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CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Existence and multiplicity of solutions for problems involving the Dirac operator

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FERNANDA SOMAVILLA
Orientador: FRANCISCO ODAIR VIEIRA DE PAIVA
Coorientador: OLIMPIO HIROSHI MIYAGAKI

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Folha de Aprovação

Assinaturas dos membros da comissão examinadora que avaliou e aprovou a Defesa de Tese de Doutorado da candidata Fernanda Somavilla, realizada em 30/07/2019:

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Prof. Dr. Francisco Odair Vieira de Paiva
UFSCar

Rodrigo S. Rodrigues

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UFSCar

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UnB

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UFCG

... Pedras no caminho? Guardo todas, um dia vou construir um castelo...

(Fernando Pessoa)

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Resumo

Nesta tese, estudamos equações que envolvem o operador de Dirac na forma

$$-i\alpha \cdot \nabla u + a\beta u + M(x)u = F_u(x,u), \text{ em } \mathbb{R}^3,$$

onde $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, sendo α_j e β matrizes complexas 4×4, $j = 1,2,3$, e $a > 0$. Utilizando métodos variacionais e elementos da teoria de pontos críticos para problemas fortemente indefinidos obtemos resultados de existência e multiplicidade de soluções $u : \mathbb{R}^3 \to \mathbb{C}^4$ sob diferentes conjuntos de hipóteses sobre o potencial M e a não-linearidade F. Inicialmente, consideramos um problema com potencial não periódico e uma não-linearidade do tipo côncavo-convexo, não periódica, contendo funções peso que podem apresentar mudança de sinal. Em seguida, utilizando a variedade de Nehari generalizada, estudamos problemas em que a não-linearidade satisfaz condições de monotonicidade fraca e pode se relacionar com a função potencial. Dentre tais problemas, consideramos um caso periódico e, devido as hipóteses, para obter resultados de multiplicidade utilizamos o subdiferencial de Clarke e o gênero de Krasnoselskii. Finalmente, abordamos um problema com não-linearidade assintoticamente linear no infinito e potencial matricial. Neste caso, o potencial é descrito como uma soma de um potencial matricial não positivo adequado e uma matriz diagonal cujos elementos são funções em algum espaço L^σ, $\sigma > 1$, as quais podem mudar de sinal.

Abstract

In this thesis, we study equations that involving the Dirac operator and which have the form

$$-i\alpha \cdot \nabla u + a\beta u + M(x)u = F_u(x,u), \text{ in } \mathbb{R}^3,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, with α_j and β complex matrices 4×4, $j = 1, 2, 3$, and $a > 0$. Using variational methods and elements from critical point theory for strongly indefinite problems we obtain existence and multiplicity results of solutions $u : \mathbb{R}^3 \to \mathbb{C}^4$ under different sets of hypothesis about the potential M and the nonlinearity F. Firstly, we consider a problem with nonperiodic potential and concave-convex type nonlinearity, nonperiodic, which contain weight functions that can present signal change. Next, using the generalized Nehari manifold, we study problems in which nonlinearity satisfies weak monotonicity conditions and may relate to the potential function. Among such problems, we consider a periodic case and, due to the assumptions, in order to obtain the multiplicity results we use the Clarke's subdifferential and Krasnoselskii genus. Finally, we approach a problem with nonlinearity asymptotically linear at infinity and matrix potential. In this case, the potential is described by a sum of a non-positive suitable matrix potential and a diagonal matrix whose elements are function in some L^σ space, $\sigma > 1$, which can change signal.

Contents

Mathematical notations

$\alpha_j,\ j = 1,2,3$	complex matrices 4×4 defined by (2)
$\alpha = (\alpha_1, \alpha_2, \alpha_3)$	complex matrix vector
β	real matrix 4×4 defined by (2)
I_4	identity matrix 4×4
$\sigma(T)$	spectrum of an operator T
$\sigma_e(T)$	essential spectrum of an operator T
$\sigma_c(T)$	continuous spectrum of an operator T
$L^s(U) = \{u : U \to \mathbb{C} : u$ is Lebesgue mensurable, $\lVert u\rVert_{L^s} < \infty\}\quad (1 \leqslant s < \infty)$	
$\lVert u\rVert_{L^s} = \left(\int\limits_U \lvert u\rvert^s\, dx\right)^{\frac{1}{s}} \quad (1 \leqslant s < \infty)$	norm in the Lebesgue space $L^s(U)$
$L^\infty(U) = \{u : U \to \mathbb{C} : u$ is Lebesgue mensurable, $\lVert u\rVert_\infty < \infty\}$	
$\lVert u\rVert_\infty \doteq ess\sup_{x\in U}\lvert u(x)\rvert$	norm in the Lebesgue space $L^\infty(U)$
$\lVert u\rVert_{H^1}$	norm on $H^1(\mathbb{R}^3, \mathbb{C}^4)$ space
$\lVert u\rVert_{H^{\frac{1}{2}}}$	norm on $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ space
$L^2(\mathbb{R}^3, \mathbb{C}^4) \equiv L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$	described by (A.9)
$u \cdot v = \sum\limits_{i=1}^{4} u_i\overline{v_i} \quad$ for $u, v \in \mathbb{C}^4$	inner product in $\mathbb{C}^4$
$\langle f, g\rangle_{L^2} = \int\limits_{\mathbb{R}^3} f(x)\cdot g(x)dx$	inner product in $L^2(\mathbb{R}^3, \mathbb{C}^4)$
$X \hookrightarrow Y$	continuous or compact embeddings
$x_n \rightharpoonup x$ in U	weak convergence in the space U
$u_n \xrightarrow{\lVert\cdot\rVert} u$, $\lVert\cdot\rVert$ norm in U	$\lVert u_n - u\rVert \to 0$ as $n \to \infty$
$H_0 = -i\alpha\nabla + a\beta$	free Dirac operator
$H_0 + W,\quad W$ suitable potential	Dirac operator in a external field
$A = H_0 + M$	M satisfies (M_0) defined by (1.2)
$H = H_0 + M$	M satisfies (M_1) defined by (3.2)
$\lvert A\rvert, \lvert H_0\rvert, \lvert H\rvert$	absolute value of A, H_0, H operators
$\mathcal{D}(A), \mathcal{D}(H_0), \mathcal{D}(H)$	domain of an operator
$(\cdot,\cdot)_A$	inner product defined by (1.5)

$\|\|\cdot\|\|_A$	norm induced by the inner product (1.5)
$\langle\cdot,\cdot\rangle$	inner product defined by (2.4)
$E = \mathcal{D}(\|A\|^{\frac{1}{2}}) = E^+ \oplus E^0 \oplus E^-$	Hilbert space in Chapter 1
$E = \mathcal{D}(\|H_0\|^{\frac{1}{2}}) = E^+ \oplus E^-$	Hilbert space in Chapter 2
$E = \mathcal{D}(\|H\|^{\frac{1}{2}}) = E^+ \oplus E^0 \oplus E^-$	Hilbert space in Chapter 3
$P_0 : E \to L^0$	projection of E onto E^0 space
$(PS)_c$ condition, $c \in \mathbb{R}$	Palais-Smale condition - Definition C.2
$(Ce)_c$ condition, $c \in \mathbb{R}$	Cerami condition - Definition B.1
X^*	dual space of X
$\mathcal{T}_{\mathcal{P}}-$topology	topology product induced by $\mathcal{P}-$family of semi-norms (B.1)
$\mathcal{T}_{w^*}-$topology	weak topology on dual space
$\mathcal{P}-$open set	open set in $\mathcal{T}_{\mathcal{P}}-$topology
$\Phi_a \doteq \{u \in E : \Phi(u) \geqslant a\}, a \in \mathbb{R}$	
$\Phi^b \doteq \{u \in E : \Phi(u) \leqslant b\}, b \in \mathbb{R}$	
$\Phi_a^b \doteq \Phi_a \cap \Phi^b, a, b \in \mathbb{R}$	
$\Gamma_{Q,S}$	set defined by (B.5)
$\mathcal{M}$	generalized Nehari manifold defined by (2.8)
$\widehat{E}(u) \doteq E^- \oplus \mathbb{R}^+ u,\ u \in E \backslash E^-$	defined by (2.9)
$m(u) \doteq \widehat{E}(u) \cap \mathcal{M}$	defined by (2.30)
$\widehat{\Psi}^{\mathrm{o}}(u; v)$	generalized directional derivative (2.32)
$\partial\widehat{\Psi}(u)$	generalized gradient defined by (2.33)
$t_n \downarrow t_0$	t_n is going to t_0 from above (2.32)
$U_\delta(P) \doteq \{w \in S^+ : dist(w, P) < \delta\}$	defined for $P \subset E^+$ by (2.51)
$\gamma(A)$	Krasnoselskii genus defined by (2.57)
$L_1(x) \leqslant L_2(x), \quad L_1, L_2$ matrices	$\max\limits_{\xi \in \mathbb{C}^4, \|\xi\|=1} \big(L_1(x) - L_2(x)\big)\xi \cdot \overline{\xi} \leqslant 0$

Introduction

The Dirac equation has its origin in quantum mechanics and was proposed by the British theoretical physicist Paul Dirac in 1928 in an attempt to establish an equation that describes the evolution of a free particle relativistic. In its original form, this equation is given by

$$i\hbar\frac{\partial\psi}{\partial t} = D_c\psi, \tag{1}$$

where D_c is described by

$$D_c = -ic\hbar\alpha\cdot\nabla + mc^2\beta = -ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k + mc^2\beta.$$

In this expression $\partial_k = \dfrac{\partial}{\partial x_k}$, c denotes the speed of light, $m > 0$ the electron mass and $\hbar$ denotes the Planck's constant. Moreover, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where α_k, $k = 1, 2, 3$ and β are 4×4 complex matrices whose standard form (in 2×2 blocks) is

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \tag{2}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices satisfy the following anticommutation relations $\alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_{kl}I_4$, $\alpha_k\beta + \beta\alpha_k = 0$ and $\beta^2 = I_4$. Due to these relations it is possible check that D_c is a symmetric operator such that $D_c^2 = -c^2\hbar^2\Delta + m^2c^4$.

In general form, the Dirac equations are given by

$$i\hbar\frac{\partial\psi}{\partial t} = -ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi + mc^2\beta\psi + V(x)\psi + G_\psi(x,\psi) \tag{3}$$

whose solution $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$, $\psi(t, \cdot) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ is a wave function, which represents the state of a relativistic electron. The external fields are given by the real matrix potential $V(x)$ and the nonlinearity $G : \mathbb{R}^3 \times \mathbb{C}^4 \to \mathbb{R}$ represents a self-coupling nonlinearity. Assuming that $G(x, e^{i\theta}\psi) = G(x, \psi)$ for all $\theta \in [0, 2\pi]$, by the *Ansatz*

$$\psi(t, x) = e^{\frac{i\theta t}{\hbar}} u(x),$$

one can check that $\psi(x, t)$ satisfies (3) if and only if $u : \mathbb{R}^3 \to \mathbb{C}^4$ satisfies the problem

$$-i \sum_{k=1}^{3} \alpha_k \partial_k u + a\beta u + M(x)u = F_u(x, u), \tag{4}$$

where $a = mc/\hbar, M(x) = V(x)/c\hbar + \theta I_4/\hbar$ and $F_u(x, u) = G_u(x, u)/c\hbar$. We define $H_0 \doteq -i\alpha\nabla + a\beta$ the Dirac operator that will be used throughout this study. The Dirac equations are used in physics to describe the behavior of particles having spin 1/2 and also in atomic, nuclear and gravitational physics [41, 65]. More details about the definition of Dirac operator in the quantum mechanics, it domain and some of their properties are presented in Appendix A.

Many researches interested in the existence and multiplicity of solutions to problems involving the Dirac operator and different sets of hypotheses in nonlinearity have been developed. However, one of the main difficulties of the study of these equations is due to the spectral structure of the operator that makes the energy functional strongly indefinite, that is, its domain has two subspaces of infinite dimension in which the energy has opposite sign in each of them (see Appendix A). One possible alternative to overcome this fact is consider the external interactions in the problem, that is, the operator $A \doteq H_0 + W$, where W represents a suitable potential that can be vector or scalar, or tensor forces, and describe, for example, the interferences of electromagnetic fields, quark particles, and particle behaviour with anomalous electrical and magnetic moments [65]. By introducing a term that accurately describes this energy interaction one obtains interesting spectral properties for the new operator and enables different approaches to the problem.

In general, it is possible to establish three main classes of problems in relation to the potential behavior of V and nonlinearity F, as the authors observed in [36] and [73].

(I) ***autonomous systems:*** in these cases the potential V is constant and F does not depend on the variable x. As examples of this class we can cite [4, 5, 16, 50], in which the authors considered the so-called Soler model, that is

$$F(u) = \frac{1}{2}H(u\tilde{u}),\ H \in C^2(\mathbb{R},\ \mathbb{R}),\ H(0) = 0,\ \text{where } u\tilde{u} \doteq \beta u \cdot u,$$

with H satisfying suitable conditions. The authors assume $V = \omega$ with $\omega \in (-a, 0)$ and used the particular *Ansatz* for the solutions in spherical coordinates:

$$\varphi(x) = \begin{pmatrix} v(r)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iu(r)\begin{pmatrix} \cos(\theta) \\ \sin(\theta)e^{i\phi} \end{pmatrix} \end{pmatrix}, \tag{5}$$

where $r = |x|$ and (θ, ϕ) are angular parameters. Hence, the equation (4) was reduced to a EDO's system

$$\begin{cases} u' + \frac{2u}{r} & = & v[h(v^2 - u^2) - (a - \omega)] \\ v' & = & u[h(v^2 - u^2) - (a + \omega)] \end{cases} \tag{6}$$

where $h(s) = H'(s)$ and the existence of solution was obtained by shooting method, which yields an infinity of localized solutions for (6). In [40], the authors also consider the Soler model and required the main assumption

$$H'(s) \cdot s \geqslant \theta H(s) \text{ for all } s \in \mathbb{R} \text{ and some } \theta > 1.$$

Therefore, considering the space $E^s \subset H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ of functions of the form (5) and using variational methods, they obtain infinitely many solutions exploiting the inherent symmetry $F(u) = F(-u)$.

(II) ***periodic systems:*** this case occurs when V and F depend periodically on the variable x. Bartsch and Ding in [8] considered a problem whose nonlinearity, in addition to the periodicity in x, can be asymptotically quadratic or superquadratic when $|u| \to \infty$. In both situations, the results of existence and multiplicity of solutions were obtained when F is even in u. In this same way, another study that considered, in addition to the periodicity, the case of asymptotically quadratic nonlinearity in 0 and ∞ was recently developed by Ding and Liu in [28]. In it, the authors obtained existence and multiplicity of periodic solutions to the problem.

Yang and Ding in [71] also approached a periodic problem without the Ambrosetti-Rabinowitz conditions. In this case, the authors used a weak variant of the Linking Theorem and Lion's concentration-compactness principle [69] to guarantee the results of solution existence. It is interesting to note that, during the development of the work, the authors proved a lemma that, combined with the Nehari manifold arguments, is widely used to obtain stationary waves for Schrödinger equations. Despite this use, similar results were obtained for the Dirac equation without a monotone Nehari type condition.

The problem

$$-i\alpha\nabla u + a\beta u + \omega u = F_u(x, u), \text{ in } \mathbb{R}^3, \tag{7}$$

with $w \in (-a, a)$ and $F \in C^1(\mathbb{R}^3 \times \mathbb{C}^4, [0, \infty))$ $1-$periodic in x_k, $k = 1, 2, 3$, and superlinear growth was approached by Zhang, Tang and Zhang in [76]. Considering additional conditions on F and using a generalized variant Fountain Theorem, developed by Batkam and Colin [10], the authors ensure the existence of infinitely many large energy solutions. Similarly, Ding and Liu, in [31], considered $F_u(x,u) = G_u(x,u) + P_u(x,u)$ with $\omega = V_0$ constant, $G \in C^1(\mathbb{R}^3 \times \mathbb{C}^4, \mathbb{R})$ and $P \in C^1(\mathbb{R}^3 \times \mathbb{C}^4, \mathbb{R})$, where both were $1-$periodic in x_k, $k = 1, 2, 3$, and it satisfy another additional relations. In this case, the problem possesses a sequence of periodic solutions with the corresponding energy sequence large enough.

Recently, Benhassine in [13] studied the following equation

$$-i\alpha\nabla u + (a + V(x))\beta u + \omega u = F_u(x,u), \text{ in } \mathbb{R}^3, \tag{8}$$

where $w \in (-a, a)$, $V \in C^1(\mathbb{R}^3, [0,\infty))$ and $F \in C^1(\mathbb{R}^3 \times \mathbb{C}^4, [0,\infty))$ were both $1-$periodic in x_k, $k = 1, 2, 3$. His work dealt with the superquadratic and asymptotically quadratic cases with weaker conditions than those considered in [9, 25, 78]. Using the theory of critical points developed by Bartsch and Ding, he obtained infinitely many solutions geometrically distinct when F is even in u.

We can also cite as examples of this class of problems, the studies developed in [27], which is more detailed in Chapter 1, [29, 75], which use the generalized Nehari manifold method, and [78].

(III) ***nonperiodic systems:*** this class includes the cases in which V and F do not depend periodically on the variable x, and, apparently, is the class of problems that involves most of the existing works. To exemplify some of these elements, we can cite, initially, the studies developed by Ding and Wei [36] which considered a Dirac equation with superquadratic nonlinearity satisfying the Ambrosetti-Rabinowitz condition. Assuming, further, that there was the limit when $|x| \to \infty$ for both potential and nonlinearity, the authors guaranteed the existence of least energy solutions and also studied their exponential decay.

A important subclass of problems is the one that considers the following equations

$$-i\hbar\alpha\nabla u + a\beta u + V(x)u = G_u(x,u), \quad \text{in } \mathbb{R}^3. \tag{9}$$

For small $\hbar$, the solitary waves are referred as semi-classical states. The existence of solutions $u_\hbar$, $\hbar$ small, possesses important physical interest because it describe the transition from quantum to classical mechanics. Indeed, one of the basic principles of quantum mechanics is the correspondence principle, according to which, when $\hbar \to 0$, the laws of quantum mechanics must reduce to those of classical mechanics. Ding [26] considered the following problem

$$-i\hbar\alpha\nabla u + a\beta u = P(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^3, \tag{10}$$

where P has neither hypothesis of periodicity nor limit at the infinity. Supposing that

$$\inf P > 0 \qquad \text{and} \qquad \limsup_{|x|\to\infty} P(x) = \max P(x),$$

the author proved, following the ideas developed by Ackermann [1] and the Nehari manifold, that, for all $\varepsilon = \hbar > 0$ small enough, the equation possesses at least one least energy solution $w_\varepsilon \in \bigcap_{q\geqslant 2} W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$. Moreover, the set of all least energy solution is compact and there exist a maximum point x_ε of $|w_\varepsilon|$ such that

$$\lim_{\varepsilon\to 0} d(x_\varepsilon, \mathcal{P}) = 0, \tag{11}$$

where $\mathcal{P} = \{x \in \mathbb{R}^3 : P(x) = \max_{y\in\mathbb{R}^3} P(y)\}$. This fact guarantees that the concentration of solutions occurs at the maximum of the coefficient of the nonlinear external field.

Following the same idea of studying the concentration phenomenon, Ding and Xu [35] questioned when it is possible to find solutions which concentrate around local minima (or maxima) of an external potential. For this, the authors studied the following problem

$$-i\hbar\alpha\nabla u + a\beta u + V(x)u = g(|u|)u, \tag{12}$$

where V is locally Hölder continuous such that $\sup |V(x)| < a$ and F can be asymptotically linear at infinity or superlinear. Moreover, it is assumed that there is a bounded domain $\Lambda \subset \mathbb{R}^3$ such that

$$\underline{c} \doteq \min_{\Lambda} V < \min_{\partial\Lambda} V, \tag{13}$$

that is, the condition does not establish restrictions in the global behavior of the function V. This is possible because the technique employed in the development of the work, called the penalization method, modifies the original problem so that the behaviour of V out of Λ does not interfere on the conclusions obtained. Thus, it was guaranteed that, for $\hbar = \varepsilon > 0$ small enough, there exist a solution in $\bigcap_{q\geqslant 2} W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ which has an exponential decay. Moreover, it concentrates around the maxima point of V, that is, there exist $x_\varepsilon \in \Lambda$ a global maximum point of solution $|w_\varepsilon|$ such that

$$\lim_{\varepsilon\to\infty} V(x_\varepsilon) = \underline{c} \quad \text{and} \quad |w_\varepsilon| \leqslant C \exp\left(-\frac{c}{\varepsilon}|x - x_\varepsilon|\right).$$

Recently, Wang and Zhang [67] considered this same problem with $g(|u|) = |u|^p$ where $p \in (2,3)$ and focused on proving the existence of an unbounded sequence of localized bound states concentrating around the local minimum points of V. Then, supposing that

(V_1) $V \in C^1(\mathbb{R}^3, \mathbb{R})$ e $|V|_\infty < a$;

(V_2) there exist a bounded domain $\Lambda \subset \mathbb{R}^3$ with smooth boundary such that

$$\overrightarrow{\eta}(x) \cdot \nabla V(x) > 0, \; x \in \partial\Lambda,$$

where $\overrightarrow{\eta}(x)$ denotes the unit outward normal vector to $\partial\Lambda$;

holds, the authors used a penalization method due to Del-Pino e Felmer [22, 23] and a local Pohozaev type argument to obtain the conclusions.

The existence of semi-classical solutions for equations that involving critical nonlinearities was approached by Ding and Ruf [34], in which the authors considered the problem (12) with

$$g(|u|) = W(x)(f(|u|) + |u|),$$

where f has superlinear and subcritical growth as $|u| \to \infty$. It should be noted that V and W, among other properties, satisfy $V, W \in C^1(\mathbb{R}^3, \mathbb{R})$ with $V(x) \leqslant 0$ and $\inf W > 0$. In view of all the established hypotheses, a minmax value $c_\hbar$ for the energy functional associated with the problem, which depends on $\hbar$, can not be considered directly a critical value of this functional. Then, using Ackermann's ideas [1], the authors obtained a reduced energy functional for which the infimum over the classical Nehari manifold associated with this reduced functional is exactly the minmax value $c_\hbar$. The arguments presented to obtain the semi-classical solutions and to study the concentration phenomenon of these solutions still involved some auxiliary problems, among them the limit problem, as well comparisons between the minmax value $c_\hbar$ and the least energy of a class of limit problems.

Finally, we cite the study developed by Zhang, Tang and Zhang [77] that considered the following problem involving a Maxwell-Dirac system in $\mathbb{R}^3$:

$$\begin{cases} -i\alpha\nabla u + a\beta u + M(x)u - K(x)\phi u &= F_u(x,u) \\ -\Delta\phi &= 4\pi K(x)|u|^2 \end{cases} . \tag{14}$$

This class of systems describes the interaction of a particle with its self-generated electromagnetic field and plays an important role in quantum electrodynamics. In this work, it was considered a subcritical nonlinearity, which is also asymptotically quadratic nonautonomous and nonperiodic; $K \in L^\gamma$, $\gamma \in (6, \infty)$, and $K(x) > 0$ for all $x \in \mathbb{R}^3$. Moreover, M satisfies the following condition

(M) $M \in C(\mathbb{R}^3, \mathbb{R}^{4\times4})$ and there exists $h > 0$ such that $\Omega_h \doteq \{x \in \mathbb{R}^3 : \beta M < hI_4\}$ is nonempty and has finite Lebesgue measure,

which characterizes it as a indefinite and nonperiodic potential. This problem can be considered an extension from Dirac equation to Maxwell-Dirac systems, since some processes were adapted from the approach used for Dirac equations, taking into account the effects of non-local terms. In this way, the authors recovered the compactness imposing a control in the size of $F(x, u)$ in relation to the behavior of M at infinity in x. The technique used to obtain existence and multiplicity of solutions for this system is the theory of critical points developed by Bartsch and Ding [8]. Ding and Ruf in [33] also studied a Maxwell-Dirac system and were interested in obtaining multiple semi-classical solutions to the problem in which nonlinearity may be subcritical or critical.

We can also cite another references with equations that involving the Dirac operator: [9, 19, 25, 28, 30, 32, 42, 43, 72, 73, 74] and references therein. Our work, inspired by some of these studies, establishes the necessary conditions to obtain existence and multiplicity of solutions to Dirac equations in the form (4).

In Chapter 1, we consider a concave-convex problem, that is, the nonlinearity has the following form

$$F_u(x,u) = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^3, \tag{15}$$

where $2 < p < 3$ and $1 < q < p'$ being p' the conjugate exponent of p. Moreover, $\lambda > 0$ is a paremeter and the function f can presented a change signal. The vector potential $M(x)$, in this case, is nonperiodic and satisfies suitable conditions which ensure that the embedding $E = \mathcal{D}(|H|^{\frac{1}{2}}) \hookrightarrow L^p(\mathbb{R}^3, \mathbb{C}^4)$ is compact for all $p \in (2,3)$. This property is very important since we are considering the problem in an unbounded domain. Then, for $\lambda > 0$ sufficiently small, we prove, using the restriction of exponents, that any Cerami sequence associated with the energy functional is bounded and, in addition, the functional satisfies the Cerami's condition. Moreover, we obtain the conditions required by the theorems of the critical point theory for strongly indefinite functionals, (see Appendix B), which allows us to conclude the existence of multiple solutions to the problem, whose energy tends to ∞.

In Chapter 2, we study the following problem

$$-i\alpha\nabla u + a\beta u + V(x)u = g(x,u), \quad x \in \mathbb{R}^3, \tag{16}$$

under two different set of hyphoteses. In the first one, inpired by the results from [42], we consider a nonperiodic situation where the potencial vanishing at infinity and it is related with the nonlinearity, which satisfies a monotone growth condition. In the second part, following the ideas from [53, 62], we approach a periodic case, that is, the potential and the nonlinearity are periodic and the nonlinearity also present a weak monotonicity condition. In the development of both cases, we used the generalized Nehari manifold and the Clarke's subdifferential to ensure the existence of ground state solutions, since the hypothesis on growth leads us to obtain a functional that is locally Lipschitz continuous. To obtain the multiplicity of solutions we use the Krasnoselskii genus.

Finally, in the Chapter 3, we consider the equation

$$-i\alpha\nabla u + a\beta u + W(x)u = f(x,u) \quad \text{in } \mathbb{R}^3, \tag{17}$$

where $W(x) = M(x) + \lambda V(x)I_4$, $\lambda > 0$ is a parameter and the nonlinearity is asymptotically liner at infinity. In this case, M is a Coulomb type potential and V is a integrable function that can present a sign change. The Coulomb type of potential is important because it guarantees the existence of eigenvalues in the discrete spectrum of the operator $H = H_0 + M$ at interval $(-a, a)$. In order to use this spectral property, we rewrite the problem as follows

$$-i\alpha\nabla u + a\beta u + M(x)u = f(x, u) - \lambda V(x)u \quad \text{in} \quad \mathbb{R}^3. \tag{18}$$

In this case, we obtain a new energy functional that not satisfies all the conditions required in the theorems established by Bartsch and Ding [8, 25] and not allows us conclude, for example, the existence of a Cerami sequence. To overcome this difficult, considering $\lambda > 0$ small sufficiently, we use a suitable theorem due to Rabinowitz to obtain a Cerami sequence and, for the multiplicity of solutions, we apply a theorem from critical points theory, presented in Appendix B.

We conclude this work by presenting in the Appendix some historical aspects of Dirac operator and it properties about self-adjointness and spectrum. Also, we mention some results about theory of critical points presented by Bartsch and Ding that applies to the class of strongly indefinite problems in which the equations involving the Dirac operator are included.

CHAPTER 1

Multiple solutions for a nonperiodic Dirac equation with concave and convex nonlinearities

In this chapter we will study, using the variational methods, the following version of a Dirac equation

$$H_0 u + M(x)u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^3, \tag{1.1}$$

where $\lambda > 0$ is a parameter which will be defined later and $2 < p < 3$ and $1 < q < p'$ with $p' = \dfrac{p}{p-1}$ the conjugate exponent of p. This condition is a technical restriction used to show the boundedness of Cerami sequences for the functional associated to Problem (1.1).

Many authors dedicated themselves to the study of problems whose operator had the form $A = H_0 + M$, where $H_0 = -i\alpha\nabla + a\beta$ and M is a appropriate potential, vectorial or scalar. Zhang, Qin, Zhao in [72], for example, considered a vector potential $M(x)$ that satisfies

(M_0) $M : \mathbb{R}^3 \to \mathbb{R}^{4\times 4}$ is continuous and there is $r_0 > 0$ such that, for any $h > 0$

$$\Big|\{x \in \mathbb{R}^3 : |x-y| \leqslant r_0,\ \beta M(x) < h\}\Big| \to 0 \text{ if } |y| \to +\infty. \tag{1.2}$$

The relation between the matrix $\beta M(x)$ and the number $h > 0$ is defined at Appendix C. This condition ensures that the potential is nonperiodic and has some coercivity behaviour, which help us to overcome the difficulties arising from of lack of compactness of the Sobolev embedding, since the domain is the whole space. This fact do not allow conclude that the energy functional satisfies $(PS)_c$–condition and requires another approach with another arguments. Notice that, if

a) $M \in C(\mathbb{R}^3, \mathbb{R})$ and for any $b > 0$ there holds $|\Omega_b| < \infty$ where $\Omega_b = \{x \in \mathbb{R}^3 : M(x) < b\}$;

b) $M \in C(\mathbb{R}^3, \mathbb{R}^{4\times 4})$ and for any $b > 0$ there holds $|\Omega_{\beta,b}| < \infty$ where $\Omega_{\beta,b} = \{x \in \mathbb{R}^3 : \beta M(x) < b\}$;

the hypotheses (M_0) is still valid. In this case, the authors studied two possibilities for the nonlinearity $F_u(x, u)$: asymptotically quadratic and superquadratic. In both were obtained existence results and multiplicity results if $F(x, u)$ is even in u. Other different hypotheses about the vector potentials were considered by [32] and [36] in which the authors obtained existence and multiplicity results for the problems, and, in some of them, they studied the exponential decay of the solutions.

In our work, inspired by [72], we consider that $M(x)$ is a vector potential that satisfies the condition (M_0) stated above and f, g are real functions that satisfy, respectively:

(H_1) $0 \neq f \in L^{\gamma}(\mathbb{R}^3)$ where $\gamma = \dfrac{p}{p-q}$ such that

$$\int_{\mathbb{R}^3} f(x)|v|^q dx \geqslant 0 \quad \text{for all} \quad v \in E^0, \tag{1.3}$$

where $E^0 = Ker(H_0 + M)$.

(H_2) $g \in L^{\infty}(\mathbb{R}^3)$ such that $g(x) \geqslant d > 0$ for all $x \in \mathbb{R}^3$;

The relation between the exponents classifies the problem into the class of concave and convex problems and we can cite many studies that have been developed to solve problems with this type of nonlinearities and weighted functions which may or not change signal. For example, Wu, Tang, Wu in [70] studied a problem involving the Laplacian operator, a nonperiodic potential $V \in C(\mathbb{R}^N, \mathbb{R})$ and functions $f(x, u)$ and $g(x, u)$, which have indefinite signal. Under different conditions, the authors studied various problems and achieved results of existence and multiplicity of solutions for all of them. A relevant research involving the Dirac operator and concave and convex nonlinearities was developed by Ding and Liu in [27]. In this case, the authors solved the following problem

$$-i\alpha\nabla u + a\beta u + V(x)u = \xi F_u(x, u) + \eta G_u(x, u) \quad \text{ in } \mathbb{R}^3, \tag{1.4}$$

under the hypotheses

(P) $V \in C(\mathbb{R}^3, [0, \infty)), F, G \in C^1(\mathbb{R}^3 \times \mathbb{C}^4, \mathbb{R})$; the functions $V(x), F(x, u)$ and $G(x, u)$ are 1-periodic in x_k with $k = 1, 2, 3$; $F(x, u)$ and $G(x, u)$ are even in u.

In addition were considered two distinct set of hypotheses: in the first one, these hypotheses provide solutions with large norms, that is, a sequence of 1−periodic solutions (u_n) was obtained satisfying $\xi\Phi(u_n) \to \infty$ as $n \to \infty$. In the second one, it was also found a sequence of solutions (v_n) satisfying $\eta\Phi(v_n) < 0$, $\eta\Phi(v_n) \to 0$ and $||v_n||_{L^\infty(Q,\mathbb{C}^4)} \to 0$ as $n \to \infty$, that is, solutions with small energies. Here Φ is the energy functional associated with this problem, the set Q is the cube $Q = [0,1] \times [0,1] \times [0,1]$ and ξ, η are real constants.

It is important to note that the set of conditions considered in our work guarantees that $H(x,u)$ is nonperiodic and, since f can change your signal, the nonlinearity has indefinite signal. In our case, the non-periodicity of elements was a significant difficulty since this condition does not allow making restriction of domain on cube $Q = [0,1] \times [0,1] \times [0,1]$ and, therefore, we need to consider a suitable potential to conclude stronger results regarding immersions in L^P spaces, not just locally, but all over space, for $p \in [2,3)$. Moreover, this potential gives us a characterization of the H operator spectrum as unlimited sequences of eigenvalues of opposite signals, ordered by their multiplicity, as we conclude by Lemma 1.2. This fact allows us to decompose the domain into a suitable form to guarantee multiplicity of solutions through Bartsch and Ding results presented in Appendix B.

However, this decomposition also involves the elements from the kernel of H, which is nonzero, that is, the norm of an element $u \in E = \mathcal{D}(|H|^{\frac{1}{2}})$ is given by

$$u = u^+ + u^0 + u^- \in E^+ \bigoplus E^0 \bigoplus E^-.$$

Thus, although the kernel has finite dimension and even if we apply the known Hölder relation or other estimates in nonlinear terms, we need establish conditions on these elements. The condition (H_1) was necessary in the proof that guarantees one of the conditions required by the theory of critical points, since without it we would not have the guarantee that there is no sequence satisfying the relation (1.23).

Another necessary condition was the relationship between exponents, which was strongly used to demonstrate the boundedness of Cerami sequence associated with energy functional. In a standard way, it suffices to assume that the Cerami sequence has a divergent norm, to define a normalized sequence, and to use the weak convergence properties to obtain a contradiction. In our case, however, the superlinear term dominates the other terms and so we would have no contradiction, which require an adaptation in the exponents and also that the function g to be a bounded function. In addition, we have established the condition that g is positive to obtain fundamental relations, such as (1.24) and (1.39), which helped to obtain the conditions for the existence of Cerami sequence and its boundedness, respectively.

The contributions of this work are significant because the authors do not know in the literature any other study that has considered the non-periodic case involving a convex concave nonlinearity with weight functions that present signal change and potential with some coercivity condition. Moreover, we developed the analysis over the whole space $\mathbb{R}^3$ without restrictions.

Under the above conditions, through linking theorems and critical point theory to strongly indefinite functional, we have been able to prove the following result:

Theorem 1.1. *Suppose* (M_0), (H_1) *and* (H_2) *be satisfied. Then there is constant* $\Lambda > 0$ *such that, if* $\lambda \in (0, \Lambda)$, *the Problem* (1.1) *has infinitely many solutions.*

This chapter is organized as follows. In Section 1.1, we analyse the operator $A = H_0 + M$, it spectrum and some additional properties. In Section 1.2, we prove that the energy functional possesses the linking structure. In the Section 1.3, we guarantee the existence and boundedness of a Cerami sequence $(Ce)_c$ for some $c > 0$ and prove that the energy functional satisfies the Cerami condition, for all $c > 0$, which is important to conclude the existence and multiplicity of solutions. Therefore, we get all the elements to prove Theorem 1.1.

1.1 Variational setting

Let the operator $A = H_0 + M$, where $H_0 = -i\alpha\nabla + a\beta$ and $M(x)$ satisfies (M_0), which is a self-adjoint operator in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with $\mathcal{D}(A) \subset H^1(\mathbb{R}^3, \mathbb{C}^4)$. This space is a Hilbert space equipped with the inner product

$$(u, v)_A = \langle Au, Av\rangle_{L^2} + \langle u, v\rangle_{L^2} \tag{1.5}$$

and the induced norm $||\cdot||_A$.

If $B = -i\alpha\nabla + \beta$, which is still self-adjoint and $\mathcal{D}(B) = H^1(\mathbb{R}^3, \mathbb{C}^4)$, we obtain a important relation between the norms from H^1 and $||\cdot||_A$.

Lemma 1.1. *For all* $u \in \mathcal{D}(A) \subset H^1(\mathbb{R}^3, \mathbb{C}^4)$, *there exist* $d > 0$ *such that*

$$||u||_{H^1} = |||B|u||_{L^2} \leqslant d||u||_A.$$

Proof: Notice that, for all $u \in \mathcal{D}(B^2) = H^2(\mathbb{R}^3, \mathbb{C}^4)$,

$$\left|||B|u\right||^2_{L^2} = \langle Bu, Bu\rangle_{L^2} = \langle B^2u, u\rangle_{L^2} = \langle(-\Delta + 1)u, u\rangle_{L^2} = ||\nabla u||^2_{L^2} + ||u||^2_{L^2} = ||u||^2_{H^1}.$$

Since H^2 is dense in $H^1(\mathbb{R}^3, \mathbb{C}^4)$, we obtain the first equality. Let $B_1 : \mathcal{D}(A) \to L^2$ the restriction of B to the set $\mathcal{D}(A)$, which is a linear and closed operator. Indeed, consider $(u_n) \subset \mathcal{D}(A)$ such that $u_n \overset{||\cdot||_A}{\longrightarrow} u$ and $B_1u_n \overset{||\cdot||_{L^2}}{\longrightarrow} v$, as $n \to \infty$. Since $\mathcal{D}(A)$ is a Hilbert space equipped with the norm $||\cdot||_A$, we conclude that $u \in \mathcal{D}(A)$. On the other hand, B is a closed operator and, therefore,

$$B_1u_n = Bu_n \overset{||\cdot||_{L^2}}{\longrightarrow} Bu = B_1u.$$

Using the uniqueness of the limit we obtain that $v = B_1 u$ and, Theorem C.5, B_1 is a linear and continuous operator, that is,

$$||Bu||_{L^2} = ||B_1 u||_{L^2} \leqslant d||u||_A,$$

which demonstrate the result. ■

The assumption (M_0) ensures that the operator A is self-adjoint and that its spectrum has only eigenvalues of finite multiplicity as justified by [72] in Lemma 2.2. Here we will present this proof to complement our work.

Lemma 1.2. *Suppose* (M_0) *holds. Then* $\sigma(A) = \sigma_d(A)$.

Proof: Consider $h > 0$. By (M_0), there exist $r_0 > 0$ such that

$$\lim_{|y|\to\infty} \left|\{x \in \mathbb{R}^3 : |x-y| \leqslant r_0, \beta M(x) < h\}\right| = 0$$

where $|\Omega|$ denotes the Lebesgue measure. Set

$$(\beta M(x) - h)^+ = \begin{cases} \beta M(x) - h, & if \quad (\beta M(x) - h) \geqslant 0 \\ 0, & if \quad (\beta M(x) - h) < 0 \end{cases} \tag{1.6}$$

and $(\beta M(x) - h)^- = (\beta M(x) - h) - (\beta M(x) - h)^+$. Then

$$\begin{aligned} A = H_0 + M &= -i\alpha\nabla + (a+h)\beta - \beta h + \beta^2 M(x) \\ &= -i\alpha\nabla + (a+h)\beta + \beta(\beta M(x) - h) \\ &= -i\alpha\nabla + (a+h)\beta + \beta(\beta M(x) - h)^+ + \beta(\beta M(x) - h)^- \\ &\doteq A_2 + \beta(\beta M(x) - h)^-. \end{aligned}$$

Notice that for $u, v \in \mathbb{C}^4$ and β defined in (2) we have that $\beta u \cdot v = u \cdot \beta v = u\beta \cdot v$ and $\beta\alpha_k = -\alpha_k\beta$, $k = 1, 2, 3$, where $u \cdot v = \sum_{j=1}^{4} u_j \overline{v_j}$. Then

$$(-i\alpha\nabla u) \cdot \overline{\beta u} = \beta(-i\alpha\nabla u) \cdot \overline{u} = -i\left(\sum_{k=1}^{3} \beta\alpha_k\partial_k u\right) \cdot \overline{u} = i\left(\sum_{k=1}^{3} \alpha_k\beta\partial_k u\right) \cdot \overline{u} = (i\alpha\nabla u) \cdot \overline{\beta u}.$$

Consequently,

$$\begin{aligned}
\langle -i\alpha\nabla u, \beta u\rangle_{L^2} + \langle \beta u, -i\alpha\nabla u\rangle_{L^2} &= \int_{\mathbb{R}^3} (-i\alpha\nabla u)\cdot\overline{\beta u}\, dx + \int_{\mathbb{R}^3} \beta u\cdot\overline{(-i\alpha\nabla u)}\, dx \\
&= \int_{\mathbb{R}^3} (i\alpha\nabla u)\cdot\overline{\beta u}\, dx + \int_{\mathbb{R}^3} \beta u\cdot(i\alpha\nabla u)\, dx \\
&= \int_{\mathbb{R}^3} (i\alpha\nabla u)\cdot\overline{\beta u}\, dx + \int_{\mathbb{R}^3} \overline{(i\alpha\nabla u)\cdot\beta u}\, dx \\
&= \int_{\mathbb{R}^3} (i\alpha\nabla u)\cdot\overline{\beta u}\, dx + \int_{\mathbb{R}^3} (-i\alpha\nabla u)\cdot\overline{\beta u}\, dx \\
&= 0.
\end{aligned} \tag{1.7}$$

Moreover, since $\beta^2 = I$,

$$\langle(\beta(\beta M(x)-h)^+)u, \beta u\rangle_{L^2} + \langle \beta u, \beta(\beta M(x)-h)^+)u\rangle_{L^2} = 2\langle(\beta M(x)-h)^+u, u\rangle_{L^2}. \tag{1.8}$$

Hence, if $u \in \mathcal{D}(A)$,

$$\begin{aligned}
\langle A_2u, A_2u\rangle_{L2} &= ||\big(-i\alpha\nabla + \beta(\beta M(x)-h)^+\big)u||_{L^2} + (a+h)^2||u||_{L^2} \\
&\quad + \Big\langle\big(-i\alpha\nabla + \beta(\beta M(x)-h)^+\big)u, (a+h)\beta u\Big\rangle_{L2} \\
&\quad + \langle(a+h)\beta u, \big(-i\alpha\nabla + \beta(\beta M(x)-h)^+\big)u\rangle_{L2} \\
&= ||\big(-i\alpha\nabla + \beta(\beta M(x)-h)^+\big)u||_{L^2} + (a+h)^2||u||_{L^2} \\
&\quad + 2(a+h)\Big\langle(\beta M(x)-h)^+u, u\Big\rangle_{L^2} \\
&\geqslant (a+h)^2||u||^2_{L^2},
\end{aligned}$$

since (1.7) and (1.8) holds. Thus, for all $u \in \mathcal{D}(A)$,

$$||A_2u - \lambda u)||_{L^2} \geqslant (a+h)||u||^2_{L^2} - |\lambda|||u||^2_{L^2} = ((a+h) - |\lambda|)||u||^2_{L^2}, \tag{1.9}$$

and, using [47] Lemma 10.4-1, λ belongs to the resolvent set $\rho(A_2)$ if and only if $((a+h)-|\lambda|) > 0$. That is,

$$\sigma(A_2) = \mathbb{R}\backslash\rho(A_2) \subset \mathbb{R}\backslash(-(a+h), (a+h)).$$

This relation help us to prove that $\sigma_e(A) \cap (-(a+h), (a+h)) = \varnothing$.

Assume by contradiction that there exist $\nu \in \sigma_e(A)$ such that $|\nu| < a+h$. Using the Weyl's criterion, let $(u_n) \subset \mathcal{D}(A)$ with $||u_n||_{L^2} = 1$, $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ and $||(A-\nu)u_n||_{L^2} \to 0$ as $n \to \infty$. Notice that the operator $u \mapsto \beta(\beta M(x)-h)^- u$ is compact, as defined by Definition C.1. Indeed, let $(w_n) \subset \mathcal{D}(A)$ a bounded sequence and, up to subsequence, we can suppose $w_n \rightharpoonup w$, $w \in \mathcal{D}(A)$, since $\mathcal{D}(A)$ is a Hilbert space. Suppose, without loss of generality, that $w_n \rightharpoonup 0$ in $\mathcal{D}(A)$ and we will prove that

$||\beta(\beta M(x)-h)^{-}w_n||_{L^2}\to 0$ as $n\to\infty$.

Fix $R>0$ and denote $B_R(0)=\{x\in\mathbb{R}^3 : |x|<R\}$. Let (y_j) a sequence in $B_R^c(0)$ and $r_0>0$ such that

(i) $B_R^c(0)\subset\bigcup_{i=1}^{\infty}B(y_i,r_0)$;

(ii) each point x is contained in at most 2^3 such balls $B(y_i,r_0)$.

Denote $B_M\doteq\{x\in B_R^c(0):\beta M(x)<h\}$, $B_i=B(y_i,r_0)\cap B_M$ and choose $s\in(1,3)$ with s' the conjugate exponent of s. Then

$$\begin{aligned}
\int_{B_R^c(0)}|\beta(\beta M(x)-h)^{-}w_n|^2dx &\leqslant \sum_{i=1}^{\infty}\int_{B_i}|\beta(\beta M(x)-h)^{-}w_n|^2dx\\
&\leqslant \sum_{i=1}^{\infty}\left(\int_{B_i}|w_n|^{2s}dx\right)^{\frac{1}{s}}\left(\int_{B_i}|\beta(\beta M(x)-h)^{-}|^{2s'}dx\right)^{\frac{1}{s'}}
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{i=1}^{\infty}\left(\int_{B_i}|w_n|^{2s}dx\right)^{\frac{1}{s}}\left(\sup_{y_i}||\beta(\beta M(x)-h)^{-}||_M^{2s'}\int_{B_i}dx\right)^{\frac{1}{s'}}\\
&=\sum_{i=1}^{\infty}\left(\int_{B_i}|w_n|^{2s}dx\right)^{\frac{1}{s}}C_R^2|B_i|^{\frac{1}{s'}}\\
&\leqslant C_R^2\,\varepsilon_R 2^3||w_n||_{L^{2s}}^2\\
&\leqslant CC_R^2\,\varepsilon_R 2^3||w_n||_A^2,
\end{aligned}$$

since H^1 embeds into L^{2s} continuously and Lemma 1.1 holds. Notice that we denotes $C_R=\sup_{y_i}||\beta(\beta M(x)-h)^{-}||_M$ and $\varepsilon_R=\sup_{y_i}|B_i|$, which is well defined because $|B_i|=|B(y_i,r_0)\cap B_M|\leqslant|B(y_i,r_0)|<\infty$. Using the assumption (M_0) we obtain that $\varepsilon_R\to 0$ as $R\to\infty$ and, therefore,

$$\int_{B_R^c(0)}|\beta(\beta M(x)-h)^{-}w_n|^2dx\to 0 \text{ as } R\to\infty.$$

On the other hand, as $n\to\infty$,

$$\int_{B_R(0)}|\beta(\beta M(x)-h)^{-}w_n|^2dx\leqslant\left(\int_{B_R(0)}|w_n|^{2s}dx\right)^{\frac{1}{s}}\left(\int_{B_R(0)}|\beta(\beta M(x)-h)^{-}|^{2s'}dx\right)^{\frac{1}{s'}}\to 0,$$

since $H^1\hookrightarrow L^{2s}_{loc}$ is compact. Then

$$||\beta(\beta M(x)-h)^{-}w_n||_{L^2}\to 0 \text{ as } n\to\infty.$$

Hence

$$\begin{aligned}o_n(1) = ||(A - \nu I)u_n||_{L^2} &= ||(A_2 + \beta(\beta M(x) - h)^- - \nu I)u_n||_{L^2} \\ &\geqslant ||A_2 u_n||_{L^2} - ||\nu u_n||_{L^2} - ||\beta(\beta M(x) - h)^- u_n||_{L^2} \\ &\geqslant ((a + h) - |\nu|) - o_n(1),\end{aligned}$$

that is, $0 < ((a+h) - |\nu|) \leqslant o_n(1)$ which is a contradiction. Hence

$$\sigma_e(A) \cap (-(a+h), (a+h)) = \varnothing$$

and, since h is arbitrary, $\sigma(A) = \sigma_d(A)$. ∎

This fact allows conclude that A has a sequence of eigenvalue

$$... \leqslant \lambda_{-k} \leqslant ... \leqslant \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \leqslant ... \leqslant \lambda_k \leqslant ...,$$

such that $\lim\limits_{j \to \infty} \lambda_{\pm k} = \pm\infty$ and a sequence of eigenfunction $\{e_{\pm k}\}$ associated with these eigenvalues that form a orthogonal basis for $L^2(\mathbb{R}^3, \mathbb{C}^4)$. Then, the space can be decomposed into

$$L^2 = L^- \oplus L^0 \oplus L^+,$$

where A is positive definite (respectively, negative definite) in L^+ (respectively, in L^-) and $L^0 = ker(A)$.

Let $E = \mathcal{D}(|A|^{\frac{1}{2}})$ the domain of self-adjoint operator $|A|^{\frac{1}{2}}$, which is a Hilbert space equipped with the inner product

$$\langle u, v \rangle = \langle |A|^{\frac{1}{2}} u, |A|^{\frac{1}{2}} v \rangle_{L^2} + \langle P_0 u, P_0 v \rangle_{L^2}, \tag{1.10}$$

where $P_0 : E \to L^0$ the projection and $||u|| = \langle u, u \rangle^{\frac{1}{2}}$. So, the space E also has a orthogonal decomposition

$$E = E^- \oplus E^0 \oplus E^+, \tag{1.11}$$

with $E^{\pm} = E \cap L^{\pm}$ and $E^0 = L^0 = ker(A)$.

Using this structure and complex interpolation arguments it is possible to demonstrate that embedding $E \hookrightarrow L^p(\mathbb{R}^3, \mathbb{C}^4)$ is compact for all $p \in [2, 3)$. Indeed, similar to [32] and following the ideias from [72], we introduce in $\mathcal{D}(A)$ the following inner product

$$\langle\langle u, v \rangle\rangle_A = \langle Au, Av \rangle_{L^2} + \langle P_0 u, P_0 v \rangle_{L^2}$$

whose induced norm will be denoted by $|\cdot|_A$. Then, considering $\widehat{A} \doteq |A| + P_0$ we have

$\mathcal{D}(\widehat{A}) = \mathcal{D}(A)$ and

$$|u|_A = ||\widehat{A}u||_{L^2}, \forall u \in \mathcal{D}(A). \tag{1.12}$$

Moreover, using that $\mathcal{D}(A) = \mathcal{D}(\widehat{A})$ is a core of $\widehat{A}^{\frac{1}{2}}$ we obtain that

$$||u|| = \left|\left|\widehat{A}^{\frac{1}{2}}u\right|\right|_{L^2} \text{ for all } u \in E. \tag{1.13}$$

By complex interpolation theory we have $H^{\frac{1}{2}} = \mathcal{D}(|B|^{\frac{1}{2}})$ and there exists constants $d_3, d_4 > 0$ such that

$$d_3||u||_{\frac{1}{2}} \leqslant \left|\left||B|^{\frac{1}{2}}u\right|\right|_{L^2} \leqslant d_4||u||_{\frac{1}{2}} \text{ for all } u \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4). \tag{1.14}$$

With this elements we can prove that

Lemma 1.3. *The embedding $E \hookrightarrow H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ is continuous and the embedding $E \hookrightarrow L^p(\mathbb{R}^3, \mathbb{C}^4)$ is compact for all $p \in [2, 3)$.*

Proof: Notice that, by Lemma 1.1 and (1.12), there exist $d_5 > 0$ such that

$$\left|\left||B|^{\frac{1}{2}}u\right|\right|_{L^2} \leqslant d_5||\widehat{A}u||_{L^2} = ||(d_5\widehat{A})u||_{L^2}, \text{ for all } u \in \mathcal{D}(A). \tag{1.15}$$

It follows from Proposition C.1 that $\langle |B|u, u\rangle_{L^2} \leqslant \langle d_5\widehat{A}u, u\rangle_{L^2}$ for all $u \in \mathcal{D}(A)$ and therefore

$$\left|\left||B|^{\frac{1}{2}}u\right|\right|^2_{L^2} = \langle |B|u, u\rangle_{L^2} \leqslant \langle (d_5\widehat{A})u, u\rangle_{L^2} = d_5\left|\left|\widehat{A}^{\frac{1}{2}}u\right|\right|^2_{L^2} \text{ for all } u \in \mathcal{D}(A). \tag{1.16}$$

Since $\mathcal{D}(A)$ is core of $\widehat{A}^{\frac{1}{2}}$, we obtain that $\left|\left||B|^{\frac{1}{2}}u\right|\right|^2_{L^2} \leqslant d_5\left|\left|\widehat{A}^{\frac{1}{2}}u\right|\right|^2_{L^2}$ for all $u \in E$. This jointly with (1.13) shows that $\left|\left||B|^{\frac{1}{2}}u\right|\right|^2_{L^2} \leqslant d_5||u||^2$ for all $u \in E$, which together with (1.14) implies that

$$||u||_{\frac{1}{2}} \leqslant d_6||u||$$

for all $u \in E$ and prove that $E \hookrightarrow H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ is continuous.

For the second part, it suffices to prove that $E \hookrightarrow L^2$ is compact. Let

$$L_j \doteq span\{e_{-j}, ..., e_{-1}, e_1, ..., e_j\}, \quad j \in \mathbb{N}$$

and denotes $P_j : E \to L_j$ the orthogonal projection. Consider $(u_n) \subset E$ such that $u_n \rightharpoonup u$ in E and define $w_n = u_n - u$. Moreover, define $K \doteq \sup_n ||w_n||^2$, which exist because of weak convergence. Given $\varepsilon > 0$ we choose $j \in \mathbb{N}$ such that $\frac{M}{\nu_j} < \frac{\varepsilon}{2}$, where $\nu_j = |\lambda_{-j}| + \lambda_j$. Since $P_jw_n \to 0$ in L_j as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $||P_jw_n||^2 < \frac{\varepsilon}{2}$ for all $n \geqslant n_0$. Let $\{E(\tau)\}_{\tau\in\mathbb{R}}$ be the spectral family of A. It follows from

$$
\begin{aligned}
||w_n||^2 \geqslant ||(I-P_j)w_n||^2 &= \left|\left|\widehat{A}^{\frac{1}{2}}(I-P_j)w_n\right|\right|_{L^2}^2 \\
&= \left\langle |\widehat{A}|^{\frac{1}{2}}(I-P_j)w_n, |\widehat{A}|^{\frac{1}{2}}(I-P_j)w_n \right\rangle_{L^2} \\
&= \int_{-\infty}^{\lambda_j} |\tau| d\left|\left|E(\tau)(I-P_j)w_n\right|\right|_{L^2}^2 + \int_{\lambda_j}^{\infty} \tau d\left|\left|E(\tau)(I-P_j)w_n\right|\right|_{L^2}^2 \\
&\geqslant (|\lambda_{-j}| + \lambda_j)\left|\left|(I-P_j)w_n\right|\right|_{L^2}^2
\end{aligned}
$$

that

$$
||(I-P_j)w_n||_{L^2}^2 \leqslant \frac{||w_n||^2}{|\lambda_{-j}| + \lambda_j} = \frac{||w_n||^2}{\nu_j} < \frac{\varepsilon}{2}.
$$

Then

$$
||w_n||_{L^2}^2 = ||P_j w_n||_{L^2}^2 + ||(I-P_j)w_n||_{L^2}^2 < \varepsilon \quad \text{for all} \quad n \geqslant n_0.
$$

This proves that $u_n \to u$ in $L^2(\mathbb{R}^3, \mathbb{C}^4)$. ■

Remark 1.1. *It follows from the above lemma that there is a positive constant C_r, $r \in [2,3]$ that*

$$
C_r ||u||_{L^r} \leqslant ||u||
$$

for all $u \in E$.

Assuming (M_0), (H_1) and (H_2), we consider the functional $\Phi : E \to \mathbb{R}$ associated with the problem (1.1) and defined by

$$
\Phi(u) = \frac{1}{2}\Big(||u^+||^2 - ||u^-||^2\Big) - \frac{\lambda}{q}\int_{\mathbb{R}^3} f(x)|u|^q dx - \frac{1}{p}\int_{\mathbb{R}^3} g(x)|u|^p dx, \tag{1.17}
$$

which is a $C^1(E;\mathbb{R})$ functional. It is well known (see [25], [30]) that the critical points of this energy functional are the solutions of the proposed problem and therefore our objective is to study this functional in order to obtain a nontrivial critical point.

By the assumption (H_1) and the Hölder inequality, we obtain

$$
\left|\int_{\mathbb{R}^3} f(x)\ |u|^q\ dx\right| \leqslant \left(\int_{\mathbb{R}^3} |f(x)|^{\frac{p}{p-q}}\ dx\right)^{\frac{p-q}{p}} \left(\int_{\mathbb{R}^3} |u|^p\ dx\right)^{\frac{q}{p}} \leqslant C_f C_p^{-q} ||u||^q, \tag{1.18}
$$

where $C_f \doteq ||f||_{L^\gamma} > 0$. It follows immediately from Lemma C.2 that

Lemma 1.4. *Assume (H_1) and suppose $(u_n) \subset E$ such that $u_n \rightharpoonup u$ in E. Then*

$$\int_{\mathbb{R}^3} f(x)|u_n|^q \, dx = \int_{\mathbb{R}^3} f(x)|u|^q \, dx + o_n(1).$$

Proof: Notice that, by Lemma 1.3, $|| \; |u_n|^q||_{L^{\frac{p}{q}}} \leqslant C_p^{-q}||u_n||^q$, for all $n \in \mathbb{N}$. Since (u_n) is bounded in E, it follows that $(|u_n|^q) \subset L^{\frac{p}{q}}(\mathbb{R}^3)$ is a bounded sequence. Moreover, $|u_n|^q \to |u|^q$ a.e. in $\mathbb{R}^3$ and $f \in L^\gamma(\mathbb{R}^3) = (L^{\frac{p}{q}}(\mathbb{R}^3))^*$. ■

In order to obtain the critical points to the functional Φ we will use the critical points results established in the Appendix B and throughout this chapter we consider $Y = E^+$, $X = (E^0 \oplus E^-)$ and, since $H(x, u)$ is even in u, the action of group G can be considered the antipodal action.

1.2 Linking structure

At this section, we will describe the linking structure of the functional Φ which is important to ensure the existence of Cerami sequences. This concept is based on the topological notion of "linking" and was firstly introduced by Benci [11] and Rabinowitz [56]. It was later generalized by [12] to include indefinite functionals as well and, recently, this concept was extended to the infinite-dimensional setting by Bartsch and Ding [8, 9]. More details can be founded at Appendix B.

Throughout this section we consider the constant

$$\Lambda_1 \doteq \frac{q}{C_f}\left(C_p^2 \frac{(p-2)}{4(p-q)}\right)^{\frac{p-q}{p-2}} \left(\frac{p}{||g||_\infty} \frac{(2-q)}{(p-2)}\right)^{\frac{2-q}{p-2}} \tag{1.19}$$

and assume $0 < \lambda < \Lambda_1$. Remember that $1 < q < p', 2 < p < 3$ and $C_f > 0$ is defined by (1.18).

Lemma 1.5. *Let (M_0), (H_1) and (H_2) be satisfied. Then there is $\rho > 0$ such that $\kappa \doteq \inf \Phi(S) > \Phi(0) = 0$ where $S = \partial B_\rho \cap E^+$.*

Proof: For any $y \in E^+$, since $0 < \lambda < \Lambda_1$, we observe that

$$\Phi(y) > ||y||^2\left(\frac{1}{2} - \left(\Lambda_1 C_f \left(q\, C_p^q\right)^{-1} ||y||^{q-2} + ||g||_\infty \left(p\, C_p^p\right)^{-1} ||y||^{p-2}\right)\right).$$

Then, for $\rho = \left(\frac{C_p^p}{4||g||_\infty} \frac{p(2-q)}{(p-q)}\right)^{\frac{1}{p-2}}$ and $S = \{y \in E^+; ||y|| = \rho\}$ it follows that

$$\Phi(y) > \rho^2\left(\frac{1}{2} - \frac{1}{4}\right) > 0, \ \forall y \in S,$$

from which we conclude that $\inf \Phi(S) \geqslant \dfrac{\rho^2}{4} > 0$. ∎

Define $Y_n = span\{e_1, e_2, ..., e_n\}$ and $E_n = E^- \oplus E^0 \oplus Y_n$, $n \in \mathbb{N}$. Using the properties of eigenfunctions it is possible to prove that

$$\lambda_1 ||u||_{L^2}^2 \leqslant ||u||^2 \leqslant \lambda_n ||u||_{L^2}^2, \ \forall u \in Y_n.$$

Lemma 1.6. *Suppose the same conditions and $\rho > 0$ from Lemma 1.5. There is a sequence (R_n) with $R_n > \rho$ such that* $\sup \Phi(E_n) < \infty$ *and* $\sup \Phi(E_n \backslash \overline{B_n}) < \inf \Phi(B_\rho \cap E^+)$, *where $\overline{B_n} \doteq \{u \in E_n : ||u|| \leqslant R_n\}$.*

Proof: For fixed $n \in \mathbb{N}$ suppose, by contradiction, that there is a sequence $(u_j) \subset E_n$ and $M > 0$ such that $||u_j|| \to \infty$ and $\Phi(u_j) \geqslant -M$ for all $j \in \mathbb{N}$.

The normalized sequence (v_j) defined by $v_j = u_j/||u_j||$ is weakly convergent in E for $v \in E_n$ and thus, it satisfies

$$v_j^+ \to v^+ \text{ in } E^+, \ \ v_j^0 \to v^0 \text{ in } E^0 \quad \text{and} \quad v_j^- \rightharpoonup v^- \text{ in } E^- \quad \text{as} \quad j \to \infty.$$

Suppose $v^+ = v^0 = 0$. Using the relation (1.18) we obtain, as $j \to \infty$,

$$\frac{-M}{||u_j||^2} \leqslant \frac{\Phi(u_j)}{||u_j||^2} \leqslant \frac{1}{2}\Big(||v_j^+||^2 - ||v_j^-||^2\Big) + \frac{\lambda C_f}{q C_p^q} ||u_j||^{q-2}. \tag{1.20}$$

In other words,

$$0 \leqslant ||v_j^-||^2 \leqslant ||v_j^+||^2 + \frac{2\lambda}{q} C_f C_p^{-q} ||u_j||^{q-2} + \frac{2M}{||u_j||^2} = o_j(1).$$

Therefore $||v_j^-||^2 = o_j(1)$ and we obtain a contradiction since $1 = ||v_j||^2 = o_j(1)$, as $j \to \infty$.

Define $\Gamma \doteq \{x \in \mathbb{R}^3; v(x) \neq 0\}$ and notice that $|\Gamma| > 0$. By definition, $|u_j(x)| \to \infty$ for all $x \in \Gamma$ and it follows from Fatou's Lemma C.1 that

$$\liminf_{j\to\infty} \int_\Gamma \frac{g(x)}{||u_j||^2} |u_j(x)|^p dx = +\infty.$$

Thus, as $j \to \infty$ in (1.20), we obtain

$$0 \leqslant \frac{1}{2}\Big(||v^+||^2 - ||v^-||^2\Big) - \liminf_{j\to\infty} \left(\frac{1}{p} \frac{1}{||u_j||^2} \int_\Gamma g(x) |u_j|^p \, dx \right) = -\infty,$$

a contradiction. ∎

As a consequence, we have

Corollary 1.1. *Let $0 < \lambda < \Lambda_1$ and $e \in Y_n$ with $||e|| = 1$. There is $R > \rho > 0$ such that $\Phi(z) \leqslant \kappa$, for all $z \in \partial Q$, where $\kappa > 0$ is from Lemma 1.5 and $Q \doteq \{u = u^- + u^0 + te;\ t \geqslant 0,\ u^- + u^0 \in E^- \oplus E^0 \ \text{ and } ||u|| \leqslant R\}$.*

1.3 Cerami sequences

To ensure the existence of a Cerami sequence $(Ce)_c$ for the functional Φ we need first demonstrate that this functional satisfies the properties (Φ_0) and (Φ_1) stated in the Appendix B. As mentioned in Remark B.1, this conditions can be weakened and it is sufficient to prove their validity for certain values of a. Let $\delta \in \big(0, 1\big)$ and

$$a > (1 - \delta)\Big(\frac{\rho^2}{4}\Big) > 0, \tag{1.21}$$

where $\rho > 0$ was defined in the Lemma 1.5. Consider the constants

$$\Lambda_2 \doteq \frac{1}{p}\frac{q}{C_f d} \quad \text{and} \quad \Lambda_3 \doteq \frac{q}{C_f}(1 - \delta)\Big(\frac{\rho^2}{4}\Big),$$

where $d, C_f > 0$ are defined in (H_2) and (1.18), respectively. In the development of this section, suppose $0 < \lambda < \Lambda$ where

$$\Lambda = \min\{\Lambda_1, \Lambda_2, \Lambda_3\}. \tag{1.22}$$

Lemma 1.7. *Let (M_0), (H_1) and (H_2) be satisfied. The functional Φ defined by (1.17) satisfies (Φ_1) for all a that satisfies the relation (1.21)*

$$a > (1 - \delta)\Big(\frac{\rho^2}{4}\Big) > 0.$$

Proof: Assume by contradiction that there is a sequence $(u_j) \subset \Phi_a$ such that for $j \in \mathbb{N}$

$$||u_j||^2 \geqslant j||u_j^+||^2. \tag{1.23}$$

Suppose that, up to a subsequence, $||u_j|| \to \infty$ and define, for each $j \in \mathbb{N}$, the normalized sequence $(w_j) \subset E$ by $w_j = u_j/||u_j||$. From the relation (1.23) we obtain

$$||w_j^0||^2 + ||w_j^-||^2 = 1 + o_j(1) \quad \text{as} \quad j \to \infty.$$

On the other hand,

$$\frac{a}{||u_j||^2} \leqslant \frac{\Phi(u_j)}{||u_j||^2} \leqslant \frac{1}{2}\Big(||w_j^+||^2 - ||w_j^-||^2\Big) + \Lambda C_f\Big(qC_p^q\Big)^{-1}||u_j||^{q-2},$$

that implies

$$0 \leqslant ||w_j^-||^2 \leqslant ||w_j^+||^2 + \Lambda C_f \Big(qC_p^q\Big)^{-1} ||u_j||^{q-2} + \frac{a}{||u_j||^2} = o_j(1),$$

that is,

$$||w_j^-||^2 = o_j(1) \quad \text{as} \quad j \to \infty.$$

As $dim(E_0) < \infty$, it is valid that $||w^0||^2 = 1$. Thus, there is a bounded set $\Omega \subset \mathbb{R}^3$ so that

$$\int_\Omega g(x)|w^0|^2 dx > 0, \tag{1.24}$$

since by the assumption (H_2), $g(x) \geqslant d > 0$ for all $x \in \mathbb{R}^3$.

Set $\Omega_j = \{x \in \mathbb{R}^3; |u_j(x)| < 1\}$. Then

$$\begin{aligned}
\frac{a}{||u_j||^2} &\leqslant \frac{\Phi(u_j)}{||u_j||^2} - \int_\Omega g(x)|w_j|^2 dx + \int_\Omega g(x)|w_j|^2 dx \\
&\leqslant o_j(1) - \int_\Omega g(x)|w_j|^2 dx + \frac{1}{p||u_j||^2}\Bigg(\int_{\Omega\cap\Omega_j} g(x)\Big(|u_j|^2 - |u_j|^p\Big)dx \\
&\qquad + \int_{\Omega\cap\Omega_j^c} g(x)\Big(|u_j|^2 - |u_j|^p\Big)dx\Bigg) \\
&\leqslant o_j(1) - \int_\Omega g(x)|w_j|^2 dx + \frac{1}{p||u_j||^2} \int_{\Omega\cap\Omega_j} g(x)\Big(|u_j|^2 - |u_j|^p\Big)dx \\
&\leqslant o_j(1) - \int_\Omega g(x)|w_j|^2 dx + \frac{C||g||_\infty|\Omega|}{p||u_j||^2},
\end{aligned}$$

that implies, as $j \to \infty$,

$$0 \leqslant -\int_\Omega g(x)|w^0|^2 dx < 0,$$

a contradiction.

Therefore, we obtain that (u_j) is a bounded sequence, that is, there is $M > 0$ such that $||u_j|| \leqslant M$ for all $j \in \mathbb{N}$ and

$$0 \leqslant ||u_j^+||^2 \leqslant \frac{||u_j||^2}{j} \leqslant \frac{M}{j} = o_j(1) \text{ as } j \to \infty. \tag{1.25}$$

Suppose that there is a subsequence $(u_{j_k}) \subset (u_j)$ such that $||u_{j_k}||_{L^p} \geqslant 1$. Then, since

$0 < \lambda < \Lambda_2$, we obtain

$$
\begin{aligned}
\Phi(u_{j_k}) &= \frac{1}{2}||u_{j_k}^+||^2 - \frac{1}{2}||u_{j_k}^-||^2 - \frac{\lambda}{q}\int\limits_{\mathbb{R}^3} f(x)|u_{j_k}|^q dx - \frac{1}{p}\int\limits_{\mathbb{R}^3} g(x)|u_{j_k}|^p dx \\
&\leqslant \frac{1}{2}||u_{j_k}^+||^2 - \frac{1}{2}||u_{j_k}^-||^2 + \frac{\Lambda_2}{q}C_f||u_{j_k}||_{L^p}^q - \frac{d}{p}||u_{j_k}||_{L^p}^p \\
&\leqslant \frac{1}{2}||u_{j_k}^+||^2 - \frac{1}{2}||u_{j_k}^-||^2 + \frac{1}{p}(||u_{j_k}||_{L^p}^q - ||u_{j_k}||_{L^p}^p) \\
&\leqslant o_j(1),
\end{aligned}
$$

which is a contradiction because $(u_j) \subset \Phi_a$ and then $\Phi(u_{j_k}) \geqslant a > 0$. Then, every subsequence of (u_j) satisfies $||u_{j_k}||_{L^p} < 1$ and, since $0 < \lambda < \Lambda_3$,

$$
\begin{aligned}
\Phi(u_j) &= \frac{1}{2}||u_j^+||^2 - \frac{1}{2}||u_j^-||^2 - \frac{\lambda}{q}\int\limits_{\mathbb{R}^3} f(x)|u_j|^q dx - \frac{1}{p}\int\limits_{\mathbb{R}^3} g(x)|u_j|^p dx \\
&\leqslant \frac{1}{2}||u_j^+||^2 - \frac{1}{2}||u_j^-||^2 + \frac{\lambda}{q}C_f||u_{j_k}||_{L^p}^q \\
&\leqslant \frac{1}{2}||u_j^+||^2 - \frac{1}{2}||u_j^-||^2 + \frac{\Lambda_3}{q}C_f \\
&\leqslant \frac{1}{2}||u_j^+||^2 - \frac{1}{2}||u_j^-||^2 + (1-\delta)\Big(\frac{\rho^2}{4}\Big).
\end{aligned}
$$

It follows from (1.21) and $\Phi(u_j) \geqslant a$, that

$$
0 \leqslant ||u_j^-||^2 \leqslant ||u_j^+||^2 + (1-\delta)\Big(\frac{\rho^2}{4}\Big) - a \leqslant o_j(1),
$$

that is, $||u_j^-|| = o_j(1)$ as $j \to \infty$. By virtue of $dim(E^0) < \infty$, we have $u_j \to u^0$ in E and, as $j \to \infty$,

$$
\int\limits_{\mathbb{R}^3} f(x)|u_j|^q dx = \int\limits_{\mathbb{R}^3} f(x)|u^0|^q dx + o_j(1).
$$

Consequently,

$$
0 < a \leqslant \Phi(u_j) \leqslant \frac{1}{2}||u_j^+||^2 - \frac{1}{2}||u_j^-||^2 - \frac{\lambda}{q}\int\limits_{\mathbb{R}^3} f(x)|u_j|^q dx \leqslant o_j(1) - \frac{\lambda}{q}\int\limits_{\mathbb{R}^3} f(x)|u^0|^q dx,
$$

which is a contradiction by the assumption (H_1). Thus, there are no sequence that satisfies the relation (1.23). ■

For the purposes of simplification of notation, define $\Psi : E \to \mathbb{R}$ by

$$
\Psi(u) = \frac{\lambda}{q}\int\limits_{\mathbb{R}^3} f(x)|u|^q dx + \frac{1}{p}\int\limits_{\mathbb{R}^3} g(x)|u|^p dx. \tag{1.26}
$$

From Lemma 1.3 and Lemma 1.4, we obtain that if $u_j \rightharpoonup u$ in E then $\Psi(u_j) = \Psi(u) + o_j(1)$.

Lemma 1.8. *Under the same set of assumptions of Lemma 1.7, the functional $\Phi : E \to \mathbb{R}$ satisfies the property (Φ_0) for all a that satisfies the relation* (1.21)

$$a > (1-\delta)\left(\frac{\rho^2}{4}\right) > 0.$$

Proof: For the first part, suppose $(u_j) \subset \Phi_a$ a $\mathcal{T}_{\mathcal{P}}-$convergent sequence to $u \in E$, where the $\mathcal{T}_{\mathcal{P}}-$topology was described in Appendix B. Particularly, $u_j^+ \to u^+$ in norm and, from Lemma 1.7, (u_j) is a bounded sequence. Hence, up to a subsequence, $u_j \rightharpoonup u$ in E and

$$\begin{aligned} a \leqslant \liminf_{j\to\infty} \Phi(u_j) &= \liminf_{j\to\infty}\left(\frac{1}{2}\left(||u_j^+||^2 - ||u_j^-||^2\right) - \Psi(u_j)\right) \\ &\leqslant \frac{1}{2}||u^+||^2 - \frac{1}{2}||u^-||^2 - \Psi(u) = \Phi(u), \end{aligned}$$

so $u \in \Phi_a$.

To ensure the continuity of $\Phi' : (\Phi_a; \mathcal{T}_{\mathcal{P}}) \to (E^*; \mathcal{T}_{w*})$, it is sufficient to demonstrate that if $u_j \rightharpoonup u$ in E then

$$\Psi'(u_j)(w) \ \to \ \Psi'(u)(w) \ \ \forall w \in E, \tag{1.27}$$

since E is a Hilbert space and the norm $\nu : E \to [0,\infty)$, $\nu(w) = ||w||$ is C^1 with $\nu' : (\Phi_a; \mathcal{T}_{\mathcal{P}}) \to (E^*; \mathcal{T}_{w*})$ sequentially continuous.

Notice that, as mentioned previously, if $(u_j) \subset \Phi_a$ is a $\mathcal{T}_{\mathcal{P}}-$convergent sequence to $u \in E$, then (u_j) is a bounded sequence and $u_j \rightharpoonup u$ as $j \to \infty$. Suppose, firstly, $\phi \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$. Then,

$$\begin{aligned} \Big|(\Psi'(u_j) - \Psi'(u))(\phi)\Big| &= \left|\int\limits_{\mathbb{R}^3}\left[\lambda f(x)\left(|u_j|^{q-2}u_j - |u|^{q-2}u\right) + g(x)\left(|u_j|^{p-2}u_j - |u|^{p-2}u\right)\right]\phi dx\right| \\ &\leqslant \int\limits_{\mathbb{R}^3}\left[\Lambda|f(x)|\Big||u_j|^{q-2}u_j - |u|^{q-2}u\Big| + g(x)\Big||u_j|^{p-2}u_j - |u|^{p-2}u\Big|\right]|\phi|dx. \end{aligned}$$

Denote $\Sigma \doteq supp(\phi)$ the support of function ϕ. Then, by Theorem C.3, Hölder's inequality and the compact embedding at Lemma 1.3, we obtain that

$$\begin{aligned} \Lambda\int\limits_{\mathbb{R}^3}|f(x)|\Big||u_j|^{q-2}u_j - |u|^{q-2}u\Big||\phi|dx &\leqslant \tilde{C}\Lambda||f||_{L^\gamma}\int\limits_{\Sigma}|u_j - u|^{\frac{(q-1)p}{q}}|\phi|^{\frac{p}{q}}dx \\ &\leqslant \tilde{C}\Lambda||f||_{L^\gamma}||u_j - u||^{q-1}_{L^p(\Sigma)}||\phi||_{L^p(\Sigma)} \\ &= o_j(1) \end{aligned}$$

and

$$\int\limits_{\mathbb{R}^3} g(x)\Big||u_j|^{p-2}u_j - |u|^{p-2}u\Big||\phi|dx \leqslant \tilde{\mathrm{C}}||g||_\infty \int\limits_{\Sigma} |u_j - u|\Big(|u_j| + |u|\Big)^{p-2}|\phi|dx$$

$$\leqslant \tilde{\mathrm{C}}||g||_\infty\Bigg(\int\limits_{\Sigma}\Big(|u_j| + |u|\Big)^p dx\Bigg)^{\frac{p-2}{p}}\Bigg(\int\limits_{\Sigma}|u_j - u|^{\frac{p}{2}}|\phi|^{\frac{p}{2}}dx\Bigg)^{\frac{2}{p}}$$

$$\leqslant \tilde{\mathrm{C}}||g||_\infty\Bigg(\int\limits_{\Sigma}\Big(|u_j| + |u|\Big)^p dx\Bigg)^{\frac{p-2}{p}}||u_j - u||_{L^p(\Sigma)}||\phi||_{L^p(\Sigma)}$$

$$= o_j(1).$$

Then,

$$\Psi'(u_j)(\phi) \to \Psi'(u)(\phi) \ \ \forall\phi \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^4). \tag{1.28}$$

Let $w \in E$ and $\varepsilon > 0$. Using that $C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$ is a dense subset in E, there exist $(\phi_k) \subset C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$ such that $||w - \phi_k|| = o_k(1)$ as $k \to \infty$. Notice that, for all $k \in \mathbb{N}$,

$$\Big|\Big(\Psi'(u_j) - \Psi'(u)\Big)(w)\Big| \leqslant \Big|\Big(\Psi'(u_j) - \Psi'(u)\Big)(\phi_k)\Big| + \Big|\Big(\Psi'(u_j) - \Psi'(u)\Big)(w - \phi_k)\Big|. \tag{1.29}$$

Using the same arguments from the previous step, we obtain

$$\Big|\Big(\Psi'(u_j) - \Psi'(u)\Big)(w - \phi_k)\Big| \leqslant \tilde{\mathrm{C}}||f||_{L^\gamma}||u_j - u||_{L^p}^{q-1}||w - \phi_k||_{L^p}$$

$$+\tilde{\mathrm{C}}||g||_\infty\Bigg(\int\limits_{\mathbb{R}^3}\Big(|u_j| + |u|\Big)^p dx\Bigg)^{\frac{p-2}{p}}||u_j - u||_{L^p}||w - \phi_k||_{L^p}$$

$$\leqslant M||w - \phi_k||, \tag{1.30}$$

where $M = M((u_j), f, g, p, q)$, since (u_j) is bounded. Let $k_0 \in \mathbb{N}$ large enough such that

$$||w - \phi_{k_0}|| \leqslant \frac{\varepsilon}{2M.}$$

By the relation (1.28), there exist $j_0 \in \mathbb{N}$ such that if $j > j_0$,

$$\Big|\Big(\Psi'(u_j) - \Psi'(u)\Big)(\phi_{k_0})\Big| \leqslant \frac{\varepsilon}{2}. \tag{1.31}$$

Then, it follows from (1.30) and (1.31) that, if $j \geqslant j_0$,

$$\Big|\Big(\Psi'(u_j) - \Psi'(u)\Big)(w)\Big| \leqslant \Big|\Big(\Psi'(u_j) - \Psi'(u)\Big)(\phi_{k_0})\Big| + \Big|\Big(\Psi'(u_j) - \Psi'(u)\Big)(w - \phi_{k_0})\Big| < \varepsilon,$$

that is,

$$\Psi'(u_j)(w) \to \Psi'(u)(w), \quad \forall w \in E.$$

$\blacksquare$

This properties together with the deformation Lemma B.3 allows us to prove the existence of a Cerami sequence to the energy functional Φ. By Definition B.1, this sequence satisfies

$$\Phi(u_j) \to c \quad \text{and} \quad (1 + ||u_j||)\Phi'(u_j) \to 0 \text{ in } E^* \quad \text{as} \quad j \to \infty. \tag{1.32}$$

Lemma 1.9. *Suppose (M_0), $(H_1), (H_2)$ are satisfied and $0 < \lambda < \Lambda$. Then there exist Cerami sequence $(Ce)_c$ with $\kappa \leqslant c \leqslant \sup \Phi(Q)$ where $\kappa > 0$ and Q are defined in Lemma 1.5 and Corollary 1.1, respectively.*

Proof: The arguments that will be used are similar to that found in [25], Teorema 4.2. Using the Brouwer Degree Theory we obtain that $Q \doteq \{u = u^- + u^0 + te;\ t \geqslant 0,\ u^- + u^0 \in E^- \oplus E^0 \text{ e } ||u|| \leqslant R\}$ finitely link with $S = \{u \in E^+ : ||u|| = \rho\}$. Consider the set $\Gamma_{Q,S}$ described in (B.5) and, using their properties, which are also described in (B.5), we will characterize the value $c > 0$.

Indeed, let $h \in \Gamma_{Q,S}$. It follows from the property (h_3) that $\Phi(h(t,u)) \leqslant \Phi(u)$ for all $u \in Q$ and $t \in I = [0,1]$, that is, particularly, $\Phi(h(1,u)) \leqslant \Phi(u)$ for all $u \in Q$. Using the continuity of Φ we obtain that

$$\sup_{u \in Q} \Phi(h(1,u)) \leqslant \sup_{u \in Q} \Phi(u) < \infty.$$

It remains to be shown that the set $\sup\limits_{u \in Q} \Phi(h(1,u))$ is bounded below for $h \in \Gamma_{Q,S}$. Note that $h(I \times \partial Q) \cap S = \emptyset$ by the assumption (h_4) established in (B.5). Using the assumption (h_5), we have that for all $(t,u) \in I \times Q$ there exist a $\mathcal{P}-$open neighbourhood W such that $v - h(s,v)$ is contained in a finite dimensional subspace of E, for all $(s,v) \in W \cap I \times Q$. Since Q is a $\mathcal{P}-$compact set, there is a finite subcollection that still cover Q. Then, the set $\{u - h(t,u) : (t,u) \in I \times Q\}$ is contained in a finite dimensional subspace $F \subset E$. Consequently, $(t,u) \in I \times (Q \cap F)$ and

$$h(I \times (Q \cap F)) = h(I \times Q_F) \subset F.$$

On the other hand, since Q finitely links with S, it follows from the Definition B.2 that $h(t, Q_F) \cap S \neq \emptyset$, that is, there exist $u_0 \in Q$ such that $h(t,u_0) \in S$ for all $t \in I$. Then, for all $h \in \Gamma_{Q,S}$,

$$\sup_{u \in Q} \Phi(h(1,u)) \geqslant \Phi(h(1,u_0)) \geqslant \inf_{u \in S} \Phi(u).$$

This allows us to define

$$c \doteq \inf_{h \in \Gamma_{Q,S}} \sup_{u \in Q} \Phi(h(1,u)) \in [\inf \Phi(S), \sup \Phi(Q)]. \tag{1.33}$$

Now, we will prove the existence of a Cerami sequence to this level c. Suppose that there exist $\alpha > 0$ and $\varepsilon \in \Big(0, c - (1-\delta)\Big(\frac{\rho^2}{4}\Big)\Big)$, $\delta \in (0,1)$, such that

$$\Big(1 + ||u||\Big)||\Phi'(u)|| \geqslant \alpha \quad \forall u \in \Phi_{(c-\varepsilon)}^{(c+\mu\varepsilon)}, \tag{1.34}$$

where $0 < \mu < \infty$ is chosen so that $\sup \Phi(Q) \leqslant c + \mu\varepsilon < \infty$. Using the condition (Φ_1) there exist $\theta > 0$ such that

$$||u|| \leqslant \theta ||u^+|| \ \text{ for all } \ u \in \Phi_a \ \text{ where } a > (1-\delta)\Big(\frac{\rho^2}{4}\Big), \tag{1.35}$$

and, particularly, this condition is still valid for $u \in \Phi_{(c-\varepsilon)}^{(c+\mu\varepsilon)}$ by the conditions stated on $\varepsilon > 0$. In the same way, by the property (Φ_0), we have that $\Phi_{(c-\varepsilon)}$ is $\mathcal{P}-$ closed and $\Phi' : (\Phi_{c-\varepsilon}; \mathcal{T}_{\mathcal{P}}) \to (E^*; \mathcal{T}_{w^*})$ is continuous. Then, by Theorem B.3, there exist a deformation $\eta : [0,1] \times \Phi^{(c+\mu\varepsilon)} \to \Phi^{(c+\mu\varepsilon)}$ that satisfies the properties $(i) - (vii)$.

Choose $h \in \Gamma_{Q,S}$ such that $\sup \Phi(h(1,Q)) < c + \mu\varepsilon$ and define $g : I \times Q \to E$ by

$$g(t,u) = \eta(t, h(t,u)). \tag{1.36}$$

This element does exist, because otherwise, if $\sup \Phi(h(1,Q)) \geqslant c + \mu\varepsilon$ for all $h \in \Gamma_{Q,S}$ then

$$c = \inf_{h \in \Gamma_{Q,S}} \sup \Phi(h(1,Q)) \geqslant c + \mu\varepsilon,$$

which is a contradiction. Using properties $(i) - (vii)$ of η established in Theorem B.3 we obtain that g satisfies $(h_1) - (h_5)$, that is $g \in \Gamma_{Q,S}$. We will briefly comment on some points of the proof of this statement. To verify (h_1) just observe that both h and η are continuous in the $\mathcal{T}-$topology because it satisfy (h_1) and (i), respectively. To obtain (h_2) notice that we choose $0 < \mu < \infty$ such that $\sup \Phi(Q) \leqslant c + \mu\varepsilon < \infty$ and, therefore, for all $u \in Q$ we obtain $u \in \Phi^{c+\mu\varepsilon}$. Now, using (iii), we have that $g(0,u) = \eta(0, h(0,u)) = \eta(0,u)$ for all $u \in Q$. For the item (h_5), notice that $u - g(t,u) = (u - h(t,u)) + (h(t,u) - \eta(t, h(t,u)))$ for all $(t,u) \in I \times Q$. Then, just use the conditions (h_5) and (vi) to obtain the finite dimensional space of E suitable. Since $g \in \Gamma_{Q,S}$, we obtain that

$$\sup_{u \in Q} \Phi(g(1,u)) \geqslant c. \tag{1.37}$$

On the other hand, notice that $h(1,Q) \in \Phi^{(c+\mu\varepsilon)}$ and, using the property (v) of the function η, we have that

$$g(1,u) = \eta(1, h(1,u)) \in \Phi^{(c-\varepsilon)} \text{ for all } u \in Q,$$

that is, $\sup\limits_{u \in Q} \Phi(g(1,u)) \leqslant c - \varepsilon < c$, a contradiction with (1.37).

Therefore, there exist a sequence $(u_j) \in \Phi_{(c-\varepsilon)}^{(c+\mu\varepsilon)}$ such that

$$\Big(1 + ||u_j||\Big)||\Phi'(u_j)|| = o_j(1) \ \ \forall \varepsilon \in \Big(0, c - \rho^2\Big(\frac{1}{4} - \delta^2\Big)\Big) \tag{1.38}$$

and, choosing $\varepsilon > 0$ sufficiently small, it satisfies the conditions that characterize a Cerami sequence and the proof is finished. ■

In the following, let $(u_j) \subset E$ be the $(Ce)_c-$sequence at the level c obtained in the previous Lemma 1.9. Then, by definition, there is constant $M_1 > 0$ such that

$$|2\Phi(u_j) - \Phi'(u_j)(u_j)| \leqslant M_1, \ \forall j,$$

and

$$\begin{aligned}
\frac{M_1}{||u_j||^s} &\geqslant \frac{|2\Phi(u_j) - \Phi'(u_j)(u_j)|}{||u_j||^s} \\
&\geqslant \frac{1}{||u_j||^s}\left(\left(\frac{p-2}{p}\right)\int\limits_{\mathbb{R}^3} g(x)|u_j|^p dx - \lambda\left(\frac{2-q}{q}\right)\left|\int\limits_{\mathbb{R}^3} f(x)|u_j|^q dx\right|\right) \\
&\geqslant \left(\frac{1}{||u_j||^s}\right)\left(\frac{p-2}{p}\right)\int\limits_{\mathbb{R}^3} g(x)|u_j|^p dx - \Lambda\left(\frac{2-q}{q}\right) C_f C_p^{-q}||u_j||^{q-s}
\end{aligned}$$

for $s \in (q, p')$, which exist by the relation between the exponents. Thus, for all $j \in \mathbb{N}$,

$$0 \leqslant \frac{1}{||u_j||^s}\left(\frac{p-2}{p}\right)\int\limits_{\mathbb{R}^3} g(x)|u_j|^p dx \leqslant \frac{M_1}{||u_j||^s} + \Lambda\left(\frac{(2-q)C_f}{qC_p^q}\right)||u_j||^{q-s}. \tag{1.39}$$

In order to guarantee that Φ defined by (1.17) satisfies the Cerami condition, we first verify the boundedness of sequence.

Lemma 1.10. *The sequence $(u_j) \subset E$ is bounded.*

Proof: Arguing indirectly, assume, up to a subsequence, that $||u_j|| \to \infty$ as $j \to \infty$. The relation (1.39) implies that there is a constant $M_2 > 0$ and $j_0 \in \mathbb{N}$ such that

$$\int\limits_{\mathbb{R}^3} g(x)|u_j|^p dx \leqslant M_2||u_j||^s, \ j \geqslant j_0. \tag{1.40}$$

Define, for each $j \in \mathbb{N}$, the normalized sequence $v_j = \Big(u_j/||u_j||\Big)$ such that $v_j = v_j^+ + v_j^0 + v_j^- \in E^+ \oplus E^0 \oplus E^-$. After passing a subsequence, we have that $v_j \rightharpoonup v \in E$ and $v_j^0 \to v^0$, since E^0 is a finite dimensional subspace. Notice that, by definition, for

$j \in \mathbb{N}$,

$$\Phi'(u_j)(u_j^+ - u_j^-) = ||u_j^+||^2 + ||u_j^-||^2 - \Psi'(u_j)(u_j^+ - u_j^-). \tag{1.41}$$

The relation (1.18) and Hölder's inequality implies that

$$\left| \int_{\mathbb{R}^3} f(x)|u_j|^{q-2}u_j \cdot (u_j^+ - u_j^-)dx \right| \leqslant M_3||u_j||^q \tag{1.42}$$

where $M_3 = M_3(f,p,q) > 0$ is a constant. Moreover, also the Hölder's inequality and (1.40) implies that, if $j \geqslant j_0$,

$$\begin{aligned} \left| \int_{\mathbb{R}^3} g(x)|u_j|^{p-2}u_j \cdot (u_j^+ - u_j^-)dx \right| &\leqslant ||g||_\infty^{\frac{1}{p}} \left(\int_{\mathbb{R}^3} g(x)|u_j|^p dx \right)^{\frac{p-1}{p}} C_p^{-1}||u_j|| \\ &\leqslant ||g||_\infty^{\frac{1}{p}} C_p^{-1} \Big(M_2||u_j||^s \Big)^{\frac{p-1}{p}} ||u_j|| \\ &\leqslant ||g||_\infty^{\frac{1}{p}} C_p^{-1} M_2||u_j||^{s\left(\frac{p-1}{p}\right)+1}. \end{aligned} \tag{1.43}$$

Thus, applying the relations (1.42) and (1.43) we obtain

$$|\Psi'(u_j)(u_j^+ - u_j^-)| \leqslant \Lambda M_3||u_j||^q + ||g||_\infty^{\frac{1}{p}} C_p^{-1} M_2||u_j||^{s\left(\frac{p-1}{p}\right)+1}, \tag{1.44}$$

if $j \geqslant j_0$, and from this, we rewrite the (1.41) obtaining

$$\begin{aligned} ||v_j^+ + v_j^-||^2 &= -\frac{1}{||u_j||^2}\Phi'(u_j)(u_j^+ - u_j^-) + \frac{\lambda}{||u_j||^2}\Psi'(u_j)(u_j^+ - u_j^-) \\ &\leqslant o_j(1) + \Lambda M_3||u_j||^{q-2} + ||g||_\infty^{\frac{1}{p}} C_p^{-1} M_2||u_j||^{s\left(\frac{p-1}{p}\right)-1} \\ &\leqslant o_n(1), \end{aligned}$$

since $s(p-1) < p$ and $(q-2) < 0$. Then $||v_j^0|| = ||v_j^0||_{L^2} \to 1 = ||v^0||_{L^2}$.

For $R > 0$, set

$$\Omega_R = \{x \in \mathbb{R}^3 : |v^0(x)| \geqslant 2R\} \quad \text{and} \quad \Omega_{jR} = \{x \in \mathbb{R}^3 : |(v_j^+ + v_j^-)(x)| \geqslant R\}.$$

Since $v^0 \in C(\mathbb{R}^3)$ and $||v^0||_{L^2} = 1$, we obtain that $\big|\Omega_R\big| > 0$ for all R small. Moreover, as $j \to \infty$,

$$|\Omega_{jR}| \leqslant \frac{1}{R^2} \int_{\mathbb{R}^3} |(v_j^+ + v_j^-)(x)|^2 dx = o_j(1).$$

Hence, $|\Omega_R \backslash \Omega_{jR}| \to |\Omega_R|$ as $j \to \infty$. Therefore, there exist $j_0 > 0$ such that $|v_j(x)| \geqslant \frac{R}{2}$

for all $x \in \Omega_R \backslash \Omega_{jR}$ with $j \geqslant j_0$. That is,

$$2|u_j(x)| \geqslant R||u_j|| \;\text{ for all }\; j \geqslant j_0 \;\text{ and }\; x \in \Omega_R \backslash \Omega_{jR}.$$

This relation allows us to conclude that, for $j \geqslant j_0$,

$$\int_{\mathbb{R}^3} |u_j(x)|^p dx \geqslant \int_{\Omega_R \backslash \Omega_{jR}} |u_j(x)|^p dx \geqslant \left(\frac{R}{2}\right)^p ||u_j||^p |\Omega_R \backslash \Omega_{jR}|,$$

that is, $||u_j||_{L^p} \to \infty$ as $j \to \infty$. Then, using the definition of Cerami sequence,

$$\begin{aligned} M_1 &\geqslant |2\Phi(u_j) - \Phi'(u_j)(u_j)| \\ &\geqslant \left(\frac{p-2}{p}\right) \int_{\mathbb{R}^3} g(x)|u_j|^p dx - \lambda\left(\frac{2-q}{q}\right) \left| \int_{\mathbb{R}^3} f(x)|u_j|^q dx \right| \\ &\geqslant \left(\frac{p-2}{p}\right) d||u_j||_{L^p}^p - \Lambda\left(\frac{2-q}{q}\right) C_f ||u_j||_{L^p}^q, \end{aligned}$$

which is an absurd. ■

By the above Lemma 1.10, $(u_j) \subset E$ is bounded hence, without loss of generality, we may assume

$$u_n \rightharpoonup u \;\text{ in } E \;\text{ and } u_n \to u \;\text{ in }\; L^r(\mathbb{R}^3, \mathbb{C}^4) \;\text{ for }\; r \in [2, 3), \tag{1.45}$$

by Lemma 1.3. Certainly, u is a critical point of Φ.

Lemma 1.11. *Let* (M_0), $(H_1), (H_2)$ *and* $0 < \lambda < \Lambda$. *The functional* $\Phi : E \to \mathbb{R}$ *satisfies the Cerami condition* $(Ce)_c$ *at the level* $c > 0$.

Proof: It follows immediately of (1.45) and the Lemma 1.3 that, as $j \to \infty$,

$$\begin{aligned} o_j(1) &= \Big(\Phi'(u_j) - \Phi'(u)\Big)(u_j^+ - u^+) \\ &= ||u_j^+ - u^+||^2 - \Big(\Psi'(u_j) - \Psi'(u)\Big)(u_j^+ - u^+) \\ &= ||u_j^+ - u^+||^2 + o_j(1). \end{aligned}$$

Thus $||u_j^+ - u^+||^2 = o_j(1)$ and, in a similar way, $||u_j^- - u^-||^2 = o_j(1)$ as $j \to \infty$. On the other hand, how $(u_j^0) \subset E^0$ is a bounded sequence in a finite dimensional space, it has a convergent subsequence $(u_{j_k}^0)$. Therefore, $(u_{j_k}) \subset (u_j)$ is a strongly convergent subsequence and the proof is complete. ■

Proof of Theorem 1.1: Assume that $0 < \lambda < \Lambda$ where Λ was defined by (1.22). The conditions (Φ_0) and (Φ_1) holds by Lemma 1.8 and Lemma 1.7, respectively, for all a that satisfies (1.21). By Lemma 1.9, we conclude that Φ possesses a Cerami sequence $(Ce)_c$

at the level $0 < \kappa \leqslant c \leqslant \sup \Phi(Q)$ where $\kappa > 0$ and the set Q were defined in Lemma 1.5 and Corollary 1.1, respectively.

Let (u_j) this sequence, which is bounded by Lemma 1.10. Hence, without loss of generality, we may assume

$$u_j \rightharpoonup u \ \text{ as } \ j \to \infty,$$

where u is a critical point of Φ. By Lemma 1.11, the functional satisfies the Cerami condition at level $c > 0$ and, thus,

$$c + o_j(1) = \Phi(u_j) = \Phi(u) + o_j(1) \ \text{ as } \ j \to \infty.$$

Therefore, $\Phi(u) = c > 0$ and u is a nontrivial solution of the problem (1.1).

Notice that $H(x,u)$ is even in u and $\Phi(0) = 0$. The Lemma 1.5 guarantee the hypothesis (Φ_3) and the Lemma 1.6 implies (Φ_5). Lemma 1.11 shows that Φ satisfies the Cerami condition for $c > 0$, hence Φ has an unbounded sequence of critical values by the Theorem B.7. ∎

CHAPTER 2

Ground state solutions for Dirac equations with weak monotonicity conditions on the nonlinear term

In this chapter, we consider the following version of the Dirac equation

$$-i\alpha\nabla u + a\beta u + V(x)u = g(x,u), \quad x \in \mathbb{R}^3, \tag{2.1}$$

under two different sets of hypotheses. First, we assume

$$g(x,u) = K(x)f(|u|)u$$

and consider a set of conditions similar to those considered by [2] and [42] that established a relation between the potential V and the nonlinearity. That is, the continuous functions $V, K : \mathbb{R}^3 \to \mathbb{R}$ satisfy:

(VK_0) $V(x) > 0$, $K(x) > 0$ for all $x \in \mathbb{R}^3$; $V, K \in L^\infty(\mathbb{R}^3)$ and $||V||_\infty < a$;

(VK_1) if $(A_n) \subset \mathbb{R}^3$ is a sequence of Borel sets such that its Lebesgue measure $|A_n| \leqslant R$ for all $n \in \mathbb{N}$ and some $R > 0$, then

$$\lim_{r\to\infty} \int\limits_{A_n \cap B_r^c(0)} K(x)dx = 0 \;\text{ uniformly in }\; n \in \mathbb{N}. \tag{2.2}$$

Furthermore, one of the conditions below occurs

(VK_2) $\dfrac{K}{V} \in L^\infty(\mathbb{R}^3)$;

(VK_3) there exists $s \in (2,3)$ such that

$$\frac{K(x)}{V(x)^{3-s}} \to 0 \;\text{ as }\; |x| \to \infty. \tag{2.3}$$

Remark 2.1. *In order to simplify the notation, when V and K satisfy the assumptions set out above, we say that $(V, K) \in \mathcal{K}$.*

Moreover, for the continuous nonlinearity $f : \mathbb{R}^+ \to \mathbb{R}$, $\mathbb{R}^+ \doteq [0, \infty)$, we assume the following growth conditions:

(f_1) $f(0) = 0$;

(f_2) there are $c_1, c_2 > 0$ and $p \in (2, 3)$ such that $|f(s)s| \leqslant c_1|s| + c_2|s|^{p-1}$ for all $s \in \mathbb{R}^+$;

(f_3) $\lim\limits_{t\to\infty} \dfrac{F(t)}{t^2} = \infty$ where $F(t) = \int\limits_0^t f(s)s\ ds$;

(f_4) f is non-decreasing on $(0, \infty)$.

The our first main result concerns the existence of ground state solution to this non-periodic problem.

Theorem 2.1. *Let $(V, K) \in \mathcal{K}$ and suppose that $f \in C^0(\mathbb{R})$ satisfies $(f_1) - (f_4)$. Then, Problem* (2.1) *possesses a ground state solution.*

This option of to choose nonlinearity as a product between a term that only depends on x and another that only depends on u is justified because the imposed conditions establish a relationship between nonlinearity and potential function, which are not periodicals. Thus, we analyse only the operator H_0 and consider the potential as an integral part of the energy functional. The boundedness of this potential is due to the fact that, by relating to the norms present in the energy functional, we obtain opposite signal coefficients, which facilitates the definition of one of the norms in function of the other.

Now, for obtaining multiplicity of solutions to Problem (2.1) we assume

(V_0) $V \in C(\mathbb{R}^3, \mathbb{R})$, 1-periodic in $x_j, j = 1, 2, 3$ and $0 \notin \sigma(H_0 + V)$, where $\sigma(S)$ represent the spectrum of an operator S;

(G_1) g is continuous and 1-periodic in $x_j, j = 1, 2, 3$;

(G_2) there is $c > 0$ and $p \in (2, 3)$ such that $|g(x, u)| \leqslant c(1 + |u|^{p-1})$;

(G_3) $g(x, u) = o(u)$ uniformly in x, as $|u| \to 0$;

(G_4) $G(x, u)/|u|^2 \to \infty$ uniformly in x, as $|u| \to \infty$, where $G(x, u) = \int\limits_0^u g(x, s)\ ds$;

(G_5) $u \mapsto g(x, u)/|u|$ is non-decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

Notice that under these assumptions Problem (2.1) is periodic and we have the following conclusion.

Theorem 2.2. *Suppose V satisfies (V_0) and g satisfies $(G_1)-(G_5)$. Then, equation* (2.1) *possesses a ground state solution. If, moreover, g is odd in u, then Problem* (2.1) *has infinitely many pairs of geometrically distinct solutions.*

These kind of hypotheses has already been considered in some cases with the condition (f_4) or (G_5) a little stronger. In [42], for example, Figueiredo and Pimenta considered $(V, K) \in \mathcal{K}$ and $f \in C^0(\mathbb{R})$ satisfies $(f_1)-(f_3)$ and the following condition

(f_4') f is increasing on $(0, \infty)$.

In this case, unlike many authors, they used a new method of approach and through a Deformation Lemma applied in a appropriately way, obtained a ground-state solution for the problem involving the Dirac operator similar to Problem (2.1).

Szulkin and Weth, in [62], considered a nonlinear stationary Schrödinger equation

$$-\Delta u + V(x)u = f(x, u), \ x \in \mathbb{R}^N,$$

where f and V satisfy the same set of hypotheses of Theorem 2.2, up to notation, except for condition (G_5), which has been replaced by

(F_5') $u \mapsto f(x, u)/|u|$ is increasing on $(-\infty, 0)$ and on $(0, \infty)$.

The authors used the Nehari manifold and a auxiliary functional C^1 to obtain a ground state solution. Moreover, if f is odd in u, they obtained multiplicity of solutions using Krasnoselskii genus.

Considering the Dirac equation, Zhang, Zhang and Zhao [73] studied the periodic problem

$$-i\alpha\nabla u + a\beta u + V(x)u = f(x, |u|)u, \ x \in \mathbb{R}^3,$$

similar to equation (2.1), under the assumptions $(G_1)-(G_5)$ (with appropriate notation). Then using the Cerami sequences and Nehari-Pankov manifold, the authors obtained a existence result and approached the exponential decay of the solution, under the additional hypotheses that $V, f \in C^1$. Moreover, they considered the situation where V and f were asymptotically periodic in x and also ensured that the problem has at least a ground-state solution. In this study, although the hypotheses are similar to those considered in our periodic case, the authors did not obtain multiplicity results and they need require additional conditions to analyse the exponential decay.

The technique present in our development involves the definition of a a functional that is only locally Lipschitz continuous and the apply a tool suitable exactly at this class of functionals, Clarke's subdifferential. First, however, we consider the Generalized Nehari manifold which was initially instituted by Pankov [54]. Recently, Szulkin and Weth have ensured that the minimum energy associated functional points restricted to this range are also critical points of the unrestricted functional. We analyze the structure of the set $\widehat{E}(u) \cap \mathcal{M}$ which, by the monotonous growth of f, can be a point or a line segment. Thus, it was not possible to establish a homeomorphism between the manifold and the S^1 sphere in E^+, as in traditional approaches. Moreover, it was also not possible to use a deformation lemma and the theory of topological degree, similar to that used by authors Figueiredo and Pimenta in [42] since the possibility of intersection being more than one point does not allow to obtain the relations for construct a suitable homotopy. So, inspired by Paiva ideas [53] we developed some ideas from the Lipschitz continuous functionals and combine with Krasnoselskii genus to obtain multiplicity results.

2.1 Variational setting

In this section we explore the properties of the free Dirac operator, that is, we consider just $H_0 = -i\alpha\nabla + a\beta$ without external interaction forces. As mentioned in Appendix A, this operator is self-adjoint in $L^2(\mathbb{R}^3, \mathbb{C}^4)$, unbounded from above and from below. Moreover, its domain $\mathcal{D} = \mathcal{D}(H_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$ is a Hilbert space with the inner product

$$(u, v)_{\mathcal{D}} = \langle H_0 u, H_0 v\rangle_{L^2} + \langle u, v\rangle_{L^2}.$$

Let $\sigma(S), \sigma_c(S)$ and $\sigma_d(S)$ denote, respectively, the spectrum, the continuous spectrum and the discrete spectrum (that is, the set of eigenvalues of finite multiplicity) of a self-adjoint operator S. It follows from Theorem A.2 and the subsequent further comments, that the spectrum of operator H_0 is $\sigma(H_0) = (-\infty, -a] \cup [a, +\infty)$ and this structure allows us to obtain a orthogonal decomposition of $L^2(\mathbb{R}^3, \mathbb{C}^4)$ into

$$L^2(\mathbb{R}^3, \mathbb{C}^4) = L^+ \oplus L^-,$$

where H_0 is negative definite (positive definite, respectively) in L^- (L^+, respectively).

Let $E = \mathcal{D}(|H_0|^{\frac{1}{2}})$ the domain of self-adjoint operator $|H_0|^{\frac{1}{2}}$, which is a Hilbert space equipped with the inner product

$$\langle u, v\rangle = \operatorname{Re} \langle |H_0|^{\frac{1}{2}} u, |H_0|^{\frac{1}{2}} v\rangle_{L^2} \tag{2.4}$$

and norm $||u|| = \langle u, u\rangle^{\frac{1}{2}}$. Since $\sigma(H_0) = \mathbb{R}\backslash(-a, a)$, one has

$$a||u||_2^2 \leqslant ||u||^2 \quad \text{for all } u \in E.$$

It follows from the complex interpolation arguments that $E = H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ and $|| \cdot ||$ is equivalent to the usual norm of $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$. Indeed, $E = [\mathcal{D}, L^2]_{\frac{1}{2}}$ and since $\mathcal{D} = \mathcal{D}(H_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$, we have that

$$[\mathcal{D}, L^2]_{\frac{1}{2}} \simeq [H^1, L^2]_{\frac{1}{2}} = H^{\frac{1}{2}}.$$

Furthermore, using the embedding properties of the fractional space $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$, we obtain that E is continuously embedded into $L^q(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in [2,3]$ and compactly embedded into $L^q_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in [2,3)$ (see [14, 52]), that is, there is a constant $C_q > 0$ such that

$$C_q ||u||_{L^q} \leqslant ||u|| \text{ for all } u \in E, \quad q \in [2,3]. \tag{2.5}$$

Moreover, the space E also has a orthogonal decomposition

$$E = E^- \oplus E^+, \tag{2.6}$$

with $E^{\pm} = E \cap L^{\pm}$, and this sum is orthogonal with respect to both $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{L^2}$.

On E we define the following functional $\Phi : E \to \mathbb{R}$ associated with Problem (2.1)

$$\Phi(u) = \frac{1}{2}\Big(||u^+||^2 - ||u^-||^2\Big) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx - \int_{\mathbb{R}^3} G(x,u)dx, \tag{2.7}$$

which is a $C^1(E;\mathbb{R})$ functional. Similar to Chapter 1, we can follow the ideias from [25, 30]) and prove that the critical points of this energy functional are the solutions of the proposed problem. Therefore our objective is to study this functional in order to obtain nontrivial critical points. For this, let the following set introduced by Pankov [54]

$$\mathcal{M} = \{u \in E \backslash E^- : \Phi'(u)(u) = 0 \text{ and } \Phi'(u)(v) = 0, \text{ for all } v \in E^-\}, \tag{2.8}$$

which is called *generalized Nehari manifold* or *Nehari-Pankov manifold*. The assumptions (VK_0) and (f_4), at the nonperiodic case, and the conditions (V_0) and (G_5) in the periodic case guarantee that $\mathcal{M}$ contains all nontrivial critical points of Φ.

Define, as in [62], for $u \in E\backslash E^-$

$$\begin{aligned} E(u) &\doteq E^- \oplus \mathbb{R}u = E^- \oplus \mathbb{R}u^+, \\ \widehat{E}(u) &\doteq E^- \oplus \mathbb{R}^+u = E^- \oplus \mathbb{R}^+u^+, \end{aligned} \tag{2.9}$$

where $\mathbb{R}^+ = [0, \infty)$. It has been shown in [42] and [62], respectively, that if (f_4) is replaced by (f_4') and (G_5) is replaced by (F_5'), the intersection $\widehat{E} \cap \mathcal{M}$ occurs at a unique point which is the unique global maximum of $\Phi|_{\widehat{E}(u)}$. In the development of this work, we will show that $\widehat{E}(u) \cap \mathcal{M} \neq \varnothing$ and if $w \in \widehat{E}(u) \cap \mathcal{M}$ there exist $0 < \sigma_w \leqslant 1 \leqslant \tau_w$ such that $\widehat{E}(u) \cap \mathcal{M} = [\sigma_w, \tau_w]w$. In other words, this intersection is either a point or a finite line segment. We can also show that a point $\overline{w} \in [\sigma_w, \tau_w]w$ is a critical point for Φ if and only if the whole segment $[\sigma_w, \tau_w]w$ consists of critical points.

Under the assumptions of Theorem 2.2 we obtain that the functional Φ is invariant with respect to the action of $\mathbb{Z}^3$ given by the translations $k \mapsto u(\cdot - k), k \in \mathbb{Z}^3$. Hence, if $u \in E$ is solution, then so is $u(\cdot - k)$. We consider that two solutions u_1 and u_2 are called *geometrically distinct* if $u_2 \neq u_1(\cdot - k)$ for any $k \in \mathbb{Z}^3$ and $u_2 \notin [\sigma_{u_1}, \tau_{u_1}]u_1$. In Theorem 2.1 there is no $\mathbb{Z}^3-$invariance and so u_1, u_2 are *geometrically distinct* if $u_2 \notin [\sigma_{u_1}, \tau_{u_1}]u_1$.

2.2 The nonperiodic case

At this section, we consider $(V, K) \in \mathcal{K}$ and the assumptions $(f_1) - (f_4)$ holds. Then the functional (2.7) can be rewrite by

$$\Phi_I(u) = \frac{1}{2}\Big(||u^+||^2 - ||u^-||^2\Big) + \frac{1}{2}\int\limits_{\mathbb{R}^3} V(x)|u|^2 dx - \int\limits_{\mathbb{R}^3} K(x)F(|u|)dx, \tag{2.10}$$

and, for $u, v \in E$, note that

$$\begin{aligned}\Phi_I'(u)(v) &= \langle u^+, v^+\rangle - \langle u^-, v^-\rangle + \mathrm{Re}\int\limits_{\mathbb{R}^3} V(x)u\cdot v\ dx - \mathrm{Re}\int\limits_{\mathbb{R}^3} K(x)f(|u|)u\cdot v\ dx\\ &= \mathrm{Re}\ \langle u, Av\rangle_{L^2} + \mathrm{Re}\int\limits_{\mathbb{R}^3} V(x)u\cdot v\ dx - \mathrm{Re}\int\limits_{\mathbb{R}^3} K(x)f(|u|)u\cdot v\ dx.\end{aligned}$$

Here, $u\cdot v$ denotes the usual inner product in $\mathbb{C}^4$, that is $u\cdot v = \sum\limits_{i=1}^{4} u_i\overline{v_i}$. The next proposition, proven by Figueiredo and Pimenta [42], Lemma 3.4, is a compactness result which is very important and will be used later. We outline the proof to complement our studies.

Proposition 2.1. *Suppose $(V, K) \in \mathcal{K}$. If $(u_n) \subset E$ a sequence such that $u_n \rightharpoonup u$ in E as $n \to \infty$, then*
a) if (VK_2) holds, then, for all $q \in (2, 3)$,

$$\int\limits_{\mathbb{R}^3} K(x)|u_n|^q dx \to \int\limits_{\mathbb{R}^3} K(x)|u|^q dx \quad as \quad n \to \infty;$$

b) if (VK_3) holds, then,

$$\int\limits_{\mathbb{R}^3} K(x)|u_n|^s dx \to \int\limits_{\mathbb{R}^3} K(x)|u|^s dx \quad as \quad n \to \infty.$$

Proof: For the first item, assume that (VK_2) holds. Fixed $q \in (2, 3)$ and $\varepsilon > 0$, there exist $0 < t_0 < t_1$ a positive constant $C > 0$ such that

$$K(x)|t|^q \leqslant \varepsilon C(V(x)|t|^2 + |t|^3) + CK(x)\chi_{[t_0,t_1]}(|t|)|t|^3, \quad \text{for all} \quad t \in \mathbb{R}.$$

Denoting $Q(u) = \int\limits_{\mathbb{R}^3} V(x)|u|^2 dx + \int\limits_{\mathbb{R}^3} |u|^3 dx$ and $A = \{x \in \mathbb{R}^3 : t_0 \leqslant |u(x)| \leqslant t_1\}$, we have

$$\int\limits_{B_r^c(0)} K(x)|u|^q dx \leqslant \varepsilon C Q(u) + C \int\limits_{A\cap B_r^c(0)} K(x)dx \quad \text{for all } u \in E.$$

Since (u_n) is a weakly convergent sequence, by Banach-Steinhaus Theorem C.4, it is bounded in E. Using the continuous embedding $E \hookrightarrow L^2(\mathbb{R}^3, \mathbb{C}^4), E \hookrightarrow L^3(\mathbb{R}^3, \mathbb{C}^4)$ and the fact that $V \in L^\infty(\mathbb{R}^3)$, there exists $C_1 > 0$ such that $Q(u_n) \leqslant C_1$ for all $n \in \mathbb{N}$, where C_1 denotes a constant.

On the other hand, denoting $A_n = \{x \in \mathbb{R}^3 : t_0 \leqslant |u_n(x)| \leqslant t_1\}$, it follows that

$$t_0^3 |A_n| \leqslant \int\limits_{A_n} |u_n|^3 dx \leqslant C_2, \quad \text{for any } n \in \mathbb{N},$$

and then $\sup\limits_{n\in\mathbb{N}} |A_n| < +\infty$, where C_2 denotes a arbitrary constant. Using the hypothesis (VK_1) there exist a positive radius $r > 0$ large enough such that, for all $n \in \mathbb{N}$,

$$\int\limits_{A\cap B_r^c(0)} K(x)dx \leqslant \frac{\varepsilon}{t_1^3}.$$

Consequently, for all $n \in \mathbb{N}$,

$$\int\limits_{B_r^c(0)} K(x)|u_n|^q dx \leqslant C_3 \varepsilon$$

where $C_3 = C(C_1, t_1)$, and then

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^3} K(x)|u_n|^q dx = \int\limits_{\mathbb{R}^3} K(x)|u|^q dx,$$

since in $B_r(0)$ we can use the Sobolev embeddings for $q \in (2,3)$ and the continuity of K.

For the second item, define

$$g(t) = V(x)t^{2-q} + t^{3-q}, \quad \text{for every } t > 0.$$

Using the minimum value of this function and combining this fact with (VK_3), we obtain again the conditions to apply the Banach-Steinhaus Theorem C.4. We can proceed as the previous case to obtain the expected conclusion about the convergence. ∎

Immediately, using the Lebesgue Dominated Convergence Theorem, Alves and Souto [2], Lemma 2.2., obtained the following result:

Corollary 2.1. *If $u_n \rightharpoonup u$ in E, then*

$$\int_{\mathbb{R}^3} K(x)F(|u_n|)dx \to \int_{\mathbb{R}^3} K(x)F(|u|)dx \quad as \quad n \to \infty.$$

Our objective, at this moment, is to study the structure of the set $\widehat{E}(u) \cap \mathcal{M}$ and, for this, the next result is crucial.

Proposition 2.2. *Let $x \in \mathbb{R}^3$, $t \in \mathbb{R}^+$ and $u, v \in \mathbb{C}^4$ such that $f(|u|) \neq 0$. Then*

$$h_u(t,v) \doteq \text{Re } f(|u|)u \cdot \left(\frac{t^2}{2}u - \frac{1}{2}u + tv\right) + F(|u|) - F(|tu+v|) \leqslant 0. \tag{2.11}$$

Moreover, there are $0 < s_u \leqslant 1 \leqslant t_u$ such that $h_u(t,v) = 0$ if and only if $t \in [s_u, t_u]$ e $v = 0$ (the case $s_u = t_u$ not excluded).

Proof: Note that $h_u : \mathbb{R} \times \mathbb{C}^4 \to \mathbb{R}$ and, by the assumption $f(|u|) \neq 0$, we obtain $|u| \neq 0$ and, consequently, $u \neq 0$. Define $z = tu + v, t \geqslant 0$, and suppose that $Re(u \cdot z) \leqslant 0$. Then

$$\begin{aligned}
h_u(t,v) &= f(|u|)\left(\frac{t^2}{2} - \frac{1}{2}\right)|u|^2 + tf(|u|)\text{Re}(u \cdot v) + F(|u|) - F(|z|) \\
&< f(|u|)\left(\frac{t^2}{2} - \frac{1}{2}\right)|u|^2 + tf(|u|)\text{Re}(u \cdot v) + \frac{1}{2}f(|u|)|u|^2 - F(|z|) \\
&= -\frac{t^2}{2}f(|u|)|u|^2 + tf(|u|)\text{Re}(u \cdot z) - F(|z|) \\
&\leqslant 0.
\end{aligned} \tag{2.12}$$

So, we only need to analyse $Re(u \cdot z) > 0$. Obviously, $h_u(1,0) = 0$ and, moreover, for $C > 0$ large enough, if $\frac{1}{2}f(|u|) < C < \infty$,

$$\begin{aligned}
h_u(t,v) &< -\frac{t^2}{2}f(|u|)|u|^2 + tf(|u|) \text{ Re}(u \cdot z) - C|z|^2 + C|z|^2 - F(|z|) \\
&\leqslant -\frac{1}{2}f(|u|)|su - z|^2 + C|z|^2 - F(|z|) \\
&= -\frac{1}{2}f(|u|)|v|^2 + C|z|^2 - F(|z|).
\end{aligned}$$

It follows from (f_3) that $h_u(t,v) < 0$ as $|z| \to \infty$. So, there is $(t_0, v_0) \in B \doteq \{(s,w) : s \geqslant 0 \quad \text{and} \quad w \in \mathbb{C}^4\}$ such that

$$h_u(t_0, v_0) = \max_{(s,w)\in B} g(s,w) \geqslant 0 \tag{2.13}$$

and Re $(u \cdot (t_0 u + v_0)) > 0$. As

$$h_u(0,v) \leqslant -\frac{1}{2}f(|u|)|u|^2 + F(|u|) < 0,$$

the maximum value is attained at some (t_0, v_0) with $t_0 > 0$. Particularly, for $w \in \mathbb{C}^4$,

$$
\begin{aligned}
0 = (h_u)'_v(t_0, v_0)w &= t_0 f(|u|) \operatorname{Re}(u \cdot w) - f(|t_0 u + v_0|) \operatorname{Re}((t_0 u + v_0) \cdot w) \\
&= \Big(f(|u|) - f(|t_0 u + v_0|)\Big) \operatorname{Re}((t_0 u + v_0) \cdot w) - f(|u|) \operatorname{Re}(v_0 \cdot w) \quad (2.14)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
0 = (h_u)'_t(t_0, v_0) &= f(|u|) \operatorname{Re}(t_0 u \cdot u) + f(|u|) \operatorname{Re}(u \cdot v_0) - f(|t_0 u + v_0|) \operatorname{Re}((t_0 u + v_0) \cdot u) \\
&= \Big(f(|u|) - f(|t_0 u + v_0|)\Big) \operatorname{Re}((t_0 u + v_0) \cdot u)
\end{aligned}
$$

and, since $\operatorname{Re}((t_0 u + v_0) \cdot u) > 0$,

$$
\Big(f(|u|) - f(|t_0 u + v_0|)\Big) = 0.
$$

Hence, by (2.14), $f(|u|) \operatorname{Re}(v_0 \cdot w) = 0$ for all $w \in \mathbb{C}^4$. Using that $f(|u|) \neq 0$ we obtain that $v_0 = 0$ and

$$
f(|u|) - f(|t_0 u|) = 0.
$$

By (f_4) there must exist $0 < s_u \leqslant 1$ and $t_u \geqslant 1$ such that $t_0 \in [s_u, t_u]$. From this relation, we can characterize the maximum point as $(t_0, v_0) : t_0 \in [s_u, t_u]$ and $v_0 = 0$. Moreover, for $t \in [s_u, t_u]$, we have that

$$
f(|u|) = f(|tu|) \tag{2.15}
$$

and

$$
h_u(t, 0) = f(|u|)u \cdot \left(\frac{t^2}{2}u - \frac{1}{2}u\right) + F(|u|) - F(|tu|) = 0. \tag{2.16}
$$

Then, using that $g(t, 0) = g(1, 0) = 0$ for all $t \in [s_u, t_u]$ and the relation holds, we obtain the conclusion, that is, $h_u(t, v) \leqslant 0$ for all $(t, v) \in B$ and $g(t, v) = 0$ if and only if $t \in [s_u, t_u]$ and $v = 0$. ∎

Corollary 2.2. *Suppose $u \in \mathcal{M}, t \geqslant 0$ and $v \in E^-$. Then*

$$
\operatorname{Re} \int_{\mathbb{R}^3} K(x) \left[f(|u|)u \cdot \left(\frac{t^2}{2}u - \frac{1}{2}u + tv\right) + F(|u|) - F(|tu + v|) \right] dx \leqslant 0 \tag{2.17}
$$

and there are $0 < s_u \leqslant 1 \leqslant t_u$ such that the equality holds if and only if $t \in [s_u, t_u]$ and $v = 0$.

It follows from this auxiliary results the following characterization to the set $\widehat{E}(u) \cap \mathcal{M}$ when $u \in E \backslash E^-$.

Proposition 2.3. *Let $u \in E \backslash E^-$. Then:*
(i) $\widehat{E}(u) \cap \mathcal{M} \neq \varnothing$;
(ii) if $w \in \widehat{E}(u) \cap \mathcal{M}$ *there are* $0 < s_w \leqslant 1 \leqslant t_w$ *such that* $\widehat{E}(u) \cap \mathcal{M} = [s_w, t_w]w$. *Moreover,* $\Phi_I(sw) = \Phi_I(w)$, $\Phi_I'(sw) = s\Phi_I'(w)$ *for all* $s \in [s_w, t_w]$ *and* $\Phi_I(z) < \Phi_I(w)$ *for the others* $z \in \widehat{E}(u)$.

Proof: For the item (i) see [42]. We outline the proof. Define, for any $u \in E \backslash E^-$, the function $\gamma_u : \mathbb{R}^+ \times E^- \to \mathbb{R}$ by $\gamma_u(t, v) = \Phi_I(tu^+ + v)$ and notice that $\gamma_u \in C^1(\mathbb{R}^+ \times E^-, \mathbb{R})$. Let $(t, v) \in \mathbb{R}^+ \times E^-$ a critical point of γ_u, that is,

$$\frac{\partial}{\partial t}\gamma_u(t, v) = 0 \quad \text{and} \quad \frac{\partial}{\partial v}\gamma_u(t, v)w = 0 \quad \text{for all} \quad w \in E^-. \tag{2.18}$$

Then,

$$\Phi_I'(tu^+ + v)(tu^+ + v) = t\frac{\partial}{\partial t}\gamma_u(t, v) + \frac{\partial}{\partial v}\gamma_u(t, v)v = 0$$

and, for all $w \in E^-$,

$$\Phi_I'(tu^+ + v)w = 0.$$

Then, $tu^+ + v \in \mathcal{M}$. Conversely, if $(tu^+ + v) \in \mathcal{M}$, we have $(t, v) \in \mathbb{R}^+ \times E^-$ is a critical point of γ_u.

In order to obtain that $\widehat{E}(u) \cap \mathcal{M} \neq \varnothing$, $u \in E \backslash E^-$, we will prove that there exist $t_u u^+ + v_u \in \widehat{E}(u)$ such that

$$\Phi_I(t_u u^+ + v_u) = \max_{t \geqslant 0,\ v \in E^-} \Phi_I(tu^+ + v), \tag{2.19}$$

since that, if this maximum point exist, by the previous analysis, $t_u u^+ + v_u \in \mathcal{M}$ and we obtain the desired conclusion.

Assume, without loss of generality, that $u \in E^+$ and $||u|| = 1$, since $\widehat{E}(u) = \widehat{E}(u^+/||u^+||)$. The first step to obtain the maximum point is to guarantee that there exist $R > 0$ such that

$$\Phi_I(u) \leqslant 0, \quad \forall w \in \widehat{E}(u) \backslash B_R(0).$$

Arguing by contradiction, suppose that there exist $(w_n) \subset \widehat{E}(u)$ such that $||w_n|| \to +\infty$ and $\Phi_I(w_n) > 0$ for all $n \in \mathbb{N}$. Then, there exist $(t_n) \subset \mathbb{R}^+$ and $(v_n) \subset E^-$ such that $w_n = t_n u + v_n$ and we can define

$$\overline{w}_n = \frac{w_n}{||w_n||} = \frac{t_n}{||w_n||}u + \frac{v_n}{||w_n||} \doteq \overline{t}_n u + \overline{v}_n.$$

Using that $F(t) \geqslant 0$ in $\mathbb{R}^+$ it follows that

$$\begin{aligned} 0 < \frac{\Phi_I(w_n)}{||w_n||^2} &\leqslant \frac{1}{2}(\overline{t}_n^2 - ||\overline{v}_n||^2) + \frac{1}{2}||V||_{L^\infty}(\overline{t}_n^2||u||_{L_2}^2 + ||\overline{v}_n||_{L^2}^2) \\ &\leqslant \frac{1}{2}\left[\overline{t}_n^2\left(1 + \frac{||V||_{L^\infty}}{a}\right) - ||\overline{v}_n||^2\left(1 - \frac{||V||_{L^\infty}}{a}\right)\right]. \end{aligned}$$

Then, using that $t_n^2 + ||\overline{v}_n||^2 = ||\overline{w}_n||^2 = 1$, we obtain, for all $n \in \mathbb{N}$,

$$0 \leqslant ||\overline{v}_n||^2 \leqslant \frac{a + ||V||_{L^\infty}}{2a} \quad \text{and} \quad 0 < \frac{a - ||V||_{L^\infty}}{2a} \leqslant \overline{t}_n^2 \leqslant 1.$$

This implies that there exist $t_0 > 0$ and $v_0 \in E^-$ such that

$$\overline{w}_n = \overline{t}_n u + \overline{v}_n \rightharpoonup w_0 = s_0 u + v_0 \neq 0,$$

and using the Fatou's Lemma C.1, we obtain that

$$0 \leqslant \limsup_{n\to\infty} \frac{\Phi_I(w_n)}{||w_n||^2} \leqslant \frac{1}{2}(t_0^2 - ||v_0||^2) + \frac{||V||_{L^\infty}}{2a} - \liminf_{n\to\infty} \int\limits_{\{w_0(x)\neq 0\}} K(x)\frac{F(|w_n|)}{||w_n||^2}dx = -\infty,$$

which is an absurd.

Let $(u_n) \subset \widehat{E}(u)$ a maximizing sequence such that

$$\lim_{n\to\infty} \Phi_I(u_n) = \beta \doteq \max_{\widehat{E}(u)} \Phi_I,$$

which exists because Φ_I is bounded from above in $\widehat{E}(u)$. Since $0 < \beta < \infty$, it follows from the above estimates that (u_n) is bounded. So, up to subsequence, there exist $u_0 \in \widehat{E}(u)$ such that $u_n \rightharpoonup u_0$ as $n \to \infty$. Using the Corollary 2.1 and the properties of weak upper semicontinuous functions (for more details see the approach on Proposition 2.7), we obtain that $\Phi(u_0) = \beta$. Therefore, $u_0 \in \mathcal{M} \cap \widehat{E}(u)$ and conclude the proof of (i).

For the item (ii), note that if $w \in \mathcal{M}$ then

$$\Phi_I(tw + v) \leqslant \Phi_I(w) \text{ for all } t \geqslant 0,\ v \in E^-.$$

Obviously, using the assumption (VK_0), the variational properties from $\mathcal{M}$ and Corollary 2.2, it follows that

$$\begin{aligned}\Phi_I(tw + v) - \Phi_I(w) &= \frac{1}{2}\int\limits_{\mathbb{R}^3} V(x)|v|^2 dx - \frac{1}{2}||v||^2 \\ &\quad + \mathrm{Re}\int\limits_{\mathbb{R}^3} K(x)\left[f(|w|)w\left(\frac{t^2}{2}w - \frac{w}{2} + rv\right) + F(|w|) - F(|rw + v|)\right]dx \\ &\leqslant 0. \end{aligned} \tag{2.20}$$

Moreover, there exists $0 < s_w \leqslant 1 \leqslant t_w$ such that $\Phi_I(tw + v) = \Phi_I(w)$ if, and only if, $t \in [s_w, t_w]$ and $v = 0$.

Let $w \in \widehat{E}(u) \cap \mathcal{M}$ given by (i). Since $M \subset E\backslash E^-$, we obtain that $w^+ \neq 0$ and there exists $t > 0$ and $v_1 \in E^-$ such that $w = tu^+ + v_1$. Particularly,

$$u^+ = t^{-1}(w - v_1). \tag{2.21}$$

Clearly $w \in [s_w, t_w]w$. Let $z \in \widehat{E}(u) \cap \mathcal{M}$ and $z \neq w$. It follows from (2.21) that there exists $r > 0$ and $v_2 \in E^-$ such that $z = rw + v_2$, that is $z \in \widehat{E}(w)$. So, it follows from (2.20) that

$$\Phi_I(z) \leqslant \Phi_I(w) \tag{2.22}$$

and the equality holds if and only if $r \in [s_w, t_w]$ and $v_2 = 0$, that is, $\Phi_I(z) = \Phi_I(w)$ if and only if $z \in [s_w, t_w]w$. Since also $w \in \widehat{E}(z)$ and $z \in \mathcal{M}$, using again the relation (2.20), we have that $\Phi_I(w) \leqslant \Phi_I(z)$. Then $\Phi_I(w) = \Phi_I(z)$ and holds $\widehat{E}(u) \cap \mathcal{M} \subset [s_w, t_w]w$.

On the other hand, from the above arguments, if $s \in [s_w, t_w]$ then $\Phi_I(sw) = \Phi_I(w)$ and

$$\max_{z \in \widehat{E}(u)} \Phi_I(z) = \Phi_I(w) = \Phi_I(sw),$$

that is,

$$\Phi_I'(sw)(z) = 0 \text{ for all } z \in \widehat{E}(u).$$

Since $E^- \subset \widehat{E}(u)$ and $w \in \widehat{E}(u)$, it follows that

$$\Phi_I'(sw)(sw) = s\Phi_I'(sw)(w) = 0 \text{ and } \Phi_I'(sw)(v) = 0 \text{ for all } v \in E^-,$$

that is, $sw \in \widehat{E}(u) \cap \mathcal{M}$ for all $s \in [s_w, t_w]$. Hence $\widehat{E}(u) \cap \mathcal{M} = [s_w, t_w]w$. The equality $\Phi_I'(sw) = s\Phi_I'(w)$ follows from the relation $f(|sw|) = f(|w|)$ for all $s \in [s_w, t_w]$ in (2.15). ∎

Remark 2.2. *It follows from the Proposition 2.3 that if $u \in \mathcal{M}$ then $u \in \widehat{E}(u) \cap mathcal M$ and there exists $0 < s_u \leqslant 1 \leqslant t_u$ such that $\Phi_I(su) = \Phi_I(u)$ for all $s \in [s_u, t_u]$ and $\Phi_I(z) < \Phi_I(u)$ for the others $z \in \widehat{E}(u)$.*

Proposition 2.4. *There exists $\delta > 0$ such that*

$$||u^+|| \geqslant \delta \textit{ for all } u \in \mathcal{M} \tag{2.23}$$

and

$$c \doteq \inf_{u \in \mathcal{M}} \Phi_I(u) > 0. \tag{2.24}$$

Moreover, $\mathcal{M}$ is closed and $\Phi_I|_{\mathcal{M}}$ is coercive, i.e., $\Phi_I(u) \to \infty$ as $u \in \mathcal{M}$ and $||u|| \to \infty$.

Proof: The relation (2.24) will be proved firstly and then we will show that there exists $\rho, \alpha > 0$ such that

$$\Phi_I(u) \geqslant \alpha \text{ for all } u \in E^+ \cap \partial B_\rho(0). \tag{2.25}$$

By the assumptions (f_1) and (f_2), for all $\varepsilon > 0$, there is $\mu_\varepsilon > 0$ such that

$$|f(s)s| \leqslant \varepsilon|s| + \mu_\varepsilon|s|^{p-1} \text{ for all } s \in \mathbb{R}^+,$$

where $p \in (2,3)$ is like in (f_2). Then, for all $u \in E^+$,

$$\begin{aligned}
\Phi_I(u) &= \frac{1}{2}||u||^2 + \frac{1}{2}\int\limits_{\mathbb{R}^3} V(x)|u|^2dx - \int\limits_{\mathbb{R}^3} K(x)F(|u|)dx \\
&\geqslant \frac{1}{2}||u||^2 - ||K||_\infty \int\limits_{\mathbb{R}^3}\left(\frac{\varepsilon}{2}|u|^2 + \frac{\mu_\varepsilon}{p}|u|^p\right)dx \\
&\geqslant \left(\frac{1}{2} - C_1\varepsilon\right)||u||^2 - \mu_\varepsilon C_2||u||^p, \\
&\geqslant \alpha
\end{aligned}$$

just by choosing $0 < \varepsilon < (2C_1)^{-1}$, for all $u \in E^+$ such that $||u^+|| = \rho$, where

$$\rho \doteq \left(\frac{1}{2C_2\mu_\varepsilon}\Big(1 - 2C_1\varepsilon\Big)\right)^{\frac{1}{p-2}} \qquad \text{and} \qquad \alpha = \frac{\rho^2}{2}\Big(\frac{1}{2} - C_1\varepsilon\Big) > 0,$$

proving the relation (2.25). Now, by the Remark 2.2, for all $u \in \mathcal{M}$

$$\Phi_I(u) \geqslant \Phi_I\left(\frac{\rho}{||u^+||}u^+\right) \geqslant \alpha > 0,$$

which guarantee the relation (2.24).

Moreover, if $u \in \mathcal{M}$, we obtain

$$\begin{aligned}
0 < c \leqslant \Phi_I(u) &= \frac{1}{2}||u^+||^2 - \frac{1}{2}||u^-||^2 + \frac{1}{2}\int\limits_{\mathbb{R}^3} V(x)|u|^2dx - \int\limits_{\mathbb{R}^3} K(x)F(|u|)dx \\
&\leqslant \frac{1}{2}\left(\frac{a + ||V||_\infty}{a}\right)||u^+||^2 - \frac{1}{2}\left(\frac{a - ||V||_\infty}{a}\right)||u^-||^2
\end{aligned}$$

hence

$$||u^+|| \geqslant \left(\frac{2ac}{a + ||V||_\infty}\right)^{\frac{1}{2}} \tag{2.26}$$

which proves (2.23).

Since $\Phi_I(v) < 0$ for all $v \in E^-$, we obtain that $\mathcal{M}$ is closed. Finally, let us prove the coercivity. Arguing by contradiction, suppose that there exists a sequence $(u_n) \subset \mathcal{M}$ such that $||u_n|| \to \infty$ as $n \to \infty$ and $\Phi_I(u_n) \leqslant d$ for some $d \in [c, \infty)$. Let $(v_n) \subset E$, $v_n \doteq u_n/||u_n||$, which is unitary. After passing to a subsequence we have $v_n \rightharpoonup v$ in E and

$v_n(x) \to v(x)$ a.e. $x \in \mathbb{R}^3$. Then, for each $n \in \mathbb{N}$,

$$
\begin{aligned}
0 \leqslant \frac{\Phi_I(u_n)}{||u_n||^2} &= \frac{1}{2}||v_n^+||^2 - \frac{1}{2}||v_n^-||^2 + \frac{1}{2}\int\limits_{\mathbb{R}^3} V(x)|v_n|^2 dx - \int\limits_{\mathbb{R}^3} K(x)\frac{F(|u_n|)}{||u_n||^2}dx \\
&\leqslant \frac{1}{2}\left[\left(\frac{a+||V||_\infty}{a}\right)||v_n^+||^2 - \left(\frac{a-||V||_\infty}{a}\right)||v_n^-||^2\right],
\end{aligned}
$$

that is

$$
\left(\frac{a-||V||_\infty}{a}\right)||v_n^-||^2 \leqslant \left(\frac{a+||V||_\infty}{a}\right)||v_n^+||^2. \tag{2.27}
$$

Since $||v_n||^2 = ||v_n^+||^2 + ||v_n^-||^2 = 1$ we obtain that, for any $n \in \mathbb{N}$,

$$
0 \leqslant ||v_n^-||^2 \leqslant \frac{a+||V||_\infty}{2a} \quad \text{and} \quad 0 < \frac{a-||V||_\infty}{2a} \leqslant ||v_n^+||^2. \tag{2.28}
$$

On the other hand, (v_n^+) and (v_n^-) are bounded, so we may assume that there exists $v^+, v^- \in E$ such that $v_n^+ \rightharpoonup v^+$ in E^+ and $v_n^- \rightharpoonup v^-$ in E^- as $n \to \infty$. Now let us prove that $v^+ \neq 0$. On the contrary, if $v_n^+ \rightharpoonup 0$ in E^+, for all $s > 0$ fixed, $sv_n^+ \rightharpoonup 0$ in E^+ and by Corollary 2.1

$$
\begin{aligned}
d \geqslant \Phi_I(u_n) \geqslant \Phi_I\left(\frac{s}{||u_n||}u_n^+\right) &= \frac{1}{2}||sv_n^+||^2 + \frac{1}{2}\int\limits_{\mathbb{R}^3} V(x)|sv_n^+|^2 dx - \int\limits_{\mathbb{R}^3} K(x)F(|sv_n^+)|)dx \\
&\geqslant \frac{s^2}{2}\left(\frac{a-||V||_\infty}{2a}\right) + \frac{1}{2}\int\limits_{\mathbb{R}^3} V(x)|sv_n^+|^2 dx + o_n(1) \\
&\geqslant \frac{s^2}{4a}\Big(a - ||V||_\infty\Big) + o_n(1),
\end{aligned}
$$

which is a contradiction for $s > 0$ large enough. Hence, $v^+ \neq 0$ and $v_n = v_n^+ + v_n^- \rightharpoonup v \doteq v^+ + v^- \neq 0$ in$E$ as $n \to \infty$.

Let us define $\Gamma \doteq \{x \in \mathbb{R}^3; v(x) \neq 0\}$, $0 < |\Gamma| \leqslant \infty$. Note that, for all $x \in \Gamma$, $|u_n(x)| \to \infty$ and

$$
\begin{aligned}
\frac{\Phi_I(u_n)}{||u_n||^2} &= \left(\frac{1}{2}||v_n^+||^2 - \frac{1}{2}||v_n^-||^2 + \frac{1}{2}\int\limits_{\mathbb{R}^3} V(x)|v_n|^2 dx - \int\limits_{\mathbb{R}^3} K(x)\frac{F(|u_n|)}{||u_n||^2}dx\right) \\
&\leqslant \frac{1}{2} - \frac{1}{2}||v_n^-||^2 + \frac{||V||_\infty}{2a} - \int\limits_{\mathbb{R}^3} K(x)\frac{F(|u_n|)}{||u_n||^2}dx \\
&\leqslant \frac{1}{2}\Big(2 - ||v_n^-||^2\Big) - \int\limits_{\Gamma} K(x)\frac{F(|u_n|)}{||u_n||^2}dx.
\end{aligned}
$$

Hence, using Fatou's Lemma C.1, we obtain that

$$0 \leqslant \limsup_{n\to\infty} \frac{\Phi_I(u_n)}{||u_n||^2} \leqslant \frac{1}{2}\Big(2 - ||v^-||^2\Big) - \liminf_{n\to\infty} \int_\Gamma K(x)\frac{F(|u_n|)}{||u_n||^2}dx = -\infty \tag{2.29}$$

which is a contradiction and proves the result. ■

According the Proposition 2.3, for each $u \in E^+\backslash\{0\}$ there exist $w \in \widehat{E}(u) \cap \mathcal{M}$ and $0 < s_w \leqslant 1 \leqslant t_w$ such that

$$m(u) \doteq [s_w, t_w]w = \widehat{E}(u) \cap \mathcal{M} \subset E. \tag{2.30}$$

This is a multivalued map from $E^+\backslash\{0\}$ to E. However, the map $\widehat{\Psi} : E^+\backslash\{0\} \to \mathbb{R}$ given by

$$\widehat{\Psi}(u) \doteq \Phi_I(m(u)) = \max_{z\in\widehat{E}(u)} \Phi_I(z) \tag{2.31}$$

is single-valued because Φ_I is constant on $\widehat{E}(u) \cap \mathcal{M}$, by Proposition 2.3.

Proposition 2.5. *The map $\widehat{\Psi}$ is locally Lipschitz continuous.*

Proof: This argument follow the ideas from [53], Proposition 2.6 and [62], Lemma 2.11 and we outline the proof. By [17], recall that $f : X \to \mathbb{R}$ is a *locally Lipschitz continuous functional* on a Banach space X if for every $x \in X$, there exists a neighbourhood N_x of x and a constant $K_x > 0$ such that

$$|f(y) - f(z)| \leqslant K_x||y - z|| \quad \text{for all} \quad y, z \in N_x.$$

If $u_0 \in E^+\backslash\{0\}$, there exist a neighbourhood $U \subset E^+\backslash\{0\}$ of u_0 and $R > 0$ such that $\Phi(w) \leqslant 0$ for all $u \in U$ and $w \in \widehat{E}(u)$, $||w|| \geqslant R$. If not, we can find sequences (u_n), (w_n) such that $u_n \to u_0$, $w_n \in \widehat{E}(u_n)$, $\Phi(w_n) > 0$ and $||w_n|| \to \infty$. Since $u_0, u_1, u_2, ...$ is a compact set, it follows from [62], Lemma 2.5, that $\Phi(w) \leqslant 0$ for some R and all $w \in \widehat{E}(u_j), j = 0, 1, 2, ..., ||w|| \geqslant R$, which is a contradiction.

We may assume without loss of generality that U is bounded and bounded away from 0. Let U, R as above and $s_1u_1 + v_1 \in m(u_1), s_2u_2 + v_2 \in m(u_2)$, where $u_1, u_2 \in U$ and $v_1, v_2 \in E^0 \oplus E^-$. Then $||w_1||, ||w_2|| \leqslant R$ for all $w_1 \in m(u_1), w_2 \in m(u_2)$. Using the maximality property of $m(u)$ and the mean value theorem,

$$\begin{aligned}
\widehat{\Psi}(u_1) - \widehat{\Psi}(u_2) &= \Phi(s_1u_1 + v_1) - \Phi(s_2u_2 + v_2) \\
&\leqslant \Phi(s_1u_1 + v_1) - \Phi(s_1u_2 + v_1) \\
&\leqslant s_1 \sup_{t\in[0,1]} ||\Phi'(s_1(tu_1 + (1-t)u_2) + v_1)||\ ||u_1 - u_2|| \\
&\leqslant C||u_1 - u_2||,
\end{aligned}$$

where the constant C depends on R but not on the particular choice of points in $m(u_1)$, $m(u_2)$ (because $||s_1(tu_1+(1-t)u_2)+v_1|| \leqslant C_0 R$ for some $C_0 > 0$; recall that U is bounded and bounded away from 0). Similarly, $||\widehat{\Psi}(u_2) - \widehat{\Psi}(u_1) \leqslant C||u_1 - u_2||$, and the conclusion follows. ■

Therefore, instead of the derivative of $\widehat{\Psi}$ we need another tool that applies to this class of functionals. We present, now, some basic ideas and concepts from the calculus of generalized gradients that were firstly developed by F. H. Clarke and are required to develop variational methods for nondifferentiable (locally Lipschitz continuous) functionals. More details about this can be found [17, 18].

The *generalized directional derivative* of $\widehat{\Psi}$ at u in the direction v is defined by

$$\widehat{\Psi}^{o}(u; v) \doteq \limsup_{h \to 0,\ t \downarrow 0} \frac{\widehat{\Psi}(u + h + tv) - \widehat{\Psi}(u + h)}{t}, \quad u \in E^{+} \backslash \{0\},\ v, h \in E^{+}. \tag{2.32}$$

The function $v \mapsto \widehat{\Psi}^{o}(u; v)$ is subadditive, positively homogeneous (hence, it is convex) and its subdifferential $\partial \widehat{\Psi}(u)$ is called the *generalized gradient* (or *Clarke's subdifferential*) of $\widehat{\Psi}$ at u, that is,

$$\partial \widehat{\Psi}(u) \doteq \{w \in E^{+} : \widehat{\Psi}^{o}(u; v) \geqslant \langle w, v \rangle \text{ for all } v \in E^{+}\}. \tag{2.33}$$

In a general way, if we consider $f : X \to \mathbb{R}$ a locally Lipschitz functional on a Banach space X, we can cite the following properties of the Clarke's subdifferential:

Proposition 2.6 ([17], Proposition 7.1.1). *Assume that $f : X \to \mathbb{R}$ is a locally Lipschitz functional on a Banach space X. The generalized gradient*

$$\partial f(x) = \{w \in X^{*}; f^{o}(x; v) \geqslant (w, v) \quad \textit{for all} \quad v \in X\},$$

where $f^{o}(x; v)$ is the generalized directional derivative, has the following properties:

(i) For each $x \in X$, $\partial f(x)$ is convex and ω^{*} – compact subset of X^{*};

(ii) For each $w \in \partial f(x)$, we have $||w||_{X^{*}} \leqslant K_x$;

(iii) Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be locally Lipschitz continuous functionals, then

$$\partial(f + g)(x) \subset \partial f(x) + \partial g(x);$$

(iv) For each $\lambda \in \mathbb{R}$,

$$\partial(\lambda f)(x) = \lambda \partial f(x);$$

(v) The set valued mapping $x \mapsto \partial f(x)$ is upper semicontinuous in the following sense: for each $x_0 \in X, \varepsilon > 0$ and $v \in X$, there exist $\delta > 0$ such that for each $w \in \partial f(x)$ with $||x - x_0|| < \delta$, there is $w_0 \in \partial f(x_0)$ such that $|\langle w - w_0, v \rangle| < \varepsilon$;

(vi) *A functional $\lambda : X \to \mathbb{R}$ defined by*

$$\lambda(x) = \min_{w \in \partial f(x)} ||w||_{X^*}$$

is lower semicontinuous, that is, $\lim_{x \to x_0} \lambda(x) \geqslant \lambda(x_0)$;

(vii) *For each $v \in X$, we have*

$$f^o(x; v) = \max\{\langle \xi, v \rangle ; \xi \in \partial f(x)\};$$

(viii) *Let $\phi \in C^1([0,1], X)$ and let $f : X \to \mathbb{R}$ be a locally Lipschitz continuous functional, then the function $h = f \circ \phi : [0,1] \to \mathbb{R}$ is differentiable a.e. and*

$$h'(t) \leqslant \max\{\langle w; \phi'(t) \rangle : w \in \partial f(\phi(t))\} \ a.e;$$

(ix) *The set valued function $x \mapsto \partial f(x)$ is weak*-closed, that is, if (x_i) and (ξ_i) are sequences in X and X^*, respectively, such that $\xi_i \in \partial f(x_i)$, $x_i \to x$ and ξ is a cluster point (limit point) of (ξ_i) in the weak*−topology, then $\xi \in \partial f(x)$.*

In our approach we consider Since E is a Hilbert space, we may assume via duality that $\partial\widehat{\Psi}(u)$ is a subset of E^+. A point u is called a *critical point* of $\widehat{\Psi}$ if $0 \in \partial\widehat{\Psi}(u)$, i.e. $\widehat{\Psi}^o(u; v) \geqslant 0$ for all $v \in E^+$. A sequence (u_n) is called a *Palais-Smale sequence* for $\widehat{\Psi}$ (or *$(PS)-$sequence*) if $\widehat{\Psi}(u_n)$ is bounded and there exist $w_n \in \partial\widehat{\Psi}(u_n)$ such that $w_n \to 0$. Here and thereafter, the following notations will be used:

$$\begin{aligned}
&S^+ \doteq \{u \in E^+ : ||u|| = 1\}, \quad T_u S^+ \doteq \{v \in E^+ : \langle u, v \rangle = 0\}, \quad \Psi \doteq \widehat{\Psi}|_{S^+},\\
&\Psi^d \doteq \{u \in S^+ : \Psi(u) \leqslant d\}, \quad \Psi_c \doteq \{u \in S^+ : \Psi(u) \geqslant c\}, \quad \Psi_c^d \doteq \Psi_c \cap \Psi^d,\\
&K \doteq \{u \in S^+ : 0 \in \partial\widehat{\Psi}(u)\}, \quad K_c \doteq \Psi_c^c \cap K, \quad \partial\Psi(u) \doteq \partial\widehat{\Psi}(u), \text{where } u \in S^+.
\end{aligned}$$

Remark 2.3. *Notice that, if $u \in S^+$, there exist an orthogonal decomposition of E into $E = E(u) \oplus T_u S^+$. Indeed, $E(u) \cap T_u S^+ = \{0\}$. On the contrary, there exist $0 \neq w \in E(u) \cap T_u S^+$, that is, there exist $t \in \mathbb{R}$ and $v \in E^-$ such that*

$$w = tu + v \quad \textit{and} \quad \langle w, u \rangle = 0.$$

Hence

$$0 = \langle w, u \rangle = \langle tu + v, u \rangle = t||u||^2 + \langle v, u \rangle = t$$

and then $w = v \in E^-$. Since $w \in T_u S^+ \subset E^+$ we obtain that $w = 0$ which is a contradiction.

Let $z \in E \backslash \{0\}$ since, obviously, $z = 0 \in E(u) \oplus T_u S^+$. By definition, $z = z^+ + z^-$ where $z^+ \in E^+$ and $z^- \in E^-$. If $z = z^- \in E^-$ we can write $z = (0u + z) + 0 \in E(u) \oplus T_u S^+$. If *not, define* $t = \langle z^+, u\rangle \in \mathbb{R}$. *Then,* $z = (tu + z^-) + (z^+ - tu) \in E(u) \oplus T_u S^+$, *since*

$$\langle z^+ - tu, u\rangle = \langle z^+, u\rangle - t\langle u, u\rangle = 0.$$

Proposition 2.7. *(i)* $u \in S^+$ *is a critical point of* $\widehat{\Psi}$ *if and only if* $m(u)$ *consists of critical points of* Φ_I. *The corresponding critical values coincide.*
(ii) $(u_n) \subset S^+$ *is a Palais-Smale sequence of* $\widehat{\Psi}$ *if and only if there exist* $w_n \in m(u_n)$ *such that* (w_n) *is a Palais-Smale sequence for* Φ_I.

Proof: (i) Let $u \in S^+$. The first item can be rewritten by: $\widehat{\Psi}^o(u; v) \geqslant 0$ for all $v \in E^+$ if and only if $m(u)$ consists of critical points of Φ_I. By Remark 2.3, there exist a orthogonal decomposition $E = E(u) \oplus T_u S^+$ and, by definition, $\Phi_I'(w)(v) = 0$ for all $w \in m(u)$ and $v \in E(u)$. Moreover, since $\widehat{\Psi}(u) = \widehat{\Psi}(\sigma u)$, for all $\sigma > 0$, and $\widehat{\Psi}$ is locally Lipschitz continuous, is valid, for $s \in \mathbb{R}$ fixed, that

$$|\widehat{\Psi}(u + h + t(su)) - \widehat{\Psi}(u + h)| = |\widehat{\Psi}((1 + ts)u + h) - \widehat{\Psi}((1 + ts)(u + h)| \leqslant Ct\ |s|\ ||h||$$

for $h \in E^+$, $||h||$ and $t > 0$ small. Then $\widehat{\Psi}^o(u; su) = 0$ for all $s \in \mathbb{R}$. So we only need to consider $v \in T_u S^+$.

Let $s_u u + z_u \in \widehat{E}(u)$, where $s_u > 0$ and $z_u \in E^-$, denote an (arbitrarily chosen) element of $m(u)$. Using the Mean Value theorem and the maximizing property of $m(u)$, we obtain that

$$\begin{aligned}\widehat{\Psi}(u + h + tv) - \widehat{\Psi}(u + h) &= \Phi_I(s_{u+h+tv}(u + h + tv) + z_{u+h+tv}) - \Phi_I(s_{u+h}(u + h) + z_{u+h})\\ &\leqslant \Phi_I(s_{u+h+tv}(u + h + tv) + z_{u+h+tv})\\ &\quad -\Phi_I(s_{u+h+tv}(u + h) + z_{u+h+tv})\\ &= ts_{u+h+tv}\Phi_I'(s_{u+h+tv}(u + h + \theta tv) + z_{u+h+tv})v\end{aligned}$$

for some $\theta \in (0, 1)$ (here and below $h \in E^+$). Letting subsequences $h_n \to 0$ and $t_n \downarrow 0$, by the maximizing property of $m(u + h_n + t_n v)$ and the coercivity of $\Phi_I|_{\mathcal{M}}$, we conclude that $(s_n(u + h_n + t_n v) + z_n)_{n\in\mathbb{N}}$ is a bounded sequence. Then, as $n \to \infty$, we may suppose that $s_n \doteq s_{u+h_n+t_n v} \to \tilde{s} > 0$ and $z_n \doteq z_{u+h_n+t_n v} \rightharpoonup \tilde{z}$ in E^-, where $\tilde{s} > 0$ follows from (2.23). Since E is a Hilbert space, $V \in L^\infty(\mathbb{R}^3)$ and $C_c^\infty(\mathbb{R}^3)$ is a dense subset of E, we obtain that

$$\widehat{\Psi}^o(u; v) \leqslant \tilde{s}\Phi_I'(\tilde{s}u + \tilde{z})v. \tag{2.34}$$

Moreover, $\tilde{s}u + \tilde{z} \in \mathcal{M}$. Indeed, consider

$$\Gamma(w) = \frac{1}{2}||w^+||^2 - \frac{1}{2}||w^-||^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|w|^2\ dx$$

and define the $\Upsilon_u : E^- \to \mathbb{R}$ by $\Upsilon_u(v) = \Gamma(u+v)$, which is a concave function. Therefore Υ_u is weak upper semicontinuous, i.e., if $v_n \rightharpoonup v$ in E^-, then

$$\Upsilon(u+v) \geqslant \limsup_{n\to\infty} \Upsilon(u+v_n).$$

Since $s_n(u+h_n+t_nv)+z_n \rightharpoonup \tilde{s}u+\tilde{z}$ in E and $s_n(u+h_n+t_nv) \to \tilde{s}$ in E^+, as $n\to\infty$, we have that

$$\limsup_{n\to\infty} \Upsilon(s_n(u+h_n+t_nv)+z_n) \leqslant \Upsilon(\tilde{s}u+\tilde{z}).$$

Then, it follows from the Corollary 2.1 that

$$\widehat{\Psi}(u) = \lim_{n\to\infty} \Phi_I(s_n(u+h_n+t_nv)+z_n) \leqslant \Phi_I(\tilde{s}u+\tilde{z}) \leqslant \max_{w\in\widehat{E}(u)} \Phi_I(w) = \widehat{\Psi}(u).$$

This implies that $\tilde{s}u+\tilde{z} \in \widehat{E}(u)\cap\mathcal{M}$. Since $\widehat{E}(u)\cap\mathcal{M}$ may be a line segment, it is not sure that $\tilde{s}$ and $\tilde{z}$ are the same for different v. However, if $\tilde{s_1},\tilde{s_2}$ and $\tilde{z_1},\tilde{z_2}$ correspond to v_1 and v_2, then by Proposition 2.3,

$$\tilde{s_1}u+\tilde{z_1} = \tau(\tilde{s_2}u+\tilde{z_2}) \text{ and } \Phi_I'(\tilde{s_1}u+\tilde{z_1})v_2 = \tau\Phi_I'(\tilde{s_2}u+\tilde{z_2})v_2,$$

for some $\tau>0$. Taking this into account, we observe that, for all $y\in\partial\widehat{\Psi}(u)$,

$$\langle y,v\rangle \leqslant \widehat{\Psi}^{\mathrm{o}}(u;v) \leqslant \tau(v)\Phi_I'(\tilde{s}u+\tilde{z})v, \tag{2.35}$$

where τ is bounded and bounded away from 0.

It follows from this inequality that u is a critical point of $\widehat{\Psi}$ if and only if $m(u)$ consists of critical points of Φ_I. Indeed, if u is a critical point of $\widehat{\Psi}$ we obtain that $\widehat{\Psi}^{\mathrm{o}}(u;v)\geqslant 0$ for all $v\in T_uS^+$ and then $\Phi_I'(\tilde{s}u+\tilde{z})v = 0$ for all $v\in T_uS^+$. Using the orthogonal decomposition $E = E(u)\oplus T_uS^+$, this relation ensures that, for $w\in E$,

$$\begin{aligned}\Phi_I'(\tilde{s}u+\tilde{z})(w) &= \Phi_I'(\tilde{s}u+\tilde{z})((tu+w^-)+v)\\ &= \Phi_I'(\tilde{s}u+\tilde{z})(tu+w^-)+\Phi_I'(\tilde{s}u+\tilde{z})v\\ &= 0,\end{aligned}$$

where $(tu+w^-)\in E(u)$ and $v\in T_uS^+$, that is, $\tilde{s}u+\tilde{z}$ is a critical point for Φ_I. Then, by the Proposition 2.3 (ii), the claim follows. Conversely, note that if $w\in E^+$ by the orthogonal decomposition, $w = tu+v$, where $t\in\mathbb{R}$ and $v\in T_uS^+$. Then, for all $y\in\partial\widehat{\Psi}(u)$,

$$\langle y,w\rangle \leqslant \widehat{\Psi}^{\mathrm{o}}(u;w) \leqslant \widehat{\Psi}^{\mathrm{o}}(u;tu)+\widehat{\Psi}^{\mathrm{o}}(u;v) \leqslant \widehat{\Psi}^{\mathrm{o}}(u;v) \leqslant \tau(v)\Phi_I'(\tilde{s}u+\tilde{z})v = 0,$$

since we are assuming that $\tilde{s}u+\tilde{z}$ is a critical point for Φ_I. Hence $\langle y,w\rangle \leqslant 0$ for all $w\in E^+$ and this implies that $y=0$, that is, $0\in\partial\widehat{\Psi}(u)$, as desired.

(ii) The arguments are similar to the previous case. We take $y_n \in \partial\Psi(u_n)$ and $w_n \in m(u_n)$. Since $\Phi_I|_{\mathcal{M}}$ is coercive, the boundedness of $\Phi_I(m(u_n))$ implies that (w_n) is bounded. Similarly to (2.35), we see that

$$\langle y_n, v\rangle \leqslant \widehat{\Psi}^{\mathrm{o}}(u;v) \leqslant \tau_n \Phi_I'(w_n)v, \tag{2.36}$$

where $v \in E^+$ and τ_n is bounded and bounded away from 0. So the conclusion follows. ■

Remark 2.4. *If $(w_n) \subset (m(u_n))$ is a Palais-Smale sequence for Φ_I, then so is any sequence $(w_n') \subset (m(u_n))$.*

Proof of Theorem 2.1 It follows from the Proposition 2.4 that

$$c = \inf_{w\in\mathcal{M}} \Phi_I(w) = \inf_{u\in S^+} \Psi(u) > 0.$$

Using the Ekeland's variational principle, there exist a sequence $(u_n) \subset S^+$ such that $\Psi(u_n) \to c$ and

$$\Psi(w) \geqslant \Psi(u_n) - \frac{1}{n}||w - u_n|| \ \text{ for all } \ w \in S^+.$$

Let $v \in T_{u_n}S^+$ and define $z_n(t) = (u_n + tv)/||u_n + tv||$. Since $\langle u_n, v\rangle = 0$ we obtain that $||u_n + tv||^2 - 1 = O(t^2)$ as $t \to 0$. Moreover, $\widehat{\Psi}(u_n + tv) = \Psi(z_n(t))$ and

$$\widehat{\Psi}^{\mathrm{o}}(u;v) \geqslant \limsup_{t\downarrow 0} \frac{\Psi(z_n(t)) - \Psi(u_n)}{t} \geqslant -\frac{1}{n}||v|| \tag{2.37}$$

for some $v \in T_{u_n}S^+$. Note that $m(u_n)$ is bounded by coercivity of $\Phi_I|_{\mathcal{M}}$ and by the second inequality in (2.36) we obtain

$$-\frac{1}{n}||v|| \leqslant \widehat{\Psi}^{\mathrm{o}}(u;v) \leqslant \tau_n \Phi_I'(w_n)v, \tag{2.38}$$

where $w_n \in m(u_n) \subset \mathcal{M}$ and τ_n is bounded and bounded away from 0. This relation ensures that $\Phi_I'(w_n)v = o_n(1)$ as $n \to \infty$ and $v \in T_{u_n}S^+$. By the maximizing property of $m(u_n)$, it follows that $\Phi_I'(w_n)z = 0$ for all $z \in \widehat{E}(u_n)$ and, then, (w_n) is a bounded Palais-Smale sequence for Φ_I. Passing to a subsequence, we may assume that $w_n \rightharpoonup w$ in E. Note that $w^+ \neq 0$. Indeed, since (w_n) is a Palais-Smale sequence for Φ_I there exists $M > 0$ such that $\Phi_I(w_n) \leqslant M$, for all $n \in \mathbb{N}$. Then, if $w_n^+ \rightharpoonup 0$ in E^+ for all $t \geqslant 0$,

$$\begin{aligned}
M &\geqslant \Phi_I(w_n) \\
&\geqslant \Phi_I(tw_n^+) \\
&= \frac{1}{2}t^2||w_n^+||^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|tw_n^+|^2\,dx - \int_{\mathbb{R}^3} K(x)F(|tw_n^+|)dx \\
&\geqslant \frac{t^2}{2}\left(\frac{2ac}{a + ||V||_\infty}\right) + o_n(1),
\end{aligned}$$

using the Corollary 2.1 and Proposition 2.4, which is a contradiction by $t \geqslant 0$ large enough. Then $w_n \rightharpoonup w \neq 0$ in E and, by the density arguments, we obtain that $\Phi_I'(w) = 0$ and $w \in \mathcal{M}$.

It remains to show that $\Phi_I(w) = c$. By the assumptions (f_1) and (f_4),

$$\begin{aligned} c + o_n(1) &= \Phi_I(w_n) - \frac{1}{2}\Phi_I'(w_n)(w_n) \\ &= \int_{\mathbb{R}^3} K(x)\left[\frac{1}{2}f(|w_n|)w_n \cdot w_n - F(|w_n|)\right]dx \\ &\geqslant \int_{\mathbb{R}^3} K(x)\left[\frac{1}{2}f(|w|)w \cdot w - F(|w|)\right]dx + o_n(1) \\ &= \Phi_I(w) + o_n(1). \end{aligned}$$

Hence $\Phi_I(w) \leqslant c$. Since $w \in \mathcal{M}$, we have that $c = \inf_{z \in \mathcal{M}} \Phi_I(z) \leqslant \Phi_I(w)$ and hence we obtain the reverse inequality. ∎

2.3 The periodic case

Through this section, assume that (V_0) and $(G_1) - (G_5)$ are satisfied. Then, considering the operator $A_V = H_0 + V$ there exists an equivalent inner product in E and, consequently, an equivalent norm, which will be also denoted by $||\cdot||$, such that the associated functional (2.7) has the following form

$$\Phi_{II}(u) = \frac{1}{2}\Big(||u^+||^2 - ||u^-||^2\Big) - \int_{\mathbb{R}^3} G(x,u)dx,$$

and, for $u, v \in E$,

$$\begin{aligned} \Phi_{II}'(u)(v) &= \langle u^+, v^+\rangle - \langle u^-, v^-\rangle - \mathrm{Re}\int_{\mathbb{R}^3} g(x,u)v\ dx \\ &= \mathrm{Re}\langle u, Av\rangle_{L^2} - \mathrm{Re}\int_{\mathbb{R}^3} g(x,u)v\ dx. \end{aligned}$$

Remark 2.5. *Notice that if (V_0) is replaced by*

(V_0') $V = \beta M$, *where* $M \in C^1(\mathbb{R}^3, [0,\infty))$ *and* $M(x)$ *is 1-periodic in* $x_j, j = 1,2,3;$

we can study the following problem

$$-i\alpha\nabla u + (a + M)\beta u = g(x,u), \quad x \in \mathbb{R}^3. \tag{2.39}$$

Generally, if $M \in L^2_{loc}(\mathbb{R}^3, \mathbb{R})$, *the operator* $H_M = -i\alpha\nabla + (a + M)\beta$ *is self-adjoint in*

$L^2(\mathbb{R}^3, \mathbb{C}^4)$, unbounded from above and from below. As before, its domain $\mathcal{D} = \mathcal{D}(H_M)$ is a Hilbert space with an appropriated inner product and $\mathcal{D} = H^1(\mathbb{R}^3, \mathbb{C}^4)$ with equivalent norms (see [9], Lemma 3.2). Moreover, by Lemma 3.3 in [9], $\sigma(H_M) = \sigma_c(H_M) \subset (-\infty, a] \cup [a, \infty)$ and $\inf \sigma(|H_V|) \leqslant a + \sup M(\mathbb{R}^3)$, that is, we also obtain a orthogonal decomposition of $L^2(\mathbb{R}^3, \mathbb{C}^4)$ into

$$L^2(\mathbb{R}^3, \mathbb{C}^4) = L^+ \oplus L^-,$$

where H_M is negative definite (positive definite, respectively) in L^- (L^+, respectively). The domain $\tilde{E}= \mathcal{D}(|H_M|^{\frac{1}{2}})$ of self-adjoint operator $|H_M|^{\frac{1}{2}}$ is a Hilbert space equipped with the following inner product

$$\langle\langle u, v\rangle\rangle \doteq \mathrm{Re}\langle |H_M|^{\frac{1}{2}} u, |H_M|^{\frac{1}{2}} v\rangle_{L^2} \tag{2.40}$$

and $||u||_1 = \langle\langle u, u\rangle\rangle^{\frac{1}{2}}$. It follows from the complex interpolation arguments, similar those mentioned at the previous section, that $\tilde{E} = H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ with equivalent norms (see [9], Lema 3.4). Since $\tilde{E}$ is a subspace of $L^2(\mathbb{R}^3, \mathbb{C}^4)$, it also has a orthogonal decomposition

$$E = E^- \oplus E^+, \tag{2.41}$$

with $E^\pm = E \cap L^\pm$, and this sum is orthogonal with respect to both $\langle\langle \cdot, \cdot\rangle\rangle$ and $\langle \cdot, \cdot\rangle_{L^2}$. Then, the solutions of the equation (2.39) will be obtained as critical points of the functional

$$\Phi(u) = \frac{1}{2}\Big(||u^+||_1^2 - ||u^-||_1^2\Big) - \int_{\mathbb{R}^3} G(x, u)dx, \tag{2.42}$$

which as the same form as Φ_{II}.

Following the same arguments from the previous section we obtain the next results which are important to study the structure of the set $\widehat{E}(u) \cap \mathcal{M}$. Due to the similarity of the statements they will be omitted.

Proposition 2.8. *Let $x \in \mathbb{R}^3$, $t \in \mathbb{R}^+$ and $u, v \in \mathbb{C}^4$ such that $g(x, u) \neq 0$. Then*

$$h(t, v) \doteq \mathrm{Re}\ g(x, u)\left(\frac{t^2}{2}u - \frac{1}{2}u + tv\right) + G(x, u) - G(x, tu + v) \leqslant 0.$$

Moreover, there are $0 < s_u \leqslant 1 \leqslant t_u$ such that $h(t, v) = 0$ if and only if $t \in [s_u, t_u]$ e $v = 0$ (the case $s_u = t_u$ not excluded).

Corollary 2.3. *Suppose $u \in \mathcal{M}$, $s \geqslant 0$ and $v \in E^-$. Then*

$$\int_{\mathbb{R}^3} h(s, v)\ dx \leqslant 0,$$

and there are $0 < s_u \leqslant 1 \leqslant t_u$ *such that the equality holds if and only if* $s \in [s_u, t_u]$ *and* $v = 0$.

With this preliminaries results, we obtain the following characterization to the set $\widehat{E}(u) \cap \mathcal{M}$ when $u \in E \backslash E^-$.

Proposition 2.9. *Let* $u \in E \backslash E^-$. *Then:*
(i) $\widehat{E}(u) \cap \mathcal{M} \neq \varnothing$;
(ii) if $w \in \widehat{E}(u) \cap \mathcal{M}$ *there are* $0 < s_w \leqslant 1 \leqslant t_w$ *such that* $\widehat{E}(u) \cap \mathcal{M} = [s_w, t_w]w$. *Moreover,* $\Phi_{II}(sw) = \Phi_{II}(w)$, $\Phi'_{II}(sw) = s\Phi'_{II}(w)$ *for all* $s \in [s_w, t_w]$ *and* $\Phi_{II}(z) < \Phi_{II}(w)$ *for the others* $z \in \widehat{E}(u)$;
(iii) $\mathcal{M}$ *is closed,* $c \doteq \inf_{u \in \mathcal{M}} \Phi_{II}(u) > 0$ *and* $\Phi_{II}|_{\mathcal{M}}$ *is coercive, i.e.,* $\Phi_{II}(u) \to \infty$ *as* $u \in \mathcal{M}$ *and* $||u|| \to \infty$;
(iv) there exists $\delta > 0$ *such that* $||u^+|| \geqslant \delta$ *for all* $u \in \mathcal{M}$.

Following the same notations associated with the Clarke's subdifferential used at the previous section, we consider for each $u \in E^+ \backslash \{0\}$ the multivalued map $m(u) \doteq [s_w, t_w]w = \widehat{E}(u) \cap \mathcal{M} \subset E$ and the map $\widehat{\Psi} : E^+ \backslash \{0\} \to \mathbb{R}$ given by

$$\widehat{\Psi}(u) \doteq \Phi_{II}(m(u)) = \max_{z \in \widehat{E}(u)} \Phi_{II}(z),$$

which is locally Lipschitz continuous, since we can apply the same arguments used in Proposition 2.5. Therefore, with small changes at the proof of Proposition 2.7, we obtain

Proposition 2.10. *(i)* $u \in S^+$ *is a critical point of* $\widehat{\Psi}$ *if and only if* $m(u)$ *consists of critical points of* Φ_{II}. *The corresponding critical values coincide.*
(ii) $(u_n) \subset S^+$ *is a Palais-Smale sequence of* $\widehat{\Psi}$ *if and only if there exist* $w_n \in m(u_n)$ *such that* (w_n) *is a Palais-Smale sequence for* Φ_{II}.

Proof of Theorem 2.2 (*Existence*) With the same arguments presented at the proof of Theorem 2.1, we obtain a sequence $(u_n) \subset S^+$ and $w_n \in m(u_n) \subset M$ such that (w_n) is a bounded Palais-Smale sequence for Φ_{II}. Now we may proceed as [62], Theorem 1.1. Passing to a subsequence, we may assume that $w_n \rightharpoonup w$ in E. Let $y_n \in \mathbb{R}^3$ satisfy

$$\int_{B_1(y_n)} |w_n|^2 \, dx = \max_{y \in \mathbb{R}^n} \int_{B_1(y)} |w_n|^2 \, dx.$$

Since Φ_{II} and $\mathcal{M}$ are invariant under translations of the form $u \mapsto u(\cdot - k), k \in \mathbb{Z}^3$, we may suppose that (y_n) is bounded in $\mathbb{R}^3$. If

$$\int_{B_1(y_n)} |w_n|^2 \, dx \to 0 \text{ as } n \to \infty, \tag{2.43}$$

then, by the Lion's Lemma [69], Lemma 1.21, $u_n \to 0$ in $L^p(\mathbb{R}^3)$, $2 < p < 3$. Note that $(G_1) - (G_3)$ imply that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that $|g(x,u)| \leqslant \varepsilon|u| + C_\varepsilon|u|^{p-1}$ for all $u \in E$. Using this relation and the Sobolev embeddings (2.5), we infer that

$$\int_{\mathbb{R}^3} g(x, w_n) w_n^+ \, dx = o_n(||w_n^+||) \text{ as } n \to \infty,$$

hence

$$o_n(||w_n^+||) = \Phi'_{II}(w_n)(w_n^+) = ||w_n^+||^2 - \int_{\mathbb{R}^3} g(x, w_n) w_n^+ \, dx = ||w_n^+||^2 - o_n(||w_n^+||)$$

and therefore $||w_n^+|| \to 0$, contrary to Proposition 2.9, (iv). Then (2.43) cannot holds and $w_n \rightharpoonup w \neq 0$ in E. Using the same arguments from the proof of Theorem 2.1, we obtain that $\Phi'_{II}(w) = 0$, $w \in \mathcal{M}$ and $\Phi_{II}(w) = c = \inf_{z \in \mathcal{M}} \Phi_{II}(z)$, that is, w is a ground state solution for the problem (2.1). □

The remainder of this section is devoted to the proof of multiplicity of solutions and, for this, we assume that the nonlinearity $g = g(x, v)$ is odd in v. The preliminary results that will be presented here taken from [53, 62]. For $u \in S^+$, let

$$\partial^- \Psi(u) = \left\{ p \in \partial\Psi(u) : ||p|| = \min_{a \in \partial\Psi(u)} ||a|| \right\} \text{ and } \mu(u) = \inf_{z \in S^+} \{ ||\partial^-\Psi(z)|| + ||u - z|| \}.$$

Hence $K \doteq \{u \in S^+ : \partial^-\Psi(u) = 0\}$, the function μ is continuous in S^+ and $u \in K$ if and only if $\mu(u) = 0$. Indeed, notice that for $u, v, a \in S^+$ we have that

$$\mu(u) \leqslant ||\partial^-\Psi(a)|| + ||u - a|| \leqslant ||\partial^-\Psi(a)|| + ||v - a|| + ||u - v||, \tag{2.44}$$

and taking the infimum over a on the right-hand side we obtain $\mu(u) \leqslant \mu(v) + ||u - v||$. In the same way, $|\mu(u) - \mu(v)| \leqslant ||u - v||$ and hence, μ is (Lipschitz) continuous. Since $0 \leqslant \mu(u) \leqslant ||\partial^-\Psi(u)||$, it is clear that $\mu(u) = 0$ if $u \in K$. If $\mu(u) = 0$, there exist a_n such that $\partial^-\Psi(a_n) \to 0$ and $a_n \to u$. Hence, $u \in K$ since $u \mapsto ||\partial^-\Psi(u)||$ is lower semicontinuity by Proposition 2.6 (vi).

This preliminaries allows us to construct a pseudo-gradient vector field $H : S^+ \backslash K \to TS^+$ for Ψ.

Proposition 2.11. *There exists a locally Lipschitz continuous vector field $H : S^+\backslash K \to TS^+$ such that $||H(u)|| \leqslant 1$ and $\inf\{\langle p, H(u)\rangle : p \in \partial\Psi(u)\} > \frac{1}{2}\mu(u)$ for all $u \in S^+\backslash K$. If Φ_{II} is even, the H may be chosen to be odd.*

Proof: This proof can be found in [53], Proposition 2.10 and here we outline the arguments. Consider $u \in S^+ \backslash K$, define

$$v_u \doteq \frac{\partial^- \Psi(u)}{||\partial^- \Psi(u)||}$$

and

$$\chi : w \mapsto \inf_{p \in \partial \Psi(w)} \langle p, v_u - \langle v_u, w \rangle w \rangle - \frac{1}{2}\mu(w), \quad w \in S^+ \backslash K. \tag{2.45}$$

Since $\partial \Psi(u)$ is convex and $\inf\limits_{p \in \partial \Psi(u)} \langle p, v_u \rangle \geqslant ||\partial^- \Psi(u)|| \geqslant \mu(u)$ we conclude that

$$\chi(u) \geqslant \frac{1}{2}\mu(u) > 0.$$

Moreover, by Proposition 2.6 (vii),

$$\inf_{p \in \partial \Psi(w)} \langle p, v_u - \langle v_u, w \rangle w \rangle = -\widehat{\Psi}^0(w; \langle v_u, w \rangle w - v_u) \tag{2.46}$$

and $\widehat{\Psi}^0$ is upper semicontinuous in both arguments. Hence we conclude that χ is lower semicontinuous and there exists a neighbourhood U_u of u such that $\chi(w) > 0$ for all $w \in U_u$.

Take a locally finite open refinement $(U_{u_i})_{i \in I}$ (with corresponding points v_{u_i}) of the open cover $(U_u)_{u \in S^+ \backslash K}$ and a subordinated locally Lipschitz continuous partition of unity $(\lambda_i)_{i \in I}$. So, define

$$H(u) \doteq \sum_{i \in I} \lambda_i(u)(v_{u_i} - \langle v_{u_i}, u \rangle u), \quad u \in S^+ \backslash K, \tag{2.47}$$

which satisfies the required conditions. Moreover, if Φ is even, then so is Ψ and we may replace $H(u)$ by $\frac{1}{2}(H(u) - H(-u))$. ■

In order to obtain infinitely many geometrically distinct solutions for the problem, suppose, by contradiction, that this does not occur. Since to each $[s_w, t_w]w \subset \mathcal{M}$ there corresponds a unique point $u \in S^+$, the set K consists of finitely many orbits $\mathcal{O}(u) \doteq \{u(\cdot - k) : u \in K, k \in \mathbb{Z}^3\}$. We may choose a subset $\mathcal{F} \subset K$ such that $\mathcal{F} = -\mathcal{F}$ and each orbit has a unique representative in $\mathcal{F}$, that is,

$$\mathcal{F} \text{ is a finite set.} \tag{2.48}$$

Define $\check{m} : \mathcal{M} \to S^+$, $\check{m}(u) = u^+/||u^+||$. This map is Lipschitz continuous since, for all $u, v \in \mathcal{M}$,

$$||\check{m}(u) - \check{m}(v)|| = \left|\left| \frac{u^+}{||u^+||} - \frac{v^+}{||v^+||} \right|\right| \leqslant \frac{2}{||u^+||}||(u-v)^+|| \leqslant \left(\frac{2}{\delta}\right)||u - v||, \tag{2.49}$$

where $\delta > 0$ was obtained at Proposition 2.9. Moreover,

$$\kappa \doteq \inf\{||v - w|| : v, w \in K,\ v \neq w\} > 0. \tag{2.50}$$

Indeed, there exist $v_n, w_n \in \mathcal{F}$ and $k_n, l_n \in \mathbb{Z}^3$ such that $v_n(\cdot - k_n) \neq w_n(\cdot - l_n)$ for all n and

$$\lim_{n\to\infty} ||v_n(\cdot - k_n) - w_n(\cdot - l_n)|| = \kappa.$$

Define $m_n = k_n - l_n$. Since $\mathcal{F}$ is finite, after passing a subsequence, $v_n = v \in \mathcal{F}$ and $w_n = w \in \mathcal{F}$. Moreover, either $m_n = m \in \mathbb{Z}^3$ for almost n or $|m_n| \to \infty$. If $m_n = m \in \mathbb{Z}^3$ for almost n then

$$0 < ||v_n(\cdot - k_n) - w_n(\cdot - l_n)|| = ||v - w(\cdot - m)|| = k,\ \forall n.$$

In the second case, $w(\cdot - m_n) \rightharpoonup 0$ and therefore

$$\kappa = \lim_{n\to\infty} ||v - w(\cdot - m_n)|| \leqslant ||v|| = 1,$$

since $v \in K \subset S^+$. So, the relation (2.50) holds.

The next result, related with Palais-Smale sequences for Ψ, is fundamental to obtain an important property of the corresponding pseudo-gradient flow (see Proposition 2.13 below).

Proposition 2.12. *Let $d \geqslant c$. If $(v_n^1), (v_n^2) \subset \Psi^d$ are two Palais-Smale sequences for Ψ, then either $||v_n^1 - v_n^2|| \to 0$ as $n \to \infty$ or $||v_n^1 - v_n^2|| \geqslant \rho(d) > 0$, where ρ depends on d but not on the particular choice of PS-sequences in Ψ^d.*

Notice that by Proposition 2.9, to $(v_n^j) \subset \Psi^d$ there correspond Palais-Smale sequences (u_n^j) with $u_n^j \in m(v_n^j)$, $j = 1, 2$. Thus, once (u_n^j) have been chosen, we can follow similar arguments of [62], Lemma 2.14 and analyse two distinct cases: $||u_n^1 - u_n^2||_{L^p} \to 0$ and $||u_n^1 - u_n^2||_{L^p} \nrightarrow 0$ as $n \to \infty$, for $p \in (2, 3)$ defined in (G_2).

Let H the vector field constructed in Proposition 2.11 and consider the flow $\eta : \mathcal{G} \to S^+\backslash K$ given by

$$\begin{cases} \frac{d}{dt}\eta(t, w) &= -H(\eta(t, w)) \\ \eta(0, w) &= w \end{cases}$$

where

$$\mathcal{G} \doteq \{(t, w) : w \in S^+\backslash K,\ T^-(w) < w < T^+\}$$

and $(T^-(w), T^+(w))$ is the maximal existence time for the trajectory $t \mapsto \eta(t, w)$ in negative and positive direction. Notice that, by Proposition 2.6 $(viii)$ and the Proposition

2.11,

$$\frac{d}{dt}\Psi((\eta(t,w)) \leqslant \sup_{p\in\partial\Psi(\eta(t,w))}\langle p, -H(\eta(t,w))\rangle = -\inf_{p\in\partial\Psi(\eta(t,w))}\langle p, H(\eta(t,w))\rangle < -\frac{1}{2}\mu(w) < 0,$$

that is, $t \mapsto \Psi(\eta(t,w))$ is strictly decreasing.

The following result is important for deformation type arguments.

Proposition 2.13. *For each $w \in S^+\backslash K$ the limit $\lim\limits_{t\to T^+(w)} \eta(t,w)$ exists and is a critical point of Ψ.*

The proof of this result is an adaptation from a similar result contained on [62] and in their development the authors consider the possibilities $T^+(w) < \infty$ and $T^+(w) = \infty$. In the first case, they used the definition of η and the maximality property of $T^+(w)$. In the second one, using the properties of pseudo-gradient vector field H and the function μ the authors obtain that for each $\varepsilon > 0$ there exist $t_\varepsilon > 0$ such that $||\eta(t_\varepsilon, u) - \eta(t,u)|| < \varepsilon$, which guarantee that $\lim\limits_{t\to T^+(w)} \eta(t,w)$ exists.

In the following, for a subset $P \subset S^+$ and $\beta > 0$, we put

$$U_\beta(P) \doteq \{w \in S^+ : dist(w,P) < \delta\}. \tag{2.51}$$

Lemma 2.1. *Let $d \geqslant c$. Then for each $\beta > 0$ there exists $\varepsilon > 0$ such that $\Psi_{d-\varepsilon}^{d+\varepsilon} \cap K = K_d$ and*

$$\lim_{t\to T^+(w)} \Psi(\eta(t,w)) < d - \varepsilon \tag{2.52}$$

for all $w \in \Psi^{d+\varepsilon}\backslash U_\beta(K_d)$.

Proof: Since $\mathcal{F}$ is finite, the first part holds for $\varepsilon > 0$ small enough. Suppose, without loss of generality, that $U_\beta(K_d) \subset \Psi^{d+1}$ and $\beta < \rho(d+1)$ (ρ is from Proposition 2.12). Define

$$\tau \doteq \inf\left\{\mu(w) : w \in U_\beta(K_d)\backslash U_{\frac{\beta}{2}}(K_d)\right\}. \tag{2.53}$$

We claim that $\tau > 0$. Indeed, if not, there exist $(v_n^1) \subset U_\beta(K_d)\backslash U_{\frac{\beta}{2}}(K_d)$ such that $\mu(v_n^1) \to 0$. By definition of μ, there exist (w_n^1) such that

$$||v_n^1 - w_n^1|| \to 0 \quad \text{and} \quad ||\partial^-\Psi(w_n^1)|| \to 0 \quad \text{as } n \to \infty. \tag{2.54}$$

Hence, (w_n^1) is a (PS)-sequence for Ψ. Using that $\mathcal{F}$ is finite and Ψ is $\mathbb{Z}^3-$invariant, we may assume that, up to subsequence, $w_n^1 \in U_\beta(w_0)\backslash U_{\frac{\beta}{2}}(w_0)$ for some $w_0 \in K_d$. Let (v_n^2) such that $v_n^2 \to w_0$ as $n \to \infty$. Since μ is continuous and $w_0 \in K_d$, we obtain that

$\mu(v_n^2) \to 0$. Repeating the previous argument, we obtain another (PS)-sequence for Ψ, denoted by (w_n^2), such that $||v_n^2 - w_n^2|| \to 0$ as $n \to \infty$. So, we obtain

$$\frac{\beta}{2} \leqslant \limsup_{n\to\infty} ||w_n^1 - w_n^2|| \leqslant \beta < \rho(d+1), \tag{2.55}$$

which contradict the Proposition 2.12. Hence τ is positive.

Consider $\varepsilon < \left(\frac{\beta\tau}{4}\right)$ such that $\Psi_{d-\varepsilon}^{d+\varepsilon} \cap K = K_d$. By Proposition 2.13, the unique way that (2.52) can fail is that $\eta(t,w) \to v \in K_d$ as $t \to T^+(w)$ for some $w \in \Psi^{d+\varepsilon}\backslash U_\beta(K_d)$. In this case, consider

$$\begin{aligned} t_1 &= \sup\{t \in [0, T^+(w)) : \eta(t,w) \notin U_\beta(v)\} \\ t_2 &= \inf\{t \in [t_1, T^+(w)) : \eta(t,w) \in U_{\frac{\beta}{2}}(v)\}. \end{aligned}$$

Then,

$$\frac{\beta}{2} = ||\eta(t_1,w) - \eta(t_2,w)|| \leqslant \int_{t_1}^{t_2} ||H(\eta(t,w))||\, dt \leqslant (t_2 - t_1)$$

and

$$\Psi(\eta(t_2,w)) - \Psi(\eta(t_1,w)) \leqslant \int_{t_1}^{t_2} \sup_{p\in\partial\Psi(\eta(t,w))} \langle p, -H(\eta(t,w))\rangle dt \leqslant -\frac{1}{2}\int_{t_1}^{t_2} \mu(\eta(t,w))dt \leqslant -\frac{\beta\tau}{4}.$$

Hence

$$\Psi(\eta(t_2,w)) \leqslant \Psi(\eta(t_1,w)) - \frac{\beta\tau}{4} \leqslant \Psi(\eta(0,w)) - \frac{\beta\tau}{4} \leqslant d + \varepsilon - \frac{\beta\tau}{4} < d \tag{2.56}$$

and $\eta(t,w) \nrightarrow v \in K_d$, since Ψ is strictly decreasing along trajectories of η. So we obtain a contradiction and hence the proof is complete. ∎

Proof of Theorem 2.2 (*Multiplicity*) For $j \in \mathbb{N}$, we consider the family Σ_j of all closed and symmetric subsets $A \subset S^+$ with $\gamma(A) \geqslant j$, where γ denotes the usual Krasnoselskii genus (see, e.g. [55, 60]), that is,

$$\gamma(A) \doteq \min\left\{m \in \mathbb{N} :\ \text{there exist an odd continuous map } \varphi : A \to \mathbb{R}^m\backslash\{0\}\right\}. \tag{2.57}$$

Particularly, $\gamma(A) \doteq \infty$ if there does not exist a finite m and $\gamma(\varnothing) = 0$. For A and B closed and symmetric subsets, we can stablish the following important properties for the usual Krasnoselskii genus:

(i) (*Mapping property*) If there exists an odd map $f \in C(A,B)$, then $\gamma(A) \leqslant \gamma(B)$;

(ii) (*Monotonicity property*) If $A \subset B$, then $\gamma(A) \leqslant \gamma(B)$;

(iii) (*Subadditivity*) $\gamma(A \cup B) \leqslant \gamma(A) + \gamma(B)$;

(iv) (*Continuity property*) If A is compact and $0 \notin A$, then $\gamma(A) < \infty$ and there is $\beta > 0$ such that $\overline{U_\beta(A)}$ is a closed and symmetric subset and $\gamma(\overline{U_\beta(A)}) = \gamma(A)$, where $U_\beta(\cdot)$ is defined in (2.51).

Consider the nondecreasing sequence of Lusternik-Schnirelman values for Ψ defined by

$$c_k \doteq \inf\{d \in \mathbb{R} : \gamma(\Psi^d) \geqslant k\} \quad (k \in \mathbb{N}).$$

To ensure that $\mathcal{F}$ is not finite, let's show that, for all $k \in \mathbb{N}$,

$$K_{c_k} \neq \varnothing \quad \text{and} \quad c_k < c_{k+1}. \tag{2.58}$$

Define, for $k \in \mathbb{N}$, $d = c_k$. Using the property (*iv*) from the genus, there exist $\beta > 0$ such that $\gamma(\overline{U}) = \gamma(K_d)$, where $U = U_\beta(K_d)$ and $\beta < \frac{\kappa}{2}$ ($\kappa > 0$ is defined by (2.50)).

Consider $\varepsilon = \varepsilon(\beta) > 0$ such that the conditions of Lemma 2.1 holds. Hence, for $w \in \Psi^{d+\varepsilon} \backslash U$ there exist $t_0 \in [0, T^+(w))$ such that $\Psi(\eta(t_0, w)) < d - \varepsilon$ and we may define the *entrance time map* $e : \Psi^{d+\varepsilon} \backslash U \to [0, \infty)$ by

$$e(w) = \inf\left\{t \in [0, T^+(w)) : \Psi(\eta(t, w)) \leqslant d - \varepsilon\right\}. \tag{2.59}$$

Notice that $e(w) < T^+(w)$ and e is continuous and even map. Consequently,

$$h : \Psi^{d+\varepsilon} \backslash U \to \Psi^{d-\varepsilon}, \quad h(w) = \eta(e(w), w)$$

is continuous and odd. Hence, using the properties of genus, we obtain that

$$\gamma(\Psi^{d+\varepsilon}) - \gamma(U) \leqslant \gamma(\Psi^{d+\varepsilon} \backslash U) \leqslant \gamma(\Psi^{d-\varepsilon}) \leqslant k - 1.$$

Since $\gamma(U) \leqslant \gamma(\overline{U}) = \gamma(K_d)$ we obtain

$$\gamma(K_d) \geqslant \gamma(\Psi^d) - k + 1.$$

It follows from the definition of $d = c_k$ and c_{k+1} that, if $c_k < c_{k+1}$, then $\gamma(K_d) \geqslant 1$. Also, if $c_k = c_{k+1}$, then $\gamma(K_d) > 1$. On the other hand, using that $\kappa > 0$, we obtain that $\gamma(K_d) \leqslant 1$ (depending on K_d is empty or not). Therefore $\gamma(K_d) = 1$, that is, the relation (2.58) holds.

This imply that there is an infinite sequence $(\pm w_k)$ of pairs of geometrically distinct critical points of Ψ with $\Psi(w_k) = c_k$, contrary to (2.48), and the proof of Theorem 2.2 is finished. ∎

CHAPTER 3

Solutions of nonlinear Dirac equations with possibly sign-changing potentials and asymptotically linear nonlinearities

In this chapter, using variational methods, we deal the existence of solution for the following problem:

$$-i\alpha\nabla u + a\beta u + W(x)u = f(x,u) \quad \text{in } \mathbb{R}^3, \tag{3.1}$$

where $W(x) = M(x) + \lambda V(x)I_4$, I_4 denotes the 4×4 identity matrix and $\lambda > 0$ is a parameter. Moreover, the matrix $M(x)$ and the real function $V : \mathbb{R}^3 \to \mathbb{R}$ satisfy:

(M_1) $M(x) = (M_{jk}(x))_{1 \leqslant j,k \leqslant 4}$ symmetric, real, defined a.e. in $\mathbb{R}^3$ and continuous in $\mathbb{R}^3 \backslash \{0\}$ such that

$$0 > M(x) \geqslant \frac{-k}{|x|}, \qquad \text{where} \quad k < \frac{1}{2}; \tag{3.2}$$

(V_0) $V \in L^\sigma(\mathbb{R}^3, \mathbb{R})$, $\sigma \doteq \frac{p}{p-2}$, $p \in (2,3)$, such that $V_- \neq 0$.

Notice that by the assumption (M_1), for each $x \in \mathbb{R}^3 \backslash \{0\}$, the matrix $M(x)$ is negative definite, that is, their eigenvalues $\xi_j(x)$, $j = 1,2,3,4$ are negative, but not essentially distinct. Moreover, by definition, the following relation is valid:

$$\frac{-k}{|x|} \leqslant \xi_j(x) < 0, \quad \forall j = 1,2,3,4$$

and

$$\max_{1 \leqslant j \leqslant 4} \{|\xi_j(x)|\} = o(|x|) \quad \text{as} \quad |x| \to \infty.$$

Then there exist a invertible matrix $Q(x)$ and a diagonal matrix $D(x)$, whose elements are the eigenvalues of $M(x)$, such that

$$W(x) = Q(x)D(x)Q^{-1}(x) + V(x)Q(x)Q^{-1}(x) = Q(x)\big(D(x) + V(x)I_4\big)Q^{-1}(x),$$

that is, for each $x \in \mathbb{R}^3\backslash\{0\}$, the matrix $W(x)$ is also diagonalizable and its eigenvalues has the following form

$$\alpha_j(x) = \xi_j(x) + \lambda V(x), \qquad j = 1, 2, 3, 4,$$

thus enabling the potential to present a signal change.

The Coulomb type potentials are considered, for example, at the studies developed by Ding and Ruf [32]. In this case, the authors using the condition (M_1) with $\kappa < \frac{\sqrt{3}}{2}$ and demonstrated the existence and multiplicity of solutions for a problem with asymptotically linear nonlinearity when $|u| \to \infty$. The authors also considered the semi-classical case with potential $M(x) = V(x)\beta$, where $V : \mathbb{R}^3 \to \mathbb{R}$ is a real function that is nonpositive at some point. In this case, the potential is scalar and the results obtained relate the existence and multiplicity of solutions to the parameter $\varepsilon^2 := \hbar$ in the equation

$$-i\varepsilon^2\alpha\nabla u + (a + V(x))\beta u = R_u(x, |u|)u, \quad x \in \mathbb{R}^3.$$

Zhang, Zhang and Zhao [73] used the generalized Nehari manifold and variational methods to study a Dirac equation with potential and nonlinearity asymptotically periodic in x. Under suitable assumptions, the authors combined generalized linking theorems [48] and diagonal method [63], [64] to construct a bounded Cerami sequence whose weak limit is exactly the *ground-state* solution. In addition, they studied properties of these solutions, such as its exponential decay.

It is important to mention that, in this case, the potential function V may also present a signal change. In this sense, we also have the recent work developed by Chen and Jiang [19], where the authors studied the following problem

$$-i\alpha\nabla u + \beta u + V(x)u = \nabla F(u), \quad x \in \mathbb{R}^3,$$

with a potential V that can change the signal and satisfies suitable conditions in order that the essential spectrum $\sigma_e(T)$ of $T \doteq -i\alpha\nabla + \beta + V$ it be $(-\infty, 1] \cup [1, \infty)$ and the operator has infinitely many eigenvalues in $(-1, 1)$ which accumulate in 1. The nonlinearity satisfies a resonant condition in essential spectrum of T. To ensure the existence and multiplicity of solutions it was demonstrated that the functional satisfies the Cerami condition (see Definition B.1) and used a critical point theorem presented in [6].

In our approach, let rewrite the problem (3.1) as follows

$$-i\alpha\nabla u + a\beta u + M(x)u = f(x, u) - \lambda V(x)u, \; x \in \mathbb{R}^3, \tag{3.3}$$

and consider the operator $H = H_0 + M$, where $H_0 = -i\alpha\nabla + a\beta$.

Lemma 3.1. *Suppose* (M_1) *holds. Then* H *is self-adjoint with* $\mathcal{D}(H) = H^1(\mathbb{R}^3, \mathbb{C}^4)$ *and*

$$\sigma_e(H) = \mathbb{R}\backslash(-a, a) \qquad \text{and} \qquad \sigma_d(H) \cap (-a, a) \neq \varnothing, \tag{3.4}$$

where σ_e *denotes the essential spectrum and* σ_d *denotes the discrete spectrum of* H*.*

Proof: For the first part of this proof, see [36], Lemma 2.1. We outline the proof. Letting $W_k(x) = \frac{\kappa}{|x|}$, it follows from (M_1) that $||Mu||_{L^2}^2 \leqslant ||W_k u||_{L^2}^2$. Then, by the Kato's inequality,

$$||W_k u||_{L^2}^2 \leqslant 4\kappa^2 ||\nabla u||_{L^2}^2 \leqslant 4\kappa^2 ||H_0 u||_{L^2}^2.$$

Recall that by Kato-Rellich theorem (see [20] IX.2, Theorem 2), if $2\kappa < 1$, H is self-adjoint and $\sigma(H) \subset \mathbb{R}\backslash(-(1-2\kappa)a, (1-2\kappa)a)$. Using the results presented in Appendix A, Theorem A.4 and their Remarks, we obtain that

$$\sigma_e(H) = \sigma_e(H_0) = \mathbb{R}\backslash(-a, a).$$

To characterize the eigenvalue in the spectral gap $(-a, a)$, consider the minmax principle developed by Morozov and Müller, [51], Theorem 2, which use in their proof sesquilinear forms and was developed precisely for the H_0 operator when combined with Coulomb type potentials. ■

Remark 3.1. *If* $M(x) = \frac{\kappa}{|x|}$, *Kato has proved in [45], Theorem 5.10, that* $H = H_0 + M$ *defined on* $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ *is self-adjoint if* $|\kappa| < \frac{1}{2}$. *On the other hand, for* $\kappa < \frac{\sqrt{3}}{2}$, *Thaller [65], Theorem 4.4, proved that* H *is essentially self-adjoint on* $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ *and self-adjoint on* $\mathcal{D}(H_0)$.

To matrix potentials, also Thaller [65], Theorem 4.2, has considered M *a hermitian matrix such that each component* M_{ij} *is a function that satisfies* $|M_{ij}(x)| \leqslant \frac{\kappa}{2|x|} + b$ *for all* $x \in \mathbb{R}^3\backslash\{0\}$, $i, j = 1, 2, 3, 4$ *for some constants* $\kappa < 1$ *and* $b > 0$. *Then, based on the Kato-Rellich theorem, he concluded that* H *is essentially self-adjoint on* $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ *and self-adjoint on* $\mathcal{D}(H_0)$.

As some of the results for the scalar Coulomb potentials can be extended for matrix potentials, it is expected that for a hermitian matrix that satisfies $\sup_x |x||M(x)| < \kappa$ *with* $|\kappa| < \frac{\sqrt{3}}{2}$ *we also obtain the essential self-adjointness for* H. *Indeed, this is not true. Arai [3] demonstrated that for any* $\varepsilon > 0$ *there exists an Hermitian symmetric potential* $Q_\varepsilon(x)$ *satisfying* $|x||Q_\varepsilon(x)| < \frac{1}{2} + \varepsilon$ *for which the Dirac operator* H *is not essentially self-adjoint.*

About the nonlinearity $f(x, u)$, we consider the asymptotically linear case, that is, f is a Carathéodory function and it satisfies:

(F_1) $f(x, u) = o(|u|)$ as $|u| \to 0$ uniformly in $x \in \mathbb{R}^3$;

(F_2) $f(x,u) - Q(x)u = o(|u|)$ uniformly in $x \in \mathbb{R}^3$ as $|u| \to \infty$, where $Q : \mathbb{R}^3 \to (0,\infty)$ is a continuous function and there is $q_0 > \inf \sigma(H) \cap (0,\infty)$ that satisfies $Q(x) \geqslant q_0$ for all $x \in \mathbb{R}^3$;

(F_3) $\tau \doteq \limsup_{|x|\to\infty} \left(\sup_{u} \dfrac{|f(x,u)|}{|u|} \right) < a;$

(F_4) $\widehat{F}(x,t) \doteq \frac{1}{2} f(x,t)t - F(x,t) \geqslant 0$ and there are constants $D > 0$ and $R > 0$ such that

$$\widehat{F}(x,t) \geqslant D \quad \text{if} \quad |t| > R.$$

With this conditions we can state our main result.

Theorem 3.1. *Suppose* (M_1), (V_0), (F_1) - (F_4) *be satisfied. Then, there is* $\Lambda > 0$ *such that, for* $0 < \lambda < \Lambda$, *the problem (3.1) has at least a nontrivial solution. If* $F(x,u)$ *is even in* u, *equation* (3.1) *has* l *pairs of solutions, where* l *will be defined in* (3.13).

Remark 3.2. *Notice that the condition* (F_3) *is equivalent to*

(F_3) $\limsup_{|x|\to\infty} \dfrac{|f(x,u)|}{|u|} < a$ *uniformly in* u;

Our work has a significant contribution since it combines a potential, which can present a signal change, with an asymptotically linear nonlinearity at infinity. The Coulomb potential considered is very important because it represents an interaction with an eletric field due to a point charge. Observing the properties obtained in the spectral structure of operator H, the authors chose to rewrite the problem and consider the term that involves the potential as a potential on the right. This operation allowed us establish all the conditions for a new orthogonal decomposition on E space as a sum of two subspaces being one of them with finite dimension, which helped the proof of boundedness of a Cerami sequence associated with the energy functional.

However, the signal change of the potential on the right and the decomposition orthogonal $E = E^+ \oplus E^0 \oplus E^-$ does not allow the authors to obtain a condition (Φ_1) from Bartsch and Ding critical point Theory presented in Appendix B, since it has no estimates about the elements of the kernel of H. Therefore, it uses another more classic result concerning the infinite linking argument, due to Benci and Rabinowitz [55]. This result consists of adequately rewriting the functional so that each element satisfies proper conditions and, combining with linking argument, to ensure the existence of an critical nontrivial value.

Remark 3.3. *The following are examples where the conditions* $(F_1) - (F_4)$ *holds.*

a. $F(x,u) = \frac{1}{2} Q(x)u \cdot u\Big(1 - \frac{1}{\ln(e+|u|)}\Big)$;

b. $f(x,u) = g(x,|u|)u$, *where* $g(x,s)$ *is even in* s; $g(x,s) \to 0$ *as* $s \to 0$ *uniformly in* x; $g(x,s)$ *is non-decreasing for* $s \in [0,\infty)$ *and* $g(x,s) \to Q(x)$ *as* $s \to \infty$.

Remark 3.4. *By the assumptions* $(F_1) - (F_3)$ *we obtain that, for any* $\varepsilon > 0$, *given* $p \in [2, 3)$, *there is constant* $\mu_\varepsilon > 0$ *such that*

$$|f(x,t)| \leqslant \varepsilon|t| + \mu_\varepsilon |t|^{p-1} \quad \textit{and} \quad |F(x,t)| \leqslant \left(\frac{\varepsilon}{2}\right)|t|^2 + \left(\frac{\mu_\varepsilon}{p}\right)|t|^p, \ \textit{for all} \ (x,t). \tag{3.5}$$

3.1 Variational Setting

Consider the operator $H = H_0 + M$, where $H_0 = -i\alpha\nabla + a\beta$ and M satisfies (M_1). Its domain is contained in the space $H^1(\mathbb{R}^3, \mathbb{C}^4)$ and it is a self-adjoint operator in $L^2(\mathbb{R}^3, \mathbb{C}^4)$. The relations (3.4) and $a > 0$ induces a orthogonal decomposition of $L^2(\mathbb{R}^3, \mathbb{C}^4)$ into

$$L^2 = L^+ \oplus L^- \oplus L^0,$$

where H is negative definite (positive definite, respectively) in L^- (L^+, respectively) and $L^0 = Ker(H)$.

Let $E \doteq \mathcal{D}(|H|^{\frac{1}{2}})$ be the domain of self-adjoint operator $|H|^{\frac{1}{2}}$, which is a Hilbert space equipped with the inner product

$$\langle u, v\rangle \doteq \langle |H|^{\frac{1}{2}}u, |H|^{\frac{1}{2}}v\rangle_{L^2} + \langle P_0 u, P_0 v\rangle_{L^2},$$

where $P_0 : L^2 \to L^0$ denotes the projection of L^2 in the subspace L^0. This inner product induces in $E \subset L^2(\mathbb{R}^3, \mathbb{C}^4)$ a norm defined by $||u|| = \langle u, u\rangle^{\frac{1}{2}}$ and the following decomposition

$$E = E^+ \oplus E^- \oplus E^0, \qquad \text{where} \qquad E^\pm = E \cap L^\pm \quad \text{and} \quad E^0 = L^0,$$

which is orthogonal with respect to both inner products $\langle \cdot, \cdot\rangle$ and $\langle \cdot, \cdot\rangle_{L^2}$. With this properties, and using interpolation theory, Ding and Ruf [32], Lemma 3.3, proved important relations of embedding. Here, we outline the proof.

Lemma 3.2. *The embedding* $E \hookrightarrow H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ *is continuous; moreover, the embedding* $E \hookrightarrow L^s(\mathbb{R}^3, \mathbb{C}^4)$ *is continuous for all* $s \in [2, 3]$ *and* $E \hookrightarrow L^r_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ *compactly for all* $r \in [2, 3)$.

Proof: Notice that the norm $||u||_{H^1}$ of $H^1(\mathbb{R}^3, \mathbb{C}^4)$ is equivalent to the one given by $\big|\big||H_0|u\big|\big|_{L^2}$, where as usual $|H_0|$ denotes the absolute value of H_0. Hence, by complex interpolation theory, the norm $||u||_{H^{\frac{1}{2}}}$ of $H^{\frac{1}{2}}$ is equivalent to the one given by $\big|\big||H_0|u\big|\big|_{L^2}$. Remark that by the spectral structure, 0 is at most an isolated eigenvalue of finite multiplicity of H. We will define some notations that will be useful to prove just this result. Define the (strictly) positive selfadjoint operator acting in $L^2(\mathbb{R}^3, \mathbb{C}^4)$

$$\overline{H} \doteq |H| + P_0 \quad \text{with} \quad \mathcal{D}(\overline{H}) = \mathcal{D}(H).$$

where $P_0 : E \to L^0$ denotes the projection of E onto E^0. The space $\mathcal{D}(H)$ is a Hilbert space with the norm

$$||u||_H \doteq ||\overline{H}u||_{L^2} = \left(|||H|u||_{L^2}^2 + ||P_0u||_{L^2}^2\right)^{\frac{1}{2}}$$

and, since $\mathcal{D}(H) \subset H^1$, we have

$$||u||_{H^{\frac{1}{2}}} \leqslant c_1||u||_H \quad \text{for all} \quad u \in \mathcal{D}(H).$$

Therefore, by complex interpolation,

$$||u||_{H^{\frac{1}{2}}} \leqslant c_2\left|\left||H_0|^{\frac{1}{2}}u\right|\right|_{L^2} \leqslant c_3||\overline{H}^{\frac{1}{2}u}||_{L^2} = c_3||u||$$

for all $u \in \mathcal{D}(H)$, where c_1, c_2, c_3 are constants. ∎

Remark 3.5. *We will denote $C_r > 0$ the constant of embedding of E in $L^r(\mathbb{R}^3, \mathbb{C}^4)$, that is, for $u \in E$,*

$$C_r||u||_{L^r} \leqslant ||u||.$$

Remark 3.6. *From the continuous embedding $E \hookrightarrow H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$, there exist $k_1 > 0$ such that*

$$||v|| \leqslant k_1||v||_{H^{\frac{1}{2}}}, \ \forall v \in E. \tag{3.6}$$

On the other hand, since $C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$ is a dense subset of $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$, if $v \in E \subset H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ is any element, there exist a sequence $(\phi_j) \subset C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$ such that

$$||\phi_j - v||_{H^{\frac{1}{2}}} = o_j(1), \ \textit{as } j \to \infty.$$

That is,

$$||\phi_j - v|| \leqslant k_1||\phi_j - v||_{H^{\frac{1}{2}}} = o_j(1), \ \textit{as } j \to \infty.$$

The relations (3.4) guarantee that there is at least one element of the discrete spectrum of H contained in the interval $(-a, a)$ and, thus, it is possible to find $\gamma > 0$ such that

$$\tau < \gamma < a \tag{3.7}$$

and there is at least one eigenvalue of H in $[-\gamma, \gamma]$. Let η_j be the eigenvalue in $[-\gamma, \gamma]$ and f_j the respective eigenfunctions associated for $j = 1, 2, ..., n$. Setting

$$L^d = span\{f_1, f_2, ..., f_n\},$$

we have another orthogonal decomposition

$$L^2 = L^d \bigoplus L^e \tag{3.8}$$

and, consequently,

$$E = E^d \bigoplus E^e, \quad \text{where} \quad E^d = L^d \quad \text{and} \quad E^e = E \cap L^e,$$

which is a orthogonal decomposition with respect to both the inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{L^2}$.

It follows from the assumption (F_3) and the relation (3.7), that there are $\gamma_0 \in (\tau, \gamma)$ and $R_0 > 0$ such that

$$\sup_{|x| \geqslant R_0} \left(\sup_u \frac{|f(x,u)|}{|u|} \right) < \gamma_0.$$

So, if

$$D = \left\{ x \in \mathbb{R}^3 : |x| < R_0 \right\}, \tag{3.9}$$

we have that

$$\sup_u \frac{|f(x,u)|}{|u|} < \gamma_0 \quad \text{in} \quad \mathbb{R}^3 \backslash D.$$

Moreover, since $\gamma_0 < \gamma$, there exist $s > 0$ such that

$$\frac{\gamma_0}{\gamma} < s < 1 \tag{3.10}$$

and this constant will be used later.

As we said before, consider the following problem

$$-i\alpha \nabla u + a\beta u + M(x)u = f(x,u) - \lambda V(x)u, \; x \in \mathbb{R}^3, \tag{3.11}$$

whose energy functional associated is denoted by $\Phi : E \to \mathbb{R}$ and described by

$$\Phi(u) = \frac{1}{2}(||u^+||^2 - ||u^-||^2) + \frac{\lambda}{2} \int\limits_{\mathbb{R}^3} V(x)|u|^2 dx - \int\limits_{\mathbb{R}^3} F(x,u) dx, \tag{3.12}$$

that lies in $C^1(E, \mathbb{R})$. Additionally, for $u, v \in E$,

$$\Phi'(u)(v) = \langle u^+, v^+ \rangle - \langle u^-, v^- \rangle + \int\limits_{\mathbb{R}^3} V(x) u \cdot v \; dx - \int\limits_{\mathbb{R}^3} f(x,u) \cdot v \; dx,$$

where $(u \cdot v)$ denotes the inner product in $\mathbb{C}^4$, that is $u \cdot v = \sum_{i=1}^{4} u_i \overline{v_i}$. It is well known (see [25], [30]) that the critical points of this energy functional are the solutions of the proposed problem and therefore our objective is to study this functional in order to obtain a nontrivial critical points. This existence will be ensured by the theorems from critical points theory for strongly indefinite problems, which was state in Appendix B. Throughout this chapter we consider $Y = E^+$ and $X = (E^0 \oplus E^-)$.

3.2 Linking structure

At this section we obtain the linking structure to functional Φ, which will be used to guarantee the existence of critical points.

Lemma 3.3. *There is $\rho > 0$ and $\Lambda_1 > 0$ such that $k \doteq \inf \Phi(\partial B_\rho \cap E^+) > 0$ whenever $0 < \lambda < \Lambda_1$.*

Proof: It follows from (3.5) that, for any $\varepsilon > 0$, given $p \in [2, 3)$, there is constant $\mu_\varepsilon > 0$ such that

$$|F(x,u)| \leqslant \left(\frac{\varepsilon}{2}\right)|t|^2 + \left(\frac{\mu_\varepsilon}{p}\right)|t|^p$$

for all $u \in E$. Since (V_0) holds, consider $\Lambda_1 = \frac{(1-s)C_p^2}{||V||_{L^\sigma}}$ where $s < 1$ was defined by (3.10). For any $u \in E^+$ we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}||u||^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} V(x)|u|^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx \\
&\geqslant \frac{1}{2}||u||^2 - \frac{\Lambda_1}{2}||V_-||_\sigma C_p^{-2}||u||^2 - \frac{\varepsilon}{2C_2^2}||u||^2 - \frac{\mu_\varepsilon}{pC_p^p}||u||^p \\
&\geqslant \left(\frac{s}{2} - \frac{\varepsilon}{2C_2^2}\right)||u||^2 - \frac{\mu_\varepsilon}{pC_p^p}||u||^p,
\end{aligned}$$

since $||V_-||_\sigma \leqslant ||V||_\sigma$. Thus, choosing $0 < \varepsilon < \left(\frac{sC_2^2}{2}\right)$, we obtain the conclusion for $||u|| = \rho$ and ρ sufficiently small. ■

By the assumption (F_2), consider l the number of elements in the $(0, q_0) \cap \sigma_d(H)$. If we arrange these eigenvalues, counted in multiplicity, we obtain

$$0 < \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_l < q_0 \tag{3.13}$$

and, if we denotes the corresponding eigenfunctions e_j, $1 \leqslant j \leqslant l$, we can define the subspace

$$Y_0 = span\{e_1, e_2, ...e_l\}, \tag{3.14}$$

for which it is valid that

$$\mu_1||w||_{L^2}^2 \leqslant ||w||^2 \leqslant \mu_l||w||_{L^2}^2, \quad w \in Y_0.$$

Define $E_F = E^0 \oplus E^- \oplus F$, where F is any subspace $Y_0 \subset E^+$ and $\Lambda > 0$ by

$$\Lambda \doteq \min\left\{\frac{(1-s)C_p^2}{||V||_{L^\sigma}}, \frac{q_0 C_p^2}{||V||_{L^\sigma}K^2}, \frac{(q_0-\mu_l)C_p^2}{2q_0||V||_{L^\sigma}}\right\} \tag{3.15}$$

where $K > 0$ is the equivalence constant of norms in E^0

$$||w^0|| \leqslant K||w^0||_{L^2}, \quad \text{for all } w^0 \in E^0,$$

which exist since $\dim(E_0) < \infty$.

Lemma 3.4. *Let $\rho > 0$ be the constant from Lemma 3.3 and $0 < \lambda < \Lambda$. Thus there is $R_F > 0$ such that*

$$\sup \Phi(E_F) < \infty \quad \textit{and} \quad \Phi(u) < \inf \Phi(\partial B_\rho \cap E^+)$$

for all $u \in E_F$ and $||u|| \geqslant R_F$.

Proof: It is sufficient to show that

$$\Phi(u) \to -\infty, \quad \text{as} \quad ||u|| \to +\infty, \ u \in E_F.$$

Suppose, by contradiction, that there is a sequence $(u_n) \subset E_F$ with $||u_n|| \to +\infty$ and $k > 0$ such that

$$\Phi(u_n) \geqslant -k, \quad \forall n. \tag{3.16}$$

Define the sequence $(v_n) \subset E$ by $v_n = u_n/||u_n||$, which is unitary. Hence, up to a subsequence, there is $v \in E$ such that $v_n \rightharpoonup v$ in E as $n \to \infty$, that is,

$$v_n^- \rightharpoonup v^-, \ v_n^+ \to v^+ \in Y_0 \ \text{ and } \ v_n^0 \to v^0, \ \text{as } n \to \infty,$$

since $dim(E^0) < \infty$. Notice that $v \neq 0$. Indeed, if $v = 0$ we had that $v \rightharpoonup 0$ and, using the continuity from projection operator,

$$1 = ||v_n||^2 = ||v_n^0||^2 + ||v_n^-||^2 + ||v_n^+||^2 = ||v_n^-||^2 + o_n(1), \quad \text{as } n \to \infty,$$

that is,

$$||v_n^-||^2 = 1 + o_n(1), \quad \text{as } n \to \infty. \tag{3.17}$$

On the other hand, using the Hölder's inequality, as $n \to \infty$, we obtain

$$\begin{aligned}
\frac{\Phi(u_n)}{||u_n||^2} &= \frac{1}{2}\Big(||v_n^+||^2 - ||v_n^-||^2\Big) + \frac{\lambda}{2}\int\limits_{\mathbb{R}^3} V(x)|v_n|^2\,dx - \frac{1}{||u_n||^2}\int\limits_{\mathbb{R}^3} F(x,u_n)\,dx \\
&\leqslant \frac{1}{2}\Big(||v_n^+||^2 - ||v_n^-||^2\Big) + \frac{\Lambda}{2}||V||_\sigma C_p^{-2}\Big(||v_n^+||^2 + ||v_n^-||^2 + ||v_n^0||^2\Big) \\
&\leqslant \frac{1}{2}\Big(||v_n^+||^2 - ||v_n^-||^2\Big) + \frac{\Lambda}{2}||V||_\sigma C_p^{-2}||v_n^-||^2 + o_n(1).
\end{aligned}$$

By (3.16) and the above estimate, we obtain

$$0 \leqslant s||v_n^-||^2 \leqslant \Big(1 - \Lambda||V||_\sigma C_p^{-2}\Big)||v_n^-||^2 \leqslant ||v_n^+|| + \frac{2k}{||u_n||^2} + o_n(1) = o_n(1),$$

that is,

$$||v_n^-|| = o_n(1) \quad \text{as } n \to \infty,$$

a contradiction with (3.17). Then, we conclude that $v \neq 0$.

It follows from $(v_n^+)_n \subset F$ that $v^+ \in F$. Moreover, if

$$g(v) \doteq \left(1 + \frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^+||^2 - \left(1 - \frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^-||^2 + \left(\frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^0||^2 - \int\limits_{\mathbb{R}^3} Q(x)|v|^2dx,$$

we obtain that $g(v) < 0$. Indeed, since the relations (3.13) and (3.15) are valid, we have that

$$\left[\left(1 + \frac{\Lambda||V||_\sigma}{C_p^2}\right)\mu_l - q_0\right] < (\mu_l - q_0) + \left(\frac{\Lambda||V||_\sigma}{C_p^2}\right)q_0 \leqslant (\mu_l - q_0) + \frac{(q_0 - \mu_l)}{2} < 0$$

and

$$\left(\frac{\Lambda||V||_\sigma K^2}{C_p^2}\right) - q_0 \leqslant \frac{q_0}{2} - q_0 < 0.$$

Then,

$$\begin{aligned}
g(v) &\leqslant \left(1 + \frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^+||^2 - \left(1 - \frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^-||^2 + \left(\frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^0||^2 - q_0||v||_{L^2}^2 \\
&\leqslant \left(1 + \frac{\Lambda||V||_\sigma}{C_p^2}\right)\mu_l||v^+||_{L^2}^2 - \left(1 - \frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^-||^2 + \left(\frac{\Lambda||V||_\sigma}{C_p^2}\right)K^2||v^0||_{L^2}^2 - q_0||v||_{L^2}^2 \\
&= \left[\left(1 + \frac{\Lambda||V||_\sigma}{C_p^2}\right)\mu_l - q_0\right]||v^+||_{L^2}^2 - \left[\left(\frac{\Lambda||V||_\sigma K^2}{C_p^2}\right) - q_0\right]||v^0||_{L^2}^2 \\
&\quad -\left(1 - \frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^-||^2 - q_0||v^0||_{L^2}^2 \\
&< 0,
\end{aligned}$$

since $v \neq 0$, at least one of v^+, v^- or v^0 is nonzero. From this relation, we obtain that there is $\delta > 0$ such that

$$\left(1+\frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^+||^2 - \left(1-\frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^-||^2 + \Lambda||V||_\sigma||v^0||_{L^2}^2 - \int_{B_\delta} Q(x)|v|^2dx < 0. \quad (3.18)$$

Moreover, notice that also by (F_2),

$$\lim_{n\to\infty}\int_{B_\delta}\left(\frac{F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2}{||u_n||^2}\right)dx = 0. \quad (3.19)$$

Indeed, consider the set

$$A = \{x \in \mathbb{R}^3;\ v(x) \neq 0\}, \qquad 0 < |A| \leqslant \infty. \quad (3.20)$$

and notice that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left|\int_{B_\delta}\frac{F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2}{||u_n||^2}dx\right| &\leqslant \int_{B_\delta}\frac{|F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2}{|u_n|^2}\ |v_n|^2\ dx\\
&\leqslant 2\int_{B_\delta}\frac{|F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2}{|u_n|^2}\ |v_n-v|^2dx + 2\int_{B_\delta}\frac{|F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2|}{|u_n|^2}\ |v|^2\ dx\\
&\leqslant 2\int_{B_\delta}\frac{|F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2|}{|u_n|^2}\ |v_n-v|^2dx + 2\int_{B_\delta\cap A}\frac{|F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2|}{|u_n|^2}\ |v|^2\ dx\\
&= 2\int_{B_\delta}\frac{|F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2|}{|u_n|^2}\ |v_n-v|^2dx + 2\int_{B_\delta\cap A}\frac{|F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2|}{|u_n|^2}\ |v|^2\ dx\\
&\leqslant 2\int_{B_\delta}\frac{|F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2|}{|u_n|^2}\ |v_n-v|^2\ dx + 2\ \sup_A\left(\frac{|F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2|}{|u_n|^2}\right)||v||_{L^2}^2\\
&\leqslant 2C_1||v_n-v||_{L^2(B_\delta)}^2 + 2\sup_A\left(\frac{|F(x,u_n)-\frac{1}{2}Q(x)|u_n|^2|}{|u_n|^2}\right)||v||_{L^2}^2.
\end{aligned}$$

By the definition of (v_n), $|u_n(x)| \to \infty$ for all $x \in A$. Then, by the compact embedding $E \hookrightarrow L^2_{loc}(\mathbb{R}^3)$ and the assumption (F_2), follows that the first and second term on the right-hand side of this equation, respectively, converge to zero, as $n \to \infty$, that is,

$$\lim_{n\to\infty}\int_{B_\delta}\left(\frac{\frac{1}{2}F(x,u_n)-Q(x)|u_n|^2}{||u_n||^2}\right)dx = 0.$$

Therefore, using (3.16) and (3.19) we have that, as $n \to \infty$,

$$\frac{-2k}{||u_n||^2} \leqslant \left[1 + \frac{\Lambda||V||_\sigma}{C_p^2}\right]||v_n^+||^2 - \left[1 - \frac{\Lambda||V||_\sigma}{C_p^2}\right]||v_n^-||^2 + \frac{\Lambda||V||_\sigma}{C_p^2}||v_n^0||^2 - \int_{B_\delta} Q(x)|v_n|^2 dx + o_n(1).$$

Hence, using the properties of weak convergence, the compact embedding $E \hookrightarrow L^2_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ and (3.18) we obtain that, as $n \to \infty$,

$$0 \leqslant \left(1 + \frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^+||^2 - \left(1 - \frac{\Lambda||V||_\sigma}{C_p^2}\right)||v^-||^2 + \Lambda||V||_\sigma||v^0||^2_{L^2} - \int_{B_\delta} Q(x)|v|^2 dx < 0,$$

a contradiction. ∎

Corollary 3.1. *Under the assumptions of Lemma 3.4, for any $e \in Y_0$, $||e|| = 1$, there is $R_e > 0$ such that*

$$sup\ \Phi(\partial\Omega) = 0,$$

where $\Omega \doteq \{w = (w^- + w^0) + te : (w^- + w^0) \in E^- \oplus E^0, t \geqslant 0, ||w|| \leqslant R_e\}$.

3.3 Cerami condition

At this section our objective is to ensure that the functional satisfies the Cerami condition (see Definition B.1). In order that, we will verify, firstly, that any $(Ce)_c-$sequence for Φ is bounded.

Lemma 3.5. *Let Φ be the energy functional defined in (3.12) with $0 < \lambda < \Lambda$ and $(u_n) \subset E$ any $(Ce)_c-$sequence for Φ, $c \in \mathbb{R}$. Then (u_n) is bounded.*

Proof: Arguing indirectly we assume that, up to a subsequence, $||u_n|| \to \infty$ as $n \to \infty$. By definition, there is $M > 0$ and $n_1 \in \mathbb{N}$ such that, if $n \geqslant n_1$,

$$\Phi(u_n) - \frac{1}{2}\Phi'(u_n)(u_n) = \frac{1}{2}\int_{\mathbb{R}^3} f(x, u_n)u_n\ dx - \int_{\mathbb{R}^3} F(x, u_n)\ dx \leqslant M. \tag{3.21}$$

Define the unitary sequence $(v_n) \subset E$ by $v_n = u_n/||u_n||$. Up to a subsequence, we can suppose $v_n \rightharpoonup v$ in E as $n \to \infty$. By Lemma 3.2, $v_n \to v$ in $L^r_{loc}(\mathbb{R}^3)$ for all $r \in [2, 3)$ and $v_n(x) \to v(x)$ *a.e.* in $\mathbb{R}^3$.

Claim 1. *$v \in E\backslash\{0\}$ is solution of the differential equation $Hv = (Q - V)v$.*

In order to prove this claim, suppose by contradiction, that $v \equiv 0$. Using the decomposition (3.8) and $dim(L^d) < \infty$, we obtain that $v_n = v_n^d + v_n^e \rightharpoonup 0$, that is,

$$||v_n^d|| = o_n(1) \ \text{ and } \ v_n \to 0 \ \text{ in } \ L^r_{loc}(\mathbb{R}^3, \mathbb{C}^4),\ r \in [2, 3),\ \text{as } \ n \to \infty. \tag{3.22}$$

Notice that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{\Phi'(u_n)(u_n^{e^+} - u_n^{e^-})}{||u_n||^2} &= \frac{1}{||u_n||^2}\Big(\langle u_n, u_n^{e^+}\rangle + \langle u_n, u_n^{e^-}\rangle\Big) + \frac{\lambda}{||u_n||^2}\int\limits_{\mathbb{R}^3} V(x)u_n \cdot (u_n^{e^+} - u_n^{e^-})dx \\
&\quad -\frac{1}{||u_n||^2}\int\limits_{\mathbb{R}^3} f(x,u_n)\ (u_n^{e^+} - u_n^{e^-})dx \\
&= \frac{1}{||u_n||^2}\Big(||u_n^{e^+}||^2 + ||u_n^{e^-}||^2\Big) + \lambda\int\limits_{\mathbb{R}^3} V(x)v_n \cdot (v_n^{e^+} - v_n^{e^-})dx \\
&\quad -\frac{1}{||u_n||}\int\limits_{\mathbb{R}^3} f(x,u_n)(v_n^{e^+} - v_n^{e^-})dx \\
&= ||v_n^e||^2 + \lambda\int\limits_{\mathbb{R}^3} V(x)v_n \cdot (v_n^{e^+} - v_n^{e^-})dx - \frac{1}{||u_n||}\int\limits_{\mathbb{R}^3} f(x,u_n)(v_n^{e^+} - v_n^{e^-})dx,
\end{aligned}$$

that is, as $n \to \infty$,

$$\begin{aligned}
||v_n^e||^2 &= o_n(1) - \lambda\int\limits_{\mathbb{R}^3} V(x)v_n(v_n^{e^+} - v_n^{e^-})dx + \frac{1}{||u_n||}\int\limits_{\mathbb{R}^3} f(x,u_n)(v_n^{e^+} - v_n^{e^-})dx \\
&\leqslant o_n(1) + \Lambda\int\limits_{\mathbb{R}^3} |V(x)|\ |v_n|\ |v_n^{e^+} - v_n^{e^-}|dx + \int\limits_{\mathbb{R}^3} \frac{|f(x,u_n)|}{|u_n|}|v_n^{e^+} - v_n^{e^-}||v_n|dx \\
&\leqslant o_n(1) + \Lambda||V||_{L^\sigma}C_p^{-2}||v_n||\ ||v_n^{e^+} - v_n^{e^-}|| + \int\limits_{\mathbb{R}^3} \frac{|f(x,u_n)|}{|u_n|}|v_n^{e^+} - v_n^{e^-}||v_n|dx. \quad (3.23)
\end{aligned}$$

Notice that, as $n \to \infty$,

$$\int\limits_{D^c} |v_n^{e^+} - v_n^{e^-}|\ |v_n|dx \leqslant ||v_n^{e^+} - v_n^{e^-}||_{L^2}||v_n||_{L^2} \leqslant ||v_n^e||_{L^2}\Big(||v_n^e||_{L^2} + ||v_n^d||_{L^2}\Big) = \frac{1}{\gamma}||v_n^e||^2 + o_n(1),$$

since (v_n^e) is bounded and the relation (3.22) holds. Moreover,

$$\begin{aligned}
\int\limits_{\mathbb{R}^3} \frac{|f(x,u_n)|}{|u_n|}|v_n^{e^+} - v_n^{e^-}||v_n|dx &= \int\limits_{D} \frac{|f(x,u_n)|}{|u_n|}|v_n^{e^+} - v_n^{e^-}||v_n|dx + \int\limits_{D^c} \frac{|f(x,u_n)|}{|u_n|}|v_n^{e^+} - v_n^{e^-}||v_n|dx \\
&\leqslant K_1\int\limits_{D} |v_n^{e^+} - v_n^{e^-}|\ |v_n|dx + \gamma_0\int\limits_{D^c} |v_n^{e^+} - v_n^{e^-}|\ |v_n|\ dx \\
&\leqslant K_1||v_n^{e^+} - v_n^{e^-}||_{L^2(D)}\ ||v_n||_{L^2(D)} + \frac{\gamma_0}{\gamma}||v_n^e||^2 + o_n(1) \\
&= o_n(1) + \frac{\gamma_0}{\gamma}||v_n^e||^2,
\end{aligned}$$

since, by (3.9), D is a compact set and $||v_n||_{L^2(D)} = o_n(1)$, by Lemma 3.2. Then, returning to relation (3.23), we obtain

$$\Big(1 - \Lambda ||V||_{L^\sigma} C_p^{-2} - \frac{\gamma_0}{\gamma}\Big) ||v_n^e||^2 \leqslant o_n(1),$$

and, using the definition (3.15) of Λ and (3.10), follows that

$$1 - \Lambda ||V||_{L^\sigma} C_p^{-2} - \frac{\gamma_0}{\gamma} \geqslant 1 - (1-s) - \frac{\gamma_0}{\gamma} = s - \frac{\gamma_0}{\gamma} > 0.$$

Hence,

$$||v_n^e||^2 = o_n(1), \quad \text{as} \quad n \to \infty,$$

which is an absurd, since $||v_n^e||^2 + ||v_n^d||^2 = ||v_n||^2 = 1 \neq o_n(1)$. Therefore, $v \neq 0$. For the second part of the Claim 1, notice that for any $x \in \mathbb{R}^3$ such that $v(x) \neq 0$ we have $|u_n(x)| \to \infty$, as $n \to \infty$. Moreover,

$$\int\limits_{\mathbb{R}^3} \frac{f(x,u_n)}{||u_n(x)||}\ \phi(x)\ dx = \int\limits_{\mathbb{R}^3} Q(x)\ v(x)\ \phi(x)\ dx + o_n(1)$$

and

$$\int\limits_{\mathbb{R}^3} V(x)\ v_n(x) \cdot \phi(x)\ dx = \int\limits_{\mathbb{R}^3} V(x)\ v(x)\ \phi(x)\ dx + o_n(1),$$

for any $\phi \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$, as $n \to \infty$. Therefore,

$$\begin{aligned} o_n(1) = \frac{\Phi'(u_n)(\phi)}{||u_n||} &= \langle v_n^+ - v_n^-, \phi \rangle + \lambda \int\limits_{\mathbb{R}^3} V(x) v_n(x) \cdot \phi(x) dx - \int\limits_{\mathbb{R}^3} \frac{f(x,u_n)}{||u_n||} \phi(x) dx \\ &= \langle v^+ - v^-, \phi \rangle + \int\limits_{\mathbb{R}^3} V(x) v(x) \cdot \phi(x) dx - \int\limits_{\mathbb{R}^3} Q(x) v(x) \cdot \phi(x) dx + o_n(1) \end{aligned}$$

ou seja, v é solução fraca não trivial da equação $Hv = (Q - \lambda V)v$.

Returning to the proof of Lemma, following the ideas from [32], set

$$X = \{x \in \mathbb{R}^3 : v(x) \neq 0\}.$$

By the weak unique continuation property for Dirac operator one has $|X| = \infty$. There exist $\theta > 0$ and $B \subset X$ such that $v(x) \geqslant 2\theta$ for $x \in B$ and

$$\frac{2M}{D} \leqslant |B| < \infty,$$

where $D > 0$ is from (F_4). By Egoroff's theorem, there is $B' \subset B$ with

$$|B'| > \frac{M}{D} \quad \text{and} \quad v_n \to v \quad \text{uniformly on} \quad B'. \tag{3.24}$$

Set, for each $n \in \mathbb{N}$, the set

$$\Gamma_n(R, \infty) = \{x \in \mathbb{R}^3 : |u_n(x)| > R\},$$

where $R > 0$ is also from (F_4). Using (3.21), we have that

$$M \geqslant \int\limits_{\mathbb{R}^3} \left(\frac{1}{2} f(x, u_n) u_n - F(x, u_n) u_n\right) dx \geqslant \int\limits_{\Gamma_n(R,\infty)} \widehat{F}(x, u)\ dx \geqslant D\ |\Gamma_n(R, \infty))|,$$

that is,

$$|\Gamma_n(R, \infty)| \leqslant \frac{M}{D},\ n \geqslant n_1.$$

Moreover, there is $n_2 \in \mathbb{N}$ such that $B' \subset \Gamma_n(R, \infty)$ for $n \geqslant n_2$ and, hence, if $n_0 = \max\{n_1, n_2\}$,

$$|B'| \leqslant |\Gamma_n(R, \infty)| \leqslant \frac{M}{D},\ n \geqslant n_0,$$

a contradiction with (3.24), and the proof is complete. ■

The previous Lemma allows assume that if $(u_n) \subset E$ is a $(Ce)_c-$ sequence for Φ, up to subsequence, $u_n \rightharpoonup u$ in E and u is a critical point for Φ.

Lemma 3.6. *For any $\varepsilon > 0$, there are subsequence (u_{n_j}) and $r_\varepsilon > 0$ such that*

$$\lim_{j \to \infty} \int\limits_{B_j \setminus B_r} |u_{n_j}|^\alpha \leqslant \varepsilon, \quad \forall r \geqslant r_\varepsilon \text{ and } \alpha = 2, p.$$

Proof: This proof follows the ideias from [24], Lemma 5.2. Indeed, notice that, for each $j \in \mathbb{N}$, we have

$$\int\limits_{B_j} |u_n|^\alpha\ dx = \int\limits_{B_j} |u|^\alpha\ dx + o_n(1),$$

as $n \to \infty$. So, there exists $i_j \in \mathbb{N}$ such that

$$\left| \int\limits_{B_j} \Big(|u_n|^\alpha - |u|^\alpha\Big) dx \right| < \frac{1}{j}, \quad \forall n = i_j + m,\ m = 1, 2, 3, ...$$

Assume, without loss of generality, that $i_{j+1} \geqslant i_j$. Then,

$$\left| \int\limits_{B_j} \Big(|u_{n_j}|^\alpha - |u|^\alpha \Big) dx \right| < \frac{1}{j}, \quad \text{for } n = i_j + j. \tag{3.25}$$

On the other hand, as $u \in L^\alpha(\mathbb{R}^3)$, there is $r_\varepsilon > 0$ such that

$$\int\limits_{\mathbb{R}^3 \backslash B_r} |u|^\alpha \, dx < \frac{\varepsilon}{2}, \quad \forall r \geqslant r_\varepsilon. \tag{3.26}$$

Therefore,

$$\int\limits_{B_r^j} |u_{n_j}|^\alpha dx = \int\limits_{B_r^j} \Big(|u_{n_j}|^\alpha - |u|^\alpha \Big) dx + \int\limits_{B_r^j} |u|^\alpha dx < \frac{1}{j} + \frac{\varepsilon}{2} - \int\limits_{B_r} \Big(|u_{n_j}|^\alpha - |u|^\alpha \Big) dx,$$

where $B_r^j \doteq B_j \backslash B_r$ and the last inequality follow directly from (3.25) and (3.26), respectively. Thus,

$$\lim_{j \to \infty} \int\limits_{B_r^j} |u_{n_j}|^\alpha dx \leqslant \lim_{j \to \infty} \left(\frac{1}{j} + \frac{\varepsilon}{2} - \int\limits_{B_r} \Big(|u_{n_j}|^\alpha - |u|^\alpha \Big) dx \right) \leqslant \varepsilon,$$

by the strong convergence in $L^\alpha_{loc}(\mathbb{R}^3)$, $\alpha = 2, p$. ■

The next results are very similar to those appearing in [32] and we shall describe them to complement our work. Let a smooth function $\eta : [0, \infty) \to [0, 1]$ such that $\eta(s) = 1$, if $s \leqslant 1$ and $\eta(s) = 0$, if $s \geqslant 2$. Define, in $\mathbb{R}^3$, the functions

$$w_j(x) = \eta\left(\frac{2|x|}{j} \right) u(x) \quad \text{and} \quad h_j(x) = u(x) - w_j(x). \tag{3.27}$$

Lemma 3.7. *Consider w_j and h_j the functions previously defined by (3.27) and suppose (M_1), (V_0) and $(F_1) - (F_4)$ be satisfied. Then:*

(i) $||h_j|| \to 0$ as $j \to \infty$. Particularly, $||h_j||_{L^s(\mathbb{R}^3)} \to 0, \;\; s \in [2, 3)$.

(ii) For all $r > 0$

$$\lim_{j \to \infty} \left| \int\limits_{B_r} \Big(f(x, u_{n_j}) - f(x, u_{n_j} - w_j) - f(x, w_j) \Big) \, \phi(x) \, dx \right| = 0$$

uniformly in $\phi \in E$ and $||\phi|| \leqslant 1$.

(iii)

$$\lim_{j\to\infty}\left|\int\limits_{\mathbb{R}^3}\Big(f(x,u_{n_j})-f(x,u_{n_j}-w_j)-f(x,w_j)\Big)\ \phi(x)\ dx\right|=0$$

uniformly in $\phi\in E$ *and* $||\phi||\leqslant 1$.

Proof: The conclusion (i) follows from the definition of the functions and the Lemma 3.2. To prove the item (ii), consider $r>0$ fixed. Then, by the estimate (3.5) with $\varepsilon=1$, as $j\to\infty$,

$$\begin{aligned}\int\limits_{B_r}|f(x,u_{n_j}-w_j)|\ |\phi(x)|dx &\leqslant \int\limits_{B_r}|u_{n_j}-w_j|\ |\phi|dx+\mu_1\int\limits_{B_r}|u_{n_j}-w_j|^{p-1}\ |\phi|\ dx\\ &\leqslant\Big(C_2^{-1}||u_{n_j}-w_j||_{L^2(B_r)}+\frac{\mu_1}{C_p}||u_{n_j}-w_j||_{L^p(B_r)}\Big)||\phi||\\ &=o_j(1),\end{aligned}$$

since B_r is a compact set. Moreover, using the continuity of the Nemytskii operator associated to f, we have that, as $j\to\infty$,

$$\begin{aligned}\left|\int\limits_{B_r}\big[f(x,u_{n_j})-f(x,u)\big]\ \phi(x)dx\right| &\leqslant C||N_f(u_{n_j})(x)-N_f(u)(x)||_{L^{\frac{p}{p-1}}(B_r)}||\phi||_{L^p(B_r)}\\ &\leqslant C||N_f(u_{n_j})(x)-N_f(u)(x)||_{L^{\frac{p}{p-1}}(B_r)}\\ &=o_j(1)\end{aligned}$$

and

$$\left|\int\limits_{B_r}[f(x,w_j)-f(x,u)]\ \phi(x)dx\right|\leqslant C||N_f(w_j)(x)-N_f(u)(x)||_{L^{\frac{p}{p-1}}(B_r)}=o_j(1).$$

Then,

$$\lim_{j\to\infty}\left|\int\limits_{B_r}\Big(f(x,u_{n_j})-f(x,u_{n_j}-w_j)-f(x,w_j)\Big)\ \phi(x)\ dx\right|=0,$$

for all $r>0$ uniformly in $\phi\in E$ such that $||\phi||\leqslant 1$.

To demonstrate the item (iii), set $\varepsilon>0$. By Lemma 3.6, there exists $r_\varepsilon>0$ such that, $\forall r\geqslant r_\varepsilon$ and $\alpha=2,p$,

$$\limsup_{j\to\infty}||u_{n_j}||^{\alpha}_{L^{\alpha}(B_j\backslash B_r)}\leqslant\varepsilon \tag{3.28}$$

and

$$\int_{\mathbb{R}^3 \backslash B_r} |u|^\alpha \, dx < \frac{\varepsilon}{2},$$

since $u \in L^\alpha(\mathbb{R}^3, \mathbb{C}^4)$. Then,

$$\int_{B_j \backslash B_r} |w_j|^\alpha \, dx \leqslant \int_{B_j \backslash B_r} |u(x)|^\alpha \, dx \leqslant \int_{\mathbb{R}^3 \backslash B_r} |u|^\alpha \, dx \leqslant \frac{\varepsilon}{2},$$

that is,

$$\limsup_{j \to \infty} ||w_j||^\alpha_{L^\alpha(B_j \backslash B_r)} \leqslant \frac{\varepsilon}{2}. \tag{3.29}$$

Using the item (ii) and the relations (3.28), (3.29), we obtain that there are constants $\delta_1, \delta_2 > 0$ satisfying

$$\limsup_{j \to \infty} \left| \int_{\mathbb{R}^3} \Big(f(x, u_{n_j}) - f(x, u_{n_j} - w_j) - f(x, w_j) \Big) \phi \, dx \right| \leqslant \left(\frac{2\delta_1 \varepsilon^{\frac{1}{2}}}{C_2} \right) + \left(\frac{2\delta_2 \varepsilon^{\frac{p-1}{p}}}{C_p} \right),$$

uniformly in $\phi \in E$ with $||\phi|| \leqslant 1$ and any $\varepsilon > 0$, which concludes the proof. ∎

Lemma 3.8. *Under the same conditions of the Lemma 3.7, we have:*

(a) $\lim_{j \to \infty} \Phi(u_{n_j} - w_j) = c - \Phi(u)$;
(b) $\lim_{j \to \infty} ||\Phi'(u_{n_j} - w_j)|| = 0$ *in* E^*.

Proof: (a) By definition of the energy functional Φ and the assumptions (V_0) follows that

$$\begin{aligned} \Phi(u_{n_j} - w_j) = \Phi(u_{n_j}) - \Phi(w_j) + o_j(1) \\ + \int_{\mathbb{R}^3} \Big(F(x, u_{n_j}) - F(x, u_{n_j} - w_j) - F(x, w_j) \Big) dx, \quad \text{as } j \to \infty. \end{aligned}$$

Through a similar proof of item (iii) and of the Lemma 3.7, we obtain

$$\lim_{j \to \infty} \int_{\mathbb{R}^3} \Big(F(x, u_{n_j}) - F(x, u_{n_j} - w_j) - F(x, w_j) \Big) \, dx = 0$$

and, then,

$$\lim_{j \to \infty} \Phi(u_{n_j} - w_j) = c - \Phi(u).$$

For the item (b), notice that, for all $\phi \in E$,

$$\Phi'(u_{n_j} - w_j)(\phi) = \Phi'(u_{n_j})(\phi) - \Phi'(w_j)(\phi) + \int_{\mathbb{R}^3} \Big(f(x, u_{n_j}) - f(x, u_{n_j} - w_j) - f(x, w_j)\Big)\ \phi(x)dx$$

and, by (iii) of Lemma 3.7,

$$\lim_{j\to\infty} \left| \int_{\mathbb{R}^3} \Big(f(x, u_{n_j}) - f(x, u_{n_j} - w_j) - f(x, w_j)\Big)\ \phi(x)dx \right| = 0$$

uniformly in $\phi \in E$ and $||\phi|| \leqslant 1$. As (u_{n_j}) is subsequence of a Cerami sequence and u is critical point of Φ, as $j \to \infty$,

$$||\Phi'(u_{n_j} - w_j)|| = \sup_{\phi\in E, ||\phi||\leqslant 1} |\Phi'(u_{n_j} - w_j)(\phi)| \leqslant o_j(1).$$

■

Theorem 3.2. Φ *satisfies the Cerami condition.*

Proof: Consider the decomposition $E = E^e \bigoplus E^d$ and define, for $j \in \mathbb{N}$, the element $y_j = u_{n_j} - w_j = y_j^e + y_j^d$. Then, as $j \to \infty$

$$||y_j^d|| = ||u_{n_j}^d - w_j^d|| \leqslant ||u_{n_j}^d - u^d|| + ||u^d - w_j^d|| = o_j(1),$$

since $||w_j - u|| = o_j(1)$ in E and $dim\ E^d < \infty$. It follows from Lemma 3.8 that $\Phi(y_j) = c - \Phi(u) + o_j(1)$ and $||\Phi^{'}(y_j)|| = o_j(1)$, as $j \to \infty$. Then,

$$o_j(1) = \Phi^{'}(y_j)(y_j^{e^+} - y_j^{e^-}) = ||y_j^e||^2 + \int_{\mathbb{R}^3} V(x)y_j \cdot (y_j^{e^+} - y_j^{e^-})dx - \int_{\mathbb{R}^3} f(x, y_j)(y_j^{e^+} - y_j^{e^-})dx.$$

By the same arguments presented in the demonstration of Claim 1, we obtain that

$$||y_j^e||^2 \leqslant o_j(1) + \Lambda ||V||_{L^\sigma} C_p^{-2} ||y_j^e||^2 + \frac{\gamma_0}{\gamma}\ ||y_j^e||^2,$$

that is,

$$||y_j^e||^2 = o_j(1), \quad \text{as} \quad j \to \infty.$$

Therefore, as $j \to \infty$,

$$||u_{n_j} - u|| \leqslant ||y_j|| + ||h_j|| = o_j(1),$$

that is, (u_{n_j}) converges strongly to u and $\Phi(u) = c$. Hence Φ satisfies the Cerami condition.
■

3.4 Proof of the main result

The results obtained in the previous sections allow to conclude that the functional satisfies the linking geometry conditions and also the Cerami condition. In order to prove the main results of this study we will verify other important properties.

Let $E = F_1 + F_2$, where $F_1 = E^+$, $F_2 = (E^- \oplus E^0)$ and P_1, P_2 are the projections onto F_1 and F_2, respectively. For any $u \in E$, $u = u^+ + u^0 + u^-$ and, hence,

$$||u^+||^2 - ||u^-||^2 = \langle u^+ - u^-, u\rangle = \langle u^+ - u^- - u^0, u\rangle + \langle u^0, u\rangle = \langle P_1 u - P_2 u, u\rangle + ||u^0||^2,$$

which allows rewrite the energy functional Φ, described in (3.12), by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\langle P_1 u - P_2 u, u\rangle + \frac{1}{2}||u^0||^2 + \frac{\lambda}{2}\int\limits_{\mathbb{R}^3} V(x)|u|^2dx - \int\limits_{\mathbb{R}^3} F(x,u)dx \\ &= \frac{1}{2}\langle P_1 u - P_2 u, u\rangle + \varphi(u) \\ &= \frac{1}{2}\langle L_1 P_1 u + L_2 P_2 u, u\rangle + \varphi(u), \end{aligned} \tag{3.30}$$

where $L_1 = id$, $L_2 = -id$ and

$$\varphi(u) = \frac{1}{2}||u^0||^2 + \frac{\lambda}{2}\int\limits_{\mathbb{R}^3} V(x)|u|^2dx - \int\limits_{\mathbb{R}^3} F(x,u)dx. \tag{3.31}$$

The next auxiliary results will be used in the proof of the main result.

Lemma 3.9. *Suppose (M_1), (V_0), (F_1) - (F_4) be satisfied and consider $\varphi : E \to \mathbb{R}$ defined in (3.31) with $0 < \lambda < \Lambda$. Then φ' is compact, that is, if $u_n \rightharpoonup u$ in E*

$$\varphi'(u_n)(w) \to \varphi'(u)(w), \ \forall w \in E.$$

Proof: Let $(u_n) \subset E$ be a sequence such that $u_n \rightharpoonup u$ as $n \to \infty$ and $\psi \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$. Then,

$$\begin{aligned}\big|\varphi'(u_n)(\psi) - \varphi'(u)(\psi)\big| \leqslant{} & \big|\langle u_n^0 - u^0, \psi\rangle\big| + \Lambda ||V||_{L^\sigma}||u_n - u||_{L^p(\Upsilon)}||\psi||_{L^p(\Upsilon)} \\ & + \int\limits_{\Upsilon} \big|f(x,u) - f(x,u_n)\big||\psi|dx,\end{aligned}$$

where $\Upsilon = supp(\psi)$, which is a compact subset. From the compact embedding described in Lemma 3.2 and $\dim E_0 < \infty$, the terms on the right side of this equation tend to zero. Thus,

$$\varphi'(u_n)(\psi) \to \varphi'(u)(\psi), \ \forall \psi \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^4). \tag{3.32}$$

Set $w \in E$ and $\varepsilon > 0$. It follows from the Remark 3.6 that there is $(\psi_j) \subset C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$ such that $||w - \psi_j|| = o_j(1)$ as $j \to \infty$. Thus, for fixed $j \in \mathbb{N}$,

$$\left|\Big(\varphi'(u_n) - \varphi'(u)\Big)(w)\right| \leqslant \left|\Big(\varphi'(u_n) - \varphi'(u)\Big)(\psi_j)\right| + \left|\Big(\varphi'(u_n) - \varphi'(u)\Big)(w - \psi_j)\right|.$$

Using the estimate (3.5) with $\varepsilon = 1$, we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^3} \Big(|f(x,u)| + |f(x,u_n)|\Big)|w - \phi_j|dx &\leqslant \int\limits_{\mathbb{R}^3} \Big(|u| + \mu_1|u|^{p-1} + |u_n| + \mu_1|u_n|^{p-1}\Big)|w - \psi_j|dx \\ &\leqslant \big(||u||_{L^2} + ||u_n||_{L^2}\big)||w - \psi_j||_{L^2} \\ &\qquad + \mu_1\big(||u_n||_{L^p}^{p-1} + ||u||_{L^p}^{p-1}\big)||w - \psi_j||_{L^p} \\ &\leqslant M_1||w - \psi_j||\end{aligned}$$

since (u_n) is a bounded sequence. Then, assuming that $||u_n - u||_{L^p} < M_2$ for all $n \in \mathbb{N}$,

$$\begin{aligned}\left|\Big(\varphi'(u_n) - \varphi'(u)\Big)(w - \psi_j)\right| &\leqslant \left|\langle u_n^0 - u^0, w - \psi_j\rangle\right| + \lambda \int\limits_{\mathbb{R}^3} |V(x)||u_n - u|\ |w - \psi_j|dx \\ &\quad + \int\limits_{\mathbb{R}^3} \Big(|f(x,u)| + |f(x,u_n)|\Big)|w - \phi_j|dx \\ &\leqslant \left|\langle u_n^0 - u^0, w - \psi_j\rangle\right| + \frac{\Lambda||V||_{L^\sigma}}{C_p} M_2||w - \psi_j|| + M_1||w - \psi_j||. \\ &\leqslant \Big(M_3 + \frac{\Lambda||V||_{L^\sigma}}{C_p} M_2 + M_1\Big)||w - \psi_j|| \\ &\doteq M||w - \psi_j||,\end{aligned}$$

since $\dim E^0 < \infty$. That is, we obtain a constant $M = M(u_n, u, V, E^0, C_2) > 0$ such that

$$\left|\Big(\varphi'(u_n) - \varphi'(u)\Big)(w - \psi_j)\right| \leqslant M||w - \psi_j||.$$

Choose $j_0 \in \mathbb{N}$ fixed such that $||w - \psi_{j_0}|| < \left(\dfrac{\varepsilon}{2M}\right)$. For this $j_0 \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that for $n \geqslant N$

$$\left|\Big(\varphi'(u_n) - \varphi'(u)\Big)(\psi_{j_0})\right| < \frac{\varepsilon}{2},$$

and thus, for $n \geqslant N$,

$$\left|\Big(\varphi'(u_n) - \varphi'(u)\Big)(w)\right| \leqslant \left|\Big(\varphi'(u_n) - \varphi'(u)\Big)(\psi_{j_0})\right| + \left|\Big(\varphi'(u_n) - \varphi'(u)\Big)(w - \psi_{j_0})\right| < \varepsilon.$$

■

Now, for the next lemma, remember the condition (Φ_0) stated in Appendix B:

(Φ_0) $\Phi \in C^1(E, \mathbb{R}), \Phi : (E; \mathcal{T}_{\mathcal{P}}) \to \mathbb{R}$ is upper semicontinuous, that is, Φ_a is $\mathcal{P}-$closed for

$a \in \mathbb{R}$ and $\Phi' : (\Phi_a; \mathcal{T}_{\mathcal{P}}) \to (E^*; \mathcal{T}_{w^*})$ is continuous for $a \in \mathbb{R}$;

Lemma 3.10. *Suppose Φ defined by (3.12) with $0 < \lambda < \Lambda$ and (M_1), (V_0), (F_1) - (F_4) be satisfied. Then Φ satisfies (Φ_0).*

Proof: Set $(u_n) \subset \Phi_a$, $a \in \mathbb{R}$, such that (u_n) $\mathcal{P}-$converges toward u. We will prove that $u \in \Phi_a$. By definition, $u_n^+ \to u^+$ in norm as $n \to \infty$ and, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
a \leqslant \Phi(u_n) &\leqslant \frac{1}{2}\Big(||u_n^+||^2 - ||u_n^- + u_n^0||^2\Big) + \frac{1}{2}||u_n^0||^2 + \frac{\lambda}{2}\int\limits_{\mathbb{R}^3} V(x)|u_n|^2\, dx \\
&\leqslant \frac{1}{2}\Bigg[\bigg(1 + \frac{\Lambda||V||_{L^\sigma}}{C_p^2}\bigg)||u_n^+||^2 - \bigg(1 - \frac{\Lambda||V||_{L^\sigma}}{C_p^2}\bigg)||u_n^- + u_n^0||^2 + ||u_n^0||^2\Bigg] \\
&\leqslant \frac{1}{2}\Bigg[(2-s)||u_n^+||^2 - s||u_n^- + u_n^0||^2 + ||u_n^0||^2\Bigg].
\end{aligned}
$$

So

$$0 \leqslant ||u_n^- + u_n^0||^2 \leqslant \Big[(2-s)||u_n^+||^2 - 2a + ||u_n^0||^2\Big]s^{-1}.$$

Notice that, if (u_n^0) is a bounded sequence then, the right hand side of this inequality is bounded. So, we conclude that (u_n) is bounded and, up to sequence, we may suppose that $u_n \rightharpoonup u$ in E. Then, it follows from the properties of weak convergence in E and L^p-spaces and Fatou's Lemma C.1 that

$$a \leqslant \lim_{n\to\infty} \Phi(u_n) \leqslant \frac{1}{2}\Big(||u^+||^2 - ||u^-||^2\Big) + \frac{\lambda}{2}\int\limits_{\mathbb{R}^3} V(x)|u|^2 - \liminf_{n\to\infty}\int\limits_{\mathbb{R}^3} F(x,u_n)dx \leqslant \Phi(u),$$

that is, $u \in \Phi_a$.

Suppose that there exists a subsequence of (u_n^0), still denoted by (u_n^0), such that $||u_n^0|| \to \infty$. Hence, particularly, $||u_n|| \to \infty$. Define the unitary sequence $(w_n) \subset E$ by $w_n = \big(u_n/||u_n||\big)$, which, we may suppose without loss of generality, satisfies $w_n \rightharpoonup w$ in E. By the boundedness of (u_n^+), it follows that $||w_n^+||^2 = o_n(1)$ as $n \to \infty$. Moreover,

$$w_n^- \rightharpoonup w^- \ \text{ and } \ w_n^0 \to w^0 \text{ as } n \to \infty,$$

since $\dim(E^0) < \infty$. Suppose that $w^0 = 0$. Then, $||w_n^0||^2 = o_n(1)$ and

$$||w_n^-||^2 = 1 + o_n(1) \text{ as } n \to \infty. \tag{3.33}$$

However, as $n \to \infty$,

$$\frac{a}{||u_n||^2} \leqslant \frac{\Phi(u_n)}{||u_n||^2} \leqslant \frac{1}{2}||w_n^+||^2 - \frac{1}{2}||w_n^-||^2 + \frac{\Lambda}{2}\frac{||V||_{L^\sigma}}{C_p^2}||w_n||^2 \leqslant o_n(1) - \frac{s}{2}||w_n^-||^2$$

and hence,

$$||w_n^-||^2 = o_n(1)$$

which is a contradiction with (3.33). Then $w^0 \neq 0$ and, consequently, $w \neq 0$. Using the same arguments of Lemma 3.4, we obtain that there is $\delta > 0$ such that

$$-s||w^-||^2 + \frac{\Lambda||V||_{L^\sigma}}{C_p^2}||w^0||^2 - \int\limits_{B_\delta} Q(x)|w|^2dx < 0$$

noting that $w = w^- + w^0$ since $||w_n^+||^2 = o_n(1)$, as $n \to \infty$. So,

$$\begin{aligned}
\frac{a}{||u_n||^2} &\leqslant \frac{\Phi(u_n)}{||u_n||^2} \\
&\leqslant o_n(1) - \left(\frac{1}{2} - \frac{\Lambda||V||_{L^\sigma}}{2C_p^2}\right)||w_n^-||^2 + \left(\frac{\Lambda||V||_{L^\sigma}}{2C_p^2}\right)||w_n^0||^2 - \frac{1}{2}\int\limits_{B_\delta} Q(x)|w_n|^2dx \\
&\leqslant o_n(1) - \frac{1}{2}\left(s||w_n^-||^2 + \frac{\Lambda||V||_{L^\sigma}}{C_p^2}||w_n^0||^2 - \int\limits_{B_\delta} Q(x)|w_n|^2dx\right)
\end{aligned}$$

and, as $n \to \infty$,

$$0 \leqslant \frac{1}{2}\left(s||w^-||^2 + \frac{\Lambda||V||_{L^\sigma}}{C_p^2}||w^0||^2 - \int\limits_{B_\delta} Q(x)|w|^2dx\right) < 0$$

which is a contradiction. Then, (u_n^0) is a bounded sequence and, consequently (u_n) is bounded, as we mentioned earlier.

To guarantee the continuity of $\Phi' : (\Phi_a; \mathcal{T}_\mathcal{P}) \to (E^*; \mathcal{T}_{w^*})$, consider again $(u_n) \subset \Phi_a$ such that $u_n \to u$ according to the $\mathcal{T}_\mathcal{P}-$topology. Then, as establish above, $u_n \rightharpoonup u$ in E, as $n \to \infty$, and we must ensure that

$$\Phi'(u_n) \rightharpoonup \Phi'(u), \quad \text{in} \quad E^*,$$

that is

$$\Phi'(u_n)(w) = \Phi'(u)(w) + o_n(1), \quad \text{for all} \quad w \in E, \text{ as } n \to \infty, \tag{3.34}$$

since E is a Hilbert space. This relation follows from the weak convergence and Lemma 3.9. ∎

Proof of Theorem 3.1: Consider $0 < \lambda < \Lambda$, where $\Lambda > 0$ is defined by (3.15). It follows from Lemma 3.2 that the functional Φ satisfies the Cerami condition for all $c \in \mathbb{R}$. Rewriting the functional in the form (3.30), we guarantee the condition (I_1) of

Theorem B.8. The condition (I_2) follows from Lemma 3.9, hence, it is sufficient to justify the hypothesis (I_3).

Consider $e \in F_1 \backslash \{0\}$, $||e|| = 1$, the constant $\rho > 0$ from of the Lemma 3.3, the constant $R_e > 0$ and set Ω from of the Corollary 3.1. Then we have

$$\Omega \doteq \{w \in F_2 \oplus \mathbb{R}^+ e : ||w|| \leqslant R_e\} = \left[F_2 \oplus \mathbb{R}^+ e\right] \cap B_{R_e}$$

and

$$S \doteq \{z \in F_1 : ||z|| = \rho\} = F_1 \cap \partial B_\rho.$$

Then $S \subset F_1$, $\Omega \subset \left[F_2 \oplus \mathbb{R}^+ e\right]$ and, following the steps of [8], Example 4.3, we guarantee that S and $\partial\Omega$ finitely link. Moreover, it follows from Lemma 3.3 and Corollary 3.1 that

$$\Phi(z) \geqslant \rho > 0, \ \forall z \in S \text{ and } \sup \Phi(\partial\Omega) = 0,$$

obtaining the condition (I_3). Therefore, it follows from the Theorem B.8, Φ has a critical value $c \geqslant \rho > 0$, and the problem (3.1) has a nontrivial solution.

Suppose that $F(x, u)$ is even in u. Then, Φ is even in u and, by Lemma 3.10, satisfies (Φ_0). The Lemma 3.3 and Lemma 3.4 guarantee (Φ_3) and (Φ_4) of Theorem B.5 with Y_0 defined by (3.14). Thus, the problem (3.1) has at least $l = \dim(Y_0)$ pairs of nontrivial solutions. ■

APPENDIX A

The Dirac operator and some properties

The objective of this appendix is to show some aspects of the Dirac operator, it historical origin, interesting properties about the self-adjointness and it spectrum. The facts presented here can be found basically in [65] and [20]. More details and another properties can be found also in [38, 39, 45, 49, 59, 68] and the references therein.

The transition from classical mechanic to quantum mechanic can be constructed using the correspondence principle, that is, replacing classical quantities by some appropriate operators. Usually, these operators are differential or multiplication operators acting on suitable wave functions. In particular, for the energy E and the momentum $\mathbf{p}$ of a free particle, the substitution

$$E \to i\hbar\frac{\partial}{\partial t} \qquad \text{and} \qquad \mathbf{p} \to -i\hbar\nabla, \tag{A.1}$$

where $\hbar$ is the Planck constant, is familiar from the nonrelativistic theory. Moreover, (A.1) is formally Lorentz invariant and if applied to the classical relativistic energy-momentum relation, we obtain

$$E = \sqrt{c^2\mathbf{p}^2 + m^2c^4}, \tag{A.2}$$

where c denotes the speed of light and $m > 0$ the electron mass, and then, the square from Klein-Gordon equation, that is,

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \sqrt{-c^2\hbar^2\Delta + m^2c^4}\ \psi(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3, \tag{A.3}$$

where Δ denotes the Laplace operator and ψ is a wave function.

In general, the square root of a differential operator can be defined with the help of Fourier-transformations, but, in this case, due to the asymmetry of space and time derivatives Dirac found it impossible to include external electromagnetic fields in a relativistically invariant way. So he looked for another equation which can be modified in order to describe the action of electromagnetic forces and also should describe the internal structure of the electrons, the spin. That is, it provide a relativistic description of the spin particles 1/2 consistent with the requirements of the special theory of relativity and, in doing so, opened the path to application from the theory of groups to the description of particles of arbitrary spin.

The relativistic equation of Klein-Gordon

$$\hbar^2 \frac{\partial^2}{\partial t^2}\psi(t,x) = (-c^2\hbar^2\Delta + m^2c^4)\psi(t,x)$$

describes a spinless particle and, moreover, it is of second order, so was not able to do this, since a quantum mechanical evolution equation should be of first order in the time derivative. Thus, Dirac reconsidered the energy-momentum equation (A.2) and, by a linearization argument, obtained

$$E = c\sum_{i=1}^{3}\alpha_i p_i + \beta mc^2 \equiv c\ \alpha\cdot\mathbf{p} + \beta mc^2, \tag{A.4}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β do not depend on either coordinates or time and it will be determined by the relation (A.2). Indeed, by combining (A.2) and (A.4), one readily gets the following

$$\begin{gathered} \alpha_j\alpha_k + \alpha_k\alpha_j = 0, \text{ if } k \neq j \\ \alpha_k\beta + \beta\alpha_k = \mathbf{0} \\ \alpha_k^2 = \beta^2 = I_n,\ j = 1,2,3, \end{gathered} \tag{A.5}$$

where I_n and $\mathbf{0}$ are $n-$dimensional unit and zero matrices. Hence, α and β are anticommuting quantities which are most naturally represented by $n\times n$ matrices. They also have to be hermitian because the Hamiltonian and the momentum operators are hermitian, and then their eigenvalues are ±1. Moreover, $tr(\alpha_k) = tr(\beta^2\alpha_k) = -tr(\beta\alpha_k\beta) = -tr(\alpha_k)$ and $tr(\beta) = -tr(\beta)$, therefore

$$tr(\alpha_k) = tr(\beta) = 0.$$

It follows from this relation that the number of positive and negative eigenvalues has to be the same, that is, the order n of the matrices has to be an even number. Considering $n = 2$, we obtain at most three linearly independent anticommuting matrices (the fourth one is the unit matrix which commutes with all matrices): the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, the smallest possible dimension of α and β matrices is 4, that is, the relations (A.5) are satisfied by choosing

$$\beta = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & -I_2 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} \mathbf{0} & \sigma_k \\ \sigma_k & \mathbf{0} \end{pmatrix}, \quad k = 1, 2, 3, \tag{A.6}$$

where σ_k are Pauli matrices.

Using this quantities, we obtain the formulation of the Dirac equation

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) = H_0 \psi(t,x) \tag{A.7}$$

with H_0 given explicitly by the matrix-valued differential expression

$$H_0 = -i\hbar c\, \alpha \cdot \nabla + mc^2 \beta = \begin{pmatrix} mc^2 I_2 & -i\hbar c\, \sigma \cdot \nabla \\ -i\hbar c\, \sigma \cdot \nabla & -mc^2 I_2 \end{pmatrix}, \tag{A.8}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are triplets of matrices.

Remark A.1. *If $m = 0$ the mass term in* (A.4) *vanishes and only three anticommuting quantities α_i are needed. In this case, it is sufficient to use the 2×2 Pauli matrices defined above. In this case, the two component equation*

$$i\hbar \frac{\partial}{\partial t}\psi(t) = c\sigma \cdot \mathbf{p}\ \psi(t)$$

is called Weyl equation.

Consider the Hilbert space

$$L^2(\mathbb{R}^3)^4 \equiv L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) = L^2(\mathbb{R}^3, \mathbb{C}^4) = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 \tag{A.9}$$

which consists of wave functions with four components column vectors $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$, where each component ψ_i is a complex valued function of the space variable x. The Dirac operator H_0 can be defined in a natural way as a self-adjoint operator in this space (this will allow the approach of cases in which the particle is subjected to the action of a potential or of an electromagnetic field), as proved by Dautray and Lion [20]:

Theorem A.1. *The unbounded operator in $H = L^2(\mathbb{R}^3, \mathbb{C}^4)$, denoted by L and defined by*

$$L = -i \sum_{k=1}^{3} \alpha_k \frac{\partial}{\partial x_k} + \alpha_4$$

and with domain

$$\mathcal{D}(L) = \{u \in H; Lu \ (\textit{in sense of } \mathcal{D}^*(\mathbb{R}^4, \mathbb{C}^4)) \in H\}$$

is selfadjoint, and such that:

$$D(L) = H^1(\mathbb{R}^3, \mathbb{C}^4) = H^1(\mathbb{R}^3)^4. \tag{A.10}$$

(*Notice that in this case,* $\hbar = c = m = 1$ *and* $\beta = \alpha_4$.)

The suitable domain expressed in (A.10) is a natural domain for first order differential operator and, as we will prove, besides being self-adjoint in $H^1(\mathbb{R}^3)^4$, the operator is essentially self-adjoint on the dense domain $C_0^\infty(\mathbb{R}^3\backslash\{0\}, \mathbb{C}^4)$ and has an absolutely continuous spectrum. In order to obtain this conclusions, we will analyze the Dirac operator in the Fourier spaces.

Consider $\mathbf{p}$ the differential operator defined by $\mathbf{p} \equiv -i\nabla = -i(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, called momentum operator, which act component-wise on the vectors ψ. Also, consider $\mathcal{F}$ the Fourier transformation defined for integrable functions by

$$(\mathcal{F}\psi_k)(\mathbf{p}) \equiv \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} e^{-i\mathbf{p}\cdot\mathbf{x}} \psi_k(x) d^3x, \;\; k = 1, 2, 3, 4, \tag{A.11}$$

which extends to a uniquely defined unitary operator in $L^2(\mathbb{R}^3)^4$. Hence, using the matrix-valued differential expression of Dirac operator (A.8), we obtain that

$$(\mathcal{F} H_0 \mathcal{F}^{-1})(\mathbf{p}) = h(\mathbf{p}) = \begin{pmatrix} mc^2 I_2 & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & -mc^2 I_2 \end{pmatrix}. \tag{A.12}$$

For each $\mathbf{p}$, we define $p = |\mathbf{p}|$ and obtain a Hermitian $4\times 4-$matrix which has the eigenvalues

$$\lambda_1(p) = \lambda_2(p) = -\lambda_3(p) = \lambda_4(p) = \sqrt{c^2p^2 + m^2c^4} \equiv \lambda(p). \tag{A.13}$$

Moreover, there exist a unitary transformation $u(\mathbf{p})$ (see [65] to explicit form) such that $u(\mathbf{p})h(\mathbf{p})u(\mathbf{p})^{-1} = \beta\lambda(p)$ and so, the unitary transformation

$$\mathcal{W} = u\mathcal{F}$$

converts the Dirac operator H_0 into an operator of multiplication by the diagonal matrix

$$(\mathcal{W} H_0 \mathcal{W}^{-1})(p) = \beta\lambda(p) \tag{A.14}$$

in the Hilbert space $L^2(\mathbb{R}^3, d^3p)^4$. This informations help us to prove the following important result.

Theorem A.2 ([65], Theorem 1.1)**.** *The Dirac operator* H_0 *is essentially self-adjoint on the dense domain* $C_0^\infty(\mathbb{R}^3\backslash\{0\}, \mathbb{C}^4)$ *and self-adjoint on the Sobolev space* $\mathcal{D}(H_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$. *Its spectrum is purely absolutely continuous and given by*

$$\sigma(H_0) = (-\infty, -a] \cup [a, +\infty).$$

Proof: It follows from (A.14) that H_0 is unitarily equivalent to the operator $\beta\lambda(\cdot)$ of multiplication by a diagonal matrix-valued function of $\mathbf{p}$, and hence is self-adjoint on

$$\mathcal{D}(H_0) = \mathcal{W}^{-1}\mathcal{D}(\beta\lambda(\cdot)) = \mathcal{F}^{-1}u^{-1}\mathcal{D}(\lambda(\cdot)) = \mathcal{F}^{-1}\mathcal{D}(\lambda(\cdot)).$$

Notice that $u(\mathbf{p})^{-1}$ is a multiplication by a unitary matrix and does not change the domain of any multiplication operator. Recall that $H^1(\mathbb{R}^3)^4$ is defined as the inverse Fourier transform of the set

$$\{f \in L^2(\mathbb{R}^3, d^3p)^4 : (1 + |p|^2)^{\frac{1}{2}} f \in L^2(\mathbb{R}^3, d^3p)^4\}. \tag{A.15}$$

On the other hand, the domain of multiplication operator $\mathcal{D}(\lambda(\cdot))$, where $\lambda(\cdot)$ is defined by (A.13), is exactly this set (A.15). Then,

$$\mathcal{D}(H_0) = \mathcal{F}^{-1}\mathcal{D}(\lambda(\cdot)) = H^1(\mathbb{R}^3)^4.$$

Moreover, the unitary equivalence guarantee that the spectrum of H_0 is the same of from multiplication operator $\beta\lambda(\cdot)$, which is given by the range of the functions $\lambda_i(\mathbf{p})$, $i = 1, 2, 3, 4$, that is

$$\sigma(H_0) = (-\infty, -mc^2] \cup [mc^2, +\infty).$$

Finally, recall that a symmetric operator T is said to be essentially self-adjoint if its closure $\overline{T}$ is self-adjoint. Moreover, $\overline{T}$ is the unique self-adjoint extension of T [57]. To demonstrate that H_0 has this property in the set $C_0^\infty(\mathbb{R}^3\backslash\{0\}, \mathbb{C}^4)$, consider $\tilde{\mathrm{H}}_0$ the Dirac operator defined on the set $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$, which is the set of functions of rapid decrease. It is known that this set is invariant by Fourier transformation, that is, $\mathcal{F}\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$, and then the operator $\tilde{\mathrm{H}}_0$ is unitarily equivalent to the restriction of $h(\mathbf{p})$ to $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$. Since this is an essentially self-adjoint operator (its closure is the self-adjoint multiplication operator $h(\mathbf{p})$), the same is true for $\tilde{\mathrm{H}}_0$, and its closure is H_0, the self-adjoint Dirac operator.

Denotes the Dirac operator in $C_0^\infty(\mathbb{R}^3\backslash\{0\}, \mathbb{C}^4)$ by $\dot{\mathrm{H}}_0$. Our objective is prove that the closure of

$$\overline{\dot{\mathrm{H}}_0} = H_0, \tag{A.16}$$

since H_0 is the self-adjoint Dirac operator. Notice that $\dot{\mathrm{H}}_0 \subset \tilde{\mathrm{H}}_0$ and this same relation is true for their closure, that is, $\overline{\dot{\mathrm{H}}_0} \subset \overline{\tilde{\mathrm{H}}_0}$. Then, by definition,

$$\mathcal{D}(\overline{\dot{\mathrm{H}}_0}) \subset \mathcal{D}(\overline{\tilde{\mathrm{H}}_0}) \;\text{ and }\; \overline{\dot{\mathrm{H}}_0}(\psi) = \overline{\tilde{\mathrm{H}}_0}(\psi),\; \forall\psi \in \mathcal{D}(\overline{\dot{\mathrm{H}}_0}). \tag{A.17}$$

This relation guarantee that is sufficient to prove

$$\mathcal{D}(\tilde{\mathrm{H}}_0) \subset \mathcal{D}(\overline{\dot{\mathrm{H}}_0}) \tag{A.18}$$

to obtain the relation (A.16). Indeed, supposing that we have already proved the above relation, let $\psi \in \mathcal{D}(\tilde{\mathrm{H}}_0) \subset \mathcal{D}(\overline{\dot{\mathrm{H}}_0})$ and we obtain, by (A.17), $\overline{\dot{\mathrm{H}}_0}(\psi) = \overline{\tilde{\mathrm{H}}_0}(\psi) = H_0(\psi) = \tilde{\mathrm{H}}_0(\psi)$. That is, $\overline{\dot{\mathrm{H}}_0}$ is an extension of $\tilde{\mathrm{H}}_0$, which is essentially self-adjoint. By the uniqueness of self-adjoint extension, we obtain (A.16). Therefore, it remains to demonstrate that (A.18) holds.

For every $\psi \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{D}(\tilde{\mathrm{H}}_0)$, we need to find a sequence $\psi_n \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ such that

$$\lim_{n\to\infty} \psi_n = \psi, \quad \lim_{n\to\infty} \dot{\mathrm{H}}_0(\psi_n) = \tilde{\mathrm{H}}_0(\psi). \tag{A.19}$$

Define

$$\psi_n(x) = f(n^{-1}x)(1 - f(nx))\psi(x),$$

where $f \in C^\infty(\mathbb{R}^3)$ with $f(x) = 1$ for $|x| \leqslant 1$, $f(x) = 0$ for $|x| \geqslant 2$ and $0 \leqslant f(x) \leqslant 1$ for all $x \in \mathbb{R}^3$. Obviously, $\psi_n \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ and $\psi_n \to \psi$. Moreover, using the assumptions on f and suitable estimates, we obtain that $\dot{\mathrm{H}}_0(\psi_n) \to \tilde{\mathrm{H}}_0(\psi)$, which proves (A.19) and hence (A.18). ∎

Remark A.2. *Similar results about the essentially self-adjointness (self-adjointness, respect.) of H_0 in $C_0^\infty(\mathbb{R}^3)^4$ (in $\mathcal{D}(H_0)$, respect.) were proved by Kato [45], §5 Sect.4, and Jörgens [44]. The least also consider the perturbed cases $H_0 + V$ for suitable potentials.*

As mentioned by Thaller [65], considering the Hilbert $\mathcal{W}L^2(\mathbb{R}^3)^4$, where the Dirac operator is diagonal, the upper two components of wave functions belong to positive energies, while the lower components to negative energies. Hence, we define the subspace of positive energies $H^+ \subset L^2(\mathbb{R}^3)^4$ as the subspace spanned by vectors of the type

$$\psi^+ \equiv \mathcal{W}^{-1}\frac{1}{2}(1+\beta)\mathcal{W}\psi, \ \psi \in L^2(\mathbb{R}^3, d^3x).$$

Similarly, we define the vectors

$$\psi^- \equiv \mathcal{W}^{-1}\frac{1}{2}(1-\beta)\mathcal{W}\psi, \ \psi \in L^2(\mathbb{R}^3, d^3x),$$

that span the negative energy subspace H^-. Since these subspaces are orthogonal, we can write $L^2 = H = H^+ \oplus H^-$ as an orthogonal direct sum and each ψ can be written as a sum of ψ^+ and ψ^-. Moreover, considering $\phi^\pm = \frac{1}{2}(1 \pm \beta)\mathcal{W}\psi$, we have

$$(\psi^+, H_0\psi^+) = (\mathcal{W}^{-1}\phi^+, \mathcal{W}^{-1}\lambda(\cdot)\phi^+) = (\phi^+, \lambda(\cdot)\phi^+) > 0,$$

that is, H_0 acts as a positive operator on H^+. Similarly, H_0 acts as a negative operator on H^-.

We can also define the self-adjoint operator $|H_0|$ as

$$|H_0| \equiv \sqrt{H_0^2} = \sqrt{-c^2\Delta + m^2c^4}\ I,$$

where the square root can be defined as a inverse Fourier transformation of the multiplication operator $\sqrt{c^2\Delta + m^2c^4}$ in $L^2(\mathbb{R}^3, d^3p)$. Obviously,

$$H_0\psi^{\pm} = \pm|H_0|\psi^{\pm}.$$

The essential spectrum is very stable under perturbations. We present here the famous theorem of H. Weyl that was proved, e.g., in [57], Theorem XIII.14:

Theorem A.3 ([65], Theorem 4.5.)**.** *Let H_1 and H_0 be self-adjoint operators such that for one (and hence all) $z \in \mathbb{C}\backslash\mathbb{R}$ the operator*

$$(H_1 - z)^{-l} - (H_0 - z)^{-l}$$

is compact. Then

$$\sigma_e(H_1) = \sigma_e(H_0).$$

.

Theorem A.4 ([65], Theorem 4.1.)**.** *Let $H_1 = H_0 + V$ be self-adjoint, and V be H_0-bounded with*

$$\lim_{R\to\infty} ||V(H_0 - Z)^{-l}\chi(|x| \geqslant R)|| = 0. \tag{A.20}$$

Then

$$\sigma_e(H_1) = (-\infty, -mc^2] \cup [mc^2, \infty). \tag{A.21}$$

Remark A.3. *The equation* (A.20) *is a very weak decay condition on the potential. If V is a multiplication operator, it is equivalent to*

$$\lim_{R\to\infty} ||V\chi(|x| \geqslant R)(H_0 - z)^{-1}|| = 0. \tag{A.22}$$

It is possible to prove that the relations (A.20) and (A.22) are equivalent. For this, just consider a suitable differentiable function $f_R : [0, \infty) \to [0, 1]$ such that $f_R(r) = 0$ for $r < \frac{R}{2}$, $f_R(r) = 1$ for $r > R$ and $\sup_r f_R'(r) < \frac{4}{R}$. Then, $\chi(|x| \geqslant R) = f_R(|x|)\chi(|x| \geqslant R)$ and

$$||V\chi(|x| \geqslant R)(H_0 - z)^{-1}|| \leqslant ||V(H_0 - z)^{-1}c\alpha \cdot (\nabla f_R)(H_0 - z)^{-1}|| + ||Vf_R(H_0 - z)^{-1}||.$$

Hence, just make sure that both terms vanishing when $R \to \infty$.

Remark A.4. *Any potential matrix, with $V(x) \to 0$, as $|x| \to \infty$ satisfies*

$$||V\chi(|x| \geqslant R)|| = \sup_{|x|>R} |V(x)| \to 0, \text{ as } R \to \infty, \tag{A.23}$$

and hence (A.22). *But the conditions* (A.20) *and* (A.22) *are more general than* (A.23) *because they admit singularities of the potential even at large distances.*

The condition (A.20) is not optimal, mainly because there are potentials which tend to infinity, as $|x| \to \infty$, and still (A.21) holds. In particular, this occurs for unisotropic potentials as well as for magnetic fields in three dimensions.

APPENDIX B

Critical point theory

In order to find critical points for the energy functional associated with the problems we will state some results of critical points theory for strongly indefinite functionals, which was developed by Bartsch and Ding and presented in [8], [9] and [25]. Another similar results about generalized linking and critical points can be found also in [7] and [48].

Consider $E = X \oplus Y$ where X and Y Banach spaces and X is separable and reflexive. Let $||\cdot||$ the norm in X, Y and E, and P_X, P_Y denote the projections onto X and Y, respectively. Let $S \subset X^*$ a dense subset and $\mathcal{D} = \{d_s : s \in S\}$ the family of semi-metrics associated in $X \cong X^{**}$. If $\mathcal{P}$ is a family of semi-norms in E defined by

$$p_s : E \to \mathbb{R}, \quad p_s(x+y) = |s(x)| + ||y||, \quad s \in S. \tag{B.1}$$

hence, $\mathcal{P}$ induces a product topology in E described by $\mathcal{D}-$topology in X and norm topology in Y. This topology associated will be denoted by $\mathcal{T}_{\mathcal{P}}$.

Let $\Phi : E \to \mathbb{R}$ a C^1 functional. Suppose that Φ satisfies

(Φ_0) $\Phi \in C^1(E, \mathbb{R}), \Phi : (E; \mathcal{T}_{\mathcal{P}}) \to \mathbb{R}$ is upper semicontinuous, that is, Φ_a is $\mathcal{P}-$closed for $a \in \mathbb{R}$ and $\Phi' : (\Phi_a; \mathcal{T}_{\mathcal{P}}) \to (E^*; \mathcal{T}_{w*})$ is continuous for $a \in \mathbb{R}$;

Remark B.1. *These conditions can be weakened. Depending on the situation, it is required only for values in a certain interval and we can replace Φ_a by the subset like Φ_a^b. Similarly, Φ_a can be $\mathcal{P}-$closed for certain values of a only.*

This assumption can be guaranteed by the following result.

Theorem B.1 ([8], Proposition 4.1; [25], Theorem 4.1)**.** *Consider a functional $\Phi \in C^1(E, \mathbb{R})$ of the form*

$$\Phi(u) = \frac{1}{2}\Big(||y||^2 - ||x||^2\Big) - \Psi(u) \quad \textit{for} \quad u = x + y \in E = X \oplus Y \tag{B.2}$$

such that

(i) $\Psi \in C^1(E, \mathbb{R})$ *is bounded from below;*

(ii) $\Psi : (E; \mathcal{T}_w) \to \mathbb{R}$ *is sequentially lower semicontinuous, that is,* $u_n \rightharpoonup u$ *in* E *implies* $\Psi(u) \leqslant \liminf \Psi(u_n)$*;*

(iii) $\Psi' : (E; \mathcal{T}_w) \to (E^*, \mathcal{T}_{w*})$ *is sequentially continuous;*

(iv) $\nu : E \to \mathbb{R}, \quad \nu(u) = ||u||^2$ *is* C^1 *and* $\nu' : (E; \mathcal{T}_w) \to (E^*, \mathcal{T}_{w*})$ *is sequentially continuous.*

Then Φ *satisfies* (Φ_0)*.*

Consider also a additional assumption:

(Φ_1) For any $a > 0$ there is $\theta > 0$ such that $||u|| \leqslant \theta ||P_Y u||$, for all $u \in \Phi_a$.

Under these hypothesis we present the following result which will be used to establish the existence of a $(Ce)_c-$sequence for the energy functional Φ. Recall that:

Definition B.1. *The sequence* $(v_n) \subset E$ *is called a Cerami sequence, or a* $(Ce)_c-$*sequence for* $c \in \mathbb{R}$*, if*

$$\Phi(v_n) \to c \quad \textit{and} \quad (1 + ||v_n||)\Phi'(v_n) \to 0 \ \textit{in} \ E^* \quad \textit{as} \quad n \to \infty.$$

We say that Φ *satisfies Cerami's condition at level c, or* $(Ce)_c-$*condition, if any* $(Ce)_c-$*sequence for* Φ *has a convergent subsequence.*

Theorem B.2 ([9], Theorem 5.1; [25], Theorem 4.5)**.** *Let* Φ *satisfies* (Φ_0) *and* (Φ_1) *and suppose there is* $R > r > 0$ *and* $u_0 \in Y, \ \ ||u_0|| = 1$ *such that for* $S = \{z \in Y; ||z|| = r\}$*,* $Q = \{tu_0 + v \in E; v \in X, ||v|| < R, 0 < t < R\}$ *we have*

$$\kappa \doteq \inf \Phi(S) > 0 \ \textit{and} \ \sup \Phi(\partial Q) \leqslant \kappa. \tag{B.3}$$

Then Φ *has a* $(Ce)_c-$*sequence with* $\kappa \leqslant c \leqslant \sup \Phi(Q)$*.*

Is important mention that, in their approach, the authors introduced a new linking in the infinite-dimensional setting and used this to characterize the critical value obtained in their theorems, like above. To do this, they consider the following notations: given a locally convex topological vector space Z, denote $L(A) \doteq \overline{span(A)}$ for the smallest closed linear subspace containing A, ∂A the boundary of A in $L(A)$ and $A_F = A \cap F$ ($F \subset Z$ linear subspace).

Definition B.2. *Given* $Q, S \subset Z$ *with* $S \cap Q = \varnothing$*, we say that* Q *finitely links with* S *if for any finite-dimensional linear subspace* $F \subset Z$ *with* $F \cap S \neq \varnothing$ *and any continuous deformation* $h : I \times Q_F \to F + L(S)$ *with* $h(0, u) = u$ *and* $h(I \times \partial Q_F) \cap S = \varnothing$ *there holds* $h(t, Q_F) \cap S \neq \varnothing$ *for all* $t \in I = [0, 1]$*.*

Using this definition and Brouwer degree arguments, it is possible to show that $Q, S \subset E$ in Theorem B.2 are such that Q *finitely links* with S. Hence, the critical value found in that theorem can be characterized by

$$c \doteq \inf_{h \in \Gamma_{Q,S}} \sup_{u \in Q} \Phi(h(1,u)), \tag{B.4}$$

where

$$\Gamma_{Q,S} \doteq \{h \in C(I \times Q, E) : h \text{ satisfies } (h_1) - (h_5)\} \tag{B.5}$$

with

(h_1) $h : I \times (Q, \mathcal{T}_{\mathcal{P}}) \to (E, \mathcal{T}_{\mathcal{P}})$ is continuous;

(h_2) $h(0,u) = u$ for all $u \in Q$;

(h_3) $\Phi(h(t,u)) \leqslant \Phi(u)$ for all $t \in I$, $u \in Q$;

(h_4) $h(I \times \partial Q) \cap S = \varnothing$;

(h_5) each $(t,u) \in I \times Q$ has a $\mathcal{P}-$open neighbourhood W such that the set $\{v - h(s,v) : (s,v) \in W \cap (I \times Q)\}$ is contained in a finite dimensional subspace of E.

The arguments used to obtain this characterization are presented in the proof of Lemma 1.9 in this work and in [25], Theorem 4.2. To ensure the existence of Cerami sequence, a suitable deformation, obtained in the next result, was used.

Theorem B.3 ([25], Theorem 3.3)**.** *Consider $a, b \in \mathbb{R}$ with $a < b$ so that Φ_a is $\mathcal{P}-$closed and $\Phi' : (\Phi_a^b, \mathcal{T}_{\mathcal{P}}) \to (E^*, \mathcal{T}_{w*})$ is continuous. Suppose moreover that*

$$\alpha \doteq \inf\{(1 + ||u||)||\Phi'(u)|| : u \in \Phi_a^b\} > 0 \tag{B.6}$$

and

$$\textit{there exists } \gamma > 0 \ \textit{ with } \ ||u|| < \gamma ||P_Y u||, \ \forall u \in \Phi_a^b. \tag{B.7}$$

Then there exists a deformation $\eta : [0,1] \times \Phi^b \to \Phi^b$ with the properties $(i) - (vii)$:

(i) η is continuous with either the $\mathcal{P}-$topology or the norm topology on Φ^b;

(ii) for each t the map $u \to \eta(t,u)$ is a homeomorphism of Φ^b onto $\eta(t, \Phi^b)$ with the $\mathcal{P}-$topology or with the norm topology;

(iii) $\eta(0,u) = u$ for all $u \in \Phi^b$;

(iv) $\eta(t, \Phi^c) \subset \Phi^c$ for all $c \in [a,b]$ and all $t \in [0,1]$;

(v) $\eta(1, \Phi^b) \subset \Phi^a$;

(vi) *each point $u \in \Phi^b$ has a $\mathcal{P}-$neighbourhood U in Φ^b so that the set $\{v - \eta(t,v) : v \in U, 0 \leqslant t \leqslant 1\}$ is contained in a finite-dimensional subspace of E;*

(vii) *if a finite group G acts isometrically on E and if Φ is $G-$invariant, then η is equivariant in u.*

We turn our attention now to the symmetric functional ones, in search of conditions that guarantee the existence of multiple solutions for the considered problems. Let $G \doteq \{e^{2ki\pi/p} : 0 \leqslant k < p\} \cong \mathbb{Z}/p$, p a prime number, a symmetry group that acts linearly and isometrically on X and Y, hence on $E = X \times Y$ and has no fixed points in $E\backslash\{0\}$. Suppose, additionally to (Φ_0) and (Φ_1), the following conditions:

(Φ_2) Φ is $G-$invariant;

(Φ_3) there exist $r > 0$ with $\kappa \doteq \inf \Phi(S_rY) > \Phi(0) = 0$ where $S_rY \doteq \{y \in Y : ||y|| = r\}$;

(Φ_4) there exist a finite-dimensional $G-$invariant subspace $Y_0 \subset Y$ and $R > r$ such that we have for $E_0 \doteq X \times Y_0$ and $B_0 \doteq \{u \in E_0 : ||u|| \leqslant R\}$:

$$b \doteq \sup \Phi(E_0) < \infty \quad \text{and} \quad \sup \Phi(E_0\backslash B_0) < \inf \Phi(B_rY).$$

With this conditions Bartsch and Ding established the following result.

Theorem B.4 ([8], Theorem 4.6)**.** *If Φ satisfies $(\Phi_0), (\Phi_2)-(\Phi_4)$ and the $(Ce)_c-$condition for $c \in [\kappa, b]$, then it has at least $n \doteq dim(Y_0)$ $G-$orbits of critical points.*

A special case of this theorem is presented by the same authors in [9], Theorem 5.2, considering the antipodal action:

Theorem B.5. *If Φ is even, satisfies $(\Phi_0), (\Phi_3), (\Phi_4)$ and the $(Ce)_c-$condition for all $c \in [\kappa, b]$, then it has at least $n \doteq dim(Y_0)$ pairs of critical points.*

In order to obtain infinitely many critical points we need replace the hypothesis (Φ_4) by

(Φ_5) there exists an increasing sequence of finite-dimensional $G-$invariant subspaces $Y_n \subset Y$ and there exist $R_n > r$ such that we have for $B_n \doteq \{u \in X \times Y_n : ||u|| \leqslant R_n\}$:

$$\sup \Phi(X \times Y_n) < \infty \quad \text{and} \quad \sup \Phi(X \times Y_n\backslash B_n) < \beta \doteq \inf \Phi(\{u \in Y : ||u|| \leqslant r\}),$$

where $r > 0$ is from (Φ_3).

Also, suppose the additionally condition, which is a replacement of the Palais-Smale condition that is established in [25].

(Φ_6) One of the following holds:

(i) for any interval $I \subset (0,\infty)$ there is a $(Ce)_I-$attractor $\mathcal{A}$ with $P^+\mathcal{A}$ bounded and $\inf\{||P_Y(u-v)|| : u, v \in \mathcal{A}, P_Y(u-v) \neq 0\} > 0$;

(ii) Φ satisfies the $(Ce)_c-$condition for $c > 0$.

These hypotheses are sufficient to state the following result, whose proof are based on [8], Theorem 4.8:

Theorem B.6 ([9], Theorem 5.3). *Assume Φ is even with $\Phi(0) = 0$ and let (Φ_0), (Φ_1), (Φ_3), (Φ_5) and (Φ_6) be satisfied. Then Φ possesses an unbounded sequence of positive critical values.*

Also, we can mention a particular version of this multiplicity result.

Theorem B.7 ([72], Theorem 4.3). *Assume that Φ is even with $\Phi(0) = 0$ and Φ satisfies (Φ_0), (Φ_1), (Φ_3), (Φ_5). If Φ satisfies the $(Ce)_c-$condition for $c > 0$ be satisfied, hence Φ has an unbounded sequence of critical values.*

Still related to critical point theory we mention the following result due to Benci and Rabinowitz.

Theorem B.8 ([29], Theorem 2.5; [55], Theorem 5.29). *Let E be a real Hilbert space with $E = F_1 + F_2$ and $F_2 = F_1^{\perp}$. Suppose $I \in C^1(E,\mathbb{R})$, satisfies the Cerami condition $(Ce)_c$ for any $c \in \mathbb{R}$ and*

(I_1) *$I(u) = \frac{1}{2}(Lu,u) + \varphi(u)$, where $Lu = L_1P_1u + L_2P_2u$ and $L_i : F_i \to F_i$ is bounded and self-adjoint, $i = 1, 2$.*

(I_2) *φ' is compact, and*

(I_3) *there exist a subspace $\widehat{E} \subset E$ and sets $S \subset F_1$, $\Omega \subset \widehat{E}$ and constants $\rho > \omega$ such that*

(i) *$\Phi(z) \geqslant \rho$, for all $z \in S$;*

(ii) *Ω is bounded and $\Phi(z) \leqslant \omega$ for all $z \in \partial\Omega$;*

(iii) *S and $\partial\Omega$ link.*

Then Φ possesses a critical value $c \geqslant \rho$, with

$$c = \inf_{h\in\Gamma} \sup_{u\in\Omega} \Phi(h(1,u)), \tag{B.8}$$

where

$$\Gamma \doteq \{h \in C([0,1] \times E, E) : h \quad \textit{satisfies} \quad (\Gamma_1) - (\Gamma_3)\}$$

here

(Γ_1) $h(0, u) = u$,

(Γ_2) $h(t, u) = u$ *for* $u \in \partial\Omega$,

(Γ_3) $h(t, u) = e^{\theta(t,u)L}u + K(t, u)$, *where* $\theta \in C([0, 1] \times E, \mathbb{R})$ *and* K *is compact.*

APPENDIX C

Abstract results and some mathematical notations

In this chapter, we present definitions and results that are important and which will be used in the development of this work. We start with some definitions.

Definition C.1. *A linear operator $T : X \to Y$ between normed spaces X and Y is called a compact linear operator if for every bounded sequence (x_n) in X, the sequence $(T(x_n)$ has a convergent subsequence.*

Definition C.2. *The sequence $(v_n) \subset E$ is called a Palais-Smale sequence, or a $(PS)_c-$ sequence for $c \in \mathbb{R}$, if*

$$\Phi(v_n) \to c \quad and \quad \Phi^{'}(v_n) \to 0 \ in \ E^* \quad as \quad n \to \infty.$$

We say that Φ satisfies Palais-Smale condition at level c, or $(PS)_c-$condition, if any $(PS)_c-$ sequence for Φ has a convergent subsequence.

Definition C.3. *The sequence $(v_n) \subset E$ is called a Cerami sequence, or a $(Ce)_c-$sequence for $c \in \mathbb{R}$, if*

$$\Phi(v_n) \to c \quad and \quad (1 + ||v_n||)\Phi^{'}(v_n) \to 0 \ in \ E^* \quad as \quad n \to \infty.$$

We say that Φ satisfies Cerami's condition at level c, or $(Ce)_c-$condition, if any $(Ce)_c-$ sequence for Φ has a convergent subsequence.

Sometimes, in the Chapter 1, for convenience, we consider a real function $L(x)$ as a symmetric matrix $L(x)I_4$, where I_4 denotes the 4×4 identity matrix. Moreover, for two given symmetric 4×4 real matrix functions $L_1(x)$ and $L_2(x)$, we rewrite that $L_1(x) \leqslant L_2(x)$ if and only if

$$\max_{\xi \in \mathbb{C}^4, |\xi|=1} \big(L_1(x) - L_2(x)\big)\xi \cdot \overline{\xi} \leqslant 0.$$

Now we present some abstract results. Firstly, we mention the known results about convergence whose proof can be found in [58], Lemma 1.28 and Theorem 1.34, respectively.

Lemma C.1 (Fatou's Lemma). *If $f_n : X \to [0, \infty]$ is mensurable, for each integer n, then*

$$\int_X \left(\liminf_{n\to\infty} f_n \right) dx \leqslant \liminf_{n\to\infty} \int_X f_n \, dx.$$

Theorem C.1 (Lebesgue's Dominated Convergence Theorem). *Suppose $\{f_n\}$ is a sequence of complex measurable functions on X such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in X$. If there is a function $g \in L^1(X)$ such that $|f_n(x)| \leqslant g(x)$ $(n = 1, 2, 3...; x \in X)$, then $f \in L^1(X)$,*

$$\lim_{n\to\infty} \int_X |f_n - f| dx = 0,$$

and

$$\lim_{n\to\infty} \int_X f_n dx = \int_X f dx.$$

The following result about the Nemytskii operators was demonstrated by Figueiredo [21], Theorem 2.3.

Theorem C.2 (Nemytskii operator continuity). *Suppose that there is a constant $c > 0$, a function $b(x) \in L^q(\Omega)$, $1 \leqslant q \leqslant \infty$ and $r > 0$ such that*

$$|f(x, s)| \leqslant c|s|^r + b(x), \qquad \forall x \in \Omega, \quad \forall s \in \mathbb{R}. \tag{C.1}$$

Then

(a) N_f maps L^{qr} into L^q, where $N_f(v)(y) = f(y, v(y))$;

(b) N_f is continuous and bounded (that is, it maps bounded sets into bounded sets).

We also state a result about the weak convergence in L^α that was proved by Kavian [46], Lemma 4.8.

Lemma C.2. *Let $\Omega \subset \mathbb{R}^N$ an open domain and (g_n) a bounded sequence in $L^\alpha(\Omega)$, for some $1 < \alpha < \infty$, such that $g_n \to g$ q.t.p in Ω. Then $g \in L^\alpha(\Omega)$ and $g_n \rightharpoonup g$ weakly in $L^\alpha(\Omega)$.*

We also use some inequalities that will be mention before.

Theorem C.3 ([66], Lemma A.1).

(i) If $p \in [2, \infty)$ then:

$$\Big||z|^{p-2}z - |y|^{p-2}y\Big| \leqslant \beta|z-y|\Big(|z|+|y|\Big)^{p-2} \quad \forall y, z \in \mathbb{R}^N \text{ and } \beta \in \mathbb{R}. \tag{C.2}$$

(ii) If $p \in (1, 2]$ then:

$$\Big||z|^{p-2}z - |y|^{p-2}y\Big| \leqslant \beta|z-y|^{p-1} \quad \forall y, z \in \mathbb{R}^N \text{ and } \beta \in \mathbb{R}. \tag{C.3}$$

Remark C.1. *At the proof of these estimates its is possible conclude that $\beta > 0$ as $y, z \neq 0$ and, moreover, if $r \in (0, \infty)$*

$$\Big||a|^r - |b|^r\Big| \leqslant |a-b|^r \quad \forall a, b \in \mathbb{R}^N. \tag{C.4}$$

By identifying the space of the complex numbers $\mathbb{C}$ with the real vector space of dimension 2 we obtain that such conditions remain valid for complex vectors.

The following are two well-known results of Functional Analysis, which can be found in Brezis, [15], Theorem 2.2 and Theorem 2.9, respectively.

Theorem C.4 (Banach-Steinhaus, uniform boundedness principle). *Let E and F be two Banach spaces and let (T_i), $i \in I$, be a family (not necessarily countable) of continuous linear operators from E into F. Assume that*

$$\sup_{i \in I} ||T_i x|| < \infty \quad \forall x \in E.$$

Then

$$\sup_{i \in I} ||T_i||_{\mathcal{L}(E,F)} < \infty.$$

In other words, there exists a constant c such that

$$||T_i x|| \leqslant c||x|| \quad \forall x \in E, \forall i \in I.$$

Here, the norm on the space $\mathcal{L}(E, F)$ of continuous (=bounded) linear operators from E into F is defined as

$$||T||_{\mathcal{L}(E,F)} = \sup_{x \in E, ||x|| \leqslant 1} ||Tx||.$$

Theorem C.5 (Closed Graph Theorem). *Let E and F be two Banach spaces. Let T be a linear operator from E into F. Assume that the graph of T, $G(T)$, is closed in $E \times F$. Then T is continuous.*

We also mention the following result that is useful to stablish relations between inner product of some particular operators.

Proposition C.1 ([37], Proposition III 8.11)**.** *Let T be selfadjoint and nonnegative and let S be symmetric with $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $||Sf|| \leqslant ||Tf||$ for all $f \in \mathcal{D}(T)$. Then*

$$(Sf, f) \leqslant (Tf, f)$$

for all $f \in \mathcal{D}(T)$.

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