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Essays on bivariate option pricing via copula and heteroscedasticity models: a classical and bayesian approach

Lucas Pereira Lopes

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Lucas Pereira Lopes

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Ensaios sobre precificação de opções bivariadas via cópulas e modelos heterocedásticos: abordagem clássica e bayesiana

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP e ao Departamento de Estatística – DEs-UFSCar, como parte dos requisitos para obtenção do título de Mestre em Estatística – Programa Interinstitucional de Pós-Graduação em Estatística. *VERSÃO REVISADA*

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Prof. Dr. Vicente Garibay Cancho
USP

Prof. Dr. Antonio Carlos Figueiredo Pinto
PUC-RJ

Márcio Luis Lanfredi Viola
Prof. Dr. Marcio Luis Lanfredi Viola
UFSCar

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Prof. Dr. Vicente Garibay Cancho

"Education is not just about going to school and getting a degree. It's about widening your knowledge and absorbing the truth about life" - Shakuntala Devi.

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I would like to wake up in a world where all people could have the chance to study and fall in love with what they do.

RESUMO

LOPES, L. P. **Ensaios sobre precificação de opções bivariadas via cópulas e modelos heterocedásticos: abordagem clássica e bayesiana**. 2019. 94 p. Dissertação (Mestrado em Estatística – Programa Interinstitucional de Pós-Graduação em Estatística) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

Essa dissertação é composta por dois principais ensaios independentes e complementares. No primeiro discutimos a precificação de opções bivariadas sob uma perspectiva bayesiana. Neste ensaio o principal objetivo foi precificar e analizar o preço justo da opção bivariada call-on-max considerando modelos heterocedásticos para as marginais e a modelagem de dependência realizada por funções cópulas. Para a inferência, adotamos o método computacionalmente intensivo baseado em simulações Monte Carlo via Cadeia de Markov (MCMC). Um estudo de simulação examinou o viés e o erro quadrático médio dos parâmetros a posteriori. Para a ilustração da abordagem, foram utilizados preços de ações de bancos Brasileiros. Além disso, foi verificado o efeito do strike e da estrutura de dependência nos preços das opções. Os resultados mostraram que os preços obtidos pelo método utilizado difere substancialmente dos obtidos pelo modelo clássico derivado de Black e Scholes. No segundo capítulo, consideramos os modelos GARCH-in-mean com especificações assimétricas para a variância com o objetivo de acomodar as características da volatilidade dos ativos-objetos sob uma perspectiva da dinâmica do risco-neutro. Além do mais, as funções cópulas foram utilizadas para capturar as possíveis estruturas de dependência linear, não-linear e caudais entre os ativos. Para ilustrar a metodologia, utilizamos dados de duas companhias Brasileiras. Confrontando os resultados obtidos com o modelo clássico extendido de Black e Scholes, notamos que a premissa de volatilidade constante sub-precifica as opções bivariadas, especialmente dentro-do-dinheiro.

Palavras-chave: Precificação, Opções, Modelos Heterocedásticos, Copula, Inferência Bayesiana.

ABSTRACT

LOPES, L. P. **Essays on bivariate option pricing via copula and heteroscedasticity models: a classical and bayesian approach**. 2019. 94 p. Dissertação (Mestrado em Estatística – Programa Interinstitucional de Pós-Graduação em Estatística) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

This dissertation is composed of two main and independents essays, but complementary. In the first one, we discuss the option price under a bayesian perspective. This essay aims to price and analyze the fair price behavior of the call-on-max (bivariate) option considering marginal heteroscedastic models with dependence structure modeled via copulas. Concerning inference, we adopt a Bayesian perspective and computationally intensive methods based on Monte Carlo simulations via Markov Chain (MCMC). A simulation study examines the bias and the root mean squared errors of the posterior means for the parameters. Real stocks prices of Brazilian banks illustrate the approach. For the proposed method is verified the effects of strike and dependence structure on the fair price of the option. The results show that the prices obtained by our heteroscedastic model approach and copulas differ substantially from the prices obtained by the model derived from Black and Scholes. Empirical results are presented to argue the advantages of our strategy. In the second chapter, we consider the GARCH-in-mean models with asymmetric variance specifications to model the volatility of the assets-objects under the risk-neutral dynamics. Moreover, the copula functions model the joint distribution, with the objective of capturing non-linear, linear and tails associations between the assets. We aim to provide a methodology to realize a more realistic pricing option. To illustrate the methodology, we use stocks from two Brazilian companies, where our the modeling offered a proper fitting. Confronting the results obtained with the classic model, which is an extension of the Black and Scholes model, we note that considering constant volatility over time underpricing the options, especially in-the-money options.

Keywords: Pricing, Option, Heterocedastic Model, Copula, Bayesian Inference.

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1

INTRODUCTION

A financial option is a contract where the investor acquires the right, but not the obligation, to buy or sell a particular asset for a predetermined price and time, where the price is known as strike price. Therefore, a put option can be interpreted as an automobile insurance policy, allowing the investor to recover a value previously established by the asset, even if it devalued. With regard to the call option, we can compare to a paid signal on the purchase of a house, as it guarantees the fixed price and also the preference in the buy.

A practical example of the use of options is the scenario where an investor believes that a stock will go up 5% next month and he makes an options contract, and he has the right, but not the obligation, to buy the stock with an extra of 3% of the current value, for example. Therefore, when the closing date of the option (maturity date) arrives, the investor can exercise his right over the option and earn 2% (disregarding the price of the premium and others costs). However, if the stock is worth less then the investor loses the amount paid for the premium and does not exercise the option.

The development of models with the purpose of pricing options began with the authors Black and Scholes (1973) and Merton (1973). The models proposed by the authors use Brownian motion techniques to obtain the fair price of an option in the univariate case. In the multivariate case, there is the extension of the model of Black and Scholes (1973), where this approach consists of the use of Brownian geometric movement for n assets considering the constant volatility.

Another model proposed by Galichon *et al.* (2014) is an extension of the local volatility model introduced by Dupire *et al.* (1994), where the objective was to construct a stochastic correlation model. Goorbergh, Genest and Werker (2005) used a GARCH process (1,1) for each asset under the physical measure and used the transformation proposed by Duan (1995) to obtain the joint distribution under neutral risk.

To analyze and understand the price behavior of a multivariate option is necessary use the jointly model on the underlying processes. The approach used by the methods derived from the traditional Black and Scholes model is the use of the normal multivariate distribution. However, the use of this approach implies in linear associations as a measure of dependence between the assets and their symmetrical behavior, but empirical evidence shows that the joint behavior of financial assets is much more complex (LOPES; PESSANHA, 2018).

Margrabe (1978), Johnson and Shanno (1987), Nelsen (2007) and Shimko (1994) used the linear correlation coefficient to analyze and capture dependence among the underlying assets. However, Embrechts, McNeil and Straumann (2002) and Forbes and Rigobon (2002) criticize the use of this tool, where the authors highlight the stylized facts in finance, such as the heavy tails of returns distributions, their autocorrelations, groupings of volatilities over time and non-normality.

As an alternative, the use of the copulas theory allows the joint modeling of the assets in which there is a separation of the structure of dependence between the variables and their marginal distributions, where this dependence can be linear, nonlinear and even dependence on the tails. Therefore, Rosenberg (2000) and Cherubini and Luciano (2002) used the copula theory in an attempt to capture the dependence among the assets in the derivative pricing process.

In addition, many models use the premise of constant volatility over time, which is not observed in finance series (FRENCH; SCHWERT; STAMBAUGH, 1987; FRANSES; DIJK *et al.*, 2000). Thus, to make the pricing process more realistic, Duan (1995) explored the concept of pricing options considering the heteroskedasticity of the assets, where the author proposed to follow a modification of the GARCH process.

1.1 Objectives and Overview

Therefore, the main objective of this dissertation is to price and analyze the fair price behavior of bivariate options considering marginal heterocedastic models and the dependence structure modeled via copulas. The remainder of this work is organized as follows:

1. In Chapter 2 is presented the theoretical tools of the models and their most important properties for the development of the next two independent chapters.

2. In Chapter 3 the first empirical article of this dissertation is presented, where call-on-max options are computed using Bayesian inference and computationally intensive methods based on Monte Carlo simulations via the Markov Chain (MCMC) and the DGARCH model under neutral-risk meansure.

3. In Chapter 4 we consider the GARCH-in-mean models with asymmetric variance specifications to model the volatility of the assets-objects under the risk-neutral dynamics and copulas. In relation to the inferential method, the Quasi Maximum Likelihood method was used.

4. Finally, in Chapter 5 we present some general comments and possible extensions of this current work.

Each chapter contains results of Monte Carlo simulations and empirical applications. In addition, the results found will be compared with the values obtained by the classic extended models of Black and Scholes. It is important to highlight the innovative character of this work in the sense of performing the pricing in the Brazilian stock market, differing from other works found in the literature. It is justified as an innovation to be the first application in a volatile stock market (CONG, 2017; LUNDEN, 2007), a fact that can be explained by the characteristic of being an emerging market (ABUGRI, 2008; TABAK; GUERRA, 2007).

CHAPTER

2

PRELIMINARES

In this chapter, we present a brief literature review of some important topics that are covered throughout this dissertation.

2.1 Factors that Impact the Price of Options

According to the literature of options, there are six main factors influencing the price of the options, namely the current price of the underlying asset, strike price, maturity time, dividends expected over the life of the option, price volatility of stock and the risk-free interest rate (BESSADA; BARBEDO; ARAÚJO, 2005).

At first, it will not be worked with options that pay dividends, for this reason, this factor of the analysis is discarded. The current price and the strike price similarly influence the price of the option. An example is, if the call option is actually exercised at some future time, the payoff will be the value that exceeded the quotation of the asset in the market in relation to the strike price.

Therefore, as the spot price increases, the option becomes more valuable to the buyer of the option, as the margin of gain increases. In contrast, the higher strike price of the option, the lower the premium. With regard to put options the situation is the opposite. That is, as the spot price rises, the option is worth less. The higher strike price, the better the option, given that there is a right to sell the stock at a higher price (MELO, 2012).

The third variable that affects the price of an option is the maturity time. The interpretation for the buying and selling options are similar, as the tendency is for the option price to increase as the maturity period increases. There are other interpretations when considering dividends, but this will not be the case in this work.

The next variable is the volatility. This metric represents the risk inherent in the option, so the higher the risk, the higher the premium charged by the launcher, both in the call and put options. Thus, volatility increases the premiums of the options. The fifth variable is the risk-free interest rate. This rate represents the cost of opportunity to obtain an asset for a given time. An increase in the interest rate impacts on an increase in the value of the call option and the lower value of the put option.

In addition to these factors, the impact of the dependency structure between the object assets and the Moneyness concept fit in this scenario. The interpretation of the impact of the correlation depends on the option in question, for example, if Call on Min option, the strongly negative dependence between the two underlying assets produces lower prices, because when an asset decreases its price, the other is likely to be in a low level as well. On the other hand, when there is a strong positive dependence, the indices benefit when prices rise.

In the case of a Call on Max option, if the correlation between assets is positive, the price of asset 1 tends to increase when asset 2 rises, which is good for this option. However, if it maintains the positive correlation and the price of asset 1 falls the price of asset 2 also falls, which is not desirable in the case of this option. Therefore, a trade-off between the two situations is noted. Therefore, it is important to analyze this variable.

Lastly, we have the variable Moneyness. The moneyness is the ratio of the last share price observed in the spot market and the strike price has been classified into three categories: in-the-money - ITM, at-the-money -ATM and out-the-money (OTM). This metric is associated with the probability of the option presenting a positive payoff on its maturity date, or the option to be exercised. Table 1 presents its classification for the univariate case and for each type of option.

Table 1 – Classification of Options in relation to Moneyness - Univariate Case.

Classification	**Call Option**	**Put Option**
ITM	Market price > Strike	Market price < Strike
ATM	Market price = Strike	Market price = Strike
OTM	Market price < Strike	Market price > Strike

More out-the-money the option is, the less likely it is to exercise on the part of the holder and consequently the more within the money, the more likely it is to exercise. An adaptation of this concept will be used in this work with the aim of expanding to the bivariate case. Let S_1 be the market price of asset 1 and S_2 the market price of asset 2, Table 2 shows which classification will be used from now on.

Table 2 – Classification of Options in Relation to Moneyness - Bivariate Case

Classification	**Call Option**	**Put Option**
ITM	Min$(S_1, S_2) >$ Strike	Max$(S_1, S_2) <$ Strike
ATM	Max$(S_1, S_2) =$ Strike	Max$(S_1, S_2) =$ Strike
OTM	Max$(S_1, S_2) <$ Strike	Min$(S_1, S_2) >$ Strike

This extrapolation of the concepts of moneyness has as objective to analyze the effect of its classification in the final prices of the options.

2.2 Some Useful Probability Results

This chapter aims to address some mathematical definitions that will be used in the course of this dissertation.

Definition 2.2.1 (Filtration)**.** $\mathbb{F} = \{F_t\}_{t \in T}$ is a filtration if it is a increasing family of sub-σ-algebras, *i.e.*, if it is a family of σ-algebras such that for all $s < t$:

$$F_t \subset F,\, F_s \subset F_t.$$

In addition, a filtration is complete and is continuous, i.e.,

$$F_t = \bigcap_{s>t} F_s, t \in T.$$

Definition 2.2.2 (Stochastic Process)**.** A continuous-time stochastic process X assumes values in a measurable space (E, ε) and is a family of random variables $\{X_t\}$ defined in space $(\Omega, F, \mathbb{P})$, indexed in time t.

Definition 2.2.3 (Martingales)**.** A real valued adapted process (M_t) is said to be a martingale with respect to the filtration $\{F_t\}_{t \in T}$ if $\mathbb{E}(|M_t|) < \infty$ for all t and for all $s \leq t$:

$$\mathbb{E}(M_t | F_s) = M_s \text{ a.s.}$$

The martingale condition can be regarded as $\mathbb{E}(X_t|F_s)$ being a version of the process X_t:

$$\int_A \mathbb{E}(X_t|F_s) d\mathbb{P} = \int_A X_s d\mathbb{P}, \text{ A} \in F_s,$$

but by definition of conditional expectation we have:

$$\int_A \mathbb{E}(X_t|F_s) d\mathbb{P} = \int_A X_t d\mathbb{P}, \text{ A} \in F_s,$$

so that for $s \leq t$:

$$\int_A X_s d\mathbb{P} = \int_A X_t d\mathbb{P},\ \mathrm{A} \in F_s.$$

Therefore, the martingale give the information available about the process until now, the expected value is the present value. A martingale is a process constant in mean, in the sense that

$$\mathbb{E}(M_t) = \mathbb{E}(M_0) \text{ for all } t \geq 0.$$

Indeed,

$$\mathbb{E}(M_t|F_s) = M_s \text{ a.s. } s \leq t,$$

implies

$$\mathbb{E}(\mathbb{E}(M_t|F_s)) = \mathbb{E}(M_s),$$

so that by interated expectation property:

$$\mathbb{E}(M_t) = \mathbb{E}(M_s) \text{ for all } s \leq t.$$

Definition 2.2.4 (Risk-neutral Probability Measure)**.** A probability measure $\mathbb{Q}$ is called a risk-neutral probability measure if

1. $\mathbb{Q}$ is equivalente to the real world measure $\mathbb{P}$.
2. $\dfrac{S_t}{B_t} = E^Q\left(\dfrac{S_{t+r}}{B_{t+r}}|F_t\right)$ for all $t, r \in R^+$, where B_t is the deterministic price process of a risk-free asset, where $B_t = B_0 exp\left(\int_0^t r(s)ds\right)$, and $r(t)$ is the short rate.

2.3 Bivariate Black and Scholes Approach

Consider a Call-on-max European option, denoted by $f(S_1, S_2) = max(max(S_1, S_2) - K), 0)$. The classical Black and Scholes approach for option pricing with one underlying is the lognormal random walk

$$\frac{\partial S}{S} = \mu \partial t + \sigma \partial W,$$

where μ is a trend (drift rate) of the stock S, σ is the stock volatility and W is the brownian - the term ∂W represents any source of uncertainty in the historical price of the stock.

This was readily extended to a scenario containing two assets via models for each asset underlying

$$\frac{\partial S_1}{S_1} = \mu_1 \partial t + \sigma_1 \partial W_1$$

$$\frac{dS_2}{S_2} = \mu_2 \partial t + \sigma_2 \partial W_2,$$

where ∂W_i, $i = 1, 2$, is a random variable drawn from a Normal distribution with mean zero and standard deviation $\partial t^{1/2}$ so that $E(\partial W_i) = 0$ and $E(\partial W_i^2)\partial t$, but the random variables ∂W_1 and ∂W_2 are correlated by $E(\partial W_1 \partial W_2) = \rho \partial t$. Here ρ is the linear correlation coefficient between the two random walks. Let $V(S_1, S_2, T)$ be the option value. Since there are two sources of uncertainty, we construct a portfolio of one long option position, two short positions in some quantities of underlying assets:

$$\Pi = V - \Delta_1 S_1 - \Delta_2 S_2,$$

where Δ_i is a change in the value of the S_i in a short interval, $i = 1, 2$.

Consider the increment $\partial \Pi = \partial V - \Delta_1 \partial S_1 - \Delta_2 \partial S_2$. Now, apply the Ito Lemma involving two variables.

$$\partial V = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right] \partial t + \frac{\partial V}{\partial S_1} \partial S_1 + \frac{\partial V}{\partial S_2} \partial S_2.$$

The two dimensional Ito Lemma can be derived by using Taylor series and the rules of thumb: $\partial W_i^2 = \partial t$, $i = 1, 2$ and $\partial W_1 \partial W_2 = \rho \partial t$. Taking $\Delta_1 = \dfrac{\partial V}{\partial S_1}$ and $\Delta_2 = \dfrac{\partial V}{\partial S_2}$ to eliminate risk, we then have

$$\partial \Pi = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right] \partial t.$$

Then the portolio is riskless and then earn riskless return, namely

$$\partial \Pi = r\Pi = r \left(V - \frac{\partial V}{\partial S_1} S_1 - \frac{\partial V}{\partial S_2} S_2 \right) \partial t.$$

So we arrive at an equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + rS_1 \frac{\partial V}{\partial S_1} + r\frac{\partial V}{\partial S_2} S_2 - rV = 0.$$

The solution domain is $\{S_1 > 0, S_2 > 0, t \in [0, T)\}$, and the final condition is $V(S_1, S_2, T) = f(S_1, S_2)$ form a complete model.

The result displayed was the motivation of other works, and some authors provided approximations of closed formulas for the above equation. In order to compare, in this work we will use the closed formula introduced by Stulz (1982) and later extended by Johnson and Shanno (1987), Boyle, Evnine and Gibbs (1989), Boyle and Tse (1990), Rubinstein *et al.* (1991) and others. The fair price for the call-on-max option is defined by

$$
\begin{aligned}
c_{max}(S_1,S_2,K,T) = S_1 e^{-rT} M(y_1,d;\rho_1) + S_2 e^{-rT} M(y_2,-d+\sigma\sqrt{T};\rho_2) - Ke^{-rT} \\
* [1 - M(-y_1+\sigma_1\sqrt{T}, -y_2+\sigma_2\sqrt{T};\rho)],
\end{aligned}
$$

which

$$
\begin{aligned}
d &= \frac{log(S_1/S_2) + (\sigma^2/2)T}{\sigma\sqrt{T}}, \\
y_1 &= \frac{log(S_1/K) + (\sigma_1^2/2)T}{\sigma_1\sqrt{T}}, \\
y_2 &= \frac{log(S_2/K) + (\sigma_2^2/2)T}{\sigma_2\sqrt{T}}, \\
\sigma &= \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \\
\rho_1 &= \frac{\sigma_1 - \rho\sigma_2}{\sigma}, \\
\rho_2 &= \frac{\sigma_2 - \rho\sigma_1}{\sigma}.
\end{aligned}
$$

The formulas derived from the Black and Scholes model imply erroneous pricing processes because they consider the volatility constant over the maturity time of the options and consider the linear association between the underlying assets. Therefore, the next section presents the pioneer model to accommodate the heteroscedasticity of the asset-object.

2.4 Duan and Heterocedastic Approach

To meet the constant variance limitation over time of the Black and Scholes model, Duan (1995) developed a method of pricing options considering GARCH processes. It highlights three main advantages of this method.First, is a function of the risk premium embedded in the underlying asset. Second, the model is non-Markovian, i.e., do not require that underlying asset value is usually assumed to follow a diffusion process.Third, can potentially explain some well-documented systematic biases associated with BS model, where the main biases is underpricing of out-of-the-money options (BLACK, 1975; GULTEKIN; ROGALSKI; TINIC, 1982), underpricing of options on low-volatility securities (BLACK; SCHOLES, 1973; WHALEY, 1982) and underpricing of short-maturity option (WHALEY, 1982).

2.4.1 Risk Neutral Valuation for Option Pricing

The theory of Risk Neutral Valuation Relationship (RNVR) proposed by Rubinstein (1976) and Brennan and Schwartz (1979) has the objective of pricing an option contract as an expected value of the payoff function discounted under a martingale measure. Therefore, the construction of a new measure of probability allows us to price options under the hypothesis that economic agents are risk neutral. This section aims to introduce the measure of risk-neutral probability in which it is equivalent to physical measure P.

Let Q be a measure of martingale in a discrete time economy with a risk free asset and a complete filtration probability space (Ω, F, F_t, P), where F_t is an increasing information filtration at time t and P is the physical probability measure.

Definition 2.4.1 (Duan (1995))**.** The measure of probability $\mathbb{Q}$ is equivalent to measure $\mathbb{P}$ if:

1. $\mathbb{Q} \approx \mathbb{P}$, i.e., for all event X, $\mathbb{Q}(X) = 0$ and $\mathbb{P}(X) = 0$.

2. $E^{\mathbb{Q}}[S_t|F_{t-1}] = S_{t-1}$, i.e., the discounted price process S_t is a martingale under $\mathbb{Q}$.

Therefore, the martingale condition for the discounted stock price can be replaced by

$$E^Q\left[\hat{S}_t|F_{t-1}\right] = \hat{S}_{t-1} <=> E^Q\left[e^{rt}S_t|F_{t-1}\right] = e^{-(t-1)}S_{t-1} <=> E^Q\left[\frac{S_t}{S_{t-1}}|F_{t-1}\right] = e^r <=>$$
$$E^Q\left[e^{y_t}|F_{t-1}\right] = e^r.$$

Duan (1995) extended the RNVR to Locally Risk Neutral Valuation Relationship (LRNVR) by assuming a conditional Normal distriburion for the log-reutrn with an unchanged volatility after change of measure.

Definition 2.4.2 (Duan (1995))**.** A measure $\mathbb{Q}$ satisfies the local risk-neutral valuation relationship (LRNVR) if:

1. $y_t|F_{t-1}$ is normally distributes under measure $\mathbb{Q}$.

2. $E^{\mathbb{Q}}[S_t/S_{t-1}|F_{t-1}] = e^r$.

3. $Var^{\mathbb{Q}}[log(S_t/S_{t-1})|F_{t-1}] = Var^{\mathbb{P}}[log(S_t/S_{t-1})|F_{t-1}]$.

Theorem 1. The LRNVR implies that, under pricing measure $\mathbb{Q}$,

$$log\frac{S_t}{S_{t-1}} = r - \frac{1}{2}h_t + \varepsilon_t,$$

where $\varepsilon|\phi_{t-1} \sim N(0,h_t)$ and $h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \left(\varepsilon_{t-i} - \lambda\sqrt{h_{t-i}}\right)^2 + \sum_{i=1}^{p} \beta_i h_{t-i}$.

Moreover, the conditional mean and variance of y_t are:

$$m_t = E(y_t|F_{t-1}) \text{ and } h_t = Var(y_t|F_{t-1}).$$

Under LRNVR, we notice that the conditional mean of yt is dependent on the conditional volatility process, i.e., the form of m_t affects the volatility dynamics while the risk neutralized conditional mean return the same, i.e., $r - \dfrac{1}{2}h_t$.

Observe that the LRNVR can be applied only when the driving noise is normally distributed. When the LRNVR concept is present, the future prices of the asset-objects can be expressed by

$$S_i(T) = S_i(0) = exp\left[rT - 0.5\sum_{t=1}^{T} h_{i,t} + \sum_{t=1}^{T} \sqrt{h_{i,t}}\varepsilon^{i,t}\right],$$

which $S_i(0)$ is the last price of the period under review for each $i = 1,2$.

2.5 Copula Functions

The definition of copulas refers to the decomposition of a n-dimensional cumulative function F into two parts, these being their marginal cumulative distributions F_i for $i = 1,...,n$, and the copula C, where it describes the dependency part of the distribution. Thus, a copula is a multivariate distribution function in which it has uniform marginal distributions in [0,1] and was introduced by Sklar (1959).

Definition 2.5.1. Let **S** = $(S_1,...,S_n)$ a random vector with cumulative distribution F and marginal distributions F_i, $S_i \sim F_i$, $1 \leq i \leq n$ and S_i is a uniform random variable. A distribution function C with uniform marginal in [0,1] is called the copula of **S** if

$$F = C(F_1,...,F_n).$$

For the bivariate case, we have that the integral probability transform of the random variables S_1 e S_2 guarantees that they are distributed as uniform variables U_i, for $i = 1,2$:

$$F_1(S_1) \sim U_1 \quad e \quad F_2(S_2) \sim U_2.$$

Similarly, we have to $F_i^{(-1)}$ denotes the quantile transformation of F_i, denoted by

$$F_i^{(-1)}(t) = inf\{x \in \Re^1\ F_i(x) \geq t\},$$

which

$$F_i^{(-1)}(U_i) \sim F_i.$$

As $F_i(S_i) \sim U(0,1)$, C is a copula and has the following representation

$$\begin{aligned}
C(u_1,u_2) &= P(F_1(S_1) \leq u_1, F_2(S_2) \leq u_2) \\
&= P(S_1 \leq F_1^{(-1)}(u_1), S_2 \leq F_2^{(-1)}(u_2)) \\
&= F_{\mathbf{S}}(F_1^{(-1)}(u_1), F_2^{(-1)}(u_2)).
\end{aligned}$$

Therefore, C is an F copula by definition

$$\begin{aligned}
F(x_1,x_2) &= P(S_1 \leq x_1, S_2 \leq x_2) \\
&= P(F_1(S_1) \leq F_1(x_1), F_2(S_2) \leq F_2(x_2)) \\
&= C(F_1(x_1), F_2(x_2)).
\end{aligned}$$

Definition 2.5.2 (Sklar (1959))**.** Let F a bivariate distribution function together with marginal F_1 e F_2. Then there is a copula C such that for all x_1 e $x_2 \in \mathbb{R}$

$$F(x_1,x_2) = C(F_1(x_1), F_2(x_2)).$$

If F_1 e F_2 are continuous, so C it's unique. In other cases, C is uniquely defined in Rang $F_1 \times$ Rang F_2. Conversely, if C is a copula and F_1 e F_2 are real distribution functions, so $F(x_1,x_2)$ defined by the equation above is a joint distribution function with marginal F_1 and F_2.

In other words, we represent a joint probability using the marginal ones and a copula represents in a unique way the relation between S_1 and S_2, and hence copulas are known as dependency functions. In the case of continuous and differentiable marginal distributions, the joint density function of the copula is given by

$$f(x_1,x_2) = f_1(x_1) f_2(x_2) c(F_1(x_1), F_2(x_2)),$$

which $f_i(x_i)$ is the respective density for the distribution function F_i e

$$c(u_1,u_2) = \frac{\partial^2 C(u_1,u_2)}{\partial u_1 u_2},$$

is the density of the copula.

Definition 2.5.3 (Analytical Interpretation)**.** A bivariate copula is a function C with the following properties:

1. $C : [0,1]^2 \longrightarrow [0,1]$;
2. $C(u,0) = c(0,u) = 0$ e $C(u,1) = C(1,u) = u$, para todo $u \in [0,1]$;
3. C é 2-increase, *i.e.* , $C(u_2,v_2)+C(u_1,v_1)-C(u_1,v_2)-C(u_2,v_1) \geq 0$, for all u_1,u_2,v_1 e $v_2 \in [0,1]$ which $u_1 \leq u_2$ e $v_1 \leq v_2$.

2.5.1 Dependency Measures

As described earlier, a copula function describes the degree and structure of dependence among random variables. In the form of parametric copulas, its parameter describes the strength of the dependency relation and the copula is associated with two measures of dependence, being they the *Kendall* Tau and the *Spearman* Rho.

Before presenting the two measures, let us define the concept of agreement. We say that (x_i,y_i) and (x_j,y_j) are concordant if $x_i < x_j$ and $y_i < y_j$ or $x_i > x_j$ e $y_i > y_j$. Otherwise, we say that (x_i,y_i) and (x_j,y_j) are discordant if $x_i < x_j$ and $y_i > y_j$ or $x_i > x_j$ and $y_i < y_j$. Therefore, we have that pairs are concordant case $(x_i-x_j)(y_i-y_j) > 0$ and discordant when $(x_i-x_j)(y_i-y_j) < 0$.

Definition 2.5.4 (*Kendall* Tau)**.** Let X_1 e X_2 two random variables, the population version of the *Kendall* Tau is given by

$$\tau_C = P[(X_1-X_2)(Y_1-Y_2) > 0] - P[(X_1-X_2)(Y_1-Y_2) < 0],$$

which (X_2,Y_2) is an independent copy of (X_1,Y_1). Thus, for a sample, Kendall's tau is the empirical probability of the difference between the concordant pairs and the discordant pairs.

Considering a random sample of n observations given by $\{(x_i,y_i)\}_{i=1}^n$. Let n_c the number of matching pairs and n_d the number of discordant pairs. For every sample, we have $\binom{n}{2}$ pairs, and the *Kendall's* Tau is given by

$$\tau_C = \frac{n_c-n_d}{n_c+n_d} = \frac{(n_c-n_d)}{\binom{n}{2}} = \frac{(n_c-n_d)}{0,5n(n-1)}.$$

The relationship between *Kendall's* Tau and a copula function C is expressed by

$$\tau_C = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1.$$

Therefore, we can calculate the degree of dependence between random variables through copula and their estimated parameters. The proof can be viewed in Nelson (1991).

Definition 2.5.5 (*Spearman's* Rho)**.** Let two random variables X_1 and Y_1, the Spearman Rho population version is given by

$$\rho_C = 3(P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0]),$$

which $(X_1,Y_1),(X_2,Y_2)$ and (X_3,Y_3) are independent copies of (X_1,Y_1).

Its sample version can be calculated by applying the linear correlation coefficient of *Pearson* in the data converted in stations, that is, in a vector ordered in ascending order. Let $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ and $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$, the linear correlation coefficient of *Pearson* between **X** and **Y** is given by

$$\rho(X,Y) = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2}\sqrt{\sum(y_i - \bar{y})^2}}.$$

Given the copula C, we can rewrite the rho *Spearman* by

$$\rho_C = 12\int_0^1\int_0^1 C(u,v)dudv - 3.$$

2.5.2 Some Types of Copulas

There is a huge amount of copula functions in the literature (NADARAJAH; AFUECHETA; CHAN, 2018), and therefore, this work will be limited in the definition and application of the nonparametric copula denominated empirical copula and five parametric copula models, being they copula Normal, copula t-student, copula Gumbel, copula Frank and copula Joe.

The normal and t-student copulas are part of the elliptic copula family, where they are characterized by multivariate distributions functions that result from the functions of elliptic distributions. The advantage of using copulas from this family instead of multivariate distributions is the possibility of obtaining structures of non-normal dependencies, that is, a more flexible approach.

The copulas Gumbel, Frank and Joe are part of the Archimedean copula family. In practice, the main difference from this family to ellipticals is the possibility of capturing a dependence structure in the tails in cases where there is some asymmetry. Obtaining an Archimedean copula is not given direct by the Sklar theorem and multivariate distributions.

According to Nelson (1991), to express an Archimedean copula it is necessary to define the generating function of the copula in question ϕ and its pseudo-inverse function ϕ^{-1}. So, let $\phi : [0,1] \longrightarrow [0,\infty[$ which

1. $\phi(1) = 0$;

2. For all $t \in (0,1)$, $\phi^{'}(t) < 0$, that is, ϕ and decreasing;

3. For all t *in* (0,1), $\phi^{''} \geq 0$, that is, ϕ is convex.

In addition, let $\phi^{[-1]} : [0,\infty[\longrightarrow [0,1]$, which

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & \text{se } 0 \leq t \leq \phi(0), \\ 0, & \text{se } \phi(0) \leq t \leq \infty. \end{cases}$$

Case ϕ be convex, the copula function $C : [0,1]^2 \longrightarrow [0,1]$ is defined by

$$C(u,v) = \phi^{[-1]}[\phi(u) + \phi(v)], \quad 0 \leq u,v \leq 1.$$

If $\phi(0) = \infty$, so $\phi^{[-1]} = \phi^{-1}$.

Empirical Copula

Introduced by Deheuvels (1979), the nonparametric estimation of a copula function is used as a tool in the visual and exploratory fit adequacy analysis.

Definition 2.5.6. Consider the random vector (**X**,**Y**). Let $(x_k, y_k)_{k=1}^n$ an observed sample of size n obtained from (**X**,**Y**).The empirical copula C_n associated with these variables is defined by

$$C_n\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}(x \leq x_{(i)}, y \leq y_{(j)}), \quad i,j = 1,\ldots,n,$$

which $x_{(i)}$ e $y_{(j)}$ are sample order statistics.

One important result is that empirical copula converges to true copula when sample size grows (DEHEUVELS, 1979; VAART; WELLNER, 1996).

Normal Copula

The normal copula or commonly known as Gaussian copula is called this because it comes from the normal density function for $n \geq 2$. A bivariate Normal copula is expressed by

$$C(u,v) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left(-\frac{t_1^2 - 2\rho t_1 t_2 + t_2^2}{2(1-\rho^2)}\right) dt_1^2 dt_2^2,$$

which $x_1 = \Phi^{-1}(u)$, $x_2 = \Phi^{-1}(v)$, where $\Phi(.)$ denotes the cumulative function of the $N(0,1)$ and $-1 \leq \rho \leq 1$. Therefore, by definition, the functions of marginal distributions are normal standard.

This type of copula has no dependence on the tails of the distributions and is symmetric.

t-student Copula

A t-*student* copula coincides with the distribution function of bivariate t-*student*,where its form is expressed by

$$C(u,v)=\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\frac{1}{2\pi\sqrt{1-\rho^2}}\left(1+\frac{t_1^2-2\rho t_1t_2}{\nu(1-\rho^2)}\right)^{-(\nu+2)/2}dt_1dt_2,$$

which ν represents the degrees of freedom of t-*student*. As in the case of normal copula, the marginal bivariate t-student copula t-*student* coincide with standard t-*student*, where $x_1=t_\nu^{-1}(u)$ and $x_2=t_\nu^{-1}(v)$.

This type of copula does not have association in the tails, which favors its use in extreme events, such as, for example, unplanned oscillations in the stock market. However, given the symmetry of the function, the degree of dependence on the upper tail is equal to the lower tail.

Gumbel Copula

The Gumbel copula is characterized by the dependence only on the upper tail and is represented by

$$C(u,v)=exp\left(-\left[(-ln(u))^\theta+(-ln(v))^\theta\right]^{1/\theta}\right),$$

which $\theta\in[1,\infty]$. When $\theta\longrightarrow\infty$ dependence is perfectly positive and independent when $\theta=1$.

Frank Copula

The form of a Frank copula is expressed through

$$C(u,v)=-\frac{1}{\theta}ln\left(1+\frac{[exp(-\theta u)-1][exp(-\theta v)-1]}{exp(-\theta)-1}\right)$$

which $\theta\neq 0$. When $\theta\longrightarrow\infty$ we have perfect positive dependence and we have the case of independence when we $\theta\longrightarrow 0$. This copula has the same dependence on both function tails, such as elliptic copulas.

Joe Copula

The Copula Joe is expressed by

$$C(u,v)=1-\left([1-u]^\theta+[1-v]^\theta-[1-u]^\theta[1-v]^\theta\right)^{1/\theta},$$

which $1\leq\theta\leq\infty$. When $\theta=1$ we have the case of independence and the case of perfect positive dependence when $\theta\longrightarrow\infty$.

2.5.3 Graphical Representation of a Copula

In the case of bivariate copulas, the graphical representation is performed on a continuous surface in the unit cube $[0,1]^3$, where its limits are defined by the quadrilateral of vertices (0,0,0), (0,1,0), (1,0,0) e (1,1,1). The comparison of different copulas is performed through the contour curves (contours), which are sets in [0,1] given by C(u,v) = k, which k is a constant.

Figures 1, 2, 3, 4 and 5 show the densities functions for the parametric copulas defined in the previous subsection and their respective contours for a fixed parameter.

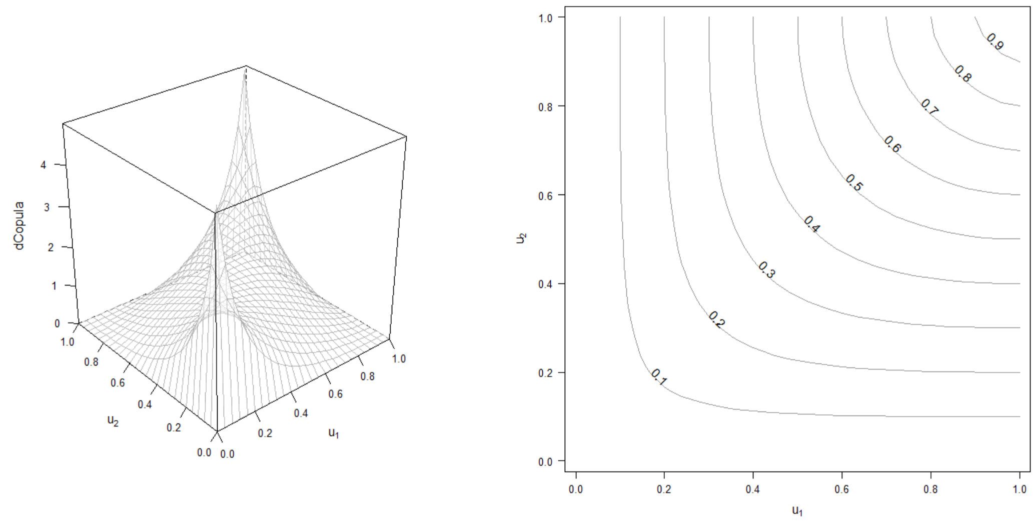

Figure 1 – Density of copula Normal left and right contour with θ = 0.7.

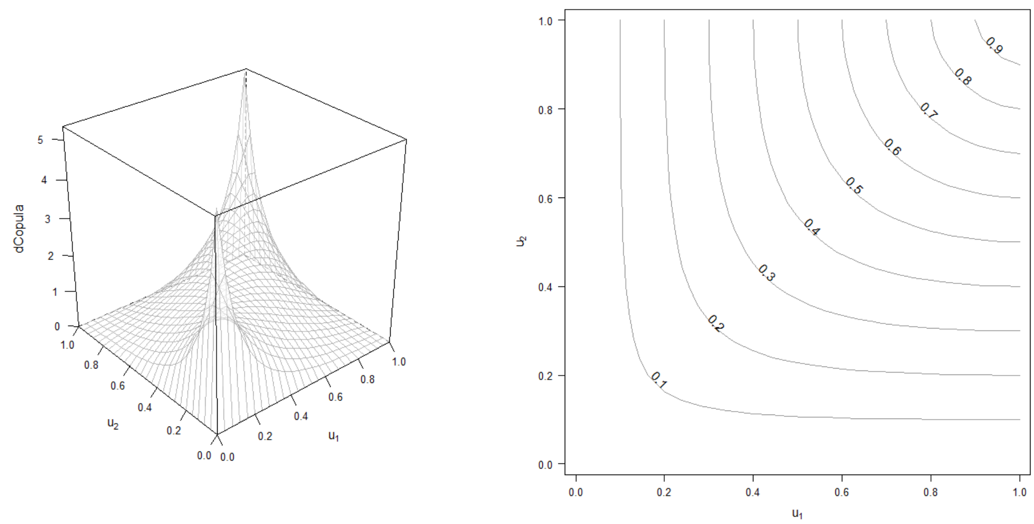

Figure 2 – Density of Copula t-*student* left and right contour with θ = 0.7 and ν = 15.

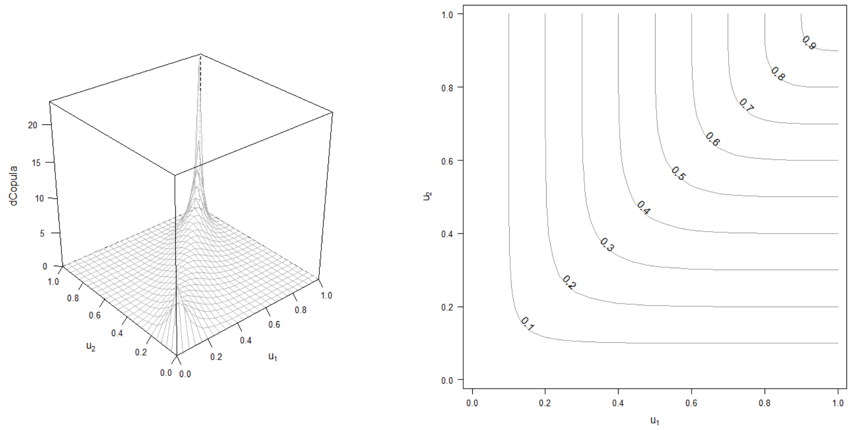

Figure 3 – Density of the Gumbel copula on the left and right contour with $\theta = 4$.

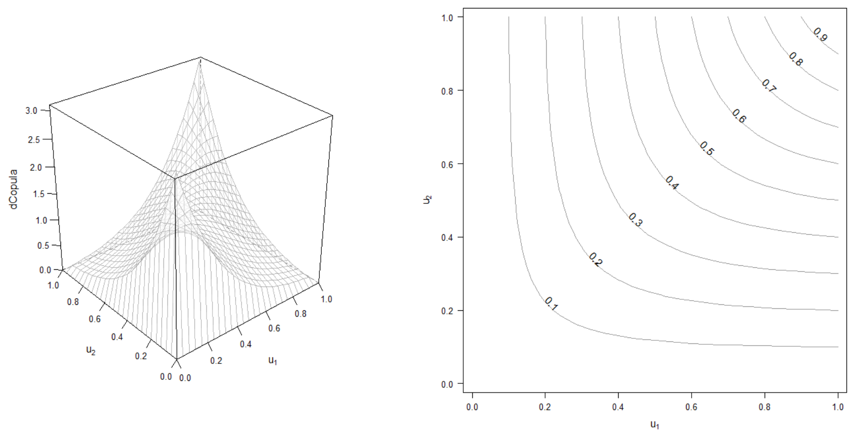

Figure 4 – Density of the Frank copula on the left and right contour with $\theta = 4$.

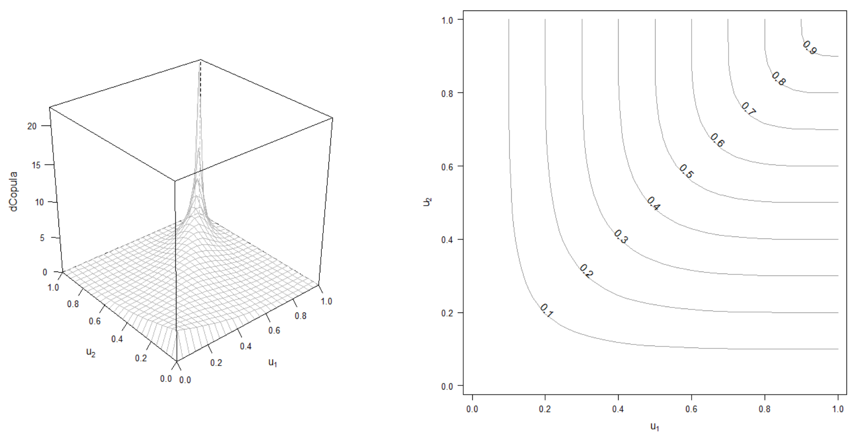

Figure 5 – Density of the Joe copula on the left and right contour with $\theta = 4$.

2.5.4 Simulating from Copulas

This subsection aims to describe the algorithms used to generate random samples from a copula model.

Normal and t-student Copula

In this case, copulas are derived from a sklar theorem and multivariate distribution, the process of generating pseudo random variables becomes particularly easier. The first step is to simulate pseudo observations of the multivariate variable intrinsic to the copula in question and in the second step to transform those results into even marginal ones through the cumulative distribution. Therefore, in both cases we will use the algorithms proposed by Schmidt (2007), which are described below.

Algorithm 1: (Cópula Normal)

1. For an arbitrary covariance matrix $\tilde{\Sigma}$ we obtain the correlation matrix Σ through the scaling of each component to obtain variance equal to 1.

2. Performs decomposition *Cholesky* $\Sigma = \mathbf{A}^{'}\mathbf{A}.$

3. Generate pseudo independent and identically distributed observations *(i.i.d)* of a standard normal $\tilde{X}_1, ..., \tilde{X}_d$.

4. Calculate $(X_1, ..., X_d)^{'} = \mathbf{X} = A\tilde{X}$ from $\tilde{X} = (\tilde{X}_1, ..., \tilde{X}_d)$.

5. Return $U_i = \Phi(X_i), i = 1, ..., d$ which Φ is the cumulative distribution function of a standard normal.

Algorithm 2: (Cópula t-*student*)

1. For an arbitrary covariance matrix $\tilde{\Sigma}$ we obtain the correlation matrix Σ through the scaling of each component to obtain variance equal to 1.

2. Generate a multivariate normal $\mathbf{X}$ with covariance matrix obtained in step 1.

3. Generate independent samples $\varepsilon \sim \chi^2_\nu$ from $\varepsilon = \sum_{i=1}^{\nu} Y_i^2$, which Y_i are sample *i.i.d.* $N(0,1)$.

4. Return $U_i = t_\nu(X_i/\sqrt{\varepsilon/\nu})$, $i = 1, ..., d$ which t_ν is the cumulative function of a univariate t-*student* distribution with degrees of freedom ν.

Archimedean Copula

For the Gumbel copula, we will follow the algorithm defined by Genest and Rivest (1993) and Nelsen (2007). Let $F_C(t)$ be the distribution function of the copula of interest with generating function ϕ:

Algorithm 3: (Copula Gumbel)

1. Generate two independent samples of uniform distribution (v_1, v_2).
2. Fix $w = F_C^{(-1)}(v_2)$, $\quad F_c(t) = t - \dfrac{\phi(t)}{\phi(t)'}$.
3. Fix $u_1 = \phi^{(-1)}[v_1\phi(w)]$ and $u_2 = \phi^{(-1)}[(1-v_1)\phi(w)]$.

Therefore, the pair of interest is (u_1, u_2). In the case of the Gumbel copula, we obtain

1. Generate two independent samples of uniform distribution (v_1, v_2).
2. Fix $F_C(w) = w\left(1 - \dfrac{ln(w)}{\theta}\right) = v_2$, and solve numerically for $0 < w < 1$.
3. Fix $u_1 = exp[v_1^{1/\theta} ln(w)]$ and $u_2 = exp[(1-v_1)^{1/\theta} ln(w)]$.

Frank Copula

Sampling from the Frank copula can be obtained through the conditional distribution approach discussed in Nelson (1991).

Algorithm 4: (Copula Frank)

1. Generate two independent samples of uniform distribution (v_1, v_2).
2. Fix $u_2 = -\dfrac{1}{\theta} ln\left(1 + \dfrac{v_2(1-e^{-\theta})}{v_2(e^{-\theta u_1} - 1) - e^{-\theta u_1}}\right)$.

Joe Copula

It is possible to generate random samples of Joe copula through the algorithm discussed in the case of Gumbel copula.

Algorithm 5: (Copula Joe)

1. Generate two independent samples of uniform distribution (v_1, v_2).

2. Fix $F_C(w) = w - \dfrac{1}{\theta} \dfrac{[ln(1-(1-w)^{\theta})][1-(1-w)^{\theta}]}{(1-w)^{\theta-1}} = v_2$ and solve numerically for $0 < w < 1$.

3. Fix $u_1 = 1 - [1 - [1 - (1-w)^{\theta}]^{v_1}]^{1/\theta}$ e $u_2 = 1 - [1 - [1 - (1-w)^{\theta}]^{1-v_1}]^{1/\theta}$.

CHAPTER

3

OPTION PRICING WITH BIVARIATE RISK-NEUTRAL DENSITY VIA COPULA AND HETEROSCEDASTIC MODEL: A BAYESIAN APPROACH

3.1 Introduction

An option is a financial derivative which the investor acquires the right, but not the obligation, to buy or sell a particular asset for a predetermined price and time, where that price is known as the exercise price. Thus, a put option may be interpreted as an auto insurance policy, where it allows the investor to recover a pre-established value for the asset, even if it has devalued. Regarding the call option, it is compared to the signal paid in the purchase of a house, as it guarantees the fixed price and also the preference in the purchase.

The elaboration of models with the purpose of pricing options began with the authors Black and Scholes (1973) and Merton (1973). The model proposed by the authors uses Brownian motion techniques to obtain the fair price of an option in the univariate case. In the multivariate case, there are several methodologies for achieving the fair price of the options, one of them being the multivariate model of Black and Scholes, where this approach consists of the use of Brownian geometric movement for n assets considering the volatility constant over time.

Tools that accommodate the co-movements between its underlying processes are needed to understand the price behavior of a multivariate option. A primary tool that is widely used by the methods derived from the traditional Black and Scholes model is the multivariate normal distribution. However, the use of this approach implies in linear associations as a measure of dependence between the assets, and empirical evidence shows that a real association between financial series is much more complex (LOPES; PESSANHA, 2018).

The works of Margrabe (1978), Johnson and Shanno (1987), Nelsen (2007) and Shimko (1994) used the linear correlation coefficient to analyze and capture dependence among the underlying assets. However, Embrechts, McNeil and Straumann (2002) and Forbes and Rigobon (2002) criticize the use of this tool, where the authors highlight the stylized facts in finance, such as the heavy tails of returns distributions, their autocorrelations, autocorrelation in squared, groupings of volatilities over time and non-normality.

As an alternative, the use of the copulas theory allows the joint modeling of the assets in which there is a separation of the structure of dependence between the variables and their marginal distributions, where this dependence can be linear, nonlinear and even dependence on the tails. Therefore, Rosenberg (2000) and Cherubini and Luciano (2002) used the copula theory in an attempt to capture the dependency among the assets in the derivative pricing process.

Besides, many models use the premise of constant volatility over time, which may not be observed in finance series (FRENCH; SCHWERT; STAMBAUGH, 1987; FRANSES; DIJK *et al.*, 2000). Thus, to make the pricing process more realistic, Duan (1995) explored the concept of option pricing considering the heteroscedasticity of the assets, where the author proposed to follow a modification of the GARCH process.

Therefore, this paper aims to price and analyze the fair price behavior of bivariate call-on-max option considering marginal heteroscedastic models and the dependence structure modeled via copulas. Besides, the results found will be compared with the values obtained by the classic extended models of Black and Scholes, known as Stulz Closed-form for a call-on-max option.

This work differs from the others found in the literature in two aspects: no studies are comparing the heteroscedastic approach with the classical one (derivations from the Black and Scholes model) for the bivariate case and, furthermore, there are no studies with this methodology considering the Brazilian stock market.

The structure of this paper is divided as follows. Section 3.2 presents the classical models and the heteroscedastic approach for pricing call-on-max option. Section 3.3 gives the Bayesian inference procedure. Section 3.4 presents a simulation study. Section 3.5 shows the application of the methodology in real data of the Brazilian stock market. Finally, Section 3.6 gives some final remarks on this work.

3.2 Conceptual Framework and Model Formulation

In this chapter, we introduce the Stulz (1982) model, which is an extension of the Black and Scholes model for the bivariate case for the call-on-max option and the Duan (1995) model, where the author considers the heteroskedasticity of the underlying assets of the option. Besides, we will introduce how to use the copula theory to model the joint distribution of assets, to capture non-linear dependence between the assets.

3.2.1 Call-on-max Option

A European option call on the maximum of two risky assets (call-on-max) is defined based on the maximum price between two assets. The payoff function of this option is given by

$$g(S(T)) = max[max(S_1(T), S_2(T)) - K, 0],$$

where S_i is the price of the $i-th$ asset *(i = 1,2)* at the maturity date T and K is the strike price or exercise price. In this work will be discussed two methodologies for obtaining $g(S(T))$, where the first approach below considers the heteroskedasticity of the object assets and the non-linear correlation structure. For comparing this methodology, the second approach is the closed-formula proposed by Stulz (1982), where it was the first approach for pricing the call-on-max option, where the author considered the volatility constant over the maturity time of the option and the linear correlation between the assets.

3.2.2 First Approach: Duan Model and Copulas

To introduce heteroscedasticity, we will use the fundamental theorem of asset pricing described by Delbaen and Schachermayer (1994). This theorem states that since the stock price $S_i(T)$ *(i=1,2)* is free from arbitrage and present in a complete market (HULL, 1991), there exists a measure of probability $\mathbb{Q}$ such that the discounted price of the stock, $e^{-r(T-t)}S_i(T)$, is a martingale under $\mathbb{Q}$ and $\mathbb{Q}$ is equivalent to the real world probability measure $\mathbb{P}$.

The fair price of the call-on-max option depends on the dependency structure among the object assets since its price is defined as an expected value (by definition and ownership of a martingale measure, for more details, see Madan and Milne (1991)). Therefore, we define the following theorem to perform the pricing.

Definition 3.2.1. Let S_1 and S_2 be two stocks traded in a complete and free arbitrary market. In addition, be t the present date, T the maturity date and r the fixed risk-free rate yield, then the option price considering the payoff function $g(S_1, S_2) = max[max(S_1(T), S_2(T)) - K, 0]$ is

$$v(t, S_1, S_2) = e^{-r(T-t)}E^{\mathbb{Q}}[max[max(S_1(T), S_2(T)) - K, 0]|F_t] \tag{3.1}$$
$$= e^{-r(T-t)} \int_0^{\infty} \int_0^{\infty} max[max(S_1(T), S_2(T)) - K, 0] f^{\mathbb{Q}}_{S_1,S_2}(x_1, x_2)dx_1dx_2, \tag{3.2}$$

which $f^{\mathbb{Q}}_{S_1,S_2}$ is the the joint density function of the two measures under neutral risk probability $\mathbb{Q}$, which in this work will be modeled by copula functions, and F_t is a filtering containing all information about the assets up to time t.

Thus, we will express the joint density function using the marginal densities $f_{S_1}(x_1)$ e $f_{S_2}(x_2)$ by means of copula functions as follows

$$f^{\mathbb{Q}}_{S_1,S_2}(x_1, x_2) = c^{\mathbb{Q}}(F^{\mathbb{Q}}_{S_1}(x_1), F^{\mathbb{Q}}_{S_2}(x_2)) f^{\mathbb{Q}}_{S_1}(x_1) f^{\mathbb{Q}}_{S_2}(x_2),$$

which $c^{\mathbb{Q}} = \dfrac{\partial^2 C^{\mathbb{Q}}(x_1,x_2)}{\partial x_1 \partial x_2}$, which $C^{\mathbb{Q}}(.)$ is a copula function.

Copulas are useful tools in constructing joint distributions (SHARIFONNASABI *et al.*, 2018). That is, copula is a multidimensional distribution function in which the marginal distributions are uniform in $[0,1]$. A bivariate copula is a function that satisfies $C : I^2 \longrightarrow I \in [0,1]$ that satisfies the following conditions

$$C(x_1,0) = C(0,x_1) = 0 \quad and \quad C(x_1,1) = C(1,x_1) = x_1, \quad x_1 \in I,$$

and the 2-increasing condition

$$C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0,$$

for all u_1,u_2,v_1 and $v_2 \in [0,1]$ such $u_1 \leq u_2$ and $v_1 \leq v_2$.

One of the most famous theorems in copula theory is the Sklar theorem. According to Sklar's theorem (SKLAR, 1959), any bivariate cumulative distribution H_{S_1,S_2} can be represented as a function of the marginal distributions F_{S_1} and F_{S_2}. Besides, if the marginal distributions are continuous, the copula exists, is unique and is given by

$$H_{S_1,S_2}(x_1,x_2) = C(F_{S_1}(x_1), F_{S_2}(x_2)),$$

which $C(u,v) = P(U \leq u, V \leq v)$, $U = F_{S_1}(x_1)$ and $V = F_{S_2}(x_2)$.

In the case of continuous and differentiable marginal distributions, the joint density function of the copula is given by

$$f(x_1,x_2) = f_{S_1}(x_1) f_{S_2}(x_2) c(F_{S_1}(x_1), F_{S_2}(x_2)),$$

which $f_{S_1}(x_1)$ and $f_{S_2}(x_2)$ are the density for the distribution function $F_{S_1}(x_1)$ and $F_{S_2}(x_2)$, respectively, and

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial uv},$$

is the density of copula. For further details about copulas, see Nelsen (2007) and Sanfins, Valle *et al.* (2012). In this work we will use the Normal, t-Student, Gumbel, Frank and Joe copulas. Details are given in an annex at the end of this paper.

Therefore, to construct a joint process of neutral risk for the bivariate distribution of the option, the marginal processes are derived first. Duan (1995) defined an option pricing model considering that the variance of the asset-object is not constant over time.

Definition 3.2.2. Let r a fixed risk-free interest rate and $\lambda > 0$. Under the Duan GARCH process (DGARCH) the log returns, $x_t = log\left(\frac{S_t}{S_{t-1}}\right) = log\,(s_t) - log\,(s_{t-1})$, for $t = 1, ..., n.$, are given by

$$x_t = r + \lambda\sqrt{h_t} + \frac{1}{2}h_t + \sqrt{h_t}\varepsilon_t, \quad \varepsilon_t \sim N(0,1), \tag{3.3}$$

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2 h_{t-j} + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{3.4}$$

which the parameters $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta \geq 0$ and $\sum_{j=1}^{q} \alpha_j + \sum_{j=1}^{p} \beta_j < 1$, which the latter condition guarantees that the process variance will not explode, i.e. to maintain the stationarity of the process. The parameter λ can be interpreted as the risk premium.

To apply the DGARCH model in the option pricing process, Duan (1995) defined the concept of *locally risk-neutral valuation relationship* (LRNVR), where it transforms the model of equation 3.4 into a neutral risk measure $\mathbb{Q}$. For more details on the transformation of the real-world measure $\mathbb{P}$ to the neutral risk measure $\mathbb{Q}$, see Duan (1995).

Definition 3.2.3. A measure $\mathbb{Q}$ satisfies the LRNVR if a measure $\mathbb{Q}$ is absolutely continuous in respect to the measure $\mathbb{P}$ (real world). Under $\mathbb{Q}$ we have

$$E^{\mathbb{Q}}\left[\frac{S_t}{S_{t-1}}|F_t\right] = e^r \quad and \quad Var^{\mathbb{Q}}(x_t|F_t) = Var^{\mathbb{P}}(x_t|F_t).$$

This definition shows that the conditional variance is the same for both measures so that we can use the parameters of equation 3.4 under $\mathbb{P}$. With this definition, Duan showed that under local measurement of neutral risk $\mathbb{Q}$, the previously defined DGARCH process becomes

$$x_t = r - \frac{1}{2}h_t + \sqrt{h_t}\varepsilon^*, \quad \varepsilon^* \sim N(0,1), \tag{3.5}$$

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j (\varepsilon_{t-j}^* - \lambda\sqrt{h_{t-j}})^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{3.6}$$

and in this work, as in Duan (1995) and Zhang and Guegan (2008), the orders p = *1* and q = *1* will be used. The construction and derivation of the Duan model is based on the premise of normality of the errors, but it is possible to consider other distributions, as in Fonseca *et al.* (2012). These extensions are being studied in a different manuscript.

When the concept of *locally risk-neutral valuation relationship* is present, the futures prices of the individual assets can be expressed by

$$S_i(T) = S_i(0)exp\left[rT - 0.5\sum_{t=1}^{T} h_{i,t} + \sum_{t=1}^{T}\sqrt{h_{i,t}}\varepsilon_{i,t}^{*}\right],$$

which $S_i(0)$ is the last price of the period under analysis for each $i = 1,2$.

To obtain the expected value of the continuous function given by equation 3.1 of a bivariate vector (S_1,S_2) with cumulative distribution function $H(x_1,x_2)$, we will use Monte Carlo integration expressed by

$$E[g(S_1,S_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(S_1,S_2)dH(x_1,x_2),$$

which can be approximated by following the algorithm below:

1. Generate n observations of bivariate random vector (S_1,S_2);
2. For each observation i, calculate $g_i = g(x_{1i},x_{2i})$, for $i = 1,2,...,n$;
3. $E[g(S_1,S_2)] \approx \frac{1}{n}\sum_{i=1}^{n} g_i$.

To generate n samples of the specific copula we will use the algorithms proposed by Schmidt (2007) and Nelsen (2007). Therefore, under the probability measure of neutral risk $\mathbb{Q}$, the fair price of the option with payoff function *g(.)* at the maturity time T is given by

$$v(t,S_1,S_2) = \frac{e^{-r(T-t)}}{N}\sum_{i=1}^{N} g(S_{1,i}(T),S_{2,i}(T)). \tag{3.7}$$

In order to compare the consistency of the results obtained by the duan model and copulas approach, we will examine the prices generated by applying the closed formula of Stulz (1982), where it is a derivation of the Black and Scholes model for the bivariate case, where the author considers that the active objects follow a geometric Brownian motion, as in Black and Scholes (1973) and Merton (1973).

3.2.3 Second Approach: Stulz Closed-Form Solution

The closed formula proposed by Stulz (1982) has two significant limitations, being that the volatility of the asset-object is considered constant throughout the time of maturity and the joint distribution is a bivariate normal, which implies a linear correlation between the assets. The fair price for the call-on-max option is set by

$$\begin{aligned} c_{max}(S_1,S_2,K,T) = S_1 e^{-rT} M(y_1,d;\rho_1) + S_2 e^{-rT} M(y_2,-d+\sigma\sqrt{T};\rho_2) - \\ Ke^{-rT} * [1 - M(-y_1+\sigma_1\sqrt{T}, -y_2+\sigma_2\sqrt{T};\rho)], \end{aligned}$$

where

$$d = \frac{ln(S_1/S_2)+(\sigma^2/2)T}{\sigma\sqrt{T}}, \quad y_1 = \frac{ln(S_1/K)+(\sigma_1^2/2)T}{\sigma_1\sqrt{T}}, \quad y_2 = \frac{ln(S_2/K)+(\sigma_2^2/2)T}{\sigma_2\sqrt{T}},$$
$$\sigma = \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}, \quad \rho_1 = \frac{\sigma_1-\rho\sigma_2}{\sigma} \quad and \quad \rho_2 = \frac{\sigma_2-\rho\sigma_1}{\sigma},$$

where S_i is the price of stock i, K the strike price, T the time for the option to expire in years, r the risk-free interest rate, σ_i the stock volatility of asset i, ρ the linear correlation between the two assets, $N(x)$ the cumulative function of the standard normal distribution and $M(a,b;\rho)$ the cumulative function of the bivariate normal distribution in *(a,b)* with linear correlation coefficient ρ.

3.3 Bayesian Inference

Given a 2-dimensional copula, $C(u_1,u_2)$, and two univariate distributions, $F_{S_1}(x_1)$ and $F_{S_2}(x_2)$, the joint density function is given by

$$f(x_1,x_2) = c(F_{S_1}(x_1),F_{S_2}(x_2))\prod_{i=1}^{2} f_{S_i}(x_i),$$

where f_{S_i} represents the marginal density functions and c is the density funcion of the copula which is given by

$$c(u_1,u_2) = \frac{f(F_{S_1}^{-1}(u_1),F_{S_2}^{-1}(u_2))}{\prod_{i=1}^{2} f_{S_i}(F_{S_i}^{-1}(u_i))}.$$

The marginal distribution for each x_{it} is given by $u_{it} = F_{S_i}(x_{it}) = F_{\varepsilon_i}([x_{it}-\mu_{it}]/\sqrt{h_{it}})$, where $F_{\varepsilon_i}(.)$ denotes the univariate distribution function of ε_{it} (AUSIN; LOPES, 2010; ROSSI; EHLERS; ANDRADE, 2012). Therefore, the joint density of x_t is then given by,

$$f(x_{1t},x_{2t}) = c(u_{1t},u_{2t})\prod_{i=1}^{2} f_{S_i}(x_{it}) = c(u_{1t},u_{2t})\prod_{i=1}^{2}\frac{1}{\sqrt{h_{it}}}f_{\varepsilon_i}\left(\frac{x_{it}-\mu_{it}}{\sqrt{h_{it}}}\right),$$

where $f_{\varepsilon_i}(.)$ is the marginal density function of each ε_{it} and μ_{it} is the mean of duan process.

Now, given a bivariate density function $f(.)$ with joint distribution function $F(.)$ and corresponding marginal densities $f_{S_i}(.)$ the copula density is obtained and then,

$$f(x_{1t},x_{2t}) = \frac{f(F_{S_1}^{-1}(u_{1t}),F_{S_2}^{-1}(u_{2t}))}{\prod_{i=1}^{2} f_{S_i}(F_{S_i}^{-1}(u_{it}))}\prod_{i=1}^{2}\frac{1}{\sqrt{h_{it}}}f_{\varepsilon_i}\left(\frac{x_{it}-\mu_{it}}{\sqrt{h_{it}}}\right).$$

In this work we will use bayesian inference, which is an approach that describes the model parameters by probability distributions. It offers a natural way to introduce parameter uncertainty in the estimation of volatilities. We design here a two-step Bayesian algorithm, for more details in Ausin and Lopes (2010). In the first step, we estimate each marginal series independentyly considering a univariate Duan GARCH model under measure $\mathbb{P}$ given in equation 3.3, where $x_{it}|h_{it} \sim N(r+\lambda\sqrt{h_t}-1/2h_t, h_t)$, for *i=1,2*. For each marginal series, we have four parameters to estimate $\theta_i = (\alpha_{0,i}, \alpha_{1,i}, \beta_i, \lambda_i)$, for *i = 1,2*, and the log-likelihood is given by

$$l(\theta_i|x_t) = -\frac{n}{2}\left[log(2\pi)+\frac{1}{n}\sum_{t=1}^{n}\left[log(h_t)+\frac{(x_t-r-\lambda\sqrt{h_t}+1/2h_t)^2}{h_t}\right]\right].$$

Therefore, we define an MCMC algorithm for sample from the posteriori distribution of θ_i for each series with a Gibbs sampling scheme, where each parameter is updated using a Metropolis-Hastings. For each element of the Monte Carlo sample of size *N*, we can obtain a set of residuals,

$$\alpha_{0,i}^{(n)}, \alpha_{1,i}^{(n)}, \beta_i^{(n)}, \lambda_i^{(n)} \Longrightarrow \varepsilon_{it}^{(n)} = \frac{x_{it}-\mu_i^{(n)}}{\sqrt{h_{it}^{(n)}}},$$

for *t = 1,...,T*, and for *n = 1,...,N*, where $\mu_t = r+\lambda\sqrt{h_t}-1/2h_t$ denote the mean process.

Thus, we can estimate the residual for each time *t* for each series as follows,

$$\hat{\varepsilon}_{it} = \frac{1}{N}\sum_{n=1}^{N}\varepsilon_{it}^{(n)},$$

for *i=1,2*.

To estimate the copula parameters, θ_c, we plug in these estimations in the likelihood of specific copula using

$$\hat{U}_{it} = F^{-1}(F(\hat{\varepsilon}_{it})),$$

and obtaining the following likelihood functions for θ_c,

$$l(\theta_c|x_t) = \sum_{i=1}^{n} log\, c_\theta(\hat{U}_{it}),$$

where c_θ is the density of the copula displayed in annex, θ is a vector of the parameters of the copula and $\hat{U}_i$ refers to the pseudo uniform sample.

Now, we construct another Markov Chain to sample from the posterior distribution of θ_c using Metropolis-Hasting steps, as in Ausin and Lopes (2010) and Rossi, Ehlers and Andrade (2012).

3.3.1 Prior Distributions

In the Bayesian approach we need to specify prior distributions for the vector of parameters which define the marginal Duan GARCH model, i.e. $\alpha_{0,i}, \alpha_{1,i}, \beta_i$ and λ_i, *i=1,2* plus the parameters in the copula functions, i.e. ρ_i in the Normal, Gumbel, Frank and Joe Copulas and ρ_i and ν_i in the t copula. Following Ausin and Lopes (2010), for each parameter we assume a uniform prior over their respective domains imposing the stationary condition, i.e., $\alpha_{1,i} + \beta_i \leq 1$. We shall adopt these prior choices in the simulation studies of Section 3.4.

3.3.2 Selection Criteria for Marginal and Joint Models

In order to verify if the distribution of the residues follows a standard normal distribution the Kolmogorov-Smirnov and Shapiro-Wilk tests will be used for a random sample. The Kolmogorov-Smirnov (KS) test for a random sample is used to compare a dataset through its empirical distribution function F(**x**) with a known cumulative function G(**x**). The null hypothesis is that $x \sim G$, and the KS statistic is defined by $D_{KS} = max(|F(x) - G(x)|)$. The Shapiro-Wilk (SW) test statistic is $W = (\sum_{i=1}^{n} a_i x_{(i)})^2 / \sum_{i=1}^{n} (x_i - \bar{x})^2$, where $x_{(i)}$ is the *i-th* order statistic, $\bar{x}$ is the sample mean and the constants a_i is given by $(a_1, ..., a_n) = m^T V^{-1} / (m^T V^{-1} V^{-1} m)^{0.5}$, where $m = (m_1, ..., m_n)^T$, and $m_1, ..., m_n$ are the expected values of the order statistics of independent and identically distributed random variables sampled from the standard normal distribution, and V is the covariance matrix of those order statistics.

The Ljung-Box test (LB) will be performed to test whether residuals from marginal distributions have independent increments. Considering the null hypothesis that the residuals do not have autocorrelation, the Ljung-box test statistic is given by $Q = N(N+2)\sum_{k=1}^{M} \dfrac{\rho_k^2}{N-k}$, which N is the sample size, M is the number of autocorrelated lags and ρ_k is the autocorrelation in lag k. Moreover, under the null hypothesis, the test statistic follows asymptotically a distribution $\chi^2(M)$.

In order to make the choice of the best copula model in the bivariate distribution fitted, the Expected Akaike Information Criteria (EAIC), Expected Bayesian Information Criterion (EBIC) and Deviance Information Criteria (DIC) will be adopted. These are given by $EAIC = E[D(\theta_M)] + 2np_M$, $EBIC = E[D(\theta_M)] + log(n)np_M$ and $DIC = 2E[D(\theta_M)] - D(E[\theta_M])$ respectively, where np_M represents the number of parameters in model M, θ_M is the set os parameters in model M, n is the sample size and $D(.)$ is the deviance function defined as minus twice the log-likelihood function. For more details see, Spiegelhalter *et al.* (2002).

3.4 Simulation Study

In this chapter, we illustrate the proposed methodology with artificial time series. The simulation study main concern is to assess the bias, mean squared error (MSE) and coverage probabilities of the posterior means for the parameters of marginals and copula obtained by two-step Bayesian algorithm described previously.

First, we simulate the innovation distribution (ε_{1t} and ε_{2t}) through a copula with a fix parameter ρ. For show the proposed simulation study will be used the Frank copula, where it obtains good fitted to financial series in several works in the literature (KLUGMAN; PARSA, 1999; CHERUBINI; LUCIANO, 2002; HÜRLIMANN, 2004). Then we simulate bivariate time series Duan GARCH processes with these copula-dependent innovations for each sample size (n = 250, 500 and 1000) and fixed interest rate r at 7% per annum with the following univariate models,

$$
\begin{aligned}
x_{1t} &= r - \frac{1}{2}h_{1t} + \sqrt{h_{1t}}\varepsilon_{1t}, \quad \varepsilon_{1t} \sim N(0,1),\\
x_{2t} &= r - \frac{1}{2}h_{2t} + \sqrt{h_{2t}}\varepsilon_{2t}, \quad \varepsilon_{2t} \sim N(0,1),\\
h_{1t} &= 0.012 + 0.17(\varepsilon_{1t-1} - 0.12\sqrt{h_{1t-1}})^2 + 0.81h_{1t-1},\\
h_{2t} &= 0.01 + 0.15(\varepsilon_{2t-1} - 0.1\sqrt{h_{2t-1}})^2 + 0.8h_{2t-1},
\end{aligned}
$$

and the Frank copula parameter is fixed in ρ = 2.

The priors were chosen following Ausin and Lopes (2010), as described in the previous subsection. For each setup, we generated 500 (replication) bivariate time series. The proposed two-stage MCMC algorithm is run for 20,000 iterations with first 10,000 as burn-in iterations. The code was made in R.

The Table 3 presents the true values, posterior mean, posterior median, highest posterior density (HPD) interval 95%, size of HPD interval, bias, MSE and coverage probabilities for each model parameter obtained from MCMC outputs. Observe that, the bias and MSEs decrease tending to zero when the sample size increases. We also noticed that the posterior means are very close to the posterior medians. Furthermore, the amplitude of the HPD interval tends to decrease as the sample size increases. The coverages are closer to the nominal ones for increasing sample sizes. Therefore, through this simulation study, the asymptotic properties of the model are satisfactorily verified.

3.5 Application to Brazilian Stock Market Data

In this chapter our methodology is illustrated on real Brazilian stock market data, specifically the stock price of Banco do Brasil (BBAS3) and Itau (ITUB4), where the prices of the option will be compared with the results of the methodology proposed by Stulz (1982) presented in chapter 2. The data is from 03/Jul/2014 to 22/Mar/2017, containing 754 daily observations and here S_0 is R$ is the stock price at 22/Jul/2017. Data was collected on the Google

Finance website. Table 4 presents the descriptive statistics of log-return data, where it is given by $x_{it} = log\left(S_{it}/S_{it-1}\right) = log\left(S_{it}\right) - log\left(S_{it-1}\right)$ for $t = 1,...,n$ and $i = 1,2$.

Table 3 – Simulation Results to n = 250, 500 and 1000.

Sample Size	**Parameter**	**True Value**	**Posterior Mean**	**Posterior Median**	**HPD 95%**	**Size of HPD**	**Bias**	**MSE**	**C.P**
n = 250	α_{01}	0.012	0.0593	0.0525	[0.0082;0.1291]	0.1209	-0.0473	0.0022	0.9266
	α_{11}	0.170	0.2093	0.2024	[0.0941;0.3361]	0.2420	-0.0393	0.0015	0.9725
	β_1	0.810	0.6560	0.6690	[0.4337;0.8449]	0.4112	0.1540	0.0237	0.8716
	λ_1	0.120	0.1364	0.1337	[0.0299;0.2430]	0.2131	-0.0164	0.0003	0.9358
	α_{02}	0.010	0.0441	0.0411	[0.0081;0.0877]	0.0796	-0.0341	0.0012	0.7982
	α_{12}	0.150	0.1989	0.1900	[0.0700;0.3453]	0.2752	-0.0489	0.0024	0.9725
	β_2	0.800	0.5677	0.5757	[0.2938;0.8141]	0.5203	0.2323	0.0540	0.7890
	λ_2	0.100	0.1120	0.1084	[0.0172;0.2112]	0.1940	-0.0120	0.0001	0.9633
	ρ	2.000	1.9317	1.9314	[1.1733;2.7030]	1.5297	0.0683	0.0047	0.9639
n = 500	α_{01}	0.012	0.0232	0.0217	[0.0065;0.0429]	0.0363	-0.0112	0.0001	0.9662
	α_{11}	0.170	0.1878	0.1849	[0.1177;0.2639]	0.1462	-0.0178	0.0003	0.9595
	β_1	0.810	0.7655	0.7698	[0.6712;0.8504]	0.1792	0.0445	0.0020	0.9189
	λ_1	0.120	0.1193	0.1183	[0.0368;0.2012]	0.1644	0.0007	0.0000	0.9595
	α_{02}	0.010	0.0191	0.0195	[0.0048;0.0368]	0.0320	-0.0091	0.0001	0.9797
	α_{12}	0.150	0.1749	0.1706	[0.0929;0.2643]	0.1714	-0.0249	0.0006	0.9527
	β_2	0.800	0.7755	0.7345	[0.5777;0.8547]	0.2770	0.0245	0.0006	0.9392
	λ_2	0.100	0.1024	0.1010	[0.0268;0.1784]	0.1516	-0.0024	0.0000	0.9257
	ρ	2.000	1.9413	1.9412	[1.4021;2.4781]	1.0760	0.0587	0.0035	0.9502
n = 1000	α_{01}	0.012	0.0149	0.0145	[0.0066;0.0241]	0.0176	-0.0029	0.0000	0.9548
	α_{11}	0.170	0.1757	0.1747	[0.1322;0.2195]	0.0872	-0.0057	0.0000	0.9582
	β_1	0.810	0.8017	0.8031	[0.7539;0.8474]	0.0936	0.0083	0.0001	0.9481
	λ_1	0.120	0.1209	0.1210	[0.0338;0.1511]	0.1173	-0.0009	0.0000	0.9502
	α_{02}	0.010	0.0138	0.0130	[0.0056;0.0226]	0.0170	-0.0038	0.0000	0.9409
	α_{12}	0.150	0.1606	0.1582	[0.1045;0.2211]	0.1166	-0.0106	0.0001	0.9620
	β_2	0.800	0.7980	0.7929	[0.6787;0.8505]	0.1718	0.0020	0.0000	0.9492
	λ_2	0.100	0.0933	0.0928	[0.0342;0.1495]	0.1154	0.0067	0.0000	0.9591
	ρ	2.000	1.9861	1.9845	[1.5515;2.3128]	0.7612	0.0139	0.0002	0.9481

Table 4 – Summary descriptive statistics of the daily log returns.

	Min.	Median	Mean	Max.	S.D.	Skewness	Kurtosis
Banco do Brasil	-0.2378	0.0000	0.0008	0.1342	0.0322	-0.2236	7.6770
Itau	-0.0909	0.0003	0.0007	0.1036	0.0215	0.2432	4.7957

As expected, the mean returns of the two stocks are close to zero, means are close to medians, and the returns have kurtosis greater than 3. The skewness presents a different result for the series, where the Banco do Brasil obtained left (negative) asymmetry and the Itau right asymmetry (positive). Figure 6 shows the behavior of the original series and the log-returns, respectively. Similar variability becomes apparent, as indicated by the standard deviation (S.D.) in the descriptive statistics table. This result is expected, given that the two companies are from the same sector industry.

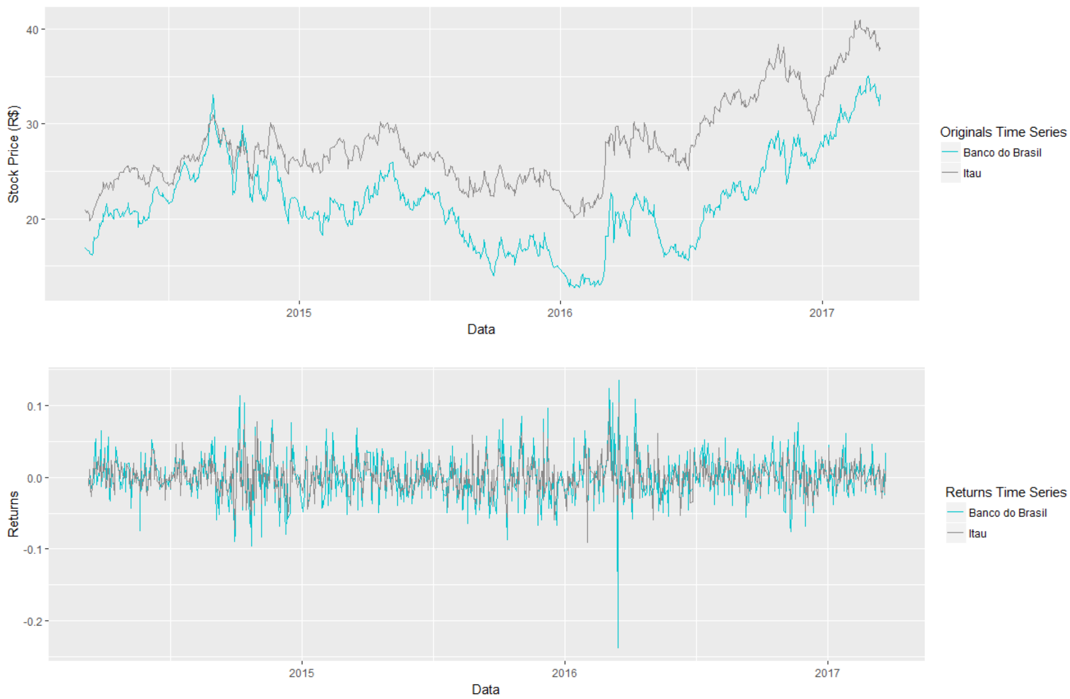

Figure 6 – Original Time Series of Prices and Log-returns.

The scatterplot and histograms provide us a visual analysis of the log-returns dispersions and are shown in Figure 7. Concerning the joint dispersion of the log-returns, we observed the greatest agglomeration around the point of origin (0,0) and a smaller concentration, but not insignificant, in the tails, which is corroborated by the histograms.

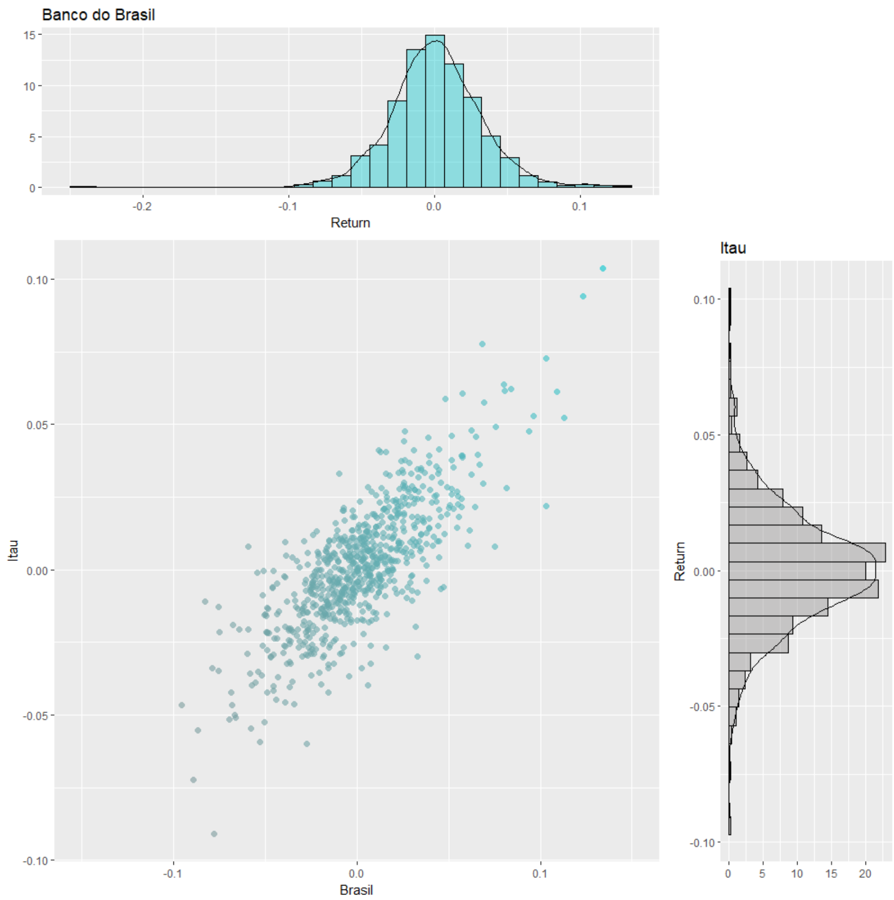

Figure 7 – Histograms and Scatterplot of the log-returns.

Prior distributions for marginals and joint distributions were equal to ones specified in the simulation study. We considered two chains of 100 000 iterations and the first 40 000 were ignored to avoid the influence of first value, i.e., as burn-in. The resulting samples are checked for absence of convergence using the test and the graphics analysis proposed by Geweke *et al.* (1991).

Table 5 shows the values of the Geweke's statistic for each parameter obtained for marginals. Using statistical convergence diagnostics, we can not prove convergence, but these provide evidence for no lack of convergence, since, if the samples are drawn from the stationary distribution of the chain, the Geweke's statistic has an asymptotically standard normal distribution. Also, Figure 8 and Figure 9 show the traces of the posterior samples of each model parameter. These indicate a good mixing performance of the Markov chain as it moves fluidly through all possible states.

Table 5 – Values of the Geweke's statistic for each parameter obtained.

	α_{01}	α_{11}	β_1	λ_1	α_{02}	α_{12}	β_2	λ_2
Chain 1	-0.2564	1.0583	-0.4843	1.3618	-0.7262	-1.4010	1.2620	-0.6024
Chain 2	-1.3027	-0.3860	1.5931	0.7212	1.3068	0.8917	-1.1425	-0.9654

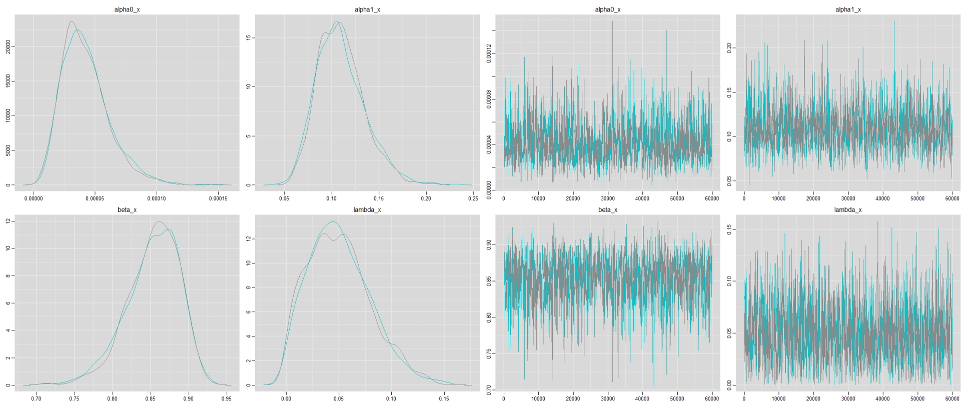

Figure 8 – Densities and Convergence diagrams of the posterior samples of each parameter for the Banco do Brasil.

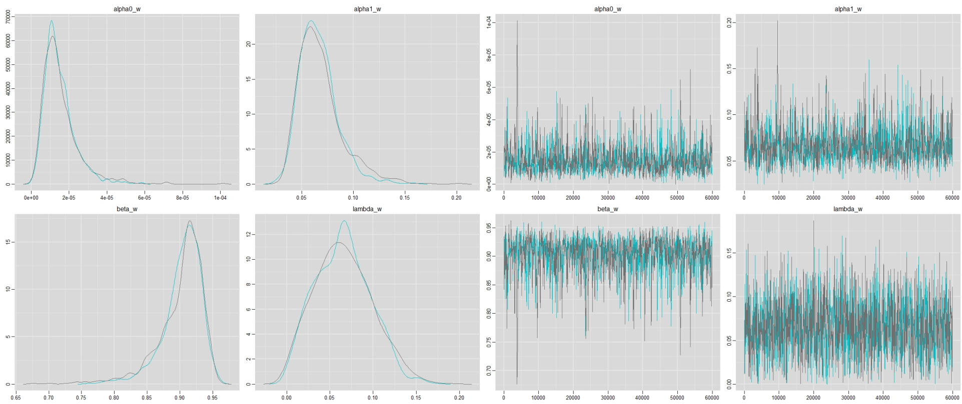

Figure 9 – Densities and Convergence diagrams of the posterior samples of each parameter for the Itau.

Table 6 presents the posterior means together with their 95% HPD credibility intervals for the marginals process and their standard desviation *(s.d)*.

Table 6 – Parameter estimation results.

Parameter	**Posterior Mean**	**S.D.**	**HPD 95%**
α_{01}	0.00004	0.00001	[0.00001;0.00008]
α_{11}	0.10940	0.02495	[0.06025;0.15737]
β_1	0.85390	0.03123	[0.77720;0.91061]
λ_1	0.05035	0.02944	[0.00018;0.10332]
α_{02}	0.00001	0.00000	[0.00000;0.00003]
α_{12}	0.06698	0.02022	[0.03591;0.10247]
β_2	0.90230	0.03278	[0.84367;0.95212]
λ_2	0.06529	0.03260	[0.00871;0.12427]

The DGARCH model assumes that the residues follow a standard normal distribution and that they have independent increments. Table 7 shows the results of KS, Shapiro-Wilk and

LB test for the significance level of 5%. As we can see, we do not reject the null hypothesis that the residues follow a normal distribution and have independent increments.

Table 7 – Kolmogorov-Smirnov, Shapiro-Wilk and Ljung-Box test.

Test	Banco do Brasil	Itau
KS statistic (p-value)	0.0325 (p-value = 0.4553)	0.0428 (p-value = 0.2792)
Shapiro statistic (p-value)	0.9464 (p-value = 0.3193)	0.8639 (p-value = 0.1613)
LB statistic (p-value)	2.7249 (p-value = 0.1249)	0.0791 (p-value = 0.7502)

Figure 10 shows the normal quantile plot for the standardized residuals of the fitted DGARCH(1,1) model for each series. In this particular case, the Gaussian assumption is not perfect, but acceptable for this paper. Here, it is worth mentioning that the normality premise was used due to the original construction of the Duan model (DUAN, 1995). Other asymmetric distributions could and can be used, but it is necessary to obtain the model under measure $\mathbb{Q}$, which may complicate the calculations, for example, to have to use the conditional Esscher transform or Radon-Nikodym derivative method (ROMBOUTS; STENTOFT, 2015; FENGLER; MELNIKOV, 2018).

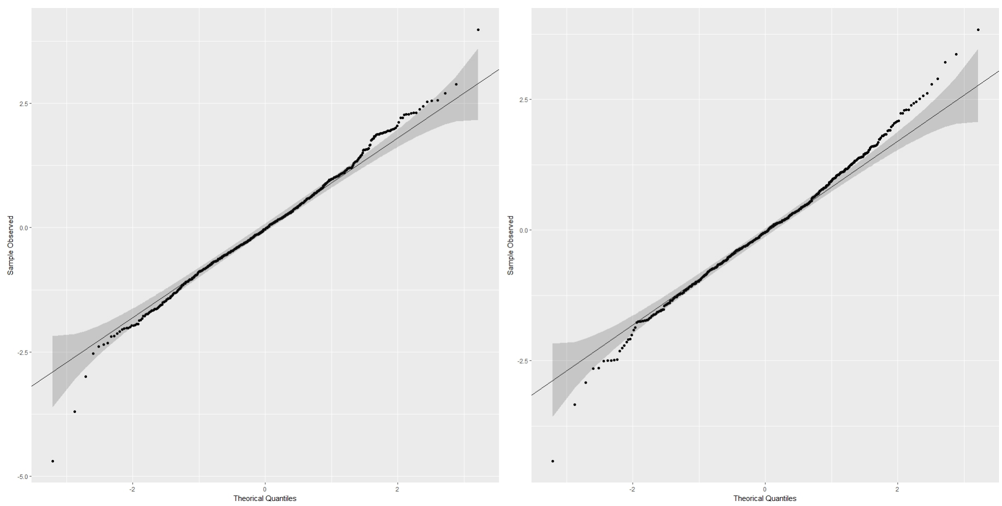

Figure 10 – QQ-plot for the standardized residuals - Banco do Brasil (left) and Itau (right).

Therefore, we conclude that there was a good fit of the DGARCH(1,1) model for both series and, thus, we can follow in the joint modeling through the copulas theory. Table 8 presents the posterior means (mean), *s.d*, 95% HPD credibility intervals and their corresponding EAIC, EBIC and DIC criteria for copulas fitted.

Table 8 – Copulas Fitted.

Copula	Mean	S.D.	HPD 95%	EAIC	EBIC	DIC
Normal	0.7596	0.0123	[0.7354;0.7843]	16839.35	16838.51	16836.48
t-student	0.7724	0.0146	[0.7435;0.7996]	16672.18	16676.8	16670.18
(ν)	[4.9530]	[0.5216]	[3.9518;5.9857]			
Gumbel	2.1183	0.0661	[2.0729;2.1387]	17928.41	17937.65	17926.3
Frank	7.2524	0.3146	[6.6251;7.8433]	16652.83	16657.45	16651.82
Joe	2.2792	0.0796	[2.0294;2.5283]	19743.26	19742.41	19740.39

Table 8 shows that the best copula according to the selection criteria was Frank, followed by t-student, Normal, Gumbel and Joe copulas. The Geweke criterion for Frank's copula obtained the values of 0.2482 and 0.6438 for the first and second chain, respectively, showing that there is no evidence of non-convergence. Figure 11 shows the density and convergence diagram of its parameter.

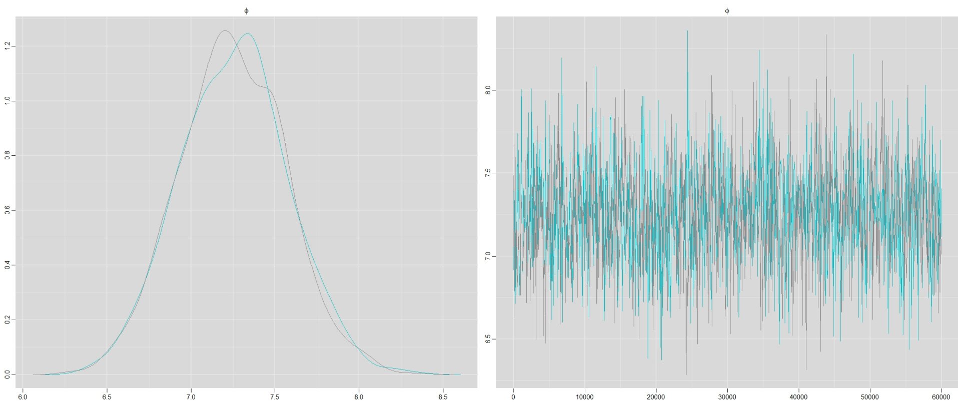

Figure 11 – Density and Convergence diagram of the posterior samples of Frank Copula.

For the sake of space, we omit here the tables and figures with the same results considering the marginal processes but changing the copula structure, which we obtained the same satisfactory results presented for the Frank copula.

3.5.1 Fixed parameters used in Black and Scholes models

To make a comparison of the results of the methodology discussed in this work with the classic Stulz model, we will define some necessary parameters presented in chapter 2. The parameters, their interpretations, and their values are given below.

1. *Interest Rate*: An annualized rate expresses the annual interest rate takes into account the effect of compound interest. That is, the average daily interest rate, annualized based on 252 traded days. In this wotk we used the value 7% per year was chosen in an attempt to standardize the rate over the maturity period according to the SELIC rate presented by

Central Bank of Brazil. However, other rates can be used as a proxy for market expectations, such as the CDI curve.

2. S_i: Stock price of asset i, i=1,2. And we observe $S_1 = R\$33.05$ and $S_2 = R\$38.05$.

3. T: Time of maturity. That is, 1/2 means half a year. We adopt one year.

4. σ_i: The annualized volatility of the stock i. Volatility is the annualized expression of the average variability of the stock return. As the returns were calculated on a daily basis, to obtain volatility, the standard deviation obtained by multiplying the squared root of the annual term used, which in this work is 252 days, should be annualized. We calculate and obtained $\sigma_1 = 43.44\%$ and $\sigma_2 = 30.19\%$ for the last year of each stock price.

5. ρ: The coefficient of linear correlation between the returns of the two assets in the last year. We calculate ρ = 0.7374.

6. K: The strike price of the option. The chosen had as a criterion the use of ATM (moneyness) defined below.

Moneyness is the difference between the strike price and the asset value and is classified into three categories: in-the-money (ITM), at-the-money (ATM) and out-the-money (OTM). The more out-the-money the option is, the less likely it is to exercise on the part of the holder and consequently the more in-the-money, the more likely it is to exercise. Let S_1 be the market price of asset 1 and S_2 the market price of asset 2, Table 9 shows which classification will be used from now on.

Table 9 – Classification of Moneyness.

Classification	**Call Option**	**Put Option**
ITM	$Min(S_1, S_2)$ >Strike	$Max(S_1, S_2)$ <Strike
ATM	$Max(S_1, S_2)$ = Strike	$Max(S_1, S_2)$ = Strike
OTM	$Max(S_1, S_2)$ <Strike	$Min(S_1, S_2)$ >Strike

As we are interested in calculating a call option, we have that an option will be ATM when striking (K) = $R\$38.05$. This extrapolation of the concepts of moneyness to the bivariate case aims to analyze the effect of its classification on the final prices of the options.

3.5.2 Comparison of Methodologies

The call and put of multivariate options are traded over the counter, that is, from individual to individual. Moreover, for this reason, there is no series in which we can check their prices for comparison of fit of models concerning their errors. However, the comparison of models with different assumptions is and can be performed, to efficiently price an option with realistic characteristics.

In this paper we will perform the fitted of several models and compare them, mainly in relation to the models from Stulz (1982), where the latter is the most widely used, widespread and with more credibility in the literature and the market for call-on-max option, because it is a derivation of the model of Black and Scholes (ZHANG; GUEGAN, 2008). Table 10 shows the price of the options considering the call-on-max option defined in chapter 2 and the parameters presented previously, considering all the copulas and the Stulz model. Moreover, 100,000 Monte Carlo simulations were performed to obtain the fair price of the option in equation 3.7.

Table 10 – Option Pricing Call-on-max (R$) with K=R$38.05 and T = 1 year.

Model	**Option Price**
Stulz Model	R$ 5.755644
Normal Copula	R$ 5.563650
t Copula	R$ 5.593955
Gumbel Copula	R$ 5.666805
Frank Copula	R$ 5.600540
Joe Copula	R$ 5.875227

Two strong arguments to give credibility to the results obtained by the copulas are: 1) the marginals process and copulas obtained good joint fitted of the series, and 2) the dependencies derived from these models take into account the non-linear dependence between the observations, which is inherent in the universe of finance. Therefore, the difference obtained between these models and Stulz model brings with it these two arguments that make modeling more realistic.

To analyze the effect of strike price, Table 11 shows the values of the call-on-max option for all models varying strike from R$ 31.00 to R$ 42.00. We verified the same behavior in all the models, that is when we increase the strike price the value of the option decreases. This result was expected because, according to the logic of the options contract, if we expect to buy an option for a higher price on the maturity date, the price of its premium tends to be lower (CHIOU; TSAY, 2008). Figure 12 shows the same results in graphic form.

Table 11 – Option Prices Call on Max (R$) varying Strike.

Strike	Stulz	Normal	t	Gumbel	Frank	Joe
31	9.77469	10.98429	11.02635	11.03995	11.01208	11.36508
32	9.10702	10.10753	10.14828	10.16287	10.13688	10.48244
33	8.47138	9.25741	9.29584	9.31168	9.28815	9.62346
34	7.86822	8.43997	8.47662	8.49324	8.47240	8.79502
35	7.29764	7.66072	7.69669	7.71314	7.69550	8.00356
36	6.75950	6.92642	6.96056	6.97811	6.96281	7.25351
37	6.25337	6.24172	6.27292	6.29216	6.27855	6.54970
38	5.77858	5.60880	5.63776	5.65781	5.64567	5.89643
39	5.33430	5.03038	5.05693	5.07829	5.06646	5.29689
40	4.91953	4.50697	4.53031	4.55341	4.54263	4.75068
41	4.53313	4.03815	4.05868	4.08241	4.07271	4.25778
42	4.17390	3.62216	3.63895	3.66288	3.65373	3.81750

Note that Joe copula obtained the highest values when we varied the variable strike (K). The result is justified because this copula obtained the worst fitted according to the criteria of selection of models used. Besides, the classical Stulz model also obtained discrepant results from the others, a finding that can be based on the constant volatility over time of the option and the modeling through the normal bivariate distribution. The normal, t-student, Gumbel and Frank copulas obtained results very close to each other, an expected result because these distributions obtained very close results in EBIC, EAIC and DIC metrics.

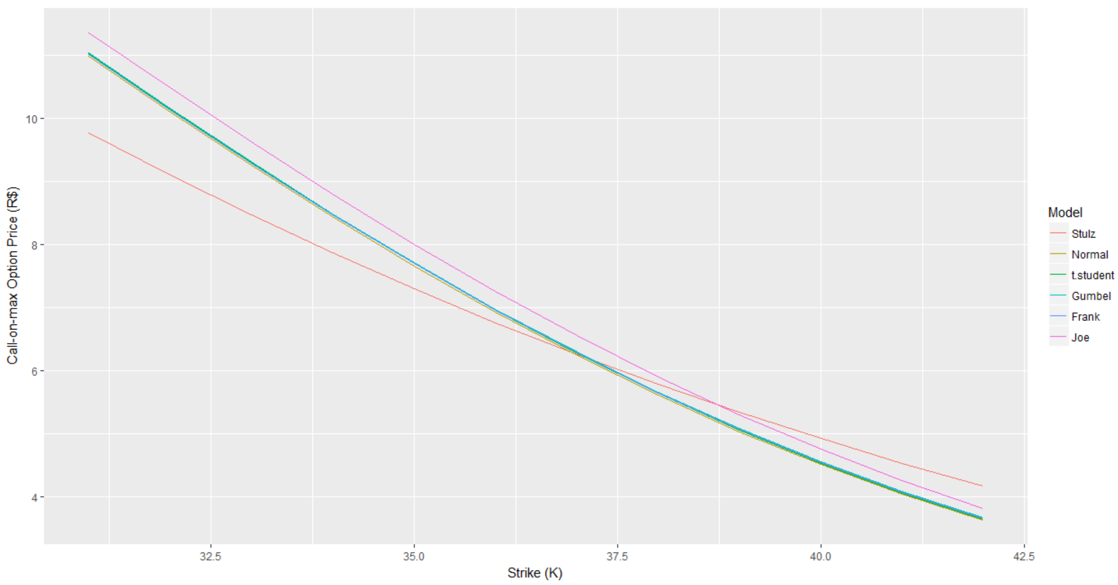

Figure 12 – Option Prices Call on Max (R$) varying Strike.

It should be noted that at the beginning of the graph, when the strike price is at R$ 31.00, there is the most significant difference between the models and that, with the increasing strike, this difference becomes smaller. The finds of Hull and White (1987), Johnson and Shanno (1987) and Kang and Brorsen (1993) corroborate this result, where the authors empirically demonstrated that, in the in-the-money options, the Black and Scholes (BS) model (Stulz model is derived from BS Model) underestimates the options. This result highlights the importance of the joint distribution to capture the dependency structure in the pricing process.

To analyze the impact of the dependency parameter on the final price of the option we adopted the following criterion. The choice of copula for this analysis was based on the good fitted shown in Table 8 and on the selection of a copula that presents negative and positive dependence. So, the analysis of this subsection is based on the t-student copula with 4.9530 degrees of freedom. Besides, authors such as Zhang and Guegan (2008) and Lopes and Pessanha (2018) have found empirical results that t-student copula has a good fit and functional characteristics in the joint modeling of stock returns.

Figure 13 emphasized that when the dependence is negative, the values are higher than with positive dependence. This result corroborates with those found by Chiou and Tsay (2008) for the call-on-max option using the American and Taiwanese indices. An intuitive interpretation is that the values of this option tend to be smaller when the underlying assets move in the same direction as when in opposite directions.

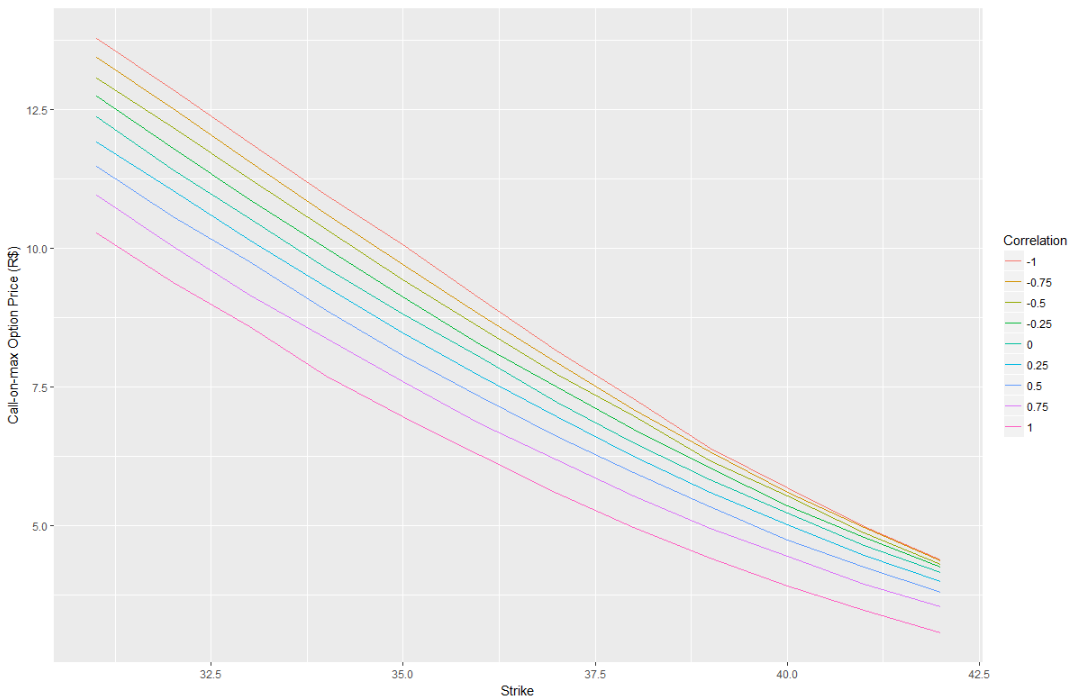

Figure 13 – Option Prices Call on Max (R$) vs. Correlation.

Furthermore, this figure represents the importance of copula selection to represent the joint structure and especially the importance of a good inferential approach, where a high discrepancy between the values is observed, varying the correlation coefficient of the t-student copula, which is between -1 and 1.

3.6 Final Remarks

In this paper, we proposed to accommodate the heteroskedasticity of the assets-objects through the marginal model Duan GARCH and to capture the structure of dependency between them through copulas functions. To compare and analyze the method proposed in this work with Stulz's already consolidated model, we price the call-on-max option for two Brazilian companies stock prices.

As a result, there was evidence that the DGARCH(1,1) model fitted well to the data of the Banco do Brasil and Itau stocks prices, as we did not reject the normality of residues using the KS and SW tests and its increments were not autocorrelated through the Ljung-box test. These results are prerequisites for transforming the data into uniform distribution to adjust the copula functions.

Besides, we verified the good fit of the copula functions, especially Frank and t-student, and these two copulas obtained the values closest to each other. Another result shown in this paper, which corroborates with the options literature, is the effect of a strike at the fair price. Besides, we illustrate the impact of choosing the dependency structure on the final options prices.

CHAPTER

4

GARCH-IN-MEAN MODELS WITH ASYMMETRIC VARIANCE PROCESSES FOR BIVARIATE EUROPEAN OPTION EVALUATION

4.1 Introduction

Multivariate options are excellent tools to manage a portfolio's risk. The first works that had as objective the pricing of options in the univariate case were Black and Scholes (1973) and Merton (1973). Through these works, other authors have used the same theory, *i.e.*, asset-objects follow a Brownian geometric motion and have proposed bi and multivariate models, such as Stulz (1982), Margrabe (1978), Johnson and Shanno (1987), Nelson (1991) and Shimko (1994). However, models derived from Brownian geometric motion methods have the assumptions that the volatilities of the assets are constant over time.

To carry out the pricing with more realistic assumptions, researchers have developed other models. For instance, we use the generalized autoregressive conditional heteroskedasticity (GARCH) family of models, because of its ability to incorporate the stylized facts about asset return dynamics. This kind of modeling is popular in economics and finance (ALMEIDA; HOTTA, 2014). Furthermore, with Black Scholes (BS) models assumptions, any contingent claim can be perfectly replicated by its underlying asset and a riskless bond, so the price of a contingent claim is merely the cost of the replicating portfolio. However, using GARCH-type models, it is generally not possible to construct a perfect replicating portfolio, as the volatility of asset returns is permitted to vary over time. It is necessary to define a risk-neutral measure to use the GARCH-type models to consider a general market equilibrium (LIU; LI; NG, 2015).

The model of Duan (1995) derives a measure of risk-neutral through the standard GARCH model, which the author shows the potential of it concerning the Black and Scholes approach. However, one of the main limitations of the standard GARCH model is the inability to incorporate the effect of asymmetry caused by unplanned returns (NELSON, 1991). Introduced by Black (1976), this effect implies that volatility tends to grow more when there is an unanticipated drop in returns (*i.e.* "bad news") than when there is an unanticipated increase of the same magnitude in returns (*i.e.* "good news"). This effect, also known as a leverage effect, has been included in the GARCH-type models, such as the exponential GARCH (EGARCH), the non-linear asymmetric GARCH (NGARCH) and the Glosten, Jagannathan, and Runkle GARCH (GJR-GARCH) models. It can be used to price options by deriving their risk-neutral measure.

Furthermore, to understand the price behavior of a multivariate option, it is necessary to use tools that accommodate the co-movements between its underlying processes. A primary tool that is widely used by the methods derived from the traditional Black and Scholes model is the multivariate normal distribution modeling. However, the use of such an approach implies in linear associations as a measure of dependence between the assets. However, empirical evidence presents that a real association between financial series is much more complex (LOPES; PESSANHA, 2018).

Therefore, this paper aims to price bivariate options by overcoming two of the above constraints of the classical approach, where asset-objects are modeled marginally by deriving their risk-neutral considering the GARCH, EGARCH, NGARCH and GJR-GARCH models, with copula functions modeling the joint distribution models, with the objective of capturing linear, non-linear and tails dependence. The entire methodology described here may be extended to any multivariate case.

An innovative feature of the present work is the comparison among methodologies, where we consider marginal processes that capture the effect of asymmetry, usually present in financial series. A second point is the performance of a simulation study of the pricing models with the purpose of verifying the good fit of the models used in the literature. It is highlighted as a third point the comparison of the methodology exposed to the standard method, extended from the Black & Scholes model to the bivariate case. Finally, the implementation of such methods in the Brazilian stock market, which is characterized as a volatile and unstable market concerning developed markets. Then, compared with the previous papers, the approach in the present paper makes the dynamic pricing more reasonable and tractable.

The paper organization follows. Section 4.2 presents the conceptual framework and the models. Section 4.3 shows the bivariate model methodology and the inference procedures. Section 4.4 presents the results of the proposed method under an artificial and a real data set. Finally, Section 4.5 ends the paper with concluding remarks.

4.2 Conceptual Framework and Models Specification

4.2.1 Option Pricing

A European option call on the maximum of two risky assets (call-on-max) is defined based on the maximum price between two assets. The payoff function of this option is given by

$$g(S(T)) = max[max(S_1(T), S_2(T)) - K, 0],$$

where S_i is the price of the $i-th$ asset *(i = 1,2)* at the maturity date T and K is the strike price or exercise price.

To introduce heteroscedasticity, we use the fundamental theorem of asset pricing (DELBAEN; SCHACHERMAYER, 1994). This theorem states that once the stock prices $S_1(T)$ and $S_2(T)$ is free from arbitrage and present in a complete market (HULL, 1991), there is a measure of probability $\mathbb{Q}$ such that the discounted price of the payoff function, $e^{-r(T-t)}g(S_1(T), S_2(T))$, is a martingale under $\mathbb{Q}$ and $\mathbb{Q}$ is equivalent to the real world probability measure $\mathbb{P}$. Therefore, we define the following definition to perform the pricing.

Thus, we express the joint density function using the marginal densities $f_{S_1}(x_1)$ e $f_{S_2}(x_2)$ by means of copula functions as follows

$$f^{\mathbb{Q}}_{S_1,S_2} = c^{\mathbb{Q}}(F^{\mathbb{Q}}_{S_1}, F^{\mathbb{Q}}_{S_2}) f^{\mathbb{Q}}_{S_1}(x_1) f^{\mathbb{Q}}_{S_2}(x_2),$$

which $c^{\mathbb{Q}} = \dfrac{\partial^2 C^{\mathbb{Q}}(x_1, x_2)}{\partial x_1 \partial x_2}$, which $C^{\mathbb{Q}}(.)$ is a copula function.

Copulas are useful tools for constructing joint distributions (SHARIFONNASABI *et al.*, 2018). That is, copula is a multidimensional distribution function in which the marginal distributions are uniform in $[0,1]$. A bivariate copula is a function $C : I^2 \longrightarrow I \in [0,1]$ that satisfies the following conditions: $C(x_1, 0) = C(0, x_1) = 0 \quad and \quad C(x_1, 1) = C(1, x_1) = x_1, \quad x_1 \in I$ and the 2-increasing condition $C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0$, for all u_1, u_2, v_1 and $v_2 \in [0,1]$ such $u_1 \leq u_2$ and $v_1 \leq v_2$.

One of the most famous theorems in copula theory is the Sklar theorem. According to Sklar's theorem (SKLAR, 1959), any bivariate cumulative distribution H_{S_1,S_2} can be represented as a function of the marginal distributions F_{S_1} and F_{S_2}. Besides, whether the marginal distributions are continuous, the copula exists, is unique and is given by

$$H_{S_1,S_2}(x_1, x_2) = C(F_{S_1}(x_1), F_{S_2}(x_2)),$$

which $C(u, v) = P(U \leq u, V \leq v)$, $U = F_{S_1}(x_1)$ and $V = F_{S_2}(x_2)$.

In the case of continuous and differentiable marginal distributions, the joint density function of the copula is given by

$$f(x_1, x_2) = f_{S_1}(x_1) f_{S_2}(x_2) c(F_{S_1}(x_1), F_{S_2}(x_2)),$$

which $f_{S_1}(x_1)$ and $f_{S_2}(x_2)$ are the density for the distribution function $F_{S_1}(x_1)$ and $F_{S_2}(x_2)$, respectively, and

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial uv},$$

is the density of copula. For further details about copulas, see Nelsen (2007) and Sanfins, Valle *et al.* (2012). In this work, we consider the Normal, t-Student, Gumbel, Frank and Joe copulas. Therefore, to construct a joint process of risk-neutral for the bivariate distribution of the option, the marginal processes are derived first.

4.2.2 GARCH-in-mean Specification under $\mathbb{P}$

Instead of deriving the bivariate risk-neutral distribution directly, each marginal process is proposed to transform separately. Duan (1995) defined an option pricing model considering that the variance of the asset-object is not constant over time. To implement non-constant volatility over the maturity time of the option, we use in this work the generalized autoregressive conditional heteroskedastic (GARCH) models. Bollerslev (1986) introduced the GARCH model by modifying the ARCH model presented by Engle (1982). The use of GARCH models in pricing leads to the correction of some biases in the model of Black and Scholes (1973), including return skewness and leptokurtic behavior.

On the other hand, GARCH-in-mean refers to the inclusion of an extra term m_t in the conditional mean of the model introduced by Bollerslev (1986). An intuitive idea to use these models in derivative pricing is that conditional variance is not constant over time and hence the conditional mean of market returns is a linear function of conditional variance. Another definite reason to work with the GARCH-in-mean models is that these models explain the presence of conditional left skewness observed in stock returns.

The general GARCH-M*(p,q)* model for the return $y_t = log(S_t/S_{t-1})$, where S_t is the specific stock price at time t, is defined as

$$y_t = m_t + \sqrt{h_t}\varepsilon_t \quad and \quad h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i h_{t-i}\phi(\varepsilon_{t-i}) + \sum_{i=1}^{q} \beta_i h_{t-i}. \tag{4.1}$$

where ε_t is a sequence of independent and identically distributed *(i.i.d)* random variables with normal distribution $N(0,1)$; the conditional mean return m_t is assumed to be an F_t-predictable process. In many studies, m_t is assumed to be a function of the conditional variance h_t of the return and a risk premium quantifier at time t; the function $\phi(.)$ describes the impact of random shock of return ε_t on the conditional variance h_t and $\alpha_0 > 0, \alpha_i$ and $\beta_i \geq 0$.

The conditional mean and variance of y_t are $m_t = E[y_t|F_{t-1}]$ and $h_t = Var[y_t|F_{t-1}]$. The effect of past innovations ε_{t-1} under the conditional variance h_t have different impacts depending on the function $\phi(\varepsilon_{t-1})$, and consequently we have different extensions of the GARCH model. For example, considering $p = q = 1$, when $\phi(\varepsilon_{t-1}) = \varepsilon_{t-1}^2$, the sign of ε_{t-1} there is no effect over

h_t, and we have the traditional GARCH proposed by Bollerslev (1986). Thus, the innovations have a symmetric effect on the conditional variance, expressed by

$$h_t = \alpha_0 + \alpha_1 h_{t-1}\varepsilon_{t-1}^2 + \beta_1 h_{t-1}. \tag{4.2}$$

Following Liu, Li and Ng (2015), Duan (1995) and Chiou and Tsay (2008), $m_t = r + \lambda\sqrt{h_t} - k_{\varepsilon_t}(\sqrt{h_t})$, which $k_{\varepsilon_t}(\sqrt{h_t})$ is the cumulate generating function of the innovation ε_t e λ is the premium risk parameter. When ε_t follows a normal distribution, we have $k_{\varepsilon_t}(\sqrt{h_t}) = \frac{1}{2}h_t$.

Because standard GARCH models given by equation 4.2 respond in the same way to positive and adverse events, such models cannot correctly capture the leverage effect. Other forms of the GARCH model, such as EGARCH, NGARCH, and GJR-GARCH, include the asymmetry effect, can thus be used in option pricing and are used in the present work. Nelson (1991) proposed the exponential GARCH (EGARCH) model. The author assumes that the dynamic of the logarithm of the conditional variance of EGARCH(1,1) is given by

$$ln\, h_t = \alpha_0 + \alpha_1(|\varepsilon_{t-1}| + \gamma_1\varepsilon_{t-1}) + \beta_1 ln(h_{t-1}), \tag{4.3}$$

where α_0, α_1, β_1 and γ_1 are constant parameters and ε forms a sequence of independent standard normal random variables representing random shocks. The EGARCH model does not require such parameter restrictions since the conditional variance is expressed as the exponential of a function. Including the random shock term in absolute value and with a parameter γ_1, the author made volatility a function of both magnitude and sign of the shock.

Engle (1982) introduced the non-linear asymmetric GARCH (NGARCH), which takes into account the leverage effect. In their model, the dynamic of the conditional variance of NGARCH(1,1) is given by

$$h_t = \alpha_0 + \alpha_1 h_{t-1}(\varepsilon_{t-1} - \gamma_1)^2 + \beta_1 h_{t-1}, \tag{4.4}$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and γ_1 is a non-negative parameter that captures the negative correlation between return and volatility innovations. Since the parameter α_1 is typically non-negative, a positive γ_1 means that negative random shocks increase volatility more than positive random shockes of similar magnitude. Hence, the NGARCH allows for the levarage through its paramater γ_1.

Another model that takes into account the asymmetry effect of news on volatility is the GJR-GARCH introduced by Glosten, Jagannathan and Runkle (1993). According to this model, the conditional variance dynamic of GJR-GARCH(1,1) is given by

$$h_t = \alpha_0 + \alpha_1 h_{t-1}\varepsilon_{t-1}^2 + \beta_1 h_{t-1} + \gamma_1 h_{t-1} max(0, -\varepsilon_{t-1})^2, \tag{4.5}$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $\gamma_1 \geq 0$ are constant parameters. This model allows for the leverage effect by adding the extra term $\gamma_1 h_{t-1} max(0, -\varepsilon_{t-1})^2$ when ε_t is negative since γ_1 is typically non-negative.

All the models presented above are in the physical measure ($\mathbb{P}$ measure). Now, we discuss their representations in the risk-neutral measure ($\mathbb{Q}$ measure), a prerequisite for pricing options under heteroscedasticity.

4.2.3 Risk-neutral with GARCH-in-mean process

The concept of risk-neutral valuation relationship (RNVR) has a fundamental role in the process of pricing options. This principle has as the base an asset, which is priced according to the discount of the expected value of a payoff function under a martingale measure, *i.e.*, that the economic agents are risk-neutral.

To apply this pricing methodology, we assume that a measure of martingale $\mathbb{Q}$ exists in a discrete economy time, with interest rate and a probability space $(\Omega, \mathbb{F}, \mathbb{F}_t, \mathbb{P})$, where $\mathbb{P}$ is a measure of physical probability and $\mathbb{F}_t$ is a filtering at time t.

Definition 4.2.2 (Duan (1995))**.** The measure of probability $\mathbb{Q}$ is equivalent to measure $\mathbb{P}$ if:

1. $\mathbb{Q} \approx \mathbb{P}$, i.e., for all event X, $\mathbb{Q}(X) = 0$ and $\mathbb{P}(X) = 0$.

2. $E^{\mathbb{Q}}[S_t|F_{t-1}] = S_{t-1}$, i.e., the discounted price process S_t is a martingale under $\mathbb{Q}$.

Brennan and Schwartz (1979) represents a starting point by providing conditions which ensure the existence of the risk-neutral measure. Duan (1995) proposes an extension of RNVR, referred to as Locally Risk-Neutral Valuation Relationship (LRNVR) by assuming a conditional Gaussian distribution for the log-returns with unchanged volatility after the change of measure.

Definition 4.2.3 (Duan (1995))**.** A measure $\mathbb{Q}$ satisfies the local risk-neutral valuation relationship (LRNVR) if:

1. $y_t|F_{t-1}$ is normally distributes under measure $\mathbb{Q}$.

2. $E^{\mathbb{Q}}[S_t/S_{t-1}|F_{t-1}] = e^r$.

3. $Var^{\mathbb{Q}}[log(S_t/S_{t-1})|F_{t-1}] = Var^{\mathbb{P}}[log(S_t/S_{t-1})|F_{t-1}]$.

In the previous definition, the conditional variance under the two measures is required to be equal. This requirement is necessary to estimate the conditional variance under $\mathbb{P}$ and use the framework to obtain the option pricing under $\mathbb{Q}$. This property and the fact of the risk-free rate can replace the conditional mean, yield a well-specified model that does not locally depend on preferences. Duan (1995) proved this latter fact. Here we reduce all preference consideration to the unit risk premium λ. Since $\mathbb{Q}$ is absolutely continuous for $\mathbb{P}$, the almost certain relationship under $\mathbb{P}$ also holds true under $\mathbb{Q}$. Duan (1995) and Duan *et al.* (2006) shows that under the risk-neutral measure $\mathbb{Q}$ given by LRNVR, the asset return dynamic becomes $y_t = r - \frac{1}{2}h_t + \sqrt{h_t}\tilde{\varepsilon}_t, \quad \tilde{\varepsilon}_t \sim N(0,1) \quad and$

GARCH(1,1): $h_t = \alpha_0 + \alpha_1 h_{t-1}(\tilde{\varepsilon}_{t-1} - \lambda_1)^2 + \beta_1 h_{t-1}$.

EGARCH(1,1): $h_t = \alpha_0 + \alpha_1[|\tilde{\varepsilon}_{t-1} - \lambda_1| + \gamma_1(\tilde{\varepsilon}_{t-1} - \lambda_1)] + \beta_1 log(h_{t-1})$.

NGARCH(1,1): $h_t = \alpha_0 + \alpha_1 h_{t-1}(\tilde{\varepsilon}_{t-1} - \gamma_1 - \lambda_1)^2 + \beta_1 h_{t-1}$.

GJR-GARCH(1,1): $h_t = \alpha_0 + h_{t-1}[\beta_1 + \alpha_1(\tilde{\varepsilon}_{t-1} - \lambda_1)^2 + \gamma_1 max(0, -\tilde{\varepsilon}_{t-1} + \lambda_1)^2]$.

Under LRNVR, the form of m_t just affects the volatility dynamics while the risk-neutralized conditional mean return remains the same, *i.e.*, $r - \frac{1}{2}h_t$. Now, we have all the variance specification in the risk-neutral measure. According to the equations above, the final asset price is derived from the following corollary.

Corollary 1. When the locally risk-neutral valuation relationship holds, the terminal price for the *i-th (i=1,2)* asset can be expressed as

$$S_{i,T} = S_{i,t} exp[(T-t)r - \frac{1}{2}\sum_{s=t+1}^{T} h_{i,s} + \sum_{s=t+1}^{T}\sqrt{h_{i,s}}\tilde{\varepsilon}_{i,s}].$$

Therefore, under the locally risk-neutral probability measure $\mathbb{Q}$, the option with exercise price K at maturity T has the value

$$v(t, S_1, S_2) = e^{-r(T-t)}E^{\mathbb{Q}}[max[max(S_1(T), S_2(T)) - K, 0]].$$

Due to the complexity of the GARCH process, analytical solution for the GARCH-in-mean Copula option-pricing model, in general, is not available. Therefore, we work with numerical methods to price the option described in the next section.

4.3 Methodology and Inference

In this chapter, we present the procedure to obtain the price of a bivariate option using the asymmetric variance process by GARCH-in-mean under risk-neutral, copulas theory and Monte Carlo simulations. Chiou and Tsay (2008) and Zhang and Guegan (2008) have inspired this approach.

Given y_1 and y_2, two vectors containing the log-returns for the two stocks, we consider the following steps,

1. For each y_i $(i = 1,2)$, use quasi-maximum likelihood described in the next subsection to estimates the parameters α_0, α_1, β_1 and λ in equation 4.2 and α_0, α_1, β_1, γ and λ for each marginals given in equation (4.3), (4.4) and (4.5). Thus, the problem is to maximize with respect to the parameters.

$$l(\theta,h_t) = -\frac{n}{2}\left[log(2\pi) + \frac{1}{n}\sum_{t=1}^{n}\left[log(h_t) + \frac{(y_{it} - m_{it})^2}{h_t}\right]\right],$$

   which m_{it} is the mean of GARCH-in-mean given by $r + \lambda\sqrt{h_t} - 1/2h_t$, which r is the fixed risk-free rate yield and h_t corresponds to each variance specification proposed in section 2.2.

2. Use the estimated parameters to calculate h_t for each specification and ε_t in equation (4.1) with $m_t = r + \lambda\sqrt{h_t} - 1/2h_t$ for each stock.

3. Therefore, the proposed technique is that the objective copula and the risk-neutral copula are assumed to be the same. To fit the copulas, we transform the data into uniformly distributed random variables. Thus we transform the ε_i $(i = 1,2)$, obtained in Step 2 for each stock into uniformly distributed variables, by $u_i = \Phi(\varepsilon_i)$, where $\Phi(.)$ is the standard normal cumulative distribution function.

4. Fit a copula to pairs $[u_1,u_2]$ using maximum likelihood, *i.e.*, estimate the copula parameters θ_c

$$\theta_c = arg\,max_{\theta_c}\sum_{t=1}^{n} log[c((u_{1,t},u_{2,t});\theta_c)],$$

   where θ_c are the parameters for the specific copula function C and the c is the density function for the given copula in annex.

5. Now, using the Monte Carlo simulation, we obtain the option price. In the first step generate a sample $\{u^*_{1,t},u^*_{2,t}\}_{t=1}^{T}$ from a uniform marginal distribution from one specific copula using the algorithm proposed by Nelsen (2007). Here T is the time to maturity for the option.

6. For each time step, transform the generated margins to standard normal margins, in the risk-neutral measure, by $\tilde{\varepsilon}_{i,t} = \Phi^{-1}(u^*_{i,t})$, for $i = 1,2$.

7. Working with $\tilde{\varepsilon}_{i,t}$ calculate the conditional variances under risk-neutral and the parameters estimated in step 1. The two future stock prices at time T are

$$S_{i,T} = S_{i,t}exp[(T-t)r - \frac{1}{2}\sum_{s=t+1}^{T} h_{i,s} + \sum_{s=t+1}^{T}\sqrt{h_{i,s}}\tilde{\varepsilon}_{i,s}].$$

8. Now, repeat Steps 5 to 7 for N runs. Thus we obtain the Monte Carlo option price as

$$v(t,S_1,S_2) = \frac{e^{-r(T-t)}}{N}\sum_{i=1}^{N} max[max(S_{1,i}(T),S_{2,i}(T)) - K,0].$$

4.3.1 Quasi-Maximum Likelihood Estimation

The assumption of conditional normality is not always appropriate in financial data. However, Weiss (1986) and Bollerslev and Wooldridge (1992) shows that even when normality is inappropriately assumed, maximizing the normalized log-likelihood results in quasi-maximum likelihood estimates (QMLEs) that are consistent and asymptotically normally distributed. Besides, the authors claim that the conditional mean and variance functions of the GARCH models are correctly specified.

In particular, a robust covariance matrix conditional non-normality for the parameter estimates is consistently estimated by $\mathbf{A}(\hat{\theta})^{-1}\mathbf{B}(\hat{\theta})\mathbf{A}(\hat{\theta})^{-1}$, where $\mathbf{A}(\hat{\theta})$ and $\mathbf{B}(\hat{\theta})$ are the Hessian Matrix and the outer product of the gradients, respectively, calculated for θ. The coefficient standard errors, computed from the square roots of the diagonal elements, are sometimes called Bollerslev-Wooldridge standard errors. For more details, see Bollerslev and Wooldridge (1992).

4.3.2 Model Selection

We notice that for each time series we have four specification for variance processes, *i.e.*, GARCH(1,1), EGARCH(1,1), NGARCH(1,1) and GJR-GARCH(1,1). Choosing an adequate model is the essence of data analysis, which ultimately returns with good forecasting results.

In this paper, for model selection, we use five different criteria. The first one is Akaike Information Criterion (AIC) (AKAIKE, 1973), which is given by $AIC = -2log(\ell) + 2k$, where ℓ is the maximized value of the likelihood function and k is the number of free parameters in the model. The second one is the BIC developed by Schwarz *et al.* (1978), which is given by $BIC = -2log(\ell) + klog(n)$, where n is the number of observations. The third criteria is the Hannan-Quinn proposed by Hannan and Quinn (1979). The criteria is given by $HQ = -2log(\ell) + 2klog(log(n))$. The fourth criteria is the Akaike Information Corrected Criterion (AICc), developed by Hurvich and Tsai (1989). The AICc is $-2log(\ell) + 2kn/(n-k-1)$. The fifth criteria is the CAIC (Consistent Akaike Information Criteria) given by $-2log(\ell) + klog(n) + 1$.

Following Genest, Rémillard and Beaudoin (2009), we use the goodness-of-fit test, which is based on a comparison of the distance between the estimated and empirical copula by using the Cramer Von Mises statistic method, to compare the copula models.

The goodness-of-fit test employed is defined bellow, tests the null hypothesis that data is fitted by C_{θ_n}, a copula with vector of parameters θ,

$$S_n = \int_{[0,1]^d} \mathbb{C}_n(u)^2 dC_n(U),$$

which $C_n(U) = 1/n \sum_{i=1}^{n} \mathbb{I}(U_{i1} \leq u_1; U_{i2} \leq u_2)$ is known as the empirical copula; $U_j = (U_{1j}, \ldots, U_{ij})$ are the pseudo-observations; $u = (u_1, u_2) \in [0,1]^2$; $\mathbb{C}_n = \sqrt{n}(C_n - C_{\theta_n})$ is the empirical process that assess the distance between the empirical copula and the estimation C_{θ_n} and n is the number of observations.

We chose this procedure because it can deal with non-linearity, asymmetry, serial dependence and also the well-known heavy-tails of financial assets (RIGHI; CERETTA *et al.*, 2011). Furthermore, we make the comparison of the adjusted copula with the empirical copula by the diagonal method (SUNGUR; YANG, 1996). Besides, the AIC, AICc, CAIC, BIC and HQ criteria are also used to support decision making in choosing the model.

4.4 Data Analysis

In this chapter, we illustrated the proposed methodology under two data sets. We used the software R for implementing the entire methods exposed here. The codes are available from the authors. The first one is artificial data, where we know the parameter values, and then we can verify if the methodology is reliable. The second data set is the Brazilian stock market data.

4.4.1 Artificial Data

We consider here 1000 replications of two correlated time-series for each sample size (*n= 250, 500* and *1000*) generated from same parameter structure with the Frank (θ = 8) and marginals as follows:

GARCH(1,1):

$$h_{1,t} = 0.02 + 0.15h_{t-1}(\tilde{\varepsilon}_{t-1} - 0.12)^2 + 0.8h_{t-1},$$
$$h_{2,t} = 0.03 + 0.2h_{t-1}(\tilde{\varepsilon}_{t-1} - 0.08)^2 + 0.7h_{t-1},$$

EGARCH(1,1):

$$h_{1,t} = -0.3057 + 0.1223[|\tilde{\varepsilon}_{t-1} - 0.12| + (-0.5057)(\tilde{\varepsilon}_{t-1} - 0.12)] + 0.98ln(h_{t-1}),$$
$$h_{2,t} = -0.3057 + 0.1223[|\tilde{\varepsilon}_{t-1} - 0.12| + (-0.5057)(\tilde{\varepsilon}_{t-1} - 0.12)] + 0.98ln(h_{t-1}),$$

NGARCH(1,1):

$$h_{1,t} = 0.012 + 0.15h_{t-1}(\tilde{\varepsilon}_{t-1} - 0.5 - 0.12)^2 + 0.8h_{t-1},$$
$$h_{2,t} = 0.03 + 0.2h_{t-1}(\tilde{\varepsilon}_{t-1} - 0.2 - 0.08)^2 + 0.7h_{t-1},$$

GJR-GARCH(1,1):

$$h_{1,t} = 0.00961 + h_{t-1}[0.93 + 0.024(\tilde{\varepsilon}_{t-1} - 0.065)^2 + 0.059max(0, -\tilde{\varepsilon}_{t-1} + 0.065)^2],$$
$$h_{2,t} = 0.00961 + h_{t-1}[0.93 + 0.024(\tilde{\varepsilon}_{t-1} - 0.065)^2 + 0.059max(0, -\tilde{\varepsilon}_{t-1} + 0.065)^2].$$

For each configuration, we calculated the average of the quasi-maximum likelihood estimates (QMLEs), as well as the robust standard deviation (S.D.) of the QMLEs, the size of confidence intervals 95% (C.I.), coverage probability (C.P.), bias and mean squared error (MSE) of the QMLEs. Table 12, 13, 14 and 15 show the simulation results, for GARCH, NGARCH, EGARCH, and GJR-GARCH, respectively.

Table 12 – Parameter estimation of both artificial time-series for each GARCH process.

	Parameter	$\alpha_{0,1}$	$\alpha_{1,1}$	β_1	λ_1	$\alpha_{0,2}$	$\alpha_{1,2}$	β_2	λ_2	θ
	Real Value	0.02	0.15	0.8	0.12	0.03	0.2	0.7	0.08	8
n = 250	**Mean**	0.0355	0.1479	0.7511	0.1192	0.0429	0.2020	0.6442	0.0834	7.9349
	S.D.	0.1695	0.4596	0.9783	0.1798	0.0337	0.0927	0.1983	0.0786	0.5777
	Size Int.	0.1213	0.2297	0.5095	0.2345	0.1347	0.2784	0.6594	0.1919	2.4122
	C.P.	0.9880	0.9490	0.9560	0.9760	0.9480	0.9289	0.9480	0.9750	0.9229
	Bias	-0.0155	0.0021	0.0489	0.0009	-0.0129	-0.0020	0.0558	-0.0034	0.0652
	MSE	0.0002	0.0000	0.0024	0.0000	0.0002	0.0000	0.0031	0.0000	0.0042
n = 500	**Mean**	0.0255	0.1492	0.7830	0.1198	0.0347	0.1998	0.6806	0.0825	7.9836
	S.D.	0.0126	0.0403	0.0615	0.0479	0.0156	0.0562	0.0917	0.0523	0.4092
	Size Int.	0.0493	0.1462	0.2403	0.1838	0.0541	0.1992	0.3349	0.1572	1.6726
	C.P.	0.9720	0.9500	0.9570	0.9470	0.9470	0.9269	0.9289	0.9720	0.9399
	Bias	-0.0055	0.0008	0.0170	0.0003	-0.0047	0.0002	0.0194	-0.0025	0.0164
	MSE	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000	0.0004	0.0000	0.0003
n = 1000	**Mean**	0.0223	0.1500	0.7931	0.1181	0.0323	0.2004	0.6908	0.0780	8.0054
	S.D.	0.0089	0.0314	0.0437	0.0403	0.0112	0.0413	0.0659	0.0401	0.2896
	Size Int.	0.0282	0.0990	0.1442	0.1251	0.0360	0.1308	0.1885	0.1111	1.2808
	C.P.	0.9600	0.9580	0.9570	0.9530	0.9439	0.9550	0.9550	0.9600	0.9469
	Bias	-0.0023	0.0000	0.0069	0.0019	-0.0023	-0.0004	0.0092	0.0020	-0.0054
	MSE	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000

We observe that the averages of the quasi-maximum likelihood estimates are close to the true values as the sample size increases, as well as decreasing the standard deviations in all the models. We also note low bias and MSEs as the sample size increases. Concerning the size of the confidence interval, we noticed they are getting smaller as the sample size increases. Besides, the empirical coverages are closer to the nominal ones for all four models. With this results, we noticed that all the models have good asymptotic properties.

Table 13 – Parameter estimation of both artificial time-series for each NGARCH process.

	Parameter	$\alpha_{0,1}$	$\alpha_{1,1}$	β_1	λ_1	γ_1	$\alpha_{0,2}$	$\alpha_{1,2}$	β_2	λ_2	γ_2	θ
	Real Value	0.012	0.15	0.8	0.12	0.5	0.03	0.2	0.7	0.08	0.2	8
n = 250	**Mean**	0.0266	0.1431	0.7656	0.1162	0.5517	0.0416	0.1962	0.6503	0.0798	0.3461	7.9343
	S.D.	0.1107	0.7069	0.7901	0.5111	6.3509	0.0444	0.1538	0.2345	0.1130	1.1610	0.5773
	Size Int.	0.0923	0.2061	0.3251	0.2349	0.9699	0.1046	0.2832	0.5296	0.1851	0.9900	2.5200
	C.P.	0.9860	0.9580	0.9620	0.9820	0.9730	0.9620	0.9429	0.9499	0.9870	0.9960	0.9289
	Bias	-0.0146	0.0069	0.0344	0.0038	-0.0517	-0.0116	0.0038	0.0497	0.0002	-0.0461	0.0657
	MSE	0.0002	0.0000	0.0012	0.0000	0.0027	0.0001	0.0000	0.0025	0.0000	0.0021	0.0043
n = 500	**Mean**	0.0162	0.1435	0.7894	0.1178	0.5345	0.0348	0.1972	0.6796	0.0811	0.3250	7.9601
	S.D.	0.0277	0.0744	0.1240	0.0964	0.4538	0.0149	0.0510	0.0778	0.0542	0.1790	0.4083
	Size Int.	0.0296	0.1412	0.1665	0.1859	0.8008	0.0553	0.2005	0.2963	0.1601	0.9699	1.7088
	C.P.	0.9860	0.9399	0.9520	0.9730	0.9620	0.9540	0.9299	0.9520	0.9740	0.9640	0.9269
	Bias	-0.0042	0.0065	0.0106	0.0022	-0.0345	-0.0048	0.0028	0.0204	-0.0011	-0.0250	0.0399
	MSE	0.0000	0.0000	0.0001	0.0000	0.0012	0.0000	0.0000	0.0004	0.0000	0.0006	0.0016
n = 1000	**Mean**	0.0142	0.1479	0.7934	0.1174	0.5167	0.0323	0.1973	0.6921	0.0789	0.3135	7.9780
	S.D.	0.0132	0.0425	0.0577	0.0535	0.1983	0.0121	0.0406	0.0625	0.0404	0.1285	0.2890
	Size Int.	0.0167	0.0971	0.1066	0.1322	0.4957	0.0336	0.1274	0.1810	0.1291	0.6677	1.1894
	C.P.	0.9520	0.9469	0.9600	0.9640	0.9590	0.9580	0.9479	0.9540	0.9640	0.9590	0.9479
	Bias	-0.0022	0.0021	0.0066	0.0026	-0.0167	-0.0023	0.0027	0.0079	0.0011	-0.0135	0.0220
	MSE	0.0000	0.0000	0.0000	0.0000	0.0003	0.0000	0.0000	0.0001	0.0000	0.0002	0.0005

Table 14 – Parameter estimation of both artificial time-series for each EGARCH process.

	Parameter	$\alpha_{0,1}$	$\alpha_{1,1}$	β_1	λ_1	γ_1	$\alpha_{0,2}$	$\alpha_{1,2}$	β_2	λ_2	γ_2	θ
	Real Value	-0.3067	0.1223	0.98	0.12	-0.5057	-0.3067	0.1223	0.98	0.12	-0.5057	8
n = 250	**Mean**	-0.2852	0.0876	0.9793	0.1204	-0.5080	-0.2874	0.0891	0.9793	0.1204	-0.5065	7.9388
	S.D.	0.8405	1.1478	0.0244	0.2555	0.3996	0.6754	0.7820	0.0242	0.2362	0.2318	0.5781
	Size Int.	0.3182	0.2010	0.0180	0.2353	0.1900	0.3250	0.2215	0.0176	0.2367	0.1876	2.5276
	C.P.	0.9139	0.8759	0.8829	0.9570	0.9249	0.9149	0.8679	0.8749	0.9640	0.9269	0.9139
	Bias	-0.0205	0.0347	0.0007	-0.0004	0.0023	-0.0183	0.0332	0.0007	-0.0004	0.0008	0.0612
	MSE	0.0004	0.0012	0.0000	0.0000	0.0000	0.0003	0.0011	0.0000	0.0000	0.0000	0.0037
n = 500	**Mean**	-0.2960	0.1070	0.9798	0.1205	-0.5060	-0.2968	0.1082	0.9798	0.1198	-0.5067	7.9560
	S.D.	0.0649	0.0374	0.0035	0.0515	0.0408	0.0607	0.0493	0.0026	0.0571	0.0437	0.4085
	Size Int.	0.1790	0.1478	0.0081	0.1756	0.1239	0.1818	0.1592	0.0084	0.1707	0.1248	1.6859
	C.P.	0.9069	0.9118	0.9009	0.9289	0.9139	0.9179	0.9278	0.9309	0.9379	0.9199	0.9339
	Bias	-0.0097	0.0153	0.0002	-0.0005	0.0003	-0.0089	0.0141	0.0002	0.0002	0.0010	0.0440
	MSE	0.0001	0.0002	0.0000	0.0000	0.0000	0.0001	0.0002	0.0000	0.0000	0.0000	0.0019
n = 1000	**Mean**	-0.3033	0.1171	0.9799	0.1212	-0.5061	-0.3039	0.1176	0.9799	0.1210	-0.5054	7.9865
	S.D.	0.0391	0.0226	0.0020	0.0362	0.0242	0.0456	0.0233	0.0026	0.0457	0.0448	0.2892
	Size Int.	0.1116	0.0842	0.0047	0.1231	0.0733	0.1098	0.0838	0.0048	0.1286	0.0707	1.2940
	C.P.	0.9459	0.9409	0.9591	0.9429	0.9599	0.9449	0.9689	0.9339	0.9419	0.9489	0.9579
	Bias	-0.0024	0.0052	0.0001	-0.0012	0.0004	-0.0018	0.0047	0.0001	-0.0010	-0.0003	0.0135
	MSE	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002

Table 15 – Parameter estimation of both artificial time-series for each GJR-GARCH process.

	Parameter	$\alpha_{0,1}$	$\alpha_{1,1}$	β_1	λ_1	γ_1	$\alpha_{0,2}$	$\alpha_{1,2}$	β_2	λ_2	γ_2	θ
	Real Value	0.00961	0.024	0.93	0.065	0.059	0.00961	0.024	0.93	0.065	0.059	8
n = 250	**Mean**	0.0582	0.0306	0.8326	0.0741	0.0548	0.0524	0.0334	0.8346	0.0725	0.0563	7.9335
	S.D.	3.1908	1.2611	7.8901	1.5276	1.5033	6.3504	7.0343	1.9827	6.7417	7.2567	0.5774
	Size Int.	0.4059	0.0949	0.9637	0.1953	0.1697	0.3703	0.1123	0.9627	0.1965	0.1697	2.5486
	C.P.	0.9970	0.9970	0.9880	0.9800	0.9990	0.9870	0.9990	0.9790	0.9840	0.9990	0.9199
	Bias	-0.0486	-0.0066	0.0974	-0.0091	0.0042	-0.0428	-0.0094	0.0954	-0.0075	0.0027	0.0665
	MSE	0.0024	0.0000	0.0095	0.0001	0.0000	0.0018	0.0001	0.0091	0.0001	0.0000	0.0044
n = 500	**Mean**	0.0219	0.0260	0.9045	0.0672	0.0572	0.0224	0.0262	0.9047	0.0682	0.0563	7.9695
	S.D.	0.2226	0.2865	0.7740	0.2871	0.2983	0.4872	0.4015	1.3715	0.4177	0.4554	0.4087
	Size Int.	0.0753	0.0611	0.2013	0.1516	0.1141	0.0845	0.0634	0.2057	0.1496	0.1141	1.7545
	C.P.	0.9790	0.9760	0.9610	0.9730	0.9680	0.9730	0.9670	0.9720	0.9790	0.9440	0.9239
	Bias	-0.0122	-0.0020	0.0255	-0.0022	0.0018	-0.0128	-0.0022	0.0253	-0.0032	0.0027	0.0305
	MSE	0.0001	0.0000	0.0007	0.0000	0.0000	0.0002	0.0000	0.0006	0.0000	0.0000	0.0009
n = 1000	**Mean**	0.0134	0.0251	0.9225	0.0651	0.0564	0.0137	0.0252	0.9211	0.0661	0.0577	7.9674
	S.D.	0.0160	0.0537	0.0640	0.0981	0.0590	0.0198	0.0521	0.0759	0.0901	0.0618	0.2888
	Size Int.	0.0299	0.0466	0.0818	0.1209	0.0902	0.0325	0.0476	0.0988	0.1188	0.0875	1.2196
	C.P.	0.9560	0.9590	0.9410	0.9570	0.9420	0.9510	0.9561	0.9440	0.9520	0.9492	0.9499
	Bias	-0.0037	-0.0011	0.0075	-0.0001	0.0026	-0.0041	-0.0012	0.0089	-0.0011	0.0013	0.0326
	MSE	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0011

4.4.2 Brazilian Data

In principle, price data are not available, since the call-on-max option is typically traded over-the-counter. For this reason, we cannot test the valuation models empirically. However, comparing models with different assumptions can be implemented, as in Zhang and Guegan (2008), Liu, Li and Ng (2015) and Chiou and Tsay (2008). In this chapter, we carry on the illustration of the proposed methodology on a real data set concerning the two stock prices of Brazilian companies.

We choose the companies Bradespar (BRAP4) and Vale S.A. (VALE3) with the aim of investigating two companies that could have a high correlation. The Brazilian company Bradespar admits the shareholdings that the bank Bradesco had in non-financial companies, among them: VCB, Vale, Scopus, and Globo. Thus, Bradespar's stocks price would be directly related to the stocks of Vale S.A., where the company holds the latter's stock control at 17.4 %. The analyzed period is from 07/01/2015 to 07/17/2018, containing 753 observations.

Figure 14 shows the high positive association between the two series, evidencing the requirement subject is financial options using these stocks, given its high correlation. Table 16 shows the similarity between the returns series, both concerning the minimum, mean, median, maximum, standard deviation (S.D.) and kurtosis, but the VALE3 series has a slightly more pronounced positive asymmetry than the BRAP4 series. As evidenced in section 4.2, asymmetry is present in financial series, a feature that symmetric GARCH processes have no potential to discriminate between positive and negative asymmetry.

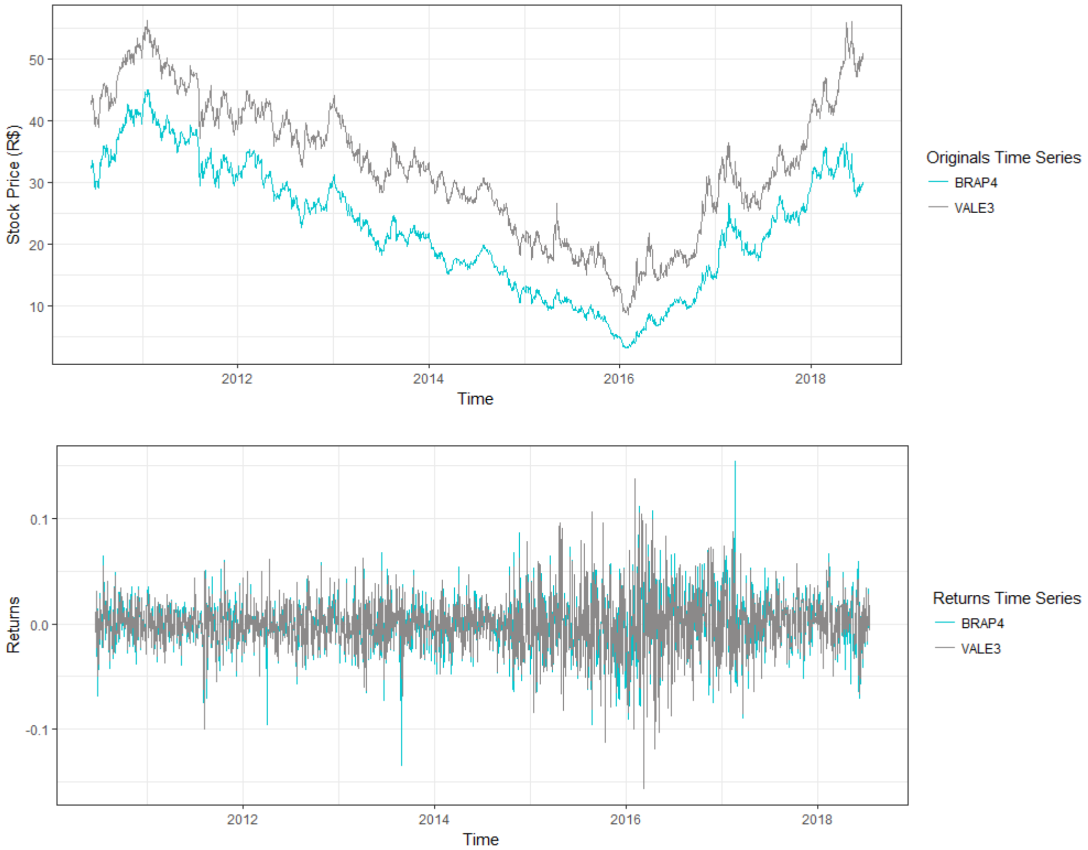

Figure 14 – Original Series and Returns.

Table 16 – Descriptive statistics of returns.

Serie	Minimum	Mean	Median	Maximum	S.D.	Kurtusis	Skewness
BRAP4	-0.134	0.000	0.000	0.153	0.027	0.050	5.150
VALE3	-0.156	0.000	0.000	0.137	0.026	0.047	5.702

Before present the estimated coefficients of time series models, we focus on the analysis of the best model according to the selection criteria. Given the flexibility of the use of models based on copula functions, we select for each marginal the best model according to the selection criteria defined in Section 3.2. According to Table 17, all criteria corroborate that the model GARCH best fit the BRAP4 series, evidencing that there is no asymmetry present in this series, while, the best model for the VALE3 series is the EGARCH (evidencing the asymmetry) - even Table 16 evidencing the asymmetry of the series by the coefficient. This result is in agreement with the statement in Table 16, where the VALE3 stock had an asymmetric coefficient more pronounced than BRAP4.

Table 17 – Selection Criteria for Marginals.

BRAP4	**GARCH**	**NGARCH**	**EGARCH**	**GJR-GARCH**
AIC	-3071.3128	-3069.3172	-3070.1057	-3069.3678
AICc	-3071.2591	-3069.2367	-3070.0252	-3069.2872
CAIC	-3048.8271	-3041.2102	-3041.9987	-3041.2607
BIC	-3052.8271	-3046.2102	-3046.9987	-3046.2607
HQ	-3064.1903	-3060.4142	-3061.2027	-3060.4647
VALE3	**GARCH**	**NGARCH**	**EGARCH**	**GJR-GARCH**
AIC	-3151.2693	-3150.0533	-3153.7289	-3151.7989
AICc	-3151.2156	-3149.9728	-3153.6484	-3151.7183
CAIC	-3123.6918	-3121.9463	-3128.7836	-3125.6219
BIC	-3128.6918	-3126.9463	-3132.7836	-3130.6219
HQ	-3144.1468	-3141.1503	-3144.8258	-3142.8958

Table 18 shows the coefficients estimated via QMLEs and their respective robust standard errors. According to this result, we noticed that the best model for the BRAP4 series was the GARCH model, where it does not have an asymmetry parameter. We view in this model the high persistence, that is, $\alpha_1 + \beta_1$ very close to one, suggesting that the volatility can be persistent (strong temporal dependence), which opens options of models to analyze series with this feature. The best model for the VALE3 series was the EGARCH, where it presented a parameter of positive asymmetry, that is, a positive shock decreases its volatility.

Table 18 – Estimated coefficients and corresponding robust standard errors for marginals.

BRAP4	**GARCH**	**NGARCH**	**EGARCH**	**GJR-GARCH**
$\hat{\alpha}_0$	6.9191e-06 (4.8306e-06)	6.8024e-06 (4.7823e-06)	-0.1399 (0.0455)	7.0316e-06 (4.9614e-06)
$\hat{\alpha}_1$	0.0479 (0.0125)	0.0477 (0.0127)	0.1005 (0.0248)	0.0507 (0.0175)
$\hat{\beta}$	0.9454 (0.0138)	0.9457 (0.0140)	0.9914 (0.0050)	0.9446 (0.0144)
$\hat{\lambda}$	0.0568 (0.0358)	0.0560 (0.0365)	0.0560 (0.0359)	0.0576 (0.0364)
$\hat{\gamma}$	-	0.0121 (0.1628)	-0.0618 (0.0235)	4.2924e-03 (0.0180)
VALE3	**GARCH**	**NGARCH**	**EGARCH**	**GJR-GARCH**
$\hat{\alpha}_0$	3.7157e-06 (2.9974e-06)	3.0711e-06 (2.9524e-06)	-0.1081 (0.0386)	2.5848e-06 (2.9020e-06)
$\hat{\alpha}_1$	0.0434 (0.0116)	0.0428 (0.0111)	0.0969 (0.0221)	0.0555 (0.0152)
$\hat{\beta}$	0.9519 (0.0121)	0.9522 (0.0117)	0.9957 (0.0004)	0.9554 (0.0113)
$\hat{\lambda}$	0.0579 (0.0357)	0.0671 (0.0365)	0.0762 (0.0387)	0.0679 (0.0363)
$\hat{\gamma}$	-	0.1771 (0.1854)	0.1433 (0.1438)	0.0278 (0.0171)

We consider the Kolmogorov-Smirnov, Jarque-Bera, Shapiro-Wilk, and Anderson-Darling tests to verify the assumption of normality of the residuals for the fitted models. Table 19 shows their p-values. All tests did not reject the null hypothesis at 5% that residuals follow a standard normal distribution. Besides, to verify that the increments are independent, Table 19 also shows the result of the Ljung-Box test with lag = 1, where, for all fitted models we do not reject the null hypothesis at 5 % that the residuals are independent.

Table 19 – Tests of Normality and Independent Increments for residuals.

BRAP4	GARCH	NGARCH	EGARCH	GJR-GARCH
Kolmogorov-Smirnov	0.9315	0.9403	0.9514	0.9343
Jarque-Bera	0.1159	0.1142	0.2351	0.1225
Shapiro-Wilk	0.2571	0.2572	0.3802	0.2633
Anderson-Darling	0.6680	0.6725	0.6572	0.6652
Ljung-Box	0.4940	0.4938	0.4988	0.4944
VALE3	**GARCH**	**NGARCH**	**EGARCH**	**GJR-GARCH**
Kolmogorov-Smirnov	0.8737	0.8752	0.8761	0.8733
Jarque-Bera	0.2059	0.1680	0.2548	0.1433
Shapiro-Wilk	0.1752	0.1895	0.2644	0.1718
Anderson-Darling	0.2288	0.2627	0.3426	0.2697
Ljung-Box	0.1927	0.2079	0.2145	0.2152

Figure 15 shows the QQ-plots for the two best models for the series, that is, on the left panel is the GARCH for the BRAP4 series and on the right panel the EGARCH for the VALE3 series, corroborating with the tests in the Table 19, evidencing the non-rejection of the normality of the residuals.

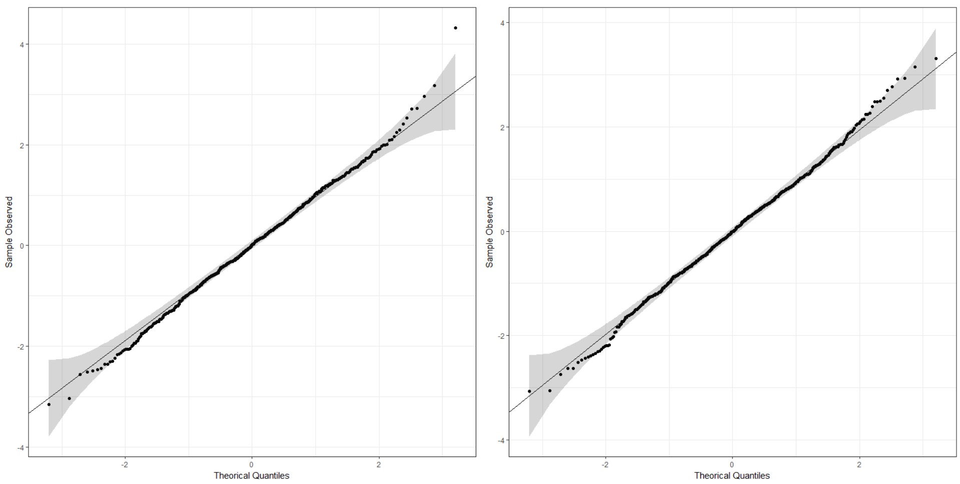

Figure 15 – QQ-plots of residuals - GARCH BRAP4 (left panel) and EGARCH VALE3 (right panel).

Figure 16 illustrates the individual behavior of each set of residual fitted through the histograms and the joint behavior through the scatterplot in the center of the figure. As expected, the series has a highly positive association behavior, which is evidenced in the adjustment of the copulas given in Table 20, where the normal and t-student copulas obtained high and positive values of their parameters (-1 $\leq \theta \leq$ 1).

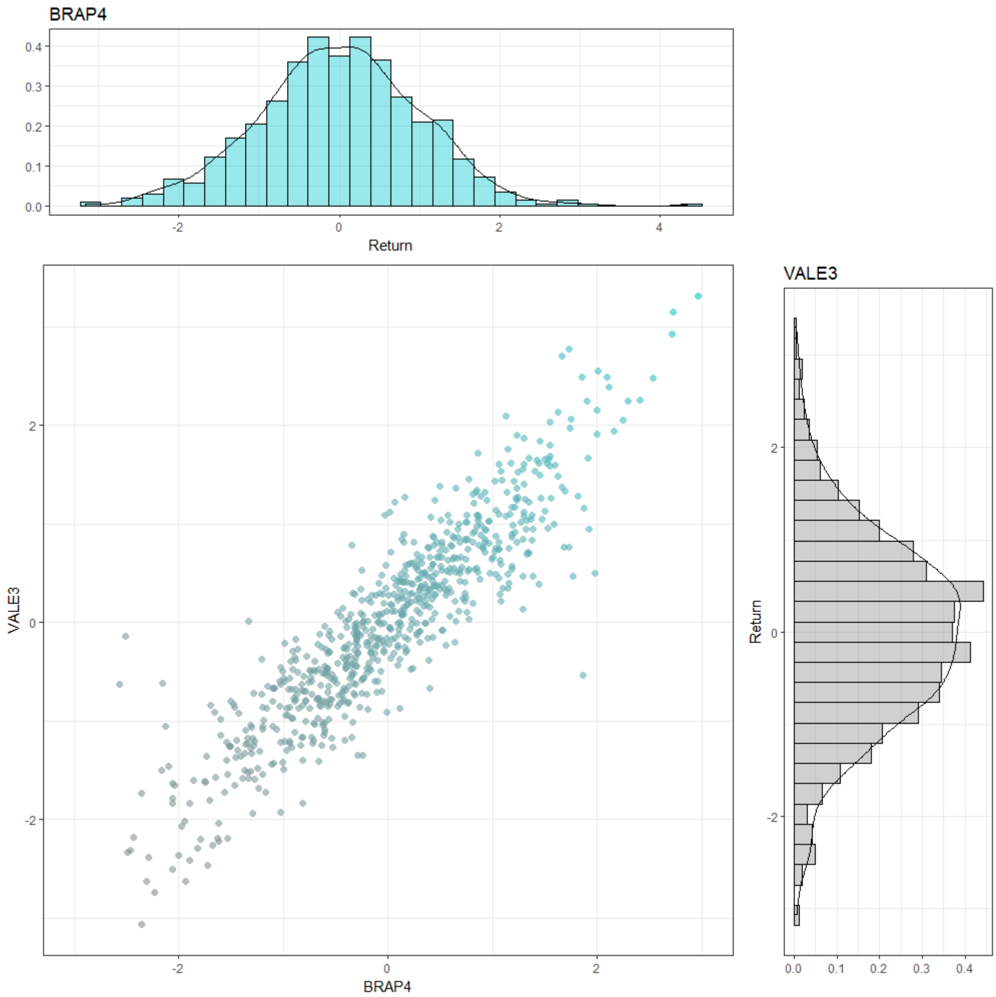

Figure 16 – Scatterplot and Histograms of residuals - GARCH BRAP4 and EGARCH VALE3.

Table 20 – Estimated coefficients and corresponding standard errors (in parentheses) for copulas.

	Normal	t-student	Gumbel	Frank	Joe
$\hat{\theta}$	0.9059 (0.0048)	0.9133 (0.0053)	3.4082 (0.1040)	14.0430 (0.4965)	4.0173 (0.1423)

The degree of freedom of the t-student copula and its respective standard deviation were 7.63401 and 1.7263.

According to the selection criteria adopted, the best copula for this dataset was the t-student copula, though the results found for the t-student copula are very similar to the one observed for the Frank copula. The empirical copula and the copula adjusted by the diagonal method, where the excellent fit of the two copulas is noted, corroborate this result.

Table 21 – Selection model of Copulas.

	Normal	t-student	Gumbel	Frank	Joe
AIC	-1290.6171	-1334.1487	-1231.5615	-1310.8730	-970.63696
AICc	-1290.6117	-1334.1327	-1231.5562	-1310.8676	-970.63162
CAIC	-1285.9957	-1324.9059	-1226.9401	-1306.2516	-966.01556
BIC	-1288.8365	-1330.5875	-1229.7809	-1309.0924	-968.85635
HQ	-1284.9957	-1322.9059	-1225.9401	-1305.2516	-965.01556

The result of the Cramer Von Mises method was 0.0025, 0.0023, 0.0042, 0.0018 and 0.01122, for Normal, t-Student, Gumbel, Frank and Joe copula, respectively. The result shows that Frank copula yields the smallest distance between fitted and empirical copula. We note that there is a minimal difference between the Frank and t-Student copula. Therefore, these two copulas are considered in this work as the best fittings.

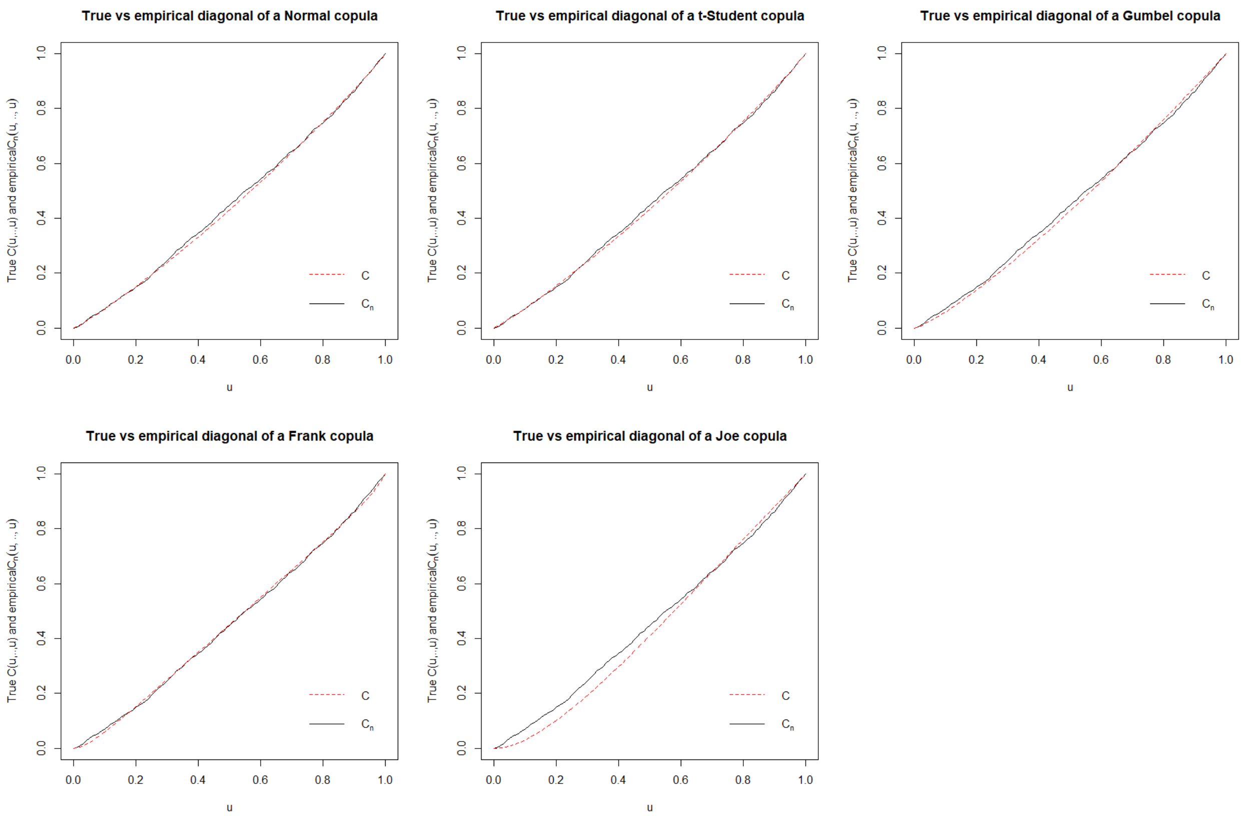

Figure 17 – Comparing the empirical copula and the true copula on the diagonal.

Given the good fitting of the marginals obtained via time series models and the good joint fitting via copulas, we now calculate and analyze the option prices considering the call-on-max payoff function. To perform the comparison process, as a benchmark, we compare the results through the methodology proposed with the classical method, which is a Black & Scholes extension for the bivariate case Haug (2007), where this model considers the volatility constant over time and the linear dependence structure from the bivariate normal distribution.

The entire study was performed with 100 000 Monte Carlo simulations, 7 % a.a. interest rate and maturity time of one year. According to Table 20, as expected, the same behavior is observed for all models, *i.e.*, as the strike variable increases it is likely that, in a call option, the price of the option becomes cheaper. We note that the classical model obtained the lowest values for all strike values. Geske and Roll (1984), Black (1975) and MacBeth and Merville (1980) corroborate this result for the univariate case, where the authors showed that the models that consider constant volatility over time underpricing the options, especially in-the-money (ITM) options. That is, a call option's strike price is below the market price in the univariate case. In this work we define ITM options when the strike price is less than the minimum between the two assets.

Moreover, in Table 22, we can see that the t-student and Frank copula models have the closest results to each other. The similarity in the excellent fit of the data can explain this result. We noticed the values obtained through normal copula obtained high results. The inability of the normal copula to capture observations in the tails of the distribution, a recurring fact in finances, can explain this result. The copula Joe obtained higher values mainly when the strike was smaller than 40, approaching the model of the normal copula. The Gumbel copula was the one that received the lowest values between the models.

Table 22 – Prices of a call-on-max option under various Strikes values (R$).

Strike	**Classic**	**Normal**	**t-Student**	**Gumbel**	**Frank**	**Joe**
20	31.1182	32.3619	32.2648	32.2532	32.2711	32.5083
22	29.2693	30.5241	30.4283	30.4148	30.4329	30.6440
24	27.4764	28.7327	28.6402	28.6228	28.6425	28.8293
26	25.7468	26.9951	26.9054	26.8845	26.9045	27.0694
28	24.0867	25.3171	25.2295	25.2061	25.2258	25.3714
30	22.5003	23.7025	23.6169	23.5918	23.6110	23.7401
32	20.9906	22.1572	22.0733	22.0457	22.0646	22.1787
34	19.5594	20.6831	20.6028	20.5704	20.5889	20.6899
36	18.2071	19.2840	19.2055	19.1678	19.1867	19.2758
38	16.9331	17.9601	17.8810	17.8399	17.8602	17.9371
40	15.7360	16.7112	16.6316	16.5866	16.6093	16.6745
42	14.6140	15.5377	15.4571	15.4103	15.4349	15.4901
44	13.5643	14.4379	14.3578	14.3081	14.3346	14.3826
46	12.5842	13.4087	13.3304	13.2795	13.3062	13.3498
48	11.6705	12.4495	12.3723	12.3199	12.3477	12.3877
50	10.8198	11.5571	11.4799	11.4270	11.4576	11.4945
52	10.0288	10.7290	10.6522	10.5987	10.6321	10.6678
54	9.2941	9.9621	9.8872	9.8339	9.8675	9.9020
56	8.6122	9.2533	9.1807	9.1278	9.1613	9.1933
58	7.9798	8.6001	8.5283	8.4762	8.5101	8.5392
60	7.3937	7.9981	7.9271	7.8762	7.9117	7.9372

Figure 18 shows the behavior of the option price (z-axis) varying the maturity from 1 to 12 months (y-axis, in days) and strike (R$ 40.00 to R$ 60.00). We note that the higher the maturity the values differ little between strike prices, which does not happen when the option has a short maturity, where we indicate that setting at 50 maturity days there is a relatively significant difference varying the price of the strike. For example, Table 23 presents the prices for considering maturity = one month, six months and one year and strike = 20, 40 and 60.

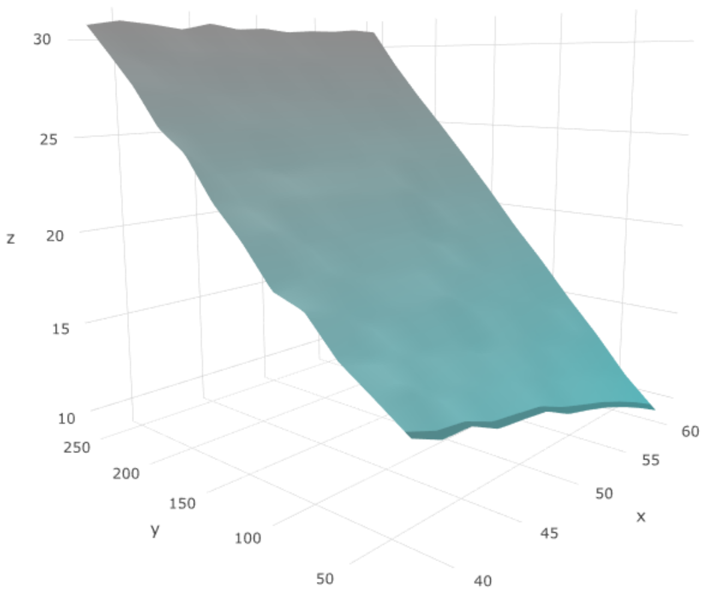

Figure 18 – Price (R$) behavior of the call-on-max option ranging from Maturity to Strike.

Table 23 – Prices (R$) of a call-on-max option varying some Maturity time and Strike (R$).

Maturity\Strike	R$ 20.00	R$ 40.00	R$ 60.00
One Month	30.9671	10.9067	0.5537
Six Months	30.9190	13.6608	4.1864
One Year	30.7311	15.6953	7.0992

Another fundamental aspect in the management of options risks is to know the levels of dependence between stocks. Therefore, Figure 19 presents the price behavior of the call-on-max option for the t-student copula by varying its degrees of dependence. This result corroborates with those found by Chiou and Tsay (2008) for the call-on-max option using the American and Taiwanese indices. An intuitive interpretation is: the values of this option tend to be smaller when the underlying assets move in the same direction as when in opposite directions.

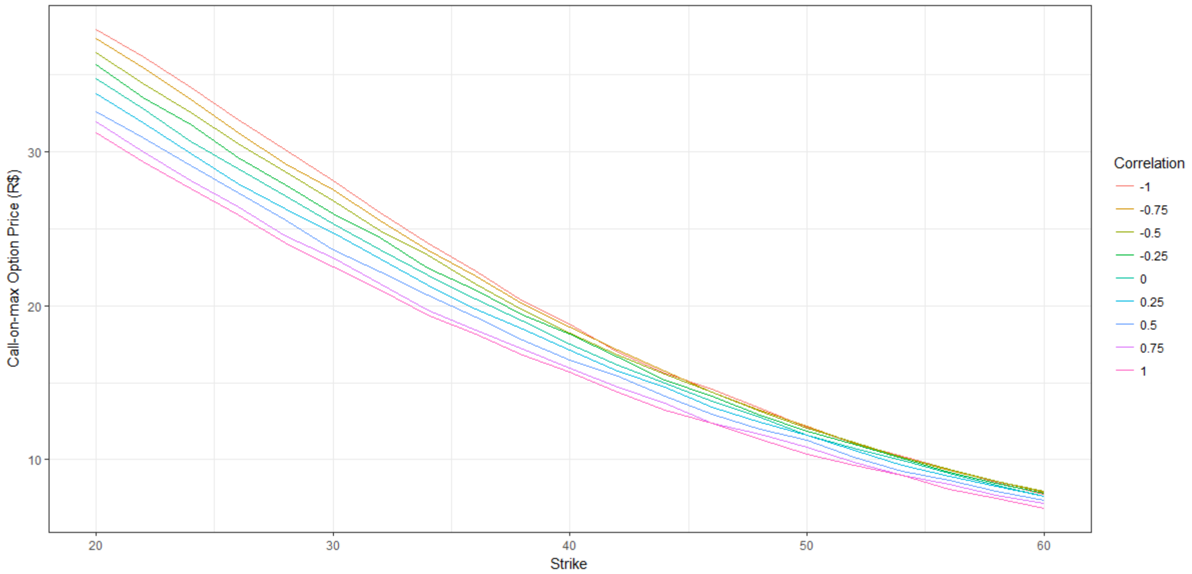

Figure 19 – Behavior of the call-on-max option price by varying the copula parameter.

Besides, Figure 19 further shows that in-the-money options have the most substantial differences between dependency levels than out-the-money options (*i.e.*, when the strike is higher than the maximum between the two assets). Therefore, it was empirically verified the importance of a good joint fit of the stocks, and above all, the calculation of the correlation between the assets. Moreover, by employing the copulas functions, it is possible to capture linear, non-linear and caudal associations. Recalling, the traditional models derived from a Brownian geometric movement consider bivariate normal to price call-on-max options for two assets, and consequently, the linear correlation coefficient as the measure of association.

4.5 Concluding Remarks

In this paper, we propose an analysis and comparison among pricing models that consider the volatility of underlying assets and in the presence of dependence between copula framework. The model is an adequate methodology to realize a more realistic pricing option. To consider the modeling of asymmetry present in financial series, we examined three models that are extensions of the GARCH model under the neutral risk measure $\mathbb{Q}$, a pre-requisite to price options (NGARCH, EGARCH, and GJR-GARCH). Therefore, through the flexibility of the copula functions, we chose which marginal processes fit best with each stock and thus proceeded in the joint fitted.

Two databases illustrate the application of the methodology. The first one was an artificial database with the objective of carrying out a simulation study and the second a database of two Brazilian companies. The simulation study showed that all models presented good asymptotic properties. Besides that, in the real time-series of two Brazilian stock companies, the model offered a proper fitting and the results obtained were confronted with the classic model, which is an extension of the Black and Scholes model.

CHAPTER 5

FINAL REMARKS

In this dissertation we investigated two approach to bivariate option pricing. Overall, the contributions of the proposed methods in the present work are as follows: (1) using the best copula makes the model more suitable; (2) the heteroscedastic model approach and the capture of dependence via copulas bring more realistic support for the modeling of financial assets and consequently more credibility; (3) a comparison of methodologies highlights the role of risk management; (4) due to the good marginal and joint fitted, in addition to the values obtained in relation to the classical consolidated model, there are arguments to believe that the differences obtained between the best models, through the copulas and the extension of the conventional method, are improvements in the calculation of the fair value; (5) the empirical relevance of such alternatives is apparent given the evidence of non-joint-normality in financial emerging markets; (6) it is possible to use the same tooling to obtain the fair price for various payoff functions, this is not verified in the case of extensions of the Black and Scholes model, as presented in Haug (2007), because for each option one formula is required; (7) it is an empirical study providing evidence and corroborating the use of techniques that consider the modeling of non-normality in financial markets, especially considering this approach in emerging markets; (8) lastly, there was no previous study to this dissertation evidencing the simulation study approach in pricing options.

In future work, we shall address the following issues:

- Adoption of other copula functions, such as power variance function (PVF) family copulas;
- Consider recents and advanced marginals processes;
- Even with extensions to asymmetric models, we often have financial series with heavy tails, which should derive a risk-neutral measure Q for these models, such as considering the non-normality of the residuals;
- Propose a multivariate model to pricing option, with the mix of the past items;

- The use of non-parametric copulas, copulas with dependency parameter varying in time;
- Trivariate and/or multivariate case using Vine copulas;
- Comparison of different sectors stocks. In this work we use stocks with very similar prices. With the goal of hedge management, it would be interesting to work with different sectors;
- The use of the predictive density to calculate the option price and so on.

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