Dumar Andres Ospina Morales

An Introduction to Conformal Invariance in String Theory

São Carlos, SP
2018

Dumar Andres Ospina Morales

An Introduction to Conformal Invariance in String Theory

Dissertação de Mestrado apresentada à Coordenação de Pós-Graduação em Física do Centro de Ciências Exatas e de Tecnologia (CCET) da Universidade Federal de São Carlos para a obtenção do título de Mestre em Ciências.

Universidade Federal de São Carlos (UFSCar)
Centro de Ciências Exatas e de Tecnologia (CCET)
Programa de Pós-Graduação em Física/CCET

Orientador: Prof. Dr. Fernando David Marmolejo Schmidtt

São Carlos, SP
2018

UNIVERSIDADE FEDERAL DE SÃO CARLOS

Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Física

Folha de Aprovação

Assinaturas dos membros da comissão examinadora que avaliou e aprovou a Defesa de Dissertação de Mestrado do candidato Dumar Andres Ospina Morales, realizada em 11/10/2018:

David M. Schmidtt

Prof. Dr. Fernando David Marmolejo Schmidtt
UFSCar

Prof. Dr. Javier Fernando Ramos Caro
UFSCar

Ferreira

Prof. Dr. Luiz Agostinho Ferreira
USP

Acknowledge

This study was finance in part by the Coordenação de Aperfeiçoamento de Pessoal de Nivel Superior-Brasil (CAPES)- Finance code 001, to which I would like to acknowledge.

I would like also to acknowledge Prof. Dr. Fernando David Marmolejo Schmidtt for having accepting me as his student and for spending a lot of his time in my formative process with weekly discussions about the topics presented in this work.

Abstract

In this work we study the formalism of Conformal Field Theory (CFT) and apply it to the theory of strings and superstrings. We also use the BRST quantization technique to obtain the string spectrum together with the celebrated critical dimension for the bosonic string, i.e. the $d = 26$. By performing a supersymmetric extension of the Polyakov action of the bosonic string we construct a superstring, which now includes fermionic degrees of freedom as well. The gauged fixed action of the latter theory leads to the critical dimension $d = 10$ after imposing the condition of the vanishing of the conformal anomaly at the quantum level.

Resumo

Neste trabalho estudamos o formalismo da Teoria do Campo Conformal (CFT) e aplicamos à teoria das cordas e supercordas. Também usamos a técnica de quantização BRST para obter o espectro de cordas junto com a célebre dimensão crítica para a corda bosônica, ou seja, a $d = 26$. Ao realizar uma extensão supersimétrica da ação de Polyakov da corda bosônica, construímos uma supercorda, que agora inclui também graus de liberdade fermiônicos. A ação do calibre fixo da última teoria leva à dimensão crítica $d = 10$ após impor a condição do desaparecimento da anomalia conformal no nível quântico.

Contents

Chapter 1

Introduction

Historical remarks

In 1966 Patashinskii, Pakrovskii and Kadanoff suggested that the fluctuations of some statistical systems are scale-invariant at a phase transition point, further, in 1970, Polyakov pointed out that the scale invariance may be seen as a particular case of conformal invariance[1]. In turn, conformally invariant quantum field theories describe the critical behavior of systems at second order phase transitions, the typical example is the Ising model in two dimensions. Therefore, the study of the conformal symmetries caught the interest of the physics community in the attempt to give a better explanation about these critical phenomenas raised in the quantum statistical mechanics. Two dimensional conformal field theories also provide the dynamical setup in string theory. In that context conformal invariance imposes constraints on the allowed space-time (which is called critical) dimension and the possible internal degrees of freedom. By these reasons, a classification of two dimensional conformal field theories could provide useful information on the set of different consistent first-quantized string theories that can be constructed.

Conformal invariance is therefore an extension of scale invariance, a symmetry under local dilations of space. Belavin, Polyakov and Zamolodchikov combined the representation theory of the Virasoro algebra with the idea of an algebra of local operators and showed how to construct completely solvable conformal theories, known as minimal models, in 1984[2]. Almost at the same time, in the attempt to unify all the forces of the nature into a single one, born string theory, in which two-dimensional scale invariance appears naturally. String theory, therefore, originated in the late 1960s at a time when no consistent field theories could describe strong and weak interactions. It arose as an attempt to explain the observed spectrum of hadrons and their interactions. In fact, Veneziano found in 1969 a formula for the scattering amplitude of four particles[3]. Afterwards, Nambu and Susskind showed that the dynamical object from which the Veneziano formula can be derived is a relativistic string. Nonetheless, String theory was discarded as a candidate theory of strong interactions, because the existence of a critical dimension, which we will show in this work to be 26 for the bosonic string and 10 for the superstring. Another obstacle for the interpretation of string theory as a theory of strong interactions was the existence of a massless spin two particle which is not present in the hadronic world. In 1974 Scherk and Schwarz suggested to interpret this massless spin two particle as the graviton, that is, the field quantum of gravity. This means that the tension of a string is related to the characteristic mass scale of gravity. They also pointed out that at low energies this "graviton" interacts according to the covariance laws of general relativity. This results imply that string theory could be, in principle, a candidate for a quantum theory of gravity.

Motivations

Here, we state a list of reasons which were considered at the time of choosing String theory as our subject of discussion in this work. Most of these reasons has to do with the introductory historical remarks mentioned above. The first reason has to do with the importance that string theory currently plays in the theoretical physic, it is the best proposal to couple gravity with quantum field theory, as mentioned, a manifestation of this will appear when we study the spectrum of the bosonic string. The second reason is more pragmatic, most of the methods used in string theory can be used to resolve another physical problems, this is for instance the case of the conformal field theory or also the study of supersymmetric theories, these are treatments that also serve for models in statistical mechanics, among others. The Faddeev-Popov procedure and BRST quantization are another examples of methods that we use in this work but their application extends to another theories endowed with a gauge symmetry. No less important is the fact that we will use recurrently the tools of quantum field theory in this work. A final reason which motivated this dissertation has to do with the mathematical structure of the string theory, in particular, because it is the best arena to use the machinery of some of the modern mathematics.

The residual symmetry

It is pertinent to explain how the conformal invariance, which will be defined in the chapter 2, will manifest itself in the bosonic string theory. We begin with the Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}, \qquad \det g_{\alpha\beta} = g. \tag{1.1}$$

An explanation is in order: Gravity couples with field theory, in string theory, through the degrees of freedom of a two-parameter depending field, $g_{\alpha\beta}\left(\sigma^0,\sigma^1\right)$[1], where the two parameters σ^0 and σ^1 parameterize the *world-sheet*. By the way, the world-sheet is the two-dimensional analogue of the world-line for the point particle when propagates in a $d-$dimensional Minkowski spacetime. This field is manipulated as being a metric, however it does not correspond exactly to a metric. Then, in analogy with a point-particle, where there are d coordinates $x^\mu\left(\tau\right)$ on the target space (Minkowski spacetime), we have in string theory d scalar fields $X^\mu\left(\sigma^0,\sigma^1\right)$. The parameter α' is related to the tension of the string, namely, if T is the tension of the string, then $\alpha' = \frac{1}{2\pi T}$.

Polyakov action presents one global symmetry, Poincaré invariance, and two local symmetries. The first of these local symmetries is the invariance under reparametrizations, $\sigma^\alpha \longrightarrow \tilde{\sigma}^\alpha\left(\sigma^0,\sigma^1\right)$, in infinitesimal form corresponding to $\tilde{\sigma}^\alpha = \sigma^\alpha + \xi^\alpha\left(\sigma^0,\sigma^1\right)$. The second one is the Weyl invariance, that is, invariance under point-wise rescaling of the metric $g_{\alpha\beta} \longrightarrow \Omega^2\left(\sigma^0,\sigma^1\right)g_{\alpha\beta}$, for $\Omega = e^{2\omega\left(\sigma^0,\sigma^1\right)}$, this corresponds to the infinitesimal transformations of the form $\delta g_{\alpha\beta} \longrightarrow 2\omega\left(\sigma^0,\sigma^1\right)g_{\alpha\beta}$. These local invariances allow for the gauge choice $g_{\alpha\beta} = \Omega^2\left(\sigma^0,\sigma^1\right)\eta_{\alpha\beta}$, where $\eta_{\alpha\beta} = \mathrm{diag}(-1,1)$. Weyl invariance allows to perform a transformation such that we set $g_{\alpha\beta} = \eta_{\alpha\beta}$. This gauge choice is called the conformal gauge. If we use light-cone coordinates, $\sigma^\pm = \sigma^0 \pm \sigma^1 \Longrightarrow 2\partial_\pm = \partial_{\sigma^0} \pm \partial_{\sigma^1}$, we can note that the action (1.1) still has a local symmetry, this action is invariant under reparametrizations of the form $\sigma^\pm \longrightarrow \tilde{\sigma}^\pm\left(\sigma^\pm\right)$, which correspond, in infinitesimal form, to $\tilde{\sigma}^\pm = \sigma^\pm + \xi^\pm\left(\sigma^\pm\right)$, this symmetry is called a residual symmetry. If we perform a Wick rotation ($\sigma^0 \longrightarrow -i\sigma^2$), the metric $g_{\alpha\beta}$ acquires a $(+,+)$ signature with parameters σ^1 and σ^2. The light-cone coordinates $\sigma^\pm$ can be promoted to complex coordinates z and $\bar{z}$, and the residual symmetry transformations acquire the form $z' = z + \xi\left(z\right)$ and $\bar{z}' = \bar{z} + \bar{\xi}\left(\bar{z}\right)$. These functions $\xi\left(z\right)$ and $\bar{\xi}\left(\bar{z}\right)$ are holomorphic and antiholomorphic mappings from the complex plane onto itself. This is how conformal symmetry appears in string theory.

About this work

The topics in this dissertation are presented in the following form: Chapter 2 spans the main topics of this work, therefore, we are going to explain how it is organized. In section 2.1 the conformal group is presented and the generators are derived. In section 2.2 we see what conformal invariance implies on classical fields, we also introduce the energy-momentum tensor. Section 2.3 is devoted to explain how conformal invariance affects the correlation functions and we derive the Ward identity. In the first three sec-

[1]This metric has $(-,+)$ signature , if we make a Wick rotation ($\sigma^0 \longrightarrow -i\sigma^2$) we obtain a metric $g_{\alpha\beta}$, with parameters σ^1 and σ^2.

tions mentioned before, we work in dimension $d \geq 3$. In sections 2.4 and 2.5 we repeat the process of the previous sections but in the $d = 2$ case, we will see that there are infinite generators of the conformal transformations in this dimension. However, between this infinite group we distinguish the global conformal group, which is formed only by the one to one invertible mappings of the complex plane onto itself, primary fields are introduced as well. Also, the residual symmetry of the Polyakov action is identified with the two dimensional conformal transformations. In section 2.6 we study two systems which have to do with the bosonic string. The first one is the matter or XX system, this system is nothing but the gauge-fixed Polyakov action after performing a Wick rotation. The second system is the b, c ghost system, which is presented in a general way, later we will see how particular cases of this system arise in string theory. In section 2.7 we introduce the concept of central charge or conformal anomaly, the Schwarzian derivative is presented and it is showed how the energy-momentum tensor transforms as a primary field only under the global conformal transformations. Section 2.8 and 2.9 are devoted to present CFT in an operator formalism, we calculate the Virasoro algebra and review the Hilbert space for the CFT. In section 2.10 and 2.11 we apply the operator formalism to the matter and the ghost systems respectively. Section 2.12 is devoted to carry out the Faddeev-Popov procedure on the Polyakov action, we will see then that a particular b, c ghost system arises as the Faddeev-Popov determinant. The BRST quantization, which is a method of quantizing gauge theories, is presented in section 2.13[2]. We find the celebrated critical dimension for the bosonic string theory, this critical dimension results to be $d = 26$. Section 2.14 is devoted to obtain the spectrum of the bosonic string by imposing the physical state conditions over the excited states. At level zero this leads us to the Tachyon in both, open a closed string, at level one this leads us to the Photon state in the open string and to the Graviton, Dilaton and Kalb-Ramond states in the closed string case. The appearance of the Tachyon and its meaning won't be considered in this work.

Despite the above mentioned, the bosonic string does not present fermion states in its spectrum, namely it is called a toy model. This is one of the main motivations to introduce fermionic degrees of freedom to the theory by coupling to Polyakov action (1.1) two kind of spinors fields. The first spinor field Ψ^μ is a two-dimensional Majorana spinor corresponding to the supersymmetric partner of the X^μ field. The second is the gravitino χ_α which is the supersymmetric partner of $g_{\alpha\beta}$. There is an analogue to the conformal gauge for these supersymmetric theory, which is called the superconformal gauge and where we set $\chi_\alpha = 0$ in addition to the usual conformal gauge. As in the bosonic case, this gauge choice will give rise to a residual symmetry, the su-

[2]"BRST quantization is a refinement of the Faddeev-Popov procedure that makes explicit the fact that the quantization is independent of a choice of a particular gauge", P. Deligne, P. Etingof, D. S. Freed, L. S. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, E. Witten, Quantum Fields and Strings: A course for mathematicians, Vol 1.

perconformal invariance. The new non-gauge-fixed action won't be presented in this work and we will begin to work directly with the gauge-fixed (superstring) action in the section 3.1.
The superstring action is invariant under supersymmetry transformations, they are transformations interchanging bosonic and fermionic degrees of freedom, the generator of such transformations is a field T_F with conformal dimension $\frac{3}{2}$. Since the matter part or Polyakov action has been augmented by adding fermionic degrees of freedom, new ghost terms will appear, thus giving rise to a β, γ system. We don't implement the BRST procedure in full for the superstring action. However, by demanding the vanishing of the central charge of a system formed by the matter, b, c and β, γ systems, we obtain the critical dimension for the superstring action, this result to be $d = 10$. In the section 3.2 we make a slightly discussion about the periodicity conditions over the fermionic fields and how different sectors arise, they are called the Ramond and Neveu-Schwarz sectors. The algebra satisfied by the Virasoro modes (the modes of the energy momentum tensor) and the modes of T_F, will be called the Ramond or Neveu-Schwarz algebra depending on the sector where we are on.

This work is just a concise review about the formalism and some topics of interest to me, but it does not pretend to make some new contribution. It is for this reason that it will not be presented some specific application of String theory per se.

Chapter 2

Conformal invariance and string action

In this chapter the basic concepts and tools in conformal field theory (CFT) are introduced, it is made a short review of the conformal group in dimension $d \geq 3$, to go after to the case of interest in string theory, the $d = 2$ case. By doing so, it is presented recurrence expressions to calculate the Noether's currents, energy momentum-tensor, Ward identities and the operator product expansion OPE. We use these tools to find the TT OPE in the matter $XX-$system (gauge-fixed action of the bosonic string) and the $b, c-$ghost system, (this arises as the Faddeev-Popov ghost system) and leads to the concept of central charge, which we will show to be related to the Weyl anomaly. We present the BRST quantization procedure, which leads to the critical dimension $d = 26$, in which the BRST charge satisfy the nilpotency condition, also, it allows to decouple non physical states from the string-spectrum.

The main reference used to develop conformal field theory are the books by Di Francesco [4], Ginsparg lectures [5], Lüst and Theisen [6] and the last version of this with Blumenhagen [7] and also the book by Polchinski [8]. Another useful references used were [9], [10], [11], [12] and [13], it was of great help the readings [14] and [15].

2.1 The conformal group

A conformal transformation of the coordinates in the $d-$ dimensional spacetime is an invertible map $\boldsymbol{x} \longrightarrow \boldsymbol{x}'$, which leaves the metric tensor invariant up a scale factor

$$g'_{\mu\nu}\left(\boldsymbol{x}'\right) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}\left(\boldsymbol{x}\right) = \Lambda\left(\boldsymbol{x}\right) g_{\mu\nu}\left(\boldsymbol{x}\right), \tag{2.1}$$

where we have denoted by $g_{\mu\nu}$ the metric tensor in the space-time of dimension d. The transformation only preserves angles not lengths. Considering the definition eq.(2.1) and an infinitesimal transformation $x^{\mu} \longrightarrow x'^{\mu} = x^{\mu} + \epsilon^{\mu}\left(\boldsymbol{x}\right)$ on it, we see that the metric at first order in ϵ, changes as,

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} - \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) \tag{2.2}$$

the requirement for the transformation to be conformal is then

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f\left(x\right) g_{\mu\nu}. \tag{2.3}$$

By taking the trace on both sides we have that

$$\begin{aligned} f\left(x\right) g_{\mu\nu} g^{\mu\nu} &= g^{\mu\nu} \partial_\mu \epsilon_\nu + g^{\mu\nu} \partial_\nu \epsilon_\mu, \\ f\left(x\right) &= \frac{2}{d} \partial_\sigma \epsilon^\sigma. \end{aligned} \tag{2.4}$$

We assume, for simplicity, that the conformal transformation is an infinitesimal deformation of the Euclidean metric $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}\left(1, 1, ..., 1\right)$. By applying an extra derivative ∂_ρ on eq.(2.3), permuting the indices and taking a linear combination, we arrive at

$$2\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\mu\rho} \partial_\nu f + \eta_{\rho\nu} \partial_\mu f - \eta_{\mu\nu} \partial_\rho f, \tag{2.5}$$

by contracting with $\eta^{\mu\nu}$ on both sides of the equation above, we get

$$2\partial^2 \epsilon_\rho = \left(2 - d\right) \partial_\rho f. \tag{2.6}$$

Applying ∂_ν on this expression and ∂^2 on eq.(2.3), we find on the one side

$$\begin{aligned} \partial^2 \partial_\nu \epsilon_\mu &= \frac{\left(2-d\right)}{2} \partial_\nu \partial_\mu f, \\ \partial^2 \partial_\mu \epsilon_\nu &= \frac{\left(2-d\right)}{2} \partial_\mu \partial_\nu f, \\ \partial^2 \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) &= \left(2-d\right) \partial_\mu \partial_\nu f, \end{aligned}$$

and from eq.(2.3)

$$\partial^2 \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) = \eta_{\mu\nu} \partial^2 f,$$

so that

$$\left(2-d\right) \partial_\mu \partial_\nu f = \eta_{\mu\nu} \partial^2 f. \tag{2.7}$$

Finally, contracting with $\eta^{\mu\nu}$, we have

$$\begin{aligned} \left(2-d\right) \partial^2 f &= d \partial^2 f, \\ \left(d-1\right) \partial^2 f &= 0. \end{aligned} \tag{2.8}$$

From eqs. (2.3)-(2.8), we can derive the explicit form of a conformal transformation in d dimensions. First, if $d = 1$, the equations above do not impose any constraint on the function f, because the notion of angle does not exist, and then any smooth transformation is a conformal transformation in one dimension. The case $d = 2$ will be studied in detail later. At the moment we concern in the cases $d \geq 3$. Equations (2.8) and (2.7) imply that $\partial_\mu \partial_\nu f = 0$, that is, the function f is at least lineal in the coordinates:

$$f(\boldsymbol{x}) = A + B_\mu x^\mu, \qquad (A,\, B_\mu \text{ constants}). \tag{2.9}$$

By substituting this in eq.(2.5), we see that $\partial_\mu \partial_\nu \epsilon_\rho$ is constant, which means that ϵ_μ is, at least, quadratic in the coordinates. Therefore we can write

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho, \qquad c_{\mu\nu\rho} = c_{\mu\rho\nu}. \tag{2.10}$$

We treat each power separately. The constant term a_μ amounts to an infinitesimal translation. Substitution of the linear term into (2.3), after used (2.4), yields

$$\begin{aligned} \partial_\mu \left(a_\nu + b_{\nu\rho} x^\rho\right) + \partial_\nu \left(a_\mu + b_{\mu\gamma} x^\gamma\right) &= \tfrac{2}{d} \partial_\sigma \left(a^\sigma + b^\sigma{}_\beta x^\beta\right) \eta_{\mu\nu}, \\ b_{\nu\mu} + b_{\mu\nu} &= \tfrac{2}{d} b^\sigma{}_\sigma \eta_{\mu\nu}. \end{aligned} \tag{2.11}$$

It is the sum of an antisymmetric part and a pure trace:

$$b_{\mu\nu} = \alpha \eta_{\mu\nu} + m_{\mu\nu}, \qquad m_{\mu\nu} = -m_{\nu\mu}. \tag{2.12}$$

[1]The pure trace represents an infinitesimal scale transformation, while the antisymmetric part is an infinitesimal rigid rotation. Finally, using (2.4) and by substituting the quadratic term in (2.10) into (2.5) yields

$$\begin{aligned} \partial_\mu \partial_\nu \epsilon_\rho &= \frac{1}{d} \left(\eta_{\mu\rho} \partial_\nu \partial_\sigma \epsilon^\sigma + \eta_{\rho\nu} \partial_\mu \partial_\sigma \epsilon^\sigma - \eta_{\mu\nu} \partial_\rho \partial_\sigma \epsilon^\sigma\right), \\ \partial_\mu \partial_\nu \left(c_{\rho\alpha\beta} x^\alpha x^\beta\right) &= \frac{1}{d} \left(\eta_{\mu\rho} \partial_\nu + \eta_{\rho\nu} \partial_\mu - \eta_{\mu\nu} \partial_\rho\right) \partial_\sigma \left(c^\sigma{}_{\lambda\gamma} x^\lambda x^\gamma\right), \\ c_{\rho\nu\mu} &= \eta_{\mu\rho} \frac{1}{d} c^\sigma{}_{\sigma\nu} + \eta_{\rho\nu} \frac{1}{d} c^\sigma{}_{\sigma\mu} - \eta_{\mu\nu} \frac{1}{d} c^\sigma{}_{\sigma\rho}, \end{aligned}$$

that is

$$c_{\rho\nu\mu} = \eta_{\mu\rho} b_\nu + \eta_{\rho\nu} b_\mu - \eta_{\mu\nu} b_\rho \quad , \qquad b_\mu = \frac{1}{d} c^\sigma{}_{\sigma\mu}. \tag{2.13}$$

[1]Since we can decompose all tensor as the sum of a symmetric part and an antisymmetric part, we have

$$b_{\mu\nu} = \frac{1}{2}\left(b_{\mu\nu} + b_{\nu\mu}\right) + \frac{1}{2}\left(b_{\mu\nu} - b_{\nu\mu}\right) = \frac{1}{d} b^\sigma{}_\sigma \eta_{\mu\nu} + \frac{1}{2}\left(b_{\mu\nu} - b_{\nu\mu}\right).$$

The corresponding infinitesimal transformation is then

$$\begin{aligned} x'^{\mu} &= x^{\mu} + c^{\mu}{}_{\nu\rho} x^{\nu} x^{\rho} \\ &= x^{\mu} + \eta^{\mu\sigma} \left(\eta_{\rho\sigma} b_{\nu} + \eta_{\sigma\nu} b_{\rho} - \eta_{\rho\nu} b_{\sigma} \right) x^{\nu} x^{\rho} \\ &= x^{\mu} + \eta^{\mu\sigma} \left(2 \left(\boldsymbol{b} \cdot \boldsymbol{x} \right) x_{\sigma} - b_{\sigma} x^{2} \right) \\ &= x^{\mu} + 2 \left(\boldsymbol{b} \cdot \boldsymbol{x} \right) x^{\mu} - b^{\mu} x^{2}, \end{aligned} \tag{2.14}$$

which is called special conformal transformation (SCT). The finite transformations corresponding to those above are the following:

$$\begin{array}{ccccc} \text{translation} & x'^{\mu} & = & x^{\mu} + a^{\mu}, \\ \text{dilatation} & x'^{\mu} & = & \alpha x^{\mu}, \\ \text{rigid rotation} & x'^{\mu} & = & M^{\mu}{}_{\nu} x^{\nu}, \\ \text{SCT} & x'^{\mu} & = & \frac{x^{\mu} - b^{\mu} x^{2}}{1 - 2\boldsymbol{b} \cdot \boldsymbol{x} + b^{2} x^{2}}. \end{array} \tag{2.15}$$

We can check that the infinitesimal version of the last expression is indeed (2.14). The SCT can also be expressed as (if from (2.15) we take the term $x'^{\mu} x'_{\mu}$)

$$\frac{x'^{\mu}}{x'^{2}} = \frac{x^{\mu}}{x^{2}} - b^{\mu}, \tag{2.16}$$

in other words, the SCTs can be understood as an inversion of x^{μ} plus a translation of b^{μ}, and following again by an inversion. Now, we are going to recall some definitions.

2.1.1 Noether's theorem and Ward identity

In this section we give a precise meaning of the symmetries in the context of a general field theory and derive Noether's theorem, which states that to every continuous symmetry in a field theory there is a conserved current.

Consider a collection of fields, which we collectively denote by $\boldsymbol{\Phi}$. The action functional will depend in general on $\boldsymbol{\Phi}$ and its derivative:

$$S = \int d^{d} x \mathcal{L} \left(\boldsymbol{\Phi}, \partial_{\mu} \boldsymbol{\Phi} \right). \tag{2.17}$$

We study the effect, over this action, of a transformation affecting the position and the fields

$$\boldsymbol{x} \longrightarrow \boldsymbol{x}', \qquad \boldsymbol{\Phi}(\boldsymbol{x}) \longrightarrow \boldsymbol{\Phi}'(\boldsymbol{x}') = \mathcal{F}(\boldsymbol{\Phi}(\boldsymbol{x})). \tag{2.18}$$

Such that, the new field $\boldsymbol{\Phi}'$ at $\boldsymbol{x}'$ is expressed as a function of the old field $\boldsymbol{\Phi}$ at $\boldsymbol{x}$. The change of the action under the transformation (2.18) is obtained by substituting $\boldsymbol{\Phi}'(\boldsymbol{x})$

by $\mathbf{\Phi}(x)$, the argument x is the same in both cases. The new action is

$$\begin{aligned}
S' &= \int d^d x \mathcal{L}\left(\mathbf{\Phi}'(x), \partial_\mu \mathbf{\Phi}'(x)\right) \\
&= \int d^d x' \mathcal{L}\left(\mathbf{\Phi}'(x'), \partial'_\mu \mathbf{\Phi}'(x')\right) \\
&= \int d^d x' \mathcal{L}\left(\mathcal{F}(\mathbf{\Phi}(x)), \partial'_\mu \mathcal{F}(\mathbf{\Phi}(x))\right) \\
&= \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}(\mathbf{\Phi}(x)), \left(\frac{\partial x^\nu}{\partial x'^\mu}\right) \partial_\nu \mathcal{F}(\mathbf{\Phi}(x))\right).
\end{aligned} \tag{2.19}$$

In the second line we have performed a change of integration variables $x \longrightarrow x'$ according to the transformation (2.18), which allows us to express $\mathbf{\Phi}'(x')$ in terms of $\mathbf{\Phi}(x)$ in the third line. In the last line we express x' in terms of x. We now study the effect, on the action, of an infinitesimal transformations of the general form

$$x'^\mu = x^\mu + \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}, \qquad \mathbf{\Phi}'(x') = \mathbf{\Phi}(x) + \epsilon_a \frac{\delta \mathcal{F}(x)}{\delta \epsilon_a}. \tag{2.20}$$

Here $\{\epsilon_a\}$ is a set of infinitesimal parameters. The generators G_a of a symmetry transformation is usually defined by the following expression for the infinitesimal transformation at the same point:

$$\delta_\epsilon \mathbf{\Phi} \equiv \mathbf{\Phi}'(x) - \mathbf{\Phi}(x) = -i\epsilon_a G_a \mathbf{\Phi}(x). \tag{2.21}$$

It can be related to (2.20) by noting that at first order in ϵ_a we have

$$\begin{aligned}
\mathbf{\Phi}'(x') &= \mathbf{\Phi}\left(x'^\mu - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}\right) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}\left(x'^\mu - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}\right) \\
&= \mathbf{\Phi}(x') - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a} \frac{\partial}{\partial x^\mu} \mathbf{\Phi}(x') + \epsilon_a \frac{\delta \mathcal{F}(x')}{\delta \epsilon_a} - \mathcal{O}\left(\epsilon_a^2\right) + \ldots \quad .
\end{aligned} \tag{2.22}$$

From (2.21) follows that

$$iG_a \mathbf{\Phi}(x) = \frac{\delta x^\mu}{\delta \epsilon_a} \partial_\mu \mathbf{\Phi}(x) - \frac{\delta \mathcal{F}(x)}{\delta \epsilon_a}. \tag{2.23}$$

Noether's theorem states that for each continuous symmetry of the action one may associate a current which is classically conserved. Given a symmetry, the action is invariant under the transformation (2.20) only if the parameters ϵ_a are independent of the position. However, we will suppose that the infinitesimal transformation (2.20) is not rigid (ϵ_a depending on the position), in order to derive the Noether's theorem.

From (2.20) we can write

$$\frac{\partial x'^{\nu}}{\partial x^{\mu}} = \frac{\partial}{\partial x^{\mu}}\left(x^{\nu} + \epsilon_a \frac{\delta x^{\nu}}{\delta \epsilon_a}\right) = \delta^{\nu}_{\mu} + \partial_{\mu}\left(\epsilon_a \frac{\delta x^{\nu}}{\delta \epsilon_a}\right). \tag{2.24}$$

The determinant of this matrix may be calculated to first order from the formula

$$\det\left(\mathbb{I} + E\right) \approx 1 + \mathrm{Tr}\left(E\right), \qquad \left(E \text{ small}\right). \tag{2.25}$$

We obtain

$$\left|\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right| \approx 1 + \mathrm{Tr}\left(\partial_{\mu}\left(\epsilon_a \frac{\delta x^{\nu}}{\delta \epsilon_a}\right)\right) = 1 + \partial_{\mu}\left(\epsilon_a \frac{\delta x^{\mu}}{\delta \epsilon_a}\right). \tag{2.26}$$

[2]With the help of these results above, the transformed action S', eq.(2.19), may be written as

$$\begin{aligned} S' &= \int d^d x \left(1 + \partial_{\mu}\left(\epsilon_a \frac{\delta x^{\mu}}{\delta \epsilon_a}\right)\right) \\ &\quad \times \mathcal{L}\left(\mathbf{\Phi}\left(\boldsymbol{x}\right) + \epsilon_a \frac{\delta \mathcal{F}\left(\boldsymbol{x}\right)}{\delta \epsilon_a}, \left(\delta^{\nu}_{\mu} - \partial_{\mu}\left(\epsilon_a \frac{\delta x^{\nu}}{\delta \epsilon_a}\right)\right)\left(\partial_{\nu}\mathbf{\Phi}\left(\boldsymbol{x}\right) + \partial_{\nu}\left(\epsilon_a \frac{\delta \mathcal{F}\left(\boldsymbol{x}\right)}{\delta \epsilon_a}\right)\right)\right). \end{aligned} \tag{2.27}$$

The variation $\delta S = S' - S$ of the action contains terms with no derivatives of ϵ_a. These terms are zero if the action is symmetric under rigid transformations, which is assumed. Then δS involves just the first derivative of ϵ_a, obtained by expanding the Lagrangian to first order in ϵ_a. We write,

$$\begin{aligned} S' &= \int d^d x \left(1 + \partial_{\mu}\left(\epsilon_a \frac{\delta x^{\mu}}{\delta \epsilon_a}\right)\right) \\ &\quad \times \mathcal{L}\left(\mathbf{\Phi}\left(\boldsymbol{x}\right) + \epsilon_a \frac{\delta \mathcal{F}\left(\boldsymbol{x}\right)}{\delta \epsilon_a}, \partial_{\mu}\mathbf{\Phi}\left(\boldsymbol{x}\right) + \partial_{\mu}\left(\epsilon_a \frac{\delta \mathcal{F}\left(\boldsymbol{x}\right)}{\delta \epsilon_a}\right) - \partial_{\mu}\left(\epsilon_a \frac{\delta x^{\nu}}{\delta \epsilon_a}\right)\partial_{\nu}\mathbf{\Phi}\left(\boldsymbol{x}\right)\right) \\ &= \int d^d x \left(1 + \partial_{\mu}\epsilon_a \frac{\delta x^{\mu}}{\delta \epsilon_a} + \epsilon_a \partial_{\mu}\frac{\delta x^{\mu}}{\delta \epsilon_a}\right)\left\{\mathcal{L}\left(\mathbf{\Phi}\left(\boldsymbol{x}\right), \partial_{\mu}\mathbf{\Phi}\left(\boldsymbol{x}\right)\right) + \epsilon_a \frac{\delta \mathcal{F}\left(\boldsymbol{x}\right)}{\delta \epsilon_a}\frac{\partial \mathcal{L}}{\partial \mathbf{\Phi}} \right. \\ &\quad + \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\mathbf{\Phi}\right)}\left(\partial_{\mu}\left(\frac{\delta \mathcal{F}\left(\boldsymbol{x}\right)}{\delta \epsilon_a}\right) - \partial_{\mu}\left(\frac{\delta x^{\nu}}{\delta \epsilon_a}\right)\partial_{\nu}\mathbf{\Phi}\left(\boldsymbol{x}\right)\right)\epsilon_a \\ &\quad \left. + \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\mathbf{\Phi}\right)}\left(\frac{\delta \mathcal{F}\left(\boldsymbol{x}\right)}{\delta \epsilon_a} - \frac{\delta x^{\nu}}{\delta \epsilon_a}\partial_{\nu}\mathbf{\Phi}\left(\boldsymbol{x}\right)\right)\partial_{\mu}\epsilon_a\right\}. \end{aligned}$$

Then, as mentioned above, all the terms proportional to ϵ_a vanishes if S is invariant,

[2]The inverse Jacobian matrix may be obtained to first order simply by inverting the sign of the transformation parameter

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\nu}_{\mu} - \partial_{\mu}\left(\epsilon_a \frac{\delta x^{\nu}}{\delta \epsilon_a}\right).$$

thus the remaining is

$$\begin{aligned} S' &= \int d^d x \left\{ \mathcal{L}\left(\mathbf{\Phi}(x), \partial_\mu \mathbf{\Phi}(x)\right) \right\} \\ &\quad + \int d^d x \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathbf{\Phi}\right)} \frac{\delta \mathcal{F}(x)}{\delta \epsilon_a} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathbf{\Phi}\right)} \frac{\delta x^\nu}{\delta \epsilon_a} \partial_\nu \mathbf{\Phi}(x) + \frac{\delta x^\mu}{\delta \epsilon_a} \mathcal{L} \right) \partial_\mu \epsilon_a \\ &= S + \delta S, \end{aligned}$$

which means that

$$\delta S = -\int d^d x \, j_a^\mu \partial_\mu \epsilon_a, \tag{2.28}$$

where

$$j_a^\mu = \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathbf{\Phi}\right)} \partial_\nu \mathbf{\Phi}(x) - \delta_\mu^\nu \mathcal{L} \right) \frac{\delta x^\nu}{\delta \epsilon_a} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \mathbf{\Phi}\right)} \frac{\delta \mathcal{F}(x)}{\delta \epsilon_a}. \tag{2.29}$$

The j_a^μ is the associated current of the infinitesimal transformation (2.20). It is called a current in the sense that this satisfies a continuity equations, as we will see below. Integration by parts yields

$$\delta S = \int d^d x \left(\partial_\mu j_a^\mu \right) \epsilon_a. \tag{2.30}$$

Noether's theorem follows; if the field configuration obeys the classical equations of motion, the action is stationary against some variation of the field. Therefore, δS should vanish for any position dependence parameters $\epsilon_a(x)$. This implies the conservation law

$$\partial_\mu j_a^\mu = 0, \tag{2.31}$$

which is a continuity equation, as anticipated. The expression (2.29) for the conserved current is called "canonical", because there are other admissible expressions. In fact, we may freely add to it the divergence of an antisymmetric tensor without affecting its conservation:

$$j_a^\mu \longrightarrow j_a^\mu + \partial_\nu B_a^{\nu\mu} \quad , \quad B_a^{\nu\mu} = -B_a^{\mu\nu}, \tag{2.32}$$

where, $\partial_\mu \partial_\nu B_a^{\nu\mu} = 0$ by antisymmetry.

We will now work with correlation functions, they are the diagonal matrix elements of a product of local operators corresponding to the vacuum vector. However, one can also compute correlation functions with respect to any pair of states, they are arbitrary matrix elements of the product of local operators. These correlation functions are related with the scattering amplitudes between various asymptotic states.

Again, we consider a classical field theory involving a collection of fields $\mathbf{\Phi}$ with an action $S\left[\mathbf{\Phi}\right]$ that is invariant under a transformation of the type (2.18). Then, the general correlation function of its associated quantum field theory is

$$\left\langle \mathbf{\Phi}\left(x_1\right)...\mathbf{\Phi}\left(x_n\right)\right\rangle = \frac{1}{Z}\int \mathcal{D}\mathbf{\Phi}\,\mathbf{\Phi}\left(x_1\right)...\mathbf{\Phi}\left(x_n\right)e^{-S[\mathbf{\Phi}]}, \tag{2.33}$$

where Z is the vacuum functional. We assume that symmetry is not anomalous in the sense that $\mathcal{D}\mathbf{\Phi}' = \mathcal{D}\mathbf{\Phi}$. The consequence under the transformation (2.18) is obtained as follows,

$$\begin{aligned}
\left\langle \mathbf{\Phi}\left(x'_1\right)...\mathbf{\Phi}\left(x'_n\right)\right\rangle &= \frac{1}{Z}\int \mathcal{D}\mathbf{\Phi}\,\mathbf{\Phi}\left(x'_1\right)...\mathbf{\Phi}\left(x'_n\right)e^{-S[\mathbf{\Phi}]}\\
&= \frac{1}{Z}\int \mathcal{D}\mathbf{\Phi}'\,\mathbf{\Phi}'\left(x'_1\right)...\mathbf{\Phi}'\left(x'_n\right)e^{-S\left[\mathbf{\Phi}'\right]}\\
&= \frac{1}{Z}\int \mathcal{D}\mathbf{\Phi}\,\mathcal{F}\left(\mathbf{\Phi}\left(x_1\right)\right)...\mathcal{F}\left(\mathbf{\Phi}\left(x_n\right)\right)e^{-S[\mathbf{\Phi}]}\\
&= \left\langle \mathcal{F}\left(\mathbf{\Phi}\left(x_1\right)\right)...\mathcal{F}\left(\mathbf{\Phi}\left(x_n\right)\right)\right\rangle ,
\end{aligned} \tag{2.34}$$

where the mapping $\mathcal{F}$ is as in Eq.(2.18). In going from the first to second line in (2.34), we have just renamed the dummy integration variable $\mathbf{\Phi} \longrightarrow \mathbf{\Phi}'$, without performing a real change of integration variables. In going from the second to the third line, we have performed a change of functional integration variables, in which $\mathbf{\Phi}'\left(x'\right)$ is expressed in terms of $\mathbf{\Phi}\left(x\right)$. The action is invariant under such a change, by hypothesis. Also, we need to assume that the Jacobian of this change of variable does not depend on the field $\mathbf{\Phi}$.

At quantum level the consequences of a symmetry of the action and the measure are reflected through constraints on the correlation functions. These constraints are called Ward identities, we will now demonstrate them. An infinitesimal transformation may be written in terms of the generators as (Eq.2.21)

$$\mathbf{\Phi}'\left(x\right) = \mathbf{\Phi}\left(x\right) + \delta\mathbf{\Phi}\left(x\right) = \mathbf{\Phi}\left(x\right) - i\epsilon_a G_a \mathbf{\Phi}\left(x\right). \tag{2.35}$$

We make a change of functional integration variables in the correlation function (2.33) in the form of the above infinitesimal transformation, with ϵ_a now a function of x. The action is not invariant under such local transformation, its variation is given by (2.30). By simplicity, we denote by $\boldsymbol{X}$ the collection $\mathbf{\Phi}\left(x_1\right)...\mathbf{\Phi}\left(x_n\right)$ of fields within the correlation function and by $\delta_\epsilon \boldsymbol{X}$ its variation under the transformation, we can write

$$\begin{aligned}
\left\langle \boldsymbol{X}\right\rangle &= \frac{1}{Z}\int \mathcal{D}\mathbf{\Phi}'\,\left(\boldsymbol{X} + \delta_\epsilon \boldsymbol{X}\right)e^{-\{S[\mathbf{\Phi}]+\delta S\}}\\
&= \frac{1}{Z}\int \mathcal{D}\mathbf{\Phi}'\,\left(\boldsymbol{X} + \delta_\epsilon \boldsymbol{X}\right)e^{-\left\{S[\mathbf{\Phi}]+\int d^d x\left(\partial_\mu j^\mu_a\right)\epsilon_a(x)\right\}}
\end{aligned} \tag{2.36}$$

We again assume that $\mathcal{D}\mathbf{\Phi}' = \mathcal{D}\mathbf{\Phi}$. When expanding to first order in $\epsilon_a(\boldsymbol{x})$, the above expression becomes

$$\begin{aligned}
\langle \boldsymbol{X} \rangle &= \left\{ \frac{1}{Z} \int \mathcal{D}\mathbf{\Phi}' \, \boldsymbol{X} e^{-S[\mathbf{\Phi}]} + \frac{1}{Z} \int \mathcal{D}\mathbf{\Phi}' \, \delta_\epsilon \boldsymbol{X} \mathrm{e}^{-S[\mathbf{\Phi}]} \right\} e^{-\int d^d x \, (\partial_\mu j_a^\mu)\epsilon_a} \\
&= \left\{ \frac{1}{Z} \int \mathcal{D}\mathbf{\Phi}' \, \boldsymbol{X} e^{-S[\mathbf{\Phi}]} + \frac{1}{Z} \int \mathcal{D}\mathbf{\Phi}' \, \delta_\epsilon \boldsymbol{X} \mathrm{e}^{-S[\mathbf{\Phi}]} \right\} \left(1 - \int d^d x \, \left(\partial_\mu j_a^\mu\right) \epsilon_a \right) \\
&= \frac{1}{Z} \int \mathcal{D}\mathbf{\Phi}' \, \boldsymbol{X} e^{-S[\mathbf{\Phi}]} + \frac{1}{Z} \int \mathcal{D}\mathbf{\Phi}' \, \delta_\epsilon \boldsymbol{X} \mathrm{e}^{-S[\mathbf{\Phi}]} \\
&\quad - \left\{ \frac{1}{Z} \int \mathcal{D}\mathbf{\Phi}' \, \boldsymbol{X} e^{-S[\mathbf{\Phi}]} + \frac{1}{Z} \int \mathcal{D}\mathbf{\Phi}' \, \delta_\epsilon \boldsymbol{X} \mathrm{e}^{-S[\mathbf{\Phi}]} \right\} \int d^d x \, \left(\partial_\mu j_a^\mu\right) \epsilon_a, \\
\cancel{\langle \boldsymbol{X} \rangle} &\approx \cancel{\langle \boldsymbol{X} \rangle} + \langle \delta \boldsymbol{X} \rangle - \frac{1}{Z} \int d^d x \int \mathcal{D}\mathbf{\Phi}' \, \boldsymbol{X} \left(\partial_\mu j_a^\mu\right) \epsilon_a e^{-S[\mathbf{\Phi}]} \\
\langle \delta \boldsymbol{X} \rangle &= \frac{1}{Z} \int d^d x \int \mathcal{D}\mathbf{\Phi}' \, \boldsymbol{X} \left(\partial_\mu j_a^\mu\right) \epsilon_a(\boldsymbol{x}) \, e^{-S[\mathbf{\Phi}]}.
\end{aligned}$$

Now, we can express $\boldsymbol{X}\left(\partial_\mu j_a^\mu\right)$ as $\partial_\mu\left(j_a^\mu \boldsymbol{X}\right)$, since the insertion points in $\boldsymbol{X}$, that is $(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n)$, are in general, different to the point $\boldsymbol{x}$. Then we have

$$\begin{aligned}
\langle \delta \boldsymbol{X} \rangle &= \frac{1}{Z} \int d^d x \int \mathcal{D}\mathbf{\Phi}' \, \partial_\mu \left(j_a^\mu \boldsymbol{X}\right) e^{-S[\mathbf{\Phi}]} \epsilon_a(\boldsymbol{x}) \\
&= \int d^d x \partial_\mu \left\langle j_a^\mu \boldsymbol{X} \right\rangle \epsilon_a(\boldsymbol{x}) \, .
\end{aligned} \tag{2.37}$$

The variation $\delta \boldsymbol{X}$ is explicitly given by (as the variation of a product), since $\delta\mathbf{\Phi}\left(\boldsymbol{x}_j\right) = -i\epsilon_a\left(\boldsymbol{x}_j\right) G_a \mathbf{\Phi}\left(\boldsymbol{x}_j\right)$;

$$\begin{aligned}
\delta \boldsymbol{X} &= -i \sum_{j=1}^{n} \epsilon_a\left(\boldsymbol{x}_j\right) \left(\mathbf{\Phi}(\boldsymbol{x}_1) ... G_a \mathbf{\Phi}\left(\boldsymbol{x}_j\right) ... \mathbf{\Phi}(\boldsymbol{x}_n)\right) \\
&= -i \int d^d x \epsilon_a(\boldsymbol{x}) \sum_{j=1}^{n} \left(\mathbf{\Phi}(\boldsymbol{x}_1) ... G_a \mathbf{\Phi}\left(\boldsymbol{x}_j\right) ... \mathbf{\Phi}(\boldsymbol{x}_n)\right) \delta\left(\boldsymbol{x} - \boldsymbol{x}_j\right) .
\end{aligned} \tag{2.38}$$

Since (2.37) holds for any infinitesimal function, we obtain the following local relation

$$\partial_\mu \left\langle j_a^\mu \boldsymbol{X} \right\rangle = -i \sum_{j=1}^{n} \left\langle \mathbf{\Phi}(\boldsymbol{x}_1) ... G_a \mathbf{\Phi}\left(\boldsymbol{x}_j\right) ... \mathbf{\Phi}(\boldsymbol{x}_n) \right\rangle \delta\left(\boldsymbol{x} - \boldsymbol{x}_j\right) . \tag{2.39}$$

This is the Ward identity for the current j_a^μ. We integrate the Ward identity (2.39) over a region of space-time that includes all the points $\boldsymbol{x}_j$. On the left-hand side (l.h.s), we obtain a surface integral, via divergence theorem

$$\int_V d^d x \partial_\mu \left\langle j_a^\mu \boldsymbol{X} \right\rangle = \int_{\partial V} d\xi_\mu \left\langle j_a^\mu \boldsymbol{X} \right\rangle = \int_{\partial V} d\xi_\mu \left\langle j_a^\mu \mathbf{\Phi}(\boldsymbol{x}_1) ... \mathbf{\Phi}(\boldsymbol{x}_n) \right\rangle \tag{2.40}$$

where $d\xi_\mu$ is an outward-directed differential orthogonal to the boundary ∂V of the domain of integration. Therefore, as a operator equation[3] we can express the Ward identity for a single operator field $\mathbf{\Phi}\left(\boldsymbol{x}_0\right)$ as

$$\int_{\partial V} d\xi_\mu j_a^\mu \mathbf{\Phi}\left(\boldsymbol{x}_0\right) = \delta\mathbf{\Phi}\left(\boldsymbol{x}_0\right). \tag{2.41}$$

We will use later the above result to see how a conformal transformation affect an operator conformal field.

2.1.2 Generators of the Conformal group

Now, we use the results above to calculate the generators of the conformal group. We suppose by the moment that the fields are not affected by the transformation (that is, $\mathcal{F}\left(\mathbf{\Phi}\right) = \mathbf{\Phi} \Longrightarrow \frac{\delta\mathcal{F}(\boldsymbol{x})}{\delta\epsilon_a} = 0$), therefore, in particular, from (2.14) we can see that

$$x'^\mu = x^\mu + 2b^\alpha x_\alpha x^\mu - b^\mu x^2 = x^\mu + b^\alpha\left(2x_\alpha x^\mu - \delta^\mu_\alpha x^2\right), \tag{2.42}$$

which, according to (2.20), allows us to identify $\frac{\delta x^\mu}{\delta\epsilon_a} = 2x_\alpha x^\mu - \delta^\mu_\alpha x^2$ by promoting $\epsilon_a \longrightarrow b^\alpha$, thus, by using (2.23) we have that the generator of the special conformal transformations K_μ is such that

$$iK_\mu\mathbf{\Phi}\left(\boldsymbol{x}\right) = \left(2x_\mu x^\nu - \delta^\nu_\mu x^2\right)\partial_\nu\mathbf{\Phi}\left(\boldsymbol{x}\right) = \left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right)\mathbf{\Phi}\left(\boldsymbol{x}\right). \tag{2.43}$$

By repeating this process for all the infinitesimal versions of the transformations (2.15), we obtain the generators

$$\begin{array}{rccl} \text{translation} & P_\mu & = & -i\partial_\mu, \\ \text{dilatation} & D & = & -ix^\mu\partial_\mu, \\ \text{rigid rotation} & L_{\mu\nu} & = & i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right), \\ \text{SCT} & K_\mu & = & -i\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right). \end{array} \tag{2.44}$$

[3]When we say "as an operator equation" we means that the expression referred just has sense within a correlation function.

They obey the following commutation rules, thus defining the conformal algebra

$$\begin{aligned}
\left[D, P_\mu\right] &= iP_\mu, \\
\left[D, K_\mu\right] &= -iK_\mu, \\
\left[K_\mu, P_\nu\right] &= 2i\left(\eta_{\mu\nu}D - L_{\mu\nu}\right), \\
\left[K_\rho, L_{\mu\nu}\right] &= i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right), \\
\left[P_\rho, L_{\mu\nu}\right] &= i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right), \\
\left[L_{\mu\nu}, L_{\rho\sigma}\right] &= i\left(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}\right).
\end{aligned} \tag{2.45}$$

In order the put the above commutation rules into a simpler way, we define the following combinations of generators:

$$\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu}, & J_{-1,\mu} &= \tfrac{1}{2}\left(P_\mu - K_\mu\right), \\
J_{-1,0} &= D, & J_{0,\mu} &= \tfrac{1}{2}\left(P_\mu + K_\mu\right),
\end{aligned} \tag{2.46}$$

where $J_{ab} = -J_{ba}$ and $a,\, b \in \{-1, 0, 1, ..., d\}$. These new generators obey the commutation relations of $SO\left(d+1, 1\right)$:

$$\left[J_{ab}, J_{cd}\right] = i\left(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\right), \tag{2.47}$$

where the metric η_{ab} is $\mathrm{diag}\left(-1, 1, 1, ..., 1\right)$ if the space-time is Euclidean (otherwise an additional component, say η_{dd}, is negative). This shows the isomorphism between the conformal group in d dimensions and the $SO\left(d+1, 1\right)$ group, which has $\frac{1}{2}\left(d+2\right)\left(d+1\right)$ elements.

We now build functions $\Gamma\left(x_i\right)$ of the N points $\boldsymbol{x}_i$ that are left invariant under every type of conformal transformations. Translational and rotational invariances imply that Γ can depend on the distances $\left|\boldsymbol{x}_i - \boldsymbol{x}_j\right|$ between pairs of different points. Scale invariance implies that only ratios of such a distances, such as

$$\frac{\left|\boldsymbol{x}_i - \boldsymbol{x}_j\right|}{\left|\boldsymbol{x}_k - \boldsymbol{x}_l\right|}$$

are allowed. Finally, under a special conformal transformation, the distance separating two points $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ becomes in

$$\left|\boldsymbol{x}'_i - \boldsymbol{x}'_j\right| = \frac{\left|\boldsymbol{x}_i - \boldsymbol{x}_j\right|}{\left(1 - 2\boldsymbol{b}\cdot\boldsymbol{x}_i + b^2 x_i^2\right)^{1/2}\left(1 - 2\boldsymbol{b}\cdot\boldsymbol{x}_j + b^2 x_j^2\right)^{1/2}} \tag{2.48}$$

therefore, it is impossible to construct and invariant Γ with only just 2 or 3 points (by the scale invariance). The simplest possibilities are then the following functions of four points:

$$\frac{|x_1 - x_2|\,|x_3 - x_4|}{|x_1 - x_3|\,|x_2 - x_4|} \qquad \frac{|x_1 - x_2|\,|x_3 - x_4|}{|x_2 - x_3|\,|x_1 - x_4|}. \tag{2.49}$$

Such expressions are called *anharmonic ratios* or *cross-ratios*. With N distinct points, $N(N-3)/2$ independent anharmonic ratios can be constructed.

2.2 Conformal invariance in classical field theory

We define the effect of conformal transformation on classical fields. Also, we show how, in certain theories, complete conformal invariance is a consequence of the scale and the Poincaré invariance.

2.2.1 Representation of the Conformal Group in d Dimensions

Given an infinitesimal conformal transformation parametrized by ϵ_g, we look for a representation matrix T_g such that a multicomponent field (classical) $\mathbf{\Phi}(x)$ transforms as

$$\mathbf{\Phi}'(x') = \left(1 - i\epsilon_g T_g\right)\mathbf{\Phi}(x) \approx e^{-i\epsilon_g T_g}\mathbf{\Phi}(x). \tag{2.50}$$

The generator T_g must be added to the space-time part in (2.44) to obtain the full generator of the symmetry, as in eq.(2.23). In order to find the allowed form of these generators, we start by studying the subgroup of the Poincaré group that leaves the point $x = 0$ invariant, that is, the Lorentz group. We then introduce a matrix representation $S_{\mu\nu}$ to define the action of infinitesimal Lorentz transformations on the fields $\mathbf{\Phi}(0)$:

$$L_{\mu\nu}\mathbf{\Phi}(0) = S_{\mu\nu}\mathbf{\Phi}(0). \tag{2.51}$$

$S_{\mu\nu}$ is a spin operator associated with the field $\mathbf{\Phi}$. Next, by use of the commutation relations of the Poincaré group, we translate the operator to a non-zero value of x

$$e^{ix^\rho P_\rho} L_{\mu\nu} e^{-ix^\rho P_\rho} = S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \tag{2.52}$$

The translation above is explicitly calculate by using the Hausdorff formula (A and B are two operators[4]):

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2!}\left[[B, A], A\right] + \frac{1}{3!}\left[[[B, A], A], A\right] + ... \tag{2.53}$$

This allows us to write the action of the generators:

$$\begin{aligned} P_\mu \mathbf{\Phi}(x) &= -i\partial_\mu \mathbf{\Phi}(x), \\ L_{\mu\nu}\mathbf{\Phi}(x) &= i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\mathbf{\Phi}(x) + S_{\mu\nu}\mathbf{\Phi}(x). \end{aligned} \tag{2.54}$$

We proceed in the same way for the full conformal group. The subgroup that leaves the origin $x = 0$ invariant is generated by rotations, dilatations and special conformal transformations. We denote $S_{\mu\nu}$, $\tilde{\Delta}$, and κ_μ the respective values of the generators $L_{\mu\nu}$, D, and K_μ at $x = 0$. These must form a matrix representation of the reduced algebra

$$\begin{aligned} \left[\tilde{\Delta}, S_{\mu\nu}\right] &= 0, \\ \left[\tilde{\Delta}, \kappa_\mu\right] &= -i\kappa_\mu, \\ \left[\kappa_\mu, \kappa_\nu\right] &= 0, \\ \left[\kappa_\sigma, S_{\mu\nu}\right] &= i\left(\eta_{\sigma\nu}\kappa_\mu - \eta_{\sigma\mu}\kappa_\nu\right), \\ \left[S_{\mu\nu}, S_{\rho\sigma}\right] &= i\left(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho}\right). \end{aligned} \tag{2.55}$$

The commutations (2.45) allows us to translate the generators, by using again the Hausdorff formula (2.53) :

$$\begin{aligned} e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho} &= D + x^\mu P_\mu, \\ e^{ix^\rho P_\rho} K_\mu e^{-ix^\rho P_\rho} &= K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu} + 2x_\mu\left(x^\nu P_\nu\right) - x^2 P_\mu. \end{aligned} \tag{2.56}$$

[4]We have then

$$\begin{aligned} e^{ix^\rho P_\rho} L_{\mu\nu} e^{-ix^\rho P_\rho} &= L_{\mu\nu}| + \left[L_{\mu\nu}, -ix^\rho P_\rho\right] + \frac{1}{2}\left[\left[L_{\mu\nu}, -ix^\rho P_\rho\right], -ix^\rho P_\rho\right] + ... \\ &= S_{\mu\nu} - i\cancelto{0}{\left[L_{\mu\nu}, x^\rho\right]} P_\rho - ix^\rho\left[L_{\mu\nu}, P_\rho\right] + \frac{1}{2}\left[\left[L_{\mu\nu}, -ix^\rho P_\rho\right], -ix^\rho P_\rho\right] + ... \\ &= S_{\mu\nu} - ix^\rho i\left(\eta_{\rho\nu}P_\mu - \eta_{\rho\mu}P_\nu\right) + \frac{1}{2}\left[\left[L_{\mu\nu}, -ix^\rho P_\rho\right], -ix^\rho P_\rho\right] + ... \\ &= S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu - \frac{1}{2}\cancelto{0}{\left[x_\mu P_\nu - x_\nu P_\mu, -ix^\rho P_\rho\right]} + ... \\ &= S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \end{aligned}$$

since x^ρ is not an operator in this case, it is rather a extended value of x^ρ, a function, we have then $\left[L_{\mu\nu}, x^\rho\right] = 0$.

Therefore,

$$
\begin{aligned}
D\mathbf{\Phi}(x) &= \left(-ix^\nu\partial_\nu + \tilde{\Delta}\right)\mathbf{\Phi}(x), \\
K_\mu\mathbf{\Phi}(x) &= \left\{\kappa_\mu + 2x_\mu\tilde{\Delta} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu\partial_\nu + ix^2\partial_\mu\right\}\mathbf{\Phi}(x).
\end{aligned}
\tag{2.57}
$$

If we demand that the field $\mathbf{\Phi}(x)$ belong to an irreducible representation of the Lorentz group, then, by the Schur's lemma, any matrix that commutes with all the generators $S_{\mu\nu}$ must be a multiple of the identity. In consequence, the matrix $\tilde{\Delta}$ is a multiple of the identity and the algebra (2.55) forces all the matrices κ_μ to vanish. $\tilde{\Delta}$ is then simply a number, manifestly equal to $-i\Delta$, where Δ is the scale dimension of the field $\mathbf{\Phi}$, it is defined by

$$
\boldsymbol{x}' = \lambda\boldsymbol{x} \longrightarrow \mathbf{\Phi}'(\lambda\boldsymbol{x}) = \lambda^{-\Delta}\mathbf{\Phi}(\boldsymbol{x}). \tag{2.58}
$$

In principle, we can derive from the above the change in $\mathbf{\Phi}$ under a finite conformal transformation. However, we just will give the result for spinless fields $\left(S_{\mu\nu} = 0\right)$. Under a conformal transformation $\boldsymbol{x} \longrightarrow \boldsymbol{x}'$, a spinless field $\phi(\boldsymbol{x})$ transform as

$$
\phi(\boldsymbol{x}) \longrightarrow \phi'\left(\boldsymbol{x}'\right) = \left|\frac{\partial\boldsymbol{x}'}{\partial\boldsymbol{x}}\right|^{-\Delta/d}\phi(\boldsymbol{x}), \tag{2.59}
$$

where $\left|\frac{\partial x'}{\partial x}\right|$ is the Jacobian of the conformal transformation of the coordinates. A field transforming as that above is called "quasi-primary".

2.2.2 The energy momentum tensor

For an arbitrary local transformation of the coordinates $x'^\mu \longrightarrow x^\mu + \epsilon^\mu(\boldsymbol{x})$, the action changes according to (2.28) as follows:

$$
\delta S = -\int d^dx T^{\mu\nu}\partial_\mu\epsilon_\nu = -\frac{1}{2}\int d^dx T^{\mu\nu}\left(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right), \tag{2.60}
$$

where $T^{\mu\nu}$ is the **energy-momentum tensor**, assumed to be symmetric. For an infinitesimal translation (in flat space), this is the associated conserved current and, according to eq.(2.29), takes the canonical form

$$
T_c^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\mathbf{\Phi}\right)}\partial^\nu\mathbf{\Phi} - \eta^{\mu\nu}\mathcal{L}, \tag{2.61}
$$

after identifying in (2.20) $\frac{\delta\mathcal{F}(x)}{\delta\epsilon^\alpha} = 0$ and $\frac{\delta x^\mu}{\delta\epsilon^\alpha} = \delta^\mu_\alpha$, the index c stands for "canonical". The definition (2.3) of an infinitesimal conformal transformation implies that the variation corresponding to the action is (using 2.4)

$$\delta S=-\frac{1}{2}\int d^{d}xT^{\mu\nu}f\left(x\right)\eta_{\mu\nu}=-\frac{1}{d}\int d^{d}xT^{\mu}_{\ \mu}\partial_{\rho}\epsilon^{\rho} \tag{2.62}$$

Thus, the tracelessness of the energy-momentum tensor implies the conformal invariance of the action. Note also that, from eq.(2.2), we have $\delta g_{\mu\nu}=\partial_{\mu}\epsilon_{\nu}+\partial_{\nu}\epsilon_{\mu}$, and since

$$\delta S=\int d^{d}x\frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu}\Longrightarrow\delta S=-\frac{1}{2}\int d^{d}xT^{\mu\nu}\delta g_{\mu\nu}\Longrightarrow T^{\mu\nu}=-2\frac{\delta S}{\delta g_{\mu\nu}}.$$

However, for a general curved space $g_{\mu\nu}\left(x\right)$, some factors must be included, and the energy momentum tensor bears the more general form

$$T_{\alpha\beta}=-\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\alpha\beta}},\qquad\delta S=-\frac{1}{4\pi}\int d^{2}\sigma\sqrt{g}\delta g^{\alpha\beta}T_{\alpha\beta}. \tag{2.63}$$

Under certain conditions, the energy momentum tensor of a theory with scale invariance can be made traceless. If this is possible, then it follows from above that the full conformal invariance is a consequence of the scale invariance and the Poincaré invariance. In order to show that, we first consider a generic field theory with scale invariance in dimension $d>2$. We now compute the conserved current associated with the infinitesimal dilation

$$x'^{\mu}=\left(1+\alpha\right)x^{\mu},\qquad\qquad\mathcal{F}\left(\phi\right)=\left(1-\alpha\Delta\right)\mathbf{\Phi}, \tag{2.64}$$

so that

$$\frac{\delta x^{\nu}}{\delta\epsilon_{D}}=\frac{\delta x^{\nu}}{\delta\alpha}=x^{\nu},\qquad\qquad\frac{\delta\mathcal{F}}{\delta\epsilon_{D}}=\frac{\delta\mathcal{F}}{\delta\alpha}=-\Delta\phi.$$

Then, from (2.29) we have

$$j^{\mu}_{D}=T^{\mu}_{c\nu}x^{\nu}+\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\Delta\phi. \tag{2.65}$$

Since by hypothesis this current is conserved, it satisfies

$$\partial_{\mu}j^{\mu}_{D}\ =\ T^{\mu}_{c\mu}+\Delta\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\phi\right)=0. \tag{2.66}$$

We now define the *virial* of the field $\mathbf{\Phi}$;

$$V^{\mu}=\frac{\partial\mathcal{L}}{\eta^{\mu\rho}\partial\left(\partial_{\mu}\mathbf{\Phi}\right)}\left(\eta^{\mu\rho}\Delta+iS^{\mu\rho}\right)\mathbf{\Phi}=\frac{\partial\mathcal{L}}{\partial\left(\partial^{\rho}\mathbf{\Phi}\right)}\left(\eta^{\mu\rho}\Delta+iS^{\mu\rho}\right)\mathbf{\Phi}, \tag{2.67}$$

where $S^{\mu\rho}$ is the spin operator of the field $\mathbf{\Phi}$. We also assume that the virial is the divergence of other tensor $\sigma^{\alpha\mu}$:

$$V^{\mu} = \partial_{\alpha}\sigma^{\alpha\mu}. \tag{2.68}$$

Then, we define

$$\begin{aligned} \sigma_{+}^{\mu\nu} &= \frac{1}{2}\left(\sigma^{\mu\nu} + \sigma^{\nu\mu}\right), \\ x^{\lambda\rho\mu\nu} &= \frac{2}{d-2}\left\{\eta^{\lambda\rho}\sigma_{+}^{\mu\nu} - \eta^{\lambda\mu}\sigma_{+}^{\rho\nu} - \eta^{\lambda\nu}\sigma_{+}^{\mu\rho} + \eta^{\mu\nu}\sigma_{+}^{\lambda\rho} \right. \\ &\qquad \left. - \frac{1}{d-1}\left(\eta^{\lambda\rho}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\rho\nu}\right)\sigma_{+\alpha}^{\alpha}\right\} \end{aligned} \tag{2.69}$$

and we consider the following modified energy-momentum tensor:

$$T^{\mu\nu} = T_{c}^{\mu\nu} + \partial_{\rho}B^{\rho\mu\nu} + \frac{1}{2}\partial_{\lambda}\partial_{\rho}x^{\lambda\rho\mu\nu}. \tag{2.70}$$

The first two terms of the above expression constitute the Belinfante tensor or the symmetrized energy-momentum tensor. The last term is an addition that will make $T^{\mu\nu}$ traceless. By the symmetry properties of $x^{\lambda\rho\mu\nu}$, this additional term doesn't spoil the conservation law:

$$\partial_{\mu}\partial_{\lambda}\partial_{\rho}x^{\lambda\rho\mu\nu} = 0. \tag{2.71}$$

It would not be so if the $x^{\lambda\rho\mu\nu}$ term had a part completely symmetric in the first three indices, but this is not the case. This new term doesn't spoil the symmetry of the Belinfante tensor either, since the part of $x^{\lambda\rho\mu\nu}$ antisymmetric in μ, ν is

$$x^{\lambda\rho\mu\nu} - x^{\lambda\rho\nu\mu} = \frac{2}{(d-2)(d-1)}\left\{\eta^{\lambda\mu}\eta^{\rho\nu} - \eta^{\lambda\nu}\eta^{\rho\mu}\right\}\sigma_{+\alpha}^{\alpha}. \tag{2.72}$$

Finally, the trace of the new term is

$$\frac{1}{2}\partial_{\lambda}\partial_{\rho}x^{\lambda\rho\mu}{}_{\mu} = \partial_{\lambda}\partial_{\rho}\sigma_{+}^{\lambda\rho} = \partial_{\mu}V^{\mu}, \tag{2.73}$$

since $\partial_{\rho}B^{\rho\mu}{}_{\mu} = -i\partial_{\rho}\left(\frac{\delta\mathcal{L}}{\delta(\partial^{\mu}\mathbf{\Phi})}S^{\mu\rho}\mathbf{\Phi}\right)$, it follows from (2.66) and (2.67) that

$$T^{\mu}_{\ \mu} = \partial_{\mu}j_{D}^{\mu}. \tag{2.74}$$

[5]Therefore, scale invariance implies that the modified energy-momentum tensor (2.70) is traceless, provided the virial satisfies the condition (2.68). This relation also means that the dilation current can generally be written as

$$j_D^{\mu} = T^{\mu}{}_{\nu} x^{\nu}. \tag{2.75}$$

[6]This argument holds only for dimensions more than two, since $x^{\lambda\rho\mu\nu}$ is defined just for $d > 2$. However, this result also holds in two dimensions, but for other reasons. For instance, by using the Schwinger function, the expectation value of the square of the trace of the energy-momentum tensor. This vanishes in two dimensions.

2.3 Conformal Invariance in Quantum Field Theory

We now see how conformal transformations affect the correlation functions, this is done through the Ward identities, introduced before.

2.3.1 Correlation functions

Correlation functions of quasi primary fields (2.59), transform according to (2.34) as

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/d} \langle \phi_1(x_1')\,\phi_2(x_2')\rangle\,. \tag{2.76}$$

If we specialize to a scale transformation $x \longrightarrow \lambda x$, we obtain

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \left(\lambda^d\right)^{\Delta_1/d} \left(\lambda^d\right)^{\Delta_2/d} \langle \phi_1(\lambda x_1)\,\phi_2(\lambda x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\,\phi_2(\lambda x_2)\rangle\,. \tag{2.77}$$

Rotational and translational invariance require that

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = f(|x_1 - x_2|)\,, \tag{2.78}$$

[5]That is

$$\begin{aligned}
T^{\mu}{}_{\mu} &= T^{\mu}_{c\mu} + \partial_\rho B^{\rho\mu}{}_{\mu} + \frac{1}{2}\partial_\lambda\partial_\rho x^{\lambda\rho\mu}{}_{\mu} \\
&= \partial_\mu j_D^{\mu} - \Delta\partial^{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\mathbf{\Phi})}\mathbf{\Phi}\right) - i\partial_\rho\left(\frac{\delta\mathcal{L}}{\delta(\partial^{\mu}\mathbf{\Phi})}S^{\mu\rho}\mathbf{\Phi}\right) + \partial_\mu V^{\mu} \\
&= \partial_\mu j_D^{\mu} - \Delta\partial^{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\mathbf{\Phi})}\mathbf{\Phi}\right) - i\partial_\rho\left(\frac{\delta\mathcal{L}}{\delta(\partial^{\mu}\mathbf{\Phi})}S^{\mu\rho}\mathbf{\Phi}\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^{\rho}\mathbf{\Phi})}\left(\eta^{\mu\rho}\Delta + iS^{\mu\rho}\right)\mathbf{\Phi}\right).
\end{aligned}$$

Then, we rename the dummy indices, such that just the first term doesn't cancel out, we obtain (2.74)

[6]The above relation is obvious since $T^{\mu}{}_{\nu}$ is also conserved we have

$$\partial_\mu j_D^{\mu} = \partial_\mu\left(T^{\mu}{}_{\nu}x^{\nu}\right) = T^{\mu}{}_{\nu}\partial_\mu x^{\nu} = T^{\mu}{}_{\nu}\delta^{\nu}_{\mu} = T^{\mu}{}_{\mu}.$$

where $f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$ because of (2.77). In other words

$$\langle \phi_1(x_1)\, \phi_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \tag{2.79}$$

where C_{12} are constant coefficients.
It remains to use invariance under special conformal transformations. Recall that, for such transformations

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{(1 - 2b \cdot x + b^2 x^2)^d} = \gamma^{-d}. \tag{2.80}$$

Given the transformations (2.48) for the distances $|x_1 - x_2|$, the covariance of the correlation function (2.79) implies that

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \tag{2.81}$$

with

$$\gamma_i = 1 - 2b \cdot x_i + b^2 x_i^2. \tag{2.82}$$

This constraint is satisfied only if $\Delta = \Delta_1 = \Delta_2$, that is

$$\langle \phi_1(x_1)\, \phi_2(x_2) \rangle = \begin{cases} \frac{C_{12}}{|x_1 - x_2|^{2\Delta}} & if \quad \Delta = \Delta_1 = \Delta_2, \\ 0 & if \quad \Delta_1 \neq \Delta_2. \end{cases} \tag{2.83}$$

Similarly, covariance under the rotations, translations and dilations forces a general three-point function to have the following form:

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3) \rangle = \frac{C_{123}^{(abc)}}{x_{12}^a x_{23}^b x_{13}^c}, \tag{2.84}$$

where $x_{ij} = |x_i - x_j|$ and with a, b, c such that

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3. \tag{2.85}$$

[7]Under conformal transformations, eq.(2.84) becomes

$$\frac{C_{123}^{(abc)}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_1\gamma_3)^{c/2}}{x_{12}^a x_{23}^b x_{13}^c}.$$

[7]A sum (over a, b, c) of such terms is also acceptable, as long as the equality (2.85) is satisfied.

In order to this expression to be of the same form as eq.(2.84), all the factors involving the transformation parameter b^{μ} must disappear, which leads to the following set of constraints

$$\frac{a}{2}+\frac{c}{2}=\Delta_1,\qquad \frac{a}{2}+\frac{b}{2}=\Delta_2,\qquad \frac{b}{2}+\frac{c}{2}=\Delta_1. \tag{2.86}$$

The solution to these constraints is unique

$$a=\Delta_1+\Delta_2-\Delta_3,\qquad b=\Delta_2+\Delta_3-\Delta_1,\qquad c=\Delta_3+\Delta_1-\Delta_2. \tag{2.87}$$

Therefore, the correlation functions of three quasi-primary fields is made of a single term of the form (2.84), that is

$$\langle\phi_1\left(x_1\right)\phi_2\left(x_2\right)\phi_3\left(x_3\right)\rangle=\frac{C_{123}^{(abc)}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{13}^{\Delta_3+\Delta_1-\Delta_2}}. \tag{2.88}$$

With four points or more, it is also possible to construct conformal invariants, the anharmonic ratios (2.49). The $n-$ point function made have an arbitrary dependence (that is, not fixed by conformal invariance) on these ratios. For example, the four-point function may take the following form:

$$\langle\phi_1\left(x_1\right)...\phi_4\left(x_4\right)\rangle=f\left(\frac{x_{12}x_{34}}{x_{13}x_{24}},\frac{x_{12}x_{34}}{x_{33}x_{14}}\right)\prod_{i<j}^{4}x_{ij}^{\Delta/3-\Delta_i-\Delta_j}, \tag{2.89}$$

where we have defined $\Delta=\sum_{i=1}^{4}\Delta_i$.

2.3.2 Ward Identities

Let us now find the Ward identities (2.39) associated with the generators (2.54) and (2.57). First, we see that the Ward identity associated with the translation invariance is:

$$\partial_\mu\left\langle T^{\mu}{}_{\nu}X\right\rangle=-\sum_{j=1}^{n}\delta\left(x-x_j\right)\left\langle\phi\left(x_1\right)...\partial_\nu\phi\left(x_j\right)...\phi\left(x_n\right)\right\rangle=-\sum_{j=1}^{n}\delta\left(x-x_j\right)\partial_{\nu,i}\left\langle X\right\rangle. \tag{2.90}$$

This identity holds even after a modification of the energy-momentum tensor, as in eq.(2.70). Consider now a Ward identity associated with the Lorentz invariance (or rotational). Once the energy-momentum tensor has been symmetrized, the associated current $j^{\mu\nu\rho}$ has the form

$$j^{\mu\nu\rho}=T^{\mu\nu}x^{\rho}-T^{\mu\rho}x^{\nu}. \tag{2.91}$$

The generator of Lorentz transformations is given by

$$L^{\rho\nu} = i\left(x^{\rho}\partial^{\nu} - x^{\nu}\partial^{\rho}\right) + S^{\rho\nu}, \tag{2.92}$$

in turn, the Ward identity is

$$\partial_{\mu}\left\langle\left(T^{\mu\nu}x^{\rho} - T^{\mu\rho}x^{\nu}\right)\boldsymbol{X}\right\rangle = \sum_{j=1}^{n}\delta\left(\boldsymbol{x}-\boldsymbol{x}_{j}\right)\left[\left(x_{j}^{\nu}\partial_{j}^{\rho} - x_{j}^{\rho}\partial_{j}^{\nu}\right)\langle\boldsymbol{X}\rangle - iS_{j}^{\nu\rho}\langle\boldsymbol{X}\rangle\right], \tag{2.93}$$

where $S_{i}^{\rho\nu}$ is the spin generator for the $i-$th field of the set $\boldsymbol{X}$. The derivative on the left hand side (l.h.s) of the above equation may act either on the energy-momentum tensor or on the coordinates. Using the Ward identity (2.90) we reduce the above to

$$\left\langle\left(T^{\rho\nu} - T^{\nu\rho}\right)\boldsymbol{X}\right\rangle = -i\sum_{i}\delta\left(\boldsymbol{x}-\boldsymbol{x}_{i}\right)S_{i}^{\nu\rho}\langle\boldsymbol{X}\rangle, \tag{2.94}$$

[8]which is the Ward identity associated with Lorentz invariance. Finally, we consider the Ward identity associated with scale invariance. We will assume that the dilation current j_{D}^{μ} may be written as in (2.75), which supposes that the energy-momentum tensor has been modified to be traceless. Since the generator of dilations is $D = -ix^{\nu}\partial_{\nu} - i\Delta$ for a field of scale dimension Δ, the Ward identity is

$$\partial_{\mu}\left\langle T^{\mu}{}_{\nu}x^{\nu}\boldsymbol{X}\right\rangle = -\sum_{i}\delta\left(\boldsymbol{x}-\boldsymbol{x}_{i}\right)\left[x_{i}^{\nu}\frac{\partial}{\partial x_{i}^{\nu}}\langle\boldsymbol{X}\rangle + \Delta_{i}\langle\boldsymbol{X}\rangle\right]. \tag{2.95}$$

Here again the derivative ∂_{μ} can act on $T^{\mu}{}_{\nu}$ and on x^{ν}. Using eq.(2.90), this equation reduces to

$$\left\langle T^{\mu}{}_{\mu}\boldsymbol{X}\right\rangle = -\sum_{i}\delta\left(\boldsymbol{x}-\boldsymbol{x}_{i}\right)\Delta_{i}\langle\boldsymbol{X}\rangle. \tag{2.96}$$

---

[8]The eq.(2.94) is showed as follows: We take the l.h.s of (2.93), that is

$$\begin{aligned}
\partial_{\mu}\left\langle\left(T^{\mu\nu}x^{\rho} - T^{\mu\rho}x^{\nu}\right)\boldsymbol{X}\right\rangle &= \partial_{\mu}\left\langle\left(T^{\mu\nu}x^{\rho}\right)\boldsymbol{X}\right\rangle - \partial_{\mu}\left\langle\left(T^{\mu\rho}x^{\nu}\right)\boldsymbol{X}\right\rangle \\
&= x^{\rho}\partial_{\mu}\left\langle T^{\mu\nu}\boldsymbol{X}\right\rangle + \left\langle\left(T^{\mu\nu}\delta_{\mu}^{\rho}\right)\boldsymbol{X}\right\rangle \\
&\qquad - x^{\nu}\partial_{\mu}\left\langle T^{\mu\rho}\boldsymbol{X}\right\rangle - \left\langle\left(T^{\mu\rho}\delta_{\mu}^{\nu}\right)\boldsymbol{X}\right\rangle \\
&= x^{\rho}\partial_{\mu}\left\langle T^{\mu\nu}\boldsymbol{X}\right\rangle - x^{\nu}\partial_{\mu}\left\langle T^{\mu\rho}\boldsymbol{X}\right\rangle \\
&\qquad + \left\langle T^{\rho\nu}\boldsymbol{X}\right\rangle - \left\langle T^{\nu\rho}\boldsymbol{X}\right\rangle \\
&\overset{(2.90)}{=} -\sum_{j=1}^{n}x^{\rho}\delta\left(\boldsymbol{x}-\boldsymbol{x}_{j}\right)\partial_{i}^{\nu}\langle\boldsymbol{X}\rangle + \sum_{j=1}^{n}x^{\nu}\delta\left(\boldsymbol{x}-\boldsymbol{x}_{j}\right)\partial_{i}^{\rho}\langle\boldsymbol{X}\rangle \\
&\qquad + \left\langle\left(T^{\rho\nu} - T^{\nu\rho}\right)\boldsymbol{X}\right\rangle \\
&= \sum_{j=1}^{n}\delta\left(\boldsymbol{x}-\boldsymbol{x}_{j}\right)\left(x_{i}^{\nu}\partial_{i}^{\rho} - x_{i}^{\rho}\partial_{i}^{\nu}\right)\langle\boldsymbol{X}\rangle + \left\langle\left(T^{\rho\nu} - T^{\nu\rho}\right)\boldsymbol{X}\right\rangle.
\end{aligned}$$

A delta function property was used to obtain the last line, that is $f(\boldsymbol{x})\,\delta\left(\boldsymbol{x}-\boldsymbol{x}_{i}\right) = f\left(\boldsymbol{x}_{i}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}_{i}\right)$. By comparing the result above with the r.h.s of (2.93) we obtain (2.94).

Eqs. (2.90), (2.94) and (2.96) are the three Ward identities associated with the conformal invariance. As a final calculation we will compute the Ward identity for a SCT, but first we calculate its canonical conserved current according to (2.29), that is

$$K^{\mu}_{\alpha} = T^{\mu\beta}\left(2x_{\alpha}x_{\beta} - \eta_{\alpha\beta}x^2\right). \tag{2.97}$$

The generator of the SCT is given in (2.57), then

$$K^{\alpha} = -i\left\{2x^{\alpha}\Delta - ix_{\nu}S^{\alpha\nu} + 2x^{\alpha}x^{\nu}\partial_{\nu} - x^2\partial^{\alpha}\right\}. \tag{2.98}$$

From (2.39), we have

$$\begin{aligned}\partial_{\mu}\left\langle T^{\mu\beta}\left(2x^{\alpha}x_{\beta} - \delta^{\alpha}_{\beta}x^2\right)\boldsymbol{X}\right\rangle &= -\sum_{i=1}^{n}\delta\left(\boldsymbol{x}-\boldsymbol{x}_i\right)\langle\phi\left(\boldsymbol{x}_1\right)\dots \\ &\quad \left\{2x_i^{\alpha}\Delta_i - ix_{\nu,i}S_i^{\alpha\nu} + 2x_i^{\alpha}x_i^{\nu}\partial_{\nu,i} - x_i^2\partial_i^{\alpha}\right\}\phi\left(\boldsymbol{x}_i\right)\dots\phi\left(\boldsymbol{x}_n\right)\Big\rangle.\end{aligned} \tag{2.99}$$

The derivative of the l.h.s above is

$$\begin{aligned}\partial_{\mu}\left\langle T^{\mu\beta}\left(2x^{\alpha}x_{\beta} - \delta^{\alpha}_{\beta}x^2\right)\boldsymbol{X}\right\rangle &= 2\partial_{\mu}\left\langle T^{\mu}{}_{\beta}\left(x^{\alpha}x^{\beta}\right)\boldsymbol{X}\right\rangle - \partial_{\mu}\left\langle T^{\mu\alpha}\left(x^2\right)\boldsymbol{X}\right\rangle \\ &= 2\left\langle T^{\mu}{}_{\beta}\left(\delta^{\alpha}_{\mu}x^{\beta} + x^{\alpha}\delta^{\beta}_{\mu}\right)\boldsymbol{X}\right\rangle + 2\left\langle\left(x^{\alpha}x^{\beta}\right)\partial_{\mu}\left(T^{\mu}{}_{\beta}\boldsymbol{X}\right)\right\rangle \\ &\quad -2\left\langle T^{\mu\alpha}x_{\mu}\boldsymbol{X}\right\rangle - \left\langle x^2\partial_{\mu}\left(T^{\mu\alpha}\boldsymbol{X}\right)\right\rangle \\ &= 2x_{\beta}\left\langle\left(T^{\alpha\beta} - T^{\beta\alpha}\right)\boldsymbol{X}\right\rangle + 2x^{\alpha}\left\langle T^{\mu}{}_{\mu}\boldsymbol{X}\right\rangle \\ &\quad +2\left(x^{\alpha}x^{\beta}\right)\partial_{\mu}\left\langle T^{\mu}{}_{\beta}\boldsymbol{X}\right\rangle - x^2\left\langle\partial_{\mu}T^{\mu\alpha}\boldsymbol{X}\right\rangle.\end{aligned}$$

Now, we use the identities (2.90, 2.94,2.96), and obtain

$$\begin{aligned}\partial_{\mu}\left\langle T^{\mu\beta}\left(2x^{\alpha}x_{\beta} - \delta^{\alpha}_{\beta}x^2\right)\boldsymbol{X}\right\rangle &= -i2\sum_i x_{\beta}\delta\left(\boldsymbol{x}-\boldsymbol{x}_i\right)S_i^{\beta\alpha}\langle\boldsymbol{X}\rangle - 2\sum_i x^{\alpha}\delta\left(\boldsymbol{x}-\boldsymbol{x}_i\right)\Delta_i\langle\boldsymbol{X}\rangle \\ &\quad -2\sum_{j=1}^{n}x^{\alpha}x^{\beta}\delta\left(\boldsymbol{x}-\boldsymbol{x}_j\right)\partial_{\beta,i}\langle\boldsymbol{X}\rangle + \sum_{j=1}^{n}x^2\delta\left(\boldsymbol{x}-\boldsymbol{x}_j\right)\partial_i^{\alpha}\langle\boldsymbol{X}\rangle \\ &= -\sum_i\delta\left(\boldsymbol{x}-\boldsymbol{x}_i\right)\left(-i2x_{\beta,i}S_i^{\alpha\beta} + 2x_i^{\alpha}\Delta_i + 2x_i^{\alpha}x_i^{\beta}\partial_{\beta,i} - x_i^2\partial_i^{\alpha}\right)\langle\boldsymbol{X}\rangle.\end{aligned} \tag{2.100}$$

By renaming the dummy index β as ν it yields exactly to the right hand side of (2.99). This means that the Ward identity (2.99) is a consequence of translations, rotations and dilations, or Ward identities, i.e. (2.90, 2.94,2.96) respectively.

2.4 The Conformal Group in Two Dimensions

We now show that in dimension $d = 2$, there exists an infinite number of coordinate transformations which are locally conformal, they are analytic or holomorphic mappings from the complex plane onto itself. Among this infinite set of mappings one must distinguish the global conformal group, made of invertible mappings of the complex plane into itself.

We begin with the identification $g_{\mu\nu} = \delta_{\mu\nu}$ and $d = 2$, in eqs.(2.3,2.4), we see that the conformal transformation satisfy

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \partial_\rho \epsilon^\rho \delta_{\mu\nu} = (\partial_1 \epsilon_1 + \partial_2 \epsilon_2)\, \delta_{\mu\nu}, \tag{2.101}$$

which yields to the following equations

$$\delta_{\mu\nu} = \delta_{21} = \delta_{12}, \qquad \partial_1 \epsilon_2 + \partial_2 \epsilon_1 = 0 \Longrightarrow \partial_1 \epsilon_2 = -\partial_2 \epsilon_1 \tag{2.102}$$

and

$$\delta_{\mu\nu} = \delta_{11} = \delta_{22}, \qquad \partial_1 \epsilon_1 + \cancel{\partial_1 \epsilon_1} = \cancel{\partial_1 \epsilon_1} + \partial_2 \epsilon_2 \Longrightarrow \partial_1 \epsilon_1 = \partial_2 \epsilon_2, \tag{2.103}$$

which are recognized as the Cauchy-Riemann equations. A complex function whose real an imaginary part satisfy eqs.(2.102,2.103) is called a holomorphic function. This motivates the use of complex coordinates z and $\bar{z}$, with the following translation rules

$$\begin{array}{rclcrcl} z &=& \sigma^1 + i\sigma^2, & \qquad & \sigma^1 &=& \frac{1}{2}(z+\bar{z}), \\ \bar{z} &=& \sigma^1 - i\sigma^2, & & \sigma^2 &=& \frac{1}{2i}(z-\bar{z}), \\ \partial_z &=& \frac{1}{2}(\partial_1 - i\partial_2), & & \partial_1 &=& \partial_z + \partial_{\bar{z}}, \\ \partial_{\bar{z}} &=& \frac{1}{2}(\partial_1 + i\partial_2), & & \partial_2 &=& i(\partial_z - \partial_{\bar{z}}). \end{array} \tag{2.104}$$

We will sometimes write $\partial_z = \partial$ and $\partial_{\bar{z}} = \bar{\partial}$ when there is no ambiguity about the differentiation variable. In term of the coordinates z and $\bar{z}$, the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \tag{2.105}$$

where the index μ takes the values z and $\bar{z}$, in that order. The metric tensor allows to transform a covariant holomorphic index into a contravariant antiholomorphic index and vice versa. The antisymmetric tensor $\varepsilon_{\mu\nu}$ in holomorphic form is

$$\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2i} \\ \frac{1}{2i} & 0 \end{pmatrix}, \qquad \varepsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}. \tag{2.106}$$

In terms of the complex functions

$$w\left(z,\bar{z}\right)=w^{1}\left(z,\bar{z}\right)+iw^{2}\left(z,\bar{z}\right),\qquad \bar{w}\left(z,\bar{z}\right)=w^{1}\left(z,\bar{z}\right)-iw^{2}\left(z,\bar{z}\right). \tag{2.107}$$

The Cauchy-Riemann condition becomes

$$\partial_{\bar{z}}w\left(z,\bar{z}\right)=0\qquad\text{and}\qquad\partial_{z}\bar{w}\left(z,\bar{z}\right)=0, \tag{2.108}$$

whose solution is any holomorphic mapping

$$z\longrightarrow w\left(z\right),\qquad \bar{z}\longrightarrow\bar{w}\left(\bar{z}\right). \tag{2.109}$$

The conformal group in two dimension is therefore the set of holomorphic maps. This set is infinite dimensional, since a infinite number of parameters (the coefficients of a Laurent series) is needed to specify all holomorphic functions in some neighborhood. The physical space is the two-dimensional submanifold (called the "real surface") defined by the condition $z^{*}=\bar{z}$, but in general z and $\bar{z}$ are two different coordinates.

2.4.1 Global Conformal Transformations

In order to form a group, the mappings that satisfy (2.102) and (2.103) must be invertible, and must map the whole plane into itself. *Global conformal transformations* satisfy these requirements, while *local conformal transformations* are those no everywhere well-defined. The set of global conformal transformations form what is called the "special conformal group" or complex Möbius group. The complete set of such mappings is

$$w\left(z\right)=\frac{az+b}{cz+d}\qquad\text{with}\qquad ad-bc=1, \tag{2.110}$$

where a, b, c and d are complex numbers, when these parameters are real the mapping (2.110) represents the modular group. These mappings are called projective transformations, and to each of them we can associate the matrix

$$A=\left(\begin{array}{cc}a & b\\ c & d\end{array}\right). \tag{2.111}$$

The composition of two maps $w_{1}\circ w_{2}$ correspond to the matrix multiplication $A_{2}A_{1}$. Therefore, what we call the global conformal group in two dimensions is isomorphic to the group of complex invertible 2×2 matrices with unit determinant, or $SL\left(2,\mathbb{C}\right)/\mathbb{Z}_{2}$. The quotient $\mathbb{Z}_{2}$ is caused by the fact that the transformation (2.110) is not sensitive to simultaneous change of sign of all parameters a, b, c, d. It is known that $SL\left(2,\mathbb{C}\right)/\mathbb{Z}_{2}$ is isomorphic to the Lorentz group in four dimensions, that is, $SO\left(3,1\right)$. Therefore, about the conformal group, we have learned nothing new since the previous section: the global conformal group is the 6-parameter[9] (3 complex) pseudo-orthogonal group $SO(3,1)$. The transformations eq.(2.110) are the only globally defined invertible holomorphic mappings. The condition $ad - bc = 1$ is just the chosen normalization, strictly it is just required that $ad - bc \neq 0$.

2.4.2 The Witt algebra

Any holomorphic infinitesimal transformation may be expressed as

$$\begin{aligned} z' &= z + \epsilon(z) = z + \sum_{-\infty}^{\infty} c_n z^{n+1}, && (2.112) \\ \bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{-\infty}^{\infty} \bar{c}_n \bar{z}^{n+1}, && (2.113) \end{aligned}$$

where, by hypothesis, the infinitesimal mapping admits a Laurent expansion around $z = 0$. The effect of such mapping on a spinless and a dimensionless field $\phi(z, \bar{z})$ living on the plane, is

$$\begin{aligned} \phi'(z', \bar{z}') &= \phi(z, \bar{z}) = \phi(z' - \epsilon, \bar{z}' - \bar{\epsilon}) \\ &= \phi(z', \bar{z}') - \epsilon(z')\,\partial'\phi(z', \bar{z}') - \bar{\epsilon}(\bar{z}')\,\bar{\partial}'\phi(z', \bar{z}') && (2.114) \end{aligned}$$

or

$$\begin{aligned} \delta\phi &= -\sum_{-\infty}^{\infty} c_n z^{n+1}\partial\phi(z, \bar{z}) - \sum_{-\infty}^{\infty} \bar{c}_n \bar{z}^{n+1}\bar{\partial}\phi(z, \bar{z}) \\ &= \sum_{-\infty}^{\infty} \left\{ c_n l_n \phi(z, \bar{z}) + \bar{c}_n \bar{l}_n \phi(z, \bar{z}) \right\}, && (2.115) \end{aligned}$$

where they were introduced the operators

$$l_n = -z^{n+1}\partial, \qquad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial}. \qquad (2.116)$$

These are the generators of the conformal transformations and obey the following commutator relations:

$$[l_n, l_m] = (n - m)\, l_{n+m}, \qquad [\bar{l}_n, \bar{l}_m] = (n - m)\, \bar{l}_{n+m}, \qquad [l_n, \bar{l}_m] = 0. \qquad (2.117)$$

[9] It is just 6 real parameters because the condition $\det A = 1$ let fix one complex parameter, say, d.

[10]Thus, the conformal algebra is the sum of two isomorphic algebras. The algebra (2.117) is called the Witt algebra. Each of these infinite dimensional algebras contains a finite subalgebra generated by l_{-1}, l_0 and l_1. This is the subalgebra associated with the global conformal group. We see from the definition (2.116) that $l_{-1} = -\partial_z$ generates translations on the complex plane, that $l_0 = -z\partial_z$ generates dilations and rotations, and that $l_1 = -z^2\partial_z$ generates special conformal transformations. In particular, $l_0 + \bar{l}_0$ generates dilations on the real surface, and $i\left(l_0 - \bar{l}_0\right)$ generates rotations.

2.4.3 Primary Fields

For a given field with scaling dimension Δ and planar spin s, we define the "(anti)holomorphic conformal dimension $\left(\bar{h}\right) h$" as

$$\Delta = h + \bar{h}, \qquad s = h - \bar{h}. \tag{2.118}$$

Under a conformal map $z \longrightarrow w(z)$, $\bar{z} \longrightarrow \bar{w}(\bar{z})$, a quasi-primary field transforms as

$$\phi'(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}). \tag{2.119}$$

For infinitesimal mappings $w = z + \epsilon(z)$ and $\bar{w} = \bar{z} + \bar{\epsilon}(\bar{z})$, the variation of quasi-primary fields is obtained from (2.119) by expanding at first order in ϵ and $\bar{\epsilon}$, we obtain

$$\delta_{\epsilon,\bar{\epsilon}}\phi \equiv \phi'(z,\bar{z}) - \phi(z,\bar{z}) = -h\phi\partial_z\epsilon - \epsilon\partial_z\phi - \bar{h}\phi\partial_{\bar{z}}\bar{\epsilon} - \bar{\epsilon}\partial_{\bar{z}}\phi\ . \tag{2.120}$$

A field whose variation under any local conformal transformation in two dimensions is given by (2.119) (or, equivalently, (2.120)) is called primary. All primary fields are also quasi-primary, but the reverse is not true. Quasi-primary fields may transform according to (2.119) only under an element of the global conformal group $SL(2,\mathbb{C})$.

[10]This is shown as follow

$$\begin{aligned}
[l_n, l_m]\phi &= l_n l_m\phi - l_m l_n\phi \\
&= \left(z^{n+1}\partial\right)\left(z^{m+1}\partial\right)\phi - \left(z^{m+1}\partial\right)\left(z^{n+1}\partial\right)\phi \\
&= \left(z^{n+1}\right)\left((m+1)z^m\partial + z^{m+1}\partial^2\right)\phi - \left(z^{m+1}\partial\right)\left((n+1)z^n\partial + z^{n+1}\partial^2\right)\phi \\
&= \left((m+1)z^{n+m+1}\partial + z^{n+m+2}\partial^2\right)\phi - \left((n+1)z^{n+m+1}\partial + z^{n+m+2}\partial^2\right)\phi \\
&= (m+1)z^{n+m+1}\partial\phi - (n+1)z^{n+m+1}\partial\phi \\
&= -(n-m)z^{n+m+1}\partial\phi \\
&= (n-m)l_{n+m}\phi.
\end{aligned}$$

In the same way it can be shown the other relations.

2.4.4 Correlation Functions

Expressed in terms of holomorphic and antiholomorphic coordinates the identity (2.34) of n primary fields ϕ_i, with conformal dimensions h_i and $\bar{h}_i$, becomes under conformal transformations

$$\langle \phi_1 (w_1, \bar{w}_1) ... \phi_n (w_n, \bar{w}_n) \rangle = \prod_{i=1}^{n} \left(\frac{dw}{dz} \right)_{w=w_i}^{-h_i} \left(\frac{d\bar{w}}{d\bar{z}} \right)_{\bar{w}=\bar{w}_i}^{-\bar{h}_i} \langle \phi_1 (z_1, \bar{z}_1) ... \phi_n (z_n, \bar{z}_n) \rangle . \tag{2.121}$$

This relation fixes the form of two- and three-point functions. The difference here is the possibility of nonzero spin, incorporated in the difference $h_i - \bar{h}_i$. Let us express the relations (2.83) and (2.88) in terms of complex coordinates, taking spin into account when imposing rotation invariance. The distance x_{ij} is equal to $\left(z_{ij}\bar{z}_{ij}\right)^{1/2}$ and eq.(2.83) becomes

$$\langle \phi_1 (z_1, \bar{z}_1) \phi_2 (z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \qquad \text{if} \quad \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h}. \end{cases} \tag{2.122}$$

The two-point function vanishes if the conformal dimensions of the two fields are different. The additional condition on the conformal dimensions come from rotation invariance; the sum of spin within a correlator should be zero. For the three-point function, eq.(2.88) becomes

$$\langle \phi_1 (z_1, \bar{z}_1) \phi_2 (z_2, \bar{z}_2) \phi_3 (z_3, \bar{z}_3) \rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}. \tag{2.123}$$

Again, the sum of the spins of the holomorphic part cancels that of the antiholomorphic part, thus ensuring rotational invariance.

As before, global conformal invariance does not fix the precise form of the four-point correlation function and beyond, because of the existence of the anharmonic ratios. However, in two dimensions the number of anharmonic ratios is reduced, since the four points of the ratio are forced to lie in the same plane, which leads to an additional linear relation between them. Indeed, we have

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad 1 - \eta = \frac{z_{14}z_{23}}{z_{13}z_{24}}, \qquad \frac{\eta}{1-\eta} = \frac{z_{12}z_{34}}{z_{14}z_{23}}. \tag{2.124}$$

The four-point function may depend on η and $\bar{\eta}$ in an arbitrary way, provided the result is real. The general expression (2.89) translates into

$$\langle \phi_1 (x_1) ... \phi_4 (x_4) \rangle = f (\eta, \bar{\eta}) \prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}, \tag{2.125}$$

where $h = \sum_{i=1}^{4} h_i$ and $\bar{h} = \sum_{i=1}^{4} \bar{h}_i$.

2.5 Ward Identities

2.5.1 Holomorphic form of the Ward Identities

In section 2.3 we have derived a set of Ward identities associated with translational (2.90), rotational (2.94) and scale invariance (2.96). In doing so, we used the canonical definition of the energy-momentum tensor, with suitable modifications needed to make it symmetric and traceless. Recall that the traceless of the energy-momentum tensor implies the conformal invariance of the action. Let us assemble these three Ward identities:

$$\begin{aligned}
\frac{\partial}{\partial x^{\mu}} \left\langle T^{\mu}_{\ \nu} (x)\, X \right\rangle &= -\sum_{i=1}^{n} \delta (x - x_i) \frac{\partial}{\partial x_i^{\nu}} \langle X \rangle , \\
\varepsilon_{\mu\nu} \langle T^{\mu\nu} (x)\, X \rangle &= -i \sum_{i=1}^{n} s_i \delta (x - x_i) \langle X \rangle , \\
\left\langle T^{\mu}_{\ \mu} (x)\, X \right\rangle &= -\sum_{i=1}^{n} \delta (x - x_i) \Delta_i \langle X \rangle .
\end{aligned} \tag{2.126}$$

In the second equation we have used the specific form of the spin generators, $S_i^{\mu\nu} = s_i \varepsilon^{\mu\nu}$, in two dimensions, where $\varepsilon_{\mu\nu}$ is the two-dimensional antisymmetric tensor and s_i is the spin of the field ϕ_i. We have written the identity (2.94) according to

$$\langle (T^{\rho\nu} - T^{\nu\rho})\, X \rangle = \left\langle \left(\delta^{\rho}_{\alpha} \delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha} \delta^{\rho}_{\beta} \right) T^{\alpha\beta} X \right\rangle = \varepsilon^{\rho\nu} \varepsilon_{\alpha\beta} \left\langle T^{\alpha\beta} X \right\rangle .$$

We wish to rewrite these identities in terms of complex coordinates (eq.(2.104)) as complex components. We use expression (2.105) and (2.106) for the metric tensor and antisymmetric tensor respectively. For the delta function we use the identity

$$\delta (x) = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z} = \frac{1}{\pi} \partial_{z} \frac{1}{\bar{z}}. \tag{2.127}$$

This identity is justified as follows. We first express the divergence theorem in terms of complex coordinates, ((2.106))

$$\int_V d^2 x \partial_{\mu} F^{\mu} = \frac{1}{2i} \oint_{\partial V} \left(dz F^{\bar{z}} - d\bar{z} F^{z} \right) . \tag{2.128}$$

Here the contour ∂V circles counterclockwise. If $F^{\bar{z}}$ (F^{z}) is holomorphic (antiholomorphic), then Cauchy's theorem may be applied; otherwise the contour ∂V must stay fixed. We consider then a holomorphic function $f(z)$ and check the correctness of the first representation in eq.(2.127) as follows

$$\begin{aligned}\int_V d^2x\delta(\boldsymbol{x}) f(z) &= \frac{1}{\pi}\int_V d^2x f(z)\,\partial_{\bar{z}}\frac{1}{z}\\ &= \frac{1}{\pi}\int_V d^2x\partial_{\bar{z}}\left(\frac{f(z)}{z}\right)\\ &= \frac{1}{2\pi i}\oint_{\partial V} dz\left(\frac{f(z)}{z}\right) = f(0)\,. \end{aligned} \tag{2.129}$$

In the second equation we have used the assumption that $f(z)$ is analytic within V, in the third equation we used the form (2.128) of Gauss theorem with $F^{\bar{z}} = \frac{f(z)}{z\pi}$ and $F^z = 0$, an in the last step we used the Cauchy's theorem. A similar proof may be applied to the second representation in eq.(2.127), but with an antiholomorphic function $\bar{f}(\bar{z})$. The Ward identities are then explicitly written as

$$2\pi\partial_z\langle T_{\bar{z}z}\boldsymbol{X}\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}\boldsymbol{X}\rangle = -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle\boldsymbol{X}\rangle\,, \tag{2.130}$$

$$2\pi\partial_z\langle T_{\bar{z}\bar{z}}\boldsymbol{X}\rangle + 2\pi\partial_{\bar{z}}\langle T_{z\bar{z}}\boldsymbol{X}\rangle = -\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle\boldsymbol{X}\rangle\,, \tag{2.131}$$

$$2\langle T_{z\bar{z}}\boldsymbol{X}\rangle + 2\langle T_{\bar{z}z}\boldsymbol{X}\rangle = -\sum_{i=1}^{n}\delta(\boldsymbol{x}-\boldsymbol{x}_i)\,\Delta_i\langle\boldsymbol{X}\rangle\,, \tag{2.132}$$

$$-2\langle T_{z\bar{z}}\boldsymbol{X}\rangle + 2\langle T_{\bar{z}z}\boldsymbol{X}\rangle = -\sum_{i=1}^{n}\delta(\boldsymbol{x}-\boldsymbol{x}_i)\,s_i\langle\boldsymbol{X}\rangle\,. \tag{2.133}$$

[11]The $\boldsymbol{x}_i$ points are now described by the $2n$ complex coordinates $(w_i, \bar{w}_i)$, on which the set of primary fields generally depends. If we add and subtract the last two equations of the above, we find

$$\begin{aligned} 2\pi\langle T_{\bar{z}z}\boldsymbol{X}\rangle &= -\textstyle\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}h_i\langle\boldsymbol{X}\rangle\,,\\ 2\pi\langle T_{z\bar{z}}\boldsymbol{X}\rangle &= -\textstyle\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\bar{h}_i\langle\boldsymbol{X}\rangle\,, \end{aligned} \tag{2.134}$$

[11]We will illustrate this process by computing explicitly the relations (2.130,2.131). We begin with the equation (2.90),

and by considering the form (2.105) for the metric, we have

$$\partial_\mu\left\langle T^{\mu}{}_{\nu}\boldsymbol{X}\right\rangle = \partial_\mu\langle g^{\mu\gamma}T_{\gamma\nu}\boldsymbol{X}\rangle = \partial_z\langle g^{z\bar{z}}T_{\bar{z}\nu}\boldsymbol{X}\rangle + \partial_{\bar{z}}\langle g^{\bar{z}z}T_{z\nu}\boldsymbol{X}\rangle = -\sum_i\delta(\boldsymbol{x}-\boldsymbol{x}_i)\,\partial_\nu\langle\boldsymbol{X}\rangle\,.$$

Since $\partial_\nu = \frac{\partial}{\partial x_i^\nu} \longrightarrow \left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_i}\right) = (\partial_{w_i}, \partial_{\bar{w}_i})$, we have then the following two equations, after employ (2.127)

$$\begin{aligned} \partial_z\langle g^{z\bar{z}}T_{\bar{z}z}\boldsymbol{X}\rangle + \partial_{\bar{z}}\langle g^{\bar{z}z}T_{zz}\boldsymbol{X}\rangle &= -\textstyle\sum_i\frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle\boldsymbol{X}\rangle\,,\\ \partial_z\langle g^{z\bar{z}}T_{\bar{z}\bar{z}}\boldsymbol{X}\rangle + \partial_{\bar{z}}\langle g^{\bar{z}z}T_{z\bar{z}}\boldsymbol{X}\rangle &= -\textstyle\sum_i\frac{1}{\pi}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle\boldsymbol{X}\rangle\,. \end{aligned}$$

Since $g^{z\bar{z}} = g^{\bar{z}z} = 2$, we get the desired relations. The last equation is deduced by using the form (2.191) for $\varepsilon^{\mu\nu}$.

where we have chosen the representation (2.127) appropriated to each case and used the definition (2.118) of the holomorphic and antiholomorphic conformal dimensions. Inserting these relations into eq.(2.130), we get

$$\begin{aligned} \partial_{\bar{z}}\left\{\langle T(\bar{z},z)\,\boldsymbol{X}\rangle-\textstyle\sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle+\frac{1}{(z-w_i)^2}h_i\langle\boldsymbol{X}\rangle\right]\right\} &= 0, \\ \partial_{z}\left\{\langle \bar{T}(\bar{z},z)\,\boldsymbol{X}\rangle-\textstyle\sum_{i=1}^{n}\left[\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle+\frac{1}{(\bar{z}-\bar{w}_i)^2}\bar{h}_i\langle\boldsymbol{X}\rangle\right]\right\} &= 0, \end{aligned} \tag{2.135}$$

[12]where we have introduced a rescaled energy-momentum tensor

$$T=-2\pi T_{zz} \qquad \bar{T}=-2\pi T_{\bar{z}\bar{z}}. \tag{2.136}$$

Thus the expression between braces in (2.135) are respectively holomorphic and antiholomorphic; we may write

$$\langle T(z)\,\boldsymbol{X}\rangle=\sum_{i=1}^{n}\left\{\frac{1}{z-w_i}\partial_{w_i}\langle\boldsymbol{X}\rangle+\frac{1}{(z-w_i)^2}h_i\langle\boldsymbol{X}\rangle\right\}+\text{reg.}, \tag{2.137}$$

where reg. means holomorphic function of z, regular at $z=w_i$. There is a similar expression for the antiholomorphic counterpart with reg. depending on $\bar{z}$ and regular at $\bar{z}=\bar{w}_i$.

2.5.2 The Conformal Ward Identity

Let us now express the three Ward identities (2.126) into a single relation. In order to do so we consider an arbitrary conformal coordinate variation $\epsilon^{\nu}(\boldsymbol{x})$. We write

$$\begin{aligned} \partial_\mu\left(\epsilon_\nu T^{\mu\nu}\right) &= \epsilon_\nu\partial_\mu T^{\mu\nu}+\left(\partial_\mu\epsilon_\nu\right)T^{\mu\nu} \\ &= \epsilon_\nu\partial_\mu T^{\mu\nu}+\frac{1}{2}\left(\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu\right)T^{\mu\nu}+\frac{1}{2}\left(\partial_\mu\epsilon_\nu-\partial_\nu\epsilon_\mu\right)T^{\mu\nu} \\ &\overset{(2.3)}{=} \epsilon_\nu\partial_\mu T^{\mu\nu}+\frac{1}{2}\left(\partial_\rho\epsilon^\rho\eta_{\mu\nu}\right)T^{\mu\nu}+\frac{1}{2}\left(\delta^\alpha_\mu\delta^\beta_\nu-\delta^\beta_\nu\delta^\alpha_\mu\right)\partial_\alpha\epsilon_\beta T^{\mu\nu} \\ &= \epsilon_\nu\partial_\mu T^{\mu\nu}+\frac{1}{2}\partial_\rho\epsilon^\rho T^{\mu}{}_{\mu}+\frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\varepsilon_{\mu\nu}T^{\mu\nu}, \end{aligned} \tag{2.138}$$

where the relation $\varepsilon^{\alpha\beta}\varepsilon_{\mu\nu}=\delta^\alpha_\mu\delta^\beta_\nu-\delta^\beta_\nu\delta^\alpha_\mu$ has been used. We note that $\frac{1}{2}\left(\partial_\rho\epsilon^\rho\right)$ is the local scale factor $f(\boldsymbol{x})$ of eq.(2.3) and $\frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta$ is a local rotation angle. Integrating both sides of (2.138), the three Ward identities (2.126) derived in Sect. 2.3.2 may be

[12]Note that we have effectuated the derivatives $\partial\left(\frac{1}{z-w_i}\right)$ and $\partial_{\bar{z}}\left(\frac{1}{\bar{z}-\bar{w}_i}\right)$.

encapsulated into

$$\begin{aligned}\int_V d^2x\partial_\mu\left(\epsilon_\nu T^{\mu\nu}\boldsymbol{X}\right) &= \int_V d^2x\epsilon^\nu\left(\boldsymbol{x}\right)\left(-\sum_{i=1}^{n}\delta\left(\boldsymbol{x}-\boldsymbol{x}_i\right)\frac{\partial}{\partial x_i^\nu}\boldsymbol{X}\right) \\ &\quad+\int_V d^2xf\left(\boldsymbol{x}\right)\left(-\sum_{i=1}^{n}\delta\left(\boldsymbol{x}-\boldsymbol{x}_i\right)\Delta_i\boldsymbol{X}\right) \\ &\quad+\int_V d^2x\left(\frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\right)\left(-i\sum_{i=1}^{n}s_i\delta\left(\boldsymbol{x}-\boldsymbol{x}_i\right)\boldsymbol{X}\right).\end{aligned} \tag{2.139}$$

By comparing with equation (2.38) we see that eq.(2.139) is the sum of the three variations of $\delta\boldsymbol{X}$, that is, translations, dilations and rotations, where in each case $\epsilon_a\left(\boldsymbol{x}\right)$ and G_a take the respective form. This allows us to identify

$$\delta_\epsilon\left\langle\boldsymbol{X}\right\rangle=\int_V d^2x\partial_\mu\left\langle T^{\mu\nu}\left(x\right)\epsilon_\nu\left(\boldsymbol{x}\right)\boldsymbol{X}\right\rangle, \tag{2.140}$$

where $\delta_\epsilon\left\langle\boldsymbol{X}\right\rangle$ is the variation of $\boldsymbol{X}$ under a local conformal transformation. Here the integral is taken over a domain V containing the position of all the fields in the string $\boldsymbol{X}$. Applying Gauss's theorem (2.128) to $F^\mu=\left\langle T^{\mu\nu}\left(x\right)\epsilon_\nu\left(x\right)\boldsymbol{X}\right\rangle$, one finds

$$\delta_{\epsilon,\bar{\epsilon}}\left\langle\boldsymbol{X}\right\rangle=\frac{i}{2}\oint_{\partial V}\left\{-dz\left\langle T^{\bar{z}\bar{z}}\epsilon_{\bar{z}}\boldsymbol{X}\right\rangle+d\bar{z}\left\langle T^{zz}\epsilon_z\boldsymbol{X}\right\rangle\right\}. \tag{2.141}$$

We defined $\epsilon=\epsilon^z$ and $\bar{\epsilon}=\epsilon^{\bar{z}}$. The terms $\left\langle T_{\bar{z}z}\boldsymbol{X}\right\rangle$ and $\left\langle T_{z\bar{z}}\boldsymbol{X}\right\rangle$ do not contribute to the contour integrals, since the contour do not exactly go through the positions contained in $\boldsymbol{X}$, and since these expressions vanish outside these points, according to eq.(2.96). Finally, substituting the definition (2.136), we obtain the so called *conformal Ward identity*:

$$\delta_{\epsilon,\bar{\epsilon}}\left\langle\boldsymbol{X}\right\rangle=-\oint_C\frac{dz}{2\pi i}\epsilon\left(z\right)\left\langle T\left(z\right)\boldsymbol{X}\right\rangle+\oint_C\frac{d\bar{z}}{2\pi i}\bar{\epsilon}\left(\bar{z}\right)\left\langle\bar{T}\left(\bar{z}\right)\boldsymbol{X}\right\rangle. \tag{2.142}$$

[13] If the fields in $\boldsymbol{X}$ are primary, one can use the equation (2.137) and its antiholomorphic counterpart in (2.142), to obtain:

$$\delta_\epsilon\left\langle\boldsymbol{X}\right\rangle=-\sum_i\left\{\epsilon\left(w_i\right)\partial_{w_i}+\partial\epsilon\left(w_i\right)h_i\right\}\left\langle\boldsymbol{X}\right\rangle. \tag{2.143}$$

[13]Note the following relations, from the definitions (2.136) $T^{zz}=g^{\gamma z}g^{\sigma z}T_{\gamma\sigma}=4T_{\bar{z}\bar{z}}=-\frac{2}{\pi}\bar{T}\left(\bar{z}\right)$, in the same way $T^{\bar{z}\bar{z}}=-\frac{2}{\pi}T\left(z\right)$. Similarly, we have that $\epsilon_\nu=g_{\nu\mu}\epsilon^\mu=g_{\nu z}\epsilon^z+g_{\nu\bar{z}}\epsilon^{\bar{z}}=g_{\nu z}\epsilon\left(z\right)+g_{\nu\bar{z}}\bar{\epsilon}\left(\bar{z}\right)$, thus $\epsilon_z=\frac{1}{2}\bar{\epsilon}$ and $\epsilon_{\bar{z}}=\frac{1}{2}\epsilon$.

We recover formula (2.120) for a variation of a primary field under an infinitesimal holomorphic conformal mapping:

$$\delta_\epsilon \phi = -\left\{\epsilon\partial + \partial\epsilon h\right\}\phi. \tag{2.144}$$

We now apply the conformal ward identity to the global conformal transformations (the $SL(2,\mathbb{C})$ mapping of eq.(2.110)). The variation $\delta_\epsilon \langle X\rangle$ must vanish for infinitesimal $SL(2,\mathbb{C})$ mappings, since they constitute a true symmetry of the theory. Such infinitesimal mappings have the form

$$f(w_i) = \frac{(1+\alpha)w_i + \beta}{\gamma w_i + 1 - \alpha} = w_i + \epsilon(w_i), \tag{2.145}$$

where α, β and γ are infinitesimal. At first order, the coordinate variation $\epsilon(w_i)$ is

$$\epsilon(w_i) = \beta + 2\alpha w_i - \gamma w_i^2. \tag{2.146}$$

For α, β and γ arbitrary, this implies the following three relations on correlators of primary fields according to(2.143):

$$\begin{aligned}\beta\sum_i \partial_{w_i}\left\langle \phi_1(w_1,\bar{w}_1)\ldots\phi_n(w_n,\bar{w}_n)\right\rangle &= 0,\\ 2\alpha\sum_i \left(w_i\partial_{w_i}+h_i\right)\left\langle \phi_1(w_1,\bar{w}_1)\ldots\phi_n(w_n,\bar{w}_n)\right\rangle &= 0,\\ -\gamma\sum_i\left(w_i^2\partial_{w_i}+2w_ih_i\right)\left\langle \phi_1(w_1,\bar{w}_1)\ldots\phi_n(w_n,\bar{w}_n)\right\rangle &= 0.\end{aligned} \tag{2.147}$$

The Ward identity (2.142) summarizes the consequences of local conformal symmetry on correlation functions. The application of (2.142) rests on the assumption that the energy-momentum tensor is everywhere well-defined or regular.

2.6 Free Fields Examples and the OPE

2.6.1 The Free Boson, the XX CFT

Here we are going to study the free massless boson X^μ $(\mu = 0,1,\ldots,d-1)$ system, with the following action:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\eta^{ab}\partial_a X^\mu\partial_b X_\mu \longrightarrow \frac{1}{2\pi\alpha'}\int d^2 z\partial X^\mu\bar{\partial}X_\mu. \tag{2.148}$$

[14]It is known that α' is the only one free parameter in string theory[15]. The equations of motion are $\partial_{\bar{z}}\partial_z X^\mu = 0$. The two point function, or propagator, can be calculated through the path integral formalism, that is as follows

$$\begin{aligned} 0 &= \int \mathcal{D}X \frac{\delta}{\delta X_\mu(z,\bar{z})}\left[\mathrm{e}^{-S} X^\nu(w,\bar{w})\right] \\ &= \int \mathcal{D}X \left[-\frac{\delta S}{\delta X_\mu(z,\bar{z})} X^\nu(w,\bar{w}) + \frac{\delta X^\nu(w,\bar{w})}{\delta X_\mu(z,\bar{z})}\right]\mathrm{e}^{-S} \\ &= \int \mathcal{D}X \left[\frac{1}{\pi\alpha'}\partial_{\bar{z}}\partial_z X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) + \eta^{\mu\nu}\delta^2(z-w,\bar{z}-\bar{w})\right]\mathrm{e}^{-S} \\ &= \frac{1}{\pi\alpha'}\partial_{\bar{z}}\partial_z \langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w})\rangle + \eta^{\mu\nu}\left\langle \delta^2(z-w,\bar{z}-\bar{w})\right\rangle, \end{aligned}$$

where it was used the fact that the path integral of a total derivative is zero. Thus

$$\partial_{\bar{z}}\partial_z X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) = -\pi\alpha'\eta^{\mu\nu}\delta^2(z-w,\bar{z}-\bar{w}) \tag{2.149}$$

holds inside a general expectation value or as an operator equation. We now define normal ordering of a general operator $\mathcal{A}$, denoted $:\mathcal{A}:$, as follows,

$$\begin{aligned} :X^\mu(z,\bar{z}): &= X^\mu(z,\bar{z}), \\ :X^\mu(z,\bar{z}) X^\nu(w,\bar{w}): &= X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2. \end{aligned} \tag{2.150}$$

The point of this definition is the property

$$\begin{aligned} \partial_z\partial_{\bar{z}} :X^\mu(z,\bar{z}) X^\nu(w,\bar{w}): &= \partial_z\partial_{\bar{z}} X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) + \frac{\alpha'}{2}\eta^{\mu\nu}\partial_z\partial_{\bar{z}}\ln|z-w|^2 \\ &\overset{(2.149)}{=} -\pi\alpha'\eta^{\mu\nu}\delta^2(z-w,\bar{z}-\bar{w}) + \pi\alpha'\eta^{\mu\nu}\delta^2(z-w,\bar{z}-\bar{w}), \\ &= 0, \end{aligned} \tag{2.151}$$

where we used a result derived from the eq.(2.127), that is

$$\partial_z\partial_{\bar{z}}\ln|z|^2 = \partial_z\partial_{\bar{z}}(\ln z + \ln\bar{z}) = \partial_{\bar{z}}\left(\frac{1}{z}\right) + \partial_z\left(\frac{1}{\bar{z}}\right) = 2\pi\delta^2(z,\bar{z}). \tag{2.152}$$

[14]Henceforth we use latin indices for the two-dimensional metric $\eta^{ab} = \mathrm{diag}(1,1)$ to differentiate it from that $\eta^{\mu\nu}$ of the $d-$dimensional Minkowski target spacetime.

[15]As mentioned in the introduction, the parameter $\alpha'$and the tension of the string T are related through the relationship $\alpha' = 1/2\pi T$. A classical string has two free parameter, namely, its tension and its linear mass density. However, in string theory, the string is assumed to be massless and the remaining only free parameter is the tension. More about this discussion in the Barton Swiebach book [10].

Then we have

$$\begin{aligned}\langle X^{\mu}(z,\bar{z})\, X^{\nu}(w,\bar{w})\rangle &= -\frac{\alpha'}{2}\eta^{\mu\nu}\left\{\ln|z-w|^{2}\right\}+\text{const.}\\ &= -\frac{\alpha'}{2}\eta^{\mu\nu}\left\{\ln(z-w)+\ln(\bar{z}-\bar{w})\right\}+\text{const..}\end{aligned} \quad (2.153)$$

The holomorphic and antiholomorphic components can be separated by taking the derivatives $\partial_z X$ and $\partial_{\bar{z}} X$:

$$\begin{aligned}\langle\partial_z X^{\mu}(z,\bar{z})\,\partial_w X^{\nu}(w,\bar{w})\rangle &= -\tfrac{\alpha'}{2}\tfrac{\eta^{\mu\nu}}{(z-w)^{2}},\\ \langle\partial_{\bar{z}} X^{\mu}(z,\bar{z})\,\partial_{\bar{w}} X^{\nu}(w,\bar{w})\rangle &= -\tfrac{\alpha'}{2}\tfrac{\eta^{\mu\nu}}{(\bar{z}-\bar{w})^{2}}.\end{aligned} \quad (2.154)$$

In the following we will concentrate in the holomorphic fields $\partial X^{\mu}\equiv\partial_z X^{\mu}$.

2.6.2 The Operator Product Expansion OPE

The operator product expansion, or OPE, is the representation of a product of operators (at positions z and w respectively) by a sum of terms. Each of them being a single operator, well defined as $z\longrightarrow w$, multiplied by a $c-$number (c for classical) function of $z-w$, possibly diverging as $z\longrightarrow w$, and which embodies the infinite fluctuations as the two positions tend toward each other. We can represent that in the form

$$\mathcal{A}_i(z)\,\mathcal{A}_j(w)=\sum_k c^{k}_{ij}(z-w)\,\mathcal{A}_k(w)\,. \quad (2.155)$$

This holds inside a general expectation value, indeed

$$\left\langle\mathcal{A}_i(z)\,\mathcal{A}_j(w)\ldots\right\rangle=\sum_k c^{k}_{ij}(z-w)\left\langle\mathcal{A}_k(w)\ldots\right\rangle. \quad (2.156)$$

The ... represent operator insertions at points far away from z and w. The terms are conventionally arranged in order of decreasing size in the limit as $z\longrightarrow w$. The definition of normal ordering for arbitrary number of field is given recursively as

$$:X^{\mu_1}(z_1,\bar{z}_1)\ldots X^{\mu_n}(z_n,\bar{z}_n):=X^{\mu_1}(z_1,\bar{z}_1)\ldots X^{\mu_n}(z_n,\bar{z}_n)+\sum\text{subtractions.} \quad (2.157)$$

For example,

$$\begin{aligned} :X^{\mu_1}(z_1,\bar z_1)X^{\mu_2}(z_2,\bar z_2)X^{\mu_3}(z_3,\bar z_3): \;&=\; X^{\mu_1}(z_1,\bar z_1)X^{\mu_2}(z_2,\bar z_2)X^{\mu_3}(z_3,\bar z_3)\\ &+\frac{\alpha'}{2}\eta^{\mu_1\mu_2}\ln|z_{12}|^2 X^{\mu_3}(z_3,\bar z_3)\\ &+\frac{\alpha'}{2}\eta^{\mu_1\mu_3}\ln|z_{13}|^2 X^{\mu_2}(z_2,\bar z_2)\\ &+\frac{\alpha'}{2}\eta^{\mu_3\mu_2}\ln|z_{23}|^2 X^{\mu_1}(z_1,\bar z_1), \end{aligned} \tag{2.158}$$

where $z_{ij}=z_i-z_j$. The definition (2.150) can be put in compact form

$$:\mathfrak{F}:=\exp\left(\frac{\alpha'}{4}\int d^2z_1d^2z_2\ln|z_{12}|^2\frac{\delta}{\delta X^\mu(z_1,\bar z_1)}\frac{\delta}{\delta X_\mu(z_2,\bar z_2)}\right)\mathfrak{F}, \tag{2.159}$$

for any functional $\mathfrak{F}$ of X. We can see that, in effect, equation (2.159) reproduces (2.53), that is

$$\begin{aligned} :X^\mu(z,\bar z)X^\nu(w,\bar w): \;&=\; \exp\left(\frac{\alpha'}{4}\int d^2z_1'd^2z_2'\ln|z_{12}'|^2\frac{1}{\eta_{\alpha\beta}}\frac{\delta}{\delta X^\alpha(z_1',\bar z_1')}\frac{\delta}{\delta X^\beta(z_2',\bar z_2')}\right)\\ &\qquad\times X^\mu(z,\bar z)X^\nu(w,\bar w)\\ &=\left(1+\frac{\alpha'}{4}\int d^2z_1d^2z_2\ln|z_{12}|^2\eta^{\alpha\beta}\frac{\delta}{\delta X^\alpha(z_1,\bar z_1')}\frac{\delta}{\delta X^\beta(z_2,\bar z_2)}+\ldots\right)\\ &\qquad\times X^\mu(z,\bar z)X^\nu(w,\bar w)\\ &= X^\mu(z,\bar z)X^\nu(w,\bar w)\\ &\quad+\frac{\alpha'}{2}\int d^2z_1d^2z_2\ln|z_{12}|^2\eta^{\alpha\beta}\delta(z_1-z,\bar z_1-\bar z)\,\delta(z_2-w,\bar z_2-\bar w)\,\delta^\mu_\alpha\delta^\nu_\beta\\ &= X^\mu(z,\bar z)X^\nu(w,\bar w)+\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2. \end{aligned} \tag{2.160}$$

By acting on both sides of (2.159) with the inverse exponential we obtain

$$\begin{aligned} \mathfrak{F} &= \exp\left(-\frac{\alpha'}{4}\int d^2z_1d^2z_2\ln|z_{12}|^2\frac{\delta}{\delta X^\mu(z_1,\bar z_1)}\frac{\delta}{\delta X_\mu(z_2,\bar z_2)}\right):\mathfrak{F}:\\ &= :\mathfrak{F}:+\sum\text{contractions}. \end{aligned} \tag{2.161}$$

That is what we will call OPE, where a contraction is $-\frac{\alpha'}{2}\eta^{\mu_i\mu_j}\ln|z_{ij}|^2$ for the XX system and in general it corresponds to the propagator of the theory. The generalization of this expression for the OPE of any pair of operators is

$$:\mathfrak{F}::\mathfrak{G}:=:\mathfrak{F}\mathfrak{G}:+\sum\text{cross}-\text{contractions}, \tag{2.162}$$

for arbitrary functionals $\mathfrak{F}$ and $\mathfrak{G}$ of X. Cross-contractions stands for the propagators formed only between fields of $\mathfrak{F}$ with those of $\mathfrak{G}$. We make recurrent use of this last expression. We will use "$\sim$" instead of "$=$" when writing the OPEs, which means equal up to nonsingular terms. In that sense, we have according to (2.162,2.137), the OPE of the energy-momentum tensor with a primary field $\phi(w,\bar{w})$ of conformal dimensions h and $\bar{h}$,

$$
\begin{aligned}
T(z)\,\phi(w,\bar{w}) &\sim \frac{1}{(z-w)^2}h\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w})\,, && (2.163)\\
\bar{T}(\bar{z})\,\phi(w,\bar{w}) &\sim \frac{1}{(\bar{z}-\bar{w})^2}\bar{h}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w})\,. && (2.164)
\end{aligned}
$$

2.6.3 The XX energy-momentum tensor

From the definition of the OPE and (2.154) we then have

$$
\partial X^\mu(z)\,\partial X^\nu(w) \sim -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2}. \tag{2.165}
$$

The OPE reflects the bosonic character of the fields: exchanging the two factors doesn't affect the correlator. The energy-momentum tensor associated with the free massless boson (according to eq.(2.61)) is:

$$
\begin{aligned}
T_{ab} &= -g_{ab}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial^a X^\mu)}\partial_b X^\mu \\
&= \frac{1}{2\pi\alpha'}\left(\partial_a X^\mu\partial_b X_\mu - \frac{1}{2}g_{ab}\partial_c X^\mu\partial^c X_\mu\right).
\end{aligned} \tag{2.166}
$$

Its quantum version (2.136) in complex coordinates is[16]

$$
T(z) = -\frac{1}{\alpha'}:\partial X^\mu\partial X_\mu:. \tag{2.167}
$$

Like all composite fields, the energy momentum tensor has to be normal ordered, in order to ensure the vanishing of its vacuum expectation value. The OPE of $T(z)$ with

[16] That is as follows

$$
\begin{aligned}
2\pi T_{zz} &= \frac{1}{\alpha'}\left(\partial_z X^\mu\partial_z X_\mu - \frac{1}{2}\cancel{g_{zz}}g^{cd}\partial_c X^\mu\partial_d X_\mu\right) = \frac{1}{\alpha'}\partial_z X^\mu\partial_z X_\mu = -T(z)\,,\\
2\pi T_{\bar{z}\bar{z}} &= \frac{1}{\alpha'}\left(\partial_{\bar{z}} X^\mu\partial_{\bar{z}} X_\mu - \frac{1}{2}\cancel{g_{\bar{z}\bar{z}}}g^{cd}\partial_c X^\mu\partial_d X_\mu\right) = \frac{1}{\alpha'}\partial_{\bar{z}} X^\mu\partial_{\bar{z}} X_\mu = -\bar{T}(\bar{z})\,,\\
T_{\bar{z}z} &= \frac{1}{2\pi\alpha'}\left(\partial_{\bar{z}} X^\mu\partial_z X_\mu - \frac{1}{2}g_{\bar{z}z}g^{\bar{z}z}\partial_{\bar{z}} X^\mu\partial_z X_\mu - \frac{1}{2}g_{\bar{z}z}g^{z\bar{z}}\partial_z X^\mu\partial_{\bar{z}} X_\mu\right)\\
&= \frac{1}{2\pi\alpha'}\left(\partial_{\bar{z}} X^\mu\partial_z X_\mu - \partial_{\bar{z}} X^\mu\partial_z X_\mu\right) = T_{z\bar{z}} = 0,
\end{aligned}
$$

∂X^μ may be calculated by considering cross contractions according to (2.162):

$$\begin{aligned} T(z)\,\partial X^\mu(w) &= -\frac{1}{\alpha'} : \partial X_\nu(z)\,\partial X^\nu(z) : \partial X^\mu(w) \\ &\sim -\frac{2}{\alpha'}\partial X_\nu(z)\,\langle \partial X^\nu(z)\,\partial X^\mu(w)\rangle \\ &\sim \frac{\partial X^\mu(z)}{(z-w)^2}. \end{aligned} \tag{2.168}$$

By expanding $\partial X^\mu(z)$ around w, we arrive at the OPE

$$\begin{aligned} T(z)\,\partial X^\mu(w) &\sim \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)(z-w)}{(z-w)^2} + \frac{1}{2}\frac{\partial^3 X^\mu(w)(z-w)^2}{(z-w)^2} + \text{reg.} \\ &\sim \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{(z-w)} + \text{reg..} \end{aligned} \tag{2.169}$$

We will always expand the OPE around the insertion point of the second field, it is possible because by definition the OPE show the behavior of one field at a point near to the other field. According to eqs.(2.163), eq.(2.169) shows that ∂X^μ is a primary field with conformal dimension $h = 1$. Now, we calculate the OPE of the energy-momentum tensor with itself. To calculate that, we consider the cross-contractions of the fields in $T(z)$ and those in $T(w)$. We have therefore

$$\begin{aligned} T(z)\,T(w) &= \frac{1}{\alpha'^2} : \partial X^\mu(z)\,\partial X_\mu(z) :: \partial X^\mu(z)\,\partial X_\mu(z) : \\ &\sim \frac{1}{\alpha'^2}\left\{ 4\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2}\right) : \partial X_\mu(z)\,\partial X_\nu(w) : \right. \\ &\qquad\qquad \left. +2\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2}\right)\left(-\frac{\alpha'}{2}\frac{\eta_{\mu\nu}}{(z-w)^2}\right)\right\} \\ &\sim \frac{1}{\alpha'^2}\left\{ -2\alpha' \frac{: \partial X^\mu(z)\,\partial X_\mu(w) :}{(z-w)^2} + \frac{\alpha'^2}{2}\frac{\eta^\mu{}_\mu}{(z-w)^4}\right\} \\ &\sim -\frac{2}{\alpha'}\frac{: \partial X^\mu(w)\,\partial X_\mu(w) :}{(z-w)^2} - \frac{2}{\alpha'}\frac{: \partial^2 X^\mu(w)\,\partial X_\mu(w) :}{(z-w)} + \frac{1}{2}\frac{d}{(z-w)^4} \\ &\sim \frac{d/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \end{aligned} \tag{2.170}$$

where we have done four single cross-contractions and two double cross-contractions and also, we expanded $\partial X^\mu(z)$ around w. We immediately see that the energy-momentum tensor is not strictly a primary field, because of the anomalous term $\frac{d/2}{(z-w)^4}$, which does not appear in eq.(2.163).

2.6.4 The Ghost System

Another theory of interest is the ghost system, which consists in considering a family of CFTs with anticommuting fields b and c with action

$$S = \frac{1}{2\pi}\int d^2z\, b\bar{\partial}c. \tag{2.171}$$

This is conformally invariant for b and c transforming as tensors of weights $(h_b, 0)$ and $(h_c, 0)$ such that

$$h_b = \lambda \qquad , \qquad h_c = 1-\lambda, \tag{2.172}$$

for any given constant λ. The operator equations of motion are

$$\begin{aligned}
\delta_b S &= \frac{1}{2\pi}\int d^2z\,(\delta b)\,\bar{\partial}c \longrightarrow \bar{\partial}c = 0,\\
\delta_c S &= \frac{1}{2\pi}\int d^2z\, b\bar{\partial}\,(\delta c)\\
&= -\frac{1}{2\pi}\int d^2z\,\left(\bar{\partial}b\right)\delta c \longrightarrow \bar{\partial}b = 0
\end{aligned}$$

and the propagator may be calculated in the same way as the matter system, that is

$$\begin{aligned}
0 &= \int \mathcal{D}b\mathcal{D}c\frac{\delta}{\delta c\,(z,\bar{z})}\left[\mathrm{e}^{-S}c\left(z',\bar{z}'\right)\right]\\
&= \int \mathcal{D}b\mathcal{D}c\left[-\frac{\delta S}{\delta c\,(z,\bar{z})}\mathrm{e}^{-S}c\left(z',\bar{z}'\right) - \mathrm{e}^{-S}\frac{\delta c\,(z',\bar{z}')}{\delta c\,(z,\bar{z})}\right]\\
&= \int \mathcal{D}b\mathcal{D}c\left[\frac{\bar{\partial}b\,(z,\bar{z})}{2\pi}c\left(z',\bar{z}'\right) - \delta^2\left(z-z',\bar{z}-\bar{z}'\right)\right]\mathrm{e}^{-S}\\
&= \partial_{\bar{z}}\left\langle b\,(z,\bar{z})\,c\left(z',\bar{z}'\right)\right\rangle - 2\pi\left\langle\delta^2\left(z-z',\bar{z}-\bar{z}'\right)\right\rangle.
\end{aligned}$$

Summarizing,

$$\bar{\partial}c = \bar{\partial}b = 0, \qquad \partial_{\bar{z}}b\,(z,\bar{z})\,c\,(0,0) = 2\pi\delta^2\,(z,\bar{z}) = \partial_{\bar{z}}\frac{1}{z}. \tag{2.173}$$

The OPEs are

$$b\,(z)\,c\,(w) \sim \frac{1}{z-w}, \qquad c\,(z)\,b\,(w) \sim \frac{1}{z-w}, \tag{2.174}$$

where in the second OPE there have been two sign flips, one from anti-commutations and one from $z \longleftrightarrow w$. Other OPEs are nonsingular:

$$b\,(z)\,b\,(w) = \mathcal{O}\,(z-w) \qquad , \qquad c\,(z)\,c\,(w) = \mathcal{O}\,(z-w)\,. \tag{2.175}$$

Before to continue, an interesting result can be derived from eq.(2.60) for the variation of the metric in complex coordinates, that is as follows

$$\begin{aligned}
\delta S &= -\int d^2 z g^{\mu\rho} T_{\rho\nu} \partial_\mu \epsilon^\nu \\
&= -\int d^2 z \left(g^{\bar{z}z} T_{zz} \partial_{\bar{z}} \epsilon^z + g^{z\bar{z}} T_{\bar{z}\bar{z}} \partial_z \epsilon^{\bar{z}} \right) \\
&\overset{(2.136)}{=} \frac{1}{2\pi} \int d^2 z \left(T(z) \bar{\partial} \epsilon^z + \bar{T}(\bar{z}) \partial \epsilon^{\bar{z}} \right) \\
&= \frac{1}{2\pi} \int d^2 z \left(T(z) \bar{\partial} \epsilon + \bar{T}(\bar{z}) \partial \bar{\epsilon} \right)
\end{aligned} \tag{2.176}$$

[17]Therefore, this tells us that $T(z)$ is the generator of the conformal transformations, and the eq.(2.144) tells us how a infinitesimal conformal transformation $\epsilon(z)$, affect an arbitrary primary field, we use that on the fields b and c to obtain

$$\delta b = -\lambda (\partial \epsilon) b(z) - \epsilon \partial b(z), \qquad \text{and} \qquad \delta c = -(1-\lambda)(\partial\epsilon) c(z) - \epsilon \partial c(z). \tag{2.177}$$

Noether's methods gives the energy momentum by considering these variations (2.177) over the action (2.171), that is

$$\begin{aligned}
\delta S &= \frac{1}{2\pi} \int d^2 z \left\{ \delta b \bar{\partial} c + b \bar{\partial} \delta c \right\} \\
&= \frac{-1}{2\pi} \int d^2 z \left\{ (\lambda \partial \epsilon b + \epsilon \partial b) \bar{\partial} c + b \bar{\partial} \left((1-\lambda)(\partial\epsilon) c + \epsilon \partial c \right) \right\} \\
&= \frac{-1}{2\pi} \int d^2 z \left\{ \cancel{\lambda \partial \epsilon b \bar{\partial} c} + \epsilon \partial b \bar{\partial} c + (1-\lambda) b \bar{\partial} \partial \epsilon c \right. \\
&\qquad \left. + b \partial \epsilon \bar{\partial} c - \cancel{\lambda b \partial \epsilon \bar{\partial} c} + b \bar{\partial} \epsilon \partial c + \epsilon b \bar{\partial} \partial c \right\}.
\end{aligned}$$

We replace

$$\begin{aligned}
(1-\lambda) b \bar{\partial} \partial \epsilon c &= (1-\lambda) \partial \left(b \bar{\partial} \epsilon c \right) - (1-\lambda) \partial b \bar{\partial} \epsilon c - (1-\lambda) b \bar{\partial} \epsilon \partial c \\
b \partial \epsilon \bar{\partial} c &= \partial \left(b \epsilon \bar{\partial} c \right) - \partial b \epsilon \bar{\partial} c - b \epsilon \partial \bar{\partial} c.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\delta S &= \frac{-1}{2\pi} \int d^2 z \left\{ -(1-\lambda) \partial b c + \lambda b \partial c \right\} \bar{\partial} \epsilon \\
&= \frac{1}{2\pi} \int d^2 z \left\{ (1-\lambda) \partial b c - \lambda b \partial c \right\} \bar{\partial} \epsilon,
\end{aligned}$$

so that

$$T(z) = (1-\lambda) : (\partial b) c : -\lambda : b \partial c :=: (\partial b) c : -\lambda \partial (: bc :). \tag{2.178}$$

[17]we have defined for convenience $\epsilon(z) = 2\epsilon^z$ and $\bar{\epsilon}(\bar{z}) = 2\epsilon^{\bar{z}}$.

The OPE of T with b and c has the standard tensor form (2.163). In another hand, the TT OPE is of form (2.170). However, in this case the coefficient of the order fourth pole is a constant $c/2$, where

$$c = -3\left(2\lambda - 1\right)^2 + 1 \tag{2.179}$$

There is a corresponding antiholomorphic theory

$$S = \frac{1}{2\pi}\int d^2z\,\bar{b}\partial\bar{c}, \tag{2.180}$$

which is the same as above with $z \longleftrightarrow \bar{z}$. The bc theory has a ghost number symmetry under the transformations $b \longrightarrow e^{-i\epsilon}b$ and $c \longrightarrow e^{i\epsilon}c$, which imply the infinitesimal translations $\delta b = -i\epsilon b$ and $\delta c = i\epsilon c$. The corresponding Noether current is

$$\begin{aligned}
\delta S &= \frac{1}{2\pi}\int d^2z\,\left(\delta b\bar{\partial}c + b\bar{\partial}\delta c\right)\\
&= \frac{1}{2\pi}\int d^2z\,\left(-i\epsilon b\bar{\partial}c + b\bar{\partial}\left(i\epsilon c\right)\right)\\
&= \frac{1}{2\pi}\int d^2z\,\left(-\cancel{i\epsilon b\bar{\partial}c} + b\left(\bar{\partial}i\epsilon\right)c + \cancel{bi\epsilon\bar{\partial}c}\right)\\
&= \frac{-1}{2\pi}\int d^2z\,\left(-bc\right)\bar{\partial}i\epsilon.
\end{aligned}$$

That is

$$j\left(z\right) = -:b\left(z\right)c\left(z\right):. \tag{2.181}$$

The above is the holomorphic component, antiholomorphic part vanishes. When there are both holomorphic and antiholomorphic bc fields, the ghost number are separately conserved. We now calculate the OPE of $T\left(z\right)$ with $b\left(z\right)$ and $c\left(z\right)$, by doing so we won't use (2.162) as we have done so far, rather, we use the conformal Ward identity (2.142) and compare that with their variations (2.177). Thus we have

$$\begin{aligned}
-\oint\frac{dz}{2\pi i}\epsilon\left(z\right)T\left(z\right)b\left(w\right) &\overset{(2.142)}{=} \delta b\left(w\right)\\
&\overset{(2.177)}{=} -\lambda\left(\partial\epsilon\right)b\left(w\right) - \epsilon\partial b\left(w\right).
\end{aligned}$$

By the Residue Cauchy's theorem, we know that, in order to the equality above be satisfied, the $T\left(z\right)b\left(w\right)$ OPE must be

$$T\left(z\right)b\left(w\right) \sim \frac{\partial b\left(w\right)}{z-w} + \lambda\frac{b\left(w\right)}{\left(z-w\right)^2}. \tag{2.182}$$

In the same way for

$$T(z)\,c(w) \sim (1-\lambda)\frac{c(w)}{(z-w)^2}+\frac{\partial c(w)}{z-w}. \tag{2.183}$$

The current (2.181) is not a tensor, to show this we calculate the $T(z)\,j(w)$ OPE, in this case we use (2.162) because a priori we don't know the conformal weight of $j(z)$, thus we have

$$\begin{aligned}
:T(z)::j(w): &= (\lambda-1):(\partial b(z))\,c(z)::bc(w):+\lambda:b\partial c(z)::bc(w):, \\
T(z)\,j(w) &= (\lambda-1)\frac{\partial b(z)\,c(w)}{z-w}+(1-\lambda)\frac{c(z)\,b(w)}{(z-w)^2}+\frac{(1-\lambda)}{(z-w)^3} \\
&\quad -\lambda\frac{b(z)\,c(w)}{(z-w)^2}+\lambda\frac{\partial c(z)\,b(w)}{z-w}-\frac{(\lambda)}{(z-w)^3}+\text{reg.} \\
&\sim \frac{(1-2\lambda)}{(z-w)^3}-\frac{\partial b(w)\,c(w)}{z-w}+(\lambda-1)\frac{b(w)\,c(w)}{(z-w)^2}-\frac{b(w)\,\partial c(w)}{(z-w)} \\
&\quad -\lambda\frac{b(w)\,c(w)}{(z-w)^2} \\
&\sim \frac{(1-2\lambda)}{(z-w)^3}-\frac{\partial(b(w)\,c(w))}{z-w}-\frac{b(w)\,c(w)}{(z-w)^2} \\
&\sim \frac{1-2\lambda}{(z-w)^3}+\frac{j(w)}{(z-w)^2}+\frac{\partial j(w)}{z-w}.
\end{aligned} \tag{2.184}$$

According to the conformal Ward identity (2.142) this implies the transformation law

$$\begin{aligned}
\delta j &= -\oint\frac{dz}{2\pi i}\epsilon(z)\,T(z)\,j(w) \\
&= -\oint\frac{dz}{2\pi i}\epsilon(z)\left(\frac{1-2\lambda}{(z-w)^3}+\frac{j(w)}{(z-w)^2}+\frac{\partial j(w)}{z-w}\right) \\
&= -\frac{1-2\lambda}{2}\partial^2\epsilon(z)-\partial\epsilon(z)\,j(w)-\epsilon(z)\,\partial j(w),
\end{aligned} \tag{2.185}$$

where we see that the ghost number current transforms as a primary field just for $\lambda=\frac{1}{2}$, here $\epsilon(z)$ is an infinitesimal conformal transformation. We now use the above results to calculate explicitly the TT OPE and, in this way, we verify the equation

(2.179). We have then

$$
\begin{aligned}
:T(z)::T(w): \;=\; & \{(1-\lambda):(\partial b(z))\,c(z):-\lambda:b(z)\,\partial c(z):\} \\
& \times\{(1-\lambda):(\partial b(w))\,c(w):-\lambda:b(w)\,\partial c(w):\} \\
=\; & (1-\lambda)^2\left\{\frac{:(\partial b(z))\,c(w):}{(z-w)^2}-\frac{:c(z)\,\partial b(w):}{(z-w)^2}-\frac{1}{(z-w)^2}\cdot\frac{1}{(z-w)^2}\right\} \\
& -(1-\lambda)\,\lambda\left\{\frac{:\partial b(z)\,\partial c(w):}{z-w}-\frac{2:c(z)\,b(w):}{(z-w)^3}-\frac{1}{z-w}\cdot\frac{2}{(z-w)^3}\right\} \\
& -(1-\lambda)\,\lambda\left\{-\frac{2:b(z)\,c(w):}{(z-w)^3}+\frac{:\partial c(z)\,\partial b(w):}{z-w}-\frac{2}{(z-w)^3}\cdot\frac{1}{z-w}\right\} \\
& +\lambda^2\left\{-\frac{:b(z)\,\partial c(w):}{(z-w)^2}+\frac{:\partial c(z)\,b(w):}{(z-w)^2}-\frac{1}{(z-w)^2}\cdot\frac{1}{(z-w)^2}\right\}+\text{reg.}.
\end{aligned}
$$

We expand around w, so we get, after canceling some terms,

$$
\begin{aligned}
T(z)\,T(w) \;\sim\; & \frac{-6\lambda^2+6\lambda-1}{(z-w)^4}+\frac{2(1-\lambda):\partial b(w)\,c(w):}{(z-w)^2}-\frac{2\lambda:b(w)\,\partial c(w):}{(z-w)^2} \\
& +\frac{(1-\lambda):\partial^2 b(w)\,c(w):}{z-w}+\frac{(1-\lambda):\partial b(w)\,\partial c(w):}{z-w} \\
& -\lambda\frac{:\partial b(w)\,\partial c(w):}{z-w}-\lambda\frac{:b(w)\,\partial^2 c(w):}{z-w} \\
\sim\; & \frac{-12\lambda^2+12\lambda-2}{2(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w},
\end{aligned}
\tag{2.186}
$$

which allows us to identify the coefficient of the order four pole $c=-12\lambda^2+12\lambda-2$, thus coinciding with (2.179). The bc theory for $\lambda=2$ will arise as the Faddeev-Popov ghosts from gauge fixing the Polyakov string.

2.7 The Central charge

The specific models treated in the last section lead us to the following general OPE of the energy- momentum tensor with itself:

$$
T(z)\,T(w)\sim\frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T}{(z-w)},
\tag{2.187}
$$

where the constant c depends on the specific model under study, it is equal to d for the free boson and -26 for the ghost system (if $\lambda=2$). This model dependent constant is called the "**central charge**" or "the conformal anomaly". Except for this anomalous term, the OPE (2.187) simply means that T is a quasi-primary field with conformal dimension $h = 2$. The value of the central charge is determined by the short-distance behavior of the theory but not by symmetry. For free fields, as seen in the previous section, it is determined by applying the OPE on the normal ordered energy-momentum tensor with itself. The central charge is somehow an extensive measure of the number of degrees of freedom of the system. Therefore, if we consider a more general theory containing the matter plus the ghost systems, the central charge will be

$$c = d - 26. \tag{2.188}$$

If we demand the vanishing of the central charge, as we will do, we find the celebrated critical dimension $d = 26$, of the bosonic string.

2.7.1 Transformation of the Energy-Momentum Tensor

By comparing the OPE (2.187) with the general form (2.163), we see that the energy momentum tensor does not exactly transform like a primary field of dimension 2, contrary to that we expect classically. According to the conformal Ward identity (2.142) the variation of T under a local conformal transformation is

$$\begin{aligned}
\delta_\epsilon T(w) &= -\frac{1}{2\pi i}\oint_C dz \epsilon(z) T(z) T(w) \\
&= \frac{1}{2\pi i}\oint_C dz \epsilon(z) \left\{ \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \right\} \\
&= -\frac{c}{12}\partial^3 \epsilon(w) - 2T(w)\partial\epsilon(w) - \epsilon(w)\partial T(w).
\end{aligned} \tag{2.189}$$

The "exponentiation" of this infinitesimal variation to a finite transformation $z \longrightarrow w(z)$ is

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} \left[T(z) - \frac{c}{12}\{w; z\}\right], \tag{2.190}$$

where we have introduced the *Schwarzian derivative*:

$$\{w; z\} = \frac{\left(\frac{d^3 w}{dz^3}\right)}{\left(\frac{dw}{dz}\right)} - \frac{3}{2}\left(\frac{\frac{d^2 w}{dz^2}}{\frac{dw}{dz}}\right)^2. \tag{2.191}$$

We will just verify it for infinitesimal transformations. For a infinitesimal map $w(z) = z + \epsilon(z)$, the Schwarzian derivative becomes, at first order in ϵ,

$$\begin{aligned} \{w;z\} &= \{z+\epsilon;z\} = \frac{\partial^3\epsilon}{(1+\partial\epsilon)} - \frac{3}{2}\left(\frac{\partial^2\epsilon}{1+\partial\epsilon}\right)^2 \\ &= \partial^3\epsilon\,(1-\partial\epsilon) - \frac{3}{2}\mathcal{O}^2(\epsilon) + \ldots \\ &\approx \partial^3\epsilon. \end{aligned} \tag{2.192}$$

The infinitesimal version of (2.190) is therefore, at first order in ϵ,

$$\begin{aligned} T'(z+\epsilon) &= (1+\partial\epsilon)^{-2}\left[T(z) - \frac{c}{12}\partial^3\epsilon\right] \\ T'(z) + \epsilon\partial T'(z) &\approx (1-2\partial\epsilon)\left[T(z) - \frac{c}{12}\partial^3\epsilon\right] \\ &\approx T(z) - \frac{c}{12}\partial^3\epsilon - 2\partial\epsilon T(z) \end{aligned} \tag{2.193}$$

or

$$\delta_\epsilon T(w) = T'(z) - T(z) = -\frac{c}{12}\partial^3\epsilon - 2\partial\epsilon T(z) - \epsilon\partial T(z), \tag{2.194}$$

which, indeed, coincides with eq.(2.189). To confirm the validity of the transformation law (2.190), we must verify the following group property: The result of two successive transformations $z \longrightarrow w \longrightarrow u$ should coincide with what is obtained from the single transformation from $z \longrightarrow u$, that is

$$\begin{aligned} T''(u) &= \left(\frac{du}{dw}\right)^{-2}\left[T'(w) - \frac{c}{12}\{u;w\}\right] \\ &= \left(\frac{du}{dw}\right)^{-2}\left[\left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w;z\}\right] - \frac{c}{12}\{u;w\}\right] \\ &= \left(\frac{du}{dw}\right)^{-2}\left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\left\{\{w;z\} + \left(\frac{dw}{dz}\right)^2\{u;w\}\right\}\right] \\ &= \left(\frac{du}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{u;z\}\right]. \end{aligned} \tag{2.195}$$

The last equality requires the following relation between Schwarzian derivatives:

$$\{u;z\} = \{w;z\} + \left(\frac{dw}{dz}\right)^2\{u;w\}. \tag{2.196}$$

[18]If we set $u = z$, we find that (with $\{z;z\} = 0$)

$$\{w;z\} = -\left(\frac{dw}{dz}\right)^2 \{z;w\} \tag{2.197}$$

and this relation allows us to write the transformation law (2.190) as

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} T(z) + \frac{c}{12}\{z;w\}. \tag{2.198}$$

We can also verify that the Schwarzian derivative of the Global conformal map

$$w(z) = \frac{az+b}{cz+d}, \qquad ad - bc = 1, \tag{2.199}$$

[18]A demonstration of (2.196) is in order. We denote $\frac{d^n u}{dz^n} = u_z^{(n)}$

$$\{u;z\} = \frac{u_z^{(3)}}{u_z^{(1)}} - \frac{3}{2}\left(\frac{u_z^{(2)}}{u_z^{(1)}}\right)^2.$$

Then, we must use the following results

$$\begin{aligned}
u_z^{(1)} &= w_z^{(1)} u_w^{(1)}, \\
u_z^{(2)} &= w_z^{(2)} u_w^{(1)} + \left(w_z^{(1)}\right)^2 u_w^{(2)}, \\
u_z^{(3)} &= w_z^{(3)} u_w^{(1)} + 3 w_z^{(1)} w_z^{(2)} u_w^{(2)} + \left(w_z^{(1)}\right)^3 u_w^{(3)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\{u;z\} &= \frac{w_z^{(3)} u_w^{(1)} + 3 w_z^{(1)} w_z^{(2)} u_w^{(2)} + \left(w_z^{(1)}\right)^3 u_w^{(3)}}{w_z^{(1)} u_w^{(1)}} - \frac{3}{2}\left(\frac{w_z^{(2)} u_w^{(1)} + \left(w_z^{(1)}\right)^2 u_w^{(2)}}{w_z^{(1)} u_w^{(1)}}\right)^2 \\
&= \frac{w_z^{(3)}}{w_z^{(1)}} - \frac{3}{2}\left(\frac{w_z^{(2)}}{w_z^{(1)}}\right)^2 + \left(w_z^{(1)}\right)^2 \left(\frac{u_w^{(3)}}{u_w^{(1)}} - \frac{3}{2}\left(\frac{u_w^{(2)}}{u_w^{(1)}}\right)^2\right) \\
&= \{w;z\} + \left(\frac{dw}{dz}\right)^2 \{u;w\}.
\end{aligned}$$

vanishes[19]. This need to be so, because $T(z)$ is a quasi-primary field. The Schwarzian derivative in (2.190) is the only possible addition to the tensor transformations.

2.7.2 The Weyl anomaly

The trace of the energy-momentum tensor vanishes at the classical level for a conformal invariant theory. We will see that for this symmetry to hold at the quantum level, the total central charge of the theory must vanishes. We can always put any two dimensional metric in the form $g_{\alpha\beta} = e^{2\omega(\sigma^1,\sigma^2)}\delta_{\alpha\beta}$. So the only non-vanishing Christoffel symbols are

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = \partial_1\omega, \quad \text{and} \quad \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\Gamma^2_{11} = \partial_2\omega. \tag{2.200}$$

In another hand, we have that for an orthogonal metric,

$$R_{\alpha\beta} = \partial_\gamma \Gamma^\gamma{}_{\alpha\beta} - \partial_\beta \Gamma^\gamma{}_{\gamma\alpha}, \qquad R = g^{\alpha\beta}R_{\alpha\beta}, \tag{2.201}$$

which are the Ricci curvature tensor and the scalar curvature respectively. So that

$$R_{11} = R_{22} = -\nabla^2\omega \qquad \text{and} \qquad R = g^{11}R_{11} + g^{22}R_{22} = -2e^{-2\omega}\nabla^2\omega. \tag{2.202}$$

Let us go to see how this curvature is related to the central charge and in general to the energy-momentum tensor trace $\langle T^\alpha{}_\alpha \rangle$. We know that in complex coordinates the trace of $T_{\alpha\beta}$ is given by $g^{\alpha\beta}T_{\alpha\beta} = 2g^{z\bar{z}}T_{z\bar{z}}$, so that, in order to study its behavior at quantum level, we need to find a expression for the $T_{z\bar{z}}T_{w\bar{w}}$ OPE. This leads us to express the conservation equation $\partial^\alpha T_{\alpha\beta} = 0$ in complex coordinates, that is

$$\partial^\alpha T_{\alpha\beta} = g^{\alpha\gamma}\partial_\gamma T_{\alpha\beta} = 0 \Longrightarrow g^{\bar{z}z}\partial_z T_{\bar{z}\beta} + g^{z\bar{z}}\partial_{\bar{z}} T_{z\beta} = 2\partial_z T_{\bar{z}\beta} + 2\partial_{\bar{z}} T_{z\beta} = 0,$$

and

$$\partial_z T_{\bar{z}\bar{z}} = -\partial_{\bar{z}} T_{z\bar{z}} \qquad \text{and} \qquad \partial_{\bar{z}} T_{zz} = -\partial_z T_{\bar{z}z}. \tag{2.203}$$

[19]That is as follows, we have that

$$\frac{dw}{dz} = \frac{1}{(cz+d)^2}, \qquad \frac{d^2w}{dz^2} = \frac{-2c}{(cz+d)^3}, \qquad \frac{d^3w}{dz^3} = \frac{6c^2}{(cz+d)^4}.$$

From (2.191)

$$\{w;z\} = \frac{6c^2(cz+d)^2}{(cz+d)^4} - \frac{3}{2}\left(\frac{-2c(cz+d)^2}{(cz+d)^3}\right)^2 = \frac{6c^2}{(cz+d)^2} - \frac{3}{2}\frac{4c^2}{(cz+d)^2} = 0.$$

Then we use the $T_{zz}T_{ww}$ OPE, that is

$$\begin{aligned}
\partial_z T_{\bar{z}z}\left(\bar{z},z\right)\partial_w T_{\bar{w}w}\left(\bar{w},w\right) &= \partial_{\bar{z}} T_{zz}\left(\bar{z},z\right)\partial_{\bar{w}} T_{ww}\left(\bar{w},w\right), \\
&\Downarrow \\
\partial_{\bar{z}}\partial_{\bar{w}}\left(T_{zz}\left(\bar{z},z\right)T_{ww}\left(\bar{w},w\right)\right) &= \partial_{\bar{z}}\partial_{\bar{w}}\left[\frac{c/2}{\left(z-w\right)^4}+\frac{2T\left(w\right)}{\left(z-w\right)^2}+\frac{\partial T\left(w\right)}{z-w}+...\right]
\end{aligned} \tag{2.204}$$

Classically the right hand side above must be zero, but now we must consider the effect of the poles. We can see that the only non-vanishing term is the first one, the others vanish because the OPE has meaning, only inside a correlator and the expectation value of $T\left(z\right)$ is zero. Therefore we have

$$\begin{aligned}
\partial_{\bar{z}}\partial_{\bar{w}}\frac{1}{\left(z-w\right)^4} &= \frac{1}{6}\partial_{\bar{z}}\partial_{\bar{w}}\left(\partial_z^2\partial_w\frac{1}{z-w}\right) \\
&= \frac{1}{6}\partial_z^2\partial_w\partial_{\bar{w}}\partial_{\bar{z}}\left(\frac{1}{z-w}\right) \\
&= \frac{2\pi}{6}\partial_z^2\partial_w\partial_{\bar{w}}\delta\left(z-w,\bar{z}-\bar{w}\right).
\end{aligned} \tag{2.205}$$

Inserting that result in (2.204), we have

$$\begin{aligned}
\partial_z\partial_w\left(T_{\bar{z}z}\left(\bar{z},z\right)T_{\bar{w}w}\left(\bar{w},w\right)\right) &= \frac{\pi}{6}c\partial_z^2\partial_w\partial_{\bar{w}}\delta\left(z-w,\bar{z}-\bar{w}\right) \\
&= \partial_z\partial_w\left(\frac{c\pi}{6}\partial_z\partial_{\bar{w}}\delta\left(z-w,\bar{z}-\bar{w}\right)\right), \\
&\Downarrow \\
T_{\bar{z}z}\left(\bar{z},z\right)T_{\bar{w}w}\left(\bar{w},w\right) &= \frac{c\pi}{6}\partial_z\partial_{\bar{w}}\delta\left(z-w,\bar{z}-\bar{w}\right).
\end{aligned} \tag{2.206}$$

Then, there is a singular behavior when $z \longrightarrow w$. We assume that in flat space $\langle T^\alpha_{\ \alpha}\rangle = 0$. From the definition of the energy-momentum tensor (2.63) we consider the following correlation function under an infinitesimal metric change

$$\begin{aligned}
\delta\left\langle T^\alpha_{\ \alpha}\left(\sigma\right)\right\rangle &= \delta\int\mathcal{D}\phi e^{-S}T^\alpha_{\ \alpha}\left(\sigma\right) \\
&= \int\mathcal{D}\phi e^{-S}T^\alpha_{\ \alpha}\left(\sigma\right)\left(-\delta_g S\right) \\
&= \frac{1}{4\pi}\int\mathcal{D}\phi e^{-S}\left(T^\alpha_{\ \alpha}\left(\sigma\right)\int d^2\sigma'\sqrt{g}\delta g^{\beta\gamma}T_{\beta\gamma}\left(\sigma'\right)\right).
\end{aligned} \tag{2.207}$$

We consider now an infinitesimal Weyl rescaling $g_{\alpha\beta} = e^{2\omega}\delta_{\alpha\beta}$, so that $\delta g_{\alpha\beta} = 2\omega\delta_{\alpha\beta}$ and $\delta g^{\alpha\beta} = -2\omega\delta^{\alpha\beta}$ (we consider also that $g = e^{2\omega} \sim 1$). Thus, we may write (2.207) as follows

$$\begin{aligned}\delta\left\langle T^{\alpha}_{\ \alpha}(\sigma)\right\rangle &= \frac{1}{4\pi}\int \mathcal{D}\phi e^{-S}\left(T^{\alpha}_{\ \alpha}(\sigma)\int d^2\sigma'\left(-2\omega\delta^{\beta\gamma}\right)T_{\beta\gamma}(\sigma')\right)\\ &= -\frac{1}{2\pi}\int \mathcal{D}\phi e^{-S}\left(T^{\alpha}_{\ \alpha}(\sigma)\int d^2\sigma'\omega(\sigma')\,T^{\beta}_{\ \beta}(\sigma')\right).\end{aligned} \tag{2.208}$$

We now transform the OPE (2.206) to Euclidean coordinates through the usual transformation rules eq.(2.104). Thus we have

$$\begin{aligned}T_{\alpha\beta}(z,\bar{z}) &= \frac{\partial\sigma^{\mu}}{\partial z^{\alpha}}\frac{\partial\sigma^{\nu}}{\partial z^{\beta}}T_{\mu\nu}\left(\sigma^1,\sigma^2\right),\\ T_{\bar{z}z}(z,\bar{z}) &= \frac{1}{4}T_{11}+\frac{1}{4}T_{22}=\frac{1}{4}T^{\alpha}_{\ \alpha}(\sigma),\end{aligned}$$

which means that

$$T_{\bar{z}z}(\bar{z},z)\,T_{\bar{w}w}(\bar{w},w)=\frac{1}{16}T^{\alpha}_{\ \alpha}(\sigma)\,T^{\beta}_{\ \beta}(\sigma'). \tag{2.209}$$

In another hand we get

$$\begin{aligned}\partial_z\partial_{\bar{w}}\delta(z-w,\bar{z}-\bar{w}) &= \frac{1}{2}(\partial_1-i\partial_2)\frac{1}{2}\left(\partial_1'+i\partial_2'\right)\frac{1}{2}\delta\left(\sigma^1-\sigma'^1\right)\delta\left(\sigma^2-\sigma'^2\right)\\ &= \left(\partial_1\partial_1'+i\partial_1\partial_2'-i\partial_2\partial_1'+\partial_2\partial_2'\right)\frac{1}{8}\delta^{(2)}(\sigma-\sigma').\end{aligned}$$

Using the delta property $\partial_{\sigma'}\delta(\sigma-\sigma')=-\partial_{\sigma}\delta(\sigma-\sigma')$, we have that

$$\partial_z\partial_{\bar{w}}\delta(z-w,\bar{z}-\bar{w})=-\frac{1}{8}(\partial_1\partial_1+\partial_2\partial_2)\,\delta^{(2)}(\sigma-\sigma')=-\frac{1}{8}\nabla^2\delta^{(2)}(\sigma-\sigma'). \tag{2.210}$$

Therefore, we can write the OPE (2.206) in Euclidean coordinates as follows

$$T^{\alpha}_{\ \alpha}(\sigma)\,T^{\beta}_{\ \beta}(\sigma') = -\frac{c\pi}{3}\nabla^2\delta^{(2)}(\sigma-\sigma'). \tag{2.211}$$

Substituting in (2.208) we have

$$\begin{aligned}\delta\left\langle T^{\alpha}_{\ \alpha}(\sigma)\right\rangle &= -\frac{1}{2\pi}\int \mathcal{D}\phi e^{-S}\left(-\frac{c\pi}{3}\int d^2\sigma'\omega(\sigma')\,\nabla^2\delta^{(2)}(\sigma-\sigma')\right)\\ &= \frac{c}{6}\int \mathcal{D}\phi e^{-S}\nabla^2\omega(\sigma)=\frac{c}{6}\nabla^2\omega(\sigma).\end{aligned} \tag{2.212}$$

Now in eq.(2.202) we take $e^{-2\omega}\sim 1$, because we are considering an infinitesimal ω, so that $R=-2\nabla^2\omega$, which allows to identify

$$\delta\left\langle T^{\alpha}_{\ \alpha}(\sigma)\right\rangle=-\frac{c}{12}R. \tag{2.213}$$

Therefore, the central charge is associated to the Weyl anomaly, for this reason we demand its vanishing in flat space in order to this symmetry to survive quantization.

2.8 The Operator Formalism of Conformal Field Theory

2.8.1 Radial Quantization

So far, we have used complex coordinates according to (2.104) with the form $-iw = \sigma^2 - i\sigma^1$, or equivalent $i\bar{w} = \sigma^2 + i\sigma^1$.

In the closed string case we impose a periodicity condition on the spatial coordinate σ^1, i.e.

$$\sigma^1 \sim \sigma^1 + 2\pi \qquad \text{while} \qquad -\infty < \sigma^2 < \infty, \tag{2.214}$$

where σ^2 is the time parameter, such that, the complex coordinates define a infinite cylinder.

The coordinate transformation

$$z = e^{-iw} = e^{-i\sigma^1+\sigma^2}, \qquad \bar{z} = e^{i\bar{w}} = e^{i\sigma^1+\sigma^2} \tag{2.215}$$

maps the cylinder to the complex plane. This choice of space and time leads to the so-called radial quantization of two dimensional CFTs. In terms of the w coordinates, times corresponds to translations of $\sigma^2 = \mathrm{Im}\,(w)$. In terms of z times runs radially, the infinite past $\left(\sigma^2 \longrightarrow -\infty\right)$ is situated at the origin $z = 0$, whereas the remote future $\left(\sigma^2 \longrightarrow \infty\right)$ lies on the point at infinity on the Riemann sphere. Within radial quantization, states of the form

$$|\phi_{in}\rangle = \lim_{z,\bar{z}\longrightarrow 0} \phi\,(z,\bar{z})\,|0\rangle \tag{2.216}$$

stand for asymptotic "in" states at the limit when an interaction is attenuated, where $|0\rangle$ is the vacuum state of the Hilbert space. The corresponding bra, called the asymptotic "out" state, is defined is concordance with the definition of Hermitian conjugation

$$[\phi\,(z,\bar{z})]^\dagger \equiv \bar{z}^{-2h} z^{-2\bar{h}} \phi\,(1/\bar{z}, 1/z)\,, \tag{2.217}$$

where by assumption ϕ is a quasi-primary field of dimensions h and $\bar{h}$. Thus we have

$$\begin{aligned} \langle\phi_{out}| &= \lim_{z,\bar{z}\longrightarrow 0} \langle 0|\,\phi\,(z,\bar{z})^\dagger \\ &= \lim_{z,\bar{z}\longrightarrow 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0|\,\phi\,(1/\bar{z}, 1/z)\,. \end{aligned} \tag{2.218}$$

This definition is in concordance with eq.(2.122) and therefore

$$\langle \phi_{out} | = | \phi_{in} \rangle^{\dagger} . \tag{2.219}$$

2.8.2 Mode Expansions

A conformal field $\phi(z,\bar{z})$ of dimensions $(h,\bar{h})$ may be mode expanded as follows:

$$\phi(z,\bar{z}) = \sum_{m,n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n}, \qquad \phi_{m,n} = \oint \frac{dz}{2\pi i} z^{m+h-1} \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+\bar{h}-1}\phi(z,\bar{z}) . \tag{2.220}$$

[20]A hermitian conjugation on the real surface ($\bar{z} = z^*$) leads to

$$\phi(z,\bar{z})^{\dagger} = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \bar{z}^{-m-h} z^{-n-\bar{h}} \phi^{\dagger}_{m,n}. \tag{2.221}$$

Thus, in order to (2.221) be compatible with (2.217), it must be satisfied that

$$\phi^{\dagger}_{m,n} = \phi_{-m,-n}. \tag{2.222}$$

If the "in" and "out" states are well-defined, the vacuum must satisfy the conditions

$$\phi_{m,n}|0\rangle = 0, \qquad m \geq 1-h, \quad n \geq 1-\bar{h}. \tag{2.223}$$

[21]There are a similar regularity condition for the "out" state. We will simplify the notation by dropping the dependence of fields upon the antiholomorphic coordinate. Thus, (2.220) will be written in the following simplified form:

$$\phi(z,\bar{z}) = \sum_{m\in\mathbb{Z}} z^{-m-h}\phi_m \qquad , \qquad \phi_m = \oint \frac{dz}{2\pi i} z^{m+h-1}\phi(z,\bar{z}) . \tag{2.224}$$

However, it must be kept in mind that the antiholomorphic dependence is always there. Alternatively, the "in" state can be defined as

$$\lim_{z,\bar{z}\longrightarrow 0} \phi(z,\bar{z})|0\rangle = \phi_{-h,-\bar{h}}|0\rangle = \oint \frac{dz}{2\pi i} z^{-1} \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{-1}\phi(z,\bar{z})|0\rangle = \phi(0,0)|0\rangle . \tag{2.225}$$

and similarly for the out state.

[20]This is simply the Fourier expansions of the field.

[21]In the equation (2.223) the conditions $(m > -h, \quad n > -\bar{h})$ is due to the fact that the "in" and "out states" are defined at the limit $\bar{z}, z \longrightarrow 0$, so that, according to the definition (2.216) and the expression for the modes $\phi_{m,n}$ in 2.220 we must have

$$|\phi_{in}\rangle = \lim_{z,\bar{z}\longrightarrow 0} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n}|0\rangle .$$

The above limit diverges for $(m > -h, \quad n > -\bar{h})$ unless the condition (2.223) be satisfied.

2.8.3 Radial Ordering and Operator Product Expansion

In radial quantization, the time ordering appearing in the definition of correlation functions becomes a "radial ordering", explicitly defined by

$$\mathcal{R}\left(\phi_1(z)\,\phi_2(w)\right)=\begin{cases}\phi_1(z)\,\phi_2(w) & \text{if} \quad |z|>|w|\,,\\ \phi_2(w)\,\phi_1(z) & \text{if} \quad |w|>|z|\,.\end{cases} \tag{2.226}$$

If the two fields are fermions, a minus sign is added in front of the second expression. Thus, the OPE's written previously have an operator meaning only if $|z|>|w|$. One of the advantages of radial quantization is that the commutator of two fields can be expressed in terms of their OPE . In order to show that, let $\mathfrak{a}(z)$ and $\mathfrak{b}(z)$ be two holomorphic fields, and consider the following integral

$$\oint_w dz\,\mathfrak{a}(z)\,\mathfrak{b}(w)\,. \tag{2.227}$$

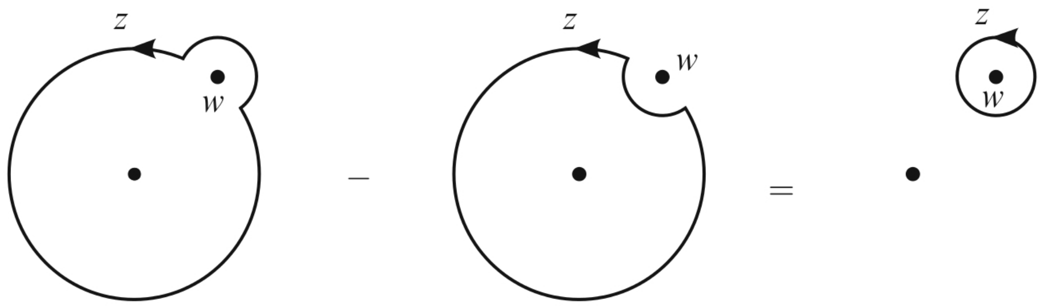

Figure 2.1: *Integration contour deformation (image taken from [7]).*

In order to this integral to have an operator meaning within correlation functions, it must satisfy (2.226). Therefore, we split the integration contour into two fixed-time circles (see Fig. 2.1) . Then

$$\oint_{C_w} dz\,\mathfrak{a}(z)\,\mathfrak{b}(w)=\oint_{C_1} dz\,\mathfrak{a}(z)\,\mathfrak{b}(w)-\oint_{C_2} dz\,\mathfrak{b}(w)\,\mathfrak{a}(z)=\left[\mathcal{A},\mathfrak{b}(w)\right], \tag{2.228}$$

where

$$\mathcal{A}=\oint dz\,\mathfrak{a}(z) \tag{2.229}$$

and C_1 and C_2 are circles centered around the origin of radii respectively equal to $|w|+\epsilon$ and $|w|-\epsilon$, ϵ being infinitesimal. Our integral is now seen to be a commutator. If $\mathfrak{a}$ and $\mathfrak{b}$ are fermions, the commutator is replaced by an anticommutator. The integral (2.227) is evaluated by substituting the OPE of $\mathfrak{a}(z)$ with $\mathfrak{b}(w)$. Thus, the commutator

$[\mathcal{A},\mathcal{B}]$ of two operators,

$$\mathcal{A}=\oint dz\,\mathfrak{a}\left(z\right), \qquad \mathcal{B}=\oint dz\,\mathfrak{b}\left(z\right), \tag{2.230}$$

is obtained by integrating eq.(2.228) over w, that is

$$[\mathcal{A},\mathcal{B}]=\oint_{\mathcal{C}_0} dw \oint_{\mathcal{C}_w} dz\,\mathfrak{a}\left(z\right)\mathfrak{b}\left(w\right). \tag{2.231}$$

Henceforth, contour integrals without a specified contour will be understood as integrate along circles centered at the origin. Otherwise, the contour will be indicated.

2.9 The Virasoro Algebra and Hilbert space

2.9.1 Conformal Generators

By applying (2.228) and (2.231) to the conformal identity (2.142), we find

$$-\,\delta_\epsilon\phi\left(w\right)=\frac{1}{2\pi i}\oint dz\,\epsilon\left(z\right)T\left(z\right)\phi\left(w\right), \tag{2.232}$$

where $\epsilon\left(z\right)$ is the holomorphic component of an infinitesimal conformal transformation. We then define the conformal charge by

$$Q_\epsilon=\frac{1}{2\pi i}\oint dz\,\epsilon\left(z\right)T\left(z\right). \tag{2.233}$$

Then, by (2.228), the conformal Ward identity translates into

$$\delta_\epsilon\phi\left(w\right)=-\left[Q_\epsilon,\phi\left(w\right)\right], \tag{2.234}$$

which means that the operator Q_ϵ is the generator of conformal transformations. We now expand the energy momentum tensor according to (2.220) and obtain

$$\begin{aligned} T\left(z\right)=&\ \textstyle\sum_{n\in\mathbb{Z}} z^{-n-2}L_n, & L_n=&\ \oint \tfrac{dz}{2\pi i}z^{n+1}T\left(z\right),\\ \bar{T}\left(\bar{z}\right)=&\ \textstyle\sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n, & \bar{L}_n=&\ \oint \tfrac{d\bar{z}}{2\pi i}\bar{z}^{n+1}\bar{T}\left(\bar{z}\right). \end{aligned} \tag{2.235}$$

We also expand the infinitesimal conformal change $\epsilon\left(z\right)$ as follows

$$\epsilon\left(z\right)=\sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n. \tag{2.236}$$

The expression (2.233) for the conformal charge becomes

$$\begin{aligned} Q_\epsilon &= \frac{1}{2\pi i}\oint dz \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n T(z) \\ &= \sum_{n\in\mathbb{Z}} \epsilon_n \underbrace{\frac{1}{2\pi i}\oint dz\, z^{n+1} T(z)}_{L_n} \\ &= \sum_{n\in\mathbb{Z}} \epsilon_n L_n. \end{aligned} \tag{2.237}$$

The mode operators L_n and $\bar{L}_n$ of the energy-momentum tensor are the generators of the local conformal transformations on the Hilbert space, exactly like l_n and $\bar{l}_n$ of (2.116) are the generators of conformal mappings on the space of functions. The generators of $SL(2,\mathbb{C})$ in the Hilbert space are $L_{\pm1}$, L_0 (and their antiholomorphic counterparts). The operator $L_0+\bar{L}_0$ generates the dilations $(z,\bar{z}) \longrightarrow \lambda(z,\bar{z})$, that is, time translations in radial quantization. Therefore, $L_0+\bar{L}_0$ must be proportional to the Hamiltonian of the system. The operators L_n obey the algebra (2.117), except for a new term depending on the central charge,

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n\left(n^2-1\right)\delta_{m+n}, \\ [L_n, \bar{L}_m] &= 0, \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n\left(n^2-1\right)\delta_{m+n}. \end{aligned} \tag{2.238}$$

These relations may be derived from the mode expansion (2.235), the OPE (2.187) and (2.231)[22] as follows,

$$\begin{aligned} [L_n, L_m] &= \frac{1}{(2\pi i)^2}\oint dw\, w^{m+1}\oint_w dz\, z^{n+1} T(z)T(w) \\ &= \frac{1}{(2\pi i)^2}\oint dw\, w^{m+1}\oint_w dz\, z^{n+1}\left\{\frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{(z-w)}+\text{reg.}\right\} \\ &= \oint \frac{dw}{2\pi i} w^{m+1}\left\{\frac{c}{12}n(n+1)(n-1)w^{n-2}+2(n+1)T(w)w^n+w^{n+1}\partial T(w)\right\} \\ &= \oint \frac{dw}{2\pi i}\left\{\frac{c}{12}n\left(n^2-1\right)w^{m+n-1}+2(n+1)T(w)w^{m+n+1}+w^{m+n+2}\partial T(w)\right\}. \end{aligned}$$

[22] Here is useful to recall the residue theorem for a regular function $f(z)$, we have

$$\oint_{z_0}\frac{dz}{2\pi i}\frac{f(z)}{(z-z_0)^{m+1}} = \frac{1}{m!}\frac{d^m}{dz^m}(f(z))(z=z_0).$$

By using the result $\frac{1}{2\pi i}\oint dz\, z^k = \delta_{k,-1}$, we have

$$\begin{aligned}
[L_n, L_m] &= \frac{c}{12}n\left(n^2-1\right)\delta_{m+n} + 2\,(n+1)\,\frac{1}{(2\pi i)}\oint dw\, T\,(w)\, w^{m+n+1} \\
&\qquad\qquad + \frac{1}{(2\pi i)}\oint dw\, w^{m+n+2}\partial T\,(w) \\
&= \frac{c}{12}n\left(n^2-1\right)\delta_{m+n} + 2\,(n+1)\,L_{n+m} - (m+n+2)\,\frac{1}{(2\pi i)}\oint dw\, w^{m+n+1}T\,(w) \\
&= \frac{c}{12}n\left(n^2-1\right)\delta_{m+n} + 2\,(n+1)\,L_{n+m} - (m+n+2)\,L_{n+m} \\
&= \frac{c}{12}n\left(n^2-1\right)\delta_{m+n} + (n-m)\,L_{n+m},
\end{aligned} \tag{2.239}$$

where, in the third term of the second line, we have integrated by parts. This is the **Virasoro algebra**. For the $[\bar{L}_n, \bar{L}_m]$ the process is identical and, $[L_n, \bar{L}_m] = 0$ follows from the OPE $T\,(z)\,\bar{T}\,(\bar{w}) \sim 0$.

2.9.2 The Hilbert Space

The vacuum state $|0\rangle$ must be invariant under $SL\,(2,\mathbb{C})$ transformations. This means that $L_{\pm 1}\,|0\rangle = L_0\,|0\rangle = 0$, with the same condition applying for the antiholomorphic counterparts, fixing the ground state energy to zero. This, in turn, can be recovered from the condition that $T\,(z)\,|0\rangle$ and $\bar{T}\,(\bar{z})\,|0\rangle$ are well-defined as $z, \bar{z} \longrightarrow 0$, which implies, for the holomorphic component that

$$\lim_{z\longrightarrow 0} T\,(z)\,|0\rangle = \lim_{z\longrightarrow 0}\sum_{n\in\mathbb{Z}} z^{-n-2}L_n\,|0\rangle = 0. \tag{2.240}$$

In the sum above, the terms diverging in the limit $z \longrightarrow 0$ are all of those for which $n \geq -1$. Then, this is precisely the requirement

$$L_n\,|0\rangle = 0, \qquad \bar{L}_n\,|0\rangle = 0, \qquad\qquad n \geq -1, \tag{2.241}$$

which includes as a sub-condition the invariance of the vacuum $|0\rangle$ with respect to the global conformal group. It also implies the vanishing of the vacuum expectation value of the energy momentum tensor:

$$\langle 0\,|\bar{T}\,(\bar{z})|\,0\rangle = \langle 0\,|T\,(z)|\,0\rangle = 0. \tag{2.242}$$

Primary fields, when acting on the vacuum, create asymptotic states, which are eigenstates of the Hamiltonian, as can be shown from (2.163,2.228)

$$\begin{aligned}
[L_n, \phi(w,\bar{w})] &= \frac{1}{2\pi i}\oint_w dz\, z^{n+1} T(z)\,\phi(w,\bar{w}) \\
&= \frac{1}{2\pi i}\oint_w dz\, z^{n+1}\left\{\frac{h}{(z-w)^2} + \frac{\partial_w}{(z-w)} + \text{reg.}\right\}\phi(w,\bar{w}) \\
&= h(n+1)\, w^n \phi(w,\bar{w}) + w^{n+1}\partial_w \phi(w,\bar{w}), \qquad n \geq -1. \quad (2.243)
\end{aligned}$$

By analogy

$$[\bar{L}_n, \phi(w,\bar{w})] = \bar{h}(n+1)\,\bar{w}^n \phi(w,\bar{w}) + \bar{w}^{n+1}\partial_{\bar{w}}\phi(w,\bar{w}), \qquad n \geq -1. \quad (2.244)$$

Applying these relations to the asymptotic state $|h,\bar{h}\rangle = \phi(0,0)|0\rangle$, we have

$$L_0\phi(w,\bar{w})|0\rangle \overset{(2.241)}{=} [L_0, \phi(w,\bar{w})]|0\rangle = (h\phi(w,\bar{w}) + w\partial_w\phi(w,\bar{w}))|0\rangle, \quad (2.245)$$

which implies

$$L_0\phi(0,0)|0\rangle = L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle, \qquad \bar{L}_0\phi(0,0)|0\rangle = \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle. \quad (2.246)$$

Thus $|h,\bar{h}\rangle$ is an eigenstate of the Hamiltonian $(L_0 + \bar{L}_0 \longrightarrow H)$. Likewise, we have from eqs. (2.241,2.243)

$$L_n|h,\bar{h}\rangle = 0, \qquad \bar{L}_n|h,\bar{h}\rangle = 0 \qquad \text{if} \qquad n > 0. \quad (2.247)$$

Excited states above the asymptotic state $|h,\bar{h}\rangle$ may be obtained by applying creation operators. Explicitly, if we expand the holomorphic field $\phi(w)$ in modes, according to (2.224), then we find according to the prescription (2.231) that

$$\begin{aligned}
[L_n, \phi_m] &= \frac{1}{(2\pi i)^2}\oint dw \oint_w dz\, z^{n+1} T(z)\, w^{m+h-1}\phi(w) \\
&= \frac{1}{(2\pi i)^2}\oint dw \oint_w dz\, z^{n+1}\left\{\frac{h\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\phi(w,\bar{w})}{(z-w)} + \text{reg.}\right\} w^{m+h-1} \\
&= \frac{1}{2\pi i}\oint dw \left\{(hn+h)\, w^{n+m+h-1}\phi(w,\bar{w}) + w^{n+m+h}\partial_w\phi(w,\bar{w})\right\}.
\end{aligned}$$

Integrating by parts yields to

$$
\begin{aligned}
[L_n, \phi_m] &= \frac{1}{2\pi i} \oint dw \left\{ (hn+h)\, w^{n+m+h-1} \phi(w,\bar{w}) - (n+m+h)\, w^{n+m+h-1} \phi(w,\bar{w}) \right\} \\
&= \frac{1}{2\pi i} \oint dw \left[n(h-1) - m \right] w^{n+m+h-1} \phi(w,\bar{w}) \\
&= \left[n(h-1) - m \right] \phi_{n+m}. \qquad (2.248)
\end{aligned}
$$

Of which a special case is

$$[L_0, \phi_m] = -m\phi_m. \qquad (2.249)$$

This means that the operators ϕ_m act as raising and lowering operator for the eigenstates of L_0 : Each application of $\phi_m^\dagger = \phi_{-m}, \quad (m > 0)$ increase the conformal dimension of the state by m. The generators $L_{-m}, \quad (m > 0)$ also increase the conformal dimension, since the Virasoro algebra (2.238) gives

$$[L_0, L_{-m}] = mL_{-m}. \qquad (2.250)$$

This means that excited states may be obtained by successive applications of these operators on the asymptotic state $|h\rangle$:

$$L_{-k_1} L_{-k_2} ... L_{-k_n} |h\rangle, \qquad 1 \le k_1 \le k_2 \le ... \le k_n. \qquad (2.251)$$

By convention, the L_{k_i} appear in increasing order of the k_i, a different ordering can always be brought by applying the commutation rules (2.238). The state (2.251) is an eigenstate of L_0 with eigenvalue.

$$h' = h + k_1 + k_2 + ... + k_n \equiv h + N. \qquad (2.252)$$

The states (2.251) are called descendants of level N of the asymptotic state $|h\rangle$. The number of distinct, linearly independent states at level N is simply the number $p(N)$ of partitions of the integer N. The generating function of the partition numbers is

$$\frac{1}{\varphi(q)} \equiv \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)\, q^n, \qquad (2.253)$$

where $\varphi(q)$ is the Euler function. The effect of a conformal transformation on a state is obtained by acting on it with a suitable function of generators L_m. The subset of the full Hilbert space generated by the asymptotic state $|h\rangle$ and its descendants is closed under the action of the Virasoro generators and form the so-called "**Verma module**".

2.10 The XX matter system revisited

Here we apply the results above to study the mode expansion of the $XX-$matter system. If we calculate the TX OPE, we obtain by using (2.153)

$$\begin{aligned} T(z)\,X^{\mu}(w) &= -\frac{1}{\alpha'} : \partial X_{\nu}(z)\,\partial X^{\nu}(z) : X^{\mu}(w) \\ &= -\frac{2}{\alpha'}\partial X_{\nu}(z)\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-w}\right) + \text{reg.} \\ &\sim \frac{\partial X^{\mu}(w)}{z-w}. \end{aligned} \tag{2.254}$$

According to (2.163) this means that X^{μ} is not exactly a primary field, however, from the OPE (2.169) we see that ∂X^{μ} and $\bar{\partial} X^{\mu}$ are, in fact, primary fields of conformal dimensions $(h, \bar{h}) = (1,1)$. From (2.153) we see that the holomorphic and antiholomorphic part of X^{μ} decouple, therefore, according to (2.220) we may expand in Laurent series as follows

$$\partial X^{\mu}(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty} z^{-m-1}\alpha_m^{\mu} \quad , \quad \bar{\partial} X^{\mu}(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty} \bar{z}^{-m-1}\bar{\alpha}_m^{\mu}. \tag{2.255}$$

Equivalently,

$$\alpha_m^{\mu} = \sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi} z^{-m}\partial X^{\mu}(z), \qquad \bar{\alpha}_m^{\mu} = -\sqrt{\frac{2}{\alpha'}}\oint \frac{d\bar{z}}{2\pi}\bar{z}^{-m}\bar{\partial} X^{\mu}(\bar{z}). \tag{2.256}$$

Single-valuedness of X^{μ} and its periodicity condition for the closed string, imply that $\bar{\alpha}_0^{\mu} = \alpha_0^{\mu}$. The Noether current for space-time translations, $\delta X^{\mu} = a^{\mu}$, is $\frac{i}{\alpha'}\partial_a X^{\mu}$ so the space-time momentum is

$$\begin{aligned} p^{\mu} &= \frac{1}{2\pi i}\oint_C \left(dz j^{\mu} - d\bar{z}\bar{j}^{\mu}\right) \\ &= \frac{1}{\alpha'}\left(\oint_C \frac{dz}{2\pi}\partial X^{\mu}(z) - \oint_C \frac{d\bar{z}}{2\pi}\bar{\partial} X^{\mu}(\bar{z})\right) \\ &= \frac{1}{\alpha'}\left(\sqrt{\frac{\alpha'}{2}}\alpha_0^{\mu} + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^{\mu}\right) \\ &= \sqrt{\frac{2}{\alpha'}}\alpha_0^{\mu}. \end{aligned} \tag{2.257}$$

Integrating the expansion (2.255) gives

$$\int dz\,\partial X^{\mu}(z,\bar{z}) = \int dz\,\partial X_R^{\mu}(z) = X_R^{\mu}(z) = x_z^{\mu} - i\frac{\alpha'}{2}p^{\mu}\ln z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{z^{-m}}{m}\alpha_m^{\mu}.$$

In the same way

$$\int d\bar{z}\,\bar{\partial}X^{\mu}(z,\bar{z}) = X_L^{\mu}(\bar{z}) = x_{\bar{z}}^{\mu} - i\frac{\alpha'}{2}p^{\mu}\ln\bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\bar{z}^{-m}}{m}\bar{\alpha}_m^{\mu},$$

such that

$$X^{\mu}(z,\bar{z}) = X_R^{\mu}(z) + X_L^{\mu}(\bar{z}) = x^{\mu} - i\frac{\alpha'}{2}p^{\mu}\ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m^{\mu}}{z^m} + \frac{\bar{\alpha}_m^{\mu}}{\bar{z}^m}\right). \quad (2.258)$$

From the contour argument and the XX OPE (2.165), one derives

$$\left[\alpha_m^{\mu},\alpha_n^{\nu}\right] = \left[\bar{\alpha}_m^{\mu},\bar{\alpha}_n^{\nu}\right] = m\delta_{m+n}\eta^{\mu\nu}, \qquad \left[x^{\mu},p^{\nu}\right] = i\eta^{\mu\nu}, \quad (2.259)$$

for instance, by using eq.(2.231), this follows that

$$\begin{aligned}
\left[\alpha_m^{\mu},\alpha_n^{\nu}\right] &= \frac{2}{\alpha'}\oint\frac{dz}{2\pi}\oint_{C_w}\frac{dw}{2\pi}z^{-m}w^{-n}\partial X^{\mu}(z)\,\partial X^{\nu}(w) \\
&\overset{(2.154)}{=} \frac{2}{\alpha'}\oint\frac{dz}{2\pi}\oint_{C_w}\frac{dw}{2\pi}z^{-m}w^{-n}\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-w}\right) \\
&= m\delta_{m+n}\eta^{\mu\nu}.
\end{aligned} \quad (2.260)$$

In a similar way we can obtain the other relations.

2.10.1 Vertex Operators

According to eq.(2.254) the canonical scaling dimension of the boson $X(z,\bar{z})$ vanishes, then, it is possible to construct an infinite variety of local fields related to $X(z,\bar{z})$ without introducing a scale, namely the so-called vertex operators:

$$\mathcal{V}_k(z,\bar{z}) = :e^{ik\cdot X(z,\bar{z})}:. \quad (2.261)$$

The normal ordering has the following meaning, in terms of the operators appearing in (2.258)

$$\begin{aligned}
\mathcal{V}_k(z,\bar{z}) &= :\mathrm{e}^{ik\left\{x_0 - \frac{i}{2}p\ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n z^{-n} + \bar{\alpha}_n\bar{z}^{-n}\right)\right\}}: \\
&= \mathrm{e}^{\left\{ikx_0 + \sqrt{\frac{\alpha'}{2}}\sum_{n>0}\frac{1}{n}k(\alpha_{-n}z^n + \bar{\alpha}_{-n}\bar{z}^n)\right\}}\mathrm{e}^{\left\{\frac{\alpha'}{2}kp\ln(z\bar{z}) - \sqrt{\frac{\alpha'}{2}}\sum_{n>0}\frac{1}{n}k\left(\alpha_n z^{-n} + \bar{\alpha}_n\bar{z}^{-n}\right)\right\}}.
\end{aligned} \quad (2.262)$$

Within each exponential, the different operators commutes. We will now demonstrate that these fields are primary, with holomorphic and antiholomorphic dimensions

$$h\left(k\right)=\bar{h}\left(k\right)=\frac{\alpha'}{4}k^{2}. \tag{2.263}$$

We first calculate the OPE of ∂X with $\mathcal{V}_k$. Then, by expressing $\mathcal{V}_k$ in power series, yields

$$\partial X\left(z,\bar{z}\right)\mathcal{V}_{k}\left(w,\bar{w}\right)=\sum_{n=0}\frac{\left(ik\right)^{n}}{n!}\partial X\left(z,\bar{z}\right):X\left(w,\bar{w}\right)^{n}:. \tag{2.264}$$

This is needed to use a previous result, the OPE (2.165). We will omit the antiholomorphic coordinate in the notation, thus we have

$$\begin{aligned}
\partial X\left(z\right)\mathcal{V}_{k}\left(w\right) &= \sum_{n=1}\frac{\left(ik\right)^{n}}{n!}\left(n\left\langle \partial X\left(z\right)X\left(w\right)\right\rangle :X\left(w\right)^{n-1}:\right)\\
&\sim \sum_{n=1}\frac{\left(ik\right)^{n}}{n\left(n-1\right)!}\left(n\left(-\frac{\alpha'}{2}\frac{1}{z-w}\right):X\left(w\right)^{n-1}:\right)\\
&\sim -\frac{ik\alpha'}{2}\frac{1}{z-w}\sum_{n=1}\frac{\left(ik\right)^{n-1}}{\left(n-1\right)!}:X\left(w\right)^{n-1}:\\
&\sim -\frac{ik\alpha'}{2}\frac{1}{z-w}\sum_{m=0}\frac{\left(ik\right)^{m}}{m!}:X\left(w\right)^{m}:\\
&\sim -\frac{ik\alpha'}{2}\frac{\mathcal{V}_{k}\left(w\right)}{z-w}.
\end{aligned} \tag{2.265}$$

Next, we calculate the OPE of $\mathcal{V}_p$ with the energy momentum tensor. We can compute this OPE just by considering the cross contractions between the fields in $T\left(z\right)$ and those in $\mathcal{V}_p\left(w\right)$, that is

$$\begin{aligned}
T\left(z\right)\mathcal{V}_{p}\left(w\right) &= -\frac{1}{\alpha'}\sum_{n=0}\frac{\left(ip\right)^{n}}{n!}:\partial X\left(z\right)\partial X\left(z\right)::X\left(w\right)^{n}:\\
&\sim -\frac{1}{\alpha'}\sum_{n=0}\frac{\left(ip\right)^{n}}{n!}\Big\{n:\partial X\left(z\right)X\left(w\right)^{n-1}:\left\langle \partial X\left(z\right)X\left(w\right)\right\rangle \\
&\qquad +n:\partial X\left(z\right)X\left(w\right)^{n-1}:\left\langle \partial X\left(z\right)X\left(w\right)\right\rangle \\
&\qquad\quad +\frac{n!}{\left(n-2\right)!}\left\langle \partial X\left(z\right)X\left(w\right)\right\rangle \left\langle \partial X\left(z\right)X\left(w\right)\right\rangle :X\left(w\right)^{n-2}:\Big\}\\
&\sim -\frac{1}{\alpha'}\sum_{n=0}\frac{\left(ip\right)^{n}}{n!}\Big\{2n:\partial X\left(z\right)X\left(w\right)^{n-1}:\left(-\frac{\alpha'}{2}\frac{1}{z-w}\right)\\
&\qquad +n\left(n-1\right)\left(-\frac{\alpha'}{2}\frac{1}{z-w}\right)^{2}:X\left(w\right)^{n-2}:\Big\},
\end{aligned}$$

where we have used that the number of variation of n fields taking in pairs without repeated elements is $\frac{n!}{(n-2)!} = n(n-1)$. Therefore

$$\begin{aligned} T(z)\,\mathcal{V}_p(w) &\sim -\frac{\alpha'}{4}\left(\frac{1}{z-w}\right)^2 \sum_{n=2} \frac{(ip)^n}{(n-2)!} : X(w)^{n-2} : \\ &\qquad + \frac{1}{z-w}\sum_{n=1}\frac{(ip)^n}{n!}\left(n : \partial X(z)\, X(w)^{n-1} :\right) \\ &\sim \frac{\alpha' p^2}{4}\frac{1}{(z-w)^2}\sum_{m=0}\frac{(ip)^m}{m!} : X(w)^m : \\ &\qquad + \frac{1}{z-w}\sum_{n=1}\frac{(ip)^n}{n!}\left(n : \partial X(w)\, X(w)^{n-1} :\right) \\ &\sim \frac{\alpha'}{4}p^2\frac{\mathcal{V}_p(w)}{(z-w)^2} + \frac{\partial_w \mathcal{V}_p(w)}{z-w}. \end{aligned} \tag{2.266}$$

We have replaced $\partial X(z)$ by $\partial X(w)$ in the last equation since the difference between the two leads to a regular term. Thus we see that by the form of this OPE, $\mathcal{V}_p$ is primary, with conformal weight indicated by (2.263). The OPE with $\bar{T}$ has exactly the same form. In order to calculate the OPE of products of vertex operators, we may use the following relation for a single harmonic oscillator:

$$: e^{B_1} :: e^{B_2} :=: e^{B_1+B_2} : e^{\langle B_1 B_2 \rangle}, \tag{2.267}$$

where $B_i = \alpha_i a + \beta_i a^\dagger$ is some linear combination of annihilation and creation operators. In particular, we may write

$$: e^{aX(z,\bar{z})} :: e^{bX(w,\bar{w})} :=: e^{aX(z,\bar{z})+bX(w,\bar{w})} : e^{ab\langle X(z,\bar{z})X(w,\bar{w})\rangle}. \tag{2.268}$$

If we apply that to the Vertex operators, and use the equation (2.153), for the correlator $\langle X(z,\bar{z})\, X(w,\bar{w})\rangle = -\frac{\alpha'}{2}\ln|z-w|^2$, we have

$$\begin{aligned} \mathcal{V}_p(z,\bar{z})\,\mathcal{V}_k(w,\bar{w}) &= : e^{ipX(z,\bar{z})} :: e^{ikX(w,\bar{w})} : \\ &= : e^{i(pX(z,\bar{z})+kX(w,\bar{w}))} : e^{-pk\langle X(z,\bar{z})X(w,\bar{w})\rangle} \\ &\sim : e^{i(pX(z,\bar{z})+kX(w,\bar{w}))} : e^{\frac{2\alpha' pk}{2}\ln|z-w|}. \end{aligned}$$

Now, we expand the first exponential of the r.h.s above, around $(w,\bar{w})$. That is

$$\begin{aligned}\mathcal{V}_p(z,\bar{z})\,\mathcal{V}_k(w,\bar{w}) &\sim |z-w|^{\alpha' pk}\Big(:e^{i(p+k)X(w,\bar{w})}+ip\,(z-w)\,\partial X e^{i(p+k)X(w,\bar{w})} \\ &\qquad +ip\,(\bar{z}-\bar{w})\,\bar{\partial}Xe^{i(p+k)X(w,\bar{w})}+\ldots:\Big) \\ &\sim |z-w|^{\alpha' pk}\,\mathcal{V}_{p+k}(w,\bar{w})+ip\,|z-w|^{\alpha' pk}\Big((z-w):\partial Xe^{i(p+k)X(w,\bar{w})}: \\ &\qquad +(\bar{z}-\bar{w}):\bar{\partial}Xe^{i(p+k)X(w,\bar{w})}:\Big)+\ldots\end{aligned} \tag{2.269}$$

However, we have seen, (2.122), that invariance under global conformal group force the fields within a non-zero two-point correlation function to have the same conformal dimension. Furthermore, the requirement that the correlation function $\langle\mathcal{V}_p(z,\bar{z})\,\mathcal{V}_k(w,\bar{w})\rangle$ does not grow with the distance imposes the constraint $pk<0$, which, after fixing $\alpha'=2$, leaves $p=-k$ as the only possibility

$$\mathcal{V}_p(z,\bar{z})\,\mathcal{V}_{-p}(w,\bar{w})\sim|z-w|^{-2p^2}+\ldots\quad . \tag{2.270}$$

From now on, the normal ordering of the vertex operator will not be explicitly written but always be implicit.

2.10.2 The Fock Space

Since p^μ commutes with all the α_n^μ and $\bar{\alpha}_n^\mu$, these operators cannot change the values of p^μ and the Fock space is built upon a one parameter family of vacuum $|0;k^\mu\rangle = |0\rangle\otimes|k^\mu\rangle$, where k^μ is the continuous eigenvalue of $\alpha_0^\mu=\sqrt{\frac{\alpha'}{2}}p^\mu$. As mentioned above, the conformal modes α_n^μ and $\bar{\alpha}_n^\mu$ are annihilation operators for $n>0$ and creation operators for $n<0$:

$$\alpha_n^\mu|k^\mu\rangle=\bar{\alpha}_n^\mu|k^\mu\rangle=0\qquad(n>0)\qquad\text{and}\qquad\alpha_0^\mu|k^\mu\rangle=\bar{\alpha}_0^\mu|k^\mu\rangle=k^\mu|k^\mu\rangle\,. \tag{2.271}$$

Explicitly, we are using a notation such that $\alpha_n^\mu|k^\mu\rangle$ stands for $\mathbb{I}\otimes\alpha_n^\mu(|0\rangle\otimes|k^\mu\rangle)$. The holomorphic energy-momentum tensor is given by (2.167) and by using (2.255) we get

$$\begin{aligned}T(z) &= -\frac{1}{\alpha'}:\left(-i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n z^{-n-1}\right)\cdot\left(-i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}}\alpha_m z^{-m-1}\right): \\ &= \frac{1}{2}\sum_{n,m\in\mathbb{Z}}z^{-n-m-2}:\alpha_n\cdot\alpha_m:,\end{aligned} \tag{2.272}$$

which implies, by using (2.235)

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}:\alpha_{n-m}\cdot\alpha_m:+\delta_{n,0}a^X, \tag{2.273}$$

where a^X is a normal ordering constant to be determined, it shifts the vacuum energy. However, it is zero in this case because

$$2L_0 |0;0\rangle = [L_1, L_{-1}] |0;0\rangle = (L_1 L_{-1} - L_{-1} L_1) |0;0\rangle = 0. \tag{2.274}$$

As mentioned above, the operator $L_0 + \bar{L}_0$ generate the dilations on the plane, which correspond to time translations on the cylinder, therefore the Hamiltonian may be written as follows

$$H = L_0 + \bar{L}_0. \tag{2.275}$$

The mode operators α_m play a role vis-a-vis L_0 similar to L_m, because of the commutation $\left[L_0, \alpha^\mu_{-m}\right] = m\alpha^\mu_{-m}$. This means that its effect on the conformal dimension (the eigenvalue of L_0) is the same that of L_m. From the equations (2.246) and (2.273) we see that $|k^\mu\rangle$ has conformal dimension $\frac{k^2}{2}$, since $L_0 |k^\mu\rangle = \frac{\alpha_0^2}{2} |k^\mu\rangle = \frac{k^2}{2} |k^\mu\rangle$. The elements of the Fock space are obtained by acting on $|k^\mu\rangle$ with the creation operators α^μ_{-n} and $\bar{\alpha}^\mu_{-n}$ $(n > 0)$:

$$\alpha^{n_1}_{-1}\alpha^{n_2}_{-2}...\bar{\alpha}^{m_1}_{-1}\bar{\alpha}^{m_2}_{-2}... |k^\mu\rangle, \qquad n_i, m_j \geq 0. \tag{2.276}$$

These states are eigenstates of L_0 and $\bar{L}_0$ with conformal dimensions

$$h = \frac{k^2}{2} + \sum_j j n_j, \qquad \bar{h} = \frac{k^2}{2} + \sum_j j m_j \quad . \tag{2.277}$$

Each vacuum $|k^\mu\rangle$ may be obtained from the $SL(2, \mathbb{C})$ invariant vacuum $|0\rangle$ by application of the vertex operator $\mathcal{V}_k(z, \bar{z}) =: e^{ik\cdot X(z,\bar{z})}:$. We now show it explicitly, that is

$$|k^\mu\rangle = \mathcal{V}_k(0) |0\rangle. \tag{2.278}$$

We will proceed by showing that $\mathcal{V}_k(0)|0\rangle$ is an eigenstate of p^μ with eigenvalue k^μ. For this we rewrite p^μ according to eq.(2.257) as

$$\begin{aligned}
p^\mu \mathcal{V}_k(0) |0\rangle &= 2\oint \frac{dz}{2\pi i} \frac{i}{\alpha'} \partial X^\mu(z) : e^{ik\cdot X(0,0)} : |0\rangle \\
&\overset{(2.265)}{=} 2\oint \frac{dz}{2\pi i} \frac{i}{\alpha'} \left(-\frac{ik^\mu \alpha'}{2} \frac{\mathcal{V}_k(0)}{z} \right) |0\rangle \\
&= \oint \frac{dz}{2\pi i} \left(\frac{1}{z}\right) k^\mu \mathcal{V}_k(0) |0\rangle \\
&= k^\mu \mathcal{V}_k(0) |0\rangle.
\end{aligned} \tag{2.279}$$

Thus, equations (2.279) allows us to identify the state $\mathcal{V}_k(0)|0\rangle$ as $|k^\mu\rangle$ provided we set $\alpha' = 2$, such that $\alpha^\mu_0 = p^\mu$.

2.11 *bc* CFT System revisited

Let us now study the mode expansion and the Fock space for the b, c system.

2.11.1 *bc* Mode Expansion

The fields b and c have the Laurent expansions

$$b\left(z\right)=\sum_{m=-\infty}^{\infty}z^{-m-\lambda}b_m \qquad , \qquad c\left(z\right)=\sum_{m=-\infty}^{\infty}z^{-m-1+\lambda}c_m. \tag{2.280}$$

These are only Laurent expansions if λ is an integer, which we will assume for now. The OPE gives the anticommutators

$$\{b_m, c_n\}=\delta_{m+n}\Longrightarrow\{b_0, c_0\}=1, \qquad b_0^2=c_0^2=0. \tag{2.281}$$

Consider first the states that are annihilate by all of the $n>0$ operators. The b_0, c_0 oscillator algebra generates two such ground states $|\downarrow\rangle$ and $|\uparrow\rangle$, with the properties

$$\begin{array}{rclcrcl} b_0\left|\downarrow\right\rangle &=& 0, & & b_0\left|\uparrow\right\rangle &=& \left|\downarrow\right\rangle, \\ c_0\left|\downarrow\right\rangle &=& \left|\uparrow\right\rangle, & & c_0\left|\uparrow\right\rangle &=& 0, \\ b_n\left|\downarrow\right\rangle &=& b_n\left|\uparrow\right\rangle &=& c_n\left|\downarrow\right\rangle &=& c_n\left|\uparrow\right\rangle = 0, \qquad n>0. \end{array} \tag{2.282}$$

The most general state is obtained by acting on these states with the $n<0$ modes at most once each because these anticommute. It is conventional to group b_0 with lowering operators and c_0 with raising operators, so we will single out $|\downarrow\rangle$ as the ghost vacuum $|0\rangle$. In string theory we will have a bc and a $\bar{b}\bar{c}$ theory, each with $\lambda=2$. The closed string spectrum thus includes a product of two copies of the above. The states $|\downarrow\rangle$ and $|\uparrow\rangle$ are not, however, the $SL\left(2,\mathbb{C}\right)$ invariant vacuum $|0\rangle_{b,c}$, because this vacuum state must satisfy ($\lambda=2$)

$$b_n\left|0\right\rangle_{b,c}=0 \quad \forall n\geq-1, \qquad c_m\left|0\right\rangle_{b,c}=0 \quad \forall m\geq 2. \tag{2.283}$$

According to the regularity condition (2.223). Since it is not annihilated by all the negative frequency modes, in fact, we can note that $b_{-1}\left|\downarrow\right\rangle$ satisfy the conditions above, and can be identified as $|0\rangle_{b,c}$, then

$$c_1\left|0\right\rangle_{b,c}=c_1b_{-1}\left|\downarrow\right\rangle=\left|\downarrow\right\rangle-b_{-1}c_1\left|\downarrow\right\rangle=\left|\downarrow\right\rangle. \tag{2.284}$$

From (2.282) it yields

$$c_0\left|\downarrow\right\rangle=\left|\uparrow\right\rangle=c_0c_1\left|0\right\rangle_{b,c}. \tag{2.285}$$

We note also that $\langle\downarrow|\downarrow\rangle = \langle\uparrow\left|b_0^2\right|\uparrow\rangle = 0$ and $\langle\uparrow|\uparrow\rangle = \langle\downarrow\left|c_0^2\right|\downarrow\rangle = 0$, however $\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = {}_{b,c}\langle 0\left|c_{-1}c_0c_1\right|0\rangle_{b,c} \neq 0$, therefore, we choose a normalization such that

$$ {}_{b,c}\langle 0\left|c_{-1}c_0c_1\right|0\rangle_{b,c} = 1. \tag{2.286}$$

The Virasoro generators are found from (2.178,2.280,2.235) as follows

$$\begin{aligned} T(z) &= (1-\lambda):(\partial b)\,c:-\lambda:b\partial c: \\ &\overset{(2.280)}{=} (1-\lambda):\left(-\sum_{n=-\infty}^{\infty}(n+\lambda)\,z^{-n-\lambda-1}b_n\right)\sum_{k=-\infty}^{\infty}z^{-k-1+\lambda}c_k: \\ &\qquad -\lambda:\sum_{n=-\infty}^{\infty}z^{-n-\lambda}b_n\left(\sum_{k=-\infty}^{\infty}(\lambda-k-1)\,z^{-k-2+\lambda}c_k\right): \\ &= \sum_{n,k=-\infty}^{\infty}z^{-n-k-2}\left(:(\lambda-1)(n+\lambda)\,b_nc_k-\lambda\,(\lambda-k-1)\,b_nc_k:\right) \\ &= \sum_{n,k=-\infty}^{\infty}z^{-n-k-2}\left(\lambda\,(n+k)-n\right):b_nc_k:. \end{aligned}$$

Now we rename the index k by $n+k=m \Longrightarrow k=m-n$, and we have

$$\begin{aligned} T(z) &= \sum_{n,m=-\infty}^{\infty}z^{-m-2}\,(\lambda m-n):b_nc_{m-n}: \\ &= \sum_{m=-\infty}^{\infty}z^{-m-2}\sum_{n=-\infty}^{\infty}(\lambda m-n):b_nc_{m-n}:, \end{aligned}$$

which, from (2.235) allows us to identify

$$L_m = \sum_{n=-\infty}^{\infty}(\lambda m-n):b_nc_{m-n}:+\delta_{m,0}a^g. \tag{2.287}$$

Note that if we rename a index by $-k=m-n$, we have, for $\lambda=2$

$$\begin{aligned} L_m &= \sum_{n=-\infty}^{\infty}(m+m-n):b_nc_{m-n}:+\delta_{m,0}a^g \\ &= \sum_{k=-\infty}^{\infty}(m-k):b_{k+m}c_{-k}:+\delta_{m,0}a^g. \end{aligned} \tag{2.288}$$

As for the $X-$system, the L_0 includes a normal ordering constant. This constant can be determined by the physical condition $(L_0-a^g)\,|\downarrow\rangle=0$, then

$$2L_0\,|\downarrow\rangle = (L_1L_{-1}-L_{-1}L_1)\,|\downarrow\rangle = L_1L_{-1}\,|\downarrow\rangle - L_{-1}\cancelto{0}{L_1|\downarrow\rangle}.$$

It is not hard to see that $L_1 |\downarrow\rangle = 0$ because (2.281,2.282). In another hand, we have

$$L_{-1} |\downarrow\rangle = \sum_{n=-\infty}^{\infty} (-\lambda - n)\, b_n c_{-1-n} |\downarrow\rangle .$$

The sum above has just one possibility that does not annihilate $|\downarrow\rangle$, that is $n = -1$, because this casts a term $b_{-1}c_0$. From eq.(2.287) we have then

$$\begin{aligned} 2L_0 |\downarrow\rangle &= L_1 L_{-1} |\downarrow\rangle \\ &= (1-\lambda) \sum_{n=-\infty}^{\infty} (\lambda - n)\, b_n c_{1-n} b_{-1} c_0 |\downarrow\rangle \\ &\overset{(2.281)}{=} (1-\lambda)\, \lambda b_0 c_0 |\downarrow\rangle - (1-\lambda) \sum_{n=-\infty}^{\infty} (\lambda - n)\, b_{-1} b_n c_0 c_{1-n} |\downarrow\rangle . \end{aligned}$$

In the second term of the last line we have used the fact that $\{c_m, c_n\} = 0$ and $\{b_m, b_n\} = 0$. Then, using a last time (2.281) for the last term, we can see that it vanishes, therefore, using also (2.282)

$$2L_0 |\downarrow\rangle = (1-\lambda)\, \lambda b_0 c_0 |\downarrow\rangle = (1-\lambda)\, \lambda |\downarrow\rangle . \tag{2.289}$$

Thus $a^g = \frac{1}{2}(1-\lambda)\lambda$ and

$$L_m = \sum_{n=-\infty}^{\infty} (m\lambda - n) : b_n c_{m-n} : + \frac{1}{2}(1-\lambda)\, \lambda \delta_{m,0}. \tag{2.290}$$

From the mode expansions eq.(2.280) ($\lambda = 2$) we obtain the mode expansion of the ghost number current eq.(2.181), that is

$$j(z) = - : b(z)\, c(z) := \sum_n z^{-n-1} j_n, \qquad \text{where} \qquad j_n = \sum_m : c_{n-m} b_m : . \tag{2.291}$$

The ghost charge is given by the contour integral of $j(z)$

$$N^g = \frac{1}{2\pi i} \oint_{C_0} dz\, j(z) = \frac{1}{2\pi i} \sum_n \oint_{C_0} dz \left(z^{-n-1} j_n \right) = j_0 = \sum_m : c_{-m} b_m : . \tag{2.292}$$

We must add a normal ordering constant which is compute by opening the sum above, as follows

$$\begin{aligned}
N^g &= \sum_{n=-\infty}^{\infty} c_{-n}b_n \\
&= \sum_{n>0} c_{-n}b_n + \sum_{n>0} c_n b_{-n} + c_0 b_0 \\
&\overset{(2.281)}{=} \sum_{n>0} c_{-n}b_n + \sum_{n>0} \left(1 - b_{-n}c_n\right) + c_0 b_0 \\
&= \sum_{n>0} \left(c_{-n}b_n - b_{-n}c_n\right) + \sum_{n>0} 1 + c_0 b_0.
\end{aligned}$$

We regularize $\sum_{n>0} 1$ through the Riemann zeta functions, that is, $\zeta\left(0\right) = \sum_{n=1} \frac{1}{n^0} = -\frac{1}{2}$, which leads to

$$N^g = \sum_{n>0} \left(c_{-n}b_n - b_{-n}c_n\right) + c_0 b_0 - \frac{1}{2}. \tag{2.293}$$

It satisfies

$$\left[N^g, b_m\right] = -b_m \quad , \quad \left[N^g, c_m\right] = c_m \tag{2.294}$$

and so counts the number of c minus the number of b excitations. The ground states have ground number $\pm\frac{1}{2}$:

$$N^g \left|\downarrow\right\rangle = -\frac{1}{2}\left|\downarrow\right\rangle \quad , \quad N^g \left|\uparrow\right\rangle = \frac{1}{2}\left|\uparrow\right\rangle. \tag{2.295}$$

This depends on the value of the ordering constant and can be shown as follows

$$\mathbb{I} \otimes N_g \left|0\right\rangle = \mathbb{I} \otimes \left(c_0 b_0 - \frac{1}{2}\right) \left|0, k^\mu\right\rangle \otimes \left|\uparrow\right\rangle = \left(1 - \frac{1}{2}\right) \left|0, k^\mu\right\rangle \otimes \left|\uparrow\right\rangle = \frac{1}{2}\left|0\right\rangle, \tag{2.296}$$

$$\mathbb{I} \otimes N_g \left|0\right\rangle = \mathbb{I} \otimes \left(c_0 b_0 - \frac{1}{2}\right) \left|0, k^\mu\right\rangle \otimes \left|\downarrow\right\rangle = -\frac{1}{2} \left|0, k^\mu\right\rangle \otimes \left|\downarrow\right\rangle = -\frac{1}{2}\left|0\right\rangle. \tag{2.297}$$

2.12 Path integral quantization

In order to carry out the path integral quantization of the Polyakov action $S\left[X, g\right]$, we must consider the path integral,

$$Z \equiv \int \mathcal{D}X\mathcal{D}g\, \mathrm{e}^{-S[X,g]}. \tag{2.298}$$

However this expression is ill-defined, because when we are integrating over all possible configurations of $g_{\mu\nu}$ contained in the integration measure $\mathcal{D}g$. We are integrating over conformal related surfaces, which contribute with the same information, so we must redefine the expression above by dividing by the volume of the local gauge symmetry group. That is

$$Z \equiv \int \frac{\mathcal{D}X\mathcal{D}g}{V_{\text{diff}\times\text{Weyl}}} \mathrm{e}^{-S[X,g]}. \tag{2.299}$$

In order to obtain the correct measure, we follow the Faddeev-Popov procedure. The idea is to separate the path integral into an integral over the gauge group times an integral along the gauge slice, and to divide by the volume of the gauge group. The Faddeev-Popov determinant is the Jacobian of this change of variables. So that the integral runs along a slice parameterized by the X^μ alone, after fixing the metric. We will represent by ζ a combined coordinate and Weyl transformation,

$$\zeta : \quad g \longrightarrow g^\zeta, \qquad g^\zeta_{ab}\left(\sigma'\right) = e^{2\omega(\sigma)} \frac{\partial\sigma^c}{\partial\sigma'^a} \frac{\partial\sigma^d}{\partial\sigma'^b} g_{cd}\left(\sigma\right). \tag{2.300}$$

We define the Faddeev-Popov measure Δ_{FP} by

$$1 = \int \mathcal{D}g \; \delta\left(g - \hat{g}\right) \longrightarrow 1 = \Delta_{FP}\left(g\right) \int \mathcal{D}\zeta \; \delta\left(g - \hat{g}^\zeta\right), \tag{2.301}$$

where $\hat{g}_{ab}$ is a "fiducial" metric, a simple choice is $\hat{g}_{ab}\left(\sigma\right) = \delta_{ab}$ or the conformal gauge $\hat{g}_{ab}\left(\sigma\right) = e^{2\omega(\sigma)}\delta_{ab}$. In (2.301) $\mathcal{D}\zeta$ is a gauge invariant measure of the diff$\times$Weyl group. The delta function is actually a **delta functional**, requiring $g_{ab} = \hat{g}^\zeta_{ab}$ at every point. Inserting (2.301) into the functional (2.299), it yields

$$Z\left[\hat{g}\right] = \int \frac{\mathcal{D}\zeta\mathcal{D}X\mathcal{D}g}{V_{\text{diff}\times\text{Weyl}}} \Delta_{FP}\left(g\right) \delta\left(g - \hat{g}^\zeta\right) \exp\left(-S\left[X, g\right]\right). \tag{2.302}$$

We will denote explicitly the dependence of Z on the choice of fiducial metric. Carry out the integration over g_{ab}, and also rename the dummy variable $X \longrightarrow X^\zeta$, to obtain

$$Z\left[\hat{g}\right] = \int \frac{\mathcal{D}\zeta\mathcal{D}X^\zeta}{V_{\text{diff}\times\text{Weyl}}} \Delta_{FP}\left(\hat{g}^\zeta\right) \exp\left(-S\left[X^\zeta, \hat{g}^\zeta\right]\right). \tag{2.303}$$

Now we show the gauge invariance of $\Delta_{FP}\left(\hat{g}^\zeta\right)$ as follows

$$\Delta_{FP}\left(g^\zeta\right)^{-1} = \int \mathcal{D}\zeta' \; \delta\left(g^\zeta - \hat{g}^{\zeta'}\right) = \int \mathcal{D}\zeta' \; \delta\left(g - \hat{g}^{\zeta^{-1}\cdot\zeta'}\right) = \int \mathcal{D}\zeta'' \; \delta\left(g - \hat{g}^{\zeta''}\right) = \Delta_{FP}\left(g\right)^{-1}, \tag{2.304}$$

where $\zeta'' = \zeta^{-1} \cdot \zeta'$. In the second equality we have used the gauge invariance of the delta function and, in the third, the invariance of the measure.
Now we use the gauge invariance of Δ_{FP}, of $\mathcal{D}X^\zeta$, and of the action to obtain

$$Z\left[\hat{g}\right] = \int \frac{\mathcal{D}\zeta\mathcal{D}X}{V_{\text{diff}\times\text{Weyl}}} \Delta_{FP}\left(\hat{g}\right) \exp\left(-S\left[X, \hat{g}\right]\right). \tag{2.305}$$

Finally, nothing in the integrand depend on ζ, so the integral over ζ just produces the volume of the gauge group and cancels the denominator, leaving

$$Z[\hat{g}] = \int \mathcal{D}X \Delta_{FP}(\hat{g}) \exp(-S[X,\hat{g}]) . \tag{2.306}$$

Thus, $\Delta_{FP}(\hat{g})$ is the correct measure on the slice. To evaluate (2.301) for the Faddeev-Popov measure, let us pretend that (2.300) are two infinitesimal transformations. Therefore, for ζ near the identity we can expand

$$\begin{aligned} \delta g_{ab} &= g - \hat{g}^{\zeta} = 2\delta\omega g_{ab} - \nabla_a \delta\sigma_b - \nabla_b \delta\sigma_a \\ &= \left(2\delta\omega - \nabla_c \delta\sigma^c\right) g_{ab} - 2\left(P_1 \delta\sigma\right)_{ab} , \end{aligned} \tag{2.307}$$

The term $2\delta\omega g_{ab}$ corresponds the Weyl rescaling and the remaining to the reparametrization. In the equation above we have defined a differential operator P_1 that takes vectors into traceless symmetric 2-tensors

$$2\left(P_1 \delta\sigma\right)_{ab} = \nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a - g_{ab} \nabla_c \delta\sigma^c . \tag{2.308}$$

We will use the functional delta function

$$\int \mathcal{D}F \; \mathrm{e}^{i \int d^2\sigma \sqrt{g} F(\sigma) G(\sigma)} = \delta[G] = \prod_{\sigma} \delta\left(G(\sigma)\right) . \tag{2.309}$$

Near the identity the inverse determinant becomes

$$\begin{aligned} \Delta_{FP}(\hat{g})^{-1} &= \int \mathcal{D}\zeta \; \delta\left(\hat{g} - \hat{g}^{\zeta}\right) \\ &= \int \mathcal{D}\delta\omega \mathcal{D}\delta\sigma \; \delta\left[-\delta g_{ab}\right] \\ &= \int \mathcal{D}\delta\omega \mathcal{D}\delta\sigma \; \delta\left[-\left(2\delta\omega - \hat{\nabla} \cdot \delta\sigma\right) \hat{g}_{ab} + 2\left(\hat{P}_1 \delta\sigma\right)_{ab}\right] \\ &= \int \mathcal{D}\delta\omega \mathcal{D}\beta \mathcal{D}\delta\sigma \; \exp\left\{2\pi i \int d^2\sigma \hat{g}^{1/2} \beta^{ab} \left[-\left(2\delta\omega - \hat{\nabla} \cdot \delta\sigma\right) \hat{g}_{ab} + 2\left(\hat{P}_1 \delta\sigma\right)_{ab}\right]\right\} . \end{aligned}$$

Integrating in $\delta\omega$ we have

$$\Delta_{FP}(\hat{g})^{-1} = \int \mathcal{D}\beta \mathcal{D}\delta\sigma \, \delta\left[-2\beta^{ab} \hat{g}_{ab}\right] \exp\left\{2\pi i \int d^2\sigma \hat{g}^{1/2} \beta^{ab} \left[\hat{\nabla} \cdot \delta\sigma \hat{g}_{ab} + 2\left(\hat{P}_1 \delta\sigma\right)_{ab}\right]\right\} .$$

The delta functional $\delta\left[-2\beta^{ab} \hat{g}_{ab}\right]$ forces β^{ab} to be traceless. Thus, we now integrate over the functional $\mathcal{D}\beta'$, which are the traceless symmetric tensors. This is made by effecting a change of variables which divides β^{ab} in a traceless part and a pure trace part. The pure trace part integral is a constant after effecting the integration over the functional delta $\delta\left[-2\beta^{ab}\hat{g}_{ab}\right]$. We have

$$\Delta_{FP}\left(\hat{g}\right)^{-1}=\int\mathcal{D}\beta'\mathcal{D}\delta\sigma\,\exp\left\{4\pi i\int d^2\sigma\hat{g}^{1/2}\beta'^{ab}\left(\hat{P}_1\delta\sigma\right)_{ab}\right\}. \tag{2.310}$$

We now have a representation of $\Delta_{FP}\left(\hat{g}\right)^{-1}$ as a functional integral over a vector field $\delta\sigma^a$ and over a traceless symmetric tensor β'^{ab}. We can invert this path integral by replacing each bosonic field with a corresponding Grassmann **ghost** field. Namely,

$$\delta\sigma^a\longrightarrow c^a,\qquad\qquad \beta'_{ab}\longrightarrow b_{ab}, \tag{2.311}$$

with b^{ab} as β'^{ab} being traceless. Thus,

$$\Delta_{FP}\left(\hat{g}\right)=\int\mathcal{D}b\mathcal{D}c\,\exp\left(-S_g\right), \tag{2.312}$$

where the ghost action S_g, with a convenient normalization for the fields, is

$$S_g=\frac{1}{2\pi}\int d^2\sigma\sqrt{\hat{g}}b_{ab}\hat{\nabla}^a c^b=\frac{1}{2\pi}\int d^2\sigma\sqrt{\hat{g}}b_{ab}\left(\hat{P}_1 c\right)^{ab}. \tag{2.313}$$

Locally on the world-sheet, the path integral is now

$$Z\left[\hat{g}\right]=\int\mathcal{D}X\mathcal{D}b\mathcal{D}c\exp\left(-S_X-S_g\right). \tag{2.314}$$

In the conformal gauge, $\hat{g}_{ab}\left(\sigma\right)=e^{2\omega(\sigma)}\delta_{ab}$. By using complex coordinates as defined in eq.(2.104) we have a new metric $g_{ab}\left(z,\bar{z}\right)$, such that

$$g_{ab}\left(z,\bar{z}\right)=\frac{1}{2}e^{2\omega}\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad g^{ab}\left(z,\bar{z}\right)=2e^{-2\omega}\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sqrt{g}=\frac{1}{2}e^{2\omega}. \tag{2.315}$$

The only non-vanishing Christoffel symbols are

$$\Gamma^z_{zz}=\partial_z\omega\left(z,\bar{z}\right)\qquad\text{and}\qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}}=\partial_{\bar{z}}\omega\left(z,\bar{z}\right). \tag{2.316}$$

In this way we have the ghost action (2.313)

$$
\begin{aligned}
S_g &= \frac{1}{2\pi}\int d^2\sigma \hat{g}^{1/2} b_{ab} g^{ac}\left(\partial_c c^b + \Gamma^b_{cd} c^d\right) \\
&= \frac{1}{2\pi}\int d^2 z \left(\frac{1}{2}e^{2\omega}\right)\left(b_{zb}g^{zc} + b_{\bar{z}b}g^{\bar{z}c}\right)\left(\partial_c c^b + \Gamma^b_{cz}c^z + \Gamma^b_{c\bar{z}}c^{\bar{z}}\right) \\
&= \frac{1}{2\pi}\int d^2 z \left(\frac{1}{2}e^{2\omega}\right)\Big(b_{zb}g^{z\bar{z}}\partial_{\bar{z}}c^b + b_{zb}g^{z\bar{z}}\Gamma^b_{\bar{z}z}c^z + b_{zb}g^{z\bar{z}}\Gamma^b_{\bar{z}\bar{z}}c^{\bar{z}} \\
&\qquad + b_{\bar{z}b}g^{\bar{z}z}\partial_z c^b + b_{\bar{z}b}g^{\bar{z}z}\Gamma^b_{zz}c^z + b_{\bar{z}b}g^{\bar{z}z}\Gamma^b_{z\bar{z}}c^{\bar{z}}\Big).
\end{aligned}
$$

Since $0 = g^{ab}b_{ab} \Longrightarrow b_{\bar{z}z} = b_{z\bar{z}} = 0$. We have

$$
\begin{aligned}
S_g &= \frac{1}{2\pi}\int d^2 z \left(\frac{1}{2}e^{2\omega}\right)\Big(b_{zz}\left(2e^{-2\omega}\right)\partial_{\bar{z}}c^z + b_{zz}g^{z\bar{z}}\cancel{\Gamma^z_{\bar{z}z}}c^z + b_{zz}g^{z\bar{z}}\cancel{\Gamma^z_{\bar{z}\bar{z}}}c^{\bar{z}} \\
&\qquad + b_{\bar{z}\bar{z}}\left(2e^{-2\omega}\right)\partial_z c^{\bar{z}} + b_{\bar{z}\bar{z}}g^{\bar{z}z}\cancel{\Gamma^{\bar{z}}_{zz}}c^z + b_{\bar{z}\bar{z}}g^{\bar{z}z}\cancel{\Gamma^{\bar{z}}_{z\bar{z}}}c^{\bar{z}}\Big) \\
&= \frac{1}{2\pi}\int d^2 z \left(b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\right).
\end{aligned}
\tag{2.317}
$$

Notice that $\omega(\sigma)$ does not appears in the final form. As anticipated, this is a bc CFT with $(h_b, h_c) = (2, -1)$ and a $\bar{b}\bar{c}$ CFT with $\left(\bar{h}_b, \bar{h}_c\right) = (2, -1)$.

2.13 BRST Quantization

To consider the most general possible variation of the gauge condition, we must allow δg_{ab} to depend on the fields in the path integral. We consider the path integral provided with a local symmetry. The path integral fields $X^\mu(\sigma)$ and $g_{ab}(\sigma)$ are denoted by ϕ_i. Here i labels the fields and also the coordinate σ. The gauge invariance is $\epsilon^\alpha\delta_\alpha$ (that is an "operator variation"), where α includes the coordinates. We assume the gauge parameter ϵ^α to be real. The gauge transformation satisfy the algebra

$$
\left[\delta_\alpha, \delta_\beta\right] = f^\gamma{}_{\alpha\beta}\delta_\gamma. \tag{2.318}
$$

Now fix the gauge by imposing the conditions

$$
F^A(\phi) = 0. \tag{2.319}
$$

Once again A includes the coordinate. Following the Faddeev-Popov procedure

$$
\int \frac{\mathcal{D}\phi_i}{V_{\text{gauge}}} \mathrm{e}^{-S_1} \longrightarrow \int \mathcal{D}\phi_i \mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha \mathrm{e}^{-S_1 - S_2 - S_3}, \tag{2.320}
$$

where S_1 is the original gauge invariant action, S_2 is the gauge fixing action, given by

$$S_2 = -i \int d^2\sigma \sqrt{g} B_A F^A \left(\phi_i; \sigma\right). \tag{2.321}$$

Last, S_3 is the Faddeev-Popov action

$$S_3 = b_A c^\alpha \delta_\alpha F^A \left(\phi\right). \tag{2.322}$$

We have introduced the field B_A to procedure an integral representation of the gauge-fixing $\delta\left(F^A\right)$, that is a functional, so by using the eq.(2.309)

$$\int \mathcal{D}B \; \mathrm{e}^{i \int d^2\sigma \sqrt{g} B_A F^A(\phi_i;\sigma)} = \delta\left(F^A\right). \tag{2.323}$$

This total action is invariant under Becchi-Rouet-Stora-Tyutin (BRST) transformation

$$\delta_B \phi_i = -i\epsilon c^\alpha \delta_\alpha \phi_i, \tag{2.324}$$
$$\delta_B B_A = 0, \tag{2.325}$$
$$\delta_B b_A = \epsilon B_A, \tag{2.326}$$
$$\delta_B c^\alpha = \frac{i}{2}\epsilon f^\alpha{}_{\beta\gamma} c^\beta c^\gamma, \tag{2.327}$$

where ϵ must be taken to be anticommuting. The original action S_1 is invariant by itself, since the action of δ_B on ϕ_i is just a gauge transformation with parameter $i\epsilon c^\alpha$. The variation of S_2 cancels the variation of b_A in S_3, while the variations of $\delta_\alpha F^A$ and c^α in S_3 cancel. In order to see the invariance of the full action in (2.320) under a BRST transformation we see that the following results holds

$$\begin{aligned}
\delta_B\left(b_A F^A\left(\phi\right)\right) &= \left(\delta_B b_A\right) F^A\left(\phi\right) + b_A\left(\delta_B F^A\left(\phi\right)\right) \\
&= \epsilon B_A F^A\left(\phi\right) + b_A\left(-i\epsilon c^\alpha \delta_\alpha F^A\left(\phi\right)\right) \\
&= i\epsilon\left(-iB_A F^A\left(\phi\right) + b_A c^\alpha \delta_\alpha F^A\left(\phi\right)\right) \\
&= i\epsilon\left(S_2 + S_3\right),
\end{aligned} \tag{2.328}$$

where in third line was used the anticommuting character of ϵ and b_A. Then we note that

$$\delta_B\left(S_1 + S_2 + S_3\right) = \delta_B\left(S_1 - \frac{i}{\epsilon}\delta_B\left(b_A F^A\left(\phi\right)\right)\right) = 0. \tag{2.329}$$

The first term in th r.h.s vanishes because S_1 is invariant under BRST transformations by itself, as mentioned above, the second term vanishes by the nilpotency of the BRST transformation, that is, $\delta_B^2 = 0$, which can be checked explicitly from the eqs. (2.324), (2.325), (2.326) and (2.327). Thus, the total action is BRST invariant by construction.

The BRST symmetry is used to derive the physical spectrum in the gauge theory. A quantum amplitude is of the form

$$\langle f \mid i \rangle = \int \mathcal{D}\phi_i \mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha e^{-S_1-S_2-S_3}, \tag{2.330}$$

for any initial i and final f states. If this amplitude is physical, it must be independent of the gauge-fixing made to calculate the path integral. We consider a small change in the gauge condition δF. Then, the change in S_2 and S_3 actions gives

$$\begin{aligned}
\epsilon\delta \langle f \mid i \rangle &= -\epsilon \int \mathcal{D}\phi_i \mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha e^{-(S_1+S_2+S_3)} \delta\left(S_1+S_2+S_3\right) \\
&= -\epsilon \int \mathcal{D}\phi_i \mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha e^{-(S_1+S_2+S_3)} \left(\cancel{\delta S_1} + \frac{i\epsilon}{i\epsilon}\delta\left(S_2+S_3\right)\right) \\
&= -\epsilon \int \mathcal{D}\phi_i \mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha e^{-(S_1+S_2+S_3)} \frac{1}{i\epsilon}\delta\left(\delta_B\left(b_A F^A(\phi)\right)\right) \\
&= i \int \mathcal{D}\phi_i \mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha e^{-(S_1+S_2+S_3)} \delta_B\left(b_A \delta F^A(\phi)\right) \\
&= i\left\langle f \left| \delta_B\left(b_A \delta F^A(\phi)\right) \right| i \right\rangle \\
&= -\epsilon \left\langle f \left| \left\{ \mathcal{Q}_B, b_A \delta F^A(\phi) \right\} \right| i \right\rangle .
\end{aligned} \tag{2.331}$$

In the last line we have written the BRST variation as an anticommutator with the corresponding conserved charge $\mathcal{Q}_B$. Since $\delta F^A(\phi)$ is arbitrary, it must be required that all physical states $|\psi\rangle$ must satisfy

$$\left\langle \psi \left| \left\{ \mathcal{Q}_B, b_A \delta F^A(\phi) \right\} \right| \psi' \right\rangle = 0. \tag{2.332}$$

Therefore, the physical states must be BRST invariant

$$\mathcal{Q}_B |\psi\rangle = \mathcal{Q}_B |\psi'\rangle = 0, \tag{2.333}$$

where we assumed $\mathcal{Q}_B^\dagger = \mathcal{Q}_B$. As we mentioned the BRST variation is a nilpotent operation, so that

$$\mathcal{Q}_B^2 = 0. \tag{2.334}$$

Which implies that a state of the form

$$\mathcal{Q}_B |\chi\rangle \tag{2.335}$$

will be annihilated by $\mathcal{Q}_B$ for any χ and so is physical. However, it is orthogonal to all physical states (say $|\psi\rangle$) including itself

$$\langle\psi| \left(\mathcal{Q}_B |\chi\rangle\right) = \left(\langle\psi| \mathcal{Q}_B\right) |\chi\rangle = 0. \tag{2.336}$$

Such states are called null states or spurious. All amplitudes involving states of the form (2.335) vanish. Two states are said to be physically equivalent if

$$\left|\psi'\right\rangle = \left|\psi\right\rangle + \mathcal{Q}_B \left|\chi\right\rangle . \tag{2.337}$$

The true physical space is identified with a set of equivalence classes, where states which differ by a null state belong to the same equivalence class. Physical states belong to the cohomology of $\mathcal{Q}_B$. In cohomology, states annihilated by $\mathcal{Q}_B$ are "BRST closed" and states of the form (2.335) are "BRST exact". Therefore, the BRST Hilbert space $\mathcal{H}_{BRST}$ is given by taking the quotient between the Hilbert space formed by the BRST closed states $\mathcal{H}_{closed}$ and the Hilbert space for the BRST exact states $\mathcal{H}_{exact}$, that is

$$\mathcal{H}_{BRST} = \frac{\mathcal{H}_{closed}}{\mathcal{H}_{exact}}. \tag{2.338}$$

2.13.1 BRST Quantization of the Bosonic string

In string theory, the total BRST invariant action is

$$S = S_P + S_{ghost} + S_{GF}, \tag{2.339}$$

where

$$S_P = \frac{1}{2\pi\alpha'} \int d^2 z \, \partial X^\mu \bar{\partial} X_\mu, \tag{2.340}$$

$$S_{ghost} = \frac{1}{2\pi} \int d^2 z \left(b\bar{\partial} c + \bar{b} \partial \bar{c} \right), \tag{2.341}$$

and

$$S_{GF} = \frac{i}{4\pi} \int d^2\sigma \sqrt{g} B^{ab} \left(\delta_{ab} - g_{ab} \right). \tag{2.342}$$

Equations (2.340) and (2.341) results after integration over B^{ab}. If we consider the variation of the metric δg_{ab} we obtain an equation of motion relating B_{ab} to the energy-momentum tensor $T^X_{ab} + T^{ghost}_{ab}$ because of the definition of the energy-momentum tensor itself, as a variation of the action with respect to the metric, that is as follows

$$\begin{aligned} \delta_g S &= \delta_g \left(S_1 + S_2 + S_3 \right) \\ &= \delta_g S_P - i\delta_g \int d^2\sigma \sqrt{g} B_A F^A \left(\phi_i ; \sigma \right) + \delta_g S_{ghost}. \end{aligned}$$

Then, we have that $F^A \longrightarrow F^{ab} = (g_{ab} - \delta_{ab})$, we must use further $\delta g = -gg_{ab}\delta g^{ab}$, then

$$\begin{aligned}
\delta_g S &= \delta_g S_P + \delta_g \frac{i}{4\pi}\int d^2\sigma\sqrt{g}B^{cd}\left(\delta_{cd} - g_{cd}\right) + \delta_g S_{ghost} \\
&= -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\delta g^{ab}T^X_{ab} - \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\delta g^{ab}T^g_{ab} \\
&\quad +\frac{i}{4\pi}\int d^2\sigma\left(-\frac{1}{2}\sqrt{g}g_{ab}\delta g^{ab}\right)B^{cd}\left(\delta_{cd} - g_{cd}\right) \\
&\quad +\frac{i}{4\pi}\int d^2\sigma\sqrt{g}B^{cd}\left(-\delta g_{cd}\right) \\
&= -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left\{T^X_{ab} + T^g_{ab} + \frac{i}{2}g_{ab}B^{cd}\left(\delta_{cd} - g_{cd}\right) + iB_{ab}\right\}\delta g^{ab} \\
&= \frac{i}{4\pi}\int d^2\sigma\sqrt{g}\left\{iT^X_{ab} + iT^g_{ab} + \frac{1}{2}g_{ab}B_{cd}F^{cd} - B_{ab}\right\}\delta g^{ab},
\end{aligned}$$

where we have used eq.(2.63). Thus we have

$$\frac{1}{2}g_{ab}B_{cd}F^{cd} + B_{ab} = i\left(T^X_{ab} + T^{ghost}_{ab}\right),$$

after gauge fixing we have $F^{cd} = 0$, therefore

$$B_{zz} = i\left(T^X_{zz} + T^{ghost}_{zz}\right), \qquad B_{\bar{z}\bar{z}} = i\left(T^X_{\bar{z}\bar{z}} + T^{ghost}_{\bar{z}\bar{z}}\right). \tag{2.343}$$

The corresponding BRST infinitesimal transformations eqs.(2.324-2.327) for the bosonic string theory become

$$\delta_B X^\mu = i\epsilon\left(c\partial + \bar{c}\bar{\partial}\right)X^\mu, \tag{2.344}$$

$$\delta_B b = i\epsilon\left(T^X + T^{ghost}\right), \tag{2.345}$$

$$\delta_B \bar{b} = i\epsilon\left(\bar{T}^X + \bar{T}^{ghost}\right), \tag{2.346}$$

$$\delta_B c = i\epsilon c\partial c, \tag{2.347}$$

$$\delta_B \bar{c} = i\epsilon\bar{c}\bar{\partial}\bar{c}. \tag{2.348}$$

After integration over the auxiliary field $B_A = B_{ab}$, the path integral (2.320) becomes

$$\int \mathcal{D}X^\mu\mathcal{D}b\,\mathcal{D}\bar{b}\,\mathcal{D}c\,\mathcal{D}\bar{c}e^{-S_P - S_{ghost}} \tag{2.349}$$

where S_P and S_{ghost} are given by (2.340) and (2.341) respectively. Therefore, we now use the Noether procedure to find the BRST current associated to the transformations

(2.344-2.348). Thus, we write

$$\delta_B S = \delta_B S_P + \delta_B S_{ghost}. \tag{2.350}$$

We promote the symmetry parameter ϵ to depend on the world-sheet variables z and $\bar{z}$. For the Polyakov action we have

$$\begin{aligned}
\delta_B S_P &= \frac{1}{2\pi\alpha'}\int d^2z\ \left(\partial\delta_B X^\mu \bar{\partial}X_\mu + \partial X^\mu \bar{\partial}\delta_B X_\mu\right) \\
& \frac{i}{2\pi\alpha'}\int d^2z\ \left\{\partial\left(\epsilon\left(c\partial+\bar{c}\bar{\partial}\right)X^\mu\right)\bar{\partial}X_\mu\right. \\
&\qquad\qquad\left. +\ \partial X_\mu\bar{\partial}\left(\epsilon\left(c\partial+\bar{c}\bar{\partial}\right)X^\mu\right)\right\} \\
&= \frac{i}{2\pi\alpha'}\int d^2z\ \left\{\left(\partial\epsilon c\partial X^\mu + \partial\epsilon\bar{c}\bar{\partial}X^\mu + \epsilon\left(\partial\left(c\partial X^\mu\right) + \cancel{\partial\left(\bar{c}\bar{\partial}X^\mu\right)}\right)\right)\bar{\partial}X_\mu\right. \\
&\qquad\left. +\ \partial X_\mu\left(\bar{\partial}\epsilon c\partial X^\mu + \bar{\partial}\epsilon\bar{c}\bar{\partial}X^\mu + \epsilon\left(\cancel{\bar{\partial}\left(c\partial X^\mu\right)} + \bar{\partial}\left(\bar{c}\bar{\partial}X^\mu\right)\right)\right)\right\} \\
&= \frac{i}{2\pi\alpha'}\int d^2z\ \left\{\partial\epsilon\left(\bar{c}\bar{\partial}X^\mu\bar{\partial}X_\mu\right) + \bar{\partial}\epsilon\left(c\partial X^\mu\partial X_\mu\right) + \left(\partial\epsilon c\partial X^\mu\bar{\partial}X_\mu + \epsilon\partial\left(c\partial X^\mu\right)\bar{\partial}X_\mu\right)\right. \\
&\qquad\qquad\left. +\ \left(\bar{\partial}\epsilon\bar{c}\bar{\partial}X^\mu\partial X_\mu + \partial X_\mu\epsilon\bar{\partial}\left(\bar{c}\bar{\partial}X^\mu\right)\right)\right\}.
\end{aligned} \tag{2.351}$$

In another hand, for the ghost action we write

$$\begin{aligned}
\delta_B S_{ghost} &= \frac{1}{2\pi}\int d^2z\left\{\delta_B b\bar{\partial}c + b\bar{\partial}\delta_B c + \delta_B\bar{b}\partial\bar{c} + \bar{b}\partial\delta_B\bar{c}\right\} \\
&= \frac{i}{2\pi}\int d^2z\left\{\epsilon\left(T^X + T^{ghost}\right)\bar{\partial}c + b\bar{\partial}\left(\epsilon c\partial c\right) + \epsilon\left(\bar{T}^X + \bar{T}^{ghost}\right)\partial\bar{c} + \bar{b}\partial\left(\epsilon\bar{c}\bar{\partial}\bar{c}\right)\right\} \\
&= \frac{i}{2\pi}\int d^2z\left\{\left(bc\partial c\right)\bar{\partial}\epsilon + \left(\bar{b}\bar{c}\bar{\partial}\bar{c}\right)\partial\epsilon\right\}.
\end{aligned} \tag{2.352}$$

We have used the equations of motion $\bar{\partial}c = 0 = \partial\bar{c}$. The last two terms in (2.351) are a total derivative since $\bar{\partial}\partial X_\mu = \partial\bar{\partial}X_\mu = 0$ because of the equation of motion. Therefore, after consider the anti-commutativity of the terms $(\partial_\alpha\epsilon)\,c^\beta$, we write the variation of the total actions as

$$\begin{aligned}
\delta_B S &= \frac{i}{2\pi}\int d^2z\left\{-\frac{1}{\alpha'}\bar{c} : \bar{\partial}X^\mu\bar{\partial}X_\mu : \partial\epsilon - \frac{1}{\alpha'}c : \partial X^\mu\partial X_\mu : \bar{\partial}\epsilon\right. \\
&\qquad\qquad\left. + : bc\partial c : \bar{\partial}\epsilon + : \bar{b}\bar{c}\bar{\partial}\bar{c} : \partial\epsilon\right\} \\
&= \frac{i}{2\pi}\int d^2z\left\{\left(-\bar{c}\frac{1}{\alpha'} : \bar{\partial}X^\mu\bar{\partial}X_\mu : + : \bar{b}\bar{c}\bar{\partial}\bar{c} :\right)\partial\epsilon\right. \\
&\qquad\qquad\left.\left(-c\frac{1}{\alpha'} : \partial X^\mu\partial X_\mu : + : bc\partial c :\right)\bar{\partial}\epsilon\right\} \\
&= \frac{i}{2\pi}\int d^2z\left\{\left(\bar{c}\bar{T}^X + : \bar{b}\bar{c}\bar{\partial}\bar{c} :\right)\partial\epsilon + \left(cT^X + : bc\partial c :\right)\bar{\partial}\epsilon\right\}.
\end{aligned} \tag{2.353}$$

We thus identify the holomorphic and anti-holomorphic parts of the BRST current

$$j_B = cT^X + :bc\partial c: + \frac{3}{2}\partial^2 c, \tag{2.354}$$
$$\bar{j}_B = \bar{c}\bar{T}^X + :\bar{b}\bar{c}\bar{\partial}\bar{c}: + \frac{3}{2}\bar{\partial}^2\bar{c}. \tag{2.355}$$

The final term in the current is a total derivative and does not contribute to the BRST charge, it has made been added by hand to make the BRST current a tensor which transform as a primary field. From the ghost energy-momentum tensor (2.178), which we now denote T^g, in contrast to the matter energy momentum tensor T^X ($XX-$system), we have for $\lambda = 2$,

$$\begin{aligned} T^g(z) &= -:\partial b(z)\, c(z): -2:b(z)\,\partial c(z):, \\ \frac{1}{2}\left(T^g(z) + :\partial b(z)\, c(z):\right) &= -:b(z)\,\partial c(z):. \end{aligned}$$

We see that

$$\begin{aligned} j_B(z) &= c(z)\,T^X(z) - c(z)\,b(z)\,\partial c(z) + \frac{3}{2}\partial^2 c(z) \\ &= c(z)\,T^X(z) + c(z)\,\frac{1}{2}\left(T^g(z) + \partial b(z)\, c(z)\right) + \frac{3}{2}\partial^2 c(z) \\ &= c(z)\,T^X(z) + \frac{1}{2}c(z)\,T^g(z) + \frac{3}{2}\partial^2 c(z), \end{aligned} \tag{2.356}$$

where we have used the fact that $c(z)\,c(z) = 0$, because $c(z)$ is a Grassmann field. There is a similar form for $\bar{j}_B$. The form (2.356) for the BRST current is very useful if we want to extend it to the superstring theory.

Now we are gonna calculate the OPEs of the BRST current with the ghost fields and the bosonic fields. We will use the previous result for the OPEs of bc system with $\lambda = 2$. Therefore we have

$$\begin{aligned} j_B(z)\,b(w) &= :c(z)\,T^X(z):b(w) + :b(z)\,c(z)\,\partial c(z):b(w) + \frac{3}{2}\partial^2 c(z)\,b(w) \\ &\sim T^X(z)\,\langle c(z)\,b(w)\rangle + b(z)\,c(z)\,\langle \partial c(z)\,b(w)\rangle \\ &\qquad - b(z)\,\partial c(z)\,\langle c(z)\,b(w)\rangle + \frac{3}{2}\left\langle \partial^2 c(z)\,b(w)\right\rangle \\ &\sim \frac{T^X(z)}{z-w} - \frac{b(z)\,c(z)}{(z-w)^2} - \frac{b(z)\,\partial c(z)}{z-w} + \frac{3}{(z-w)^3} + \ldots \quad . \end{aligned}$$

We expand the depending on z operator up to regular terms, that is

$$\begin{aligned}
j_B(z)\, b(w) &\sim \frac{T^X(w)}{z-w} - \frac{b(w)\, c(w)}{(z-w)^2} - \frac{\partial(b(w)\, c(w))}{z-w} - \frac{b(w)\, \partial c(w)}{z-w} + \frac{3}{(z-w)^3} + \ldots \\
&\sim \frac{T^X(w)}{z-w} + \frac{j^g(w)}{(z-w)^2} + \frac{T^{ghost}(w)}{z-w} + \frac{3}{(z-w)^3} + \ldots \\
&\sim \frac{T^{X+g}(w)}{z-w} + \frac{j^g(w)}{(z-w)^2} + \frac{3}{(z-w)^3} + \ldots \quad . \qquad (2.357)
\end{aligned}$$

In the same way we can compute

$$\begin{aligned}
j_B(z)\, c(w) &= :c(z)\, T^X(z): c(w) + :b(z)\, c(z)\, \partial c(z): c(w) + \frac{3}{2}\partial^2 c(z)\, c(w) \\
&\sim c(z)\, \partial c(z) \langle b(z)\, c(w) \rangle + \ldots \\
&\sim \frac{c(z)\, \partial c(z)}{z-w} + \ldots \\
&\sim \frac{c(w)\, \partial c(w)}{z-w} + \ldots \quad . \qquad (2.358)
\end{aligned}$$

Clearly $\bar{j}_B(z)\, b(w) \sim$ regular terms. We now calculate the OPE with a general bosonic primary field $\phi(w)$, so we have

$$\begin{aligned}
j_B(z)\, \phi(w) &= :c(z)\, T^X(z): \phi(w) + :b(z)\, c(z)\, \partial c(z): \phi(w) + \frac{3}{2}\partial^2 c(z)\, \phi(w) \\
&\sim c(z) \left\langle T^X(z)\, \phi(w) \right\rangle \\
&\sim \frac{hc(z)\, \phi(w)}{(z-w)^2} + \frac{c(z)\, \partial\phi(w)}{z-w} + \ldots \\
&\sim \frac{hc(w)\, \phi(w)}{(z-w)^2} + \frac{h\partial c(w)\, \phi(w)}{z-w} + \frac{c(w)\, \partial\phi(w)}{z-w} + \ldots \\
&\sim \frac{hc(w)\, \phi(w)}{(z-w)^2} + \frac{1}{z-w}\left(h\partial c(w)\, \phi(w) + c(w)\, \partial\phi(w)\right). \qquad (2.359)
\end{aligned}$$

We define now the BRST charge $\mathcal{Q}_B$ as usually

$$\mathcal{Q}_B = \frac{1}{2\pi i} \oint \left(dz\, j_B - d\bar{z}\, \bar{j}_B\right). \qquad (2.360)$$

From the OPE (2.357) we calculate the following anticommutator

$$\begin{aligned}\{\mathcal{Q}_B, b_n\} &= \oint_w \frac{dz}{2\pi i}\oint_0 \frac{dw}{2\pi i} j_B(z)\, b(w)\, w^{n+1} \\ &= \oint_0 \frac{dw}{2\pi i} w^{n+1} \oint_w \frac{dz}{2\pi i}\left(\frac{T^{X+g}(w)}{z-w} + \frac{j^g(w)}{(z-w)^2} + \frac{3}{(z-w)^3} + \ldots\right) \\ &= \oint_0 \frac{dw}{2\pi i} w^{n+1} T^{X+g}(w) \\ &= L_n^X + L_n^g. \end{aligned} \tag{2.361}$$

From (2.360) we find the mode expansion of the BRST charge,

$$\begin{aligned}\mathcal{Q}_B &= \frac{1}{2\pi i}\oint \left(dz\, j_B(z) - d\bar{z}\, \bar{j}_B(\bar{z})\right) \\ &= \oint \frac{dz}{2\pi i}\left(\sum_{m,n}\frac{c_m L_n^X}{z^{m+n+1}} - \sum_{p,k,l}(l-1)\frac{b_p c_k c_l}{z^{p+k+l+1}}\right) - \text{antiholomorphic part} \\ &= \sum_{m,n} c_m L_n^X \delta_{m+n} - \sum_{p,k,l}(l-1)\, b_p c_k c_l \delta_{p+k+l} + \text{antiholomorphic part} \\ &= \sum_m c_m L_{-m}^X - \sum_{k,l}(l-1)\, b_{-k-l} c_k c_l + \text{antiholomorphic part} \\ &= \sum_m c_m L_{-m}^X - \frac{1}{2}\sum_{k,l}(l-1)\, b_{-k-l} c_k c_l - \frac{1}{2}\sum_{k,l}(l-1)\, b_{-k-l} c_k c_l + \text{anti. part} \\ &= \sum_m c_m L_{-m}^X - \frac{1}{2}\sum_{k,l}(l-1)\, b_{-k-l} c_k c_l - \frac{1}{2}\sum_{l,k}(k-1)\, b_{-l-k} c_l c_k + \text{anti. part} \\ &= \sum_m c_m L_{-m}^X - \frac{1}{2}\sum_{k,l}(l-1)\, b_{-k-l} c_k c_l + \frac{1}{2}\sum_{l,k}(k-1)\, b_{-l-k} c_k c_l + \text{anti. part} \\ &= \sum_m c_m L_{-m}^X + \frac{1}{2}\sum_{k,l}(k-l)\, b_{-k-l} c_k c_l + \text{anti. part} \\ &= \sum_m c_m L_{-m}^X + \frac{1}{2}\sum_{m,n}(m-n) : c_m c_n b_{-m-n} : + \text{anti. part} \quad . \end{aligned} \tag{2.362}$$

Then, including the antiholomorphic part explicitly and the normal ordering, we have

$$\begin{aligned}\mathcal{Q}_B &= \sum_m \left(c_m L_{-m}^X + \bar{c}_m \bar{L}_{-m}^X\right) \\ &\quad + \sum_{m,n}\frac{(m-n)}{2} : \left(c_m c_n b_{-m-n} + \bar{c}_m \bar{c}_n \bar{b}_{-m-n}\right) : + a^B\left(c_0 + \bar{c}_0\right), \end{aligned} \tag{2.363}$$

where the normal ordering constant a^B is such that $a^B = a^g = -1$. Comparing the above expression with (2.288), we also can write the mode expansion of $\mathcal{Q}_B$ as

$$\mathcal{Q}_B = \sum_m \left(c_m L^X_{-m} + \bar{c}_m \bar{L}^X_{-m} \right) + \frac{1}{2} \sum_m \left(c_m L^g_{-m} + \bar{c}_m \bar{L}^g_{-m} \right) + a^B \left(c_0 + \bar{c}_0 \right). \tag{2.364}$$

Now we will show that the charge $\mathcal{Q}_B$ is not nilpotent when $c^X \neq 26$. That is, we need to show

$$\{ \mathcal{Q}_B, \mathcal{Q}_B \} = 0 \qquad \text{only if} \qquad c^X = 26. \tag{2.365}$$

Therefore we have to use the definition (2.360) and compute the $j_B(z)\, j_B(w)$ OPE since the $\bar{j}_B(\bar{z})\, \bar{j}_B(\bar{w})$ will be the same but replacing the corresponding antiholomorphic components. Thus we have

$$\begin{aligned}
j_B(z)\, j_B(w) &= \left(c(z)\, T^X(z) + :b(z)\, c(z)\, \partial c(z): + \frac{3}{2} \partial^2 c(z) \right) \\
&\quad \times \left(c(w)\, T^X(w) + :b(w)\, c(w)\, \partial c(w): + \frac{3}{2} \partial^2 c(w) \right) \\
&= :c(z)\, c(w):\, T^X(z)\, T^X(w) + c(z)\, T^X(z) :b(w)\, c(w)\, \partial c(w): + \\
&\quad :b(z)\, c(z)\, \partial c(z): c(w)\, T^X(w) + \left(:b(z)\, c(z)\, \partial c(z):\right) \left(:b(w)\, c(w)\, \partial c(w):\right) \\
&\quad + \frac{3}{2} :b(z)\, c(z)\, \partial c(z): \partial^2 c(w) + \frac{3}{2} \partial^2 c(z) :b(w)\, c(w)\, \partial c(w): + \ldots \quad .
\end{aligned} \tag{2.366}$$

We will calculate term by term in order to be clear with these count. Therefore

$$\begin{aligned}
:c(z)\, c(w):\, T^X(z)\, T^X(w) &= :c(z)\, c(w): \left(\frac{c^X/2}{(z-w)^4} + \frac{2T^X(w)}{(z-w)^2} + \frac{\partial T^X(w)}{(z-w)} \right) \\
&\sim \partial c(w)\, c(w) \left(\frac{c^X/2}{(z-w)^3} + \frac{2T^X(w)}{z-w} \right) \\
&\quad + \frac{1}{4} \partial^2 c(w)\, c(w)\, \frac{c^X}{(z-w)^2} \\
&\quad + \frac{1}{12} \partial^3 c(w)\, c(w) \left(\frac{c^X}{z-w} \right).
\end{aligned} \tag{2.367}$$

In the second line we performed a Taylor expansion. The next two terms are

$$\begin{aligned}
c(z)\, T^X(z) :b(w)\, c(w)\, \partial c(w): &= T^X(z) \langle c(z)\, b(w) \rangle :c(w)\, \partial c(w): \\
&\sim \frac{T^X(w)\, c(w)\, \partial c(w)}{z-w}
\end{aligned} \tag{2.368}$$

and

$$
\begin{aligned}
:b(z)\,c(z)\,\partial c(z):c(w)\,T^X(w) &= :c(z)\,\partial c(z):\langle b(z)\,c(w)\rangle\,T^X(w) \\
&\sim \frac{T^X(w)\,c(w)\,\partial c(w)}{z-w}.
\end{aligned}
\tag{2.369}
$$

Now we go with the fourth term in (2.366), that is

$$
\begin{aligned}
:b(z)\,c(z)\,\partial c(z)::b(w)\,c(w)\,\partial c(w): &= -\frac{:c(z)\,\partial c(z)\,b(w)\,\partial c(w):}{z-w}+\frac{:c(z)\,\partial c(z)\,b(w)\,c(w):}{(z-w)^2} \\
&\quad -\frac{:b(z)\,c(z)\,c(w)\,\partial c(w):}{(z-w)^2}-\frac{:b(z)\,\partial c(z)\,c(w)\,\partial c(w):}{z-w} \\
&\quad +\frac{:\partial c(z)\,\partial c(w):}{(z-w)^2}-\frac{:\partial c(z)\,c(w):}{(z-w)^3} \\
&\quad +\frac{:c(z)\,\partial c(w):}{(z-w)^3}-\frac{:c(z)\,c(w):}{(z-w)^4} \\
&\sim +\frac{\partial c(w)\,\partial c(w)}{(z-w)^2}+\frac{\partial^2 c(w)\,\partial c(w)}{z-w} \\
&\quad -\frac{\partial c(w)\,c(w)}{(z-w)^3}-\frac{\partial^2 c(w)\,c(w)}{(z-w)^2}-\frac{1}{2}\frac{\partial^3 c(w)\,c(w)}{z-w} \\
&\quad +\frac{c(w)\,\partial c(w)}{(z-w)^3}+\frac{1}{2}\frac{\partial^2 c(w)\,\partial c(w)}{z-w} \\
&\quad -\frac{\partial c(w)\,c(w)}{(z-w)^3}-\frac{1}{2}\frac{\partial^2 c(w)\,c(w)}{(z-w)^2}-\frac{1}{6}\frac{\partial^3 c(w)\,c(w)}{z-w},
\end{aligned}
$$

where we have used the fact that $b(w)$ and $c(w)$ are Grassmann fields, and then its square and those of their derivatives vanish. This gives

$$
\begin{aligned}
:b(z)\,c(z)\,\partial c(z)::b(w)\,c(w)\,\partial c(w): \sim\ & \frac{3}{2}\frac{\partial^2 c(w)\,\partial c(w)}{z-w}-3\frac{\partial c(w)\,c(w)}{(z-w)^3} \\
& -\frac{3}{2}\frac{\partial^2 c(w)\,c(w)}{(z-w)^2}-\frac{2}{3}\frac{\partial^3 c(w)\,c(w)}{z-w}.
\end{aligned}
\tag{2.370}
$$

Now the fifth term,

$$\begin{aligned}
\frac{3}{2}:b(z)\,c(z)\,\partial c(z):\partial^{2}c(w) &= \frac{3}{2}:c(z)\,\partial c(z):\left\langle b(z)\,\partial^{2}c(w)\right\rangle \\
&= \frac{3}{2}:c(z)\,\partial c(z):\frac{2}{(z-w)^{3}} \\
&\sim 3\frac{c(w)\,\partial c(w)}{(z-w)^{3}}+3\frac{\partial\left(c(w)\,\partial c(w)\right)}{(z-w)^{2}}+\frac{3}{2}\frac{\partial^{2}\left(c(w)\,\partial c(w)\right)}{z-w} \\
&\sim 3\frac{c(w)\,\partial c(w)}{(z-w)^{3}}+3\frac{c(w)\,\partial^{2}c(w)}{(z-w)^{2}}+\frac{3}{2}\frac{\partial c(w)\,\partial^{2}c(w)}{z-w} \\
&\qquad+\frac{3}{2}\frac{c(w)\,\partial^{3}c(w)}{z-w}. \qquad (2.371)
\end{aligned}$$

Finally, the last term,

$$\begin{aligned}
\frac{3}{2}\partial^{2}c(z):b(w)\,c(w)\,\partial c(w): &= :c(w)\,\partial c(w):\frac{3}{2}\left\langle \partial^{2}c(z)\,b(w)\right\rangle \\
&\sim 3\frac{c(w)\,\partial c(w)}{(z-w)^{3}}. \qquad (2.372)
\end{aligned}$$

Replacing all these results in (2.366) we get

$$\begin{aligned}
j_{B}(z)\,j_{B}(w) \sim{}& \frac{\partial c(w)\,c(w)\,\left(c^{X}/2\right)}{(z-w)^{3}}+\frac{1}{4}\partial^{2}c(w)\,c(w)\,\frac{c^{X}}{(z-w)^{2}} \\
&+\partial^{3}c(w)\,c(w)\left(\frac{c^{X}/12}{z-w}\right)-3\frac{\partial c(w)\,c(w)}{(z-w)^{3}} \\
&-\frac{3}{2}\frac{\partial^{2}c(w)\,c(w)}{(z-w)^{2}}-\frac{2}{3}\frac{\partial^{3}c(w)\,c(w)}{z-w}+3\frac{c(w)\,\partial c(w)}{(z-w)^{3}} \\
&+3\frac{c(w)\,\partial^{2}c(w)}{(z-w)^{2}}+\frac{3}{2}\frac{c(w)\,\partial^{3}c(w)}{z-w}+3\frac{c(w)\,\partial c(w)}{(z-w)^{3}} \\
\sim{}& \frac{\partial c(w)\,c(w)}{(z-w)^{3}}\left(\frac{c^{X}}{2}-9\right)+\frac{\partial^{2}c(w)\,c(w)}{(z-w)^{2}}\left(\frac{c^{X}}{4}-\frac{3}{2}-3\right) \\
&+\frac{\partial^{3}c(w)\,c(w)}{z-w}\left(\frac{c^{X}}{12}-\frac{2}{3}-\frac{3}{2}\right) \\
\sim{}& \frac{\partial c(w)\,c(w)}{(z-w)^{3}}\left(\frac{c^{X}-18}{2}\right)+\frac{\partial^{2}c(w)\,c(w)}{(z-w)^{2}}\left(\frac{c^{X}-18}{4}\right)+\frac{\partial^{3}c(w)\,c(w)}{z-w}\left(\frac{c^{X}-26}{12}\right).
\end{aligned} \qquad (2.373)$$

Since

$$\left\{\mathcal{Q}_{B},\mathcal{Q}_{B}\right\}=\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\,j_{B}(z)\,j_{B}(w)+\oint\frac{d\bar{z}}{2\pi i}\oint\frac{d\bar{w}}{2\pi i}\,\bar{j}_{B}(\bar{z})\,\bar{j}_{B}(\bar{w}), \qquad (2.374)$$

we note that $\mathcal{Q}_B$ is nilpotent only if the simple pole in (2.373) does not exist, that is, if and only if $c^X = d = 26$. This is the celebrated critical dimension.

2.14 BRST Cohomology of the string

2.14.1 Little group

Consider a group G acting on a space X, then some elements $g \in G$ may fix a point $x \in X$, these elements form a subgroup G_x, called isotropy group or **little group**, defined by

$$G_x = \{g \in G : \quad gx = x\} . \tag{2.375}$$

For instance, consider the group $SO\,(3)$ of all rotations of a sphere S^2, and $x = (0, 0, 1)$. The rotations which does not change x correspond to action of the circle group S^1 on the equator, that is $SO\,(2)$. Now, consider a massive particle moving into a $d-$dimensional Minkowski space. Since any massive particle necessary move slower that the light, we can make a Lorentz boost a go to its rest frame. In this frame the particle momentum is $k^\mu = (m, 0, ..., 0)$ with $k^2 = -m^2$ whose little group is $SO\,(d-1)$. This means that massive string excitations can be classified by representations of $SO\,(d-1)$. In massless particles case, since they satisfy $k^2 = 0$, we can choose a frame in which its momentum is $k^\mu = (E, 0, ..., E)$. The little group of this vector is the group of motion in $(d-2)\,-$dimensional Euclidean space, $E\,(d-2)$. Massless string states form, however, representations of its connected component $SO\,(d-2) \subset E\,(d-2)$.

2.14.2 open string spectrum

Let us now look at the BRST cohomology at the lowest levels of the string. The inner product is defined by specifying

$$\begin{aligned}
\left(\alpha^\mu_m\right)^\dagger &= \alpha^\mu_{-m}, \qquad \left(\bar{\alpha}^\mu_m\right)^\dagger = \bar{\alpha}^\mu_{-m},\\
\left(b^\mu_m\right)^\dagger &= b^\mu_{-m}, \qquad \left(\bar{b}^\mu_m\right)^\dagger = \bar{b}^\mu_{-m},\\
\left(c^\mu_m\right)^\dagger &= c^\mu_{-m}, \qquad \left(\bar{c}^\mu_m\right)^\dagger = \bar{c}^\mu_{-m}.
\end{aligned} \tag{2.376}$$

The Hermiticity of the BRST charge requires that the ghosts fields be Hermitian as well. The Hermiticity of the ghosts zero modes forces the inner products of the ground states to take the form

$$\text{open string:} \qquad \left\langle 0; k\, |c_0|\, 0; k' \right\rangle = (2\pi)^{26}\, \delta^{26}\left(k - k'\right), \tag{2.377}$$

$$\text{closed string:} \qquad \left\langle 0; k\, |\bar{c}_0 c_0|\, 0; k' \right\rangle = i\,(2\pi)^{26}\, \delta^{26}\left(k - k'\right). \tag{2.378}$$

Here $|0;k\rangle$ denotes $|0;k\rangle \otimes |\downarrow\rangle$ with momentum k. The c_0 and $\bar{c}_0$ insertions are necessary for a nonzero result. The factor of i is needed in the ghost zero modes inner product for Hermiticity. Inner product of general states are then obtained by using the commutations relations and the adjoint (2.376).

We begin with the open string spectrum. We will claim that physical states must satisfy the additional condition

$$b_0 |\psi\rangle = 0. \tag{2.379}$$

From eqs.(2.333) and (2.361) this also implies

$$L_0 |\psi\rangle = \{\mathcal{Q}_B, b_0\} |\psi\rangle = 0. \tag{2.380}$$

The operator L_0 is

$$L_0 = L_0^X + L_0^g = \alpha' \left(p^2 + m^2\right). \tag{2.381}$$

From equation (2.290) we have

$$\begin{aligned} L_0^g &= -\sum_{n=-\infty}^{\infty} n : b_n c_{-n} : + a^g \\ &= -\sum_{n=1}^{\infty} n : b_n c_{-n} : - \sum_{n=-\infty}^{1} n : b_n c_{-n} : + a^g \\ &= \underbrace{\sum_{n=1}^{\infty} n c_{-n} b_n}_{N_c} + \underbrace{\sum_{n=1}^{\infty} n b_{-n} c_n}_{N_b} + a^g, \end{aligned} \tag{2.382}$$

then

$$\begin{aligned} L_0 &= \frac{\alpha_0^2}{2} + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{n\,\mu} + a^X + \sum_{n=1}^{\infty} n c_{-n} b_n + \sum_{n=1}^{\infty} n b_{-n} c_n + a^g \\ &= \alpha' p^2 + N_X + N_b + N_c - 1, \end{aligned} \tag{2.383}$$

[23]where $N_X = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$. It is not hard to show that $\left[N_b, b_m^\dagger\right] = m b_m^\dagger$, $\left[N_c, c_m^\dagger\right] = m c_m^\dagger$ and $\left[N_X, \alpha_{-m}^{\mu}\right] = m \alpha_{-n}^{\mu}$. Therefore,

$$\alpha' m^2 = N_b + N_c + N_X - 1 = N - 1, \tag{2.384}$$

where $N = N_b + N_c + N_X$. Thus, the L_0 condition (2.380) determines the mass spectrum of the string. BRST invariance with (2.379) implies that every string state is on the mass shell. We will denote $\hat{\mathcal{H}}$ the space of sates satisfying (2.379) and (2.380).

[23]For the open string $\alpha_0 = \sqrt{2\alpha'}p$, while for closed string $\alpha_0 = \sqrt{\frac{\alpha'}{2}}p$.

The inner products (2.378) are not well defined because the ghost zero modes give zero, while $\delta^{26}(k-k')$ contains a factor $\delta(0)$ since the momentum is restricted to the mass shell. Therefore, we use in $\hat{\mathcal{H}}$ a reduced inner product $\langle||\rangle$ in which we simply ignore the X^0 and ghost zero modes. Let us now see the first levels of the $d=26$ flat spacetime string. At the lowest level, $N=0$, we have

$$m^2\,|0;\boldsymbol{k}\rangle = -\frac{1}{\alpha'}\,|0;\boldsymbol{k}\rangle\,, \qquad -k^2 = -\frac{1}{\alpha'}. \tag{2.385}$$

[24]This state is BRST invariant,

$$\begin{aligned}\mathcal{Q}_B\,|0;\boldsymbol{k}\rangle &= \left(c_0 L_0^X - c_0\right)|0;\boldsymbol{k}\rangle \\ &= c_0\left(\alpha' k^2 - 1\right)|0;\boldsymbol{k}\rangle \\ &= 0,\end{aligned} \tag{2.386}$$

because of the mass shell condition. Therefore, there are no exact states at this level, so each of these states correspond to a cohomology class. The associated particle with these states is called the Tachyon. At the level, $N=1$, there are $26+2$ states,

$$|\psi_1\rangle = \left(\xi_\mu \alpha^\mu_{-1} + \beta b_{-1} + \gamma c_{-1}\right)|0;\boldsymbol{k}\rangle\,, \qquad -k^2 = 0, \tag{2.387}$$

depending on a 26-vector ξ_μ and two constants, β and γ. The norm of this state is

$$\begin{aligned}\langle\psi_1|\psi_1\rangle &= \left\langle 0;\boldsymbol{k}\right|\left(\eta_{\mu\nu}\xi^{*\nu}\alpha^\mu_1 + \beta^* b_1 + \gamma^* c_1\right)\left(\eta_{\rho\sigma}\xi^\rho\alpha^\sigma_{-1} + \beta b_{-1} + \gamma c_{-1}\right)\left|0;\boldsymbol{k}'\right\rangle \\ &= \left\langle 0;\boldsymbol{k}\right|\left(\eta_{\mu\nu}\eta_{\rho\sigma}\xi^{*\nu}\xi^\rho\alpha^\mu_1\alpha^\sigma_{-1} + \beta^*\gamma b_1 c_{-1} + \gamma^*\beta c_1 b_{-1}\right)\left|0;\boldsymbol{k}'\right\rangle \\ &= \left\langle 0;\boldsymbol{k}\right|\left(\eta_{\mu\nu}\eta_{\rho\sigma}\xi^{*\nu}\xi^\rho\left(\eta^{\mu\sigma} + \alpha^\sigma_{-1}\alpha^\mu_1\right) + \beta^*\gamma\left(1 - c_{-1}b_1\right) + \gamma^*\beta\left(1 - b_{-1}c_1\right)\right)\left|0;\boldsymbol{k}'\right\rangle \\ &= \left\langle 0;\boldsymbol{k}\right|\left(\eta_{\mu\nu}\delta^\mu_\rho\xi^{*\nu}\xi^\rho + \beta^*\gamma + \gamma^*\beta\right)\left|0;\boldsymbol{k}'\right\rangle \\ &= \left(\xi^*\cdot\xi + \beta^*\gamma + \gamma^*\beta\right)\left\langle 0;\boldsymbol{k}|0;\boldsymbol{k}'\right\rangle.\end{aligned} \tag{2.388}$$

In the second line we have written only the non commuting or non anticommuting products, in third line we have used the equations (2.150) and (2.281) to put the annihilation operators to the right. Going to an orthogonal basis, there are 26 positive norm states

$$\xi_i\alpha^i_{-1}\,|0;\boldsymbol{k}\rangle \quad \text{with} \quad i\in\{1,...,25\}\,, \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\left(b_{-1}+c_{-1}\right)|0;\boldsymbol{k}\rangle\,, \tag{2.389}$$

[24]Note also that the state $c_0\,|0;\boldsymbol{k}\rangle$ obeys the mass-shell conditions, however, since $b_0c_0\,|0;\boldsymbol{k}\rangle = |0;\boldsymbol{k}\rangle$ we excluded this state from the physical spectrum.

and 2 negative norm states

$$\alpha^0_{-1}\,|0;\boldsymbol{k}\rangle \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\,(b_{-1}-c_{-1})\,|0;\boldsymbol{k}\rangle\,. \tag{2.390}$$

The BRST condition is, from (2.363) (holomorphic part)[25]

$$\begin{aligned}
0 = \mathcal{Q}_B\,|\psi_1\rangle &= \Big(c_{-1}L^X_1 + c_0 L^X_0 + c_1 L^X_{-1} \\
&\quad + c_0 c_{-1} b_1 + c_1 c_0 b_{-1} - c_0\big)\,|\psi_1\rangle \\
&= \left(c_{-1}\alpha_0\cdot\alpha_1 + c_1\alpha_0\cdot\alpha_{-1}\right)|\psi_1\rangle \\
&= \sqrt{2\alpha'}\,\left(c_{-1}k\cdot\alpha_1 + c_1 k\cdot\alpha_{-1}\right) \\
&\qquad \times\left(\xi\cdot\alpha_{-1} + \beta b_{-1} + \gamma c_{-1}\right)|0;\boldsymbol{k}\rangle \\
&= \sqrt{2\alpha'}\,\left(c_{-1}k\cdot\xi + \beta k\cdot\alpha_{-1}\right)|0;\boldsymbol{k}\rangle\,,
\end{aligned} \tag{2.391}$$

where we selected only the non commuting terms with the operators forming the state $|\psi_1\rangle$. Physical states therefore must satisfy $k\cdot\xi = \beta = 0$. Thus, from (2.387), the physical states are those of the form

$$|\psi_1\rangle = \left(\xi_\mu\alpha^\mu_{-1} + \gamma c_{-1}\right)|0;\boldsymbol{k}\rangle\,, \qquad k^2 = 0, \qquad k\cdot\xi = 0. \tag{2.392}$$

In particular, we can consider massless open string states with momentum spacetime $k^\mu = (E, E, ..., 0)$, therefore, an orthogonal basis satisfying the physical conditions $k\cdot\xi = \beta = 0$ is

$$c_{-1}\,|0;\boldsymbol{k}\rangle\,, \quad k\cdot\alpha_{-1}\,|0;\boldsymbol{k}\rangle = E\left(-\alpha^0_{-1}+\alpha^1_{-1}\right)|0;\boldsymbol{k}\rangle\,, \quad \alpha^i_{-1}\,|0;\boldsymbol{k}\rangle \quad i\in\{2,...,25\}\,. \tag{2.393}$$

The two first states above are exact, since they are of the form (2.391) and therefore have zero norm, then we exclude these two states from the proper physical spectrum. In turn, for the momentum spacetime $k^\mu = (E, E, ..., 0)$ a basis for the physical states is

$$\alpha^i_{-1}\,|0;\boldsymbol{k}\rangle\,, \qquad i\in\{2,...,25\}\,. \tag{2.394}$$

We have then that the proper physical states at $N = 1$ are the cohomology classes of states of the form

$$\xi\cdot\alpha_{-1}\,|0;\boldsymbol{k}\rangle\,, \qquad k^2 = 0, \qquad k\cdot\xi = 0. \tag{2.395}$$

25

$$\begin{aligned}
\mathcal{Q}_B &= \sum_m c_m L^X_{-m} + \sum_{m,n}\frac{(m-n)}{2} : c_m c_n b_{-m-n} : + a^B c_0 \\
&= ... + c_{-2}L^X_2 + c_{-1}L^X_1 + c_0 L^X_0 + c_1 L^X_{-1} + c_2 L^X_{-2} ... + c_1 c_0 b_{-1} + c_0 c_{-1} b_1 - c_{-2}c_0 b_2 + ... + a^B c_0
\end{aligned}$$

We say that two states $\xi \cdot \alpha_{-1} |0;\boldsymbol{k}\rangle$ and $\xi' \cdot \alpha_{-1} |0;\boldsymbol{k}'\rangle$ are in the same cohomology if there exists a constant β' such that $\xi'_\mu = \xi_\mu + \beta' k_\mu$.

Since these states are massless with polarization vector ξ_μ such that $k \cdot \xi = 0$, it is natural identify these states with the states of the photon.

2.14.3 Closed string spectrum

The generalization to the closed string is simple. We restrict attention to the space $\hat{\mathcal{H}}$ of states satisfying

$$b_0 |\psi\rangle = \bar{b}_0 |\psi\rangle = 0, \tag{2.396}$$

implying also

$$L_0 |\psi\rangle = \bar{L}_0 |\psi\rangle = 0, \tag{2.397}$$

where we have

$$L_0 = \frac{\alpha'}{4}\left(p^2 + m^2\right), \qquad \bar{L}_0 = \frac{\alpha'}{4}\left(p^2 + \bar{m}^2\right), \tag{2.398}$$

and

$$\frac{\alpha'}{4} m^2 = N_b + N_c + N_X - 1 = N - 1, \tag{2.399}$$

$$\frac{\alpha'}{4} \bar{m}^2 = \bar{N}_b + \bar{N}_c + \bar{N}_X - 1 = \bar{N} - 1. \tag{2.400}$$

Physical conditions imply the level matching condition, $N = \bar{N}$. At the lowest state, $N = \bar{N} = 0$, we have again the Tachyonic state with mass $m^2 = -4/\alpha$.

At the next level, $N = \bar{N} = 1$, the most general state takes the form

$$\begin{aligned} |\psi_1\rangle = & \left(G_{\mu\nu}\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1} + e\cdot\alpha_{-1}\bar{b}_{-1} + \bar{e}\cdot\bar{\alpha}_{-1}b_{-1}\right. \\ & + f\cdot\alpha_{-1}\bar{c}_{-1} + \bar{f}\cdot\bar{\alpha}_{-1}c_{-1} + \beta b_{-1}\bar{b}_{-1} \\ & \left. + \gamma c_{-1}\bar{c}_{-1} + \zeta b_{-1}\bar{c}_{-1} + \eta\bar{b}_{-1}c_{-1}\right)|0;\boldsymbol{k}\rangle, \qquad -k^2 = 0, \end{aligned} \tag{2.401}$$

so we have $(26)^2 + 4(26) + 4 = 784$ states, of these 104 are negative norm states. The BRST invariant conditions are obtained in analogy to (2.391) but including the antiholomorphic modes, thus, it is obtained (see [16]) the general form for the exact states

$$\begin{aligned} 0 = \mathcal{Q}_B |\psi_1\rangle = & \sqrt{\frac{\alpha'}{2}}\left\{\left(e_\mu k_\nu + \bar{e}_\nu k_\mu\right)\alpha^\mu_{-1}\alpha^\nu_{-1} + \beta\left(k\cdot\alpha_1\bar{b}_{-1} - k\cdot\bar{\alpha}_1 b_{-1}\right)\right. \\ & + \left(\zeta k_\mu + G_{\mu\nu}k^\nu\right)\alpha^\mu_{-1}\bar{c}_{-1} + \left(\eta k_\mu + G_{\mu\nu}k^\nu\right)\bar{\alpha}^\mu_{-1}c_{-1} \\ & \left. + e\cdot k b_{-1}\bar{c}_{-1} + \bar{e}\cdot k\bar{b}_{-1}c_{-1} + \left(f_\mu + \bar{f}_\mu\right)k^\mu\bar{c}_{-1}c_{-1}\right)|0;\boldsymbol{k}\rangle . \end{aligned} \tag{2.402}$$

BRST invariance amounts 104 conditions, just the same as the number of negative norm states. If we set $k^\mu = (E, E, ..., 0)$, the BRST conditions lead to a basis of 680 physical non-negative norm states, however, between them there are spurious states of zero norm with a general form given by (2.402), provided we remove the $\alpha^i_{-1}\bar{\alpha}^j_{-1}$ terms. Therefore, the most general physical state, at this level, has the form

$$\xi_{ij}\alpha^i_{-1}\bar{\alpha}^j_{-1}\,|0;\boldsymbol{k}\rangle\,, \qquad i \in \{2, ..., 25\}\,, \tag{2.403}$$

and transforms under the transverse rotation group $SO\,(d-2) = SO\,(24)$ as a 2-tensor, however, this representation is reducible and can be decomposed in irreducible representations as follows

$$\xi_{ij} = \underbrace{\xi_{(ij)}}_{\text{symmetric traceless}} + \underbrace{\xi_{[ij]}}_{\text{antisymmetric}} + \underbrace{\xi^{(0)}}_{\text{trace part}}\,. \tag{2.404}$$

That is

$$\begin{aligned}\alpha^i_{-1}\bar{\alpha}^j_{-1}\,|0;\boldsymbol{k}\rangle = \alpha^{[i}_{-1}\bar{\alpha}^{j]}_{-1}\,|0;\boldsymbol{k}\rangle + \left[\alpha^{(i}_{-1}\bar{\alpha}^{j)}_{-1} - \tfrac{1}{24}\delta^{ij}\alpha^k_{-1}\bar{\alpha}^k_{-1}\right]|0;\boldsymbol{k}\rangle \\ +\tfrac{1}{24}\delta^{ij}\alpha^k_{-1}\bar{\alpha}^k_{-1}\,|0;\boldsymbol{k}\rangle\,,\end{aligned} \tag{2.405}$$

where indices in parentheses and brackets are symmetrized and anti-symmetrized, respectively. The symmetric traceless tensor $\xi_{(ij)}$ is identified with the *graviton*, a massless spin two particle. The antisymmetric tensor $\xi_{[ij]} \equiv B_{ij}$ with the degrees of freedom of a tensor field called the *Kalb-Ramond field*. Last, the remaining massless scalar field $\xi^{(0)}$ is called the *Dilaton.*

Chapter 3

Superstrings

We now present a brief revision of the superstring action, the goal of this short chapter is to show that the critical dimension reduces to $d = 10$. In doing so we see that the superstring action is invariant under supersymmetric transformation, that is, transformations that mix bosonic and fermionic degrees of freedom. The generator of such transformations is treated in a similar way to the energy-momentum tensor and its OPE relations with T and itself generate the Ramond algebra and the Neveu-Schwarz algebra, depending on the periodicity of the fermionic fields. This idea arises naturally if we try to include spacetime fermions in the spectrum, and by guesswork we are led to superconformal symmetry. In this chapter we discuss the (1,1) superconformal algebra.

The main references to study the super conformal field theory is [19], standard literature for superstring are [17] and [18], some physical concepts and mathematical definitions were studied from [20] and the final chapters of Lüst and Theisen books.

3.1 The superconformal algebra

In the bosonic string theory the physical state condition

$$L_0 |\psi\rangle = 0 \tag{3.1}$$

and also from $\bar{L}_0 |\psi\rangle = 0$ in the closed string, implies the mass-shell condition

$$p_\mu p^\mu + m^2 = 0, \tag{3.2}$$

which, is the Klein-Gordon equation in momentum space. This is one way to motivate the following generalization, we require that physical condition implies the Dirac equation

$$ip_\mu \Gamma^\mu + m = 0. \tag{3.3}$$

For the bosonic string, L_0 and $\bar{L}_0$ are the center of mass modes of the world-sheet energy-momentum tensor $(T_B, \bar{T}_B)$. A subscript "B" for bosonic has been added to distinguish these for the fermionic currents, which we introduce now. Then, we need new conserved quantities T_F, $\bar{T}_F$, whose center-of-mass modes give the Dirac equation, and which play the same role as T_B, and $\bar{T}_B$ in the bosonic string. Noting further that the space-time momenta p^μ are the center of mass modes of the world-sheet current $\left(\partial X^\mu, \bar{\partial} X^\mu\right)$, and it is natural to guess that the gamma matrices with algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{3.4}$$

are the center-of-mass modes of an anticommuting world-sheet field ψ^μ, that is, the gamma matrices will be proportional to the zero modes of ψ^μ. This previous analysis leads us to consider the world-sheet action

$$S = \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \bar{\psi}^\mu \partial \bar{\psi}_\mu\right). \tag{3.5}$$

The fields ψ^μ and $\bar{\psi}^\mu$ are respectively holomorphic and antiholomorphic and the OPEs are

$$\psi^\mu(z)\,\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}, \qquad \bar{\psi}^\mu(\bar{z})\,\bar{\psi}^\nu(\bar{w}) \sim \frac{\eta^{\mu\nu}}{\bar{z}-\bar{w}}. \tag{3.6}$$

This OPEs are justified because the fermionic part of this action correspond to a bc system with $\lambda = \frac{1}{2}$, therefore $h_b = h_c = \frac{1}{2}$, for which the central charge $c = 1$. Then, we use the notation $b \longrightarrow \psi$, $c \longrightarrow \bar{\psi}$. For this case the bc CFT can be split in two in a conformally invariant way,

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \qquad \bar{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \tag{3.7}$$

and the ghost action takes the form

$$S = \frac{1}{2\pi}\int d^2z\, b\bar{\partial}c \longrightarrow \frac{1}{2\pi}\int d^2z\,\left(\psi_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\psi_2\right). \tag{3.8}$$

Each ψ theory has central charge $\frac{1}{2}$. Thus we have

$$\begin{aligned} T_B(z) &= \left(1-\frac{1}{2}\right) :(\partial b)\,c: -\frac{1}{2} :b\partial c: \\ &= \frac{1}{4}(\partial\psi_1 + i\partial\psi_2)(\psi_1 - i\psi_2) - \frac{1}{4}(\psi_1 + i\psi_2)(\partial\psi_1 - i\partial\psi_2) \\ &= -\frac{1}{2}\psi_1\partial\psi_1 - \frac{1}{2}\psi_2\partial\psi_2. \end{aligned} \tag{3.9}$$

Therefore, we have d holomorphic ψ^μ theories, with $c = \frac{d}{2}$, and the same number of antiholomorphic theories. The holomorphic energy-momentum tensor is then

$$T_B = -\frac{1}{2}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu. \tag{3.10}$$

The world-sheet supercurrents

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu(z)\,\partial X_\mu(z), \qquad \bar{T}_F(\bar{z}) = i\sqrt{\frac{2}{\alpha'}}\bar{\psi}^\mu(\bar{z})\,\bar{\partial} X_\mu(\bar{z}), \tag{3.11}$$

are also holomorphic and antiholomorphic, respectively. In the following, the factors $\sqrt{\frac{2}{\alpha'}}$ will be eliminated by working in units where $\alpha' = 2$. The normal ordering is implicit. This gives the desired result: The modes ψ_0^μ and $\bar{\psi}_0^\mu$ will satisfy the gamma matrix algebra, as mentioned above, and the center of mass of T_F and $\bar{T}_F$ will have the form of Dirac operators. We can see how these supercurrents act on the matter fields by computing the OPEs, this gives

$$T_F(z)\,X^\mu(w) \sim -i\frac{\psi^\mu(w)}{z-w}, \quad \text{and} \quad T_F(z)\,\psi^\mu(w) \sim i\frac{\partial X^\mu(w)}{z-w}. \tag{3.12}$$

We can write the Ward identity (2.41) in complex coordinates as

$$\begin{aligned} \delta\phi(z,\bar{z}) &= \int_{\partial V} d\xi_\mu j_\epsilon^\mu \phi(z,\bar{z}) \\ &= \int_{\partial V} \left(\varepsilon_{\mu\nu} ds^\nu j_\epsilon^\mu\right)\phi(z,\bar{z}) \\ &= \frac{1}{2i}\int_{\partial V}\left(d\bar{z} j_\epsilon^z - dz j_\epsilon^{\bar{z}}\right)\phi(z,\bar{z}), \end{aligned} \tag{3.13}$$

where we have used a differential ds^ν parallel to the contour ∂V, defined by $d\xi_\mu = \varepsilon_{\mu\nu} ds^\nu$. From these OPEs and the Ward identity it follows that the currents

$$j_\epsilon(z) \equiv \pi j_{z\,\epsilon} = \epsilon(z)\,T_F(z), \qquad \bar{j}_\epsilon(\bar{z}) \equiv \pi j_{\bar{z}\,\epsilon} = \bar{\epsilon}(\bar{z})\,\bar{T}_F(\bar{z}) \tag{3.14}$$

generate the superconformal transformations

$$\delta X^\mu(z,\bar{z}) = \epsilon(z)\,\psi^\mu(z) + \bar{\epsilon}(\bar{z})\,\bar{\psi}^\mu(\bar{z}), \tag{3.15}$$

$$\delta\psi^\mu(z) = -\epsilon(z)\,\partial X^\mu(z), \tag{3.16}$$

$$\delta\bar{\psi}^\mu(\bar{z}) = -\bar{\epsilon}(\bar{z})\,\bar{\partial} X^\mu(\bar{z}). \tag{3.17}$$

This transformation mixes the commuting fields X^μ with the anticommuting fields ψ^μ and $\bar{\psi}^\mu$, so the parameter $\epsilon(z)$ must be anticommuting. As with conformal symmetry the parameters are holomorphic or antiholomorphic functions. The fact that this is a symmetry of the action (3.5) follows at once because the current is (anti)holomorphic, and so conserved.

The commutator of two superconformal transformations is a conformal transformation,

$$\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1} = \delta_{\xi}, \qquad \xi(z) = -2\epsilon_1(z)\,\epsilon_2(z) \tag{3.18}$$

as can be checked by acting on the various fields. We now do that:

$$\begin{aligned}
\left[\delta_{\epsilon_1}, \delta_{\epsilon_2}\right] X^{\mu}(z,\bar{z}) &= \delta_{\epsilon_1}\epsilon_2(z)\,\psi^{\mu}(z) - \delta_{\epsilon_2}\epsilon_1(z)\,\psi^{\mu}(z) \\
&= -\epsilon_2(z)\,\delta_{\epsilon_1}\psi^{\mu}(z) + \epsilon_1(z)\,\delta_{\epsilon_2}\psi^{\mu}(z) \\
&= -\epsilon_2(z)\left(-\epsilon_1(z)\,\partial X^{\mu}(z)\right) + \epsilon_1(z)\left(-\epsilon_2(z)\,\partial X^{\mu}(z)\right) \\
&= \epsilon_2(z)\,\epsilon_1(z)\,\partial X^{\mu}(z) - \epsilon_1(z)\,\epsilon_2(z)\,\partial X^{\mu}(z) \\
&= -2\epsilon_1(z)\,\epsilon_2(z)\,\partial X^{\mu}(z)\,.
\end{aligned} \tag{3.19}$$

Since $X^{\mu}(z,\bar{z})$ has conformal dimension $h = 0$, we know that this transform under an infinitesimal conformal transformation, with parameter $\xi(z)$, according to $\delta X^{\mu}(z) = \xi(z)\,\partial X^{\mu}(z)$. Similarly, the commutator of a conformal and superconformal transformation is a superconformal transformation. The conformal and superconformal transformations thus close to form the superconformal algebra. The OPEs of T_F with itself and with T_B closes. That is, only T_F and T_B appears in the singular terms:

$$T_B(z)\,T_B(w) \sim \frac{\frac{3d}{4}}{(z-w)^4} + \frac{2T_B(w)}{(z-w)^2} + \frac{\partial T_B(w)}{z-w}, \tag{3.20}$$

$$T_B(z)\,T_F(w) \sim \frac{\frac{3}{2}T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w}, \tag{3.21}$$

$$T_F(z)\,T_F(w) \sim \frac{d}{(z-w)^3} + \frac{2T_B(w)}{z-w} \tag{3.22}$$

and similarly for the antiholomorphic currents.

We proceed now to show explicitly (3.21), that is

$$
\begin{aligned}
:T_B(z)::T_F(w): &= -\frac{i}{2}:\left(\eta_{\mu\nu}\partial X^\mu\partial X^\nu+\eta_{\mu\nu}\psi^\mu\partial\psi^\nu\right)(z)::\eta_{\rho\sigma}\psi^\rho\partial X^\sigma(w): \\
&= -\frac{i}{2}:\eta_{\mu\nu}\partial X^\mu\partial X^\nu(z)::\eta_{\rho\sigma}\psi^\rho\partial X^\sigma(w):-\frac{i}{2}:\eta_{\mu\nu}\psi^\mu\partial\psi^\nu(z)::\eta_{\rho\sigma}\psi^\rho\partial X^\sigma(w): \\
&= -\frac{i}{2}\eta_{\mu\nu}\eta_{\rho\sigma}:\partial X^\mu(z)\,\psi^\rho(w):\langle\partial X^\nu(z)\,\partial X^\sigma(w)\rangle \\
&\qquad -\frac{i}{2}\eta_{\mu\nu}\eta_{\rho\sigma}:\partial X^\nu(z)\,\psi^\rho(w):\langle\partial X^\mu(z)\,\partial X^\sigma(w)\rangle \\
&\quad -\frac{i}{2}\eta_{\mu\nu}\eta_{\rho\sigma}:\psi^\mu(z)\,\langle\partial\psi^\nu(z)\,\psi^\rho(w)\rangle\,\partial X^\sigma(w): \\
&\qquad +\frac{i}{2}\eta_{\mu\nu}\eta_{\rho\sigma}:\partial\psi^\nu(z)\,\langle\psi^\mu(z)\,\psi^\rho(w)\rangle\,\partial X^\sigma(w): \\
&= -\frac{i}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\left(-\frac{\eta^{\nu\sigma}:\partial X^\mu(z)\,\psi^\rho(w):}{(z-w)^2}-\frac{\eta^{\mu\sigma}:\partial X^\nu(z)\,\psi^\rho(w):}{(z-w)^2}\right) \\
&\quad -\frac{i}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\left(-\frac{\eta^{\nu\rho}:\psi^\mu(z)\,\partial X^\sigma(w):}{(z-w)^2}-\frac{\eta^{\mu\rho}:\partial\psi^\nu(z)\,\partial X^\sigma(w):}{z-w}\right) \\
&= \frac{i}{2}\left(\frac{\eta_{\mu\nu}\delta^\nu_\rho:\partial X^\mu(z)\,\psi^\rho(w):}{(z-w)^2}+\frac{\eta_{\mu\nu}\delta^\mu_\rho:\partial X^\nu(z)\,\psi^\rho(w):}{(z-w)^2}\right) \\
&\quad +\frac{i}{2}\left(\frac{\eta_{\mu\nu}\delta^\nu_\sigma:\psi^\mu(z)\,\partial X^\sigma(w):}{(z-w)^2}+\frac{\eta_{\mu\nu}\delta^\mu_\sigma:\partial\psi^\nu(z)\,\partial X^\sigma(w):}{z-w}\right) \\
&\sim i\frac{:\partial X^\mu(w)\,\psi_\mu(w):}{(z-w)^2}+i\frac{:\partial^2X^\mu(w)\,\psi_\mu(w):}{(z-w)}+\frac{i}{2}\frac{:\psi^\mu(w)\,\partial X_\mu(w):}{(z-w)^2} \\
&\quad +\frac{i}{2}\frac{:\partial\psi^\mu(w)\,\partial X_\mu(w):}{(z-w)}+\frac{i}{2}\frac{:\partial\psi^\mu(w)\,\partial X_\mu(w):}{z-w} \\
&\sim \frac{3}{2}i\frac{:\psi^\mu(w)\,\partial X_\mu(w):}{(z-w)^2}+i\frac{:\psi^\mu(w)\,\partial^2X_\mu(w):}{(z-w)}+i\frac{:\partial\psi^\mu(w)\,\partial X_\mu(w):}{(z-w)} \\
&\sim \frac{3}{2}\frac{i:\psi^\mu(w)\,\partial X_\mu(w):}{(z-w)^2}+\frac{\partial\left(:i\psi^\mu(w)\,\partial X_\mu(w):\right)}{(z-w)} \\
&\sim \frac{3}{2}\frac{T_F(w)}{(z-w)^2}+\frac{\partial T_F(w)}{(z-w)}.
\end{aligned}
$$

The T_BT_F OPE implies that T_F is a tensor of weight $\left(\frac{3}{2},0\right)$. Each bosonic field ∂X^μ contributes 1 to the central charge and each fermion, ψ^μ, contributes $\frac{1}{2}$, for a total

$$c=\left(1+\frac{1}{2}\right)d=\frac{3}{2}d \tag{3.23}$$

We will impose this enlarged algebra with T_F and $\bar{T}_F$ as well as T_B and $\bar{T}_B$, on the states as a constraint algebra, it must annihilate physical states in the sense of BRST quantization.

More generally, the $N=1$ superconformal algebra in operator product form is

$$T_B(z)\,T_B(w) \sim \frac{\frac{c}{2}}{(z-w)^4}+\frac{2T_B(w)}{(z-w)^2}+\frac{\partial T_B(w)}{z-w} \tag{3.24}$$

$$T_B(z)\,T_F(w) \sim \frac{\frac{3}{2}T_F(w)}{(z-w)^2}+\frac{\partial T_F(w)}{z-w} \tag{3.25}$$

$$T_F(z)\,T_F(w) \sim \frac{\frac{2}{3}c}{(z-w)^3}+\frac{2T_B(w)}{z-w} \tag{3.26}$$

Here, $N=1$ refers to the number of $\left(\frac{3}{2},0\right)$ currents. In the present case there is also an antiholomorphic copy of the same algebra, so we have an $(N,\bar{N})=\left(\frac{3}{2},\frac{3}{2}\right)$ superconformal field theory (SCFT).

3.1.1 Superconformal ghosts and critical dimension

The matter system studied in the previous chapter was the result of a gauge fixation. In the same way, the superstring action presented above do, a detailed construction of the non gauge-fixed action is presented in [21, 22]. Therefore, in the path integral quantization, the gauge fixation will give rise an extra term for the ghost action, due to the new fermionic terms incorporated above. This new contribution is

$$S_{\beta,\gamma}=\frac{1}{2\pi}\int d^2z\beta\partial_{\bar{z}}\gamma, \tag{3.27}$$

where the equations of motion are

$$\partial_{\bar{z}}\gamma=\partial_{\bar{z}}\beta=0. \tag{3.28}$$

In general, we can consider β and γ as commuting holomorphic fields of conformal weights $h_\beta=\lambda-\frac{1}{2}$ and $h_\gamma=\frac{3}{2}-\lambda$, very similar to the b,c system. The OPE is compute as in the b,c system but, since the statistical are changed, some signs are different

$$\beta(z_1)\,\gamma(z_2)=-\gamma(z_1)\,\beta(z_2)\sim-\frac{1}{z_1-z_2}. \tag{3.29}$$

The energy-momentum tensor is (eq.(2.178))

$$T_{\beta\gamma}(z)=\left(\frac{3}{2}-\lambda\right):(\partial\beta)\,\gamma:-\left(\lambda-\frac{1}{2}\right):\beta\partial\gamma:. \tag{3.30}$$

The central charge is simply

$$c=3\,(2\lambda-2)^2-1. \tag{3.31}$$

For $\lambda = 2$, weights $\left(h_\beta, h_\gamma\right) = \left(\frac{3}{2}, -\frac{1}{2}\right)$, we recover the β, γ theory as the Faddeev-Popov ghost from gauge-fixing the superstring.

Then, we consider a general free SCFT which combines an anticommuting bc theory with a commuting $\beta\gamma$ system, with weights

$$h_b = \lambda, \quad h_c = 1 - \lambda, \qquad h_\beta = \lambda - \frac{1}{2}, \quad h_\gamma = \frac{3}{2} - \lambda. \tag{3.32}$$

The action is

$$S_{ghost} = \frac{1}{2\pi} \int d^2z \left(b\bar{\partial}c + \beta\bar{\partial}\gamma\right). \tag{3.33}$$

Since the generator of the supersymmetry transformations is T_F, we have

$$\delta_\epsilon S = \frac{1}{2\pi} \int d^2z T_{Fghost} \bar{\partial}\epsilon, \tag{3.34}$$

where ϵ is an infinitesimal supersymmetry transformation. We then consider the following superconformal infinitesimal variations

$$\delta_\epsilon \beta = \frac{1}{2}\epsilon b, \qquad \delta_\epsilon b = \frac{1}{2}\epsilon\partial\beta + \frac{3}{2}\left(\partial\epsilon\right)\beta, \qquad \delta_\epsilon c = \frac{1}{2}\epsilon\gamma, \qquad \delta_\epsilon\gamma = \frac{1}{2}\epsilon\partial c - \left(\partial\epsilon\right)c. \tag{3.35}$$

We have by considering these variations on S_{ghost} that

$$\begin{aligned}
\delta_\epsilon S &= \frac{1}{2\pi} \int d^2z \left(\delta b\bar{\partial}c + b\bar{\partial}\delta c + \delta\beta\bar{\partial}\gamma + \beta\bar{\partial}\delta\gamma\right) \\
&= \frac{1}{2\pi} \int d^2z \left\{ \left(\frac{1}{2}\epsilon\partial\beta + \frac{3}{2}\partial\epsilon\beta\right)\bar{\partial}c + b\bar{\partial}\left(\frac{1}{2}\epsilon\gamma\right) + \left(\frac{1}{2}\epsilon b\right)\bar{\partial}\gamma + \beta\bar{\partial}\left(\frac{1}{2}\epsilon\partial c - \partial\epsilon c\right)\right\} \\
&= \frac{1}{2\pi} \int d^2z \left\{\frac{1}{2}\epsilon\partial\beta\bar{\partial}c + \frac{1}{2}\partial\epsilon\beta\bar{\partial}c + \frac{1}{2}b\bar{\partial}\epsilon\gamma + \frac{1}{2}\bar{\partial}\epsilon\beta\partial c + \frac{1}{2}\epsilon\beta\bar{\partial}\partial c - \bar{\partial}\partial\epsilon\beta c\right\}.
\end{aligned}$$

Now, we replace the following terms

$$\begin{aligned}
-\bar{\partial}\partial\epsilon\beta c &= -\partial\left(\bar{\partial}\epsilon\beta c\right) + \bar{\partial}\epsilon\partial\beta c + \bar{\partial}\epsilon\beta\partial c, \\
\frac{1}{2}\epsilon\beta\bar{\partial}\partial c &= \frac{1}{2}\partial\left(\epsilon\beta\bar{\partial}c\right) - \frac{1}{2}\partial\epsilon\beta\bar{\partial}c - \frac{1}{2}\epsilon\partial\beta\bar{\partial}c.
\end{aligned}$$

Then,

$$\delta_\epsilon S = \frac{1}{2\pi} \int d^2z \left(\frac{1}{2}b\gamma - \frac{3}{2}\beta\partial c - \partial\beta c\right)\bar{\partial}\epsilon. \tag{3.36}$$

Thus, we have

$$T_F = \frac{1}{2}b\gamma - \frac{3}{2}\beta\partial c - \left(\partial\beta\right)c. \tag{3.37}$$

In another hand, T_B for the total ghost system is simply the sum of the b, c part plus the β, γ part, that is (for $\lambda = 2$)

$$T_B = -(\partial b)\, c - 2b\partial c + \frac{1}{2}(\partial \beta)\,\gamma - \frac{3}{2}\beta\partial\gamma. \tag{3.38}$$

In general, for an arbitrary entire λ , we have

$$\begin{aligned} T_B &= (\partial b)\, c - \lambda\partial (bc) + (\partial\beta)\,\gamma - \left(\lambda - \frac{1}{2}\right)\partial(\beta\gamma)\,, & (3.39)\\ T_F &= -\frac{1}{2}(\partial\beta)\, c + \frac{2\lambda - 1}{2}\partial(\beta c) - 2b\gamma. & (3.40)\end{aligned}$$

The central charge is

$$c_{ghost} = \left[-3\,(2\lambda-1)^2 + 1\right] + \left[3\,(2\lambda-2)^2 - 1\right] = 9 - 12\lambda. \tag{3.41}$$

There is a corresponding antiholomorphic theory. With $\lambda = 2$, the ghost central charge is then $c_{ghost} = -26 + 11 = -15$.

Taking a total energy-momentum tensor, formed by (3.38) T_{Bghost} and the (3.10) energy-momentum tensors, we have

$$T_B^{total} = -2b\partial c - (\partial b)\, c - \frac{3}{2}\beta\partial\gamma - \frac{1}{2}(\partial\beta)\,\gamma - \frac{1}{2}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu. \tag{3.42}$$

The total central charge of this system is the sum of the central charge of each system, since the matter system contribute with a central charge $c_{matter} = d + \frac{d}{2} = \frac{3}{2}d$, we have that the condition to the total central charge $c = c_{ghost} + c_{matter}$ vanishes, implies

$$\frac{3}{2}d - 15 = 0 \Longrightarrow d = 10. \tag{3.43}$$

Hence, we obtain $d = 10$, as the celebrated critical dimension of the superstring.

3.2 Ramond and Neveu-Schwarz sectors and the superalgebra

We won't deepen in the study of the spectrum of the $X^\mu\psi^\mu$ super conformal field theory. However, we going to show how the different periodicity conditions give rise to different sector which will have different Fock spaces. We start with the cylinder coordinate $w = \sigma^1 + i\sigma^2$. The matter fermion action

$$\frac{1}{4\pi}\int d^2w\,\left(\psi^\mu\partial_{\bar{w}}\psi_\mu + \bar{\psi}^\mu\partial_w\bar{\psi}_\mu\right) \tag{3.44}$$

must be invariant under the periodic identification of the cylinder, $w \cong w + 2\pi$. There are two possible periodicity conditions for ψ^μ,

$$\begin{aligned} \text{Ramond}(R): \qquad \psi^\mu (w + 2\pi) &= +\psi^\mu (w), & (3.45)\\ \text{Neveu Schwarz}(NS): \qquad \psi^\mu (w + 2\pi) &= -\psi^\mu (w), & (3.46) \end{aligned}$$

where the sign must be the same for all μ. Similarly there are two possible periodicities for $\bar{\psi}^\mu$. Summarizing, we will write

$$\begin{aligned} \psi^\mu (w + 2\pi) &= e^{2\pi i a}\psi^\mu (w), & (3.47)\\ \bar{\psi}^\mu (\bar{w} + 2\pi) &= e^{-2\pi i \bar{a}}\bar{\psi}^\mu (\bar{w}), & (3.48) \end{aligned}$$

where a and $\bar{a}$ take the values 0 and $\frac{1}{2}$. We are interested just in theories with X^μ periodic. The supercurrents has then the same periodicity as the corresponding ψ,

$$\begin{aligned} T_F (w + 2\pi) &= e^{2\pi i a} T_F (w), & (3.49)\\ \bar{T}_F (\bar{w} + 2\pi) &= e^{-2\pi i \bar{a}} \bar{T}_F (\bar{w}). & (3.50) \end{aligned}$$

Thus, there are four different ways to put on a cylinder, we will denote this by $(a, \bar{a})$ or by NS-NS, NS-R, R-NS, and R-R. Each of which will lead to a different Hilbert space. To study the spectrum in a given sector we must expand in Fourier modes,

$$\psi^\mu (w) = i^{-\frac{1}{2}} \sum_{r \in \mathbb{Z}+a} \psi^\mu_r e^{irw}, \qquad \bar{\psi}^\mu (\bar{w}) = i^{\frac{1}{2}} \sum_{r \in \mathbb{Z}+\bar{a}} \bar{\psi}^\mu_r e^{-ir\bar{w}}. \qquad (3.51)$$

On each side the sum runs over integers in the R sector and over $\left(\mathbb{Z} + \frac{1}{2}\right)$ in the NS sector. Let us also write these as Laurent expansion. After replacing $e^{-iw} \longrightarrow z$ we must transform the fields according to eq.(2.119)

$$\psi^\mu (z) = \left(\frac{\partial w}{\partial z}\right)^{1/2} \psi^\mu (w) = i^{\frac{1}{2}} z^{-1/2} \psi^\mu (w). \qquad (3.52)$$

The frame is indicated implicitly by the argument of the field. The Laurent expansion are then

$$\psi^\mu (w) = \sum_{r \in \mathbb{Z}+a} \psi^\mu_r z^{-r-\frac{1}{2}}, \qquad \bar{\psi}^\mu (\bar{w}) = \sum_{r \in \mathbb{Z}+\bar{a}} \bar{\psi}^\mu_r \bar{z}^{-r-\frac{1}{2}}. \qquad (3.53)$$

Notice that in the NS sector the branch cut in $z^{-\frac{1}{2}}$ offsets the original anti-periodicity, while in the R sector it introduces a branch cut. Let us also recall the corresponding bosonic expansions

$$\partial X^{\mu}(z) = -i \sum_{m=-\infty}^{\infty} z^{-m-1}\alpha_m^{\mu} \quad , \quad \bar{\partial} X^{\mu}(\bar{z}) = -i \sum_{m=-\infty}^{\infty} \bar{z}^{-m-1}\bar{\alpha}_m^{\mu}, \tag{3.54}$$

where we have set $\alpha' = 2$, such that $\alpha_0 = \bar{\alpha}_0 = p^{\mu}$.

The OPE and the Laurent expansions give the anticommutators

$$\left\{\psi_r^{\mu}, \psi_s^{\nu}\right\} = \left\{\bar{\psi}_r^{\mu}, \bar{\psi}_s^{\nu}\right\} = \delta_{r+s}\eta^{\mu\nu}, \qquad \left[\alpha_m^{\mu}, \alpha_n^{\nu}\right] = \left[\bar{\alpha}_m^{\mu}, \bar{\alpha}_n^{\nu}\right] = m\delta_{m+n}\eta^{\mu\nu}. \tag{3.55}$$

For T_F and T_B the Laurent expansions are

$$T_F(z) = \sum_{r\in\mathbb{Z}+a} z^{-m-\frac{3}{2}}G_r, \qquad \bar{T}_F(\bar{z}) = \sum_{r\in\mathbb{Z}+\bar{a}} \bar{z}^{-m-\frac{3}{2}}\bar{G}_r, \tag{3.56}$$

$$T_B(z) = \sum_{m=-\infty}^{\infty} z^{-m-2}L_m, \qquad \bar{T}_B(\bar{z}) = \sum_{m=-\infty}^{\infty} \bar{z}^{-m-2}\bar{L}_m. \tag{3.57}$$

From the OPEs eqs. (3.24-3.26) and the usual CFT contour calculation gives the mode algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n}, \tag{3.58}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}, \tag{3.59}$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r}. \tag{3.60}$$

This is called the Ramond algebra for r, s integers and as the Neveu-Schwarz algebra for r, s half integer. The antiholomorphic fields give a second copy of these algebras.

The superconformal generators in either sectors are

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}}^{\infty} :\alpha_{m-n}\cdot\alpha_n: + \frac{1}{2}\sum_{r\in\mathbb{Z}+a}^{\infty}\left(r - \frac{m}{2}\right) :\psi_{m-r}\cdot\psi_r: + a^m\delta_{m,0}, \tag{3.61}$$

$$G_r = \sum_{n\in\mathbb{Z}}^{\infty} :\alpha_n\cdot\psi_{r-n}: . \tag{3.62}$$

The normal ordered constant can be obtained by any method from the bosonic case. Before, we see that unitarity for representations of the Virasoro algebra amounts the following conditions, for $n > 0$

$$\begin{aligned} \langle h\,|L_n L_{-n}|\,h\rangle &= \langle h\,|[L_n, L_{-n}]|\,h\rangle \\ &= \left\langle h\left|\left(2nL_0 + \frac{c}{12}n\left(n^2-1\right)\right)\right|h\right\rangle \\ &= \left\{2nh + \frac{c}{12}n\left(n^2-1\right)\right\}\langle h|h\rangle \geq 0. \end{aligned} \tag{3.63}$$

For $n = 1$, we have that $h \geq 0$, and by taking n large we have $c \geq 0$. Now, for the superconformal case by considering $(r > 0)$, we have

$$\begin{aligned}\langle h | G_r G_{-r} | h \rangle &= \langle h | \{G_r, G_{-r}\} | h \rangle \\ &= \langle h | 2L_0 | h \rangle + \left\langle h \left| \frac{c}{3}\left(r^2 - \frac{1}{4}\right) \right| h \right\rangle \\ &= \left(2h + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\right) \langle h | h \rangle \geq 0, \end{aligned} \tag{3.64}$$

from which we find that (NS $\left(r \geq \frac{1}{2}\right)$ and R $(r \geq 0)$)

$$h \geq 0 \qquad \text{NS}, \tag{3.65}$$

$$h \geq \frac{c}{24} \qquad \text{R}. \tag{3.66}$$

Since $c = \frac{3}{2}d$, we have $\frac{c}{24} = \frac{d}{16}$, therefore

$$\text{R:} \quad a^m = \frac{d}{16}, \qquad \text{NS:} \quad a^m = 0. \tag{3.67}$$

We already know that the $|h\rangle$ are eigenstates of L_0 with eigenvalue h. According to equations (2.198) and (2.215) when we map from the cylinder to the complex plane we have $e^{-iw} = z = (-i)^{-n} \frac{d^n z}{dw^n}$ and $w = i \ln z$, so that $\frac{dw}{dz} = \frac{i}{z}$, thus we have $\{z; w\} = \frac{1}{2}$, and then

$$T_{cyl}(w) = -z^2 T(z) + \frac{c}{24}. \tag{3.68}$$

This implies that

$$(L_0)_{\text{cyl.}} = (L_0)_{\text{pl.}} - \frac{c}{24}, \tag{3.69}$$

where $(L_0)_{\text{cyl.}}$ is the translation operator on the cylinder. Therefore, each periodic boson contributes $-\frac{1}{24}$ because $c = 1$. Each periodic fermion contribute $\frac{1}{24}$ and each antiperiodic fermion $-\frac{1}{48}$. We are going to explain the last sentences. For a periodic fermion or R sector, using (3.69) and the results (3.65,3.66), we have

$$\left(L_{0\psi}\right)_{\text{cyl.}} = \left(L_{0\psi}\right)_{\text{pl.}} + \frac{1}{16} - \frac{(1/2)}{24} = \left(L_{0\psi}\right)_{\text{pl.}} + \frac{1}{24}, \tag{3.70}$$

while for the NS sector we have simply

$$\left(L_{0\psi}\right)_{\text{cyl.}} = \left(L_{0\psi}\right)_{\text{pl.}} - \frac{(1/2)}{24} = \left(L_{0\psi}\right)_{\text{pl.}} - \frac{1}{48}. \tag{3.71}$$

Thus, since the modes L_0 are proportional to the Hamiltonian, what the expressions above say us is that there is a shift of the energy when we map from the cylinder to the complex plane.

Chapter 4

Conclusions

The main goal of this work was to study string theory using the formalism of two dimensional conformal field theory. We made a detailed study of the bosonic string theory, which was made by applying the tools of CFT to the gauge-fixed Polyakov action, also called matter system, and the ghost system, which was the result of the Faddeev-Popov procedure. We have presented the operator formalism of conformal field theory in a quite general way, such that this could be applied for those systems different than the string ones. We focused in explaining and to make the counts of the BRST quantization, trying to give a greater detail than the one found in the standard literature about it. This allowed us to find the critical dimension as 26 by demanding the nilpotency of the BRST charge, also, by demanding the vanishing of the central charge. In a slight different way, it was presented the procedure to obtain the physical spectrum of the bosonic string.

It was also presented a basic introduction to the the superstring theory, this study began with a quite general introduction of the superconformal field theory as a supersymmetric extension of the conformal ones. Afterwards, this superconformal formalism was applied to the superstring action, by doing so, we presented the superstring action, which couple fermionic degrees of freedom and it is invariant under local supersymmetry transformations. This superstring action is also the result of a gauge-fixing, in turn, gives rise to new ghosts extra terms. By demanding the vanishing of the conformal anomaly of the gauge-fixed superstring theory led us to the critical dimension $d = 10$. We have not explored the superstring spectrum but have motivated it by presenting the two sectors of a general superconformal theory. As an aside, it is good to mention that in this dissertation were used different ways to calculate the operator product and the energy-momentum tensor, such that we have analyzed alternative procedure and techniques to work out all the systems.

Bibliography

[1] A.M Polyakov, "Conformal symmetry of critical fluctuations", JETP Letters, Vol 12, 381 (1970). 1

[2] A.A Belavin, A.M Polyakov and A.B Zamolodchikov, "Infinite conformal symmetry in two-dimensional quantum field theory", Nuclear Physics B241 (1984) 333. 1

[3] G. Veneziano, "Construction of a crossing-symmetric, Regge behaved amplitude for linearly rising trajectories", Nuovo Cimento A57 (1968) 190. 1

[4] Philippe Di Francesco, Pierre Mathieu, and David Sénéchali. Conformal Field Theory, Springer, (1997). 6

[5] Paul H. Ginsparg. Applied Conformal Field Theory, (1988), https://arxiv.org/pdf/hep-th/9108028.pdf. 6

[6] D. Lüst, S. Theisen, "Lectures Notes in Physic-Lectures on String Theory", Springer-Verlag, Berlin-Heidelberg, (1989), ISBN 3-540-51882-7. 6

[7] R. Blumenhagen, D. Lüst, S. Theisen, "Basic Concepts of String Theory", Springer-Verlag, Berlin-Heidelberg, (2013), ISBN 978-3-642-29497-6 . 6, 55

[8] J. Polchinski, "String Theory Volume I, "An Introduction to the Bosonic String", Cambridge University Press 2001, (2005), ISBN-13 978-0-511-33709-3. 6

[9] A.M. Polyakov "Quantum Geometry of Bosonic Strings ", Phys.Lett., 103B, 207–210, (1981). 6

[10] B. Zweibach, "A First Course in String Theory-Second Edition", Cambridge University Press, New York, (2009), ISBN-13 978-0-511-47932-8 6, 37

[11] G. Arutyunov, "Lectures on String Theory ", http://stringworld.ru/files/Arutyunov_G._Lectures_on_string_theory.pdf. 6

[12] D. Tong, "String Theory", Cambridge University (2009), http://www.damtp.cam.ac.uk/user/tong/string/string.pdf. 6

[13] T. Weigand, "Introduction to String Theory ", Institut für Theoretische Physik, Universität Heidelberg (2015),`http://www.thphys.uni-heidelberg.de/~weigand/Strings15-16/Strings.pdf`. 6

[14] R.N Huamán, "An introduction to the superstring in flat and $AdS_5 \times S^5$ backgrounds ", Catálogo USP, (2014). 6

[15] M.B. Green, J.H Schwarz, E. Witten, "Super String Theory Volume 1", Cambridge University Press, (1987), ISBN 0-521-32384-3. 6

[16] Bram M. Wouters, "BRST quantization and string theory spectra", University of Amsterdam, (2010). 90

[17] A.M. Polyakov "Quantum Geometry of Fermionic Strings ", Phys.Lett., 103B, 211– 213, (1981). 92

[18] D. Friedman, E. Martinec, S. Shenker, "Conformal Invariance, Supersymmetry and String Theory" Nuclear Physic **B271**, 93 (1986). 92

[19] J. Polchinski, "String Theory Volume II, "Superstring Theory and Beyond", Cambridge University Press 1998, ISBN-0 521-63304-4. 92

[20] P. Deligne, D. Kazhdan, P. Etingf, J.W. Morgan, D.S. Freed, D.R. Morrison, L.C. Jeffrey, E. Witten, "Quantum Fields and Strings: A course for Mathematicians, 2 Vol" American Mathematical Society, Providence, (1999). 92

[21] L. Brink, P. Di Vecchia and P. Howe, "A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String " Phys.Lett., 65B, 471–474, (1976). 97

[22] S. Deser, B. Zumino, "A Complete Action For The Spinning String" Phys.Lett., 65B, 369– 373, (1976).

97