y', z']$. Then with a little patience one can use the Piecewise geodesic proposition 2.28 to see that $\gamma = [w, y, x, x', y', w']$ is a geodesic segment. Since it is non-degenerate we have $C_h \cap C_{h'} = \emptyset$. If $\tilde{\gamma}$ is the bridge from C_h to $C_{h'}$, we have $\tilde{\gamma} \subset \gamma$ by definition. But from the construction of γ one can see that $\gamma = \tilde{\gamma}$. So, $[x, x']$ is contained in the bridge from C_h to $C_{h'}$. From Proposition 2.38, we have $[x, x'] \subset C_{hh'}$ is in the translation axis of the hyperbolic element hh', as desired. □

2.5 Filters and ultrafilters

Here we give a minimal approach to the notions of filters, ultrafilters and the ultralimit of a sequence of metric spaces. All of this is going to be needed in chapters 8 and 9.

This section is all based on [13] and [62].

Definition 2.57 (Filters)**.** A filter $\mathcal{F}$ in a nonempty set X is a collection of subsets of X such that:

- For every $A \subset X$, if $A \supset B$ for some $B \in \mathcal{F}$ then $A \in \mathcal{F}$;
- If $B_1, ..., B_n \in \mathcal{F}$ then $B_1 \cap ... \cap B_n \in \mathcal{F}$;
- $\emptyset \notin \mathcal{F}$.

We say $(X, \mathcal{F})$ is a filtered space.

Of course, from the definition we can see that the finite intersection of elements of $\mathcal{F}$ is never empty. Also, any nonempty filter $\mathcal{F}$ contains as an element the whole set X, for X contains some element $A \in \mathcal{F}$ and therefore $X \in \mathcal{F}$ by the first item.

Example 2.58. The two examples we're going to use are:

1) If (X, τ) is a topological space and $x_0 \in A$, the collection

$$\mathcal{F}_{x_0} = \{A \subset X \mid \text{there is } B \in \tau \text{ such that } x_0 \in B \subset A\}$$

is a filter in X and is called the *neighborhood filter on* x.

2) If X is any infinite set, the collection

$$\mathcal{F} = \{A \subset X \mid X - A \text{ is finite}\}$$

is a filter in X and is called the *finite complement filter.* In particular, for the natural numbers $\mathbb{N}$, the finite complement filter is also called the *Fréchet filter* in $\mathbb{N}$.

Filters can be used for a quite general definition of limit:

Definition 2.59 (Limit of a map over a filter)**.** Let $(X, \mathcal{F})$ be a filtered space, (Y, τ) be a topological space and $f : X \to Y$ be any map. We say that a point $y \in Y$ is the limit of f over the filter $\mathcal{F}$ and denote $y = \lim_{\mathcal{F}} f$ if, for every open set $A \in \tau$ of Y containing y, there is $B \in \mathcal{F}$ such that $f(B) \subset A$. If $X = \mathbb{N}$, we denote $x_n = f(n)$ and use the notation $y = \lim_{\mathcal{F}} x_n$ instead of the previous one.

It is straightforward to see that this definition generalizes, for example, the well-known notions of limit of a sequence and limit of a map in the topological sense. In fact, if (X, τ') and (Y, τ) are both topological spaces, if $x_0 \in X$ and $\mathcal{F}_{x_0}$ is the neighborhood filter on x_0, then for any $y \in Y$,

$$y = \lim_{\mathcal{F}_{x_0}} f \Leftrightarrow y = \lim_{x \to x_0} f(x).$$

Similarly, let $X = \mathbb{N}$ and let $\mathcal{F}$ be the Fréchet filter on $\mathbb{N}$. If $(x_n)_n$ is any sequence in a topological space (X, τ) and $x \in X$, we easily see that

$$x = \lim_{\mathcal{F}} x_n \Leftrightarrow x = \lim_{n \to \infty} x_n.$$

Because of the above fact, the convergence of a sequence can be thought in terms of the existence of some good elements of a filter in $\mathbb{N}$. So, the "bigger" a filter is (i.e., the more elements it has), the more chance we have of finding such elements and therefore the more convergence we have (of course, with respect to that filter). Keeping this intuitive notion in mind, we go in the direction of sequences in a compact space, in particular bounded sequences. For we know that, in general, not every sequence in a compact space converges; in particular, there are bounded sequences in $\mathbb{R}$ with no global limit points (for example, any non-constant periodic sequence). An interesting question then arises: are there "big enough" filters in $\mathbb{N}$ in order to guarantee the convergence of such sequences? Fortunately, the answer is positive. These are the ultrafilters, as we will see now.

For the definition of ultrafilters below, note that the collection of all filters in a set X is partially ordered by $\mathcal{F} \leq \mathcal{F}' \Leftrightarrow \mathcal{F} \subset \mathcal{F}'$.

Definition 2.60 (Ultrafilters)**.** We say that a filter $\mathcal{U}$ in a set X is an ultrafilter if it is a maximal element in the collection of all filters of X; that is, if $\mathcal{U}$ is a filter and satisfies the following:

$$\text{If } \mathcal{F} \text{ is a filter in } X \text{ and } \mathcal{U} \subset \mathcal{F}, \text{ then } \mathcal{U} = \mathcal{F}.$$

Ultrafilters are the answer to our previous question. Their maximality will guarantee they are "big enough" in the sense we asked, as we will see next. A first and important observation is that every ultrafilter $\mathcal{U}$ contains the set X as an element. Indeed, they are nonempty, because if $\mathcal{U} = \emptyset$, then the collection $\mathcal{F} = \{X\}$ is a filter in X and would contain $\mathcal{U}$ properly, a contradiction with the maximality of $\mathcal{U}$. Since $\mathcal{U} \neq \emptyset$, we have $X \in \mathcal{U}$, as observed right after Definition 2.57. As a standard and straightforward application of Zorn's lemma, one can also show

Proposition 2.61. *If $\mathcal{F}$ is any filter in a set X, there is an ultrafilter $\mathcal{U}$ in X such that $\mathcal{F} \subset \mathcal{U}$.*

Another important property is

Proposition 2.62. *Let $\mathcal{U}$ be an ultrafilter in X. If $A_1, ..., A_n \subset X$ are such that $A_1 \cup ... \cup A_n \in \mathcal{U}$, then $A_i \in \mathcal{U}$ for some $1 \leq i \leq n$. In particular, if $X = A_1 \cup ... \cup A_n$ then $A_i \in \mathcal{U}$ for some $1 \leq i \leq n$.*

Demonstração. It is enough to show the property for only two subsets, say, A and B, for the general case follows by trivial induction. Suppose then by contradiction that $A, B \subset X$ are such that $A \cup B \in \mathcal{U}$ but $A \notin \mathcal{U}$ and $B \notin \mathcal{U}$. then it is easy to see that the collection

$$\mathcal{F} = \{S \subset X \mid A \cup S \in \mathcal{U}\}$$

would be a filter and would properly contain $\mathcal{U}$, contradicting the maximality of $\mathcal{U}$. This finishes the first part. The particular case $X = A_1 \cup ... \cup A_n$ follows from the first part and from the fact $X \in \mathcal{U}$ we showed after Definition 2.60. □

Now we present the property we were talking about before:

Proposition 2.63. *Let $(X,\mathcal{U})$ be an (ultra)filtered space, (Y, τ) be a compact and Hausdorff topological space and let $f : X \to Y$ be any map. Then there is a unique $y \in X$ such that $y = \lim_{\mathcal{U}} f$.*

Demonstração. For the existence, suppose by contradiction that every $y \in Y$ is not a limit point of f. Then, by definition, there exists an open set A_y of Y containing y such that there is no $B \in \mathcal{U}$ with $f(B) \subset A_y$. In particular, we have

$$f^{-1}(A_y) \notin \mathcal{U} \text{ for every } y \in Y, \tag{2.1}$$

for $f(f^{-1}(A_y)) \subset A_y$. We have $Y = \bigcup_{y\in Y} A_y$ an open cover of the compact set Y, so there are $A_{y_1}, ..., A_{y_n}$ such that $Y = A_{y_1} \cup ... \cup A_{y_n}$. Therefore,

$$X = f^{-1}(Y) = f^{-1}(A_{y_1}) \cup ... \cup f^{-1}(A_{y_n}).$$

Since $\mathcal{U}$ is an ultrafilter in X, by the previous proposition we must have $f^{-1}(A_{y_i}) \in \mathcal{U}$ for some i. This is a contradiction with property 2.1. This shows existence.

To show uniqueness, suppose by contradiction there are distinct elements $y_1 \neq y_2$ such that $y_1 = \lim_{\mathcal{U}} f$ and $y_2 = \lim_{\mathcal{U}} f$. Since Y is Hausdorff, let $A_1, A_2 \in \tau$ containing y_1 and y_2, respectively, such that $A_1 \cap A_2 = \emptyset$. By hypothesis, there are $B_1, B_2 \in \mathcal{U}$ such that $f(B_1) \subset A_1$ and $f(B_2) \subset A_2$. Then

$$B_1 \cap B_2 \subset f^{-1}(A_1) \cap f^{-1}(A_2) = f^{-1}(A_1 \cap A_2) = f^{-1}(\emptyset) = \emptyset$$

and $B_1 \cap B_2 = \emptyset$, which is a contradiction because every finite intersection of elements of a filter is nonempty. This concludes the proof. □

In particular, we have the very useful corollary below.

Corollary 2.64. *If* $(\mathbb{N}, \mathcal{U})$ *is ultrafiltered and* $(x_n)_n$ *is a bounded sequence in* $\mathbb{R}$*, then there is an unique* $y \in \mathbb{R}$ *such that* $y = \lim_{\mathcal{U}} x_n$*.*

Proposition 2.65. *Let* ω *be an ultrafilter of* $\mathbb{N}$ *containing the Fréchet filter (Proposition 2.61). If* $x = \lim_{n\to\infty} x_n$ *for a sequence* $\{x_n\}$ *in a topological space* X*, then* $x = \lim_\omega x_n$*. In other words, standard convergence implies* ω*-convergence.*

Demonstração. Let us represent the sequence $(x_n)_n$ by $f : \mathbb{N} \to X$ with $f(n) = x_n$. Let A be open in X containing x. By hypothesis we have n_0 such that $x_n \in A$ for every $n \geq n_0$, so we have $f(B) \subset A$ for $B = \mathbb{N} - \{1, 2, ..., n_0 - 1\}$. But since $\{1, 2, ..., n_0 - 1\}$ is finite we have B as an element of the Fréchet filter and therefore by hypothesis $B \in \omega$. This shows by definition that $x = \lim_\omega x_n$, as desired. □

Below we give the basic properties about limits that we're going to need in our context. The proofs are omitted, for they the exact same fashion of the well known ones from basic analysis.

Proposition 2.66 (Basic properties)**.** *Let* $\mathcal{F}$ *be a filter in* $\mathbb{N}$*. The following sentences are true:*

- *(Sum and scalar product) Let* $(x_n)_n$ *and* $(y_n)_n$ *be sequences in a real normed vector space* V *and let* $\lambda \in \mathbb{R}$*. Suppose* $x = \lim_{\mathcal{F}} x_n$*,* $y = \lim_{\mathcal{F}} y_n$*. Then* $\lambda x = \lim_{\mathcal{F}} \lambda x_n$ *and* $x + y = \lim_{\mathcal{F}}(x_n + y_n)$*;*

- *(Order preserving) Let* $(x_n)_n$*,* $(y_n)_n$*,* x *and* y *be as above and suppose* $V = \mathbb{R}$*. If* $x_n \leq y_n$ *for every* n*, then* $x \leq y$*, that is,* $\lim_{\mathcal{F}} x_n \leq \lim_{\mathcal{F}} y_n$*;*

- *(Direct products) Let* $(x_n)_n$ *and* $(y_n)_n$ *be sequences in a real normed vector spaces* V *and* W*, respectively. Suppose* $x = \lim_{\mathcal{F}} x_n$ *and* $y = \lim_{\mathcal{F}} y_n$*. Then, in the direct product* $V \times W$ *we have* $(x, y) = \lim_{\mathcal{F}}(x_n, y_n)$*;*

- *(Continuous maps) Let* X, Y *be topological spaces,* $f : X \to Y$ *a continuous map and suppose* $x = \lim_{\mathcal{F}} x_n$ *for some sequence* $(x_n)_n$ *in* X*. Then* $f(x) = \lim_{\mathcal{F}} f(x_n)$*.*

2.6 Ultralimits

Let us now construct a space that we will call the ultralimit of a sequence of pointed metric spaces. From now on, we let ω be any ultrafilter of $\mathbb{N}$ containig the Fréchet filter (Example 2.58 and Proposition 2.61) and fix it. For every $n \geq 1$, let (X_n, d_n, p_n) be a pointed metric space, that is, (X_n, d_n) is a metric space and $p_n \in X_n$. Denote by $\prod_{n\geq 1} X_n$ the set of all sequences $(x_n)_n$ such that $x_n \in X_n$ for every n, and define

$$X_\infty = \{(x_n)_n \in \prod_{n\geq 1} X_n \mid \text{there is } C \geq 0 \text{ such that } d_n(x_n, p_n) \leq C \text{ for every } n \geq 1\}.$$

It is easy to see from the definition and using the triangle inequality that for every two elements $(x_n)_n, (y_n)_n \in X_\infty$, the real sequence $(d_n(x_n, y_n))_n$ is bounded and therefore by Proposition 2.63 there is a unique (so, well-defined) real number $\lim_\omega d_n(x_n, y_n)$. So, we denote $x = (x_n)_n$ and $y = (y_n)_n$ and define $d_\infty : X_\infty \times X_\infty \to \mathbb{R}$ by putting

$$d_\infty(x, y) = \lim_\omega d_n(x_n, y_n).$$

This is what we call a pseudo-distance, that is, it satisfies the following properties for every $x, y, z \in X_\infty$: $d_\infty(x, x) = 0$, $d_\infty(x, y) = d_\infty(y, x)$ and $d_\infty(x, z) \leq d_\infty(x, y) + d_\infty(y, z)$. All of these can be easily verified using the properties in the previous part. For example, the triangle inequality follows by the "sum" and "order preserving" properties:

$$\begin{aligned} d_\infty(x, z) &= \lim_\omega d_n(x_n, z_n) \\ &\leq \lim_\omega (d_n(x_n, y_n) + d_n(y_n, z_n)) \\ &= \lim_\omega d_n(x_n, y_n) + \lim_\omega d_n(y_n, z_n) \\ &= d_\infty(x, y) + d_\infty(y, z). \end{aligned}$$

The only property d_∞ lacks to be a distance is "$d_\infty(x, y) = 0 \Rightarrow x = y$". So, to create a metric space we need a quotient of X_∞. Define the following relation in X_∞: $x \sim y \Leftrightarrow d_\infty(x, y) = 0$. It is an equivalence relation, whose equivalence classes will be denoted by $[x] = [(x_n)_n]$, for any $x = (x_n)_n \in X_\infty$.

Definition 2.67. The (ω)-ultralimit of a given a sequence $(X_n, d_n, p_n)_n$ of pointed metric spaces is the space (X_ω, d_ω), where X_ω is the quotient

$$X_\omega = X_\infty / \sim = \{[x] \mid x \in X_\infty\}$$

given by the relation above, and $d_\omega : X_\omega \times X_\omega \to \mathbb{R}$ is defined as

$$d_\omega([x], [y]) = d_\infty(x, y) = \lim_\omega d_n(x_n, y_n).$$

Of course we could expect that the ultralimit inherits some properties of the metric spaces involved. Below we present some of them. The first one is about hyperbolicity:

Proposition 2.68 (Properties of ultralimits). *Let $(X_n, d_n, p_n)_n$ be a sequence of pointed metric spaces.*

1) *If each X_n is geodesic, then X_ω is geodesic;*

2) *If each X_n is (δ_n)-hyperbolic (Definition 2.17) and there is $\delta \geq 0$ such that $\lim_\omega \delta_n = \delta$, then X_ω is (δ)-hyperbolic;*

3) *If each X_n is (δ_n)-hyperbolic (Definition 2.17) and $\lim_\omega \delta_n = 0$, then X_ω is an $\mathbb{R}$-tree.*

Demonstração. Item 1) is a straightforward construction that we left to the curious reader, and item 3) is a consequence of item 2) and of Corollary 2.32. We will show item 2) by using practically all the basic properties of ω-limits (Proposition 2.66). Indeed, let $x, y, z, w \in X_\omega$ and let us show that $\langle x, z\rangle_w \geq \min\{\langle x, y\rangle_w, \langle y, z\rangle_w\} - \delta$ (remember Definition 2.17). For every n, since x_n, y_n, z_n, w_n are elements of the (δ_n)-hyperbolic space X_n, we have

$$\langle x_n, z_n\rangle_{w_n} \geq \min\{\langle x_n, y_n\rangle_{w_n}, \langle y_n, z_n\rangle_{w_n}\} - \delta_n. \tag{2.2}$$

Now, by linearity of ω-limits we have

$$\begin{aligned}
\langle x, z\rangle_w &= \frac{1}{2}\left[d_\omega(x, w) + d_\omega(y, w) - d_\omega(x, z)\right] \\
&= \frac{1}{2}\left[\lim_\omega d_n(x_n, w_n) + \lim_\omega d_n(y_n, w_n) - \lim_\omega d_n(x_n, z_n)\right] \\
&= \lim_\omega \frac{1}{2}\left[d_n(x_n, w_n) + d_n(y_n, w_n) - d_n(x_n, z_n)\right] \\
&= \lim_\omega \langle x_n, z_n\rangle_{w_n}.
\end{aligned}$$

and the same is similarly true for $\langle x, y\rangle_w$ and $\langle y, z\rangle_w$. Denote by $f : \mathbb{R}^2 \to \mathbb{R}$ the continuous map $f(x, y) = \min\{x, y\}$. By using again the properties of Proposition 2.66 and all information above we finally get

$$\begin{aligned}
\min\{\langle x, y\rangle_w, \langle y, z\rangle_w\} - \delta &= \min\left\{\lim_\omega \langle x_n, y_n\rangle_{w_n}, \lim_\omega \langle y_n, z_n\rangle_{w_n}\right\} - \lim_\omega \delta_n \\
&= f\left(\lim_\omega \langle x_n, y_n\rangle_{w_n}, \lim_\omega \langle y_n, z_n\rangle_{w_n}\right) - \lim_\omega \delta_n \\
&= f\left(\lim_\omega \left(\langle x_n, y_n\rangle_{w_n}, \langle y_n, z_n\rangle_{w_n}\right)\right) - \lim_\omega \delta_n \\
&= \lim_\omega f\left(\langle x_n, y_n\rangle_{w_n}, \langle y_n, z_n\rangle_{w_n}\right) - \lim_\omega \delta_n \\
&= \lim_\omega \left(\min\left\{\langle x_n, y_n\rangle_{w_n}, \langle y_n, z_n\rangle_{w_n}\right\} - \delta_n\right) \\
&\leq \lim_\omega \langle x_n, z_n\rangle_{w_n} \\
&= \langle x, z\rangle_w,
\end{aligned}$$

which concludes the proof. □

The second property is about actions:

Proposition 2.69. *Let $(X_n, d_n, p_n)_n$, X_∞ and X_ω be as above. Suppose a group G acts by isometries on each (X_n, d_n) and that*

For every $g \in G$, there is $C = C(g) \geq 0$ such that $d_n(gp_n, p_n) \leq C$ for every n.

Then G acts naturally by isometries on both X_∞ and X_ω.

Demonstração. We will omit some easy verifying details. We naturally define $G \curvearrowright X_\infty$ by putting $g(x_n)_n = (gx_n)_n$. This element is in X_∞. In fact, let $K \geq 0$ such that $d_n(x_n, p_n) \leq K$. Then, for every n,

$$d_n(gx_n, p_n) \leq d_n(gx_n, gp_n) + d_n(gp_n, p_n) = d_n(x_n, p_n) + d_n(gp_n, p_n) \leq K + C,$$

as desired. It is easy to see that the action is by isometries by "passing the limit". Now, we define $G \curvearrowright X_\omega$ by putting $g[(x_n)_n] = [(gx_n)_n]$, which is in X_ω by what we already observed. Let us see that this is well defined on classes: if $[(x_n)_n] = [(y_n)_n]$, then $\lim_\omega d_n(x_n, y_n) = 0$, which implies $\lim_\omega d_n(gx_n, gy_n) = \lim_\omega d_n(x_n, y_n) = 0$, so $[(gx_n)_n] = [(gy_n)_n]$, as desired. It is therefore a well defined action, and it is by isometries because

$$d_\omega(g[(x_n)_n], g[(y_n)_n]) = d_\infty((gx_n)_n, (gy_n)_n) = d_\infty((x_n)_n, (y_n)_n) = d_\omega([(x_n)_n], [(y_n)_n]).$$

□

The last property is about the limit of a sequence of quasi-isometric embeddings. We will omit the proof since it involves the exact same strategies we are already dealing with:

Proposition 2.70. *Let $(X_n, d_n, p_n)_n$ and X_ω as in Definition 2.67. Suppose for each $n \geq 1$ there are maps $f_n : X_n \to X_n$ that are (λ_n, ϵ_n)-QI-embeddings and that $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lim_{n\to\infty} \epsilon_n = \epsilon$ for some $\lambda \geq 1$ and $\epsilon \geq 0$. If $\{d_n(f_n(p_n), p_n) \mid n \geq 1\}$ is a bounded set, then the f_n give rise to a map*

$$f_\omega : X_\omega \to X_\omega, \quad [(x_n)_n] \mapsto [(f_n(x_n))_n]$$

which is a (λ, ϵ)-QI-embedding.

2.7 Convergence actions by homeomorphisms

The topics of this section, especially Proposition 2.84, will be useful to the construction of relatively hyperbolic groups of Chapter 9. Based on [15], we define the notion of convergence action and show that it is equivalent to the existence of some properly discontinuous action on an 3-unordered configuration space.

In the following pages, let (M, d) be a compact metric space (with at least 4 points, something that will be eventually required) and suppose an infinite group G acts by homeomorphisms on M. By a *distinct sequence* $(g_n)_n$ in G we mean any sequence where the g_n are pairwise distinct elements of G.

Definition 2.71. We say a sequence $(g_n)_n \subset G$ of elements of G is a collapsing sequence if there are $x, y \in M$ such that the sequence of restrictions $(g_n|_{M-\{x\}})$ converge uniformly on compact subsets to the constant map $z \in M - \{x\} \mapsto y$. Explicitly, this is equivalent to say that for any compact $K \subset M - \{x\}$ and any $\epsilon > 0$, there is n_0 such that $d(g_n z, y) < \epsilon$ for any $n \geq n_0$ and $z \in K$. We can also say that $(g_n)_n$ collapses on x, y and write $g_n|_{M-\{x\}} \twoheadrightarrow y$.

Definition 2.72. We say the action $G \curvearrowright M$ above is a convergence action, and that G is a convergence group, if every distinct sequence (g_n) in G contains a collapsing subsequence.

It will be useful to rewrite the definition above in terms of convergence of sequences:

Proposition 2.73. *If $(g_n)_n$ is any sequence and $x, y \in M$, then $g_n|_{M-\{x\}} \twoheadrightarrow y$ if, and only if, the following statement is true:*

If $(g_{n_k})_k$ is a subsequence and $(z_{n_k})_k \subset M$ with $z_{n_k} \to z \neq x$ and $g_{n_k} z_{n_k} \to z' \in M$, then $z' = y$.

Demonstração. We will show that the negations of the assertions above are equivalent. Suppose $(g_n)_n$ does not collapse on x, y. Then, by definition, there must be a compact $K \subset M - \{x\}$ and $\epsilon > 0$ such that for any n_0 there is $n \geq n_0$ such that K is not entirely mapped by g_n into $B(y, \epsilon)$. In particular there must be a subsequence $(g_{n_k})_k$ and points $(z_{n_k})_k \subset K$ such that $g_{n_k} z_{n_k} \in M - B(y, \epsilon)$ (note that $M - B(y, \epsilon)$ is compact, for it is a closed subset of the compact space M). Since K is compact, there must be a subsequence of $(z_{n_k})_k$ - that will still be denoted by $(z_{n_k})_k$ - converging to $z \in K$. Now, the asociated subsequence $(g_{n_k} z_{n_k})_k$ inside the compact $M - B(y, \epsilon)$ must also have a subsequence - still denoted by $(g_{n_k} z_{n_k})_k$ - converging to $z' \in M - B(y, \epsilon)$. In particular, we got $z_{n_k} \to z \neq x$ (for $x \notin K$) and $g_{n_k} z_{n_k} \to z' \neq y$, which is what we wanted. Suppose, on the other hand, we have such a situation and let us show $(g_n)_n$ does not collapse on x, y. Since $y \neq z'$ and $x \neq z$, let $\epsilon, \delta > 0$ such that $z' \notin B(y, \epsilon)$ and $x \notin \overline{B}(z, \delta)$. Let $K = \overline{B}(z, \delta)$ (it is compact, for it is closed and M is compact). If $(g_n)_n$ collapsed on x, y, there would be in particular k_0 such that $g_{n_k}(K) \subset B(y, \epsilon)$ for any $k \geq k_0$. In particular, since $z_{n_k} \in K$ for sufficiently large k, we would have $g_{n_k} z_{n_k} \in B(y, \epsilon)$ for sufficiently large k, and so $g_{n_k} z_{n_k}$ could not converge to z', a contradiction. So $(g_n)_n$ cannot collapse on x, y, as we desired. □

Remember the well-known notion of a properly discontinuous action (see, for example, [14]):

Definition 2.74. If a group G acts on a topological space W, we say it is a properly discontinuous action if for every compact subsets $K, L \subset W$, the set $\{g \in G \mid gK \cap L \neq \emptyset\}$ is finite.

Let us now start again with the compact metric space M where an infinite group G acts by homeomorphisms. We are going to construct a space $\Theta(M)$ with an induced action by homeomorphisms $G \curvearrowright \Theta(M)$. We will show that this new action is properly discontinuous if and only if $G \curvearrowright M$ is a convergence action. The space $\Theta(M)$ is well known, and for more details of the following construction we refer [15].

Consider the cartesian product $M^3 = M \times M \times M$, with product topology, and the "fat" diagonal $\Delta = \{(x, y, z) \in M^3 \mid \operatorname{card}\{x, y, z\} \leq 2\}$, so that

$$M^3 - \Delta = \{(x, y, z) \in M^3 \mid \operatorname{card}\{x, y, z\} = 3\}.$$

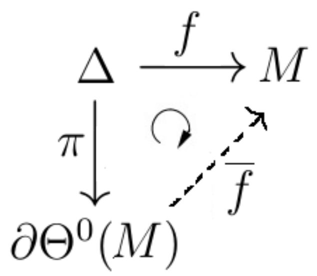
Define an equivalence relation on M^3 in the following way: declare $(x, y, z) \sim (x', y', z')$ if either $(x, y, z) = (x', y', z')$ or there are two coordinates in (x, y, z) and other two in (x', y', z') coinciding (in other words, four of the elements x, y, z, x', y', z' are the same, two in the first and two in the second collections, for example $(x, w, w) \sim (w, y', w)$). If $\pi : M^3 \to M^3/\sim$ is the quotient projection, denote by $\Theta^0(M) = \pi(M^3 - \Delta)$ and $\partial\Theta^0(M) = \pi(\Delta)$, so the quotient space is $\Theta^0(M) \cup \partial\Theta^0(M)$. It is obvious that the relation $\sim$ is trivial in $M^3 - \Delta$, so $\Theta^0(M) \simeq M^3 - \Delta$, but the relation is not trivial on Δ.

Lemma 2.75. *$\partial\Theta^0(M) \simeq M$.*

Demonstração. Let $f : \Delta \to M$ be defined by

$$f(x,y,z) = \begin{cases} x, & \text{if } x = y, \\ x, & \text{if } x = z, \\ y, & \text{if } y = z. \end{cases}$$

It is obviously well defined, surjective and continuous by the Pasting Lemma. It is also easily seen to be an open map, by the product topology properties. Since $f(x,y,z) = f(x',y',z') \Leftrightarrow \pi(x,y,z) = \pi(x',y',z')$, by a known topology lemma there must be a (bijective) and continuous map $\overline{f} : \partial\Theta^0(M) \to M$ such that the following diagram commutes:



We just have to check that the inverse map $g : M \to \partial\Theta^0(M)$ is continuous. But this is easy: if $A \subset \partial\Theta^0(M)$ is open, then $g^{-1}(A) = f(\pi^{-1}(A))$, which is open in M because π is continuous and f is an open map. □

Let the symmetric group S_3 act on M^3 by permuting coordinates. It is clear that $(x,y,z) \sim (x',y',z')$ implies $\sigma(x,y,z) \sim \sigma(x',y',z')$ for any permutation σ, so S_3 also acts on the quotient space $\Theta^0(M) \cup \partial\Theta^0(M)$, by "permutting coordinates" on $\Theta^0(M) \simeq M^3 - \Delta$ and trivially on $\partial\Theta^0(M) \simeq M$. Because of this, we can quotient the space $\Theta^0(M) \cup \partial\Theta^0(M)$ by this action and obtain a space

$$\Theta^T(M) = \Theta(M) \cup \partial\Theta(M)$$

(here the letter T means "total"), where the two subsets on the right are the respective quotients of $\Theta^0(M)$ and $\partial\Theta^0(M)$ by the action of S_3. We still have $\partial\Theta(M) \simeq M$ by the same reasons above, and the space $\Theta(M)$ is exactly the space of unordered triples, the 3-configuration space of M. To clear notation, we then denote an element of $\Theta(M)$ by a set $\{x,y,z\}$ of cardinality 3. Since M is compact, $\partial\Theta(M) \simeq M$ is compact and a closed subset of the (also compact) space $\Theta^T(M)$, while $\Theta(M)$ is open. This is our ambient space.

If G acts on M by homeomorphisms, let us create an action $G \curvearrowright \Theta^T(M)$ on the total space. Given $g \in G$, consider it as a homeomorphism $g : M \to M$. By abuse of notation, it induces a homeomorphism $g : M^3 \to M^3$ acting as g on each coordinate. It is easy to see that $(x,y,z) \sim (x',y',z')$ implies $(gx,gy,gz) \sim (gx',gy',gz')$, so we have an induced homeomorphism $g : \Theta^0(M) \cup \partial\Theta^0(M) \to \Theta^0(M) \cup \partial\Theta^0(M)$ which clearly passes to the quotient by the action of S_3, giving rise to a homeomorphism $g : \Theta^T(M) \to \Theta^T(M)$, which acts like $g : M \to M$ on its invariant subset $\partial\Theta(M)$ and is of the form $g\{x,y,z\} = \{gx,gy,gz\}$ on the (also invariant) configuration space $\Theta(M)$. These are the induced actions $G \curvearrowright \Theta^T(M)$ and $G \curvearrowright \Theta(M)$.

Definition 2.76. We say an action by homeomorphisms $G \curvearrowright M$ of an infinite group G onto a compact metric space M is properly discontinuous on triples if the induced action $G \curvearrowright \Theta(M)$ is properly discontinuous.

Our goal is to show

Theorem 2.77. *Let G be an infinite group acting by homeomorphisms on a compact metric space M. Then the action is properly discontinuous on triples if and only if it is a convergence action (Definition 2.72).*

In order to do this, we first rewrite the "properly discontinuous on triples" property in terms of convergence of sequences (like we did for collapsing sequences), in order to get a common language:

Proposition 2.78. *The action above is properly discontinuous on triples if and only if the following condition is satisfied: let $(g_n)_n$ be a distinct sequence in G and $(x_n)_n, (y_n)_n$ and $(z_n)_n$ be sequences in M such that $x_n \to x$, $y_n \to y$, $z_n \to z$, $g_n x_n \to x'$, $g_n y_n \to y'$ and $g_n z_n \to z'$ for elements $x, y, z, x', y', z' \in M$. If $\operatorname{card}\{x, y, z\} = 3$, then $\operatorname{card}\{x', y', z'\} \leq 2$.*

Demonstração. Here we assume the basic properties about convergence in a configuration space. We will show that the negations of the assertions above are equivalent. Suppose first that the condition is false. So there must be a distinct sequence $(g_n)_n$ in G, sequences $(x_n)_n, (y_n)_n$ and $(z_n)_n$ in M such that $x_n \to x$, $y_n \to y$, $z_n \to z$, $g_n x_n \to x'$, $g_n y_n \to y'$ and $g_n z_n \to z'$ for elements $x, y, z, x', y', z' \in M$, with $\operatorname{card}\{x, y, z\} = 3 = \operatorname{card}\{x', y', z'\}$. Then we have $\{x_n, y_n, z_n\} \to \{x, y, z\}$ $g_n\{x_n, y_n, z_n\} \to \{x', y', z'\}$ inside $\Theta(M)$, which is open in $\Theta^T(M)$. Since $\partial\Theta(M)$ is compact and $\{x, y, z\}, \{x', y', z'\} \notin \partial\Theta(M)$, there are compact neighborhoods $K, L \subset \Theta(M)$ (that is, disjoint from $\partial\Theta(M)$) containing $\{x, y, z\}$ and $\{x', y', z'\}$, respectively. By the two convergences above in $\Theta(M)$ and compactness, there must be n_0 such that $\{x_n, y_n, z_n\} \in K$ and $g_n\{x_n, y_n, z_n\} \in L$ for $n \geq n_0$, thus $g_n\{x_n, y_n, z_n\} \in g_n K \cap L$ for $n \geq n_0$. Therefore, the set $\{g \in G \mid gK \cap L \neq \emptyset\}$ is infinite and the action is not properly discontinuous, as desired. Suppose now the action is not properly discontinuous. Then there are compacts $K, L \subset \Theta(M)$ such that $\{g \in G \mid gK \cap L \neq \emptyset\}$ is infinite, and therefore a distinct sequence $(g_n)_n$ and $(k_n)_n \subset K$ and $(l_n)_n \subset L$ such that $g_n k_n = l_n$ (and we can write $k_n = \{x_n, y_n, z_n\}$ for every n). By compactness, we find convergent subsequences of $(k_n)_n$ and $(l_n)_n$, so to clear notation we can assume $k_n \to k \in K$ and $l_n \to l \in L$. If $k = \{x, y, z\}$ and $l = \{x', y', z'\}$, then since K and L are inside $\Theta(M)$ we have $\operatorname{card}\{x, y, z\} = 3 = \operatorname{card}\{x', y', z'\}$ and since $k_n \to k$, without loss of generality we can assume $x_n \to x$, $y_n \to y$ and $z_n \to z$ (after all, what matters is that the three points are distinct). Similarly, since $\{g_n x_n, g_n y_n, g_n z_n\} = g_n\{x_n, y_n, z_n\} = g_n k_n = l_n \to l = \{x', y', z'\}$, we can also assume $g_n x_n \to x'$, $g_n y_n \to y'$ and $g_n z_n \to z'$. Therefore, the condition is false, as desired. This completes the proof. □

Given the property above, the first half of Theorem 2.77 is the following:

Lemma 2.79. *If $G \curvearrowright M$ is a convergence action, then it is properly discontinuous on triples.*

Demonstração. Let us show that the condition of Proposition 2.78 is satisfied. That is, let $(g_n)_n$ be a distinct sequence in G and $(x_n)_n, (y_n)_n$ and $(z_n)_n$ be sequences in M such that $x_n \to x$, $y_n \to y$, $z_n \to z$, $g_n x_n \to x'$, $g_n y_n \to y'$ and $g_n z_n \to z'$ for elements $x, y, z, x', y', z' \in M$. Suppose card$\{x,y,z\}=3$ and let us show card$\{x',y',z'\}\leq 2$. By the definition of convergence action, $(g_n)_n$ must contain a collapsing subsequence. Since convergence passes to every subsequence, we can assume without loss of generality that $(g_n)_n$ collapses on a,b for some points $a,b\in M$. Now, since card$\{x,y,z\}=3$, the point a must be at most one of these points, so assume $x\neq a\neq y$, for example. Then, since $g_n|_{M-\{a\}}\twoheadrightarrow b$ and $x,y\in M-\{a\}$, in particular (considering the compacts $\{x\}$ and $\{y\}$ and using the definition of convergence action) we must have $g_nx_n\to b$ and $g_ny_n\to b$, so by uniqueness of limits we have $x'=b=y'$ and card$\{x',y',z'\}\leq 2$, as we wanted. □

To estabilish the second half of Theorem 2.77, we follow [15] and subdivide the proof into some lemmas. Suppose from now on that the action $G\curvearrowright M$ is properly discontinuous on triples. We shall use the notation $x_n\to\{a,b\}$ for a sequence $(x_n)_n$ in M and $a,b\in M$ if for every open neighborhoods U,V of a and b, respectively, there is n_0 such that $x_n\in U\cup V$ for any $n\geq n_0$.

Lemma 2.80. *Suppose $x_n\to x$, $y_n\to y$, $z_n\to z$ with* card$\{x,y,z\}=3$ *and that $g_nx_n\to a$, $g_ny_n\to a$ and $g_nz_n\to b\neq a$. Then for any sequence $w_n\to w\neq z$ we have $g_nw_n\to\{a,b\}$.*

Demonstração. Since $x\neq y$, w must be different from at least one of them, so suppose $w\neq y$. The sequence $(g_nw_n)_n$ is inside the compact M, so it may be supposed to converge to a point c without loss of generality. We then have the three sequences $(y_n)_n$, $(z_n)_n$ and $(w_n)_n$ converging to three distinct points y,z and w, with $g_ny_n\to a$, $g_nz_n\to b\neq a$ and $g_nw_n\to c$. If $g_nw_n\not\twoheadrightarrow\{a,b\}$ we would have infinite elements g_nw_n outside of some union $U\cup V$ of neighborhoods of a and b, so in particular $c\notin\{a,b\}$. Therefore, we would have card$\{a,b,c\}=3$ and the action would not be properly discontinuous on triples by Proposition 2.78, a contradiction. This shows the lemma. □

Lemma 2.81. *Suppose $x_n\to x$, $y_n\to y$, $z_n\to z$ and $w_n\to w$ with* card$\{x,y,z,w\}=4$ *and that $g_nx_n\to a$, $g_ny_n\to a$, $g_nz_n\to b$ and $g_nw_n\to b$. Then $b=a$.*

Demonstração. Let $c\in M-\{a,b\}$ and define the sequence $(u_n)_n$ by putting $u_n=g_n^{-1}c$. By compactness we know we can assume u_n to converge to a point $u\in M$. Now, since card$\{x,y,z,w\}=4$, u is different from at least three of them, so suppose $u\notin\{x,y,z\}$ for example (the other 3 cases are equally similar). If we had $b\neq a$, then by applying Lemma 2.80 to the sequences $(x_n)_n$, $(y_n)_n$, $(z_n)_n$ and $(u_n)_n$ we would have $g_nu_n\to\{a,b\}$, so $c=g_nu_n$ would be either a or b, a contradiction. Then $b=a$. □

Lemma 2.82. *Suppose $x,y,z\in M$ with* card$\{x,y,z\}=3$ *and $z_n\to z$. Suppose $g_nx\to a$, $g_ny\to a$ and $g_nz_n\to b\neq a$. Then $g_n|_{M-\{z\}}\twoheadrightarrow a$.*

Demonstração. First let us show that the maps $g_n|_{M-\{z\}}$ converge pointwise to a. Let $w\in M-\{z\}$ and let us show $g_nw\to a$. If $w=x$ or $w=y$ we are done by hypothesis, so we assume $w\notin\{x,y\}$. By applying Lemma 2.80 to the sequences $x_n=x$, $y_n=y$, the sequence $(z_n)_n$ and $w_n=w$, we get $g_nw=g_nw_n\to\{a,b\}$. If $g_nw\not\twoheadrightarrow a$, then we can easily see that a subsequence $(g_{n_k}w_{n_k})_k$ would converge to b, so by applying Lemma 2.81 to the sequences $(x_{n_k})_k$, $(y_{n_k})_k$, $(z_{n_k})_k$ and $(w_{n_k})_k$ we would get $b=a$, a contradiction. Thus $g_nw\to a$, and the convergence is pointwise.

Now, let us show $g_n|_{M-\{z\}} \twoheadrightarrow a$ by showing that the condition of Proposition 2.73 is satisfied. So, let $(g_{n_k})_k$ be a subsequence and $(w_{n_k})_k \subset M$ with $w_{n_k} \to w \neq z$ and $g_{n_k}w_{n_k} \to w' \in M$, and let us show $w' = a$. By applying Lemma 2.80 to the sequences $x_{n_k} = x$, $y_{n_k} = y$ and the sequences $(z_{n_k})_k$ and $(w_{n_k})_k$, we get $g_{n_k}w_{n_k} \to \{a, b\}$. Suppose by contradiction $g_{n_k}w_{n_k} \not\twoheadrightarrow a$. Then we would have a subsequence (also denoted by $g_{n_k}w_{n_k}$) converging to b. In the case $w \notin \{x, y\}$ we apply Lemma 2.81 to the sequences $x_{n_k} = x$, $y_{n_k} = y$, $(z_{n_k})_k$ and w_{n_k} and get $b = a$, a contradiction, so assume $w \in \{x, y\}$, say $w = x$. In this case we cannot apply the same lemma for the same sequences, so let $x' \notin \{y, z, w\}$ be a new element and put $x'_{n_k} = x'$. By the pointwise convergence we have $g_{n_k}x'_{n_k} \to a$, so now we have all hypotheses and can apply Lemma 2.81 to $(x'_{n_k})_k$, $(y_{n_k})_k$, $(z_{n_k})_k$ and $(w_{n_k})_k$ to get $b = a$, a contradiction. Therefore, $g_{n_k}w_{n_k} \to a$ and $w' = a$ by uniqueness of limits. This completes the proof. □

We are now ready to complete the proof of Theorem 2.77:

Lemma 2.83. *If $G \curvearrowright M$ is properly discontinuous on triples, then it is a convergence action.*

Demonstração. Let $(g_n)_n$ be a distinct sequence in G and let us find a collapsing subsequence. Choose three distinct points $x, y, z \in M$. Define the constant sequences $x_n = x$, $y_n = y$ and $z_n = z$. Since the sequences $(g_nx_n)_n, (g_ny_n)_n$ and $(g_nz_n)_n$ are inside the compact M there must be common converging subsequences $(g_{n_k}x_{n_k})_k \to a$, $(g_{n_k}y_{n_k})_k \to c$ and $(g_{n_k}z_{n_k})_k \to b$ for some $a, c, b \in M$. Since the action is properly discontinuous we have $\text{card}\{a, c, b\} \leq 2$ by Proposition 2.78, so suppose $c = a$ for example. If $b \neq a$ then by applying Lemma 2.82 to x, y and $(z_{n_k})_k$ we immediately get $g_{n_k}|_{M-\{z\}} \twoheadrightarrow a$ and we found a collapsing subsequence, so assume $b = a$. in this case, let $c \in M - \{a\}$ and define $w_{n_k} = g_{n_k}^{-1}c$ for any k. By compactness, we can assume $w_{n_k} \to w \in M$. Since $\text{card}\{x, y, z\} = 3$, w is at most one of them, and since all three g_{n_k}-sequences converge to the same point a, we can assume $w \notin \{x, y\}$. Since $(g_{n_k}w_{n_k})_k = c$ obviously converges to $c \neq a$, we apply Lemma 2.82 for x, y and $(w_{n_k})_k$ get $g_{n_k}|_{M-\{w\}} \twoheadrightarrow a$ and we found again a collapsing subsequence, as desired. This completes the proof of the lemma and therefore of Theorem 2.77. □

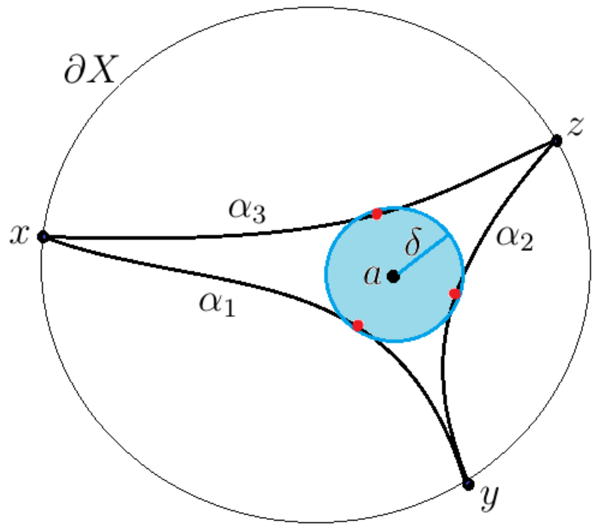
The purpose (and the main example) of convergence actions in this thesis is the following: let G be an infinite and finitely generated group and X be a proper geodesic hyperbolic space. Suppose G acts properly discontinuously on X by isometries. By Proposition 2.15, G acts by homeomorphisms on ∂X by $g \cdot c(\infty) = gc(\infty)$, which is a compact metric space by Theorem 2.13 and Proposition 2.14. We claim the following:

Proposition 2.84. *The action $G \curvearrowright \partial X$ is a convergence action.*

For the proof of this proposition, we will need the following Lemma, which has a straightforward proof. Remember a map between topological spaces is said to be *proper* if the inverse image of any compact subset is compact.

Lemma 2.85. *Let G be a group acting by homeomorphisms on two topological spaces X and Y and let $f : X \to Y$ be a proper, continuous, surjective and G-equivariant map (that is, $g \cdot f(x) = f(g \cdot x)$ for any $x \in X$, $g \in G$). Then $G \curvearrowright X$ is properly discontinuous if and only if $G \curvearrowright Y$ is properly discontinuous.* □

Let us give a sketch for a proof of Proposition 2.84. Let X be δ-hyperbolic and define $Y \subset X \times \Theta(\partial X)$ as the subset consisting of the pairs $(a, \{x, y, z\}) \in X \times \Theta(\partial X)$ such that there are geodesic lines α_1, α_2 and α_3 connecting x to y, y to z and z to x, respectively, such that $d(\alpha_i, a) \leq \delta$ for $i = 1, 2, 3$ (see next figure).



There is an obvious action by homeomorphisms $G \curvearrowright X \times \Theta(\partial X)$ that is given by $g \cdot (a, \{x, y, z\}) = (ga, \{gx, gy, gz\})$. The subset Y is G-invariant, for if $(a, \{x, y, z\}) \in Y$ and α_1, α_2 and α_3 are the geodesic lines satisfying definition above, then the geodesic lines $g\alpha_1$, $g\alpha_2$ and $g\alpha_3$ connect gx to gy, gy to gz and gz to gx, respectively, and are such that $d(g\alpha_i, ga) = d(\alpha_i, a) \leq \delta$ for $i = 1, 2, 3$, as desired. Therefore we have an action by homeomorphisms $G \curvearrowright Y$. Let $\pi_1 : Y \to X$ and $\pi_2 : Y \to \Theta(\partial X)$ be the natural projections, obviously continuous and G-equivariant maps (by construction).

Lemma 2.86. *π_1 is proper.*

Demonstração. Let $K \subset X$ be a compact and let us show $\pi_1^{-1}(K)$ is compact, by showing it is sequentially compact. Let $(a_n, \{x_n, y_n, z_n\})_n$ be a sequence in $\pi_1^{-1}(K)$ (in particular, in Y). Since K is compact, we can assume without loss of generality that $a_n \to a$ for some $a \in K$. For any n there are geodesic lines α_n, β_n and γ_n connecting x_n to y_n, y_n to z_n and z_n to x_n, respectively, such that $d(\alpha_n, a_n), d(\beta_n, a_n), d(\gamma_n, a_n) \leq \delta$. Since $a_n \to a$, by a similar construction of Proposition 2.9 we can find subsequences $(\alpha_{n_k})_k$, $(\beta_{n_k})_k$ and $(\gamma_{n_k})_k$ converging (uniformly on compact sets) to geodesic lines α, β and γ. Of course the endpoints of α_{n_k}, β_{n_k} and γ_{n_k} converge in ∂X to the respective endpoints of α, β and γ, so α, β and γ form a geodesic "triangle" in $X \cup \partial X$ with "vertices" $x, y, z \in \partial X$. By construction, $(a_{n_k}, \{x_{n_k}, y_{n_k}, z_{n_k}\}) \to (a, \{x, y, z\})$, so if we show $(a, \{x, y, z\}) \in Y$ we are done. We're just left to show $d(\alpha, a), d(\beta, a), d(\gamma, a) \leq \delta$, so let us show $d(\alpha, a) \leq \delta$ for example. For any $k \geq 1$, there is by hypothesis $p_{n_k} \in \alpha_{n_k}$ such that $d(p_{n_k}, a_{n_k}) \leq \delta$, so it is easy to see $(p_{n_k})_k$ is bounded and therefore contains a converging subsequence (still denoted by the same subindexes) $p_{n_k} \to p$. By the uniform convergence we have $p \in \alpha$. Now, for any $\epsilon > 0$, $d(p, a) \leq d(p, p_{n_k}) + d(p_{n_k}, a_{n_k}) + d(a_{n_k}, a) \leq \epsilon + \delta + \epsilon$ for some large enough n_k. This implies $d(\alpha, a) \leq d(p, a) \leq \delta$, as desired. □

We can finally complete the proof of Proposition 2.84. The map π_1 can similarly be shown to be surjective, and π_2 can also be shown to be surjective and proper. Now we just have to apply Lemma 2.85. Since $G \curvearrowright X$ is properly discontinuous, by Lemma 2.85 we have $G \curvearrowright Y$ properly discontinuous. Again, by Lemma 2.85, $G \curvearrowright \Theta(\partial X)$ is properly discontinuous, so by 2.77 $G \curvearrowright \partial X$ is a convergence action, as desired.

Capítulo 3

Σ^1 invariant and property R_∞

Geometric invariants detect essential properties of groups in the same sense as algebraic invariants detect properties of topological spaces in Algebraic Topology. Keeping this comparison in mind, they behave with functorial properties as well. This means that isomorphic groups have the same geometric invariants, in some sense we will define later. Also, homomorphisms between groups will induce morphisms between the invariants, as one could expect. One of these is the first BNS-invariant Σ^1, for finitely generated groups. In this chapter we will define it, show some of its properties and explain how it can be used to detect property R_∞.

3.1 The character sphere

We will denote the character sphere of a finitely generated group G by $S(G)$. It is our "work place" for the chapter, that is, the ambient space where we will define the geometric invariant Σ^1. In this section we will define this sphere and show that it is really (homeomorphic to) a finite dimensional euclidean sphere.

Let G be a finitely generated group. Denote by G' the commutator subgroup $G' = \langle\{ghg^{-1}h^{-1} \mid g,h \in G\}\rangle$ and by G^{ab} the abelianized group of G, that is, $G^{ab} = G/G'$. Note that this is well defined since G' is a characteristic subgroup of G, in particular a normal subgroup. By basic facts of Algebra, G^{ab} is a finitely generated abelian group and therefore by the Structure Theorem for finitely generated abelian groups (see [56], Chapter 2) we have an isomorphism

$$G^{ab} \simeq \mathbb{Z} \oplus ... \oplus \mathbb{Z} \oplus \mathbb{Z}_{m_1} \oplus ... \oplus \mathbb{Z}_{m_k}$$

for some $1 \leq k$ and $1 \leq m_j$ for all $1 \leq j \leq k$, where the $\mathbb{Z}$-factors (and also the $\mathbb{Z}_{m_j}$-factors) may not appear in the isomorphism. Let n be the number of $\mathbb{Z}$-factors above, which we call the free rank of G^{ab}. Suppose $n \geq 1$. We will denote the projection on the quocient $\pi : G \to G/G' = G^{ab}$ by $g \mapsto \overline{g}$. Let $x_1, ..., x_n, y_1, ..., y_k$ be elements of G representing this isomorphism:

$$G^{ab} = \langle\overline{x_1}\rangle \oplus ... \oplus \langle\overline{x_n}\rangle \oplus \langle\overline{y_1}\rangle \oplus ... \oplus \langle\overline{y_k}\rangle,$$

that is, $\langle\overline{x_j}\rangle \simeq \mathbb{Z}$ for $1 \leq j \leq n$ and $\langle\overline{y_j}\rangle \simeq \mathbb{Z}_{m_j}$ for $1 \leq j \leq k$. Let

$$Hom(G, \mathbb{R}) = \{\chi : G \to \mathbb{R} \mid \chi \text{ is a homomorphism}\},$$

where $\mathbb{R}$ is the additive group of real numbers. $Hom(G,\mathbb{R})$ is a real vector space with the natural operations $(\chi+\chi')(g) = \chi(g)+\chi'(g)$ and $(r\chi)(g) = r\chi(g)$. Next we show that is a n-dimensional real vector space.

Lemma 3.1. *We have a $\mathbb{R}$-linear isomorphism $Hom(G,\mathbb{R}) \simeq \mathbb{R}^n$.*

Demonstração. Let

$$\begin{aligned} T : Hom(G,\mathbb{R}) &\longrightarrow \mathbb{R}^n \\ \chi &\longmapsto (\chi(x_1),...,\chi(x_n)). \end{aligned}$$

It is easy to see that it is a linear operator. For surjectivity, let $(a_1,...,a_n) \in \mathbb{R}^n$. For each $1 \leq j \leq n$, consider the homomorphisms $\varphi_j : \langle \overline{x_j} \rangle \to \mathbb{R}$ with $\varphi_j(\overline{x_j}^k) = ka_j$. Consider also the zero homomorphisms $z_j : \langle \overline{y_j} \rangle \to \mathbb{R}$ for all $1 \leq j \leq k$. Then, by the Universal Property of the direct sum (see [56]) there is a homomorphism

$$\begin{aligned} \varphi : G^{ab} &\longrightarrow \mathbb{R} \\ (\overline{x_1}^{r_1},...,\overline{x_n}^{r_n},\overline{y_1}^{s_1},...,\overline{y_k}^{s_k}) &\longmapsto r_1\varphi_1(\overline{x_1}) + ... + r_n\varphi_n(\overline{x_n}) + s_1z_1(\overline{y_1}) + ... + s_kz_k(\overline{y_k}) \\ &= r_1a_1 + ... + r_na_n, \end{aligned}$$

so $\varphi(\overline{x_j}) = a_j$ for all j. Now, $\chi = \varphi \circ \pi$ is a homomorphism in $Hom(G,\mathbb{R})$ such that

$$T(\chi) = (\chi(x_1),...,\chi(x_n)) = (\varphi(\overline{x_1}),...,\varphi(\overline{x_n})) = (a_1,...,a_n),$$

as desired. For injectivity, suppose $T(\chi) = 0$, that is, $\chi(x_j) = 0$ for all $1 \leq j \leq n$. Since $\chi(ghg^{-1}h^{-1}) = \chi(g) + \chi(h) - \chi(g) - \chi(h) = 0$ for all generators $ghg^{-1}h^{-1}$ of G', we have $\chi|_{G'} = 0$ and then there is a homomorphism $\overline{\chi} : G^{ab} \to \mathbb{R}$ with $\overline{\chi} \circ \pi = \chi$. For all $1 \leq j \leq n$, $\overline{\chi}(\overline{x_j}) = \chi(x_j) = 0$. Since $\langle \overline{y_j} \rangle \simeq \mathbb{Z}_{m_j}$ we have $\overline{y_j}^{m_j} = 1$ (identity element) and then

$$m_j\overline{\chi}(\overline{y_j}) = \overline{\chi}(\overline{y_j}^{m_j}) = \overline{\chi}(1) = 0,$$

from where we get $\overline{\chi}(\overline{y_j}) = 0$ for all j, since $m_j \geq 1$. The homomorphism $\overline{\chi}$ vanishes in all the generators $\overline{x_j}$ and $\overline{y_j}$ of G^{ab}, then $\overline{\chi} = 0$ and we get $\chi = \overline{\chi} \circ \pi = 0 \circ \pi = 0$, which concludes the proof. □

If we make $Hom(G,\mathbb{R})$ inherit the norm from $\mathbb{R}^n$ putting $\|\chi\| = \|T(\chi)\| = \sqrt{\chi(x_1)^2 + ... + \chi(x_n)^2}$, then because $\|\chi\| = \|T(\chi)\|$ it turns out that the linear isomorphism T become a homeomorphism with the norm-induced topologies. Now we intend to show that $S(G)$ is homeomorphic to the euclidean sphere S^{n-1}. We will use the following standard lemma which can be found in [76], Theorem 22.2 at page 142:

Lemma 3.2 (Quotient map Lemma)**.** *Let $X, \overline{X}$ be topological spaces with a quocient map $p : X \to \overline{X}$. Let Z be another topological space and $f : X \to Z$ a continuous map. If f is fiber-*

preserving on p *(that is,* $p(x) = p(x') \Rightarrow f(x) = f(x')$*) then there is a unique continuous map* $\overline{f} : \overline{X} \to Z$ *such that* $\overline{f} \circ p = f$*.*

Lemma 3.3. *Let* X, Y *be topological spaces with homeomorphism* $T : X \to Y$*. If a group* G *acts in* X*, then there is an action of* G *in* Y *such that the quocient spaces* $X/G \simeq Y/G$ *are homeomorphic.*

Demonstração. This proof is straightforward. Define the action putting

$$g \cdot y = T(g \cdot T^{-1}(y)) \in Y$$

for $g \in G$ and $y \in Y$. It is an action because

$$1 \cdot y = T(1 \cdot T^{-1}(y)) = T(T^{-1}(y)) = y,$$

and

$$\begin{aligned} g \cdot (g' \cdot y) &= g \cdot (T(g' \cdot T^{-1}(y))) \\ &= T(g \cdot T^{-1}(T(g' \cdot T^{-1}(y)))) \\ &= T(g \cdot (g \cdot T^{-1}(y))) \\ &= T((gg') \cdot T^{-1}(y)) = (gg') \cdot y. \end{aligned}$$

Denote by $p_X : X \to X/G$ and $p_Y : Y \to Y/G$ the respective projections such that $p_X(x) = p_X(x') \Leftrightarrow g \cdot x = x'$ and $p_Y(y) = p_Y(y') \Leftrightarrow g \cdot y = y'$ for some $g \in G$. Note that $p_Y \circ T$ is fiber-preserving on p_X, because

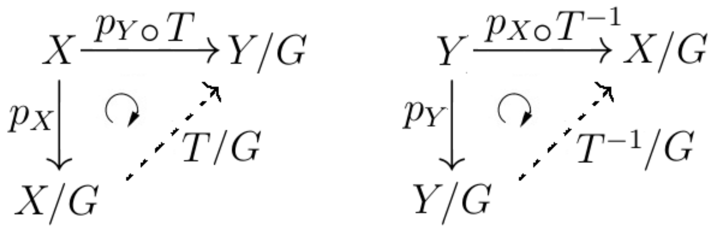
$$\begin{aligned} p_X(x) = p_X(x') &\Rightarrow g \cdot x = x' \\ &\Rightarrow g \cdot T(x) = T(g \cdot T^{-1}(T(x))) = T(g \cdot x) = T(x') \\ &\Rightarrow p_Y \circ T(x) = p_Y(T(x)) = p_Y(T(x')) = p_Y \circ T(x'), \end{aligned}$$

then by the Quotient map Lemma there exists a continuous map $T/G : X/G \to Y/G$ such that $T/G \circ p_X = p_Y \circ T$. Similarly, $p_X \circ T^{-1}$ is fiber-preserving on p_Y, because

$$\begin{aligned} p_Y(y) = p_Y(y') &\Rightarrow g \cdot y = y' \\ &\Rightarrow T(g \cdot T^{-1}(y)) = y' \\ &\Rightarrow g \cdot T^{-1}(y) = T^{-1}T(g \cdot T^{-1}(y)) = T^{-1}(y') \\ &\Rightarrow p_X \circ T^{-1}(y) = p_X(T^{-1}(y)) = p_X(T^{-1}(y')) = p_X \circ T^{-1}(y'), \end{aligned}$$

then there also exists a continuous map $T^{-1}/G : Y/G \to X/G$ such that $T^{-1}/G \circ p_Y = p_X \circ T^{-1}$.

To finish, note that these maps are each other inverses. Indeed, for any $z \in Y/G$ write



$z = p_Y(y)$. Then

$$T/G \circ T^{-1}/G(z) = (T/G)(T^{-1}/G)p_Y(y) = (T/G)p_X T^{-1}(y) = p_Y T T^{-1}(y) = p_Y(y) = z.$$

Similarly, given $w \in X/G$ write $w = p_X(x)$. Then

$$T^{-1}/G \circ T/G(w) = (T^{-1}/G)T/G p_X(x) = (T^{-1}/G)p_Y T(x) = p_X T^{-1} T(x) = p_X(x) = w,$$

therefore $T/G : X/G \to Y/G$ given by $T/G([x]) = [T(x)]$ is a homeomorphism, as desired. □

Let us denote by $\mathbb{R}_+$ the multiplicative group of the positive real numbers. $\mathbb{R}_+$ acts in the set $Hom(G, \mathbb{R}) - \{0\}$ in the natural way: $r \cdot \chi = r\chi$. There is our character sphere:

Definition 3.4. Given a finitely generated group G, the character sphere of G is the orbit space $S(G) = (Hom(G, \mathbb{R}) - \{0\})/\mathbb{R}_+$ of the natural $\mathbb{R}_+$-action on $Hom(G, \mathbb{R}) - \{0\}$. In other words,

$$S(G) = \{[\chi] \mid \chi \in Hom(G, \mathbb{R}) - \{0\}\}$$

with the relation $[\chi] = [\chi'] \Leftrightarrow r\chi = \chi'$ for some $r > 0$. The $\chi \neq 0$ are called the characters of G.

Definition 3.5. For any subgroup $H \leq G$ of a finitely generated group G we define

$$S(G, H) = \{[\chi] \in S(G) \mid \chi|_H = 0\}$$

and call it the sub sphere relative to H.

Proposition 3.6. *If the free rank of G^{ab} is n with generators $\overline{x_1}, ..., \overline{x_n}$ then $S(G) \simeq S^{n-1}$ with homeomorphism*

$$\begin{aligned} H : S(G) &\longrightarrow S^{n-1} \\ [\chi] &\longmapsto \frac{(\chi(x_1), ..., \chi(x_n))}{\|(\chi(x_1), ..., \chi(x_n))\|}. \end{aligned}$$

Demonstração. The restriction of the homeomorphism $T : Hom(G, \mathbb{R}) \to \mathbb{R}^n$ gives a homeomorphism $T : Hom(G, \mathbb{R}) - \{0\} \to \mathbb{R}^n - \{0\}$. Since $\mathbb{R}_+$ acts on $Hom(G, \mathbb{R}) - \{0\}$, by the previous lemma we obtain that

$$S(G) = (Hom(G, \mathbb{R}) - \{0\})/\mathbb{R}_+ \simeq \mathbb{R}^n - \{0\}/\mathbb{R}_+$$

by the homeomorphism $[\chi] \mapsto [T(\chi)]$, where the action of $\mathbb{R}_+$ on $\mathbb{R}^n - \{0\}$ is given by

$$r \cdot (a_1, ..., a_n) = T(r \cdot T^{-1}(a_1, ..., a_n)) = T(r\chi) = (r\chi(x_1), ..., r\chi(x_n)) = (ra_1, ..., ra_n).$$

But with this action, we know that $\mathbb{R}^n - \{0\}/\mathbb{R}_+ \simeq S^{n-1}$ with homeomorphism $[P] \mapsto \frac{P}{\|P\|}$. Then the composition of both homeomorphisms leads us to the desired one:

$$\begin{aligned} H : S(G) &\longrightarrow (\mathbb{R}^n - \{0\})/\mathbb{R}_+ \longrightarrow S^{n-1} \\ [\chi] &\longmapsto [T(\chi)] = [(\chi(x_1), ..., \chi(x_n))] \longmapsto \frac{(\chi(x_1), ..., \chi(x_n))}{\|(\chi(x_1), ..., \chi(x_n))\|}. \end{aligned}$$

□

3.2 Σ^1 invariant and properties

Although we already have a great survey on the basic theory of Sigma invariants in [92], we intend to develop it from the ground up, in order to produce a self-contained chapter. Here we focus on the first BNS invariant Σ^1. There are higher invariants Σ^n for each $n \geq 1$ (see [7]).

For the next definition the reader must remember the definitions of the Cayley graph and of the subgraph induced by a subset of vertices (definitions 1.21 and 1.34, respectively) in Chapter 1.

Definition 3.7. Let G be a finitely generated group, choose a finite generating subset S and consider the Cayley graph $\Gamma = \Gamma(G, S)$. Given $[\chi] \in S(G)$, let

$$G_\chi = \{g \in G \mid \chi(g) \geq 0\} \subset V(\Gamma)$$

be a collection of vertices inside Γ and let $\Gamma_\chi = \Gamma_{G_\chi}$, that is, the subgraph of Γ induced by the vertices G_χ. The Σ^1-invariant of G (and S) is

$$\Sigma^1(G, S) = \{[\chi] \in S(G) \mid \Gamma_\chi \text{ is connected}\}.$$

Note that if $r > 0$ then $G_\chi = G_{r\chi}$ and therefore $\Gamma_\chi = \Gamma_{r\chi}$. So the previous definition does not depend on the class representative $\chi \in [\chi]$ chosen and therefore $\Sigma^1(G, S)$ is well defined. Later we are going to show that Σ^1 does not depend on the finite generator set S choosen either. Because of this, from now on we will denote $\Sigma^1(G, S)$ only by $\Sigma^1(G)$. In short, a point $[\chi] \in S(G)$ is in $\Sigma^1(G)$ if any two vertices in Γ_χ can be connected by a path inside Γ_χ or, equivalently, if any vertex in Γ_χ can be connected to the identity vertex 1 inside Γ_χ.

Example 3.8 ($\Sigma^1(\mathbb{Z} \oplus \mathbb{Z}) = S(\mathbb{Z} \oplus \mathbb{Z})$)**.** The Cayley graph $\Gamma(\mathbb{Z} \oplus \mathbb{Z}, \{(1, 0), (0, 1)\})$ is the known infinite grid in $\mathbb{R}^2$. Any character χ can be seen as the restriction of a (non-vanishing) linear map $T_\chi : \mathbb{R}^2 \to \mathbb{R}$. $\ker(T_\chi)$ is one-dimensional and partition $\mathbb{R}^2$ into two connected half-planes. It turns out that Γ_χ is exactly the intersection of the infinite grid with the half-plane $\{T_\chi \geq 0\}$, so one can see that it is connected and then $[\chi] \in \Sigma^1(\mathbb{Z} \oplus \mathbb{Z})$.

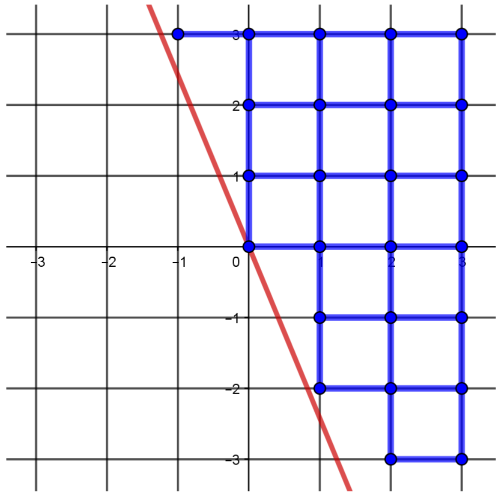


Figura 3.1: the blue grid is a finite portion of the infinite connected subgraph Γ_χ

Example 3.9 (The Baumslag-Solitar groups)**.** This example is dedicated to the Baumslag-Solitar groups. They were first defined in [4] and are a very important and vastly studied class of groups in Geometric Group Theory. Here we are going to define them and give the reader some intuitions on their Cayley graphs and Σ^1 to help understanding the generalizations in Chapter 5. This example is based on [92] and [12]. If $m \neq 0 \neq n$, we define

$$BS(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle.$$

For $m = 1$ and $n \geq 1$, the groups $G = BS(1,n)$ are solvable and we call them the Solvable BS groups. The relation $tat^{-1} = a^n$ implies $ta = a^n t$ and $ta^{-1} = a^{-n}t$, so we can make all the t-letters go right in a word. Similarly, we have $at^{-1} = t^{-1}a^n$ and $a^{-1}t^{-1} = t^{-1}a^{-n}$, so we can also make all the t^{-1}-letters go left in a word. So every element assumes the form $t^{-k}a^r t^s$ for $k, s \geq 0$ and $r \in \mathbb{Z}$. Define $V = \langle t \rangle \leq G$ and $U = \ker(\psi) \lhd G$, where $\psi : G \to \mathbb{Z}$ with $\psi(g) = (g)^t$ is the homomorphism that sends $g \in G$ to the sum $(g)^t$ of all the t-exponents of g. Then writing $t^{-k}a^r t^s = t^{s-k}t^{-s}a^r t^s$ we have that every element is of the form $t^{k'}u$ for some $k' \in \mathbb{Z}$ and $u = t^{-s}a^r t^s \in \ker(\psi) = U$. Then $VU = G$. Since $V \cap U = \{t^k \in V \mid k = \psi(t^k) = 0\} = \{1\}$, by Section 1.3 in [88] we have that $G = U \rtimes V$ is the semidirect product of its subgroups U and V. Now, t is torsion-free and then $V \simeq \mathbb{Z}$. On the other hand, using the Reidemeister-Schreier Theorem 1.50 we obtain that U has a presentation

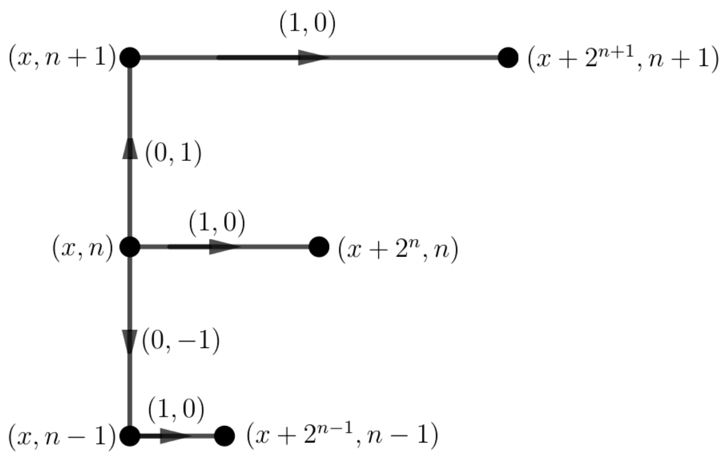
$$U = \langle a_j,\ j \in \mathbb{Z} \mid a_j^n = a_{j+1},\ j \in \mathbb{Z} \rangle,$$

then U is isomorphic to the group of n-adic fractions $\mathbb{Z}[\frac{1}{n}] = \{\frac{s}{n^r} \in \mathbb{Q} \mid s \in \mathbb{Z},\ r \geq 0\}$ under isomorphism $a_j \mapsto n^j$ with inverse $\frac{s}{n^r} \longmapsto a^s_{-r}$. Then $BS(1,n) \simeq \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$ (see [12]).

Now let us visualize the Cayley graph. Consider $G = BS(1,2)$ and generators $S = \{a, t\}$. We have $BS(1,2) \simeq \mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z}$. The generator a corresponds to $(1,0)$ and t corresponds to $(0,1)$. The $\mathbb{Z}$-action is given by $n \cdot x = 2^n x,\ n \in \mathbb{Z},\ x \in \mathbb{Z}[\frac{1}{2}]$. Then the operation is

$$(x,n)(x',m) = (x + 2^n x', n+m).$$

The vertices are $\mathbb{Z}[\frac{1}{2}] \times \mathbb{Z} \subset \mathbb{R}^2$. Since $(x,n)(0,\pm 1) = (x, n\pm 1)$ and $(x,n)(\pm 1, 0) = (x \pm 2^n, n)$, there are four possible directions to walk, and the size of the horizontal movies is 2^n, depending on the vertex height n.



Since in G^{ab} we have $a^2 = tat^{-1} = tt^{-1}a = a$ and so $a = 1$, we have $G^{ab} = \langle t \rangle \simeq \mathbb{Z}$ and then by Proposition 3.6 there is a homeomorphism $S(G) \simeq S^0$ with $[\chi] \mapsto \frac{\chi(t)}{\|\chi(t)\|}$. So $S(G) = \{[\chi], [-\chi]\}$, where $\chi(a) = 0$ and $\chi(t) = 1$. Looking to the isomorphism, this means that $\chi(x,n) = n$. We assert that $\Sigma^1(G) = \{[-\chi]\}$. To see that Γ_χ is not connected observe that the vertices of Γ_χ are the (x,n) such that $n \geq 0$. Then, as we told before, the size of all the horizontal movies inside Γ_χ is at least $2^0 = 1$. So it is impossible, for example, to connect the vertices $(0,0)$ and $(\frac{1}{2}, 0)$ of Γ_χ inside Γ_χ, because the horizontal distance between them is $\frac{1}{2}$. Therefore $[\chi] \notin \Sigma^1(G)$. On the other hand, $[-\chi] \in \Sigma^1(G)$. Indeed, given an arbitrary vertex $(x,n) = (\frac{s}{2^r}, n)$ in $\Gamma_{-\chi}$ (that is, $n \leq 0$) let us connect it to the vertex $(0,0)$. First, connect (x,n) to $(x,-r)$ by going vertically (note that we didn't leave $\Gamma_{-\chi}$ because both n and $-r$ are non-positive). Now, the horizontal moves at height $-r$ have size 2^{-r}. Since $x = \frac{s}{2^r}$ is a multiple of 2^{-r}, by going horizontally (again, not leaving $\Gamma_{-\chi}$) we can connect $(x,-r)$ to $(0,-r)$, after $|s|$ moves. Finally, it is easy to connect $(0,-r)$ to $(0,0)$ by going up vertically. So $\Gamma_{-\chi}$ is connected and $\Sigma^1(G) = \{[-\chi]\}$, as desired.

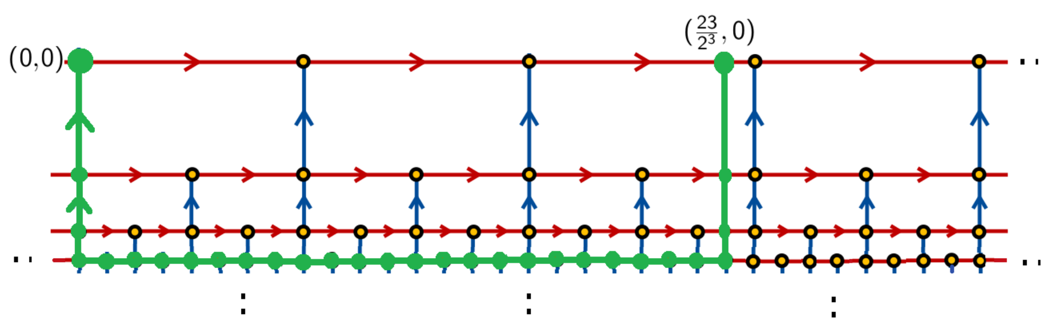


Figura 3.2: connecting $(\frac{23}{8}, 0)$ to $(0,0)$

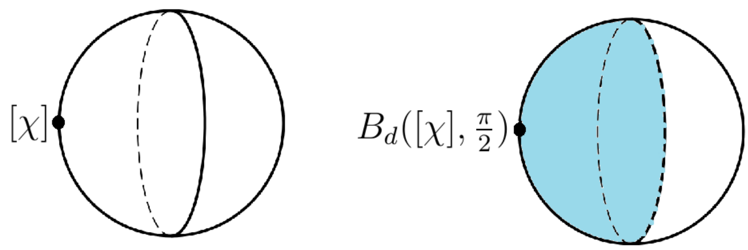
Before we explore the properties of Σ^1, let us define another geometric invariant called Ω^1. The Ω^n invariants ($n \geq 1$) were first defined by N. Koban in [63] (2006) and are analogous of the Σ^n ones. The definitions of Ω^n are nontrivial. However, when $n = 1$, a great characterization of

Ω^1 in terms of Σ^1 was given in [63] and allows us to define it in a much easier way:

Definition 3.10 ([63], Theorem 3.1, pg 1977)**.** Let G be a finitely generated group and identify $S(G)$ with an euclidean sphere (see Proposition 3.6). The first Omega invariant of G is defined as

$$\Omega^1(G) = \{[\chi] \in S(G) \mid B_d([\chi], \pi/2) \subset \Sigma^1(G)\},$$

where $B_d([\chi], \pi/2)$ is the open ball centered in $[\chi]$ with ray $\frac{\pi}{2}$, where d is the natural geodesic distance on euclidean spheres.



It was shown in [63] that $\Omega^1(G)$ is closed in $S(G)$. Also, we have the following fact shown in [64]:

Definition 3.11. Let $A \subset S^n$ and $B \subset S^m$. The spherical join of A and B in S^{n+m+1} is

$$A \circledast B = \left\{ \frac{((1-t)a, tb)}{\|((1-t)a, tb)\|} \mid a \in A, b \in B, t \in [0,1] \right\} \subset S^{n+m+1}.$$

Theorem 3.12 ([64], main result)**.** *If G, H are finitely generated groups, then $\Omega^1(G \times H) = \Omega^1(G) \circledast \Omega^1(H)$ is the spherical join of $\Omega^1(G)$ and $\Omega^1(H)$ in $S(G \times H)$.*

Since the Ω^1-invariant can be determined by the knowledge of Σ^1, we will easily compute them for some groups as corollaries in some later cases in the thesis, in case somebody needs them at some point.

Some fundamental properties

Now we will show three fundamental properties of the invariant Σ^1: its independence from the generating set, its invariance under automorphism and a geometric criterion. Let us introduce some notation that is going to be used in the rest of the work. The notation is based on [92].

Remember that the notation for a path in a Cayley graph $\Gamma = \Gamma(G, S)$ is $p = (g, s_1 ... s_n) \in P(\Gamma)$. Given a G-character χ, the path valuation function is

$$\begin{aligned} \nu_\chi : P(\Gamma) &\longrightarrow \mathbb{R} \\ p = (g, s_1 ... s_n) &\longmapsto \min\{\chi(g), \chi(gs_1), \chi(gs_1 s_2), ..., \chi(gs_1 s_2 ... s_n)\}. \end{aligned}$$

From the definition of Γ_χ we get that a path $p \in P(\Gamma)$ runs inside Γ_χ if and only if all its vertices lie in Γ_χ, or equivalently, $\nu_\chi(p) \geq 0$. There are three basic "path operations":

- The group G acts on $P(\Gamma)$ by putting $g' \cdot (g, s_1...s_n) = (g'g, s_1...s_n)$. This corresponds to translating the entire path by g' in Γ. Using the definition of ν_χ and basic minimum properties we obtain $\nu_\chi(g' \cdot p) = \chi(g') + \nu_\chi(p)$;
- If $p = (g, s_1...s_n)$ we define the inverse path of p by $p^{-1} = (gs_1...s_n, {s_n}^{-1}...{s_1}^{-1})$, that walks in the exactly opposite direction. Of course we have $\nu_\chi(p) = \nu_\chi(p^{-1})$;
- If $p = (g, s_1...s_n)$ and $p' = (g', r_1...r_m)$ are such that $gs_1...s_n = g'$, the concatenation of p and p' is defined by $pp' = (g, s_1...s_n r_1...r_m)$. As one could expect, we have $\nu_\chi(pp') = \min\{\nu_\chi(p), \nu_\chi(p')\}$.

Independence from generating set

The first fundamental property of Σ^1 is

Proposition 3.13 (Independence property)**.** *Let G be a finitely generated group and S_1, S_2 two finite generating sets for G. Then $\Sigma^1(G, S_1) = \Sigma^1(G, S_2)$.*

Demonstração. It suffices to show that, given a finite generating set S and $z \in G$ we have $\Sigma^1(G, S) = \Sigma^1(G, S \cup \{z\})$, that is, we can add one new element in S without changing Σ^1. Indeed, if we had shown this, then by adding in S_1, one by one, all the elements of S_2 we obtain $\Sigma^1(G, S_1) = \Sigma^1(G, S_1 \cup S_2)$. Similarly, $\Sigma^1(G, S_2) = \Sigma^1(G, S_2 \cup S_1)$ and then $\Sigma^1(G, S_1) = \Sigma^1(G, S_1 \cup S_2) = \Sigma^1(G, S_2)$, as desired.

Let S be a finite generating set of G and let $z \in G$. Let $S' = S \cup \{z\}$, $\Gamma = \Gamma(G, S)$ and $\Gamma' = \Gamma(G, S')$. We have to show that $\Sigma^1(G, S) = \Sigma^1(G, S')$. Let $[\chi] \in S(G)$. We must show that Γ_χ is connected if and only if Γ'_χ is connected. Now, Γ is a subgraph of Γ'. The two subgraphs Γ_χ and Γ'_χ of Γ' have exactly the same set G_χ of vertices, but Γ'_χ may have more edges, because of the new generator z. So if Γ_χ is connected, Γ'_χ is connected as well. The difficult is the converse: suppose now Γ'_χ is connected. Let $g, h \in G$ two vertices in Γ_χ and let us connect them inside Γ_χ. Since $z \in G = \langle S \rangle$, write $z = s_1...s_n$ with $s_i \in S^{\pm 1}$. Since $\chi \neq 0$, choose $t \in S^{\pm 1}$ such that $\chi(t) > 0$. Then there is $k \geq 0$ such that

$$\chi(t^k) \geq -\min\{0, \nu_\chi(1, s_1...s_n), \nu_\chi(1, {s_n}^{-1}...{s_1}^{-1})\}.$$

Let $g' = t^{-k}gt^k$ and $h' = t^{-k}ht^k$. Both are vertices in Γ'_χ, because $\chi(g') = \chi(g) \geq 0$ and $\chi(h') = \chi(h) \geq 0$. Since Γ'_χ is connected by hypothesis, take a path $p' = (g', w')$ from g' to h' inside Γ'_χ. p' is not a path in Γ because it involves the letter z. We solve this problem in the following way: w' is a word in $S'^{\pm 1}$, so write $w' = w_1 z^{e_1}...w_m z^{e_m}$, w_i words in $S^{\pm 1}$ and $e_i = \pm 1$. Replacing z^{e_i} by $(s_1...s_n)^{e_i}$ we define $w = w_1(s_1...s_n)^{e_1}...w_m(s_1...s_n)^{e_m}$.

Now $p = (g', w)$ is a path in Γ that is not necessarily inside Γ_χ, but its translation $t^k \cdot p$ is. Indeed,

$$\begin{aligned}
\nu_\chi(p) &= \nu_\chi(g', w_1(s_1...s_n)^{e_1}...w_m(s_1...s_n)^{e_m}) \\
&= \nu_\chi((g', w_1)(g'w_1, (s_1...s_n)^{e_1})(g'w_1z^{e_1}, w_2)...(g'w_1z^{e_1}...w_m, (s_1...s_n)^{e_m})) \\
&= \min\{\nu_\chi(g', w_1), \nu_\chi(g'w_1, (s_1...s_n)^{e_1}), \nu_\chi(g'w_1z^{e_1}, w_2), ..., \nu_\chi(g'w_1z^{e_1}...w_m, (s_1...s_n)^{e_m})\}.
\end{aligned}$$

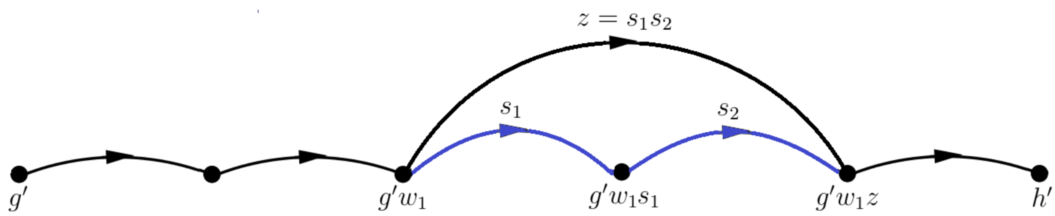


Figura 3.3: exchanging p' by p for $z^{e_1} = z = s_1s_2$

Now, by construction all the paths $(g', w_1), (g'w_1z^{e_1}, w_2)...(g'w_1z^{e_1}...w_{m-1}z^{e_{m-1}}, w_m)$ are subpaths of p' and then have non-negative ν_χ-value. Then

$$\nu_\chi(p) \geq \min\{0, \nu_\chi(g'w_1, (s_1...s_n)^{e_1}), \nu_\chi(g'w_1z^{e_1}w_2, (s_1...s_n)^{e_2}), ..., \nu_\chi(g'w_1z^{e_1}...w_m, (s_1...s_n)^{e_m})\}.$$

But every path $(g'w_1z^{e_1}...z^{e_{i-1}}w_i, (s_1...s_n)^{e_i})$ above starts (again, by construction) with a vertex in p'. Then $\nu_\chi(g'w_1z^{e_1}...z^{e_{i-1}}w_i, (s_1...s_n)^{e_i}) = \chi(g'w_1z^{e_1}...z^{e_{i-1}}w_i) + \nu_\chi(1, (s_1...s_n)^{e_i}) \geq \nu_\chi(1, (s_1...s_n)^{e_i})$. Then

$$\nu_\chi(p) \geq \min\{0, \nu_\chi(1, s_1...s_n), \nu_\chi(1, {s_n}^{-1}...{s_1}^{-1})\}$$

and therefore

$$\nu_\chi(t^k \cdot p) = \chi(t^k) + \nu_\chi(p) \geq \chi(t^k) + \min\{0, \nu_\chi(1, s_1...s_n), \nu_\chi(1, {s_n}^{-1}...{s_1}^{-1})\} \geq 0,$$

as desired. Now, $t^k \cdot p$ is a path in Γ_χ connecting $t^kg' = gt^k$ and $t^kh' = ht^k$. But it is easy to connect g to gt^k and h to ht^k in Γ_χ. The composition of these three paths connects g to h in Γ_χ, which finishes the proof.

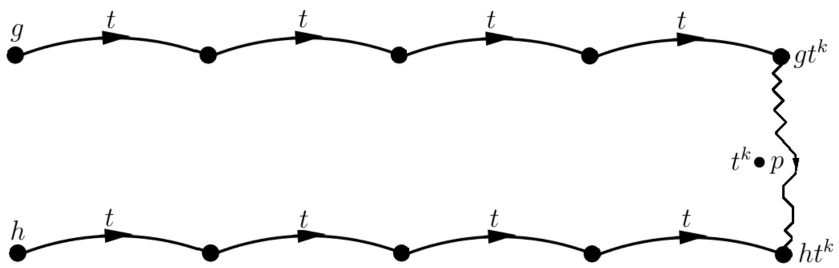


Figura 3.4: connecting g to h

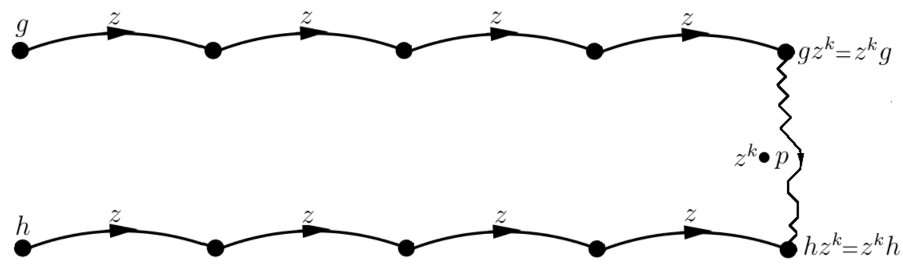
□

The freedom to choose any finite generating set of G for computing $\Sigma^1(G)$ leads us to many interesting consequences. We show two of them. The first one deals with the center $Z(G) = \{z \in G \mid zg = gz \ \forall \ g \in G\}$.

Corollary 3.14. *If $[\chi] \in S(G)$ is such that $\chi|_{Z(G)} \neq 0$, then $[\chi] \in \Sigma^1(G)$.*

Demonstração. Let $z \in Z(G)$ such that $\chi(z) \neq 0$ and suppose $\chi(z) > 0$, exchanging by z^{-1} if necessary. By the independence property, choose a finite generating set S of G containing z (if

$z \notin S$, just take the generating set $S \cup \{z\}$). Let us show that Γ_χ is connected by connecting two arbitrary vertices $g, h \in \Gamma_\chi$. Since Γ is always connected, choose a path p in Γ from g to h. Choose $k \geq 0$ such that $\chi(z^k) \geq -\nu_\chi(p)$. Then the path $z^k \cdot p$ connects $z^k g = gz^k$ to $z^k h = hz^k$ and lies inside Γ_χ, because $\nu_\chi(z^k \cdot p) = \chi(z^k) + \nu_\chi(p) \geq 0$. Now, it is easy to connect g to h inside Γ_χ, in a similar way we did in the previous proof.



□

Corollary 3.15. *If G is a finitely generated abelian group, then $\Sigma^1(G) = S(G)$.*

Demonstração. We have $Z(G) = G$. Then for every $[\chi] \in S(G)$, $\chi|_{Z(G)} = \chi \neq 0$, so by the previous corollary $[\chi] \in \Sigma^1(G)$. □

Invariance under automorphisms

Let us concentrate on the second fundamental property. Let G, H be finitely generated groups with an isomorphism $\varphi : G \to H$. Consider the pullback

$$\varphi^* : Hom(H, \mathbb{R}) \longrightarrow Hom(G, \mathbb{R})$$
$$\chi \longmapsto \chi \circ \varphi,$$

which is a linear map between two finite dimensional vector spaces and therefore continuous (basic fact of Functional Analysis) with inverse $\varphi^{-1}*$, that is, a homeomorphism. Restrict it to $\varphi^* : Hom(H, \mathbb{R}) - \{0\} \longrightarrow Hom(G, \mathbb{R}) - \{0\}$. We have

$$[\chi] = [\chi'] \text{ in } S(H) \Rightarrow \chi' = r\chi,\ r > 0 \Rightarrow \varphi^*(\chi') = \chi' \circ \varphi = r\chi \circ \varphi = r\varphi^*(\chi) \Rightarrow [\varphi^*(\chi)] = [\varphi^*(\chi')],$$

which shows that the quotient map

$$\varphi^* : S(H) \longrightarrow S(G)$$
$$[\chi] \longmapsto [\chi \circ \varphi]$$

is well defined. It is also continuous with continuous inverse φ^{-1^*}, so it is a homeomorphism between the character spheres.

Now, fix a finite generating set S for G, and fix $\varphi(S) \subset H$ as a finite generating set for H.

Lemma 3.16. $\varphi^*(\Sigma^1(H,\varphi(S))) = \Sigma^1(G,S)$.

Demonstração. We will first show the equivalence $[\chi] \in \Sigma^1(H,\varphi(S)) \Leftrightarrow \varphi^*[\chi] = [\chi \circ \varphi] \in \Sigma^1(G,S)$ for any $[\chi] \in S(G)$. By definition,

$$\begin{cases} [\chi] \in \Sigma^1(H,\varphi(S)) \Leftrightarrow \Gamma_\chi = \Gamma(H,\varphi(S))_\chi \text{ is connected,} \\ [\chi \circ \varphi] \in \Sigma^1(G,S) \Leftrightarrow \Gamma_{\chi\circ\varphi} = \Gamma(G,S)_{\chi\circ\varphi} \text{ is connected.} \end{cases}$$

But is easy to see that the isomorphism $\varphi : G \to H$ easily induces the graph isomorphism

$$\begin{aligned} \varphi_* : \Gamma(G,S) &\longrightarrow \Gamma(H,\varphi(S)) \\ g &\longmapsto \varphi(g) \\ (g,s) &\longmapsto (\varphi(g),\varphi(s)) \end{aligned}$$

that maps the vertices and edges of $\Gamma_{\chi\circ\varphi}$ on the vertices and edges of Γ_χ. Indeed,

$$\begin{aligned} \varphi_*(V(\Gamma_{\chi\circ\varphi})) &= \varphi_*(\{g \in G \mid \chi \circ \varphi(g) \geq 0\}) \\ &= \{\varphi(g) \in H \mid \chi \circ \varphi(g) \geq 0\} \\ &= \{h \in H \mid \chi(h) \geq 0\} \\ &= V(\Gamma_\chi) \end{aligned}$$

$$\begin{aligned} \varphi_*(E(\Gamma_{\chi\circ\varphi})) &= \varphi_*(\{(g,s) \mid g, gs \in V(\Gamma_{\chi\circ\varphi})\}) \\ &= \varphi_*(\{(g,s) \mid \chi(\varphi(g)), \chi(\varphi(gs)) \geq 0\}) \\ &= \{(\varphi(g),\varphi(s)) \mid \chi(\varphi(g)), \chi(\varphi(g)\varphi(s)) \geq 0\} \\ &= \{(h,\varphi(s)) \mid h, h\varphi(s) \in V(\Gamma_\chi)\} \\ &= E(\Gamma_\chi). \end{aligned}$$

Then $\Gamma_{\chi\circ\varphi} \simeq \Gamma_\chi$. In particular, $\Gamma_{\chi\circ\varphi}$ is connected if and only if Γ_χ is, which shows the equivalence. Now let us show the equality of the lemma. The $(\subset)$ part follows directly from the $(\Rightarrow)$ part of the equivalence. Finally, let $[\xi] \in \Sigma^1(G,S)$. Since φ^* is bijective write $[\xi] = \varphi^*[\chi]$ for some $[\chi] \in S(H)$. Since $\varphi^*[\chi] = [\xi] \in \Sigma^1(G,S)$, from the $(\Leftarrow)$ part of the equivalence we have $[\chi] \in \Sigma^1(H,\varphi(S))$ and therefore $[\xi] \in \varphi^*(\Sigma^1(H,\varphi(S)))$, as desired. □

Definition 3.17. Let G be finitely generated and $A \subset S(G)$. We say that A is invariant (or invariant by automorphisms) in $S(G)$ if for all automorphism φ of G we have $\varphi^*(A) = A$ (see the induced homeomorphism φ^* above), or, equivalently, if $[\chi \circ \varphi] \in A$ for all $[\chi] \in A$.

Theorem 3.18 (Invariance under automorphism property)**.** *$\Sigma^1(G)$ and $\Sigma^1(G)^c$ are invariant subsets of $S(G)$.*

Demonstração. Let $\varphi \in Aut(G)$. This is the special case of the above lemma for $G = H$. Fix the finite generator sets S and $\varphi(S)$ of G. Since Σ^1 does not depend on the finite generator set choosen, we have

$$\varphi^*(\Sigma^1(G)) = \varphi^*(\Sigma^1(G, \varphi(S))) = \Sigma^1(G, S) = \Sigma^1(G).$$

Since φ^* is bijective we also have

$$\varphi^*(\Sigma^1(G)^c) = \varphi^*(S(G) - \Sigma^1(G)) = \varphi^*(S(G)) - \varphi^*(\Sigma^1(G)) = S(G) - \Sigma^1(G) = \Sigma^1(G)^c.$$

□

From the invariance above, two special characteristic subgroups of G arise:

Corollary 3.19. *If G is a finitely generated group, then the subgroups*

$$N = \bigcap_{[\chi]\in\Sigma^1(G)} \ker(\chi) \quad \textit{and} \quad N' = \bigcap_{[\chi]\in\Sigma^1(G)^c} \ker(\chi)$$

are characteristic subgroups of G. Furthermore, G/N and G/N' are abelian, finitely generated and torsion-free groups (and so isomorphic to $\mathbb{Z}^k$ for some $k \geq 0$, see [56]).

Demonstração. Let us show that N is characteristic, the proof for N' being similar. Let $\varphi \in Aut(G)$ and $g \in N$. We must show that $\varphi(g) \in N$, so take $[\chi] \in \Sigma^1(G)$ and let us show that $\varphi(g) \in \ker(\chi)$. Since $[\chi] \in \Sigma^1(G)$, by the Theorem above we have $[\chi \circ \varphi] = \varphi^*[\chi] \in \Sigma^1(G)$. Now, since $g \in N$, in particular $g \in \ker(\chi \circ \varphi)$. Then $\chi(\varphi(g)) = 0$, that is, $\varphi(g) \in \ker(\chi)$, as desired.

Now we show the other properties for G/N, the proof for G/N' being also similar. G/N is obviously finitely generated because it is the quotient of a finitely generated group. Let $\overline{g}, \overline{h} \in G/N$. Since $\chi(ghg^{-1}h^{-1}) = \chi(g) + \chi(h) - \chi(g) - \chi(h) = 0$ for all $[\chi] \in \Sigma^1(G)$ (in fact, this is true for every $[\chi] \in S(G)$), we have $ghg^{-1}h^{-1} \in N$ and by definition of quotient $\overline{ghg^{-1}h^{-1}} = \overline{1}$, or $\overline{g}\overline{h} = \overline{h}\overline{g}$, that is, G/N is abelian. To finish, let $\overline{g} \neq \overline{1}$ in G/N (or $g \notin N$) such that $\overline{g}^k = \overline{1}$ for some power $k \geq 0$, and let us show that $k = 0$. Since $\overline{g^k} = \overline{g}^k = \overline{1}$ we have $g^k \in N$, or $k\chi(g) = \chi(g^k) = 0$ for every $[\chi] \in \Sigma^1(G)$. Since $g \notin N$ we have $\chi_0(g) \neq 0$ for some $[\chi_0] \in \Sigma^1(G)$. But since $k\chi_0(g) = 0$ we must have $k = 0$. So G/N is torsion-free. □

Geometric criterion

To show the third property we need the following

Definition 3.20. Let G be a finitely generated group with finite generating set S and let $\Gamma = \Gamma(G, S)$ be its Cayley graph. If $I \subset \mathbb{R}$ is any interval, let $G_\chi^I = \{g \in G \mid \chi(g) \in I\}$. We denote by Γ_χ^I the subgraph of Γ induced by the vertices in G_χ^I.

By definition, $\Gamma_\chi = \Gamma_\chi^{[0,\infty)}$ and $\Gamma_{-\chi} = \Gamma_\chi^{(-\infty,0]}$. Also, it is easy to see that an element of g acts on these subgraphs by translation, that is, $g \cdot \Gamma_\chi^I = \Gamma_\chi^{\chi(g)+I}$. All these translation actions are isomorphisms between the subgraphs.

Lemma 3.21. *If $\Gamma_\chi^{[a_0,\infty)}$ is connected for some $a_0 \in \mathbb{R}$ then $\Gamma_\chi^{[a,\infty)}$ is connected for all $a \in \mathbb{R}$ (in particular, Γ_χ is connected, that is, $[\chi] \in \Sigma^1(G)$).*

Demonstração. We show that $\Gamma_\chi^{[a,\infty)}$ is connected in the two cases: $a < a_0$ and $a > a_0$. The case $a = a_0$ is done by hypothesis.

($a < a_0$) Since $\chi \neq 0$, take a generator $t \in S^{\pm}$ such that $\chi(t) > 0$ and a sufficient large $k \geq 1$ such that $a + \chi(t^k) \geq a_0$. Now, let g, g' be two vertices in $\Gamma_\chi^{[a,\infty)}$. Since $\chi(t) > 0$, the paths $p = (g, t^k)$ and $p' = (g', t^k)$ remains inside $\Gamma_\chi^{[a,\infty)}$ and connects g, g' respectively with the vertices gt^k and $g't^k$ which are inside $\Gamma_\chi^{[a_0,\infty)}$, because $\chi(gt^k) = \chi(g)+\chi(t^k) \geq a+\chi(t^k) \geq a_0$ and $\chi(g't^k) = \chi(g') + \chi(t^k) \geq a + \chi(t^k) \geq a_0$. Since $\Gamma_\chi^{[a_0,\infty)}$ is connected by hypothesis, connect the vertices gt^k and $g't^k$ by a path $\tilde{p}$ inside $\Gamma_\chi^{[a_0,\infty)}$ (therefore inside $\Gamma_\chi^{[a,\infty)}$). So obviously g and g' can be connected inside $\Gamma_\chi^{[a,\infty)}$, as desired.

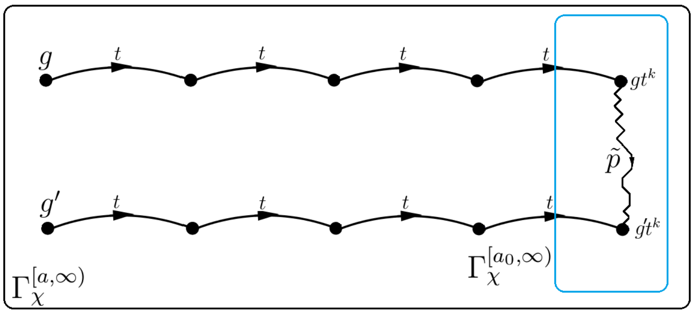


Figura 3.5: $a < a_0$ case

($a > a_0$) Fix $g \in G$ such that $\chi(g) + a < a_0$. But $g \cdot \Gamma_\chi^{[a,\infty)} = \Gamma_\chi^{\chi(g)+[a,\infty)} = \Gamma_\chi^{[\chi(g)+a,\infty)}$. Since $\chi(g) + a < a_0$, $g \cdot \Gamma_\chi^{[a,\infty)}$ is connected by the previous item. Therefore, $\Gamma_\chi^{[a,\infty)} \simeq g \cdot \Gamma_\chi^{[a,\infty)}$ is connected, as desired.

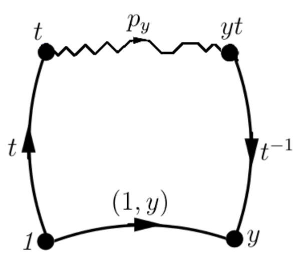
□

Theorem 3.22 (Geometric criterion for Σ^1). *Let G be a finitely generated group with finite generating set S and denote $Y = S^{\pm}$. Let $[\chi] \in S(G)$ and choose $t \in Y$ such that $\chi(t) > 0$. Then the following are equivalent:*

1) Γ_χ is connected (or $[\chi] \in \Sigma^1(G)$);

2) For every $y \in Y = S^{\pm}$, there exists a path p_y from t to yt in Γ such that

$$\nu_\chi(p_y) > \nu_\chi((1, y)) \quad (\textit{or} \ \ \nu_\chi(p_y) - \nu_\chi((1, y)) > 0). \tag{3.1}$$



Demonstração.

1) ⇒ 2) Suppose $\Gamma_\chi = \Gamma_\chi^{[0,\infty)}$ connected and let $y \in Y$. We must build the path p_y from t to yt satisfying 3.1. Let $r_y = \min\{\chi(t), \chi(yt)\}$. By Lemma 3.21, $\Gamma_\chi^{[r_y,\infty)}$ is also connected. Since

the vertices t and yt are inside $\Gamma_\chi^{[r_y,\infty)}$ by definition of r_y, we can connect them by a path p_y inside $\Gamma_\chi^{[r_y,\infty)}$ (therefore, $\nu_\chi(p_y) \geq r_y$). This path satisfies 3.1. In fact,

$$\begin{aligned} \nu_\chi(p_y) \geq r_y &= \min\{\chi(t), \chi(yt)\} \\ &= \chi(t) + \min\{0, \chi(y)\} \\ &= \chi(t) + \nu_\chi((1, y)) \\ &> \nu_\chi((1, y)). \end{aligned}$$

2) ⇒ 1) To show that Γ_χ is connected, let g be any vertex of Γ_χ and let us build a path from 1 to g inside Γ_χ. Write all the existing paths p_y by $p_y = (t, w_y)$, where each w_y is a word in Y. Let

$$d = \min\{\nu_\chi(p_y) - \nu_\chi((1, y)) \mid y \in Y\} > 0.$$

Since the whole graph Γ is always connected, take a path $p_0 \in P(\Gamma)$ from 1 to g. If $\nu_\chi(p_0) \geq 0$ we are done. If not, then, starting from p_0, we will modify it (without changing the extremities 1 and g) until we get a path with $\nu_\chi \geq 0$. To do so, define

$$\begin{aligned} T :& P(\Gamma) \longrightarrow P(\Gamma) \\ & p = (h, y_1 y_2 \ldots y_n) \longmapsto T(p) = (h, t w_{y_1} w_{y_2} \ldots w_{y_n} t^{-1}). \end{aligned}$$

Note that $T(p)$ has the same extremities of p. Note also that p can be written as the path concatenation $p = (h, y_1)(hy_1, y_2)\ldots(hy_1 y_2 \ldots y_{n-1}, y_n)$ and $T(p)$ can be written as the concatenation

$$T(p) = (h, t)(h \cdot p_{y_1})(hy_1 \cdot p_{y_2})(hy_1 y_2 \cdot p_{y_3})\ldots(hy_1 y_2 \ldots y_{n-1} \cdot p_{y_n})(y_1 \ldots y_n t, t^{-1})$$

as one can see in the example.

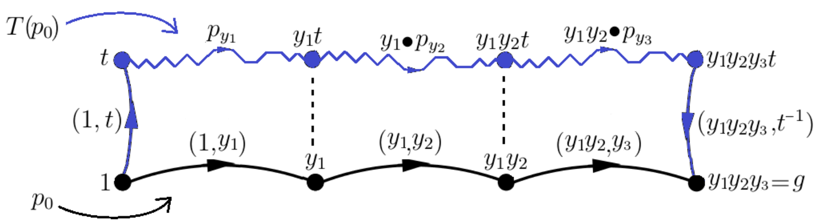


Figura 3.6: Applying the path transformation T

Write $p_0 = (1, y_1 y_2 \ldots y_n)$, $y_i \in Y$, so $y_1 \ldots y_n = g$. We claim that $\nu_\chi(T(p_0)) \geq \min\{0, \nu_\chi(p_0) + d\}$. In fact, using first the concatenation equation (before Proposition 3.13) and then using that $\nu_\chi(1, t) = 0$ and $\nu_\chi(gt, t^{-1}) = \chi(g) \geq 0$ we have

$$\begin{aligned}\nu_\chi(T(p_0)) &= \min\{\nu_\chi((1,t)), \nu_\chi(p_{y_1}), \nu_\chi(y_1 \cdot p_{y_2}), ..., \nu_\chi(y_1...y_{n-1} \cdot p_{y_n}), \nu_\chi(gt, t^{-1})\} \\ &\geq \min\{0, \nu_\chi(p_{y_1}), \nu_\chi(y_1 \cdot p_{y_2}), ..., \nu_\chi(y_1...y_{n-1} \cdot p_{y_n})\}. \\ &= \min\{0, \nu_\chi(p_{y_1}), \chi(y_1) + \nu_\chi(p_{y_2}), ..., \chi(y_1y_2...y_{n-1}) + \nu_\chi(p_{y_n})\}.\end{aligned}$$

Now, by definition of d we have $\nu_\chi(p_{y_i}) \geq \nu_\chi(1, y_i) + d$ for $1 \leq i \leq n$. So

$$\begin{aligned}\nu_\chi(T(p_0)) &\geq \min\{0, \nu_\chi(p_{y_1}), \chi(y_1) + \nu_\chi(p_{y_2}), ..., \chi(y_1y_2...y_{n-1}) + \nu_\chi(p_{y_n})\}. \\ &\geq \min\{0, \nu_\chi(1, y_1) + d, \chi(y_1) + \nu_\chi(1, y_2) + d, ..., \chi(y_1y_2...y_{n-1}) + \nu_\chi(1, y_n) + d\}. \\ &= \min\{0, \nu_\chi(1, y_1) + d, +\nu_\chi(y_1, y_2) + d, ..., +\nu_\chi(y_1y_2...y_{n-1}, y_n) + d\}.\end{aligned}$$

But all the paths $(1, y_1), (y_1, y_2), ..., (y_1y_2...y_{n-1}, y_n)$ above are pieces of p_0 (see the figure) so they have bigger χ-value than p_0 and then

$$\begin{aligned}\nu_\chi(T(p_0)) &\geq \min\{0, \nu_\chi(1, y_1) + d, +\nu_\chi(y_1, y_2) + d, ..., +\nu_\chi(y_1y_2...y_{n-1}, y_n) + d\}. \\ &\geq \min\{0, \nu_\chi(p_0) + d, \nu_\chi(p_0) + d, ..., \nu_\chi(p_0) + d\} \\ &= \min\{0, \nu_\chi(p_0) + d\},\end{aligned}$$

which shows the claim. By induction one can easily prove that $\nu_\chi(T^k(p_0)) \geq \min\{0, \nu_\chi(p_0) + kd\}$ for $k \geq 1$. Then $\nu_\chi(p_0) + kd \geq 0$ for some large enough k and so

$$\nu_\chi(T^k(p_0)) \geq \min\{0, \nu_\chi(p_0) + kd\} = 0.$$

This means that $T^k(p_0)$ connects 1 and g inside Γ_χ, completing the proof.

□

Theorem 3.22 above characterizes $\Sigma^1(G)$ in terms of a finite number of equations having the form $\nu_\chi(p_y) - \nu_\chi((1, y)) > 0$ (equation 3.1). But the maps $\chi \mapsto \nu_\chi(p_y) - \nu_\chi((1, y))$ seem to be continuous, so this system of equations should be an open condition for $\Sigma^1(G)$. We formalize this in the following

Corollary 3.23 (Openess of $\Sigma^1(G)$)**.** *If G is a finitely generated group, $\Sigma^1(G)$ is an open set of $S(G)$.*

Demonstração. Fix a finite generating set S for G and denote $Y = S^{\pm}$ as above. Let $[\chi_0] \in \Sigma^1(G)$ and fix $t \in Y$ such that $\chi_0(t) > 0$. Write $Y = \{y_1, ..., y_n\}$ and fix the existing paths $p_{y_1}, ..., p_{y_n}$ of Theorem 3.22. We must find a set $A \subset S(G)$ which is open in $S(G)$ and such that $[\chi_0] \in A \subset \Sigma^1(G)$. We have the system

$$\begin{cases} \nu_{\chi_0}(p_{y_1}) - \nu_{\chi_0}((1, y_1)) > 0, \\ \vdots \\ \nu_{\chi_0}(p_{y_n}) - \nu_{\chi_0}((1, y_n)) > 0. \end{cases} \qquad (*)$$

Let $f_i : Hom(G, \mathbb{R}) \to \mathbb{R}$ with $f_i(\chi) = \nu_\chi(p_{y_i}) - \nu_\chi((1, y_i))$, $1 \le i \le n$, and let $\tau : Hom(G, \mathbb{R}) \to \mathbb{R}$ with $\tau(\chi) = \chi(t)$. Since the numbers $\nu_\chi(p_{y_i})$ and $\nu_\chi((1, y_i))$ are given in terms of the $\chi(y)$ for $y \in Y$ and the topology of $Hom(G, \mathbb{R})$ is also given in terms of some of them, the f_i are continuous, and since $t \in Y$, so it is τ. This implies that $W = \bigcap_{i=1}^n f_i^{-1}((0, \infty)) \cap \tau^{-1}((0, \infty))$ is open in $Hom(G, \mathbb{R})$. Since $W \subset Hom(G, \mathbb{R}) - \{0\}$ (because $0 \notin \tau^{-1}((0, \infty))$, for example), W is open in $Hom(G, \mathbb{R}) - \{0\}$. Now let $p : Hom(G, \mathbb{R}) - \{0\} \to S(G)$ be the natural quotient map and let $A = p(W) \subset S(G)$. We will show that A satisfy what we wanted:

- $[\chi_0] \in A$. If we show that $\chi_0 \in W$, then $[\chi_0] = p(\chi_0) \in p(W) = A$. But $\tau(\chi_0) = \chi_0(t) > 0$, so $\chi_0 \in \tau^{-1}((0, \infty))$, and the equations $(*)$ show that $\chi_0 \in \bigcap_{i=1}^n f_i^{-1}((0, \infty))$.

- $A \subset \Sigma^1(G)$. Let $[\chi] \in A = p(W)$. Then $[\chi] = [\chi']$ for $\chi' \in W$. We have $\chi = r\chi'$ for some $r > 0$, then $\chi(t) = r\chi'(t) > 0$ and we can use the same t and the same paths p_{y_i} to apply Theorem 3.22 for $[\chi]$. Since $\chi' \in \bigcap_{i=1}^n f_i^{-1}((0, \infty))$, we have

$$\nu_\chi(p_{y_i}) - \nu_\chi((1, y_i)) = \nu_{r\chi'}(p_{y_i}) - \nu_{r\chi'}((1, y_i)) = r(\nu_{\chi'}(p_{y_i}) - \nu_{\chi'}((1, y_i))) > 0,$$

  so the n equations 3.1 are satisfied also for χ and therefore by the Geometric Criterion 3.22, $[\chi] \in \Sigma^1(G)$.

- A open in $S(G)$. By definition of quotient topology, we just have to show that $p^{-1}(A) = p^{-1}p(W)$ is open in $Hom(G, \mathbb{R}) - \{0\}$. But W is open, so let us show that $p^{-1}p(W) = W$. Obviously, $W \subset p^{-1}p(W)$. Now, let $\chi \in p^{-1}p(W)$, that is, $[\chi] = [\chi']$ for some $\chi' \in W$. Write $\chi = r\chi'$. Again, we have $\tau(\chi) = \chi(t) = r\chi'(t) > 0$ and $f_i(\chi) = \nu_\chi(p_{y_i}) - \nu_\chi((1, y_i)) = r(\nu_{\chi'}(p_{y_i}) - \nu_{\chi'}((1, y_i))) > 0$, so $\chi \in W$ by definition.

□

To finish, we just cite Theorem $A4.1$ of [92], one of the reasons why the Σ invariant is so important: it detects the finite generation of normal subgroups $N \lhd G$ containing the commutator G':

Theorem 3.24. *Let G be a finitely generated group and $N \lhd G$ with $G' \subset N$. Then N is finitely generated if and only if $S(G, N) \subset \Sigma^1(G)$. In particular, G' is finitely generated if and only if $\Sigma^1(G) = S(G)$.*

Other properties

Next we show some other important properties of Σ^1 under quotients, finite index subgroups, direct and amalgamated products and HNN extensions. Each of them will be useful to the present work at some point. Most proofs can be found in [92], although we give here some additional details.

Quotients

Proposition 3.25. *Let G be a finitely generated group, $N \lhd G$ a normal subgroup with projection homomorphism $\pi : G \to G/N$ and let $[\chi] \in S(G/N)$. If $[\chi \circ \pi] \in \Sigma^1(G)$ then $[\chi] \in \Sigma^1(G/N)$. The converse is also true if N is finitely generated.*

Demonstração. Let S be a finite generating set for G, and fix $\pi(S)$ as a finite generating set for G/N. Denote $\Gamma = \Gamma(G, S)$, $\overline{\Gamma} = \Gamma(G/N, \pi(S))$ and the induced subgraphs $\Gamma_{\chi\circ\pi} = \Gamma(G, S)_{\chi\circ\pi}$ and $\overline{\Gamma}_\chi = \Gamma(G/N, \pi(S))_\chi$. There is a natural surjective graph homomorphism

$$\begin{aligned} \Upsilon : \Gamma &\longrightarrow \overline{\Gamma} \\ g &\longmapsto \overline{g} \\ (g, s) &\longmapsto (\overline{g}, \overline{s}) \end{aligned}$$

such that

$$V(\Gamma_{\chi\circ\pi}) = \{g \in G \mid \chi \circ \pi(g) = \chi(\overline{g}) \geq 0\} = \Upsilon^{-1}(\{\overline{g} \in G/N \mid \chi(\overline{g}) \geq 0\}) = \Upsilon^{-1}(V(\overline{\Gamma}_\chi))$$

and

$$\begin{aligned} E(\Gamma_{\chi\circ\pi}) &= \{(g, s) \mid g, gs \in V(\Gamma_{\chi\circ\pi})\} \\ &= \{(g, s) \mid g, gs \in \Upsilon^{-1}(V(\overline{\Gamma}_\chi))\} \\ &= \Upsilon^{-1}(\{(\overline{g}, \overline{gs}) \mid \overline{g}, \overline{gs} \in V(\overline{\Gamma}_\chi)\}) \\ &= \Upsilon^{-1}(E(\overline{\Gamma}_\chi)). \end{aligned}$$

Then $\Gamma_{\chi\circ\pi} = \Upsilon^{-1}(\overline{\Gamma}_\chi)$ and this implies that the restriction homomorphism $\Upsilon : \Gamma_{\chi\circ\pi} \to \overline{\Gamma}_\chi$ is also surjective, for $\Upsilon(\Gamma_{\chi\circ\pi}) = \Upsilon(\Upsilon^{-1}(\overline{\Gamma}_\chi)) = \overline{\Gamma}_\chi$. Now the first part of the proposition is easy: if $[\chi \circ \pi] \in \Sigma^1(G)$, $\Gamma_{\chi\circ\pi}$ is connected. Then its image $\overline{\Gamma}_\chi$ under Υ is connected, that is, $[\chi] \in \Sigma^1(G/N)$.

Now suppose N is finitely generated by a fixed finite subset $Z \subset N$. By the independence theorem of Σ^1, we can add Z to the generator set of G without changing $\Sigma^1(G)$, so denote $\Gamma = \Gamma(G, S \cup Z)$ and $\Gamma_{\chi\circ\pi} = \Gamma(G, S \cup Z)_{\chi\circ\pi}$. Supppose $[\chi] \in \Sigma^1(G/N)$ ($\overline{\Gamma}_\chi$ is connected) and let us show that $[\chi \circ \pi] \in \Sigma^1(G)$ ($\Gamma_{\chi\circ\pi}$ is connected). Let g be any vertex in $\Gamma_{\chi\circ\pi}$ and let us connect 1 to g inside $\Gamma_{\chi\circ\pi}$. Since $\chi(\overline{g}) = \chi \circ \pi(g) \geq 0$, $\overline{1}$ and $\overline{g}$ are vertices of the connected graph $\overline{\Gamma}_\chi$. Connect $\overline{1}$ to $\overline{g}$ by a path $p = (\overline{1}, \overline{s_1} ... \overline{s_m})$ in $\overline{\Gamma}_\chi$, $\overline{s_i} \in \pi(S)^\pm$. Note that $\nu_\chi(p) \geq 0$ and $\overline{g} = \overline{s_1} ... \overline{s_m} = \overline{s_1 ... s_m}$. Because of this, we have $g(s_1 ... s_m)^{-1} \in N$, so write $g = ns_1 ... s_m$ for $n \in N = \langle Z \rangle$ and write $n = z_1 ... z_k$, $z_i \in \mathbb{Z}^\pm$. So, $z_1 ... z_k s_1 ... s_m$ is a word in the set $S \cup Z$, $\gamma = (1, z_1 ... z_k s_1 ... s_m)$ is a path from 1 to $z_1 ... z_k s_1 ... s_m = ns_1 ... s_m = g$ and because $z_i \in N$ we have $\chi \circ \pi(z_i) = \chi(\overline{1}) = 0$, which implies

$$\begin{aligned} \nu_{\chi\circ\pi}(\gamma) &= \min\{0, \chi \circ \pi(z_1), ..., \chi \circ \pi(z_1 ... z_k), \chi \circ \pi(z_1 ... z_k s_1), ..., \chi \circ \pi(z_1 ... z_k s_1 ... s_m)\} \\ &= \min\{0, \chi \circ \pi(s_1), ..., \chi \circ \pi(s_1 ... s_m)\} \\ &= \nu_\chi(p) \geq 0 \end{aligned}$$

and finishes the proof. □

Finite index subgroups

With respect to finite index subgroups H, Σ^1 behaves better than one could expect. First we need a

Lemma 3.26. *Let $\chi : G \to \mathbb{R}$ be a homomorphism of groups and $H \leq G$ a subgroup of finite index in G. Then $\chi = 0 \Longleftrightarrow \chi|_H = 0$.*

Demonstração. If $\chi = 0$ it is obvious that $\chi|_H = 0$. Suppose $\chi|_H = 0$ and let $G = g_0H \sqcup g_1H \sqcup ... \sqcup g_nH$ be a finite coset decomposition of G ($g_0 = 1$). For each $g \in G$ we have $g = g_ih$ for some i and some $h \in H$, so $\chi(g) = \chi(g_ih) = \chi(g_i) + \chi(h) = \chi(g_i)$. Then $\operatorname{im}(\chi) \subset \{\chi(g_0), \chi(g_1), ..., \chi(g_n)\}$ is a finite subgroup of $\mathbb{R}$. But since $\mathbb{R}$ is torsion-free, every nontrivial subgroup must be infinite. Then $\operatorname{im}(\chi) = 0$ and $\chi = 0$, as desired. □

Because of this lemma, if $[\chi] \in S(G)$ then the expression $[\chi|_H] \in S(H)$ makes sense because $\chi|_H \neq 0$. If $i : H \to G$ is the inclusion, we have the well defined map $i^* : S(G) \to S(H)$ with $[\chi] \mapsto [\chi \circ i] = [\chi|_H]$. Here is our main property:

Proposition 3.27. *Let G be a finitely generated group with finite generating set S, $H \leq G$ a finite index subgroup and $[\chi] \in S(G)$. Then $[\chi] \in \Sigma^1(G) \Longleftrightarrow [\chi|_H] \in \Sigma^1(H)$.*

Demonstração. A transversal T of G mod H is a collection $T \subset G$ of coset representatives ($G = \sqcup_{t \in T} Ht$) with $1 \in T$. Fix such a transversal T (finite, in our case). We may assume that $\chi(t) \leq 0$ for every $t \in T$. In fact, if some $t \in T$ is such that $\chi(t) > 0$ ($t \neq 1$), by the above Lemma 3.26 since $\chi|_H \neq 0$ we can find $h \in H$ such that $\chi(ht) = \chi(h) + \chi(t) \leq 0$. Since $Hht = Ht$, then replacing t by ht in the collection T we still have a transversal. By doing all the necessary replacements we get the desired transversal T. Now, given $g \in G$, denote by $\overline{g}$ the (unique) element of T such that $Hg = H\overline{g}$. We claim that H is generated by the set $W = \{ts\overline{ts}^{-1} \mid t \in T, s \in S^{\pm}\}$. Indeed, given $h \in H$, write $h = s_1...s_m$ for $s_i \in S^{\pm}$. Let $t_1 = \overline{s_1} \in T$ and $t_i = \overline{t_{i-1}s_i} \in T$ for $2 \leq i \leq m$. We have $t_m = 1$. Indeed, given $g_1, g_2 \in G$, we have $Hg_1g_2 = H\overline{g_1}g_2 = H\overline{\overline{g_1}g_2}$, so by uniqueness $\overline{\overline{g_1}g_2} = \overline{g_1g_2}$. Using this, we have $t_1 = \overline{s_1}$, $t_2 = \overline{t_1s_2} = \overline{\overline{s_1}s_2} = \overline{s_1s_2}$ and recursively we get $t_m = \overline{t_{m-1}s_m} = \overline{\overline{s_1...s_{m-1}}s_m} = \overline{s_1...s_m} = \overline{h} = 1$. Then

$$h = s_1...s_m = (1s_1{t_1}^{-1})(t_1s_2{t_2}^{-1})...(t_{m-1}s_m{t_m}^{-1}),$$

but all the parenthesis $(t_{i-1}s_it_i) = (t_{i-1}s_i\overline{t_{i-1}s_i}^{-1})$ are by definition letters (elements) of W, so h is generated by W, which shows the claim. Denote then $\Gamma_\chi = \Gamma(G,S)_\chi$, $\Gamma_H = \Gamma(H,W)$ and $\Gamma_{H\chi} = \Gamma(H,W)_{\chi|_H}$.

($\Rightarrow$) Suppose Γ_χ connected and let us connect 1 to any vertex $h \in \Gamma_{H\chi}$ inside $\Gamma_{H\chi}$. Since $\chi(h) \geq 0$, connect by hypothesis 1 to h inside Γ_χ by a path $p = (1, w)$ where $w = s_1..s_m$, $s_i \in S^{\pm}$ is a word in S. Define the word

$$u_w = (1s_1{t_1}^{-1})(t_1s_2{t_2}^{-1})...(t_{m-1}s_m{t_m}^{-1}),$$

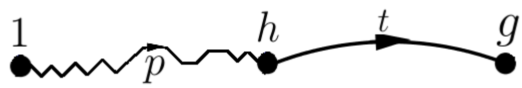
where $t_1 = \overline{s_1} \in T$ and $t_i = \overline{t_{i-1}s_i} \in T$ for $2 \leq i \leq m$, like before. We have $t_m = 1$ and u_w is a word in W. Then $p' = (1, u_w)$ is a path in Γ_H from 1 to

$(1s_1{t_1}^{-1})(t_1s_2{t_2}^{-1})...(t_{m-1}s_m{t_m}^{-1}) = s_1...s_m{t_m}^{-1} = h{t_m}^{-1} = h$ and such that the χ-values in all of its vertices $(1s_1{t_1}^{-1})(t_1s_2{t_2}^{-1})...(t_{i-1}s_i{t_i}^{-1})$ are non-negative, because

$$\chi((1s_1{t_1}^{-1})(t_1s_2{t_2}^{-1})...(t_{i-1}s_i{t_i}^{-1})) = \chi(s_1s_2...s_i{t_i}^{-1}) = \chi(s_1s_2...s_i) - \chi(t_i) \geq \chi(s_1s_2...s_i) \geq 0.$$

This shows that $[\chi|_H] \in \Sigma^1(H)$.

($\Leftarrow$) Conversely, suppose $[\chi|_H] \in \Sigma^1(H)$. Since W generates H and $G = \sqcup_{t\in T} Ht$, every $g \in G$ can be written as $g = ht = w_1...w_kt$, $w_i \in W^{\pm}$, so G is generated by the finite set $W \cup T$. By the independence property, we can fix this finite set of generators for G. Let us show that $[\chi] \in \Sigma^1(G)$. Let g be a vertex in $\Gamma_\chi = \Gamma(G, W \cup T)_\chi$ and write $g = ht$ for $h \in H$ and $t \in T$. Since $\chi(h) = \chi(gt^{-1}) = \chi(g) - \chi(t) \geq \chi(g) \geq 0$, h is a vertex of $\Gamma_{H\chi}$ and therefore by hypothesis there is a path $p = (1, w)$ inside $\Gamma_{H\chi}$ from 1 to h, w being a word in W. Now, the concatenated path $p' = (1, wt)$ goes from 1 to $wt = ht = g$ and lies inside Γ_χ, because $\nu_\chi(p') = \min\{\nu_\chi(p), \chi(g)\} \geq 0$.



This shows that $[\chi] \in \Sigma^1(G)$ and concludes the proof. □

Note that the above proposition allows us to compute $\Sigma^1(G)$ in terms of $\Sigma^1(H)$ (if we know the latter, we can compute the former). The converse is not true in general, because the proposition only deals with homomorphisms of $Hom(H, \mathbb{R})$ which are restrictions of homomorphisms of $Hom(G, \mathbb{R})$. So, a special hypothesis about extension of characters on H is enough to show the converse:

Corollary 3.28. *Let G be a finitely generated group and $H \leq G$ a finite index subgroup with inclusion $i : H \to G$. Suppose that any homomorphism $\chi : H \to \mathbb{R}$ can be extended to a homomorphism $\hat{\chi} : G \to \mathbb{R}$ (that is, $\hat{\chi}|_H = \chi$). Then*

$$\Sigma^1(H) = i^*(\Sigma^1(G)) \text{ and } \Sigma^1(H)^c = i^*(\Sigma^1(G)^c).$$

Demonstração. If $[\chi] \in \Sigma^1(G)$ we have $i^*[\chi] = [\chi|_H] \in \Sigma^1(H)$ by Proposition 3.27. This shows that $i^*(\Sigma^1(G)) \subset \Sigma^1(H)$. Conversely, if $[\chi] \in \Sigma^1(H)$, let $\hat{\chi}$ be the extension of χ to G. Since $[\hat{\chi}|_H] = [\chi] \in \Sigma^1(H)$, again by Proposition 3.27 we must have $[\hat{\chi}] \in \Sigma^1(G)$. Then $[\chi] = i^*[\hat{\chi}] \in i^*(\Sigma^1(G))$ and so $\Sigma^1(H) = i^*(\Sigma^1(G))$. The second part is analogous. □

Corollary 3.29. *Let G be a finitely generated subgroup. If G has a finite index abelian subgroup H then $\Sigma^1(G) = S(G)$.*

Demonstração. Since G is finitely generated and H has finite index in G, by Corollary 1.51 H is also finitely generated. Since H is abelian we have $\Sigma^1(H) = S(H)$ by Corollary 3.15. Now let $[\chi] \in S(G)$. Since $[\chi|_H] \in S(H) = \Sigma^1(H)$, by Proposition 3.27 we have $[\chi] \in \Sigma^1(G)$, as desired. □

Direct products

Let $G = G_1 \times G_2$ be a direct product of two finitely generated groups. For $i = 1, 2$, choose and fix a finite generating set X_i for G_i containing the identity element 1 of G_i. Then $X = (X_1 \times \{1\}) \cup (\{1\} \times X_2)$ is a finite generating set for G. Fix X. Let $\pi_i : G \to G_i$ and $j_i : G_i \to G$ be the natural projections and injections, respectively, and consider the linear pullbacks

$$\begin{aligned} {j_i}^* : Hom(G, \mathbb{R}) &\longrightarrow Hom(G_i, \mathbb{R}) \\ \chi &\longmapsto \chi \circ j_i, \end{aligned}$$

$$\begin{aligned} {\pi_i}^* : Hom(G_i, \mathbb{R}) &\longrightarrow Hom(G, \mathbb{R}) \\ \chi &\longmapsto \chi \circ \pi_i. \end{aligned}$$

Since ${j_i}^* \circ {\pi_i}^* = (\pi_i \circ j_i)^* = Id^* = Id$, the ${\pi_i}^*$ are injective. Consider the restrictions ${\pi_i}^* : Hom(G_i, \mathbb{R}) - \{0\} \longrightarrow Hom(G, \mathbb{R}) - \{0\}$. To take the quotient applications, note that

$$\begin{aligned} [\chi] = [\chi'] &\Leftrightarrow r\chi = \chi',\ r > 0 \\ &\Leftrightarrow r\chi(g_i) = \chi'(g_i)\ \forall\ g_i \in G_i \\ &\Leftrightarrow r\chi\pi_i(g_1, g_2) = \chi'\pi_i(g_1, g_2)\ \forall\ g_1 \in G_1, g_2 \in G_2 \\ &\Leftrightarrow r(\chi \circ \pi_i) = \chi' \circ \pi_i,\ r > 0 \\ &\Leftrightarrow [\chi \circ \pi_i] = [\chi' \circ \pi_i], \end{aligned}$$

which shows that the applications

$$\begin{aligned} {\pi_i}^* : S(G_i) &\longrightarrow S(G) \\ [\chi] &\longmapsto [\chi \circ \pi_i] \end{aligned}$$

are well defined and injective. We are ready to show the following theorem, which can be found in [92]:

Theorem 3.30 (Σ^1 for direct products)**.** *If $G = G_1 \times G_2$ is the direct product of two finitely generated groups, then*

$$\Sigma^1(G)^c = {\pi_1}^*(\Sigma^1(G_1)^c) \cup {\pi_2}^*(\Sigma^1(G_2)^c).$$

Demonstração. To simplify we will denote $G_1 \times \{1\}$ and $\{1\} \times G_2$ only by G_1 and G_2. The proof is divided into easier steps:

1) ${\pi_1}^*(\Sigma^1(G_1)^c) = S(G, G_2) - \Sigma^1(G)$.
(⊂) Let $[\chi] \in {\pi_1}^*(\Sigma^1(G_1)^c)$, that is, $[\chi] = {\pi_1}^*([\chi_1]) = [\chi_1 \circ \pi_1]$ for $[\chi_1] \notin \Sigma^1(G_1)$. Since $\chi_1 \circ \pi_1(1, g_2) = \chi_1(1) = 0$ for all $(1, g_2) \in G_2$, we have $[\chi] \in S(G, G_2)$. Let us show that $[\chi] \notin \Sigma^1(G)$. If $[\chi]$ was in $\Sigma^1(G)$, let us show that $[\chi_1] \in \Sigma^1(G_1)$, contradiction. Let $g_1 \in G_1$ be a vertex in Γ_{χ_1}, that is, $\chi_1(g_1) \geq 0$. Then $g = (g_1, 1)$ is in Γ_χ because $\chi(g) = \chi_1(g_1) \geq 0$. By hypothesis, $[\chi] \in \Sigma^1(G)$, so connect 1 to g inside Γ_χ by a path $p = (1, w)$ (remember our path notation given before) with w a word in X. By definition of X, w is a product of letters either of the form $(y, 1)$ (whose product is $(g_1, 1)$) and of the form $(1, z)$ (whose product is $(1, 1)$). Since $[\chi] \in S(G, G_2)$, the letters $(1, z)$ does not contribute to the χ-values. So

$$\begin{aligned} 0 \leq \nu_\chi(p) &= \min\{\chi(y_{11}, 1), \chi(y_{11}y_{12}, 1), ..., \chi(y_{11}y_{12}...y_{1k}, 1)\} \\ &= \min\{\chi_1(y_{11}), \chi_1(y_{11}y_{12}), ..., \chi_1(y_{11}y_{12}...y_{1k})\} \\ &= \nu_{\chi_1}(p_1), \end{aligned}$$

where $p_1 = (1, y_{11}y_{12}...y_{1k})$. Then p_1 is a path in Γ_{χ_1} connecting 1 and $y_{11}y_{12}...y_{1k} = g_1$ and we have $[\chi_1] \in \Sigma^1(G_1)$, a contradiction. Then $[\chi] \in S(G, G_2) - \Sigma^1(G)$.

(⊃) Let $[\chi] \in S(G, G_2) - \Sigma^1(G)$. Let $\chi_1 = \chi \circ j_1$. We have $\chi_1 \neq 0$, because if $0 = \chi_1(g_1) = \chi(g_1, 1)$ for all $g_1 \in G_1$ we would have $\chi(g_1, g_2) = \chi(g_1, 1) + \chi(1, g_2) = 0 + 0 = 0$ for all $(g_1, g_2) \in G$ (since $[\chi] \in S(G, G_2)$) and then we would have $\chi = 0$, contradiction. So, $[\chi_1] \in S(G_1)$ and ${\pi_1}^*([\chi_1]) = [\chi_1 \circ \pi_1] = [\chi \circ j_1 \circ \pi_1] = [\chi]$, because

$$\chi \circ j_1 \circ \pi_1(g_1, g_2) = \chi \circ j_1(g_1) = \chi(g_1, 1) = \chi(g_1, 1) + \chi(1, g_2) = \chi(g_1, g_2)$$

for all $(g_1, g_2) \in G$. We are just left to show that $[\chi_1] \notin \Sigma^1(G_1)$. Again, if $[\chi_1] \in \Sigma^1(G_1)$ let us show that $[\chi] \in \Sigma^1(G)$, a contradiction. Let $g = (g_1, g_2) \in G$ be a vertex in Γ_χ, that is, $0 \leq \chi(g) = \chi_1 \circ \pi_1(g) = \chi_1(g_1)$. Then $g_1 \in \Gamma_{\chi_1}$. Since $[\chi_1] \in \Sigma^1(G_1)$, we can connect 1 to g_1 inside Γ_{χ_1} by a path $p_1 = (1, y_{11}...y_{1k})$. Write $g_2 = y_{21}...y_{2h}$ as a word in the generator set X_2 and define the path

$$p = (1, (y_{11}, 1)(y_{12}, 1)...(y_{1k}, 1)(1, y_{21})(1, y_{22})...(1, y_{2h})).$$

It is a path connecting 1 to $(y_{11}, 1)(y_{12}, 1)...(y_{1k}, 1)(1, y_{21})(1, y_{22})...(1, y_{2h}) = (y_{11}y_{12}...y_{1k}, y_{21}y_{22}...y_{2h}) = (g_1, g_2) = g$ and using that $[\chi] \in S(G, G_2)$ we have $\nu_\chi(p) = \nu_{\chi_1}(p_1) \geq 0$, so $[\chi] \in \Sigma^1(G)$, the desired contradiction. So $[\chi] \in {\pi_1}^*(\Sigma^1(G_1)^c)$.

2) ${\pi_2}^*(\Sigma^1(G_2)^c) = S(G, G_1) - \Sigma^1(G)$. Similar argument from item 1).

3) $\Sigma^1(G)^c \subset S(G, G_1) \cup S(G, G_2)$.
Let us prove by contradiction. Suppose $[\chi] \notin S(G, G_1) \cup S(G, G_2)$, that is, $\chi|_{G_1} \neq 0 \neq \chi|_{G_2}$ and let us show that $[\chi] \in \Sigma^1(G)$. Let $g = (g_1, g_2)$ be a vertex in Γ_χ, that is, $0 \leq \chi(g_1, g_2) = \chi(g_1, 1) + \chi(1, g_2)$. Then $\chi(g_1, 1) \geq 0$ or $\chi(1, g_2) \geq 0$. Without loss of generality, suppose $\chi(g_1, 1) \geq 0$ (the other case is similar) and let us connect 1 to g inside Γ_χ. Write $g_1 = y_{11}...y_{1r}$ and $g_2 = y_{21}...y_{2s}$, $y_{1i} \in X_1$, $y_{2i} \in X_2$. Since $\chi|_{G_2} \neq 0$, take

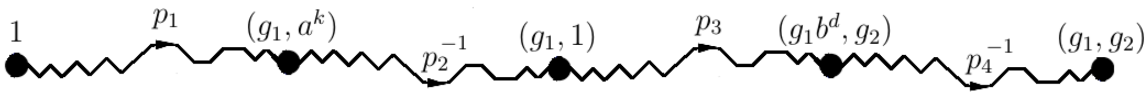
$(1,a) \in \{1\} \times X_2{}^{\pm}$ with $\chi(1,a) > 0$ and a sufficient large $k \geq 1$ such that

$$\chi((1,a)^k) + \min\{\chi(y_{11},1), \chi(y_{11}y_{12},1), ..., \chi(y_{11}y_{12}...y_{1r},1)\} \geq 0.$$

Because of this inequality, the path $p_1 = (1, (1,a)^k(y_{11},1)(y_{12},1)...(y_{1r},1))$ connects 1 to $(y_{11}y_{12}...y_{1r}, a^k) = (g_1, a^k)$ inside Γ_χ. Also, since $\chi(g_1,1) \geq 0$ and $\chi(1,a) > 0$, the path $p_2 = ((g_1,1),(1,a)^k)$ connects $(g_1,1)$ to (g_1,a^k) inside Γ_χ. Now, using that $\chi|_{G_1} \neq 0$, take $(b,1) \in X_1{}^{\pm} \times \{1\}$ with $\chi(b,1) > 0$ and a sufficient large $d \geq 1$ such that

$$\chi((b,1)^d) + \min\{\chi(g_1,y_{21}), \chi(g_1,y_{21}y_{22}), ..., \chi(g_1,y_{21}y_{22}...y_{2s})\} \geq 0.$$

Because of this inequality, the path $p_3 = ((g_1,1),(b,1)^d(1,y_{21})(1,y_{22})...(1,y_{2s}))$ connects $(g_1,1)$ to $(g_1b^d, y_{21}...y_{2s}) = (g_1b^d, g_2)$ inside Γ_χ. Also, since $\chi(g_1,g_2) \geq 0$ and $\chi(b,1) > 0$, the path $p_4 = ((g_1,g_2),(b,1)^d)$ connects (g_1,g_2) to (g_1b^d,g_2) inside Γ_χ. A concatenation of these four paths connects 1 to (g_1,g_2) inside Γ_χ. Then $[\chi] \in \Sigma^1(G)$, as desired.



Now we finish the proof. By items 1) and 2), we have

$$\begin{aligned} \pi_1{}^*(\Sigma^1(G_1)^c) \cup \pi_2{}^*(\Sigma^1(G_2)^c) &= (S(G,G_2) - \Sigma^1(G)) \cup (S(G,G_1) - \Sigma^1(G)) \\ &= (S(G,G_1) \cup S(G,G_2)) - \Sigma^1(G). \end{aligned}$$

Now, $(S(G,G_1) \cup S(G,G_2)) - \Sigma^1(G) = S(G) - \Sigma^1(G)$. Indeed, the inclusion $(\subset)$ is obvious, and $(\supset)$ follows easily from item 3). This completes the proof. □

Amalgamated products and HNN extensions

Now we will exhibit some behavior properties of the invariant Σ^1 concerning amalgamated products and HNN extensions (see definitions 1.64 and 1.65). The two next propositions can be found in Lemma 2.1 of [19] and will be especially useful to deal with GBS and GBS_n groups later.

Proposition 3.31. *Let $\tilde{G} = G *_A H$ be the amalgamated product of two finitely generated groups G, H and let $[\chi] \in S(\tilde{G})$. If $\chi|_A \neq 0$, $[\chi|_G] \in \Sigma^1(G)$ and $[\chi|_H] \in \Sigma^1(H)$ then $[\chi] \in \Sigma^1(\tilde{G})$.*

Demonstração. Fix finite generating sets R and S for G and H, respectively, and fix the finite generating set $R \cup S$ for $\tilde{G}$. We denote $\Gamma_\chi = \Gamma(\tilde{G}, R \cup S)$, $\Gamma_{G\chi} = \Gamma(G,R)_{\chi|_G}$ and $\Gamma_{H\chi} = \Gamma(H,S)_{\chi|_H}$. The last two subgraphs are connected by hypothesis and we want to show connectivity of the first. Let $\tilde{g} \in \tilde{G}_\chi$, say, $\tilde{g} = g_1h_1...g_nh_n$, $g_i \in G, h_i \in H$, and let us connect 1 to $\tilde{g}$ inside Γ_χ. We proceed by induction on n:

- $(n = 1)$. To simplify, write $\tilde{g} = gh$, $g \in G, h \in H$. Since $\chi|_A \neq 0$, fix $a \in A$ such that $\chi(a) > 0$ and sufficiently large so that $\chi(ga) = \chi(g) + \chi(a) \geq 0$, denoting $g' = ga$. Now, choose another $a' \in A$ with $\chi(a') > 0$ and large enough so that $\chi(a^{-1}ha') = \chi(a^{-1}h) + \chi(a') \geq 0$, denoting $h' = a^{-1}ha'$. Since $A \subset G \cap H$, we have $g' \in G$ and $h' \in H$. By construction, both have nonnegative χ-values, then by hypothesis there are paths p_1 from 1 to g' in $\Gamma_{G\chi}$ and p_2 from 1 to h' in $\Gamma_{H\chi}$ (note that these paths are also in Γ_χ). Since $\chi(g') \geq 0$, the translated path $g' \cdot p_2$ is in Γ_χ (for $\nu_\chi(g' \cdot p_2) = \chi(g') + \nu_\chi(p_2) \geq 0 + 0 = 0$) and goes from g' to $g'h' = gaa^{-1}ha' = \tilde{g}a'$. Finally, connect a' to 1 by a path p_3 in $\Gamma_{G\chi}$. The translated path $\tilde{g} \cdot p_3$ is in Γ_χ (for $\nu_\chi(\tilde{g} \cdot p_3) = \chi(\tilde{g}) + \nu_\chi(p_3) \geq 0 + 0 = 0$) and goes from $\tilde{g}a'$ to $\tilde{g}$. The concatenation of $p_1, g' \cdot p_2$ and $\tilde{g} \cdot p_3$ connects 1 to $\tilde{g}$ inside Γ_χ, as desired.

- (induction). Suppose the claim is true for $n - 1 \geq 1$ and let $\tilde{g} = g_1h_1...g_nh_n \in \tilde{G}_\chi$. The strategy is the same from above. Denote $g_0 = g_1h_1...g_{n-1}h_{n-1}$. Find $a \in A$ with $\chi(a) > 0$ and $\chi(g_0a) \geq 0$. Since $g_0a = g_1h_1...g_{n-1}(h_{n-1}a)$ is an alternated product of length $n-1$, by induction there is a path p_1 in Γ_χ from 1 to g_0a. Now, choose $a' \in A$ with $\chi(a') > 0$ and $\chi(a^{-1}g_ma') \geq 0$. Since $a^{-1}g_ma' \in G$ and $[\chi|_G] \in \Sigma^1(G)$, there is a path p_2 from 1 to $a^{-1}g_ma'$ in $\Gamma_{G\chi}$ (therefore also in Γ_χ). The translated path $g_0a \cdot p_2$ is in Γ_χ (for $\nu_\chi(g_0a \cdot p_2) = \chi(g_0a) + \nu_\chi(p_2) \geq 0 + 0 = 0$) and goes from g_0a to $g_0aa^{-1}g_ma' = g_0g_ma'$. Take $a'' \in A$ with $\chi(a'') > 0$ and $\chi(a'^{-1}h_ma'') \geq 0$. Since $a'^{-1}h_ma'' \in H$ and $[\chi|_H] \in \Sigma^1(H)$, there is a path p_3 from 1 to $a'^{-1}h_ma''$ in $\Gamma_{H\chi}$ (therefore also in Γ_χ). The translated path $g_0g_ma' \cdot p_3$ is in Γ_χ (for $\nu_\chi(g_0g_ma' \cdot p_3) = \chi(g_0g_ma') + \nu_\chi(p_3) \geq 0 + 0 = 0$) and goes from g_0g_ma' to $g_0g_ma'a'^{-1}h_ma'' = \tilde{g}a''$. To finish, since $a'' \in A \subset G$, $\chi(a'') > 0$ and $[\chi|_G] \in \Sigma^1(G)$, connect a'' to 1 by a path p_4 in $\Gamma_{G\chi}$ (and so in Γ_χ). The concatenation path $p_1(g_0a \cdot p_2)(g_0g_ma' \cdot p_3)(\tilde{g} \cdot p_4)$ connects 1 to $\tilde{g}$ inside Γ_χ, as desired.

□

The same thing can be done for HNN extensions:

Proposition 3.32. *Let $\tilde{G} = \left\langle X, t \mid R, tat^{-1}\ \theta(a)^{-1},\ a \in A \right\rangle$ be an HNN extension of a finitely generated group $G = \langle X \mid R \rangle$ and let $[\chi] \in S(\tilde{G})$. If $\chi|_A \neq 0$ and $[\chi|_G] \in \Sigma^1(G)$ then $[\chi] \in \Sigma^1(\tilde{G})$.*

Demonstração. Fix a finite generating set S for G and fix the finite generating set $S \cup \{t\}$ for $\tilde{G}$. Denote $\Gamma = \Gamma(\tilde{G}, S \cup \{t\}), \Gamma_\chi = \Gamma(\tilde{G}, S \cup \{t\})_\chi$ and $\Gamma_{G\chi} = \Gamma(G, S)_{\chi|_G} \leq \Gamma_\chi$. We first show a useful property: "for every $\tilde{g} \in \tilde{G}, b_1, b_2 \in G$, there is a path p in Γ from $\tilde{g}b_1$ to $\tilde{g}b_2$ such that $\nu_\chi(p) \geq \min\{\chi(\tilde{g}b_1), \chi(\tilde{g}b_2)\}$". Indeed, suppose without loss of generality that $\chi(b_1) \leq \chi(b_2)$. Then $\chi({b_1}^{-1}b_2) = -\chi(b_1) + \chi(b_2) \geq 0$ and so ${b_1}^{-1}b_2 \in \Gamma_{G\chi}$. By hypothesis, there is a path p' from 1 to ${b_1}^{-1}b_2$ in $\Gamma_{G\chi}$. The translated path $\tilde{p} = b_1 \cdot p'$ goes from b_1 to $b_1{b_1}^{-1}b_2 = b_2$ and $\nu_\chi(\tilde{p}) = \chi(b_1) + \nu_\chi(p') \geq \chi(b_1) = \min\{\chi(b_1), \chi(b_2)\}$. Now the path $p = \tilde{g} \cdot \tilde{p}$ connects $\tilde{g}b_1$ to $\tilde{g}b_2$ and $\nu_\chi(p) = \chi(\tilde{g}) + \nu_\chi(\tilde{p}) \geq \chi(\tilde{g}) + \min\{\chi(b_1), \chi(b_2)\} = \min\{\chi(\tilde{g}b_1), \chi(\tilde{g}b_2)\}$, which shows the property.

Now we show that $[\chi] \in \Sigma^1(\tilde{G})$. Let $\tilde{g} \in \Gamma_\chi$ and let us connect 1 to $\tilde{g}$ in Γ_χ. We can write $\tilde{g} = g_0t^{\epsilon_1}g_1...t^{\epsilon_n}g_n$ for $g_i \in G$ and $\epsilon_i = \pm 1$. Like in 3.31, we proceed by induction on n:

- ($n = 0$). This is the case $\tilde{g} = g_0 \in G$. Since $[\chi|_G] \in \Sigma^1(G)$, we can connect 1 to $\tilde{g}$ inside $\Gamma_{G\chi}$, in particular inside Γ_χ, as desired.

- (induction). Suppose the claim is true for $n - 1 \geq 0$ and let $\tilde{g} = g_0t^{\epsilon_1}g_1...t^{\epsilon_n}g_n$. Denote $g' = g_0t^{\epsilon_1}g_1...t^{\epsilon_{n-1}}g_{n-1}$ (so $\tilde{g} = g't^{\epsilon_n}g_n$). Like in 3.31, since $\chi|_A \neq 0$ choose $a \in A$ such that $\chi(g'a) = \chi(g') + \chi(a) \geq 0$ and $\chi(g'at^{\epsilon_n}) = \chi(g') + \chi(a) + \chi(t^{\epsilon_n}) \geq 0$. There are two cases. If $\epsilon_n = 1$, using the presentation for $\tilde{G}$ write

$$\tilde{g} = g'tg_n = g'taa^{-1}g_n = g'\theta(a)ta^{-1}g_n.$$

But $g'\theta(a) = g_0t^{\epsilon_1}g_1...t^{\epsilon_{n-1}}(g_{n-1}\theta(a))$ is an alternated product of length $n-1$ such that $\chi(g'\theta(a)) = \chi(g') + \chi(\theta(a)) = \chi(g') + \chi(a) = \chi(g'a) \geq 0$ (because $\chi(\theta(a)) = \chi(tat^{-1}) = \chi(t) + \chi(a) - \chi(t) = \chi(a)$), so by induction we can connect 1 to $g'\theta(a)$ by a path p_1 in Γ_χ. Now, connect $g'\theta(a)$ to $g'\theta(a)t$ by the obvious path $(g'\theta(a), t)$, which is in Γ_χ because $\chi(g'\theta(a)) \geq 0$ and $\chi(g'\theta(a)t) = \chi(g'at) \geq 0$. Finally, use the property of the first paragraph (for $g'\theta(a)t \in \tilde{G}$ and $b_1 = 1, b_2 = a^{-1}g_n \in G$) to connect $g'\theta(a)t$ to $g'\theta(a)ta^{-1}g_n = \tilde{g}$ by a path p_2 with $\nu_\chi(p_2) \geq \min\{\chi(g'\theta(a)t), \chi(\tilde{g})\} \geq 0$. The concatenation of these three paths is the desired path. In the case $\epsilon_n = -1$ we write

$$\tilde{g} = g't^{-1}g_n = g'aa^{-1}t^{-1}g_n = g'at^{-1}\theta(a^{-1})g_n.$$

Similarly, we use induction hypothesis to connect 1 to $g'a$ by p_1 in Γ_χ, then we connect $g'a$ to $g'at^{-1}$ by $(g'a, t^{-1})$ (which is in Γ_χ because $\chi(g'a) \geq 0$ and $\chi(g'at^{-1}) \geq 0$) and use the property in the first paragraph to connect $g'at^{-1}$ and $g'at^{-1}\theta(a^{-1})g_n = \tilde{g}$ in Γ_χ. Concatenating these paths we finish the proposition.

□

There are other two useful properties of Σ^1 concerning amalgamated products and HNN extensions. However, since their proof involves a theorem about reduced forms in both cases we will only enunciate them and use later. Both correspond to the two items of Proposition $C2.13$, at page 136 of [92].

Proposition 3.33. *Let $\tilde{G} = G *_A H$ be the amalgamated product of two finitely generated groups G, H and let $[\chi] \in S(\tilde{G})$. Suppose also that $G \supsetneq A \subsetneq H$, that is, A is a proper subgroup of both G and H. If $[\chi] \in \Sigma^1(\tilde{G})$ then $\chi|_A \neq 0$.*

Proposition 3.34. *Let $\tilde{G} = \left\langle X, t \mid R, tat^{-1}\ \theta(a)^{-1},\ a \in A \right\rangle$ be an HNN extension of a finitely generated group $G = \langle X \mid R \rangle$ and let $[\chi] \in S(\tilde{G})$. Suppose also that the inclusion $l : A \hookrightarrow G$ and the monomorphism $\theta : A \hookrightarrow G$ are both proper (not surjective). If $[\chi] \in \Sigma^1(\tilde{G})$ then $\chi|_A \neq 0$.*

3.3 Property R_∞ under Σ^1 invariant

The invariance under automorphisms of the Sigma invariant is the fundamental key to the implications of this chapter. The use of Σ^1 to detect twisted conjugacy and R_∞ properties was first noted by D. Gonçalves and D. Kochloukova in their paper [42] (2010), where they were able to use Σ^1 to show property R_∞, for example, for the generalized Thompson's groups $F_{n,0}$ and their finite direct products, as well as finding many classes of groups G in which one can guarantee the existence of a finite index subgroup $H \leq Aut(G)$ whose automorphisms have an infinite number of twisted conjugacy classes.

In this section, we present these theoretic results. We also give a simple proof for the (known) fact that this result of D. Gonçalves and D. Kochloukova is also compatible with finite direct products.

Definition 3.35. Let G be a finitely generated group. A character $[\chi] \in S(G)$ is called rational (or discrete) if $[\chi] = [\chi']$ for some homomorphism $\chi' : G \to \mathbb{R}$ such that $\text{im}(\chi') = \mathbb{Z} \subset \mathbb{R}$.

This definition is essential for the next

Lemma 3.36 ([42], Lemma 3.1)**.** *Let G be a finitely generated group such that*

$$\Sigma^1(G)^c = \{[\chi_1], ..., [\chi_m]\}$$

is finite, nonempty and contains only rational points, and let $\varphi \in Aut(G)$. Let $N = \cap_{i=1}^m \ker(\chi_i)$, $V = Hom(G/N, \mathbb{R})$ and denote by $\theta : V \to V$ the linear map induced by φ. Suppose $\{\overline{\chi_1}, ..., \overline{\chi_m}\}$ is a basis for V. Then θ permutes the $\overline{\chi_i}$, where each class representative χ_i is chosen so that the coordinates of the $\overline{\chi_i}$ are integers with greatest common divisor 1*.*

Demonstração. Let us first identify the objects. By Corollary 3.19, N is a characteristic subgroup of G. Furthermore, G/N is an abelian, finitely generated and torsion-free group, so we identify $G/N \simeq \mathbb{Z}^s = \langle \overline{g_1}, ..., \overline{g_s} \rangle$. By definition, all the χ_i vanish in N, so we have the induced homomorphisms $\overline{\chi_i} : G/N \to \mathbb{R}$ with $\overline{\chi_i}(\overline{g}) = \chi_i(g)$. Since $G/N \simeq \mathbb{Z}^s$ we have $V = Hom(G/N, \mathbb{R}) \simeq Hom(\mathbb{Z}^s, \mathbb{R}) \simeq \mathbb{R}^s$ a real vector space of dimension s, and then $s = m$ because $\{\overline{\chi_1}, ..., \overline{\chi_m}\}$ is a basis for V. The isomorphism $V \simeq \mathbb{R}^m$ is given by $\alpha \mapsto (\alpha(\overline{g_1}), ..., \alpha(\overline{g_m}))$ and we call this vector the "coordinates" of $\alpha \in V$. Since N is characteristic we also have the induced group automorphism $\overline{\varphi} : G/N \to G/N$ such that $\overline{\varphi}(\overline{g}) = \overline{\varphi(g)}$ which induces the linear transpose isomorphism $\theta = (\overline{\varphi})^T : V \to V$ with $\theta(\alpha) = \alpha \circ \overline{\varphi}$ (see the diagram).

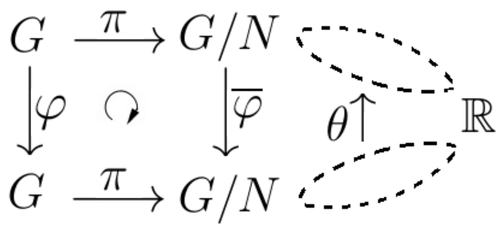


Figura 3.7: The dashed elipse is $V = Hom(G/N, \mathbb{R})$

.

Now we show the lemma. Since the $[\chi_i]$ are rational, up to multiplying each χ_i by a positive real number (which does not change the class $[\chi_i]$) we may suppose that $\text{im}(\chi_i) = \mathbb{Z}$ and therefore the coordinates $\{\chi_i(g_1), ..., \chi_i(g_m)\}$ are integer. Because of the property $\gcd(\frac{a}{\gcd(a,b)}, \frac{b}{\gcd(a,b)}) = 1$ we can divide χ_i, if necessary, by the greatest common divisor d of all its non-vanishing integer coordinates $\chi_i(g_j)$, and then the coordinates of χ_i/d will have greatest common divisor 1. Then, up to switching χ_i by χ_i/d we may suppose there are integers $k_{i1}, ..., k_{im}$ such that

$$k_{i1}\chi_i(g_1) + ... + k_{im}\chi_i(g_m) = 1, \ 1 \le i \le m. \tag{3.2}$$

Now, since $\varphi^* : S(G) \to S(G)$ is a homeomorphism and $\Sigma^1(G)^c$ is invariant under φ^* (Theorem 3.18), the restriction $\varphi^*|_{\Sigma^1(G)^c} : \Sigma^1(G)^c \to \Sigma^1(G)^c$ is a bijection. Then there is a permutation

$\pi \in S_m$ (symmetric group on m elements) such that $[\chi_i \circ \varphi] = \varphi^*[\chi_i] = [\chi_{\pi(i)}]$ for each i, or $\chi_i \circ \varphi = r_i \chi_{\pi(i)}$ for some $r_i > 0$. This implies

$$\theta(\overline{\chi_i}) = \overline{\chi_i} \circ \overline{\varphi} = \overline{\chi_i \circ \varphi} = \overline{r_i \chi_{\pi(i)}} = r_i \overline{\chi_{\pi(i)}}, \; 1 \leq i \leq m. \tag{3.3}$$

We just have to show that $r_i = 1$ for each i. We only know that $r_i > 0$ at the moment.

First let us show that $r_i \in \mathbb{Z}$ for each i. If we identify $G/N \simeq \mathbb{Z}^m$ then $\overline{\varphi} \in Aut(\mathbb{Z}^m) \simeq Gl_m(\mathbb{Z})$ can be identified with an invertible $m \times m$ integer matrix A. So, write

$$\overline{\varphi} = A = \begin{bmatrix} a_{11} & ... & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & ... & a_{mm} \end{bmatrix}.$$

Writing the $\overline{\chi_i}$ in coordinates we have

$$\theta(\chi_i(g_1), ..., \chi_i(g_m)) = \theta(\overline{\chi_i}) = r_i \overline{\chi_{\pi(i)}} = r_i(\chi_{\pi(i)}(g_1), ..., \chi_{\pi(i)}(g_m)). \tag{3.4}$$

On the other hand, we know from linear algebra that the matrix of a transpose map is exactly the transpose matrix of the map. Then we can identify $\theta = A^T$ and

$$\theta(\chi_i(g_1), ..., \chi_i(g_m)) = \begin{bmatrix} a_{11} & ... & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1m} & ... & a_{mm} \end{bmatrix} \begin{bmatrix} \chi_i(g_1) \\ \vdots \\ \chi_i(g_m) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^m a_{j1} \chi_i(g_j) \\ \vdots \\ \sum_{j=1}^m a_{jm} \chi_i(g_j) \end{bmatrix}. \tag{3.5}$$

Putting together 3.4 and 3.5 we have

$$(r_i \chi_{\pi(i)}(g_1), ..., r_i \chi_{\pi(i)}(g_m)) = \left(\sum_{j=1}^m a_{j1} \chi_i(g_j), ..., \sum_{j=1}^m a_{jm} \chi_i(g_j) \right) \tag{3.6}$$

and using 3.2 (for $\pi(i)$, not for i) and 3.6 we finally get

$$\begin{aligned} r_i &= r_i(k_{\pi(i)1} \chi_{\pi(i)}(g_1) + ... + k_{\pi(i)m} \chi_{\pi(i)}(g_m)) \\ &= k_{\pi(i)1}(r_i \chi_{\pi(i)}(g_1)) + ... + k_{\pi(i)m}(r_i \chi_{\pi(i)}(g_m)) \\ &= k_{\pi(i)1} \left(\sum_{j=1}^m a_{j1} \chi_i(g_j) \right) + ... + k_{\pi(i)m} \left(\sum_{j=1}^m a_{jm} \chi_i(g_j) \right) \\ &\in \mathbb{Z}, \end{aligned}$$

then $r_i \in \{1, 2, 3, ...\}$, as desired.

Now we show that $r_i = 1$ for each i. Let $k = m! = \text{card}(S_m)$. The order $o(\pi)$ of $\pi \in S_m$ must divide k, so write $k = o(\pi)n$ for some integer $n \geq 1$. We have $\pi^k = \pi^{o(\pi)n} = (\pi^{o(\pi)})^n = Id^n = Id$. We will successively apply θ in some fixed $\overline{\chi_i}$. We have $\theta(\overline{\chi_i}) = r_i \overline{\chi_{\pi(i)}}$, $\theta^2(\overline{\chi_i}) = \theta(r_i \overline{\chi_{\pi(i)}}) = r_i \theta(\overline{\chi_{\pi(i)}}) = r_i r_{\pi(i)} \overline{\chi_{\pi^2(i)}}$ and recursively we get that $\theta^k(\overline{\chi_i}) = r_i r_{\pi(i)} ... r_{\pi^{k-1}(i)} \overline{\chi_{\pi^k(i)}} = \lambda_i \overline{\chi_i}$, where $\lambda_i = r_i r_{\pi(i)} ... r_{\pi^{k-1}(i)}$ is a positive integer multiple of r_i. So it is enough to show that all the $\lambda_i = 1$. Because $\theta^k(\overline{\chi_i}) = \lambda_i \overline{\chi_i}$ for $1 \leq i \leq m$, the matrix of θ^k over the basis $\{\overline{\chi_1}, ..., \overline{\chi_m}\}$ is exactly

$$\begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_m \end{bmatrix}$$

whose determinant is $\lambda_1 ... \lambda_m$. On the other hand, the matrix of θ^k (over another basis) is also the matrix $(A^T)^k$ which is an integer matrix because A is. Since θ^k is an automorphism its determinant must be ± 1. But the determinant is independent from the basis chosen. Then $\lambda_1 ... \lambda_m = \pm 1$. Since all the λ_i are positive we have $\lambda_1 ... \lambda_m = 1$ and since they are integer we have $\lambda_i = 1$ for all i, as desired. □

The next two theorems are the main results of this section. Theorem 3.38 is more important for us, since it is easier to be applied.

Theorem 3.37 ([42], Theorem 3.2)**.** *Let G be a finitely generated group such that*

$$\Sigma^1(G)^c = \{[\chi_1], ..., [\chi_m]\}$$

is finite, nonempty and contains only rational points. Let $N = \cap_{i=1}^m \ker(\chi_i)$ and $V = Hom(G/N, \mathbb{R})$. If $\{\overline{\chi_1}, ..., \overline{\chi_m}\}$ is a basis for V, then G has property R_∞.

Demonstração. Let $\varphi \in Aut(G)$ and let us show that $R(\varphi) = \infty$. By the above lemma, if we denote by $\theta : V \to V$ the linear map induced by φ, then by choosing the correct class representatives χ_i we have $\theta(\overline{\chi_i}) = \overline{\chi_{\pi(i)}}, 1 \leq i \leq m$ for some permutation $\pi \in S_m$. Using linearity and then rearranging the terms we obtain

$$\theta(\overline{\chi_1} + ... + \overline{\chi_m}) = \theta(\overline{\chi_1}) + ... + \theta(\overline{\chi_m}) = \overline{\chi_{\pi(1)}} + ... + \overline{\chi_{\pi(m)}} = \overline{\chi_1} + ... + \overline{\chi_m}.$$

Since $\{\overline{\chi_1}, ..., \overline{\chi_m}\}$ is a basis for V the vector $\overline{\chi_1} + ... + \overline{\chi_m}$ is nontrivial and therefore by definition it is an eigenvector of θ with eigenvalue 1. Since the matrix of θ is the transpose matrix of $\overline{\varphi}$ and matrix transposition does not alter eigenvalues, $\overline{\varphi}$ also has eigenvalue 1. Then there is a nontrivial element $\overline{g} \in G/N$ such that $\overline{\varphi}(\overline{g}) = \overline{g}$, or $(\overline{\varphi} - Id)(\overline{g}) = 0$ (additive notation). Identifying $\overline{\varphi} - Id$ with its matrix, we have $\det(\overline{\varphi} - Id) = 0$ and therefore $R(\overline{\varphi}) = \infty$ by Example 1.3. Then applying item 1) of Lemma 1.7 to the diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/N \\ \downarrow{\scriptstyle\varphi} & \circlearrowright & \downarrow{\scriptstyle\overline{\varphi}} \\ G & \xrightarrow{\pi} & G/N \end{array}$$

we get $R(\varphi) = \infty$, as desired. □

The only reason we needed that $\{\overline{\chi_1}, ..., \overline{\chi_m}\}$ was a basis for V in Theorem 3.37 above is that we had to guarantee that the vector $\overline{\chi_1} + ... + \overline{\chi_m}$ was nontrivial. This will also happen if the points $[\chi_1], ..., [\chi_m]$ are all inside some open half-space of $S(G)$. Because of this, with a similar proof (see [42]) we can show the following

Theorem 3.38. *Let G be a finitely generated group such that*

$$\Sigma^1(G)^c = \{[\chi_1], ..., [\chi_m]\}$$

is finite, nonempty and contains only rational points. If $\{[\chi_1], ..., [\chi_m]\}$ is contained in an open half-space of $S(G)$, then G has property R_∞.

Corollary 3.39 ([42],Corollary 3.4)**.** *Let G be a finitely generated group such that $\Sigma^1(G)^c = \{[\chi_1], ..., [\chi_m]\}$ is nonempty, finite and consisting only of rational points. Then there is a finite index normal subgroup $H \lhd Aut(G)$ such that $R(\varphi) = \infty$ for each $\varphi \in H$.*

Demonstração. Let $N = \cap_{i=1}^m \ker(\chi_i)$ and $V = Hom(G/N, \mathbb{R})$. Consider the group $Aut(V)$ of linear automorphisms of V with the operation $*$ defined as $T * S = S \circ T$. Let

$$\Upsilon : Aut(G) \longrightarrow Aut(V)$$
$$\varphi \longmapsto \tilde{\varphi}$$

where $\tilde{\varphi}$ is the automorphism θ defined in Theorem 3.37 above. Υ is a group homomorphism. In fact, given $\varphi, \psi \in Aut(G)$ and $\alpha \in V$, we have

$$\widetilde{\varphi \circ \psi}(\alpha) = \alpha \circ \overline{\varphi \circ \psi} = \alpha \circ \overline{\varphi} \circ \overline{\psi} = \tilde{\varphi}(\alpha) \circ \overline{\psi} = \tilde{\psi}(\tilde{\varphi}(\alpha)) = (\tilde{\psi} \circ \tilde{\varphi})(\alpha),$$

So $\widetilde{\varphi \circ \psi} = \tilde{\psi} \circ \tilde{\varphi}$, and this implies

$$\Upsilon(\varphi \circ \psi) = \widetilde{\varphi \circ \psi} = \tilde{\psi} \circ \tilde{\varphi} = \Upsilon(\psi) \circ \Upsilon(\varphi) = \Upsilon(\varphi) * \Upsilon(\psi).$$

But, by Lemma 3.36, each $\tilde{\varphi}$ must permute the set $\{\overline{\chi_1}, ..., \overline{\chi_m}\}$, so there are only a finite number of possibilities and then $\text{im}(\Upsilon)$ is finite. Let $H = \ker(\Upsilon)$. By the Isomorphism Theorem we have $Aut(G)/H \simeq \text{im}(\Upsilon)$, so H is a finite index normal subgroup. Furthermore, by definition of H, if $\varphi \in H$ then $\tilde{\varphi} = Id$, which implies (as in Theorem 3.37) $\overline{\varphi} = Id$ and $R(\overline{\varphi}) = \infty$ because of Lemma 1.3. Again, by Lemma 1.7, $R(\varphi) = \infty$. So H is the desired group. □

We know that property R_∞ is not well-behaved for direct products in general. But when R_∞ comes from Theorem 3.37, then it works perfectly, as we will see.

Theorem 3.40. *Theorem 3.37 (and Theorem 3.38) are closed under direct products. That is, if $G_1, ..., G_n$ satisfy the hypothesis of Theorem* 3.37 *(or the ones of Theorem* 3.38*), then $G_1 \times ... \times G_n$ also satisfy the hypothesis of Theorem* 3.37 *(Theorem* 3.38*) and so it has R_∞.*

Demonstração. We will first show the cases of only two factors for both theorems. First, the case of Theorem 3.37. Let G, H be finitely generated groups and write

$$\Sigma^1(G)^c = \{[\chi_1], ..., [\chi_m]\},\ M = \cap_{j=1}^m \ker(\chi_j),\ V = Hom(G/M, \mathbb{R}) \text{ and}$$

$$\Sigma^1(H)^c = \{[\sigma_1], ..., [\sigma_n]\},\ N = \cap_{j=1}^n \ker(\sigma_j),\ W = Hom(G/N, \mathbb{R}).$$

To be more precise, take the $\mathbb{Z}$-basis $\overline{g_1}, ..., \overline{g_m}$ and $\overline{h_1}, ..., \overline{h_n}$ of the correspondent f.g. free abelian quotients $G/M \simeq \mathbb{Z}^m$ and $H/N \simeq \mathbb{Z}^n$. Then we have the $\mathbb{R}$-isomorphisms

$$V \xrightarrow{\simeq} \mathbb{R}^m$$
$$\varphi \longmapsto (\varphi(\overline{g_1}), ..., \varphi(\overline{g_m}))$$

and

$$W \xrightarrow{\simeq} \mathbb{R}^n$$
$$\varphi \longmapsto (\varphi(\overline{h_1}), ..., \varphi(\overline{h_n})).$$

So, the hypothesis that the $\overline{\chi_i}$ form a basis for V is equivalent to saying that the m vectors $(\overline{\chi_i}(\overline{g_1}), ..., \overline{\chi_i}(\overline{g_m})) = (\chi_i(g_1), ..., \chi_i(g_m))$ are independent in $\mathbb{R}^m$, or that the $m \times m$ matrix $A = [\chi_i(g_j)]_{ij}$ is an isomorphism. Similarly, saying that the $\overline{\sigma_i}$ form a basis for W is saying that the $n \times n$ matrix $B = [\sigma_i(h_j)]_{ij}$ is an isomorphism.

By Proposition 3.30,

$$\begin{aligned} \Sigma^1(G \times H)^c &= \pi^*(\Sigma^1(G)^c) \cup \pi'^*(\Sigma^1(H)^c), \\ &= \{[\chi_1 \circ \pi], ..., [\chi_m \circ \pi], [\sigma_1 \circ \pi'], ..., [\sigma_n \circ \pi']\}, \end{aligned}$$

where $\pi : G \times H \to G$ and $\pi' : G \times H \to H$ are the natural projections. First note that these $m+n$ points are distinct: in fact, the $[\chi_i \circ \pi]$ are pairwise distinct (because π^* is injective) as well as the $[\sigma_i \circ \pi']$. Since no $\chi_i \circ \pi$ vanish in $G \times \{1\}$ and all the $\sigma_j \circ \pi'$ vanishes there, and the opposite happens in $\{1\} \times H$, no $\chi_i \circ \pi$ can be a multiple of any of the $\sigma_j \circ \pi'$, which completes the argument. Note also that they are rational points, for $\mathrm{im}(\chi_i \circ \pi) = \mathrm{im}(\chi_i)$ is rational, as well as $\mathrm{im}(\sigma_i \circ \pi') = \mathrm{im}(\sigma_i)$.

We just have then to show that $\{\overline{\chi_1 \circ \pi}, ..., \overline{\chi_m \circ \pi}, \overline{\sigma_1 \circ \pi'}, ..., \overline{\sigma_n \circ \pi'}\}$ is a basis for $Z = Hom((G \times H)/L, \mathbb{R})$ where $L = (\cap_{i=1}^m \ker(\chi_i \circ \pi)) \cap (\cap_{i=1}^n \ker(\sigma_i \circ \pi'))$. But

$$\ker(\chi_i \circ \pi) = \ker(\chi_i) \times H, \ \ker(\sigma_i \circ \pi') = G \times \ker(\sigma_i)$$

and so

$$L = (\cap_{i=1}^m (\ker(\chi_i) \times H)) \cap (\cap_{i=1}^n (G \times \ker(\sigma_i))) = (\cap_{i=1}^m \ker(\chi_i)) \times (\cap_{i=1}^n \ker(\sigma_i)) = M \times N.$$

Then we have the isomorphisms

$$\begin{aligned}
(G\times H)/L &\xrightarrow{\simeq} G/M\times H/N \xrightarrow{\simeq} \mathbb{Z}^{m+n}\\
\overline{(g_i,1)} &\longmapsto (\overline{g_i},\overline{1})\\
\overline{(1,h_i)} &\longmapsto (\overline{1},\overline{h_i})
\end{aligned}$$

And since the $\overline{g_i}$ form a basis of G/M and the $\overline{h_i}$ form a basis of H/N we can take the $\overline{(g_i,1)}$ together with the $\overline{(1,h_i)}$ as a $\mathbb{Z}$-basis of the f.g. free-abelian group $(G\times H)/L$, obtaining the isomorphism

$$\begin{aligned}
&Z \xrightarrow{T} \mathbb{R}^{m+n}\\
&\varphi \longmapsto (\varphi\overline{(g_1,1)},...,\varphi\overline{(g_m,1)},\varphi\overline{(1,h_1)},...,\varphi\overline{(1,h_n)}).
\end{aligned}$$

But

$$\begin{aligned}
T(\overline{\chi_i\circ\pi}) &= (\overline{\chi_i\circ\pi}\overline{(g_1,1)},...,\overline{\chi_i\circ\pi}\overline{(g_m,1)},\overline{\chi_i\circ\pi}\overline{(1,h_1)},...,\overline{\chi_i\circ\pi}\overline{(1,h_n)})\\
&= (\chi_i\circ\pi(g_1,1),...,\chi_i\circ\pi(g_m,1),\chi_i\circ\pi(1,h_1),...,\chi_i\circ\pi(1,h_n))\\
&= (\chi_i(g_1),...,\chi_i(g_m),0,...,0)
\end{aligned}$$

and, similarly,

$$T(\overline{\sigma_i\circ\pi'}) = (0,...,0,\sigma_i(h_1),...,\sigma_i(h_n)),$$

so the vectors $\{\overline{\chi_1\circ\pi},...,\overline{\chi_m\circ\pi},\overline{\sigma_1\circ\pi'},...,\overline{\sigma_n\circ\pi'}\}$ in Z correspond to the $(m+n)\times(m+n)$-matrix

$$\begin{bmatrix} A_{m\times m} & 0\\ 0 & B_{n\times n}\end{bmatrix}$$

which is an isomorphism because A and B are isomorphisms. Then they form a basis for Z, as desired.

Now, the case of Theorem 3.37. Suppose that both

$$\Sigma^1(G)^c = \{[\chi_1],...,[\chi_m]\},\ \text{and}\ \Sigma^1(H)^c = \{[\sigma_1],...,[\sigma_n]\}$$

consist of rational points contained in open hemispheres H_v of $S(G)$ and H_w of $S(H)$, respectively. As we already know, $\Sigma^1(G\times H)^c$ is nonempty, finite and of rational points. To see that it is in an open hemisphere of $S(G\times H)$, we identify $S(G)=S^{n-1}$, $S(H)=S^{m-1}$ and $S(G\times H)=S^{n+m-1}$. Under this identification, the maps π^* and π'^* assume the form $\pi^*(x)=(x,0)$ and $\pi'^*(y)=(0,y)$ and preserve inner products, because $\langle\pi^*(x),\pi^*(z)\rangle = \langle(x,0),(z,0)\rangle = \langle x,z\rangle$, similar for π'^*. So, if $\Sigma^1(G)^c\subset H_v$ and $\Sigma^1(H)^c\subset H_w$, we claim that $\Sigma^1(G\times H)^c\subset H_{(v,w)}$ in $S(G\times H)$. Indeed, $\langle\pi^*(x),(v,w)\rangle = \langle(x,0),(v,w)\rangle = \langle x,v\rangle > 0$ for all $x\in\Sigma^1(G)^c$ and similarly $\langle\pi'^*(y),(v,w)\rangle = \langle(0,y),(v,w)\rangle = \langle y,w\rangle > 0$ for all $y\in\Sigma^1(H)^c$. Since

$\Sigma^1(G \times H)^c = \pi^*(\Sigma^1(G)^c) \cup \pi'^*(\Sigma^1(H)^c)$, this shows the claim and finishes the two factor case.

The general case follows by trivial induction: suppose Theorem 3.37 (Theorem 3.38) is valid for n factors and let $G_1, ..., G_{n+1}$ satisfy its hypothesis. Let $G = G_1$ and $H = G_2 \times ... \times G_{n+1}$. By hypothesis, G and H satisfy Theorem 3.37 (Theorem 3.38). Then, by the previous case, $G_1 \times ... \times G_{n+1} = G \times H$ satisfy Theorem 3.37 (Theorem 3.38), as desired. □

Open questions

1) Could some other equivalent definitions (or characterizations) of $\Sigma^1(G)$ be used in the investigation of property R_∞? In this thesis, we have dealt with Definition 3.7 for its simplicity. But, for example, we have Brown's characterization in [17] in terms of the possible existence of "non-trivial and abelian" actions of G on $\mathbb{R}$-trees. This would correspond, in the language we used in Sections 2.3 and 2.4, to fixed-end actions with no invariant lines. We do not want to get into details here, but it seems like this definition could have connections with hyperbolic or relatively hyperbolic groups G. The reason is that there are natural constructions of actions of G on $\mathbb{R}$-trees, as we will see in chapters 8 and 9.

2) Could the higher invariants Σ^n, $n \geq 1$, be computed and used to determine property R_∞? The reader may read [7] to know the definitions. By their apparent complexity and the lack of literature in the computation of these invariants, we decided to restrict our attention only to Σ^1. However, since they are known to be invariant under automorphisms, Theorem 3.37 would equally work for them.

Capítulo 4

Invariant convex polytopes and property R_∞

In this chapter we go in the same kind of direction of the last section of Chapter 3: we show that the existence of some invariant closed convex polytopes in $S(G)$ can also guarantee property R_∞ of a finitely generated group G (Theorem 4.28). The intuitive idea that gave rise to this result is that the induced homeomorphism $\varphi^* : S(G) \to S(G)$ of an automorphism $\varphi : G \to G$ seemed like to map geodesics to geodesics (not linearly). So, we conjectured that if φ^* fixed a polytope, then it should map vertices to vertices. This turned out to be true.

Like we already said, the key fact to guarantee property R_∞ by the previous chapter is that Σ^1 is an invariant subset of $S(G)$; so we start this chapter by rewriting Theorem 3.38 by replacing $(\Sigma^1)^c$ with an arbitrary invariant subset of $S(G)$. With the same proof we get

Theorem 4.1. *Let G be a finitely generated group. Suppose there is a nonempty and finite subset $A \subset S(G)$ which is invariant in $S(G)$, consisting only of rational points and contained in an open half-space of $S(G)$. Then G has property R_∞.*

In the rest of the chapter we will deal only with convex polytopes, although sometimes we will call them simply by polytopes for simplification. Most of the results obtained here are surely false for the not-convex ones.

4.1 Convex polytopes in Euclidean spaces

Since polytopes are not among the main goals of the project, our approach here is minimal. For more details about convexity and convex polytopes in $\mathbb{R}^d$, see [52], which was the basic literature for this section.

Definition 4.2. Let $\mathbb{R}^d$, $d \geq 1$ be the d-dimensional euclidean space. We say that a subset $K \in \mathbb{R}^d$ is convex if every straight path between two points of K is contained in K. In other words, K is convex if for all $P, Q \in K$ and $t \in [0, 1]$ we have $(1 - t)P + tQ \in K$.

Definition 4.3. For any subset $A \in \mathbb{R}^d$, the convex hull of A in $\mathbb{R}^d$ is the smallest convex subset of $\mathbb{R}^d$ which contains A. We denote it by $conv(A)$.

Observation 4.4. It is easy to see that $conv(A)$ is also the intersection of all the convex subsets of $\mathbb{R}^d$ containing A. Another description (see [52]) is

$$conv(A) = \{t_1a_1 + ... + t_na_n \mid n \geq 1,\ a_i \in A,\ t_i \geq 0,\ \sum t_i = 1\}.$$

In particular, if $A = \{a_1, ..., a_n\}$ is finite, then we denote $conv(A)$ by $conv(a_1, ..., a_n)$ and we have

$$conv(a_1, ..., a_n) = \{t_1a_1 + ... + t_na_n \mid t_i \geq 0,\ \sum t_i = 1\}.$$

In the special case of only two points a_1, a_2, since $t_1 + t_2 = 1$ we have $t_1 = 1 - t_2$ and so

$$conv(a_1, a_2) = \{(1 - t)a_1 + ta_2 \mid t \in [0, 1]\}.$$

Definition 4.5. A closed halfspace in $\mathbb{R}^d$ is a set of the form $H = \{x \in \mathbb{R}^d \mid \langle x, v\rangle \geq \beta\}$ for some $v \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$. Here the dot product of two vectors $x = (x_1, ..., x_d)$ and $v = (\alpha_1, ..., \alpha_d)$ is $\langle x, v\rangle = \alpha_1x_1 + ... + \alpha_dx_d$. It follows that the set H is characterized by the $x \in \mathbb{R}^d$ satisfying the equation

$$\alpha_1x_1 + ... + \alpha_dx_d \geq \beta.$$

The boundary of H in $\mathbb{R}^d$ is

$$\partial H = \{x \in \mathbb{R}^d \mid \alpha_1x_1 + ... + \alpha_dx_d = \beta\}.$$

Definition 4.6. A convex polytope K in $\mathbb{R}^d$ is a finite intersection $K = \cap_{i=1}^n H_i$ of closed halfspaces of $\mathbb{R}^d$ which is also a bounded subset in $\mathbb{R}^d$. It is always a convex and compact subspace of $\mathbb{R}^d$. Since it is and also a submanifold of $\mathbb{R}^d$ (with boundary), there is a well defined dimension $dim(K)$. We say that K is a r-polytope if $dim(K) = r$. From now on, we may omit the word "convex" since we are dealing only with convex polytopes.

Definition 4.7. If $K = \cap_{i=1}^n H_i$ is a convex polytope, we say that the family $\{H_1, ..., H_n\}$ is irredundant if for every $1 \leq i \leq n$ we have $K \subsetneq \cap_{j\neq i} H_j$, that is, if K cannot be written as the intersection of a proper subfamily of the H_i. If K is a convex d-polytope in $\mathbb{R}^d$ and $\{H_1, ..., H_n\}$ is irredundant, the facets of K are the subsets $F_i = (\partial H_i) \cap K$.

Observation 4.8. If the family $\{H_1, ..., H_n\}$ is not irredundant then by a simple recursive argument we can write $K = \cap_{j=1}^k H_{i_j}$ where $\{H_{i_1}, ..., H_{i_k}\}$ is a proper irredundant subfamily of $\{H_1, ..., H_n\}$. So, from now on, we will always suppose that the family of closed halfspaces defining K is irredundant.

Observation 4.9. It follows from the definition that a convex polytope K is characterized by the points $x \in \mathbb{R}^d$ such that

$$\begin{cases} \alpha_{11}x_1 + ... + \alpha_{1d}x_d \geq \beta_1, \\ \qquad\vdots \\ \alpha_{n1}x_1 + ... + \alpha_{nd}x_d \geq \beta_n. \end{cases}$$

for some $\alpha_{ij}, \beta_i \in \mathbb{R}$. We can abbreviate this system by the expression "$f_i(x) \geq \beta_i,\ i = 1, ..., n$", where the $f_i : \mathbb{R}^d \to \mathbb{R}$ are linear maps given by $f_i(x) = \alpha_{i1}x_1 + ... + \alpha_{id}x_d$. With this notation, a point $x \in K$ is in a facet F_i if and only if $f_i(x) = \beta_i$. It is then easy to show that the boundary of K in $\mathbb{R}^d$ is

$$\partial K = F_1 \cup ... \cup F_n$$

and, if K is a d-polytope (that is, with maximal dimension), then it has nonempty interior given by

$$int(K) = \{x \in \mathbb{R}^d \mid f_i(x) > \beta_i,\ i = 1, ..., n\}.$$

Of course, $K = \partial K \sqcup int(K)$.

The main properties we need to know about d-polytopes in $\mathbb{R}^d$ are below.

Lemma 4.10. *Let K be a d-polytope in $\mathbb{R}^d$ and $P, Q \in K$. If the straight path $\{(1-t)P+tQ \mid t \in [0,1]\}$ from P to Q is contained in ∂K then there is a facet F_i containing P and Q.*

Demonstração. Let K be characterized by the system "$f_i(x) \geq \beta_i,\ i = 1, ..., n$". Suppose by contradiction that there is no facet F_i containing both P and Q. This means that for every i, there are only three possibilities:

- $f_i(P) = \beta_i$ and $f_i(Q) > \beta_i$, or
- $f_i(P) > \beta_i$ and $f_i(Q) = \beta_i$, or
- $f_i(P) > \beta_i$ and $f_i(Q) > \beta_i$.

Let $R = (P + Q)/2$. Then R is contained in the straight path from P to Q. But for all i, since f_i is linear we have

$$f_i(R) = f_i\left(\frac{P+Q}{2}\right) = \frac{f_i(P) + f_i(Q)}{2}$$

and in any of the three cases above is easy to see that $f_i(R) > \beta_i$ and so $R \in int(K)$. Therefore, the straight path from P to Q is not contained in ∂K, contradiction. □

Lemma 4.11. *If K is a d-polytope in $\mathbb{R}^d$, every facet of K contains a point which does not belong to any other facet.*

Demonstração. Let K be characterized by the system "$f_j(x) \geq \beta_j,\ j = 1, ..., n$" and let F_i be one of its facets. Since K is d-dimensional, let P be a point of $int(K)$, with $f_j(P) > \beta_j$ for all j. Also, since the H_j are irredundant, we have $K \subsetneq \cap_{j \neq i} H_j$, so let Q be a point in $(\cap_{j \neq i} H_j) - H_i$, that is, $f_j(Q) \geq \beta_j$ for all $j \neq i$ and $f_i(Q) < \beta_i$. Let $\gamma_{P,Q}(t) = (1-t)P + tQ$ be the straight path from P to Q and consider the continuous composition $f_i \circ \gamma_{P,Q} : [0,1] \to \mathbb{R}$. Since $(f_i \circ \gamma_{P,Q})(0) = f_i(P) > \beta_i$ and $(f_i \circ \gamma_{P,Q})(1) = f_i(Q) < \beta_i$, by the Intermediate Value Theorem there is $t_0 \in (0,1)$ such that $(f_i \circ \gamma_{P,Q})(t_0) = \beta_i$. Then the point $Z = \gamma_{P,Q}(t_0)$ is such that $f_i(Z) = \beta_i$ and if $j \neq i$ we have $f_j(Z) = f_j((1-t_0)P + t_0 Q) = (1-t_0) f_j(P) + t_0 f_j(Q) > \beta_j$ (since $f_j(P) > \beta_j$ and $f_j(Q) \geq \beta_j$). This means that $Z \in K$ and F_i is the only facet of K containing Z. □

Lemma 4.12. *Any d-polytope in $\mathbb{R}^d$ contains at least $d + 1$ facets.*

Demonstração. We will first show that a finite intersection of only d closed halfspaces of $\mathbb{R}^d$ cannot be bounded and therefore cannot be a polytope. Let $K = \cap_{i=1}^{d} H_i$ be defined by the system

$$\begin{cases} \alpha_{11}x_1 + ... + \alpha_{1d}x_d \geq \beta_1, \\ \quad\quad \vdots \\ \alpha_{d1}x_1 + ... + \alpha_{dd}x_d \geq \beta_d \end{cases}$$

and let $P = (p_1, ..., p_d) \in K$, so we have the equations $f_i(P) = \gamma_i \geq \beta_i$ for $i = 1, ..., d$. Since the H_i are irredundant, the system of equations

$$\begin{cases} \alpha_{11}x_1 + ... + \alpha_{1d}x_d = \gamma_1, \\ \quad\quad \vdots \\ \alpha_{d-1,1}x_1 + ... + \alpha_{d-1,d}x_d = \gamma_{d-1} \end{cases}$$

for an arbitrary point $x = (x_1, ..., x_d) \in \mathbb{R}^d$ is equivalent to x having only one free coordinate x_i and all the other ones linearly dependent of x_i, that is, $x_j = L_j(x_i)$ for affine functions $L_j : \mathbb{R} \to \mathbb{R}$. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ given by

$$\gamma(t) = (L_1(t), ..., L_{i-1}(t), t, L_{i+1}(t), ..., L_d(t)).$$

Since the L_j are affine maps, γ is a straight line. By the above paragraph we also know that $\gamma(t)$ satisfies the system of $d-1$ equations above for all t. So, to see if a point $\gamma(t)$ is in K we only have to see whether $f_d(\gamma(t)) \geq \beta_d$. Because of this, let us analyze the composition $f_d \circ \gamma$, which is also an affine map: we have $f_d(\gamma(p_i)) = f_d(P) = \gamma_d \geq \beta_d$. If the derivative $(f_d \circ \gamma)'(p_i) \geq 0$, then it is a non-descending map and for every $t \geq p_i$ we still have $f_d(\gamma(t)) \geq f_d(\gamma(p_i)) = \gamma_d \geq \beta_d$, which implies the entire semi-straight line $\gamma[p_i, \infty)$ is inside K. On the other hand, if $(f_d \circ \gamma)'(p_i) < 0$, then it is a descending map and for every $t < p_i$ we have $f_d(\gamma(t)) > f_d(\gamma(p_i)) = \gamma_d \geq \beta_d$, which implies the entire semi-straight line $\gamma(-\infty, p_i]$ is inside K. So in any way K is not bounded and therefore is not a polytope.

Finally, if $r < d$, a finite intersection of r closed halfspaces contains a finite intersection of d closed halfspaces and therefore also cannot be bounded. This completes the lemma. □

The vertices of a polytope should be the "least interior" points, which are not even in the 1-dimensional interiors of the polytope. Then, following [52], we define:

Definition 4.13. A point P of a d-polytope $K \subset \mathbb{R}^d$ is a vertex of K if P is not in the interior of any straight path contained in K. In other words, P is a vertex of K if for any straight line $\gamma : \mathbb{R} \to \mathbb{R}^d$ containing P (say, $\gamma(t_0) = P$), there is no $\epsilon > 0$ such that $\gamma(t_0 - \epsilon, t_o + \epsilon) \subset K$. The set of vertices of K is denoted by $V(K)$.

Note that, since every straight line can be reparametrized by any translation of $\mathbb{R}$, it is enough to suppose $t_0 = 0$ in the definition above. Now we characterize the vertices of K in a way that will be useful for us:

Lemma 4.14. *Let $K \subset \mathbb{R}^d$ be a d-polytope and $P \in K$. Then P is a vertex of K if, and only if, P belongs to (at least) d distinct facets of K.*

Demonstração. Let K be defined by the system "$f_i(x) \geq \beta_i,\ i = 1, ..., n$" as described above (we already know that $n \geq d + 1$) and let $P = (p_1, ..., p_d)$.

($\Rightarrow$) Suppose P is only in $r \leq d-1$ distinct facets of K, say, $f_i(P) = \beta_i$ for $i = 1, ..., r$ and $f_i(P) > \beta_i$ for $i = r+1, ..., n$ (after reordering the f_i if necessary). Since the H_i are irredundant, the system of r equations above "$f_i(x) = \beta_i$ for $i = 1, ..., r$" for $x = (x_1, ..., x_d) \in \mathbb{R}^d$ is equivalent to having $d-r$ free coordinates and the other r coordinates of x being affinely dependent of them, say,

$$f_i(x) = \beta_i \text{ for } 1 \leq i \leq r \Leftrightarrow x = (x_1, ..., x_{d-r}, L_{d-r+1}(x_1, ..., x_{d-r}), ..., L_d(x_1, ..., x_{d-r}))$$

(after reordering the coordinates if necessary), where the $L_j : \mathbb{R}^{d-r} \to \mathbb{R}$ are affine maps. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ with

$$\gamma(t) = (t, p_2, ..., p_{d-r}, L_{d-r+1}(t, p_2, ..., p_{d-r}), ..., L_d(t, p_2..., p_{d-r})).$$

Since the L_j are affine maps, this is a straight line passing through $\gamma(p_1) = P$ and by construction we know that for all t, $\gamma(t)$ satisfies $f_i(\gamma(t)) = \beta_i$, $1 \leq i \leq r$. Now, since $f_i(P) > \beta_i$ for $i = r+1, ..., n$ and these f_i are continuous, there is an open ball $B(P, \delta)$ such that $f_i(x) > \beta_i$ for $i = r+1, ..., n$ and for every $x \in B(P, \delta)$. By the continuity of γ, let $\epsilon > 0$ be such that $\gamma(p_1 - \epsilon, p_1 + \epsilon) \in B(P, \delta)$. Then by construction we have $\gamma(p_1 - \epsilon, p_1 + \epsilon) \subset K$ and therefore P is not a vertex of K.

($\Leftarrow$) Suppose P is in d distinct facets of K, that is, $f_i(P) = \beta_i$ for $i = 1, ..., d$ and $f_i(P) \geq \beta_i$ for $i = d+1, ..., n$ (after reordering the f_i if necessary). Since the H_i are irredundant, the system of d equations "$f_i(x) = \beta_i$ for $i = 1, ..., d$" in $\mathbb{R}^d$ has unique solution in $\mathbb{R}^d$, which is P. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a straight line containing P, say, $\gamma(t) = P + tv$ for $v \in \mathbb{R}^d - \{0\}$ and consider the affine composition maps $f_i \circ \gamma : \mathbb{R} \to \mathbb{R}$. There must be $1 \leq i \leq d$ such that $(f_i \circ \gamma)'(0) \neq 0$; otherwise, all the maps $f_i \circ \gamma$ would be constant with value $f_i \circ \gamma(0) = \beta_i$ and therefore all $\gamma(t)$ would be a solution of the system of equations above, a contradiction. Let $1 \leq i \leq d$ such that $(f_i \circ \gamma)'(0) \neq 0$. If $(f_i \circ \gamma)'(0) > 0$, then $f_i \circ \gamma$ is an ascending map and $f_i \circ \gamma(-\epsilon) < f_i \circ \gamma(0) = \beta_i$ for all $\epsilon > 0$, which implies $\gamma(-\epsilon, \epsilon) \not\subset K$ for all $\epsilon > 0$. Similarly, if $(f_i \circ \gamma)'(0) < 0$, then $f_i \circ \gamma$ is an descending map and $f_i \circ \gamma(\epsilon) < f_i \circ \gamma(0) = \beta_i$ for all $\epsilon > 0$, which also implies $\gamma(-\epsilon, \epsilon) \not\subset K$ for all $\epsilon > 0$. This shows that P is a vertex of K. □

Corollary 4.15. *Every d-polytope of $\mathbb{R}^d$ has a finite number of vertices.*

Demonstração. We showed above that $V(K)$ is contained in the set of all possible intersections of d (or more) facets of K. But in the part "($\Leftarrow$)" we showed that the systems of d equations "$f_i(x) = \beta_i$ for $i = 1, ..., d$" which characterizes a vertex in K have unique solutions, so all these d-intersections give rise to at most one vertex. Since there are a finite number of facets, $V(K)$ must be finite. □

Proposition 4.16. *Let $K \subset \mathbb{R}^d$ be a d-polytope and $f : K \to K$ a homeomorphism. If for any $P, Q \in K$, $f(conv(P,Q)) = conv(f(P), f(Q))$, then f maps vertices to vertices, that is, $f(V(K)) = V(K)$.*

Demonstração. Let K be defined by the system "$f_i(x) \geq \beta_i$, $i = 1, ..., n$". Since f is a homeomorphism, it must map the boundary ∂K to itself, and so $f(F_1 \cup ... \cup F_n) = F_1 \cup ... \cup F_n$. Suppose by contradiction that a vertex $P \in K$ is mapped to a non-vertex point $f(P) \in K$

(but obviously $P, f(P) \in \partial K$). If a point $Q \in K$ belongs to any facet of K containing P (say, F), then $conv(Q,P) \subset F$, since every facet is easily seen to be convex. Then $conv(f(Q), f(P)) \subset f(F) \subset \partial K$ by hypothesis, which implies that the whole straight path joining $f(Q)$ and $f(P)$ is contained in the boundary ∂K. By Lemma 4.10, $f(Q)$ must be in a facet which also contains $f(P)$. This argument shows that all the facets containing P must be mapped into the facets containing $f(P)$. By Lemma 4.14, there are at least d facets containing P, say, $F_1, ..., F_d$ and at most $d-1$ facets containing $f(P)$, say, $F_{i_1}, ..., F_{i_{d-1}}$. Then we have

$$f(F_1 \cup ... \cup F_d) \subset F_{i_1} \cup ... \cup F_{i_{d-1}}.$$

We continue: since there are at least $d+1$ facets, let $Z \in \partial K$ be a point outside $F_{i_1} \cup ... \cup F_{i_{d-1}}$, say, $Z \in F_{i_d}$. By Lemma 4.11, we can suppose F_{i_d} is the only facet containing Z. Since f is surjective, $Z = f(W)$, so W must be a boundary point outside $F_1 \cup ... \cup F_d$, say, $W \in F_{d+1}$. Using again 4.14, we can show that all the facets containing W must be mapped into the facets containing Z (only the facet F_{i_d}). In particular, $f(F_{d+1}) \subset F_{i_d}$ and so

$$f(F_1 \cup ... \cup F_{d+1}) \subset F_{i_1} \cup ... \cup F_{i_d}.$$

If $d+1 = n$, we stop. If not, we keep doing this same argument and adding facets to both sides of the above expression. After finite steps we will have

$$f(F_1 \cup ... \cup F_n) \subset F_{i_1} \cup ... \cup F_{i_{n-1}},$$

so $f(\partial K) \subsetneq \partial K$, contradiction. □

4.2 Convex polytopes in Euclidean spheres and induced homeomorphisms

Let G be a finitely generated group whose abelianized group G^{ab} has free rank n. Consider the homeomorphism

$$\begin{aligned} H : S(G) &\longrightarrow S^{n-1} \\ [\chi] &\longmapsto \frac{(\chi(x_1), ..., \chi(x_n))}{\|(\chi(x_1), ..., \chi(x_n))\|}, \end{aligned}$$

where the $x_i \in G$ are the free-abelian generators of G^{ab}. Given $\varphi \in Aut(G)$, we have the induced homeomorphism $\varphi^* : S(G) \to S(G)$ with $\varphi^*[\chi] = [\chi \circ \varphi]$. Let $\varphi^S : S^{n-1} \to S^{n-1}$ be the composition $\varphi^S = H \circ \varphi^* \circ H^{-1}$.

By the definition above, it is easy to see that a subset $K \subset S(G)$ is invariant in $S(G)$ (that is, invariant under φ^* for all $\varphi \in Aut(G)$) if and only if $H(K)$ is invariant under all φ^S. Before defining a polytope in $S(G)$, let us show a useful property of φ^S:

Definition 4.17. Let $A \subset S^n \subset \mathbb{R}^{n+1}$ and suppose A is contained in an open hemisphere of S^n,

$$\begin{array}{ccc} S(G) & \xrightarrow{H} & S^{n-1} \\ \varphi^* \downarrow & \circlearrowright & \downarrow \varphi^S \\ S(G) & \xrightarrow{H} & S^{n-1} \end{array}$$

say, $A \subset O(v) = \{x \in S^n \mid \langle x, v\rangle > 0\}$ for some $v \in \mathbb{R}^{n+1} - \{0\}$. We say that A is (spherically) convex if for any $a_1, a_2 \in A$, the geodesic path from a_1 to a_2 is contained in A, that is, if $\gamma_{a_1,a_2}(t) = \frac{(1-t)a_1+ta_2}{\|(1-t)a_1+ta_2\|} \in A$ for all $t \in [0,1]$.

Definition 4.18. Let $A \subset S^n \subset \mathbb{R}^{n+1}$ and suppose $A \subset O(v)$ as above. The convex hull of A in S^n is the smallest convex subset of $O(v)$ which contains A. We denote this set by $conv(A)$. If $A = \{a_1, ..., a_m\}$ is finite, we denote $conv(A)$ by $conv(a_1, ..., a_m)$.

Proposition 4.19. *Let $A \subset S^n \subset \mathbb{R}^{n+1}$ and suppose $A \subset O(v)$ as above. Then*

$$conv(A) = \left\{ \frac{t_1a_1 + ... + t_ma_m}{\|t_1a_1 + ... + t_ma_m\|} \mid m \geq 1, a_i \in A, (t_1, ..., t_m) \in [0, \infty)^m - \{0\} \right\}.$$

Demonstração. It is straightforward to verify that the set on the right is spherically convex and contains A, so $(\subset)$ is valid. To show $(\supset)$, let C be any convex set in $O(v)$ containing A and let us show that C must contain the set on the right. We will show this by induction on the number of nonvanishing vectors in the elements $\frac{t_1a_1+...+t_ma_m}{\|t_1a_1+...+t_ma_m\|}$. For $m = 1$, $\frac{t_1a_1}{\|t_1a_1\|} = \frac{t_1a_1}{t_1\|a_1\|} = \frac{a_1}{\|a_1\|} = a_1 \in C$, because $A \subset C$ by hypothesis. Assume that we showed that any element of the form $\frac{t_1a_1+...+t_ma_m}{\|t_1a_1+...+t_ma_m\|}$ (with the $t_i > 0$) is in C and consider and element $\frac{t_1a_1+...+t_{m+1}a_{m+1}}{\|t_1a_1+...+t_{m+1}a_{m+1}\|}$ with the $t_i > 0$. By hypothesis, the element $\frac{t_1a_1+...+t_ma_m}{\|t_1a_1+...+t_ma_m\|}$ is an element $c \in C$, so we denote $\lambda = \|t_1a_1 + ... + t_ma_m\|$ and write $t_1a_1 + ... + t_ma_m = \lambda c$. Then

$$\frac{t_1a_1 + ... + t_{m+1}a_{m+1}}{\|t_1a_1 + ... + t_{m+1}a_{m+1}\|} = \frac{\lambda c + t_{m+1}a_{m+1}}{\|\lambda c + t_{m+1}a_{m+1}\|}.$$

Now, let $\lambda' = \frac{1}{\lambda+t_{m+1}} > 0$. Then $\lambda'\lambda = 1 - \lambda' t_{m+1}$ and we get

$$\frac{\lambda c + t_{m+1}a_{m+1}}{\|\lambda c + t_{m+1}a_{m+1}\|} = \frac{\lambda'\lambda c + \lambda' t_{m+1}a_{m+1}}{\|\lambda'\lambda c + \lambda' t_{m+1}a_{m+1}\|} = \frac{(1 - \lambda' t_{m+1})c + \lambda' t_{m+1}a_{m+1}}{\|(1 - \lambda' t_{m+1})c + \lambda' t_{m+1}a_{m+1}\|} \in C,$$

since $c, a_{m+1} \in C$ and C is convex. This completes the proof. □

The t_i above are called the coefficients of $P = \frac{t_1a_1+...+t_ma_m}{\|t_1a_1+...+t_ma_m\|}$. As in Euclidean spaces, the following happens:

Lemma 4.20. *Let $O(v) \subset S^n$ as above. The image of the geodesic path $\gamma_{P,Q}$ joining two points of $O(v)$ is the convex hull of $\{P, Q\}$.*

Demonstração. By definition,

$$\mathrm{im}(\gamma_{P,Q}) = \left\{ \frac{(1-t)P + tQ}{\|(1-t)P + tQ\|} \mid t \in [0, 1] \right\}$$

and

$$conv(P,Q) = \left\{ \frac{t_1P + t_2Q}{\|t_1P + t_2Q\|} \mid (t_1,t_2) \in [0,\infty)^2 - \{0\} \right\},$$

so it is obvious that $\mathrm{im}(\gamma_{P,Q}) \subset conv(P,Q)$. On the other hand, given $(t_1,t_2) \in [0,\infty)^2 - \{0\}$, let

$$\lambda = \frac{1}{t_1 + t_2} > 0.$$

Then $\lambda t_1 = 1 - \lambda t_2$ and

$$\frac{t_1P + t_2Q}{\|t_1P + t_2Q\|} = \frac{\lambda t_1P + \lambda t_2Q}{\|\lambda t_1P + \lambda t_2Q\|} = \frac{(1-t)P + tQ}{\|(1-t)P + tQ\|} \in conv(P,Q)$$

for $t = \lambda t_2 \in [0,1]$, as desired. □

Lemma 4.21. *Let $A \subset O(v)$ and suppose $\varphi^S(A) \subset O(w)$ for some w. Then $\varphi^S(conv(A)) = conv(\varphi^S(A))$, that is, the homeomorphism $\varphi^S : S^{n-1} \to S^{n-1}$ maps convex hulls to convex hulls. In particular, it maps geodesic paths to geodesic paths.*

Demonstração. Let us show that $\varphi^S(conv(A)) \subset conv(\varphi^S(A))$. If we show this, then since $(\varphi^{-1})^S = (\varphi^S)^{-1}$ we can similarly show that $(\varphi^S)^{-1}(conv(\varphi^S(A))) \subset conv(A)$, or $conv(\varphi^S(A)) \subset \varphi^S(conv(A))$; therefore $\varphi^S(conv(A)) = conv(\varphi^S(A))$ and we are done.

Let $P \in conv(A)$ and write $P = \frac{t_1a_1+...+t_ma_m}{\|t_1a_1+...+t_ma_m\|}$ for some $a_i \in A$ and $t_i \geq 0$. For each a_i, since $H : S(G) \to S^{n-1}$ is surjective we write $a_i = H[\chi_i] = \frac{(\chi_i(x_1),...,\chi_i(x_n))}{\|(\chi_i(x_1),...,\chi_i(x_n))\|}$ for some $[\chi_i] \in S(G)$ and, up to multiplying the representative χ_i by $\frac{1}{\|(\chi_i(x_1),...,\chi_i(x_n))\|}$ we can actually suppose $a_i = H[\chi_i] = (\chi_i(x_1),...,\chi_i(x_n))$. Remember then that, by definition,

$$\begin{aligned}
\varphi^S(a_i) &= H \circ \varphi^* \circ H^{-1}(a_i) \\
&= H \circ \varphi^*[\chi_i] \\
&= H[\chi_i \circ \varphi] \\
&= \frac{(\chi_i \circ \varphi(x_1),...,\chi_i \circ \varphi(x_n))}{\|(\chi_i \circ \varphi(x_1),...,\chi_i \circ \varphi(x_n))\|} \\
&= \frac{1}{\lambda_i}(\chi_i \circ \varphi(x_1),...,\chi_i \circ \varphi(x_n)),
\end{aligned}$$

where $\lambda_i = \|(\chi_i \circ \varphi(x_1),...,\chi_i \circ \varphi(x_n))\| > 0$. Now we compute $\varphi^S(P)$. But

$$\begin{aligned}
H[t_1\chi_1 + ... + t_m\chi_m] &= \frac{((t_1\chi_1 + ... + t_m\chi_m)(x_1),...,(t_1\chi_1 + ... + t_m\chi_m)(x_n))}{\|((t_1\chi_1 + ... + t_m\chi_m)(x_1),...,(t_1\chi_1 + ... + t_m\chi_m)(x_n))\|} \\
&= \frac{(t_1\chi_1(x_1) + ... + t_m\chi_m(x_1),...,t_1\chi_1(x_n) + ... + t_m\chi_m(x_n))}{\|(t_1\chi_1(x_1) + ... + t_m\chi_m(x_1),...,t_1\chi_1(x_n) + ... + t_m\chi_m(x_n))\|} \\
&= \frac{t_1a_1 + ... + t_ma_m}{\|t_1a_1 + ... + t_ma_m\|} \\
&= P.
\end{aligned}$$

By denoting

$$\lambda = \|(t_1(\chi_1 \circ \varphi)(x_1) + ... + t_m(\chi_m \circ \varphi)(x_1),...,t_1(\chi_1 \circ \varphi)(x_n) + ... + t_m(\chi_m \circ \varphi)(x_n))\|,$$

we have

$$\begin{aligned}
\varphi^S(P) &= H\circ\varphi^*\circ H^{-1}(P)\\
&= H\circ\varphi^*[t_1\chi_1+...+t_m\chi_m]\\
&= H[(t_1\chi_1+...+t_m\chi_m)\circ\varphi]\\
&= H[t_1(\chi_1\circ\varphi)+...+t_m(\chi_m\circ\varphi)]\\
&= \frac{(t_1(\chi_1\circ\varphi)(x_1)+...+t_m(\chi_m\circ\varphi)(x_1),...,t_1(\chi_1\circ\varphi)(x_n)+...+t_m(\chi_m\circ\varphi)(x_n))}{\lambda}\\
&= \frac{t_1}{\lambda}((\chi_1\circ\varphi)(x_1),...,(\chi_1\circ\varphi)(x_n))+...+\frac{t_m}{\lambda}((\chi_m\circ\varphi)(x_1),...,(\chi_m\circ\varphi)(x_n))\\
&= \frac{\lambda_1t_1}{\lambda}\frac{1}{\lambda_1}((\chi_1\circ\varphi)(x_1),...,(\chi_1\circ\varphi)(x_n))+...+\frac{\lambda_mt_m}{\lambda}\frac{1}{\lambda_m}((\chi_m\circ\varphi)(x_1),...,(\chi_m\circ\varphi)(x_n))\\
&= \frac{\lambda_1t_1}{\lambda}\varphi^S(a_1)+...+\frac{\lambda_mt_m}{\lambda}\varphi^S(a_m)\\
&= \frac{\frac{\lambda_1t_1}{\lambda}\varphi^S(a_1)+...+\frac{\lambda_mt_m}{\lambda}\varphi^S(a_m)}{\|\frac{\lambda_1t_1}{\lambda}\varphi^S(a_1)+...+\frac{\lambda_mt_m}{\lambda}\varphi^S(a_m)\|} \quad \text{(since the above vector is already unitary)}\\
&\in conv(\varphi^S(A)),
\end{aligned}$$

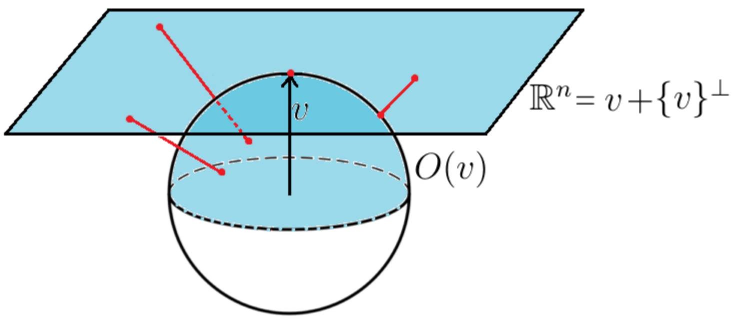
as desired. Since the convex hull of two points is the geodesic path between them (by the above lemma), φ^S must map geodesic paths to geodesic paths. □

Observation 4.22. Note that the coeficients of $\varphi^S(P)$ may not be the same of the ones of P. For example, the middle point $\frac{P+Q}{\|P+Q\|}$ between P and Q may not be mapped to the middle point $\frac{\varphi^S(P)+\varphi^S(Q)}{\|\varphi^S(P)+\varphi^S(Q)\|}$. However, we showed that it is certainly mapped into the geodesic path from $\varphi^S(P)$ to $\varphi^S(Q)$.

If $S^n \subset \mathbb{R}^{n+1}$, let $O(v) = \{x \in S^n \mid \langle x, v\rangle > 0\}$ be an open hemisphere of S^n for some $v \in S^n$. Consider the affine n-space $v + \{v\}^\perp = \{v + w \mid \langle w, v\rangle = 0\} \subset \mathbb{R}^{n+1}$. One can show that there is a homeomorphism

$$\theta_v : v + \{v\}^\perp \to O(v) \text{ with } \theta_v(P) = \frac{P}{\|P\|},$$

the inverse map given by $P \mapsto \frac{\|v\|^2}{\langle P,v\rangle}P$. From now on we identify $\mathbb{R}^n = v + \{v\}^\perp$.



It is straightforward to show from the definitions that $\theta_v : \mathbb{R}^n \to O(v)$ maps convex hulls of $\mathbb{R}^n$ to convex hulls of $O(v)$. More precisely, $\theta_v(conv(A)) = conv(\theta_v(A))$.

Definition 4.23. A closed hemisphere in S^n is a set having the form $C(w) = \{p \in S^n \mid \langle p, w\rangle \geq 0\}$ for some $w \in S^n$. A polytope $K \subset S^n$ is a finite intersection of closed hemispheres in S^n.

Lemma 4.24. *Let K be a polytope contained in an open hemisphere $O(v)$ of S^n and consider the projection homeomorphism $\theta_v : \mathbb{R}^n \to O(v)$. Then ${\theta_v}^{-1}(K)$ is a polytope in $\mathbb{R}^n$.*

Demonstração. We are identifying $v + \{v\}^\perp = \mathbb{R}^n$. Let $\{e_1, ..., e_n\}$ be an orthogonal basis of the n-vector space $\{v\}^\perp$ (so the identification $\mathbb{R}^n = v + \{v\}^\perp$ can be seen as $(\lambda_1, ..., \lambda_n) \mapsto v + \lambda_1 e_1 + ... + \lambda_n e_n$). Write $e_i = (e_{i1}, ..., e_{i,n+1})$ for each i. First, let us see what is the inverse image under θ_v of a closed half space $C(w)$ of S^n. Let $z = v + \lambda_1 e_1 + ... + \lambda_n e_n \in v + \{v\}^\perp$. Then

$$z \in {\theta_v}^{-1}(C(w)) \Leftrightarrow \left\langle \frac{z}{\|z\|}, w \right\rangle \geq 0 \Leftrightarrow \langle z, w \rangle \geq 0.$$

But if $v = (v_1, ..., v_{n+1})$ and $w = (w_1, ..., w_{n+1})$,

$$\langle z, w \rangle = \lambda_1(e_{11}w_1 + ... + e_{1,n+1}w_{n+1}) + ... + \lambda_n(e_{n1}w_1 + ... + e_{n,n+1}w_{n+1}) + \langle v, w \rangle.$$

So, under the identification, $v + \{v\}^\perp$ is characterized by the $(\lambda_1, ..., \lambda_n)$ such that

$$\langle (\lambda_1, ..., \lambda_n), (e_{11}w_1 + ... + e_{1,n+1}w_{n+1}, ..., e_{n1}w_1 + ... + e_{n,n+1}w_{n+1}) \rangle \geq -\langle v, w \rangle,$$

which is a closed half space of $\mathbb{R}^n$. This shows that the inverse image under θ_v of closed half spaces of S^n are closed half spaces of $\mathbb{R}^n$. Now we easily show the proposition. Since $K \subset O(v)$ is by definition a finite intersection of closed half spaces of S^n and θ_v is a bijection, ${\theta_v}^{-1}(K)$ is a finite intersection of closed halfspaces of $\mathbb{R}^n$. Furthermore, K is closed (finite intersection of closed subsets) inside the compact S^n, and therefore compact. Then, because θ_v is a homeomorphism, ${\theta_v}^{-1}(K)$ is compact and therefore bounded, so it is a polytope. □

Definition 4.25. Let K be a polytope contained in an open hemisphere $O(v)$ of S^n. The vertices of K are $V(K) = \theta_v(V({\theta_v}^{-1}(K)))$, that is, the vertices of K are the projections of the vertices of the polytope ${\theta_v}^{-1}(K)$. The dimension of K is also defined to be the dimension of ${\theta_v}^{-1}(K)$.

4.3 Property R_∞ under convex polytopes

Definition 4.26. Let G be a finitely generated group with homeomorphism $H : S(G) \to S^{n-1}$. We say that $K \subset S(G)$ is a r-polytope in $S(G)$ if $H(K)$ is a r-polytope in S^{n-1}. In this case, we define the vertices of K as $V(K) = H^{-1}(V(H(K)))$, that is, $[\chi]$ is a vertex of K if $H[\chi]$ is a vertex of $H(K)$. We say that K is contained in an open hemisphere of $S(G)$ if $H(K)$ is contained in an open hemisphere of S^{n-1}.

Theorem 4.27. *Let G be a finitely generated group and $K \subset S(G)$ a polytope contained in an open hemisphere of $S(G)$. Then K is invariant in $S(G)$ if and only if $V(K)$ is invariant in $S(G)$.*

Demonstração. Let $\varphi \in Aut(G)$. By hypothesis, $H(K) \subset O(v)$ for some open hemisphere of S^{n-1}, so let $\theta_v : \mathbb{R}^{n-1} \to O(v)$ be the homeomorphism. By Lemma 4.24, $K' = {\theta_v}^{-1}(H(K))$ is an r-polytope in $\mathbb{R}^{n-1}$ for some $0 \leq r \leq n-1$. It is enough to show that $H(K)$ is invariant under φ^S if and only if $V(H(K))$ is. Suppose first that $V(H(K))$ is invariant under φ^S. But in

Euclidean space, every convex polytope is the convex hull of its vertices (see [52]), so

$$conv(V(H(K))) = conv(\theta_v(V(K'))) = \theta_v(conv(V(K'))) = \theta_v(K') = H(K),$$

that is, $H(K)$ is also the convex hull of its vertices. Using this, the main hypothesis and Lemma 4.21, we have

$$\varphi^S(H(K)) = \varphi^S(conv(V(H(K)))) = conv(\varphi^S(V(H(K)))) = conv(V(H(K))) = H(K),$$

as desired. Now, suppose $\varphi^S(H(K)) = H(K)$. We know K' is an r-polytope in $\mathbb{R}^{n-1}$. If $r < n-1$, then K' is contained in a proper r-hyperspace (the translation of an r-subspace) of $\mathbb{R}^{n-1}$. Indeed, if that was not the case, then since every $r+1$ points are contained in an r-hyperspace, there are $r+2$ points of K' which are not contained in any r-hyperspace. But since K' is convex, K' must contain the convex hull of these $r+2$ points, which is an $r+1$-dimensional closed simplex. Then $dim(K') \geq r+1$, contradiction. Denote by E^r the r-hyperspace of $\mathbb{R}^n$ containing K'. Considering E^r as a linear space, there is a linear isomorphism and isometry $T : \mathbb{R}^r \to E^r$ and a r-polytope $\tilde{K} \subset \mathbb{R}^r$ such that $K' = T(\tilde{K})$. Consider the composition made by (the restrictions of) the homeomorphisms

$$\tilde{K} \xrightarrow{T} K' \xrightarrow{\theta_v} H(K) \xrightarrow{\varphi^S} H(K) \xrightarrow{\theta_v{}^{-1}} K' \xrightarrow{T^{-1}} \tilde{K}.$$

Since T maps straight paths to straight paths, θ_v maps straight paths to geodesic paths and φ^S maps geodesic paths to geodesic paths, this composition is a homeomorphism which maps straight paths to straight paths. By 4.16, it must map the vertices of $\tilde{K}$ to themselves. Then, since T maps the vertices of $\tilde{K}$ to the vertices of K', the composition

$$K' \xrightarrow{\theta_v} H(K) \xrightarrow{\varphi^S} H(K) \xrightarrow{\theta_v{}^{-1}} K'$$

must map the vertices of K' to themselves. Finally, since $V(H(K)) = \theta_v(V(K'))$, the last fact implies that φ^S must map the vertices of $H(K)$ to themselves, as desired. □

Theorem 4.28. *Let G be a finitely generated group. If there is an invariant polytope $K \subset S(G)$ contained in an open hemisphere of $S(G)$ and with rational vertices, then G has property R_∞. In particular, if $\Sigma^n(G)^c$ is such a polytope for some $n \geq 1$, then G has property R_∞.*

Demonstração. By the previous theorem, $V(K) \subset S(G)$ is finite, invariant and by definition contained in an open half-space of $S(G)$. Then the result follows directly from Theorem 4.1. □

Like we did in Theorem 3.40, we will now show that it is possible to guarantee property R_∞ for a direct product if all factors have $(\Sigma^1)^c$ as polytopes described above.

Lemma 4.29. *Let $A \subset O(v) \subset S^n$ and $B \subset O(w) \subset S^m$ and consider the inclusions $i : S^n \to S^{n+m+1}$ with $(x_1, ..., x_{n+1}) \mapsto (x_1, ..., x_{n+1}, 0, ..., 0)$ (which we abbreviate to $x \mapsto (x, 0)$) and $i' : S^m \to S^{n+m+1}$ with $(y_1, ..., y_{m+1}) \mapsto (0, ..., 0, y_1, ..., y_{m+1})$ (which we abbreviate to $y \mapsto (0, y)$). If A and B are both convex subsets, then*

$$A \circledast B = conv(i(A) \cup i'(B)).$$

Demonstração. By the definition of spherical join (3.11), ($\subset$) is immediate. Let us show ($\supset$). By Proposition 4.19, a general element of $conv(i(A) \cup i'(B))$ has the form

$$\begin{aligned}\frac{t_1(a_1,0)+...+t_r(a_r,0)+t'_1(0,b_1)+...+t'_s(0,b_s)}{\|t_1(a_1,0)+...+t_r(a_r,0)+t'_1(0,b_1)+...+t'_s(0,b_s)\|} &= \frac{(t_1a_1+...+t_ra_r, t'_1b_1+...+t'_sb_s)}{\|(t_1a_1+...+t_ra_r, t'_1b_1+...+t'_sb_s)\|}\\ &= \left(\frac{t_1}{\lambda}a_1+...+\frac{t_r}{\lambda}a_r, \frac{t'_1}{\lambda}b_1+...+\frac{t'_s}{\lambda}b_s\right)\end{aligned}$$

for $\lambda = \|(t_1a_1+...+t_ra_r, t'_1b_1+...+t'_sb_s)\|$, $(t_1,...,t_r) \in [0,\infty]^r - \{0\}$ and $(t'_1,...,t'_s) \in [0,\infty]^s - \{0\}$. Let

$$\lambda_1 = \left\|\frac{t_1}{\lambda}a_1+...+\frac{t_r}{\lambda}a_r\right\| \text{ and } \lambda_2 = \left\|\frac{t'_1}{\lambda}b_1+...+\frac{t'_s}{\lambda}b_s\right\|.$$

Then, since A and B are convex, we can write

$$\frac{\frac{t_1}{\lambda}a_1+...+\frac{t_r}{\lambda}a_r}{\lambda_1} = a \in A \text{ and } \frac{\frac{t'_1}{\lambda}b_1+...+\frac{t'_s}{\lambda}b_s}{\lambda_2} = b \in B$$

and so

$$\left(\frac{t_1}{\lambda}a_1+...+\frac{t_r}{\lambda}a_r, \frac{t'_1}{\lambda}b_1+...+\frac{t'_s}{\lambda}b_s\right) = (\lambda_1a, \lambda_2b) = \frac{(\lambda_1a, \lambda_2b)}{\|(\lambda_1a, \lambda_2b)\|},$$

the last since the vector is unitary. Finally, let $\epsilon = \frac{1}{\lambda_1+\lambda_2}$. Then

$$\frac{(\lambda_1a, \lambda_2b)}{\|(\lambda_1a, \lambda_2b)\|} = \frac{(\epsilon\lambda_1a, \epsilon\lambda_2b)}{\|(\epsilon\lambda_1a, \epsilon\lambda_2b)\|} = \frac{((1-\epsilon\lambda_2)a, \epsilon\lambda_2b)}{\|((1-\epsilon\lambda_2)a, \epsilon\lambda_2b)\|} \in A \circledast B,$$

as desired. □

Proposition 4.30. *If $K_1,...,K_m$ are polytopes contained in open hemispheres, say, $K_i \subset O(v_i) \subset S^{n_i}$, then the spherical join $K_1 \circledast ... \circledast K_m$ is a polytope contained in an open hemisphere.*

Demonstração. Let us show it by induction on m; first, the case $m = 2$. Let $K \subset O(v) \subset S^n$ and $K' \subset O(w) \subset S^m$be polytopes. It is easy to see that $K \circledast K' \subset O(v,w)$. Now, write $K = \cap_{i=1}^r C(v_i)$ and $K' = \cap_{i=1}^s C(w_i)$ for $v_i \in S^n$ and $w_i \in S^m$. Consider the polytope in S^{n+m+1}

$$K'' = (\cap_{i=1}^r C(v_i,0)) \cap (\cap_{i=1}^s C(0,w_i)).$$

We claim that $K \circledast K' = K''$. From the previous lemma, since K and K' are convex we have $K \circledast K' = conv(i(K) \cup i'(K'))$, so we will show that $conv(i(K) \cup i'(K')) = K''$. On one hand, it's easy to see that K'' is convex (is an intersection of convex sets) and contains $i(K) \cup i'(K')$, so we have ($\subset$). On the other hand, let C be a convex set containing $i(K) \cup i'(K')$ and let us show that $K'' \subset C$. If $(x,y) \in K''$ we must have $\langle x, v_i\rangle = \langle (x,y),(v_i,0)\rangle \geq 0$ for all $1 \leq i \leq r$ and $\langle y, w_i\rangle = \langle (x,y),(0,w_i)\rangle \geq 0$ for all $1 \leq i \leq s$, so $x \in K$ and $y \in K'$. Then $i(x), i'(y) \in C$ and, since C is convex,

$$(x,y) = (x,0) + (0,y) = i(x) + i'(y) = \frac{i(x)+i'(y)}{\|i(x)+i'(y)\|} \in C,$$

as desired. This shows ($\supset$) and finishes the case $m = 2$.

Now suppose the fact is valid for $m \geq 2$ and let $K_1 \circledast ... \circledast K_{m+1}$ be a spherical join of polytopes $K_i \subset O(v_i) \subset S^{n_i}$. By the induction hypothesis, $K_1 \circledast ... \circledast K_m$ is a polytope in an open hemisphere; then $K_1 \circledast ... \circledast K_{m+1} = (K_1 \circledast ... \circledast K_m) \circledast K_{m+1}$ is a polytope contained in an open hemisphere by the case $m = 2$. This completes the proof. □

Theorem 4.31. *Let $G = G_1 \times ... \times G_m$ be a direct product of finitely generated groups G_i. If, for all i, $\Sigma^1(G_i)^c$ is a polytope with rational vertices contained in an open hemisphere of $S(G_i)$, then G has the R_∞ property.*

Demonstração. We will show the case $m = 2$, for the general case follows by trivial induction. If we identify $S(G_1) \simeq S^{n_1-1}$ and $S(G_2) \simeq S^{n_2-1}$ as in Theorem 3.6, we have $S(G) = S^{n_1+n_2+1}$ and the maps $\pi_i{}^* : S(G_i) \to S(G)$ can be identified with inclusions of the form $x \mapsto (x, 0)$ and $y \mapsto (0, y)$, respectively for $i = 1, 2$. By using this fact, the hypothesis on $\Sigma^1(G_i)^c$ and the product formula (Theorem 3.30) we have that $\Sigma^1(G)^c = \pi_1{}^*(\Sigma^1(G_1)^c) \cup \pi_2{}^*(\Sigma^1(G_2)^c)$ is the union of two convex polytopes in $S(G)$. Since each one was contained in an open hemisphere of $S(G_i)$, it is easy to see that $\Sigma^1(G)^c$ is contained in an open hemisphere of $S(G)$ (see the end of the proof of Theorem 3.40). Now, since $\Sigma^1(G)^c$ is invariant in $S(G)$ and the maps φ^S send convex hulls to convex hulls (Lemma 4.21) we have $conv(\Sigma^1(G)^c)$ invariant in $S(G)$. But the $\Sigma^1(G_i)^c$ are convex and the $\pi_i{}^*$ are inclusions, so by Lemma 4.29 we have

$$conv(\Sigma^1(G)^c) = conv(\pi_1{}^*(\Sigma^1(G_1)^c) \cup \pi_2{}^*(\Sigma^1(G_2)^c)) = \Sigma^1(G_1)^c \circledast \Sigma^1(G_2)^c,$$

so $\Sigma^1(G_1)^c \circledast \Sigma^1(G_2)^c$ is invariant in $S(G)$ and contained in an open hemisphere of it. By Proposition 4.30, it is also a (rationally defined) polytope. Thus, the theorem follows from Theorem 4.28. □

Open question: are there any known groups in the literature having such invariant convex polytopes in the character sphere? In particular, are there groups with the complementar of the Σ^1 invariant being such polytopes? We know from the results of this chapter that, if the set $\Sigma^1(G)^c$ is finite, of rational points and contained in an open hemisphere, then its convex closure in $S(G)$ is such an invariant convex polytope and our Theorem 4.28 applies. But in this case we have no particular gain with respect to twisted conjugacy, for Theorem 3.38 already guarantees property R_∞ for G. The interesting situation, therefore, would be either finding invariant convex polytopes which are not convex closures of $\Sigma^1(G)^c$, or groups such that $\Sigma^1(G)^c$ are non degenerated convex polytopes. Is there any methodical way of building such groups by using group presentations?

Parte II

The Σ^1 invariant of the Generalized Solvable Baumslag-Solitar groups Γ_n and of their finite index subgroups

Capítulo 5

Generalized Solvable Baumslag-Solitar groups Γ_n

In this chapter we investigate the Σ^1 invariant of an important class of groups which generalizes the solvable Baumslag-Solitar groups $BS(1,n)$ and that we will call Γ_n. Property R_∞ is known for these groups and for every group which is quasi-isometric to some Γ_n (see the paper [94]). The techniques of the paper, nevertheless, do not involve geometric invariants. By computing $\Sigma^1(\Gamma_n)$, we obtained a new proof of R_∞ property for Γ_n.

Recently, it has been pointed out by professor Dessislava Hristova Kochloukova (which is a specialist on the subject of BNS invariants) that the groups Γ_n are metabelian (that is, they contain a normal abelian subgroup $\mathbb{Z}\left[\frac{1}{n}\right]$ with an abelian quotient $\mathbb{Z}^r$, see the first paragraphs of Chapter 6) and that a lot of good information is known about BNS invariants for this class of groups. For example, it was already known by [11] (see also [10] and [9]) that $\Sigma^1(\Gamma)^c$ is finite and could be explicitly computed by easy calculations involving the finite generation (or not) of its commutator group as a module over a monoid Q_μ inside the abelianized group $Q = G/G'$ ($G = \Gamma_n$), the action being given by conjugation. We could have followed these directions, but we decided to maintain the more elementary and geometric proofs below, for didactic reasons and also to maintain the graphic-likeness of the thesis.

Definition 5.1. Let $n \geq 2$ be a positive integer with prime decomposition $n = {p_1}^{y_1}...{p_r}^{y_r}$, the p_i being pairwise distinct. We define the solvable generalization of the Baumslag-Solitar group by

$$\Gamma_n = \left\langle a, t_1, ..., t_r \mid t_i t_j = t_j t_i,\ i \neq j,\ t_i a {t_i}^{-1} = a^{{p_i}^{y_i}},\ i = 1, ..., r \right\rangle.$$

More generally, let $n_1, ..., n_r$ be pairwise coprime positive integers and let us assume there is at least one i such that $n_i \geq 2$. Define

$$G = \Gamma_{\{n_1,...,n_r\}} = \left\langle a, t_1, ..., t_r \mid t_i t_j = t_j t_i,\ i \neq j,\ t_i a {t_i}^{-1} = a^{n_i},\ i = 1, ..., r \right\rangle.$$

In the next section we will deal with this group G. The hypothesis $n_i \geq 2$ for some i is just to turn the investigation to the interesting cases. Indeed, if $n_i = 1$ for all i then all the generators of G would commute and we would have $G \simeq \mathbb{Z}^{r+1}$ and $\Sigma^1(G) = S(G)$, by Corollary 3.15. Also, G would not have the property R_∞ by Example 1.3 and so there would be nothing to be done in this chapter. Note that for every $n \geq 1$, the group $\Gamma_n = \Gamma_{\{{p_1}^{y_1},...,{p_r}^{y_r}\}}$ is a special case of our group G.

5.1 Computation of $\Sigma^1(\Gamma_n)$ and property R_∞

In this section we intend to compute $\Sigma^1(G)$ (in particular, $\Sigma^1(\Gamma_n)$) in order to guarantee the property R_∞ for it. Note that G is torsion-free. In the abelianized G^{ab}, taking i with $n_i \geq 2$ we have $a^{n_i} = t_i a {t_i}^{-1} = a$, then $a^{n_i - 1} = 1$, and so the homeomorphism

$$\begin{aligned} H : S(G) &\longrightarrow S^{r-1} \\ [\chi] &\longmapsto \frac{(\chi(t_1), ..., \chi(t_r))}{\|(\chi(t_1), ..., \chi(t_r))\|}. \end{aligned}$$

We are going to use the geometric Σ^1-criterion given by Ralph Strebel (Theorem 3.22). In our case, we have $S = \{a, t_1, ..., t_r\}$, $Y = \{a, a^{-1}, t_1, {t_1}^{-1}, ..., t_t, {t_r}^{-1}\}$. Using Theorem 3.22, we will prove that

1) if there is $1 \leq i \leq r$ such that $\chi(t_i) < 0$, then $[\chi] \in \Sigma^1(G)$;

2) if there are $1 \leq i, j \leq r$ with $i \neq j$ and such that $\chi(t_i), \chi(t_j) > 0$, then $[\chi] \in \Sigma^1(G)$.

Let's do it:

1) if there is $1 \leq i \leq r$ such that $\chi(t_i) < 0$, then $[\chi] \in \Sigma^1(G)$.

Fix $t = {t_i}^{-1}$ and we have $\chi(t) = -\chi(t_i) > 0$. By the Geometric criterion, it suffices to exhibit $2r + 2$ paths p_y $(y \in Y)$ in $\Gamma(G, S)$ from ${t_i}^{-1}$ to $y{t_i}^{-1}$ such that $\nu_\chi(p_y) - \nu_\chi((1, y)) > 0$.

$\underline{y = a}$: since $t_i a {t_i}^{-1} = a^{n_i}$ in G we have $a{t_i}^{-1} = {t_i}^{-1}a^{n_i}$, so we take $p_a = ({t_i}^{-1}, a^{n_i})$, as in the figure. We have $\nu_\chi(p_a) = min\{\chi({t_i}^{-1}), \chi({t_i}^{-1}a), ..., \chi({t_i}^{-1}a^{n_i})\} = -\chi(t_i)$ (since $\chi(a) = 0$), $\nu_\chi((1, y)) = min\{\chi(1), \chi(a)\} = 0$ and then $\nu_\chi(p_a) - \nu_\chi((1, a)) = -\chi(t_i) > 0$, as desired.

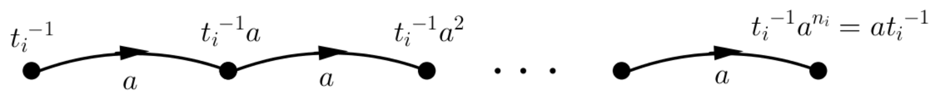


Figura 5.1: the path p_a

$\underline{y = a^{-1}}$: as in the previous item, from the relation $t_i a^{-1} {t_i}^{-1} = a^{-n_i}$ in G we have $a^{-1}{t_i}^{-1} = {t_i}^{-1}a^{-n_i}$, so we take $p_{a^{-1}} = ({t_i}^{-1}, a^{-n_i})$, as in the figure. We have $\nu_\chi(p_{a^{-1}}) = min\{\chi({t_i}^{-1}), \chi({t_i}^{-1}a^{-1}), ..., \chi({t_i}^{-1}a^{-n_i})\} = -\chi(t_i)$, $\nu_\chi((1, a^{-1})) = min\{\chi(1), \chi(a^{-1})\} = 0$ and then $\nu_\chi(p_{a^{-1}}) - \nu_\chi((1, a^{-1})) = -\chi(t_i) > 0$, as desired.

$\underline{y = t_i}$: we just take $p_{t_i} = ({t_i}^{-1}, t_i)$. So

$$\nu_\chi(p_{t_i}) - \nu_\chi((1, t_i)) = min\{\chi({t_i}^{-1}), 0\} - min\{0, \chi(t_i)\} = 0 - \chi(t_i) = -\chi(t_i) > 0,$$

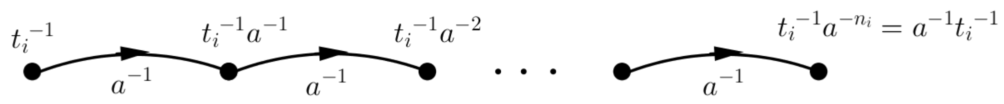


Figura 5.2: the path $p_{a^{-1}}$

as desired.

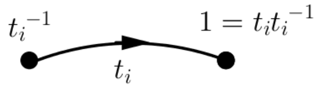


Figura 5.3: the path p_{t_i}

$y = t_i{}^{-1}$: analogously, we take $p_{t_i{}^{-1}} = (t_i{}^{-1}, t_i{}^{-1})$. Then

$$\begin{aligned}\nu_\chi(p_{t_i{}^{-1}}) - \nu_\chi((1, t_i{}^{-1})) &= min\{-\chi(t_i), -2\chi(t_i)\} - min\{0, -\chi(t_i)\} \\ &= -\chi(t_i) - 0 > 0,\end{aligned}$$

as desired.

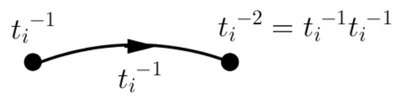


Figura 5.4: the path $p_{t_i{}^{-1}}$

$y = t_j,\ j \neq i$: Since $t_i t_j = t_j t_i$, take $p_{t_j} = (t_i{}^{-1}, t_j)$. Then,

$$\nu_\chi(p_{t_j}) = min\{-\chi(t_i), -\chi(t_i) + \chi(t_j)\} = -\chi(t_i) + min\{0, \chi(t_j)\}$$

and $\nu_\chi((1, t_j)) = min\{0, \chi(t_j)\}$, and so

$$\nu_\chi(p_{t_j}) - \nu_\chi((1, t_j)) = -\chi(t_i) + min\{0, \chi(t_j)\} - min\{0, \chi(t_j)\} = -\chi(t_i) > 0,$$

as we wanted.

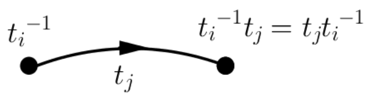


Figura 5.5: the path p_{t_j}

$y = t_j{}^{-1},\ j \neq i$: Since $t_i t_j = t_j t_i$, take $p_{t_j} = (t_i{}^{-1}, t_j{}^{-1})$. Then,

$$\nu_\chi(p_{t_j{}^{-1}}) = min\{-\chi(t_i), -\chi(t_i) - \chi(t_j)\} = -\chi(t_i) + min\{0, -\chi(t_j)\}$$

and $\nu_\chi((1,t_j{}^{-1})) = min\{0, -\chi(t_j)\}$, and so

$$\nu_\chi(p_{t_j{}^{-1}}) - \nu_\chi((1,t_j{}^{-1})) = -\chi(t_i) + min\{0,-\chi(t_j)\} - min\{0,-\chi(t_j)\} = -\chi(t_i) > 0,$$

as we wanted.

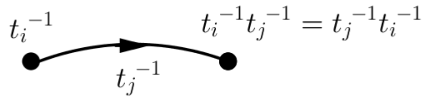


Figura 5.6: the path $p_{t_j{}^{-1}}$

Since we obtained the $2r+2$ inequalities $\nu_\chi(p_y) - \nu_\chi((1,y)) > 0$, one has $[\chi] \in \Sigma^1(G)$.

2) if there are $1 \leq i,j \leq r$ with $i \neq j$ and such that $\chi(t_i), \chi(t_j) > 0$, then $[\chi] \in \Sigma^1(G)$.

This time we fix $t = t_i$ with $\chi(t_i) > 0$. From the two relations $t_i a t_i{}^{-1} = a^{n_i}$ and $t_i a^{-1} t_i{}^{-1} = a^{-n_i}$ it is easy to prove by induction that $t_i a^k t_i{}^{-1} = a^{kn_i}$ for every $k \in \mathbb{Z}$. The same happens for j: $t_j a^k t_j{}^{-1} = a^{kn_j}$. Finally, as n_i and n_j are coprime, take integers r, s such that $rn_i + sn_j = 1$. Again, let us exhibit the $2r+2$ paths p_y from t_i to yt_i with $\nu_\chi(p_y) - \nu_\chi((1,y)) > 0$.

$\underline{y = a}$: Since $\nu_\chi((1,a)) = 0$, we have to create a path from t_i to at_i with $\nu_\chi(p_a) > 0$, that is, a path having positive χ-values in all its vertices. Based on the equation $rn_i + sn_j = 1$, we take then $p_a = (t_i, a^r t_j t_i{}^{-1} a^s t_i t_j{}^{-1})$ and use the relations of the group to guarantee that p_a ends in at_i, as one can see at the (merely illustrative) figure.

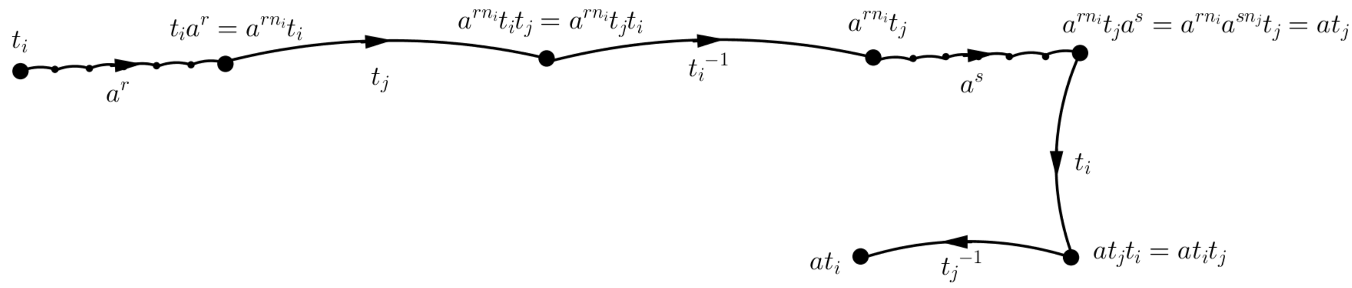


Figura 5.7: the path p_a

Again, as $\chi(a) = 0$, we have

$$\nu_\chi(p_a) = min\{\chi(t_i), \chi(t_i t_j), \chi(t_j)\} = min\{\chi(t_i), \chi(t_j)\} > 0$$

and so $\nu_\chi(p_a) - \nu_\chi((1,a)) = \nu_\chi(p_a) > 0$, as we wanted.

$\underline{y = a^{-1}}$: here, we use that $-rn_i - sn_j = -1$ to construct a similar path as before: $p_{a^{-1}} = (t_i, a^{-r} t_j t_i{}^{-1} a^{-s} t_i t_j{}^{-1})$. Then $\nu_\chi(p_{a^{-1}}) - \nu_\chi((1,a^{-1})) = min\{\chi(t_i), \chi(t_j)\} - 0 > 0$.

$\underline{y = t_i}$: take $p_{t_i} = (t_i, t_i)$. Then $\nu_\chi(p_{t_i}) = min\{\chi(t_i), 2\chi(t_i)\} = \chi(t_i)$. Also, we have $\nu_\chi((1,t_i)) = min\{0, \chi(t_i)\} = 0$ and so $\nu_\chi(p_{t_i}) - \nu_\chi((1,t_i)) = \chi(t_i) > 0$, as desired.

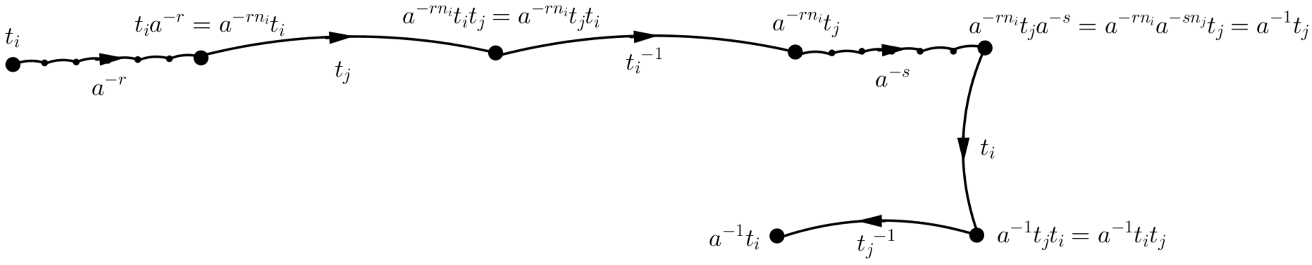


Figura 5.8: the path $p_{a^{-1}}$

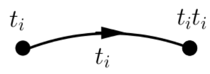


Figura 5.9: the path p_{t_i}

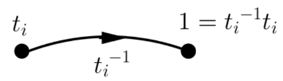


Figura 5.10: the path $p_{{t_i}^{-1}}$

$\underline{y = {t_i}^{-1}}$: we just take $p_{{t_i}^{-1}} = (t_i, {t_i}^{-1})$. Then $\nu_\chi(p_{{t_i}^{-1}}) = min\{\chi(t_i), 0\} = 0$ and $\nu_\chi((1, {t_i}^{-1})) = min\{0, -\chi(t_i)\} = -\chi(t_i)$, and so $\nu_\chi(p_{{t_i}^{-1}}) - \nu_\chi((1, {t_i}^{-1})) = 0 - (-\chi(t_i)) = \chi(t_i) > 0$.

$\underline{y = t_j,\ j \neq i}$: again, using that t_i commutes with t_j we take $p_{t_j} = (t_i, t_j)$. We have $\nu_\chi(p_{t_j}) = min\{\chi(t_i), \chi(t_i) + \chi(t_j)\} = \chi(t_i) + min\{0, \chi(t_j)\}$ and $\nu_\chi((1, t_j)) = min\{0, \chi(t_j)\}$, so $\nu_\chi(p_{t_j}) - \nu_\chi((1, t_j)) = \chi(t_i) + min\{0, \chi(t_j)\} - min\{0, \chi(t_j)\} = \chi(t_i) > 0$.

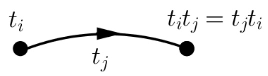


Figura 5.11: the path p_{t_j}

$\underline{y = {t_j}^{-1},\ j \neq i}$: let $p_{{t_j}^{-1}} = (t_i, {t_j}^{-1})$. Then $\nu_\chi(p_{{t_j}^{-1}}) = min\{\chi(t_i), \chi(t_i) - \chi(t_j)\} = \chi(t_i) + min\{0, -\chi(t_j)\}$ and $\nu_\chi((1, {t_j}^{-1})) = min\{0, -\chi(t_j)\}$ and so again we have $\nu_\chi(p_{{t_j}^{-1}}) - \nu_\chi((1, {t_j}^{-1})) = \chi(t_i) > 0$, as desired.

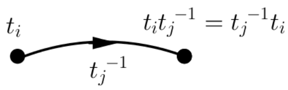


Figura 5.12: the path $p_{{t_j}^{-1}}$

Again, since we obtained $2r + 2$ inequalities $\nu_\chi(p_y) - \nu_\chi((1, y)) > 0$, one has $[\chi] \in \Sigma^1(G)$.

The two cases before cover almost the entire sphere $S(G)$, except for the r points $[\chi_1], ..., [\chi_r]$ corresponding to the points $(1, 0, ..., 0), (0, 1, ..., 0), ..., (0, 0, ..., 1)$ of S^{r-1}, that is, $\chi_i(t_i) = 1$ and $\chi_i(t_j) = 0$ if $j \neq i$. Let us determine whether they are in $\Sigma^1(G)$. There are two cases:

1) if $n_i = 1$, then $[\chi_i] \in \Sigma^1(G)$.

This is easy. In fact, if $n_i = 1$, then t_i commutes with a in G and therefore it is in the center $Z(G)$. Since $\chi_i(t_i) = 1 \neq 0$, we have $[\chi_i] \in \Sigma^1(G)$ by Corollary 3.14.

2) if $n_i \geq 2$, then $[\chi_i] \notin \Sigma^1(G)$.

To show this we will use mainly the following relations in G:

$$t_j^k a^N = a^{n_j^k N} t_j^k \text{ and } a^N t_j^{-k} = t_j^{-k} a^{n_j^k N}, \text{ if } k \geq 0, N \in \mathbb{Z}, j = 1, ..., r,$$

Since the t_j commute each other, this means that all the positive powers of the t_j can be entirely pushed to the right and the negative ones can be pushed to the left in a word, up to multiplying the powers of a by some n_j^k, $k \geq 0$.

Suppose by contradiction that $[\chi_i] \in \Sigma^1(G)$, that is, $\Gamma_{\chi_i} = \Gamma(G, S)_{\chi_i}$ is connected. Then in particular there is a path $p = (1, w)$ in Γ_{χ_i} from 1 to the vertex ${t_i}^{-1} a t_i$, with w a word in $W(a, t_1, ..., t_r)$. The first thing to do is to eliminate the letters t_i from w.

Write

$$w = {t_1}^{k_1^1} ... {t_r}^{k_r^1} a^{r_1} {t_1}^{k_1^2} ... {t_r}^{k_r^2} a^{r_2} ... {t_1}^{k_1^m} ... {t_r}^{k_r^m} a^{r_m}, \text{ with } k_j^l, r_j \in \mathbb{Z}.$$

Since p is a path in Γ_{χ_i}, all its vertices have nonnegative χ_i values, and then we have the inequalities

$$\begin{aligned}
k_i^1 &= \chi_i({t_1}^{k_1^1} ... {t_r}^{k_r^1}) \geq 0 \\
k_i^1 + k_i^2 &= \chi_i({t_1}^{k_1^1} ... {t_r}^{k_r^1} a^{r_1} {t_1}^{k_1^2} ... {t_r}^{k_r^2}) \geq 0 \\
&... \\
k_i^1 + k_i^2 + ... + k_i^{m-1} &= \chi_i({t_1}^{k_1^1} ... {t_r}^{k_r^1} a^{r_1} {t_1}^{k_1^2} ... {t_r}^{k_r^2} a^{r_2} ... {t_1}^{k_1^{m-1}} ... {t_r}^{k_r^{m-1}}) \geq 0 \\
k_i^1 + k_i^2 + ... + k_i^{m-1} + k_i^m &= \chi(w) = \chi({t_i}^{-1} a t_i) = -1 + 0 + 1 = 0.
\end{aligned}$$

We now use the relations we just mentioned: since $k_i^1 \geq 0$, push ${t_i}^{k_i^1}$ to the right in w until ${t_i}^{k_i^2}$ and we get ${t_i}^{k_i^1 + k_i^2}$. Since $k_i^1 + k_i^2 \geq 0$, push ${t_i}^{k_i^1 + k_i^2}$ to the right until ${t_i}^{k_i^3}$ and we get ${t_i}^{k_i^1 + k_i^2 + k_i^3}$. We keep doing this until we get ${t_i}^{k_i^1 + k_i^2 + ... + k_i^m}$ as the only t_i letter in w. Since $k_i^1 + k_i^2 + ... + k_i^m = 0$, we eliminated all the t_i from w. Then we can write

$$w = w_1 a^{r_1} w_2 a^{r_2} ... w_m a^{r_m}$$

where $r_j \in \mathbb{Z}$ (different from the first r_j ones) and the w_j are the words ${t_1}^{k_1^j} ... {t_{i-1}}^{k_{i-1}^j} {t_{i+1}}^{k_{i+1}^j} ... {t_r}^{k_r^j}$ above but now without t_i.

Now, since we must have $w = {t_i}^{-1} a t_i$ in G and the w_j commute with t_i, we have

$$a = t_i w {t_i}^{-1} = t_i (w_1 a^{r_1} ... w_m a^{r_m}) {t_i}^{-1} = w_1 a^{n_i r_1} ... w_m a^{n_i r_m},$$

or

$$w_1 a^{n_i r_1} ... w_{m-1} a^{n_i r_{m-1}} w_m a^{n_i r_m - 1} = 1$$

in G. From this expression we will derive a contradiction. Specifically, we will conclude that $n_i = 1$. Note that, in this expression, for all fixed j the sum of all the powers of t_j must be 0, because $\chi_j(w_1 a^{n_i r_1} ... w_{m-1} a^{n_i r_{m-1}} w_m a^{n_i r_m - 1}) = \chi_j(1) = 0$.

We can push now all the positive powers of all the t_j to the right and all the negative ones to the left in this expression. After doing this, we will obtain an expression

$$t_1^{-s_1} ... t_{i-1}^{-s_{i-1}} t_{i+1}^{-s_{i+1}} ... t_r^{-s_r} a^{\alpha_1(n_i r_1) + ... + \alpha_{m-1}(n_i r_{m-1}) + \alpha_m(n_i r_m - 1)} t_r^{s_r} ... t_{i+1}^{s_{i+1}} t_{i-1}^{s_{i-1}} ... t_1^{s_1} = 1,$$

where each α_j is 1 or a product $n_1^{l_1} ... n_{i-1}^{l_{i-1}} n_{i+1}^{l_{i+1}} ... n_r^{l_r}$. Again, note that the powers s_j appear symmetrically since the sum of the powers of each t_j is 0. Then, conjugating the expression we easily obtain

$$a^{\alpha_1(n_i r_1) + ... + \alpha_{m-1}(n_i r_{m-1}) + \alpha_m(n_i r_m - 1)} = 1,$$

and since a is torsion-free we have

$$\alpha_1(n_i r_1) + ... + \alpha_{m-1}(n_i r_{m-1}) + \alpha_m(n_i r_m - 1) = 0.$$

Putting on the left side the multiples of n_i and only α_m on the right we obtain either

$$k n_i = \alpha_m = n_1^{l_1} ... n_{i-1}^{l_{i-1}} n_{i+1}^{l_{i+1}} ... n_r^{l_r}$$

or

$$k n_i = \alpha_m = 1.$$

In the latter case, since $n_i \geq 0$ we must have $n_i = 1$, contradiction. In the former case, n_i divides α_m, so $\gcd(n_i, \alpha_m) = n_i$. On the other hand, since $\alpha_m = n_1^{l_1} ... n_{i-1}^{l_{i-1}} n_{i+1}^{l_{i+1}} ... n_r^{l_r}$ does not involve n_i and the n_j are pairwise coprime, n_i and α_m have no commom prime divisors and then $\gcd(n_i, \alpha_m) = 1$. So, $n_i = \gcd(n_i, \alpha_m) = 1$, a contradiction. Then $[\chi_i] \notin \Sigma^1(G)$ if $n_i \geq 2$, as desired, and we have proved

Theorem 5.2. *The complement $\Sigma^1(\Gamma_{\{n_1,...,n_r\}})^c$ of the group*

$$\Gamma_{\{n_1,...,n_r\}} = \langle a, t_1, ..., t_r \mid t_i t_j = t_j t_i,\ i \neq j,\ t_i a t_i^{-1} = a^{n_i},\ i = 1, ..., r \rangle$$

is given by

$$\Sigma^1(\Gamma_{\{n_1,...,n_r\}})^c = \{[\chi_i] \mid n_i \geq 2\},$$

where $\chi_i(t_i) = 1$ and $\chi_i(t_j) = 0$ if $j \neq i$. In particular, if $n = p_1^{y_1} ... p_r^{y_r}$ then

$$\Sigma^1(\Gamma_n)^c = \{[\chi_1], ..., [\chi_r]\}.$$

Now we can guarantee property R_∞ for $\Gamma_{\{n_1,...,n_r\}}$

Corollary 5.3. *The group $\Gamma_{\{n_1,...,n_r\}}$ has property R_∞. In particular, the solvable generalization Γ_n of the Baumslag-Solitar group has R_∞ property.*

Demonstração. Since $\Sigma^1(\Gamma_{\{n_1,...,n_r\}})^c$ is nonempty, finite and of rational points we can apply Theorem 3.37. Let $\Sigma^1(\Gamma_{\{n_1,...,n_r\}})^c = \{[\chi_{i_1}], ..., [\chi_{i_k}]\}$, $N = \cap_{j=1}^k \ker(\chi_{i_j})$ and $V = Hom(G/N, \mathbb{R})$. We just have to see that the natural induced maps $\overline{\chi_{i_1}}, ..., \overline{\chi_{i_k}}$ on V form a basis of V. The

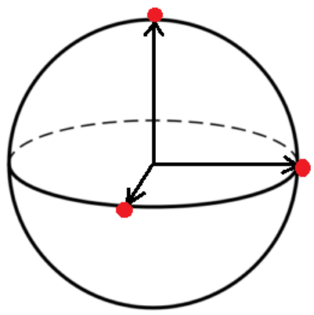


Figura 5.13: case $r = 3$ and $n_1, n_2, n_3 \geq 2$. $\Sigma^1(\Gamma_{\{n_1,n_2,n_3\}})^c$ are the red points.

relation in G/N is given by

$$\overline{g} = \overline{h} \Leftrightarrow \chi_{i_1}(g) = \chi_{i_1}(h), ..., \chi_{i_k}(g) = \chi_{i_k}(h).$$

So, it is easy to see that G/N is f.g. free abelian with basis $\overline{t_{i_1}}, ..., \overline{t_{i_k}}$. Since $\overline{\chi_{i_j}}(\overline{t_{i_j}}) = 1$ and $\overline{\chi_{i_j}}(\overline{t_{i_s}}) = 0$ if $s \neq j$, the elements $\overline{\chi_{i_1}}, ..., \overline{\chi_{i_k}}$ act exactly as a dual basis in V of the $\overline{t_{i_1}}, ..., \overline{t_{i_k}}$, so they form a basis for V. Then, by Theorem 3.37, it follows that $\Gamma_{\{n_1,...,n_r\}}$ has property R_∞. □

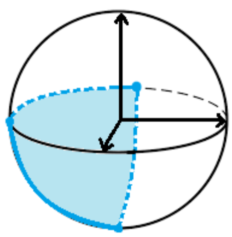
Another easy consequence is

Corollary 5.4. *If $n = p_1^{y_1}...p_r^{y_r}$, the first Ω-invariant $\Omega^1(\Gamma_n)$ of the group*

$$\Gamma_n = \left\langle a, t_1, ..., t_r \mid t_i t_j = t_j t_i,\ i \neq j,\ t_i a t_i^{-1} = a^{p_i^{y_i}},\ i = 1, ..., r \right\rangle$$

is given by

$$\Omega^1(\Gamma_n) = \{[\chi] \in S(\Gamma_n) \mid \chi(t_i) \leq 0\ \forall\ i\}.$$



5.2 Partial generalizations

The techniques we used to compute $\Sigma^1(\Gamma_{\{n_1,...,n_r\}})$ above can be used for some special generalized presentations. Here we show two of them.

Theorem 5.5. *Let*

$$G = \langle a, t, s \mid tat^{-1} = a^n,\ sas^{-1} = a^m,\ tst^{-1}s^{-1} = a^r \rangle$$

for some coprime numbers $n, m \geq 2$ and some $r \in \mathbb{Z}$. Then G has property R_∞.

Demonstração. We have

$$\begin{aligned} H : S(G) &\longrightarrow S^1 \\ [\chi] &\longmapsto \frac{(\chi(t), \chi(s))}{\|(\chi(t), \chi(s))\|}. \end{aligned}$$

Let us compute $\Sigma^1(G)$ by using the geometric criterion. Fix $X = \{a, t, s\}$ and so $Y = \{a, a^{-1}, t, t^{-1}, s, s^{-1}\}$.

- **if** $\chi(t) < 0$ **then** $[\chi] \in \Sigma^1(G)$. Fix t^{-1} such that $\chi(t^{-1}) > 0$. It is straightforward to verify that the paths $p_a = (t^{-1}, a^n)$, $p_{a^{-1}} = (t^{-1}, a^{-n})$, $p_t = (t^{-1}, t)$ and $p_{t^{-1}} = (t^{-1}, t^{-1})$ satisfy the conditions of the criterion and are similar to those we used in $\Gamma_{\{n_1,\ldots,n_r\}}$. The paths p_s and $p_{s^{-1}}$ need to be slightly different though, because t and s do not commute this time. Since $st^{-1} = t^{-1}a^r s$ and $s^{-1}t^{-1} = t^{-1}s^{-1}a^{-r}$, the paths $p_s = (t^{-1}, a^r s)$ and $p_{s^{-1}} = (t^{-1}, s^{-1}a^{-r})$ are also easily seen to satisfy the criterion, so $[\chi] \in \Sigma^1(G)$.

- **if** $\chi(s) < 0$ **then** $[\chi] \in \Sigma^1(G)$. We fix s^{-1} with $\chi(s^{-1}) > 0$. The paths satisfying the geometric criterion are analogous to the previous ones. Define $p_a = (s^{-1}, a^m)$, $p_{a^{-1}} = (s^{-1}, a^{-m})$, $p_s = (s^{-1}, s)$ and $p_{s^{-1}} = (s^{-1}, s^{-1})$. Since $s^{-1}a^{-r}t = ts^{-1}$ and $s^{-1}t^{-1}a^r = t^{-1}s^{-1}$, the paths $p_t = (s^{-1}, a^{-r}t)$ and $p_{t^{-1}} = (s^{-1}, t^{-1}a^r)$ also satisfy the criterion. So $[\chi] \in \Sigma^1(G)$.

  Note that since $ta = a^n t$, $ta^{-1} = a^{-n}t$, $ts = a^r st$ and $ts^{-1} = s^{-1}a^{-r}t$, all the positive t-letters can be pushed right in a word of G without changing its power, and since $at^{-1} = t^{-1}a^n$, $a^{-1}t^{-1} = t^{-1}a^{-n}$, $st^{-1} = t^{-1}a^r s$ and $s^{-1}t^{-1} = t^{-1}s^{-1}a^{-r}$, all the negative t-letters can go left in the same way. The same can be done with s: positive powers to the right and negative ones to the left (obviously, the other adjacent letters may be affected because both t and s are not in the center of G). This is useful for the next two items:

- **if** $\chi(t) = 1$ **and** $\chi(s) = 0$ **then** $[\chi] \notin \Sigma^1(G)$. The strategy is also somewhat similar to the one we used in the case $\Gamma_{\{n_1,\ldots,n_r\}}$. Suppose by contradiction that $[\chi] \in \Sigma^1(G)$. Then, in particular, there is a path $p = (1, w)$ in Γ_χ from 1 to $t^{-1}at$. Write

  $$w = t^{k_{11}} s^{k_{12}} a^{r_1} \ldots t^{k_{c1}} s^{k_{c2}} a^{r_c}.$$

  Since p is contained in Γ_χ, $\chi(t) = 1$ and $\chi(s) = 0$ we must have

  $$k_{11} \geq 0,\ k_{11} + k_{21} \geq 0,\ \ldots,\ k_{11} + \ldots + k_{c-1,1} \geq 0 \text{ and } k_{11} + \ldots + k_{c1} = 0.$$

  We push right $t^{k_{11}}$ until $t^{k_{21}}$, then we push right $t^{k_{11}+k_{21}}$ until $t^{k_{31}}$, and so on. Since $k_{11} + \ldots + k_{c1} = 0$, we eliminate from w all the t-letters and (after relabeling the s and a powers) we can write

  $$w = s^{k_1} a^{r_1} \ldots s^{k_c} a^{r_c}$$

  in G. But, as a vertex, w must be the end if the path p. So we have $w = t^{-1}at$ and therefore

  $$a = twt^{-1} = t(s^{k_1} a^{r_1} \ldots s^{k_c} a^{r_c})t^{-1} = (a^r s)^{k_1} a^{nr_1} \ldots (a^r s)^{k_c} a^{nr_c},$$

or

$$w' = (a^r s)^{k_1} a^{nr_1} ... (a^r s)^{k_{c-1}} a^{nr_{c-1}} (a^r s)^{k_c} a^{nr_c - 1} = 1$$

in G. Since the homomorphism $G \to \mathbb{Z} \times \mathbb{Z}$ with $w \mapsto ((w)^t, (w)^s)$ is well defined in G, we have $k_1 + ...k_c = (w')^s = (1)^s = 0$. Also, $(a^r s)a^M = a^{mM}(a^r s)$ and $a^M(a^r s)^{-1} = (a^r s)^{-1}a^{mM}$ for every $M \in \mathbb{Z}$. This means that, in w', the entire positive pieces $(a^r s)^{k_i}$ can be pushed right and the negative ones can be pushed left. After doing this, we obtain an expression of the form

$$(a^r s)^{-\lambda} a^{\alpha_1 nr_1 + ... + \alpha_{c-1} nr_{c-1} + \alpha_c(nr_c - 1)} (a^r s)^{\lambda} = 1,$$

where each α_i is either 1 or a positive power of m. By conjugating the expression,

$$a^{\alpha_1 nr_1 + ... + \alpha_{c-1} nr_{c-1} + \alpha_c(nr_c - 1)} = 1$$

which implies (since a is torsion-free in G)

$$\alpha_1 nr_1 + ... + \alpha_{c-1} nr_{c-1} + \alpha_c(nr_c - 1) = 0.$$

By putting all the multiples of n above to the left and only α_c on the right, we get either $Mn = 1$ (contradiction with the fact $n \geq 2$) or $Mn = m^Q$ a positive power of m. In the latter case, on one hand $\gcd(n, m^Q) = n$ (because n divides m^Q) and on the other hand $\gcd(n, m^Q) = 1$ because n and m are coprime. Then $n = 1$, also a contradiction.

- **if** $\chi(t) = 0$ **and** $\chi(s) = 1$ **then** $[\chi] \notin \Sigma^1(G)$. This is analogous to the previous item.

Let us identify $S(G) = S^1$ by the homeomorphism H and let $[\chi_1]$ and $[\chi_2]$ be the points of the third and fourth items above, respectively. The first two items showed that the geodesic γ in $S(G)$ between these points (that is, the closed fourth part of the circle) contains $\Sigma^1(G)^c$. We claim that γ is itself invariant in $S(G)$. In fact, if $\varphi \in Aut(G)$ and $p \in \gamma$, then by Lemma 4.21 $\varphi^*(p)$ must be in the geodesic between $\varphi^*[\chi_1]$ and $\varphi^*[\chi_2]$. But these points are in $\Sigma^1(G)^c$ by its invariance and by the third and fourth items; therefore by the two first items they must be in γ. Since γ is a convex subset we have $\varphi^*(p) \in \gamma$, which shows our claim. So we have γ an invariant convex 1-dimensional polytope with the two rational vertices $[\chi_i]$, and so the proposition follows from Theorem 4.28. □

Observation 5.6. Since n and m are coprime in the example above, it is also possible to show that $\chi(t), \chi(s) > 0 \Rightarrow [\chi] \in \Sigma^1(G)$, using the geometric criterion. Then $\Sigma^1(G)^c$ consist of two rational points inside an open halfspace and we could guarantee property R_∞ also by Theorem 4.1. We just wanted do register the usefulness of the kind of strategy we used above.

We actually tried to generalize the theorem above for all groups having the presentation

$$G = \langle \alpha, x_1, ..., x_r \mid x_i \alpha x_i^{-1} = \alpha^{P_i} \ \forall\ i,\ x_i x_j x_i^{-1} x_j^{-1} = \alpha^{R_{ij}} \ \forall\ i, j \rangle$$

for pairwise coprime integers $P_i \geq 1$ and $R_{ij} \in \mathbb{Z}$, but we were not able to show that the points $[\chi_i] \in S(G)$ defined by $\chi_i(x_i) = 1$ and $\chi_i(x_j) = 0$ (for $j \neq i$) are not in $\Sigma^1(G)$. So, because of this, we added an additional hypothesis:

Theorem 5.7. *Let*

$$G = \langle \alpha, x_1, ..., x_r \mid x_i \alpha {x_i}^{-1} = \alpha^{P_i} \ \forall\ i,\ x_i x_j {x_i}^{-1} {x_j}^{-1} = \alpha^{R_{ij}} \ \forall\ i, j \rangle$$

for pairwise coprime integers $P_i \geq 1$ and $R_{ij} \in \mathbb{Z}$. If the commutator G' is not finitely generated, then G has property R_∞.

Demonstração. Let us compute $\Sigma^1(G)$. We have $Y = \{\alpha, \alpha^{-1}, x_1, x_1^{-1}, ..., x_r, x_r^{-1}\}$. By using the Geometric Criterion and paths similar to the ones in the previous theorem, we can show that

- **if there is i such that $\chi(x_i) < 0$ then $[\chi] \in \Sigma^1(G)$.**

  Now, let us show that

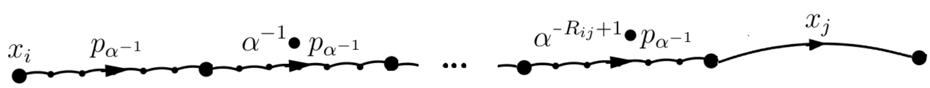
- **if there are $i \neq j$ such that $\chi(x_i), \chi(x_j) > 0$ then $[\chi] \in \Sigma^1(G)$.** In fact, fix x_i with $\chi(x_i) > 0$ and let $M, N \in \mathbb{Z}$ such that $MP_i + NP_j = 1$. Using the relations of G and that $x_i x_j = \alpha^{R_{ij}} x_j x_i$, we can see that the paths $p_\alpha = (x_i, \alpha^M x_j x_i^{-1} \alpha^N x_i x_j^{-1})$, $p_{\alpha^{-1}} = (x_i, \alpha^{-M} x_j x_i^{-1} \alpha^{-N} x_i x_j^{-1})$, $p_{x_i} = (x_i, x_i)$ and $p_{x_i^{-1}} = (x_i, x_i^{-1})$ satisfy the criterion. We just have to build p_{x_j} and $p_{x_j^{-1}}$. If $R_{ij} = 0$, they are $p_{x_j} = (x_i, x_j)$ and $p_{x_j^{-1}} = (x_i, x_j^{-1})$. If $R_{ij} > 0$, we define them as concatenations

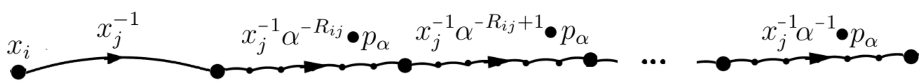
$$p_{x_j} = (p_{\alpha^{-1}})(\alpha^{-1} \cdot p_{\alpha^{-1}})...(\alpha^{-R_{ij}+1} \cdot p_{\alpha^{-1}})(\alpha^{-R_{ij}} x_i, x_j)$$

and

$$p_{x_j^{-1}} = (x_i, x_j^{-1})(x_j^{-1} \alpha^{-R_{ij}} \cdot p_\alpha)(x_j^{-1} \alpha^{-R_{ij}+1} \cdot p_\alpha)...(x_j^{-1} \alpha^{-1} \cdot p_\alpha)$$

(see figure)





If $R_{ij} < 0$, though, the paths are defined as

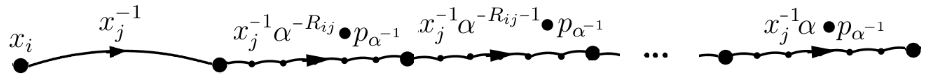
$$p_{x_j} = (p_\alpha)(\alpha \cdot p_\alpha)...(\alpha^{-R_{ij}-1} \cdot p_\alpha)(\alpha^{-R_{ij}} x_i, x_j)$$

and

$$p_{x_j^{-1}} = (x_i, x_j^{-1})(x_j^{-1} \alpha^{-R_{ij}} \cdot p_{\alpha^{-1}})(x_j^{-1} \alpha^{-R_{ij}-1} \cdot p_{\alpha^{-1}})...(x_j^{-1} \alpha \cdot \cdot p_{\alpha^{-1}}).$$

(see figure)

In any of the cases, the paths satisfy the Geometric Criterion and so $[\chi] \in \Sigma^1(G)$.



This shows that $\Sigma^1(G)^c \subset \{[\chi_1], ..., [\chi_r]\}$ where $\chi_i(x_i) = 1$ and $\chi_i(x_j) = 0$ for $j \neq i$. So, $\Sigma^1(G)^c$ is a finite set of rational points contained in an open halfspace of $S(G)$. If G' is not finitely generated, then by Theorem 3.24 $\Sigma^1(G)^c$ is also non-empty and then the result follows from Theorem 3.38. □

Open question: Could we also use Brown's definition of the BNS invariants to compute $\Sigma^1(\Gamma_n)$? Brown's characterization in [17] is given in terms of the possible existence of "non-trivial and abelian" actions of G on $\mathbb{R}$-trees. This corresponds, in the language of our Sections 2.3 and 2.4, to fixed-end actions with no invariant lines. One could start by trying to understand the case of $BS(1,2)$ (or $BS(1,3)$), with the help of [1] to understand the actions, and then by trying to generalize it to Γ_n.

Capítulo 6

Finite index subgroups of Γ_n

In this chapter we turn the investigation to the finite index subgroups H of Γ_n. The reason for doing this is that Γ_n has a nice Σ^1 invariant (Theorem 5.2) and we also have nice results relating Σ^1 of a group G and Σ^1 of a finite index subgroup H (Proposition 3.27 and Corollary 3.28). So it is natural to guess that $\Sigma^1(H)$ maybe should be nice enough to deduce property R_∞ for H, as we did in Corollary 5.3. In fact, we did get an affirmative answer to this question.

First, in Theorem 6.6 we find a nice set of generators for H using a generalization of a technique developed by Bogopolski in [12]. We also get there enough conditions (on these generators) that we may easily compute the index of H, by finding a nice collection of coset representatives of Γ_n mod H. Then, in Theorem 6.8, we find a nice presentation for H and, in Theorem 6.10, we compute $\Sigma^1(H)$ using Proposition 3.28. From this we deduce property R_∞ for H in Corollary 6.11, based on Theorem 3.38. In the last section, we show that some of these H are also Solvable Generalized Baumslag-Solitar groups and some of them are not.

Remember the definition of Γ_n: let $n \geq 2$ be a positive integer with prime decomposition $n = {p_1}^{y_1} ... {p_r}^{y_r}$, the p_i being pairwise distinct and define

$$\Gamma_n = \left\langle a, t_1, ..., t_r \mid t_i t_j = t_j t_i,\ i \neq j,\ t_i a {t_i}^{-1} = a^{{p_i}^{y_i}},\ i = 1, ..., r \right\rangle .$$

It is known that Γ_n is characterized by the following exact sequence

$$1 \to \mathbb{Z}\left[\frac{1}{n}\right] \to \Gamma_n \xrightarrow{\varphi} \mathbb{Z}^r \to 1.$$

To be more precise, if $\mathbb{Z}^r$ has the presentation $\mathbb{Z}^r = \langle t_1, ..., t_r \mid t_i t_j = t_j t_i, i \neq j \rangle$ then there is a natural epimorphism $\Gamma_n \xrightarrow{\varphi} \mathbb{Z}^r$ sending $a \mapsto 0$ and $t_i \mapsto t_i$, whose kernel is (isomorphic to) $\mathbb{Z}\left[\frac{1}{n}\right] = \langle a_j,\ j \in \mathbb{Z} \mid a_j^n = a_{j+1},\ j \in \mathbb{Z} \rangle$ and is generated by the elements

$$a_j = (t_1 ... t_r)^j a (t_1 ... t_r)^{-j}.$$

This exact sequence easily splits with the homomorphism $\mathbb{Z}^r \to \Gamma_n$ sending $t_i \to t_i$. So $\Gamma_n = \mathbb{Z}\left[\frac{1}{n}\right] \rtimes \mathbb{Z}^r$ is the semidirect product of these two subgroups, and so every element $w \in \Gamma_n$ can be written as $w = t_1^{\alpha_1} ... t_r^{\alpha_r} u$ for $u \in \mathbb{Z}\left[\frac{1}{n}\right]$ and $\alpha_i \in \mathbb{Z}$. Two more properties will be useful (and used many times): first, because of the homomorphism $\Gamma_n \xrightarrow{\varphi} \mathbb{Z}^r$, the "$t_i$-coordinates" in Γ_n are well behaved: that is, $(t_1^{\alpha_1} ... t_r^{\alpha_r} u)(t_1^{\beta_1} ... t_r^{\beta_r} u') = t_1^{\alpha_1+\beta_1} ... t_r^{\alpha_r+\beta_r} u''$ for some $u'' \in \mathbb{Z}\left[\frac{1}{n}\right]$. Second, because of the presentation of the subgroup $\mathbb{Z}\left[\frac{1}{n}\right]$, we see that any two generators a_i, a_j must be powers of the common generator $a_{min\{i,j\}}$, so they must commute. So $\mathbb{Z}\left[\frac{1}{n}\right]$ is abelian.

6.1 Generators, cosets and index

Since we will deal with generators of a subgroup, we start by remembering a general and standard argument:

Observation 6.1 (Changing generators argument). Let G be any group and $H = \langle g_1, ..., g_n\rangle \leq G$ be a finitely generated subgroup. Choose some g_i and words w, w' in the elements $g_1, ..., g_{i-1}, g_{i+1}, ..., g_n$ (except g_i). Then

$$H = \langle g_1, ..., g_n\rangle = \langle g_1, ..., g_{i-1}, wg_iw', g_{i+1}, ..., g_n\rangle,$$

that is, we can replace any generator g_i by its product wg_iw' with any words w, w' involving the other generators. Indeed, to show this it suffices to see that $wg_iw' \in \langle g_1, ..., g_n\rangle$ (which is obvious) and that $g_i \in \langle g_1, ..., g_{i-1}, wg_iw', g_{i+1}, ..., g_n\rangle$, which is true because w and w' does not involve the generator g_i.

We will also need the following lemmas:

Lemma 6.2. *If $\varphi : G \to G'$ is a group epimorphism and $H \leq G$ is a subgroup such that $\varphi(H)$ has infinite index in G', then H has infinite index in G.*

Demonstração. We first note that for every epimorphism, the index of the preimage of a group $K \leq G'$ in the group G is the same as the index of K in G'. In fact, denote by $\varphi^{-1}(K)G$ and KG' the collection of right cosets of $\varphi^{-1}(K)$ in G and of K in G', respectively. There is a natural function $\varphi^{-1}(K)G \to KG'$ with $\varphi^{-1}(K)g \to K\varphi(g)$. Since φ is an epimorphism, this is obviously surjective. Now suppose $K\varphi(g) = K\varphi(g')$. By definition, $\varphi(gg'^{-1}) = \varphi(g)\varphi(g')^{-1} \in K$, or $gg'^{-1} \in \varphi^{-1}(K)$ and then $\varphi^{-1}(K)g = \varphi^{-1}(K)g'$, so this is a bijection and we have $|G : \varphi^{-1}(K)| = |G' : K|$, as desired. Let us now show the lemma: if $|G' : \varphi(H)| = \infty$, by the previous comment we have $|G : \varphi^{-1}(\varphi(H))| = |G' : \varphi(H)| = \infty$. Since $H \leq \varphi^{-1}(\varphi(H))$, H is contained in an infinite index subgroup of G and therefore must have infinite index, as desired. □

Lemma 6.3. *Let $n, s \geq 1$ be integers. Let m be the biggest positive divisor of s such that* $\gcd(m, n) = 1$. *Then s divides mn^s.*

Demonstração. Let $n = {p_1}^{l_1}...{p_r}^{l_r}$ and $s = {p_1}^{k_1}...{p_r}^{k_r}$ be the prime decompositions of n and s, with pairwise distinct primes p_i and $0 \leq l_i, k_i$. We define $m' = {p_1}^{\alpha_1}...{p_r}^{\alpha_r}$ with

$$\alpha_i = \begin{cases} k_i, & \text{if } l_i = 0, \\ 0, & \text{if } l_i > 0. \end{cases}$$

Let us show that $m' = m$. Since $\alpha_i \leq k_i$, m' is a divisor of s, and, since $\min\{\alpha_i, l_i\} = 0$ we have $\gcd(m', n) = {p_1}^{\min\{\alpha_1, l_1\}}...{p_r}^{\min\{\alpha_r, l_r\}} = 1$. Finally, m' is the biggest number with these two properties. Indeed, suppose $\tilde{m} = {p_1}^{\beta_1}...{p_r}^{\beta_r}$ has these two properties and let us see that $\beta_i \leq \alpha_i$ for all i, from where we conclude that $\tilde{m} \leq m'$. We must have $\beta_i \leq k_i$ because $\tilde{m}$ divides s and

$\min\{\beta_i, l_i\} = 0$ because $\gcd(\tilde{m}, n) = 1$. If i is such that $l_i = 0$ we have $\beta_i \leq k_i = \alpha_i$. If i is such that $l_i > 0$ then, because $\min\{\beta_i, l_i\} = 0$, we must have $\beta_i = 0$ and then $\beta_i = 0 = \alpha_i$, as desired. By uniqueness, $m = m' = {p_1}^{\alpha_1}...{p_r}^{\alpha_r}$.

Now that we know precisely the number m we show the lemma. We have

$$mn^s = {p_1}^{\alpha_1}...{p_r}^{\alpha_r}({p_1}^{l_1}...{p_r}^{l_r})^{{p_1}^{k_1}...{p_r}^{k_r}} = {p_1}^{\alpha_1+l_1({p_1}^{k_1}...{p_r}^{k_r})}...{p_r}^{\alpha_r+l_r({p_1}^{k_1}...{p_r}^{k_r})},$$

so s will divide mn^s if and only if $k_i \leq \alpha_i + l_i({p_1}^{k_1}...{p_r}^{k_r})$ for all i. If i is such that $l_i = 0$ then by definition $\alpha_i = k_i$ and so $k_i = \alpha_i \leq \alpha_i + l_i({p_1}^{k_1}...{p_r}^{k_r})$. If i is such that $l_i > 0$ (or $l_i \geq 1$) we have

$$k_i \leq {p_i}^{k_i} \leq {p_1}^{k_1}...{p_r}^{k_r} \leq l_i({p_1}^{k_1}...{p_r}^{k_r}) \leq \alpha_i + l_i({p_1}^{k_1}...{p_r}^{k_r}),$$

which completes the proof. □

To find a good set of generators of a finite index subgroup of Γ_n, we must be able to manipulate a little bit its (not so good) generators. To do so, we have the next two lemmas:

Lemma 6.4 (Replacing j_0 by any j). *Let*

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{j_0}^{l}\rangle \leq \Gamma_n \tag{6.1}$$

be a subgroup with arbitrary integers $k_{ii}, l > 0$, $k_{ij} \geq 0$ and $q_i, l_i, j_0 \in \mathbb{Z}$. Then, up to modifying $l > 0$ by another positive integer (also called l), we can replace $a_{j_0}^l$ above by a_j^l for any choosen $j \in \mathbb{Z}$, that is, $H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_j^l\rangle$.

Demonstração. If $j \leq j_0$ we know from the presentation of $\mathbb{Z}\left[\frac{1}{n}\right]$ that a_{j_0} is a positive power of a_j, so $a_{j_0}^l$ is also a positive power of a_j and the lemma is obviously valid. Let us treat the case $j > j_0$. If we conjugate $a_{j_0}^l$ by one of the other r generators, we have $({t_i}^{k_{ii}}...{t_r}^{k_{ir}}a_{q_i}^{l_i})a_{j_0}^l({t_i}^{k_{ii}}...{t_r}^{k_{ir}}a_{q_i}^{l_i})^{-1} = a_{j_0}^{l{p_i}^{y_ik_{ii}}...{p_r}^{y_rk_{ir}}}$, using that $\mathbb{Z}\left[\frac{1}{n}\right]$ is abelian and the relations of Γ_n. By induction, we get

$$({t_i}^{k_{ii}}...{t_r}^{k_{ir}}a_{q_i}^{l_i})^{m_i}a_{j_0}^l({t_i}^{k_{ii}}...{t_r}^{k_{ir}}a_{q_i}^{l_i})^{-m_i} = a_{j_0}^{l{p_i}^{m_iy_ik_{ii}}...{p_r}^{m_iy_rk_{ir}}}$$

for every integer $m_i > 0$. By the exchanging generators argument we can replace $a_{j_0}^l$ in the expression of H by this element $a_{j_0}^{l{p_i}^{m_iy_ik_{ii}}...{p_r}^{m_iy_rk_{ir}}}$, that is, we can multiply the power l of a_{j_0} by ${p_i}^{m_iy_ik_{ii}}...{p_r}^{m_iy_rk_{ir}}$ in 6.1, and since this new power is still positive we can repeat the process recursively. So, by doing this for $i = 1, ..., r$ we can replace the power l of a_{j_0} in 6.1 by any number of the form

$$l({p_1}^{m_1y_1k_{11}}...{p_r}^{m_1y_rk_{1r}})({p_2}^{m_2y_2k_{22}}...{p_r}^{m_2y_2k_{2r}})...({p_r}^{m_ry_rk_{rr}})$$

for any $m_1, ..., m_r > 0$. By putting together the first primes in the parentheses we rewrite this as

$${p_1}^{m_1y_1k_{11}}{p_2}^{m_2y_2k_{22}}...{p_r}^{m_ry_rk_{rr}}l\lambda$$

for some integer $\lambda > 0$ depending on the m_i. In particular, for the integers $m_i = k_{11}...\widehat{k_{ii}}...k_{rr}$

we can replace the power l of a_{j_0} by

$$p_1{}^{y_1 k} p_2{}^{y_2 k} ... p_r{}^{y_r k} l\lambda = n^k l\lambda,$$

where $k = k_{11}...k_{rr}$. But $a_{j_0}^{n^k l\lambda} = a_{j_0+k}^{l\lambda}$. Since $k \geq 1$ this is a positive power of a_{j_0+1}, so we can replace $a_{j_0}^{l}$ in 6.1 by a positive power of a_{j_0+1}. By repeating this a finite number of times we reach the index $j > j_0$ we wanted and the lemma is proved. □

Lemma 6.5 (Replacing l by m)**.** *Let*

$$H = \langle t_1{}^{k_{11}}...t_r{}^{k_{1r}} a_{q_1}^{l_1}, t_2{}^{k_{22}}...t_r{}^{k_{2r}} a_{q_2}^{l_2}, ..., t_r{}^{k_{rr}} a_{q_r}^{l_r}, a_j^l\rangle \leq \Gamma_n \tag{6.2}$$

be a subgroup with arbitrary integers $k_{ii}, l > 0$, $k_{ij} \geq 0$ and $q_i, l_i, j \in \mathbb{Z}$. Let m be the biggest divisor of l such that $\gcd(m,n) = 1$. *Then we can replace a_j^l by a_j^m in the expression above, that is, $H = \langle t_1{}^{k_{11}}...t_r{}^{k_{1r}} a_{q_1}^{l_1}, t_2{}^{k_{22}}...t_r{}^{k_{2r}} a_{q_2}^{l_2}, ..., t_r{}^{k_{rr}} a_{q_r}^{l_r}, a_j^m\rangle$.*

Demonstração. We just have to show that $a_j^l \in \langle t_1{}^{k_{11}}...t_r{}^{k_{1r}} a_{q_1}^{l_1}, t_2{}^{k_{22}}...t_r{}^{k_{2r}} a_{q_2}^{l_2}, ..., t_r{}^{k_{rr}} a_{q_r}^{l_r}, a_j^m\rangle$ and $a_j^m \in H$. The first inclusion is easy, because l is a multiple of m and so a_j^l is a power of a_j^m. The second inclusion is the hard part. By Lemma 6.3, l must divide mn^l, then it must also divide $mn^{lk_{rr}}$. This implies that the number

$$\gamma = \frac{mn^{lk_{rr}} p_1{}^{y_1(k_{11}-1)lk_{rr}} ... p_{r-1}{}^{y_{r-1}(k_{r-1,r-1}-1)lk_{rr}} \prod_{j=1}^{r-1}\prod_{i=j+1}^{r} p_i{}^{y_i k_{ji} k_{rr} l}}{l}$$

is an integer. Let $A_1, ..., A_r$ be the first r generators of H in 6.2, that is, $H = \langle A_1, ..., A_r, a_j^l\rangle$. We will show that

$$A_1{}^{-lk_{rr}} ... A_{r-1}{}^{-lk_{rr}} A_r{}^{-l} (a_j^l)^\gamma A_r{}^l A_{r-1}{}^{lk_{rr}} ... A_1{}^{lk_{rr}} = a_j^m,$$

then $a_j^m \in H$, as desired. To show this, remember that conjugating $(a_j^l)^\gamma$ by an element of $\mathbb{Z}\left[\frac{1}{n}\right]$ is the same as doing nothing, since $\mathbb{Z}\left[\frac{1}{n}\right]$ is abelian. Note that $n^{lk_{rr}} = (p_1{}^{y_1}...p_r{}^{y_r})^{lk_{rr}} = p_1{}^{ly_1k_{rr}} ... p_{r-1}{}^{ly_{r-1}k_{rr}} p_r{}^{ly_r k_{rr}}$ and then

$$\begin{aligned}
\gamma &= \frac{mp_1{}^{ly_1k_{rr}}...p_{r-1}{}^{ly_{r-1}k_{rr}} p_r{}^{ly_rk_{rr}} p_1{}^{y_1(k_{11}-1)lk_{rr}} ... p_{r-1}{}^{y_{r-1}(k_{r-1,r-1}-1)lk_{rr}} \prod_{j=1}^{r-1}\prod_{i=j+1}^{r} p_i{}^{y_i k_{ji} k_{rr} l}}{l} \\
&= \frac{mp_r{}^{ly_rk_{rr}} p_1{}^{y_1 k_{11} k_{rr} l} ... p_{r-1}{}^{y_{r-1}k_{r-1,r-1}k_{rr}l} \prod_{j=1}^{r-1}\prod_{i=j+1}^{r} p_i{}^{y_i k_{ji} k_{rr} l}}{l} \\
&= \frac{p_r{}^{ly_rk_{rr}} m \prod_{j=1}^{r-1}\prod_{i=j}^{r} p_i{}^{y_i k_{ji} k_{rr} l}}{l}.
\end{aligned}$$

So

$$\begin{aligned}
A_r{}^{-l}(a_j^l)^\gamma A_r{}^l &= (t_r{}^{k_{rr}} a_{q_r}^{l_r})^{-l} (a_j)^{p_r{}^{ly_rk_{rr}} m \prod_{j=1}^{r-1}\prod_{i=j}^{r} p_i{}^{y_i k_{ji} k_{rr} l}} (t_r{}^{k_{rr}} a_{q_r}^{l_r})^l \\
&= t_r{}^{-k_{rr}l} (a_j)^{p_r{}^{ly_rk_{rr}} m \prod_{j=1}^{r-1}\prod_{i=j}^{r} p_i{}^{y_i k_{ji} k_{rr} l}} t_r{}^{k_{rr}l} \\
&= (a_j)^{m \prod_{j=1}^{r-1}\prod_{i=j}^{r} p_i{}^{y_i k_{ji} k_{rr} l}}.
\end{aligned}$$

For each $1 \leq s \leq r-1$, let $E_s = \prod_{j=1}^{s}\prod_{i=j}^{r} p_i{}^{y_i k_{ji} k_{rr} l}$. We just showed that $A_r{}^{-l}(a_j^l)^\gamma A_r{}^l =$

$(a_j)^{mE_{r-1}}$. Write now $mE_{r-1} = \prod_{i=r-1}^{r} p_i{}^{y_i k_{r-1,i} k_{rr} l} mE_{r-2}$. Then

$$\begin{aligned} A_{r-1}{}^{-lk_{rr}} A_r{}^{-l} (a_j^l)^\gamma A_r{}^l A_{r-1}{}^{lk_{rr}} &= t_{r-1}^{-k_{r-1,r-1}k_{rr}l} t_r^{-k_{r-1,r}k_{rr}l} (a_j)^{mE_{r-1}} t_r^{k_{r-1,r}k_{rr}l} t_{r-1}^{k_{r-1,r-1}k_{rr}l} \\ &= t_{r-1}^{-k_{r-1,r-1}k_{rr}l} t_r^{-k_{r-1,r}k_{rr}l} (a_j)^{\prod_{i=r-1}^{r} p_i{}^{y_i k_{r-1,i} k_{rr} l} mE_{r-2}} \\ &\quad t_r^{k_{r-1,r}k_{rr}l} t_{r-1}^{k_{r-1,r-1}k_{rr}l} \\ &= (a_j)^{mE_{r-2}}. \end{aligned}$$

Suppose by induction that, for some $2 \leq s \leq r-1$,

$$A_s^{-lk_{rr}} \dots A_{r-1}^{-lk_{rr}} A_r^{-l} (a_j^l)^\gamma A_r^l A_{r-1}^{lk_{rr}} \dots A_s^{lk_{rr}} = (a_j)^{mE_{s-1}}.$$

Write $E_{s-1} = \prod_{i=s-1}^{r} p_i{}^{y_i k_{s-1,i} k_{rr} l} E_{s-2}$. Then

$$\begin{aligned} A_{s-1}^{-lk_{rr}} \dots A_{r-1}^{-lk_{rr}} A_r^{-l} (a_j^l)^\gamma A_r^l A_{r-1}^{lk_{rr}} \dots A_{s-1}^{lk_{rr}} &= A_{s-1}^{-lk_{rr}} (a_j)^{mE_{s-1}} A_{s-1}^{lk_{rr}} \\ &= A_{s-1}^{-lk_{rr}} (a_j)^{\prod_{i=s-1}^{r} p_i{}^{y_i k_{s-1,i} k_{rr} l} mE_{s-2}} A_{s-1}^{lk_{rr}} \\ &= (a_j)^{mE_{s-2}}. \end{aligned}$$

By induction,

$$A_2^{-lk_{rr}} \dots A_{r-1}^{-lk_{rr}} A_r^{-l} (a_j^l)^\gamma A_r^l A_{r-1}^{lk_{rr}} \dots A_2^{lk_{rr}} = (a_j)^{mE_1} = (a_j)^{mp_1{}^{y_1 k_{11} k_{rr} l} \dots p_r{}^{y_r k_{1r} k_{rr} l}}$$

and finally

$$\begin{aligned} A_1^{-lk_{rr}} \dots A_{r-1}^{-lk_{rr}} A_r^{-l} (a_j^l)^\gamma A_r^l A_{r-1}^{lk_{rr}} \dots A_1^{lk_{rr}} &= A_1^{-lk_{rr}} (a_j)^{mp_1{}^{y_1 k_{11} k_{rr} l} \dots p_r{}^{y_r k_{1r} k_{rr} l}} A_1^{lk_{rr}} \\ &= t_1^{-k_{11}k_{rr}l} \dots t_r^{-k_{1r}k_{rr}l} (a_j)^{mp_1{}^{y_1 k_{11} k_{rr} l} \dots p_r{}^{y_r k_{1r} k_{rr} l}} t_r^{k_{1r}k_{rr}l} \dots t_1^{k_{11}k_{rr}l} \\ &= a_j^m, \end{aligned}$$

which shows the lemma. □

Theorem 6.6. 1) *Every finite index subgroup H of Γ_n can be written as*

$$H = \langle t_1{}^{k_{11}} \dots t_r{}^{k_{1r}} a^{l_1}, t_2{}^{k_{22}} \dots t_r{}^{k_{2r}} a^{l_2}, \dots, t_r{}^{k_{rr}} a^{l_r}, a^m \rangle \qquad (*)$$

with $k_{11} > 0$, $0 \leq k_{ji} < k_{ii}$ for all $1 \leq j < i \leq r$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m,n) = 1$ *and $H \cap \langle a \rangle = \langle a^m \rangle$.*

2) *If H is any subgroup of Γ_n given by the expression $(*)$ for $0 \leq k_{1i}, \dots, k_{i-1,i} < k_{ii}$, $l_i \in \mathbb{Z}$ and $m > 0$ such that* $\gcd(m,n) = 1$ *and $H \cap \langle a \rangle = \langle a^m \rangle$, then $T = \{t_1{}^{\beta_1} \dots t_r{}^{\beta_r} a^j \mid 0 \leq \beta_i < k_{ii},\ 0 \leq j < m\}$ is a transversal of H in Γ_n. In particular, the index of H in Γ_n is $k_{11} \dots k_{rr} m$ and H has finite index in Γ_n.*

Demonstração. (Item 1)). First, since Γ_n is finitely generated and H is finite index, by Corollary

1.51 H must be also finitely generated and we write

$$H = \langle t_1{}^{\alpha_{11}}...t_r{}^{\alpha_{1r}}v_1, ..., t_1{}^{\alpha_{m1}}...t_r{}^{\alpha_{mr}}v_m\rangle$$

for $\alpha_{ij} \in \mathbb{Z}$ and $v_i \in \mathbb{Z}\left[\frac{1}{n}\right]$. Note that $m \geq r$. Otherwise, $\varphi(H)$ would be a subgroup of $\mathbb{Z}^r$ with rank $< r$ and then would have infinite index by Lemma 1.4, so by Lemma 6.2 H would have infinite index in Γ_n. Let us denote by $\varphi_i : \Gamma_n \to \mathbb{Z}^i$ the (surjective) composition $\Gamma_n \stackrel{\varphi}{\to} \mathbb{Z}^r \stackrel{\pi}{\to} \mathbb{Z}^i$ of φ with the natural projection π of $\mathbb{Z}^r$ onto the first i coordinates. There must be at least one i such that $\alpha_{i1} \neq 0$. Otherwise, $\varphi_1(H) = 0 \leq \mathbb{Z}$ would be infinite index and by Lemma 6.2 H would be infinite index. Let $k_{11} = \gcd_{\alpha_{i1}\neq 0} \{\alpha_{i1}\}$. Since $k_{11} > 0$ is the smallest positive linear combination of the $\alpha_{i1} \neq 0$ and since the t_i-coordinates are well behaved in Γ_n, we can obtain inside H an element of the form $t_1{}^{k_{11}}...t_r{}^{k_{1r}}u_1$ for some $k_{12}, ..., k_{1r} \in \mathbb{Z}$ and $u_1 \in \mathbb{Z}\left[\frac{1}{n}\right]$, so we can write

$$H = \langle t_1{}^{\alpha_{11}}...t_r{}^{\alpha_{1r}}v_1, ..., t_1{}^{\alpha_{m1}}...t_r{}^{\alpha_{mr}}v_m, t_1{}^{k_{11}}...t_r{}^{k_{1r}}u_1\rangle. \tag{6.3}$$

Now, since all the nonzero α_{i1} are multiples of k_{11}, say, $\alpha_{i1} = d_i k_{11}$, by the changing generators argument we can replace $t_1{}^{\alpha_{i1}}...t_r{}^{\alpha_{ir}}v_i$ by $(t_1{}^{\alpha_{i1}}...t_r{}^{\alpha_{ir}}v_i)(t_1{}^{k_{11}}...t_r{}^{k_{1r}}u_1)^{-d_i} = t_2{}^{\alpha'_{i2}}...t_r{}^{\alpha'_{ir}}v'_i$ in 6.3. Then, after relabeling these new generators, we can write

$$H = \langle t_2{}^{\alpha_{12}}...t_r{}^{\alpha_{1r}}v_1, ..., t_2{}^{\alpha_{m2}}...t_r{}^{\alpha_{mr}}v_m, t_1{}^{k_{11}}...t_r{}^{k_{1r}}u_1\rangle.$$

We added a new generator and "eliminated" all the t_1 coordinates of the first m generators of H. This was the first step. We have to do this for all the other $t_2, ..., t_r$ coordinates. Suppose that, after $j-1 < r$ steps we have obtained

$$H = \langle t_j{}^{\alpha_{1j}}...t_r{}^{\alpha_{1r}}v_1, ..., t_j{}^{\alpha_{mj}}...t_r{}^{\alpha_{mr}}v_m, t_1{}^{k_{11}}...t_r{}^{k_{1r}}u_1, t_2{}^{k_{22}}...t_r{}^{k_{2r}}u_2, ..., t_{j-1}{}^{k_{(j-1)(j-1)}}...t_r{}^{k_{(j-1)r}}u_{j-1}\rangle \tag{6.4}$$

for some integer powers α, k and with $k_{11}, ..., k_{(j-1)(j-1)} > 0$. Let us describe the j^{th} step. There must be at least one i such that $\alpha_{ij} \neq 0$. Otherwise, $\varphi_j(H) = \langle t_1{}^{k_{11}}...t_j{}^{k_{1j}}, t_2{}^{k_{22}}...t_j{}^{k_{2j}}, ..., t_{j-1}{}^{k_{(j-1)(j-1)}}t_j{}^{k_{(j-1)j}}\rangle \leq \mathbb{Z}^j$ is generated by $j-1$ elements and therefore have rank at most $j-1$. By Lemma 1.4, it would be infinite index in $\mathbb{Z}^j$, so by Lemma 6.2 H would be infinite index in Γ_n. Let $k_{jj} = \gcd_{\alpha_{ij}\neq 0} \{\alpha_{ij}\} > 0$. Similarly as we did above, there must be an element of the form $t_j{}^{k_{jj}}...t_r{}^{k_{jr}}u_j$ in H (written as a product of the m first generators of 6.4). We can add this new generator to the expression 6.4. Also, since all the nonzero α_{ij} are multiples of k_{jj} and the t_j coordinate is well behaved in Γ_n, we can use the exchanging generators argument as in step 1 to eliminate the t_j letters from the first m generators. After relabeling these new m generators we obtain

$$H = \langle t_{j+1}{}^{\alpha_{1(j+1)}}...t_r{}^{\alpha_{1r}}v_1, ..., t_{j+1}{}^{\alpha_{m(j+1)}}...t_r{}^{\alpha_{mr}}v_m, t_1{}^{k_{11}}...t_r{}^{k_{1r}}u_1, t_2{}^{k_{22}}...t_r{}^{k_{2r}}u_2, ..., t_j{}^{k_{jj}}...t_r{}^{k_{jr}}u_j\rangle$$

for some integer powers α, k and with $k_{11}, ..., k_{jj} > 0$, which completes the inductive step. After r steps, we added r new generators and eliminated all the $t_1, ..., t_r$ letters from the first m generators from H, so we have

$$H = \langle v_1, ..., v_m, t_1{}^{k_{11}}...t_r{}^{k_{1r}}u_1, t_2{}^{k_{22}}...t_r{}^{k_{2r}}u_2, ..., t_r{}^{k_{rr}}u_r\rangle$$

with $k_{ii} > 0$ and $v_i, u_i \in \mathbb{Z}\left[\frac{1}{n}\right]$. Since every finitely generated subgroup of $\mathbb{Z}\left[\frac{1}{n}\right]$ is cyclic we have $\langle v_1, ..., v_m\rangle = \langle u\rangle$ for some $u \in \mathbb{Z}\left[\frac{1}{n}\right]$ and

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}u_1, {t_2}^{k_{22}}...{t_r}^{k_{2r}}u_2, ..., {t_r}^{k_{rr}}u_r, u\rangle \tag{6.5}$$

We have $({t_{r-1}}^{k_{(r-1)(r-1)}}{t_r}^{k_{(r-1)r}}u_{r-1})({t_r}^{k_{rr}}u_r) = {t_{r-1}}^{k_{(r-1)(r-1)}}{t_r}^{k_{(r-1)r}+k_{rr}}u'$, so by the exchanging generators argument we could replace the generator ${t_{r-1}}^{k_{(r-1)(r-1)}}{t_r}^{k_{(r-1)r}}u_{r-1}$ of H by this product, with the same t_{r-1}-power but with higher t_r-power $k_{(r-1)r} + k_{rr} > k_{(r-1)r}$. So, by doing this process a finite number of times we may suppose that $0 \leq k_{(r-1)r} < k_{rr}$ (or even $k_{(r-1)r} > 0$ if we wanted). Now we use this fact together with the exchanging generators argument for the $r^{th}, (r-1)^{th}$ and $(r-2)^{th}$ generators and we may similarly suppose that $0 \leq k_{r-2,r-1} < k_{r-1,r-1}$ and $0 \leq k_{r-2,r} < k_{rr}$ (or both could be positive if we wanted). By doing this recursively, we may suppose that $0 \leq k_{ji} < k_{ii}$ for all $1 \leq j < i \leq r$ in 6.5. Finally, write $u_i = a_{q_i}^{l_i}, u = a_q^l$ for $q_i, q, l_i, l \in \mathbb{Z}$. Then

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_q^l\rangle. \tag{6.6}$$

Let us show that we may assume $l > 0$ above. If $l \neq 0$ then, up to changing a_q^l by $(a_q^l)^{-1} = a_q^{-1}$ if necessary, we are done. If $l = 0$, that is,

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}\rangle, \tag{6.7}$$

we do the following: since $\mathbb{Z}^r$ is abelian, every commutator of elements in H must be in $\ker(\varphi)$ (and obviously in H). Look to all the commutators between the r generators of H in 6.7: at least one of them must be non-trivial. Otherwise, H would be a finite index abelian subgroup of Γ_n and we would have $\Sigma^1(\Gamma_n) = S(\Gamma_n)$ by Corollary 3.29, a contradiction with Theorem 5.2. Let then $a_j^{l'}$ ($l' \neq 0$) be a non-trivial commutator between two generators of H. We can add it to 6.7 and up to changing $a_j^{l'}$ by its inverse we are done. So we may assume $l > 0$ in 6.6.

Our next steps will be eliminating the subindexes q_i from the a letters in the generators of 6.6. Fix some $1 \leq i \leq r$. If $q_i \geq 0$, then $a_{q_i}^{l_i} = a^{n^{q_i}l_i}$ and by doing this substitution in 6.6 and relabeling $n^{q_i}l_i$ by l_i again we removed the subindex q_i. If $q_i < 0$, by Lemma 6.4 we replace q by q_i in 6.6, that is,

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^l\rangle$$

for some new positive integer l. Now, let m be the biggest divisor of l such that $\gcd(m, n) = 1$. By Lemma 6.5 we can replace l by m above and obtain

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^m\rangle.$$

Since $\gcd(m, n) = 1$ we also have $\gcd(m, n^{-q_i}) = 1$ and there must be $\tilde{\alpha}, \tilde{\beta} \in \mathbb{Z}$ such that $\tilde{\alpha}m + \tilde{\beta}n^{-q_i} = 1$. Then for $\alpha = l_i\tilde{\alpha}$ and $\beta = l_i\tilde{\beta}$ we have $\alpha m + \beta n^{-q_i} = l_i$, or

$$l_i - m\alpha = n^{-q_i}\beta.$$

Then, using the changing generators argument and the relations in Γ_n we have

$$\begin{aligned} H &= \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a_{q_i}^{l_i}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^{m}\rangle \\ &= \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a_{q_i}^{l_i-m\alpha}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^{m}\rangle \\ &= \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a_{q_i}^{n^{-q_i}\beta}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^{m}\rangle \\ &= \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a^{\beta}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_{q_i}^{m}\rangle \end{aligned}$$

and relabeling β by l_i, m by l and q_i by q again we have

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a_{q_1}^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a_{q_2}^{l_2}, ..., {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a^{l_i}, ..., {t_r}^{k_{rr}}a_{q_r}^{l_r}, a_q^{l}\rangle,$$

that is, we removed the subindex q_i from $a_{q_i}^{l_i}$ in 6.6. If we do this for all i we remove all the subindexes and obtain

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}, ..., {t_r}^{k_{rr}}a^{l_r}, a_q^{l}\rangle$$

for some $q \in \mathbb{Z}$. We can use Lemma 6.4 to replace q by 0 and we get

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}, ..., {t_r}^{k_{rr}}a^{l_r}, a^{l}\rangle,\ k_{ij}, l > 0,\ l_i \in \mathbb{Z}.$$

To finish, let m (a new one) be the biggest divisor of l such that $\gcd(m, n) = 1$. By Lemma 6.5, we replace a^l by a^m in the expression above. If $H \cap \langle a\rangle = \langle a^m\rangle$, we are done. If not, let $m' = \min\{k \geq 1 \mid a^k \in H\}$ (this set is not empty because it contains m). We claim that $H \cap \langle a\rangle = \left\langle a^{m'}\right\rangle$. The "($\supset$)" part is obvious. On the other hand, if some $a^l \in H$, write $l = qm' + \tilde{r}$ for some integer q and $0 \leq \tilde{r} < m'$. Then $a^{\tilde{r}} = a^{l-qm'} = (a^l)(a^{m'})^{-q} \in H$ implies $\tilde{r} = 0$ (minimality of m') and so $a^l = (a^{m'})^q \in \left\langle a^{m'}\right\rangle$, which shows the claim. Since $a^m \in H$, m is a multiple of m' and we have $\gcd(m', n) = 1$. Then, by adding $a^{m'}$ to the set of generators of H, the generator a^m becomes useless and can be removed. By relabeling m' by m, we obtain the desired result.

(Item 2)). Let H be such a subgroup. Using the same argument from item 1), we may suppose that $k_{ij} > 0$ for all i, j. Let us first show that $G = \bigcup_{{t_1}^{\beta_1}...{t_r}^{\beta_r}a^j \in T} H{t_1}^{\beta_1}...{t_r}^{\beta_r}a^j$. Every element of G is written as ${t_1}^{-\alpha_1}...{t_r}^{-\alpha_r}a^l{t_1}^{\gamma_1}...{t_r}^{\gamma_r}$ for $\alpha_i, \gamma_i \geq 0$ and $l \in \mathbb{Z}$. Since $k_{ij} > 0$ for all i, j, let $q \geq 1$ be such that $qk_{1i} - \alpha_i \geq 0$ for all i. We can write $({t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1})^q = a^{l'_1}{t_1}^{qk_{11}}...{t_r}^{qk_{1r}}$ for some integer l'_1. Since ${t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1} \in H$ we have

$$\begin{aligned} H{t_1}^{-\alpha_1}...{t_r}^{-\alpha_r}a^l{t_1}^{\gamma_1}...{t_r}^{\gamma_r} &= H({t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1})^q{t_1}^{-\alpha_1}...{t_r}^{-\alpha_r}a^l{t_1}^{\gamma_1}...{t_r}^{\gamma_r} \\ &= H(a^{l'_1}{t_1}^{qk_{11}}...{t_r}^{qk_{1r}}){t_1}^{-\alpha_1}...{t_r}^{-\alpha_r}a^l{t_1}^{\gamma_1}...{t_r}^{\gamma_r} \\ &= Ha^{l'_1}{t_1}^{qk_{11}-\alpha_1}...{t_r}^{qk_{1r}-\alpha_r}a^l{t_1}^{\gamma_1}...{t_r}^{\gamma_r} \\ &= Ha^{l'_1}a^{l''}{t_1}^{\gamma_1+qk_{11}-\alpha_1}...{t_r}^{\gamma_r+qk_{1r}-\alpha_r} \\ &= Ha^{l'}{t_1}^{\gamma'_1}...{t_r}^{\gamma'_r} \end{aligned}$$

for some integers l' and $\gamma_i' \geq 0$. Relabeling them by l and γ_i, respectively, every coset of G is of the form $Ha^l {t_1}^{\gamma_1}...{t_r}^{\gamma_r}$ for $l \in \mathbb{Z}$ and $\gamma_i \geq 0$. Now we claim that every such coset can be also written as $H{t_1}^{\gamma_1}...{t_r}^{\gamma_r}a^{l'}$ for some integer l'. In fact, because $1 = \gcd(m,n) = \gcd(m, {p_1}^{y_1}...{p_r}^{y_r})$, the prime decomposition of m does not involve any of the p_i. Then it is also true that $\gcd(m, {p_1}^{\gamma_1 y_1}...{p_r}^{\gamma_r y_r}) = 1$. Let k, k' be integers such that $km + k'{p_1}^{\gamma_1 y_1}...{p_r}^{\gamma_r y_r} = 1$. Then $l + (-lk)m = (lk'){p_1}^{\gamma_1 y_1}...{p_r}^{\gamma_r y_r}$ and relabeling $-lk$ by k and lk' by k' we get $l + km = k'{p_1}^{\gamma_1 y_1}...{p_r}^{\gamma_r y_r}$. Now since $a^m \in H$ we do

$$\begin{aligned} Ha^l{t_1}^{\gamma_1}...{t_r}^{\gamma_r} &= H(a^m)^k a^l {t_1}^{\gamma_1}...{t_r}^{\gamma_r} \\ &= Ha^{l+km}{t_1}^{\gamma_1}...{t_r}^{\gamma_r} \\ &= Ha^{k'{p_1}^{\gamma_1 y_1}...{p_r}^{\gamma_r y_r}}{t_1}^{\gamma_1}...{t_r}^{\gamma_r} \\ &= H{t_1}^{\gamma_1}...{t_r}^{\gamma_r}a^{k'} \end{aligned}$$

and relabeling k' by l' we showed the claim. So every coset is of the form $H{t_1}^{\gamma_1}...{t_r}^{\gamma_r}a^l$ with $\gamma_i \geq 0$ and $l \in \mathbb{Z}$. To transform this coset into one of the cosets in the theorem, we will apply successive algorithms, defined as follows: choose some index i. If $\gamma_i < k_{ii}$ we stop the algorithm. If $\gamma_i \geq k_{ii}$, we do

$$\begin{aligned} H{t_1}^{\gamma_1}...{t_r}^{\gamma_r}a^l &= H({t_i}^{k_{ii}}...{t_r}^{k_{ir}}a^{l_i})^{-1}{t_1}^{\gamma_1}...{t_r}^{\gamma_r}a^l \\ &= Ha^{-l_i}{t_i}^{-k_{ii}}...{t_r}^{-k_{ir}}{t_1}^{\gamma_1}...{t_r}^{\gamma_r}a^l \\ &= Ha^{-l_i}{t_{i+1}}^{-k_{i,i+1}}...{t_r}^{-k_{ir}}{t_1}^{\gamma_1}...{t_{i-1}}^{\gamma_{i-1}}{t_i}^{\gamma_i-k_{ii}}{t_{i+1}}^{\gamma_{i+1}}...{t_r}^{\gamma_r}a^l \\ &= H{t_{i+1}}^{-k_{i,i+1}}...{t_r}^{-k_{ir}}a^{l_i'}{t_1}^{\gamma_1}...{t_{i-1}}^{\gamma_{i-1}}{t_i}^{\gamma_i-k_{ii}}{t_{i+1}}^{\gamma_{i+1}}...{t_r}^{\gamma_r}a^l \end{aligned}$$

which we abbreviate to $H{t_{i+1}}^{-k_{i,i+1}}...{t_r}^{-k_{ir}}a^{l_i'}{t_1}^{\gamma_1}...{t_i}^{\gamma_i-k_{ii}}...{t_r}^{\gamma_r}a^l$. Now let $q \geq 1$ be such that $qk_{i+1,j} - k_{ij} \geq 0$ for all $i+1 \leq j \leq r$. Then

$$\begin{aligned} & H{t_{i+1}}^{-k_{i,i+1}}...{t_r}^{-k_{ir}}a^{l_i'}{t_1}^{\gamma_1}...{t_i}^{\gamma_i-k_{ii}}...{t_r}^{\gamma_r}a^l = \\ = \; & H({t_{i+1}}^{k_{i+1,i+1}}...{t_r}^{k_{i+1,r}}a^{l_{i+1}})^q{t_{i+1}}^{-k_{i,i+1}}...{t_r}^{-k_{ir}}a^{l_i'}{t_1}^{\gamma_1}...{t_i}^{\gamma_i-k_{ii}}...{t_r}^{\gamma_r}a^l \\ = \; & Ha^{l_{i+1}'}{t_{i+1}}^{qk_{i+1,i+1}}...{t_r}^{qk_{i+1,r}}{t_{i+1}}^{-k_{i,i+1}}...{t_r}^{-k_{ir}}a^{l_i'}{t_1}^{\gamma_1}...{t_i}^{\gamma_i-k_{ii}}...{t_r}^{\gamma_r}a^l \\ = \; & Ha^{l_{i+1}'}{t_{i+1}}^{qk_{i+1,i+1}-k_{i,i+1}}...{t_r}^{qk_{i+1,r}-k_{i,r}}a^{l_i'}{t_1}^{\gamma_1}...{t_i}^{\gamma_i-k_{ii}}...{t_r}^{\gamma_r}a^l \\ = \; & Ha^{l_{i+1}'}a^{l_i''}a^{l'}{t_{i+1}}^{qk_{i+1,i+1}-k_{i,i+1}}...{t_r}^{qk_{i+1,r}-k_{i,r}}{t_1}^{\gamma_1}...{t_i}^{\gamma_i-k_{ii}}...{t_r}^{\gamma_r} \\ = \; & Ha^{l''}{t_1}^{\gamma_1}...{t_{i-1}}^{\gamma_{i-1}}{t_i}^{\gamma_i-k_{ii}}{t_{i+1}}^{\gamma_{i+1}+qk_{i+1,i+1}-k_{i,i+1}}...{t_r}^{\gamma_r+qk_{i+1,r}-k_{i,r}} \\ = \; & H{t_1}^{\gamma_1}...{t_{i-1}}^{\gamma_{i-1}}{t_i}^{\gamma_i-k_{ii}}{t_{i+1}}^{\gamma_{i+1}+qk_{i+1,i+1}-k_{i,i+1}}...{t_r}^{\gamma_r+qk_{i+1,r}-k_{i,r}}a^{l'''}, \end{aligned}$$

using the claim in the last equality. By relabeling the $i+1, ..., r$ powers we have shown that

$$H{t_1}^{\gamma_1}...{t_r}^{\gamma_r}a^l = H{t_1}^{\gamma_1}...{t_{i-1}}^{\gamma_{i-1}}{t_i}^{\gamma_i-k_{ii}}{t_{i+1}}^{\gamma_{i+1}'}...{t_r}^{\gamma_r'}a^{l'}$$

for some integer l'. If $\gamma_i - k_{ii} < k_{ii}$ we stop the algorithm. If $\gamma_i - k_{ii} \geq k_{ii}$ we do all of this again. Then after finite steps our "i-algorithm" shows that

$$Ht_1{}^{\gamma_1}...t_r{}^{\gamma_r}a^l = Ht_1{}^{\gamma_1}...t_{i-1}{}^{\gamma_{i-1}}t_i{}^{\beta_i}t_{i+1}{}^{\gamma'_{i+1}}...t_r{}^{\gamma'_r}a^{l'}$$

for some $0 \leq \beta_i < k_{ii}$. Now, starting with the coset $Ht_1{}^{\gamma_1}...t_r{}^{\gamma_r}a^l$, we successively apply the "i-algorithm" for $i = 1, 2, ..., r$ and obtain exactly

$$Ht_1{}^{\gamma_1}...t_r{}^{\gamma_r}a^l = Ht_1{}^{\beta_1}...t_r{}^{\beta_r}a^{l'}$$

for $0 \leq \beta_i < k_{ii}$ and $l' \in \mathbb{Z}$. Finally, write $l' = qm + j$ for $0 \leq j < m$. Then $Ht_1{}^{\beta_1}...t_r{}^{\beta_r}a^{l'} = Ht_1{}^{\beta_1}...t_r{}^{\beta_r}a^j$ because

$$\begin{aligned} t_1{}^{\beta_1}...t_r{}^{\beta_r}a^{l'}(t_1{}^{\beta_1}...t_r{}^{\beta_r}a^j)^{-1} &= t_1{}^{\beta_1}...t_r{}^{\beta_r}a^{l'-j}t_r{}^{-\beta_r}...t_1{}^{-\beta_1} \\ &= t_1{}^{\beta_1}...t_r{}^{\beta_r}a^{mq}t_r{}^{-\beta_r}...t_1{}^{-\beta_1} \\ &= (a^m)^{qp_1^{\beta_1 y_1}...p_r{}^{\beta_r y_r}} \in H. \end{aligned}$$

This shows that $G = \bigcup_{t_1{}^{\beta_1}...t_r{}^{\beta_r}a^j \in T} Ht_1{}^{\beta_1}...t_r{}^{\beta_r}a^j$.

Now let us show that the cosets in T are all distinct. Let $Ht_1{}^{\beta_1}...t_r{}^{\beta_r}a^j = Ht_1{}^{\beta'_1}...t_r{}^{\beta'_r}a^{j'}$ for $0 \leq \beta_i, \beta'_i < k_{ii}$ and $0 \leq j, j' < m$. By definition,

$$\begin{aligned} w = a^{p_1{}^{y_1\beta_1}...p_r{}^{y_r\beta_r}(j-j')}t_1{}^{\beta_1-\beta'_1}...t_r{}^{\beta_r-\beta'_r} &= t_1{}^{\beta_1}...t_r{}^{\beta_r}a^{j-j'}t_1{}^{-\beta_1}...t_r{}^{-\beta_r}t_1{}^{\beta_1-\beta'_1}...t_r{}^{\beta_r-\beta'_r} \\ &= t_1{}^{\beta_1}...t_r{}^{\beta_r}a^j(t_1{}^{\beta'_1}...t_r{}^{\beta'_r}a^{j'})^{-1} \in H. \end{aligned}$$

Then, projecting in $\mathbb{Z}^r$,

$$(\beta_1 - \beta'_1, ..., \beta_r - \beta'_r) = \varphi(w) \in \varphi(H) = \langle (k_{11}, k_{12}, ..., k_{1r}), (0, k_{22}, ..., k_{2r}), ..., (0, ..., 0, k_{rr}) \rangle .$$

Write

$$(\beta_1 - \beta'_1, ..., \beta_r - \beta'_r) = \lambda_1(k_{11}, k_{12}, ..., k_{1r}) + \lambda_2(0, k_{22}, ..., k_{2r}) + ... + \lambda_r(0, ..., 0, k_{rr})$$

for integers λ_i. We show by induction that all the λ_i must vanish. First, since the first vector $(k_{11}, k_{12}, ..., k_{1r})$ is the only one with non-vanishing first coordinate we have $\beta_1 - \beta'_1 = \lambda_1 k_{11}$. Since $0 \leq \beta_1, \beta'_1 < k_{11}$ we must have $\beta_1 = \beta'_1$ and therefore $\lambda_1 = 0$. Suppose we have shown that $\lambda_1 = ... = \lambda_i = 0$ for some $1 \leq i < r$. Then the above equation gives

$$(0, ..., 0, \beta_{i+1} - \beta'_{i+1}, ..., \beta_r - \beta'_r) = \lambda_{i+1}(0, ..., 0, k_{i+1,i+1}, ..., k_{i+1,r}) + ... + \lambda_r(0, ..., 0, k_{rr}).$$

Since the $(i+1)^{th}$ vector $(0, ..., 0, k_{i+1,i+1}, ..., k_{i+1,r})$ is the only one with non-vanishing $(i+1)^{th}$ coordinate we have $\beta_{i+1} - \beta'_{i+1} = \lambda_{i+1}k_{i+1,i+1}$. Since $0 \leq \beta_{i+1}, \beta'_{i+1} < k_{i+1,i+1}$ we must have $\beta_{i+1} = \beta'_{i+1}$ and therefore $\lambda_{i+1} = 0$. This shows by induction that $\beta_i = \beta'_i$ for all i. We just have to show that $j = j'$. We already have $a^{p_1{}^{y_1\beta_1}...p_r{}^{y_r\beta_r}(j-j')} \in H$. Since $H \cap \langle a \rangle = \langle a^m \rangle$ (by item 1)), we have

$$p_1{}^{y_1\beta_1}...p_r{}^{y_r\beta_r}(j - j') = qm$$

for some $q \in \mathbb{Z}$. So m divides ${p_1}^{y_1\beta_1}...{p_r}^{y_r\beta_r}(j-j')$. Since $\gcd(n,m)=1$, m does not contain any of the p_i in its prime decomposition, and therefore m must divide $j-j'$. Since $0 \le j, j' < m$ we have $j = j'$, as desired. This completes the proof.

□

Observation 6.7. The hypothesis $H \cap \langle a \rangle = \langle a^m \rangle$ cannot be removed in item 2) of Theorem 6.6, that is, if $H \cap \langle a \rangle \neq \langle a^m \rangle$, then the index of H in G is not necessarily $k_{11}...k_{rr}m$, it can be smaller. For example, $H = \left\langle tsa, sa^2, a^5 \right\rangle \le \Gamma_6$ has not index 5 in Γ_6. Indeed, since $a^{18} = (tsa)(sa^2)(tsa)^{-1}(sa^2)^{-1} \in H$ and $\gcd(5,18)=1$, we have

$$H = \left\langle tsa, sa^2, a^5 \right\rangle = \left\langle tsa, sa^2, a^5, a^{18} \right\rangle = \left\langle tsa, sa^2, a \right\rangle = \langle ts, s, a \rangle = \langle t, s, a \rangle = G$$

and therefore the index is 1, that is, $k_{11}...k_{rr} = |G:H|$ in this case.

Despite this, we do know in general that $k_{11}...k_{rr} \le |G:H| \le k_{11}...k_{rr}m$, even without assuming $H \cap \langle a \rangle = \langle a^m \rangle$. In fact, we did not need this hypothesis to show that $G = \bigcup_{{t_1}^{\beta_1}...{t_r}^{\beta_r}a^j \in T} H{t_1}^{\beta_1}...{t_r}^{\beta_r}a^j$, and therefore we have $|G:H| \le k_{11}...k_{rr}m$. We also did not use this hypothesis to see that two cosets $H{t_1}^{\beta_1}...{t_r}^{\beta_r}a^j$ and $H{t_1}^{\beta_1'}...{t_r}^{\beta_r'}a^{j'}$ of T are different unless $\beta_i = \beta_i'$ for all i. In particular, all the cosets $H{t_1}^{\beta_1}...{t_r}^{\beta_r}$ of T are distinct and then $k_{11}...k_{rr} \le |G:H|$.

6.2 A presentation

Theorem 6.8. *Let H be any finite index subgroup of Γ_n (see Theorem 6.6), say,*

$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}, ..., {t_r}^{k_{rr}}a^{l_r}, a^m \rangle \qquad (*)$$

with $k_{11} > 0$, $0 \le k_{ji} < k_{ii}$ for all $1 \le j < i \le r$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m,n) = 1$ *and $H \cap \langle a \rangle = \langle a^m \rangle$. Then H has the following presentation:*

$$H \simeq \left\langle \alpha, x_1, ..., x_r \mid x_i \alpha x_i^{-1} = \alpha^{P_i},\ x_i x_j x_i^{-1} x_j^{-1} = \alpha^{R_{ij}} \right\rangle,$$

where $P_i = p_i^{y_i k_{ii}}...p_r^{y_r k_{ir}}$ $(i = 1, ..., r)$ and $R_{ij} \in \mathbb{Z}$ characterized by

$$l_i P_i (1 - P_j) - l_j P_j (1 - P_i) = R_{ij} m.$$

Demonstração. It is easy to see that $(t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i})a^m(t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i})^{-1} = a^{mP_i}$ in Γ_n, for $i = 1, ..., r$. Also, since

$$(t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i})(t_j^{k_{jj}}...t_r^{k_{jr}}a^{l_j})(t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i})^{-1}(t_j^{k_{jj}}...t_r^{k_{jr}}a^{l_j})^{-1} = a^{l_i P_i(1-P_j) - l_j P_j (1-P_i)} \in H \cap \langle a \rangle = \langle a^m \rangle,$$

we have $l_i P_i (1 - P_j) - l_j P_j (1 - P_i) = R_{ij} m$ for some integer R_{ij} and so we write $(t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i})(t_j^{k_{jj}}...t_r^{k_{jr}}a^{l_j})(t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i})^{-1}(t_j^{k_{jj}}...t_r^{k_{jr}}a^{l_j})^{-1} = a^{mR_{ij}}$. Now let

$$G = \left\langle \alpha, x_1, ..., x_r \mid x_i \alpha x_i^{-1} = \alpha^{P_i},\ x_i x_j x_i^{-1} x_j^{-1} = \alpha^{R_{ij}} \right\rangle.$$

The group G has the relations

$$x_i\alpha = \alpha^{P_i}x_i,\ x_i\alpha^{-1} = \alpha^{-P_i}x_i,\ x_ix_j = \alpha^{R_{ij}}x_jx_i,\ x_ix_j^{-1} = x_j^{-1}\alpha^{-R_{ij}}x_i,$$

which shows that, for every fixed i, all the x_i-letters in a word with positive power can be pushed right as much as we want. Similarly, the relations

$$\alpha x_i^{-1} = x_i^{-1}\alpha^{P_i},\ \alpha^{-1}x_i^{-1} = x_i^{-1}\alpha^{-P_i},\ x_jx_i^{-1} = x_i^{-1}\alpha^{R_{ij}}x_j,\ x_j^{-1}x_i^{-1} = x_i^{-1}x_j^{-1}\alpha^{-R_{ij}}$$

show that all the x_i-letters in a word with negative power can be pushed left as much as we want. Because of this, we claim that any element of G is of the form $x_1^{-\lambda_1}...x_r^{-\lambda_r}\alpha^M x_r^{\delta_r}...x_1^{\delta_1}$ for $\lambda_i, \delta_i \geq 0$ and $M \in \mathbb{Z}$. Indeed, let

$$w = x_1^{s_{11}}...x_r^{s_{1r}}\alpha^{r_1}...x_1^{s_{c1}}...x_r^{s_{cr}}\alpha^{r_c}$$

be any element of G. Push all the x_1-letters of w with positive (resp. negative) power to the right (resp. left) extremity of w. Then $w = x_1^{-\lambda_1}w'x_1^{\delta_1}$ for some word w' which does not involve the letter x_1. Now, push all the x_2-letters of w' with positive (resp. negative) power to the right (resp. left) extremity of w'. Then $w = x_1^{-\lambda_1}x_2^{-\lambda_2}w''x_2^{\delta_2}x_1^{\delta_1}$ for some word w'' which does not involve the letters x_1 and x_2. By doing this recursively we show the claim.

Now let us show that $G \simeq H$. Define $\theta : G \to \Gamma_n$ by putting $\theta(\alpha) = a^m$ and $\theta(x_i) = t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i}$ for $i = 1, ..., r$. We first check that θ is a group homomorphism:

$$\theta(x_i)\theta(\alpha)\theta(x_i)^{-1} = (t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i})a^m(t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i})^{-1} = a^{mP_i} = \theta(\alpha)^{P_i},$$

and

$$\theta(x_i)\theta(x_j)\theta(x_i)^{-1}\theta(x_j)^{-1} = a^{mR_{ij}} = \theta(\alpha)^{R_{ij}},$$

as desired. Also, by construction, $\mathrm{im}(\theta) = H \leq \Gamma_n$. So $\theta : G \to H$ is surjective and we only need to show that θ is also injective. Indeed, let $w = x_1^{-\lambda_1}...x_r^{-\lambda_r}\alpha^M x_r^{\delta_r}...x_1^{\delta_1} \in G$ such that $\theta(w) = 1$. Then

$$(t_1^{k_{11}}...t_r^{k_{1r}}a^{l_1})^{-\lambda_1}...(t_r^{k_{rr}}a^{l_r})^{-\lambda_r}a^{mM}(t_r^{k_{rr}}a^{l_r})^{\delta_r}...(t_1^{k_{11}}...t_r^{k_{1r}}a^{l_1})^{\delta_1} = 1.$$

By projecting both sides of equation above on the t_1-coordinate by the homomorphism $w \mapsto (w)^{t_1}$, we get $k_{11}(\delta_1 - \lambda_1) = 0$ and so $\delta_1 = \lambda_1$. Then by conjugating the above equation on both sides by $(t_1^{k_{11}}...t_r^{k_{1r}}a^{l_1})^{\lambda_1}$ we get

$$(t_2^{k_{22}}...t_r^{k_{2r}}a^{l_2})^{-\lambda_2}...(t_r^{k_{rr}}a^{l_r})^{-\lambda_r}a^{mM}(t_r^{k_{rr}}a^{l_r})^{\delta_r}...(t_2^{k_{22}}...t_r^{k_{2r}}a^{l_2})^{\delta_2} = 1.$$

Again, by projecting both sides of equation above on the t_2-coordinate by the homomorphism $w \mapsto (w)^{t_2}$, we get $k_{22}(\delta_2 - \lambda_2) = 0$ and so $\delta_2 = \lambda_2$. Then by conjugating the above equation on both sides by $(t_2^{k_{22}}...t_r^{k_{2r}}a^{l_2})^{\lambda_2}$ we get

$$(t_3^{k_{33}}...t_r^{k_{3r}}a^{l_3})^{-\lambda_3}...(t_r^{k_{rr}}a^{l_r})^{-\lambda_r}a^{mM}(t_r^{k_{rr}}a^{l_r})^{\delta_r}...(t_3^{k_{33}}...t_r^{k_{3r}}a^{l_3})^{\delta_3} = 1.$$

By doing this recursively we get $\delta_i = \lambda_i$ for $i = 1, ..., r$ and

$$a^{mM} = 1.$$

Then $M = 0$ (since a is torsion free and $m > 0$). So

$$w = x_1^{-\lambda_1} ... x_r^{-\lambda_r} \alpha^0 x_r^{\lambda_r} ... x_1^{\lambda_1} = 1,$$

as desired. This completes the proof. □

6.3 Σ^1 invariant and property R_∞

Let H be a finite index subgroup of Γ_n, say,

$$H = \langle {t_1}^{k_{11}} ... {t_r}^{k_{1r}} a^{l_1}, {t_2}^{k_{22}} ... {t_r}^{k_{2r}} a^{l_2}, ..., {t_r}^{k_{rr}} a^{l_r}, a^m \rangle \qquad (*)$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m, n) = 1$ and $H \cap \langle a \rangle = \langle a^m \rangle$. We intend to apply Theorem 3.38 to guarantee property R_∞ to H. To do so, we first need to have an idea of $S(H)$. Because of this, we use Theorem 6.8 to identify H with its presentation

$$H = \left\langle \alpha, x_1, ..., x_r \mid x_i \alpha x_i^{-1} = \alpha^{P_i},\ x_i x_j x_i^{-1} x_j^{-1} = \alpha^{R_{ij}} \right\rangle,$$

for $P_i = p_i^{y_i k_{ii}} ... p_r^{y_r k_{ir}}$ $(i = 1, ..., r)$ and some $R_{ij} \in \mathbb{Z}$. Here, $\alpha = a^m$ and $x_i = {t_i}^{k_{ii}} ... {t_r}^{k_{ir}} a^{l_i}$. Since all the $p_i^{y_i}$ are ≥ 2, obviously the P_i also are ≥ 2 and so it is easy to see that α must have torsion in the abelianized group H^{ab}. The x_i are torsion-free, though. So we have the homeomorphism

$$\begin{aligned} S(H) &\longrightarrow S^{r-1} \\ [\chi] &\longmapsto \frac{(\chi(x_1), ..., \chi(x_r))}{\|(\chi(x_1), ..., \chi(x_r))\|}. \end{aligned}$$

Now, we will compute $\Sigma^1(H)$ inside this sphere by using Corollary 3.28. So we need the following

Lemma 6.9. *Let H be a finite index subgroup of Γ_n, say,*

$$H = \langle {t_1}^{k_{11}} ... {t_r}^{k_{1r}} a^{l_1}, {t_2}^{k_{22}} ... {t_r}^{k_{2r}} a^{l_2}, ..., {t_r}^{k_{rr}} a^{l_r}, a^m \rangle \qquad (*)$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m, n) = 1$ *and* $H \cap \langle a \rangle = \langle a^m \rangle$. *Then every homomorphism $\xi : H \to \mathbb{R}$ can be extended to a homomorphism $\chi : \Gamma_n \to \mathbb{R}$ (that is, $\chi|_H = \xi$).*

Demonstração. Since H is generated by the elements $x_i = {t_i}^{k_{ii}} ... {t_r}^{k_{ir}} a^{l_i}$ for $i = 1, ..., r$, the equation $\chi|_H = \xi$ is equivalent to a system of r equations

$$\begin{cases} \chi({t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}) = \xi({t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}), \\ \chi({t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}) = \xi({t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}), \\ \vdots \\ \chi({t_r}^{k_{rr}}a^{l_r}) = \xi({t_r}^{k_{rr}}a^{l_r}). \end{cases}$$

So to create such an extension χ we just have to define $\chi(a) = 0$ and define the real numbers $\chi(t_i)$ satisfying equations (1) to (r) above. Equation (r) is equivalent to

$$k_{rr}\chi(t_r) = \xi(t_r^{k_{rr}}a^{l_r}),$$

so if we define $\chi(t_r) = \frac{1}{k_{rr}}\xi(t_r^{k_{rr}}a^{l_r})$, equation (r) is satisfied. Similarly, equation $(r-1)$ is equivalent to

$$k_{r-1,r-1}\chi(t_{r-1}) + k_{r-1,r}\chi(t_r) = \xi(t_{r-1}^{k_{r-1,r-1}}t_r^{k_{r-1,r}}a^{l_{r-1}}),$$

so if we define $\chi(t_{r-1}) = \frac{1}{k_{r-1,r-1}}\xi(t_{r-1}^{k_{r-1,r-1}}t_r^{k_{r-1,r}}a^{l_{r-1}}) - \frac{k_{r-1,r}}{k_{r-1,r-1}}\chi(t_r)$, equation $(r-1)$ is satisfied. We proceed recursively: suppose that, for some $1 \leq i \leq r-1$ we have defined $\chi(t_{i+1}), ..., \chi(t_r)$ satisfying equations $(i+1), ..., (r)$. Equation (i) is equivalent to

$$k_{ii}\chi(t_i) + k_{i,i+1}\chi(t_{i+1}) + ... + k_{ir}\chi(t_r) = \xi(t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i}),$$

so if we define

$$\chi(t_i) = \frac{1}{k_{ii}}\xi(t_i^{k_{ii}}...t_r^{k_{ir}}a^{l_i}) - \frac{k_{i,i+1}}{k_{ii}}\chi(t_{i+1}) - ... - \frac{k_{ir}}{k_{ii}}\chi(t_r),$$

equation (i) is satisfied. After doing this to all i, we are done. □

Theorem 6.10. *Let H be a finite index subgroup of Γ_n, say,*

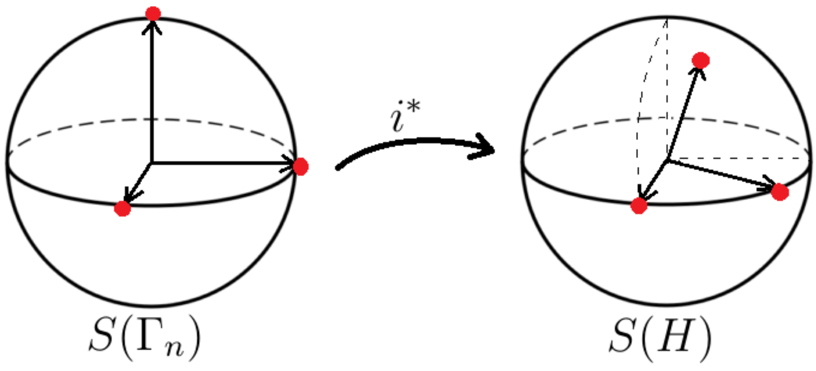
$$H = \langle {t_1}^{k_{11}}...{t_r}^{k_{1r}}a^{l_1}, {t_2}^{k_{22}}...{t_r}^{k_{2r}}a^{l_2}, ..., {t_r}^{k_{rr}}a^{l_r}, a^m\rangle \qquad (*)$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m,n) = 1$ and $H \cap \langle a\rangle = \langle a^m\rangle$, and let $\alpha = a^m$ and $x_i = {t_i}^{k_{ii}}...{t_r}^{k_{ir}}a^{l_i}$ be its generators. Then $\Sigma^1(H)^c = \{[\xi_1], ..., [\xi_r]\}$, where $\xi_i(x_j) = k_{ji}$ if $j \leq i$ and $\xi_i(x_j) = 0$ if $j > i$. In other words, if we identify $S(H) \simeq S^{r-1}$ as we did above, then

$$\Sigma^1(H)^c = \left\{\frac{(k_{11}, 0, 0, ..., 0)}{\|(k_{11}, 0, 0, ..., 0)\|}, \frac{(k_{12}, k_{22}, 0, ..., 0)}{\|(k_{12}, k_{22}, 0, ..., 0)\|}, ..., \frac{(k_{1r}, k_{2r}, k_{3r}, ..., k_{rr})}{\|(k_{1r}, k_{2r}, k_{3r}, ..., k_{rr})\|}\right\}.$$

Demonstração. Because of Lemma 6.9, by Corollary 3.28 we have $\Sigma^1(H)^c = i^*(\Sigma^1(\Gamma_n)^c)$, where $i^* : S(\Gamma_n) \to S(H)$ is given by $i^*[\chi] = [\chi|_H]$. By Theorem 5.2, $\Sigma^1(H)^c = \{[\chi_1|_H], ..., [\chi_r|_H]\}$. Since we are identifying H with its presentation we must compute $\chi_i(x_j)$. Remember that $\chi_i(t_j) = 1$ if $i = j$ and $\chi_i(t_j) = 0$ if $i \neq j$. If $j > i$, the generator $x_j = {t_j}^{k_{jj}}...{t_r}^{k_{jr}}a^{l_j}$ does not involve t_i and so $\chi_i(x_j) = 0$. If $j \leq i$, we have

$$\chi_i(x_j) = \chi_i({t_j}^{k_{jj}}...{t_r}^{k_{jr}}a^{l_j}) = k_{jj}\chi_i(t_j) + ... + k_{jr}\chi_i(t_r) = k_{ji},$$



which completes the first claim. It is easy to see that the image of $[\chi_i|_H]$ (which we denote by $[\xi_i]$) under the homeomorphism $S(H) \simeq S^{r-1}$ described above is $\frac{(k_{1i},\ldots,k_{ii},0,\ldots,0)}{\|(k_{1i},\ldots,k_{ii},0,\ldots,0)\|}$. This completes the proof. $\square$

Corollary 6.11. *All finite index subgroups of Γ_n have property R_∞.*

Demonstração. Let H be a finite index subgroup of Γ_n and describe it as in Theorem 6.10 above. Then $\Sigma^1(H)^c$ is a nonempty and finite set. Since the k_{ij} are integer, these points are all rational, and since $k_{ij} \geq 0$, $\Sigma^1(H)^c$ is contained, for example, in the open (geodesic) half space $B_d\left(\frac{(1,1,\ldots,1)}{\|(1,1,\ldots,1)\|}, \frac{\pi}{2}\right)$. So, by Theorem 3.38, H has property R_∞. $\square$

6.4 Finite index subgroups that are not Γ_k

In [12] it was shown that every finite index subgroup of a solvable Baumslag-Solitar group $BS(1,n)$ is also (isomorphic to) a solvable Baumslag-Solitar group $BS(1,n^k)$ for some $k \geq 1$. Since the groups Γ_n are generalizations of these groups, a natural question arises:

Are all finite index subgroups of Γ_n also (isomorphic to) another Γ_k for some $k \geq 2$?

In this section we show that this question has a negative answer. Below, we will define a specific type of finite index subgroup of Γ_n which can be shown to be (or not) isomorphic to Γ_k, depending on the powers used. This leads us to an infinite number of finite index subgroups which are examples (they are some Γ_k) and also an infinite number of counterexamples (which are not any Γ_k).

Let Γ_n be described as before and let

$$H = \langle t_1^{k_{11}} t_2^{k_{12}} \ldots t_r^{k_{1r}}, t_2^{k_{22}} \ldots t_r^{k_{2r}}, \ldots, t_r^{k_{rr}}, a^m \rangle$$

with $k_{11} > 0$, $0 \leq k_{ji} < k_{ii}$ for all $1 \leq j < i \leq r$ and $m > 0$ such that $\gcd(m,n) = 1$ (this is the description of an arbitrary finite index subgroup of Γ_n with the condition $l_i = 0$ for $1 \leq i \leq r$). It is also obvious that $H \cap \langle a \rangle = \langle a^m \rangle$. We will show that

$$H \simeq \Gamma_k \text{ for some } k \geq 2 \Leftrightarrow k_{ij} = 0 \text{ for all } 1 \leq i < j \leq r.$$

Suppose first that $k_{ij} = 0$ for all $1 \leq i < j \leq r$. Then from Theorem 6.8 we immediately get that $H \simeq \Gamma_k$ for $k = p_1^{y_1 k_{11}} ... p_r^{y_r k_{rr}}$. Suppose now that $H \simeq \Gamma_k$ for some $k \geq 2$ and write $k = q_1^{z_1} ... q_s^{z_s}$, $q_1 < q_2 < ... < q_s$, $z_i \geq 1$ the prime decomposition of k. Then we have a homeomorphism $\Sigma^1(H)^c \simeq \Sigma^1(\Gamma_k)^c$. By theorems 5.2 and 6.10, $s = \text{card}(\Sigma^1(\Gamma_k)^c) = \text{card}(\Sigma^1(H)^c) = r$, so $k = q_1^{z_1} ... q_r^{z_r}$. By Theorem 6.8, H has the presentation

$$H = \langle \alpha, x_1, ..., x_r \mid x_i \alpha x_i^{-1} = \alpha^{n_i},\ x_i x_j = x_j x_i \text{ for all } i, j \rangle,$$

where $n_i = p_i^{y_i k_{ii}} ... p_r^{y_r k_{ir}}$. There is also a split exact sequence

$$1 \to \ker(\pi) \to H \xrightarrow{\pi} \mathbb{Z}^r \to 1$$

where $\pi(x_i) = e_i$, $\pi(\alpha) = 0$ and $\ker(\pi)$ abelian. In particular, every element of H can be written as $x_1^{\lambda_1} ... x_r^{\lambda_r} u$ for some $\lambda_i \in \mathbb{Z}$ and $u \in \ker(\pi)$. Since $H \simeq \Gamma_k$, then there must be $r+1$ elements inside H (which are the images of the analogous $r+1$ elements in Γ_k), say, $X_i = x_1^{k'_{i1}} ... x_r^{k'_{ir}} u_i$, $1 \leq i \leq r$ and $A = x_1^{\tilde{k_1}} ... x_r^{\tilde{k_r}} \tilde{u}$ for some $k'_{ij}, \tilde{k_i} \in \mathbb{Z}$ and $u_i, \tilde{u} \in \ker(\pi)$, such that

$$H = \langle X_1, ..., X_r, A \rangle$$

and

$$X_i A X_i^{-1} = A^{q_i^{z_i}} \text{ for all } 1 \leq i \leq r.$$

By projecting any of these equations on $\mathbb{Z}^r$ we obtain $\tilde{k_1} = ... = \tilde{k_r} = 0$ and so $A = \tilde{u} = x_1^{-\lambda_1} ... x_r^{-\lambda_r} \alpha^M x_r^{\lambda_r} ... x_1^{\lambda_1}$ for some $\lambda_i \geq 0$ and $M \neq 0$. By replacing this in the r equations above and using that $\ker(\pi)$ is abelian and the x_i commute each other, we obtain the r equations in H

$$x_1^{k'_{i1}} ... x_r^{k'_{ir}} \alpha^M x_r^{-k'_{ir}} ... x_1^{-k'_{i1}} = \alpha^{M q_i^{z_i}} \qquad (i)$$

for each $1 \leq i \leq r$. If a power k'_{ij} is nonnegative we can use a relation of H to conjugate α^M. If it is negative, though, then since all the x_i commute we can push the two x_j from the left side to the right side of equation (i) and use the (now positive) power $-k'_{ij}$ to conjugate $\alpha^{M q_i^{z_i}}$. So every equation (i) will always imply an equality of a power of α^M with a power of $\alpha^{M q_i^{z_i}}$. Since H is torsion-free and $M \neq 0$, this implies an equality of prime decompositions (we will call this a prime equation) which depends on the signal of the k'_{ij}. Note that the right side always involve a positive power of the prime q_i. The left side, on the other hand, can involve (at most) the prime numbers $p_1, ..., p_r$. By uniqueness of prime decomposition, we must then have $q_i \in \{p_1, ..., p_r\}$, for all i. Since $q_1 < ... < q_r$ we must then have $q_i = p_i$ for all i, so $k = p_1^{z_1} ... p_r^{z_r}$.

We claim that $k'_{i1} \geq 0$ for all i. Indeed, if some $k'_{i1} < 0$, then x_1 goes to the right side of (i) and we have a prime equation with a positive power $p_1^{y_1 k_{11}(-k'_{i1})}$ on the right and only (possibly) $p_2, ..., p_r$ on the left (since $n_2, ..., n_r$ don't involve p_1), a contradiction. This shows the claim.

We claim that $k'_{ij} = 0$ if $i > j$. We will show this by induction on j. For $j = 1$ and for every $i > 1$, the fact $k'_{i1} \geq 0$ implies a prime equation with $p_1^{y_1 k_{11} k'_{i1}}$ on the left and no p_1 on the right (again, because $n_2, ..., n_r$ and $p_i^{z_i}$ don't involve p_1). So $y_1 k_{11} k'_{i1} = 0$ and therefore $k'_{i1} = 0$, since the two first numbers are positive. This completes the proof for $j = 1$. Now let $1 \leq j < r - 1$ and suppose the fact is valid for any $1 \leq j' \leq j$. Let us show it for $j + 1$. For any $i > j + 1$ we have also $i > 1, 2, ..., j$, so by induction $k'_{i1} = k'_{i2} = ... = k'_{ij} = 0$ and equation (i) becomes

$$x_{j+1}^{k'_{i,j+1}}...x_r^{k'_{ir}}\alpha^M x_r^{-k'_{ir}}...x_{j+1}^{-k'_{i,j+1}} = \alpha^{Mp_i^{z_i}}.$$

With the same argument we used for k'_{i1} before we can show that $k'_{i,j+1} \geq 0$. Because of this, (i) implies a prime equation with $p_{j+1}^{y_{j+1}k_{j+1,j+1}k'_{i,j+1}}$ on the left and no p_{j+1} on the right. Then $y_{j+1}k_{j+1,j+1}k'_{i,j+1} = 0$ and so $k'_{i,j+1} = 0$, which shows the claim.

Equations (i) then become

$$x_i^{k'_{ii}}...x_r^{k'_{ir}}\alpha^M x_r^{-k'_{ir}}...x_i^{-k'_{ii}} = \alpha^{Mp_i^{z_i}}. \qquad (i)$$

Again, since $n_{i+1}, ..., n_r$ do not involve p_i we must have $k'_{ii} \geq 0$ for all i. We claim that $k'_{ii} = 1$ for all i. In fact, by the last claim and by hypothesis we have

$$\begin{aligned} \mathbb{Z}^r &= \pi(H) \\ &= \pi\langle X_1, ..., X_r, A\rangle \\ &= \langle \pi(X_1), ..., \pi(X_r), \pi(A)\rangle \\ &= \langle (k'_{11}, k'_{12}, ..., k'_{1r}), (0, k'_{22}, ..., k'_{2r}), ..., (0, 0, ..., k'_{rr})\rangle. \end{aligned}$$

Then the fact that e_1 belongs to the subgroup above implies $k'_{11} = \pm 1$. Since $k'_{11} \geq 0$ we must have $k'_{11} = 1$. Now suppose $k'_{11} = ... = k'_{ii} = 1$ for $1 \leq i < r$ and let us show that $k'_{i+1,i+1} = 1$. Since e_{i+1} belongs to the sugroup above we have

$$e_{i+1} = \alpha_1(k'_{11}, k'_{12}, ..., k'_{1r}) + \alpha_2(0, k'_{22}, ..., k'_{2r}) + ... + \alpha_r(0, 0, ..., k'_{rr})$$

for some $\alpha_j \in \mathbb{Z}$. Let c_j be the j^{th} coordinate of the right element above. Then by the previous equation we must have $0 = c_1 = \alpha_1 k'_{11} = \alpha_1$. This implies $c_2 = \alpha_2 k'_{22} = \alpha_2$ and then $0 = c_2 = \alpha_2$. Recursively, we get $\alpha_1 = ... = \alpha_i = 0$. This implies $c_{i+1} = \alpha_{i+1}k'_{i+1,i+1}$. Then, because the $(i+1)^{th}$ coordinate of e_{i+1} is 1 we have $1 = c_{i+1} = \alpha_{i+1}k'_{i+1,i+1}$. Then $k'_{i+1,i+1} = \pm 1$, which implies $k'_{i+1,i+1} = 1$, since it is non negative. This shows the claim.

Fix any $1 \leq i < r$. We claim that $k'_{ij} = 0$ if $i < j$. We'll show this by induction on j, starting from $j = i + 1$. We know that equation (i) is

$$x_i^{k'_{ii}}...x_r^{k'_{ir}}\alpha^M x_r^{-k'_{ir}}...x_i^{-k'_{ii}} = \alpha^{Mp_i^{z_i}}. \qquad (i)$$

Since n_i and n_{i+1} are the only numbers involving p_{i+1}, if $k'_{i,i+1} < 0$ we would have $p_{i+1}^{y_{i+1}k_{i,i+1}k'_{ii}} = p_{i+1}^{y_{i+1}k_{i+1,i+1}(-k'_{i,i+1})}$, so

$$k_{i,i+1} = k_{i,i+1}k'_{ii} = k_{i+1,i+1}(-k'_{i,i+1}) \geq k_{i+1,i+1},$$

which is a contradiction with our first description of the group H. Then $k'_{i,i+1} \geq 0$ and equation (i) implies

$$p_{i+1}^{y_{i+1}(k_{i,i+1}k'_{ii}+k_{i+1,i+1}k'_{i,i+1})} = 1.$$

Since $y_{i+1} > 0$ we must have $k_{i,i+1}k'_{ii} + k_{i+1,i+1}k'_{i,i+1} = 0$. Since all these numbers are non negative and $k_{i+1,i+1} = 1$, we have $k'_{i,i+1} = 0$, as desired. Suppose now $k'_{i,i+1} = ... = k'_{ij} = 0$ for some $i+1 \leq j < r$ and let us show that $k'_{i,j+1} = 0$. By induction, equation (i) becomes

$$x_i^{k'_{ii}} x_{j+1}^{k'_{i,j+1}} ... x_r^{k'_{ir}} \alpha^M x_r^{-k'_{ir}} ... x_{j+1}^{-k'_{i,j+1}} x_i^{-k'_{ii}} = \alpha^{Mp_i^{z_i}}. \qquad (i)$$

Because of this, n_i and n_{j+1} are the only exponents which involve p_{j+1} in the equation above. By the same type of argument we used above for $i+1$, we show that $k'_{i,j+1} \geq 0$ and then also $k'_{i,j+1} = 0$. This shows the claim that $k'_{ij} = 0$ for all $i < j$. Then k'_{ij} is 1 if $i = j$ and 0 otherwise. The equations (i) become $x_i \alpha^M x_i^{-1} = \alpha^{Mp_i^{z_i}}$. This implies

$$p_i^{y_i k_{ii}} p_{i+1}^{y_{i+1} k_{i,i+1}} ... p_r^{y_r k_{ir}} = p_i^{z_i},$$

which implies $k_{i,i+1} = ... = k_{ir} = 0$. Since i is arbitrary, we showed that $k_{ij} = 0$ for any $1 \leq i < j \leq r$, as desired.

Parte III

Hyperbolic and relatively hyperbolic groups: an investigation of R_∞

Capítulo 7

Actions on hyperbolic spaces and property R_∞

A note on self-containment of Part III: The last part of this thesis is dedicated to the study of property R_∞ in a more geometric fashion - instead of combinatorial. The reader probably noticed that all the combinatorial theory developed in the previous chapters is self-contained, with only a few exceptions in the combinatorial preliminaries. The geometric preliminaries and chapters 7 through 9, however, could not be done this way. Otherwise, we would have many hundreds of preliminary pages on geometric group theory, hyperbolic groups, metric and geodesic spaces, quasi-isometry invariants and so on. Instead, we give the necessary definitions (so that the reader knows what we are talking about) and only state many well-known results, giving references to proofs in the literature. After all, scientists depend on each other.

This chapter is a theoretical preparation for the results of chapters 8 and 9. Here we show how some actions of a group G on hyperbolic spaces can be used to guarantee property R_∞ for G. We divide the chapter in two parts, considering whether the order of the projection $\pi(\varphi)$ of an automorphism φ is finite or not in the quotient $Out(G) = \frac{Aut(G)}{Inn(G)}$. The first part is based, for example, on [33], with some adaptations and clarifications. The second part is a detailed proof of a generalized version of a result by G. Levitt and M. Lustig (see [68], section 3), a key result to chapters 8 and 9.

7.1 Finite order case

Let G be a group and denote by $\pi : Aut(G) \to Out(G)$ the natural projection. Let $\varphi \in Aut(G)$ such that $\pi(\varphi)$ has finite order (say, $m \geq 1$) in $Out(G)$. Define the group

$$G_\varphi = \left\langle G, t \mid t^m = 1, \ \ tgt^{-1} = \varphi(g), \ \ \forall\, g \in G \right\rangle .$$

It is straightforward to see that G_φ is the semidirect product $G \rtimes_\varphi \langle t \rangle = G \rtimes_\varphi \mathbb{Z}_m$. In fact, G is normal in G_φ by the relations $tgt^{-1} = \varphi(g) \in G$ and $t^{-1}gt = \varphi^{-1}(g) \in G$; also, we have the relations $tg = \varphi(g)t$ and $t^{-1}g = \varphi^{-1}(g)t$ for every $g \in G$, so all t-letters can be moved to the right in a word of G_φ and so $G_\varphi = G\langle t \rangle$. Finally, to see that $G \cap \langle t \rangle = \{1\}$, suppose there is an element $g = t^r$ (with $0 \leq r < m$) in $G \cap \langle t \rangle$ and let us show $r = 0$ (and therefore $g = 1$). Since, for every $\tilde{g} \in G$,

$$\varphi^r(\tilde{g}) = t^r \tilde{g} t^{-r} = g\tilde{g}g^{-1} = \gamma_g(\tilde{g}),$$

we have $\varphi^r = \gamma_g \in Inn(G)$ and so $\pi(\varphi)^r = 1$ in $Out(G)$. Since the order of $\pi(\varphi)$ is m and $0 \leq r < m$, we must have $r = 0$, as desired.

To show the main property in the case of finite order, we need a lemma of T. Delzant. To state it, note that if a group G acts on a hyperbolic geodesic space X, there is a well defined induced action $G \curvearrowright \partial X$ by putting $g \cdot r(\infty) = (g \cdot r)(\infty)$.

Definition 7.1. We say an action by isometries of a group G on a hyperbolic geodesic space X is non-elementary if all items below are satisfied:

1) There is an element $g \in G$ whose action in X has infinite order;

2) There is not a global fixed point in ∂X;

3) There is not a global invariant pair in ∂X.

Lemma 7.2 ([68], Lemma 3.4)**.** *Let G be a group acting non-elementary on a hyperbolic geodesic space X. If $K \lhd G$ is such that G/K is abelian, then every coset Kg of K in G has an infinite number of usual conjugacy classes.*

Using Delzant's Lemma above we obtain

Proposition 7.3. *Let G be a group and $\varphi \in Aut(G)$ such that $\pi(\varphi)$ has finite order in $Out(G)$. If G_φ acts non-elementary on a hyperbolic geodesic space X, then $R(\varphi) = \infty$.*

Demonstração. Given $g, h \in G$, we claim that $g \sim_\varphi h$ if and only if gt and ht are conjugate in G_φ. In fact, if $g \sim_\varphi h$, let $z \in G$ such that $zg\varphi(z)^{-1} = h$. Then, by the relations in G_φ,

$$zgtz^{-1} = zg\varphi(z)^{-1}t = ht$$

and therefore gt and ht are conjugated. On the other hand, if they are conjugated by any element $zt^r \in G_\varphi$, then $z\varphi^r(g)tz^{-1} = zt^r g t^{-r} t z^{-1} = (zt^r)gt(zt^r)^{-1} = ht$ and therefore

$$z\varphi^r(g)\varphi(z)^{-1} = (z\varphi^r(g)\varphi(z)^{-1}t)t^{-1} = (z\varphi^r(g)tz^{-1})t^{-1} = htt^{-1} = h,$$

so $\varphi^r(g) \sim_\varphi h$ are φ-conjugated. But since both equalities $g = g\varphi(g)\varphi(g)^{-1}$ and $g = g\varphi^{-1}(g)\varphi^{-1}(g)^{-1}$ are true, we have $g \sim_\varphi \varphi^{\pm 1}(g)$, so by easy induction we can show $g \sim_\varphi \varphi^k(g)$ for every integer k. In particular, $g \sim_\varphi \varphi^r(g)$, so by transitivity $h \sim_\varphi g$, as desired.

Because of the fact above, the number of Reidemeister classes $R(\varphi)$ is exactly the number of conjugacy classes of elements of the form gt for $g \in G$. That is, it is the number of conjugacy classes in the coset Gt of G_φ. Now we use Delzant's lemma: by hypothesis, G_φ acts non-elementary on a hyperbolic geodesic space X and the normal subgroup $G \lhd G_\varphi$ is such that $\frac{G_\varphi}{G} = \frac{G \rtimes \mathbb{Z}_m}{G} \simeq \mathbb{Z}_m$ is abelian. It follows from Delzant's Lemma that every coset Gz for $z \in G_\varphi$ has an infinite number of conjugacy classes. In particular the coset Gt has, so $R(\varphi) = \infty$ and the proof is complete. $\square$

7.2 Levitt and Lustig's infinite order case

This section is dedicated to show Theorem 7.4 below, which is a slight generalization of Levitt and Lustig's result (see [68], section 3). Their result is the main part of the paper [68] on the matter of showing R_∞ for non-elementary hyperbolic groups. Although that paper is known as the one who shows R_∞ for non-elementary hyperbolic groups (which is true), I would like to point out the fact that their proof there relies on Paulin's equally complex Theorem 8.9 (see Chapter 8), so I would personally say that R_∞ for hyperbolic groups is a result by Levitt and Lustig and with a good contribution by Paulin.

Theorem 7.4. *Let G be a finitely generated group and $\varphi \in Aut(G)$ such that $\pi(\varphi)$ has infinite order in $Out(G)$. Suppose there is a non-trivial, small and irreducible action by isometries of G on an $\mathbb{R}$-tree (T,d), whose translation length function l satisfies the following:*

$$\text{there is } \lambda \geq 1 \text{ such that } \; l \circ \varphi = \lambda l,$$

and such that there is a unique map $h_\varphi : T \to T$ with $d(h_\varphi(x), h_\varphi(y)) = \lambda d(x,y)$ for every $x, y \in T$ and $h_\varphi(g \cdot x) = \varphi(g) \cdot h_\varphi(x)$ for every $(g,x) \in G \times T$. Then $R(\varphi) = \infty$.

First we observe that the dilation h_φ is not alone: if $H' = \langle \{\varphi\} \cup Inn(G) \rangle = Inn(G)\, \langle \varphi \rangle$ and $\psi \in H'$, we actually have a unique dilation map $h_\psi : T \to T$ such that $h_\psi(g \cdot x) = \psi(g) \cdot h_\psi(x)$ for every $x \in T$. In fact, write $\psi = \gamma_g \varphi^n \in H'$ ($g \in G, n \in \mathbb{Z}$) and define the two following actions:

$$G \times (T,d) \overset{\diamond}{\to} (T,d) \text{ with } g' \diamond x = \psi(g') \cdot x$$

and we also consider the action $G \curvearrowright (T,d)$ but with dilated metric:

$$G \times (T, \lambda^n d) \overset{\bullet}{\to} (T, \lambda^n d) \text{ with } g' \bullet x = g' \cdot x$$

These actions are also irreducible (since $G \curvearrowright T$ is), in particular semi-simple and not shifts. They also have the same translation length function. Indeed, for every $g' \in G$,

$$\begin{aligned} l_{(T,d,\diamond)}(g') &= \inf_{x \in T} d(x, g' \diamond x) \\ &= \inf_{x \in T} d(x, \psi(g') \cdot x) \\ &= l(\psi(g')) \\ &= l(g\varphi^n(g')g^{-1}) \\ &= l(\varphi^n(g')) \\ &= \lambda^n l(g'), \end{aligned}$$

and

$$l_{(T,\lambda^n d,\bullet)}(g') = \inf_{x \in T} \lambda^n d(x, g' \bullet x) = \inf_{x \in T} \lambda^n d(x, g' \cdot x) = \lambda^n \inf_{x \in T} d(x, g' \cdot x) = \lambda^n l(g').$$

Therefore by Theorem 2.55 there is a unique G-equivariant isometry $h_\psi : (T, \lambda^n d) \to (T,d)$. Its isometry gives us $d(h_\psi(x), h_\psi(y)) = \lambda^n d(x,y)$ for every $x, y \in T$, that is, h is an affine map of

(T,d) with λ^n a dilation coefficient. Equivariance gives us $h_\psi(g'\cdot x) = h_\psi(g' \bullet x) = g' \diamond h_\psi(x) = \psi(g')\cdot h_\psi(x)$ for every $g' \in G$, $x \in T$. The map h_ψ is then the unique affine map that makes the following diagram commute for every $g \in G$:

$$\begin{array}{ccc} T & \xrightarrow{\;g\cdot\;} & T \\ {\scriptstyle h_\psi}\downarrow & \circlearrowright & \downarrow{\scriptstyle h_\psi} \\ T & \xrightarrow{\;\psi(g)\cdot\;} & T \end{array}$$

Because of this uniqueness of diagram it is straightforward to see that $h_{\psi\psi'} = h_\psi h_{\psi'}$ for $\psi, \psi' \in H'$ and that $h_{\gamma_g} = g\cdot$. With this it is also easy to show that $h_{\gamma_g\varphi} = gh_\varphi$ and $h_{\gamma_g\varphi\gamma_g^{-1}} = gh_\varphi g^{-1}$. This will be used later. By Proposition 1.10, to show $R(\varphi) = \infty$ it is enough to show $S(\pi(\varphi)) = \infty$. So, we will show that $S(\pi(\varphi)) = \infty$ in the following two cases:

Case 1: $\lambda = 1$.

In this case, the λ-dilation h_φ becomes an isometry of (T,d) and has a well defined characteristic set C_{h_φ}. Let $\alpha \subset C_{h_\varphi}$ be any non-degenerate geodesic segment. Since the action is minimal and irreducible, by Theorem 2.56 there is a hyperbolic element $g \in G$ such that $\alpha \subset C_g$. If the isometry h_φ is hyperbolic we can also suppose without loss of generality (replacing g by g^{-1} if necessary) that the orientations of C_g and C_{h_φ} are the same on α. Then, by Proposition 2.39 we have $\|gh_\varphi\| = \|g\| + \|h_\varphi\|$. Similarly, for any $n \geq 1$ we have

$$\|g^n h_\varphi\| = \|g^n\| + \|h_\varphi\| = n\|g\| + \|h_\varphi\|.$$

We claim the automorphisms $\gamma_{g^n}\varphi$, $n \geq 1$, are pairwise non-isogredient, which gives us $S(\pi(\varphi)) = \infty$ and therefore $R(\varphi) = \infty$, completing case 1. In fact, if n, n' are such that $\gamma_{g^n}\varphi$ and $\gamma_{g^{n'}}\varphi$ are isogredient let us show that $n = n'$. There is by definition $g' \in G$ such that $\gamma_{g'}\gamma_{g^n}\varphi\gamma_{g'}^{-1} = \gamma_{g^{n'}}\varphi$. Then, by what we observed in the beginning we have $g'g^n h_\varphi g'^{-1} = g^{n'}h_\varphi$. Since the translation length function is invariant under conjugation of isometries, we have

$$n\|g\| + \|h_\varphi\| = \|g^n h_\varphi\| = \|g^{n'} h_\varphi\| = n'\|g\| + \|h_\varphi\|,$$

therefore $n = n'$ since $\|g\| > 0$. Case 1 is done.

Case 2: $\lambda > 1$.

This case is at least “significantly” harder. We also want to show $S(\pi(\varphi)) = \infty$. We divide the proof into 3 steps:

Step 1: fix a special point P and prove some general properties. We fix the point P in the following way: if the λ-dilation h_φ has a fixed point (easily seen to be unique), let P be this point, and this is called the first situation. If h_φ has no fixed points, then by [69] there is a h_φ-open-eigenray $\rho : (0,\infty) \to T$, that is, an open ray such that $h_\varphi(\mathrm{im}(\rho)) = \mathrm{im}(\rho)$ and $h_\varphi(\rho(t)) = \rho(\lambda t)$ for every $t > 0$. In this case, we let P be any point in the ray ρ, and this is called the second situation. Note that, in any of the two situations, $P \in [h_\varphi^{-1}(P), h_\varphi(P)]$, with $h_\varphi(P) = P$ in the first situation and $h_\varphi(P) \neq P$ in the second one. It is useful to note that the second situation gives rise to a fixed point in the metric completion $\overline{T}$ of T:

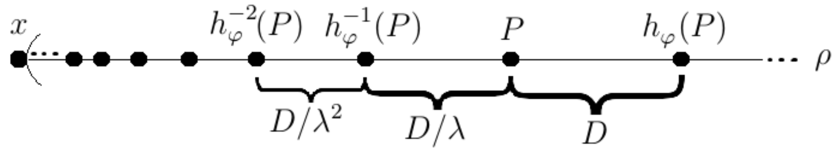
Proposition 7.5 ([78], Theorem 2.5). *If (X, d) is any metric space, then X is isometric to a dense subset of a complete metric space, denoted by $(\overline{X}, \overline{d})$ and called the metric completion of X. Furthermore, any two such metric completions are isometric, so $\overline{X}$ is well defined.* □

We usually identify X as a subspace of $\overline{X}$ and d as the restriction of $\overline{d}$.

Let $(\overline{T}, \overline{d})$ be the metric completion of T. It's straightforward to see that one can naturally extend the λ-dilation h_φ of T to a λ-dilation of $\overline{T}$, that will also be called h_φ. Let $D = d(P, h_\varphi(P))$. We have $d(h_\varphi^{-n-1}(P), h_\varphi^{-n}(P)) = \frac{1}{\lambda^{n+1}} d(P, h_\varphi(P)) = \frac{1}{\lambda^{n+1}} D$ for every $n \geq 1$, by induction. Now, since the series $\sum_{j=1}^{\infty} \frac{1}{\lambda^j}$ converges with sum $\frac{1}{1-\frac{1}{\lambda}} - 1 = \frac{1}{\lambda-1}$ (for $\lambda > 1$), the sequence $(x_n)_n = (h_\varphi^{-n}(P))_n$ is Cauchy, for

$$\begin{aligned}\overline{d}(x_n, x_{n+k}) &\leq \overline{d}(x_n, x_{n+1}) + ... + \overline{d}(x_{n+k-1}, x_{n+k}) \\ &= \frac{1}{\lambda^{n+1}} D + ... + \frac{1}{\lambda^{n+k}} D \\ &= D \sum_{j=n+1}^{n+k} \frac{1}{\lambda^j} \\ &\leq D \sum_{j=n+1}^{\infty} \frac{1}{\lambda^j} \to 0 \quad \text{if } n \to \infty.\end{aligned}$$

Therefore, since $\overline{T}$ is complete, $(x_n)_n$ converges to a point $x \in \overline{T}$, that is clearly seen to correspond to the origin of the ray ρ.



The distance between P and x is

$$\overline{d}(P, x) = \sum_{j=1}^{\infty} D \frac{1}{\lambda^j} = \frac{D}{\lambda - 1}.$$

The point x is a fixed point of h_φ. In fact, since $h_\varphi(x_n) \to h_\varphi(x)$ (by continuity), we have

$$D \frac{1}{\lambda^n} = \overline{d}(x_n, x_{n-1}) = \overline{d}(x_n, h_\varphi(x_n)) \to \overline{d}(x, h_\varphi(x)),$$

so $\overline{d}(x, h_\varphi(x)) = 0$ since $\frac{1}{\lambda^n} \to 0$ as $n \to \infty$. Since h_φ is a λ-dilation in $\overline{T}$ with $\lambda > 1$, it has at most one fixed point, so x turns out to be the only fixed point of h_φ in the whole completion $\overline{T}$. Facts similar to these will be used again soon.

After fixing the point P and before we move to the next step, let us state a lemma and show another two, that will be useful to step 3.

Lemma 7.6 ([68], Lemma 3.6). *Suppose $l \circ \varphi = \lambda l$, where l is the translation length function of a non-trivial small action by isometries of a finitely generated group G on an $\mathbb{R}$-tree T. If $\lambda > 1$, then every stabilizer of an arc is finite.* □

Lemma 7.7. *If ρ is any open eigenray ray of h_φ and $t > 0$, then $Stab(\rho(0,t)) = Stab(\rho(0,\infty))$. In other words, the stabilizer subgroup of any inicial open segment of any open eigenray of h_φ is actually the stabilizer of the whole ray ρ.*

Demonstração. Let us first show that $Stab(\rho(0,\lambda t)) = \varphi(Stab(\rho(0,t)))$. For ($\subset$), let $g \in Stab(\rho(0,\lambda t))$ and let us show that $\varphi^{-1}(g) \in Stab(\rho(0,t))$. Given $x = \rho(s) \in \rho(0,t)$, we have $\lambda s \in (0,\lambda t)$ and so by hypothesis

$$h_\varphi\varphi^{-1}(g)\cdot\rho(s) = g\cdot h_\varphi(\rho(s)) = g\cdot\rho(\lambda s) = \rho(\lambda s) = h_\varphi(\rho(s)),$$

so $\varphi^{-1}(g)\cdot\rho(s) = \rho(s)$ and $\varphi^{-1}(g) \in Stab(\rho(0,t))$, as desired. To show ($\supset$), let $g \in Stab(\rho(0,t))$ and let us show $\varphi(g) \in Stab(\rho(0,\lambda t))$. If $x = \rho(s) \in \rho(0,\lambda t)$, then $\lambda^{-1}s \in (0,t)$, so

$$\varphi(g)\rho(s) = \varphi(g)\rho(\lambda\lambda^{-1}s) = \varphi(g)h_\varphi\rho(\lambda^{-1}s) = h_\varphi g\rho(\lambda^{-1}s) = h_\varphi\rho(\lambda^{-1}s) = \rho(\lambda\lambda^{-1}s) = \rho(s),$$

so $\varphi(g) \in Stab(\rho(0,\lambda t))$, as desired.

Now let us show the lemma. By Lemma 7.6, the subgroups $Stab(\rho(0,t))$ and $Stab(\rho(0,\lambda t)) = \varphi(Stab(\rho(0,t)))$ are both finite with $Stab(\rho(0,\lambda t)) \subset Stab(\rho(0,t))$ (for $(0,t) \subset (0,\lambda t)$). Since they are isometric by the isomorphism φ, we must have $Stab(\rho(0,t)) = Stab(\rho(0,\lambda t))$. Recursively we can actually show that $Stab(\rho(0,t)) = Stab(\rho(0,\lambda^n t))$ for every $n \geq 1$. With this it is easy to see that $Stab(\rho(0,t)) = Stab(\rho(0,\infty))$. In fact, ($\supset$) is obvious, and if $g \in Stab(\rho(0,t))$ and $x = \rho(s) \in \rho(0,\infty)$, just take n such that $s < \lambda^n t$, so $x \in \rho(0,\lambda^n t)$ and since $g \in Stab(\rho(0,t)) = Stab(\rho(0,\lambda^n t))$ we have $gx = x$; so, ($\subset$) is valid and the lemma is complete. □

Lemma 7.8. *If ρ and ρ' are two open eigenrays of h_φ and $g \in G$ takes an initial segment $\rho(0,t)$ to an initial segment $\rho'(0,t)$, then g takes the whole ray ρ to ρ'.*

Demonstração. Let us show that the element $g^{-1}\varphi(g) \in G$ fixes the segment $\rho(0,t)$. In fact, let $\rho(s) \in \rho(0,t)$. Since $\lambda > 1$, the element $\rho(\lambda^{-1}s)$ is also in $\rho(0,t)$, so we have

$$g^{-1}\varphi(g)\rho(s) = g^{-1}\varphi(g)h_\varphi\rho(\lambda^{-1}s) = g^{-1}h_\varphi g\rho(\lambda^{-1}s) = g^{-1}h_\varphi\rho'(\lambda^{-1}s) = g^{-1}\rho'(s) = \rho(s),$$

as desired. By Lemma 7.7, $g^{-1}\varphi(g)$ fixes the whole ray ρ, which means $g^{-1}\varphi(g)x = x$ or $gx = \varphi(g)x$ for every x in the ray ρ (in particular, $\rho'(0,t) = g\cdot\rho(0,t) = \varphi(g)\cdot\rho(0,t)$). Applying this fact to the element $\varphi(g)$ we get $\varphi(g)x = \varphi^2(g)x$ for every x in the ray ρ and, recursively, $\varphi^n(g)x = gx$ for every such x and every $n \geq 1$. On the other hand we have

$$\varphi(g)\cdot\rho(0,\lambda t) = \varphi(g)h_\varphi\rho(0,t) = h_\varphi g\rho(0,t) = h_\varphi\rho'(0,t) = \rho'(0,\lambda t)$$

and recursively we get $\varphi^n(g)\cdot\rho(0,\lambda^n t) = \rho'(0,\lambda^n t)$ for every n. Finally, by using this and the fact that the action of g and $\varphi^n(g)$ coincides on the whole ray ρ, we get

$$\begin{aligned} g\cdot\rho(0,\infty) &= g\cdot(\cup_{n=1}^{\infty}\rho(0,\lambda^n t)) \\ &= \cup_{n=1}^{\infty} g\cdot\rho(0,\lambda^n t) \\ &= \cup_{n=1}^{\infty}\varphi^n(g)\cdot\rho(0,\lambda^n t) \\ &= \cup_{n=1}^{\infty}\rho'(0,\lambda^n t) \\ &= \rho'(0,\infty), \end{aligned}$$

and the lemma is proved. □

Step 2: find two special hyperbolic isometries $u, v \in G$. Precisely, they have to satisfy all of the items below:

a) $\langle u, v\rangle \simeq F_2$ is a free subgroup of G;

b) uP and vP both belong to the same path-connected component (or "path-component") of $T - \{P\}$, say, T^+;

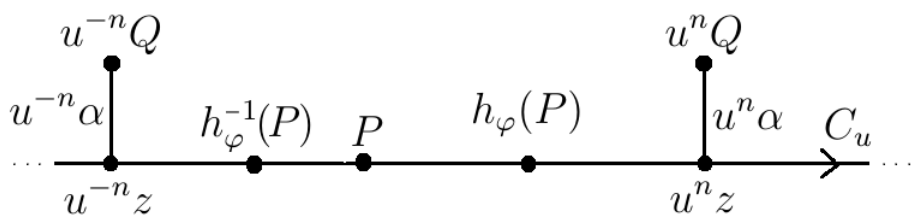
c) $u^{-1}P$ and $v^{-1}P$ both belong to another path-component T^- of $T - \{P\}$ that is different from T^+;

d) If $h_\varphi(P) \neq P$, then $h_\varphi^{\pm 1}(P) \in T^{\pm 1}$;

e) If $h_\varphi(P) = P$, then $h_\varphi(T^+) \neq T^-$;

In the second situation we have $h_\varphi(P) \neq P$ and P is an interior point of the non-degenerate segment $[h_\varphi^{-1}(P), h_\varphi(P)]$. In this case u, v must satisfy only $a), b), c)$ and $d)$. We do the following: since the action is irreducible, by Theorem 2.56 we let u be a hyperbolic element such that $[h_\varphi^{-1}(P), h_\varphi(P)] \subset C_u$ and suppose without loss of generality that the orientations of $[h_\varphi^{-1}(P), h_\varphi(P)]$ and C_u coincide. We know C_u is properly contained in T; otherwise, T would be a line and the action would then be either dihedral or a shift (with a fixed end), a contradiction with Proposition 2.48, since the action is irreducible. So let $Q \notin C_u$ and let $\alpha = [Q, z]$ be the bridge from Q to C_u. Denote by "$<$" the total order defined in C_u by its orientation. Since u translate the bridge α by uniform distances let $n \geq 1$ such that

$$u^{-n}z < h_\varphi^{-1}(P) < P < h_\varphi(P) < u^n z \text{ (see the figure).}$$



By construction and by properties of bridges, we have $[u^{-n}Q, u^nQ] = [u^{-n}Q, u^{-n}z, h_\varphi^{-1}(P), P, h_\varphi(P), u^nz, u^nQ]$. Now, again by Theorem 2.56, let v be a hyperbolic element such that $[u^{-n}Q, u^nQ] \subset C_v$, with same orientation of C_u in the intersection

$C_u \cap C_v = [u^{-n}z, u^n z] \supset [h_\varphi^{-1}(P), h_\varphi(P)]$. Let us see that u and v satisfy $a), b), c)$ and $d)$. First, since $C_g = C_{g^m}$ and $\|g^m\| = m\|g\|$ for every $m \geq 1$ and every hyperbolic element g, we can suppose without loss of generality that $length(C_u \cap C_v) < \min\{\|u\|, \|v\|\}$. Then, by Proposition 2.51 we have $a)$. Now, by the basic properties of $\mathbb{R}$-trees it is easy to see that two points $x, y \in T$ are in a different path-component of $T - \{P\}$ if and only if $[x, P] \cap [P, y] = \{P\}$. So, if we denote by T^+ (by T^-) the path-component of $T - \{P\}$ containing $h_\varphi(P)$ (containing $h_\varphi^{-1}(P)$), we easily have $T^+ \neq T^-$ and $b), c)$ and $d)$ being satisfied by construction.

Let us treat the first situation, where $h_\varphi(P) = P$ is the unique fixed point of h_φ in T. Let us find u and v satisfying $a), b), c)$ and $e)$. Remember h_φ is a homeomorphism of T for it is continuous with a continuous inverse $h_{\varphi^{-1}}$. So, since $h_\varphi(P) = P$, the map $h_\varphi : T - \{P\} \to T - \{P\}$ is a homeomorphism and therefore a bijection on the set of path-components of $T - \{P\}$ (we know there are at least two such components, for we are removing a point from an $\mathbb{R}$-tree). We have two subcases, a "good" one and a "bad" one.

Good subcase: suppose first that either $T - \{P\}$ has exactly two path-components which are both fixed by h_φ or that it has at least three path-components. In the former case we just let T^+ and T^- be the two components and we already have $h_\varphi(T^+) = T^+ \neq T^-$; in the latter case we use the incredibly easy lemma:

Lemma 7.9. *If X is any set with $\operatorname{card}(X) \geq 3$ and $f : X \to X$ is any bijection, there are two distinct elements $x_1 \neq x_2$ in X with $f(x_1) \neq x_2$.* □

Since h_φ is a bijection on the path-components of $T - \{P\}$ we can choose by the lemma two distinct components $T^+ \neq T^-$ such that $h_\varphi(T^+) \neq T^-$, as well as we did in the 2-component case. Now we do the following: let $x^+ \in T^+$ and $x^- \in T^-$. Since $T^+ \neq T^-$ we have $[x^+, P] \cap [P, x^-] = \{P\}$, so P is an interior point of $[x^-, x^+] = [x^-, P, x^+]$. Now we apply the same construction we did in the situation $h_\varphi(P) \neq P$ (but using the geodesic $[x^-, x^+]$ instead of $[h_\varphi^{-1}(P), h_\varphi(P)]$) and we get the desired elements u, v satisfying $a), b), c)$ and $e)$.

Bad subcase: the last subcase of the case $h_\varphi(P) = P$ we have to deal is the "bad" one, where T has exactly 2 components which are not fixed (therefore permuted) by h_φ. To find u, v satisfying $a), b), c)$ and $e)$ in this subcase we need the following

Lemma 7.10. *Suppose $h_1, h_2 : T \to T$ are two λ-affine maps with respective unique fixed points $P_1 \neq P_2$. Also, suppose*

- *$h_1(P_2)$ is not in the path-component of $T - \{P_1\}$ containing P_2;*
- *$h_2(P_1)$ is not in the path-component of $T - \{P_2\}$ containing P_1.*

Then $h_2 h_1^{-1}$ is a hyperbolic isometry of T such that $[P_1, P_2] \subset C_{h_2 h_1^{-1}}$.

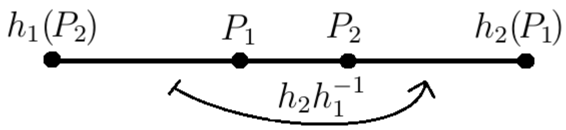
Demonstração. The isometry part is easy, for

$$d(h_2 h_1^{-1}(x), h_2 h_1^{-1}(y)) = \lambda d(h_1^{-1}(x), h_1^{-1}(y)) = \lambda \frac{1}{\lambda} d(x, y) = d(x, y).$$

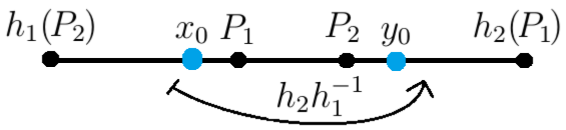
Note that we still don't know if the characteristic set $C_{h_2 h_1^{-1}}$ is a line for we still don't know if $h_2 h_1^{-1}$ is hyperbolic. By the two last hypotheses, $[h_2(P_1), P_2] \cap [P_2, P_1] = \{P_2\}$ and $[h_1(P_2), P_1] \cap$

$[P_1, P_2] = \{P_1\}$. Since $P_1 \neq P_2$, by Proposition 2.28 we have the geodesic $[h_2(P_1), h_1(P_2)] = [h_2(P_1), P_2, P_1, h_1(P_2)]$. Since $h_1^{-1}(P_1) = P_1$ (for $d(h_1^{-1}(P_1), P_1) = \frac{1}{\lambda}d(P_1, h_1(P_1)) = 0$), we have

$$h_2h_1^{-1}[h_1(P_2), P_1] = [h_2h_1^{-1}h_1(P_2), h_2h_1^{-1}(P_1)] = [P_2, h_2(P_1)] \text{ (see the figure).}$$



By Proposition 2.36, the middle point of the segment $[h_1(P_2), h_2h_1^{-1} \cdot h_1(P_2)] = [h_1(P_2), P_2]$ (say, x_0) belongs to $C_{h_2h_1^{-1}}$. Since $[h_1(P_2), P_1]$ is bigger (λ times the size) than $[P_1, P_2]$, x_0 is in the interior of $[h_1(P_2), P_1]$. Similarly, the middle point y_0 of the segment $[P_1, h_2(P_1)]$ belongs to $C_{h_2h_1^{-1}}$ and is in the interior of $[P_2, h_2(P_1)]$.



Since $C_{h_2h_1^{-1}}$ is connected, $[x_0, y_0] \subset C_{h_2h_1^{-1}}$ and in particular $[P_1, P_2] \subset C_{h_2h_1^{-1}}$. Finally, since P_1 is a point of $C_{h_2h_1^{-1}}$ which is not fixed by $h_2h_1^{-1}$ (for $h_2h_1^{-1}(P_1) = h_2(P_1) \neq P_1$), $h_2h_1^{-1}$ must be hyperbolic and the lemma is done. □

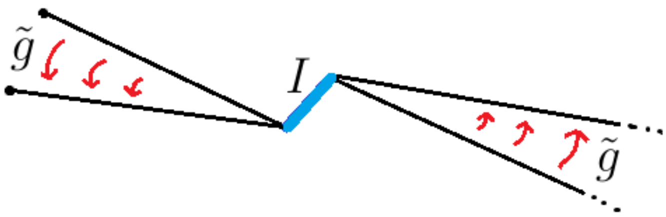
Let h_φ be in the bad subcase. There must be a hyperbolic element $g \in G$ such that $P \notin C_g$, for if $P \in C_g$ for every hyperbolic the action would be reducible by Proposition 2.45, a contradiction. Fix such g and let $\varphi' = \gamma_g \circ \varphi$. We know $h_{\varphi'} = gh_\varphi$ and so the dilation coefficient of $h_{\varphi'}$ is also λ. We also have $R(\varphi') = R(\varphi)$, since

$$\Re(\varphi) \to \Re(\varphi'), \quad [x]_\varphi \mapsto [xg^{-1}]_{\varphi'}$$

is easily seen to be a bijection with inverse $[x]_{\varphi'} \mapsto [xg]_\varphi$. Thus, if we show that $h_{\varphi'}$ is not in the bad subcase (only subcase we still haven't found the desired u and v), then we can find u, v for $h_{\varphi'}$ and proceed the proof with $h_{\varphi'}$ instead of h_φ to show $R(\varphi') = \infty$ and we will be done, because $R(\varphi) = R(\varphi') = \infty$. Suppose then by contradiction that $h_{\varphi'}$ is also in the bad subcase, that is, it has a unique fixed point P' and it permutes the only two path-components of $T - \{P'\}$. We claim we can apply the lemma above for h_φ and $h_{\varphi'}$. In fact, they are both λ-affine maps. Their fixed points are distinct. Indeed, $h_{\varphi'}(P) = gh_\varphi(P) = gP \neq P$ (for g is hyperbolic and has no fixed points), so P is not a fixed point of $h_{\varphi'}$ and therefore $P \neq P'$, as desired. Furthermore, since both are in the bad subcase, P' and $h_\varphi(P')$ are in distinct path-components of $T - \{P\}$ and similarly P and $h_{\varphi'}(P)$ are in distinct path-components of $T - \{P'\}$. By the lemma above we have $h_{\varphi'}h_\varphi^{-1}$ a hyperbolic isometry such that $[P, P'] \subset C_{h_{\varphi'}h_\varphi^{-1}}$, in particular $P \in C_{h_{\varphi'}h_\varphi^{-1}}$. But $h_{\varphi'}h_\varphi^{-1} = gh_\varphi h_\varphi^{-1} = g$, so we would have $P \in C_{h_{\varphi'}h_\varphi^{-1}} = C_g$, a contradiction. So $h_{\varphi'}$ cannot be in the bad case and step 2 is complete. We fix u and v satisfying $a)$ through $e)$ and proceed with the proof.

Step 3: prove that $R(\varphi) = \infty$. Remember that it is enough to show that $S(\pi(\varphi)) = \infty$. Suppose by contradiction that $S(\pi(\varphi)) = K < \infty$. By the works of M. Bestvina and M.

Feighn (see [6]), there is $N_0 \geq 1$ such that, for every $Q \in T$, the natural action of $Stab(Q)$ on $\pi_0(T - \{Q\})$ (the set of path-components of $T - \{Q\}$) has at most N_0 orbits. Also, by Lemma 7.6, arc stabilizers are finite groups, so let $I = [P, uP] \cap [P, vP]$ (which is a non-trivial segment because of a)) and let $s = \text{card}(Stab(I)) < \infty$. The general idea is to use u, v to construct a big enough collection of pairwise distinct rays, all of them containing I (within a same distance from their respective origins, see the figure) and such that all of them are mapped to each other by some isometry $\tilde{g}$. This will give us a number of $s + 1$ such isometries $\tilde{g}$, all belonging to $Stab(I)$, a contradiction.



For every $m \geq 1$, consider the set $W = W(m)$ of the words w written in the letters u, v (u^{-1} and v^{-1} not allowed here) such that each of the two letters appears exactly m times in w (so the length of w is exactly $2m$). Since $\langle u, v\rangle \simeq F_2$, the obvious map $W \to G$ (that considers a word as an element of G) is injective, so we can consider $W \subset G$. It is easy to see that $\text{card}(W) = \binom{2m}{m}$. Since $\text{card}(W(m+1)) = \binom{2(m+1)}{m+1} \geq 2\binom{2m}{m}$, $\text{card}(W) \to \infty$ as $m \to \infty$, so in particular we can fix m big enough such that

$$\text{card}(W) > KN_0(s+3).$$

The elements $w \in W$ have many interesting properties. First, write $w = \sigma_1 ... \sigma_{2m}$, for $\sigma_i \in \{u, v\}$. Let us show that $[P, wP] = [P, \sigma_1 P, \sigma_1\sigma_2 P, ..., \sigma_1\sigma_2...\sigma_{2m}P]$ by induction. Indeed, note that, by construction, C_{σ_1} and C_{σ_2} intercept in a non-trivial segment containing P in its interior and with same orientation. Thus, by Proposition 2.39, $\|\sigma_1\sigma_2\| = \|\sigma_1\| + \|\sigma_2\|$. Furthermore, in the proof of that proposition, we can see that $C_{\sigma_1\sigma_2} \cap C_{\sigma_1}$ also contains the same non-trivial segment with P in its interior and same orientation, so

$$d(P, \sigma_1\sigma_2 P) = \|\sigma_1\sigma_2\| = \|\sigma_1\| + \|\sigma_2\| = d(P, \sigma_1 P) + d(P, \sigma_2 P) = d(P, \sigma_1 P) + d(\sigma_1 P, \sigma_1\sigma_2 P),$$

thus $[P, \sigma_1\sigma_2 P] = [P, \sigma_1 P, \sigma_1\sigma_2 P]$ by Proposition 2.28. Recursively we show by induction until the m^{th} step: suppose we have shown that $[P, \sigma_1\sigma_2...\sigma_{2m-1}P] = [P, \sigma_1 P, \sigma_1\sigma_2 P, ..., \sigma_1\sigma_2...\sigma_{2m-1}P]$, that $\|\sigma_1...\sigma_{2m-1}\| = \|\sigma_1\| + ... + \|\sigma_{2m-1}\|$ and that $\sigma_1\sigma_2...\sigma_{2m-1}$ is hyperbolic whose characteristic set contains a non-trivial segment with P in its interior and whose orientation coincides with the one on both C_u and C_v in the intersection with them. We repeat the same argument above, replacing σ_1 by $\sigma_1\sigma_2...\sigma_{2m-1}$ and σ_2 by σ_{2m}. Proposition 2.28 gives us

$$\|\sigma_1...\sigma_{2m}\| = \|\sigma_1...\sigma_{2m-1}\| + \|\sigma_{2m}\| = \|\sigma_1\| + \|\sigma_2\| + ... + \|\sigma_{2m}\|,$$

and the proof of Proposition 2.39 gives us that $P \in C_{\sigma_1 ... \sigma_{2m}}$. Thus,0

$$\begin{aligned} d(P, \sigma_1 ... \sigma_{2m} P) &= \|\sigma_1 ... \sigma_{2m}\| \\ &= \|\sigma_1\| + \|\sigma_2\| + ... + \|\sigma_{2m}\| \\ &= d(P, \sigma_1 P) + d(P, \sigma_2 P) ... + d(P, \sigma_{2m} P) \\ &= d(P, \sigma_1 P) + d(\sigma_1 P, \sigma_1\sigma_2 P) + ... + d(\sigma_1\sigma_2 ... \sigma_{2m-1} P, \sigma_1\sigma_2 ... \sigma_{2m} P), \end{aligned}$$

and Proposition 2.28 gives us $[P, wP] = [P, \sigma_1 P, \sigma_1\sigma_2 P, ..., \sigma_1\sigma_2 ... \sigma_{2m} P]$, as desired. A second property of every $w \in W$ is that

$$[P, wh_\varphi(P)] = [P, \sigma_1 P, \sigma_1\sigma_2 P, ..., wP, wh_\varphi(P)].$$

In fact, in the case $h_\varphi(P) = P$ this is true by the property above. In the case $h_\varphi(P) \neq P$, by the property of $\mathbb{R}$-trees and by the previous property, it is enough to show that $[P, wP] \cap [wP, wh_\varphi(P)] = \{wP\}$. The end of the segment $[P, wP]$ is in $w(C_u \cap C_v)$ with a positive orientation. Since the beginning of $[P, h_\varphi(P)]$ has positive orientation in $C_u \cap C_v$ (by d)), the beginning of $[wP, wh_\varphi(P)]$ also has positive orientation in $w(C_u \cap C_v)$, so $[P, wP] \cap [wP, wh_\varphi(P)] = \{wP\}$ and the second property is proved. In particular, if we define $L = d(P, wh_\varphi(P))$, then by this property and by definition of W we have

$$\begin{aligned} L &= d(P, wh_\varphi(P)) \\ &= d(P, wP) + d(wP, wh_\varphi(P)) \\ &= \|\sigma_1\| + ... + \|\sigma_{2m}\| + d(P, h_\varphi(P)) \\ &= m\|u\| + m\|v\| + d(P, h_\varphi(P)) \end{aligned}$$

and so L is independent of w (it is the same for every $w \in W$). Another thing to register is that $I \subset [P, \sigma_1 P] \subset [P, wh_\varphi(P)]$ for every $w \in W$. To show a third property, let us show the simple

Lemma 7.11. *Any bijective λ-affine map $h : T \to T$ on an $\mathbb{R}$-tree takes geodesic to geodesics, that is, $h([x, y]) = [h(x), h(y)]$ for $x, y \in T$.*

Demonstração. It is enough to show the part ($\subset$), for then, since h^{-1} is a $\frac{1}{\lambda}$-affine map we apply the first part to h^{-1} and obtain $h^{-1}[h(x), h(y)] \subset [h^{-1}h(x), h^{-1}h(y)] = [x, y]$, or $[h(x), h(y)] \subset h([x, y])$, completing the proof. Let us show ($\subset$). For every three points x, y, z in T, it is easy to see (by taking the bridge from z to $[x, y]$) that $d(x, z) + d(z, y) = d(x, y) + 2d(z, [x, y])$. Therefore, $z \in [x, y] \Leftrightarrow d(x, y) = d(x, z) + d(z, y)$. Now, let $z \in [x, y]$ and let us show that $h(z) \in [h(x), h(y)]$. But $d(h(x), h(y)) = \lambda d(x, y) = \lambda(d(x, z) + d(z, y)) = \lambda d(x, z) + \lambda d(z, y) = d(h(x), h(z)) + d(h(z), h(y))$, so $h(z) \in [h(x), h(y)]$ and the lemma is finished. □

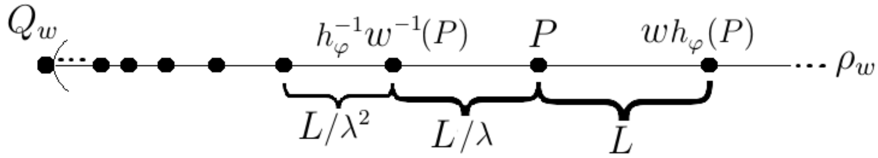
A third property of the elements $w \in W$ is that $[P, wh_\varphi(P)] \cap [wh_\varphi(P), (wh_\varphi)^2(P)] = \{wh_\varphi(P)\}$. Let us show this in the two cases: if $h_\varphi(P) = P$, showing that $[P, wP] \cap [wP, wh_\varphi wP] = \{wP\}$ is equivalent to show that $wP \in [P, wh_\varphi wP]$, or that $P \in [w^{-1}P, h_\varphi wP]$. By the first property we already know that $wP \in T^+$ (it is easy to see that actually $\tilde{w}P \in T^+$ for any word $\tilde{w}$ in the positive letters u and v) and similarly $w^{-1}P \in T^-$. Since $h_\varphi(T^+) \neq T^-$ by e), $h_\varphi wP$ cannot be in T^- and so the points $w^{-1}P$ and $h_\varphi wP$ are in different path-components of $T-\{P\}$, which implies $P\in[w^{-1}P,h_\varphi wP]$, as desired. In the case $h_\varphi(P)\neq P$, showing this property is equivalent to showing that $wh_\varphi(P)\in[P,wh_\varphi wh_\varphi(P)]$, or that $h_\varphi(P)\in[w^{-1}P,h_\varphi wh_\varphi(P)]$, or even (by the above lemma) that $P\in h_\varphi^{-1}[w^{-1}P,h_\varphi wh_\varphi(P)]=[h_\varphi^{-1}w^{-1}P,wh_\varphi(P)]$. Again, showing this last assertion is equivalent to show that the points $h_\varphi^{-1}w^{-1}P$ and $wh_\varphi(P)$ are in different path-components of $T-P$. On one hand, since $h_\varphi(P)\in C_u$ (on the positive side of P), we have $h_\varphi(P)\in[P,u^nP]$ for some big $n\geq 1$ and thus $wh_\varphi(P)\in[wP,wu^nP]$. But both points wP and wu^nP are in T^+, so $wh_\varphi(P)\in T^+$. On the other hand, since h_φ^{-1} has negative orientation in $C_u\cap C_v$, it also has negative orientation in $w^{-1}(C_u\cap C_v)$. Since $w^{-1}P\in T^-$, we have $h_\varphi^{-1}w^{-1}P\in T^-\neq T^+$, and this completes the proof of the third property.

Now we are ready to construct the rays we talked about. By the third property, for any $n\in\mathbb{Z}$ we have

$$\begin{aligned}
& [(wh_\varphi)^nP,(wh_\varphi)^{n+1}(P)]\cap[(wh_\varphi)^{n+1}(P),(wh_\varphi)^{n+2}(P)]=\\
= \;& (wh_\varphi)^n[P,wh_\varphi(P)]\cap(wh_\varphi)^n[wh_\varphi(P),(wh_\varphi)^2(P)]\\
= \;& (wh_\varphi)^n([P,wh_\varphi(P)]\cap[wh_\varphi(P),(wh_\varphi)^2(P)])=(wh_\varphi)^n\{wh_\varphi(P)\}\\
= \;& \{(wh_\varphi)^{n+1}(P)\},
\end{aligned}$$

which means $[P,wh_\varphi(P)]$ is a fundamental domain for the action of wh_φ on its $\mathbb{Z}$-orbit

$$\rho_w=\cup_{n\in\mathbb{Z}}(wh_\varphi)^n[P,wh_\varphi(P)]\quad\text{(see the figure).}$$



As we observed in step 1 (applying it now to the λ-dilation wh_φ), the sequence $((wh_\varphi)^{-n}P)_{n\geq1}$ converges to a point $Q_w\in\overline{T}$, that is the unique fixed point of wh_φ in $\overline{T}$. We also know that the distance $d(Q_w,P)=\frac{L}{\lambda-1}$ is fixed and independent of $w\in W$. A last observation before proceeding is that the ray ρ_w is obviously isometric to $(0,\infty)$ and $[P,wh_\varphi(P)]$ is the unique fundamental domain of ρ_w whose length is exactly L. In fact, identify $\rho_w=(0,\infty)$ and let h be a λ-dilation on $(0,\infty)$ with a fundamental domain $[a,b]$ with $b-a=L$. Let $[x,y]$ be any other fundamental domain of $(0,\infty)$ such that $y-x=L$ and let us show $[x,y]=[a,b]$. Write $x=a+k$ for some $k\in\mathbb{R}$. We have $d(b,h(x))=d(h(a),h(a+k))=\lambda d(a,a+k)=\lambda|k|$, so $y=h(x)=b+\lambda k$. Then

$$L=d(x,y)=d(a+k,b+\lambda k)=b+\lambda k-a-k=L+(\lambda-1)k,$$

which implies $(\lambda-1)k=0$ and then $k=0$. So $a=x$ and $b=y+\lambda k=y$, as desired. So far, we have built the rays ρ_w, eigenrays of wh_φ for any $w\in W$, all of them containing I within the same distance $\frac{L}{\lambda-1}$ from the origin Q_w and with fundamental domain $[P,wh_\varphi(P)]$ of uniform length L. They are also pairwise distinct, for, by the first property, if $w\neq w'$, the segments

$[P, wP]$ and $[P, w'p]$ must go in different directions at some point (see the figure).

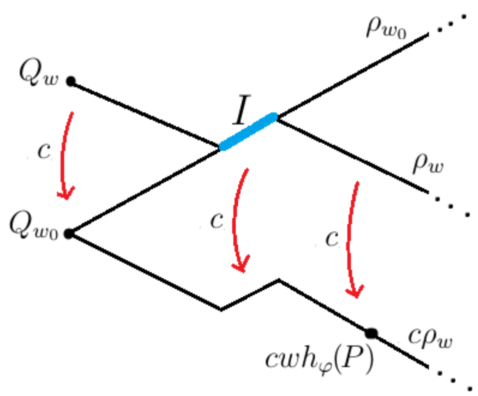
Note that the affine maps wh_φ are the induced affine maps of the isomorphisms $\gamma_w\varphi$ of G. So, let A be the set of isogredience classes $[\gamma_g\varphi]_{isogr}$ of φ (the set we supposed to be finite with K elements) and let $f : W \to A$ with $f(w) = [\gamma_w\varphi]_{isogr}$. We use the following easy and intuitive principle:

Lemma 7.12 (Generalized Pigeonhole Principle)**.** *Let X, Y be two finite sets and let $f : X \to Y$ be any map. If $\mathrm{card}(Y) \leq n$ and $\mathrm{card}(X) \geq kn$ for some $k, n \geq 1$, then there are k different elements in X with same image, that is, there is $X' \subset X$ with $\mathrm{card}(X') = k$ and $f(x) = f(z)$ for any $x, z \in X'$.* □

Since $\mathrm{card}(A) = K$ and $\mathrm{card}(W) > N_0(s+3)K$, by the Generalized Pigeonhole Principle there is $W' \subset W$ with $\mathrm{card}(W') = N_0(s+3)$ such that for every $w, w' \in W'$, $\gamma_w\varphi$ and $\gamma_{w'}\varphi$ are isogredient. Fix $w_0 \in W'$. For any $w \in W'$, since $\gamma_w\varphi$ and $\gamma_{w_0}\varphi$ are isogredient there is $c = c(w) \in G$ such that $\gamma_c(\gamma_w\varphi)\gamma_c^{-1} = \gamma_{w_0}\varphi$. Then

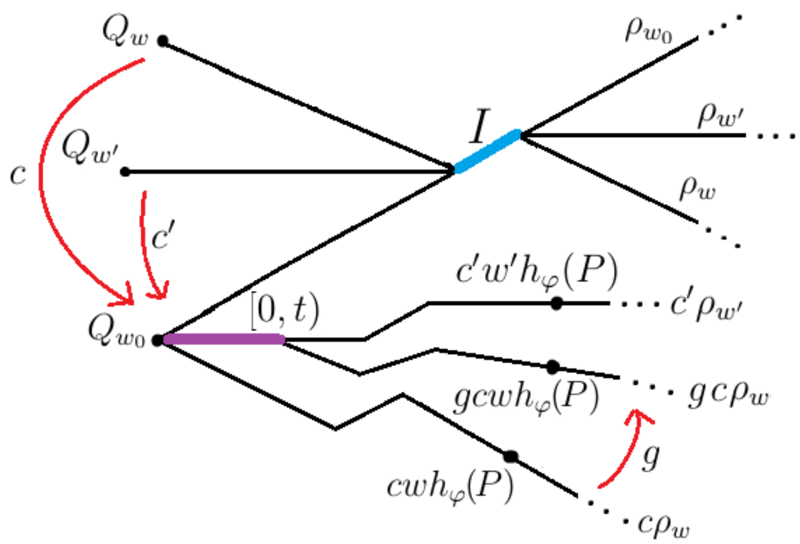
$$\gamma_{cw\varphi(c)^{-1}}\varphi = \gamma_c\gamma_w\gamma_{\varphi(c)^{-1}}\varphi = \gamma_c\gamma_w\varphi\gamma_{c^{-1}} = \gamma_{w_0}\varphi.$$

The induced affine maps of the two automorphisms in the equation above must then coincide, so $cw\varphi(c)^{-1}h_\varphi = w_0h_\varphi$ or $cwh_\varphi c^{-1} = w_0h_\varphi$. With this we can see that c maps the origin Q_w of ρ_w to the origin Q_{w_0} of ρ_{w_0}. Indeed, since Q_w and Q_{w_0} are the unique fixed points of wh_φ and w_0h_φ, respectively, and since $w_0h_\varphi(cQ_w) = cwh_\varphi(Q_w) = cQ_w$, uniqueness gives us $cQ_w = Q_{w_0}$. The ray ρ_w is then mapped by $c = c(w)$ to a ray $c\rho_w$ starting at Q_{w_0}. In the same way we can see that $c\rho_w$ is also an eigenray of w_0h_φ, for if $cx \in c\rho_w$ (for some $x \in \rho_w$) then $w_0h_\varphi(cx) = cwh_\varphi(x) \in c\rho_w$. Thus, we have the following configuration:

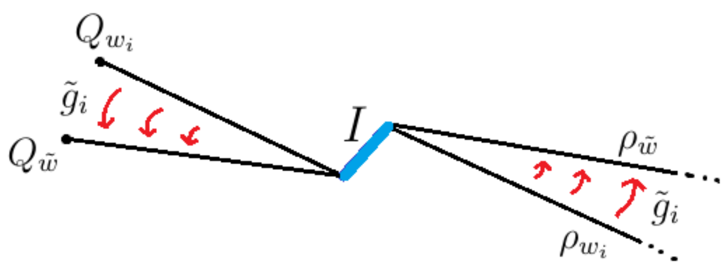


Consider the action of $Stab(Q_{w_0})$ on the set $\pi_0(T-\{Q_{w_0}\})$ of path-components of $T-\{Q_{w_0}\}$), which has at most N_0 orbits. The set $Orb(\pi_0(T-\{Q_{w_0}\}))$ of orbits under this action has at most N_0 elements. Let $f' : W' - \{w_0\} \to Orb(\pi_0(T - \{Q_{w_0}\}))$ be the map that associates $w \in W' - \{w_0\}$ to the orbit of $\pi_0(T - \{Q_{w_0}\})$ containing the point $cwh_\varphi(P)$. Since $\mathrm{card}(W' -$

$\{w_0\}) = \text{card}(W') - 1 = N_0(s+3) - 1 \geq N_0(s+2)$, the Generalized Pigeonhole Principle gives us $W'' \subset W' - \{w_0\}$ with $\text{card}(W'') = s+2$ and such that $f'(w) = f'(w'')$ for any $w, w' \in W''$, that is, the path-components containing the points $cwh_\varphi(P)$ and $c'w'h_\varphi(P)$ are in the same $Stab(Q_{w_0})$-orbit (here, $c' = c(w')$). This means there must be $g = g(w, w') \in Stab(Q_{w_0})$ such that the path-components of $gcwh_\varphi(P)$ and $c'w'h_\varphi(P)$ inside $T - \{Q_{w_0}\}$ are the same, which is equivalent to say that the rays $gc\rho_w$ and $c'\rho_{w'}$ coincide in a non-degenerate interval $[0, t)$. See the figure:



As we already know, the rays $c\rho_w$ and $c'\rho_{w'}$ are eigenrays of the same λ-affine map w_0h_φ. Since $gc\rho_w(0,t) = c'\rho_{w'}(0,t)$, by Lemma 7.8 g maps the whole ray $c\rho_w$ onto $c'\rho_{w'}$, so $\tilde{g}\rho_w = \rho_{w'}$ for $\tilde{g} = c'^{-1}gc$. Since this can be done for any $w, w' \in W''$ and since $\text{card}(W'') = s+2$, write $W'' = \{\tilde{w}, w_1, ..., w_{s+1}\}$ and denote by $\tilde{g}_i$ the element such that $\tilde{g}_i\rho_{w_i} = \rho_{\tilde{w}}$ we just constructed.



The elements $\tilde{g}_i$ must be pairwise distinct. In fact, if $\tilde{g}_i = \tilde{g} = \tilde{g}_j$ for $1 \leq i, j \leq s+1$, then $\tilde{g}\rho_{w_i} = \tilde{g}_i\rho_{w_i} = \rho_{\tilde{w}} = \tilde{g}_j\rho_{w_j} = \tilde{g}\rho_{w_j}$, so $\rho_{w_i} = \rho_{w_j}$ and therefore $i = j$, since the rays ρ_{w_i} are pairwise distinct (fact already shown). Finally, since $d(I, Q_{\tilde{w}}) = \frac{L}{\lambda - 1} = d(I, Q_{w_i})$, the elements $\tilde{g}_i$ must fix the segment I, so $\tilde{g}_i \in Stab(I)$ for $1 \leq i \leq s+1$. We then found $s+1$ distinct elements inside a set $Stab(I)$ of cardinality s. This is a contradiction, so $S(\pi(\varphi)) = \infty$ and therefore $R(\varphi) = \infty$, which completes the Theorem. □

Capítulo 8

Property R_∞ for hyperbolic groups

In this chapter we give the details of the already known proof or property R_∞ for non-elementary hyperbolic groups given by G. Levitt and M. Lustig in [68]. Knowing this proof will also be especially useful for chapter 9.

The key ingredient of the above proof is an exhibition of Paulin's result ([81], Theorem A, corresponding to Theorem 8.9 here). This theorem implies that infinite-order automorphisms of non-elementary hyperbolic groups satisfy the conditions of Levitt and Lustig's result (given in more general terms by our Theorem 7.4). Then, by Theorem 7.4, these automorphisms have infinite Reidemeister numbers. So - as we have said in the previous chapter - although Levitt and Lustig's paper [68] is known as the one who shows R_∞ for non-elementary hyperbolic groups (which is true), I would like to point out the fact that their proof there relies on Paulin's equally complex Theorem 8.9, so I would personally say that R_∞ for hyperbolic groups is a result by Levitt, Lustig and with a good contribution by Paulin.

Hyperbolic groups were first defined in 1987 by Gromov [50]. This is a large (and also largely studied) class of groups in geometric group theory. For other equivalent definitions of hyperbolic groups, see [55].

Definition 8.1. Let G be a group and $\delta \geq 0$. We say G is δ-hyperbolic if G is finitely generated and if its Cayley graph $\Gamma(G, S)$ is a δ-hyperbolic space (for some finite generating set S). We say G is hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.

If G is hyperbolic as above, it follows that, for every other finite generating set S', we have $\Gamma(G, S) \overset{QI}{\sim} \Gamma(G, S')$ by Proposition 2.20 and therefore $\Gamma(G, S')$ is also a hyperbolic space by Proposition 2.23. So, it does not matter which generating set we choose for hyperbolicity. Similarly, with the same argument, if two f.g. groups G and H are quasi-isometric (i.e., have quasi-isometric Cayley graphs) and G is hyperbolic, then H is a hyperbolic group.

There are some trivial examples of hyperbolic groups. For example, finite groups are hyperbolic because their Cayley graphs, having finite diameter, are always hyperbolic spaces. Another example is $\mathbb{Z}$, whose usual Cayley graph is homeomorphic to $\mathbb{R}$, which is an (0-hyperbolic) $\mathbb{R}$-tree. Let us go just a little bit further: suppose a group G is virtually ciclic, i.e., it contains a finite index cyclic subgroup H. Then, by Proposition 2.22 we have $G \overset{QI}{\sim} H$ and therefore by 2.23 G is hyperbolic, since H is. All these groups are relatively simple and not interesting for some areas inside geometric group theory. Because of this, we usually make the following distinction:

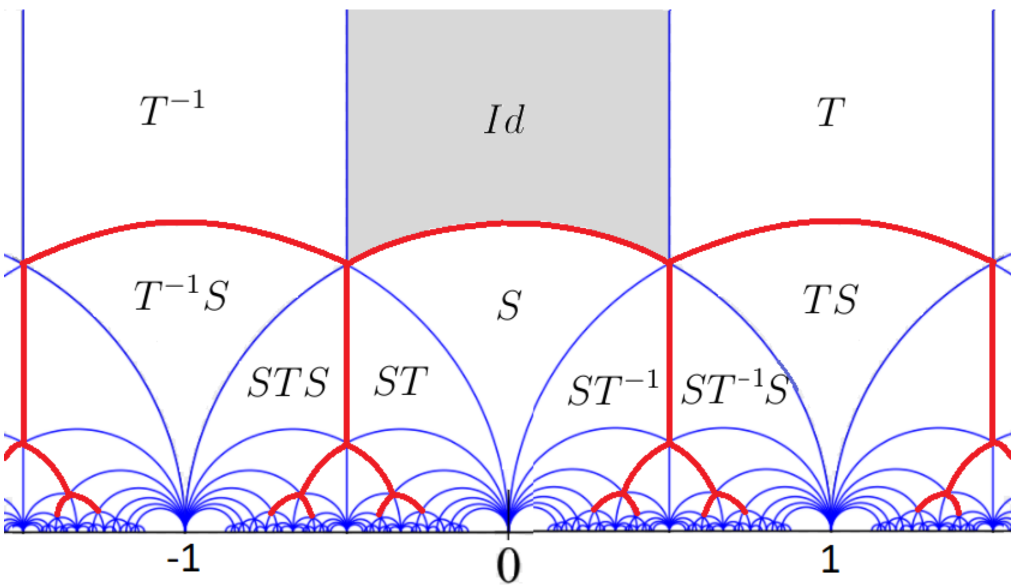
Definition 8.2. We say a hyperbolic group G is non-elementary if G is infinite and is not virtually cyclic.

Example 8.3. Here are some basic examples of non-elementary hyperbolic groups. First, all finitely generated free groups F_n of rank $n \geq 2$ are non-elementary hyperbolic groups. In fact, we know from Proposition 1.22 that the Cayley graph of such groups are (combinatorial) trees, in particular 0-hyperbolic spaces, as desired. Therefore, by Propositions 2.22 and 2.23, every virtually-F_n group is also non-elementary hyperbolic. For example, direct products $F_n \times \mathbb{Z}_m$ and semidirect products $F_n \rtimes S_n$ for the natural permutation action of S_n on F_n. Other examples of virtually-F_n groups are the fundamental groups $\pi_1(G, \Gamma, T)$ (see chapter 1) of finite graphs of groups (G, Γ) whose vertex and edge groups G_P and G_y are all finite. In fact, they are virtually-F_n by [86] (see pages 120-122) and so hyperbolic.

One of them is the special linear group $G = SL_2(\mathbb{Z})$, or modular group, as some would say. This is the group of the square integer matrices of size 2 with determinant 1. One can show that G is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. G acts on the (hyperbolic) upper half plane $\mathcal{H} = \{z = x + yi \in \mathbb{C} \mid y > 0\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

The half plane contains a G-invariant tree (in red), which is the 1-skeleton of a tessellation of the hyperbolic plane, so we could very likely expect G to be hyperbolic. Fundamental domains of the action are given in the figure, where the action can be geometrically seen.



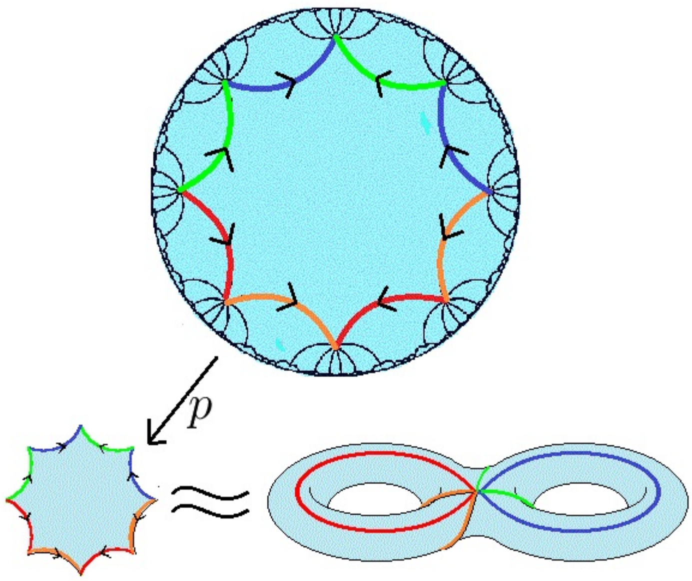
The group G can be shown to have the following presentation:

$$G = \langle S, ST \mid S^4 = 1,\ (ST)^6 = 1,\ S^2 = (ST)^3 \rangle \simeq \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6,$$

which is clearly the fundamental group of a segment with $\mathbb{Z}_4$ and $\mathbb{Z}_6$ as vertex groups and $\mathbb{Z}_2$ as edge group. So, as we already said, G is virtually-F_n by [86]. In this case, G contains an index

6 copy of F_2, which is the subgroup consisting of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that a, d are odd and b, c are even. Since G contains F_2, it must be a non-elementary hyperbolic group.

Example 8.4. A last and elegant example of hyperbolic group is every fundamental group G of a closed hyperbolic n-manifold X, with $n \geq 2$. In fact, let $\tilde{X} = \mathcal{H}^n$ be the universal covering of X, which is the well-known n-dimensional simply connected hyperbolic space. It is known that $G = \pi_1(X)$ is isomorphic to the covering transformation group of $\tilde{X}$ (see [76]), a group of isometries of $\tilde{X}$ on which G acts properly discontinuously. Since the orbit space $\tilde{X}/G \simeq X$ is compact, the action is cocompact and therefore G is a hyperbolic group (see [14]), for its Cayley graph is quasi-isometric to $\tilde{X}$. If G was virtually cyclic, its Cayley graph would be quasi-isometric to $\mathbb{R} = \mathcal{H}^1$, so $\mathcal{H}^n = \tilde{X} \overset{QI}{\sim} \mathcal{H}^1$ and it is known that this implies $n = 1$, a contradiction. Therefore G is non-elementary, as desired. One of these groups is the fundamental group of the bitorus, that is, $G = \pi_1(T^2 \sharp T^2)$. It is widely known that $G = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$. The universal cover of the bitorus is the hyperbolic Poincare (open) disk $\mathcal{H}^2$ and a model of the covering map p can be seen in the figure. This map produces a tesselation of $\mathcal{H}^2$ by (hyperbolic) regular octagons, such that each vertex has exactly 8 octagons adjacent to it, therefore called a $\{8, 8\}$ tesselation of $\mathcal{H}^2$. The Cayley graph of G consists exactly of the edges inside the disk. It is quasi-isometric to $\mathcal{H}^2$ and so hyperbolic.



Non-elementary hyperbolic groups are way more interesting than the elementary ones and have some “non-abelian-like” properties, such as

Proposition 8.5. *[47] Every non-elementary hyperbolic group has a finite center.*

Proposition 8.6. *[47] Every non-elementary hyperbolic group contains a non-abelian free subgroup of rank* 2*.*

Another useful property is:

Proposition 8.7. *[21] Every hyperbolic group is finitely presented.*

A key ingredient to show property R_∞ for non-elementary hyperbolic groups is a theorem due to Frédéric Paulin ([81], 1997). It creates, from a non-elementary hyperbolic group and a special subgroup of automorphisms, a very special type of action on an $\mathbb{R}$-tree, allowing us to use the results on the previous chapter.

To state the theorem, we will use the notion of amenability for groups. Pier's work ([84], 1984) contains a great number of definitions that are equivalent to amenability. Since it is not vital for the strategy of the proof, we will postpone the definition and only state one of the characterizations later, when necessary.

Definition 8.8 (Affine action). Let $G \curvearrowright (X, d)$ be any action of a group G on a metric space X. We say the action is affine if there is a multiplicative homomorphism $\lambda : G \to (0, \infty)$ such that

$$d(gx, gy) = \lambda(g)d(x, y) \ \forall \ (x, y, g) \in X \times X \times G.$$

Theorem 8.9 (Paulin's Theorem). *Let G be a non-elementary hyperbolic group and denote $\pi : Aut(G) \to Out(G) = \frac{Aut(G)}{Inn(G)}$ the natural projection. If $H \leq Aut(G)$ is amenable such that $\pi(H)$ has infinite center, then there is an $\mathbb{R}$-tree T and an affine action $(G \rtimes H) \curvearrowright T$ whose restriction to G is a non-trivial, minimal and small action by isometries.*

8.1 Proof of Paulin's Theorem 8.9

The proof we are going to present is the main result of [81]. Since the proof is a bit long, we divided it in three steps:

- **Step 1**: *create a non-trivial and small action by isometries of G on an $\mathbb{R}$-tree X_ω.* the space X_ω will be defined as an ultralimit (see Chapter 2) of a sequence of hyperbolic spaces X_n on which we know G acts and whose hyperbolicity constants converge to 0. So it will be an $\mathbb{R}$-tree.
- **Step 2**: *Extend the action above to a well-defined action $(G \rtimes H) \curvearrowright X_\omega$.* Using that every $\varphi \in H$ induces a map f_n on each X_n, we put them together to a well defined homeomorphism $f_\varphi : X_\omega \to X_\omega$. We show these maps are coherent with the previous action and give rise to $(G \rtimes H) \curvearrowright X_\omega$.
- **Step 3**: *Modify "a little bit" the $\mathbb{R}$-tree X_ω and the action $(G \rtimes H) \curvearrowright X_\omega$ above to obtain the affine action $(G \rtimes H) \curvearrowright T$ desired.* Here we create a linear action of H on a closed convex cone of length functions and use amenability to find a special one, called l, of some action of G on some $\mathbb{R}$-tree T. The special properties l satisfies makes it possible to extend this action to all $G \rtimes H$.

Step 1: *create a non-trivial and small action by isometries of G on an $\mathbb{R}$-tree X_ω.*

By hypothesis, let $(\psi_n)_{n\geq 1}$ be a sequence of elements of H such that the $\pi(\psi_n)$ are pairwise distinct elements in the center of $\pi(H)$. Let $S = \{s_1, ..., s_k\}$ be a finite generating set for G such that $S = S^{-1}$. By Proposition 8.6, G contains a copy of the free group F_2 as a subgroup, so we may assume $\langle s_1, s_2 \rangle \simeq F_2$.

Let $|.| : G \to \mathbb{R}$ be the natural geodesic length and $d : \Gamma(G, S) \times \Gamma(G, S) \to \mathbb{R}$ be the natural geodesic metric on the Cayley graph of G (see Chapter 2). With this metric, G acts by isometries on $\Gamma(G,S)$, by extending by linearity the left multiplication action of G on itself. For now, let us restrict d to the set G of vertices, which also becomes a metric space on which G acts by isometries. Then, for each $n \geq 1$, define

$$\lambda_n = \inf_{g\in G}\{\max_{s\in S}\{d(g,\psi_n(s)g)\}\}.$$

Since d only takes integer values on $G \times G$, λ_n is attained by (at least) one point (element) of G, which we call p_n. Then

$$\lambda_n = \max_{s\in S}\{d(p_n,\psi_n(s)p_n)\}.$$

We first claim that for every $g \in G$ and $n \geq 1$, $d(p_n,\psi_n(g)p_n) \leq |g|\lambda_n$. In fact, given g, write $g = s_1...s_m$ as a word in S with minimal length (and so $|g| = m$). Then, by using a finite number of times the triangular inequality, the fact ψ_n is an automorphism and the fact that the action is isometric we have

$$\begin{aligned} d(p_n,\psi_n(g)p_n) &\leq d(p_n,\psi_n(s_1)p_n) + d(\psi_n(s_1)p_n,\psi_n(s_1)\psi_n(s_2)p_n) + ... \\ &\quad ... + d(\psi_n(s_1)...\psi_n(s_{m-1})p_n,\psi_n(g)p_n) \\ &= d(p_n,\psi_n(s_1)p_n) + d(p_n,\psi_n(s_2)p_n) + ... + d(p_n,\psi_n(s_m)p_n) \\ &\leq \lambda_n + \lambda_n + ... + \lambda_n \ (m \text{ times}) \\ &= |g|\lambda_n, \end{aligned}$$

as we claimed. We will need this information later.

Now we claim that $\lim_{n\to\infty}\lambda_n = \infty$. In fact, suppose by contradiction that it is false. Then there is a bounded subsequence of $(\lambda_n)_n$, which we will still call by $(\lambda_n)_n$ just for simplicity on the argument. Then there is $K \geq 0$ such that $\lambda_n \leq K$ for every n, and by the definitions of $|.|$, d and of λ_n we have

$$|\gamma_{p_n^{-1}}\psi_n(s)| = |p_n^{-1}\psi_n(s)p_n| = d(p_n,\psi_n(s)p_n) \leq \lambda_n \leq K \text{ for every } s \in S \text{ and } n \geq 1.$$

This means for every $n \geq 1$ and $s \in S$, the elements $\gamma_{p_n^{-1}}\psi_n(s)$ - as vertices of the Cayley graph - are all contained in the closed ball $\overline{B}_\Gamma(1,K)$ with center the identity element 1 and ray K. But in a Cayley graph of a finitely generated group, every such ball contains only a finite number of vertices. So there is a finite set V of vertices such that $\gamma_{p_n^{-1}}\psi_n(s) \in V$ for every $n \geq 1$ and $s \in S$. Since $S = \{s_1,...,s_m\}$, consider the map

$$f : \mathbb{N} \to V^m, \quad n \mapsto (\gamma_{p_n^{-1}}\psi_n(s_1),...,\gamma_{p_n^{-1}}\psi_n(s_m)).$$

Since V^m is finite and $\mathbb{N}$ is infinite, there are $n_1 \neq n_2$ such that $f(n_1) = f(n_2)$. By the definition of f and because $G = \langle s_1,...,s_m\rangle$, we immediately have $\gamma_{p_{n_1}^{-1}}\psi_{n_1} = \gamma_{p_{n_2}^{-1}}\psi_{n_2}$. Since inner automorphisms are mapped to Id by π, we have $\pi(\psi_{n_1}) = \pi(\psi_{n_2})$, a contradiction, for we chose all the ψ_n so that their projection are pairwise distinct. This shows our claim.

Let us create the sequence of pointed metric spaces. For every $n \geq 1$, define (X_n,d_n,p_n) by putting $X_n = G$, p_n the vertex considered above and $d_n = \frac{d}{\lambda_n}$. For this last definition of d_n, we must note that $\lambda_n > 0$, for since $\psi_n(s_1) \neq 1$, for example, we have $\lambda_n \geq d(p_n,\psi_n(s_1)p_n) \geq 1$.

Now, let ω be an ultrafilter of $\mathbb{N}$ containing the Fréchet filter (Proposition 2.61). Define the actions $G \curvearrowright X_n$ by putting

$$g \cdot x_n = \psi_n(g)x_n \text{ (multiplication in G)}$$

for every $x_n \in X_n$. So, G acts by isometries on each X_n. For every g, let $C(g) = |g|$. By what we have shown before,

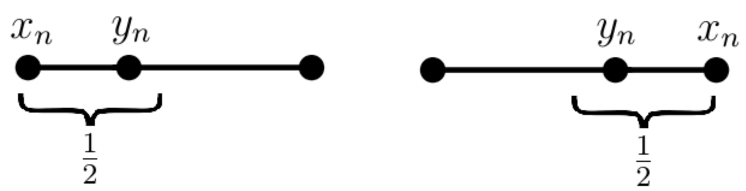
$$d_n(p_n, g \cdot p_n) = \frac{d(p_n, \psi_n(g)p_n)}{\lambda_n} \leq |g| = C(g)$$

for every n. By Proposition 2.69, these actions induce an action by isometries $G \curvearrowright X_\omega$, where X_ω is the ω-ultralimit of the sequence $(X_n, d_n, p_n)_n$. The action is given by $g \cdot [(x_n)_n] = [(g \cdot x_n)_n] = [(\psi_n(g)x_n)_n]$.

To finish step 1, let us show that X_ω is an $\mathbb{R}$-tree. We will do this by showing X_ω is isometric to an $\mathbb{R}$-tree Y_ω. So, let us define the sequence $(Y_n, \tilde{d}_n, p_n)_n$ by putting $Y_n = \Gamma(G, S)$ (so $X_n = G \subset Y_n$ is the set of vertices of Y_n), $\tilde{d}_n = \frac{d}{\lambda_n}$ (where d is now the metric on the whole Cayley graph) and putting p_n as the same points we chose for X_n. Since $\Gamma(G, S)$ is δ-hyperbolic with its distance d (for G is hyperbolic), it is easy to see that the Y_n are $\frac{\delta}{\lambda_n}$-hyperbolic. Since $\lim_{n\to\infty} \frac{\delta}{\lambda_n} = 0$ (for $\lim_{n\to\infty} \lambda_n = \infty$), it follows from Proposition 2.68 that Y_ω is an $\mathbb{R}$-tree. Now let us define the isometry. Since the X_n are subsets of the Y_n and the metrics d_n on the X_n are easily seen to be restrictions of the metrics $\tilde{d}_n$ on the Y_n, it follows that X_ω can be seen as a metric subspace of Y_ω. So, the natural inclusion

$$i : X_\omega \to Y_\omega, \quad i[(x_n)_n] = [(x_n)_n]$$

is well defined and an isometric embedding of X_ω in Y_ω. We just have to see i is surjective. Let $[(y_n)_n] \in Y_\omega$. For every n, the point y_n is in the Cayley graph $Y_n = \Gamma(G, S)$. Since the length of every edge is 1, there is a vertex x_n such that $d(x_n, y_n) \leq \frac{1}{2}$.



Let us see that $[(x_n)_n] \in X_\omega$, that is, the sequence $d_n(x_n, p_n)$ is uniformely bounded. By hypothesis there is $C \geq 0$ such that $\tilde{d}_n(y_n, p_n) \leq C$ for every n. Since $\lambda_n \geq 1$ for every n we also have $\frac{1}{\lambda_n} \leq 1$. Then, for every n,

$$d_n(x_n, p_n) = \tilde{d}_n(x_n, p_n) \leq \tilde{d}_n(x_n, y_n) + \tilde{d}_n(y_n, p_n) \leq \frac{1}{2\lambda_n} + C \leq \frac{1}{2} + C,$$

as desired. Finally, to see that $i[(x_n)_n] = [(x_n)_n] = [(y_n)_n]$ in Y_ω, we just have to check if $\lim_\omega \tilde{d}_n(x_n, y_n) = 0$. But this is clear since we just saw that $\tilde{d}_n(x_n, y_n) \leq \frac{1}{2\lambda_n} \to 0$ as $n \to \infty$. Thus, $X_\omega \simeq Y_\omega$ is an $\mathbb{R}$-tree. By applying the results of [82], we guarantee the action $G \curvearrowright X_\omega$ above is non-trivial and small, which completes step 1.

Step 2: *Extend the action above to a well-defined action $(G \rtimes H) \curvearrowright X_\omega$.*

To create such action, we must create an action $H \curvearrowright X_\omega$ and show that it is compatible (in the semidirect product sense) with the one from step 1. So, given $\varphi \in H$, let us first create maps $f_n : X_n \to X_n$ (or $f_{\varphi,n}$ if we need to specify) for every n to use Proposition 2.70. Since $\pi(\psi_n)$ is in the center of $\pi(H)$ we have $\pi(\varphi)\pi(\psi_n)\pi(\varphi)^{-1}\pi(\psi_n)^{-1} = Id$, so $\psi_n\varphi = \gamma_{y_n}\varphi\psi_n$ for some inner automorphism γ_{y_n} and some $y_n \in G$ (also denoted by $y_{\varphi,n}$ if needed). Define then

$$f_n = f_{n,\varphi} : X_n \to X_n \text{ by } f_n(x) = y_n\varphi(x).$$

Let $C = C(\varphi) = \sup_{s\in S}\{\max\{|\varphi(s)|, |\varphi^{-1}(s)|\}\}$. We claim f_n is C-bi-Lipschitz with the distance d_n (same number C for every n). In fact, we first see that $\varphi : G \to G$ is C-bi-Lipschitz by showing two inequalities. Given $g \in G$, represent g by a reduced word $w = s_1...s_m$ in S with minimal length, so that $|g| = m$. If w_i are reduced words representing $\varphi(s_i)$ with minimal length, then $w_1...w_n$ is a (not necessarily reduced or minimal) word representing $\varphi(g)$, from where we have

$$\begin{aligned} |\varphi(g)| &\leq length(w_1...w_n) \\ &= length(w_1) + ... + length(w_m) \\ &= |\varphi(s_1)| + ... + |\varphi(s_m)| \\ &\leq C + ... + C \ (m \text{ times}) \\ &= C|g|. \end{aligned}$$

This is true for every $g \in G$. Now, if $g, g' \in G$,

$$d(\varphi(g), \varphi(g')) = |\varphi(g)^{-1}\varphi(g')| = |\varphi(g^{-1}g')| \leq C|g^{-1}g'| = Cd(g, g'),$$

sp we have the first inequality. For the second one, note that $C(\varphi) = C(\varphi^{-1})$ and that the inequality above works for every element in H, so applying it for φ^{-1} we have

$$d(g, g') = d(\varphi^{-1}(\varphi(g)), \varphi^{-1}(\varphi(g'))) \leq C(\varphi^{-1})d(\varphi(g), \varphi(g')) = C(\varphi)d(\varphi(g), \varphi(g')),$$

so $\varphi : G \to G$ is C-bi-Lipschitz. By using these two inequalities and the fact that d is invariant under left multiplication in G we get that f_n is $\frac{C}{\lambda_n}$-bi-Lipschitz (because $d_n = \frac{d}{\lambda_n}$), and since $\lambda_n \geq 1$, f_n is obviously also C-bi-Lipschitz, as desired. If we show that the set $\{d_n(p_n, f_n(p_n)) \mid n \geq 1\}$ is bounded, then by Proposition 2.70 the f_n will give rise to a C-bi-Lipschitz map $f_\varphi : (X_\omega, d_\omega) \to (X_\omega, d_\omega)$, which defines our desired action. This is the next

Lemma 8.10. *The set* $\{d_n(p_n, f_n(p_n)) \mid n \geq 1\}$ *is bounded.*

Demonstração. Since $d_n = \frac{d}{\lambda_n}$, we have to find $D \geq 0$ such that $d(p_n, f_n(p_n)) \leq D\lambda_n$ for every n. Before this, let us show that there is $D' \geq 0$ such that, for every n,

$$\sup_{s\in S} d(f_n(p_n), \psi_n(s)f_n(p_n)) \leq D'\lambda_n \quad \text{and} \quad \sup_{s\in S} d(p_n, \psi_n(s)p_n) \leq D'\lambda_n.$$

The second inequality is not hard to be satisfied because $\sup_{s\in S} d(p_n, \psi_n(s)p_n) = \lambda_n$, so it suffices to choose any $D' \geq 1$. For the first one, we use many properties obtained so far to note that, for every $s \in S$ and n,

$$\begin{aligned} d(f_n(p_n), \psi_n(s)f_n(p_n)) &= d(y_n\varphi(p_n), \psi_n(s)y_n\varphi(p_n)) \\ &= d(y_n\varphi(p_n), \psi_n\varphi(\varphi^{-1}(s))y_n\varphi(p_n)) \\ &= d(y_n\varphi(p_n), y_n\varphi\psi_n(\varphi^{-1}(s))\varphi(p_n)) \\ &= d(\varphi(p_n), \varphi\psi_n(\varphi^{-1}(s))\varphi(p_n)) \\ &= d(\varphi(p_n), \varphi(\psi_n(\varphi^{-1}(s))p_n)) \\ &\leq Cd(p_n, \psi_n(\varphi^{-1}(s))p_n) \\ &\leq C\lambda_n|\varphi^{-1}(s)| \\ &\leq CC\lambda_n, \end{aligned}$$

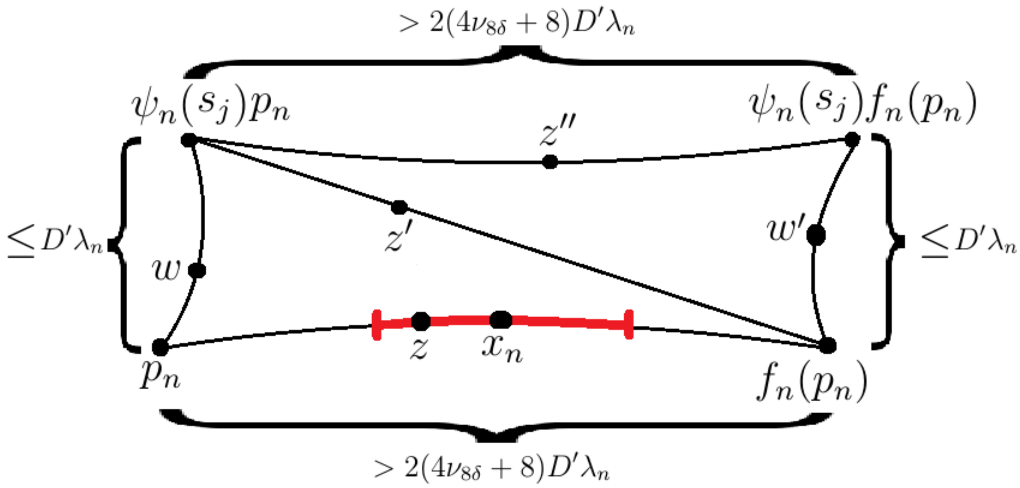
therefore $\sup_{s\in S} d(f_n(p_n), \psi_i(s)f_n(p_n)) \leq C^2\lambda_n$, and so the number $D' = \max\{1, C^2\}$ satisfies the two inequalities we wanted. Now let us find the number D of the lemma. It is enough to find $\tilde{D} \geq 0$ such that $d(p_n, f_n(p_n)) \leq \tilde{D}\lambda_n$ for every n but a finite set of indexes $F \subset \mathbb{N}$. For if we find such $\tilde{D}$, it is straightforward to see that $D = \max\{\tilde{D}, \max_{n\in F}\{\frac{d(p_n, f_n(p_n))}{\lambda_n}\}\}$ satisfies the lemma. To find $\tilde{D}$, let $\delta \geq 0$ such that the Cayley graph $X_S = \Gamma(G, S)$ is δ-hyperbolic. Define

$$\nu_{8\delta} = \mathrm{card}(\{g \in G \mid |g| \leq 8\delta\})$$

as the number of vertices in the closed ball $\overline{B}_\Gamma(1, 8\delta)$ and $\tilde{D} = 2(4\nu_{8\delta} + 8)D'$. We claim $\tilde{D}$ satisfies what we want. Suppose by contradiction this is false. Then there is an infinite set of indexes $B \subset \mathbb{N}$ such that, for every $n \in B$, $d(p_n, f_n(p_n)) > \tilde{D}\lambda_n$, or

$$d(p_n, f_n(p_n)) > 2(4\nu_{8\delta} + 8)D'\lambda_n.$$

For every $n \in B$, let $I_n = [p_n, f_n(p_n)]$ be the geodesic in X_S and let x_n be the middle point of this geodesic segment. Now let $n \in B$ be a fixed index such that $D'\lambda_n > \delta$ (for $\lim_{n\to\infty} \lambda_n = \infty$). Let us see that every segment $I' \subset I_n$ centered in x_n and with diameter less than $2(4\nu_{8\delta}+5)D'\lambda_n$ must be in a distance less than 2δ of both sets $\psi_n(s_1)I_n$ and $\psi_n(s_2)I_n$ (see the figure).



In fact, let $j \in \{1, 2\}$ and $z \in I'$. Then $d(z, p_n) \geq 3D'\lambda_n$, for since $d(p_n, x_n) > (4\nu_{8\delta} + 8)D'\lambda_n > 3D'\lambda_n$ by hypothesis, the only chance of $d(z, p_n) < 3D'\lambda_n$ happening is if $z \in [p_n, x_n]$.

But then we would have

$$d(p_n, x_n) = d(p_n, z) + d(z, x_n) < 3D'\lambda_n + (4\nu_{8\delta} + 5)D'\lambda_n = (4\nu_{8\delta} + 8)D'\lambda_n$$

and therefore $d(p_n, f_n(p_n)) = 2d(p_n, x_n) < 2(4\nu_{8\delta} + 8)D'\lambda_n$, a contradiction with the number n we chose. Now, the fact $d(z, p_n) \geq 3D'\lambda_n$ implies

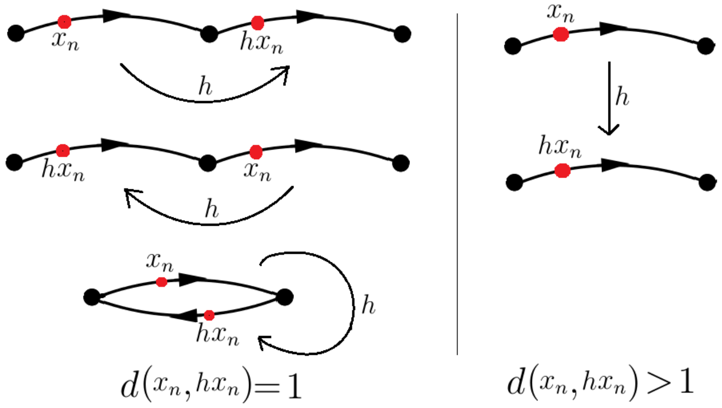
$$d(z, w) \geq 2D'\lambda_n \text{ for every } w \in [p_n, \psi_n(s_j)p_n],$$

for if not we would have $d(z, p_n) \leq d(z, w) + d(w, p_n) < 2D'\lambda_n + D'\lambda_n = 3D'\lambda_n$, another contradiction. So, $d(z, w) \geq 2D'\lambda_n > D'\lambda_n > \delta$ for every $w \in [p_n, \psi_n(s_j)p_n]$. If we consider the geodesic triangle $\Delta(p_n, f_n(p_n), \psi_n(s_j)p_n)$, δ-hyperbolicity implies there must be $z' \in [f_n(p_n), \psi_n(s_j)p_n]$ with $d(z, z') \leq \delta$. Now we similarly think in terms of the triangle $\Delta(f_n(p_n), \psi_n(s_j)p_n, \psi_n(s_j)f_n(p_n))$: for every $w' \in [f_n(p_n), \psi_n(s_j)f_n(p_n)]$ we must have $d(z', w') > \delta$, for if not we would have

$$d(z, f_n(p_n)) \leq d(z, z') + d(z', w') + d(w', f_n(p_n)) \leq \delta + \delta + D'\lambda_n < 3D'\lambda_n,$$

a contradiction because one can show $d(z, f_n(p_n)) \geq 3D'\lambda$ in the same way we showed $d(z, p_n) \geq 3D'\lambda$. So, by hyperbolicity again, there must be $z'' \in [\psi_n(s_j)p_n, \psi_n(s_j)f_n(p_n)] = \psi_n(s_j)I$ such that $d(z', z'') \leq \delta$. Therefore $d(z, z'') \leq 2\delta$ and z has a bounded 2δ-distance from $\psi_n(s_j)I$, as desired.

With this, one can show that $d(x_n, [\psi_n(s_1)^r, \psi_n(s_2)^r]x_n) \leq 8\delta$ for every $r = 1, ..., \nu_{8\delta}+1$. Now, since $\langle s_1, s_2 \rangle$ is a free group, the commutators $[s_1^r, s_2^r]$ are pairwise distinct in G, and because ψ_n is injective, the elements $h_r = [\psi_n(s_1)^r, \psi_n(s_2)^r] = \psi_n([s_1^r, s_2^r])$ must be pairwise distinct. Because of the action of G on its Cayley graph X_S, the $\nu_{8\delta}+1$ points $\{h_r \cdot x_n \mid 1 \leq r \leq \nu_{8\delta}+1\}$ are pairwise distinct inside the set $\overline{B}(x_n, 8\delta) \cap G \cdot x_n$. This is a contradiction because this set contains at most $\nu_{8\delta}$ points. In fact, if x_n is a vertex it contains exactly $\nu_{8\delta}$ points, because of the symmetry of the Cayley graph. If it is not a vertex, then the distance between any two points of the orbit $G \cdot x_n$ is at least 1 (see next figure for the only 4 possible cases); therefore $\text{card}(\overline{B}(x_n, 8\delta) \cap G \cdot x_n) \leq \nu_{8\delta}$. This finishes the lemma.



□

Coming back to the main line of proof of step 2, we got a C-bi-Lipschitz map $f_\varphi : X_\omega \to X_\omega$ for every $\varphi \in H$. To show this defines an action $H \curvearrowright X_\omega$ (by putting $\varphi \cdot x = f_\varphi(x)$), we must show $f_{Id_G} = Id_{X_\omega}$ and $f_{\varphi\varphi'} = f_\varphi f_{\varphi'}$. The first one is easy to see. To show the second one, note that

$$\psi_n\varphi\varphi' = \gamma_{y_{\varphi,n}}\varphi\psi_n\varphi' = \gamma_{y_{\varphi,n}}\varphi\gamma_{y_{\varphi',n}}\varphi'\psi_n = \gamma_{y_{\varphi,n}}\gamma_{\varphi(y_{\varphi',n})}\varphi\varphi'\psi_n = \gamma_{y_{\varphi,n}\varphi(y_{\varphi',n})}\varphi\varphi'\psi_n.$$

On the other hand, by definition we know that $\psi_n\varphi\varphi' = \gamma_{y_{\varphi\varphi',n}}\varphi\varphi'\psi_n$. Putting the equations together and canceling the bijections $\varphi\varphi'\psi_n$ on the right, we get $\gamma_{y_{\varphi\varphi',n}} = \gamma_{y_{\varphi,n}\varphi(y_{\varphi',n})}$ or, equivalently.

$$\gamma_{y^{-1}_{\varphi\varphi',n}y_{\varphi,n}\varphi(y_{\varphi',n})} = Id,$$

so the element $z_n = y^{-1}_{\varphi\varphi',n}y_{\varphi,n}\varphi(y_{\varphi',n})$ is by definition in the center of G. Now, for every n and every $x \in X_n$,

$$\begin{aligned}
d(f_{\varphi\varphi',n}(x), f_{\varphi,n}f_{\varphi',n}(x)) &= d(y_{\varphi\varphi',n}\varphi\varphi'(x), y_{\varphi,n}\varphi(y_{\varphi',n}\varphi'(x))) \\
&= d(y_{\varphi\varphi',n}\varphi\varphi'(x), y_{\varphi,n}\varphi(y_{\varphi',n})\varphi\varphi'(x)) \\
&= d(y_{\varphi\varphi',n}\varphi\varphi'(x), y_{\varphi\varphi',n}z_n\varphi\varphi'(x)) \\
&= d(y_{\varphi\varphi',n}\varphi\varphi'(x), y_{\varphi\varphi',n}\varphi\varphi'(x)z_n) \\
&= d(1, z_n) \\
&= |z_n|.
\end{aligned}$$

By Proposition 8.5, $Z(G)$ is finite and there is $Q \geq 0$ such that $|z_n| \leq Q$ for every n. Then $d(f_{\varphi\varphi',n}(x), f_{\varphi,n}f_{\varphi',n}(x)) \leq Q$ for every n and this easily implies $f_{\varphi\varphi'} = f_\varphi f_{\varphi'}$, because for every $[(x_n)_n] \in X_\omega$,

$$\begin{aligned}
d_\omega(f_{\varphi\varphi'}[(x_n)_n], f_\varphi f_{\varphi'}[(x_n)_n]) &= d_\omega([(f_{\varphi\varphi',n}(x_n))_n], [(f_{\varphi,n}f_{\varphi',n}(x_n))_n]) \\
&= \lim_\omega d_n(f_{\varphi\varphi',n}(x_n), f_{\varphi,n}f_{\varphi',n}(x_n)) \\
&= \lim_\omega \frac{d(f_{\varphi\varphi',n}(x_n), f_{\varphi,n}f_{\varphi',n}(x_n))}{\lambda_n} \\
&\leq \lim_\omega \frac{Q}{\lambda_n} \\
&= 0.
\end{aligned}$$

So we have an action $H \curvearrowright X_\omega$. It is important to note that since $f_\varphi f_{\varphi^{-1}} = f_{\varphi\varphi^{-1}} = f_{Id} = Id$, the f_φ are bijections, continuous (bi-Lipschitz) with continuous inverse maps $f_{\varphi^{-1}}$, so they are all homeomorphisms of X_ω. To complete step 2, we are only left to show that this action is coherent with the action $G \curvearrowright X_\omega$ from step 1 in the semidirect product sense. By the definition of semidirect product, this means we have to show that

$$\varphi(g) \cdot (f_\varphi(x)) = f_\varphi(g \cdot x)$$

for every $g \in G$, $\varphi \in H$ and $x = [(x_n)_n] \in X_\omega$. But for every n,

$$\begin{aligned} f_n(\psi_n(g)x_n) &= y_n\varphi(\psi_n(g)x_n) \\ &= y_n\varphi\psi_n(g)\varphi(x_n) \\ &= y_n\varphi\psi_n(g)y_n^{-1}y_n\varphi(x_n) \\ &= \gamma_{y_n}\varphi\psi_n(g)f_n(x_n) \\ &= \psi_n\varphi(g)f_n(x_n), \end{aligned}$$

and this implies

$$\begin{aligned} d_\omega(\varphi(g)\cdot(f_\varphi(x)), f_\varphi(g\cdot x)) &= d_\omega(\varphi(g)\cdot[(f_n(x_n))_n], f_\varphi[(\psi_n(g)x_n)_n]) \\ &= d_\omega([(\psi_n\varphi(g)f_n(x_n))_n], [(f_n(\psi_n(g)x_n))_n]) \\ &= \lim_\omega d_n(\psi_n\varphi(g)f_n(x_n), f_n(\psi_n(g)x_n)) \\ &= \lim_\omega 0 \\ &= 0, \end{aligned}$$

which completes step 2.

Step 3: *Modify "a little bit" the $\mathbb{R}$-tree X_ω and the action $(G \rtimes H) \curvearrowright X_\omega$ above to obtain the affine action $(G \rtimes H) \curvearrowright T$ desired.*

By the definition of what should be an affine action $(G \rtimes H) \curvearrowright T$, it suffices to find an $\mathbb{R}$-tree (T, d) (this metric d will have "nothing" to do with the one from step 1) and a non-trivial small action by isometries $G \curvearrowright T$ whose translation length function l is such that: for every $\varphi \in H$, there is $\lambda(\varphi) > 0$ (with $\varphi \in H \mapsto \lambda(\varphi) \in (0, \infty)$ a homomorphism) with $l \circ \varphi^{-1} = \lambda(\varphi)l$ and also a unique affine map $h_\varphi : T \to T$ with $\lambda(\varphi)$ a dilation coefficient $(d(h_\varphi(x), h_\varphi(y)) = \lambda(\varphi)d(x, y))$ and such that $h_\varphi(g \cdot x) = \varphi(g) \cdot h_\varphi(x)$. Why would this give us an affine action? In fact, the last equality is to guarantee all the maps h_φ combine with $G \curvearrowright T$ to a well define action $(G \rtimes H) \curvearrowright T$. The dilation coefficients $\lambda(\varphi)$ are the affine part of the definition, together with the coefficients $\lambda(g) = 1$ for $g \in G$. The fact $l \circ \varphi^{-1} = \lambda(\varphi)l$ is an extra information we will get while we try to find the affine maps h_φ.

First, since $G \curvearrowright X_\omega$ is non-trivial, let $T' \subset X_\omega$ be the unique minimal G-invariant subtree (Proposition 2.54). We claim that T' is also H-invariant, therefore $(G \rtimes H)$-invariant. Let $\varphi \in H$. Since f_φ is a homeomorphism, $f_\varphi(T')$ is also a subtree of X_ω, for it is connected and so convex, by uniqueness of geodesics. It is also G-invariant, for if $f_\varphi(x) \in f_\varphi(T')$ is any element and $g \in G$, then $g \cdot f_\varphi(x) = f_\varphi(\varphi^{-1}(g) \cdot x) \in f_\varphi(T')$, as desired. But by Proposition 2.54, every G-invariant subtree of X_ω must contain T', so $T' \subset f_\varphi(T')$. By using the same argument to φ^{-1} we get $T' \subset f_{\varphi^{-1}}(T')$, or (because $f_\varphi f_{\varphi^{-1}} = Id$) $f_\varphi(T') \subset T'$. So $\varphi \cdot T' = f_\varphi(T') = T'$ and T' is H-invariant, which gives an action $(G \rtimes H) \curvearrowright T'$. Denote then by d' the restriction of d_ω in X_ω to T'. This action is clearly still non-trivial, small and by isometries. We will need from now on a lot of convex cones:

Definition 8.11. A subset $C \subset V$ in a real vector space V is a convex cone if

a) If $x \in C$ and $t > 0$, then $tx \in C$;

b) If $x, y \in C$, then $x + y \in C$.

It is easy to see that every convex cone is a convex subset in the natural meaning.

Denote by $\mathbb{R}^{T' \times T'}$ the set of all maps $\delta : T' \times T' \to \mathbb{R}$. Given a map $\delta \in \mathbb{R}^{T' \times T'}$ which is a metric, define the following conditions:

i) The topology τ_δ induced in T' by the metric δ coincides with the topology $\tau_{d'}$ induced on T' by d' (that is, $\tau_\delta = \tau_{d'}$);

ii) The action $G \curvearrowright T'$, thought as an action $G \curvearrowright (T', \delta)$, is also isometric (that is, $\delta(g \cdot x, g \cdot y) = \delta(x, y)$ for $x, y \in T'$).

Define

$$D(T') = \{\delta \in \mathbb{R}^{T' \times T'} \mid \delta \text{ is a metric on } T' \text{ satisfying } i) \text{ and } ii)\}.$$

Of course $D(T') \neq \emptyset$ for d' is an element of it.

Lemma 8.12. *$D(T')$ is a convex cone of $\mathbb{R}^{T' \times T'}$. Furthermore, every $\delta \in D(T')$ induces on the set T' an $\mathbb{R}$-tree structure (T', δ) for which the already known action $G \curvearrowright T'$ is still non-trivial, small and by isometries with the exact same characteristic sets, that is, $C_{(g,T',\delta)} = C_{(g,T',d')}$ for every $g \in G$.*

This Lemma will open us up possibilities of choices of different metrics on T' on which the action of G is still good enough.

Demonstração. Let us first show that $D(T')$ is a convex cone by showing conditions $a)$ and $b)$ of the definition. First, let $\delta \in D(T')$ and $t > 0$ and let us show $t\delta \in D(T')$ by showing it satisfies $i)$ and $ii)$. Since $B_\delta(x, r) = B_{t\delta}(x, tr)$ for every $x \in T'$ and $r \geq 0$, the collections of open balls of the metric spaces (T', δ) and $(T', t\delta)$ coincide, so the topologies τ_δ and $\tau_{t\delta}$ are the same. Therefore $\tau_{t\delta} = \tau_\delta = \tau_{d'}$ and $t\delta$ satisfies $i)$. Moreover, $t\delta$ also satisfies $ii)$ because $(t\delta)(g \cdot x, g \cdot y) = t\delta(g \cdot x, g \cdot y) = t\delta(x, y) = (t\delta)(x, y)$. Therefore $t\delta \in D(T')$ and we have $a)$. Now, let $\delta_1, \delta_2 \in D(T')$ and let us show $\delta_1 + \delta_2 \in D(T')$. It is easy to see their sum satisfy $ii)$, because

$$(\delta_1 + \delta_2)(g \cdot x, g \cdot y) = \delta_1(g \cdot x, g \cdot y) + \delta_2(g \cdot x, g \cdot y) = \delta_1(x, y) + \delta_2(x, y) = (\delta_1 + \delta_2)(x, y).$$

Item $i)$ is not much harder. In fact, let us show that $\tau_{\delta_1+\delta_2} = \tau_{\delta_1}$, for this will give us $\tau_{\delta_1+\delta_2} = \tau_{\delta_1} = \tau_{d'}$, by hypothesis. To see $\tau_{\delta_1+\delta_2} \subset \tau_{\delta_1}$, let $B_{\delta_1+\delta_2}(x, r)$ be an open ball. Since $\tau_{\delta_1} = \tau_{d'} = \tau_{\delta_2}$, let $r' > 0$ such that $B_{\delta_1}(x, r') \subset B_{\delta_2}(x, r/2)$. Now let $\tilde{r} = \min\{r/2, r'\}$. For this $\tilde{r}$ we have $B_{\delta_1}(x, \tilde{r}) \subset B_{\delta_1+\delta_2}(x, r)$, for if $z \in B_{\delta_1}(x, \tilde{r})$, then since $\tilde{r} \leq r'$, z is also an element of $B_{\delta_1}(x, r') \subset B_{\delta_2}(x, r/2)$ and we have

$$(\delta_1 + \delta_2)(z, x) = \delta_1(z, x) + \delta_2(z, x) < \tilde{r} + \frac{r}{2} \leq \frac{r}{2} + \frac{r}{2} = r,$$

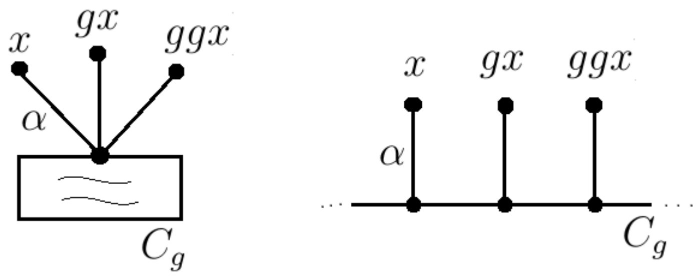
and this shows $\tau_{\delta_1+\delta_2} \subset \tau_{\delta_1}$. To see $\tau_{\delta_1} \subset \tau_{\delta_1+\delta_2}$ just note that we always have $B_{\delta_1+\delta_2}(x, r) \subset B_{\delta_1}(x, r)$. So, item $i)$ is also true and $\delta_1 + \delta_2 \in D(T')$. This shows $b)$ and so $D(T')$ is a convex cone.

Now, let $\delta \in D(T')$. Condition $ii)$ says exactly that the action $G \curvearrowright (T', \delta)$ is still by isometries. By condition $i)$, the topologies of (T', δ) and (T', d') coincide, so to show the action

$G \curvearrowright (T', \delta)$ inherits all those properties of $G \curvearrowright (T', d')$ we just have to show that non-triviality, smallness and characteristic sets of an action depend only on the topology of T' (instead of the metric). First, because of the topological characterization of trees in Proposition 2.30, (T', δ) is also an $\mathbb{R}$-tree. Since G is finitely generated, proposition 2.42 says that triviality of an action of G on T' is characterized by the existence of a global fixed point, so it depends at most on the topology of T' and therefore $G \curvearrowright (T', \delta)$ is also non-trivial. Since smallness of actions is given in terms of stabilizers of topological arcs and the collection of such arcs on (T', d') and (T', δ) coincide, $G \curvearrowright (T', \delta)$ is also small. Finally, to show the characteristic sets of $G \curvearrowright (T', \delta)$ are the same of $G \curvearrowright (T', d')$, let us show a topologic characterization of a characteristic set C_g, for any $g \in G$. We claim

$$C_g = \{x \in T' \mid gx \text{ is in the topological arc from } x \text{ to } ggx\}.$$

Indeed, $(\subset)$ is clear from Proposition 2.36, both in the elliptic and hyperbolic cases. To show $(\supset)$, let $x \notin C_g$ and let us show gx is not in the topological arc between x and ggx. Let α be the bridge from x to C_g. Then $g\alpha$ and $gg\alpha$ are the bridges from gx and ggx to C_g, respectively. By using Proposition 2.36 again, we can easily identify the arc between x and ggx and it is clear that gx is not there, as the figure illustrates (elliptic case on the left, hyperbolic on the right). This completes the proof.



□

Of course, each element δ of $D(T')$ gives rise to the action $G \curvearrowright (T', \delta)$ and therefore to a translation length function $l_{(T',\delta)} = l_\delta$ on G (Definition 2.41). Denote by $SLF(G) \subset \mathbb{R}^G - \{0\}$ the set of all translation length functions of non-trivial and small actions by isometries of G (on any $\mathbb{R}$-trees), and define $\theta : D(T') \to SLF(G)$ by putting $\theta(\delta) = l_{(T',\delta)} = l_\delta$. Consider the image $C(T') = \theta(D(T'))$. Denote also by $P(G)$ the projective space of $\mathbb{R}^G$, that is, the quotient of $\mathbb{R}^G - \{0\}$ by the equivalence relation $x \sim y \Leftrightarrow x = \lambda y$ for some $\lambda \neq 0$ (possibly negative). Equip $P(G)$ with the quotient topology and denote by $P : R^G - \{0\} \to P(G)$ the natural projection.

Lemma 8.13. $C(T')$ *and its closure* $\overline{C(T')}$ *are both convex cones of* $\mathbb{R}^G$. *Moreover,* $\overline{C(T')} \subset SLF(G)$ *and* $P(\overline{C(T')})$ *is compact and convex.*

Demonstração. To show $b)$ for $C(T')$, let $\theta(\delta_1), \theta(\delta_2) \in C(T')$. For every $g \in G$, let $x \in C_g$ and we have $l_{\delta_i}(g) = \delta_i(x, g \cdot x)$ and $l_{\delta_1+\delta_2}(g) = (\delta_1 + \delta_2)(x, g \cdot x)$ (since C_g is the same set for δ_1, δ_2

and for $\delta_1 + \delta_2$, by the previous lemma). So

$$\begin{aligned}\theta(\delta_1+\delta_2)(g) &= l_{\delta_1+\delta_2}(g)\\ &= (\delta_1+\delta_2)(x, g\cdot x)\\ &= \delta_1(x,g\cdot x)+\delta_2(x,g\cdot x)\\ &= l_{\delta_1}(g)+l_{\delta_2}(g)\\ &= \theta(\delta_1)(g)+\theta(\delta_2)(g)\\ &= (\theta(\delta_1)+\theta(\delta_2))(g).\end{aligned}$$

Therefore, $\theta(\delta_1)+\theta(\delta_2)=\theta(\delta_1+\delta_2)\in C(T')$, since $\delta_1+\delta_2\in D(T')$ (previous lemma). Item $a)$ is similar, so $C(T')$ is a convex cone. It is straightforward to check that the closure of a convex cone is also a convex cone, so it is the closure $\overline{C(T')}$.

Let us show $\overline{C(T')}\subset SLF(G)$. Since $C(T')\subset SLF(G)$ by definition, it is enough to show $SLF(G)$ is closed in $\mathbb{R}^G$, for then $\overline{C(T')}\subset\overline{SLF(G)}=SLF(G)$. By [83], $P(SLF(G))$ is compact in the Hausdorff space $P(G)$, so it is closed and therefore $P^{-1}(P(SLF(G)))$ is closed in $\mathbb{R}^G$. But $SLF(G)$ is closed under multiplication by positive scalars, for if l is a translation function of a small action of G on an $\mathbb{R}$-tree (T,d) and $t>0$, then tl is the translation function of the same action of G on (T,td) which is still non-trivial and small. No we show $SLF(G)$ is closed: suppose $(x_n)_n$ is a sequence in $SLF(G)$ converging to a point $x\in\mathbb{R}^G$. Then $(P(x_n))_n$ is a sequence in the compact set $P(SLF(G))$ converging to $P(x)$, so $P(x)\in P(SLF(G))$ and so $P(x)=P(y)$ for some $y\in SLF(G)$. Write $x=\lambda y$ for $\lambda\neq 0$. If $\lambda>0$, then by what we just observed we have $x\in SLF(G)$, as desired. If $\lambda<0$ we get a contradiction. In fact, since y is in particular a non-trivial length function we have $y(g)>0$ for some $g\in G$, so $x(g)<0$. But since $x_n\to x$ and $\mathbb{R}^G$ has the product topology, in particular $x_n(g)\to x(g)$, so we would have $x_n(g)<0$ for some n. This is a contradiction because x_n is a translation length function and therefore must assume only non-negative values.

Finally, let us show $P(\overline{C(T')})$ is compact and convex. For compacity, since $P(\overline{C(T')})\subset P(SLF(G))$ is contained in a compact space, it is enough to show it is closed. By the quotient topology of $P(G)$, it is then enough to check if $P^{-1}P(\overline{C(T')})$ is closed in $\mathbb{R}^G$. But $\overline{C(T')}$ is a convex cone, contained in the "half-space" of $\mathbb{R}^G$ of non-negative coordinates, so by the definition of P we have $P^{-1}P(\overline{C(T')})=\overline{C(T')}\cup(-\overline{C(T')})$ is the union of two closed spaces and therefore is closed. The projection of any convex cone inside a half-space of $\mathbb{R}^G$ is clearly a convex set inside a hemisphere of $P(G)$, so $P(\overline{C(T')})$ is convex. This completes the lemma. □

Now, to find a special length function of a special action of G to satisfy our theorem, we want to use the following:

Theorem 8.14 ([48], Theorem 3.3.5)**.** *A group H is amenable if and only if every time it acts affinely and separately continuously on a compact convex set Ω of a locally convex space E, there is a global fixed point $p\in\Omega$, that is, $h\cdot p=p$ for every $h\in H$.*

So we want to create a special action of our group H on the compact convex set $P(\overline{C(T')})$ to find a global fixed point $P(l)$, that will give rise to a translation length function l of a small and non-trivial action by isometries of G on an $\mathbb{R}$-tree (T,d). So first we will make H act on $\overline{C(T')}$. Define an action $H\curvearrowright\mathbb{R}^G$ by $\varphi\bullet x=x\circ\varphi^{-1}$.

Lemma 8.15. *The sets $C(T')$ and $\overline{C(T')}$ are both invariant under the action above.*

Demonstração. Let us define an auxiliary action $H \curvearrowright D(T')$ by putting

$$(\varphi * \delta)(a,b) = \delta(f_\varphi(a), f_\varphi(b))$$

for every $\varphi \in H$, $\delta \in D(T')$ and $a, b \in T'$. Since f_φ is a homeomorphism, it is easy to see that $\varphi * \delta$ is a metric. Let us show $\varphi * \delta \in D(T')$ by showing it satisfies *i*) and *ii*). For *i*), it is enough to show $\tau_{\varphi*\delta} = \tau_\delta$. For ($\supset$), let $B_\delta(x,r)$ be any open ball. Since f_φ^{-1} is continuous at $a = f_\varphi(x)$, there is $r' > 0$ such that $\delta(b,a) < r' \Rightarrow \delta(f_\varphi^{-1}(b), f_\varphi^{-1}(a)) < r$. This implies $B_{\varphi*\delta}(x,r') \subset B_\delta(x,r)$, for if $y \in B_{\varphi*\delta}(x,r')$ then $\delta(f_\varphi(y), f_\varphi(x)) = (\varphi * \delta)(y,x) < r'$ and so $\delta(y,x) = \delta(f_\varphi^{-1}(f_\varphi(y)), f_\varphi^{-1}(f_\varphi(x))) < r$. Part ($\subset$) is similar: if $B_{\varphi*\delta}(x,r)$ is any open ball, then since f_φ is continuous at x there is $r' > 0$ such that $\delta(y,x) < r' \Rightarrow \delta(f_\varphi(y), f_\varphi(x)) < r$, and this implies $B_\delta(x,r') \subset B_{\varphi*\delta}(x,r)$. Now, to show *ii*), we use the fact $f_\varphi(gx) = \varphi(g)f_\varphi(x)$, already shown before, and the fact that δ satisfies *ii*). Then we have

$$(\varphi * \delta)(gx, gy) = \delta(f_\varphi(gx), f_\varphi(gy)) = \delta(\varphi(g)f_\varphi(x), \varphi(g)f_\varphi(y)) = \delta(f_\varphi(x), f_\varphi(y)) = (\varphi * \delta)(x,y),$$

as desired. This shows $\varphi * \delta \in D(T')$ and the action $H \curvearrowright D(T')$ is well defined.

Let us show now $C(T')$ is invariant under the action. Let $l_{(T',\delta)} = \theta(\delta)$ be any element of $C(T') = \theta(D(T'))$ and let $\varphi \in H$. It is enough to show $\varphi^{-1} \bullet l_{(T',\delta)} \in C(T')$, for H is a group. For any $g \in G$, fix a point $x \in C_g$. Then $gx \in [x, ggx]$, and since f_φ is a homeomorphism, it takes topological arcs to topological arcs. Then

$$\varphi(g)f_\varphi(x) = f_\varphi(gx) \in f_\varphi([x, ggx]) = [f_\varphi(x), f_\varphi(ggx)] = [f_\varphi(x), \varphi(g)\varphi(g)f_\varphi(x)],$$

which implies $f_\varphi(x) \in C_{\varphi(g)}$ (by the topological characterization of characteristic sets we showed above). Because of this,

$$\begin{aligned}
(\varphi^{-1} \bullet l_{(T',\delta)})(g) &= (l_{(T',\delta)} \circ \varphi)(g) \\
&= l_{(T',\delta)}(\varphi(g)) \\
&= \delta(f_\varphi(x), \varphi(g)f_\varphi(x)) \\
&= \delta(f_\varphi(x), f_\varphi(gx)) \\
&= (\varphi * \delta)(x, gx) \\
&= l_{(T',\varphi*\delta)}(g),
\end{aligned}$$

so $\varphi^{-1} \bullet l_{(T',\delta)} = l_{(T',\varphi*\delta)} = \theta(\varphi * \delta) \in C(T')$ since $\varphi * \delta \in D(T')$. This shows $C(T')$ is invariant.

Finally, showing the invariance of $\overline{C(T')}$ is straightforward: if $x \in \overline{C(T')}$ and $\varphi \in H$, let $(x_n)_n$ be a sequence in $C(T')$ converging to x. The map $\mathbb{R}^G \to \mathbb{R}^G$ given by $y \mapsto \varphi \bullet y$ is a coordinate permutation (thinking of y as a G-uple), so it is easily seen to be a homeomorphism on the product topology. It follows that $\varphi \bullet x_n \to \varphi \bullet x$. But by the invariance of $C(T')$, $\varphi \bullet x_n \in C(T')$ for every n, so $\varphi \bullet x \in \overline{C(T')}$, as desired. This completes the lemma. □

Now the action $H \curvearrowright \overline{C(T')}$ respects the projective relation, for if $x' = \lambda x$ for $\lambda \neq 0$ then $\varphi \bullet x' = x' \circ \varphi^{-1} = (\lambda x) \circ \varphi^{-1} = \lambda(x \circ \varphi^{-1}) = \lambda(\varphi \bullet x)$. So the action gives rise to $H \curvearrowright P(\overline{C(T')})$

by putting $\varphi \bullet [x] = [\varphi \bullet x]$. Since $P(\overline{C(T')})$ is compact and convex, Theorem 8.14 gives us a global fixed point $P(l) = [l]$, that is, $[l \circ \varphi^{-1}] = [\varphi \bullet l] = \varphi \bullet [l] = [l]$ for every $\varphi \in H$, or

$$l \circ \varphi^{-1} = \lambda(\varphi) l$$

for some $\lambda = \lambda(\varphi)$, for every $\varphi \in H$. To be even more specific we can write

$$l(\varphi^{-1}(g)) = \lambda l(g)$$

for every $g \in G$. Since $l \in \overline{C(T')} \subset SLF(G)$, l is a translation length function of a non-trivial and small action by isometries of G on some arbitrary $\mathbb{R}$-tree (T, d), which is finally the tree we were looking for in the statement of our theorem. Let us denote the action $G \curvearrowright (T, d)$ by denoting the action of $g \in G$ on $x \in T$ by $g \cdot x$. We can assume without loss of generality that T is minimal: if needed, just consider a minimal subtree and call it T again. Since G is non-elementary, by 8.6 we already know it contains a free subgroup of rank 2, which acts freely and properly discontinuously on T, so by Theorem 2.52 this action is irreducible.

We are just left to extend this action to an affine action $(G \rtimes H) \curvearrowright (T, d)$. By what we observed in the beginning of step 3, we are just left to show that $\lambda = \lambda(\varphi) > 0$ for every $\varphi \in H$, that the association $\varphi \in H \mapsto \lambda(\varphi) \in (0, \infty)$ is a homomorphism and that there is a unique affine map $h_\varphi : T \to T$, with $\lambda(\varphi)$ a dilation coefficient, such that $h_\varphi(g \cdot x) = \varphi(g) \cdot h_\varphi(x)$. The two first facts are easy. Given $\varphi \in H$, since $l \neq 0$ there is $g \in G$ with $l(g) > 0$. Then $0 \leq l(\varphi^{-1}(g)) = \lambda(\varphi) l(g)$ and $\lambda(\varphi) \neq 0$ implies $\lambda(\varphi) > 0$. Now, let $\varphi, \psi \in H$. Then

$$\lambda(\psi \circ \varphi) l = l \circ (\psi \circ \varphi)^{-1} = l \circ \varphi^{-1} \circ \psi^{-1} = (\lambda(\varphi) l) \circ \psi^{-1} = \lambda(\varphi)(l \circ \psi^{-1}) = \lambda(\varphi)\lambda(\psi) l = \lambda(\psi)\lambda(\varphi) l,$$

so for the g above, $\lambda(\psi \circ \varphi) l(g) = \lambda(\psi)\lambda(\varphi) l(g)$, which implies $\lambda(\psi \circ \varphi) = \lambda(\psi)\lambda(\varphi)$. For the last fact, we invoke Theorem 2.55. Given $\varphi \in H$, denote $\lambda = \lambda(\varphi^{-1})$. Based on the action $G \curvearrowright (T, d)$, $(g, x) \mapsto g \cdot x$ above we define:

$$G \times (T, d) \xrightarrow{\diamond} (T, d) \text{ with } g \diamond x = \varphi(g) \cdot x$$

and we also consider the action $G \curvearrowright (T, d)$ but with dilated metric:

$$G \times (T, \lambda d) \xrightarrow{\bullet} (T, \lambda d) \text{ with } g \bullet x = g \cdot x$$

Since G is non-elementar, these actions are irreducible, in particular semi-simple and not shifts. They also have the same translation length function. Indeed, for every $g \in G$,

$$l_{(T,d,\diamond)}(g) = \inf_{x \in T} d(x, g \diamond x) = \inf_{x \in T} d(x, \varphi(g) \cdot x) = l(\varphi(g)) = \lambda l(g)$$

and

$$l_{(T,\lambda d,\bullet)}(g) = \inf_{x \in T} \lambda d(x, g \bullet x) = \inf_{x \in T} \lambda d(x, g \cdot x) = \lambda \inf_{x \in T} d(x, g \cdot x) = \lambda l(g).$$

Therefore by Theorem 2.55 there is a unique G-equivariant isometry $h_\varphi : (T, \lambda d) \to (T, d)$. Its isometry gives us

$$d(h_\varphi(x), h_\varphi(y)) = \lambda d(x, y)$$

for every $x, y \in T$, that is, h is an affine map of (T, d) with λ a dilation coefficient. Equivariance gives us

$$h_\varphi(g \cdot x) = h_\varphi(g \bullet x) = g \diamond h_\varphi(x) = \varphi(g) \cdot h_\varphi(x)$$

for every $g \in G$, $x \in T$, which was the last assertion to be shown. This completes the proof of Theorem 8.9. □

8.2 R_∞ for non-elementary hyperbolic groups

This section is dedicated to finally show that

Theorem 8.16. *Every non-elementary hyperbolic group has property R_∞.*

In fact, let $\varphi \in Aut(G)$ and let us show $R(\varphi) = \infty$. We divide the proof in two possible cases:

Case 1: $\pi(\varphi)$ has finite order in $Out(G)$.

Let $m \geq 1$ be the order of $\pi(\varphi)$. By Proposition 7.3, it suffices to show G_φ acts non-elementary on a hyperbolic geodesic space X. Since $G_\varphi/G \simeq \mathbb{Z}_m$ is finite, G has finite index in G_φ. Therefore, $G_\varphi \overset{QI}{\sim} G$ by Proposition 2.22, so G_φ is hyperbolic by Proposition 2.23 and therefore by definition its Cayley graph must be a hyperbolic space where G_φ acts. Also, there is a nonabelian free group F_2 such that $F_2 \leq G \leq G_\varphi$, so G_φ must be a non-elementar hyperbolic group. Then, to finish case 1), we're just left to show that the action of any non-elementar hyperbolic group on its Cayley graph is non-elementary:

Lemma 8.17. *Let G be any non-elementary hyperbolic group and $G \curvearrowright \Gamma(G, S)$ be its natural action on its Cayley graph $\Gamma = \Gamma(G, S)$. Then the action is non-elementary.*

Demonstração. The action is combinatorially given by $g \cdot \tilde{g} = g\tilde{g}$ for any vertex $\tilde{g} \in G = V(\Gamma)$ and $g \cdot (\tilde{g}, s) = (g\tilde{g}, s)$ for every (oriented) edge $(\tilde{g}, s) \in G \times S = E(\Gamma)$. As we observed in Example 1.36, this action is (combinatorially) without inversions and free. Topologically, one can think of the action on the vertices being linearly extended to the topological edges between them. Let us check it is non-elementary by checking items 1) to 3) of Definition 7.1:

1) By Proposition 8.6, there must be in particular $g \in G$ with infinite order in G. We have to show that g has infinite order as an isometry of Γ. If $g^n = Id$ as an isometry (for $n \neq 0$), then by applying it to the vertex $1 \in \Gamma$ we get $g^n = Id(1) = 1$ in G, contradiction. This shows 1).

2) Let $x \in \partial\Gamma$. Showing x is not a global fixed point is showing $\{x\} \subsetneq G \cdot x$, that is, x is properly contained on its G-orbit. It is known (see, for example, the great survey [61]) that $\partial\Gamma = \partial G$ by definition and that since G is non-elementary, $\partial\Gamma$ is an infinite compact metrizable space with no isolated points, in particular infinite and Hausdorff. Because of this, no single point in $\partial\Gamma$ can be dense in it. But it is also known (see [61] again) that $G \cdot x$ is dense in $\partial\Gamma$, so $\{x\} \subsetneq G \cdot x$, as desired.

3) Similar to item 2): let $\{p, q\}$ be any pair ($p \neq q$) and let us show $\{p, q\} \subsetneq G\{p, q\}$. Since $\partial\Gamma$ is infinite and Hausdorff, let $z \notin \{p, q\}$ and let U be an open neighborhood of z not intercepting $\{p, q\}$. Since $G \cdot p$ is dense in $\partial\Gamma$, let $g \in G$ such that $gp \in U$. In particular, we have $gp \notin \{p, q\}$ and therefore $g \cdot \{p, q\} = \{gp, gq\} \neq \{p, q\}$. This shows 3) and completes the proof. □

Case 2: $\pi(\varphi)$ has infinite order in $Out(G)$.

Let $H = \langle\varphi\rangle \leq Aut(G)$. Since the order of $\pi(\varphi)$ is infinite, φ also has infinite order, so $H \simeq \mathbb{Z}$ is amenable (it is a well-known fact that $\mathbb{Z}$ is amenable) and $\pi(H) = \langle\pi(\varphi)\rangle \simeq \mathbb{Z}$ has an infinite center. By Paulin's Theorem, there is an $\mathbb{R}$-tree (T, d) and an affine action $(G \rtimes H) \curvearrowright T$ whose restriction to G is a non-trivial, minimal, small and irreducible action by isometries. By what we observed in that proof, this gives a translation length function l of the action and a homomorphism $\psi \in H \mapsto \lambda(\psi) \in (0, \infty)$ such that $l \circ \psi^{-1} = \lambda(\psi)l$, such that for every $\psi \in H$ there is a unique affine map $h_\psi : T \to T$, with $\lambda(\psi)$ a dilation coefficient and also such that $h_\psi(g \cdot x) = \psi(g) \cdot h_\psi(x)$ for every $(g, x) \in G \times T$.

In particular, since $\lambda(\varphi^{-1})\lambda(\varphi) = \lambda(Id) = 1$, we either have $\lambda(\varphi^{-1}) \geq 1$ or $\lambda(\varphi) \geq 1$. In the former case, we define $\lambda = \lambda(\varphi^{-1}) \geq 1$ and, since $l \circ \varphi = \lambda l$, we can apply Theorem 7.4 for φ, which gives us $R(\varphi) = \infty$, as desired. In the latter case, we define $\lambda = \lambda(\varphi) \geq 1$ and, since $l \circ \varphi^{-1} = \lambda l$, we can apply Theorem 7.4 for the element φ^{-1}, which gives us $R(\varphi^{-1}) = \infty$, and this gives us $R(\varphi) = \infty$ by Proposition 1.8. This completes the proof of Theorem 8.16. $\square$

Capítulo 9

Relatively hyperbolic groups and property R_∞

This chapter contains a careful analysis of Fel'shtyn's strong claim (item (2) of Theorem 3.3 in [31]) that every non-elementary relatively hyperbolic group has property R_∞. To be more precise, this chapter provides an exploration of what should be a natural and complete proof of R_∞ for these groups, according to the sketch of proof Fel'shtyn provides in [31]. The basic (and good) idea of the sketch is to generalize to the relative case the proof of R_∞ for non-elementary hyperbolic groups, given implicitly by G. Levitt and M. Lustig in [68] (and exhibited in our previous chapter). However, as I will argument throughout this chapter, my conclusion is that such a proof, based on that sketch, is at least more complicated than it looks like and at most incomplete.

The aim of doing this analysis is to locate where are the valid arguments and the exact difficulties in such a proof, in order to make an advance on the discussion of property R_∞ for non-elementary relatively hyperbolic groups, given their importance for geometric group theory and the importance of the claim for the R_∞ subject.

9.1 Locating the difficulty

For the reader who has not read the paper [31] yet, we reproduce Fel'shtyn's claim and sketch of proof below:

Theorem 9.1 ([68], Theorem 3.3, item (2)). *Non-elementary relatively hyperbolic groups have the R_∞ property.*

Fel'shtyn's sketch of proof: Theorem 3.2 applies if G is a Gromov-hyperbolic group or relatively hyperbolic group and if φ has finite order in $Out(G)$. In fact, in this case, G_φ contains G as a subgroup of finite index, thus is quasi-isometric to G, and by quasi-isometry invariance, it is itself a Gromov-hyperbolic or relatively hyperbolic group. Now let assume that an automorphism of a hyperbolic or relatively hyperbolic group has infinite order in $Out(G)$. We describe the main steps of the proof in this case (see [30, 68] for details). By [81] and [5], Φ preserves some $\mathbb{R}$-tree T with nontrivial minimal small action of G (recall that an action of G is small if all arc stabilizers are virtually cyclic; the action of G on T is always irreducible (no global fixed point, no invariant line, no invariant end)). This means that there is an $\mathbb{R}$-tree T equipped with an isometric action of G whose length function satisfies $l \cdot \Phi = \lambda l$ for some $\lambda \geq 1$.

Step 1. Suppose $\lambda = 1$. Then the Reidemeister number $R(\phi)$ is infinite.

Step 2. Suppose $\lambda > 1$. Assume arc stabilizers are finite, and there exists $N_0 \in \mathbb{N}$ such that, for every $Q \in T$, the action of $StabQ$ on $\pi_0(T - Q)$ has at most N_0 orbits. Then the Reidemeister number $R(\phi)$ is infinite.

Step 3. If $\lambda > 1$, then T has finite arc stabilizers. If $\lambda > 1$ then from work of Bestvina and Feighn [6] it follows that there exists $N_0 \in \mathbb{N}$ such that, for every $Q \in T$, the action of $StabQ$ on $\pi_0(T - (Q))$ has at most N_0 orbits. □

The expansion and investigation of the sketch of proof above will be done in Section 9.3. For now, let us divide the sketch in sentences and just make brief comments about them. These comments are being made based on my (already done) detailed investigation in 9.3.

"Theorem 3.2 applies if G is a Gromov-hyperbolic group or relatively hyperbolic group and if φ has finite order in $Out(G)$. In fact, in this case, G_φ contains G as a subgroup of finite index, thus is quasi-isometric to G, and by quasi-isometry invariance, it is itself a Gromov-hyperbolic or relatively hyperbolic group."

This argument turned out to be true. The "Theorem 3.2" quoted corresponds to our Proposition 7.3, which works for every group acting non-elementary on a hyperbolic space. The quasi-isometry invariance also works for relatively hyperbolic groups, as we will see. Now, look at these two sentences:

"Now let assume that an automorphism of a hyperbolic or relatively hyperbolic group has infinite order in $Out(G)$. We describe the main steps of the proof in this case (see [30, 68] for details)."

"Step 1 ... Step 2 ... Step 3."

The language used in the first sentence above indicates that the proof for the relatively hyperbolic case should be the same as the proof for the hyperbolic case. So, if we want to write it in details, we should try to adapt the proof of Levitt and Lustig in [68]. The three steps Fel'shtyn describes by the end of his sketch (second sentence above) are just the steps of Levitt and Lustig's proof. As we told in the beggining of section 7.2, by studying their result we noted that it could be applied to any finitely generated group G, given the existence of some *"super special"* action they describe. So, we decided to state and show their result within this more general context, and that is our Theorem 7.4. To summarize, in these sentences Fel'shtyn wants to apply "our" Theorem 7.4 for a relatively hyperbolic group. So far, everything is good and the proof will work ***provided that*** we guarantee the existence of that *"super special"* action of G (check Theorem 7.4).

Let us comment on the two remaining sentences of the sketch. They are:

"By [81] and [5], Φ preserves some $\mathbb{R}$-tree T with nontrivial minimal small action of G (recall that an action of G is small if all arc stabilizers are virtually cyclic; the action of G on T is always irreducible (no global fixed point, no invariant line, no invariant end))."

"This means that there is an $\mathbb{R}$-tree T equipped with an isometric action of G whose length function satisfies $l \cdot \Phi = \lambda l$ for some $\lambda \geq 1$."

By what we argued above, these two sentences should be the ones who guarantee the existence of the *"super special"* action of G on an R-tree, described in the statement of Theorem 7.4. In the hyperbolic case, that action is proved to exist exactly by Paulin's Theorem 8.9, whose proof consists of three consecutive steps (which shouldn't be confused with the three steps Fel'shtyn describes in his sketch: from now one, those can be forgotten). Without further details, Paulin's steps can be summarized as:

1) Create a nontrivial, minimal, irreducible and small action of G on some $\mathbb{R}$-tree T;

2) Extend the action above to an action of $G \rtimes \langle \varphi \rangle$ on T;

3) Modify the action above to get the *"special action"* required.

As the reader can see, step 1 coincides with Fel'shtyn's claim in the first sentence above. Can that step be adapted to the relative case? We showed it can, indeed, in Step 1 of Case 2 of Theorem 9.27.

Therefore, Fel'shtyn's second sentence above *should* be the one that guarantees (or at least indicates) that steps 2 and 3 of Paulin's proof can also be adapted to the relatively hyperbolic case, guaranteeing the existence of the *"super special"* action. But that's not what it does, and that is where the difficulty is. By the expression "This means" followed by the description of the *"super special"* action in Fel'shtyn's second sentence, it looks like it is being assumed that steps 2 and 3 of Paulin's proof can be adapted for any relatively hyperbolic group, with at least the same kind of proof and no difficulties. The question is: can they? As the reader may see in our section 9.4, if we assume that Paulin's step 2 can be adapted to relatively hyperbolic groups, then step 3 also follows, with the same proof as the one in the hyperbolic case. The problem is with step 2. By trying to naturally adapt step 2 to the relative case, a new obstacle rose, which we couldn't overcome: basically, it is the non-compactness of the fundamental domain of the action of G, which makes it difficult to guarantee that the automorphism φ acts naturally by a quasi-isometry on X and therefore by a homeomorphism on the $\mathbb{R}$-tree T. In the hyperbolic case (see Chapter 8), the fundamental domain is just a point; the action on X is φ itself, which is easily seen to be bi-Lipschitz and in particular a quasi-isometry, so it induces a homeomorphism on T.

Our conclusion, therefore, is that Paulin's step 2 is the strong obstacle for Fel'shtyn's sketch of proof. To be more precise, the difficult is showing that an (infinite order) automorphism φ of a relatively hyperbolic group G acts by homeomorphisms on the $\mathbb{R}$-tree T induced by X, or that it acts by quasi-isometries on X. This is stated in Lemma 9.29. We do not think this Lemma can be proved for a general relatively hyperbolic group, for, in trying to prove it, we found a counterexample; that is, we found a relatively hyperbolic group and an infinite order automorphism which does not act on X by a quasi-isometry. So, it looks like the actions of the automorphisms on X are not good enough as they should be so that Fel'sthyn proof could work (see Section 9.5).

On the other hand, since Lemma 9.29 is the only obstacle we found, we have a complete proof for the following positive result: every non-elementary relatively hyperbolic group G in which all infinite order automorphisms act by quasi-isometries on the space X must have the R_∞ property (see Corollary 9.31). However, we do not have much information on how to find these groups.

The rest of the chapter is organized as follows: in Section 9.2, we define geometrically finite actions, and use them to define relatively hyperbolic groups in Section 9.3. In that section, we also explore some properties of these groups (which will be useful in our context), especially the fundamental domain of their action on the related hyperbolic space X. Then, Section 9.4 is where we exhibit what would be a proof of property R_∞ for non-elementary relatively hyperbolic groups by following Fel'shtyn's sketch, that is, by adapting the proof of the hyperbolic case we give in chapter 8. That proof would be complete if it wasn't for Lemma 9.29, which we believe is not true in general. To convince the reader of this, in Section 9.5 we show an example where Lemma 9.29 does not work.

We decided to maintain the incomplete proof in this thesis, anyway, for two reasons: first, to give the reader an idea of what could be a proof of R_∞ for non-elementary relatively hyperbolic groups. Second, if we assume that Lemma 9.29 works for some particular relatively hyperbolic group G, we get property R_∞ for G (Corollary 9.31). A last comment is: we do not necessarily believe Fel'shtyn's claim is false. We haven't found any non-elementary relatively hyperbolic groups without property R_∞. Although we believe such a counterexample could be found, we also think that Lemma 9.29 could be somehow avoided to a similar and complete proof. A third possible option, of course, would be to find a totally different proof of R_∞ for non-elementary relatively hyperbolic groups; but that's beyond my capacity for now. Maybe you, dear reader, can help me someday.

9.2 Geometrically finite actions

Relatively hyperbolic groups were first defined by M. Gromov in his 1987 paper [50] on hyperbolic groups. Since then, many other definitions were given, for example by Bowditch [16], Farb [29], Drutu-Osin-Sapir [27], Osin [80] and others. All of these definitions are known to coincide when the groups and subgroups involved are finitely generated and infinite. Because of the geometric language used and our familiarity with the author, we will use the notions of relative hyperbolicity given by Bowditch in [16], comparing with some of its equivalences.

Let G be an infinite and finitely generated group and X be a proper geodesic hyperbolic space. Suppose G acts properly discontinuously on X by isometries. By Proposition 2.84, G acts as a convergence group on the compact metric space $M = \partial X$. For $g \in G$, denote by $fix(g) = \{x \in \partial X \mid gx = x\}$ the set of fixed points of g in $M = \partial X$. The next definition should not be confused with "elliptic" and "hyperbolic" isometries of an $\mathbb{R}$-tree from Chapter 2.

Definition 9.2. We say an element $g \in G$ is

- *elliptic* if it has finite order in G;
- *parabolic* if it has infinite order in G and $\mathrm{card}(fix(g)) = 1$;
- *loxodromic* if it has infinite order in G and $\mathrm{card}(fix(g)) = 2$.

Of course these are mutually exclusive definitions. Furthermore, they form a partition of the elements of G:

Proposition 9.3. *Every element of G is either elliptic, parabolic or loxodromic.*

Demonstração. Let $g \in G$. If g has finite order in G, it is elliptic. Suppose then it has infinite order. We just have to show that $0 < \mathrm{card}(fix(g)) \leq 2$. Consider the sequence $(g^n)_n$. Since

$G \curvearrowright M$ is a convergence action, we have a collapsing subsequence, so we can assume without loss of generality that $g_n|_{M-\{x\}} \twoheadrightarrow y$ for $x, y \in M$. Let $z \in M - \{x, g^{-1}x\}$. Since $z \neq x$, $g^n z \to y$. Then $g^{n+1}z = gg^n z \to gy$. But $gz \neq x$, so we also have $g^{n+1}z = g^n gz \to y$. By uniqueness, $gy = y$ and so $0 < \text{card}(fix(g))$. If g had three distinct fixed points, say, x, y and z, then $\{x, y, z\}$ would be an element of $\Theta(M)$ fixed by g. In particular, the set $\{h \in G \mid h\{x, y, z\} = \{x, y, z\}\}$ would contain all powers $g^n, n \in \mathbb{Z}$ of g and be infinite, so the action $G \curvearrowright \Theta(M)$ would not be properly discontinuous, a contradiction with Theorem 2.77. So $\text{card}(fix(g)) \leq 2$ and we're done. □

It is known in this context that parabolic and loxodromic elements do not live together very well. For a proof, we refer [95].

Proposition 9.4. *Let G and M be as above. Suppose every element of an infinite subgroup $H \leq G$ fixes a point $p \in M$. Then H either consists entirely of elliptic and parabolic elements, or it consists entirely of elliptic and loxodromic elements. In the latter case, H also fixes another point $q \in M - \{p\}$ and is virtually cyclic.* □

With this in hands we can go in the direction of defining a geometrically finite action.

Definition 9.5. Let G act as a convergence group on a compact metric space M. We say a subgroup $H \leq G$ is *parabolic* if it is infinite, with no loxodromic elements and if it has a global fixed point in M ($x \in M$ such that $hp = p$ for every $h \in H$). This fixed point is unique by definition and will be called the associated *parabolic point*.

Proposition 9.6. *If $p \in M$ is a parabolic point, the group $Stab_G(p) = \{g \in G \mid gp = p\}$ is parabolic.*

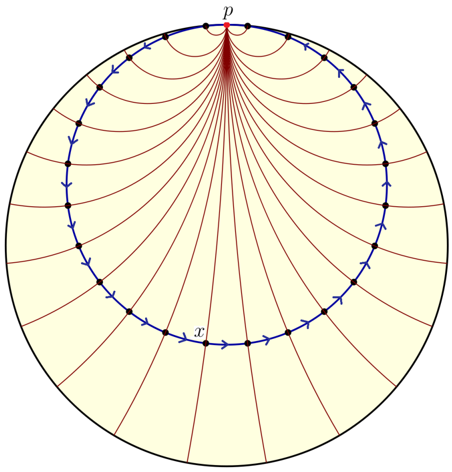
Demonstração. Let $H \leq G$ be the parabolic subgroup whose parabolic point is p. Since $H \leq Stab_G(p)$ and H is infinite, $Stab_G(p)$ is infinite. Since by definition every element of it fixes p, by Proposition 9.4 it either consists entirely of elliptic and parabolic elements (case 1), or it consists entirely of elliptic and loxodromic elements (case 2). If H contains a parabolic element, then so does $Stab_G(p)$ and then it must be in case 1, therefore not containing any loxodromics, so it is a parabolic group, as desired. Suppose therefore H consists only of elliptic elements. If $Stab_G(p)$ was in case 2, then by Proposition 9.4 it is virtually cyclic. So, $H \leq Stab_G(p)$ is virtually cyclic, in particular finitely generated. Since every element of H is elliptic (finite order), H is a finite group, a contradiction, for it is parabolic. Therefore $Stab_G(p)$ must be in case 2 and we are done. □

Observation 9.7. There is therefore a one-to-one correspondence between the parabolic points p in M and the groups $Stab(p) \leq G$, which are the maximal parabolic subgroups of G.

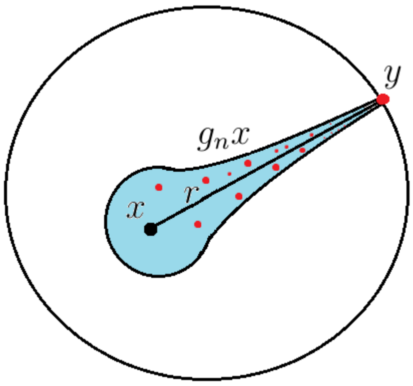
If $H \leq G$ is a parabolic group with associated parabolic point p, then the action $G \curvearrowright M$ by homeomorphisms can be restricted to the action $H \curvearrowright M - \{p\}$, which gives rise to the quotient topological orbit space $M - \{p\}/H$.

Definition 9.8 (bounded parabolic point)**.** We say a parabolic group $H \leq G$ with associated parabolic point p is *bounded* if the orbit space $M - \{p\}/H$ is compact. We say a parabolic point p is *bounded* if the parabolic group $Stab_G(p)$ is bounded.

Example 9.9. Let $\mathbb{Z}$ act on the upper half plane hyperbolic model by translations: $n \cdot (x, y) = (x+n, y)$. This is a properly discontinuous action by isometries. If we think of this action on the Poincare disk model $\mathcal{H}^2$, it is easy to see that the north pole p is the unique global fixed point in $\partial\mathcal{H}^2 \simeq S^1$ (so $\mathbb{Z} = Stab(p)$), and that the quotient space $(S^1 - \{p\})/\mathbb{Z} \simeq \mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1, which is compact. Therefore, $\mathbb{Z}$ is a bounded parabolic subgroup of itself in this action. Next figure illustrates the $\mathbb{Z}$-orbit of $x \in \mathcal{H}^2$.



Definition 9.10 (conical limit point)**.** We say a point $y \in M = \partial X$ is a *conical limit point* if there is a point $x \in X$ and a sequence $(g_n)_n$ in G such that $g_n x \to y$ in $\overline{X} = X \cup \partial X$ and that $d(g_n x, r) \leq K$ for every $n \geq 1$, for some geodesic ray r representing y and some constant $K \geq 0$.



Definition 9.11. [geometrically finite action] Let G act as a convergence group on the compact metric space $M = \partial X$ as above. We say the action is *geometrically finite* (or that G acts as a geometrically finite convergence group on M) if every point of M is either a conical limit point or a bounded parabolic point.

9.3 Defining a relatively hyperbolic group

Now we follow [16] to define our notion:

Definition 9.12 (Relatively hyperbolic groups)**.** Let G be an infinite and finitely generated group and X be a proper geodesic hyperbolic space. Suppose G acts properly discontinuously on X by isometries, and let $\mathcal{H}$ be a collection of finitely generated subgroups of G. We say G is hyperbolic relative to $\mathcal{H}$ (or just that G is a relatively hyperbolic group) if the induced (convergence) action $G \curvearrowright \partial X$ is a geometrically finite action (Definition 9.11) and the subgroups of the collection $\mathcal{H}$ are exactly the maximal parabolic subgroups of G (Observation 9.7).

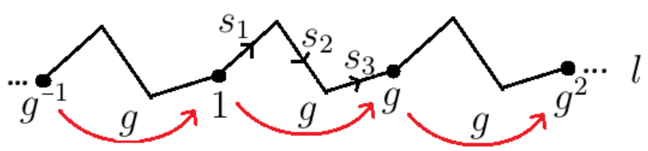
The subgroups in the collection $\mathcal{H}$ are called the *peripheral* subgroups of G. Some authors in the literature use the notation $(G, \mathcal{H})$ and call it a relatively hyperbolic structure. We may sometimes use this notation. B.H. Bowditch shows in [16] the equivalence of definition above with the following one:

Definition 9.13. Let G be an infinite and finitely generated group and suppose G acts on a connected hyperbolic graph K. Let $\mathcal{H}$ be a collection of finitely generated subgroups of G. We say G is hyperbolic relative to $\mathcal{H}$ if the following conditions are satisfied: for each $n \geq 1$, each edge of K is contained in only finitely many circuits of length n; there are finitely many G-orbits of edges, each edge stabilizer is finite and the subgroups of the collection $\mathcal{H}$ are exactly the infinite vertex stabilizers of K.

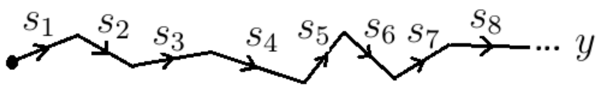
Although the first definition is better for doing theory, both of them can be used to obtain a few examples. Let's contemplate some of them before studying the main properties of these groups.

Example 9.14. Every hyperbolic group G is relatively hyperbolic with respect to the empty collection $\mathcal{H} = \emptyset$ (note that we do not assume $\mathcal{H} \neq \emptyset$ in any of the definitions above), so in particular all examples given in 8.3 and 8.4 are relatively hyperbolic. In fact, let us check this by the second definition. The natural action (by left translations) of G on its Cayley graph $\Gamma = \Gamma(G, S)$ is properly discontinuous (it is in fact free) and cocompact, the quotient being a bouquet with $\operatorname{card} S$ petals, for Γ is $\operatorname{card} S$-regular. In particular, there are only $\operatorname{card} S$ G-orbits of edges. Also, for any $n \geq 1$, since Γ is regular it is enough to check if each edge $(1, s)$ is contained in a finite number of circuits of length n. But each such circuit gives rise to a different relation $w = 1$ in the presentation of G, with the word w starting with s and having length n. Since G is finitely presented (Proposition 8.7), there is only a finite number of such circuits. Since the action is free, the edge stabilizers are all trivial and therefore finite. Finally, both collections $\mathcal{H}$ and the set of infinite vertex stabilizers of K are empty, so they coincide and G is relatively hyperbolic with respect to $\mathcal{H} = \emptyset$.

Therefore, hyperbolic groups acting on their Cayley graphs correspond by the first definition to the case where there are no parabolic points, subgroups or isometries (for $\mathcal{H} = \emptyset$). In fact, every element $g = s_1 ... s_k \in G$ (seen as an isometry of Γ and a homeomorphism of $\partial\Gamma$) is either elliptic or loxodromic, in which case the fixed points in $\partial\Gamma$ correspond to the two "ends" of the geodesic line l whose vertices are the subwords $s_1 ... s_i$ ($1 \leq i \leq k$) and its g^n-translations for $n \in \mathbb{Z}$. The action of g on l is just translation by a distance of $|g| = k$ (see figure).



We can also see that every hyperbolic group is relatively hyperbolic by using the first definition. In fact, let $y \in \partial\Gamma$ and let us show y is a conical limit point. By definition of the Cayley graph, y is the endpoint of a geodesic ray $r : [0, \infty) \to \Gamma$ such that $r(n) = s_1...s_n$ is a vertex for every n and $s_i \in S$ for $i \geq 1$. Then for $g_n = s_1...s_n$ and the point $x = 1$ of Γ we obviously have $g_n x \to y$, and $d(g_n x, r) = 0$ (for $g_n x = s_1...s_n \in r$). That is, y is a "degenerated" conical limit point.



Example 9.15. Let (G, Γ) be a graph of groups with Γ a finite graph, G_y finite for every edge y in Γ and G_P finitely generated for every vertex P in Γ. Let $\tilde{G} = \pi_1(G, \Gamma, T)$ be the fundamental group of (G, Γ) and let us check $\tilde{G}$ is relatively hyperbolic according to the second definition above. By the theory of [86], there is a combinatorial tree X (which is 0-hyperbolic) on which $\tilde{G}$ acts without inversion, and the orbit quotient space is $\tilde{G}/X \simeq \Gamma$. Of course, for each $n \geq 1$, each edge of X is contained in a finite number of circuits of length n (exactly 0 such circuits). The number of $\tilde{G}$-orbits of edges in X is exactly the number of edges of $\tilde{G}/X \simeq \Gamma$, which is finite, and the edge stabilizers G_y are finite by hypothesis. By the second definition, then, $\tilde{G}$ is hyperbolic relative to the collection $\mathcal{H} = \{G_P \mid G_P \text{ is infinite}\}$ of infinite vertex stabilizers.

This example includes a wide class of groups, such as any finite amalgamation $*_A G_i$ of finitely generated groups $G_1, ..., G_n$ over a common finite subgroup A, in particular any such finite free product $G_1 * ... * G_n$.

In the particular case of all G_P being also finite, $\tilde{G}$ is then hyperbolic relative to the empty collection $\mathcal{H} = \emptyset$, so it is a hyperbolic group and we get again the last class of examples in 8.3. For more complex and different classes of relatively hyperbolic groups, we refer [5].

Let's comment about some known properties of relatively hyperbolic groups $(G, \mathcal{H})$ and its action on X, based on [16]. It is known that a conical limit point cannot be a parabolic point, so the parabolic points are all bounded. From this and from the first definition it follows that the groups in $\mathcal{H}$ are the groups of the form $Stab(p)$, where p is a bounded parabolic point of ∂X. It is also known that the intersection of any two such groups is a finite subgroup. In [96], Tukia shows that there are only a finite number of conjugacy classes of the groups in $\mathcal{H}$. This is equivalent to the geometrical fact that there are only a finite number of G-orbits of the parabolic points in ∂X. In fact, if $H = stab(p)$ and $g \in G$, it is easy to see that $gHg^{-1} = stab(gp)$ is a parabolic group and gp a parabolic point. Therefore, conjugating an element of $\mathcal{H}$ by g corresponds to walking from p to gp. It is also known that the collection of bounded parabolic points is countable.

We would rather have an even more geometric characterization of the geometrically finite action of relatively hyperbolic groups on their spaces. To present the characterization given by Bowditch in [16], we must deal with horospheres and horoballs (definitions are based on [55]).

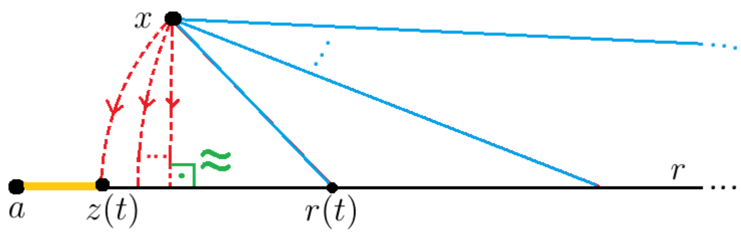
Definition 9.16. Let X be a geodesic space and r be a geodesic ray in X. The *Busemann function* (or just horofunction) $b_r : X \to \mathbb{R}$ associated to r is given by

$$b_r(x) = \lim_{t \to \infty} (d(x, r(t)) - t).$$

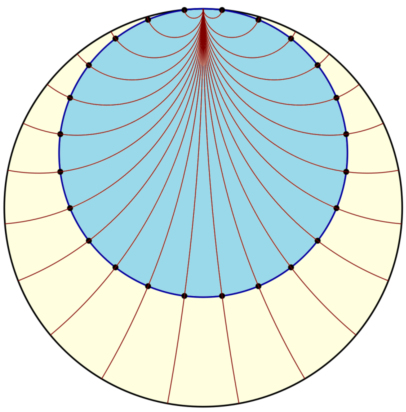
If $p = r(\infty) \in \partial X$ is the endpoint of r, a horosphere of X on p is a level set of the form $S(p) = b_r^{-1}\{k\}$ for some $k \in \mathbb{R}$. A horoball on p is a sublevel set of the form $B(p) = b_r^{-1}(-\infty, k]$.

Observation 9.17. The limit above exists for every x. In fact, let us show the map $t \mapsto d(x, r(t)) - t$ is non-increasing and bounded below. If $0 \leq t' < t$, then $d(x, r(t)) \leq d(x, r(t')) + d(r(t'), r(t)) = d(x, r(t')) + t - t'$, so $d(x, r(t')) - t' \geq d(x, r(t)) - t$, as desired. Also, $t - t' = d(r(t), r(t')) \leq d(r(t), x) + d(x, r(t'))$. In particular for $t' = 0$ we obtain $t \leq d(r(t), x) + d(x, r(0))$, or $d(r(t), x) - t \geq -d(x, r(0))$, so it is bounded below.

Example 9.18. In the Euclidean spaces $\mathbb{R}^n$, let $r(t) = a + tv$ be a geodesic ray, for some $a \in \mathbb{R}^n$ and a unitary vector $v \in \mathbb{R}^n$. Then $b_r(x) = \langle a - x, v \rangle$. Let's have an intuitive idea of this in $\mathbb{R}^2$ (see next figure). For a fixed t, the number $t - d(x, r(t)) = d(a, r(t)) - d(x, r(t))$ can be obtained by "lying" the vector $x - r(t)$ over r fixing the point $r(t)$ (the red arcs represent this motion) and then computing the size of the orange geodesic $[a, z(t)]$. As t gets bigger, the point $z(t)$ tends to be the orthogonal projection of x on r and the size of $[a, z(t)]$ therefore tends to $\langle x - a, v \rangle$. By "multiplying the argument above by -1" we get what we desired. It follows that horospheres of r are the orthogonal lines to r, and horoballs are the closed half spaces determined by these lines (the ones containing $r(t)$ for all sufficiently large t).



Example 9.19. In the Poincare disk $\mathcal{H}^2$, horoballs and horospheres look like the Euclidean ones, with the difference that their "center" is the boundary point $p = r(\infty)$. Since this is our main intuitive model of a hyperbolic space, we can intuitively think of a horosphere $S(p)$ as the set of points of X which are all equidistant from p in $\overline{X}$. Next figure shows a horoball $B(p)$ in $\mathcal{H}^2$ and some geodesics tending to the north pole p. All of them cross the horophere $S(p)$ orthogonally at points equidistant from p.



Note that if we take the numbers k in $B(p) = b_r^{-1}(-\infty, k]$ to be arbitrarily large negative numbers, the balls $B(p)$ tend to be arbitrarily small.

From now on, suppose only that the induced action of G on ∂X is a convergence action, that is, not necessarily a geometrically finite action. Horoballs and horospheres are used by Bowditch in this context to describe parabolic points of ∂X in terms of the geometrical behavior of the H-action of the associated parabolic groups H around them.

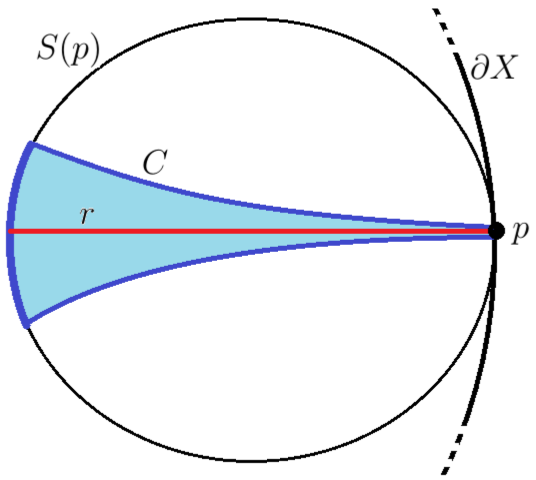
Proposition 9.20. *If H is a parabolic group with p the parabolic point associated, then there is a H-invariant horofunction around p.*

This means $b_r(x) = b_r(hx)$ for every $(h, x) \in H \times X$. In particular, there is a collection of H-invariant horoballs and horospheres $B(p)$ around p. This H-invariance can be perfectly visualized in the figures of Examples 9.9 and 9.19. Using this fact, Bowditch obtained:

Proposition 9.21. *Let H and p as above, and let $B(p)$ be an H-invariant horoball with associated horosphere $S(p)$. Then H is a bounded parabolic subgroup if and only if the quotient space $S(p)/H$ is compact.*

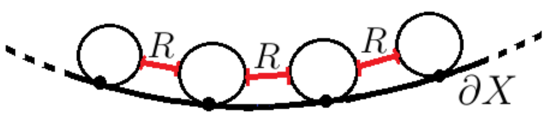
Definition 9.22. If H, p and $B(p)$ are as above, a *cusp region* for p is the space $B(p)/H$.

We can easily visualize a cusp region by lifting it to X. Since $S(p)/H$ is compact, Bowditch observes that every point x of $B(p)$ is a bounded distance from a H-image of a ray r tending to p inside the cusp. The cusp region C is also shown to be quasi-isometric to this ray. See next figure.



Bounded parabolic points can also be separated by sufficiently far away horoballs:

Proposition 9.23. *Let $\mathcal{P}$ be a G-invariant collection of bounded parabolic points of ∂X. Then, for any $R \leq 0$, there is a G-invariant collection $\mathcal{B} = \{B(p) \mid p \in \mathcal{P}\}$ of horoballs around the points of $\mathcal{P}$ which is R-separated, that is, $d(B(p), B(p')) \geq R$ for any $p \neq p'$ in $\mathcal{P}$.*



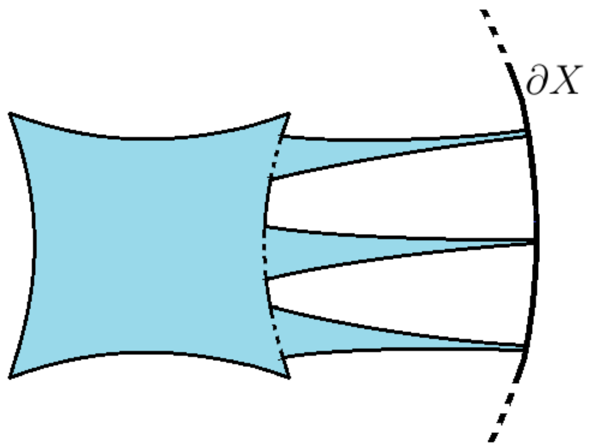
Now, let $\mathcal{P}$ be a G-invariant collection of parabolic points of ∂X and suppose $\mathcal{B} = \{B(p) \mid p \in \mathcal{P}\}$ is a G-invariant and R-separated collection of horoballs around them, for some $R \geq 0$. We denote $Y(\mathcal{B}) = X - \cup_{p \in \mathcal{P}}\, int B(p)$, which is clearly a closed and G-invariant subset of X. With all these tools in hands, Bowditch shows the following characterization:

Theorem 9.24. *Let G be an infinite and finitely generated group acting by isometries on a proper hyperbolic geodesic space X and let $\mathcal{P}$ be a $G-$invariant collection of parabolic points of ∂X. Then the (convergence) action $G \curvearrowright \partial X$ is geometrically finite (and $\mathcal{P}$ is the set of all bounded parabolic points) if and only if there is $R \geq 0$ and an R-separated G-invariant collection $\mathcal{B} = \{B(p) \mid p \in \mathcal{P}\}$ of horoballs such that the quotient space $Y(\mathcal{B})/G$ is compact.*

Now we can finish the section by showing an intuitive drawing of a fundamental domain of the action of a relatively hyperbolic group $(G, \mathcal{H})$ on its hyperbolic space X. Let $\mathcal{P}$ be the collection of all (bounded) parabolic points of ∂X, which we know is G-invariant. Bowditch shows that, given $R \geq 0$, there's an R-separated G-invariant collection $\mathcal{B} = \{B(p) \mid p \in \mathcal{P}\}$ of horoballs. By the above theorem, the quotient space $Y(\mathcal{B})/G$ is compact. We have

$$X/G = (Y(\mathcal{B}) \cup (\cup_{p\in\mathcal{P}}\ intB(p)))/G = (Y(\mathcal{B})/G) \cup (\cup_{p\in\mathcal{P}}\ intB(p)/G).$$

To quotient all the interiors $intB(p)$ by the action of G is obviously equivalent to quotient only one representative $intB(p)$ for each G-orbit of the horoballs (or the parabolic points). We know there are only a finite number of such orbits, so we need to quotient by G only a finite number of such horoballs, say, $B(p_1), ..., B(p_n)$. But if we quotient each ball $B(p_i)$ by G, in particular we are taking the quotient by its associated parabolic stabilizer subgroup $H(p) = Stab(p)$, so we obtain exactly a cusp region. Then X/G is the union of a compact subspace with a finite number of cusps. By lifting this to X, we get the following intuitive idea of a fundamental domain F (the expert reader might forgive me for the eventually unrealistic picture):

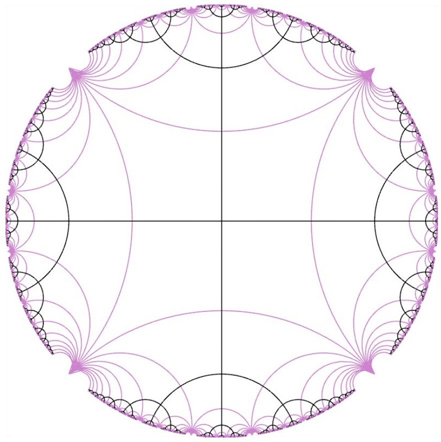


Here, the square represents a compact set $K \subset X$ and we have one cusp for each bounded parabolic point representing its G-orbit. Since K is compact, Bowditch observes that this implies X/G is quasi-isometric to a finite wedge of geodesic rays.

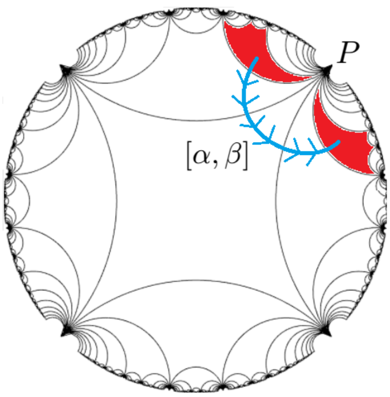
We finish this section by giving a good and visual example of a relatively hyperbolic group, the hyperbolic space associated and the fundamental domain of the action.

Example 9.25. This is a standard example of a relatively hyperbolic group. Let Y be a once punctured torus and $G = \pi_1(Y)$ be its topological fundamental group. Instead of thinking of Y in $\mathbb{R}^3$, we may use the standard square representation, with the usual edge identifications of the torus, with the only difference that the four vertices are now removed. With these adaptations, let us imagine it as a hyperbolic square. The universal cover of Y can be thought as the open

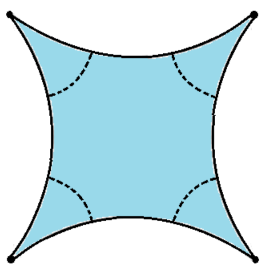
Poincare disk $X = \mathcal{H}^2$. It is known that G then acts properly discontinuously and by isometries on X, with fundamental domain being the infinite (and vertexless) hyperbolic squares given in the figure. As $G \simeq F_2$ is free of rank 2, one can imagine the Cayley graph of G inside X, with each vertex being the "center" of one of the hyperbolic squares. This way, the action of G on X is easy to imagine, for an element g takes the central square isometrically onto the square whose center is the vertex g of the Cayley graph.



All points in $\partial X \simeq S^1$ that are limits of geodesics in the Cayley graph can be easily seen to be conical limit points (see the last paragraph of Example 9.14). It is known that these points form a dense subset of ∂X. The others are bounded parabolical points, fixed by cyclic subgroups of G. For example, the point $P = e^{i\frac{\pi}{4}}$ is fixed by the cyclic subgroup $\langle[\alpha, \beta]\rangle$ generated by the commutator $[\alpha, \beta]$ of the two generators of G, which performs a "rotation around P" of the fundamental squares, as the figure shows.



So, G is, by Definition 9.12, hyperbolic relative to this collection of cyclic subgroups. Note that the fundamental domain of the action satisfies Theorem 9.24, for it can clearly be seen as the union of a compact hyperbolic octagon and four cusp regions.



9.4 An almost complete proof of R_∞

As we told before, in this section we exhibit what would be a proof of property R_∞ for non-elementary relatively hyperbolic groups by following Fel'shtyn's sketch and, most importantly, by assuming Lemma 9.29 to be true. This proof would be complete if it wasn't for Lemma 9.29, which we believe is not true in general, as we show in Section 9.5.

Definition 9.26. Let G be a relatively hyperbolic group and X the space on which it acts (first definition). A subgroup $H \leq G$ is called *elementary* if either H is finite, or if there is a point z in ∂X fixed by all H, or if there is a set $\{z, w\}$ of distinct points in ∂X which is invariant under H.

It is known that the subgroup H above is elementary if and only if H is either finite, or parabolic, or virtually cyclic. Consider then the special case of a hyperbolic group G and G itself as a subgroup. Since there are no parabolic subgroups (in particular G is not parabolic), then G is non elementary (according to the above definition) if and only if it is a non-elementary hyperbolic group according to the previous chapter. Therefore, the above definition extends the one of non-elementary hyperbolic groups. For example, the group $\mathbb{Z}$ of Example 9.9 is elementary as a hyperbolic group (for it is cyclic), but it can also be seen as an elementary relatively hyperbolic group, for it has a global fixed point p in $\partial\mathcal{H}^2$.

From now on, let us assume Lemma 9.29 to be true. The rest of this section is dedicated to show

Theorem 9.27. *Assuming Lemma 9.29 to be true, every non-elementary relatively hyperbolic group has property R_∞.*

Let G be a non-elementary relatively hyperbolic group with respect to a finite collection of subgroups $\mathcal{H}$ and $\varphi \in Aut(G)$. Let us show $R(\varphi) = \infty$. As we did in the hyperbolic chapter, we divide the proof in two possible cases:

Case 1: $\pi(\varphi)$ has finite order in $Out(G)$.

Let $m \geq 1$ be the order of $\pi(\varphi)$. By Proposition 7.3, it suffices to show G_φ acts non-elementary on a hyperbolic geodesic space X. Since $G_\varphi/G \simeq \mathbb{Z}_m$ is finite, G has finite index in G_φ. Therefore, $G_\varphi \overset{QI}{\sim} G$ by Proposition 2.22. But in [28], Drutu shows that relative hyperbolicity is a quasi-isometric invariant, so G_φ is relatively hyperbolic, so by definition it acts by isometries on a hyperbolic space X. Also, since it contains G, it must be non-elementary. Therefore, to finish case 1, we're only left to show:

Lemma 9.28. *Let $(G, \mathcal{H})$ be any non-elementary relatively hyperbolic group with X the associated hyperbolic space. Then the action $G \curvearrowright X$ is non-elementary.*

Demonstração. Let us check items 1) to 3) from Definition 7.1 are satisfied. Items 2) and 3) come directly from the definition of G being non-elementary. To check item 1), let $g \in G$ have infinite order in G (since G is finitely generated and infinite, there must be such element; otherwise, G would be finite). Let us show it has infinite order as an isometry of X. If $g^n = Id_X$ for some $n \geq 1$, then $g^{kn} = Id^k = Id$ for every $k \in \mathbb{Z}$. Fix $x \in X$. We have $g^{kn}x = x$. But since the action of G is properly discontinuous, in particular there should be only a finite number of elements in G fixing x, a contradiction, for the elements $g^{kn} \in G$ are pairwise distinct. Thus, g has infinite order as an isometry and the lemma is complete. □

Case 2: $\pi(\varphi)$ has infinite order in $Out(G)$.

We want to follow the three similar steps to the ones in Theorem 8.9. As step 1, let us construct a non-trivial and small action by isometries of G on an $\mathbb{R}$-tree X_ω, in the same way we did there, but now with a few adaptations. Let $H = \langle\varphi\rangle$. Since $\pi(H) = \langle\pi(\varphi)\rangle \simeq \mathbb{Z}$ has an infinite center (itself), in particular for every n we can take $\psi_n \in H$ such that the projections $\pi(\psi_n)$ are pairwise distinct and all inside the center of $\pi(H)$. Fix a finite generating set S for G. For every $n \geq 1$, define

$$\lambda_n = \inf_{x\in X}\{\max_{s\in S} d(x, \psi_n(s)x)\},$$

and by definition let $x_n \in X$ such that $\lambda_n \leq \max_{s\in S} d(x_n, \psi_n(s)x_n) < \lambda_n + \frac{1}{n}$. We claim $\lim_{n\to\infty} \lambda_n = \infty$. In fact, suppose by contradiction that this is false. Then there is a bounded subsequence of it, that we will still denote $(\lambda_n)_n$ by simplicity of notation. Let then $R = 2 + \sup_n \lambda_n < \infty$.

By the geometric characterization of the action in the previous section, there's an R-separated G-invariant collection $\mathcal{B} = \{B(p) \mid p \in \mathcal{P}\}$ of horoballs around the collection $\mathcal{P}$ of bounded parabolic points of ∂X. Also, there is a connected fundamental domain F of the action which is the union of a compact set with a finite number of cusp regions, each one "converging" to a bounded parabolic point representing a different G-orbit of $\mathcal{P}$. Since F is a fundamental domain, for every n let $g_n \in G$ such that $g_n x_n \in F$. The sequence $(g_n x_n)_n$ is either bounded in X or not. We are going to derive a contradiction from each situation.

Suppose first that $(g_n x_n)_n$ is bounded. We have $d(g_n x_n, y) \leq K$ for every n, for some point $y \in X$ and some $K \geq 0$. Equivalently, $g_n x_n \in \overline{B}(y, K)$ for every n. Now, for every $s \in S$ and every n,

$$d(g_n\psi_n(s)g_n^{-1} \cdot g_n x_n, g_n x_n) = d(g_n\psi_n(s)x_n, g_n x_n) = d(\psi_n(s)x_n, x_n) < \lambda_n + \frac{1}{n} < R,$$

so

$$d(g_n\psi_n(s)g_n^{-1} \cdot g_n x_n, y) \leq d(g_n\psi_n(s)g_n^{-1} \cdot g_n x_n, g_n x_n) + d(g_n x_n, y) < R + K,$$

and therefore $g_n\psi_n(s)g_n^{-1} \cdot g_n x_n \in \overline{B}(y, R+K)$. But since $g_n x_n \in \overline{B}(y, K)$, we already know that $g_n\psi_n(s)g_n^{-1} \cdot g_n x_n \in g_n\psi_n(s)g_n^{-1}\overline{B}(y, K)$, so

$$g_n\psi_n(s)g_n^{-1}\overline{B}(y, K) \cap \overline{B}(y, R+K) \neq \emptyset$$

for every $s \in S$ and every n. Since X is proper, such closed balls are compact. Therefore, since the action $G \curvearrowright X$ is properly discontinuous, the set $W = \{g_n\psi_n(s)g_n^{-1} \mid s \in S, n \geq 1\}$ is finite. Now, write $S = \{s_1, ..., s_m\}$ and consider the map

$$f : \mathbb{N} \to W^m, \quad n \mapsto (g_n\psi_n(s_1)g_n^{-1}, ..., g_n\psi_n(s_m)g_n^{-1}).$$

Since $\mathbb{N}$ is infinite and W^m is finite, there must be $n_1 \neq n_2$ such that $f(n_1) = f(n_2)$. By definition, this implies $g_{n_1}\psi_{n_1}(s_i)g_{n_1}^{-1} = g_{n_2}\psi_{n_2}(s_i)g_{n_2}^{-1}$ for every $1 \leq i \leq m$, or $\gamma_{g_{n_1}}\psi_{n_1}(s_i) = \gamma_{g_{n_2}}\psi_{n_2}(s_i)$ for every $1 \leq i \leq m$. Since S generates G, we must have $\gamma_{g_{n_1}}\psi_{n_1} = \gamma_{g_{n_2}}\psi_{n_2}$, which implies

$$\pi(\psi_{n_1}) = \pi(\gamma_{g_{n_1}}\psi_{n_1}) = \pi(\gamma_{g_{n_2}}\psi_{n_2}) = \pi(\psi_{n_2})$$

in $Out(G)$, a contradiction, as desired.

Suppose now $(g_n x_n)_n \subset F$ is not bounded. Since the cusps of F are the only not bounded parts and there are a finite number of them, there must be a subsequence (still called $(g_n x_n)_n$) converging to the boundary of X inside some cusp, so $g_n x_n$ must converge to the bounded parabolic point p at the end of it. In particular, for some large enough n, $g_n x_n$ must be inside the horoball $B(p)$ around p. Fix such number n. We claim every element of the form $g_n \psi_n(g) g_n^{-1}$ ($g \in G$) sends the point $g_n x_n \in B(p)$ to another point inside $B(p)$. This can be easily proved by induction on $k = |g|$ (the length of g as a word in S), so to facilitate notation we will show only steps $k = 1$ and $k = 2$. First, let $s \in S$. Since $g_n x_n \in B(p)$ and the collection $\mathcal{B}$ is G-invariant, the element $g_n \psi_n(s) g_n^{-1} \cdot g_n x_n$ must be in some horoball of the collection. But we have $d(g_n \psi_n(s) g_n^{-1} \cdot g_n x_n, g_n x_n) < R$ (shown above) and the horoballs of $\mathcal{B}$ are pairwise R-separated, so the only horoball $g_n \psi_n(s) g_n^{-1} \cdot g_n x_n$ can be in is $B(p)$. For $k = 2$, let $s_1, s_2 \in S$ be any two elements. By induction, we have $g_n \psi_n(s_1) g_n^{-1} \cdot g_n x_n \in B(p)$. But it is clear that

$$d(g_n \psi_n(s_1 s_2) g_n^{-1} \cdot g_n x_n, g_n \psi_n(s_1) g_n^{-1} \cdot g_n x_n) = d(g_n \psi_n(s_2) g_n^{-1} \cdot g_n x_n, g_n x_n) < R.$$

Since the horoballs are G-invariant and R-separated, again the only horoball $g_n \psi_n(s_1 s_2) g_n^{-1} \cdot g_n x_n$ can be is $B(p)$. Induction follows similarly. With this, we showed the entire group $g_n \psi_n(G) g_n^{-1}$ sends a point of $B(p)$ to a point of $B(p)$, so it must be contained in the subgroup $Stab(p)$ which is the subgroup of G that keeps invariant the whole horoball $B(p)$. Finally, since ψ_n is an automorphism we have $g_n \psi_n(G) g_n^{-1} = g_n G g_n^{-1} = G$, so $G \leq Stab(p)$ and p must be a global fixed point of G in ∂X, a contradiction, since G is non-elementary.

Now we build the action on an $\mathbb{R}$-tree: since $\lambda_n \to \infty$, in particular we can assume $\lambda_n \geq 1$ for every $n \geq 1$. Define then the sequence of pointed metric spaces (X_n, d_n, x_n) with $X_n = X$,$d_n = \frac{d}{\lambda_n}$ and the points x_n above. Let G act on X_n by $g \cdot x = \psi_n(g)x$, clearly an action by isometries, for $G \curvearrowright X$ is. We have to show that for every $g \in G$, there is $C(g) \geq 0$ such that $d_n(x_n, g \cdot x_n) \leq C(g)$ for every n. Write $g = s_1 ... s_m$ (so that $|g| = m$). Then

$$\begin{aligned}
d(x_n, g \cdot x_n) &= d(x_n, \psi_n(g) x_n) \\
&= d(x_n, \psi_n(s_1) ... \psi_n(s_m) x_n) \\
&\leq d(x_n, \psi_n(s_1) x_n) + ... + d(\psi_n(s_1) ... \psi_n(s_{m-1}) x_n, \psi_n(s_1) ... \psi_n(s_m) x_n) \\
&= d(x_n, \psi_n(s_1) x_n) + d(x_n, \psi_n(s_2) x_n) + ... + d(x_n, \psi_n(s_m) x_n) \\
&\leq (\lambda_n + 1/n) + (\lambda_n + 1/n) + ... + (\lambda_n + 1/n) \\
&= m(\lambda_n + 1/n) = |g|(\lambda_n + 1/n) \\
&\leq 2|g|\lambda_n,
\end{aligned}$$

therefore

$$d_n(x_n, g \cdot x_n) = \frac{d(x_n, g \cdot x_n)}{\lambda_n} \leq 2|g|,$$

so $C(g) = 2|g|$ satisfies our desired condition. By Proposition 2.69, the actions $G \curvearrowright X_n$ induce an action by isometries $G \curvearrowright X_\omega$, where X_ω is the ω-ultralimit of the sequence $(X_n, d_n, x_n)_n$. Since each (X_n, d_n) is a $\frac{\delta}{\lambda_n}$-hyperbolic space and $\frac{\delta}{\lambda_n} \to 0$ (for $\lambda_n \to \infty$), X_ω is an $\mathbb{R}$-tree by Proposition 2.68. By [82], we know this action is non-trivial and small, as desired. This was step 1.

Step 2 is to extend the action above to an action $(G \rtimes H) \curvearrowright X_\omega$ such that the restriction $H \curvearrowright X_\omega$ is an action by homeomorphisms. Remember also that $H = \langle\varphi\rangle$ in our case. Because of this, we can see that it is enough to find one homeomorphism $f_\varphi : X_\omega \to X_\omega$ such that $f_\varphi(gx) = \varphi(g)f_\varphi(x)$ for every $x \in X_\omega$ and $g \in G$. Indeed, from that we can define $f_{\varphi^n} = (f_\varphi)^n$ for $n \in \mathbb{Z}$, and with a simple proof by induction we show that $f_{\varphi^n}(gx) = \varphi^n(g)f_{\varphi^n}(x)$ for $n \in \mathbb{Z}, x \in X_\omega, g \in G$, which gives the desired action $(G \rtimes H) \curvearrowright X_\omega$.

Let us find then such map f_φ. As in the last chapter, the idea is to combine maps $f_n : X_n \to X_n$ together to get f_φ as an ultralimit map. Let us get a little intuition to define this map in our case. If we read step 2 of Theorem 8.9, we see that, if we were in the particular case $H = \langle\varphi\rangle$, the maps there would be all the same: $f_n(x) = \varphi(x)$ for all n and all $x \in G$ (remember that points were elements of G there). Intuitively, if we think of that action being a transitive action, the points x are all individual fundamental domains of the action and the maps f_n (or "the map" f) is taking fundamental domains to their respective image by φ. This is how we will define our maps $f_n = f : X \to X$ for every n. Let F be the fundamental domain of the action $G \curvearrowright X$ (see the end of Section 9.2). Given $x \in X$, there are unique elements $y \in F$ and $g \in G$ such that $x = gy$. We then define $f(x) = \varphi(g)y$. Geometrically, we have the same situation of the hyperbolic case: each fundamental domain gF is being mapped isometrically to the corresponding fundamental domain $\varphi(g)F$ in X. It is then clear that $f : X \to X$ is bijective. We claim it is a quasi-isometry. In fact, since it is bijective, its image is obviously cobounded, so we just have to show the two inequalities from Definition 2.19. But if we started with the automorphism φ^{-1}, the same definition above would give rise to an inverse map for f. So it is easy to see that to show the claim it is enough to show only one inequality:

Lemma 9.29. *There are $A, B \geq 0$ such that $d(f(x), f(x')) \leq Ad(x, x') + B$ for every $x, x' \in X$.*

Because of this, there are $A, B \geq 0$ such that $f : X \to X$ is a (A, B)-QI. Then, the maps $f = f_n : X_n \to X_n$ are easily seen by definition to be $(A, B/\lambda_n)$-QI. If we show the set $\{d_n(f(x_n), x_n) \mid n \geq 1\}$ is bounded, we can apply Proposition 2.70 to the maps $f_n = f$. Since the action $G \curvearrowright X$ is properly discontinuous, the proof of Lemma 8.10 can be easily repeated in our case, since it only demands the hyperbolicity of the space in question, which we have here. Therefore, by Proposition 2.70, the maps $f_n = f$ give rise to a map $f_\varphi : X_\omega \to X_\omega$ by $f_\varphi([(y_n)_n]) = [(f_n(y_n))_n] = [(f(y_n))_n]$. Since the $f_n : X_n \to X_n$ are $(A, B/\lambda_n)$-QI and $\lim \lambda_n = \infty$, we also have by Proposition 2.70 that f_φ is a $(A, 0)$-QI map, or an A-bi-Lipschitz map, in particular continuous. Now, if we started with the automorphism φ^{-1}, it is straightforward to check that this construction would give rise to an inverse map $f_{\varphi^{-1}}$ for f_φ. In other words, $f_{\varphi^{-1}} = f_\varphi^{-1}$, and f_φ is an homeomorphism of X_ω, as we wanted.

To finish step 2, we are just left to show that $f_\varphi(gy) = \varphi(g)f_\varphi(y)$ for any $y = [(y_n)_n] \in X_\omega$ and $g \in G$. But the starting map $f : X \to X$ we defined satisfies this. In fact, if $x \in X$ and $g \in G$, let $h \in G$ and $y \in F$ be the unique elements such that $x = hy$. Then $gx = ghy$ with $y \in F$ and, by definition, $f(gx) = f(ghy) = \varphi(gh)y = \varphi(g)\varphi(h)y = \varphi(g)f(x)$. So, for any $y = [(y_n)_n] \in X_\omega$,

$$f_\varphi(gy) = [(f(gy_n))_n] = [(\varphi(g)f(y_n))_n] = \varphi(g)[(f(y_n))_n] = \varphi(g)f_\varphi(y),$$

and step 2 is complete.

Step 3 is, again, to "modify" the $\mathbb{R}$-tree X_ω and the action $(G \rtimes H) \curvearrowright X_\omega$ above to obtain an affine action $(G \rtimes H) \curvearrowright T$ on some $\mathbb{R}$-tree T. As we saw in the beginning of step 3 in Theorem 8.9, the existence of this affine action will give us in particular an action $G \curvearrowright T$ satisfying all hypotheses of Theorem 7.4, which in turn gives us that $R(\varphi) = \infty$, finishing our proof.

Fortunately, the entire proof of step 3 of Theorem 8.9 is applicable to our case. In fact, as the reader can easily check, the only hypotheses we use in that proof (besides the existence of the action built on step 2, of course) are:

1) the restriction $G \curvearrowright X_\omega$ is by isometries, non-trivial and small;

2) the restriction $H \curvearrowright X_\omega$ is by homeomorphisms;

3) H is amenable;

4) G is finitely generated;

5) G contains a non-abelian free group.

All these hypotheses are satisfied in our case: item 1) comes from step 1, item 2) comes from step 2, the group $H = \langle \varphi \rangle$ is known to be amenable, G is by hypothesis finitely generated and, since it is by hypothesis non-elementary, it is known to contain a copy of F_2. So, step 3 can be reproduced here and, as we argued above, this shows that $R(\varphi) = \infty$ and finishes the proof of Theorem 9.27. □

We could just restate the proof of Case 2 (the infinite order case) above in the following ways:

Corollary 9.30. *Let G be a non-elementary relatively hyperbolic group and (X, d) be the space G acts on (Definition 9.12). Write $\pi : Aut(G) \to Out(G)$ as the usual projection. Let $\varphi \in Aut(G)$ such that $\pi(\varphi)$ has infinite order in $Out(G)$ and denote by $f = f_\varphi : X \to X$ the map defined in the proof of Theorem 9.27 above. If there are $A, B \geq 0$ such that $d(f(x), f(x')) \leq Ad(x, x') + B$ for every $x, x' \in X$, then $R(\varphi) = \infty$.* □

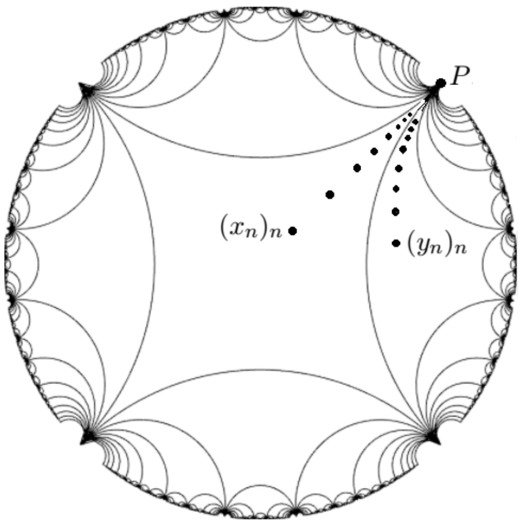
Corollary 9.31. *Let G be a non-elementary relatively hyperbolic group and (X, d) be the space G acts on (Definition 9.12). Suppose that, for every automorphism $\varphi \in Aut(G)$ such that $\pi(\varphi)$ has infinite order in $Out(G)$, there are $A, B \geq 0$ such that $d(f(x), f(x')) \leq Ad(x, x') + B$ for every $x, x' \in X$. Then G has property R_∞.* □

9.5 A counterexample to Lemma 9.29

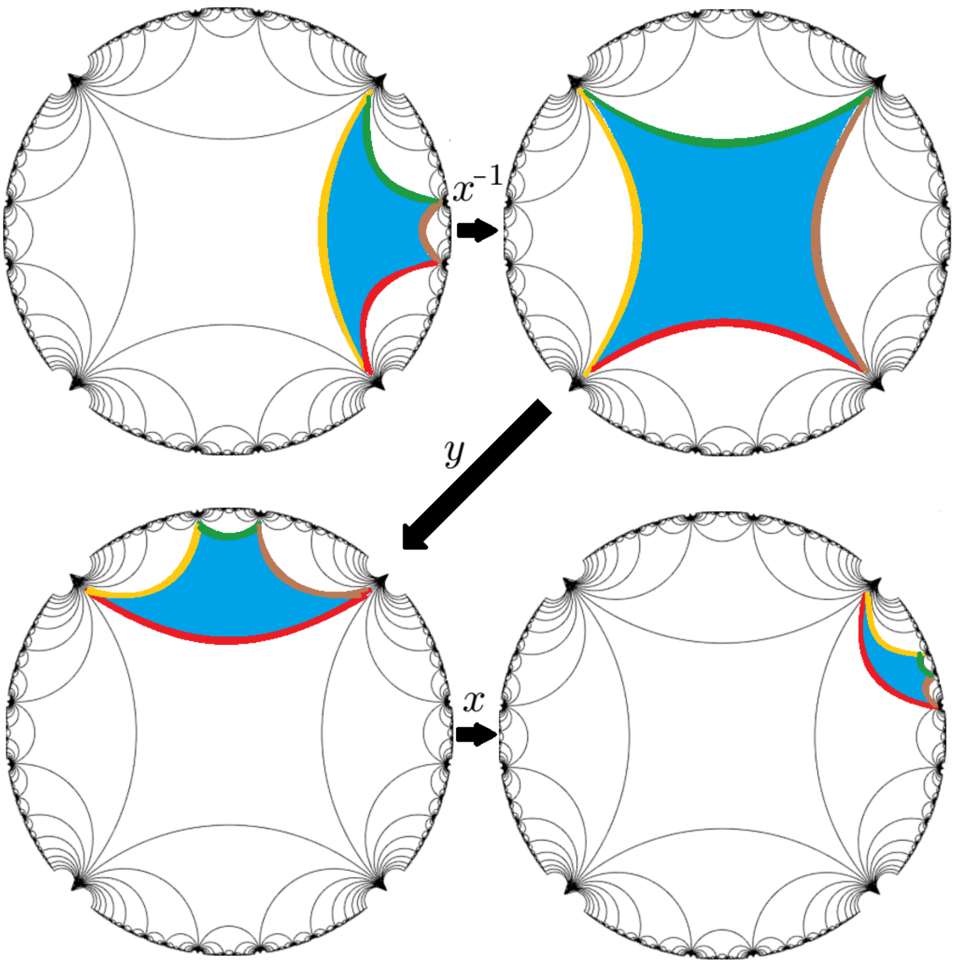
To finish the chapter, let us show why we believe Lemma 9.29 is not true in general, that is, for any relatively hyperbolic group.

Let G and X be the group and space of Example 9.25, respectively. G is a non-elementary relatively hyperbolic group acting on the Poincare disk (X, d), with hyperbolic squares as fundamental domains. Since $G = \pi_1(Y)$ and Y is homotopically equivalent to the "figure 8", G is isomorphic to the free group F_2 on two generators. Let x and y be the two corresponding generators of G and consider the automorphism φ with $\varphi(x) = xy$ and $\varphi(y) = y$. We have an infinite order automorphism φ of G, which induces a bijection $f : X \to X$ by permuting the fundamental domains of the action according to the map φ. To be more precise, let F be the fixed fundamental domain of the action $G \curvearrowright X$. Given $x \in X$, there are unique elements $y \in F$ and $g \in G$ such that $x = gy$. We then define $f(x) = \varphi(g)y$. That is, each fundamental domain gF is being mapped isometrically to the corresponding fundamental domain $\varphi(g)F$ in X.

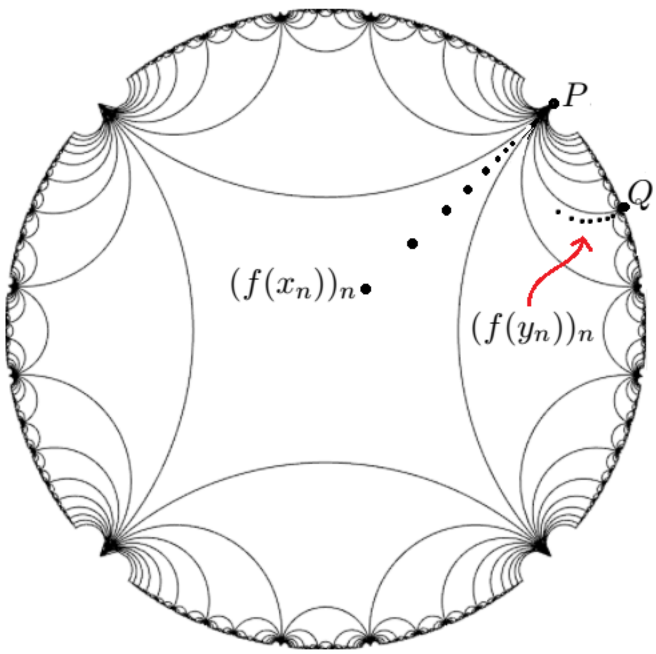
We want to (intuitively) convince the reader that f cannot be a quasi isometry, or that the numbers A and B of Lemma 9.29 cannot exist. Remember that these numbers are defined to be such that $d(f(x), f(x')) \leq Ad(x, x') + B$ for every $x, x' \in X$. Therefore, it is enough for us to find two sequences $(x_n)_n$ and $(y_n)_n$ in X such that $\{d(x_n, y_n) \mid n \geq 1\}$ is bounded but $\{d(f(x_n), f(y_n)) \mid n \geq 1\}$ is not. If, in particular, $(x_n)_n$ and $(y_n)_n$ are sequences of points inside geodesic lines contained in fundamental domains and converging to boundary points (and, therefore, so will be the sequences $(f(x_n))_n$ and $(f(y_n))_n$), then by the definition of boundary 2.4 it is easy to see that the previous condition we want is equivalent to say that $(x_n)_n$ and $(y_n)_n$ converge to the same boundary point P but $(f(x_n))_n$ and $(f(y_n))_n$ converge to distinct boundary points. Now, finding such sequences is easy. Let $x_n = P - \left(\frac{1}{n}, \frac{1}{n}\right)$, where $P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is the boundary point of Example 9.25. Of course, $(x_n)_n$ is in the (straigth) geodesic inside the central "fundamental square" F and converges to P. It is known about the Poincare disk that the right-sided edge of this square (which is also the left sided edge of the square xF) is contained in a circle C of $\mathbb{R}^2 = \mathbb{C}$. Now, for every n, let y_n be the inversion of x_n with respect to the circle C. Since $(x_n)_n$ is in a geodesic and converges to $P \in C$, the sequence $(y_n)_n$ is also in a geodesic and converges to P. Since $(x_n)_n$ is entirely in the square F, $(y_n)_n$ is entirely in the square xF.



Since $\varphi(1) = 1$, the square F is mapped by f onto itself ($\varphi(1)F = 1F = F$), so it remains unmoved and $f(x_n) = x_n$. Therefore, the sequence $(f(x_n))_n$ still converges to P. On the other hand, we will show that $(f(y_n))_n$ converges to $Q \neq P$. Since f maps xF to $\varphi(x)F = xyF$, its restriction to xF can be written as the composition of the translations of xF by x^{-1}, y and then x, in that order, so that f is $xF \overset{x^{-1}}{\mapsto} F \overset{y}{\mapsto} yF \overset{x}{\mapsto} xyF$. These translations are illustrated in the following figure, where the image of an edge has the same color of it:



Because of this, one can easily see that the sequence $(f(y_n))_n$ converges to the boundary point $Q \neq P$, as in the next figure:



This shows that Lemma 9.29 does not hold in this case and completes our example.

Open questions

1) Is there anything we missed in our understanding of Fel'shtyn's sketch of proof? We do not believe so, but humility is important.

2) Are there non-elementary relatively hyperbolic groups that fit in the case of Corollary 9.31? If so, how common (or rare) these groups are with respect to the family of relatively hyperbolic groups? Is there a methodical way to build such examples? Since it was quite easy to build our counterexample of Lemma 9.29, we believe that such a group (if any) must be of a very specific type.

3) Is there a way to avoid Lemma 9.29 in the proof of Theorem 9.27? In light of our counterexample, we believe that a proof of R_∞ for relatively hyperbolic groups would be, at least, not so similar to the one we proposed and, at most, completely different.

4) Does any non-elementary relatively hyperbolic group has property R_∞? We think a good family for a possible counterexample would be the groups described in Example 9.15, assuming they are not hyperbolic, of course.

5) Can we use either Brown's characterization of Σ^1 in [17] or Cashen and Levitt's Theorem 11.4 to compute the BNS invariants for some relatively hyperbolic groups, in order to obtain extra information about property R_∞?

Parte IV

Appendix

Capítulo 10

Nilpotent quotients of Γ_n

Given a group G with property R_∞, we define the R_∞ nilpotency index of G to be the smallest number $c \geq 1$ such that the quotient $G/\gamma_{c+1}(G)$ has property R_∞ (if such number exists) and to be infinite (or $c = \infty$) elsewhere, that is, if none of the quotients $G/\gamma_{c+1}(G)$ have property R_∞. In this chapter, we show that the groups Γ_n have infinite R_∞ nilpotency index. In contrast to chapters 5 and 6, we shall see that the Σ^1 invariant cannot be used here, so our technique here is different and that is the reason why this chapter is in the appendix.

Let us describe the motivation for our work in this chapter. The R_∞ nilpotency index is a quite recent notion, defined in 2016 for the first time in the paper [23]. There, the authors D. L. Gonçalves and K. Dekimpe compute the index for the surface group of a genus $g > 1$ surface and showed that it is 4 in the orientable case and $2(g-1)$ in the non-orientable one. Remember that surface groups are known to have property R_∞ since they are hyperbolic groups (see Chapter 8). But even before this very recent definition, knowing whether some nilpotent quotient has R_∞ was already an active topic in combinatorial and geometric group theory. For example, in [24] (2013), the same authors (among other results) generalized a result by Roman'kov [85] (2011) and showed that the free groups F_r have R_∞ nilpotency index $2r$ (but did not use this terminology). But our most direct motivation was the paper [22] (2020), where the two authors described the R_∞ nilpotency index of all Baumslag-Solitar groups $BS(m,n)$. So, in the same way knowing information of the Σ^1 invariant for $BS(1,n)$ was the motivation to investigate $\Sigma^1(\Gamma_n)$ (Chapter 5), knowing which nilpotent quotients of $BS(1,n)$ had R_∞ led us to investigate the same aspects for the groups Γ_n. That being said, it is clear that paper [22] is the basis of the first computations of our chapter.

As in Chapter 5, let $n \geq 2$ be an integer with prime decomposition $n = {p_1}^{y_1}...{p_r}^{y_r}$, the p_i being pairwise distinct and $y_i > 0$. We consider again the Generalized Solvable Baumslag-Solitar group

$$\Gamma_n = \left\langle a, t_1, ..., t_r \mid t_i t_j = t_j t_i,\ i \neq j,\ t_i a {t_i}^{-1} = a^{{p_i}^{y_i}},\ i = 1, ..., r \right\rangle .$$

For any $c \geq 1$, we will denote

$$\Gamma_{n,c} = \frac{\Gamma_n}{\gamma_{c+1}(\Gamma_n)},$$

where γ_{c+1} is the $(c+1)^{th}$ term of the lower central series of Γ_n (Definition 1.57). We know it is a nilpotent group with nilpotency class $\leq c$, for $\gamma_{c+1}(\Gamma_{n,c}) = \gamma_{c+1}(\frac{\Gamma_n}{\gamma_{c+1}(\Gamma_n)}) = \frac{\gamma_{c+1}(\Gamma_n)}{\gamma_{c+1}(\Gamma_n)} = 1$. In this chapter, the torsion subgroup of a nilpotent group G will be denoted by $\tau G = \{g \in$

$G \mid g^n = 1$ for some $n \geq 1\}$.

Lemma 10.1. *Let* $m = \gcd(p_1^{y_1} - 1, ..., p_r^{y_r} - 1)$. *Then* $a^{m^k} \in \gamma_{k+1}(\Gamma_n)$ *for all* $k \geq 1$.

Demonstração. Induction on k. First, $k = 1$. By using the group relations, note that, for any $1 \leq i \leq r$, $a^{p_i^{y_i}-1} = t_i a t_i^{-1} a^{-1} = [t_i, a] \in \gamma_2(\Gamma_n)$. Since this is true for any i and m is an integer combination of the $p_i^{y_i} - 1$, we have $a^m \in \gamma_2(\Gamma_n)$. Now, suppose the lemma is true for some $k \geq 1$. Then

$$a^{(p_i^{y_i}-1)m^k} = a^{p_i^{y_i} m^k} a^{-m^k} = t_i a^{m^k} t_i^{-1} a^{-m^k} = [t_i, a^{m^k}] \in \gamma_{k+2}(\Gamma_n).$$

Again, since this is true for any i and m is an integer combination of the $p_i^{y_i} - 1$, we have $a^{mm^k} = a^{m^{k+1}} \in \gamma_{k+2}(\Gamma_n)$, as desired. This completes the proof. □

10.1 Torsion and lower central series

To compute the torsion of the groups $\Gamma_{n,c}$, we need the following

Lemma 10.2. *Let G be a nilpotent group of class $\leq c$ and denote $\gamma_i = \gamma_i(G)$. If the quotients $\gamma_2/\gamma_3, ..., \gamma_c/\gamma_{c+1}$ are finite, then γ_2 is a torsion subgroup of G.*

Demonstração. Let $g_2 \in \gamma_2$ and let us find $k \geq 1$ such that $g_2^k = 1$. Since γ_2/γ_3 is finite there is $k_3 \geq 1$ such that $g_2^{k_3}\gamma_3 = \gamma_3$, or $g_2^{k_3} \in \gamma_3$. Now, we have the element $g_2^{k_3}\gamma_4 \in \gamma_3/\gamma_4$. Since this is a finite group, there is $k_4 \geq 1$ such that $g_2^{k_3 k_4}\gamma_4 = \gamma_4$, or $g_2^{k_3 k_4} \in \gamma_4$. If we proceed recursively we obtain $g_2^k \in \gamma_{c+1}$ for $k = k_3 k_4 ... k_{c+1} \geq 1$. Since $\gamma_{c+1} = 1$ we have $g_2^k = 1$, as desired. □

Proposition 10.3. $\tau\Gamma_{n,c} = \langle \overline{a}, \gamma_2(\Gamma_{n,c})\rangle$, *where* $\overline{a} = a\gamma_{c+1} = a\gamma_{c+1}(\Gamma_n) \in \Gamma_{n,c}$.

Demonstração. In the case $c = 1$ we have $\Gamma_{n,1}$ is the abelianized group of Γ_n, so

$$\Gamma_{n,1} = \left\langle \overline{a}, \overline{t_1}, ..., \overline{t_r} \mid \overline{t_i t_j} = \overline{t_j t_i},\ \overline{t_i}\overline{a} = \overline{a}\overline{t_i},\ \overline{a}^{p_i^{y_i}-1} = 1 \right\rangle \simeq \mathbb{Z}_m \times \mathbb{Z}^r,$$

so $\tau\Gamma_{n,1} = \langle \overline{a}\rangle = \langle \overline{a}, \gamma_2(\Gamma_{n,1})\rangle$, since $\gamma_2(\Gamma_{n,1}) = 1$.

Now let us show the proposition in the case $c \geq 2$. For ($\subset$), let $x\gamma_{c+1} \in \tau\Gamma_{n,c}$. This means $x^k\gamma_{c+1} = (x\gamma_{c+1})^k = \gamma_{c+1}$ for some $k \geq 1$. Since $c \geq 2$, we have $x^k \in \gamma_{c+1} \subset \gamma_2$, so $x\gamma_2 \in \tau\Gamma_{n,1} = \langle \overline{a}\rangle$. Write then $x = a^l g_2$ for $l \in \mathbb{Z}$ and $g_2 \in \gamma_2 = \gamma_2(\Gamma_n)$. This gives $x\gamma_{c+1} = (a\gamma_{c+1})^l (g_2\gamma_{c+1}) \in \langle \overline{a}, \gamma_2(\Gamma_{n,c})\rangle$, as we wanted. To show ($\supset$), we note that by Lemma 10.1 we get $\overline{a}^{m^c} = 1$ in $\Gamma_{n,c}$, so $\overline{a} \in \tau\Gamma_{n,c}$. So, we just need to show that $\gamma_2(\Gamma_{n,c})$ is a torsion subgroup of $\Gamma_{n,c}$. To do this, we invoke Lemma 10.2, by which we know it is enough to show the quotients

$$\frac{\gamma_2(\Gamma_{n,c})}{\gamma_3(\Gamma_{n,c})}, ..., \frac{\gamma_c(\Gamma_{n,c})}{\gamma_{c+1}(\Gamma_{n,c})}$$

are all finite. But for every $2 \leq i \leq c$, by the known Isomorphism Theorem for quotients, we have

$$\frac{\gamma_i(\Gamma_{n,c})}{\gamma_{i+1}(\Gamma_{n,c})} = \frac{\gamma_i(\Gamma_n)/\gamma_{c+1}(\Gamma_n)}{\gamma_{i+i}(\Gamma_n)/\gamma_{c+1}(\Gamma_n)} \simeq \frac{\gamma_i(\Gamma_n)}{\gamma_{i+i}(\Gamma_n)} = \gamma_i/\gamma_{i+1},$$

so let us show that $\gamma_2/\gamma_3, ..., \gamma_c/\gamma_{c+1}$ are finite by induction. By Proposition 1.61, we know they are abelian groups, generated by their i-fold comutator cosets.

The group γ_2/γ_3 is generated by the elements $[t_i, a]\gamma_3$, $1 \leq i \leq r$ and by $[t_i, t_j]\gamma_3 = 1\gamma_3 = \gamma_3$, which are trivial. By Proposition 1.60, we get

$$[t_i, a]^m \gamma_3 = [t_i, a^m]\gamma_3 = \gamma_3,$$

since $a^m \in \gamma_2$ (Lemma 10.1). So all generators of γ_2/γ_3 have torsion. Since it is finitely generated and abelian, it must be a finite group.

Finally, suppose by induction that γ_i/γ_{i+1} is finite for some $i \geq 2$. By Proposition 1.61, $\gamma_{i+1}/\gamma_{i+2}$ is then generated by the elements of the form $[x, y]\gamma_{i+2}$ with $x \in \Gamma_n$ and $y \in \gamma_i$. Since γ_i/γ_{i+1} is finite, let $k = k(x, y) \geq 1$ with $(y\gamma_{i+1})^k = \gamma_{i+1}$, or $y^k \in \gamma_{i+1}$. Then

$$[x, y]^k \gamma_{i+2} = [x, y^k]\gamma_{i+2} = \gamma_{i+2}.$$

By the same argument we just used, this implies $\gamma_{i+1}/\gamma_{i+2}$ is finite and completes the proof. □

Proposition 10.4. $\gamma_k(\Gamma_{n,c}) = \left\langle \overline{a}^{m^{k-1}} \right\rangle$ *for all* $k \geq 2$ *and* $c \geq 1$.

Demonstração. First, we will show that

$$\frac{\gamma_k(\Gamma_{n,c})}{\gamma_{k+1}(\Gamma_{n,c})} = \left\langle \overline{a}^{m^{k-1}} \gamma_{k+1}(\Gamma_{n,c}) \right\rangle. \tag{10.1}$$

For $k = 2$, by Proposition 1.61, $\frac{\gamma_2(\Gamma_{n,c})}{\gamma_3(\Gamma_{n,c})}$ is generated by the cosets $[\overline{t_i}, \overline{a}]\gamma_3(\Gamma_{n,c})$. Since $[\overline{t_i}, \overline{a}] = \overline{a}^{p_i^{y_i}-1}$, we have

$$\frac{\gamma_2(\Gamma_{n,c})}{\gamma_3(\Gamma_{n,c})} = \left\langle \overline{a}^{p_1^{y_1}-1}\gamma_3(\Gamma_{n,c}), ..., \overline{a}^{p_r^{y_r}-1}\gamma_3(\Gamma_{n,c}) \right\rangle = \langle \overline{a}^m \gamma_3(\Gamma_{n,c})\rangle$$

(remember that $m = \gcd(p_1^{y_1} - 1, ..., p_r^{y_r} - 1)$). Suppose now 10.1 is true for some $k \geq 2$. We know $\frac{\gamma_{k+1}(\Gamma_{n,c})}{\gamma_{k+2}(\Gamma_{n,c})}$ is generated by the cosets $[x, z]\gamma_{k+2}(\Gamma_{n,c})$, where $x \in \Gamma_{n,c}$ and $z \in \gamma_k(\Gamma_{n,c})$. By induction, we can write $z = \overline{a}^{\alpha m^{k-1}} w_{k+1}$ for some $w_{k+1} \in \gamma_{k+1}(\Gamma_{n,c})$ and $\alpha \in \mathbb{Z}$. Then, by using Proposition 1.60 we get

$$\begin{aligned}
[x, z]\gamma_{k+2}(\Gamma_{n,c}) &= [x, \overline{a}^{\alpha m^{k-1}} w_{k+1}]\gamma_{k+2}(\Gamma_{n,c}) \\
&= [x, \overline{a}^{m^{k-1}}]^\alpha [x, w_{k+1}]\gamma_{k+2}(\Gamma_{n,c}) \\
&= [x, \overline{a}^{m^{k-1}}]^\alpha \gamma_{k+2}(\Gamma_{n,c}),
\end{aligned}$$

so the quotient $\frac{\gamma_{k+1}(\Gamma_{n,c})}{\gamma_{k+2}(\Gamma_{n,c})}$ is actually generated only by the cosets $[x, \overline{a}^{m^{k-1}}]\gamma_{k+2}(\Gamma_{n,c})$. Since $[\overline{a}, \overline{a}^{m^{k-1}}]$ is obviously trivial, the quotient group is generated only by the generators $[\overline{t_i}, \overline{a}^{m^{k-1}}]\gamma_{k+2}(\Gamma_{n,c})$. Since $[\overline{t_i}, \overline{a}^{m^{k-1}}] = \overline{a}^{(p_i^{y_i}-1)m^{k-1}}$, we obtain

$$\frac{\gamma_{k+1}(\Gamma_{n,c})}{\gamma_{k+2}(\Gamma_{n,c})} = \left\langle \overline{a}^{(p_1^{y_1}-1)m^{k-1}}\gamma_{k+2}(\Gamma_{n,c}), ..., \overline{a}^{(p_r^{y_r}-1)m^{k-1}}\gamma_{k+2}(\Gamma_{n,c}) \right\rangle = \left\langle \overline{a}^{\beta}\gamma_{k+2}(\Gamma_{n,c}) \right\rangle,$$

where

$$\beta = \gcd((p_1^{y_1} - 1)m^{k-1}, ..., (p_r^{y_r} - 1)m^{k-1}) = m^{k-1} \gcd(p_1^{y_1} - 1, ..., p_r^{y_r} - 1) = m^k,$$

and this shows 10.1. Now, let us show the proposition. The ($\supset$) part is a direct consequence of Lemma 10.1. Let us show ($\subset$). In the case $c < k$, we have $\gamma_k(\Gamma_{n,c}) = 1 \subset \left\langle \overline{a}^{m^{k-1}} \right\rangle$. Suppose then $c \geq k$ and let $x \in \gamma_k(\Gamma_{n,c})$. Since $x\gamma_{k+1}(\Gamma_{n,c}) \in \left\langle \overline{a}^{m^{k-1}} \gamma_{k+1}(\Gamma_{n,c}) \right\rangle$ (by 10.1), write $x = \overline{a}^{j_k m^{k-1}} x_{k+1}$ for $j_k \in \mathbb{Z}$ and $x_{k+1} \in \gamma_{k+1}(\Gamma_{n,c})$. By using 10.1 again, we write $x_{k+1} = \overline{a}^{j_{k+1} m^k} x_{k+2}$ for $j_{k+1} \in \mathbb{Z}$ and $x_{k+2} \in \gamma_{k+2}(\Gamma_{n,c})$. We can do this recursively to obtain

$$\begin{aligned} x &= \overline{a}^{j_k m^{k-1}} \overline{a}^{j_{k+1} m^k} ... \overline{a}^{j_c m^{c-1}} x_{c+1} \\ &= \overline{a}^{m^{k-1}(j_k + j_{k+1} m + ... + j_c m^{c-k})} \\ &\in \left\langle \overline{a}^{m^{k-1}} \right\rangle, \end{aligned}$$

and the proof is complete. □

By Lemma 10.1 and the two propositions above, we easily get

Corollary 10.5. $\tau\Gamma_{n,c} = \langle \overline{a} \rangle$ *and* $\text{card}(\tau\Gamma_{n,c}) \leq m^c$. □

10.2 An isomorphism for $\Gamma_{n,c}$

The next step is to find a presentation to $\Gamma_{n,c}$, so we will find an isomorphism between $\Gamma_{n,c}$ and a more known group. Keep in mind all notations we have used above, such as $n, p_i^{y_i}, c, r$ and m. In this chapter, we will use the notation $\mathbb{Z}_{m^c} = \left\langle x \mid x^{m^c} = 1 \right\rangle$ and $\mathbb{Z}^r = \langle s_1, ..., s_r \mid s_i s_j = s_j s_i \rangle$. We define the group

$$G_{n,c} = \mathbb{Z}_{m^c} \rtimes \mathbb{Z}^r,$$

where the group action $\mathbb{Z}^r \curvearrowright \mathbb{Z}_{m^c}$ is given by $s_i x s_i^{-1} = x^{p_i^{y_i}}, i \leq i \leq r$.

Observation 10.6. Note that the actions defined above are all automorphisms of $\mathbb{Z}_{m^c}$. In fact, since $1 = p_i^{y_i} - (p_i^{y_i} - 1) = p_i^{y_i} - k_i m$ for some $k_i \in \mathbb{Z}$, we have $\gcd(p_i^{y_i}, m) = 1$ and so $\gcd(p_i^{y_i}, m^c) = 1$ for any $c \geq 1$, by elementary number theory. This implies the map $x \mapsto x^{p_i^{y_i}}$ induced by s_i is an automorphism of $\mathbb{Z}_{m^c}$, for it has an inverse given by $x \mapsto x^\beta$, where $\beta p_i^{y_i} = 1 \mod m^c$. Second, all such automorphisms commute, for $\mathbb{Z}_{m^c}$ is cyclic. These two facts show that there is a well defined homomorphism $\mathbb{Z}^r \to Aut(\mathbb{Z}_{m^c})$, so this semidirect product is well defined.

We will show that $\Gamma_{n,c} \simeq G_{n,c}$. To do this, we need this:

Lemma 10.7. *$G_{n,c}$ is nilpotent of class $\leq c$.*

Demonstração. Since $[s_i, x] = x^{p_i^{y_i} - 1} \in \langle x \rangle$ for every i, we have $\gamma_2(G_{n,c}) \subset \langle x^m \rangle$. Similarly, since $[s_i, x^m] = x^{(p_i^{y_i} - 1)m} \in \left\langle x^{m^2} \right\rangle$ for every i, in particular we have $[s_i, z] \in \left\langle x^{m^2} \right\rangle$ for every $z \in \gamma_2(G_{n,c})$, so it is easy to see that $\gamma_3(G_{n,c}) \subset \left\langle x^{m^2} \right\rangle$. Recursively, we can show that $\gamma_k(G_{n,c}) \subset \left\langle x^{m^{k-1}} \right\rangle$ for every $k \geq 2$. In particular, $\gamma_{c+1}(G_{n,c}) \subset \left\langle x^{m^c} \right\rangle = 1$, since $x^{m^c} = 1$ in $\mathbb{Z}_{m^c}$. This shows the lemma. □

Corollary 10.8. *$\tau G_{n,c}$ is a subgroup of $G_{n,c}$. Moreover, $\tau G_{n,c} = \mathbb{Z}_{m^c} = \langle x \rangle$ and so* $\text{card}(\tau G_{n,c}) = m^c$.

Theorem 10.9. *$\Gamma_{n,c} \simeq G_{n,c}$.*

Demonstração. Let $f : \Gamma_n \to G_{n,c}$ be the map $f(a) = x$ and $f(t_i) = s_i$. Since

$$f(t_i)f(a)f(t_i)^{-1} = s_i x s_i^{-1} = x^{p_i^{y_i}} = f(a)^{p_i^{y_i}},$$

f is a well defined group homomorphism. Since $f(\gamma_i(\Gamma_n)) \subset \gamma_i(G_{n,c})$ (in fact, that is true for any group homomorphism), f induces the morphism (also denoted by f)

$$f : \Gamma_{n,c} = \frac{\Gamma_n}{\gamma_{c+1}(\Gamma_n)} \to \frac{G_{n,c}}{\gamma_{c+1}(G_{n,c})} = G_{n,c}$$

given by $f(\overline{a}) = x$ and $f(\overline{t_i}) = s_i$. It is obviously surjective. We are just left to show that $\ker(f) = 1$, and to do that we will make use of the torsion subgroups. Since $f(\tau\Gamma_{n,c}) \subset \tau G_{n,c}$ (again, that is true for any homomorphisms between nilpotent groups), there is the restriction morphism $f_\tau : \tau\Gamma_{n,c} \to \tau G_{n,c}$. By corollaries 10.5 and 10.8, we can actually write $f_\tau : \langle\overline{a}\rangle \to \langle x\rangle$. Since $f_\tau(\overline{a}) = x$, it is clearly surjective. Now, f_τ is a surjective map from a finite set of $\leq m^c$ elements (Corollary 10.5) to a finite set with exactly m^c elements (Corollary 10.8), so we must have $\mathrm{card}(\langle x\rangle) = m^c$ and f_τ an isomorphism. In particular, $\ker(f_\tau) = 1$. We claim that $\ker(f) \subset \tau\Gamma_{n,c}$. In fact, let $z \in \ker(f)$. By using the normal form of the elements in Γ_n, write

$$z = \overline{t_1}^{k_1} ... \overline{t_r}^{k_r} \overline{t_1}^{-\alpha_1} ... \overline{t_r}^{-\alpha_r} \overline{a}^l \overline{t_r}^{\alpha_r} ... \overline{t_1}^{\alpha_1},$$

for $k_i, l \in \mathbb{Z}$ and $\alpha_i \geq 0$. So

$$1 = f(z) = s_1^{k_1} ... s_r^{k_r} s_1^{-\alpha_1} ... s_r^{-\alpha_r} x^l s_r^{-\alpha_r} ... s_1^{-\alpha_1}.$$

Since $x \in \tau G_{n,c} \lhd G_{n,c}$ we have $s_1^{-\alpha_1} ... s_r^{-\alpha_r} x^l s_r^{-\alpha_r} ... s_1^{-\alpha_1} \in \tau G_{n,c} = \langle x\rangle$, so $1 = f(z) = s_1^{k_1} ... s_r^{k_r} x^{l'}$ for some $l' \in \mathbb{Z}$. By projecting this equality under the natural homomorphism $G_{n,c} \to \mathbb{Z}^r$ we get $1 = s_1^{k_1} ... s_r^{k_r}$, which implies $k_i = 0$ for every i. Therefore

$$z = \overline{t_1}^{-\alpha_1} ... \overline{t_r}^{-\alpha_r} \overline{a}^l \overline{t_r}^{\alpha_r} ... \overline{t_1}^{\alpha_1} \in \tau G_{n,c},$$

since $\overline{a} \in \tau\Gamma_{n,c} \lhd \Gamma_{n,c}$, which shows the claim. Finally, this gives

$$\ker(f) = \ker(f) \cap \tau\Gamma_{n,c} = \ker(f_\tau) = 1$$

and the theorem is proved. □

Corollary 10.10. *For any $c \geq 1$, the nilpotent quotient $\Gamma_{n,c}$ has the following presentation:*

$$\Gamma_{n,c} = \left\langle x, s_1, ..., s_r \mid x^{m^c} = 1,\ s_i s_j = s_j s_i,\ s_i x s_i^{-1} = x^{p_i^{y_i}} \right\rangle.$$

□

As we told in the introduction of the chapter, let us note that the Σ^1 invariant cannot be directly used here. In fact, since $\Gamma_{n,c} \simeq \mathbb{Z}_{m^c} \rtimes \mathbb{Z}^r$, $\mathbb{Z}^r$ is a finite index abelian subgroup and we have $\Sigma^1(\Gamma_{n,c}) = S(\Gamma_{n,c})$ by Corollary 3.29, so none of the Σ^1 theorems of chapters 3 and 4 can be applied. Our approach, then, was more combinatorial and matricial, as you shall see.

10.3 Reidemeister numbers

Because of the theorem above, in the rest of this chapter we will make the following identifications

$$\Gamma_{n,c} = G_{n,c} = \mathbb{Z}_{m^c} \rtimes \mathbb{Z}^r = \langle x \rangle \rtimes \langle s_1, ..., s_r \rangle .$$

It's also worth pointing out that we will restrict us to investigate Reidemeister numbers of $\Gamma_{n,c}$ only in the case $r \geq 2$, for, if $r = 1$, then Γ_n is by definition a Baumslag-Solitar group $BS(1, n)$ and its Reidemeister numbers were studied in [22]. Let $\varphi \in Aut(\Gamma_{n,c})$. Since $\varphi(\tau\Gamma_{n,c}) \subset \tau\Gamma_{n,c}$, we have an induced automorphism

$$\overline{\varphi} : \frac{\Gamma_{n,c}}{\tau\Gamma_{n,c}} = \mathbb{Z}^r \to \mathbb{Z}^r = \frac{\Gamma_{n,c}}{\tau\Gamma_{n,c}}.$$

From now on, we will use the usual identification $Aut(\mathbb{Z}^r) = GL_r(\mathbb{Z})$ which sees an automorphism of $\mathbb{Z}^r$ as its (integer invertible) matrix with respect to the coordinates s_i. So, if $\overline{\varphi}(s_i) = s_1^{\alpha_{1i}}...s_r^{\alpha_{ri}}$, we will identify

$$\overline{\varphi} = (a_{ij})_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix} = [A_1 \cdots A_r] , \quad \text{where } A_i = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ri} \end{bmatrix} \in \mathbb{Z}^r.$$

Proposition 10.11. *If $\varphi \in Aut(\Gamma_{n,c})$, the following are equivalent:*

(1) $R(\varphi) = \infty$;

(2) $R(\overline{\varphi}) = \infty$;

(3) $\det(\overline{\varphi} - Id) = 0$;

(4) $\overline{\varphi}$ *has* 1 *as an eigenvalue.*

Demonstração. Items (2), (3) and (4) are all equivalent (see Example 1.3), so we just have to show that (1) and (2) are equivalent. We have the following commutative diagram:

$$\begin{array}{ccc} \Gamma_{n,c} & \xrightarrow{\pi} & \mathbb{Z}^r \\ \downarrow \varphi & \circlearrowright & \downarrow \overline{\varphi} \\ \Gamma_{n,c} & \xrightarrow{\pi} & \mathbb{Z}^r \end{array}$$

So, if $R(\overline{\varphi}) = \infty$, by Lemma 1.7 we get $R(\varphi) = \infty$. Let us show (1) $\Rightarrow$ (2). To simplify the computation, let us use the following notation in this proof: given $y = (y_1, ..., y_r) \in \mathbb{Z}^r$ (either a row or a column vector), we will denote the element $s_1^{y_1}...s_r^{y_r} \in \Gamma_{n,c}$ by S^y, and the scalar product of $k \in Z$ by y is denoted by ky. With this notation, it turns out that any element of $\Gamma_{n,c}$ is of the form $S^y x^\beta$ for some $y \in \mathbb{Z}^r$ and $\beta \in \mathbb{Z}$. Suppose then that $R(\overline{\varphi}) = d < \infty$ and write $\mathcal{R}(\overline{\varphi}) = \{[v_1]_{\overline{\varphi}}, ..., [v_d]_{\overline{\varphi}}\}$ for $v_i \in \mathbb{Z}^r$ or, equivalently (Example 1.3), $\frac{\mathbb{Z}^r}{\text{im}(\overline{\varphi}-Id)} = \{\overline{v_1}, ..., \overline{v_d}\}$ (where $\overline{v_i} = v_i + \text{im}(\overline{\varphi} - Id)$). Write $\varphi(x) = x^\mu$ (for some $\mu \in \mathbb{Z}$ with $\gcd(\mu, m^c) = 1$) and $\varphi(s_i) = S^{A_i} x^{\beta_i}$,

$\beta_i \in \mathbb{Z}$. Given that the s_i-coordinates behave well in the $\Gamma_{n,c}$, for any $k = (k_1, ..., k_r) \in \mathbb{Z}^r$ and $l \in \mathbb{Z}$ we have

$$\begin{aligned} \varphi(S^k x^l) &= \varphi(s_1)^{k_1} ... \varphi(s_r)^{k_r} \varphi(x)^l \\ &= (S^{A_1} x^{\beta_1})^{k_1} ... (S^{A_r} x^{\beta_r})^{k_r} x^{\mu l} \\ &= S^{k_1 A_1 + ... + k_r A_r} x^{\theta}, \text{ for some } \theta \in \mathbb{Z} \\ &= S^{\overline{\varphi}(k)} x^{\theta}. \end{aligned}$$

This implies that, for any $j \in \mathbb{Z}$ and $y \in \mathbb{Z}^r$,

$$\begin{aligned} (S^k x^l)(S^y x^j)\varphi(S^k x^l)^{-1} &= S^k x^l S^y x^j x^{-\theta} S^{-\overline{\varphi}(k)} \\ &= S^{y+k-\overline{\varphi}(k)} x^{\tilde{\theta}}, \text{ for } \tilde{\theta} \in \mathbb{Z} \\ &= S^{y+(Id-\overline{\varphi})(k)} x^{\tilde{\theta}}. \end{aligned}$$

This means that, if two vectors $y, y' \in \mathbb{Z}^r$ are such that $\overline{y} = \overline{y'} \in \frac{\mathbb{Z}^r}{\text{im}(\overline{\varphi}-Id)}$, then every element $S^y x^j$ is φ-conjugated to some element $S^{y'} x^{\theta}$ for some $0 \leq \theta < m^c$. Since $\frac{\mathbb{Z}^r}{\text{im}(\overline{\varphi}-Id)} = \{\overline{v_1}, ..., \overline{v_d}\}$, every element $S^y x^j$ is φ-conjugated to some $S^{v_i} x^{\theta}$, $1 \leq i \leq d$, $0 \leq \theta < m^c$, so

$$R(\varphi) \leq dm^c < \infty$$

and the proposition is proved. □

In the rest of the chapter we will use the following notation: we know that $\gcd(p_i^{y_i}, m^c) = 1$. This means that $p_i^{y_i}$ is an invertible element in the commutative ring $\mathbb{Z}_{m^c}$ (now thought in the abelian notation $\mathbb{Z}_{m^c} = \{0, 1, ..., m^c - 1\}$). So, as in commutative algebra, we will naturally denote by $p_i^{-y_i}$ the inverse element $(p_i^{y_i})^{-1} \in \mathbb{Z}_{m^c}$ and, similarly, we define $p_i^{-ky_i}$ as $(p_i^{ky_i})^{-1}$ for any $k \geq 0$, so it makes sense to write $p_i^{ky_i}$ for any $k \in \mathbb{Z}$, thinking of it as an invertible element of the ring $\mathbb{Z}_{m^c}$. We are saying this to avoid a possible misinterpretation of $p_i^{-y_i}$ as $\frac{1}{p_i^{y_i}} \in \mathbb{Q}$, for example. With this notation, it is clear that $s_i^k x s_i^{-k} = x^{p_i^{ky_i}}$ for any $k \in \mathbb{Z}$.

Proposition 10.12. *$\Gamma_{n,c}$ has not property R_∞ if and only if there is $M = (a_{ij})_{ij} \in Gl_r(\mathbb{Z})$ such that*

- $\det(M - Id) \neq 0$;
- *for any $1 \leq i \leq r$,*

$$p_1^{a_{1i}y_1} p_2^{a_{2i}y_2} .. p_r^{a_{ri}y_r} = p_i^{y_i} \mod m^c. \qquad (M, c, i)$$

Demonstração. Suppose first that $\Gamma_{n,c}$ has not property R_∞. Let $\varphi \in Aut(\Gamma_{n,c})$ such that $R(\varphi) < \infty$. Let $M = \overline{\varphi} \in Gl_r(\mathbb{Z})$, and write $M = (a_{ij})_{ij}$. By Proposition 10.11, we have $\det(M - Id) \neq 0$. Since $\varphi(\tau\Gamma_{n,c}) \subset \tau\Gamma_{n,c}$, we have $\varphi(x) = x^{\mu}$ for some $\mu \in \mathbb{Z}$ such that $\gcd(\mu, m^c) = 1$. Let us show that for any $1 \leq i \leq r$ the equation (M, c, i) holds. For any such i, since φ is a homomorphism of $\Gamma_{n,c}$ it must satisfy $\varphi(s_i)\varphi(x)\varphi(s_i)^{-1} = \varphi(x)^{p_i^{y_i}}$, so

$$s_1^{a_{1i}}...s_r^{a_{ri}}x^{\mu}s_r^{-a_{ri}}...s_1^{-a_{1i}} = x^{\mu p_i^{y_i}}$$

or, equivalently,

$$x^{\mu p_1^{a_{1i}y_1}...p_r^{a_{ri}y_r}} = x^{\mu p_i^{y_i}}.$$

Then $\mu p_1^{a_{1i}y_1}...p_r^{a_{ri}y_r} = \mu p_i^{y_i}$ mod m^c, and since $\gcd(\mu, m^c) = 1$, we have $p_1^{a_{1i}y_1}...p_r^{a_{ri}y_r} = p_i^{y_i}$ mod m^c, which is exactly (M, c, i). This shows the "if" part. Suppose now that there is such a matrix $M = (a_{ij})_{ij}$ and let us show $\Gamma_{n,c}$ has not R_∞. Define $\varphi : \Gamma_{n,c} \to \Gamma_{n,c}$ by $\varphi(x) = x$ and $\varphi(s_i) = s_1^{a_{1i}}s_2^{a_{2i}}...s_r^{a_{ri}}$. Let us check that φ is a well defined homomorphism:

$$\varphi(s_i)\varphi(x)\varphi(s_i)^{-1} = s_1^{a_{1i}}s_2^{a_{2i}}...s_r^{a_{ri}}xs_r^{-a_{ri}}...s_2^{-a_{2i}}s_1^{-a_{1i}} = x^{p_1^{a_{1i}y_1}...p_r^{a_{ri}y_r}} = \varphi(x)^{p_1^{a_{1i}y_1}...p_r^{a_{ri}y_r}} = \varphi(x)^{p_i^{y_i}},$$

the last equality being true by (M, c, i). Also, since the s_i commute, we obviously have

$$\varphi(s_i)\varphi(s_j) = s_1^{a_{1i}}...s_r^{a_{ri}}s_1^{a_{1j}}...s_r^{a_{rj}} = s_1^{a_{1j}}...s_r^{a_{rj}}s_1^{a_{1i}}...s_r^{a_{ri}} = \varphi(s_j)\varphi(s_i).$$

Finally,

$$\varphi(x)^{m^c} = x^{m^c} = 1,$$

so φ is in fact a homomorphism. Let us now construct an inverse homomorphism. Let $N = M^{-1} \in GL_r(\mathbb{Z})$ and write $N = (b_{ij})_{ij}$. Let us show that, for any $1 \le i \le r$, N satisfies the equation (N, c, i), that is $p_1^{b_{1i}y_1}p_2^{b_{2i}y_2}..p_r^{b_{ri}y_r} = p_i^{y_i}$ mod m^c. Since $MN = Id$, for any $1 \le i, j \le r$ we have

$$\prod_{k=1}^{r} a_{ik}b_{kj} = (MN)_{ij} = Id_{ij} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Fix i. We do the following: for each fixed $1 \le j \le r$, we raise both sides of equation (M, c, j) to the power of b_{ji} and obtain

$$p_1^{a_{1j}b_{ji}y_1}p_2^{a_{2j}b_{ji}y_2}...p_r^{a_{rj}b_{ji}y_r} = p_j^{b_{ji}y_j} \quad mod\ m^c$$

Now, if we do the product of all the r equations above (on both sides, of course) and rearrange the left side according to the primes we get

$$p_1^{(a_{11}b_{1i}+...+a_{1r}b_{ri})y_1}p_2^{(a_{21}b_{1i}+...+a_{2r}b_{ri})y_2}...p_r^{(a_{r1}b_{1i}+...+a_{rr}b_{ri})y_r} = p_1^{b_{1i}y_1}p_2^{b_{2i}y_2}...p_r^{b_{ri}y_r} \quad mod\ m^c,$$

or

$$p_1^{(\prod_k a_{1k}b_{ki})y_1}p_2^{(\prod_k a_{2k}b_{ki})y_2}...p_r^{(\prod_k a_{rk}b_{ki})y_r} = p_1^{b_{1i}y_1}p_2^{b_{2i}y_2}...p_r^{b_{ri}y_r} \quad mod\ m^c,$$

or even

$$p_1^{\delta_{1i}y_1}p_2^{\delta_{2i}y_2}...p_r^{\delta_{ri}y_r} = p_1^{b_{1i}y_1}p_2^{b_{2i}y_2}...p_r^{b_{ri}y_r} \quad mod\ m^c,$$

which results in

$$p_i^{y_i} = p_1^{b_{1i}y_1}p_2^{b_{2i}y_2}...p_r^{b_{ri}y_r} \quad mod\ m^c,$$

which is exactly (N, c, i), as we wanted. Now define $\psi : \Gamma_{n,c} \to \Gamma_{n,c}$ by $\psi(x) = x$ and $\psi(s_i) = s_1^{b_{1i}}s_2^{b_{2i}}...s_r^{b_{ri}}$. As we did with φ, the fact that N satisfies (N, c, i) for all i gives us that ψ is a group homomorphism. Of course we have $\varphi(\psi(x)) = x$. Also,

$$\begin{aligned}
\varphi(\psi(s_i)) &= \varphi(s_1^{b_{1i}} s_2^{b_{2i}} ... s_r^{b_{ri}}) \\
&= \varphi(s_1)^{b_{1i}} \varphi(s_2)^{b_{2i}} ... \varphi(s_r)^{b_{ri}} \\
&= (s_1^{a_{11}} ... s_r^{a_{r1}})^{b_{1i}} (s_1^{a_{12}} ... s_r^{a_{r2}})^{b_{2i}} ... (s_1^{a_{1r}} ... s_r^{a_{rr}})^{b_{ri}} \\
&= s_1^{(a_{11}b_{1i}+a_{12}b_{2i}+...+a_{1r}b_{ri})} s_2^{(a_{21}b_{1i}+a_{22}b_{2i}+...+a_{2r}b_{ri})} ... s_r^{(a_{r1}b_{1i}+a_{r2}b_{2i}+...+a_{rr}b_{ri})} \\
&= s_1^{\prod_k a_{1k}b_{ki}} s_2^{\prod_k a_{2k}b_{ki}} ... s_r^{\prod_k a_{rk}b_{ki}} \\
&= s_1^{\delta_{1i}} s_2^{\delta_{2i}} ... s_r^{\delta_{ri}} \\
&= s_i.
\end{aligned}$$

Similarly, we show that $\psi\varphi = Id$ by using that $NM = Id$, so $\varphi \in Aut(\Gamma_{n,c})$. Since $\overline{\varphi} = M$ we have $\det(\overline{\varphi} - Id) = \det(M - Id) \neq 0$ by hypothesis, so $R(\varphi) < \infty$ by Proposition 10.11. This completes the proof. □

For the next theorem, we will need the following

Lemma 10.13. *Let $x, m \geq 2$. If $x = 1$ mod m, then $x^{m^k} = 1$ mod m^{k+1} for any $k \geq 0$.*

Demonstração. Induction on k. Note that the case $k = 0$ is obvious. For $k = 1$, write by hypothesis $x = qm + 1$ for some $q \in \mathbb{Z}$ and, by the known Binomial Theorem we have

$$\begin{aligned}
x^m - 1 &= (qm+1)^m - 1 \\
&= \binom{m}{0} q^m m^m + \binom{m}{1} q^{m-1} m^{m-1} + ... + \binom{m}{m-2} q^2 m^2 + \binom{m}{m-1} qm + 1 - 1 \\
&= \binom{m}{0} q^m m^m + \binom{m}{1} q^{m-1} m^{m-1} + ... + \binom{m}{m-2} q^2 m^2 + \binom{m}{m-1} qm
\end{aligned}$$

Note that all summands above are obviously multiples of m^2 - except for the last one, which is also a multiple of m^2 because $\binom{m}{m-1} = m$. This completes the case $k = 1$. Suppose now the lemma is true for some $k \geq 1$ and let us show it for $k+1$. Write by hypothesis $x^{m^k} = qm^{k+1} + 1$ for some $q \in \mathbb{Z}$. Using this and the Binomial Theorem again we get

$$\begin{aligned}
x^{m^{k+1}} - 1 &= (x^{m^k})^m - 1 \\
&= (qm^{k+1} + 1)^m - 1 \\
&= \binom{m}{0} q^m m^{m(k+1)} + ... + \binom{m}{m-2} q^2 m^{2(k+1)} + \binom{m}{m-1} qm^{k+1} + 1 - 1 \\
&= \binom{m}{0} q^m m^{m(k+1)} + ... + \binom{m}{m-2} q^2 m^{2(k+1)} + \binom{m}{m-1} qm^{k+1}
\end{aligned}$$

Again, all summands above can be seen to be multiples of m^{k+2}. In fact, for $2 \leq i \leq m$ we have $i(k+1) = ik + i \geq k + 2$, and these numbers $i(k+1)$ are exactly the powers of m on the summands above - except for the last one, which is also a power of m^{k+2}, for it is $\binom{m}{m-1} qm^{k+1} = mqm^{k+1} = qm^{k+2}$. This completes the proof. □

Theorem 10.14. *Let $n \geq 2$ have prime decomposition $n = {p_1}^{y_1}...{p_r}^{y_r}$, the p_i being pairwise distinct and $y_i > 0$. Suppose $r \geq 2$, that is, there are at least two primes involved. Then the nilpotent quotient group $\Gamma_{n,c} = \Gamma_n/\gamma_{c+1}(\Gamma_n)$ does not have property R_∞ for any $c \geq 1$. In other words, the R_∞ nilpotency index of Γ_n is infinite.*

Demonstração. Let $m = \gcd(p_1^{y_1} - 1, ..., p_r^{y_r} - 1)$, as we have done in this chapter. If $m = 1$, then none of the groups $\Gamma_{n,c}$ have property R_∞. This is because $\Gamma_{n,c} \simeq \mathbb{Z}^r$ for any c in this case (see Theorem 10.9), and we know $\mathbb{Z}^r$ has not R_∞. So, from now on, suppose $m \geq 2$. Of course $\Gamma_{n,1}$ does not have property R_∞, for it is a finitely generated abelian group. Now, for any fixed $c \geq 2$, we will use Proposition 10.12, that is, for any $r \geq 2$, we will find a matrix $M = (a_{ij})_{ij} \in Gl_r(\mathbb{Z})$ with $\det(M - Id) \neq 0$ and satisfying equations (M, c, i) for $1 \leq i \leq r$. We will look for a particular family of matrices M, that is,

$$M = m^k N + Id.$$

Here, k will be some suitable positive number, $N = (j_{\alpha\beta})_{\alpha\beta}$ will be some integer $r \times r$ matrix with determinant 1 and $m^k N = (m^k j_{\alpha\beta})_{\alpha\beta}$ is the natural scalar product of a number by a matrix. The first thing to observe is that any such matrix M satisfies all the equations (M, c, i) for some big enough $k \geq 1$. Let us see that. It is easy to see that, for such M, the equations (M, c, i) become exactly

$$(p_1^{j_{1i}y_1} p_2^{j_{2i}y_2} ... p_r^{j_{ri}y_r})^{m^k} = 1 \mod m^c. \qquad (M, c, i)$$

For us to use the previous lemma, the term inside the parenthesis in the above equation must be congruent to 1 modulo m, so we claim this is true. Since m divides each number $p_s^{y_s} - 1$ $(1 \leq s \leq r)$ by definition, we have $p_s^{y_s} = 1 \mod m$, so by the multiplicative property of integer congruence we have

$$\begin{aligned} p_1^{j_{1i}y_1} p_2^{j_{2i}y_2} ... p_r^{j_{ri}y_r} &= 1^{j_{1i}} 1^{j_{2i}} ... 1^{j_{ri}} \mod m \\ &= 1 \mod m, \end{aligned}$$

which shows our claim. Now let $k = c-1$. By the above lemma we have $(p_1^{j_{1i}y_1} p_2^{j_{2i}y_2} ... p_r^{j_{ri}y_r})^{m^k} = 1 \mod m^c$, so for every i equation (M, c, i) is satisfied for such M.

It is then enough for us to find, for any $r \geq 2$, an integer matrix N which makes $det(M) = 1$ and $det(M - Id) \neq 0$. Since $M = m^k N + Id$, we have

$$det(M - Id) = det(m^k N) = m^{rk} det(N),$$

so for $det(M - Id)$ to be non-zero it suffices us to have $det(N) \neq 0$. We claim therefore that, for any $r \geq 2$, there is a matrix N such that $det(N) = 1$ and $det(M) = det(m^k N + Id) = 1$. We will show the cases $r = 2, 3, 4$ separately and then show the case $r \geq 4$ by induction.

For $r = 2$, it is straightforward to see that $det(M) = m^{2k} \det(N) + m^k tr(N) + 1$ so $det(M) = 1$ if and only if $m^k \det(N) + tr(N) = 0$. Let us find then N with $\det(N) = 1$ and $tr(N) = -m^k$, and we will be done. Now, this is an easy task: the matrix

$$N = \begin{bmatrix} 1 & -(m^k+2) \\ 1 & -(m^k+1) \end{bmatrix}$$

fits the requirements. For $r = 3$, if we define

$$N = \begin{bmatrix} 1 & -(m^k+2) & m^k+1 \\ 1 & -(m^k+1) & m^k \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M = m^k N + Id = \begin{bmatrix} m^k+1 & -m^k(m^k+2) & m^k(m^k+1) \\ m^k & -m^k(m^k+1)+1 & m^{2k} \\ 0 & m^k & 1 \end{bmatrix},$$

it is easy to see that $det(N) = 1 = Det(M)$. For $r = 4$, the matrices

$$N = \begin{bmatrix} 1 & -(m^k+2) & m^k+1 & -(m^k+1) \\ 1 & -(m^k+1) & m^k & -m^k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$M = m^k N + Id = \begin{bmatrix} m^k+1 & -m^k(m^k+2) & m^k(m^k+1) & -m^k(m^k+1) \\ m^k & -m^k(m^k+1)+1 & m^{2k} & -m^{2k} \\ 0 & m^k & 1 & 0 \\ 0 & 0 & m^k & 1 \end{bmatrix},$$

satisfy $det(N) = 1 = Det(M)$. Note the recursion here: the matrices $N = N_{r+1}$ and $M = M_{r+1}$ of size $r+1$ always contain the matrices $N = N_r$ and $M = M_r$ of size r in their left superior corner. We will keep doing this for $r \geq 4$. The induction will be the following: we will show that, for any even number $r \geq 4$, we can find such matrices N and M with $det(N) = 1$ and $det(M) = 1$ for $r+1$ and $r+2$. Let us show this claim to $r = 4$, that is, let us find the matrices M_r and N_r for the cases $r = 5$ and $r = 6$. For $r = 5$,

$$N = \left[\begin{array}{cccc|c} 1 & -(m^k+2) & m^k+1 & -(m^k+1) & m^k+1 \\ 1 & -(m^k+1) & m^k & -m^k & m^k \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

and

$$M = m^kN + Id = \left[\begin{array}{cccc|c} m^k+1 & -m^k(m^k+2) & m^k(m^k+1) & -m^k(m^k+1) & m^k(m^k+1) \\ m^k & -m^k(m^k+1)+1 & m^{2k} & -m^{2k} & m^{2k} \\ 0 & m^k & 1 & 0 & 0 \\ 0 & 0 & m^k & 1 & 0 \\ \hline 0 & 0 & 0 & m^k & 1 \end{array}\right]$$

satisfy $det(N) = 1 = det(M)$. For $r = 6$, the matrices

$$N = \left[\begin{array}{ccccc|c} 1 & -(m^k+2) & m^k+1 & -(m^k+1) & m^k+1 & -(m^k+1) \\ 1 & -(m^k+1) & m^k & -m^k & m^k & -m^k \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

and

$$M = \left[\begin{array}{ccccc|c} m^k+1 & -m^k(m^k+2) & m^k(m^k+1) & -m^k(m^k+1) & m^k(m^k+1) & -m^k(m^k+1) \\ m^k & -m^k(m^k+1)+1 & m^{2k} & -m^{2k} & m^{2k} & -m^{2k} \\ 0 & m^k & 1 & 0 & 0 & 0 \\ 0 & 0 & m^k & 1 & 0 & 0 \\ 0 & 0 & 0 & m^k & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & m^k & 1 \end{array}\right]$$

satisfy $det(N) = 1 = det(M)$ (this can be checked by developing the determinant using the last column of the matrices). The reader can easily see the induction step now. Suppose that, for some even number $r \geq 4$, the square matrices

$$N_r = \left[\begin{array}{ccccccc} 1 & -(m^k+2) & m^k+1 & -(m^k+1) & \cdots & m^k+1 & -(m^k+1) \\ 1 & -(m^k+1) & m^k & -m^k & \cdots & m^k & -m^k \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

and

$$M_r = \begin{bmatrix} m^k+1 & -m^k(m^k+2) & m^k(m^k+1) & -m^k(m^k+1) & \cdots & m^k(m^k+1) & -m^k(m^k+1) \\ m^k & -m^k(m^k+1)+1 & m^{2k} & -m^{2k} & \cdots & m^{2k} & -m^{2k} \\ 0 & m^k & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & m^k & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & m^k & 1 \end{bmatrix}$$

have both determinant 1. For $r+1$, the matrices

$$N_{r+1} = \left[\begin{array}{ccccc|c} & & & & & m^k+1 \\ & & & & & m^k \\ & & N_r & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 \end{array}\right] \text{ and } M_{r+1} = \left[\begin{array}{ccccc|c} & & & & & m^k(m^k+1) \\ & & & & & m^{2k} \\ & & M_r & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ \hline 0 & 0 & \cdots & 0 & m^k & 1 \end{array}\right]$$

Have determinant 1. In fact, developing the determinant of N_{r+1} by the last column we get $det(N_{r+1}) = (-1)^{r+1+1}(m^k+1).1 + (-1)^{r+1+2}m^k.1 = m^k+1-m^k = 1$, using that the two submatrices are upper triangular with 1 in all diagonal entries and that r is even. Similarly, we develop the determinant of M_{r+1} by the last column. Using that r is even, that the first two submatrices that appear are upper triangular and that $det(M_r) = 1$ we get

$$\begin{aligned} det(M_{r+1}) &= (-1)^{r+1+1}m^k(m^k+1)m^{rk} + (-1)^{r+1+2}m^{2k}(m^k+1)m^{(r-1)k} + (-1)^{r+1+r+1}det(M_r) \\ &= m^{(r+1)k}(m^k+1) - m^{(r+1)k}(m^k+1) + 1 \\ &= 1. \end{aligned}$$

Finally, for $r+2$, the matrices

$$N_{r+2} = \left[\begin{array}{ccccc|c} & & & & & -(m^k+1) \\ & & & & & -m^k \\ & & N_{r+1} & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 \end{array}\right] \text{ and } M_{r+2} = \left[\begin{array}{ccccc|c} & & & & & -m^k(m^k+1) \\ & & & & & -m^{2k} \\ & & M_{r+1} & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ \hline 0 & 0 & \cdots & 0 & m^k & 1 \end{array}\right]$$

Have determinant 1. Indeed, developing the determinant of N_{r+2} by the last column we get $det(N_{r+2}) = (-1)^{r+2+1}(-(m^k+1)).1 + (-1)^{r+2+2}(-m^k).1 = m^k+1-m^k = 1$, this time using that the two submatrices are upper triangular with 1 in all diagonal entries and that r is even. Similarly, we develop the determinant of M_{r+2} by the last column. Using that r is even, that the first two submatrices that appear are upper triangular and that $det(M_{r+1}) = 1$ we get

$$\begin{aligned} det(M_{r+2}) &= (-1)^{r+3}(-m^k(m^k+1))m^{(r+1)k} + (-1)^{r+4}(-m^{2k})(m^k+1)m^{rk} + (-1)^{2r+4}.1 \\ &= m^{(r+2)k}(m^k+1) - m^{(r+2)k}(m^k+1) + 1 \\ &= 1. \end{aligned}$$

This completes the induction step and finishes our proof. □

Open question: is it possible to use similar techniques to compute the R_∞ nilpotency index for GBS groups? Indeed, this was one of the suggestions of the authors in [22] (2020), the main paper inspiring our chapter. By this moment, with other combinatorial techniques, we have already started this investigation and obtained some particular and interesting computations. In particular, we know GBS groups with finite and with infinite R_∞ nilpotency indexes, but the research is still far away from its end.

Capítulo 11

GBS and GBS_n groups and Σ^1 invariant

We dedicate this chapter to investigate the behaviour of the so called GBS (and GBS_n) groups concerning Σ^1 and Ω^1 and to try to guarantee R_∞ or other twisted conjugacy properties for these groups. The reason for this investigation is that property R_∞ has already been shown to any non-elementary GBS group (see [67], Proposition 2.7 at pg. 486) and, in fact, for any group that is quasi-isometric to a GBS group (see [93]), but geometric invariants have not been used. So, could Σ-theory be applied to determine property R_∞ for GBS and GBS_n groups? This turned out to be not so effective as it was in chapters 5 and 6, and that is the reason why this chapter in the appendix. The problem is that Σ^1 is symmetric inside the character sphere, as we shall see. However, we have good results here. In fact, by using a result from Cashen and Levitt in [19], we algorithmically classify the possible shapes of the Σ-invariant of GBS (and GBS_n) groups, given the associated finite graph of groups. We then use this to get some partial twisted conjugacy results (not necessarily R_∞ results) on some special cases.

We start by showing two quite general properties of Σ^1 concerning fundamental groups of finite graphs of groups. The first one is Corollary 2.2 in [19]. The proof technique is interesting enough to be shown below.

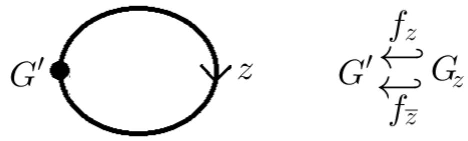
Proposition 11.1. *Let G be the fundamental group of a finite and connected graph of groups (G, Γ), with each vertex and edge group being finitely generated. Let $[\chi] \in S(G)$. If $\chi|_{G_y} \neq 0$ for each edge group G_y and $[\chi|_{G_P}] \in \Sigma^1(G_P)$ for each vertex group G_P then $[\chi] \in \Sigma^1(G)$.*

Demonstração. We divide the proof into 3 steps.

- Γ a finite tree. We show the proposition by induction on the number n of (geometric) edges of Γ. For $n = 1$, Γ is a segment of groups and G is an amalgamated product of two finitely generated groups (Definition 1.64). In this case what we want to show is exactly Proposition 3.31, already shown. Now suppose the claim to be true for $n - 1 \geq 1$ and let Γ be a finite tree with n edges. By Proposition 1.29, let P_0 be a terminal vertex of Γ associated to an edge y_0 and let $\Gamma' = \Gamma - P_0$. Let (G, Γ') be the restriction of (G, Γ) to Γ' and G' be its fundamental group. From Example 1.66 we have $G = G' *_{G_{y_0}} G_{P_0}$. By hypothesis, $\chi|_{G_y} \neq 0$ for each edge y of Γ (in particular for every edge of Γ') and $[\chi|_{G_P}] \in \Sigma^1(G_P)$ for each vertex P of Γ (in particular for every vertex of Γ'). Since

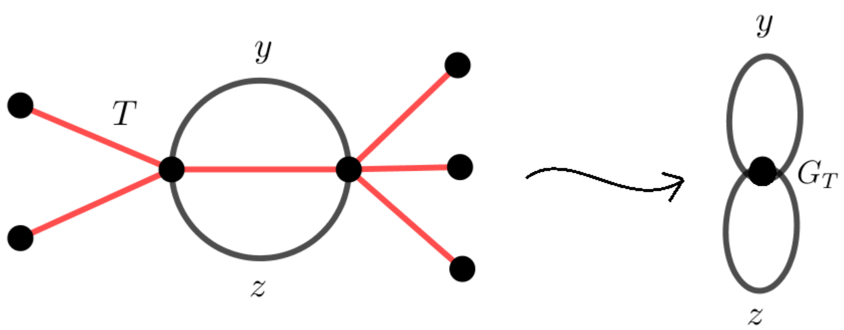
Γ' is a tree with $n-1$ edges, by induction hypothesis the claim is true for (G, Γ') and so $[\chi|_{G'}] \in \Sigma^1(G')$. Since $\chi|_{G_{y_0}} \neq 0$ and $[\chi|_{G_{P_0}}] \in \Sigma^1(G_{P_0})$ by hypothesis, we apply Proposition 3.31 for the amalgam $G' *_{G_{y_0}} G_{P_0}$ and conclude that $[\chi] \in \Sigma^1(G)$, as desired.

- Γ a finite bouquet. We show the proposition by induction on the number n of petals (edges) of Γ. For $n = 1$, Γ is a loop of groups and G is an HNN extension of a finitely generated group (Definition 1.65). In this case what we want to show is exactly Proposition 3.32, already shown. Now suppose the claim to be true for $n - 1 \geq 1$ and let Γ be a finite bouquet with n petals. Fix one petal z and denote by Γ' the "sub bouquet" obtained by removing z and $\overline{z}$ from Γ. Denote by G' the fundamental group of Γ'. From Example 1.67 we have $G = \pi_1(Z)$, where Z is the loop of groups defined there and shown in the figure below.



By hypothesis, $\chi|_{G_y} \neq 0$ for each edge y of Γ (in particular for every edge of Γ') and $[\chi|_{G_P}] \in \Sigma^1(G_P)$. Since Γ' is a bouquet with $n-1$ edges, by induction hypothesis the claim is true for (G', Γ') and so $[\chi|_{G'}] \in \Sigma^1(G')$. Now, $\chi|_{G_z} \neq 0$ and $[\chi|_{G'}] \in \Sigma^1(G')$, so we can apply Proposition 3.32 for the loop of groups (HNN extension) Z to conclude that $[\chi] \in \Sigma^1(G)$, as desired.

- General case. Let Γ be any connected finite graph and fix a maximal tree T of Γ. Let (G, T) be the restriction of (G, Γ) to T and denote by G_T its fundamental group. By hypothesis, $\chi|_{G_y} \neq 0$ for each edge y of Γ (in particular for every edge of T) and $[\chi|_{G_P}] \in \Sigma^1(G_P)$ for each vertex P of Γ (in particular for every vertex of T). Since T is a finite tree, by the first case we have $[\chi|_{G_T}] \in \Sigma^1(G_T)$. Now let (H, W) be the following graph of groups: define $W = \Gamma/T$ as the contraction of the maximal tree T inside Γ, that is, H is a bouquet whose vertex we call P_0 and whose edges are exactly the edges y of Γ outside T. Define the vertex group as $H_{P_0} = G_T$ and the edge groups as $H_y = G_y$.



Define the morphisms as

$$\begin{cases} H_y = G_y \overset{f_{\overline{y}}}{\hookrightarrow} G_{o(y)} \leq G_T = H_{P_0}, \\ H_y = G_y \overset{f_y}{\hookrightarrow} G_{t(y)} \leq G_T = H_{P_0}. \end{cases}$$

Then by the rebuilding argument (Example 1.67) the fundamental group of (H, W) is exactly G. But W is a finite bouquet. Since $\chi|_{H_y} = \chi|_{G_y} \neq 0$ for every edge y of W and $[\chi|_{H_{P_0}}] = [\chi|_{G_T}] \in \Sigma^1(G_T) = \Sigma^1(H_{P_0})$, by the second case we have $[\chi] \in \Sigma^1(G)$, and the proof is complete.

□

The second and last property works like a partial converse to the previous one. Following [19]:

Definition 11.2. We say that a graph of groups (G, Γ) is not an ascending HNN extension if either Γ is not a loop or Γ is a loop but the monomorphisms $G_y \overset{f_y}{\hookrightarrow} G_P$ and $G_y \overset{f_{\overline{y}}}{\hookrightarrow} G_P$ are both proper (not surjective). We say that (G, Γ) is reduced if for every segment y of Γ the monomorphisms $G_y \overset{f_y}{\hookrightarrow} G_{t(y)}$ and $G_y \overset{f_{\overline{y}}}{\hookrightarrow} G_{o(y)}$ are both proper.

Proposition 11.3. *Let G be the fundamental group of a finite, connected and reduced graph of groups (G, Γ) which is not an ascending HNN extension, and with each vertex and edge group being finitely generated. If $[\chi] \in \Sigma^1(G)$ then $\chi|_{G_y} \neq 0$ for each edge group G_y of (G, Γ).*

Demonstração. Again, we divide the proof into steps:

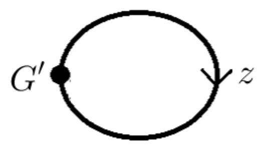
- Γ is a segment or a loop. In this case, G is either an amalgamated product or an HNN extension. In the former case, then because Γ is reduced we have exactly the hypotheses of Proposition 3.33, and we are done. In the latter case, then because (G, Γ) is not an ascending HNN extension we have exactly the hypotheses of Proposition 3.34, and we are done again.

Now let us show the general case, supposing that Γ contains at least two edges (the one-edged case is treated above). Fix an arbitrary edge y_0 of Γ and let us show that $\chi|_{G_{y_0}} \neq 0$. There are only two cases (see Lemma 1.25):

- $\Gamma - y_0$ connected. In this case, let $(G, \Gamma - y_0)$ be the restriction of (G, Γ) to $\Gamma - y_0$ and let G' be its fundamental group. By the reconstruction argument 1.67, G is exactly the fundamental group of the loop of groups (H, W) with $H_P = G'$, $H_z = G_{y_0}$ and morphisms

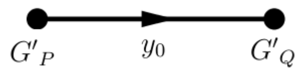
$$\begin{cases} H_z = G_{y_0} \overset{f_{\overline{y_0}}}{\hookrightarrow} G_{o(y_0)} \overset{l}{\hookrightarrow} G' = H_P, \\ H_z = G_{y_0} \overset{f_{y_0}}{\hookrightarrow} G_{t(y_0)} \overset{l}{\hookrightarrow} G' = H_P, \end{cases} \tag{11.1}$$

where l are the respective inclusion morphisms.



We already have $[\chi] \in \Sigma^1(G)$, G being the fundamental group of (H, W). We want now to use the loop case of the proposition to (H, W). To do so, we must guarantee that both morphisms in 11.1 are proper. If y_0 is a segment in Γ, then the morphisms $f_{\overline{y_0}}$ and f_{y_0} are themselves proper, so are $l \circ f_{\overline{y_0}}$ and $l \circ f_{y_0}$. If y_0 is a loop, say, from P to P, it will be enough to show that the inclusion $l : G_P \to G'$ is proper. Since Γ has at least two edges, let y be another edge different from y_0 starting in P. If y is a segment, say, from P to some vertex Q, then, since Γ is reduced, there is an element $g \in G_Q \leq G'$ which is not in $G_y = G_P \cap G_Q$, then $g \in G' - G_P$ and l is proper, as desired. If y is a loop, also from P to P, then the stable letter t_y is by definition in $G' - G_P$ and l is again proper. Then, applying Proposition 3.34 to (H, W) we get $\chi|_{G_{y_0}} \neq 0$, as we wanted.

- $\Gamma - y_0$ with two components Γ_P and Γ_Q (see Proposition 1.25). In this case, if T is a maximal tree of Γ, then $T \cap \Gamma_P$ and $T \cap \Gamma_Q$ are maximal trees of Γ_P and Γ_Q. Let (G, Γ_P) and (G, Γ_Q) be the restriction of (G, Γ) to Γ_P and Γ_Q with fundamental groups G'_P and G'_Q, respectively. We have $G = G'_P *_{G_{y_0}} G'_Q$, that is, G is the fundamental group of the following segment of groups



with monomorphisms $G_{y_0} \overset{f_{\overline{y_0}}}{\hookrightarrow} G_P \leq G'_P$ and $G_{y_0} \overset{f_{y_0}}{\hookrightarrow} G_Q \leq G'_Q$. Since $f_{\overline{y_0}}$ and f_{y_0} are proper, this segment of groups is reduced and since $[\chi] \in \Sigma^1(G)$, then by Proposition 3.33 we have $\chi|_{G_{y_0}} \neq 0$, as desired. This completes the proof.

□

Putting together the last two propositions we obtain a precious weapon for the rest of the chapter. This theorem is actually Corollary 2.10 in [19]:

Theorem 11.4. *Let G be the fundamental group of a finite, connected and reduced graph of groups (G, Γ) which is not an ascending HNN extension, and with each vertex and edge group being finitely generated. If $\Sigma^1(G_P) = S(G_P)$ for each vertex P then*

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \chi|_{G_y} \neq 0 \textit{ for each edge group } G_y\}.$$

Demonstração. If $[\chi] \in \Sigma^1(G)$, then since (G, Γ) is reduced and not an ascending HNN extension we have $\chi|_{G_y} \neq 0$ for each edge group G_y by Proposition 11.2. On the other hand, suppose $[\chi] \in S(G)$ is such that $\chi|_{G_y} \neq 0$ for each edge group G_y. Since $\Sigma^1(G_P) = S(G_P)$ for each vertex P, we have also $[\chi|_{G_P}] \in \Sigma^1(G_P)$ and then by Proposition 11.1 we obtain $[\chi] \in \Sigma^1(G)$, as desired. □

It is worth observing a special case: when the monomorphisms maps onto finite index subgroups.

Corollary 11.5. *Let G be as in Theorem 11.4. Assume also that all the edge monomorphisms $f_{\overline{y}}$ and f_y maps all the edge groups G_y onto finite index subgroups $f_{\overline{y}}(G_y)$ of $G_{o(y)}$ and $f_y(G_y)$ of $G_{t(y)}$. Then*

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \chi|_{G_{P_0}} \neq 0\}$$

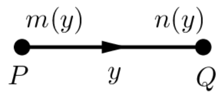
where G_{P_0} is any fixed vertex group.

Demonstração. We will identify each G_y with its isomorphic images $f_{\overline{y}}(G_y)$ and $f_y(G_y)$. Given any edge y in Γ, since G_y has finite index in both $G_{o(y)}$ and $G_{t(y)}$, by Lemma 3.26 we have $\chi|_{G_{o(y)}} \neq 0 \Longleftrightarrow \chi|_{G_y} \neq 0 \Longleftrightarrow \chi|_{G_{t(y)}} \neq 0$. Now, let P be any vertex of Γ. If T is a maximal tree for Γ, connect P and P_0 by a geodesic p in T. By repeating the same finite index argument we just used for all the edges of p we get that $\chi|_{G_P} \neq 0 \Longleftrightarrow \chi|_{G_{P_0}} \neq 0$. Then the Σ^1 condition"$\chi|_{G_y} \neq 0$ for each edge group G_y" in Theorem 11.4 can be replaced just by "$\chi|_{G_{P_0}} \neq 0$", as desired. □

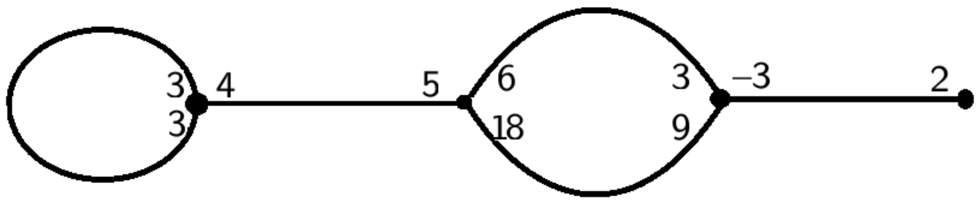
11.1 *GBS* groups

Definition 11.6. A graph of $\mathbb{Z}'s$ is a graph of groups (G, Γ) with $G_P = \mathbb{Z}$ and $G_y = \mathbb{Z}$ for all vertices P and all edges y of Γ.

For every edge y we then have two monomorphisms $f_y : G_y = \mathbb{Z} \hookrightarrow \mathbb{Z} = G_{t(y)}$ and $f_{\overline{y}} : G_y = \mathbb{Z} \hookrightarrow \mathbb{Z} = G_{o(y)}$ that are uniquely determined by the nonzero integers $n(y) = f_y(1)$ and $m(y) = f_{\overline{y}}(1)$. The notation will be the following:



An example of graph of $\mathbb{Z}'s$ is given by the next figure.



Definition 11.7. A *GBS* group G is the fundamental group of a finite connected graph of $\mathbb{Z}'s$.

Let T be a maximal tree and E^+ an orientation of Γ. If $G_P = \langle a_P \rangle \simeq \mathbb{Z}$, we call a_P the vertex letter associated to the vertex P. Let $y_1, ..., y_k$ be the edges of E^+ outside T with associated stable letters $t_1, ..., t_k$. Then a presentation for the *GBS* group G is

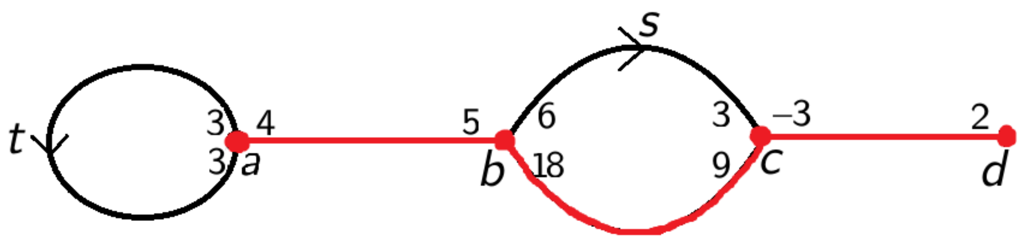
$$G = \langle a_P, t_1, ..., t_k,\ P \in V(\Gamma) \mid a_{t(y)}^{n(y)} = a_{o(y)}^{m(y)},\ t_i a_{t(y_i)}^{n(y_i)} t_i^{-1} = a_{o(y_i)}^{m(y_i)},\ y \in E^+ \cap E(T), 1 \leq i \leq k \rangle.$$

To clarify, the next figure shows the graph (G, Γ), the maximal tree T and the associated vertex and stable letters. The associated presentation for G is

$$G = \langle a, b, c, d, t, s \mid a^4 = b^5, b^{18} = c^9, c^{-3} = d^2, ta^3t^{-1} = a^3, sc^3s^{-1} = b^6 \rangle.$$

The definition implies that for every oriented edge y (inside or outside T) we have the following relation in the abelianized G^{ab}:

$$a_{t(y)}^{n(y)} = a_{o(y)}^{m(y)}.$$



Definition 11.8. If (G,Γ) is a graph of $\mathbb{Z}$'s, we say that a closed path $\gamma = y_1, ..., y_k$ is killing if $n(y_1)...n(y_k) \neq m(y_1)...m(y_k)$.

Lemma 11.9 (Removing non-killing closed paths)**.** *If $\gamma = y_1, ..., y_k$ $(k \geq 3)$ is a killing closed path in (G,Γ) and $y_i, ..., y_j$ is a proper closed subpath in γ such that $n(y_i)...n(y_j) = m(y_i)...m(y_j)$, then the path $\gamma' = y_1, ..., y_{i-1}, y_{j+1}, ..., y_k$ obtained by removing this subpath is also a killing closed path.*

Demonstração. Let $y_i, ..., y_j$ be a proper closed subpath in γ such that $n(y_i)...n(y_j) = m(y_i)...m(y_j)$. Since $n(y_1)...n(y_k) \neq m(y_1)...m(y_k)$, then by canceling $n(y_i)...n(y_j)$ on the left side and $m(y_i)...m(y_j)$ on the right side of this inequality we continue with an inequality, which says exactly that the closed path γ' obtained by removing this subpath from γ is still killing. □

Corollary 11.10 (Removing backtrackings)**.** *If $\gamma = y_1, ..., y_k$ $(k \geq 3)$ is a killing closed path in (G,Γ) and $y_{i+1} = \overline{y_i}$ is a backtracking in γ, the path $\gamma' = y_1, ..., y_{i-1}, y_{i+2}, ..., y_k$ obtained by removing this backtracking is also a killing closed path.*

Demonstração. If $y_{i+1} = \overline{y_i}$ is a backtracking we have $m(y_{i+1}) = n(y_i)$ and $n(y_{i+1}) = m(y_i)$ by definition and then $n(y_i)n(y_{i+1}) = m(y_i)m(y_{i+1})$. Then applying the previous lemma to the closed subpath $y_i, \overline{y_i}$ we are done. □

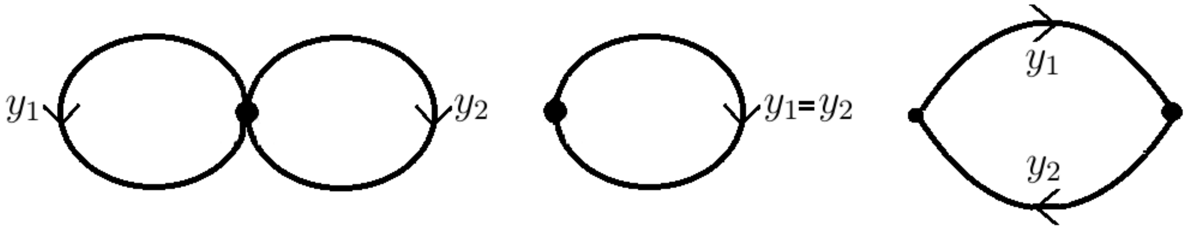
Lemma 11.11. *There are no killing contractible closed paths.*

Demonstração. Let γ be a closed contractible path with length $2k$ (every contractible path must have even length by definition) and let us show by induction on k that γ is not killing. If $k = 1$ we must have $\gamma = y, \overline{y}$, with $n(y) = m(\overline{y})$ and $n(\overline{y}) = m(y)$, so $n(y)n(\overline{y}) = m(\overline{y})m(y) = m(y)m(\overline{y})$ and γ is not killing. Suppose this is true for k and let γ be a contractible closed path of length $2(k+1)$. Suppose by contradiction that γ is killing. Let $y_{i+1} = \overline{y_i}$ be a backtracking of γ. Then, by the previous lemma, the path obtained by removing this backtracking of γ is killing. Since it is also contractible and of length $2k$, by induction hypothesis it is not killing and we have a contradiction. This concludes the proof. □

Lemma 11.12. *Every killing closed path in (G,Γ) contains a killing circuit.*

Demonstração. First we show this lemma for all killing closed paths of length 1 or 2. Since contractible closed paths are not killing, we just have to analyze the non-contractible ones. If the length is 1 the path itself is a killing circuit. If the length is 2, there are only three kinds of non-contractible killing closed paths:

In the two left figures it is easy to see that if both loops are not killing then the entire path is not killing, so at least one of them must be a killing loop. In the right figure, the path itself is a killing circuit, as desired.



Now we show the lemma. Let $\gamma = y_1, ..., y_k$ be a killing closed path, with $k \geq 3$, and suppose by contradiction that γ does not contain any killing circuits. If there is a backtracking in γ, remove it with the "Removing backtrackings" Corollary 11.10, and we get a new killing closed path with no killing circuits. If we repeat this process a finite number of times, we may suppose that γ is a killing closed path without backtrackings. Note that if, during this process, we obtain a path with length 1 or 2, then by the first paragraph we find a killing circuit in γ, a contradiction, and we are done. So in the rest of the proof we may suppose all the paths obtained have length at least 3. Now, since γ has no backtrackings, it is not contractible by definition, therefore by Proposition 1.27 it must contain a circuit c. By assumption, c is not killing. So, if c is the entire path γ we are done with a contradiction, since γ is killing. If c is a proper circuit, by the "Removing non-killing closed paths" Lemma 11.9 we can remove c from γ and continue with a killing closed path γ'.

Repeat all the algorithm above to the path γ'. Each time we do this we find a new circuit in γ and remove it. Then, since γ obviously contains a finite number of circuits, after a finite number of steps we will obtain a killing closed path $\tilde{\gamma}$ with no more circuits. But by Proposition 1.27 again, $\tilde{\gamma}$ is contractible and therefore cannot be killing by Lemma 11.11, a contradiction. $\square$

Before we show the main theorem of this section, we will highlight what we already obtained about Σ^1 for *GBS* groups.

Corollary 11.13. *Let G be a GBS group associated to the reduced graph of $\mathbb{Z}'s$ (G, Γ). If G is not a solvable Baumslag Solitar group $BS(1, n)$, then*

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \chi(a) \neq 0\}$$

where a is any fixed vertex letter.

Demonstração. We want to use Corollary 11.5. Remember that since $\mathbb{Z}$ is abelian we have $\Sigma^1(G_P) = S(G_P)$ for each vertex P by Corollary 3.15. Also, since all nontrivial subgroups of $\mathbb{Z}$ have the form $n\mathbb{Z}$ and are of finite index, all the monomorphisms f_y and $f_{\overline{y}}$ map the edge groups G_y onto finite index subgroups of the vertex groups G_P. Finally, if Γ was an ascending *HNN* extension then G would be by definition some solvable Baumslag Solitar group $BS(1, n)$ for $n \neq 0$, contradiction. Then, applying Corollary 11.5, we obtain that $[\chi] \in \Sigma^1(G) \Longleftrightarrow \chi(G_{P_0}) \neq 0$ for some fixed vertex group G_{P_0}. Since $G_{P_0} = \mathbb{Z} = \langle a \rangle$ for some vertex letter a, we have $\chi(G_{P_0}) \neq 0 \Longleftrightarrow \chi(a) \neq 0$, as desired. $\square$

Now we use the concept of killing circuits to compute the dimension of the character sphere of a reduced *GBS* group. Also, we use Corollary 11.13 to determine the possible shapes of Σ^1 for these groups. This is the main result of the section:

Theorem 11.14. *Let G be a GBS group associated to the reduced graph of $\mathbb{Z}'s$ (G,Γ) with topological rank $k \geq 0$ and orientation E^+. Assume that G is not a solvable Baumslag-Solitar group $BS(1,n)$ and fix a vertex letter a. There are only three distinct cases:*

- *If Γ is a tree, then $\Sigma^1(G) = S(G) = \{[\chi],[-\chi]\}$, where $\chi(a) = 1$;*
- *If there is a killing circuit in Γ then $S(G) \simeq S^{k-1}$ and $\Sigma^1(G) = \emptyset$.*
- *If Γ is not a tree and there is not a killing circuit in Γ, then $S(G) \simeq S^k$ and $\Sigma^1(G)$ is the disjoint union of two antipodal open hemispheres in $S(G) \simeq S^k$.*

Observation 11.15. The character sphere and the Sigma invariant have already been computed for the solvable Baumslag Solitar groups $BS(1,n)$ (see Example 3.9).

Demonstração. First of all, note that, if b, b' are two adjacent vertex letters in an edge y, then in G^{ab} we have the relation $b^{n(y)} = b'^{m(y)}$, therefore $n(y)\chi(b) = m(y)\chi(b')$ or $\chi(b) = \frac{m(y)}{n(y)}\chi(b')$ for every character χ. This means that their χ-value are dependent. Since $\frac{m(y)}{n(y)} \neq 0$, in particular $\chi(b) = 0 \Leftrightarrow \chi(b') = 0$. Since Γ is connected, all the χ-values in the vertex letters depend only of the value $\chi(a)$. So all the vertex letters can contribute with at most one dimension in the sphere $S(G)$, depending if a is torsion-free or not in G^{ab}. Also, since the topological rank is k, let $y_1, ..., y_k$ be the oriented edges (if any) outside a maximal tree T of Γ chosen, with $t_1, ..., t_k$ associated stable letters. It is obvious from the presentation that $t_1, ..., t_k$ are always torsion-free in G^{ab} (if any).

If Γ is a tree, there are no stable letters and the only generators of G are the vertex letters. Then, as we told, every character χ depends uniquely on the value $\chi(a)$. If $\chi(a) = 0$ then we would have $\chi = 0$, a contradiction. Then $\chi(a) \neq 0$ for every character and a is torsion-free in G^{ab}. This gives the unique dimension of $Hom(G,\mathbb{R})$ and we have the homeomorphism $S(G) \to S^0$ with $[\chi] \mapsto \frac{\chi(a)}{|\chi(a)|}$. Now, by Corollary 11.13, $[\chi] \in \Sigma^1(G)$ if and only if $\chi(a) \neq 0$, then by the argument above $\Sigma^1(G) = S(G) = \{[\chi],[-\chi]\}$ is the whole 0-sphere.

If there is a killing circuit $\gamma = y_1, ..., y_s$ in Γ, let $P_0 = o(y_1)$, $P_i = t(y_i)$ for $1 \leq i \leq s$ and let a_i be the vertex letters associated to the P_i, $0 \leq i \leq s$. Then $P_s = P_0$ and $a_s = a_0$. By the first paragraph, we have the following relations in G^{ab}:

$$a_0^{n(y_1)} = a_1^{m(y_1)}, a_1^{n(y_2)} = a_2^{m(y_2)}, ..., a_{s-1}^{n(y_s)} = a_s^{m(y_s)}.$$

Then $a_2^{m(y_2)m(y_1)} = a_1^{n(y_2)m(y_1)} = a_1^{m(y_1)n(y_2)} = a_0^{n(y_1)n(y_2)}$, and recursively we obtain $a_s^{m(y_s)...m(y_1)} = a_0^{n(y_s)...n(y_1)}$, or, since $a_0 = a_s$, we have

$$a_0^{n(y_1)...n(y_s)-m(y_1)...m(y_s)} = 1 \text{ in } G^{ab}.$$

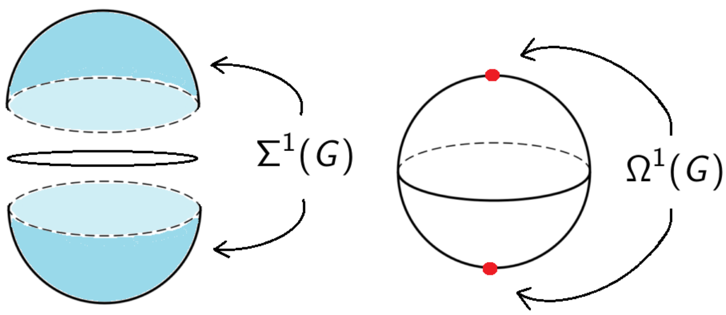
Since γ is killing the exponent $n(y_1)...n(y_s) - m(y_1)...m(y_s)$ is non-zero and then a_0 have torsion in G^{ab}, or, equivalently, $\chi(a_0) = 0$ for every character χ. By the first paragraph again, $\chi(a) = 0$ for every character. This means that the vertex letter a_0 does not contribute with a dimension in the sphere $S(G)$, only the stable letters contribute. Then the homeomorphism is $S(G) \to S^{k-1}$ with $[\chi] \mapsto \frac{(\chi(t_1),...,\chi(t_k))}{\|(\chi(t_1),...,\chi(t_k))\|}$. Since $\chi(a) = 0$ for every character we have $\Sigma^1(G) = \emptyset$ by Corollary 11.13, as desired.

Now, let us show that if Γ is not a tree and there are no killing circuits in Γ then a is torsion-free in G^{ab}. By definition, the unique relations in G^{ab} are the commutators (which does not generate any torsion on the generators) and the ones having the form $a_{t(y)}^{n(y)} = a_{o(y)}^{m(y)}$ for all oriented edges. Since these relations only appear between adjacent edges, the only way to obtain a relation of the form $a^{\beta} = a^{\delta}$ in G^{ab} is if we have a closed path in the vertex P associated to a. Furthermore: if this closed path is not killing, then similarly to the previous paragraph we would only obtain a relation of the form $a^{n(y_1)...n(y_s)-m(y_1)...m(y_s)} = 1$ with a zero exponent $n(y_1)...n(y_s) - m(y_1)...m(y_s)$, that is, a useless relation $a^0 = 1$. Then if a have torsion in G^{ab} we must have a killing closed path in Γ. By the previous lemma, we must have a killing circuit in Γ, a contradiction. Finally, since Γ is not a tree we must have stable letters and since a is also torsion-free we have the homeomorphism $S(G) \to S^k$ with $[\chi] \mapsto \frac{(\chi(a),\chi(t_1),...,\chi(t_k))}{\|(\chi(a),\chi(t_1),...,\chi(t_k))\|}$. By Corollary 11.13 we know that $[\chi] \in \Sigma^1(G) \Leftrightarrow \chi(a) \neq 0$, so the points in $\Sigma^1(G)$ correspond to the points in the sphere S^k with non-zero first coordinate $\chi(a)$, that is, the disjoint union of the two antipodal open hemispheres, as we wanted. □

As a consequence of this, we have all the possible shapes of the Ω^1-invariants:

Theorem 11.16. *Let G be a GBS group associated to the reduced graph of $\mathbb{Z}'s$ (G, Γ) with topological rank $k \geq 0$ and orientation E^+. Assume that G is not a solvable Baumslag-Solitar group $BS(1,n)$ and fix a vertex letter a. There are only three distinct cases:*

- *If Γ is a tree, then $\Omega^1(G) = S(G)$;*
- *If there is a killing circuit in Γ then $\Omega^1(G) = \emptyset$.*
- *If Γ is not a tree and there is not a killing circuit in Γ, then $\Omega^1(G)$ consists of two antipodal rational points.*



Demonstração. The two first cases are obvious because we have $\Sigma^1(G) = S(G)$ and $\Sigma^1(G) = \emptyset$, respectively. In the third case, since $\Sigma^1(G)$ is the disjoint union of two antipodal open hemispheres in $S(G) \simeq S^k$ (given respectively by $\{\chi(a) > 0\}$ and $\{\chi(a) < 0\}$), it follows from Definition 3.10 that $\Omega^1(G)$ consists of only two antipodal points $[\chi], [-\chi]$ (where $\chi(a) = 1$ and χ vanishes the other generators). □

There is a special case in which we can guarantee an infinite Reidemeister number for at least "half" of the automorphisms of G:

Corollary 11.17. *Let G be a GBS group associated to the reduced graph of $\mathbb{Z}$'s (G, Γ) which has rank $k = 1$ and does not contain any killing circuits. Suppose that G is not a solvable Baumslag-Solitar group. Then there exists a normal subgroup $H \lhd Aut(G)$ with index 2 such that $R(\varphi) = \infty$ for every automorphism $\varphi \in H$.*

Demonstração. Let t be the stable letter associated with the unique circuit of (G, Γ) and fix a vertex letter a. By 11.14 we have the homeomorphism

$$\begin{aligned} S(G) &\longrightarrow S^1 \\ [\chi] &\longmapsto \frac{(\chi(a), \chi(t))}{\|(\chi(a), \chi(t))\|} \end{aligned}$$

and $[\chi] \in \Sigma^1(G)$ if and only if $\chi(a) \neq 0$. Then the points $[\chi]$ in the complement $\Sigma^1(G)^c$ corresponds only to the two antipodal points $(0, 1)$ and $(0, -1)$ of S^1. By Corollary 3.39, there is a normal subgroup $H \lhd Aut(G)$ with finite index such that $R(\varphi) = \infty$ for every automorphism $\varphi \in H$. But, in the proof of that corollary, one can see that the index of H is the number of possible permutations of the points in $\Sigma^1(G)^c$, which is 2 in this case. Then H has index 2 and we conclude the corollary. □

Based on Theorem 11.14 we also obtain the impossibility of finite generation of a family of subgroups of some "bouquet" GBS groups.

Corollary 11.18. *Let G be a Bouquet GBS group, that is, G is a GBS group associated to some finite bouquet Γ with $r \geq 2$ petals. Let $G = \langle a, t_1, ..., t_r \mid t_i a^{m_i} {t_i}^{-1} = a^{n_i} \rangle$ be its presentation for integers $n_i \neq 0 \neq m_i$ and $1 \leq i \leq r$. Suppose there is at least one i with $m_i = 1$ and at least one j such that $n_j \neq m_j$. Then the normal closure*

$$N = \ll t_i t_j {t_i}^{-1} {t_j}^{-1} \mid 1 \leq i, j \leq r \gg \; \lhd G$$

is not finitely generated. Moreover, the subgroup $H = \langle t_i t_j {t_i}^{-1} {t_j}^{-1} \mid 1 \leq i, j \leq r \rangle$ is not normal in G.

Demonstração. Since there are no segments in Γ, it is reduced, and since $r \geq 2$, Γ is not a solvable Baumslag Solitar group, so we can apply Theorem 11.14. On one hand, the fact $n_j \neq m_j$ for some j means that some petal is a killing loop in Γ, so $\Sigma^1(G) = \emptyset$ by Theorem 11.14. On the other hand, by the definition of N the quotient group G/N has the following presentation:

$$G/N = \langle a, t_1, ..., t_r \mid t_i t_j = t_j t_i, \; t_i a^{m_i} {t_i}^{-1} = a^{n_i} \rangle .$$

Now, let i be the index such that $m_i = 1$. Then we have the relation $t_i a {t_i}^{-1} = a^{n_i}$ in G/N. The generator t_i is torsion-free in the abelianized of G/N, so let $[\chi] \in S(G/N)$ be any character with $\chi(t_i) < 0$. Then by using the Geometric Criterion 3.22 and the same path construction of the proof of Theorem 5.2 we can show that $[\chi] \in \Sigma^1(G/N)$. If N was finitely generated, then by Proposition 3.25 we would have $[\chi \circ \pi] \in \Sigma^1(G) = \emptyset$, a contradiction. This shows the first claim of the corollary. If H was normal in G we would have by definition $H = \ll H \gg = N$, then N would be finitely generated because H is, which contradicts what we have just shown. This completes the proof. □

The corollary above is interesting for at least two reasons. First, the Σ^1 invariant is most known to be able to provide information about subgroups of a group G containing G'. Our corollary, however, shows that Σ^1 can give information about subgroups which doesn't necessarily contain G', as it is the case of N above (because G/N is not abelian). Second, it shows in particular that every generalized solvable Baumslag-Solitar group Γ_n can be seen as a quotient of a GBS group by some infinitely generated subgroup.

11.2 Generalizing for GBS_n groups

Based on the generality of the useful Theorem 11.4 we had the idea of trying to generalize the results in the previous section to the GBS_n groups.

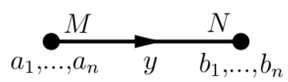
Definition 11.19. Given $n \geq 1$, a graph of $\mathbb{Z}^n$'s is a graph of groups (G, Γ) with $G_P = \mathbb{Z}^n$ and $G_y = \mathbb{Z}^n$ for all vertices P and all edges y of Γ. Here, $\mathbb{Z}^n = \mathbb{Z} \oplus ... \oplus \mathbb{Z}$ is the direct sum of n copies of $\mathbb{Z}$.

Definition 11.20. Given $n \geq 1$, a GBS_n group is the fundamental group of a finite connected graph of $\mathbb{Z}^n$'s.

Observation 11.21. Note that a GBS group is then a GBS_1 group.

For every edge y we have two monomorphisms $f_y : G_y = \mathbb{Z}^n \hookrightarrow \mathbb{Z}^n = G_{t(y)}$ and $f_{\overline{y}} : G_y = \mathbb{Z}^n \hookrightarrow \mathbb{Z}^n = G_{o(y)}$ that, due to the linear-like behavior of $\mathbb{Z}^n$, are uniquely determined by the n images $f_y(e_i)$ (respectively, $f_{\overline{y}}(e_i)$) of the free-abelian generators $e_1, ..., e_n$ of $\mathbb{Z}^n$. This time, to get a presentation for the GBS_n group G of the graph of $\mathbb{Z}^n$'s (G, Γ) we must choose n vertex letters $a_1, ..., a_n$ for each vertex P, corresponding to the n generators of $G_P = \langle a_1, ..., a_n \rangle \simeq \mathbb{Z}^n$. Also, given an orientation E^+ and a maximal tree T of Γ, choose one stable letter t_y for each oriented edge y outside T. These vertex and stable letters are the generators of G. The relations are: all the commutators $a_i a_j = a_j a_i$ between two vertex letters associated to the same vertex P (because they commute in G_P), all the relations $f_y(e_i) = f_{\overline{y}}(e_i), 1 \leq i \leq n$ for the oriented edges y inside T and all the relations $t_y f_y(e_i) {t_y}^{-1} = f_{\overline{y}}(e_i), 1 \leq i \leq n$ for the ones outside T.

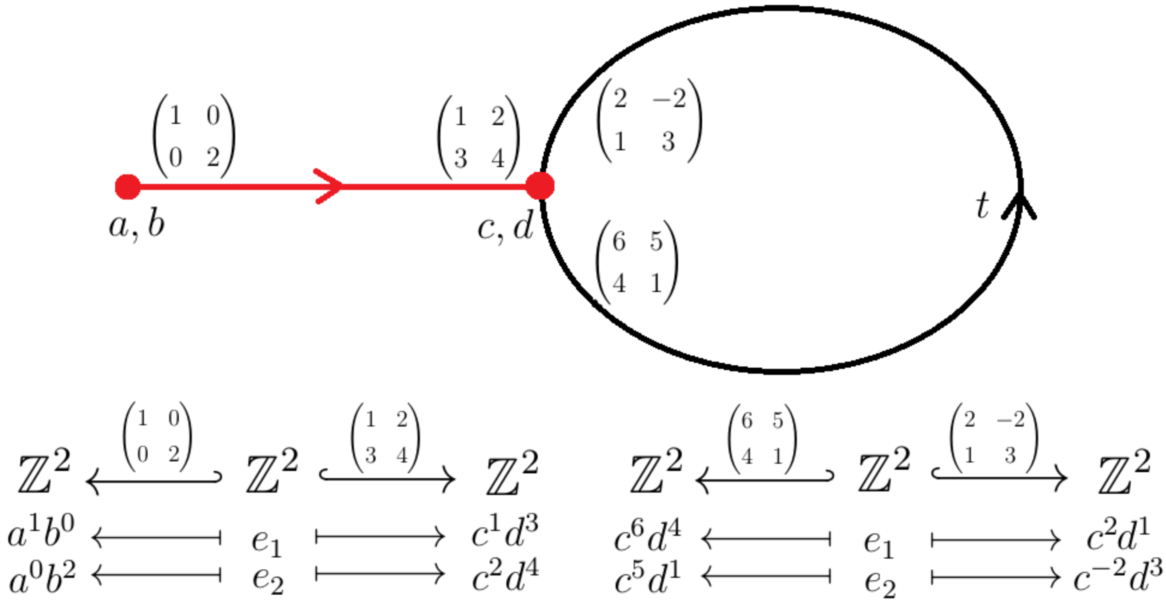
For each edge y, if we define the $n \times n$ matrices $M = \begin{bmatrix} f_{\overline{y}}(e_1) & ... & f_{\overline{y}}(e_n) \end{bmatrix}$ and $N = \begin{bmatrix} f_y(e_1) & ... & f_y(e_n) \end{bmatrix}$ whose i^{th} columns are the n-vectors $f_{\overline{y}}(e_i)$ (respectively, $f_y(e_i)$), as in linear algebra, and if the a_i and b_i are the corresponding vertex letters, we can use the following notation:



We can see these monomorphisms $f_y : \mathbb{Z}^n \to \mathbb{Z}^n$ as restrictions of the linear maps $\mathbb{R}^n \to \mathbb{R}^n$ with corresponding integer matrix N. Because of this, f_y being injective is equivalent to the vectors $f_y(e_i)$ being linearly independent vectors in $\mathbb{R}^n$, which is equivalent to $\det(N) \neq 0$. Moreover, the index of $\mathrm{im}(f_y)$ in $\mathbb{Z}^n = G_{t(y)}$ is exactly $|\det(N)|$, and the index of $\mathrm{im}(f_{\overline{y}})$ in

$\mathbb{Z}^n = G_{o(y)}$ is exactly $|\det(M)|$ (see Example 1.3), so both are always finite index subgroups. To summarize all of this,

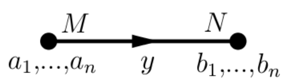
$$f_y : \mathbb{Z}^n \to \mathbb{Z}^n \text{ injective} \Leftrightarrow N \text{ with l.i. columns} \Leftrightarrow N \text{ with l.i. lines} \Leftrightarrow \det(N) \neq 0.$$



$$G = \langle a, b, c, d, t \mid ab = ba, cd = dc, a = cd^3, b^2 = c^2d^4, tc^2dt^{-1} = c^6d^4, tc^{-2}d^3t^{-1} = c^5d \rangle.$$

Figura 11.1: An example of graph of $\mathbb{Z}^2$'s, its corresponding monomorphisms and the associated GBS_2 group presentation.

Let y be the edge as we defined previously.



Note that y may be a loop; in this case, $a_i = b_i$ for all i. Let $M = (m_{ij})_{ij}$ and $N = (\eta_{ij})_{ij}$ be the associated matrices. Then, by definition of the presentation of G, y gives rise to exactly the following relations in the abelianized group G^{ab}:

$$\begin{cases} a_1{}^{m_{11}} a_2{}^{m_{21}} ... a_n{}^{m_{n1}} = b_1{}^{\eta_{11}} b_2{}^{\eta_{21}} ... b_n{}^{\eta_{n1}}, \\ a_1{}^{m_{12}} a_2{}^{m_{22}} ... a_n{}^{m_{n2}} = b_1{}^{\eta_{12}} b_2{}^{\eta_{22}} ... b_n{}^{\eta_{n2}}, \\ \qquad\qquad ... \\ a_1{}^{m_{1n}} a_2{}^{m_{2n}} ... a_n{}^{m_{nn}} = b_1{}^{\eta_{1n}} b_2{}^{\eta_{2n}} ... b_n{}^{\eta_{nn}}. \end{cases}$$

Then, applying an arbitrary character χ to these equations we get that every character must satisfy the system

$$\begin{cases} m_{11}\chi(a_1) + m_{21}\chi(a_2) + ... + m_{n1}\chi(a_n) = \eta_{11}\chi(b_1) + \eta_{21}\chi(b_2) + ... + \eta_{n1}\chi(b_n), \\ m_{12}\chi(a_1) + m_{22}\chi(a_2) + ... + m_{n2}\chi(a_n) = \eta_{12}\chi(b_1) + \eta_{22}\chi(b_2) + ... + \eta_{n2}\chi(b_n), \\ \qquad\qquad\qquad\qquad ... \\ m_{1n}\chi(a_1) + m_{2n}\chi(a_2) + ... + m_{nn}\chi(a_n) = \eta_{1n}\chi(b_1) + \eta_{2n}\chi(b_2) + ... + \eta_{nn}\chi(b_n). \end{cases}$$

Since all the coordinates $\chi(a_i), \chi(b_i)$ are real numbers, this is equivalent to the real homogeneous linear system

$$Ax = 0,$$

where $x = (\chi(a_1), ..., \chi(a_n), \chi(b_1), ..., \chi(b_n))$ is the column vector of variables and

$$A = \begin{bmatrix} m_{11} & m_{21} & ... & m_{n1} & -\eta_{11} & -\eta_{21} & ... & -\eta_{n1} \\ m_{12} & m_{22} & ... & m_{n2} & -\eta_{12} & -\eta_{22} & ... & -\eta_{n2} \\ & & & & ... & & & \\ m_{1n} & m_{2n} & ... & m_{nn} & -\eta_{1n} & -\eta_{2n} & ... & -\eta_{nn} \end{bmatrix} = \begin{bmatrix} M^T & \mid & -N^T \end{bmatrix}.$$

Now, we know that all the lines and columns of M and N are by definition linearly independent. So, by applying the Gaussian elimination process to the matrix A we can obtain an equivalent reduced matrix having the form

$$A' = \begin{bmatrix} 1 & 0 & ... & 0 & -\alpha_{11} & -\alpha_{12} & ... & -\alpha_{1n} \\ 0 & 1 & ... & 0 & -\alpha_{21} & -\alpha_{22} & ... & -\alpha_{2n} \\ & & & & ... & & & \\ 0 & 0 & ... & 1 & -\alpha_{n1} & -\alpha_{n2} & ... & -\alpha_{nn} \end{bmatrix}$$

for some $\alpha_{ij} \in \mathbb{R}$. Then the equivalent linear system $A'x = 0$ can be written in the coordinate form

$$\begin{cases} \chi(a_1) = \alpha_{11}\chi(b_1) + \alpha_{12}\chi(b_2) + ... + \alpha_{1n}\chi(b_n), \\ \chi(a_2) = \alpha_{21}\chi(b_1) + \alpha_{22}\chi(b_2) + ... + \alpha_{2n}\chi(b_n), \\ \qquad\qquad ... \\ \chi(a_n) = \alpha_{n1}\chi(b_1) + \alpha_{n2}\chi(b_2) + ... + \alpha_{nn}\chi(b_n), \end{cases}$$

which is finally equivalent to the system

$$\overline{a} = B_y\overline{b},$$

where $\overline{a} = (\chi(a_1), ..., \chi(a_n))$, $\overline{b} = (\chi(b_1), ..., \chi(b_n))$ are the column vectors and $B_y = (\alpha_{ij})_{ij}$ acts like a change of basis matrix, allowing us to write the coordinates $\chi(a_i)$ in terms of the $\chi(b_i)$. Note that this change of basis matrix is the same for every character χ and it is invertible, since it was obtained by applying Gaussian elimination to the invertible (over $\mathbb{R}$) matrix $-N^T$. The linear system above is the only obstruction y can impose to the $\chi(a_i)$ and $\chi(b_i)$, by definition. We can then define:

Definition 11.22. Fix a vertex P of Γ with vertex letters $a_1, ..., a_n$. If $\gamma = y_1, ..., y_k$ is a closed path in P, the matrix associated to γ is $M_\gamma = B_{y_1}B_{y_2}...B_{y_k}$.

If γ is a closed path in P as in the above definition, then by induction on the argument "$\overline{a} = B_y\overline{b}$" above we get that $\overline{a} = B_{y_1}B_{y_2}...B_{y_k}\overline{a}$, that is, $M_\gamma\overline{a} = \overline{a}$, or $(M_\gamma - Id)\overline{a} = 0$. This equation can kill some coordinates $\chi(a_i)$ of the vector $\overline{a} = (\chi(a_1), ..., \chi(a_n))$, that is, can imply that $\chi(a_i)$ is dependent of the other $\chi(a_j)$. From linear algebra we know that the number of dependent variables of a homogeneous linear system is the rank of the matrix associated, because it is exactly the number of pivots in its reduced Gaussian form. So γ will kill exactly $rk(M_\gamma - Id)$ coordinates $\chi(a_i)$. This is the basic principle to understanding the following theorem, which computes the dimension of the character sphere:

Theorem 11.23. *Let G be a GBS_n group associated to the graph of $\mathbb{Z}^n$'s (G, Γ), with orientation E^+ and maximal tree T. Fix any vertex P of Γ with vertex letters $a_1, ..., a_n$. Let $y_1, ..., y_k$ be the oriented edges outside T with stable letters $t_1, ..., t_k$ associated. For each $1 \leq i \leq k$, let γ_i be a closed path in P that rounds the circuit containing y_i once. Then $S(G) \simeq S^{n-r+k-1}$, where*

$$r = rk \begin{bmatrix} M_{\gamma_1} - Id \\ \vdots \\ M_{\gamma_k} - Id \end{bmatrix}.$$

The homeomorphism is given by

$$\begin{aligned} S(G) &\longrightarrow S^{n-r+k-1} \\ [\chi] &\longmapsto \frac{(\chi(a_{i_1}), ..., \chi(a_{i_{n-r}}), \chi(t_1), ..., \chi(t_k))}{\|(\chi(a_{i_1}), ..., \chi(a_{i_{n-r}}), \chi(t_1), ..., \chi(t_k))\|} \end{aligned}$$

where $a_{i_1}, ..., a_{i_{n-r}}$ are the vertex letters which freely generates G^{ab}.

Demonstração. First of all, remember that every edge y generates a linear dependence $\overline{a} = B_y\overline{b}$ between the coordinates $\chi(a_i)$ and $\chi(b_i)$ of its two collection of vertex letters, for each character χ. Since Γ is connected, by fixing the vertex P with the vertex letters a_i we get that all the vertex letter coordinates are linearly dependent only on $\chi(a_1), ..., \chi(a_n)$. The stable letters $\chi(t_1), ..., \chi(t_k)$ are always torsion-free in G^{ab}. So we are just left to see how many coordinates $\chi(a_i)$ are linearly independent, or equivalently, how many a_i are needed to freely generate G^{ab}. Let us call the number of linear dependent coordinates $\chi(a_i)$ as the number of kills. Since the relations in G^{ab} are given only between adjacent vertex letters by definition, the kills can only be obtained by closed paths in P (as we justified in the GBS case), by the linear systems $(M_\gamma - Id)\overline{a} = 0$ we showed. So, at first, we should compute the kills from every closed path in P. What we are going to do from now on is showing that the kills are all coming only from the paths $\gamma_1, ..., \gamma_k$.

Let $\gamma = y_1, ..., y_k$ be a closed path in P. Suppose $y_i, \overline{y_i}$ is a backtracking in γ. From linear algebra we know that the inverse of a basis change matrix is the basis change matrix in the opposite direction. So $B_{\overline{y_i}} = B_{y_i}^{-1}$ and then

$$M_\gamma = B_{y_1}...B_{y_{i-1}}B_{y_i}B_{\overline{y_i}}B_{y_{i+2}}...B_{y_k} = B_{y_1}...B_{y_{i-1}}B_{y_{i+2}}...B_{y_k} = M_{\gamma'},$$

where γ' is the closed path obtained by removing this backtracking from γ. Since the same associated matrix generates the same kills, it is enough compute the kills from γ'. By induction, we can remove all the backtrackings from γ and remain with the same kills. This shows that it is enough considering closed paths without backtrackings. Furthermore, if the path γ is contained in T then it is contractible. So the matrices B_{y_i} must cancel pairwise and $M_\gamma = Id$, which implies that the linear system $(M_\gamma - Id)\overline{a} = 0$ becomes $0 = 0$ and does not lead to any kill. So it is enough to consider closed paths without backtrackings which are not contained in T.

Now, Γ is a finite graph. So by [86], as a topological space, Γ has the same homotopy type of a finite bouquet Υ. This homotopy equivalence is obtained by contracting the tree T to the point P. So the number of "petals" in the bouquet is exactly the number of edges outside T, that is, k. Since the fundamental group is invariant by homotopy equivalence we get an isomorphism $\pi_1(\Gamma) \to \pi_1(\Upsilon)$ which maps the paths γ_i exactly to the k bouquet petals l_i.

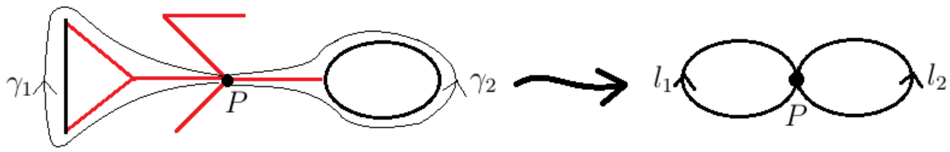


Figura 11.2: the contraction maps the γ_i to the petals l_i

But the l_i are the generators of $\pi_1(\Upsilon) \simeq *_{i=1}^{k}\mathbb{Z}$. So by going back in the isomorphism, every closed path γ in Γ is a finite concatenation of the γ_i and its inverses. If γ is not inside T, this concatenation is non-trivial.

Now we show that the kills of a concatenation of closed paths in P are consequences of the individual path kills. Let $\gamma = \sigma_1\sigma_2$ be a concatenation of two closed paths in P. Then $M_\gamma = M_{\sigma_1}M_{\sigma_2}$, by definition of M_γ. The kills generated by γ come from the system $M_\gamma\overline{a} = \overline{a}$. But this system is a consequence of the systems $M_{\sigma_i}\overline{a} = \overline{a}$. In fact, if these two systems are satisfied then

$$M_\gamma\overline{a} = M_{\sigma_1}M_{\sigma_2}\overline{a} = M_{\sigma_1}\overline{a} = \overline{a}.$$

This argument obviously work for a finite concatenation. Since the γ_i (and its inverses) generate all closed paths in P by finite concatenations, we only have to compute the kills from the γ_i and its inverses.

Finally, we show that the kills coming from a closed path and from its inverse path are the same. Indeed, we already know that $B_{\overline{y}} = B_y^{-1}$ for every edge. Then, if $\gamma = y_1, ..., y_k$ is closed, we have $\gamma^{-1} = \overline{y_k}, ..., \overline{y_1}$ and then $M_{\gamma^{-1}} = B_{\overline{y_k}}...B_{\overline{y_1}} = B_{y_k}^{-1}...B_{y_1}^{-1} = M_\gamma^{-1}$. Then the kills coming from the paths γ and γ^{-1} are, respectively, $M_\gamma\overline{a} = \overline{a}$ and $M_\gamma^{-1}\overline{a} = \overline{a}$. Since M_γ and M_γ^{-1} are invertible matrices, these systems are equivalent and therefore the kills are the same, so it is enough to compute only the kills from the γ_i. All these arguments showed that the number of dependent variables $\chi(a_i)$ come only from the equations $(M_{\gamma_i} - Id)\overline{a} = 0$ for $1 \leq i \leq k$, or the linear system

$$\begin{bmatrix} M_{\gamma_1} - Id \\ \vdots \\ M_{\gamma_k} - Id \end{bmatrix} \begin{bmatrix} \chi(a_1) \\ \vdots \\ \chi(a_k) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

From linear algebra we already commented that the number of dependent variables of a homogeneous linear system is the rank of the matrix associated, because it is exactly the number of pivots in its reduced Gaussian form. So the number of free variables $\chi(a_i)$ is exactly $n-r$ where r is the rank of the matrix above. Then the free generators of G^{ab} are some $a_{i_1}, ..., a_{i_{n-r}}$ and $t_1, ..., t_k$, and the theorem follows from Theorem 3.6. □

Following Corollary 11.13, the analogous situation for Σ^1 of GBS_n groups is the following:

Corollary 11.24. *Let G be a GBS_n group associated to the reduced graph of $\mathbb{Z}^n$'s (G,Γ) which is not an ascending HNN extension. Then*

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid (\chi(a_1), ..., \chi(a_n)) \neq (0, ..., 0)\}$$

where $a_1, ..., a_n$ are fixed vertex letters of a vertex P_0.

Demonstração. We want to use Corollary 11.5 again. Since $\mathbb{Z}^n$ is abelian we have $\Sigma^1(G_P) = S(G_P)$ for each vertex P by Corollary 3.15. The monomorphisms f_y and $f_{\overline{y}}$ maps the edge groups onto finite index subgroups of the vertex groups, the index being the absolute value of the determinant of the matrix associated, like we already commented in this section. Applying Corollary 11.5, we obtain that $[\chi] \in \Sigma^1(G) \iff \chi(G_{P_0}) \neq 0$ for some fixed vertex group G_{P_0}. Since $G_{P_0} = \mathbb{Z}^n = \langle a_1, ..., a_n \rangle$ for some vertex letters $a_1, ..., a_n$, we have $\chi(G_{P_0}) \neq 0 \iff (\chi(a_1), ..., \chi(a_n)) \neq (0, ..., 0)$, as desired. □

Four corollaries arise from the previous corollary and Theorem 11.23:

Corollary 11.25. *Let G be a GBS_n group associated to the reduced graph of $\mathbb{Z}^n$'s (G,Γ), with Γ a tree. Then $S(G) \simeq S^{n-1}$ and we have $\Sigma^1(G) = S(G)$ and $\Omega^1(G) = S(G)$.*

Demonstração. We are in a particular case of Theorem 11.23, where $k = 0$ and $r = 0$. If $a_1, ..., a_n$ are the vertex letters of the choosen vertex P, then because $k = 0$ the real numbers $\chi(a_1), ..., \chi(a_n)$ are the only coordinates determining a character $[\chi] \in S(G)$, so $\chi(a_i) \neq 0$ for at least one a_i. By Corollary 11.24, we have $\Sigma^1(G) = S(G)$ and therefore $\Omega^1(G) = S(G)$. □

Corollary 11.26. *Let G be a GBS_n group associated to the reduced graph of $\mathbb{Z}^n$'s (G,Γ). Suppose that (G,Γ) is not an ascending HNN extension. If $r = n$ (see Theorem 11.23), then $\Sigma^1(G) = \emptyset$ and $\Omega^1(G) = \emptyset$.*

Demonstração. By Theorem 11.23 we have exactly $n-r$ free a_i-coordinates determining a character $[\chi] \in S(G)$. In our case, we have no free coordinates, that is, $\chi(a_i) = 0$ for $1 \leq i \leq n$, for every character $[\chi]$. It follows directly from Corollary 11.24 that $\Sigma^1(G) = \emptyset$ and therefore $\Omega^1(G) = \emptyset$. □

Corollary 11.27. *Let G be a GBS_n group associated to the reduced graph of $\mathbb{Z}^n$'s (G,Γ). Suppose that (G,Γ) is not a tree and it is not an ascending HNN extension. If $n-r = 1$ (see Theorem 11.23), then $\Omega^1(G)$ consists of two antipodal rational points.*

Demonstração. We have exactly one free a_i-coordinate determining whether a character $[\chi] \in S(G)$ is in $\Sigma^1(G)$. Then $\Sigma^1(G)$ assumes the shape of the third case of Theorem 11.14, that is, $\Sigma^1(G)$ is the disjoint union of two antipodal open hemispheres in $S(G)$. It follows then easily that $\Omega^1(G)$ consists of two antipodal and rational points $[\chi], [-\chi]$, where $\chi(a_i) = 1$ and χ vanishes the other generators. □

Observation 11.28. The last three corollaries above are the respective generalizations of the three cases of Theorem 11.14. The next one is a generalization of Corollary 11.17.

Corollary 11.29. *Let G be a GBS_n group associated to the reduced graph of $\mathbb{Z}^n$'s (G, Γ). Suppose that (G, Γ) is not an ascending HNN extension. If $r < n$ and $k = 1$ (see Theorem 11.23), then there exists a normal subgroup $H \lhd Aut(G)$ with index 2 such that $R(\varphi) = \infty$ for every automorphism $\varphi \in H$.*

Demonstração. By 11.23 we have the homeomorphism

$$\begin{aligned} S(G) &\longrightarrow S^{n-r} \\ [\chi] &\longmapsto \frac{(\chi(a_{i_1}), ..., \chi(a_{i_{n-r}}), \chi(t_1))}{\|(\chi(a_{i_1}), ..., \chi(a_{i_{n-r}}), \chi(t_1))\|}. \end{aligned}$$

From Corollary 11.24 we know that $[\chi] \in \Sigma^1(G)$ if and only if $\chi(a_{i_j}) \neq 0$ for some $1 \leq j \leq n-r$. Then the points $[\chi]$ in the complement $\Sigma^1(G)^c$ corresponds only to the two antipodal points $(0, ..., 0, 1)$ and $(0, ..., 0, -1)$ in the sphere S^{n-r}. By Corollary 3.39, there is a normal subgroup $H \lhd Aut(G)$ with finite index such that $R(\varphi) = \infty$ for every automorphism $\varphi \in H$. But, in the proof of that corollary, one can see that the index of H is the number of possible permutations of the points in $\Sigma^1(G)^c$, which is 2 in this case. Then H has index 2 and we conclude the corollary. □

Open question: is it possible to use Cashen and Levitt's Theorem 11.4 to compute some examples of the Σ^1 invariant of hyperbolic and relatively hyperbolic groups? As we see in chapters 8 and 9, examples of these groups can be easily constructed as fundamental groups of graphs of groups.

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