UNIVERSIDADE FEDERAL DE SÃO CARLOS

PPGM – DM – UFSCAR

A mean-field game model of economic growth: an essay in regularity theory

Lucas Fabiano Lima

Supervisor: Prof. Dr. Edgard Almeida Pimentel

Thesis approved in public session to obtain the MSc degree in

Mathematics

Committee

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Assinaturas dos membros da comissão examinadora que avaliou e aprovou a Defesa de Dissertação de Mestrado do candidato Lucas Fabiano Lima, realizada em 20/12/2016:

Prof. Dr. Edgard Almeida Pimentel
UFSCar

Prof. Dr. Cesar Rogerio de Oliveira
UFSCar

Prof. Dr. Olivaine Santana de Queiroz
UNICAMP

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Resumo

Nesta dissertação são apresentadas algumas estimativas a priori para soluções de sistemas mean-field games (MFG), definidos em domínios limitados $\Omega \subset \mathbb{R}^d$. Tais estimativas são aplicadas em um modelo mean-field específico, que descreve o acúmulo de riqueza e capital.

No Capítulo 1, é apresentada uma breve introdução histórica sobre os mean-field games. Nesta introdução, exploramos sua relação com a teoria dos jogos, cujos alicerces foram construídos por economistas e matemáticos ao longo do século XX. O objetivo do capítulo é transmitir ao leitor, as origens da Teoria Econômica contemporânea, que surgem à partir da utilização da Matemática na formulação e resolução de problemas econômicos. Tal abordagem é motivada principalmente pelo rigor e clareza da Matemática em tais circunstâncias.

No Capítulo 2, apresentamos um problema de controle ótimo em que cada agente é suposto ser indistinguível, racional e inteligente. Indistinguível no sentido de que cada um é governado pela mesma equação diferencial estocástica. Racional no sentido de que todos os esforços do agente são no sentido de maximizar um funcional de recompensa e, inteligente no sentido de que são capazes de resolver um problema de controle ótimo. Descreve-se este problema de controle ótimo, e apresenta-se a derivação heurística dos mean-field games; obtém-se através de um Teorema de Verificação, a equação de Hamilton-Jacobi (HJ) associada, e em seguida, obtém-se a equação de Fokker-Planck. De posse destas equações, apresentamos alguns resultados preliminares, como uma fórmula de representação para soluções da equação de HJ e alguns resultados de regularidade.

No Capítulo 3, descreve-se um problema específico de controle ótimo e apresenta-se a respectiva derivação heurística culminando na descrição de um MFG com condições não periódicas na fronteira; esta abordagem é original na literatura de MFG. O restante do capítulo é dedicado à exposição de dois tipos bem conhecidos de estimativas: a fórmula de Hopf-Lax e estimativa de Primeira Ordem. Uma observação relevante, é a de que o trabalho em obter-se estimativas a priori é aumentado substancialmente neste caso, devido ao fato de lidarmos com estimativas para os termos de fronteira com normas em L^p.

No Capítulo 4, apresenta-se o modelo de jogo do tipo mean-field de acúmulo de capital e riqueza, o que deixa claro a relevância do estudo dos MFG em um domínio limitado. À luz dos resultados obtidos no Capítulo 3, encerramos o Capítulo 4 com as estimativas do tipo Hopf-Lax e de Primeira Ordem.

Três apêndices encerram o texto desta dissertação de mestrado; estes reúnem material elementar sobre Cálculo Estocástico e Análise Funcional.

Palavras-chave: Mean-field games, Equação de Hamilton-Jacobi, Equação de Fokker-Planck, estimativas a priori, domínios limitados, método adjunto não-linear.

Abstract

In this thesis, we present a priori estimates for solutions of a mean-field game (MFG) defined over a bounded domain $\Omega \subset \mathbb{R}^d$. We propose an application of these results to a model of capital and wealth accumulation.

In Chapter 1, an introduction to mean-field games is presented. We also put forward some of the motivation from Economics and discuss previous developments in the theory of differential games. These comments aim at indicating the connection between mean-field games theory, its applications and the realm of Mathematical Analysis.

In Chapter 2, we present an optimal control problem. Here, the agents are supposed to be undistinguishable, rational and intelligent. Undistinguishable means that every agent is governed by the same stochastic differential equation. Rational means that all efforts of the agent is to maximize a payoff functional. Intelligent means that they are able to solve an optimal control problem. Once we describe this (stochastic) optimal control problem, we produce a heuristic derivation of the mean-field games system, which is summarized in a Verification Theorem; this gives rise to the Hamilton-Jacobi equation (HJ). After that, we obtain the Fokker-Plank equation (FP). Finally, we present a representation formula for the solutions to the (HJ) equation, together with some regularity results.

In Chapter 3, a specific optimal control problem is described and the associated MFG is presented. This MFG is prescribed in a bounded domain $\Omega \subset \mathbb{R}^d$, which introduces substantial additional challenges from the mathematical view point. This is due to estimates for the solutions at the boundary in L^p. The rest of the chapter puts forward two well known tips of estimates: the so-called Hopf-Lax formula and the First Order Estimate.

In Chapter 4, the wealth and capital accumulation mean-field game model is presented. The relevance of studying MFG in a bounded domain then becomes clear. In light of the results obtained in Chapter 3, we close Chapter 4 with the Hopf-Lax formula, and the First Order estimates.

Three appendices close this thesis. They gather elementary material on Stochastic Calculus and Functional Analysis.

Keywords: Mean-field games, Hamilton-Jacobi equation, Fokker-Plank equation, a priori estimates, bounded domains, non-linear adjoint method.

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Contents

Chapter 1

Introduction

1.1 Game Theory and Mean-field Games

The introduction of mathematical methods in economic theory gave rise to a new research area, called mathematical economics. It consists in applications of mathematical methods to represent economic theories and to analyse problems from economic realm. One of the greatest advantages of such area is to make possible for the economists to make clear, specific and positive statements about their problems of interest. A further advantage of the mathematical formalism is that it enables authors to formulate and solve models in a unified language.

The formal modelling in economics started in the beginning of the XIX century, with advances of differential calculus, to represent and justify the economic behavior originated in utility maximization. Economics became more mathematical as a subject during the first half of the XX century, see for example [25]. However, the introduction of new and generalized techniques around the IIGWW period, as game theory, greatly expanded the mathematical formulation of economic problems.

Nowadays, in modern economic theory, mathematical methods are central, even while not all economists agree that the behavior of agents can be reduced to a precise mathematical formulation. In any case, utility maximization principle and game-theoretical equilibria explain, at least partially, many economic phenomena.

One of the fundamentals characteristics of economic models is the concept of competing agents, as illustrated in the works of A. Cournot and L. Walras. Indeed, Walras, also known as the founder of the *École de Lausanne*, refers to Cournot in 1873, as the first author to seriously recur to the mathematical formalism in investigating economic problems. Cournot duopoly model is one of the earliest formulations of a non-cooperative game. This pioneering work sets up the foundations of contemporary game theory. Walras developed a first theory of a competitive market (general) equilibrium. W. S. Jevons and C. Menger are also well known for their influence on the presence of mathematical formalism in economics.

The mathematical formulation of economic problems has attracted the attention of notable mathematicians, including E. Borel and J. von Neumann. In the mid-20th century, in the paper [20], J. Nash developed a concept of equilibrium that is fundamental in modern game theory. The arguments in that paper rely on a fixed-point theorem, due to S. Kakutani. For his "*contributions to the analysis of equilibria in the theory of non-cooperative games"*, J. Nash (together with C. Harsanyi and R. Selten) was awarded the Nobel Prize in Economics in 1994.

Another Nobel Prize Laureate, R. Aumann, introduced in 1964 the idea of an economy with a continuum of players, which are atomized in nature [4]. In that paper, Aumann argues that only for an economy with infinitely many participants it is reasonable to assume that the actions of individual agents are negligible in determining the overall outcome.

In 1995, the Noble Prize was awarded to R. Lucas, for the development and applications of the hypothesis of rational expectations, in the early 70's, [19]. This hypothesis states that economic agents' predictions of economically relevant quantities are not systematically wrong. More precisely, the subjective probabilities as perceived by the agents agree with the empirical probabilities. After the introduction of this framework, an important paradigm in economic theory is based on three hypotheses: efficient markets, rational expectations, and representative agent.

It is only around the 90's that alternatives to the representative agent model begin to be considered in mainstream economics. The idea of heterogeneous agents, as suggested in the works of S. Aiyagari [1], T. Bewley [5], M. Huggett [13] and P. Krussel and A. Smith [14], points out in an alternative direction. In this formulation, agents in the economy are charaterized by different levels of the model's variables. For example, individuals can have distinct income or wealth levels.

In the theory of mean-field games (MFG), the concept of Nash equilibrium and the rational expectation hypothesis are combined to produce mathematical models for large systems, with infinitely many indistinguishable rational players. The term *indistinguishable* refers to a setting where agents share common structures of the model, though they are allowed to have heterogeneous states. In other terms, the MFG theory enables us to investigate the solution concept of Nash equilibrium, for a large population of heterogeneous agents, under the hypothesis of rational expectations.

The formalism of mean-field games was developed in a series of papers by J.-M. Lasry and P.-L. Lions [16], [17], [18], and M. Huang, R. Malhamé and P. Caines [11] and [12]. Methods and techniques to study differential games with a large population of rational players are introduced in these papers. The agent's preferences does not depend only on their states (e.g., wealth, capital) but also on the distribution of the remaining individuals in the population. It is fair to say that mean-field games theory studies generalized Nash equilibria for these systems. Typically, these models are formulated in terms of partial differential equations, namely a transport or Fokker-Planck equation for the distribution of the agents coupled with a Hamilton-Jacobi equation.

An important research direction in the theory of MFG concerns the study of the existence and regularity of solutions. Well-posedness in the class of smooth solutions was investigated, both in the stationary and in the time-dependent setting. In [9] we can find results related to this research direction but concerning certain problems; In particular in [9] the authors consider problems in the periodic setting.

1.2 Mean-field games and economic theory

The traditional approach to economic models often involves the simplifying assumption that all agents are identical (called representative agent assumption). However, heterogeneous agent problems allow the study of questions in which the differences among agents are of primary relevance. Matters such as wealth distribution or income inequality are inherently associated with differences among agents. In different problems, the representative agent assumption can not adequately capture the effect of heterogeneity.

Mean-field games theory aims at modelling large populations of rational, heterogeneous agents. The analytical spectrum of the MFG framework accommodates preference structures and effects that depend on the whole distribution of the population. A rational agent is an agent with defined preferences that she seeks to optimize. In the vast majority of cases, these preferences can be modelled through a utility functional. Rationality means that the agent always acts optimally, seeking to maximizes her utility. Finally, MFG are closely linked to the assumption of rational expectations. The prediction of future quantities by the agents is an essential part of any economic model. The rational expectation hypothesis states that predictions by the agents of the value of relevant variables do not differ systematically from equilibrium conditions. This hypothesis has several advantages: firstly, it can be a good approximation to reality, as agents who act in a non-rational way will be driven out in a competitive market; secondly, it produces well-defined and relatively tractable mathematical problems; finally, because of this, it is possible to make quantitative and qualitative predictions that can be compared to real data. In mean-field game models, the actions of the agents are determined by looking at objective functionals involving expectated values with respect to probability measures that are consistent with the equilibrium behavior of the model. This contrasts with the adaptative expectations approach, where the model for future behavior of the agents is built on their past actions.

Computational methods based in agents are very common and useful tools to study heterogeneous-agents economic problems. Unfortunately, numerical methods are not able to provide analytical models from which qualitative properties can be derived. In modern macroeconomics, an important role is played by dynamic stochastic general equilibrium models. These aim to understand the fundamental questions such as uniqueness, existence of economic equilibrium. However, many MFG problems arising in mathematical economics raise issues that cannot be dealt with current results. In this sense, this contributes to the literature by studying MFG with non trivial boundary conditions, in a bounded domain $\Omega \subset \mathbb{R}^d$.

1.3 General assumptions and main results

In this thesis, we study the following mean-field game system

$$\begin{cases} -u_t + H(x, Du) = \Delta u + g[m] & \text{in } \Omega \times [0, T]\,, \\ m_t - \operatorname{div}(D_p H m) = \Delta m & \text{in } \Omega \times [0, T]\,, \end{cases} \tag{1.1}$$

equipped with boundary conditions,

$$\begin{cases} u(x, T) = u_T(x) & \text{in } \Omega, \\ m(x, 0) = m_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

and

$$\begin{cases} u(x, t) = h(x, t) & \text{on } \partial\Omega \times [0, T], \\ m(x, t) = k(x, t) & \text{on } \partial\Omega \times [0, T]. \end{cases} \tag{1.3}$$

Here, $T > 0$ is a fixed terminal instant, Ω is a bounded subset of $\mathbb{R}^d$, and $h, k : \partial\Omega \times [0, T] \to \mathbb{R}$ are real functions with suitable regularity. The map $g[m]$ is the mean-field hypothesis (MFh, for short) and encodes the dependence of the value function on the measure. Some typical examples in the literature are the following:

$$g[m](x, t) := m^{\alpha}(x, t) \quad \forall (x, t) \in \Omega \times [0, T], \tag{1.4}$$

$$g[m](x, t) := \frac{-1}{m^{\alpha}(x, t)} \quad \forall (x, t) \in \Omega \times [0, T], \tag{1.5}$$

$$g[m](x, t) := \ln[m](x, t) \quad \forall (x, t) \in \Omega \times [0, T]. \tag{1.6}$$

The formulation in (1.4) is inspired by the notion of hyperbolic absolute risk aversion utility function. The singular formulation is (1.5), it is inspired in Alt-Philips [2] formulation of the stochastic problem. In (1.6) it is the considered the logarithmic utility.

The derivation of the problem (1.1)-(1.3) is presented further in this work/thesis. In this thesis we work under several assumptions on the Hamiltonian H, as well as on the boundary conditions which cover a wide range of examples and applications. As an instance of the latter, we investigate the following model of Economic Growth.

$$\begin{cases} V_t(a,k,t) + H(a,k,r,p,\delta,D_aV,D_kV) + \Delta V = 0, \\ m_t + ((D_{q_a}H)m)_a + ((D_{q_k}H)m)_k = \Delta m, \end{cases} \tag{1.7}$$

in $\Omega \times [0,T] = (\underline{a},\overline{a}) \times (0,\overline{k}) \times [0,T]$, with initial-terminal boundary conditions:

$$\begin{cases} V(a,k,T) = V_T(a,k) \quad \text{in} \quad \Omega, \\ m(a,k,0) = m_0(a,k) \quad \text{in} \quad \Omega, \end{cases} \tag{1.8}$$

and

$$\begin{cases} V(a,k,t) = f(a,k,t) \qquad \text{on} \quad \partial\Omega \times [0,T], \\ m(a,k,t) = 0 \qquad\qquad\quad \text{on} \quad \partial\Omega \times [0,T]. \end{cases} \tag{1.9}$$

The state of the model, $x = [a_t, k_t] \in \mathbb{R}^2$, comprises the wealth a_t and the stock of capital k_t of each agent, over time. The bounds $\{\overline{a}, \underline{a}, \overline{k}\}$, encodes the boundaries for wealth and stock of capital in a finite time $0 < t < T$. In addition,

1. p is the price level of the economy,
2. $\delta \in (0,1)$ is a constant that measures the depreciation of the capital,
3. r is a fixed interest rate.

It is reasonable to suppose that $a \in (\underline{a},\overline{a})$, because the levels of wealth in Economy are bounded in finite horizon T. In addition, assuming $k \in (0,\overline{k})$ is reasonable because the stock of capital is also bounded from below; also, negative values of k are not expected to be verified.

We proceed by listing general hypotheses on the Hamiltonian H, the MFGh g as well as on the boundary conditions.

A 1. *The Hamiltonian* $H : \Omega \times \mathbb{R}^d \to \mathbb{R}$

1. is of class $\mathscr{C}^2(\Omega \times \mathbb{R}^d)$;
2. for fixed x, the map $p \mapsto H(x,p)$ is a strictly convex function;

3. satisfies the coercivity condition

$$\lim_{|p|\to\infty} \frac{H(x,p)}{|p|} = \infty,$$

and without loss of generality we suppose further that $H(x,p) \geq 1$.

A 2. *The mean-field-hypothesis,* $g : \mathbb{R}_0^+ \to \mathbb{R}$*, is a non-negative increasing function. Therefore, there exists* $G : \mathbb{R}_0^+ \to \mathbb{R}$*, convex and increasing, such that* $g = G'$*.*

We define the Legendre transform of H by

$$L(x,v) := \sup_p(-p.v - H(x,p)). \tag{1.10}$$

Then if we set

$$\widehat{L}(x,p) = D_pH(x,p)p - H(x,p), \tag{1.11}$$

by standard properties of the Legendre transform

$$\widehat{L}(x,p) = L(x, -D_pH(x,p)). \tag{1.12}$$

A 3. *For some constants,* $c, C > 0$

$$\widehat{L}(x,p) \geq cH(x,p) - C. \tag{1.13}$$

Next we impose conditions on the first and second order derivatives of H.

A 4. H *satisfies the following bounds*

$$|D_xH|, |D^2_{xx}H| \leq CH + C, \tag{1.14}$$

where $C > 0$.

The next assumption addresses the initial conditions

A 5. *The initial conditions satisfy* $(u_T, m_0) \in \mathscr{C}^\infty(\Omega \times [0,T])$ *with* $m_0 \geq \kappa_0$ *for some* $\kappa_0 \in \mathbb{R}^+$ *and*

$$\int_\Omega m_0(x)dx = 1. \tag{1.15}$$

A 6. *H satisfies the sub-quadratic growth conditions;*

$$H(x,p) \leq C|p|^{\gamma} + C, \tag{1.16}$$

$$|D_p H(x,q)| \leq C|q|^{\gamma-1} + C, \tag{1.17}$$

$$|\operatorname{div}(D_p H(x,q))| \leq C|q|^{\gamma-1} + C, \tag{1.18}$$

for some $1 < \gamma < 2$ and $C > 0$.

Finally we obtain the following results for a mean-field game model in a bounded domain. We emphasize the fact that such results are new, although expected; in this sense the next theorems constitute the main contribution of this work to the mean-field game theory.

Our first result is a representation formula, in the spirit of the Hopf-Lax theory.

Theorem 1.1 (Stochastic Hopf-Lax Formula in bounded domain)**.** *Suppose that A1 holds. Let (u,m) be a solution to* (1.1)-(1.3)*. Then for any solution to*

$$\begin{cases} \xi_t + \operatorname{div}(b.\xi) &= \Delta\xi \quad \textit{in } \Omega \times (\tau, T), \\ \xi(x,\tau) &= \xi_\tau(x) \quad \textit{in } \Omega, \\ \xi(x,t) &= 0 \quad \textit{in } \partial\Omega \times (\tau, T), \end{cases}$$

and for any smooth vector field $b : \Omega \subset \mathbb{R}^d \times (\tau, T) \to \mathbb{R}^d$. We have the following inequality:

$$\begin{aligned} \int_\Omega u(x,\tau)\xi_\tau(x)dx \quad &\leq \quad \int_\tau^T \int_\Omega (L(b,x) + g[m])\,\xi(x,t)dxdt + \int_\Omega u_T(x)\xi(x,T)dx \\ &+ \quad \|h\|_{L^r(\tau,T),L^p(\partial\Omega))} \left(C + C\|\xi_\tau\|_{L^2(\Omega)}\right). \end{aligned}$$

Theorem 1.1 is instrumental in producing a pivotal class of estimates in the realm of MFG theory; Next we present the so-called First Order Estimates.

Theorem 1.2. *Let (u,m) be a solution to* (1.1)-(1.3)*, then, there exists $C > 0$ such that*

$$\begin{aligned} c\int_0^T\!\!\int_\Omega H(x,Du)mdx\,dt + \int_0^T\!\!\int_\Omega G(m)dxdt \quad &\leq \quad C + \|h\|_{L^r(0,T;L^p(\partial\Omega))} + C\ osc\ u(\cdot,T) \\ &\quad + CT + C\|h\|_{L^p(\Omega)}\left[1 + \|Du\|^{\frac{\gamma-1}{1-\alpha}}_{L^{(\gamma-1)jr}}\right], \end{aligned}$$

where $G' = g$.

Also, we especialize the former results to the case of the wealth and capital accumulation model described in (1.7)-(1.9). The former theorems become:

Theorem 1.3. *Suppose that A1 holds. Let (V,m) be a solution to* (1.7)-(1.9)*. Then for any solution* *to*

$$\begin{cases} \xi_t + \operatorname{div}(b.\xi) &= \Delta\xi \quad \text{in } \Omega \times (\tau, T), \\ \xi(x,\tau) &= \xi_\tau(x) \quad \text{in } \Omega, \\ \xi(x,t) &= 0 \quad \text{in } \partial\Omega \times (\tau, T), \end{cases}$$

where b is any smooth vector field $b : \Omega \subset \mathbb{R}^2 \times (t,T) \to \mathbb{R}^2$, we have the following upper bound:

$$\begin{aligned} \int_0^k \int_{\underline{a}}^{\overline{a}} V(k,a,\tau)\xi_\tau(a,k)dadk &\leq \int_\tau^T \int_0^k \int_{\underline{a}}^{\overline{a}} u(c_t)\xi(a,k,t)dadkdt + \|V_T\|_{L^\infty(\Omega)} + C \\ &+ C\|f\|_{L^r(\tau,T;L^p(\partial\Omega))} \end{aligned}$$

Theorem 1.4. *Let (V,m) be a solution to* (1.7)-(1.9)*, then, there exists $C > 0$ such that*

$$\begin{aligned} c\int_0^T\!\!\int_\Omega H(a,k,D_aV,D_kV)mdx\;dt &\leq C + \|f\|_{L^r(0,T;L^p(\partial\Omega))} + C\;osc\,V(\cdot,T) \\ &+ CT + C\|f\|_{L^r(0,T;L^p(\Omega))}\left[1 + \|DV\|^{\frac{\gamma-1}{1-\alpha}}_{L^{(\gamma-1)jr}}\right], \end{aligned}$$

The proofs of Theorems 1.1 and 1.2 are presented in Chapter 3. The proofs for Theorems 1.3 and the 1.4 are presented in Chapter 4. In the next chapter we describe an optimal control problem and associated MFG is obtained through the heuristic derivation. Some estimates for the Hamilton-Jacobi equation are presented.

Chapter 2

Second order MFG

In this chapter, we produce a heuristic derivation of a (reduced) second order MFG; it consists of a system of two PDE's, namely: a Hamilton-Jacobi and a Fokker-Planck equation. Some of the basic concepts of stochastic calculus necessary to our derivation are put forward in the Appendix A. For the sake of presentation, we proceed by considering the heuristic derivation in $\mathbb{R}^d$. Suitable adaptations to our setting are mentioned later (Chapter 3).

2.1 Hamilton-Jacobi equation

Consider a very large group of agents, where each one of them is fully characterized by a point $x \in \mathbb{R}^d$. Each agent can decide to change its state, and this is done by applying a control $v \in \mathbb{R}^d$. However, the agents are subject to independent random forces that are modeled by a Brownian motion (white noise). This is, briefly speaking, a formal way to describe "unexpected events". To make matters precise, fix $T > 0$ and fix a stochastic basis $\mathcal{P} = (\Omega, \{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{P})$, supporting a d-dimensional Brownian motion B_t. Let $\sigma > 0$. In this simplified model, the trajectory of the agent is governed by the stochastic differential equation (SDE):

$$\begin{cases} d\mathbf{x}_t &= \mathbf{v}_t dt + \sqrt{\sigma} dB_t, \\ \mathbf{x}_{t_0} &= \mathbf{x}, \end{cases} \tag{2.1}$$

where $\mathbf{v}_t$ is a progressively measurable control with respect to the filtration $\mathcal{F}_t$, or simply $\mathcal{F}_t$-progressively measurable.

Definition 2.1. *A control is progressively measurable with respect to the filtration $\mathcal{F}_t$ if we have that for each $0 \le s \le t$, the map*

$$(s, \omega) \mapsto \mathbf{v}(s, \omega)$$

is measurable with respect to $\mathcal{B}([t_0, t]) \times \mathcal{F}_t$.

Consider a Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$. By selecting the control $\mathbf{v}$ in a progressively measurable way, the agent seeks to maximize a payoff functional given by

$$J(\mathbf{v}, \mathbf{x}; t) := \mathbb{E}^{\mathbf{x}} \left[\int_t^T L(\mathbf{x}_s, \mathbf{v}_s; m(\mathbf{x}_s, s)) ds + \Psi(\mathbf{x}_T) \right], \tag{2.2}$$

where m represents the population's density and $t \in [0, T]$. In (2.2), $\mathbb{E}^{\mathbf{x}}$ denotes the expectation operator, given that $\mathbf{x}_t = \mathbf{x}$. Furthermore, $\Psi : \mathbb{R}^d \to \mathbb{R}$ is the terminal cost of the system. Observe that we are supposing that the state of the agent belongs to $\mathbb{R}^d$; if we study the case when $\mathbb{R}^d$ is replaced by a bounded subset $\Omega \in \mathbb{R}^d$, suitable adaptations, on the payoff functional are required. These are described in Chapter 3.

The Legendre transform of L is

$$H(x, p; m) = \sup_{v \in \mathbb{R}^d} (p \cdot v + L(x, v; m)). \tag{2.3}$$

We are interested in the value function of this problem, denoted by u, which is determined by

$$u(x, t) := \sup_{\mathbf{v} \in \mathcal{V}} J(\mathbf{v}, \mathbf{x}; t), \tag{2.4}$$

where $\mathcal{V}$ is the class of admissible controls. The function $u : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}$ is called the *value function* associated with the (stochastic) optimal control problem (2.1)-(2.2). We can start to build a conection between this optimal control problem and the theory of partial differential equations with the following result:

Theorem 2.1. *If the value function* u *of the optimal control problem* (2.1)-(2.2) *is twice differentiable with respect to* x *and differentiable with respect to* t*, then it solves*

$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t); m) + \frac{\mathrm{Tr}\left(\sigma^T \sigma D_x^2 u(x,t)\right)}{2} = 0 & \text{in } \mathbb{R}^d \times [t_0, T) \\ u(x,T) = \Psi(x) & \text{in } \mathbb{R}^d. \end{cases} \tag{2.5}$$

For a proof of this result, see [8]. This result motivates the following question:

"Let w *be a solution to* (2.5)*; what conditions should be imposed on* w *to ensure that it is the value function of the problem* (2.1)-(2.2)*?"*

We can give an answer to this question in rigorous form by means of a Verification Theorem. Before we proceed, we introduce the definition of the infinitesimal generator of a stochastic process and state the Dynikin's formula. See Appendix A.

Definition 2.2 (Infinitesimal Generator)**.** *Let* $\mathbf{x}_t$ *be a solution to* (2.1)*, adapted to a fixed stochastic basis. The infinitesimal generator of the stochastic process* $\mathbf{x}_t$ *is the operator* A *defined by*

$$A(f)(x_0,t_0)=\lim_{t\downarrow t_0}\frac{E^{x(t_0)=x_0}(f(\mathbf{x}_t,t))-f(x_0,t_0)}{t-t_0}.$$

The set of functions f *for which the limit above exists and is finite for all* x*, is called the domain of* A *and is denoted by* $\mathcal{D}(A)$*.*

Example 2.1 (Itô Diffusion)**.** *Consider the following SDE:*

$$d\mathbf{x}_s=h(\mathbf{x}_s,\mathbf{v}_s,s)ds+\sigma(\mathbf{x}_s,\mathbf{v}_s,s)dB_s,$$

where h *and* σ *satisfy certain growth conditions as in Appendix A for* $b=h$*. Assume that* $\mathbf{v}$ *is a Markovian control; it means that* $\mathbf{v}_s$ *is a random variable whose probability does not depends on the past of the system. Then,* (2.1) *is called a Markov diffusion. In this case,* $A^{\mathbf{v}}$ *is given by*

$$A^{\mathbf{v}}f(x,t)=\frac{\partial}{\partial t}f(x,t)+h\cdot f_x(x,t)+\frac{\mathrm{Tr}\left(\sigma^T\sigma D_x^2 f(x,t)\right)}{2}.$$

This example is a particular case of Theorem A.2; before we state and prove a Verification Theorem, we present the Dynkin's formula.

Proposition 2.1 (Dynkin's formula)**.** *Let* $\mathbf{x}_s$ *be a Markov diffusion with infinitesimal generator* A*. Assume that* $\mathbf{x}_{t_0}=\mathbf{x}$*. If* $f\in\mathcal{D}(A)$*, then*

$$E^{(x,t_0)}(f(x_t,t))-f(x,t_0)=E^{(x,t_0)}\left(\int_{t_0}^{t}Af(x_s,s)ds\right), \tag{2.6}$$

for every $t\geq t_0$*.*

For the proof of Dynkin's formula we suggest [8]. We observe that (2.6) is the stochastic counterpart of the Mean Value Theorem.

Theorem 2.2 (Verification Theorem)**.** *Let* w *be a solution to* (2.5)*. Assume that* w *is differentiable with respect to the time variable and twice differentiable with respect to the space variable. Then,* $w\geq u$*. In addition, if there exists* $\mathbf{v}^*$ *such that*

$$\mathbf{v}^*\in\operatorname{argmax}\left[A^{\mathbf{v}}w(\mathbf{x}_s^*,s)+L(\mathbf{x}_s^*,\mathbf{v}_s;m)\right] \tag{2.7}$$

we have $w=u$*.*

Proof. Applying the Dynkin's formula for w in the point $(\mathbf{x}_T, T)$ we have:

$$E^{(x,t)}(w(\mathbf{x}_T, T)) - w(x,t) \quad = \quad E^{(x,t)}\left(\int_t^T A^v w(x_s, s)ds\right).$$

i.e.,

$$-w(x,t) \quad = \quad E^{(x,t)}\left(\int_t^T w_t + v\cdot D_x w + \frac{\mathrm{Tr}\left(\sigma^T\sigma D_x^2 w\right)}{2}ds - w(x_T, T)\right).$$

Because w is a solution to (2.5), we have that $w(x_T, T) = \Psi(x_T)$; indeed,

$$\begin{aligned} -w(x,t) \quad &= \quad E^{(x,t)}\left[\int_t^T w_t + vD_x w + \frac{\mathrm{Tr}\left(\sigma^T\sigma D_x^2 w\right)}{2} + L(x_s, v_s; m) - L(x_s, v_s; m)ds - \Psi(x_T)\right] \\ &\leq \quad E^{(x,t)}\left[\int_t^T w_t + H(x, D_x w; m) + \frac{\mathrm{Tr}\left(\sigma^T\sigma D_x^2 w\right)}{2} - L(x_s, v_s; m)ds - \Psi(x_T)\right]. \end{aligned} \tag{2.8}$$

Using the equation (2.5), we have

$$w(x,t) \quad \geq \quad E^{(x,t)}\left(\int_t^T L(x_s, v_s; m)ds + \Psi(x_T)\right).$$

The inequality above is true for all admissible control $v \in \mathcal{V}$. In particular, the definition of value function yields $u \leq w$. If we assume (2.7) is in force, i.e., in case there exists an optimal control v^*, the inequality (2.8) becomes an equality with v^* and x^* replacing v_s and x_s. □

By the Verification Theorem we obtain that under certain assumptions on the solution w of the Hamilton-Jacobi equation (2.5), we have that w is the value function of the optimal control problem (2.1)-(2.2), and in addition, the optimal control $\mathbf{v}^*$ is given in feedback form by

$$\mathbf{v}^* = D_p H(x, D_x u(x,t); m).$$

These assumptions require w to be a classical solution to the problem. We observe that much weaker requirements can be made; for example as in the theory of viscosity solutions, see [8].

2.2 Fokker-Planck equation

In this section, we examine the Fokker-Planck equation. Consider a population of agents whose state is $\mathbf{x} \in \mathbb{R}^d$. Assume further that the state of each agent in the population is governed by the stochastic differential equation in (2.1). Under the assumption of uncorrelated noise, the evolution of the population's density is determined by a Fokker-Planck equation. To discuss the derivation of this equation, we depend once more on the notion of infinitesimal generator of a (Markov) process.

Let A be the generator of a Markov process $\mathbf{x}_t$. The formal adjoint of A, denoted by A^*, acts on functions in a suitable regularity class and is determined by the identity

$$\int_{\mathbb{R}^d\times[0,T]} \phi(x,t)Af(x,t)dx \quad = \quad \int_{\mathbb{R}^d\times[0,T]} f(x,t)A^*\phi(x,t)dx, \tag{2.9}$$

for every $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^d\times[0,T])$.

Example 2.2 (Markov diffusions)**.** *The infinitesimal generator of a Markov diffusion is given in Example* (2.1)*. It has the form*

$$A^{\mathbf{v}}[f](x,t) = \frac{\partial}{\partial t}f(x,t) + h\cdot f_x(x,t) + \frac{\mathrm{Tr}\left(\sigma^T\sigma D_x^2 f\right)}{2}.$$

Therefore, A^* can be obtained by applying the equation (2.9) *as follows:*

$$\int_{\mathbb{R}^d\times[0,T]} \phi(x)A^{\mathbf{v}}[f](x,t)dxdt \quad = \quad \int_{\mathbb{R}^d\times[0,T]} f(x,t)A^*\phi(x,t)dxdt, \tag{2.10}$$

for every $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^d\times[0,T])$. Using the definition of $A^{\mathbf{v}}$ we have

$$\int_{\mathbb{R}^d\times[0,T]} \phi(x,t)\left[\frac{\partial}{\partial t}f(x,t) + h\cdot f_x(x,t) + \frac{\mathrm{Tr}\left(\sigma^T\sigma D_x^2 f\right)}{2}\right]dxdt$$

$$= \underbrace{\int_{\mathbb{R}^d\times[0,T]} \phi(x,t)\frac{\partial}{\partial t}f(x,t)dxdt}_{\mathbf{I}} \quad + \quad \underbrace{\int_{\mathbb{R}^d\times[0,T]} \phi(x,t)h(x,v,t)f(x,t)dxdt}_{\mathbf{II}}$$
$$+ \quad \underbrace{\int_{\mathbb{R}^d\times[0,T]} \phi(x,t)\left[\frac{\mathrm{Tr}\left(\sigma^T\sigma D_x^2 f\right)}{2}\right]dxdt}_{\mathbf{III}}$$

Using integration by parts we obtain:

for **I***:*

$$\int_{\mathbb{R}^d\times[0,T]} \phi(x,t)\frac{\partial}{\partial t}f(x,t)dxdt = -\int_{\mathbb{R}^d\times[0,T]} \frac{\partial}{\partial t}\phi(x,t)f(x,t)dxdt;$$

for **II***:*

$$\int_{\mathbb{R}^d\times[0,T]} \phi(x,t)h(x,v,t)f(x,t)dxdt = -\int_{\mathbb{R}^d\times[0,T]} f(x,t)\,\mathrm{div}(\phi(x,t)h(x,v,t))dxdt;$$

for **III***:*

$$\begin{aligned}\int_{\mathbb{R}^d\times[0,T]}\phi(x,t)\left[\frac{\mathrm{Tr}\left(\sigma^T\sigma D_x^2 f\right)}{2}\right]dxdt &= \int_{\mathbb{R}^d\times[0,T]}\phi(x,t)\left[\frac{\left((\sigma^T\sigma)_{i,j}f(x,t)\right)_{x_ix_j}}{2}\right]dxdt\\ &= \int_{\mathbb{R}^d\times[0,T]}f(x,t)\frac{\left((\sigma^T\sigma)_{i,j}\phi(x,t)\right)_{x_ix_j}}{2}dxdt.\end{aligned}$$

Using the results from ***I, II*** *and* ***III*** *in the equation* (2.10)*, we conclude that*

$$(A^{\mathbf{v}})^*[m](x,t) = -\frac{\partial}{\partial t}m(x,t) - \mathrm{div}(h(x,v,t)m(x,t)) + \frac{\left((\sigma^T\sigma)_{i,j}m(x,t)\right)_{x_ix_j}}{2}.$$

A fundamental result, that we can find in [3] for example, states that the evolution of the population's density, given an initial configuration m_0, is described by the equation:

$$\begin{cases} A^*[m](x,t) = 0, \\ m(x,t_0) = m_0(x). \end{cases} \tag{2.11}$$

Example (2.2) builds upon (2.11) to yield the Fokker-Plank equation

$$m_t(x,t) + \mathrm{div}(h(x,v,t)m(x,t)) = \frac{\left((\sigma^T\sigma)_{i,j}m(x,t)\right)_{x_ix_j}}{2}. \tag{2.12}$$

2.3 Second order mean-field games

Here, we combine elements from the two previous sections to derive a model second-order mean-field game system.

Consider a large population of agents whose state $\mathbf{x}_t \in \mathbb{R}^d$ is governed by (2.1). Assume further that each agent in this population faces the same optimization problem, given by (2.2). The Verification Theorem 2.2 shows that the solution to the Hamilton-Jacobi equation (2.5) is the value function. Moreover, the optimal control $\mathbf{v}^*$ is given in feedback form by

$$\mathbf{v}^* = D_pH(\mathbf{x}, D_xu(\mathbf{x},t); m).$$

On the other hand, the agents' population evolves according to (2.12). By setting $h \equiv \mathbf{v}$, we obtain

$$m_t(x,t) + \mathrm{div}(\mathbf{v}m(x,t)) = \frac{\left((\sigma^T\sigma)_{i,j}m(x,t)\right)_{x_ix_j}}{2}.$$

Under the assumption of rationality of the agents, the population is driven by the optimal control; hence, it evolves according to

$$m_t(x,t) + \operatorname{div}(D_pH(\mathbf{x}, D_xu(\mathbf{x},t); m)m(x,t)) = \frac{\big((\sigma^T\sigma)_{i,j}m(x,t)\big)_{x_ix_j}}{2}.$$

Therefore, the MFG system associated with (2.1)-(2.2) is:

$$\begin{cases} u_t + H(x, D_xu; m) + \frac{\operatorname{Tr}\sigma^T\sigma D_x^2 u}{2} = 0, & (x,t) \in \mathbb{R}^d \times [t_0, T) \\ m_t + \operatorname{div}(D_pH(x, D_xu; m)m) = \frac{\big((\sigma^T\sigma)_{i,j}m\big)_{x_ix_j}}{2}, & (x,t) \in \mathbb{R}^d \times (t_0, T], \end{cases} \tag{2.13}$$

equipped with the initial-terminal conditions

$$\begin{cases} u(x,T) &= u_T(x), \quad x \in \mathbb{R}^d, \\ m(x,t_0) &= m_0(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{2.14}$$

2.4 A formula inspired by the adjoint method

Consider a (non-linear) differential operator $\mathcal{L} : Y \to Y$ and the corresponding homogeneous equation

$$\mathcal{L}[u](x) = 0. \tag{2.15}$$

It is possible to associate with (2.15) a linear equation that encodes important information about the solutions of (2.15). The idea presented in [7] is to construct a new method to improve the standard viscosity solution approach to Hamilton-Jacobi PDE's; this is called adjoint method and consists in examining the solution ξ of the adjoint equation to the formal linearization of the HJ equation. This procedure leads to a natural phase space kinetic formulation and also a new compensated compactness technique. Here, inspired by this idea, we calculate the adjoint of the formal linearization to obtain a representation formula for the solutions of the HJ equation. We start by considering then the linearized operator L of $\mathcal{L}$, determined by

$$\lim_{h\to 0} \frac{\mathcal{L}(f+h) - \mathcal{L}(f) - L(f)h}{h} = 0.$$

Next, we compute its formal adjoint, in the L^2 sense:

$$\int_\Omega \phi(x)L[v](x)dx = \int_\Omega v(x)L^*[\phi](x)dx. \tag{2.16}$$

The equation

$$L^*[\rho](x) = 0$$

is called the *adjoint* equation to (2.15), and ρ is called the adjoint variable. In what follows, we consider specific operators $\mathcal{L}$ and produce a few elementary results that illustrate the adjoint methods. For obvious reasons, we specialize $\mathcal{L}$ to be the H-J operator studied so far. Hence

$$\mathcal{L}[u](x,t) := u_t + H(x, Du) + \Delta u - g[m],$$

Where $Du = D_x u$ for $x \in \mathbb{R}^d$. To obtain the formal linearization of the Hamilton-Jacobi equation, we produce a linear operator $L_u : \mathscr{C}_0^\infty(\Omega) \to \mathscr{C}_0^\infty(\Omega)$ related, to $\mathcal{L}(u)$. We start considering the following limit

$$\begin{aligned}
\lim_{h\to 0} \frac{\mathcal{L}(u+h.v) - \mathcal{L}(u)}{h} &= \lim_{h\to 0} \frac{1}{h}(u+h.v)_t + H(x, D(u+h.v)) + \Delta(u+h.v) - g[m] \\
&\quad -u_t - H(x, Du) - \Delta u + g[m] \\
&= \lim_{h\to 0} \frac{1}{h} h.v_t + H(x, D(u+hv)) + h\Delta v - H(x, Du) \\
&= \lim_{h\to 0} \frac{h.v_t + H(x, D(u+h.v)) + h.\Delta v - H(x, Du)}{h} \\
&= v_t + \Delta v + \lim_{h\to 0} \frac{H(x, D(u+hv)) - H(x, Du)}{h} \\
&= v_t + \Delta v + D_p H(x, Du) Dv.
\end{aligned}$$

Now we have the linear operator

$$L_u[v] = v_t + \Delta v + D_p H(x, Du) Dv,$$

for every function v in a suitable function space. Then we will obtain the formal adjoint of L_u in the L^2 sense; as we know from functional analysis, $L^2(\Omega \times [0,T])$ with the following inner product is a Hilbert space:

$$\langle f, g \rangle := \int_{\Omega\times[0,T]} f.\bar{g} dxdt = \int_{\Omega\times[0,T]} f.g dxdt \quad, \forall f, g \in L^2(\Omega \times [0,T]).$$

Searching for the adjoint of L_u consists in finding an linear operator L_u^* satisfying

$$\langle L_u[v], \phi \rangle = \langle v, L_u^*[\phi], \rangle \qquad\qquad \forall v \in L^2(\Omega \times [0,T]), \phi \in \mathscr{C}_0^\infty(\Omega \times [0,T]).$$

Using the definition of the inner product in L^2 we have,

$$\int_{\Omega\times[0,T]} \phi L_u(v)dxdt = \int_{\Omega\times[0,T]} L_u^*(\phi).vdxdt \tag{2.17}$$

But if we expand the left-hand side of (2.17) we obtain:

$$\begin{aligned}\int_{\Omega\times[0,T]} \phi(v_t + \Delta v + D_pH(x,Du)Dv)dxdt = &+ \underbrace{\int_{\Omega\times[0,T]} \phi v_t dxdt}_{\mathbf{I}} + \underbrace{\int_{\Omega\times[0,T]} \phi\Delta v dxdt}_{\mathbf{II}} \\ &+ \underbrace{\int_{\Omega\times[0,T]} \phi[D_pH(x,Du)Dv]dxdt}_{\mathbf{III}}\end{aligned}$$

Using the integration by parts formula we obtain the following expressions:

for **I**:

$$\int_{\Omega\times[0,T]} \phi v_t dxdt = -\int_{\Omega\times[0,T]} v\phi_t dxdt;$$

for **II**:

$$\int_{\Omega} \phi\Delta v dxdt = -\int_{\Omega} D\phi Dv dxdt = \int_{\Omega\times[0,T]} v\Delta\phi dxdt$$

for **III**:

$$\int_{\Omega\times[0,T]} \phi D_pH(x,Du).Dvdxdt = -\int_{\Omega\times[0,T]} v\,\mathrm{div}(\phi D_pH(x,Du))dxdt.$$

Remark 2.1. *Observe that the equation in the item* **II** *means that the Laplacian operator is self-adjoint, i.e. its adjoint in L^2 is itself.*

It follows that:

$$\begin{aligned}\int_{\Omega\times[0,T]} L_u^*[\phi]vdxdt &= -\int_{\Omega\times[0,T]} v\phi_t dxdt + \int_{\Omega\times[0,T]} v\Delta\phi dxdt - \int_{\Omega\times[0,T]} v\,\mathrm{div}(\phi D_pH(x,Du)) = \\ &= \int_{\Omega\times[0,T]} v[-\phi_t + \Delta\phi - \mathrm{div}(\phi D_pH(x,Du))]dxdt\end{aligned}$$

So we conclude

$$L_u^*[\phi] = -\phi_t + \Delta\phi - \mathrm{div}(\phi D_pH(x,Du)).$$

Hence

$$L_u^*[\rho] = 0$$

is, indeed,

$$\rho_t + \operatorname{div}(D_pH(x, Du)\rho) = \Delta\rho. \tag{2.18}$$

This is the Fokker-Plank equation describing the evolution of a population that solves the optimal control problem. Together with the HJ equation it gives rise to the MFG-system to be studied in the next chapter.

Fokker-Planck equation and conservation of mass In the case that we are searching for solutions to the Fokker-Planck equation in $\Omega_T = \mathbb{T}^d \times [0, T]$ where $\mathbb{T}^d$ is the d-dimensional torus, we can verify that such solutions have an interesting property, this is the context of the next proposition:

Proposition 2.2 (Conservation of mass)**.** *Let m be a solution to* (2.18)*, then:*

$$\int_\Omega m(x,t)dx = 1 \qquad \forall t \in [0, T].$$

Proof. We have:

$$\frac{d}{dt}\int_\Omega m(x,t)dx = \int_\Omega m_t(x,t)dx = \underbrace{\int_\Omega \operatorname{div}(D_pHm)dx}_{\mathbf{I}} + \underbrace{\int_\Omega \Delta m dx}_{\mathbf{II}}$$

Considering $\Omega = \mathbb{T}^d$ and using the integration by parts formula we obtain:

for **I**:

$$\int_\Omega \operatorname{div}(D_pHm).1dx = \int_{\partial\Omega} mD_pHdS - \int_\Omega (D_pHm)D(1)dx = 0$$

and for **II**:

$$\int_\Omega \Delta m dx = -\int_\Omega D(1)Dmdx + \int_{\partial\Omega} Dm\nu dS = 0.$$

With these equalities we conclude that the integral of $m(x, \tau)$ for any instant $\tau \geq 0$ is constant, and moreover, the constant is one, given that m_0 is the initial density and

$$\int_\Omega m_0 dx = 1$$

. □

Remark 2.2. *When we consider Ω arbitrary, the equality in* **II** *becomes an inequality because $\int_{\partial\Omega} Dm\nu dS \leq 0$. In our problem it means that the population in the system is non-increasing over time. For more details see [23].*

In the sequel, we equip (2.18) with appropriate initial conditions. This choice is arbitrary and motivated by the amount of information we can extract. We start by fixing an initial time $\tau \in [t_0, T)$ and a point x_0 in $\mathbb{R}^d$. Then, set

$$\rho(x, \tau) = \delta_{x_0}(x), \tag{2.19}$$

where $\delta_{x_0}(x)$ is the Dirac delta centered at x_0. The aforementioned choice leads to the next lemma.

Lemma 2.1 (Representation formula for u)**.** *Let u be a solution to* (2.5)*. Assume that ρ solves the adjoint equation* (2.18) *with initial condition* (2.19)*. Then*

$$\begin{aligned} u(x,t) &= \int_\tau^T \int_{\mathbb{R}^d} (H(x, Du; m) - D_pH(x, Du) \cdot Du)\rho dx \\ &\quad + \int_{\mathbb{R}^d} u_T(x)\rho(x,T)dx + \int_\tau^T \int \mathbb{R}^d g[m]dxdt. \end{aligned}$$

Proof. Multiply (2.5) by ρ, (2.18) by u, sum them and integrate by parts to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} u\rho dx = \int_\tau^T \int_{\mathbb{R}^d} (H(x, Du) - D_pH(x, Du) \cdot Du)\rho dx + \int \mathbb{R}^d g[m]dx.$$

By integrating with respect to t we get:

$$\begin{aligned} u(x_0,\tau) &= \int_\tau^T \int_{\mathbb{R}^d} (H(x, Du) - D_pH(x, Du) \cdot Du)\rho dx \\ &\quad + \int_{\mathbb{R}^d} u_T(x)\rho(x,T)dx + \int_\tau^T \int_{\mathbb{R}^d} g[m]dxdt. \end{aligned}$$

Since x_0 and τ are chosen arbitrarily, the former computation finishes the proof. □

Similar ideas yield a representation formula for the directional derivatives of the value function.

Lemma 2.2 (Representation formula for u_ξ)**.** *Let u be a solution to* (2.5)*. Assume that ρ solves the adjoint equation* (2.18) *with initial condition* (2.19)*. Fix a direction ξ in $\mathbb{R}^d$. Then*

$$u_\xi(x,t) = \int_\tau^T \int_{\mathbb{R}^d} D_\xi H(x, Du; m)dxdt + \int_{\mathbb{R}^d} (u_T)_\xi(x)\rho(x,T)dx. \tag{2.20}$$

Proof. Differentiate (2.5) in the ξ direction and multiply it by ρ. Then, multiply (2.18) by u_ξ, sum them and integrate by parts to get:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u_\xi(x,t)\rho(x,t)dx = \int_{\mathbb{R}^d} D_\xi H(x, Du)\rho(x,t)dx. \tag{2.21}$$

Integrating the former equality with respect to t and noticing that x_0 and t are arbitrary, we obtain the result. □

In Lemmas (2.1) and (2.2) we can obtain information about the values of u and u_ξ in a given point of the domain.In what follows, we show that under general assumptions on H, we can get uniform upper bound for the solutions of (2.5).

Corollary 2.1 (Upper bounds for u)**.** *Let u be a solution to* (2.5)*. Assume that ρ solves the adjoint equation* (2.18) *with initial condition* (2.19)*. Assume further that there exists $C > 0$ so that*

$$H - D_pH \cdot p \leq -C. \tag{2.22}$$

Hence,

$$u(x,t) \leq \|u_T\|_{L^\infty(\mathbb{R}^d)} + \int_\tau^T \int_{\mathbb{R}^d} g[m]dxdt, \tag{2.23}$$

for every $(x,t) \in \mathbb{R}^d \times [t_0, T]$.

Proof. It follows from Lemma (2.1), by using (2.22) and taking the supremum in the right-hand side of the representation formula. □

Remark 2.3. *The assumption* (2.22) *seems to be artificial, but it is not. If we study the case the Hamiltonian is suppose to be quadratic, i.e.,*

$$H(x,p) = \frac{|p|^2}{2}$$

we have that

$$H - D_pH(x,p) \cdot p = \frac{|p|^2}{2} - \frac{|p|^2}{2} = -\frac{|p|^2}{2} < 0.$$

We conclude this chapter with a corollary about the Lipschitz regularity of the value function u. In order to have a better presentation,we assume that D_xH is uniformly bounded, i.e., there exists a constant $C > 0$ such that

$$|D_xH(x,p)| \leq C.$$

Corollary 2.2 (Lipschitz regularity for u)**.** *Let u be a solution to* (2.5)*. Assume that ρ solves the adjoint equation* (2.18) *with initial condition* (2.19)*. Assume further that tehre exists $C > 0$ so that*

$$|D_xH(x,p)| \leq C.$$

Hence,

$$Du \in L^{\infty}(\mathbb{R}^d \times [t_0, T]).$$

Proof. Lemma (2.2) yields:

$$u_\xi(x,t) = \int_\tau^T \int_{\mathbb{R}^d} D_x H(x, Du) dx dt + \int_{\mathbb{R}^d} (u_T)_\xi(x) \rho(x,T) dx;$$

Take absolute values on both sides of the previous equality and use (2.24) to conclude the proof.

□

Note that these results were obtained under the assumption of solutions defined over $\mathbb{R}^d \times [0,T]$. What will be done on the next chapter, is to explore the same structure of the problem, i.e. a MFG, but with solutions over a different domain, $\Omega \times [0,T]$, where Ω is a bounded subset of $\mathbb{R}^d$.

Chapter 3

General model: a second order mean-field game

In this chapter we prove some estimates for a mean-field game that comes from the heuristic derivation presented before. In this chapter, however, we work in a bounded domain Ω contained in the Euclidean space $\mathbb{R}^d$. Here, Ω always represents a bounded subset of $\mathbb{R}^d$, where $\partial\Omega$ is of class $\mathscr{C}^1$.

3.1 The General Model

3.1.1 Hamilton-Jacobi equation for a specific case

Following the ideas presented in the previous chapter, we can obtain a mean-field game from heuristic derivation associated with an optimal control problem, and more, if we impose some conditions in this optimal control problem we can obtain a different Hamilton-Jacobi equation. Having this information in hands,it is natural from a mathematical approach to this problem, to investigate the relations between the conditions of an optimal control problem and the respectively hypothesis in the correspondent MFG obtained from a heuristic derivation.

One interesting problem in this sense, trying to obtain a mean-field game in a bounded domain, i.e., search for solutions of a MFG (which regularity assumed as smooth as it possible), whose domain is a bounded subset of $\mathbb{R}^d$. The motivation to investigate this kind of problem comes from models that better approximates concrete situation in economic models, like the one presented in Chapter 4. There, the Wealth and Capital accumulation are bounded processes, and the Fokker-Plank equation, which describes the density of a population, has to vanish at the boundary to encode the hypothesis of zero-mass in such situation.

To obtain a MFG with these characteristics we describe an optimal control problem, whose agent's state is determined by a point $x \in \Omega \subset \mathbb{R}^d$. Where Ω is an bounded subset of $\mathbb{R}^d$. As in Chapter 2, the trajectory of the agent is governed by the SDE:

$$\begin{cases} d\mathbf{x}_t = vdt + \sqrt{2}dB_t \\ \mathbf{x}_0 = x \in \Omega \subset \mathbb{R}^d, \end{cases} \tag{3.1}$$

In this optimal control problem, the agent faces the payoff functional:

$$J(\mathbf{x}, \mathbf{v}; t) := E^{\mathbf{x}} \left[\int_t^{T \wedge \tau} L(\mathbf{x}_s, \mathbf{v}_s) + g[m] ds + \Psi(\mathbf{x}_T, T) \right], \tag{3.2}$$

where the Lagrangian $L : \Omega \subset \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \to \mathbb{R}$ is:

$$L(x, v; m) := L(x, v) + g[m], \quad \forall x, v \in \mathbb{R}^d, \text{and} \quad m \in \mathbb{R}. \tag{3.3}$$

Note that $L(x, v)$ and $L(x, v; m)$ are denoted by the same letter L for simplicity. The formulation in (3.3) is called additive with respect to the measure m. The terminal payoff of the system is Ψ, which is given by:

$$\Psi(\mathbf{x}_\tau, \tau) = \begin{cases} u_T(x), & if \quad \tau \geq T \\ h(x, \tau), & if \quad \tau < T. \end{cases} \tag{3.4}$$

For any instance $\tau > 0$. Each case in (3.4) reflects a distinct situation; if $T \leq \tau$ the state remained in the interior of Ω until $t = T$. In case $\tau \leq T$, the system has reached the boundary at instant τ. From heuristic derivation we obtain the Hamilton-Jacobi equation as follows:

$$-u_t + H(x, Du) = \Delta u + g[m] \qquad \text{in } \Omega \times [0, T]$$

with the following boundary conditions,

$$\begin{cases} u(x, T) = u_T(x) & \text{in } \Omega, \\ u(x, t) = h(x, t) & \text{on } \partial\Omega \times [0, T], \end{cases}$$

where h is a function whose regularity is prescribed in the following section.

3.1.2 The associated Fokker-Plank equation

To compose a MFG system, we need to work to obtain a second equation, which one is able to describe the evolution of the population of agents, to do so, we remark two of them. The first one, as we can see in Example (2.2), is to obtain the adjoint operator of the infinitesimal generator of a Markov diffusion. In a similar way (using adjoint linear operators), we can obtain the Fokker-Plank equation so that it is derived from Hamilton-Jacobi equation. In what follows we choose to work in this second approach to obtain the Fokker-Plank equation. Approximating non-linear structures by linear ones is a very common strategy in Mathematics. One example of such strategy is to approximate the values of a differentiable function by a linear one, in order to obtain the infinitesimal small errors. It can be done using the Taylor's formula, which uses the best approximation on each point evaluating the derivative of suitable order of the function. In problems which involving non-linear PDE's, it is common trying to search for structures of operators in space functions to better understand the behavior of a solution to these equations. In our case, the Hamilton-Jacobi is non-linear, and it is obtained as the heuristic derivation of an optimal control problem. An interesting fact is that we can obtain the Fokker-Plank equation searching for the formal adjoint of the linearized Hamilton-Jacobi equation. This process is described as follows: to linearize the Hamilton-Jacobi equation consists in finding a linear operator $L(u)[\cdot]$ or $L_u : \mathscr{C}_0^\infty(\Omega) \to \mathscr{C}_0^\infty(\Omega)$ related a non-linear on $N(u)$, defined as:

$$N(u) := -u_t + H(x, Du) - \Delta u - g[m]. \tag{3.5}$$

We start considering the following limit

$$\begin{aligned}
\lim_{h\to 0} \frac{N(u+h.v) - N(u)}{h} &= \lim_{h\to 0} \frac{1}{h} - (u+h.v)_t + H(x, D(u+h.v)) - \Delta(u+h.v) - g[m] \\
&\quad +u_t - H(x, Du) + \Delta u + g[m] = \\
&= \lim_{h\to 0} \frac{1}{h} - h.v_t + H(x, D(u+hv)) - h\Delta v - H(x, Du) = \\
&= \lim_{h\to 0} \frac{-h.v_t + H(x, D(u+h.v)) - h.\Delta v - H(x, Du)}{h} \\
&= -v_t - \Delta v + \lim_{h\to 0} \frac{H(x, D(u+hv)) - H(x, Du)}{h} = \\
&= -v_t - \Delta v + D_p H(x, Du) Dv.
\end{aligned}$$

Now we have the linear operator

$$L_u[v] = -v_t - \Delta v + D_p H(x, Du) Dv \tag{3.6}$$

for every fucntion v in a suitable function space. Then we will obtain the formal adjoint of L_u in the L^2 sense; as we know from functional analysis, $L^2(\Omega)$ with the following inner product is a Hilbert space:

$$\langle f, g \rangle := \int_\Omega f.\overline{g} = \int_\Omega f.gdx \quad ,\forall f, g \in L^2(\Omega) \tag{3.7}$$

Searching for the adjoint of L_u consists in finding an linear operator L_u^* satisfying

$$\langle L_u[v], \phi \rangle = \langle v L_u^*[\phi] \rangle \qquad ,\forall v \in L^2(\Omega), \phi \in \mathscr{C}_0^\infty(\Omega). \tag{3.8}$$

Using the definition of the inner product in L^2 we have,

$$\int_{\Omega\times[0,T]} \phi L_u(v)dxdt \quad = \quad \int_{\Omega\times[0,T]} L_u^*(\phi).vdxdt \tag{3.9}$$

But if we expand the left-hand side of (3.9) we obtain:

$$\begin{aligned}\int_{\Omega\times[0,T]} \phi(-v_t - \Delta v + D_pH(x,Du)Dv)dxdt = \quad & - \underbrace{\int_{\Omega\times[0,T]} \phi v_t dxdt}_{\mathbf{I}} - \underbrace{\int_{\Omega\times[0,T]} \phi \Delta v dxdt}_{\mathbf{II}} \\ & + \underbrace{\int_{\Omega\times[0,T]} \phi[D_pH(x,Du)Dv]dxdt}_{\mathbf{III}}\end{aligned}$$

Using the integration by parts formula we obtain the following expressions:

for **I**:

$$-\int_{\Omega\times[0,T]} \phi v_t dxdt = +\int_{\Omega\times[0,T]} v\phi_t dxdt$$

for **II**:

$$-\int_{\Omega\times[0,T]} \phi \Delta v dx = \int_{\Omega\times[0,T]} D\phi Dv dxdt = -\int_{\Omega\times[0,T]} v\Delta\phi dxdt$$

for **III**:

$$\int_{\Omega\times[0,T]} \phi D_pH(x,Du).Dvdxdt = -\int_{\Omega\times[0,T]} v\,\mathrm{div}(\phi D_pH(x,Du))dxdt$$

Remark 3.1. *Observe that the equation in the item* **II** *means that the Laplacian operator is self-adjoint, i.e. its adjoint in $L^2(\Omega), \langle , \rangle$) is itself.*

It follows that:

$$\begin{aligned}\int_{\Omega\times[0,T]} L_u^*[\phi]vdxdt \quad &= \quad \int_{\Omega\times[0,T]} v\phi_t dxdt - \int_{\Omega\times[0,T]} v\Delta\phi dxdt - \int_{\Omega\times[0,T]} v\,\mathrm{div}(\phi D_pH(x,Du)) = \\ &= \quad \int_{\Omega\times[0,T]} v[\phi_t - \Delta\phi - \mathrm{div}(\phi D_pH(x,Du))]dxdt\end{aligned}$$

So we conclude

$$L_u^*[\phi] = \phi_t - \Delta\phi - \mathrm{div}(\phi D_p H(x, Du))$$

Hence

$$L_u^*[m] = 0$$

$$m_t - \mathrm{div}(D_p H(x, Du)m) = \Delta m. \tag{3.10}$$

is the Fokker-Plank equation describing the evolution of a population that solves the optimal control problem that, with H-J equation composing a the MFG-system studied in this Chapter.

3.1.3 The Mean-Field Game in a bounded domain

The coupling of those two equations obtained above gives us the MFG problem that is at the center of our estimates. Observe that the boundary conditions in a bounded domain reflect the fact that we are trying to establish bounds for the wealth and capital of an agent in a situation as the stock market. For example, the condition $m \equiv 0$ encodes that there is no player on the boundary, that is, the density is null on the boundaries. Another consideration is about the function h that traduces the behavior of a solution to Hamilton-Jacobi equation on the boundary; h encodes the cost of reaching the boundary at times $\tau \leq T$ and is supposed be non trivial. The m_0 is the initial distribution of the population, and u_T the final payoff for a player. We have

$$\begin{cases} -u_t + H(x, Du) = \Delta u + g[m] & \text{in } \Omega \times [0, T] \\ m_t - \mathrm{div}(D_p H m) = \Delta m & \text{in } \Omega \times [0, T], \end{cases} \tag{3.11}$$

equipped with boundary conditions,

$$\begin{cases} u(x, T) = u_T(x) & \text{in } \Omega, \\ m(x, 0) = m_0(x) & \text{in } \Omega, \end{cases} \tag{3.12}$$

and

$$\begin{cases} u(x, t) = h(x, t) & \text{on } \partial\Omega \times [0, T] \\ m(x, t) = 0 & \text{on } \partial\Omega \times [0, T]. \end{cases} \tag{3.13}$$

The null mass on the boundary impacts on the conservation of mass In Chapter 1 we presented a brief discussion about the conservation of mass property of F-P equation on that case. Here the hypothesis that $m \equiv 0$ on $\partial\Omega \times [0,T]$ has an direct impact on that property as we can check through the following inequality:

$$\begin{aligned} \frac{d}{dt}\int_\Omega m(x,t)dx &= \int_\Omega \operatorname{div}(D_pH(x,Du)m)dx + \int_\Omega \operatorname{div}(Dm)dx \\ &= \int_{\partial\Omega} Dm \cdot \nu \, dS \leq 0 \end{aligned}$$

This inequality occurs because we are assuming that an agent has only two possible states in the final instant: Remained in the game (in the domain) or not. With this we can conclude that the population is non-increasing over time.

3.2 Hopf-Lax estimate

In the sequel, we explore some estimates for the MFG; although well known, these estimates hold a substantial degree of originality, since the domains treated here are bounded and the boundary conditions are not periodic (as in the majority of papers in the literature). We begin with a concept of solution to the MFG (3.11).

Definition 3.1. *We say that a pair of functions $(u,m) \in \mathscr{C}^\infty(\Omega \times [0,T])$ is a solution to the MFG* (3.11) *if u satisfies the Hamilton-Jacobi equation and m satisfies the Fokker-Plank equation in* (3.11) *with conditions* (3.12)-(3.13) *both in the classical sense.*

Remark 3.2. *Note that we are supposing the best regularity possible, because existence of such solutions are not the objective in this work, the main question is; "If such solution exists, what kind of a priori information can be obtained from the problem". Answering this kind of question, is important to justify the study of existence and uniqueness of solutions.*

Theorem 3.1 (Hopf-Lax Formula)**.** *Suppose that A1 holds. Let (u,m) be a solution to* (3.11)*. Then for any solution to*

$$\begin{cases} \xi_t + \operatorname{div}(b.\xi) = \Delta\xi & \text{in } \Omega \times (\tau,T), \\ \xi(x,\tau) = \xi_\tau(x) & \text{in } \Omega, \\ \xi(x,t) = 0 & \text{in } \partial\Omega \times (\tau,T), \end{cases} \tag{3.14}$$

and for any smooth vector field $b : \Omega \subset \mathbb{R}^d \times (t,T) \to \mathbb{R}^d$, we have the following upper bound:

$$\begin{aligned}\int_\Omega u(x,\tau)\xi_\tau(x)dx \quad &\leq \quad \int_\tau^T \int_\Omega (L(b,x)+g[m])\,\xi(x,t)dxdt + \int_\Omega u_T(x)\xi(x,T)dx \\ &+ \quad \|h\|_{L^r(\tau,T;L^p(\partial\Omega))}\,\left(C + C\|\xi_\tau\|_{L^2(\Omega)}\right)\end{aligned} \tag{3.15}$$

To prove the Theorem 3.1, we will need some previous results that are organized as lemmas and propositions in what follows.

Lemma 3.1. *Let (u,m) be a solution to* (3.11)*, and ξ a solution to* (3.14)*, we have the following upper bound:*

$$\int_\tau^T \int_{\partial\Omega} uD\xi\;\nu dSdt \quad \leq \quad \|h\|_{L^r(\tau,T;L^p(\partial\Omega))}\,\|D\xi\|_{L^s(\tau,T;L^q(\partial\Omega))}, \tag{3.16}$$

where

$$\frac{1}{p}+\frac{1}{q}=1 \qquad \textit{and} \qquad \frac{1}{r}+\frac{1}{s}=1 \tag{3.17}$$

Proof. Applying the Hölder's inequality and the definitions of norm in L^p and in anisotropic Lebesgue spaces we have:

$$\begin{aligned}\int_\tau^T \int_{\partial\Omega} uD\xi\;\nu dxdt \quad &\leq \quad \int_\tau^T \int_{\partial\Omega} |uD\xi|\;|\nu| dSdt \\ &\leq \quad \int_\tau^T \left[\|u\|_{L^p(\partial\Omega)}\;\|D\xi\|_{L^q(\partial\Omega)}\right]dt \\ &\leq \quad \|h\|_{L^r(\tau,T;L^p(\partial\Omega))}\,\|D\xi\|_{L^s(\tau,T;L^q(\partial\Omega))},\end{aligned} \tag{3.18}$$

where

$$\frac{1}{p}+\frac{1}{q}=1 \qquad \text{and} \qquad \frac{1}{r}+\frac{1}{s}=1$$

□

Proposition 3.1. *Suppose that A1 holds. Let (u,m) a solution to* (3.11)*. Then, for any smooth vector field $b : \Omega \subset \mathbb{R}^d \times (t,T) \to \mathbb{R}^d$, and any solution* (3.14) *on this conditions, we have the following upper bound:*

$$\begin{aligned}\int_\Omega u(x,\tau)\xi_\tau(x)dx \quad &\leq \quad \int_\tau^T \int_\Omega (L(b,x)+g[m])\,\xi(x,t)dxdt \int_\Omega u_T(x)\xi(x,T)dx \\ &+ \quad \|h\|_{L^r(\tau,T;L^p(\partial\Omega))}\,\|D\xi\|_{L^s(\tau,T;L^q(\partial\Omega))}\end{aligned}$$

where

$$\frac{1}{p}+\frac{1}{q}=1 \qquad \textit{and} \qquad \frac{1}{r}+\frac{1}{s}=1$$

Since that $\xi_T \geq 0$ and $\int_\Omega \xi_T(x)dx = 1$.

Proof. Multiplying the first equation in (3.14) by $-u$ and the first equation in (3.11) by ξ we obtain:

$$\begin{aligned} -\xi u_t + \xi H(x, Du) &= \xi\Delta u + \xi g[m], \\ -u\xi_t - u\ \operatorname{div}(b.\xi) &= -u\Delta\xi. \end{aligned}$$

Adding these equations we obtain:

$$\begin{aligned} -\xi u_t - u\xi_t + \xi H(x, Du) - u\ \operatorname{div}(b.\xi) &= \xi\Delta u - u\Delta\xi + \xi g[m], \text{i.e.,} \\ -\frac{d}{dt}(\xi u) + \xi H(x, Du) - u\ \operatorname{div}(b.\xi) &= \xi\Delta u - u\Delta\xi + \xi g[m]. \end{aligned}$$

Integrating with respect to the Lebesgue measure in Ω we have

$$\begin{aligned} -\frac{d}{dt}\int_\Omega (\xi u)dx &= \underbrace{-\int_\Omega \left[\xi H(x, Du) - u\ \operatorname{div}(b.\xi)\right] dx}_{\mathbf{C}} + \underbrace{\int_\Omega \xi\Delta u dx}_{\mathbf{A}} - \underbrace{\int_\Omega u\Delta\xi dx}_{\mathbf{B}} \\ &+ \int_\Omega \xi g[m]dx \end{aligned} \tag{3.19}$$

Now, let us evaluate **A** and **B** using integration by parts in **A** we obtain:

$$\begin{aligned} \int_\Omega \xi\Delta u dx = \int_\Omega \xi\ \operatorname{div}(Du)dx &= -\int_\Omega D\xi Du dx \\ &= -\int_\Omega D\xi Du dx. \end{aligned}$$

The integral vanishes at the boundary because of the conditions prescribed in (3.14). Proceeding in the same way with **B** we have:

$$-\int_\Omega u\Delta\xi dx = -\int_\Omega u \operatorname{div}(D\xi)dx = \int_\Omega DuD\xi dx - \int_{\partial\Omega} uD\xi\nu dS$$

Then **A**+ **B** $= -\int_{\partial\Omega} uD\xi\nu dS$. Applying the same techniques we obtain a simplification of the term **C**:

$$\int_\Omega \left[\xi H(x, Du) - u\ \operatorname{div}(b.\xi)\right] dx = \int_\Omega \left(H(x, Du) + Du.b\right)\xi dx.$$

So, with this simplifications for A, B and C, (3.19) becomes:

$$-\frac{d}{dt}\int_\Omega (\xi u)dx = -\int_\Omega \left(H + Du.b\right)\xi dx - \int_{\partial\Omega} uD\xi\nu dS + \int_\Omega \xi g[m]dx$$

By definition of $L(b,x) = \sup_p(-pb - H(x,p))$ we have:

$$-\frac{d}{dt}\int_\Omega (\xi(x,t)u(x,t))dx \quad \leq \quad \int_\Omega (L(b,x)+g[m])\,\xi(x,t)dx - \int_{\partial\Omega} u(x,t)D\xi(x,t)\;\nu dS + \int_\Omega \xi g[m]dx.$$

Integrating with respect to the Lebesgue measure in (τ,T) :

$$\begin{aligned}\int_\Omega u(x,\tau)\xi(x,\tau)dx \quad &\leq \quad \int_\tau^T\int_\Omega (L(b,x)+g[m])\,\xi(x,t)dxdt \\ &- \quad \int_\tau^T\int_{\partial\Omega} u(x,t)\;D\xi(x,t)\;\nu dxdt + \int_\Omega u(x,T)\xi(x,T)dx. \qquad (3.20)\end{aligned}$$

We will now proceed to find an upper bound for the boundary term that appears on the right-hand side of inequality (3.20). It can be done using Lemma 3.1. Rewriting the inequality (3.20) with the result in Lemma 3.1 we obtain

$$\begin{aligned}\int_\Omega u(x,\tau)\xi_\tau(x)dx \quad &\leq \quad \int_\tau^T\int_\Omega (L(b,x)+g[m])\,\xi(x,t)dxdt + \int_\Omega u_T(x)\xi(x,T)dx \\ &+ \quad \|u\|_{L^r(\tau,T;L^p(\partial\Omega))}\,\|D\xi\|_{L^s(\tau,T;L^q(\partial\Omega))} \\ \\ &\leq \quad \int_\tau^T\int_\Omega (L(b,x)+g[m])\,\xi(x,t)dxdt + \int_\Omega u_T(x)\xi(x,T)dx \\ &+ \quad \|h\|_{L^r(\tau,T);L^p(\partial\Omega)}\,\|D\xi\|_{L^s(\tau,T);L^q(\partial\Omega)} \\ \\ &\leq \quad \int_\tau^T\int_\Omega (L(b,x)+g[m])\,\xi(x,t)dxdt\int_\Omega u_T(x)\xi(x,T)dx \\ &+ \quad \|h\|_{L^r(\tau,T;L^p(\partial\Omega))}\,\|D\xi\|_{L^s(\tau,T;L^q(\partial\Omega))} \qquad (3.21)\end{aligned}$$

where

$$\frac{1}{p}+\frac{1}{q}=1 \qquad \text{and} \qquad \frac{1}{r}+\frac{1}{s}=1 \qquad (3.22)$$

and we have used that $\xi_T \geq 0$ and $\int_\Omega \xi_T(x)dx = 1$. In the path to prove the Theorem 3.1, we need to work on an estimate for the term $\|D\xi\|_{L^s(\tau,T);L^q(\partial\Omega)}$. This is the subject of the next Section. □

3.2.1 Estimates and Results Needed

We state some well-know results in Lemmas 3.2, 3.3 and 3.4, whose proofs can be found in : [15] *(Appendix D)*, [21] and [6] *(Theorem 1,p.275)* respectively.

Proposition 3.2. *Let* $\xi \in C^\infty([0,\tau]\times\Omega)$ *be a solution to* (3.14). *If* $f = \mathrm{div}(b\cdot\xi), \xi_\tau \in L^2([0,T]\times\Omega)$, *then exists a $C > 0$ such that:*

$$\|\xi(\cdot,t)\|_{L^2(\Omega)} \leq C\|\xi_\tau\|^2_{L^2(\Omega)}.$$

Proof. Let ξ as in the Propositon's statement, using the chain rule, integration by parts formula, and the Cauchy's inequality wiht ε we can evaluate the following:

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{\xi^2}{2}dx &= \int_\Omega \frac{2\xi\ \xi_t}{2}dx = \int_\Omega \xi(\Delta\xi - \operatorname{div}(b\cdot\xi))dx = \int_\Omega \xi(x,t)\ \Delta\xi(x,t) - \int_\Omega \xi(x,t)\operatorname{div}(b\cdot\xi)dx = \\
&= -\int_\Omega D\xi.D\xi dx + \int_\Omega D\xi(x,t)\cdot b.\xi dx \\
&\leq -\int_\Omega |D\xi|^2 dx + \frac{\varepsilon}{2}\int \xi^2(x)dx + \frac{1}{2}\int_\Omega b^2\xi^2(x,t)dx \\
&\leq -\frac{1}{2}\int |D\xi|^2(x)dx + \frac{1}{2}\int_\Omega b^2\xi^2(x,t)dx.
\end{aligned}$$

So we conclude:

$$\frac{d}{dt}\int_\Omega \frac{\xi^2(x,t)}{2} \leq \frac{1}{2}\int_\Omega |D\xi|^2 dx + \frac{1}{2}\int_\Omega b^2(x,t)\xi^2(x,t)dx$$

i.e,

$$\frac{d}{dt}\int_\Omega \xi^2(x,t) \leq -\int_\Omega |D\xi|^2 + |b^2(x,t)|\int_\Omega \xi^2 dx.$$

Integrating in (τ,t) we have:

$$\int_\tau^t\int_\Omega \xi^2(x,s)dxds \quad \leq \quad \int_\Omega \xi_\tau^2(x)dx + \sup_{x\in\Omega}|b|^2\int_\tau^t\int_\Omega \xi^2(x,s)ds.$$

Applying the Gronwall's Inequality, we obtain

$$\begin{aligned}
\|\xi(\cdot,t)\|^2_{L^2(\Omega)} &\leq \|\xi_\tau\|^2_{L^2(\Omega)}\exp t\sup_{x\in\Omega}|b|^2 \\
&\leq C\|\xi_\tau\|^2_{L^2(\Omega)}.
\end{aligned}$$

□

Lemma 3.2. *Let v be a solution to* (3.14)*, for every integer s such that $1 < s < \infty$ we have the following upper bounds:*

$$\|v_t\|_{L^a(\Omega)} + \|D^2 v\|_{L^a(\Omega)} \leq C\|\operatorname{div}(b.\xi)\|_{L^a(\Omega)}$$

for $a \in (1,\infty)$.

Lemma 3.3 (Gagliardo-Nirenberg interpolation Inequality)**.** *Let $u : \Omega \to \mathbb{R}$ lying in $L^q(\mathbb{R}^n)$ with*

m^{th} derivative in $L^r(\mathbb{R}^n)$ also having j^{th} derivative in $L^p(\mathbb{R}^n)$ be defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, then

$$\|D^j u\|_{L^p(\Omega)} \leq C_1 \|D^m u\|^{\alpha}_{L^r(\Omega)} \|u\|^{1-\alpha}_{L^q(\Omega)} + C_2 \|u\|_{L^s(\Omega)},$$

where $s > 0$ is arbitrary, and

$$\frac{1}{p} = \frac{j}{d} + \alpha \left(\frac{1}{r} - \frac{m}{d} \right) + \frac{1-\alpha}{q}, \tag{3.23}$$

with $\frac{j}{m} \leq \alpha \leq 1$. Naturally the constants C_1 and C_2 depends upon the domain Ω as the parameters.

Lemma 3.4 (The Poincaré-Wirtinger inequality)**.** *Let Ω be a bounded, connected, open subset of $\mathbb{R}^d$, with a $\mathscr{C}^1$ boundary $\partial\Omega$. Assume $1 \leq p \leq \infty$. Then there exist constants C_1 and C_2, depending on d, p and Ω, such that:*

$$\|u\|_{L^p(\Omega)} \leq C_1 \|Du\|_{L^p(\Omega)} + C_2.$$

for each function $u \in W^{1,p}(\Omega)$. More specifically, the constant C_2 is the average of u over Ω. Then, if $u \in L^1(\Omega)$.

Lemma 3.5. *Let ξ be a solution to* (3.14)*, then we have the following inequality:*

$$\|D^2 \xi\|_{L^r(\Omega)} \leq C_b + C_b \|D\xi\|_{L^r(\Omega)}$$

Proof. Follows from a immediate application of Lemma (3.2) as we can see:

$$\begin{aligned} \|D^2 \xi\|_{L^r(\Omega)} &\leq \|\operatorname{div}(b.\xi)\|_{L^r(\Omega)} \\ &\leq \|\operatorname{div}(b.\xi)\|_{L^s(\Omega)} + \|b \cdot D\xi\|_{L^r(\Omega)} \\ &\leq C_b + C_b \|D\xi\|_{L^r(\Omega)} \end{aligned}$$

□

Lemma 3.6. *Let ξ be a solution to* (3.14)*, then exists a constant C depending upon b and ξ_τ such that:*

$$\|D\xi\|_{L^p(\Omega)} \leq C + C\|\xi_\tau\|_{L^2(\Omega)}.$$

where

$$\frac{1}{p} = \frac{1}{d} + \alpha\left(\frac{1}{r} - \frac{m}{d}\right) + \frac{1-\alpha}{q} \tag{3.24}$$

with

$$\frac{1}{2} \leq \alpha \leq 1.$$

Proof. Applying Gagliardo-Nirenberg interpolation inequality, (Lemma 3.3) and Proposition 3.2 we obtain that

$$\|D\xi\|_{L^p(\Omega)} \leq C_1\|D^2\xi\|^{1-\alpha}_{L^r(\Omega)}\|\xi\|^{\alpha}_{L^q(\Omega)} + C_2\|\xi\|_{L^2(\Omega)}$$

for

$$\frac{1}{p} = \frac{1}{d} + \alpha\left(\frac{1}{r} - \frac{2}{d}\right) + \frac{1-\alpha}{q} \qquad where \quad \alpha \in \left[\frac{1}{2}, 1\right], \quad 1 \leq q \leq 2, \text{and } s > 0.$$

$$\begin{aligned}\|D\xi\|_{L^p(\Omega)} &\leq C_1\|D^2\xi\|^{1-\alpha}_{L^r(\Omega)}\|\xi_\tau\|^{\alpha}_{L^2(\Omega)} + \|\xi_\tau\|_{L^2(\Omega)} \\ &\leq C\|D\xi\|^{1-\alpha}_{L^r(\Omega)}\|\xi_\tau\|^{\alpha}_{L^2(\Omega)} + \|\xi_\tau\|_{L^2(\Omega)},\end{aligned}$$

as $r \leq p$ we have that

$$\|D\xi\|_{L^p(\Omega)} \leq C + C\|\xi_\tau\|_{L^2(\Omega)}.$$

□

Lemma 3.7. *Let ξ be a solution to* (3.14)*, then we have the following upper bound for $D\xi$ in L^p norm on the boundary of Ω.*

$$\|D\xi\|_{L^p(\partial\Omega)} \leq C + C\|D\xi\|_{L^r(\Omega)}.$$

Proof. By the Lemmas 3.5, and 3.6, and applying the Theorem C.4

$$
\begin{aligned}
\|D\xi\|_{L^p(\partial\Omega)} &\leq C\|D\xi\|_{W^{1,p}(\Omega)} = C\left(\sum_{|\alpha|\leq 1}\int_\Omega \mid D^\alpha(D\xi)\mid^p\right)^{\frac{1}{p}} \\
&\leq \|D\xi\|_{L^p(\Omega)} + \|D^2\xi\|_{L^p(\Omega)} \\
&\leq C + C\|\xi_\tau\|_{L^2(\Omega)} + C_b + C_b\|D\xi\|_{L^p(\Omega)} \\
&\leq C + C\|\xi_\tau\|_{L^2(\Omega)}
\end{aligned}
$$

□

We Conclude this section by gathering the former computation in order to proof of Theorem 3.1.

Proof of Theorem 3.1: Applying the Lemma 3.7, in the inequality (3.21) we obtain that

$$
\begin{aligned}
\int_\Omega u(x,\tau)\xi_\tau(x)dx &\leq \int_\tau^T\int_\Omega (L(b,x)+g[m])\,\xi(x,t)dxdt + \int_\Omega u_T(x)\xi(x,T)dx \\
&+ \|h\|_{L^r(\tau,T),L^p(\partial\Omega))}\left(C + C\|\xi_\tau\|_{L^2(\Omega)}\right) \qquad (3.25)
\end{aligned}
$$

which establishes the result.

3.3 First Order Estimate

Next we apply the Hopf-Lax estimate to produce a cornerstone of the regularity theory for MFG problems. That is the so-called First Order Estimate. We begin by defining the *oscillation of a given function* f as follows:

$$\text{osc } f := \sup f - \inf f.$$

Theorem 3.2. *Let* (u,m) *be a solution to* (3.11)*, then, there exists* $C>0$ *such that*

$$
\begin{aligned}
c\int_0^T\!\!\int_\Omega H(x,Du)mdx\,dt + \int_0^T\!\!\int_\Omega G(m)dxdt &\leq C + \|h\|_{L^r(0,T;L^p(\partial\Omega))} + C\,osc\,u(\cdot,T) \\
&+ CT + C\|h\|_{L^p(\Omega)}\left[1+\|Du\|^{\frac{\gamma-1}{1-\alpha}}_{L^{(\gamma-1)jr}}\right],
\end{aligned}
$$

where $G' = g$.

The proof of Theorem 3.2 is presented next.

Proposition 3.3. *Let* (u,m) *be a solution to* (3.11)*, if we assume that* $b\equiv 0$ *on* (3.14)*, then we have the following upper bound*

$$\begin{aligned} C\int_0^T\int_\Omega H(x,D_xu)mdxdt \quad &\leq \quad CT-\int_0^T\int_{\partial\Omega} uDm\nu\ dSdt+\int_0^T\int_\Omega g[m](x,t)(\mu(x,t)-m(x,t))dxdt \\ &\quad +\int_\Omega u(x,T)(\mu(x,t)-m(x,T))dx+C\|h\|_{L^r(\tau,T);L^p(\Omega)}+C. \end{aligned} \tag{3.26}$$

Proof. We multiply the first equation in (3.11) by m and the second equation by $-u$. Then sum then integrate by parts to obtain

$$\begin{aligned} -\frac{d}{dt}\int_\Omega umdx \quad &+ \quad \int_\Omega mH(x,Du)dx+\int_\Omega u\operatorname{div}(D_pH(x,Du).m)dx= \\ &= \quad \underbrace{\int_\Omega m\Delta udx}_{\mathbf{A}}-\underbrace{\int_\Omega u\Delta mdx}_{\mathbf{B}}+\int_\Omega mg[m]dx \end{aligned}$$

Applying the integration by parts formula in **A** we obtain:

$$\int_\Omega m\Delta udx=\int_\Omega m\operatorname{div}(Du)dx=-\int_\Omega Dm\cdot Dudx$$

and for **B** we obtain:

$$-\int_\Omega u\Delta mdx=-\int_\Omega u\operatorname{div}(Dm)dx=\int_\Omega Dm\cdot Dudx-\int_{\partial\Omega} uDm\cdot\nu\ dS$$

Then $A+B=-\int_{\partial\Omega} uDm\cdot\nu\ dS$. Hence,

$$\begin{aligned} -\frac{d}{dt}\int_\Omega umdx \quad &+ \quad \int_\Omega \left(H(x,Du)-D_xu\cdot D_pH\right)mdx= \\ &= \quad -\int_{\partial\Omega} uDm\cdot\nu\ dS+\int_\Omega mg[m]dx \end{aligned}$$

Integrating in $[0,T]$

$$\begin{aligned} -\int_\Omega u_T(x)m_T(x)dx \quad &+ \quad \int_0^T\int_\Omega \left(H(x,Du)-Du\cdot D_pH\right)mdxds= \\ &= \quad -\int_0^T\int_{\partial\Omega} uDm\cdot\nu dSdt+\int_0^T\int_\Omega mg[m]dxdt \\ &\qquad -\int_\Omega u(x,0)m_0(x)dx \end{aligned} \tag{3.27}$$

If $\hat{L}(x,p)=D_pH(x,p)\cdot p-H(x,p)$,

$$\hat{L}(x,p)=L(x,-D_pH(x,p))=\sup_p(p\cdot D_pH(x,p)-H(x,p))$$

$$\hat{L}(x,p)=D_xu\cdot D_pH(x,p)-H(x,p).$$

Multiplying (3.27) by (-1), using the definition of $\hat{L}$ and the assumption A3 the Hamiltonian H we obtain:

$$\begin{aligned}\int_0^T\int_\Omega (CH(x,D_xu)-C)mdxdt \quad &\leq \quad -\int_0^T\int_{\partial\Omega} uDm\nu \; dSdt - \int_0^T\int_\Omega mg[m]dxdt \\ &+\int_\Omega u(x,0)m(x,0)-u(x,T)m(x,T)dx\end{aligned}$$

Integrating the constants, and observing the conservation of mass property from m we obtain

$$\begin{aligned}\int_0^T\int_\Omega H(x,D_xu)mdxdt \quad &\leq \quad -\int_0^T\int_{\partial\Omega} uDm\nu \; dSdt - \int_0^T\int_\Omega mg[m]dxdt \\ &+\int_\Omega u(x,0)m(x,0)-u(x,T)m(x,T)dx + CT \qquad (3.28)\end{aligned}$$

If we put $b \equiv 0$ in (3.14), we will have a solution to the heat equation denoted by $\mu(x,0) = m(x,0) = m_0$. Using the Lax-Hopf formula in this case and applying on(3.28) we obtain

$$\begin{aligned}C\int_0^T\int_\Omega H(x,D_xu)mdxdt \quad &\leq \quad CT - \int_0^T\int_{\partial\Omega} uDm\nu \; dSdt + \int_0^T\int_\Omega g[m](x,t)(\mu(x,t)-m(x,t))dxdt \\ &+\int_\Omega u(x,T)(\mu(x,t)-m(x,T))dx + C\|h\|_{L^r(0,T;L^p(\Omega))} + C\end{aligned}$$

□

3.3.1 Integral estimates on the boundary

The following steps will be in the direction of finding an upper bound for the following term.

$$\int_0^T\int_{\partial\Omega} uDm\nu \; dSdt \qquad (3.29)$$

We rewrite the Fokker-Plank equation as follows,

$$m_t - \Delta m = \mathrm{div}(D_pHm),$$

where

$$D_pHm = [mD_{p_1}H(x,p), \ldots, mD_{p_d}H(x,p)]$$

and

$$D(D_pHm) = \begin{bmatrix} (D_{p_1}Hm)_{x_1} & \ldots & (D_{p_1}Hm)_{x_d} \\ \vdots & \ldots & \vdots \\ (D_{p_d}Hm)_{x_1} & \ldots & (D_{p_d}Hm)_{x_d} \end{bmatrix}.$$

So

$$TrD(D_pHm) = \sum_{i=1}^{d} D_{p_i}Hx_im + D_{p_i}Hm_{x_i} = m \ \operatorname{div}(D_pH) + D_pHDm.$$

Now we can rewrite the Fokker-Plank equation as

$$m_t - \Delta m = m \ \operatorname{div}(D_pH) + D_pHDm. \tag{3.30}$$

In equation (3.30) we have to find an upper bound to

$$\|m \ \operatorname{div}(D_pH) + D_pHDm\|_{L^j(\Omega)},$$

for some $1 \leq j \leq \infty$. For it we will assume the hypothesis A1 and A6 for the Hamiltonian H.

Lemma 3.8. *Let (u, m) be a solution to*(3.30)*, and assume that the Hamiltonian H satisfies A1 and A6. Then there exists $C > 0$ so that*

$$\|\operatorname{div}(D_pH)m + D_pH \cdot Dm\|^j_{L^j(\Omega)} \leq C\left(1 + \|Dm\|^j_{L^{sj}(\Omega)}\right)\left(1 + \|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)j}(\Omega)}\right).$$

for $1 < \gamma < 2$.

Proof. Using Minkonsky's inequality:

$$\|\operatorname{div}(D_pH)m + D_pH \cdot Dm\|^j_{L^j(\Omega)} \leq C\Bigg(\underbrace{\|\operatorname{div}(D_pH)m\|^j_{L^j(\Omega)}}_{\mathbf{I}} + \underbrace{\|D_pH \cdot Dm\|^j_{L^j(\Omega)}}_{\mathbf{II}}\Bigg) \tag{3.31}$$

for **II**:

$$\begin{aligned}
\|D_pH \cdot Dm\|^j_{L^j(\Omega} &= \int_\Omega |D_pH|^j.|Dm|^j dx \leq \int_\Omega |(C + C|Du|^{\gamma-1})|^j|Dm|^j \\
&\leq \int_\Omega C|Dm|^j dx + \int_\Omega C|Du|^{(\gamma-1)j}|Dm|^j dx \\
&\leq C\int_\Omega |Dm|^j dx + C\||Du|^{(\gamma-1)j}\|_{L^r(\Omega)}\||Dm|^j\|_{L^s(\Omega)} \\
&\leq C\|Dm\|^j_{L^j(\Omega)} + C\|Du\|^{(\gamma-1)j}_{L^{(\gamma-1)jr}(\Omega)}\|Dm\|^j_{L^{js}(\Omega)}
\end{aligned}$$

where $r^{-1}+s^{-1}=1$. On the other hand, for **I**:

$$\begin{aligned}\|\operatorname{div}(D_pH)m\|^j_{L^j(\Omega)} &= \int_\Omega |\operatorname{div}(D_pH)|^j|m|^j dx \le \int_\Omega |\left(C+C|Du|^j\right)^{(\gamma-1)}||m|^j dx\\ &\le \int_\Omega C|m|^j dx + \int_\Omega C|Du|^{j(\gamma-1)}|m|^j dx\\ &\le C\|m\|^j_{L^j(\Omega)} + C\||Du|^{(\gamma-1)j}\|_{L^r(\Omega)}\||m|^j\|_{L^s(\Omega)}\\ &\le C\|m\|^j_{L^j(\Omega)} + C\|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)j}(\Omega)}\|m\|^j_{L^{sj}(\Omega)}\\ &\le C\|m\|^j_{L^{sj}(\Omega)}(1+C\|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)j}(\Omega)})\end{aligned}$$

Then, combining the upper bounds for **I** and **II** in (3.31), and using Poincaré-Winteger inequality as in Lemma 3.4 for $\|m\|^j_{L^{js}(\Omega)}$ we have:

$$\begin{aligned}\|\operatorname{div}(D_pH)m+D_pH\cdot Dm\|^j_{L^j(\Omega)} &\le C\|m\|^j_{L^{sj}(\Omega)}\left(1+C\|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)j}(\Omega)}\right)\\ &\quad +C\|Dm\|^j_{L^{js}(\Omega)}\left(1+C\|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)j}(\Omega)}\right)\\ &\le C\left(1+C\|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)j}(\Omega)}\right)+\\ &\qquad +C\|Dm\|^j_{L^{sj}(\Omega)}\left(1+C\|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)j}(\Omega)}\right)\\ &\le C\left(1+\|Dm\|^j_{L^{sj}(\Omega)}\right)\left(1+\|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)j}(\Omega)}\right).\end{aligned}$$

□

Lemma 3.9. *Let (u,m) be a solution to* (3.11)*; H satisfying A1- A1, A4 and A6, we have the following upper bound for the gradient of m:*

$$\|Dm\|_{L^p(\Omega)} \le C + C\|Du\|^{\frac{(\gamma-1)\alpha}{(1-\alpha)}}_{L^{(\gamma-1)jr}(\Omega)},$$

where $q=1$, $js<p$ and

$$\frac{1}{p}=\frac{1}{d}+\alpha\left(\frac{1}{j}-\frac{2}{d}\right)+\frac{(1-\alpha)}{q},$$

for $\alpha\in\left(\frac{1}{2},1\right)$. And constants $C>0$.

Proof. Applying the Lemma 3.2, (the estimate for $\|D^2m\|_{L^p(\Omega)}$) and the result obtained in Lemma 3.8 we have:

$$\|D^2m\|_{L^{j(\gamma-1)}(\Omega)} \le C\left(1+\|Dm\|_{L^{sj}(\Omega)}\right)\left(1+\|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)j}(\Omega)}\right). \tag{3.32}$$

Now, to estimate $\|Dm\|_{L^p(\Omega)}$ we proceed by applying Gagliardo-Nierenberg (Lemma 3.3), the inequality (3.32) and Hölder's Inequality as follows:

$$\begin{aligned}\|Dm\|_{L^p(\Omega)} &\leq C_1\|D^2m\|^{\alpha}_{L^j(\Omega)}\|m\|^{1-\alpha}_{L^q(\Omega)}+C_2\|m\|_{L^{\tilde{q}}(\Omega)}\\ &\leq C_1\left(1+\|Dm\|_{L^{sj}(\Omega)}\right)^{\alpha}\left(1+\|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)}(\Omega)}\right)^{\alpha}\|m\|^{1-\alpha}_{L^q(\Omega)}+C_2\|m\|_{L^{\tilde{q}}(\Omega)},\end{aligned}$$

where $q=1$, $s<1$ and $\widetilde{q}=1$. As m is a probability measure we have

$$\begin{aligned}\|Dm\|_{L^p(\Omega)} &\leq C_1\left(1+\|Dm\|_{L^{sj}(\Omega)}\right)^{\alpha}\left(1+\|Du\|^{(\gamma-1)}_{L^{r(\gamma-1)j}(\Omega)}\right)^{\alpha}\|m\|^{1-\alpha}_{L^{\tilde{q}}(\Omega)}+C_2\\ &\leq C_1+C_1\|Du\|^{(\gamma-1)\alpha}_{L^{r(\gamma-1)j}(\Omega)}+\underbrace{C\|Du\|^{(\gamma-1)\alpha}_{L^{r(\gamma-1)j}(\Omega)}\|Dm\|^{\alpha}_{L^{sj}(\Omega)}}_{\mathbf{A}}\end{aligned}$$

Applying Young with ϵ on **A** we obtain:

$$C\|Du\|^{(\gamma-1)\alpha}_{L^{r(\gamma-1)j}(\Omega)}\|Dm\|^{\alpha}_{L^{sj}(\Omega)}\leq C\|Du\|^{\frac{(\gamma-1)\alpha}{1-\alpha}}_{L^{r(\gamma-1)j}(\Omega)}+C\varepsilon\|Dm\|_{L^{sj}(\Omega)}.$$

As Ω is bounded and we are supposing $sj<p$ we have that.

$$(1-\epsilon C)\|Dm\|_{L^p(\Omega)}\leq C_1++C\|Du\|^{\frac{(\gamma-1)\alpha}{1-\alpha}}_{L^{(\gamma-1)jr}(\Omega)}$$

Here we are using $p=j$, $q=1$, $js<p$ and

$$\frac{1}{p}=\frac{1}{d}+\alpha\left(\frac{1}{j}-\frac{2}{d}\right)+\frac{(1-\alpha)}{q}$$

Then we have that:

$$\|Dm\|_{L^p(\Omega)}\leq C(\epsilon)+\|Du\|^{\frac{(\gamma-1)\alpha}{(1-\alpha)}}_{L^{(\gamma-1)jr}(\Omega)}$$

With $C(\epsilon)=C+\left(\frac{\epsilon}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}\alpha$.

□

Lemma 3.10. *Let m be a solution to* (3.30)*, then we have that*

$$\|Dm\|_{L^q(\partial\Omega)} \quad\leq\quad C\left[1+\|Du\|^{\frac{(\gamma-1)}{1-\alpha}}_{L^{(\gamma-1)jr}(\Omega)}\right]$$

Proof. Now we use the Trace Theorem C.4, and the inequality (3.32) and Lemma 3.9 to evaluate the therm

$$\|Dm\|_{L^q(\partial\Omega)}.$$

We have,

$$
\begin{aligned}
\|Dm\|_{L^j(\partial\Omega)} &\leq C\|Dm\|_{W^{1,j}(\Omega)} \leq \left(\int_\Omega \|Dm\|^j\right)^{\frac{1}{j}} + \left(\int_\Omega \|D^2m\|^j\right)^{\frac{1}{j}} \\
&\leq \|Dm\|_{L^j(\Omega)} + \|D^2m\|_{L^j(\Omega)} \\
&\leq \|Dm\|_{L^j(\Omega)} + C\left(1 + \|Dm\|_{L^{sj}(\Omega)}\right)\left(1 + \|Du\|^{(\gamma-1)j}_{L^{r(\gamma-1)j}(\Omega)}\right).
\end{aligned}
$$

if $q \geq sj$, Then:

$$
\begin{aligned}
\|Dm\|_{L^q(\partial\Omega)} &\leq \|Dm\|_{L^{sj}(\Omega)}\left[1 + C\left(1 + \|Du\|^{(\gamma-1)}_{L^{(\gamma-1)jr}(\Omega)}\right)\right] + C\left(1 + \|Du\|^{(\gamma-1)}_{L^{(\gamma-1)jr}(\Omega)}\right) \\
&\leq \left[\widetilde{C}_1 + \widetilde{C}_2\|Du\|^{\frac{(\gamma-1)\alpha}{1-\alpha}}_{L^{(\gamma-1)jr}(\Omega)}\right]\left[1 + C\left(1 + \|Du\|^{(\gamma-1)}_{L^{(\gamma-1)jr}(\Omega)}\right)\right] \\
&\quad + C\left(1 + \|Du\|^{(\gamma-1)}_{L^{(\gamma-1)jr}(\Omega)}\right) \\
&\leq 1 + \widetilde{C}_2 C\left(1 + \|Du\|^{(\gamma-1)}_{L^{(\gamma-1)jr}(\Omega)}\right) + \widetilde{C}_2\|Du\|^{\frac{(\gamma-1)\alpha}{1-\alpha}}_{L^{(\gamma-1)jr}(\Omega)} + \\
&\quad + C\left(1 + \|Du\|^{(\gamma-1)}_{L^{(\gamma-1)jr}(\Omega)}\right)
\end{aligned}
\tag{3.33}
$$

We can apply The Young with epsilon inequality (B.3) to obtain that:

$$
\|Dm\|_{L^q(\partial\Omega)} \leq C\left[1 + \|Du\|^{\frac{(\gamma-1)}{1-\alpha}}_{L^{(\gamma-1)jr}(\Omega)}\right]
$$

□

Lemma 3.11. *Let* (u, m) *be a solution to* (3.11)*, then we have the following upper bound*

$$
\int_{\partial\Omega} uDm\nu \, dS \leq \|h\|_{L^p(\partial\Omega)}\|Dm\|_{L^q(\partial\Omega)}
$$

with $p^{-1} + q^{-1} = 1.$

Proof. The proof follows direct from aplication of Hölder's inequality to the therm on the left-hand side (3.29) and the function h on the boundary of $\Omega \times [0, T]$ as in (3.11).

$$
\begin{aligned}
\int_{\partial\Omega} uDm\nu \, dS &\leq \int_{\partial\Omega} |u|\|Dm\| dS \leq \|u\|_{L^p(\partial\Omega)}\|Dm\|_{L^q(\partial\Omega)} \\
&\leq \|h\|_{L^p(\partial\Omega)}\|Dm\|_{L^q(\partial\Omega)}
\end{aligned}
$$

with $p^{-1} + q^{-1} = 1.$ □

Proof of Theorem 3.2 : Using Lemma 3.10, in (3.11), and after using this estimate obtained in inequality (3.26) from Proposition 3.3 and integrating the therms, unsing A2 we obtain the result.

Chapter 4

A Model of wealth and capital accumulation

In this chapter, we present a model that describes the wealth and capital accumulation denoted by a_t and k_t, respectively. Such phenomena involves other variables like consumption, investment, interest rate etc. The accumulation of wealth and capital of a single agent will be described by an SDE; the stochastic term in this SDE represents the unexpected events in the economy dynamics, whereas constants σ_a and σ_k represent the volatility of wealth and capital. In [10], this kind of problem is presented in a different domain.

Description of the optimal control problem The state of the model, $\mathbf{x} = [\mathbf{a}_t, \mathbf{k}_t] \in \mathbb{R}^2$, comprises the wealth $\mathbf{a}_t$ and the stock of capital $\mathbf{k}_t$, over time. These evolve according to the following SDE:

$$d\mathbf{x}_t = b(x, z)dt + \sqrt{2\sigma}dB_t, \tag{4.1}$$

where $b : \Omega \times \mathbb{R}^5 \to \mathbb{R}^2$ is a smooth vector field to be detailed later, z represents a number of variables of interest, $x \in \mathbb{R}^2$, and B_t is a two-dimensional Brownian motion. The matrix $\sqrt{2\sigma}$ is given by

$$\sqrt{2\sigma} := \begin{bmatrix} \sqrt{2\sigma_a} & 0 \\ 0 & \sqrt{2\sigma_k} \end{bmatrix}$$

We notice that (4.1) is a vectorial equation composed of two unidimensional stochastic differential equations. These are

$$\begin{cases} d\mathbf{a}_t = (-c_t - i_t + r\mathbf{a}_t + \rho\mathbf{k}_t)dt + \sqrt{2\sigma_a}\ dB_t^a, \\ d\mathbf{k}_t = (-i_t - \delta\mathbf{k}_t)dt + \sqrt{2\sigma_k}\ dB_t^k, \end{cases} \tag{4.2}$$

where,

1. i_t and c_t are respectively the investment and the consumption of the agent over time,
2. $\delta \in (0,1)$ is a constant that measures the depreciation of the capital,
3. r is a fixed interest rate,
4. σ_a is the volatility of wealth,
5. σ_k is the volatility of the capital.

Agent's preferences are represented by an instantaneous utility function u. Its intertemporal counterpart is the following utility functional:

$$J(c, i, a, k, t) := \mathbb{E}^{\mathbf{x}} \left[\int_0^{t \wedge T} u(c_s)ds + \Psi(x_T, T)\chi_{\{T<\tau\}} + f(x_\tau, \tau)\chi_{\{\tau<T\}} \right]. \tag{4.3}$$

Then,

$$V(a, k, t) = \sup_{c,i} J(c, i, a, k, t).$$

is value function of the state-constrained problem described by (4.2)-(4.3). Define the Hamiltonian as

$$H(a, k, r, p, \delta, q_a, q_k) = \sup_{c,i} \left((-c - i + ra + \rho k)q_a + (i - \delta k)q_k + u(c) \right).$$

Then, after a change of variables, we know that a (viscosity) solution to

$$V_t(a, k, t) + H(a, k, r, p, \delta, V_a, V_k) + \Delta V = 0, \tag{4.4}$$

Also, the feedback optimal control is given by

$$\begin{cases} D_{q_a} H(a, k, r, \rho, \delta, V_a(a, k, t), V_k(a, k, t)) = -c^*(a, k, t) - i^*(a, k, t) + \rho k \\ D_{q_k} H(a, k, r, \rho, \delta, V_a(a, k, t), V_k(a, k, t)) = i^*(a, k, t) - \delta k. \end{cases} \tag{4.5}$$

Using the adjoint of the formal linearized Hamilton-Jacobi equation we obtain the associated Fokker-Planck equation, describing the evolution of the population density, given an initial configuration m_0. This is given by:

$$m_t + ((-c - i + ra + \rho k)m)_a + ((i - \delta k)m)_k = \Delta m. \tag{4.6}$$

The coupling of (4.4) and (4.6) yields the following MFG:

$$\begin{cases} V_t(a,k,t) + H(a,k,r,p,\delta,V_a,V_k) + \Delta V = 0 \\ m_t + ((D_{q_a}H)m)_a + ((D_{q_k}H)m)_k = \Delta m, \end{cases} \tag{4.7}$$

in $\Omega \times [0,T] = (\underline{a}, \overline{a}) \times (0, \overline{k}) \times [0,T]$, with initial-terminal boundary conditions:

$$\begin{cases} V(a,k,T) = V_T(a,k) \quad \text{in} \quad \Omega \\ m(a,k,0) = m_0(a,k) \quad \text{in} \quad \Omega, \end{cases} \tag{4.8}$$

and

$$\begin{cases} V(a,k,t) = f(a,k,t) \quad \text{on} \quad \partial\Omega \times [0,T] \\ m(a,k,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,T]. \end{cases} \tag{4.9}$$

Theorem 4.1. *Suppose that A1 holds. Let (V,m) be a solution to* (4.7)-(4.9)*. Then for any solution to*

$$\begin{cases} \xi_t + \operatorname{div}(b.\xi) &= \Delta\xi \quad \text{in } \Omega \times (\tau,T), \\ \xi(x,\tau) &= \xi_\tau(x) \quad \text{in } \Omega, \\ \xi(x,t) &= 0 \quad \text{in } \partial\Omega \times (\tau,T), \end{cases}$$

where b is any smooth vector field $b : \Omega \subset \mathbb{R}^2 \times (t,T) \to \mathbb{R}^2$, we have the following upper bound:

$$\begin{aligned} \int_0^k \int_{\underline{a}}^{\overline{a}} V(k,a,\tau)\xi_\tau(a,k)dadk &\leq \int_\tau^T \int_0^k \int_{\underline{a}}^{\overline{a}} u(c_t)\xi(a,k,t)dadkdt + \|V_T\|_{L^\infty(\Omega)} + C \\ &+ C\|f\|_{L^r(\tau,T;L^p(\partial\Omega))} \end{aligned}$$

Proof. The proof follows from direct application of Theorem 3.1 to the case (4.7)-(4.9). □

Theorem 4.2. *Let (V,m) be a solution to* (4.7)-(4.9)*, then, there exists $C > 0$ such that*

$$\begin{aligned} c\int_0^T\int_\Omega H(a,k,D_aV,D_kV)mdx\,dt &\leq C + \|f\|_{L^r(0,T;L^p(\partial\Omega))} + C\,osc\,V(\cdot,T) \\ &+ CT + C\|f\|_{L^r(0,T;L^p(\Omega))}\left[1 + \|DV\|_{L^{(\gamma-1)jr}}^{\frac{\gamma-1}{1-\alpha}}\right], \end{aligned}$$

Proof. The proof follows from direct application of Theorem 3.2 to the case (4.7)-(4.9). □

Appendix A

Elementary notions in stochastic calculus

In this appendix, we introduce the basic concepts of stochastic calculus to develop the theory for a stochastic mean-field game. We begin defining what is a stochastic basis, then the necessary concepts to build the notion of the Itô's Integral, Diffusion processes and the associated infinitesimal generator.

A.1 Stochastic Basis

Definition A.1. *Let Ω be a set. A collection $\mathcal{U}$ of subsets of Ω is called a σ-algebra if it satisfies:*

i) $\Omega \in \mathcal{U}$; $A \in U$ implies $A^c = \Omega \backslash A$ (the complement of A in Ω) belongs to $\mathcal{U}$.

ii) $\mathcal{U}$ is stable under intersections and unions of elements of $\mathcal{U}$.

Definition A.2. *A probability measure on Ω is a function $P : \mathcal{U} \to [0, 1]$ with:*

i) $P(\emptyset) = 0$, $P(\Omega) = 1$ and which,

ii) Is σ-additive i.e., for any sequence $\{A_n\} \subset \mathcal{U}$ such that $A = \bigcup_{n \geqslant 0} A_n \in \mathcal{U}$ and $A_i \cap A_j = \emptyset$, for $i \neq j$, one has

$$P(A) = \sum_{n \geq 0} P(A_n).$$

With these concepts we have:

Definition A.3. *A probability space is a triple $(\Omega, \mathcal{U}, P)$, $\mathcal{U}$ is a σ-algebra and P is a probability measure on Ω.*

We are considering *complete probability spaces* which are the probability spaces where $\mathcal{U}$ contains all P-*null sets*; this condition is formally stated as :

Definition A.4. *$N \subset \Omega$ is a P-null set if there exists $B \in \mathcal{U}$ such that $N \subset B$ and $P(B) = 0$.*

An *Euclidian space* is a subset of $\mathbb{R}^d$, $d \in \mathbb{N}$, endowed with the topology induced by the euclidean metric of $\mathbb{R}^d$. It will be denoted by E.

The *Borel σ-algebra of E* (i.e., the smallest σ-algebra which contains all open sets of E) will be indicated by $\mathcal{E}$.

Definition A.5. *A Random variable in a probability space is a function $\mathbf{x} : \Omega \to E$ such that $\mathbf{x}^{-1}(A) \in \mathcal{U}$ for every $A \in E$; it is common to say that $\mathbf{x}$ is $\mathcal{U}$-measurable.*

The smallest σ-algebra which makes $\mathbf{x}$ measurable is $\sigma(\mathbf{x}) = \{\mathbf{x}^{-1}(B),\ B \in \mathcal{E}\}$.

A.1.1 Expected Value, Variance

Integration with respect to a measure. If $(\Omega, \mathcal{U}, P)$ is a probability space and $\mathbf{x} = \sum_{i=1}^{k} a_i \chi_{A_i}$ is a real-valued simple random variable, we define the *integral* of $\mathbf{x}$ by

$$\int_\Omega \mathbf{x}\, dP \quad := \quad \sum_{i=1}^{k} a_i P(A_i). \tag{A.1}$$

If next $\mathbf{x}$ is a *nonnegative* random variable, we define

$$\int_\Omega \mathbf{x}\, dP \quad := \quad \sup_{\mathbf{y} \le \mathbf{x}, \mathbf{y} \text{ simple}} \int_\Omega \mathbf{y}\, dP. \tag{A.2}$$

Finally if $\mathbf{x} : \Omega \to \mathbb{R}$ is a random variable, we write

$$\int_\Omega \mathbf{x}\, dP \quad := \quad \int_\Omega \mathbf{x}^+\, dP - \int_\Omega \mathbf{x}^-\, dP, \tag{A.3}$$

provided at least one of the integrals on the right is finite. Here $\mathbf{x}^+ = \max(\mathbf{x}, 0)$ and $\mathbf{x}^- = \max(-\mathbf{x}, 0)$; so that $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$.

Next, suppose $\mathbf{x} : \Omega \to \mathbb{R}^n$ is a vector-valued random variable, $\mathbf{x} = (\mathsf{x}^1, \mathsf{x}^2, \ldots, \mathsf{x}^n)$. Then we write

$$\int_\Omega \mathbf{x}\, dP \quad = \quad \left(\int_\Omega \mathsf{x}^1\, dP, \int_\Omega \mathsf{x}^2\, dP, \ldots, \int_\Omega \mathsf{x}^n\, dP \right). \tag{A.4}$$

We will assume the usual rules for these integrals.

Definition A.6. *We call*

$$\mathbb{E}(\mathbf{x}) \quad := \quad \int_\Omega \mathbf{x}\, dP, \tag{A.5}$$

the expected value (or mean value) of **x**.

Definition A.7. *We call*

$$V(\mathbf{x}) \quad := \quad \int_\Omega |\mathbf{x} - \mathbb{E}(\mathbf{x})|^2\, dP \tag{A.6}$$

the variance of **x**, *where* $|\cdot|$ *denotes the Euclidean norm.*

A.1.2 Distribution Functions

Let $(\Omega, \mathcal{U}, P)$ be a probability space and suppose $\mathbf{x} : \Omega \to \mathbb{R}^n$ is a random variable.
Notation: Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. Then

$$x \le y$$

means that $x_i \le y_i$ for $i = 1, \dots, n$.

Definition A.8. *i) The distribution function of* **x** *is the function* $F_{\mathbf{x}} : \mathbb{R}^n \to [0, 1]$ *defined by*

$$F_{\mathbf{x}}(x) \quad := \quad P(\mathbf{x} \le x) \qquad \text{for all } x \in \mathbb{R}^n, \tag{A.7}$$

ii) If $\mathbf{x}_1, \dots, \mathbf{x}_m : \Omega \to \mathbb{R}^n$ *are random variables, their joint distribution function is* $F_{\mathbf{x}_1,\dots,\mathbf{x}_m} : (\mathbb{R}^n)^m \to [0, 1]$,

$$F_{\mathbf{x}_1,\dots,\mathbf{x}_m}(x_1, \dots, x_m) \quad := \quad P(\mathbf{x}_1 \le x_1, \dots, \mathbf{x}_m \le x_m) \qquad \text{for all } x_i \in \mathbb{R}^n, i = 1, \dots, m.$$

Definition A.9. *Suppose* $\mathbf{x} : \Omega \to \mathbb{R}^n$ *is a random variable and* $F = F_{\mathbf{x}}$ *its distribution function. If there exists a nonnegative, integrable function* $f : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$F(x) = F(x_1, \dots, x_n) \quad = \quad \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y_1, \dots, y_n) dy_n \dots dy_1, \tag{A.8}$$

then f *is called the density function of* **x**.

It follows that

$$P(\mathbf{x} \in B) \quad = \quad \int_B f(x) dx \qquad \text{for all } B \in \mathcal{E} \tag{A.9}$$

The formula above is important as the expression on the right-hand-side is an ordinary integral, and can often be explicitly calculated. With the definition of density function of a random variable we can introduce the concept of *distribution of a random variable*. The distribution of a random variable depends solely on its density function.

Definition A.10. *If* $\mathbf{x} : \Omega \to \mathbb{R}$ *has density*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp^{-\frac{|x-m|^2}{2\sigma^2}} \qquad (x \in \mathbb{R}), \tag{A.10}$$

we say that $\mathbf{x}$ *has a Gaussian (or normal) distribution, with mean* m *and variance* σ^2*.*

In this case let us write

$$\mathbf{x} \text{ is } N(m, \sigma^2) \text{ random variable.}$$

Lemma A.1. *Let* $\mathbf{x} : \Omega \to \mathbb{R}^n$ *be a random variable, and assume that its distribution function* $F = F_{\mathbf{x}}$ *has the density* f*. Suppose* $g : \mathbb{R}^n \to \mathbb{R}$*, and*

$$\mathbf{y} = g(\mathbf{x})$$

is integrable. Then

$$E(\mathbf{y}) = \int_{\mathbb{R}^n} g(x) f(x) dx \tag{A.11}$$

In particular,

$$E(\mathbf{x}) = \int_{\mathbb{R}^n} x f(x)\, dx \qquad \text{and} \qquad V(\mathbf{x}) = \int_{\mathbb{R}^n} |x - E(\mathbf{x})|^2 f(x)\, dx \tag{A.12}$$

Proof. Suppose first g is a simple function on $\mathbb{R}^n$:

$$g = \sum_{i=1}^{m} b_i \chi_{B_i} \qquad (B_i \in \mathcal{E}) \tag{A.13}$$

Then

$$E(g(\mathbf{x})) = \sum_{i=1}^{m} b_i \int_{\Omega} \chi_{B_i}(\mathbf{x})\, dP \quad = \quad \sum_{i=1}^{m} b_i P(\mathbf{x} \in B_i). \tag{A.14}$$

But also

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)f(x)dx &= \sum_{i=1}^{m} b_i \int_{B_i} f(x)dx \\ &= \sum_{i=1}^{m} b_i P(\mathbf{x} \in B_i).\end{aligned}$$

Consequently the formula holds for all simple functions g and, by approximation, it holds therefore for general function g. □

Hence we can compute $E(\mathbf{x})$, $V(\mathbf{x})$, etc. in terms of integrals over $\mathbb{R}^n$. This is an important observation, since as mentioned before the probability space $(\Omega, \mathcal{U}, P)$ is "unobservable": all that we "see" are the values $\mathbf{x}$ takes on in $\mathbb{R}^n$. Indeed, *all quantities of interest in probability theory can be computed in $\mathbb{R}^n$ in terms of the density f.*

Example A.1. *If $\mathbf{x}$ is $N(m, \sigma^2)$, then*

$$E(\mathbf{x}) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x e^{-\frac{(x-m)^2)}{2\sigma^2}} dx \quad = \quad m$$

and

$$V(\mathbf{x}) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} (x-m)^2 e^{-\frac{(x-m)^2)}{2\sigma^2}} dx \quad = \quad \sigma^2.$$

Therefore m is indeed the mean, and σ^2 the variance.

To conclude this section defining what is a stochastic basis we now introduce a fundamental concept in stochastic calculus:

Definition A.11. *A Stochastic Process is a collection $\mathbf{x} = \{\mathbf{x}_t,\ t \in T\}$ of random variables taking values in E, but we can also see it as a function $\mathbf{x} = \mathbf{x}(t, \omega) : T \times \Omega \to E$. where $T \subset \mathbb{R}^+$, and $t \in T$ is meant to represent an evolution parameter, usually taken to be time. For each $\omega \in \Omega$ the map $t \mapsto \mathbf{x}_t(\omega)$ is a trajectory or the sample path of the stochastic process. A stochastic Process $\{\mathbf{x}_t,\ t \in T\}$ is called continuous or almost surely continuous(a.s.) if its trajectories(or paths) are continuous.*

Definition A.12. *i) A a filtration $\{\mathcal{F}_t,\ t \in T\}$ is a family of sub-σ-algebras of $\mathcal{U}$ increasing in time,i.e., if $s < t$ the $\mathcal{F}_s \subset \mathcal{F}_t$. We add to the filtration $\{\mathcal{F}_t,\ t \in T\}$ the σ-algebra $\mathcal{F}_\infty = \bigvee_{t \in T} \mathcal{F}_t$, where the right hand side stands for the minimal σ-algebra, which contains all $\mathcal{F}_t$.*

ii) We define for every $t \in T$ the following σ-algebra $\mathcal{F}_{t+} = \bigcap_{t<s\in T} \mathcal{F}_s$.

Definition A.13. *The Filtration $\{\mathcal{F}_t,\ t \in T\}$ is right-continuous if for every $t \in T$, $\mathcal{F}_t = \mathcal{F}_{t+}$.*

Assume we are given a stochastic process $X = \{\mathbf{x}_t,\ t \in T\}$ and a filtration $\{\mathcal{F}_t,\ t \in T\}$ on $(\Omega, \mathcal{F}, P)$. We say that the process X is *adapted* to the filtration $\{\mathcal{F}_t,\ t \in T\}$ if, for every $t \in T$ fixed, the random variable $\mathbf{x}_t$ is $\mathcal{F}_t$-measurable on Ω; equivalently, we say that X is adapted to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. It is always possible to construct a filtration with respect to which the process is adapted, by setting $\mathcal{F}_t^X = \sigma(x_s,\ s \leq t)$; $\mathcal{F}_t^X$ is called the *natural filtration* of X. A stochastic process X is adapted to a filtration $\{\mathcal{F}_t,\ t \in T\}$ if and only if one has $\mathcal{F}_t^X \subset \mathcal{F}_t, t \in T$.

For technical difficulties, we shall always require that the filtration satisfies in addition the so-called *standard assumptions*:

1. the filtration is right continuous, that is, for any $t \in T : \mathcal{F}_t = \mathcal{F}_{t+}$.

2. the filtration is complete, that is, for any $t \in T$, $\mathcal{F}_t$ contains all P*-null sets*.

Now we include the notion of filtration in a probability space to obtain the definition of Stochastic Basis.

Definition A.14. *Stochastic Basis: A filtered probability space or a stochastic basis is the quadruplet $(\Omega, \mathcal{U}, \{\mathcal{F}_t\}, P)$, where the filtration $\{\mathcal{F}_t\}$ verifies the standard assumptions.*
We say that a stochastic process $X = \{\mathbf{x}_t,\ t \in T\}$ is defined on a stochastic basis $(\Omega, \mathcal{U}, \{\mathcal{F}_t\}, P)$, we also require that it is adapted to $\{\mathcal{F}_t\}$.

A.2 Itô's Integral

Motivation: If we fix $b_0 \in \mathbb{R}^d$ and consider

$$\begin{cases} \dot{\mathbf{x}}_t &= b(\mathbf{x}_t) + G(\mathbf{x}_t).\xi(t)(t > 0) \\ \mathbf{x}_0 &= b_0, \end{cases} \tag{A.15}$$

where $b_0 \in \mathbb{R}^d$, $G : \mathbb{R}^d \to M_{d \times m}$ and $\xi(.) \in \mathbb{R}^m$ is called a *"White Noise"*. To fully understand this equation, we need

1. To define the "white noise" $\xi(.)$ in a rigourous way.

2. Understand what does it mean to solve this equation.

A.2.1 Some Heuristics

Let us first study (A.15) in the case $d = m$, $\mathbf{x}_0 = 0$ $b \equiv 0$, and $G \equiv I$. The solution to (A.15) in this setting turns out to be the *d-dimensional Brownian motion*, denoted by B. Thus we may symbolically write

$$\dot{B} = \xi(),$$

thereby asserting that "white noise" is the time derivative of the Brownian motion. Now return to the general case of the equation (A.15), writing $\frac{d}{dt}$ instead of the dot:

$$\frac{d\mathbf{x}_t}{dt} = b(\mathbf{x}_t) + G(\mathbf{x}_t)\frac{dB(t)}{dt}$$

and multiply by "dt":

$$\begin{cases} d\mathbf{x}_t &= b(\mathbf{x}_t)dt + G(\mathbf{x}_t)dB_t(t > 0) \\ \mathbf{x}_0 &= b_0 \end{cases}$$

This expression, properly interpreted, is a *stochastic differential equation*. We say that $x(\cdot)$ solves (SDE) provided

$$\mathbf{x}_t = x_0 + \int_0^t b(x_s)ds + \int_0^t G(x_s)dB \tag{A.16}$$

So we can say that solving (A.15) is equivalent to search for stochastic processes that satisfy the expression (A.16). To check if a specific process is a solution, we need to define the meaning of the second integral, and for which integrands we can compute it. In this direction, we begin by setting a special case of stochastic process, which is the "White noise" and answer (1).

Definition A.15. *An m-dimensional Brownian motion $B = \{B_t,\ t \geq 0\}$ defined on a stochastic basis $(\Omega, \mathcal{U}, \{\mathcal{F}_t,\ t \geq 0\}, P)$ is a continuous stochastic process such that*

1. $B_0 = 0$ *a.s.(means that $B_0(\omega) = 0\ \forall A \in \mathcal{U}$ except in P-null sets);*
2. *for every $0 \leq s < t$ the random variable $B_t - B_s$ is independent of $\mathcal{F}_s$;*
3. *for every $0 \leq s < t$ the random variable $B_t - B_s$ has a Gaussian law $N(0, (t-s)I_d)$, where I_d is the (m-dimensional) identity matrix.*

Remark A.1. *In this work, if the filtration $\{\mathcal{F}_t\}$ is not explicitly mentioned, we can assume that we are using the completed natural filtration $\{\mathcal{F}_t^B\}$ of B. In [24] it is shown that the completed natural filtration of any Brownian motion is right continuous and so it satisfies the standard assumptions. A Brownian Motion B defined on $(\Omega, \mathcal{U}, \{\mathcal{F}_t^B,\ t \geq 0\}, P)$ is also called a natural Brownian Motion.*

A.2.2 The 1-dimensional Itô's Integral

Let us define the Itô's Integral for the 1-dimensional case, i.e., $d = m = 1$. With this purpose in mind we begin defining a specific set of stochastic process, which will be the set of integrands for Itô's Integral.

Definition A.16. *A real-valued stochastic process $\{G_t,\ t \in [0, T]\}$ on a stochastic basis $(\Omega, \mathcal{U}, \{\mathcal{F}_t,\ t \in [0, T]\}, P)$ adapted with respect to filtration $\{\mathcal{F}_t\}$ for every $t \in [0, T]$ is called progressively mensurable.*

The idea is that for each time $t \geq 0$, the random variable G_t "depends upon only the information available in the σ-algebra $\mathcal{F}_t$".

Definition A.17. *(i) We denote by $\mathbb{L}^2(0,T)$ the space of all real-valued, progressively measurable stochastic processes $G(\cdot)$ such that*

$$E\left(\int_0^T G^2 dt\right) < \infty.$$

- *Likewise, $\mathbb{L}^1(0,T)$ is the space of all real-valued, progressively measurable processes $F(\cdot)$ such that*

$$E\left(\int_0^T \mid F \mid dt\right) < \infty.$$

Definition A.18. *A process $G \in \mathbb{L}^2(0,T)$ is called a step process if there exists a partition $P = \{0 = t_0 < t_1 < \ldots < t_m = T\}$ such that*

$$G(t) \equiv G_k \qquad \text{for} \qquad t_k \leq t < t_{k+1} \quad (K = 0, \ldots, m-1)$$

then each G_k is an $\mathcal{F}(t_k)$-measurable random varable, since G is progressively measurable.

Definition A.19. *Let $G \in \mathbb{L}^2(0,T)$ be a step process, as above, Then*

$$\int_0^T GdB := \sum_{k=0}^{m-1} G_k(B_{t_{k+1}} - B_{t_k})$$

is the Itô stochastic integral of G on the interval $(0,T)$.

Lemma A.2 (Properties of the stochastic integral for step processes)**.** *We have for all constants $a, b \in \mathbb{R}$ and for all step processes $G, H \in \mathbb{L}^2(0,T)$:*

(i)

$$\int_0^T aG + bHdB = a\int_0^T GdB + b\int_0^T HdB$$

(ii)

$$E\left(\int_0^T GdB\right) = 0$$

(iii)

$$E\left(\left(\int_0^T GdB\right)^2\right) = 0$$

To define the Itô integral of an arbitrary process $G \in \mathbb{L}^2(0,T)$, we will approximate it by step processes in $\mathbb{L}^2(0,T)$, and then take limits.

Lemma A.3 (Approximation by step processes)**.** *If $G \in \mathbb{L}^2(0,T)$, there exists a sequence of bounded step processes $G^n \in \mathbb{L}^2(0,T)$ such that*

$$E\left(\int_0^T \mid G - G^n \mid^2 dt\right) \to 0.$$

Definition A.20. *If $G \in \mathbb{L}^2(0,T)$, take step processes G^n as above. Then*

$$E\left(\left(\int_0^T G^n - G^m dB\right)^2\right) \overset{\text{(A.2)}}{=} E\left(\int_0^T (G^n - G^m)^2 dt\right) \to 0 \quad \text{as } n, m \to 0,$$

and so the limit

$$\int_0^T GdB := \lim_{n\to\infty} \int_0^T G^n dB$$

exists in $L^2(\Omega)$.

A.2.3 The Multi-dimensional Itô's Integral

Let $B_t = (B_t^1, \ldots, B_t^m)$ be an m-dimensional Brownian motion, where each B_t^i is a 1-dimensional Brownian Motion (as in (A.15)) for $i = 1, \ldots, m$. Now we have to define a new a integrand's space for a m-dimensional Brownian Motion, which is similar to the set of progressive mensurable stochastic process defined above for the 1-dimensional case of Itô's Integral.

Definition A.21. *(i) An $\mathbb{M}^{d\times m}$-valued stochastic process $G = ((G^{ij}))$ belongs to $\mathbb{L}^2_{d\times m}(0,T)$ if*

$$G^{ij} \in \mathbb{L}^2(0,T) \qquad (i = 1, \ldots, d; \quad j = 1, \ldots, m)$$

(ii) An $\mathbb{R}^d$-valued stochastic process $F = (F^1, F^2, \ldots, F^d)$ belongs to $\mathbb{L}^1_n(0,T)$ if

$$F^i \in \mathbb{L}^1(0,T) \qquad (i = 1, \ldots, d).$$

Definition A.22. *If $G \in \mathbb{L}_{d\times m}(0,T)$, then*

$$\int_0^T GdB$$

is an $\mathbb{R}^d$-valued random variable, whose, i-th component is

$$\sum_{j=1}^m \int_0^T G^{ij} dB^j \qquad (i = 1, \ldots, d).$$

Approximating by step processes as before, we can establish the following Lemma.

Lemma A.4. *If $G \in \mathbb{L}^2_{d\times m}$, then*

$$E\left(\int_0^T GdB\right) = 0,$$

and

$$E\left(\left|\int_0^T GdB\right|^2\right) = E\left(\int_0^T |G|^2 dt\right),$$

where $|G|^2 := \sum_{\substack{0\le i\le d \\ 1\le j\le m}} |G^{ij}|^2$.

Definition A.23. *If $\mathbf{x}_t = (\mathbf{x}_t^1, \ldots, \mathbf{x}_t^d)$ is an $\mathbb{R}^d$-valued stochastic process such that*

$$\mathbf{x}_r = \mathbf{x}_s + \int_s^r Fdt + \int_s^r GdB$$

for some $F \in \mathbb{L}^1_d(0,T)$, $G \in \mathbb{L}^2_{d\times m}(0,T)$ and all $0 \le s \le r \le T$, we say $\mathbf{x}(\cdot)$ has the stochastic differential

$$d\mathbf{x} = Fdt + gdB.$$

Remark A.2. *It means that;*

$$d\mathbf{x}^i = F^i dt + \sum_{j=1}^m G^{ij} dB^j \qquad \text{for} \quad i = 1, \ldots, d.$$

Theorem A.1 (Itô's formula in n-dimensions)**.** *Suppose that $dx = Fdt + GdB$, as above. Let $u : \mathbb{R}^d \times [0,T]$ be continous, with continuous partial derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, (i,j = 1,\ldots,d)$. Then*

$$d(u(\mathbf{x}_t, t)) = \frac{\partial u}{\partial t}dt + \sum_{i=1}^d \frac{\partial u}{\partial x_i}dx^i + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 u}{\partial x_i \partial x_j}\sum_{l=1}^m G^{il}G^{jl}dt, \tag{A.17}$$

where the argument of the partial derivatives of u is $(\mathbf{x}_t, t)$.

We can write the equation A.17 as

$$u_t(\mathbf{x}(t), t) + D_x u dX + \frac{\mathrm{Tr}(GD_x^2 u(\mathbf{x}(t), t)G^T)}{2},$$

where $D_x u = \left(\frac{\partial u}{\partial x_i}, \ldots, \frac{\partial u}{\partial x_d}\right)$ is the gradient of u in the x -variables, $D_x^2 u = \left((\frac{\partial^2 u}{\partial x_i \partial x_j})\right)$ is the Hessian matrix, G^T is the transposed matrix of G and Tr is the trace function.

An alternative notation: When

$$d\mathbf{x} = Fdt + GdB,$$

we sometimes write

$$H^{ij} := \sum_{k=1}^m G^{ik}G^{jk}.$$

Then Itôs formula reads

$$du(\mathbf{x},t) = \left(\frac{\partial u}{\partial t} + F.D_x u + \frac{1}{2}H : D^2 u\right) dt + Du.GdB,$$

where

$$F.D_x u = \sum_{i=1}^{d} F^i \frac{\partial u}{\partial x_i}$$

$$H : D_x^2 = \sum_{i,j=1}^{d} H^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j},$$

$$D_x u.GdB = \sum_{i=1}^{d} \sum_{k=1}^{m} \frac{\partial u}{\partial x_i} G^{ik} dB^k.$$

We may formally compute

$$d(u(\mathbf{x},t)) = \frac{\partial u}{\partial t}dt + \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} d\mathbf{x}^i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 u}{\partial x_i \partial x_j} d\mathbf{x}^i d\mathbf{x}^j \tag{A.18}$$

and then simplify the term "$d\mathbf{x}^i d\mathbf{x}^j$" by expanding it out and using the formal multiplication rules

$$(dt)^2 = 0 \qquad dtdB^k = 0 \qquad dB^k dB^l = \delta_{kl} dt \qquad (k,l = 1,\ldots,m). \tag{A.19}$$

A.3 The Infinitesimal Generator

Definition A.24. *An Itô diffusion is a stochastic process*

$$\mathbf{x}_t(\omega) = \mathbf{x}(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^d,$$

satisfying a stochastic differential equation of the form

$$d\mathbf{x}_t \quad = \quad b(\mathbf{x}_t)dt + \sigma(\mathbf{x}_t)dB_t, \ \ t \geq s; \ \mathbf{x}_s = x \tag{A.20}$$

where B_t is a m-dimensional Brownian motion and $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ satisfy the following condition:

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq C|x - y|; \qquad x, y \in \mathbb{R}^d, \tag{A.21}$$

where $|\sigma|^2 = \sum |\sigma_{ij}|^2$, for some $C > 0$, fixed.

Remark A.3. *Under certain conditions this equation has a unique solution, and it is denoted by*

$\mathbf{x}_t = \mathbf{x}_t^{s,x}; t \geq s$. *In* (A.20) *we assumed that* b *and* σ *do not depend on* t *but on* x *only, for the general case reduces to his one by setting* $(x,t) =: \tilde{x} \in \mathbb{R}^{d+1}$.

Definition A.25. *Let* $\{\mathbf{x}_t\}$ *be a Itô diffusion in* $\mathbb{R}^d$. *The infinitesimal generator* A *of* $\mathbf{x}_t$ *is defined by*

$$Af(x) = \lim_{t \downarrow 0} \frac{E^x[f(\mathbf{x}_t)] - f(x)}{t}; \quad x \in \mathbb{R}^d. \tag{A.22}$$

The set of functions $f : \mathbb{R}^d \to \mathbb{R}$ *such that the limit exists at* x *is denoted by* $\mathcal{D}_A(x)$*, while* $\mathcal{D}_A$ *denotes the set of functions for which the limit exists for all* $x \in \mathbb{R}^d$.

To find the relation between A and the coefficients b,σ in the stochastic differential equation (**A.20**) defining $\mathbf{x}_t$, we need the following result, which is useful.

Lemma A.5. *Let* $\mathbf{y}_t = \mathbf{y}_t^x$ *be a stochastic process in* $\mathbb{R}^d$ *such that*

$$\mathsf{Y}_t^x(\omega) \quad = \quad x + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s(\omega),$$

where B *is m-dimensional. Let* $f \in C_0^2(\mathbb{R}^d)$*, and let* $[0,T) \subset \mathbb{R}$*. Assume that* $u(t,\omega)$ *and* $v(t,\omega)$ *are bounded on the set of* (t,ω) *such that* $\mathbf{y}(t,\omega)$ *belongs to the support of* f*. Then*

$$E^x[f(\mathbf{x}_T)] = f(x) + E^x\left[\int_0^T \left(\sum_i^d b_i(\mathbf{x}_s)\frac{\partial f}{\partial x_i}(\mathbf{x}_s) + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^T)_{ij}(\mathbf{x}_s)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_s)\right)ds\right]$$

where E^x *is the expectation w.r.t. stochastic process* $\mathbf{y}_t$*, such that* $\mathbf{y}_0 = x$.

Theorem A.2 (The infinitesimal generator of a diffusion)**.** *Let* $\mathbf{x}_t$ *be a Itô diffusion*

$$d\mathbf{x}_t = b(\mathbf{x}_t)dt + \sigma(\mathbf{x}_t)dB_t$$

if $f \in C_0^2(\mathbb{R}^d)$ *then* $f \in \mathcal{D}_A$ *and*

$$Af(x) = \sum_{i=1}^d b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

Remark A.4. *In Øksendal [22] you can find the demonstration of Lemma A.5 and the justification for the Theorem A.2. Here we present a proof that makes clear how both results are linked and the idea to obtain the theorem starting from the lemma's proof.*

Proof. Put $\mathbf{z} = f(\mathbf{y})$; by applying Itôs formula in the form (A.18) and simplifying the notation supressing the index t we have

$$d\mathbf{z} = \frac{\partial f}{\partial t}(\mathbf{y})dt + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(\mathbf{y})d\mathbf{y}^i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{y})d\mathbf{y}^i d\mathbf{y}^j, \tag{A.23}$$

where

$$\begin{cases} d\mathbf{y}^i &= u_i(t,\omega)dt + \underbrace{\sum_{j=1}^{m} v_{ij}dB^j}_{(vdB)_i}, \qquad i = 1,\ldots,d. \\ \\ d\mathbf{y}^j &= u_j(t,\omega)dt + \underbrace{\sum_{i=1}^{m} v_{ji}dB^i}_{(vdB)_j}, \qquad j = 1,\ldots,d. \end{cases}$$

Now, using these equations above and (A.19) we calculate $d\mathbf{y}^i d\mathbf{y}^j$:

$$d\mathbf{y}^i d\mathbf{y}^j = u_i u_j + (vdB)_i (vdB)_j \tag{A.24}$$

where $(vdB)_i(vdB)_j$ can be expanded, and observing that $dB^k dB^l = \delta_{kl}dt$ we have:

$$\begin{aligned} (vdB)_i(vdB)_j &= \left(\sum_{k=1}^{m} v_{ik}dB_k\right)\left(\sum_{l=1}^{m} v_{jl}dB_l\right) \\ &= \left(\sum_{k=1}^{m} v_{ik}v_{jk}\right)dt = (vv^T)_{ij}dt. \end{aligned} \tag{A.25}$$

Putting A.25 and the expression of $d\mathbf{y}^i$ in A.23 we have

$$\begin{aligned} d\mathbf{z} &= \sum_{i=1}^{d} u_i \frac{\partial f}{\partial x_i}(\mathbf{y})dt + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(\mathbf{y})(vdB)_i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{y})(vdB)_i(vdB)_j \\ &= \left(\sum_{i=1}^{d} u_i \frac{\partial f}{\partial x_i}(\mathbf{y}) + \frac{1}{2}\sum_{i,j=1}^{d}(vv^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{y})\right)dt + \sum_{i,j=1}^{d,m} v_{ij}\frac{\partial f}{\partial x_i}(\mathbf{y})dB^j. \end{aligned}$$

Which means:

$$\begin{aligned} f(\mathbf{y}_t) = f(\mathbf{y}_0) &+ \int_0^t \left(\sum_{i}^{d} u_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}(vv^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{y})\right)ds \\ &+ \sum_{i=1}^{d}\sum_{k=1}^{m}\int_0^t v_{ik}\frac{\partial f}{\partial x_i}(\mathbf{y})dB_k. \end{aligned} \tag{A.26}$$

Hence

$$\begin{aligned} E^x[f(\mathbf{y}_T)] = f(x) \quad &+ \quad E^x\left[\int_0^T \left(\sum_i^d u_i \frac{\partial f}{\partial x_i}(\mathbf{y}) + \frac{1}{2}\sum_{i,j=1}^d (vv^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{y})\right) ds\right] \\ &+ \quad \sum_{i=1}^d \sum_{k=1}^m E^x\left[\int_0^T v_{ik}\frac{\partial f}{\partial x_i}(\mathbf{y})dB_k\right]. \end{aligned} \tag{A.27}$$

In our context, we can replace the functions $u(t, \omega)$ and $v(t, \omega)$ respectively by $b(\mathbf{x}_t)$ and $\sigma(\mathbf{x}_t)$, so (A.27) becomes

$$\begin{aligned} E^x[f(\mathbf{x}_T)] = f(x) \quad &+ \quad E^x\left[\int_0^T \left(\sum_i^d b_i(\mathbf{x}_s)\frac{\partial f}{\partial x_i}(\mathbf{x}_s) + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^T)_{ij}(\mathbf{x}_s)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_s)\right) ds\right] \\ &+ \quad \sum_{i=1}^d \sum_{k=1}^m E^x\left[\int_0^T \sigma_{ik}(\mathbf{x}_s)\frac{\partial f}{\partial x_i}(\mathbf{x}_s)dB^k\right]. \end{aligned} \tag{A.28}$$

But in the last summand in (A.28) we have that $\sigma = (\sigma_{ij})$ is constant, so we can take σ^* such that $\sigma^* \geq \sum_{i=1}^d \sum_{k=1}^m \sigma_{ik}(\mathbf{x}_s)\frac{\partial f}{\partial x_i}(\mathbf{x}_s)$ for . Which give us:

$$\begin{aligned} \sum_{i=1}^d \sum_{k=1}^m E^x\left[\int_0^T \sigma_{ik}(\mathbf{x}_s)\frac{\partial f}{\partial x_i}(\mathbf{x}_s)dB_k\right] \quad &\leq \quad \sigma^* E^x[\int_0^T dB] \\ &= \quad \sigma^* E^x[B(T)] = 0. \end{aligned} \tag{A.29}$$

Combining (A.28) and (A.29) we get

$$E^x[f(\mathbf{x}_T)] = f(x) \quad + \quad E^x\left[\int_0^T \left(\sum_i^d b_i(\mathbf{x}_s)\frac{\partial f}{\partial x_i}(\mathbf{x}_s) + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^T)_{ij}(\mathbf{x}_s)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_s)\right) ds\right].$$

If we interpret the equation above as a differential equation we can interpret the left-hand-side as the right-derivative of the function $E^x[f(\mathbf{x}_t)]$, using the notation $\frac{d}{dt}$ we have:

$$\frac{dE^x[f(\mathbf{x}_t)]}{dt} \quad = \quad \sum_{i=1}^d b_i(\mathbf{x}_t)\frac{\partial f}{\partial x_i}(\mathbf{x}_t) + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^T)_{ij}(\mathbf{x}_t)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_t). \tag{A.30}$$

If we evaluate this right-derivative in $0 \in (0,T)$ we conclude:

$$\begin{aligned}\lim_{t\downarrow 0} \frac{E^x[f(\mathbf{x}_{0+t})] - E^x[f(\mathbf{x}_0)]}{t} &= \sum_{i=1}^{d} b_i(\mathbf{x}_0)\frac{\partial f}{\partial x_i}(\mathbf{x}_0) + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^T)_{ij}(\mathbf{x}_0)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)\\ \lim_{t\downarrow 0} \frac{E^x[f(\mathbf{x}_t)] - f(x)}{t} &= \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x). \qquad \text{(A.31)}\end{aligned}$$

□

Appendix B

Notation, Inequalities and Calculus facts

In this chapter, we present the notation and collect some important calculus results that are used along the text. Some proofs are omitted for thy can be easily found in classical books as [6], on which this appendix is largely based.

B.1 Notation

B.1.1 Function spaces.

(i) $C(U) = \{u : U \to \mathbb{R} \mid u$ continuous$\}$

$C(\overline{U}) = \{u \in C(U) \mid u$ is uniformly continuous on bounded subsets of$U\}$

$C^k(U) = \{u : U \to \mathbb{R} \mid u$is k-times continuosly differentiable$\}$

$C^k(\overline{U}) = \{u \in C^k(U) \mid D^\alpha u$ is uniformly continuous on bounded subsets of U for all $|\alpha| \leq k\}$.

ii) $C^\infty(U) = \{u : U \to \mathbb{R} \mid u$is infinitely differentiable$\} = \bigcap_{k=0}^{\infty} C^k(U)$

$C^\infty(\overline{U}) = \bigcap_{k=0}^{\infty} C^k(\overline{U})$.

iii) $C_c(U), C_c^k(U)$, etc, denote these functions in $C(U), C^k(U)$, etc, with *compact support*.

iv) $L^p(U) = \{u : U \to \mathbb{R} \mid$ is Lebesgue measurable, $\|u\|_{L^p(U)} < \infty\}$, where

$$\|u\|_{L^p(U)} = \left(\int_U |f|^p dx \right)^{\frac{1}{p}} \qquad (1 \leq p < \infty).$$

$L^\infty(U) = \{u : U \to \mathbb{R} \mid u$ is Lebesgue measurable, $\|u\|_{L^\infty(U)} < \infty\}$, where

$$\|u\|_{L^\infty(U)} = \text{ess} \sup_U |u|.$$

$L^p_{loc}(U) = \{u : U \to \mathbb{R} \mid v \in L^p(V)$ for each $V \subset\subset U\}$.

v) $\|Du\|_{L^p(U)} = \||Du|\|_{L^p(U)}$
$\|D^2u\|_{L^p(U)} = \||D^2u|\|_{L^p(U)}$.

vi) $W^{k,p}(U), H^k(U)$, etc. $(k = 0, 1, 2, \ldots, 1 \leq p \leq \infty)$ denote Sobolev spaces.

vii) $C^{k,\beta}(U), C^{k,\beta}(\overline{U})$ $(k = 0, \ldots, 0 < \beta \leq 1)$ denote Hölder spaces.ø

B.1.2 Vector-valued functions

i) If now $m > 1$ and $\mathbf{u} : U \to \mathbb{R}^m$,$\mathbf{u} = (u^1, \ldots, u^m)$, we define

$$D^\alpha \mathbf{u} = (D^\alpha u^1, \ldots, D^\alpha u^m) \text{for each multiindex} \alpha$$

Then
$$D^k \mathbf{u} = \{D^\alpha \mathbf{u} \mid |\alpha| = k\}.$$

and
$$|D^k \mathbf{u}| = \left(\sum_{|\alpha|=k} |D^\alpha \mathbf{u}|^2 \right)^{\frac{1}{2}},$$

as before.

ii) In the special case $k = 1$, we write

$$D(\mathbf{u}) = \begin{pmatrix} \frac{\partial u^1}{\partial x_1} & \cdots & \frac{\partial u^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u^m}{\partial x_1} & \cdots & \frac{\partial u^m}{\partial x_n} \end{pmatrix}_{m \times n} = \textit{gradient matrix}.$$

iii) If $m = n$, we have
$$\operatorname{div} \mathbf{u} = \operatorname{Tr}(D\mathbf{u}) = \sum_{i=1}^n u^i_{x_i} = \text{divergence of } \mathbf{u}.$$

iv) The spaces $C(U; \mathbb{R}^m), L^p(U; \mathbb{R}^m)$, etc., consist of those functions $\mathbf{u} : U \to \mathbb{R}^m$,$\mathbf{u} = (u^1, \ldots, u^m)$, with $u^i \in C(U)$,$L^p(U)$, etc.$(i = 1, \ldots, m)$.

B.2 Inequalities

B.2.1 Convex functions

Definition B.1. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex provided*

$$f(\tau x + (1-\tau)y) \leq \tau f(x) + (1-\tau)f(y) \tag{B.1}$$

for all $x, y \in Rn$ and each $0 \leq \tau \leq 1.$

Theorem B.1. *(Jensen's inequality). Assume $f : \mathbb{R} \to \mathbb{R}$ is convex, and $U \subset \mathbb{R}^n$ is open, bounded. Let $u : U \to \mathbb{R}$ be a summable. Then*

$$f\left(⨍_U u dx\right) \leq ⨍_U f(u) dx. \tag{B.2}$$

Proof. Since f is convex, for each $p \in \mathbb{R}$ there exists $r \in \mathbb{R}$ such that

$$f(q) \geq f(p) + r(q-p) \qquad \text{for all } q \in \mathbb{R}.$$

Let $p = ⨍_U u dx$,$q = u(x)$:

$$f(u(x)) \geq f\left(⨍_U u dx\right) + r\left(u(x) - ⨍_U u dx\right).$$

Integrate with respect to x over U. □

B.2.2 Elementary inequalities

Following is a collection of elementary, but fundamental, inequalities. Thes estimates are continually employed throughout the text.

Cauchy's inequality. Let a, b be a real numbers then

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2} \qquad (a, b \in \mathbb{R}).$$

Proof. It comes from $0 \leq (a-b)^2 = a^2 - 2ab + b^2$. □

Cauchy's inequality with ϵ.

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon} \qquad (a, b > 0, \epsilon > 0).$$

Proof. Write

$$ab = ((2\epsilon)^{\frac{1}{2}} a)) \left(\frac{b}{(2\epsilon)^{\frac{1}{2}}} \right)$$

and apply Cauchy's inequality. □

Young's inequality. Let $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$.Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad (a, b > 0).$$

Proof. The mapping $x \mapsto e^x$ is convex, and consequently

$$ab = e^{\log a + \log b} = e^{\frac{1}{p} \log a^p + \frac{1}{q} \log b^q} \leq \frac{1}{p} e^{\log a^p} + \frac{1}{q} e^{\log b^q} = \frac{a^p}{p} + \frac{b^q}{q}.$$

□

Young's inequality with ϵ.

$$ab \leq \epsilon a^p + C(\epsilon) b^q \qquad (a, b > 0, \epsilon > 0) \tag{B.3}$$

for $C(\epsilon) = (\epsilon p)^{\frac{-q}{p}} q^{-1}$.

Proof. Write $ab = \left((\epsilon p)^{\frac{1}{p}} a\right) \left(\frac{b}{(\epsilon p)^{\frac{1}{p}}}\right)$ and apply Young's inequality. □

Hölder's inequality. Assume $1 \leq p, q \leq \infty, \frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(U), v \in L^q(U)$, we have

$$\int_U |uv| dx \leq \|u\|_{L^p(U)} \|v\|_{L^q(U)}.$$

Proof. By homogeneity, we may assume $\|u\|_{L^p} = \|v\|_{L^q} = 1$. Then Young's inequality implies for $1 < p, q < \infty$ that

$$\int_U |uv| dx \leq \frac{1}{p} \int_U |u|^p dx + \frac{1}{q} \int_U |v|^q dx = 1 = \|u\|_{L^p} \|v\|_{L^q}.$$

□

Minkowski's inequality. Assume $1 \leq p \leq \infty$ and $u, v \in L^p(U)$. Then

$$\|u + v\|_{L^p(U)} \leq \|u\|_{L^p(U)} + \|v\|_{L^p(U)}.$$

Proof.

$$\begin{aligned}
\|u+v\|^p_{L^p(U)} &= \int_U |u+v|^p dx \le \int_U |u+v|^{p-1}(|u|+|v|)dx \\
&\le \left(\int_U |u+v|^p dx\right)^{\frac{p-1}{p}} \left(\left(\int_U |u|^p dx\right)^{\frac{1}{p}} + \left(\int_U |v|^p dx\right)^{\frac{1}{p}}\right) \\
&= \|u+v\|^{p-1}_{L^p(U)}(\|u\|_{L^p(U)} + \|v\|_{L^p(U)})
\end{aligned}$$

□

General Hölder inequality. Let $1 \le p_1, \ldots, p_m \le \infty$, with $\frac{1}{p_1} + \frac{1}{p_2} + \ldots \frac{1}{p_m} = 1$, and assume $u_k \in L^{p_k}(U)$ for $k = 1, \ldots, m$. then

$$\int_U |u_1 \ldots u_m| dx \le \prod_{k=1}^m \|u_i\|_{L^{p_k}(U)}.$$

Proof. Indution, using Hölder's inequality. □

Interpolation inequality for L^p-norms Assume $1 \le s \le r \le t \le \infty$ and

$$\frac{1}{r} = \frac{\theta}{s} + \frac{(1-\theta)}{t}.$$

Suppose also $u \in L^s(U) \cap L^t(U)$. Then $u \in L^r(U)$, and

$$\|u\|_{L^r(U)} \le \|u\|^{\theta}_{L^s(U)} \|u\|^{1-\theta}_{L^t(U)}.$$

Proof. We compute

$$\begin{aligned}
\int_U \|u\|^r dx &= \int_U |u|^{\theta r} |u|^{(1-\theta)r)} dx \\
&\le \left(\int_U |u|^{\theta r \frac{s}{\theta r}} dx\right)^{\frac{\theta r}{s}} \left(\int_U |u|^{(1-\theta)r \frac{t}{(1-\theta)r}} dx\right)^{\frac{(1-\theta)r}{t}}.
\end{aligned}$$

Here was used the Hölder's inequality, which applies since $\frac{\theta r}{s} + \frac{(1-\theta)r}{t} = 1.$ □

Cauchy-Schwarz inequality.

$$|x \cdot y| \le |x||y| \qquad (x, y \in \mathbb{R}^n).$$

Proof. Let $\epsilon > 0$ and note

$$0 \leq |x \pm \epsilon y|^2 = |x|^2 \pm 2\epsilon x \cdot y + \epsilon^2 |y|^2.$$

Consequently

$$\pm x \cdot y \leq \frac{1}{2\epsilon}|x|^2 + \frac{\epsilon}{2}|y|^2.$$

Minimize the right-hand side by setting $\epsilon = |\frac{x}{y}|$, provided $y \neq 0$. □

Gronwall's inequality (differential form). Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e t the differential inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \Psi(t), \tag{B.4}$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0, T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds} \left[\eta(0) + \int_0^t \Psi(s)ds\right] \tag{B.5}$$

for all $0 \leq t \leq T$. In particular if

$$\eta' \leq \phi\eta \qquad \text{on } [0, T] \text{ and } \eta(0) = 0,$$

then

$$\eta \equiv= 0 \qquad \text{on } [0, T].$$

Proof. From (B.4) we see

$$\frac{d}{ds}\left(\eta(s)e^{-\int_0^s \phi(r)dr}\right) = e^{-\int_0^s \phi(r)dr}(\eta'(s) - \phi(s)\eta(s)) \leq e^{-\int_0^s \phi(r)dr}\Psi(s)$$

for a.e. $0 \leq s \leq T$. Consequently for each $0 \leq t \leq T$, we have

$$\eta(t)e^{-\int_0^t \phi(r)dr} \leq \eta(0) + \int_0^t e^{-\int_0^s \phi(r)dr}\Psi(s)ds \leq \eta(0) + \int_0^t \Psi(s)ds.$$

This implies the inequality (B.5). □

Gronwall's inequality(integral form). Let $\xi(t)$ be a nonnegative, summable function on $[0, T]$ which satisfies for a.e. t the integral inequality

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2 \tag{B.6}$$

for constants $C_1, C_2 \geq 0$. Then

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t})$$

for a.e. $0 \leq t \leq T$. In particular if

$$\xi(t) \leq C_1 \int_0^t \xi(s) ds$$

for a.e. $0 \leq t \leq T$, then

$$\xi(t) = 0 \qquad \text{a.e.}$$

Proof. Let $\eta(t) := \int_0^t \xi(s) ds$; then $\eta \leq C_1 \eta + C_2$ a.e. in $[0, T]$. According to the differential form of Gronwall's inequality above:

$$\eta(t) \leq e^{C_1 t}(\eta(0) + C_2 t) = C_2 t e^{C_1 t}.$$

Then (B.6) implies

$$\xi(t) \leq C_1 \eta(t) + C_2 \leq C_2(1 + C_1 t e^{C_1 t}).$$

□

B.3 Calculus Facts

B.3.1 Boundaries

Let $U \subset \mathbb{R}^n$ be open and bounded, $k \in 1, 2, \ldots$.

Definition B.2. *We say ∂U is C^k if for each point $x^0 \in \partial U$ there exist $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ sucha that- upon relabeling and reorienting the coordinates axes if necessary- we have*

$$U \cap B[x^0, r] = \{x \in B[x^0, r] \mid x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$

Likewise ∂U is C^∞ if ∂U is C^k for $k = 1, 2, \ldots$, and ∂U is analytic if the mapping γ is analytic.

Definition B.3. *i) If ∂U is C^1, the along ∂U is defined the outward pointing unit normal vector field*

$$\nu = (\nu^1, \ldots, \nu^n).$$

The unit normal at any point $x^0 \in \partial U$ is $\nu(x^0) = \nu = (\nu_1, \ldots, \nu_n)$.

ii) Let $u \in C^1(\overline{U})$. *We call*

$$\frac{\partial u}{\partial \nu} := \nu \cdot Du$$

the (outward) normal derivative of u.

B.3.2 Gauss-Green Theorem.

In this section we assume U is boundade, open subset of $\mathbb{R}^n$, and ∂U is C^1

Theorem B.2. *(Gauss-Green Theorem). Suppose* $u \in C^1(\overline{U})$. *Then*

$$\int_U u_{x_i} dx = \int_{\partial U} u\nu^i \, dS \qquad (i = 1, \ldots, n). \tag{B.7}$$

Theorem B.3. *(Integration-by-parts formulas).*

i) Let $u, v \in C^1(\overline{U})$. *Then*

$$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} uv\nu^i \, dS \qquad (i = 1, \ldots, n). \tag{B.8}$$

ii)

Proof. Apply Theorem (B.2) to uv. □

Theorem B.4. *(Green's formulas) Let* $u, v \in C^2(\overline{U})$. *Then*

i) $\int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} \, dS,$

ii) $\int_U Dv \cdot Du dx = -\int_U u \Delta v dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u \, dS,$

iii) $\int_U \Delta v - v \Delta u dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} dS$

Proof. Using (B.8), with u_{x_i} in place u and $v \equiv 1$, we see

$$\int_U u_{x_i x_i} dx = \int_{\partial U} u_{x_i} \nu^i \, dS. \tag{B.9}$$

Sum $i = 1, \dots, n$ to estabilish $i)$. To derive (ii), we employ (B.8) with $v = u_{x_i}$

$$
\begin{aligned}
\int_U Du \cdot Dv dx &= \int_U \sum_{i=1}^{n} u_{x_i} v_{x_i} dx = \sum_{i=1}^{n} \int_U u_{x_i} v_{x_i} dx = \\
&= \sum_{i=1}^{n} \left(\int_{\partial U} u v_{x_i} \nu^i \, dS - \int_U u v_{x_i x_i} dx \right) = \\
&= \int_{\partial U} \sum_{i=1}^{n} u v_{x_i} \nu^i \, dS - \int_U \sum_{i=1}^{n} u v_{x_i x_i} dx = \\
&= \int_{\partial U} u \sum_{i=1}^{n} v_{x_i} \nu^i \, dS - \int_U u \sum_{i} v_{x_i x_i} dx = \\
&= \int_{\partial U} u Dv \cdot \nu \, dS - \int_U u \Delta v dx = \\
&= \int_{\partial U} u Dv . \nu \, dS - \int_U u \operatorname{div}(Dv) dx
\end{aligned}
\tag{B.10}
$$

Write (ii) with u and v interchanged and then subtract, to obtain (iii). □

The (B.10) will be referred as Integration-by-parts formula too.

Appendix C

Sobolev Spaces

C.1 Linear Functional Analysis

C.1.1 Banach spaces

Let X denote a real linear space.

Definition C.1. *A mapping* $\| \quad \| : X \to [0, \infty)$ *is called a norm if*

i) $\|u + v\| \leq \|u\| + \|v\|$ *for all*$u, v \in X$.

ii) $\|\lambda u\| = |\lambda| \|u\|$ *for all* $u \in X, \lambda \in \mathbb{R}$.

iii) $\|u\| = 0$ *if and only if* $u = 0$.

Hereafter we assume X is a normed linear space.

Definition C.2. *We say a sequence* $\{u_k\}_{k=1}^{\infty} \subset X$ *converges to* $u \in X$, *written*

$$u_k \to u,$$

if

$$\lim_{k \to \infty} \|u_k - u\| = 0.$$

Definition C.3. *i) A sequence* $\{u_k\}_{k=1}^{\infty} \subset X$ *is called a Cauchy sequence provided for each* $\epsilon > 0$ *there exists* $N > 0$ *such that*

$$\|u_k - u_l\| < \epsilon \qquad \text{for all } k, l \geq N.$$

ii) X *is complete if each Cauchy sequence in* X *converges; that is whenever* $\{u_k\}_{k=1}^{\infty}$ *is a Cauchy sequence, there exists* $u \in X$ *such that* $\{u_k\}_{k=1}^{\infty}$ *converges to* u.

iii) A Banach space X is a complete, normed linear space.

iv) We say X is separable if X contains a countable dense subset.

Example C.1. *L^p spaces. Assume U is an open subset of $\mathbb{R}^n$, and $1 \leq p \leq \infty$. If $f : U \to \mathbb{R}$ is a measurable, we define*

$$\|f\|_{L^p(U)} := \begin{cases} \left(\int_U |f|^p dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \operatorname{ess\,sup}_U |f| & \text{if } p = \infty \end{cases}$$

We define $L^p(U)$ to be the linear space of all measurable functions $f : U \to \mathbb{R}$ for which $\|f\|_{L^p(U)} < \infty$. The $L^p(U)$ is a Banach space, provided we identify two functions which agree a.e. We can also generalize the space L^p to vector functions $u : U \to \mathbb{R}^n$. Let U be an open subset of $\mathbb{R}^n$, and $1 \leq p \leq \infty$. If $f = (f^1, \dots f^n) : U \to \mathbb{R}^n$ where f_i for each $i = 1, \dots n$ is a measurable function, we define

$$\|f\|_{L^p(U,\mathbb{R}^n)} := \begin{cases} \left(\int_U \left(\sum_{i=1}^{n} |f^i|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \operatorname{ess\,sup}_U \|f\| & \text{if } p = \infty \end{cases}$$

Then we have that $L^p(U,\mathbb{R}^n) := \{f : U \to \mathbb{R}^n \mid \|f\|_{L^p(U,\mathbb{R}^n)} < \infty\}$ Another example of Banach space are the Sobolev spaces, that deserves a more complete description.

Example C.2. *Another example of a normed space functions are the Anisotropic Lebesgue spaces or Strichartz spaces. Let $\Omega \subset \mathbb{R}^d$ and $(0,T) \subset \mathbb{R}$ with Lebesgue measure. For each measurable function $f : \Omega \times (0,T) \to \mathbb{R}$ define*

$$\|f(t,x)\|_{L^p(0,T;L^q(\Omega))} := \left[\int_{\Omega_t} \left(\int_{\Omega_x} |f(t,x)|^q dx\right)^{\frac{p}{q}} dt\right]^{\frac{1}{p}}. \tag{C.1}$$

C.2 Hölder Spaces

Before turning to Sobolev spaces, we first discuss the simpler *Hölder spaces.* Assume $U \in \mathbb{R}^n$ is open and $0 < \gamma \leq 1$. We have previously considered the class of Lipschitz continuous functions $u : U \to \mathbb{R}$, which by definition satisfy the estimate

$$|u(x) - u(y)| \leq C|x - y| \qquad (x, y \in U), \tag{C.2}$$

for some constant C. The above inequality implies u is continuous, and more importantly provides a uniform modulus of continuity. It turns out to be useful to consider also functions u satisfying a variant of it, namely

$$|u(x) - u(y)| \leq C|x - y|^\gamma \qquad (x, y \in U) \tag{C.3}$$

for some constant C. Such function is said to be *Hölder continuous with expoent* γ.

Definition C.4. *i) If $u : U \to \mathbb{R}$ is bounded and continuous, we write*

$$\|u\|_{C(\overline{U})} := \sup_{x \in U} |u(x)|.$$

ii) The γ^{th}-Hölder seminorm of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\overline{U})} := \sup_{x,y \in U_{x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\},$$

and the $\gamma^{th}-$Hölder norm is

$$\|u\|_{C^{0,\gamma}(\overline{U})} := \|u\|_{C(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}.$$

Definition C.5. *The Hölder space*

$$C^{k,\gamma)}(\overline{U})$$

consists of all functions $u \in C^k(\overline{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\overline{U})} \quad := \quad \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} \tag{C.4}$$

is finite.

So the space $C^{k,\gamma}(\bar{(}U)$ consists of those functions u that are $k-$times consinuously differentiable and whose $k^{th}-$partial derivatives are Hölder continuous with exponent γ. Such functions are well-behaved, and furthermore the space $C^{k,\gamma}(\overline{U})$ itself possesses a good mathematical structure:

Theorem C.1 (Hölder spaces as function spaces)**.** *The space of functions $C^{k,\gamma}(\overline{U})$ is a Banach space.*

The Hölder spaces introduced above, are unfortunately not often suitable settings for the elementary theory of Partial Differential Equations, as we usually cannot make good enough analytic estimates to demonstrate that the solutions actually belong to such spaces, we must strike a balance, by designing spaces comprising functions which have some, but not too great, smoothness properties.

Definition C.6. *Let $\mathscr{C}_0^\infty(U)$ denote the space of infinitely differentiable functions $\phi : u \to \mathbb{R}$, with compact support in U. We will call a function ϕ belonging to $\mathscr{C}_0^\infty(U)$ a test function.*

Definition C.7. *Suppose $u, v \in L^1_{loc}(U)$, and α is a multiindex. We say that v is the $\alpha^{th}-$weak partial derivate of u written*

$$D^\alpha u = v,$$

provided

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx \tag{C.5}$$

for all test functions $\phi \in \mathscr{C}_0^\infty(U)$.

Lemma C.1 (Uniqueness of weak derivatives)**.** *A weak $\alpha^{th}-$partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

Proof. Assume that v, $\tilde{v} \in L^1_{\text{loc}}(U)$ satisfy

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx = (-1)^{|\alpha|} \int_U \tilde{v} \phi dx$$

for all $\phi \in \mathscr{C}_0^\infty(U)$. Then

$$\int_U (v - \tilde{v}) \phi dx = 0 \tag{C.6}$$

for all $\phi \in \mathscr{C}_0^\infty(U)$; whence $v - \tilde{v} = 0$ a.e.. □

Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. We define now certain function spaces, whose members have weak derivatives of various orders lying in various L^p spaces.

Definition C.8. *The Sobolev space*

$$W^{k,p}(U)$$

consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sens and belongs to $L^p(U)$.

Definition C.9. *If $u \in W^{k,p}(U)$, we define its norm to be*

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{\frac{1}{p}} & (1 \leq p < \infty) \\ \sum_{|\alpha| \leq k} \textit{ess} \sup_U |D^\alpha u| & (p = \infty). \end{cases}$$

Definition C.10. *We denote by*

$$W_0^{k,p}(U)$$

the closure of $\mathscr{C}_0^\infty(U)$ *in* $W^{k,p}(U)$*. Thus* $u \in W_0^{k,p}(U)$ *if and only if there exist functions* $u_m \in \mathscr{C}_0^\infty(U)$ *such that* $u_m \to u$ *in* $W^{k,p}(U)$*. We interpret* $W_0^{k,p}(U)$ *as comprising those functions* $u \in W^{k,p}(U)$ *such that*

$$\text{“}D^\alpha u = 0 \text{ on } \partial U\text{” for all } |\alpha| \le k-1.$$

C.2.1 Elementary properties

Next we verify certain properties of weak derivatives. Note very carefully that whereas thes various rules are obiously tru for smooth functions, functions in Sobolev space are not necessarily smooth: We must always rely solely upon the definition of weak derivatives.

Theorem C.2 (Properties of weak derivatives)**.** *Assume* $u, v \in W^{k,p}(U)$,$|\alpha| \le k$*. Then*

i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$ *and* $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ *for all multiindex* α, β *with* $|\alpha| + |\beta| \le k.$

ii) For each $\lambda, \mu \in \mathbb{R}$*,* $\lambda u + \mu v \in W^{k,p}(U)$ *and* $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v, |\alpha| \le k.$

iii) If V *is an open subset of* U*, then* $u \in W^{k,p}(V)$*.*

iv) If $\zeta \in \mathscr{C}_0^\infty(U)$*, the* $\zeta u \in W^{k,p}(U)$ *and*

$$D^\alpha(\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \qquad \text{(Leibniz' formula)}$$

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

Proof. To prove (i), first fix $\phi \in \mathscr{C}_0^\infty(U)$. Then $D^\beta \phi \in \mathscr{C}_0^\infty(U)$, and so

$$\begin{aligned}
\int_U D^\alpha u d^\beta \phi dx &= (-1)^{|\alpha|} \int_U u D^{\alpha+\beta} \phi dx \\
&= (-1)^{|\alpha|)} (-1)^{|\alpha+\beta|} \int_U D^{\alpha+\beta} u \phi dx \\
&= (-1)^{|\beta|} \int_U D^{\alpha+\beta} u \phi dx. \qquad \text{(C.7)}
\end{aligned}$$

Thus $D^\beta(D^\alpha u) = D^{\alpha+\beta}u$ in the weak sense. Assertions (ii) and (iii) are easy, and the formula in (iv) is proved by induction on $|\alpha|$. □

Theorem C.3 (Sobolev spaces as function spaces)**.** *For each* $k = 1, \dots$ *and* $1 \le p \le \infty$*, the Sobolev space* $W^{k,p}$ *is a Banach space.*

It is awkward to return continually to the definition of weak derivatives. To study the deeper properties of Sobolev spaces, we therefore need to develop some systematic procedures for approximating a function in a Sobolev space by smooth functions.

C.2.2 Traces

In this subsection we present some results that answer the question about the possibility of assigning “boundary values" along ∂U to a function $u \in W^{1,p}(U)$, assuming that ∂U is C^1. If $u \in C(\overline{U})$, then clearly u has values on ∂U in the usual sense. The problem is that a typical function $u \in W^{1,p}(U)$ is not in general continuous and, even worse, is only defined a.e. in U. Since ∂U has $n-$dimensional Lebesgue measure zero, there is no direct meaning we can give to the expression “ u restricted to ∂U". The notion of a *trace operator* resolves this problem. For this subsection we take $1 \leq p < \infty$.

Theorem C.4 (Trace Theorem)**.** *Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(U) \to L^p(\partial U)$$

such that

i) $Tu = u|_{\partial U}$ *if* $u \in W^{1,p}(U) \cap C(\overline{U})$ *and*

ii)

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)},$$

for each $u \in W^{1,p}(U)$, with the constant C depending only on p and U.

Definition C.11. *We call Tu the trace of u on ∂U.*

Corollary C.1. *Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator*

$$\tilde{T} : W^{1,p}(U, \mathbb{R}^n) \to L^p(U, \mathbb{R}^n)$$

such that

i) $Tu = u|_{\partial U}$ *if* $u \in W^{1,p}(U, \mathbb{R}^n) \cap C(\overline{U}, \mathbb{R}^n)$ *and*

ii)

$$\|Tu\|_{L^p(\partial U, \mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U, \mathbb{R}^n)},$$

for each $u \in W^{1,p}(U)$, with the constant C depending only on p and U.

C.2.3 Sobolev Inequalities

Our goal in this subsection is to discover embeddings of various Sobolev spaces into others. The crucial analytic tools here will be certain so-called “Sobolev-type inequalities", which will prove below for smooth functions. Thes will then establish the estimates for arbitrary functions in the various relevant Sobolev spaces. To clarify the presentetion we will consider first only Sobolev space $W^{1,p}(U)$ and ask for the following basic question: *“If a function u belongs to $W^{1,p}(U)$ does u automatically belong to certain other spaces?* The answer will be “yes", but which other spaces depends upon whether.

(1)

$$1 \leq p < n,$$

(2)

$$p = n,$$

(3)

$$n < p \leq \infty.$$

We study case (1) in(C.5), case (3) in (C.8)

Definition C.12. *If $1 \leq p < n$ the Sobolev conjugate of p is*

$$p^* := \frac{np}{n-p}.$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \qquad p^* > p.$$

Theorem C.5. *(The Gagliardo-Nirenberg-Sobolev Inequality) Assume $1 \leq p < n$. There exists a constant C, depending only on p and n such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \tag{C.8}$$

for all $u \in C_0^1(\mathbb{R}^n)$.

Theorem C.6. *(Estimates for $W^{1,p}$,$1 \leq p < n$) Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}, \tag{C.9}$$

the constant C depending only on p, n and U.

Theorem C.7. *(Poincaré's inequality)(Estimates for $W_0^{1,p}$, $1 \leq p < n$.) Assume U is a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

for each $q \in [1, p^*]$, the constant C depending only on p, q, n and U.

Theorem C.8. *(Morrey's inequality) Assume $n < p \leq \infty$. Then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{C.10}$$

for all $u \in C^1(\mathbb{R}^n)$, where

$$\gamma := 1 - \frac{n}{p}.$$

Theorem C.9. *(General Sobolev inequalities). Let U be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W^{k,p}(U)$.*

i) If

$$k < \frac{n}{p},$$

then $u \in L^q(U)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

We have in addition the estimate

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)}, \tag{C.11}$$

the constant C depending only on k, p, n and U.

ii) If

$$k > \frac{n}{p},$$

then $u \in C^{k-\left[\frac{n}{p}-1,\gamma\right]}(\overline{U})$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

We have in addition the estimate

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{U})} \leq C\|u\|_{W^{k,p}(U)}, \tag{C.12}$$

the constant C *depending only on* k, p, n, γ *and* U*.*

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