Superfícies Mínimas e a Teoria Min-Max de Almgren–Pitts

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São Carlos, 13 de Junho de 2019.

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Disssertação de Mestrado

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Resumo

Primeiro, apresentamos o conceito básico de superfícies mínimas e desenvolvemos alguns resultados na teoria geral de superfícies mínimas.

Na segunda parte, estamos interessados na abordagem Min-Max Simon–Smith para provar a existência de superfícies mínimas em 3-variedades riemannianas compactas (COLDING; DE LELLIS, 2003). Isso é feito usando o conceito de varifolds, que é estudado em Teoria Geométrica da Medida.

Na terceira parte, consideramos superfícies mínimas min-max em 3-variedades e provamos alguns resultados de rigidez sob a hipótese de curvaturas escalar e de Ricci positivas (MARQUES; NEVES, 2012). Uma ferramenta importante aqui é o chamado fluxo de Ricci.

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Part I

Preliminaries

1 Minimal Surfaces

1.1 First variation formula

Let (M, g) be a riemannian 3-manifold and $\Sigma \subset M$ a compact surface possibly with boundary (all we do in this section can be easily adapted to a submanifold Σ^k of a manifold M^n). Consider (x_1, x_2) local coordinates on Σ given by a parametrization $\boldsymbol{x} : U \to \Sigma$ and let

$$g_{ij}(x) = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), \text{ for } 1 \leq i, j \leq 2$$

be the components of $g|_\Sigma$. Since the matrix $g_{ij}(x)$ is symmetric, positive definite and non-degenerated, we have $\det g_{ij}(x) > 0$ for all $x \in U$. We define the area of $R = \boldsymbol{x}(U)$ as

$$|R| = \int_R d\Sigma := \iint_U \sqrt{\det g_{ij}(x)}\, dx_1 dx_2,$$

where $\iint dx_1 dx_2$ is just the Riemann integral on $\mathbb{R}^2$. Using the change of variables theorem, one can show that the area of R is well-defined, i.e. it does not depend on the parametrization $\boldsymbol{x} : U \to R$. Then, covering Σ by parametrizations and using a partition of unity in the usual way, we can define the area of Σ:

$$|\Sigma| = \int_\Sigma d\Sigma.$$

This defines the area of Σ even if it is not orientable. If Σ is orientable, we can also look to $d\Sigma = \sqrt{\det g_{ij}(x)}$ as a differentiable 2-form on Σ.

A (smooth) variation of Σ is a smooth map $F : \Sigma \times (-\epsilon, \epsilon) \to M$ such that each $F_t := F(t, \cdot) : \Sigma \to M$ is an embedding and $F_0(x) = \mathrm{Id}_\Sigma : \Sigma \to \Sigma$.

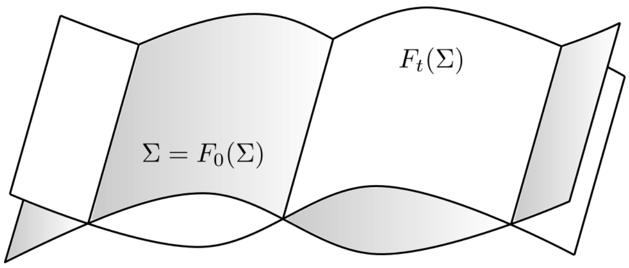


Figure 1

We denote $\Sigma_t = F_t(\Sigma)$ and we are interested in the derivative of the function $f(t) = |\Sigma_t|$. Of course, we are considering the case in which $|\Sigma| < +\infty$ (and hence $|\Sigma_t| < +\infty$).

DEFINITION 1.1 (DIVERGENCE)**.** *Let X be an arbitrary vector field on $\Sigma \subset M$ (not necessarily tangent). We define the **divergence** as*

$$\operatorname{div}_\Sigma X(p) = \sum_{i=1}^{2} g(\nabla_{e_i} X, e_i) \tag{1.1}$$

where $\{e_1, e_2\}$ is an orthonormal basis for $T_p\Sigma$ and ∇ is the Levi-Civita connection with respect to the riemannian metric g.

THEOREM 1.2 (FIRST VARIATION FORMULA I)**.**

$$\frac{d}{dt}|\Sigma_t| = \int_{\Sigma_t} \operatorname{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) d\Sigma_t \tag{1.2}$$

Proof. Let $\boldsymbol{x} : U \to \Sigma$ be a parametrization of $R = \boldsymbol{x}(U)$. Then $\boldsymbol{x}_t := F_t \circ \boldsymbol{x} : U \to \Sigma_t$ is a parametrization of $R_t = F_t(R)$. We have

$$|R_t| = \iint_U \sqrt{\det g^t_{ij}(x)}\, dx_1 dx_2,$$

where $g^t_{ij}(x) = g(\frac{\partial}{\partial x^t_i}, \frac{\partial}{\partial x^t_j})$ and $\frac{\partial}{\partial x^t_i}$ are the coordinate vectors of $\boldsymbol{x}_t$. Thus $\frac{\partial}{\partial x^t_i} = (F_t)_* \frac{\partial}{\partial x_i}$ and we use the notation $\partial_i F_t$ for it. Note that

$$\frac{\partial}{\partial t} \det g_t = \operatorname{tr}(g_t^{-1} \partial_t g_t) \det g_t,$$

where $g_t^{-1} = (g_t^{ij}) = (g^t_{ij})^{-1}$. Then

$$\frac{\partial}{\partial t} \det g_t = \sum_{i,j} (g_t^{ij} \partial_t g^t_{ij}) \det g_t.$$

We can compute $\partial_t g^t_{ij}$ using the compatibility of ∇ with the metric g

$$\partial_t g^t_{ij} = \partial_t g(\partial_i F_t, \partial_j F_t) = g(\nabla_{\partial_t F} \partial_i F_t, \partial_j F_t) + g(\partial_i F_t, \nabla_{\partial_t F} \partial_j F_t).$$

Now, since $[\partial_t F, \partial_i F_t] = 0$, by symmetry we have $\nabla_{\partial_t F} \partial_i F_t = \nabla_{\partial_i F_t} \partial_t F$. Putting this together, we have

$$\frac{\partial}{\partial t} \det g_t = 2 \sum_{i,j} g_t^{ij} g(\nabla_{\partial_i F_t} \partial_t F, \partial_j F_t) \det g_t$$

We can change the coordinates on U such that, at the point $x \in U$, $\{\partial_1 F_t, \partial_2 F\}$ is an orthonormal basis of $T_{F_t(x)}\Sigma_t$. In this coordinate system, at the point x, $(g^t_{ij}) = (g_t^{ij}) = \mathbb{I}$. Thus

$$\begin{aligned}\frac{\partial}{\partial t} \det g_t &= 2 \sum_{i=1}^{2} g(\nabla_{\partial_i F_t} \partial_t F, \partial_i F_t) \\ &= 2 \operatorname{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) \det g_t.\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}|R_t| = \frac{\partial}{\partial t}\iint_U \sqrt{\det g_{ij}^t}\, dx_1 dx_2 = \iint_U \operatorname{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right)\sqrt{\det g_{ij}^t}\, dx_1 dx_2$$
$$= \int_{R_t} \operatorname{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) d\Sigma_t.$$

We cover Σ by parametrizations and take a partition of unity subordinated to the collection of their open domains. Using $F_t : \Sigma \to \Sigma_t$, this yields a partition of unity of Σ_t which is essentially "the same", i.e. it does not depend on t: if φ is a function from the partition, then $\varphi(F(x,t)) = \varphi(x)$. Then, summing up everything gives

$$\frac{d}{dt}|\Sigma_t| = \int_{\Sigma_t} \operatorname{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) d\Sigma_t.$$

□

We want to study surfaces that are critical points for area. We need to to introduce a important geometric concept.

DEFINITION 1.3. *For each $p \in \Sigma$, define the* ***second fundamental form*** *of $\Sigma \subset M$ as*

$$B(X,Y) = \nabla_X Y - (\nabla_X Y)^T = (\nabla_X Y)^N,$$

where X, Y are vector fields tangent to Σ. B is a symmetric tensor. If $\{e_1, e_2\}$ is an orthonormal basis for $T_p\Sigma$, define the ***mean curvature vector*** *as*

$$\vec{H} = \operatorname{tr} B = \sum_{i=1}^{2} (\nabla_{e_i} e_i)^N.$$

LEMMA 1.4. $\operatorname{div}_\Sigma X = \operatorname{div}_\Sigma X^T - g(X^N, \vec{H})$.

Proof. Write X in its tangent and normal components $X = X^T + X^N$. We have

$$\operatorname{div}_\Sigma X = \operatorname{div}_\Sigma X^T + \sum_{i=1}^{2} g(\nabla_{e_i} X^N, e_i).$$

Since X^N is normal and e_i is tangent to Σ,

$$0 = e_i g(X^N, e_i) = g(\nabla_{e_i} X^N, e_i) + g(X^N, \nabla_{e_i} e_i).$$

Thus,

$$\operatorname{div}_\Sigma X = \operatorname{div}_\Sigma X^T - \sum_{i=1}^{2} g(X^N, \nabla_{e_i} e_i) = \operatorname{div}_\Sigma X^T - g(X^N, \sum_{i=1}^{2} \nabla_{e_i} e_i)$$
$$= \operatorname{div}_\Sigma X^T - g(X^N, \sum_{i=1}^{2} (\nabla_{e_i} e_i)^N) = \operatorname{div}_\Sigma X^T - g(X^N, \vec{H}).$$

□

For the first variation formula II, we will use the divergence theorem on Σ.

THEOREM 1.5 (DIVERGENCE THEOREM)**.** *Suppose $\Sigma \subset M$ is compact possibly with boundary. If X is a vector field tangent to Σ, $\nu \in T_p\Sigma$ is the outward unit vector field normal to $\partial\Sigma$ and $d\sigma$ is the length element of $\partial\Sigma$, then*

$$\int_\Sigma \operatorname{div}_\Sigma X \, d\Sigma = \int_{\partial\Sigma} g(X, \nu) \, d\sigma.$$

If $\partial\Sigma = \varnothing$, then the integral is zero.

THEOREM 1.6 (FIRST VARIATION FORMULA II)**.**

$$\frac{d}{dt}|\Sigma_t| = -\int_{\Sigma_t} g(\frac{\partial F}{\partial t}, \vec{H}_t) \, d\Sigma_t + \int_{\partial\Sigma_t} g(\frac{\partial F}{\partial t}, \nu_t) \, d\sigma_t.$$

In particular, if $X = \frac{\partial F}{\partial t}$ vanishes on $\partial\Sigma$ at $t = 0$, then

$$\left.\frac{d}{dt}\right|_{t=0} |\Sigma_t| = -\int_\Sigma g(X, \vec{H}) \, d\Sigma.$$

Proof. We just need to apply Lemma 1.4 and the Divergence Theorem to the first variation formula I:

$$\begin{aligned}\frac{d}{dt}|\Sigma_t| &= \int_{\Sigma_t} \operatorname{div}_{\Sigma_t} \frac{\partial F}{\partial t} \, d\Sigma_t = \int_{\Sigma_t} \operatorname{div}_{\Sigma_t} \frac{\partial F}{\partial t}^T \, d\Sigma_t - \int_{\Sigma_t} g(\frac{\partial F}{\partial t}^N, \vec{H}_t) \, d\Sigma_t \\ &= -\int_{\Sigma_t} g(\frac{\partial F}{\partial t}, \vec{H}_t) \, d\Sigma_t + \int_{\partial\Sigma_t} g(\frac{\partial F}{\partial t}, \nu_t) \, d\sigma_t.\end{aligned}$$

□

This formula leads to an important corollary and definition of the main object of our study.

COROLLARY 1.7. $\left.\frac{d}{dt}\right|_{t=0} |\Sigma_t| = 0$ *for any $X = 0$ with $X = 0$ on $\partial\Sigma$ if and only if $\vec{H} = 0$.*

DEFINITION 1.8. *$\Sigma \subset M$ is said to be a **minimal surface** if $\vec{H} = 0$.*

REMARK 1.9.

1. *Definition 1.8 makes sense even if Σ has infinite area.*
2. *Suppose $\vec{H} \neq 0$ somewhere in the interior of Σ. Take a positive function $f : \Sigma \to \mathbb{R}$ with $f = 0$ on $\partial\Sigma$ and $f > 0$ in the same point of Σ in which $\vec{H} \neq 0$. Then $X = f\vec{H}$ is zero on $\partial\Sigma$ and $\left.\frac{d}{dt}\right|_0 |\Sigma_t| = -\int_\Sigma g(f\vec{H}, \vec{H}) \, d\Sigma = -\int_\Sigma f g(\vec{H}, \vec{H}) \, d\Sigma < 0$. This shows that $|\Sigma_t|$ decreases when we vary Σ by the mean curvature vector field.*
3. *Consider that $X = \frac{\partial F}{\partial t} = 0$ on ∂M at $t = 0$. Decompose $X = X^N + X^T$ in its normal and tangent components. By the first variation formula II,*

$$\left.\frac{d}{dt}\right|_0 |\Sigma_t| = -\int_\Sigma g(X, \vec{H}) \, d\Sigma = -\int_\Sigma g(X^N, \vec{H}) + g(X^T, \vec{H}) \, d\Sigma = -\int_\Sigma g(X^N, \vec{H}) \, d\Sigma.$$

Thus $\left.\frac{d}{dt}\right|_0 |\Sigma_t|$ depends only on the normal component of the vector field along Σ given by $X = \frac{\partial F}{\partial t}$. So instead of working with variations given in the form $F : (-\epsilon, \epsilon) \times \Sigma \to M$ with $\frac{\partial F}{\partial t}(0, \cdot) \equiv 0$ on $\partial\Sigma$ we can work simply with the normal vectors fields on Σ with $X = 0$ on the boundary.

1.2 EXAMPLES

EXAMPLE 1.10 (GEODESICS)**.** Let us look to smooth regular curves $\gamma : I \to M$ instead of surfaces. We have $g(\nabla_{\frac{\gamma'}{|\gamma'|}}\frac{\gamma'}{|\gamma'|}, \frac{\gamma'}{|\gamma'|}) = \frac{\gamma'}{|\gamma'|}g(\frac{\gamma'}{|\gamma'|}, \frac{\gamma'}{|\gamma'|}) = \frac{\gamma'}{|\gamma'|}1 = 0$. Therefore $\vec{H} = (\nabla_{\frac{\gamma'}{|\gamma'|}}\frac{\gamma'}{|\gamma'|})^N = \nabla_{\frac{\gamma'}{|\gamma'|}}\frac{\gamma'}{|\gamma'|}$. Thus γ is a geodesic if and only if $\vec{H} = 0$, i.e. geodesics can be seen as minimal "surfaces" of dimension 1.

EXAMPLE 1.11 (MINIMAL SURFACES IN $\mathbb{R}^3$)**.** We consider the case of minimal surfaces given by the graph of a smooth function $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$. Since every surface in $\mathbb{R}^3$ is locally the graph of a smooth function on one of its three coordinated planes, there is no loss of generality in doing so. Let $\varphi : \Omega \to \mathrm{graph}(u)$ be the parametrization given by $\varphi(x) = (x, u(x))$. Use the notation

$$\frac{\partial \varphi}{\partial x_i} = \left(\frac{\partial}{\partial x_i}, \frac{\partial u}{\partial x_i}\right),\ i = 1, 2.$$

The Euclidean metric of $\mathbb{R}^3$ restricted to $\mathrm{graph}(u)$ is given by

$$g_{ij} = g\left(\frac{\partial \varphi}{\partial x_i}, \frac{\partial \varphi}{\partial x_j}\right) = \delta_{ij} + \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j},\ 1 \leq i, j \leq 2,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. We have

$$\begin{aligned}\det(g_{ij}) &= \left(1 + \frac{\partial u}{\partial x_1}^2\right)\left(1 + \frac{\partial u}{\partial x_2}^2\right) - \left(\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\right)^2 = 1 + \frac{\partial u}{\partial x_1}^2 + \frac{\partial u}{\partial x_2}^2 \\ &= 1 + |\nabla u|^2.\end{aligned}$$

Thus,

$$|\mathrm{graph}(u)| = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx.$$

Consider variations of $\mathrm{graph}(u)$ given by functions $u_t = u + tv$, for any fixed function $v : \Omega \to \mathbb{R}$ with $v = 0$ on $\partial\Omega$. By integration by parts we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} |\mathrm{graph}(u_t)| &= \left.\frac{d}{dt}\right|_{t=0} \int_\Omega \sqrt{1 + |\nabla(u + tv)|^2}\, dx = \int_\Omega \left\langle \nabla v, \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right\rangle dx \\ &= -\int_\Omega v\, \mathrm{div}_{\mathbb{R}^2}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) dx + \int_{\partial\Omega} v \left\langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \nu \right\rangle d\sigma \\ &= -\int_\Omega v\, \mathrm{div}_{\mathbb{R}^2}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) dx,\end{aligned}$$

since $v = 0$ on $\partial\Omega$. Therefore $\mathrm{graph}(u)$ is a minimal surface if and only if u satisfies

$$\mathrm{div}_{\mathbb{R}^2}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0. \tag{1.3}$$

This is the minimal surface equation. It is equivalent to the second order elliptic quasi-linear p.d.e given by

$$\sum_{i,j=1}^{2}\left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1 + |\nabla u|^2}\right)\partial_i\partial_j u = 0. \tag{1.4}$$

We now present some basic examples of minimal surfaces in $\mathbb{R}^3$:

- The plane: $\{(x, y, z) \in \mathbb{R}^3 \,:\, z = 0\}$.
- The catenoid: $\{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 = \cosh z\}$. It is a bograph obtained by the revolution of the curve $y = \cosh z$ over the z-axis.

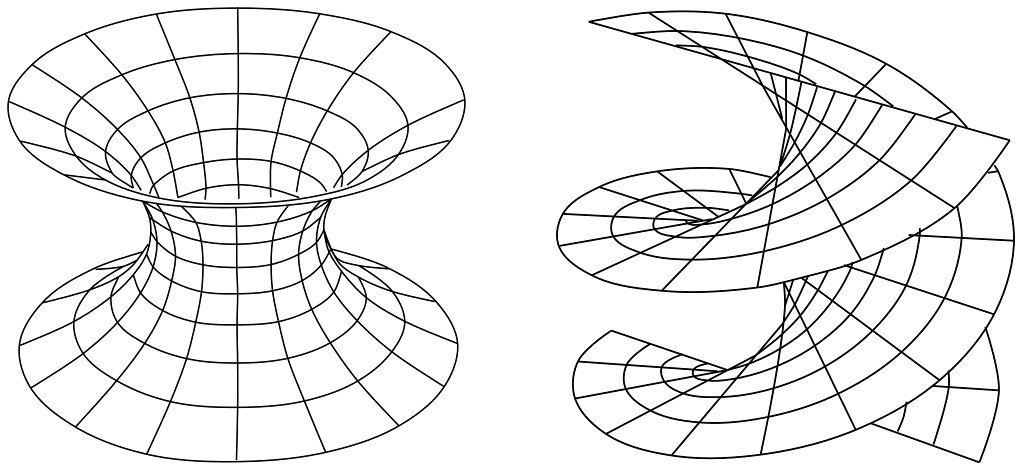


Figure 2 – The catenoid and the helicoid

- The helicoid: $(u, v) \mapsto (u \cos v, u \sin v, v)$. This is a multigraph over $\mathbb{R}^2 \backslash \{0\}$ for the function $\arctan(y/x)$.
- The Scherk's surface: $\operatorname{graph}(u)$, where $u(x, y) = \log\left(\frac{\cos x}{\cos y}\right)$ for $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Scherk's surface is a doubly periodic minimal surface.

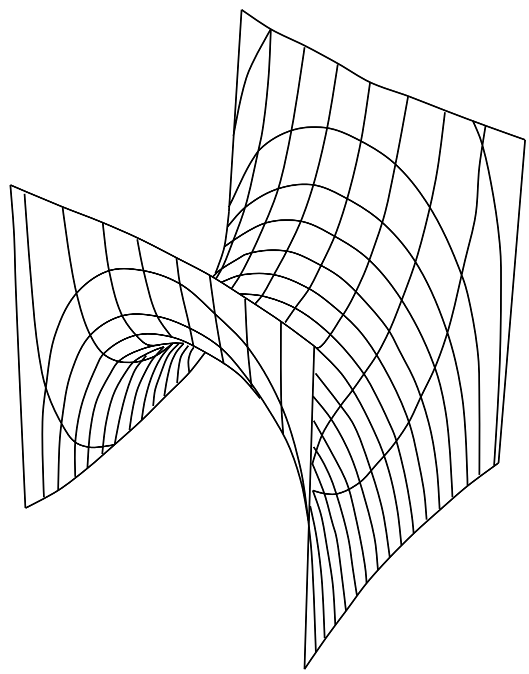


Figure 3 – Scherk's surface

These are important examples when we try to classify the minimal surfaces in $\mathbb{R}^3$. For example, we have

THEOREM 1.12 (MEEKS AND ROSENBERG, 2005)**.** *The only complete embedded simply connected minimal surfaces in $\mathbb{R}^3$ are the plane and the helicoid.*

1.3 THE MAXIMUM PRINCIPLE

Let $\Omega \subset \mathbb{R}^n$ be a domain and $L : C^\infty(\Omega) \to C^\infty(\Omega)$ be a second order differential operator

$$Lv(x) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j v(x) + \sum_{i=1}^{n} b_i(x)\partial_i v(x) + c(x)v(x), \quad x \in \Omega,$$

where $a_{ij} = a_{ji}$, b_i and c are smooth functions on Ω. We say L is ***elliptic*** if $(a_{ij}(x))_{ij}$ is positive definite, for all $x \in \Omega$. This is equivalent to say that

$$0 < \lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,\, x \in \Omega$$

where $\lambda(x)$ and $\Lambda(x)$ are the minimum and maximum eigenvalues of $(a_{ij}(x))_{ij}$ respectively. Note that if $\lambda > 0$ on Ω, then L is elliptic. If there is $\lambda_0 > 0$ such that $0 < \lambda_0 \leq \lambda(x)$ for all $x \in \Omega$, we say L is ***strictly elliptic***. In addition, if $\Lambda(x)/\lambda(x)$ is bounded on Ω, then L is called ***uniformly elliptic***. Now we introduce the strong maximum principle for uniformly elliptic operators.

THEOREM 1.13 (STRONG MAXIMUM PRINCIPLE)**.** *Let L be uniformly elliptic, $c = 0$ and $Lv \geq 0$ (≤ 0) in a domain Ω (not necessarily bounded). Then if v achieves its maximum (minimum) in the interior of Ω, v is a constant.*

Proof. This is Theorem 3.5 from (GILBARG; TRUDINGER, 2001) □

The next proposition shows that if $u_1, u_2 : \Omega \to \mathbb{R}$ are solutions for the minimal surface equation, then their difference $v = u_2 - u_1$ is a solution for an uniformly elliptic equation, in a divergent form.

PROPOSITION 1.14. *0.2Let $u_1, u_2 : \Omega \to \mathbb{R}$ be solutions for the minimal surface equation on the compact domain $\Omega \subset \mathbb{R}^2$. Then there is a map $A : \Omega \to M_{2\times 2}(\mathbb{R})$ and there is a number $\mu > 0$ such that*

1. *the eigenvalues of $A(x)$ satisfies $0 < \mu \leq \lambda_1(x) \leq \lambda_2(x) \leq \frac{1}{\mu}$, for all $x \in \Omega$;*
2. *$v = u_2 - u_1$ is a solution to $\operatorname{div}(A(x)\nabla v(x)) = 0$.*

Proof. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ as $F(x) = \frac{x}{(1+|x|^2)^{1/2}}$. Then $\operatorname{div}(F(\nabla u)) = 0$ is just the minimal surface equation. By the fundamental theorem of calculus and the chain rule, we get

$$F(\nabla u_2) - F(\nabla u_1) = \int_0^1 \frac{d}{dt} F(\nabla u_1 + t\nabla(u_2 - u_1))\, dt$$

$$
\begin{aligned}
&= \int_0^1 dF_{\nabla u_1 + t\nabla(u_2-u_1)} \cdot \nabla(u_2-u_1)\,dt \\
&= \left(\int_0^1 dF_{\nabla u_1 + t\nabla(u_2-u_1)}\,dt\right) \cdot \nabla(u_2-u_1).
\end{aligned}
$$

Thus if we define

$$
A(x) = \int_0^1 dF_{\nabla u_1(x) + t\nabla v(x)}\,dt
$$

we have $\operatorname{div}(A\nabla v) = \operatorname{div}(F(\nabla u_2) - F(\nabla u_2)) = \operatorname{div}(F(\nabla u_2)) - \operatorname{div}(F(\nabla u_1)) = 0$.

Now, we show that $A = A(x)$ is positive definite for all $x \in \Omega$, i.e. $\langle w, Aw\rangle > 0$, for all $w \in \mathbb{R}^2 \backslash \{0\}$. Before we prove this, let $w \in \mathbb{R}^2$ with $|w| = 1$ and $y \in \mathbb{R}^2$. We have

$$
\begin{aligned}
dF_y \cdot w &= \left.\frac{d}{dt}\right|_{t=0} F(y+tw) = \left.\frac{d}{dt}\right|_{t=0} \frac{(y+tw)}{(1+|y+tw|^2)^{1/2}} \\
&= \left.\frac{w(1+|y+tw|^2)^{1/2} - (y+tw)\frac{1}{2}(1+|y+tw|^2)^{-1/2}2\langle y+tw, w\rangle}{1+|y+tw|^2}\right|_{t=0} \\
&= \frac{w}{(1+|y|^2)^{1/2}} - \frac{\langle y, w\rangle y}{(1+|y|^2)^{3/2}}.
\end{aligned}
$$

By Cauchy-Schwarz we have $\langle y, w\rangle^2 \le |y|^2|w|^2 = |y|^2$. Then

$$
\begin{aligned}
\langle w, dF_y w\rangle &= \frac{1}{(1+|y|^2)^{1/2}} - \frac{\langle y, w\rangle^2}{(1+|y|^2)^{3/2}} \ge \frac{1}{(1+|y|^2)^{1/2}} - \frac{|y|^2}{(1+|y|^2)^{3/2}} \\
&= \frac{1+|y|^2-|y|^2}{(1+|y|^2)^{3/2}} = \frac{1}{(1+|y|^2)^{3/2}} > 0.
\end{aligned}
$$

This shows the matrix dF_y is positive definite, for all $y \in \mathbb{R}^2$. Thus, A is a weighted average of positive definite matrices and hence it is also positive definite. Therefore A has positive eigenvalues.

Now, if $\{e_1, e_2\}$ is the canonical basis for $\mathbb{R}^2$, we also have

$$
a_{ij} := \langle e_i, dF_y e_j\rangle = \frac{\delta_{ij}}{(1+|y|^2)^{1/2}} - \frac{\langle y, e_i\rangle\langle y, e_j\rangle}{(1+|y|^2)^{3/2}}.
$$

Since $(a_{ij})_{ij}$ is the matrix of dF_y in the canonical basis and $a_{ij} = a_{ji}$, this shows that dF_y is symetric, for all $y \in \mathbb{R}^2$. Again, $A(x)$ inherits this property. Thus we can use the Rayleigh quotient method for eigenvalues. We know in particular that

$$
\begin{aligned}
\lambda_1(x) &= \inf_{w \in \mathbb{R}^2\backslash\{0\}} \frac{\langle A(x)w, w\rangle}{\langle w, w\rangle} = \inf_{|w|=1} \langle A(x)w, w\rangle, \\
\lambda_2(x) &= \sup_{w \in \mathbb{R}^2\backslash\{0\}} \frac{\langle A(x)w, w\rangle}{\langle w, w\rangle} = \sup_{|w|=1} \langle A(x)w, w\rangle.
\end{aligned}
$$

Let $M = \sup_{x\in\Omega} |\nabla u_1(x)|$ and $N = \sup_{x\in\Omega} |\nabla v(x)|$. If $|w| = 1$, we have

$$
\begin{aligned}
\langle A(x)w, w\rangle &= \left\langle \left(\int_0^1 dF_{\nabla u_1(x)+t\nabla v(x)}\,dt\right) w,\ w \right\rangle = \int_0^1 \langle dF_{\nabla u_1(x)+t\nabla v(x)} \cdot w, w\rangle\,dt \\
&\ge \int_0^1 \frac{1}{(1+|\nabla u_1(x) + t\nabla v(x)|^2)^{3/2}}\,dt
\end{aligned}
$$

$$\geq \int_0^1 \frac{1}{(1+(|\nabla u_1(x)|+t|\nabla v(x)|)^2)^{3/2}}\,dt$$
$$\geq \int_0^1 \frac{1}{(1+(M+tN)^2)^{3/2}}\,dt := \mu_1 > 0$$

Thus $0 < \mu_1 \leq \lambda_1(x)$, for all $x \in \Omega$. On the other hand,

$$\langle A(x)w, w\rangle = \int_0^1 \langle dF_{\nabla u_1(x)+t\nabla v(x)} \cdot w, w\rangle\,dt \leq \int_0^1 1\,dt = 1.$$

Thus $\lambda_2(x) \leq 1$. Putting $\mu = \min\{1, \mu_1\}$, we have $0 < \mu \leq \lambda_1(x) \leq \lambda_2(x) \leq \frac{1}{\mu}$. □

REMARK 1.15. *If $A(x) = (a_{ij}(x))_{ij}$ and $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, observe that*

$$Lv(x) := \operatorname{div}(A(x)\nabla v(x)) = \sum_{i,j=1}^n a_{ij}(x)\,\partial_i\partial_j v(x) + \sum_{j=1}^n \left(\sum_{i=1}^n \partial_i a_{ij}(x)\right)\partial_j v(x).$$

So Proposition 1.14 is precisely saying that $v = u_2 - u_1$ is a solution for $Lv = 0$, for an uniformly elliptic operator, namely $L = \operatorname{div}(A\nabla)$. As a consequence, we have the maximum principle for minimal surfaces.

THEOREM 1.16 (STRONG MAXIMUM PRINCIPLE FOR MINIMAL SURFACES)**.** *Let Σ_1 and Σ_2 be complete connected minimal surfaces in $\mathbb{R}^3$. If Σ_1 lies in one side of Σ_2 and $\Sigma_1 \cap \Sigma_2 \neq \varnothing$, then $\Sigma_1 = \Sigma_2$.*

Proof. Let $u_1, u_2 : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be solutions for the minimal surface equation with $u_1 \leq u_2$ and suppose there is some $p \in \Omega$ such that $u_1(p) = u_2(p)$. By the previous proposition, we can apply the maximum principle to $v = u_2 - u_1$. Since $v \geq 0$ and $v(p) = 0$, p is a minimum of v. Therefore, v must be constant and $u_1 = u_2$. Since every surface is locally a graph of a function, this proves the theorem. □

1.4 SECOND VARIATION FORMULA

In this section, M will denote a compact orientable riemannian 3-manifold without boundary. Let $\Sigma \subset M$ be a connected embedded compact surface. If Σ is minimal ($\frac{d}{dt}\big|_0 |\Sigma_t| = 0$ for any variation Σ_t) and we want to know if it is a local minimum of area for a given variation Σ_t, we need to study $\frac{d^2}{dt^2}\Big|_0 |\Sigma_t|$. Then the Jacobi operator comes into play.

By Remark 1.9 (iii), in order to compute $\frac{d}{dt}\big|_0 |\Sigma_t|$ for variations of Σ with fixed boundary we just need to consider the normal vector fields X along Σ with $X = 0$ on the boundary $\partial\Sigma$. We call these vector fields ***admissible***. If X is one such vector field and Σ_t is the associated smooth variation, we denote

$$[\delta\Sigma](X) = \frac{d}{dt}\bigg|_0 |\Sigma_t|, \quad [\delta^2\Sigma](X,X) = \frac{d^2}{dt^2}\bigg|_0 |\Sigma_t|.$$

Now we look more closely at these vector fields.

We divide the situation in two cases: Σ is orientable or non-orientable. Since the ambient manifold M is orientable, the orientability of Σ is equivalent to two-sidedness. A surface $\Sigma \subset M$ is said to be two-sided on M if there is a section $\nu : \Sigma \to N\Sigma$ with $\nu \neq 0$ everywhere on Σ (here $N\Sigma$ is the Σ normal bundle). Otherwise Σ is said to be one-sided. Note that two-sidedness is an extrinsic concept, while orientability is intrinsic to the surface Σ. But in the case that M is orientable, these concepts coincide.

Suppose first that Σ is orientable. Then we can consider a normal unitary vector field ν on Σ. Since Σ is a surface and M is 3-dimensional, each fiber on $N\Sigma$ is 1-dimensional. Thus any admissible vector field X on Σ can be written in the form $X = \phi\nu$ for some smooth function $\phi : \Sigma \to \mathbb{R}$ with $\phi = 0$ on $\partial\Sigma$. These will be the admissible functions on Σ in the case Σ is orientable.

If Σ is non-orientable we introduce a new surface $\tilde{\Sigma}$ which is orientable and can be used to study variations of Σ. This is the so called orientable double cover of Σ (see Appendix A for more details). In this case, if Σ_t is given by a smooth variation of Σ, then we can associate a smooth variation $\tilde{\Sigma}_t$ of the orientable double cover $\tilde{\Sigma}$ such that

$$|\Sigma_t| = \frac{1}{2}|\tilde{\Sigma}_t|.$$

Thus, in order to compute the derivatives of $|\Sigma_t|$, we can always suppose that Σ is orientable. However, we still need to say which will be the admissible functions on $\tilde{\Sigma}$, because not every function on it comes from a smooth variation of Σ (see Appendix A). The admissible functions in this case are those smooth functions $\phi : \tilde{\Sigma} \to \mathbb{R}$ such that $\phi = 0$ on $\partial\tilde{\Sigma}$ and $\phi = -\phi \circ \tau$, where $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ is a certain isometry involution of $\tilde{\Sigma}$ such that $\tilde{\Sigma}/\{\mathbb{1}, \tau\} = \Sigma$.

THEOREM 1.17 (SECOND VARIATION FORMULA). *Let $F : (-\epsilon, \epsilon) \times \Sigma \to M$ be a smooth variation of Σ and $\{e_1, e_2\}$ a orthonormal bases in $T\Sigma$. If $|\nabla^\perp X|^2$ denotes $\sum_{i=1}^2 |(\nabla_{e_i} X)^\perp|^2$, then*

$$\left.\frac{d^2}{dt^2}\right|_0 |\Sigma_t| = \int_\Sigma \Bigg(\sum_{i=1}^2 R^M(X, e_i, X, e_i) + \operatorname{div}_\Sigma(\nabla_X X) + |\nabla^\perp X|^2 - \sum_{i,j=1}^2 g(\nabla_{e_i} X, e_j) g(\nabla_{e_j} X, e_i) + (\operatorname{div}_\Sigma X)^2 \Bigg)\, d\Sigma.$$

Proof. Using the same notation as in Theorem 1.2, recall that

$$\frac{d}{dt}\, d\Sigma_t = \operatorname{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) d\Sigma_t = \left(\sum_{i,j=1}^2 g^{ij} g(\nabla_{\partial_i F}\partial_t F, \partial_j F)\right) d\Sigma_t.$$

We want to compute $\frac{d^2}{dt^2} d\Sigma_t$. Before doing that, we do some remarks.

Observe first that $\partial_t g^{-1} = -g^{-1}(\partial_t g)g^{-1}$, hence $\partial_t g^{ij} = -\sum_{k,l=1}^2 g^{ik}(\partial_t g_{kl})g^{lj}$ and we have

$$\partial_t g_{kl} = \partial_t g(\partial_k F, \partial_l F) = g(\nabla_{\partial_k F}\partial_t F, \partial_l F) + g(\partial_k F, \nabla_{\partial_l F}\partial_t F).$$

On the other hand, if R^M denotes the riemannian curvature tensor of M, then

$$\begin{aligned} R^M(\partial_t F, \partial_i F)X &:= \nabla_{\partial_t F}\nabla_{\partial_i F}X - \nabla_{\partial_i F}\nabla_{\partial_t F}X - \nabla_{[\partial_t F, \partial_i F]}X \\ &= \nabla_{\partial_t F}\nabla_{\partial_i F}X - \nabla_{\partial_i F}\nabla_{\partial_t F}X, \end{aligned}$$

since $[\partial_i F, \partial_t F] = 0$. Then we have

$$\begin{aligned} &\partial_t g(\nabla_{\partial_i F}\partial_t F, \partial_j F) = g(\nabla_{\partial_t F}\nabla_{\partial_i F}\partial_t F, \partial_j F) + g(\nabla_{\partial_i F}\partial_t F, \nabla_{\partial_t F}\partial_j F) \\ &= g(\nabla_{\partial_t F}\nabla_{\partial_i F}\partial_t F - \nabla_{\partial_i F}\nabla_{\partial_t F}\partial_t F, \partial_j F) + g(\nabla_{\partial_i F}\nabla_{\partial_t F}\partial_t F, \partial_j F) \\ &\quad + g(\nabla_{\partial_i F}\partial_t F, \nabla_{\partial_t F}\partial_j F) \\ &= g(R^M(\partial_t F, \partial_i F)\partial_t F, \partial_j F) + g(\nabla_{\partial_i F}\nabla_{\partial_t F}\partial_t F, \partial_j F) + g(\nabla_{\partial_i F}\partial_t F, \nabla_{\partial_j F}\partial_t F). \end{aligned}$$

Thus

$$\begin{aligned} &\partial_t \operatorname{div}_{\Sigma_t}(\partial_t F) = \\ &= \sum_{i,j=1}^2 \left(-\sum_{k,l=1}^2 g^{ik}(g(\nabla_{\partial_k F}\partial_t F, \partial_l F) + g(\partial_k F, \nabla_{\partial_l F}\partial_t F))g^{lj} \right) g(\nabla_{\partial_i F}\partial_t F, \partial_j F) \\ &\quad + \sum_{i,j=1}^2 g^{ij}(g(R^M(\partial_t F, \partial_i F)\partial_t F, \partial_j F) + g(\nabla_{\partial_i F}\nabla_{\partial_t F}\partial_t F, \partial_j F) + g(\nabla_{\partial_i F}\partial_t F, \nabla_{\partial_j F}\partial_t F)). \end{aligned}$$

Take the basis $\{e_1, e_2\}$ in $T\Sigma$ to be orthonormal and denote $X = \partial_t F$ at $t = 0$. Noticing that $\partial_i F = e_i$, the expression above at $t = 0$ simplifies to

$$\begin{aligned} \left.\frac{d}{dt}\right|_0 \operatorname{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) &= -\sum_{i,j=1}^2 (g(\nabla_{e_i}X, e_j) + g(\nabla_{e_j}X, e_i))g(\nabla_{e_i}X, e_j) \\ &\quad + \sum_{i=1}^2 g(R^M(X, e_i)X, e_i) + \operatorname{div}_\Sigma(\nabla_X X) + \sum_{i=1}^2 g(\nabla_{e_i}X, \nabla_{e_i}X). \end{aligned}$$

If $(\nabla_{e_i}^\perp X) = (\nabla_{e_i}X)^\perp$ denotes the component of $\nabla_{e_i}X$ normal to Σ, then we have

$$\begin{aligned} \sum_{i=1}^2 g(\nabla_{e_i}X, \nabla_{e_i}X) &= \sum_{i=1}^2 g(\nabla_{e_i}^\perp X, \nabla_{e_i}^\perp X) + \sum_{i,j,k=1}^2 g(g(\nabla_{e_i}X, e_j)e_j, g(\nabla_{e_i}X, e_k)e_k) \\ &= |\nabla^\perp X|^2 + \sum_{i,j=1}^2 g(\nabla_{e_i}X, e_j)^2, \end{aligned}$$

with $|\nabla^\perp X|^2 := \sum_{i=1}^2 g(\nabla_{e_i}^\perp X, \nabla_{e_i}^\perp X)$. Therefore

$$\begin{aligned} \left.\frac{d}{dt}\right|_0 \operatorname{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) &= \sum_{i=1}^2 R^M(X, e_i, X, e_i) + \operatorname{div}_\Sigma(\nabla_X X) + |\nabla^\perp X|^2 \\ &\quad - \sum_{i,j=1}^2 g(\nabla_{e_i}X, e_j)g(\nabla_{e_j}X, e_i). \end{aligned}$$

Then

$$\frac{d}{dt}\bigg|_0 \operatorname{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) d\Sigma_t = \left(\frac{d}{dt}\bigg|_0 \operatorname{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right)\right) d\Sigma + \operatorname{div}_\Sigma(X)\,\frac{d}{dt}\bigg|_0 d\Sigma_t$$

gives the theorem. □

We are interested in computing $\frac{d^2}{dt^2}\big|_0 |\Sigma_t|$ when $\frac{d}{dt}\big|_0 |\Sigma_t| = 0$ for any variation F, i.e. when Σ is a minimal surface. For this purpose, we have seen that we only need to consider variations with $X = \frac{\partial F}{\partial t}\big|_0$ normal to Σ. We also will always suppose the boundary is fixed, i.e. $X = 0$ on ∂M.

THEOREM 1.18 (SECOND VARIATION FORMULA FOR MINIMAL SURFACES). *Let Σ be an orientable minimal surface embedded in M and ν be a global unit normal vector field along Σ. Consider a smooth variation F of Σ such that $X := \frac{\partial F}{\partial t}\big|_0 = \phi\nu$, with $\phi \in C^\infty(\Sigma)$, and $X = 0$ on $\partial\Sigma$. Then*

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_0 |\Sigma_t| &= \int_\Sigma |\operatorname{grad}_\Sigma \phi|^2 - \operatorname{Ric}(\nu,\nu)\phi^2 - |A|^2\phi^2\, d\Sigma \\ &= -\int_\Sigma \phi(\Delta_\Sigma\phi + \operatorname{Ric}(\nu,\nu)\phi + |A|^2\phi)\, d\Sigma,\end{aligned}$$

where $\Delta_\Sigma = \operatorname{div}_\Sigma \operatorname{grad}_\Sigma$, Ric is the Ricci tensor of M and A is second fundamental form of Σ.

Proof. We analyze the terms in the formula given by the previous theorem:

$$(*)\sum_{i=1}^{2} R^M(\phi\nu, e_i, \phi\nu, e_i) = \phi^2\sum_{i=1}^{2} -R^M(e_i,\nu,\nu,e_i) =: -\phi^2\operatorname{Ric}(\nu,\nu),$$

$$\begin{aligned}(*) -\sum_{i,j=1}^{2} g(\nabla_{e_i}\phi\nu, e_j)g(\nabla_{e_j}\phi\nu, e_i) &= -\sum_{i,j=1}^{2} g(\phi\nu, \nabla_{e_i}e_j)g(\phi\nu,\nabla_{e_j}e_i)\\ = -\sum_{i,j=1}^{2} g(\phi\nu,\nabla_{e_i}e_j)g(\phi\nu,\nabla_{e_i}e_j) &= -\phi^2\sum_{i,j=1}^{2} g(\nabla_{e_i}e_j,\nu)^2 = -\phi^2|A|^2,\end{aligned}$$

$$\begin{aligned}(*)\int_\Sigma \operatorname{div}_\Sigma(\nabla_X X)d\Sigma &= \int_\Sigma \operatorname{div}_\Sigma(\nabla_X X)^T - g(X,\vec{H})d\Sigma = \int_\Sigma \operatorname{div}_\Sigma(\nabla_X X)^T\\ &= \int_{\partial\Sigma} g((\nabla_X X)^T,\eta)d\sigma = 0,\end{aligned}$$

$$(*)\int_\Sigma (\operatorname{div}_\Sigma(X))^2 d\Sigma = \int_\Sigma(\operatorname{div}_\Sigma X^T - g(X,\vec{H}))^2 d\Sigma = 0,$$

In the third $(*)$ we used Lemma 1.4, $\vec{H} = 0$, the divergence theorem and $X = 0$ on $\partial\Sigma$. In the fourth $(*)$ we used Lemma 1.4 and $X^T = 0$, $\vec{H} = 0$. It only remains to show that

$$\int_\Sigma |\nabla^\perp X|^2\, d\Sigma = \int_\Sigma |\operatorname{grad}_\Sigma\phi|^2\, d\Sigma = -\int_\Sigma \phi\Delta_\Sigma\phi\, d\Sigma.$$

First notice that $g(\nabla_{e_i}\nu,\nu) = \frac{1}{2}e_i(g(\nu,\nu)) = \frac{1}{2}e_i(1) = 0$, for $i = 1, 2$. Then

$$|\nabla^\perp X|^2 = \sum_{i=1}^{2} g(\nabla^\perp_{e_i}X, \nabla^\perp_{e_i}X) = \sum_{i=1}^{2} g(g(\nabla_{e_i}X,\nu)\nu, g(\nabla_{e_i}X,\nu)\nu) = \sum_{i=1}^{2} g(\nabla_{e_i}X,\nu)^2$$

$$= \sum_{i=1}^{2} (\phi g(\nabla_{e_i}\nu, \nu) + g(e_i(\phi)\nu, \nu))^2 = \sum_{i=1}^{2} e_i(\phi)^2 = |\operatorname{grad}_\Sigma \phi|^2.$$

Finally, we have

$$\int_\Sigma \operatorname{div}_\Sigma(\phi \operatorname{grad}_\Sigma \phi)\, d\Sigma = \int_{\partial\Sigma} g(\phi \operatorname{grad}_\Sigma \phi, \eta)\, d\sigma = 0,$$

since $\phi = 0$ on $\partial\Sigma$. On the other hand,

$$\begin{aligned}
\operatorname{div}_\Sigma(\phi \operatorname{grad}_\Sigma \phi) &= \sum_{i=1}^{2} g(\nabla_{e_i}\phi \operatorname{grad}_\Sigma \phi, e_i) \\
&= \phi \sum_{i=1}^{2} g(\nabla_{e_i} \operatorname{grad}_\Sigma \phi, e_i) + \sum_{i=1}^{2} g(e_i(\phi) \operatorname{grad}_\Sigma \phi, e_i) \\
&= \phi \operatorname{div}_\Sigma \operatorname{grad}_\Sigma \phi + \sum_{i=1}^{2} e_i(\phi) g(\operatorname{grad}_\Sigma \phi, e_i) \\
&= \phi\Delta_\Sigma\phi + \sum_{i=1}^{2} e_i(\phi)^2 = \phi\Delta_\Sigma\phi + |\operatorname{grad}_\Sigma \phi|^2.
\end{aligned}$$

Thefore

$$\int_\Sigma |\operatorname{grad}_\Sigma \phi|^2\, d\Sigma = -\int_\Sigma \phi\Delta_\Sigma\phi\, d\Sigma.$$

□

1.5 STABILITY, JACOBI OPERATOR

DEFINITION 1.19. *We say that a minimal surface Σ is **stable** (resp. **strictly stable**) if $\frac{d^2}{dt^2}\Big|_0 |\Sigma_t| \geq 0$ (resp. > 0), for any smooth variation Σ_t of Σ.*

REMARK 1.20. *Notice that if Σ is a stable minimal surface, then **there is no** smooth variation Σ_t of $\Sigma_0 = \Sigma$ such that $t = 0$ is a local maximum of the area function $t \mapsto |\Sigma_t|$. If Σ is strictly stable, then it is a local minimum of the area functional, for any given variation of Σ.*

The following result follows immediately from the second variation formula.

PROPOSITION 1.21 (STABILITY INEQUALITY)**.** *Let Σ be a minimal surface. If Σ is orientable, then Σ is stable if and only if*

$$\int_\Sigma (\operatorname{Ric}(\nu,\nu) + |A|^2)\phi^2\, d\Sigma \leq \int_\Sigma |\operatorname{grad}_\Sigma \phi|^2\, d\Sigma, \text{ for every } \phi \in C^\infty(\Sigma).$$

If Σ is non-orientable, then Σ is stable if and only if

$$\int_{\tilde\Sigma} (\operatorname{Ric}(\nu,\nu) + |A|^2)\phi^2\, d\tilde\Sigma \leq \int_{\tilde\Sigma} |\operatorname{grad}_{\tilde\Sigma} \phi|^2\, d\tilde\Sigma, \text{ for every } \phi \in C^\infty(\tilde\Sigma) \text{ with } \phi\circ\tau = -\phi.$$

COROLLARY 1.22. *Suppose* $\mathrm{Ric} > 0$ *on* M*, i.e.* $\mathrm{Ric}(v, v) > 0$*, for all* $v \in TM$*. Then no embedded closed orientable minimal surface in* M *can be stable.*

Proof. Let Σ be an embedded closed orientable minimal surface in M. Then, since Σ has empty boundary, $\phi \equiv 1$ is admissible. If Σ were stable, the stability inequality would give

$$0 < \int_\Sigma (\mathrm{Ric}(\nu,\nu) + |A|^2)\phi^2\, d\Sigma \leq \int_\Sigma |\operatorname{grad}_\Sigma \phi|^2\, d\Sigma = 0,$$

a contradiction. □

REMARK 1.23. *Note that the same argument cannot be applied to the non-orientable case, since* $\phi \equiv 1$ *on* $\tilde{\Sigma}$ *is not admissible (*$\phi \circ \tau \neq -\phi$*).*

DEFINITION 1.24 (JACOBI OPERATOR)**.** *Let* $\Sigma \subset M$ *be an embedded minimal surface. We define the* **Jacobi operator** *as the following linear differential operator, according to the respective case:*

1. Σ *orientable:* $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ *given by* $L\phi = \Delta_\Sigma \phi + (\mathrm{Ric}(\nu,\nu) + |A|^2)\phi$*;*
2. Σ *non-orientable:* $L : \tilde{C}^\infty(\tilde{\Sigma}) \to \tilde{C}^\infty(\tilde{\Sigma})$ *given by* $L\phi = \Delta_{\tilde{\Sigma}}\phi + (\mathrm{Ric}(\nu,\nu) + |A|^2)\phi$ *and with* $\tilde{C}^\infty(\tilde{\Sigma}) := \{\phi \in C^\infty(\tilde{\Sigma}) \,:\, \phi \circ \tau = -\phi\}$.

We say that $\lambda \in \mathbb{R}$ *is an* **eigenvalue** *with associated* **eigenfunction** $\phi \in C^\infty(\Sigma)$ *(resp.* $\tilde{C}^\infty(\tilde{\Sigma})$*) if* ϕ *is not identically zero and* $L\phi + \lambda\phi = 0$*. The set*

$$\mathrm{Spec}(L) := \{\lambda \in \mathbb{R} \,:\, \lambda \textit{ is an eigenvalue of } L\}$$

is called the **spectrum of** L*. For each eigenvalue* λ*, we have the associated* **eigenspace**

$$V_\lambda := \{\phi \in D \,:\, L\phi + \lambda\phi = 0\},$$

with $D = C^\infty(\Sigma)$ *(resp.* $D = \tilde{C}^\infty(\tilde{\Sigma})$*).*

REMARK 1.25.

1. *If* Σ *is an orientable minimal surface and* Σ_t *is a variation with variational vector field* $X = \phi\nu$*, then*
$$\left.\frac{d^2}{dt^2}\right|_0 |\Sigma_t| = -\int_\Sigma \phi L\phi\, d\Sigma.$$
2. *If* Σ *is a non-orientable minimal surface and* Σ_t *is a smooth variation of* Σ *with associated variational vector field* $\tilde{X} = \phi\nu$ *on the orientable double cover* $\tilde{\Sigma}$*, then*
$$\left.\frac{d^2}{dt^2}\right|_0 |\Sigma_t| = \frac{1}{2}\left.\frac{d^2}{dt^2}\right|_0 |\tilde{\Sigma}_t| = -\frac{1}{2}\int_{\tilde{\Sigma}} \phi L\phi\, d\tilde{\Sigma}.$$

PROPOSITION 1.26 (JACOBI OPERATOR SPECTRUM)**.** *Let* Σ *be an embedded compact orientable minimal surface in* M*. Then*

1. $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ *is self-adjoint, i.e.* $\langle \phi_1, L\phi_2\rangle = \langle L\phi_1, \phi_2\rangle$*, where* $\langle \phi_1, \phi_2\rangle = \int_\Sigma \phi_1\phi_2\, d\Sigma$ *is the inner product in* $C^\infty(\Sigma)$*. Hence, eigenfunctions associated to distinct eigenvalues are always orthonormal;*

2. $\mathrm{Spec}(L) = \{\lambda_1 < \lambda_2 < \cdots < \lambda_n < \ldots \uparrow +\infty\}$;

3. $\dim V_\lambda < \infty$, *for every* $\lambda \in \mathrm{Spec}(L)$. *The number* $\dim V_\lambda$ *is called the* **multiplicity of** λ;

4. $\dim V_{\lambda_1} = 1$ *and if* $\phi \in V_{\lambda_1}$, *then* $\phi(p) \neq 0$, *for every* $p \in \Sigma$. *Since* Σ *is connected, this means that* $\phi > 0$ *or* $\phi < 0$. *Moreover, every eigenfunction associated to another eigenvalue necessarily changes sign on* Σ;

5. *there is an orthonormal basis of eigenfunctions of* L *for* $L^2(\Sigma)$, *the Hilbert space of the functions* $\phi : \Sigma \to \mathbb{R}$ *such that* $\exists \int_\Sigma \phi^2 d\Sigma < +\infty$. *More precisely, write* $\mathrm{Spec}(L) = \{\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \ldots \uparrow +\infty\}$, *repeating the eigenvalues according to its multiplicity. Then let* $\{\phi_i \in C^\infty(\Sigma)\}_{i\in\mathbb{N}}$ *be such that* ϕ_i *is an eigenfunction with eigenvalue* λ_i *and* $\langle \phi_i, \phi_j \rangle = 0$ *if* $i \neq j$. *Then, any* $\phi \in L^2(\Sigma)$ *can be written as* $\phi = \sum_{i=1}^\infty \langle \phi, \phi_i \rangle \phi_i$.

PROPOSITION 1.27. *Let* $\Sigma \subset M$ *be an embedded compact non-orientable minimal surface in* M. *Then*

1. $L : \tilde{C}^\infty(\tilde{\Sigma}) \to \tilde{C}^\infty(\tilde{\Sigma})$ *is self-adjoint;*

2. $\mathrm{Spec}(L) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \uparrow +\infty\}$;

3. $\dim V_\lambda < \infty$, *for every* $\lambda \in \mathrm{Spec}(L)$;

4. *there is an orthonormal basis of eigenfunctions of* L *for* $\tilde{L}^2(\tilde{\Sigma})$, *the Hilbert space of functions* $\phi : \tilde{\Sigma} \to \mathbb{R}$ *such that* $\exists \int_{\tilde{\Sigma}} \phi^2 \, d\tilde{\Sigma} < +\infty$ *and* $\phi \circ \tau = -\phi$.

The previous theorems lead to the definition of index of a minimal surface. This concept measures how far a minimal surface is from being stable.

DEFINITION 1.28. *The* **Morse index of** Σ, *denoted by* $\mathrm{ind}(\Sigma)$ *is the number of negative eigenvalues of the Jacobi operator associated to* Σ *counted with multiplicities.*

PROPOSITION 1.29. Σ *is stable if and only if* $\mathrm{ind}(\Sigma) = 0$.

Proof. Suppose Σ is orientable (the proof for the non-orientable case is the same). If Σ is stable and λ is an eigenvalue of L with eigenfunction ϕ, then putting $X = \phi\nu$ we have

$$0 \leq [\delta^2\Sigma](X, X) = -\int_\Sigma \phi L\phi \, d\Sigma = \lambda \int_\Sigma \phi^2 \, d\Sigma \implies 0 \leq \lambda.$$

Therefore $\mathrm{ind}(\Sigma) = 0$. Conversely, suppose $\mathrm{ind}(\Sigma) = 0$ and let $X = \phi\nu$ an admissible vector field on Σ. Then if $\{\phi_i\}_{i\in\mathbb{N}}$ is the L^2-orthonormal basis of eigenfunctions of L, we have

$$\phi = \sum_{i=1}^\infty b_i \phi_i, \text{ with } b_i = \int_\Sigma \phi\phi_i \, d\Sigma,$$

and then

$$[\delta^2\Sigma](X, X) = -\int_\Sigma \phi L\phi \, d\Sigma = \sum_{i=1}^\infty \lambda_i b_i^2 \geq 0,$$

since $\lambda_i \geq 0$ for all $i \in \mathbb{N}$. $\square$

2 Varifolds

In this chapter we introduce the concept of varifolds, from Geometric Measure Theory (for details, see e.g. (SIMON, 2014)), as well as some of their properties. These concepts are some of the main ingredients in the proof of Simon–Smith's Theorem (which we discuss in the next chapter).

2.1 Radon measures

Definition 2.1. *1. Let X be any set. A function $\mu : \wp(X) \to \mathbb{R}$ is said to be an outer measure on X if $\mu(\varnothing) = 0$ and $\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$, whenever $A \subset \cup_{j=1}^{\infty} A_n$.*

2. *If X is a topological space and $x \in X$, we say that $\mu(x) = 0$ if there is some open neighborhood $U \subset X$ of x such that $\mu(U) = 0$. Then the* **support of** *μ in X will be $\operatorname{supp}(\mu) = X \backslash A$, where $A = \{x \in X \,:\, \mu(x) = 0\}$.*

3. *Given a measure μ on X and a subset $A \subset X$, we define a new measure $\mu \llcorner A$ on X by $\mu \llcorner A(B) = \mu(B \cap A)$.*

Note that if μ is an outer measure on X, then $\mu(A) \leq \mu(B)$, whenever $A \subset B \subset X$. Also, since $\varnothing \subset A$, for any set A, we have that $\mu(A) \geq 0$.

We consider the Caratheodory's notion of measurability:

Definition 2.2. *A subset $A \subset X$ is μ-measurable if $\mu(S) = \mu(S \backslash A) + \mu(S \cap A)$, for any $S \subset X$.*

Remark 2.3. *Notice that A is μ-measurable if and only if $\mu(S) \geq \mu(S \backslash A) + \mu(S \cap A)$, for any $S \subset X$.*

A collection $\mathcal{S}$ of subsets of X is said to be a σ-*algebra* if:

1. $\varnothing, X \in \mathcal{S}$;
2. $A \in \mathcal{S} \implies X \backslash A \in \mathcal{S}$
3. $A_n \in \mathcal{S} \implies \cup_{n=1}^{\infty} A_n \in \mathcal{S}$.

Observe that, by (2) and (3), we also have $\cap_{n=1}^{\infty} A_n = X \backslash (\cup_{n=1}^{\infty} X \backslash A_n) \in \mathcal{S}$, whenever $A_n \in \mathcal{S}$.

If $\{\mathcal{S}_\alpha\}_{\alpha \in A}$ is any family of σ-algebras on X, then $\mathcal{S} = \cap_{\alpha \in A} \mathcal{S}_\alpha$ is again a σ-algebra on X. Since $\wp(X)$ is a σ-algebra, the family of σ-algebras which contain a given collection $\mathcal{C}$ of subsets of X is never empty. This allows us to talk about the σ-algebra generated by $\mathcal{C}$. It is defined as the intersection of all σ-algebras which contain $\mathcal{C}$ and therefore it is the least σ-algebra which contains $\mathcal{C}$. If X is a topological space, then the σ-algebra generated by the collection of open sets in X is called the ***Borel σ-algebra*** on X and its elements are usually referred to as ***Borel sets*** of X.

PROPOSITION 2.4. *The collection $\mathcal{M}$ of all μ-measurable sets is a σ-algebra on X.*

DEFINITION 2.5. *Let X be a set and $\mu : \wp(X) \to \mathbb{R}$ be an outer measure.*

1. *μ is said to be* **regular** *if for every $A \in \wp(X)$ there is a μ-measurable $B \in \wp(X)$ such that $A \subset B$ and $\mu(A) = \mu(B)$;*
2. *if X is a topological space, then μ is said to be* **Borel-regular** *if every Borel set of X is μ-measurable and, for any $A \in \wp(X)$, there is some $B \in \wp(X)$ with $A \subset B$ and $\mu(A) = \mu(B)$;*
3. *if X is a Hausdorff space, then μ is said to be* **Radon** *if*

$$\mu \text{ is Borel-regular and } \mu(K) < \infty, \text{ for every compact } K \subset X, \tag{R1}$$

$$\mu(A) = \inf_{U \text{ open}, A \subset U} \mu(U), \text{ for every } A \in \wp(X), \tag{R2}$$

$$\mu(U) = \sup_{K \text{ compact}, K \subset U} \mu(K), \text{ for every } U \in \wp(X) \text{ open}. \tag{R3}$$

Let X be a metric space and $s > 0$ a real number. For $\delta > 0$, set

$$\mathscr{F}_\delta = \{\{C_i\}_{i\in\mathbb{N}} \,:\, C_i \in \wp(X),\ \mathrm{diam} C_i < \delta\}.$$

Then, for $A \in \wp(X)$, put

$$\mathcal{H}^s_\delta(A) := \inf\left\{\omega_s \sum_{i=1}^{\infty} \left(\frac{\mathrm{diam} C_i}{2}\right)^s \,:\, \{C_i\}_{i\in\mathbb{N}} \in \mathscr{F}_\delta,\ A \subset \bigcup_{i=1}^{\infty} C_i\right\}.$$

Here, if $s \in \mathbb{N}$, then ω_s denotes the volume of the unit sphere $\mathbb{S}^{s-1}$ in $\mathbb{R}^s$. Otherwise, ω_s is any fixed positive number. Notice that, if $\delta_1 \leq \delta_2$, then $\mathscr{F}_{\delta_1} \subset \mathscr{F}_{\delta_2}$. Therefore, $\mathcal{H}^s_{\delta_2}(A) \leq \mathcal{H}^s_{\delta_1}(A)$ and the limit $\lim_{\delta\downarrow 0} \mathcal{H}^s_\delta(A)$ exists, although it can be $+\infty$.

DEFINITION 2.6 (HAUSDORFF MEASURE)**.** *Let X be a metric space and $s \geq 0$ a real number. The s-dimensional* **Hausdorff measure on** *X is the outer measure $\mathcal{H}^s : \wp(X) \to \mathbb{R}$ given by*

$$\mathcal{H}^s(A) = \begin{cases} \mathrm{card}(A), & \text{if } s = 0; \\ \lim_{\delta\downarrow 0} \mathcal{H}^s(A), & \text{if } s > 0. \end{cases} \text{ for } A \in \wp(X).$$

If $A \subset X$, then $\dim_{\mathcal{H}}(A) := \inf\{s \geq 0 \,:\, \mathcal{H}^s(A) = 0\}$ is the ***Hausdorff dimension*** *of A.*

REMARK 2.7. *Fractal sets are examples of sets with non-integer Hausdorff dimension.*

PROPOSITION 2.8. *Let X be a metric space.*

1. *If X is a locally compact and any open set in X is a countable union of compact sets (in particular, if X is a riemannian manifold), then $\mathcal{H}^s$ is a Radon measure on X, for every $s > 0$;*

2. *If $s \in \mathbb{N}$, then $\mathcal{H}^s = \mathcal{L}^s$, the Lebesgue measure on $\mathbb{R}^s$;*

3. *If $X = M$ is a riemannian manifold, and $\Sigma \subset M$ is an embedded k-submanifold, then $\mathcal{H}^k(\Sigma) = |\Sigma|$, the k-dimensional volume of Σ.*

2.2 VARIFOLDS, WEAK TOPOLOGY

Now we introduce a generalization of the concept of submanifolds which has good compactness properties. This is the concept of varifolds. Before we define what a varifold is, we briefly talk about grassmannians.

Let $\mathbb{E}$ be a vector space with dimension $n < \infty$. For each integer $1 \leq k \leq n$, we define the *k-grassmannian of* $\mathbb{E}$ as the set $G_k(\mathbb{E})$ of all k-dimensional subspaces of $\mathbb{E}$. Each $G_k(\mathbb{E})$ has a natural differentiable structure that turns it into a compact smooth manifold of dimension $k(n-k)$. Notice that the grassmannian $G_1(\mathbb{E})$ is simply the projective space $\mathbb{P}(\mathbb{E})$.

Then, if M is a n-dimensional smooth manifold, we denote by $G_k(M)$ the bundle with base M and fibers $G_k(T_xM)$, $x \in M$. We call it the k-grassmannian bundle over M. We denote an element of $G_k(M)$ by (x, π), with $x \in M$ and $\pi \in G_k(T_xM)$. Of course, the dimension of $G_k(M)$ is $nk(n-k)$. Since each fiber is compact, $G_k(M)$ is compact provided that the base M is compact.

Now we define varifolds.

DEFINITION 2.9 (VARIFOLDS). *Let M be a smooth n-manifold and $1 \leq k \leq n$. Any Radon measure on the k-grassmannian bundle $G_k(M)$ is called a (k-dimensional)* **varifold** *on M. We denote by $\mathcal{V}_k(M)$ the set of all k-dimensional varifolds on M. Given a varifold $V \in \mathcal{V}_k(M)$, one defines the* **mass of** *V as the unique measure $\|V\|$ on M satisfying*

$$\int_M \varphi(x)\, d\|V\| = \int_{G_k(M)} \varphi(x)\, dV, \ \ \forall \varphi \in \mathcal{C}_c(M),$$

where $\mathcal{C}_c(M)$ denotes the set of continuous functions of compact support in M.

Suppose M is a riemannian manifold and let $\Sigma \subset M$ be a k-submanifold. One can define a varifold $V_\Sigma \in \mathcal{V}_k(M)$ by

$$\int_{G_k(M)} \varphi(x, \pi)\, dV_\Sigma = \int_\Sigma \varphi(x, T_x\Sigma)\, d\Sigma,$$

for all continuous functions $\varphi \in \mathcal{C}_c(G_k(M))$. This is how we look at Σ as a varifold. More generally, we can use rectifiable sets with multiplicity instead of submanifolds in order to induce varifolds. A subset $R \subset M$ is said to be a k-dimensional rectifiable set if $R = \bigcup_{i=0}^\infty N_i$, with $\mathcal{H}^k(N_0) = 0$ and each N_i, $i > 0$, a closed subset of some $\mathcal{C}^1$ k-submanifold Σ_i of M. Rectifiable sets have a notion of tangent spaces and, if $R \subset M$ is k-rectifiable, then the tangent space at $x \in R$, denoted by T_xR, exists for $\mathcal{H}^k$-a.e.

point $x \in R$. Then, if R is a k-rectifiable, $\mathcal{H}^k$-measurable subset of M, and $\theta : R \to \mathbb{R}^+$ is a measurable function, then one can define a k-varifold V on M by

$$\int_{G_k(M)} \varphi(x, \pi)\, dV = \int_R \theta(x)\varphi(x, T_x R)\, d\mathcal{H}^k, \quad \forall \varphi \in \mathcal{C}_c(G_k(M)).$$

We denote $V = v(R, \theta)$ and $v(R, \theta)$ is a k-rectifiable varifold with multiplicity θ. If θ is integer valued for $\mathcal{H}^k$-a.e. $x \in R$, we say also that $v(R, \theta)$ is an integral varifold.

Since we will be working mostly with 2-dimensional varifolds, we drop the k from our notations.

We endow $\mathcal{V}(M)$ with the topology given by the following convergence notion (see B.1), called the **weak convergence**: we say that a net $(V_\lambda)_{\lambda \in D}$ in $\mathcal{V}(M)$ converges weakly to $V \in \mathcal{V}(M)$ if

$$\int_{G(M)} \varphi(x, \pi)\, dV_\lambda \to \int_{G(M)} \varphi(x, \pi)\, dV, \quad \forall \varphi \in \mathcal{C}_c(G(M)).$$

In this case, we write $V_\lambda \rightharpoonup V$. For $c \geq 0$, denote $\mathcal{V}^c(M) := \{V \in \mathcal{V}(M) : \|V\|(M) \leq c\}$. We have the following important result about the weak topology on $\mathcal{V}(M)$.

THEOREM 2.10. *If M is compact, then $\mathcal{V}^c(M)$ is metrizable and compact, for any $c \geq 0$.*

Proof. For a detailed proof, see B.2 in the Appendix. □

2.3 STATIONARY VARIFOLDS

Let $\psi : M \to M'$ a diffeomorphism between riemannian manifolds. For any varifold V in M induced by a submanifold $\Sigma \subset M$, we can define a varifold $\psi_\sharp V$ in M' as the varifold induced by $\psi(\Sigma)$. This notion can be generalized for any varifold $V \in \mathcal{V}(M)$ by

$$\int_{G(M')} \varphi(y, \sigma)\, d\psi_\sharp V = \int_{G(M)} \varphi(\psi(x), d\psi_x(\pi))|J\psi(x, \pi)|\, dV, \quad \forall \varphi \in \mathcal{C}_c(G(M')).$$

where $J\psi(x, \pi)$ denotes $\det((d\psi_x)|_\pi)$. The map $\psi_\sharp : \mathcal{V}(M) \to \mathcal{V}(M')$ is called the ***pushforward*** with respect to ψ.

Let V be a varifold on M and X a vector field on M with compact support. Let $F : (-\epsilon, \epsilon) \times M \to M$ be the isotopy induced by X, i.e. $\frac{\partial F}{\partial t} = X(F)$. We define the ***first variation of*** V with respect to X as

$$[\delta V](X) = \left.\frac{d}{dt}\right|_0 \|F_{t\sharp} V\|(M).$$

The next proposition shows that the definition of first variation coincides with the usual definition when V is a submanifold.

PROPOSITION 2.11. *If we look to a submanifold Σ as a varifold, then*

$$[\delta\Sigma](X) = \int_\Sigma \mathrm{div}_\Sigma X \, d\Sigma = \left.\frac{d}{dt}\right|_0 |F_t(\Sigma)|.$$

We also have $[\delta V](\lambda X) = \lambda[\delta V](X)$, for any $\lambda \in \mathbb{R}$.

Proof. The mass of Σ viewed as a varifold is simply its volume:

$$\|\Sigma\|(M) = \int_M d\|\Sigma\| = \int_{G(M)} d\Sigma = \int_\Sigma d\mathcal{H}^k = \int_\Sigma d\Sigma = |\Sigma|.$$

Therefore

$$[\delta\Sigma](X) = \left.\frac{d}{dt}\right|_0 \|F_{t\sharp}\Sigma\|(M) = \left.\frac{d}{dt}\right|_0 |F_t(\Sigma)|,$$

and then we use the classic first variation formula for submanifolds.

If $\frac{\partial F}{\partial t} = X(F)$, then $\tilde{F}(t,p) := F(\lambda t, p)$ is such that $\frac{\partial \tilde{F}}{\partial t} = \lambda X(\tilde{F})$. Thus, writing $s(t) = \lambda t$,

$$\begin{aligned}[\delta V](\lambda X) &= \left.\frac{d}{dt}\right|_0 \|\tilde{F}(t,\cdot)_\sharp V\|(M) = \left.\frac{d}{dt}\right|_0 \|F(s(t),\cdot)_\sharp V\|(M) \\ &= \left(\left.\frac{d}{ds}\right|_{s(0)} \|F(s,\cdot)_\sharp V\|(M) \right) \left.\frac{d}{dt}\right|_0 s(t) = \lambda[\delta V](X).\end{aligned}$$

□

With the definition of first variation, we can generalize the notion of minimal submanifolds.

DEFINITION 2.12 (STATIONARITY). *A varifold V in M is said to be* **stationary** *if $[\delta V](X) = 0$, for every vector field X on M with compact support.*

A central question when we deal with varifolds, is to know when a varifold is rectifiable. The concept of density is a useful tool in this direction. Before, we need to introduce the monotonicity formula for varifolds.

THEOREM 2.13 (MONOTONICITY). *Let M^n be a riemannian manifold, $V \in \mathcal{V}_k(M)$ and $p \in M$. Then there exists $r_0 > 0$ and $A = A(p) > 0$ such that*

$$r \mapsto e^{Ar^2} \frac{\|V\|(B_r(p))}{r^k} \quad \textit{is non-decreasing on } r, \textit{ for } r < r_0.$$

Here, $B_r(p)$ denotes the open ball of radius r centered at p.

This theorem implies that

$$\exists \lim_{r\downarrow 0} e^{Ar^2} \frac{\|V\|(B_r(p))}{r^k}, \quad \forall p \in M.$$

But then,

$$\lim_{r\downarrow 0} \frac{\|V\|(B_r(p))}{r^k} = \lim_{r\downarrow 0} e^{-Ar^2} e^{Ar^2} \frac{\|V\|(B_r(p))}{r^k} = (\lim_{r\downarrow 0} e^{-Ar^2})(\lim_{r\downarrow 0} e^{Ar^2} \frac{\|V\|(B_r(p))}{r^k})$$

$$= \lim_{r \downarrow 0} e^{Ar^2} \frac{\|V\|(B_r(p))}{r^k}$$

shows that $\lim_{r \downarrow 0} r^{-k} \|V\|(B_r(p))$ also exists, for every $p \in M$. Now, we can define density.

DEFINITION 2.14 (DENSITY). *Let M be a riemannian manifold, $V \in \mathcal{V}_k(M)$ and $p \in M$. The limit*

$$\Theta(V, x) = \lim_{r \downarrow 0} \frac{\|V\|(B_r(p))}{\omega_k r^k}$$

is called the **density of V at p**. *Here ω_k is the volume of the k-dimensional unit ball in $\mathbb{R}^k$.*

THEOREM 2.15 (5.5 OF (SIMON, 2014), P. 215). *If V is a k-dimensional varifold in M with $\Theta(V, p) > 0$, for $\|V\|$-a.e. $p \in M$, then V is rectifiable.*

We also have

THEOREM 2.16 (CONSTANCY THEOREM, 4.1 OF (SIMON, 2014), P. 213). *Let M be a riemannian manifold and V a k-varifold on M. If V is a stationary integral varifold and $\operatorname{supp}(\|V\|) \subset \bigcup_{i=1}^n \Sigma_i$, where each Σ_i is a connected $\mathcal{C}^2$ k-submanifold of M, then $V = \bigcup_{i=1}^n n_i \Sigma_i$, i.e. $V = v(R, \theta)$ with $R = \bigcup_{i=1}^n \Sigma_i$ and $\theta \equiv n_i$ on Σ_i.*

2.4 TANGENT VARIFOLDS

Tangent varifolds are the natural generalization of tangent planes for smooth surfaces. Before we define tangent varifolds, we need to recall the concept of dilation in a manifold. Let M be a smooth manifold, $x \in M$ and $0 < \rho < \operatorname{inj}(x)$. Here, $\operatorname{inj}(x)$ is the injectivity radius of M at x, i.e. for any $0 < \rho < \operatorname{inj}(x)$, the exponential map $\exp_x : \mathcal{B}_\rho^x \subset T_x M \to B_\rho(x) \subset M$ is a diffeomorphism. If M is compact, then $\operatorname{Inj}(M) = \inf\{\operatorname{inj}(x) : x \in M\} > 0$. The ***dilation around x with factor ρ*** is the map $\mathfrak{D}_\rho^x : B_\rho(x) \to \mathcal{B}_1^x$ given by $\mathfrak{D}_\rho^x(z) = \exp_x^{-1}(z)/\rho$. If $M = \mathbb{R}^n$, then $\mathfrak{D}_\rho^x$ is the usual dilation $y \longmapsto (y - x)/\rho$.

DEFINITION 2.17 (TANGENT VARIFOLD). *If $V \in \mathcal{V}(M)$, then we denote by V_ρ^x the dilated varifold in $\mathcal{V}(\mathcal{B}_1^x)$ given by $(\mathfrak{D}_\rho^x)_\sharp V$. Any limit $V' \in \mathcal{V}(\mathcal{B}_1^x)$ of a sequence $V_{s_n}^x$ of dilated varifolds with $s_n \downarrow 0$, is said to be a varifold* **tangent to V at x**. *The set of all tangent varifolds to V at x is denoted by $T(x, V)$.*

If $V = \Sigma$ is a smooth submanifold in M and $x \in V$, then $T_x\Sigma \cap \mathcal{B}_1^x$ is the only varifold tangent to V at x. Of course, we identify $T(x, V)$ and $T_x\Sigma$ in this case.

It is well known that if the varifold V is stationary, then any tangent varifold to V is a stationary *Euclidean* cone (see section 42 of (SIMON, 2014)), i.e. a stationary varifold in $T_x M$ which is invariant under the dilations $y \in T_x M \longmapsto y/\rho \in T_x M$.

Now, we state two technical lemmas which are going to be used in the next chapter.

LEMMA 2.18. *Let U be an open subset of a three-manifold M and W a 2-dimensional stationary varifold in $\mathcal{V}(U)$. If $K \subset\subset U$ is a smooth strictly convex set and $x \in (\mathrm{supp}\|W\|) \cap \partial K$, then*

$$(B_r(x) \backslash \overline{K}) \cap \mathrm{supp}\|W\| \neq \varnothing, \quad \textit{for every } r > 0.$$

LEMMA 2.19. *Let M be a compact three-manifold, $x \in M$ and V a 2-dimensional stationary integer rectifiable varifold in M. Denote by $T \subset M$ the set given by*

$$T = \{y \in \mathrm{supp}\|V\| \,:\, T(y, V) \textit{ consists of a plane transversal to } \partial B_{d(x,y)}(x)\}.$$

If $\rho < \mathrm{Inj}(M)$, then T is dense in $(\mathrm{supp}\|V\|) \cap B_\rho(x)$.

Part II

The Min-Max Construction of Minimal Surfaces

3 Min-max minimal surfaces

The goal of this chapter is to give some ideas on the proof of the following important existence theorem for minimal surfaces:

Theorem. (Simon–Smith) *Let M be a closed riemannian three-manifold. For any saturated set of sweepouts Λ, there is a min-max sequence obtained from Λ which converges in the varifold sense to smooth embedded minimal surface with area $W(M,\Lambda)$ (counted with multiplicity).*

We begin with some basic definitions from Min–Max Theory which will make the statement above clearer. The method used here is usually called Simon–Smith method. This is a version of the Almgren–Pitts method for min-max minimal surfaces.

3.1 Basic definitions

We begin this section giving a rather general definition. We introduce the Hausdorff distance. This measures how "similar" two subsets of a metric space are, taking into account their geometry and position inside the metric space.

Definition 3.1 (Hausdorff distance)**.** *Let (X,d) be a metric space. If $\epsilon > 0$ and $A \subset X$, denote*

$$A^\epsilon = \{x \in X \,:\, \exists a \in A,\ d(x,a) < \epsilon\} = \bigcup_{a \in A} B_\epsilon(a).$$

The **Hausdorff distance** *between A and B, subsets of X, is then defined by*

$$d_H(A,B) = \inf\{\epsilon \in \mathbb{R}_+ \,:\, B \subset A^\epsilon \text{ and } A \subset B^\epsilon\}.$$

Here, we consider $\inf \varnothing = +\infty$.

Proposition 3.2. *Let X denote a metric space and A, B, C any subsets of X. Then*

1. $d_H(\varnothing, A) = +\infty$, *if* $A \neq \varnothing$;
2. $d_H(A,B) \geq 0$;
3. $d_H(A,B) = 0$ *if, and only if,* $\overline{A} = \overline{B}$;
4. $d_H(A,B) = d_H(B,A)$;
5. $d_H(A,B) \leq d_H(A,C) + d_H(C,B)$;

Remark 3.3. *This proposition shows that d_H behaves like a distance function on the collection $\mathcal{C}\ell(X)$ of closed subsets of X. The only reason why it is not an usual distance function is the fact that $d_H(A,B)$ can be $+\infty$. However, this is sufficient to define a topology on $\mathcal{C}\ell(X)$ in the usual way, by taking open balls $B_\epsilon(A) = \{B \in \mathcal{C}\ell(X) \,:\, d_H(A,B) < \epsilon\}$. Of course, this will be called the* **Hausdorff topology**.

Proof. 1. Immediate from the definition of d_H;

2. It is trivial since, if $A \neq \varnothing$, there is no $\epsilon > 0$ so that $A \subset \varnothing^\epsilon = \varnothing$.

3. Suppose $d_H(A,B) = 0$. This implies $A \subset B^\epsilon$ and $B \subset A^\epsilon$ for all $\epsilon > 0$. Let $a \in A$ and $\epsilon > 0$. Since $a \in B^\epsilon$, there is some $b \in B$ such that $d(a,b) < \epsilon$. Therefore, $B_\epsilon(a) \cap B \neq \varnothing$, for every $\epsilon > 0$. This shows $A \subset \overline{B}$ and thus $\overline{A} \subset \overline{B}$. The same arguments shows that $\overline{B} \subset \overline{A}$. Suppose now $\overline{A} = \overline{B}$. Observe that the conclusion will follow if we show $A^\epsilon = \overline{A}^\epsilon$, for all $\epsilon > 0$. Of course $A^\epsilon \subset \overline{A}^\epsilon$, because $A \subset \overline{A}$. Let $a \in \overline{A}^\epsilon$. Then there is some $a' \in \overline{A}$ such that $d(a,a') < \epsilon$. Since $a' \in \overline{A}$, we have $B_{\epsilon - d(a,a')}(a') \cap A \neq \varnothing$, thus there is some $a'' \in A$ so that $d(a', a'') < \epsilon - d(a,a')$. By the triangle inequality, we have $d(a,a'') < \epsilon$. This shows that $a \in A^\epsilon$ and thus $\overline{A}^\epsilon \subset A^\epsilon$.

4. Denote $D(A,B) = \{\epsilon > 0 \,:\, A \subset B^\epsilon,\ B \subset A^\epsilon\}$ so that $d_H(A,B) = \inf D(A,B)$. Then the result follows if we show $D(A,C) + D(C,B) \subset D(A,B)$. Let $\epsilon_1 + \epsilon_2 \in D(A,C) + D(C,B)$. We need to show that $\epsilon_1 + \epsilon_2 \in D(A,B)$, i.e. $A \subset B^{\epsilon_1+\epsilon_2}$ and $B \subset A^{\epsilon_1+\epsilon_2}$. Let $a \in A$. Since $\epsilon_1 \in D(A,C)$, there is some $c \in C$ so that $d(a,c) < \epsilon_1$. In the same way, there is some $b \in B$ such that $d(c,b) < \epsilon_2$. Thus, we have found a $b \in B$ such that $d(a,b) \leq d(a,c) + d(c,b) < \epsilon_1 + \epsilon_2$. This shows $A \subset B^{\epsilon_1+\epsilon_2}$. By the same reason, we have $B \subset A^{\epsilon_1+\epsilon_2}$ and the result follows.

□

REMARK 3.4. *One can generalize d_H in order to measure only how "geometrically similar" two subsets of a metric space are, i.e. regardless of their position in the space. This is done in the following way. Denote by $\mathrm{I}(X)$ the collection of all isometries $\Phi : X \to X$, i.e. Φ is bijective and $d(\Phi(x), \Phi(y)) = d(x,y)$ for every $x, y \in X$. Then, for subsets A and B of X, set*

$$d_{HG}(A,B) = \inf\{d_H(A, \Phi(B)) \,:\, \Phi \in I(X)\}.$$

Going even further, one can try to define the distance between any two metric spaces. If (X,d) and (Y,ρ) are metric spaces, denote by $I(X,Y)$ the collection of preserving distance functions, i.e. $\Phi : X \to Y$ such that $\rho(\Phi(x), \Phi(x')) = d(x,x')$, for any $x, x' \in X$. Then, if X_1, X_2 and Y are metric spaces, one puts

$$d^Y_{HG}(X_1, X_2) = \inf\{d_{HG}(\Phi_1(X_1), \Phi_2(X_2)) \,:\, \Phi_i \in I(X_i, Y)\}.$$

Finally, one puts

$$\tilde{d}_{HG}(X_1, X_2) = \inf_Y d^Y_{HG}(X_1, X_2),$$

where Y go through the class of all metric spaces. These are called the ***Hausdorff-Gromov*** *distances. The last one is delicate to define formally. We will only deal with the Hausdorff distance d_H.*

Let M be a compact riemannian three-manifold, possibly with connected boundary. If $\Sigma \subset M$, we will denote its two-dimensional Hausdorff measure by $\mathcal{H}^2(\Sigma)$ (see Definition 2.6). If Σ is a surface, then $\mathcal{H}^2(\Sigma) = |\Sigma|$. We will denote $I = [0,1] \subset \mathbb{R}$.

Next, we give the main definitions of Min-Max Theory.

DEFINITION 3.5 (SWEEPOUTS). *A family $\{\Sigma_t\}_{t\in I}$ of closed subsets of M with finite $\mathcal{H}^2$-measure is said to be a* **sweepout** *if there are finite sets $T \subset I$ and $P \subset M$ such that*

1. *$t \in I \mapsto \mathcal{H}^2(\Sigma_t)$ is continuous;*
2. *Σ_t converges to Σ_{t_0} in the Hausdorff topology, as $t \to t_0$, i.e. $\lim\limits_{t\to t_0} d_H(\Sigma_t, \Sigma_{t_0}) = 0$;*
3. *if $t \in I \backslash T$, then Σ_t is a closed surface;*
4. *if $t \in T$, then either $\Sigma_t \backslash P$ is a surface in M or else $\mathcal{H}^2(\Sigma_t) = 0$;*
5. *Σ_t varies smoothly in $I \backslash T$, i.e. for each $(a, b) \subset I \backslash T$, Σ_t is given by the smooth variation $F : (a, b) \times \Sigma \to M$ of some closed surface $\Sigma \subset M$;*
6. *if $\tau \in T$ and $\mathcal{H}^2(\Sigma_\tau) > 0$, then Σ_t converges smoothly to Σ_τ in $M \backslash P$ as $t \to \tau$, i.e. if $\epsilon > 0$ is sufficiently small such that $\mathcal{H}^2(\Sigma_t) > 0$ for $t \in (\tau - \epsilon, \tau + \epsilon)$, then $\Sigma_t \backslash P$ is given by a smooth variation of $\Sigma_\tau \backslash P$ in $M \backslash P$;*
7. *if $\partial M \neq \varnothing$, then we require $\Sigma_0 = \partial M$, $\Sigma_t \subset \operatorname{int}(M)$ for $t > 0$ and $\{\Sigma_t\}_{t\in I}$ foliates a neighborhood of ∂M, i.e. if ν denotes the unit outward vector field normal to ∂M, then there exists a smooth function $\omega : [0, \epsilon] \times \partial M \to \mathbb{R}$, satisfying $\omega(0, x) = 0$ and $\frac{\partial \omega}{\partial t}(0, x) > 0$, such that*
$$\Sigma_t = \{\exp_x(-\omega(t, x)\nu(x))\, ;\, x \in \partial M\},$$
for any $t \in [0, \epsilon]$.

Each Σ_t is called a **slice** *in the sweepout $\{\Sigma_t\}_{t\in I}$.*

EXAMPLE 3.6. Let M be the three-dimensional sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. Then the family $\{\Sigma_t\}_{t\in I}$ defined by
$$\Sigma_t = \{x \in \mathbb{S}^3 \,:\, x_4 = 2t - 1\}$$
is a sweepout of $\mathbb{S}^3$ with $T = \{0, 1\}$ and $P = \{-e_4, e_4\}$. The slices are points if $t \in T$ and two-dimensional spheres for $t \in I \backslash T$.

EXAMPLE 3.7. In one dimension less, the level sets of the height function is a sweepout of the torus T^2, with T and P consisting of four points. Notice that these are critical points of the height function.

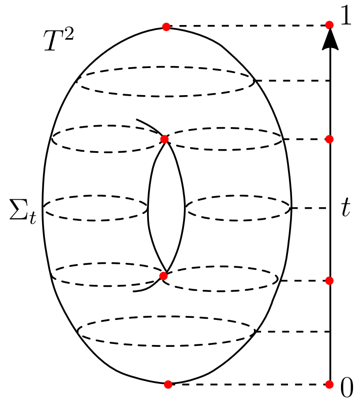


Figure 4 – A sweepout of the torus

Now, we introduce a natural way of generating sweepouts from a given one. Maybe the first idea that one can think of is to consider the image of a sweepout under diffeomorphisms of M. Actually, one can consider isotopies. Before we do that, we ask some technical restrictions.

We denote by Diff_0 the set of diffeomorphisms of M which are isotopic to the identity map, i.e. Diff_0 is the set of all diffeomorphisms $\psi : M \to M$ for which there is a smooth map $\Psi : I \times M \to M$ so that $\Psi_0 = \mathbb{1}_M$, $\Psi_1 = \psi$ and $\Psi_t : M \to M$ is a diffeomorphism, for all $t \in I$. If $\partial M \neq \varnothing$, we also require the isotopies to leave some neighborhood of ∂M fixed, i.e. there is an open set $U \subset M$, $\partial M \subset U$, such that $\Psi(t,x) = x$, for all $(t,x) \in I \times U$. In both cases ($\partial M = \varnothing$, $\partial M \neq \varnothing$), we denote the set of such isotopies by $\mathfrak{I}_0(M)$.

Let $\{\Sigma_t\}_{t\in I}$ be a sweepout of M and $\Psi : I \times M \to M$ a smooth map such that $\Psi_t \in \mathrm{Diff}_0$, for all $t \in I$. We denote such a map Ψ by saying "$\{\Psi_t\}_{t\in I}$ is a smooth one parameter family of diffeomorphisms". The family $\{\Psi_t(\Sigma_t)\}_{t\in I}$ is a sweepout of M. This is a natural fact to imagine but somewhat cumbersome to prove, so we skip the proof of it.

DEFINITION 3.8 (SATURATED SET OF SWEEPOUTS)**.** *A collection Λ of sweepouts of M is said to be* **saturated** *if*

$$\{\Sigma_t\}_{t\in I} \in \Lambda \implies \{\Psi_t(\Sigma_t)\}_{t\in I} \in \Lambda,$$

for every smooth one parameter family $\{\Psi_t\}_{t\in I}$ of diffeomorphisms of M.

REMARK 3.9. *We will only work with saturated sets Λ for which there is some $N_0 = N_0(\Lambda)$ such that the set P (in the definition of sweepouts) has at most N_0 points for any $\{\Sigma_t\}_{t\in I} \in \Lambda$.*

DEFINITION 3.10 (WIDTH)**.** *Let Λ be a set of sweepouts of M (not necessarily saturated). We define the* **width of M with respect to Λ** *by*

$$W(M,\Lambda) = \inf_{\{\Sigma_t\}\in\Lambda} \sup_{t\in I} \mathcal{H}^2(\Sigma_t).$$

REMARK 3.11. *Notice that, since $t \in I \mapsto \mathcal{H}^2(\Sigma_t)$ is continuous and I compact, in fact*

$$\sup_{t\in I} \mathcal{H}^2(\Sigma_t) = \max_{t\in I} \mathcal{H}^2(\Sigma_t),$$

i.e. there is a $\tau \in I$ such that $\mathcal{H}^2(\Sigma_\tau) = \sup_{t\in I} \mathcal{H}^2(\Sigma_t)$. In this case, Σ_τ is called the **maximal slice** *of the sweepout $\{\Sigma_t\}_{t\in I}$. So one can think of the Λ-width of M as the infimum over the areas of all maximal slices of sweepouts in Λ*

EXAMPLE 3.12. Again, an example in one dimension less, since it is easier to make figures. Let Λ be the set containing two sweepouts $\{\Sigma_t^1\}_{t\in I}$ and $\{\Sigma_t^2\}_{t\in I}$. Of course, such Λ is not saturated.

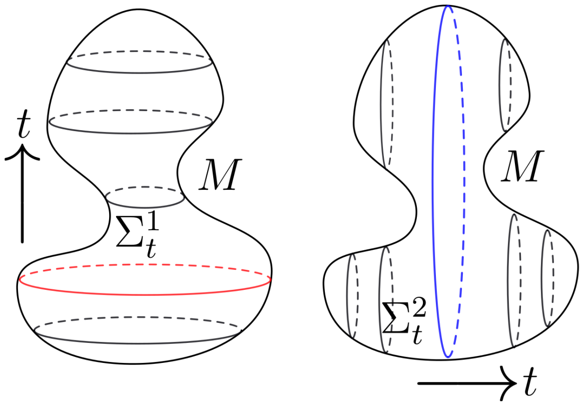


Figure 5 – $W(M,\Lambda)$ is given by the red slice

The red and blue slices in the figure indicates the respective maximal slices. In this case, since Λ is finite,

$$W(M,\Lambda) = \min\{\max_{t\in I} \mathcal{H}^1(\Sigma_t^1),\ \max_{t\in I} \mathcal{H}^1(\Sigma_t^2)\},$$

and this is given by the red slice, in the specific case of the figure.

DEFINITION 3.13 (MINIMIZING AND MIN-MAX SEQUENCES). *A sequence $\{\Sigma_t^n\}_{t\in I}$ of sweepouts in Λ is said to be a* **minimizing sequence** *if $\lim\limits_{n\to\infty} \sup_{t\in I} \mathcal{H}^2(\Sigma_t^n) = W(M,\Lambda)$. Given a minimizing sequence $\{\Sigma_t^n\}_{t\in I}$, a sequence of slices $\{\Sigma_{t_n}^n\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} \mathcal{H}^2(\Sigma_{t_n}^n) = W(M,\Lambda)$ is said to be a* **min-max sequence** *in Λ.*

3.2 SIMON–SMITH THEOREM

In this first part, we work with a "pull-tight" procedure, giving a high level of technical details for the proof of Simon–Smith's Theorem (namely Theorem 3.14 below). But after the pull-tight procedure the proof gets too technical and we will continue only with the ideas behind the proof.

THEOREM 3.14 (SIMON–SMITH). *Let M be a closed riemannian three-manifold. For any saturated set of sweepouts Λ, there is a min-max sequence obtained from Λ which converges in the varifold sense to smooth embedded minimal surface with area $W(M,\Lambda)$ (counted with multiplicity).*

Throughout this section, M will denote a closed (compact without boundary) riemannian three-manifold and Λ a saturated set of sweepouts in M. We also denote

$$m_0 = \inf_{\{\Sigma_t\}\in\Lambda} \sup_{t\in I} \mathcal{H}^2(\Sigma_t)$$

and $\mathscr{V} = \{V \in \mathcal{V}(M) : \|V\|(M) \le 4m_0\}$. By Theorem 2.10, we know that $\mathscr{V}$ with the weak topology is metrizable and compact. Let us denote by $\mathfrak{d}$ a metric on this space. Also, we denote by $\mathcal{V}_\infty$ the set of all stationary varifolds in $\mathscr{V}$, i.e. $V \in \mathcal{V}_\infty$ iff $\|V\|(M) \le 4m_0$ and $[\delta V](X) = 0$ for every vector field X on M. Notice that $\mathcal{V}_\infty \neq \varnothing$, since the null varifold $0 \in \mathcal{V}_\infty$.

PROPOSITION 3.15. *$\mathcal{V}_\infty$ is closed in $\mathscr{V}$ (and hence compact).*

Proof. Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{V}_\infty$ with $V_n \rightharpoonup V$. First, of course $V \in \mathscr{V}$, since

$$\|V_n\|(M) = \int_M d\|V_n\| = \int_{G(M)} dV_n \longrightarrow \int_{G(M)} dV = \int_M d\|V\| = \|V\|(M),$$

and $\|V_n\|(M) \leq 4m_0$ implies $\|V\|(M) \leq 4m_0$. Let X be a vector field on M and F the flow generated by X. Then

$$\begin{aligned} 0 = [\delta V_n](X) = \left.\frac{d}{dt}\right|_0 \|F_{t\sharp}V_n\|(M) = \left.\frac{d}{dt}\right|_0 \int_{G(M)} |JF_t(x,\pi)| dV_n \\ \longrightarrow \left.\frac{d}{dt}\right|_0 \int_{G(M)} |JF_t(x,\pi)| dV = \left.\frac{d}{dt}\right|_0 \|F_{t\sharp}V\|(M) = [\delta V](X). \end{aligned}$$

This shows that $[\delta V](X) = 0$ for every vector field X, thus $V \in \mathcal{V}_\infty$. Therefore, $\mathcal{V}_\infty$ is closed. □

The goal of this section is to prove the following theorem.

THEOREM 3.16. *There exists a minimizing sequence $\{\Sigma_t^n\}_{t\in I} \in \Lambda$ such that, if $\{\Sigma_{t_n}^n\}$ is a min-max sequence obtained from it, then $\mathfrak{d}(\Sigma_{t_n}^n, \mathcal{V}_\infty) \to 0$.*

REMARK 3.17. *Note that if $\{\Sigma_{t_n}^n\}$ is a min-max sequence, then $\Sigma_{t_n}^n \in X$, for all n sufficiently big. Thus the limit $\mathfrak{d}(\Sigma_{t_n}^n, \mathcal{V}_\infty) \to 0$ makes sense.*

We do this by proving several claims. Denote $\mathscr{V}' = \{V \in \mathcal{V}(M) : \|V\|(M) \leq 3m_0\}$. Of course $\mathscr{V}' \subset \mathscr{V}$ and $\mathscr{V}'$ is compact. The idea is to build a continuous map $\Psi : \mathscr{V}' \to \mathfrak{I}_0(M)$ such that

- if V is stationary, then Ψ_V is the trivial isotopy;
- if V is not stationary, then Ψ_V decreases the mass of V.

Such a map is called a ***shortening process*** or a ***pull-tight*** of varifolds that are not stationary.

Step 1: A map from $\mathscr{V}$ to the space of vector fields

For each $k \in \mathbb{Z}$, define the annular neighborhood of $\mathcal{V}_\infty$

$$\mathcal{V}_k := \{V \in \mathscr{V} : 2^{-k} \leq \mathfrak{d}(V, \mathcal{V}_\infty) \leq 2^{-k+1}\}.$$

Since $\mathscr{V}$ is compact and $f : \mathscr{V} \to \mathbb{R}$, $f(V) = \mathfrak{d}(V, \mathcal{V}_\infty)$ is continuous, there is some $A > 0$ such that $\mathfrak{d}(V, \mathcal{V}_\infty) \leq A$, for all $V \in \mathscr{V}$. Therefore, there is some $k_0 \in \mathbb{Z}$ such that $\mathcal{V}_{k_0} \neq \varnothing$ and $\mathcal{V}_k = \varnothing$, for all $k < k_0$.

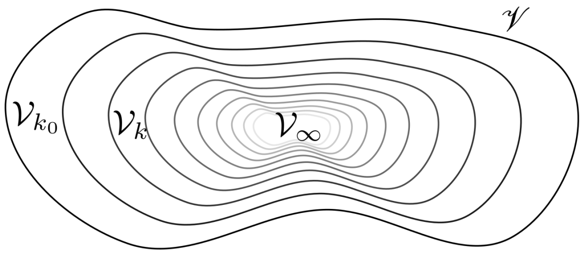


Figure 6

CLAIM 3.18. *There exist $c(k) > 0$ and $\phi : \mathcal{V}_k \to \mathcal{X}(M)$, which we denote by $\phi(V) = X_V$, such that $\|X_V\|_\infty \leq 1$ and $[\delta V](X_V) \leq -c(k)$ for all $V \in \mathcal{V}_k$.*

Suppose the claim to be false. Then, for every $c > 0$, there is a $V_c \in \mathcal{V}_k$ such that, for every $X \in \mathcal{X}(M)$ with $\|X\|_\infty \leq 1$, we have $-c < [\delta V](X) < c$ (note that, in this context, this is the negative of $[\delta V](X) \leq -c$, since $[\delta V](-X) = -[\delta V](X)$ by 2.11). Thus, taking $c = \frac{1}{n}$, $n \in \mathbb{N}$, we get a sequence $V_n \in \mathcal{V}_k$ such that for every $X \in \mathcal{X}(M)$, $\|X\|_\infty \leq 1$, we have $-\frac{1}{n} < [\delta V_n](X) < \frac{1}{n}$. Since $\mathcal{V}_k$ is compact, we can suppose that V_n converges to some $V_0 \in \mathcal{V}_k$. But then, for every $X \in \mathcal{X}(M)$, we have

$$[\delta V_0](X) = \|X\|_\infty [\delta V_0]\left(\frac{1}{\|X\|_\infty} X\right) = \|X\|_\infty \lim_{n\to\infty} [\delta V_n]\left(\frac{1}{\|X\|_\infty} X\right) = 0.$$

Therefore, V_0 is stationary and $V_0 \in \mathcal{V}_k$, a contradiction. Thus, there is some $c(k) > 0$ and $\phi : \mathcal{V}_k \to \mathcal{X}(M)$, $\phi(V) =: X_V$, such that $c(k) \leq |[\delta V](X_V)|$ and $\|X_V\|_\infty \leq 1$. Since $[\delta V](-X_V) = -[\delta V](X)$, we can change ϕ if necessary so that $[\delta V](X_V) \leq -c(k)$. This proves the Claim 3.18.

CLAIM 3.19. *We can choose the $c(k)$s in the previous claim so that $j < k \implies c(k) \leq c(j)$.*

Indeed, if $j < k_0$, then $\mathcal{V}_j = \varnothing$, then by vacuity we can choose $c(j) = c(k_0)$, for all $j < k_0$. Now, if $[\delta V](X_V) \leq -c(k_0 + 1)$ for all $V \in \mathcal{V}_{k_0+1}$, then

$$[\delta V](X_V) \leq -c(k_0 + 1) \leq -\min\{c(k_0), c(k_0 + 1)\}, \quad \forall V \in \mathcal{V}_{k_0+1}.$$

Thus, if we can change $c(k_0 + 1)$ by $c'(k_0 + 1) = \min\{c(k_0), c(k_0 + 1)\} > 0$ and the property is still true, but now $c'(k_0+1) \leq c(k_0)$. We can continue indutively in this way so that

$$0 < \cdots \leq c(k_0 + 2) \leq c(k_0 + 1) \leq c(k_0) = c(k_0 - 1) = c(k_0 - 2) = \ldots$$

and this proves the claim.

Now, we build a continuous map $\chi : \mathscr{V} \to \mathcal{X}(M)$.

Since for each fixed $V \in \mathcal{V}_k$, the map $W \in \mathscr{V} \mapsto [\delta W](X_V) \in \mathbb{R}$ is continuous, there is some radius $r_V > 0$ such that $[\delta W](X_V) \leq -c(k)/2$ for every W in the ball $U_{r_V}(V)$. Then, since $U_{r_V/2}(V)$, $V \in \mathcal{V}_k$ is an open cover for the compact set $\mathcal{V}_k$, we are able to find balls $\{U_i^k\}_{i=1,\ldots,N(k)}$ and vector fields X_i^k with $\|X_i^k\|_\infty \leq 1$ such that

1. The balls $\tilde{U}_i^k$ concentric to U_i^k with half the radii cover $\mathcal{V}_k$;
2. If $W \in U_i^k$, then $[\delta W](X_i^k) \leq -c(k)/2$;
3. The balls U_i^k are disjoint from from $\mathcal{V}_j$ if $|j-k| \geq 2$;

For the last item, just take every r_V smaller than $\min\{\mathfrak{d}(\mathcal{V}_k, \mathcal{V}_{k-2}), \mathfrak{d}(\mathcal{V}_k, \mathcal{V}_{k+2})\} > 0$ from the very beginning. Hence, $\{U_i^k : k \in \mathbb{Z}, i = 1, \ldots, N(k)\}$ is a locally finite open cover of $\mathscr{V}\backslash\mathcal{V}_\infty$, which is a metric space. So we can subordinate a continuous partition of unit φ_i^k to this family $\{U_i^k\}_{k,i}$. Then define

$$\begin{array}{rcl} \chi: \mathscr{V} & \to & \mathcal{X}(M) \\ V & \mapsto & \chi_V = \sum_{k,i} \varphi_i^k(V) X_i^k \end{array}.$$

Notice that χ is well defined over all X instead of only $\mathscr{V}\backslash\mathcal{V}_\infty$, because each $\varphi_i^k : \mathscr{V}\backslash\mathcal{V}_\infty \to [0,1]$ is zero outside $U_i^k \subset X\backslash\mathcal{V}_\infty$ and then can be extended as zero over all $\mathscr{V}\backslash U_i^k$ without losing continuity. Such χ is continuous and $\|\chi_V\|_\infty \leq 1$, for every $V \in \mathscr{V}$.

Step 2: A map from $\mathscr{V}'$ to the space of isotopies

CLAIM 3.20. *There exist continuous functions $\rho, \gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\mathfrak{d}(W,V) < \rho(\mathfrak{d}(V,\mathcal{V}_\infty)) \implies [\delta W](\chi_V) < -\gamma(\mathfrak{d}(V,\mathcal{V}_\infty)), \quad \textit{for every } V \in \mathscr{V}\backslash\mathcal{V}_\infty,$$

with $\lim_{t\downarrow 0} \rho(t) = 0 = \lim_{t\downarrow 0} \gamma(t)$ and ρ, γ are strictly increasing.

For $V \in \mathcal{V}_k$, let $r(V)$ be the radius of the smallest ball $\tilde{U}_i^j$ which contains it. Since there are finitely many such balls that touch $\mathcal{V}_k$ (namely, at most $N(k-1)+N(k)+N(k+1)$), we have that $r(V) > r(k) > 0$, for every $V \in \mathcal{V}_k$ and some $r(k)$ depending only on k. We have that $U_{r(V)}(V)$ is contained in every other ball U_i^j which contains V.

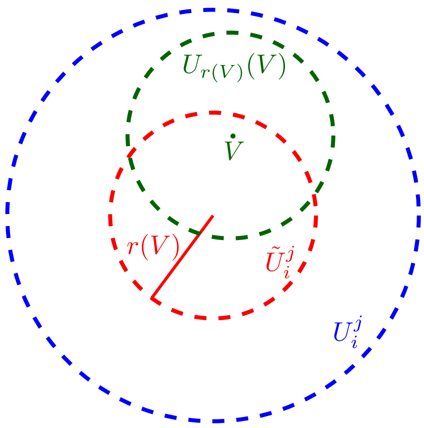


Figure 7

If $W \in U_{r(V)}(V)$, then by (2) and (3), $[\delta W](\chi_V) \leq -\frac{1}{2}c(k-1)$. Indeed

$$[\delta W](\chi_V) = \sum_{\substack{j\in\{k-1,k,k+1\} \\ i\in\{1,\ldots,N(j)\}}} \varphi_i^j(V)[\delta W](\chi_i^j)$$

$$\leq \sum_{\substack{j\in\{k-1,k,k+1\}\\ i\in\{1,\ldots,N(j)\}}} \varphi_i^j(V)\frac{1}{2}\min\{-c(k-1),-c(k),-c(k+1)\} = -\frac{1}{2}c(k-1).$$

Thus, if $V \in \mathcal{V}_k$ and $\mathfrak{d}(W,V) \leq r(k)$, then $[\delta W](\chi_V) \leq -g(k)$, where $g(k) = \frac{1}{2}c(k-1)$. With the same idea we used in the choice of the $c(k)$s, we can choose the $r(k)$s so that $r(k+1) \leq r(k)$, for all $k \in \mathbb{Z}$. Then, just take $\gamma, \rho : \mathbb{R}_+ \to \mathbb{R}_+$ any continuous functions such that

$$\rho(t) < r(k) \text{ and } \gamma(t) < g(k),\ \forall t \in I_k := [2^{-k}, 2^{-k+1}].$$

Of course, we can choose ρ and γ so that $\lim_{t\downarrow 0}\rho(t) = 0 = \lim_{t\downarrow 0}\gamma(t)$ and ρ, γ are strictly increasing (since $r(k+1) \leq r(k)$ and $g(k+1) \leq g(k)$, for all k).

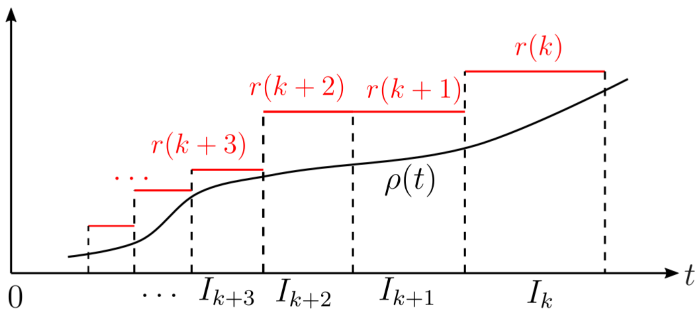


Figure 8

These functions fit our purposes. Indeed, if $V \notin \mathcal{V}_\infty$, then $V \in \mathcal{V}_k$ for some $k \in \mathbb{Z}$ and $\mathfrak{d}(V,\mathcal{V}_\infty) \in I_k$. It follows that $\mathfrak{d}(W,V) < \rho(\mathfrak{d}(V,\mathcal{V}_\infty)) < r(k) \implies [\delta W](\chi_V) \leq -g(k) < -\gamma(\mathfrak{d}(V,\mathcal{V}_\infty))$. This proves the claim.

Let $\Phi_V : [0,+\infty) \times M \to M$ be the flow generated by χ_V, i.e.

$$\frac{\partial \Phi_V}{\partial t}(t,x) = \chi_V(\Phi_V(t,x)).$$

For each t and V we denote by $\Phi_V^t : M \to M$ the diffeomorphism given by $\Phi_V^t(x) = \Phi_V(t,x)$.

For each $V \in \mathscr{V}$, define the curve $\alpha_V : [0,+\infty) \to \mathcal{V}(M)$ by

$$\alpha_V(t) = \Phi^t_{V\sharp}V.$$

For simplicity, in the following we will denote $\|V\|(M)$ just by $\|V\|$.

CLAIM 3.21. *For every $V \in \mathscr{V}'\backslash\mathcal{V}_\infty$, there is some $0 < T_V \leq 1$ such that*

$$\|\alpha_V(T_V)\| - \|V\| < -T_V\gamma(\mathfrak{d}(V,\mathcal{V}_\infty)).$$

Let $V \in \mathscr{V}'\backslash\mathcal{V}_\infty$. Then in particular $\|V\| \leq 3m_0$. Since $\|\cdot\| : \mathcal{V}(M) \to \mathbb{R}$ is continuous, there is a $0 < t_V \leq 1$ such that $\|\alpha_V(t)\| < 4m_0$, for all $t \in [0,t_V]$. Therefore, the restriction $\alpha_V : [0, t_V] \to \mathscr{V}$ is well defined. Since we also have $\mathfrak{d}(V, \mathcal{V}_\infty) > 0$, there is some $0 < T_V \leq t_V$ such that $\alpha_V([0, T_V]) \subset U_r(V)$, with $r := \rho(\mathfrak{d}(V, \mathcal{V}_\infty)) > 0$. Then, it follows from Claim 3.20 that, for every $t \in [0, T_V]$,

$$\frac{d}{dt}\|\alpha_V(t)\| = \frac{d}{dt}\|\Phi^t_{V\sharp}V\| = [\delta\alpha_V(t)](\chi_V) < -\gamma(\mathfrak{d}(V, \mathcal{V}_\infty)).$$

It follows from the fundamental theorem of calculus that

$$\|\alpha_V(T_V)\| - \|\alpha_V(0)\| = \|\alpha_V(T_V)\| - \|V\| = \int_0^{T_V} [\delta\alpha_V(t)](\chi_V)dt < -T_V\gamma(\mathfrak{d}(V, \mathcal{V}_\infty)).$$

This proves the claim.

Using a procedure similar to step 1, we can choose T_V depending continuously on V i.e. there is a continuous function $T : \mathscr{V}'\backslash\mathcal{V}_\infty \to [0, 1]$ such that

$$\|\alpha_V(T(V))\| - \|V\| < -T(V)\gamma(\mathfrak{d}(V, \mathcal{V}_\infty)).$$

Then we can also choose $T : \mathbb{R}_+ \to [0, 1]$ depending only on $\mathfrak{d}(V, \mathcal{V}_\infty)$ and $\lim_{\delta\downarrow 0} T(\delta) = 0$. If we define $G : \mathbb{R}_+ \to \mathbb{R}_+$ by $G(\delta) = T(\delta)\gamma(\delta)$, we have the following claim:

CLAIM 3.22. *There are continuous functions $T : \mathbb{R}_+ \to [0, 1]$ and $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

1. *if $\delta = \mathfrak{d}(V, \mathcal{V}_\infty) > 0$, $\|V\| \leq 3m_0$ and $V' = \Phi_V^{T(\delta)}{}_\sharp V$, then $\|V'\| \leq \|V\| - G(\delta)$;*
2. $\lim\limits_{\delta\downarrow 0} T(\delta) = 0 = \lim\limits_{\delta\downarrow 0} G(\delta)$.

Since $\mathcal{V}_k = \varnothing$ for $k < k_0$, by vacuity we can suppose the T is constant on $[2^{-(k_0-1)}, +\infty)$. Since $\lim\limits_{\delta\downarrow 0} G(\delta) = 0 = \lim\limits_{\delta\downarrow 0} T(\delta)$, we can extend T and G to continuous functions $T : [0, +\infty) \to [0, +\infty)$ and $G : [0, +\infty) \to [0, +\infty)$ by defining $T(0) = 0 = G(0)$.

CLAIM 3.23. *There is a strictly increasing continuous function $L : [0, +\infty) \to [0, +\infty)$, with $L(0) = 0$ and $L(t) \leq G(t)$, for all $t \in [0, +\infty)$.*

Let $\ell_{k_0} := \min\{G(t) \,:\, t \in I_{k_0}\} > 0$. Then, if ℓ_k is defined, define

$$0 < \ell_{k+1} := \frac{1}{2}\min\{\ell_k, \min\{G(t) \,:\, t \in I_k\}\} < \ell_k.$$

Then we have a step function $\ell : (0, 2^{-(k_0-1)}) \to \mathbb{R}_+$ given by $\ell(t) = \ell_k$ if $t \in I_k$ and such that $\ell(t) < G(t)$ for all $t \in (0, 2^{-(k_0-1)})$. Then define $L : [0, 2^{-k_0+1}] \to [0, +\infty)$ by

$$L(t) = \begin{cases} 0, & \text{if } t = 0, \\ \ell_{k+1} + \frac{(\ell_k - \ell_{k+1})}{2^{-(k+1)}}(t - 2^{-(k+1)}), & \text{if } t \in I_k, k \geq k_0 \end{cases}$$

This is strictly increasing, continuous, $L(0) = 0$ and $L(t) \leq G(t)$, for $t \in [0, 2^{-k_0+1}]$. Then, since $G(t) = G(t) = T(t)\gamma(t)$ is itself strictly increasing for $t > 2^{-k_0+1}$, we do not face any difficulty to extend L to $[0,+\infty)$ with the required properties. This proves the claim.

Finally, we define the pull-tight $\Psi : \mathscr{V}' \to \mathfrak{I}_0(M)$. For $V \in \mathscr{V}'$, let $\Psi(V) = \Psi_V \in \mathfrak{I}_0(M)$ be given by

$$\Psi_V(t,x) = \Phi_V(tT(\mathfrak{d}(V,\mathcal{V}_\infty),x), \quad t \in I = [0,1].$$

Of course, Ψ is continuous and

- If $V \in \mathscr{V}' \cap \mathcal{V}_\infty$, then $\Psi^1_{V\sharp}V = V$. Indeed

$$\Psi^1_{V\sharp}V = \Phi_V^{1T(\mathfrak{d}(V,\mathcal{V}_\infty))}{}_\sharp V = \Phi_V^{T(0)}{}_\sharp V = \Phi^0_{V\sharp}V = \mathbb{1}_{M\sharp}V = V.$$

- If $V \in \mathscr{V}' \backslash \mathcal{V}_\infty$, then $\|\Psi^1_{V\sharp}V\| \le \|V\| - L(\mathfrak{d}(V,\mathcal{V}_\infty))$. Indeed,

$$\|\Psi^1_{V\sharp}V\| = \|\Phi_V^{T(\mathfrak{d}(V,\mathcal{V}_\infty)}{}_\sharp V\| \le \|V\| - G(\mathfrak{d}(V,\mathcal{V}_\infty)) \le \|V\| - L(\mathfrak{d}(V,\mathcal{V}_\infty)).$$

Observe that, since L is strictly increasing, the more V is far from $\mathcal{V}_\infty$ the more $\|V\|$ is decreased by Ψ^1_V.

We would like to apply the pull-tight Ψ on minimizing sequences obtained from Λ to get "better" minimizing sequences in Λ. Let $\{\Sigma^n_t\}_{t\in I} \in \Lambda$ be a minimizing sequence. Since $\lim_{n\to+\infty} \sup_{t\in I} \mathcal{H}^2(\Sigma^n_t) = m_0$, we can suppose that $\mathcal{H}^2(\Sigma^n_t) \le 3m_0$ (thus $\Sigma^n_t \in \mathscr{V}'$) for all $t \in I$, $n \in \mathbb{N}$. Then

$$\Gamma^n_t = \Psi_{\Sigma^n_t}(1,\Sigma^n_t), \quad t \in I, n \in \mathbb{N}$$

defines a sequence of sweepouts $\{\Gamma^n_t\}_{t\in I}$ of M such that

$$\mathcal{H}^2(\Gamma^n_t) \le \mathcal{H}^2(\Sigma^n_t) - L(\mathfrak{d}(\Sigma^n_t,\mathcal{V}_\infty)).$$

However, $\{\Gamma^n_t\}_{t\in I}$ does not necessarily belong to Λ. Indeed, the pull-tight Ψ is only *continuous*. This implies that, for each fixed n, the one parameter family $\{\Psi^n_t\}_{t\in I}$ of diffeomorphisms of M defined by

$$\Psi^n_t := \Psi_{\Sigma^n_t}(1,\cdot), \quad t \in I,$$

may not be *smooth* on t. Thus, since the definition of saturated sets of sweepouts requires the family to be smooth, we cannot guarantee $\{\Gamma^n_t\}_{t\in I} \in \Lambda$. We overcome this technical issue by approximating $\{\Psi^n_t\}_{t\in I}$ by a smooth one parameter family $\{\tilde{\Psi}^n_t\}_{t\in I}$. First, observe that

$$\begin{aligned}\frac{\partial}{\partial s}\Psi_{\Sigma^n_t}(s,x) &= \frac{\partial}{\partial s}\Phi_{\Sigma^n_t}(sT(\Sigma^n_t),x) = T(\Sigma^n_t)\chi_{\Sigma^n_t}(\Phi_{\Sigma^n_t}(sT(\Sigma^n_t),x)) \\ &= T(\Sigma^n_t)\chi_{\Sigma^n_t}(\Psi_{\Sigma^n_t}(s,x))\end{aligned}$$

where $T(\Sigma_t^n)$ denotes $T(\mathfrak{d}(\Sigma_t^n, \mathcal{V}_\infty))$. Therefore, the one parameter family of isotopies $\{\Psi_{\Sigma_t^n}\}$, $t \in I$ is generated by the one parameter family of vector fields $h_t^n = T(\Sigma_t^n)\chi_{\Sigma_t^n}$, $t \in I$. We think of h^n as a continuous map

$$h^n : I \to \mathcal{X}(M), \text{ with the topology of } \mathcal{C}^k \text{ seminorms.}$$

Then h^n can be approximated by a *smooth* map $\tilde{h}^n : I \to \mathcal{X}(M)$. Consider the *smooth* one parameter family of isotopies $\tilde{\Psi}_t^n$ generated by the vector fields $\tilde{h}_t^n$. Then, let $\tilde{\Gamma}_t^n := \tilde{\Psi}_t^n(1, \Sigma_t^n)$. Now, since $\{\tilde{\Psi}_t^n(1, \cdot)\}_{t\in I}$ is a smooth one parameter family of diffeomorphisms, we have $\{\tilde{\Gamma}_t^n\}_{t\in I} \in \Lambda$, for all $n \in \mathbb{N}$.

If we take our approximation so that $\sup_{t\in I} \|h_t^n - \tilde{h}_t^n\|_{\mathcal{C}^1}$ is sufficiently small, then

$$\mathcal{H}^2(\tilde{\Gamma}_t^n) < \mathcal{H}^2(\Gamma_t^n) + e^{-n}, \text{ for all } t \in I.$$

Hence,

$$\mathcal{H}^2(\tilde{\Gamma}_t^n) < \mathcal{H}^2(\Sigma_t^n) - L(\mathfrak{d}(\Sigma_t^n, \mathcal{V}_\infty)) + e^{-n}, \quad \forall t \in I.$$

Doing this for each $n \in \mathbb{N}$, we define a sequence $\{\tilde{\Gamma}_t^n\}_{t\in I} \in \Lambda$ such that

$$m_0 \leq \sup_{t\in I} \mathcal{H}^2(\tilde{\Gamma}_t^n) \leq \sup_{t\in I} \mathcal{H}^2(\Sigma_t^n) + e^{-n}.$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \sup_{t\in I} \mathcal{H}^2(\tilde{\Gamma}_t^n) = m_0.$$

Therefore $\{\tilde{\Gamma}_t^n\}_{t\in I}$ is a minimizing sequence in Λ as well. Note that the construction yields a continuous and increasing function $\lambda : [0 + \infty) \to [0, +\infty)$ such that

$$\lambda(0) = 0 \quad \text{and} \quad \mathfrak{d}(\Sigma_t^n, \mathcal{V}_\infty) \geq \lambda(\tilde{\Gamma}_t^n, \mathcal{V}_\infty).$$

Finally, we prove Theorem 3.16, which says that *there exists a minimizing sequence* $\{\Sigma_t^n\}_{t\in I} \in \Lambda$ *such that, if* $\{\Sigma_{t_n}^n\}$ *is a min-max sequence, then* $\mathfrak{d}(\Sigma_{t_n}^n, \mathcal{V}_\infty) \to 0$.

Proof of Theorem 3.16. Let $\{\Sigma_t^n\}_{t\in I}$ be a minimizing sequence in Λ so that $\sup_{t\in I} \mathcal{H}^2(\Sigma_t^n) \leq m_0 + e^{-n}$. Then let $\{\tilde{\Gamma}_t^n\}_{t\in I} \in \Lambda$ be the minimizing sequence constructed from $\{\Sigma_t^n\}_{t\in I}$ as above, by the pull-tight procedure.

CLAIM 3.24. *Let* $\{\tilde{\Gamma}_{t_n}^n\}_{n\in\mathbb{N}}$ *be a min-max sequence obtained from* $\{\tilde{\Gamma}_t^n\}_{t\in I}$*. For every* $\epsilon > 0$*, there exist* $\delta > 0$ *and* $N \in \mathbb{N}$ *such that*

$$\text{if } \begin{pmatrix} n > N \\ \text{and } \mathcal{H}^2(\tilde{\Gamma}_{t_n}^n) > m_0 - \delta \end{pmatrix}, \quad \text{then } \mathfrak{d}(\tilde{\Gamma}_{t_n}^n, \mathcal{V}_\infty) < \epsilon.$$

Let $\epsilon > 0$ be given. Then $L(\lambda(\epsilon)) > 0$. Take $\delta > 0$ and $N \in \mathbb{N}$ such that $\delta + 2e^{-N} < L(\lambda(\epsilon))$. We claim that this choice works. Suppose it does not. Then there are $n > N$ and t_n such that $\mathcal{H}^2(\tilde{\Gamma}_{t_n}^n) > m_0 - \delta$ but $\epsilon < \mathfrak{d}(\tilde{\Gamma}_{t_n}^n, \mathcal{V}_\infty)$. Then we have $\lambda(\epsilon) \leq$

$\lambda(\mathfrak{d}(\tilde{\Gamma}^n_{t_n}, \mathcal{V}_\infty)) \leq \mathfrak{d}(\Sigma^n_{t_n}, \mathcal{V}_\infty)$. Since L is strictly increasing, this implies $L(\lambda(\epsilon)) \leq L(\mathfrak{d}(\Sigma^n_{t_n}, \mathcal{V}_\infty))$. Then

$$m_0 + 2e^{-n} < m_0 + 2e^{-N} < m_0 - \delta + L(\lambda(\epsilon)) < \mathcal{H}^2(\tilde{\Gamma}^n_{t_n}) + L(\mathfrak{d}(\Sigma^n_{t_n}, \mathcal{V}_\infty))$$
$$< \mathcal{H}^2(\Sigma^n_{t_n}) + e^{-n} \implies m_0 + e^{-n} < \mathcal{H}^2(\Sigma^n_{t_n}).$$

This is a contradiction, since $\sup_{t \in I} \mathcal{H}^2(\Sigma^n_t) \leq m_0 + e^{-n}$. This proves the claim.

Now, we prove that $\{\tilde{\Gamma}^n_t\}_{t\in I}$ is a minimizing sequence as asserted by the theorem. Let $\{\tilde{\Gamma}^n_{t_n}\}$ be a min-max sequence obtained from $\{\tilde{\Gamma}^n_t\}_{t\in I}$. We need to find $n_0 \in \mathbb{N}$ such that, $n \geq n_0 \Rightarrow \mathfrak{d}(\tilde{\Gamma}^n_{t_n}, \mathcal{V}_\infty) < \epsilon$. Let $\delta > 0$ and $N \in \mathbb{N}$ be as in Claim 3.24. Since $\{\tilde{\Gamma}^n_{t_n}\}$ is a min-max sequence, there is some $n_0' \in \mathbb{N}$ such that $n \geq n_0' \Rightarrow |\mathcal{H}^2(\tilde{\Gamma}^n_{t_n}) - m_0| < \delta \Rightarrow m_0 - \delta < \mathcal{H}^2(\tilde{\Gamma}^n_{t_n})$. Thus, if

$$n_0 > \max\{n_0', N\},$$

then

$$n \geq n_0 \implies \begin{pmatrix} n > N \\ \text{and } \mathcal{H}^2(\tilde{\Gamma}^n_{t_n}) > m_0 - \delta \end{pmatrix} \implies \mathfrak{d}(\tilde{\Gamma}^n_{t_n}, \mathcal{V}_\infty) < \epsilon.$$

This proves that $\lim_{n\to\infty} \mathfrak{d}(\tilde{\Gamma}^n_{t_n}, \mathcal{V}_\infty) = 0$. □

We have the important corollary:

COROLLARY 3.25. *There is a min-max sequence $\{\Sigma^n_{t_n}\}_{n\in\mathbb{N}} \in \Lambda$ which converges in the varifold sense to a stationary varifold V with $\|V\| = m_0$.*

Proof. Let $\{\Sigma^n_t\}_{t\in I} \in \Lambda$ be the minimizing sequence given by Theorem 3.16 and $\{\Sigma^n_{t_n}\}_{n\in\mathbb{N}}$ a min-max sequence obtained from it. Then $\mathfrak{d}(\Sigma^n_{t_n}, \mathcal{V}_\infty) \to 0$. Since $\mathcal{V}_\infty$ is compact,

$$\mathfrak{d}(\Sigma^n_{t_n}, \mathcal{V}_\infty) = \min\{\mathfrak{d}(\Sigma^n_{t_n}, V)\,;\, V \in \mathcal{V}_\infty\} = \mathfrak{d}(\Sigma^n_{t_n}, V_n),$$

for some $V_n \in \mathcal{V}_\infty$ and up to a subsequence we can suppose $V_n \rightharpoonup V$. Then, letting $n \to \infty$ in

$$0 \leq \mathfrak{d}(\Sigma^n_{t_n}, V) \leq \mathfrak{d}(\Sigma^n_{t_n}, V_n) + \mathfrak{d}(V_n, V)$$

shows that $\mathfrak{d}(\Sigma^n_{t_n}, V) \to 0$. Of course, $m_0 = \lim_{n\to\infty} \mathcal{H}^2(\Sigma^n_{t_n}) = \lim_{n\to\infty} \|\Sigma^n_{t_n}\| = \|V\|$. □

REMARK 3.26. *If we want the varifold V obtained in this way to be nontrivial, we need to guarantee that the saturated set is such that $m_0 > 0$. We can do so if Λ is generated by the family of level sets of a Morse function on M.*

THEOREM 3.27. *Let M be a closed riemannian three-manifold, $\{\Sigma_t\}_{t\in I}$ a sweepout given by level sets of a Morse function on M and Λ the smallest saturated set that contains $\{\Sigma_t\}_{t\in I}$. Then $m_0 = W(M, \Lambda) > 0$.*

Proof. For the proof of this theorem, we will use the following isoperimetric inequality for compact manifolds (see (DRUET, 2002)).

THEOREM 3.28 (ISOPERIMETRIC INEQUALITY)**.** *Let (M,g) be a smooth compact riemannian manifold without boundary of dimension $n \geq 2$. Let Θ be the collection of all open subsets $\Omega \subset M$ with finite perimeter. There exists $C = C(M) > 0$ such that*

$$\frac{|\partial B^n|}{|B^n|^{\frac{n-1}{n}}}|\Omega|^{\frac{n-1}{n}} \leq |\partial\Omega| + C|\Omega|, \quad \forall \Omega \in \Theta.$$

Here, B^n denotes the unit ball in $\mathbb{R}^n$, $|\Omega| = \mathcal{H}^n(\Omega)$ and $|\partial\Omega| = \lim_{\epsilon\downarrow 0} \frac{\mathcal{H}^n(\Omega^\epsilon)-\mathcal{H}^n(\Omega)}{\epsilon}$ is the perimeter of Ω. If Ω has smooth boundary $\partial\Omega$, then $|\partial\Omega| = \mathcal{H}^{n-1}(\partial\Omega)$.

Now, we prove Theorem 3.27. Let f be a Morse function on M. We can suppose that 0 and 1 are the minimum and maximum of f, i.e. $f : M \to I$ with f surjective. Let $\{\Sigma_t\}_{t\in I}$ be sweepout given by the level sets of f, i.e. $\Sigma_t = f^{-1}(\{t\})$, and let Λ be the smallest saturated set that contains it. Then

$$\Lambda = \{\{\Gamma_t\}_{t\in I} \ : \ \Gamma_t = \psi(t, \Sigma_t) \text{ for some } \psi \in \mathcal{C}^\infty(I\times M, M) \text{ with } \psi_t \in \mathrm{Diff}_0 \text{ for all } t\}.$$

Denote $U_0 = f^{-1}(\{0\})$, $U_t = f^{-1}([0,t))$ for $0 < t < 1$ and $U_1 = f^{-1}(I)$. Then for an isotopy ψ as above, denote $\Omega_t = \psi(t, U_t)$. Of course $\Gamma_t = \partial\Omega_t$. Thus, since $\{\Gamma_t\}_{t\in I}$ is a sweepout, $\partial\Omega_t$ is a surface, a surface in $M\backslash P$ for a finite set P or $\mathcal{H}^2(\partial\Omega_t) = 0$. In all cases $|\partial\Omega_t| = \mathcal{H}^2(\Gamma_t)$. In particular, $\Omega_t \in \Theta$ for all $t \in I$. The function

$$\begin{array}{rccc} g: & I & \to & \mathbb{R} \\ & t & \mapsto & |\Omega_t| \end{array}$$

is continuous. Since 0 is a critical value of f (global minimum) and f is a Morse function, we have that Ω_0 is a finite set, hence $g(0) = 0$. On the other hand, $\Omega_1 = M$, thus $g(1) = |M|$. Denote $A = \frac{4\pi}{(\frac{4}{3}\pi)^{2/3}}$ and let $C > 0$ be as in the isoperimetric inequality. Let $\alpha = \frac{1}{2}\min\{\frac{A^3}{C^3}, |M|\}$. Since $0 < \alpha < |M|$ and $g(0) = 0$, $g(1) = |M|$ there is $s \in (0,1)$ such that $g(s) = \alpha$. Notice that

$$\alpha < \frac{A^3}{C^3} \Rightarrow C^3\alpha < A^3 \Rightarrow C^3\alpha^3 < A^3\alpha^2 \Rightarrow C\alpha < A\alpha^{\frac{2}{3}} \Rightarrow 0 < A\alpha^{\frac{2}{3}} - C\alpha.$$

Then it follows from the isoperimetric inequality that

$$A|\Omega_s|^{\frac{2}{3}} \leq \mathcal{H}^2(\Gamma_s) + C|\Omega_s| \Rightarrow \quad 0 < A\alpha^{\frac{2}{3}} - C\alpha \leq \mathcal{H}^2(\Gamma_s).$$

Hence,

$$0 < A\alpha^{\frac{2}{3}} - C\alpha \leq \sup_{t\in I}\mathcal{H}^2(\Gamma_t), \quad \forall\{\Gamma_t\}_{t\in I} \in \Lambda \Rightarrow 0 < A\alpha^{\frac{2}{3}} - C\alpha \leq m_0.$$

□

Now, we do some comments and overview on the concepts and ideas behind the next steps of the proof for Simon–Smith Theorem, not giving much details. A full-length proof can be found in (COLDING; DE LELLIS, 2003).

At this point, the proof of the Simon–Smith's Theorem consists in proving that the stationary varifold obtained from Theorem 3.16 and Corollary 3.25 is in fact an embedded smooth minimal surface.

A stationary varifold can be quite far from an embedded minimal surface. To get regularity for varifolds produced by min-max sequences the concept of almost minimizing surfaces is needed. A surface Σ is said to be almost minimizing if any path of surfaces $\{\Sigma_t\}_{t\in I}$ starting at Σ with Σ_1 much smaller than Σ (in terms of area) must necessarily pass through a surface with large area, compared to Σ. More precisely,

DEFINITION 3.29 (ALMOST MINIMIZING). *Given $\epsilon > 0$, an open set $U \subset M$ and a closed set $\Sigma \subset M$, we say that Σ is ϵ-**almost minimizing in** U (or simply ϵ-a.m. in U) if there is no isotopy ψ supported in U such that*

$$\mathcal{H}^2(\psi(t,\Sigma)) \leq \mathcal{H}^2(\Sigma) + \frac{\epsilon}{8} \text{ for all } t \in I;$$
$$\mathcal{H}^2(\psi(1,\Sigma)) \leq \mathcal{H}^2(\Sigma) - \epsilon.$$

*A sequence of closed sets $\{\Sigma^n\}$ is said to be **a.m. in** U if each Σ^n is ϵ_n-a.m. in U for some $\epsilon_n \downarrow 0$.*

REMARK 3.30. *In the definition above, we use closed sets instead of surfaces in order to include slices of sweepouts.*

Using a version of the arguments of Pitts (PITTS, 1981), Colding and De Lellis prove the following (cf. (COLDING; DE LELLIS, 2003))

PROPOSITION 3.31. *There exists a continuous function $r : M \to \mathbb{R}_+$ and a min-max sequence $\{\Sigma_j\}$ such that:*

1. *$\{\Sigma_j\}$ is a.m. in every annulus An centered at x and with outer radius at most $r(x)$;*
2. *In any such annulus, Σ_j is smooth when j is sufficiently large;*
3. *Σ_j converges to a stationary varifold V in M, as $j \uparrow \infty$.*

In the proof of this proposition, the varifold V is taken as in Corollary 3.25.

Let $\{\Sigma_j\}$ and V be as in Proposition 3.31 above. One proves that if $\{\Sigma_j\}$ is a.m. on a certain annulus An, then there is a stationary varifold V' such that

1. V and V' have the same mass and $V = V'$ on $M \backslash \mathrm{An}$;
2. V' is a stable minimal surface inside An.

Such V' is said to be a **replacement** for V. This replacement property and a compactness property for stable minimal surfaces are used to prove that V is an integer rectifiable varifold (cf. (COLDING; DE LELLIS, 2003, Lemma 6.4)).

For V as in Proposition 3.31, one can construct a further replacement V'' also for V'. One proves that if we can replace sufficiently many times, then V must be regular. Then the last part of the proof of Simon–Smith is dedicated to construct such replacements.

3.3 Simon–Smith with boundary

In this section, we prove a version of Simon–Smith's Theorem for manifolds with boundary. If $\partial M \neq \varnothing$ and ν is the outward unit normal vector field along ∂M, then we define the scalar mean curvature $H(\partial M)$ of the boundary by $\vec{H} = -H(\partial M)\nu$. In this fashion, if $H(\partial M) > 0$, then $\vec{H}$ points into M. The theorem is

Theorem 3.32. *Let (M, g) be a compact three-manifold with connected boundary such that $H(\partial M) > 0$. If Λ is a saturated set of sweepouts of M with $|\partial M| < W(M, \Lambda)$, then there is a min-max sequence obtained from Λ that converges in the varifold sense to an embedded minimal surface Σ (possibly disconnected) contained in the interior of M. The area of Σ is equal to $W(M, \Lambda)$, if counted with multiplicities.*

We prove some lemmas before Theorem 3.32.

Lemma 3.33. *Let M be a compact riemannian manifold with boundary. In a neighborhood of ∂M, the metric can be written as $g = dr^2 + g_r$ on $[0, 2a] \times \partial M$ for some $a > 0$, where ∂M is identified with $\{0\} \times \partial M$.*

Remark 3.34. *Here, $g = dr^2 + g_r$ means that if*

$$u_i = (t_i, v_i) \in \mathbb{R} \times (T_x \partial M) \cong T_{(r,x)}([0, 2a] \times \partial M), \quad i = 1, 2$$

are two tangent vectors at the level $r \in [0, 2a]$, then

$$g(u_1, u_2) = t_1 t_2 + g_r(v_1, v_2),$$

for some riemannian metric g_r on ∂M.

Proof. Let η be the normal unitary, inward vector field on ∂M. Since M is compact, $L = \inf_{x \in M} \operatorname{inj}(x) > 0$. Define

$$\begin{array}{rccc} F: & [0, L] \times \partial M & \longrightarrow & M \\ & (r, x) & \longmapsto & \exp_x(r\eta(x)). \end{array}$$

We show that $dF_{(0,x)}$ is an isomorphism, for all $(0, x) \in [0, L] \times \partial M$. Let $v \in T_x \partial M$ be given by a curve $\alpha : I \to \partial M$, $\alpha(0) = x$, $\alpha'(0) = v$. Then $\bar{\alpha}(t) = (0, \alpha(t))$ is a curve in $[0, L] \times \partial M$ with $\bar{\alpha}(0) = (0, x)$ and $w := \bar{\alpha}'(0) = (0, v)$. We have

$$dF_{(0,x)} w = \left.\frac{d}{dt}\right|_0 F(\bar{\alpha}(t)) = \left.\frac{d}{dt}\right|_0 \exp_{\alpha(t)}(0\eta(\alpha(t))) = \left.\frac{d}{dt}\right|_0 \alpha(t) = v.$$

Now, if we take $\beta(t) = (at, x)$, $s \neq 0$, we have $\beta(0) = (0, x)$, $u := \beta'(0) = (s, 0)$ and

$$dF_{(0,x)} u = \left.\frac{d}{dt}\right|_0 F(\beta(t)) = \left.\frac{d}{dt}\right|_0 \exp_x(st\eta(x)) = s\eta(x).$$

This shows that $dF_{(0,x)}$ sends a basis of $T_{(0,x)}[0, L] \times \partial M$ with form $\{(0, v_1), (0, v_2), (a, 0)\}$ onto a basis of $T_x M$, since η is normal. Therefore $dF_{(0,x)}$ is an isomorphism.

Then the inverse function theorem gives us an open cover of $\{0\} \times \partial M$ such that F restricted to each of these open sets is a diffeomorphism onto its image. Since ∂M is compact, we can find $a > 0$ such that F restricted to $[0, 2a] \times \partial M$ is a diffeomorphism onto its image, say U.

Now, we turn F into an isometry. If g is the metric on M, define a metric $\tilde{g}$ on $[0, a] \times \partial M$ by

$$\tilde{g}(u, w) = g(dF\, u, dF\, w).$$

We only have to show that $\tilde{g}$ has the form stated in the theorem. First, observe that at the point $(0, x)$,

$$\tilde{g}((s, 0), (0, v)) = g(dF_{(0,x)}(s, 0), dF_{(0,x)}(0, v)) = g(s\eta(x), v) = 0,$$

for any $s \in \mathbb{R}$ and $v \in T_x\partial M$.

Consider the differentiable function $f : [0, 2a] \to \mathbb{R}$ given by

$$f(r) = g(dF_{(r,x)}(s, 0), dF_{(r,x)}(0, v)).$$

We have just showed above that $f(0) = 0$. Consider $\beta(t) = (t+r, x)$ and $\bar{\alpha}(t) = (r, \alpha(t))$ with $\bar{\alpha}(0) = (r, x)$, $\bar{\alpha}'(0) = (0, v)$. We have that

$$\begin{aligned} f'(r) &= \frac{\partial}{\partial r} g(dF_{(r,x)}(1, 0), dF_{(r,x)}(0, v)) \\ &= g\left(\frac{D}{dr} dF_{(r,x)}(1, 0), dF_{(r,x)}(0, v)\right) + g\left(dF_{(r,x)}(1, 0), \frac{D}{dr} dF_{(r,x)}(0, v)\right) \\ &= g\left(dF_{(r,x)}(1, 0), \frac{D}{dr} dF_{(r,x)}(0, v)\right) = (*), \end{aligned}$$

since $\frac{D}{dr} dF_{(r,x)}(1, 0) = \frac{D}{dr} \frac{d}{dt}\big|_0 F(\beta(t)) = \frac{D}{dr} \frac{d}{dt}\big|_0 \exp_x((t+r)\eta(x)) = \frac{D}{dr} d(\exp_x)_{r\eta(x)}\eta(x) = \frac{D}{dr}\frac{d}{dr} \exp_x(r\eta(x)) = 0$, because $r \mapsto \exp_x(r\eta(x))$ is a geodesic. Now,

$$\begin{aligned} \frac{D}{dr} dF_{(r,x)}(0, v) &= \frac{D}{dr} \frac{d}{dt}\bigg|_0 F(\bar{\alpha}(t)) = \frac{D}{dt}\bigg|_0 \frac{d}{dr} F(\bar{\alpha}(t)) = \frac{D}{dt}\bigg|_0 \frac{d}{dr} \exp_{\alpha(t)}(r\eta(\alpha(t))) \\ &= \frac{D}{dt}\bigg|_0 d(\exp_{\alpha(t)})_{r\eta(\alpha(t))}\eta(\alpha(t)). \end{aligned}$$

Then, it follows from the Gauß lemma that

$$\begin{aligned} (*) &= g\left(d(\exp_x)_{r\eta(x)}\eta(x), \frac{D}{dt}\bigg|_0 d(\exp_{\alpha(t)})_{r\eta(\alpha(t))}\eta(\alpha(t))\right) \\ &= \frac{1}{2} \frac{d}{dt}\bigg|_0 g\left(d(\exp_{\alpha(t)})_{r\eta(\alpha(t))}\eta(\alpha(t)), d(\exp_{\alpha(t)})_{r\eta(\alpha(t))}\eta(\alpha(t))\right) \\ &= \frac{1}{2} \frac{d}{dt}\bigg|_0 g(\eta(\alpha(t)), \eta(\alpha(t))) = \frac{1}{2} \frac{d}{dt}\bigg|_0 1 = 0. \end{aligned}$$

This proves that $f(r) = 0$ for all $r \in [0, 2a]$, no matter which vector v we choose in the definition of f. This proves that the mixed terms in the following computation are zero, for any level $r \in [0, 2a]$. If $(s, u), (t, v) \in T_{(r,x)}[0, 2a] \times \partial M$, then

$$\begin{aligned} \tilde{g}((s, u), (t, v)) &= \tilde{g}((s, 0), (t, 0)) + \tilde{g}((s, 0), (0, v)) + \tilde{g}((0, u), (t, 0)) + \tilde{g}((0, u), (0, v)) \\ &= st\tilde{g}((1, 0), (1, 0)) + \tilde{g}((0, u), (0, v)) \\ &= stg(dF_{(r,x)}(1, 0), dF_{(r,x)}(1, 0)) + \tilde{g}((0, u), (0, v)) \end{aligned}$$

$$
\begin{aligned}
&= stg(d(\exp_x)_{r\eta(x)}\eta(x), d(\exp_x)_{r\eta(x)}\eta(x)) + \tilde{g}((0,u),(0,v)) \\
&= stg(\eta(x), \eta(x)) + \tilde{g}((0,u),(0,v)) = st + g_r(u,v),
\end{aligned}
$$

with the metric g_r on ∂M is defined by $g_r(u,v) = g(dF_{(r,x)}(0,u), dF_{(r,x)}(0,v))$. $\square$

Thus, if $\partial M \neq \varnothing$, we identify the neighborhood given by Lemma 3.33 with $[0,2a] \times \partial M$ for some $a > 0$ and introduce some notations. We denote $C_r = \{r\} \times \partial M$ and $M_r = M \backslash ([0,r) \times \partial M)$. Observe that $C_0 = \{0\} \times \partial M = \partial M$, under the identification. Also, since $-\frac{\partial}{\partial r}$ is an extension of ν to $[0,2a] \times \partial M$, we can extend the function $H(\partial M)$ to a function H on $[0,2a] \times \partial M$ defined by the equation $\vec{H}_r(x) = H(r,x)\frac{\partial}{\partial r}(x)$, where $\vec{H}_r(x)$ is the mean curvature vector of C_r at (r,x). We also denote by $A = A_{(r,x)}$ the second fundamental form of C_r at (r,x), i.e. $A_{(r,x)}(u,v) = g(\nabla_u v, \frac{\partial}{\partial r})$, where $u, v \in T_{(r,x)}C_r$. Using this notation, we have the following lemma.

LEMMA 3.35. *If $H(\partial M) > 0$ then for any $\{\Sigma_t\} \in \Lambda$ and $t_0 \in (0,1)$, there exists a smooth one-parameter family of diffeomorphisms $(F_t)_{0 \le t \le 1}$ of M so that*

1. *$F_0 = \mathbb{1}_M$;*
2. *$F_t = \mathbb{1}_M$ in a neighborhood U of ∂M;*
3. *$|F_t(\Sigma_t)| \le |\Sigma_t|$;*
4. *for any $t \ge t_0$, we have $F_t(\Sigma_t) \subset M_{a/2}$.*

Proof. Let $\{\Sigma_t\} \in \Lambda$ and $t_0 \in (0,1)$. Since $\{\Sigma_t\}$ is a sweepout, the function $t \mapsto d(\Sigma_t, \partial M)$ is continuous. Also, since Σ_t and ∂M are compact and disjoint for $t > 0$, we have that $d(\Sigma_t, \partial M) > 0$ for $t > 0$. Thus, since $[t_0/2, 1]$ is compact, there is some $\eta > 0$ such that $d(\Sigma_t, \partial M) \ge 2\eta$ for all $t \in [t_0/2, 1]$. We can also suppose that η is sufficiently small so that $\eta \le a/8$.

We denote $A = A_{(r,x)}$ the second fundamental form of C_r at (r,x), i.e. $A(u,v) = g(\nabla_u v, \frac{\partial}{\partial r})$, with $u, v \in T_{(r,x)}[0,2a] \times \partial M$. The function on $[0,2a] \times \partial M$ defined by $(r,x) \mapsto |A| = |A_{(r,x)}| = \sup\{|A_{(r,x)}(u,v)| : |u| \le 1, |v| \le 1\}$ is continuous and then, since $[0,2a] \times \partial M$ is compact, we have $c := \sup\{|A| : (r,x) \in [0,2a] \times \partial M\} < \infty$.

Choose a nonnegative real function ϕ so that $\phi' \le -c\phi$, $\phi(r) > 0$ for $r < a$, and $\phi(r) = 0$ for $r \ge a$. We can do this, for example, by taking a bump function α nonnegative and nonincreasing such that $\alpha(r) > 0$ for $r < a$ and $\alpha(r) = 0$ for $r \ge a$ and then putting $\phi(r) = \alpha(r)\exp(-cr)$. Then, choose also another nonnegative bump function κ such that $\kappa(r) = 0$ for $r \le \eta$ and $\kappa(r) = 1$ for $r \ge 2\eta$. We consider ϕ and κ to be defined on $[0,2a] \times \partial M$ without changing the notation, i.e. $\phi(r,x) = \phi(r)$ and $\kappa(r,x) = \kappa(r)$.

Denote by $(\tilde{F}_t)_{0 \le t < \infty}$ the one-parameter family of diffeomorphisms generated by the vector field $X = \kappa(r)\phi(r)\frac{\partial}{\partial r}$. Notice that X is a vector field on the entire M, since $X = 0$ for $a \le r \le 2a$ (i.e. we can automatically extend X outside $[0,2a] \times \partial M$).

CLAIM 3.36. *For every surface $L \subset M_{2\eta}$, the function $t \mapsto |\tilde{F}_t(L)|$ is nonincreasing. In particular, $|\tilde{F}_t(L)| \le |L|$ if $t \ge 0$.*

By the first variation formula, we have

$$\frac{d}{dt}|\tilde{F}_t(L)| = \int\limits_{\tilde{F}_t(L)} \operatorname{div}_{\tilde{F}_t(L)} X \, d\mu.$$

Thus, it suffices to show that for any orthonormal basis $\{e_1, e_2\}$ we have $\sum_{i=1}^2 g(\nabla_{e_i} X, e_i) \leq 0$ (since $\operatorname{div}_{\tilde{F}_t(L)} X = \sum_{i=1}^2 g(\nabla_{e_i} X, e_i)$ for a particular choice of $\{e_1, e_2\}$, namely, a basis for the plane tangent to $\tilde{F}_t(L)$). Notice that $\kappa \equiv 1$ in $M_{2\eta}$ and $\tilde{F}_t(L) \subset M_{2\eta}$. Without loss of generality, we can assume that e_1 is orthogonal to $\frac{\partial}{\partial r}$ (and thus e_1 is tangent to C_r). We denote e_1^* a unit vector tangent to C_r and orthogonal to e_1. We also denote by π the projection of an arbitrary vector in M into the tangent space at C_r. Under these conditions and notations, we have

$$\begin{aligned}
\sum_{i=1}^2 g\left(\nabla_{e_i} X, e_i\right) &= \sum_{i=1}^2 g\left(\nabla_{e_i} \phi \frac{\partial}{\partial r}, e_i\right) = \sum_{i=1}^2 g\left(e_i(\phi)\frac{\partial}{\partial r} + \phi \nabla_{e_i} \frac{\partial}{\partial r}, e_i\right) \\
&= g\left(e_2(\phi)\frac{\partial}{\partial r}, e_2\right) + \phi \sum_{i=1}^2 g\left(\nabla_{e_i}\frac{\partial}{\partial r}, e_i\right) = g\left((d\phi \cdot e_2)\frac{\partial}{\partial r}, e_2\right) + \phi \sum_{i=1}^2 g\left(\nabla_{e_i}\frac{\partial}{\partial r}, e_i\right)
\end{aligned}$$

Writing $e_2 = e_2^T + e_2^N$ with e_2^T tangent to C_r, $e_2^N = g(e_2, \frac{\partial}{\partial r})\frac{\partial}{\partial r}$ normal to C_r, and noticing that ϕ does not vary along C_r, we have

$$d\phi \cdot e_2 = d\phi(e_2^T) + d\phi(e_2^N) = d\phi(e_2^N) = g\left(e_2, \frac{\partial}{\partial r}\right) d\phi \frac{\partial}{\partial r} = g\left(e_2, \frac{\partial}{\partial r}\right)\phi'$$

Thus

$$\sum_{i=1}^2 g(\nabla_{e_i} X, e_i) = \phi' g\left(e_2, \frac{\partial}{\partial r}\right)^2 + \phi \sum_{i=1}^2 g\left(\nabla_{e_i}\frac{\partial}{\partial r}, e_i\right).$$

Now, since $\pi(e_1) = e_1$, $\pi(e_2) = e_2 - g(e_2, \frac{\partial}{\partial r})\frac{\partial}{\partial r}$, $g(\nabla_{e_2}\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 0$ and $\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0$, we have

$$\begin{aligned}
\sum_{i=1}^2 A(\pi(e_i), \pi(e_i)) &= \sum_{i=1}^2 g\left(\nabla_{\pi(e_i)}\pi(e_i), \frac{\partial}{\partial r}\right) \\
&= g\left(\nabla_{e_1} e_1, \frac{\partial}{\partial r}\right) + g\left(\nabla_{e_2 - g(e_2, \frac{\partial}{\partial r})\frac{\partial}{\partial r}}\left(e_2 - g\left(e_2, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}\right), \frac{\partial}{\partial r}\right) \\
&= g\left(\nabla_{e_1} e_1, \frac{\partial}{\partial r}\right) - g\left(e_2 - g\left(e_2, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}, \nabla_{e_2 - g(e_2, \frac{\partial}{\partial r})\frac{\partial}{\partial r}}\frac{\partial}{\partial r}\right) \\
&= g\left(\nabla_{e_1} e_1, \frac{\partial}{\partial r}\right) - g\left(e_2 - g\left(e_2, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}, \nabla_{e_2}\frac{\partial}{\partial r}\right) \\
&= g\left(\nabla_{e_1} e_1, \frac{\partial}{\partial r}\right) - g\left(e_2, \nabla_{e_2}\frac{\partial}{\partial r}\right) + g\left(e_2, \frac{\partial}{\partial r}\right) \cancelto{0}{g\left(\frac{\partial}{\partial r}, \nabla_{e_2}\frac{\partial}{\partial r}\right)} \\
&= -g\left(e_1, \nabla_{e_1}\frac{\partial}{\partial r}\right) + \nabla_{e_1} \cancelto{0}{g\left(e_1, \frac{\partial}{\partial r}\right)} - g\left(e_2, \nabla_{e_2}\frac{\partial}{\partial r}\right) \\
&= -g\left(e_1, \nabla_{e_1}\frac{\partial}{\partial r}\right) - g\left(e_2, \nabla_{e_2}\frac{\partial}{\partial r}\right),
\end{aligned}$$

and thus we have

$$\sum_{i=1}^{2} g(\nabla_{e_i} X, e_i) = \phi' g\left(e_2, \frac{\partial}{\partial r}\right)^2 - \phi \sum_{i=1}^{2} A(\pi(e_i), \pi(e_i)).$$

Notice that $\pi(e_2) = g(e_2, e_1^*)e_1^*$, and $g(e_2, e_1^*)^2 + g(e_2, \frac{\partial}{\partial r})^2 = 1$, since $|e_2| = 1$ and $\{e_1, e_1^*, \frac{\partial}{\partial r}\}$ is an orthonormal basis. Then

$$\sum_{i=1}^{2} A(\pi(e_i), \pi(e_i)) = g\left(\nabla_{e_1} e_1, \frac{\partial}{\partial r}\right) + g\left(\nabla_{g(e_2,e_1^*)e_1^*} g(e_2, e_1^*)e_1^*, \frac{\partial}{\partial r}\right)$$
$$= g\left(\nabla_{e_1} e_1, \frac{\partial}{\partial r}\right) + g(e_2, e_1^*) g\left(g(e_2, e_1^*)\nabla_{e_1^*} e_1^* + e_1^*(g(e_2, e_1^*))e_1^*, \frac{\partial}{\partial r}\right)$$
$$= g\left(\nabla_{e_1} e_1, \frac{\partial}{\partial r}\right) + g(e_2, e_1^*)^2 g\left(\nabla_{e_1^*} e_1^*, \frac{\partial}{\partial r}\right)$$
$$= g\left(\nabla_{e_1} e_1, \frac{\partial}{\partial r}\right) + \left(1 - g\left(e_2, \frac{\partial}{\partial r}\right)^2\right) g\left(\nabla_{e_1^*} e_1^*, \frac{\partial}{\partial r}\right) = -g\left(e_2, \frac{\partial}{\partial r}\right)^2 A(e_1^*, e_1^*) + H$$

Then, we finally get

$$\begin{aligned}\sum_{i=1}^{2} g(\nabla_{e_i} X, e_i) &= (\phi' + A(e_1^*, e_1^*)\phi) g\left(e_2, \frac{\partial}{\partial r}\right)^2 - \phi H \\ &\leq (\phi' + c\phi) g\left(e_2, \frac{\partial}{\partial r}\right)^2 - \phi H \leq 0.\end{aligned}$$

This proves the claim.

Notice that $\tilde{F}_t$ is the identity in M_a and $X = \phi(r)\frac{\partial}{\partial r}$ with $\phi(r) > 0$ for $2\eta \leq r < a$. Thus $\lim_{t\to\infty} \tilde{F}_t(r, x) = (a, x)$ for all $x \in \partial M$ and $r \in [2\eta, a)$. Notice also that each C_r is taken onto some other $C_{r'}$ since the field X is orthogonal to each C_r and $|X|$ is constant along C_r. Therefore, since $\lim_{t\to\infty} \tilde{F}_t(r, x) = (a, x)$ for $2\eta \leq r < a$, there must be some $T > 0$ such that $\tilde{F}_T(C_{2\eta}) = C_{a/2}$. Choose a smooth nonnegative function $h : [0, 1] \to \mathbb{R}$ such that $h(t) = 0$ for $t \leq t_0/2$ and $h(t) = T$ for $t \geq t_0$.

We claim that $F_t := \tilde{F}_{h(t)}$ fulfill the conditions in the statement of the lemma. We have $F_0 = \tilde{F}_0 = \mathbb{1}_M$. Since $X = 0$ outside M_η, we have that $F_t = \tilde{F}_{h(t)} = \mathbb{1}_M$ outside M_η and this proves (ii). To prove (iii), recall that at the beginning of the proof, we have set that $d(\Sigma_t, \partial M) \geq 2\eta$ for $t \in [t_0/2, 1]$. Thus, if $t \geq t_0/2$, then $\Sigma_t \subset M_{2\eta}$. In this case, it follows from the claim that $|F_t(\Sigma_t)| = |\tilde{F}_{h(t)}(\Sigma_t)| \leq |\Sigma_t|$. If $t \leq t_0/2$ then the inequality is trivial because $F_t = \tilde{F}_0 = \mathbb{1}_M$. Finally, we prove (iv). If $t \geq t_0$, we have $F_t = \tilde{F}_T$. In this case, since $\Sigma_t \subset M_{2\eta}$, we conclude that $F_t(\Sigma_t) \subset \tilde{F}_T(M_{2\eta}) = M_{a/2}$. This finishes the proof of the lemma. □

LEMMA 3.37. *If $H(\partial M) > 0$ and $|\partial M| < W(M, \Lambda)$, then there exist $a > 0$, $\delta > 0$ with $|\partial M| < W(M, \Lambda) - 2\delta$ and a minimizing sequence $\{\Sigma_t^n\} \in \Lambda$ such that $|\Sigma_t^n| \geq W(M, \Lambda) - \delta \implies d(\Sigma_t^n, \partial M) \geq a/2$.*

Proof. Notice that we are in a particular case of Lemma 3.35. Let $m_0 = W(M, \Lambda)$ and choose $0 < \delta < \frac{1}{2}(m_0 - |\partial M|)$. This is possible since $|\partial M| < m_0$. Choose $a > 0$ as in the proof of Lemma 3.35.

Let $\{\Sigma_t\} \in \Lambda$. By the definition of sweepouts for manifolds with boundary, there is a smooth function $\omega : [0, \epsilon_0] \times \partial M \to \mathbb{R}$, satisfying $\omega(0, x) = 0$ and $\frac{\partial \omega}{\partial t}(0, x) > 0$, such that $\Sigma_t = \{\exp_x(-\omega(t, x)\nu(x)) : x \in \partial M\}$. Thus there exists $\epsilon > 0$ such that the map $\Psi : [0, 2\epsilon] \times \partial M \to M$ given by $\Psi(t, x) = \exp_x(-\omega(t, x)\nu(x))$ is a diffeomorphism onto a neighborhood of ∂M. Since the area varies continuously and $\Sigma_0 = \partial M$, we can take ϵ sufficiently small so that $|\Sigma_t| \leq |\partial M| + \delta$ for $t \in [0, 2\epsilon]$. Now, choose $t_0 = \epsilon$ in Lemma 3.35 and then consider the sweepout $\{\Gamma_t\} \in \Lambda$ given by $\Gamma_t = F_t(\Sigma_t)$. We claim that

- $\sup_{t \in I} \mathcal{H}^2(\Gamma_t) \leq \sup_{t \in I} \mathcal{H}^2(\Sigma_t)$ (immediate from Lemma 3.35)
- if $|\Gamma_t| \geq m_0 - \delta$, then $\Gamma_t \subset M_{a/2}$.

To prove the second item, we show that the condition $m_0 - \delta \leq |\Gamma_t|$ implies $t \geq t_0$ and the conclusion follows from item (iv) of the lemma. Suppose by contradiction that $t < t_0 = \epsilon$. In this case, $m_0 - \delta \leq |\Gamma_t| \leq |\Sigma_t| \leq |\partial M| + \delta \implies m_0 \leq |\partial M| + 2\delta < |\partial M| + m_0 - |\partial M| \implies m_0 < m_0$, a contradiction. This proves the second item.

These two items together show that, if $\{\Sigma_t^n\}$ is a minimizing sequence, then the corresponding $\{\Gamma_t^n\}$ is also minimizing and has the property

$$|\Gamma_t^n| \geq m_0 - \delta \implies d(\Gamma_t^n, \partial M) \geq a/2.$$

□

THEOREM 3.38. *Let (M, g) be a compact three-manifold with connected boundary such that $H(\partial M) > 0$. If Λ is a saturated set of sweepouts of M with $|\partial M| < W(M, \Lambda)$, then there is a min-max sequence obtained from Λ that converges in the varifold sense to an embedded minimal surface Σ (possibly disconnected) contained in the interior of M. The area of Σ is equal to $W(M, \Lambda)$, if counted with multiplicities.*

Proof. The first step is to modify the pull-tight procedure from the case of empty boundary to find a min-max sequence $\{\Gamma_{t_n}^n\}_{n\in\mathbb{N}} \in \Lambda$ uniformly distant from ∂M which converges to a stationary varifold V with $d(\text{supp}V, \partial M) > 0$.

Let $a > 0$, $\delta > 0$ and $\{\Sigma_t^n\}$ be as in Lemma 3.37. Consider $(\hat{M}, \hat{g})$ a closed (compact without boundary) extension of (M, g). Let $\mathscr{V} = \{V \in \mathcal{V}(\hat{M}) : \|V\| \leq 4m_0\}$ and $\mathscr{V}' = \{V \in \mathcal{V}(\hat{M}) : \|V\| \leq 3m_0\}$, where $m_0 = W(M, \Lambda)$. Then repeat the construction of Section 3.2 but with

$$\mathcal{V}_\infty^* = \{V \in \mathscr{V} : V \text{ is stationary}\} \cup \{V \in \mathscr{V} : \|V\| \leq m_0 - \delta\}$$

instead of $\mathcal{V}_\infty$. Then, not only the stationary varifolds are going to be fixed under the pull-tight but also those varifolds with mass bounded above by $m_0 - \delta$. We can also require that the map $\chi : \mathscr{V} \to \mathcal{X}(\hat{M})$ in the construction is such that $\|\chi_V\|_\infty \leq a/4$, for all $V \in \mathscr{V}$. We obtain a *continuous* map $\Psi : \mathscr{V}' \to \mathfrak{I}_0(\hat{M})$ such that

- If $V \in \mathscr{V}' \cap \mathcal{V}_\infty^*$, then $\Psi_{V\sharp}^1 V = V$;
- If $V \notin \mathcal{V}_\infty^*$, then $\|\Psi_{V\sharp}^1 V\| \leq \|V\| - L(\mathfrak{d}(V, \mathcal{V}_\infty^*))$ for some increasing continuous function $L : [0, +\infty) \to [0, +\infty)$ with $L(0) = 0$;
- for every $V \in \mathscr{V}'$, $\frac{\partial \Psi_V}{\partial s} = T(V)\chi_V$ for some continuous function $T : \mathscr{V}' \to [0, 1]$ with $T(V) = 0$ if $V \in \mathcal{V}_\infty^*$.

Fix $n \in \mathbb{N}$. Since $\{\Sigma_t^n\}_{t\in I}$ is minimizing, we can suppose $\Sigma_t^n \in \mathscr{V}'$ for all $t \in I$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we can choose a *smooth* one-parameter family of isotopies $\{\tilde{\Psi}_t^n\}_{t\in I}$ which approximates $\{\Psi_{\Sigma_t^n}\}_{t\in I}$ so that

$$\left\| \frac{\partial \tilde{\Psi}_t^n}{\partial s} - \frac{\partial \Psi_{\Sigma_t^n}}{\partial s} \right\|_\infty < \frac{a}{8} \quad \text{and} \quad \mathcal{H}^2(\Gamma_t^n) \le \mathcal{H}^2(\Sigma_t^n) - L(\mathfrak{d}(\Sigma_t^n, \mathcal{V}_\infty^*)) + e^{-n},$$

where $\Gamma_t^n := \tilde{\Psi}_t^n(1, \Sigma_t^n)$. We do one more requirement for our approximation. Consider the continuous function $f_n : I \to [0, +\infty)$, $f_n(t) = \mathcal{H}^2(\Sigma_t^n)$. Then, $f_n^{-1}(A)$ is an open subset of I, with $A = [0, m_0 - \delta)$. We know from real analysis that any open subset of $\mathbb{R}$ is a countable union of open disjoint intervals. Since I is compact, $f_n^{-1}(A)$ is a finite such union. Since $\Sigma_0^t = \partial M$ and $|\partial M| < m_0 - 2\delta < m_0 - \delta$, we have that $0 \in f_n^{-1}(A)$. So, there are $0 = t_1 < t_2 < t_3 < \cdots < t_{2k(n)} \le 1$ such that

$$f_n^{-1}(A) = [0, t_2) \cup (t_3, t_4) \cup \cdots \cup (t_{2k(n)-1}, t_{2k(n)}),$$

where the last interval may be closed or not in $t_{2k(n)}$, if $t_{2k(n)} = 1$. For $t \in f_n^{-1}(A)$, we have $\frac{\partial \Psi_{\Sigma_t^n}}{\partial s} = 0$ and therefore $\Psi_{\Sigma_t^n}$ depends smoothly on t, for $t \in f_n^{-1}(A)$. Thus, we can require our approximation $\tilde{\Psi}_t^n$ to be such that

$$\tilde{\Psi}_t^n = \Psi_{\Sigma_t^n}, \quad \text{for } t \in f_n^{-1}(A).$$

CLAIM 3.39. *For every $n \in \mathbb{N}$, $t \in I$, we have $\Gamma_t^n \subset M$.*

Fix $n \in \mathbb{N}$. If $\mathcal{H}^2(\Sigma_t^n) < m_0 - \delta$, then $\Gamma_t^n = \Sigma_t^n \subset M$. So suppose $\mathcal{H}^2(\Sigma_t^n) \ge m_0 - \delta$. Suppose also, by contradiction, that there exists $p \in \Gamma_t^n \cap \hat{M} \backslash M$. Recall $\{\Sigma_t^n\}_{t\in I}$ is as in Lemma 3.37, thus $\hat{d}(\Sigma_t^n, \partial M) = d(\Sigma_t^n, \partial M) \ge a/2$. Since $p \notin M$, for any smooth path $\gamma : I \to \hat{M}$ joining some point of Σ_t^n to p we must have $\ell(\gamma) > a/2$ (every such path must intersect ∂M). But since $\Gamma_t^n = \Psi_t^n(1, \Sigma_t^n)$, there is some $x \in \Sigma_t^n$ such that $p = \Psi_t^n(1, x)$. Then consider the smooth path $\gamma : I \to \hat{M}$ given by $\gamma(s) = \tilde{\Psi}_t^n(s, x)$ joining x to p. We have

$$\ell(\gamma) = \int_0^1 \|\gamma'(s)\|\, ds \le \int_0^1 \left\| \frac{\partial \tilde{\Psi}_t^n}{\partial s} \right\|_\infty ds \le \int_0^1 \left(\left\| \frac{\partial \tilde{\Psi}_t^n}{\partial s} - T(\Sigma_t^n)\chi_{\Sigma_t^n} \right\|_\infty + \left\| T(\Sigma_t^n)\chi_{\Sigma_t^n} \right\|_\infty \right) ds$$
$$< \int_0^1 \frac{a}{8} + \frac{a}{4}\, ds = \frac{3a}{8} < \frac{a}{2},$$

a contradiction. Thus, $\Gamma_t^n \subset M$ for all $t \in I$.

Now, let $\varphi : M \to [0, 1]$ be a smooth function so that $\varphi(p) = 0$ for $p \in [0, a/16) \times \partial M$ and $\varphi(p) = 1$, for $p \in M_{a/8}$. Also, denote

$$\hat{\chi}_t^n = \frac{\partial \tilde{\Psi}_t^n}{\partial s} \in \mathcal{X}(\hat{M}).$$

Then define $\chi_t^n \in \mathcal{X}(M)$ by $\chi_t^n = \varphi \hat{\chi}_t^n$ and let $\{\Phi_t^n\}_{t\in I}$ be the smooth one-parameter family of isotopies of M generated by $\{\chi_t^n\}_{t\in I}$. All of these isotopies leave $U = [0, a/16) \times \partial M$ fixed and $\Phi_t^n(1, \cdot) \in \mathrm{Diff}_0$ for all $t \in I$. Then, $\{\Phi_t^n(1, \Sigma_t^n)\}_{t\in I} \in \Lambda$. We have

$$\mathcal{H}^2(\Sigma_t^n) < m_0 - \delta \implies \hat{\chi}_t^n = 0 \implies \Phi_t^n(1, \Sigma_t^n) = \Sigma_t^n = \Gamma_t^n;$$

$$\mathcal{H}^2(\Sigma_t^n) \geq m_0 - \delta \implies \Sigma_t^n \subset M_{a/2} \implies \Phi_t^n(1, \Sigma_t^n) = \tilde{\Psi}_t^n(1, \Sigma_t^n) = \Gamma_t^n.$$

Therefore, $\{\Gamma_t^n\}_{t\in I} \in \Lambda$, for all $n \in \mathbb{N}$.

Now, the property $\mathcal{H}^2(\Gamma_t^n) \leq \mathcal{H}^2(\Sigma_t^n) - L(\mathfrak{d}(\Sigma_t^n, \mathcal{V}_\infty^*)) + e^{-n}$ implies that $\{\Gamma_t^n\}_{t\in I}$ is also minimizing and $\mathfrak{d}(\Gamma_{t_n}^n, \mathcal{V}_\infty^*) \to 0$ for any min-max sequence $\{\Gamma_{t_n}^n\}_{n\in\mathbb{N}}$. Since $\mathcal{H}^2(\Gamma_{t_n}^n) \to m_0 > m_0 - \delta$, we necessarily have $\mathfrak{d}(\Gamma_{t_n}^n, \mathcal{V}_\infty) \to 0$. Thus, up to a subsequence, there is some stationary varifold $V \in \mathcal{V}(M)$ such that $\Gamma_{t_n}^n \rightharpoonup V$. For some $n_0 \in \mathbb{N}$, we have $n > n_0 \implies \mathcal{H}^2(\Sigma_{t_n}^n) > m_0 - \delta \implies \Sigma_{t_n}^n \subset M_{a/2}$. Then, arguing with paths like above, we have the triangle inequality below

$$\begin{aligned} n > n_0 \implies \frac{a}{2} &\leq d(\Sigma_{t_n}^n, \partial M) \leq d(\Sigma_{t_n}^n, \Gamma_{t_n}^n) + d(\Gamma_{t_n}^n, \partial M) < \frac{3a}{8} + d(\Gamma_{t_n}^n, \partial M) \\ \implies \frac{a}{8} &< d(\Gamma_{t_n}^n, \partial M). \end{aligned}$$

This implies $\operatorname{supp} V \subset \{x \in M \,:\, d(x, \partial M) > a/8\}$.

The second step is to proceed as in before to find some subsequence $\Gamma_{t_n}^n$ which is almost minimizing in every annulus centered at a point $x \in M$ and with outer radius smaller than $r(x)$. We can require that $r(x) < a/16$ for all $x \in \partial M$. Since $d(\Gamma_{t_n}^n, \partial M) > a/8$ for $n > n_0$, this implies that whenever $B_{r(x)}(x) \cap \partial M \neq \varnothing$ we have $B_{r(x)}(x) \cap \Gamma_{t_n}^n = \varnothing$ for $n > n_0$. Therefore all arguments apply.

The third and final step is to prove the regularity of V. This can be done exactly as in because all arguments from (COLDING; DE LELLIS, 2003) are local and only take place in annular regions of small radius which do not intersect ∂M. $\square$

Part III

Rigidity of Min-Max Minimal Spheres

4 Genus and index

In this chapter, (M, g) will denote a connected compact orientable riemannian three-manifold without boundary and Λ will denote a saturated set of sweepouts such that no sweepout $\{\Sigma_t\}_{t\in I}$ in Λ contains a nonorientable surface. All surfaces are considered to be closed (compact without boundary). We denote by $g(\Sigma)$ the genus of Σ and $\mathrm{ind}(\Sigma)$ the Morse index of Σ (see Definition 1.28). Also, there is no special reason why we should use the interval $I = [0, 1]$ in our definitions, we can consider any closed interval $[a, b]$ instead.

4.1 The $(\star)_h$-condition

The next result is useful to prove that certain min-max minimal surfaces have Morse index one.

Proposition 4.1. *If $\{\Sigma_t\}_{t\in[-1,1]} \in \Lambda$ is a sweepout such that*

1. *Σ_0 is an embedded surface and there is a smooth variation $F : [-\epsilon, \epsilon] \times \Sigma_0 \to M$ such that $\Sigma_t = F_t(\Sigma_0)$, for all $t \in [-\epsilon, \epsilon]$;*
2. *$\mathcal{H}^2(\Sigma_t) < \mathcal{H}^2(\Sigma_0)$, for all $t \neq 0$;*
3. *$|\Sigma_0| = W(M, \Lambda)$;*

then Σ_0 is a minimal surface of index one.

Proof. First, we prove that Σ_0 is a minimal surface. If not, then $\vec{H} \neq 0$ at some point of Σ_0. Let X be any ambient vector field which is zero outside a tubular neighborhood of Σ_0 and is equal to $\vec{H}$ on Σ_0. Denote by $\{G_s\}_{s\in\mathbb{R}}$ the one-parameter family of diffeomorphisms generated by X and define the function $f : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ by $f(t, s) = \mathcal{H}^2(G_s(\Sigma_t))$. From (i) we have that f is smooth on $[-\epsilon, \epsilon] \times \mathbb{R}$. It follows from (ii) that

$$\frac{\partial f}{\partial t}(0,0) = \left.\frac{d}{dt}\right|_0 |G_0(\Sigma_t)| = \left.\frac{d}{dt}\right|_0 |\Sigma_t| = 0.$$

On the other hand, it follows from Remark 1.9 (ii) that

$$\frac{\partial f}{\partial s}(0,0) = \left.\frac{d}{ds}\right|_0 |G_s(\Sigma_0)| < 0.$$

Thus, the Taylor expansion of f around $(0, 0)$ has the form

$$f(t,s) = |\Sigma_0| + As + r(t,s), \quad \text{with } A < 0, \quad \lim_{(t,s)\to(0,0)} \frac{r(t,s)}{|(t,s)|} = 0.$$

Claim 4.2. *There exists $\delta > 0$ such that $f(t, \delta) < |\Sigma_0|$ for all $t \in [-1, 1]$.*

If the claim is false, then for all $\delta > 0$ there exists $t_\delta \in [-1,1]$ such that $f(t_\delta,\delta) \geq |\Sigma_0| = f(0,0)$. Take $\delta = \frac{1}{n}$, $n \in \mathbb{N}$. Since $[-1,1]$ is compact, we can suppose that $t_n \to t_0 \in [-1,1]$. Then

$$f(t_n,\frac{1}{n}) \geq |\Sigma_0| \overset{n\to\infty}{\Longrightarrow} f(t_0,0) \geq |\Sigma_0| \implies \mathcal{H}^2(\Sigma_{t_0}) \geq |\Sigma_0|$$

and then (ii) implies $t_0 = 0$. From $f(t_n,\frac{1}{n}) - |\Sigma_0| \geq 0$ follows that

$$0 \leq A\frac{1}{n} + r(t_n,\frac{1}{n}), \quad \forall n \in \mathbb{N}.$$

Thus,

$$0 < -A \leq \frac{-A\frac{1}{n}}{|(t_n,\frac{1}{n})|} \leq \frac{r(t_n,\frac{1}{n})}{|(t_n,\frac{1}{n})|}$$

is a contradiction with $\lim\limits_{n\to\infty} \frac{r(t_n,\frac{1}{n})}{|(t_n,\frac{1}{n})|} = 0$. This proves Claim 4.2. The sweepout $\{G_\delta(\Sigma_t)\}_{t\in[-1,1]}$ is in Λ, but Claim 4.2 says that

$$\sup_{t\in[-1,1]} \mathcal{H}^2(\Sigma_t) = \max_{t\in[-1,1]} \mathcal{H}^2(\Sigma_t) < |\Sigma_0| = W(M,\Lambda) = \inf_{\{\Gamma_t\}\in\Lambda} \sup_{t\in[-1,1]} \mathcal{H}^2(\Gamma_t),$$

a contradiction. This proves that Σ_0 is a minimal surface.

It remains to prove that $\operatorname{ind}(\Sigma_0) = 1$. Notice that (i) and (ii) imply that $\operatorname{ind}(\Sigma_0) \geq 1$. Let ν be a unit normal vector field along Σ_0 and $\phi_0 \in \mathcal{C}^\infty(\Sigma_0)$ such that

$$\left.\frac{\partial F}{\partial t}\right|_0 = \phi_0\nu := Z.$$

If $\operatorname{ind}(\Sigma_0) > 1$, we can choose orthonormal eigenfunctions $\phi_1,\phi_2 \in \mathcal{C}^\infty(\Sigma_0)$ for the Jacobi operator with negative eigenvalues (see Proposition 1.26). There exists a non-trivial linear combination of ϕ_1 and ϕ_2, say $\phi_3 = a\phi_1 + b\phi_2$, which is orthogonal to $L\phi_0 \in \mathcal{C}^\infty(\Sigma_0)$, i.e.

$$\int_{\Sigma_0} \phi_3 L\phi_0 \, d\Sigma_0 = 0, \quad \phi_3 \neq 0.$$

This can be done for any vector space with inner product, a pair of linear independent vectors and a third one. Consider the normal vector field $\tilde{X} = \phi_3\nu$ along Σ_0 and extend it smoothly to be zero outside a tubular neighborhood of Σ_0. Let $\{\tilde{F}_s\}_{s\in\mathbb{R}}$ be the smooth one-parameter family of diffeomorphisms generated by $\tilde{X}$. Then, consider $\tilde{f} : [-1,1]\times\mathbb{R} \to \mathbb{R}$ given by $\tilde{f}(t,s) = \mathcal{H}^2(\tilde{F}_s(\Sigma_t))$. Again, $\tilde{f}$ is smooth on $[-\epsilon,\epsilon]\times\mathbb{R}$. Since Σ_0 is a minimal surface, we have

$$\frac{\partial \tilde{f}}{\partial t}(0,0) = 0 = \frac{\partial \tilde{f}}{\partial s}(0,0).$$

From (ii), we have

$$\frac{\partial^2 \tilde{f}}{\partial t^2}(0,0) < 0.$$

Doing basically the same computation in the second variation formula, we have

$$\frac{\partial^2 \tilde{f}}{\partial s \partial t}(0,0) = -\int_{\Sigma_0} \phi_3 L\phi_0 \, d\Sigma_0 = 0.$$

By the choice of ϕ_3,

$$\frac{\partial^2 \tilde{f}}{\partial s^2}(0,0) = -\int_{\Sigma_0} \phi_3 L\phi_3 \, d\Sigma_0 = -\int_{\Sigma_0} (a\phi_1 + b\phi_2)(a(-\lambda_1)\phi_1 + b(-\lambda_2)\phi_2) \, d\Sigma_0$$
$$= \lambda_1 a^2 + \lambda_2 b^2 < 0,$$

since the eigenvalues λ_1 and λ_2 are negative. Then the Taylor expansion of $\tilde{f}$ around $(0,0)$ has the form

$$\tilde{f}(t,s) = |\Sigma_0| + At^2 + Bs^2 + r(t,s), \quad \text{with } A, B < 0, \quad \lim_{(t,s)\to(0,0)} \frac{r(s,t)}{|(t,s)|^2} = 0$$

CLAIM 4.3. *There exists $\delta > 0$ such that $\tilde{f}(t,\delta) < |\Sigma_0|$ for all $t \in [-1,1]$.*

As before, if this is not true, we find a sequence $t_n \to 0$ in $[-1,1]$ such that $\tilde{f}(t_n, \frac{1}{n}) \geq |\Sigma_0| = \tilde{f}(0,0)$. Thus

$$At_n^2 + B\frac{1}{n^2} + r(t_n, \frac{1}{n}) \geq 0, \quad \forall n \in \mathbb{N}.$$

Then, denoting $C = \min\{-A, -B\} > 0$, we have

$$0 < C = \frac{C(t_n^2 + \frac{1}{n^2})}{t_n^2 + \frac{1}{n^2}} \leq \frac{-At_n^2 - B\frac{1}{n^2}}{t_n^2 + \frac{1}{n^2}} \leq \frac{r(t_n, \frac{1}{n})}{t_n^2 + \frac{1}{n^2}}.$$

This is a contradiction with $\lim_{n\to\infty} \frac{r(t_n,\frac{1}{n})}{t_n^2 + \frac{1}{n^2}} = 0$, thus Claim 4.3 is true. But such a claim is a contradiction with $|\Sigma_0| = W(M,\Lambda)$, as before (because $\{\tilde{F}_\delta(\Sigma_t)\}_{t\in[-1,1]} \in \Lambda$). Therefore, $\operatorname{ind}(\Sigma_0) = 1$. □

DEFINITION 4.4 (HEEGAARD SPLITTING). *A closed orientable surface $\Sigma \subset M$ is said to be a* **Heegaard splitting** *if $M \backslash \Sigma = A \cup B$ with $A \cap B = \varnothing$ and $\overline{A}$ and $\overline{B}$ are handlebodies, i.e. diffeomorphic to a solid ball with handles attached. The* **Heegaard genus of** M *is the lowest possible genus of a Heegaard splitting of M.*

REMARK 4.5. *Every closed orientable three-manifold M has a Heegaard splitting. Indeed, if M is compact, then M has a finite triangulation (by tetrahedra) T. Let σ be the 1-skeleton structure of T and A be an open ϵ-tubular neighborhood of σ. Notice that $\overline{A}$ is homeomorphic to a solid sphere with finite handles and $\overline{B}$ as well, with $B = M \backslash \overline{A}$. Then, ∂A can be deformed into a smooth embedded closed surface Σ. Since M is orientable, T can be oriented and hence Σ is orientable and a Heegaard splitting of M.*

DEFINITION 4.6 ($(\star)_h$-CONDITION). *If $h \geq 0$ is an integer, we denote by $\mathscr{E}_h$ the collection of all connected embedded minimal surfaces $\Sigma \subset M$ with $g(\Sigma) \leq h$. We say that (M,g) satisfies the $(\star)_h$-***condition** *if*

1. *M does not contain embedded nonorientable surfaces;*
2. *no surface in $\mathscr{E}_h$ is stable.*

REMARK 4.7. *If M has positive Ricci curvature and does not contain embedded nonorientable surfaces, then it follows from Corollary 1.22 that M satisfies the $(\star)_h$-condition for all h. Lens spaces $L(p,q)$ with odd p and the Poincaré homology sphere are some examples.*

LEMMA 4.8. *If M is a smooth connected orientable three-manifold and $\Sigma \subset M$ is an embedded connected orientable closed (as a set) surface, then $M \backslash \Sigma$ consists of one or two components.*

Proof. Since Σ and M are orientable, we can consider a unit normal vector field ν along Σ. Since M is connected, any point $p \in M \backslash \Sigma$ can be joined to any other point $q \in \Sigma$ by a smooth path. Let $\gamma : [a, b] \to M$ be a smooth path with $\gamma(a) = p \in M \backslash \Sigma$ and $\gamma(b) = q \in \Sigma$. Since Σ is closed, $\gamma^{-1}(\Sigma)$ is closed in $[a, b]$ and there is some $t_0 \in (a, b]$ such that $\gamma([a, t_0)) \subset M \backslash \Sigma$ and $\gamma(t_0) \in \Sigma$. Thus, *every* point $p \in M \backslash \Sigma$ can be joined by a smooth path to *some* point $q' \in \Sigma$, with q' being the only contact point of Σ and the trace of the path. Moreover, we can suppose that the path touches Σ transversely, i.e. if u is the vector tangent to the path at the contact point, then $g(u, \nu) \neq 0$ (if the path is tangent at the contact point, since Σ is embedded, we can "fix" the path in a small neighborhood of the point in order to turn it transversal).

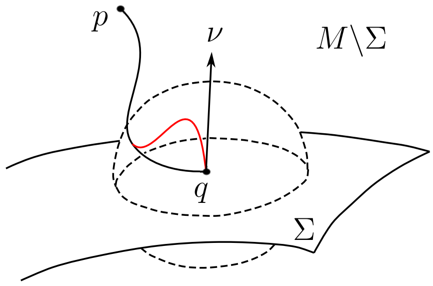


Figure 9

CLAIM 4.9. *Suppose that $p \in M \backslash \Sigma$ can be joined to* **some** *point $q \in \Sigma$ by a path $\gamma : [a, b] \to M$ with $\gamma([a, b)) \subset M \backslash \Sigma$, $\gamma(a) = p$, $\gamma(b) = q$ and $g(\gamma'(b), \nu) < 0$. Then p can be joined to* **every** *point of Σ in this fashion.*

Let Σ^+ be the set of points of Σ that are attained by paths beginning at p as in the claim. By assumption, $\Sigma^+ \neq \varnothing$. If $q \in \Sigma^+$, then taking a small adapted coordinated neighborhood U of q, we see that every $q' \in U \cap \Sigma$ is in Σ^+. Therefore, Σ^+ is open in Σ. If $q \notin \Sigma^+$, then, by the same argument, no point in a neighborhood of q as before can be in Σ^+. Thus, $\Sigma \backslash \Sigma^+$ is open in Σ. Hence, Σ^+ is open and closed in Σ and since Σ is connected, we must have $\Sigma^+ = \Sigma$. This proves the claim.

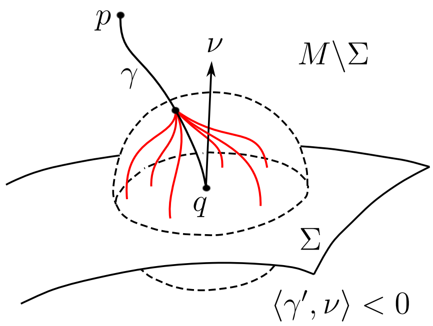


Figure 10

Let A^+ be the set of points $p \in M \backslash \Sigma$ that can be joined to some (and, therefore, to every) point of Σ by a path $\gamma : [a, b] \to M$ with $\gamma([a, b)) \subset M \backslash \Sigma$, $\gamma(a) = p$, $\gamma(b) \in \Sigma$ e $g(\gamma'(b), \nu) < 0$. Changing the last condition by $g(\gamma'(b), \nu) > 0$ we define A^-. What we did in the beginning of the proof shows that $M \backslash \Sigma = A^+ \cup A^-$.

CLAIM 4.10. *A^+ and A^- are connected components of $M\backslash\Sigma$.*

Let $p \in A^+$ be fixed. If $p' \in M\backslash\Sigma$ can be joined to p by a path contained in $M\backslash\Sigma$, then concatenating this path with that one which joins p to the surface Σ and then turning it smooth, we obtain a smooth path γ joining p' to Σ with $g(\gamma', \nu) < 0$. Therefore, $p' \in A^+$. Hence, if C_p is the connected component of $M\backslash\Sigma$ which contains p, we have $C_p \subset A^+$. Now, let $p, p' \in A^+$. Then p and p' can be joined to Σ by smooth paths γ_1 and γ_2 with $g(\gamma_i', \nu) < 0$, $i = 1, 2$. By claim 4.9, we can suppose that γ_1 and γ_2 have the same point of contact $q \in \Sigma$. Again taking a coordinated neighborhood U of q as before, we see that the ends of $\gamma_1 \cap U$ and $\gamma_2 \cap U$ are in the same "hemisphere" of U (otherwise, it they would not be able to hit the point q with the same sign). Therefore, working inside this hemisphere, it is possible to concatenate γ_1 and $-\gamma_2$, "unstick" from Σ this part of the path and finally, to turn it smooth, obtaining a smooth path joining p to p' completely contained in $M\backslash\Sigma$. This shows that $A^+ = C_p$. The proof that A^- is a connected component of $M\backslash\Sigma$ is the same. This proves the claim.

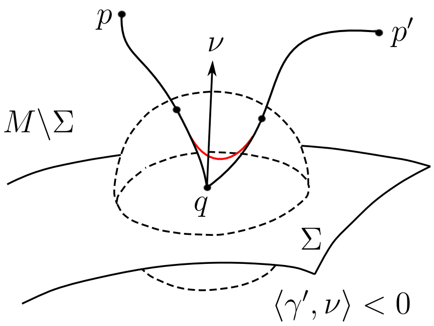


Figure 11

Since $M\backslash\Sigma = A^+ \cup A^-$ and A^+, A^- are connected components, we have that $M\backslash\Sigma$ has at most two components.

□

LEMMA 4.11. *If (M, g) satisfies the $(\star)_h$-condition, then any surface $\Sigma \in \mathscr{E}_h$ is a Heegaard splitting.*

Proof. Let $\Sigma \in \mathscr{E}_h$. We first prove that Σ must separate M in two components. By the previous lemma, $M\backslash\Sigma$ has at most two components. Suppose, by contradiction, that $M\backslash\Sigma$ is connected. Let $\phi \in \mathcal{C}^\infty(\Sigma)$ be an eigenfunction for the lowest eigenvalue λ of the Jacobi operator. Since Σ is unstable, we have $\lambda < 0$. By Proposition 1.26 (iv), we can take ϕ strictly positive. Let X be a vector field in M such that $X = \phi\nu$ on Σ, where ν is a unit normal vector field along Σ. Let $\{F_t\}_{t\in\mathbb{R}}$ be the smooth one-parameter family of diffeomorphisms generated by X. Since Σ does not separate M, for $t > 0$ sufficiently small we have that

$$M\backslash(F_t(\Sigma) \cup F_{-t}(\Sigma)) = A_t \cup B_t,$$

where A_t, B_t are disjoint open regions with $\Sigma \subset B_t$.

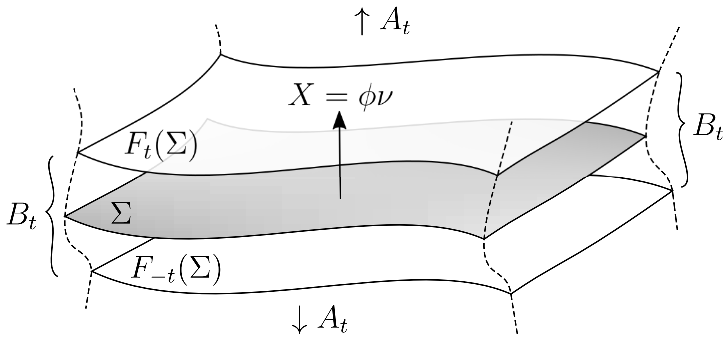


Figure 12

From (HUISKEN; POLDEN, 1999, Theorem 3.2), we have

$$\frac{\partial}{\partial t}\bigg|_0 g(\vec{H}(F_t(\Sigma)), \nu_t) = L\phi = -\lambda\phi > 0,$$

where ν_t is the unit normal vector field along $F_t(\Sigma)$ which varies smoothly on t and such that $\nu_0 = \nu$. Since $\vec{H}(F_0(\Sigma)) = \vec{H}(\Sigma) = 0$, this formula tells us that, for small $t > 0$,

$$g(\vec{H}(F_{-t}(\Sigma)), \nu_{-t}) < 0 \quad \text{and} \quad g(\vec{H}(F_t(\Sigma)), \nu_t) > 0,$$

i.e. the mean curvature vector of ∂A_t points into A_t (∂A_t is said to be *mean convex*). It follows from (MEEKS; SIMON; S. T. YAU, 1982, Lemma 4, p. 657) that we can minimize area in the isotopy class of one of the boundary components of ∂A_t, say $F_t(\Sigma)$, to obtain an embedded stable minimal surface Σ' in A_t. Since $\Sigma \in \mathscr{E}_h$, we have $g(\Sigma') = g(F_t(\Sigma)) = g(\Sigma) \leq h$. Thus, $\Sigma' \in \mathscr{E}_h$ is stable. But this is impossible since M satisfies the $(\star)_h$-condition. Therefore, $M \backslash \Sigma$ must have two components.

To prove that Σ is a Heegaard splitting, we will use the following characterization for handlebodies:

PROPOSITION 4.12. *(MEEKS; SIMON; S. T. YAU, 1982, Proposition 1, p. 650) Let N be a compact three-dimensional Riemannian manifold with non-empty boundary. N is a handlebody if and only if for every compact surface Σ in the interior of N and for every positive number ϵ there exists a surface Σ' isotopic to Σ such that $|\Sigma'| < \epsilon$. Actually N will be a handlebody if and only if the isotopy class of a surface parallel to a boundary component contains surfaces of arbitrarily small area.*

Suppose Σ is not a Heegaard splitting. Then, some of the two components of $M \backslash \Sigma$, say N, is not a handlebody. Since $\Sigma = \partial N$ is unstable, and N is not a handlebody, we can minimize area in its isotopy class and obtain a stable minimal surface Σ' in the interior of N. Since Σ and Σ' are isotopic, $g(\Sigma') = g(\Sigma) \leq h$ and $\Sigma' \in \mathscr{E}_h$. Again, this is impossible, since M satisfies the $(\star)_h$-condition. This proves the lemma. □

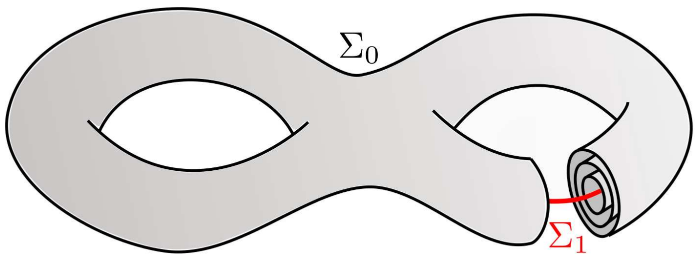
LEMMA 4.13. *If (M, g) satisfies the $(\star)_h$-condition, then any $\Sigma \in \mathscr{E}_h$ must intersect every other embedded minimal surface.*

Proof. Let Σ_1 and Σ_2 be embedded minimal surfaces with $\Sigma_1 \in \mathscr{E}_h$. Suppose by contradiction that $\Sigma_1 \cap \sigma_2 = \varnothing$. By the previous lemma, Σ_1 is a Heegaard splitting. Then, there is some region $A \subset M$, homeomorphic to a handlebody with $\Sigma_1 = \partial A$ and $\Sigma_2 \subset A$. Then there is a region B of M such that $\partial C = \Sigma_1 \cup \Sigma_2$. But then C is not a handlebody. Since Σ_1 is unstable, it follows again from the characterization of handlebodies above that we can minimize area in the isotopy class of Σ_1, to obtain a stable embedded minimal surface $\Sigma' \subset C$ with $g(\Sigma') \leq h$. Again, this contradicts the fact that M satisfies the $(\star)_h$-condition. The lemma is proved. $\square$

Let Σ be a Heegaard splitting of M. There is a natural class of sweepouts $\{\Sigma_t\}_{t\in[-1,1]}$ associated to Σ. Each $\{\Sigma_t\}_{t\in[-1,1]}$ in this class satisfies

1. $\Sigma_0 = \Sigma$ and Σ_t is isotopic to Σ for all $-1 < t < 1$;

2. if N_1 and N_2 are the connected components of $M \backslash \Sigma$ then, up to a change of N_1 and N_2, $\{\Sigma_t\}_{t\in[-1,0]}$ foliates N_1 and $\{\Sigma_t\}_{t\in[0,1]}$ foliates N_2, with Σ_{-1} and Σ_1 being graphs.



The smallest saturated set that contains this class of sweepouts is denoted by Λ_Σ and we call it the ***saturated set associated to*** Σ. If $g(\Sigma) = h$, we define the ***large saturated set associated to*** Σ, denoted by Λ^h, as the union of all saturated sets associated with Heegaard splittings of genus h.

The goal of this section is to prove the following theorem:

THEOREM 4.14. *Suppose (M, g) satisfies the $(\star)_h$-condition, where h is the Heegaard genus of M. Then there is an orientable embedded minimal surface $\Sigma_0 \subset M$ with $g(\Sigma_0) = h$ and $\mathrm{ind}(\Sigma_0) = 1$ such that*

$$|\Sigma_0| = \inf_{S\in\mathscr{E}_h} |S| = W(M, \Lambda_{\Sigma_0}) = W(M, \Lambda^h).$$

To prove this theorem, we will use some results without proof. The first one concerns on compactness properties for minimal surfaces and is proved in (WHITE, 1987). Before we state it, we give some definitions and notations.

Denote by $B_1(M)$ the unit sphere bundle on M, i.e. the fiber at $x \in M$ is $B_1(p) = \{\nu \in T_xM \ : \ \|v\| = 1\}$. We denote a point of $B_1(M)$ by (x, ν), with $x \in M$ and $\nu \in B_1(x)$. Consider a function $\Phi : B_1(M) \to \mathbb{R}$. This defines a functional on orientable surfaces in M by

$$\Phi(\Sigma) = \int_\Sigma \Phi(x, \nu(x))\, d\Sigma,$$

where Σ is the surface and ν is the unit normal vector field along Σ. If Φ is even, i.e. $\Phi(x,\nu) \equiv \Phi(x,-\nu)$, then $\Phi(\Sigma)$ is defined even if Σ is nonorientable. We say that Σ is Φ-stationary if

$$\left.\frac{d}{dt}\right|_0 \Phi(\Sigma_t) = 0,$$

for any variation $\{\Sigma_t\}_{t\in[-\epsilon,\epsilon]}$, with $\Sigma_0 = \Sigma$. Of course, $\Phi \equiv 1$ is the area functional, and in this case Σ is Φ-stationary if and only if it is a minimal surface.

If Γ is an embedded curve in M, then $\|\Gamma\|_{2,\alpha}$ denotes the $\mathcal{C}^{2,\alpha}$-norm of Γ parametrized by arc length and

$$\|\Gamma\|^*_{2,\alpha} = \|\Gamma\|_{2,\alpha} + \max\left\{\frac{d_\Gamma(x,y)}{d_M(x,y)} \,:\, x,y\in\Gamma,\, x\neq y\right\},$$

where d_Γ and d_M are the intrinsic geodesic distance on Γ and M, respectively. Now we state the compactness result in its generality. We will need just a particular case of it.

THEOREM 4.15. *(WHITE, 1987, Theorem 3, p. 251) Let M be a compact 3-manifold with (possibly empty) boundary. Let Σ_i be a sequence of connected immersed Φ_i stationary surfaces (with or without boundary), where*

$$\|\Phi_i - \Phi\|_{1,\alpha} + \|D_2\Phi_i - D_2\Phi\|_{1,\alpha} \to 0 \text{ as } i\to\infty.$$

Suppose that the area and the genus of Σ_i and the total curvature of $\partial\Sigma_i$ are uniformly bounded, and that the $\partial\Sigma_i$ converges as sets (i.e. in the Hausdorff metric) to some set Γ. Then

1. *There is a finite set $S\subset M$ and a subsequence $\Sigma_{i'}$ that converges uniformly in $\mathcal{C}^{2,\beta}$ ($\beta<\alpha$) on compact subsets of $\Omega = M\backslash(S\cup\Gamma)$ to a Φ-stationary surface Σ.*
2. *If $\|\partial\Sigma_i\|^*_{2,\alpha}$ is bounded, then we can let $\Omega = M\backslash S$.*

Now, suppose that each Σ_i is embedded, Φ_i is even and

$$\|\Phi_i - \Phi\|_{2,\alpha} + \|D_2\Phi_i - D_2\Phi\|_{2,\alpha} \to 0 \text{ as } i\to\infty.$$

Then

3. *$(\Sigma\cup S)\backslash\Gamma$ is a regular embedded surface.*
4. *If ∂M is strictly convex, $\partial\Sigma_i\subset\partial M$ is not empty, and $\|\partial\Sigma_i\|^*_{2,\alpha}$ is uniformly bounded, then S is empty.*

Our surfaces are going to be embedded, all boundaries empty and the sequence of functionals is going to be constant equal to the area functional. Thus, all hypotheses in this theorem are going to be trivially satisfied.

Now we introduce the second compactness result. Let $\mathscr{M}_n$ be the set of embedded closed minimal surfaces Σ in M with Euler characteristc $\chi(\Sigma)\geq n$ and consider the weak topology on $\mathscr{M}_n$ induced as a subspace of $\mathcal{V}_2(M)$ (space of 2-varifolds on M). We have

THEOREM 4.16. *(ANDERSON, 1985, Theorem 4.2, p. 103) The boundary $\partial\mathscr{M}_n = \overline{\mathscr{M}}_n \backslash \mathscr{M}_n$ of $\mathscr{M}_n$ is contained in $\mathscr{M}_{n/2}$, counted with multiplicity ≥ 2.*

The next and last result we state before we prove Theorem 4.14 gives a genus bound for minimal surfaces obtained by the min-max method.

THEOREM 4.17. *(LELLIS; PELLANDINI, 2009, Theorem 0.6) Let Λ be a saturated set of sweepouts in M and $\{\Sigma^n_{t_n}\}$, Σ be the min-max sequence and minimal surface produced in the proof of the Simon–Smith Theorem. Let $\Sigma = \sum_{i=1}^N n_i\Gamma_i$ where the Γ_i's are connected components of Σ without multiplicity and $n_i \in \mathbb{N}\backslash\{0\}$. Denoting $\mathcal{O} = \{i : \Gamma_i$ is orientable$\}$ and $\mathcal{N} = \{i : \Gamma_i$ is nonorientable$\}$, we have*

$$\sum_{i\in\mathcal{O}} g(\Gamma_i) + \frac{1}{2}\sum_{i\in\mathcal{N}}(g(\Gamma_i) - 1) \leq g_0 := \liminf_{n\uparrow\infty}\liminf_{\tau\to t_n} g(\Sigma^n_\tau).$$

Now, we prove Theorem 4.14. We recall its statement:

THEOREM 4.14. *Suppose (M, g) satisfies the $(\star)_h$-condition, where h is the Heegaard genus of M. Then there is an orientable embedded minimal surface $\Sigma_0 \subset M$ with $g(\Sigma_0) = h$ and $\operatorname{ind}(\Sigma_0) = 1$ such that*

$$|\Sigma_0| = \inf_{S\in\mathscr{E}_h} |S| = W(M, \Lambda_{\Sigma_0}) = W(M, \Lambda^h).$$

Proof of Theorem 4.14. By the Simon–Smith Theorem, we know that (M, g) has at least one embedded minimal surface (we just need to consider a saturated set Λ of sweepouts such that $W(M, \Lambda) > 0$ and we can do this by consider a sweepout given by level sets of a Morse function on M). Therefore, the set $\mathscr{E}_\infty$ of all embedded minimal surfaces in M is nonempty. Let

$$h' = \min\{g(S) : S \in \mathscr{E}_\infty\}.$$

CLAIM 4.18. *There is an embedded minimal surface Σ_0 in M such that $g(\Sigma_0) = h'$ and $|\Sigma_0| = \inf_{S\in\mathscr{E}_{h'}} |S|$.*

Let $S_n \in \mathscr{E}_{h'}$ be a sequence such that $\lim_{n\to\infty} |S_n| = \inf_{S\in\mathscr{E}_{h'}} |S|$. Notice that by the definition of h' and $\mathscr{E}_{h'}$, we have $g(S_n) = h'$, for all $n \in \mathbb{N}$. Then by Theorem 4.15, there is an embedded minimal surface Σ_0 in M and a subsequence S_{n_k} such that $S_{n_k} \to \Sigma_0$. Then

$$|\Sigma_0| = \lim_{n\to\infty} |S_n| = \inf_{S\in\mathscr{E}_{h'}} |S|.$$

Now we prove that $g(\Sigma_0) = h'$. First, not that all surfaces involved are orientable, because M satisfies the $(\star)_h$-condition. Thus $\chi(S_n) = 2(1-g(S_n)) = 2(1-h')$ and hence $S_n \in \mathscr{M}_{2(1-h')}$, for all $n \in \mathbb{N}$. This implies $\Sigma_0 \in \overline{\mathscr{M}}_{2(1-h')}$. We have two possibilities: either $\Sigma_0 \in \mathscr{M}_{2(1-h')}$ or $\Sigma_0 \in \partial\mathscr{M}_{2(1-h')}$. If $\Sigma_0 \in \mathscr{M}_{2(1-h')}$, then

$$2(1 - g(\Sigma_0)) = \chi(\Sigma_0) \geq 2(1 - h') \Rightarrow g(\Sigma_0) \leq h' \Rightarrow g(\Sigma) = h',$$

by the definition of h'. Now if $\Sigma_0 \in \partial\mathscr{M}_{2(1-h')}$, then by Theorem 4.16, we have $\Sigma_0 \in \mathscr{M}_{1-h'}$. Then

$$2(1 - g(\Sigma_0)) = \chi(\Sigma_0) \geq 1 - h' \Rightarrow g(\Sigma_0) \leq \frac{h'+1}{2}.$$

If $h' = 0$, then $g(\Sigma_0) \le \frac{1}{2} \Rightarrow g(\Sigma_0) = 0 = h'$. If $h' \ge 1$, then

$$g(\Sigma_0) \le \frac{h'+1}{2} \le h',$$

and again by the definition of h' we have $g(\Sigma_0) = h'$. This proves Claim 4.18.

CLAIM 4.19. $h' = h$.

Let Σ be a Heegaard splitting of M with least possible genus $g(\Sigma) = h$ and consider the saturated set Λ_Σ associated to Σ. Then applying Theorem 4.17 and noticing that M does not contain embedded nonorientable surfaces, we have a minimal surface $\Sigma' = \sum_{i=1}^n n_i\Gamma_i$ attained as a limit of a min-max sequence $\{\Sigma^n_{t_n}\} \in \Lambda_\Sigma$ and we have

$$\sum_{i=1}^N g(\Gamma_i) \le \liminf_{n\to\infty} \liminf_{\tau\to t_n} g(\Sigma^n_\tau) = g(\Sigma) = h.$$

This implies that each connected component Γ_i is in $\mathscr{E}_h$. It follows from Lemma 4.13 that $N = 1$, i.e. Σ' has only one connected component. Thus

$$h' \le g(\Sigma') \le h.$$

Now, since $g(\Sigma_0) = h' \le h$, Lemma 4.11 tell us that Σ_0 is a Heegaard splitting of M. But then

$$h \le g(\Sigma_0) = h'.$$

This proves that $h = h'$.

CLAIM 4.20. $|\Sigma_0| = W(M, \Lambda_{\Sigma_0}) = W(M, \Lambda^h)$. *Moreover,* Σ_0 *is contained in a sweepout* $\{\Sigma_t\}_{t\in[-1,1]} \in \Lambda_{\Sigma_0}$ *such that*

1. *there is a smooth variation* $F : [-\epsilon, \epsilon] \times \Sigma_0 \to M$ *such that* $\Sigma_t = F_t(\Sigma_0)$*, for all* $t \in [-\epsilon, \epsilon]$*;*
2. $\mathcal{H}^2(\Sigma_t) < \mathcal{H}^2(\Sigma_0)$*, for all* $t \ne 0$*;*

Notice that once we prove Claim 4.20, it follows from Proposition 4.1 that $\operatorname{ind}(\Sigma_0) = 1$ and the theorem is proved.

Since Σ_0 is a Heegaard splitting, $M\backslash\Sigma_0 = N_1 \cup N_2$, with $N_1 \cap N_2 = \varnothing$, $\overline{N}_1$ and $\overline{N}_2$ handlebodies. Let ν be the unit normal vector field along Σ_0 which points into N_1. Let $\phi \in \mathcal{C}^\infty(\Sigma)$ be an eigenfunction associated to the lowest eigenvalue λ of the Jacobi operator L. By Proposition 1.26, we can take ϕ strictly positive on Σ_0. Since M satisfies the $(\star)_h$-condition, Σ_0 is unstable and $\lambda < 0$. Let X be a vector field on M with $X = \phi\nu$ on M and let $\{F_t\}_{t\in\mathbb{R}}$ be the one-parameter family of diffeomorphisms generated by X. Denote $\Sigma_t = F_t(\Sigma_0)$. Since ϕ is strictly positive, there is some $\epsilon_1 > 0$ such that $\Sigma_t \subset N_1$ (resp. N_2) for all $0 < t < \epsilon_1$ (resp. $-\epsilon_1 < t < 0$). We have

$$\left.\frac{\partial}{\partial t}\right|_0 g(\vec{H}(\Sigma_t), \nu_t) = L\phi = -\lambda\phi > 0.$$

Thus there is $\epsilon_2 > 0$ such that the mean curvature vector $\vec{H}(\Sigma_t)$ points into N_1 (resp. into N_2) for all $0 < t < \epsilon_2$ (resp. $-\epsilon_2 < t < 0$). Also, $\left.\frac{d}{dt}\right|_0 |\Sigma_t| = 0$ (Σ_0 is minimal) and

$$\left.\frac{d^2}{dt^2}\right|_0 |\Sigma_t| = -\int_{\Sigma_0} \phi L\phi \, d\Sigma_0 = \lambda \int_{\Sigma_0} \phi^2 \, d\Sigma_0 < 0.$$

Thus, there exists $\epsilon_3 > 0$ such that

$$|\Sigma_t| < |\Sigma_0|, \quad \text{for all } 0 < t < \epsilon_3.$$

Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\} > 0$. The surface Σ_ϵ bounds a handlebody $N \subset N_1$. Consider a sweepout $\{\tilde{\Sigma}_t\}_{t\in[0,1]}$ of N such that $\tilde{\Sigma}_t = \Sigma_{t+\epsilon}$ for all $t \in [0, \delta]$ and some small $\delta > 0$. Let $\tilde{\Lambda}$ be the saturated set of sweepouts in N generated by $\{\tilde{\Sigma}_t\}_{t\in[0,1]}$. If $W(N, \tilde{\Lambda}) > |\partial N|$, since $H(\partial N) > 0$, it follows from Theorem 3.32 that there exists an embedded minimal surface $\Sigma' \subset \text{int}(N)$ and therefore disjoint from Σ_0. This is a contradiction with Lemma 4.13. Therefore $W(N, \tilde{\Lambda}) \leq |\partial N| < |\Sigma_0|$. Since $W(N, \tilde{\Lambda}) = \inf_{\{\Gamma_t\}_{t\in[0,1]}\in\tilde{\Lambda}} \sup_{t\in[0,1]} |\Gamma_t|$, we can find $\{\Gamma_t\}_{t\in[0,1]} \in \tilde{\Lambda}$ such that

$$\sup_{t\in[0,1]} |\Gamma_t| < |\Sigma_0|.$$

Because N has nonempty boundary, recall that in the definition of saturated sets we ask the one-parameter families of diffeormophisms to leave some neighborhood of the boundary fixed. Because of this, we still have

$$\Gamma_t = \Sigma_{t+\epsilon}, \quad \text{for all } t \in [0, \delta'],$$

and some small $\delta' > 0$. Arguing in the same way for $\Sigma_{-\epsilon}$, we are able to build a sweepout $\{\Sigma_t\}_{t\in[-1,1]}$ of M which is in Λ_{Σ_0} and satisfies (i) and (ii) of the Claim 4.20.

Now we prove that $|\Sigma_0| = W(M, \Lambda_{\Sigma_0})$. We have

$$W(M, \Lambda_{\Sigma_0}) \leq \sup_{t\in[0,1]} |\Sigma_t| = |\Sigma_0|.$$

By the Simon–Smith Theorem, there is an embedded minimal surface $S_0 = \sum_{i=1}^N n_i\Gamma_i$ embedded in M, such that $|S_0| = W(M, \Lambda_{\Sigma_0})$. Each connected component Γ_i has to be orientable (there are no nonorientable embedded surfaces) and by Theorem 4.17,

$$g(\Gamma_i) \leq \sum_{i=1}^N g(\Gamma_i) \leq g(\Sigma_0) = h.$$

Therefore, any connected component Γ_i is in $\mathscr{E}_h$ and $|\Gamma_i| \leq |S_0| = W(M, \Lambda) \leq |\Sigma_0|$. But recall that

$$|\Sigma_0| = \inf_{S\in\mathscr{E}_h} |S| \leq |\Gamma_i|.$$

Therefore $|\Sigma_0| = W(M, \Lambda_{\Sigma_0})$. Now, since $\Lambda_{\Sigma_0} \subset \Lambda^h$, we have $W(M, \Lambda^h) \leq W(M, \Lambda_{\Sigma_0}) = |\Sigma_0|$. Now, by the same argument used above, we can find a surface $S_0 \in \mathscr{E}_h$ such that

$$|\Sigma_0| = \inf_{S\in\mathscr{E}_h} |S| \leq |S_0| \leq W(M, \Lambda^h) \leq |\Sigma_0|,$$

and we have $|\Sigma_0| = W(M, \Lambda^h)$. This proves Claim 4.20 and the theorem follows. □

4.2 NONORIENTABLE SURFACES CASE

The nonorientable closed surface $N_{\tilde{h}}$ of genus $\tilde{h}$ is defined as a sphere with $\tilde{h}$ disks removed and $\tilde{h}$ Möbius strips (with boundary) attached. We say that $N_{\tilde{h}}$ is a sphere with $\tilde{h}$ *cross-caps*. The surface $N_{\tilde{h}}$ is also obtained as the connected sum of $\tilde{h}$ projective planes $\mathbb{RP}^2$. The Euler characteristic of $N_{\tilde{h}}$ is given by $\chi(N_{\tilde{h}}) = 2 - \tilde{h}$. Every closed nonorientable surface is homeomorphic to $N_{\tilde{h}}$, for some $\tilde{h} \geq 1$.

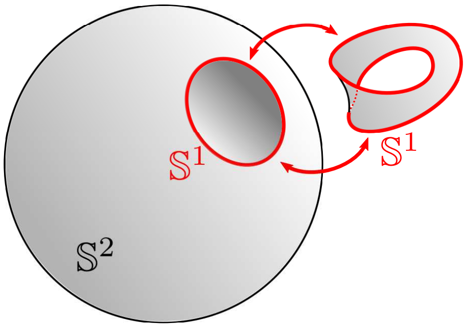


Figure 13 – N_1 is obtained by cutting a disk from $\mathbb{S}^2$ and gluing a Möbius strip along its boundary.

In this section, (M, g) still is a compact orientable riemannian three-manifold but now we suppose that M contains an embedded nonorientable closed surface. Let $\tilde{h}$ be the lowest genus of an embedded nonorientable closed surface in M, i.e. there is an embedding of $N_{\tilde{h}}$ into M and every embedded nonorientable closed surface in M has genus greater than or equal to $\tilde{h}$. Let $\mathscr{F}$ denote the set of all embedded surfaces in M diffeomorphic to $N_{\tilde{h}}$ and

$$\mathscr{A}(M, g) = \inf\{|S| \,:\, S \in \mathscr{F}\}.$$

LEMMA 4.21. $\mathscr{A}(M, g) > 0$.

Proof. This follows from the following result:

LEMMA 4.22. *(MEEKS; SIMON; S. T. YAU, 1982, Lemma 1, p. 625, adapted) Let $\rho_0 > 0$ be such that $\exp_{x_0} : B_{\rho_0} \subset T_{x_0}M \to \mathcal{B}_{\rho_0}(x_0)$ is a diffeomorphism for all $x_0 \in M$ (such ρ_0 exists, since M is compact). There is a number $\delta \in (0, 1)$ such that: if $\Sigma \subset M$ is a closed embedded surface and*

$$|\Sigma \cap \mathcal{B}_{\rho_0}(x_0)| \leq \delta^2\rho_0^2 \quad \textit{for all } x_0 \in M,$$

then there is a unique compact $K_\Sigma \subset M$ with $\partial K_\Sigma = \Sigma$ and

$$\mathrm{vol}(K_\Sigma \cap \mathcal{B}_{\rho_0}(x_0)) \leq \delta^2\rho_0^3, \quad x_0 \in M.$$

Suppose $\mathscr{A}(M, g) = 0$. Then, we can find surfaces $\Sigma \in \mathscr{F}$ with arbitrarily small area. Therefore, we can find a surface $\Sigma \in \mathscr{F}$ such that $|\Sigma \cap \mathcal{B}_{\rho_0}(x_0)| \leq \delta^2\rho_0^2$ for all $x_0 \in M$. In particular, the lemma above guarantees the existence of a compact region K_Σ with $\partial K_\Sigma = \Sigma$. But then the outward (to K_Σ) unit normal vector field is well defined on Σ and yields a non-vanishing normal vector field in $\Sigma \subset M$. Thus Σ is two-sided. Since M is orientable and Σ non-orientable, this is a contradiction. □

PROPOSITION 4.23. *There exists an embedded stable minimal surface $\Sigma \in \mathscr{F}$ with $|\Sigma| = \mathscr{A}(M, g)$.*

Proof. Since $\mathscr{A}(M,g) = \inf\{|S| \,:\, S \in \mathscr{F}\}$, we can find a sequence of surfaces $\Sigma_k \in \mathscr{F}$ such that

$$|\Sigma_k| \leq \mathscr{A}(M,g) + \frac{1}{k}.$$

Let $\mathscr{I}(\Sigma_k)$ denote the collection of all embedded surfaces isotopic to Σ_k. Since $\mathscr{I}(\Sigma_k) \subset \mathscr{F}$, we have

$$|\Sigma_k| \leq \mathscr{A}(M,g) + \frac{1}{k} \leq \inf_{S \in \mathscr{I}(\Sigma_k)} |S| + \frac{1}{k}.$$

By (MEEKS; SIMON; S. T. YAU, 1982, Theorem 1, p. 624), a subsequence of Σ_k converges weakly to a disjoint union of smooth embedded minimal surfaces $\Sigma^{(1)}, \ldots, \Sigma^{(R)}$ with positive integer multiplicities $n_1, \ldots, n_R$ and

$$\sum_{j=1}^{R} n_j |\Sigma^{(j)}| \leq \mathscr{A}(M,g). \tag{4.1}$$

We define surfaces $S_k^{(1)}, \ldots, S_k^{(R)}$ as follows:

$$\begin{cases} S_k^{(j)} = \bigcup_{r=1}^{m_j} \{x \in M \,:\, d(x, \Sigma^{(j)}) = \frac{r}{k}\}, & \text{if } n_j = 2m_j \text{ is even,} \\ S_k^{(j)} = \Sigma^{(j)} \cup \bigcup_{r=1}^{m_j} \{x \in M \,:\, d(x, \Sigma^{(j)}) = \frac{r}{k}\}, & \text{if } n_j = 2m_j + 1 \text{ is odd.} \end{cases}$$

By (MEEKS; SIMON; S. T. YAU, 1982, Remark 3.27, p. 635), we can find embedded closed surfaces $S_k^{(0)}$ and $\tilde{\Sigma}_k$ with the following properties:

1. the surface $S_k = \bigcup_{j=0}^{R} S_k^{(j)}$ is isotopic to $\tilde{\Sigma}_k$ for k sufficiently large;
2. the surface $\tilde{\Sigma}_k$ is obtained from Σ_{q_k} by γ_0-reduction (cf. (MEEKS; SIMON; S. T. YAU, 1982, Section 3) for precise definition);
3. $S_k^{(0)} \cap \left(\bigcup_{j=1}^{R} S_k^{(j)} \right) = \varnothing$ and $\lim_{k\to\infty} |S_k^{(0)}| = 0$;
4. $g(\tilde{\Sigma}_k) \leq g(\Sigma_{q_k}) = \tilde{h}$ (this is a property of γ_0-reduction, (MEEKS; SIMON; S. T. YAU, 1982, Inequality (3.2), p. 629)).

In particular, we have that $\Sigma_{q_k} \backslash \tilde{\Sigma}_k$ has closure A diffeomorphic to the standard closed annulus $\{x \in \mathbb{R}^2 \,:\, \frac{1}{2} \leq |x| \leq 1\}$ (this is one of the conditions in the definition of $\tilde{\Sigma}_k$ to be obtained from Σ_{q_k} by γ_0-reduction). Then, since Σ_{q_k} is homeomorphic to $N_{\tilde{h}}$ (by assumption), this implies that $\tilde{\Sigma}_k$ must contain all the cross-caps in Σ_{q_k}, i.e. one of the connected components of $\tilde{\Sigma}_k$ is a nonorientable embedded surface. Let k be sufficiently large such that (i) holds. Thus, for this k, one of the components of S_k, say E_k, is a nonorientable surface. From (iv), we have $g(E_k) \leq \tilde{h}$ and since $\tilde{h}$ is minimal, E_k must be homeomorphic to $N_{\tilde{h}}$. Since $E_k \in \mathscr{F}$, we have $|E_k| \geq \mathscr{A}(M,g) > 0$. Then, by (iii), we can choose k even larger so that $|S_k^{(0)}| < \mathscr{A}(M,g)$, and hence $|S_k^{(0)}| < |E_k|$. Therefore, E_k is not contained in $S_k^{(0)}$. Since E_k is connected component of S_k and $S_k^{(0)} \cap \left(\bigcup_{j=1}^{R} S_k^{(j)} \right) = \varnothing$, this implies that E_k is a connected component of $S_k^{(i)}$, for some integer $i \in \{1, \dots, R\}$. Then, by the definition of $S_k^{(i)}$, we have two possibilities for E_k: either E_k is homeomorphic to $\Sigma^{(i)}$ or E_k is an orientable double cover of $\Sigma^{(i)}$. But E_k is nonorientable. Hence, E_k is homeomorphic to $\Sigma^{(i)}$ and $\Sigma^{(i)} \in \mathscr{F}$. Thus, $\mathscr{A}(M, g) \leq |\Sigma^{(i)}|$. But from (4.1), we have

$$|\Sigma^{(i)}| \leq \mathscr{A}(M, g).$$

Hence, $\Sigma^{(i)}$ is the desired minimizer. □

In the next result, we adopt the following notation. If M does not admit nonorientable surfaces then $\tilde{h}$ denotes the Heegaard genus of M. Otherwise, $\tilde{h}$ is the lowest possible genus of all nonorientable embedded closed surfaces in M, as above.

COROLLARY 4.24. *Let (M, g) be a compact orientable three-manifold. There exists an embedded minimal surface $\Sigma \subset M$ with $\operatorname{ind}(\Sigma) \leq 1$ and $g(\Sigma) \leq \tilde{h}$.*

Proof. If M admits nonorientable embedded closed surfaces, then the result follows from Proposition 4.23. Thus we can suppose that M does not admit such surfaces. If M has a stable minimal surface of genus less than or equal to h then the result follows immediately. The remaining case is when M satisfies the $(\star)_h$-condition and the result follows from Theorem 4.14. □

COROLLARY 4.25. *For any riemannian metric on $\mathbb{S}^3$, there exists an embedded minimal sphere Σ in $\mathbb{S}^3$ of index at most one.*

Proof. Alexander's Duality implies that $\mathbb{R}^n$ does not contain embedded closed nonorientable surfaces (cf. (HATCHER, 2002, p. 256)). Since $\mathbb{S}^3 \subset \mathbb{R}^4$ is embedded, this implies that $\mathbb{S}^3$ does not contain embedded closed nonorientable surfaces. Also, one easily verifies that the sphere

$$S = \{(x_1, x_2, x_3, x_4) \in \mathbb{S}^3 \,:\, x_4 = 0\}$$

is a Heegaard splitting of $\mathbb{S}^3$. Therefore, the Heegaard genus of $\mathbb{S}^3$ must be zero and the result follows from the previous corollary. □

5 Ricci flow and rigidity results

5.1 Ricci flow

Let (M, g) be a compact riemannian three-manifold and denote by $\mathrm{Ric}(g)$ the Ricci tensor of M with respect to the metric g, i.e. $\mathrm{Ric}(g)$ is the 2-tensor defined by

$$\mathrm{Ric}(g)(u, v) = \mathrm{tr}(z \mapsto R_g(z, v)u),$$

where R_g is the curvature tensor of (M, g)

$$R_g(X, Y)Z = \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z,$$

and ∇^g is the Levi-Civita connection with respect to g. If $g(t)$, $t \in [0, T)$, is a smooth curve in the space of riemannian metrics on M, then $\frac{\partial}{\partial t} g(t)$ and $\mathrm{Ric}(g)$ are 2-tensors on M and we can ask $g(t)$ to satisfy

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g), \quad g(0) = g$$

This is the Ricci flow equation. The Ricci flow was first introduced by Hamilton in his paper (HAMILTON, 1982) and it has been useful in solving many important problems in geometry, as for example the Poincaré conjecture solution by Grigori Perelman. If $g(t)$ is a solution to the Ricci flow equation, then the manifold $(M, g(t))$ becomes "rounder" and "rounder" as t increases.

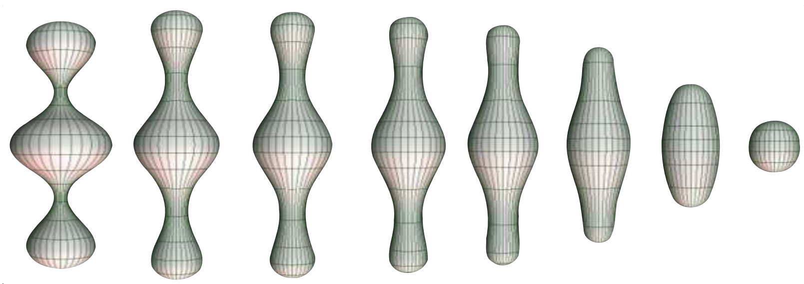


Figure 14 – Ricci flow on a 2D-sphere.

Now, we introduce some basic results on the Ricci flow (see e.g. (SHERIDAN, 2006)).

Theorem 5.1. *(SHERIDAN, 2006, Theorem 5.4, p. 45) Given a smooth Riemannian metric g on a closed manifold M, there exists a maximal time interval $[0, T)$ such that a solution $g(t)$ of the Ricci flow, with $g(0) = g$, exists and is smooth on $[0, T)$, and this solution is unique.*

Proposition 5.2. *Positive Ricci curvature is preserved under the Ricci flow. More precisely, if (M, g) has positive Ricci curvature and $g(t)$ is a solution to the Ricci flow equation with $g(0) = g$, then $(M, g(t))$ has positive Ricci curvature, for all $t \in [0, T)$.*

Recall the definition of scalar curvature at a point $p \in M$:

$$R(p) = \operatorname{tr}\operatorname{Ric} = \sum_{i=1}^{3} \operatorname{Ric}(e_i, e_i),$$

if $\{e_1, e_2, e_3\}$ is an orthonormal basis for T_pM.

LEMMA 5.3. *(SHERIDAN, 2006, Lemma 6.2, p. 48) Suppose that $g(t)$ is a solution of the Ricci flow. Then, if $R(t)$ denotes the scalar curvature of $(M, g(t))$, we have the following evolution equation:*

$$\frac{\partial}{\partial t} R = \Delta R + 2|\operatorname{Ric}|^2,$$

where $|\operatorname{Ric}|^2 = \sum_{i=1}^{3} \operatorname{Ric}(e_i, e_i)^2$ if $\{e_1, e_2, e_3\}$ is an orthonormal basis in TM.

REMARK 5.4. *From Lemma 5.3, a direct but rather long computation shows that*

$$\frac{\partial}{\partial t} R = \Delta R + \frac{2}{3}R^2 + 2|\overset{\circ}{\operatorname{Ric}}|^2,$$

where $|\overset{\circ}{\operatorname{Ric}}|^2 = \sum_{i=1}^{3} (\operatorname{Ric}(e_i, e_i) - \frac{1}{3}R)^2$, for an orthonormal basis $\{e_1, e_2, e_3\}$.

THEOREM 5.5 (SCALAR MAXIMUM PRINCIPLE). *(SHERIDAN, 2006, Theorem 3.2, p. 32) Let $(M, g(t))$ be a closed manifold with a time-dependent Riemannian metric $g(t)$. Suppose that $u : M \times [0, T) \to \mathbb{R}$ satisfies*

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + g(t)\left(X(t), \operatorname{grad}_{g(t)} u\right) + F(u)$$
$$u(x, 0) \leq C, \quad \text{for all } x \in M,$$

for some constant C, where $X(t)$ is a time-dependent vector field on M and $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz function. Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is the solution of the associated ode, which is formed by neglecting the Laplacian and gradient terms:

$$\frac{d\phi}{dt} = F(\phi), \quad \phi(0) = C.$$

Then $u(x, t) \leq \phi(t)$ for all $x \in M$ and $t \in [0, T)$ such that $\phi(t)$ exists.

5.2 RIGIDITY

The goal of this section (and of this work) is to prove the following rigidity result:

THEOREM 5.18. *Suppose that M has positive Ricci curvature and $R \geq 6$. Then there exists an embedded minimal surface Σ, with $\operatorname{ind}(\Sigma) \leq 1$, such that*

$$|\Sigma| \leq 4\pi.$$

Moreover,

$$\inf_{\Sigma \in \mathscr{J}} |\Sigma| = 4\pi$$

if and only if g has constant sectional curvature one and $M = \mathbb{S}^3$.

Here $\mathscr{J}$ is the collection of all embedded minimal surfaces $\Sigma \subset M$ with $\operatorname{ind}(\Sigma) \leq 1$. From this theorem we also prove some other interesting rigidity results, which we introduce later on.

We begin by showing that the function $t \mapsto W(M, \Lambda, g(t))$ is continuous, where $g(t)$ is the Ricci flow solution for (M, g).

LEMMA 5.6. *Let (M, g) be a compact riemannian three-manifold and Λ bet a saturated set of sweepouts of M. If $g(t)$, $t \in [0, T)$, is the solution of the Ricci flow with $g(0) = g$, then the function $f : [0, t_0] \to \mathbb{R}$ defined by $f(t) = W(M, \Lambda, g(t))$ is Lipschitz continuous for any $t_0 \in [0, T)$.*

Proof. Let $\boldsymbol{U}(M)$ denote the g-unit bundle on M, i.e. the bundle of base M and fiber at $p \in M$ given by $B_p = \{v \in T_pM \ : \ g(v, v) = 1\}$. $\boldsymbol{U}(M)$ is compact and has the property that any $v \in TM$ is written as $v = \lambda u$, for some $u \in \boldsymbol{U}(M)$ and $\lambda \in \mathbb{R}$. Let $t_0 \in [0, T)$ and consider the function

$$\begin{array}{rccc} \xi: & \boldsymbol{U}(M) \times [0, t_0] & \longrightarrow & \mathbb{R} \\ & (u, t) & \longmapsto & \frac{-2 \operatorname{Ric}(g(t))(u,u)}{g(t)(u,u)}. \end{array}$$

Since ξ is continuous and $\boldsymbol{U}(M) \times [0, t_0]$ is compact, there is some $C > 0$ such that

$$|\xi(u, t)| \leq C, \ \ \forall (u, t) \in \boldsymbol{U}(M) \times [0, t_0].$$

For any fixed $u \in \boldsymbol{U}(M)$, the function $t \in [0, t_0] \mapsto \ln g(t)(u, u)$ is smooth, and

$$\left| \frac{d}{dt} \ln g(t)(u, u) \right| = \left| \frac{1}{g(t)(u, u)} \frac{\partial}{\partial t} g(t)(u, u) \right| = |\xi(u, t)| \leq C.$$

Then it follows from the Mean Value Theorem that, for any $t_1, t_2 \in [0, t_0]$,

$$\begin{aligned} &- C|t_2 - t_1| \leq \ln g(t_2)(u, u) - \ln g(t_1)(u, u) \leq C|t_2 - t_1| \\ \Rightarrow &e^{-C|t_2 - t_1|} g(t_1)(u, u) \leq g(t_2)(u, u) \leq e^{C|t_2 - t_1|} g(t_2)(u, u). \end{aligned}$$

Notice that this implies

$$e^{-C|t_2 - t_1|} g(t_1)(v, v) \leq g(t_2)(v, v) \leq e^{C|t_2 - t_1|} g(t_2)(v, v),$$

for any $t_1, t_2 \in [0, t_0]$ and $v \in TM$. In this case, we write simply

$$e^{-C|t_2 - t_1|} g(t_1) \leq g(t_2) \leq e^{C|t_2 - t_1|} g(t_2), \quad \text{for all } t_1, t_2 \in [0, t_0].$$

Given $\delta > 0$, let $\{\Sigma_s\}_{s \in [-1,1]} \in \Lambda$ be a sweepout such that

$$\sup_{s \in [-1,1]} \mathcal{H}^2_{g(t_1)}(\Sigma_s) \leq W(M, \Lambda, g(t_1)) + \delta.$$

It is simple to prove that, if g_1 and g_2 are riemannian metrics on M such that $g_1 \leq k g_2$, for some $k > 0$, then $\mathcal{H}^2_{g_1} \leq k \mathcal{H}^2_{g_2}$. Then, $g(t_2) \leq e^{C|t_2 - t_1| g(t_1)}$ implies

$$W(M, \Lambda, g(t_2)) \leq \sup_{s \in [-1,1]} \mathcal{H}^2_{g(t_2)}(\Sigma_s) \leq e^{C|t_2 - t_1|} \sup_{s \in [-1,1]} \mathcal{H}^2_{g(t_1)}(\Sigma_s)$$

$$\leq e^{C|t_2-t_1|}(W(M,\Lambda,g(t_1))+\delta).$$

In the same way, we prove that

$$e^{-C|t_2-t_1|}W(M,\Lambda,g(t_1)) \leq W(M,\Lambda,g(t_2))+\delta.$$

Letting $\delta \to 0$, we obtain

$$e^{-C|t_2-t_1|}W(M,\Lambda,g(t_1)) \leq W(M,\Lambda,g(t_2)) \leq e^{C|t_2-t_1|}W(M,\Lambda,g(t_1)).$$

Thus,

$$|\ln W(M,\Lambda,g(t_2)) - \ln W(M,\Lambda,g(t_1))| \leq C|t_2-t_1|.$$

This shows that $t \longmapsto \ln W(M,\Lambda,g(t))$ is Lipschitz on $[0,t_0]$. Since the exponential function is Lipschitz on bounded intervals and composition of Lipschitz functions is Lipschitz, we have that

$$t \longmapsto W(M,\Lambda,g(t)) = e \circ \ln W(M,\Lambda,g(t_0))$$

is Lipschitz on $[0,t_0]$. $\square$

PROPOSITION 5.7. *Let h be the Heegaard genus of M and $(M,g(t))$ be the Ricci flow with $g(0)=g$. Assume that $(M,g(t))$ satisfies the $(\star)_h$-condition for all $0 \leq t < T'$, for some $T' \leq T$. Then*

$$W(M,\Lambda^h,g(t)) \geq W(M,\Lambda^h,g) - \left(16\pi - 8\pi\left[\frac{h}{2}\right]\right)t$$

for all $0 \leq t < T'$.

Proof. Suppose the assertion is false. Then there exists $\tau \in (0,T')$ such that

$$W(M,\Lambda^h,g(\tau)) < W(M,\Lambda^h,g) - \left(16\pi - 8\pi\left[\frac{h}{2}\right]\right)\tau.$$

Let $\epsilon > 0$ be such that

$$W(M,\Lambda^h,g(\tau)) < W(M,\Lambda^h,g) - \left(16\pi - 8\pi\left[\frac{h}{2}\right]\right)\tau - 2\epsilon\tau$$

and define

$$A = \left\{t \in [0,T') \,:\, W(M,\Lambda^h,g(t)) < W(M,\Lambda^h,g) - \left(16\pi - 8\pi\left[\frac{h}{2}\right] + \epsilon\right)t - \epsilon\tau\right\}.$$

Of course, $\tau \in A$ and hence $A \neq \varnothing$ (notice that $0 \notin A$). Denote

$$t_0 = \inf A.$$

We claim that $t_0 \in (0,\tau)$. Indeed, if $t_0 = 0$, then there is a sequence $t_n \to 0$ with $t_n \in A$ and hence

$$W(M,\Lambda^h,g(t_n)) < W(M,\Lambda^h,g) - \left(16\pi - 8\pi\left[\frac{h}{2}\right] + \epsilon\right)t_n - \epsilon\tau$$

for all $n \in \mathbb{N}$. Since $t \longmapsto W(M, \Lambda^h, g(t))$ is continuous (previous lemma), letting $n \to \infty$ we get

$$W(M, \Lambda^h, g) \leq W(M, \Lambda^h, g) - \epsilon\tau \implies 0 \leq -\epsilon\tau,$$

a contradiction. Thus $t_0 > 0$. Since $\tau \in A$, of course $t_0 \leq \tau$. Since

$$f(t) := W(M, \Lambda^h, g(t)) + \left(16\pi - 8\pi \left[\frac{h}{2}\right] + \epsilon\right) t$$

is continuous and $f(\tau) < W(M, \Lambda^h, g) - \epsilon\tau$, we have that $f(t) < W(M, \Lambda^h, g) - \epsilon\tau$ for every t in some open interval $(\tau - \delta, \tau + \delta)$, $\delta > 0$. Then $\tau - \frac{\delta}{2} \in A$ and hence $t_0 < \tau$. Therefore $t_0 \in (0, \tau)$.

Arguing with sequences and continuity as we did above, we get

$$W(M, \Lambda^h, g(t_0)) \leq W(M, \Lambda^h, g) - \left(16\pi - 8\pi \left[\frac{h}{2}\right] + \epsilon\right) t_0 - \epsilon\tau.$$

For all $t \in [0, t_0)$ we have $t \notin A$, then

$$W(M, \Lambda^h, g(t)) \geq W(M, \Lambda^h, g) - \left(16\pi - 8\pi \left[\frac{h}{2}\right] + \epsilon\right) t - \epsilon\tau, \quad \forall t \in [0, t_0)$$
$$\implies -W(M, \Lambda^h, g(t)) \leq -W(M, \Lambda^h, g) - \left(16\pi - 8\pi \left[\frac{h}{2}\right] + \epsilon\right)(-t) + \epsilon\tau, \quad \forall t \in [0, t_0).$$

Summing up these two inequalities, we have

$$W(M, \Lambda^h, g(t_0)) - W(M, \Lambda^h, g(t)) \leq -\left(16\pi - 8\pi \left[\frac{h}{2}\right] + \epsilon\right)(t_0 - t), \tag{5.1}$$

for all $t \in [0, t_0)$. Since $(M, g(t_0))$ satisfies the $(\star)_h$-condition, let $\{\Sigma_s\}_{s \in [-1,1]}$ be the sweepout in M given by Theorem 4.14 (in particular, $\Sigma_0 = W(M, \Lambda^h, g(t_0))$ and Σ_0 is minimal). Set $f(s,t) = |\Sigma_s|_{g(t)}$. A standard computation shows that

$$\frac{\partial f}{\partial t}(0, t_0) = \left.\frac{d}{dt}\right|_{t_0} |\Sigma_0|_{g(t)} = -\int_{\Sigma_0} R - \operatorname{Ric}(\nu, \nu)\, d\Sigma_0,$$

where the geometric quantities are computed with respect to $g(t_0)$. Then by Lemma C.1 and the Gauß–Bonnet Theorem,

$$\begin{aligned}\frac{\partial f}{\partial t}(0, t_0) &= -\int_{\Sigma_0} \operatorname{Ric}(\nu, \nu) + |A|^2 + 2K\, d\Sigma_0 = -4\pi\chi(\Sigma_0) - \int_{\Sigma_0} \operatorname{Ric}(\nu, \nu) + |A|^2\, d\Sigma_0 \\ &= -8\pi(1 - h) - \int_{\Sigma_0} \operatorname{Ric}(\nu, \nu) + |A|^2\, d\Sigma_0.\end{aligned}$$

Furthermore, Σ_0 is orientable and has index one, so we can apply Proposition C.2 to get

$$\begin{aligned}\frac{\partial f}{\partial t}(0, t_0) &\geq 8\pi(h - 1) - 8\pi\left(\left[\frac{h+1}{2}\right] + 1\right) \\ &= -16\pi + 8\pi \left[\frac{h}{2}\right].\end{aligned}$$

Since f is smooth in a neighborhood of $(0, t_0)$, we have

$$\begin{aligned} f(s,t) &\le f(s,t_0) - \left(16\pi - 8\pi\left[\frac{h}{2}\right] + \frac{\epsilon}{2}\right)(t - t_0) \\ &\le W(M, \Lambda^h, g(t_0)) - \left(16\pi - 8\pi\left[\frac{h}{2}\right] + \frac{\epsilon}{2}\right)(t - t_0), \end{aligned}$$

for all (s, t) close to $(0, t_0)$ with $t \le t_0$. Since $s \longmapsto f(s, t_0)$ has a unique maximum at $s = 0$, by continuity we have that, for all t sufficiently close to t_0,

$$\sup_{s\in[-1,1]} f(s,t) \le W(M, \Lambda^h, g(t_0)) - \left(16\pi - 8\pi\left[\frac{h}{2}\right] + \frac{\epsilon}{2}\right)(t - t_0).$$

Since $\{\Sigma_s\}_{s\in[-1,1]} \in \Lambda^h$, by the definition of width and of f, we have

$$W(M, \Lambda^h, g(t)) \le W(M, \Lambda^h, g(t_0)) - \left(16\pi - 8\pi\left[\frac{h}{2}\right] + \frac{\epsilon}{2}\right)(t - t_0) \tag{5.2}$$

for such t's. Summing up inequalities (5.1) and (5.2), we get

$$0 \le -\frac{\epsilon}{2}(t_0 - t),$$

for all $t \le t_0$ with t sufficiently close to t_0, a contradiction. □

COROLLARY 5.8. *Suppose that (M, g) has positive Ricci curvature and that it contains no nonorientable embedded surface. Let h be the Heegaard genus of M. Then*

$$W(M, \Lambda^h, g(t)) \ge W(M, \Lambda^h, g) - \left(16\pi - 8\pi\left[\frac{h}{2}\right]\right)t$$

for all $0 \le t < T$.

Proof. Recall that if (M, g) has positive Ricci curvature then it contains no stable closed embedded minimal surface (Corollary 1.22). Also, positive Ricci curvature is preserved by the Ricci flow. Therefore, $(M, g(t))$ satisfies the $(\star)_h$-condition for all $0 \le t < T$. The result follows from Proposition 5.7. □

THEOREM 5.9. *Suppose that (M, g) has positive Ricci curvature and that it contains no nonorientable embedded surface. Let h be the Heegaard genus of M. If $R \ge 6$, then*

$$W(M, \Lambda^h, g) \le 4\pi - 2\pi\left[\frac{h}{2}\right] \le 4\pi.$$

Moreover, $W(M, \Lambda^h, g) = 4\pi$ if and only if g has constant sectional curvature one and $M = \mathbb{S}^3$.

REMARK 5.10. *If (M, g) has positive Ricci curvature, it follows from (HAMILTON, 1982) that M is diffeomorphic to a spherical space form, which are Seifert-fibered over a base of genus of zero with at most three exceptional fibers. It follows from (SCHULTENS, 1996) that these spaces have Heegaard genus at most 2. Therefore, $0 \le \left[\frac{h}{2}\right] \le 1$ in the theorem above.*

Proof. Let $(g(t))_{0\leq t<T}$ denote the maximal solution of the Ricci flow with $g(0) = g$. It follows from Corollary 5.8 that

$$W(M,\Lambda^h,g(t)) \geq W(M,\Lambda^h,g) - \left(16\pi - 8\pi\left[\frac{h}{2}\right]\right)t. \tag{5.3}$$

CLAIM 5.11. $\lim\limits_{t\to T} W(M,\Lambda^h,g(t)) = 0.$

Since $(M,g(t))$ satisfies the $(\star)_h$-condition for every $0\leq t<T$. Then, for each $0\leq t<T$, let $\{\Sigma_s^t\}_{s\in[-1,1]}$ be the sweepout of M given by Theorem 4.14. By Proposition C.2, we have

$$0\leq \min_M R(g(t))W(M,\Lambda^h,g(t)) = \min_M R(g(t))|\Sigma_0^t| \leq 24\pi + 16\pi\left(\frac{h}{2} - \left[\frac{h}{2}\right]\right).$$

Now, from (HAMILTON, 1982, Theorem 15.1) we have

$$\lim_{t\to T}\min_M R(g(t)) = +\infty,$$

and thus since $0\leq \min\limits_M R(g(t))W(M,\Lambda^h,g(t))$ is bounded above, we need to have

$$\lim_{t\to T} W(M,\Lambda^h,g(t)) = 0.$$

Combining Claim 5.11 and inequality 5.3, we get

$$W(M,\Lambda^h,g) \leq \left(16\pi - 8\pi\left[\frac{h}{2}\right]\right)T. \tag{5.4}$$

From Lemma 5.3, we have the evolution equation for the scalar curvature:

$$\frac{\partial R}{\partial t} = \Delta R + \frac{2}{3}R^2 + 2|\mathring{\mathrm{Ric}}|^2.$$

Therefore,

$$\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{3}R^2.$$

Now we apply the Scalar Maximum Principle (Theorem 5.5) with $u = -R$, $X(t)\equiv 0$, $F(u) = -\frac{2}{3}u^2$ and $C = k_1 := -\min\limits_M R(g(t_1))$. The associated ode is

$$\frac{d\phi}{dt} = -\frac{2}{3}\phi^2, \quad \phi(t_1) = -k_1,$$

which has solution

$$\phi(t) = \frac{-3k_1}{3-2k_1(t-t_1)}, \quad \text{for all } t_1\leq t<T.$$

The principle then gives

$$\min_M R(g(t)) \geq \frac{3k_1}{3-2k_1(t-t_1)}, \quad \text{for all } t_1\leq t<T. \tag{5.5}$$

Choosing $t_1 = 0$ and $k_1 = 6$, we obtain

$$\min_M R(g(t)) \geq \frac{6}{1-4t} \tag{5.6}$$

and hence $T \leq \frac{1}{4}$ and it follows from (5.4) that

$$W(M, \Lambda^h, g) \leq 4\pi - 2\pi \left[\frac{h}{2}\right].$$

If $W(M, \Lambda^h, g) = 4\pi$, then we must have $T = \frac{1}{4}$ and $h \in \{0, 1\}$. First, we show that g must be Einstein, i.e. $\mathrm{Ric}(X,Y) = \lambda g(X,Y)$, for some constant $\lambda \in \mathbb{R}$. Since M is three-dimensional, this implies that M has constant sectional curvature (see (CARMO, 1988, Ex. 10, p. 120)). Since, in (5.5), we must have $3 - 2k_1(t - t_1) \neq 0$ for all $t_1 \leq t < \frac{1}{4}$ and for $t = t_1$ this quantity is positive (equal to 3), we have that $3 - 2k_1(t - t_1) > 0$, for all $t_1 \leq t < \frac{1}{4}$. Letting $t \to \frac{1}{4}$, we get

$$3 - 2k_1\left(\frac{1}{4} - t_1\right) \geq 0 \implies \min_M R(g(t_1)) \leq \frac{6}{1-4t_1},$$

for all $0 \leq t_1 < \frac{1}{4}$ (recall $k_1 = \min_M R(g(t_1))$ by definition). Together with (5.6), this implies

$$\min_M R(g(t)) = \frac{6}{1-4t},$$

and the maximum principle implies that g must be Einstein.

CLAIM 5.12. $h = 0$.

Suppose $h = 1$. Then it follows from Theorem 4.14 that M contains a minimal embedded torus T which realizes the width and any other embedded minimal torus must have area bigger than $|T| = 4\pi$. It is a classical fact that manifolds with Heegaard genus one are either Lens spaces $L(p,q)$ or $\mathbb{S}^2 \times \mathbb{S}^1$ (see (STILLWELL, 2012, Section 8.3.4) or (SAVELIEV, 1999, Theorem 1.6)). Thus M contains a flat torus of area $2\pi^2/p < \pi$ (projection of Clifford torus), which is a contradiction. This proves that $h = 0$.

CLAIM 5.13. $M = \mathbb{S}^3$.

By Theorem 4.14, M contains an embedded minimal sphere S. Now, we use the following theorem

THEOREM 5.14. *(FRANKEL, 1966, p. 69) Let M_{n+1} be complete, connected, and have positive Ricci curvature. Let Σ_n be a compact immersed minimal hypersurface. Then the natural homomorphism of fundamental groups: $\pi_1(\Sigma_n) \to \pi_1(M_{n+1})$ is surjective.*

Since $\pi_1(S) = 0$, by the theorem above we conclude that $\pi_1(M) = 0$, i.e M is simply connected. Then it follows from Poincaré conjecture that $M = \mathbb{S}^3$. □

If M contains nonorientable embedded surfaces, we consider the invariant $\mathscr{A}(M,g)$ as defined in Section 4.2.

LEMMA 5.15. *If $g(t), t \in [0,T)$, is the solution of the Ricci flow with $g(0) = g$, then the function $f : [0,t_0] \to \mathbb{R}$ defined by $f(t) = \mathscr{A}(M, g(t))$ is Lipschitz continuous for any $t_0 \in [0,T)$.*

Proof. Analogous to Lemma 5.6. □

PROPOSITION 5.16. *For all* $0 \leq t < T$ *we have*

$$\mathscr{A}(M, g(t)) \geq \mathscr{A}(M, g) - 8\pi t.$$

Proof. By Proposition 4.23, there exists an embedded stable minimal surface $\Sigma \in \mathscr{F}$ such that $|\Sigma|_{g(t_0)} = \mathscr{A}(M, g(t_0))$. A straightforward computation shows that

$$\left.\frac{d}{dt}\right|_{t_0} |\Sigma|_{g(t)} = -\int_\Sigma R - \mathrm{Ric}(\nu, \nu)\, d\Sigma.$$

Then using Lemma C.1 and Proposition C.2 (ii), we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t_0} |\Sigma|_{g(t)} &= -2\int_\Sigma K\, d\Sigma - \int_\Sigma \mathrm{Ric}(\nu,\nu) + |A|^2\, d\Sigma = -\int_{\tilde\Sigma} K\, d\tilde\Sigma - \int_\Sigma \mathrm{Ric}(\nu,\nu) + |A|^2\, d\Sigma \\ &\geq -2\pi g(\tilde\Sigma) - 4\pi(g(\tilde\Sigma)+1) = 4\pi(g(\tilde\Sigma)+1) - 4\pi(g(\tilde\Sigma)-1) = -8\pi.\end{aligned}$$

Using this estimate, we can argue exactly as we did in Proposition 5.7. □

THEOREM 5.17. *Let* (M, g) *be of positive Ricci curvature and suppose* M *contains embedded nonorientable surfaces. If* $R \geq 6$*, then* $\mathscr{A}(M, g) \leq 2\pi$*.*

Proof. Let $(g(t))_{0\leq t<T}$ denote a maximal solution of the Ricci flow equation with $g(0) = g$. From Proposition 5.16, we have

$$\mathscr{A}(M, g(t)) \geq \mathscr{A}(M, g) - 8\pi t.$$

Reasoning like in Theorem 5.9, we have

$$\lim_{t\to T} \mathscr{A}(M, g(t)) = 0.$$

Therefore $\mathscr{A}(M, g) \leq 8\pi T$. But we know from the proof of Theorem 5.9 that $T \leq \frac{1}{4}$. Hence $\mathscr{A}(M, g) \leq 2\pi$. □

Let $\mathscr{J}$ be the collection of all embedded minimal surfaces $\Sigma \subset M$ with $\mathrm{ind}(\Sigma) \leq 1$. Now we prove the main theorems of this work.

THEOREM 5.18. *Suppose that* M *has positive Ricci curvature and* $R \geq 6$*. Then there exists an embedded minimal surface* Σ*, with* $\mathrm{ind}(\Sigma) \leq 1$*, such that*

$$|\Sigma| \leq 4\pi.$$

Moreover,

$$\inf_{\Sigma\in\mathscr{J}} |\Sigma| = 4\pi$$

if and only if g *has constant sectional curvature one and* $M = \mathbb{S}^3$*.*

Proof. Suppose M has embedded nonorientable surfaces. It follows from Proposition 4.23 that there exists an embedded minimal surface $\Sigma \in \mathscr{F}$ with $\operatorname{ind}(\Sigma) = 0$ and $|\Sigma| = \mathscr{A}(M, g)$. Then, it follows from Theorem 5.17 that $|\Sigma| \leq 2\pi < 4\pi$.

Suppose now that M does not contain nonorientable embedded surfaces. Let h be the Heegaard genus of M. Then (M, g) satisfies the $(\star)_h$-condition. By Theorem 4.14, there is an embedded minimal surface $\Sigma' \subset M$ with $\operatorname{ind}(\Sigma') = 1$ and such that $|\Sigma'| = W(M, \Lambda^h, g)$. Theorem 5.9 implies $|\Sigma'| \leq 4\pi$.

If $\inf\limits_{\Sigma \in \mathscr{J}} |\Sigma| = 4\pi$, then it follows from the previous arguments that M does not contain nonorientable embedded surfaces and $W(M, \Lambda^h, g) = 4\pi$. Hence, by Theorem 5.9, g has constant sectional curvature one and $M = \mathbb{S}^3$. □

Now, we consider the case in which M is diffeomorphic to the three-sphere $\mathbb{S}^3$, whose Heegaard genus is zero. We take Λ^0 to be the smallest saturated set that contains the family $\{\Sigma_t\}$ of level sets given by the height function $x_4 : \mathbb{S}^3 \subset \mathbb{R}^4 \to \mathbb{R}$. We define the *width of* $(\mathbb{S}^3, g)$ to be

$$W(\mathbb{S}^3, g) = W(\mathbb{S}^3, \Lambda^0, g).$$

THEOREM 5.19. *Assume that* $(\mathbb{S}^3, g)$ *has no stable embedded minimal spheres. If* $R \geq 6$*, there exists an embedded minimal sphere* Σ*, of index one, such that*

$$W(\mathbb{S}^3, g) = |\Sigma| = \inf_{S \in \mathscr{E}_0} |S| \leq 4\pi.$$

Moreover, $W(\mathbb{S}^3, g) = 4\pi$ *if and only if* g *has constant sectional curvature one.*

Proof. $(\mathbb{S}^3, g)$ satisfies the $(\star)_0$-condition because the three-sphere contains no nonorientable embedded surface. By Theorem 4.14, there is an embedded minimal sphere Σ with $\operatorname{ind}(\Sigma) = 1$ and $|\Sigma| = \inf\limits_{S \in \mathscr{E}_0} |S| = W(\mathbb{S}^3, g)$. From Proposition C.2 (i), we have

$$6|\Sigma| \leq 24\pi + 16\pi \left(\frac{g(\Sigma)}{2} - \left[\frac{g(\Sigma)}{2} \right] \right) = 24\pi \implies |\Sigma| \leq 4\pi.$$

Suppose $|\Sigma| = 4\pi$. As in Theorem 5.9, we show that g is Einstein. Let $g(t)$, $t \in [0, \epsilon)$ be a solution of Ricci flow with $g(0) = g$. The maximum principle applied to the evolution equation of the scalar curvature gives us

$$\min_M R(g(t)) \geq \frac{6}{1 - 4t} > 0, \quad \text{for } 0 \leq t < \frac{1}{4}.$$

It follows from Proposition C.2 (iii) that any embedded stable minimal surface in $(\mathbb{S}^3, g(t))$ would have to be a *sphere*. We are assuming that $(\mathbb{S}^3, g(0))$ has no stable embedded minimal spheres. Hence, $(\mathbb{S}^3, g(0))$ does not contain stable embedded minimal surfaces.

CLAIM 5.20. $(\mathbb{S}^3, g(t))$ *has no stable embedded minimal surfaces, for all* $t \in [0, \epsilon)$*, provided* $\epsilon > 0$ *is small enough.*

Suppose the claim is false. Then, for all $n \in \mathbb{N}$, with $n > 4$, there is a stable embedded minimal sphere $\Sigma_n \subset (\mathbb{S}^3, g(1/n))$ with area

$$|\Sigma_n| \leq \frac{8\pi}{\min_M R(g(1/n))} = \frac{4\pi}{3}\left(1 - \frac{4}{n}\right) < \frac{4\pi}{3}.$$

Now we use (without proof) the following result:

PROPOSITION 5.21. *Suppose that (M, g) is a compact riemannian 3-manifold that contains no stable embedded minimal surfaces. Given a constant $C > 0$, there exists a $\mathcal{C}^{3,\alpha}$ neighborhood $\mathscr{U}$ of g so that every metric g' in $\mathscr{U}$ contains no stable minimal surface of area smaller than C.*

Take $C = \frac{4\pi}{3}$ and let $\mathscr{U}$ be the neighborhood given above. For n sufficiently large, $g(1/n)$ is in $\mathscr{U}$, since $g(t)$ is continuous. But then Σ_n is a stable minimal surface of area smaller than C, which is a contradiction. This proves the claim.

Therefore $(\mathbb{S}^3, g(t))$ satisfies the $(\star)_0$-condition for all $t \in [0, \epsilon)$ and we are in the hypothesis of Proposition 5.7. Thus

$$W(\mathbb{S}^3, g(t)) \geq W(\mathbb{S}^3, g) - \left(16\pi - 8\pi \left[\frac{0}{2}\right]\right) t = 4\pi - 16\pi t = 4\pi(1 - 4t).$$

We know from Theorem 4.14 that $W(\mathbb{S}^3, g(t))$ is the area of an index one embedded minimal sphere in $(\mathbb{S}^3, g(t))$, say Σ_t. Again, by Proposition C.2 (i), we have

$$\begin{aligned} &\min_M R(g(t)) 4\pi(1 - 4t) \leq \min_M R(g(t)) W(\mathbb{S}^3, g(t)) = \min_M R(g(t)) |\Sigma_t| \leq 24\pi \\ &\implies \min_M R(g(t)) \leq \frac{6}{1 - 4t}. \end{aligned}$$

Therefore

$$\min_M R(g(t)) = \frac{6}{1 - 4t}$$

and the maximum principle tell us that g is Einstein. Since the dimension is three, g has constant sectional curvature. $\square$

We finalize this monograph with one more rigidity result, which is just a corollary from the previous ones.

THEOREM 5.22. *Let g be a metric on $\mathbb{S}^3$ with scalar curvature $R \geq 6$. If g does not have constant sectional curvature one, then there exists an embedded minimal sphere Σ, of index zero or one, with $|\Sigma| < 4\pi$.*

Proof. If $(\mathbb{S}^3, g)$ contains a stable embedded minimal sphere Σ, then $|\Sigma| \leq \frac{4\pi}{3}$ by Proposition C.2 (iii). If not, $W(\mathbb{S}^3, g) < 4\pi$ and the result follows from Theorem 4.14. $\square$

A Orientable double cover

Let $\Sigma \subset M$ be a compact embedded, connected non-orientable surface. Let $N\Sigma$ denote the normal bundle of Σ and $\pi : N\Sigma \to \Sigma$ the projection onto Σ. Since Σ is compact and embedded, Σ admits a uniform tubular neighborhood, i.e. there is an open neighborhood U of Σ and $\epsilon > 0$ such that the map $\Phi : \tilde{U} \to U$ given by $\Phi(v) = \exp_{\pi(v)}(v)$ is well defined and it is a diffeomorphism from $\tilde{U} = \{v \in N\Sigma : |v| < \epsilon\}$ onto U. We have that $\tilde{\Sigma} := \{v \in N\Sigma : |v| = \epsilon/2\}$ is a submanifold of $\tilde{U} \subset N\Sigma$ (pre-image of a regular value). Consider

$$V = \left\{ v \in N\Sigma : \frac{\epsilon}{4} < |v| < \frac{3\epsilon}{4} \right\} \subset U.$$

V is an open neighborhood of $\tilde{\Sigma}$ in $N\Sigma$. Then define $\Psi : V \to \tilde{U}$ by $\Psi(v) = (4|v| - 2\epsilon)\frac{v}{|v|}$. One verifies that Ψ is well defined, surjective and a local diffeomorphism. Finally, we define $\Pi : V \to U$ by $\Pi = \Phi \circ \Psi$. Note that $\Pi(\tilde{\Sigma}) = \Sigma$. Since Π is a local diffeomorphism, we can transform Π into a local isometry, by defining the metric $\tilde{g}$ on V to be

$$\tilde{g}(u, v) = g(\Pi_* u, \Pi_* v), \quad u, v \in TV.$$

Proposition A.1. *$\tilde{\Sigma}$ is connected.*

Proof. We claim that if $v_1, v_2 \in \tilde{\Sigma}$, then v_1 is either in the component of v_2 or in the component of $-v_2$. Denote $p_i = \pi(v_i)$. since Σ is connected, there is a path $\gamma : [0,1] \to \Sigma$ contained in Σ joining p_1 to p_2. By taking a coordinated neighborhood U_1 of p_1, we are able to construct a normal section $\nu_1 : U_1 \to N\Sigma$ with $\nu_1(p_1) = v_1$ and $|\nu_1| = \epsilon$, i.e. $\nu_1(U_1) \subset \tilde{\Sigma}$. Now we cover $\gamma([0,1])$ we a finite number of connected coordinated neighborhoods U_j such that $U_j \cap U_k = \varnothing$ if $|j - k| \geq 2$ and $U_j \cap U_{j+1}$ is connected. Consider normal sections $\nu_j : U_j \to \tilde{\Sigma}$. Since $U_j \cap U_{j+1}$ is connected and $N_p\Sigma$ is 1-dimensional, we have $\nu_j = \nu_{j+1}$ or $\nu_j = -\nu_{j+1}$ all over $U_j \cap U_{j+1}$. Then, changing ν_j for $-\nu_j$ if necessary, we obtain a normal section $\nu : \bigcup_j U_j \to \tilde{\Sigma}$ such that $\nu(p_1) = v_1$. Now $\nu \circ \gamma : [0,1] \to \tilde{\Sigma}$ is a path joining v_1 to $\nu(\gamma(1)) = \nu(p_2)$. But $\nu(p_2) \in (N_{p_2}\Sigma) \cap \tilde{\Sigma} = \{v_2, -v_2\}$. Thus, $\nu \circ \gamma$ is a path joining v_1 to v_2 or $-v_2$. This proves the claim. Therefore, in particular, $\tilde{\Sigma}$ has at most two components. Suppose by contradiction that $\tilde{\Sigma}$ has two components, $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$. Take $v \in \tilde{\Sigma}_1 \cap N_p\Sigma$. We claim that we can extend v to a global normal section $\nu : \Sigma \to \tilde{\Sigma}_1 \subset N\Sigma$ contradicting the fact that Σ is one-sided. Indeed, if $q \in \Sigma$, by taking a path γ joining p and q we can uniquely extend v along γ just as we did before. Then v and $\nu(\gamma(1))$ are in the same component $\tilde{\Sigma}_1$. Choose another path $\tilde{\gamma}$ joining p and q. Then necessarily $\tilde{\nu}(\tilde{\gamma}(1)) = \nu(\gamma(1))$, otherwise $\nu(\gamma(1))$ and $-\nu(\gamma(1))$ are in the same component $\tilde{\Sigma}_1$. This cannot happen, because we just proved that any $w \in \tilde{\Sigma}$ is in the same component of $\nu(\gamma(1))$ or $-\nu(\gamma(1))$ and if these components are the same, $\tilde{\Sigma}$ must be connected. This shows that $\nu(q)$ can be defined without dependence of the underlying path γ. Since Σ is connected, this defines the global normal section $\nu : \Sigma \to N\Sigma$ desired. $\square$

COROLLARY A.2. *V is connected.*

Proof. This follows since any $v \in V \subset N\Sigma$ can be connected to $\tilde{\Sigma}$ by a path along the fiber of v and, by the previous proposition, $\tilde{\Sigma}$ is connected. □

PROPOSITION A.3. *$\tilde{\Sigma}$ is orientable.*

Proof. We show that the image $\Phi(\tilde{\Sigma})$ in M is orientable and since Φ is a diffeomorphism the result follows. To show this, we construct a nowhere vanishing normal vector field on $\Phi(\tilde{\Sigma})$ and since the ambient M is orientable we are done. For each $p \in \Phi(\tilde{\Sigma})$ we have $\Phi^{-1}(p) \in \tilde{\Sigma} \subset N\Sigma$. Define

$$X(p) = \left.\frac{d}{dt}\right|_0 \exp_{\Pi(p)}((1+t)\Phi^{-1}(p)) = d\Phi_{\Phi^{-1}(p)} \cdot \Phi^{-1}(p) \in T_pM.$$

Now, consider any vector tangent to $\Phi(\tilde{\Sigma})$ given by a smooth path $\alpha : I \to \Phi(\tilde{\Sigma})$, $\alpha(0) = p$. Then $\alpha(s) = \exp_{\Pi(\alpha(s))}(\Phi^{-1}(\alpha(s)))$. If we denote $\alpha_t(s) = \exp_{\Pi(\alpha(s))}((1+t)\Phi^{-1}(\alpha(s)))$ then $\alpha_0(s) = \alpha(s)$ and $\alpha_{-1}(s) = \Pi(\alpha(s))$ is a path on Σ. Define the following function of t:

$$f(t) = g\left(\frac{d}{dt}\exp_{\Pi(p)}((1+t)\Phi^{-1}(p)),\ \left.\frac{d}{ds}\right|_0 \alpha_t(s)\right).$$

Note that f is well defined since both vectors are based on $\exp_{\Pi(p)}((1+t)\Phi^{-1}(p))$. Now, we look at the derivative of f (using Gauß's lemma in the last equations):

$$\begin{aligned}
f'(t) &= g\left(\frac{D}{dt}\frac{d}{dt}\exp_{\Pi(p)}((1+t)\Phi^{-1}(p)),\ \left.\frac{d}{ds}\right|_0 \alpha_t(s)\right)\\
&\quad + g\left(\frac{d}{dt}\exp_{\Pi(p)}((1+t)\Phi^{-1}(p)),\ \frac{D}{dt}\left.\frac{d}{ds}\right|_0 \alpha_t(s)\right)\\
&= g\left(\frac{d}{dt}\exp_{\Pi(p)}((1+t)\Phi^{-1}(p)),\ \left.\frac{D}{ds}\right|_0 \frac{d}{dt}\alpha_t(s)\right)\\
&= g\left(\frac{d}{dt}\exp_{\Pi(p)}((1+t)\Phi^{-1}(p)),\ \left.\frac{D}{ds}\right|_0 (d\exp_{\Pi(\alpha(s))})_{(1+t)\Phi^{-1}(\alpha(s))} \cdot \Phi^{-1}(\alpha(s))\right)\\
&= \frac{1}{2}\left.\frac{d}{ds}\right|_0 g\big((d\exp_{\Pi(\alpha(s))})_{(1+t)\Phi^{-1}(\alpha(s))} \cdot \Phi^{-1}(\alpha(s)),\\
&\qquad\qquad (d\exp_{\Pi(\alpha(s))})_{(1+t)\Phi^{-1}(\alpha(s))} \cdot \Phi^{-1}(\alpha(s))\big)\\
&= \frac{1}{2}\left.\frac{d}{ds}\right|_0 g(\Phi^{-1}(\alpha(s)), \Phi^{-1}(\alpha(s))) = \frac{1}{2}\left.\frac{d}{ds}\right|_0 \epsilon = 0.
\end{aligned}$$

Thus f must be constant. On one hand we have

$$\begin{aligned}
f(-1) &= g\left(\left.\frac{d}{dt}\right|_{-1} \exp_{\Pi(p)}((1+t)\Phi^{-1}(p)),\ \left.\frac{d}{ds}\right|_0 \alpha_{-1}(s)\right)\\
&= g\left((d\exp_{\Pi(p)})_0 \cdot \Phi^{-1}(p),\ \left.\frac{d}{ds}\right|_0 \alpha_{-1}(s)\right) = g\left(\Phi^{-1}(p),\ \left.\frac{d}{ds}\right|_0 \alpha_{-1}(s)\right)\\
&= 0,
\end{aligned}$$

since $\Phi^{-1}(p) \in N_{\Pi(p)}\Sigma$ and $\alpha'_{-1}(0) \in T_{\Pi(p)}\Sigma$. On the other hand,

$$f(0) = g(X(p), \alpha'(0)).$$

This proves that $X(p) \in N_p(\Phi\tilde{\Sigma})$. Using Gauß's lemma again, we have

$$g(X(p), X(p)) = g(\Phi^{-1}(p), \Phi^{-1}(p)) = \frac{\epsilon}{2} > 0.$$

Therefore, $X : \Phi(\tilde{\Sigma}) \to N(\Phi\tilde{\Sigma})$ is a nowhere vanishing section and $\Phi(\tilde{\Sigma})$ is two-sided, hence orientable. □

PROPOSITION A.4. *$\Pi : V \to U$ is an isometric double covering map.*

Proof. Since we already noticed that Π is a local isometry, we just need to show that for any point $p \in U$, the pre-image $\Pi^{-1}(\{p\})$ consists of exact two points. Thus, suppose $v \in V$ is such that $\Pi(v) = p$. Then

$$\begin{aligned} \Psi(v) = \Phi^{-1}(p) &\implies (4|v| - 2\epsilon)\frac{v}{|v|} = \Phi^{-1}(p) \implies |4|v| - 2\epsilon| = |\Phi^{-1}(p)| \\ &\implies (4|v| - 2\epsilon)^2 - |\Phi^{-1}(p)|^2 = 0 \\ &\implies (4|v| - 2\epsilon + |\Phi^{-1}(p)|)(4|v| - 2\epsilon - |\Phi^{-1}(p)|) = 0 \\ &\implies |v| = \frac{2\epsilon - |\Phi^{-1}(p)|}{4} \quad \text{or} \quad |v| = \frac{2\epsilon + |\Phi^{-1}(p)|}{4}. \end{aligned}$$

Note that these equations are specifically at the fiber $N_{\pi(\Phi^{-1}p)}\Sigma$ which is 1-dimensional and we are searching for solutions in $V = \{v \in N\Sigma : \frac{\epsilon}{4} < v < \frac{3\epsilon}{4}\}$.

If $|\Phi^{-1}(p)| = 0$ (which means $p \in \Sigma$), then we have the equation $|v| = \frac{\epsilon}{2}$, which has exactly two solutions, say v_0 and $-v_0$, both of them lying at $\tilde{\Sigma}$. If $|\Phi^{-1}(p)| \neq 0$, then the first equation has two solutions, $v = \pm\frac{(2\epsilon - |\Phi^{-1}(p)|)}{4}\frac{\Phi^{-1}(p)}{|\Phi^{-1}(p)|}$, but only $v = -\frac{(2\epsilon - |\Phi^{-1}(p)|)}{4}\frac{\Phi^{-1}(p)}{|\Phi^{-1}(p)|}$ is such that $\Psi(v) = \Phi^{-1}(p)$. Similarly, the second equation has two solutions, but only one of them is admissible, namely $v = \frac{2\epsilon + |\Phi^{-1}(p)|}{4}\frac{\Phi^{-1}(p)}{|\Phi^{-1}(p)|}$. □

Note that $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$, $\tau(v) = -v$ an isometry of $\tilde{\Sigma}$ and thus $\{Id_{\tilde{\Sigma}}, \tau\}$ is a discrete subgroup of $\mathrm{Isomt}(\tilde{\Sigma})$.

The next result shows why it is interesting to consider the orientable double cover $\tilde{\Sigma}$ in the case that Σ is non-orientable.

PROPOSITION A.5. *Let $\Sigma \subset M$ be a non-orientable embedded compact surface and $F : (-\epsilon, \epsilon) \times \Sigma \to M$ be a smooth variation of Σ. Then, if ϵ is small enough, there is a smooth variation $\tilde{\Sigma_t}$ of $\tilde{\Sigma}$ in V such that $|\tilde{\Sigma_t}| = 2|\Sigma_t|$ for all $t \in (-\epsilon, \epsilon)$.*

Proof. Let $\Pi : V \to U$ be as before and let ϵ be small enough such that $\Sigma_t \subset U$ for all $t \in I_\epsilon := (-\epsilon, \epsilon)$. We are not going to trouble ourselves constructing $\tilde{F} : I_\epsilon \times \tilde{\Sigma} \to V$. Instead of doing this, we only introduce the surfaces $\tilde{\Sigma_t} := \Pi^{-1}(\Sigma_t)$. Since each Σ_t is embedded, we have that $\tilde{\Sigma_t}$ is also an embedded surface in V, because Π is a local diffeomorphism.

Since Σ_t is compact, we can take $\mathscr{U} = \{\Omega_i\}_{i=1}^n$ a finite open cover of Σ_t such that each Ω_i is a coordinated neighborhood and there are $\tilde{\Omega}_i^1$ and $\tilde{\Omega}_i^2$ disjoint open subsets of $\tilde{\Sigma_t}$

such that $\Pi : \tilde{\Omega}_i^j \to \Omega_i$ is an isometry. Of course $\tilde{\mathscr{U}} = \{\tilde{\Omega}_i^1, \tilde{\Omega}_i^2\}_{i=1}^n$ is a finite open cover of $\tilde{\Sigma}_t$. Let $\{\varphi_i : \Sigma_t \to [0,1]\}_{i=1}^n$ be a partition of unity subordinated to $\mathscr{U}$. Define $\tilde{\varphi}_i^j : \tilde{\Omega}_i^j \to [0,1]$ by $\tilde{\varphi}_i^j(p) = \varphi_i(\Pi(p))$. Since $\tilde{\Omega}_i^1$ and $\tilde{\Omega}_i^2$ are disjoint for all i, we have that $\{\tilde{\varphi}_i^j\}$ is a partition of unity subordinated to $\tilde{\mathscr{U}}$. Then we have

$$|\tilde{\Sigma}_t| = \sum_{i=1}^n \int_{\tilde{\Omega}_i^1} \tilde{\varphi}_i^1 \, d\tilde{\Sigma}_t + \sum_{i=1}^n \int_{\tilde{\Omega}_i^1} \tilde{\varphi}_i^2 \, d\tilde{\Sigma}_t = \sum_{i=1}^n \int_{\Omega_i} \varphi_i \, d\Sigma_t + \sum_{i=1}^n \int_{\Omega_i} \varphi_i \, d\Sigma_t = 2|\Sigma_t|.$$

□

We summarize the construction of the orientable double cover in the following theorem.

THEOREM A.6. *Let M be an orientable 3-riemannian manifold. If $\Sigma \subset M$ is an embedded compact, connected non-orientable surface, then there is an open neighborhood U of Σ, a 3-riemannian manifold V, a surface $\tilde{\Sigma}$ in V and a map $\Pi : V \to U$ such that:*

1. *$\tilde{\Sigma}$ is embedded, compact, connected and orientable;*
2. *$\Pi : V \to U$ is a local isometry and a double covering map with $\Pi(\tilde{\Sigma}) = \Sigma$;*
3. *there is an isometry $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ such that $\Sigma \simeq \tilde{\Sigma}/\{Id_{\tilde{\Sigma}}, \tau\}$;*
4. *if Σ_t is a smooth variation of Σ in M, then there is smooth variation $\tilde{\Sigma}_t$ of $\tilde{\Sigma}$ in V such that $|\tilde{\Sigma}_t| = 2|\Sigma_t|$.*

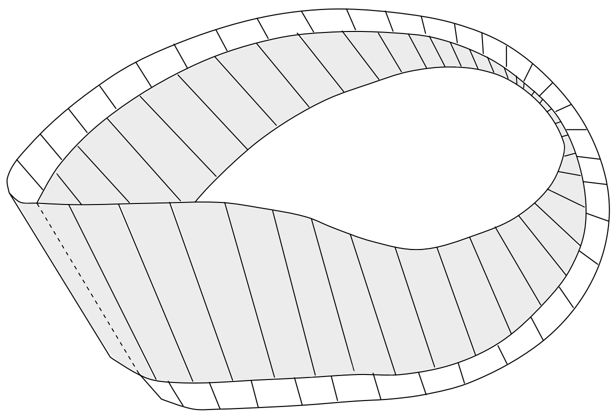


Figure 15 – The orientable double cover of the Möbius strip

So we can understand the variation of area of Σ by looking to certain variations of $\tilde{\Sigma}$. We said before that we only have to study the normal vector fields along Σ with $X = 0$ on $\partial\Sigma$.

Let Σ_t be a variation given by $F : I_\epsilon \times \Sigma \to M$ such that $X = \frac{\partial F}{\partial t}$ is normal to Σ and vanishes on its boundary at $t = 0$. Denote by $\tilde{F}$ the associated variation of $\tilde{\Sigma}$ in V and $\tilde{X} = \frac{\partial \tilde{F}}{\partial t}$ the corresponding vector field (which is normal along $\tilde{\Sigma}$ at $t = 0$ and vanishes on its boundary). Since $\tilde{\Sigma}$ is orientable, we can consider a global unit normal vector field $\tilde{\nu}$ along $\tilde{\Sigma}$. Then, $\tilde{X} = \phi\tilde{\nu}$, for some $\phi \in C^\infty(\tilde{\Sigma})$. Recall we constructed

$\tilde{\Sigma}$ in an open subset of $N\Sigma$. Let $p \in \Sigma$ and $v \in \tilde{\Sigma}$ such that $\Pi(v) = p$. A direct computation shows that $\tilde{X}(v) = \tilde{X}(\tau(v))$ and $\tilde{\nu}(v) = -\tilde{\nu}(\tau(v))$. Hence it follows that $\phi(v) = -\phi(\tau(v))$, for all $v \in \tilde{\Sigma}$.

Conversely, if $\phi \in C^\infty(\tilde{\Sigma})$ is such that $\phi \circ \tau = -\phi$ (and zero on the boundary of $\tilde{\Sigma}$) and we define $\tilde{X} = \phi\tilde{\nu}$, then $d\Pi_v\tilde{X}(v) = d\Pi_{\tau(v)}\tilde{X}(\tau(v))$ doesn't depend on the choice of v or $\tau(v)$. Thus we can define an admissible vector field on Σ given by $X(p) = d\Pi_v\tilde{X}(v)$.

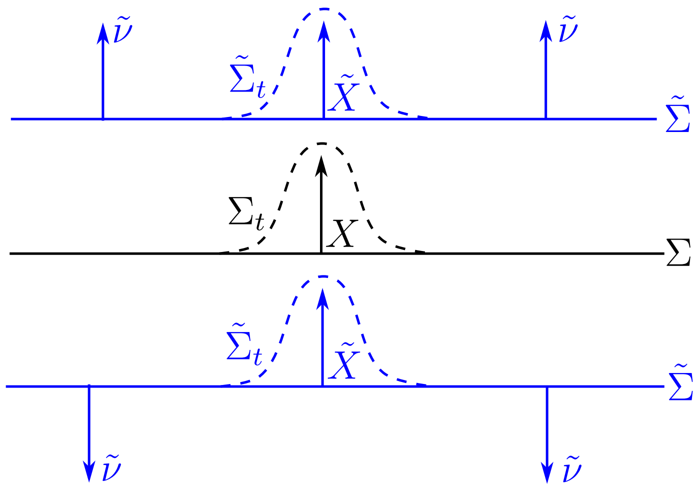


Figure 16

B Metrizability of the space of varifolds

The goal of this appendix is to show that, if M is compact, then the space $\mathcal{V}_k^c(M)$ of k-dimensional varifolds with mass bounded above by $c > 0$ is metrizable and compact, namely, Theorem 2.10. In the first section, we discuss in more details the concept of convergence notion and the relations between convergence notions and topologies. This is important because we defined a topology on the space of varifolds using a convergence notion, namely, the weak convergence. In the second section, we prove Theorem 2.10.

B.1 Convergence notions

Now, we briefly present the concept of nets, a generalization of sequences.

Definition B.1. *A nonempty set D with a relation $\prec$ is said to be* ***directed*** *if*

1. *$a \prec a$, for all $a \in D$;*
2. *if $a \prec b$ and $b \prec c$, then $a \prec c$;*
3. *if $a, b \in D$, then there is $c = c(a, b) \in D$ such that $a \prec c$ e $b \prec c$.*

Definition B.2. *A map $h : D \to D'$ between directed sets is said to be* ***cofinal monotone*** *if*

1. *if $a, b \in D$ with $a \prec b$, then $h(a) \prec' h(b)$;*
2. *for any $a' \in D'$, there is $a \in D$ such that $a' \prec' h(a)$.*

Definition B.3. *Let X be a set. Any map $\phi : D \to X$ from a directed set D into X is called a* ***net*** *in X. We say that a net $\phi' : D' \to X$ is a subnet of $\phi : D \to X$ if there is a cofinal monotone map $h : D' \to D$ such that $\phi' = \phi \circ h$. Given a net ϕ we denote $\phi(\lambda) = x_\lambda$ and $\phi = (x_\lambda)_{\lambda \in D}$.*

If (X, τ) is a topological space, then there is a natural convergence notion on X. If D is a directed set, then we say a net $(x_\lambda)_{\lambda \in D}$ converges to $x \in X$ and write $x_\lambda \to x$ if

$$\forall U \in \tau,\ x \in U,\ \exists \lambda_0 \in D : \lambda_0 \prec \lambda \implies x_\lambda \in U. \tag{B.1}$$

We can ask ourselves: does a "convergence notion" on X induce a topology on X? If so, does the new convergence notion induced by this topology (as above) coincide with the notion originally given? Before answering these questions, we need first to think about what we would like to be a "convergence notion". For example, consider the following convergence notion on $\mathbb{R}$:

"$x_\lambda \to x$ if and only if x_λ is irrational for all $\lambda \in D$".

With this notion, the constant sequence $x_n = 0\ \forall n \in \mathbb{N}$ does *not* converge to 0, ***não*** converge a 0, and this is not the case with any convergence notion induced by a topology. The next proposition will motivate a good definition of *convergence notion* that will avoid such pathologies.

PROPOSITION B.4. *Let (X, τ) be a topological space. If D is a directed set, $(x_\lambda)_{\lambda \in D}$ is a net in X and $x \in X$ is fixed, then the convergence notion of nets defined at (B.1) satisfies:*

1. *if $x_\lambda = x$, for each $\lambda \in D$, then $(x_\lambda)_{\lambda \in D}$ converges to x;*
2. *if $(x_\lambda)_{\lambda \in D}$ converges to x, then every sub-net $(x_\lambda)_{\lambda \in D}$ converges to x;*
3. *if every sub-net of $(x_\lambda)_{\lambda \in D}$ has a sub-net which converges to x, then $(x_\lambda)_{\lambda \in D}$ converges to x;*
4. *(diagonal principle) if $(x_\lambda)_{\lambda \in D}$ converges to x and, for each $\lambda \in D$, a net $(x^\lambda_\gamma)_{\gamma \in D_\lambda}$ converges to x_λ, then the net $(x^\lambda_\gamma)_{(\lambda,\gamma) \in \mathscr{D}}$, with $\mathscr{D} = \{(\lambda, \gamma) \in D \times \bigcup_{\lambda \in D} D_\lambda \,:\, \lambda \in D,\ \gamma \in D_\lambda\}$ ordered lexicographically first in $\lambda \in D$ and then in $\gamma \in D_\lambda$, admits a sub-net that converges to x.*

$$\begin{array}{ccccccc}
x_1^1 & x_2^1 & \dots & x_j^1 & \to & x_1 \\
x_2^1 & x_2^2 & \dots & x_j^2 & \to & x_2 \\
\vdots & \vdots & \ddots & \vdots & & \vdots \\
x_n^1 & x_2^n & \dots & x_j^n & \to & x_n \\
\vdots & \vdots & & \vdots & & \downarrow \\
 & & & & & x
\end{array}$$

Figure 17 – The diagonal principle for sequences

Proof. The proofs of (1), (2) and (3) follow essentially the same steps of the analogous proofs in the case of sequences in metric spaces. Let's prove the diagonal principle. Consider the net $(x^\lambda_\gamma)_{(\lambda,\gamma) \in \mathscr{D}}$ as described in (4). Denote by $\mathscr{U}(x)$ the collection of all open sets in X containing x and consider $\mathscr{E} = \{(U, \lambda, \gamma) \,:\, U \in \mathscr{U}(x),\ x_\lambda,\ x^\lambda_\gamma \in U\}$. Since $x_\lambda \to x$ and $x^\lambda_\gamma \to x_\lambda$, for every $U \in \mathscr{U}(x)$, there are λ and γ such that $(U, \lambda, \gamma) \in \mathscr{E}$. Thus, $\mathscr{E} \neq \varnothing$. The relation

$$(U, \lambda, \gamma) \prec (U', \lambda', \gamma') \iff \begin{cases} U' \subset U \text{ and } \lambda \prec \lambda'; \\ \text{or: } U' \subset U,\ \lambda = \lambda' \text{ and } \gamma \prec \gamma'. \end{cases}$$

turns $\mathscr{E}$ into a directed set. In fact, it is clear that $\prec$ is reflexive and transitive in $\mathscr{E}$. If $(U, \lambda, \gamma),\ (U', \lambda', \gamma') \in \mathscr{E}$, then let $\lambda_1 \in D$ be such that $\lambda_1 \prec \tilde{\lambda} \implies x_{\tilde{\lambda}} \in U \cap U'$. Since D is directed, there exists a $\tilde{\lambda} \in D$ such that $\lambda, \lambda', \lambda_1 \prec \tilde{\lambda}$. There exists also $\tilde{\gamma} \in D_{\tilde{\lambda}}$ with $x^{\tilde{\lambda}}_{\tilde{\gamma}} \in U \cap U'$. Thus, $(U \cap U', \tilde{\lambda}, \tilde{\gamma}) \in \mathscr{E}$ and $(U, \lambda, \gamma) \prec (U \cap U', \tilde{\lambda}, \tilde{\gamma})$, $(U', \lambda', \gamma') \prec (U \cap U', \tilde{\lambda}, \tilde{\gamma})$ because $U \cap U' \subset U,\ U'$ and $\lambda, \lambda' \prec \tilde{\lambda}$.

Consider the map

$$\begin{array}{cccc} h: & \mathscr{E} & \to & \mathscr{D} \\ & (U, \lambda, \gamma) & \mapsto & (\lambda, \gamma) \end{array}.$$

We claim h is cofinal monotone and the sub-net $(x_{h(\delta)})_{\delta\in\mathscr{E}}$ converges to x. The fact of h to be monotone follows immediately from the definition of $\prec$ in $\mathscr{E}$. Let $(\lambda,\gamma)\in\mathscr{D}$. We have that (X,λ,γ) is trivially in $\mathscr{E}$, hence there is a $\delta=(X,\lambda,\gamma)\in\mathscr{E}$ such that $(\lambda,\gamma)\prec h(\delta)$ (indeed, $(\lambda,\gamma)=h(\delta)$, thus h is more than cofinal, it is surjective). This proves that $(x_{h(\delta)})_{\delta\in\mathscr{E}}$ is indeed a sub-net of $(x^{\lambda}_{\gamma})_{(\lambda,\gamma)\in\mathscr{D}}$. Let us show that $x_{h(\delta)}\to x$. Consider $U\in\mathscr{U}(x)$. Since $x_\lambda\to x$, there exists $\lambda_0\in D$ such that $\lambda_0\prec\lambda \implies x_\lambda\in U$. Since $x^{\lambda_0}_{\gamma}\to x_{\lambda_0}$, there exists $\gamma_0\in D_{\lambda_0}$ such that $\gamma_0\prec\gamma \implies x^{\lambda_0}_{\gamma}\in U$. Take $\delta_0=(U,\lambda_0,\gamma_0)\in\mathscr{E}$. If $\delta=(U',\lambda,\gamma)\in\mathscr{E}$ with $\delta_0\prec\delta$ then

$$\begin{cases} U'\subset U \text{ and } \lambda_0\prec\lambda \implies x_{h(\delta)}=x^{\lambda}_{\gamma}\in U'\subset U;\\ \text{or: } U'\subset U,\ \lambda=\lambda_0 \text{ and } \gamma_0\prec\gamma \implies x_{h(\delta)}=x^{\lambda_0}_{\gamma} \text{ with } \gamma_0\prec\gamma \implies x_{h(\delta)}\in U.\end{cases}$$

So we proved:

$$\text{for all } U\in\mathscr{U}(x), \text{ there is a } \delta_0\in\mathscr{E} \text{ such that } \delta_0\prec\delta \implies x_{h(\delta)}\in U.$$

Thus, $x_{h(\delta)}\to x$. □

Now, we give the definition of convergence notion.

DEFINITION B.5. *Let X be a set. A rule $\mathscr{C}$ saying which nets in X converge, is said to be a* ***convergence notion*** *if it satisfies (1), (2), (3) e (4) from Proposition B.4.*

REMARK B.6. *If (x_λ) converges to x with respect to the convergence notion $\mathscr{C}$, we say $(x_\lambda)_{\lambda\in D}$ is $\mathscr{C}$-convergent to x and write $\mathscr{C}: x_\lambda\to x$ or simply $x_\lambda\to x$, if there is no risk of confusion. If (X,τ) is a topological space, we denote by $C(\tau)$ the convergence notion induced by τ.*

We will see now how a convergence notion induces a topology. Let X be a set and $\mathscr{C}$ a convergence notion in X.

Consider the map $u_{\mathscr{C}}:\wp(X)\to\wp(X)$ defined by

$$u_{\mathscr{C}}(E)=\{x\in X\ :\ \exists(x_\lambda)_{\lambda\in D}\subset E,\ \mathscr{C}: x_\lambda\to x\}. \tag{B.2}$$

PROPOSITION B.7. *Given a convergence notion $\mathscr{C}$ in X, the collection $T(\mathscr{C})=\{X\backslash u_{\mathscr{C}}(E) : E\in\wp(X)\}$ is a topology in X, called the* ***topology induced by*** *$\mathscr{C}$. Moreover, if $\overline{E}$ is the closure of E with respect to $T(\mathscr{C})$, then $\overline{E}=u_{\mathscr{C}}(E)$, for every $E\in\wp(X)$.*

Proof. It suffices to show that $u=u_{\mathscr{C}}:\wp(X)\to\wp(X)$ satisfies conditions (1), (2), (3) and (4) from the following theorem:

THEOREM B.8 (DUGUNDJI (DUGUNDJI, 1966), PAGE 73)**.** *Let X be a set and $u:\wp(X)\to\wp(X)$ a map with the following properties:*

1. *$u(\varnothing)=\varnothing$;*
2. *$A\subset u(A)$, for each $A\in\wp(X)$;*
3. *$u\circ u(A)=u(A)$, for each $A\in\wp(X)$;*
4. *$u(A\cup B)=u(A)\cup u(B)$, for each $A,B\in\wp(X)$.*

Then he family $\mathscr{T} = \{X \backslash u(A) : A \in \wp(X)\}$ is a topology in X and given the closure notion in $\mathscr{T}$, i.e. $\overline{A} := \{x \in X : U \in \mathscr{T}, x \in U \implies U \cap A \neq \varnothing\}$, we have $\overline{A} = u(A)$ for each $A \in \wp(X)$.

So, let us prove that $u = u_{\mathscr{C}}$ has such properties.

1. If there were $x \in u(\varnothing)$, then by definition it would exist a net $(x_\lambda)_{\lambda \in D} \subset \varnothing$ with $\mathscr{C} : x_\lambda \to x$, but the first condition is impossible, since D, being a directed set, is nonempty. Thus, $u(\varnothing) = \varnothing$.

2. If $x \in A$, then $(x_n)_{n \in \mathbb{N}}$ with $x_n := x$ for all $n \in \mathbb{N}$ is a net in A $\mathscr{C}$-convergent to x, by axiom (1) of convergence notion. Thus, by the definition of u, $x \in u(A)$. Therefore, $A \subset u(A)$.

3. By (2), $u(A) \subset u \circ u(A)$. We need to show $u \circ u(A) \subset u(A)$. Let $x \in u \circ u(A)$. By definition, there is a net $(x_\lambda)_{\lambda \in D} \subset u(A)$ with $\mathscr{C} : x_\lambda \to x$. Since each $x_\lambda \in u(A)$, there are nets $(x^\lambda_\gamma)_{\gamma \in D_\lambda} \subset A$ with $\mathscr{C} : x^\lambda_\gamma \to x_\lambda$. From the diagonal principle, we obtain a net $(y_j)_{j \in J} \subset A$ with $y_j \to x$. Therefore, $x \in u(A)$.

4. Since $A, B \subset A \cup B$, it follows from (2) that $u(A) \cup u(B) \subset u(A \cup B)$. It remais to show $u(A \cup B) \subset u(A) \cup u(B)$. If $x \in u(A \cup B)$, then there exists $(x_\lambda)_{\lambda \in D} \subset A \cup B$ with $\mathscr{C} : x_\lambda \to x$. We claim that, switching A and B if we need, for all $\lambda \in D$, there is $\lambda \prec \gamma(\lambda) \in D$ such that $x_\gamma \in A$. In fact, if it were otherwise, there would be $\lambda_1, \lambda_2 \in D$ such that, for all $\lambda \in D$ with $\lambda_1 \prec \lambda$, $\lambda_2 \prec \lambda$, $x_\lambda \notin A$ and $x_\lambda \notin B$. Since D is directed, there is such λ, but $x_\lambda \in A \cup B$, a contradiction. This yields a sub-net $(x_\gamma)_{\gamma \in \gamma(D)}$ of $(x_\lambda)_{\lambda \in D} \subset A$ that, by axiom (2), also is $\mathscr{C}$-convergent to x. Therefore, $x \in u(A) \subset u(A) \cup u(B)$.

□

The following theorem shows that a convergence notion completely determines its topology and vice-versa. In other words, to work with topologies is equivalent to work with convergence notions.

THEOREM B.9. *1. The convergence notion induced by a topology (as in B.1) induces this same topology, i.e.if τ is a topology, then $T(C(\tau)) = \tau$;*

2. The topology induced by a convergence notion induces this same convergence notion, i.e. if $\mathscr{C}$ is a convergence notion, then $C(T(\mathscr{C})) = \mathscr{C}$.

Proof. 1. Let (X, τ) be a topological space. By the previous theorem, it suffices to show that the closure notion $u = u_{C(\tau)}$ coincides with the original closure notion given by τ. Let $A \in \wp(X)$ and let's show that $u(A) = \overline{A}$. If $x \in u(A)$, then there is a net $(x_\lambda)_{\lambda \in D} \subset A$ with $C(\tau) : x_\lambda \to x$. Let $U \in \tau$ be any open set with $x \in U$. By the definition of $C(\tau)$, there is $\lambda_0 \in D$ such that $\lambda_0 \prec \lambda \implies x_\lambda \in U$. Therefore, $x_{\lambda_0} \in U \cap A \implies U \cap A \neq \varnothing$. Then $x \in \overline{A}$ and $u(A) \subset \overline{A}$. Suppose now $x \in \overline{A}$ and denote $\mathscr{V}(x) = \{U \in \tau : x \in U\}$. By definition of $\overline{A}$, we have that for all $U \in \mathscr{V}(x)$, there is a $x_U \in U \cap A$. The relation $U \prec V$ if $V \subset U$ in $\mathscr{V}(x)$ turns it into a directed set. We have thus a net $(x_U)_{U \in \mathscr{V}(x)} \subset A$. We claim that $C(\tau) : x_U \to x$. In fact, let $U \in \mathscr{V}(x)$. We have that, if $U \prec V$, then $V \subset U$ and

$x_V \in V \subset U \implies x_V \in U$. Therefore, $C(\tau) : x_U \to x$ and this shows that $x \in u(A)$ and $\overline{A} \subset u(A)$.

2. Let X be a set and $\mathscr{C}$ a convergence notion on X. Consider $(x_\lambda)_{\lambda\in D}$ a net in X. Suppose, by contradiction, that $\mathscr{C} : x_\lambda \to x$ but $C(T(\mathscr{C})) : x_\lambda \not\to x$. This means that there is a $U \in T(\mathscr{C})$ with $x \in U$ such that, for all $\lambda \in D$, there is $\lambda \prec \gamma(\lambda) \in D$ with $x_\gamma \notin U$. By definition, $U = X \backslash u_{\mathscr{C}}(E)$, for some $E \in \wp(X)$. This gives us a sub-net $(x_\gamma)_{\gamma\in\gamma(D)} \subset u(E)$ that also is $\mathscr{C}$-convergent to x, by axiom (2). Applying the diagonal principle, we then obtain a net $(y_j)_{j\in J} \subset E$ which is $\mathscr{C}$-convergent to x. Thus, $x \in u(E)$, a contradiction, since $x \in U = X\backslash u(E)$. Suppose now that $C(T(\mathscr{C})) : x_\lambda \to x$ but $\mathscr{C} : x_\lambda \not\to x$. This will lead us to a contradiction. If $\mathscr{C} : x_\lambda \not\to x$, then by axiom (3), $(x_\lambda)_{\lambda\in D}$ has a sub-net $(y_\gamma)_{\gamma\in D'}$ that has no $\mathscr{C}$-convergent to x sub-net. Note that $C(T(\mathscr{C})) : y_\gamma \to x$, and thus for all $j \in D'$, $x \in \overline{A(j)}$, where $A(j) = \{y_\gamma : j \prec \lambda\}$ (the upper bar is for closure with respect to $T(\mathscr{C})$). But, by proposition B.7, $\overline{A(j)} = u_{\mathscr{C}}(A(j))$. Thus, $x \in \overline{A(j)} = u_{\mathscr{C}}(A(j))$ implies that, for each $j \in D'$, there is a net $(x^j_\delta)_{\delta\in D_j} \subset A(j)$ with $\mathscr{C} : x^j_\delta \to x$. From diagonal principle again, we yield a sub-net $(x_\alpha)_{\alpha\in A}$ of $(x^j_\delta)_{(j,\delta)\in\mathscr{D}}$ which is $\mathscr{C}$-convergent to x. To simplify notation, suppose this sub-net is $(x^j_\delta)_{(j,\delta)\in\mathscr{D}}$ itself. If for each D_j we choose any $\delta_j \in D_j$, we obtain a sub-net $(x^j_{\delta_j})_{j\in D'}$ of $(x^j_\delta)_{(j,\delta)\in\mathscr{D}}$ that, by axiom (2), is also $\mathscr{C}$-convergent to x. However, $x^j_{\delta_j} \in A(j)$ and because of this $(x^j_{\delta_j})_{j\in D'}$ is a sub-net of $(y_\gamma)_{\gamma\in D'}$, a contradiction, since $(y_\gamma)_{\gamma\in D'}$ has no sub-nets that are $\mathscr{C}$-convergent to x. This proves that $C(T(\mathscr{C})) = \mathscr{C}$.

□

Given a net convergence notion $\mathscr{C}$ in X, we can restrict this notion to sequences. If $A \subset X$, we define the *sequential closure of* A as the set of all limit points of sequences in A. More precisely, we define $u^{\mathbb{N}}_{\mathscr{C}} : \wp(X) \to \wp(X)$ by

$$u^{\mathbb{N}}_{\mathscr{C}}(A) = \{x \in X : \exists (x_n)_{n\in\mathbb{N}} \subset A,\ \mathscr{C} : x_n \to x\}.$$

REMARK B.10. *Of course* $A \subset u^{\mathbb{N}}_{\mathscr{C}}(A) \subset u_{\mathscr{C}}(A)$*, for every* $A \in \wp(X)$.

The following proposition shows that the convergence notion for *sequences* is enough to study metric spaces, i.e. for metric spaces, there is no need to work with general nets, only sequences.

PROPOSITION B.11. *The topology of a metric space is completely determined by its convergence notion of* sequences. *More precisely, if* (X, d) *is a metric space and* τ_d *is its topology, then* $\tau_d = T(C(\tau_d)) = T^{\mathbb{N}}(C(\tau_d))$*, where*

$$T^{\mathbb{N}}(C(\tau_d)) = \{X \backslash u^{\mathbb{N}}_{C(\tau_d)}(A) : A \in \wp(X)\}.$$

Proof. By simplicity, denote $u = u_{C(\tau_d)}$ and $u^{\mathbb{N}} = u^{\mathbb{N}}_{C(\tau_d)}$. By theorem B.9 and remark above, it suffices to show that $u(A) \subset u^{\mathbb{N}}(A)$, for every $A \in \wp(X)$. Let $A \in \wp(X)$ and $x \in u(A)$. By definition, there is a net $(x_\lambda)_{\lambda\in D} \subset A$ such that $C(\tau_d) : x_\lambda \to x$. Thus, for every $n \in \mathbb{N}$, there is $\lambda_n \in D$ such that $\lambda_n \prec \lambda \implies x_\lambda \in B_{1/n}(x)$, where $B_{1/n}(x)$ is the open ball centered at x with radius $1/n$. We claim that the sequence defined by $y_n = x_{\lambda_n}$ is in A (immediate) and is $C(\tau_d)$-convergent to x, hence $x \in u^{\mathbb{N}}(A)$. Let

$U \in \tau_d$ with $x \in U$. There is some $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Take $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. If $n_0 \leq n$, then $\frac{1}{n} \leq \frac{1}{n_0} < \epsilon$ and hence $y_n = x_{\lambda_n} \in B_{1/n}(x) \subset B_{1/n_0}(x) \subset U$. Therefore, $n_0 \leq n \implies y_n \in U$, i.e. $C(\tau_d) : y_n \to x$. □

B.2 WEAK TOPOLOGY FOR VARIFOLDS

We defined the following convergence notion in the set of k-dimensional varifolds $\mathcal{V}_k(M)$: a net $(V_\lambda)_{\lambda\in D}$ in $\mathcal{V}_k(M)$ *weakly converges* to $V \in \mathcal{V}_k(M)$, written $V_\lambda \rightharpoonup V$, if

$$\int_{G_k(M)} f \, dV_\lambda \longrightarrow \int_{G_k(M)} f \, dV \text{ in } \mathbb{R}, \tag{B.3}$$

for all $f \in \mathcal{C}_c(G_k(M))$ (continuous functions with compact support). For simplicity, we drop the k from $\mathcal{V}_k(M)$ and $G_k(M)$. Since weak convergence of varifolds is given in terms of usual convergence of nets in the real line, it is immediate that weak convergence is a convergence notion in the sense of definition B.5. Therefore, if we denote by $\mathscr{C}_w$ the weak convergence notion on $\mathcal{V}(M)$, we have the topology $\mathscr{T}_w := T(\mathscr{C}_w)$ induced by this convergence notion. We call this topology the ***weak topology*** on $\mathcal{V}(M)$.

For each $f \in \mathcal{C}_c(G(M))$, we can define a function

$$\begin{array}{rccc} \varphi_f : & \mathcal{V}(M) & \to & \mathbb{R} \\ & V & \mapsto & \displaystyle\int_{G(M)} f \, dV. \end{array} \tag{B.4}$$

Each of these functions is continuous, since $V_\lambda \rightharpoonup V$ implies $\varphi_f(V_\lambda) \to \varphi_f(V)$, by definition of weak convergence.

Our goal is to show that the subset of $\mathcal{V}(M)$ of varifolds with uniformly bounded mass is metrizable, if M is compact. We begin with the following proposition:

PROPOSITION B.12. *If M is a compact manifold, then the Banach space $(\mathcal{C}(G(M)), \| \, \|_\infty)$ is separable, i.e. it has a countable dense subset.*

Proof. Since M is compact and each fiber $G(T_xM)$ of $G(M)$ is compact, we have that the grassmannian bundle $G(M)$ is compact. Thus, every continuous function $f : G(M) \to \mathbb{R}$ has compact support and is bounded. Therefore, $\mathcal{C}(G(M)) = \mathcal{C}_b(G(M))$ (every continuous function from $G(M)$ is bounded) and $(\mathcal{C}(G(M)), \| \, \|_\infty)$ is in fact a Banach space. To prove that $\mathcal{C}(G(M))$ is separable, we are going to use the following theorem:

THEOREM B.13 (STONE-WEIERSTRASS, COROLLARY 35, (ROYDEN, 1988), P. 213)**.** *Every continuous function on a compact set $X \subset \mathbb{R}^n$ can be uniformly approximated on X by a polynomial (on the coordinates of $\mathbb{R}^n$).*

By Whitney's Theorem, $G(M)$ can be embedded into some $\mathbb{R}^N$. Thus, we can consider $G(M)$ as a compact submanifold of $\mathbb{R}^N$. By Theorem B.13, the set $\mathscr{P}$ of polynomials (on the coordinates of $\mathbb{R}^N$) with domain restricted to $G(M)$ is dense in $(\mathcal{C}(G(M)), \| \|_\infty)$. We claim that the *countable* set $P \subset \mathscr{P}$ of polynomials with rational cofficients is dense in $\mathcal{C}(G(M))$. In fact, let $f \in \mathcal{C}(G(M))$ and $\epsilon > 0$. Since $\overline{\mathscr{P}} = \mathcal{C}(G(M))$, there is $p \in \mathscr{P}$ such that $\|f - p\|_\infty < \epsilon/2$. Write $p(x_1, \ldots, x_N) = \sum_{j=1}^k a_j m_j(x_1, \ldots, x_N)$, with $a_j \in \mathbb{R}$ and each m_j a monomial, with $m_j \neq m_i$, if $j \neq i$. Take $M > \max\{\|m_j\|_\infty : j = 1, \ldots, k\}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there are $b_1, \ldots, b_k \in \mathbb{Q}$ such that $|a_j - b_j| < \epsilon/(2kM)$, for all $j = 1, \ldots, k$. Denote $q = \sum_{j=1}^k b_j m_j \in P$. For every $(x_1, \ldots, x_N) \in G(M)$ we have

$$|(p - q)(x_1, \ldots, x_N)| \leq \sum_{j=1}^k |a_j - b_j| \|m_j\|_\infty < \sum_{j=1}^k \frac{\epsilon}{2kM} M = \frac{\epsilon}{2}.$$

Thus, $\|p - q\|_\infty \leq \epsilon/2$. It follows from the triangular inequality that $\|f - q\|_\infty < \epsilon$. Therefore, P is dense in $\mathcal{C}(G(M))$. This finishes the proof. $\square$

From now on suppose M is always a compact manifold.

Let $\{h_n\}_{n\in\mathbb{N}}$ be a dense subset of $B_1 = \{f \in \mathcal{C}(G(M)) : \|f\|_\infty \leq 1\}$ (such subset exists, by Proposition B.12). By simplicity, denote $\varphi_n := \varphi_{h_n}$, with φ_{h_n} defined at (B.4). Define

$$\begin{array}{rccl} \mathfrak{d}: & \mathcal{V}(M) \times \mathcal{V}(M) & \to & [0, +\infty) \\ & (V, W) & \mapsto & \sum\limits_{n=1}^{\infty} 2^{-n} |\varphi_n(V) - \varphi_n(W)|. \end{array} \tag{B.5}$$

Note that $\mathfrak{d}$ depends on the choice of $\{h_n\}_{n\in\mathbb{N}}$.

PROPOSITION B.14. *The function $\mathfrak{d}$ is well defined and is distance function on $\mathcal{V}(M)$.*

Proof. Of course $0 \leq \mathfrak{d}$, thus in order to show that $\mathfrak{d}$ is well defined, we need to show that $\mathfrak{d} < +\infty$. Let $V, W \in \mathcal{V}(M)$. Since $G(M)$ is compact and V, W are Radon measures, we have $\|V\|, \|W\| < +\infty$. Thus,

$$\begin{aligned} \mathfrak{d}(V, W) &= \sum_{n=1}^{\infty} 2^{-n} |\varphi_n(V) - \varphi_n(W)| \leq \sum_{n=1}^{\infty} 2^{-n} (|\varphi_n(V)| + |\varphi_n(W)|) \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \left(\int_{G(M)} |h_n| \, dV + \int_{G(M)} |h_n| \, dW \right) \\ &\leq \sum_{n+1}^{\infty} 2^{-n} \left(\int_{G(M)} dV + \int_{G(M)} dW \right) \\ &= \sum_{n=1}^{\infty} 2^{-n} (\|V\| + \|W\|) = \|V\| + \|W\| < +\infty. \end{aligned}$$

Let us show now that $\mathfrak{d}$ is a distance function. It is immediate from the definition that $\mathfrak{d}(V, V) = 0$ and that $\mathfrak{d}$ is symmetric. Triangle inequality follows from:

$$\mathfrak{d}(V, W) = \sum_{n=1}^{\infty} 2^{-n} |\varphi_n(V) - \varphi_n(W)| \leq \sum_{n=1}^{\infty} 2^{-n} (|\varphi_n(V) - \varphi_n(Z)| + |\varphi_n(Z) - \varphi_n(W)|)$$

$$= \mathfrak{d}(V,Z) + \mathfrak{d}(Z,W).$$

It only remains to show that $\mathfrak{d}(V,W) = 0 \Rightarrow V = W$. If $\mathfrak{d}(V,W) = 0$, then

$$\int_{G(M)} h_n \, dV = \int_{G(M)} h_n \, dW,$$

for all $n \in \mathbb{N}$. Consider any $f \in B_1 \subset \mathcal{C}(G(M))$. Since $\{h_n\}_{n\in\mathbb{N}}$ is dense B_1, there is a sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\lim_{k\to\infty} \|h_{n_k} - f\|_\infty = 0$. This implies

$$\int_{G(M)} f \, dV = \lim_{k\to\infty} \int_{G(M)} h_{n_k} \, dV = \lim_{k\to\infty} \int_{G(M)} h_{n_k} \, dW = \int_{G(M)} f \, dW,$$

for any $f \in B_1$.

Let $A \subset G(M)$ be a Borel set and $\chi_A : G(M) \to \mathbb{R}$ the characteristic function of A. Taking a sequence $\{f_k\}_{k\in\mathbb{N}} \subset B_1$ converging to χ_A, we get

$$V(A) = \int_{G(M)} \chi_A \, dV = \lim_{k\to\infty} \int_{G(M)} f_k \, dV = \lim_{k\to\infty} \int_{G(M)} f_k \, dW = \int_{G(M)} \chi_A \, dW = W(A).$$

Thus, $V(A) = W(A)$, for all Borel set $A \subset G(M)$.

Finally, if $S \subset G(M)$ is any set, since V and W are Radon measures, it follows that $V(S) = \inf\{V(U) : S \subset U,\, U \text{ aberto}\} = \inf\{W(U) : S \subset U,\, U \text{ aberto}\} = W(S)$. This proves that $V = W$. □

Now, we finally show that the space of varifolds of uniformly bounded mass is metrizable. Next, we use a compactness theorem for Radon measures to prove that this space is also compact. This will prove Theorem 2.10.

THEOREM B.15. *The weak topology coincide with the topology induced by $\mathfrak{d}$ on $\mathcal{V}^c(M) := \{V \in \mathcal{V}(M) : \|V\| \le c\}$, for each $c \ge 0$.*

Proof. To prove the theorem, it suffices to show that the notions of weak and metric convergences coincide. Let $(V_\lambda)_{\lambda\in D}$ be a net in $\mathcal{V}(M)$ and $V \in \mathcal{V}(M)$. Suppose first that $V_\lambda \rightharpoonup V$. Let us show that $\mathfrak{d}(V_\lambda, V) \to 0$. Consider $\epsilon > 0$. Fix $m \in \mathbb{N}$. Then

$$\begin{aligned}
\mathfrak{d}(V_\lambda, V) &= \sum_{n=1}^{m} 2^{-n}|\varphi_n(V_\lambda) - \varphi_n(V)| + \sum_{n=m+1}^{\infty} 2^{-n}|\varphi_n(V_\lambda) - \varphi_n(V)| \\
&\le \sum_{n=1}^{m} 2^{-n}|\varphi_n(V_\lambda) - \varphi_n(V)| + \sum_{n=m+1}^{\infty} 2^{-n}2c \\
&= \sum_{n=1}^{m} 2^{-n}|\varphi_n(V_\lambda) - \varphi_n(V)| + \sum_{n=m}^{\infty} 2^{-n}c = \sum_{n=1}^{m} 2^{-n}|\varphi_n(V_\lambda) - \varphi_n(V)| + 2^{-m+1}c.
\end{aligned}$$

Thus, if we choose m such that $2^{-m+1}c < \epsilon/2$, we have

$$\mathfrak{d}(V_\lambda, V) < \sum_{n=1}^{m} 2^{-n}|\varphi_n(V_\lambda) - \varphi_n(V)| + \epsilon/2.$$

Now, since $V_\lambda \rightharpoonup V$, for each $n = 1, \ldots, m$ there is a $\lambda_n \in D$ such that $\lambda_n \prec \lambda \implies |\varphi_n(V_\lambda) - \varphi_n(V)| < \epsilon/(2m)$. Since D is directed, we can find a $\lambda_0 \in D$ with $\lambda_n \prec \lambda_0$, for every $n = 1, \ldots, m$. Then,

$$\begin{aligned}\lambda_0 \prec \lambda \implies \mathfrak{d}(V_\lambda, V) &< \sum_{n=1}^{m} 2^{-n}|\varphi_n(V_\lambda) - \varphi_n(V)| + \frac{\epsilon}{2} \\ &< \sum_{n=1}^{m} 2^{-n}\frac{\epsilon}{2m} + \frac{\epsilon}{2} < m\frac{\epsilon}{2m} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Therefore $\mathfrak{d}(V_\lambda, V) \to 0$.

Now, suppose $\mathfrak{d}(V_\lambda, V) \to 0$. We must show that $V_\lambda \rightharpoonup V$. Let $f \in \mathcal{C}(G(M))$ and let's show that $\varphi_f(V_\lambda) \to \varphi_f(V)$. We can suppose that $f \neq 0$. Then, since $G(M)$ is compact, $\sup_{x \in G(M)} f(x) = \|f\|_\infty = a < +\infty$. Thus $\frac{f}{a} \in B_1$. Since $\{h_n\}_{n\in\mathbb{N}}$ is dense in B_1, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \{h_n\}_{n\in\mathbb{N}}$ such that $\|f_k - \frac{f}{a}\|_\infty \to 0$. Let $\epsilon > 0$. Since $\mathfrak{d}(V_\lambda, V) \to 0$, there exists a $\lambda_0 \in D$ such that $\lambda_0 \prec \lambda$ implies $\mathfrak{d}(V_\lambda, V) < \epsilon/a$. Then

$$\lambda_0 \prec \lambda \implies |\varphi_{f_k}(V_\lambda) - \varphi_{f_k}(V)| \leq \mathfrak{d}(V_\lambda, V) < \frac{\epsilon}{a}.$$

Now, note that

$$\lim_{k\to\infty} \varphi_{f_k}(W) = \lim_{k\to\infty} \int_{G(M)} f_k \, dW = \int_{G(M)} \frac{f}{a} \, dW = \frac{1}{a}\varphi_f(W), \ \ \forall W \in \mathcal{V}(M).$$

Thus, making $k \to \infty$ in the inequality above, we get

$$\lambda_0 \prec \lambda \implies |\varphi_f(V_\lambda) - \varphi_f(V)| < \epsilon.$$

This shows that $V_\lambda \rightharpoonup V$ and finishes the proof. □

THEOREM B.16. *For each $c > 0$, the space $\mathcal{V}^c(M)$ is compact.*

Proof. This result will follow straightforward from the following theorem, which we won't prove.

THEOREM B.17 (COMPACTNES THEOREM FOR RADON MEASURES, (SIMON, 2014), P. 37)**.** *Suppose $\{\mu_k\}$ is a sequence of Radon measures on the locally compact, σ-compact Hausdorff space X with the property $\sup_k \mu_k(K) < \infty$ for each compact $K \subset X$. Then there is a subsequence $\{\mu_{k'}\}$ which converges to a Radon measure μ on X in the sense that*

$$\lim \mu_{k'}(f) = \mu(f) \text{ for each } f \in \mathcal{K}(X),$$

where $\mathcal{K}(X)$ denotes the set of continuous functions $f : X \to \mathbb{R}$ with compact support on X and where we use the notation

$$\mu(f) = \int_X f \, d\mu, \ \ f \in \mathcal{K}(X).$$

If we take $X = G(M)$, this becomes a result about varifolds. Since $G(M)$ is a compact smooth manifold (since we are considering M to be compact), $G(M)$ is trivially locally compact and σ-compact (this last condition means that the space is a union of countably many compact subspaces).

Take any sequence $\{V_k\} \subset \mathcal{V}^c(M)$. Then for any compact $K \subset G(M)$, we have $\sup_k V_k(K) \leq \sup_k V_k(G(M)) = \sup_k \|V_k\|(M) \leq c < +\infty$ and we can apply Theorem B.17. Thus, there is a subsequence $\{V_{k'}\}$ of $\{V_k\}$ and a varifold $V \in G(M)$ such that $V_{k'} \rightharpoonup V$. In particular,

$$\|V\|(M) = \lim \|V_{k'}\|(M) \leq c,$$

and therefore $V \in \mathcal{V}^c(M)$. This shows that $\mathcal{V}^c(M)$ is sequentially compact. Since we already proved that $\mathcal{V}^c(M)$ is metrizable and sequentially compactness and compactness are equivalent for metric spaces, this shows that $\mathcal{V}^c(M)$ is compact. $\square$

Theorems B.15 and B.16 together prove Theorem 2.10.

C Geometric lemmas

Lemma C.1. *Let (M,g) be a riemannian three-manifold and Σ an embedded two-sided surface in M. Then*

$$R - 2\mathrm{Ric}(\nu,\nu) - |A|^2 = 2K - H^2.$$

Proof. The lemma will follow from the Gauß equation

$$\langle \boldsymbol{R}(X,Y)Z,W\rangle = \langle \boldsymbol{r}(X,Y)Z,W\rangle - \langle A(Y,W),A(X,Z)\rangle + \langle A(X,W),A(Y,Z)\rangle,$$

where $\boldsymbol{r}$ denotes the curvature tensor of Σ. Let $\{e_1,e_2,\nu\}$ be an orthonormal basis for T_pM, $p\in\Sigma$. Doing $Y=W=e_1$, $Y=W=e_2$ and summing the equations, we obtain

$$\begin{aligned}&\langle \boldsymbol{R}(X,e_1)Z,e_1\rangle + \langle \boldsymbol{R}(X,e_2)Z,e_2\rangle = \langle \boldsymbol{r}(X,e_1)Z,e_1\rangle - \langle A(e_1,e_1),A(X,Z)\rangle\\ &+\langle A(X,e_1),A(e_1,Z)\rangle + \langle \boldsymbol{r}(X,e_2)Z,e_2\rangle - \langle A(e_2,e_2),A(X,Z)\rangle + \langle A(X,e_2),A(e_2,Z)\rangle.\end{aligned}$$

Denote $A(X,Y)=h(X,Y)\nu$, and then $H := \text{traço}(A) = h(e_1,e_1)+h(e_2,e_2)$. Summing to both sides of the equation the term $\langle \boldsymbol{R}(X,\nu)Z,\nu\rangle$, in the left-hand side we get traço$(Y\mapsto R(X,Y)Z) = \mathrm{Ric}(X,Z)$. Noticing that $\langle \boldsymbol{r}(X,e_1)Z,e_1\rangle + \langle \boldsymbol{r}(X,e_2)Z,e_2\rangle =$ traço$(Y\mapsto \boldsymbol{r}(X,Y)Z) = \mathrm{ric}(X,Z)$, we get:

$$\begin{aligned}\mathrm{Ric}(X,Z) &= \mathrm{ric}(X,Z) - (h(e_1,e_1)+h(e_2,e_2))h(X,Z) + h(X,e_1)h(e_1,Z)\\ &\quad + h(X,e_2)h(e_2,Z) + \langle \boldsymbol{R}(X,\nu)Z,\nu\rangle\\ &= \mathrm{ric}(X,Z) - Hh(X,Z) + \sum_{i=1}^{2} h(X,e_i)h(e_i,Z) + \langle \boldsymbol{R}(X,\nu)Z,\nu\rangle.\end{aligned}$$

Now, doing $X=Z=e_1$, $X=Z=e_2$ and then summing up the equations, we get

$$\begin{aligned}\sum_{j=1}^{2}\mathrm{Ric}(e_j,e_j) &= \sum_{j=1}^{2}\mathrm{ric}(e_j,e_j) - H\sum_{j=1}^{2}h(e_j,e_j) + \sum_{i,j=1}^{2}h(e_i,e_j)^2 + \sum_{j=1}^{2}\langle \boldsymbol{R}(e_j,\nu)e_j,\nu\rangle\\ &= r - H^2 + |A|^2 + \sum_{j=1}^{2}\langle \boldsymbol{R}(\nu,e_j)\nu,e_j\rangle\\ &= r - H^2 + |A|^2 + \mathrm{Ric}(\nu,\nu).\end{aligned}$$

Finally, summing the term $\mathrm{Ric}(\nu,\nu)$ to both sides of the equation and using that $r = 2K$, we get

$$R = 2K - H^2 + |A|^2 + 2\mathrm{Ric}(\nu,\nu).$$

□

Proposition C.2. *Let M be a compact riemannian 3-manifold with scalar curvature $R\geq k_0$.*

1. *If Σ is an embedded orientable minimal surface of index one, then*
$$\int_\Sigma \mathrm{Ric}(\nu,\nu)+|A|^2\,d\Sigma \le 8\pi\left(\left[\frac{g(\Sigma)+1}{2}\right]+1\right)$$
and
$$k_0|\Sigma| \le 24\pi + 16\left(\frac{g(\Sigma)}{2}-\left[\frac{g(\Sigma)}{2}\right]\right).$$
2. *If Σ is stable and nonorientable, then*
$$\int_\Sigma \mathrm{Ric}(\nu,\nu)+|A|^2 d\Sigma \le 4\pi(g(\tilde\Sigma)+1)$$
and
$$k_0|\Sigma| \le 12\pi + 4\pi g(\tilde\Sigma).$$
3. *Suppose $k_0 >$. If Σ is stable and orientable, then it is a sphere with $k_0|\Sigma| \le 8\pi$. Equality implies that $R = k_0$ on Σ.*

Here $[x]$ denotes the integer part of x and $\tilde\Sigma$ is the orientable double cover of Σ.

Proof. 1. If Σ is orientable of index one, then there is a conformal map $\phi : \Sigma \to \mathbb{S}^2$ such that
$$\int_\Sigma \mathrm{Ric}(\nu,\nu)+|A|^2\,d\Sigma \le 8\pi\deg(\phi).$$
See (S. YAU, 1986, p. 127). Furthermore, we can choose ϕ such that
$$\deg(\phi) \le \left[\frac{g(\Sigma)+1}{2}\right]+1,$$
as in (RITORÉ; ROS, 1992, p. 299). The first inequality follows.

Now, it follows from Lemma C.1 and the Gauß–Bonnet Theorem that
$$\begin{aligned}\frac{k_0}{2}|\Sigma| &\le \int_\Sigma \frac{R}{2}\,d\Sigma = \int_\Sigma \mathrm{Ric}(\nu,\nu)+\frac{|A|^2}{2}+K\,d\Sigma \le 8\pi\left(\left[\frac{g(\Sigma)+1}{2}\right]+1\right)+2\pi\chi(\Sigma)\\ &\le 8\pi\left(\left[\frac{g(\Sigma)+1}{2}\right]+1\right)+4\pi(1-g(\Sigma)) = 12\pi+8\pi\left(\left[\frac{g(\Sigma)+1}{2}\right]-\frac{g(\Sigma)}{2}\right).\end{aligned}$$
Now, to finish the proof we just need to show that
$$\left[\frac{n+1}{2}\right]-\frac{n}{2} = \frac{n}{2}-\left[\frac{n}{2}\right],\quad \forall n=0,1,2,\ldots \tag{$\star$}$$
which is equivalent to
$$\left[\frac{n+1}{2}\right] = n-\left[\frac{n}{2}\right],\quad \forall n=0,1,2,\ldots$$
Of course, this is true for $n=0,1$. Suppose the equation holds for $n=k$. Then
$$\left[\frac{(k+1)+1}{2}\right] = \left[\frac{k}{2}+1\right] = 1+\left[\frac{k}{2}\right] = 1+k-\left[\frac{k+1}{2}\right],$$
and thus the equation holds for $n=k+1$ and $(\star)$ follows by induction. Then we have
$$k_0|\Sigma| \le 24\pi+16\pi\left(\frac{g(\Sigma)}{2}-\left[\frac{g(\Sigma)}{2}\right]\right).$$

2. The first inequality follows from (ROSS, 1997, Lemma 2 and identity (2')). The second is a consequence from Lemma C.1 and the Gauß–Bonnet Theorem:

$$
\begin{aligned}
\frac{k_0}{2}|\Sigma| = k_0|\tilde{\Sigma}| &\leq \int_{\tilde{\Sigma}} R \, d\tilde{\Sigma} = \int_{\tilde{\Sigma}} 2\,\mathrm{Ric}(\nu,\nu) + |A|^2 + 2K \, d\tilde{\Sigma} \\
&\leq \int_{\tilde{\Sigma}} \mathrm{Ric}(\nu,\nu) + |A|^2 \, d\tilde{\Sigma} + 2\pi(1 - g(\tilde{\Sigma})) \leq 4\pi(g(\tilde{\Sigma}) + 1) + 2\pi(1 - g(\tilde{\Sigma})) \\
&\leq 6\pi + 2\pi g(\tilde{\Sigma}) \\
\Rightarrow k_0|\Sigma| &\leq 12\pi + 4\pi g(\tilde{\Sigma}).
\end{aligned}
$$

3. This is proven in (BRAY; BRENDLE; NEVES, 2010) (see identity (4)).

□

Bibliography

ANDERSON, M. T. Curvature estimates for minimal surfaces in 3-manifolds. In: 1. ANNALES scientifiques de l'École Normale Supérieure. 1985. vol. 18, pp. 89–105.

BRAY, H.; BRENDLE, S.; NEVES, A. Rigidity of area-minimizing two-spheres in three-manifolds. In: vol. 18.

CARMO, M.P. do. **Geometria Riemanniana**. Instituto de Matemática Pura e Aplicada, 1988. (Projeto Euclides).

COELHO, F. U.; LOURENÇO, M. L. **Um curso de álgebra linear**. São Paulo: EDUSP, 2013.

COLDING, T.; DE LELLIS, C. The min-max construction of minimal surfaces. In: SURVEYS in Differential Geometry. Boston: International Press, 2003. Vol. 8.

DRUET, O. Isoperimetric Inequalities on Compact Manifolds. **Geometriae Dedicata**, vol. 90, no. 1, pp. 217–236, Mar. 2002.

DUGUNDJI, J. **Topology**. Allyn and Bacon, 1966. (Allyn and Bacon series in advanced mathematics).

FRANKEL, T. On the Fundamental Group of a Compact Minimal Submanifold. **Annals of Mathematics**, Annals of Mathematics, vol. 83, no. 1, pp. 68–73, 1966.

GILBARG, D.; TRUDINGER, N.S. **Elliptic Partial Differential Equations of Second Order**. Springer Berlin Heidelberg, 2001. (Classics in Mathematics).

HAMILTON, R. S. Three-manifolds with positive Ricci curvature. **J. Differential Geom.**, Lehigh University, vol. 17, no. 2, pp. 255–306, 1982.

HASS, J.; SCOTT, P. The Existence of Least Area Surfaces in 3-Manifolds. **Transactions of The American Mathematical Society - TRANS AMER MATH SOC**, vol. 310, pp. 87–87, Nov. 1988.

HATCHER, A. **Algebraic Topology**.
https://pi.math.cornell.edu/~hatcher/AT/AT.pdf. 2002.

HUISKEN, G.; POLDEN, A. Geometric evolution equations for hypersurfaces. In: CALCULUS of Variations and Geometric Evolution Problems: Lectures given at the 2nd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Cetraro, Italy, June 15–22, 1996. Berlin, Heidelberg: Springer Berlin Heidelberg, 1999. pp. 45–84.

LELLIS, C.; PELLANDINI, F. Genus bounds for minimal surfaces arising from min-max constructions. **Journal für die reine und angewandte Mathematik (Crelles Journal)**, vol. 2010, May 2009.

MARQUES, F. C.; NEVES, A. Rigidity of min-max minimal spheres in three-manifolds. **Duke Math. J.**, Duke University Press, vol. 161, no. 14, pp. 2725–2752, Nov. 2012.

MEEKS, W.; SIMON, L.; YAU, S. T. Embedded Minimal Surfaces, Exotic Spheres, and Manifolds with Positive Ricci Curvature. **Annals of Mathematics**, vol. 116, no. 3, pp. 621–659, 1982.

PITTS, J. T. **Existence and Regularity of Minimal Surfaces on Riemannian Manifolds. (MN-27)**. Princeton University Press, 1981.

RITORÉ, M.; ROS, A. Stable constant mean curvature tori and the isoperimetric problem in three space forms. **Commentarii Mathematici Helvetici**, vol. 67, pp. 293–305, Dec. 1992.

ROSS, M. The second variation of nonorientable minimal submanifolds. **Transactions of the American Mathematical Society**, vol. 349, no. 8, pp. 3093–3104, 1997.

ROYDEN, H. L. **Real Analysis**. Macmillan, 1988. (Mathematics and statistics).

SAVELIEV, N. **Lectures on the Topology of 3-manifolds: An Introduction to the Casson Invariant**. New Orleans, LA: Walter de Gruyter, 1999. (De Gruyter textbook).

SCHULTENS, J. The stabilization problem for Heegaard splittings of Seifert fibered spaces. **Topology and its Applications**, vol. 73, no. 2, pp. 133–139, 1996.

SHERIDAN, N. **Hamilton's Ricci flow**. Honour thesis
https://web.math.princeton.edu/~nsher/ricciflow.pdf. 2006.

SIMON, L. **Introduction to Geometric Measure Theory**.
http://web.stanford.edu/class/math285/ts-gmt.pdf. 2014.

STILLWELL, J. **Classical topology and combinatorial group theory**. New York: Springer Science & Business Media, 2012. vol. 72.

WHITE, B. Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals. **Inventiones mathematicae**, Springer, vol. 88, no. 2, pp. 243–256, 1987.

YAU, S.T. **Nonlinear Analysis in Geometry: Lectures Given at the ETH-Zurich**. Geneva: L'Enseignement mathématique, Université de Genève, 1986. (Monographies de l'Enseignement mathématique).