Universidade Federal de São Carlos
Centro de Ciências Exatas e Tecnologia
Programa de Pós-Graduação em Matemática

Positively Curved Killing Foliations via Deformations

Francisco Carlos Caramello Junior

São Carlos - SP
May 2018

Universidade Federal de São Carlos

Centro de Ciências Exatas e Tecnologia

Programa de Pós-Graduação em Matemática

Positively Curved Killing Foliations via Deformations

Francisco Carlos Caramello Junior

Supervisor: Prof. Dr. Dirk Töben
Co-supervisor: Prof. Dr. Luiz R. Hartmann Jr.

Dissertation submitted to PPGM/UFSCar as partial fulfillment of the requirements for the degree of Doctor of Science.

This version contains the corrections and modifications suggested by the doctoral committee during the dissertation defense on March 22, 2018.

This work was supported by the Brazilian Federal Agency for Support and Evaluation of Graduate Education within the Brazilian Ministry of Education.

São Carlos - SP

May 2018

Caramello Jr., Francisco Carlos

Positively curved Killing foliations via deformations / Francisco Carlos Caramello Junior. -- 2018.
105 f. : 30 cm.

Tese (doutorado) - Universidade Federal de São Carlos, campus São Carlos, São Carlos.
Orientador: Dirk Töben. Coorientador: Luiz Roberto Hartmann Junior.
Banca examinadora: Dirk Töben, Luiz Roberto Hartmann Junior, Marcos Martins Alexandrino da Silva, Llohann Dallagnol Sperança, Fernando Manfio.
Inclui referências

1. Geometria Diferencial. 2. Folheações Riemannianas. 3. Deformações. I. Orientador. II. Universidade Federal de São Carlos. III. Título.

UNIVERSIDADE FEDERAL DE SÃO CARLOS

Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Matemática

Folha de Aprovação

Assinaturas dos membros da comissão examinadora que avaliou e aprovou a Defesa de Tese de Doutorado do candidato Francisco Carlos Caramello Junior, realizada em 22/03/2018:

Prof. Dr. Dirk Toben
UFSCar

Prof. Dr. Luiz Roberto Hartmann Junior
UFSCar

Prof. Dr. Marcos Martins Alexandrino da Silva
USP

Prof. Dr. Llohann Dallagnol Sperança
UNIFESP

Prof. Dr. Fernando Manfio
USP

Acknowledgements

I owe my deepest gratitude to my supervisor, Prof. Dirk Töben, for the crucial ideas, the many insightful suggestions and the essential contributions he gave to this work, for his patience, encouragement and continuous support towards my formation and for the friendship we ended up developing through this process. I am also thankful to my co-supervisor, Prof. Luiz Hartmann, for sharing expertise and valuable career advices and for the suggestions and constructive criticism that significantly improved the first drafts of this dissertation. I thank Prof. Marcos Alexandrino as well, for his genuine solicitude and friendly hospitality during my occasional visits to IME.

I am grateful to my fiancée, Thais, for keeping me such a sweet company through both the brightest and the darkest hours of these years, for the great adventures that we shared and are yet to share together and, above all, for the caring love that we sustain toward each other. I express here as well my profound gratitude to my mother, Adriana, for the unconditional love, the unceasing assistance and the moral guidance she always provided me. I also thank my other family members, for the support they gave me along the way, and my fellow doctoral students, for the cooperation, the stimulating discussions and, of course, all the fun we have had in these four years.

This work had the indispensable financial support of CAPES, for which I am genuinely grateful.

"Though this be madness, yet there is method in't."

Hamlet, Act 2, Scene 2.

Resumo

Mostramos que uma variedade admitindo uma folheação de Killing com curvatura seccional transversa positiva e defeito máximo se fibra sobre quocientes finitos de esferas ou espaços projetivos complexos com pesos. Este resultado é obtido deformando-se a folheação em uma folheação fechada enquanto preservamos propriedades geométricas transversas, o que nos permite aplicar resultados da geometria Riemanniana de orbifolds ao espaço das folhas. Mostramos também que a característica de Euler básica é preservada por tais deformações, o que nos provê algumas obstruções topológicas para folheações Riemannianas.

Abstract

We show that a manifold admitting a Killing foliation with positive transverse curvature and maximal defect fibers over finite quotients of spheres or weighted complex projective spaces. This result is obtained by deforming the foliation into a closed one, while maintaining transverse geometric properties, which allows us to apply results from the Riemannian geometry of orbifolds to the space of leaves. We also show that the basic Euler characteristic is preserved by such deformations, which provides us some topological obstructions for Riemannian foliations.

List of Figures

1.1 A foliation is locally defined by submersions . . . 17

1.2 λ-Kronecker foliations . . . 18

1.3 The flow of a foliate field preserves the foliation . . . 19

1.4 Sliding along the leaves . . . 21

1.5 The 1-dimensional foliations of $\mathbb{S}^3$ (via stereographic projection) . . . 25

1.6 The Molino construction . . . 30

1.7 The orbits of the Molino sheaf are the closures of the leaves . . . 32

2.1 The space W . . . 36

2.2 Homothetic transformation with respect to N . . . 41

3.1 A homothopic deformation . . . 47

3.2 Lift of γ to the Darboux cover . . . 48

5.1 Construction of the vector field W . . . 66

5.2 The frame E . . . 67

6.1 The holonomy groupoid . . . 80

A.1 The $\mathbb{Z}_3$-$\mathbb{Z}_4$-football . . . 88

A.2 A smooth local lift . . . 90

A.3 Visualizing the stratification of $\mathbb{CP}^3[\lambda]$ as a 3-simplex . . . 93

A.4 Tangent space and tangent cone at x . . . 96

Contents

Introduction **10**

1 Preliminaries **15**

1.1 Foliations 15

1.2 Holonomy 20

1.3 Foliations of Orbifolds 22

1.4 Basic Cohomology 22

1.5 Riemannian Foliations 24

1.6 Taut Foliations and Harmonic Forms 26

1.7 Molino Theory 28

1.8 Molino Sheaf and Killing Foliations 31

2 Transverse Killing Vector Fields and the Structural Algebra **34**

2.1 Elementary Properties of Transverse Killing Vector Fields 34

2.2 The Canonical Stratification 42

2.3 The Structural Algebra and Finite Coverings 43

3 Deformations of Killing Foliations **46**

3.1 A Suitable Deformation 50

4 Direct Applications of the Deformation Technique **55**

4.1 A Closed Leaf Theorem 55

4.2 Positive Transverse Curvature and the Defect 57

4.3 Large Defect and Isolated Closed Leaves 59

5 Transverse Symmetries and the Basic Euler Characteristic **61**
5.1 Localization of the Basic Euler Characteristic 63
5.2 The Basic Euler Characteristic Under Deformations 69
5.3 Foliations Admitting a Killing Vector Field with Large Zero Set . . . 71
5.4 Symmetry Rank and the Basic Euler Characteristic 75

6 A Topological Obstruction **79**
6.1 Classifying Spaces of Holonomy Groupoids 79
6.2 A Topological Obstruction for Riemannian Foliations 82

A Orbifolds **85**
A.1 Definition and Examples 85
A.2 Smooth Maps 89
A.3 Quotient Orbifolds 90
A.4 Associated Pseudogroups 93
A.5 Orbibundles 95
A.6 Integration and de Rham Cohomology 97
A.7 Actions on Orbifolds 98
A.8 Riemannian Orbifolds 99
A.9 Orbifold Homotopy and (Co)Homology 102

Bibliography **104**

Index **111**

Introduction

The theory of foliations was initiated by H. Poincaré [Poi92] and I. Bendixson [Ben01] at the turn of the twentieth century, who studied the behavior of the solutions of first order ordinary differential equations on the plane. This theory was later developed and generalized to higher-dimensional systems by C. Ehresmann, G. Reeb, W. Thurston and many others, and nowadays accounts, via the Frobenius Theorem, for the qualitative behavior of first order partial differential equations on manifolds. Informally, a foliation is Riemannian if its leaves are locally equidistant. The precise definition involves the extra structure of a Riemannian metric that allows us to locally define the foliation as fibers of a Riemannian submersion. These objects, first presented by B. Reinhart [Rei59], are therefore approachable by the techniques of Riemannian geometry and form a very relevant class of foliations, whose research has been quite active since their introduction [see Ton97, Appendix D].

The structural theory for Riemannian foliations, due mainly to P. Molino, asserts that a complete Riemannian foliation $\mathcal{F}$ admits a locally constant sheaf of Lie algebras of germs of local transverse Killing vector fields $\mathscr{C}_{\mathcal{F}}$ whose action describes the dynamics of $\mathcal{F}$, in the sense that, for each leaf $L_x \in \mathcal{F}$,

$$T_x\overline{L}_x = \{X_x \mid X \in (\mathscr{C}_{\mathcal{F}})_x\} \oplus T_xL_x,$$

where $\overline{L}_x$ denotes the closure of L_x (see Sections 1.7 and 1.8). From this it follows that the partition $\overline{\mathcal{F}} := \{\overline{L} \mid L \in \mathcal{F}\}$ of M is a singular foliation, in the sense that the dimension of the leaf closures may vary.

In this work we are primarily interested in the so-called Killing foliations, that is, complete Riemannian foliations that have a globally constant sheaf $\mathscr{C}$. In other words, if $\mathcal{F}$ is a Killing foliation then there exists transverse Killing vector fields $X_1, \dots, X_d$ such that $T\overline{\mathcal{F}} = T\mathcal{F} \oplus \langle X_1, \dots, X_d \rangle$. This class of foliations includes Riemannian foliations on simply-connected manifolds and foliations induced by isometric actions on compact manifolds.

In Chapter 1 we summarize the prerequisites from foliation theory used in this work. Chapter 2 contains some basic results and technical lemmas that are mostly direct generalizations of known results. In Chapter 3 we present the main tool for our study of Killing foliations: a deformation method that preserves transverse geometric properties of the foliation. Chapter 4 is dedicated to direct applications of this deformation technique. In Chapter 5 we study the behavior of the basic Euler characteristic, an algebraic invariant of a foliation, under such deformations. Finally, in Chapter 6 we use the deformation technique to obtain an obstruction for Riemannian foliations on manifolds with finite fundamental group and non-zero Euler characteristic. Appendix A surveys the basics of the differential geometry of orbifolds that we use.

Let us now introduce in more detail the main results obtained in this work. In the 1990s, K. Grove proposed that a classification of Riemannian manifolds with positive sectional curvature and large isometry group should be pursued. Together with C. Searle, they introduced the symmetry rank as a way to interpret what it is meant by a "large" isometry group and showed in [GS94] that a closed Riemannian manifold with positive sectional curvature and maximal symmetry rank is diffeomorphic to either a sphere, a real or complex projective space or a lens space. A generalization of this result for Alexandrov spaces was obtained recently in [HS17]. We show the following transverse analogue of the Grove–Searle result.

Theorem A. *Let $\mathcal{F}$ be a q-codimensional, transversely orientable Killing foliation of a compact manifold M. If the transverse sectional curvature $\sec_{\mathcal{F}}$ of $\mathcal{F}$ is positive, then*

$$\operatorname{codim}(\overline{F}) \geq \left\lceil \frac{\operatorname{codim}(\mathcal{F}) - 1}{2} \right\rceil$$

and, if equality holds, there is a homotopic deformation of $\mathcal{F}$ into a closed Rieman-

nian foliation $\mathcal{G}$ such that $M/\mathcal{G}$ is homeomorphic to either

(i) a quotient of a sphere $\mathbb{S}^q/\Lambda$, where Λ is a finite subgroup of the centralizer of the maximal torus in $\mathrm{O}(q+1)$, or

(ii) a quotient of a weighted complex projective space $|\mathbb{CP}^{q/2}[\lambda]|/\Lambda$, where Λ is a finite subgroup of the torus acting linearly on $\mathbb{CP}^{q/2}[\lambda]$ (this case occurs only when q is even).

This is Theorem 4.4 in the text. We obtain the closed foliation $\mathcal{G}$ by the already mentioned deformation technique that preserves some properties of the transverse geometry of $\mathcal{F}$, given by the result below.

Theorem B. *Let $(\mathcal{F}, \mathrm{g}^T)$ be a Killing foliation of a closed manifold M satisfying $\sec_{\mathcal{F}} > c$. Then there is a homotopic deformation of $\mathcal{F}$ into a closed Riemannian foliation $\mathcal{G}$ which can be chosen arbitrarily close to $\mathcal{F}$ and satisfying $\sec_{\mathcal{G}} > c$. Moreover, the deformation occurs within the closures of the leaves of $\mathcal{F}$ and $M//\mathcal{G}$ admits an effective isometric action of a torus $\mathbb{T}^d$, where $d = \dim(\mathscr{C}_{\mathcal{F}}(M))$ is the defect of $\mathcal{F}$, such that $M/\overline{\mathcal{F}} \cong (M/\mathcal{G})/\mathbb{T}^d$. In particular, $\mathrm{symrank}(\mathcal{G}) \geq d$.*

This result appears in the text as Theorem 3.5. The proof is based on a theorem by A. Haefliger and E. Salem [HS90] that expresses $\mathcal{F}$ as the pullback of a homogeneous foliation on an orbifold. Theorem B enables us to apply results from the theory of Riemannian orbifolds to the study of Killing foliations, by passing from $\mathcal{F}$ to $M//\mathcal{G}$. Besides Theorem A, another application of this technique is the following "closed leaf" result, that generalizes [Osh01, Theorem 2] by allowing non-compact ambient manifolds. It is obtained by reducing the problem to an application of the Synge–Weinstein Theorem for orbifolds (Theorem A.14).

Theorem C. *Let $\mathcal{F}$ be an even-codimensional complete Riemannian foliation of a manifold M satisfying $|\pi_1(M)| < \infty$. If $\sec_{\mathcal{F}} \geq c > 0$ then $\mathcal{F}$ possesses a closed leaf.*

A foliation $\mathcal{F}$ on M defines a subcomplex of the De Rham complex of M that consists of those forms that can be projected into the local quotients of $\mathcal{F}$. The cohomology of this subcomplex, the basic cohomology of $\mathcal{F}$, therefore generalizes the usual De Rham cohomology (see Section 1.4). In particular, consider $\chi_B(\mathcal{F})$ the basic Euler characteristic of $\mathcal{F}$, the alternate sum of the dimensions of the basic cohomology groups. We show that for a Killing foliation $\mathcal{F}$ of a compact manifold this invariant localizes to the stratum $\Sigma^{\dim(\mathcal{F})}$ that consists of the closed leaves of $\mathcal{F}$.

Theorem D. *If $\mathcal{F}$ is a Killing foliation of a closed manifold M, then $\chi_B(\mathcal{F}) = \chi(\Sigma^{\dim(\mathcal{F})}/\mathcal{F})$.*

Equivalently, using the language of transverse actions (see Section 1.8), we can write $\mathcal{F}|_{\Sigma^{\dim(\mathcal{F})}} = \mathcal{F}^{\mathfrak{g}_{\mathcal{F}}}$, where $\mathcal{F}^{\mathfrak{g}_{\mathcal{F}}}$ denotes the fixed-point set of the transverse action of the structural algebra $\mathfrak{g}_{\mathcal{F}}$ of $\mathcal{F}$, so the formula in Theorem D becomes $\chi_B(\mathcal{F}) = \chi_B(\mathcal{F}^{\mathfrak{g}_{\mathcal{F}}})$, in analogy with the localization of the classical Euler characteristic to the fixed-point set a torus action. Theorem D is a special case of a stronger result: we prove that

$$\chi_B(\mathcal{F}) = \chi_B(\mathcal{F}|_{\mathrm{Zero}(\overline{X})})$$

for any transverse Killing vector field $\overline{X} \in \mathfrak{iso}(\mathcal{F})$ (see Theorem 5.7).

Theorem D enables us to show that the basic Euler characteristic is preserved by the deformations devised in Theorem 3.5 (see Theorem 5.11). Using this we obtain the following transverse analogue of the partial answer to Hopf's conjecture by T. Püttmann and C. Searle for manifolds with large symmetry rank [see PS01, Theorem 2], by reducing it to an orbifold version of this same result that we also prove (see Corollary 5.16).

Theorem E. *Let $\mathcal{F}$ be an even-codimensional, transversely orientable Killing foliation of a closed manifold M. If $\sec_{\mathcal{F}} > 0$ and*

$$\mathrm{codim}(\overline{\mathcal{F}}) \leq \frac{3\,\mathrm{codim}(\mathcal{F})}{4} + 1,$$

then $\chi_B(\mathcal{F}) > 0$.

This is Theorem 5.17. Finally, we combine the results of A. Haefliger on classifying spaces of holonomy groupoids [Hae86] with the invariance of χ_B by deformations to obtain the following topological obstruction for Riemannian foliations.

Theorem F. *Let M be a closed manifold satisfying $|\pi_1(M)| < \infty$ and $\chi(M) \neq 0$. Then any Riemannian foliation of M is closed.*

This appears in the text as Corollary 6.5. It is shown in [Ghy84, Théorème 3.5] that a Riemannian foliation of a closed simply-connected manifold M satisfying $\chi(M) \neq 0$ must have a closed leaf, a fact that also follows from the Poincaré-Hopf index theorem. Notice that Theorem F shows, in fact, that all leaves of such a foliation must be closed.

Chapter 1

Preliminaries

For the reader's convenience, in this chapter we collect some of the prerequisites on Riemannian foliations that will be used in this work. We assume the reader is familiar with the principal elements of the theory of smooth[1] manifolds, as in [Lee13] or [War83] for example, Riemannian geometry, references being [Lee97], [Pet06] or [Sak97], and Lie group (actions) theory, as in [AB15], [Bre72], [DK00], and [War83]. We also assume some background in algebraic topology, for which we refer to [Hat02] and [Spa81], and sheaf theory language, that can be found in [Bre97]. The readers acquainted with the theory of Riemannian foliations, as presented in [MM03], [Mol88] or [Ton97], may skip this chapter probably without compromising the reading.

The brief exposition we do here follows mainly the references [Mol88], [MM03], [Ton97] and [Asa+14]. Further references are [CC00], [GW09] and [CN85].

1.1 Foliations

Let M be a smooth n-dimensional manifold. A **foliation atlas** of dimension $p \in \mathbb{N}$ for M is a smooth atlas $\{\phi_i : U_i \to \mathbb{R}^p \times \mathbb{R}^{n-p}\}_{i\in I}$ for M for which the changes of charts are locally of the form $\phi_{ij}(x, y) = (\phi^1_{ij}(x, y), \phi^2_{ij}(y))$, with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$. Each U_i is partitioned into **plaques**, which are the

[1]For us, "smooth" will always mean "of differentiability class C^∞". Also, we will often omit the word "smooth" when it is clear from the context that differentiability is needed.

connected components of the sets $\phi_i^{-1}(\mathbb{R}^p \times \{y\})$. These plaques glue together to form immersed connected submanifolds of M that we call **leaves**. The partition $\mathcal{F}$ of M into leaves is a (regular) **foliation** of M of dimension p. We call $q = n - p$ the **codimension** of $\mathcal{F}$ and, for $x \in M$, we denote the leaf containing x by L_x.

It is not difficult to prove [see MM03, Section 1.2] that the following objects can be used as alternative definitions for $\mathcal{F}$.

- A smooth involutive distribution Δ of rank p of TM. In this case $\Delta_x = T_x L_x$.
- An integrable subbundle $T\mathcal{F} \subset TM$ of rank p, called the **tangent bundle** of $\mathcal{F}$. Here, of course, we have $T_x\mathcal{F} = T_x L_x$.
- A subsheaf of Lie algebras $\mathfrak{X}_{\mathcal{F}} \subset \mathfrak{X}_M$ with constant span dimension p. In this case, for $U \subset M$ open, $\mathfrak{X}_{\mathcal{F}}(U)$ is the space of vector fields in U tangent to the leaves. We denote $\mathfrak{X}_{\mathcal{F}}(M)$ simply by $\mathfrak{X}(\mathcal{F})$.
- A locally trivial differential graded ideal $\mathfrak{I} \subset \Omega^*(M)$ of rank q. Here, if $U \subset M$ is an open subset on which $\mathfrak{I}$ is trivial, then $\mathfrak{I}|_U$ is generated by q linearly independent 1-forms $\alpha^1, \dots, \alpha^q$ and
$$T_x L_x = \bigcap_{i=1}^{q} \ker(\alpha_x^i).$$
- An open cover $\{U_i\}_{i \in I}$ of M, submersions $\pi_i : U_i \to \overline{U}_i$, with $\overline{U}_i \subset \mathbb{R}^q$ open, and diffeomorphisms $\gamma_{ij} : \pi_j(U_i \cap U_j) \to \pi_i(U_i \cap U_j)$ satisfying $\gamma_{ij} \circ \pi_j|_{U_i \cap U_j} = \pi_i|_{U_i \cap U_j}$ for all $i, j \in I$. The collection $\{\gamma_{ij}\}$ is a **Haefliger cocycle** representing $\mathcal{F}$ and each U_i is a **simple open set** for $\mathcal{F}$ (see Figure 1.1). We will assume without loss of generality that the fibers $\pi_i^{-1}(\overline{x})$ are connected.

Example 1.1. Let $(M, \mathcal{F})$ be a foliation and $f : N \to M$ a smooth map transverse to $\mathcal{F}$, that is, f is transverse to each leaf. Then f defines a foliation $f^*(\mathcal{F})$ on N as follows. If $(U_i, \pi_i, \gamma_{ij})$ is a cocycle representing $\mathcal{F}$, then $f^*(\mathcal{F})$ is given by the cocycle $(V_i, \pi_i', \gamma_{ij})$, where $V_i = f^{-1}(U_i)$ and $\pi_i' = \pi_i \circ f|_{V_i}$. Observe that $Tf^*(\mathcal{F}) = \mathrm{d}f^{-1}(T\mathcal{F})$ and that $\operatorname{codim}(f^*(\mathcal{F})) = \operatorname{codim}(\mathcal{F})$.

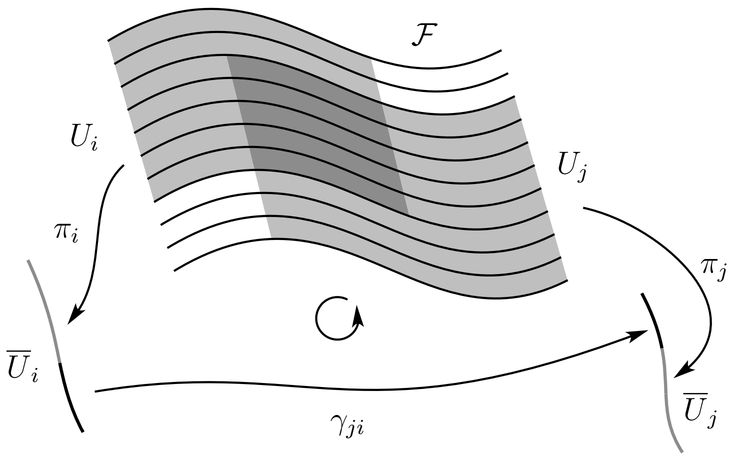


Figure 1.1: A foliation is locally defined by submersions

Example 1.2. Lie group actions constitute a main source of foliations. Precisely, recall that when $\mu : G \times M \to M$ is a smooth action, each orbit Gx is the image of an injective immersion $G/G_x \to M$ [see, for instance, AB15, Proposition 3.14]. Thus, if we suppose that $\dim(G_x)$ is a constant function of x, it follows that the connected components of orbits of G decompose M into immersed submanifolds of constant dimension. This decomposition $\mathcal{F}$ is easily seen to be a foliation, because $T_x(Gx) = \mathrm{d}(\mu_x)_e(\mathfrak{g})$, so the fields $V^* \in \mathfrak{X}(M)$, $V \in \mathfrak{g}$, induced by the action generate $T\mathcal{F}$, showing that this is an involutive distribution.

A specific example is the following. Consider the flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For each $\lambda \in (0, +\infty)$, we have a smooth $\mathbb{R}$-action

$$\begin{array}{rcl} \mathbb{R} \times \mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \\ (t, [x, y]) & \longmapsto & [x + t, y + \lambda t] \end{array}$$

with $\dim(\mathbb{R}_{[x,y]}) \equiv 0$. The resulting foliation is the **λ-Kronecker foliation** of the torus, $\mathcal{F}(\lambda)$. Observe that when λ is irrational each leaf is dense in $\mathbb{T}^2$, while a rational λ yields closed leaves (see Figure 1.2).

When a foliation $\mathcal{F}$ is given by the action of a Lie group we say that $\mathcal{F}$ is **homogeneous**.

Example 1.3. Another class of examples of foliations comes from suspensions of homomorphisms, a useful construction originally due to A. Haefliger [Hae62]. Let

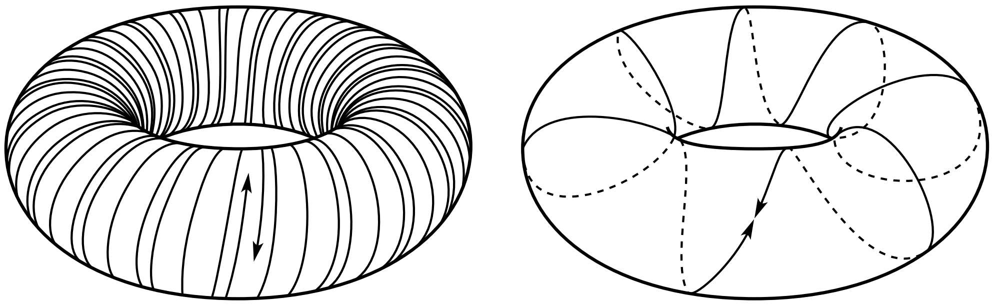


Figure 1.2: λ-Kronecker foliations

B and T be smooth manifolds, let $h : \pi_1(B, x_0) \to \mathrm{Diff}(T)$ be a homomorphism and denote by $\rho : \widehat{B} \to B$ the projection of the universal covering space of B. On $\widetilde{M} := T \times \widehat{B}$, the fibers of the first projection $\widetilde{M} \to T$ determine a foliation $\widetilde{\mathcal{F}}$. Define an action of $\pi_1(B, x_0)$ on $\widetilde{M}$ by setting, for $[\gamma] \in \pi_1(B, x_0)$,

$$[\gamma]\cdot(t, \hat{b}) = \Big(h\left([\gamma]^{-1}\right)(t), \hat{b}\cdot[\gamma]\Big),$$

where $\hat{b}\cdot[\gamma]$ denotes the image of $\hat{b}$ by the deck transformation associated to $[\gamma]$. There is a manifold structure on $M = \widetilde{M}/\pi_1(B, x_0)$ [see Mol88, p. 28] such that the orbit projection $\pi : \widetilde{M} \to M$ is a covering map and, if $\tau : M \to B$ is given by $\tau(\pi(t, \tilde{b})) = \rho(\hat{b})$, then it is the projection of a fiber bundle with total space M, base B, fiber T and structural group $h(\pi_1(B, x_0))$. The action of $\pi_1(B, x_0)$ preserves the leaves of $\widetilde{\mathcal{F}}$, so projecting through π we obtain a foliation $\mathcal{F}$ on M with $\mathrm{codim}(\mathcal{F}) = \dim(T)$, constructed by **suspension of the homomorphism** h.

For example, the Kronecker foliation $\mathcal{F}(\lambda)$ (see Example 1.2) can be obtained by the suspension of the homomorphism $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z} \to \mathrm{Diff}(\mathbb{S}^1)$ given by $k \mapsto e^{-2\pi \mathrm{i} \lambda k}$.

As the Kronecker foliation shows, a leaf L of a foliation $\mathcal{F}$ need not to be closed as a subspace of the ambient manifold M. We denote the set of leaf closures by $\overline{\mathcal{F}} := \{\overline{L} \mid L \in \mathcal{F}\}$. Understanding $\overline{\mathcal{F}}$ is part of the study of the dynamics of the foliation. In the simple case when $\overline{\mathcal{F}} = \mathcal{F}$, that is, when all the leaves of $\mathcal{F}$ are closed, we say that $\mathcal{F}$ is a **closed** foliation.

A foliation $(M, \mathcal{F})$ is **tangentially orientable** if $T\mathcal{F}$ is orientable, and **transversely orientable** if its **normal bundle** $\nu\mathcal{F} := TM/T\mathcal{F}$ is orientable. In this

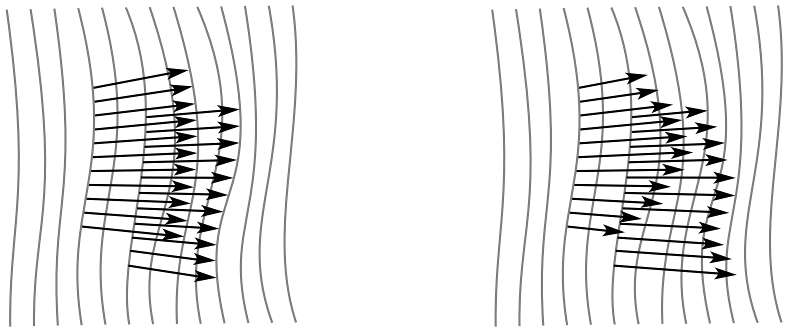


Figure 1.3: The flow of a foliate field preserves the foliation

case, choices of orientations for $T\mathcal{F}$ and $\nu\mathcal{F}$ give, respectively, an **tangential orientation** and a **transverse orientation** for $\mathcal{F}$. It is always possible to choose an orientable finite covering space $\widehat{M}$ of M such that the lifted foliation $\widehat{\mathcal{F}}$ is transversely (and hence also tangentially) orientable [see CC00, Proposition 3.5.1]. In terms of a Haefliger cocycle, $\mathcal{F}$ is transversely oriented if and only if there is a cocycle $\{(U_i, \pi_i, \gamma_{ij})\}$ representing $\mathcal{F}$ that satisfies $\det(\mathrm{d}\gamma_{ij}) > 0$ as a function on $\pi_j(U_i \cap U_j)$, for all $i, j \in I$.

Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be foliations. A **foliate morphism** between $(M, \mathcal{F})$ and $(N, \mathcal{G})$ is a map $f : M \to N$ that sends leaves of $\mathcal{F}$ into leaves of $\mathcal{G}$. When there is such f, the foliations $\mathcal{F}$ and $\mathcal{G}$ are often said to be **congruent**. In particular, we may consider $\mathcal{F}$-foliate diffeomorphisms $f : M \to M$. The infinitesimal counterparts of this notion are the **foliate vector fields** of $\mathcal{F}$, that is, the vector fields in the subalgebra

$$\mathfrak{L}(\mathcal{F}) = \{X \in \mathfrak{X}(M) \mid [X, \mathfrak{X}(\mathcal{F})] \subset \mathfrak{X}(\mathcal{F})\}.$$

If $X \in \mathfrak{L}(\mathcal{F})$ and $\pi : U \to \overline{U}$ is a submersion locally defining $\mathcal{F}$, then $X|_U$ is π-related to some vector field $\overline{X}_{\overline{U}} \in \mathfrak{X}(\overline{U})$. In fact, this characterizes $\mathfrak{L}(\mathcal{F})$ [see Mol88, Section 2.2]. Another characterization is that the local flows of the fields in $\mathfrak{L}(\mathcal{F})$ send leaves to leaves (see Figure 1.3).

The Lie algebra $\mathfrak{L}(\mathcal{F})$ also has the structure of a module, whose coefficient ring consists of the **basic functions** of $\mathcal{F}$, that is, functions $f \in C^\infty(M)$ such that $Xf = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$. We denote this ring by $\Omega^0_B(\mathcal{F})$. A smooth function is basic if and only if it is constant on each leaf and also if and only if it factors through each submersion $\pi : U \to \overline{U}$ locally defining $\mathcal{F}$ to a smooth function on the quotient $\overline{U}$ [see Mol88, Section 2.1].

The quotient of $\mathfrak{L}(\mathcal{F})$ by the ideal $\mathfrak{X}(\mathcal{F})$ yields the Lie algebra $\mathfrak{l}(\mathcal{F})$ of **transverse vector fields**. For $X \in \mathfrak{L}(\mathcal{F})$, we denote its induced transverse field by $\overline{X} \in \mathfrak{l}(\mathcal{F})$. Note that each $\overline{X}$ defines a unique section of $\nu\mathcal{F}$ and that $\mathfrak{l}(\mathcal{F})$ is also a $\Omega^0_B(\mathcal{F})$-module.

1.2 Holonomy

Let $(M, \mathcal{F})$ be a foliation represented by the cocycle $\{(U_i, \pi_i, \gamma_{ij})\}$. The pseudogroup of local diffeomorphisms (see Section A.4) generated by $\gamma = \{\gamma_{ij}\}$ acting on

$$T_\gamma := \bigsqcup_i \overline{U}_i$$

is the **holonomy pseudogroup** of $\mathcal{F}$ associated to γ, that we denote by $\mathscr{H}_\gamma$. If δ is another Haefliger cocycle defining $\mathcal{F}$ then $\mathscr{H}_\delta$ is equivalent to $\mathscr{H}_\gamma$, so we can define, up to equivalence, the holonomy pseudogroup of $\mathcal{F}$. We will write $(T_\mathcal{F}, \mathscr{H}_\mathcal{F})$ to denote both this equivalence class and a specific representative in it, for it seldom leads to confusion. It is clear that $T_\mathcal{F}/\mathscr{H}_\mathcal{F}$ is precisely the $M/\mathcal{F}$ of $\mathcal{F}$ endowed with the quotient topology.

Example 1.4. If $(M, \mathcal{F})$ is given by the suspension of $h : \pi_1(B, x_0) \to \mathrm{Diff}(T)$ (see Example 1.3) we can choose a cocycle $\{(U_i, \pi_i, \gamma_{ij})\}$ representing $\mathcal{F}$ where each U_i is the domain of a trivialization of $\tau : M \to B$ and $\pi_i : U_i \to T$ is the trivial projection. Then $\mathscr{H}_\mathcal{F}$ is just the pseudogroup generated by $h(\pi_1(B, x_0)) < \mathrm{Diff}(T)$, encoding the recurrence of the leaves on T.

If $L := L_x = L_y$, choose a path $c : [0, 1] \to L$ joining x to y. Fix a cocycle $\{(U_i, \pi_i, \gamma_{ij})\}$ representing $\mathcal{F}$ and a subdivision $0 = t_1 < \cdots < t_{m+1} = 1$ such that $s([t_k, t_{k+1}]) \subset U_{i_k}$ for some U_{i_k}. Then, there is a diffeomorphism

$$\gamma_{i_m i_{(m-1)}} \circ \gamma_{i_{(m-1)} i_{(m-2)}} \circ \cdots \circ \gamma_{i_2 i_1} = \gamma_{i_m i_1}$$

between small enough neighborhoods of $\overline{x} = \pi_1(x)$ and $\overline{y} = \pi_m(y)$. If we identify T_γ

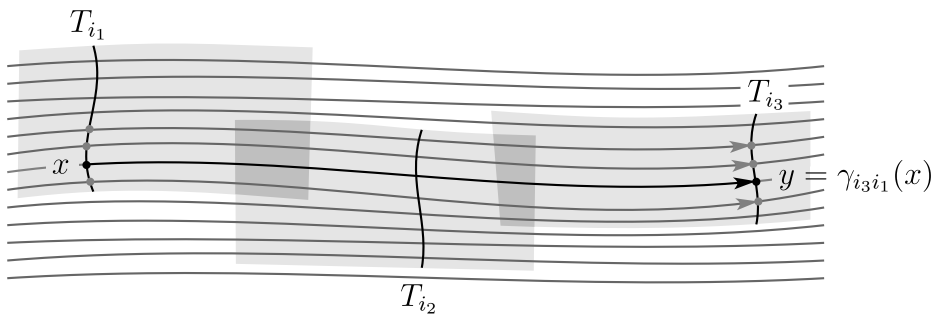


Figure 1.4: Sliding along the leaves

with a total transversal $\bigsqcup_i T_i$ for $\mathcal{F}$ containing x and y, this becomes the "sliding along the leaves" notion [Mol88, Section 1.7] (see Figure 1.4). Let us denote the germ of $\gamma_{i_m i_1}$ at $\overline{x}$ by h_c. This germ actually depends only on the $\partial[0,1]$-relative homotopy class of c [see CC00, Proposition 2.3.2], hence, if we consider in particular the **holonomy group** of L at x, that is, the group

$$\mathrm{Hol}_x(L) = \{h_c \mid c : [0,1] \to L \ \text{ is a loop}\},$$

we have a surjective homomorphism $h : \pi_1(L, x) \to \mathrm{Hol}_x(L)$.

As the isomorphism class of $\mathrm{Hol}_x(L)$ does not depend on x, we often omit x in this notation. In particular, we can say that L is a **leaf without holonomy** (or a **generic leaf**) when $\mathrm{Hol}(L) = 0$. It follows immediately from the surjectivity of h that simply-connected leaves are without holonomy. Also, it can be shown that leaves without holonomy are generic, in the sense that $\{x \in M \mid \mathrm{Hol}_x(L) = 0\}$ is residual in M [see CC00, Theorem 2.3.12].

Leaf holonomy plays the same role of the stabilizer in the case of group actions. In particular, there is the following analogue to Proposition A.4 [see, for example, MM03, Theorem 2.15].

Proposition 1.5. *Let $\mathcal{F}$ be a q-codimensional foliation of M whose every leaf is compact and with finite holonomy. Then $M/\mathcal{F}$ has a canonical q-dimensional orbifold structure. Relative to this structure, the local group of a leaf in $M/\mathcal{F}$ is its holonomy group.*

We will denote the orbifold obtained this way by $M/\!/\mathcal{F}$ in order to distinguish it from the topological space $M/\mathcal{F}$, similarly to our notation for quotient orbifolds (see Section A.3).

Note that if $\mathrm{Hol}_x(L)$ is finite it is possible to identify it with a subgroup of $\mathrm{Diff}(T)$, where T is a small local transversal of $\mathcal{F}$ passing through x. With this in mind we can state the famous Reeb Stability Theorem as follows [see MM03, Theorem 2.9; CC00, Theorem 2.4.3][2].

Theorem 1.6 (Generalized local Reeb stability)**.** *Let $\mathcal{F}$ be a smooth foliation with a compact leaf L_x. If $\mathrm{Hol}_x(L)$ is finite then there is a saturated tubular neighborhood $\mathrm{pr} : \mathrm{Tub}(L_x) \to L_x$ restricted to which $\mathcal{F}$ is congruent to the foliation given by the suspension of $h : \pi_1(L, x) \to \mathrm{Hol}_x(L) < \mathrm{Diff}(T)$, where $T = \mathrm{pr}^{-1}(x)$.*

In particular, for every $y \in \mathrm{Tub}(L)$ the projection $\mathrm{pr} : L_y \to L_x$ is a finitely-sheeted covering map, the number of sheets being the index $|\mathrm{Hol}_x(L_x) : \mathrm{Hol}_y(L_y)|$.

1.3 Foliations of Orbifolds

Let $\mathcal{O}$ be an orbifold with atlas $\mathcal{A} = \{(\widetilde{U}_i, H_i, \phi_i)\}$ and associated pseudogroup $(U_{\mathcal{A}}, \mathscr{H}_{\mathcal{A}})$ (see Appendix A). Following [HS90, Section 3.2], we define a smooth foliation $\mathcal{F}$ of $\mathcal{O}$ as a smooth foliation of $U_{\mathcal{A}}$ which is invariant by $\mathscr{H}_{\mathcal{A}}$. The atlas can be chosen so that on each $\widetilde{U}_i$ the foliation is given by a surjective submersion with connected fibers onto a manifold T_i. The holonomy pseudogroup of $\mathcal{F}$, therefore, will be generated by the local diffeomorphisms of the disjoint union $\bigsqcup_{i\in I} T_i$ that are projections of elements of $\mathscr{H}_{\mathcal{A}}$.

1.4 Basic Cohomology

Let $(M, \mathcal{F})$ be a smooth foliation. A covariant tensor field ξ on M is **$\mathcal{F}$-basic** if $\xi(X_1, \dots, X_k) = 0$, whenever some $X_i \in \mathfrak{X}(\mathcal{F})$, and $\mathcal{L}_X \xi = 0$ for all $X \in \mathfrak{X}(\mathcal{F})$. In particular, we say that a differential form $\omega \in \Omega^i(M)$ is **basic** when it is basic as a tensor field. By Cartan's formula, ω is basic if, and only if, $i_X\omega = 0$ and $i_X(d\omega) = 0$ for all $X \in \mathfrak{X}(\mathcal{F})$. These are the differential forms that project to differential forms

[2]See also [CC00, Theorem 3.1.5] for the correspondence between foliated bundles and suspensions.

in the local quotients $\overline{U}$ and are invariant by the holonomy pseudogroup of $\mathcal{F}$ [see Mol88, Proposition 2.3]. We denote the $\Omega^0_B(\mathcal{F})$-module of basic i-forms of $\mathcal{F}$ by $\Omega^i_B(\mathcal{F})$. Then

$$\Omega^*_B(\mathcal{F}) := \bigoplus_{i=0}^{q} \Omega^i_B(\mathcal{F})$$

is the $\wedge$-graded **algebra of basic forms** of $\mathcal{F}$.

By definition, $\Omega^*_B(\mathcal{F})$ is closed under the exterior derivative, so we can consider the complex

$$\cdots \xrightarrow{d} \Omega^{i-1}_B(\mathcal{F}) \xrightarrow{d} \Omega^i_B(\mathcal{F}) \xrightarrow{d} \Omega^{i+1}_B(\mathcal{F}) \xrightarrow{d} \cdots .$$

The cohomology groups of this complex are the **basic cohomology groups** of $\mathcal{F}$, that we denote by $H^i_B(\mathcal{F})$. A foliate map $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ pulls basic forms on N back to basic forms on M and hence induces a linear map $f^* : H^i_B(\mathcal{G}) \to H^i_B(\mathcal{F})$.

We define, when the dimensions $\dim(H^i_B(\mathcal{F}))$ are all finite, the **basic Euler characteristic** of $\mathcal{F}$ as the alternate sum

$$\chi_B(\mathcal{F}) = \sum_i (-1)^i \dim(H^i_B(\mathcal{F})).$$

In analogy with the manifold case, we say that $b^i_B(\mathcal{F}) := \dim(H^i_B(\mathcal{F}))$ are the **basic Betti numbers** of $\mathcal{F}$. When $\mathcal{F}$ is the trivial foliation by points we recover the classical Euler characteristic and Betti numbers of M.

Since we have an identification between $\mathcal{F}$-basic forms and $\mathscr{H}_{\mathcal{F}}$-invariant forms on $T_{\mathcal{F}}$ and an identification between differential forms on an orbifold $\mathcal{O}$ and $\mathscr{H}_{\mathcal{O}}$-invariant forms on $U_{\mathcal{O}}$, Proposition 1.5 gives us the following.

Proposition 1.7. *Let $(M, \mathcal{F})$ be a foliation such that every leaf is compact and with finite holonomy. Then the projection $\pi : M \to M//\mathcal{F}$ induces an isomorphism of differential complexes $\pi^* : \Omega^*(M//\mathcal{F}) \to \Omega^*_B(\mathcal{F})$. In particular, $H^*_B(\mathcal{F}) \cong H^*_{\mathrm{dR}}(M//\mathcal{F})$.*

1.5 Riemannian Foliations

Let $\mathcal{F}$ be a smooth foliation of M. A **transverse metric** for $\mathcal{F}$ is a symmetric, positive, $\mathcal{F}$-basic $(2,0)$-tensor field g^T on M. In this case $(M, \mathcal{F}, \mathrm{g}^T)$ is called a **Riemannian foliation**. On the other hand, a Riemannian metric g on M is **bundle-like** for $\mathcal{F}$ if for any open set U and any $Y, Z \in \mathfrak{L}(\mathcal{F}|_U)$ perpendicular to the leaves we have $\mathrm{g}(Y,Z) \in \Omega^0_B(\mathcal{F}|_U)$. In this case, setting

$$\mathrm{g}^T(X,Y) := \mathrm{g}(X^\perp, Y^\perp)$$

defines a transverse metric for $\mathcal{F}$, where we write $X = X^\top + X^\perp$ with respect to the decomposition $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$. Conversely, given g^T one can always choose a bundle-like metric on M that induces it [see Mol88, Proposition 3.3]. With a bundle-like metric chosen, we will identify the bundles $\nu\mathcal{F} \equiv T\mathcal{F}^\perp$.

Example 1.8. If a foliation $\mathcal{F}$ on M is given by the action of a Lie group G (see Example 1.2) and g is a Riemannian metric on M such that G acts by isometries, then g is bundle-like for $\mathcal{F}$ [see MM03, Remark 2.7(8)]. In other words, a foliation induced by an isometric action is Riemannian.

Example 1.9. The 1-dimensional Riemannian foliations of the euclidean sphere $\mathbb{S}^n$ where classified in [GG88, Theorem 5.4]. They exist only if n is odd, say $n = 2k+1$, and are all homogeneous, given (up to isometric congruence) by $\mathbb{R}$-actions of the type[3]

$$t \cdot (z_0, \dots, z_k) = (e^{2\pi \mathrm{i} \lambda_0 t} z_0, \dots, e^{2\pi \mathrm{i} \lambda_k t} z_k),$$

where $\lambda_i \in (0,1]$. In particular, such an action correspond to a closed riemannian 1-foliation $\mathcal{F}$ of $\mathbb{S}^n$ precisely when all λ_i are rational, say $\lambda_i = p_i/q_i$. Notice that in this case we can equivalently assume that $\lambda_i \in \mathbb{N}$, by changing the parameter t to $\mathrm{lcm}(q_1, \dots, q_k)t$, hence $\mathbb{S}^n /\!/ \mathcal{F} = \mathbb{CP}^k[\lambda_0, \dots, \lambda_k]$ (see Example A.5).

Let us visualize these foliations in the case of the 3-dimensional sphere, that is, for $k = 1$. Consider the action of $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ on $\mathbb{S}^3$ by $(t_0, t_1) \cdot (z_0, z_1) = (t_0 z_0, t_1 z_1)$. This action has two singular orbits, $\mathbb{T}^2(1,0)$ and $\mathbb{T}^2(0,1)$, that are

[3] Here $z_i \in \mathbb{S}^n \subset \mathbb{C}^{k+1}$.

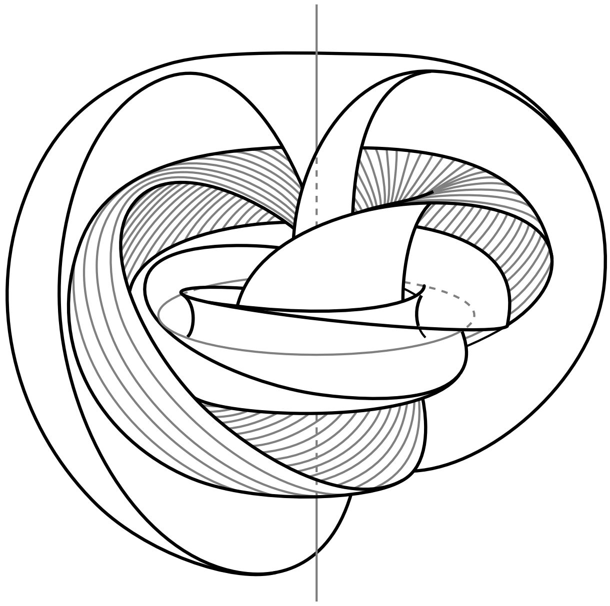


Figure 1.5: The 1-dimensional foliations of $\mathbb{S}^3$ (via stereographic projection)

diffeomorphic to $\mathbb{S}^1$. The other orbits are tori and coincide with the distance tubes of the two singular orbits. The 1-dimensional Riemannian foliations of $\mathbb{S}^3$, up to congruence, can be identified with the 1-dimensional Lie subalgebras of $\mathbb{R}^2 \cong \mathrm{lie}(\mathbb{T}^2)$ via the induced action of the corresponding 1-parameter subgroup. They restrict to Kronecker foliations on each regular $\mathbb{T}^2$-orbit (see Figure 1.5).

Example 1.10. Let (T, g) be a Riemannian manifold. A foliation $\mathcal{F}$ defined by the suspension of a homomorphism (see Examples 1.3 and 1.4) $h : \pi_1(B, x_0) \to \mathrm{Iso}(T)$ is naturally a Riemannian foliation [see also Mol88, Section 3.7].

It follows from the definition that g^T projects to Riemannian metrics on the local quotients $\overline{U}_i$ of a Haefliger cocycle $\{(U_i, \pi_i, \gamma_{ij})\}$ defining $\mathcal{F}$ [see Mol88, Section 3.2; MM03, Remark 2.7(2)]. The holonomy pseudogroup $\mathscr{H}_{\mathcal{F}}$ then becomes a pseudogroup of local isometries of $T_{\mathcal{F}}$ and, by choosing a bundle-like metric on M, the submersions defining $\mathcal{F}$ become Riemannian submersions. We will say that $\mathcal{F}$ has positive **transverse sectional curvature** when $(T_{\mathcal{F}}, \mathrm{g}^T)$ has positive sectional curvature. In this case we denote $\sec_{\mathcal{F}} > 0$. The notions of negative, non-positive and non-negative transverse sectional curvature are defined analogously, as well as the corresponding notions for the **transverse Ricci curvature**, denoted $\mathrm{Ric}_{\mathcal{F}}$.

Example 1.11. By the description via Haefliger cocycles, the pullback of a Riemannian foliation is obviously a Riemannian foliation (see Example 1.1).

Bundle-like metrics can be characterized in terms of its geodesics [Rei59, see]. A metric g is bundle-like for $(M, \mathcal{F})$ if and only if a geodesic that is perpendicular to a leaf in one point remains perpendicular to all the leaves it intersects. Moreover, he showed that geodesic segments perpendicular to the leaves project to geodesic segments in the local quotients $\overline{U}$. It follows that the leaves of a Riemannian foliation are locally equidistant.

It is shown in [ASH85, Théorème 0] that the basic cohomology of Riemannian foliations on compact manifolds have finite dimension.

Theorem 1.12 (Alaoui–Sergiescu–Hector)**.** *Let $\mathcal{F}$ be a Riemannian foliation of a closed manifold M. Then* $\dim(H^i_B(\mathcal{F})) < \infty$.

As remarked in [GT16, Proposition 3.11], the hypothesis that M is compact can be relaxed to $M/\mathcal{F}$ being compact, provided that $\mathcal{F}$ is a **complete** Riemannian foliation, that is, M is a complete Riemannian manifold with respect to some bundle-like metric for $\mathcal{F}$. Hence, if this is the case, $\chi_B(\mathcal{F})$ is always defined.

The following transverse analogue of the Bonnet–Myers Theorem due to J. Hebda will be useful [Heb86, Theorem 1].

Theorem 1.13 (Hebda)**.** *Let $\mathcal{F}$ be a complete Riemannian foliation satisfying* $\mathrm{Ric}_{\mathcal{F}} \geq c > 0$. *Then $M/\mathcal{F}$ is compact and* $H^1_B(\mathcal{F}) \cong 0$.

1.6 Taut Foliations and Harmonic Forms

Another way to approach the basic cohomology of Riemannian foliations is via the basic laplacian Δ_B [we refer to Ton97, Chapter 7, for an introduction]. Let $\mathcal{F}$ be a transversely oriented Riemannian foliation of a compact oriented manifold M endowed with a bundle-like metric g. Consider the scalar product $\langle \cdot, \cdot \rangle_B$ in $\Omega^i_B(\mathcal{F})$ given by the restriction of the usual scalar product in $\Omega^i(M)$. The **basic laplacian** is the operator $\Delta_B : \Omega^i_B(\mathcal{F}) \to \Omega^i_B(\mathcal{F})$ given by $\Delta_B = d\delta + \delta d$, where δ is the formal adjoint of d with respect to $\langle \cdot, \cdot \rangle_B$. We denote by $\mathcal{H}^i_B(\mathcal{F})$ the space of **harmonic** basic i-forms, that is, basic i-forms α satisfying $\Delta_B \alpha = 0$.

There is a version of the Hodge decomposition theorem for Δ_B that gives an orthogonal decomposition $\Omega^i_B(\mathcal{F}) \cong \mathrm{Im}(d) \oplus \mathrm{Im}(\delta) \oplus \mathcal{H}^i_B(\mathcal{F})$ [see Ton97, Theorem 7.22] and provides an isomorphism $H^i_B(\mathcal{F}) \cong \mathcal{H}^i_B(\mathcal{F})$ [see Ton97, Theorem 7.51], which leads to duality theorems for the basic cohomology. Poincaré duality in its expected form, however, is only available for the so-called taut foliations, that we now introduce.

We say that a p-dimensional Riemannian foliation $\mathcal{F}$ of M is **taut** when M admits a metric for which all the leaves are minimal submanifolds[4]. Rummler's criterion [see Rum79; or CC00, Theorem 10.5.9] characterizes tautness by the presence of a form $\omega \in \Omega^p(M)$ such that $\omega|_L$ is non singular for each leaf $L \in \mathcal{F}$ and $d\omega(X_i, \dots, X_{p+1}) = 0$ whenever at least p of the vector fields X_i are tangent to $\mathcal{F}$. Another characterization is in terms of the vanishing of the cohomology class of the mean curvature form $\kappa_{\mathcal{F}} \in \Omega^1(M)$ [see Ton97, Chapter 3, for the definition]. Precisely, there is the following result[5] by J. López [Lóp92, Theorem 6.4].

Theorem 1.14 (López)**.** *A q-codimensional Riemannian foliation $\mathcal{F}$ of a closed manifold is taut if and only if $[\kappa_{\mathcal{F}}] = 0$. Moreover, when $\mathcal{F}$ is transversely oriented it is taut if and only if $H^q_B(\mathcal{F}) \neq 0$.*

From the last part we see that tautness is necessary for Poincaré duality to hold on $H^*_B(\mathcal{F})$. Basic Hodge decomposition arguments apply to this end, if this is the case, as was proved by F. Kamber and Ph. Tondeur [KT83] (with the additional hypothesis of orientability of M) and later by A. Alaoui and G. Hector [AH86] and V. Sergiescu [Ser85] (via homological techniques), giving us the following.

Theorem 1.15 (Kamber–Tondeur, Alaoui–Hector, Sergiescu)**.** *Let $\mathcal{F}$ be a transversely oriented taut Riemannian foliation of codimension q of a closed manifold M. Then $H^i_B(\mathcal{F}) \cong H^{q-i}_B(\mathcal{F})$.*

[4] This can be achieved with a bundle-like metric [see KT82, Corollary 2.31].

[5] To apply López's result directly to $[\kappa_{\mathcal{F}}]$, we observe that there is a bundle-like metric for $\mathcal{F}$ for which the mean curvature form is basic [see Dom98] and that the class $[(\kappa_{\mathcal{F}})_B]$ of the basic component of $\kappa_{\mathcal{F}}$ is invariant under changes of the bundle-like metric [Lóp92, Theorem 5.2].

In particular, if $\operatorname{codim}(\mathcal{F})$ is odd then $\chi_B(\mathcal{F}) = 0$. This, however, holds for any (not necessarily taut) odd-codimensional transversely oriented Riemannian foliation of a compact manifold, as shown by G. Habib and K. Richardson [see HR13, Corollary 3.3].

1.7 Molino Theory

Molino theory consists of structure theorems for Riemannian foliations developed by P. Molino and others in the decade of 1980. In this section we summarize it, following mostly the brief presentations in [GT16, Section 4.1] and [Töb14, Section 3.2]. A thorough introduction can be found in [Mol88].

Let $(M, \mathcal{F}, \mathrm{g}^T)$ be a q-codimensional Riemannian foliation. Recall that g^T induces a Riemannian metric $\pi_*(\mathrm{g}^T)$ on the local quotient $\overline{U}$ of each foliation chart (U, π). Consider the pullback to U of the Levi-Civita connection on $\overline{U}$ determined this metric. By uniqueness, the pullbacks obtained this way glue together to a well-defined connection ∇^B on TM, the **canonical basic Riemannian connection**, where "basic" means that the local foliate fields in $\mathfrak{L}(\mathcal{F}|_U)$ are parallel along the leaves with respect to ∇^B. Note that ∇^B induces a covariant derivative on $\mathfrak{l}(\mathcal{F}|_U)$, that we also denote by ∇^B.

Let $\pi^{\curlywedge} : M^{\curlywedge}_{\mathcal{F}} \to M$ be the principal $\mathrm{O}(q)$-bundle of $\mathcal{F}$-transverse orthonormal frames[6], which we call the **Molino bundle** of $\mathcal{F}$. The normal bundle $\nu\mathcal{F}$ is associated to $M^{\curlywedge}_{\mathcal{F}}$ and so the basic Riemannian connection ∇^B on $\nu\mathcal{F}$ induces a connection form $\omega_{\mathcal{F}}$ on $M^{\curlywedge}_{\mathcal{F}}$. This connection form in turn defines an $\mathrm{O}(q)$-invariant horizontal distribution $\mathcal{H} := \ker(\omega_{\mathcal{F}})$ on $M^{\curlywedge}_{\mathcal{F}}$ that allows us to horizontally lift the leaves of $\mathcal{F}$, defining this way an $\mathrm{O}(q)$-invariant foliation $\mathcal{F}^{\curlywedge}$ of $M^{\curlywedge}_{\mathcal{F}}$. By construction, $\omega_{\mathcal{F}}$ is $\mathcal{F}^{\curlywedge}$-basic, that is, $i_X(\omega_{\mathcal{F}}) = 0$ and $\mathcal{L}_X \omega_{\mathcal{F}} = 0$ for all $X \in \mathfrak{X}(\mathcal{F}^{\curlywedge})$, so we may regard it as a map $\omega_{\mathcal{F}} : \nu\mathcal{F}^{\curlywedge} \to \mathfrak{so}(q)$.

A practical way to think of $\mathcal{F}^{\curlywedge}$ is the following: if $x = \pi^{\curlywedge}(x^{\curlywedge})$ and $y = \pi^{\curlywedge}(y^{\curlywedge})$

[6]When $\mathcal{F}$ is transversely orientable $M^{\curlywedge}_{\mathcal{F}}$ consists of two $\mathrm{SO}(q)$-invariant connected components that correspond to the possible orientations. In this case we will assume that one component was chosen and, by abuse of notation, denote it also by $M^{\curlywedge}_{\mathcal{F}}$. Everything stated in this section then will carry over to this case by changing $\mathrm{O}(q)$ to $\mathrm{SO}(q)$.

then $y^\curlywedge$ belongs to the leaf $L_{x^\curlywedge} \in \mathcal{F}^\curlywedge$ if and only if the orthonormal frame $y^\curlywedge$ of $\nu_y\mathcal{F}$ is the parallel transport of the frame $x^\curlywedge$, with respect to ∇^B, along some smooth path in L_x from x to y.

We now define a transverse metric for $\mathcal{F}^\curlywedge$. We lift the transverse metric g^T on $\nu\mathcal{F}$ to an $\mathrm{O}(q)$-invariant metric on the $\mathrm{O}(q)$-invariant transverse horizontal distribution $\nu\mathcal{H} := \mathcal{H}/T\mathcal{F}^\curlywedge$. The pullback $(\pi^\curlywedge)^*(\mathrm{g}^T)$ coincides with the pullback of the standard scalar product on $\mathbb{R}^q$ by the **fundamental 1-form** $\theta_{\mathcal{F}} : \nu\mathcal{F}^\curlywedge \to \mathbb{R}^q$ defined by [see Mol88, p. 70, p. 148]

$$\theta_{\mathcal{F}}(X_{x^\curlywedge}) = (x^\curlywedge)^{-1}(\mathrm{d}\pi^\curlywedge(X_{x^\curlywedge})),$$

where $x^\curlywedge$ is an orthonormal basis of $\nu_x\mathcal{F}$ regarded as an isomorphism $x^\curlywedge : \mathbb{R}^q \to \nu_x\mathcal{F}$ and $X_{x^\curlywedge} \in \nu_{x^\curlywedge}\mathcal{F}^\curlywedge$. Moreover $\theta_{\mathcal{F}}$ is $\mathcal{F}^\curlywedge$-basic [Mol88, Lemma 2.1(i)], so we get an $\mathcal{F}^\curlywedge$-basic, $\mathrm{O}(q)$-equivariant map $\omega_{\mathcal{F}} \oplus \theta_{\mathcal{F}} : \nu\mathcal{F}^\curlywedge \to \mathfrak{so}(q) \oplus \mathbb{R}^q$. By the discussion above, the pullback of the sum of an arbitrary (which is unique up to scalar λ) bi-invariant scalar product on $\mathfrak{so}(q)$ with the standard scalar product on $\mathbb{R}^q$ by $\omega_{\mathcal{F}} \oplus \theta_{\mathcal{F}}$ yields an $\mathrm{O}(q)$-invariant $\mathcal{F}^\curlywedge$-transverse metric $(\mathrm{g}^T)^\curlywedge$ with respect to which $\mathcal{F}^\curlywedge$ is a Riemannian foliation. Moreover, $\pi^\curlywedge : M^\curlywedge_{\mathcal{F}} \to M$ becomes a transversely Riemannian submersion, that is, $\mathrm{d}\pi^\curlywedge$ is surjective and restricts to an isometry $\mathrm{d}\pi^\curlywedge : (\nu\mathcal{H})_{x^\curlywedge} \to \nu_x\mathcal{F}$ for each $x^\curlywedge \in M^\curlywedge_{\mathcal{F}}$. We now fix λ by requiring that the fibers of $\pi^\curlywedge$ satisfy $\mathrm{vol}((\pi^\curlywedge)^{-1}(x)) = 1$.

The advantage of lifting $\mathcal{F}$ to $\mathcal{F}^\curlywedge$ is that the latter admits a global transverse parallelism, that is, $\nu\mathcal{F}^\curlywedge$ is parallelizable by fields in $\mathfrak{l}(\mathcal{F}^\curlywedge)$ [see Mol88, p. 82, p.148]. If we assume that $\mathcal{F}$ is complete, then those fields admit complete representatives[7] in $\mathfrak{L}(\mathcal{F}^\curlywedge)$ [see GT16, Section 4.1]. Now, from the theory of transversely parallelizable foliations, it follows that the partition $\overline{\mathcal{F}^\curlywedge}$ of $M^\curlywedge_{\mathcal{F}}$ is a **simple foliation**, that is, $W := M^\curlywedge_{\mathcal{F}}/\overline{\mathcal{F}^\curlywedge}$ is a manifold and $\overline{\mathcal{F}^\curlywedge}$ is given by the fibers of a locally trivial fibration $b : M^\curlywedge_{\mathcal{F}} \to W$ [see Mol88, Proposition 4.1'], the **basic fibration**. Since $\mathcal{F}^\curlywedge$ is $\mathrm{O}(q)$-invariant, by continuity so is $\overline{\mathcal{F}^\curlywedge}$, hence the action of $\mathrm{O}(q)$ on $M^\curlywedge_{\mathcal{F}}$ descends to an action on W such that b is now $\mathrm{O}(q)$-equivariant. A leaf closure $\overline{L} \in \overline{\mathcal{F}}$ is the image by $\pi^\curlywedge$ of a leaf closure of $\mathcal{F}^\curlywedge$, which implies that each leaf closure is an embedded

[7] Compare this with the definition of complete Riemannian foliations of Molino [Mol88, Remark on p. 88].

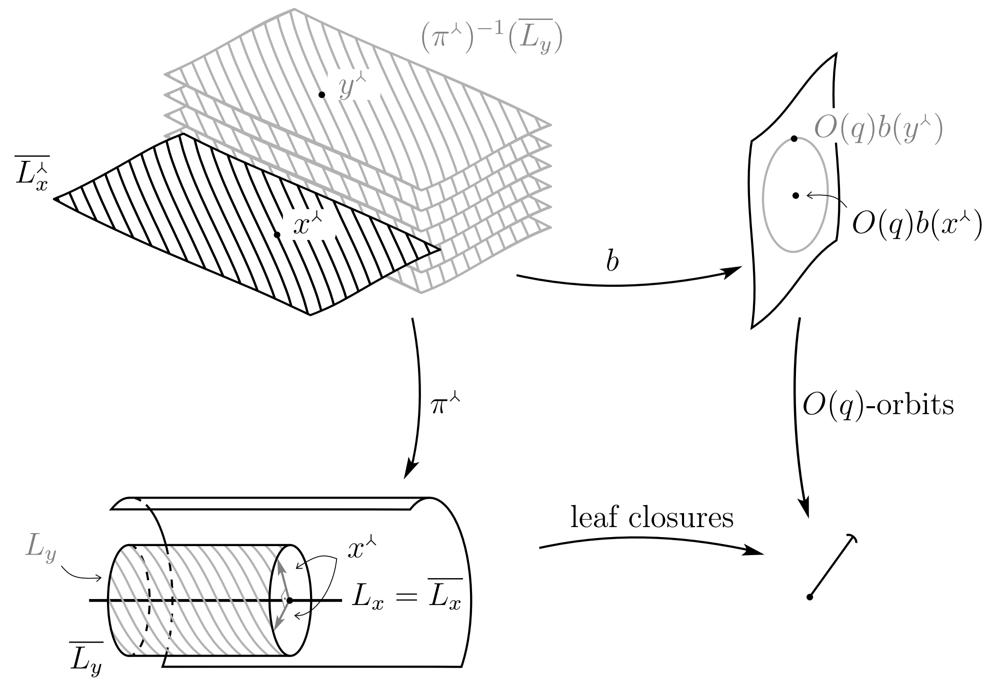


Figure 1.6: The Molino construction

submanifold of M [Mol88, Lemma 5.1][8], that also corresponds to an $\mathrm{O}(q)$-orbit in W. This induces an identification $M/\overline{\mathcal{F}} \equiv W/\mathrm{O}(q)$ and gives a commutative diagram (see Figure 1.6[9])

$$\begin{array}{ccc} (M^\curlywedge_{\mathcal{F}}, \mathcal{F}^\curlywedge, \mathrm{O}(q)) & \xrightarrow{\ b\ } & (W, \mathrm{O}(q)) \\ \downarrow{\scriptstyle \pi^\curlywedge} & & \downarrow \\ (M, \mathcal{F}) & \longrightarrow & M/\overline{\mathcal{F}} \equiv W/\mathrm{O}(q). \end{array}$$

Fix $L^\curlywedge \in \mathcal{F}^\curlywedge$, denote $J = \overline{L^\curlywedge}$, consider the foliation $(J, \mathcal{F}^\curlywedge|_J)$ and define $\mathfrak{g}_{\mathcal{F}} := \mathfrak{l}(\mathcal{F}^\curlywedge|_J)$. Then $\mathcal{F}^\curlywedge|_J$ is a complete Lie $\mathfrak{g}_{\mathcal{F}}$-foliation in the terminology of E. Fedida [Fed71], whose work establishes that such foliations are **developable**, that is, they lift to simple foliations of the universal covering spaces [see Mol88, Theorem 4.1]. The restriction of $\mathcal{F}^\curlywedge$ to the closure of a different leaf is isomorphic to $(J, \mathcal{F}^\curlywedge|_J)$, so $\mathfrak{g}_{\mathcal{F}}$ is an algebraic invariant of $\mathcal{F}$, called its **structural algebra**. We say that $d := \dim(\mathfrak{g}_{\mathcal{F}})$ is the **defect** of $\mathcal{F}$, motivated by the results in the next section.

[8]Molino's results are usually stated for a compact M, but completeness of $\mathcal{F}$ is sufficient [see GT16, Section 4.1; Töb14, Section 3.2].

[9]Here $M^\curlywedge$ is 4-dimensional so, unfortunately, we cannot have a realistic picture of $\mathcal{F}^\curlywedge$.

1.8 Molino Sheaf and Killing Foliations

Let $(M, \mathcal{F}, \mathrm{g}^T)$ be a q-codimensional Riemannian foliation. A field $X \in \mathfrak{X}(M)$ is a **Killing vector field for** g^T if $\mathcal{L}_X \mathrm{g}^T = 0$. These fields form a Lie subalgebra of $\mathfrak{L}(\mathcal{F})$ [see Mol88, Lemma 3.5] and there is, thus, a corresponding Lie algebra of **transverse Killing vector fields**, that we will denote by $\mathfrak{iso}(\mathcal{F}, \mathrm{g}^T)$. We will omit the transverse metric when it is clear from the context, writing just $\mathfrak{iso}(\mathcal{F})$. Of course, these are the transverse fields that project to Killing vector fields on the local quotients of $\mathcal{F}$.

Now let $(M, \mathcal{F}, \mathrm{g}^T)$ be a complete Riemannian foliation. Consider on $M^{\curlywedge}_{\mathcal{F}}$ the sheaf of Lie algebras $\mathscr{C}_{\mathcal{F}^{\curlywedge}}$ that, to an open set $U^{\curlywedge} \subset M^{\curlywedge}_{\mathcal{F}}$, associates the Lie algebra $\mathscr{C}_{\mathcal{F}^{\curlywedge}}(U^{\curlywedge})$ of the transverse fields in $U^{\curlywedge}$ that commute with all the global fields in $\mathfrak{l}(\mathcal{F}^{\curlywedge})$. Each field in $\mathscr{C}_{\mathcal{F}^{\curlywedge}}(U^{\curlywedge})$ is the natural lift of a $\mathcal{F}$-transverse Killing vector field on $\pi^{\curlywedge}(U^{\curlywedge})$ [see Mol88, Proposition 3.4]. The pushforward $\pi^{\curlywedge}_*(\mathscr{C}_{\mathcal{F}^{\curlywedge}})$ will be called the **Molino sheaf**[10] of $\mathcal{F}$, that we denote simply by $\mathscr{C}_{\mathcal{F}}$. From what we just saw, it is the sheaf of the Lie algebras consisting of the local transverse Killing vector fields that lift to local sections of $\mathscr{C}_{\mathcal{F}^{\curlywedge}}$.

The sheaf $\mathscr{C}_{\mathcal{F}}$ is Hausdorff [Mol88, Lemma 4.6] and its stalk $(\mathscr{C}_{\mathcal{F}})_x$ on each point is isomorphic to the Lie algebra $\mathfrak{g}_{\mathcal{F}}^{-1}$ opposed to $\mathfrak{g}_{\mathcal{F}}$ [Mol88, Proposition 4.4]. The main motivation for the study of $\mathscr{C}_{\mathcal{F}}$ is that its orbits describe the closures of the leaves of $\mathcal{F}$ [Mol88, Theorem 5.2], in the sense that

$$\{X_x \mid X \in (\mathscr{C}_{\mathcal{F}})_x\} \oplus T_x L_x = T_x \overline{L}_x.$$

Since $\mathscr{C}_{\mathcal{F}}$ is locally constant, this means that for a small open set U, fixing a basis $\overline{X}_1, \dots, \overline{X}_d$ for $\mathscr{C}_{\mathcal{F}}(U)$ we have $T\overline{L}|_U = TL|_U \oplus \langle X_1, \dots, X_d \rangle$ for any $L \in \mathcal{F}$, where $X_1, \dots, X_d \in \mathfrak{L}(\mathcal{F})$ are representatives for that basis (see Figure 1.7).

Using these facts one shows [see Mol88, Proposition 6.2] that $\overline{\mathcal{F}}$ is a **singular Riemannian foliation**, meaning that the module $\mathfrak{X}(\overline{\mathcal{F}})$ of smooth vector fields tangent to the leaf closures is transitive on each leaf closure (that is, $\overline{\mathcal{F}}$ is a singular

[10] In Molino's terminology this is the **commuting sheaf** [Mol88], also sometimes referred to as the **central transverse sheaf** [Mol79].

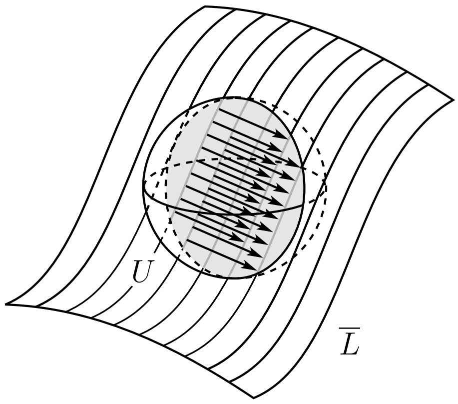


Figure 1.7: The orbits of the Molino sheaf are the closures of the leaves

foliation in the terminology of H. Sussmann [Sus73] and P. Stefan [Ste74]) and that M admits a Riemannian metric adapted to $\overline{\mathcal{F}}$, that is, such that every geodesic that is perpendicular at some point to a leaf closure remains perpendicular to every leaf closure that it intersects.

An interesting subclass of complete Riemannian foliations is the one consisting of foliations $\mathcal{F}$ for which $\mathscr{C}_{\mathcal{F}}$ is *globally* constant. Such foliations will be called **Killing foliations**, following [Moz85]. In other words, if $\mathcal{F}$ is a Killing foliation then there exists $\overline{X}_1, \dots, \overline{X}_d \in \mathfrak{iso}(\mathcal{F})$ such that $T\overline{\mathcal{F}} = T\mathcal{F} \oplus \langle X_1, \dots, X_d \rangle$. A complete Riemannian foliation is a Killing foliation if and only if $\mathscr{C}_{\mathcal{F}^\curlywedge}$ is globally constant, and in this case $\mathscr{C}_{\mathcal{F}^\curlywedge}(M^\curlywedge_{\mathcal{F}})$ is the center of $\mathfrak{l}(\mathcal{F}^\curlywedge)$. Hence $\mathscr{C}_{\mathcal{F}}(M)$ is central in $\mathfrak{l}(\mathcal{F})$ (but not necessarily the full center) . It follows that the structural algebra of a Killing foliation is Abelian, because we have $\mathfrak{g}_{\mathcal{F}}^{-1} \cong \mathscr{C}_{\mathcal{F}}(M) \cong (\mathscr{C}_{\mathcal{F}})_x$ for each $x \in M$.

Example 1.16. A complete Riemannian foliation $\mathcal{F}$ of a simply-connected manifold is automatically a Killing foliation [see Mol88, Proposition 5.5], since in this case $\mathscr{C}_{\mathcal{F}}$ cannot have holonomy.

Example 1.17. Homogeneous Riemannian foliations provide another important class of examples [see Mol84, Lemme III]. In fact, if $\mathcal{F}$ is a Riemannian foliation of a compact manifold M given by the action of $H < \mathrm{Iso}(M)$, then $\mathcal{F}$ is a Killing foliation and $\mathscr{C}_{\mathcal{F}}(M)$ consists of the transverse Killing vector fields induced by the action of $\overline{H} \subset \mathrm{Iso}(M)$.

Specific examples in this class are, therefore, the λ-Kronecker foliations (see Example 1.2) and the Riemannian 1-foliations of the round sphere (see Example 1.9).

Riemannian foliations defined by the method of suspension[11] are, in general, not Killing. A specific example can be seen in Example 4.2.

In the terminology of transverse actions introduced in [GT16, Section 2], a Killing foliation $\mathcal{F}$ admits an effective isometric transverse action of $\mathfrak{g}_{\mathcal{F}}$ given by the isomorphism $\mathfrak{g}_{\mathcal{F}} \ni V \mapsto \overline{V}^* \in \mathscr{C}_{\mathcal{F}}(M) < \mathfrak{iso}(\mathcal{F})$, such that the singular foliation everywhere tangent to the distribution of varying rank $\mathfrak{g}_{\mathcal{F}} \cdot \mathcal{F}$ defined by $(\mathfrak{g}_{\mathcal{F}} \cdot \mathcal{F})_x := \{\overline{V}^*_x \mid V \in \mathfrak{g}_{\mathcal{F}}\} \oplus T_x\mathcal{F}$ is $\overline{\mathcal{F}}$ [see GT16, Theorem 2.2]. For short, we write this as $\mathfrak{g}_{\mathcal{F}} \cdot \mathcal{F} = \overline{\mathcal{F}}$, that is, we denote by $\mathfrak{g}_{\mathcal{F}} \cdot \mathcal{F}$ both the distribution and its associated singular foliation.

[11] See Examples 1.3 and 1.10.

Chapter 2

Transverse Killing Vector Fields and the Structural Algebra

In this chapter we prove some basic facts about the zero sets of transverse Killing vector fields which will later be used in our study of Killing foliations. Although some of these results were not found in the current literature, they consist mainly of direct generalizations of known properties of Killing vector fields on Riemannian manifolds [see, for instance, Pet06, Chapter 7]. We also study the behavior of the structural algebra when a Riemannian foliation is lifted to a finitely-sheeted covering space.

2.1 Elementary Properties of Transverse Killing Vector Fields

Let us begin by establishing some notation. If $(M, \mathcal{F}, \mathrm{g}^T)$ is a Riemannian foliation and $\overline{X} \in \mathfrak{iso}(\mathcal{F}, \mathrm{g}^T)$ is a transverse Killing vector field (see Setion 1.8), we denote by $\mathrm{Zero}(\overline{X})$ the set where $\overline{X}$ vanishes, that is,

$$\mathrm{Zero}(\overline{X}) := \{x \in M \mid \overline{X}_x = 0\}.$$

Recall that if $\pi : U \to \overline{U}$ is a local trivialization of $\mathcal{F}$, then $\overline{X}|_U$ projects to a Killing vector field $\overline{X}_{\overline{U}}$ on the Riemannian manifold $(\overline{U}, \pi_*(\mathrm{g}^T))$. It is known that for any $\overline{x} \in \overline{U}$, the field $\overline{X}_{\overline{U}}$ is completely determined by $(\overline{X}_{\overline{U}})_{\overline{x}}$ and by the skew-symmetric linear map $(\nabla^{\overline{U}}(\overline{X}_{\overline{U}}))_{\overline{x}}$, where $\nabla^{\overline{U}}$ is the Levi-Civita connection of $\overline{U}$ [see, for example, Pet06, Propositions 27 and 28]. Hence it follows that, for any $x \in M$, the field $\overline{X} \in \mathfrak{iso}(\mathcal{F})$ is also uniquely determined by $\overline{X}_x$ and by the skew-symmetric linear map

$$(\nabla^B \overline{X})_x : \nu_x \mathcal{F} \to \nu_x \mathcal{F},$$

where ∇^B denotes the canonical basic Riemannian connection of $(M, \mathcal{F}, \mathrm{g}^T)$ (see Section 1.7).

We say that an $\mathcal{F}$-saturated submanifold $N \subset M$ is **horizontally totally geodesic** if it projects to totally geodesic submanifolds in the local quotients $\overline{U}$ of $\mathcal{F}$.

Proposition 2.1. *Let $(M, \mathcal{F}, \mathrm{g}^T)$ be a Riemannian foliation and let $\overline{X} \in \mathfrak{iso}(\mathcal{F})$ be a transverse Killing vector field. Then each connected component N of* $\mathrm{Zero}(\overline{X})$ *is an even-codimensional closed submanifold of M saturated by the leaves of $\mathcal{F}$ (and hence $\overline{\mathcal{F}}$). Moreover, N is horizontally totally geodesic and if $\mathcal{F}$ is transversely orientable then $(N, \mathcal{F}|_N)$ is transversely orientable.*

Proof. As $\overline{X}$ is transverse, if $x \in \mathrm{Zero}(\overline{X})$ then $L_x \subset \mathrm{Zero}(\overline{X})$, hence each connected component N is saturated by leaves of $\mathcal{F}$. Furthermore, it is clear that $\mathrm{Zero}(\overline{X})$ is closed, so $\overline{L_x} \subset \mathrm{Zero}(\overline{X})$ and N is also saturated by the leaves of the singular foliation $\overline{\mathcal{F}}$.

Let $x \in N$ and let $\pi : U \to \overline{U}$ be a local trivialization of $\mathcal{F}$ with $x \in U$. If $\overline{X}_{\overline{U}}$ is the Killing vector field on $\overline{U}$ induced by $\overline{X}|_U$, then

$$\mathrm{Zero}(\overline{X}) \cap U = \pi^{-1}(\mathrm{Zero}(\overline{X}_{\overline{U}})).$$

In particular, $N \cap U = \pi^{-1}(\overline{N})$, where $\overline{N}$ is a connected component of $\mathrm{Zero}(\overline{X}_{\overline{U}})$. We know that $\overline{N}$ is a totally geodesic submanifold of $\overline{U}$ of even codimension [see Kob72, Theorem 5.3] so, as π is trivially transverse to $\overline{N}$, it follows that $N \cap U$ is a horizontally totally geodesic submanifold of M. Observe that this also shows that N is path-connected. If y is another point of N, we choose a path γ in N connecting

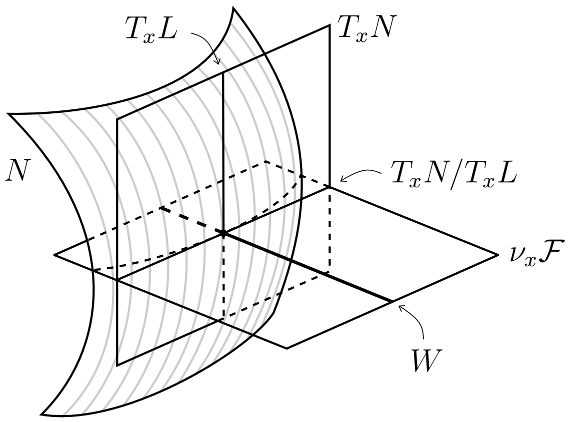


Figure 2.1: The space W

x to y and a chain $U = U_1, U_2, \ldots, U_k \ni y$ of simple open sets covering $\mathrm{Im}(\gamma)$. Then each $N \cap U_i$ is a horizontally totally geodesic submanifold and, as

$$N \cap (U_i \cap U_{i+1}) = (N \cap U_i) \cap U_{i+1} = (N \cap U_{i+1}) \cap U_i,$$

it follows that $\dim(N \cap U) = \dim(N \cap U_k)$. Thus N is a horizontally totally geodesic submanifold of M. Moreover, π is a Riemannian submersion, hence

$$\mathrm{codim}(N) = \mathrm{codim}(N \cap U) = \mathrm{codim}(\overline{N})$$

is even.

It remains to show that if $\mathcal{F}$ is transversely orientable then $(N, \mathcal{F}|_N)$ is also transversely orientable. Indeed,

$$(\nabla^B \overline{X})_x : \nu_x\mathcal{F} \to \nu_x\mathcal{F}$$

is skew-symmetric and vanishes on T_xN/T_xL_x, so it preserves the g^T-orthogonal complement (see Figure 2.1)

$$W = \left(\frac{T_xN}{T_xL_x}\right)^{\perp} \leq \nu_x\mathcal{F}.$$

Choosing the adequate orthonormal basis, $(\nabla^B \overline{X})_x|_{W_x}$ has a matrix representation of the type

$$\begin{bmatrix} 0 & \alpha_1 & & & \\ -\alpha_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \alpha_l \\ & & & -\alpha_l & 0 \end{bmatrix},$$

where $l = \dim(W_x)/2 = \operatorname{codim}(N)/2$. That is, W_x decomposes into 2-dimensional invariant subspaces E_j, $j = 1, \dots, l$. The eigenvalues $\pm i\alpha_j$ remain constant on N, because on each simple open set $U \ni x$ these are the eigenvalues of $(\nabla^{\overline{U}}(\overline{X}_{\overline{U}}))_{\overline{x}}$, that are known to remain constant on $\overline{N}$ [see Kob72, p. 61]. Therefore we can decompose the bundle $W = (TN/T\mathcal{F})^{\perp}$ into subbundles

$$W = E_1 \oplus \cdots \oplus E_l,$$

where E_j corresponds to the eigenvalue $-\alpha_j^2$ of $(\nabla^B \overline{X}|_W)^2$. Now define an endomorphism J of W by

$$\mathrm{J}|_{E_j} = \frac{1}{\alpha_j} \nabla^B \overline{X}.$$

Then clearly $\mathrm{J}^2 = -\mathrm{Id}$, so J defines a complex structure on W. In particular, we have that W is orientable. As

$$\left.\frac{TM}{T\mathcal{F}}\right|_N = \frac{TN}{T\mathcal{F}} \oplus W,$$

the result follows. □

Recall that the symmetry rank of a Riemannian manifold is the rank of its isometry group. In analogy, if $(M, \mathcal{F}, \mathrm{g}^T)$ is a Riemannian foliation we define the **transverse symmetry rank** of $\mathcal{F}$ by

$$\operatorname{symrank}(\mathcal{F}) := \max_{\mathfrak{a}} \Big\{ \dim(\mathfrak{a}) \Big\},$$

where $\mathfrak{a}$ runs over all the Abelian subalgebras of $\mathfrak{iso}(\mathcal{F})$. Fix a subalgebra $\mathfrak{a} < \mathfrak{iso}(\mathcal{F})$ satisfying $\dim(\mathfrak{a}) = \operatorname{symrank}(\mathcal{F})$. We will denote by $\mathcal{Z}(\mathfrak{a})$ the set consisting of all proper connected components of the zero sets of the transverse Killing vector fields in $\mathfrak{a}$. We have the following properties.

Proposition 2.2. *For any $N, N' \in \mathcal{Z}(\mathfrak{a})$ the following holds:*

(i) Every transverse Killing vector field in $\mathfrak{a}$ is tangent[1] *to N and, therefore, the restriction to N of the fields in $\mathfrak{a}$ yields a commutative Lie algebra $\mathfrak{a}|_N$ of transverse Killing vector fields of $\mathcal{F}|_N$.*

(ii) If N is maximal in $\mathcal{Z}(\mathfrak{a})$ with respect to set inclusion, then

$$\dim(\mathfrak{a}|_N) = \dim(\mathfrak{a}) - 1.$$

(iii) Each connected component of $N \cap N'$ also belongs to $\mathcal{Z}(\mathfrak{a})$.

Proof. Let N and N' be connected components of the transverse Killing vector fields $\overline{X}$ and $\overline{X}'$ in $\mathfrak{a}$, respectively.

(i) Choose any $\overline{Y} \in \mathfrak{a}$. As we have that $\overline{\mathcal{L}_Y X} = \overline{[Y, X]} = 0$ and that $Y \in \mathfrak{L}(\mathcal{F})$ is a Killing vector field for g^T, it follows that

$$\begin{aligned} 0 &= \mathcal{L}_Y \mathrm{g}^T(\overline{X}, \overline{X}) = Y \mathrm{g}^T(\overline{X}, \overline{X}) - 2\mathrm{g}^T(\overline{[Y, X]}, \overline{X}) \\ &= Y \|\overline{X}\|_T^2. \end{aligned}$$

That is, the flow of Y preserves the level sets of $\|\overline{X}\|_T^2$ and, hence, in particular, it preserves $\mathrm{Zero}(\overline{X})$. Therefore $\overline{Y}$ is tangent to N. It is now clear that $\mathfrak{a}|_N$ is a commutative Lie algebra of transverse Killing vector fields of $\mathcal{F}|_N$.

(ii) Suppose that $N \in \mathcal{Z}(\mathfrak{a})$ is maximal and that $\overline{Y} \in \mathfrak{a}$ vanishes on N. Fix $x \in N$ and consider the commuting skew-symmetric applications

$$(\nabla^B \overline{X})_x, (\nabla^B \overline{Y})_x : \nu_x \mathcal{F} \to \nu_x \mathcal{F}.$$

They both vanish on $T_x N / T_x L_x$, so, again, as in the proof of Proposition 2.1, they preserve the g^T-orthogonal complement $W = (T_x N / T_x L_x)^{\perp}$ (see Figure 2.1), which

[1] That is, any foliate field representing a field in $\mathfrak{a}$ is tangent to N.

therefore decomposes as

$$W = E_1 \oplus \cdots \oplus E_l,$$

where[2] $l = \dim(W)/2$ and each E_i is a 2-dimensional subspace invariant by both $(\nabla^B \overline{X})_x$ and $(\nabla^B \overline{Y})_x$. Now fix any index i. As the space of skew-symmetric transformations E_i is one-dimensional, there exists some linear combination $\alpha(\nabla^B \overline{X})_x + \beta(\nabla^B \overline{Y})_x$ that vanishes on E_i. In particular,

$$\ker(\nabla^B(\alpha\overline{X} + \beta\overline{Y}))_x \supset \frac{T_x N}{T_x L_x} \oplus E_i$$

and, since $\ker(\nabla^B(\alpha\overline{X} + \beta\overline{Y}))_x$ completely determines the connected component of $\mathrm{Zero}(\alpha\overline{X}+\beta\overline{Y})$ containing x, it follows that $\alpha\overline{X}+\beta\overline{Y}$ vanishes on a set that contains N properly. This violates the maximality of N unless $\overline{Y}$ is a multiple of $\overline{X}$, hence $\dim(\mathfrak{a}|_N) = \dim(\mathfrak{a}) - 1$.

(iii) Let C be a connected component of $N \cap N'$. First, by considering local trivializations $\pi : U \to \overline{U}$ of $\mathcal{F}$ and observing that

$$N \cap N' \cap U = (N \cap U) \cap (N' \cap U) = \pi^{-1}(\overline{N}) \cap \pi^{-1}(\overline{N}') = \pi^{-1}(\overline{N} \cap \overline{N}'),$$

where $\overline{N}$ and $\overline{N}'$ are connected components of the zero sets of the Killing vector fields induced on $\overline{U}$ by $\overline{X}$ and $\overline{X}'$ respectively, it follows as in the proof of Proposition 2.1 that C is a submanifold, since each connected component of $\overline{N}\cap\overline{N}'$ is a submanifold of $\overline{U}$ [see Pet06, Proposition 30 (4)].

It is clear that $\alpha\overline{X}+\beta\overline{X}'$ vanishes on C for any $\alpha, \beta \in \mathbb{R}$. Let C' be the connected component of $\mathrm{Zero}(\alpha\overline{X}_1 + \beta\overline{X}_2)$ containing C. For $x \in C$ we have that the maps $(\nabla^B \overline{X}_1)_x$ and $(\nabla^B \overline{X}_2)_x$ vanish simultaneously on

$$\left(\frac{T_x N}{T_x L_x}\right) \cap \left(\frac{T_x N'}{T_x L_x}\right) = \frac{T_x C}{T_x L_x}$$

but not on any (nonzero) vector on the orthogonal complement $W = (T_x C/T_x L_x)^\perp$. Again, we decompose W into 2-dimensional subspaces E_i that remain invariant under both the commuting skew-symmetric maps $(\nabla^B \overline{X}_1)_x|_W$ and $(\nabla^B \overline{X}_2)_x|_W$. Since

[2] Recall from Proposition 2.1 that $\dim(W) = \mathrm{codim}(N)$ is even.

these maps do not vanish simultaneously on the subspaces E_i, it is possible to choose $\alpha, \beta \in \mathbb{R}$ such that $(\nabla^B(\alpha\overline{X} + \beta\overline{X}'))_x$ does not vanish on W. Hence

$$\frac{T_x C'}{T_x L_x} = \ker\Big((\nabla^B(\alpha\overline{X} + \beta\overline{X}'))_x\Big) = \frac{T_x C}{T_x L_x},$$

which shows that $C = C'$. □

We end this section with a basic technical result that we will use later. Consider a Riemannian foliation $\mathcal{F}$ of a closed manifold M endowed with a bundle-like metric, let $\overline{X} \in \mathfrak{iso}(\mathcal{F})$ and fix N a connected component of $\mathrm{Zero}(\overline{X})$. Recall from Proposition 2.1 that N is a compact submanifold, so it is known that for a sufficiently small $\varepsilon > 0$, if $B_\varepsilon(N)$ is the bundle of open balls consisting of vectors perpendicular to N and of length less than ε, then the restriction

$$\exp^\perp : B_\varepsilon(N) \longrightarrow \mathrm{Tub}_\varepsilon(N)$$

of the exponential map is a diffeomorphism onto a tubular neighborhood $\mathrm{Tub}_\varepsilon(N)$ of N in M. If $\|X_x\| = \varepsilon' < \varepsilon$ and $y = \exp^\perp(X_x)$ then clearly $d(y, N) = \varepsilon$. Moreover, the orthogonal projection $\pi_N : \mathrm{Tub}_\varepsilon(N) \to N$ is a locally trivial fibration whose typical fiber is $\pi_N^{-1}(x) = \exp^\perp((B_\varepsilon(N))_x)$. We will assume, decreasing ε if necessary, that for each $y \in \mathrm{Tub}_\varepsilon(N)$ the leaf L_y is transverse to $\pi_N^{-1}(\pi(y))$. The set of points $\mathrm{S}^N_{\varepsilon'}$ in $\mathrm{Tub}_\varepsilon(N)$ which are at a distance ε' from N is the **tube** of radius ε' centered on N. It is precisely the image by $\exp^\perp$ of the bundle of spheres of radius ε' in $B_\varepsilon(N)$, so it is a submanifold of M.

Let $\eta_\lambda : TN^\perp \to TN^\perp$ denote the product by a scalar $\lambda > 0$. Then on the union of the tubes $\mathrm{S}^N_{\varepsilon'}$ such that $\lambda\varepsilon' < \varepsilon$ we define a **homothetic transformation** $h_\lambda := \exp^\perp \circ \eta_\lambda \circ (\exp^\perp)^{-1}$ of proportionality constant λ (see Figure 2.1).

Lemma 2.3. *With the notation above, π_N and h_λ are foliate maps.*

Proof. Since $\overline{\mathcal{F}}$ is a singular Riemannian foliation and N is $\overline{\mathcal{F}}$-saturated (see Proposition 2.1), a geodesic perpendicular to N remains perpendicular to all the leaf closures that it intersects, therefore if $d(y, N) = \varepsilon'$ it follows that $T_y\overline{L}_y$ is contained in the tangent space $T_y\mathrm{S}^N_{\varepsilon'}$ of the tube of radius ε' centered on N. In particular, any

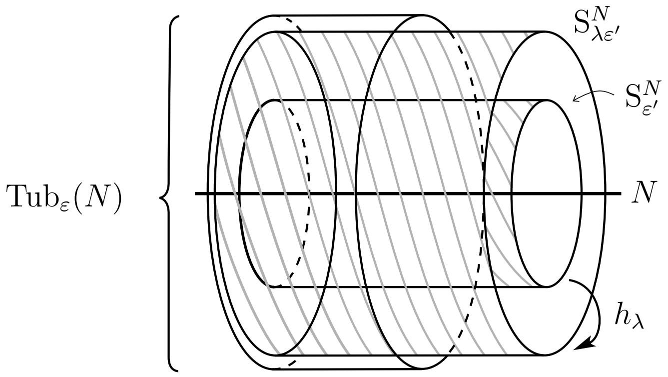


Figure 2.2: Homothetic transformation with respect to N

leaf closure in $\overline{\mathcal{F}}$ (and hence any leaf of $\mathcal{F}$) intersecting $\mathrm{Tub}_\varepsilon(N)$ is contained in a tube centered on N.

Now let P be a relatively compact, connected, open neighborhood of $x = \pi(y)$ in L_x and define $\mathrm{Tub}_\varepsilon(P)$ and $\mathrm{S}^P_{\varepsilon'}$ just as we did for N. Then $\mathrm{Tub}_\varepsilon(P)$ is a distinguished tubular neighborhood, in the sense of [Mol88, p. 193]. By the same argument we gave above, we conclude that the connected component P_y of $L_y \cap \mathrm{Tub}_\varepsilon(P)$ containing y, called the **plaque** of $\mathcal{F}$ passing through y in $\mathrm{Tub}_\varepsilon(P)$, is entirely contained in $\mathrm{S}^P_{\varepsilon'}$ and, so, in $\mathrm{S}^P_{\varepsilon'} \cap \mathrm{S}^N_{\varepsilon'}$, where π_P clearly coincides with π_N. Thus π_N maps plaques to plaques and is therefore foliate.

The proof that h_λ is foliate consists of an adaptation of the proof of Molino's homothetic transformation lemma [Mol88, Lemma 6.2], so we will just sketch it. It is sufficient to consider λ close to 1 so that $y_\lambda := h_\lambda(y)$ belongs to a distinguished neighborhood of y and hence there are equidistant neighborhoods $\Xi \ni y$ and $\Xi_\lambda \ni y_\lambda$ in P^y and P^{y_λ} respectively, such that the shortest distance from a point $y'_\lambda \in \Xi_\lambda \subset \mathrm{S}^N_{\lambda\varepsilon'}$ to $\Xi \subset \mathrm{S}^N_{\varepsilon'}$ is realized by the geodesic joining y'_λ to its projection y' in Ξ. But $d(y', y'_\lambda) = d(y, y_\lambda) = \varepsilon'|1-\lambda|$ and such a distance between a point in $\mathrm{S}^N_{\varepsilon'}$ and a point in $\mathrm{S}^N_{\lambda\varepsilon'}$ can only be realized if $y'_\lambda = h_\lambda(y')$. Therefore $\Xi_\lambda = h_\lambda(\Xi)$, hence the result. $\square$

2.2 The Canonical Stratification

As we already saw in the Sections 1.7 and 1.8, given a complete Riemannian foliation $\mathcal{F}$ of M, the partition $\overline{\mathcal{F}}$ defined by the closures of the leaves of $\mathcal{F}$ is a singular Riemannian foliation[3] of M, whose **dimension** we define by

$$\dim(\overline{\mathcal{F}}) = \max_{L\in\mathcal{F}} \Big\{ \dim(\overline{L}) \Big\}.$$

We define the **codimension** of $\overline{\mathcal{F}}$ by $\operatorname{codim}(\overline{\mathcal{F}}) = \dim(M) - \dim(\mathcal{F})$. Notice that the relationship between $\overline{\mathcal{F}}$, the Molino sheaf $\mathscr{C}_{\mathcal{F}}$ and the structural algebra $\mathfrak{g}_{\mathcal{F}}$ (see Section 1.8) enables us to write the defect of $\mathcal{F}$ as

$$d = \dim(\mathfrak{g}_{\mathcal{F}}) = \dim(\overline{\mathcal{F}}) - \dim(\mathcal{F}) = \operatorname{codim}(\mathcal{F}) - \operatorname{codim}(\overline{\mathcal{F}}).$$

Now for s satisfying $\dim(\mathcal{F}) \leq s \leq \dim(\overline{\mathcal{F}})$, let us denote by Σ^s the subset of points $x \in M$ such that $\dim(\overline{L}_x) = s$. Then we get a decomposition

$$M = \bigcup_{x\in M} \Sigma_x,$$

called the **canonical stratification** of $\mathcal{F}$, where Σ_x is the connected component of Σ^s that contains x. Each component Σ_x is an embedded submanifold [Mol88, Lemma 5.3] called a **stratum** of $\mathcal{F}$. Moreover, the restriction $\overline{\mathcal{F}}|_{\Sigma_x}$ now has constant dimension and forms a (regular) Riemannian foliation [Mol88, Lemma 5.3]. The **regular stratum** $\Sigma^{\dim(\overline{\mathcal{F}})}$ is an open, connected and dense subset of M, and each other stratum $\Sigma_x \neq \Sigma^{\dim(\overline{\mathcal{F}})}$ is called **singular** and satisfies $\operatorname{codim}(\Sigma_x) \geq 2$ [see Mol88, p. 197]. In particular, $\operatorname{codim}(\mathcal{F}|_{\Sigma_x}) < \operatorname{codim}(\mathcal{F})$. The subset $\Sigma^{\dim(\mathcal{F})}$ will be called the **stratum of the closed leaves**, even though it is not, in general, a canonical stratum (for it can be disconnected).

Of course, the canonical stratification is closely related to the structural Lie algebra $\mathfrak{g}_{\mathcal{F}}$. Consider $\mathcal{F}$ a q-codimensional Killing foliation of a closed manifold M. W. Mozgawa shows in [Moz85, Théorème] that there are $r = \min_{L\in\mathcal{F}}\{\dim(\overline{L})\}$

[3] See [Mol88, Chapter 6] for details.

vector fields in $\mathfrak{g}_{\mathcal{F}}$, linearly independent at every point of M, defining a $(q-r)$-codimensional foliation $\mathcal{F}'$ of M such that $\overline{\mathcal{F}'} = \overline{\mathcal{F}}$ and $\mathcal{F}'$ has at least one closed leaf. When $r = 0$ we also have the following.

Proposition 2.4. *Let $(M, \mathcal{F})$ be a Killing foliation. Then there exists a transverse Killing vector field $\overline{X} \in \mathscr{C}_{\mathcal{F}}(M)$ such that*

$$\mathrm{Zero}(\overline{X}) = \Sigma^{\dim(\mathcal{F})}.$$

Proof. Choose an enumerable cover $\{U_i\}$ of $M \setminus \Sigma^{\dim(\mathcal{F})}$ by simple open sets. Since there are no closed leaves in U_i, the algebra $\mathscr{C}_{\mathcal{F}}(U_i) = \mathscr{C}_{\mathcal{F}}(M)|_{U_i}$ projects on the quotient $\overline{U}_i$ to an Abelian algebra $\mathfrak{c}_i$ of Killing vector fields whose orbits have dimension at least 1. It is known that the set of 1-dimensional subalgebras of $\mathfrak{c}_i$ having trivial isotropy at each point of $\overline{U}_i$ is residual in the Grassmannian $\mathrm{Gr}^1(\mathfrak{c}_i)$ [see Moz85, Lemme]. It is clear that the isotropy of $\mathscr{C}_{\mathcal{F}}(M)$ at $x \in U_i$ corresponds to the isotropy of $\mathfrak{c}_i$ at $\overline{x}$, hence, since we have only countable many open sets U_i, it follows that the subset of 1-dimensional subalgebras of $\mathscr{C}_{\mathcal{F}}(M)$ having trivial isotropy at each point of $M \setminus \Sigma^{\dim(\mathcal{F})}$ is residual in $\mathrm{Gr}^1(\mathscr{C}_{\mathcal{F}}(M))$. In other words, a generic $\overline{X} \in \mathscr{C}_{\mathcal{F}}(M) < \mathfrak{iso}(\mathcal{F})$ satisfies $\mathrm{Zero}(\overline{X}) = \Sigma^{\dim(\mathcal{F})}$. □

We can now use Proposition 2.1 to conclude the following.

Corollary 2.5. *Let $(M, \mathcal{F})$ be a Killing foliation. Each connected component of $\Sigma^{\dim(\mathcal{F})}$ is a horizontally totally geodesic, closed submanifold of M of even codimension, and $\mathcal{F}|_{\Sigma^{\dim(\mathcal{F})}}$ is transversely orientable when $\mathcal{F}$ is.*

2.3 The Structural Algebra and Finite Coverings

We now study the behavior of the structural algebra when a Riemannian foliation is lifted to a finitely-sheeted covering space.

Lemma 2.6. *Let $\mathcal{F}$ be a smooth foliation of a smooth manifold M and let $\rho : \widehat{M} \to M$ be a finitely-sheeted covering map. Then $\overline{\rho(\widehat{L})} = \rho(\overline{\widehat{L}})$ for any $\widehat{L} \in \widehat{\mathcal{F}}$, where $\widehat{\mathcal{F}} = \rho^*(\mathcal{F})$. In particular, $\rho : \overline{\widehat{L}} \to \overline{\rho(\widehat{L})}$ is a finitely-sheeted covering map.*

Proof. Notice that, as ρ is a finitely-sheeted covering, it is a closed map. Hence, if $\widehat{L} \in \widehat{\mathcal{F}}$ projects to $L \in \mathcal{F}$, then $\overline{\rho(\widehat{L})} \subset \rho(\overline{\widehat{L}})$. The converse inclusion follows directly from the continuity of ρ, so

$$\overline{L} = \overline{\rho(\widehat{L})} = \rho(\overline{\widehat{L}}).$$

It is clear that $\rho|_{\rho^{-1}(\rho(\overline{\widehat{L}}))} : \rho^{-1}(\rho(\overline{\widehat{L}})) \to \rho(\overline{\widehat{L}})$ is a finitely-sheeted covering, therefore $\rho : \overline{\widehat{L}} \to \overline{L}$ is also a finitely-sheeted covering, since $\overline{\widehat{L}}$ is a connected component of $\rho^{-1}(\rho(\overline{\widehat{L}}))$. □

Proposition 2.7. *Let $\mathcal{F}$ be a complete Riemannian foliation of M and let $\rho : \widehat{M} \to M$ be a finitely-sheeted covering map. Then $\mathscr{C}_{\rho^*(\mathcal{F})} \cong \rho^*(\mathscr{C}_{\mathcal{F}})$.*

Proof. Let us denote $\widehat{\mathcal{F}} := \rho^*(\mathcal{F})$. We can identify $\widehat{M}^{\pitchfork}_{\widehat{\mathcal{F}}}$ with the pullback bundle $(\rho^{\pitchfork})^*(M^{\pitchfork}_{\mathcal{F}})$, so we have a commutative diagram

$$\begin{array}{ccc} \widehat{M}^{\pitchfork}_{\widehat{\mathcal{F}}} & \xrightarrow{\rho^{\pitchfork}} & M^{\pitchfork}_{\mathcal{F}} \\ \hat{\pi}\downarrow & & \downarrow\pi \\ \widehat{M} & \xrightarrow{\rho} & M, \end{array}$$

where the horizontal arrows are finitely-sheeted covering maps. Moreover, $(\rho^{\pitchfork})^*(\mathscr{C}_{\mathcal{F}^{\pitchfork}})$ can be identified with the lift $\hat{\pi}^*(\rho^*(\mathscr{C}_{\mathcal{F}}))$, hence, since the Molino sheaves of $\mathcal{F}$ and $\widehat{\mathcal{F}}$ are defined in terms of the sheaves of the correspondent lifted foliations, it remains to show that $(\rho^{\pitchfork})^*(\mathscr{C}_{\mathcal{F}^{\pitchfork}}) = \mathscr{C}_{\widehat{\mathcal{F}}^{\pitchfork}}$.

Indeed, $\mathscr{C}_{\widehat{\mathcal{F}}^{\pitchfork}}$ commutes with the Lie algebra $\mathfrak{l}(\widehat{\mathcal{F}}^{\pitchfork})$ of $\widehat{\mathcal{F}}^{\pitchfork}$-transverse fields, hence, in particular, it commutes with $(\rho^{\pitchfork})^*(\mathfrak{l}(\mathcal{F}^{\pitchfork}))$, so $\mathscr{C}_{\widehat{\mathcal{F}}^{\pitchfork}}$ is a subsheaf of $(\rho^{\pitchfork})^*(\mathscr{C}_{\mathcal{F}^{\pitchfork}})$. This implies that if we consider an open subset $U \subset \widehat{M}^{\pitchfork}_{\widehat{\mathcal{F}}}$ where both $\mathscr{C}_{\widehat{\mathcal{F}}^{\pitchfork}}$ and $(\rho^{\pitchfork})^*(\mathscr{C}_{\mathcal{F}^{\pitchfork}})$ are constant and such that $\rho^{\pitchfork}|_U$ is a diffeomorphism over its image, then $\rho^{\pitchfork}_*(\mathscr{C}_{\widehat{\mathcal{F}}^{\pitchfork}}(U)) < \mathscr{C}_{\mathcal{F}^{\pitchfork}}(\rho^{\pitchfork}(U))$. Now, by Lemma 2.6, the leaf closures in $\widehat{\mathcal{F}}^{\pitchfork}$ and $\mathcal{F}^{\pitchfork}$ have the same dimension, therefore we must have

$$\rho^{\pitchfork}_*(\mathscr{C}_{\widehat{\mathcal{F}}^{\pitchfork}}(U)) = \mathscr{C}_{\mathcal{F}^{\pitchfork}}(\rho^{\pitchfork}(U)),$$

thus $\mathscr{C}_{\widehat{\mathcal{F}}^{\pitchfork}}(U) = (\rho^{\pitchfork})^*(\mathscr{C}_{\mathcal{F}^{\pitchfork}})(U)$ for any small enough U. □

Corollary 2.8. *Let $\mathcal{F}$ be a complete Riemannian foliation of M and let $\rho : \widehat{M} \to M$ be a finitely-sheeted covering. Then $\mathfrak{g}_{\mathcal{F}} \cong \mathfrak{g}_{\widehat{\mathcal{F}}}$, where $\widehat{\mathcal{F}}$ is the lifted foliation of $\widehat{M}$. In particular, if $|\pi_1(M)| < \infty$, then $\mathfrak{g}_{\mathcal{F}}$ is Abelian.*

Chapter 3

Deformations of Killing Foliations

There are several notions of deformations of foliations available in the literature [see, for instance, CC00, Section 3.6]. Here we will be interested in deformations of the following type: two smooth foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of a manifold M are C^∞-**homotopic** if there is a smooth foliation $\mathcal{F}$ of $M \times [0,1]$ such that $M \times \{t\}$ is saturated by leaves of $\mathcal{F}$, for each $t \in [0,1]$, and

$$\mathcal{F}_i = \mathcal{F}|_{M\times\{i\}},$$

for $i = 0, 1$ (see Figure 3.1). Note that, in particular, $\dim(\mathcal{F}_0) = \dim(\mathcal{F}_1)$. In this case we will also say that $\mathcal{F}$ is a **homotopic deformation** of $\mathcal{F}_0$ into $\mathcal{F}_1$.

Example 3.1. Consider, for $\lambda_1, \lambda_2 \in \mathbb{R}$, the Kronecker foliations $\mathcal{F}(\lambda_i)$ of $\mathbb{T}^2$ (see Example 1.2). Clearly, a homotopic deformation between $\mathcal{F}(\lambda_1)$ and $\mathcal{F}(\lambda_2)$ is given by $\mathcal{F}((1-t)\lambda_1 + t\lambda_2)$, $t \in [0,1]$. Notice that if λ_1 is irrational and we choose $\lambda_2 \in \mathbb{Q}$, then we obtain a deformation of a foliation with dense leaves into a closed foliation. We are primarily interested in deformations with this property.

A more involved example is the following.

Example 3.2. Let us see how to deform a given Riemannian foliation $\mathcal{F}$ of a closed simply-connected manifold M into a closed foliation $\mathcal{G}$. The ideas here come from the article [Ghy84] by E. Ghys, where it is shown that $\mathcal{F}$ can be arbitrarily approximated by such a foliation $\mathcal{G}$ [Ghy84, Théorème 3.3]. Later we will see how a theorem by

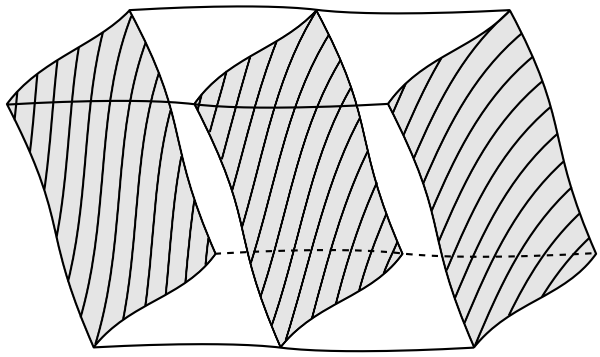


Figure 3.1: A homothopic deformation

A. Haefliger and E. Salem [HS90] can be used in order to obtain this result for Killing foliations (without assuming that M is simply-connected). Nevertheless, Ghys' approach is rather constructive, so we have chosen to include it here along with some additional details.

The desired C^∞-homotopy will come from a deformation between the lifted foliation $\mathcal{F}^\curlywedge$ and a foliation $\mathcal{G}^\curlywedge$ such that the restriction of $\mathcal{G}^\curlywedge$ to the closure of a leaf of $\mathcal{F}^\curlywedge$ is given by a fibration over a torus $\mathbb{T}^d$. This fact appears in [Ghy84, Remarque 3.4] without a proof, and the author uses it to induce an action of $\mathbb{T}^d$ on $M/\mathcal{G}$ with quotient $M/\overline{\mathcal{F}}$. This useful feature will also be present in the aforementioned generalization, but let us prove it here directly.

As M is simply-connected, we have that $\mathfrak{g}_\mathcal{F} \cong \mathbb{R}^d$ and that the restriction of $\mathcal{F}^\curlywedge$ to a leaf closure $J = \overline{L^\curlywedge}$ is a Lie $\mathbb{R}^d$-foliation (see Section 1.7), that is,

$$\mathcal{F}^\curlywedge|_J = \bigcap_{i=1}^{d} \ker(\alpha_i),$$

where $\alpha_1, \dots, \alpha_d$ are closed 1-forms. In other words, we have that $\alpha = (\alpha_1, \dots, \alpha_d)$ is a $\mathbb{R}^d$-valued Maurer–Cartan form on J that defines $\mathcal{F}^\curlywedge|_J$. We now use the Darboux cover construction, following the presentation in [MM03, Section 4.3.2]. Consider the connection η on the principal $\mathbb{R}^d$-bundle $J \times \mathbb{R}^d$ given by

$$\eta_{(x,g)}(\xi, v) = \alpha_x(\xi) + v,$$

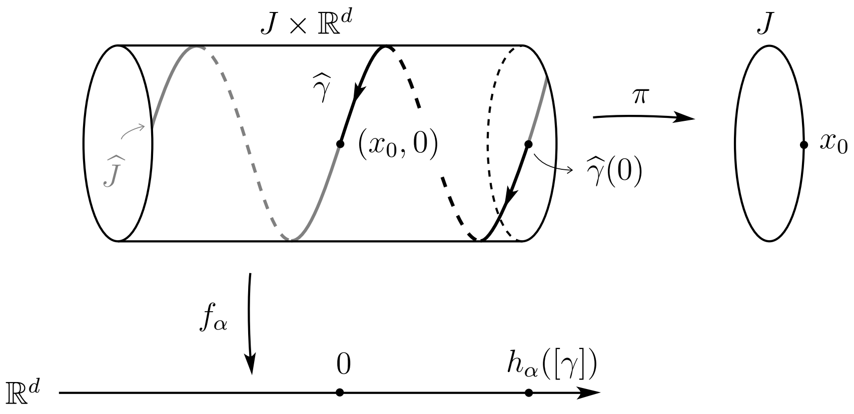


Figure 3.2: Lift of γ to the Darboux cover

where we identify $T_g\mathbb{R} \equiv \mathbb{R}^d$. Then η is flat, so it determines a foliation $\mathcal{F}_\eta$ on $J \times \mathbb{R}^d$. Choose $x_0 \in J$ and take $\widehat{J} \in \mathcal{F}_\eta$ the leaf containing $(x_0, 0)$. Then $\pi := \mathrm{pr}_1|_{\widehat{J}} : \widehat{J} \to J$ is a covering map and $f_\alpha := \mathrm{pr}_2|_{\widehat{J}} : \widehat{J} \to \mathbb{R}^d$ is a submersion. By Fedida's Theorem [Mol88, Theorem 4.1], the foliation $\pi^*(\mathcal{F}^\curlywedge|_J)$ coincides with the foliation determined by the fibers of f_α, that is, $\mathcal{F}^\curlywedge|_J$ is developable over $\mathbb{R}^d$. Now the action $\cdot$ of $\pi_1(J, x_0)$ on $\widehat{J}$ defines an homomorphism $h_\alpha : \pi_1(J, x_0) \to \mathbb{R}^d$, whose image is[1]

$$H_\alpha = \{g \in \mathbb{R}^d \mid \widehat{J}g = \widehat{J}\}.$$

By definition we have the identity $\hat{x} \cdot [\gamma] = \hat{x}h_\alpha([\gamma])$ for all $\hat{x} \in \widehat{J}$ and all classes $[\gamma] \in \pi_1(J, x_0)$, which shows the h_α-equivariancy of f_α, that is,

$$f_\alpha(\hat{x} \cdot [\gamma]) = f_\alpha(\hat{x}) + h_\alpha([\gamma]).$$

Denote by $\overline{H_\alpha}$ the closure of H_α in $\mathbb{R}^d$. Then $\mathbb{R}^d/\overline{H_\alpha}$ is a Lie group and, by its h_α-equivariancy, f_α induces a submersion $\overline{f_\alpha} : J \to \mathbb{R}^d/\overline{H_\alpha}$ whose fibers are the closures of the leaves of $\mathcal{F}^\curlywedge|_J$. In our case specifically $\mathcal{F}^\curlywedge|_J$ has dense leaves, so $\overline{H_\alpha} = \mathbb{R}^d$ and $\overline{\mathcal{F}^\curlywedge|_J}$ is the trivial foliation whose only leaf is J.

Let $\gamma : [0, 1] \to J$ be a smooth path with $\gamma(1) = x_0$ and let $\hat{\gamma}$ be the unique lifting of γ to $\widehat{J}$ satisfying $\hat{\gamma}(1) = (x_0, 0)$ (see Figure 3.2). Then we may write $\hat{\gamma} = (\gamma, \tau)$, τ being a smooth path in $\mathbb{R}^d$. We have $\tau(1) = 0$ and $\tau(0) = f_\alpha(\hat{\gamma}(0))$. As $\hat{\gamma}([0, 1]) \subset \widehat{J} \in \mathcal{F}_\eta$, it follows that

[1]Juxtaposition denotes the natural action of $\mathbb{R}^d$ on $J \times \mathbb{R}^d$

$$0 = \eta\left(\frac{d\gamma}{dt}, \frac{d\tau}{dt}\right) = \alpha\left(\frac{d\gamma}{dt}\right) + \frac{d\tau}{dt},$$

and so

$$\tau(0) = \int_0^1 \alpha\left(\frac{d\gamma}{dt}\right) dt = \int_{[0,1]} \gamma^*\alpha.$$

In particular, for $[\gamma] \in \pi_1(J, x_0)$, we have

$$\begin{aligned} h_\alpha([\gamma]) &= f_\alpha(\hat{\gamma}(0)) = \int_{[0,1]} \gamma^*\alpha \\ &= \left(\int_0^1 \alpha_1\left(\frac{d\gamma}{dt}\right) dt, \ldots, \int_0^1 \alpha_d\left(\frac{d\gamma}{dt}\right) dt\right). \end{aligned}$$

The homomorphism h_α has Abelian image, so it factors to $\overline{h_\alpha} : H_1(J, \mathbb{Z})/T \to \mathbb{R}^d$, where T is the torsion subgroup of $H_1(J, \mathbb{Z})$. Denoting $\overline{h_\alpha} = (\overline{h_\alpha^1}, \ldots, \overline{h_\alpha^d})$, the equation above shows that $\ell_{\mathrm{dR}}[\alpha_i] = \overline{h_\alpha^i}$, where $\ell_{\mathrm{dR}} : H^1_{\mathrm{dR}}(J) \to H^1(J, \mathbb{R})$ is De Rham's isomorphism.

Take $[\vartheta_1], \ldots, [\vartheta_r]$ a free Abelian basis of $H^1(J, \mathbb{Z}) \subset H^1(J, \mathbb{R})$. Then,

$$\ell_{\mathrm{dR}}([\alpha_i]) = \sum_{j=1}^{r} c_i^j [\vartheta_j],$$

$c_i^j \in \mathbb{R}$. If $\sigma_1, \ldots, \sigma_r$ are smooth loops representing the basis of $H_1(J, \mathbb{Z})/T$ dual to $\big([\vartheta_j]\big)_{j=1}^r$, it follows that

$$H_\alpha = \overline{h_\alpha}\big(H_1(J, \mathbb{Z})/T\big) = \left\langle \big(c_1^j, \ldots, c_d^j\big)_{j=1}^r \right\rangle,$$

because $c_i^j = \int_{\sigma_j} \alpha_i$.

Now for each $i \in \{1, \ldots, r\}$ we choose $[u_i] \in H^1_{\mathrm{dR}}(J)$ such that $[\alpha_i] + [u_i]$ is a rational cohomology class and $[u_i]$ is small enough so that the set $\{\alpha_i + u_i\}_{i=1}^r$ is still linearly independent at every point of J. Then for each $t \in [0, 1]$, $\alpha(t) = (\alpha_1 + tu_1, \ldots, \alpha_d + tu_d)$ is a new Maurer-Cartan form that defines a foliation $\mathcal{G}_t^\curlywedge$ of J. Moreover, because $[\alpha_i] + [u_i]$ are rational, we now have

$$H_{\alpha(1)} = \left\langle \left(\int_{\sigma_j} (\alpha_1 + u_1), \ldots, \int_{\sigma_j} (\alpha_d + u_d)\right)_{j=1}^r \right\rangle$$

with $\int_{\sigma_j}(\alpha_i + u_i) \in \mathbb{Q}$ for all i and j. Then $H_{\alpha(1)}$ is a finitely generated subgroup of $\mathbb{Q}^d$, hence $\overline{H_{\alpha(1)}} = H_{\alpha(1)}$. It follows that $H_{\alpha(1)}$ is a lattice in $\mathbb{R}^d$ and

$$\overline{f_{\alpha(1)}} : J \to \mathbb{R}^d / H_{\alpha(1)} \cong \mathbb{T}^d \tag{3.1}$$

is a fibration defining the closed foliation $\mathcal{G}^\curlywedge := \mathcal{G}_1^\curlywedge$.

As noted in [Ghy84, p. 211], because M is simply-connected it is possible to reduce the structural group of the basic fibration $b : M_\mathcal{F}^\curlywedge \to W$ to the connected component of the identity $\mathrm{Diff}(J, \mathcal{F}^\curlywedge|_J)_0$ of the group of diffeomorphisms of J preserving $\mathcal{F}^\curlywedge|_J$. By [Ghy84, Lemme 3.2], every element in $\mathrm{Diff}(J, \mathcal{F}^\curlywedge|_J)_0$ preserves the forms α_i, so they can be transported without ambiguity to the different fibers of b. Let us denote by $\alpha_{i,w}$ the forms on $b^{-1}(w)$ obtained this way.

The reduction of the structural group of b to $\mathrm{Diff}(J, \mathcal{F}^\curlywedge|_J)_0$ also provides a canonical isomorphism between $H^1_{\mathrm{dR}}(b^{-1}(w))$ and $H^1_{\mathrm{dR}}(J)$. If we denote by $u_{i,w}$ the harmonic form representing the class $[u_i] \in H^1_{\mathrm{dR}}(b^{-1}(w))$, we obtain, for each $t \in [0,1]$ and in each fiber $b^{-1}(w)$, a foliation $(\mathcal{G}_t^\curlywedge)_w$ such that the leaves of $(\mathcal{G}_1^\curlywedge)_w$ are all closed. As $\mathrm{SO}(q)$ acts isometrically on $M_\mathcal{F}^\curlywedge$, the harmonic forms $u_{i,w}$ are $\mathrm{SO}(q)$-invariant. The isometries induced by the $\mathrm{SO}(q)$-action restricted to J are elements of $\mathrm{Diff}_0(J, \mathcal{F}^\curlywedge|_J)$, so the forms $\alpha_{i,w}$ are also preserved. Therefore $\mathcal{G}_t^\curlywedge$ is $\mathrm{SO}(q)$-invariant and, passing to the quotient, we obtain, for each $t \in [0,1]$, a foliation $\mathcal{G}_t$ of M such that $\mathcal{G} := \mathcal{G}_1$ is arbitrarily close to $\mathcal{F}$ and all its leaves are closed.

It is clear that $\mathcal{G}_t$ defines a homotopic deformation between $\mathcal{F}$ and $\mathcal{G}$. Note that, by construction, the deformation respects $\overline{\mathcal{F}}$, that is, the deformation of each leaf $L \in \mathcal{F}$ occurs within its closure $\overline{L}$.

3.1 A Suitable Deformation

The aforementioned theorem by A. Haefliger and E. Salem that we will now use is the following [see HS90, Theorem 3.4].

Theorem 3.3 (Haefliger–Salem). *There is a bijection between the set of equivalence classes of Killing foliations $\mathcal{F}$ with compact leaf closures of M and the set of equiv-*

alence classes[2] *of quadruples* $(\mathcal{O}, \mathbb{T}^N, H, \mu)$ *where* $\mathcal{O}$ *is an orbifold,* μ *is an effective action of* $\mathbb{T}^N$ *on* $\mathcal{O}$ *and* H *is a dense contractible subgroup of* $\mathbb{T}^N$ *whose action on* $\mathcal{O}$ *is locally free. This bijection associates* $\mathcal{F}$ *to a canonical realization* $(\mathcal{O}, \mathcal{F}_H)$ *of the classifying space*[3] *of its holonomy pseudogroup, where* $\mathcal{F}_H$ *is the foliation of* $\mathcal{O}$ *by the orbits of* H*. In particular, there is a smooth map* $\Upsilon : M \to \mathcal{O}$ *such that* $\Upsilon^*(\mathcal{F}_H) = \mathcal{F}$*.*

Remark 3.4. In [HS90, Theorem 3.4] the map $\Upsilon : M \to \mathcal{O}$ is smooth "in the orbifold sense" [HS90, p. 715], so we need to check that it is also a smooth map between orbifolds according to the definition we adopted (see Section A.2), to justify the above statement of the theorem. Indeed, a smooth map "in the orbifold sense" from a manifold M on an orbifold $\mathcal{O}$ is an equivalence class of 1-cocycles defined over an open cover of M with values on the Lie groupoid $\mathrm{G}_{\mathcal{O}}$ of germs of the pseudogroup of changes of charts of $\mathcal{O}$ (see Chapter 6 and also Remark A.8), two cocycles being equivalent if they are restrictions of a third one. This clearly defines a morphism between the groupoids G_M and $\mathrm{G}_{\mathcal{O}}$, which in turn corresponds to a Chen–Ruan good map $\Upsilon : M \to \mathcal{O}$ [see LU04, Proposition 5.1.7] that is, in particular, a smooth map in our sense (see Remark A.3).

If $L \in \mathcal{F}$ is a generic leaf, then $\mathscr{H}_{\mathcal{F}|_{\overline{L}}}$ is equivalent to the pseudogroup generated by a dense subgroup $\Gamma < \mathbb{R}^d$, where $d = \dim(\mathfrak{g}_{\mathcal{F}})$ is the defect of $\mathcal{F}$. In this case $N = \operatorname{rank}(\Gamma)$ and

$$\mathbb{T}^N \cong \frac{\Gamma \otimes \mathbb{R}}{\Gamma \otimes \mathbb{Z}}.$$

Note that, in particular, we must have $N \geq d$.

Assume that M is closed, so $\mathcal{F}$, in particular, has compact leaf closures. Theorem 3.3 can be used to deform $\mathcal{F}$ as follows. Let $\mathfrak{h}$ be the Lie algebra of H and consider a Lie subalgebra $\mathfrak{k} < \operatorname{Lie}(\mathbb{T}^N) \cong \mathbb{R}^N$, with $\dim(\mathfrak{k}) = \dim(\mathfrak{h})$, such that its corresponding Lie subgroup $K < \mathbb{T}^N$ is closed. We can suppose $\mathfrak{k}$ close enough to $\mathfrak{h}$, as points in the Grassmannian $\mathrm{Gr}^{\dim(\mathfrak{h})}(\operatorname{Lie}(\mathbb{T}^N))$, so that the action $\mu|_K$ remains

[2] In this paragraph, two foliations $\mathcal{F}$ and $\mathcal{F}'$ are **equivalent** if their holonomy pseudogroups are equivalent (see Section A.4) and two quadruples $(\mathcal{O}, \mathbb{T}^N, H, \mu)$ and $(\mathcal{O}', \mathbb{T}^{N'}, H', \mu')$ are **equivalent** if there is an isomorphism between $\mathbb{T}^N$ and $\mathbb{T}^{N'}$ and a diffeomorphism of $\mathcal{O}$ onto $\mathcal{O}'$ that conjugates the actions μ and μ'

[3] See Section 6.1, also [Hae86, Section 3].

locally free. Because K is a closed subgroup, the leaves of the foliation $\mathcal{G}_K$ defined by the orbits of K are all closed. Taking $\mathfrak{k}$ even closer to $\mathfrak{h}$ if necessary, we can suppose that Υ remains transverse to $\mathcal{G}_K$, so $\mathcal{G} := \Upsilon^*(\mathcal{G}_K)$ is the desired approximation of $\mathcal{F}$. In this sense, $\mathcal{F}$ can be arbitrarily approximated by such a closed foliation $\mathcal{G}$. Moreover, this allows us to suppose that a submanifold $T \subset M$ that is a total transversal for $\mathcal{F}$ is also a total transversal for $\mathcal{G}$, so that there are representatives of the holonomy pseudogroups $\mathscr{H}_{\mathcal{F}}$ and $\mathscr{H}_{\mathcal{G}}$ both acting on T. The same goes for $\mathscr{H}_{\mathcal{F}_H}$ and $\mathscr{H}_{\mathcal{G}_K}$, they both act on the transversal $S = \Upsilon(T)$.

Theorem 3.3 states that $(\mathcal{O}, \mathcal{F}_H)$ is a realization of the classifying space of the holonomy pseudogroup $(T, \mathscr{H}_{\mathcal{F}})$. Roughly speaking, this classifying space is a space with a foliation such that the holonomy covering of each leaf is contractible and whose holonomy pseudogroup is equivalent to $(T, \mathscr{H}_{\mathcal{F}})$ (see Section 6.1). In particular, $(T, \mathscr{H}_{\mathcal{F}})$ is equivalent to $(S, \mathscr{H}_{\mathcal{F}_H})$, the equivalence being generated by restrictions of Υ to open sets in T where it becomes a diffeomorphism. This induces a correspondence between the $\mathcal{F}$-basic tensor fields on M and the $\mathcal{F}_H$-basic tensor fields on $\mathcal{O}$. Now choose a Riemannian metric on $\mathcal{O}$ with respect to which $\mathbb{T}^N$ acts by isometries and consider the normal bundle $\nu\mathcal{F}_H \subset T\mathcal{O}$, which is well defined because the leaves of $\mathcal{F}_H$ are strong suborbifolds of $\mathcal{O}$ [see GG+17, p. 6]. Since H preserves $T\mathcal{F}_H$ it also preserves $\nu\mathcal{F}_H$. By density, it follows that $\mathbb{T}^N$ (and, in particular, K) preserves $\nu\mathcal{F}_H$.

Let $\xi \in \mathcal{T}^*(\mathcal{F})$ and let ξ_H be the corresponding $\mathcal{F}_H$-basic tensor field on $\mathcal{O}$, that is, ξ_H is H-invariant (hence $\mathbb{T}^N$-invariant), and $\xi_H(X_1, \dots, X_k) = 0$ whenever some $X_i \in T\mathcal{F}_H$. Define a tensor field ξ_K on $\mathcal{O}$ by declaring that $\xi_K = \xi_H$ on $\nu\mathcal{F}_H$ and that ξ_K vanishes on $T\mathcal{G}_K$. Then from what we saw above it follows that ξ_K is K-invariant. Since ξ_K vanishes in $T\mathcal{G}_K$ by construction, it is therefore $\mathcal{G}_K$-basic. Notice that $\xi_H \mapsto \xi_K$ defines an isomorphism between $\mathcal{T}^*(\mathcal{F}_H)$ and the subalgebra $\mathcal{T}^*(\mathcal{G}_K)^{\mathbb{T}^N}$ of $\mathbb{T}^N$-invariant $\mathcal{G}_K$-basic fields. The association $\xi \mapsto \Upsilon^*(\xi_K)$ defines an injection $\iota : \mathcal{T}^*(\mathcal{F}) \hookrightarrow \mathcal{T}^*(\mathcal{G})$.

Since M is complete, $\mathscr{H}_{\mathcal{F}}$ is a complete pseudogroup of local isometries[4] with respect to g^T [see Sal88, Proposition 2.6]. By [Sal88, Proposition 2.3], its closure

[4]In the sense of [Sal88, Definition 2.1].

$\overline{\mathscr{H}_{\mathcal{F}}}$ in the C^1-topology is also a complete pseudogroup of local isometries, whose orbits are the closures of the orbits of $\mathscr{H}_{\mathcal{F}}$, which in turn correspond to leaf closures in $\overline{\mathcal{F}}$. The same goes for the orbits of $\overline{\mathscr{H}_{\mathcal{F}_H}}$: they correspond to the closures of the leaves of $\mathcal{F}_H$, that is, the orbits of $\mathbb{T}^N$. Hence, by the equivalence $\mathscr{H}_{\mathcal{F}} \cong \mathscr{H}_{\mathcal{F}_H}$, we have $M/\overline{\mathcal{F}} \cong \mathcal{O}/\mathbb{T}^N$. It is clear that the $\mathbb{T}^N$-action projects to an action of the torus $\mathbb{T}^N/K \cong \mathbb{T}^{N-\dim(\mathfrak{k})}$ on the orbifold $\mathcal{O}/\!/\mathcal{G}_K$. Therefore

$$\frac{M}{\overline{\mathcal{F}}} \cong \frac{|\mathcal{O}|}{\mathbb{T}^N} \cong \frac{|\mathcal{O}|/\mathcal{G}_K}{\mathbb{T}^{N-\dim(\mathfrak{k})}}.$$

Observe that, since $N = \dim(\overline{\mathcal{F}_H})$, we have $N - \dim(\mathfrak{k}) = N - \dim(\mathcal{F}_H) = \operatorname{codim}(\mathcal{F}) - \operatorname{codim}(\overline{\mathcal{F}}) = d$, where d is the defect of $\mathcal{F}$. On the other hand, $\mathcal{O}/\!/\mathcal{G}_K \cong M/\!/\mathcal{G}$ by construction, so we get a smooth $\mathbb{T}^d$ action on $M/\!/\mathcal{G}$ satisfying[5]

$$\frac{M}{\overline{\mathcal{F}}} \cong \frac{M/\mathcal{G}}{\mathbb{T}^d}.$$

If we denote the canonical projection $M \to M/\!/\mathcal{G}$ by $\pi^{\mathcal{G}}$, we see that $\pi^{\mathcal{G}}_* \circ \iota$ defines an isomorphism between $\mathcal{T}^*(\mathcal{F})$ and $\mathcal{T}^*(M/\!/\mathcal{G})^{\mathbb{T}^d}$. In fact, it suffices to see this for $\mathcal{T}^*(\mathcal{F}_H)$ and $\mathcal{T}^*(\mathcal{O}/\!/\mathcal{G}_K)^{\mathbb{T}^d}$. If $\xi_H \in \mathcal{T}^*(\mathcal{F}_H)$, we saw that ξ_K is $\mathbb{T}^N$-invariant, hence it projects to a $\mathbb{T}^d$-invariant tensor field on $\mathcal{O}/\!/\mathcal{G}_K$. On the other hand, a tensor field in $\mathcal{T}^*(\mathcal{O}/\!/\mathcal{G}_K)^{\mathbb{T}^d}$ lifts to a $\mathcal{G}_K$-basic tensor field ξ_K on $\mathcal{O}$ which is $\mathbb{T}^N$-invariant and, thus, corresponds to some element in $\mathcal{T}^*(\mathcal{F}_H)$. In particular, the $\mathbb{T}^d$-action on $M/\!/\mathcal{G}$ is isometric with respect to $\pi^{\mathcal{G}}_* \circ \iota(\mathrm{g}^T)$.

Of course, we could further consider a smooth path $\mathfrak{h}(t)$, for $t \in [0,1]$, on the Grassmannian $\mathrm{Gr}^{\dim(\mathfrak{h})}(\mathrm{Lie}(\mathbb{T}^N))$ connecting $\mathfrak{h}$ to $\mathfrak{k}$ such that the action $\mu|_{H(t)}$ of each corresponding Lie subgroup $H(t)$ is locally free. Then $\mathcal{F}_t := \Upsilon^*(\mathcal{G}_{H(t)})$ defines a C^∞-homotopic deformation of $\mathcal{F}$ into $\mathcal{G}$. In this case we have an injection $\iota_t : \mathcal{T}^*(\mathcal{F}) \hookrightarrow \mathcal{T}^*(\mathcal{F}_t)$, for each t, such that $\iota_t(\xi)$ is a smooth time-dependent tensor field on M for each $\xi \in \mathcal{T}^*(\mathcal{F})$, that is, $\iota(\xi)$ is smooth as a map $[0,1] \times M \to \bigotimes^* TM$. Since both the deformation $\mathcal{F}_t$ and $\iota_t(\mathrm{g}^T)$ depend smoothly on t, the transverse sectional curvature $\sec_{\mathcal{F}_t}$ with respect to $\iota_t(\mathrm{g}^T)$ is a smooth function on t. We have proved the following.

[5]Compare this with (3.1) in Example 3.2.

Theorem 3.5. *Let* $(\mathcal{F}, \mathrm{g}^T)$ *be a Killing foliation of a closed manifold* M *satisfying* $\sec_{\mathcal{F}} > c$*. Then there is a homotopic deformation of* $\mathcal{F}$ *into a closed Riemannian foliation* $\mathcal{G}$ *which can be chosen arbitrarily close to* $\mathcal{F}$ *and satisfying* $\sec_{\mathcal{G}} > c$*. Moreover, the deformation occurs within the closures of the leaves of* $\mathcal{F}$ *and* $M//\mathcal{G}$ *admits an effective isometric action of a torus* $\mathbb{T}^d$*, where* d *is the defect of* $\mathcal{F}$*, such that* $M/\overline{\mathcal{F}} \cong (M/\mathcal{G})/\mathbb{T}^d$*. In particular,* $\mathrm{symrank}(\mathcal{G}) \geq d$*.*

Remark 3.6. Similarly, if $\sec_{\mathcal{F}} < c$ then we can choose $\mathcal{G}$ satisfying $\sec_{\mathcal{G}} < c$, and analogous properties hold for $\mathrm{Ric}_{\mathcal{F}}$, although we will not use them here.

In the next chapter we will see some applications of this result.

Chapter 4

Direct Applications of the Deformation Technique

The deformation devised in Theorem 3.5 allows us to use results from the Riemannian geometry of orbifolds in the study of the transverse geometry of $\mathcal{F}$, by passing from $\mathcal{F}$ to $M//\mathcal{G}$. In this chapter we will illustrate this idea.

4.1 A Closed Leaf Theorem

Our first application will be a generalization of a theorem by G. Oshikiri [Osh01, Theorem 2] that asserts that an even-codimensional Riemannian foliation on a complete Riemannian manifold M satisfying $\sec_M \geq c > 0$ must have a closed leaf. Our contribution here will be to change the hypothesis on $\sec_M$ to a transverse one, in particular allowing non-compact ambient manifolds M, provided[1] $|\pi_1(M)| < \infty$.

Theorem 4.1. *Let $(\mathcal{F}, \mathrm{g}^T)$ be an even-codimensional complete Riemannian foliation of a manifold M satisfying $|\pi_1(M)| < \infty$. If $\sec_{\mathcal{F}} \geq c > 0$, then $\mathcal{F}$ possesses a closed leaf.*

Proof. Suppose that $\mathcal{F}$ has no closed leaves and consider the lift $\widehat{\mathcal{F}}$ of $\mathcal{F}$ to the universal covering space $\rho : \widehat{M} \to M$ of M. Then, endowed with $\rho^*(\mathrm{g}^T)$, $\widehat{\mathcal{F}}$ is a Killing foliation also satisfying $\sec_{\widehat{\mathcal{F}}} \geq c > 0$. By Proposition 2.7, $\widehat{\mathcal{F}}$ has no closed

[1]Recall that, if $\sec_M \geq c > 0$, one already has $|\pi_1(M)| < \infty$ by the Bonnet–Myers Theorem.

leaves as well. By Theorem 1.13 it follows that $\widehat{M}/\overline{\widehat{\mathcal{F}}}$ is compact, therefore the holonomy pseudogroup $(\widehat{T}, \mathscr{H}_{\widehat{\mathcal{F}}})$ of $\widehat{\mathcal{F}}$ is a complete pseudogroup of local isometries [see Sal88, Proposition 2.6] whose space of orbit closures is compact (as it coincides with $\widehat{M}/\overline{\widehat{\mathcal{F}}}$). Moreover, since there is a surjective homomorphism [for details, see Sal88, Section 1.11]

$$\pi_1\big(\widehat{M}\big) \to \pi_1\big(\mathscr{H}_{\widehat{\mathcal{F}}}\big),$$

we have that $\mathscr{H}_{\widehat{\mathcal{F}}}$ is 1-connected. We can now apply [HS90, Theorem 3.7] to conclude that there exists a Riemannian foliation $\mathcal{F}'$ of a simply-connected compact manifold M' with $(T', \mathscr{H}_{\mathcal{F}'})$ equivalent to $(\widehat{T}, \mathscr{H}_{\widehat{\mathcal{F}}})$. In particular, $\mathcal{F}'$ is Killing and also satisfies $\sec_{\mathcal{F}'} \geq c > 0$, since it is endowed with the transverse metric $(\mathrm{g}^T)'$ on T' induced by $\rho^*(\mathrm{g}^T)$ via the equivalence. Furthermore, $\mathcal{F}'$ has no closed leaves, otherwise $\mathscr{H}_{\mathcal{F}'}$ would have a closed orbit, contradicting $\mathscr{H}_{\mathcal{F}'} \cong \mathscr{H}_{\widehat{\mathcal{F}}}$.

We now apply Theorem 3.5 to $(M', \mathcal{F}')$, deforming it into an even-codimensional closed Riemannian foliation $\mathcal{G}'$ with $\sec_{\mathcal{G}'} > 0$. Since M' is simply-connected we have that $\mathcal{G}'$ is transversely orientable, hence $M'/\!/\mathcal{G}'$ is orientable. Fixing an orientation for it, we have an even-dimensional, compact, oriented Riemannian orbifold $M'/\!/\mathcal{G}'$ with positive sectional curvature admitting an effective isometric action of a torus $\mathbb{T}^d$, where $d = \dim(\mathfrak{g}_{\mathcal{F}'}) > 0$. Moreover, this action has no fixed points, since $\mathcal{F}'$ has no closed leaves and $M'/\overline{\mathcal{F}'} = (M'/\mathcal{G}')/\mathbb{T}^d$. It is possible to choose a 1-parameter subgroup $\mathbb{S}^1 < \mathbb{T}^d$ such that $(M'/\!/\mathcal{G}')^{\mathbb{S}^1} = (M'/\!/\mathcal{G}')^T$ [see AP93, Lemma 4.2.1], that is, without fixed points. In particular, this gives us an isometry of $M'/\!/\mathcal{G}'$ that has no fixed points, which contradicts the Synge–Weinstein Theorem for orbifolds (Theorem A.14). □

The hypothesis $|\pi_1(M)| < \infty$ cannot be removed, as the following example, also due to Oshikiri, shows [see Osh01, Example on p. 530].

Example 4.2. Consider $(\mathbb{S}^2, \mathring{\mathrm{g}})$, the sphere with its standard round metric, and $\Sigma_2 := \mathbb{T}^2 \# \mathbb{T}^2$. Present $\pi_1(\Sigma_2)$ as

$$\pi_1(\Sigma_2) = \big\langle a_1, a_2, b_1, b_2 \;\big|\; a_1 a_2 a_1^{-1} a_2^{-1} b_1 b_2 b_1^{-1} b_2^{-1} \big\rangle$$

and choose $\phi, \psi \in \mathrm{SO}(3)$ two axial-independent rotations of irrational angles α and β, respectively. Now consider the homomorphism

$$h : \pi_1(\Sigma_2) \to \mathrm{SO}(3) = \mathrm{Iso}(\mathbb{S}^2)$$

defined by $h(a_1) = \phi$, $h(a_2) = \psi$ and $h(b_1) = h(b_2) = \mathrm{Id}$. The Riemannian foliation $\mathcal{F}$ defined by the suspension of h (see Examples 1.3, 1.4 and 1.10) clearly has positive transverse sectional curvature, since its transverse metric coincides with $\mathring{\mathrm{g}}$ (when we identify $\mathbb{S}^2$ with a total transversal), and every leaf of $\mathcal{F}$ is dense, since the orbits of $\mathscr{H}_{\mathcal{F}} \cong h(\pi_1(B, x_0))$ are dense in $\mathbb{S}^2$ because α and β are irrational.

4.2 Positive Transverse Curvature and the Defect

In the 1990s, K. Grove proposed that a classification of Riemannian manifolds with positive sectional curvature and large isometry group should be pursued. Later, together with C. Searle, they introduced the symmetry rank (see Section 2.1) as a way to interpret what it is meant by a "large" isometry group and, with this notion, they obtained in [GS94] that $\mathrm{symrank}(M) \leq \lfloor (\dim(M) + 1)/2 \rfloor$ and that equality holds only if M is diffeomorphic to either a sphere, a real or complex projective space, or a lens space, provided that M is a closed Riemannian manifold with positive sectional curvature. A generalization of the Grove–Searle classification for Alexandrov spaces was obtained recently in [HS17]. The following is the respective corollary for orbifolds [HS17, Corollary E].

Theorem 4.3 (Harvey–Searle)**.** *Let $\mathcal{O}$ be an n-dimensional, closed Riemannian orbifold with positive sectional curvature admitting an isometric, effective $\mathbb{T}^k$-action. Then $k \leq \lfloor (n+1)/2 \rfloor$ and, in case of equality, $|\mathcal{O}|$ is homeomorphic to either $\mathbb{S}^n/\Lambda$, where Λ is a finite subgroup of the centralizer of the maximal torus in $\mathrm{O}(n+1)$, or, only in the case that n is even, to $|\mathbb{CP}^{n/2}[\lambda]|/\Lambda$, where Λ is a finite subgroup of the linearly acting torus.*

By combining this result with Theorem 3.5 we obtain a similar classification, up to deformations, of the leaf spaces of Killing foliations with maximal defect and positive transverse curvature.

Theorem 4.4. *Let $\mathcal{F}$ be a q-codimensional, transversely orientable Killing foliation of a closed manifold M. If $\sec_{\mathcal{F}} > 0$ then*

$$\operatorname{codim}(\overline{F}) \geq \left\lceil \frac{\operatorname{codim}(\mathcal{F}) - 1}{2} \right\rceil$$

and, if equality holds, there is a homotopic deformation of $\mathcal{F}$ into a closed Riemannian foliation $\mathcal{G}$ such that $M/\mathcal{G}$ is homeomorphic to either

(i) $\mathbb{S}^q/\Lambda$, *where Λ is a finite subgroup of the centralizer of the maximal torus in* $\mathrm{O}(q+1)$, *or*

(ii) $|\mathbb{CP}^{q/2}[\lambda]|/\Lambda$, *where Λ is a finite subgroup of the linearly acting torus (this case occurs only when q is even).*

Proof. Let $\mathcal{G}$ be a closed foliation, given by Theorem 3.5, satisfying $\sec_{\mathcal{G}} > 0$. Then the leaf space of $\mathcal{G}$ is a q-dimensional Riemannian orbifold $M/\!/\mathcal{G}$ whose sectional curvature is bounded below by c. By the Bonnet–Myers Theorem for orbifolds (Theorem A.12) it follows that $M/\!/\mathcal{G}$ is compact. Since $\mathcal{F}$ is transversely orientable, $M/\!/\mathcal{G}$ is orientable, so it has no topological boundary. Therefore $M/\!/\mathcal{G}$ is closed.

Furthermore, since $M/\!/\mathcal{G}$ admits an isometric action of the torus $\mathbb{T}^d$, it follows directly from Theorem 4.3 that $d \leq \lfloor (q+1)/2 \rfloor$ and, in case of equality, that $M/\mathcal{G}$ is homeomorphic to one of the listed spaces. We can easily rewrite $d \leq \lfloor (q+1)/2 \rfloor$ as $\operatorname{codim}(\overline{F}) \geq \lceil (\operatorname{codim}(\mathcal{F}) - 1)/2 \rceil$, since $d = \operatorname{codim}(\mathcal{F}) - \operatorname{codim}(\overline{\mathcal{F}})$. □

By passing to the universal cover we obtain a corollary for Riemannian foliations on positively curved manifolds.

Corollary 4.5. *Let $(M, \mathcal{F})$ be a Riemannian foliation. If $\sec_M \geq c > 0$ with respect to a bundle-like metric for $\mathcal{F}$, then*

$$\operatorname{codim}(\overline{F}) \geq \left\lceil \frac{\operatorname{codim}(\mathcal{F}) - 1}{2} \right\rceil$$

and, if equality holds, then the universal cover $\widehat{M}$ of M fibers over $\mathbb{S}^n$ or $|\mathbb{CP}^{n/2}[\lambda]|$, *meaning that it admits a closed foliation $\mathcal{G}$ such that $\widehat{M}/\mathcal{G}$ is homeomorphic to one of those spaces.*

Proof. Let $(\widehat{M}, \widehat{\mathcal{F}})$ be the lifted foliation of the universal covering space of M. Since $\sec_M \geq c > 0$ it follows from the Bonnet–Myers Theorem that M is compact and $|\pi_1(M)| < \infty$. Hence, using Proposition 2.7 and Theorem 4.4, we obtain that

$$\operatorname{codim}(\overline{\mathcal{F}}) = \operatorname{codim}(\overline{\widehat{\mathcal{F}}}) \geq \left\lceil \frac{\operatorname{codim}(\widehat{\mathcal{F}}) - 1}{2} \right\rceil = \left\lceil \frac{\operatorname{codim}(\mathcal{F}) - 1}{2} \right\rceil,$$

since $\widehat{\mathcal{F}}$ is a Killing foliation and $\sec_{\widehat{\mathcal{F}}} \geq \sec_{\widehat{M}}$ by O'Neill's equation.

In case of equality, let $\mathcal{G}$ be the closed foliation of $\widehat{M}$ given by Theorem 4.4. The fact that $\widehat{M}$ is simply-connected guarantees[2] that $\pi_1^{\mathrm{orb}}(\widehat{M}/\!/\mathcal{G}) = 0$, thus excluding the possibility that $\widehat{M}/\!/\mathcal{G}$ is a quotient of $\mathbb{S}^n$ or $\mathbb{CP}^{n/2}[\lambda]$ by a (non trivial) finite group Λ. □

4.3 Large Defect and Isolated Closed Leaves

In [GT16, Proposition 8.1] it is shown[3] that if a Killing foliation $\mathcal{F}$ of codimension q admits a closed leaf, then $2d \leq q$, where d is the defect of $\mathcal{F}$. When q is even this is in line with what we have obtained above under the hypothesis $\sec_{\mathcal{F}} \geq c > 0$ (see also Theorem 4.1). If q is odd our defect bound becomes $\lfloor (q+1)/2 \rfloor = (q+1)/2$, so if there exists such an odd-codimensional $\mathcal{F}$ satisfying $2d = q+1$ (and the hypotheses in Theorem 4.4), it cannot have closed leaves. Moreover, [GT16, Proposition 8.1] also states that if there is an *isolated* closed leaf, then q is even. When $q = 2d$ we have the following partial converse.

Proposition 4.6. *Let $\mathcal{F}$ a Killing foliation of a closed manifold M. If* $\operatorname{codim}(\mathcal{F}) = 2\operatorname{codim}(\overline{\mathcal{F}})$*, then $\mathcal{F}$ has only finitely many*[4] *(hence isolated) closed leaves.*

Proof. Let d be the defect of $\mathcal{F}$ and denote $q = \operatorname{codim}(\mathcal{F})$, so $\operatorname{codim}(\mathcal{F}) = 2\operatorname{codim}(\overline{\mathcal{F}})$ becomes $q = 2d$. If $\mathcal{F}$ has no closed leaves there is nothing to prove, so suppose that

[2]See Section 6.1.

[3]The authors assume that $\mathcal{F}$ is transversely oriented, but this hypothesis can be removed using Corollary 2.8.

[4]Note that this does not exclude the case where $\mathcal{F}$ has no closed leaves at all.

$\Sigma^{\dim(\mathcal{F})}$ is non-empty. To prove that the closed leaves must be isolated, consider a closed foliation $\mathcal{G}$ of M given by Theorem 3.5. Note that, in view of Proposition 2.7, we may suppose without loss of generality that $\mathcal{F}$ and $\mathcal{G}$ are transversely orientable. Then, $\mathcal{O} := M//\mathcal{G}$ is a q-dimensional, closed, orientable orbifold admitting a $\mathbb{T}^d$-action such that $M/\overline{\mathcal{F}} \cong |\mathcal{O}|/\mathbb{T}^d$. In particular, the fixed-point set $|\mathcal{O}|^{\mathbb{T}^d}$ is non-empty, so $(\mathcal{O}, \mathbb{T}^d)$ is a torus orbifold. It follows from Proposition A.11 that $|\mathcal{O}|^{\mathbb{T}^d}$ consists of finitely many isolated points. Since the closed leaves of $\mathcal{F}$ correspond to the inverse image of the points in $|\mathcal{O}|^{\mathbb{T}^d}$ by the quotient projection, the result follows. □

Corollary 4.7. *Let $\mathcal{F}$ be a Riemannian foliation of a closed manifold M satisfying $|\pi_1(M)| < \infty$. If $\operatorname{codim}(\mathcal{F}) = 2\operatorname{codim}(\overline{\mathcal{F}})$, then $\mathcal{F}$ has only finitely many closed leaves.*

Proof. The lift $\widehat{\mathcal{F}}$ of $\mathcal{F}$ to the universal covering space of M is a Killing foliation and satisfies $\operatorname{codim}(\widehat{\mathcal{F}}) = \operatorname{codim}(\mathcal{F}) = 2\operatorname{codim}(\overline{\mathcal{F}}) = 2\operatorname{codim}(\overline{\widehat{\mathcal{F}}})$, by Corollary 2.8. Therefore $\widehat{\mathcal{F}}$ (and, consequently, $\mathcal{F}$) has finitely many closed leaves. □

Chapter 5

Transverse Symmetries and the Basic Euler Characteristic

A well know conjecture by H. Hopf states that any even-dimensional, compact Riemannian manifold M with positive sectional curvature must have positive Euler characteristic. In dimension 2, for example, provided that the Ricci curvature is bounded below by a positive constant, it follows that $|\pi(M)| < \infty$ and, thus, that the conjecture holds, for this implies $H_1(M, \mathbb{R}) = 0$ by the Hurewicz Theorem. In dimension 4, passing to the orientable double cover $\widehat{M}$ and using Poincaré duality, we obtain $H_3(\widehat{M}, \mathbb{R}) \cong H_1(\widehat{M}, \mathbb{R}) = 0$ and, since $\chi(\widehat{M}) = 2\chi(M)$, we see that Hopf conjecture also holds. For dimensions larger than 4 the full conjecture is still an open problem. It holds, however, when the symmetry rank of M is sufficiently large. For example, if $\dim(M) = 6$ and M admits a non-trivial Killing vector field X, then, by the lower-dimensional cases, $0 < \chi(\mathrm{Zero}(X)) = \chi(M)$. This last equality comes from a theorem by P. Conner [see Con57], which also states that $\sum b_{2i}(M) \geq \sum b_{2i}(\mathrm{Zero}(X))$ and $\sum b_{2i+1}(M) \geq \sum b_{2i+1}(\mathrm{Zero}(X))$, where $b_i(M)$ is the ith Betti number of M. For dimensions bigger than 6, there is the following result by T. Püttmann and C. Searle [see PS01, Theorem 2; also Pet06, Theorem 44].

Theorem 5.1 (Püttmann–Searle). *Let M be an n-dimensional complete Riemannian manifold such that $\sec_M \geq c > 0$. If n is even and $\mathrm{symrank}(M) \geq n/4 - 1$, then $\chi(M) > 0$.*

Our goal in this chapter is to obtain a transverse version of this result for Killing foliations. Let us begin by trying to directly generalize the low-codimensional cases from the manifold counterparts that we saw above. Suppose that $\mathcal{F}$ is a 2-codimensional complete Riemannian foliation on M and that $\operatorname{Ric}_{\mathcal{F}} \geq c > 0$. By Theorem 1.13 it follows that $H^1_B(\mathcal{F}) \cong 0$. Furthermore, since $\Omega^0_B(\mathcal{F})$ consists of the basic functions, which are precisely the functions $f : M \to \mathbb{R}$ that are constant on the closures of the leaves, we have $H^0_B(\mathcal{F}) = H^0_{\mathrm{dR}}(M)$, thus

$$\chi_B(\mathcal{F}) = b^0_B(\mathcal{F}) + b^2_B(\mathcal{F}) = \dim(H^0_{\mathrm{dR}}(M)) + b^2_B(\mathcal{F}) > 0.$$

Summing up, we obtain the following.

Proposition 5.2. *Let $\mathcal{F}$ be a 2-codimensional complete Riemannian foliation satisfying $\operatorname{Ric}_{\mathcal{F}} \geq c > 0$. Then $\chi_B(\mathcal{F}) > 0$.*

In order to adapt the proof of the Hopf conjecture for 4-manifolds to the case of 4-codimensional foliations we need Poincaré duality to hold for the basic cohomology complex. As discussed in Section 1.6, this happens when M is compact and $\mathcal{F}$ is taut and transversely oriented. On the other hand, Theorem 1.14 characterizes tautness by the vanishing of the mean curvature class $[\kappa_{\mathcal{F}}] \in H^1_B(\mathcal{F})$, and we know that $H^1_B(\mathcal{F}) \cong 0$ when $\operatorname{Ric}_{\mathcal{F}} \geq c > 0$. That is, we have the following [see Lóp92, Corollary 6.5].

Lemma 5.3. *Let $\mathcal{F}$ be a transversely oriented Riemannian foliation of a closed manifold such that $\operatorname{Ric}_{\mathcal{F}} > 0$. Then $H^i_B(\mathcal{F}) \cong H^{q-i}_B(\mathcal{F})$.*

This gives us a sufficient condition for the adaptation of the 4-dimensional case.

Proposition 5.4. *Let $\mathcal{F}$ be a 4-codimensional transversely orientable Riemannian foliation of a closed manifold M satisfying $\operatorname{Ric}_{\mathcal{F}} > 0$. Then $\chi_B(\mathcal{F}) \geq 2$.*

Proof. Since $H^1_B(\mathcal{F}) \cong 0$ by Theorem 1.13 and $H^1_B(\mathcal{F}) \cong H^3_B(\mathcal{F})$ by Lemma 5.3, we have

$$\chi_B(\mathcal{F}) = 1 + b^2_B(\mathcal{F}) + 1 \geq 2.$$ □

5.1 Localization of the Basic Euler Characteristic

We will prove in this section that the basic Euler characteristic of a Killing foliation $(M, \mathcal{F})$ localizes to the zero set of any transverse Killing vector field and hence, in particular, to the stratum of closed leaves.

A useful tool to us will be a basic version of Hopf's index theorem, due to V. Belfi, E. Park and K. Richardson, that appears in [BPR03]. For the reader's convenience we will briefly introduce this result here. Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold M endowed with a bundle-like metric g for $\mathcal{F}$ and let $X \in \mathfrak{L}(\mathcal{F})$. If X is tangent to a leaf $L \in \mathcal{F}$ at some point then it is tangent to L at every point, because it is foliate. Thus, by continuity, it must be tangent to any leaf in $J := \overline{L}$. In other words, X is tangent to L if, and only if, $\overline{X} = 0$ on J. Such a leaf closure is called a **critical leaf closure** of X, in the terminology of [BPR03, p. 325], and plays the analogous role of a critical point in the classical Hopf index theorem. The field X is called **$\mathcal{F}$-non-degenerate** if for every critical leaf closure J of X and each $x \in J$, the linear map

$$\begin{aligned} X_{J,x} : T_xJ^{\perp} &\longrightarrow T_xJ^{\perp} \\ Y &\longmapsto [X, \widetilde{Y}]^{\perp}_x, \end{aligned}$$

where $\widetilde{Y}$ is any vector field satisfying $\widetilde{Y}_x = Y$, is an isomorphism. This property ensures that the critical leaf closures of X are isolated and that the determinant of $X_{J,x}$ is either positive or negative everywhere on J [see BPR03, p. 325]. The **index** of X at J is then well defined as

$$\operatorname{ind}_J(X) = \begin{cases} 1 & \text{if } \det(X_{J,x}) \text{ is positive on } J, \\ -1 & \text{if } \det(X_{J,x}) \text{ is negative on } J. \end{cases}$$

As the classical critical point is replaced here by a critical leaf closure, a basic index formula must take topological data of the critical leaf closures into account. Also, orientation issues are circumvented by considering a twisted De Rham complex, as follows. If $X_{J,x} = P_x\Theta_x$ is the polar decomposition[1] of $X_{J,x}$, the **orientation line**

[1] That is, $P_x = \sqrt{X^*_{J,x}X_{J,x}}$ is symmetric and positive, and $\Theta_x = P_x^{-1}X_{J,x}$ is an isometry.

bundle $\mathrm{Or}_J(X)$ of X at J is the orientation bundle of the subbundle of $TJ^\perp$ given by the eigenspaces of Θ_x corresponding to the eigenvector -1. Consider the space $\Omega^*(J, \mathrm{Or}_J(X))$ of differential forms on J with values in $\mathrm{Or}_J(X)$. There is a well-defined differential d on this space [see, for example, BT82, Chapter I, §7] and it is possible to make sense of interior multiplication $i(X)$ on $\Omega^*(J, \mathrm{Or}_J(X))$ by vector fields $X \in \mathfrak{X}(J)$ [see BPR03, p. 337 for details]. The space $\Omega^*_B(J, \mathcal{F}, \mathrm{Or}_J(X))$ of **basic differential forms with values in** $\mathrm{Or}_J(X)$ may be defined now as the subcomplex of forms ω in $\Omega^*(J, \mathrm{Or}_J(X))$ satisfying $i(X)\omega = 0$ and $i(X)d\omega = 0$ for any $X \in \mathfrak{X}(\mathcal{F}|_J)$.

The relevant topological data of the critical leaf closures is contained in the cohomology groups $H^*_B(J, \mathcal{F}, \mathrm{Or}_J(X))$ of the complex $\Omega^*_B(J, \mathcal{F}, \mathrm{Or}_J(X))$, particularly in the **basic Euler characteristic with values in** $\mathrm{Or}_J(X)$, defined as

$$\chi_B(J, \mathcal{F}, \mathrm{Or}_J(X)) := \sum_k (-1)^k \dim \left[H^k_B(J, \mathcal{F}, \mathrm{Or}_J(X))\right].$$

Example 5.5. Simple examples of $\mathcal{F}$-non-degenerate foliate vector fields are given by gradients of basic Bott–Morse functions. Those are basic functions $f \in \Omega^0_B(\mathcal{F})$ that descend to non-degenerate $\mathscr{H}_{\mathcal{F}}$-invariant functions on $T_{\mathcal{F}}$, as defined in [Lóp93, p. 1]. We are assuming that M is compact, thus $M/\mathcal{F}$ is compact and it follows by [Lóp93, Theorem 4.1] that $\mathcal{F}$ always admits basic Bott–Morse functions, hence $\mathcal{F}$-non-degenerate foliate vector fields always exist.

If $X = \operatorname{grad} f$ then $X_{J,x}$ is simply $(\operatorname{Hess} f_x|_{T_x J^\perp})^\sharp$, that is, $X_{J,x}(Y)$ is the vector satisfying $\operatorname{Hess} f_x(Y, Z) = \mathrm{g}_x(X_{J,x}(Y), Z)$ for all $Z \in T_x J^\perp$. Also, in this case $\mathrm{Or}_J(X)$ is the orientation bundle of the subbundle of negative directions of f.

As mentioned above, we are interested in the following result [see BPR03, Theorem 3.18].

Theorem 5.6 (Basic Hopf Index Theorem)**.** *Let $\mathcal{F}$ be a Riemannian foliation of a closed manifold M. If $X \in \mathfrak{L}(\mathcal{F})$ is $\mathcal{F}$-non-degenerate, then*

$$\chi_B(\mathcal{F}) = \sum_J \mathrm{ind}_J(X) \chi_B(J, \mathcal{F}, \mathrm{Or}_J(X)),$$

the sum ranging over all critical leaf closures J of $\mathcal{F}$.

We can now proceed to the localization theorem for the basic Euler characteristic.

Theorem 5.7. *Let $\mathcal{F}$ be a Riemannian foliation of a closed manifold M. If $\overline{X} \in \mathfrak{iso}(\mathcal{F})$, then*

$$\chi_B(\mathcal{F}) = \chi_B(\mathcal{F}|_{\mathrm{Zero}(\overline{X})}).$$

Proof. We are going to construct a suitable vector field $W \in \mathfrak{L}(\mathcal{F})$ and apply Theorem 5.6. Let $N_1, \dots, N_k$ be the connected components of $\mathrm{Zero}(\overline{X})$. Recall from Proposition 2.1 that these components are embedded submanifolds of M. For each $i \in \{1, \dots, k\}$, choose $f_i \in \Omega^0_B(\mathcal{F}|_{N_i})$ a basic Bott–Morse function and $\mathrm{Tub}_\varepsilon(N_i)$ a saturated tubular neighborhood of N_i of radius $\varepsilon > 0$ with orthogonal projection $\pi_i : \mathrm{Tub}_\varepsilon(N_i) \to N_i$. Assume ε sufficiently small so that the tubular neighborhoods are pairwise disjoint.

By Lemma 2.3 each π_i is a foliate map, so we have that

$$\widetilde{f_i} := f_i \circ \pi_i : \mathrm{Tub}_{\varepsilon/2}(N_i) \longrightarrow \mathbb{R}$$

is a basic function. Consider ϕ_i a basic bump function for $\mathrm{Tub}_{\varepsilon/4}(N_i)$ satisfying $\mathrm{supp}(\phi_i) \subset \mathrm{Tub}_{\varepsilon/2}(N_i)$ and define $Y_i := \phi_i \,\mathrm{grad}(\widetilde{f_i})$. Then Y_i is foliate and if v is a vector in the vertical bundle of π_i then

$$\mathrm{g}(Y_i, v) = \mathrm{g}(\phi_i \,\mathrm{grad}(\widetilde{f_i}), v) = \phi_i \mathrm{d}(f_i \circ \pi_i)v = \phi_i \mathrm{d}f_i(\mathrm{d}\pi_i v) = 0,$$

hence Y_i is π_i-horizontal.

Now let $\{\varphi, \psi\}$ be a basic partition of unity subordinate to the saturated open cover

$$M = \left[\bigsqcup_i \mathrm{Tub}_\varepsilon(N_i)\right] \cup \left[M \Big\backslash \overline{\bigsqcup_i \mathrm{Tub}_{\varepsilon/2}(N_i)}\right]$$

and define $Z_i := \varphi \,\mathrm{grad}(d^2_{N_i})$, where $d_{N_i} : M \to \mathbb{R}$ is the distance function to N_i. Then clearly Z_i is π_i-vertical and $\mathrm{supp}(Z_i) \subset \mathrm{Tub}_{\varepsilon/2}(N_i)$. Moreover, note that at each instant t_0 its flow $\Phi^{Z_i}_{t_0}$ restricted to a tube $\mathrm{S}^{N_i}_{\varepsilon'}$ is a homothetic transformation. Therefore, by Lemma 2.3, we conclude that $\Phi^{Z_i}_{t_0}$ preserves $\mathcal{F}$, hence Z_i is foliate.

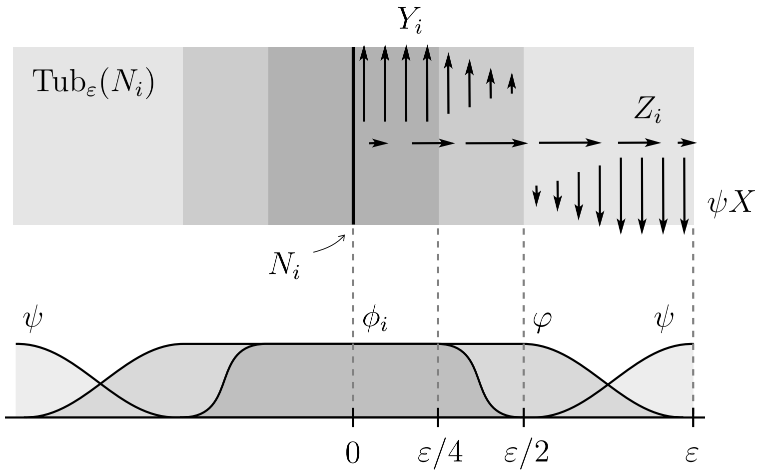


Figure 5.1: Construction of the vector field W

The vector field that we are interested in is

$$W := \psi X + \sum_{i=1}^{k} (Y_i + Z_i),$$

where $X \in \mathfrak{L}(\mathcal{F})$ is a fixed representative for $\overline{X}$ and it is understood that Y_i and Z_i are extended by zero outside their original domains (see Figure 5.1). On $M \setminus \mathrm{Tub}_\varepsilon(\mathrm{Zero}(\overline{X}))$ we have $\overline{W} = \overline{X} \neq 0$. It is easy to prove, by passing to local quotients, that X is π_i-horizontal on each $\mathrm{Tub}_\varepsilon(N_i)$. Therefore, on $\mathrm{Tub}_\varepsilon(N_i) \setminus \mathrm{Tub}_{\varepsilon/2}(N_i)$ we have $\overline{W} = \psi\overline{X} + \overline{Z}_i \neq 0$, because Z_i is π-vertical. Thus, the critical leaf closures of W must be within $\mathrm{Tub}_{\varepsilon/2}(N_i)$, where we have $W = Y_i + Z_i$. Since Y_i is π-horizontal and $\mathrm{Zero}(Z_i|_{\mathrm{Tub}_{\varepsilon/2}(N_i)}) = N_i$, we conclude that the critical leaf closures of $W|_{\mathrm{Tub}_\varepsilon(N_i)}$ coincide with those of Y_i (and are, therefore, within N_i).

Let $J = \overline{L}$ be one of those critical leaf closures of $W|_{\mathrm{Tub}_\varepsilon(N_i)}$ and let $(x_1, \dots, x_p)$ be normal coordinates on a neighborhood $V \subset L$. Now choose an orthonormal frame

$$E = (E_1, \dots, E_r, E_{r+1}, \dots, E_{\overline{q}}, E_{\overline{q}+1}, \dots, E_q)$$

for $TL^\perp$ on V such that $(E_1, \dots, E_r)$ is an orthonormal frame for $TJ^\perp \cap TN_i$ and $(E_{r+1}, \dots, E_{\overline{q}})$ is an orthonormal frame for $TN_i^\perp$. In particular, $(E_1, \dots, E_{\overline{q}})$ forms an orthonormal frame for $TJ^\perp$ (see Figure 5.2). Applying $\exp_L^\perp$ we get coordinates

$$(x_1, \dots, x_p, y_1, \dots, y_r, y_{r+1}, \dots, y_{\overline{q}}, y_{\overline{q}+1}, \dots, y_q)$$

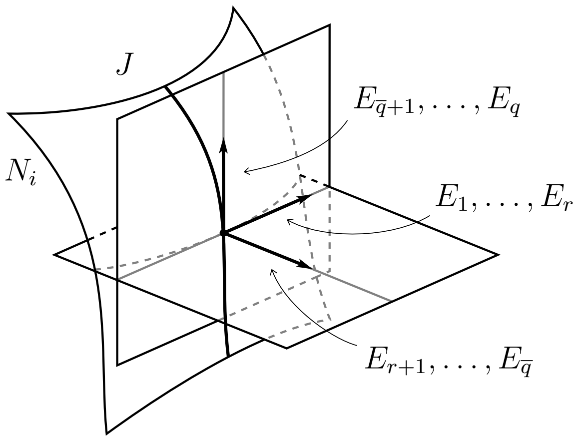


Figure 5.2: The frame E

for a tubular neighborhood $\mathrm{Tub}_\delta(V)$ such that $(x_1, \dots, x_p, y_1, \dots, y_r, y_{\overline{q}+1}, \dots, y_q)$ are local coordinates for N_i. Choose $\delta < \varepsilon/4$ so that $\phi_i|_{\mathrm{Tub}_\delta(V)} \equiv 1 \equiv \varphi|_{\mathrm{Tub}_\delta(V)}$. The expression of $W|_{\mathrm{Tub}_\delta(V)} = \mathrm{grad}(\widetilde{f}_i) + \mathrm{grad}(d^2_{N_i})$ in those coordinates is simply

$$W = \sum_{j=1}^{r} \frac{\partial \widetilde{f}_i}{\partial y_j} \frac{\partial}{\partial y_j} + \sum_{j=r+1}^{\overline{q}} 2y_j \frac{\partial}{\partial y_j}$$

and at each $x \in V$ we have the matrix representation[2]

$$W_{J,x} \equiv \left[\begin{array}{ccc|ccc} \frac{\partial^2 f_i}{\partial y_1 \partial y_1}(x) & \cdots & \frac{\partial^2 f_i}{\partial y_1 \partial y_r}(x) & & & \\ \vdots & \ddots & \vdots & & 0 & \\ \frac{\partial^2 f_i}{\partial y_r \partial y_1}(x) & \cdots & \frac{\partial^2 f_i}{\partial y_r \partial y_r}(x) & & & \\ \hline & & & 2 & & \\ & 0 & & & \ddots & \\ & & & & & 2 \end{array} \right]$$

on $TJ^\perp$. Therefore W is $\mathcal{F}$-non-degenerate.

On the other hand, consider $(N_i, \mathcal{F}|_{N_i})$ and the restriction $W|_{N_i}$. Using the coordinates $(y_1, \dots, y_r, y_{\overline{q}+1}, \dots, y_q)$ constructed above we get, at the critical leaf

[2]Note that $\dfrac{\partial^2 \widetilde{f}_i}{\partial y_j \partial y_k}(x) = \dfrac{\partial^2 f_i}{\partial y_j \partial y_k}(x)$ for $1 \leq j, k \leq r$.

closure J of $\mathcal{F}|_{N_i}$,

$$(W|_{N_i})_{J,x} \equiv \begin{bmatrix} \dfrac{\partial^2 f_i}{\partial y_1 \partial y_1}(x) & \dots & \dfrac{\partial^2 f_i}{\partial y_1 \partial y_r}(x) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f_i}{\partial y_r \partial y_1}(x) & \dots & \dfrac{\partial^2 f_i}{\partial y_r \partial y_r}(x) \end{bmatrix}.$$

In particular, $\mathrm{ind}_J(W) = \mathrm{ind}_J(W|_{N_i})$. Moreover, this also shows that we can identify $\mathrm{Or}_J(W) \equiv \mathrm{Or}_J(W|_{N_i})$, since the negative directions of $W_{J,x}$ and $(W|_{N_i})_{J,x}$ coincide. Hence $\chi_B(J, \mathcal{F}, \mathrm{Or}_J(W)) = \chi_B(J, \mathcal{F}|_{N_i}, \mathrm{Or}_J(W|_{N_i}))$.

We now apply Theorem 5.6 to $(M, \mathcal{F})$ and to each $(N_i, \mathcal{F}|_{N_i})$, obtaining

$$\begin{aligned} \chi_B(\mathcal{F}) &= \sum_J \mathrm{ind}_J(W)\chi_B(J, \mathcal{F}, \mathrm{Or}_J(W)) \\ &= \sum_i \sum_J \mathrm{ind}_J(W|_{N_i})\chi_B(J, \mathcal{F}, \mathrm{Or}_J(W|_{N_i})) \\ &= \sum_i \chi_B(\mathcal{F}|_{N_i}). \end{aligned}$$

It is clear that basic cohomology is additive under disjoint unions, so the result follows. □

Corollary 5.8. *If $\mathcal{F}$ is a Killing foliation of a closed manifold, then $\chi_B(\mathcal{F}) = \chi(\Sigma^{\dim(\mathcal{F})}/\mathcal{F})$. In particular, if $\mathcal{F}$ has no closed leaves, then $\chi_B(\mathcal{F}) = 0$.*

Proof. By Proposition 2.4 we can choose a transverse Killing vector field $\overline{X} \in \mathfrak{iso}(\mathcal{F})$ such that $\mathrm{Zero}(\overline{X}) = \Sigma^{\dim(\mathcal{F})}$, thus Theorem 5.7 gives us $\chi_B(\mathcal{F}) = \chi_B(\mathcal{F}|_{\Sigma^{\dim(\mathcal{F})}})$. Now Proposition 1.7 yields $\chi_B(\mathcal{F}|_{\Sigma^{\dim(\mathcal{F})}}) = \chi(\Sigma^{\dim(\mathcal{F})}//\mathcal{F})$ and by Stake's isomorphism (Theorem A.9) we have the result. □

The above consequence generalizes [GL13, Corollary 1], where it is obtained from the study of the index of transverse operators, under additional assumptions. Notice that, using the language of transverse actions (see Section 1.8) we can write $\mathcal{F}|_{\Sigma^{\dim(\mathcal{F})}} = \mathcal{F}^{\mathfrak{g}_{\mathcal{F}}}$. The formula in Corollary 5.8 then becomes

$$\chi_B(\mathcal{F}) = \chi_B(\mathcal{F}^{\mathfrak{g}_{\mathcal{F}}}),$$

in analogy with the localization of the classical Euler characteristic to the fixed-point set a torus action [see, for instance, Bre72, Theorem 10.9]. The following result can be seen, thus, as a transverse version of the theorem by P. Conner [Con57] mentioned in the beginning of this chapter[3].

Corollary 5.9. *Let $\mathcal{F}$ be a transversely orientable Killing foliation of a closed manifold M. Then*

(i) $\sum_i b_B^{2i}(\mathcal{F}) \geq \sum_i b_B^{2i}(\mathcal{F}|_{\Sigma^{\dim(\mathcal{F})}})$,

(ii) $\sum_i b_B^{2i+1}(\mathcal{F}) \geq \sum_i b_B^{2i+1}(\mathcal{F}|_{\Sigma^{\dim(\mathcal{F})}})$.

Proof. It is shown in [GT16, Theorem 5.5] that

$$\sum_i b_B^i(\mathcal{F}) \geq \sum_i b_B^i(\mathcal{F}|_{\Sigma^{\dim(\mathcal{F})}}). \tag{5.1}$$

On the other hand, $\chi_B(\mathcal{F}) = \chi(\Sigma^{\dim(\mathcal{F})}/\mathcal{F})$ gives us

$$\sum_i (-1)^i b_B^i(\mathcal{F}) = \sum_i (-1)^i b_B^i(\mathcal{F}|_{\Sigma^{\dim(\mathcal{F})}}). \tag{5.2}$$

By adding (5.1) to (5.2) we get the first item and by subtracting (5.2) from (5.1) we get the second one. □

5.2 The Basic Euler Characteristic Under Deformations

We will show now that the basic Euler characteristic is preserved by the deformation given in Theorem 3.5. This is a somewhat surprising fact, because the basic Betti numbers are not preserved, as the following example suggests.

[3] Conner's Theorem is originally stated for the set of fixed points of a torus action instead of Zero(X). We recover the result for Zero(X) by considering the action of the closure, in Iso(M), of the 1-parameter subgroup generated by the flow of X.

Example 5.10. For $\lambda \in [0,1]$, consider the $\mathbb{R}$ action on $M = \mathbb{S}^3 \times \mathbb{S}^1$ given by

$$(t, ((z_1, z_2), z)) \longmapsto ((e^{2\pi \mathrm{i}(1-\lambda)t} z_1, e^{2\pi \mathrm{i}(1-\lambda)t} z_2), e^{2\pi \mathrm{i}\lambda t} z).$$

Then $\mathbb{R}$ acts locally freely and by isometries for each λ, yielding a Riemannian foliation $\mathcal{F}(\lambda)$. For $\lambda = 0$ this action decomposes into the product of the standard Hopf action[4] of $\mathbb{S}^1$ on $\mathbb{S}^3$ with the trivial action on $\mathbb{S}^1$. On the other hand, for $\lambda = 1$ we get the product of the trivial action on the first factor $\mathbb{S}^3$ with the action of $\mathbb{S}^1$ on itself as the second factor of M.

The deformation $\mathcal{F}(\lambda)$ is of the type given by Theorem 3.5. Indeed, for an irrational λ the generic leaf of $\mathcal{F}(\lambda)$ is contractible, so the Haefliger–Salem construction (see Theorem 3.3) of the classifying space of $\mathscr{H}_{\mathcal{F}(\lambda)}$ is trivial: $M = \mathcal{O}$ and $\mathbb{T}^N = \mathbb{S}^1 \times \mathbb{S}^1$, acting on M by

$$((t_1, t_2), ((z_1, z_2), z)) \longmapsto (t_1 \cdot (z_1, z_2), t_2 z),$$

where $\cdot$ denotes the Hopf action.

Now notice that

$$H_B^*(\mathcal{F}(0)) = H^*(M/\!/\mathcal{F}(0)) = H^*(\mathbb{S}^2 \times \mathbb{S}^1),$$

while

$$H_B^*(\mathcal{F}(1)) = H^*(M/\!/\mathcal{F}(1)) = H^*(\mathbb{S}^3).$$

In particular, $b_B^i(\mathcal{F}(0)) = 1$ for all $i \in \{0, \dots, 3\}$, and $b_B^i(\mathcal{F}(1)) = 1$ for $i = 0, 3$ and vanishes otherwise. However, we do have

$$\chi_B(\mathcal{F}(0)) = 0 = \chi_B(\mathcal{F}(1)).$$

Theorem 5.11. *Let $\mathcal{F}$ be a Killing foliation of a closed manifold M and suppose*

[4] That is, the action given by (A.1) with $\lambda_i \equiv 1$.

that $\mathcal{G}$ is obtained from $\mathcal{F}$ by a deformation as in Theorem 3.5. Then

$$\chi_B(\mathcal{F}) = \chi_B(\mathcal{G}).$$

Proof. It is sufficient to prove this under the assumption that $\mathcal{G}$ is closed. From Theorem 3.5 we have that $M/\overline{\mathcal{F}} = (M/\mathcal{G})/\mathbb{T}^d$. In particular, the fixed-point set $(M/\mathcal{G})^{\mathbb{T}^d}$ corresponds to the closed leaves of $\mathcal{F}$, that is $(M/\mathcal{G})^{\mathbb{T}^d} = \Sigma^{\dim(\mathcal{F})}/\mathcal{F}$. Hence

$$\chi_B(\mathcal{G}) = \chi(M/\mathcal{G}) = \chi\left((M/\mathcal{G})^{\mathbb{T}^d}\right) = \chi(\Sigma^{\dim(\mathcal{F})}/\mathcal{F}) = \chi_B(\mathcal{F}),$$

where the second equality follows from the theory of continuous torus actions [see Bre72, Theorem 10.9][5] and the last one from Theorem 5.7. □

5.3 Foliations Admitting a Killing Vector Field with Large Zero Set

The Grove–Searle classification of positively curved manifolds with maximal symmetry rank mentioned in Section 4.2 is achieved in [GS94] by reducing it to the classification of positively curved manifolds admitting a Killing vector field X such that $\mathrm{Zero}(X)$ has a connected component N with $\mathrm{codim}(N) = 2$ [GS94, Theorem 1.2]. In this section we will study, analogously, the leaf spaces of closed Riemannian foliations that admit a transverse Killing vector field $\overline{X}$ such that $\mathrm{codim}(N) = 2$ for some connected component N of $\mathrm{Zero}(\overline{X})$. We do so because it will be useful later in our pursuit of a transverse version of Theorem 5.1. We begin with the following transverse version of Frankel's Theorem.

Lemma 5.12. *Let $(M, \mathcal{F}, \mathfrak{g}^T)$ be a complete Riemannian foliation with $\sec_{\mathcal{F}} \geq c > 0$ and let N and N' be $\mathcal{F}$-saturated, horizontally totally geodesic, compact submanifolds such that $\mathrm{codim}(\mathcal{F}|_N) + \mathrm{codim}(\mathcal{F}|_{N'}) \geq \mathrm{codim}(\mathcal{F})$. Then $N \cap N' \neq \emptyset$.*

Proof. We will adapt the proof of the classical Frankel's Theorem [Fra61, Theorem 1]. Suppose $N \cap N' = \emptyset$ and choose $\gamma : [0, l] \to M$ a unit speed geodesic, with

[5]Note that both $M/\mathcal{F}$ and $(M/\mathcal{F})/T$ are "finitistic", as defined in [Bre72, p. 133], because M is compact.

respect to some bundle-like metric that induces $\eth^T$, that minimizes the distance between N and N'. Now let $\{\pi_i : U_i \to \overline{U}_i\}$, $i = 1, \ldots, k$, be local trivializations of $\mathcal{F}$ such that $\gamma([0,l]) \subset \bigcup U_i$, $\gamma(0) \in U_1$ and $\gamma(l) \in U_k$. Then $\overline{N} := \pi_1(U_1 \cap N)$ and $\overline{N}' := \pi_k(U_k \cap N')$ are totally geodesic submanifolds of $\overline{U}_1$ and $\overline{U}_k$ (with the metric induced by g^T), respectively, and γ projects to a geodesic $\overline{\gamma}$ on the pseudogroup $(T_{\mathcal{F}}, \mathscr{H}_{\mathcal{F}})$ that is perpendicular to $\overline{N}$ and $\overline{N}'$ at its endpoints.

Consider

$$V := \Big\|_0^l (T_{\overline{\gamma}(0)}\overline{N}),$$

the parallel transport being with respect to the Levi-Civita connection on the local quotients. Then $V \subset (\overline{\gamma}'(l))^{\perp}$ and

$$\begin{aligned} \dim(V) + \dim(T_{\overline{\gamma}'(l)}\overline{N}') &= \operatorname{codim}(\mathcal{F}|_N) + \operatorname{codim}(\mathcal{F}|_{N'}) \\ &\geq \operatorname{codim}(\mathcal{F}) = \dim(T_{\overline{\gamma}'(l)}\overline{U}_k) = \dim(\overline{\gamma}'(l)^{\perp}) + 1, \end{aligned}$$

so $\dim(V \cap T_{\overline{\gamma}'(l)}\overline{N}') \geq 1$. Therefore it is possible for us to choose a parallel field $\overline{X}$ along $\overline{\gamma}$ that is tangent to $\overline{N}$ and $\overline{N}'$ at either end and is normal to $\overline{\gamma}'(t)$ for all $t \in [0,l]$.

Let $V(s,t)$ be a variation yielding $\overline{X}$ as its variational field. Since $\overline{N}$ and $\overline{N}'$ are totally geodesic, it follows that

$$\mathrm{g}^T\left(\overline{\gamma}', \nabla_{\frac{\partial V}{\partial s}} \frac{\partial V}{\partial s}(0,l)\right) - \mathrm{g}^T\left(\overline{\gamma}', \nabla_{\frac{\partial V}{\partial s}} \frac{\partial V}{\partial s}(0,0)\right) = 0.$$

Furthermore, as $\overline{X}$ is parallel, $\left\|\frac{\nabla \overline{X}^{\perp}}{\mathrm{d}t}\right\| = 0$. The second variation formula for the length operator ℓ then reduces to

$$\mathrm{d}^2\ell_{\overline{\gamma}}(\overline{X}, \overline{X}) = -\int_0^l \sec_{\mathcal{F}}(\overline{X}, \overline{\gamma}')\, dt < 0.$$

This means that we can find a perturbation of $\overline{\gamma}$ with shorter length. Since geodesics on the local quotients can be lifted to geodesics of M that are orthogonal to the leaves and have the same length, this contradicts the fact that γ minimizes the distance between N and N'. □

By using Proposition A.15 we see that Frankel's Theorem also holds for orbifolds.

Theorem 5.13. *Let $\mathcal{F}$ be a q-codimensional, closed Riemannian foliation of a closed manifold M. Suppose that q is even,* $\sec_{\mathcal{F}} > 0$ *and* $\overline{X} \in \mathfrak{iso}(\mathcal{F})$ *satisfies* $\operatorname{codim}(N) = 2$ *for some connected component N of* $\operatorname{Zero}(\overline{X})$*. Then $M/\mathcal{F}$ is homeomorphic to the quotient space of either $\mathbb{S}^q$ or $|\mathbb{CP}^{q/2}[\lambda]|$ by the action of a finite group.*

Proof. Denote $\mathcal{O} := M/\!/\mathcal{F}$. Then $\overline{X}$ induces $\overline{X}_{\mathcal{O}} \in \mathfrak{iso}(\mathcal{O})$. Let T be the closure of the subgroup generated by the flow of $\overline{X}_{\mathcal{O}}$ in $\operatorname{Iso}(\mathcal{O})$. It is clear that T is a torus and that $\overline{N} := \pi(N)$ is a connected component of the fixed-point set $\mathcal{O}^T$. Choose a closed 1-parameter subgroup $\mathbb{S}^1 < T$ such that $\mathcal{O}^{\mathbb{S}^1} = \mathcal{O}^T$ [AP93, Lemma 4.2.1]. Then $|\mathcal{O}|$, with the distance function induced by g^T, is a positively curved Alexandrov space admitting fixed-point homogeneous action of $\mathbb{S}^1$, in the terminology of [HS17, Section 6]. By [HS17, Theorem 6.5], there is a unique orbit $\mathbb{S}^1 x$ at maximal distance from $\overline{N}$, the "soul" orbit, and there is an $\mathbb{S}^1$-equivariant homeomorphism

$$|\mathcal{O}| \cong \frac{\nu_x * \mathbb{S}^1}{\mathbb{S}^1_x},$$

where $*$ denotes the join operation, ν_x is the space of normal directions to $\mathbb{S}^1 x$ at x and $\mathbb{S}^1_x$ acts on the left on $\nu_x * \mathbb{S}^1$, the action on ν_x being the isotropy action and the action on $\mathbb{S}^1$ being the inverse action on the right. Notice that ν_x can be identified with $\mathbb{S}^{\operatorname{codim}(\mathbb{S}^1 x)-1}/\Gamma_x$, where $\mathbb{S}^{\operatorname{codim}(\mathbb{S}^1 x)-1}$ is the unit sphere in $T_x(\mathbb{S}^1 x)^{\perp} \subset T_x\mathcal{O}$ and Γ_x the local group of $\mathcal{O}$ at x. By [GG+17, Proposition 2.12 and Corollary 2.13] we can choose an orbifold chart $(\widetilde{U}, \Gamma_x, \phi)$ and an extension

$$0 \longrightarrow \Gamma_x \longrightarrow \widetilde{\mathbb{S}}^1_x \xrightarrow{\rho} \mathbb{S}^1_x \to 0$$

acting on $\widetilde{U}$, let us denote this action by μ, with $\widetilde{U}/\widetilde{\mathbb{S}}^1_x = U/\mathbb{S}^1_x$. We now consider separately the cases when $\mathbb{S}^1_x$ is a finite cyclic group $\mathbb{Z}_r$ and when $\mathbb{S}^1_x = \mathbb{S}^1$.

Suppose that $\mathbb{S}^1_x \cong \mathbb{Z}_r$. Then $\dim(\mathbb{S}^1 x) = 1$, hence $\nu_x \cong \mathbb{S}^{q-2}/\Gamma_x$, and $\widetilde{\mathbb{S}}^1_x$ is finite. Recall that there is an isometry $\mathbb{S}^m * \mathbb{S}^n \cong \mathbb{S}^{m+n+1}$ when we realize $\mathbb{S}^m * \mathbb{S}^n$ via the map

$$\begin{aligned} \mathbb{S}^m \times \mathbb{S}^n \times [0,1] &\longrightarrow \mathbb{S}^{m+n+1} \subset \mathbb{R}^{m+n+2} \\ (s_1, s_2, t) &\longmapsto \left(\cos\left(\tfrac{\pi}{2}t\right) s_1, \sin\left(\tfrac{\pi}{2}t\right) s_2\right). \end{aligned}$$

If we define an isometric action of $\widetilde{\mathbb{S}}^1_x$ on $\mathbb{S}^q \cong \mathbb{S}^{q-2} * \mathbb{S}^1$ via this map by

$$\left(g, \left(\cos\left(\frac{\pi}{2}t\right) s_1, \sin\left(\frac{\pi}{2}t\right) s_2\right)\right) \longmapsto \left(\cos\left(\frac{\pi}{2}t\right) \mathrm{d}(\mu^g)_{\tilde{x}} s_1, \sin\left(\frac{\pi}{2}t\right) s_2 \rho(g)^{-1}\right),$$

we get

$$|\mathcal{O}| \cong \frac{\mathbb{S}^{q-2}/\Gamma_x * \mathbb{S}^1}{\mathbb{Z}_r} \cong \frac{\mathbb{S}^q}{\widetilde{\mathbb{S}}^1_x},$$

which exhibits $|\mathcal{O}|$ as a finite quotient of a sphere.

Now suppose $\mathbb{S}^1_x \cong \mathbb{S}^1$. Then $\dim(\mathbb{S}^1 x) = 0$ and $\nu_x \cong \mathbb{S}^{q-1}/\Gamma_x$, so, similarly, we have

$$|\mathcal{O}| \cong \frac{\mathbb{S}^{q-1}/\Gamma_x * \mathbb{S}^1}{\mathbb{S}^1} \cong \frac{\mathbb{S}^{q-1} * \mathbb{S}^1}{\widetilde{\mathbb{S}}^1_x} \cong \frac{\mathbb{S}^{q+1}}{\widetilde{\mathbb{S}}^1_x}.$$

In this case x is a fixed point of the $\mathbb{S}^1$-action and therefore corresponds to a leaf L of $\mathcal{F}$ where $\overline{X}$ vanishes. We claim that x is an isolated fixed point. Indeed, if this was not the case, since $q = \operatorname{codim}(\mathcal{F})$ is even, the connected component N_L of $\operatorname{Zero}(\overline{X})$ containing L would satisfy $\operatorname{codim}(N_L) \leq q-2$ (see Proposition 2.1), hence

$$\operatorname{codim}(\mathcal{F}|_N) + \operatorname{codim}(\mathcal{F}|_{N_L}) = 2q - \operatorname{codim}(N) - \operatorname{codim}(N_L) \geq q,$$

so, by Lemma 5.12, $N_L \cap N \neq \emptyset$, which translates to $x \in \overline{N}$, absurd. It follows, therefore, that $\widetilde{\mathbb{S}}^1_x$ acts almost freely on the first join factor $\mathbb{S}^{q-1}$ that corresponds to the space of normal directions to $\mathbb{S}^1 x$ and, hence, that its induced action on $\mathbb{S}^{q+1}$ is also almost free.

Let $\mathcal{E}$ be the Riemannian foliation of $\mathbb{S}^{q+1}$ given by the connected components of the orbits of $\widetilde{\mathbb{S}}^1_x$. Notice that $\widetilde{\mathbb{S}}^1_x$ may be disconnected, but the connected component $(\widetilde{\mathbb{S}}^1_x)_0$ of the identity is a circle whose action defines the same foliation $\mathcal{E}$. Thus, denoting by Λ the finite group $\widetilde{\mathbb{S}}^1_x/(\widetilde{\mathbb{S}}^1_x)_0$, it follows that

$$\frac{\mathbb{S}^{q+1}}{\widetilde{\mathbb{S}}^1_x} \cong \frac{\mathbb{S}^{q+1}/(\widetilde{\mathbb{S}}^1_x)_0}{\Lambda} = \frac{\mathbb{S}^{q+1}/\mathcal{E}}{\Lambda},$$

where the action of Λ identifies the points in $\mathbb{S}^{q+1}/\mathcal{E}$ corresponding to the same $\widetilde{\mathbb{S}}^1_x$-orbit. In view of the classification of Riemannian 1-foliations of the sphere [GG88, Theorem 5.4], and since $\mathcal{E}$ is closed, we obtain $|\mathcal{O}| \cong |\mathbb{CP}^{n/2}[\lambda]|/\Lambda$. □

Notice that, since any Riemannian orbifold can be realized as the leaf space of a closed Riemannian foliation (Proposition A.15), Theorem 5.13 yields a classification of the underlying spaces of even-dimensional, positively curved orbifolds admitting a Killing vector field with a 2-codimensional zero set (compare with Theorem 4.3).

5.4 Symmetry Rank and the Basic Euler Characteristic

Recall that we obtained a transverse version of Conner's Theorem in Corollary 5.9. Notice, however, that a complete analogue of this theorem should be stated for the zero set of any transverse Killing vector field, in place of $\Sigma^{\dim(\mathcal{F})}$. Unfortunately, the result in [GT16] that we use in the proof of Corollary 5.9 cannot be adapted to show this for a general Killing foliation. The following version for closed foliations, however, will be useful.

Proposition 5.14. *Let $\mathcal{F}$ be a closed Riemannian foliation of a closed manifold M and let $\overline{X} \in \mathfrak{iso}(\mathcal{F})$. Then*

$$\sum_i b_B^{2i+k}(\mathcal{F}) \geq \sum_i b_B^{2i+k}(\mathcal{F}|_{\mathrm{Zero}(\overline{X})}),$$

for $k = 0, 1$.

Proof. Let us denote $\mathcal{O} := M//\mathcal{F}$. By Proposition 1.7 and by Satake's isomorphism (Theorem A.9) we have

$$b_B^i(\mathcal{F}) = \dim(H^i_{\mathrm{dR}}(\mathcal{O})) = \dim(H^i(|\mathcal{O}|, \mathbb{R})).$$

Furthermore, since $|\mathcal{O}|$ is paracompact and locally contractible, its singular cohomology coincides with its Čech cohomology, so we also have[6]

$$b_B^i(\mathcal{F}) = \operatorname{rank}(\check{H}^i(|\mathcal{O}|, \mathbb{R})).$$

Now consider the closure of the subgroup generated by the flow of the induced Killing vector field $\overline{X}_{\mathcal{O}} \in \mathfrak{iso}(\mathcal{O})$. This subgroup is a torus $T < \operatorname{Iso}(\mathcal{O})$ and $\operatorname{Zero}(\overline{X}_{\mathcal{O}}) = |\mathcal{O}|^T$, the fixed-point set of its action. From the theory of continuous torus actions [see Bre72, Theorem 10.9] it follows that

$$\sum_i \operatorname{rank}(\check{H}^{2i+k}(|\mathcal{O}|, \mathbb{R})) \geq \sum_i \operatorname{rank}(\check{H}^{2i+k}(|\mathcal{O}|^T, \mathbb{R})). \qquad \square$$

We are now in position to prove a transverse analogue of Theorem 5.1 for Killing foliations. We will prove it first for closed foliations and then use Theorem 5.11.

Lemma 5.15. *Let $\mathcal{F}$ be a q-codimensional, transversely orientable, closed Riemannian foliation of a closed manifold M and let $N \in \mathcal{Z}(\mathfrak{a})$, where $\mathfrak{a} < \mathfrak{iso}(\mathcal{F})$ is any Abelian Lie subalgebra such that $\dim(\mathfrak{a}) = \operatorname{symrank}(\mathcal{F})$. If q is even, $\sec_{\mathcal{F}} > 0$ and $\operatorname{symrank}(\mathcal{F}) \geq q/4 - 1$, then $\chi_B(\mathcal{F}|_N) > 0$. In particular, $\chi_B(\mathcal{F}) > 0$.*

Proof. We proceed by induction on q. Notice that $\operatorname{codim}(\mathcal{F}|_N) = \operatorname{codim}(\mathcal{F}) - \operatorname{codim}(N)$ is always even and $\sec_{\mathcal{F}|_N} > 0$, by Proposition 2.1. For $q < 6$ the result follows directly from Propositions 5.2 and 5.4.

For the induction step, take a maximal $N' \in \mathcal{Z}(\mathfrak{a})$ such that $N \subset N'$. We now have two cases.

If $\operatorname{codim}(N') \geq 4$, then $\operatorname{codim}(\mathcal{F}|_{N'}) \leq \operatorname{codim}(\mathcal{F}) - 4$. Therefore, by Proposition 2.2,

$$\begin{aligned} \operatorname{symrank}(\mathcal{F}|_{N'}) &= \dim(\mathfrak{a}|_{N'}) = \dim(\mathfrak{a}) - 1 = \operatorname{symrank}(\mathcal{F}) - 1 \\ &\geq \frac{\operatorname{codim}(\mathcal{F}) - 4}{4} - 1 \geq \frac{\operatorname{codim}(\mathcal{F}|_{N'})}{4} - 1, \end{aligned}$$

so $(N', \mathcal{F}|_{N'})$ satisfy the induction hypothesis and $\chi_B(\mathcal{F}|_N) > 0$, because $N \in \mathcal{Z}(\mathfrak{a}|_{N'})$.

[6] Here we see $\mathbb{R}$ as a constant presheaf on $M/\mathcal{F}$.

If $\operatorname{codim}(N') = 2$, then using Theorem 5.13 we have that

$$M/\mathcal{F} \cong \begin{cases} |\mathbb{CP}^{\frac{q}{2}}[\lambda]|/\Lambda \text{ or} \\ \mathbb{S}^q/\Lambda, \end{cases}$$

where Λ is a finite group in either case. Now, by [see Bre72, Theorem 7.2],

$$\check{H}^*(M/\mathcal{F}, \mathbb{R}) \cong \begin{cases} \check{H}^*(\mathbb{CP}^{\frac{q}{2}}[\lambda], \mathbb{R})^\Lambda \text{ or} \\ \check{H}^*(\mathbb{S}^q, \mathbb{R})^\Lambda. \end{cases}$$

Since all odd Betti numbers of both $\mathbb{CP}^{q/2}[\lambda]$ and $\mathbb{S}^q$ vanish, we obtain $b_{2i+1}(M/\mathcal{F}) = b_B^{2i+1}(\mathcal{F}) = 0$ for all $i > 0$. From Proposition 5.14,

$$0 \geq \sum_i b_B^{2i+1}(\mathcal{F}|_{\mathrm{Zero}(\overline{X})}) \geq \sum_i b_B^{2i+1}(\mathcal{F}|_N),$$

so $b_B^{2i+1}(\mathcal{F}|_N)$ also vanishes for all i. In particular, $\chi_B(\mathcal{F}|_N) > 0$. □

The orbifold analogue of Theorem 5.1 follows immediately.

Corollary 5.16. *Let $\mathcal{O}$ be a compact, orientable n-dimensional Riemannian orbifold such that* $\sec_{\mathcal{O}} > 0$. *If n is even and* $\operatorname{symrank}(\mathcal{O}) \geq n/4 - 1$, *then* $\chi(|\mathcal{O}|) > 0$.

Proof. The frame bundle[7] $\mathcal{O}^{\perp}_{\mathbb{C}}$ of $\mathcal{O}$ is also compact, thus, by Proposition A.15, $\mathcal{O}$ is the leaf space of a closed, transversely orientable, n-codimensional Riemannian foliation $\mathcal{F}$ of a closed manifold, satisfying $\sec_{\mathcal{F}} = \sec_{\mathcal{O}} > 0$. Moreover, the identification $\mathfrak{iso}(\mathcal{O}) \cong \mathfrak{iso}(\mathcal{F})$ gives us that $\operatorname{symrank}(\mathcal{F}) \geq n/4 - 1$, therefore $\mathcal{F}$ satisfies the hypotheses of Lemma 5.15. From Proposition 1.7 and Satake's isomorphism (Theorem A.9) it follows that

$$\chi(|\mathcal{O}|) = \sum_i (-1)^i \dim(H^i_{\mathrm{dR}}(\mathcal{O})) = \chi_B(\mathcal{F}).$$ □

Finally, we reach the main objective of this section.

[7]See p. 101.

Theorem 5.17. *Let $\mathcal{F}$ be an even-codimensional, transversely orientable Killing foliation of a closed manifold M. If* $\sec_{\mathcal{F}} > 0$ *and*

$$\operatorname{codim}(\overline{\mathcal{F}}) \leq \frac{3\operatorname{codim}(\mathcal{F})}{4} + 1,$$

then $\chi_B(\mathcal{F}) > 0$.

Proof. By Theorem 3.5, we can deform $\mathcal{F}$ into a closed Riemannian foliation $\mathcal{G}$ such that $\sec_{\mathcal{G}} > 0$ and $\operatorname{symrank}(\mathcal{G}) \geq d = \operatorname{codim}(\mathcal{F}) - \operatorname{codim}(\overline{\mathcal{F}})$. Notice that we can rewrite $\operatorname{codim}(\overline{\mathcal{F}}) \leq (3/4)\operatorname{codim}(\mathcal{F}) + 1$ as $d \geq \operatorname{codim}(\mathcal{F})/4 - 1$, therefore by Lemma 5.15 and Theorem 5.11,

$$\chi_B(\mathcal{F}) = \chi_B(\mathcal{G}) > 0. \qquad \square$$

We stress that the basic cohomology of a foliation $\mathcal{F}$ is an invariant that does not depend on any additional structure, such as a transverse metric, hence Theorem 5.17 can be seen as an obstruction result. The deformation technique also provides a topological obstruction when combined with the fibration property of the Euler characteristic, as we show in the next section.

Chapter 6

A Topological Obstruction

Our goal in this chapter is to prove that any Riemannian foliation of a closed manifold M satisfying $|\pi_1(M)| < \infty$ and $\chi(M) \neq 0$ is closed. This will follow from the invariance of the basic Euler characteristic under the deformations (Theorem 5.11) combined with an analog of the product property of the classical Euler characteristic of fibrations for closed foliations. For this last part we are going to use the results in [Hae86] on classifying spaces of topological groupoids, so we will briefly introduce these objects and results here.

6.1 Classifying Spaces of Holonomy Groupoids

Recall that a **groupoid** G is a small category with inverses [see MM03, for an introduction], that is, G consists of a set of objects Obj(G), a set of arrows Hom(G) and five structural maps involving these sets, namely, the source, target, composition, unity and inverse maps. We say that G is a **topological groupoid** when both these sets are topological spaces and all the structural maps are continuous. Among many applications, groupoids provide another framework to study holonomy, as we can see in the following examples.

Example 6.1. Let $(M, \mathcal{F})$ be a smooth foliation and consider the set of triples (x, c, y), where $x, y \in L \in \mathcal{F}$ and c is a path on L joining x to y. We identify $(x, c_1, y) \sim (x, c_2, y)$ when $h_{c_1} = h_{c_2}$ (see Section 1.2). The **holonomy groupoid** of $\mathcal{F}$ is the groupoid $\mathrm{G}_{\mathcal{F}}$ on M whose space of morphisms $\mathrm{Hom}(\mathrm{G}_{\mathcal{F}})$ consists of

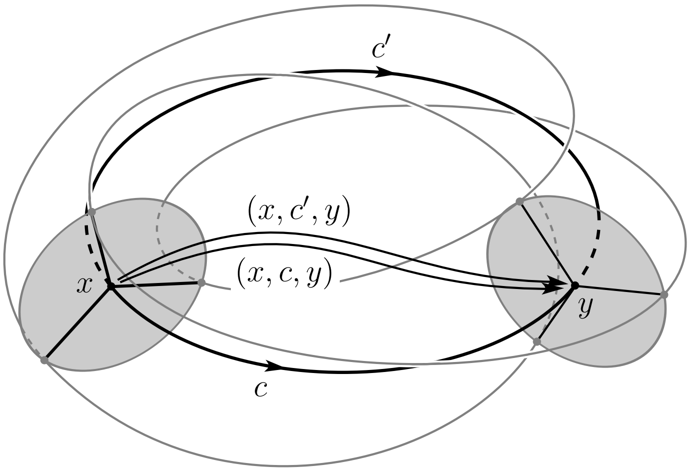


Figure 6.1: The holonomy groupoid

the set of equivalence classes of triples (x, c, y) (see Figure 6.1). It is possible to define on $\text{Hom}(\text{G}_{\mathcal{F}})$ an (in general non-Hausdorff) smooth structure of dimension $\dim(M) + \dim(\mathcal{F})$ that turns $\text{G}_{\mathcal{F}}$ into a **Lie groupoid**, that is, with this structure the source map is a smooth submersion with Hausdorff fibers and all the other structural maps are smooth [see MM03, Proposition 5.6; also Win83].

Example 6.2. Another way we can associate a Lie groupoid to a foliation $\mathcal{F}$ is by considering the groupoid of germs of a representative $(T_{\mathcal{F}}, \mathscr{H}_{\mathcal{F}})$ of the holonomy pseudogroup of $\mathcal{F}$, that we will denote by $\text{G}^T_{\mathcal{F}}$. Note that now we have $\text{Obj}(\text{G}^T_{\mathcal{F}}) = T_{\mathcal{F}}$.

Let G be a topological groupoid and let X be a topological space with an open cover $\{U_i\}_{i \in I}$. A **G-valued** 1**-cocycle** over $\{U_i\}$ consists of continuous maps

$$\begin{cases} \pi_i : U_i \longrightarrow \text{Obj}(\text{G}) \\ \gamma_{ij} : \pi_i(U_i) \cap \pi_j(U_j) \longrightarrow \text{Hom}(\text{G}) \end{cases}$$

satisfying $\text{source}(\gamma_{ij}(x)) = \pi_i(x)$ and $\text{target}(\gamma_{ij}(x)) = \pi_j(x)$, for $x \in U_i \cap U_j$, and verifying the 1-cocycle condition

$$\gamma_{ik}(x) = \gamma_{ij}(x)\gamma_{jk}(x)$$

for $x \in U_i \cap U_j \cap U_k$. The maps π_i are usually identified with γ_{ii} via the map unity : Obj(G) $\to$ Hom(G). Another G-valued 1-cocycle over a (possibly) different open cover of X is **equivalent** to $\{\gamma_{ij}\}$ if both these cocycles are restrictions of a third one, defined over the union of the two coverings. An equivalence class of such G-valued 1-cocycles is a G-**structure** on X. Following [Hae86], we denote the set of G-structures on X by $H^1(X, \mathrm{G})$. A continuous map $f : Y \to X$ induces a map $f^* : H^1(X, \mathrm{G}) \to H^1(Y, \mathrm{G})$ that associates the class of $\{\pi_i, \gamma_{ij}\}$ to the class of the 1-cocycle $\{\pi_i \circ f, \gamma_{ij}\}$ defined over the open cover $\{f^{-1}(U_i)\}$ of Y.

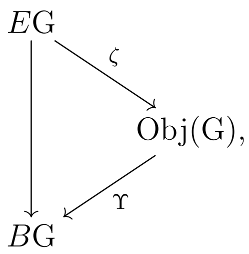
Example 6.3. Let G denote the groupoid of germs of local diffeomorphisms of $\mathbb{R}^q$. It is clear from the definition via Haefliger cocycles (see Section 1.1) that a q-codimensional smooth foliation $(M, \mathcal{F})$ can be seen as a G-structure on M represented by a 1-cocycle over a covering $\{U_i\}$ consisting of simple open sets such that $\pi_i : U_i \to \overline{U_i} \subset \mathbb{R}^q$ are local submersions defining $\mathcal{F}$ and γ_{ij} are the germs of the elements in $\mathscr{H}_{\mathcal{F}}$.

Note that, this way, $\mathcal{F}$ also determines an element of $H^1(M, \mathrm{G}^T_{\mathcal{F}})$. It canonically determines an element of $H^1(M, \mathrm{G}_{\mathcal{F}})$ too, represented by the 1-cocycle over the trivial cover $\{M\}$ with the maps $\pi = \mathrm{Id} : M \to M$ and $\gamma = \text{unity} : M \to \mathrm{Hom}(\mathrm{G}_{\mathcal{F}})$.

Similarly to the case of the classifying space of a topological group, the infinite join construction of J. Milnor [see, for instance, Hus94, Chapter 4, Section 11] can be carried over for a topological groupoid G, yielding a universal G-principal bundle $E\mathrm{G} \to B\mathrm{G}$ [Hae86, Théoème 3.1.1]. The precise definition of G-principal bundles over a space X can be seen in [Hae86, Section 2.2]. For us it is sufficient to know that the isomorphism classes of these objects are in one-to-one correspondence with the G-structures on X [see Hae86, §2.2.3], so we will use these notions interchangeably. In particular, G canonically defines a G-structure on Obj(G), for source : Hom(G) $\to$ Obj(G) is trivially a G-principal bundle.

The space $B\mathrm{G}$ is a **classifying space** for G in the sense that for any countable G-structure γ on a topological space X there is a continuous map $\Upsilon : X \to B\mathrm{G}$ such that $\alpha = \Upsilon^*(\omega)$, where ω is the G-structure on $B\mathrm{G}$ corresponding to the isomorphism class of the G-principal bundle $E\mathrm{G} \to B\mathrm{G}$.

In particular, it follows from this construction that the map $\Upsilon : \mathrm{Obj}(\mathrm{G}) \to B\mathrm{G}$ classifying G as a G-structure on Obj(G) is homotopy equivalent to $E\mathrm{G} \to B\mathrm{G}$, meaning that there is a commutative diagram



where ζ is a homotopy equivalence [Hae86, Coloraire 3.1.4].

6.2 A Topological Obstruction for Riemannian Foliations

When $(M, \mathcal{F})$ is Riemannian, $E\mathrm{G}_{\mathcal{F}} \to B\mathrm{G}_{\mathcal{F}}$ is a locally trivial fibration whose fiber is a generic leaf of $\mathcal{F}$ [Hae86, Corollaire 3.1.5]. We further remark that, since $\mathrm{G}_{\mathcal{F}}$ and $\mathrm{G}_{\mathcal{F}}^T$ are equivalent [see Hae86, p. 81], it follows from [Hae86, Corollaire 3.1.3] that $B\mathrm{G}_{\mathcal{F}}$ and $B\mathrm{G}_{\mathcal{F}}^T$ are homotopy equivalent. When $\mathcal{F}$ is a closed foliation, $B\mathrm{G}_{\mathcal{F}}^T$ coincides with the classifying space of the orbifold $M/\!/\mathcal{F}$ (see Section A.9) and the above results show that, at least from the homotopy theoretic point o view, closed foliations behave essentially like fibrations. We can now combine these facts with the invariance of the basic Euler characteristic under deformations (see Theorem 5.11) to obtain the following.

Theorem 6.4. *Let $\mathcal{F}$ be a Riemannian foliation of a simply-connected, closed manifold M. If $\dim(\mathfrak{g}_{\mathcal{F}}) \geq 1$, then $\chi(M) = 0$.*

Proof. Suppose there is a closed leaf $L \in \mathcal{F}$. If we choose a closed foliation $\mathcal{G}$ near $\mathcal{F}$ via Theorem 3.5, then we know that L is also a leaf of $\mathcal{G}$, because the deformation respects $\overline{\mathcal{F}}$. Fix a generic leaf $\widehat{L} \in \mathcal{G}$ near L. Since M is paracompact, the $\mathrm{G}_{\mathcal{G}}$-structure on M defined by $\mathcal{G}$ is countable, thus, by the results in [Hae86] presented above, we have a locally trivial fibration $E\mathrm{G}_{\mathcal{G}} \to B\mathrm{G}_{\mathcal{G}}$ with fiber $\widehat{L}$ and a commutative diagram

$$\begin{array}{ccccc} & & E\mathrm{G}_{\mathcal{G}} & & \\ & & \Big\downarrow & \overset{\zeta}{\searrow} & \\ & & & & M, \\ & & & \underset{\Upsilon}{\swarrow} & \\ B\mathcal{O} & \xrightarrow{\;h\;} & B\mathrm{G}_{\mathcal{G}} & & \end{array}$$

where $\mathcal{O}$ denotes $M//\mathcal{G}$ and both ζ and h are homotopy equivalences. By the homotopy exact sequence of the fibration $E\mathrm{G}_{\mathcal{G}} \to B\mathrm{G}_{\mathcal{G}}$ we see that $B\mathrm{G}_{\mathcal{G}}$ is also simply-connected, hence the Euler characteristic satisfies the product property [see, for instance, Spa81, Theorem 9.3.1]

$$\chi(M) = \chi(E\mathrm{G}_{\mathcal{G}}) = \chi(\widehat{L})\chi(B\mathrm{G}_{\mathcal{G}}).$$

Moreover, by Theorem A.16 we have $H^*(B\mathcal{O}, \mathbb{R}) \cong H^*(|\mathcal{O}|, \mathbb{R})$, so

$$\chi(M) = \chi(\widehat{L})\chi(|\mathcal{O}|) = \chi(\widehat{L})\chi_B(\mathcal{G}) = \chi(\widehat{L})\chi_B(\mathcal{F}), \tag{6.1}$$

where we also use Proposition 1.7 and Theorem 5.11.

On the other hand, local Reeb stability (Theorem 1.6) asserts that the restriction of the orthogonal projection $\mathrm{Tub}_\varepsilon(L) \to L$ to $\widehat{L}$ is a $k(\mathcal{G})$-sheeted covering map $\widehat{L} \to L$, where $k(\mathcal{G}) = |\mathrm{Hol}(L)| < \infty$ (the holonomy being with respect to $\mathcal{G}$). Hence $\chi(\widehat{L}) = k(\mathcal{G})\chi(L)$ and equation (6.1) can be rewritten as

$$\chi(M) = k(\mathcal{G})\chi(L)\chi_B(\mathcal{F}). \tag{6.2}$$

Notice, however, that for a sequence $\mathcal{G}_i$ of closed foliations approaching $\mathcal{F}$ we must have

$$\lim_{i\to\infty} k(\mathcal{G}_i) = \infty = k(\mathcal{F}),$$

so, in particular, we can change the number $k(\mathcal{G})$ by varying $\mathcal{G}$. Indeed, let Hx be the closed orbit on the orbifold $\mathcal{O}$ associated to $\mathcal{F}$ (see Theorem 3.3) corresponding to L. The stabilizer $\mathbb{T}^N_x$ is transverse to H in $\mathbb{T}^N$ and, since H is dense, there are infinitely many elements in $H \cap \mathbb{T}^N_x$. If $\mathcal{G} := \Upsilon^*(\mathcal{G}_K)$, for a closed subgroup $K < \mathbb{T}^N$, it suffices to choose another closed subgroup K' near H such that $|\mathbb{T}^N_x \cap K| \neq |\mathbb{T}^N_x \cap K'|$ and define $\mathcal{G}' := \Upsilon^*(\mathcal{G}_{K'})$. This would violate equation (6.2), unless $\chi(M) = 0$.

If $\mathcal{F}$ has no closed leaves then $\chi_B(\mathcal{F}) = 0$, by Corollary 5.8, and equation (6.1) (for any generic leaf of a closed approximation $\mathcal{G}$) again yields $\chi(M) = 0$. □

Corollary 6.5. *Let M be a closed manifold satisfying $|\pi_1(M)| < \infty$ and $\chi(M) \neq 0$. Then any Riemannian foliation of M is closed.*

Proof. Let $(M, \mathcal{F})$ be such a foliation and let $\widehat{\mathcal{F}}$ be the lift of $\mathcal{F}$ to the universal covering $\widehat{M}$ of M. We have $\chi(\widehat{M}) = |\pi_1(M)|\chi(M) > 0$, so $\widehat{\mathcal{F}}$ is a closed foliation, otherwise it would contradict Theorem 6.4. It follows from Corollary 2.8 that $\mathcal{F}$ is also closed. □

We can use Theorem 5.1 to obtain a corollary for Riemannian foliations of positively curved Riemannian manifolds with large symmetry rank. Indeed, in the following corollary we use a sharper version of Theorem 5.1 due to X. Rong and X. Su [see RS05, Theorem A].

Corollary 6.6. *Any Riemannian foliation of a complete, even-dimensional Riemannian manifold M^{2n} satisfying $\sec_M \geq c > 0$ and $\operatorname{symrank}(M) > (\dim(M) - 4)/8$ is closed.*

Combining equation 6.1 together with Corollary 6.5 and Theorem 4.1 we also have the following.

Corollary 6.7. *Let $\mathcal{F}$ be a non-closed Riemannian foliation of a closed manifold M such that $|\pi_1(M)| < \infty$. If $\chi_B(\mathcal{F}) \neq 0$ then $\Sigma^{\dim(\mathcal{F})} \neq \emptyset$ and $\chi(L) = 0$ for any closed leaf $L \in \mathcal{F}$.*

Appendix A

Orbifolds

In this appendix we present the basics of the differential geometry of orbifolds, following mostly [ALR07], [BG08], [KL14], [CR02] and [MM03], which can be used for further reading on the subject[1].

Orbifolds, first defined by I. Satake [Sat56] as V-manifolds[2] are amongst the simplest generalizations of manifolds that include singularities. They are topological spaces locally modeled on quotients of $\mathbb{R}^n$ by a finite group action, and appear naturally in many areas of mathematics, such as algebraic geometry, differential geometry (including, of course, foliation theory), and string theory. Although we will adopt the local charts description of orbifolds, we mention that there are many different ways to approach them in the literature, for example as Lie groupoids (see Remark A.8 and also [MM03] for an introduction to groupoids in this context) and as Deligne–Mumford stacks [see Ler10].

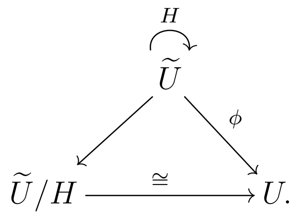
A.1 Definition and Examples

Let X be a topological space and fix $n \in \mathbb{N}$. An **orbifold chart** $(\widetilde{U}, H, \phi)$ of dimension n for an open set $U \subset X$ consists of a connected open subset $\widetilde{U} \subset \mathbb{R}^n$,

[1] Although slightly different, the definitions of orbifold in [ALR07], [BG08], [KL14] and [MM03], as well as the definition adopted here, are all equivalent and encompass [see CR02, p. 67] the original definition in [Sat56], dropping the requirement that the fixed-point set of the chart groups have codimension at least 2. Moreover, the broader definition of non-effective orbifolds in [CR02] encompasses and is compatible to all the previous ones.

[2] The term “orbifold” was introduced by W. Thurston [Thu02], after a vote by his students.

a finite group $H < \mathrm{Diff}(\widetilde{U})$ and a continuous H-invariant map $\phi : \widetilde{U} \to X$ that induces a homeomorphism between $\widetilde{U}/H$ and U:



An **embedding** $\lambda : (\widetilde{U}_1, H_1, \phi_1) \hookrightarrow (\widetilde{U}_2, H_2, \phi_2)$ between two orbifold charts is a smooth embedding $\lambda : \widetilde{U}_1 \hookrightarrow \widetilde{U}_2$ that satisfies $\phi_2 \circ \lambda = \phi_1$. Note that for every chart $(\widetilde{U}, H, \phi)$, each h in the **chart group** H is, in particular, an embedding $(\widetilde{U}, H, \phi \circ h) \hookrightarrow (\widetilde{U}, H, \phi)$.

An **orbifold atlas** for X is a collection $\mathcal{A} = \{(\widetilde{U}_i, H_i, \phi_i)\}_{i \in I}$ of orbifold charts that cover X and are locally compatible in the following sense: for any two charts $(\widetilde{U}_i, H_i, \phi_i)$, $i = 1, 2$, and $x \in U_1 \cap U_2$, there is an open neighborhood $U_3 \subset U_1 \cap U_2$ containing x and an orbifold chart $(\widetilde{U}_3, H_3, \phi_3)$ for U_3 that admits embeddings in $(\widetilde{U}_i, H_i, \phi_i)$, $i = 1, 2$. We say that an atlas $\mathcal{A}$ **refines** an atlas $\mathcal{B}$ when every chart in $\mathcal{A}$ admits an embedding in some chart in $\mathcal{B}$. Two atlases are **equivalent** if they have a common refinement. As in the manifold case, an orbifold atlas is always contained in a unique maximal one.

An n-dimensional **smooth orbifold** $\mathcal{O}$ consists of a Hausdorff paracompact topological space $|\mathcal{O}|$ together with an **orbifold structure**, that is, an equivalence class $[\mathcal{A}]$ of n-dimensional orbifold atlases for $|\mathcal{O}|$. We will say that an orbifold chart is a **chart of** $\mathcal{O}$ when it is an element of some atlas in $[\mathcal{A}]$. Observe that if the groups H_i are all trivial for some atlas in $[\mathcal{A}]$, then $\mathcal{O}$ is locally Euclidean and, therefore, a manifold. We say that $\mathcal{O}$ is **orientable** if for some atlas $\mathcal{B} = \{(\widetilde{U}_i, H_i, \phi_i)\} \in [\mathcal{A}]$ we can choose an orientation for each $\widetilde{U}_i$ that makes every embedding between charts of $\mathcal{B}$ orientation-preserving. Of course, with such orientations chosen $(\mathcal{O}, \mathcal{B})$ is an **oriented** orbifold.

We stress that with this definition orbifolds are automatically **effective**, because we required the chart groups H to be subgroups of $\mathrm{Diff}(\widetilde{U})$. In some appearances of orbifolds it is more natural and useful to consider a (possibly non-effective) action of H on $\widetilde{U}$ [see, for example, CR02].

Below we list some technical consequences of the definition.

(i) If $(\widetilde{U}, H, \phi)$ is a chart for U and $U' \subset U$ is connected, then $(\widetilde{U}', H', \phi')$ is a compatible chart for U', where $\widetilde{U}'$ is a connected component of $\phi^{-1}(U')$, $H' := H_{\widetilde{U}'} < H$ is the subgroup that preserves $\widetilde{U}'$ and $\phi' := \phi|_{\widetilde{U}'}$ [see CR02, Lemma 4.1.1].

(ii) Complementing item (i), if $(\widetilde{U}_i, H_i, \phi_i)$, $i = 1, 2$, are compatible charts, $\widetilde{U}_1$ is simply-connected and $\phi_1(\widetilde{U}_1) \subset \phi_2(\widetilde{U}_2)$, then there is an embedding $\lambda : (\widetilde{U}_1, H_1, \phi_1) \hookrightarrow (\widetilde{U}_2, H_2, \phi_2)$ [see ALR07, p. 3; Sat57, footnote 2].

(iii) For two embeddings $\lambda, \lambda' : (\widetilde{U}_1, H_1, \phi_1) \hookrightarrow (\widetilde{U}_2, H_2, \phi_2)$ there exists a unique $h \in H_2$ such that $\lambda' = h \circ \lambda$ [MP97, Proposition A.1].

(iv) As a consequence of item (iii), any embedding $\lambda : (\widetilde{U}_1, H_1, \phi_1) \hookrightarrow (\widetilde{U}_2, H_2, \phi_2)$ induces a monomorphism $\overline{\lambda} : H_1 \to H_2$ [see ALR07, p. 3].

Let $x \in |\mathcal{O}|$ and consider a chart $(\widetilde{U}, H, \phi)$ satisfying $x = \phi(\widetilde{x}) \in U$. The **local group** Γ_x at x is the isomorphism class[3] of the isotropy subgroup $H_{\widetilde{x}} < H$. It is independent of both the chart and the lifting $\widetilde{x}$ [see ALR07, p. 4], and for every $x \in |\mathcal{O}|$ we can always find a compatible chart $(\widetilde{U}, \Gamma_x, \phi)$ **around** x, that is, such that $\phi^{-1}(x)$ consists of a single point $\tilde{x}$. We denote by Σ_Γ the subset of $|\mathcal{O}|$ of the points with local group Γ. The decomposition

$$|\mathcal{O}| = \bigsqcup \Sigma,$$

where each Σ is a connected component of some Σ_Γ called a **stratum**, is the **canonical stratification** of $\mathcal{O}$. The subset Σ_0 of **regular points** is an open, connected and dense manifold, called the **regular stratum** of $\mathcal{O}$ and also denoted by $\mathcal{O}_{\text{reg}}$. The subset $|\mathcal{O}| \setminus \mathcal{O}_{\text{reg}}$ is a closed subset of $|\mathcal{O}|$ with empty interior, called the **singular locus** of $\mathcal{O}$.

[3] We will denote both the isomorphism class and a representative of it by Γ_x, when the meaning is clear from the context.

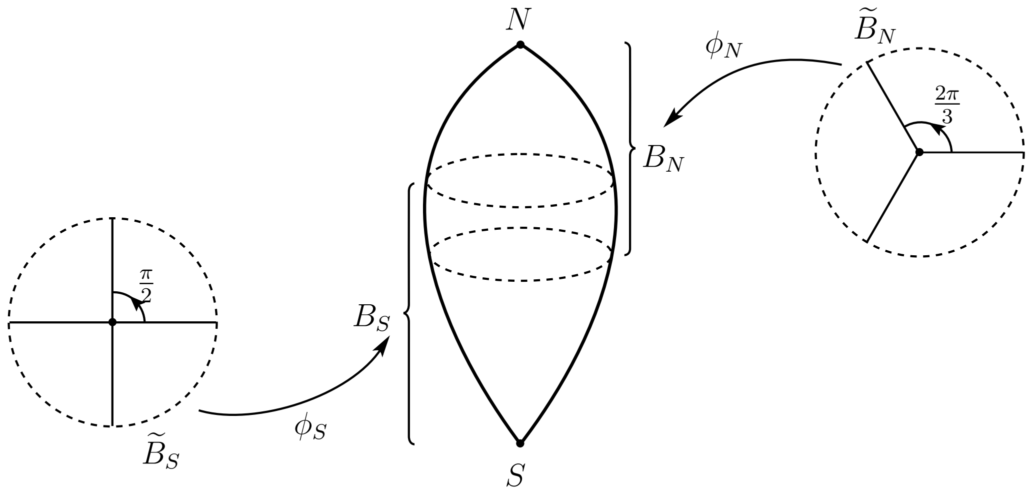


Figure A.1: The $\mathbb{Z}_3$-$\mathbb{Z}_4$-football

Example A.1. On the sphere $\mathbb{S}^2$, consider open balls B_N and B_S centered in the north and the south poles respectively, such that $\mathbb{S}^2 = D_N \cup D_S$, and let $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$. Over D_N we choose the orbifold chart $(\widetilde{B}_N, \mathbb{Z}_p, \phi_N)$, where $\widetilde{B}_N$ is an open ball in $\mathbb{R}^2$ centered at the origin on which $\mathbb{Z}_p$ acts by rotations and $\phi_N(0) = N$. Similarly, over D_S we choose the chart $(\widetilde{B}_S, \mathbb{Z}_q, \phi_S)$. The resulting orbifold is called the **$\mathbb{Z}_p$-$\mathbb{Z}_q$-football** (see Figure A.1). More details on this example can be seen in [Sch56, Example 1.7.6]. In the special case $q = 1$ the south pole becomes a regular point, and the resulting orbifold is called the **$\mathbb{Z}_p$-teardrop**. Of course, the $\mathbb{Z}_1$-$\mathbb{Z}_1$-football is just the regular sphere.

Orbifolds with boundary are defined similarly to the manifold case, by requiring the sets $\widetilde{U}$ in the charts to be open subsets of $[0, \infty) \times \mathbb{R}^{n-1}$. These objects will not be used in this work, but it is worth to note that it is possible that $\partial\mathcal{O} = \emptyset$ while $|\mathcal{O}|$ is homeomorphic to a topological manifold with non-empty boundary. In fact, it is often useful to consider the **topological boundary** of an orbifold (without boundary) $\mathcal{O}$, defined as the union of the closures of all the singular strata of codimension 1. An orbifold is called **closed** when its underlying space is compact and its topological boundary is empty. In particular, a compact orientable orbifold $\mathcal{O}$ is closed, because if its topological boundary was non-empty then there would be a **mirror point** in $\mathcal{O}$, that is, $x \in |\mathcal{O}|$ with Γ_x generated by a reflection, hence $\mathcal{O}$

would not be orientable.

A.2 Smooth Maps

Let $\mathcal{O}$ and $\mathcal{P}$ be orbifolds and let $|f| : |\mathcal{O}| \to |\mathcal{P}|$ be a continuous map. We say that $|f|$ is **smooth** at $x \in |\mathcal{O}|$ when, given charts $(\widetilde{U}, \Gamma_x, \phi)$ and $(\widetilde{V}, \Gamma_{|f|(x)}, \psi)$ around x and $|f|(x)$, respectively, such that $|f|(U) \subset V$, there exists a **smooth local lift** of $|f|$ at x, that is, a homomorphism $\overline{f}_x : \Gamma_x \to \Gamma_{|f|(x)}$ together with a smooth $\overline{f}_x$-equivariant map $\widetilde{f}_x : \widetilde{U} \to \widetilde{V}$ such that

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\widetilde{f}_x} & \widetilde{V} \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ U & \xrightarrow[|f|]{} & V \end{array}$$

commutes (see Figure A.2). A **smooth map** $f : \mathcal{O} \to \mathcal{P}$ consists of a continuous map $|f| : |\mathcal{O}| \to |\mathcal{P}|$ that is smooth at every $x \in |\mathcal{O}|$.

Example A.2. We stress that it is not always the case that two different local lifts at x differ by composition with some element of $\Gamma_{|f|(x)}$. For example, consider the action of $\mathbb{Z}_4$ on $\mathbb{R} \times \mathbb{C}$ consisting of multiplication by $\mathrm{i} = \sqrt{-1}$ on $\mathbb{C}$ and let $\mathcal{O}$ be the corresponding quotient orbifold. Define $\widetilde{f}_1, \widetilde{f}_2 : \mathbb{R} \to \mathbb{R} \times \mathbb{C}$ by $\widetilde{f}_1(t) = (t, e^{-t^{-2}})$ and

$$\widetilde{f}_2(t) = \begin{cases} (t, e^{-t^{-2}}) & \text{if} \quad t \leq 0, \\ (t, \mathrm{i}e^{-t^{-2}}) & \text{if} \quad t > 0. \end{cases}$$

It is clear that $\widetilde{f}_1$ and $\widetilde{f}_2$ are local lifts of the same underlying map $|f| : \mathbb{R} \to |\mathcal{O}|$ and that they do not differ by an element of $\mathbb{Z}_4$.

A smooth map $f : \mathcal{O} \to \mathcal{P}$ is a **diffeomorphism** if it admits a smooth inverse. In this case we clearly have $\Gamma_x \cong \Gamma_{|f|(x)}$ for all $x \in |\mathcal{O}|$, that is, diffeomorphisms must preserve the orbifold stratification. Orbifolds and smooth maps in this sense form a category, but there are some relevant refinements of the notion of morphism between orbifolds, as we comment below.

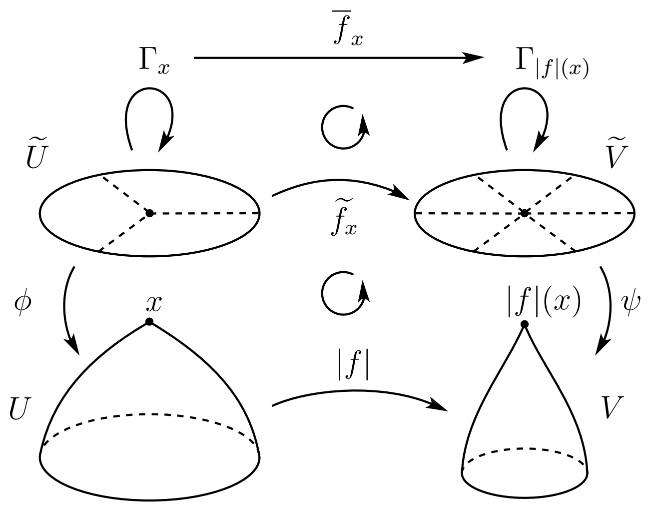


Figure A.2: A smooth local lift

Remark A.3. Smooth maps between orbifolds were first introduced in [Sat56] as continuous maps admitting smooth local liftings (without the accompanying homomorphism between the chart groups). Later it was realized that this notion was insufficient to coherently define pullbacks of (the orbifold analogues of) bundles and sheaves. To overcome this, more subtle notions of morphisms between orbifolds were introduced, like the Moerdijk–Pronk *strong maps* [MP97], that match the definition of groupoid homomorphisms when the orbifolds are seen as Lie groupoids (see Remark A.8), and the equivalent[4] notion of *good maps* by W. Chen and Y. Ruan [CR02]. Every good or strong map is, nevertheless, a smooth map as defined here. Further notions of smooth maps between orbifolds are also investigated in [BB12].

A.3 Quotient Orbifolds

Orbifolds appear naturally as quotients of smooth Lie group actions. More precisely, we have the following result.

Proposition A.4. *Suppose that a Lie group G acts properly[5], effectively and almost freely on a smooth manifold M. Then the quotient space M/G has a natural orbifold structure.*

Proof. Let us just sketch the proof. As $G \times M \ni (g, x) \mapsto (gx, x) \in M \times M$ is a

[4]See [LU04, Proposition 5.1.7].

[5]That is, $G \times M \ni (g, x) \mapsto (gx, x) \in M \times M$ is a smooth proper map.

proper map between locally compact spaces, it is also closed, hence $R := \{(x, y) \in M \times M \mid G(x) = G(y)\}$ is closed. As the quotient projection $\pi : M \to M/G$ is an open map, it follows that $\pi((M \times M) \setminus R)$, the complement of the diagonal in $M/G \times M/G$, is open. Therefore M/G is Hausdorff. Moreover, it is clearly paracompact.

Now, for any $x \in M$ there is a slice [see AB15, Theorem 3.49] $S_x = \exp^{\perp}(B_\varepsilon(0))$ (with respect to a suitable Riemannian metric on M) on which the finite isotropy subgroup G_x acts. Defining $\mathrm{Tub}(Gx) := G(S_x)$, the Slice Theorem [see, for instance, AB15, Theorem 3.57] asserts that $\mathrm{Tub}(Gx)/G \cong S_x/G_x$, so $(B_\varepsilon(0), G_x, \pi \circ \exp^{\perp})$ is an orbifold chart around $\pi(x)$, where we consider the linearized action of G_x on $B_\varepsilon(0)$ via the isotropy representation $G_x < \mathrm{GL}(T_x S_x)$. □

We will denote the **quotient orbifold** obtained this way by $M//G$ in order to differentiate it from its underlying topological space M/G. That is, we have

$$|M//G| = M/G.$$

In the special case that G is discrete, we say that the orbifold $\mathcal{O} = M//G$ is **good** or **developable**, otherwise $\mathcal{O}$ is called a **bad** orbifold. Clearly, the $\mathbb{Z}_p$-$\mathbb{Z}_q$-football is a good orbifold when $p = q$, and it is possible to prove that it is bad when $p \neq q$. In fact, it is a special case of another class of (in general) bad [see ALR07, Example 1.53] orbifolds that we now describe.

Example A.5. Fix $\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{N}^{n+1}$ satisfying $\gcd(\lambda_0, \dots, \lambda_n) = 1$. We now modify the standard action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1} \setminus \{0\}$ by adding weights given by λ. Precisely, let $z \in \mathbb{C}^\times$ act by

$$z \cdot (z_0, \dots, z_n) = (z^{\lambda_0} z_0, \dots, z^{\lambda_n} z_n). \tag{A.1}$$

The quotient orbifold $(\mathbb{C}^{n+1} \setminus \{0\})//\mathbb{C}^\times$ is called a **weighted projective space**. We denote it by $\mathbb{CP}^n[\lambda_0, \dots, \lambda_n]$ to emphasize the weights, or simply by $\mathbb{CP}^n[\lambda]$ when the exact weights are not relevant. Weighted projective spaces play the same role in the category of orbifolds as the usual complex projective space plays in the category of smooth manifolds. As the later, they can also be seen as algebraic varieties and, so, they exemplify how orbifolds can appear in algebraic geometry [see, for example, Dol82].

Observe that $\mathbb{CP}^1[p,q]$ is simply the $\mathbb{Z}_p$-$\mathbb{Z}_q$-football (see Example A.1) and that $\mathbb{CP}^n[1,\dots,1]$ is just the usual complex projective space. Also, note that the above action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1}\backslash\{0\}$ restricts to an action of $\mathbb{S}^1 < \mathbb{C}^\times$ on $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ with the same quotient, so that we could equivalently define $\mathbb{CP}^n[\lambda_0,\dots,\lambda_n] := \mathbb{S}^{2n+1}/\!/\mathbb{S}^1$. Comparing this with Example 1.9 we see that a weighted complex projective space is the quotient space of a closed Riemannian 1-foliation of a sphere.

Charts for $\mathbb{CP}^n[\lambda_0,\dots,\lambda_n]$ that are compatible with the induced orbifold structure as a quotient can be constructed as follows [see BG08, Section 4.5, for further details]. Cover the space $|\mathbb{CP}^n[\lambda]|$ with the $n+1$ open sets

$$U_i = \Big\{[z_0 : \cdots : z_n] \in \mathbb{CP}^n[\lambda] \;\Big|\; z_i \neq 0\Big\},$$

where we use homogeneous coordinates as in the manifold case, that is, $[z_0 : \cdots : z_n]$ denotes the orbit of $(z_0,\dots,z_n)$. The charts will be $(\widetilde{U}_i, G_i, \varphi_i)$, where $\widetilde{U}_i = \mathbb{C}^n$ with affine coordinates $(w_{i0},\dots,\widehat{w}_{ii},\dots,w_{in})$ satisfying $w_{ij}^{\lambda_i} = z_j^{\lambda_i}/z_i^{\lambda_j}$, and the maps $\varphi_i : \widetilde{U}_i \to U_i$ are given by

$$\varphi_i(w_{i0},\dots,\widehat{w}_{ii},\dots,w_{in}) = [w_{i0}^{\lambda_i},\dots,w_{i(i-1)}^{\lambda_i},1,w_{i(i+1)}^{\lambda_i},\dots,w_{in}^{\lambda_i}].$$

The chart groups $G_i \cong \mathbb{Z}_{\lambda_i}$ are simply the groups of λ_ith roots of the unity acting on $\widetilde{U}_i$ by multiplication.

The singular locus of $\mathbb{CP}^n[\lambda]$ consist of copies of $\mathbb{CP}^k[\lambda_k]$, $0 \leq k \leq n$, that correspond to some of the coordinate $(k+1)$-dimensional subspaces of $\mathbb{C}^{n+1}$. We can visualize this stratification as an n-simplex, where each k-face correspond to a copy of $\mathbb{CP}^k[\lambda']$ (see Figure A.3). Precisely, the subset

$$\Big\{[z_0 : \cdots : z_n] \in \mathbb{CP}^n[\lambda_0,\dots,\lambda_n] \;\Big|\; z_j = 0 \text{ for } j \neq i_1,\dots,i_k\Big\}$$

is singular if and only if $l := \gcd(\lambda_{i_1},\dots,\lambda_{i_k}) > 1$. In this case the local group at a

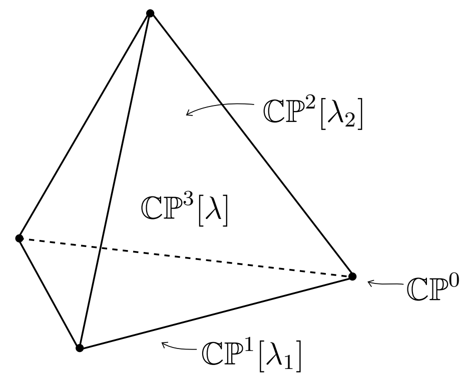


Figure A.3: Visualizing the stratification of $\mathbb{CP}^3[\lambda]$ as a 3-simplex

generic point in this singular subset is $\mathbb{Z}_l$.

Notice that the linear action of $\mathbb{T}^{n+1}$ on $\mathbb{C}^{n+1}$ descends to an action on $\mathbb{CP}^n[\lambda]$.

A.4 Associated Pseudogroups

Let S be a smooth manifold. Recall that a **pseudogroup** $\mathscr{H}$ of local diffeomorphisms of S consists of a set of diffeomorphisms $h : U \to V$, where U and V are open sets of S, such that

(i) $\mathrm{Id}_U \in \mathscr{H}$ for any open set $U \subset S$,

(ii) $h \in \mathscr{H}$ implies $h^{-1} \in \mathscr{H}$,

(iii) if $h_1 : U_1 \to V_1$ and $h_2 : U_2 \to V_2$ are in $\mathscr{H}$, then their composition

$$h_2 \circ h_1 : h_1^{-1}(V_1 \cap U_2) \longrightarrow h_2(V_1 \cap U_2)$$

also belongs to $\mathscr{H}$, and

(iv) if $U \subset S$ is open and $k : U \to V$ is a diffeomorphism such that U admits an open cover $\{U_i\}$ with $k|_{U_i} \in \mathscr{H}$ for all i, then $k \in \mathscr{H}$.

The **$\mathscr{H}$-orbit** of $x \in S$ consists of the points $y \in S$ for which there is some $h \in \mathscr{H}$ satisfying $h(x) = y$. The quotient by the corresponding equivalence relation, endowed with the quotient topology, is the **space of orbits** of $\mathscr{H}$, that we denote $S/\mathscr{H}$.

Example A.6. Let $\mathcal{O}$ be an orbifold with atlas $\mathcal{A} = \{(\widetilde{U}_i, H_i, \phi_i)\}$. We define $U_{\mathcal{A}} := \bigsqcup_{i\in I} \widetilde{U}_i$ and $\phi := \{\phi_i\}_{i\in I} : U_{\mathcal{A}} \to |\mathcal{O}|$, that is, $x \in \widetilde{U}_i \subset U_{\mathcal{A}}$ implies $\phi(x) = \phi_i(x)$. A **change of charts** of $\mathcal{A}$ is a diffeomorphism $h : V \to W$, with $V, W \subset U_{\mathcal{A}}$ open sets, such that $\phi \circ h = \phi|_V$. Note that embeddings between charts of $\mathcal{A}$ and, in particular, the elements of the chart groups H_i are changes of charts.

The collection of all changes of charts of $\mathcal{A}$ form a pseudogroup $\mathscr{H}_{\mathcal{A}}$ of local diffeomorphisms of $U_{\mathcal{A}}$, and ϕ induces a homeomorphism $U_{\mathcal{A}}/\mathscr{H}_{\mathcal{A}} \to |\mathcal{O}|$.

Let $\mathscr{H}$ and $\mathscr{K}$ be pseudogroups of local diffeomorphisms of S and T, respectively. A (smooth) **equivalence** between $\mathscr{H}$ and $\mathscr{K}$ is a maximal collection Φ of diffeomorphisms from open sets of S to open sets of T such that $\{\mathrm{Dom}(\varphi) \mid \varphi \in \Phi\}$ covers S, $\{\mathrm{Im}(\varphi) \mid \varphi \in \Phi\}$ covers T and, for all $\varphi, \psi \in \Phi$, $h \in \mathscr{H}$ and $k \in \mathscr{K}$, we have $\psi^{-1} \circ k \circ \varphi \in \mathscr{H}$, $\psi \circ h \circ \varphi^{-1} \in \mathscr{K}$ and $k \circ \varphi \circ h \in \Phi$, whenever these compositions make sense.

Example A.7. Let $\mathcal{A}_i$, $i = 1, 2$, be two equivalent atlases for an orbifold $\mathcal{O}$, with common refinement $\mathcal{B}$. Then for each chart $(\widetilde{V}_k, H_k, \phi_k)$ of $\mathcal{B}$ we have an embedding λ_i^k in some chart of $\mathcal{A}_i$. The maximal collection of diffeomorphisms generated by $\{\lambda_2^k \circ (\lambda_1^k)^{-1}\}$ is an equivalence between $\mathscr{H}_{\mathcal{A}_1}$ and $\mathscr{H}_{\mathcal{A}_2}$. Thus, up to equivalence, we can define the **pseudogroup $\mathscr{H}_{\mathcal{O}}$ of changes of charts of $\mathcal{O}$**. We will sometimes abuse the notation and use $(U_{\mathcal{O}}, \mathscr{H}_{\mathcal{O}})$ to mean one of its representatives, when it is clear that a different choice would not affect the results.

In fact, changes of charts can be used as an alternative notion of compatibility between the charts in an orbifold atlas, yielding therefore yet another definition for orbifolds (still equivalent to our definition [see MM03, Proposition 2.13]). This is the approach adopted by A. Haefliger [see, for example, GH06; HS90].

Remark A.8. The pseudogroup $\mathscr{H}_{\mathcal{O}}$ is also relevant in enabling us to associate to $\mathcal{O}$ a Lie groupoid (see Chapter 6). We refer to [MM03, Chapter 5] and [ALR07, Section 1.4] for an introduction to orbifolds from this point of view. More precisely, if $\mathscr{H}_{\mathcal{A}} \in \mathscr{H}_{\mathcal{O}}$ for an atlas $\mathcal{A}$ for $\mathcal{O}$, consider $\mathrm{G}_{\mathcal{A}}$ the groupoid of germs of elements in $\mathscr{H}_{\mathcal{A}}$. Then $\mathrm{G}_{\mathcal{A}}$ is a proper, effective, *étale* Lie groupoid, and for a different compatible atlas $\mathcal{B}$, the groupoid $\mathrm{G}_{\mathcal{B}}$ is Morita equivalent to $\mathrm{G}_{\mathcal{A}}$ [see MM03, Proposition 5.29]. Hence we can associate to $\mathcal{O}$ a unique Morita equivalence class of proper Lie groupoids $\mathrm{G}_{\mathcal{O}}$. Conversely, any proper, effective, *étale* Lie groupoid $G_1 \rightrightarrows G_0$ defines an orbifold structure on its coarse moduli space G_0/G_1, with G_{G_0/G_1} Morita equivalent to $G_1 \rightrightarrows G_0$ [see MM03, Corollary 5.31]. We also refer to [BG08; Moe02; MP97; Ler10] for more details on this correspondence between orbifolds and Lie groupoids.

A.5 Orbibundles

As in the manifold case, there is a general notion of a vector bundle over an orbifold that allow us to define objects like vector fields and differential forms in this context and to carry over many other useful constructions. Let us begin with the general definition.

Let $\mathcal{E}$ and $\mathcal{B}$ be orbifolds and V be a k-dimensional vector space. A **rank-k vector orbibundle** over $\mathcal{B}$ with total space $\mathcal{E}$ and fiber V is a smooth map $\pi : \mathcal{E} \to \mathcal{B}$, with $|\pi|$ surjective, such that for all $x \in |\mathcal{B}|$ there is an orbifold chart $(\widetilde{U}, \Gamma_x, \varphi)$ around x, a linear action of Γ_x on V and a diffeomorphism $(V \times \widetilde{U})//\Gamma_x \to \mathcal{E}|_{|\pi|^{-1}(U)}$ such that

$$\begin{array}{ccc} (V \times \widetilde{U})//\Gamma_x & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \pi \\ \widetilde{U}//\Gamma_x & \longrightarrow & \mathcal{B} \end{array}$$

is commutative. A **section** of π is simply a smooth map $s : \mathcal{B} \to \mathcal{E}$ satisfying $\pi \circ s = \mathrm{Id}_{\mathcal{E}}$. There exists also the more general notion of a fiber orbibundle which has a general orbifold as fiber [see, for example, KL14, p. 7].

A first specific case will naturally be the following. Let $\mathcal{O}$ be a n-dimensional orbifold and let $(\widetilde{U}, H, \phi)$ be a chart of $\mathcal{O}$. Then we have a smooth action of H on $T\widetilde{U}$ by $h(\tilde{x}, v) = (h(\tilde{x}), \mathrm{d}h_{\tilde{x}}v)$, that gives an orbifold chart for $TU := T\widetilde{U}/H$. These charts glue together to form a $(2n)$-dimensional orbifold $T\mathcal{O}$, called the **tangent bundle** of $\mathcal{O}$ [see ALR07, p. 10]. We have an equivariant projection $(\mathbb{R}^n \times \widetilde{U}) \cong T\widetilde{U} \to \widetilde{U}$, which in turn induces a natural projection $p : T\widetilde{U}//H \to \widetilde{U}//H$, showing that $T\mathcal{O} \to \mathcal{O}$ is a rank-n vector orbibundle. The **tangent space** at $x = \varphi(\tilde{x})$, denoted $T_x\mathcal{O}$, is the vector space $T_{\tilde{x}}\widetilde{U}$ together with the induced linear action of Γ_x.

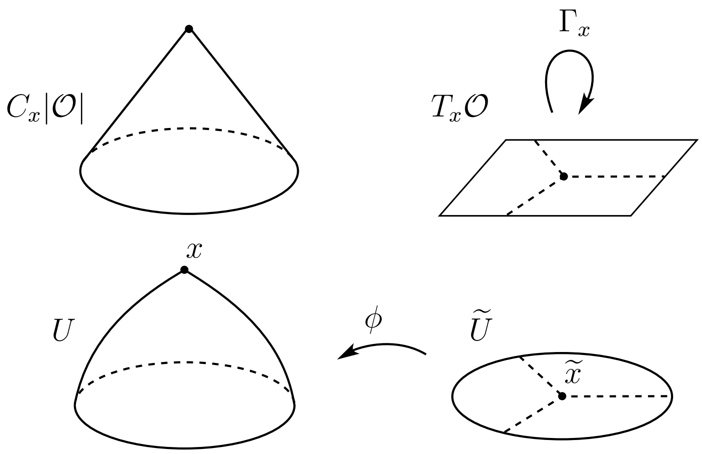


Figure A.4: Tangent space and tangent cone at x

The **tangent cone** at x is $C_x|\mathcal{O}| := T_{\tilde{x}}\widetilde{U}/\Gamma_x$ (see Figure A.4), which coincides with the fiber $|p|^{-1}(x)$ [see ALR07, p. 11].

Of course, we define a **vector field** on $\mathcal{O}$ to be a section of $T\mathcal{O}$. We denote the $C^\infty(\mathcal{O})$-module of the smooth vector fields in $\mathcal{O}$ by $\mathfrak{X}(\mathcal{O})$. In terms of a chart $(\widetilde{U}, H, \phi)$ and the induced chart for TU, a vector field restricts to an H-invariant vector field in $\widetilde{U}$. In fact, we have a natural correspondence between vector fields on $\mathcal{O}$ and $\mathscr{H}_{\mathcal{O}}$-invariant vector fields on $U_{\mathcal{O}}$. Hence, by naturality $\mathfrak{X}(\mathcal{O})$ is closed under the Lie bracket.

A smooth map $f : \mathcal{O} \to \mathcal{P}$ gives rise to its **differential**, a smooth bundle map $\mathrm{d}f : T\mathcal{O} \to T\mathcal{P}$. In terms of a local lifting $f_{UV} : (\widetilde{U}, H, \phi) \to (\widetilde{V}, K, \psi)$, for $\tilde{x} \in U$ with $\phi(\tilde{x}) = x$ we have a $\overline{f_{UV}}$-equivariant map $\mathrm{d}(f_{UV})_{\tilde{x}} : T_{\tilde{x}}\widetilde{U} \to T_{f_{UV}(\tilde{x})}\widetilde{V}$, which gives a linear map $\mathrm{d}f_x : T_x\mathcal{O} \to T_{f(x)}\mathcal{P}$. Using this map we define, in complete analogy to the manifold case, the notions of immersions, submersions, transversality etc. We refer to [BB12; KL14] for results on the differential topology of orbifolds.

Likewise, by gluing the local data we obtain (j,i)**-tensor orbibundles** over $\mathcal{O}$, denoted $\bigotimes_j^i \mathcal{O}$, which have as suborbibundles the i**th exterior orbibundles** $\bigwedge^i \mathcal{O}$. Smooth sections of $\bigotimes_j^i \mathcal{O}$ and $\bigwedge^i \mathcal{O}$ yields us (j,i)**-tensor** fields and i**-forms** on $\mathcal{O}$, respectively. We denote the space of i-forms on $\mathcal{O}$ by $\Omega^i(\mathcal{O})$. Again, in a local chart $(\widetilde{U}, H, \phi)$ these are just H-invariant tensor fields in $\widetilde{U}$ and we have a correspondence between tensor fields on $\mathcal{O}$ and $\mathscr{H}_{\mathcal{O}}$-invariant tensor fields on $U_{\mathcal{O}}$. In particular, by naturality there is a well-defined **exterior derivative**

$$d : \Omega^i(\mathcal{O}) \longrightarrow \Omega^{i+1}(\mathcal{O}).$$

Similarly, if X and ξ are $\mathscr{H}_{\mathcal{O}}$-invariant vector and $(0,i)$-tensor fields on $U_{\mathcal{O}}$, respectively, then $\mathcal{L}_X \xi$ will also be $\mathscr{H}_{\mathcal{O}}$-invariant, that is, we can define Lie derivatives for the corresponding objects on $\mathcal{O}$.

A.6 Integration and de Rham Cohomology

Given an orbifold $\mathcal{O}$, the cohomology groups of the complex

$$\cdots \xrightarrow{d} \Omega^{i-1}(\mathcal{O}) \xrightarrow{d} \Omega^i(\mathcal{O}) \xrightarrow{d} \Omega^{i+1}(\mathcal{O}) \xrightarrow{d} \cdots$$

are the **de Rham cohomology groups** of $\mathcal{O}$, that we denote by $H^i_{\mathrm{dR}}(\mathcal{O})$. As in the manifold case, these groups are invariant under homotopy equivalence. This follows from the result below, which can be seen as version of the de Rham Theorem for orbifolds, that asserts that they are isomorphic to the real singular cohomology groups of the quotient space $|\mathcal{O}|$ [see Sat56, Theorem 1; ALR07, Theorem 2.13].

Theorem A.9 (Satake)**.** *Let $\mathcal{O}$ be an orbifold. Then $H^i_{\mathrm{dR}}(\mathcal{O}) \cong H^i(|\mathcal{O}|, \mathbb{R})$.*

On an oriented n-orbifold $\mathcal{O}$ we can also define integration. Let $(\widetilde{U}, H, \phi)$ be an orbifold chart for U. Given a compactly supported n-form $\omega \in \Omega^n(\widetilde{U}/\!/H)$, i.e, an H-invariant compactly supported n-form $\tilde{\omega} \in \Omega^n(\widetilde{U})$, we define

$$\int_U \omega := \frac{1}{|H|} \int_{\widetilde{U}} \tilde{\omega}.$$

In general, because orbifolds admit partitions of unity [see KL14, Lemma 2.11; CR02, Lemma 4.2.1], for a compactly supported n-form $\omega \in \Omega^n(\mathcal{O})$ we can define

$$\int_{\mathcal{O}} \omega := \sum_j \int_{U_j} \xi_j \omega,$$

where $\{\xi_j\}$ is a partition of unity subordinated to the cover $\{U_j\}$ of $|\mathcal{O}|$ coming from a fixed oriented atlas $\{(\widetilde{U}_j, H_j, \phi_j)\}$. This definition is independent of the choices involved [see ALR07, p. 35].

All the machinery involving differential forms on manifolds such as Stokes' Theorem and Mayer–Vietoris arguments generalize to orbifolds. A particular case is that, if $\mathcal{O}$ is a n-dimensional compact oriented orbifold, $\omega \otimes \eta \mapsto \int_{\mathcal{O}} \omega \wedge \eta$ induces a non-degenerate pairing

$$H^i_{\mathrm{dR}}(\mathcal{O}) \otimes H^{n-i}_{\mathrm{dR}}(\mathcal{O}) \longrightarrow \mathbb{R},$$

which gives us Poincaré duality on $H^*_{\mathrm{dR}}(\mathcal{O})$.

A.7 Actions on Orbifolds

Let G be a Lie group and $\mathcal{O}$ be an orbifold. We say that a smooth orbifold map $\mu : G \times \mathcal{O} \to \mathcal{O}$ is a **smooth action** of G on $\mathcal{O}$ if $|\mu| : G \times |\mathcal{O}| \to |\mathcal{O}|$ is a continuous action. This definition is equivalent to the one given in [HS90, Section 3] in terms of the charts: for each $g_0 \in G$ and $x_0 \in |\mathcal{O}|$ there are charts $(\widetilde{U}, H, \phi)$ covering x_0 and $(\widetilde{V}, K, \psi)$ covering $|\mu|(g_0, x_0)$, together with a neighborhood A of g_0 in G and a smooth map $\tilde{\mu} : A \times \widetilde{U} \to \widetilde{V}$ such that $\psi(\tilde{\mu}(g, \tilde{u})) = |\mu|(g, \phi(\tilde{u}))$ for all $g \in A$ and all $\tilde{u} \in \widetilde{U}$. Moreover, for each $g \in A$, $\tilde{u} \mapsto \tilde{\mu}(g, \tilde{u})$ is a diffeomorphism from $\widetilde{U}$ onto an open subset of $\widetilde{V}$.

We say that μ is **effective** when $|\mu|$ is effective, and that μ is **proper** when the map

$$\begin{array}{rcl} G \times |\mathcal{O}| & \longrightarrow & |\mathcal{O}| \times |\mathcal{O}| \\ (g, x) & \longmapsto & (|\mu|(g, x), x) \end{array}$$

is proper.

For each fixed $g \in G$, a smooth action defines a diffeomorphism $\mu^g : \mathcal{O} \to \mathcal{O}$. In particular, each orbit Gx is contained within the single stratum $\Sigma_{\Gamma_x} \subset |\mathcal{O}|$, of which it is a submanifold [see GG+17, Lemma 2.11]. We have the following useful relation between the action of local groups of the orbifold and the isotropy groups of the action [see GG+17, Proposition 2.12 and Corollary 2.13].

Proposition A.10 (Galaz-García et al)**.** *Let $\mathcal{O}$ be an orbifold with a smooth effective* *action by a compact lie group G and let $x \in |\mathcal{O}|$. Then there is a chart $(\widetilde{U}, \widetilde{G}_x, \phi)$ around x such that U is G_x-invariant and $\widetilde{G}_x$ is an extension*[6] *of G_x by Γ_x satisfying $\widetilde{U}/\widetilde{G}_x = U/G_x$. Moreover, Γ_x injects as a normal subgroup of $\widetilde{G}_x$, commutes with any of its connected subgroups and satisfies $\widetilde{G}_x/\Gamma_x = G_x$.*

Following the manifold case, an orbifold $\mathcal{O}$ of even dimension $2n$ is a **torus orbifold** when it is closed, oriented and admits a smooth effective action of a torus $\mathbb{T}^n$ with non-empty fixed-point set. We will use the following result [GG+17, Lemma 3.3].

Proposition A.11 (Galaz-García et al)**.** *The fixed-point set of a torus orbifold consists of finitely many (hence isolated) points.*

A.8 Riemannian Orbifolds

A **Riemannian metric** on an orbifold $\mathcal{O}$ is a symmetric, positive tensor field $\mathrm{g} \in \bigotimes_0^2(\mathcal{O})$. The pair $(\mathcal{O}, \mathrm{g})$ is then a **Riemannian orbifold**. As already mentioned for tensor fields in general, in a chart $(\widetilde{U}, H, \phi)$ a Riemannian metric yields an H-invariant Riemannian metric on $\widetilde{U}$, and so we have an induced inner product $\langle \cdot, \cdot \rangle_x := \mathrm{g}_x(\cdot, \cdot)$ on $T_x\mathcal{O}$ for each x. Also, the embeddings between charts of $\mathcal{O}$ become isometries and the pseudogroup $\mathscr{H}_\mathcal{O}$ becomes, with the induced Riemannian metric in $U_\mathcal{O}$, a pseudogroup of local isometries. Hence, the Levi-Civita connection ∇ on $U_\mathcal{O}$ is invariant by the changes of charts and thus can be seen as a covariant derivative on $T\mathcal{O}$. We define the curvature tensor of $\mathcal{O}$ and its derived curvature notions as the curvature tensor of ∇ and the respective derived curvatures.

If $\mathcal{O}$ is Riemannian and $\gamma : [a, b] \to \mathcal{O}$ is a piecewise smooth curve, its **length** can be defined as

$$\ell(\gamma) := \int_a^b \|\gamma'(t)\| dt.$$

We induce the length structure $d(x, y) = \inf \ell(\gamma)$ on $|\mathcal{O}|$, where the infimum is taken amongst all piecewise smooth curves connecting x and y. We say that $\mathcal{O}$ is **complete** when $(|\mathcal{O}|, d)$ is a complete metric space. Lower bounds in Alexandrov

[6]That is, there is a short exact sequence $0 \to \Gamma_x \to \widetilde{G}_x \to G_x \to 0$.

curvature are preserved upon quotienting by finite groups of isometries [see BBI01, Proposition 10.2.4], hence, if the sectional curvature of $\mathcal{O}$ is bounded below by $c \in \mathbb{R}$, then the Alexandrov curvature of $|\mathcal{O}|$ is also bounded below by c. This way we can apply the results of metric geometry to orbifolds.

As in the manifold case, any (smooth) orbifold admits a Riemannian metric [see MM03, Proposition 2.20; CR02, Lemma 4.2.3]. Many results in the Riemannian geometry of manifolds generalize to orbifolds, as can be seen, for example, in [Bor93; Bor92; KL14, Section 2.5; CR02, Section 4.2; Yer14, Section 2.3]. We point out the following version of the Bonnet–Myers Theorem for orbifolds due to J. Borzellino [see Bor93, Corollary 21; also Yer14, Corollary 2.3.4, for a simpler proof].

Theorem A.12 (Bonnet–Myers Theorem for orbifolds)**.** *Let $\mathcal{O}$ be a complete n-dimensional Riemannian orbifold satisfying* $\mathrm{Ric}_{\mathcal{O}} \geq n-1$. *Then* $\mathrm{diam}(|\mathcal{O}|) \leq r$. *In particular, $|\mathcal{O}|$ is compact.*

A smooth curve $\gamma : I \to \mathcal{O}$ is a **geodesic** if $\nabla_{\gamma'}\gamma' = 0$, that is, if it lifts in local charts to curves satisfying the geodesic equation. When $\mathcal{O}$ is complete, for any $x \in |\mathcal{O}|$ and $v \in C_x|\mathcal{O}|$ there is a unique geodesic $\gamma : \mathbb{R} \to \mathcal{O}$ with $|\gamma|(0) = x$ and $\gamma'(0) = v$ [see KL14, Lemma 2.16]. We can therefore define the exponential map as in the manifold case, obtaining a smooth (orbifold) map $\exp_x : T_x\mathcal{O} \to \mathcal{O}$. We stress that geodesics passing through the singular set of $\mathcal{O}$ fail to be locally minimizing [see Bor93, Proposition 15].

A diffeomorphism $f : (\mathcal{O}, \mathrm{g}^{\mathcal{O}}) \to (\mathcal{P}, \mathrm{g}^{\mathcal{P}})$ is an **isometry** if $f^*(\mathrm{g}^{\mathcal{P}}) = \mathrm{g}^{\mathcal{O}}$. We have the following analogue of the Myers–Steenrod Theorem due to A. Bagaev an N. Zhukova [BZ07, Theorem 1].

Theorem A.13 (Myers–Steenrod Theorem for orbifolds)**.** *Let $\mathcal{O}$ be an n-dimensional Riemannian orbifold. With the compact-open topology, the isometry group* $\mathrm{Iso}(\mathcal{O})$ *of $\mathcal{O}$ has a Lie group structure with which it acts smoothly and properly on $\mathcal{O}$.*

There is also the following orbifold version of the Synge–Weinstein Theorem due to D. Yeroshkin [see Yer14, Theorem 2.3.5].

Theorem A.14 (Synge–Weinstein Theorem for orbifolds)**.** *Let $\mathcal{O}$ be a compact, oriented, n-dimensional orbifold with positive sectional curvature and let $f \in$* $\mathrm{Iso}(\mathcal{O})$.

Suppose that f preserves orientation if n is even and reverses orientation if n is odd. Then f has a fixed point.

A vector field X on $\mathcal{O}$ is a **Killing vector field** if $\mathcal{L}_X \mathrm{g} = 0$, which means that the local flows of X act by isometries. When $\mathcal{O}$ is complete, X generates a one-parameter subgroup of $\mathrm{Iso}(\mathcal{O})$ and it follows that the Lie algebra $\mathfrak{iso}(\mathcal{O})$ of Killing vector fields is the Lie algebra of $\mathrm{Iso}(\mathcal{O})$.

Another construction that the presence of a Riemannian metric on $\mathcal{O}$ enables us to do is that of the orthonormal frame bundle of $\mathcal{O}$, as follows. If $(\widetilde{U}, H, \phi)$ is a chart of $\mathcal{O}$, consider the orthogonal frame bundle $\widetilde{U}^{\curlywedge}$ with the induced action of H by $h \cdot (\tilde{x}, B) = (h(\tilde{x}), \mathrm{d}h_{\tilde{x}} B)$. This is actually a free action, therefore $\widetilde{U}^{\curlywedge}/H$ is a manifold that inherits a proper, effective and almost free $\mathrm{O}(n)$-action from the action of $\mathrm{O}(n)$ on $\widetilde{U}^{\curlywedge}$. Taking the quotient by this action we obtain the natural projection $\widetilde{U}^{\curlywedge}/H \to U$. The manifolds $\widetilde{U}^{\curlywedge}/G$ glue together to form a manifold $\mathcal{O}^{\curlywedge}$, the **orthonormal frame bundle** of $\mathcal{O}$. With the natural projection, it defines an orbibundle $\mathcal{O}^{\curlywedge} \to \mathcal{O}$ [see ALR07, Section 1.3; MM03, Section 2.4, for more details].

An orientation of $\mathcal{O}$ corresponds to a decomposition $\mathcal{O}^{\curlywedge} = \mathcal{O}^{\curlywedge}_{+} \sqcup \mathcal{O}^{\curlywedge}_{-}$. Then $\mathcal{O}^{\curlywedge}_{+}$ is an $\mathrm{SO}(n)$-orbibundle over $\mathcal{O}$. A key point of the construction above is that the quotient orbifold $\mathcal{O}^{\curlywedge}_{+}/\!/\mathrm{SO}(n)$ is isomorphic to $\mathcal{O}$ [see MM03, Proposition 2.22]. The orientability is needed so that we have an action of the *connected* Lie group $\mathrm{SO}(n)$ inheriting $\mathcal{O}$ as a quotient, which ensures that the holonomy of an orbit matches the corresponding isotropy group. A similar construction can be carried over for non-orientable orbifolds by first taking the complexification $\widetilde{U}^{\curlywedge} \otimes \mathbb{C}$, which leads to an $\mathrm{U}(n)$-orbibundle $\mathcal{O}^{\curlywedge}_{\mathbb{C}}$ over $\mathcal{O}$ [see MM03, Proposition 2.23]. Moreover, the Riemannian metric g on $\mathcal{O}$ induces a Riemannian metric on $\mathcal{O}^{\curlywedge}_{\mathbb{C}}$ such that $\mathrm{U}(n)$ acts by isometries and $\mathcal{O}$ is isometric to the quotient $\mathcal{O}^{\curlywedge}_{\mathbb{C}}/\!/\mathrm{U}(n)$ [see AB15, Proposition 5.21]. Hence there is the following converse to Propositions A.4 and 1.5.

Proposition A.15. *Every Riemannian orbifold is isometric to the quotient space of an almost free isometric action of a compact connected Lie group.*

A.9 Orbifold Homotopy and (Co)Homology

Satake's isomorphism (Theorem A.9) shows that De Rham's cohomology does not capture information on the smooth structure of an orbifold. An alternative cohomology theory that is sensitive to the orbifold structure can be defined via the classifying space $B\mathcal{O}$ of an orbifold $\mathcal{O}$, that we now introduce following [BG08, Section 4.3]. Let $E\mathrm{O}(n) \to B\mathrm{O}(n)$ be the universal principal $\mathrm{O}(n)$-bundle [see, for instance, Hus94, Chapter 4] and choose a riemannian metric on $\mathcal{O}$. The **classifying space** of $\mathcal{O}$ is the space

$$B\mathcal{O} := \mathcal{O}^{\curlywedge} \times_{\mathrm{O}(n)} E\mathrm{O}(n).$$

Note that if we consider $\mathcal{O}$ as a Lie groupoid $\mathrm{G}_{\mathcal{O}}$ (see Remark A.8), we can write simply $\mathcal{O}^{\curlywedge} = U_{\mathcal{O}}^{\curlywedge}/\mathrm{G}_{\mathcal{O}}$, where $U_{\mathcal{O}}$, as in Example A.6, coincides with the space of objects $\mathrm{Obj}(\mathrm{G}_{\mathcal{O}})$ (see Chapter 6). Define

$$E\mathcal{O} := U_{\mathcal{O}}^{\curlywedge} \times_{\mathrm{O}(n)} E\mathrm{O}(n).$$

The $\mathrm{G}_{\mathcal{O}}$-action on $U_{\mathcal{O}}^{\curlywedge}$ commutes with the action of $\mathrm{O}(n)$, so $\mathrm{G}_{\mathcal{O}}$ acts on both the total space $E\mathcal{O}$ (the action on the factor $E\mathrm{O}(n)$ being trivial) and the base space of the fiber bundle $E\mathcal{O} \to U_{\mathcal{O}}$. Upon quotienting by these actions one obtains a commutative diagram

$$\begin{array}{ccc} E\mathcal{O} & \longrightarrow & B\mathcal{O} \\ \big\downarrow & & \big\downarrow p \\ U_{\mathcal{O}} & \longrightarrow & |\mathcal{O}|. \end{array}$$

A fiber of p over a regular point of $|\mathcal{O}|$ is $E\mathrm{O}(n)$, while a fiber over a singular point x is an Eilenberg–MacLane space $K(\Gamma_x, 1)$, that is, a connected topological space satisfying $\pi_1(K(\Gamma_x, 1)) \cong \Gamma_x$ and $\pi_i(K(\Gamma_x, 1)) = 0$ for all $i > 1$.

The **orbifold homology, cohomology and homotopy groups** of $\mathcal{O}$ are, by definition, the homology, cohomology and homotopy groups of $B\mathcal{O}$, that we denote, respectively, by $H_i^{\mathrm{orb}}(\mathcal{O}, R)$, $H^i_{\mathrm{orb}}(\mathcal{O}, R)$ and $\pi_i^{\mathrm{orb}}(\mathcal{O})$, for some unital ring R. When $\mathcal{O}$ is a manifold these groups coincide with the usual ones. Moreover, this definition of $\pi_1^{\mathrm{orb}}(\mathcal{O})$ is equivalent to the one in [see HD84] via homotopy of loops on $\mathcal{O}$.

Although these (co)homology groups are, in general, sensitive to the orbifold structure of $\mathcal{O}$, the study of the Leray spectral sequence of p leads to the following result [see BG08, Corollary 4.3.8; also Hae86, Proposition 4.2.3].

Theorem A.16. *If R is a field, then $H_i^{\mathrm{orb}}(\mathcal{O}, R) \cong H_i(|\mathcal{O}|, R)$ and $H^i_{\mathrm{orb}}(\mathcal{O}, R) \cong H^i(|\mathcal{O}|, R)$.*

Bibliography

[AB15] M. Alexandrino and R. Bettiol. *Lie Groups and Geometric Aspects of Isometric Actions*. Springer, 2015.

[AH86] A. Alaoui and G. Hector. “Décomposition de Hodge basique pour un feuilletage riemannien”. In: *Annales de l’Institut Fourier* 36.3 (1986), pp. 207–227.

[ALR07] A. Adem, J. Leida, and Y. Ruan. *Orbifolds and Stringy Topology*. Cambridge Tracts in Mathematics 171. Cambridge University Press, 2007.

[AP93] C. Allday and V. Puppe. *Cohomological Methods in Transformation Groups*. Cambridge Studies in Advanced Mathematics 32. Cambridge University Press, 1993.

[Asa+14] M. Asaoka et al. *Foliations: Dynamics, Geometry and Topology*. Advanced Courses in mathematics CRM Barcelona. Birkhäuser, 2014.

[ASH85] A. Alaoui, V. Sergiescu, and G. Hector. “La cohomologie basique d’un feuilletage riemannien est de dimension finie”. In: *Mathematische Zeitschrift* 188 (1985), pp. 593–599.

[BB12] J. Borzellino and V. Brunsden. “Elementary orbifold differential topology”. In: *Topology and its Applications* 159.17 (2012), pp. 3583–3589.

[BBI01] D. Burago, Y. Burago, and S. Ivanov. *A Course in Metric Geometry*. Graduate Studies in Mathematics 33. American Mathematical Society, 2001.

[Ben01] I. Bendixson. “Sur les courbes définies par des équations différentielles”. In: *Acta Mathematica* 24.1 (1901), pp. 1–88.

[BG08] C. Boyer and K. Galicki. *Sasakian Geometry*. Oxford Mathematical Monographs. Oxford University Press, 2008.

[Bor92] J. Borzellino. "Riemannian geometry of orbifolds". University of California, Los Angeles, 1992.

[Bor93] J. Borzellino. "Orbifolds of maximal diameter". In: *Indiana University Mathematics Journal* 42.1 (1993), pp. 37–53.

[BPR03] V. Belfi, E. Park, and K. Richardson. "A Hopf index theorem for foliations". In: *Differential Geometry and its Applications* 18.3 (2003), pp. 319–341.

[Bre72] G. Bredon. *Introduction to Compact Transformation Groups*. Pure and Applied Mathematics 46. Academic press, 1972.

[Bre97] G. Bredon. *Sheaf Theory*. Vol. 170. Graduate Texts in Mathematics. Springer-Verlag, 1997.

[BT82] R. Bott and W. Tu. *Differential Forms in Algebraic Topology*. Graduate Texts in Mathematics. Springer-Verlag, 1982.

[BZ07] A. Bagaev and N. Zhukova. "The isometry groups of riemannian orbifolds". In: *Siberian Mathematical Journal* 48.4 (2007), pp. 579–592.

[CC00] A. Candel and L. Conlon. *Foliations I*. Graduate Studies in Mathematics 23. American Mathematical Society, 2000.

[CN85] C. Camacho and A. Neto. *Geometric Theory of Foliations*. Birkhäuser, 1985.

[Con57] P. Conner. "On the action of the circle group". In: *Michigan Mathematical Journal* 4 (1957), pp. 241–247.

[CR02] W. Chen and Y. Ruan. "Orbifold Gromov-Witten theory". In: *Orbifolds in Mathematics and Physics*. Ed. by A. Adem, J. Morava, and Y. Ruan. Contemporary Mathematics 310. American Mathematical Society, 2002, pp. 25–85.

[DK00] J. Duistermaat and J. Kolk. *Lie Groups*. Universitext. Springer, 2000.

[Dol82] I. Dolgachev. “Weighted projective varieties”. In: *Group Actions and Vector Fields*. Ed. by J. Carrel. Lecture Notes in Mathematics 956. Springer, 1982, pp. 34–71.

[Dom98] D. Domínguez. “Finiteness and tenseness theorems for riemannian foliations”. In: *American Journal of Mathematics* 120.6 (1998), pp. 1237–1276.

[Fed71] E. Fedida. “Sur les feuilletages de Lie”. In: *Comptes Rendus de l’Académie des Sciences* 272 (1971), pp. 999–1001.

[Fra61] T. Fraenkel. “Manifolds with Positive Curvature”. In: *Pacific Journal of Mathematics* 11 (1961), pp. 165–174.

[GG88] D. Gromoll and K. Grove. “The low-dimensional metric foliations of euclidean spheres”. In: *Journal of Differential Geometry* 28 (1988), pp. 143–156.

[GG+17] F. Galaz-García et al. “Torus orbifolds, slice-maximal torus actions and rational ellipticity”. In: *International Mathematics Research Notices* (2017), doi: 10.1093/imrn/rnx064.

[GH06] K. Guruprasad and A. Haefliger. “Closed geodesics on orbifolds”. In: *Topology* 45 (2006), pp. 611–641.

[Ghy84] E. Ghys. “Feuilletages riemanniens sur les varietes simplement connexes”. In: *Annales de l’Institut Fourier* 34 (1984), pp. 203–223.

[GL13] A. Gorokhovsky and J. Lott. “The index of a transverse Dirac-type operator: the case of abelian Molino sheaf”. In: *Journal für die Reine und Angewandte Mathematik (Crelles Journal)* 678 (2013), pp. 125–162.

[GS94] K. Grove and C. Searle. “Positivelly curved manifolds with maximal symmetry rank”. In: *Journal of Pure and Applied Algebra* 91 (1994), pp. 137–142.

[GT16] O. Goertsches and D. Töben. “Equivariant basic cohomology of riemannian foliations”. In: *Journal für die Reine und Angewandte Mathematik (Crelles Journal)* (2016), doi:10.1515/crelle–2015–0102.

[GW09] D. Gromoll and G. Walschap. *Metric Foliations and Curvature*. Vol. 268. Progress in Mathematics. Birkhäuser, 2009.

[Hae62] A. Haefliger. “Variétés feuilletées”. In: *Annali della Scuola Normale Superiore di Pisa* 16 (1962), pp. 367–397.

[Hae86] A. Haefliger. “Groupoïdes d’holonomie et classifiants”. In: *Astérisque* 116 (1986), pp. 321–331.

[Hat02] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[HD84] A. Haefliger and Q. Du. “Appendice: une présentation du groupe fondamental d’une orbifold”. In: *Astérisque* 116 (1984), pp. 98–107.

[Heb86] J. Hebda. “Curvature and focal points in riemannian foliations”. In: *Indiana University Mathematics Journal* 35 (1986), pp. 321–331.

[HR13] G. Habib and K. Richardson. “Modified differentials and basic cohomology for riemannian foliations”. In: *Journal of Geometric Analysis* 23 (2013), pp. 1314–1342.

[HS17] J. Harvey and C. Searle. “Orientation and symmetries of Alexandrov spaces with applications in positive curvature”. In: *Journal of Geometric Analysis* 27.2 (2017), pp. 1636–1666.

[HS90] A. Haefliger and E. Salem. “Riemannian foliations on simply connected manifolds and actions of tori on orbifolds”. In: *Illinois Journal of Mathematics* 34 (1990), pp. 706–730.

[Hus94] D. Husemoller. *Fibre Bundles*. Third Edition. Graduate Texts in Mathematics 20. Springer-Verlag, 1994.

[KL14] B. Kleiner and J. Lott. “Geometrization of three-dimensional orbifolds via Ricci flow”. In: *Astérisque* 365 (2014), pp. 101–177.

[Kob72] S. Kobayashi. *Transformation Groups in Differential Geometry*. Reprint of the 1972 Edition. Classics in Mathematics. Springer-Verlag, 1972.

[KT82] F. Kamber and Ph. Tondeur. “Harmonic foliations”. In: *Harmonic Maps*. Ed. by R. Knill, M. Kalka, and H. Sealey. Lecture Notes in Mathematics 949. Springer, 1982, pp. 87–121.

[KT83] F. Kamber and Ph. Tondeur. “Duality for riemannian foliations”. In: *Singularities.* Ed. by P. Orlik. Proceedings of Symposia in Pure Mathematics 40. American Mathematical Society, 1983, pp. 609–618.

[Lee13] J. Lee. *Introduction to Smooth Manifolds.* Second Edition. Graduate Texts in Mathematics 218. Springer, 2013.

[Lee97] J. Lee. *Riemannian Manifolds. An Introduction to Curvature.* Graduate Texts in Mathematics 176. Springer-Verlag, 1997.

[Ler10] E. Lerman. “Orbifolds as stacks?” In: *L'Enseignement Mathématique* 56.3-4 (2010), pp. 315–363.

[LU04] E. Lupercio and B. Uribe. “Gerbes over orbifolds and twisted K-theory”. In: *Communications in Mathematical Physics* 245.3 (2004), pp. 449–489.

[Lóp92] J. López. “The basic component of the mean curvature of riemannian foliations”. In: *Annals of Global Analysis and Geometry* 10 (1992), pp. 179–194.

[Lóp93] J. López. “Morse inequalities for pseudogroups of local isometries”. In: *Journal of Differential Geometry* 37 (1993), pp. 603–638.

[MM03] I. Moerdijk and J. Mrčun. *Introduction to Foliations and Lie Groupoids.* Cambridge studies in advanced mathematics 91. Cambridge University Press, 2003.

[Moe02] I. Moerdijk. “Orbifolds as groupoids: an introduction”. In: *Orbifolds in Mathematics and Physics.* Ed. by A. Adem, J. Morava, and Y. Ruan. Contemporary Mathematics 310. American Mathematical Society, 2002, pp. 205–222.

[Mol79] P. Molino. “Feuilletages riemanniens sur les variétés compactes; champs de Killing transverses”. In: *Comptes Rendus de l'Académie des Sciences* 289 (1979), pp. 421–423.

[Mol84] P. Molino. “Desingularisation des feuilletages riemanniens”. In: *American Journal of Mathematics* 106.5 (1984), pp. 1091–1106.

[Mol88] P. Molino. *Riemannian Foliations*. Progress in Mathematics 73. Birkhäuser, 1988.

[Moz85] W. Mozgawa. “Feuilletages de Killing”. In: *Collectanea Mathematica* 36.3 (1985), pp. 285–290.

[MP97] I. Moerdijk and D. Pronk. “Orbifolds, sheaves and groupoids”. In: *K-Theory* 12.1 (1997), pp. 3–21.

[Osh01] G. Oshikiri. “On transverse Killing fields of metric foliations of manifolds with positive curvature”. In: *Manuscripta Mathematica* 104 (2001), pp. 527–531.

[Pet06] P. Petersen. *Riemannian Geometry*. Second Edition. Graduate Texts in Mathematics 171. Springer, 2006.

[Poi92] H. Poincaré. *Sur les courbes définies par une équation différentielle*. 1892.

[PS01] T. Püttmann and C. Searle. “The Hopf conjecture for manifolds with low cohomogeneity or high symmetry rank”. In: *Proceedings of the American Mathematical Society* 130.1 (2001), pp. 163–166.

[Rei59] B. Reinhart. “Foliated manifolds with bundle-like metrics”. In: *Annals of Mathematics* 69 (1959), pp. 119–132.

[RS05] X. Rong and X. Su. “The Hopf conjecture for positively curved manifolds with abelian group actions”. In: *Communications in Contemporary Mathematics* 7 (2005), pp. 121–136.

[Rum79] H. Rummler. “Quelques notions simples en géométrie riemanniennes et leurs applications aux feuilletage compacts”. In: *Commentarii Mathematici Helvetici* 54 (1979), pp. 224–239.

[Sak97] T. Sakai. *Riemannian Geometry*. Translations of Mathematical Monographs 149. American Mathematical Society, 1997.

[Sal88] E. Salem. “Riemannian foliations and pseudogroups of isometries”. In: *Riemmanian Foliations*. Ed. by P. Molino. Progress in Mathematics 73. Birkhäuser, 1988, pp. 265–296.

[Sat56] I. Satake. "On a generalization of the notion of manifold". In: *Proceedings of the National Academy of Sciences of the United States of America* 42.6 (1956), pp. 359–363.

[Sat57] I. Satake. "The Gauss-Bonnet theorem for V-manifolds". In: *Journal of the Mathematical Society of Japan* 9.4 (1957), pp. 464–492.

[Sch56] A. Schmeding. "The diffeomorphism group of a non-compact orbifold". In: *Dissertationes Mathematicae* 507 (1956), pp. 1–179.

[Ser85] V. Sergiescu. "Cohomologie basique et dualité pour les feuilletages riemannien". In: *Annales de l'Institut Fourier* 33.3 (1985), pp. 137–158.

[Spa81] E. Spanier. *Algebraic Topology.* New Edition. McGraw-Hill, 1981.

[Ste74] P. Stefan. "Accessible sets, orbits and foliations with singularities". In: *Proceedings of the London Mathematical Society* 29 (1974), pp. 699–713.

[Sus73] H. Sussmann. "Orbits of families of vector fields and integrability of distributions". In: *Transactions of the American Mathematical Society* 180 (1973), pp. 171–188.

[Thu02] W. Thurston. *The Geometry and Topology of Three-Manifolds.* Electronic version 1.1 of lecture notes. 2002. URL: `www.msri.org/publications/books/gt3m`.

[Ton97] Ph. Tondeur. *Geometry of Foliations.* Monographs in Mathematics 90. Birkhäuser, 1997.

[Töb14] D. Töben. "Localization of basic characteristic classes". In: *Annales de l'Institut Fourier* 64.2 (2014), pp. 537–570.

[War83] F. Warner. *Foundations of Differentiable Manifolds and Lie Groups.* Graduate Texts in Mathematics 94. Springer-Verlag, 1983.

[Win83] H. Winkelnkemper. "The graph of a foliation". In: *Annals of Global Analysis and Geometry* 1.3 (1983), pp. 51–75.

[Yer14] D. Yeroshkin. "Riemannian orbifolds with non-negative curvature". University of Pennsylvania, 2014.

Index

G-structure, 81

basic
  Betti numbers, 23
  cohomology, 23
  connection, 28
  differential form, 22
  Euler characteristic, 23
  fibration, 29
  function, 19
  laplacian, 26
  tensor field, 22
bundle-like metric, 24

classifying space
  of a groupoid, 81
  of an orbifold, 102
closed foliation, 18
cocycle
  with values on a groupoid, 80
codimension of a foliation, 16
critical leaf closure, 63

defect, 30
  as difference of codimensions, 42
deformation
  of foliations, 46
differential form
  basic, 22
    harmonic, 27
  on an orbifold, 96

foliate
  morphism, 19
  vector field, 19
foliation, 16
  atlas, 15
  closed, 18
  codimension, 16
  homogeneous, 17
  homotopic deformtion, 46
  Kronecker, 17
  normal bundle, 18
  orientable, 18
  Riemannian, 24
    complete, 26
    defect, 30
    singular, 31
    taut, 27
  simple, 29

smooth, 16
stratification, 42
tangent bundle, 16
football orbifold, 88

geodesic
Riemannian orbifold, 100
groupoid, 79
holonomy, 79
topological, 79

Haefliger cocycle, 16
holonomy
group, 21
groupoid, 79
pseudogroup, 20
homogeneous foliation, 17
homothetic transformation, 40
homotopic
deformation, 46
foliations, 46
horizontally totally geodesic
submanifold, 35

index of a transverse vector field, 63

Killing
foliation, 32
transverse vector field, 31
vector field
for a transverse metric, 31
on a Riemannian orbifold, 101
Kronecker foliation, 17

leaf, 16
critical closure, 63
generic, 21
holonomy, 21
space, 20
without holonomy, 21

Molino
bundle, 28
sheaf, 31

orbibundle, 95
exterior, 96
section, 95
tensor, 96
orbifold, 86
as a groupoid, 94
atlas, 86
bad, 91
change of charts, 94
chart, 85
around a point, 87
group, 86
classifying space, 102
cohomology, 102
de Rham cohomology, 97
diffeomorphism, 89
differential form, 96
effective, 86
football, 88
fundamental group, 102
good, 91

homology, 102
homotopy groups, 102
local group, 87
orientation, 86
quotient, 91
Riemannian, 99
singular locus, 87
smooth, 86
smooth action on, 98
smooth map, 89
differential, 96
stratification, 87
structure, 86
tangent bundle, 95
tangent cone, 96
teardrop, 88
tensor field, 96
orientable
foliation, 18

plaque, 15, 41
pseudogroup, 93
equivalence, 94
holonomy, 20
of an orbifold, 94
orbit, 93

quotient orbifold, 91

Riemannian
foliation, 24
complete, 26
defect, 30
singular, 31
stratification, 42
taut, 27
metric
bundle-like, 24
orbifold, 99
as a metric space, 99
complete, 99
frame bundle, 101
geodesic, 100
isometry, 100
Killing vector field, 101

simple open set, 16
singular Riemannian foliation, 31
codimension, 42
dimension, 42
smooth action on an orbifold, 98
stratification
of a foliation, 42
of an orbifold, 87
stratum
regular, 42
orbifold, 87
singular, 42
structural algebra, 30
suspension of homomorphisms, 18
symmetry rank, 37

teardrop orbifold, 88
tensor field
basic, 22

on an orbifold, 96

transverse

index, 63

non-degenerate vector field, 63

orientation, 19

Ricci curvature, 25

Riemannian metric, 24

sectional curvature, 25

symmetry rank, 37

vector field, 20

tube, 40

vector field

on an orbifold, 96

transverse, 20

Killing, 31

non-degenerate, 63

weighted projective space, 91