UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Limites Singulares para Equações do tipo Rosenau-KdV-RLW e Benney-Lin: Existência e Convergência de Soluções

DANILO DE JESUS FERREIRA

São Carlos - SP
Setembro de 2017

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Tese apresentada ao Programa de Pós-Graduação em Matemática da UFSCar como parte dos requisitos necessários para a obtenção do título de Doutor em Matemática.

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Resumo

Consideramos as aproximações

$$\partial_t u + \partial_x f(u) + \beta b_1 \partial_x^3 u + \beta b_2 \partial_t \partial_x^2 u + \beta^2 c \partial_t \partial_x^4 u = \epsilon \partial_x^2 u \tag{1}$$

e

$$\partial_t u + \partial_x f(u) + \beta \partial_x^2 u + \beta^2 b \partial_x^3 u + \beta^3 c \partial_x^4 u + \beta^5 d \partial_x^5 u = \epsilon \partial_x^2 u \tag{2}$$

das equações de Rosenau-KdV-RLW e Benney-Lin e, suplementando-as com uma condição inicial

$$u(0,x) = u_{\epsilon,\beta,0}(x), \tag{3}$$

estabelecemos a existência de soluções globais $u_{\epsilon,\beta}$ para os problemas (1)–(3) e (2)–(3). Além disso, estudamos o comportamento limite da sequência $u_{\epsilon,\beta}$ quando os parâmetros ϵ e β são mantidos em balanço e tendem a zero, e mostramos que a função limite consiste da única solução de entropia da lei de conservação associada

$$\partial_t u + \partial_x f(u) = 0.$$

As ferramentas utilizadas serão a Teoria da Compacidade Compensada desenvolvida por Tartar-Murat [22, 23, 27, 28] e a teoria de DiPerna [10, 11] sobre Soluções de Entropia com Valores em Medida, juntamente com uma série de estimativas uniformes sobre a sequência $u_{\epsilon,\beta}$ obtidas no decorrer do texto.

Abstract

We consider the approximations

$$\partial_t u + \partial_x f(u) + \beta b_1 \partial_x^3 u + \beta b_2 \partial_t \partial_x^2 u + \beta^2 c \partial_t \partial_x^4 u = \epsilon \partial_x^2 u \tag{4}$$

and

$$\partial_t u + \partial_x f(u) + \beta \partial_x^2 u + \beta^2 b \partial_x^3 u + \beta^3 c \partial_x^4 u + \beta^5 d \partial_x^5 u = \epsilon \partial_x^2 u \tag{5}$$

of the Rosenau-KdV-RLW and Benney-Lin equations and supplementing them with an initial condition

$$u(0, x) = u_{\epsilon,\beta,0}(x) \tag{6}$$

we establish the global existence of solutions $u_{\epsilon,\beta}$ for the problems (4)–(6) and (5)–(6). Moreover, we study the limiting behaviour of the sequence $u_{\epsilon,\beta}$ when the parameters ϵ and β are kept in balance and tend to zero, and we prove that the limit function consists of the unique entropy solution of the conservation law

$$\partial_t u + \partial_x f(u) = 0.$$

The tools used will be the Compensated Compactness Theory developed by Tartar-Murat [22, 23, 27, 28] and DiPerna's theory [10, 11] on Entropy Measure-Valued Solutions together with a number of uniform estimates on the sequence $u_{\epsilon,\beta}$ obtained during the text.

Sumário

Introdução

Neste trabalho estudaremos a existência de soluções globais $u_{\epsilon,\beta}$ para os problemas de Cauchy

$$\begin{aligned} &\partial_t u + \partial_x f(u) + \beta b_1 \partial_x^3 u + \beta b_2 \partial_t \partial_x^2 u + \beta^2 c \partial_t \partial_x^4 u = \epsilon \partial_x^2 u \\ &u(0,x) = u_{\epsilon,\beta,0}(x) \end{aligned} \tag{7}$$

e

$$\begin{aligned} &\partial_t u + \partial_x f(u) + \beta \partial_x^2 u + \beta^2 b \partial_x^3 u + \beta^3 c \partial_x^4 u + \beta^5 d \partial_x^5 u = \epsilon \partial_x^2 u \\ &u(0,x) = u_{\epsilon,\beta,0}(x), \end{aligned} \tag{8}$$

e investigaremos a convergência da sequência $u_{\epsilon,\beta}$ para uma solução da lei de conservação associada

$$\partial_t u + \partial_x f(u) = 0 \tag{9}$$

quando os parâmetros ϵ e β são mantidos em balanço e tendem a zero.

Em (7) e (8) os parâmetros ϵ e β são números reais positivos, b_1, b_2, c e etc. são os coeficientes das equações, $u_{\epsilon,\beta,0}$ é uma aproximação (no caso $u_0 \in L^1 \cap L^p, p = 2, 4$) de um dado inicial u_0 e o fluxo f é uma função sub-quadrática, i.e., $|f'(u)| \leq C_0(1 + |u|)$.

Baseando-se nos trabalhos [15] e [16] estabeleceremos a existência de soluções globais mediante um resultado local e um processo recursivo de extensão. Na extensão, faremos uso de uma série de estimativas a priori, fundamentais para a validez do procedimento.

Com as soluções em mãos, estudaremos em seguida a convergência das mesmas para uma solução de (9) utilizando duas ferramentas de convergência: a Teoria da Compacidade Compensada desenvolvida por Tartar [27, 28] e Murat [22, 23], e a teoria de DiPerna das Soluções de Entropia com Valores em Medida [11]. Mais precisamente, usaremos adaptações destas teorias para o ambiente L^p feitas por Schonbek [24] e Szepessy [26].

Quando $\epsilon = 0$ e $\beta = 1$ em (7) e (8), resgatamos respectivamente as famosas equações de Rosenau-KdV-RLW e de Benney-Lin:

$$\partial_t u + a\partial_x u + k\partial_x u^n + b_1 \partial_x^3 u + b_2 \partial_t \partial_x^2 u + c\partial_t \partial_x^4 u = 0, \quad f(u) = a + ku^n \tag{10}$$

$$\partial_t u + \frac{1}{2}\partial_x u^2 + \partial_x^2 u + b\partial_x^3 u + c\partial_x^4 u + d\partial_x^5 u = 0, \quad f(u) = u^2/2. \tag{11}$$

A primeira delas funciona como um modelo de captação da dinâmica de ondas rasas dispersivas, enquanto a segunda, derivada primeiramente por Benney [3] e posteriormente por Lin [20], descreve a evolução unidimensional de ondas pequenas em vários problemas em dinâmica dos fluidos.

Em [6, 7] os autores consideraram dois casos particulares de (10). Assumindo algumas condições sobre os dados iniciais e utilizando as adaptações (mencionadas acima) ao ambiente L^p, eles mostraram que as soluções $u_{\epsilon,\beta}$ das aproximações

$$\partial_t u + \partial_x u^2 + \beta^2 \partial_t \partial_x^4 u = \epsilon \partial_x^2 u,$$
$$\partial_t u + \partial_x u^2 + \beta \partial_x^3 u - \beta \partial_t \partial_x^2 u + \beta^2 \partial_t \partial_x^4 u = \epsilon \partial_x^2 u,$$

convergem localmente para a única solução de entropia da equação de Burgers

$$\partial_t u + \partial_x u^2 = 0$$

em todo espaço L^r para $r \in [1, 4)$. Seguindo a mesma linha resultados similares foram obtidos em [8] considerando a aproximação

$$\partial_t u + \frac{1}{2}\partial_x u^2 + \beta^2 \partial_x^2 u + k^2 \beta^2 \partial_x^3 u + \beta^3 \partial_x^4 u = \epsilon \partial_x^2 u$$

da equação de Kuramoto-Sivashinsky-Korteweg de Vries ($b = c = 1$ e $d = 0$ em (11)). Em todos estes casos os parâmetros ϵ, β eram mantidos em balanço e tendiam a zero.

A título de informação, os autores estabeleceram em [5] a boa colocação global do problema de Cauchy para a equação de Benney-Lin

$$\partial_t u + \frac{1}{2}\partial_x u^2 + \partial_x^3 u + \beta(\partial_x^2 u + \partial_x^4 u) + \epsilon \partial_x^5 u = 0$$

quando o dado inicial pertencia a algum espaço de Sobolev H^s com $s \geq 0$. Para obter este importante resultado eles fizeram uso da Teoria de Interpolação Não-Linear. Além disso, eles analisaram o comportamento limite das soluções $u_{\epsilon,\beta}$ nos casos: (i) ϵ fixo e $\beta \to 0$ e (ii) β fixo e $\epsilon \to 0$.

Nosso objetivo neste trabalho será estender para os problemas (7) e (8) os resultados mencionados acima obtidos em [6, 7, 8], além de demonstrar a existência de soluções globais dos mesmos. Obteremos soluções nos espaços $C([0, \infty), H^l(\mathbb{R}))$ para $l \geq 1$ inteiro e mostraremos que as mesmas convergem localmente para a única solução de entropia de (9) em todo L^r com (i) $r \in [1, 2)$ se o dado inicial estiver em $L^1 \cap L^2$ e (ii) $r \in [1, 4)$ se o dado inicial estiver em $L^1 \cap L^4$.

O texto está estruturado em três capítulos. O primeiro deles reune os pré-requisitos necessários para o entendimento do trabalho, enquanto os Capítulos 2 e 3 abordam, seguindo o roteiro descrito no parágrafo anterior, os problemas (7) e (8) respectivamente. Por fim, incluímos dois pequenos apêndices contendo as demonstrações de alguns fatos afirmados no texto.

Capítulo 1

Pré-requisitos

Apresentaremos neste capítulo as ferramentas necessárias para o entendimento de todo o trabalho. Nele reuniremos as principais propriedades da transformada de Fourier, algumas estimativas nos espaços de Sobolev H^s, além de vários resultados elementares sobre Compacidade Compensada e medidas de Young.

1.1 A Transformada de Fourier

Dada $f \in L^1(\mathbb{R}^n)$, sua *transformada de Fourier* é a função $\widehat{f}$ definida por

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx \quad (\xi \in \mathbb{R}^n).$$

A inversa desta função, como sabemos, é dada por

$$f^\vee(x) = \widehat{f}(-x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi \quad (x \in \mathbb{R}^n).$$

A fim de estender esta definição para todo o espaço $L^2(\mathbb{R}^n)$, necessitaremos do seguinte

Teorema 1.1 (Plancherel). *Se $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ então $\widehat{f}, f^\vee \in L^2(\mathbb{R}^n)$ e*

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\widehat{f}\|_{L^2(\mathbb{R}^n)} = \|f^\vee\|_{L^2(\mathbb{R}^n)}. \tag{1.1}$$

A relação (1.1) nos permite definir a transformada de Fourier de elementos do $L^2(\mathbb{R}^n)$ da seguinte maneira: dada $f \in L^2(\mathbb{R}^n)$, pomos $\widehat{f} = \lim_{k\to\infty} \widehat{f_k}$, sendo $\{f_k\}_{k\in\mathbb{N}} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ uma sequência convergindo para f em $L^2(\mathbb{R}^n)$. É claro que $\widehat{f}$ está bem definida e que as igualdades em (1.1) continuam válidas. Além disso, temos a importante

Proposição 1.1. *Se $f, g \in L^2(\mathbb{R}^n)$, as seguintes propriedades são válidas:*

(i) $\int_{\mathbb{R}^n} f(x)\bar{g}(x)dx = \int_{\mathbb{R}^n} \widehat{f}(x)\overline{\widehat{g}}(x)dx$, $(\bar{g}(x) = \overline{g(x)})$;

(ii) $\widehat{\partial^\alpha f} = (ix)^\alpha \widehat{f}$ *para todo multi-índice α tal que* $\partial^\alpha f \in L^2(\mathbb{R}^n)$, $(\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n})$;

(iii) $\partial^\alpha \widehat{f} = [(-ix)^\alpha f]^\frown$ *para todo multi-índice* α *tal que* $x^\alpha f \in L^2(\mathbb{R}^n)$*;*

(iv) $(\widehat{f})^\vee = (f^\vee)^\wedge = f$.

Todas estas propriedades serão utilizadas livremente no texto e podem ser encontradas em [12].

1.2 Algumas Estimativas em H^s

Teorema 1.2. *Seja* $f : \mathbb{R} \to \mathbb{R}$ *uma função suave tal que* $f(0) = 0$. *Para qualquer inteiro* $s \geq 0$, *se* $u \in H^s(\mathbb{R})$ *e*

$$\|u\|_{L^\infty(\mathbb{R})} \leq K, \tag{1.2}$$

então $f(u) \in H^s(\mathbb{R})$ *e*

$$\|f(u)\|_{H^s(\mathbb{R})} \leq C_K \|u\|_{H^s(\mathbb{R})},$$

sendo $C_K > 0$ *uma constante que depende apenas de* K.

Teorema 1.3. *Sejam* $f : \mathbb{R} \to \mathbb{R}$ *uma função suave e* $u, v \in H^s(\mathbb{R})$ *com* $s \geq 1$ *inteiro. Se* v *satisfizer* (1.2)*, então* $f(v)u \in H^s(\mathbb{R})$ *e*

$$\|f(v)u\|_{H^s(\mathbb{R})} \leq C_{K,s} \|u\|_{H^s(\mathbb{R})} (|f(0)| + \|v\|_{H^s(\mathbb{R})}),$$

sendo $C_{K,s} > 0$ *uma constante que depende apenas de* K *e* s.

As demonstrações dos resultados acima encontram-se, respectivamente, nas páginas 22 e 30 da referência [29]. O corolário seguinte é uma consequência imediata do Teorema 1.3 e pode ser encontrado na página 10 de [30].

Corolário 1.1. *Assuma as hipóteses do Teorema 1.3 e suponha que* u *e* v *satisfaçam* (1.2)*. Nestas condições,*

$$\|f(u) - f(v)\|_{H^s(\mathbb{R})} \leq C_{K,s} \|u - v\|_{H^s(\mathbb{R})} (|f'(0)| + \|u\|_{H^s(\mathbb{R})} + \|v\|_{H^s(\mathbb{R})}),$$

sendo $C_{K,s} > 0$ *uma constante que depende apenas de* K *e* s.

Finalizaremos esta seção apresentando uma versão dos famosos mergulhos de Sobolev. Antes disso, daremos algumas definições.

Dizemos que uma função $f : \mathbb{R} \to \mathbb{R}$ se *anula no infinito*, se para todo $\epsilon > 0$ o conjunto $\{x \in \mathbb{R}; |f(x)| \geq \epsilon\}$ é compacto. Denotaremos o espaço de tais funções por $C_0(\mathbb{R})$:

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}); f \textit{ se anula no infinito}\}.$$

Mais geralmente, podemos considerar os espaços

$$C_0^k(\mathbb{R}) = \{f \in C^k(\mathbb{R}); f^{(j)} \in C_0(\mathbb{R}), j = 0, \ldots, k\} \quad k = 0, 1, 2, \ldots.$$

Lema 1.1 (Sobolev)**.** *Se* $s > k + 1/2$*, então* $H^s(\mathbb{R}) \subset C_0^k(\mathbb{R})$.

O Lema de Sobolev será muito importante nos capítulos seguintes e sua demonstração pode ser encontrada [14].

1.3 Compacidade Compensada

Seguindo Schonbek [24], apresentaremos alguns resultados de compacidade para uma sequência de soluções aproximadas $\{u_k\}_{k\in\mathbb{N}}$ pertencentes a um subconjunto limitado de $L^p(\Omega)$ de uma equação diferencial parcial escalar da forma

$$\partial_t u + \partial_x f(u) = 0, \tag{1.3}$$

onde Ω é um aberto limitado de $\mathbb{R}^2$, f é uma função dada e a sequência $\{u_k\}_{k\in\mathbb{N}}$ satisfaz a condição de entropia

$$\partial_t \eta(u_k) + \partial_x q(u_k) \in [\textit{conjunto compacto de } H^{-1}(\Omega)] \tag{1.4}$$

para todo par de funções (η, q) tal que $q' = \eta' f'$. Recordemos que um par de funções (η, q) é chamado um *par de entropia–fluxo de entropia* com respeito a (1.3) se ele satisfizer a condição $q' = \eta' f'$. Este teorema é uma extensão de um resultado obtido por Tartar em [27] para soluções aproximadas pertencentes a um subconjunto aberto limitado de $L^\infty(\Omega)$.

Teorema 1.4. *Sejam $\Omega \subset \mathbb{R}^2$ um aberto limitado e $f \in C^1(\mathbb{R})$ uma função satisfazendo a condição de crescimento*

$$|f'(u)| \leq C_0(1 + |u|^{p-1}) \textit{ para algum } p \in (1, \infty).$$

Além disso, seja $\{u_k\}_{k\in\mathbb{N}}$ uma sequência de soluções aproximadas de (1.3)*, uniformemente limitada em $L^p(\Omega)$, satisfazendo a condição de entropia* (1.4) *para todo par de entropia–fluxo de entropia (η, q) com $\eta \in C_c^2(\mathbb{R})$ convexa em algum intervalo limitado não-vazio. Nestas condições, existe uma subsequência $\{u_{k_j}\}_{j\in\mathbb{N}}$ tal que*

$$u_{k_j} \rightharpoonup \tilde{u} \;\textit{ e }\; f(u_{k_j}) \rightharpoonup f(\tilde{u}) \;\textit{ em } \mathscr{D}'(\Omega)$$

e $\tilde{u} \in L^p(\Omega)$ é uma solução fraca de (1.3)*.*

Corolário 1.2. *Sejam $\Omega, \{u_k\}_{k\in\mathbb{N}}$ e f como no Teorema 1.4. Se $f'' > 0$ então*

$$u_{k_j} \to \tilde{u} \textit{ fortemente em } L^q(\Omega) \textit{ para todo } q \in (1, p).$$

Finalizaremos esta seção com o importante lema de Murat [23]:

Lema 1.2 (Murat)**.** *Sejam $\Omega \subset \mathbb{R}^n$ um aberto limitado e $\{f_k\}_{k\in\mathbb{N}}$ uma sequência satisfazendo*

(i) $\{f_k\}_{k\in\mathbb{N}} \in [\textit{conjunto limitado de } W^{-1,\infty}(\Omega)]$;

(ii) $\{f_k\}_{k\in\mathbb{N}} \in [\textit{conjunto compacto de } H^{-1}(\Omega)] + [\textit{conjunto limitado de } \mathcal{M}(\Omega)]$.

Então

(iii) $\{f_k\}_{k\in\mathbb{N}} \in [\textit{conjunto compacto de } H^{-1}(\Omega)]$.

No lema acima,

$$W^{-1,\infty}(\Omega) = \{T \in \mathcal{D}'(\Omega); T = f_0 + \sum_{i=1}^{n} \partial f_i/\partial x_i, f_0, \ldots, f_n \in L^\infty(\Omega)\},$$

$$H^{-1}(\Omega) = \{T \in \mathcal{D}'(\Omega); T = f_0 + \sum_{i=1}^{n} \partial f_i/\partial x_i, f_0, \ldots, f_n \in L^2(\Omega)\}$$

e $\mathcal{M}(\Omega)$ denota o espaço das medidas Radon em Ω, i.e., o espaço dos funcionais lineares em $C_c(\Omega), \phi \mapsto \langle \mu, \phi \rangle$, tais que para todo compacto $K \subset \Omega$ existe uma constante $C_K > 0$ satisfazendo

$$|\langle \mu, \phi \rangle| \leq C_K \|\phi\|_{L^\infty(\Omega)} \text{ para toda } \phi \in C_c(\Omega) \text{ com } supp(\phi) \subset K.$$

1.4 Medidas de Young e Soluções de Entropia

Esta seção contém alguns fatos básicos sobre medidas de Young e soluções de entropia com valores em medidas. O seu conteúdo pode ser encontrado em [19] e [26]. A seguir suporemos $p \in (1, \infty)$ e denotaremos (de agora em diante) a semirreta positiva $(0, \infty)$ por $\mathbb{R}_+$.

Lema 1.3. *Seja $\{u_k\}_{k\in\mathbb{N}}$ uma sequência uniformemente limitada em $L^\infty(\mathbb{R}_+; L^p(\mathbb{R}))$. Então existe uma subsequência $\{u_{k'}\}_{k'\in\mathbb{N}}$ e uma aplicação $\nu : \mathbb{R}_+ \times \mathbb{R} \to Prob(\mathbb{R})$ tal que, para toda função $g \in C(\mathbb{R})$ satisfazendo*

$$g(u) = O(1 + |u|^r) \tag{1.5}$$

para algum $r \in [0, p)$, a seguinte representação é válida

$$\lim_{k'\to\infty} \iint_{\mathbb{R}_+\times\mathbb{R}} g(u_{k'}(t,x))\phi(t,x)dtdx = \iint_{\mathbb{R}_+\times\mathbb{R}} \int_{\mathbb{R}} g(\lambda)d\nu_{(t,x)}(\lambda)\phi(t,x)dtdx \tag{1.6}$$

para toda $\phi \in L^1(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R})$.

A função ν é chamada uma *medida de Young* associada com a sequência $\{u_k\}_{k\in\mathbb{N}}$ e é definida por

$$\langle \nu_{(\cdot)}, \phi(\lambda) \rangle = \int_{\mathbb{R}} \phi(\lambda) d\nu_{(\cdot)}(\lambda).$$

O próximo resultado revela a relação entre a medida ν e a convergência forte da sequência inicial.

Lema 1.4. *Seja ν uma medida de Young associada com uma sequência $\{u_k\}_{k\in\mathbb{N}}$ uniformemente limitada em $L^\infty(\mathbb{R}_+; L^p(\mathbb{R}))$. Para $u \in L^\infty(\mathbb{R}_+; L^p(\mathbb{R}))$, as seguintes afirmações são equivalentes:*

(i) $\lim_{k\to\infty} u_k = u$ *em* $L^\infty(\mathbb{R}_+; L^r_{loc}(\mathbb{R}))$ *para algum* $r \in [1, p)$*;*

(ii) $\nu_{(t,x)}(\lambda) = \delta_{u(t,x)}(\lambda)$ *para quase todo* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

Agora consideremos o problema de Cauchy

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{1.7}$$

Dizemos que uma função u é uma *solução de entropia* de (1.7) se ela for uma solução fraca de (1.7) e satisfazer a condição de entropia

$$\partial_t \eta(u) + \partial_x q(u) \leq 0$$

no sentido distribucional para todo par de entropia–fluxo de entropia (η, q) com η convexa, i.e.,

$$\int_0^\infty \int_{\mathbb{R}} \left[\partial_t \eta(u(t,x)) + \partial_x q(u(t,x))\right] \phi(t,x) dtdx \leq 0$$

para toda $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ não-negativa.

Mais ainda, se f satisfizer a condição de crescimento (1.5) e $u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, uma medida de Young ν associada com uma sequência $\{u_k\}_{k\in\mathbb{N}}$, uniformemente limitada em $L^\infty(\mathbb{R}_+; L^p(\mathbb{R}))$, é chamada uma *solução de entropia m-v* de (1.7) se

$$\lim_{T\to 0^+} \frac{1}{T} \int_0^T \int_I \left\langle \nu_{(t,x)}, |\lambda - u_0(x)| \right\rangle dtdx = 0 \tag{1.8}$$

para todo compacto $I \subset \mathbb{R}$, e

$$\partial_t \left\langle \nu_{(\cdot)}, |\lambda - \alpha| \right\rangle + \partial_x \left\langle \nu_{(\cdot)}, sgn(\lambda - \alpha)(f(\lambda) - f(\alpha)) \right\rangle \leq 0 \tag{1.9}$$

no sentido distribucional para todo $\alpha \in \mathbb{R}$.

Em relação a sequência $\{u_k\}$, as condições (1.8) e (1.9) são conhecidas (respectivamente) como *consitência forte com o dado inicial* e *consistência fraca com as desigualdades de entropia.* Esta última faz uso das famosas entropias de Kruzkov $(|\lambda - \alpha|, |f(\lambda) - f(\alpha)|)_{\lambda,\alpha}$ introduzidas em seu célebre trabalho [17].

Relacionados à estas definições estão os teoremas abaixo.

Teorema 1.5. *Assuma que f satisfaça (1.5) e que $u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Se ν for uma solução de entropia m-v do problema de Cauchy (1.7), então existe uma função $w \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}) \cap L^p(\mathbb{R}))$ tal que*

$$\nu_{(t,x)} = \delta_{w(t,x)}$$

para quase todo $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

Teorema 1.6. *Assuma que f satisfaça (1.5) e que $u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Então existe uma única solução de entropia $u \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R})) \cap L^p(\mathbb{R}))$ para o problema de Cauchy (1.7) satisfazendo*

$$\|u(t,\cdot)\|_{L^r(\mathbb{R})} \leq \|u_0\|_{L^r(\mathbb{R})}$$

para quase todo $t \in \mathbb{R}_+$ e todo $r \in [1,p]$. A aplicação com valores em medida $\nu_{(t,x)} = \delta_{u(t,x)}$ é a única solução de entropia m-v de (1.7).

Combinando o Lema 1.4 e os Teoremas 1.5 e 1.6 obtemos o importante

Corolário 1.3. *Assuma que f satisfaça (1.5) e que $u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Sejam $\{u_k\}_{k\in\mathbb{R}}$ uma sequência uniformemente limitada em $L^\infty(\mathbb{R}_+; L^p(\mathbb{R}))$, e ν uma medida de Young associada com esta sequência. Nestas condições, se ν for uma solução de entropia m-v de (1.7), então*

$$\lim_{k\to\infty} u_k = u \text{ em } L^\infty(\mathbb{R}_+; L^r_{loc}(\mathbb{R}))$$

para todo $r \in [1, p)$, sendo $u \in L^\infty(\mathbb{R}_+; L^p(\mathbb{R}))$ a única solução de entropia de (1.7).

Capítulo 2

Equação de Rosenau-KdV-RLW Generalizada

Estudaremos neste capítulo uma equação tipo Rosenau-KdV-RLW. Mostraremos a existência de soluções globais para dados iniciais no espaço H^5 e analisaremos o comportamento das soluções quando os parâmetros tendem a zero.

2.1 Existência de Soluções

Estabeleceremos nesta seção a existência de soluções globais para o seguinte problema de Cauchy

$$\partial_t u + \partial_x f(u) + \beta b_1 \partial_x^3 u + \beta b_2 \partial_t \partial_x^2 u + \beta^2 c \partial_t \partial_x^4 u = \epsilon \partial_x^2 u \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{2.1}$$

$$u(0,x) = u_{\epsilon,\beta,0}(x) \quad x \in \mathbb{R} \tag{2.2}$$

sendo $f : \mathbb{R} \to \mathbb{R}$ uma função suave, ϵ e β números reais no intervalo $(0,1)$ e b_1, b_2 e c constantes satisfazendo $b_1 \in \mathbb{R}, b_2 \leq 0$ e $c > 0$. Em toda nossa discussão $\mathcal{F}$ denotará a transformada de Fourier com relação a x, e $\mathcal{F}^{-1}$ a sua inversa. Assim, formalmente

$$\mathcal{F}(\partial_t u + \partial_x f(u) + \beta b_1 \partial_x^3 u + \beta b_2 \partial_t \partial_x^2 u + \beta^2 c \partial_t \partial_x^4 u) = \mathcal{F}(\epsilon \partial_x^2 u)$$

e daí

$$(1 + \beta |b_2| \xi^2 + \beta^2 c \xi^4) \partial_t \mathcal{F}(u) + (\epsilon \xi^2 - i\beta b_1 \xi^3) \mathcal{F}(u) = -\mathcal{F}(\partial_x f(u)). \tag{2.3}$$

Multiplicando (2.3) pelo fator integrante

$$\exp\left\{ \frac{(\epsilon \xi^2 - i\beta b_1 \xi^3)t}{1 + \beta |b_2| \xi^2 + \beta^2 c \xi^4} \right\}$$

segue-se que

$$\partial_t \left[\exp\left\{ \frac{(\epsilon \xi^2 - i\beta b_1 \xi^3)t}{1 + \beta |b_2| \xi^2 + \beta^2 c \xi^4} \right\} \mathcal{F}(u) \right] = -\exp\left\{ \frac{(\epsilon \xi^2 - i\beta b_1 \xi^3)t}{1 + \beta |b_2| \xi^2 + \beta^2 c \xi^4} \right\} \frac{\mathcal{F}(\partial_x f(u))}{1 + \beta |b_2| \xi^2 + \beta^2 c \xi^4}$$

e portanto integrando a expressão acima em $(0,t)$ obtemos

$$\mathcal{F}(u) = \exp\left\{\frac{-(\epsilon\xi^2 - i\beta b_1\xi^3)t}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}\right\}\mathcal{F}(u_{\epsilon,\beta,0}) - \int_0^t \frac{\exp\left\{\frac{-(\epsilon\xi^2 - i\beta b_1\xi^3)(t-s)}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}\right\}\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}ds. \quad (2.4)$$

Definindo

$$Q(t,\xi) = \exp\left\{\frac{-(\epsilon\xi^2 - i\beta b_1\xi^3)t}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}\right\}$$

e

$$G(t)u = \mathcal{F}^{-1}\left(Q(t,\cdot)\mathcal{F}u(\cdot)\right),$$

a partir de (2.4) segue-se que a equação integral da solução é

$$u(t) = G(t)u_{\epsilon,\beta,0} - \int_0^t G(t-s)\mathcal{F}^{-1}\left(\frac{\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}\right)ds.$$

Dado $u_{\epsilon,\beta,0} \in H^2(\mathbb{R})$ consideremos o espaço de Banach

$$X_T = \{u \in C([0,T],H^2(\mathbb{R})); \|u(t) - G(t)u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})} \leq \|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}, t \in [0,T]\}$$

com a norma

$$\|u\|_{X_T} = \sup_{0\leq t\leq T}\|u(t)\|_{H^2(\mathbb{R})},$$

e definamos o seguinte operador em X_T:

$$\Lambda u(t) = G(t)u_{\epsilon,\beta,0} - \int_0^t G(t-s)\mathcal{F}^{-1}\left(\frac{\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}\right)ds.$$

O lema a seguir nos fornecerá informações locais sobre o operador Λ. Na sua demonstração (e na do Teorema 2.1) denotaremos com C_0 as constantes que dependem apenas dos parâmetros ϵ, β e dos coeficientes b_1, b_2 e c em (2.1).

Lema 2.1. *Supondo $u_{\epsilon,\beta,0} \in H^2(\mathbb{R})$, existe $T = T(u_{\epsilon,\beta,0}) > 0$ tal que as seguintes afirmações são válidas:*

(i) $\Lambda u \in X_T$ e $\|\Lambda u(t)\|_{H^2(\mathbb{R})} \leq 2\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}$ para todo $t \in [0,T]$ se $u \in X_T$;

(ii) Λ é uma contração em X_T.

Demonstração. Assumiremos (sem perda de generalidade) $f(0) = 0$ e seja $u \in X_T$. Dado $t \in [0,T]$ (T será escolhido a posteriori), observe que

$$\begin{aligned}
\|G(t)u_{\epsilon,\beta,0}\|^2_{H^2(\mathbb{R})} &= \sum_{k=0}^{2}\int_{\mathbb{R}}\left|\mathcal{F}^{-1}\left((-i\xi)^kQ(t,\xi)\mathcal{F}(u_{\epsilon,\beta,0})\right)\right|^2d\xi \\
&= \sum_{k=0}^{2}\int_{\mathbb{R}}\left|(i\xi)^kQ(t,\xi)\mathcal{F}(u_{\epsilon,\beta,0})\right|^2d\xi \quad (2.5)\\
&\leq \sum_{k=0}^{2}\int_{\mathbb{R}}|\mathcal{F}(\partial_x^k u_{\epsilon,\beta,0})|^2d\xi
\end{aligned}$$

$$= \sum_{k=0}^{2} \int_{\mathbb{R}} |\partial_x^k u_{\epsilon,\beta,0}|^2 d\xi$$
$$= \|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}^2$$

já que $|Q| \leq 1$. Logo, $G(t)u_{\epsilon,\beta,0} \in H^2(\mathbb{R})$ e consequentemente $\|u(t)\|_{H^2(\mathbb{R})} \leq 2\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}$. Além disso, pelo Lema de Sobolev $u(t)$ se anula no infinito, e portanto

$$\begin{aligned} u^2(t,y) &= 2\int_{-\infty}^{y} u(t,x)\partial_x u(t,x)dx \\ &\leq 2\|u(t)\|_{L^2(\mathbb{R})}\|\partial_x u(t)\|_{L^2(\mathbb{R})} \\ &\leq \|u(t)\|_{H^2(\mathbb{R})}^2, \end{aligned}$$

donde

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq 2\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})} \qquad t \in [0,T]. \tag{2.6}$$

A condição (2.6) juntamente com o Teorema 1.2 e o Corolário 1.1 garantem a existência de uma constante $K_0 > 0$ (dependente apenas da cota $2\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}$) tal que

$$\|f(u(t))\|_{H^2(\mathbb{R})} \leq K_0\|u(t)\|_{H^2(\mathbb{R})} \tag{2.7}$$

e

$$\|f(u(t)) - f(v(t))\|_{H^2(\mathbb{R})} \leq K_0\|u(t)-v(t)\|_{H^2(\mathbb{R})}(|f'(0)| + \|u(t)\|_{H^2(\mathbb{R})} + \|v(t)\|_{H^2(\mathbb{R})}) \tag{2.8}$$

se $u, v \in X_T$.

Agora observe que

$$p(\xi) = \frac{\xi^2}{(1+\beta|b_2|\xi^2 + \beta^2 c\xi^4)^2} \leq 1 + (\beta^2 c)^{-1} \quad (\xi \in \mathbb{R}) \tag{2.9}$$

pois $p(\xi) \leq 1$ se $|\xi| \leq 1$, e para $|\xi| \geq 1$

$$p(\xi) \leq \frac{\xi^4}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \leq \frac{\xi^4}{\beta^2 c\xi^4} = (\beta^2 c)^{-1}.$$

Assim, (2.7) e (2.9) implicam que

$$\begin{aligned} \|\Lambda u(t) - G(t)u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})} &\leq \int_0^t \left\{ \sum_{k=0}^{2} \int_{\mathbb{R}} \left| \frac{(i\xi)^k Q(t-s,\xi)\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right|^2 d\xi \right\}^{1/2} ds \\ &\leq \int_0^t \left\{ \sum_{k=0}^{2} \int_{\mathbb{R}} \left| \frac{(i\xi)^k \mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right|^2 d\xi \right\}^{1/2} ds \\ &= \int_0^t \left\{ \sum_{k=0}^{2} \int_{\mathbb{R}} \frac{\xi^2|\mathcal{F}(\partial_x^k f(u(s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2} d\xi \right\}^{1/2} ds \end{aligned}$$

$$
\begin{aligned}
&\leq (1+(\beta^2 c)^{-1})^{1/2} \int_0^t \left\{ \sum_{k=0}^{2} \int_{\mathbb{R}} |\mathcal{F}(\partial_x^k f(u(s)))|^2 d\xi \right\}^{1/2} ds \\
&= (1+(\beta^2 c)^{-1})^{1/2} \int_0^t \|f(u(s))\|_{H^2(\mathbb{R})} ds \\
&\leq (1+(\beta^2 c)^{-1})^{1/2} K_0 \int_0^t \|u(s)\|_{H^2(\mathbb{R})} ds \\
&\leq 2(1+(\beta^2 c)^{-1})^{1/2} K_0 \|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})} t \\
&\leq \|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}
\end{aligned}
$$

se

$$
0 < T \leq \frac{1}{2(1+(\beta^2 c)^{-1})^{1/2} K_0}. \tag{2.10}
$$

Segue de (2.5) que $\Lambda u(t) \in H^2(\mathbb{R})$ e $\|\Lambda u(t)\|_{H^2(\mathbb{R})} \leq 2\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}$.

A seguir mostraremos que $\Lambda u \in C([0,T], H^2(\mathbb{R}))$. Dados $0 \leq t_0 \leq t < T$, temos

$$
\begin{aligned}
\|\Lambda u(t) - \Lambda u(t_0)\|_{H^2(\mathbb{R})} &\leq \|G(t)u_{\epsilon,\beta,0} - G(t_0)u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})} \\
&\quad + \left\| \int_0^t G(t-s)\mathcal{F}^{-1}\left(\frac{\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4}\right) ds \right. \\
&\quad \left. - \int_0^{t_0} G(t_0-s)\mathcal{F}^{-1}\left(\frac{\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4}\right) ds \right\|_{H^2(\mathbb{R})} \\
&\leq \|G(t)u_{\epsilon,\beta,0} - G(t_0)u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})} \\
&\quad + \left\| \int_{t_0}^t G(s)\mathcal{F}^{-1}\left(\frac{\mathcal{F}(\partial_x f(u(t-s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4}\right) ds \right\|_{H^2(\mathbb{R})} \\
&\quad + \left\| \int_0^{t_0} G(s)\mathcal{F}^{-1}\left(\frac{\mathcal{F}[\partial_x(f(u(t-s)) - f(u(t_0-s)))]}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4}\right) ds \right\|_{H^2(\mathbb{R})} \\
&= A_1 + A_2 + A_3.
\end{aligned}
$$

Em primeiro lugar, observemos que

$$
q(\xi) = \left| \frac{\epsilon\xi^2 - i\beta b_1 \xi^3}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right|^2 \ \textit{é limitada} \tag{2.11}
$$

pois

$$
q(\xi) = \frac{\epsilon^2\xi^4 + \beta^2 b_1^2 \xi^6}{(1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2} \leq \epsilon^2 + \beta^2 b_1^2
$$

para $|\xi| \leq 1$, e

$$
q(\xi) \leq \frac{(\epsilon^2 + \beta^2 b_1^2)\xi^8}{\beta^4 c^2 \xi^8} = (\epsilon^2 + \beta^2 b_1^2)(\beta^4 c^2)^{-1}
$$

para $|\xi| \geq 1$.

Logo a desigualdade do Valor Médio e (2.11) implicam que

$$
\begin{aligned}
A_1^2 &= \sum_{k=0}^{2} \int_{\mathbb{R}} |Q(t,\xi) - Q(t_0,\xi)|^2 |\mathcal{F}(\partial_x^k u_{\epsilon,\beta,0})|^2 d\xi \\
&\leq \sum_{k=0}^{2} \int_{\mathbb{R}} \left| \frac{\epsilon\xi^2 - i\beta b_1 \xi^3}{1+\beta|b_2|\xi^2 + \beta^2 c\xi^4} \right|^2 |t-t_0|^2 |\mathcal{F}(\partial_x^k u_{\epsilon,\beta,0})|^2 d\xi \\
&\leq C_0 |t-t_0|^2 \sum_{k=0}^{2} \int_{\mathbb{R}} |\mathcal{F}(\partial_x^k u_{\epsilon,\beta,0})|^2 d\xi \\
&= C_0 \|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}^2 |t-t_0|^2,
\end{aligned} \tag{2.12}
$$

e por (2.7) e (2.9)

$$
\begin{aligned}
A_2 &\leq \int_{t_0}^{t} \left\| G(s)\mathcal{F}^{-1} \left(\frac{\mathcal{F}(\partial_x f(u(t-s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right) \right\|_{H^2(\mathbb{R})} ds \\
&= \int_{t_0}^{t} \left\{ \sum_{k=0}^{2} \int_{\mathbb{R}} \left| Q(s,\xi) \frac{(i\xi)^k \mathcal{F}(\partial_x f(u(t-s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right|^2 dx \right\}^{1/2} ds \\
&\leq \int_{t_0}^{t} \left\{ \sum_{k=0}^{2} \int_{\mathbb{R}} \left| \frac{(i\xi)^k \mathcal{F}(\partial_x f(u(t-s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right|^2 dx \right\}^{1/2} ds \\
&= \int_{t_0}^{t} \left\{ \sum_{k=0}^{2} \int_{\mathbb{R}} \frac{\xi^2 |\mathcal{F}(\partial_x^k f(u(t-s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2} d\xi \right\}^{1/2} ds \\
&\leq C_0 \int_{t_0}^{t} \left\{ \sum_{k=0}^{2} \int_{\mathbb{R}} |\partial_x^k f(u(t-s))|^2 d\xi \right\}^{1/2} ds \\
&= C_0 \int_{t_0}^{t} \|f(u(t-s))\|_{H^2(\mathbb{R})} ds \\
&\leq C_0 K_0 \int_{t_0}^{t} \|u(t-s)\|_{H^2(\mathbb{R})} ds \\
&\leq C_0 K_0 \|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})} |t-t_0|.
\end{aligned} \tag{2.13}
$$

Além disso, por (2.8) e (2.9) temos

$$
\begin{aligned}
A_3 &\leq \int_{0}^{t_0} \left\| G(s)\mathcal{F}^{-1} \left(\frac{\mathcal{F}[\partial_x(f(u(t-s)) - f(u(t_0-s)))]}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right) \right\|_{H^2(\mathbb{R})} ds \\
&\leq \int_{0}^{t_0} \left\{ \sum_{k=0}^{2} \int_{\mathbb{R}} \frac{\xi^2 |\mathcal{F}[\partial_x^k(f(u(t-s)) - f(u(t_0-s)))]|^2}{(1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2} d\xi \right\}^{1/2} ds \\
&\leq C_0 \int_{0}^{t_0} \|f(u(t-s)) - f(u(t_0-s))\|_{H^2(\mathbb{R})} ds \\
&\leq C_0 K_0 \int_{0}^{t_0} \|u(t-s) - u(t_0-s)\|_{H^2(\mathbb{R})} (|f'(0)| + \|u(t-s)\|_{H^2(\mathbb{R})} + \|u(t_0-s)\|_{H^2(\mathbb{R})}) ds \\
&\leq C_0 K_0 (|f'(0)| + 4\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}) \int_{0}^{t_0} \|u(t-s) - u(t_0-s)\|_{H^2(\mathbb{R})} ds.
\end{aligned}
$$

A condição $u \in C([0,T]; H^2(\mathbb{R}))$ implica que a última integral converge para zero e consequentemente $A_3 \to 0$ quando $t \to t_0^+$. Este fato, juntamente com (2.12) e (2.13) nos permite concluir que

$$\lim_{t \to t_0^+} \|\Lambda u(t) - \Lambda u(t_0)\|_{H^2(\mathbb{R})} = 0.$$

Sendo o caso $0 < t \leq t_0 \leq T$ análogo segue-se que $\Lambda u \in C([0,T]; H^2(\mathbb{R}))$.

Para finalizar a demonstração provaremos que Λ é uma contração em X_T. Sejam então u e v dois elementos de X_T. Dado $t \in [0,T]$, (2.8) e (2.9) implicam que

$$\begin{aligned}
\|\Lambda u(t) - \Lambda v(t)\|_{H^2(\mathbb{R})} &\leq \int_0^t \left\| G(t-s)\mathcal{F}^{-1}\left(\frac{\mathcal{F}[\partial_x(f(u(s)) - f(v(s))]}{1+\beta|b_2|\xi^2 + \beta^2 c\xi^4}\right)\right\|_{H^2(\mathbb{R})} ds \\
&\leq \int_0^t \left\{ \sum_{k=0}^{2} \int_{\mathbb{R}} \frac{\xi^2 |\mathcal{F}[\partial_x^k(f(u(s)) - f(v(s))]|^2}{(1+\beta|b_2|\xi^2 + \beta^2 c\xi^4)^2} d\xi \right\}^{1/2} ds \\
&\leq (1 + (\beta^2 c)^{-1})^{1/2} \int_0^t \|f(u(s)) - f(v(s))\|_{H^2(\mathbb{R})} ds \\
&\leq (1 + (\beta^2 c)^{-1})^{1/2} K_0(|f'(0)| + 4\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}) \int_0^t \|u(s) - v(s)\|_{H^2(\mathbb{R})} ds \\
&\leq (1 + (\beta^2 c)^{-1})^{1/2} K_0(|f'(0)| + 4\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}) \|u - v\|_{X_T} t \\
&\leq \frac{1}{2}\|u - v\|_{X_T}
\end{aligned}$$

se

$$0 < T \leq \frac{1}{2(1 + (\beta^2 c)^{-1})^{1/2} K_0(|f'(0)| + 4\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})})} \tag{2.14}$$

mostrando que Λ é uma contração. Portanto o resultado será válido se escolhermos (qualquer) T satisfazendo as condições (2.10) e (2.14). □

Considerando T como no lema anterior, podemos então enunciar o seguinte

Teorema 2.1. *Se $u_{\epsilon,\beta,0} \in H^5(\mathbb{R})$, o problema de Cauchy (2.1)–(2.2) admite uma solução*

$$u \in C([0,T], H^5(\mathbb{R})) \cap C^1((0,T], H^5(\mathbb{R}))$$

tal que $\|u(t)\|_{H^2(\mathbb{R})} \leq 2\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}$ para todo $t \in [0,T]$.

Demonstração. O lema anterior juntamente com o Teorema do Ponto Fixo de Banach garantem a existência de um único elemento $u \in X_T$ tal que $\Lambda u = u$. Assim, u será uma solução da equação integral

$$u(t) = G(t)u_{\epsilon,\beta,0} - \int_0^t G(t-s)\mathcal{F}^{-1}\left(\frac{\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2 + \beta^2 c\xi^4}\right) ds, \tag{2.15}$$

e cumpre a condição $\|u(t)\|_{H^2(\mathbb{R})} \leq 2\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}$. Sua norma em $H^5(\mathbb{R})$ satisfaz

$$\begin{aligned}
\|u(t)\|_{H^5(\mathbb{R})} &\leq \|G(t)u_{\epsilon,\beta,0}\|_{H^5(\mathbb{R})} + \int_0^t \left\| G(t-s)\mathcal{F}^{-1}\left(\frac{\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2 + \beta^2 c\xi^4}\right)\right\|_{H^5(\mathbb{R})} ds \\
&= A_1 + A_2.
\end{aligned}$$

Estimando como em (2.5)

$$A_1 = \|G(t)u_{\epsilon,\beta,0}\|_{H^5(\mathbb{R})} \leq \|u_{\epsilon,\beta,0}\|_{H^5(\mathbb{R})}.$$

Em seguida, usando (2.7) e a estimativa (devido a (2.9))

$$\begin{aligned}
\sum_{k=0}^{5}\int_{\mathbb{R}} \frac{\xi^2|\mathcal{F}(\partial_x^k f(u(s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2c\xi^4)^2}d\xi &\leq C_0\|f(u(s))\|_{H^2(\mathbb{R})}^2 + \sum_{k=3}^{5}\int_{\mathbb{R}} \frac{\xi^2|\mathcal{F}(\partial_x^k f(u(s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2c\xi^4)^2}d\xi \\
&= C_0\|f(u(s))\|_{H^2(\mathbb{R})}^2 + \sum_{k=0}^{2}\int_{\mathbb{R}} \frac{\xi^8|\mathcal{F}(\partial_x^k f(u(s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2c\xi^4)^2}d\xi \qquad (2.16)\\
&\leq C_0\|f(u(s))\|_{H^2(\mathbb{R})}^2 + (\beta^4c^2)^{-1}\|f(u(s))\|_{H^2(\mathbb{R})}^2 \\
&\leq C_0\|f(u(s))\|_{H^2(\mathbb{R})}^2,
\end{aligned}$$

segue-se que

$$\begin{aligned}
A_2 &\leq \int_0^t \left\{\sum_{k=0}^{5}\int_{\mathbb{R}} \frac{\xi^2|\mathcal{F}(\partial_x^k f(u(s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2c\xi^4)^2}d\xi\right\}^{1/2} ds \\
&\leq C_0\int_0^t \|f(u(s))\|_{H^2(\mathbb{R})}ds \\
&\leq C_0K_0\int_0^t \|u(s)\|_{H^2(\mathbb{R})}ds \\
&\leq C_0K_0\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}T,
\end{aligned}$$

e portanto $u(t) \in H^5(\mathbb{R})$ para todo $t \in [0,T]$.
A prova de $u \in C([0,T], H^5(\mathbb{R}))$ é análoga à prova de $\Lambda u \in C([0,T], H^2(\mathbb{R}))$ estabelecida no Lema 2.1. A única diferença é a necessidade da estimativa (2.16) na argumentação.

Agora mostraremos que $\partial_t u \in C((0,T]; H^5(\mathbb{R}))$. Em primeiro lugar, observemos que a partir de (2.3)

$$\mathcal{F}(\partial_t u) = -\left(\frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}\right)\mathcal{F}(u) - \frac{\mathcal{F}(\partial_x f(u))}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}.$$

Logo, substituindo (2.4) na igualdade acima resulta que

$$\begin{aligned}
\partial_t u(t) = &- \mathcal{F}^{-1}\left[\left(\frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}\right)Q(t,\xi)\mathcal{F}(u_{\epsilon,\beta,0})\right] \\
&+ \int_0^t \mathcal{F}^{-1}\left[\left(\frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}\right)Q(t-s,\xi)\frac{\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}\right]ds \\
&- \mathcal{F}^{-1}\left(\frac{\mathcal{F}(\partial_x f(u(t)))}{1+\beta|b_2|\xi^2+\beta^2c\xi^4}\right)
\end{aligned}$$

donde

$$
\begin{aligned}
\|\partial_t u(t)\|_{H^5(\mathbb{R})} \le & \left\| \mathcal{F}^{-1}\left[\left(\frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4}\right) Q(t,\xi)\mathcal{F}(u_{\epsilon,\beta,0})\right]\right\|_{H^5(\mathbb{R})} \\
& + \int_0^t \left\| \mathcal{F}^{-1}\left[\left(\frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4}\right) Q(t-s,\xi)\frac{\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4}\right]\right\|_{H^5(\mathbb{R})} ds \\
& + \left\| \mathcal{F}^{-1}\left(\frac{\mathcal{F}(\partial_x f(u(t)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4}\right)\right\|_{H^5(\mathbb{R})} \\
= & B_1 + B_2 + B_3.
\end{aligned}
$$

A partir de (2.11)

$$
\begin{aligned}
B_1^2 &\le \sum_{k=0}^{5}\int_{\mathbb{R}} \left|\frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4}\right|^2 |\mathcal{F}(\partial_x^k u_{\epsilon,\beta,0})|^2 d\xi \\
&\le C_0 \sum_{k=0}^{5}\int_{\mathbb{R}} |\mathcal{F}(\partial_x^k u_{\epsilon,\beta,0})|^2 d\xi \\
&= C_0\|u_{\epsilon,\beta,0}\|^2_{H^5(\mathbb{R})}.
\end{aligned}
$$

Além disso, de (2.7), (2.9) e (2.16) resulta que

$$
\begin{aligned}
B_2 &\le C_0\int_0^t \left\{\sum_{k=0}^{5}\int_{\mathbb{R}} \frac{\xi^2|\mathcal{F}(\partial_x^k f(u(s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2} d\xi\right\}^{1/2} ds \\
&\le C_0\int_0^t \|f(u(s))\|_{H^2(\mathbb{R})} ds \\
&\le C_0 K_0\int_0^t \|u(s)\|_{H^2(\mathbb{R})} ds \\
&\le C_0 K_0 \|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})} T,
\end{aligned}
$$

e

$$
\begin{aligned}
B_3^2 &= \sum_{k=0}^{5}\int_{\mathbb{R}} \frac{\xi^2|\mathcal{F}(\partial_x^k f(u(t)))|^2}{(1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2} d\xi \\
&\le C_0\|f(u(s))\|^2_{H^2(\mathbb{R})} \\
&\le C_0 K_0^2\|u(s)\|^2_{H^2(\mathbb{R})} \\
&\le C_0 K_0^2\|u_{\epsilon,\beta,0}\|^2_{H^2(\mathbb{R})}.
\end{aligned}
$$

Consequentemente

$$\|\partial_t u(t)\|_{H^5(\mathbb{R})} < \infty$$

para todo $t \in (0,T]$.

Agora sejam $0 < t_0 \leq t < T$. Então

$$\begin{aligned}
&\|\partial_t u(t) - \partial_t u(t_0)\|_{H^5(\mathbb{R})} \\
&\quad \leq \left\| \mathcal{F}^{-1}\left[[Q(t,\xi) - Q(t_0,\xi)] \left(\frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right) \mathcal{F}(u_{\epsilon,\beta,0}) \right] \right\|_{H^5(\mathbb{R})} \\
&\quad + \left\| \int_0^t \mathcal{F}^{-1}\left[Q(t-s,\xi) \left(\frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right) \frac{\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right] ds \right. \\
&\quad - \left. \int_0^{t_0} \mathcal{F}^{-1}\left[Q(t_0-s.\xi) \left(\frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right) \frac{\mathcal{F}(\partial_x f(u(s)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right] ds \right\|_{H^5(\mathbb{R})} \\
&\quad + \left\| \mathcal{F}^{-1}\left[\frac{\mathcal{F}(\partial_x f(u(t))) - \mathcal{F}(\partial_x f(u(t_0)))}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right] \right\|_{H^5(\mathbb{R})} \\
&\quad \leq \left\{ \sum_{k=0}^5 \int_{\mathbb{R}} |Q(t,\xi) - Q(t_0,\xi)|^2 \left| \frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right|^2 |\mathcal{F}(\partial_x^k u_{\epsilon,\beta,0})|^2 d\xi \right\}^{1/2} \\
&\quad + \int_{t_0}^t \left\{ \sum_{k=0}^5 \int_{\mathbb{R}} \left| \frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right|^2 \frac{\xi^2|\mathcal{F}(\partial_x^k f(u(t-s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2} d\xi \right\}^{1/2} ds \\
&\quad + \int_0^{t_0} \left\{ \sum_{k=0}^5 \int_{\mathbb{R}} \left| \frac{\epsilon\xi^2 - i\beta b_1\xi^3}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right|^2 \frac{\xi^2|\mathcal{F}(\partial_x^k (f(u(t-s)) - f(u(t_0-s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2} d\xi \right\}^{1/2} ds \\
&\quad + \left\{ \sum_{k=0}^5 \frac{\xi^2|\mathcal{F}(\partial_x^k (f(u(t)) - f(u(t_0))))|^2}{(1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2} d\xi \right\}^{1/2} \\
&\quad = D_1 + D_2 + D_3 + D_4.
\end{aligned}$$

Pela desigualdade do Valor Médio e (2.11)

$$\begin{aligned}
D_1^2 &\leq \sum_{k=0}^5 \int_{\mathbb{R}} \left| \frac{\epsilon\xi^2 - i\beta b_1\xi^2}{1+\beta|b_2|\xi^2+\beta^2 c\xi^4} \right|^4 |t-t_0|^2 |\mathcal{F}(\partial_x^k u_{\epsilon,\beta,0})|^2 d\xi \\
&\leq C_0 |t-t_0|^2 \|u_{\epsilon,\beta,0}\|^2_{H^5(\mathbb{R})}
\end{aligned}$$

e por (2.7), (2.11) e (2.16)

$$\begin{aligned}
D_2 &\leq C_0 \int_0^t \left\{ \sum_{k=0}^5 \int_{\mathbb{R}} \frac{\xi^2|\mathcal{F}(\partial_x^k f(u(t-s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2} d\xi \right\}^{1/2} ds \\
&\leq C_0 \int_{t_0}^t \|f(u(t-s))\|_{H^2(\mathbb{R})} ds \\
&\leq C_0 K_0 \int_{t_0}^t \|u(t-s)\|_{H^2(\mathbb{R})} ds \\
&\leq C_0 K_0 \|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})} |t-t_0|.
\end{aligned}$$

As estimativas para D_3 e D_4 são feitas utilizando (2.8), (2.11) e (2.16):

$$
\begin{aligned}
D_3 &\leq C_0 \int_0^{t_0} \left\{ \sum_{k=0}^{5} \int_{\mathbb{R}} \frac{\xi^2 |\mathcal{F}(\partial_x^k (f(u(t-s)) - f(u(t_0-s)))|^2}{(1+\beta|b_2|\xi^2 + \beta^2 c\xi^4)^2} d\xi \right\}^{1/2} ds \\
&\leq C_0 \int_0^{t_0} \|f(u(t-s)) - f(u(t_0-s))\|_{H^2(\mathbb{R})} ds \\
&\leq C_0 K_0 (|f'(0)| + 4\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}) \int_0^{t_0} \|u(t-s) - u(t_0-s)\|_{H^2(\mathbb{R})} ds,
\end{aligned}
$$

e

$$
\begin{aligned}
D_4 &\leq C_0 \|f(u(t)) - f(u(t_0))\|_{H^2(\mathbb{R})} \\
&\leq C_0 K_0 (|f'(0)| + 4\|u_{\epsilon,\beta,0}\|_{H^2(\mathbb{R})}) \|u(t) - u(t_0)\|_{H^2(\mathbb{R})}.
\end{aligned}
$$

Uma vez que $u \in C([0,T]; H^2(\mathbb{R})), D_3$ e $D_4 \to 0$ quando $t \to t_0^+$, e portanto

$$
\lim_{t \to t_0^+} \|\partial_t u(t) - \partial_t u(t_0)\|_{H^5(\mathbb{R})} = 0.
$$

Sendo o caso $0 < t \leq t_0 \leq T$ tratado analogamente, podemos concluir a continuidade de $\partial_t u$ em todo o intervalo $(0, T]$. □

Observe que $u(t) \in C_0^4(\mathbb{R})$ pelo Lema de Sobolev e que a estimativa feita em (2.16) não funcionaria nos espaços $H^N(\mathbb{R})$ para $N > 5$, pois o grau do polinômio $p(\xi) = (1+\beta|b_2|\xi^2+\beta^2 c\xi^4)^2$ é 8.

A fim de estender esta solução para todo $t \geq 0$ precisaremos estabelecer dois lemas técnicos. Começaremos supondo a existência de uma constante $C_0 > 0$ satisfazendo

$$
\|u_{\epsilon,\beta,0}\|_{L^2(\mathbb{R})}^2 + \beta^{1/2} \|\partial_x u_{\epsilon,\beta,0}\|_{L^2(\mathbb{R})}^2 + \beta^{3/2} \|\partial_x^2 u_{\epsilon,\beta,0}\|_{L^2(\mathbb{R})}^2 + \beta^{5/2} \|\partial_x^3 u_{\epsilon,\beta,0}\|_{L^2(\mathbb{R})}^2 \leq C_0. \tag{2.17}
$$

Lema 2.2. *Assuma a condição* (2.17). *Se u for uma solução do problema de Cauchy* (2.1)-(2.2) *em $[0, t_1] \times \mathbb{R}$ tal que $u \in C([0, t_1], H^5(\mathbb{R}))$, então existe uma constante $C_0 > 0$ (independente do tempo) tal que*

$$
\|u(t,.)\|_{L^2(\mathbb{R})}^2 + \beta|b_2| \|\partial_x u(t,.)\|_{L^2(\mathbb{R})}^2 + \beta^2 c \|\partial_x^2 u(t,.)\|_{L^2(\mathbb{R})}^2 + 2\epsilon \int_0^t \|\partial_x u(s,.)\|_{L^2(\mathbb{R})}^2 ds \leq C_0
$$

para todo $t \in [0, t_1]$.

Demonstração. Multiplicando (2.1) por u temos

$$
u\partial_t u + u\partial_x f(u) + \beta b_1 u \partial_x^3 u + \beta b_2 u \partial_t \partial_x^2 u + \beta^2 c u \partial_t \partial_x^4 u = \epsilon u \partial_x^2 u.
$$

Integrando cada termo da expressão acima em $\mathbb{R}$ e usando o fato que cada $u(t)$ se anula no infinito obtemos as seguintes igualdades:

$$
\int_{\mathbb{R}} u\partial_t u dx = \frac{1}{2} \int_{\mathbb{R}} \frac{d}{dt}(u^2) dx = \frac{1}{2} \frac{d}{dt} \|u(t,\cdot)\|_{L^2(\mathbb{R})}^2;
$$

$$\begin{aligned}\int_{\mathbb{R}} u\partial_x f(u)dx &= -\int_{\mathbb{R}} \partial_x u f(u)dx = -\int_{\mathbb{R}} \partial_x u g'(u)dx \\ &= -\int_{\mathbb{R}} \partial_x g(u)dx = 0,\end{aligned}$$

onde $g' = f$;

$$\begin{aligned}\beta b_1 \int_{\mathbb{R}} u\partial_x^3 u dx &= -\beta b_1 \int_{\mathbb{R}} \partial_x u \partial_x^2 u dx = \frac{\beta b_1}{2}\int_{\mathbb{R}} \partial_x(\partial_x u)^2 dx \\ &= 0;\end{aligned}$$

$$\begin{aligned}\beta b_2 \int_{\mathbb{R}} u\partial_t\partial_x^2 u dx &= -\beta b_2 \int_{\mathbb{R}} \partial_x u \partial_t \partial_x u dx = \frac{-\beta b_2}{2}\int_{\mathbb{R}} \frac{d}{dt}((\partial_x u)^2) dx \\ &= \frac{\beta|b_2|}{2}\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})};\end{aligned}$$

$$\begin{aligned}\beta^2 c \int_{\mathbb{R}} u\partial_t\partial_x^4 u dx &= -\beta^2 c \int_{\mathbb{R}} \partial_x u \partial_t \partial_x^3 u dx = \beta^2 c \int_{\mathbb{R}} \partial_x^2 u \partial_t \partial_x^2 u dx \\ &= \frac{\beta^2 c}{2}\int_{\mathbb{R}} \frac{d}{dt}((\partial_x^2 u)^2) dx = \frac{\beta^2 c}{2}\frac{d}{dt}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})};\end{aligned}$$

$$\epsilon \int_{\mathbb{R}} u\partial_x^2 u dx = -\epsilon \int_{\mathbb{R}} (\partial_x u)^2 dx = -\epsilon\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Assim,

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta|b_2|\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 c\frac{d}{dt}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ + 2\epsilon\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} = 0.\end{aligned} \tag{2.18}$$

Integrando (2.18) em $(0,t)$ com $t \in (0,t_1]$ e utilizando (2.17), segue-se que

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ \beta|b_2|\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 c\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &+ 2\epsilon\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \\ &= \|u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta|b_2|\|\partial_x u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta^2 c\|\partial_x^2 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \\ &\leq C_0\end{aligned} \tag{2.19}$$

estabelecendo o resultado. □

Assumiremos agora a condição

$$|f'(u)| \leq C_0(1+|u|) \qquad u \in \mathbb{R}. \tag{2.20}$$

Lema 2.3. *Assuma as condições* (2.17) *e* (2.20) *e suponha* $\beta = O(\epsilon^4)$. *Se* u *for uma solução do problema de Cauchy* (2.1)-(2.2) *em* $[0,t_1]\times\mathbb{R}$ *tal que* $u \in C([0,t_1], H^5(\mathbb{R}))$, *então existe uma constante* $C_0 > 0$ *(independente do tempo) tal que*

$$\|u\|_{L^\infty([0,t_1]\times\mathbb{R})} \leq C_0\beta^{-1/4} \tag{2.21}$$

e

$$\begin{aligned} \beta\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2|b_2|\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^3 c\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ + \beta\epsilon\int_0^t \|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \le C_0 \end{aligned} \tag{2.22}$$

para todo $t \in [0, t_1]$.

Demonstração. Multiplicando (2.1) por $-\beta^{1/2}\partial_x^2 u$ e integrando cada termo obtido em $\mathbb{R}$, obtemos as seguintes igualdades:

$$\begin{aligned} -\beta^{1/2}\int_{\mathbb{R}} \partial_x^2 u \partial_t u dx &= \frac{\beta^{1/2}}{2}\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}; \\ -\beta^{1/2}\int_{\mathbb{R}} \partial_x^2 u \partial_x f(u) dx &= -\beta^{1/2}\int_{\mathbb{R}} \partial_x u \partial_x^2 u f'(u) dx; \\ -\beta^{3/2} b_1 \int_{\mathbb{R}} \partial_x^3 u \partial_x^2 u dx &= 0; \\ -\beta^{3/2} b_2 \int_{\mathbb{R}} \partial_t \partial_x^2 u \partial_x^2 u dx &= \frac{\beta^{3/2}|b_2|}{2}\frac{d}{dt}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}; \\ -\beta^{5/2} c \int_{\mathbb{R}} \partial_t \partial_x^4 u \partial_x^2 u dx &= \frac{\beta^{5/2} c}{2}\frac{d}{dt}\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})}; \\ -\beta^{1/2}\epsilon \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 u dx &= -\beta^{1/2}\epsilon\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Portanto,

$$\begin{aligned} \beta^{1/2}\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^{3/2}|b_2|\frac{d}{dt}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^{5/2} c\frac{d}{dt}\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ + 2\beta^{1/2}\epsilon\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} = 2\beta^{1/2}\int_{\mathbb{R}} \partial_x u \partial_x^2 u f'(u) dx. \end{aligned} \tag{2.23}$$

Pela condição (2.20) e a desigualdade

$$ab \le \frac{1}{2}(a^2 + b^2) \quad a, b \in \mathbb{R}$$

temos

$$\begin{aligned} \beta^{1/2}\int_{\mathbb{R}} \partial_x u \partial_x^2 u f'(u) dx &\le \beta^{1/2}\int_{\mathbb{R}} |\partial_x u \partial_x^2 u f'(u)| dx \\ &= \beta^{1/2}\int_{\mathbb{R}} |\epsilon^{1/2}\partial_x^2 u||\epsilon^{-1/2}\partial_x u f'(u)| dx \\ &\le \frac{\beta^{1/2}\epsilon}{2}\int_{\mathbb{R}} |\partial_x^2 u|^2 dx + \frac{\beta^{1/2}}{2\epsilon}\int_{\mathbb{R}} |\partial_x u f'(u)|^2 dx \\ &\le \frac{\beta^{1/2}\epsilon}{2}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{C_0\beta^{1/2}}{\epsilon}\int_{\mathbb{R}} |\partial_x u|^2(1 + |u|)^2 dx \\ &\le \frac{\beta^{1/2}\epsilon}{2}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{C_0\beta^{1/2}}{\epsilon}\int_{\mathbb{R}} |\partial_x u|^2(1 + |u|^2) dx \\ &\le \frac{\beta^{1/2}\epsilon}{2}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{C_0\beta^{1/2}}{\epsilon}(1 + \|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})})\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{2.24}$$

Substituindo (2.24) em (2.23) e usando a hipótese $\beta = O(\epsilon^4)$ segue-se que

$$\begin{aligned}
&\beta^{1/2}\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^{3/2}|b_2|\frac{d}{dt}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^{5/2}c\frac{d}{dt}\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + \beta^{1/2}\epsilon\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad \leq \frac{C_0\beta^{1/2}}{\epsilon}(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})})\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad \leq C_0\epsilon(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})})\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{2.25}$$

Uma integração de (2.25) em $(0,t)$ com $t\in(0,t_1]$, (2.17) e o Lema 2.2 fornecem

$$\begin{aligned}
&\beta^{1/2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^{3/2}|b_2|\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^{5/2}c\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + \beta^{1/2}\epsilon\int_0^t\|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad \leq C_0\epsilon(1+\|u\|^2_{L^\infty[0,t_1]\times\mathbb{R})})\int_0^t\|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \beta^{1/2}\|\partial_x u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})}\\
&\quad + \beta^{3/2}|b_2|\|\partial_x^2 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta^{5/2}c\|\partial_x^3 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})}\\
&\quad \leq C_0(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})}).
\end{aligned} \tag{2.26}$$

A seguir mostraremos (2.21). Em primeiro lugar, observe que (2.26) e o Lema 2.2 implicam em

$$\begin{aligned}
u^2(t,x) &\leq 2\|u(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}\\
&\leq C_0\beta^{-1/4}(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})})^{1/2},
\end{aligned} \tag{2.27}$$

donde

$$\|u\|^4_{L^\infty([0,t_1]\times\mathbb{R})} \leq C_0\beta^{-1/2}(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})}). \tag{2.28}$$

Pondo $y = \|u\|_{L^\infty([0,t_1]\times\mathbb{R})}$, (2.28) equivale à

$$y^4 \leq C_0\beta^{-1/2}(1+y^2), \tag{2.29}$$

a qual por sua vez acarreta (2.21). Com efeito, se $y\in[0,1]$, obviamente $y\leq\beta^{-1/4}$. Se for $y>1$ (note que $y<\infty$ pois $u\in C([0,t_1],H^5(\mathbb{R}))$), (2.29) implica em $y^4\leq C_0\beta^{-1/2}(1+y^2)\leq 2C_0\beta^{-1/2}y^2$. Logo, $y^2\leq 2C_0\beta^{-1/2}$ e consequentemente $y\leq(2C_0)^{1/2}\beta^{-1/4}$. Assim, $y\leq\max\{1,(2C_0)^{1/2}\}\beta^{-1/4}$ o que estabelece a afirmação.

Agora, a partir de (2.21) e (2.26)

$$\begin{aligned}
&\beta^{1/2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^{3/2}|b_2|\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^{5/2}c\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + \beta^{1/2}\epsilon\int_0^t\|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad \leq C_0(1+C_0\beta^{-1/2})\\
&\quad \leq C_0\beta^{-1/2},
\end{aligned}$$

e daí (2.22) resulta imediatamente

$$\begin{aligned}
&\beta\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2|b_2|\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^3 c\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + \beta\epsilon\int_0^t\|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \leq C_0.
\end{aligned}$$

□

Observe que se $b_2 \neq 0$ (lembre que $b_2 \leq 0$), (2.21) será uma consequência imediata do Lema 2.2 e a primeira desigualdade em (2.27).

Podemos enfim estender nossa solução.

Teorema 2.2. *Assuma a condição* (2.20) *e suponha* $\beta = O(\epsilon^4)$. *Dado* $u_{\epsilon,\beta,0} \in H^5(\mathbb{R})$, *o problema de Cauchy* (2.1)–(2.2) *admite uma solução* $u \in C([0,\infty), H^5(\mathbb{R}))$.

Demonstração. O Teorema 2.1 assegura a existência de um número real $T > 0$ (garantido através das condições (2.10) e (2.14)) e uma solução $u \in C([0,T], H^5(\mathbb{R}))$ dada por (2.15). Assim, pelos Lemas 2.2 e 2.3 existe uma constante $C_0 > 0$ independente de T tal que $\|u_T\|_{H^2(\mathbb{R})} \leq C_0$ sendo $u_T(\cdot) = u(T, \cdot)$. Agora considere o espaço

$$X_S = \{u \in C([T, T+S], H^2(\mathbb{R})); \|u(t) - G(t-T)u_T\|_{H^2(\mathbb{R})} \leq \|u_T\|_{H^2(\mathbb{R})},\ t \in [T, T+S]\}$$

e o seguinte operador nele definido:

$$\Lambda u(t) = G(t-T)u_T - \int_T^t G(t-s)\mathcal{F}^{-1}\left(\frac{\mathcal{F}(\partial_x f(u(s)))}{1 + \beta|b_2|\xi^2 + \beta^2 c\xi^4}\right) ds. \tag{2.30}$$

Dado $u \in X_S$ temos $\|u(t)\|_{L^\infty(\mathbb{R})} \leq 2\|u_T\|_{H^2(\mathbb{R})}$ de modo que $\|u(t)\|_{L^\infty(\mathbb{R})} \leq 2C_0$ para todo $t \in [T, T+S]$. Logo pelo Teorema 1.2 e o Corolário 1.1 existirá uma constante $K_1 > 0$ dependendo apenas da cota $2C_0$ (e consequentemente independente de T) tal que (2.7) e (2.8) se verificam para quaisquer $u, v \in X_S$. Portanto argumentando como na demonstração de (2.10) e (2.14) e fixando (qualquer) S no intervalo

$$(0, \min\{\alpha^{-1}, [\alpha(|f'(0)| + 4C_0)]^{-1}\}] \quad \alpha = 2(1 + (\beta^2 c)^{-1})^{1/2} K_1$$

o Teorema 2.1 nos garante uma solução $u \in C([T, T+S], H^5(\mathbb{R}))$ dada por (2.30) (e portanto uma solução $u \in C([0, T+S], H^5(\mathbb{R}))$) e, além disso, este S servirá para todas as etapas seguintes uma vez que as constantes C_0 e K_1 não dependerão dos dados iniciais devido aos Lemas 2.2 e 2.3. Isto nos permite estender a solução para todo $t \geq 0$ procedendo recursivamente. □

O papel dos Lemas 2.2 e 2.3 é limitar uniformemente a norma $H^2(\mathbb{R})$ dos dados iniciais das etapas de extensão e a norma $\|u(t)\|_{L^\infty(\mathbb{R})}$ em $[T, +\infty)$ de modo que a magnitude de S possa ser fixada.

O Lema 2.3 é necessário apenas quando $b_2 = 0$, pois neste caso o Lema 2.2 não nos permite limitar uniformemente o termo $\|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}$. Contudo se ocorrer $b_2 \neq 0$, como o Lema 2.2 utiliza apenas a hipótese (mais fraca)

$$\|u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta\|\partial_x u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \leq C_0, \tag{2.31}$$

o Teorema 2.2 continuará válido sem as condições (2.20) e $\beta = O(\epsilon^4)$ desde que assumamos (2.31).

Em toda nossa discussão acima poderíamos trocar o espaço $H^2(\mathbb{R})$ por $H^3(\mathbb{R})$. Com efeito, os Lemas 2.2 e 2.3 ainda nos permitiriam limitar uniformemente $\|u(t)\|_{L^\infty(\mathbb{R})}$ e a norma $H^3(\mathbb{R})$ dos dados iniciais de cada etapa de extensão. Desta forma, utilizando a estimativa (análoga à (2.16))

$$
\begin{aligned}
\sum_{k=0}^{6}\int_{\mathbb{R}}\frac{\xi^2|\mathcal{F}(\partial_x^k f(u(s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2c\xi^4)^2}d\xi &\leq C_0\|f(u(s))\|^2_{H^3(\mathbb{R})}+\sum_{k=4}^{6}\int_{\mathbb{R}}\frac{\xi^2|\mathcal{F}(\partial_x^k f(u(s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2c\xi^4)^2}d\xi\\
&= C_0\|f(u(s))\|^2_{H^3(\mathbb{R})}+\sum_{k=1}^{3}\int_{\mathbb{R}}\frac{\xi^8|\mathcal{F}(\partial_x^k f(u(s)))|^2}{(1+\beta|b_2|\xi^2+\beta^2c\xi^4)^2}d\xi\\
&\leq C_0\|f(u(s))\|^2_{H^3(\mathbb{R})}+(\beta^4c^2)^{-1}\|f(u(s))\|^2_{H^3(\mathbb{R})}\\
&\leq C_0\|f(u(s))\|^2_{H^3(\mathbb{R})}
\end{aligned}
$$

obteríamos o seguinte

Teorema 2.3. *Assuma a condição* (2.20) *e suponha* $\beta = O(\epsilon^4)$. *Dado* $u_{\epsilon,\beta,0} \in H^6(\mathbb{R})$, *o problema de Cauchy* (2.1)–(2.2) *admite uma solução* $u \in C([0,\infty), H^6(\mathbb{R}))$.

2.2 Estimativas a priori e Convergênia em L^2

Para cada $\epsilon, \beta \in (0,1)$ consideremos o problema de Cauchy (2.1)–(2.2) com coeficientes b_1, b_2 e c satisfazendo $b_1 \in \mathbb{R}, b_2 \leq 0$ e $c > 0$, e f suave tal que

$$
|f'(u)| \leq C_0(1+|u|^p) \text{ para algum } p \in [0,1). \tag{2.32}
$$

Assumindo $u_{\epsilon,\beta,0} \in C_c^\infty(\mathbb{R})$, seja $u_{\epsilon,\beta} \in C([0,\infty), H^5(\mathbb{R}))$ uma solução de (2.1)–(2.2) (garantida pelo Teorema 2.2). Suponhamos ainda que $u_{\epsilon,\beta,0}$ seja uma aproximação da função real

$$
u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \tag{2.33}
$$

tal que

$$
u_{\epsilon,\beta,0} \to u_0 \text{ em } L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \text{ quando } \epsilon, \beta \to 0; \tag{2.34}
$$

e

$$
\begin{cases}
\|u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + (\beta^{1/2}+\epsilon^2)\|\partial_x u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \leq C_0\\
(\beta^{3/2}+\beta\epsilon^2)\|\partial_x^2 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta^{5/2}\|\partial_x^3 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \leq C_0;
\end{cases} \tag{2.35}
$$

sendo $C_0 > 0$ uma constante independente de ϵ e β.

O principal resultado desta seção é o seguinte

Teorema 2.4. *Nas condições acima, se* $\beta = o(\epsilon^4)$ *e*

$$
\|u_{\epsilon,\beta}(t,\cdot)\|_{L^{r_0}(\mathbb{R})} \leq \|u_{\epsilon,\beta,0}\|_{L^{r_0}(\mathbb{R})} \quad \text{para algum } r_0 \in (1,2)
$$

então a sequência (inteira) $u_{\epsilon,\beta}$ *converge para uma função* $u \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}))$ *em todo espaço* $L^r_{loc}(\mathbb{R}_+ \times \mathbb{R})$ *com* $r \in [1,2)$, *sendo* u *a única solução de entropia de*

$$
\partial_t u + \partial_x f(u) = 0 \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{2.36}
$$

$$
u(0,x) = u_0(x) \quad x \in \mathbb{R}. \tag{2.37}
$$

Para demonstrar o teorema acima necessitaremos de algumas estimativas sobre as soluções $u_{\epsilon,\beta}$. Os dois primeiros lemas abaixo são na realidade os Lemas 2.2 e 2.3.

Lema 2.4. *Assumindo a condição* (2.35)*, existe uma constante* $C_0 > 0$ *(independente de* ϵ, β*) tal que*

$$\|u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta|b_2|\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 c\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\epsilon\int_0^t \|\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \leq C_0$$

para todo $t > 0$*.*

Lema 2.5. *Assumindo* (2.35) *e* $\beta = O(\epsilon^4)$*, existe uma constante* $C_0 > 0$ *(independente de* ϵ, β*) tal que*

$$\|u_{\epsilon,\beta}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \leq C_0\beta^{-1/4}.$$

Além do mais,

$$\beta\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2|b_2|\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^3 c\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta\epsilon\int_0^t \|\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \leq C_0$$

para todo $t > 0$*.*

Lema 2.6. *Assumindo* (2.35) *e* $\beta = O(\epsilon^4)$*, as seguintes afirmações são válidas:*

(i) as famílias $\{\beta^{1/4}\epsilon\partial_x u_{\epsilon,\beta}\}_{\epsilon,\beta}$ *e* $\{\beta^{3/4}\epsilon\partial_x^2 u_{\epsilon,\beta}\}_{\epsilon,\beta}$ *são limitadas em* $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}))$*.*

(ii) as famílias $\{\beta^{3/4}\epsilon^{1/2}\partial_t\partial_x u_{\epsilon,\beta}\}_{\epsilon,\beta}$*,* $\{\beta^{1/4}\epsilon^{1/2}\partial_t u_{\epsilon,\beta}\}_{\epsilon,\beta}$*,* $\{\beta^{7/4}\epsilon^{1/2}\partial_t\partial_x^3 u_{\epsilon,\beta}\}_{\epsilon,\beta}$ *e* $\{\beta^{5/4}\epsilon^{1/2}\partial_t\partial_x^2 u_{\epsilon,\beta}\}_{\epsilon,\beta}$ *são limitadas em* $L^2(\mathbb{R}_+\times\mathbb{R})$*.*

Demonstração. Multiplicando (2.1) por

$$-\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} + \epsilon\partial_t u_{\epsilon,\beta}$$

e integrando cada termo obtido em $\mathbb{R}$ obtemos

$$\int_{\mathbb{R}} (-\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} + \epsilon\partial_t u_{\epsilon,\beta})\partial_t u_{\epsilon,\beta} dx = \beta\epsilon\|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \epsilon\|\partial_t u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$\int_{\mathbb{R}} (-\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} + \epsilon\partial_t u_{\epsilon,\beta})\partial_x f(u_{\epsilon,\beta}) dx = -\beta\epsilon\int_{\mathbb{R}} \partial_x u_{\epsilon,\beta}\partial_t\partial_x^2 u_{\epsilon,\beta} f'(u_{\epsilon,\beta}) dx + \epsilon\int_{\mathbb{R}} \partial_x u_{\epsilon,\beta}\partial_t u_{\epsilon,\beta} f'(u_{\epsilon,\beta}) dx,$$

$$\begin{aligned}\beta b_1\int_{\mathbb{R}} (-\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} + \epsilon\partial_t u_{\epsilon,\beta})\partial_x^3 u_{\epsilon,\beta} dx &= -\beta^2\epsilon b_1\int_{\mathbb{R}} \partial_t\partial_x^2 u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta} dx + \beta\epsilon b_1\int_{\mathbb{R}} \partial_t u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta} dx \\ &= -\beta^2\epsilon b_1\int_{\mathbb{R}} \partial_t\partial_x^2 u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta} dx - \beta\epsilon b_1\int_{\mathbb{R}} \partial_t\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta} dx,\end{aligned}$$

$$\beta b_2\int_{\mathbb{R}}(-\beta\epsilon\partial_t\partial_x^2u_{\epsilon,\beta}+\epsilon\partial_tu_{\epsilon,\beta})\partial_t\partial_x^2u_{\epsilon,\beta}dx=\beta^2\epsilon|b_2|\|\partial_t\partial_x^2u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta\epsilon|b_2|\|\partial_t\partial_xu_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$\begin{aligned}\beta^2c\int_{\mathbb{R}}(-\beta\epsilon\partial_t\partial_x^2u_{\epsilon,\beta}+\epsilon\partial_tu_{\epsilon,\beta})\partial_t\partial_x^4u_{\epsilon,\beta}dx&=\beta^3\epsilon c\int_{\mathbb{R}}(\partial_t\partial_x^3u_{\epsilon,\beta})^2dx+\beta^2\epsilon c\int_{\mathbb{R}}(\partial_t\partial_x^2u_{\epsilon,\beta})^2dx\\&=\beta^3\epsilon c\|\partial_t\partial_x^3u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\epsilon c\|\partial_t\partial_x^2u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}$$

e

$$\epsilon\int_{\mathbb{R}}(-\beta\epsilon\partial_t\partial_x^2u_{\epsilon,\beta}+\epsilon\partial_tu_{\epsilon,\beta})\partial_x^2u_{\epsilon,\beta}dx=-\frac{\beta\epsilon^2}{2}\frac{d}{dt}\|\partial_x^2u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}-\frac{\epsilon^2}{2}\frac{d}{dt}\|\partial_xu_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Portanto,

$$\begin{aligned}&\beta\epsilon(1+|b_2|)\|\partial_t\partial_xu_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\epsilon\|\partial_tu_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\epsilon(c+|b_2|)\|\partial_t\partial_x^2u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad+\beta^3\epsilon c\|\partial_t\partial_x^3u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta\epsilon^2}{2}\frac{d}{dt}\|\partial_x^2u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\epsilon^2}{2}\frac{d}{dt}\|\partial_xu_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&=\beta\epsilon\int_{\mathbb{R}}\partial_xu_{\epsilon,\beta}\partial_t\partial_x^2u_{\epsilon,\beta}f'(u_{\epsilon,\beta})dx-\epsilon\int_{\mathbb{R}}\partial_xu_{\epsilon,\beta}\partial_tu_{\epsilon,\beta}f'(u_{\epsilon,\beta})dx\\&\quad+\beta^2\epsilon b_1\int_{\mathbb{R}}\partial_t\partial_x^2u_{\epsilon,\beta}\partial_x^3u_{\epsilon,\beta}dx+\beta\epsilon b_1\int_{\mathbb{R}}\partial_t\partial_xu_{\epsilon,\beta}\partial_x^2u_{\epsilon,\beta}dx.\end{aligned}\tag{2.38}$$

A partir do Lema 2.5 temos as seguintes estimativas:

$$\begin{aligned}&\beta\epsilon\int_{\mathbb{R}}\partial_xu_{\epsilon,\beta}\partial_t\partial_x^2u_{\epsilon,\beta}f'(u_{\epsilon,\beta})dx\le\beta\epsilon\int_{\mathbb{R}}|\partial_xu_{\epsilon,\beta}\partial_t\partial_x^2u_{\epsilon,\beta}f'(u_{\epsilon,\beta})|dx\\&=\epsilon\int_{\mathbb{R}}|\beta(c+|b_2|)^{1/2}\partial_t\partial_x^2u_{\epsilon,\beta}||(c+|b_2|)^{-1/2}\partial_xu_{\epsilon,\beta}f'(u_{\epsilon,\beta})|dx\\&\le\frac{\beta^2\epsilon(c+|b_2|)}{2}\int_{\mathbb{R}}|\partial_t\partial_x^2u_{\epsilon,\beta}|^2dx+\frac{\epsilon}{2(c+|b_2|)}\int_{\mathbb{R}}|\partial_xu_{\epsilon,\beta}f'(u_{\epsilon,\beta})|^2dx\\&\le\frac{\beta^2\epsilon(c+|b_2|)}{2}\|\partial_t\partial_x^2u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+C_0\epsilon\int_{\mathbb{R}}|\partial_xu_{\epsilon,\beta}|^2(1+|u_{\epsilon,\beta}|)^2dx\\&\le\frac{\beta^2\epsilon(c+|b_2|)}{2}\|\partial_t\partial_x^2u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+C_0\epsilon\beta^{-1/2}\|\partial_xu_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})};\end{aligned}$$

$$\begin{aligned}&\epsilon\int_{\mathbb{R}}\partial_xu_{\epsilon,\beta}\partial_tu_{\epsilon,\beta}f'(u_{\epsilon,\beta})dx\le\epsilon\int_{\mathbb{R}}|\partial_xu_{\epsilon,\beta}\partial_tu_{\epsilon,\beta}f'(u_{\epsilon,\beta})|dx\\&\le\frac{\epsilon}{2}\int_{\mathbb{R}}|\partial_tu_{\epsilon,\beta}|^2dx+\frac{\epsilon}{2}\int_{\mathbb{R}}|\partial_xu_{\epsilon,\beta}f'(u_{\epsilon,\beta})|^2dx\\&\le\frac{\epsilon}{2}\|\partial_tu_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+C_0\epsilon\beta^{-1/2}\|\partial_xu_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})};\end{aligned}$$

$$\begin{aligned}&\beta^2\epsilon b_1\int_{\mathbb{R}}\partial_t\partial_x^2u_{\epsilon,\beta}\partial_x^3u_{\epsilon,\beta}dx=|\beta^2\epsilon b_1\int_{\mathbb{R}}\partial_t\partial_x^3u_{\epsilon,\beta}\partial_x^2u_{\epsilon,\beta}dx|\\&\le\epsilon\int_{\mathbb{R}}|c^{1/2}\beta^{3/2}\partial_t\partial_x^3u_{\epsilon,\beta}||b_1c^{-1/2}\beta^{1/2}\partial_x^2u_{\epsilon,\beta}|dx\\&\le\frac{\beta^3\epsilon c}{2}\|\partial_t\partial_x^3u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta\epsilon b_1^2}{2c}\|\partial_x^2u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})};\end{aligned}$$

$$
\begin{aligned}
\beta\epsilon b_1 \int_{\mathbb{R}} \partial_t\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}dx &\le \epsilon \int_{\mathbb{R}} |\beta^{1/2}\partial_t\partial_x u_{\epsilon,\beta}||\beta^{1/2}b_1\partial_x^2 u_{\epsilon,\beta}|dx \\
&\le \frac{\beta\epsilon}{2}\|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta\epsilon b_1^2}{2}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Substituindo todas estas desigualdades em (2.38) resulta que

$$
\begin{aligned}
&\beta\epsilon(\frac{1}{2}+|b_2|)\|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\epsilon}{2}\|\partial_t u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2\epsilon}{2}(c+|b_2|)\|\partial_t\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad + \frac{\beta^3\epsilon c}{2}\|\partial_t\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta\epsilon^2}{2}\frac{d}{dt}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\epsilon^2}{2}\frac{d}{dt}\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le C_0\epsilon\beta^{-1/2}\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0\beta\epsilon\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R}),}
\end{aligned} \tag{2.39}
$$

Agora integrando (2.39) em $(0,t)$, e em seguida utilizando os Lemas 2.4 e 2.5 além da condição (2.35), obtemos

$$
\begin{aligned}
&\beta\epsilon(\frac{1}{2}+|b_2|)\int_0^t \|\partial_s\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \frac{\epsilon}{2}\int_0^t \|\partial_s u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \\
&\quad + \frac{\beta^2\epsilon}{2}(c+|b_2|)\int_0^t \|\partial_s\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \frac{\beta^3\epsilon c}{2}\int_0^t \|\partial_s\partial_x^3 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \\
&\quad + \frac{\beta\epsilon^2}{2}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\epsilon^2}{2}\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le C_0\epsilon\beta^{-1/2}\int_0^t \|\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + C_0\beta\epsilon\int_0^t \|\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \\
&\quad + \frac{\beta\epsilon^2}{2}\|\partial_x^2 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \frac{\epsilon^2}{2}\|\partial_x u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \\
&\le C_0\beta^{-1/2} + C_0 \\
&\le C_0\beta^{-1/2}
\end{aligned}
$$

e portanto

$$
\begin{aligned}
&\beta^{3/2}\epsilon(\frac{1}{2}+|b_2|)\int_0^t \|\partial_s\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \frac{\beta^{1/2}\epsilon}{2}\int_0^t \|\partial_s u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \\
&\quad + \frac{\beta^{5/2}\epsilon}{2}(c+|b_2|)\int_0^t \|\partial_s\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \frac{\beta^{7/2}\epsilon c}{2}\int_0^t \|\partial_s\partial_x^3 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \\
&\quad + \frac{\beta^{3/2}\epsilon^2}{2}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^{1/2}\epsilon^2}{2}\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le C_0
\end{aligned}
$$

finalizando a demonstração. □

Daremos agora a

Demonstração (do Teorema 2.4). De acordo com o Corolário 1.3 será suficiente verificar as condições (1.8) e (1.9). Isto será feito seguindo Lefloch [19].

Começamos observando que pelo Lema 2.4, $\{u_{\epsilon,\beta}\}_{\epsilon,\beta}$ é limitada em $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}))$, e portanto o Lema 1.3 garante a existência de uma subsequência $\{u_{\epsilon_k,\beta_k}\}_{k\in\mathbb{N}}$ e uma medida de Young $\nu : \mathbb{R}_+ \times \mathbb{R} \to \text{Prob}(\mathbb{R})$ tal que

$$\lim_{k\to\infty} \int_0^\infty \int_{\mathbb{R}} g(u_{\epsilon_k,\beta_k})\phi(t,x)dtdx = \int_0^\infty \int_{\mathbb{R}} \left\langle \nu_{(t,x)}, g \right\rangle \phi(t,x)dtdx, \tag{2.40}$$

para toda $\phi \in L^1(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R})$ e toda $g \in C(\mathbb{R})$ satisfazendo $g(u) = O(1 + |u|^r)$ para algum $r \in [0, 2)$. De posse da representação acima, estamos aptos à provar que

$\nu_{(\cdot)}$ ***satisfaz (1.8).***

Considere a função $g(\lambda) = |\lambda|^{r_0}$ e defina

$$G(\lambda, \lambda_0) = g(\lambda) - g(\lambda_0) - g'(\lambda_0)(\lambda - \lambda_0).$$

Então (Vide Apêndice A)

$$G(\lambda, \lambda_0) \geq \frac{r_0(r_0 - 1)}{4} \frac{(\lambda - \lambda_0)^2}{(1 + |\lambda| + |\lambda_0|)^{2-r_0}} \quad (\lambda, \lambda_0) \in \mathbb{R}^2 \tag{2.41}$$

e $g''(\lambda) = r_0(r_0 - 1)|\lambda|^{r_0-2}$ se $\lambda \neq 0$. Dados $I \subset \mathbb{R}$ compacto e $T > 0$, considere as medidas

$$\mu_1, \mu_2 : X \to \mathbb{R} \qquad X = [0, T] \times I \times \mathbb{R}$$

dadas por

$$\begin{aligned} d\mu_1(t, x, \lambda) &= (T\text{m}(I))^{-1} d\nu_{(t,x)}(\lambda)dtdx, \\ d\mu_2(t, x, \lambda) &= d\nu_{(t,x)}(\lambda)dtdx, \end{aligned}$$

onde m denota a medida de Lebesgue em $\mathbb{R}$. Então $\mu_1, \mu_2 \geq 0$ e $\mu_1(X) = 1$. Sendo g convexa em $(0, \infty)$, a desigualdade de Jensen, a desigualdade de Holder (com expoentes conjugados $p = 2/r_0$ e $q = 2/(2 - r_0)$) e a estimativa (2.41) implicam em

$$\begin{aligned} \left\{ \frac{1}{T\text{m}(I)} \int_0^T \int_I \left\langle \nu_{(t,x)}, |\lambda - u_0(x)| \right\rangle dtdx \right\}^{r_0} &= \left\{ \int_X |\lambda - u_0(x)| d\mu_1 \right\}^{r_0} \\ &\leq \int_X |\lambda - u_0(x)|^{r_0} d\mu_1 = \frac{1}{T\text{m}(I)} \int_X |\lambda - u_0(x)|^{r_0} d\mu_2 \\ &\leq \frac{1}{T\text{m}(I)} \left\{ \int_X \frac{|\lambda - u_0(x)|^2}{(1 + |\lambda| + |u_0(x)|)^{2-r_0}} d\mu_2 \right\}^{r_0/2} \left\{ \int_X (1 + |\lambda| + |u_0(x)|)^{r_0} d\mu_2 \right\}^{\frac{2-r_0}{2}} \\ &\leq \frac{C_I}{T} \left\{ \int_0^T \int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x)) \right\rangle dtdx \right\}^{r_0/2} \left\{ \int_X (1 + |\lambda| + |u_0(x)|)^{r_0} d\mu_2 \right\}^{\frac{2-r_0}{2}}. \end{aligned} \tag{2.42}$$

Agora observe que

$$\begin{aligned} \int_X (1 + |\lambda| + |u_0(x)|)^{r_0} d\mu_2 &\leq 2^{2(r_0-1)} \int_X (1 + |\lambda|^{r_0} + |u_0(x)|^{r_0}) d\nu_{(t,x)}(\lambda)dtdx \\ &= \text{m}(I)T + T \int_I |u_0(x)|^{r_0} dx + \int_0^T \int_I \left\langle \nu_{(t,x)}, |\lambda|^{r_0} \right\rangle dtdx \\ &\leq C_I T + \lim_{k\to\infty} \int_0^T \int_I |u_{\epsilon_k,\beta_k}|^{r_0} dtdx. \end{aligned} \tag{2.43}$$

Pelo Lema 2.4

$$\int_0^T \int_I |u_{\epsilon_k,\beta_k}|^2 dtdx \leq C_0 T \qquad (C_0 \text{ independente de } \epsilon_k \text{ e } \beta_k)$$

e daí

$$\begin{aligned}\int_0^T \int_I |u_{\epsilon_k,\beta_k}|^{r_0} dtdx &\leq [T\mathrm{m}(I)]^{1-r_0/2} \left\{ \int_0^T \int_I |u_{\epsilon_k\beta_k}|^2 dtdx \right\}^{r_0/2} \\ &\leq C_I T.\end{aligned} \tag{2.44}$$

Substituindo (2.44) em (2.43) obtemos

$$\int_X (1 + |\lambda| + |u_0(x)|)^{r_0} d\mu_2 \leq C_I T. \tag{2.45}$$

A partir de (2.42) e (2.45) segue-se que

$$\frac{1}{T}\int_0^T \int_I \left\langle \nu_{(t,x)}, |\lambda - u_0(x)| \right\rangle dtdx \leq C_I \left\{ \frac{1}{T}\int_0^T \int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x)) \right\rangle dtdx \right\}^{1/2}. \tag{2.46}$$

Em seguida seja $\{\phi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ tal que $\phi_n \to g'(u_0)$ em $L^{r_0'}(\mathbb{R})$ onde $1/r_0 + 1/r_0' = 1$. Para cada $n \in \mathbb{N}$ temos

$$\begin{aligned}\int_0^T\int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x)) \right\rangle dtdx &= \int_0^T\int_I \left\langle \nu_{(t,x)}, |\lambda|^{r_0} - |u_0(x)|^{r_0} - g'(u_0(x))(\lambda - u_0(x)) \right\rangle dtdx \\ &= \int_0^T\int_I \left\langle \nu_{(t,x)}, |\lambda|^{r_0} - |u_0(x)|^{r_0} \right\rangle dtdx + \int_0^T\int_I \left\langle \nu_{(t,x)}, u_0(x) - \lambda \right\rangle \phi_n(x) dx \\ &\quad + \int_0^T\int_I \left\langle \nu_{(t,x)}, \lambda - u_0(x) \right\rangle (\phi_n(x) - g'(u_0(x))) dtdx.\end{aligned} \tag{2.47}$$

Ora, (2.40), (2.34) e a condição

$$\|u_{\epsilon,\beta}(t,\cdot)\|_{L^{r_0}(\mathbb{R})} \leq \|u_{\epsilon,\beta,0}\|_{L^{r_0}(\mathbb{R})}$$

implicam em

$$\begin{aligned}\int_0^T \int_I \left\langle \nu_{(t,x)}, |\lambda|^{r_0} - |u_0(x)|^{r_0} \right\rangle dtdx &= \int_0^T \int_I \left\langle \nu_{(t,x)}, |\lambda|^{r_0} \right\rangle dtdx - T\int_I |u_0(x)|^{r_0} dx \\ &= \lim_{k\to\infty} \int_0^T \int_I |u_{\epsilon_k,\beta_k}(t,x)|^{r_0} dtdx - T\int_I |u_0(x)|^{r_0} dx \\ &\leq \lim_{k\to\infty} T\|u_{\epsilon_k,\beta_k,0}\|_{L^{r_0}(\mathbb{R})}^{r_0} - T\int_I |u_0(x)|^{r_0} dx \\ &= T\|u_0\|_{L^{r_0}(\mathbb{R})}^{r_0} - T\int_I |u_0(x)|^{r_0} dx \\ &= T\int_{\mathbb{R}\setminus I} |u_0(x)|^{r_0} dx\end{aligned} \tag{2.48}$$

e

$$
\begin{aligned}
&\int_0^T\!\!\int_I \left\langle \nu_{(t,x)}, \lambda - u_0(x)\right\rangle (\phi_n(x) - g'(u_0(x)))dtdx \\
&\quad = \int_0^T\!\!\int_I \left\langle \nu_{(t,x)}, \lambda\right\rangle (\phi_n(x) - g'(u_0(x)))dtdx - T\int_I u_0(x)(\phi_n(x) - g'(u_0(x)))dtdx \\
&\quad \le \| \left\langle \nu_{(t,x)}, \lambda\right\rangle \|_{L^{r_0}((0,T)\times I)} \|\phi_n - g'(u_0)\|_{L^{r_0'}((0,T)\times I)} + T\|u_0\|_{L^{r_0}(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})} \\
&\quad \le T^{1/r_0'}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})} \left\{\int_0^T\int_I \left\langle \nu_{(t,x)}, |\lambda|^{r_0}\right\rangle dtdx\right\}^{1/r_0} + T\|u_0\|_{L^{r_0}(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})} \\
&\quad = T^{1/r_0'}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})} \left\{\lim_{k\to\infty}\int_0^T\int_I |u_{\epsilon_k,\beta_k}(t,x)|^{r_0} dtdx\right\}^{1/r_0} \\
&\qquad + T\|u_0\|_{L^{r_0}(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})} \\
&\quad \le T^{1/r_0'}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})} \left\{\lim_{k\to\infty} T\|u_{\epsilon_k,\beta_k,0}\|^{r_0}_{L^{r_0}(\mathbb{R})}\right\}^{1/r_0} \\
&\qquad + T\|u_0\|_{L^{r_0}(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})} \\
&\quad = 2T\|u_0\|_{L^{r_0}(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})}.
\end{aligned} \tag{2.49}
$$

Substituindo (2.48) e (2.49) em (2.47) segue-se que

$$
\begin{aligned}
\int_0^T\!\!\int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x))\right\rangle dtdx &\le \int_0^T\!\!\int_I \left\langle \nu_{(t,x)}, u_0(x) - \lambda\right\rangle \phi_n(x)dtdx + T\int_{\mathbb{R}\setminus I} |u_0(x)|^{r_0}dx \\
&\quad + 2T\|u_0\|_{L^{r_0}(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})}.
\end{aligned} \tag{2.50}
$$

Agora seja $\{K_i\}_{i\in\mathbb{N}}$ uma sequência de compactos tal que

$$
I \subset K_1 \subset K_2 \subset \cdots \quad \text{e} \quad \bigcup_{i\in\mathbb{N}} K_i = \mathbb{R}.
$$

Então, sendo $G \ge 0$

$$
\int_0^T\!\!\int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x))\right\rangle dtdx \le \int_0^T\!\!\int_{K_i} \left\langle \nu_{(t,x)}, G(\lambda, u_0(x))\right\rangle dtdx
$$

e portanto

$$
\begin{aligned}
\frac{1}{T}\int_0^T\!\!\int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x))\right\rangle dtdx &\le \frac{1}{T}\int_0^T\!\!\int_{K_i} \left\langle \nu_{(t,x)}, u_0(x) - \lambda\right\rangle \phi_n(x)dtdx \\
&\quad + \int_{\mathbb{R}\setminus K_i} |u_0(x)|^{r_0}dx + 2\|u_0\|_{L^{r_0}(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})}.
\end{aligned}
$$

Como $u_0 \in L^{r_0}(\mathbb{R})$,

$$
\lim_{i\to\infty}\int_{\mathbb{R}\setminus K_i} |u_0(x)|^{r_0}dx = 0
$$

e consequentemente

$$\frac{1}{T}\int_0^T\!\!\int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x))\right\rangle dtdx \leq \frac{1}{T}\int_0^T\!\!\int_{\mathbb{R}} \left\langle \nu_{(t,x)}, u_0(x)-\lambda\right\rangle \phi_n(x)dtdx + 2\|u_0\|_{L^{r_0}(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})}. \tag{2.51}$$

A partir de (2.46) e (2.51) concluímos que

$$\frac{1}{T}\int_0^T\!\!\int_I \left\langle \nu_{(t,x)}, |\lambda - u_0(x)|\right\rangle dtdx \leq C_I\left\{\frac{1}{T}\int_0^T\!\!\int_{\mathbb{R}} \left\langle \nu_{(t,x)}, u_0(x)-\lambda\right\rangle \phi_n(x)dtdx + 2\|u_0\|_{L^{r_0}(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})}\right\}^{1/2}. \tag{2.52}$$

A seguir mostraremos que

$$\lim_{T\to 0}\frac{1}{T}\int_0^T\int_{\mathbb{R}} \left\langle \nu_{(t,x)}, u_0(x)-\lambda\right\rangle \phi_n(x)dtdx \leq 0 \qquad n = 1,2,3,\ldots. \tag{2.53}$$

De fato, fixado $n\in\mathbb{N}$

$$\frac{1}{T}\int_0^T\int_{\mathbb{R}} \left\langle \nu_{(t,x)}, u_0(x)-\lambda\right\rangle \phi_n(x)dtdx = \lim_{k\to\infty}\frac{1}{T}\int_0^T\int_{\mathbb{R}}(u_0(x) - u_{\epsilon_k,\beta_k}(t,x))\phi_n(x)dtdx. \tag{2.54}$$

Por outro lado,

$$\begin{aligned}\frac{1}{T}\int_0^T\int_{\mathbb{R}}(u_0(x) - u_{\epsilon_k,\beta_k}(t,x))\phi_n(x)dtdx &= \int_{\mathbb{R}}(u_0(x) - u_{\epsilon_k,\beta_k,0}(x))\phi_n(x)dx\\ &\quad - \frac{1}{T}\int_0^T\int_{\mathbb{R}}\left(\int_0^t \partial_s u_{\epsilon_k,\beta_k}(s,x)ds\right)\phi_n(x)dtdx\\ &= I_k^n + J_k^n,\end{aligned}$$

onde

$$I_k^n = \int_{\mathbb{R}}(u_0(x) - u_{\epsilon_k,\beta_k,0}(x))\phi_n(x)dx$$

e

$$J_k^n = -\frac{1}{T}\int_0^T\int_{\mathbb{R}}\left(\int_0^t \partial_s u_{\epsilon_k,\beta_k}(s,x)ds\right)\phi_n(x)dtdx.$$

Como $|I_k^n| \leq \|u_0 - u_{\epsilon_k,\beta_k,0}\|_{L^1(\mathrm{supp}(\phi_n))}\|\phi_n\|_{L^\infty(\mathrm{supp}(\phi_n))}$ e $u_{\epsilon_k,\beta_k,0}\to u_0$ em $L^1_{loc}(\mathbb{R})$, $|I_k^n|\to 0$ quando $k\to\infty$. Agora estimaremos cada termo de J_k^n observando que

$$\begin{aligned}J_k^n &= -\frac{1}{T}\int_0^T\int_{\mathbb{R}}\left(\int_0^t \partial_s u_{\epsilon_k,\beta_k}(s,x)ds\right)\phi_n(x)dtdx\\ &= -\frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \left[\epsilon_k\partial_x^2 u_{\epsilon_k,\beta_k}(s,x) - \partial_x f(u_{\epsilon_k,\beta_k}(s,x)) - \beta_k b_1\partial_x^3 u_{\epsilon_k,\beta_k}(s,x)\right.\\ &\qquad \left. - \beta_k b_2\partial_s\partial_x^2 u_{\epsilon_k,\beta_k}(s,x) - \beta_k^2 c\partial_s\partial_x^4 u_{\epsilon_k,\beta_k}(s,x)\right]\phi_n(x)dsdtdx.\end{aligned}$$

Pelo Lema 2.4

$$
\begin{aligned}
-\frac{\epsilon_k}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_x^2 u_{\epsilon_k,\beta_k}(s,x)\phi_n(x)dsdtdx &= -\frac{\epsilon_k}{T}\int_0^T\int_{\mathbb{R}}\int_0^t u_{\epsilon_k,\beta_k}(s,x)\partial_x^2\phi_n(x)dsdtdx \\
&\leq \frac{\epsilon_k}{T}\int_0^T\int_{\mathbb{R}}\int_0^t |u_{\epsilon_k,\beta_k}(s,x)\partial_x^2\phi_n(x)|dsdtdx \\
&\leq \frac{\epsilon_k}{T}\int_0^T\int_0^t \|u_{\epsilon_k,\beta_k}(s,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^2\phi_n\|_{L^2(\mathbb{R})}dsdt \\
&\leq \frac{C_0\epsilon_k}{T}\int_0^T\int_0^t dsdt \\
&= C_0\epsilon_k T,
\end{aligned}
$$

e

$$
\begin{aligned}
\frac{\beta_k b_1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_x^3 u_{\epsilon_k,\beta_k}(s,x)\phi_n(x)dsdtdx &= \frac{-\beta_k b_1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t u_{\epsilon_k,\beta_k}(s,x)\partial_x^3\phi_n(x)dsdtdx \\
&\leq \frac{\beta_k|b_1|}{T}\int_0^T\int_0^t \|u_{\epsilon_k,\beta_k}(s,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^3\phi_n\|_{L^2(\mathbb{R})}dsdt \\
&\leq \frac{C_0\beta_k}{T}\int_0^T\int_0^t dsdt \\
&\leq C_0\beta_k T.
\end{aligned}
$$

Devido ao Lema 2.6,

$$
\begin{aligned}
\frac{\beta_k b_2}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_s\partial_x^2 u_{\epsilon_k,\beta_k}(s,x)\phi_n(x)dsdtdx &= \frac{\beta_k|b_2|}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_s\partial_x u_{\epsilon_k,\beta_k}(s,x)\partial_x\phi_n(x)dsdtdx \\
&\leq \frac{\beta_k|b_2|}{T}\int_0^T \|\partial_s\partial_x u_{\epsilon_k,\beta_k}\|_{L^2((0,t)\times\mathbb{R})}\|\partial_x\phi_n\|_{L^2((0,t)\times\mathbb{R})}dt \\
&\leq \frac{C_0\beta_k^{1/4}\epsilon_k^{-1/2}}{T}\int_0^T t^{1/2}dt \\
&\leq C_0\beta_k^{1/4}\epsilon_k^{-1/2}T^{1/2}
\end{aligned}
$$

e

$$
\begin{aligned}
\frac{\beta_k^2 c}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_s\partial_x^4 u_{\epsilon_k,\beta_k}(s,x)\phi_n(x)dsdtdx &= -\frac{\beta_k^2 c}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_s\partial_x^3 u_{\epsilon_k,\beta_k}(s,x)\partial_x\phi_n(x)dsdtdx \\
&\leq \frac{\beta_k^2 c}{T}\int_0^T \|\partial_s\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2((0,t)\times\mathbb{R})}\|\partial_x\phi_n\|_{L^2((0,t)\times\mathbb{R})}dt \\
&\leq \frac{C_0\beta_k^{1/4}\epsilon_k^{-1/2}}{T}\int_0^T t^{1/2}dt \\
&\leq C_0\beta_k^{1/4}\epsilon_k^{-1/2}T^{1/2}.
\end{aligned}
$$

Observando que $|f(u)| \leq C_0(1+|u|^2)$ e usando o Lema 2.4 segue-se que

$$
\begin{aligned}
\frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_x f(u_{\epsilon_k,\beta_k}(s,x))\phi_n(x)dsdtdx &= -\frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t f(u_{\epsilon_k,\beta_k}(s,x))\partial_x\phi_n(x)dsdtdx \\
&\leq \frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t |f(u_{\epsilon_k,\beta_k}(s,x))\partial_x\phi_n(x)|dsdtdx \\
&= \frac{1}{T}\int_0^T\int_{\mathrm{supp}(\phi_n)}\int_0^t |f(u_{\epsilon_k,\beta_k}(s,x))\partial_x\phi_n(x)|dsdtdx \\
&\leq \frac{C_0}{T}\int_0^T\int_{\mathrm{supp}(\phi_n)}\int_0^t (1+|u_{\epsilon_k,\beta_k}(s,x)|^2)dsdtdx \\
&\leq \frac{C_0}{T}m(\mathrm{supp}(\phi_n))\int_0^T\int_0^t dsdt + \frac{C_0}{T}\int_0^T\int_0^t \|u_{\epsilon_k,\beta_k}(s,\cdot)\|^2_{L^2(\mathbb{R})}dsdt \\
&\leq C_0T + \frac{C_0}{T}\int_0^T\int_0^t dsdt \\
&\leq C_0T.
\end{aligned}
$$

Assim,

$$|J_k^n| \leq C_0(\epsilon_kT + \beta_kT + \beta_k^{1/4}\epsilon_k^{-1/2}T^{1/2} + T),$$

e pela hipótese $\beta = o(\epsilon^4)$,

$$\lim_{k\to\infty} |J_k^n| \leq C_0T. \tag{2.55}$$

Logo (2.54) e (2.55) implicam que

$$\frac{1}{T}\int_0^T\int_{\mathbb{R}} \left\langle \nu_{(t,x)}, u_0(x)-\lambda\right\rangle \phi_n(x)dtdx \leq C_0T$$

e a partir deste fato (2.53) segue imediatamente.
Portanto resulta de (2.52) e (2.53) que

$$\lim_{T\to 0}\frac{1}{T}\int_0^T\int_I \left\langle \nu_{(t,x)}, |\lambda-u_0(x)|\right\rangle dtdx \leq C_I\left\{2\|u_0\|_{L^{r_0}(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^{r_0'}(\mathbb{R})}\right\}^{1/2},$$

e fazendo $n\to\infty$ obtemos

$$\lim_{T\to 0}\frac{1}{T}\int_0^T\int_I \left\langle \nu_{(t,x)}, |\lambda-u_0(x)|\right\rangle dtdx = 0$$

estabelecendo a afirmação.

Passemos agora à segunda etapa da demonstração.

$\nu_{(\cdot)}$ ***satisfaz (1.9).***

Seja (η, q) um par de entropia–fluxo de entropia $\eta, q : \mathbb{R}\to\mathbb{R}$ com $\eta \in C^2(\mathbb{R})$ convexa, η' e η'' limitadas e q dada por

$$q(u) = \int_0^u f'(t)\eta'(t)dt. \tag{2.56}$$

Multiplicando (2.1) por $\eta'(u_{\epsilon,\beta})$ e fazendo uso da igualdade $q' = f'\eta'$ tem-se

$$\begin{aligned}
\partial_t \eta(u_{\epsilon,\beta}) + \partial_x q(u_{\epsilon,\beta}) &= \epsilon \eta'(u_{\epsilon,\beta}) \partial_x^2 u_{\epsilon,\beta} - \beta b_1 \eta'(u_{\epsilon,\beta}) \partial_x^3 u_{\epsilon,\beta} - \beta b_2 \eta'(u_{\epsilon,\beta}) \partial_t \partial_x^2 u_{\epsilon,\beta} \\
&\quad - \beta^2 c \eta'(u_{\epsilon,\beta}) \partial_t \partial_x^4 u_{\epsilon,\beta} \\
&= \sum_{i=1}^{8} I_{i,\epsilon,\beta},
\end{aligned} \tag{2.57}$$

onde

$$\begin{aligned}
I_{1,\epsilon,\beta} &= \epsilon \partial_x (\eta'(u_{\epsilon,\beta}) \partial_x u_{\epsilon,\beta}); \\
I_{2,\epsilon,\beta} &= -\epsilon \eta''(u_{\epsilon,\beta}) (\partial_x u_{\epsilon,\beta})^2; \\
I_{3,\epsilon,\beta} &= -\beta b_1 \partial_x (\eta'(u_{\epsilon,\beta}) \partial_x^2 u_{\epsilon,\beta}); \\
I_{4,\epsilon,\beta} &= \beta b_1 \eta''(u_{\epsilon,\beta}) \partial_x u_{\epsilon,\beta} \partial_x^2 u_{\epsilon,\beta}; \\
I_{5,\epsilon,\beta} &= -\beta b_2 \partial_x (\eta'(u_{\epsilon,\beta}) \partial_t \partial_x u_{\epsilon,\beta}); \\
I_{6,\epsilon,\beta} &= \beta b_2 \eta''(u_{\epsilon,\beta}) \partial_x u_{\epsilon,\beta} \partial_t \partial_x u_{\epsilon,\beta}; \\
I_{7,\epsilon,\beta} &= -\beta^2 c \partial_x (\eta'(u_{\epsilon,\beta}) \partial_t \partial_x^3 u_{\epsilon,\beta}); \\
I_{8,\epsilon,\beta} &= \beta^2 c \eta''(u_{\epsilon,\beta}) \partial_x u_{\epsilon,\beta} \partial_t \partial_x^3 u_{\epsilon,\beta}.
\end{aligned}$$

Sendo η convexa, $\eta'' \geq 0$. Logo, se $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ for uma função não-negativa

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} [\partial_t \eta(u_{\epsilon_k,\beta_k}) + \partial_x q(u_{\epsilon_k,\beta_k})] \, \phi dt dx &= \epsilon_k \int_0^\infty \int_{\mathbb{R}} [\partial_x (\eta'(u_{\epsilon_k,\beta_k}) \partial_x u_{\epsilon_k,\beta_k})] \, \phi dt dx \\
&- \epsilon_k \int_0^\infty \int_{\mathbb{R}} \eta''(u_{\epsilon_k,\beta_k}) (\partial_x u_{\epsilon_k,\beta_k})^2 \phi dt dx - \beta_k b_1 \int_0^\infty \int_{\mathbb{R}} \partial_x (\eta'(u_{\epsilon_k,\beta_k}) \partial_x^2 u_{\epsilon_k,\beta_k}) \phi dt dx \\
&+ \beta_k b_1 \int_0^\infty \int_{\mathbb{R}} \eta''(u_{\epsilon_k,\beta_k}) \partial_x u_{\epsilon_k,\beta_k} \partial_x^2 u_{\epsilon_k,\beta_k} \phi dt dx - \beta_k b_2 \int_0^\infty \int_{\mathbb{R}} \partial_x (\eta'(u_{\epsilon_k,\beta_k}) \partial_t \partial_x u_{\epsilon_k,\beta_k}) \phi dt dx \\
&+ \beta_k b_2 \int_0^\infty \int_{\mathbb{R}} \eta''(u_{\epsilon_k,\beta_k}) \partial_x u_{\epsilon_k,\beta_k} \partial_t \partial_x u_{\epsilon_k,\beta_k} \phi dt dx - \beta_k^2 c \int_0^\infty \int_{\mathbb{R}} \partial_x (\eta'(u_{\epsilon_k,\beta_k}) \partial_t \partial_x^3 u_{\epsilon_k,\beta_k}) \phi dt dx \\
&+ \beta_k^2 c \int_0^\infty \int_{\mathbb{R}} \eta''(u_{\epsilon_k,\beta_k}) \partial_x u_{\epsilon_k,\beta_k} \partial_t \partial_x^3 u_{\epsilon_k,\beta_k} \phi dt dx \\
\leq &- \epsilon_k \int_0^\infty \int_{\mathbb{R}} \eta'(u_{\epsilon_k,\beta_k}) \partial_x u_{\epsilon_k,\beta_k} \partial_x \phi dt dx + \beta_k b_1 \int_0^\infty \int_{\mathbb{R}} \eta'(u_{\epsilon_k,\beta_k}) \partial_x^2 u_{\epsilon_k,\beta_k} \partial_x \phi dt dx \\
&+ \beta_k b_1 \int_0^\infty \int_{\mathbb{R}} \eta''(u_{\epsilon_k,\beta_k}) \partial_x u_{\epsilon_k,\beta_k} \partial_x^2 u_{\epsilon_k,\beta_k} \phi dt dx + \beta_k b_2 \int_0^\infty \int_{\mathbb{R}} \eta'(u_{\epsilon_k,\beta_k}) \partial_t \partial_x u_{\epsilon_k,\beta_k} \partial_x \phi dt dx \\
&+ \beta_k b_2 \int_0^\infty \int_{\mathbb{R}} \eta''(u_{\epsilon_k,\beta_k}) \partial_x u_{\epsilon_k,\beta_k} \partial_t \partial_x u_{\epsilon_k,\beta_k} \phi dt dx + \beta_k^2 c \int_0^\infty \int_{\mathbb{R}} \eta'(u_{\epsilon_k,\beta_k}) \partial_t \partial_x^3 u_{\epsilon_k,\beta_k} \partial_x \phi dt dx \\
&+ \beta_k^2 c \int_0^\infty \int_{\mathbb{R}} \eta''(u_{\epsilon_k,\beta_k}) \partial_x u_{\epsilon_k,\beta_k} \partial_t \partial_x^3 u_{\epsilon_k,\beta_k} \phi dt dx \\
\leq & \, \epsilon_k \|\eta'\|_{L^\infty(\mathbb{R})} \int_0^\infty \int_{\mathbb{R}} |\partial_x u_{\epsilon_k,\beta_k} \partial_x \phi| dt dx + \beta_k |b_1| \|\eta'\|_{L^\infty(\mathbb{R})} \int_0^\infty \int_{\mathbb{R}} |\partial_x^2 u_{\epsilon_k,\beta_k} \partial_x \phi| dt dx \\
&+ \beta_k |b_1| \|\eta''\|_{L^\infty(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \int_0^\infty \int_{\mathbb{R}} |\partial_x u_{\epsilon_k,\beta_k} \partial_x^2 u_{\epsilon_k,\beta_k}| dt dx
\end{aligned}$$

$$
\begin{aligned}
&+ \beta_k|b_2|\|\eta'\|_{L^\infty(\mathbb{R})}\int_0^\infty\!\!\int_{\mathbb{R}}|\partial_t\partial_x u_{\epsilon_k,\beta_k}\partial_x\phi|dtdx\\
&+ \beta_k|b_2|\|\eta''\|_{L^\infty(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\int_0^\infty\!\!\int_{\mathbb{R}}|\partial_x u_{\epsilon_k,\beta_k}\partial_t\partial_x u_{\epsilon_k,\beta_k}|dtdx\\
&+ \beta_k^2 c\|\eta'\|_{L^\infty(\mathbb{R})}\int_0^\infty\!\!\int_{\mathbb{R}}|\partial_t\partial_x^3 u_{\epsilon_k,\beta_k}\partial_x\phi|dtdx\\
&+ \beta_k^2 c\|\eta''\|_{L^\infty(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\int_0^\infty\!\!\int_{\mathbb{R}}|\partial_x u_{\epsilon_k,\beta_k}\partial_t\partial_x^3 u_{\epsilon_k,\beta_k}|dtdx\\
\le\ & C_0\epsilon_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x\phi\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x\phi\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k\|\partial_t\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x\phi\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_t\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k^2\|\partial_t\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x\phi\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k^2\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_t\partial_x^3 u_{\epsilon_k\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
\le\ & C_0\epsilon_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k\|\partial_t\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_t\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k^2\|\partial_t\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&+ C_0\beta_k^2\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_t\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}.
\end{aligned}
$$

A partir dos Lemas 2.4, 2.5 e 2.6 obtemos

$$
\begin{aligned}
\epsilon_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \epsilon_k^{1/2}\left\{\epsilon_k\int_0^\infty\|\partial_x u_{\epsilon_k,\beta_k}(t,\cdot)\|^2_{L^2(\mathbb{R})}dt\right\}^{1/2}\\
&\le C_0\epsilon_k^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
\beta_k\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta_k^{1/2}\epsilon_k^{-1/2}\|\beta_k^{1/2}\epsilon_k^{1/2}\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\le C_0\beta_k^{1/2}\epsilon_k^{-1/2},
\end{aligned}
$$

$$
\begin{aligned}
\beta_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\le C_0\beta_k^{1/2}\epsilon_k^{-1/2}\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\le C_0\beta_k^{1/2}\epsilon_k^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\beta_k\|\partial_t\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta_k^{1/4}\epsilon_k^{-1/2}\|\beta_k^{3/4}\epsilon_k^{1/2}\partial_t\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\le C_0\beta_k^{1/4}\epsilon_k^{-1/2},
\end{aligned}
$$

$$\begin{aligned}\beta_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_t\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\leq C_0\beta_k^{1/4}\epsilon_k^{-1/2}\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\ &\leq C_0\beta_k^{1/4}\epsilon_k^{-1},\end{aligned}$$

$$\begin{aligned}\beta_k^2\|\partial_t\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta_k^{1/4}\epsilon_k^{-1/2}\|\beta_k^{7/4}\epsilon_k^{1/2}\partial_t\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\ &\leq C_0\beta_k^{1/4}\epsilon_k^{-1/2}\end{aligned}$$

e

$$\begin{aligned}\beta_k^2\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_t\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\leq C_0\beta_k^{1/4}\epsilon_k^{-1/2}\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\ &\leq C_0\beta_k^{1/4}\epsilon_k^{-1}.\end{aligned}$$

Combinando estas estimativas e usando o fato de que $\epsilon_k, \beta_k \in (0,1)$ obtemos

$$\int_0^\infty\int_{\mathbb{R}} [\partial_t\eta(u_{\epsilon_k,\beta_k}) + \partial_x q(u_{\epsilon_k,\beta_k})]\,\phi dtdx \leq C_0\epsilon_k^{1/2} + C_0\beta_k^{1/4}\epsilon_k^{-1},$$

donde

$$\int_0^\infty\int_{\mathbb{R}} [\eta(u_{\epsilon_k,\beta_k})\partial_t\phi + q(u_{\epsilon_k,\beta_k})\partial_x\phi]\,dtdx \geq -C_0\epsilon_k^{1/2} - C_0\beta_k^{1/4}\epsilon_k^{-1}.$$

Fazendo $k\to\infty$ na expressão acima, a condição $\beta = o(\epsilon^4)$ (crucial nesta passagem) nos diz que

$$\int_0^\infty\int_{\mathbb{R}} \left[\left\langle \nu_{(\cdot)}, \eta(\lambda)\right\rangle \partial_t\phi + \left\langle \nu_{(\cdot)}, q(\lambda)\right\rangle \partial_x\phi\right] dtdx \geq 0$$

e portanto

$$\partial_t \left\langle \nu_{(\cdot)}, \eta(\lambda)\right\rangle + \partial_x \left\langle \nu_{(\cdot)}, q(\lambda)\right\rangle \leq 0 \tag{2.58}$$

no sentido distribucional. A desigualdade (1.9) é obtida a partir de (2.58) e de uma regularização padrão da função $\phi_\alpha(u) = |u-\alpha|$ para todo $\alpha\in\mathbb{R}$ (Vide Apêndice B).

Mostramos então que ν é uma solução de entropia m-v para o problema (2.36)–(2.37), e portanto o Corolário 1.3 nos diz que $u_{\epsilon,\beta}\to u$ em $L^\infty(\mathbb{R}_+, L^r_{loc}(\mathbb{R}))$ quando $\epsilon\to 0$ para todo $r\in[1,2)$, sendo $u\in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}))$ a única solução de entropia de (2.36)–(2.37). □

Agora consideremos o problema (2.1)–(2.2) com f suave satisfazendo

$$|f'(u)| \leq C_0(1+|u|) \quad u\in\mathbb{R}, \tag{2.59}$$

$u_{\epsilon,\beta,0}\in C_c^\infty(\mathbb{R})$ satisfazendo (2.35) e seja $u_{\epsilon,\beta}\in C([0,\infty), H^5(\mathbb{R}))$ uma solução do mesmo.

Teorema 2.5. *Nas condições acima, se $\beta = O(\epsilon^4)$, existirão uma subsequência $\{u_{\epsilon_k,\beta_k}\}_{k\in\mathbb{N}}$ com $\epsilon_k,\beta_k\to 0$ e uma função $u\in L^2_{loc}(\mathbb{R}_+\times\mathbb{R})$ tais que $u_{\epsilon_k,\beta_k}\rightharpoonup u, f(u_{\epsilon_k,\beta_k})\rightharpoonup f(u)$ em $\mathscr{D}'(\mathbb{R}_+\times\mathbb{R})$ e u é uma solução fraca de*

$$\partial_t u + \partial_x f(u) = 0 \ \text{ em } \mathbb{R}_+\times\mathbb{R}. \tag{2.60}$$

Além disso, $u_{\epsilon_k,\beta_k}\to u$ fortemente em $L^r_{loc}(\mathbb{R}_+\times\mathbb{R})$ para todo $r\in[1,2)$ se $f''>0$.

Demonstração. Seja (η, q) um par de entropia–fluxo de entropia $\eta, q : \mathbb{R} \to \mathbb{R}$ com $\eta \in C_c^2(\mathbb{R})$ convexa em algum intervalo limitado não-vazio e q dada por (2.56). Argumentando como em (2.57) temos a seguinte decomposição

$$\partial_t \eta(u_{\epsilon,\beta}) + \partial_x q(u_{\epsilon,\beta}) = \sum_{i=1}^{8} I_{i,\epsilon,\beta}. \tag{2.61}$$

As afirmações abaixo nos fornecerão informações sobre cada elemento $I_{i,\epsilon,\beta}$.

Afirmação 1. $I_{i,\epsilon,\beta} \to 0$ em $H^{-1}(\mathbb{R}_+ \times \mathbb{R})$ quando $\epsilon \to 0$ $(i = 1, 3, 5, 7)$.

De fato, devido ao Lema 2.4

$$\begin{aligned}\|\epsilon\eta'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \epsilon^2 \int_0^\infty\!\!\int_{\mathbb{R}} |\eta'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}|^2 dtdx \\ &\leq C_0\epsilon\Big(\epsilon\int_0^\infty \|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}dt\Big) \\ &\leq C_0\epsilon.\end{aligned}$$

Logo, se $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$

$$\begin{aligned}|\langle I_{1,\epsilon,\beta}, \phi\rangle| &= \left|\int_0^\infty\!\!\int_{\mathbb{R}} \epsilon\eta'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x\phi dtdx\right| \\ &\leq \|\epsilon\eta'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x\phi\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \\ &\leq C_0\epsilon^{1/2},\end{aligned}$$

e consequentemente $I_{1,\epsilon,\beta} \to 0$ em $H^{-1}(\mathbb{R}_+ \times \mathbb{R})$ quando $\epsilon \to 0$. Analogamente, a hipótese $\beta = O(\epsilon^4)$ e os Lemas 2.5, 2.6 implicam em

$$\begin{aligned}\|\beta b_1\eta'(u_{\epsilon,\beta})\partial_x^2 u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta^2 b_1^2 \int_0^\infty\!\!\int_{\mathbb{R}} |\eta'(u_{\epsilon,\beta})\partial_x^2 u_{\epsilon,\beta}|^2 dtdx \\ &\leq C_0\beta^2\int_0^\infty \|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}dt \\ &\leq C_0\epsilon^3\Big(\beta\epsilon\int_0^\infty \|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}dt\Big) \\ &\leq C_0\epsilon^3,\end{aligned}$$

$$\begin{aligned}\|\beta b_2\eta''(u_{\epsilon,\beta})\partial_t\partial_x u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta^2 b_2^2 \int_0^\infty\!\!\int_{\mathbb{R}} |\eta''(u_{\epsilon,\beta})\partial_t\partial_x u_{\epsilon,\beta}|^2 dtdx \\ &\leq C_0\beta^2\int_0^\infty \|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}dt \\ &= C_0\epsilon^{-1}\beta^{1/2}\|\beta^{3/4}\epsilon^{1/2}\partial_t\partial_x u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} \\ &\leq C_0\epsilon^{-1}\beta^{1/2} \\ &\leq C_0\epsilon\end{aligned}$$

e

$$\begin{aligned}
\|\beta^2 c\eta'(u_{\epsilon,\beta})\partial_t\partial_x^3 u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta^4 c^2 \int_0^\infty\!\!\int_{\mathbb{R}} |\eta'(u_{\epsilon,\beta})\partial_t\partial_x^3 u_{\epsilon,\beta}|^2 dtdx \\
&\le C_0\beta^4 \int_0^\infty\!\!\int_{\mathbb{R}} |\partial_t\partial_x^3 u_{\epsilon,\beta}|^2 dtdx \\
&= C_0\epsilon^{-1}\beta^{1/2}\|\beta^{7/4}\epsilon^{1/2}\partial_t\partial_x^3 u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} \\
&\le C_0\epsilon^{-1}\beta^{1/2} \\
&\le C_0\epsilon.
\end{aligned}$$

As convergências $I_{3,\epsilon,\beta}, I_{5,\epsilon,\beta}, I_{7,\epsilon,\beta} \to 0$ em $H^{-1}(\mathbb{R}_+\times\mathbb{R})$ quando $\epsilon \to 0$ seguem imediatamente.

Afirmação 2. $I_{i,\epsilon,\beta}$ são limitadas $L^1(\mathbb{R}_+\times\mathbb{R})$ $(i = 2, 4, 6, 8)$.

A verificação deste fato é simples. Com efeito, a partir da hipótese $\beta = O(\epsilon^4)$ e dos Lemas 2.4, 2.5 e 2.6, obtemos

$$\begin{aligned}
\|I_{2,\epsilon,\beta}\|_{L^1(\mathbb{R}_+\times\mathbb{R})} &= \int_0^\infty\!\!\int_{\mathbb{R}} |\epsilon\eta''(u_{\epsilon,\beta})(\partial_x u_{\epsilon,\beta})^2| dtdx \\
&\le C_0\Big(\epsilon\int_0^\infty \|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} dt\Big) \\
&\le C_0,
\end{aligned}$$

$$\begin{aligned}
\|I_{4,\epsilon,\beta}\|_{L^1(\mathbb{R}_+\times\mathbb{R})} &= \int_0^\infty\!\!\int_{\mathbb{R}} |\beta b_1\eta''(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}| dtdx \\
&\le C_0\beta \int_0^\infty\!\!\int_{\mathbb{R}} |\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}| dtdx \\
&= C_0 \int_0^\infty\!\!\int_{\mathbb{R}} |\epsilon^{1/2}\partial_x u_{\epsilon,\beta}||\beta\epsilon^{-1/2}\partial_x^2 u_{\epsilon,\beta}| dtdx \\
&\le C_0\epsilon \int_0^\infty\!\!\int_{\mathbb{R}} |\partial_x u_{\epsilon,\beta}|^2 dtdx + C_0\beta^2\epsilon^{-1} \int_0^\infty\!\!\int_{\mathbb{R}} |\partial_x^2 u_{\epsilon,\beta}|^2 dtdx \\
&\le C_0 + C_0\epsilon^2\Big(\beta\epsilon\int_0^\infty \|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} dt\Big) \\
&\le C_0 + C_0\epsilon^2 \\
&\le C_0,
\end{aligned}$$

$$\begin{aligned}
\|I_{6,\epsilon,\beta}\|_{L^1(\mathbb{R}_+\times\mathbb{R})} &= \int_0^\infty\!\!\int_{\mathbb{R}} |\beta b_2\eta''(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_t\partial_x u_{\epsilon,\beta}| dtdx \\
&\le C_0\beta \int_0^\infty\!\!\int_{\mathbb{R}} |\partial_x u_{\epsilon,\beta}\partial_t\partial_x u_{\epsilon,\beta}| dtdx \\
&= C_0 \int_0^\infty\!\!\int_{\mathbb{R}} |\beta^{1/4}\epsilon^{-1/2}\partial_x u_{\epsilon,\beta}||\beta^{3/4}\epsilon^{1/2}\partial_t\partial_x u_{\epsilon,\beta}| dtdx
\end{aligned}$$

$$
\begin{aligned}
&\leq C_0\beta^{1/2}\epsilon^{-1}\int_0^\infty \|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}dt + C_0\|\beta^{3/4}\epsilon^{1/2}\partial_t\partial_x u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\leq C_0\epsilon\int_0^\infty \|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}dt + C_0\\
&\leq C_0
\end{aligned}
$$

e

$$
\begin{aligned}
\|I_{8,\epsilon,\beta}\|_{L^1(\mathbb{R}_+\times\mathbb{R})} &= \int_0^\infty\int_{\mathbb{R}} |\beta^2 c\eta''(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_t\partial_x^3 u_{\epsilon,\beta}|dtdx\\
&\leq C_0\beta^2\int_0^\infty\int_{\mathbb{R}} |\partial_x u_{\epsilon,\beta}\partial_t\partial_x^3 u_{\epsilon,\beta}|dtdx\\
&= C_0\int_0^\infty |\beta^{1/4}\epsilon^{-1/2}\partial_x u_{\epsilon,\beta}||\beta^{7/4}\epsilon^{1/2}\partial_t\partial_x^3 u_{\epsilon,\beta}|dtdx\\
&\leq C_0\beta^{1/2}\epsilon^{-1}\int_0^\infty \|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}dt + C_0\|\beta^{7/4}\epsilon^{1/2}\partial_t\partial_x^3 u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\leq C_0\epsilon\int_0^\infty \|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}dt + C_0\\
&\leq C_0.
\end{aligned}
$$

Agora defina

$$T_{\epsilon,\beta} = I_{1,\epsilon,\beta} + I_{3,\epsilon,\beta} + I_{5,\epsilon,\beta} + I_{7,\epsilon,\beta}$$

e

$$\mu_{\epsilon,\beta} = I_{2,\epsilon,\beta} + I_{4,\epsilon,\beta} + I_{6,\epsilon,\beta} + I_{8,\epsilon,\beta},$$

de modo que

$$\partial_t\eta(u_{\epsilon,\beta}) + \partial_x q(u_{\epsilon,\beta}) = T_{\epsilon,\beta} + \mu_{\epsilon,\beta}.$$

As Afirmações 1 e 2 nos dizem (respectivamente) que $\{T_{\epsilon,\beta}\}_{\epsilon,\beta}$ é compacto em $H^{-1}(\mathbb{R}_+\times\mathbb{R})$ e $\{\mu_{\epsilon,\beta}\}_{\epsilon,\beta}$ é limitado de $\mathcal{M}(\mathbb{R}_+\times\mathbb{R})$. Sendo η uma função de suporte compacto, claramente $\{\partial_t\eta(u_{\epsilon,\beta}) + \partial_x q(u_{\epsilon,\beta})\}_{\epsilon,\beta}$ é uma sequência limitada em $W^{-1,\infty}(\mathbb{R}_+\times\mathbb{R})$. Portanto, pelo Lema de Murat, a sequência de distribuições $\{\partial_t\eta(u_{\epsilon,\beta})+\partial_x q(u_{\epsilon,\beta})\}_{\epsilon,\beta}$ pertence a um subconjunto compacto de $H^{-1}(\Omega)$ se Ω for um subconjunto aberto limitado de $\mathbb{R}_+\times\mathbb{R}$. A primeira parte do teorema é uma consequência imediata do Teorema 1.4 e de um argumento diagonal padrão, e a segunda resulta do Corolário 1.2. □

2.3 Estimativas a priori e Convergência em L^4

Para cada $\epsilon,\beta\in(0,1)$, sejam $u_{\epsilon,\beta}\in C([0,\infty),H^5(\mathbb{R}))$ uma solução de (2.1)–(2.2) onde b_1, b_2 e c são tais que $b_1\in\mathbb{R}, b_2<0$ e $c>0$, f é uma função suave satisfazendo (2.59) e $u_{\epsilon,\beta,0}\in C_c^\infty(\mathbb{R})$. Suponhamos também que $u_{\epsilon,\beta,0}$ seja uma aproximação da função real

$$u_0\in L^1(\mathbb{R})\cap L^4(\mathbb{R}) \tag{2.62}$$

tal que

$$u_{\epsilon,\beta,0} \to u_0 \text{ em } L^1(\mathbb{R}) \cap L^4(\mathbb{R}) \text{ quando } \epsilon, \beta \to 0 \tag{2.63}$$

e

$$\begin{cases} \|u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \|u_{\epsilon,\beta,0}\|^4_{L^4(\mathbb{R})} + (\beta^{1/2}+\epsilon^2)\|\partial_x u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \leq C_0 \\ (\beta^{3/2} + \beta\epsilon^2)\|\partial_x^2 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + (\beta^{5/2} + \beta^2\epsilon^2)\|\partial_x^3 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \leq C_0 \end{cases} \tag{2.64}$$

com $C_0 > 0$ independente de ϵ e β.

Provaremos nesta seção o seguinte

Teorema 2.6. *Nas condições acima, se $\beta = O(\epsilon^4)$ existe uma função $u \in L^\infty(\mathbb{R}_+, L^4(\mathbb{R}))$ tal que $u_{\epsilon,\beta} \to u$ em $L^r_{loc}(\mathbb{R}_+ \times \mathbb{R})$ para todo $r \in [1, 4)$, sendo u a única solução de entropia de*

$$\partial_t u + \partial_x f(u) = 0 \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \tag{2.65}$$

$$u(0, x) = u_0(x) \quad x \in \mathbb{R}. \tag{2.66}$$

Estabeleceremos a seguir algumas estimativas sobre cada $u_{\epsilon,\beta}$.

Lema 2.7. *Assumindo* (2.64)*, existe uma constante $C_0 > 0$ (independente de ϵ e β) tal que*

$$\|u_{\epsilon,\beta}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \leq C_0\beta^{-1/4}. \tag{2.67}$$

Demonstração. Isto resulta imediatamente do Lema 2.4 uma vez que $b_2 \neq 0$. □

Lema 2.8. *Suponhamos* (2.64) *e $\beta = O(\epsilon^4)$. Então as seguintes afirmações são válidas:*

(i) a família $\{u_{\epsilon,\beta}\}$ é limitada em $L^\infty(\mathbb{R}_+, L^4(\mathbb{R}))$;

(ii) as famílias $\{\epsilon\partial_x u_{\epsilon,\beta}\}_{\epsilon,\beta}$, $\{\beta^{1/2}\epsilon\partial_x^2 u_{\epsilon,\beta}\}_{\epsilon,\beta}$ e $\{\beta\epsilon\partial_x^3 u_{\epsilon,\beta}\}_{\epsilon,\beta}$ são limitadas em $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}))$;

(iii) as famílias $\{(\beta\epsilon)^{1/2}\partial_t\partial_x u_{\epsilon,\beta}\}_{\epsilon,\beta}$, $\{\beta\epsilon^{1/2}\partial_t\partial_x^2 u_{\epsilon,\beta}\}_{\epsilon,\beta}$, $\{\beta^{3/2}\epsilon^{1/2}\partial_t\partial_x^3 u_{\epsilon,\beta}\}_{\epsilon,\beta}$, $\{\epsilon^{3/2}\partial_x^2 u_{\epsilon,\beta}\}_{\epsilon,\beta}$ e $\{\epsilon^{1/2}u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\}_{\epsilon,\beta}$ são limitadas em $L^2(\mathbb{R}_+ \times \mathbb{R})$.

Demonstração. Multiplicando (2.1) por

$$u^3_{\epsilon,\beta} - A\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} - B\epsilon^2\partial_x^2 u_{\epsilon,\beta}$$

onde A e B são constantes positivas escolhidas a posteriori, e integrando cada termo da expressão resultante, obtemos

$$\int_{\mathbb{R}} (u^3_{\epsilon,\beta} - A\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} - B\epsilon^2\partial_x^2 u_{\epsilon,\beta})\partial_t u_{\epsilon,\beta} dx = \frac{1}{4}\frac{d}{dt}\|u_{\epsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}$$
$$+A\beta\epsilon\|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{B\epsilon^2}{2}\frac{d}{dt}\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 - A\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} - B\epsilon^2\partial_x^2 u_{\epsilon,\beta})\partial_x f(u_{\epsilon,\beta})dx = -A\beta\epsilon\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx$$
$$-B\epsilon^2\int_{\mathbb{R}}\partial_x^2 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx,$$

$$\beta b_1\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 - A\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} - B\epsilon^2\partial_x^2 u_{\epsilon,\beta})\partial_x^3 u_{\epsilon,\beta}dx = -3\beta b_1\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}dx$$
$$-A\beta^2\epsilon b_1\int_{\mathbb{R}}\partial_x^3 u_{\epsilon,\beta}\partial_t\partial_x^2 u_{\epsilon,\beta}dx,$$

$$\beta b_2\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 - A\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} - B\epsilon^2\partial_x^2 u_{\epsilon,\beta})\partial_t\partial_x^2 u_{\epsilon,\beta}dx = -3\beta b_2\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_t\partial_x u_{\epsilon,\beta}dx$$
$$+A\beta^2\epsilon|b_2|\|\partial_t\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \frac{B|b_2|}{2}\beta\epsilon^2\frac{d}{dt}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2,$$

$$\beta^2 c\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 - A\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} - B\epsilon^2\partial_x^2 u_{\epsilon,\beta})\partial_t\partial_x^4 u_{\epsilon,\beta}dx = -3\beta^2 c\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_t\partial_x^3 u_{\epsilon,\beta}dx$$
$$+A\beta^3\epsilon c\|\partial_t\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \frac{Bc}{2}\beta^2\epsilon^2\frac{d}{dt}\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2$$

e

$$\epsilon\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 - A\beta\epsilon\partial_t\partial_x^2 u_{\epsilon,\beta} - B\epsilon^2\partial_x^2 u_{\epsilon,\beta})\partial_x^2 u_{\epsilon,\beta}dx = -3\epsilon\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2$$
$$-\frac{A}{2}\beta\epsilon^2\frac{d}{dt}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 - B\epsilon^3\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2.$$

Assim,

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{4}\|u_{\epsilon,\beta}(t,\cdot)\|_{L^4(\mathbb{R})}^4 + \frac{B}{2}\epsilon^2\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{2}(A+B|b_2|)\beta\epsilon^2\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2\right.\\
&\quad\left. + \frac{Bc}{2}\beta^2\epsilon^2\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2\right] + A\beta\epsilon\|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + A\beta^2\epsilon|b_2|\|\partial_t\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2\\
&\quad + Ac\beta^3\epsilon\|\partial_t\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + 3\epsilon\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + B\epsilon^3\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2\\
&= 3b_1\beta\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}dx + Ab_1\beta^2\epsilon\int_{\mathbb{R}}\partial_x^3 u_{\epsilon,\beta}\partial_t\partial_x^2 u_{\epsilon,\beta}dx \qquad (2.68)\\
&\quad + 3b_2\beta\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_t\partial_x u_{\epsilon,\beta}dx + 3c\beta^2\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_t\partial_x^3 u_{\epsilon,\beta}dx\\
&\quad + A\beta\epsilon\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx + B\epsilon^2\int_{\mathbb{R}}\partial_x^2 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx.
\end{aligned}$$

A partir da hipótese $\beta = O(\epsilon^4)$ existe uma constante positiva D_0 (especificada posteriormente) tal que

$$\beta \leq D_0\epsilon^4. \qquad (2.69)$$

Então, usando (2.67) e (2.69) segue-se que

$$
\begin{aligned}
3b_1\beta\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}dx &\le 3\beta|b_1|\|u_{\epsilon,\beta}\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}|dx\\
&\le C_0\beta^{1/2}\int_{\mathbb{R}}|\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}|dx\\
&= \beta^{1/2}\int_{\mathbb{R}}\left|2C_0B^{-1/2}D_0^{1/4}\epsilon^{-1/2}\partial_x u_{\epsilon,\beta}\right|\left|\frac{1}{2}B^{1/2}D_0^{-1/4}\epsilon^{1/2}\partial_x^2 u_{\epsilon,\beta}\right|dx\\
&\le \beta^{1/2}\left\{C_0B^{-1}D_0^{1/2}\epsilon^{-1}\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{B}{8}D_0^{-1/2}\epsilon\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right\}\\
&\le C_0B^{-1}D_0\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{B}{8}\epsilon^3\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
3b_2\beta\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_t\partial_x u_{\epsilon,\beta}dx &\le \beta\int_{\mathbb{R}}\left|3b_2A^{-1/2}\epsilon^{-1/2}u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\right|\left|A^{1/2}\epsilon^{1/2}\partial_t\partial_x u_{\epsilon,\beta}\right|dx\\
&\le C_0A^{-1}\beta\epsilon^{-1}\|u_{\epsilon,\beta}\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{A}{2}\beta\epsilon\|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C_0A^{-1}\beta^{1/2}\epsilon^{-1}\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{A}{2}\beta\epsilon\|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C_0A^{-1}D_0^{1/2}\epsilon\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{A}{2}\beta\epsilon\|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}
\end{aligned}
$$

e

$$
\begin{aligned}
&3\beta^2c\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_t\partial_x^3 u_{\epsilon,\beta}dx\\
&\quad\le c\int_{\mathbb{R}}\left|6A^{-1/2}\beta^{1/2}\epsilon^{-1/2}u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\right|\left|\frac{A^{1/2}}{2}\beta^{3/2}\epsilon^{1/2}\partial_t\partial_x^3 u_{\epsilon,\beta}\right|dx\\
&\quad\le C_0A^{-1}\beta^{1/2}\epsilon^{-1}\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{Ac}{8}\beta^3\epsilon\|\partial_t\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad\le C_0A^{-1}D_0^{1/2}\epsilon\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{Ac}{8}\beta^3\epsilon\|\partial_t\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Como

$$|f'(u)|\le C_0(1+|u|)$$

segue-se que

$$
\begin{aligned}
A\beta\epsilon\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx &= A\beta\epsilon\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\epsilon,\beta}f'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}dx\\
&\le C_0A\beta\epsilon\int_{\mathbb{R}}|\partial_t\partial_x^2 u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}|dx+C_0A\beta\epsilon\int_{\mathbb{R}}|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_t\partial_x^2 u_{\epsilon,\beta}|dx\\
&= A\epsilon\int_{\mathbb{R}}\left|\frac{\beta}{2}|b_2|^{1/2}\partial_t\partial_x^2 u_{\epsilon,\beta}\right|\left|2C_0|b_2|^{-1/2}\partial_x u_{\epsilon,\beta}\right|dx\\
&\quad+A\epsilon\int_{\mathbb{R}}\left|2C_0|b_2|^{-1/2}u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\right|\left|\frac{\beta}{2}|b_2|^{1/2}\partial_t\partial_x^2 u_{\epsilon,\beta}\right|dx
\end{aligned}
$$

$$\leq \frac{A|b_2|}{8}\beta^2\epsilon\|\partial_t\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0A\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$+ C_0A\epsilon\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{A|b_2|}{8}\beta^2\epsilon\|\partial_t\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$= \frac{A|b_2|}{4}\beta^2\epsilon\|\partial_t\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0A\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0A\epsilon\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

e

$$B\epsilon^2\int_{\mathbb{R}}\partial_x^2 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx = B\epsilon^2\int_{\mathbb{R}}\partial_x^2 u_{\epsilon,\beta} f'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}dx$$
$$\leq C_0B\epsilon^2\int_{\mathbb{R}}|\partial_x^2 u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}|dx + C_0B\epsilon^2\int_{\mathbb{R}}|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}|dx$$
$$= B\epsilon^2\int_{\mathbb{R}}\left|\frac{\epsilon^{1/2}}{2}\partial_x^2 u_{\epsilon,\beta}\right|\left|2C_0\epsilon^{-1/2}\partial_x u_{\epsilon,\beta}\right|dx + B\epsilon^2\int_{\mathbb{R}}\left|2C_0\epsilon^{-1/2}u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\right|\left|\frac{\epsilon^{1/2}}{2}\partial_x^2 u_{\epsilon,\beta}\right|dx$$
$$\leq \frac{B}{8}\epsilon^3\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0B\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$+ C_0B\epsilon\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{B}{8}\epsilon^3\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$= \frac{B}{4}\epsilon^3\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0B\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0B\epsilon\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Além disso,

$$Ab_1\beta^2\epsilon\int_{\mathbb{R}}\partial_x^3 u_{\epsilon,\beta}\partial_t\partial_x^2 u_{\epsilon,\beta}dx = -Ab_1\beta^2\epsilon\int_{\mathbb{R}}\partial_x^2 u_{\epsilon,\beta}\partial_t\partial_x^3 u_{\epsilon,\beta}dx$$
$$\leq A\beta\epsilon\int_{\mathbb{R}}\left|2b_1c^{-1/2}\partial_x^2 u_{\epsilon,\beta}\right|\left|\frac{\beta}{2}c^{1/2}\partial_t\partial_x^3 u_{\epsilon,\beta}\right|dx$$
$$\leq C_0A\beta\epsilon\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{Ac}{8}\beta^3\epsilon\|\partial_t\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Substituindo estas estimativas em (2.68) temos

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{4}\|u_{\epsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{B}{2}\epsilon^2\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}(A+B|b_2|)\beta\epsilon^2\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right.\\
&\quad\left. + \frac{Bc}{2}\beta^2\epsilon^2\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right] + \frac{A}{2}\beta\epsilon\|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{5B}{8}\epsilon^3\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + \frac{3A|b_2|}{4}\beta^2\epsilon\|\partial_t\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3Ac}{4}\beta^3\epsilon\|\partial_t\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + \epsilon\big[3 - C_0(A+B+A^{-1}D_0^{1/2})\big]\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\leq C_0(A+B+B^{-1}D_0)\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0A\beta\epsilon\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{2.70}$$

Nosso próximo passo será encontrar constantes positivas A e B tais que

$$3 - C_0(A+B+A^{-1}D_0^{1/2}) > 0. \tag{2.71}$$

Considerando a função polinomial

$$p(T) = T^2 + (B - 3C_0^{-1})T + D_0^{1/2}$$

a condição (2.71) equivale a $p(A) < 0$ para alguma constante $A > 0$. Pondo $B = 2C_0^{-1}$, então $B > 0$ e $p(T) = T^2 - C_0^{-1}T + D_0^{1/2}$. Escolhendo $D_0 > 0$ de modo que $D_0 < (2C_0)^{-4}$, o discrimante $\Delta = C_0^{-2} - 4D_0^{1/2}$ é positivo e a função p possui dois zeros $0 < T_1 < T_2$. Portanto, (2.71) é verificada quando $A \in (T_1, T_2)$. Fixado um $A \in (T_1, T_2)$ defina $K_1 = 3 - C_0(A + B + A^{-1}D_0^{1/2})$. Então $K_1 > 0$ e a partir de (2.70)

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{4}\|u_{\epsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{B}{2}\epsilon^2\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}(A + B|b_2|)\beta\epsilon^2\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right.\\
&\qquad \left. + \frac{Bc}{2}\beta^2\epsilon^2\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right] + \frac{A}{2}\beta\epsilon\|\partial_t\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad + \frac{3A|b_2|}{4}\beta^2\epsilon\|\partial_t\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3Ac}{4}\beta^3\epsilon\|\partial_t\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad + K_1\epsilon\|u_{\epsilon,\beta}(t,\cdot)\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{5B}{8}\epsilon^3\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad \leq K_2\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + K_3\beta\epsilon\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned} \tag{2.72}$$

sendo K_2 e K_3 duas constantes positivas.

Logo, integrando (2.72) em $(0, t)$ e usando (2.64) além dos Lemas 2.4 e 2.5 segue-se que

$$\begin{aligned}
&\frac{1}{4}\|u_{\epsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{B}{2}\epsilon^2\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}(A + B|b_2|)\beta\epsilon^2\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + \frac{Bc}{2}\beta^2\epsilon^2\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{A}{2}\beta\epsilon\int_0^t \|\partial_s\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad + \frac{3A|b_2|}{4}\beta^2\epsilon\int_0^t \|\partial_s\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \frac{3Ac}{4}\beta^3\epsilon\int_0^t \|\partial_s\partial_x^3 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + K_1\epsilon\int_0^t \|u_{\epsilon,\beta}(s,\cdot)\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \frac{5B}{8}\epsilon^3\int_0^t \|\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\leq K_2\epsilon\int_0^t \|\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + K_3\beta\epsilon\int_0^t \|\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \frac{1}{4}\|u_{\epsilon,\beta,0}\|^4_{L^4(\mathbb{R})}\\
&\quad + \frac{B}{2}\epsilon^2\|\partial_x u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \frac{1}{2}(A + B|b_2|)\beta\epsilon^2\|\partial_x^2 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \frac{Bc}{2}\beta^2\epsilon^2\|\partial_x^3 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})}\\
&\leq C_0
\end{aligned} \tag{2.73}$$

estabelecendo o lema. □

Passemos enfim à

Demonstração (do Teorema 2.6). Pelo Lema 2.8 $\{u_{\epsilon,\beta}\}_{\epsilon,\beta}$ é limitada em $L^\infty(\mathbb{R}_+; L^4(\mathbb{R}))$. Assim, o Lema 1.3 garante a existência de uma subsequência $\{u_{\epsilon_k,\beta_k}\}_{k\in\mathbb{N}}$ e uma medida de Young $\nu : \mathbb{R}_+ \times \mathbb{R} \to \text{Prob}(\mathbb{R})$ satisfazendo (2.40) para toda $g \in C(\mathbb{R})$ tal que $g(u) = O(1 + |u|^r)$ com $r \in [0, 4)$.

Argumentando como na demonstração do Teorema 2.4 verificaremos as condições (1.8) e (1.9).

$\nu_{(\cdot)}$ satisfaz (1.8).

Considere a função $g(\lambda) = \lambda^2$ e defina

$$G(\lambda, \lambda_0) = g(\lambda) - g(\lambda_0) - g'(\lambda_0)(\lambda - \lambda_0).$$

Pela Fórmula de Taylor, $G(\lambda,\lambda_0)=(\lambda-\lambda_0)^2$. Agora sejam $I\subset\mathbb{R}$ um compacto e $\mu:(0,T)\times I\times\mathbb{R}\to\mathbb{R}$ a medida de probabilidade dada por $d\mu(t,x,\lambda)=(T\mathrm{m}(I))^{-1}\,d\nu_{(t,x)}(\lambda)dtdx$. Então pela desigualdade de Jensen

$$\begin{aligned}
&\left\{(T\mathrm{m}(I))^{-1}\int_0^T\int_I\left\langle\nu_{(t,x)},|\lambda-u_0(x)|\right\rangle dtdx\right\}^2=\left\{\int_{(0,T)\times I\times\mathbb{R}}|\lambda-u_0(x)|d\mu(t,x,\lambda)\right\}^2\\
&\le\int_{(0,T)\times I\times\mathbb{R}}|\lambda-u_0(x)|^2d\mu(t,x,\lambda)=(T\mathrm{m}(I))^{-1}\int_0^T\int_I\left\langle\nu_{(t,x)},|\lambda-u_0(x)|^2\right\rangle dtdx\\
&=(T\mathrm{m}(I))^{-1}\int_0^T\int_I\left\langle\nu_{(t,x)},G(\lambda,u_0(x))\right\rangle dtdx,
\end{aligned}$$

e portanto

$$\frac{1}{T}\int_0^T\int_I\left\langle\nu_{(t,x)},|\lambda-u_0(x)|\right\rangle dtdx\le C_I\left\{\frac{1}{T}\int_0^T\int_I\left\langle\nu_{(t,x)},G(\lambda,u_0(x))\right\rangle dtdx\right\}^{1/2}. \tag{2.74}$$

Em seguida seja $\{\phi_n\}_{n\in\mathbb{N}}\subset C_c^\infty(\mathbb{R})$ tal que $\phi_n\to g'(u_0)$ em $L^2(\mathbb{R})$. Para cada $n\in\mathbb{N}$ temos

$$\begin{aligned}
&\int_0^T\int_I\left\langle\nu_{(t,x)},G(\lambda,u_0(x))\right\rangle dtdx=\int_0^T\int_I\left\langle\nu_{(t,x)},|\lambda|^2-|u_0(x)|^2-g'(u_0(x))(\lambda-u_0(x))\right\rangle dtdx\\
&=\int_0^T\int_I\left\langle\nu_{(t,x)},|\lambda|^2-|u_0(x)|^2\right\rangle dtdx+\int_0^T\int_I\left\langle\nu_{(t,x)},u_0(x)-\lambda\right\rangle\phi_n(x)dtdx\\
&+\int_0^T\int_I\left\langle\nu_{(t,x)},\lambda-u_0(x)\right\rangle(\phi_n(x)-g'(u_0(x)))dtdx.
\end{aligned}\tag{2.75}$$

Ora, (2.40), (a igualdade em) (2.19), (2.63) e (2.64) implicam que

$$\begin{aligned}
&\int_0^T\int_I\left\langle\nu_{(t,x)},|\lambda|^2-|u_0(x)|^2\right\rangle dtdx=\int_0^T\int_I\left\langle\nu_{(t,x)},|\lambda|^2\right\rangle dtdx-T\int_I|u_0(x)|^2dx\\
&=\lim_{k\to\infty}\int_0^T\int_I|u_{\epsilon_k\beta_k}(t,x)|^2dtdx-T\int_I|u_0(x)|^2dx\\
&\le\lim_{k\to\infty}T\left\{\|u_{\epsilon_k,\beta_k,0}\|^2_{L^2(\mathbb{R})}+C_0\beta_k^{1/2}\right\}-T\int_I|u_0(x)|^2dx\\
&=T\|u_0\|^2_{L^2(\mathbb{R})}-T\int_I|u_0(x)|^2dx\\
&=T\int_{\mathbb{R}\setminus I}|u_0(x)|^2dx
\end{aligned}\tag{2.76}$$

e

$$\begin{aligned}
&\int_0^T\int_I\left\langle\nu_{(t,x)},\lambda-u_0(x)\right\rangle(\phi_n(x)-g'(u_0(x)))dtdx\\
&=\int_0^T\int_I\left\langle\nu_{(t,x)},\lambda\right\rangle(\phi_n(x)-g'(u_0(x)))dtdx-T\int_I u_0(x)(\phi_n(x)-g'(u_0(x))dx\\
&\le\|\left\langle\nu_{(t,x)},\lambda\right\rangle\|_{L^2((0,T)\times I)}\|\phi_n-g'(u_0)\|_{L^2((0,T)\times I)}+T\|u_0\|_{L^2(\mathbb{R})}\|\phi_n-g'(u_0)\|_{L^2(\mathbb{R})}
\end{aligned}$$

$$
\begin{aligned}
&\leq T^{1/2}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}\left\{\int_0^T\int_I \left\langle \nu_{(t,x)}, \lambda^2\right\rangle dtdx\right\}^{1/2} + T\|u_0\|_{L^2(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}\\
&\leq T^{1/2}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}\left\{\lim_{k\to\infty}\int_0^T\int_I |u_{\epsilon_k,\beta_k}(t,x)|^2 dtdx\right\}^{1/2}\\
&\quad + T\|u_0\|_{L^2(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}\\
&\leq T^{1/2}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}\left\{\lim_{k\to\infty} T(\|u_{\epsilon_k,\beta_k,0}\|^2_{L^2(\mathbb{R})} + C_0\beta_k^{1/2})\right\}^{1/2}\\
&\quad + T\|u_0\|_{L^2(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}\\
&= 2T\|u_0\|_{L^2(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.77}
$$

Substituindo (2.76) e (2.77) em (2.75) segue-se que

$$
\begin{aligned}
\int_0^T\int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x))\right\rangle dtdx &\leq \int_0^T\int_I \left\langle \nu_{(t,x)}, u_0(x) - \lambda\right\rangle \phi_n(x)dtdx + T\int_{\mathbb{R}\setminus I}|u_0(x)|^2dx\\
&\quad + 2T\|u_0\|_{L^2(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

Agora seja $\{K_i\}_{i\in\mathbb{N}}$ uma sequência de compactos tal que

$$
I \subset K_1 \subset K_2 \subset \cdots \quad \text{e} \quad \bigcup_{i\in\mathbb{N}} K_i = \mathbb{R}.
$$

Então, sendo $G \geq 0$,

$$
\int_0^T\int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x))\right\rangle dtdx \leq \int_0^T\int_{K_i} \left\langle \nu_{(t,x)}, G(\lambda, u_0(x))\right\rangle dtdx
$$

e portanto

$$
\begin{aligned}
\frac{1}{T}\int_0^T\int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x))\right\rangle dtdx &\leq \frac{1}{T}\int_0^T\int_{K_i} \left\langle \nu_{(t,x)}, u_0(x) - \lambda\right\rangle \phi_n(x)dtdx\\
&\quad + \int_{\mathbb{R}\setminus K_i}|u_0(x)|^2dx + 2\|u_0\|_{L^2(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

Como $u_0 \in L^2(\mathbb{R})$,

$$
\lim_{i\to\infty}\int_{\mathbb{R}\setminus K_i}|u_0(x)|^2dx = 0,
$$

e daí

$$
\begin{aligned}
\frac{1}{T}\int_0^T\int_I \left\langle \nu_{(t,x)}, G(\lambda, u_0(x))\right\rangle dtdx &\leq \frac{1}{T}\int_0^T\int_{\mathbb{R}} \left\langle \nu_{(t,x)}, u_0(x) - \lambda\right\rangle \phi_n(x)dtdx\\
&\quad + 2\|u_0\|_{L^2(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.78}
$$

A partir de (2.78) e (2.74) concluímos que

$$
\begin{aligned}
&\frac{1}{T}\int_0^T\int_I \left\langle \nu_{(t,x)}, |\lambda - u_0(x)|\right\rangle dtdx\\
&\qquad \leq C_I\left\{\frac{1}{T}\int_0^T\int_{\mathbb{R}} \left\langle \nu_{(t,x)}, u_0(x) - \lambda\right\rangle \phi_n(x)dtdx + 2\|u_0\|_{L^2(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}\right\}^{1/2}.
\end{aligned}
\tag{2.79}
$$

Mostraremos a seguir que

$$\lim_{T\to 0}\frac{1}{T}\int_0^T\int_{\mathbb{R}}\left\langle \nu_{(t,x)}, u_0(x)-\lambda\right\rangle \phi_n(x)dtdx \le 0 \qquad n=1,2,3,\dots. \tag{2.80}$$

De fato, fixado $n\in\mathbb{N}$

$$\frac{1}{T}\int_0^T\int_{\mathbb{R}}\left\langle \nu_{(t,x)}, u_0(x)-\lambda\right\rangle \phi_n(x)dtdx = \lim_{k\to\infty}\frac{1}{T}\int_0^T\int_{\mathbb{R}}(u_0(x)-u_{\epsilon_k,\beta_k}(t,x))\phi_n(x)dtdx. \tag{2.81}$$

Por outro lado, note que

$$\begin{aligned}\frac{1}{T}\int_0^T\int_{\mathbb{R}}(u_0(x)-u_{\epsilon_k,\beta_k}(t,x))\phi_n(x)dtdx &= \int_{\mathbb{R}}(u_0(x)-u_{\epsilon_k,\beta_k,0}(x))\phi_n(x)dx\\ &-\frac{1}{T}\int_0^T\int_{\mathbb{R}}\left(\int_0^t\partial_s u_{\epsilon_k,\beta_k}(s,x)ds\right)\phi_n(x)dtdx\\ &= I_k^n+J_k^n,\end{aligned}$$

onde

$$I_k^n=\int_{\mathbb{R}}(u_0(x)-u_{\epsilon_k,\beta_k,0}(x))\phi_n(x)dx$$

e

$$J_k^n=-\frac{1}{T}\int_0^T\int_{\mathbb{R}}\left(\int_0^t\partial_s u_{\epsilon_k,\beta_k}(s,x)ds\right)\phi_n(x)dtdx.$$

Como

$$|I_k^n|\le \|u_0-u_{\epsilon_k,\beta_k,0}\|_{L^1(\mathrm{supp}(\phi_n))}\|\phi_n\|_{L^\infty(\mathrm{supp}(\phi_n))}$$

e $u_{\epsilon_k,\beta_k,0}\to u_0$ em $L^1_{loc}(\mathbb{R})$, resulta que $|I_k^n|\to 0$ quando $k\to\infty$. Agora estimaremos cada termo de J_k^n observando que

$$\begin{aligned}J_k^n &= -\frac{1}{T}\int_0^T\int_{\mathbb{R}}\left(\int_0^t\partial_s u_{\epsilon_k,\beta_k}(s,x)ds\right)\phi_n(x)dtdx\\ &= -\frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t\left[\epsilon_k\partial_x^2u_{\epsilon_k,\beta_k}(s,x)-\partial_x f(u_{\epsilon_k,\beta_k}(s,x))-\beta_k b_1\partial_x^3u_{\epsilon_k,\beta_k}(s,x)\right.\\ &\left.-\beta_k b_2\partial_s\partial_x^2u_{\epsilon_k,\beta_k}(s,x)-\beta_k^2c\partial_s\partial_x^4u_{\epsilon_k,\beta_k}(s,x)\right]\phi_n(x)dsdtdx.\end{aligned}$$

Pelo Lema 2.4 e observando que

$$|f(u)|\le C_0(1+|u|^2)$$

temos

$$\begin{aligned}-\frac{\epsilon_k}{T}\int_0^T\int_{\mathbb{R}}\int_0^t\partial_x^2u_{\epsilon_k,\beta_k}(s,x)\phi_n(x)dsdtdx &= -\frac{\epsilon_k}{T}\int_0^T\int_{\mathbb{R}}\int_0^t u_{\epsilon_k,\beta_k}(s,x)\partial_x^2\phi_n(x)dsdtdx\\ &\le \frac{\epsilon_k}{T}\int_0^T\int_{\mathbb{R}}\int_0^t|u_{\epsilon_k,\beta_k}(s,x)\partial_x^2\phi_n(x)|dsdtdx\end{aligned}$$

$$\leq \frac{\epsilon_k}{T}\int_0^T\int_0^t \|u_{\epsilon_k,\beta_k}(s,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^2\phi_n\|_{L^2(\mathbb{R})}dsdt$$
$$\leq \frac{C_0\epsilon_k}{T}\int_0^T\int_0^t dsdt$$
$$\leq C_0\epsilon_k T,$$

e

$$\frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_x f(u_{\epsilon_k,\beta_k}(s,x))\phi_n(x)dsdtdx = -\frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t f(u_{\epsilon_k,\beta_k}(s,x))\partial_x\phi_n(x)dsdtdx$$
$$\leq \frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t |f(u_{\epsilon_k,\beta_k}(s,x))\partial_x\phi_n(x)|dsdtdx$$
$$= \frac{1}{T}\int_0^T\int_{\mathrm{supp}(\phi_n)}\int_0^t |f(u_{\epsilon_k,\beta_k}(s,x))\partial_x\phi_n(x)|dsdtdx$$
$$\leq \frac{C_0}{T}\int_0^T\int_{\mathrm{supp}(\phi_n)}\int_0^t (1+|u_{\epsilon_k,\beta_k}(s,x)|^2)dsdtdx$$
$$\leq \frac{C_0}{T}m(\mathrm{supp}(\phi_n))\int_0^T\int_0^t dsdt + \frac{C_0}{T}\int_0^T\int_0^t \|u_{\epsilon_k,\beta_k}(s,\cdot)\|^2_{L^2(\mathbb{R})}dsdt$$
$$\leq C_0T + \frac{C_0}{T}\int_0^T\int_0^t dsdt$$
$$\leq C_0T.$$

Pela hipotése

$$\beta \leq D_0\epsilon^4 \tag{2.82}$$

temos

$$\frac{\beta_k b_1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_x^3 u_{\epsilon_k,\beta_k}(s,x)\phi_n(x)dsdtdx = \frac{-\beta_k b_1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t u_{\epsilon_k,\beta_k}(s,x)\partial_x^3\phi_n(x)dsdtdx$$
$$\leq \frac{\beta_k|b_1|}{T}\int_0^T\int_0^t \|u_{\epsilon_k,\beta_k}(s,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^3\phi_n\|_{L^2(\mathbb{R})}dsdt$$
$$\leq \frac{C_0\epsilon_k^4}{T}\int_0^T\int_0^t dsdt$$
$$\leq C_0\epsilon_k T.$$

Pelo Lema 2.8 e (2.82)

$$\frac{\beta_k b_2}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_s\partial_x^2 u_{\epsilon_k,\beta_k}(s,x)\phi_n(x)dsdtdx = \frac{\beta_k|b_2|}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_s\partial_x u_{\epsilon_k,\beta_k}(s,x)\partial_x\phi_n(x)dsdtdx$$

$$\leq \frac{\beta_k|b_2|}{T}\int_0^T \|\partial_s\partial_x u_{\epsilon_k,\beta_k}\|_{L^2((0,t)\times\mathbb{R})}\|\partial_x\phi_n\|_{L^2((0,t)\times\mathbb{R})}dt$$
$$\leq \frac{C_0}{T}\beta_k^{1/2}\epsilon_k^{-1/2}\int_0^T t^{1/2}dt$$
$$\leq C_0\epsilon_k T^{1/2}$$

e

$$\frac{\beta_k^2 c}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_s\partial_x^4 u_{\epsilon_k,\beta_k}(s,x)\phi_n(x)dsdtdx = -\frac{\beta_k^2 c}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_s\partial_x^3 u_{\epsilon_k,\beta_k}(s,x)\partial_x\phi_n(x)dsdtdx$$
$$\leq \frac{\beta_k^2 c}{T}\int_0^T \|\partial_s\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2((0,t)\times\mathbb{R})}\|\partial_x\phi_n\|_{L^2((0,t)\times\mathbb{R})}dt$$
$$\leq \frac{C_0}{T}\beta_k^{1/2}\epsilon_k^{-1/2}\int_0^T t^{1/2}dt$$
$$\leq C_0\epsilon_k T^{1/2}.$$

Assim,

$$|J_k^n| \leq C_0(\epsilon_k T + \epsilon_k T^{1/2} + T),$$

e portanto

$$\lim_{k\to\infty}|J_k^n| \leq C_0 T. \tag{2.83}$$

A partir de (2.81) e (2.83) obtemos

$$\frac{1}{T}\int_0^T\int_{\mathbb{R}}\left\langle \nu_{(t,x)}, u_0(x)-\lambda\right\rangle \phi_n(x)dtdx \leq C_0 T$$

e fazendo $T\to 0$ estabelecemos (2.80).

Resulta então de (2.79) e (2.80) que

$$\lim_{T\to 0}\frac{1}{T}\int_0^T\int_I \left\langle \nu_{(t,x)}, |\lambda - u_0(x)|\right\rangle dtdx \leq C_I\left\{2\|u_0\|_{L^2(\mathbb{R})}\|\phi_n - g'(u_0)\|_{L^2(\mathbb{R})}\right\}^{1/2} \tag{2.84}$$

e fazendo $n\to\infty$ em (2.84) concluímos que

$$\lim_{T\to 0}\frac{1}{T}\int_0^T\int_I \left\langle \nu_{(t,x)}, |\lambda - u_0(x)|\right\rangle dtdx = 0.$$

$\nu_{(\cdot)}$ satisfaz (1.9).

Seja (η,q) um par de entropia–fluxo de entropia $\eta, q:\mathbb{R}\to\mathbb{R}$ com $\eta\in C^2(\mathbb{R})$ convexa, η' e η'' limitadas e q dada por (2.56). Em vista de (2.57),

$$\partial_t\eta(u_{\epsilon,\beta})+\partial_x q(u_{\epsilon,\beta}) = \sum_{i=1}^{8} I_{i,\epsilon,\beta}$$

e daí

$$\int_0^\infty\int_{\mathbb{R}}\left[\partial_t\eta(u_{\epsilon_k,\beta_k}) + \partial_x q(u_{\epsilon_k,\beta_k})\right]\phi dtdx$$
$$\leq C_0\epsilon_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}$$
$$+ C_0\beta_k\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}$$
$$+ C_0\beta_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}$$
$$+ C_0\beta_k\|\partial_t\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}$$
$$+ C_0\beta_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_t\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}$$
$$+ C_0\beta_k^2\|\partial_t\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}$$
$$+ C_0\beta_k^2\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_t\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}$$

para toda função não-negativa $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$. Em seguida, usando (2.82) e os Lemas 2.4 e 2.8 obtemos

$$\begin{aligned}\epsilon_k \|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \epsilon_k^{1/2} \lim_{T\to\infty} \left\{ \epsilon_k \int_0^T \|\partial_x u_{\epsilon_k,\beta_k}(t,\cdot)\|^2_{L^2(\mathbb{R})} dt \right\}^{1/2} \\ &\leq C_0 \epsilon_k^{1/2},\end{aligned}$$

$$\begin{aligned}\beta_k \|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\leq D_0 \epsilon_k^{5/2} \|\epsilon_k^{3/2} \partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \\ &\leq C_0 \epsilon_k^{5/2} \\ &\leq C_0 \epsilon_k^{1/2},\end{aligned}$$

$$\begin{aligned}\beta_k \|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\leq C_0 \epsilon_k^{5/2} \|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \\ &\leq C_0 \epsilon_k^{1/2},\end{aligned}$$

$$\begin{aligned}\beta_k \|\partial_t \partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\leq D_0 \epsilon_k^{3/2} \|\beta_k^{1/2} \epsilon_k^{1/2} \partial_t \partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \\ &\leq C_0 \epsilon_k^{3/2} \\ &\leq C_0 \epsilon_k^{1/2},\end{aligned}$$

$$\begin{aligned}\beta_k \|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \|\partial_t \partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\leq C_0 \epsilon_k^{3/2} \|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \\ &\leq C_0 \epsilon_k^{1/2},\end{aligned}$$

$$\begin{aligned}\beta_k^2 \|\partial_t \partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\leq D_0 \epsilon_k^{3/2} \|\beta_k^{3/2} \epsilon_k^{1/2} \partial_t \partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \\ &\leq C_0 \epsilon_k^{3/2} \\ &\leq C_0 \epsilon_k^{1/2}\end{aligned}$$

e

$$\begin{aligned}\beta_k^2 \|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \|\partial_t \partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\leq C_0 \epsilon_k^{3/2} \|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \\ &\leq C_0 \epsilon_k^{1/2}.\end{aligned}$$

Combinando estas estimativas

$$\int_0^\infty \int_{\mathbb{R}} \left[\partial_t \eta(u_{\epsilon_k,\beta_k}) + \partial_x q(u_{\epsilon_k,\beta_k})\right] \phi dtdx \leq C_0 \epsilon_k^{1/2}$$

e daí

$$\int_0^\infty \int_{\mathbb{R}} \left[\eta(u_{\epsilon_k,\beta_k}) \partial_t \phi + q(u_{\epsilon_k,\beta_k}) \partial_x \phi\right] dtdx \geq -C_0 \epsilon_k^{1/2}. \tag{2.85}$$

Fazendo $k \to \infty$ em (2.85) segue-se que

$$\int_0^\infty \int_{\mathbb{R}} \left[\left\langle \nu_{(\cdot)}, \eta(\lambda) \right\rangle \partial_t \phi + \left\langle \nu_{(\cdot)}, q(\lambda) \right\rangle \partial_x \phi\right] dtdx \geq 0,$$

donde

$$\partial_t \left\langle \nu_{(\cdot)}, \eta(\lambda) \right\rangle + \partial_x \left\langle \nu_{(\cdot)}, q(\lambda) \right\rangle \leq 0 \tag{2.86}$$

no sentido distribucional. A desigualdade (1.9) é obtida a partir de (2.86) e de uma regularização padrão da função $\phi_\alpha(u) = |u - \alpha|$ para todo $\alpha \in \mathbb{R}$.

Mostramos então que ν é uma solução de entropia m-v de (2.65)–(2.66), e portanto o Corolário 1.3 implica que $u_{\epsilon,\beta} \to u$ in $L^\infty(\mathbb{R}_+, L^r_{loc}(\mathbb{R}))$ quando $\epsilon \to 0$ para todo $r \in [1,4)$, sendo $u \in L^\infty(\mathbb{R}_+, L^4(\mathbb{R}))$ a única solução de entropia do mesmo problema. $\square$

Capítulo 3

Equação de Benney-Lin Generalizada

Neste capítulo estudaremos uma equação do tipo Benney-Lin. Além da existência de soluções globais suaves, mostraremos a convergência das mesmas para soluções de entropia quando os parâmetros tendem a zero.

3.1 Existência de Soluções

Estabeleceremos nesta seção a existência de soluções globais para o seguinte problema de Cauchy

$$\partial_t u + \partial_x f(u) + \beta \partial_x^2 u + \beta^2 b \partial_x^3 u + \beta^3 c \partial_x^4 u + \beta^5 d \partial_x^5 u = \epsilon \partial_x^2 u \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{3.1}$$

$$u(0,x) = u_{\epsilon,\beta,0}(x) \quad x \in \mathbb{R} \tag{3.2}$$

sendo $f : \mathbb{R} \to \mathbb{R}$ uma função suave, ϵ e β números reais no intervalo $(0,1)$ e b, c e d constantes satisfazendo $b, d \in \mathbb{R}$ e $c > 0$. Novamente, $\mathcal{F}$ denotará a transformada de Fourier em x e $\mathcal{F}^{-1}$ a sua inversa. Assim, formalmente

$$\mathcal{F}(\partial_t u + \partial_x f(u) + \beta \partial_x^2 u + \beta^2 b \partial_x^3 u + \beta^3 c \partial_x^4 u + \beta^5 d \partial_x^5 u) = \mathcal{F}(\epsilon \partial_x^2 u)$$

e daí

$$\partial_t \mathcal{F}(u) + \left[(\epsilon - \beta)\xi^2 + \beta^3 c \xi^4 + i(\beta^5 d \xi^5 - \beta^2 b \xi^3) \right] \mathcal{F}(u) = -\mathcal{F}(\partial_x f(u)). \tag{3.3}$$

Definindo

$$Q(t,\xi) = \exp\left\{ - \left[(\epsilon - \beta)\xi^2 + \beta^3 c \xi^4 + i(\beta^5 d \xi^5 - \beta^2 b \xi^3) \right] t \right\},$$

$$G(t)u = \mathcal{F}^{-1}\left(Q(t,\cdot) \mathcal{F} u(\cdot) \right)$$

e argumentando como no capítulo anterior, segue-se que a equação integral da solução é

$$u(t) = G(t) u_{\epsilon,\beta,0} - \int_0^t G(t-s) \partial_x f(u(s)) ds.$$

Dado $u_{\epsilon,\beta,0} \in H^1(\mathbb{R})$ consideremos o espaço de Banach

$$X_T = \{ u \in C([0,T], H^1(\mathbb{R})); \|u(t) - G(t) u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})} \leq \|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}, t \in [0,T] \}$$

com a norma

$$\|u\|_{X_T} = \sup_{0\le t\le T} \|u(t)\|_{H^1(\mathbb{R})},$$

e definamos o seguinte operador em X_T:

$$\Lambda u(t) = G(t)u_{\epsilon,\beta,0} - \int_0^t G(t-s)\partial_x f(u(s))ds.$$

Na demonstração do lema a seguir (e na do Teorema (3.1)) denotaremos com C_0 a constantes que dependem apenas dos parâmetros ϵ, β e dos coeficientes b, c e d em (3.1).

Lema 3.1. *Supondo $u_{\epsilon,\beta,0} \in H^1(\mathbb{R})$ e $\beta < \epsilon$, existe $T = T(u_{\epsilon,\beta,0}) > 0$ tal que as seguintes afirmações são válidas:*

(i) $\Lambda u \in X_T$ e $\|\Lambda u(t)\|_{H^1(\mathbb{R})} \le 2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$ para todo $t \in [0,T]$ se $u \in X_T$;

(ii) Λ é uma contração em X_T.

Demonstração. Assumiremos (sem perda de generalidade) $f(0) = 0$. Dado $t \in [0,T]$ (T será escolhido a posteriori), obtemos $\|G(t)u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})} \le \|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$ uma vez que a condição $\beta < \epsilon$ implica em $|Q| \le 1$. Logo $\|u(t)\|_{H^1(\mathbb{R})} \le 2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$ e consequentemente

$$\|u(t)\|_{L^\infty(\mathbb{R})} \le 2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})} \quad t \in [0,T]. \tag{3.4}$$

A condição (3.4) juntamente com o Teorema 1.2 e o Corolário 1.1 garantem a existência de uma constante $K_0 > 0$ (dependendo apenas da cota $2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$) tal que

$$\|f(u(t))\|_{H^1(\mathbb{R})} \le K_0\|u(t)\|_{H^1(\mathbb{R})} \tag{3.5}$$

e

$$\|f(u(t)) - f(v(t))\|_{H^1(\mathbb{R})} \le K_0\|u(t)-v(t)\|_{H^1(\mathbb{R})}(|f'(0)| + \|u(t)\|_{H^1(\mathbb{R})} + \|v(t)\|_{H^1(\mathbb{R})}) \tag{3.6}$$

se $u, v \in X_T$.

Assim, utilizando (3.5) e a limitação

$$t\exp\{-\alpha t\} \le (\alpha e)^{-1} \ \ (\alpha > 0) \tag{3.7}$$

para todo $t \ge 0$ obtemos

$$\begin{aligned}
\|\Lambda u(t) - G(t)u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})} &\le \int_0^t \left\{ \sum_{k=0}^1 \int_{\mathbb{R}} \left|(i\xi)^k Q(t-s,\xi)\mathcal{F}(\partial_x f(u(s)))\right|^2 d\xi \right\}^{1/2} ds \\
&\le \int_0^t \left\{ \sum_{k=0}^1 \int_{\mathbb{R}} \exp\{-2(t-s)(\epsilon-\beta)\xi^2\} \left|(i\xi)^k \mathcal{F}(\partial_x f(u(s)))\right|^2 d\xi \right\}^{1/2} ds \\
&= \int_0^t \left\{ \sum_{k=0}^1 \int_{\mathbb{R}} \xi^2 \exp\{-2(t-s)(\epsilon-\beta)\xi^2\} \left|\mathcal{F}(\partial_x^k f(u(s)))\right|^2 d\xi \right\}^{1/2} ds \\
&\le [2e(\epsilon-\beta)]^{-1/2} \int_0^t (t-s)^{-1/2} \|f(u(s))\|_{H^1(\mathbb{R})} ds
\end{aligned}$$

$$
\begin{aligned}
&\leq 2K_0[2e(\epsilon-\beta)]^{-1/2}\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}\int_0^t (t-s)^{-1/2}ds\\
&= 2\sqrt{2}K_0[e(\epsilon-\beta)]^{-1/2}\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}t^{1/2}\\
&\leq \|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}
\end{aligned}
$$

se

$$
0 < T \leq \frac{e(\epsilon-\beta)}{8K_0^2}. \tag{3.8}
$$

Em particular, $\Lambda u(t) \in H^1(\mathbb{R})$ e $\|\Lambda u(t)\|_{H^1(\mathbb{R})} \leq 2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$ para todo $t \in [0,T]$.

A seguir mostraremos que $\Lambda u \in C([0,T], H^1(\mathbb{R}))$. De fato, dado $t_0 \in [0,T)$, para todo $t_0 \leq t < T$ temos

$$
\begin{aligned}
\|\Lambda u(t)-\Lambda u(t_0)\|_{H^1(\mathbb{R})} &\leq \|G(t)u_{\epsilon,\beta,0}-G(t_0)u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})} + \int_{t_0}^t \|G(s)\partial_x f(u(t-s))\|_{H^1(\mathbb{R})}\,ds\\
&\quad + \int_0^{t_0} \|G(s)\partial_x\left[f(u(t-s))-f(u(t_0-s))\right]\|_{H^1(\mathbb{R})}\,ds\\
&= A_1 + A_2 + A_3.
\end{aligned}
$$

Como

$$
A_1^2 = \sum_{k=0}^{1}\int_{\mathbb{R}} |Q(t,\xi)-Q(t_0,\xi)|^2|\mathcal{F}(\partial_x^k u_{\epsilon,\beta,0})|^2 d\xi,
$$

segue-se que $A_1 \to 0$ quando $t \to t_0^+$ pelo Teorema da Convergência Dominada. Além disso, utilizando (3.5) e (3.7)

$$
\begin{aligned}
A_2 &\leq \int_{t_0}^t \left\{\sum_{k=0}^{1}\int_{\mathbb{R}} \xi^2 \exp\left\{-2s(\epsilon-\beta)\xi^2\right\}|\mathcal{F}(\partial_x^k f(u(t-s)))|^2 d\xi\right\}^{1/2} ds\\
&\leq [2e(\epsilon-\beta)]^{-1/2}\int_{t_0}^t s^{-1/2}\|f(u(t-s))\|_{H^1(\mathbb{R})}ds\\
&\leq 2K_0[2e(\epsilon-\beta)]^{-1/2}\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}\int_{t_0}^t s^{-1/2}ds\\
&= 2\sqrt{2}K_0[e(\epsilon-\beta)]^{-1/2}\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}(t^{1/2}-t_0^{1/2})
\end{aligned}
$$

estabelecendo a convergência $A_2 \to 0$ quando $t \to t_0^+$. Mais ainda, por (3.6) e (3.7)

$$
\begin{aligned}
A_3 &\leq \int_0^{t_0} \left\{\sum_{k=0}^{1}\int_{\mathbb{R}} \xi^2 \exp\left\{-2s(\epsilon-\beta)\xi^2\right\}\left|\mathcal{F}\left[\partial_x^k(f(u(t-s))-f(u(t_0-s)))\right]\right|^2 d\xi\right\}^{1/2} ds\\
&\leq [2e(\epsilon-\beta)]^{-1/2}\int_0^{t_0} s^{-1/2}\|f(u(t-s))-f(u(t_0-s))\|_{H^1(\mathbb{R})}ds\\
&\leq K_0[2e(\epsilon-\beta)]^{-1/2}\left[|f'(0)|+4\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}\right]\int_0^{t_0} s^{-1/2}\|u(t-s)-u(t_0-s)\|_{H^1(\mathbb{R})}ds.
\end{aligned}
$$

A condição $u \in C([0,T], H^1(\mathbb{R}))$ implica que a última integral converge para zero e consequentemente $A_3 \to 0$ quando $t \to t_0^+$. Disto resulta que

$$\lim_{t \to t_0^+} \|\Lambda u(t) - \Lambda u(t_0)\|_{H^1(\mathbb{R})} = 0$$

e sendo o caso $t_0 \in (0,T]$ análogo, $\Lambda u \in C([0,T], H^1(\mathbb{R}))$. Para finalizar a demonstração provaremos que Λ é uma contração em X_T. Sejam então u e v dois elementos de X_T. Dado $t \in [0,T]$, através de (3.6) e (3.7) obtemos

$$\begin{aligned}
\|\Lambda u(t) - \Lambda v(t)\|_{H^1(\mathbb{R})} &\leq \int_0^t \|G(t-s)\partial_x[f(u(s)) - f(v(s))]\|_{H^1(\mathbb{R})} ds \\
&\leq \int_0^t \left\{ \sum_{k=0}^{1} \int_{\mathbb{R}} \xi^2 \exp\left\{-2(t-s)(\epsilon - \beta)\xi^2\right\} |\mathcal{F}[\partial_x^k (f(u(s)) - f(v(s)))]|^2 d\xi \right\}^{1/2} ds \\
&\leq [2e(\epsilon - \beta)]^{-1/2} \int_0^t (t-s)^{-1/2} \|f(u(s)) - f(v(s))\|_{H^1(\mathbb{R})} ds \\
&\leq K_0 [2e(\epsilon - \beta)]^{-1/2} [|f'(0)| + 4\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}] \int_0^t (t-s)^{-1/2} \|u(s) - v(s)\|_{H^1(\mathbb{R})} ds \\
&\leq \sqrt{2} K_0 [e(\epsilon - \beta)]^{-1/2} [|f'(0)| + 4\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}] \|u - v\|_{X_T} t^{1/2} \\
&\leq \frac{1}{2} \|u - v\|_{X_T}
\end{aligned}$$

se

$$0 < T \leq \frac{e(\epsilon - \beta)}{8K_0^2 [|f'(0)| + 4\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}]^2} \tag{3.9}$$

mostrando que Λ é uma contração. Logo, o resultado será válido se escolhermos (qualquer) T satisfazendo as condições (3.8) e (3.9). □

Considerando T como no lema anterior podemos então enunciar o

Teorema 3.1. *Se $u_{\epsilon,\beta,0} \in H^1(\mathbb{R})$ e $\beta < \epsilon$, o problema de Cauchy* (3.1)–(3.2) *admite uma solução*

$$u \in C([0,T], H^1(\mathbb{R})) \cap C((0,T], H^l(\mathbb{R})) \qquad l = 1, 2, 3, \ldots$$

tal que $\|u(t)\|_{H^1(\mathbb{R})} \leq 2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$ para todo $t \in [0,T]$. Além disso, se $u_{\epsilon,\beta,0} \in H^l(\mathbb{R})$ para algum inteiro $l \geq 1$, então $u \in C([0,T], H^l(\mathbb{R}))$.

Demonstração. Sendo X_T um espaço de Banach, o Lema 3.1 juntamente com o Teorema do Ponto Fixo de Banach garantem a existência de um único elemento $u \in X_T$ tal que $\Lambda u = u$. Assim, u será uma solução da equação integral

$$u(t) = G(t)u_{\epsilon,\beta,0} - \int_0^t G(t-s)\partial_x f(u(s)) ds \tag{3.10}$$

e satisfaz $\|u(t)\|_{H^1(\mathbb{R})} \leq 2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$ para todo $t \in [0,T]$. Dado $l \geq 1$ inteiro, nossa próxima etapa será mostrar que $u \in C((0,T], H^{l+1}(\mathbb{R}))$ se $u \in C((0,T], H^l(\mathbb{R}))$. Fixando $t_0 \in (0,T)$, será suficiente provar que $u \in C([t_0, T], H^{l+1}(\mathbb{R}))$. Escolhendo $0 < t_1 < t_0$ e utilizando as propriedades $G(0) = I$ e $G(t+s) = G(t)G(s)$ para $t, s \geq 0$ podemos escrever u da seguinte maneira:

$$u(t) = G(t-t_1)u(t_1) - \int_{t_1}^{t} G(t-s)\partial_x f(u(s))ds \ \ (t \geq t_1). \tag{3.11}$$

O primeiro fato a ser estabelecido é o seguinte:

$$\sup_{t_0 \leq t \leq T} \|u(t)\|_{H^{l+1}(\mathbb{R})} < \infty. \tag{3.12}$$

De fato, dado $t \in [t_0, T]$ temos

$$\begin{aligned}
\|u(t)\|_{H^{l+1}(\mathbb{R})} &\leq \|G(t-t_1)u(t_1)\|_{H^{l+1}(\mathbb{R})} + \int_{t_1}^{t} \|G(t-s)\partial_x f(u(s))\|_{H^{l+1}(\mathbb{R})}ds \\
&= A_1 + A_2.
\end{aligned}$$

Pela estimativa (3.7)

$$\begin{aligned}
A_1^2 &= \sum_{k=0}^{l+1} \int_{\mathbb{R}} \left|(i\xi)^k Q(t-t_1,\xi)\mathcal{F}(u(t_1))\right|^2 d\xi \\
&\leq \sum_{k=0}^{l+1} \int_{\mathbb{R}} \xi^{2k} \exp\{-2(t-t_1)(\epsilon-\beta)\xi^2\}|\mathcal{F}(u(t_1))|^2 d\xi \\
&\leq \int_{\mathbb{R}} |\mathcal{F}(u(t_1))|^2 d\xi + \sum_{k=1}^{l+1} \int_{\mathbb{R}} [\xi^2 \exp\{-2k^{-1}(t-t_1)(\epsilon-\beta)\xi^2]^k |\mathcal{F}(u(t_1))|^2 d\xi \\
&\leq \|u(t_1)\|^2_{L^2(\mathbb{R})} + \sum_{k=1}^{l+1} [2k^{-1}(t-t_1)(\epsilon-\beta)e]^{-k} \|u(t_1)\|^2_{L^2(\mathbb{R})} \\
&\leq \|u(t_1)\|^2_{L^2(\mathbb{R})} \left\{1 + \sum_{k=1}^{l+1} k^k [2(t_0-t_1)(\epsilon-\beta)e]^{-k}\right\},
\end{aligned}$$

e como

$$\sup_{t_1 \leq t \leq T} \|u(t)\|_{H^l(\mathbb{R})} < \infty$$

uma vez que $u \in C((0,T], H^l(\mathbb{R}))$, o Teorema 1.2 juntamente com (3.7) implicam que

$$\begin{aligned}
A_2 &= \int_{t_1}^{t} \left\{\sum_{k=0}^{l+1} \int_{\mathbb{R}} |(i\xi)^k Q(t-s,\xi)\mathcal{F}(\partial_x f(u(s)))|^2 d\xi\right\}^{1/2} ds \\
&\leq \int_{t_1}^{t} \left\{\|\partial_x f(u(s))\|^2_{L^2(\mathbb{R})} + \sum_{k=1}^{l+1} \int_{\mathbb{R}} |(i\xi)^k Q(t-s,\xi)\mathcal{F}(\partial_x f(u(s)))|^2 d\xi\right\}^{1/2} ds \\
&= \int_{t_1}^{t} \left\{\|f(u(s))\|^2_{H^l(\mathbb{R})} + \sum_{k=1}^{l+1} \int_{\mathbb{R}} \xi^4 \exp\{-2(t-s)\beta^3 c\xi^4\}|\mathcal{F}(\partial_x^{k-1} f(u(s))|^2 d\xi\right\}^{1/2} ds \\
&\leq \int_{t_1}^{t} \left\{\|f(u(s))\|^2_{H^l(\mathbb{R})} + [2(t-s)\beta^3 ce]^{-1} \|f(u(s))\|^2_{H^l(\mathbb{R})}\right\}^{1/2} ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_1}^{t} \|f(u(s))\|_{H^l(\mathbb{R})} \left\{1 + [2(t-s)\beta^3 ce]^{-1}\right\}^{1/2} ds \\
&\le \int_{t_1}^{t} \|f(u(s))\|_{H^l(\mathbb{R})} \left\{1 + [2(t-s)\beta^3 ce]^{-1/2}\right\} ds \qquad (a+b)^{1/2} \le a^{1/2} + b^{1/2} \ (a,b \ge 0) \\
&\le K_0 \int_{t_1}^{t} \|u(s)\|_{H^l(\mathbb{R})} \left\{1 + [2(t-s)\beta^3 ce]^{-1/2}\right\} ds \\
&\le K_0 \sup_{t_1 \le t \le T} \|u(t)\|_{H^l(\mathbb{R})} \int_{t_1}^{t} \left\{1 + [2(t-s)\beta^3 ce]^{-1/2}\right\} ds \\
&\le K_0 \sup_{t_1 \le t \le T} \|u(t)\|_{H^l(\mathbb{R})} \left\{(t-t_1) + \sqrt{2}[\beta^3 ce]^{-1/2}(t-t_1)^{1/2}\right\} \\
&\le K_0 \sup_{t_1 \le t \le T} \|u(t)\|_{H^l(\mathbb{R})} \left\{(T-t_1) + \sqrt{2}[\beta^3 ce]^{-1/2}(T-t_1)^{1/2}\right\}
\end{aligned}
$$

sendo K_0 uma constante que depende apenas da cota $2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$. A estimativa (3.12) segue imediatamente.

Agora dado $t_2 \in [t_0, T)$, para $t_2 \le t < T$ temos

$$
\begin{aligned}
\|u(t) - u(t_2)\|_{H^{l+1}(\mathbb{R})} &\le \|G(t-t_1)u(t_1) - G(t_2-t_1)u(t_1)\|_{H^{l+1}(\mathbb{R})} \\
&\quad + \int_{t_2-t_1}^{t-t_1} \|G(s)\partial_x f(u(t-s))\|_{H^{l+1}(\mathbb{R})} ds \\
&\quad + \int_{0}^{t_2-t_1} \|G(s)\partial_x [f(u(t-s)) - f(u(t_2-s))]\|_{H^{l+1}(\mathbb{R})} ds \\
&= A_1 + A_2 + A_3.
\end{aligned}
$$

Em primeiro lugar, observe que

$$
\begin{aligned}
A_1^2 &= \sum_{k=0}^{l+1} \int_{\mathbb{R}} |Q(t-t_1,\xi) - Q(t_2-t_1,\xi)|^2 |\mathcal{F}(\partial_x^k u(t_1))|^2 d\xi \\
&= \int_{\mathbb{R}} |Q(t-t_1,\xi) - Q(t_2-t_1,\xi)|^2 |\mathcal{F}(u(t_1))|^2 d\xi \qquad (3.13) \\
&\quad + \sum_{k=0}^{l} \int_{\mathbb{R}} |\xi(Q(t-t_1,\xi) - Q(t_2-t_1,\xi))|^2 |\mathcal{F}(\partial_x^k u(t_1))|^2 d\xi.
\end{aligned}
$$

O primeiro termo (da segunda igualdade) acima tende a zero quando $t \to t_2^+$ pelo Teorema da Convergência Dominada. Agora definindo

$$
\phi_t^{(k)}(\xi) = |\xi(Q(t-t_1,\xi) - Q(t_2-t_1,\xi))|^2 |\mathcal{F}(\partial_x^k u(t_1))|^2 \quad k = 0, 1, \ldots, l
$$

claramente $\phi_t^{(k)}(\xi) \to 0$ quando $t \to t_2^+$. Além disso, (3.7) e a condição $t_1 < t_0 \le t_2 \le t$ implicam em

$$
\begin{aligned}
\phi_t^{(k)}(\xi) &\le 2\xi^2 \left\{|Q(t-t_1,\xi)|^2 + |Q(t_2-t_1,\xi)|^2\right\} \left|\mathcal{F}(\partial_x^k u(t_1))\right|^2 \\
&\le 2\xi^2 \left\{\exp\{-2(t-t_1)(\epsilon-\beta)\xi^2\} + \exp\{-2(t_2-t_1)(\epsilon-\beta)\xi^2\}\right\} \left|\mathcal{F}(\partial_x^k u(t_1))\right|^2 \\
&\le 4\xi^2 \exp\{-2(t_0-t_1)(\epsilon-\beta)\xi^2\} \left|\mathcal{F}(\partial_x^k u(t_1))\right|^2 \\
&\le 2[(t_0-t_1)(\epsilon-\beta)e]^{-1} \left|\mathcal{F}(\partial_x^k u(t_1))\right|^2.
\end{aligned}
$$

Portanto utilizando a hipótese $u(t_1) \in H^l(\mathbb{R})$, o segundo somatório em (3.13) tende a zero quando $t \to t_2^+$ pelo Teorema da Convergência Dominada, e consequentemente $A_1 \to 0$ quando $t \to t_2^+$.

Quanto ao termo A_2, o Teorema 1.2, a estimativa (3.7) e a relação $t_1 \leq t - s \leq T$ para $t_2 - t_1 \leq s \leq t - t_1$ implicam que

$$\begin{aligned}
A_2 &\leq \int_{t_2-t_1}^{t-t_1} \left\{ \sum_{k=0}^{l+1} \int_{\mathbb{R}} |(i\xi)^k Q(s,\xi)\mathcal{F}(\partial_x f(u(t-s)))|^2 d\xi \right\}^{1/2} ds \\
&\leq \int_{t_2-t_1}^{t-t_1} \left\{ \|\partial_x f(u(s))\|_{L^2(\mathbb{R})}^2 + \sum_{k=1}^{l+1} \int_{\mathbb{R}} \xi^4 \exp\{-2s\beta^3 c\xi^4\} |\mathcal{F}(\partial_x^{k-1} f(u(t-s)))|^2 d\xi \right\}^{1/2} ds \\
&\leq \int_{t_2-t_1}^{t-t_1} \left\{ \|\partial_x f(u(s))\|_{L^2(\mathbb{R})}^2 + [2s\beta^3 ce]^{-1} \|f(u(t-s))\|_{H^l(\mathbb{R})}^2 \right\}^{1/2} ds \\
&\leq \int_{t_2-t_1}^{t-t_1} \|f(u(t-s))\|_{H^l(\mathbb{R})} \left\{1 + [2s\beta^3 ce]^{-1}\right\}^{1/2} ds \\
&\leq K_0 \int_{t_2-t_1}^{t-t_1} \|u(t-s)\|_{H^l(\mathbb{R})} \left\{1 + [2s\beta^3 ce]^{-1/2}\right\} ds \\
&\leq K_0 \sup_{t_1 \leq t \leq T} \|u(t)\|_{H^l(\mathbb{R})} \int_{t_2-t_1}^{t-t_1} \left\{1 + [2s\beta^3 ce]^{-1/2}\right\} ds \\
&= K_0 \sup_{t_1 \leq t \leq T} \|u(t)\|_{H^l(\mathbb{R})} \left\{ (t-t_2) + \sqrt{2}[\beta^3 ce]^{-1/2}((t-t_1)^{1/2} - (t_2-t_1)^{1/2}) \right\}
\end{aligned}$$

onde novamente K_0 é uma constante que depende apenas da cota $2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$. Assim, $A_2 \to 0$ quando $t \to t_2^+$. Analogamente, como $t_1 \leq t_2 - s, t - s \leq T$ para $0 \leq s \leq t_2 - t_1$, o Corolário 1.1 e (3.7) implicam que

$$\begin{aligned}
A_3 &\leq \int_0^{t_2-t_1} \left\{ \sum_{k=0}^{l+1} |(i\xi)^k Q(s,\xi)\mathcal{F}(\partial_x[f(u(t-s)) - f(u(t_2-s))])|^2 d\xi \right\}^{1/2} ds \\
&\leq \int_0^{t_2-t_1} \Bigg\{ \|\partial_x[f(u(t-s)) - f(u(t_2-s))]\|_{L^2(\mathbb{R})}^2 \\
&\quad + \sum_{k=1}^{l+1} \xi^4 \exp\{-2s\beta^3 c\xi^4\} |\mathcal{F}(\partial_x^{k-1}[f(u(t-s)) - f(u(t_2-s))])|^2 d\xi \Bigg\}^{1/2} ds \\
&\leq \int_0^{t_2-t_1} \|f(u(t-s)) - f(u(t_2-s))\|_{H^l(\mathbb{R})} \left\{1 + [2s\beta^3 ce]^{-1}\right\}^{1/2} ds \\
&\leq K_l[|f'(0)| + 2 \sup_{t_1 \leq t \leq T} \|u(t)\|_{H^l(\mathbb{R})}] \int_0^{t_2-t_1} \|u(t-s) - u(t_2-s)\|_{H^l(\mathbb{R})} \left\{1 + [2s\beta^3 ce]^{-1/2}\right\} ds
\end{aligned}$$

onde K_l é uma constante dependente da cota $2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$ e do expoente l. Utilizando a hipótese $u \in C((0,T], H^l(\mathbb{R}))$, a última integral converge para zero pelo Teorema da Convergência Dominada, e daí $A_3 \to 0$ quando $t \to t_2^+$. Portanto,

$$\lim_{t \to t_2^+} \|u(t) - u(t_2)\|_{H^{l+1}(\mathbb{R})} = 0$$

e sendo o caso $t_2 \in (t_0, T]$ análogo, concluímos que $u \in C([t_0, T], H^{l+1}(\mathbb{R}))$. Finalmente, usando a expressão (3.10) para a solução u, a última afirmação do teorema é estabelecida seguindo as mesmas linhas do procedimento acima. Mais precisamente, basta observar que

$$u_{\epsilon,\beta,0} \in H^{l+1}(\mathbb{R}) \ \ e \ \ u \in C([0,T], H^l(\mathbb{R})) \Rightarrow u \in C([0,T], H^{l+1}(\mathbb{R}))$$

para todo inteiro $l \geq 1$. □

Uma consequência imediata deste resultado é que $u(t) \in C_0^\infty(\mathbb{R})$ para todo $t \in (0, T]$ devido ao Lema de Sobolev.

Um refinamento do teorema acima é fornecido pelo

Corolário 3.1. *Se $u_{\epsilon,\beta,0} \in H^1(\mathbb{R})$ e $\beta < \epsilon$, a solução do Teorema 3.1 satisfaz $u \in C^1((0,T], H^l(\mathbb{R}))$ para todo $l \in \mathbb{N}$.*

Demonstração. Com as notações da demonstração do Teorema 3.1, sejam $0 < t_1 < t_0 \leq T$ e u dada por (3.11). Nosso objetivo será mostrar que $\partial_t u \in C([t_0, T], H^l(\mathbb{R}))$. Escrevendo

$$R(\xi) = (\epsilon - \beta)\xi^2 + \beta^3 c\xi^4 + i(\beta d\xi^5 - \beta^2 b\xi^3)$$

e utilizando (3.3) e (3.11), obtemos (argumentando como no capítulo anterior)

$$\begin{aligned}\partial_t u(t) = & -\mathcal{F}^{-1}\left[R(\xi)Q(t-t_1,\xi)\mathcal{F}(u(t_1))\right] \\ & + \int_{t_1}^{t} \mathcal{F}^{-1}\left[R(\xi)Q(t-s,\xi)\mathcal{F}(\partial_x f(u(s)))\right] ds \\ & - \partial_x f(u(t))\end{aligned}$$

para todo $t \geq t_1$. Logo, se $t \in [t_0, T]$

$$\begin{aligned}\|\partial_t u(t)\|_{H^l(\mathbb{R})} \leq & \|\mathcal{F}^{-1}\left[R(\xi)Q(t-t_1,\xi)\mathcal{F}(u(t_1))\right]\|_{H^l(\mathbb{R})} \\ & + \int_{t_1}^{t} \|\mathcal{F}^{-1}\left[R(\xi)Q(t-s,\xi)\mathcal{F}(\partial_x f(u(s)))\right]\|_{H^l(\mathbb{R})} ds \\ & + \|\partial_x f(u(t))\|_{H^l(\mathbb{R})} \\ = & A_1 + A_2 + A_3.\end{aligned}$$

Observando que

$$|R(\xi)|^2 \leq C_0(\xi^4 + \xi^6 + \xi^8 + \xi^{10})$$

temos

$$\begin{aligned}A_1^2 & \leq \sum_{k=0}^{l} \int_{\mathbb{R}} |R(\xi)\mathcal{F}(\partial_x^k u(t_1))|^2 d\xi \\ & \leq C_0 \sum_{k=0}^{l} \int_{\mathbb{R}} (\xi^4 + \xi^6 + \xi^8 + \xi^{10})|\mathcal{F}(\partial_x^k u(t_1))|^2 d\xi\end{aligned}$$

$$
\begin{aligned}
&= C_0 \sum_{k=0}^{l} \sum_{i=2}^{5} \int_{\mathbb{R}} \left\{ |\mathcal{F}(\partial_x^{k+i} u(t_1))|^2 \right\} d\xi \\
&\leq 4C_0 \sum_{k=0}^{l+5} \int_{\mathbb{R}} |\mathcal{F}(\partial_x^{k} u(t_1))|^2 d\xi \\
&= 4C_0 \|u(t_1)\|^2_{H^{l+5}(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
A_2 &\leq \int_{t_1}^{t} \left\{ \sum_{k=0}^{l} \int_{\mathbb{R}} |R(\xi)\mathcal{F}(\partial_x^{k+1} f(u(s)))|^2 d\xi \right\}^{1/2} ds \\
&\leq C_0 \int_{t_1}^{t} \left\{ \sum_{k=0}^{l} \int_{\mathbb{R}} (\xi^4 + \xi^6 + \xi^8 + \xi^{10}) |\mathcal{F}(\partial_x^{k+1} f(u(s)))|^2 d\xi \right\}^{1/2} ds \\
&= C_0 \int_{t_1}^{t} \left\{ \sum_{k=0}^{l} \sum_{i=2}^{5} \int_{\mathbb{R}} |\mathcal{F}(\partial_x^{k+i+1} f(u(s)))|^2 d\xi \right\}^{1/2} ds \\
&\leq 2C_0 \int_{t_1}^{t} \left\{ \sum_{k=0}^{l+6} \int_{\mathbb{R}} |\mathcal{F}(\partial_x^{k} f(u(s)))|^2 d\xi \right\}^{1/2} ds \\
&= 2C_0 \int_{t_1}^{t} \|f(u(s))\|_{H^{l+6}(\mathbb{R})} ds \\
&\leq 2C_0 K_0 \sup_{t_1 \leq t \leq T} \|u(t)\|_{H^{l+6}(\mathbb{R})} (T - t_1)
\end{aligned}
$$

e

$$
A_3 \leq \|f(u(s))\|_{H^{l+1}(\mathbb{R})} \leq K_0 \|u(s)\|_{H^{l+1}(\mathbb{R})} \leq K_0 \sup_{t_1 \leq t \leq T} \|u(t)\|_{H^{l+1}(\mathbb{R})}
$$

sendo K_0 uma constante dependendo da cota $2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$. Disto resulta que

$$
\sup_{t_0 \leq t \leq T} \|\partial_t u(t)\|_{H^l(\mathbb{R})} < \infty
$$

uma vez que $u \in C((0,T], H^l(\mathbb{R}))(l = 1, 2, 3, \ldots)$ pelo Teorema 3.1.

Agora dado $t_2 \in [t_0, T)$, para $t_2 \leq t < T$ temos

$$
\begin{aligned}
\|\partial_t u(t) - \partial_t u(t_2)\|_{H^l(\mathbb{R})} &\leq \|\mathcal{F}^{-1} \left[R(\xi)(Q(t - t_1, \xi) - Q(t_2 - t_1, \xi))\mathcal{F}(u(t_1)) \right] \|_{H^l(\mathbb{R})} \\
&+ \int_{t_2 - t_1}^{t - t_1} \|\mathcal{F}^{-1} \left[R(\xi) Q(s, \xi) \mathcal{F}(\partial_x f(u(t - s))) \right] \|_{H^l(\mathbb{R})} ds \\
&+ \int_{0}^{t_2 - t_1} \|\mathcal{F}^{-1} \left[R(\xi) Q(s, \xi) \mathcal{F}(\partial_x (f(u(t - s)) - f(u(t_2 - s)))) \right] \|_{H^l(\mathbb{R})} ds \\
&+ \|\partial_x f(u(t)) - \partial_x f(u(t_2))\|_{H^l(\mathbb{R})} \\
&= A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

Ora,

$$
\begin{aligned}
A_1^2 &\leq C_0 \sum_{k=0}^{l} \int_{\mathbb{R}} (\xi^4+\xi^6+\xi^8+\xi^{10}) |Q(t-t_1,\xi)-Q(t_2-t_1,\xi)|^2 |\mathcal{F}(\partial_x^k u(t_1))|^2 d\xi \\
&= C_0 \sum_{k=0}^{l} \sum_{i=2}^{5} \int_{\mathbb{R}} |Q(t-t_1,\xi)-Q(t_2-t_1,\xi)|^2 |\mathcal{F}(\partial_x^{k+i} u(t_1))|^2 d\xi \\
&\leq 4C_0 \sum_{k=0}^{l+5} \int_{\mathbb{R}} |Q(t-t_1,\xi)-Q(t_2-t_1,\xi)|^2 |\mathcal{F}(\partial_x^k u(t_1))|^2 d\xi,
\end{aligned}
$$

$$
\begin{aligned}
A_2 &\leq C_0 \int_{t_2-t_1}^{t-t_1} \left\{ \sum_{k=0}^{l} \int_{\mathbb{R}} (\xi^4+\xi^6+\xi^8+\xi^{10}) |\mathcal{F}(\partial_x^{k+1} f(u(t-s)))|^2 d\xi \right\}^{1/2} ds \\
&= C_0 \int_{t_2-t_1}^{t-t_1} \left\{ \sum_{k=0}^{l} \sum_{i=2}^{5} \int_{\mathbb{R}} |\mathcal{F}(\partial_x^{k+i+1} f(u(t-s)))|^2 d\xi \right\}^{1/2} ds \\
&\leq 2C_0 \int_{t_2-t_1}^{t-t_1} \left\{ \sum_{k=0}^{l+6} \int_{\mathbb{R}} |\mathcal{F}(\partial_x^{k} f(u(t-s)))|^2 d\xi \right\}^{1/2} ds \\
&= 2C_0 \int_{t_2-t_1}^{t-t_1} \|f(u(t-s))\|_{H^{l+6}(\mathbb{R})} ds \\
&\leq 2K_0 C_0 \sup_{t_1 \leq t \leq T} \|u(t)\|_{H^{l+6}(\mathbb{R})} (t-t_2),
\end{aligned}
$$

$$
\begin{aligned}
A_3 &\leq C_0 \int_{0}^{t_2-t_1} \left\{ \sum_{k=0}^{l} \int_{\mathbb{R}} (\xi^4+\xi^6+\xi^8+\xi^{10}) |\mathcal{F}(\partial_x^{k+1} [f(u(t-s))-f(u(t_2-s))])|^2 d\xi \right\}^{1/2} ds \\
&= C_0 \int_{0}^{t_2-t_1} \left\{ \sum_{k=0}^{l} \sum_{i=2}^{5} \int_{\mathbb{R}} |\mathcal{F}(\partial_x^{k+i+1} [f(u(t-s))-f(u(t_2-s))])|^2 d\xi \right\}^{1/2} ds \\
&\leq 2C_0 \int_{0}^{t_2-t_1} \left\{ \sum_{k=0}^{l+6} \int_{\mathbb{R}} |\mathcal{F}(\partial_x^{k} [f(u(t-s))-f(u(t_2-s))])|^2 d\xi \right\}^{1/2} ds \\
&= 2C_0 \int_{0}^{t_2-t_1} \|f(u(t-s))-f(u(t_2-s))\|_{H^{l+6}(\mathbb{R})} ds \\
&\leq 2K_{l+6} C_0 [|f'(0)| + 2 \sup_{t_1 \leq t \leq T} \|u(t)\|_{H^{l+6}(\mathbb{R})}] \int_{0}^{t_2-t_1} \|u(t-s)-u(t_2-s)\|_{H^{l+6}(\mathbb{R})} ds,
\end{aligned}
$$

e

$$
\begin{aligned}
A_4 &= \left\{ \sum_{k=0}^{l} \int_{\mathbb{R}} |\partial_x^{k+1} [f(u(t))-f(u(t_2))]|^2 d\xi \right\}^{1/2} \\
&\leq \|f(u(t))-f(u(t_2))\|_{H^{l+1}(\mathbb{R})} \\
&\leq K_{l+1} [|f'(0)| + 2 \sup_{t_1 \leq t \leq T} \|u(t)\|_{H^{l+1}(\mathbb{R})}] \|u(t)-u(t_2)\|_{H^{l+1}(\mathbb{R})}
\end{aligned}
$$

onde K_0 depende da cota $2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$ e os K_l dependem da dimensão l e da cota $2\|u_{\epsilon,\beta,0}\|_{H^1(\mathbb{R})}$. Pelo Teorema 3.1, $u \in C((0,T], H^l(\mathbb{R}))$ para todo $l \in \mathbb{N}$, e portanto cada A_i converge para zero quando $t \to t_2^+$, donde

$$\lim_{t \to t_2^+} \|\partial_t u(t) - \partial_t u(t_2)\|_{H^l(\mathbb{R})} = 0.$$

Sendo o caso $t_2 \in (t_0, T]$ análogo, concluímos que $\partial_t u \in C([t_0,T], H^l(\mathbb{R}))$ finalizando a demonstração. □

A fim de estender nossa solução para toda a semirreta positiva necessitaremos de alguns lemas técnicos. Começaremos supondo a existência de uma constante positiva C_0 satisfazendo

$$\|u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta\|\partial_x u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \leq C_0. \tag{3.14}$$

Lema 3.2. *Assuma a condição* (3.14) *e suponha* $\beta \leq \epsilon^2/2$. *Se* u *for uma solução do problema de Cauchy* (3.1)-(3.2) *em* $[0,t_1] \times \mathbb{R}$ *tal que* $u \in C([0,t_1], H^1(\mathbb{R}))$ *e* $u(t) \in C_0^5(\mathbb{R})$ *para todo* $t \in (0,t_1]$, *então existe uma constante* $C_0 > 0$ *(independente do tempo) tal que*

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \epsilon\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + 2\beta^3 c\int_0^t \|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \leq C_0$$

para todo $t \in [0,t_1]$.

Demonstração. Multiplicando (3.1) por u e integrando em $\mathbb{R}$ teremos a seguinte cadeia de igualdades:

$$\int_{\mathbb{R}} u\partial_t u dx = \frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})};$$

$$\int_{\mathbb{R}} u\partial_x f(u) dx = 0;$$

$$\beta\int_{\mathbb{R}} u\partial_x^2 u dx = -\beta\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})};$$

$$\beta^2 b\int_{\mathbb{R}} u\partial_x^3 u dx = 0;$$

$$\beta^3 c\int_{\mathbb{R}} u\partial_x^4 u dx = \beta^3 c\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})};$$

$$\beta^5 d\int_{\mathbb{R}} u\partial_x^5 u dx = 0;$$

$$\epsilon\int_{\mathbb{R}} u\partial_x^2 u dx = -\epsilon\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Assim,

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\epsilon\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^3 c\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} = 2\beta\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Sendo $\beta \leq \epsilon^2/2$ e $\epsilon \in (0,1)$,

$$2\beta\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq \epsilon\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

e portanto

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \epsilon\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^3 c\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq 0. \tag{3.15}$$

Integrando (3.15) em $(0,t)$ e usando (3.14)

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \epsilon\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + 2\beta^3 c\int_0^t \|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \leq \|u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \leq C_0$$

estabelecendo o resultado. □

Agora suponhamos

$$|f'(u)| \leq C_0(1+|u|) \quad u \in \mathbb{R}. \tag{3.16}$$

Lema 3.3. *Assuma as condições* (3.14) *e* (3.16) *e suponha* $\beta \leq \epsilon^2/2$. *Se* u *for uma solução do problema de Cauchy* (3.1)-(3.2) *em* $[0,t_1]\times\mathbb{R}$ *tal que* $u \in C([0,t_1],H^1(\mathbb{R}))$ *e* $u(t) \in C_0^5(\mathbb{R})$ *para todo* $t \in (0,t_1]$, *então existe uma constante* $C_0 > 0$ *(independente do tempo) tal que*

$$\|u\|_{L^\infty([0,t_1]\times\mathbb{R})} \leq C_0\beta^{-1/2}$$

e

$$\beta^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3}{4}\beta^2\epsilon\int_0^t \|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + 2\beta^5 c\int_0^t \|\partial_x^3 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \leq C_0$$

para todo $t \in [0,t_1]$.

Demonstração. Multiplicando (3.1) por $-\beta\partial_x^2 u$ e integrando em $\mathbb{R}$ temos

$$-\beta\int_{\mathbb{R}} \partial_t u\partial_x^2 u dx = \frac{\beta}{2}\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$-\beta\int_{\mathbb{R}} \partial_x f(u)\partial_x^2 u ds = -\beta\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^2 u dx,$$

$$-\beta^2\int_{\mathbb{R}} \partial_x^2 u\partial_x^2 u dx = -\beta^2\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$-\beta^3 b\int_{\mathbb{R}} \partial_x^3 u\partial_x^2 u dx = 0,$$

$$-\beta^4 c\int_{\mathbb{R}} \partial_x^4 u\partial_x^2 u dx = \beta^4 c\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$-\beta^6 d\int_{\mathbb{R}} \partial_x^5 u\partial_x^2 u dx = 0$$

e

$$-\beta\epsilon\int_{\mathbb{R}} \partial_x^2 u\partial_x^2 u dx = -\beta\epsilon\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Assim,

$$\beta\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta\epsilon\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^4 c\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$= 2\beta^2\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^2 u dx.$$

Sendo $\beta \leq \epsilon^2/2$ e $\epsilon \in (0,1)$,

$$2\beta^2\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq \beta\epsilon\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

donde

$$\beta\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta\epsilon\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^4 c\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq 2\beta\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^2 u dx. \tag{3.17}$$

Devido a (3.16) temos

$$2\beta\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^2 u dx \leq 2C_0\beta\int_{\mathbb{R}}|\partial_x u\partial_x^2 u|dx + 2C_0\beta\int_{\mathbb{R}}|u\partial_x u\partial_x^2 u|dx$$

e observando que

$$2C_0\beta\int_{\mathbb{R}}|\partial_x u\partial_x^2 u|dx = \beta\int_{\mathbb{R}}|4C_0\epsilon^{-1/2}\partial_x u||\frac{1}{2}\epsilon^{1/2}\partial_x^2 u|dx$$
$$\leq C_0\beta\epsilon^{-1}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta\epsilon}{8}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$\leq C_0\epsilon\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta\epsilon}{8}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

e

$$2C_0\beta\int_{\mathbb{R}}|u\partial_x u\partial_x^2 u|dx = \beta\int_{\mathbb{R}}|4C_0\epsilon^{-1/2}u\partial_x u||\frac{1}{2}\epsilon^{1/2}\partial_x^2 u|dx$$
$$\leq C_0\beta\epsilon^{-1}\|u\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta\epsilon}{8}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$\leq C_0\epsilon\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta\epsilon}{8}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

obtemos

$$2\beta\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^2 u dx \leq C_0\epsilon\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})}) + \frac{\beta\epsilon}{4}\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{3.18}$$

Substituindo (3.18) em (3.17) segue-se que

$$\beta\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3}{4}\beta\epsilon\|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^4 c\|\partial_x^3 u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$\leq C_0\epsilon\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})})$$

e integrando a expressão acima em $(0, t)$ e utilizando o Lema 3.2 obtemos

$$\begin{aligned}\beta\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3}{4}\beta\epsilon\int_0^t \|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + 2\beta^4 c\int_0^t \|\partial_x^3 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \\ \leq C_0(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})})\epsilon\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \beta\|\partial_x u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \\ \leq C_0(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})}).\end{aligned} \tag{3.19}$$

Agora note que pelo Lema 3.2 e (3.19)

$$\begin{aligned}u^2(t,y) &\leq 2\|u(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \\ &\leq C_0\beta^{-1/2}(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})})^{1/2}\end{aligned}$$

donde

$$\|u\|^4_{L^\infty([0,t_1]\times\mathbb{R})} \leq C_0\beta^{-1}(1+\|u\|^2_{L^\infty([0,t_1]\times\mathbb{R})}).$$

Argumentando como no capítulo anterior vemos que

$$\|u\|_{L^\infty([0,t_1]\times\mathbb{R})} \leq C_0\beta^{-1/2}$$

para alguma constante positiva C_0 independente de t. Logo,

$$\beta\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3}{4}\beta\epsilon\int_0^t \|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + 2\beta^4 c\int_0^t \|\partial_x^3 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \leq C_0\beta^{-1}$$

e portanto

$$\beta^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3}{4}\beta^2\epsilon\int_0^t \|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + 2\beta^5 c\int_0^t \|\partial_x^3 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \leq C_0. \qquad \square$$

Diferentemente do capítulo anterior, nosso resultado global garante a suavidade (na variável x) da solução u.

Teorema 3.2. *Assuma a condição* (3.16) *e suponha* $\beta \leq \epsilon^2/2$. *Dado* $u_{\epsilon,\beta,0} \in H^1(\mathbb{R})$, *o problema de Cauchy* (3.1)-(3.2) *admite uma solução*

$$u \in C([0,\infty), H^1(\mathbb{R})) \cap C((0,\infty), H^l(\mathbb{R})) \quad l = 1, 2, 3, \ldots.$$

Além disso, se $u_{\epsilon,\beta,0} \in H^l(\mathbb{R})$ *para algum inteiro* $l \geq 1$ *então* $u \in C([0,\infty), H^l(\mathbb{R}))$.

Demonstração. O Teorema 3.1 assegura a existência de um número real $T > 0$ (garantido através das condições (3.8) e (3.9)) e uma solução $u \in C([0,T], H^1(\mathbb{R})) \cap C((0,T], H^l(\mathbb{R}))$ dada por (3.10). Logo pelos Lemas 3.2 e 3.3 existe uma constante $C_0 > 0$ independente de T tal que $\|u_T\|_{H^1(\mathbb{R})} \leq C_0$ onde $u_T(\cdot) = u(T,\cdot)$. Em seguida considere o espaço

$$X_S = \{u \in C([T, T+S], H^1(\mathbb{R})); \|u(t) - G(t-T)u_T\|_{H^1(\mathbb{R})} \leq \|u_T\|_{H^1(\mathbb{R})}, t \in [T, T+S]\}$$

e o seguinte operador em X_S:

$$\Lambda u(t) = G(t-T)u_T - \int_T^t G(t-s)\partial_x f(u(s))ds. \tag{3.20}$$

Dado $u \in X_S$ temos $\|u(t)\|_{H^1(\mathbb{R})} \leq 2\|u_T\|_{H^1(\mathbb{R})}$ de modo que $\|u(t)\|_{L^\infty(\mathbb{R})} \leq 2C_0$ para todo $t \in [T, T+S]$. Logo pelo Teorema 1.2 e o Corolário 1.1 existirá uma constante $K_1 > 0$ dependendo apenas da cota $2C_0$ (e consequentemente independente de T) tal que (3.5) e (3.6) se verificam para quaisquer $u, v \in X_S$. Assim, argumentando como na demonstração de (3.8) e (3.9) e fixando (qualquer) S no intervalo

$$(0, \min\{\alpha, \alpha[(|f'(0)| + 4C_0)]^{-2}\}], \quad \alpha = \frac{e(\epsilon - \beta)}{8K_1^2}$$

o Teorema 3.1 nos garante uma solução $u \in C([T, T+S], H^l(\mathbb{R}))$ dada por (3.20) (e portanto uma solução $u \in C([0, T+S], H^1(\mathbb{R})) \cap C((0, T+S], H^l(\mathbb{R})))$ e, além disso, este S servirá para todas as etapas seguintes uma vez que as constantes C_0 e K_1 não dependerão dos dados iniciais devido aos Lemas 3.2 e 3.3. Então, procedendo recursivamente obtemos uma solução

$$u \in C([0, \infty), H^1(\mathbb{R})) \cap C((0, \infty), H^l(\mathbb{R}))$$

para todo $l \in \mathbb{N}$. □

O papel dos Lemas 3.2 e 3.3 é limitar uniformemente a norma $H^1(\mathbb{R})$ dos dados iniciais das etapas de extensão e a norma $\|u(t)\|_{L^\infty(\mathbb{R})}$ em $[T, \infty)$ de modo que a magnitude de S possa ser fixada.

3.2 Estimativas a priori e Convergência em L^2

Para cada $\epsilon, \beta \in (0,1)$ consideremos o problema (3.1)–(3.2) com coeficientes b, c e d tais que $b, d \in \mathbb{R}$ e $c > 0$ e f suave satisfazendo (3.16). Suponhamos que $u_{\epsilon,\beta,0} \in C_c^\infty(\mathbb{R})$ satisfaça a condição

$$\|u_{\epsilon,\beta,0}\|_{L^2(\mathbb{R})}^2 + \beta\|\partial_x u_{\epsilon,\beta,0}\|_{L^2(\mathbb{R})}^2 + \beta^4\|\partial_x^2 u_{\epsilon,\beta,0}\|_{L^2(\mathbb{R})}^2 \leq C_0 \tag{3.21}$$

com $C_0 > 0$ independente de ϵ e β e seja $u_{\epsilon,\beta} \in C((0,\infty), H^6(\mathbb{R}))$ uma solução deste problema (garantida pelo Teorema 3.2).

A seguir estabeleceremos algumas estimativas sobre a sequência de soluções $u_{\epsilon,\beta}$. Os dois primeiros lemas abaixo são na realidade os Lemas 3.2 e 3.3.

Lema 3.4. *Assuma a condição* (3.21) *e suponha* $\beta \leq \epsilon^2/2$. *Existe uma constante* $C_0 > 0$ *(independente de* ϵ *e* β*) tal que*

$$\|u_{\epsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \epsilon\int_0^t \|\partial_x u_{\epsilon,\beta}(s,\cdot)\|_{L^2(\mathbb{R})}^2 ds + 2\beta^3 c\int_0^t \|\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0$$

para todo $t \geq 0$.

Lema 3.5. *Assuma as condições* (3.16) *e* (3.21) *e suponha* $\beta \leq \epsilon^2/2$. *Existe uma constante* $C_0 > 0$ *(independente de* ϵ *e* β*) tal que*

$$\|u_{\epsilon,\beta}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \leq C_0\beta^{-1/2}.$$

Além disso,

$$\beta^2\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3}{4}\beta^2\epsilon\int_0^t \|\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + 2\beta^5 c\int_0^t \|\partial_x^3 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \leq C_0$$

para todo $t \geq 0$.

Lema 3.6. *Assuma as condições* (3.16) *e* (3.21) *e suponha* $\beta \leq \epsilon^2/2$. *Existe uma constante* $C_0 > 0$ *(independente de* ϵ *e* β*) tal que*

$$\beta^5\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^5\epsilon\int_0^t \|\partial_x^3 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \beta^8 c\int_0^t \|\partial_x^4 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \leq C_0$$

para todo $t \geq 0$.

Demonstração. Multiplicando (3.1) por $\beta^4\partial_x^4 u_{\epsilon,\beta}$ e integrando cada termo obtido em $\mathbb{R}$, teremos

$$\beta^4\int_{\mathbb{R}} \partial_t u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx = \frac{\beta^4}{2}\frac{d}{dt}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$\beta^4\int_{\mathbb{R}} \partial_x f(u_{\epsilon,\beta})\partial_x^4 u_{\epsilon,\beta}dx = \beta^4\int_{\mathbb{R}} f'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx,$$

$$\beta^5\int_{\mathbb{R}} \partial_x^2 u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx = -\beta^5\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$\beta^6 b\int_{\mathbb{R}} \partial_x^3 u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx = 0,$$

$$\beta^7 c\int_{\mathbb{R}} \partial_x^4 u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx = \beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$\beta^9 d\int_{\mathbb{R}} \partial_x^5 u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx = 0$$

e

$$\beta^4\epsilon\int_{\mathbb{R}} \partial_x^2 u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx = -\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Assim,

$$\begin{aligned}\beta^4\frac{d}{dt}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ = 2\beta^5\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\beta^4\int_{\mathbb{R}} f'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx.\end{aligned}$$

Como $\beta \leq \epsilon^2/2$ e $\epsilon \in (0,1)$,

$$2\beta^5\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq \beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

e portanto

$$\begin{aligned}\beta^4\frac{d}{dt}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \leq -2\beta^4\int_{\mathbb{R}} f'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx.\end{aligned}\tag{3.22}$$

Agora observe que por (3.16)

$$2\beta^4\int_{\mathbb{R}}|f'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx\leq 2C_0\beta^4\int_{\mathbb{R}}|\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx+2C_0\beta^4\int_{\mathbb{R}}|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx,$$

e como

$$\begin{aligned}2C_0\beta^4\int_{\mathbb{R}}|\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx&=\beta^{1/2}\int_{\mathbb{R}}|2C_0c^{-1/2}\partial_x u_{\epsilon,\beta}||\beta^{7/2}c^{1/2}\partial_x^4 u_{\epsilon,\beta}|dx\\&\leq C_0\beta^{1/2}\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{1}{2}\beta^{15/2}c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\leq C_0\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{1}{2}\beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\end{aligned}$$

e

$$\begin{aligned}2C_0\beta^4\int_{\mathbb{R}}|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx&=\beta^{1/2}\int_{\mathbb{R}}|2C_0c^{-1/2}u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}||\beta^{7/2}c^{1/2}\partial_x^4 u_{\epsilon,\beta}|dx\\&\leq C_0\beta^{1/2}\|u_{\epsilon,\beta}\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{1}{2}\beta^{15/2}c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\leq C_0\epsilon\|u_{\epsilon,\beta}\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R}}+\frac{1}{2}\beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\end{aligned}$$

obtemos

$$\begin{aligned}2\beta^4\int_{\mathbb{R}}|f'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx\leq C_0\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}(1+\|u_{\epsilon,\beta}\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})})\\+\beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{3.23}$$

Resulta de (3.22) e (3.23) que

$$\begin{aligned}\beta^4\frac{d}{dt}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \leq C_0\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}(1+\|u_{\epsilon,\beta}\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})}).\end{aligned}$$

Integrando esta última desigualdade em $(0,t)$ e usando (3.21) e os Lemas 3.4 e 3.5 segue-se que

$$\begin{aligned}\beta^4\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^4\epsilon\int_0^t\|\partial_x^3 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+\beta^7 c\int_0^t\|\partial_x^4 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\ \leq C_0(1+\|u_{\epsilon,\beta}\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})})\epsilon\int_0^t\|\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+\beta^4\|\partial_x^2 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})}\\ \leq C_0(1+\|u_{\epsilon,\beta}\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})})\\ \leq C_0\beta^{-1}\end{aligned}$$

e consequentemente

$$\beta^5\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^5\epsilon\int_0^t\|\partial_x^3 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+\beta^8 c\int_0^t\|\partial_x^4 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\leq C_0. \qquad \square$$

Nosso primeiro resultado é o seguinte

Teorema 3.3. *Assuma as condições* (3.16) *e* (3.21). *Se* $\beta \leq \epsilon^2/2$, *existirão uma subsequência* $\{u_{\epsilon_k,\beta_k}\}_k$ *com* $\epsilon_k, \beta_k \to 0$ *e uma função* $u \in L^2_{loc}(\mathbb{R}_+ \times \mathbb{R})$ *tais que* $u_{\epsilon_k,\beta_k} \rightharpoonup u$, $f(u_{\epsilon_k,\beta_k}) \rightharpoonup f(u)$ *em* $\mathscr{D}'(\mathbb{R}_+ \times \mathbb{R})$ *e* u *é uma solução fraca de*

$$\partial_t u + \partial_x f(u) = 0 \quad em \quad \mathbb{R}_+ \times \mathbb{R}.$$

Além disso, $u_{\epsilon_k,\beta_k} \to u$ *fortemente em* $L^r_{loc}(\mathbb{R}_+ \times \mathbb{R})$ *para todo* $r \in [1,2)$ *se* $f'' > 0$.

Demonstração. Seja (η, q) um par de entropia–fluxo de entropia $\eta, q : \mathbb{R} \to \mathbb{R}$ com $\eta \in C^2_c(\mathbb{R})$ convexa em algum intervalo limitado não-vazio e q dada por

$$q(u) = \int_0^u f'(t)\eta'(t)dt.$$

Multiplique a equação (3.1) por $\eta'(u_{\epsilon,\beta})$. A relação $q' = f'\eta'$ nos fornece a seguinte decomposição:

$$\begin{aligned} \partial_t \eta(u_{\epsilon,\beta}) + \partial_x q(u_{\epsilon,\beta}) &= (\epsilon - \beta)\eta'(u_{\epsilon,\beta})\partial_x^2 u_{\epsilon,\beta} - \beta^2 b\eta'(u_{\epsilon,\beta})\partial_x^3 u_{\epsilon,\beta} - \beta^3 c\eta'(u_{\epsilon,\beta})\partial_x^4 u_{\epsilon,\beta} \\ &\quad - \beta^5 d\eta'(u_{\epsilon,\beta}))\partial_x^5 u_{\epsilon,\beta} \\ &= \sum_{i=1}^{8} I_{i,\epsilon,\beta} \end{aligned} \tag{3.24}$$

onde

$$\begin{aligned} I_{1,\epsilon,\beta} &= (\epsilon - \beta)\partial_x(\eta'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}); \\ I_{2,\epsilon,\beta} &= -(\epsilon - \beta)\eta''(u_{\epsilon,\beta})(\partial_x u_{\epsilon,\beta})^2; \\ I_{3,\epsilon,\beta} &= -\beta^2 b\partial_x(\eta'(u_{\epsilon,\beta})\partial_x^2 u_{\epsilon,\beta}); \\ I_{4,\epsilon,\beta} &= \beta^2 b\eta''(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}; \\ I_{5,\epsilon,\beta} &= -\beta^3 c\partial_x(\eta'(u_{\epsilon,\beta})\partial_x^3 u_{\epsilon,\beta}); \\ I_{6,\epsilon,\beta} &= \beta^3 c\eta''(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta}; \\ I_{7,\epsilon,\beta} &= -\beta^5 d\partial_x(\eta'(u_{\epsilon,\beta})\partial_x^4 u_{\epsilon,\beta}); \\ I_{8,\epsilon,\beta} &= \beta^5 d\eta''(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}. \end{aligned}$$

As afirmações abaixo nos darão informações sobre cada elemento $I_{i,\epsilon,\beta}$.

Afirmação 1. $I_{i,\epsilon,\beta} \to 0$ em $H^{-1}(\mathbb{R}_+ \times \mathbb{R})$ quando $\epsilon \to 0$ $(i = 1, 3, 5, 7)$.

De fato, pelo Lema 3.4

$$\begin{aligned} \|(\epsilon - \beta)\eta'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+ \times \mathbb{R})} &= (\epsilon - \beta)^2 \int_0^\infty \int_{\mathbb{R}} |\eta'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}|^2 dxdt \\ &\leq C_0 \epsilon^2 \int_0^\infty \|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} dt \\ &= C_0 \epsilon \left(\epsilon \int_0^\infty \|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} dt \right) \\ &\leq C_0 \epsilon. \end{aligned}$$

Assim, se $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$

$$\begin{aligned}
|\langle I_{1,\epsilon,\beta}, \phi\rangle| &= \left|\int_0^\infty \int_{\mathbb{R}} (\epsilon - \beta)\eta'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x \phi dt dx\right| \\
&\leq \|(\epsilon-\beta)\eta'(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x\phi\|_{L^2(\mathbb{R}_+\times\mathbb{R})} \\
&\leq C_0\epsilon^{1/2},
\end{aligned}$$

e consequentemente $I_{1,\epsilon,\beta} \to 0$ em $H^{-1}(\mathbb{R}_+ \times \mathbb{R})$ quando $\epsilon \to 0$.

Analogamente, usando os Lemas 3.4, 3.5 e 3.6 obtemos

$$\begin{aligned}
\|\beta^2 b\eta'(u_{\epsilon,\beta})\partial_x^2 u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta^4 b^2 \int_0^\infty \int_{\mathbb{R}} |\eta'(u_{\epsilon,\beta})\partial_x^2 u_{\epsilon,\beta}|^2 dt dx \\
&\leq C_0\beta^4 \int_0^\infty \|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} dt \\
&\leq C_0\epsilon\left(\beta^3 \int_0^\infty \|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} dt\right) \\
&\leq C_0\epsilon,
\end{aligned}$$

$$\begin{aligned}
\|\beta^3 c\eta'(u_{\epsilon,\beta})\partial_x^3 u_{\epsilon,\beta}|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta^6 c^2 \int_0^\infty \int_{\mathbb{R}} |\eta'(u_{\epsilon,\beta})\partial_x^3 u_{\epsilon,\beta}|^2 dt dx \\
&\leq C_0\epsilon\left(\beta^5 \int_0^\infty \|\partial_x^3 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\right) \\
&\leq C_0\epsilon
\end{aligned}$$

e

$$\begin{aligned}
\|\beta^5 d\eta'(u_{\epsilon,\beta})\partial_x^4 u_{\epsilon,\beta}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta^{10} d^2 \int_0^\infty \int_{\mathbb{R}} |\eta'(u_{\epsilon,\beta})\partial_x^4 u_{\epsilon,\beta}|^2 dt dx \\
&\leq C_0\beta^2\left(\beta^8 \int_0^\infty \|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} dt\right) \\
&\leq C_0\epsilon.
\end{aligned}$$

Logo, $I_{3,\epsilon,\beta}, I_{5,\epsilon,\beta}, I_{7,\epsilon,\beta} \to 0$ em $H^{-1}(\mathbb{R}_+ \times \mathbb{R})$ quando $\epsilon \to 0$.

Afirmação 2. $I_{i,\epsilon,\beta}$ é limitado em $L^1(\mathbb{R}_+ \times \mathbb{R})$ $(i = 2, 4, 6, 8)$.

De fato, a partir dos Lemas 3.4, 3.5 e 3.6

$$\begin{aligned}
\|I_{2,\epsilon,\beta}\|_{L^1(\mathbb{R}_+\times\mathbb{R})} &\leq (\epsilon+\beta)\int_0^\infty \int_{\mathbb{R}} |\eta''(u_{\epsilon,\beta})(\partial_x u_{\epsilon,\beta})^2| dx dt \\
&\leq C_0\epsilon \int_0^\infty \|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} dt \\
&\leq C_0,
\end{aligned}$$

$$
\begin{aligned}
\|I_{4,\epsilon,\beta}\|_{L^1(\mathbb{R}_+\times\mathbb{R})} &= \beta^2|b| \int_0^\infty \int_{\mathbb{R}} |\eta''(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}| dtdx \\
&\leq C_0\beta^2 \int_0^\infty \int_{\mathbb{R}} |\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}| dtdx \\
&\leq C_0 \left\{\epsilon \int_0^\infty \|\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\right\}^{1/2} \left\{\beta^3 \int_0^\infty \|\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\right\}^{1/2} \\
&\leq C_0,
\end{aligned}
$$

$$
\begin{aligned}
\|I_{6,\epsilon,\beta}\|_{L^1(\mathbb{R}_+\times\mathbb{R})} &= \beta^3 c \int_0^\infty \int_{\mathbb{R}} |\eta''(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta}| dtdx \\
&\leq C_0\beta^3 \int_0^\infty \int_{\mathbb{R}} |\partial_x u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta}| dtdx \\
&\leq C_0 \left\{\epsilon \int_0^\infty \|\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\right\}^{1/2} \left\{\beta^5 \int_0^\infty \|\partial_x^3 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\right\}^{1/2} \\
&\leq C_0
\end{aligned}
$$

e

$$
\begin{aligned}
\|I_{8,\epsilon,\beta}\|_{L^1(\mathbb{R}_+\times\mathbb{R})} &= \beta^5|d| \int_0^\infty \int_{\mathbb{R}} |\eta''(u_{\epsilon,\beta})\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}| dtdx \\
&\leq C_0\beta^5 \int_0^\infty \int_{\mathbb{R}} |\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}| dtdx \\
&\leq C_0 \left\{\epsilon \int_0^\infty \|\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\right\}^{1/2} \left\{\beta^8 \int_0^\infty \|\partial_x^4 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\right\}^{1/2} \\
&\leq C_0.
\end{aligned}
$$

Agora defina

$$T_{\epsilon,\beta} = I_{1,\epsilon,\beta} + I_{3,\epsilon,\beta} + I_{5,\epsilon,\beta} + I_{7,\epsilon,\beta}$$

e

$$\mu_{\epsilon,\beta} = I_{2,\epsilon,\beta} + I_{4,\epsilon,\beta} + I_{6,\epsilon,\beta} + I_{8,\epsilon,\beta}$$

de modo que

$$\partial_t \eta(u_{\epsilon,\beta}) + \partial_x q(u_{\epsilon,\beta}) = T_{\epsilon,\beta} + \mu_{\epsilon,\beta}.$$

As Afirmações 1 e 2 nos dizem (respectivamente) que $\{T_{\epsilon,\beta}\}_{\epsilon,\beta}$ é compacto em $H^{-1}(\mathbb{R}_+ \times \mathbb{R})$ e $\{\mu_{\epsilon,\beta}\}_{\epsilon,\beta}$ é limitado em $\mathcal{M}(\mathbb{R}_+ \times \mathbb{R})$. Claramente $\{\partial_t \eta(u_{\epsilon,\beta}) + \partial_x q(u_{\epsilon,\beta})\}_{\epsilon,\beta}$ é uma sequência limitada em $W^{-1,\infty}(\mathbb{R}_+ \times \mathbb{R})$ uma vez que η tem suporte compacto. Portanto, pelo Lema de Murat a sequência de distribuições $\{\partial_t \eta(u_{\epsilon,\beta}) + \partial_x q(u_{\epsilon,\beta})\}_{\epsilon,\beta}$ pertence a um subconjunto compacto de $H^{-1}(\Omega)$ se Ω for um subconjunto aberto limitado de $\mathbb{R}_+ \times \mathbb{R}$. A primeira parte do teorema é uma consequência imediata do Teorema 1.4 e de um argumento diagonal padrão; a segunda segue do Corolário 1.2. □

Um resultado análogo ao Teorema 2.4 pode ser obtido se assumirmos (2.32), (2.33), (2.34) e (3.21), além das hipóteses

$$\beta \leq \epsilon^2/2 \quad e \quad \|u_{\epsilon,\beta}(t,\cdot)\|_{L^{r_0}(\mathbb{R})} \leq \|u_{\epsilon,\beta,0}\|_{L^{r_0}(\mathbb{R})} \;\; \textit{para algum } r_0 \in (1,2).$$

A demonstração é feita utilizando os Lemas 3.4, 3.5 e 3.6.

3.3 Estimativas a priori e Convergência em L^4

Nesta seção, além da condição (3.16) com f suave (e $b, d \in \mathbb{R}$ e $c > 0$), assumiremos também que $u_{\epsilon,\beta,0} \in C_c^\infty(\mathbb{R})$ seja uma aproximação da função real

$$u_0 \in L^1(\mathbb{R}) \cap L^4(\mathbb{R})$$

satisfazendo

$$u_{\epsilon,\beta,0} \to u_0 \;\text{ em }\; L^1(\mathbb{R}) \cap L^4(\mathbb{R}) \;\text{ quando }\; \epsilon, \beta \to 0 \tag{3.25}$$

e

$$\|u_{\epsilon,\beta,0}\|^4_{L^4(\mathbb{R})} + \|u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta\|\partial_x u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta^4\|\partial_x^2 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \leq C_0 \tag{3.26}$$

sendo $C_0 > 0$ uma constante independente de ϵ, β e seja $u_{\epsilon,\beta} \in C((0,\infty), H^6(\mathbb{R}))$ uma solução deste problema.

O lema a seguir nos dará algumas informações sobre a sequência $u_{\epsilon,\beta}$.

Lema 3.7. *Assumamos as condições* (3.16) *e* (3.26)*. Se*

$$\beta \leq D_0\epsilon^2 \tag{3.27}$$

para alguma constante $D_0 \in (0, 1/2)$ suficientemente pequena, então as seguintes afirmações são válidas:

(i) a família $\{u_{\epsilon,\beta}\}_{\epsilon,\beta}$ é limitada em $L^\infty(\mathbb{R}_+, L^4(\mathbb{R}))$;

(ii) a família $\{\beta^2\partial_x^2 u_{\epsilon,\beta}\}_{\epsilon,\beta}$ é limitada em $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}))$;

(iii) as famílias $\{\beta^2\epsilon^{1/2}\partial_x^3 u_{\epsilon,\beta}\}_{\epsilon,\beta}, \{\beta^{7/2}\partial_x^4 u_{\epsilon,\beta}\}_{\epsilon,\beta}$ e $\{\epsilon^{1/2}u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\}_{\epsilon,\beta}$ são limitadas em $L^2(\mathbb{R}_+ \times \mathbb{R})$.

Demonstração. Multiplicando (3.1) por

$$u^3_{\epsilon,\beta} + B\beta^4\partial_x^4 u_{\epsilon,\beta}$$

onde B é constante positiva escolhida a posteriori, e integrando a expressão obtida em $\mathbb{R}$ teremos as seguintes igualdades:

$$\int_{\mathbb{R}} (u^3_{\epsilon,\beta} + B\beta^4\partial_x^4 u_{\epsilon,\beta})\partial_t u_{\epsilon,\beta}dx = \frac{1}{4}\frac{d}{dt}\|u_{\epsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{B\beta^4}{2}\frac{d}{dt}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 + B\beta^4\partial_x^4 u_{\epsilon,\beta})\partial_x f(u_{\epsilon,\beta})dx = B\beta^4\int_{\mathbb{R}}\partial_x^4 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx,$$

$$\begin{aligned}\beta\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 + B\beta^4\partial_x^4 u_{\epsilon,\beta})\partial_x^2 u_{\epsilon,\beta}dx = &-3\beta\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &- B\beta^5\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}$$

$$\beta^2 b\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 + B\beta^4\partial_x^4 u_{\epsilon,\beta})\partial_x^3 u_{\epsilon,\beta}dx = -3\beta^2 b\int_{\mathbb{R}}u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}dx,$$

$$\begin{aligned}\beta^3 c\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 + B\beta^4\partial_x^4 u_{\epsilon,\beta})\partial_x^4 u_{\epsilon,\beta}dx = &-3\beta^3 c\int_{\mathbb{R}}u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta}dx\\ &+ B\beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}$$

$$\beta^5 d\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 + B\beta^4\partial_x^4 u_{\epsilon,\beta})\partial_x^5 u_{\epsilon,\beta}dx = -3\beta^5 d\int_{\mathbb{R}}u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx$$

e

$$\begin{aligned}\epsilon\int_{\mathbb{R}}(u_{\epsilon,\beta}^3 + B\beta^4\partial_x^4 u_{\epsilon,\beta})\partial_x^2 u_{\epsilon,\beta}dx = &-3\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &- B\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Logo,

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{4}\|u_{\epsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{B\beta^4}{2}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right\} + B\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad + B\beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 3\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &= -B\beta^4\int_{\mathbb{R}}\partial_x^4 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx + 3\beta\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad + B\beta^5\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 3\beta^2 b\int_{\mathbb{R}}u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}dx\\ &\quad + 3\beta^3 c\int_{\mathbb{R}}u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta}dx + 3\beta^5 d\int_{\mathbb{R}}u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx.\end{aligned}$$

Por hipótese, existe uma constante $D_0 \in (0, 1/2)$ (a ser escolhida posteriormente) tal que

$$\beta \leq D_0\epsilon^2. \tag{3.28}$$

Em particular,

$$\beta \leq \epsilon^2/2 \tag{3.29}$$

já que $\epsilon \in (0, 1)$. Logo,

$$B\beta^5\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq \frac{B\beta^4\epsilon}{2}\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

e

$$3\beta\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq \frac{3}{2}\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

donde

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{4}\|u_{\epsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{B\beta^4}{2}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right\} + \frac{1}{2}B\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad + \frac{3}{2}\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + B\beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad \leq -B\beta^4\int_{\mathbb{R}}\partial_x^4 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx + 3\beta^2 b\int_{\mathbb{R}} u^2_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}dx \\
&\quad + 3\beta^3 c\int_{\mathbb{R}} u^2_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta}dx + 3\beta^5 d\int_{\mathbb{R}} u^2_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx.
\end{aligned} \tag{3.30}$$

Pela condição (3.16) temos

$$\begin{aligned}
B\beta^4\int_{\mathbb{R}}\partial_x^4 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx &\leq B\beta^4\int_{\mathbb{R}}|\partial_x^4 u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}f'(u_{\epsilon,\beta})|dx \\
&\leq C_0B\beta^4\int_{\mathbb{R}}|\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx + C_0B\beta^4\int_{\mathbb{R}}|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx.
\end{aligned}$$

Agora, a partir de (3.29)

$$\begin{aligned}
C_0B\beta^4\int_{\mathbb{R}}|\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx &= B\int_{\mathbb{R}}|2C_0\beta^{1/2}c^{-1/2}\partial_x u_{\epsilon,\beta}||\frac{1}{2}\beta^{7/2}c^{1/2}\partial_x^4 u_{\epsilon,\beta}|dx \\
&\leq C_0B\beta\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{B\beta^7 c}{8}\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\leq C_0B\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{B\beta^7 c}{8}\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

e

$$\begin{aligned}
C_0B\beta^4\int_{\mathbb{R}}|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx &= B\int_{\mathbb{R}}|2C_0\beta^{1/2}c^{-1/2}u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}||\frac{1}{2}\beta^{7/2}c^{1/2}\partial_x^4 u_{\epsilon,\beta}|dx \\
&\leq C_0B\beta\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{B\beta^7 c}{8}\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\leq C_0B\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{B\beta^7 c}{8}\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

donde

$$\begin{aligned}
B\beta^4\int_{\mathbb{R}}\partial_x^4 u_{\epsilon,\beta}\partial_x f(u_{\epsilon,\beta})dx &\leq C_0B\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{B\beta^7 c}{4}\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad + C_0B\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{3.31}$$

Além disso, utilizando (3.28) e o Lema 3.5 segue-se que

$$\begin{aligned}3\beta^2 b\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}dx &\leq C_0\beta^2\|u_{\epsilon,\beta}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\int_{\mathbb{R}}|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^2 u_{\epsilon,\beta}|dx\\ &\leq C_0\int_{\mathbb{R}}|\beta^{1/2}B^{-1/2}\epsilon^{-1/2}u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}||\beta B^{1/2}\epsilon^{1/2}\partial_x^2 u_{\epsilon,\beta}|dx\\ &\leq C_0B^{-1}\beta\epsilon^{-1}\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+C_0B\beta^2\epsilon\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\leq C_0B^{-1}D_0\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+C_0B\beta^2\epsilon\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}\tag{3.32}$$

$$\begin{aligned}3\beta^3 c\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta}dx &\leq C_0\beta^3\|u_{\epsilon,\beta}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\int_{\mathbb{R}}|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^3 u_{\epsilon,\beta}|dx\\ &\leq \int_{\mathbb{R}}|2C_0\beta^{1/2}B^{-1/2}\epsilon^{-1/2}u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}||\frac{1}{2}B^{1/2}\beta^2\epsilon^{1/2}\partial_x^3 u_{\epsilon,\beta}|dx\\ &\leq C_0\beta B^{-1}\epsilon^{-1}\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{1}{8}B\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}dx\\ &\leq C_0D_0B^{-1}\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{1}{8}B\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}dx\end{aligned}\tag{3.33}$$

e

$$\begin{aligned}3\beta^5 d\int_{\mathbb{R}} u_{\epsilon,\beta}^2\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}dx &\leq C_0\beta^5\|u_{\epsilon,\beta}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\int_{\mathbb{R}}|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}\partial_x^4 u_{\epsilon,\beta}|dx\\ &\leq \int_{\mathbb{R}}|2C_0\beta^{1/2}B^{-1/2}c^{-1/2}u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}||\frac{1}{2}\beta^4B^{1/2}c^{1/2}\partial_x^4 u_{\epsilon,\beta}|dx\\ &\leq C_0B^{-1}\beta\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{B\beta^8c}{8}\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\leq C_0B^{-1}D_0\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{B\beta^7c}{8}\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{3.34}$$

Assim, substituindo as estimatinas (3.31)–(3.34) em (3.30) resulta que

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{4}\|u_{\epsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}+\frac{B\beta^4}{2}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right\}\\ &\quad+\frac{3}{8}B\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{5}{8}B\beta^7c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad+\left(\frac{3}{2}-C_0(B+B^{-1}D_0)\right)\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\leq C_0B\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+C_0B\beta^2\epsilon\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{3.35}$$

Procuraremos a seguir uma constante $B > 0$ tal que

$$\frac{3}{2}-C_0(B+B^{-1}D_0)>0.\tag{3.36}$$

Considerando a função polinomial

$$p(T)=T^2-3(2C_0)^{-1}T+D_0,$$

a condição (3.36) equivale a $p(B) < 0$ para alguma constante $B > 0$. Escolhendo $D_0 \in (0, 1/2)$ de modo que $D_0 < 9(4C_0)^{-2}$, o discrimante $\Delta = 9(2C_0)^{-2} - 4D_0$ é positivo e a função p possui dois zeros $0 < T_1 < T_2$. Portanto, (3.36) é verificada quando $B \in (T_1, T_2)$. Fixando um $B \in (T_1, T_2)$ defina $K_1 = 3/2 - C_0(B + B^{-1}D_0)$. Então $K_1 > 0$ e a partir de (3.35) obtemos

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{4}\|u_{\epsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{B\beta^4}{2}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right\} + \frac{3}{8}B\beta^4\epsilon\|\partial_x^3 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad + \frac{5}{8}B\beta^7 c\|\partial_x^4 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + K_1\epsilon\|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad \le K_2\epsilon\|\partial_x u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + K_3\beta^2\epsilon\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

sendo K_2 e K_3 duas constantes positivas.

Por fim, integrando em $(0,t)$ a desigualdade acima, (3.26) e os Lemas 3.4 e 3.5 nos permitem concluir que

$$\begin{aligned}
&\frac{1}{4}\|u_{\epsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{B\beta^4}{2}\|\partial_x^2 u_{\epsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3}{8}B\beta^4\epsilon\int_0^t \|\partial_x^3 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \\
&\quad + \frac{5}{8}B\beta^7 c\int_0^t \|\partial_x^4 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + K_1\epsilon\int_0^t \|u_{\epsilon,\beta}\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \\
&\le K_2\epsilon\int_0^t \|\partial_x u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + K_3\beta^2\epsilon\int_0^t \|\partial_x^2 u_{\epsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \frac{1}{4}\|u_{\epsilon,\beta,0}\|^4_{L^4(\mathbb{R})} \\
&\quad + \frac{B\beta^4}{2}\|\partial_x^2 u_{\epsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \\
&\le C_0
\end{aligned}$$

finalizando a demonstração. □

Podemos então enunciar o seguinte

Teorema 3.4. *Assuma as condições* (3.16), (3.25) *e* (3.26). *Se*

$$\beta \le D_0\epsilon^2 \tag{3.37}$$

para alguma constante $D_0 \in (0, 1/2)$ *suficientemente pequena, então existe uma função* $u \in L^\infty(\mathbb{R}_+, L^4(\mathbb{R}))$ *tal que* $u_{\epsilon,\beta} \to u$ *em* $L^r_{loc}(\mathbb{R}_+ \times \mathbb{R})$ *para todo* $r \in [1, 4)$, *sendo* u *a única solução de entropia de*

$$\begin{aligned}
&\partial_t u + \partial_x f(u) = 0 \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\
&u(0,x) = u_0(x) \quad x \in \mathbb{R}.
\end{aligned}$$

Demonstração. Argumentando como na demonstração do Teorema 2.6 é suficiente verificar que

$$\lim_{k\to\infty} |J_n^k| \le C_0 T \qquad n = 1, 2, 3, \ldots \tag{3.38}$$

e

$$\int_0^\infty \int_{\mathbb{R}} [\partial_t \eta(u_{\epsilon_k,\beta_k}) + \partial_x q(u_{\epsilon_k,\beta_k})]\phi dtdx \le C_0\epsilon_k^r \quad \textit{para algum } r \in \mathbb{Q} \cap (0,\infty) \tag{3.39}$$

sendo $\{u_{\epsilon_k,\beta_k}\}_k$ uma subsequência satisfazendo (2.40) para toda $g \in C(\mathbb{R})$ tal que $g(u) = O(1 + |u|^r)$ com $r \in [0,4)$, $\{\phi_n\}_n \in C_c^\infty(\mathbb{R})$ tal que $\phi_n \to 2u_0$ em $L^2(\mathbb{R})$, (η, q) um par de entropia–fluxo de entropia com $\eta \in C^2(\mathbb{R})$ convexa, η e η' limitadas, $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ não-negativa e

$$J_n^k = -\frac{1}{T}\int_0^T \int_{\mathbb{R}} \left(\int_0^t \partial_s u_{\epsilon_k,\beta_k}(s,x)ds \right) \phi_n(x)dtdx.$$

Para verificar (3.38) começamos observando que

$$\begin{aligned} J_n^k = -\frac{1}{T}\int_0^T \int_{\mathbb{R}} \int_0^t & \left[-\partial_x f(u_{\epsilon_k,\beta_k}) + (\epsilon_k - \beta_k)\partial_x^2 u_{\epsilon_k,\beta_k} - \beta_k^2 b \partial_x^3 u_{\epsilon_k,\beta_k} - \beta_k^3 c \partial_x^4 u_{\epsilon_k,\beta_k} \right. \\ & \left. -\beta_k^5 d \partial_x^5 u_{\epsilon_k,\beta_k} \right] \phi_n(x) dsdtdx. \end{aligned}$$

Agora pelo Lema 3.4 e a relação (3.37) segue-se que

$$\begin{aligned} \frac{1}{T}\int_0^T \int_{\mathbb{R}} \int_0^t \partial_x f(u_{\epsilon_k,\beta_k})\phi_n(x)dsdtdx &= -\frac{1}{T}\int_0^T \int_{\mathbb{R}} \int_0^t f(u_{\epsilon_k,\beta_k}(s,x))\partial_x \phi_n(x)dsdtdx \\ &\le \frac{C_0}{T}\int_0^T \int_{\mathbb{R}} \int_0^t (1 + |u_{\epsilon_k,\beta_k}(s,x)|^2)|\partial_x \phi_n(x)|dsdtdx \\ &\le \frac{C_0}{T}\int_0^T \int_{\text{supp}(\phi_n)} \int_0^t (1 + |u_{\epsilon_k,\beta_k}(s,x)|^2)dsdtdx \\ &\le C_0 T + \frac{C_0}{T}\int_0^T \int_0^t dsdt \\ &\le C_0 T, \end{aligned}$$

$$\begin{aligned} \frac{(\epsilon_k - \beta_k)}{T}\int_0^T \int_{\mathbb{R}} \int_0^t \partial_x^2 u_{\epsilon_k,\beta_k}\phi_n(x)dsdtdx &= \frac{(\epsilon_k - \beta_k)}{T}\int_0^T \int_{\mathbb{R}} \int_0^t u_{\epsilon_k,\beta_k}(s,x)\partial_x^2 \phi_n(x)dsdtdx \\ &\le \frac{2\epsilon_k}{T}\int_0^T \int_0^t \|u_{\epsilon_k,\beta_k}(s,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^2 \phi_n(x)\|_{L^2(\mathbb{R})}dsdt \\ &\le \frac{C_0 \epsilon_k}{T}\int_0^T \int_0^t dsdt \\ &\le C_0 \epsilon_k T, \end{aligned}$$

$$\begin{aligned} \frac{\beta_k^2 b}{T}\int_0^T \int_{\mathbb{R}} \int_0^t \partial_x^3 u_{\epsilon_k,\beta_k}\phi_n(x)dsdtdx &= -\frac{\beta_k^2 b}{T}\int_0^T \int_{\mathbb{R}} \int_0^t u_{\epsilon_k,\beta_k}(s,x)\partial_x^3 \phi_n(x)dsdtdx \\ &\le \frac{\beta_k^2}{T}\int_0^T \int_0^t \|u_{\epsilon_k,\beta_k}(s,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^3 \phi_n(x)\|_{L^2(\mathbb{R})}dsdt \\ &\le \frac{C_0}{T}\epsilon_k \int_0^T \int_0^t dsdt \\ &\le C_0 \epsilon_k T, \end{aligned}$$

$$
\begin{aligned}
\frac{\beta_k^3 c}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_x^4 u_{\epsilon_k,\beta_k}\phi_n(x)dsdtdx &= \frac{\beta_k^3 c}{T}\int_0^T\int_{\mathbb{R}}\int_0^t u_{\epsilon_k,\beta_k}(s,x)\partial_x^4\phi_n(x)dsdtdx\\
&\leq \frac{\beta_k^3 c}{T}\int_0^T\int_0^t \|u_{\epsilon_k,\beta_k}(s,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^4\phi_n(x)\|_{L^2(\mathbb{R})}dsdt\\
&\leq \frac{C_0}{T}\epsilon_k\int_0^T\int_0^t dsdt\\
&\leq C_0\epsilon_k T,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\beta_k^5 d}{T}\int_0^T\int_{\mathbb{R}}\int_0^t \partial_x^5 u_{\epsilon_k,\beta_k}\phi_n(x)dsdtdx &= -\frac{\beta_k^5 d}{T}\int_0^T\int_{\mathbb{R}}\int_0^t u_{\epsilon_k,\beta_k}(s,x)\partial_x^5\phi_n(x)dsdtdx\\
&\leq \frac{\beta_k^5|d|}{T}\int_0^T\int_0^t \|u_{\epsilon_k,\beta_k}(s,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^5\phi_n(x)\|_{L^2(\mathbb{R})}dsdt\\
&\leq \frac{C_0}{T}\epsilon_k\int_0^T\int_0^t dsdt\\
&\leq C_0\epsilon_k T
\end{aligned}
$$

e portanto

$$\lim_{k\to\infty}|J_n^k|\leq C_0T.$$

Quanto a (3.39), a partir de (3.24) é fácil ver que

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}}\left[\partial_t\eta(u_{\epsilon_k,\beta_k})+\partial_x q(u_{\epsilon_k,\beta_k})\right]\phi dtdx\\
&\qquad\leq C_0\epsilon_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}+C_0\beta_k^3\|\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\qquad+C_0\beta_k^3\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}+C_0\beta_k^2\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\qquad+C_0\beta_k^2\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}+C_0\beta_k^5\|\partial_x^4 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\qquad+C_0\beta_k^5\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x^4 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}.
\end{aligned}
$$

Assim, utilizando (3.37) e os Lemas 3.4 e 3.7 obtemos

$$
\begin{aligned}
\epsilon_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \epsilon_k^{1/2}\|\epsilon_k^{1/2}\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\leq C_0\epsilon_k^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
\beta_k^2\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta_k^{1/2}\|\beta_k^{3/2}\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\leq C_0\beta_k^{1/2}\\
&\leq C_0\epsilon_k^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
\beta_k^2\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x^2 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\leq C_0\beta_k^{1/2}\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\leq C_0\epsilon_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\
&\leq C_0\epsilon_k^{1/2},
\end{aligned}
$$

$$\begin{aligned}\beta_k^3\|\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta_k\epsilon_k^{-1/2}\|\beta_k^2\epsilon_k^{1/2}\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\ &\le C_0\beta_k\epsilon_k^{-1/2}\\ &\le C_0\epsilon_k^{1/2},\end{aligned}$$

$$\begin{aligned}\beta_k^3\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x^3 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\le C_0\beta_k\epsilon_k^{-1/2}\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\ &\le C_0\epsilon_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\ &\le C_0\epsilon_k^{1/2},\end{aligned}$$

$$\begin{aligned}\beta_k^5\|\partial_x^4 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &= \beta_k^{3/2}\|\beta_k^{7/2}\partial_x^4 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\ &\le C_0\beta_k^{3/2}\\ &\le C_0\epsilon_k^{1/2}\end{aligned}$$

e

$$\begin{aligned}\beta_k^5\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\|\partial_x^4 u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})} &\le C_0\beta_k^{3/2}\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\ &\le C_0\epsilon_k\|\partial_x u_{\epsilon_k,\beta_k}\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\\ &\le C_0\epsilon_k^{1/2}\end{aligned}$$

estabelecendo (3.39). □

Apêndice A

Verificação de (2.41)

Afirmação. Sejam $r \in (1,2)$ e $g(\lambda) = |\lambda|^r$. Se

$$Q(\lambda,\lambda_0) = g(\lambda) - g(\lambda_0) - g'(\lambda_0)(\lambda - \lambda_0)$$

então

$$Q(\lambda,\lambda_0) \geq \frac{r(r-1)}{4}\frac{(\lambda-\lambda_0)^2}{(1+|\lambda|+|\lambda_0|)^{2-r}} \quad (\lambda,\lambda_0) \in \mathbb{R}^2.$$

Demonstração. Começamos observando que

$$g'(\lambda) = \begin{cases} r|\lambda|^{r-1} & ; \quad \lambda \geq 0 \\ -r|\lambda|^{r-1} & ; \quad \lambda < 0 \end{cases}$$

e que g' é diferenciável (apenas) em $\mathbb{R}\backslash\{0\}$ com $g''(\lambda) = r(r-1)|\lambda|^{r-2}$. Note também que g é uma função par e g' é uma função ímpar. Além do mais, sendo a afirmação óbvia para $\lambda = \lambda_0$, suporemos $\lambda \neq \lambda_0$. Se for $\lambda_0 = 0$ (donde $\lambda \neq 0$) então

$$\begin{aligned} \frac{r(r-1)}{4}\frac{\lambda^2}{(1+|\lambda|)^{2-r}} &\leq \frac{r(r-1)}{2}\frac{\lambda^2}{(1+|\lambda|)^{2-r}} \\ &\leq \frac{\lambda^2}{|\lambda|^{2-r}} = Q(\lambda,0), \end{aligned}$$

e portanto podemos também assumir $\lambda_0 \neq 0$. O restante da demonstração será dividida em vários casos.

Caso 1: $\lambda_0 > 0$

1.1: $0 < \lambda_0 < \lambda$

Pela Fórmula de Taylor existe $c \in (\lambda_0, \lambda)$ tal que

$$g(\lambda) = g(\lambda_0) + g'(\lambda_0)(\lambda-\lambda_0) + \frac{g''(c)}{2}(\lambda-\lambda_0)^2$$

ou seja,

$$Q(\lambda,\lambda_0) = \frac{r(r-1)}{2}c^{r-2}(\lambda-\lambda_0)^2.$$

Como $0 < c < 1 + \lambda + \lambda_0$ e $r - 2 < 0$ segue-se que $(1 + \lambda + \lambda_0)^{r-2} < c^{r-2}$, e portanto

$$\begin{aligned} Q(\lambda, \lambda_0) &\geq \frac{r(r-1)}{2}(1 + \lambda + \lambda_0)^{r-2}(\lambda - \lambda_0)^2 \\ &\geq \frac{r(r-1)}{4}\frac{(\lambda - \lambda_0)^2}{(1 + \lambda + \lambda_0)^{2-r}}. \end{aligned}$$

1.2: $0 < \lambda < \lambda_0$

Neste caso $-\lambda_0 < -\lambda < 0$ e existe $c \in (-\lambda_0, -\lambda)$ tal que

$$g(-\lambda) = g(-\lambda_0) + g'(-\lambda_0)(-\lambda + \lambda_0) + \frac{g''(c)}{2}(-\lambda + \lambda_0)^2$$

donde

$$Q(\lambda, \lambda_0) = \frac{r(r-1)}{2}|c|^{r-2}(\lambda - \lambda_0)^2.$$

Logo, argumentando como no Caso 1.1 concluímos que

$$\begin{aligned} Q(\lambda, \lambda_0) &\geq \frac{r(r-1)}{2}(1 + \lambda + \lambda_0)^{r-2}(\lambda - \lambda_0)^2 \\ &\geq \frac{r(r-1)}{4}\frac{(\lambda - \lambda_0)^2}{(1 + \lambda + \lambda_0)^{2-r}}. \end{aligned}$$

1.3: $\lambda = 0$

Como

$$Q(0, \lambda_0) = (r-1)\lambda_0^r$$

e

$$\begin{aligned} \frac{r(r-1)}{4}\frac{\lambda_0^2}{(1 + \lambda_0)^{2-r}} &\leq \frac{r(r-1)}{2}\frac{\lambda_0^2}{(1 + \lambda_0)^{2-r}} \\ &\leq (r-1)\lambda_0^r \end{aligned}$$

segue-se que

$$Q(0, \lambda_0) \geq \frac{r(r-1)}{4}\frac{\lambda_0^2}{(1 + \lambda_0)^{2-r}}.$$

1.4: $\lambda < 0 < \lambda_0$

Por definição

$$Q(\lambda, \lambda_0) = g(\lambda) - g(\lambda_0) - g'(\lambda_0)(\lambda - \lambda_0)$$

e

$$Q(\lambda, 0) + Q(0, \lambda_0) = g(\lambda) - g(\lambda_0) + g'(\lambda_0)\lambda_0.$$

Agora observando que $0 < \lambda_0 < \lambda_0 - \lambda$ e $g'(\lambda_0) = r\lambda_0^{r-1} > 0$ obtemos

$$-g'(\lambda_0)(\lambda - \lambda_0) \geq g'(\lambda_0)\lambda_0$$

e portanto

$$Q(\lambda, \lambda_0) \geq Q(\lambda, 0) + Q(0, \lambda_0).$$

Mas

$$Q(\lambda, 0) \geq \frac{r(r-1)}{2} \frac{\lambda^2}{(1+|\lambda|)^{2-r}}$$

e

$$Q(0, \lambda_0) \geq \frac{r(r-1)}{2} \frac{\lambda_0^2}{(1+\lambda_0)^{2-r}},$$

donde

$$\begin{aligned} Q(\lambda, \lambda_0) &\geq \frac{r(r-1)}{2} \frac{\lambda^2}{(1+|\lambda|)^{2-r}} + \frac{r(r-1)}{2} \frac{\lambda_0^2}{(1+\lambda_0)^{2-r}} \\ &\geq \frac{r(r-1)}{2} \frac{\lambda^2+\lambda_0^2}{(1+|\lambda|+\lambda_0)^{2-r}} \\ &\geq \frac{r(r-1)}{4} \frac{(\lambda-\lambda_0)^2}{(1+|\lambda|+\lambda_0)^{2-r}}, \end{aligned}$$

pois $(\lambda - \lambda_0)^2 \leq 2(\lambda^2 + \lambda_0^2)$.

Caso 2: $\lambda_0 < 0$

Por um lado, o Caso 1 nos diz que

$$Q(-\lambda, -\lambda_0) \geq \frac{r(r-1)}{4} \frac{(\lambda-\lambda_0)^2}{(1+|\lambda|+|\lambda_0|)^{2-r}} \qquad (\lambda \in \mathbb{R})$$

uma vez que $-\lambda_0 > 0$. Por outro lado,

$$Q(-\lambda, -\lambda_0) = Q(\lambda, \lambda_0)$$

para todo par $(\lambda, \lambda_0) \in \mathbb{R}^2$. Logo,

$$Q(\lambda, \lambda_0) = Q(-\lambda, -\lambda_0) \geq \frac{r(r-1)}{4} \frac{(\lambda-\lambda_0)^2}{(1+|\lambda|+|\lambda_0|)^{2-r}} \qquad (\lambda \in \mathbb{R})$$

o que finaliza a demonstração. □

Apêndice B

Verificação de (1.9)

Afirmação. Conforme a Seção 2.2,

$$\partial_t \left\langle \nu_{(\cdot)}, |\lambda - \alpha| \right\rangle + \partial_x \left\langle \nu_{(\cdot)}, sgn(\lambda - \alpha)(f(\lambda) - f(\alpha)) \right\rangle \leq 0$$

no sentido distribucional para todo $\alpha \in \mathbb{R}$.

Demonstração. Considere uma função $\omega \in C_c^\infty(\mathbb{R})$ tal que $\omega \geq 0, \text{supp}(\omega) \subset B[0,1]$ e $\int_{\mathbb{R}} \omega(x)dx = 1$, e defina para cada $\delta > 0$ a função suavizante $\omega_\delta(x) = \delta^{-1}\omega(\delta^{-1}x)$. Fixado $\alpha \in \mathbb{R}$, definamos

$$j_\delta(x) = sgn * \omega_\delta(x),$$
$$\eta_\delta^\alpha(x) = \int_\alpha^x j_\delta(s - \alpha)ds$$

e

$$q_\delta^\alpha(x) = \int_\alpha^x f'(s)(\eta_\delta^\alpha)'(s)ds$$

onde sgn denota a função sinal dada por $sgn(x) = x/|x|$ se $x \neq 0$ e $sgn(0) = 0$.

Afirmamos que cada $(\eta_\delta^\alpha, q_\delta^\alpha)$ é uma par de entropia-fluxo de entropia com $\eta_\delta^\alpha \in C^\infty(\mathbb{R})$ convexa e $(\eta_\delta^\alpha)', (\eta_\delta^\alpha)''$ limitadas. De fato, $j_\delta \in C^\infty(\mathbb{R})$ (donde $\eta_\delta^\alpha \in C^\infty(\mathbb{R})$) e $j_\delta' = sgn * \omega_\delta' = 2\omega_\delta$. Agora como $(\eta_\delta^\alpha)''(x) = j_\delta'(x - \alpha) = 2\omega_\delta(x - \alpha)$ e w_δ é não-negativa, η_δ^α é convexa. Além disso, $(\eta_\delta^\alpha)'$ e $(\eta_\delta^\alpha)''$ são limitadas pois $|(\eta_\delta^\alpha)'| \leq 1$ e $|(\eta_\delta^\alpha)''| \leq C_\omega \delta^{-1}$, e claramente $(q_\delta^\alpha)' = f'(\eta_\delta^\alpha)'$. Assim, como estabelecido em (2.58) obtemos

$$\partial_t \left\langle \nu_{(\cdot)}, \eta_\delta^\alpha(\lambda) \right\rangle + \partial_x \left\langle \nu_{(\cdot)}, q_\delta^\alpha(\lambda) \right\rangle \leq 0 \quad (\delta > 0) \tag{B.1}$$

no sentido distribucional.

Observe também que $j_\delta \to sgn$ em $\mathbb{R}\backslash\{0\}, \eta_\delta^\alpha(\lambda) \to |\lambda - \alpha|$ e $q_\delta^\alpha(\lambda) \to sgn(\lambda - \alpha)(f(\lambda) - f(\alpha))$ para todo $\lambda \in \mathbb{R}$ quando $\delta \to 0$.

Dada $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ com $\phi \geq 0$, defina $h_\delta^\alpha(\lambda, t, x) = \eta_\delta^\alpha(\lambda)\partial_t\phi(t, x)$. Então $h_\delta^\alpha(\lambda, t, x) \to |\lambda - \alpha|\partial_t\phi(t, x)$ quando $\delta \to 0$ e $|h_\delta^\alpha(\lambda, t, x)| \leq |\lambda - \alpha||\partial_t\phi(t, x)|$. Logo, a representação (1.6) e o

Lema 2.4 implicam que

$$\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}|\lambda-\alpha||\partial_t\phi(t,x)|d\nu_{(t,x)}(\lambda)dtdx = \lim_{k\to\infty}\int_0^\infty\int_{\mathbb{R}}|u_{\epsilon_k,\beta_k}(t,x)-\alpha||\partial_t\phi(t,x)|dtdx$$
$$\leq C_0 + \lim_{k\to\infty}\int_0^\infty\int_{\mathbb{R}}|u_{\epsilon_k,\beta_k}(t,x)||\partial_t\phi(t,x)|dtdx$$
$$\leq C_0 + \lim_{k\to\infty}\int_0^\infty \|u_{\epsilon_k,\beta_k}(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_t\phi(t,\cdot)\|_{L^2(\mathbb{R})}dt$$
$$\leq C_0 + C_0\int_0^\infty \|\partial_t\phi(t,\cdot)\|_{L^2(\mathbb{R})}dt < \infty,$$

e portanto

$$\lim_{\delta\to 0}\int_0^\infty\int_{\mathbb{R}}\partial_t\left\langle\nu_{(t,x)},\eta_\delta^\alpha(\lambda)\right\rangle\phi(t,x)dtdx = -\lim_{\delta\to 0}\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}h_\delta^\alpha(\lambda,t,x)d\nu_{(t,x)}(\lambda)dtdx$$
$$= -\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}|\lambda-\alpha|\partial_t\phi(t,x)d\nu_{(t,x)}(\lambda)dtdx$$
$$= \int_0^\infty\int_{\mathbb{R}}\partial_t\left\langle\nu_{(t,x)},|\lambda-\alpha|\right\rangle\phi(t,x)dtdx$$

pelo Teorema da Convergência Dominada. Analogamente, se $g_\delta^\alpha(\lambda,t,x) = q_\delta^\alpha(\lambda)\partial_x\phi(t,x)$ então $g_\delta^\alpha(\lambda,t,x) \to sgn(\lambda-\alpha)(f(\lambda)-f(\alpha))\partial_x\phi(t,x)$ quando $\delta\to 0$, e utilizando (2.32) obtemos $|g_\delta^\alpha(\lambda,t,x)| \leq C_0(1+|\lambda|^{p+1})|\partial_x\phi(t,x)|$. Como $p+1<2$, a desigualdade de Holder (com expoentes $r = 2/(p+1)$ e $r' = 2/(1-p)$) juntamente com (1.6) e o Lema 2.4 implicam que

$$\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\lambda|^{p+1})|\partial_x\phi(t,x)|d\nu_{(t,x)}(\lambda)dtdx = C_0 + \lim_{k\to\infty}\int_0^\infty\int_{\mathbb{R}}|u_{\epsilon_k,\beta_k}(t,x)|^{p+1}|\partial_x\phi(t,x)|dtdx$$
$$\leq C_0 + \lim_{k\to\infty}\int_0^\infty \|u_{\epsilon_k,\beta_k}(t,\cdot)\|_{L^2(\mathbb{R})}^{p+1}\|\partial_x\phi(t,\cdot)\|_{L^{2/(1-p)}(\mathbb{R})}dt$$
$$\leq C_0 + C_0\int_0^\infty \|\partial_x\phi(t,\cdot)\|_{L^{2/(1-p)}(\mathbb{R})}dt < \infty,$$

e daí

$$\lim_{\delta\to 0}\int_0^\infty\int_{\mathbb{R}}\partial_x\left\langle\nu_{(t,x)},q_\delta^\alpha(\lambda)\right\rangle\phi(t,x)dtdx = -\lim_{\delta\to 0}\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}g_\delta^\alpha(\lambda,t,x)d\nu_{(t,x)}(\lambda)dtdx$$
$$= -\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}sgn(\lambda-\alpha)(f(\lambda)-f(\alpha))\partial_x\phi(t,x)d\nu_{(t,x)}(\lambda)dtdx$$
$$= \int_0^\infty\int_{\mathbb{R}}\partial_x\left\langle\nu_{(t,x)},sgn(\lambda-\alpha)(f(\lambda)-f(\alpha))\right\rangle\phi(t,x)dtdx$$

pelo Teorema da Convergência Dominada.

O resultado é agora obtido fazendo $\delta\to 0$ em (B.1). □

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