UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Characterization of the continuity of strongly singular Calderón–Zygmund type operators in Hardy spaces.

Claudio Henrique Machado Vasconcelos Filho

São Carlos - SP

Abril de 2023

UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Characterization of the continuity of strongly singular Calderón–Zygmund type operators in Hardy spaces.

Claudio Henrique Machado Vasconcelos Filho

Orientador: Prof. Dr. Tiago Henrique Picon

Tese apresentada ao Programa de Pós-Graduação em Matemática da Universidade Federal de São Carlos como parte dos requisitos para a obtenção do Título de Doutor em Matemática.

São Carlos - SP

Abril de 2023

UNIVERSIDADE FEDERAL DE SÃO CARLOS

Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Matemática

Folha de Aprovação

Defesa de Tese de Doutorado do candidato Claudio Henrique Machado Vasconcelos Filho, realizada em 04/04/2023.

Comissão Julgadora:

Prof. Dr. Tiago Henrique Picon (USP)

Prof. Dr. Gustavo Hoepfner (UFSCar)

Prof. Dr. Lucas da Silva Oliveira (UFRGS)

Prof. Dr. Laurent Moonens (Paris-Sud)

Prof. Dr. Emmanuel Russ (UGA)

O Relatório de Defesa assinado pelos membros da Comissão Julgadora encontra-se arquivado junto ao Programa de Pós-Graduação em Matemática.

À minha família.

Acknowledgement

Agradeço a Deus pelas oportunidades e pela força ao longo desse caminho.

À toda minha família pelo apoio incondicional, em especial aos meus pais, tios, irmãs e avós Angela, Lúcia e Luiz (*in memoriam*).

Ao Prof. Tiago Henrique Picon pelo direcionamento ao realizar esse trabalho, confiança, apoio, ajuda e por sempre me incentivar a seguir em frente e acreditar em meu potencial. Minha eterna gratidão e respeito.

Ao Prof. Laurent Moonens, que tão bem me recebeu na Université Paris-Saclay, pelos ensinamentos em GMT e paciência. Merci beaucoup!

À Prof. Galia Dafni pelas oportunidades, ensinamentos e apoio ao longo de grande parte da realização desse trabalho. Thank you!

Aos colaboradores Chun Ho Lau, Marcelo de Almeida e Mateus Sousa, pessoas nas quais tive a grande oportunidade de trabalhar e aprender.

A todos os professores que tive desde a graduação em Matemática Aplicada a Negócios na USP-Ribeirão Preto. Em especial a Profa. Maria Aparecida Bená por me apresentar a Análise Matemática desde muito cedo e me apoiar. Agradeço também aos professores do PPGM-UFSCar pelos ensinamentos.

Aos amigos que a matemática me proporcionou conhecer e compartilhar tantos conhecimentos e momentos de alegria e consolo: Fernanda Elias, Estefani, Carolinne, Rafael, Victor Hugo, Ronaldo, Luiz e Renan.

Aos amigos não-matemáticos que compartilhei momentos tão inesquecíveis na Maison du Brésil: Anita, Maria, Gabriel, Thaís e Thaysa.

À Aurélie, Alexane e Marianne por me receberem tão bem em Montréal, tornando meus últimos meses no Canadá inesquecíveis.

Ao Charles por estar ao meu lado pacientemente e me apoiar durante a reta final deste trabalho, independente de qual parte do mundo eu esteja. Merci Beaucoup!

À comissão avaliadora composta pelos professores Gustavo Hoepfner, Lucas Oliveira, Laurent Moonens e Emmanuel Russ pela leitura desse trabalho e sugestões.

À CAPES pelo apoio financeiro durante o doutorado. Ao programa CAPES PrInt por incentivar a internacionalização de estudantes de doutorado e pelo apoio financeiro durante a visita à Université Paris-Saclay. Aos programas MITACS e PBEEE do governo Canadense pelo apoio financeiro durante visita à Concordia University.

Resumo

Nesta tese caracterizamos a continuidade de operadores de Calderón–Zygmund fortemente singulares do tipo σ em espaços de Hardy, espaços de Hardy com peso e espaços de Hardy–Morrey no contexto do resultado apresentado por Coifman e Meyer em [64, Capítulo 7, Proposição 4] para operadores clássicos de Calderón–Zygmund. Em particular, consideramos condições integrais do tipo Hörmander sobre o núcleo associado a tais operadores. Exemplos de operadores dessa natureza incluem operadores pseudo-diferenciais $OpS^{m}_{\sigma,\nu}(\mathbb{R}^n)$ e operadores associados a δ-núcleos do tipo σ, introduzidos por Álvarez e Milman em [5].

O método para a obtenção das propriedades de continuidade remete a decomposição atômica e molecular de tais espaços. Em particular, para os espaços de Hardy locais $h^p(\mathbb{R}^n)$ no qual $0 < p \leqslant 1$, apresentamos uma nova definição para átomos e moléculas assumindo condições de cancelamento mais fracas e apropriadas, estendendo e unificando trabalhos anteriores apresentados em [19, 22, 23, 50]. Como aplicação, provamos também uma versão não-homogênea da desigualdade de Hardy em $h^p(\mathbb{R}^n)$ e condições necessárias e suficientes para a continuidade de operadores do tipo Calderón-Zygmund não-homogêneo nestes espaços.

Palavras-chave: Espaços de Hardy, espaços de Hardy locais, pesos na classe de Muckenhoupt, decomposição molecular, operadores de Calderón–Zygmund, operadores pseudo-diferenciais.

Abstract

In this thesis, we characterize the continuity of strongly singular Calderón-Zygmund operators of type σ in Hardy spaces, weighted Hardy spaces and Hardy–Morrey spaces in the spirit of Coifman-Meyer's result [64, Chapter 7, Proposition 4]. In particular, we consider weaker integral Hörmander-type conditions on the kernel. Calderón–Zygmund operators of this type include appropriate classes of pseudodifferential operators $OpS^{m}_{\sigma,\nu}(\mathbb{R}^n)$ and operators associated to standard δ-kernels of type σ introduced by Álvarez and Milman in [5].

The method to obtain the boundedness properties refers to the atomic and molecular decomposition of such spaces. In particular, in order to obtain it for local Hardy spaces $h^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$, we present a new approach to atoms and molecules assuming weaker cancellation conditions, extending and unifying previous results presented in [19, 22, 23, 50]. As applications, we prove a non-homogeneous version of Hardy's inequality in $h^p(\mathbb{R}^n)$ and improved necessary and sufficient conditions for the continuity of inhomogeneous Calderón-Zygmund type operators on these spaces.

Keywords: Hardy spaces, local Hardy spaces, Muckenhoupt weights, molecular decomposition, Calderón–Zygmund operators, pseudodifferential operators.

Contents

List of Symbols

$\mathbb{N}$	set of natural numbers;
$\mathbb{Z}$	set of integers numbers;
$\mathbb{R}$	set of real numbers;
$\mathbb{C}$	set of complex numbers;
$\mathbb{R}^n$	n-dimensional Euclidean space;
$\mathbb{Z}_+$	set of non-negative integers;
$\mathbb{Z}_+^n$	$\mathbb{Z}_+^n = \{(a_1, \cdots, a_n) : a_j \in \mathbb{Z}_+ \text{ for all } j = 1, 2, \cdots, n\}$;
$\lvert A \rvert$	Lebesgue measure of the set $A \subset \mathbb{R}^n$;
$B(x_0, r)$	ball in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ and radius $r > 0$;
A^c	complement of the set A in $\mathbb{R}^n$ (also denoted by $\mathbb{R}^n \backslash A$);
χ_A	characteristic function of the set A;
$Q(x_Q, \ell_Q)$	cube in $\mathbb{R}^n$ centered in x_Q with side-length ℓ_Q;
$⨍_A f(x)dx$	$⨍_A f(x)dx = \frac{1}{\lvert A \rvert} \int_A f(x)dx$;
f_A	$f_A = ⨍_A f(x)dx$;

X^*	dual space of X;
$C(\mathbb{R}^n)$	space of continuous functions on $\mathbb{R}^n$
$\mathcal{M}_\varphi$	maximal function;
$\mathfrak{m}_\varphi$	local maximal function;
$\mathcal{M}_\mathcal{F}$	grand maximal function;
$\mathcal{M}_\varphi^*$	non-tangential maximal function;
M	Hardy-Littlewood maximal operator;
$\ell_p(\mathbb{R})$	space of sequences os real numbers belonging to ℓ^p;
$\mathcal{P}_N(\mathbb{R}^n)$	space os polynomials of degree less than or equal to N on $\mathbb{R}^n$;
A_t	set of measurable functions in the Muckenhoupt class;
$C^k(\mathbb{R}^n)$	space of functions f with $\partial^\alpha f$ continuous for all $\|\alpha\| \leqslant k$;
$C^\infty(\mathbb{R}^n)$	space of smooth functions $\bigcap_{k=1}^\infty C^k$ on $\mathbb{R}^n$
$C_c^\infty(\mathbb{R}^n)$	space of smooth functions $\bigcap_{k=1}^\infty C^k$ with compact support on $\mathbb{R}^n$;
$\mathcal{S}(\mathbb{R}^n)$	space of Schwartz functions on $\mathbb{R}^n$;
$\mathcal{S}'(\mathbb{R}^n)$	space of tempered distributions on $\mathbb{R}^n$;
$L^p(\mathbb{R}^n)$	Lebesgue space over $\mathbb{R}^n$;
$L^p_{loc}(\mathbb{R}^n)$	space of functions that lie in $L^p(K)$ for any compact set $K \subset \mathbb{R}^n$;
$L^p_w(\mathbb{R}^n)$	weighted Lebesgue space over $\mathbb{R}^n$;
$\mathcal{M}_q^\lambda(\mathbb{R}^n)$	Morrey space on $\mathbb{R}^n$

$H^p(\mathbb{R}^n)$	real Hardy space on $\mathbb{R}^n$;
$H^p_w(\mathbb{R}^n)$	weighted real Hardy space on $\mathbb{R}^n$;
$h^p(\mathbb{R}^n)$	real local Hardy space on $\mathbb{R}^n$;
$\mathcal{HM}^\lambda_q(\mathbb{R}^n)$	Hardy-Morrey spaces on $\mathbb{R}^n$;
$\dot{\Lambda}_\alpha(\mathbb{R}^n)$	homogeneous Lipschitz space on $\mathbb{R}^n$;
$\Lambda_\alpha(\mathbb{R}^n)$	non-homogeneous Lipschitz space on $\mathbb{R}^n$;
$BMO(\mathbb{R}^n)$	space of functions of bounded mean oscillation function on $\mathbb{R}^n$;
$bmo(\mathbb{R}^n)$	local BMO space on $\mathbb{R}^n$;
$L^{q,\psi}_k(\mathbb{R}^n)$	$\psi-$ generalized Campanato spaces on $\mathbb{R}^n$;

Introduction

The theory of Hardy spaces were introduced by Hardy [39] in the setting of analytic functions in the unitary disc. With the development of the Calderón–Zygmund theory (see [12]), C. Fefferman and Stein in [32] extended the classical theory of Hardy spaces in the complex plane to the Euclidean space $\mathbb{R}^n$, providing a definition for $H^p(\mathbb{R}^n)$ in terms of several maximal representations. To be precise, given $0 < p < \infty$, the space $H^p(\mathbb{R}^n)$ is defined as the set of tempered distributions f such that $\mathcal{M}_\varphi f \in L^p(\mathbb{R}^n)$, where $\mathcal{M}_\varphi$ denotes a special type of maximal function, equipped with the "norm" $\|f\|_{H^p} := \|\mathcal{M}_\varphi f\|_{L^p}$ (it will be a quasi-norm when $0 < p < 1$ and a norm otherwise). This characterization has led to deep interesting consequences and applications. The first one is that when $p > 1$ the spaces $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with equivalent norms, $H^1(\mathbb{R}^n) \subsetneq L^1(\mathbb{R}^n)$ with continuous embedding and when $0 < p \leqslant 1$, the spaces $H^p(\mathbb{R}^n)$ represents a richer functional space when compared to $L^p(\mathbb{R}^n)$ due to the existence of non-trivial dual, since

$$(H^p(\mathbb{R}^n))^* = \begin{cases} \dot{\Lambda}_{n\left(\frac{1}{p}-1\right)}(\mathbb{R}^n) & \text{if } 0 < p < 1, \\ BMO(\mathbb{R}^n) & \text{if } p = 1, \end{cases}$$

where $\dot{\Lambda}_\alpha(\mathbb{R}^n)$ denotes the homogeneous Lipschitz/Zygmund space (depending if $n(1/p-1)$ is integer or not) and $BMO(\mathbb{R}^n)$ is the space of bounded mean oscillation functions, introduced by John and Niremberg in [49]. Moreover, in several settings the space $H^1(\mathbb{R}^n)$ represents a good subspace for $L^1(\mathbb{R}^n)$. For instance, some singular integral operators, such as Riesz transform, fails to be bounded in $L^1(\mathbb{R}^n)$ but have nice continuity properties in $H^1(\mathbb{R}^n)$.

An important tool in the study of Hardy spaces for $0 < p \leqslant 1$, specially when dealing with the difficulties of working with the H^p norm, is the atomic decomposition. It allows one to express a tempered distribution $f \in H^p(\mathbb{R}^n)$ in terms of compactly supported L^∞ functions as

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \text{ in } H^p \text{ norm and consequently in } \mathcal{S}', \text{ with } \|f\|_{H^p} \approx \inf \left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \right\},$$

where $\{\lambda_j\}_j \in \ell^p(\mathbb{C})$, the infimum is taken over all such representations and each a_j satisfies some size conditions depending on p and the support, and vanishing moment conditions $\int a_j(x) x^\alpha dx = 0$ for all $|\alpha| \leqslant N_p := \lfloor n(1/p - 1) \rfloor$. This decomposition was first presented for $H^1(\mathbb{R}^n)$ in [32], to deal with the duality problem and it was extended for $p < 1$ and $n = 1$ by Coifman in [15]. The generalization for the $n-$dimensional case was proved by Latter in [53]. This decomposition has been extensively used over the years to simplify the proof of several properties of $H^p(\mathbb{R}^n)$ and provides an important method to prove the boundedness of linear operators acting on them. For instance, to show that a linear and continuous operator $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ can be extended to a bounded operator in $H^p(\mathbb{R}^n)$, it suffices to show that Ta is uniformly bounded on H^p norm.

Unfortunately, it is unreasonable to believe, in general, that the image of an operator by an atom is also an atom, since it may not preserve the compact support. This lead to consider rough atoms, called molecules, which satisfy equivalent properties of an atom, like the uniform control of its H^p norm, but the compact support is not required. These functions are more general than atoms and a decomposition of $H^p(\mathbb{R}^n)$ in terms of it still holds. The molecular theory on Hardy spaces was first studied by Coifman [14] in order to characterize the Fourier transform of distributions on $H^p(\mathbb{R})$, by Coifman, Taibleson and Weiss in the subsequent works [16, 75] and has also been extensively explored in more general setting (see for instance [76] for the molecular approach in Triebel-Lizorkin spaces). One of the fundamental applications of molecules, is that they provide another simple method to prove boundedness of operators in Hardy spaces, just by showing that T maps atoms into molecules.

Even though the Hardy spaces $H^p(\mathbb{R}^n)$ represents, in certain aspects, a good substitute for $L^p(\mathbb{R}^n)$ when $0 < p \leqslant 1$, there are still some unsatisfactory points about them. For instance, the spaces $H^p(\mathbb{R}^n)$ are not closed under multiplication by test functions, since it may not preserve the global vanishing moment condition. Moreover, they do not contain, in general, functions in the Schwartz space and they are not well defined in manifolds. For this reason, Goldberg in [37] introduced a local version of Hardy spaces, denoted by $h^p(\mathbb{R}^n)$ and called in the literature as localized or inhomogeneous Hardy spaces. These spaces are equivalent to $L^p(\mathbb{R}^n)$ when $p > 1$ and $H^p(\mathbb{R}^n)$ are embedded in $h^p(\mathbb{R}^n)$. Moreover, they satisfy the desired property: if $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $f \in h^p(\mathbb{R}^n)$ then $\varphi f \in h^p(\mathbb{R}^n)$. From a comparison lemma between $H^p(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$ (see [37, Lemma 4]), an analogous atomic decomposition was naturally established. These atoms are called Goldberg's atoms and vanishing moment conditions are only required for atoms supported in small balls (the size condition is analogous). In [19, Appendix B], Dafni showed that this local vanish moments conditions can be replaced by an approximate one, given in terms of a positive power of the radius in which the support of the atom is contained, that is, there exists a universal constant $C > 0$ such that

$$\left| \int_{B(x_0,r)} a(x)(x-x_0)^\alpha dx \right| \leqslant C\, r^\eta \tag{1}$$

for all $|\alpha| \leqslant N_p$ and some $\eta > 0$, where supp $(a) \subset B(x_0, r)$. In a subsequent work, Dafni and Hue [23] improved this estimate for $p = 1$, requiring in this case

$$\left| \int_{B(x_0,r)} a(x) dx \right| \leqslant \left[\log\left(1 + \frac{C}{r}\right) \right]^{-1}. \tag{2}$$

The study of singular integrals operators started back in the 50's in the works of Calderón and Zygmund (see for instance [12]). These operators arises naturally when studying some partial differential equations and were extensively studied over the past years. In its initial formulation, also refereed as first generation of Caderón–Zygmund operators, they were represented as translation invariant convolution operators and the generalization to the non-convolution setting is due to Coifman and Meyer [63, 64], and are refereed as standard Calderón–Zygmund operatos. Motivated by the study of multipliers operators associated to symbol of the type $e^{i|\xi|^\sigma}/|\xi|^\beta$, studied in the works [40, 78], and *weakly strongly singular convolution operators* introduced by C. Fefferman in [31], Álvarez and Milman in [5] introduced strongly singular non-convolution Calderón–Zygmund operators, extending classical standard Calderón–Zygmund operators. These operators are more singular near the diagonal, in comparison to standard Calderón–Zygmund operators, and the real variables methods developed by Calderón and Zygmund cannot be applied directly in this context. Moreover, besides the $L^2(\mathbb{R}^n)$ boundedness, the continuity from $L^q(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ in which

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n} \quad \text{for some } (1-\sigma)\frac{n}{2} \leqslant \beta < \frac{n}{2}$$

is also required. In the mentioned work, Alvarez and Milman showed that strongly singular Calderón–Zygmund operators whose kernel satisfies the Hölder type regularity

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leqslant C\frac{|y-z|^{\delta}}{|x-z|^{n+\frac{\delta}{\sigma}}}$$

for all $|x-z| \geqslant 2|y-z|^{\sigma}$, some $0 < \sigma \leqslant 1$ and $0 < \delta \leqslant 1$, are bounded on $H^p(\mathbb{R}^n)$ to itself for every $\frac{n}{n+1} < p_0 < p \leqslant 1$, where p_0 depends on the parameters of the operator, under the cancellation condition $T^*(1) = 0$, which essentially means that the image of an atom by the operator has integral zero. This condition is also necessary (see for instance [64] for the standard case) and it is related to the cancellation property required in $H^p(\mathbb{R}^n)$. In a subsequent work [6], the authors proved L^p inequalities for $1 \leqslant p < \infty$ under the $\sigma-$Hörmander condition on the kernel, a natural extension to the strongly singular case of the well known Hörmander condition, given by

$$\sup_{\substack{|y-z| \leqslant 1 \\ z \in \mathbb{R}^n}} \int_{|x-z| \geqslant 2|y-z|^{\sigma}} |K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| dx \leqslant C$$

and

$$\sup_{\substack{|y-z| > 1 \\ z \in \mathbb{R}^n}} \int_{|x-z| \geqslant 2|y-z|} |K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| dx \leqslant C.$$

Even though this condition is strong enough to show L^p inequalities for $p \geqslant 1$, the question for Hardy spaces when $0 < p \leqslant 1$ is more delicate and remains open even for standard operators (see [81] for some progress in $H^1(\mathbb{R}^n)$).

In this thesis we continue the program of Álvarez and Milman investigating continuity properties of strongly singular Calderón–Zygmund type operators in Hardy spaces. In particular, we provide a characterization of the continuity of such operators for all $0 < p \leqslant 1$, replacing the Hölder regularity of the kernel by an appropriate integral conditions on annulus and cancellation conditions.

Starting with $H^p(\mathbb{R}^n)$, in [67], we have shown that the continuity in $H^p(\mathbb{R}^n)$ remains true under an intermediate L^s-Hörmander type condition on annulus, called D_s condition, and given by

$$\sup_{\substack{r>1\\ z\in\mathbb{R}^n}} \sup_{|y-z|<r} \left(\int_{C_j(z,r)} |K(x,y)-K(x,z)|^s + |K(y,x)-K(z,x)|^s dx \right)^{1/s} \lesssim |C_j(z,r)|^{\frac{1}{s}-1}\, 2^{-j\delta}$$

and

$$\sup_{\substack{0<r<1\\ z\in\mathbb{R}^n}} \sup_{|y-z|<r} \left(\int_{C_j(z,r^\rho)} |K(x,y)-K(x,z)|^s + |K(y,x)-K(z,x)|^s dx \right)^{1/s} \lesssim |C_j(z,r^\rho)|^{\frac{1}{s}-1+\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}},$$

for $0<\rho\leqslant\sigma\leqslant 1\leqslant s<\infty$ and $\delta>0$. In particular, we have obtained the following continuity result:

Theorem A. *Let $T:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$ be a linear and continuous operator such that:*

(i) T extends to a continuous operator from $L^2(\mathbb{R}^n)$ to itself;

(ii) There exists $1\leqslant s_1<\infty$ such that T is associated to a kernel satisfying D_{s_1} condition;

(iii) T extends to a continuous operator from $L^q(\mathbb{R}^n)$ to $L^{s_2}(\mathbb{R}^n)$, for some $1<s_2<\infty$ and

$$\frac{1}{q}=\frac{1}{s_2}+\frac{\beta}{n}, \quad \textit{where} \quad n(1-\sigma)\left(1-\frac{1}{s_2}\right)\leqslant\beta<n\left(1-\frac{1}{s_2}\right).$$

Under such conditions, if $T^*(x^\alpha)=0$ for all $\alpha\in\mathbb{Z}^n_+$ such that $|\alpha|\leqslant\lfloor\delta\rfloor$, $p<s_1$ and $s_1\leqslant s_2$, then T can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to itself for every $p_0<p\leqslant 1$, where

$$\frac{1}{p_0}:=\frac{1}{s_2}+\frac{\beta\left[\dfrac{\delta}{\sigma}+n\left(1-\dfrac{1}{s_2}\right)\right]}{n\left(\dfrac{\delta}{\sigma}-\delta+\beta\right)}.$$

Conversely, if T is a bounded operator from $H^p(\mathbb{R}^n)$ to itself for every $p_0<p\leqslant 1$, then $T^*(x^\alpha)=0$ for every $|\alpha|\leqslant N_{p_0}$.

In the previous theorem, small values of p can be reached assuming big values of δ, which means more decay on the integral conditions of the kernel. This decay can also be expressed in terms of regularity of the kernel, assuming it satisfies the *derivative D_s condition with decay δ*, given by

$$\sup_{\substack{|y-z|<r\\ r>1}} \left(\int_{C_j(z,r)} |\partial_y^\gamma K(x,y) - \partial_y^\gamma K(x,z)|^s + |\partial_y^\gamma K(y,x) - \partial_y^\gamma K(z,x)|^s dx \right)^{1/s} \lesssim r^{-\lfloor \delta \rfloor} |C_j(z,r)|^{\frac{1}{s}-1} 2^{-j\delta}$$

and

$$\sup_{\substack{|y-z|<r\\ 0<r<1}} \left(\int_{C_j(z,r^\rho)} |\partial_y^\gamma K(x,y) - \partial_y^\gamma K(x,z)|^s + |\partial_y^\gamma K(y,x) - \partial_y^\gamma K(z,x)|^s dx \right)^{1/s}$$
$$\lesssim r^{-\lfloor \delta \rfloor} |C_j(z,r^\rho)|^{\frac{1}{s}-1+\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}}$$

for every $|\gamma| = \lfloor \delta \rfloor$.

We have also investigate the following extensions of Theorem A:

- In Theorem 2.4, we consider kernels satisfying D_s condition with decay δ;
- In Theorem 2.5, we extend it for kernels with $\theta-$modulus of continuity, inspired by the work of Yabuta in [79];
- In Theorem 2.10, we consider the continuity in the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$, when w is an appropriate Muckenhoupt weight;
- In Theorem 4.4, we prove the continuity in Hardy-Morrey spaces $\mathcal{HM}^\lambda_q(\mathbb{R}^n)$, considered in the work [26].

Examples of operators satisfying D_s conditions for $1 < s \leqslant 2$ have been considered in Proposition 2.3, where we show that the kernel of $OpS^m_{\sigma,\nu}(\mathbb{R}^n)$ with $0 < \sigma \leqslant 1$, $0 \leqslant \nu < 1$, $\nu \leqslant \sigma$ and $m \leqslant -n(1-\sigma)/2$ satisfies the D_s condition with derivatives for $1 < s \leqslant 2$, extending the classical case with $s = 1$, proved by Álvarez and Hounie in [4]. Our method to prove Theorem A and its extensions is based in the molecular theory of these spaces.

Inhomogeneous versions of standard Caderón–Zygmund operators have been considered by Ding, Han and Zhu in [29], where for some $\mu > 0$ the following strong decay on the kernel at the infinity is assumed

$$|K(x,y)| \leqslant C \ \min\left\{\frac{1}{|x-y|^n}, \ \frac{1}{|x-y|^{n+\mu}}\right\}, \quad \text{for every } x \neq y.$$

Necessary and sufficient conditions for the boundedness of such operators in $h^p(\mathbb{R}^n)$ have been established in [29, Theorem 1.1] for $\frac{n}{n+1} < p_{\delta,\mu} < p < 1$ under a similar T^* condition of the homogeneous case. As expected, the full cancellation is not necessary, so the sufficient condition for the boundedness is that $T^*(1) \in \dot{\Lambda}_{n(1/p-1)}$, which follows by a molecular approach of $h^p(\mathbb{R}^n)$ given by Komori in [50]. The necessity is that $T^*(1) \in \Lambda_{n(1/p-1)}$ and follows by duality argument. In order to investigate extensions of the previous continuity result for $0 < p \leqslant 1$, the first step is to extend the molecular theory of Komori, which holds only for $n/(n+1) < p < 1$.

In contrast to $H^p(\mathbb{R}^n)$, the molecular theory of $h^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$ was still not completely well established. Some initial formulation can be found in [7, Definition 2.4] for the setting of Chébli-Trimèche hypergroups, in which the vanish moment condition for molecules concentrated on small balls have been replaced by one like (1). Later, as mentioned before, Komori in [50] defined molecules for $n/(n+1) < p < 1$ replacing the vanishing moment condition by an uniform control of its size

$$\left| \int_{\mathbb{R}^n} M(x)dx \right| \leqslant C, \tag{3}$$

which is weaker then a power of its radius. However, estimate (3) holds trivially for the case $p = 1$. More recently, Dafni and Liflyand [22] studied molecules for $h^1(\mathbb{R})$, requiring a cancellation condition of the type (2).

Motivated by the previous works, in [21], we have established a new atomic and molecular characterization of $h^p(\mathbb{R}^n)$ for all $0 < p \leqslant 1$ in which the vanish moments of atoms and molecules for small balls are not required and are replaced by controls like (2) and (3), depending on the values of p. The key is to introduce inhomogeneous cancellation conditions on both atoms and molecules, by giving different cancellation properties when $p = n/(n+k)$ for $k \in \mathbb{Z}_+$ and $n/(n+k+1) < p < n/(n+k)$. Our cancellation is the following: suppose that supp $(a) \subset B(x_0, r)$, it satisfies the standard size condition and

$$\left|\int a(x)(x-x_0)^{\alpha}dx\right| \leqslant \begin{cases} \omega, & \text{if } |\alpha| < n(1/p-1), \\ \left[\log\left(1+\frac{1}{\omega r}\right)\right]^{-\frac{1}{p}}, & \text{if } |\alpha| = N_p = n(1/p-1). \end{cases}$$

for some $\omega \geqslant 0$ (for $\omega = 0$ the size is 0 and corresponds to the homogeneous case). Note that if $n/(n+k+1) < p < n/(n+k)$, then $N_p < n(1/p-1)$ and hence an uniform bound on the size of the moment condition of the atom is enough. This is not the case when $p = n/(n+k)$, where the log control is assumed for the moment of the order N_p. For molecules M concentrated in $B(x_0, r)$, the same estimate is imposed. This extend Komori's molecular approach for $0 < p \leqslant n/(n+1)$ and $p = 1$, giving appropriate bounds when $p = n/(n+k)$ for $k \in \mathbb{Z}_+$. Moreover, we use this molecular approach to prove the following inhomogeneous version of Hardy's inequality for any $0 < p \leqslant 1$, i.e. if $f \in h^p(\mathbb{R}^n)$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|^p}{(1+|\xi|)^{n(2-p)}} d\xi \leqslant C\|f\|_{h^p}^p$$

(see Theorem 1.3).

Using the molecular theory without cancellation developed in [21], we have obtained in the mentioned work and [20] the following characterization result for such operators:

Theorem B. *Let $0 < p \leqslant 1$ and T be a strongly singular inhomogeneous Calderón–Zygmund operator associated to a kernel satisfying the integral condition*

$$\sup_{\substack{0<r<1 \\ z\in\mathbb{R}^n}} \sup_{|y-z|<r} \left(\int_{C_j(z,r^{\rho})} |K(x,y)-K(x,z)|^s + |K(y,x)-K(z,x)|^s dx\right)^{1/s} \lesssim |C_j(z,r^{\rho})|^{\frac{1}{s}-1+\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}},$$

for some $\delta > 0$ and $1 \leqslant s \leqslant s_2$ with $p < s$. Then, T can be extended to a bounded operator from $h^p(\mathbb{R}^n)$ to itself for $\max\left\{\frac{n}{n+\mu},\ p_0\right\} < p \leqslant 1$, if, and only if there exists a constant $C > 0$ such that

$$f = T^*[(\cdot - x_0)^{\alpha}] \quad \textit{satisfies} \quad \left(⨍_B |f(x) - P_B^{N_p}(f)(x)|^2 dx\right)^{1/2} \leqslant C\,\Psi_{p,\alpha}(r), \tag{4}$$

for every ball $B = B(x_0, r) \subset \mathbb{R}^n$ such that $r < 1$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leqslant N_p$, where $P_B^{N_p}(f)$ is the *polynomial of degree* $\leqslant N_p$ *that has the same moments as* f *over* B *up to order* N_p *and*

$$\Psi_{p,\alpha}(t) := \begin{cases} t^{n\left(\frac{1}{p}-1\right)} & \text{if } |\alpha| < n\,(1/p-1)\,, \\ t^{n\left(\frac{1}{p}-1\right)}\left[\log\left(1+\dfrac{1}{t}\right)\right]^{-\frac{1}{p}} & \text{if } |\alpha| = n\,(1/p-1) = N_p. \end{cases}$$

In particular, the local condition (4) can also be replaced by the stronger one

$$\begin{cases} T^{*}[(x-x_0)^{\alpha}] \in \dot{\Lambda}_{n\left(\frac{1}{p}-1\right)}(\mathbb{R}^n) & \text{if } |\alpha| < n\,(1/p-1) \\ T^{*}[(x-x_0)^{\alpha}] \in L^{2,\Psi_p}_{N_p}(\mathbb{R}^n) & \text{if } |\alpha| = n\,(1/p-1) = N_p, \end{cases}$$

where $L^{2,\Psi_p}_{N_p}(\mathbb{R}^n)$ denotes the $\Psi-$Campanato space. Moreover, the necessity part of the previous theorem can be extended for more general operators T having the property that it maps atoms supported in $B \subset \mathbb{R}^n$ into what we call *pseudo-molecules*, which are tempered distributions $\mathfrak{M}$ that can be decomposed as $\mathfrak{M} = g + h$, in which $g \in h^p(\mathbb{R}^n)$ with supp $(g) \subset B$ and $h \in H^p(\mathbb{R}^n)$. Molecules are typical examples of pseudo-molecules. We have obtained the following result in [20]:

Theorem C. *Let* $0 < p \leqslant 1$ *and* T *to be a linear and bounded operator on* $h^p(\mathbb{R}^n)$ *that maps each* $(h^p, 2)$ *atom in* $h^p(\mathbb{R}^n)$ *into a pseudo-molecule centered in the same ball as the support of the atom. Then, the local Campanato-cancellation condition* (4) *must hold.*

The organization of this thesis is as follows. In Chapter 1, we describe all the results related to Hardy spaces. In particular, in Section 1.1 we prove a molecular decomposition for $H^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$ based on the size conditions presented in [5]. In Section 1.2 we prove an approximate atomic and molecular decomposition for $h^p(\mathbb{R}^n)$ and the inhomogeneous Hardy's inequality. In Section 1.3 we describe results for the weighted Hardy space $H^p_w(\mathbb{R}^n)$, in particular a molecular decomposition analogous as the one did in Section 1.1. Chapter 2 is devoted to present and prove the continuity results for strongly singular Calderón–Zygmund operators. In particular, in Section 2.1 we describe the tools to prove Theorem A, which will be proved in the Subsection 2.1.1, and describe its extensions in the subsequent Subsections 2.1.2 and 2.1.3. In Section 2.4 we present the proof of the continuity from $L^{\infty}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$, under the same conditions of Theorem A. In Chapter 3, we prove the necessity and sufficiency of Theorem B and provide the proof of Theorem C in Section 3.2. In Chapter 4, we provide the definition and basic properties of Hardy–Morrey spaces and we prove the analogous version of Theorem A into this setting.

Chapter 1

A new approach to atoms and molecules in Hardy spaces

The goal of this chapter is to present new tools that will be used during this work, regarding Hardy spaces defined in the Euclidean space $\mathbb{R}^n$, denoted by $H^p(\mathbb{R}^n)$, and its non-homogeneous version $h^p(\mathbb{R}^n)$.

Given $0 < p < \infty$, the real Hardy spaces $H^p(\mathbb{R}^n)$ were introduced by C. Fefferman and Stein in [32] and are characterized in terms of smooth maximal functions as follows: Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, we define the maximal operator by

$$\mathcal{M}_{\varphi} f(x) := \sup_{t>0} |f * \varphi_t(x)| = \sup_{t>0} |\langle f, \varphi_t(\cdot - x)\rangle|,$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$.

Definition 1.1. *Let $0 < p < \infty$. We say that a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $H^p(\mathbb{R}^n)$ if there exists $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \varphi \neq 0$ such that $\mathcal{M}_{\varphi} f \in L^p(\mathbb{R}^n)$.*

We endowed $H^p(\mathbb{R}^n)$ with the functional $\|f\|_{H^p} := \|\mathcal{M}_{\varphi} f\|_{L^p}$, which defines a quasi-norm for $0 < p < 1$ and a norm for $p \geqslant 1$ (for simplicity we always refer to it as a norm for $0 < p < \infty$). It is well known that $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ when $p > 1$, with equivalent norms, and $H^1(\mathbb{R}^n) \subsetneq L^1(\mathbb{R}^n)$ with continuous inclusion. Moreover, $H^p(\mathbb{R}^n)$ is a complete metric space with the distance $d(f,g) = \|f-g\|_{H^p}^p$ for any

$f, g \in H^p(\mathbb{R}^n)$ and $0 < p \leqslant 1$. Even though we state the definition of Hardy spaces for a particular choice of φ, it can be also defined using any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\int \varphi \neq 0$, resulting on equivalent norms. Elements of $H^p(\mathbb{R}^n)$ also satisfy a cancellation property depending on p. This cancellation can be elucidated in the following sense: If $f \in H^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} f(x)x^\alpha = 0$ whenever $|\alpha| \leqslant n(1/p - 1)$. For the proof of these properties and other general details on Hardy spaces see [35, 41, 73, 77].

An important class of examples of functions in Hardy spaces was introduced in [32] for $H^1(\mathbb{R}^n)$ and are called atoms. These functions allows one, for $0 < p \leqslant 1$, to express every tempered distribution in $H^p(\mathbb{R}^n)$ in terms of it. The 1-dimensional case was proved in [15] and the n-dimensional case generalized in [53].

Definition 1.2. *Let $0 < p \leqslant 1$ and $1 \leqslant s \leqslant \infty$ with $p < s$. We say that a measurable function a is a (p, s) atom in H^p if there exist a ball $B := B(x_0, r) \subset \mathbb{R}^n$ such that*

$$(i)\ \ supp\ (a) \subset B; \quad (ii)\ \ \|a\|_{L^s} \leqslant |B|^{\frac{1}{s}-\frac{1}{p}}; \quad (iii)\ \ \int a(x)x^\alpha dx = 0 \text{ for all } |\alpha| \leqslant N_p,$$

where $N_p := \lfloor n(1/p - 1) \rfloor$. For the limit case $s = \infty$, the condition (ii) is understood by $\|a\|_{L^\infty} \leqslant |B|^{-1/p}$.

It can be shown that (p, s) atoms are elements of $H^p(\mathbb{R}^n)$ and moreover there exists a constant $C = C(n, p, s) > 0$, independently of the atom, such that $\|a\|_{H^p} \leqslant C$ (see [35, Chapter III, Corollary 4.5]).

In the works [15, 53], the atomic decomposition for $H^p(\mathbb{R}^n)$ has been stated in terms of (p, ∞) atoms, however, since the atomic spaces generated by such atoms are equivalent for any $1 \leqslant s \leqslant \infty$ (see the proof of [35, Chapter III, Theorem 4.10 p. 283]), one can choose the most convenient one.

Theorem 1.1 ([35, Theorem 4.10 p. 283])**.** *Let $0 < p \leqslant 1 \leqslant s \leqslant \infty$ with $p < s$. If $f \in H^p(\mathbb{R}^n)$, then there exist a sequence $\{a_j\}_j$ of (p, s) atoms in H^p and $\{\lambda_j\}_j \in \ell^p(\mathbb{C})$ such that*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \ \ with \ \ \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \lesssim \|f\|_{H^p},$$

where the convergence is given in H^p norm. Conversely, if $f \in \mathcal{S}'(\mathbb{R}^n)$ is such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$, in the sense of distributions, with $\{a_j\}_j$ a sequence of (p, s) atoms in H^p and $\{\lambda_j\}_j \in \ell^p(\mathbb{C})$, then $f \in H^p(\mathbb{R}^n)$

and $\|f\|_{H^p} \lesssim \left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{1/p}$. *In particular* $\|f\|_{H^p} \approx \inf\left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{1/p}$, *where the infimum is taken over all such atomic representations.*

The previous theorem turned out to be very convenient since several properties and applications of Hardy spaces can be reduced in some sense of showing it for atoms. For instance, if $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a linear and continuous operator, its extension and continuity in $H^p(\mathbb{R}^n)$ can be established by just verifying that $\|Ta\|_{H^p} \leqslant C$ uniformly whenever a is a (p, s) atom in H^p. Bownik in [10] showed this is not always true by exhibiting an example of a linear operator that maps $(1, \infty)$ atoms uniformly, i.e. $\|Ta\|_{H^1} \leqslant C$ independently of a, but has not bounded extension in $H^1(\mathbb{R}^n)$. This example is in some sense pathological, see for instance the works [62, 70, 80] where this question was addressed with more details.

1.1 Molecular decomposition

In this section we describe the molecular theory of $H^p(\mathbb{R}^n)$, which originates from the works [14, 16, 75]. As formulated in [75], given $0 < p \leqslant 1 \leqslant s < \infty$ with $p < s$, let

$$\varepsilon > \max\left\{\frac{N_p}{n},\ \frac{1}{p} - 1\right\},\quad a = 1 - \frac{1}{p} + \varepsilon, \text{ and } b = 1 - \frac{1}{s} + \varepsilon. \tag{1.1}$$

A measurable function $\mathfrak{M}$ is called a (p, s, ε) molecule in H^p centered at $x_0 \in \mathbb{R}^n$ if $\mathfrak{M}$ and $\mathfrak{M}\,|\cdot - x_0|^{nb}$ belong to $L^s(\mathbb{R}^n)$,

$$\|\mathfrak{M}\|_{L^s}^{a/b}\ \ \|\mathfrak{M}\,|\cdot - x_0|^{nb}\|_{L^s}^{1-a/b} < \infty \ \text{ and } \int_{\mathbb{R}^n}\mathfrak{M}(x)x^{\alpha}dx = 0 \quad \text{for all } |\alpha| \leqslant N_p. \tag{1.2}$$

The molecules we will present in the next definition follow the formulation presented in [5, Definition 1.1] for $n/(n+1) < p \leqslant 1$, where a particular size of the L^s norm of the molecule is given, and it is a particular case of the one described above.

Definition 1.3. *Let* $0 < p \leqslant 1 \leqslant s < \infty$ *with* $p < s$ *and* $\lambda > n\left(\frac{s}{p} - 1\right)$. *A measurable function* M *is a* (p, s, λ) *molecule in* H^p *if there exist a ball* $B = B(x_0, r) \subset \mathbb{R}^n$ *and a universal constant* $C > 0$ *such that*

M1. $\|M\|_{L^s(B)} \leqslant C\,|B|^{\frac{1}{s}-\frac{1}{p}}$;

M2. $\|M\,|\cdot - x_0|^{\frac{\lambda}{s}}\|_{L^s(B^c)} \leqslant C\,|B|^{\frac{\lambda}{ns}+\frac{1}{s}-\frac{1}{p}}$;

M3. $\displaystyle\int M(x)x^\alpha dx = 0$ *for all* $|\alpha| \leqslant N_p$.

Considering $\lambda = n(s-1+\varepsilon s)$, we can see that the lower bound imposed on ε in (1.1) implies $\lambda > n(s/p-1)$. Morevover, the size condition imposed on (M1) and (M2) implies the uniform estimate (1.2).

If a is a (p,s) atom supported in $B \subset \mathbb{R}^n$, then it is a (p,s,λ) molecule centered in the same ball.

Remark 1.1.

(i) We may equivalently replace (M1) and (M2) simultaneously by global estimates, namely, for a universal constant $C > 0$

$$\text{M1'.} \quad \int_{\mathbb{R}^n} |M(x)|^s dx \leqslant C\,|B|^{1-\frac{s}{p}} \quad \text{and} \quad \text{M2'.} \quad \int_{\mathbb{R}^n} |M(x)|^s |x-x_0|^\lambda dx \leqslant C\,|B|^{\frac{\lambda}{n}+1-\frac{s}{p}}$$

or integrating in any dilation of B, that is, (M1) in $L^s(B(x_0,cr))$ and (M2) in $L^s(B(x_0,cr)^c)$ for some $c > 1$.

(ii) If (M2) holds for a given λ, then the analogous estimate holds for any $\lambda' < \lambda$ with the same constant $C > 0$. In fact, for $B = B(x_0,r)$

$$\int_{B^c} |M(x)|^s\,|x-x_0|^{\lambda'} dx \leqslant r^{\lambda'-\lambda} \int_{B^c} |M(x)|^s\,|x-x_0|^{\lambda} dx \leqslant C\,r^{\lambda'+n\left(1-\frac{s}{p}\right)}.$$

Combining with (M1), we also have the corresponding global bound on $\|M\,|\cdot - x_0|^{\frac{\lambda'}{s}}\|_{L^s(\mathbb{R}^n)}$.

Next, we show that condition (M3) is well defined when M satisfies (M1) and (M2). This is the analogous result of [5, Lemma 1.1] for $n/(n+1) < p \leqslant 1$.

Proposition 1.1. *Let M to be a function satisfying (M1) and (M2). Then, $M(x)x^\alpha$ is an absolutely integrable function for every $|\alpha| \leqslant N_p$.*

Proof. Suppose M satisfies (M1) and (M2) with respect to the ball $B = B(x_0,r)$. Split

$$\int_{\mathbb{R}^n} |M(x)\,x^\alpha| dx = \int_B |M(x)\,x^\alpha| dx + \int_{B^c} |M(x)\,x^\alpha| dx.$$

For the first integral, from Hölder inequality and (M1) we get

$$\int_B |M(x)\, x^\alpha| dx \leqslant \|x^\alpha\|_{L^\infty(B)}\, |B|^{1-\frac{1}{s}}\, \|M\|_{L^s(B)} \lesssim (r+|x_0|)^{|\alpha|}\, r^{n\left(1-\frac{1}{p}\right)} < \infty.$$

For the second,

$$\int_{B^c} |M(x)\, x^\alpha| dx \leqslant \sum_{|\gamma|\leqslant|\alpha|} C_{\alpha,\gamma}\, |x_0|^{|\alpha|-|\gamma|} \int_{B^c} |M(x)|\, |x-x_0|^{\frac{\lambda}{s}+\left(|\gamma|-\frac{\lambda}{s}\right)} dx$$
$$\leqslant \|M\,|\cdot - x_0|^{\frac{\lambda}{s}}\|_{L^s} \sum_{|\gamma|\leqslant|\alpha|} C_{\alpha,\gamma,x_0} \left(\int_{B^c} |x-x_0|^{\left(|\gamma|-\frac{\lambda}{s}\right)\frac{s}{s-1}} \right)^{1-\frac{1}{s}} \lesssim \sum_{|\gamma|\leqslant|\alpha|} C_{\alpha,\gamma,x_0}\, r^{|\gamma|+n\left(1-\frac{1}{p}\right)} < \infty$$

where the integrability of $|x-x_0|^{\left(|\gamma|-\frac{\lambda}{s}\right)\frac{s}{s-1}}$ on B^c follows by $\lambda > n\,(s/p-1)$ and $|\gamma| \leqslant |\alpha| \leqslant n(1/p-1)$.

□

In order to provide a molecular decomposition for distributions in $H^p(\mathbb{R}^n)$, it suffices to show that it can be decomposed into atoms. This fact is covered by the next result, whose proof is standard and follows the same ideas as in [75, Theorem 2.9] (or [35, Theorem 7.16]). We present a detailed proof since its ideas are going to be used in subsequent sections.

Proposition 1.2. *Let M to be a (p,s,λ) molecule in H^p. Then, there exists a sequence $\{a_j\}_j$ of (p,s) atoms in H^p and $\{\gamma_j\}_j \in \ell^p(\mathbb{C})$ such that*

$$M = \sum_{j=1}^{\infty} \gamma_j\, a_j \;\text{ in } \mathcal{S}'(\mathbb{R}^n).$$

In particular, there exists a constant $C > 0$, independent of M, such that $\|M\|_{H^p} \leqslant C$.

Proof. Suppose M is a (p,s,λ) molecule in H^p associated to a ball $B = B(x_0,r) \subset \mathbb{R}^n$ and define for any $j \in \mathbb{Z}_+$ the sets $B_j = B(x_0, 2^j r)$, $E_0 = B$, $E_j = B_j \backslash B_{j-1}$ if $j \geqslant 1$, and the function $M_j(x) = M(x)\chi_{E_j}(x)$. We may assume without loss of generality that $x_0 = 0$. For a given $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| \leqslant N_p$, let $\mathcal{P}_{N_p}$ denote the finite dimensional vector space of polynomials in $\mathbb{R}^n$ with degree at most N_p and $\mathcal{P}_{N_p,j}$ its restriction on the set E_j. By the Gram-Schmidt orthogonalization process on the Hilbert space

$\mathcal{H} = L^2(E_j, |E_j|^{-1}dx)$ (considering $\mathcal{P}_{N_p,j}$ as a subspace of $\mathcal{H}$ with respect to the base $\{x^\beta\}_{|\beta|\leqslant N_p}$), there exist polynomials $\phi_\gamma^j \in \mathcal{P}_{N_p,j}$ uniquely determined such that

$$\frac{1}{|E_j|}\int_{E_j} \phi_\gamma^j(x)\; x^\beta dx = \delta_{\gamma,\beta} = \begin{cases} 1, & \text{if } \gamma = \beta \\ 0, & \text{if } \gamma \neq \beta. \end{cases} \tag{1.3}$$

In addition, these polynomials satisfy the estimate $(2^j r)^{|\gamma|}|\phi_\gamma^j(x)| \leqslant C$ uniformly on j for every $x \in E_j$ (see [75, p. 77]). Let

$$M_j^\gamma = \frac{1}{|E_j|}\int_{E_j} M(x)x^\gamma dx, \quad P_j(x) = \sum_{|\gamma|\leqslant N_p} M_j^\gamma\, \phi_\gamma^j(x),$$

and split

$$M = \sum_{j=0}^{\infty} M_j = \sum_{j=0}^{\infty}(M_j - P_j) + \sum_{j=0}^{\infty} P_j \quad \text{in } L^s(\mathbb{R}^n).$$

We will show that for each $j \in \mathbb{Z}_+$, $(M_j - P_j)$ is a multiple of a (p, s) atom in H^p and P_j can be written as a finite linear combination of (p, ∞) atoms.

Starting with $(M_j - P_j)$, since M_j and P_j are both supported in $E_j \subset B_j$, then supp $(M_j - P_j) \subset B_j$ and also for all $|\alpha| \leqslant N_p$ it has the right cancellation property since

$$\begin{aligned}\int_{\mathbb{R}^n} [M_j(x) - P_j(x)]\, x^\alpha dx &= \int_{E_j}\left[M_j(x) - \sum_{|\gamma|\leqslant N_p} M_j^\gamma\, \phi_\gamma^j(x)\right]x^\alpha dx \\ &= \int_{E_j} M(x)x^\alpha dx - \sum_{|\gamma|\leqslant N_p}\left(\int_{E_j} M(y)y^\gamma dy\right)⨍_{E_j} \phi_\gamma^j(x)x^\alpha dx = 0.\end{aligned}$$

In order to estimate the L^s norm of $M_j - P_j$, we do it separately. From condition (M1) and (M2), we get for M_j that

$$\|M_j\|_{L^s} \leqslant (2^{j-1}r)^{-\frac{\lambda}{s}}\left(\int_{E_j} |M(x)|^s|x - x_0|^\lambda dx\right)^{1/s} \lesssim |B_j|^{\frac{1}{s}-\frac{1}{p}}\; (2^j)^{-\frac{\lambda}{s}+n\left(\frac{1}{p}-\frac{1}{s}\right)}. \tag{1.4}$$

For P_j, using that $(2^j r)^{|\gamma|}|\phi_\gamma^j(x)| \leqslant C$ we can bound its size by

$$\begin{aligned}
|P_j(x)| &\leqslant \sum_{|\gamma|\leqslant N_p} |\phi_\gamma^j(x)| \fint_{E_j} |M(y)|\ |y|^{|\gamma|}\ dy \\
&\lesssim \left(\sum_{|\gamma|\leqslant N_p} (2^j r)^{|\gamma|} |\phi_\gamma^j(x)| \right) \fint_{E_j} |M(y)| dy \\
&\lesssim \fint_{E_j} |M_j(y)| dy \leqslant |E_j|^{-\frac{1}{s}}\ \|M_j\|_{L^s},
\end{aligned}$$

which implies that $\|P_j\|_{L^s} \lesssim \|M_j\|_{L^s}$. From this and (1.4) we obtain

$$\|M_j - P_j\|_{L^s} \lesssim \|M_j\|_{L^s} \lesssim |B_j|^{\frac{1}{s}-\frac{1}{p}}\ (2^j)^{-\frac{\lambda}{s}+n\left(\frac{1}{p}-\frac{1}{s}\right)}.$$

Finally, we write $(M_j - P_j)(x) = d_j\, a_j(x)$ where

$$a_j(x) = \frac{M_j(x) - P_j(x)}{\|M_j - P_j\|_{L^s}} |B_j|^{\frac{1}{s}-\frac{1}{p}} \quad \text{and} \quad d_j = \|M_j - P_j\|_{L^s}\, |B_j|^{\frac{1}{p}-\frac{1}{s}}.$$

By the previous considerations, each a_j is a (p, s) atom in H^p supported on B_j and the scalars $\{d_j\}_j \in \ell^p(\mathbb{R})$, since $\lambda > n\,(s/p - 1)$ yields

$$\begin{aligned}
\sum_{j=0}^{\infty} |d_j|^p = \sum_{j=0}^{\infty} \|M_j - P_j\|_{L^s}^p\, |B_j|^{1-\frac{p}{s}} &\leqslant \sum_{j=0}^{\infty} \left[|B_j|^{\frac{1}{s}-\frac{1}{p}}\ (2^j)^{-\frac{\lambda}{s}+n\left(\frac{1}{p}-\frac{1}{s}\right)} \right]^p |B_j|^{1-\frac{p}{s}} \\
&\simeq \sum_{j=0}^{\infty} (2^j)^{-\frac{\lambda p}{s}+n\left(1-\frac{p}{s}\right)} < \infty.
\end{aligned}$$

We show now that P_j is a finite linear combination of (p, ∞) atoms. Define for each $j \in \mathbb{Z}_+$ and $|\gamma| \leqslant N_p$

$$N_\gamma^j := |E_k| \sum_{k=j}^{\infty} M_k^\gamma = \sum_{k=j}^{\infty} \int_{E_k} M(x) x^\gamma dx \ \text{ and } \ \psi_\gamma^j(x) := N_\gamma^{j+1} \left[|E_{j+1}|^{-1} \phi_\gamma^{j+1}(x) - |E_j|^{-1} \phi_\gamma^j(x) \right].$$

Then, we can rewrite P_j as

$$\begin{aligned}\sum_{j=0}^{\infty} P_j(x) &= \sum_{j=0}^{\infty} \sum_{|\gamma|\leqslant N_p} (M_j^{\gamma}|E_j|)\,(|E_j|^{-1}\phi_{\gamma}^{j}(x)) = \sum_{j=0}^{\infty} \sum_{|\gamma|\leqslant N_p} (N_{\gamma}^{j} - N_{\gamma}^{j+1})(|E_j|^{-1}\phi_{\gamma}^{j}(x)) \\ &= \sum_{j=0}^{\infty} \sum_{|\gamma|\leqslant N_p} \left[N_{\gamma}^{j}|E_j|^{-1}\phi_{\gamma}^{j}(x) - N_{\gamma}^{j+1}|E_j|^{-1}\phi_{\gamma}^{j}(x)\right] \\ &= \sum_{|\gamma|\leqslant N_p} \sum_{j=0}^{\infty} \psi_{\gamma}^{j}(x) + \sum_{|\gamma|\leqslant N_p} \sum_{j=0}^{\infty} \left[N_{\gamma}^{j}|E_j|^{-1}\phi_{\gamma}^{j}(x) - N_{\gamma}^{j+1}|E_{j+1}|^{-1}\phi_{\gamma}^{j+1}(x)\right] \qquad (1.5) \\ &= \sum_{j=0}^{\infty} \sum_{|\gamma|\leqslant N_p} \psi_{\gamma}^{j}(x),\end{aligned}$$

since noticing that $N_{\gamma}^{0} = \displaystyle\int M(x)x^{\gamma}dx = 0$,

$$\sum_{j=0}^{\infty} \left[N_{\gamma}^{j}|E_j|^{-1}\phi_{\gamma}^{j}(x) - N_{\gamma}^{j+1}|E_{j+1}|^{-1}\phi_{\gamma}^{j+1}(x)\right] = N_{\gamma}^{0}\,|E_0|^{-1}\phi_{\gamma}^{0}(x) = 0 \qquad (1.6)$$

for all $|\gamma| \leqslant N_p$. We show that ψ_{γ}^{j} is a multiple of a (p,∞) atom in H^p. By definition supp $(\psi_{\gamma}^{j}) \subset E_j \subset B_{j+1}$ and moreover

$$\int_{\mathbb{R}^n} \psi_{\gamma}^{j}(x)x^{\beta}dx = N_{\gamma}^{j+1}\left(\fint_{E_{j+1}} \phi_{\gamma}^{j+1}(x)x^{\beta}dx - \fint_{E_j} \phi_{\gamma}^{j}(x)x^{\beta}dx\right) = 0 \qquad (1.7)$$

for all $|\beta| \leqslant N_p$. In order to estimate $\|\psi_{\gamma}^{j}\|_{L^{\infty}}$, by Hölder inequality and (1.4) we have

$$\begin{aligned}|N_{\gamma}^{j+1}| = \left|\sum_{k=j+1}^{\infty} \int_{E_k} M(x)x^{\gamma}dx\right| &\leqslant \sum_{k=j+1}^{\infty} (2^k r)^{|\gamma|} \int_{E_k} |M_k(x)|dx \\ &\leqslant \sum_{k=j+1}^{\infty} (2^k r)^{|\gamma|}\, \|M_k\|_{L^s}\, |E_k|^{1-\frac{1}{s}} \\ &\lesssim r^{|\gamma|+n\left(1-\frac{1}{p}\right)} (2^{j+1})^{|\gamma|-\frac{\lambda}{s}+n\left(1-\frac{1}{s}\right)} \sum_{k=0}^{\infty} (2^k)^{|\gamma|-\frac{\lambda}{s}+n\left(1-\frac{1}{s}\right)} \\ &\lesssim r^{|\gamma|+n\left(1-\frac{1}{p}\right)} (2^{j+1})^{|\gamma|-\frac{\lambda}{s}+n\left(1-\frac{1}{s}\right)} \\ &\simeq |B_{j+1}|^{1-\frac{1}{p}} (2^{j+1}r)^{|\gamma|} (2^{j+1})^{-\frac{\lambda}{s}+n\left(\frac{1}{p}-\frac{1}{s}\right)}\end{aligned}$$

since $|\gamma| \leqslant n\,(1/p-1)$ and $\lambda > n\,(s/p-1)$. From the uniform estimate $(2^j r)^{|\gamma|}|\phi_{\gamma}^{j}(x)| \leqslant C$ and the previous control it follows

$$|N_\gamma^j|E_j|^{-1}\phi_\gamma^j(x)| \lesssim |B_j|^{-\frac{1}{p}}(2^j)^{-\frac{\lambda}{s}+n\left(\frac{1}{p}-\frac{1}{s}\right)}.$$

Denote $\psi_\gamma^j(x) = h_j\ b_{j\gamma}(x)$, where

$$h_j = (2^j)^{-\frac{\lambda}{s}+n\left(\frac{1}{p}-\frac{1}{s}\right)} \quad \text{and} \quad b_{j\gamma}(x) = (2^j)^{\frac{\lambda}{s}-n\left(\frac{1}{p}-\frac{1}{s}\right)}\psi_\gamma^j(x).$$

It is clear that $b_{j\gamma}$ is a multiple of a (p,∞) atom since supp $(b_{j\gamma}) \subset B_j$, $\|b_{j\gamma}(x)\|_{L^\infty} \lesssim |B_j|^{-\frac{1}{p}}$ and the moment condition follows immediately from (1.7). Finally, note that $\{h_j\}_j \in \ell^p(\mathbb{R})$ since

$$\sum_{j=0}^{\infty}|h_j|^p = \sum_{j=0}^{\infty}(2^j)^{-\frac{\lambda p}{s}+n\left(1-\frac{p}{s}\right)} < \infty.$$

Summarizing, we have shown that $M = \sum_{j=0}^{\infty}\gamma_j\, a_j$ in $L^s(\mathbb{R}^n)$, where $\{a_j\}_j$ are (p,s) atoms in H^p and $\{\gamma_j\}_j \in \ell^p(\mathbb{R})$ (with constant independent of M). Moreover, the series converges in $H^p(\mathbb{R}^n)$. Indeed, given $\varepsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ with $N_1 < N_2$ such that

$$\left\|\sum_{j=0}^{N_2}\gamma_j\, a_j - \sum_{j=0}^{N_1}\gamma_j\, a_j\right\|_{H^p}^p \leqslant \sum_{j=N_1}^{N_2}\|\gamma_j\, a_j\|_{H^p}^p \lesssim \sum_{j=N_1}^{N_2}|\gamma_j|^p < \varepsilon.$$

Thus, the sequence of partial sums is Cauchy in $H^p(\mathbb{R}^n)$ and the convergence follows by the completeness of $H^p(\mathbb{R}^n)$. Since convergence in $H^p(\mathbb{R}^n)$ implies in $\mathcal{S}'(\mathbb{R}^n)$, by the uniqueness of the limit, $S_N = \sum_{j=1}^{N}\gamma_j\, a_j \to M$ in $H^p(\mathbb{R}^n)$ as $N \to \infty$. Therefore

$$\begin{aligned}\|M\|_{H^p} = \left\|M - \sum_{j=0}^{N}\gamma_j\, a_j + \sum_{j=0}^{N}\gamma_j\, a_j\right\|_{H^p} &\leqslant \left\|M - \sum_{j=0}^{N}\gamma_j\, a_j\right\|_{H^p} + \left\|\sum_{j=0}^{N}\gamma_j\, a_j\right\|_{H^p} \\ &\leqslant \epsilon + \left(\sum_{j=0}^{\infty}|\gamma_j|^p\right)^{1/p} < \varepsilon + C,\end{aligned}$$

which implies $\|M\|_{H^p} \leqslant C$ taking $\varepsilon \to 0$. □

As a consequence, we have the following molecular decomposition of $H^p(\mathbb{R}^n)$:

Corollary 1.1. *Let $0 < p \leqslant 1 \leqslant s < \infty$ with $p < s$. Then, $f \in H^p(\mathbb{R}^n)$ if and only if there exists a sequence $\{M_j\}_j$ of (p, s, λ) molecules and $\{\gamma_j\}_j \in \ell^p(\mathbb{C})$ such that $f = \sum_{j=1}^{\infty} \gamma_j M_j$ in H^p norm (and consequently in $\mathcal{S}'$), and moreover*

$$\|f\|_{H^p} \approx \inf \left(\sum_{j=1}^{\infty} |\gamma_j|^p \right)^{1/p},$$

where the infimum is taken over all such representations.

As an application, if $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a linear and continuous operator that takes (p, s) atoms in H^p into (p, s', λ) molecules, then T is bounded in $H^p(\mathbb{R}^n)$. In fact, let $f \in H^p(\mathbb{R}^n)$ and from the atomic decomposition (Theorem 1.1) we write

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \text{ in } H^p \text{ and } \mathcal{S}'.$$

Since T is also continuous in $\mathcal{S}'(\mathbb{R}^n)$, we can show that

$$T\left(\sum_{j=1}^{N} \lambda_j a_j \right) \to Tf \text{ as } N \to \infty \text{ in } H^p.$$

From the hypothesis that Ta_j is a (p, s', λ) molecule from each j, we get from from Proposition 1.2 that for N sufficiently large and an arbitrary $\varepsilon > 0$

$$\begin{aligned}
\|Tf\|_{H^p} &\leqslant \left\| Tf - T\left(\sum_{j=1}^{N} \lambda_j a_j \right) \right\|_{H^p} + \left\| T\left(\sum_{j=1}^{N} \lambda_j a_j \right) \right\|_{H^p} \\
&\leqslant \varepsilon + \sum_{j=1}^{N} |\lambda_j| \, \|Ta_j\|_{H^p} \\
&\leqslant \varepsilon + C \left(\sum_{j=1}^{N} |\lambda_j|^p \right)^{1/p} \\
&\leqslant \varepsilon + C\|f\|_{H^p}.
\end{aligned}$$

1.2 Local Hardy spaces

The local Hardy spaces, denoted by $h^p(\mathbb{R}^n)$ were introduced by Goldberg in [37] as an alternative to deal with some localization problems in $H^p(\mathbb{R}^n)$. For instance, such spaces are not closed under multiplication by test functions, since this localization may not satisfy global vanishing moment conditions; it does not contain $\mathcal{S}(\mathbb{R}^n)$, are not well defined in manifolds, and in general, pseudodifferential operators are not bounded on $H^p(\mathbb{R}^n)$ without a strong hypothesis on the cancellation.

As it is for the homogeneous case, the spaces $h^p(\mathbb{R}^n)$ for $p > 0$ can be described by maximal functions.

Definition 1.4 ([37, Theorem 1]). *Let $0 < p < \infty$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int \varphi \neq 0$. We say that $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $h^p(\mathbb{R}^n)$ if $\mathfrak{m}_\varphi f \in L^p(\mathbb{R}^n)$, in which*

$$\mathfrak{m}_\varphi f(x) := \sup_{0<t<1} |f * \varphi_t(x)|$$

denotes the local maximal function. We denote its norm by $\|f\|_{h^p} := \|\mathfrak{m}_\varphi f\|_{L^p}$.

For $p \geqslant 1$, the functional $\|\cdot\|_{h^p}$ defines a norm and for $0 < p < 1$ a *quasi-norm*. As before, we refer to it always as a norm for simplicity. The local Hardy spaces is also a complete Banach space with the distance $d(f, g) = \|f - g\|_{h^p}^p$ and when $p > 1$ it is equal to $L^p(\mathbb{R}^n)$ with equivalent norms. We have the continuous embedding $\mathcal{S}(\mathbb{R}^n) \subset h^1(\mathbb{R}^n) \subsetneq L^1(\mathbb{R}^n)$ and so the latter containment is dense. The homogeneous Hardy space $H^p(\mathbb{R}^n)$ is strictly contained in $h^p(\mathbb{R}^n)$, since for instance $C_c^\infty(\mathbb{R}^n) \subset h^p(\mathbb{R}^n)$. The supremum over $0 < t < 1$ in the previous definition can also be replaced by $0 < t < T$ for any constant $T > 0$, resulting on equivalent norms.

In [37, Lemma 4], Goldberg showed that $H^p(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$ are related in the following way: If φ is a Schwartz function with integral one satisfying vanishing moments and $f \in h^p(\mathbb{R}^n)$, then $f - \varphi * f \in H^p(\mathbb{R}^n)$. This yields to $h^p(\mathbb{R}^n)$ a similar atomic decomposition to homogeneous case, except that vanishing moment conditions are required only for small atoms. Next we present the definition of an L^s atoms for $h^p(\mathbb{R}^n)$ (see [37, p. 36-37] for the case $s = \infty$).

Definition 1.5. *Let $0 < p \leqslant 1 \leqslant s \leqslant \infty$ with $p < s$. A measurable function a is called a (p, s) atom in h^p if there exists a ball $B = B(x_0, r) \subset \mathbb{R}^n$ such that*

$$(i)\ supp(a) \subset B; \quad (ii)\ \|a\|_{L^s} \leqslant |B|^{\frac{1}{s}-\frac{1}{p}}; \quad (iii) \text{ if } r < 1, \int a(x)x^\alpha dx = 0 \text{ for all } |\alpha| \leqslant N_p\ .$$

The next theorem states the atomic decomposition for $h^p(\mathbb{R}^n)$.

Theorem 1.2 ([37, Lemma 5])**.** *Let $f \in h^p(\mathbb{R}^n)$. Then, there exists a sequence $\{a_j\}_j$ of (p, s) atoms in h^p and $\{\lambda_j\}_j \in \ell^p(\mathbb{C})$ such that*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \ \text{ in } \mathcal{S}' \text{ and } h^p, \text{ and } \ \|f\|_{h^p} \approx \inf \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p},$$

where the infimum is taken over all such atomic representations.

1.2.1 Atoms and molecules

Conditions *(i)* and *(ii)* in Definition 1.5 alone show the size of the moment condition can be bounded by

$$\left| \int_B a(x)(x - x_0)^\alpha dx \right| \leqslant r^{|\alpha|} \|a\|_{L^s} |B|^{1-\frac{1}{s}} \lesssim \ r^{|\alpha|-n\left(\frac{1}{p}-1\right)}. \tag{1.8}$$

This implies that, for any (p, s) atom in h^p supported in a ball B with radius $r \geqslant 1$, the α-th moment is bounded by a constant depending only on p, s and n, for every $|\alpha| \leqslant N_p$. Note that when considering vanishing moments, there is no need to incorporate the center of the ball in the polynomial.

Inhomogeneous cancellation conditions for $h^p(\mathbb{R}^n)$ atoms, like (1.8), were previously introduced by Dafni in [19, Appendix B], where Goldberg's vanishing moment conditions on atoms supported in balls $B = B(x_0, r)$ with $r < 1$ were relaxed to the condition

$$\left| \int a(x)(x - x_0)^\alpha dx \right| \lesssim \ r^\eta \text{ for all } |\alpha| \leqslant N_p \text{ and some } \eta > 0$$

for every $0 < p \leqslant 1$. For the case $p = 1$ and in the setting of metric measure spaces, it was shown by Dafni and Yue [23, Definition 7.3] that the previous r-power condition can be weakened to a log-type one, i.e., there exists $C > 0$ such that

$$\left|\int a(x)dx\right| \leqslant \left[\log\left(1+\frac{C}{r}\right)\right]^{-1}. \tag{1.9}$$

This type of cancellation condition was further used by Dafni and Liflyand [22] to give a molecular decomposition and prove Goldberg's version of Hardy's inequality for $h^1(\mathbb{R})$, in dimension one. Similar approximate moments conditions using powers of the radius have also been considered recently in [11, 61].

In the next definition we provide a notion of atoms which, in the same spirit as before, also does not distinguish between the size of the radius of the ball that contain its support, and vanishing moment conditions are not required. Instead, the cancellation condition imposed is related to p in the following way: if $p \neq n/(n+k)$ for every $k \in \mathbb{Z}_+$, which in other words means that $N_p < n(1/p-1)$, it suffices to bound the size of the moments up to order $|\alpha| \leqslant N_p$ by a constant; on the other hand, if $p = n/(n+k)$, a log-type control like (1.9) is needed for the $\alpha-$th moment such that $|\alpha| = N_p = n(1/p-1)$.

Definition 1.6. *Let $0 < p \leqslant 1 \leqslant s \leqslant \infty$ with $p < s$, $\omega \geqslant 0$, and define $\varphi_p : (0,\infty) \to (0,\infty)$ by*

$$\varphi_p(t) := \left[\log\left(1+\frac{1}{\omega t}\right)\right]^{-1/p},$$

where $\varphi_p(t) = 0$ in the limiting case $\omega = 0$. We say that a measurable function a is a (p, s, ω) atom in h^p if there exists a ball $B = B(x_0, r) \subset \mathbb{R}^n$ such that the standard support and size condition are satisfied ((i) and (ii) in Definition 1.5) and moreover

$$(iii)' \quad \left|\int_B a(x)(x-x_0)^\alpha dx\right| \leqslant \begin{cases} \omega, & \text{if } |\alpha| < n(1/p-1), \\ \\ \varphi_p(r), & \text{if } |\alpha| = N_p = n(1/p-1). \end{cases}$$

The previous definition covers the one in [23, Definition 7.3] for the case $p = 1$, the one in [50, Lemma 3] for the case $n/(n+1) < p < 1$, and when $\omega = 0$ we have the (p, s) atoms for the homogeneous Hardy space $H^p(\mathbb{R}^n)$.

Remark 1.2. Combining (1.8) for $r \geqslant 1$ with the fact that $\varphi_p(r) \leqslant [\log(1+\omega^{-1})]^{-1/p} = C_{p,\omega}$ for $r < 1$, we have that for any $r > 0$ the moments satisfy

$$\left| \int_B a(x)(x-x_0)^\alpha dx \right| \leqslant C_{p,\omega} \quad \text{for } |\alpha| \leqslant N_p.$$

Next we show the space $h^p(\mathbb{R}^n)$ can also be characterized in terms of (p,s,ω) atoms. Let $h^p_{at,\omega}(\mathbb{R}^n)$ to be the atomic space generated by (p,s,ω) atoms, that is, $f \in h^p_{at,\omega}(\mathbb{R}^n)$ if there exists a sequence $\{a_j\}_j$ of (p,s,ω) atoms in h^p and $\{\lambda_j\}_j \in \ell^p(\mathbb{C})$ such that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \text{ in } \mathcal{S}'(\mathbb{R}^n), \text{ equipped with the norm } \|f\|_{h^p_{at,\omega}} := \inf \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p},$$

where the infimum is taken over all such representations. We want to show that this atomic space generates $h^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$. Consider first the following lemma:

Lemma 1.1. *If a is a (p,s,ω) atom in h^p, then $\|a\|_{h^p} \leqslant C$, where the constant $C > 0$ depends only on p, s and ω.*

Proof. Let a be a (p,s,ω) atom supported in $B = B(x_0, r)$. Split

$$\|\mathfrak{m}_\phi a\|^p_{L^p} = \int_{2B} |\mathfrak{m}_\phi a(x)|^p dx + \int_{(2B)^c} |\mathfrak{m}_\phi a(x)|^p dx.$$

To estimate the first integral, we recall the pointwise control of the local maximal function by the Hardy-Littlewood Maximal function $\mathfrak{m}_\phi a(x) \leqslant \mathcal{M}_\phi a(x) \leqslant C_\phi Ma(x)$. Then, since M bounded from $L^s(\mathbb{R}^n)$ to itself for $1 < s \leqslant \infty$, it follows that

$$\int_{2B} |\mathfrak{m}_\phi a(x)|^p dx \leqslant C_\phi |2B|^{1-\frac{p}{s}} \|Ma\|^p_{L^s} \leqslant C_{\phi,p} |2B|^{1-\frac{p}{s}} \|a\|^p_{L^s} \leqslant C_{\phi,s,p,n} r^{n\left(1-\frac{p}{s}\right)} r^{n\left(\frac{p}{s}-1\right)} = C_{\phi,s,p,n}.$$

Note the last estimate holds for all $0 < p \leqslant 1$. For $s = 1$ and $p < 1$ we use that M satisfies weak $(1,1)$

type inequality to estimate

$$\begin{aligned}\int_{2B}|\mathfrak{m}_\phi a|^p dx &\lesssim \int_{2B}|Ma(x)|^p dx\\ &= p\left\{\int_0^{r^{-\frac{n}{p}}}\lambda^{p-1}\left|\{x\in 2B:\ |Ma(x)|>\lambda\}\right|d\lambda+\int_{r^{-\frac{n}{p}}}^{\infty}\lambda^{p-1}\left|\{x\in 2B:\ |Ma(x)|>\lambda\}\right|d\lambda\right\}\\ &\lesssim p\,|2B|\int_0^{r^{-\frac{n}{p}}}\lambda^{p-1}d\lambda+p\,\|a\|_{L^1}\int_{r^{-\frac{n}{p}}}^{\infty}\lambda^{p-2}d\lambda\simeq p.\end{aligned}$$

Now we deal with the estimate on $(2B)^c$. Note first that since $\phi\in\mathcal{S}(\mathbb{R}^n)$, for any $N>0$, that will be chosen conveniently, and $\alpha\in\mathbb{Z}_+^n$ we have

$$|\partial^\alpha\phi(x)|\leqslant C_{\alpha,N}\,|x|^{-N}. \tag{1.10}$$

From Taylor expansion of the function $y\mapsto\phi_t(x-y)$ up to the order N_p, we write for some $c\in\ (0,1)$

$$|\phi_t{*}a(x)|=\left|\int\sum_{|\alpha|\leqslant N_p-1}C_\alpha\,\partial^\alpha\phi_t(x{-}x_0)\ (x_0{-}y)^\alpha a(y)dy+\int\sum_{|\alpha|=N_p}C_\alpha\,\partial^\alpha\phi_t(x{-}x_0{+}c(x_0{-}y))\ (x_0{-}y)^\alpha a(y)dy\right|.$$

Using (1.10) in the previous estimate we get

$$\begin{aligned}|\phi_t*a(x)| &\leqslant \sum_{|\alpha|\leqslant N_p-1}C_\alpha|\partial^\alpha\phi_t(x-x_0)|\left|\int_{\mathbb{R}^n}a(y)(x_0-y)^\alpha dy\right|\\ &\qquad+\sum_{|\alpha|=N_p}C_\alpha\left|\int_{\mathbb{R}^n}|\partial^\alpha\phi_t(x-x_0+c(x_0-y))|\,a(y)(x_0-y)^\alpha dy\right|\\ &\leqslant \sum_{|\alpha|\leqslant N_p-1}C_\alpha\,t^{-n-|\alpha|}\left|\frac{x-x_0}{t}\right|^{-N}\left|\int_{\mathbb{R}^n}a(y)(x_0-y)^\alpha dy\right|\\ &\qquad+\sum_{|\alpha|=N_p}C_\alpha\left|\int_{\mathbb{R}^n}t^{-n-|\alpha|}\left|\frac{x-x_0+c(x_0-y)}{t}\right|^{-N}a(y)(x_0-y)^\alpha dy\right|\\ &\leqslant \sum_{|\alpha|\leqslant N_p}C_\alpha\,t^{-n-|\alpha|+N}\,|x-x_0|^{-N}\left|\int_{\mathbb{R}^n}a(y)(x_0-y)^\alpha dy\right|\end{aligned} \tag{1.11}$$

since as $|x-x_0| \geqslant 2r$ and $|y-x_0| \leqslant r$, we have $|x-x_0+c(x_0-y)| \geqslant |x-x_0|/2$. Let

$$\int_{(2B)^c} |\mathfrak{m}_\phi\, a(x)|^p dx = \int_{2r<|x-x_0|\leqslant 2} |\mathfrak{m}_\phi\, a(x)|^p dx + \int_{|x-x_0|\geqslant 2} |\mathfrak{m}_\phi\, a(x)|^p dx = I_1 + I_2.$$

Estimate of I_1. In this case we may assume $0<r<1$ since otherwise the region of integration is empty. By (1.11) with $N = n + |\alpha|$

$$\int_{2r<|x-x_0|\leqslant 2} |\mathfrak{m}_\phi\, a(x)|^p dx \lesssim \sum_{|\alpha|\leqslant N_p} \left| \int_{\mathbb{R}^n} a(y)(y-x_0)^\alpha dy \right|^p \int_{2r<|x-x_0|\leqslant 2} |x-x_0|^{-np-|\alpha|p} dx.$$

In the case where $p \neq n/(n+k)$ for any $k \in \mathbb{Z}_+$, that is $N_p < n(1/p-1)$, we have that $-np-|\alpha|p > -n$ for all $|\alpha| \leqslant N_p$ and therefore the integral over $|x-x_0| \leqslant 2$ is convergent and uniformly bounded. This together with (iii)' gives a bound which is a constant multiple of ω. The same bound also works when $p = n/(n+k)$ for some $k \in \mathbb{Z}_+$ and $|\alpha| < N_p$. When $|\alpha| = N_p = n(1/p-1)$, we have $-np-|\alpha|p = -n$ and therefore

$$\int_{2r<|x-x_0|\leqslant 2} |x-x_0|^{-np-|\alpha|p} dx = \log\left(\frac{1}{r}\right).$$

Using condition (iii)' again, this time with the log bound on the moments, we get since $0<r\leqslant 1$

$$\int_{2r<|x-x_0|\leqslant 2} |\mathfrak{m}_\phi\, a(x)|^p dx \lesssim \log\left(\frac{1}{r}\right) \log\left(1+\frac{1}{\omega r}\right) \lesssim \log\left(1+\frac{1}{r}\right) \log\left(1+\frac{1}{\omega r}\right) = C.$$

Estimate of I_2. In this case we consider $N = n+N_p+1$. Since the supremum is taken over $t \in (0,1)$, we have $t^{-|\alpha|+N_p+1} \leqslant 1$ for all $|\alpha| \leqslant N_p$. Thus, using Remark 1.2 to bound uniformly the moment condition of the atom and the fact that $p(n+N_p+1) > n$ to get the integrability over $|x-x_0| \geqslant 2$ we get

$$\begin{aligned}\int_{|x-x_0|\geqslant 2} |\mathfrak{m}_\phi\, a(x)|^p dx &\lesssim \int_{|x-x_0|\geqslant 2} \left(\sup_{0<t<1} \sum_{|\alpha|\leqslant N_p} C_\alpha\, t^{-n-|\alpha|} \left|\frac{x-x_0}{t}\right|^{-n-N_p-1} \left| \int_{\mathbb{R}^n} a(y)(y-x_0)^\alpha dy \right| \right)^p dx \\ &\lesssim \sum_{|\alpha|\leqslant N_p} \left| \int_{\mathbb{R}^n} a(y)(y-x_0)^\alpha dy \right|^p \int_{|x-x_0|\geqslant 2} |x-x_0|^{-p(n+N_p+1)} dx \leqslant C_{n,p,w}.\end{aligned}$$

$\square$

Remark 1.3. An inspection of the previous proof shows that we could have chosen any function φ_p such that $\varphi_p(t) \leqslant C\,[\log(1/t)]^{1/p}$.

Proposition 1.3. *For any* $0 < p \leqslant 1$ *we have* $h^p(\mathbb{R}^n) = h^p_{at,\omega}(\mathbb{R}^n)$ *with equivalent norms.*

Proof. Using Goldberg's atomic decomposition it follows that $h^p(\mathbb{R}^n) \subset h^p_{at,\omega}(\mathbb{R}^n)$ continuously, since (p, s) atoms in h^p supported in small balls are automatically (p, s, ω) atoms. Moreover, if $\omega > 0$, by (1.8) it follows that (p, s) atoms supported in large balls (without vanishing moment) are (p, s, ω) atoms up to multiplication by a constant depending on ω, n, p and s. Indeed, since $r > 1$

$$\left[\log\left(1 + \frac{1}{r\omega}\right)\right]^{-1/p} > \left[\log\left(1 + \frac{1}{\omega}\right)\right]^{-1/p}.$$

This implies

$$\begin{aligned}\left|\int a(x)(x - x_0)^\alpha dx\right| &\leqslant r^{|\alpha| - n\left(\frac{1}{p} - 1\right)} \leqslant 1 \\ &= \left[\log\left(1 + \frac{1}{\omega}\right)\right]^{-1/p} \left[\log\left(1 + \frac{1}{\omega}\right)\right]^{1/p} \\ &\leqslant C_{n,p,\omega} \left[\log\left(1 + \frac{1}{r\omega}\right)\right]^{-1/p}.\end{aligned}$$

On the other direction, to show that $h^p_{at,\omega}(\mathbb{R}^n) \subset h^p(\mathbb{R}^n)$, that is, every infinite linear combination of (h^p, s, ω) atoms lies in $h^p(\mathbb{R}^n)$ with norm bounded by a constant times $\|\{\lambda_j\}_{j \in \mathbb{N}}\|_{\ell^p}$ it suffices to use the sub-linearity of the local maximal function and the convergence in $\mathcal{S}'(\mathbb{R}^n)$ together with Lemma 1.1. ∎

From the proof of Lemma 1.1 and Remark 1.3, we see the sufficiency of the log-decay on the last $\alpha-$th moment of the atom when $p = \frac{n}{n+k}$ for $k \in \mathbb{Z}_+$ and moreover, the proof does not work assuming an uniform estimate on its size. Next we show that assuming the support and size conditions (i) and (ii) in Definition 1.6, we cannot replace (iii)' by a uniform bound on the moments when $p = n/(n+k)$ for some $k \in \mathbb{Z}_+$. That is, a strong decay on the highest order of the moments is necessary for this case. It is still an open question if this log-decay is optimal. The construction of the next counterexample is inspired by [42, Lemma 3.1].

Example 1.1. Let $n = 1$ and $p = 1/(k+1)$ for some $k \in \mathbb{Z}_+$ (hence $N_p = k$). Take $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ to be any bounded function with $\|\varphi\|_{L^\infty} \leqslant 1$. For any $0 < r < 2^{-k}$, we will construct a function a, depending on r, satisfying the support and size conditions (i) and (ii) of a (p, ∞) atom supported in the interval $I_k = [-2^{k-1}r,\, 2^{k-1}r]$. Moreover, we will show

$$\int a(t)\, t^\ell dt = 0 \;\text{ for all } 0 \leqslant \ell \leqslant k-1 \quad \text{and} \quad \int a(t)\, t^k dt = C_k \varphi(r) \tag{1.12}$$

for a positive constant C_k independent of r. Finally, we will see that $\|a\|_{h^p} \gtrsim \varphi(r)|\log r|$. As a result, letting r tend to 0, we conclude that the norm of a can remain bounded only if $\varphi(r) = O(1/|\log r|)$.

We start by defining the even function

$$a_0(t) = \left(2^k r\right)^{-k-1} \varphi(r) \chi_{\left[-\frac{r}{2}, \frac{r}{2}\right]}(t).$$

Now we construct a_1 by translating the previous function a_0 by $r/2$ units to the right half-line and then extending it to $[-r, 0]$ in such a way that the resulting function is odd. We define a_2 proceeding in the same way, translating a_1 by r units to the right and extending it to $[-2r, 0]$ as an even function. In general, we construct a_m by translating a_{m-1} $(m-1)$ units to the right and then extending it to $[-mr, 0]$ in an even way if m is even and oddly otherwise (see Figure 1.1). Inductively, we describe this process writing

$$a_{m+1}(t) = \begin{cases} a_m(t - 2^{m-1}r) - a_m(-t - 2^{m-1}r), & \text{if } m \text{ is even;} \\ a_m(t - 2^{m-1}r) + a_m(-t - 2^{m-1}r), & \text{if } m \text{ is odd.} \end{cases}$$

Observe that a_m is an even function if m is even and it is an odd function if m is odd. In addition,

$$\text{supp}\,(a_m) \subset [-2^{m-1}r,\, 2^{m-1}r] \;\text{ and }\; \|a_m\|_{L^\infty} \leqslant (2^k r)^{-k-1} \|\varphi\|_{L^\infty} \leqslant \left|[-2^{k-1}r,\, 2^{k-1}r]\right|^{-k-1}.$$

When $m = k$, this shows that the function $a = a_k$ satisfies conditions (i) and (ii) of Definition 1.6.

We want to show that (1.12) holds for $a = a_k$. The first identity in (1.12) follows from [42, Lemma 3.1]. For the sake of completeness, we show both by proving the following identities for any m:

$$\int a_m(t)\, t^\ell dt = 0 \;\text{ for all } 0 \leqslant \ell \leqslant m-1 \quad \text{and} \quad \int a_m(t)\, t^m dt = C_{m,k}\, r^{m-k} \varphi(r), \tag{1.13}$$

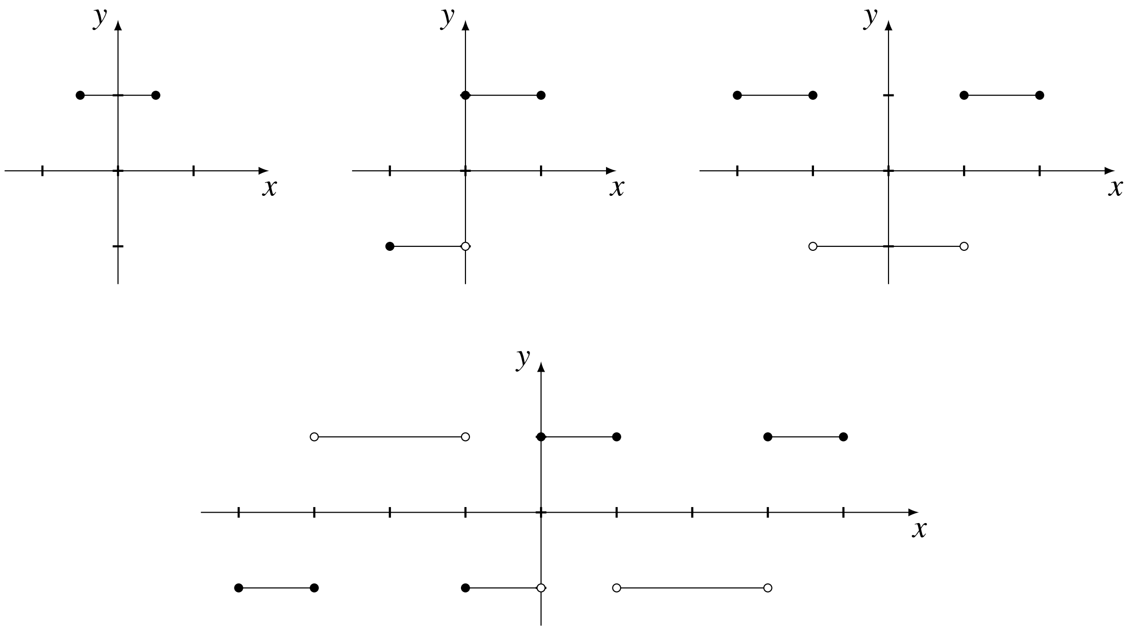

Figure 1.1: Construction of a_0, a_1, a_2 and a_3 respectively.

where $C_{m,k} > 0$ is a constant independent of r. We proceed to prove the first identity by induction on m, starting with $m = 1$. By the oddness of a_1, we immediately get $\int a_1(t)dt = 0$. Now assuming the vanishing moments for a_m for every $0 \leqslant \ell \leqslant m-1$, we show it for a_{m+1} for every $0 \leqslant \ell \leqslant m$. Suppose without loss of generality that m is odd (the same argument works if m is even). By construction, a_{m+1} is even and the vanishing moments will immediately hold for every odd ℓ and $1 \leqslant \ell \leqslant m$. Suppose ℓ is even. Using the definition of a_{m+1} and the fact that $a_m(-t-2^{m-1}r) = 0$ when $t \in [0, 2^m r]$ we get

$$\begin{aligned}\int a_{m+1}(t)t^\ell dt &= 2\int_0^{2^m r} a_{m+1}(t)t^\ell dt = 2\int_0^{2^m r} \left[a_m(t-2^{m-1}r) + a_m(-t-2^{m-1}r)\right] t^\ell dt \\ &= 2\int_0^{2^m r} a_m(t-2^{m-1}r)t^\ell dt = 2\int_{-2^{m-1}r}^{2^{m-1}r} a_m(t)(t+2^{m-1}r)^\ell dt \\ &= \sum_{\gamma \leqslant \ell-1<m} C_{\ell,\gamma,r,m} \int_{-2^{m-1}r}^{2^{m-1}r} a_m(t)\, t^\gamma dt = 0.\end{aligned}$$

In the last two steps we have used the fact that $a_m(t)t^\ell$ is odd to eliminate the integral of the highest order term in the binomial expansion, followed by the induction hypothesis.

For the second identity in (1.13), we will follow the same procedure. Starting from $m = 0$, it holds with $C_{0,k} = 2^{-k(k+1)}$. Indeed

$$\int a_0(t)dt = \left(2^k r\right)^{-k-1} \varphi(r) r = 2^{-k(k+1)} \varphi(r) r^{-k} = C_{0,k}\, \varphi(r) r^{-k}.$$

Assuming that $\int a_m(t)\, t^m dt = C_{m,k}\, \varphi(r)\, r^{m-k}$ for some $C_{m,k} > 0$ independent of r, we write, as above

$$\begin{aligned}\int a_{m+1}(t) t^{m+1} dt &= 2\int_0^{2^m r} a_{m+1}(t) t^{m+1} dt = 2\int_{-2^{m-1}r}^{2^{m-1}r} a_m(t)(t + 2^{m-1}r)^{m+1} dt \\ &= 2\int_{-2^{m-1}r}^{2^{m-1}r} a_m(t)(m+1)t^m\, 2^{m-1} r\, dt = (m+1)2^m r \int a_m(t) t^m dt \\ &= (m+1)2^m\, C_{m,k}\, \varphi(r)\, r^{m+1-k}.\end{aligned}$$

Here again we have used the fact that $a_m(t)t^{m+1}$ is odd to eliminate the integral of the highest order term in the binomial expansion, as well as the vanishing moments of a_m of order ℓ for all $\ell < m$, followed by the induction hypothesis. This proves the induction step with $C_{m+1,k} = (m+1)2^m\, C_{m,k}$.

We have shown so far that $a = a_k$ satisfies the conditions of Definition 1.6 for a (p, ∞) atom with the bound on the highest-order moment in (iii)' replaced by $C_k\, \varphi(r)$. We want now to estimate its h^p norm. We will do this by testing against an element f of the dual space $(h^p(\mathbb{R}))^* = \Lambda^k(\mathbb{R})$ if $0 < p < 1$ and $(h^1(\mathbb{R}))^* = bmo(\mathbb{R})$. Fix a cutoff function $\eta \in C^\infty(\mathbb{R})$ with supp $(\eta) \subset (-1, 1)$ which is equal to 1 on $[-1/2, 1/2]$, and let f be given by $f(t) = t^k \log(|t|)\, \eta(t)$. Recall that we are assuming $r < 2^{-k}$, so that $\eta = 1$ on the support of a, and we have, by (1.12),

$$\begin{aligned}\left|\int a(t) f(t) dt - C_k \varphi(r)\, \log(r)\right| &= \left|\int a(t)\, t^k\, [\log(|t|) - \log(r)]\, dt\right| = \left|2\int_0^{2^{k-1}r} a(t)\, t^k\, \log\left(\frac{t}{r}\right) dt\right| \\ &= \left|2r^{k+1} \int_0^{2^{k-1}} a(ur) u^k \log(u) du\right| \leqslant 2r^{k+1} \|a\|_{L^\infty} \int_0^{2^{k-1}} u^k |\log(u)| dt \\ &\leqslant \widetilde{C}_k,\end{aligned}$$

where $\widetilde{C}_k$ is independent of r. This shows

$$\varphi(r)|\log(r)| \leqslant C_k^{-1}\left(\widetilde{C}_k + \left|\int a(t) f(t) dt\right|\right) \lesssim 1 + \|f\|_{(h^p)^*} \|a\|_{h^p} \lesssim 1 + \|a\|_{h^p}$$

with constants depending on k but independent of r.

□

Now we turn our attention to present a new class of molecules satisfying analogous cancellation conditions of (p, s, ω) atoms and a molecular decomposition of $h^p(\mathbb{R}^n)$ for the full range $0 < p \leqslant 1$ in terms of such molecules.

Definition 1.7. *Let $0 < p \leqslant 1 \leqslant s < \infty$ with $p < s$ and $\lambda > n(s/p - 1)$. Suppose ω and φ_p be as in Definition 1.6. We say that a measurable function M is a (p, s, λ, ω) molecule in h^p if there exists a ball $B = B(x_0, r) \subset \mathbb{R}^n$ such that the size conditions (M1) and (M2) of Definition 1.3 are satisfied and moreover*

$$\text{M3.} \quad \left|\int_{\mathbb{R}^n} M(x)(x - x_0)^\alpha dx\right| \leqslant \begin{cases} \omega, & \text{if } |\alpha| < n(1/p - 1), \\ \\ \varphi_p(r), & \text{if } |\alpha| = N_p = n(1/p - 1). \end{cases}$$

We call the molecule "normalized" if the constant appearing in (M1) and (M2) is $C = 1$.

Choosing $s = 1$, the previous definition covers the molecules introduced by Komori in [50, Definition 4.4] for $n/(n+1) < p < 1$. In particular, our definition not only extends it for $0 < p \leqslant \frac{n}{n+1}$ and $p = 1$, but also provides an appropriate bound for the size of the moment condition when $p = n/(n+k)$ for $k \in \mathbb{Z}^+$.

Remark 1.4. As in (1.8), assuming only conditions (M1) and (M2), we can derive the same estimate on the moments of M. In fact, for any $j \in \mathbb{Z}_+$ let $C_j = \{x \in \mathbb{R}^n : \; 2^j r \leqslant |x - x_0| < 2^{j+1} r\}$. Then, since $|\alpha| \leqslant n(1/p - 1)$ and $\lambda > n(s/p - 1)$ we have

$$\begin{aligned} \left|\int M(x)(x - x_0)^\alpha dx\right| &\lesssim r^{|\alpha|} |B|^{1-\frac{1}{s}} \|M\|_{L^s(B)} + \sum_{j=0}^{\infty} (2^j r)^{|\alpha| - \frac{\lambda}{s}} |C_j|^{1-\frac{1}{s}} \, \|M \, | \cdot - x_0|^{\frac{\lambda}{s}} \|_{L^s(B^c)} \\ &\lesssim r^{|\alpha| - n\left(\frac{1}{p} - 1\right)} + r^{|\alpha| - n\left(\frac{1}{p} - 1\right)} \sum_{j=0}^{\infty} (2^j)^{|\alpha| - \frac{\lambda}{s} + n\left(1 - \frac{1}{s}\right)} \\ &\lesssim r^{|\alpha| - n\left(\frac{1}{p} - 1\right)}. \end{aligned}$$

This shows that (M3) holds automatically, with some constant $C_{n,s,\omega}$ in place ω, for all balls with $r \geqslant 1$.

Next we show that we can decompose a (p,s,λ,ω) molecule in terms of (p,s,ω) atoms in h^p with uniformly bounded norm in $h^p(\mathbb{R}^n)$.

Proposition 1.4. *If M is a normalized (p,s,λ,ω) molecule in h^p, then $\|M\|_{h^p} \lesssim 1$, with the constant depending on the parameters p,n,s,λ,ω but not on M, i.e., independent of M.*

Proof. Following the proof of Proposition 1.2, we can decompose M as follows (see in particular (1.5) and (1.6))

$$\begin{aligned} M &= \sum_{j=0}^{\infty} (M_j - P_j) + \sum_{j=0}^{\infty} \sum_{|\gamma|\leqslant N_p} \psi_\gamma^j + \sum_{|\gamma|\leqslant N_p} N_\gamma^0 \, |E_0|_\gamma^{-1} \, \phi_\gamma^0 \\ &= \sum_{j=0}^{\infty} d_j \, a_j + \sum_{j=0}^{\infty} \sum_{|\gamma|\leqslant N_p} h_j \, b_{j\gamma} + \sum_{|\gamma|\leqslant N_p} N_\gamma^0 \, |E_0|_\gamma^{-1} \, \phi_\gamma^0, \end{aligned}$$

in which a_j and $b_{j\gamma}$ are (p,s) and (p,∞) atoms in H^p (and hence (p,s,ω) atoms in h^p) with coefficients belonging to $\ell^p(\mathbb{C})$. It remains to deal with the third sum (which is zero in the homogeneous case). Let

$$a_\omega(x) = \sum_{|\gamma|\leqslant N_p} N_\gamma^0 \, |E_0|^{-1} \, \phi_\gamma^0(x).$$

We have that supp $(a_\omega) \subset E_0 = B$ and, proceeding as in Remark 1.4, conditions (M1) and (M2) give

$$|N_\gamma^0| = \left| \int_{\mathbb{R}^n} M(x)(x-x_0)^\gamma dx \right| \lesssim r^{|\gamma|+n\left(1-\frac{1}{p}\right)}.$$

Using the previous estimate and the fact that $r^{|\gamma|}|\phi_\gamma^0(x)| \leqslant C$, we get the desired size condition:

$$\begin{aligned} \left\| \sum_{|\gamma|\leqslant N_p} N_\gamma^0 \, |E_0|^{-1} \, \phi_\gamma^0 \right\|_{L^s} &\leqslant \sum_{|\gamma|\leqslant N_p} |N_\gamma^0| \, |E_0|^{-1} \left(\int_{E_0} |\phi_\gamma^0(x)|^s dx \right)^{1/s} \\ &\leqslant \sum_{|\gamma|\leqslant N_p} |N_\gamma^0| \, |E_0|^{\frac{1}{s}-1} \, r^{-|\gamma|} \lesssim r^{n\left(\frac{1}{s}-\frac{1}{p}\right)}. \end{aligned}$$

It remains to show the estimate on the moment conditions of a_ω, which are the same as those of M and hence follows immediately from (M3). Indeed, for any $|\alpha| \leqslant N_p$,

$$\begin{aligned}\int a_\omega(x)\,(x-x_0)^\alpha dx &= \sum_{|\gamma|\leqslant N_p} N_\gamma^0 \left(|E_0|^{-1} \int_{E_0} \phi_\gamma^0(x)(x-x_0)^\alpha dx \right) \\ &= N_\alpha^0 = \int_{\mathbb{R}^n} M(x)(x-x_0)^\alpha dx.\end{aligned}$$

Thus, a_ω is a multiple of a (p, s, ω) atom. ∎

Remark 1.5. From the previous proof we see that the condition on the moments required on the molecules are automatically the same as the one imposed on the atoms. In this sense, from the approximate atoms developed in [19, Appendix B] we could also define molecules with the cancellation condition

$$\left| \int M(x)(x-x_0)^\alpha dx \right| \lesssim r^\eta \text{ for all } |\alpha| \leqslant N_p \text{ and some } \eta > 0$$

for every $0 < p \leqslant 1$. This observation covers the molecules defined recently in [11, Definition 2.2].

Since (p, s) atoms in h^p are automatically (p, s, λ, ω) molecules for any choice of λ and ω in Definition 1.7, we can combine the atomic decomposition in Theorem 1.2 with Proposition 1.4 (see also the remarks preceding Proposition 1.1) to get:

Corollary 1.2. *Let $0 < p \leqslant 1$. Then, $f \in h^p(\mathbb{R}^n)$ if and only if there exists a sequence $\{M_j\}_{j\in\mathbb{N}}$ of (p, s, λ, ω) molecules in h^p and a sequence $\{d_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{C})$ such that $f = \sum_{j=1}^{\infty} d_j M_j$ in the sense of distributions and in h^p norm, and $\|f\|_{h^p}$ is comparable to the infimum of $\|\{d_j\}_{j\in\mathbb{N}}\|_{\ell^p}$ over all such representations.*

1.2.2 Application: Inhomogeneous Hardy's inequality

It is well known that if $0 < p \leqslant 1$ and $f \in H^p(\mathbb{R}^n)$, then its Fourier transform is a continuous function and satisfies the pointwise inequality

$$|\widehat{f}(\xi)| \leqslant C\,|\xi|^{n\left(\frac{1}{p}-1\right)}\,\|f\|_{H^p} \tag{1.14}$$

(see [35, Corollary 7.21 p. 339]). Moreover, a weaker integral estimate, known as Hardy's inequality also holds for this setting, given by

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|^p}{|\xi|^{n(2-p)}} d\xi \leqslant C\|f\|_{H^p}^p \tag{1.15}$$

(see [35, Corollary 7.23 p. 342]).

For local Hardy spaces, the inhomogeneous version of inequality (1.14) has been proved by Hounie and Kapp in [45, Proposition 5.1]. On the other hand, for the Hardy's inequality (1.15), it was originally stated without proof by Goldberg in [36, Theorem 2']. When $p = n = 1$, the corresponding inequality was proved by Dafni and Liflyand [22, Theorem 1] using the approximate molecules and the characterization of $h^1(\mathbb{R})$ in terms of the local Hilbert transform.

In this section, we apply the molecular theory without cancellation developed in the previous section to extend [22, Theorem 1] and prove a inhomogeneous version of Hardy's inequality on $h^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$ and any dimension. Our main theorem is the following:

Theorem 1.3. *For any* $0 < p \leqslant 1$*, there exists a constant* $C > 0$ *such that for every* $f \in h^p(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|^p}{(1+|\xi|)^{n(2-p)}} d\xi \leqslant C\|f\|_{h^p}^p.$$

The main ingredient to extend this result for $0 < p \leqslant 1$ and any dimension is a pointwise control of the Fourier transform of (p, s, λ, ω) molecules on $h^p(\mathbb{R}^n)$, as proved in Lemma 1.2 below. This result resembles the decay of the Fourier transform for standard atoms on $H^p(\mathbb{R}^n)$ (see for instance [35, Theorem 7.20 p. 337]), but we need to take into consideration the non-vanishing moments. Moreover, due to the weaker decay at infinity of the molecules, namely condition (M2) compared to compact support of an atom, we cannot get unlimited smoothness of the Fourier transform. The parameter λ is what determines this limitation.

Lemma 1.2. *Let* $0 < p \leqslant 1 \leqslant s < \infty$ *with* $p < s$ *and* $\lambda > n\,(s/p - 1)$*. Suppose* M *satisfies conditions*

(M1) and (M2) with respect to the ball $B = B(x_0, r) \subset \mathbb{R}^n$. Then, the Fourier transform of M satisfies

$$|\widehat{M}(\xi)| \lesssim |\xi|^\gamma r^{\gamma - n\left(\frac{1}{p}-1\right)} + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|} \left| \int_{\mathbb{R}^n} M(x)(x-x_0)^\alpha dx \right| \tag{1.16}$$

for any $\gamma \in \left(n\left(\frac{1}{p}-1\right), \frac{\lambda}{s} - \frac{n}{s'}\right)$ and $N \in \mathbb{Z}_+$ satisfying $N < \gamma \leqslant N+1$.

Proof. Since the absolute value of the Fourier transform is preserved under translation of the function, we may assume for simplicity $x_0 = 0$. For $\xi = 0$, we see that equality (1.16) holds by considering $\alpha = 0$ term in the sum on the right-hand-side, so we may assume $\xi \neq 0$.

Suppose first that $\gamma = N + 1 < \lambda/s - n/s'$. Denoting $\varphi(x) = e^{-2\pi i x\cdot\xi}$, we write

$$P_{N,\varphi,0}(x) = \sum_{|\alpha|\leqslant N} C_\alpha\, (\partial^\alpha \varphi)(0) x^\alpha$$

its Taylor polynomial of order N centered at 0. Using the formula for the remainder, we get for $t \in (0, 1)$,

$$\begin{aligned}
|\widehat{M}(\xi)| &= \left| \int_{\mathbb{R}^n} M(x)\left[\varphi(x) - P_{N,\varphi,0}(x)\right] dx + \sum_{|\alpha|\leqslant N} C_\alpha\, (\partial^\alpha\varphi)(0) \int_{\mathbb{R}^n} M(x) x^\alpha dx \right| \\
&\leqslant \left| \int_{\mathbb{R}^n} M(x) \sum_{|\alpha|=N+1} C_\alpha\, (\partial^\alpha\varphi)(tx) x^\alpha dx \right| + \sum_{|\alpha|\leqslant N} C_\alpha\, |2\pi\xi|^{|\alpha|} \left| \int_{\mathbb{R}^n} M(x) x^\alpha dx \right| \\
&\lesssim |\xi|^{N+1} \int_{\mathbb{R}^n} |M(x)|\, |x|^{N+1} dx + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|} \left| \int_{\mathbb{R}^n} M(x) x^\alpha dx \right|.
\end{aligned} \tag{1.17}$$

Similarly to Remark 1.4, from conditions (M1) and (M2) of the molecule and Hölder's inequality, one has

$$\begin{aligned}
\int_{\mathbb{R}^n} |M(x)|\, |x|^{N+1} dx &\leqslant r^{\frac{n}{s'}+N+1} \|M\|_{L^s(B)} + \|M\, |\cdot|^{\frac{\lambda}{s}}\|_{L^s(B^c)} \|\, |\cdot|^{-\frac{\lambda}{s}+N+1}\|_{L^{s'}(B^c)} \\
&= r^{\frac{n}{s'}+N+1} \|M\|_{L^s(B)} + r^{-\frac{\lambda}{s}+N+1+\frac{n}{s'}} \|M\, |\cdot|^{\frac{\lambda}{s}}\|_{L^s(B^c)} \\
&\lesssim r^{N+1-n\left(\frac{1}{p}-1\right)},
\end{aligned}$$

where the convergence of the integral follows from the assumption that $N + 1 < \lambda/s - n/s'$. This gives the result in the case $\gamma = N + 1$.

Now suppose $\gamma < N+1$. Recalling that $\xi \neq 0$, we write

$$\widehat{M}(\xi) = \int_{|x|\geqslant|\xi|^{-1}} e^{-2\pi i x\cdot\xi} M(x)dx + \int_{|x|\leqslant|\xi|^{-1}} e^{-2\pi i x\cdot\xi} M(x)dx := I_1 + I_2.$$

We estimate the first integral using Hölder's inequality, together with the global (M2) condition for $M(x)|x|^{\lambda'/s}$ with $\lambda' = s(\gamma + n/s') < \lambda$ (see Remark 1.1 (ii)), as follows:

$$|I_1| \leqslant \int_{|x|\geqslant|\xi|^{-1}} |M(x)|dx \leqslant \|M|\cdot|^{\frac{\lambda'}{s}}\|_{L^s(\mathbb{R}^n)} \||\cdot|^{-\frac{\lambda'}{s}}\|_{L^{s'}(|x|\geqslant|\xi|^{-1})} \leqslant r^{\gamma-n\left(\frac{1}{p}-1\right)}|\xi|^{\gamma}.$$

For the second integral, we again proceed via the Taylor expansion of $\varphi(x) = e^{-2\pi i x\cdot\xi}$, to get, as in (1.17)

$$\begin{aligned}
|I_2| &\lesssim |\xi|^{N+1}\int_{|x|\leqslant|\xi|^{-1}} |M(x)|\,|x|^{N+1}dx + \sum_{|\alpha|\leqslant N}|\xi|^{|\alpha|}\left|\int_{|x|\leqslant|\xi|^{-1}} M(x)x^{\alpha}dx\right| \\
&= |\xi|^{N+1}\int_{|x|\leqslant|\xi|^{-1}} |M(x)|\,|x|^{\frac{\lambda'}{s}}|x|^{N+1-\frac{\lambda'}{s}}dx + \sum_{|\alpha|\leqslant N}|\xi|^{|\alpha|}\left|\int_{\mathbb{R}^n} M(x)x^{\alpha}dx - \int_{|x|\geqslant|\xi|^{-1}} M(x)x^{\alpha}dx\right| \\
&\leqslant |\xi|^{N+1}\|M|\cdot|^{\frac{\lambda'}{s}}\|_{L^s(\mathbb{R}^n)}\||\cdot|^{N+1-\frac{\lambda'}{s}}\|_{L^{s'}(|x|\leqslant|\xi|^{-1})} \\
&\qquad + \sum_{|\alpha|\leqslant N}|\xi|^{|\alpha|}\int_{|x|\geqslant|\xi|^{-1}}|M(x)||x|^{\alpha}dx + \sum_{|\alpha|\leqslant N}|\xi|^{|\alpha|}\left|\int_{\mathbb{R}^n} M(x)x^{\alpha}dx\right| \\
&\lesssim |\xi|^{N+1}r^{\frac{\lambda'}{s}-\frac{n}{s'}-n\left(\frac{1}{p}-1\right)}|\xi|^{-\left(N+1-\frac{\lambda'}{s}+\frac{n}{s'}\right)} + \sum_{|\alpha|\leqslant N}|\xi|^{|\alpha|}\|M|\cdot|^{\frac{\lambda'}{s}}\|_{L^s(\mathbb{R}^n)}\||\cdot|^{|\alpha|-\frac{\lambda'}{s}}\|_{L^{s'}(|x|\geqslant|\xi|^{-1})} \\
&\qquad + \sum_{|\alpha|\leqslant N}|\xi|^{|\alpha|}\left|\int_{\mathbb{R}^n} M(x)x^{\alpha}dx\right| \\
&\lesssim r^{\gamma-n\left(\frac{1}{p}-1\right)}|\xi|^{\gamma} + \sum_{|\alpha|\leqslant N}|\xi|^{|\alpha|}\left|\int_{\mathbb{R}^n} M(x)x^{\alpha}dx\right|.
\end{aligned}$$

Here we have used that $\gamma = \lambda'/s - n/s' < N+1$ for the local integrability and that $|\alpha| \leqslant N < \gamma = \lambda'/s - n/s'$ implies $s'(|\alpha| - \lambda'/s) < -n$. This concludes the case $\gamma < N+1$.

□

For a molecule, the above estimate on the Fourier transform and the control on the moments allow us to prove the following refined version of Hardy's inequality:

Lemma 1.3. *Let $1 \leqslant s \leqslant 2$ with $p < s$ and M is a (p, s, λ, ω) molecule in h^p associated to the ball $B = B(x_0, r)$. Then, for any $a > 0$,*

$$\int_{\mathbb{R}^n} \frac{|\widehat{M}(\xi)|^p}{(a\omega + |\xi|)^{n(2-p)}} d\xi \leqslant C_{a,\omega,p}. \tag{1.18}$$

In the homogeneous case, that is $\omega = 0$, we recover Hardy's inequality for $H^p(\mathbb{R}^n)$ given in (1.15). For $\omega > 0$, choosing $a = \omega^{-1}$ we see that Goldberg's Hardy inequality holds uniformly for molecules with a constant depending on ω and p.

Proof. To show (1.18) we split the integral in the following way:

$$\int_{\mathbb{R}^n} \frac{|\widehat{M}(\xi)|^p}{(a\omega + |\xi|)^{n(2-p)}} d\xi = \int_{|\xi| < r^{-1}} + \int_{|\xi| > r^{-1}} := I_1 + I_2.$$

Estimate of I_2. Applying Hölder and Hausdorff-Young inequalities, one gets

$$\begin{aligned} \int_{|\xi| > r^{-1}} \frac{|\widehat{M}(\xi)|^p}{(a\omega + |\xi|)^{n(2-p)}} d\xi &\leqslant \|\widehat{M}\|^p_{L^{s'}(\mathbb{R}^n)} \left(\int_{|\xi| > r^{-1}} |\xi|^{-\frac{n(2-p)}{1-p/s'}} d\xi \right)^{1-\frac{p}{s'}} \\ &\lesssim \|M\|^p_{L^s(\mathbb{R}^n)} r^{n(2-p)-n\left(1-\frac{p}{s'}\right)} \left(\int_{|\xi| > 1} |\xi|^{-\frac{n(2-p)}{1-p/s'}} d\xi \right)^{1-\frac{p}{s'}} \simeq C. \end{aligned}$$

Here we have used condition (M1), and the integrability of the second term follows since

$$1 > p\left(1 - \frac{1}{s'}\right) \quad \Leftrightarrow \quad -\frac{n(2-p)}{1-p/s'} < -n.$$

Estimate of I_1. Taking $N = N_p$ and $\gamma \in \left(n\left(\frac{1}{p} - 1\right), \frac{\lambda}{s} - \frac{n}{s'}\right) \cap (N_p, N_p + 1\,]$ in Lemma 1.2, one has

$$\begin{aligned} I_1 &\lesssim r^{p\left[\gamma - n\left(\frac{1}{p} - 1\right)\right]} \int_{|\xi| < r^{-1}} |\xi|^{p\gamma} (a\omega + |\xi|)^{n(p-2)} d\xi \\ &\quad + \sum_{|\alpha| \leqslant N_p} \left| \int_{\mathbb{R}^n} M(x)(x - x_0)^\alpha dx \right|^p \int_{|\xi| < r^{-1}} |\xi|^{|\alpha| p} (a\omega + |\xi|)^{n(p-2)} d\xi := I_3 + I_4. \end{aligned}$$

For I_3, using that $(a\omega + |\xi|)^{n(p-2)} \leqslant |\xi|^{n(p-2)}$ we get

$$\begin{aligned}|I_3| &\leqslant r^{p\left[\gamma - n\left(\frac{1}{p}-1\right)\right]} \int_{|\xi|<r^{-1}} |\xi|^{n(p-2)+p\gamma} d\xi \\ &\simeq r^{p\gamma+n(p-1)}\, r^{-p\gamma-n(p-2)-n} = 1,\end{aligned}$$

where the integrability follows from $\gamma > n(1/p-1)$.

For I_4, using the approximate moment conditions (M3) of the molecule when $\omega > 0$, we get

$$\begin{aligned}&\sum_{|\alpha|\leqslant N_p} \left|\int_{\mathbb{R}^n} M(x)(x-x_0)^\alpha dx\right|^p \int_{|\xi|<r^{-1}} |\xi|^{|\alpha|p}(a\omega+|\xi|)^{n(p-2)} d\xi \\ &\quad= \sum_{|\alpha|\leqslant N_p} \left|\int_{\mathbb{R}^n} M(x)(x-x_0)^\alpha dx\right|^p (a\omega)^{np-n+|\alpha|p} \int_{|\xi|<(a\omega r)^{-1}} |\xi|^{|\alpha|p}(1+|\xi|)^{n(p-2)} d\xi \\ &\quad\leqslant \sum_{|\alpha|\leqslant N_p} \left|\int_{\mathbb{R}^n} M(x)(x-x_0)^\alpha dx\right|^p (a\omega)^{np-n+|\alpha|p} \int_1^{1+(a\omega r)^{-1}} t^{p|\alpha|+np-n-1} dt \\ &\quad\leqslant \sum_{|\alpha|<n\left(\frac{1}{p}-1\right)} \omega^p\,(a\omega)^{p\left[|\alpha|-n\left(\frac{1}{p}-1\right)\right]} \int_1^\infty t^{p\left[|\alpha|-n\left(\frac{1}{p}-1\right)\right]-1} dt \\ &\qquad\qquad + \sum_{\substack{|\alpha|=n\left(\frac{1}{p}-1\right)\\ n\left(\frac{1}{p}-1\right)\in\mathbb{Z}}} \left[\log\left(1+\frac{1}{\omega r}\right)\right]^{-1} \int_1^{1+(a\omega r)^{-1}} t^{-1} dt \\ &\quad\leqslant C_{a,\omega,p} + \left[\log\left(1+\frac{1}{\omega r}\right)\right]^{-1} \log\left(1+\frac{1}{a\omega r}\right) \simeq C_{a,\omega,p}.\end{aligned}$$

□

We now proceed to the proof of Theorem 1.3.

Proof. Let $f \in h^p(\mathbb{R}^n)$. Since the molecular decomposition of f presented in Corollary 1.2 converges in $\mathcal{S}'(\mathbb{R}^n)$, and moreover the Fourier transform is continuous in $\mathcal{S}'(\mathbb{R}^n)$ it follows that

$$\widehat{f} = \sum_{j=1}^\infty d_j \widehat{M}_j.$$

Noticing that

$$\sum_{j=1}^{\infty} |d_j| \leqslant \left(\sum_{j=1}^{\infty} |d_j|^p \right)^{1/p} \leqslant C \, \|f\|_{h^p},$$

we get the desired result after aplying Lemma 1.3. ∎

1.3 Weighted Hardy spaces

In this section we introduce weights in the Muckenhoupt class and the weighted Hardy spaces associated to it. In particular, we are interested in proving a molecular decomposition of such spaces.

A non-negative measurable function w belongs to the Muckenhoupt class A_t for $1 < t < \infty$ if

$$[w]_{A_t} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B w(x)dx \left(\frac{1}{|B|} \int_B w^{\frac{1}{1-t}}(x)dx \right)^{t-1} < \infty,$$

where the supremum is taking over all balls in $\mathbb{R}^n$. We say $w \in A_1$ if

$$[w]_{A_1} := \sup_{B \subset \mathbb{R}^n} \frac{1}{w(x)\chi_B(x)} \frac{1}{|B|} \int_B w(y)dy < \infty. \tag{1.19}$$

We set $A_\infty = \bigcup_{t \geqslant 1} A_t$. It is known that $A_{t_1} \subset A_{t_2}$ with $[w]_{A_{t_2}} \leqslant [w]_{A_{t_1}}$ for $t_1 < t_2$ and if $w \in A_t$ for some $1 < t < \infty$, there exists $1 < s < t$ such that $w \in A_s$. Then, we define the critical index for the weight w to be $t_w := \inf \{t : \; w \in A_t\}$.

We say the weight w satisfies the reverse Hölder inequality with index $1 < r < \infty$, denoted by $w \in RH_r$, if there exists a constant $C > 0$ such that for any $B \subset \mathbb{R}^n$

$$\left(\frac{1}{|B|} \int_B w^r(x)dx \right)^{1/r} \leqslant \frac{C}{|B|} \int_B w(x)dx.$$

A simple Hölder inequality shows that if $w \in RH_r$ then $w \in RH_s$ for all $1 < s < r$ and hence we define the critical index for the Hölder reverse inequality to be $r_w = \sup \{r : \; w \in RH_r\}$. If $r_w = \infty$, it means that $w \in RH_r$ for all $1 < r < \infty$. We also denote by $w(A) := \int_A w$ for any subset $A \subset \mathbb{R}^n$. The following lemma relates Muchenhoupt weights with the Lebesgue measure of a set.

Lemma 1.4 ([35, Chapter IV.2 - Theorem 2.1])**.** *If $w \in A_t \cap RH_r$ for some $t \geqslant 1$ and $r > 1$, then there* *exists constants* C_1, $C_2 > 0$ *such that*

$$C_1 \left(\frac{|A|}{|B|}\right)^t \leqslant \frac{w(A)}{w(B)} \leqslant C_2 \left(\frac{|A|}{|B|}\right)^{1-\frac{1}{r}}$$

for any subsets $A \subset B$ *of* $\mathbb{R}^n$.

Moreover, if we denote by $kB = B(x_0, kr)$ the $k-$th dilation of the ball B, then $w(kB)$ satisfies

Lemma 1.5 ([35, Chapter IV.2 - Lemma 2.2])**.** *If* $w \in A_t$ *for some* $t \geqslant 1$, *then there exists a constant* $C > 0$ *such that for any ball* $B = B(x_0, r) \subset \mathbb{R}^n$ *it follows that* $w(kB) \leqslant C\, k^{nt}\, w(B)$.

For any $0 < p < \infty$ and $w \in A_\infty$, we define the weighted Lebesgue space, denoted by $L^p_w(\mathbb{R}^n) := L^p(\mathbb{R}^n, w(x)dx)$, as the set of all measurable functions such that

$$\|f\|_{L^p_w} := \left(\int |f(x)|^p w(x)dx\right)^{1/p} < \infty.$$

Then, we can define the weighted Hardy spaces analogous as before.

Definition 1.8. *Let* $0 < p < \infty$ *and* $w \in A_\infty$. *We say that* $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to the weighted Hardy space, denote by* $H^p_w(\mathbb{R}^n)$, *if there exists* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *satisfying* $\int \varphi \neq 0$ *such that* $\mathcal{M}_\varphi f \in L^p_w(\mathbb{R}^n)$.

For the characterization in terms of several maximal characterizations, see [74, Chapter VI]. We set the H^p_w norm as $\|f\|_{H^p_w} := \|\mathcal{M}_\varphi f\|_{L^p_w}$. In contrast to the non-weighted theory, given $w \in A_\infty$ we may have $H^p_w(\mathbb{R}^n) \neq L^p_w(\mathbb{R}^n)$ when $p > 1$, since the Hardy space associated to a general weight may contain measures and distributions. However, if $w \in A_p$, then $H^p_w(\mathbb{R}^n) = L^p_w(\mathbb{R}^n)$ ([74, Theorem 1 p. 86]). We refer to [34, 74] for further details and properties of weighted Hardy spaces.

The atomic theory for the weighted Hardy spaces where first developed in [34] for $0 < p \leqslant 1$, dimension $n = 1$ and weights $w \in A_q$ for $q > p$. The n-dimensional case, including also $0 < p < \infty$ and general weight classes can be consulted in [74, Chapter VIII]. See also [71, Section 2.2.1], where atoms with a slightly different size conditions were considered. Here we deal only with the case $0 < p \leqslant 1$.

Definition 1.9. *Let $0 < p \leqslant 1 \leqslant s \leqslant \infty$ with $p < s$ and $w \in A_\infty$. A measurable function a is a (p, w, s) atom in H^p_w if there exist a ball $B = B(x_0, r) \subset \mathbb{R}^n$ such that*

$$(i)\ \ supp\ (a) \subset B, \quad (ii)\ \ \|a\|_{L^s} \leqslant w(B)^{\frac{1}{s}-\frac{1}{p}} \quad (iii)\ \ \int a(x)x^\alpha dx = 0$$

for all $\alpha \in \mathbb{Z}^n_+$ such that $|\alpha| \leqslant N_{w,p} := \left\lfloor n\left(\frac{t_w}{p} - 1\right)\right\rfloor$.

We have the following atomic decomposition theorem for $H^p_w(\mathbb{R}^n)$:

Theorem 1.4. *Let $0 < p \leqslant 1 \leqslant s \leqslant \infty$ with $p < s$ and $w \in A_\infty$. If $f \in H^p_w(\mathbb{R}^n)$, then there exist $\{\lambda_j\}_j \in \ell^p(\mathbb{C})$ and $\{a_j\}_j$ a sequence of (p, w, ∞) atoms in H^p_w such that*

$$f = \sum_{j=1}^{\infty} \lambda_j\, a_j \ \text{ in } H^p_w \text{ norm and } \left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} \lesssim \|f\|_{H^p_w}.$$

Conversely, if $\{a_j\}_j$ is a sequence of (p, w, ∞) atoms and $\{\lambda_j\}_j \in \ell^p(\mathbb{C})$, then $f = \sum_{j=1}^{\infty} \lambda_j\, a_j$ belongs to $H^p_w(\mathbb{R}^n)$ and $\|f\|_{H^p_w} \lesssim \left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p}$. Moreover,

$$\|f\|_{H^p_w} \approx \inf\left\{\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all atomic representations of f.

The molecular structure of weighted Hardy spaces were first studied in [57] for $w \in A_1$ and later in [54] for more general Muckenhoupt classes. In both works, the notion of molecules is a weighted analogous version of the original work of Taibleson and Weiss, discussed earlier in this chapter. Other notion where the integral estimates were replaced by a pointwise one can be found in [71]. We present now a definition based on the same estimates given in Definition 1.3.

Definition 1.10. *Let $0 < p \leqslant 1 \leqslant s < \infty$ with $p < s$ and $w \in A_t$ for some $1 \leqslant t < \infty$. We say that a measurable function M is a (p, w, s, λ) molecule in H^p_w for $t \leqslant s$ and*

$$\lambda > \max\left\{n\left(\frac{s}{p} - 1\right),\ s\left(N_{w,p} + n\right)\frac{r_w}{r_w - 1} - n\right\}, \tag{1.20}$$

if there exists a ball $B = B(x_0, r) \subset \mathbb{R}^n$ and a constant $C > 0$ such that:

M1. $\|M\|_{L^s_w(B)} \leqslant C\, w(B)^{\frac{1}{s}-\frac{1}{p}}$;

M2. $\left\| M\, w\left(B_{|\cdot - x_0|}\right)^{\frac{\lambda}{sn}} \right\|_{L^s_w(B^c)} \leqslant C\, w(B)^{\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}}$*, where $B_{|x-x_0|}$ denotes the ball $B(x_0, |x - x_0|)$;*

M3. $\displaystyle\int M(x) x^\alpha dx = 0$ *for all $|\alpha| \leqslant N_{w,p}$.*

Remark 1.6. Let us provide a few comments about the lower bound of λ in (1.20). Since $N_{w,p} \leqslant n(t_w/p - 1)$, we always have

$$s\,(N_{w,p} + n)\,\frac{r_w}{r_w - 1} - n \leqslant n\left(\frac{s}{p}\,\frac{t_w\, r_w}{r_w - 1} - 1\right) \quad \text{and} \quad n\left(\frac{s}{p} - 1\right) < n\left(\frac{s}{p}\,\frac{t_w\, r_w}{r_w - 1} - 1\right).$$

Thus, we could replace the lower bound in (1.20) by the stronger one

$$\lambda > n\left(\frac{s}{p}\,\frac{t_w\, r_w}{r_w - 1} - 1\right). \tag{1.21}$$

Note that this is stronger only when p is such that $N_{p,w} < n(t_w/p - 1)$. Indeed, for $N_{p,w} = n(t_w/p - 1)$, condition (1.21) is the right one.

This notion allows us to recover the non-weighted molecules (Definition 1.3) considering $w \equiv 1 \in A_1$. Also, just as in Remark 1.1 (i), condition (M1) on B and (M2) on B^c can both be replaced simultaneously by global ones.

The next proposition is the weighted counterpart of Proposition 1.2.

Proposition 1.5. *Suppose M is a (p, w, s, λ) molecule. Then $\|M\|_{H^p_w} \leqslant C$ uniformly.*

Proof. This proof follows [54, Theorem 1] and Proposition 1.2. We keep the same notation and point out here only the differences. Split

$$M = \sum_{j=0}^{\infty} M_j = \sum_{j=0}^{\infty} (M_j - P_j) + \sum_{j=0}^{\infty} P_j$$

in $L^s_w(\mathbb{R}^n)$. We show here that $(M_j - P_j)$ is a multiple of a (p, w, s) atom and P_j a finite linear combination of (p, w, ∞) atoms for each j.

We focus on estimating the L^s_w norm of M_j. For M_0, condition (M1) immediately gives $\|M_0\|_{L^s_w} \leqslant w(B_0)^{\frac{1}{s}-\frac{1}{p}}$. Consider the continuous and strictly decreasing function

$$g(z) = n\left(\frac{s}{p}\cdot\frac{t_w z}{z-1} - 1\right) \quad \text{for } z > 1.$$

Since by hypothesis $\lambda > g(r_w)$, there exists $1 < \delta < r_w$ such that $w \in RH_\delta$ and $\lambda > g(\delta) > g(r_w)$. Consider now $j \in \mathbb{Z}_+$ and $x \in E_j$. Since $B_{|x-x_0|} \subset B_j$, Lemma 1.4 implies

$$\left[\frac{w\left(B_{|x-x_0|}\right)}{w\left(B_j\right)}\right]^{-\frac{\lambda}{n}} \leqslant C\left(\frac{|B_j|}{|B_{|x-x_0|}|}\right)^{\frac{t\lambda}{n}} \leqslant C_{t,n,\lambda}$$

where the constant does not depend on r nor j. Then,

$$\int_{E_j} |M(x)|^s w(x)dx = \int_{E_j} |M(x)|^s \left[\frac{w\left(B_{|x-x_0|}\right)}{w\left(B_j\right)}\right]^{\frac{\lambda}{n}} \left[\frac{w\left(B_{|x-x_0|}\right)}{w\left(B_j\right)}\right]^{-\frac{\lambda}{n}} w(x)dx \lesssim w(B_j)^{-\frac{\lambda}{n}}\, w(B)^{\frac{\lambda}{n}+1-\frac{s}{p}}.$$

Hence, again by Lemma 1.4

$$\begin{aligned}\|M_j\|_{L^s_w} &\lesssim w(B_j)^{-\frac{\lambda}{sn}}\, w(B)^{\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}} = w(B_j)^{\frac{1}{s}-\frac{1}{p}}\left[\frac{w(B)}{w(B_j)}\right]^{\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}} \\ &\lesssim w(B_j)^{\frac{1}{s}-\frac{1}{p}}\left(2^j\right)^{-n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)}. \end{aligned} \tag{1.22}$$

On the other hand, since $t \leqslant s$, in particular $w \in A_s$ and

$$\begin{aligned}|P_j(x)| &\leqslant \left(\sum_{|\gamma|\leqslant d} (2^j r)^{|\gamma|}|\phi^j_\gamma(x)|\right)\frac{1}{|E_j|}\int_{E_j}|M(y)|w^{\frac{1}{s}}(y)w^{-\frac{1}{s}}(y)dy \\ &\lesssim |E_j|^{\frac{1}{s'}-1}\|M_j\|_{L^s_w}\left(\frac{1}{|B_j|}\int_{B_j} w^{-\frac{s'}{s}}(y)dy\right)^{\frac{1}{s'}} \\ &\lesssim \|M_j\|_{L^s_w}\, w(B_j)^{-\frac{1}{s}}. \end{aligned} \tag{1.23}$$

From (1.22) and (1.23) we obtain

$$\|M_j - P_j\|_{L^s_w} \leqslant 2\|M_j\|_{L^s_w} \lesssim w(B_j)^{\frac{1}{s}-\frac{1}{p}}\left(2^j\right)^{-n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)}.$$

From these estimates, writing $(M_j - P_j)(x) = d_j\, a_j(x)$ where $d_j = \|M_j - P_j\|_{L^s_w}\, w(B_j)^{\frac{1}{p}-\frac{1}{s}}$ and

$$a_j(x) = \frac{M_j(x) - P_j(x)}{\|M_j - P_j\|_{L^s_w}}\, w(B_j)^{\frac{1}{t}-\frac{1}{p}},$$

we get that a_j is a (p,w,s) atom and

$$\sum_{j=0}^{\infty} |d_j|^p = \sum_{j=0}^{\infty} \|M_j - P_j\|^p_{L^s_w}\, w(B_j)^{1-\frac{p}{s}} \lesssim \sum_{j=0}^{\infty} (2^j)^{-np\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)} < \infty \tag{1.24}$$

provided that $\lambda > n\,(s/p - 1)$.

We show now the claim for P_j. By Hölder inequality and (1.22) we have

$$\begin{aligned}
|N^{j+1}_\gamma| &= \left|\sum_{k=j+1}^{\infty} \int_{E_k} M(x)x^\gamma dx\right| \lesssim \sum_{k=j+1}^{\infty} (2^k r)^{|\gamma|} \int_{E_k} |M_k(x)|dx \\
&\leqslant \sum_{k=j+1}^{\infty} (2^k r)^{|\gamma|}\, \|M_k\|_{L^s_w}\, |B_k|\, w(B_k)^{-\frac{1}{s}} \\
&\lesssim \sum_{k=j+1}^{\infty} (2^k r)^{|\gamma|+n} w(B_k)^{-\frac{1}{p}} (2^k)^{-n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)} \\
&= (2^j r)^{|\gamma|+n} (2^j)^{-n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)} \sum_{k=0}^{\infty} w(B_{k+j+1})^{-\frac{1}{p}} (2^k)^{|\gamma|+n-n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)}.
\end{aligned}$$

Using Lemma 1.4 we obtain

$$w(B_{k+j+1})^{-\frac{1}{p}} = \left[\frac{w(B_{j+1})}{w(B_{k+j+1})}\right]^{\frac{1}{p}} w(B_{j+1})^{-\frac{1}{p}} \lesssim (2^k)^{-\frac{n}{p}\left(1-\frac{1}{\delta}\right)}\, w(B_{j+1})^{-\frac{1}{p}}$$

and hence since $\lambda > g(\delta)$, for all $|\gamma| \leqslant N_{w,p}$ one has

$$\begin{aligned}
|N^{j+1}_\gamma| &\lesssim (2^j r)^{|\gamma|+n} (2^j)^{-n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)} w(B_{j+1})^{-\frac{1}{p}} \sum_{k=0}^{\infty} (2^k)^{|\gamma|+n-n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}\right)} \\
&\lesssim (2^j r)^{|\gamma|+n} (2^j)^{-n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)} w(B_{j+1})^{-\frac{1}{p}}.
\end{aligned}$$

Using that $(2^j r)^{|\gamma|}|\phi^j_\gamma(x)| \leqslant C$ uniformly and the previous control it follows

$$|N^j_\gamma |E_j|^{-1} \phi^j_\gamma(x)| \lesssim w(B_j)^{-\frac{1}{p}}\, (2^j)^{-n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)}.$$

Denote by $\psi^j_\gamma(x) = h_j\, b_{j\gamma}(x)$ where $h_j = (2^j)^{-n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)}$ and $b_{j\gamma}(x) = k_j \psi^j_\gamma(x)$ for $k_j = (2^j)^{n\left(1-\frac{1}{\delta}\right)\left(\frac{\lambda}{sn}+\frac{1}{s}-\frac{1}{p}\right)}$. It is clear that $b_{j\gamma}$ is a multiple of a (p,w,∞) atom since supp $(B_{j\gamma}) \subset B_j$, $\|b_{j\gamma}\|_{L^\infty} \lesssim w(B_j)^{-\frac{1}{p}}$ and the moment condition follows immediately from (1.7). In addition, just as in (1.24) one has $\sum_{j=0}^{\infty} |h_j|^p < \infty$. ∎

CHAPTER

2

Strongly Singular Calderón–Zygmund operators

We start defining Calderón–Zygmund operators within a more general framework, which were first considered by Coifman and Meyer in [17, Chapter IV], in connection with pseudodifferential operators. Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ to be a linear and continuous operator. From the Schwartz Kernel Theorem (see for instance in [43, Theorem 5.2.1 p. 128]), it is guaranteed the existence of a tempered distribution $W \in \mathcal{S}'(\mathbb{R}^{2n})$, called distributional kernel of T, satisfying

$$\langle T(\varphi), \phi \rangle = \langle W, \varphi \otimes \psi \rangle, \quad \text{for all} \;\; \varphi,\, \psi \in \mathcal{S}(\mathbb{R}^n), \tag{2.1}$$

in which $\varphi \otimes \psi(x,y) = \varphi(x)\psi(y)$. We are interested in operators T as described before in which its distributional kernel coincides with a locally integrable function K defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, where $\Delta = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \; x = y\}$, satisfying certain regularity conditions described in the next definition.

Definition 2.1. *We say that a locally integrable function K defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ is a standard kernel if there exists constants C_1, $C_2 > 0$ and $0 < \delta \leqslant 1$ such that*

$$|K(x,y)| \leqslant \frac{C_1}{|x-y|^n} \;\; \textit{for all} \;\; x \neq y \tag{2.2}$$

and

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leqslant C_2 \, \frac{|y-z|^{\delta}}{|x-z|^{n+\delta}}, \;\; \textit{for all} \; |x-z| \geqslant 2|y-z|. \tag{2.3}$$

When this is the case, the representation of T in the distributional sense (2.1) can be written in terms of an absolute convergent integral given by

$$\langle T(\varphi),\psi\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\varphi(x)\psi(y)dxdy, \quad \text{whenever } \operatorname{supp}(\varphi)\cap\operatorname{supp}(\psi)=\varnothing.$$

Under the additional assumption that T extends to a bounded operator on $L^2(\mathbb{R}^n)$, one can show [38, Proposition 4.1.9] that T can also be represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad \text{for all } f\in L_c^\infty(\mathbb{R}^n) \text{ and } x\notin \operatorname{supp}(f),$$

where $L_c^\infty(\mathbb{R}^n)$ denotes the set of bounded functions with compact support.

Definition 2.2. *Let $T : \mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$ to be a linear and continuous operator associated to a standard kernel K. We say that T is a standard Calderón–Zygmund operator if it can be extended to a bounded operator from $L^2(\mathbb{R}^n)$ to itself.*

In the convolution setting, the L^2 continuity assumed in the previous definition can be derived by assuming $|\widehat{K}(\xi)|\leqslant C$. Necessary and sufficient conditions for the L^2 boundedness of non-convolution Calderón–Zygmund operators were studied by David and Journé in the celebrated $T(1)-$Theorem [24].

Classical examples of standard Calderón–Zygmund operators are pseudodifferential operators of order zero in the Hörmander class $OpS^0_{1,0}(\mathbb{R}^n)$ [17, Theorem 19 p. 87], the Cauchy integral and Calderón commutators [30, p. 99]. For more examples see [3, Section 3].

The classical methods developed by Calderón and Zygmund to obtain L^p inequalities for singular integrals easily apply to Calderón–Zygmund operators, yielding the following well known result:

Theorem 2.1 ([17, Theorem 20 p. 89])**.** *Let T to be a standard Calderón–Zygmund operator. Then, T extends to a bounded operator from $L^p(\mathbb{R}^n)$ to itself when $1<p<\infty$ and satisfies weak $(1,1)$ inequality.*

The proof of the previous theorem relies on the following steps. First, applying the Calderón–Zygmund decomposition [30, Theorem 2.11] and the conditions on the kernel it is possible to show that T satisfies weak $(1,1)$ inequality. This fact together with the boundedness on $L^2(\mathbb{R}^n)$, one gets from Marcinkiewicz

Interpolation Theorem [30, Theorem 2.4] that T is bounded on $L^p(\mathbb{R}^n)$ for every $1 < p \leqslant 2$. Since the adjoint T^* of T is also a Calderón–Zygmund operator (associated to the kernel $\widetilde{K}(x,y) = K(y,x)$) we get the continuity for $p > 2$ by duality. For a detailed proof see [30, Chapter 5].

When considering Hardy spaces $H^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$, Álvarez and Milman showed the following:

Theorem 2.2 ([5, Theorem 1.2])**.** *If T is a standard Calderón–Zygmund operator satisfying $T^*(1) = 0$, then T extended to a bounded operator from $H^p(\mathbb{R}^n)$ to itself provided that* $\dfrac{n}{n+1} < \dfrac{n}{n+\delta} < p \leqslant 1$.

The precise definition of $T^*(1) = 0$ will be provided in Section 2.1. Given $m \in \mathbb{N}$, the range $\dfrac{n}{n+m+1} < p \leqslant \dfrac{n}{n+m}$ can be reached under more cancellation on the operator, namely $T^*(x^\alpha) = 0$, and imposing more regularity on the kernel, that is, assuming that

$$|\partial_y^\alpha K(x,y)| \leqslant C\,|x-y|^{-n-|\alpha|}, \text{ for all } |\alpha| \leqslant m+1.$$

Strongly singular Calderón–Zygmund operators were first motivated by the following multiplier operator. Let $0 < \sigma < 1$, $0 < \beta \leqslant n\sigma/2$ and $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi \equiv 0$ in a neighborhood of the origin and $\psi \equiv 1$ outside a bounded set and consider

$$(T_{\sigma,\beta} f)\widehat{\ }(\xi) = \frac{e^{i|\xi|^\sigma}}{|\xi|^\beta}\psi(\xi)\widehat{f}(\xi), \text{ if } f \in C_c^\infty(\mathbb{R}^n).$$

These operators are not Mihlin-Hörmander type multipliers and provides examples of pseudodifferential operators in the Hörmander class $OpS^{-\beta}_{\sigma,\nu}(\mathbb{R}^n)$ with $0 < \sigma < 1$, $0 \leqslant \nu < 1$ and $\nu \leqslant \sigma$. The L^p boundedness were established by Hirschman and Wainger in [40, 78], where they showed that if

$$\left|\frac{1}{2} - \frac{1}{p}\right| < \frac{\beta}{n}\left[\frac{\dfrac{n}{2} + \dfrac{\frac{n\sigma}{2} - \beta}{1-\sigma}}{\beta + \dfrac{\frac{n\sigma}{2} - \beta}{1-\sigma}}\right] := p_{n,\beta,\sigma},$$

then $T_{\sigma,\beta}$ is bounded on $L^p(\mathbb{R}^n)$ and it is unbounded if $|1/2 - 1/p| > p_{n,\beta,\sigma}$. In order to further investigate the endpoint case, C. Fefferman in [31] showed that if $1/2 - 1/p = p_{n,\beta,\sigma}$, a weaker result holds: $T_{\sigma,\beta}$ maps $L^p(\mathbb{R}^n)$ into the Lorentz space $L^{p,p'}(\mathbb{R}^n)$. The proof relies on showing the weak $(1,1)$ inequality and then it follows by an interpolation argument. Note the case $p = 1$ occurs when $\beta = n\sigma/2$. Based on this special case, he defined a class of convolution operators namely weakly-strongly singular integrals, given by kernels that will be presented in Definition 2.3. These operators include the particular case $T_{\sigma,\frac{n\sigma}{2}}$ and are expressed in terms of kernels that are, as expected, more singular at the diagonal but still nice enough to obtain good continuity properties.

Definition 2.3. *Let $K \in L^1_{loc}(\mathbb{R}^n\backslash\{0\})$ and $0 < \sigma \leqslant 1$. We say K is a weakly-strongly singular kernel if there exists $C > 0$ such that for all $x \in \mathbb{R}^n$ and $|y| < 1$ one has*

$$(i)\ \ |\widehat{K}(x)| \leqslant C(1+|x|)^{-\frac{n(1-\sigma)}{2}}; \quad (ii)\ \ \int_{|x|>2|y|^{\sigma}} |K(x)-K(x-y)|dx \leqslant C.$$

It has been shown in [31, Theorem 2'] that if K is a weakly strongly singular kernel, then the convolution operator associated to it satisfies the weak $(1,1)$ inequality. The endpoint case $L^{\infty}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$ were considered in [32, Theorem 1] by C. Fefferman and Stein and weighted inequalities in [13].

Motivated by convolution weakly strongly singular Calderón–Zygmund operators, Álvarez and Milman in [5] introduced its non-convolution version (see Definition 2.4 bellow) and showed that it falls into the scope of more general classes of pseudodifferential operators in the Hörmander class.

Remind that for standard convolution operators, condition $|\widehat{K}(\xi)| \leqslant C$ implies the L^2 continuity. Replacing this uniform control on the Fourier transform of the kernel by condition *(i)* of Definition 2.3, what type of strong inequality it would imply? Write

$$\widehat{K*f}(\xi) = (1+|\xi|)^{-\beta}\widehat{K}(\xi)\widehat{f}(\xi)(1+|\xi|)^{\beta}$$

and consider a function g such that $\widehat{g}(\xi) = \widehat{K}(\xi)\widehat{f}(\xi)(1+|\xi|)^{\beta}$. Applying the Bessel Potential G_{β} on g and calculating its Fourier transform we get

$$[G_{\beta}(g)]^{\wedge}(\xi) = (1+|\xi|)^{\beta}\,\widehat{g}(\xi) = \widehat{K*f}(\xi),$$

which implies that $G_{\beta}(g)(x) = K * f(x)$. From [38, Corollary 1.2.6 (b)] we have

$$\|T^{*}f\|_{L^{q'}} = \|G_{\beta}(g)\|_{L^{q'}} \lesssim \|g\|_{L^2} = \|\widehat{g}\|_{L^2} \lesssim \|\widehat{K}(\xi)\widehat{f}(\xi)(1+|\xi|)^{\beta}\|_{L^2} \lesssim \|\widehat{f}\|_{L^2} = \|f\|_{L^2}$$

in which $\frac{1}{q'} = \frac{1}{2} - \frac{\beta}{n}$. This imply by duality that T is bounded from $L^q(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ for $\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n}$. Thus, roughly speaking, this stronger decay at the Fourier transform of the kernel can be interpreted as a suitable correction of the L^2 continuity due to action of kernels that are more singular at the diagonal. Therefore we have the following natural extension in the non-convolution setting of weakly-strongly singular integrals.

Definition 2.4 ([5, Definition 2.1]). *We say that a continuous function K defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash \Delta$ is a δ-kernel of type σ for $0 < \delta \leqslant 1$ and $0 < \sigma \leqslant 1$ if there exists a constant $C > 0$ such that*

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leqslant C \frac{|y-z|^\delta}{|x-z|^{n+\frac{\delta}{\sigma}}} \tag{2.4}$$

for all $|x-z| \geqslant 2|y-z|^\sigma$. A linear and continuous operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is called a strongly singular Calderón–Zygmund operator if its distributional kernel restricted away of the diagonal is a δ-kernel of type σ, in the sense

$$\langle Tf, g \rangle = \int\int K(x,y) f(y) g(x) dy dx, \quad \text{for all} \quad f, g \in \mathcal{S}(\mathbb{R}^n) \quad \text{with disjoint supports,}$$

and satisfies the following boundedness properties:

(i) T has bounded extension from $L^2(\mathbb{R}^n)$ to itself;

(ii) T and T^* extend to continuous operators from $L^q(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, in which

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n} \quad \text{for some} \quad (1-\sigma)\frac{n}{2} \leqslant \beta < \frac{n}{2}.$$

These non-convolution operators recover the classical Calderón–Zygmund operators as a limit case $\sigma = 1$ and $\beta = 0$. Examples will be discussed in Section 2.3.

A natural question arises on investigating the boundedness properties of these operators in a wide range of functional spaces and the relation between these properties and the condition imposed on the kernel. This is the question we are going to consider in the following sections.

2.1 Continuity in $H^p(\mathbb{R}^n)$

The boundedness of standard Calderón–Zygmund operators (i.e. $\sigma = 1$) on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ can be actually shown under a more general integral condition on the kernel, known as Hörmander condition and given by

$$\sup_{z\in\mathbb{R}^n} \int_{|x-z|\geqslant 2|y-z|} |K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|dx \leqslant C$$

(see [30, Theorem 5.10]). It is a simple calculation to verify that (2.3) implies Hörmander condition. The question of whether or not this condition is sufficient to guarantee the boundedness in Hardy spaces is more delicate and still not completely known; only the case $H^1(\mathbb{R}^n)$ has been investigated so far and a negative answer was provided in [81, Theorem 2], where the authors constructed a kernel satisfying Hörmander condition in which the linear operator associated to it is bounded on $L^2(\mathbb{R}^n)$, but is not bounded in $H^1(\mathbb{R}^n)$.

In [6], the authors considered an integral Hörmander-type condition of kernels in the strongly singular setting. We say that a kernel $K(x,y)$ satisfies the $\sigma-$Hörmander condition if

$$\sup_{\substack{|y-z|\leqslant 1\\ z\in\mathbb{R}^n}} \int_{|x-z|\geqslant 2|y-z|^\sigma} |K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|dx \leqslant C \tag{2.5}$$

and

$$\sup_{\substack{|y-z|>1\\ z\in\mathbb{R}^n}} \int_{|x-z|\geqslant 2|y-z|} |K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|dx \leqslant C. \tag{2.6}$$

It is a simple calculation to see that $\delta-$kernels of type σ satisfy (2.5) and (2.6). In this case, we say that a strongly singular Calderón–Zygmund operator is associated to a kernel satisfying $\sigma-$Hörmander condition if it is an operator in the sense of Definition 3.2 where condition (2.4) is replaced by (2.5) and (2.6).

Moreover, the authors also improved the continuity results for $L^p(\mathbb{R}^n)$ when $1 \leqslant p < \infty$ assuming the $\sigma-$Hörmander condition. Using analogous ideas of C. Fefferman in [31, Theorem 2'], they were able to show that:

Theorem 2.3 ([6, Theorem 4.1])**.** *If T is a strongly singular Calderón–Zygmund operator associated to a kernel satisfying $\sigma-$Hörmander condition, then T satisfies weak $(1,1)$ type inequality.*

Using interpolation between the weak $(1,1)$ inequality and the $L^2(\mathbb{R}^n)$ continuity one can get the bondedness in $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. It is also straight forward to verify that we can further obtain the boundedness from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ using this condition, as shown in the next proposition.

Proposition 2.1. *Let T be a strongly singular Calderón–Zygmund operator associated to a kernel satisfying $\sigma-$Hörmander condition. Then, T is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

Proof. Let a to be a $(1,2)$ atom in H^1 supported on $B = B(x_0, r) \subset \mathbb{R}^n$. If $r \leqslant 1$ we split

$$\int_{\mathbb{R}^n} |Ta(x)|dx = \int_{2B^\sigma} |Ta(x)|dx + \int_{(2B^\sigma)^c} |Ta(x)|dx := I_1 + I_2. \tag{2.7}$$

To control I_1, we use the continuity from $L^q(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ to obtain

$$\int_{2B^\sigma} |Ta(x)|dx \leqslant |2B^\sigma|^{\frac{1}{2}} \|Ta\|_{L^2} \lesssim r^{\frac{\sigma n}{2}} \|a\|_{L^q} \lesssim r^{n\left[\frac{1}{q}-\left(1-\frac{\sigma}{2}\right)\right]} \lesssim 1$$

since $1/q > 1 - \sigma/2$ and $0 < r \leqslant 1$. To estimate I_2, since a has vanish integral one can write

$$|Ta(x)| = \left| \int_B [K(x,y) - K(x,x_0)]\, a(y)dy \right| \leqslant r^{-n} \int_B |K(x,y) - K(x,x_0)|\, dy.$$

Hence, by Fubini Theorem and (2.5)

$$\int_{(2B^\sigma)^c} |Ta(x)|dx \lesssim r^{-n} \int_B \int_{(2B^\sigma)^c} |K(x,y) - K(x,x_0)|dxdy \lesssim 1.$$

The case $r > 1$ is analogous splitting the integral (2.7) over $2B$ and $(2B)^c$ and using the L^2 continuity together with (2.6). □

Remark 2.1. If T is a strongly singular Calderón–Zygmund operator of convolution type, the continuity from $H^1(\mathbb{R}^n)$ to itself follows from the previous proposition. In fact, since T is a convolution operator,

$R_j(T) = T(R_j)$ for every $1 \leqslant j \leqslant n$. Hence, using characterization of $H^1(\mathbb{R}^n)$ in terms of Riesz Transforms (see [32, p. 123]), the previous proposition and the fact that R_j is a bounded opertor on $H^1(\mathbb{R}^n)$ for every $j = 1, \cdots, n$, it follows

$$\|Tf\|_{H^1} = \|Tf\|_{L^1} + \sum_{j=1}^{n} \|R_j(Tf)\|_{L^1} = \|Tf\|_{L^1} + \sum_{j=1}^{n} \|T(R_jf)\|_{L^1} \lesssim \|f\|_{H^1} + \sum_{j=1}^{n} \|R_jf\|_{H^1} \lesssim \|f\|_{H^1}.$$

Unfortunately, it is still an open question if $\sigma-$Hörmander condition is sufficient to guarantee the bondedness of strongly singular Calderón–Zygmund operators for $H^p(\mathbb{R}^n)$ when $0 < p \leqslant 1$. However, some progress can be made assuming an L^s integral condition on annulus, which we call in this work D_s condition, weaker then (2.4) but stronger then $\sigma-$Hörmander condition.

Definition 2.5. *Let $0 < \rho \leqslant \sigma \leqslant 1 \leqslant s < \infty$ and $\delta > 0$. We say the kernel $K(x, y)$ associated to T satisfies the D_s condition if*

$$\sup_{\substack{r>1 \\ z\in\mathbb{R}^n}} \sup_{|y-z|<r} \left(\int_{C_j(z,r)} |K(x,y) - K(x,z)|^s + |K(y,x) - K(z,x)|^s dx \right)^{1/s} \lesssim |C_j(z,r)|^{\frac{1}{s}-1} 2^{-j\delta} \tag{2.8}$$

and

$$\sup_{\substack{0<r<1 \\ z\in\mathbb{R}^n}} \sup_{|y-z|<r} \left(\int_{C_j(z,r^\rho)} |K(x,y) - K(x,z)|^s + |K(y,x) - K(z,x)|^s dx \right)^{1/s} \lesssim |C_j(z,r^\rho)|^{\frac{1}{s}-1+\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}}, \tag{2.9}$$

where $C_j(z,\tilde{r}) = \{x \in \mathbb{R}^n : \ 2^j\tilde{r} < |x - z| \leqslant 2^{j+1}\tilde{r}\}$.

It is easy to verify that D_{s_1} condition is stronger than D_{s_2} if $s_1 > s_2$. In that sense, D_1 is more general and closest of $\sigma-$Hörmander condition. A standard calculation also shows that for every $\rho \leqslant \sigma$, $\delta-$kernels of type σ satisfy D_s conditions for every $1 \leqslant s < \infty$. By simplicity, we use the nomenclature D_s condition omitting the dependence of σ, ρ, and δ. If necessary to emphasize the decay δ, we write D_s condition with decay δ (see for instance Proposition 2.3).

Estimates of this type are slightly different of $D_{s,\alpha}$ conditions considered in [27, Definition 1.1 p.12] and they are naturally related to kernels associated to pseudodifferential operators in the Hörmander class

$OpS^m_{\sigma,\nu}(\mathbb{R}^n)$ with $0 < \sigma \leqslant 1$ and $0 \leqslant \nu < 1$. In particular, it has been shown in [4, Theorem 5.1] that if $\nu \leqslant \sigma$ and $m \leqslant -n(1-\sigma)/2$, the kernel $K(x,y)$ associated to $T \in OpS^m_{\sigma,\nu}(\mathbb{R}^n)$ satisfies the D_1 condition with $\delta = 1$, that is

$$\sup_{\substack{r>1 \\ z\in\mathbb{R}^n}} \sup_{|y-z|<r} \int_{C_j(z,r)} |K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| dx \lesssim 2^{-j\delta}, \tag{2.10}$$

and

$$\sup_{\substack{0<r<1 \\ z\in\mathbb{R}^n}} \sup_{|y-z|<r} \int_{C_j(z,r^\rho)} |K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| dx \lesssim |C_j(z,r^\rho)|^{\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}}. \tag{2.11}$$

The D_1 condition have already been explored in [2, Theorem 5.2] and [6, Theorem 3.9] to deal with continuity from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$.

Lets turn our attention to the continuity of strongly singular Calderón–Zygmund operators from $H^p(\mathbb{R}^n)$ to itself for $0 < p \leqslant 1$ considering D_s conditions. As it is well known from standard operators, the continuity in this case can be expressed in terms of the so called T^* condition, which dictates the amount of cancellation needed at the image of the operator. For instance, Álvarez and Milman have shown in [5, Theorems 1.1 and 2.2] that a sufficient condition for the continuity in $H^p(\mathbb{R}^n)$, for $n/(n+1) < p \leqslant 1$, is $T^*(1) = 0$, which means that

$$\int_{\mathbb{R}^n} Tf(x)dx = 0, \text{ whenever } f \text{ is an } L^2 \text{ function supported on a ball with vanish integral.}$$

In the convolution setting, such condition is immediately true, since

$$\int_{\mathbb{R}^n} Tf(x)dx = \widehat{Tf}(0) = \widehat{K*f}(0) = \widehat{K}(0)\cdot\widehat{f}(0) = 0.$$

To deal with the range $0 < p \leqslant n/(n+1)$, more vanishing moments conditions are required on the image of the operator, which leads us to the definition of $T^*(x^\alpha)$, for α depending on p.

Definition 2.6 ([64, p. 23])**.** *Let $m \in \mathbb{Z}^+$ and*

$$L^2_{c,m}(\mathbb{R}^n) = \left\{ g \in L^2(\mathbb{R}^n) : \text{ g has compact support and } \int x^\alpha g(x)dx = 0 \text{ for all } |\alpha| \leqslant m \right\}.$$

We say that an operator T satisfies $T^*(x^\alpha) = 0$ for $|\alpha| \leqslant m$ if

$$\int x^\alpha T f(x)dx = 0, \quad \text{for all } f \in L^2_{c,m}(\mathbb{R}^n). \tag{2.12}$$

Remark 2.2. If a is a (p,s) atom in H^p for $2 \leqslant s \leqslant \infty$, then $a \in L^2_{c,N_p}(\mathbb{R}^n)$. Moreover, given $f \in L^2_{c,N_p}(\mathbb{R}^n)$, $a = \dfrac{|B|^{\frac{1}{2}-\frac{1}{p}} f}{\|f\|_{L^2}}$ is a $(p,2)$ atom, that is, f is a multiple of an atom.

Next we show that $T^*(x^\alpha)$ is well defined for operators whose kernel satisfies D_1 condition, and consequently for every kernel satisfying the D_s condition for $1 < s < \infty$.

Proposition 2.2. *Let T be a linear and bounded operator on $L^2(\mathbb{R}^n)$ whose kernel associated to it satisfies the D_1 condition. Then $x^\alpha T f \in L^1(\mathbb{R}^n)$ for all $f \in L^2_{c,m}(\mathbb{R}^n)$ with $m = \lfloor \delta \rfloor$.*

Proof. Suppose without loss of generality that supp $(f) \subset B(0,r)$. If $r \geqslant 1$, write

$$\int_{\mathbb{R}^n} |x^\alpha T f(x)|dx = \int_{B(0,2r)} |x^\alpha T f(x)|dx + \int_{\mathbb{R}^n \setminus B(0,2r)} |x^\alpha T f(x)|dx.$$

From Hölder inequality and the L^2 boundedness of T we get

$$\int_{B(0,2r)} |x^\alpha T f(x)|dx \leqslant \|x^\alpha\|_{L^\infty(B(0,2r))} \, |B(0,2r)|^{\frac{1}{2}} \, \|T f\|_{L^2} \lesssim r^{|\alpha|+\frac{n}{2}} \|f\|_{L^2} < \infty.$$

For the second integral, since $f \in L^2_{c,m}(\mathbb{R}^n)$ we may estimate

$$|T f(x)| = \left| \int_{B(0,r)} [K(x,y) - K(x,0)] f(y)dy \right| \leqslant \int_{B(0,r)} |K(x,y) - K(x,0)||f(y)|dy.$$

Then,

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B(0,2r)} |x^\alpha Tf(x)|dx &\leqslant \int_{B(0,r)} |f(y)| \int_{\mathbb{R}^n\setminus B(0,2r)} |x|^{|\alpha|}\ |K(x,y)-K(x,0)|dxdy \\
&\leqslant \sum_{j=0}^{\infty} (2^j r)^{|\alpha|} \int_{B(0,r)} |f(y)| \int_{C_j(0,r)} |K(x,y)-K(x,0)|dxdy \\
&\lesssim \sum_{j=0}^{\infty} (2^j r)^{|\alpha|} \|f\|_{L^2}\ |B(0,r)|^{\frac{1}{2}}\ 2^{-j\delta} \\
&\leqslant r^{|\alpha|+\frac{n}{2}}\ \|f\|_{L^2}\ \sum_{j=0}^{\infty} (2^j)^{|\alpha|-\delta} < \infty
\end{aligned}$$

since $|\alpha| < \delta$. For $r < 1$ we write

$$\int_{\mathbb{R}^n} |x^\alpha Tf(x)|dx = \int_{B(0,2r^\rho)} |x^\alpha Tf(x)|dx + \int_{\mathbb{R}^n\setminus B(0,2r^\rho)} |x^\alpha Tf(x)|dx,$$

for some $0 < \rho \leqslant \sigma < 1$. The estimate of the first integral is the same as the previous case and for the second

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B(0,2r^\rho)} |x^\alpha Tf(x)|dx &\leqslant \sum_{j=0}^{\infty} (2^j r^\rho)^{|\alpha|} \int_{B(0,r)} |f(y)| \int_{C_j(0,r^\rho)} |K(x,y)-K(x,0)|dxdy \\
&\lesssim r^{|\alpha|\rho+\frac{n}{2}+\delta-\frac{\rho\delta}{\sigma}}\ \|f\|_{L^2}\ \sum_{j=0}^{\infty} (2^j)^{|\alpha|-\frac{\delta}{\sigma}} < \infty
\end{aligned}$$

since $|\alpha| < \delta$ implies $|\alpha| < \delta/\sigma$. □

In order to provide a complete understanding on how the parameters in the next continuity result are related to the hypothesis assumed on the kernel and on the operator, we state it in a more general framework, although with a more complicated notation. The result is the following:

Theorem A. *Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear and continuous operator such that:*

(i) T extends to a continuous operator from $L^2(\mathbb{R}^n)$ to itself;

(ii) There exists $1 \leqslant s_1 < \infty$ such that T is associated to a kernel satisfying D_{s_1} condition;

(iii) T extends to a continuous operator from $L^q(\mathbb{R}^n)$ to $L^{s_2}(\mathbb{R}^n)$, for some $1 < s_2 < \infty$ and

$$\frac{1}{q} = \frac{1}{s_2} + \frac{\beta}{n}, \quad \textit{where} \quad n(1-\sigma)\left(1 - \frac{1}{s_2}\right) \leqslant \beta < n\left(1 - \frac{1}{s_2}\right).$$

Under such conditions, if $T^*(x^\alpha) = 0$ for all $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| \leqslant \lfloor \delta \rfloor$, $p < s_1$ and $s_1 \leqslant s_2$, then T can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to itself for every $p_0 < p \leqslant 1$, where

$$\frac{1}{p_0} := \frac{1}{s_2} + \frac{\beta\left[\frac{\delta}{\sigma} + n\left(1 - \frac{1}{s_2}\right)\right]}{n\left(\frac{\delta}{\sigma} - \delta + \beta\right)}. \tag{2.13}$$

Conversely, if T is a bounded operator from $H^p(\mathbb{R}^n)$ to itself for every $p_0 < p \leqslant 1$, then $T^*(x^\alpha) = 0$ for every $|\alpha| \leqslant N_{p_0}$.

This result extends [5, Theorem 2.2] with additional advantage of considering kernels associated to weaker integral conditions. In addition, our approach enables us to include the D_1 condition only for $p < 1$, which represents the closest of Hörmander condition we are able to provide a satisfactory answer to continuity results in Hardy spaces.

In contrast to condition (2.4), although any upper bound on δ is assumed on the D_s condition, examples of operators whose associated kernel satisfies it with $\delta > 1$ will be considered in Section 2.3 with a suitable refinement of D_s conditions, assuming control of derivatives (see (2.20) and (2.21)).

The conclusion of Theorem A for $p = p_0$ is still not known, however, if $s_2 = 2$ and under D_1 condition, [6, Theorem 3.9] asserts that T can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $p_0 \leqslant p \leqslant 1$. The inclusion of p_0 in this case is a quite interesting result since it is known that if T is a standard Calderón–Zygmund operator ($\sigma = 1$), then there exists $f \in H^{\frac{n}{n+\delta}}(\mathbb{R})$ such that $Tf \notin L^{\frac{n}{n+\delta}}(\mathbb{R})$ (see [2, Theorem 1.2]). Restricting ourselves to the case where T is a convolution operator, the continuity from $H^{p_0}(\mathbb{R}^n)$ to itself holds (see [5, Theorem 2.3]).

2.1.1 Proof of Theorem A

Let $t > \max\{s_1, q\}$ and a a (p,t) atom in H^p supported on $B = B(x_0, r) \subset \mathbb{R}^n$. In particular, a is (p, s_1) and (p,q) atoms. From Lemma 1.2, it suffices to show that Ta is an (p, λ, s_1) molecule in H^p for an appropriate range of λ. Suppose first $r > 1$. Since T is bounded from $L^t(\mathbb{R}^n)$ to itself we have

$$\int_{2B} |Ta(x)|^{s_1} dx \leqslant |2B|^{1-\frac{s_1}{t}} \|Ta\|_{L^t}^{s_1} \lesssim |2B|^{1-\frac{s_1}{t}} \|a\|_{L^t}^{s_1} \lesssim r^{n\left(1-\frac{s_1}{p}\right)}, \tag{2.14}$$

which proves (M1a). Note the previous estimate also covers the case $s_1 = 1$. To estimate (M2a), the moment condition of the atom allow us to write

$$\int_{(2B)^c} |Ta(x)|^{s_1} |x-x_0|^{\lambda} dx = \sum_{j=0}^{\infty} \int_{C_j} \left| \int_B [K(x,y) - K(x,x_0)] a(y) dy \right|^{s_1} |x-x_0|^{\lambda} dx,$$

where $C_j = C_j(x_0, r)$. Then, applying Minkowski inequality for integrals, Hölder's inequality and D_{s_1} condition we can control the previous sum as

$$\begin{aligned}
&\sum_{j=0}^{\infty} \left\{ \left[\int_{C_j} \left(\int_B |K(x,y) - K(x,x_0)|\, |a(y)|\, |x-x_0|^{\frac{\lambda}{s_1}} dy \right)^{s_1} dx \right]^{\frac{1}{s_1}} \right\}^{s_1} \\
&\quad \leqslant \sum_{j=0}^{\infty} \left\{ \int_B |a(y)| \left[\int_{C_j} |K(x,y) - K(x,x_0)|^{s_1} |x-x_0|^{\lambda} dx \right]^{\frac{1}{s_1}} dy \right\}^{s_1} \\
&\quad \leqslant \sum_{j=0}^{\infty} (2^j r)^{\lambda} \left\{ \int_B |a(y)| \left[\int_{C_j} |K(x,y) - K(x,x_0)|^{s_1} dx \right]^{\frac{1}{s_1}} dy \right\}^{s_1} \\
&\quad \lesssim \sum_{j=0}^{\infty} (2^j r)^{\lambda} (2^j r)^{-n(s_1-1)} 2^{-js_1\delta} \|a\|_{L^1}^{s_1} \\
&\quad \lesssim \sum_{j=0}^{\infty} (2^j r)^{\lambda} (2^j r)^{-n(s_1-1)} 2^{-js_1\delta} r^{s_1 n\left(1-\frac{1}{p}\right)} \\
&\quad \simeq r^{\lambda + n\left(1-\frac{s_1}{p}\right)} \sum_{j=0}^{\infty} 2^{j[\lambda - n(s_1-1) - s_1\delta]} \simeq r^{\lambda + n\left(1-\frac{s_1}{p}\right)}
\end{aligned} \tag{2.15}$$

assuming $\lambda < n(s_1 - 1) + s_1\delta$.

Now, we move on to the case $r \leqslant 1$. The estimate of $2B$ will be the same since it does not depend on the kernel. For the one outside the double ball, we will do it for $\mathbb{R}^n$. Let $0 < \rho \leqslant \sigma \leqslant 1$ a parameter to be determined precisely later and to simplify the notation consider $2B^\rho := B(x_0, 2r^\rho)$ and $C_j^\rho := C_j(x_0, r^\rho)$. We will split the global integral into

$$\int_{\mathbb{R}^n} |Ta(x)|^{s_1} |x - x_0|^\lambda dx = \int_{2B^\rho} |Ta(x)|^{s_1} |x - x_0|^\lambda dx + \int_{(2B^\rho)^c} |Ta(x)|^{s_1} |x - x_0|^\lambda dx.$$

For the first one, since $s_1 \leqslant s_2$ and T is bounded from $L^q(\mathbb{R}^n)$ to $L^{s_2}(\mathbb{R}^n)$ we have

$$\begin{aligned} \int_{2B^\rho} |Ta(x)|^{s_1} |x - x_0|^\lambda dx &\lesssim r^{\lambda\rho} \int_{2B^\rho} |Ta(x)|^{s_1} dx \lesssim r^{\lambda\rho + n\rho\left(1 - \frac{s_1}{s_2}\right)} \|Ta\|_{L^{s_2}}^{s_1} \\ &\lesssim r^{\lambda\rho + n\rho\left(1 - \frac{s_1}{s_2}\right)} \|a\|_{L^q}^{s_1} \\ &\lesssim r^{\lambda + n\left(1 - \frac{s_1}{p}\right)} r^{-\lambda(1-\rho) + n\left[\rho\left(1 - \frac{s_1}{s_2}\right) + \frac{s_1}{q} - 1\right]} \\ &\lesssim r^{\lambda + n\left(1 - \frac{s_1}{p}\right)}, \end{aligned} \tag{2.16}$$

assuming

$$\lambda \leqslant n\left(\frac{s_1}{s_2} - 1\right) + \frac{s_1 \beta}{1 - \rho}.$$

For the second integral, using the same argument as before and D_{s_1} condition we get

$$\begin{aligned} \int_{(2B^\rho)^c} |Ta(x)|^{s_1} |x - z|^\lambda dx &\leqslant \sum_{j=0}^{\infty} (2^j r^\rho)^\lambda \left\{ \int_B |a(y)| \left[\int_{C_j^\rho} |K(x,y) - K(x,x_0)|^{s_1} dx \right]^{\frac{1}{s_1}} dy \right\}^{s_1} \\ &\lesssim \sum_{j=0}^{\infty} (2^j r^\rho)^\lambda \left(|C_j^\rho|^{\frac{1}{s_1} - 1 + \frac{\delta}{n}\left(\frac{1}{\rho} - \frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}} \right)^{s_1} \|a\|_{L^1}^{s_1} \\ &\lesssim r^{\lambda + \left(1 - \frac{s_1}{p}\right)} r^{-\lambda(1-\rho) + n\left[s_1 + \frac{s_1\delta}{n} - s_1\rho\left(1 - \frac{1}{s_1} + \frac{\delta}{n\sigma}\right) - 1\right]} \sum_{j=0}^{\infty} 2^{j\left[\lambda - n(s_1 - 1) - \frac{s_1\delta}{\sigma}\right]} \\ &\lesssim r^{\lambda + \left(1 - \frac{s_1}{p}\right)}, \end{aligned} \tag{2.17}$$

assuming $\lambda < n(s_1 - 1) + s_1\delta/\sigma$ and choosing ρ to be such that

$$-\lambda(1-\rho) + n\left[s_1 + \frac{s_1\delta}{n} - s_1\rho\left(1 - \frac{1}{s_1} + \frac{\delta}{n\sigma}\right) - 1\right] = -\lambda(1-\rho) + n\left[\rho\left(1 - \frac{s_1}{s_2}\right) + \frac{s_1}{q} - 1\right],$$

that is,

$$s_1 + \frac{s_1\delta}{n} - \rho\left(s_1 - 1 + \frac{s_1\delta}{n\sigma}\right) = \rho\left(1 - \frac{s_1}{s_2}\right) + \frac{s_1}{q} \Leftrightarrow \rho = \frac{n\left(1-\frac{1}{q}\right)+\delta}{n\left(1-\frac{1}{s_2}\right)+\frac{\delta}{\sigma}}. \tag{2.18}$$

As pointed out in Definition 2.5, it is also important to show that $\rho \leqslant \sigma$. In fact,

$$\begin{aligned}
\beta \geqslant n(1-\sigma)\left(1-\frac{1}{s_2}\right) &\Leftrightarrow 1 - \frac{1}{q} \leqslant \sigma\left(1-\frac{1}{s_2}\right) \\
&\Leftrightarrow n\left(1-\frac{1}{q}\right) + \delta \leqslant n\sigma\left(1-\frac{1}{s_2}\right) + \delta \\
&\Leftrightarrow \rho = \frac{n\left(1-\frac{1}{q}\right)+\delta}{n\left(1-\frac{1}{s_2}\right)+\frac{\delta}{\sigma}} \leqslant \sigma.
\end{aligned}$$

Summing up, to obtain the desired estimates, we have imposed the following conditions on λ:

(a) $\lambda > n\left(\frac{s_1}{p} - 1\right)$ (b) $\lambda < n(s_1-1) + \frac{s_1\delta}{\sigma}$

(c) $\lambda < n(s_1-1) + s_1\delta$ (d) $\lambda \leqslant n\left(\frac{s_1}{s_2} - 1\right) + \frac{s_1\beta}{1-\rho}$.

Since $0 < \sigma \leqslant 1$, it immediately follows that (c) $\Rightarrow$ (b). We also have (d) $\Rightarrow$ (c) since

$$\begin{aligned}
\beta < n\left(1-\frac{1}{s_2}\right) &\Leftrightarrow \beta < \frac{\delta\left(\frac{1}{\sigma}-1\right)\left[n\left(1-\frac{1}{s_2}\right)+\delta\right]}{\delta\left(\frac{1}{\sigma}-1\right)} \\
&\Leftrightarrow s_2\beta\delta\left(\frac{1}{\sigma}-1\right) < \delta\left(\frac{1}{\sigma}-1\right)\left[(s_2-1)n + s_2\delta\right] \\
&\Leftrightarrow \beta\left[n(s_2-1) + \frac{s_2\delta}{\sigma}\right] - \beta\left[(s_2-1)n + s_2\delta\right] < \delta\left(\frac{1}{\sigma}-1\right)\left[(s_2-1)n + s_2\delta\right] \\
&\Leftrightarrow \beta\left[n(s_2-1) + \frac{s_2\delta}{\sigma}\right] < \left[(s_2-1)n + s_2\delta\right]\ \left(\beta + \frac{\delta}{\sigma} - \delta\right) \\
&\Leftrightarrow \frac{s_2\beta\left[n\left(1-\frac{1}{s_2}\right)+\frac{\delta}{\sigma}\right]}{\beta+\frac{\delta}{\sigma}-\delta} < (s_2-1)n + s_2\delta \\
&\Leftrightarrow \frac{s_2\beta}{1-\rho} < (s_2-1)n + s_2\delta
\end{aligned}$$

and this implies

$$n\left(\frac{s_1}{s_2}-1\right) + \frac{s_1\beta}{(1-\rho)} < n(s_1-1) + s_1\delta.$$

Therefore, the lower and upper bound for λ is given by (a) and (d) respectively. This implies a lower bound for p given by

$$n\left(\frac{s_1}{p}-1\right) < n\left(\frac{s_1}{s_2}-1\right) + \frac{s_1\beta}{1-\rho} \Leftrightarrow \frac{1}{p} < \frac{1}{s_2} + \frac{\beta\left[\frac{\delta}{\sigma}+n\left(1-\frac{1}{s_2}\right)\right]}{n\left(\frac{\delta}{\sigma}-\delta+\beta\right)} = \frac{1}{p_0}.$$

Note that $n/(n+\delta) < p_0 < p \leqslant 1$, thus $n(1/p-1) < \delta$. Since $T^*(x^\alpha) = 0$ for $|\alpha| \leqslant \lfloor\delta\rfloor$, then (M3) holds for every $|\alpha| \leqslant N_p$.

For the converse, suppose that T maps continuously $H^p(\mathbb{R}^n)$ into itself for all $p_0 < p \leqslant 1$ and let $f \in L^2_{c,N_{p_0}}(\mathbb{R}^n)$. Using only the conditions on the kernel and the L^2 continuity of the operator we can follow the proof of Proposition 2.2 and get that $Tf \in L^1(\mathbb{R}^n)$. Moreover, by the boundedness hypothesis $Tf \in L^1(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$. Hence, by [73, Sec. 5.4 (c) p.128] it follows that

$$\int_{\mathbb{R}^n} Tf(x)x^\alpha dx = 0, \text{ for all } |\alpha| \leqslant N_p \text{ and } p_0 < p \leqslant 1.$$

Therefore it will also holds for N_{p_0} since $p \searrow p_0$ and consequently $N_p = N_{p_0}$ for p sufficiently close to p_0.

Remark 2.3. The authors in [5] used a refined version of the molecular decomposition to prove the analogous continuity result for $\frac{n}{n+1} < p \leqslant 1$ assuming Hölder regularity (see [5, Lemma 2.1]). As we have seen before, it is not needed and we can prove it using the standard molecular decomposition.

Remark 2.4. In the proof of Theorem A we have shown that when a is an atom, Ta is a molecule and hence $\|Ta\|_{H^p} \leqslant C$ uniformly. This suggests that T extends to a bounded operator from $H^p(\mathbb{R}^n)$ to itself, however since the atomic decomposition may not be unique, an additional argument is needed. This can be done using an approximation argument of the operator T. For a precise description on how to extend continuously a Calderón–Zygmund operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ bounded in $L^2(\mathbb{R}^n)$ to the space $H^p(\mathbb{R}^n)$ see [9, Chapter 1.9] and also [64]. The proof has been done for $\sigma = 1$ but the same arguments can be adapted when $0 < \sigma < 1$.

2.1.2 Extensions of Theorem A

It is also natural to ask the validity of Theorem A if one want to investigate the continuity from $H^{p_1}(\mathbb{R}^n)$ to $H^{p_2}(\mathbb{R}^n)$. Under the same assumptions *(i)-(iii)*, for $p_1 \leqslant p_2$, it is enough to show that T maps (p_1, t) atoms in H^{p_1} into (p_2, λ, s_1) molecules in H^{p_2}. For $r \geqslant 1$, conditions (M1) and (M2) can be verified using the $L^2(\mathbb{R}^n)$ continuity, the fact that $p_1 \leqslant p_2$ and the condition on the kernel. On the other hand, if $0 < r < 1$, condition (M1) can be proved using the continuity from $L^q(\mathbb{R}^n)$ to $L^{s_2}(\mathbb{R}^n)$ and

$$\frac{1}{p_2} > \frac{1}{p_1} - \frac{\beta}{n}.$$

For (M2), using the same choice of ρ as in the proof of Theorem A we get that

$$\lambda \leqslant n\left(\frac{s_1}{s_2} - 1\right) + \frac{ns_1}{(1-\rho)}\left(\frac{\beta}{n} + \frac{1}{p_2} - \frac{1}{p_1}\right).$$

This bound together with the lower bound from the molecular decomposition one gets

$$\frac{1}{p_2} < \frac{1}{s_2} + \frac{\left[\beta + n\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\right]\left[\frac{\delta}{\sigma} + n\left(1 - \frac{1}{s_2}\right)\right]}{n\left(\frac{\delta}{\sigma} - \delta + \beta\right)},$$

that can be rewritten as

$$\frac{1}{p_2} > \frac{\beta + \delta\left(\frac{1}{\sigma} - 1\right)}{s_2(\beta - n - \delta) + n} + \frac{\left(\frac{\beta}{n} - \frac{1}{p_1}\right)\left[n\left(1 - \frac{1}{s_2}\right) + \frac{\delta}{\sigma}\right]}{\beta - \delta - n\left(1 - \frac{1}{s_2}\right)}.$$

Therefore, under the assumptions *(i)* to *(iii)*, $p_1 < s_1 \leqslant s_2$ and the same cancellation condition, T maps $H^{p_1}(\mathbb{R}^n)$ into $H^{p_2}(\mathbb{R}^n)$ continuously for every $0 < p_1 \leqslant p_2 \leqslant 1$ such that

$$\frac{1}{p_2} > \max\left\{\frac{1}{p_1} - \frac{\beta}{n}, \frac{\beta + \delta\left(\frac{1}{\sigma} - 1\right)}{s_2(\beta - n - \delta) + n} + \frac{\left(\frac{\beta}{n} - \frac{1}{p_1}\right)\left[n\left(1 - \frac{1}{s_2}\right) + \frac{\delta}{\sigma}\right]}{\beta - \delta - n\left(1 - \frac{1}{s_2}\right)}\right\}.$$

Now we consider *strongly singular Calderón–Zygmund operators of type* σ associated kernels satisfying derivative conditions and we show an analogous version of Theorem A under such conditions.

Let $\delta > 0$ and $K \in C^{\lfloor\delta\rfloor}(\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,x) : x \in \mathbb{R}^n\})$ satisfying

$$|\partial_y^\alpha K(x,y) - \partial_y^\alpha K(x,z)| + |\partial_y^\alpha K(y,x) - \partial_y^\alpha K(z,x)| \leqslant C \frac{|y-z|^{\delta-\lfloor\delta\rfloor}}{|x-z|^{n+\frac{\delta}{\sigma}}}, \tag{2.19}$$

for $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = \lfloor\delta\rfloor$, $|x-z| \geqslant 2|y-z|^\sigma$ and $0 < \sigma \leqslant 1$. The condition (2.19) is a natural generalization of derivative conditions usually assumed on standard $\delta-$kernels (see [35, p. 320] and [73, p. 117]).

In the same way, we may incorporate derivatives of the kernel in the integral D_s condition (2.8) and (2.9). We say that a kernel satisfies the *derivative D_s condition with decay δ*, if for every $|\gamma| = \lfloor\delta\rfloor$ it follows that

$$\sup_{\substack{|y-z|<r \\ r>1}} \left(\int_{C_j(z,r)} |\partial_y^\gamma K(x,y) - \partial_y^\gamma K(x,z)|^s + |\partial_y^\gamma K(y,x) - \partial_y^\gamma K(z,x)|^s dx \right)^{1/s} \lesssim r^{-\lfloor\delta\rfloor} |C_j(z,r)|^{\frac{1}{s}-1} 2^{-j\delta} \tag{2.20}$$

and

$$\sup_{\substack{|y-z|<r \\ 0<r<1}} \left(\int_{C_j(z,r^\rho)} |\partial_y^\gamma K(x,y) - \partial_y^\gamma K(x,z)|^s + |\partial_y^\gamma K(y,x) - \partial_y^\gamma K(z,x)|^s dx \right)^{1/s} \lesssim r^{-\lfloor\delta\rfloor} |C_j(z,r^\rho)|^{\frac{1}{s}-1+\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}}. \tag{2.21}$$

We announce the following self-improvement of Theorem A:

Theorem 2.4. *Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear and continuous operator satisfying assumptions (i) and (iii) from Theorem A and*

(ii)' For some $1 \leqslant s_1 < \infty$, T is associated to a kernel satisfying the derivative D_{s_1} condition with decay $\delta > 0$.

If $T^*(x^\alpha) = 0$ for all $|\alpha| \leqslant \lfloor\delta\rfloor$, $p < s_1$ and $s_1 \leqslant s_2$, then T is bounded from $H^p(\mathbb{R}^n)$ to itself for $p_0 < p \leqslant 1$, where p_0 is given by (2.13). *Conversely, if T is bounded from $H^p(\mathbb{R}^n)$ to itself for $p_0 < p \leqslant 1$, then $T^*(x^\alpha) = 0$ for every $|\alpha| \leqslant N_{p_0}$.*

The proof of the previous theorem is analogous of Theorem A since Taylor's formula allows us to write

$$Ta(x) = \int_{B(x_0,r)} R(x,y)a(y)dy \quad \text{in which} \quad R(x,y) = \sum_{|\gamma|=M} \frac{(y-x_0)^\gamma}{\gamma!} \left[\partial_y^\gamma K(x,\xi_y) - \partial_y^\gamma K(x,z)\right]$$

for some ξ_y in the line segment between y and x_0. Examples of operators satisfying such kernel conditions will be discussed in Section 2.3.

2.1.3 Dini-type conditions

Yabuta considered in [79, Definition 2.1] a generalization of standard Calderón–Zygmund operators introducing a θ-modulus of continuity on the kernel. Instead of kernels satisfying the pointwise estimate (2.3), it was considered

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \lesssim \theta\left(\frac{|y-z|}{|x-z|}\right) |y-z|^{-n}, \ \text{for all} \ |x-z| \geqslant 2|y-z|,$$

where θ is a non-negative and non-decreasing function satisfying the Dini condition $\int_0^1 \theta(t)t^{-1}dt < \infty$. Moreover, it has been shown that this Dini condition imposed on the function θ is sufficient to show standard L^p and BMO boundedness properties (see [79, Theorem 2.4]). These kernels are related to general classes of pseudodifferential operators beyond Hörmander class, see for instance [79, Theorems 3.1 and 3.2].

Inspired by this work, in this section we introduce a generalization of strongly singular Calderón–Zygmund operators of type σ assuming an analogous θ-modulus of continuity of the kernel. This has its own interests and can lead to new paths in connection to pseudodifferential operators associated to rough symbols.

Definition 2.7. *Let $\theta : (0,\infty) \to (0,\infty)$ be an increasing function and $0 < \sigma \leqslant 1$. We say that a continuous function $K(x,y)$ defined on $\mathbb{R}^{2n}$ away the diagonal is a $\theta-$kernel of type σ if*

$$|K(x,y)| \lesssim \frac{1}{|x-y|^n} \quad \textit{for all } x \neq y$$

and

$$|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)|\lesssim \theta\left(\frac{|y-z|}{|x-z|^{\frac{1}{\sigma}}}\right)|y-z|^{-n} \tag{2.22}$$

for all $|x-z|\geqslant 2|y-z|^{\sigma}$. *A linear and continuous operator* $T:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$ *is called a strongly singular* θ*-Calderón–Zygmund operator if it is associated to a* $\theta-$*kernel of type* σ *and satisfies the boundedness properties (i) and (iii) of Theorem A.*

Remark 2.5. Considering $\theta(t)=t^{\delta}$ for some $0<\delta\leqslant 1$, we recover condition (2.4) on the kernel.

In the next theorem, we investigate the continuity of such operators in $H^p(\mathbb{R}^n)$.

Theorem 2.5. *Let* $0<p\leqslant 1$ *and* T *a strongly singular* $\theta-$*Calderón–Zygmund operator. Suppose that for some* $\delta>0$ *and* $1\leqslant s_1<\infty$ *with* $p<s_1$ *the function* θ *satisfies*

$$\int_0^1 \frac{[\theta(t)]^{s_1}}{t^{1+\delta s_1}}dt<\infty \tag{2.23}$$

and $T^*(x^{\alpha})=0$ *for every* $|\alpha|\leqslant\lfloor\delta\rfloor$. *Then* T *is a bounded operator on* $H^p(\mathbb{R}^n)$ *to itself for every* $p_0<p\leqslant 1$*, where* p_0 *is as in* (2.13)*. Conversely, if* T *is bounded from* $H^p(\mathbb{R}^n)$ *to itself for* $p_0<p\leqslant 1$*, then* $T^*(x^{\alpha})=0$ *for every* $|\alpha|\leqslant N_{p_0}$.

Conditions like (2.23) have already been considered in the literature to obtain boundedness of standard $\theta-$Calderón–Zygmund operators. For instance, in [51, Theorem 1.2], the same condition with $s_1=1$ has been used in the setting of weighted Hardy spaces (see also [68, Theorems 8 and 9] for similar ones in weak-Hardy spaces). Conditions like $\displaystyle\int_0^1\frac{[\theta(t)]^a}{t}dt<\infty$ for $a>0$ have also been considered in the literature (see [59] and their cited papers) and is usually referred as $a-$Dini condition.

Every increasing function θ such that $\theta(t)\lesssim[\log(1+t)]^{\frac{1}{s_1}}t^{\delta}$ satisfies condition (2.23), since $\displaystyle\int_0^1\frac{\log(1+t)}{t}dt=-\frac{\pi}{6}<\infty.$

Next we present the proof of Theorem 2.5.

Proof. Let a be a (p,∞) atom in H^p supported on $B=B(x_0,r)\subset\mathbb{R}^n$ and we show that Ta is a (p,λ,s_1) molecule. Since conditions (M1) and (M3) rely only on the continuity and cancellation properties of T, the proofs will be the same. We show (M2) and suppose first $r > 1$. Since θ is increasing, by (2.22) it follows that

$$|Ta(x)| \leqslant \int_B |K(x,y) - K(x,x_0)|\,|a(y)|dy \lesssim r^{n\left(1-\frac{1}{p}\right)}\theta\left(\frac{r}{|x-x_0|^{\frac{1}{\sigma}}}\right)|x-x_0|^{-n}.$$

Therefore

$$\begin{aligned}
\int_{(2B)^c} |Ta(x)|^{s_1}|x-x_0|^{\lambda}dx &\leqslant r^{s_1 n\left(1-\frac{1}{p}\right)} \int_{(2B)^c} \left[\theta\left(\frac{(2r)^{\frac{1}{\sigma}}}{|x-x_0|^{\frac{1}{\sigma}}}\right)\right]^{s_1} |x-x_0|^{\lambda - s_1 n}dx \\
&= r^{\lambda+n\left(1-\frac{s_1}{p}\right)} \int_{|w|>1} \left[\theta\left(|w|^{-\frac{1}{\sigma}}\right)\right]^{s_1} |w|^{\lambda-s_1 n}dw \\
&= r^{\lambda+n\left(1-\frac{s_1}{p}\right)} \int_1^{\infty} \left[\theta\left(u^{-\frac{1}{\sigma}}\right)\right]^{s_1} u^{\lambda-n(s_1-1)-1}du \\
&\lesssim r^{\lambda+n\left(1-\frac{2}{p}\right)} \int_0^1 \frac{[\theta(t)]^{s_1}}{t^{1+\delta s_1}} t^{-\sigma\lambda+\sigma n(s_1-1)+\delta s_1}dt \\
&\lesssim r^{\lambda+n\left(1-\frac{s_1}{p}\right)},
\end{aligned}$$

since $\lambda < n(s_1-1) + \frac{\delta s_1}{\sigma}$. For $r < 1$, with similar arguments

$$\begin{aligned}
\int_{(2B^{\rho})^c} |Ta(x)|^{s_1}|x-x_0|^{\lambda}dx &\lesssim r^{s_1 n\left(1-\frac{1}{p}\right)} \int_{(2B^{\rho})^c} \left[\theta\left(\frac{r}{|x-x_0|^{\frac{1}{\sigma}}}\right)\right]^{s_1} |x-x_0|^{\lambda-s_1 n}dx \\
&= r^{s_1 n\left(1-\frac{1}{p}\right)+\rho[\lambda-n(s_1-1)]} \int_{|w|>1} \left[\theta\left(\frac{r^{1-\frac{\rho}{\sigma}}}{|w|^{\frac{1}{\sigma}}}\right)\right]^{s_1} |w|^{\lambda-s_1 n}dw \\
&= r^{s_1 n\left(1-\frac{1}{p}\right)+\sigma(\lambda-ns_1+n)+\rho-1} \int_0^{r^{1-\frac{\rho}{\sigma}}} \frac{[\theta(t)]^{s_1}}{t^{1+\delta s_1}} t^{-\sigma\lambda+\sigma n(s_1-1)+\delta s_1}dt \\
&\lesssim r^{\rho\lambda+n\left[\rho\left(1-\frac{s_1}{s_2}\right)+s_1\left(\frac{1}{q}-\frac{1}{p}\right)\right]},
\end{aligned}$$

where in the last integral we estimate $t \leqslant r^{1-\frac{\rho}{\sigma}}$ and we choose ρ as in (2.18). □

Remark 2.6. Condition (2.23) can be refined for one related to (2.8) and (2.9). Let $I = (2^{-\frac{1}{\sigma}}, 1)$, $I_j^{\rho} = r^{1-\frac{\rho}{\sigma}}2^{-\frac{j}{\sigma}} \times I$ and $I_j = r^{1-\frac{1}{\sigma}}2^{-\frac{j}{\sigma}} \times I$. If θ satisfies

$$\left(\int_{I_j^{\rho}} \frac{[\theta(t)]^{s_1}}{t}dt\right)^{\frac{1}{s_1}} \lesssim |I_j^{\rho}|^{\delta} \text{ if } r<1 \text{ and } \left(\int_{I_j} \frac{[\theta(t)]^{s_1}}{t}dt\right)^{\frac{1}{s_1}} \lesssim (2^j)^{-\delta} \text{ if } r>1,$$

then θ-kernels of type σ satisfy the D_{s_1} condition.

2.2 Continuity in $H^p_w(\mathbb{R}^n)$

In this section, we investigate the continuity of strongly singular Calderón–Zygmund operators in weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ when w belongs to some Muckenhoupt classes. In particular, we provide an analogous version of Theorem A for this setting.

When T is a convolution operator associated to standard kernels, the following result was proved by Lee and Lin in [54] when $w \in A_1$ (see (1.19))

Theorem 2.6 ([54, Theorem 4]). *Let $w \in A_1$ and $K \in L^1_{loc}(\mathbb{R}^n \backslash \{0\})$ such that*

$$|K(x-y) - K(x)| \leqslant C \frac{|y|^\delta}{|x|^{n+\delta}}, \quad \text{for all } |x| \geqslant C|y| \text{ and some } 0 < \delta \leqslant 1.$$

Assume also that the convolution operator associated to K (denoted by T) is bounded on $L^2_w(\mathbb{R}^n)$. If the reverse Hölder exponent satisfies $r_w > \frac{n+\delta}{\delta}$, then T is bounded on $H^p_w(\mathbb{R}^n)$ to itself for every $\frac{n}{n+\delta} < p \leqslant 1$.

Later on, using discrete Littlewood-Paley decomposition methods, the authors in [60, Theorem 1.1] extended the previous theorem for $w \in A_\infty$ and $0 < p < \infty$ assuming regularity conditions on the kernel.

In the non-convolution setting, Hart and Oliveira in [66] obtained the following continuity result for a limited range of Muckenhoupt weight classes, depending on p and the regularity of the kernel.

Theorem 2.7 ([66, Theorem 2.10]). *Let T be a standard Calderón–Zygmund operator associated to a kernel satisfying*

(i) $\left|\partial^\alpha_x \partial^\beta_y K(x,y)\right| \leqslant C\,|x-y|^{-n-|\alpha|-|\beta|}$ *for all $x \neq y$ and $|\alpha|$, $|\beta| \leqslant L$;*

(ii) $\left|\partial^\alpha_x \partial^\beta_y K(x,y) - \partial^\alpha_x \partial^\beta_y K(x,z)\right| \leqslant C \dfrac{|y-z|^\delta}{|x-y|^{n+|\alpha|+L+\delta}}$ *for all $|x-y| > 2|y-z|$ and $|\alpha| \leqslant |\beta| = L$;*

(iii) $\left|\partial^\alpha_x \partial^\beta_y K(x,y) - \partial^\alpha_x \partial^\beta_y K(z,y)\right| \leqslant C \dfrac{|x-z|^\delta}{|x-y|^{n+|\beta|+L+\delta}}$ *for all $|x-y| > 2|x-z|$ and $|\beta| \leqslant |\alpha| = L$.*

If $T^*(y^\alpha) = 0$ *for every* $|\alpha| < \lfloor L/2 \rfloor$*, then* T *extends to a bounded operator from* $H^p_w(\mathbb{R}^n)$ *to itself provided that*

$$\frac{n}{n + \lfloor L/2 \rfloor + L/2 + \delta/2} < p < \infty \quad \text{and} \quad w \in A_{p\left(\frac{n+\lfloor L/2\rfloor + L/2+\delta/2}{n}\right)}.$$

More recently, using a different approach, Cruz-Uribe, Moen and Nguyen [18] established the following result for weights in A_∞.

Theorem 2.8 ([18, Theorem 1.9])**.** *Let* T *be a standard Calderón–Zygmund operator associated to a kernel satisfying*

$$\left|\partial_y^\beta K(x,z) - \partial_y^\beta K(x,y)\right| \leqslant C\,\frac{|y-z|^\delta}{|x-z|^{n+L+1+\delta}}$$

for all $|x-z| \geqslant 2|y-z|$*, some* $0 < \delta \leqslant 1$ *and* $|\beta| = L+1$*, and suppose* $T^*(x^\beta) = 0$ *in the sense that* $\int x^\beta T a(x) dx = 0$*, for all* (p, w, ∞) *atoms with vanishing moments up the order* $L+1$ *and* $|\beta| \leqslant L$*. If* $w \in A_\infty$ *then* T *maps continuously* $H^p_w(\mathbb{R}^n)$ *into itself for*

$$L = N_{w,p} := \left\lfloor n\left(\frac{t_w}{p} - 1\right)\right\rfloor. \tag{2.24}$$

From (2.24) we get that $\dfrac{n\,t_w}{n+L+1} < p \leqslant \dfrac{n\,t_w}{n+L}$. So in the previous theorem there exists an implicit relation between p and the Muckenhoupt class the weight belongs, just as in Theorem 2.7. Moreover, a careful inspection in the proof of previous theorem, to be precise [18, Lemma 7.2], shows the assumption $L+1 = 0$ can not be assumed. Hence, both Theorems 2.7 and 2.8 does not cover standard Calderón–Zygmund operators associated to $\delta-$kernels and even more generally, kernels satisfying integral-type conditions.

Our first result in this setting is to show that strongly singular Calderón–Zygmund operators are bounded from $H^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$, where w belongs to a special class of Muckenhoupt weight.

Theorem 2.9. *Let* $0 < p \leqslant 1 \leqslant t < \infty$*,* $1 < s_1 < \infty$ *and* T *a strongly singular Calderón–Zygmund operator whose kernel satisfies a* D_{s_1} *condition and the boundedness properties (i) and (iii) of Theorem A. If*

$$w \in A_t \cap RH_d \quad \text{with} \quad t \leqslant \frac{1}{p_0} \quad \text{and} \quad d = \max\left\{\frac{s_1}{p(s_1-1)}, \frac{2}{2-p}, \frac{s_2}{s_2-p}\right\},$$

then T maps continuously $H^p_w(\mathbb{R}^n)$ into $L^p_w(\mathbb{R}^n)$ for every $p_0\, t \leqslant p \leqslant 1$ in which p_0 is given by (2.13).

Remark 2.7. The previous theorem does not cover the case $s_1 = 1$. We also observe that when considering condition (2.4) on the kernel, assumption $w \in RH_{\frac{s_1}{p(s_1-1)}}$ can be dropped.

In particular, the previous result cover and extend [55, Theorem 2] due to Li and Lu, proved for the case $w \in A_1$, $s_2 = 2$ and kernels satisfying (2.4). Moreover, restricting ourselves in the unweighted setting we recover [2, Theorem 5.1]. Indeed, with the same notation of the reference, it suffices to consider $s_0' = q$ and $q_0' = s_2$. We point out that for the unweighted case, inspection of the previous proof shows that condition D_1 is sufficient.

Proof. Let a to be a (p, w, ∞) atom in H^p_w supported in $B = B(x_0, r)$. We will show that Ta is uniformly bounded in L^p_w norm. Let $2B = B(x_0, 2r)$ and $C_j = C_j(x_0, r)$. Suppose first $r > 1$ and split

$$\|Ta\|^p_{L^p_w} = \int_{2B} |Ta(x)|^p w(x)dx + \sum_{j=1}^{\infty} \int_{C_j} |Ta(x)|^p w(x)dx.$$

The first integral can be uniformly estimated from Hölder inequality with exponent $2/p$, the L^2 continuity and $w \in RH_{\frac{2}{2-p}}$. In fact,

$$\begin{aligned}
\int_{2B} |Ta(x)|^p w(x)dx &\leqslant \left(\int_{\mathbb{R}^n} |Ta(x)|^2 dx\right)^{\frac{p}{2}} \left(\frac{1}{|2B|}\int_{2B} w^{\frac{2}{2-p}}(x)dx\right)^{1-\frac{p}{2}} |2B|^{1-\frac{p}{2}} \\
&\lesssim \|a\|^p_{L^2}\, w(2B)\, |2B|^{-\frac{p}{2}} \lesssim \frac{w(2B)}{w(B)}\, |B|^{\frac{p}{2}}\, |2B|^{-\frac{p}{2}} \\
&\lesssim \frac{|2B|}{|B|}\, |B|^{\frac{p}{2}}\, |2B|^{-\frac{p}{2}} \lesssim 1.
\end{aligned}$$

Before estimating the second integral, note that since $w \in RH_{\frac{s_1}{p(s_1-1)}}$ it follows

$$\begin{aligned}\int_{C_j} |K(x,y)-K(x,x_0)| w^{\frac{1}{p}}(x)dx &\leqslant \left(\int_{C_j} |K(x,y)-K(x,x_0)|^{s_1}dx\right)^{\frac{1}{s_1}} \left(\int_{B_{j+1}} w^{\frac{s_1}{p(s_1-1)}}(x)dx\right)^{1-\frac{1}{s_1}} \\ &\lesssim |C_j|^{\frac{1}{s_1}-1} 2^{-j\delta} |B_{j+1}|^{1-\frac{1}{s_1}} \left(\frac{1}{|B_{j+1}|}\int_{B_{j+1}} w^{\frac{s_1}{p(s_1-1)}}(x)dx\right)^{1-\frac{1}{s_1}} \\ &\lesssim 2^{-j\delta}\left(\frac{|B_{j+1}|}{|C_j|}\right)^{1-\frac{1}{s_1}} |B_{j+1}|^{-\frac{1}{p}} w(B_{j+1})^{\frac{1}{p}}. \end{aligned} \tag{2.25}$$

Then,

$$\begin{aligned}\sum_{j=1}^{\infty}\int_{C_j} |Ta(x)|^p w(x)dx &\leqslant \sum_{j=1}^{\infty}\int_{C_j}\left(\int_B |K(x,y)-K(x,x_0)|\,|a(y)|dy\right)^p w(x)dx \\ &\leqslant \sum_{j=1}^{\infty} w(B)^{-1}\left(\int_{C_j}\int_B |K(x,y)-K(x,x_0)|\,w^{\frac{1}{p}}(x)dydx\right)^p |C_j|^{1-p} \\ &= \sum_{j=1}^{\infty} w(B)^{-1}\left(\int_B\left[\int_{C_j} |K(x,y)-K(x,x_0)|\,w^{\frac{1}{p}}(x)dx\right]dy\right)^p |C_j|^{1-p} \\ &\lesssim \sum_{j=1}^{\infty}\left[\frac{w(B)}{w(B_{j+1})}\right]^{-1} |C_j|^{1-p}\,|B|^p\left(\frac{|B_{j+1}|}{|C_j|}\right)^{p\left(1-\frac{1}{s}\right)} |B_{j+1}|^{-1}\,2^{-jp\delta} \\ &\lesssim \sum_{j=1}^{\infty} |C_j|^{1-p-p\left(1-\frac{1}{s}\right)}\,|B|^{p-t}\,|B_{j+1}|^{t-1+p\left(1-\frac{1}{s}\right)} 2^{-jp\delta} \\ &\lesssim \sum_{j=1}^{\infty} 2^{j[nt-p(n+\delta)]} \lesssim 1 \end{aligned}$$

since $p > nt/(n+\delta)$. Lets consider now the case $0 < r \leqslant 1$. In the same way, we split

$$\|Ta\|^p_{L^p_w} = \int_{2B^\rho} |Ta(x)|^p w(x)dx + \sum_{j=1}^{\infty}\int_{C^\rho_j} |Ta(x)|^p w(x)dx$$

where $2B^\rho = B(x_0, 2r^\rho)$, $C^\rho_j = C_j(x_0, r^\rho)$ and $0 < \rho \leqslant \sigma \leqslant 1$ will be chosen conveniently later. For the first integral we use Hölder inequality with exponent s_2/p, the continuity of T from $L^q(\mathbb{R}^n)$ to $L^{s_2}(\mathbb{R}^n)$

and $w \in RH_{\frac{s_2}{s_2-p}}$ to get

$$\begin{aligned}\int_{2B^\rho} |Ta(x)|^p w(x)dx &\leqslant \|Ta\|^p_{L^{s_2}} \left(\frac{1}{|2B^\rho|}\int_{2B^\rho} w^{\frac{s_2}{s_2-p}}(x)dx\right)^{1-\frac{p}{s_2}} |2B^\rho|^{1-\frac{p}{s_2}} \\ &\lesssim \frac{w(2B^\rho)}{w(B)} |B|^{\frac{p}{q}} |2B^\rho|^{-\frac{p}{s_2}} \lesssim |B|^{\frac{p}{q}-t} |2B^\rho|^{t-\frac{p}{s_2}} \\ &\lesssim r^{n\left[\frac{p}{q}-t+\rho\left(t-\frac{p}{s_2}\right)\right]} \lesssim 1\end{aligned}$$

for $\rho \geqslant \rho_1 := \frac{t-p\left(\frac{1}{s_2}+\frac{\beta}{n}\right)}{t-\frac{p}{s_2}}$.

For the second integral, proceeding just like in (2.25), it follows from $w \in RH_{\frac{s_1}{p(s_1-1)}}$ and the D_{s_1} condition that

$$\int_{C^\rho_j} |K(x,y)-K(x,x_0)| w^{\frac{1}{p}}(x)dx \lesssim |C^\rho_j|^{\frac{1}{s_1}-1+\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} |B^\rho_{j+1}|^{1-\frac{1}{s_1}-\frac{1}{p}} w(B^\rho_{j+1})^{\frac{1}{p}} 2^{-j\frac{\delta}{\rho}}.$$

Then,

$$\begin{aligned}\sum_{j=1}^{\infty}\int_{C^\rho_j} |Ta(x)|^p w(x)dx &\lesssim \sum_{j=1}^{\infty}\left[\frac{w(B)}{w(B^\rho_{j+1})}\right]^{-1} |B|^p |C^\rho_j|^{\frac{p}{s_1}+1-2p+\frac{p\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} |B^\rho_{j+1}|^{p-\frac{p}{s_1}-1} 2^{-j\frac{p\delta}{\rho}} \\ &\lesssim \sum_{j=0}^{\infty} |B|^{p-t} |C^\rho_j|^{1-2p+\frac{p}{s_1}+\frac{p\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} |B^\rho_{j+1}|^{p\left(1-\frac{1}{s_1}\right)-1+t} 2^{-j\frac{p\delta}{\rho}} \\ &\lesssim r^{-\rho\left[n(p-t)+\frac{p\delta}{\sigma}\right]+p\delta+n(p-t)} \sum_{j=1}^{\infty} 2^{j\left[nt-p\left(n+\frac{\delta}{\sigma}\right)\right]} \lesssim 1\end{aligned}$$

in which $\rho \leqslant \rho_2 := \frac{p(n+\delta)-nt}{p\left(n+\frac{\delta}{\sigma}\right)-nt} \leqslant \sigma$. The restriction $\rho_1 \leqslant \rho_2$ implies that uniform estimate holds for every $p \geqslant p_w$. Hence, given $f \in H^p_w(\mathbb{R}^n)$, by standard arguments one has

$$\|Tf\|^p_{L^p_w} \leqslant \sum_{j\in\mathbb{N}} |\lambda_j|^p \|Ta\|^p_{L^p_w} \lesssim \|f\|^p_{H^p_w},$$

which concludes the proof. □

Remark 2.8.

(i) If $s_1 \leqslant s_2$, then $w \in RH_{\frac{s_1}{p(s_1-1)}}$ implies that $w \in RH_{\frac{s_2}{s_2-p}}$ and the reverse Hölder exponent in this case

is $d = \max\left\{\frac{s_1}{p(s_1-1)},\ \frac{2}{2-p}\right\}$. In fact,

$$\frac{1}{s_2} - \frac{1}{s_1} < \frac{1}{p} - 1 \Leftrightarrow 1 - \frac{1}{s_1} < \frac{1}{p} - \frac{1}{s_2} \quad \Leftrightarrow p\left(1 - \frac{1}{s_1}\right) < \frac{1}{p} - \frac{1}{s_2} \Leftrightarrow \frac{s_1}{p(s_1-1)} > \frac{s_2}{s_2-p}.$$

(ii) If one assumes that T is continuous on $L^2_w(\mathbb{R}^n)$, restriction $w \in RH_{\frac{2}{2-p}}$ can be dropped. In fact, from Hölder inequality with exponent $2/p$ one was

$$\int_{2B} |Ta(x)|^p w(x)dx \leqslant \left(\int_{R^n} |Ta(x)|^2 w(x)dx\right)^{\frac{p}{2}} \left(\int_{2B} w(x)dx\right)^{1-\frac{p}{2}} \lesssim \|a\|^p_{L^2_w} w(B)^{1-\frac{p}{2}} \lesssim 1.$$

Replacing the target space for $H^p_w(\mathbb{R}^n)$, we can use the molecular decomposition of weighted Hardy spaces presented in Section 1.3, to extend the previous result and obtain a generation of Theorem 2.6.

Theorem 2.10. *Let $0 < p \leqslant 1$ and T a strongly singular Calderón–Zygmund operator associated to a kernel satisfying a D_{s_1} condition and the continuity hypothesis (i) and (iii) of Theorem A. If $w \in A_t \cap RH_d$ for*

$$1 \leqslant t < \min\left\{s_1,\ s_2,\ \frac{1}{p_0} \cdot \frac{1}{t_w}\left(1 - \frac{1}{r_w}\right)\right\} \quad \textit{and} \quad d = \max\left\{\frac{s_2}{s_2-t},\ \frac{s_1}{s_1-t}\right\},$$

and $T^*(x^\alpha) = 0$ for all $|\alpha| \leqslant N_{w,p}$, then T extends to a bounded operator from $H^p_w(\mathbb{R}^n)$ to itself provided that

$$p_0 \cdot t \cdot \frac{t_w\, r_w}{r_w - 1} < p \leqslant 1. \tag{2.26}$$

Proof. Let a be a (p, w, ∞) atom supported on $B = B(x_0, r) \subset \mathbb{R}^n$. Under the hypothesis on the weight w, we show that Ta is an (p, w, t, λ) molecule in H^p_w for any

$$n\left(\frac{t}{p} \cdot \frac{t_w\, r_w}{r_w - 1} - 1\right) < \lambda \leqslant n\left(\frac{1}{s_2} - 1\right) + \frac{\beta}{1-\rho}. \tag{2.27}$$

Condition (M1) follows using the fact that the operator is in particular bounded from $L^{s_2}(\mathbb{R}^n)$ to itself, $1 \leqslant t < s_2$ and $w \in RH_{\frac{s_2}{s_2-t}}$. In fact,

$$\int_{2B} |Ta(x)|^t w(x)dx \leqslant \|Ta\|^t_{L^{s_2}} \left(\frac{1}{|2B|}\int_{2B} w^{\frac{s_2}{s_2-t}}(x)dx\right)^{1-\frac{t}{s_2}} |2B|^{1-\frac{t}{s_2}} \quad \lesssim \|a\|^t_{L^{s_2}} |2B|^{-\frac{t}{s_2}} w(B) \lesssim w(B)^{1-\frac{t}{p}} \tag{2.28}$$

For (M2) we will consider two cases. Suppose first $0 < r < 1$ and let $0 < \rho \leqslant \sigma$. Split

$$\int_{\mathbb{R}^n} |Ta(x)|^t w(B_{|x-x_0|})^{\frac{\lambda}{n}} w(x)dx = \int_{2B^\rho} |Ta(x)|^t w(B_{|x-x_0|})^{\frac{\lambda}{n}} w(x)dx + \int_{(2B^\rho)^c} |Ta(x)|^t w(B_{|x-x_0|})^{\frac{\lambda}{n}} w(x)dx.$$

For the estimate on $2B^\rho$, first note that for any $|x-x_0| \leqslant 2r^\rho$, Lemma 1.4 guarantee that

$$\left[\frac{w(B_{|x-x_0|})}{w(2B^\rho)}\right]^{\frac{\lambda}{n}} \leqslant \left(\frac{|x-x_0|}{r^\rho}\right)^{\frac{\lambda t}{n}} \leqslant C_{n,\lambda,t}.$$

Hence, proceeding like in (2.28), but now using the continuity from $L^q(\mathbb{R}^n)$ to $L^{s_2}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\int_{2B^\rho} |Ta(x)|^t w(B_{|x-x_0|})^{\frac{\lambda}{n}} w(x)dx &= \int_{2B^\rho} |Ta(x)|^t \left[\frac{w(B_{|x-x_0|})}{w(2B^\rho)}\right]^{\frac{\lambda}{n}} w(2B^\rho)^{\frac{\lambda}{n}} w(x)dx \\
&\lesssim w(2B^\rho)^{\frac{\lambda}{n}} \int_{2B^\rho} |Ta(x)|^t w(x)dx \\
&\lesssim w(2B^\rho)^{\frac{\lambda}{n}+1} w(B)^{-\frac{t}{p}} r^{nt\left(\frac{1}{q}-\frac{\rho}{s_2}\right)} \\
&\lesssim w(B)^{\frac{\lambda}{n}+1-\frac{t}{p}} \left[\frac{w(B)}{w(2B^\rho)}\right]^{-\frac{\lambda}{n}-1} r^{nt\left(\frac{1}{q}-\frac{\rho}{s_2}\right)} \\
&\lesssim w(B)^{\frac{\lambda}{n}+1-\frac{t}{p}} r^{-\lambda t(1-\rho)+nt\left[\frac{1}{q}-1+\rho\left(1-\frac{1}{s_2}\right)\right]} \qquad (2.29) \\
&\lesssim w(B)^{\frac{\lambda}{n}+1-\frac{t}{p}}
\end{aligned}$$

since

$$\lambda \leqslant n\left(\frac{1}{s_2}-1\right) + \frac{\beta}{1-\rho} \quad \text{and} \quad 0 < r < 1.$$

We estimate now the integral on $(2B^\rho)^c$. Since $t < s_1$, we apply Hölder inequality, $w \in RH_{\frac{s_1}{s_1-t}}$ and the D_{s_1} condition to obtain

$$\begin{aligned}
\int_{C_j^\rho} |K(x,y)-K(x,x_0)|^t w(x)dx &\leqslant \left(\int_{C_j^\rho} |K(x,y)-K(x,x_0)|^{s_1} dx\right)^{\frac{t}{s_1}} \left(\int_{2^{j+1}B^\rho} w^{\frac{s_1}{s_1-t}}(x)dx\right)^{1-\frac{t}{s_1}} \\
&\lesssim w(2^{j+1}B^\rho)\, |C_j^\rho|^{\frac{t}{s_1}-t+\frac{t\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} 2^{-\frac{jt\delta}{\rho}} |2^{j+1}B^\rho|^{-\frac{t}{s_1}} \\
&\simeq w(2^{j+1}B^\rho)\, r^{-\rho nt+t\delta\left(1-\frac{\rho}{\sigma}\right)} (2^j)^{-nt-\frac{t\delta}{\sigma}}. \qquad (2.30)
\end{aligned}$$

By Lemmata 1.5 and 1.4 we get

$$w(2B^\rho)^{\frac{\lambda}{n}}\, w(2^{j+1}B^\rho)w(B)^{-\frac{\lambda}{n}-1} \lesssim (2^j)^{nt} \left[\frac{w(B)}{w(2B^\rho)}\right]^{-\frac{\lambda}{n}-1} \lesssim (2^j)^{nt}\, r^{-nt(1-\rho)\left(\frac{\lambda}{n}-1\right)}. \tag{2.31}$$

Hence, from (2.30) and (2.31)

$$\begin{aligned}
&\int_{C_j^\rho} |K(x,y)-K(x,x_0)|^t\, w(B_{|x-x_0|})^{\frac{\lambda}{n}} w(x)dx \\
&\qquad = w(2B^\rho)^{\frac{\lambda}{n}} \int_{C_j^\rho} |K(x,y)-K(x,x_0)|^t \left[\frac{w(2B^\rho)}{w(B_{|x-x_0|})}\right]^{-\frac{\lambda}{n}} w(x)dx \\
&\qquad \lesssim w(2B^\rho)^{\frac{\lambda}{n}}\, (2^j)^{\lambda t} \int_{C_j^\rho} |K(x,y)-K(x,x_0)|^t w(x)dx \\
&\qquad \lesssim w(2B^\rho)^{\frac{\lambda}{n}}\, w(2^{j+1}B^\rho)\, r^{-\rho nt+t\delta\left(1-\frac{\rho}{\sigma}\right)}\, (2^j)^{\lambda t-nt-\frac{t\delta}{\sigma}} \\
&\qquad \lesssim w(B)^{\frac{\lambda}{n}+1}\, r^{-\lambda t(1-\rho)-nt+t\delta\left(1-\frac{\rho}{\sigma}\right)}\, (2^j)^{\lambda t-\frac{t\delta}{\sigma}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{(2B^\rho)^c} |Ta(x)|^t w(B_{|x-x_0|})^{\frac{\lambda}{n}} w(x)dx \\
&\quad \leqslant \sum_{j=0}^{\infty} \left\{ \int_B |a(y)| \left[\int_{C_j^\rho} |K(x,y)-K(x,x_0)|^t\, w(B_{|x-x_0|})^{\frac{\lambda}{n}} w(x)dx\right]^{\frac{1}{t}} dy \right\}^t \\
&\quad \lesssim w(B)^{\frac{\lambda}{n}+1-\frac{t}{p}}\, r^{-\lambda t(1-\rho)+t\delta\left(1-\frac{\rho}{\sigma}\right)} \sum_{j=0}^{\infty} (2^j)^{\lambda t-\frac{t\delta}{\sigma}} \\
&\quad \lesssim w(B)^{\frac{\lambda}{n}+1-\frac{t}{p}}
\end{aligned}$$

assuming $\lambda < \frac{\delta}{\sigma}$ and choosing ρ to be such that

$$-\lambda t(1-\rho) + nt\left[\frac{1}{q} - 1 + \rho\left(1-\frac{1}{s_2}\right)\right] = -\lambda t(1-\rho) + t\delta\left(1-\frac{\rho}{\sigma}\right),$$

which gives us the same choice of ρ as in (2.18). Then, the estimate follows proceeding like in (2.29).

Now, we sketch the proof of (M2) for the case $r \geqslant 1$. Under the same hypothesis of the previous case we get

$$\int_{C_j} |K(x,y)-K(x,x_0)|^t w(x)dx \lesssim w(2^{j+1}B)\,|C_j|^{\frac{t}{s_1}-t}\,2^{-jt\delta}\,|2^{j+1}B|^{-\frac{t}{s_1}} \lesssim w(B)\,r^{-nt}\,(2^j)^{-t\delta}.$$

Then, using this estimate one can show

$$\int_{C_j} |K(x,y)-K(x,x_0)|^t\, w(B_{|x-x_0|})^{\frac{\lambda}{n}} w(x)dx \lesssim w(B)^{\frac{\lambda}{n}+1}\,r^{-nt}\,(2^j)^{\lambda t-t\delta}$$

and this implies

$$\int_{(2B)^c} |Ta(x)|^t w(B_{|x-x_0|})^{\frac{\lambda}{n}} w(x)dx \lesssim w(B)^{\frac{\lambda}{n}+1-\frac{t}{p}} \sum_{j=0}^{\infty} (2^j)^{\lambda t-t\delta} \lesssim w(B)^{\frac{\lambda}{n}+1-\frac{t}{p}}$$

since $\lambda < \delta$.

Therefore, from (2.27) the continuity will holds for every p such that

$$\frac{1}{p} < \left[\frac{1}{s_2} + \frac{\beta\left[n\left(1-\frac{1}{s_2}\right)+\frac{\delta}{\sigma}\right]}{n\left(\frac{\delta}{\sigma}-\delta+\beta\right)}\right] \times \frac{1}{t\,t_w}\left(1-\frac{1}{r_w}\right).$$

□

Remark 2.9.

(i) Restriction $t < \frac{1}{p_0}\cdot\frac{1}{t_w}\left(1-\frac{1}{r_w}\right)$ is necessary to guarantee that the critical index (2.26) is less than 1. Even though this restriction depends on $t_w,\ r_w$ and the parameters of the operator, assuming $\delta > 0$ large enough will be sufficient to give more flexibility on the class of weights considered. Note that the same phenomenon occurs in Theorem 2.7.

(ii) The previous theorem does not cover operators associated to kernels satisfying condition D_1. One way to include this, is to consider kernels satisfying the following weighted inequality

$$\sup_{\substack{|y-z|\leqslant r\\ r\geqslant 1}} \left(\int_{C_j} |K(x,y)-K(x,z)|^s w(x)dx\right)^{1/s} \lesssim w(C_j)^{\frac{1}{s}}\,|C_j|^{-1}\,(2^j)^{-\delta}$$

and

$$\sup_{\substack{|y-z|<r\\0<r<1}}\left(\int_{C_j^\rho}|K(x,y)-K(x,z)|^s w(x)dx\right)^{1/s}\lesssim w(C_j^\rho)^{\frac{1}{s}}\,|C_j^\rho|^{-1+\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)}2^{-\frac{j\delta}{\rho}}.$$

Under such hypothesis we can also drop restrictions $t < s_1$ and $RH_{\frac{s_1}{s_1-t}}$. When $w = 1$, these are the classical D_s condition.

2.3 Pseudodifferential operators and D_s conditions

It is well understood that pseudodifferential operators $OpS^{-n(1-\sigma)}_{\sigma,\nu}(\mathbb{R}^n)$ for $0 < \sigma \leqslant 1$ and $0 \leqslant \nu < 1$ have distributional kernels satisfying the pointwise estimate (2.4) with $\delta = 1$ (see for instance [4, Remark (d) p. 4]). On the other hand, integral estimates are more suitable when dealing with this type of operators and using them we have the advantage of finding a wider set of examples. In [5, Section 3], even though the main theorem relies on the pointwise estimate (2.4) of the kernel, the authors have shown that $OpS^{-m}_{\sigma,\nu}(\mathbb{R}^n)$ for $0 < \nu \leqslant \sigma < 1$ and $n(1-\sigma)/2 \leqslant m < n/2$ satisfy the following Hörmander-type condition:

$$\int_{|x|\geqslant 2r^\sigma}|K(x+z,x-y)-K(x+z,x)|dx+\int_{|x|\geqslant 2r^\sigma}|K(x-y,x+z)-K(x,x+z)|dx\leqslant C$$

for all $z \in \mathbb{R}^n$, $|y| \leqslant r$ and $r > 0$. This represents a weaker condition, but as mentioned before, the continuity on $H^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$ assuming it is still not known.

In this section, we present classes of pseudodifferential operators satisfying the hypothesis of Theorem A. We start showing the derivative D_s condition for $1 \leqslant s \leqslant 2$, extending the case $s = 1$ and $|\gamma| = 0$ proved in [4, Theorem 2.1].

Proposition 2.3. *Let $\delta > 0$ and $T \in OpS^m_{\sigma,\nu}(\mathbb{R}^n)$ with $0 < \sigma \leqslant 1$, $0 \leqslant \nu < 1$, $\nu \leqslant \sigma$ and $m \leqslant -n(1-\sigma)/2$. If $1 \leqslant s \leqslant 2$, then T satisfies the derivative D_s condition with decay $\lfloor\delta\rfloor + 1$. In particular, when $0 < \delta < 1$ it satisfies integral conditions* (2.9) *and* (2.8) *with decay* 1.

It follows from [4, Theorem 3.5] that T maps continuously $L^q(\mathbb{R}^n)$ into $L^{s_2}(\mathbb{R}^n)$ where $\dfrac{1}{q} = \dfrac{1}{s_2} + \dfrac{\beta}{n}$

and $n(1-\sigma)\left(1-\frac{1}{s_2}\right) \leqslant \beta < n\left(1-\frac{1}{s_2}\right)$ since:

$$\text{(i)} \quad m \leqslant -\beta - n(1-\sigma)\left(\frac{1}{s_2}-\frac{1}{2}\right) \text{ if } 1 < q \leqslant s_2 \leqslant 2;$$

$$\text{(ii)} \quad m \leqslant -\beta \text{ if } 1 < q \leqslant 2 \leqslant s_2;$$

$$\text{(iii)} \quad m \leqslant -\frac{n}{2}(1-\sigma) \text{ if } 2 \leqslant q \leqslant s_2.$$

Note that $m \leqslant -n(1-\sigma)/2$ in all the cases and since $0 \leqslant \nu \leqslant \sigma < 1$ we have that $T \in OpS^m_{\sigma,\nu}(\mathbb{R}^n)$ is bounded from $L^2(\mathbb{R}^n)$ to itself. For the $T^*(x^\alpha) = 0$ condition for pseudodifferential operators see for instance [76, p. 154]

Before presenting the proof of Proposition 2.3, consider the following lemma concerning the L^2 continuity of pseudodifferential operators.

Lemma 2.1 ([44, Theorem 1]). *Let $m \leqslant -n \max\{0, (\nu-\sigma)/2\}$ and $a \in S^m_{\sigma,\nu}(\mathbb{R}^n)$ such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leqslant C\,(1+|\xi|)^{m-\sigma|\beta|+|\alpha|\nu}$$

for all $|\alpha|, |\beta| \leqslant \lfloor n/2 \rfloor + 1$. Then, the pseudodifferential operator associated to the symbol $a(x,\xi)$ is bounded on $L^2(\mathbb{R}^n)$ with norm proportional to the constant C.

We proceed now to the proof of Proposition 2.3.

Proof. Let $T \in OpS^m_{\sigma,\nu}(\mathbb{R}^n)$ and K its distributional kernel. We denote by $\widetilde{K}(x,y) = \partial_y^\gamma K(x,y)$ for $|\gamma| = \lfloor \delta \rfloor$. In order to obtain the derivative D_s condition for $1 \leqslant s < 2$, it suffices to prove it for $s = 2$ and then it follows by Hölder inequality. We claim that under the restriction $m \leqslant -n[(1-\sigma)/2 + \lambda]$ in which $\lambda = \max\{0, (\nu-\sigma)/2\}$ it follows for $C_j = C_j(z,r)$ that

$$\sup_{\substack{|y-z| \leqslant r \\ r \geqslant 1}} \left(\int_{C_j} |\widetilde{K}(x,y) - \widetilde{K}(x,z)|^2 dx\right)^{1/2} \lesssim r^{-\lfloor \delta \rfloor}\, |C_j|^{-\frac{1}{2}}\, 2^{-j(\lfloor \delta \rfloor + 1)},$$

and $C_j^\rho = C_j(z, r^\rho)$

$$\sup_{\substack{|y-z|\leqslant r\\ 0<r<1}} \left(\int_{C_j^\rho} |\widetilde{K}(x,y) - \widetilde{K}(x,z)|^2 dx \right)^{1/2} \lesssim r^{-\lfloor\delta\rfloor} \, |C_j^\rho|^{-\frac{1}{2}+\frac{\lfloor\delta\rfloor+1}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} \, 2^{-\frac{j}{\rho}(\lfloor\delta\rfloor+1)}$$

The analogous estimate for the adjoint $\widetilde{K}(y,x)$ will be treated in the end assuming $m \leqslant -n(1-\sigma)/2$.

The proof consists an adaptation of [4, Theorem 2.1], for the case $s = 1$ and $\lfloor\delta\rfloor = 0$. Assume without loss of generality that the symbol $p(x,\xi)$ associated to T vanishes for $|\xi| \leqslant 1$ and consider $\psi \in C_c^\infty(\mathbb{R})$ a non-negative function such that supp $(\psi) \subset [1/2, 1]$ and

$$\int_0^\infty \psi\left(\frac{1}{t}\right) \, \frac{1}{t} \, dt = \int_1^2 \psi\left(\frac{1}{t}\right) \, \frac{1}{t} \, dt = 1. \tag{2.32}$$

Define $K(x,y,t) = (2\pi)^{-n} \int e^{i(x-y)\,\xi} p(x,\xi)\psi\left(\frac{|\xi|}{t}\right) d\xi$ and consequently

$$\widetilde{K}(x,y,t) = (-i)^{\lfloor\delta\rfloor}(2\pi)^{-n} \int e^{i(x-y)\,\xi} p(x,\xi)\psi\left(\frac{|\xi|}{t}\right) \xi^\gamma d\xi.$$

By the standard representation of the kernel of a pseudodifferential operator we get that

$$\widetilde{K}(x,y) = (-i)^{\lfloor\delta\rfloor}(2\pi)^{-n} \int e^{i(x-y)\,\xi} \xi^\gamma p(x,\xi) d\xi,$$

and from (2.32) we may write it as

$$\widetilde{K}(x,y) = \int_0^\infty \widetilde{K}(x,y,t) \, \frac{dt}{t} \; = \int_1^\infty \widetilde{K}(x,y,t) \, \frac{dt}{t}. \tag{2.33}$$

In fact,

$$\begin{aligned}
\int_0^\infty \widetilde{K}(x,y,t) \, \frac{dt}{t} &= (-i)^{\lfloor\delta\rfloor}(2\pi)^{-n} \int_0^\infty \int_{|\xi|>1} e^{i(x-y)\cdot\xi} \, \xi^\gamma \, p(x,\xi)\psi\left(\frac{|\xi|}{t}\right) d\xi \; dt/t \\
&= (-i)^{\lfloor\delta\rfloor}(2\pi)^{-n} \int_{|\xi|>1} e^{i(x-y)\cdot\xi} \, \xi^\gamma \, p(x,\xi) \left(\int_0^\infty \psi\left(\frac{|\xi|}{t}\right) \, \frac{1}{t} \, dt \right) d\xi \\
&= (-i)^{\lfloor\delta\rfloor}(2\pi)^{-n} \int_{|\xi|>1} e^{i(x-y)\cdot\xi} \, \xi^\gamma \, p(x,\xi) d\xi = \widetilde{K}(x,y).
\end{aligned}$$

Consider first $0 < r < 1$. From Minkowski inequality for integrals

$$\begin{aligned}\int_{C_j^\rho} |\widetilde{K}(x,y)-\widetilde{K}(x,y)|^2 dx &\leqslant \int_{C_j^\rho} \left(\int_1^\infty |\widetilde{K}(x,y,t) - \widetilde{K}(x,z,t)| \frac{dt}{t} \right)^2 dx \\ &= \left\{ \left[\int_{C_j^\rho} \left(\int_1^\infty |\widetilde{K}(x,y,t) - \widetilde{K}(x,z,t)| \frac{dt}{t} \right)^2 dx \right]^{\frac{1}{2}} \right\}^2 \\ &\leqslant \left\{ \int_1^\infty \left(\int_{C_j^\rho} |\widetilde{K}(x,y,t) - \widetilde{K}(x,z,t)|^2 dx \right)^{\frac{1}{2}} \frac{dt}{t} \right\}^2 .\end{aligned}$$

Let $\Gamma(t) = \|\widetilde{K}(\cdot,y,t) - \widetilde{K}(\cdot,z,t)\|_{L^2(C_j^\rho)}$ and from the previous estimate

$$\left(\int_{C_j^\rho} |\widetilde{K}(x,y) - \widetilde{K}(x,z)|^2 dx \right)^{1/2} \leqslant \int_1^{r^{-1}} \Gamma(t) \frac{dt}{t} + \int_{r^{-1}}^\infty \Gamma(t) \frac{dt}{t} = I_1 + I_2. \tag{2.34}$$

Lets deal first with I_1, in which the estimate relies on the assumption $tr < 1$. Throughout this proof, let $N \in \mathbb{Z}_+$ to be a constant that will be chosen conveniently later. Note that

$$\Gamma(t) \leqslant \left(\int |\widetilde{K}(x,y,t) - \widetilde{K}(x,z,t)|^2 (1 + t^{2\sigma}|x-z|^2)^N dx \right)^{1/2} \sup_{x \in C_j^\rho} (1 + t^{2\sigma}|x-z|^2)^{-\frac{N}{2}}.$$

We claim that for $m \leqslant -n[(1-\sigma)/2 + \lambda]$

$$\left(\int_{\mathbb{R}^n} |\widetilde{K}(x,y,t) - \widetilde{K}(x,z,t)|^2 (1 + t^{2\sigma}|x-z|^2)^N dx \right)^{1/2} \lesssim (tr) t^{\frac{\sigma n}{2} + [\delta]} \text{ for } tr \leqslant 1 \tag{2.35}$$

and

$$\sup_{x \in C_j^\rho} (1 + t^{2\sigma}|x-z|^2)^{-\frac{N}{2}} \leqslant \left[1 + t^{2\sigma}(2^j r^\rho)^2\right]^{-\frac{N}{2}}.$$

Using these estimates and the change of variables $\omega = t^{\sigma} 2^j r^{\rho}$ we obtain

$$\begin{aligned}
\int_1^{r^{-1}} \Gamma(t)\frac{dt}{t} &\lesssim \int_1^{r^{-1}} r\, t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor}\left[1+t^{2\sigma}(2^j r^{\rho})^2\right]^{-\frac{N}{2}} dt \\
&\lesssim r^{1-\frac{\rho n}{2}-\frac{\rho}{\sigma}(1+\lfloor\delta\rfloor)}\,(2^j)^{-\frac{n}{2}-\frac{1+\lfloor\delta\rfloor}{\sigma}} \int_{2^j r^{\rho}}^{2^j r^{\rho-\sigma}} \frac{\omega^{\frac{n}{2}-1+\frac{1+\lfloor\delta\rfloor}{\sigma}}}{(1+\omega^2)^{\frac{N}{2}}} d\omega \\
&\lesssim |C_j^{\rho}|^{-\frac{1}{2}+\frac{1}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)}\, 2^{-\frac{j}{\rho}} \int_0^{\infty} \frac{\omega^{\frac{n}{2}+\frac{1}{\sigma}-1}}{(1+\omega^2)^{\frac{N}{2}}} d\omega \\
&\lesssim r^{-\lfloor\delta\rfloor}\, |C_j^{\rho}|^{-\frac{1}{2}+\frac{\lfloor\delta\rfloor+1}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)}\, 2^{-\frac{j}{\rho}(\lfloor\delta\rfloor+1)},
\end{aligned} \tag{2.36}$$

since

$$\int_0^{\infty} \frac{\omega^{\frac{n}{2}-1+\frac{1+\lfloor\delta\rfloor}{\sigma}}}{(1+\omega^2)^{\frac{N}{2}}} d\omega < \infty \text{ for } N > \frac{n}{2} + \frac{1+\lfloor\delta\rfloor}{\sigma}.$$

Lets us give an idea of the proof of (2.35). Using integration by parts, for $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| \leqslant N$ we can write

$$\begin{aligned}
&t^{\sigma|\alpha|}(x-z)^{\alpha}[\widetilde{K}(x,y,t) - \widetilde{K}(y,z,t)] \\
&\quad = \sum_{|\beta|\leqslant|\alpha|} C_{\alpha,\beta}\, t^{\sigma|\alpha|+\lfloor\delta\rfloor} \int e^{i(x-z)\cdot\xi} |\xi|^{n(1-\sigma)/2+\sigma|\beta|}\, \partial_\xi^{\beta}\left[(e^{i(z-y)\cdot\xi}-1)p(x,\xi)\right] \\
&\qquad\qquad \times |\xi|^{-n(1-\sigma)/2-\sigma|\beta|}\, \partial_\xi^{\alpha-\beta}\left[\psi\left(\frac{|\xi|}{t}\right)\left(\frac{\xi}{t}\right)^{\gamma}\right] d\xi.
\end{aligned} \tag{2.37}$$

Since $|e^{i(z-y)\cdot\xi}-1| \leqslant tr$ and $|\partial_\xi^{\beta} e^{i(z-y)\cdot\xi}| \lesssim |\xi|^{-|\beta|}\,(tr)^{|\beta|}$ one can show that if $\chi \in C_c^{\infty}(\mathbb{R}^+)$ is a function such that $\chi = \psi$ on the support of ψ, then

$$\left\{ |\xi|^{n(1-\sigma)/2+\sigma|\beta|}\partial_\xi^{\beta}\left[(e^{i(z-y)\cdot\xi}-1)p(x+z,\xi)\right]\chi\left(\frac{|\xi|}{t}\right) :\ |y-z| < r,\ z \in \mathbb{R}^n \right\}$$

is a bounded subset of $S_{\sigma,\mu}^{m+n(1-\sigma)/2}(\mathbb{R}^n)$ with bounds being less than or equal to Ctr. Therefore, since $m \leqslant -n\lambda$, by Lemma 2.1 the family of symbols above defines pseudodifferential operators bounded on

$L^2(\mathbb{R}^n)$ with norm proportional to Ctr. Therefore, from this consideration and (2.37) we get

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |\widetilde{K}(x,y,t)-\widetilde{K}(x,z,t)|^2(1+t^{2\sigma}|x-z|^2)^N dx\right)^{1/2} \\ \lesssim tr\sum_{|\alpha|\leqslant N}\sum_{|\beta|\leqslant|\alpha|} C_\alpha \left\| t^{\sigma|\alpha|+\lfloor\delta\rfloor}\,|\xi|^{-n(1-\sigma)/2-\sigma|\beta|}\partial_\xi^{\alpha-\beta}\left[\psi\left(\frac{|\xi|}{t}\right)\left(\frac{\xi}{t}\right)^\gamma\right]\right\|_{L^2} \\ \lesssim (tr)t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor}.\end{aligned}$$

On the other hand, to control I_2 we use Minkowski inequality to estimate

$$\Gamma(t)\leqslant \|\widetilde{K}(\cdot,y,t)\|_{L^2(C_j^\rho)}+\|\widetilde{K}(\cdot,z,t)\|_{L^2(C_j^\rho)}.$$

If $x\in C_j^\rho$ and $|y-z|<r<1$, then $|x-y|\geqslant|x-z|-|y-z|\geqslant 2^{j-1}r^\rho$ and

$$\|\widetilde{K}(\cdot,y,t)\|_{L^2(C_j^\rho)}\leqslant\left(\int_{\mathbb{R}^n}|\widetilde{K}(x,y,t)|^2(t^{2\sigma}|x-y|^2)^N dx\right)^{1/2}\sup_{|x-y|>2^{j-1}r^\rho}(t^{2\sigma}|x-y|^2)^{-\frac{N}{2}}.$$

We claim that

$$\left(\int_{\mathbb{R}^n}|\widetilde{K}(x,y,t)|^2(t^{2\sigma}|x-y|^2)^N dx\right)^{1/2}\lesssim t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor} \tag{2.38}$$

and the second term can be estimated by $(t^\sigma 2^{j-1}r^\rho)^{-N}$. Thus

$$\|\widetilde{K}(\cdot,y,t)\|_{L^2(C_j^\rho)}\lesssim t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor}(t^\sigma 2^{j-1}r^\rho)^{-N}.$$

Analogously $\|\widetilde{K}(\cdot,z,t)\|_{L^2(C_j^\rho)}\lesssim t^{\sigma n/2+\lfloor\delta\rfloor}(t^\sigma 2^{j-1}r^\rho)^{-N}$. Using these estimates and assuming $N>\dfrac{n}{2}+\dfrac{\lfloor\delta\rfloor+1}{\sigma}$ we obtain

$$\begin{aligned}\int_{r^{-1}}^\infty\Gamma(t)\frac{dt}{t}&\lesssim\int_{r^{-1}}^\infty t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor-\sigma N-1}(2^jr^\rho)^{-N}dt\lesssim(2^jr^\rho)^{-\frac{n}{2}-\frac{1+\lfloor\delta\rfloor}{\sigma}}\,r\\ &= r^{1-\frac{\rho}{\sigma}-\frac{\rho n}{2}-\frac{\rho\lfloor\delta\rfloor}{\sigma}}(2^j)^{-\frac{n}{2}-\frac{1+\lfloor\delta\rfloor}{\sigma}}\\ &\lesssim r^{-\lfloor\delta\rfloor}\,|C_j^\rho|^{-\frac12+\frac{\lfloor\delta\rfloor+1}{n}\left(\frac1\rho-\frac1\sigma\right)}\,2^{-\frac{j}{\rho}(\lfloor\delta\rfloor+1)}.\end{aligned} \tag{2.39}$$

It just remains to show now (2.38). In the same spirit as previously, taking $|\alpha| = N$ we may write

$$t^{\sigma|\alpha|}(x-y)\widetilde{K}(x,y,t) \simeq \sum_{|\beta|\leqslant|\alpha|} t^{\sigma|\alpha|+\lfloor\delta\rfloor} \int e^{i(x-y)\cdot\xi}\partial_\xi^\beta p(x,\xi)\partial_\xi^{\alpha-\beta}\left[\psi\left(\frac{|\xi|}{t}\right)\left(\frac{\xi}{t}\right)^\gamma\right]d\xi.$$

Since the class of symbols $\left\{|\xi|^{n(1-\sigma)/2+\sigma|\beta|}\partial_\xi^\beta p(x+y,\xi) : y \in \mathbb{R}^n\right\}$ are a bounded subset of $S^{m+n(1-\sigma)/2}_{\sigma,\mu}(\mathbb{R}^n)$, it follows directly that the family of pseudodifferential associated to it is uniformly bounded on $L^2(\mathbb{R}^n)$. Therefore

$$\left(\int_{\mathbb{R}^n}|\widetilde{K}(x,y,t)|^2(t^{2\sigma}|x-y|^2)^N dx\right)^{1/2} \lesssim \sum_{|\beta|\leqslant|\alpha|} t^{\sigma|\alpha|+\lfloor\delta\rfloor}\left\||\xi|^{-\frac{n(1-\sigma)}{2}-\sigma|\beta|}\partial_\xi^{\alpha-\beta}\left[\psi\left(\frac{|\xi|}{t}\right)\left(\frac{\xi}{t}\right)^\gamma\right]\right\|_{L^2} \lesssim t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor}.$$

Now we consider the case $r > 1$. Since we can estimate $\|\widetilde{K}(\cdot,y,t)\|_{L^2(C_j)}$ and $\|\widetilde{K}(\cdot,z,t)\|_{L^2(C_j)}$ in the same way as before, we obtain for $N > \max\left\{\frac{n}{2}+\frac{\lfloor\delta\rfloor}{\sigma}, \frac{n}{2}+\lfloor\delta\rfloor+1\right\}$,

$$\begin{aligned}\left(\int_{C_j}|\widetilde{K}(x,y)-\widetilde{K}(x,z)|^2dx\right)^{1/2} &\leqslant \int_1^\infty\left(\|\widetilde{K}(\cdot,y,t)\|_{L^2(C_j)}+\|\widetilde{K}(\cdot,z,t)\|_{L^2(C_j)}\right)\frac{dt}{t}\\ &\lesssim (2^j r)^{-N}\int_{r^{-1}}^\infty t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor-\sigma N-1}dt\\ &\lesssim r^{-\frac{n}{2}-\lfloor\delta\rfloor-(1-\sigma)-\lfloor\delta\rfloor(1-\sigma)}\,(2^j)^{-\frac{n}{2}-\lfloor\delta\rfloor-1}\\ &\lesssim r^{-\lfloor\delta\rfloor}\,|C_j(z,r)|^{-\frac{1}{2}}\,2^{-j(1+\lfloor\delta\rfloor)}.\end{aligned} \tag{2.40}$$

Now we deal with estimates of the adjoint. Suppose first $0 < r < 1$ and write

$$\begin{aligned}(-i)^{-|\gamma|}(2\pi)^n[\widetilde{K}(y,x,t)-\widetilde{K}(z,x,t)] &= \int e^{-i(x-y)\cdot\xi}[p(y,\xi)-p(z,\xi)]\,\psi\left(\frac{|\xi|}{t}\right)\xi^\gamma\,d\xi\\ &\quad+\int e^{ix\cdot\xi}(e^{iy\xi}-e^{iz\cdot\xi})p(z,\xi)\,\psi\left(\frac{|\xi|}{t}\right)\xi^\gamma d\xi\\ &:= f(x-y,y,z,t)+g(x,y,z,t).\end{aligned}$$

Then

$$\left(\int_{C_j^\rho}|\widetilde{K}(y,x,t)-\widetilde{K}(z,x,t)|^2dx\right)^{1/2} \lesssim \|g(\cdot,y,z,t)\|_{L^2(C_j^\rho)}+\|f(\cdot-y,y,z,t)\|_{L^2(C_j^\rho)}$$

and we will obtain analogous estimates for the L^2 norm as presented before. Suppose first $tr < 1$ and note that

$$|g(x,y,z,t)|^2(1+t^{2\sigma}|x|^2)^N = \sum_{|\alpha|\leqslant N}\left[|g(x,y,z,t)|\,(t^{\sigma}|x|)^{|\alpha|}\right]^2. \tag{2.41}$$

Considering $G(\xi,y,z,t) = (e^{iy\cdot\xi} - e^{iz\cdot\xi})p(z,\xi)\,\psi\,(|\xi|/t)\;\xi^{\gamma}$ and taking the Fourier transform in the first variable we have the identity $\widehat{G}(x,y,z,t) = (2\pi)^{-n}g(x,y,z,t)$. In addition, from mean value inequality it follows for $|y-z| \leqslant r$ and $tr < 1$ that

$$\left|\partial_{\xi}^{\beta}[(e^{iy\cdot\xi} - e^{iz\cdot\xi})p(z,\xi)]\right| \lesssim (tr)t^{m-\sigma|\beta|}. \tag{2.42}$$

Then, from (2.41) and (2.42)

$$\begin{aligned}
\left(\int_{\mathbb{R}^n}|g(x,y,z,t)|^2(1+t^{2\sigma}|x|^2)^N dx\right)^{1/2} &\lesssim \sum_{|\alpha|\leqslant N} t^{\sigma|\alpha|}\|\widehat{G}(\cdot,y,z,t)|x|^{|\alpha|}\|_{L^2} \\
&\lesssim \sum_{|\alpha|\leqslant N} t^{\sigma|\alpha|}\|\widehat{\partial_{\xi}^{\alpha}G}(\cdot,y,z,t)\|_{L^2} \\
&\leqslant \sum_{|\alpha|\leqslant N}\sum_{|\beta|\leqslant|\alpha|} t^{\sigma|\alpha|+\lfloor\delta\rfloor}\left\|\partial_{\xi}^{\beta}[(e^{iy\cdot\xi} - e^{iz\cdot\xi})p(z,\xi)]\,\partial_{\xi}^{\alpha-\beta}\left[\psi\left(\frac{|\xi|}{t}\right)\left(\frac{\xi}{t}\right)^{\gamma}\right]\right\|_{L^2} \\
&\leqslant \sum_{|\alpha|\leqslant N}\sum_{|\beta|\leqslant|\alpha|} t^{\sigma|\alpha|+\lfloor\delta\rfloor}\,(tr)\,t^{m-\sigma|\beta|}\,t^{|\beta|-|\alpha|}\,t^{\frac{n}{2}} \\
&\lesssim (tr)t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor}
\end{aligned}$$

since $m \leqslant -n(1-\sigma)/2$. The estimate for f follows by the same steeps as the one presented for g. We proceed as before replacing G by $G'(\xi,y,z,t) = [p(y,\xi) - p(z,\xi)]\,\psi\,(|\xi|/t)\;\xi^{\gamma}$ and using the estimate

$$\left|\partial_{\xi}^{\alpha}[p(y,\xi) - p(z,\xi)]\right| \leqslant (tr)\,t^{m-\sigma|\alpha|}.$$

Thus, the conclusion follows in the same way as did in (2.36). If we drop the assumption $tr < 1$ we proceed as follows. First, write

$$\begin{aligned} g(x,y,z,t) &= \int e^{-i(x-y)\cdot\xi} p(z,\xi)\,\psi\left(\frac{|\xi|}{t}\right)\,\xi^\gamma d\xi - \int e^{-i(x-z)\cdot\xi} p(z,\xi)\,\psi\left(\frac{|\xi|}{t}\right)\,\xi^\gamma d\xi \\ &:= g_1(x,y,z,t) - g_2(x,y,z,t). \end{aligned}$$

We will obtain the L^2 estimate for g_1 and g_2. In the same way as before

$$\|g_2(\cdot,y,z,t)\|_{L^2(C_j^\rho)} \leqslant \left(\int |g_2(x,y,z,t)|^2\,[(x-z)^2 t^{2\sigma}]^N dx\right)^{1/2} \sup_{x\in C_j^\rho} [(x-z)t^\sigma]^{-N}$$

and consider $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| = N$. Integration by parts gives us

$$g_2(x,y,z,t)\,(x-z)^\alpha t^{\sigma|\alpha|} = C\,t^{\sigma|\alpha|}\widehat{\partial_\xi^\alpha G}(x-z,y,z,t),$$

where $G(\xi,y,z,t) = p(z,\xi)\,\psi\,(|\xi|/t)\,\xi^\gamma$. Using that

$$\begin{aligned} \left|\partial_\xi^\alpha G(\xi,y,z,t)\right| &\leqslant t^{\lfloor\delta\rfloor} \sum_{|\beta|\leqslant|\alpha|} \left|\partial_\xi^\beta p(z,\xi)\right| \left|\partial_\xi^{\alpha-\beta}\left[\psi\left(\frac{|\xi|}{t}\right)\left(\frac{\xi}{t}\right)^\gamma\right]\right| \\ &\leqslant \sum_{|\beta|\leqslant|\alpha|} |\xi|^{m-\sigma|\beta|}\,t^{|\beta|-|\alpha|+\lfloor\delta\rfloor} \\ &\leqslant \sum_{|\beta|\leqslant|\alpha|} t^{m-\sigma|\alpha|+\lfloor\delta\rfloor}\,t^{(1-\sigma)(|\beta|-|\alpha|)} \lesssim t^{m-\sigma|\alpha|+\lfloor\delta\rfloor} \end{aligned}$$

we get

$$\begin{aligned} \|g_2(\cdot,y,z,t)\,(x-z)^\alpha t^{\sigma|\alpha|}\|_{L^2(C_j^\rho)} &\leqslant \|t^{\sigma|\alpha|}\widehat{\partial_\xi^\alpha G}(\cdot-z,y,z,t)\|_{L^2} \\ &= \|t^{\sigma|\alpha|}\partial_\xi^\alpha G(\cdot-z,y,z,t)\|_{L^2} \\ &\lesssim t^{\sigma|\alpha|}\,t^{m-\sigma|\alpha|+\frac{n}{2}+\lfloor\delta\rfloor} \lesssim t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor} \end{aligned}$$

since $m \leqslant -(1-\sigma)n/2$. On the other hand, the same estimate for g_1 is valid. Indeed

$$\|g_1(\cdot,y,z,t)\|_{L^2(C_j^\rho)} \leqslant \left(\int |g_1(x,y,z,t)|^2\,[(x-y)^2 t^{2\sigma}]^N dx\right)^{1/2} \sup_{x\in C_j^\rho} [(x-y)t^\sigma]^{-N}.$$

The control of the integral is analogous as in the previous case and for supremum term note that since

$r < 1$, $x \in C_j^\rho$ and $|y - z| < r$ we get $|x - y| > 2^{j-1}r^\rho$ and thus

$$\sup_{x \in C_j^\rho} [(x - y)t^\sigma]^{-N} \leqslant (2^{j-1}r^\rho t^\sigma)^{-N}.$$

From that point we proceed as in (2.39) and obtain the desired estimates for g. The same argument applies to f if we split

$$f(x - y, y, z, t) = \int e^{-i(x-y)\cdot\xi} p(y, \xi)\psi\left(\frac{|\xi|}{t}\right) \xi^\gamma \, d\xi - \int e^{-i(x-y)\cdot\xi} p(z, \xi)\psi\left(\frac{|\xi|}{t}\right) \xi^\gamma \, d\xi$$

and estimate exactly in the same way as did for g.

The case $r \geqslant 1$ is analogous as the previous and we obtain that

$$\|g(\cdot, y, z, t)\|_{L^2(C_j)} \lesssim t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor} (2^j r t^\sigma)^{-N} \quad \text{and} \quad \|f(\cdot - y, y, z, t)\|_{L^2(C_j)} \lesssim t^{\frac{\sigma n}{2}+\lfloor\delta\rfloor} (2^j r t^\sigma)^{-N}.$$

Thus, the desired estimate follows as in (2.40). ∎

In [28, Theorem 2], the authors have shown the boundedness of $OpS^m_{\sigma,\delta}(\mathbb{R}^n)$, for $0 < \sigma \leqslant 1$, $0 \leqslant \nu < 1$ and $\sigma n - (n + 1) < m \leqslant -(n + 1)(1 - \sigma)$ in $H^1_w(\mathbb{R}^n)$ for

$$w \in A_t \quad \text{with} \quad t \in \left[1, \frac{1 + n + m}{n\sigma}\right) \subset [1, 2),$$

under the assumption $T^*(1) = 0$. In view of Proposition 2.3 and Theorem 2.10, we obtain the continuity of $OpS^m_{\sigma,\delta}(\mathbb{R}^n)$ in $H^1_w(\mathbb{R}^n)$ for every $m \leqslant -\frac{n}{2}(1 - \sigma)$ and $w \in A_t \cap RH_{\frac{2}{2-t}}$ with $1 < t < 2$ (in this case, we use the kernel of T satisfies D_2 condition).

2.4 $L^\infty(\mathbb{R}^n) - BMO(\mathbb{R}^n)$ boundedness

In section section, we point out other related result for the sake of completeness. The boundedness from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$ was proved in [5, Theorem 2.1] for the case $s_2 = 2$ and kernels satisfying (2.4) with the aditional assumption that T^* also satisfies condition (i). Further, the authors showed in [6, Corollary 3.3] that the same result, in the vector valued setting, remains true under the following general hypothesis: the kernel of T satisfies D_1 condition and for some $1 < p < q \leqslant \infty$ and $p/q \leqslant \sigma \leqslant 1$, T satisfies

$$|B(z,r)|^{-\frac{1}{q}} \|Tf\|_{L^q[B(z,r)]} \leqslant C\, |B(z,r^\sigma)|^{-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)} \quad \text{if } r < 1$$

and

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leqslant C\, \|f\|_{L^p(\mathbb{R}^n)} \quad \text{if } r > 1.$$

We show the following:

Theorem 2.11. *Let T be a strongly singular Calderón–Zygmund operator satisfying (i) and (iii) of Theorem A and D_1 condition. Assume also that (iii) holds for T^*. Then T is continuous from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.*

Proof. The proof is classical we only outline the main ideas. Given $f \in L^\infty(\mathbb{R}^n)$, we show that for any ball $B \subset \mathbb{R}^n$ there exist a constant a_B (may depend on B) such that

$$\sup_B \fint_B |Tf(x) - a_B|dx \lesssim \|f\|_{L^\infty}$$

with implicit constant independent of $B = B(x_0, r) \subset \mathbb{R}^n$. Suppose $r \leqslant 1$ and let $2B^\sigma = B(x_0, 2r^\sigma)$. Split f into

$$f = f\chi_{2B^\sigma} + f\chi_{(2B^\sigma)^c} := f_1 + f_2.$$

Since $T^* : L^q(\mathbb{R}^n) \to L^{s_2}(\mathbb{R}^n)$ is bounded, then $T : L^{s_2'}(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n)$ will also be bounded for $\dfrac{1}{s_2'} = 1 - \dfrac{1}{s_2}$ and $\dfrac{1}{q'} = \dfrac{1}{s_2'} - \dfrac{\beta}{n}$. In particular, since $f_1 \in L^{s_2'}(\mathbb{R}^n)$ then Tf_1 is well defined, belongs to $L^{q'}(\mathbb{R}^n)$ and

$$\int_B |Tf_1(x)|dx \leqslant |B|^{\frac{1}{q}} \|Tf_1\|_{L^{q'}} \lesssim |B|^{\frac{1}{q}} \|f_1\|_{L^{s_2'}} \lesssim |B|^{\frac{1}{q} + \frac{\sigma}{s_2'}} \|f\|_{L^\infty}.$$

Since $1/q + \sigma/s_2' - 1 \geqslant 0$ and $r \leqslant 1$ we have $|B|^{\frac{1}{q} + \frac{\sigma}{s_2'} - 1} \lesssim 1$. Then

$$\fint_B |Tf_1(x)|dx \leqslant C\, \|f\|_{L^\infty}. \tag{2.43}$$

For f_2 we use condition (2.9) to show

$$\begin{aligned}\int_{\mathbb{R}^n} |K(x,y)-K(z,y)|\ |f_2(y)|dy &\leqslant \|f\|_{L^\infty} \int_{|y-z|>2r^\sigma} |K(x,y)-K(z,y)|dy \\ &= \|f\|_{L^\infty} \sum_{j=0}^{\infty} \int_{C_j^\sigma} |K(x,y)-K(z,y)|dy \\ &\leqslant \|f\|_{L^\infty} \sum_{j=0}^{\infty} 2^{-\frac{j\delta}{\sigma}} \lesssim \|f\|_{L^\infty},\end{aligned}$$

and from the previous estimate

$$⨍_B |Tf_2(x) - Tf_2(z)|dx \leqslant C\,\|f\|_{L^\infty}. \tag{2.44}$$

Hence, we choose $a_Q := Tf_2(z)$ and from (2.43) and (2.44) we conclude

$$⨍_B |Tf(x) - Tf_2(z)|dx \leqslant ⨍_B |Tf_1(x)| + |Tf_2(x) - Tf_2(z)|dx \leqslant C\,\|f\|_{L^\infty}.$$

The proof for $r > 1$ is analogous if we split f in $2B$ and $(2B)^c$ and use the L^2 boundedness of T together with (2.8). □

Remark 2.10. The hypothesis on T^* on the previous theorem may be weakened to the condition

$$|B(z,r)|^{-\frac{1}{q}} \int_{B(z,r)} |Tf(x)|dx \leqslant C\,\|f\|_{L^{s_2'}}$$

that is, T maps continuously $L^{s_2'}(\mathbb{R}^n)$ into $\mathcal{M}^1_\lambda(\mathbb{R}^n)$ where $1/\lambda = 1/s_2' - \beta/n$ and

$$\mathcal{M}^1_\lambda(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) :\ \sup_{0<r\leqslant 1} |B(z,r)|^{\frac{1}{\lambda}-1} \int_{B(z,r)} |f(x)|dx < \infty \right\}$$

denotes the local Morrey-space with $\lambda > 1$.

Chapter 3

Inhomogeneous Calderón–Zygmund operators

In this chapter we consider a non-homogeneous version of Calderón–Zygmund operators, imposing a strong decay on the size of kernel at infinity. We follow the terminology of Ding, Han and Zhu in [29] for the standard case.

Definition 3.1. *We say that a locally integrable function K defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash \Delta$ is called a (μ, δ, σ) inhomogeneous kernel for $\mu > 0$, $0 < \delta \leqslant 1$ and $0 < \sigma \leqslant 1$ if there exists $C > 0$ such that*

$$|K(x,y)| \leqslant C \min\left\{\frac{1}{|x-y|^n}, \frac{1}{|x-y|^{n+\mu}}\right\}, \quad \textit{for every } x \neq y, \tag{3.1}$$

and

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leqslant C\frac{|y-z|^\delta}{|x-z|^{n+\frac{\delta}{\sigma}}}$$

for all $|x-z| \geqslant 2|y-z|^\sigma$. When $\sigma = 1$, we simply call it a (μ, δ) standard inhomogeneous kernel.

In comparison with the kernels studied in the previous chapter, we assume in (3.1) an extra decay of the kernel at the infinity. Such condition is natural when considering pseudodifferential operators in the Hörmander class $OpS^m_{\sigma,\nu}(\mathbb{R}^n)$ with $0 < \sigma \leqslant 1$ and $0 \leqslant \nu < 1$. For instance, it has been shown in [4, Theorem 1.1 (a)] that the kernel associated to this class of operators satisfies the pseudo local property: there exist $N_0 \in \mathbb{Z}_+$ such that $|K(x,y)| \lesssim |x-y|^{-N}$ for every $N \geqslant N_0$ and $x \neq y$. Then, (3.1) follows immediately. Moreover, condition (3.1) has already been explored in earlier works for non-homogeneous spaces (see for instance [76, Theorem 3.2.49] for non-homogeneous Triebel–Lizorkin spaces).

Definition 3.2. *We say that a linear and continuous operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a strongly singular inhomogeneous Calderón–Zygmund operator if the following properties are satisfied:*

(i) T extends to a continuous operator from $L^2(\mathbb{R}^n)$ to itself;

(ii) T extends to a continuous operator from $L^q(\mathbb{R}^n)$ to $L^{s_2}(\mathbb{R}^n)$, for some $1 < s_2 < \infty$ and

$$\frac{1}{q} = \frac{1}{s_2} + \frac{\beta}{n}, \quad \textit{where} \quad n(1-\sigma)\left(1 - \frac{1}{s_2}\right) \leqslant \beta < n\left(1 - \frac{1}{s_2}\right).$$

(iii) T is associated to a (μ, δ, σ) inhomogeneous kernel and it is given (formally) by

$$\langle Tf, g \rangle = \int\int K(x,y) f(y) g(x) dy dx, \quad \textit{for all } \; f, g \in \mathcal{S}(\mathbb{R}^n) \textit{ with disjoint supports.}$$

When $\sigma = 1$, we call it standard inhomogeneous Calderón–Zygmund operator and condition (ii) can be dropped, that is, only the continuity on $L^2(\mathbb{R}^n)$ is required.

In the same spirit of the previous chapter, we assume an integral weaker condition on the kernel. With the same parameters and notation introduced earlier in Definition 2.5, we say the kernel K satisfies a *local D_s condition* if (2.9) holds for $0 < r < 1$, that is

$$\sup_{\substack{|y-z|<r \\ 0<r<1}} \left(\int_{C_j(z,r^\rho)} |K(x,y) - K(x,z)|^s + |K(y,x) - K(z,x)|^s dx \right)^{1/s} \lesssim |C_j(z,r^\rho)|^{\frac{1}{s} - 1 + \frac{\delta}{n}\left(\frac{1}{\rho} - \frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}}$$

for any $0 < \rho \leqslant \sigma \leqslant 1$ together with the inhomogeneous size control (3.1).

Regarding the L^p continuity of such operators, in Theorem 2.3 we pointed out that strongly singular Calderón–Zygmund operators satisfy weak $(1,1)$-type inequality when the kernel satisfies a $\sigma-$Hörmander condition (2.5) and (2.6), which is weaker than D_1. For the inhomogeneous case, we can see that the size condition (3.1) implies

$$\sup_{r>1} \int_{|x-z|>Cr} |K(x,y) - K(x,z)| dx \leqslant 2(Cr)^{-\mu} \leqslant 2C^{-\mu}.$$

This means that assuming (3.1) and (2.5) (local $\sigma-$Hörmander condition), one can shows that strongly singular inhomogeneos Calderón–Zygmund operators satisfy weak $(1,1)$-type inequality. This, together with the L^2 continuity and interpolation we conclude the bondedness on $L^p(\mathbb{R}^n)$ for every $1 < p < \infty$. The same conclusion holds for standard inhomogeneous Calderón–Zygmund operators.

The continuity of pseudodifferential operators in local Hardy spaces have been considered for instance in [37, 42, 45] among others. For standard inhomogeneous Calderón–Zygmund operators, necessary and sufficient conditions for the continuity on $h^p(\mathbb{R}^n)$ have been considered in [29] when $\frac{n}{n+1} < p < 1$. In the mentioned work, the authors obtained the following result:

Theorem 3.1 ([29, Theorem 1.1])**.** *If T is a standard inhomogeneous Calderón–Zygmund operator such that $T^*(1) \in \dot{\Lambda}_{n\left(\frac{1}{p}-1\right)}(\mathbb{R}^n)$, then T is bounded on $h^p(\mathbb{R}^n)$ provided that* $\max\left\{\frac{n}{n+\mu}, \frac{n}{n+\delta}\right\} < p < 1$. *Conversely, if T is bounded on $h^p(\mathbb{R}^n)$ for $\frac{n}{n+1} < p < 1$, then $T^*(1) \in \Lambda_{n\left(\frac{1}{p}-1\right)}(\mathbb{R}^n)$.*

For the sufficiency part, they used Komori's molecular approach [50, Definition 4.4] for $n/(n+1) < p < 1$. The necessity follows by the estimate

$$\left| \int f(x)dx \right| \lesssim \|f\|_{h^p}, \quad \text{for every } f \in L^2(\mathbb{R}^n) \cap h^p(\mathbb{R}^n), \tag{3.2}$$

together with a duality argument.

The goal of this chapter is to extend the previous theorem for $0 < p \leqslant 1$ and both standard and strongly singular inhomogeneous operators. In particular, we will assume an appropriate T^* inhomogeneous cancellation condition expressed in terms of the Campanato-type spaces, that we describe in the sequence. Given $k \in \mathbb{Z}_+$, $1 \leqslant s < \infty$, $\psi : (0,\infty) \to (0,\infty)$ and $B = B(x_0, r) \subset \mathbb{R}^n$, we define the

ψ-generalized Campanato spaces as

$$L_k^{s,\psi}(\mathbb{R}^n) := \Bigg\{ f \in L^s_{c,k}(\mathbb{R}^n) : \text{ there exists } C > 0 \text{ such that for all } B \subset \mathbb{R}^n, \\ \left(\fint_B |f(y) - (P_B^k f)(y)|^s dy \right)^{1/s} \leqslant C\,\psi(r) \Bigg\},$$

where $P_B^k f(y)$ is the unique polynomial of degree less than or equal to k that has the same moments as f over B up to order k. We equip it with the functional

$$\|f\|_{L_k^{s,\psi}} := \sup_{B \subset \mathbb{R}^n} \frac{1}{\psi(r)} \left(\fint_B |f(y) - (P_B^k f)(y)|^s dy \right)^{1/s}.$$

The space $L_k^{s,\psi}(\mathbb{R}^n)$ is considered as a quotient space of the above classes of functions modulo all polynomials of degree less than or equal to k. There are several identifications of ψ-generalized Campanato spaces with other well known fuctions spaces in Harmonic Analysis. For instance, if $k = 0$, $1 \leqslant s < \infty$ and $\psi \equiv 1$, then $L_0^{s,\psi}(\mathbb{R}^n) \cong BMO(\mathbb{R}^n)$, and if $\psi(t) = t^\gamma$, then $L_{\lfloor\gamma\rfloor}^{s,\psi}(\mathbb{R}^n) \cong \dot{\Lambda}_\gamma(\mathbb{R}^n)$. In addition, following the proof of John-Nirenberg inequality (see [73] for instance) or the one for Morrey-Campanato spaces in [56], we can see that if ψ is an increasing function and $k \in \mathbb{Z}_+$, then for all $1 \leqslant s < \infty$ we have $L_k^{s,\psi}(\mathbb{R}^n) \cong L_k^{1,\psi}(\mathbb{R}^n)$. We refer to [35, Chapter III Section 5] for a detailed discussion on the relation between Campanato, Lipschitz and Zygmund spaces, and also to [69] for an exposition of Campanato spaces on different domains and their generalizations.

We will make use of the same T^* notation previously introduced in Definition 2.6 for the homogeneous case.

Definition 3.3. *Let $m \in \mathbb{Z}_+$ and $L^2_{c,m}(\mathbb{R}^n)$ as in Definition 2.6. For every $\alpha \in \mathbb{Z}^n_+$ such that $|\alpha| \leqslant m$ and $x_0 \in \mathbb{R}^n$, define $T^*\left((x - x_0)^\alpha\right)$ in the distributional sense by*

$$\langle T^*\left((x - x_0)^\alpha\right), g\rangle = \langle (x - x_0)^\alpha, Tg\rangle = \int_{\mathbb{R}^n} (x - x_0)^\alpha Tg(x)dx, \quad \textit{for all } g \in L^2_{c,m}(\mathbb{R}^n). \tag{3.3}$$

We have proved in Proposition 2.2 the well definition of (3.3) for standard and strongly singular Calderón–Zygmund operators (see also [64, p. 23]). The next proposition extends it to the inhomogeneous case considered in this chapter.

Proposition 3.1. *Let T be a linear and bounded operator on $L^2(\mathbb{R}^n)$ whose associated kernel satisfies the local D_1 condition, that is* (2.9) *and* (3.1) *with $s = 1$. Then $(x - x_0)^\alpha Tg(x) \in L^1(\mathbb{R}^n)$ for all $g \in L^2_{c,m}(\mathbb{R}^n)$ provided that $m = \min\{\mu, \delta\}$.*

Proof. Let $g \in L^2_{c,m}(\mathbb{R}^n)$, fix a ball $B = B(x_0, r)$ such that supp $(g) \subset B$, and write

$$\int_{\mathbb{R}^n} |(x - x_0)^\alpha Tg(x)|dx = \int_{2B} |(x - x_0)^\alpha Tg(x)|dx + \int_{(2B)^c} |(x - x_0)^\alpha Tg(x)|dx.$$

From the boundedness of T on $L^2(\mathbb{R}^n)$ we get

$$\int_{2B} |(x - x_0)^\alpha Tg(x)|dx \leqslant \|(\cdot - x_0)^\alpha\|_{L^\infty(2B)} \, |2B|^{\frac{1}{2}} \, \|Tg\|_{L^2} \lesssim r^{|\alpha| + \frac{n}{2}} \|g\|_{L^2} < \infty.$$

For the second integral, suppose first $r \geqslant 1$. Splitting the integral on $(2B)^c$ into an appropriate annulus decomposition $C_j = C_j(x_0, r)$, the estimation follows by (3.1), Hölder inequality and the fact that $x \in C_j$ and $y \in B$ implies $|x - y| \geqslant |x - x_0|/2$:

$$\begin{aligned}
\int_{(2B)^c} |x - x_0|^{|\alpha|} |Tg(x)|dx &\leqslant \sum_{j \in \mathbb{N}} (2^{j+1} r)^{|\alpha|} \int_{C_j} \int_B |K(x, y)| \, |g(y)| dy dx \\
&\lesssim \|g\|_{L^2} \, r^{|\alpha| + \frac{n}{2}} \sum_{j \in \mathbb{N}} (2^j)^{|\alpha|} \int_{C_j} |x - x_0|^{-n-\mu} dx \\
&\lesssim \|g\|_{L^2} \, r^{|\alpha| + \frac{n}{2} - \mu} \sum_{j \in \mathbb{N}} (2^j)^{|\alpha| - \mu} < \infty
\end{aligned}$$

since $|\alpha| < \mu$. For $r < 1$, we use condition (2.9) and proceed like in the proof of Proposition 2.2. ∎

Remark 3.1. Note that the hypotheses on T together with the proof of the previous proposition imply that for every ball B, the dual pairing (3.3) defines $f = T^* ((\cdot - x_0)^\alpha)$ as an element of $\left(L^2_{N_p}(B)\right)^*$, which can be identified with the quotient space of $L^2(B)$ by the polynomials of degree up to N_p.

In our main result, we extend both necessary and sufficient part of Theorem 3.1 for the full range $0 < p \leqslant 1$ and for both standard and strongly singular inhomogeneous Calderón-Zygmund operators. In particular, it conciliate the Lipschitz-type cancellation condition when $p \neq n/(n+k)$ for any $k \in \mathbb{Z}_+$ and introduce a local Campanato-type condition when $p = n/(n+k)$ for some $k \in \mathbb{Z}_+$.

Theorem B. *Let* $0 < p \leqslant 1$ *and* T *be a strongly singular inhomogeneous Calderón–Zygmund operator associated to a kernel satisfying the local integral integral condition* (2.9) *for some* $\delta > 0$ *and* $1 \leqslant s \leqslant s_2$ *with* $p < s$. *Then,* T *can be extended to a bounded operator from* $h^p(\mathbb{R}^n)$ *to itself for* $\max\left\{\frac{n}{n+\mu},\, p_0\right\} < p \leqslant 1$, *where* p_0 *is given by* (2.13), *if, and only if there exists a constant* $C > 0$ *such that*

$$f = T^*[(\cdot - x_0)^\alpha] \quad \textit{satisfies} \quad \left(\fint_B |f(x) - P_B^{N_p}(f)(x)|^2 dx\right)^{1/2} \leqslant C\,\Psi_{p,\alpha}(r), \tag{3.4}$$

for every ball $B = B(x_0, r) \subset \mathbb{R}^n$ *such that* $r < 1$ *and* $\alpha \in \mathbb{Z}_+^n$ *with* $|\alpha| \leqslant N_p$, *where* $P_B^{N_p}(f)$ *is the polynomial of degree* $\leqslant N_p$ *that has the same moments as* f *over* B *up to order* N_p *and*

$$\Psi_{p,\alpha}(t) := \begin{cases} t^{n\left(\frac{1}{p}-1\right)} & \textit{if } |\alpha| < n\,(1/p-1)\,, \\ t^{n\left(\frac{1}{p}-1\right)}\left[\log\left(1+\dfrac{1}{t}\right)\right]^{-\frac{1}{p}} & \textit{if } |\alpha| = n\,(1/p-1) = N_p. \end{cases}$$

Remark 3.2. Note that $f = T^*[(\cdot - x_0)^\alpha]$ is required to satisfy the Campanato-type condition (3.4) only for balls B such that $|B| < 1$, but the condition makes sense for any ball due to Remark 3.1. Thus, one can replace (3.4) by the stronger condition: for every $x_0 \in \mathbb{R}^n$

$$\begin{cases} T^*[(x-x_0)^\alpha] \in \dot{\Lambda}_{n\left(\frac{1}{p}-1\right)}(\mathbb{R}^n) & \text{if } |\alpha| < n\,(1/p-1) \\ T^*[(x-x_0)^\alpha] \in L_{N_p}^{2,\Psi_p}(\mathbb{R}^n) & \text{if } |\alpha| = n\,(1/p-1) = N_p. \end{cases}$$

When $p = 1$, condition (3.4) is

$$\left(\fint_B |f(x) - f_B|^2 dx\right)^{1/2} \leqslant C\left[\log\left(1+\frac{1}{r}\right)\right]^{-1}.$$

The function space characterized by the above condition is known in the literature as $LMO(\mathbb{R}^n)$ and is connected with $h^1(\mathbb{R}^n)$ in the following sense: in [52], the authors have shown that $b \in LMO(\mathbb{R}^n)$ is a necessary and sufficient condition for the commutator $[b, OpS^m_{\sigma,\nu}]$ to be bounded on $h^1(\mathbb{R}^n)$ to itself for $0 < \sigma \leqslant 1$, $0 \leqslant \nu < 1$, $\delta \leqslant \sigma$ and $-(n+1) < m \leqslant -(n+1)(1-\sigma)$. For other results relating

$LMO(\mathbb{R}^n)$-type spaces and both $H^1(\mathbb{R}^n)$ and $h^1(\mathbb{R}^n)$, we refer to [72].

3.1 Proof of Theorem B

3.1.1 Sufficiency

To prove the sufficiency part of Theorem B, we are going to use the molecular decomposition of $h^p(\mathbb{R}^n)$ presented in Section 1.2.1. Let a to be a $(p,2)$ atom in h^p (as in Definition 1.5) supported in $B := B(x_0, r)$. We will show that Ta is a (p, s, λ, ω) molecule for λ satisfying

$$n\left(\frac{s}{p}-1\right) < \lambda < \min\left\{ s\mu + n(s-1),\ n\left(\frac{s}{s_2}-1\right) + s\beta\left[\frac{n\left(1-\frac{1}{s_2}\right)\frac{\delta}{\sigma}}{\beta+\frac{\delta}{\sigma}-\delta}\right]\right\}.$$

Given $1 \leqslant s < \infty$, we choose $t > s$ and condition (M1) will follow from the $L^t(\mathbb{R}^n)$ continuity, as in (2.14). To show (M2) we split it in two cases, depending on the size of the radius of the ball. If $r \geqslant 1$, from condition (3.1), it follows that for $|x - x_0| > 2r$ and $|y - x_0| < r$ we have $|K(x,y)| \lesssim |x - x_0|^{-n-\mu}$. Then

$$|Ta(x)| \leqslant \int_B |K(x,y)|\,|a(y)|dy \lesssim \|a\|_{L^2}\,|B|^{\frac{1}{2}}\,|x-x_0|^{-n-\mu} \lesssim r^{-n\left(\frac{1}{p}-1\right)}|x-x_0|^{-n-\mu}.$$

Since $\lambda/s - n/s' < \mu$, we have $\lambda - s(n+\mu) < -n$ and therefore

$$\int_{(2B)^c} |Ta(x)|^s\,|x-x_0|^{\lambda}dx \lesssim r^{-sn\left(\frac{1}{p}-1\right)}\int_{(2B)^c}|x-x_0|^{\lambda-s(n+\mu)}dx \lesssim r^{\lambda+n\left(1-\frac{s}{p}\right)}\,r^{-s\mu} \lesssim r^{\lambda+n\left(1-\frac{s}{p}\right)}.$$

Condition (M3) follows from Remark 1.4 for this case. Suppose now that $r < 1$. To show the global estimate (M2), we will recall the same idea and notation used on the proof of Theorem A. Consider $0 < \rho \leqslant \sigma \leqslant 1$, where ρ is given by (2.18). Splitting the integral of $\mathbb{R}^n$ in $2B^{\rho}$ and $(2B^{\rho})^c$, we follow the same estimates as (2.16) and (2.17) since

$$\lambda \leqslant n\left(\frac{s}{s_2}-1\right) + \frac{s\beta}{1-\rho}.$$

Finally, in order to verify that (M3) holds, note that for $r < 1$ the function a is in particular a $(p,2)$ atom in H^p and has vanishing moments up to the order N_p. From condition (3.4), setting $f = T^*[(\cdot - x_0)^\alpha]$, we have, by (3.3),

$$
\begin{aligned}
\left|\int Ta(x)(x-x_0)^\alpha dx\right| &= |\langle Ta, (\cdot - x_0)^\alpha\rangle| = |\langle T^*\left[(\cdot - x_0)^\alpha\right], a\rangle| \\
&\leqslant \int_B |f(x) - P_B^{N_p}(f)(x)|\, |a(x)| dx \\
&\leqslant \left(\int_B |f(x) - P_B^{N_p}(f)(x)|^2 dy\right)^{1/2} \|a\|_{L^2(B)} \\
&\lesssim \Psi_{p,\alpha}(r)\, |B|^{\frac{1}{2}}\, \|a\|_{L^2(B)} \\
&\lesssim \Psi_{p,\alpha}(r)\, r^{-n\left(\frac{1}{p}-1\right)} \\
&\leqslant \begin{cases} C_{n,p} & \text{if } \alpha < n(1/p-1), \\ \varphi_p(r) & \text{if } |\alpha| = n(1/p-1) = N_p. \end{cases}
\end{aligned}
$$

Therefore, Ta is a (p, s, λ, ω) molecule provided that $\max\left\{\dfrac{n}{n+\mu},\ p_0\right\} < p \leqslant 1$.

Remark 3.3. In the spirit of Theorem 2.4, we can also consider in the previous theorem the local derivative D_s condition (2.21).

3.1.2 Necessity

In order to show that condition (3.4) is necessary for the boundedness of inhomogeneous Calderón–Zygmund-type operators, it will be more convenient to use a characterization of $h^p(\mathbb{R}^n)$ in terms of the grand maximal function (see for instance [8, Section 2]), and also replace the restriction $0 < t < 1$ in the definition of the maximal function by $0 < t < T$ for some $T < \infty$, which yields equivalent norms. Given $0 < T < \infty$ and $x \in \mathbb{R}^n$, consider the family

$$
\mathcal{F}_k^{T,x} = \left\{\phi \in C^\infty(\mathbb{R}^n) : \text{ supp } (\phi) \subset B(x,t),\ 0 < t < T \text{ and } \|\partial^\alpha \phi\|_{L^\infty} \leqslant t^{-n-|\alpha|} \text{ for all } |\alpha| \leqslant k\right\}.
$$

We define the *local grand maximal function* associated to the family $\mathcal{F}_k^{T,x}$ by

$$
m_{\mathcal{F}_k}(f)(x) = \sup_{\phi \in \mathcal{F}_k^{T,x}} |\langle f, \phi\rangle|,
$$

where by $\langle\, ,\, \rangle$ we mean the pairing in the distributional sense.

Lemma 3.1. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. If $k \in \mathbb{N}$ is such that $\frac{n}{n+k} < p \leqslant \frac{n}{n+k-1}$ (i.e. $k = N_p + 1$), then*

$$\|m_{\mathcal{F}_k}(f)\|_{L^p} \leqslant C_{n,p,T}\, \|f\|_{h^p}, \tag{3.5}$$

where $C_{n,1,T} \lesssim 1 + \log_+ T$ and $C_{n,p,T} \lesssim \max\{1,\, T^{n(1/p-1)}\}$ for $p < 1$.

Proof. Since the atomic decomposition (1.2) converges in the sense of distributions and $m_{\mathcal{F}_k}$ is sub-linear, it suffices to prove that if a is a (p, ∞) atom in h^p, then $\|m_{\mathcal{F}_k}(a)\|_{L^p} \leqslant C$. Indeed, writing $f = \sum_{j=1}^{\infty} \lambda_j a_j$, this gives

$$\|m_{\mathcal{F}_k}(f)\|_{L^p} \leqslant \left(\sum_{j=1}^{\infty} |\lambda|^p \, \|m_{\mathcal{F}_k}(a)\|^p_{L^p} \right)^{1/p} \leqslant C \left(\sum_{j=1}^{\infty} |\lambda|^p \right)^{1/p} \leqslant C\, \|f\|_{h^p}.$$

Fix a (p, ∞) atom a supported on $B = B(x_0, r) \subset \mathbb{R}^n$ and split

$$\|m_{\mathcal{F}_k}(a)\|^p_{L^p} = \int_{2B} [m_{\mathcal{F}_k}(a)(x)]^p dx + \int_{(2B)^c} [m_{\mathcal{F}_k}(a)(x)]^p dx.$$

To deal with the first integral, note that for any $\phi \in \mathcal{F}_k^{T,x}$ one has

$$\left| \int a(y)\phi(y)dy \right| \leqslant \|a\|_{L^\infty} \, \|\phi\|_{L^\infty} \, |B(x_0, r) \cap B(x, t)| \leqslant C_n\, r^{-\frac{n}{p}}.$$

Then,

$$\int_{2B} [m_{\mathcal{F}_k}(a)(x)]^p dx \leqslant C_{n,p}\, r^{-n}\, |2B| \simeq C_{n,p}.$$

When $x \notin 2B$, note that $\int a(y)\phi(y)dy$ vanishes unless $B(x,t) \cap B(x_0, r) \neq \varnothing$ and this implies $t > r$ and moreover

$$|x - x_0| \leqslant t + r \leqslant t + \frac{|x - x_0|}{2} \quad \Rightarrow \quad \frac{|x - x_0|}{2} \leqslant t.$$

Hence $r \leqslant \frac{|x-x_0|}{2} \leqslant t < T$. Thus, if $r \geqslant 1$ we have

$$\left| \int a(y)\phi(y)dy \right| \leqslant \|a\|_{L^1} \|\phi\|_{L^\infty} \leqslant C_n\, r^{n\left(1-\frac{1}{p}\right)} t^{-n} \leqslant C_n |x - x_0|^{-n},$$

and therefore

$$\int_{(2B)^c} [m_{\mathcal{F}_k}(a)(x)]^p dx \lesssim \int_{2r<|x-x_0|<2T} |x-x_0|^{-np} dx \lesssim \int_{2<|x-x_0|<2T} |x-x_0|^{-np} dx < \infty.$$

Note that the integral on the right has order $\log T$ when $p = 1$ and $T^{n(1-p)}$ when $p < 1$. For $0 < r < 1$, we have the standard $H^p(\mathbb{R}^n)$ argument. Using the moment conditions of a up to the order $N_p = k - 1$ and the Taylor expansion of $\phi \in \mathcal{F}_k^{T,x}$ to write

$$\begin{aligned}\left|\int a(y)\phi(y)dy\right| &= \left|\int \left[\phi(y) - \sum_{|\alpha|\leqslant k-1} C_\alpha\, \partial^\alpha \phi(x-x_0)(y-x_0)^\alpha\right] a(y)dy\right| \\ &\leqslant \sum_{|\alpha|=k} C_\alpha\, \|\partial^\alpha \phi\|_{L^\infty}\, r^{|\alpha|+n}\, \|a\|_{L^\infty} \\ &\leqslant C_n\, t^{-n-k}\, r^{k+n\left(1-\frac{1}{p}\right)}.\end{aligned}$$

Then,

$$\int_{(2B)^c} [m_{\mathcal{F}_k}(a)(x)]^p dx \leqslant C_{n,p}\, r^{kp+np-n} \int_{|x-x_0|>2r} |x-x_0|^{p(-k-n)} dx < \infty,$$

since $p > n/(n+k)$. ◻

Remark 3.4. Since $m_\varphi f \leqslant C\, m_{\mathcal{F}_k} f$, it is also possible to show the other direction of (3.5) and therefore we have a characterization.

The next result is a strengthening of (3.2) for $f \in h^p(\mathbb{R}^n)$ supported in small balls and when a higher amount of moments are considered. In particular, a more appropriate logarithmic bound, depending on the support, is provided when $p = \frac{n}{n+k}$ for some $k \in \mathbb{Z}_+$, that is $n\left(\frac{1}{p} - 1\right) \in \mathbb{Z}_+$.

Proposition 3.2. *Let $g \in h^p(\mathbb{R}^n)$ be supported in $B(x_0, r)$ for some $x_0 \in \mathbb{R}^n$ and $0 < r < 1$. Then for $\alpha \in \mathbb{Z}_+^n$, the moments $\langle g, (\cdot - x_0)^\alpha\rangle$ are well-defined and satisfy*

$$|\langle g, (\cdot - x_0)^\alpha\rangle| \leqslant \begin{cases} C_{\alpha,p}\, \|g\|_{h^p} & \text{if } |\alpha| < n(1/p-1); \\ C_{\alpha,p}\, \|g\|_{h^p} \left[\log\left(1+\dfrac{1}{r}\right)\right]^{-1/p} & \text{if } |\alpha| = n(1/p-1) = N_p. \end{cases} \tag{3.6}$$

Remark 3.5. Note that condition (3.6) for $|\alpha| = n(1/p - 1) = N_p$ gets stronger as $r \to 0$.

Proof. Since g is a compactly supported tempered distribution, it acts on $C^\infty(\mathbb{R}^n)$ and therefore we can define $\langle g, (\cdot - x_0)^\alpha\rangle$ unambiguously for any multi-index $\alpha \in \mathbb{Z}^n_+$, and $\langle g, (\cdot - x_0)^\alpha\rangle = \langle g, \phi\rangle$ for all $\phi \in C^\infty(\mathbb{R}^n)$ such that $\phi(y) = (y - x_0)^\alpha$ on the support of g.

By a translation argument we may assume that $x_0 = 0$. For each unit vector on $v \in \mathbb{S}^{n-1}$ and $\alpha \in \mathbb{Z}^n_+$ such that $|\gamma| \leqslant N_p$, we choose $\phi_0^{v,\gamma}$ to be a fixed function satisfying the following conditions:

(i) $\phi_0^{v,\gamma} \in C_c^\infty(\mathbb{R}^n)$ with support in $B\left(\frac{v}{2}, 2\right)$ and $\|\partial^\beta \phi_0^{v,\gamma}\|_{L^\infty} \leqslant 2^{|\beta|-2n}$ for all $|\beta| \leqslant N_p + 1$;

(ii) $\phi_0^{v,\gamma}(y) = C_\alpha\, y^\gamma$ for all $|y| < 1$ for some constant C_α depending only on n and α;

(iii) $\displaystyle\int \phi_0^{v,\gamma}(y)dy \neq 0$.

Let $x \in \mathbb{R}^n$ such that $|x| > \dfrac{r}{2}$ and define

$$\phi^{x,\gamma}(y) = \frac{1}{|x|^n}\, \phi_0^{\frac{x}{|x|},\gamma}\left(\frac{y}{2|x|}\right).$$

We claim $\phi^{x,\gamma} \in \mathcal{F}_k^{T,x}$ for $T = 2$ and $k \leqslant N_p + 1$. Indeed, note first that supp $(\phi^{x,\gamma}) \subset B(x,t)$ for $t = 4|x|$ since if $|y - x| > t$ we have

$$\left|\frac{y}{2|x|} - \frac{x}{2|x|}\right| = \frac{|y-x|}{2|x|} > \frac{t}{2|x|} = 2$$

and then $\phi_0^{\frac{x}{|x|},\gamma}(y/2|x|) = 0$. Moreover, for $|\beta| \leqslant N_p + 1$, by assumption *(i)*,

$$\left\|\partial^\beta \phi^{x,\gamma}\right\|_{L^\infty} = 2^{-|\beta|}|x|^{-n-|\beta|}\left\|\partial^\beta \phi_0^{\frac{x}{|x|},\gamma}\right\|_{L^\infty} \leqslant t^{-n-|\beta|}.$$

On the support of g, $|y| < r$ and $|x| > \dfrac{r}{2}$, so $\dfrac{|y|}{2|x|} < 1$ and by assumption *(ii)*, $\phi^{x,\alpha}(y) = \dfrac{C_\alpha\, y^\alpha}{|x|^{n+|\alpha|}}$. Hence

$$m_{\mathcal{F}_k}(g)(x) = \sup_{\phi \in \mathcal{F}_k^{T,x}} |\langle g, \phi\rangle| \geqslant |\langle g, \phi^{x,\alpha}\rangle| = C_\alpha |x|^{-n-|\alpha|}\left|\langle g, (\cdot - x_0)^\alpha\rangle\right|.$$

When $|\alpha| = n(1/p-1) = N_p$, this gives

$$\begin{aligned}\|g\|_{h^p}^p &\geqslant \int_{\frac{r}{2}<|x|<\frac{r+1}{2}} [m_{\mathcal{F}_k}(g)(x)]^p dx \\ &\geqslant C_\alpha \left|\langle g, (\cdot - x_0)^\alpha\rangle\right|^p \int_{\frac{r}{2}<|x|\leqslant\frac{r+1}{2}} |x|^{-p(n+|\alpha|)} dx \\ &\geqslant C_\alpha \left|\langle g, (\cdot - x_0)^\alpha\rangle\right|^p \log\left(1+\frac{1}{r}\right).\end{aligned}$$

For $|\alpha| < n(1/p-1)$, we consider $1 < |x| < 3/2$. Since in particular $|x| > r/2$, the same calculations as above give

$$\|g\|_{h^p}^p \geqslant \int_{1<|x|<\frac{3}{2}} [m_{\mathcal{F}_k}(g)(x)]^p dx \geqslant C_\alpha \left|\langle g, (\cdot - x_0)^\alpha\rangle\right|^p \int_{1<|x|\leqslant\frac{3}{2}} |x|^{-p(n+|\alpha|)} dx = C_{n,\alpha,p} \left|\langle g, (\cdot - x_0)^\alpha\rangle\right|^p.$$

□

We will now show that the above result can be extended to a class of h^p distributions called *pre-molecules*, which satisfies the size conditions of a molecule, but without any assumption on the cancellation.

Definition 3.4. *Let $0 < p \leqslant 1 \leqslant s < \infty$ with $p < s$, $\lambda > n(s/p-1)$, and $C > 0$. We say that a measurable function M is a (p,s,λ,C) pre-molecule in h^p if there exist a ball $B = B(x_0,r) \subset \mathbb{R}^n$ and a constant $C > 0$ such that:*

M1. $\|M\|_{L^s(B)} \leqslant C\,|B|^{\frac{1}{s}-\frac{1}{p}}$;

M2. $\|M\,|\cdot - x_0|^{\frac{\lambda}{s}}\|_{L^s(B^c)} \leqslant C\,|B|^{\frac{\lambda}{ns}+\frac{1}{s}-\frac{1}{p}}$.

From the previous proposition, we have the following control on the moments of a pre-molecule associated to small balls:

Proposition 3.3. *Let $0 < p \leqslant 1$ and M a pre-molecule in h^p associated to the ball $B = B(x_0,r)$ with*

$0 < r < 1$. *Then for* $\alpha \in \mathbb{Z}_+^n$ *and* $|\alpha| \leqslant N_p$

$$\left|\int M(x)\,(x-x_0)^\alpha dx\right| \lesssim \begin{cases} \|M\|_{h^p} + C_{n,p,\lambda} & \text{if } |\alpha| < n(1/p-1); \\ \dfrac{\|M\|_{h^p} + C_{n,p,\lambda}}{\left[\log\left(\frac{1}{r}\right)\right]^{1/p}} & \text{if } |\alpha| = n(1/p-1) = N_p. \end{cases} \tag{3.7}$$

Proof. From the proof of the molecular decomposition (Proposition 1.4), we see that we can decompose the pre-molecule as

$$M = \sum_{j=1}^{\infty} c_j\, a_j \;+\; a_B,$$

where $\{a_j\}$ are $(p, 2)$ atoms in H^p (i.e., have full cancellation) supported in $B(x_0, 2^j r)$, $\sum |c_j|^p \leqslant C_{n,p,\lambda}$, and $a_B \in L^2(B)$. Hence, by Proposition 3.2,

$$\begin{aligned} \left|\int M(x)\,(x-x_0)^\alpha dx\right| &= \left|\int \left(\sum_{j=0}^{\infty} d_j\, a_j(x) \;+\; a_B(x)\right)(x-x_0)^\alpha dx\right| \\ &= \left|\sum_{j=0}^{\infty} d_j \int a_j(x)\,(x-x_0)^\alpha dx + \int a_B(x)\,(x-x_0)^\alpha dx\right| \\ &= \left|\int a_B(x)\,(x-x_0)^\alpha dx\right| \\ &\lesssim \begin{cases} \|a_B\|_{h^p} & \text{if } |\alpha| < n(1/p-1); \\ \|a_B\|_{h^p}\left[\log\left(1+\dfrac{1}{r}\right)\right]^{-1/p} & \text{if } |\alpha| = n(1/p-1) = N_p, \end{cases} \end{aligned} \tag{3.8}$$

where (3.8) follows since $\displaystyle\sum_{j=1}^{\infty}\int d_j\, a_j(x)(x-x_0)^\alpha dx$ converges absolutely. In fact, recall that supp $(a_j) \subset B(x_0, 2^j r)$ and $|\alpha| = N_p = n(1/p-1)$. Then

$$\int |a_j(x)||x-x_0|^{N_p} dx \leqslant C_n\,(2^j r)^{N_p}\,(2^j r)^{n\left(1-\frac{1}{p}\right)} = C_{n,p}$$

where the implicit constant is independent of j and r. Using this estimate we get

$$\sum_{j=1}^{\infty}\int |d_j||a_j(x)||x-x_0|^{N_p} dx \leqslant C_{n,p}\sum_{j=1}^{\infty}|d_j| \leqslant C_{n,p}\left(\sum_{j=1}^{\infty}|d_j|^p\right)^{1/p} < \infty.$$

Moreover, from triangle inequality we can also derive the following relation between the h^p norm of

a_B and M:

$$\|a_B\|_{h^p} = \left\| M - \sum_{j=0}^{\infty} d_j\, a_j(x) \right\|_{h^p} \leqslant \|M\|_{h^p} + \left(\sum_{j=0}^{\infty} |d_j|^p \right)^{1/p} \|a_j\|_{h^p} \leqslant \|M\|_{h^p} + C_{n,p,\lambda}.$$

Therefore,

$$\left| \int M(x)\,(x - x_0)^{\alpha} dx \right| \lesssim \begin{cases} \|M\|_{h^p} + C_{n,p,\lambda} & \text{if } |\alpha| < n(1/p - 1); \\ \dfrac{\|M\|_{h^p} + C_{n,p,\lambda}}{\left[\log\left(\frac{1}{r}\right)\right]^{1/p}} & \text{if } |\alpha| = n(1/p - 1) = N_p. \end{cases}$$

∎

Now we go to the proof of the necessity part of Theorem B.

Proof. From the proof presented in Section 3.1.1, only the conditions imposed on the kernel and the boundedness assumptions on the operator implies that Ta is a pre-molecule when a is an atom.

Given a ball $B = B(x_0, r) \subset \mathbb{R}^n$ with $0 < r < 1$ and $g \in L^2_{N_p}(B)$ with $\|g\|_{L^2(B)} \leqslant 1$, let $f(x) = T^*\left[(x - x_0)^{\alpha}\right]$ and $a_g(x) = g(x)\,|B|^{\frac{1}{2} - \frac{1}{p}}$. Note that a_g is a $(p, 2)$ atom in H^p supported on B and $Ta_g \in h^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ from the boundedness assumptions. Since Ta_g is a pre-molecule, using estimate (3.7) and the boundedness assumption

$$
\begin{aligned}
\left(\fint_B |f(y)-P_{N_p}(f)(y)|^2 dy\right)^{1/2} &= |B|^{-\frac{1}{2}} \left(\int_B |f(y)-P_{N_p}(f)(y)|^2 dy\right)^{1/2}\\
&= |B|^{-\frac{1}{2}} \sup_{\substack{g\in L^2_{N_p}(B)\\ \|g\|_{L^2(B)}\leqslant 1}} \left|\int f(y)g(y)dy\right|\\
&= |B|^{\frac{1}{p}-1} \sup_{\substack{g\in L^2_{N_p}(B)\\ \|g\|_{L^2(B)}\leqslant 1}} \left|\int f(y)a_g(y)dy\right|\\
&= |B|^{\frac{1}{p}-1} \sup_{\substack{g\in L^2_{N_p}(B)\\ \|g\|_{L^2(B)}\leqslant 1}} |\langle T^*[(x-x_0)^\alpha],\, a_g\rangle|\\
&= |B|^{\frac{1}{p}-1} \sup_{\substack{g\in L^2_{N_p}(B)\\ \|g\|_{L^2(B)}\leqslant 1}} |\langle (x-x_0)^\alpha,\, T(a_g)\rangle|\\
&= |B|^{\frac{1}{p}-1} \sup_{\substack{g\in L^2_{N_p}(B)\\ \|g\|_{L^2(B)}\leqslant 1}} \left|\int_{\mathbb{R}^n} Ta_g(x)\,(x-x_0)^\alpha dx\right|\\
&\leqslant \begin{cases} |B|^{\frac{1}{p}-1} \sup\limits_{\substack{g\in L^2_{N_p}(B)\\ \|g\|_{L^2(B)}\leqslant 1}} \|Ta_g\|_{h^p} & \text{if } |\alpha|<n(1/p-1);\\[2ex] |B|^{\frac{1}{p}-1} \sup\limits_{\substack{g\in L^2_{N_p}(B)\\ \|g\|_{L^2(B)}\leqslant 1}} \dfrac{\|Ta_g\|_{h^p}}{\left[\log\left(1+\frac{1}{r}\right)\right]^{1/p}} & \text{if } |\alpha|=n(1/p-1)=N_p. \end{cases}\\
&\leqslant \begin{cases} C_{n,p}\, r^{n\left(\frac{1}{p}-1\right)} & \text{if } |\alpha|<\ n(1/p-1),\\ C_{n,p}\, r^{n\left(\frac{1}{p}-1\right)}\left[\log\left(1+\frac{1}{r}\right)\right]^{-1/p} & \text{if } |\alpha|=n(1/p-1)=N_p. \end{cases}\\
&= C_{n,p}\ \Psi_{p,\alpha}(r). \qquad (3.9)
\end{aligned}
$$

□

3.2 Necessary condition for the boundedness of linear operators in $h^p(\mathbb{R}^n)$

As we have seen before, a necessary condition for the boundedness of inhomogeneous Calderón–Zygmund operators is based on the fact that such operators maps atoms into what we called pre-molecules. As pointed out, this property comes from the conditions assumed on the kernel, without any further assumption. This relation means that the size conditions on the molecules are motivated by the behavior of the kernel associated to the operator. Conversely, for homogeneous Triebel–Lizorkin spaces $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$, where $1 < p, q < \infty$ and $\alpha > 0$, the authors in [33, Theorem 1.16] showed that if a continuous operator maps smooth atoms into smooth molecules, then the kernel of this operator satisfies Calderón–Zygmund estimates. In [76, Theorems 3.2.34 and 3.2.35], the author includes the case $p = 1$ and $\alpha = 0$, which covers $\dot{F}^{0}_{1,2}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, however no explicit proof is presented.

In this section we describe how the necessary condition from the boundedness presented on Theorem B can be stated for more general operators. This can be achieved once the operator has the property of mapping atoms into objects called pseudo-molecules, that we present in the sequel.

Definition 3.5. *Fix some constant $\mathfrak{C} > 0$. We say that $\mathfrak{M} \in \mathcal{S}'(\mathbb{R}^n)$ is a pseudo-molecule in h^p associated to the ball $B \subset \mathbb{R}^n$ if $\mathfrak{M} = g + h$ in $\mathcal{S}'(\mathbb{R}^n)$, where $g \in h^p(\mathbb{R}^n)$ is such that supp $(g) \subset B$, $h \in H^p(\mathbb{R}^n)$, and*

$$\|g\|_{h^p} + \|h\|_{H^p} \leqslant \mathfrak{C}.$$

Next, we prove that pseudo-molecules satisfies the analogous moment estimates of Proposition 3.3.

Proposition 3.4. *Let $0 < p \leqslant 1$ and $\mathfrak{M}$ a pseudo-molecule in h^p associated to the ball $B = B(x_0, r)$ with $0 < r < 1$. Then for $\alpha \in \mathbb{Z}^n_+$, $|\alpha| \leqslant N_p$, the moments $\langle \mathfrak{M}, (\cdot - x_0)^\alpha \rangle$ are well-defined and satisfy*

$$|\langle \mathfrak{M}, (\cdot - x_0)^\alpha \rangle| \lesssim \begin{cases} C_{\alpha,p}\, \mathfrak{C} & \text{if } |\alpha| < n(1/p - 1); \\ \\ C_{\alpha,p}\, \mathfrak{C} \left[\log\left(1 + \dfrac{1}{r}\right)\right]^{-1/p} & \text{if } |\alpha| = n(1/p-1) = N_p. \end{cases} \tag{3.10}$$

Proof. Write $\mathfrak{M} = g + h$, as in Definition 3.5. Since $h \in H^p(\mathbb{R}^n)$ satisfies vanishing moment conditions up the order N_p, we have $\langle h, (\cdot - x_0)^\alpha \rangle = 0$ (where the pairing here is the one between $H^p(\mathbb{R}^n)$ and its dual space $\dot{\Lambda}_{n(1/p-1)}(\mathbb{R}^n)$). For $g \in h^p(\mathbb{R}^n)$ such that supp $(g) \subset B$, the moments $\langle g, (\cdot - x_0)^\alpha \rangle$ can be defined as in Proposition 3.2. Thus we can set

$$\langle \mathfrak{M}, (\cdot - x_0)^\alpha \rangle := \langle g, (\cdot - x_0)^\alpha \rangle + \langle h, (\cdot - x_0)^\alpha \rangle = \langle g, (\cdot - x_0)^\alpha \rangle.$$

If $\mathfrak{M}$ has an alternative decomposition $g' + h'$ satisfying the conditions of Definition 3.5, then we must have that $g - g' \in H^p(\mathbb{R}^n)$ and therefore the moments of g' are the same as those of g. Therefore, the estimates (3.10) follow immediately from (3.6). ∎

We obtain the following theorem:

Theorem C. *Let $0 < p \leqslant 1$ and T to be a linear and bounded operator on $h^p(\mathbb{R}^n)$ that maps each $(p, 2)$ atom in h^p into a pseudo-molecule centered in the same ball as the support of the atom. Then, the local Campanato-cancellation condition* (3.4) *must hold.*

As pointed out before, since pre-molecules are in particular pseudo-molecules, we obtain as a corollary the following result:

Corollary 3.1. *Let $0 < p \leqslant 1$ and $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ a linear and continuous operator that maps each $(p, 2)$ atom in h^p supported on $B \subset \mathbb{R}^n$ into a pre-molecule centered also in B. Then, T is bounded on $h^p(\mathbb{R}^n)$ to itself if, and only if, the local Companato-cancellation condition* (3.4) *holds.*

Proof. The necessity of the local Campanato-cancellation condition for the continuity in $h^p(\mathbb{R}^n)$ follows by Theorem C. For the sufficiency, we will show that T maps $(p, 2)$ atoms into a $(p, 2, \omega)$ molecule. Since by hypothesis T maps $(p, 2)$ atoms into a (p, s, λ, C) pre-molecule, we have that the size conditions (M1) and (M2) of a $(p, 2, \omega)$ molecule are satisfied, so it remains to show that the cancellation condition (M3) holds. For this, we can proceed as in the estimate (3.9) did in the proof the necessity of Theorem B (see moreover the next proof, where we show that $T^*[(\cdot - x_0)^\alpha]$ are well defined in this setting). ∎

In Remark 3.1, to justify the well-definition of the local T^* Campanato-condition (3.4) when T is an inhomogeneous Calderón–Zygmund operator, we strongly rely on the kernel estimates of these operators. Since in the previous theorem we are no longer requiring that T is an inhomogeneous Calderón–Zygmund operator, we will specify in the beginning of the next proof how to make sense of condition (3.4) in this more general framework.

Proof. We show first that the local Campanato condition is well defined in this context. Fix $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leqslant N_p$ and a ball $B = B(x_0, r) \subset \mathbb{R}^n$ with $r < 1$. We want to show that $T^*\left[(\cdot - x_0)^\alpha\right]$ is well defined locally and can be identified with f in $(L^2_{N_p}(B))^*$, the quotient space of $L^2(B)$ by the subspace $\mathcal{P}_{N_p}$. We then have

$$\|f\|_{(L^2_{N_p}(B))^*} := \sup_{\substack{\psi \in L^2_{N_p}(B) \\ \|\psi\|_{L^2(B)} \leqslant 1}} |\langle f, \psi \rangle| = \inf_{P \in \mathcal{P}_{N_p}} \|f - P\|_{L^2(B)} = \|f - P_B^{N_p}(f)\|_{L^2(B)}. \tag{3.11}$$

Given a $\psi \in L^2_{N_p}(B)$ with $\|\psi\|_{L^2(B)} \leqslant 1$, let

$$a(x) = \psi(x)\, |B|^{\frac{1}{2} - \frac{1}{p}}.$$

Note that a is a $(p, 2)$ atom in H^p supported on B. By the boundedness assumptions on T we have that $\|Ta\|_{h^p} \lesssim \|a\|_{h^p} \leqslant C$ independent of a and $\mathfrak{M} = Ta$ is a pseudo-molecule, where the choice of the constant $\mathfrak{C}$ in Definition 3.5 should be consistent with the norm of T. Thus by (3.10),

$$\begin{aligned} |\langle T^*[(\cdot - x_0)^\alpha],\, a\rangle| &:= |\langle (\cdot - x_0)^\alpha,\, Ta\rangle| \\ &\leqslant \begin{cases} C_{\alpha,p}\, \mathfrak{C} & \text{if } |\alpha| < n(1/p - 1), \\ C_{\alpha,p}\, \mathfrak{C}\left[\log\left(1 + \frac{1}{r}\right)\right]^{-1/p} & \text{if } |\alpha| = n(1/p - 1) = N_p. \end{cases} \end{aligned}$$

Replacing a by ψ, we see that the left-hand-side defines a bounded linear functional $f \in (L^2_{N_p}(B))^*$ with

$$|\langle f, \psi\rangle| = |B|^{\frac{1}{p} - \frac{1}{2}}\, |\langle T^*[(\cdot - x_0)^\alpha],\, a\rangle| \leqslant \begin{cases} C_{\alpha,p} |B|^{\frac{1}{p} - \frac{1}{2}}\, \mathfrak{C} & \text{if } |\alpha| < n(1/p - 1), \\ C_{\alpha,p} |B|^{\frac{1}{p} - \frac{1}{2}}\, \mathfrak{C}\left[\log\left(1 + \frac{1}{r}\right)\right]^{-1/p} & \text{if } |\alpha| = n(1/p - 1) = N_p. \end{cases}$$

Thus by (3.11), we have

$$\begin{aligned}
\left(\fint_B |f - P_{N_p}(f)|^2\right)^{1/2} &= |B|^{-\frac{1}{2}} \left(\int_B |f - P_{N_p}(f)|^2\right)^{1/2} \\
&= |B|^{-\frac{1}{2}} \sup_{\substack{\psi \in L^2_{N_p}(B) \\ \|\psi\|_{L^2(B)} \leqslant 1}} |\langle f, \psi \rangle| \\
&\leqslant \begin{cases} C_{n,p}\, r^{n\left(\frac{1}{p}-1\right)} & \text{if } |\alpha| < n(1/p - 1), \\ C_{n,p}\, r^{n\left(\frac{1}{p}-1\right)} \left[\log\left(1 + \frac{1}{r}\right)\right]^{-1/p} & \text{if } |\alpha| = n(1/p - 1) = N_p. \end{cases} \\
&= C_{n,p}\, \Psi_{p,\alpha}(r).
\end{aligned}$$

□

CHAPTER 4

Boundedness of Calderón-Zygmund-type operators on Hardy-Morrey spaces

The goal of this chapter is to provide elementary facts about Hardy-Morrey spaces and extend the continuity results of Chapter 2 to this setting. Even though many of the results regarding the space are not entirely new, we write down the detailed proofs of the statements that were claimed somewhere in the literature with no rigorous proof.

The theory of Morrey spaces was developed in [65] to deal with some problems in elliptic partial differential equations. These spaces describe the local regularity of locally integrable functions by considering particular averages on cubes (or equivalently on balls) and represent a refinement, for local scales, of the classical Lebesgue spaces. For a reference in this subject, we refer to [1].

Given $1 < q \leqslant \lambda < \infty$ and $J \subset \mathbb{R}^n$ a (dyadic) cube, the Morrey space $\mathcal{M}_q^\lambda(\mathbb{R}^n)$ are defined as the set of all $f \in L^q_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}_q^\lambda} := \sup_{J \subset \mathbb{R}^n} |J|^{\frac{1}{\lambda} - \frac{1}{q}} \left(\int_J |f(x)|^q dx \right)^{1/q} < \infty,$$

where the supremum is taken over all cubes $J \subset \mathbb{R}^n$. As indicated, we denote the right-hand side of the previous identity as the norm in the Morrey spaces. If $q = \lambda$, then $\mathcal{M}_q^q(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, but it defers if $q < \lambda$ (see [82, p. 587] for an example).

Motivated by the maximal characterization of real Hardy spaces, in [47] and [48] the authors defined a natural extension of Morrey spaces when $0 < q \leqslant 1$, called Hardy-Morrey spaces. Among the features of these spaces, the authors pointed out that a natural atomic and molecular decomposition can be provided with the same amount of cancellation inherit from the Hardy spaces theory. This provides better estimates when dealing with some partial differential equations estimates, see for instance [48].

Definition 4.1. *Let $0 < q \leqslant \lambda < \infty$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \varphi \neq 0$. We say that $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to Hardy-Morrey space, denoted by $\mathcal{HM}_q^\lambda(\mathbb{R}^n)$, if*

$$\|f\|_{\mathcal{HM}_q^\lambda} := \|M_\varphi f\|_{\mathcal{M}_q^\lambda} < \infty,$$

where M_φ stands for the standard maximal function (previously denoted by $\mathcal{M}_\varphi$).

The functional $\|f\|_{\mathcal{HM}_q^\lambda}$ defines a quasi-norm as $0 < q < 1$ and a norm if $q \geqslant 1$. As in the Hardy space case, different choices of φ will yield equivalent norms up to a constant depending on φ.

These spaces also have equivalent maximal characterizations that were established by Jia and Wang in [47, Section 2] and we describe here. For a given $N \in \mathbb{Z}_+$, consider the finite collection of semi-norms $\mathcal{F}_N = \left\{ \|\varphi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \varphi(x) \right| \text{ such that } |\alpha|, |\beta| \leqslant N \right\}$ and set

$$\mathcal{S}_{\mathcal{F}} := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \ \|\varphi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \varphi(x) \right| \leqslant 1, \ \text{for all } \ \|\cdot\|_{\alpha,\beta} \in \mathcal{F}_N \right\}.$$

We denote the *grand maximal function* and the *non-tangential maximal function* by

$$M_{\mathcal{F}} f = \sup_{\varphi \in \mathcal{S}_{\mathcal{F}}} M_\varphi f \quad \text{and} \quad M_\varphi^* f(x) = \sup_{|x-y|<t} |\varphi_t * f(y)|$$

respectively. Then, the following characterization of $\mathcal{HM}_q^\lambda(\mathbb{R}^n)$ in terms of maximal functions was established:

Theorem 4.1 ([47, Setion 2])**.** *Let $0 < q \leqslant 1$, $q \leqslant \lambda < \infty$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The following are equivalents:*

(i) There exists $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \varphi = 1$ such that $M_\varphi f \in \mathcal{M}_q^\lambda(\mathbb{R}^n)$;

(ii) There exist a collection $\mathcal{F}_N$ *so that* $M_{\mathcal{F}} f \in \mathcal{M}_q^\lambda(\mathbb{R}^n)$;

(iii) f *is bounded distribution and* $M_\varphi^* f \in \mathcal{M}_q^\lambda(\mathbb{R}^n)$.

Moreover,

$$\|M_{\mathcal{F}} f\|_{\mathcal{M}_q^\lambda} \lesssim \|M_\varphi^* f\|_{\mathcal{M}_q^\lambda} \lesssim \|M_\varphi f\|_{\mathcal{M}_q^\lambda} \lesssim \|M_{\mathcal{F}} f\|_{\mathcal{M}_q^\lambda}.$$

In [46, Proposition 1.11], the authors considered the analogous equivalence between (i) and (ii) of the previous theorem but now allowing $0 < q \leqslant \lambda < \infty$. Moreover, they also showed in [46, Proposition 1.5] that if $1 < q \leqslant \lambda < \infty$, then $\mathcal{HM}_q^\lambda(\mathbb{R}^n) = \mathcal{M}_q^\lambda(\mathbb{R}^n)$. For the case $q = 1$, an argument presented in [58, Remark 2.1 (i)] shows that $\mathcal{HM}_1^\lambda(\mathbb{R}^n) \subset \mathcal{M}_1^\lambda(\mathbb{R}^n)$ continuously for $1 < \lambda < \infty$. Another proof of these facts using an approximate of identity in the predual of Morrey spaces can be found in [25, Proposition 2.7].

4.1 Atomic and molecular decomposition

In this section, we recall the atomic decomposition that has been proved in [47, Theorem 2.3] for L^∞ atoms. Moreover, we show that the atomic spaces generated by L^r- atoms with $1 \leqslant r < \infty$ and the one by L^∞ atoms are equivalent. This result is well known for the classical Hardy spaces and allow ones to consider the most convenient size for atoms. This will prove the assertion made by the authors in [48, Remark 2.4 (1)].

Definition 4.2 ([48, Definiton 2.2])**.** . *Let* $0 < q \leqslant 1 \leqslant r \leqslant \infty$ *with* $q < r$ *and* $q \leqslant \lambda < \infty$. *A measurable function* a_Q *is called a* (λ, q, r) *atom in* $\mathcal{HM}_q^\lambda$ *if there exists a cube* $Q = Q(x_Q, \ell_Q) \subset \mathbb{R}^n$ *such that:*

$$(i) \quad supp\ (a_Q) \subset Q \quad (ii) \quad \|a_Q\|_{L^r} \leqslant |Q|^{\frac{1}{r}-\frac{1}{\lambda}} \quad (iii) \quad \int a_Q\, x^\alpha(x)dx = 0$$

for all $\alpha \in \mathbb{Z}_+^n$ *such that* $|\alpha| \leqslant N_q := \lfloor n\,(1/q-1)\rfloor$.

To provide a better understanding of the necessity of moment conditions on Hardy-Morrey spaces, the following lemma is an extension of [25, Proposition 2.5], proved for bounded functions, and the analogous corresponding result for Hardy spaces.

Proposition 4.1. *Let $0 < q \leqslant 1 \leqslant r \leqslant \infty$ with $q < r$ and $q \leqslant \lambda < \infty$ with $\lambda \leqslant r$. If f is a compactly supported function in $L^r(\mathbb{R}^n)$ satisfying the moment condition*

$$\int_{\mathbb{R}^n} x^\alpha f(x)dx = 0 \ \text{for all} \ |\alpha| \leqslant N_q, \tag{4.1}$$

then it belongs to $\mathcal{HM}_q^\lambda(\mathbb{R}^n)$ and moreover $\|f\|_{\mathcal{HM}_q^\lambda} \lesssim \|f\|_{L^r}|Q|^{\frac{1}{\lambda}-\frac{1}{r}}$ for all cubes $Q \supseteq$ supp (f). In particular, if $f = a_Q$, then $\|a_Q\|_{\mathcal{HM}_q^\lambda} \lesssim 1$ uniformly.

Proof. Let $J \subset \mathbb{R}^n$ be an arbitrary cube, $Q = Q(x_Q, \ell)$ any cube such that supp $(f) \subseteq Q$ and denote by $Q^* = Q(x_Q, 2\ell)$. Split

$$\int_J |M_\varphi f(x)|^q dx = \int_{J\cap Q^*} |M_\varphi f(x)|^q dx + \int_{J\setminus Q^*} |M_\varphi f(x)|^q dx.$$

For the first integral consider the case where $1 < r \leqslant \infty$. We use Hölder with the boundedness of M_φ on $L^r(\mathbb{R}^n)$ to obtain

$$\int_{J\cap Q^*} |M_\varphi f(x)|^q dx \leqslant \|M_\varphi f\|_{L^r}^q \, |J\cap Q^*|^{1-\frac{q}{r}} \lesssim \|f\|_{L^r}^q \, |J\cap Q^*|^{1-\frac{q}{r}}.$$

If $r = 1$ and $0 < q < 1$, setting $R = \|f\|_{L^1}|J\cap Q^*|^{-1}$ and using that M_φ satisfies weak $(1,1)$ inequality

$$\begin{aligned}\int_{J\cap Q^*} |M_\varphi f(x)|^q dx &\simeq \int_0^\infty \omega^{q-1} \left|\{x \in J\cap Q^* : |M_\varphi f(x)| > \omega\}\right| d\omega \\ &\lesssim |J\cap Q^*| \int_0^R \omega^{q-1} d\omega + \|f\|_{L^1} \int_R^\infty \omega^{q-2} d\omega \\ &\lesssim \|f\|_{L^1}^q \, |J\cap Q^*|^{1-q}. \end{aligned} \tag{4.2}$$

To obtain the desired estimate, suppose first that $|Q| < |J|$. For all $1 \leqslant r < \infty$, since

$$\frac{q}{\lambda} - 1 \leqslant 0 \quad \text{and} \quad 1 - \frac{q}{r} > 0$$

it follows that $|J|^{\frac{q}{\lambda}-1}|J\cap Q^*|^{1-\frac{q}{r}} \leqslant |Q|^{\frac{q}{\lambda}-\frac{q}{r}}$. On the other hand, if $|J| < |Q|$, using that $\lambda \leqslant r$ it follows

$$|J|^{\frac{q}{\lambda}-1}|J\cap Q^*|^{1-\frac{q}{r}} = |J|^{\frac{q}{\lambda}-\frac{q}{r}} \left(\frac{|J\cap Q^*|}{|J|}\right)^{1-\frac{q}{r}} \leqslant |Q|^{\frac{q}{\lambda}-\frac{q}{r}}.$$

Hence,

$$|J|^{\frac{q}{\lambda}-1}\int_{J\cap Q^*}|M_\varphi f(x)|^q dx \lesssim \|f\|_{L^r}^q\,|Q|^{q\left(\frac{1}{\lambda}-\frac{1}{r}\right)}.$$

To estimate the integral on $J\backslash Q^*$, using the moment condition (4.1) we write

$$\varphi_t * f(x) = \int f(y)\,(\varphi_t(x-y) - P_{\varphi_t}(y))\,dy,$$

where $P_{\varphi_t}(y) = \sum_{|\alpha|\leqslant N_q}\dfrac{\partial^\alpha\varphi_t(x)}{\alpha!}(-y)^\alpha$ denotes the Taylor polynomial of degree N_q of the function $y \mapsto \varphi_t(x-y)$. The standard estimate of the remainder term (see [73, p. 106]) yields

$$|\varphi_t(x-y) - P_{\varphi_t}(y)| \lesssim |y-x_Q|^{N_q+1}|x-x_Q|^{-(n+N_q+1)}$$

and since supp $(f) \subseteq Q$, we get the following pointwise control:

$$|M_\varphi f(x)| \lesssim \frac{\ell^{N_q+1}}{|x-x_Q|^{n+N_q+1}}\int_Q |f(y)|dy \lesssim \frac{\ell^{N_q+1}}{|x-x_Q|^{n+N_q+1}}\,\|f\|_{L^r}\,|Q|^{1-\frac{1}{r}}.$$

If $|Q| < |J|$, since $N_q + 1 > n\,(1/q-1)$, we estimate as follows

$$\begin{aligned}|J|^{\frac{q}{\lambda}-1}\int_{J\backslash Q^*}|M_\varphi f(x)|^q dx &\lesssim \|f\|_{L^r}^q\,|Q|^{q\left(\frac{1}{\lambda}-\frac{1}{r}+\frac{N_q}{n}+\frac{1}{n}+1\right)-1}\int_{(Q^*)^c}|x-x_Q|^{-q(n+N_q+1)}dx\\ &\lesssim \|f\|_{L^r}^q\,|Q|^{\frac{q}{\lambda}-\frac{q}{r}}.\end{aligned}$$

Finally, if $|J| < |Q|$

$$|J|^{\frac{q}{\lambda}-1}\int_{J\backslash Q^*}|M_\varphi f(x)|^q dx \lesssim \|f\|_{L^r}^q\,|J|^{\frac{q}{\lambda}-1}\,|Q|^{q-\frac{q}{r}}\,\ell_Q^{-nq}\;\;|J\backslash Q^*| \lesssim \|f\|_{L^r}^q\,|Q|^{q\left(\frac{1}{\lambda}-\frac{1}{r}\right)},$$

which concludes the proof. □

Given $1 \leqslant r \leqslant \infty$, we denote the atomic space $\mathbf{at}\mathcal{HM}_q^{\lambda,r}(\mathbb{R}^n)$ by the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f = \sum_j s_{Q_j}\,a_{Q_j}$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{a_{Q_j}\}_j$ are (λ, q, r) atoms in $\mathcal{HM}_q^\lambda$ and $\{s_{Q_j}\}_j$ is a sequence of complex scalars satisfying

$$\|\{s_{Q_j}\}_j\|_{\lambda,q} := \sup_J \left\{ \left(|J|^{\frac{q}{\lambda}-1} \sum_{Q_j \subseteq J} \left(|Q_j|^{\frac{1}{q}-\frac{1}{\lambda}} |s_{Q_j}| \right)^q \right)^{\frac{1}{q}} \right\} < \infty.$$

The functional

$$\|f\|_{\mathbf{at}\mathcal{HM}_q^{\lambda,r}} := \inf \left\{ \|\{s_{Q_j}\}_j\|_{\lambda,q} : f = \sum_j s_{Q_j} a_{Q_j} \right\},$$

where the infimum is taken over all such atomic representations, defines a quasi-norm in $\mathbf{at}\mathcal{HM}_q^{\lambda,r}(\mathbb{R}^n)$. If $1 \leqslant r_1 < r_2 \leqslant \infty$, then $\mathbf{at}\mathcal{HM}_q^{\lambda,r_2}(\mathbb{R}^n)$ is continuously embedded in $\mathbf{at}\mathcal{HM}_q^{\lambda,r_1}(\mathbb{R}^n)$. The converse is the content of the next result and shows the desired equivalence between the atomic spaces.

Lemma 4.1. *Let $0 < q \leqslant 1 \leqslant r$ with $q < r$ and $q \leqslant \lambda < \infty$. Then $\boldsymbol{at}\mathcal{HM}_q^{\lambda,r}(\mathbb{R}^n) = \boldsymbol{at}\mathcal{HM}_q^{\lambda,\infty}(\mathbb{R}^n)$ with comparable quasi-norms.*

Proof. The proof is based on the analogous theorem for Hardy spaces (see [35, Theorem 4.10]). It suffices to show that for every given (λ, q, r) atom in $\mathcal{HM}_q^{\lambda}$, denoted by a_Q, can be decomposed as $\sum_j s_{Q_j} a_{Q_j}$, where $\{a_{Q_j}\}_j$ are (λ, q, ∞) atoms and $\|\{s_{Q_j}\}_j\|_{q,\lambda} \leqslant C$ independent of the atom. Consider $b_Q = |Q|^{1/\lambda} a_Q$ and since

$$\int_Q |b_Q(x)|^r dx \leqslant |Q|,$$

from Calderón–Zygmund decomposition applied for $|b_Q|^r \in L^1(Q)$ at level $\alpha^r > 0$, there exists a sequence $\{Q_j\}_j$ of disjoint dyadic cubes (subcubes of Q) such that:

(i) $|b_Q(x)| \leqslant \alpha$, for all $x \notin \bigcup_j Q_j$;

(ii) $\alpha^r \leqslant \fint_{Q_j} |b_Q(x)|^r dx \leqslant 2^n \alpha^r$;

(iii) $|\bigcup_j Q_j| \leqslant \dfrac{1}{\alpha^r} \displaystyle\int_Q |b_Q(x)|^r dx \leqslant \dfrac{|Q|}{\alpha^r}$.

Let $\mathcal{P}_{N_q}$ be the space of polynomials in $\mathbb{R}^n$ with degree at most N_q, $\mathcal{P}_{N_q,j}$ its restriction to Q_j and denote by $P_{Q_j} b \in \mathcal{P}_{N_q,j}$ such that

$$\int_{Q_j} [b_Q(x) - P_{Q_j}(b)(x)] x^\beta dx = 0 \ \text{ for all } |\beta| \leqslant N_q.$$

Now we write $b_Q = g_0 + \sum_j h_j$, where

$$g_0(x) = \begin{cases} b_Q(x) \text{ for } x \notin \bigcup_j Q_j \\ P_{Q_j}(b)(x) \text{ for } x \in Q_j \end{cases}$$

and $h_j(x) = [b_Q(x) - P_{Q_j}(b)(x)]\chi_{Q_j}(x)$. We also have the following:

(i) g_0 is bounded and $|g_0(x)| \leqslant c\alpha$ a.e. (see [73, Remark 2.1.4 p. 104]);

(ii) Each function h_j is supported in the cube Q_j and satisfies

$$\int_{\mathbb{R}^n} h_j(x)x^\beta dx = 0 \quad \text{and} \quad ⨍_{Q_j} |h_j(x)|^r dx \leqslant c\alpha^r \text{ for all } |\beta| \leqslant N_q;$$

(iii) $\sum_{j=1}^{\infty} |Q_j| \leqslant |\{x \in \mathbb{R}^n \,:\, M(|b_Q|^r)(x) > \alpha^r\}| \leqslant \frac{c}{\alpha^r}|Q|$.

These remarks implies

$$\left(⨍_{Q_j} |h_j(x)|^r dx\right)^{1/r} \leqslant \left(⨍_{Q_j} |b_Q(x)|^r dx\right)^{1/r} + \left(⨍_{Q_j} |g_0(x)|^r dx\right)^{1/r} \leqslant c\alpha.$$

For each $j_0 \in \mathbb{N}$, let $b_{j_0}(x) := (c\alpha)^{-1}h_{j_0}(x)$ and write

$$b_Q(x) = g_0(x) + (c\alpha)\sum_{j_0} b_{j_0}(x) \quad \text{in which} \quad \int_{Q_{j_0}} |b_{j_0}(x)|^r dx \leqslant |Q_{j_0}|.$$

Now, applying the previous argument for each b_{j_0} individually we obtain the identity

$$b_Q = g_0 + (c\alpha)\sum_{j_0} b_{j_0} = g_0 + c\alpha \sum_{j_0} g_{j_0} + (c\alpha)^2 \sum_{j_0,j_1} b_{j_0,j_1},$$

where

$$\int_{Q_{j_0,j_1}} |b_{j_0,j_1}(x)|^r dx \leqslant |Q_{j_0,j_1}|$$

and $\{Q_{j_0,j_1}\}_{j_1}$ is a sequence of disjoint dyadic cubes (subcubes of Q_{j_0}) such that $|g_{j_0}(x)| \leqslant c\alpha$ a.e.,

$$\alpha^r \leqslant ⨍_{Q_{j_0,j_1}} |b_{j_0}(x)|^r dx \leqslant 2^n\alpha^r \quad \text{and} \quad \Big|\bigcup_{j_1} Q_{j_0,j_1}\Big| \leqslant \frac{c}{\alpha^r}\int_{Q_{j_0}} |b_{j_0}(x)|^r dx \leqslant c\frac{|Q_{j_0}|}{\alpha^r}.$$

Applying an induction argument, we can find a family $\{Q_{i_{k-1},j}\}_j := \{Q_{j_0,\cdots,j_{k-1},j}\}_j$ of disjoint dyadic subcubes of $Q_{i_{k-1}} := Q_{j_0,\cdots,j_{k-1}}$ for $k = 1, 2, \cdots$ with $i_{k-1} = \{j_0, j_1, \cdots, j_{k-1}\}$ such that

$$\begin{aligned} b_Q &= g_0 + (c\alpha)\sum_{j_0} b_{j_0} \\ &= g_0 + c\alpha\sum_{j_0} g_{j_0} + (c\alpha)^2\sum_{j_0,j_1} b_{j_0,j_1} \\ &= g_{i_0} + c\alpha\sum_{i_1} g_{i_1} + (c\alpha)^2\sum_{i_2} g_{i_2} + \cdots + (c\alpha)^{k-1}\sum_{i_{k-1}} g_{i_{k-1}} + (c\alpha)^k\sum_{i_k} h_{i_k}, \end{aligned} \tag{4.3}$$

where $g_{i_{k-1}}(x)$ and $h_{i_k}(x)$ for $i_k = (j_0, j_1, \cdots, j_{k-1}, j)$ satisfies $|g_{i_{k-1}}(x)| \leqslant c\alpha$ a.e. $x \in \mathbb{R}^n$,

$$\alpha^r \leqslant \frac{1}{|Q_{i_{k-1},j}|}\int_{Q_{i_{k-1},j}} |h_{i_k}(x)|^r dx \leqslant 2^n\alpha^r \text{ and } \Big|\bigcup_j Q_{i_{k-1},j}\Big| \leqslant c\frac{|Q_{i_{k-1}}|}{\alpha^r}.$$

The sum at (4.3) is interpreted as $\sum_{i_{k-1}} g_{i_{k-1}} := \sum_{j_0\in\mathbb{N}}\cdots\sum_{j_{k-1}\in\mathbb{N}} g_{j_0,\cdots,j_{k-1}}$ (analogously to $\sum_{i_k} h_{i_k}$). We claim that the reminder term $(c\alpha)^k\sum_{i_k} h_{i_k}$ in (4.3) goes to zero in $L^1(\mathbb{R}^n)$ as $k \to \infty$. Indeed, writing $Q_{i_k} := Q_{i_{k-1},j}$ for some fixed j we have

$$\begin{aligned} \int_{\mathbb{R}^n} |h_{i_k}(x)| dx &= \int_{Q_{i_k}} |h_{i_k}(x)| dx \\ &\leqslant \left(\int_{Q_{i_k}} |h_{i_k}(x)|^r dx\right)^{\frac{1}{r}} |Q_{i_k}|^{1-\frac{1}{r}} \\ &= \left(\frac{1}{|Q_{i_k}|}\int_{Q_{i_k}} |h_{i_k}(x)|^r dx\right)^{\frac{1}{r}} |Q_{i_k}| \leqslant c\alpha|Q_{i_k}| \end{aligned}$$

and iterating $(k+1)$-times the previous argument one has

$$\sum_{i_k} |Q_{i_k}| \leqslant \left(\frac{c}{\alpha^r}\right)^{k+1} |Q|. \tag{4.4}$$

It follows from dominated convergence theorem that

$$\begin{aligned}\int_{\mathbb{R}^n}\Big|(c\alpha)^k\sum_{i_k}h_{i_k}(x)\Big|dx &\leqslant \sum_{i_k}(c\alpha)^k\int_{\mathbb{R}^n}|h_{i_k}(x)|dx\\ &\leqslant (c\alpha)^{k+1}\sum_{i_k}|Q_{i_k}| \lesssim \frac{\alpha^{k+1}}{\alpha^r}\sum_{i_{k-1}}|Q_{i_{k-1}}|\\ &\cdots\\ &\lesssim \frac{\alpha^{k+1}}{\alpha^{kr}}\sum_{i_0}|Q_{i_0}| \leqslant (c^2\alpha^{1-r})^{(k+1)}|Q| \to 0 \text{ as } k\to\infty,\end{aligned}$$

that is, $(c\alpha)^k\sum_{i_k}h_{i_k}(x)\to 0$ in $L^1(\mathbb{R}^n)$ as $k\to\infty$, provided that $c^2\alpha^{1-r}<1$. Therefore,

$$b_Q = g_{i_0} + c\alpha\sum_{i_1}g_{i_1} + (c\alpha)^2\sum_{i_2}g_{i_2} + \cdots + (c\alpha)^{k-1}\sum_{i_{k-1}}g_{i_{k-1}} + (c\alpha)^k\sum_{i_k}g_{i_k} + \cdots$$

in $L^1(\mathbb{R}^n)$, where $|g_{i_k}(x)|\leqslant c\alpha$ a.e. and for all $|\beta|\leqslant N_q$ we have

$$\int_{\mathbb{R}^n}x^\beta g_{i_k}(x)dx = \int_{\mathbb{R}^n}x^\beta b_{i_k}(x)dx + \sum_j\int_{Q_{i_{k-1},j}}x^\beta P_{Q_{i_k,j}}b(x)dx = \int_{\mathbb{R}^n}x^\beta b_{i_k}(x)dx = 0.$$

It is clear now that $a_{i_0} := (c\alpha)^{-1}|Q|^{-1/\lambda}g_{i_0}$ and $a_{i_k} := (c\alpha)^{-1}|Q_{i_k}|^{-1/\lambda}g_{i_k}$ are (λ,q,∞) atoms in $\mathcal{HM}_q^\lambda$, for all $k=1,2,\cdots$. Moreover, we can write

$$\begin{aligned}a_Q &= |Q|^{-\frac{1}{\lambda}}\Big\{g_0 + c\alpha\sum_{i_0}g_{i_0} + (c\alpha)^2\sum_{i_1}g_{i_1} + \cdots + (c\alpha)^k\sum_{i_{k-1}}g_{i_{k-1}} + \cdots\Big\}\\ &= s_{i_0}a_{i_0} + \sum_{i_1}s_{i_1}a_{i_1} + \sum_{i_2}s_{i_2}a_{i_2} + \cdots + \sum_{i_k}s_{i_k}a_{i_k} + \cdots \end{aligned}\tag{4.5}$$

where each coefficient $\{s_{i_k}\}_k$ is defined by $s_{i_k} = (c\alpha)^{k+1}|Q|^{-1/\lambda}|Q_{i_k}|^{1/\lambda}$. It remains to show that $\|\{s_{i_k}\}_k\|_{\lambda,q}\leqslant C$, uniformly. Fixed $J\subset\mathbb{R}^n$ a dyadic cube, we may estimate using (4.4)

$$\begin{aligned}|J|^{\frac{q}{\lambda}-1}\sum_{k=0}^{\infty}\sum_{Q_{i_k}\subseteq J}|s_{i_k}|^q|Q_{i_k}|^{1-\frac{q}{\lambda}} &= |J|^{\frac{q}{\lambda}-1}|Q|^{-\frac{q}{\lambda}}\sum_{k=0}^{\infty}(c\alpha)^{q(k+1)}\Big(\sum_{Q_{i_k}\subseteq J}|Q_{i_k}|\Big)\\ &\lesssim |J|^{\frac{q}{\lambda}-1}|Q|^{-\frac{q}{\lambda}}|J\cap Q|\sum_{k=0}^{\infty}(c\alpha)^{q(k+1)}\left(\frac{c}{\alpha^r}\right)^{k+1}\leqslant C\end{aligned}$$

provided $c^{q+1}\alpha^{q-r}<1$ (weaker than the previous one) and $q\leqslant\lambda$. Note that here we have used a refinement of (4.4) given by

$$\sum_{i_k : Q_{i_k} \subseteq J} |Q_{i_k}| \lesssim \left(\frac{c}{\alpha^r} \right)^{k+1} |J \cap Q|$$

and the uniform control $|J|^{q/\lambda - 1} |Q|^{-q/\lambda} |J \cap Q| \lesssim 1$. ∎

The previous lemma allow us to study $\mathcal{HM}_q^\lambda(\mathbb{R}^n)$ with any of the atomic spaces $\mathbf{at}\mathcal{HM}_q^{\lambda,r}(\mathbb{R}^n)$ for $1 \leqslant r \leqslant \infty$ provided $q < r$. In addition, we announce an atomic decomposition in terms of (λ, q, r) atoms in $\mathcal{HM}_q^\lambda$, which is a direct consequence of the one proved in [47, p. 100] for (λ, q, ∞) atoms and Lemma 4.1, since they are in particular (λ, q, r) atoms.

Theorem 4.2. *Let $0 < q \leqslant 1 \leqslant r \leqslant \infty$ with $q < r$ and $q < \lambda < \infty$. Then, $f \in \mathcal{HM}_q^\lambda(\mathbb{R}^n)$ if and only if there exists a collection of (λ, q, r) atoms $\{a_{Q_j}\}_j$ and a sequence of complex numbers $\{s_{Q_j}\}_j$ such that $f = \sum_j s_{Q_j} a_{Q_j}$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\|f\|_{\mathbf{at}\mathcal{HM}_q^\lambda} \approx \|f\|_{\mathcal{HM}_q^\lambda}$.*

The proof is a direct consequence of the Atomic Decomposition Theorem proved in [47, p. 100] for (λ, q, ∞) atoms, since they are in particular (λ, q, r) atoms, together with Lemma 4.1.

Remark 4.1. There is also another atomic characterization of $\mathcal{HM}_q^\lambda(\mathbb{R}^n)$ that holds for $0 < q \leqslant \lambda < \infty$. For further details see [46, Theorem 1.3].

The L^2 molecular structure of $\mathcal{HM}_q^\lambda(\mathbb{R}^n)$ have been first defined in [48, Definition 2.5 and Theorem 2.6]. In what follows we define L^r molecules as in Chapter 1.

Definition 4.3. *Let $0 < q \leqslant 1 \leqslant r < \infty$ with $q < r$, $q \leqslant \lambda < \infty$, and $s > n\,(r/q - 1)$. A function M is called a (λ, q, s, r) molecule in $\mathcal{HM}_q^\lambda$ if there exist a cube $Q = Q(x_Q, \ell)$ such that:*

M1. $\|M\|_{L^r(Q)} \lesssim |Q|^{\frac{1}{r} - \frac{1}{\lambda}}$;

M2. $\|M\,| \cdot - x_Q|^{\frac{s}{r}}\|_{L^r(Q^c)} \lesssim |Q|^{\frac{s}{nr} + \frac{1}{r} - \frac{1}{\lambda}}$;

M3. $\int_{\mathbb{R}^n} M(x) x^\alpha dx = 0$ *for all* $|\alpha| \leqslant N_q$.

Lemma 4.2. *Let M to be a (λ, q, s, r) molecule in $\mathcal{HM}_q^\lambda$ centered in $Q = Q(x_Q, \ell_Q)$. Then*

$$M = \sum_{j=0}^{\infty} d_{Q_j}\, a_{Q_j} + \sum_{j=0}^{\infty} t_{Q_j}\, b_{Q_j} \quad in \quad L^r(\mathbb{R}^n),$$

where each $\{a_{Q_j}\}_j$ and $\{b_{Q_j}\}_j$ are (λ, q, r) and (λ, q, ∞) atoms respectively. Moreover, the sequences of scalars satisfy

$$\sum_{j=0}^{\infty} |d_{Q_j}|^q |Q_j|^{1-\frac{q}{\lambda}} \lesssim |Q|^{1-\frac{q}{\lambda}} \quad and \quad \sum_{j=0}^{\infty} |t_{Q_j}|^q |Q_j|^{1-\frac{q}{\lambda}} \lesssim |Q|^{1-\frac{q}{\lambda}},$$

where $Q_j = Q(x_Q, 2^j \ell_Q)$.

Proof. The proof will follow by the same ideas of the corresponding result for Hardy spaces, as in Proposition 1.2, and we will only outline the differences. Let M to be a (λ, q, s, r) molecule centered in $Q = Q(x_Q, \ell_Q)$. For each $j \in \mathbb{Z}_+$, let Q_j be a cube centered at x_Q with sidelength $\ell_j = 2^j \ell_Q$. Consider the collection of annulus $\{E_j\}_{j \in \mathbb{Z}_+}$ given by $E_0 = Q$ and $E_j = Q_j \backslash Q_{j-1}$ for $j \geqslant 1$. Let $M_j(x) := M(x)\chi_{E_j}(x)$, $M_\gamma^j = \fint_{E_j} M_j(x) x^\gamma dx$ and consider $P_j(x) = \sum_{|\gamma| \leqslant N_q} M_\gamma^j \phi_\gamma^j(x)$. Write

$$M = \sum_{j=0}^{\infty} (M_j - P_j) + \sum_{j=0}^{\infty} P_j.$$

For every $j \in \mathbb{Z}_+$, write

$$(M_j - P_j)(x) = d_{Q_j}\, a_{Q_j}(x) \quad \text{for} \quad d_{Q_j} = \|M_j - P_j\|_{L^r} |Q_j|^{\frac{1}{\lambda} - \frac{1}{r}} \quad \text{and} \quad a_{Q_j} = \frac{M_j - P_j}{\|M_j - P_j\|_{L^r}} |Q_j|^{\frac{1}{r} - \frac{1}{\lambda}}.$$

Furthermore, from (M1) and (M2) we have

$$\|M_j - P_j\|_{L^r} \lesssim \|M_j\|_{L^r} \lesssim |Q_j|^{\frac{1}{r} - \frac{1}{\lambda}} \, (2^j)^{-\frac{s}{r} + n\left(\frac{1}{\lambda} - \frac{1}{r}\right)}.$$

By this, we get that $\{a_{Q_j}\}_j$ is a sequence of (λ, q, r) atoms supported on Q_j. Moreover, since $s > n\,(r/q - 1)$ it follows

$$\sum_{j=0}^{\infty} |d_{Q_j}|^q |Q_j|^{1-\frac{q}{\lambda}} \lesssim |Q|^{1-\frac{q}{\lambda}} \sum_{j=0}^{\infty} (2^j)^{q\left[-\frac{s}{r} + n\left(\frac{1}{q} - \frac{1}{r}\right)\right]} \lesssim |Q|^{1-\frac{q}{\lambda}}.$$

For the second sum, let $\psi_\gamma^j(x) := N_\gamma^{j+1}\left[|E_{j+1}|^{-1}\phi_\gamma^{j+1}(x) - |E_j|^{-1}\phi_\gamma^j(x)\right]$, where

$$N_\gamma^j = \sum_{k=j}^{\infty} m_\gamma^k |E_k| = \sum_{k=j}^{\infty}\int_{E_k} M(x)x^\gamma dx.$$

Then, we can represent P_j (using the vanish moments (M3)) as

$$\sum_{j=0}^{\infty} P_j(x) = \sum_{j=0}^{\infty}\sum_{|\gamma|\leqslant N_q}\psi_\gamma^j(x).$$

Since $|\gamma| \leqslant n\left(1/\lambda - 1\right)$ and $s > n\left(r/q - 1\right)$ we have

$$|N_\gamma^{j+1}| \leqslant |Q_j|^{1-\frac{1}{\lambda}}\left(2^j\ell_Q\right)^{|\gamma|}(2^j)^{-\frac{s}{r}+n\left(\frac{1}{\lambda}-\frac{1}{r}\right)} \quad \text{and then} \quad \left|N_\gamma^{j+1}|E_j|^{-1}\phi_\gamma^j(x)\right| \leqslant C|Q_j|^{-\frac{1}{\lambda}}(2^j)^{-\frac{s}{r}+n\left(\frac{1}{\lambda}-\frac{1}{r}\right)}.$$

Letting

$$\psi_\gamma^j = t_{Q_j}\, b_\gamma^j \quad \text{where} \quad t_{Q_j} = (2^j)^{-\frac{s}{r}+n\left(\frac{1}{\lambda}-\frac{1}{r}\right)} \quad \text{and} \quad b_\gamma^j(x) = (2^j)^{\frac{s}{r}-n\left(\frac{1}{\lambda}-\frac{1}{r}\right)}\,\psi_\gamma^j(x),$$

we can write $\sum_{j=0}^{\infty} P_j(x) = \sum_{j=0}^{\infty}\sum_{|\gamma|\leqslant N_q} t_{Q_j}\, b_\gamma^j(x)$ and for each $j \in \mathbb{Z}_+$ the function $b_\gamma^j(x)$ is a (λ, q, ∞) atom. Moreover from $s > n\left(r/q - 1\right)$ one has

$$\sum_{j=0}^{\infty}|t_{Q_j}|^q|Q_j|^{1-\frac{q}{\lambda}} = |Q|^{1-\frac{q}{\lambda}}\sum_{j=0}^{\infty}(2^j)^{q\left(-\frac{s}{r}+n\left(\frac{1}{q}-\frac{1}{r}\right)\right)} \lesssim |Q|^{1-\frac{q}{\lambda}}.$$

∎

Now we ready to announce our main theorem of molecule decomposition in Hardy-Morrey spaces.

Theorem 4.3. *Consider $0 < q \leqslant 1 \leqslant r < \infty$ and $q \leqslant \lambda$. Let $\left\{M_{Q_j}\right\}_j$ to be a collection of (λ, q, s, r) molecules and $\left\{s_{Q_j}\right\}_j$ a sequence of complex numbers such that $\|\{s_{Q_j}\}_j\|_{\lambda,q} < \infty$. If the series $f = \sum_j s_{Q_j} M_{Q_j}$ converges in $\mathcal{S}'(\mathbb{R}^n)$, $q < r$ and $\lambda < r$, then $f \in \mathcal{HM}_q^\lambda(\mathbb{R}^n)$ and moreover, $\|f\|_{\mathcal{HM}_q^\lambda} \lesssim \|\{s_{Q_j}\}_j\|_{\lambda,q}$ with implicit constant independent of f.*

Remark 4.2. The previous theorem covers [48, Theorem 2.6], where the case $r = 2$ was considered. The natural restriction $\lambda < r$ was omitted in the previous reference, since only a decomposition like in

Lemma 4.2 was proved. This restriction is was necessary in Proposition 4.1 to show that atoms in $\mathcal{HM}_q^\lambda$ are uniformly bounded.

Proof. Suppose $f = \sum_j s_{Q_j} M_{Q_j}$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\|\{s_{Q_j}\}_j\|_{\lambda,q} < \infty$. Since $0 < q \leqslant 1$, for a fixed dyadic cube $J \subset \mathbb{R}^n$ we have

$$\int_J |\mathcal{M}_\varphi f(x)|^q dx \leqslant \sum_{Q_j \subseteq J} |s_{Q_j}|^q \int_J |\mathcal{M}_\varphi (M_{Q_j})(x)|^q dx + \sum_{J \subset Q_j} |s_{Q_j}|^q \int_J |\mathcal{M}_\varphi (M_{Q_j})(x)|^q dx = I_1 + I_2.$$

Estimate of I_1. From Lemma 4.2, for each $j \in \mathbb{Z}_+$, there exists sequences $\{a_{Q_{j,i}}\}_i$ and $\{d_{Q_{j,i}}\}_i$ of (λ, q, r) atoms and scalars respectively such that

$$M_{Q_j} = \sum_i d_{Q_{j,i}} a_{Q_{j,i}} \quad \text{and} \quad \sum_i |d_{Q_{j,i}}|^q \, |Q_{j,i}|^{1-\frac{q}{\lambda}} \lesssim |Q_j|^{1-\frac{q}{\lambda}}.$$

It follows from this decomposition and analogous estimates as in Proposition 4.1 that

$$\begin{aligned}
\sum_{Q_j \subseteq J} |s_{Q_j}|^q \int_J |\mathcal{M}_\varphi (M_{Q_j})(x)|^q dx &\lesssim \sum_{Q_j \subseteq J} |s_{Q_j}|^q \sum_i |d_{Q_{j,i}}|^q \int_J |\mathcal{M}_\varphi (a_{Q_{j,i}})(x)|^q dx \\
&\lesssim \sum_{Q_j \subseteq J} |s_{Q_j}|^q \sum_i |d_{Q_{j,i}}|^q \, |Q_{j,i}|^{1-\frac{q}{\lambda}} \\
&\lesssim \sum_{Q_j \subseteq J} |s_{Q_j}|^q |Q_j|^{1-\frac{q}{\lambda}} \\
&\lesssim |J|^{1-\frac{q}{\lambda}} \, \|\{s_{Q_j}\}_j\|_{\lambda,q}^q.
\end{aligned}$$

Estimate of I_2. For $1 < r < \infty$, using that $\mathcal{M}_\varphi$ is bounded on $L^r(\mathbb{R}^n)$ to itself, it follows for each $j \in \mathbb{Z}_+$

and $E_{j,0} = Q_j$ and $E_{j,i} = Q(x_{Q_j}, 2^i \ell_{Q_j}) \backslash Q(x_{Q_j}, 2^{i-1}\ell_{Q_j})$ that

$$\begin{aligned} |J|^{\frac{q}{\lambda}-1} \int_J |\mathcal{M}_\varphi(M_{Q_j})(x)|^q dx &\leqslant |J|^{q\left(\frac{1}{\lambda}-\frac{1}{r}\right)} \left(\int_{\mathbb{R}^n} |\mathcal{M}_\varphi(M_{Q_j})(x)|^r dx \right)^{\frac{q}{r}} \\ &\leqslant |J|^{q\left(\frac{1}{\lambda}-\frac{1}{r}\right)} \left(\sum_i \int_{E_{j,i}} |M_{Q_j}(x)|^r dx \right)^{\frac{q}{r}} \\ &\leqslant |J|^{q\left(\frac{1}{\lambda}-\frac{1}{r}\right)} \left(\sum_i (2^i \ell_{Q_j})^{-s} \int_{E_{j,i}} |M_{Q_j}(x)|^r |x - x_{Q_j}|^s dx \right)^{\frac{q}{r}} \\ &\lesssim |J|^{q\left(\frac{1}{\lambda}-\frac{1}{r}\right)} |Q_j|^{q\left(\frac{1}{r}-\frac{1}{\lambda}\right)} \left(\sum_i 2^{-is} \right)^{\frac{q}{r}} \simeq \left(\frac{|J|}{|Q_j|} \right)^{q\left(\frac{1}{\lambda}-\frac{1}{r}\right)}. \end{aligned}$$

If $r = 1$ and $0 < q < 1$, we proceed as in (4.2) with $A = |Q_j|^{1-\frac{1}{\lambda}} |J|^{-1}$ to obtain

$$|J|^{\frac{q}{\lambda}-1} \int_J |\mathcal{M}_\varphi(M_{Q_j})(x)|^q dx \lesssim |J|^{\frac{q}{\lambda}-1} \left[|J| \int_0^A \omega^{q-1} d\omega + |Q_j|^{-1+\frac{1}{\lambda}} \int_A^\infty \omega^{q-2} d\omega \right] \lesssim \left(\frac{|J|}{|Q_j|} \right)^{q\left(\frac{1}{\lambda}-1\right)}.$$

For a fixed a dyadic cube J, there exists a subset $N \subseteq \mathbb{N}$ such that each cube $J \subset Q_j$ is uniquely determined by a dyadic cube $Q_{k,J} \in \left\{ Q \text{ dyadic} : J \subset Q_j \text{ and } \ell_Q = 2^k \ell_J \right\}$ with $k \in N$. Hence, we can write

$$\sum_{J \subset Q_j} |s_{Q_j}|^q \left(\frac{|J|}{|Q_j|} \right)^{\gamma q} = \sum_{k \in N} |s_{Q_{k,J}}|^q \, 2^{-kn\gamma q} \text{ with } \gamma := 1/\lambda - 1/r > 0.$$

Then,

$$\begin{aligned} |J|^{\frac{q}{\lambda}-1} \sum_{J \subset Q_j} |s_{Q_j}|^q |\mathcal{M}_\varphi(M_{Q_j})|^q_{L^q(J)} &\lesssim \sum_{k \in N} \left(|s_{Q_{k,J}}|^q |Q_{k,J}|^{1-\frac{q}{\lambda}} \right) |Q_{k,J}|^{\frac{q}{\lambda}-1} 2^{-kn\gamma q} \\ &\leqslant \sum_{k \in N} \left(\sum_{Q_j \subseteq Q_{k,J}} |s_{Q_j}|^q |Q_j|^{1-\frac{q}{\lambda}} \right) |Q_{k,J}|^{\frac{q}{\lambda}-1} 2^{-kn\gamma q} \\ &\lesssim \| \left\{ s_{Q_j} \right\}_j \|^q_{\lambda,q} \sum_{k \in N} 2^{-kn\gamma} \lesssim \| \left\{ s_{Q_j} \right\}_j \|^q_{\lambda,q}. \end{aligned}$$

□

4.2 Continuity in Hardy-Morrey spaces

Using the atomic and molecular theory presented in the previous section, we have the following extension of Theorem A for Hardy-Morrey spaces:

Theorem 4.4. *Let* $0 < q \leqslant 1 \leqslant r < \infty$, $q \leqslant \lambda < \infty$ *and* T *a strongly singular Calderón–Zygmund operator whose associated kernel satisfies a* D_r *condition for some* $\delta > 0$. *Under the assumptions that* $T^*(x^\alpha) = 0$ *for every* $|\alpha| \leqslant \lfloor \delta \rfloor$, $1 \leqslant r \leqslant 2$ *with* $q < r$ *and* $\lambda < r$, T *can be extended to a bounded operator from* $\mathcal{HM}_q^\lambda(\mathbb{R}^n)$ *to itself for* $q_0 < q \leqslant 1$, *where* q_0 *is given by* (2.26). *The case* $r = 1$ *holds for the range* $q_0 < q < 1$.

As an immediate corollary, we also obtain the continuity of classical non-convolution Calderón–Zygmund operators ($\sigma = 1$) associated to kernels satisfying integral conditions, or in particular standard $\delta-$kernels (2.4) with $\sigma = 1$. The same result in the convolution setting for kernels satisfying derivative conditions can be found in [48, Section 2.2].

Corollary 4.1. *Under the same hypothesis of the previous theorem, if* T *is a standard Calderón–Zygmund operator, then it is bounded from* $\mathcal{HM}_q^\lambda(\mathbb{R}^n)$ *to itself provided that* $n/(n+\delta) < q \leqslant 1$.

Proof. Let a be a (λ, q, r) atom supported in the cube Q. From Theorem 4.3, it suffices to show that Ta is a (λ, q, s, r) molecule associated to Q. Suppose first that $\ell_Q \geqslant 1$. Since T is bounded in $L^2(\mathbb{R}^n)$ to itself and $1 \leqslant r \leqslant 2$, condition $(M1)$ follows by

$$\int_{2Q} |Ta(x)|^r dx \leqslant |2Q|^{1-\frac{r}{2}} \|Ta\|_{L^2}^r \lesssim |Q|^{1-\frac{r}{2}} \|a\|_{L^2}^r \lesssim |Q|^{1-\frac{r}{\lambda}} \simeq \ell_Q^{n\left(1-\frac{r}{\lambda}\right)}. \tag{4.6}$$

For $(M2)$ using the moment condition of the atom, Minkowski inequality and D_r we get

$$\begin{aligned}\int_{2Q^c}|Ta(x)|^r|x-x_Q|^s dx &\leqslant \sum_{j=1}^{\infty}\int_{C_j(x_Q,\ell_Q)}\left|\int_Q [K(x,y)-K(x,x_Q)]a(y)dy\right|^r |x-x_Q|^s dx\\ &\leqslant \sum_{j=1}^{\infty}(2^j\ell_Q)^s\left\{\int_Q|a(y)|\left[\int_{C_j(x_Q,\ell_Q)}|K(x,y)-K(x,x_Q)|^r dx\right]^{\frac{1}{r}} dy\right\}^r\\ &\lesssim \sum_{j=1}^{\infty}(2^j\ell_Q)^{s-n(r-1)}\, 2^{-jr\delta}\, \ell_Q^{rn\left(1-\frac{1}{\lambda}\right)} \simeq \ell_Q^{r+n\left(1-\frac{r}{p}\right)}\sum_{j=1}^{\infty}2^{j[s-n(r-1)-r\delta]} \simeq \ell_Q^{s+n\left(1-\frac{r}{p}\right)},\end{aligned}$$

since $s<n(r-1)+r\delta$. We remark that for the case $r=1$, one needs to consider $(\lambda,q,s,1)$ molecules in $\mathcal{HM}_q^\lambda$ and hence $0<q<1$. Suppose now that $\ell_Q<1$. Since T is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and $1<r\leqslant 2$, condition (M1) follows by

$$\int_{2Q}|Ta(x)|^r dx \leqslant |2Q|^{1-\frac{r}{2}}\,\|Ta\|_{L^2}^r \lesssim |Q|^{1-\frac{r}{2}}\|a\|_{L^p}^r \lesssim |Q|^{1-\frac{r}{\lambda}+r\left(\frac{1}{p}-\frac{1}{2}\right)} \lesssim |Q|^{1-\frac{r}{\lambda}}.$$

To estimate the global (M2) condition, we consider $0<\rho\leqslant\sigma\leqslant 1$ a parameter that will be chosen conveniently later, denote by $2Q^\rho := Q(x_Q,2\ell_Q^\rho)$ and split the integral over $\mathbb{R}^n$ into $2Q^\rho$ and $(2Q^\rho)^c$. For $2Q^\rho$ we use the boundedness from $L^p(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ again and obtain

$$\begin{aligned}\int_{2Q^\rho}|Ta(x)|^r\,|x-x_Q|^s dx &\lesssim \ell_Q^{s\rho}|4Q^\rho|^{1-\frac{r}{2}}\|Ta\|_{L^2}^r \lesssim \ell_Q^{\rho s+n\rho\left(1-\frac{r}{2}\right)}\|a\|_{L^p}^r\\ &\lesssim \ell_Q^{\rho s+n\left[\rho-\frac{r\rho}{2}+r\left(\frac{1}{p}-\frac{1}{\lambda}\right)\right]} \lesssim \ell_Q^{s+n\left(1-\frac{r}{\lambda}\right)},\end{aligned}$$

assuming $s\leqslant -n\left(1-\frac{r}{2}\right)+\frac{nr}{1-\rho}\left(\frac{1}{p}-\frac{1}{2}\right)$. For $(2Q^\rho)^c$, we use (2.9) to obtain

$$\begin{aligned}\int_{(2Q^\rho)^c}|Ta(x)|^r|x-x_Q|^s dx &\lesssim \sum_{j=1}^{\infty}(2^j\ell_Q^\rho)^s\left\{\int_Q|a(y)|\left[\int_{C_j(x_Q,\ell_Q^\rho)}|K(x,y)-K(x,x_Q)|^r dx\right]^{\frac{1}{r}} dy\right\}^r\\ &\lesssim \sum_{j=1}^{\infty}(2^j\ell_Q^\rho)^s\,\left(|C_j(x_Q,\ell_Q^\rho)|^{\frac{1}{r}-1+\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)}\,2^{-\frac{j\delta}{\rho}}\right)^r\,\ell_Q^{rn\left(1-\frac{1}{\lambda}\right)}\\ &\simeq \ell_Q^{\rho s+n\left[r+\frac{r\delta}{n}-r\rho\left(1-\frac{1}{r}+\frac{\delta}{n\sigma}\right)-\frac{r}{\lambda}\right]}\sum_{j=1}^{\infty}2^{j\left[s-n(r-1)-\frac{r\delta}{\sigma}\right]}\\ &\lesssim \ell_Q^{\rho s+n\left[\rho\left(1-\frac{r}{2}\right)+r\left(\frac{1}{p}-\frac{1}{\lambda}\right)\right]} \leqslant \ell_Q^{s+n\left(1-\frac{r}{\lambda}\right)},\end{aligned}$$

where the convergence follows assuming $s < n(r-1) + \frac{r\delta}{\sigma}$ and we choose ρ to be such that $r + \frac{r\delta}{n} - \rho\left(r - 1 + \frac{r\delta}{n\sigma}\right) = \rho\left(1 - \frac{r}{2}\right) + \frac{r}{p} \Leftrightarrow \rho := \frac{n\left(1-\frac{1}{p}\right)+\delta}{\frac{n}{2}+\frac{\delta}{\sigma}}$. By the choice of ρ we have

$$-n\left(1-\frac{r}{2}\right) + \frac{nr}{1-\rho}\left(\frac{1}{p}-\frac{1}{2}\right) < n(r-1) + r\delta < n(r-1) + \frac{r\delta}{\sigma}.$$

In particular, collecting the restrictions on s we get

$$n\left(\frac{r}{q}-1\right) < s \leqslant -n\left(1-\frac{r}{2}\right) + \frac{nr}{1-\rho}\left(\frac{1}{p}-\frac{1}{2}\right) \quad \Rightarrow \quad \frac{1}{q} < \frac{1}{2} + \frac{\beta\left(\frac{\delta}{\sigma}+\frac{n}{2}\right)}{n\left(\frac{\delta}{\sigma}-\delta+\beta\right)} := \frac{1}{q_0}.$$

We point out that when $\sigma = 1$, only condition $s < n(r-1) + r\delta$ is imposed to verify $(M1)$ and $(M2)$. Condition $(M3)$ follows as before by $T^*(x^\alpha) = 0$. □

Remark 4.3. It could also be possible to extend Theorem B for local Hardy-Morrey spaces $hM_q^\lambda(\mathbb{R}^n)$.

Bibliography

[1] D. R. Adams, *Morrey spaces*, Lecture notes in applied and numerical Harmonic Analysis, Birkhäuser, 2015.

[2] J Álvarez, *H^p and weak H^p continuity of Calderón-Zygmund type operators*, Fourier Analysis: Analytic and Geometric Aspects, Lectures Notes in Pure and Applied Mathematics, CRC Press, 1994, pp. 17–34.

[3] J. Álvarez and M. Guzman-Partida, *The $T(1)$ theorem revisited*, Surveys in Mathematics and its Applications **13** (2018), 41–94.

[4] J. Álvarez and J. Hounie, *Estimates for the kernel and continuity properties of pseudo-differential operators*, Arkiv för Matematik **28** (1990), no. 1, 1–22.

[5] J. Álvarez and M. Milman, *H^p continuity properties of Calderón-Zygmund-type operators*, Journal of Mathematical Analysis and Applications **118** (1986), no. 1, 63–79.

[6] ______, *Vector valued inequalities for strongly singular Calderón-Zygmund operators*, Revista Matemática Iberoamericana **2** (1986), 405–426.

[7] W. R. Bloom and Z. Xu, *Fourier multipliers for local Hardy spaces on Chébli-Trimèche hypergroups*, Canadian Journal of Mathematics **50** (1998), no. 5, 897–928.

[8] A. Bonami, J. Feuto, and S. Grellier, *Endpoint for the div-curl lemma in Hardy spaces*, Publicacions Matemàtiques **54** (2010), no. 2, 341–358.

[9] M. Bownik, *Anisotropic Hardy spaces and wavelets*, vol. 164, American Mathematical Society: Memoirs of the American Mathematical Society, no. 781, American Mathematical Society, 2003.

[10] ______, *Boundedness of operators on Hardy spaces via atomic decompositions*, Proceeding of the American Mathematical Society **113** (2005), no. 12, 3535–3542.

[11] T. A. Bui and F. K. Ly, *Calderón–Zygmund operators on local Hardy spaces*, Potential Analysis (2022).

[12] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Mathematica **88** (1952), no. 85, 85–139.

[13] S. Chanillo, *Weighted norm inequalities for strongly singular convolution operators*, Transactions of the American Mathematical Society **281** (1984), no. 1, 77–107.

[14] R. Coifman, *Characterization of fourier transforms of Hardy spaces*, Proceedings of the National Academy of Sciences of the United States of America **71** (1974), 4133–4134.

[15] ______, *A real variable characterion of H^p*, Studia Mathematica **51** (1974), 269–274.

[16] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bulletin of the American Mathematical Society **83** (1977), no. 4, 589–645.

[17] R. R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudo-différentiels*, 2 ed., Astérisque 57, Société Mathématique de France, 1978.

[18] D. Cruz-Uribe, K. Moen, and H. V. Nguyen, *A new approach to norm inequalities on weighted and variable Hardy spaces*, Annales Academiae Scientiarum Fennicae **45** (2020), 175–198.

[19] G. Dafni, *Hardy spaces on strongly pseudoconvex domains in C^n and domains of finite type in C^2*, Ph.D. thesis, Princeton University, 1993.

[20] G. Dafni, C. H. Lau, T. Picon, and C. Vasconcelos, *Necessary cancellation conditions for the boundedness of operators on local Hardy spaces*, arXiv (preprint).

[21] ______, *Inhomogeneous cancellation conditions and Calderón-Zygmund type operators on h^p*, Nonlinear Analysis **225** (2022).

[22] G. Dafni and E. Liflyand, *A local Hilbert transform, Hardy's inequality and molecular characterization of Goldberg's local Hardy space*, Complex Analysis and its Synergies **5** (2019), no. 10.

[23] G. Dafni and H. Yue, *Some characterizations of local bmo and h^1 on metric measure spaces*, Analysis and Mathematical Physics **2** (2012), 285–318.

[24] G. David and J-L Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Annals of Mathematics **120** (1984), 371–397.

[25] M. de Almeida and T. Picon, *Atomic decomposition, Fourier transform decay and pseudodifferential operators on localizable Hardy-Morrey spaces*, ArXiv (preprint) (2020).

[26] M. de Almeida, T. Picon, and C. Vasconcelos, *A note on continuity of strongly singular Calderón–Zygmund operators in Hard–Morrey spaces*, Trends in Mathematics (to appear).

[27] J. de Francia, F. Ruiz, and J. Torrea, *Calderón-Zygmund theory for operator-valued kernels*, Advances in Mathematics **62** (1986), no. 1, 7 – 48.

[28] Y-L. Deng. and S-C. Long, *Pseudodifferential operators on weighted Hardy spaces*, Journal of Function Spaces **2020** (2020), 1–7.

[29] W. Ding, Y. Han, and Y. Zhu, *Bondedness of singular integral operators on local Hardy spaces and dual spaces*, Potential Analysis (2020).

[30] J. Duoandikoetxea, *Fourier Analysis*, Crm Proceedings & Lecture Notes, American Mathematical Soc., 2001.

[31] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Mathematica **124** (1970), no. 1, 9–36.

[32] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Mathematica **129** (1972), no. 1, 137–193.

[33] M. Frazier, Y.-S. Han, B. Jawerth, and G. Weiss, *The T1 Theorem for Triebel-Lizorkin spaces*, Harmonic Analysis and Partial Differential Equations, Lecture Notes in Mathematics, vol. 1384, Springer, Berlin, 1989, pp. 168–181.

[34] J. García-Cuerva, *Weighted H^p spaces*, Dissertationes Mathematicae, Panstwowe Wydawnictwo Naukowe, 1979.

[35] J. García-Cuerva and J. de Francia, *Weighted norm inequalities and related topics*, Annals of Discrete Mathematics, no. Nº 116, North-Holland, 1985.

[36] D. Goldberg, *Local Hardy Spaces*, Harmonic Analysis in Euclidean Spaces, Proceedings of the Symposium on Pure Mathematics, vol. XXXV, American Mathematical Society, 1979, pp. 245–248.

[37] ______, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), no. 1, 27–42.

[38] L. Grafakos, *Modern Fourier Analysis*, third ed., Graduate Texts in Mathematics, Springer, 2014.

[39] G. H. Hardy, *The mean value of the modulus of an analytic function*, Proceedings of the London Mathematical Society **14** (1915), no. 1, 269–277.

[40] I. Hirschman, *On multiplier transformations*, Duke Mathematical Journal **26** (1959), no. 2, 221–242.

[41] G. Hoepfner, *Hardy spaces, its variants and applications*, Lecture notes in: IV Workshop on Geometric Analysis of PDE and several complex variables, 2007.

[42] G. Hoepfner, R. Kapp, and T. Picon, *On the continuity and compactness of pseudodifferential operators on localizable Hardy spaces*, Potential Analysis (2020).

[43] L. Hörmander, *The Analysis of Linear Partial Pifferential Operators I, Distribution theory and Fourier analysis*, second ed., Springer-Verlag, 1990.

[44] J. Hounie, *On the L^2 continuity of pseudo-differential operators*, Communications in Partial Differential Equations **11** (1986), no. 7, 765–778.

[45] J. Hounie and R. A. S. Kapp, *Pseudodifferential operators on local Hardy spaces*, Journal of Fourier Analysis and Aplications **15** (2009), 153–178.

[46] T. Iida, Y. Sawano, and H. Tanaka, *Atomic decomposition for Morrey spaces*, Journal of Analysis and its Applications **33** (2014), 149–170.

[47] H. Jia and H. Wang, *Decomposition of Hardy-Morrey spaces*, Journal of Mathematical Analysis and Applications **354** (2009), 99–110.

[48] ______, *Singular integral operator, Hardy-Morrey space estimates for multilinear operators and Navier-Stokes equations*, Mathematical Methods in Applied Sciences **33** (2010), 1661–1984.

[49] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Communications on Pure and Applies Mathematics **XIV** (1961), 415–426.

[50] Y. Komori, *Calderón-Zygmund operators on $H^p(\mathbb{R}^n)$*, Scientiae Mathematicae Japonicae Online **4** (2001), 35–44.

[51] L. D. Ky, *A note on H^p_w-boundedness of Riesz transforms and θ-Calderón-Zygmund operators through molecular characterization*, Analysis in Theory and Applications **27** (2011), no. 3, 251–264.

[52] L. D. Ky and H. D. Hung, *An Hardy estimate for commutators of pseudo-differential operators*, Taiwanese Journal of Mathematics **19** (2015), no. 4, 1097–1109.

[53] R. Latter, *A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms*, Studia Mathematica **62** (1978), 93–101.

[54] M-Y. Lee and C-C. Lin, *The molecular characterization of weighted Hardy spaces*, Journal of Functional Analysis **188** (2002), 442–460.

[55] J. F. Li and S. Z. Lu, *Strongly singular integral operators on weighted Hardy space*, Acta Mathematica Sinica, English Series **22** (2006), no. 3, 767–772.

[56] W. Li, *John-Nirenberg type inequalities for the Morrey-Campanato spaces*, J. Inequal. Appl. (2008), Art. ID 239414, 5. MR 2379517

[57] X. Li and L. Peng, *The molecular characterization of weighted Hardy spaces*, Science in China Series A: Mathematics **44** (2001), 201–211.

[58] L. Liu and J. Xiao, *Restricting Riesz-Morrey-Hardy potentials*, Journal of Differential Equations **262** (2017), 5468–5496.

[59] G. Lu and P. Zhang, *Multilinear Calderón-Zygmund operators with kernels of Dini's type with applications*, Nonlinear Analysis **107** (2014), 92–117.

[60] G. Lu and Y. Zhu, *Bounds for singular integrals on weighted Hardy spaces and discrete Littlewood-Paley analysis*, Journal of Geometric Analysis **22** (2012), 666–684.

[61] F. K. Ly and V. Naibo, *Pseudo-multipliers and smooth molecules on Hermite Besov and Hermite Triebel-Lizorkin spaces*, Journal of Fourier Analysis and Applications **27** (2021), no. 57.

[62] S. Meda, P. Sjögren, and M. Vallarino, *On the $H^1 - L^1$ boundedness of operators*, Proceedings of the American Mathematical Society **136** (2008), 2921–2931.

[63] Y. Meyer, *Ondelettes et operateurs, II. Opérateurs de Calderon-Zygmund*, Actualités Mathématiques, Hermann, 1990.

[64] Y. Meyer and R. Coifman, *Wavelets: Calderón-Zygmund and multilinear operators*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1997.

[65] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Transactions of the Americal Mathematical Society **43** (1938), no. 1, 126–166.

[66] L. Oliveira and J. Hart, *Hardy space estimates for a limited ranges of Muckenhoupt weights*, Advances in Mathematics **313** (2017), 803–838.

[67] T. Picon and C. Vasconcelos, *On the continuity of strongly singular Calderón-Zygmund-type operators on Hardy spaces.*, Integral Equations and Operator Theory **95** (2023), no. 9.

[68] T. Quek and D. Yang, *Calderón-Zygmund operators on weighted weak Hardy spaces over* $\mathbb{R}^n$, Acta Mathematica Sinica, English Series **16** (2000), no. 1, 141–160.

[69] H. Rafeiro, N. Samko, and S. Samko, *Morrey-Campanato spaces: an overview*, Operator theory, pseudo-differential equations, and mathematical physics, Oper. Theory Adv. Appl., vol. 228, Birkhäuser/Springer Basel AG, Basel, 2013, pp. 293–323. MR 3025501

[70] F. Ricci and J. Verdera, *Duality in spaces of finite linear combinations of atoms*, Transactions of the American Mathematical Society **363** (2011), no. 3, 1311–1323.

[71] P. Rocha, *On the atomic and molecular decomposition of weighted Hardy spaces*, Revista de la Unión Matemática Argentina **61** (2020), no. 2, 229–247.

[72] D. A. Stegenga, *Bounded Toeplitz operators on* H^1 *and applications of the duality between* H^1 *and the functions of bounded mean oscillation*, American Journal of Mathematics **98** (1976), no. 3, 573–589.

[73] E. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Monographs in Harmonic Analysis, Princeton University Press, 1993.

[74] J. O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 1989.

[75] M. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces*, Astérisque, no. 77, Société Mathématique de France, 1980, pp. 67–149.

[76] R. H. Torres, *Boundedness results for operators with singular kernels on distribution spaces*, American Mathematical Society: Memoirs of the American Mathematical Society, American Mathematical Society, 1991.

[77] C. Vasconcelos, *Operadores de Calderón–Zygmund, pseudo-diferenciais e espaços de Hardy*, Master's thesis, Universidade Federal de São Carlos, 2018.

[78] S. Wainger, *Special trigonometric series in k dimensions*, Memoirs of the American Mathematical Society, no. 59, Amer Mathematical Society, 1965.

[79] K. Yabuta, *Generalizations of Calderón-Zygmund operators*, Studia Mathematica **82** (1985), no. 1, 17–31.

[80] ______, *A remark on the* (H^1, L^1) *boundedness*, Bulletin of the Faculty of Science, Ibaraki University (1993), no. 25, 19–21.

[81] Q. Yang, L. Yan, and D. Deng, *On Hörmander condition*, Chinese Science Bulletin **42** (1997), no. 16, 1341–1345.

[82] C. Zorko, *Morrey space*, Proceedings of the Mathematical Americal Society **98** (1986), no. 4, 586–592.