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Positive stationary solutions of Kirchhoff equations with first order terms and lack of coerciveness

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To my father and grandfather *(in memoriam)*.

Abstract

We investigate the existence of positive stationary solutions of Kirchhoff equations

$$\partial_t^2 u - \mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x, u, \partial_t u, \nabla u) \qquad (x,t) \in \Omega \times [0,T), \tag{$\mathscr{K}$}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, driven by the p-Laplace, $p > 1$, spatially inhomogeneous coefficients $\mathcal{M}$, and sources depending on first order terms with up to the natural growth. Such solutions satisfy a nonlocal elliptic PDE of the form

$$-\mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x, u, \nabla u) \quad \text{in } \Omega. \tag{$\mathcal{S}\mathscr{K}$}$$

Unlike the coercive cases where the operator in ($\mathcal{S}\mathscr{K}$) has a suitable lower order term or Dirichlet boundary condition is prescribed, a lack of coerciveness takes place if ($\mathcal{S}\mathscr{K}$) is supplied with homogeneous Neumann boundary condition. In this latter setting, we prove an existence result which play the role of a sub-supersolution principle for positive solutions of ($\mathcal{S}\mathscr{K}$). As an application, some examples showing the existence of positive stationary solutions of ($\mathscr{K}$) satisfying Neumann boundary condition are provided. To overcome the lack of coerciveness on ($\mathcal{S}\mathscr{K}$), we combine monotonicity and truncation techniques, with elliptic regularity theory, in order to construct parametric approximate problems of ($\mathcal{S}\mathscr{K}$) which are coercive, and whose solutions converge, as the parameter tends to zero, to a positive solution of ($\mathcal{S}\mathscr{K}$).

Keywords: Kirchoff, Neumann boundary, p-Laplace, homogeneous, inhomogeneous, elliptic partial differential equations, non-coercive, sub-supersolutions principle, positive solutions.

Resumo

Neste trabalho, investigamos a existência de soluções estacionárias positivas das equações de Kirchhoff da forma:

$$\partial_t^2 u - \mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x, u, \partial_t u, \nabla u) \qquad (x,t) \in \Omega \times [0,T), \tag{$\mathscr{K}$}$$

em um domínio limitado $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, p-Laplaciano com $p > 1$, $\mathcal{M}$ um coeficiente espacial não homogêneo e f dependendo de termos de primeira ordem com crescimento até o natural. Tais soluções satisfazem uma EDP elíptica não local da forma

$$-\mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x, u, \nabla u) \quad \text{em } \Omega. \tag{$\mathcal{S}\mathscr{K}$}$$

Ao contrário dos casos coercivos em que o operador em ($\mathcal{S}\mathscr{K}$) tem um termo de ordem inferior adequado ou a condição de contorno de Dirichlet, no caso de uma condição de contorno homogênea de Neumann, ocorre uma falta de coercividade em ($\mathcal{S}\mathscr{K}$). Neste último cenário, provamos um resultado de existência que desempenha o papel de um princípio de sub-supersolução para soluções positivas de ($\mathcal{S}\mathscr{K}$). Como aplicação, são fornecidos exemplos mostrando a existência de soluções estacionárias positivas de ($\mathscr{K}$) satisfazendo a condição de contorno de Neumann. Para superar a falta de coercitividade em ($\mathcal{S}\mathscr{K}$), combinamos técnicas de monotonicidade e truncamento, com a teoria da regularidade elíptica, a fim de construir problemas aproximados paramétricos de ($\mathcal{S}\mathscr{K}$) que são coercivos, e cujas soluções convergem, conforme o parâmetro tende a zero, para uma solução positiva de ($\mathcal{S}\mathscr{K}$).

Palavras-Chave: Kirchoff, Neumann, p-Laplace, homogêneo, não homogêneo, equações diferenciais parciais elípticas, não coercivo, métodos de sub-super soluções, soluções positivas.

Contents

Introduction

General hyperbolic equations with non-local coefficients of the form

$$\partial_t^2 u - \mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x, u, \partial_t u, \nabla u) \qquad (x,t) \in \Omega \times [0,T), \tag{$\mathscr{K}$}$$

supplied with initial and boundary conditions, where Δ_p is the p-Laplace, $p > 1$, $\Omega \subset \mathbb{R}^N$ is an open set, $N \geqslant 1$, and $\mathcal{M} \geqslant 0$ is a continuous function on $\Omega \times [0, \infty)$, can be seen as models for several problems studied in the literature. Many results concern the local case $\mathcal{M} \equiv 1$ with $p = 2$, with source and dissipation terms acting in Ω, with prescribed Dirichlet or flux boundary conditions. As a brief citation, Kirchhoff introduced in [59] the one dimensional equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\mathcal{P}_0}{\mathcal{H}} + \frac{\mathcal{E}}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where $\rho, \mathcal{P}_0, \mathcal{H}, \mathcal{E}$ and L are some specific constants, which describes transverse oscillations of a string by considering the effects of change of its length along the vibration. The first results about the well-posedness and global existence in higher dimensions seem to go back to Bernstein [17], Pohožaev [75], J.-L. Lions [68], Arosio & Panizzi [11], D'Ancona & Spagnolo [31], Gobbino [51], among others. For more recent work dealing with spatially homogeneous non-local terms $\mathcal{M}$, we refer to the papers of Ghisi & Gobbino [50], Nakao [73], Autuori, Pucci & Salvatori [13], Zhiji & Yunqing [84], Chueshov [28], see also Pucci & Rădulescu [76] and its references. The PDE ($\mathscr{K}$) with spatially inhomogeneous coefficients $\mathcal{M} = \mathcal{M}(x, \cdot)$ can be seen as a model describing the small vertical vibrations of an elastic string where the density of the material is not constant, as remarked in the works of J.-L. Lions [68], Límaco, Clark & Medeiros [67], and Figueiredo *et al.* [44]. The reader interested in applications or modeling aspects is referred to the monographs of Lasiecka & Triggiani [60] and Chueshov & Lasiecka [29], and many of the references therein.

A relevant issue as well is the asymptotic behavior of solutions of $(\mathscr{K})$, in particular, the existence of global attractors. In this setting, the stationary or equilibrium solutions play an important role concerning the long term dynamics of time dependent PDEs with some dissipation mechanism. The stationary solutions of the Kirchhoff equation $(\mathscr{K})$ satisfy an elliptic PDE of the form

$$-\mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x, u, \nabla u) \quad \text{in } \Omega. \tag{$\mathcal{S}\mathscr{K}$}$$

Results concerning the convergence to stationary solutions or the existence of global attractors for damped wave equations included in the model equation $(\mathscr{K})$, with $\mathcal{M} \equiv 1$ and $f = f(x, u, \partial_t u)$, can be found in the works of Lopes [71], Ghidaglia & Temam [49], Arrieta, Carvalho & Hale [12], Ball [15], among others. This topic has also been studied in Lasiecka & Triggiani [60] and Chueshov & Lasiecka [29], besides many references therein.

Another remarkable aspect regarding to hyperbolic equations which are included in the model $(\mathscr{K})$ is the source depending on first order terms. Actually, if $f = f(x, u, \partial_t u, \nabla u)$ additional difficulties impose on the stability issue, since it prevents to derive information about the influence of the integral $\int_\Omega f(x, u, \partial_t u, \nabla u)\partial_t u$ on the associated energy, involving the norm of $(u, \partial_t u)$, or the sign of its derivative. In this setting, the energy is not necessarily decreasing, and an important ingredient for obtaining the appropriate decay rates is not available. For related results about such not exhaustively explored class of problems in the local case $\mathcal{M} \equiv 1$ and $p = 2$ in $(\mathscr{K})$, we refer to the works of Cavalcanti, Lar'kin & Soriano [24], Guesmia [53], Liu & Chen [69], Zhang *et al.* [83], Aassila, Cavalcanti & Domingos Cavalcanti [1], Cavalcanti & Guesmia [25], and their references.

The effect of non-local terms on elliptic equations, in particular, the stationary solutions of $(\mathscr{K})$, have recently been studied in several works. Variational methods are the main tool which many authors have used to obtain existence or multiplicity results for spatially homogeneous coefficients $\mathcal{M}$ and source terms $f = f(x, u)$ in $(\mathcal{S}\mathscr{K})$. For examples of results in that direction, we refer to Pucci & Rădulescu [76] and most of the references quoted therein. In the case of coefficients $\mathcal{M} = \mathcal{M}(x, \cdot)$ in $(\mathcal{S}\mathscr{K})$ the literature is less extensive, and the related works, up to our knowledge, only deal with Dirichlet boundary condition. Such type of spatially inhomogeneous coefficients immerse $(\mathcal{S}\mathscr{K})$ in a non-variational setting no matter the right-hand side is independent on first order terms. Source terms $f = f(x)$ or $f = f(x, u)$ in $(\mathcal{S}\mathscr{K})$ have been studied, for instance, in the works of Chipot & Corrêa [26], Delgado *et al.* [36], Figueiredo *et al.* [44]. Singular sources have been considered in Santos, Santos & Mishra [78], whereas gradient dependent source terms in Alves & Corrêa [5] and Huy & Quan [54]. A common feature in most of the previous papers, with respect to inhomogeneous coefficients, is their results concern $\mathcal{M}(x,\tau) = a(x) + b(x)\tau^{\kappa}$, with $\kappa > 0$, and $a(\cdot)$, $b(\cdot)$ positive continuous functions over $\overline{\Omega}$.

It is well known that elliptic PDEs with sources depending on first order terms have been a topic of intensive research since some decades. Just to quote a few among the papers which have long been influential on the subject, we refer to the works of Serrin [79], Deuel & Hess [37], Brézis & Turner [23], Kazdan & Kramer [58], Amann & Crandall [6], Boccardo, Murat & Puel [22], among others. For more recent results, we quote the works of Arcoya *et al.* [8], De Figueiredo, Girardi & Matzeu [34], De Figueiredo *et al.* [35], Faria, Miyagaki & Motreanu [42], Figueiredo & Madeira [43], Papageorgiou, Rădulescu & Repovš [74], Ruiz [77], and their references. Regarding to the connections with applications, elliptic PDEs involving first order terms appear in stochastic control problems [61], Hamilton-Jacobi-Bellman equations [64], ergodic limits [61], stationary solutions in the Kardar–Parisi–Zhang model of growing interfaces [57], and many others.

Elliptic PDEs with natural growth in the gradient of the form

$$\begin{cases} -\Delta u = c(x)u + \mu(x)|\nabla u|^2 + h(x) & \text{in } \Omega, \\ \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

with coefficients satisfying certian sign or regularity conditions, have been studied by several authors. In the coercive case (i.e., $c(\cdot) \leqslant c_0 < 0$ in Ω), the existence of solution of (1) follows from the work of Boccardo, Murat & Puel [22] and its uniqueness from the results in Barles *et al.* [16]. The weakly coercive case (for instance, if $c \equiv 0$) has been treated by Abdellaoui, Dall'Aglio & Peral [3], and the so-called limit coercive case (where $c \leqslant 0$ a.e. in Ω) has been investigated in Arcoya *et al.* [8]. The non-coercive case ($c \gneqq 0$ or c changing sign) seems to be first considered by Jeanjean & Sirakov [56]. More recent improvements have been reached along the works of Arcoya *et al.* [8], de Coster & Fernández [32], de Coster, Fernández & Jeanjean [33], De Figueiredo *et al.* [35], Jeanjean & Ramos Quoirin [55], and Souplet [80], among others. In the case of Neumann boundary condition, the existence of a positive solution of (1) in some situations with the p-Laplace, $p > 1$, besides other related problems, is established below in Theorems 2.2, 2.3, 2.6.

Another class of elliptic equations with natural growth combines gradient and singular terms of the form

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^2}{u^\alpha} + h(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

Existence, uniqueness or nonexistence of solution of (2) for $\lambda \in \mathbb{R}$ and $\alpha > 0$, assuming sign and/or regularity conditions on the coefficients $\mu(\cdot)$ and $h(\cdot)$, have been considered, for instance, in the works of Boccardo [20], Giachetti & Murat [47], Arcoya *et al.* [7], Arcoya & Moreno-Mérida [9], Arcoya & Segura de Leon [10], Boccardo *et al.* [21]. Positive solutions of singular elliptic equations with other growth in the gradient and different structural hypotheses have been obtained in Faraci, Motreanu & Puglisi [40], Liu, Motreanu & Zeng [70], Figueiredo & Madeira [43], see also the references therein. Examples showing the existence of a positive solution of problems involving singular and gradient terms, but supplied with Neumann boundary condition, can be found below in Theorems 2.4, 2.5.

The literature on elliptic PDEs with gradient terms and Neumann boundary condition seems less extensive, but interest results have been proved. Actually, set the problem

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2}u + H(x, \nabla u) = 0 \quad \text{in } \Omega, \\ \mathrm{B}u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

where

$$\mathrm{B}u = \begin{cases} u & \text{(Dirichlet boundary condition)} \\ \partial_\nu u & \text{(Neumann boundary condition).} \end{cases}$$

For quasilinear operators including the case $p = 2$, and Hamiltonian $H(x, \cdot)$ convex, P.-L. Lions has proven in [63], among various results, that a solution of (3) with Dirichlet boundary conditions exists if, and only if, a $W^{1,\infty}$-subsolution exists. In such case, the solution is unique. This result, among others, have been generalized by P.-L. Lions in [65] by requiring less smoothness on the Hamiltonian, and many improvements along with related issues have been obtained by Lasry & P.-L. Lions in [61]. Several of those results have been extended to the quasilinear case $p \in (1, \infty)$ in (3) with Dirichlet boundary conditions by Leonori & Porretta in [62]. If the boundary condition in (3) is of Neumann type, then for $p = 2$ and $H(x, \nabla u) = \psi(\nabla u) - f(x)$, with ψ of class C^1 and $f \in W^{1,\infty}(\Omega)$, the existence of a unique solution of (3) has been proved in [63] for $\lambda > 0$ and Ω convex, a hypothesis removed in [65] allowing any regular bounded open set Ω. A similar result for

$p \in (1, \infty)$ and Neumann boundary condition has been obtained [62]. The main techniques adopted in [61, 62, 63, 65] rely on gradient estimates−in the spirit of previous works by Bernstein, Serrin, Lions, and Barles, see [62]−combined with the construction of suitable barriers and/or sub-supersolutions. Furthermore, one also emphasizes in [61, 62, 63, 65] the growth of the Hamiltonian $H(x, \cdot)$ in (3) may be like an arbitrary positive power.

The existence of solution of Neumann boundary value problems of the form

$$\begin{cases} \mathscr{L}u + \alpha(x)|u|^{p-2}u = g(x, u, \nabla u) \quad \text{in } \Omega, \\ \partial_\nu u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4}$$

where the coefficient $\alpha(\cdot)$ satisfies the coerciveness condition

$$\alpha \in L^\infty(\Omega), \text{ with } \alpha \geqslant 0 \text{ and } \alpha \not\equiv 0, \tag{5}$$

have been recently studied for some operators $\mathscr{L}$ and functions g. For instance, Gasiński & Papageorgiou have shown in [46] the existence of a positive solution of (4) for $\mathscr{L}u = -\mathrm{div}\,(a(u)\nabla u)$ and $p = 2$, where $0 < c_1 \leqslant a(\cdot) \leqslant c_2$ is a Lipschitz coefficient. Motreanu, Sciammetta, & Tornatore have proven in [72] a sub-supersolution theorem which is used to show the existence and multiplicity of positive solution of (4) with $\mathscr{L}u = -\mathrm{div}\,(A(x, \nabla u))$, including double phase operators like the (p, q)-Laplace and others. The existence of a positive solution of (4) with g having singular and convective terms has been shown by Papageorgiou, Rădulescu & Repovš in [74], see also their references. We also refer to the work of Zeng, Rădulescu & Winkert [82] on double phase multivalued obstacle problems with convection terms and mixed boundary conditions, and the references therein.

It is worth noticing that, to our knowledge, the only results on the existence of positive solutions of (4) with $\alpha \equiv 0$, i.e., in a lack of coerciveness setting, have been obtained by Guarnotta & Marano in [52] for a elliptic system where $\mathscr{L}u = -\Delta_p u$, $p \in (1, \infty)$. As a matter of fact, the proofs of the results in [46, 72, 74, 82] seem to strongly depend on a hypothesis like (5), which is not assumed in (2.1). Ultimately, the lack of coerciveness is the main challenge to be overcome in order to establish the existence of a positive solution of (2.1), and how to accomplish that is the main contribution of this thesis. The material herein is divided in two parts:

(1) In the first part, we deal with the Neumann problem

$$
\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x,u,\nabla u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}
$$

on bounded domains $\Omega \subset \mathbb{R}^N$, $N \geq 2$, $p \in (1,\infty)$, having smooth boundary $\partial\Omega$ with outward unit normal ν. The source term $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function (i.e., $f(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$, and $f(\cdot,s,\xi)$ is measurable for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$) satisfying

(H_f) $\quad |f(x,s,\xi)| \leqslant h(x,s)(1+|\xi|^q)$ for a.e. $x \in \Omega$, $\forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N$,

where $q \in [0,p]$ and $h : \Omega \times \mathbb{R} \to [0,\infty)$ is a Carathéodory function which is bounded on bounded sets of $\mathbb{R}$ uniformly with respect to the first variable. The coefficient $\mathcal{M}$ is a continuous function satisfying

$(H_\mathcal{M})$ There exist $m, M > 0$ such that $m \leqslant \mathcal{M}(x,s) \leqslant M$, $\quad \forall (x,s) \in \overline{\Omega} \times [0,\infty)$.

With these hypothesis we prove the following theorem in Chapter 2:

Theorem 0.1 *Assume (H_f) and $(H_\mathcal{M})$ hold. Further, suppose there exist $\overline{u}, \underline{u} \in W^{1,\infty}(\Omega)$ such that $0 \leqslant \underline{u} \leqslant \overline{u}$ a.e. in Ω, and satisfying the following conditions:*

(i) $f(x,\overline{u},\nabla\overline{u}) \leqslant 0 \leqslant f(x,\underline{u},\nabla\underline{u})$ *a.e. in* Ω.

(ii) $\displaystyle\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi\, dx \geqslant \frac{1}{M}\int_\Omega f(x,\overline{u},\nabla\overline{u})\varphi\, dx$, $\quad \forall \varphi \in W^{1,p}(\Omega)$, $\varphi \geqslant 0$ *a.e. in* Ω.

(iii) $\displaystyle\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla\varphi\, dx \leqslant \frac{1}{M}\int_\Omega f(x,\underline{u},\nabla\underline{u})\varphi\, dx$, $\quad \forall \varphi \in W^{1,p}(\Omega)$, $\varphi \geqslant 0$ *a.e. in* Ω.

Then there exists a solution $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, of (2.1) with $\underline{u} \leqslant u \leqslant \overline{u}$ a.e. in Ω.

Examples where Theorem 0.1 is applied to prove existence of a positive solution are discussed in Theorems 2.2-2.6

(2) In the second part, we deal with Neumann problems with non-linear boundary conditions of the form

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x,u,\nabla u) \quad \text{in } \Omega, \\ \mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)|\nabla u|^{p-2}\partial_\nu u = g(x,u) \quad \text{on } \partial\Omega, \end{cases} \tag{7}$$

and an interplay between f and g now takes place. The main result we prove is the following

Theorem 0.2 *Assume* (H_f) *and* $(H_{\mathcal{M}})$ *hold. Suppose there exist* $\bar{u}, \underline{u} \in W^{1,\infty}(\Omega)$ *such that* $0 \le \underline{u} \leqslant \bar{u}$ *a.e. in* Ω, *satisfying the following conditions:*

(i) $f(x,\underline{u},\nabla\underline{u}) \geqslant 0$ *and* $g(x,\underline{u}) \geqslant 0$, *a.e. in* Ω.

(ii) $\displaystyle\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla\varphi\, dx \leqslant \frac{1}{M}\int_\Omega f(x,\underline{u},\nabla\underline{u})\varphi\, dx + \frac{1}{M}\int_{\partial\Omega} g(x,\underline{u})\varphi d\mathcal{H}^{N-1}$,
$\forall\varphi \in W^{1,p}(\Omega)$, $\varphi \geqslant 0$ *a.e. in* Ω.

(iii) $\displaystyle\int_\Omega |\nabla\bar{u}|^{p-2}\nabla\bar{u}\nabla\varphi dx \geqslant \frac{1}{Q_1}\int_\Omega f(x,\bar{u},\nabla\bar{u})\varphi dx + \frac{1}{Q_2}\int_{\partial\Omega} g(x,\bar{u})\varphi d\mathcal{H}^{N-1}$,
$\forall\varphi \in W^{1,p}(\Omega)$, $\varphi \geqslant 0$ *a.e. in* Ω.

where

$$(Q_1, Q_2) = \begin{cases} (M,m), & \text{if } f(x,\bar{u},\nabla\bar{u}) \leqslant 0 \text{ and } g(x,\bar{u}) \geqslant 0, \\ (M,M), & \text{if } f(x,\bar{u},\nabla\bar{u}) \leqslant 0 \text{ and } g(x,\bar{u}) \leqslant 0, \\ (m,M), & \text{if } f(x,\bar{u},\nabla\bar{u}) \geqslant 0 \text{ and } g(x,\bar{u}) \leqslant 0. \end{cases} \tag{8}$$

Then there exists a solution $u \in C^{1,\gamma}(\overline{\Omega})$ *of* (3.1), $\gamma \in (0,1)$, *with* $0 \le \underline{u} \leqslant u \leqslant \bar{u}$ *a.e. in* Ω.

Examples where Theorem 0.2 is applied to prove existence of positive solutions are discussed in Theorems 3.2-3.6.

We remark that the requirement of $\mathcal{M}$ is bounded from above in $(H_{\mathcal{M}})$ is not essential. Indeed, to keep the method of proof as free as possible of technicalities we have introduced

$(H_{\mathcal{M}})$. At the end of each Chapter 2 and 3, we prove a version of Theorems 0.1 and 0.2 for coefficients $\mathcal{M}$ which may be unbounded from above.

Finally, the notation adopted along the work is standard. We only mention that $w_+ = \max(w(x), 0)$ and $w_- = \max(-w(x), 0)$ are the positive and negative parts of a function w, respectively, and " $\rightharpoonup$ " denotes the weak convergence of a sequence. Sometimes the symbol "dx" is omitted from some integrals, but $\int_A z$ or $\int_A z\,dx$ denote the same Lebesgue integral of a measurable function z over a Lebesgue measurable set $A \subset \mathbb{R}^N$. A constant "C" denotes a positive constant which may be different in a same line or line to line.

Chapter 1

Preliminaries

The purpose of this chapter is to establish the notation used throughout this work, stating several important classical results which will be also necessary. Unless specified, Ω is a nonempty open set in the n-dimensional Euclidean space $\mathbb{R}^N$, $N \geq 2$ and $1 < p < \infty$.

Definition 1.1 *Let $u, v \in L^1_{loc}(\Omega)$. A function v is the weak partial derivative $\partial_j u$ of u for $j = 1, 2, ..., n$, if*

$$\int_\Omega u \partial_j \varphi dx = -\int_\Omega v\varphi \quad \textit{for all functions} \quad \varphi \in \mathcal{C}_0^\infty(\Omega),$$

where $\mathcal{C}_0^\infty(\Omega)$ is the space of all infinitely differentiable functions with compact support in Ω. The functions in $\mathcal{C}_0^\infty(\Omega)$ are called test functions. If the weak partial derivative $\partial_j u$ exists, then it is uniquely defined up to a set of Lebesgue measure zero. It is worth noting that classical derivatives are always weak derivatives, but in general the converse is not true. The following example demonstrates this case.

Example 1.1 *Let $\Omega = (-1, 1)$. Let $u : \Omega \to \mathbb{R}$ be defined by $u(x) = |x|$. Integration by parts show that, for all $\varphi \in C_0^\infty(\Omega)$,*

$$\int_\Omega u(x)\varphi'(x)dx = -\int_\Omega v(x)\varphi(x)dx,$$

where

$$v(x) = \begin{cases} -1, & x < 0, \\ 0, \; x = 0, \\ 1, \; x > 0. \end{cases}$$

Hence $u(x)$ has a derivative on $(-1, 1)$ but is not differentiable at $x = 0$ in classical sense.

We use the notation $\nabla u = (\partial_1 u, \cdots, \partial_N u)$ to mean the vector whose coordinates are the weak partial derivatives of u.

Definition 1.2 ***(Sobolev Spaces)*** *The Sobolev space $W^{1,p}(\Omega)$ Consists of all functions $u \in L^p(\Omega)$ such that their distributional gradients ∇u exist and belong to $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is equipped with the norm*

$$\|u\|_{W^{1,p}(\Omega)} \doteq \left(\int_\Omega (|u|^p + |\nabla u|^p) dx \right)^{\frac{1}{p}}$$

Moreover, the Sobolev space $W_0^{1,p}(\Omega)$ of functions of $W^{1,p}(\Omega)$ with zero boundary values is the completition of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, while a function u is in $W_{loc}^{1,p}(\Omega)$ if and only if it belongs to $W^{1,p}(\Omega')$ for every subset $\Omega' \Subset \Omega$. As usual, $E \Subset \Omega$ means that the closure of E, written as $\overline{E}$, is a compact subset of Ω.

The Sobolev spaces $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ are Banach spaces.

Theorem 1.2 *If $\Omega \subset \mathbb{R}^N$ is a closed bounded domain with a C^1 boundary then $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$ in the norm $\|\cdot\|_{W^{1,p}(\Omega)}$.*

Proof See [18] p. 99. □

Proposition 1.1 *Suppose $\Omega \subset \mathbb{R}^N$ bounded. If f is λ-Hölder continuous in Ω, the it is Hölder continuous in Ω for every exponent $\mu < \lambda$*

Proof See [41] p. 3. □

Proposition 1.2 *Let the scalar-valued functions f,g be bounded, the function g being such that* $\inf |g| > 0$. *If f, g are λ-Hölder continuous then f/g is λ-Hölder continuous.*

Proof See [41] p. 15. □

Theorem 1.3 *(**Continuous Sobolev embedding**) Let $\Omega \subset \mathbb{R}^N$ be a domain satisfying the cone condition (i.e, if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C). Then the following embeddings are continuous*

(1) $W^{k+1,p}(\Omega) \hookrightarrow W^{1,q}(\Omega) \quad \forall 1 \leq q \leq \frac{Np}{N-kp}$, *with* $kp < N$

(2) $W^{k+1,p}(\Omega) \hookrightarrow W^{1,q}(\Omega) \quad \forall q \geq 1$, *with* $kp = N$

Proof See [18] p. 212-213. □

Theorem 1.4 *(**Compact Sobolev embedding**) The embeddings in Theorem 1.3 are compact for all $1 \leq q < \frac{Np}{(N-kp)}$. Moreover, if Ω is of class $C^{0.1}$ then following embeddings are also compact*

(1) $W^{k+1,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, *with* $kp > N$.

(2) $W^{k+1,p}(\Omega) \hookrightarrow C^{1,\theta}(\overline{\Omega})$, *where* $0 < \theta < k - \frac{N}{p}$, *with* $kp > N \geq (k-1)p$.

Proof See [18] p. 212-213. □

Theorem 1.5 *Let $\Omega \subset \mathbb{R}^N$ be a Lipshitz domain. Let $1 \leq p \leq N$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{N}$. Then $W^{1,p}(\Omega) \subset L^q(\Omega)$,i.e. the identity mapping from $W^{1,p}(\Omega)$ to L^q is bounded.*

Proof See [18] p. 213-214. □

Lemma 1.1 *Let $\underline{u}, \bar{u} \in W^{1,p}(\Omega)$ satisfying $\underline{u} \leqslant \bar{u}$, and let T be the truncation operator defined by*

$$Tu(x) = \begin{cases} \bar{u}(x) & if \quad u(x) \geqslant \bar{u}(x), \\ u(x) & if \quad \underline{u}(x) \leqslant u(x) \leqslant \bar{u}(x), \\ \underline{u}(x) & if \quad u(x) < \underline{u}(x). \end{cases}$$

for all $u \in W^{1,p}(\Omega)$. Then T is a bounded continous mapping from $W^{1,p}(\Omega)$(respectively, $L^p(\Omega)$) into itself.

Proof See [37]. □

Lemma 1.2 *Let Ω be a $C^{k,\alpha}$ domain in R^N(with $k \geq 1$) and let S be a bounded set in $C^{k,\alpha}(\overline{\Omega})$. Then S is precompact in $C^{j,\beta}(\overline{\Omega})$ if $j + \beta < k + \alpha$.*

Proof See [48] p. 136. $\square$

Theorem 1.6 ***(Chain rule)*** *Let $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$. If $u \in W^{1,p}(\Omega)$ with $1 \le p < \infty$ then $f \circ u \in W^{1,p}(\Omega)$, and*

$$\nabla(f \circ u) = f'(u)\nabla u.$$

Proof See [18] p. 215. $\square$

Then if we set now $u_+ \doteq \max\{u, 0\}$ and $u_- \doteq \max\{-u, 0\}$ we can set the following corollaries:

Corollary 1.7 *If $u \in W^{1,p}(\Omega)$, with $1 \le p < \infty$ then u_+, u_- and $|u| \in W^{1,p}(\Omega)$ where $|u| = u_+ + u_-$. Further, if we set*

$$\{u > 0\} \doteq \{x \in \textit{supp } u : u(x) > 0\}$$

and

$$\{u < 0\} \doteq \{x \in \textit{supp } u : u(x) < 0\}$$

we have

$$\nabla u_+ = \chi_{\{u>0\}}\nabla u \quad \textit{and} \quad \nabla u_- = \chi_{\{u<0\}}\nabla u.$$

Proof See [18] p. 216. $\square$

Corollary 1.8 *If $u, v \in W^{1,p}(\Omega)$ with $1 \le p < \infty$, then $\max\{u, v\}$ and $\min\{u, v\} \in W^{1,p}(\Omega)$.*

Proof See [18] p. 216. $\square$

Corollary 1.9 *If (u_j), $(v_j) \subset W^{1,p}(\Omega)$ $(1 \leqslant p < \infty)$ are such that $u_j \to u$ and $v_j \to v$, then $\min\{u_j, v_j\} \to \min\{u, v\}$ and $\max\{u_j, v_j\} \to \max\{u, v\}$ in $W^{1,p}(\Omega)$, as $j \to \infty$.*

Proof See [18] p. 217. $\square$

Theorem 1.10 ***(Hölder's inequality)***. *Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leqslant p \leqslant \infty$ and $1/p + 1/q = 1$. Then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg| \leqslant \|f\|_p \|g\|_q.$$

Proof See [18] p. 92. □

Theorem 1.11 ***(Young inequality)*** *Let $a, b \geqslant 0$ be real numbers and p, q real numbers with $1/p + 1/q = 1$, Then*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof See [18] p. 92. □

Corollary 1.12 *Let $a, b \geqslant 0$ be real numbers. Then*

$$ab \leqslant \varepsilon a^2 + \frac{b^2}{4\varepsilon},$$

for all $\varepsilon > 0$.

Proof See [18] p. 92. □

Theorem 1.13 *Let (f_n) be a sequence in $L^p(\Omega)$ and let $f \in L^p(\Omega)$ be such that $\|f_n - f\|_p \to 0$. Then there exist a subsequence (f_{n_k}), and a function $h \in L^p(\Omega)$, such that*

(a) $f_{n_k}(x) \to f(x)$ a.e in Ω.

(b) $|f_{n_k}(x)| \leqslant h(x)$, $\forall k$, a.e in Ω.

Proof See [18] p. 94. □

Theorem 1.14 *$L^p(\Omega)$ is reflexive for $p \in (1, \infty)$.*

Proof See [18] p. 95. □

Theorem 1.15 *The space $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for all $1 \leqslant p < \infty$.*

Proof See [18] p. 97. □

Theorem 1.16 *Assume that Ω is a separable measure space. Then $L^p(\Omega)$ is separable for any $1 \leqslant p < \infty$.*

Proof See [18] p. 98. □

Proposition 1.3 *$W^{1,p}(\Omega)$ is a Banach space for every $1 \leqslant p \leqslant \infty$. $W^{1,p}(\Omega)$ is reflexive for $1 < p < \infty$, and it is separable for $1 \leqslant p < \infty$.*

Proof See [18] p. 203. □

Theorem 1.17 ***(Dominated Convergence Theorem)*** *Let (f_n) be a sequence of functions in $L^1(\Omega)$ satisfying*

(a) $f_n(x) \to f(x)$ a.e in Ω.

(b) there is a function $g \in L^1(\Omega)$ such that for all n, $|f_n(x)| \leqslant g(x)$ a.e in Ω.

Then $f \in L^1(\Omega)$, and $\|f_n - f\|_1 \to 0$.

Proof See [18] p. 90. □

Theorem 1.18 ***(Vazquez Maximum Principle)*** *Let $u \in L^1_{loc}(\Omega)$ be such that*

- *$\Delta_p u \in L^1_{loc}(\Omega)$ in the sense of distributions in Ω;*
- *$u \geqslant 0$ a.e in Ω;*
- *$\Delta_p u \leqslant \beta(u)$ a.e in $\{x \in \Omega \ : \ 0 < u(x) < c\}$,*

where c is a positive constant and $\beta : [0, c] \to \mathbb{R}$ is a continuous non decreasing function with $\beta(0) = 0$. Under the assumption that $\beta(S) = 0$ for some $S > 0$ or

$$\int_0^1 (\beta(S)S)^{-\frac{1}{p}} dS = \infty$$

if $\beta(S) > 0$ for $S > 0$, then either $u \equiv 0$ a.e in Ω or u is strictly positive in Ω in the sense that for every compact $K \subset \Omega$ there is a constant $C(K) > 0$ such that $u \geqslant C(K)$ a.e in K. In particular, if u vanishes a.e in a set of positive measure then it must vanish a.e in Ω.

Proof See [81]. □

Lemma 1.3 ***(Hopf's Lemma)*** *Let $u \in C^1(\Omega)$ be such that $\Delta_p u \in L^2_{loc}(\Omega)$, $u \geqslant 0$ a.e in Ω, $\Delta_p u \leqslant \beta(u)$ a.e in Ω with $\beta : [0, \infty) \to \mathbb{R}$ continuous, non-decreasing , $\beta(0) = 0$ and either $\beta(s) = 0$ for some $s > 0$ or $\beta(s) > 0$ for all $s > 0$ with*

$$\int_0^1 (\beta(S)S)^{-\frac{1}{p}} dS = \infty$$

Then if u does not vanish identically on Ω, it is positive everywhere in Ω.

Moreover, if $u \in C^1(\Omega \cup \{x_0\})$ for some $x_0 \in \partial\Omega$ that satisfies an interior sphere condition and $u(x_0) = 0$, then

$$\frac{\partial u(x_0)}{\partial \nu} > 0 \tag{1.1}$$

where ν is the interior normal vector at x_0.

Proof See [81]. □

Definition 1.3 *Let E be a Banach space and $B : E \to E^*$ an operator. We say that B is pseudomonotone if $u_n \rightharpoonup u$ in E and*

$$\limsup_{n\to\infty} \langle Bu_n, u_n - u\rangle \leq 0, \tag{1.2}$$

then,

$$\liminf_{n\to\infty} \langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle, \quad \forall\, v \in E. \tag{1.3}$$

Where $\langle\cdot,\cdot\rangle$ denotes the duality between E^* and E.

Definition 1.4 *A function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function if*

- *$f(\cdot, s)$ is measurable for each $s \in \mathbb{R}$ fixed.*
- *$f(x, \cdot)$ is continuous in $\mathbb{R}$ for a.e $x \in \Omega$ fixed.*

Theorem 1.19 ***(Minty-Browder)*** *Let E be a reflexive and separable Banach space and $B : E \to E^*$ an operator satisfying*

(i) B is coercive, i.e, $\lim_{\|u\|\to\infty} \frac{\langle Bu,u\rangle}{\|u\|} = +\infty$;

(ii) B is bounded (i.e, B transforms bounded sets in E into bounded sets in E^*);

(iii) B is pseudomonotone.

Then, B is surjective, that is, $B(E) = E^*$.

Proof See [19], Theorem 5.5. □

Lemma 1.4 *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two bounded real sequences. If $\lim_{j\to\infty} b_j = b \in \mathbb{R}$ then*

$$\liminf_{j\to\infty}(a_j + b_j) = \liminf_{j\to\infty} a_j + b \quad \text{and} \quad \limsup_{j\to\infty}(a_j + b_j) = \limsup_{j\to\infty} a_j + b$$

Lemma 1.5 ***(Tartar Inequality)*** *If $\xi, \eta \in \mathbb{R}$ then*

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \geq \begin{cases} C|\xi-\eta|^p & \text{if } \ p \geq 2 \\ \tilde{C}\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}} & \text{if } \ 1 < p < 2. \end{cases}$$

Proof See [27] p. 235. □

Theorem 1.20 ***(Poincaré-Wirtinger inequality)*** *Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$ be a bounded connected open set with a Lipschitz boundary. Then there exists a constant C, depending only on Ω and p, such that for every function $u \in W^{1,p}(\Omega)$ one has*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leqslant C\|\nabla u\|_{L^p(\Omega)},$$

where

$$u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(s)ds$$

is the average value of u over Ω, and $|\Omega|$ stands for the Lebesgue measure of Ω.

Proof See [38] p. 265. □

1.1 A regularity result

Consider the elliptic equation in divergence form

$$-\text{div } A(x, u, \nabla u) = B(x, u, \nabla u) \ \text{ in } \Omega, \tag{1.4}$$

with the Neumann boundary condition

$$A(x, u, \nabla u)\nu = h(x, u) \ \text{ on } \partial\Omega. \tag{1.5}$$

The following assumptions will be assumed.

Assumption (A^k): Let $A = (A_1, A_2, \dots A_N) \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$. For every $(x, u) \in \overline{\Omega} \times \mathbb{R}$, $A(x, u, \cdot) \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$, and there exist a nonnegative constant $k \geq 0$, a non-increasing continuous function $\lambda : [0, \infty) \to (0, \infty)$, and a non-decreasing continuous function $\Lambda : [0, \infty) \to (0, \infty)$ such that for all $x, x_1, x_2 \in \overline{\Omega}$, $u, u_1, u_2 \in \mathbb{R}$, $\eta \in \mathbb{R}^N \setminus \{0\}$, and $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N$, the following conditions are satisfied:

- $A(x, u, 0) = 0$,
- $\sum_{i,j} \frac{\partial A_j}{\partial \eta_i}(x, u, \eta)\xi_i \xi j \geqslant \lambda(|u|)(k + |\eta|^2)^{\frac{p-2}{2}} |\xi|^2$,
- $\sum_{i,j} \left| \frac{\partial A_j(x, u, \eta)}{\partial \eta_i} \right| \leqslant \Lambda(|u|)(k + |\eta|^2)^{\frac{p-2}{2}}$,
- $|A(x_1, u_1, \eta) - A(x_2, u_2, \eta)| \leq \Lambda(\max\{|u_1|, |u_2|\})(|x_1 - x_2|^{\beta_1} + |u_1 - u_2|^{\beta_2})$
  $\times [(k + |\eta|^2)^{\frac{p-2}{2}} + (k + |\eta|^2)^{\frac{p-2}{2}}]|\eta|(1 + |\log(k + |\eta|^2)|$.

A typical example of the function A satisfying the assumption (A^k) is

$$A(x, u, \xi) = a(x, u)(k + |\xi|^2)^{\frac{p-2}{2}} \xi,$$

where $a(x, u)$ is Hölder continuous in (x, u) and $a(x, u) \geqslant \delta > 0$.

Assumption (B): $B : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, where $B(x, u, \eta)$ is measurable in x and continuous in (u, η), and

$$|B(x, u, \eta)| \leq \Lambda(|u|)(1 + |\eta|^p), \quad \forall \ (x, u, \eta) \ \in \ \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N. \tag{1.6}$$

Definition 1.5 *$u \in W^{1,p}(\Omega)$, is called a bounded generalized solution of the boundary value problem* (1.4)-(1.5) *if $u \in L^\infty(\Omega)$ and*

$$\int_\Omega A(x, u, \nabla u)\nabla \varphi dx = \int_\Omega B(x, u, \nabla u)\varphi dx + \int_{\partial\Omega} h(x, u)\varphi ds \tag{1.7}$$

$\forall \varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Assumption (M): Suppose that Assumptions (A^k) and (B) are satisfied. There exists a positive constant M such that for a bounded generalized solution u holds

$$\operatorname*{ess\,sup}_\Omega |u(x)| \leq M. \tag{1.8}$$

Under the previous conditions, the following result holds

Theorem 1.1 ***(Fan, Lieberman)*** *Assume Assumptions (A^k), (B) and (M) hold, and let the boundary $\partial\Omega$ of Ω be of class $C^{1,\gamma}$. Suppose $h \in C(\partial\Omega \times \mathbb{R}, \mathbb{R})$ satisfying for*

$x_1, x_2 \in \partial\Omega,\ u_1, u_2 \in \mathbb{R}$,

$$|h(x_1, u_1) - h(x_2, u_2)| \leq \Lambda(\max\{|u_1|, |u_2|\}) \left(|x_1 - x_2|^{\beta_1} + |u_1 - u_2|^{\beta_2}\right), \tag{1.9}$$

where Λ *is as in Assumption* (A^k)*. If* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *is a bounded generalized solution of the boundary value problem* (1.4)-(1.5)*, then* $u \in C^{1,\alpha}(\overline{\Omega})$*, where* α *and* $|u|_{C^{1,\alpha}(\overline{\Omega})}$ *depend only on* $p, N, \Lambda(K), K, \beta_1, \beta_2, \gamma, \sup|h(\partial\Omega \times [-M, M])|$*, and* Ω*.*

Proof See [39, 66]. □

Remark 1.1 *(i) If* $A(x, u, \nabla u) = |\nabla u|^{p-2}\nabla u$*, i.e., if the operator in* (1.4) *is the* p*-Laplace, then assumption* (A^k) *is automatically satisfied.*

(ii) Theorem 1.1 holds for variable exponents $p = p(x)$ *under appropriate conditions.*

Chapter 2

Positive stationary solutions of Kirchhoff equations with first order terms and lack of coerciveness: homogeneous Neumann boundary condition

This chapter is addressed to the existence of positive solutions of $(\mathcal{SK})$ under Neumann boundary conditions. More precisely, we consider non-variational elliptic PDEs with nonlocal terms of the form

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x,u,\nabla u) \quad \text{in } \Omega, \\ \qquad\qquad |\nabla u|^{p-2}\partial_\nu u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

on bounded domains $\Omega \subset \mathbb{R}^N$, $N \geq 2$, $p \in (1,\infty)$, having smooth boundary $\partial\Omega$ with outward unit normal ν. The source term $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function (i.e., $f(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$, and $f(\cdot,s,\xi)$ is measurable for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$) satisfying

(H_f) $\qquad |f(x,s,\xi)| \leqslant h(x,s)(1+|\xi|^q)$ for a.e. $x \in \Omega$, $\forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N$,

where $q \in [0,p]$ and $h : \Omega \times \mathbb{R} \to [0,\infty)$ is a Carathéodory function which is bounded on bounded sets of $\mathbb{R}$ uniformly with respect to the first variable. The coefficient $\mathcal{M}$ is a continuous function satisfying

$(H_{\mathcal{M}})$ There exist $m, M > 0$ such that $m \leqslant \mathcal{M}(x,s) \leqslant M, \quad \forall (x,s) \in \Omega \times [0,\infty)$.

We seek for a positive solution of (2.1) in the following sense.

Definition 2.1 *Suppose* (H_f) *and* $(H_{\mathcal{M}})$ *hold. A function* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *is called a solution of* (2.1) *if satisfies*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_\Omega \frac{f(x,u,\nabla u)}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)} \varphi \, dx, \qquad \forall \varphi \in W^{1,p}(\Omega). \tag{2.2}$$

The main result of this chapter on the existence of a positive solution of (2.1) reads as follows.

Theorem 2.1 *Assume* (H_f) *and* $(H_{\mathcal{M}})$ *hold. Further, suppose there exist* $\overline{u}, \underline{u} \in W^{1,\infty}(\Omega)$ *such that* $0 \leqslant \underline{u} \leqslant \overline{u}$ *a.e. in* Ω, *and satisfying the following conditions:*

(i) $f(x,\overline{u},\nabla\overline{u}) \leqslant 0 \leqslant f(x,\underline{u},\nabla\underline{u})$ *a.e. in* Ω.

(ii) $\displaystyle\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi \, dx \geqslant \frac{1}{M} \int_\Omega f(x,\overline{u},\nabla\overline{u}) \varphi \, dx, \quad \forall \varphi \in W^{1,p}(\Omega), \ \varphi \geqslant 0$ *a.e. in* Ω.

(iii) $\displaystyle\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi \, dx \leqslant \frac{1}{M} \int_\Omega f(x,\underline{u},\nabla\underline{u}) \varphi \, dx, \quad \forall \varphi \in W^{1,p}(\Omega), \ \varphi \geqslant 0$ *a.e. in* Ω.

Then there exists a solution $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, *of* (2.1) *with* $\underline{u} \leqslant u \leqslant \overline{u}$ *a.e. in* Ω.

Remark 2.1 (i) Theorem 2.1 can be seen as a sub-supersolution principle for the inhomogeneous non-local Neumann problem (2.1), which recovers the local case if $\mathcal{M} \equiv 1$. It is known that comparison and sub-supersolution principles do not hold in general for stationary Kirchhoff equations, if formulated the same way as in the local case, unless very specific conditions are fulfilled. This is discussed for Dirichlet problems in Figueiredo & Suárez [45]. Still under Dirichlet boundary condition, some versions of sub-supersolution principles for $(\mathcal{SK})$ with $p = 2$ can be found in [4, 26, 45] for homogeneous coefficients $\mathcal{M}$ and sources $f = f(x,u)$. The authors in [5] consider $p = 2$ and sources $f \geqslant 0$ depending on first order terms, but assuming the existence of a family of small functions (in a sense) playing the role of subsolutions.

(ii) Theorem 2.1 seems to be new in the local case $\mathcal{M} \equiv 1$, i.e., if the operator in (2.1) is the p-Laplace for all $p \in (1, \infty)$. Furthermore, in the case of spatially inhomogeneous coefficients $\mathcal{M} = \mathcal{M}(x, \cdot)$ and Neumann boundary condition on $(\mathcal{S}\mathcal{K})$, it seems also new for sources $f = f(x, u)$ not depending on first order terms.

(iii) The assumption on the coefficient $\mathcal{M}$ to be bounded from above is not essential in Theorem 2.1. Actually, in order to keep the main ideas in evidence, we have stated and proven Theorem 2.1 assuming $(H_{\mathcal{M}})$. An extension to the case of unbounded from above coefficients $\mathcal{M}$ in (2.1) will be discussed below in Section 2.4.

Regarding to the proof of Theorem 2.1, to overcome the lack of coerciveness due to the absence of a lower order term like (5) in the equation, besides the homogeneous Neumann boundary condition, we proceed as follows. We introduce parametric ϵ-approximate problems of (2.1) which are coercive. Combining monotonicity methods, truncation techniques and cutoff functions, similarly as in [37, 63, 5], we obtain $W^{1,p}(\Omega)$-solutions of the ϵ-approximate problems. After providing uniform L^∞-estimates on the ϵ-approximate solutions, which in turn lead to $C^{1,\gamma}(\overline{\Omega})$-estimates uniformly on ϵ, it is possible to pass to the limit on the ϵ_j-approximate problems for a sequence $\epsilon_j \to 0$, as $j \to \infty$. This limiting problem recovers the originally truncated problem, giving rise to a solution of (2.1).

Some examples establishing the existence of a positive solution of (2.1) are given in Section 2.3, where the following source terms, including non-Lipschitz cases, are considered:

- $f(x, u, \nabla u) = c(x)u^{p-1} - u^s + a(x)|\nabla u|^q - g(x)$, with $s \in (p-1, \infty)$, $q \in [0, p]$, $g \leqslant 0$;

  (*semipositone gradient dependent sources*)

- $f(x, u, \nabla u) = c(x)u^r - u^s + g(x, u)|\nabla u|^q$, with $0 < r < s$, $q \in [0, p]$, $g(x, \cdot)$ is continuous;

  (*sources having gradient terms with continuous coefficients*)

- $f(x, u, \nabla u) = c(x)u^m - u^s + a(x)\dfrac{|\nabla u|^q}{u^\alpha}$, with $0 < m < s$, $\alpha > 0$, $q \in [0, p]$;

  (*sources having gradient terms with singular coefficients*)

- $f(x, u, \nabla u) = \dfrac{1}{u^\theta} + a(x)\dfrac{|\nabla u|^q}{u^\beta} - c(x)u^r$, with $r, \beta, \theta > 0$, $q \in [0, p]$;

  (*sources combining singular and gradient terms with singular coefficients*)

- $f(x,u,\nabla u) = g(x) - \lambda |u|^{p-2}u - b(x)\psi\left(\nabla u\right)$, with $\psi(\cdot) \approx |\cdot|^q$ at infinity, $q \in [0,p]$;

  (*sources arising in stochastic control problems*)

where $a, b, c, g \in L^\infty(\Omega)$, which may be indefinite sign coefficients in some cases. The precise statements and proofs are contained in Section 2.3, see Theorems 2.2−2.6.

Let us briefly describe how the chapter is organized. In Section 2.1, we introduce ϵ-approximate problems of (2.1) which are coercive, and set up the limiting problem, as $\epsilon_j \to 0$. Theorem 2.1 is proved in Section 2.2. Positive solutions of (2.1) are constructed in Section 2.3 through examples using the source terms above described. In Section 2.4, an extension of Theorem 2.1 for coefficients $\mathcal{M}$ which are unbounded from above is established.

Finally, the notation adopted along the paper is standard. We only mention that $w_+ = \max(w(x), 0)$ and $w_- = \max(-w(x), 0)$ are the positive and negative parts of a function w, respectively, and " $\rightharpoonup$ " denotes the weak convergence of a sequence. Sometimes the symbol "dx" is omitted from some integrals, but $\int_A z$ or $\int_A z\,dx$ denote the same Lebesgue integral of a measurable function z over a Lebesgue measurable set $A \subset \mathbb{R}^N$. A constant "C" denotes a positive constant which may be different in a same line or line to line.

2.1 Approximate coercive problems

Let us choose a fixed $R_0 > 0$ satisfying

$$\max\left(\|\overline{u}\|_{1,\infty}, \|\underline{u}\|_{1,\infty}\right) \leqslant R_0, \tag{2.3}$$

where $\|\cdot\|_{1,\infty}$ denotes the usual norm in $W^{1,\infty}(\Omega)$. Let $T_R(\tau) = \max(-R, \min(\tau, R))$, $\forall \tau \in \mathbb{R}$, be the truncation function for $R \geqslant R_0$, and define the truncated function

$$f_R(x,\tau,\xi) \stackrel{\text{def}}{=} f(x,\tau,T_R(\xi_1),\cdots,T_R(\xi_N)), \tag{2.4}$$

for a.e. $x \in \Omega$, for all $(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^N$, where $\xi = (\xi_1,\cdots,\xi_N) \in \mathbb{R}^N$. Some properties of f_R which will be used later on are described in the following lemma.

Lemma 2.1 *Under the hypotheses in Theorem* 2.1 *one has for all* $R \geqslant R_0$:

- $f_R(\cdot,\cdot,\xi) = f(\cdot,\cdot,\xi)$ *if* $|\xi| \leqslant R$.

- $f_R(\cdot,\cdot,\cdot) \leqslant h(\cdot,\cdot)\big(1+N^q R^q\big)$.

- $f_R(\cdot,\underline{u},\nabla\underline{u}) = f(\cdot,\underline{u},\nabla\underline{u})$ *and* $f_R(\cdot,\overline{u},\nabla\overline{u}) = f(\cdot,\overline{u},\nabla\overline{u})$ *a.e in* Ω.

- $|f_R(\cdot,\cdot,\xi)| \leqslant h(\cdot,\cdot)(1+|\xi|^q), \;\; \forall \xi \in \mathbb{R}^N$.

Proof Most of the cases are straightforward from (2.3) and (2.4). We observe that

$$|T_R(s)| \leqslant |s|, \forall\ s \in \mathbb{R}, \tag{2.5}$$

so one has

$$|f_R(x,s,\xi)| \leqslant h(x,s)\left[1+\left(\sum_{i=1}^{N}|T_R(\xi_i)|^2\right)^{\frac{q}{2}}\right] \leqslant h(x,s)\left[1+\left(\sum_{i=1}^{N}|(\xi_i)|^2\right)^{\frac{q}{2}}\right] \leqslant h(x,s)(1+|\xi|^q).$$

The proof is complete. □

2.1.1 Coercive parametric ϵ-approximate problem

Tacking into account the lack of coerciveness on the Neumann problem (2.1), we introduce the parametric ϵ-approximate problem

$$\begin{cases} -\Delta_p u_\epsilon + \epsilon\,|u_\epsilon|^{p-2}u_\epsilon \;=\; F_R(u_\epsilon) - \Upsilon(u_\epsilon) \quad \text{in } \Omega, \\ |\nabla u_\epsilon|^{p-2}\partial_\nu u_\epsilon \;=\; 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

for all $\epsilon > 0$, where $F_R : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$, is given by

$$\langle F_R(u), v\rangle = \int_\Omega \frac{f_R(x, Tu, \nabla Tu)}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)}\, v\, dx, \quad \forall v \in W^{1,p}(\Omega), \tag{2.7}$$

using the truncation operator

$$Tu(x) = \begin{cases} \bar{u}(x) \;\; if \;\; u(x) \geqslant \bar{u}(x), \\ u(x) \;\; if \;\; \underline{u}(x) \leqslant u(x) \leqslant \bar{u}(x), \\ \underline{u}(x) \;\; if \;\; u(x) \leqslant \underline{u}(x), \end{cases} \tag{2.8}$$

and $\Upsilon : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is given by

$$\langle \Upsilon(u), v\rangle = \int_\Omega \frac{\upsilon(x,u)}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)}\, v\, dx, \quad \forall v \in W^{1,p}(\Omega), \tag{2.9}$$

where $\upsilon(\cdot, u) = -(\underline{u} - u)_+^\ell + (u - \bar{u})_+^\ell$, with $\ell \in (0, p-1)$ fixed.
To prove that problem (2.6) has a solution, let $\mathscr{B}_\epsilon : W^{1,p}(\Omega) \to \big(W^{1,p}(\Omega)\big)^*$ be given by

$$\langle \mathscr{B}_\epsilon u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx \, + \epsilon \int_\Omega |u|^{p-2} uv\, dx \, - \langle F_R(u), v\rangle + \langle \Upsilon(u), v\rangle. \tag{2.10}$$

for $u, v \in W^{1,p}(\Omega)$. We shall apply the surjectivity theorem for pseudo-monotone coercive operators (see [19], Theorem 5.5) to obtain a solution $u_\epsilon \in W^{1,p}(\Omega)$ of (2.6) for all $\epsilon > 0$.

Lemma 2.2 *Assume $(H_{\mathcal{M}})$ holds. Then $\mathscr{B}_\epsilon$ is bounded and coercive for all $\epsilon > 0$.*

Proof From direct calculations using $W^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$ (see Theorem 1.3)

$$|\langle F_R(u), u\rangle| \leqslant C\|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{2.11}$$

Further, there exist constants $C_1, C_2 > 0$ satisfying

$$\begin{aligned} |\langle \Upsilon(u), u\rangle| &\leqslant \int_\Omega \frac{|\upsilon(x,u)u|}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)}\, dx \leqslant \int_\Omega \frac{C_1}{m}|u| dx + \int_\Omega \frac{C_2}{m}|u|^{\ell+1} dx \\ &\leqslant C\|u\|_{W^{1,p}(\Omega)} + C\|u\|^{\ell+1}_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \end{aligned} \tag{2.12}$$

Using previous estimates and (2.10), for some constant $C > 0$ we have

$$\langle \mathscr{B}_\epsilon u, u\rangle \geqslant \|u\|^p_{1,p,\epsilon} - C\|u\|_{W^{1,p}(\Omega)} - C\|u\|^{\ell+1}_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$, where $\|\cdot\|_{1,p,\epsilon}$ denotes the norm

$$\|u\|^p_{1,p,\epsilon} = \int_\Omega |\nabla u|^p\, dx + \epsilon \int_\Omega |u|^p\, dx, \tag{2.13}$$

which is equivalent to the usual norm $\|\cdot\|_{W^{1,p}(\Omega)}$. Since $\ell \in (0, p-1)$, one has

$$\langle \mathscr{B}_\epsilon u, u\rangle \geqslant C_\epsilon \|u\|^p_{W^{1,p}(\Omega)} - C\|u\|_{W^{1,p}(\Omega)} - C\|u\|^{\ell+1}_{W^{1,p}(\Omega)} \longrightarrow \infty,$$

as $\|u\|_{W^{1,p}(\Omega)} \to \infty$. This shows $\mathscr{B}_\epsilon$ is coercive for all $\epsilon > 0$. It is not difficult to infer from $(H_{\mathcal{M}})$ and the definitions of F_R and Υ that $\mathscr{B}_\epsilon$ is bounded (i.e., transforms bounded sets into bounded sets) for all $\epsilon > 0$. The proof is complete. $\square$

Lemma 2.3 *Assume $(H_{\mathcal{M}})$ holds. Then $\mathscr{B}_\epsilon$ is a pseudo-monotone operator for all $\epsilon > 0$.*

Proof Let $(u_j) \subset W^{1,p}(\Omega)$ satisfying $u_j \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\limsup_{j\to\infty} \langle \mathscr{B}_\epsilon u_j, u_j - u\rangle \leqslant 0$. One needs to show that

$$\liminf_{j\to\infty} \langle \mathscr{B}_\epsilon u_j, u_j - v\rangle \geqslant \langle \mathscr{B}_\epsilon u, u - v\rangle \qquad \forall v \in W^{1,p}(\Omega). \tag{2.14}$$

Passing to a not relabeled subsequence satisfying $u_j \to u$ in $L^p(\Omega)$, from $(H_{\mathcal{M}})$ and the definitions of F_R and Υ, it follows that

$$\left| \int_\Omega \frac{1}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_j|^p dx\Big)} f_R(x, Tu_j, \nabla T u_j)(u_j - u)dx \right| \leqslant C \|u_j - u\|_{L^1(\Omega)} \overset{j\to\infty}{\longrightarrow} 0,$$

$$\left| \int_\Omega \frac{1}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_j|^p dx\Big)} \upsilon(x, u_j)(u_j - u) \right| \leqslant C \|\upsilon(\cdot, u_j)\|_{L^{\frac{\ell+1}{\ell}}(\Omega)} \|u_j - u\|_{L^{\ell+1}(\Omega)} \overset{j\to\infty}{\longrightarrow} 0.$$

Collecting the information above we have

$$\begin{aligned}\limsup_{j\to\infty} \langle \mathscr{B}_\epsilon u_j, u_j - u\rangle &= \limsup_{j\to\infty} \int_\Omega |\nabla u_j|^{p-2}\nabla u_j \nabla(u_j - u)dx \\ &= \limsup_{j\to\infty} \int_\Omega \big(|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u\big)\nabla(u_j - u)dx \geqslant 0.\end{aligned}$$

Thus we obtain

$$\int_\Omega \big(|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u\big)\nabla(u_j - u)dx \overset{j\to\infty}{\longrightarrow} 0,$$

so (u_j) strongly converges to u in $W^{1,p}(\Omega)$ (see [19]). Taking into account that $\mathscr{B}_\epsilon$ is continuous, we conclude that (2.14) follows. $\square$

Lemma 2.4 *Assume $(H_{\mathcal{M}})$ holds. Then (2.6) has a solution $u_\epsilon \in W^{1,p}(\Omega)$ for all $\epsilon > 0$.*

Proof From Lemmas 3.1−3.2, all hypotheses of the surjectivity theorem in [19], Theorem 5.5, are fulfilled. Hence $\mathscr{B}_\epsilon$ is surjective, and there exists $u_\epsilon \in W^{1,p}(\Omega)$ satisfying $\mathscr{B}_\epsilon u_\epsilon = 0$ for all $\varepsilon > 0$, i.e., u_ϵ is a solution of (2.6) in accordance with Definition 2.1. $\square$

2.1.2 Passing to the limit in (2.6)

We are now in position to pass to the limit in the coercive parametric problem (2.6) as a final stage before proving Theorem 2.1.

Lemma 2.5 *Assume the hypotheses in Theorem* 2.1 *hold. Then for all sequence* $\epsilon_j \to 0$, *as* $j \to \infty$, *the corresponding sequence* (u_{ϵ_j}) *of solutions of* (2.6) *strongly converges in* $W^{1,p}(\Omega)$ *to a solution* $u_R \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *of the limiting problem*

$$\begin{cases} -\Delta_p u = F_R(u) - \Upsilon(u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

where F_R *and* Υ *are given by* (2.7) *and* (2.9), *respectively.*

Proof The proof will be split into some steps.

Step 1: $u_\epsilon \geqslant 0$ a.e in Ω for all $\epsilon > 0$.

Indeed, recalling (2.13) and taking u_{ϵ_-} as test function in (2.6), we have

$$\begin{aligned} -\|u_{\epsilon_-}\|^p_{1,p,\epsilon} &= \int_\Omega \frac{1}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_\epsilon|^p dx\Big)} \big[f_R(x, Tu_\epsilon, \nabla Tu_\epsilon) - \upsilon(x, u_\epsilon)\big] u_{\epsilon_-}\, dx \\ &= \int_{\{u_\epsilon \leqslant \underline{u}\}} \frac{1}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_\epsilon|^p dx\Big)} f_R(x, \underline{u}, \nabla \underline{u}) u_{\epsilon_-} dx \\ &+ \int_{\{u_\epsilon \leqslant \underline{u}\}} \frac{1}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_\epsilon|^p dx\Big)} (\underline{u} - u_\epsilon)^\ell_+ u_{\epsilon_-} dx. \end{aligned}$$

Since last two terms in the right-hand side in previous relation are non-negative, we obtain $\|u_{\epsilon_-}\|^p_{1,p,\epsilon} = 0$. Hence $u_\epsilon = u_{\epsilon_+}$ a.e. in Ω for all $\epsilon > 0$, and Step 1 follows.

Step 2: $u_\epsilon \leqslant \overline{u}$ a.e in Ω for all $\epsilon > 0$.

Taking $v = (u_\epsilon - \overline{u})_+$ as test function in (2.6), one has

$$\int_\Omega \big[|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \nabla (u_\epsilon - \overline{u})_+ + \epsilon |u_\epsilon|^{p-2} u_\epsilon (u_\epsilon - \overline{u})_+\big] = \int_\Omega \frac{\big[f_R(x, Tu_\epsilon, \nabla Tu_\epsilon) - \upsilon(x, u_\epsilon)\big]}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_\epsilon|^p dx\Big)} (u_\epsilon - \overline{u})_+.$$

From (2.7), (2.9), and $(H_{\mathcal{M}})$, we deduce

$$\int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \nabla (u_\epsilon - \overline{u})_+ \, dx \leqslant \frac{1}{M}\int_\Omega f(x,\overline{u},\nabla\overline{u})(u_\epsilon-\overline{u})_+ \, dx - \int_\Omega \frac{(u_\epsilon-\overline{u})_+^{\ell+1}}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_\epsilon|^p dx\Big)} \, dx.$$

Thanks to hypothesis (ii) in Theorem 2.1, we have

$$\int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \nabla (u_\epsilon - \overline{u})_+ \, dx \leqslant \int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla(u_\epsilon-\overline{u})_+ \, dx - \int_\Omega \frac{(u_\epsilon-\overline{u})_+^{\ell+1}}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_\epsilon|^p dx\Big)} \, dx,$$

what ensures

$$0 \leqslant \int_\Omega \big(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon - |\nabla\overline{u}|^{p-2}\nabla\overline{u}\big)\nabla(u_\epsilon-\overline{u})_+ \, dx \leqslant -\int_\Omega \frac{(u_\epsilon-\overline{u})_+^{\ell+1}}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_\epsilon|^p dx\Big)} \, dx \leqslant 0.$$

Hence

$$\int_\Omega \frac{(u_\epsilon-\overline{u})_+^{\ell+1}}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_\epsilon|^p dx\Big)} \, dx = 0,$$

and $(u_\epsilon - \overline{u})_+ = 0$ a.e in Ω for all $\epsilon > 0$, so Step 2 follows.

Step 3: $\forall \epsilon_j \to 0$, as $j \to \infty$, (u_{ϵ_j}) strongly converges in $W^{1,p}(\Omega)$ to a solution u_R of (2.15).

Indeed, previous steps ensure (u_ϵ) is bounded in $L^\infty(\Omega)$ since $0 \leqslant u_\epsilon \leqslant \|\overline{u}\|_\infty$ a.e in Ω for all $\epsilon > 0$. Using u_ϵ as test function in (2.6), from (2.11) and (2.12) we have

$$\int_\Omega |\nabla u_\epsilon|^p dx + \epsilon \int_\Omega |u_\epsilon|^p dx \leqslant \frac{C}{m}\int_\Omega |u_\epsilon| dx + \frac{C}{m}\int_\Omega |u_\epsilon|^{\ell+1} dx \leqslant C(R, \|\overline{u}\|_\infty), \tag{2.16}$$

where $C > 0$ does not depend on $\epsilon > 0$, so (u_ϵ) is bounded in $W^{1,p}(\Omega)$. Thus, by choosing $\epsilon_j \to 0$, there exists $u_R \in W^{1,p}(\Omega)$ satisfying

- $u_{\epsilon_j} \rightharpoonup u_R$ in $W^{1,p}(\Omega)$,
- $u_{\epsilon_j} \to u_R$ in $L^p(\Omega)$,

as $j \to \infty$. After using $u_{\epsilon_j} - u$ as a test function in (2.6), and following the arguments used in (2.16) and Step 2, one infers

$$\int_\Omega \left[|\nabla u_{\epsilon_j}|^{p-2}\nabla u_{\epsilon_j} - |\nabla u_R|^{p-2}\nabla u_R\right] \nabla(u_{\epsilon_j} - u_R)\, dx \leqslant C(m, R, \|\overline{u}\|_\infty) \int_\Omega |u_{\epsilon_j} - u_R|\, dx$$

Therefore

$$\int_\Omega |\nabla(u_{\epsilon_j} - u_R)|^p\, dx \overset{j\to\infty}{\longrightarrow} 0,$$

and (u_{ϵ_j}) strongly converges to u_R in $W^{1,p}(\Omega)$. Hence it is possible to pass to the limit in (2.6) (equivalently, in the equation $\mathscr{B}_{\epsilon_j} u_{\epsilon_j} = 0$ on $\big(W^{1,p}(\Omega)\big)^*$, given by (2.10)) to obtain $u_R \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution of (2.15). The lemma is proved. $\square$

2.2 Proof of Theorem 2.1

Proof of Theorem 2.1 From steps 1 and 2 in the proof of Lemma 2.5 we have $0 \leqslant u_R \leqslant \overline{u}$ a.e. in Ω, where u_R satisfies (2.15). We claim that $\underline{u} \leqslant u_R$ a.e. in Ω. Indeed, using $(\underline{u} - u_R)_+$ as a test function in (2.15), from (2.7) and (2.9) we have

$$\int_\Omega |\nabla u_R|^{p-2}\nabla u_R \nabla(\underline{u} - u_R)_+\, dx = \int_\Omega \frac{f_R(x, \underline{u}, \nabla \underline{u})(\underline{u} - u_R)_+ + (\underline{u} - u_R)_+^{\ell+1}}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_R|^p dx\Big)}\, dx.$$

This implies on the one hand,

$$\int_\Omega |\nabla u_R|^{p-2}\nabla u_R \nabla(\underline{u} - u_R)_+ \geqslant \frac{1}{M}\int_\Omega f(x, \underline{u}, \nabla \underline{u})(\underline{u} - u_R)_+ + \int_\Omega \frac{(\underline{u} - u_R)_+^{\ell+1}}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_R|^p dx\Big)},$$

and, on the other hand, thanks to hypothesis (iii) in Theorem 2.1,

$$\int_\Omega |\nabla u_R|^{p-2}\nabla u_R \nabla(\underline{u} - u_R)_+\, dx \geqslant \int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla(\underline{u} - u_R)_+\, dx + \int_\Omega \frac{(\underline{u} - u_R)_+^{\ell+1}}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_R|^p dx\Big)}\, dx.$$

But this estimate in turn leads to

$$\int_\Omega \frac{1}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_R|^p dx\Big)} (\underline{u} - u_R)_+^{\ell+1}\, dx = 0,$$

what ensures $(\underline{u}-u_R)_+^{\ell+1}=0$ a.e. in Ω. Thus $0\leqslant \underline{u}\leqslant u_R\leqslant \overline{u}$ a.e. in Ω for all $R\geqslant R_0$, where R_0 is given by (2.3). From (2.9) it follows that $\upsilon=0$, so, from (2.15), u_R is a weak solution of the elliptic problem

$$\begin{cases} -\Delta_p u = \dfrac{f_R(\cdot,u(\cdot),\nabla u(\cdot))}{\mathcal{M}\Big(\cdot,\displaystyle\int_\Omega |\nabla u|^p dx\Big)} & \text{in } \Omega,\\ |\nabla u|^{p-2}\partial_\nu u=0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

Recall that $\|\underline{u}\|_{1,\infty}$ and $\|\overline{u}\|_{1,\infty}$ do not depend on R large, and let us denote u_R by u. Since $\|u\|_\infty$ is independent on $R\geqslant R_0$, from Lemma 2.1 we can apply the regularity results of Lieberman [66] or Fan [39] to ensure that $u\in C^{1,\gamma}(\overline{\Omega})$, where $\gamma\in(0,1)$ and $\|u\|_{C^{1,\gamma}(\overline{\Omega})}$ depend on certain ingredients, but are independent on $R\in[R_0,\infty)$. Hence we have $f_R=f$ for all $R\in[R_0,\infty)$, and u is a solution of (2.1). The proof is complete. □

2.3 Positive solutions of non-coercive BVPs: examples

We discuss some examples of non-coercive elliptic boundary value problems (BVPs, for short) as (2.1) having, at least, one positive solution. The source terms included appear in stochastic control type problems, population genetics or harvesting problems, or involve continuous or singular coefficients. Those examples extend or complement previous results in the literature available in the case $\mathcal{M}\equiv 1$ or with Dirichlet boundary conditions in (2.1).

1) Semipositone or logistic gradient dependent sources. Set the problem

$$\begin{cases} -\mathcal{M}\Big(x,\displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u \;=\; c(x)u^{p-1}-u^s+a(x)|\nabla u|^q-g(x) & \text{in } \Omega,\\ |\nabla u|^{p-2}\partial_\nu u \;=\; 0 \quad \text{on } \partial\Omega,\\ u>0 \quad \text{on } \overline{\Omega}. \end{cases} \tag{2.18}$$

The exponents s,q and the coefficients $a(\cdot)$, $c(\cdot)$, $g(\cdot)$ satisfy, respectively,

(A_1) $s\in(p-1,\infty)$, $q\in[0,p]$.

(A_2) $a,c,g\in L^\infty(\Omega)$, with $g(\cdot)\geqslant 0$, $c(\cdot)\geqslant c_0>0$ a.e. in Ω.

Previous assumptions turn (2.18) into a semipositone (or non-positone) problem, since its right-hand side, given by the function $f(x,\tau,\xi) = c(x)\tau^{p-1} - \tau^s + a(x)|\xi|^q - g(x)$ for $(x,\tau,\xi) \in \Omega \times [0,\infty) \times \mathbb{R}^N$, satisfies $f(\cdot,0,0) \leqslant 0$. The existence of positive solutions of semipositone problems with Neumann boundary condition has been obtained in [2] ($\mathcal{M} \equiv 1$ and $p = 2$ with convective terms) with sources not depending on the gradient, and in [30] ($\mathcal{M} \equiv 1$ and $p = 2$) for gradient dependent sources, see also their references. If $g \equiv 0$ in (2.18), one has the logistic equation plus a convection term, see [5, 43] for the existence of a positive solution in the case of Dirichlet boundary condition. The result we will prove regarding to (2.18) is the following:

Theorem 2.2 *Assume (A_1), (A_2) and $(H_{\mathcal{M}})$ hold. Then, for all $c_0 > 0$ sufficiently large, there exists a solution $u \in C^{1,\gamma}(\overline{\Omega})$ of (2.18), $\gamma \in (0,1)$, satisfying $u > 0$ on $\overline{\Omega}$. If $\|g\|_\infty$ is sufficiently small, such solution exists for all $c_0 > 0$.*

2) Sources having gradient terms with continuous coefficients. Set the problem

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u \;=\; c(x)u^r - u^s + g(x,u)|\nabla u|^q \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\partial_\nu u \;=\; 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{on } \overline{\Omega}. \end{cases} \tag{2.19}$$

where the exponents r, s, q, and the coefficient $c(\cdot)$, satisfy

(A_3) $r, s \in (0,\infty)$, with $r < s$, and $q \in [0,p]$.

(A_4) $c \in L^\infty(\Omega)$, with $c(\cdot) \geqslant c_0 > 0$ a.e. in Ω.

Dirichlet problems involving continuous coefficients have been studied, for instance, in [3, 6, 16, 22, 23, 34, 35, 37, 42, 43, 58, 77, 79], whereas Neumann problems in [6, 46, 52, 72]. In the case of (2.19), some examples of coefficients g, not necessarily being Lipschitz functions, are $g(x,u) = a(x)\sin u$, $g(x,u) = a(x)|u|^\kappa$, $g(x,u) = a(x)e^{|u|^\kappa}$, where $\kappa > 0$ and $a \in L^\infty(\Omega)$ is an indefinite weight. More generally, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ in (2.19) may be any Carathéodory function which is bounded on bounded sets of $\mathbb{R}$ uniformly with respect to the first variable. The existence result we will prove with respect to (2.19) is the following:

Theorem 2.3 *Assume* (A_3), (A_4), *and* $(H_{\mathcal{M}})$ *hold. Then there exists a solution* $u \in C^{1,\gamma}(\overline{\Omega})$ *of* (2.19), $\gamma \in (0,1)$ *satisfying* $u > 0$ *on* $\overline{\Omega}$.

3) Sources having gradient and singular terms. Neumann problems with first order and singular terms are contained in next two examples. Actually, we consider convective terms with singular coefficients of the form

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)\Delta_p u \;=\; c(x)u^m - u^s + a(x)\dfrac{|\nabla u|^q}{u^\alpha} \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\partial_\nu u \;=\; 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{on } \overline{\Omega}, \end{cases} \tag{2.20}$$

or sources having a combination of convective and singular terms of the form

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)\Delta_p u \;=\; \dfrac{1}{u^\theta} - c(x)u^r + a(x)\dfrac{|\nabla u|^q}{u^\beta} \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\partial_\nu u \;=\; 0 \quad \text{on } \partial\Omega. \\ u > 0 \quad \text{on } \overline{\Omega}. \end{cases} \tag{2.21}$$

The exponents $\alpha, \beta, \theta, m, r, s, q$, and the coefficients $a(\cdot)$, $c(\cdot)$ in (2.20) and (2.21), satisfy

(A_5) $\alpha \in (0,\infty)$, and $m, s \in (0,\infty)$, with $m < s$.

(A_6) $r, \beta, \theta \in (0,\infty)$.

(A_7) $q \in [0,p]$ and $a, c \in L^\infty(\Omega)$, with $c(\cdot) \geqslant c_0 > 0$ a.e. in Ω.

Positive solutions of Dirichlet problems with sources depending on singular and first order terms have been studied in [20, 47, 7, 9, 10, 21, 40, 43, 70], where natural growth in the gradient is included in some works. For Neumann problems, the existence of positive solutions with sources involving singular and gradient terms has been studied in [74, 52] for some classes of coercive or non-coercive problems, respectively. Regarding to the non-coercive problems (2.20) and (2.21) having up to the natural growth in the gradient, and with right-hand sides which are not Lipschitz functions in some cases, we prove the following:

Theorem 2.4 *Assume* (A_5), (A_7), *and* $(H_{\mathcal{M}})$ *hold. Then there exists a solution* $u \in C^{1,\sigma}(\overline{\Omega})$ *of* (2.20), $\sigma \in (0,1)$, *satisfying* $u > 0$ *on* $\overline{\Omega}$.

Theorem 2.5 *Assume* (A_6), (A_7), *and* $(H_{\mathcal{M}})$ *hold. Then there exists a solution* $u \in C^{1,\varrho}(\overline{\Omega})$ *of* (2.21), $\varrho \in (0,1)$ *satisfying* $u > 0$ *on* $\overline{\Omega}$.

4) Sources of stochastic control problems type. Let us set the problem

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)\Delta_p u + \lambda |u|^{p-2}u + b(x)\psi(\nabla u) \; = \; g(x) \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |\nabla u|^{p-2}\partial_\nu u \; = \; 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.22}$$

where $p \in (1, \infty)$ and $\lambda > 0$ is a parameter, assuming the following conditions:

(A_8) $\psi : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying $|\psi(\xi)| \leqslant C(1+|\xi|^p)$, for all $\xi \in \mathbb{R}^N$.

(A_9) $b, g \in L^\infty(\Omega)$, with $g(x) \geqslant b(x)\psi(0)$ for a.e $x \in \Omega$.

In the local case $\mathcal{M} \equiv 1$, problems of the form (2.22) appear, under some hypotheses, as models in stochastic control problems or in the study of some problems involving Hamilton-Jacobi-Bellman equations, see for instance [63, 64, 65, 61, 62]. The case $\mathcal{M} \equiv 1$ and $p = 2$ in (2.22) has been studied in [63], assuming $b \equiv 1$, $g \in W^{1,\infty}(\Omega)$, $\psi \in C^1(\Omega \times \mathbb{R}^N, \mathbb{R})$, and Ω convex, but allowing ψ to grow as any arbitrary power. Hamiltonian terms H more general than $H(x,\xi) = b(x)\psi(\xi)$ in (2.22), also growing as arbitrary powers, have been considered in [65, 61] ($\mathcal{M} \equiv 1$ and $p = 2$ with a non-divergence form operator) and [62] ($\mathcal{M} \equiv 1$ and $p \in (1,\infty)$), but requiring $H \in C^1(\Omega \times \mathbb{R}^N, \mathbb{R})$ and some further conditions. The result we shall prove with respect to (2.22) is the following:

Theorem 2.6 *Assume* (A_8) *and* $(H_{\mathcal{M}})$ *hold. Further, suppose* (A_9) *holds with* $g \not\equiv 0$ *or* $b \not\equiv 0$ *and* $\psi(0) \neq 0$. *Then there exists a solution* $u \in C^{1,\gamma}(\overline{\Omega})$ *of* (2.22), $\gamma \in (0,1)$, *not identically zero, satisfying* $u \geqslant 0$ *on* $\overline{\Omega}$.

2.3.1 Proofs of Theorems 2.2–2.6

Proof of Theorem 2.2 Set the function $f(x,\tau,\xi) = c(x)|\tau|^{p-1} - |\tau|^s + a(x)|\xi|^q - g(x)$, for $(x,\tau,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, which satisfies (H_f) by $(A_1) - (A_2)$. Note that

$$f(x,\tau,0) = c(x)\tau^{p-1} - \tau^s - g(x) \geqslant \tau^{p-1}(c_0 - \tau^{s-p+1}) - \|g\|_\infty, \qquad \forall \tau > 0.$$

Choosing a constant $0 < \tau_0 < \min\big\{1, \frac{1}{2}c_0^{\frac{1}{s-p+1}}\big\}$, we obtain $\underline{u} = \tau_0$ satisfies (i), (iii) in Theorem 2.1 provided that c_0 is sufficiently large. Further, at this point, one can see the same conclusion is achieved without any restriction on $c_0 > 0$ if it is possible to choose $\|g\|_\infty$ sufficiently small. Now, on the other hand, one has

$$f(x,\tau,0) \leqslant \tau^{p-1}(\|c\|_\infty - \tau^{s-p+1}) + \|g\|_\infty, \qquad \forall \tau > 0.$$

By choosing $\overline{u} = \tau_1 > \tau_0 > 0$ sufficiently large, we obtain $\overline{u}$ satisfies (i), (ii) in Theorem 2.1. Hence Theorem 2.1 ensures the existence of a solution $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, of (2.18) satisfying $\tau_0 \leqslant u \leqslant \tau_1$ in $\overline{\Omega}$. The proof is complete. □

Proof of Theorem 2.3 Let the function $f(x,\tau,\xi) = c(x)|\tau|^r - |\tau|^s + g(x,|\tau|)|\xi|^q$, for $(x,\tau,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. From $(A_3) - (A_4)$, f satisfies (H_f). Since

$$f(x,\tau,0) \geqslant \tau^r(c_0 - \tau^{s-r}), \qquad \forall \tau > 0,$$

it suffices to consider $\underline{u} = \tau_2 > 0$, with $\tau_2 \in (0, c_0^{\frac{1}{s-r}})$, to have (i), (iii) in Theorem 2.1 to be satisfied. Analogously, using the estimate

$$f(x,\tau,0) \leqslant \tau^r(\|c\|_\infty - \tau^{s-r}), \qquad \forall \tau > 0,$$

we infer that $\overline{u} = \tau_3 > 0$, where $\tau_3 \in \big(\max\{\tau_2, \|c\|_\infty^{\frac{1}{s-r}}\}, \infty\big)$, will satisfy (i), (ii) in Theorem 2.1. Therefore, from Theorem 2.1, there exists a solution $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, of (2.19) satisfying $\tau_2 \leqslant u \leqslant \tau_3$ in $\overline{\Omega}$. This proof is complete. □

Proof of Theorem 2.4 Let us introduce the approximate problems

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_n|^p dx\Big)\Delta_p u_n \;=\; c(x)|u_n|^m - |u_n|^s + a(x)\dfrac{|\nabla u_n|^q}{\big(|u_n| + \frac{1}{n}\big)^\alpha} & \text{in } \Omega, \\ |\nabla u_n|^{p-2}\partial_\nu u_n \;=\; 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

for all $n \geqslant 1$, in the weak sense (2.2). Their right-hand sides are given by the functions

$$f_n(x,\tau,\xi) = c(x)|\tau|^m - |\tau|^s + a(x)\frac{|\xi|^q}{\big(|\tau| + \frac{1}{n}\big)^\alpha}, \quad \forall (x,\tau,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \tag{2.24}$$

which verify (H_f) by (A_5), (A_7), for all $n \geqslant 1$. Arguing as in the proof of Theorem 2.3, one obtains constants $\kappa_1, \kappa_2 > 0$, independent on n, which allow one to apply Theorem 2.1 to ensure the existence of solutions $u_n \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, of (2.23) satisfying

$\kappa_1 \leqslant u_n \leqslant \kappa_2$ in $\overline{\Omega}$. Thus $\|u_n\|_\infty$ is uniformly bounded with respect to n, and one can modify the functions in (2.24) to verify a growth condition of the form

$$|f_n(x,s,\xi)| \leqslant C(1+|\xi|^q) \quad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N, \tag{2.25}$$

for some $C > 0$ independent on n. From the regularity results in [66, 39], there exists $\tilde{\gamma} \in (0,1)$, independent on n, such that $\|u_n\|_{C^{1,\tilde{\gamma}}(\overline{\Omega})}$ is bounded. The compactness of the imbedding $C^{1,\tilde{\gamma}}(\overline{\Omega}) \hookrightarrow C^{1,\sigma}(\overline{\Omega})$ for all $\sigma \in (0,\tilde{\gamma})$ implies the existence of $u \in C^{1,\sigma}(\overline{\Omega})$, and a not relabeled subsequence, satisfying $u_n \to u$ in $C^{1,\sigma}(\overline{\Omega})$, as $n \to \infty$. Hence $\kappa_1 \leqslant u \leqslant \kappa_2$ in $\overline{\Omega}$, and we can pass to the limit in the weak formulation of (2.23) (as in (2.2)) to conclude that u is a solution of (2.20). The proof is complete. □

Proof of Theorem 2.5 We set the approximate problems

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_n|^p dx\Big)\Delta_p u_n = \dfrac{1}{(|u| + \frac{1}{n})^\theta} + a(x)\dfrac{|\nabla u_n|^q}{(|u_n| + \frac{1}{n})^\beta} - c(x)|u_n|^r & \text{in } \Omega, \\ |\nabla u_n|^{p-2}\partial_\nu u_n = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.26}$$

for all $n \geqslant 1$, where their right-hand sides, which satisfy (H_f), are given by

$$f_n(x,\tau,\xi) = \frac{1}{(|\tau| + \frac{1}{n})^\theta} - c(x)|\tau|^r + a(x)\frac{|\xi|^q}{(|\tau| + \frac{1}{n})^\beta}, \quad \forall (x,\tau,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N. \tag{2.27}$$

Similarly as in the proof of Theorem 2.4, one obtains constants $c_1, c_2 > 0$ independent on n and, with the help of Theorem 2.1, solutions $u_n \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, of (2.26) satisfying $c_1 \leqslant u_n \leqslant c_2$ in $\overline{\Omega}$. As a consequence, $\|u_n\|_\infty$ is bounded, and f_n in (2.27) can be modified to verify an estimate like (2.25) uniformly with respect to n. This enable one to apply the regularity results in [66, 39] to infer the existence of $\bar{\gamma} \in (0,1)$, independent on n, such that $\|u_n\|_{C^{1,\bar{\gamma}}(\overline{\Omega})}$ is uniformly bounded. Extracting a subsequence if necessary, there exists $u \in C^{1,\varrho}(\overline{\Omega})$, with $\varrho \in (0,\bar{\gamma})$, such that $u_n \to u$ in $C^{1,\varrho}(\overline{\Omega})$, as $n \to \infty$. So $c_1 \leqslant u \leqslant c_2$ in $\overline{\Omega}$, and we can pass to the limit in the weak formulation of (2.26) (as in (2.2)) to obtain u is a solution of (2.21). The proof is complete. □

Proof of Theorem 2.6 The function $f(x,\tau,\xi) = g(x) - \lambda|\tau|^{p-2}\tau - b(x)\psi(\xi)$, for $(x,\tau,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, satisfies (H_f) from $(A_8) - (A_9)$. Setting $\underline{u} \equiv 0$, one has (i), (iii) in Theorem

2.1 are satisfied. Furthermore, for $\tau_1 \in \left(\left[\frac{1}{\lambda}\left(\|g\|_\infty + |\psi(0)|\,\|b\|_\infty\right)\right]^{\frac{1}{p-1}}, \infty\right)$ we have

$$f(x, \tau_1, 0) \leqslant \|g\|_\infty + |\psi(0)|\,\|b\|_\infty - \lambda \tau^{p-1} \leqslant 0,$$

so (i), (ii) in Theorem 2.1 hold. By applying Theorem 2.1, we obtain a solution $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, of (2.22) satisfying $0 \leqslant u \leqslant \tau_1$ in $\overline{\Omega}$, which necessarily verifies $u \not\equiv 0$. The proof is complete. □

2.4 Unbounded coefficients $\mathcal{M}$ in (2.1)

In this section, we prove a more general version of Theorem 2.1 removing the requirement on the non-local coefficient $\mathcal{M} : \Omega \times [0,\infty) \to [0,\infty)$ in (2.1) to be bounded from above. Let us suppose $\mathcal{M}$ is a continuous function satisfying

$(\widetilde{H_\mathcal{M}})$ There exists $m > 0$ such that $\mathcal{M}(x,\tau) \geqslant m, \quad \forall (x,\tau) \in \Omega \times [0,\infty)$.

Theorem 2.7 *Assume* (H_f) *and* $(\widetilde{H_\mathcal{M}})$ *hold. Suppose there exist* $\overline{u}, \underline{u} \in W^{1,\infty}(\Omega)$ *such that* $0 \leqslant \underline{u} \leqslant \overline{u}$ *a.e. in* Ω, *and satisfying:*

(i) $f(x,\overline{u},\nabla\overline{u}) \leqslant 0 \leqslant f(x,\underline{u},\nabla\underline{u})$ *a.e. in* Ω.

Furthermore, suppose there exists $M_0 > 0$ *such that for all* $M \geqslant M_0$ *one has:*

(ii) $\displaystyle\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi\,dx \geqslant \frac{1}{M}\int_\Omega f(x,\overline{u},\nabla\overline{u})\varphi\,dx, \quad \forall\varphi \in W^{1,p}(\Omega),\ \varphi \geqslant 0$ *a.e. in* Ω;

(iii) $\displaystyle\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla\varphi\,dx \leqslant \frac{1}{M}\int_\Omega f(x,\underline{u},\nabla\underline{u})\varphi\,dx, \quad \forall\varphi \in W^{1,p}(\Omega),\ \varphi \geqslant 0$ *a.e. in* Ω.

Then there exists a solution $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, *of* (2.1) *with* $\underline{u} \leqslant u \leqslant \overline{u}$ *a.e. in* Ω.

Proof We set the following truncated problem associated with (2.1)

$$\begin{cases} -\mathcal{M}_\eta\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x,u,\nabla u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.28}$$

where $\mathcal{M}_\eta(\cdot,\cdot)$ is the function defined by

$$\mathcal{M}_\eta(x,s) = \min\big\{\mathcal{M}(x,s), \eta\big\}, \quad \forall (x,s) \in \Omega \times [0,\infty),\ \forall \eta \in (m,\infty). \tag{2.29}$$

Since $\mathcal{M}_\eta$ is a continuous function with $m \leqslant \mathcal{M}_\eta(\cdot,\cdot) \leqslant \eta$, and (i)$-$(iii) above hold for all $\eta \in \big(\max\{m, M_0\}, \infty\big)$, we apply Theorem 2.1 to obtain a solution $u_\eta \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, of (2.28) satisfying $\underline{u} \leqslant u_\eta \leqslant \overline{u}$ a.e. in Ω. In particular, $\|u_\eta\|_\infty$ is uniformly bounded with respect to $\eta \in \big(\max\{m, M_0\}, \infty\big)$.

Claim: $\{u_\eta\}$ is uniformly bounded in $W^{1,p}(\Omega)$ with respect to $\eta \in \big(\max\{m, M_0\}, \infty\big)$.

Indeed, let $\Lambda = \sup\big\{\|u_\eta\|_\infty : \eta \in (\max\{m, M_0\}, \infty)\big\}$. We consider separately the cases $q \in [0,p)$ and $q = p$ in (H_f). In the first case, by testing the weak formulation of (2.28) with $\varphi = u_\eta$ (as in (2.2)), from Young's inequality with $\varepsilon > 0$ we have

$$\int_\Omega |\nabla u_\eta|^p\,dx \leqslant C + C\int_\Omega |\nabla u_\eta|^q |u_\eta|\,dx \leqslant C + C\varepsilon\int_\Omega |\nabla u_\eta|^p\,dx,$$

where $C = C(m, \Omega, \Lambda, \varepsilon) > 0$. Choosing ε sufficiently small, the claim follows for all $q \in [0,p)$. For the natural growth case $q = p$, we shall use a test function inspired on [22]. In fact, setting $\varphi_s = e^{su_\eta^2}u_\eta$, $s > 0$, as a test function in the weak formulation of (2.28), by a direct computation we have

$$\int_\Omega e^{su_\eta^2}(1+2su_\eta^2)|\nabla u_\eta|^p\,dx \leqslant \frac{C}{m} + \frac{C}{m}\int_\Omega |\nabla u_\eta|^p e^{su_\eta^2}|u_\eta|\,dx,$$

where $C = C(s, \Lambda, \Omega) > 0$ is a constant independent on η. For all $\varepsilon > 0$, Young's inequality implies

$$\int_\Omega e^{su_\eta^2}(1+2su_\eta^2)|\nabla u_\eta|^p\,dx \leqslant \frac{C}{m} + \frac{C}{m}\int_\Omega |\nabla u_\eta|^p e^{su_\eta^2}\left[\frac{\varepsilon}{2} + \frac{u_\eta^2}{2\varepsilon}\right]dx.$$

Now, by choosing $\varepsilon = \frac{C}{4sm}$, with $s > 0$ large in a such way that $\frac{C^2}{8sm^2} < 1$, we obtain

$$\left(1 - \frac{C^2}{8sm^2}\right)\int_\Omega |\nabla u_\eta|^p\,dx \leqslant \frac{C}{m},$$

proving the claim. Hence (2.29) and previous claim ensure

$$\mathcal{M}_{\eta_0}\Big(\cdot\,,\int_\Omega |\nabla u_{\eta_0}|^p dx\Big) = \mathcal{M}\Big(\cdot\,,\int_\Omega |\nabla u_{\eta_0}|^p dx\Big)$$

for all $\eta_0 > 0$ sufficiently large, i.e., $u = u_{\eta_0}$ is a solution of (2.1). The proof is complete. $\square$

The examples in Section 2.3 keep holding by assuming $(\widetilde{H_{\mathcal{M}}})$. Indeed, with the same proofs given in Section 2.3, but applying Theorem 2.7 rather than Theorem 2.1, one has

Corollary 2.1 *Theorems* $2.2-2.6$ *remain valid replacing* $(H_{\mathcal{M}})$ *by* $(\widetilde{H_{\mathcal{M}}})$.

Chapter 3

Positive stationary solution of Kirchhoff equations with first order terms and lack of coerciveness: nonlinear Neumann boundary conditions

This chapter is addressed to the existence of positive solutions of $(\mathcal{SK})$ under Neumann boundary conditions. To be precise, we consider non-variational elliptic PDEs of the form

$$\begin{cases} -\mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x,u,\nabla u) \quad \text{in } \Omega, \\ \mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)|\nabla u|^{p-2}\partial_\nu u = g(x,u) \quad \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

on bounded domains $\Omega \subset \mathbb{R}^N$, $N \geq 2$, having smooth boundary $\partial\Omega$ with outward unit normal ν. The source term $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function (i.e., $f(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$, and $f(\cdot,s,\xi)$ is measurable for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$) satisfying

(H_f) $\qquad |f(x,s,\xi)| \leqslant h(x,s)(1+|\xi|^q)$ for a.e. $x \in \Omega$, $\forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N$,

where $q \in [0,p]$ and $h : \Omega \times \mathbb{R} \to [0,\infty)$ is a Carathéodory function which is bounded on bounded sets of $\mathbb{R}$ uniformly with respect to the first variable. The boundary source term $g \in C^{0,\alpha}(\partial\Omega \times \mathbb{R}, \mathbb{R})$, $\alpha \in (0,1)$, is a Hölder continuous function, and the coefficient

$\mathcal{M} \in C^{0,\beta}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $\beta \in (0,1)$ is a Hölder continuous function satisfying

$(H_{\mathcal{M}})$ There exist $m, M > 0$ such that $m \leqslant \mathcal{M}(x,s) \leqslant M, \quad \forall (x,s) \in \overline{\Omega} \times [0,\infty)$.

We seek for a solution of (3.1) in the following sense.

Definition 3.1 *Suppose (H_f) and $(H_{\mathcal{M}})$ hold. A function $u \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ is a solution of* (3.1) *if satisfies*

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\Omega} \frac{f(x,u,\nabla u)}{\mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)} \varphi \, dx + \int_{\partial\Omega} \frac{g(x,u)}{\mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)} \varphi \, d\mathcal{H}^{N-1}, \tag{3.2}$$

for all $\varphi \in W^{1,p}(\Omega)$. The term $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure.

The main result of this chapter reads as follows.

Theorem 3.1 *Assume (H_f) and $(H_{\mathcal{M}})$ hold. Suppose there exist $\bar{u}, \underline{u} \in W^{1,\infty}(\Omega)$ such that $0 \leq \underline{u} \leqslant \bar{u}$ a.e. in Ω, satisfying the following conditions:*

(i) $f(x,\underline{u},\nabla\underline{u}) \geqslant 0$ *and* $g(x,\underline{u}) \geqslant 0$, *a.e. in* Ω.

(ii) $\displaystyle\int_{\Omega} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi \, dx \leqslant \frac{1}{M} \int_{\Omega} f(x,\underline{u},\nabla\underline{u}) \varphi \, dx + \frac{1}{M} \int_{\partial\Omega} g(x,\underline{u}) \varphi d\mathcal{H}^{N-1}$,
$\forall \varphi \in W^{1,p}(\Omega)$, $\varphi \geqslant 0$ *a.e. in* Ω.

(iii) $\displaystyle\int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \nabla \varphi dx \geqslant \frac{1}{Q_1} \int_{\Omega} f(x,\bar{u},\nabla\bar{u}) \varphi dx + \frac{1}{Q_2} \int_{\partial\Omega} g(x,\bar{u}) \varphi d\mathcal{H}^{N-1}$,
$\forall \varphi \in W^{1,p}(\Omega)$, $\varphi \geqslant 0$ *a.e. in* Ω.

where

$$(Q_1,Q_2) = \begin{cases} (M,m), & \textit{if} \quad f(x,\bar{u},\nabla\bar{u}) \leqslant 0 \quad \textit{and} \quad g(x,\bar{u}) \geqslant 0, \\[2ex] (M,M), & \textit{if} \quad f(x,\bar{u},\nabla\bar{u}) \leqslant 0 \quad \textit{and} \quad g(x,\bar{u}) \leqslant 0, \\[2ex] (m,M), & \textit{if} \quad f(x,\bar{u},\nabla\bar{u}) \geqslant 0 \quad \textit{and} \quad g(x,\bar{u}) \leqslant 0. \end{cases} \tag{3.3}$$

Then there exists a solution $u \in C^{1,\gamma}(\overline{\Omega})$ of (3.1)*, $\gamma \in (0,1)$, with $0 \leq \underline{u} \leqslant u \leqslant \bar{u}$ a.e. in Ω.*

Remark 3.1 *In the case $g(x,\bar{u}) \geqslant 0$ and $f(x,\bar{u},\nabla\bar{u}) \geqslant 0$ in Theorem 3.1, an estimate of the form*

$$\int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \nabla \varphi \, dx \geqslant c_1 \int_{\Omega} f(x,\bar{u},\nabla\bar{u}) \varphi \, dx + c_2 \int_{\partial\Omega} g(x,\bar{u}) \varphi d\mathcal{H}^{N-1}, \tag{3.4}$$

with c_1,$c_2 > 0$, *for all* $\varphi \in W^{1,p}(\Omega)$, $\varphi \geqslant 0$ *a.e. in* Ω, *can only holds if* $g(x,\bar{u}) \equiv f(x,\bar{u},\nabla\bar{u}) \equiv 0$.

Indeed, by taking with $\varphi \equiv 1$ *in* 3.4 *one obtains*

$$0 \geqslant c_1 \int_\Omega f(x,\bar{u},\nabla\bar{u})dx + c_2 \int_{\partial\Omega} g(x,\bar{u})d\mathcal{H}^{N-1},$$

which implies $g(x,\bar{u}) \equiv f(x,\bar{u},\nabla\bar{u}) \equiv 0$ *a.e in* Ω. *Note that this case is already included in Theorem 3.1.*

3.1 Auxiliary Problem

In this section we will work set an auxiliary problem to help with the proof of Theorem 3.1. From now on, fix $R_0 > 0$, large enough such that

$$\max\{\|\nabla\bar{u}\|_\infty, \|\nabla\underline{u}\|_\infty\} \leq R_0$$

with $\bar{u}$ and $\underline{u}$ set in Theorem 3.1.

For all $R \geq R_0$, let $\tau_R : \mathbb{R} \to \mathbb{R}$ be the truncation function

$$\tau_R(t) = \begin{cases} t, & \text{if } |t| \leq R, \\ R\frac{t}{|t|}, & \text{if } |t| \geq R, \end{cases}$$

and truncate f as

$$f_R(x,t,\xi) \doteq f(x,t,\tau_R(\xi_1),\cdots,\tau_R(\xi_N)), \;\; \forall(x,t,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^N, \; \xi = (\xi_1,\cdots,\xi_N). \tag{3.5}$$

Using this definition, we have that exists $K \in \mathbb{R}$ such that

$$|f_R(x,t,\xi)| \leq h(x,t)(1+R^\ell N^\ell) \quad \text{and} \quad |G(x,t)| \leqslant K \tag{3.6}$$

and

$$f_R(x,t,\xi) = f(x,t,\xi) \;\; \text{if} \;\; |t| \leq R,$$

then we conclude

$$f_R(x,\bar{u},\nabla\bar{u}) = f(x,\bar{u},\nabla\bar{u}) \;\; \text{and} \;\; f_R(x,\underline{u},\nabla\underline{u}) = f(x,\underline{u},\nabla\underline{u}). \tag{3.7}$$

3.2 Coercive parametric ε-approximate problem

First of all, we define (as in Lemma 1.1) the truncation operator $T : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ by

$$Tu(x) = \begin{cases} \bar{u}(x) & if \quad u(x) \geqslant \bar{u}(x), \\ u(x) & if \quad \underline{u}(x) \leqslant u(x) \leqslant \bar{u}(x), \\ \underline{u}(x) & if \quad u(x) \leqslant \underline{u}(x). \end{cases} \tag{3.8}$$

Now, $\forall \varepsilon > 0$, we introduce the auxiliary problem on the form

$$\begin{cases} -\Delta_p u_\varepsilon + \varepsilon\, |u_\varepsilon|^{p-2} u_\varepsilon \;=\; F_R(u_\varepsilon) - \Psi(u_\varepsilon) \quad \text{in } \Omega, \\ |\nabla u_\varepsilon|^{p-2} \partial_\nu u_\varepsilon \;=\; G(u) \quad \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

In (3.9), $F_R : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is given by

$$\langle F_R(u), v \rangle = \int_\Omega \frac{f_R(x, Tu, \nabla Tu)}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)}\, v\, dx, \quad \forall v \in W^{1,p}(\Omega), \tag{3.10}$$

the function $G : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is given by

$$\langle G(u), v \rangle = \int_{\partial\Omega} \frac{g(x, Tu)}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)}\, v\, d\mathcal{H}^{N-1}, \quad \forall v \in W^{1,p}(\Omega), \tag{3.11}$$

and $\Psi : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is given by

$$\langle \Psi(u), v \rangle = \int_\Omega \frac{\psi(x, u)}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)}\, v\, dx, \quad \forall v \in W^{1,p}(\Omega), \tag{3.12}$$

where $\psi(\cdot, u) = -(\underline{u} - u)_+^\ell + (u - \bar{u})_+^\ell$, and $\ell \in (0, p-1)$ is fixed. We shall prove now the existence of a solution for this auxiliary problem (3.9). To do this, we shall use Theorem 1.19.

Let $\mathscr{B}_\varepsilon : W^{1,p}(\Omega) \to \big(W^{1,p}(\Omega)\big)^*$ be the operator given by

$$\langle \mathscr{B}_\varepsilon u, v \rangle = \langle \mathcal{I}_\varepsilon(u), v \rangle - \langle F_R(u), v \rangle + \langle \Psi(u), v \rangle - \langle G(u), v \rangle, \quad \forall v \in W^{1,p}(\Omega), \tag{3.13}$$

where $\mathcal{I}_\varepsilon : W^{1,p}(\Omega) \to \big(W^{1,p}(\Omega)\big)^*$ is given by

$$\langle \mathcal{I}_\varepsilon(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, dx + \varepsilon \int_\Omega |u|^{p-2} uv \, dx, \quad \forall v \in W^{1,p}(\Omega). \tag{3.14}$$

Lemma 3.1 *Assume that $(H_\mathcal{M})$ and (H_f) hold. Then $\mathscr{B}_\varepsilon$ is bounded and coercive for all $\varepsilon > 0$.*

Proof By the definition of G, there exists $K > 0$ satisfying

$$|\langle G(u), u\rangle| \leqslant \int_{\partial\Omega} \frac{|g(x,Tu)u|}{\mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)} d\mathcal{H}^{N-1} \leqslant \int_{\partial\Omega} \frac{K}{m} |u| d\mathcal{H}^{N-1}, \quad \forall u \in W^{1,p}(\Omega).$$

Using the Sobolev embedding, we have that $W^{1,p}(\Omega) \hookrightarrow L^1(\partial\Omega)$ so

$$|\langle G(u), u\rangle| \leqslant C\|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{3.15}$$

Futhermore, there exist $K_1 > 0$ such that

$$|\langle F_R(u), u\rangle| \leqslant \int_\Omega \frac{|f_R(x, Tu, \nabla Tu)u|}{\mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)} dx, \quad \forall u \in W^{1,p}(\Omega).$$

Since Theorem 1.3 implies $W^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$, and $f_R(\cdot, Tu, \nabla Tu) \in L^\infty(\Omega)$, we have

$$|\langle F_R(u), u\rangle| \leqslant C\|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{3.16}$$

Using Hölder's inequality, there exists constants $K_2, K_3 > 0$ with

$$\begin{aligned} |\langle \Psi(u), u\rangle| &\leqslant \int_\Omega \frac{|\psi(x,u)u|}{\mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)} \, dx \leqslant \int_\Omega \frac{K_2}{m}|u| dx + \int_\Omega \frac{K_3}{m} |u|^{\ell+1} dx \\ &\leqslant C\|u\|_{W^{1,p}(\Omega)} + C\|u\|^{\ell+1}_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \end{aligned} \tag{3.17}$$

From these inequalities we obtain

$$\begin{aligned} \langle \mathscr{B}_\varepsilon u, u\rangle &= \langle \mathcal{I}_\varepsilon(u), u\rangle - \int_\Omega \frac{f_R(x,Tu,\nabla Tu) - \psi(x,u)}{\mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)} \, u \, dx - \int_{\partial\Omega} \frac{g(x,Tu)}{\mathcal{M}\Big(x, \int_\Omega |\nabla u|^p dx\Big)} \, u \, d\mathcal{H}^{N-1} \\ &\geqslant \|u\|^p_{1,p,\varepsilon} - C\|u\|_{W^{1,p}(\Omega)} - C\|u\|^{\ell+1}_{W^{1,p}(\Omega)} \end{aligned}$$

for all $u \in W^{1,p}(\Omega)$, where $\|\cdot\|_{1,p,\varepsilon}$ denotes the norm

$$\|u\|_{1,p,\varepsilon}^p = \int_\Omega |\nabla u|^p \, dx + \varepsilon \int_\Omega |u|^p \, dx, \tag{3.18}$$

which is equivalent to the usual norm $\|\cdot\|_{W^{1,p}(\Omega)}$. Since $\ell \in (0, p-1)$ we conclude

$$\langle \mathscr{B}_\varepsilon u, u\rangle \geqslant K_\varepsilon \|u\|_{W^{1,p}(\Omega)}^p - C\|u\|_{W^{1,p}(\Omega)} - C\|u\|_{W^{1,p}(\Omega)}^{\ell+1} \longrightarrow \infty,$$

as $\|u\|_{W^{1,p}(\Omega)} \to \infty$, This shows that $\mathscr{B}_\varepsilon$ is coercive for all $\varepsilon > 0$, from (3.10), (3.11) and (3.12), we observe that $\mathscr{B}_\varepsilon$ is also bounded (i.e., transforms bounded sets into bounded sets) for all $\varepsilon > 0$, proving the lemma. □

Lemma 3.2 *Assume that* $(H_\mathcal{M})$ *and* (H_f) *hold. Then the auxiliary problem* (3.9) *has a solution* $u_\varepsilon \in W^{1,p}(\Omega)$, $\forall \varepsilon > 0$.

Proof Let (u_j) be a sequence satisfying $u_j \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\limsup\limits_{j\to\infty} \langle \mathscr{B}_\varepsilon u_j, u_j - u\rangle \leqslant 0$. We will show, accordingly definition 1.3, that

$$\liminf_{j\to\infty} \langle \mathscr{B}_\varepsilon u_j, u_j - v\rangle \geqslant \langle \mathscr{B}_\varepsilon u, u - v\rangle \qquad \forall v \in W^{1,p}(\Omega). \tag{3.19}$$

Passing to a subsequence, still denoted by (u_j), we have by the Compact Sobolev Embedding (see Theorem 1.4) that $u_j \to u$ in $L^p(\Omega)$, $u_j \to u$ in $L^1(\partial\Omega)$. By $(H_\mathcal{M})$, (3.11) and (3.15), we obtain

$$\left| \int_{\partial\Omega} \frac{g(x, Tu_j)(u_j - u)}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_j|^p dx\Big)} \, d\mathcal{H}^{N-1} \right| \leqslant C\|u_j - u\|_{L^1(\partial\Omega)} \stackrel{j\to\infty}{\longrightarrow} 0.$$

On the one hand, from $(H_\mathcal{M})$ and (3.16) we have

$$\left| \int_\Omega \frac{f_R(x, Tu_j, \nabla Tu_j)(u_j - u)}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_j|^p dx\Big)} dx \right| \leqslant C\,\|u_j - u\|_{L^1(\Omega)} \stackrel{j\to\infty}{\longrightarrow} 0.$$

On the other hand, since, by (3.12) and (3.17) that exists constants $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3 > 0$ such that

$$\|\psi(x,u_j)\|_{L^{\frac{\ell+1}{\ell}}(\Omega)}^{\frac{\ell+1}{\ell}} = \int_\Omega |\psi(x,u_j)|^{\frac{\ell+1}{\ell}} dx \leqslant \int_\Omega (C_1 + C_2|u_j|^\ell)^{\frac{\ell+1}{\ell}} dx$$
$$\leqslant \tilde{C}_1 + \tilde{C}_2 \int_\Omega |u_j|^{\ell+1} \leqslant \tilde{C}_1 + \tilde{C}_2 \|u_j\|_{L^p(\Omega)} \leqslant \tilde{C}_3$$

by Hölder inequality we obtain

$$\left| \int_\Omega \frac{\psi(x,u_j)(u_j-u)}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_j|^p dx\Big)} dx \right| \leqslant \left| \int_\Omega \frac{\psi(x,u_j)(u_j-u)}{m} dx \right|$$
$$\leqslant \frac{1}{m} \|\psi(\cdot,u_j)\|_{L^{\frac{\ell+1}{\ell}}(\Omega)} \|u_j - u\|_{L^{\ell+1}(\Omega)} \overset{j\to\infty}{\longrightarrow} 0.$$

So, by using the inequalities above, the convergence of (u_j) in L^p and Lemma 1.4 we have

$$\limsup_{j\to\infty} \langle \mathscr{B}_\epsilon u_j, u_j - u\rangle = \limsup_{j\to\infty} \left[\int_\Omega |\nabla u_j|^{p-2} \nabla u_j \nabla(u_j - u) dx + \epsilon \int_\Omega |u_j|^{p-2} u_j (u_j - u) dx \right]$$
$$= \limsup_{j\to\infty} \left[\int_\Omega |\nabla u_j|^{p-2} \nabla u_j \nabla(u_j - u) dx \right]$$
$$= \limsup_{j\to\infty} \left[\int_\Omega |\nabla u_j|^{p-2} \nabla u_j \nabla(u_j - u) dx + \int_\Omega |\nabla u|^{p-2} \nabla u \nabla(u_j - u) - \int_\Omega |\nabla u|^{p-2} \nabla u \nabla(u_j - u) \right]$$

Since $\varphi \mapsto \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi$ define a bounded linear functional in $W^{1,p}(\Omega)$ for each $u \in W^{1,p}(\Omega)$, and $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, it follows that

$$\limsup_{j\to\infty} \langle \mathscr{B}_\epsilon u_j, u_j - u\rangle = \limsup_{j\to\infty} \left[\int_\Omega \left(|\nabla u_j|^{p-2} \nabla u_j - |\nabla u|^{p-2} \nabla u \right) \nabla(u_j - u) dx \right].$$

Thus Lemma 1.5 implies

$$\int_\Omega \big(|\nabla u_j|^{p-2} \nabla u_j - |\nabla u|^{p-2} \nabla u \big) \nabla(u_j - u) dx \overset{j\to\infty}{\longrightarrow} 0. \tag{3.20}$$

Claim $u_j \to u$ strongly in $W^{1,p}(\Omega)$, as $j \to \infty$.

Indeed, for $p \geqslant 2$ we have from (3.20) and Lemma 1.5

$$C \int_\Omega |\nabla u_j - \nabla u|^p dx \leqslant \int_\Omega \big(|\nabla u_j|^{p-2} \nabla u_j - |\nabla u|^{p-2} \nabla u \big) \nabla(u_j - u) dx \overset{j\to\infty}{\longrightarrow} 0.$$

If $1 < p < 2$, from (3.20), Lemma 1.5, and Hölder's inequality we have

$$
\begin{aligned}
\int_\Omega |\nabla(u_j - u)|^p dx &= \int_\Omega |\nabla(u_j - u)|^p \frac{(|\nabla u_j| + |\nabla u|)^{\frac{p(2-p)}{2}}}{(|\nabla u_j| + |\nabla u|)^{\frac{p(2-p)}{2}}} dx \\
&= \int_\Omega \frac{|\nabla(u_j - u)|^p}{(|\nabla u_j| + |\nabla u|)^{\frac{p(2-p)}{2}}} (|\nabla u_j| + |\nabla u|)^{\frac{p(2-p)}{2}} dx \\
&\leqslant \left[\left(\int_\Omega \frac{|\nabla(u_j - u)|^p}{(|\nabla u_j| + |\nabla u|)^{\frac{p(2-p)}{2}}}\right)^{\frac{2}{p}} dx\right]^{\frac{p}{2}} \left[\int_\Omega \left((|\nabla u_j| + |\nabla u|)^{\frac{p(2-p)}{2}}\right)^{\frac{-2}{p-2}} dx\right]^{\frac{-(p-2)}{2}} \\
&= \left(\int_\Omega \frac{|\nabla(u_j - u)|^2}{(|\nabla u_j| + |\nabla u|)^{2-p}} dx\right)^{\frac{p}{2}} \left(\int_\Omega |\nabla u_j| + |\nabla u|)^p\right)^{\frac{-(p-2)}{2}} \stackrel{j\to\infty}{\longrightarrow} 0
\end{aligned}
$$

since the last term is bounded. Hence the claim follows, and (u_j) strongly converges to u in $W^{1,p}(\Omega)$. Taking into account that $\mathscr{B}_\varepsilon$ is continuous, we conclude that (3.19) holds. Now all of the hypotheses of Theorem (1.19) are fulfilled. Then $\mathscr{B}_\varepsilon$ is surjective, what ensures the existence of $u_\varepsilon \in W^{1,p}(\Omega)$ satisfying $\mathscr{B}_\varepsilon u_\varepsilon = 0$, i.e., u_ε is a solution of (3.9). The proof is complete. □

3.3 Proof of Theorem 3.1

In this section we will prove a lemma on the auxiliary problem which will help us to obtain a solution for (3.1).

Lemma 3.3 *Assume the same hypotheses in Theorem* (3.1). *Then the family* (u_ε) *of solutions of* (3.9) *strongly converge in* $W^{1,p}(\Omega)$, *as* $\varepsilon \to 0$, *to a solution* $u_R \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *of the limiting problem*

$$
\begin{cases} -\Delta_p u = F_R(u) - \Psi(u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\partial_\nu u = G(u) & \text{on } \partial\Omega, \end{cases} \tag{3.21}
$$

where F_R *and* Ψ *are given by* (3.10) *and* (3.12), *respectively.*

Proof Let $u_\varepsilon = u_{\varepsilon_+} - u_{\varepsilon_-}$, be the sum of its the positive and negative parts of u_ϵ, and recalling (3.18), one can take u_{ε_-} as test function in (3.9) to obtain

$$
-\|u_{\varepsilon_-}\|^p_{1,p,\varepsilon} = \int_\Omega \frac{\big[f_R(x, Tu_\varepsilon, \nabla Tu_\varepsilon) - \psi(x, u_\varepsilon)\big] u_{\varepsilon_-}}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_\varepsilon|^p dx\Big)} dx + \int_{\partial\Omega} \frac{g(x, Tu_\varepsilon) u_{\varepsilon_-}}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_\varepsilon|^p dx\Big)} d\mathcal{H}^{N-1}.
$$

Now, from the definitions of f_R, Ψ, and T we have

$$
\begin{aligned}
-\|u_{\varepsilon_-}\|^p_{1,p,\varepsilon} = & \int_\Omega \frac{f(x,\underline{u},\nabla\underline{u})u_{\varepsilon_-}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega|\nabla u_\varepsilon|^p dx\Big)}dx + \int_{\partial\Omega}\frac{g(x,\underline{u})u_{\varepsilon_-}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega|\nabla u_\varepsilon|^p dx\Big)}d\mathcal{H}^{N-1}\\
& + \int_{\{u_\varepsilon\leqslant\underline{u}\}}\frac{(\underline{u}-u_\varepsilon)^\ell_+ u_{\varepsilon_-}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega|\nabla u_\varepsilon|^p dx\Big)}dx.
\end{aligned}
$$

Since the terms in the right-hand side are non-negative (see the hypotheses in Theorem 3.1) we have $\|u_{\varepsilon-}\|^p_{1,p,\varepsilon}=0$, i.e., $u_{\varepsilon-}=0$ a.e in Ω. Hence $u_\varepsilon = u_{\varepsilon_+}\geqslant 0$ for all $\varepsilon>0$. On the other hand, if we take $v=(u_\varepsilon-\bar{u})_+$ as test function in (3.9) and recalling (3.14), we obtain

$$
\begin{aligned}
\langle\mathcal{I}_\varepsilon(u_\varepsilon),(u_\varepsilon-\bar{u})_+\rangle = & \int_\Omega\frac{\big[f_R(x,Tu_\varepsilon,\nabla Tu_\varepsilon)-\psi(x,u_\varepsilon)\big]}{\mathcal{M}\Big(x,\displaystyle\int_\Omega|\nabla u_\varepsilon|^p dx\Big)}(u_\varepsilon-\bar{u})_+ + \int_{\partial\Omega}\frac{g(x,Tu_\varepsilon)}{\mathcal{M}\Big(x,\displaystyle\int_\Omega|\nabla u_\varepsilon|^p dx\Big)}(u_\varepsilon-\bar{u})_+\\
= & \int_\Omega\frac{\big[f_R(x,\bar{u},\nabla\bar{u})-\psi(x,u_\varepsilon)\big]}{\mathcal{M}\Big(x,\displaystyle\int_\Omega|\nabla u_\varepsilon|^p dx\Big)}(u_\varepsilon-\bar{u})_+ + \int_{\partial\Omega}\frac{g(x,\bar{u})}{\mathcal{M}\Big(x,\displaystyle\int_\Omega|\nabla u_\varepsilon|^p dx\Big)}(u_\varepsilon-\bar{u})_+.
\end{aligned}
$$

At this stage (sometimes the symbols dx and $d\mathcal{H}^{N-1}$ will be omitted), in all cases listed in Theorem 3.1 we have an estimate of the form

$$
\begin{aligned}
\int_\Omega|\nabla u_\epsilon|^{p-2}\nabla u_\varepsilon\nabla(u_\epsilon-\bar{u})_+dx \leqslant & \frac{1}{Q_1}\int_\Omega f(x,\bar{u},\nabla\bar{u})(u_\varepsilon-\bar{u})_+dx - \int_\Omega\frac{(u_\varepsilon-\bar{u})^{\ell+1}_+}{\mathcal{M}\Big(x,\displaystyle\int_\Omega|\nabla u_\varepsilon|^p dx\Big)}dx\\
& + \frac{1}{Q_2}\int_{\partial\Omega}g(x,\bar{u})(u_\varepsilon-\bar{u})_+d\mathcal{H}^{N-1} \qquad (3.22)
\end{aligned}
$$

where

$$
(Q_1,Q_2)=\begin{cases}(M,m), & \text{if } \ f(x,\bar{u},\nabla\bar{u})\leqslant 0 \ \text{ and } \ g(x,\bar{u})\geqslant 0,\\[2ex] (M,M), & \text{if } \ f(x,\bar{u},\nabla\bar{u})\leqslant 0 \ \text{ and } \ g(x,\bar{u})\leqslant 0,\\[2ex] (m,M), & \text{if } \ f(x,\bar{u},\nabla\bar{u})\geqslant 0 \ \text{ and } \ g(x,\bar{u})\leqslant 0.\end{cases} \qquad (3.23)
$$

Hypothesis (iii) in Theorem 3.1 ensures

$$
\frac{1}{Q_1}\int_\Omega f(x,\bar{u},\nabla\bar{u})(u_\varepsilon-\bar{u})_+dx+\frac{1}{Q_2}\int_{\partial\Omega}g(x,\bar{u})(u_\varepsilon-\bar{u})_+d\mathcal{H}^{N-1}\leqslant\int_\Omega|\nabla\bar{u}|^{p-2}\nabla\bar{u}\nabla(u_\epsilon-\bar{u})_+dx,
$$

what, combined with (3.22), implies

$$\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(u_\varepsilon-\bar{u})_+\,dx \leqslant \int_\Omega |\nabla \bar{u}|^{p-2}\nabla \bar{u}\nabla(u_\epsilon-\bar{u})_+dx - \int_\Omega \frac{(u_\varepsilon-\bar{u})_+^{\ell+1}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega |\nabla u_\varepsilon|^p dx\Big)}dx.$$

Therefore

$$\int_\Omega \big(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon - |\nabla \bar{u}|^{p-2}\nabla \bar{u}\big)\nabla(u_\epsilon-\bar{u})_+\,dx \leqslant -\int_\Omega \frac{(u_\epsilon-\bar{u})_+^{\ell+1}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega |\nabla u_\epsilon|^p dx\Big)}\,dx \leqslant 0.$$

Now, using Corollary 1.7 we have

$$\int_\Omega |\nabla(u_\varepsilon-\bar{u})_+|^p dx = \int_\Omega |\nabla(u_\varepsilon-\bar{u})\chi_{\{u_\varepsilon-\bar{u}\geqslant 0\}}|^p dx = \int_\Omega |\nabla(u_\varepsilon-\bar{u})|^p \chi_{\{u_\varepsilon-\bar{u}\geqslant 0\}}dx,$$

so, from Lemma 1.5, we obtain

$$0 \leqslant \int_\Omega C|\nabla(u_\epsilon-\bar{u})|^p dx \leqslant -\int_\Omega \frac{(u_\epsilon-\bar{u})_+^{\ell+1}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega |\nabla u_\epsilon|^p dx\Big)}\,dx \leqslant 0.$$

Hence

$$\int_\Omega \frac{(u_\epsilon-\bar{u})_+^{\ell+1}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega |\nabla u_\epsilon|^p dx\Big)}\,dx = 0,$$

what shows that $(u_\epsilon-\bar{u})_+ = 0$ a.e in Ω for all $\epsilon>0$, and we conclude

$$0 \leqslant u_\varepsilon \leqslant \bar{u} \text{ a.e. in } \Omega.$$

This implies

$$0 \leqslant u_\varepsilon \leqslant \|\bar{u}\|_\infty \quad \text{a.e in} \quad \Omega, \quad \forall \varepsilon > 0,$$

and (u_ε) is bounded in $L^\infty(\Omega)$. Using u_ε as test function in (3.9) and recalling (3.14) – (3.17), we have

$$|\langle \mathcal{I}_\varepsilon(u_\varepsilon), u_\varepsilon\rangle| \leqslant \tilde{K}(R, \|\bar{u}\|_\infty), \tag{3.24}$$

where $\tilde{K}>0$ does not depend on $\varepsilon>0$, so (u_ε) is bounded in $W^{1,p}(\Omega)$. Thus, by choosing $\varepsilon_j \to 0$, there exists $u_R \in W^{1,p}(\Omega)$ satisfying

- $u_{\epsilon_j} \rightharpoonup u_R$ in $W^{1,p}(\Omega)$,

- $u_{\epsilon_j} \to u_R$ in $L^p(\Omega)$,

as $j \to \infty$. Using $v = (u_{\varepsilon_j} - u)$ as a test function in (3.9), following the arguments used in (3.24) we infer

$$\int_\Omega \left[|\nabla u_{\varepsilon_j}|^{p-2}\nabla u_{\varepsilon_j} - |\nabla u_R|^{p-2}\nabla u_R\right] \nabla(u_{\varepsilon_j} - u_R)dx \leqslant \frac{K}{m}\int_\Omega |u_{\varepsilon_j} - u_R|dx.$$

Therefore

$$\int_\Omega |\nabla(u_{\varepsilon_j} - u_R)|^p dx \overset{j\to\infty}{\longrightarrow} 0,$$

so (u_{ε_j}) strongly converges to u_R in $W^{1,p}(\Omega)$. Hence it is possible to pass to the limit in (3.9) to obtain $u_R \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution of (3.21), proving the lemma. □

Remark 3.2 *As a consequence of the proof of Lemma 3.3 we have proved that* $0 \leqslant u_R \leqslant \bar{u}$ *a.e. in* Ω*, so* $\|u_R\|_\infty$ *does not depend on* R*.*

Lemma 3.4 *Assume all the hypotheses in Theorem 3.1. Then* $\underline{u} \leqslant u_R$ *a.e. in* Ω*.*

Proof Using the test function $v = (\underline{u} - u_R)_+$ in (3.21) gives

$$\begin{aligned}\int_\Omega |\nabla u_R|^{p-2}\nabla u_R \nabla(\underline{u} - u_R)_+ dx =& \int_\Omega \frac{\left[f_R(x, Tu_R, \nabla Tu_R) - \psi(x, u_R)\right]}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_R|^p dx\Big)}(\underline{u} - u_R)_+ dx \\ &+ \int_{\partial\Omega} \frac{g(x, Tu_R)}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_R|^p dx\Big)}(\underline{u} - u_R)_+ d\mathcal{H}^{N-1}.\end{aligned}$$

Thus from (3.7) and (3.12), we have

$$\begin{aligned}\int_\Omega |\nabla u_R|^{p-2}\nabla u_R \nabla(\underline{u} - u_R)_+\, dx =& \int_\Omega \frac{f(x, \underline{u}, \nabla\underline{u})(\underline{u} - u_R)_+ + (\underline{u} - u_R)_+^{\ell+1}}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_R|^p dx\Big)}\, dx \\ &+ \int_{\partial\Omega} \frac{g(x, \underline{u})}{\mathcal{M}\Big(x, \int_\Omega |\nabla u_R|^p dx\Big)}(\underline{u} - u_R)_+ d\mathcal{H}^{N-1}.\end{aligned}$$

Then, by using hypothesis (i) in Theorem 3.1, we obtain

$$\int_\Omega |\nabla u_R|^{p-2}\nabla u_R \nabla(\underline{u}-u_R)_+\,dx \geqslant \frac{1}{M}\int_\Omega f(x,\underline{u},\nabla\underline{u})(\underline{u}-u_R)_+dx + \int_\Omega \frac{(\underline{u}-u_R)_+^{\ell+1}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega |\nabla u_R|^p dx\Big)}\,dx$$
$$+\frac{1}{M}\int_{\partial\Omega} g(x,\underline{u})(\underline{u}-u_R)_+ d\mathcal{H}^{N-1}.$$

Now, hypothesis (ii) in Theorem 3.1 implies

$$\int_\Omega |\nabla u_R|^{p-2}\nabla u_R \nabla(\underline{u}-u_R)_+\,dx \geqslant \int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla(\underline{u}-u_R)_+\,dx + \int_\Omega \frac{(\underline{u}-u_R)_+^{\ell+1}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega |\nabla u_R|^p dx\Big)}\,dx.$$

so, from Corollary 1.7 and Lemma 1.5,

$$0 \geqslant \int_\Omega C|\nabla(\underline{u}-u_R)_+|^p dx + \int_\Omega \frac{(\underline{u}-u_R)_+^{\ell+1}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega |\nabla u_R|^p dx\Big)}\,dx \geqslant 0.$$

Hence

$$\int_\Omega \frac{(\underline{u}-u_R)_+^{\ell+1}}{\mathcal{M}\Big(x,\displaystyle\int_\Omega |\nabla u_R|^p dx\Big)}\,dx = 0,$$

what ensures $(\underline{u}-u_R)_+^{\ell+1}=0$ a.e. in Ω, that is, $\underline{u}\leqslant u_R$ a.e in Ω. □

Thus $0\leqslant \underline{u}\leqslant u_R\leqslant \bar{u}$ a.e. in Ω for all $R\geqslant R_0$, so, from (3.12) it follows that $\psi=0$, i.e, from (3.9)-(3.13), $u_R\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ is a, weak solution of the problem

$$\begin{cases} -\Delta_p u = \dfrac{f_R(\cdot,u(\cdot),\nabla u(\cdot))}{\mathcal{M}\Big(\cdot,\displaystyle\int_\Omega |\nabla u|^p dx\Big)} & \text{in } \Omega, \\ |\nabla u|^{p-2}\partial_\nu u = \dfrac{g(\cdot,u(\cdot))}{\mathcal{M}\Big(\cdot,\displaystyle\int_\Omega |\nabla u|^p dx\Big)} & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

Now since one has:

- $\|u\|_\infty$ is independent on R large;
- $\dfrac{f_R(\cdot,u(\cdot),\nabla u(\cdot))}{\mathcal{M}(\cdot,\int_\Omega |\nabla u|^p dx)}$ satisfies (H_f) independently on R(Lemma 2.1 of Chapter 2);
- $\dfrac{g(\cdot,\cdot)}{\mathcal{M}(\cdot,\int_\Omega |\nabla u|^p dx)} \in C^{0,\min\{\alpha,\beta\}}(\partial\Omega\times\mathbb{R},\mathbb{R})$, with α,β independent on R;

The regularity result of Theorem 1.1 ensures $u\in C^{1,\gamma}(\overline{\Omega})$, with γ and $\|u\|_{C^{1,\gamma}(\overline{\Omega})}$ independent on $R \in [R_0, \infty)$. Hence we can infer $f_R = f$ for $R \in [R_0, \infty)$, what allows to conclude that u is a solution of (3.1). The proof is complete.

3.4 Positive solutions of non-coercive BVPs: examples

In this section we will list some applications for Theorem 3.1.

1) Example 1: In the first example, we consider a problem with logistic sources of the form

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u \;=\; u^r - u^s + h(x) + a(x)|\nabla u|^\ell \quad \text{in } \Omega, \\ \mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)|\nabla u|^{p-2}\partial_\nu u \;+ u^q = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.26}$$

The coefficients satisfy, respectively,

(A_1) $0 < r < p-1 < s < \infty$, $q \in (1, \infty)$ and $\ell \in [0, p]$;

(A_2) $a, h \in L^\infty(\Omega)$ with $a(\cdot) \geqslant 0$ and $h(\cdot) \geqslant 0$.

Theorem 3.2 *Assume (A_1), (A_2) and $(H_{\mathcal{M}})$ hold. Then there exists a solution $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, of (3.26), satisfying $u \geqslant 0$ on $\overline{\Omega}$. If $h \not\equiv 0$, then $u > 0$ on $\overline{\Omega}$.*

2) Example 2: In the example, we consider the problem

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u \;= u^r - u^s + \dfrac{|\nabla u|^\ell}{u^\theta} \quad \text{in } \Omega, \\ \mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)|\nabla u|^{p-2}\partial_\nu u \;=\; u^q(c(x) - u) \quad \text{on } \partial\Omega. \end{cases} \tag{3.27}$$

The coefficients satisfy

(A_3) $r, s \in (0, \infty), r < s$, $\theta \in (0, \infty)$, $q > 1$ and $\ell \in [0, p]$;

(A_4) $c \in C^{1,\alpha}(\partial\Omega), \alpha \in (0,1), c_0 \leqslant c(x) \leqslant c_1 \;\; \forall x \in \partial\Omega,\; c_0, c_1 > 0.$

Theorem 3.3 *Assume that (A_3), (A_4) and $(H_{\mathcal{M}})$ hold. Then there exists a solution $u \in C^{1,\sigma}(\overline{\Omega})$ of (3.27), with $\sigma \in (0,1)$, satisfying $u > 0$ on $\overline{\Omega}$.*

3) Example 3: In the example, we consider a problem with singular terms in the boundary condition of the form

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)\Delta_p u \ = \ |\nabla u|^{\ell} \quad \text{in } \Omega, \\ \mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)|\nabla u|^{p-2}\partial_\nu u \ = \ \dfrac{1}{u^{\alpha}} - \dfrac{1}{u^{\beta}} \quad \text{on } \partial\Omega, \end{cases} \tag{3.28}$$

where the exponents satisfy

(A_5) $\quad 0 < \beta < \alpha < \infty$ and $\ell \in [0, p]$.

Theorem 3.4 *Assume* (A_5) *and* $(H_{\mathcal{M}})$ *hold. Then there exists a solution* $u \in C^{1,\sigma}(\overline{\Omega})$ *of* (3.28), *with* $\sigma \in (0,1)$, *satisfying* $u > 0$ *on* $\overline{\Omega}$.

4) Example 4: In the example, we consider a problem with singularity form

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)\Delta_p u \ = \ |\nabla u|^{\ell} \quad \text{in } \Omega, \\ \mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)|\nabla u|^{p-2}\partial_\nu u \ = \ \lambda c(x) u^r - u^s \quad \text{on } \partial\Omega. \end{cases} \tag{3.29}$$

(A_6) $1 < r < s < \infty$, $\lambda > 0$ and $\ell \in [0, p]$;

(A_7) $c \in C^{1,\alpha}(\partial\Omega), \alpha \in (0,1), c_0 \leqslant c(x) \leqslant c_1 \;\; \forall x \in \partial\Omega, \; c_0, c_1 > 0.$

Theorem 3.5 *Assume that* (A_6),(A_7) *and* $(H_{\mathcal{M}})$ *hold. Then there exists a solution* $u \in C^{1,\gamma}(\overline{\Omega})$ *of* (3.29), *with* $\gamma \in (0,1)$, *satisfying* $u > 0$ *on* $\overline{\Omega}$.

5) Example 5: In this example, we consider a problem with the form

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)\Delta_p u \ = \ u^r + u^s + |\nabla u|^{\ell} - \mu \sin(u^{p-1}) \quad \text{in } \Omega, \\ \mathcal{M}\Big(x, \displaystyle\int_{\Omega} |\nabla u|^p dx\Big)|\nabla u|^{p-2}\partial_\nu u \ = u^{\theta} - \mu \sin(u^{p-1}) \quad \text{on } \partial\Omega. \end{cases} \tag{3.30}$$

(A_8) $0 < r < s < \infty$ and $\ell \in [0, p]$;

(A_9) $1 < \theta < \infty$ and $\mu > \mu^*$

Where $\mu^* = \left(\frac{5\pi}{2}\right)^{\frac{r}{p-1}} + \left(\frac{5\pi}{2}\right)^{\frac{s}{p-1}} + \left(\frac{5\pi}{2}\right)^{\frac{\theta}{p-1}}$.

Theorem 3.6 *Assume that* (A_8),(A_9) *and* $(H_{\mathcal{M}})$ *hold. Then there exists a solution* $u \in C^{1,\gamma}(\overline{\Omega})$ *of* (3.30), *with* $\gamma \in (0,1)$, *satisfying* $u > 0$ *on* $\overline{\Omega}$.

6) Example 6: In this example, we consider a problem with the form

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^2 dx\Big)\Delta u \; = \; \lambda u - u^q \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad \partial_\nu u \; = u \quad \text{on } \partial\Omega. \end{cases} \tag{3.31}$$

(A_{10}) $q > 1$ and $\lambda > \frac{\lambda_1}{M}$.

Theorem 3.7 *Assume that* (A_{10}) *and* $(H_{\mathcal{M}})$ *hold. Then there exists a solution* $u \in C^{1,\gamma}(\overline{\Omega})$ *of* (3.31), *with* $\gamma \in (0,1)$, *satisfying* $u > 0$ *on* $\overline{\Omega}$.

3.5 Proofs of Theorems 3.2–3.7

Proof of Theorem 3.2: Consider the function $f(x,t,\xi) = |t|^r - |t|^s + h(x) + a(x)|\xi|^\ell$, $\forall (x,t,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, and $g(x,t) = -|t|^q$. By $(A_1), (A_2)$ we have that (H_f) is satisfied. Choosing $\underline{u} = 0$, $(i), (ii)$ in Theorem 3.1 hold. We now choose $\bar{u} = \tau_1 > 0$ such that $\tau_1^s \geqslant \tau_1^r + \|h(x)\|_\infty$, then

$$f(x,\tau_1,0) = \tau_1^r - \tau_1^s + h(x) \leqslant 0 \;\text{ and }\; g(x,\tau_1) = -|\tau_1|^q \leqslant 0, \tag{3.32}$$

and condition (iii) in Theorem 3.1 is satisfied. Therefore Theorem 3.1 ensures there exists a solution $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, with $0 \leqslant u \leqslant \tau_1$ in $\overline{\Omega}$. Furthermore, by (A_2) we have

$$\Delta_p u + \frac{1}{m} u^s \geqslant \Delta_p u + \frac{u^s}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)} = \frac{u^r + h(x) + a(x)|\nabla u|^\ell}{\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)} \geqslant 0, \tag{3.33}$$

so, by the maximum principle (see Theorem 1.18), one has $u > 0$ in Ω, if $h \not\equiv 0$.

Finally, in the former case, if $u(x_0) = 0$ for some $x_0 \in \partial\Omega$, by Hopf's lemma (see Lemma 1.1) we obtain

$$0 = -(u(x_0))^q = \mathcal{M}\Big(x_0, \int_\Omega |\nabla u|^p dx\Big)|\nabla u(x_0)|^{p-2}\partial_\nu(x_0) < 0,$$

what is impossible. Hence, if $h \not\equiv 0$, we have $u > 0$ on $\overline{\Omega}$, and the proof is complete. □

Proof of Theorem 3.3: Consider the sequence of approximate problems

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_n|^p dx\Big)\Delta_p u_n = |u_n|^r - |u_n|^s + \dfrac{|\nabla u_n|^\ell}{\big(|u_n| + \frac{1}{n}\big)^\theta} & \text{in } \Omega, \\ \mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_n|^p dx\Big)|\nabla u_n|^{p-2}\partial_\nu u_n = |u|_n^q(c(x) - |u_n|) & \text{on } \partial\Omega, \end{cases} \tag{3.34}$$

for all $n \geqslant 1$, define the functions

$$f_n(x,t,\xi) = |t|^r - |t|^s + \frac{|\xi|^\ell}{\big(|t| + \frac{1}{n}\big)^\theta} \quad \text{and} \quad g_n(x,t) = |t|^q(c(x) - |t|) \tag{3.35}$$

for all $(x,t,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. Therefore, from definition and (A_3) we have that (H_f) is satisfied. Now, choose $\underline{u} = \tau_0 \leqslant \min\{1, c_0\}$ then we obtain that

$$f_n(x,\tau_0,0) = |\tau_0|^r - |\tau_0|^s \geqslant 0 \quad \text{and} \quad g_n(x,\tau_0) = |\tau_0|^q(c(x) - |\tau_0|) \geqslant 0$$

satisfying (i) and (ii) of Theorem (3.1). Define $\bar{u} = \tau_1 \geqslant \max\{1, c_1\}$ such that $\tau_1 \geqslant 1$ then

$$f_n(x,\tau_1,0) = |\tau_1|^r - |\tau_1|^s \leqslant 0 \quad \text{and} \quad g_n(x,\tau_1) = |\tau_1|^q(c(x) - |\tau_1|) \leqslant 0,$$

satisfying (iii) of Theorem 3.1 for every $n \geqslant 1$. Then, we have constants $\tau_0, \tau_1 > 0$, independent on n, which allow one to apply Theorem 3.1 to ensure the existence of solutions $u_n \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, of (3.34) satisfying $\tau_0 \leqslant u_n \leqslant \tau_1$ in $\overline{\Omega}$. Thus $\|u_n\|_\infty$ is uniformly bounded with respect to n and one can modify the functions in (3.35) to verify a growth condition of the form

$$|f_n(x,s,\xi)| \leqslant C(1 + |\xi|^\ell) \quad \text{for a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^N, \tag{3.36}$$

for some $C > 0$ independent on n. From the regularity results in [66, 39], there exists $\tilde{\gamma} \in (0,1)$, independent on n, such that $\|u_n\|_{C^{1,\tilde{\gamma}}(\overline{\Omega})}$ is bounded. The compactness of the imbedding $C^{1,\tilde{\gamma}}(\overline{\Omega}) \hookrightarrow C^{1,\sigma}(\overline{\Omega})$ for all $\sigma \in (0,\tilde{\gamma})$ implies the existence of $u \in C^{1,\sigma}(\overline{\Omega})$, and a not relabeled subsequence, satisfying $u_n \to u$ in $C^{1,\sigma}(\overline{\Omega})$, as $n \to \infty$. Hence $\tau_0 \leqslant u \leqslant \tau_1$ in $\overline{\Omega}$, and we can pass to the limit in the weak formulation of (3.34) (as in (3.2)) to conclude that u is a solution of (3.27). The proof is complete. $\square$

Proof of Theorem 3.4: Consider the approximate problems for all $n \geq 1$

$$\begin{cases} -\mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_n|^p dx\Big)\Delta_p u_n = |\nabla u_n|^\ell \quad \text{in } \Omega, \\ \mathcal{M}\Big(x, \displaystyle\int_\Omega |\nabla u_n|^p dx\Big)|\nabla u_n|^{p-2}\partial_\nu u_n = \dfrac{1}{\big(|u_n| + \frac{1}{n}\big)^\alpha} - \dfrac{1}{\big(|u_n| + \frac{1}{n}\big)^\beta} \quad \text{on } \partial\Omega, \end{cases} \tag{3.37}$$

and, for all $(x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, the functions

$$f_n(x, t, \xi) = |\xi|^\ell \text{ and } g_n(x, t) = \frac{1}{\big(|t| + \frac{1}{n}\big)^\alpha} - \frac{1}{\big(|t| + \frac{1}{n}\big)^\beta}.$$

Note that f_n satisfy (H_f). Now we set $\underline{u} = \tau_0 > 0$ such that $\tau_0 < 1$, and observe that

$$f_n(x, \tau_0, 0) \geqslant 0 \text{ and } g_n(x, \tau_0) = \frac{1}{\big(|\tau_0| + \frac{1}{n}\big)^\alpha} - \frac{1}{\big(|\tau_0| + \frac{1}{n}\big)^\beta} \geqslant 0,$$

for all n sufficiently large. Further, (i) and (ii) in Theorem 3.1 are satisfied. Now, to check (iii) in Theorem 3.1, we choose $\bar{u} = \tau_1$ such that $\tau_1 > 1$, so we have

$$f_n(x, \tau_1, 0) \geqslant 0 \text{ and } g_n(x, \tau_1) = \frac{1}{\big(|\tau_1| + \frac{1}{n}\big)^\alpha} - \frac{1}{\big(|\tau_1| + \frac{1}{n}\big)^\beta} \leqslant 0, \ \forall n \geqslant 1.$$

So, we have obtained constants $\tau_0, \tau_1 > 0$, independent on n, which allow one to apply Theorem 3.1 to ensure the existence of solutions $u_n \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0, 1)$, of (3.37) satisfying $\tau_0 \leqslant u_n \leqslant \tau_1$ in $\overline{\Omega}$. Thus $\|u_n\|_\infty$ is uniformly bounded with respect to n and f_n verify a growth condition of the form

$$|f_n(x, s, \xi)| \leqslant C(1 + |\xi|^\ell) \quad \text{for a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^N, \tag{3.38}$$

for some $C > 0$ independent on n. From the regularity results in [66, 39], there exists $\tilde{\gamma} \in (0, 1)$, independent on n, such that $\|u_n\|_{C^{1,\tilde{\gamma}}(\overline{\Omega})}$ is bounded. The compactness of the imbedding $C^{1,\tilde{\gamma}}(\overline{\Omega}) \hookrightarrow C^{1,\sigma}(\overline{\Omega})$ for all $\sigma \in (0, \tilde{\gamma})$ implies the existence of $u \in C^{1,\sigma}(\overline{\Omega})$, and a not relabeled subsequence, satisfying $u_n \to u$ in $C^{1,\sigma}(\overline{\Omega})$, as $n \to \infty$. Hence $\tau_0 \leqslant u \leqslant \tau_1$ in $\overline{\Omega}$, and we can pass to the limit in the weak formulation of (3.37) (as in (3.2)) to conclude that u is a solution of (3.28). The proof is complete. $\square$

Proof of Theorem 3.5: First of all, consider the functions

$$f(x,t,\xi) = |\xi|^{\ell} \ \text{ and } \ g(x,t) = \lambda c(x)|t|^{r} - |t|^{s}$$

for all $(x,t,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. From a calculation, (A_6) and (A_7), we have that (H_f) is satisfied. We now set $\underline{\mathrm{u}} = \tau_0$ such that $\tau_0 < (\lambda c_0)^{\frac{1}{s-r}}$, then by (A_6) and (A_7) we have

$$f(x,\tau_0,0) \geqslant 0 \ \text{ and } \ g(x,\tau_0) \geqslant \tau_0^r(\lambda c_0 - \tau_0^{s-r}) \geqslant 0,$$

so $(i),(ii)$ in Theorem 3.1 are satisfied. Take now $\bar{u} = \tau_1$ such that $\tau_1 \geqslant (\lambda c_1)^{\frac{1}{s-r}}$, then $\tau_0 < \tau_1$ and we have by (A_6) and (A_7) that

$$f(x,\tau_1,0) \leqslant 0 \ \text{ and } \ g(x,\tau_1) \leqslant \tau_1^r(\lambda c_1 - \tau_1^{s-r}) \leqslant 0.$$

Thus (iii) in Theorem 3.1 hold, and the same Theorem implies there exists $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$ such that $\tau_0 \leqslant u \leqslant \tau_1$ and u is a solution of (3.29).

Proof of Theorem 3.6: Consider the functions

$$f(x,t,\xi) = |t|^r + |t|^s + |\xi|^{\ell} - \mu \sin(|t|^{p-1}) \ \text{ and } \ g(x,t) = |t|^{\theta} - \mu \sin(|t|^{p-1})$$

for all $(x,t,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. From (A_{10}) and (A_{11}) we have that f satisfy (H_f). We now choose $\underline{\mathrm{u}} = \pi^{\frac{1}{p-1}} \in \mathbb{R}$ then

$$f(x,\pi^{\frac{1}{p-1}},0) = \pi^{\frac{r}{p-1}} + \pi^{\frac{s}{p-1}} \geqslant 0 \ \text{ and } \ g(x,\pi^{\frac{1}{p-1}}) = \pi^{\frac{\theta}{p-1}} \geqslant 0,$$

and (i),(ii) from Theorem (3.1) are satisfied. Now take $\bar{u} = (\frac{5\pi}{2})^{\frac{1}{p-1}}$ and observe that

$$f(x,(\frac{5\pi}{2})^{\frac{1}{p-1}},0) = \left(\frac{5\pi}{2}\right)^{\frac{r}{p-1}} + \left(\frac{5\pi}{2}\right)^{\frac{s}{p-1}} - \mu \leqslant 0 \ \text{ and } \ g(x,(\frac{5\pi}{2})^{\frac{1}{p-1}}) = (\frac{5\pi}{2})^{\frac{\theta}{p-1}} - \mu \leqslant 0.$$

Satisfying the condition (iii), then Theorem (3.1) conclude that exists a positive solution $u \in C^{1,\gamma}(\overline{\Omega})$ with $\gamma \in (0,1)$ for (3.30) with $\pi^{\frac{1}{p-1}} \leqslant u \leqslant (\frac{5\pi}{2})^{\frac{1}{p-1}}$.

Proof of Theorem 3.7: First of all, consider the eigenvalue problem:

$$\begin{cases} -\Delta\varphi_1 \; = \; \lambda_1\varphi_1 \quad \text{in } \Omega, \\ \quad \partial_\nu u \; = \; \varphi_1 \quad \text{on } \partial\Omega, \end{cases} \tag{3.39}$$

With $\lambda_1 \in \mathbb{R}$ principal eigenvalue and φ_1 the principal auto function associate to λ_1, where $\varphi_1 > 0$ on $\overline{\Omega}$ and $\|\varphi_1\|_\infty = 1$

Choose $\bar{u} = C\varphi_1$, now, let us determine C. We shall use the following case on Theorem 3.1, $f(x, \bar{u}, \nabla\bar{u}) \leqslant 0$ and $g(x, \bar{u}) \geqslant 0$, so

$$\int_\Omega \nabla\bar{u}\nabla\varphi \geqslant \frac{1}{M}\int_\Omega (\lambda\bar{u} - \bar{u}^q)\varphi + \frac{1}{m}\int_{\partial\Omega} \bar{u}\varphi$$

By the eigenvalue problem 3.39,

$$\frac{C^q}{M}\int_\Omega \varphi_1^q\varphi + C\lambda_1\int_\Omega \varphi_1\varphi - \frac{\lambda C}{M}\int_\Omega \varphi_1\varphi + \left(C - \frac{C}{M}\right)\int_{\partial\Omega} \varphi_1\varphi \geqslant 0$$

Suppose now, $m \geqslant 1$ with $\delta_0 = \min_\Omega \varphi_1$, then the inequality follows as

$$C\left[\frac{C^{q-1}\delta_0^{q-1}}{M} + \lambda_1 - \frac{\lambda}{M}\right]\int_\Omega \varphi_1\varphi \geqslant 0$$

and (iii) of Theorem 3.1 its satisfied for sufficiently large C. Choose now $\underline{u} = \epsilon\varphi_1$ with $\epsilon > 0$ sufficiently small. With analogous arguments we have (ii) of Theorem 3.1 satisfied. Then, Theorem 3.1 conclude that exists a positive solution $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0, 1)$, with $\epsilon\varphi_1 \leqslant u \leqslant C\varphi_1$ on $\overline{\Omega}$.

3.6 Unbounded coefficients $\mathcal{M}$ in (3.1)

In this section we prove a more general version of Theorem 3.1 excluding the requirement the non-local coefficient $\mathcal{M}$ in (3.1) to be bounded above. The proof is reached combining an updated notion of sub-supersolutions with an additional truncation in (3.1) plus suitable estimates, through a close inspection in the arguments used in the proof of Theorem 3.1. Let us assume now the continuous function $\mathcal{M} : \overline{\Omega} \times [0, \infty) \to [0, \infty)$ satisfies

$(\widetilde{H_{\mathcal{M}}})$ There exists $m>0$ such that $\mathcal{M}(x,s)\geqslant m, \quad \forall (x,s)\in\overline{\Omega}\times[0,\infty)$.

The result we prove in this section is the following

Theorem 3.8 *Assume (H_f) and $(\widetilde{H_{\mathcal{M}}})$ hold. Suppose there exist $\bar{u},\underline{u}\in W^{1,\infty}(\Omega)$ such that $0\leq\underline{u}\leqslant\bar{u}$ a.e. in Ω, besides that suppose there exists M_0 such that for all $M\geqslant M_0$ satisfying the conditions:*

(i) $f(x,\underline{u},\nabla\underline{u})\geqslant 0$ *and* $g(x,\underline{u})\geqslant 0$ *a.e. in* Ω.

(ii) $\displaystyle\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla\varphi\,dx\leqslant\frac{1}{M}\int_\Omega f(x,\underline{u},\nabla\underline{u})\varphi\,dx+\frac{1}{M}\int_{\partial\Omega}g(x,\underline{u})\varphi d\mathcal{H}^{N-1},$
$\forall\varphi\in W^{1,p}(\Omega)$, $\varphi\geqslant 0$ *a.e. in* Ω.
(iii) $\displaystyle\int_\Omega |\nabla\bar{u}|^{p-2}\nabla\bar{u}\nabla\varphi dx\geqslant\frac{1}{Q_1}\int_\Omega f(x,\bar{u},\nabla\bar{u})\varphi dx+\frac{1}{Q_2}\int_{\partial\Omega}g(x,\bar{u})\varphi d\mathcal{H}^{N-1},$
$\forall\varphi\in W^{1,p}(\Omega)$, $\varphi\geqslant 0$ *a.e. in* Ω.
where

$$(Q_1,Q_2)=\begin{cases}(M,m), & \textit{if} \quad f(x,\bar{u},\nabla\bar{u})\leqslant 0 \quad \textit{and} \quad g(x,\bar{u})\geqslant 0,\\[2ex] (M,M), & \textit{if} \quad f(x,\bar{u},\nabla\bar{u})\leqslant 0 \quad \textit{and} \quad g(x,\bar{u})\leqslant 0,\\[2ex] (m,M), & \textit{if} \quad f(x,\bar{u},\nabla\bar{u})\geqslant 0 \quad \textit{and} \quad g(x,\bar{u})\leqslant 0.\end{cases}\tag{3.40}$$

Then there exists a solution $u\in C^{1,\gamma}(\overline{\Omega})$ of (3.1), $\gamma\in(0,1)$, with $0\leq\underline{u}\leqslant u\leqslant\bar{u}$ a.e. in Ω.

Proof Consider for $\delta\in(m,\infty)$ the following truncated problem associated with (3.1)

$$\begin{cases}-\mathcal{M}_\delta\Big(x,\displaystyle\int_\Omega|\nabla u|^p dx\Big)\Delta_p u=f(x,u,\nabla u) \quad \text{in } \Omega,\\ \mathcal{M}_\delta\Big(x,\displaystyle\int_\Omega|\nabla u|^p dx\Big)|\nabla u|^{p-2}\partial_\nu u=g(x,u) \quad \text{on } \partial\Omega,\end{cases}\tag{3.41}$$

where $\mathcal{M}_\delta(\cdot,\cdot)$ is the function defined by

$$\mathcal{M}_\delta(x,s)=\min\big\{\mathcal{M}(x,s),\delta\big\}, \quad \forall (x,s)\in\overline{\Omega}\times[0,\infty).\tag{3.42}$$

Note that $\mathcal{M}_\delta$ is a continous function with

$$m\leqslant\mathcal{M}_\delta\leqslant\delta, \quad \forall\delta\in(m,\infty).$$

We are able to reproduce, with the necessary changes, the entire proof of Theorem 3.1, obtaining a solution u_δ such that $\underline{u} \leqslant u_\delta \leqslant \bar{u}$ for every $\delta \in (m, \infty)$, what implies that the norm $\|u_\delta\|_\infty$ is uniformly bounded with respect to $\delta \in (\max\{m, M_0\}, \infty)$. The goal now is to determine an estimate to $\|u_\delta\|_{W^{1,p}(\Omega)}$ independent on δ. We will first consider the more involved natural growth case $q = p$ in (H_f). Let $\Lambda = \sup\big\{\|u_\delta\|_\infty : \delta \in (\max\{m, M_0\}, \infty)\big\}$. Choosing $\varphi_s = e^{su_\delta^2}u_\delta$, $s > 0$, as a test function in the weak formulation of (3.41), that is,

$$\int_\Omega |\nabla u_\delta|^{p-2}\nabla u_\delta \nabla\varphi_s\, dx = \int_\Omega \frac{f(x, u_\delta, \nabla u_\delta)}{\mathcal{M}_\delta\Big(x, \displaystyle\int_\Omega |\nabla u_\delta|^p dx\Big)}\, \varphi_s\, dx + \int_{\partial\Omega} \frac{g(x, u_\delta)}{\mathcal{M}_\delta\Big(x, \displaystyle\int_\Omega |\nabla u|^p dx\Big)}\, \varphi_s\, d\mathcal{H}^{N-1}.$$

Now, by (H_f) and the continuity of G we define

$$H = \sup_{(x,s)\in\Omega\times[0,\|u_\delta\|_\infty]} h(x,s) \quad \text{and} \quad G = \sup_{(x,s)\in\partial\Omega\times[0,\|u_\delta\|_\infty]} g(x,s)$$

we have

$$\begin{aligned}
\int_\Omega e^{su_\delta^2}(1 + 2su_\delta^2)|\nabla u_\delta|^p &\leqslant \frac{1}{m}\int_\Omega |h(x,u_\delta)|\varphi_s + \frac{1}{m}\int_\Omega |h(x,u_\delta)||\nabla u_\delta|^p\varphi_s + \frac{1}{m}\int_{\partial\Omega} |g(x,u_\delta)|\varphi_s d\mathcal{H}^{N-1} \\
&\leqslant \frac{H}{m}\int_\Omega \varphi_s dx + \frac{H}{m}\int_\Omega |\nabla u_\delta|^p \varphi_s dx + \frac{G}{m}\int_{\partial\Omega} \varphi_s d\mathcal{H}^{N-1} \\
&\leqslant C + \frac{H}{m}\int_\Omega |\nabla u_\delta|^p e^{su_t^2} u_\delta dx
\end{aligned}$$

Where $C = C(s, \Lambda, G, H) > 0$ is a constant independent on δ. Now, for all $\varepsilon > 0$, Young's inequality implies

$$\int_\Omega e^{su_\delta^2}(1 + 2su_\delta^2)|\nabla u_\delta|^p\, dx \leqslant C + \frac{H}{m}\int_\Omega |\nabla u_\delta|^p e^{su_\delta^2}\left[\frac{\varepsilon}{2} + \frac{u_\delta^2}{2\varepsilon}\right] dx.$$

Choosing $\varepsilon = \frac{H}{4sm}$, with $s > 0$ large enough such that $\frac{H^2}{8sm^2} < 1$, it follows that

$$\Big(1 - \frac{H^2}{8sm^2}\Big)\int_\Omega |\nabla u_\delta|^p\, dx \leqslant C.$$

Thus for the solution u_δ of (3.41) it holds that $\|u_\delta\|_{W^{1,p}(\Omega)}$ is uniformly bounded with respect to $\delta \in \big(\max\{m, M_0\}, \infty\big)$. Hence, for $\delta_0 > 0$ sufficiently large, from (3.42) we obtain

$$\mathcal{M}_{\delta_0}\Big(x, \int_\Omega |\nabla u_{\delta_0}|^p dx\Big) = \mathcal{M}\Big(x, \int_\Omega |\nabla u_{\delta_0}|^p dx\Big), \tag{3.43}$$

that is, $u = u_{\delta_0}$ is a solution of (3.1). For the case $q < p$, by testing the weak formulation of (3.41) with $\varphi = u_\delta$ (as in (3.2)), from Young's inequality with $\varepsilon > 0$ we have

$$\int_\Omega |\nabla u_\delta|^p \, dx \leqslant C + C \int_\Omega |\nabla u_\delta|^q |u_\delta| \, dx \leqslant C + C\varepsilon \int_\Omega |\nabla u_\delta|^p \, dx,$$

where $C = C(m, \Omega, G, H, \varepsilon) > 0$. Choosing ε sufficiently small, the claim follows for all $q \in [0, p)$. Thus $\|u_\delta\|_{W^{1,p}(\Omega)}$ is bounded uniformly with respect to δ, and (3.43) also holds for $q < p$. The theorem is proved. □

The examples studied in the applications section keep holding by assuming $(\widetilde{H_{\mathcal{M}}})$. Indeed, with the same proofs given in there but changing Theorem 3.1 by Theorem 3.8 one has

Corollary 3.1 *Theorems* 3.26-3.30 *remain valid replacing* $(H_{\mathcal{M}})$ *by* $(\widetilde{H_{\mathcal{M}}})$.

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