UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Lauren Maria Mezzomo Bonaldo

Existence and multiplicity of solutions for a class of elliptic equations involving nonlocal integrodifferential operator with variable exponent

São Carlos - SP
16 de março de 2020

O presente trabalho teve suporte financeiro da Capes

UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS GRADUAÇÃO EM MATEMÁTICA

Existence and multiplicity of solutions for a class of elliptic equations involving nonlocal integrodifferential operator with variable exponent

Lauren Maria Mezzomo Bonaldo
Bolsista Capes

Orientador: Olímpio Hiroshi Miyagaki

Tese apresentada ao Programa de Pós-Graduação em Matemática da UFSCar como parte dos requisitos para a obtenção do título de Doutor em Matemática

Mezzomo Bonaldo, Lauren Maria

Existence and multiplicity of solutions for a class of elliptic equations involving nonlocal integrodifferential operator with variable exponent / Lauren Maria Mezzomo Bonaldo. -- 2020.
96 f. : 30 cm.

Tese (doutorado)-Universidade Federal de São Carlos, campus São Carlos, São Carlos
Orientador: Lauren M. M. Bonaldo
Banca examinadora:
Bibliografia

1. Nonlocal integrodifferential operator. 2. Fractional Sobolev space with variable exponents. 3. Variational methods. I. Orientador. II. Universidade Federal de São Carlos. III. Título.

Ficha catalográfica elaborada pelo Programa de Geração Automática da Secretaria Geral de Informática (SIn).
DADOS FORNECIDOS PELO(A) AUTOR(A)
Bibliotecário(a) Responsável: Ronildo Santos Prado – CRB/8 7325

UNIVERSIDADE FEDERAL DE SÃO CARLOS

Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Matemática

Folha de Aprovação

Assinaturas dos membros da comissão examinadora que avaliou e aprovou a Defesa de Tese de Doutorado da candidata Lauren Maria Mezzomo Bonaldo, realizada em 16/03/2020:

Prof. Dr. Olimpio Hiroshi Miyagaki
UFJF

Prof. Dr. Sérgio Henrique Monari Soares
USP

Prof. Dr. Giovany de Jesus Malcher Figueiredo
UnB

Prof. Dr. Gustavo Ferron Madeira
UFSCar

Prof. Dr. Francisco Odair Vieira de Paiva
UFSCar

Certifico que a defesa realizou-se com a participação à distância do(s) membro(s) Giovany de Jesus Malcher Figueiredo e, depois das arguições e deliberações realizadas, o(s) participante(s) à distância está(ao) de acordo com o conteúdo do parecer da banca examinadora redigido neste relatório de defesa.

Prof. Dr. Olimpio Hiroshi Miyagaki

Agradecimentos

Agradeço primeiramente a Deus por ter me dado saúde e força para realização deste trabalho. Por sempre colocar em minha vida pessoas ímpares, demonstrando, por meio delas, todo o Seu cuidado e amor por mim.

Aos meus pais Maria e Emir obrigada pelo apoio e incentivo de sempre. Obrigada pela educação recebida, pela compreensão na minha ausência familiar, por acreditarem e confiarem em mim.

Ao meu noivo Rafael obrigada por todo seu amor, pelos momentos de paciência e apoio incondicional para chegar até aqui. Obrigada por sempre ter estado ao meu lado com muita alegria e motivação não só nos dias alegres, mas também nos dias complicados e tristes.

A minha avó Eliza(in memoriam) gratidão pelas suas orações, preocupações e carinho até sua partida.

Ao meu tio Luciano obrigada por sempre me apoiar, torcer e vibrar por cada conquista minha.

Ao meu orientador Olímpio o qual tive a honra de trabalhar. Obrigada pela confiança em mim depositada, pelas nossas inúmeras conversas, pelos seus conselhos os quais foram muito importantes para o meu amadurecimento profissional e pessoal. Enfim, muito obrigada por sua infinita generosidade em transmitir parte de seu grande conhecimento na área de Equações Diferenciais Parciais Elípticas de forma competente, acolhedora e amiga.

Ao meu colega e grande amigo Elard Juaréz Hurtado, ser humano extraordinário, exemplo de dedicação e compreensão. Sempre pronto a atender minhas dúvidas e me dando força nas dificuldades que não foram poucas. Obrigada pelos nossos inúmeros momentos de estudos, ensinamentos, conselhos e conversas. Levarei teu exemplo para a vida toda, minha eterna gratidão.

Aos meus amigos e demais familiares que vivenciaram comigo cada dificuldade desse trabalho sempre me apoiando, torcendo e propiciando momentos de descanso mais agradáveis e descontraídos. Obrigada por vocês serem minhas fortalezas, fontes de carinho e alegria.

Aos colegas do doutorado, em especial, Eric, Fernanda, Givanildo, Jéssica, Maria Carolina, Maykel, Rafaela, Renato, Renata, Rodrigo, Ronaldo e Sandra obrigada pela linda amizade que construimos, pelas trocas de experiências, pelos momentos de distração, pelos conselhos e também por partilharmos juntos os momentos de angústia e desespero.

A todos professores e funcionários do Departamento de Matemática da UFSCar, em especial, o professor Fabio Ferrari Ruffino que no início do doutorado muito me motivou e aconselhou a seguir sem medo nesta trajetória de estudo. Obrigada por vocês de alguma forma contribuirem para a realização deste trabalho.

Aos professores da Banca, Francisco Odair de Paiva, Giovany de Jesus Malcher Figueiredo, Gustavo Ferron Madeira e Sergio Henrique Monari Soares obrigada pela revisão do trabalho e pelas valiosas contribuições.

A Capes pelo apoio financeiro.

Enfim, a todos que torceram para que mais essa etapa fosse concluída. Meu muito obrigada!

Paciência e perseverança tem o efeito mágico de fazer as dificuldades desaparecerem e os obstáculos sumirem.
John Quincy Adams

Resumo

Neste trabalho, estamos interessados na existência e multiplicidade de soluções não-triviais para uma classe de problemas elípticos. O primeiro problema trata da existência de soluções fracas não-triviais para uma classe de equações elípticas que envolvem um operador integrodiferencial não-local geral $\mathscr{L}_{AK}$ com expoentes variáveis, dois parâmetros reais e duas funções peso que podem mudar de sinal em um domínio suave limitado. Considerando diferentes situações relacionadas ao crescimento das não-linearidades envolvidas no problema, provamos a existência de duas soluções distintas não-triviais para o caso de expoentes constantes e a existência de uma família contínua de autovalores no caso de expoentes variável. As provas dos principais resultados são baseadas em soluções ground state usando o método de Nehari, o princípio variacional de Ekeland e o método direto do cálculo variacional.

O segundo problema trata da existência e da multiplicidade de soluções fracas envolvendo o mesmo operador $\mathscr{L}_{AK}$, um parâmetro real positivo e expoentes variáveis sem condições de crescimento do tipo Ambrosetti e Rabinowitz em um domínio suave e limitado. Utilizando diferentes versões do Teorema do Passo da Montanha, bem como o Teorema de Fountain e o Teorema de Dual Fountain com a condição de Cerami, obtemos a existência de soluções fracas para o problema. Além disso, para o caso sublinear, ao impor algumas hipóteses adicionais à não-linearidade, obtemos a existência de infinitas soluções fracas que tendem a ser zero, na norma de Sobolev fracionário, para qualquer parâmetro positivo.

Abstract

In this work, we are interested in the existence and multiplicity of nontrivial solutions for a class of elliptic problems. The first problem deals with the existence of nontrivial weak solutions to a class of elliptic equations involving a general nonlocal integrodifferential operator $\mathscr{L}_{AK}$ with variable exponent, two real parameters, and two weight functions, which can be sign-changing in a smooth bounded domain. Considering different situations related to the growth of nonlinearities involved in problem, we prove the existence of two distinct nontrivial solutions for the case of constant exponents and the existence of a continuous family of eigenvalues in the case of variable exponents. The proofs of the main results are based on ground state solutions using the Nehari method, Ekeland's variational principle, and the direct method of the calculus of variations.

The second problem deals with the existence and multiplicity of weak solutions involving the same operator $\mathscr{L}_{AK}$, variable exponents without Ambrosetti and Rabinowitz type growth conditions and a positive real parameter in a smooth bounded domain. Using different versions of the Mountain Pass Theorem, as well as, the Fountain Theorem and Dual Fountain Theorem with Cerami condition, we obtain the existence of weak solutions for problem. Moreover, for the case sublinear, by imposing some additional hypotheses on the nonlinearity, we obtain the existence of infinitely many weak solutions which tend to be zero, in the fractional Sobolev norm, for any positive parameter.

Contents

Introdução

Nos últimos anos, o estudo de problemas elípticos envolvendo operadores integrodiferenciais não-locais tornou-se objeto de estudo de muitos pesquisadores. Problemas desse tipo têm uma base teórica muito interessante, cuja integrabilidade e estrutura analítica requerem provas com técnicas bastante delicadas conforme podemos ver em [7, 8, 41, 50]. Além disso, eles têm aplicações concretas nos mais diversos campos como otimização, finanças, teoria da probabilidade, transições de fase, mecânica continuum, processo de imagem, teoria dos jogos, deslocamento de cristais, fluxos quase-geostróficos, teoria peridinâmica, entre outras, veja [5, 6, 11, 18, 19, 35, 40, 42, 45, 63, 68, 73] e suas referências.

Nesse sentido afim de expandir os resultados em torno dessa teoria, no presente trabalho, estudamos a existência, a multiplicidade e comportamento assintótico de soluções fracas para uma classe de equações elípticas envolvendo operadores integrodiferenciais não-locais gerais com expoentes variáveis. Os resultados que demonstramos foram baseados em métodos variacionais e topológicos os quais são técnicas eficazes para obter os resultados desejados.

Na sequência vamos descrever brevemente os problemas estudados e os progressos obtidos nos Capítulos 1 e 2.

No **Capítulo 1**, estudamos resultados relativos à existência de soluções fracas para uma classe de equações elípticas envolvendo um operador integrodiferencial não-local geral com expoente variável, dois parâmetros reais, duas funções peso que podem mudar de sinal e diferentes não-linearidades críticas, a saber

$$\begin{cases} \mathscr{L}_{\mathcal{A}K} u = \lambda \mathfrak{a}(x)|u|^{\mathtt{m}_1(x)-2}u + \beta \mathfrak{b}(x)|u|^{\mathtt{m}_2(x)-2}u & \text{em } \Omega, \\ u = 0 & \text{em } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

onde $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ é um domínio suave limitado, λ e β são parâmetros reais, as funções peso $\mathfrak{a}, \mathfrak{b} : \overline{\Omega} \to \mathbb{R}$ podem mudar de sinal em Ω, $\mathfrak{m}_1$ e $\mathfrak{m}_2 \in C^+(\overline{\Omega})$, e para definir o operador integrodiferencial não-local geral $\mathscr{L}_{\mathcal{A}K}$ consideramos o expoente variável $p(x) := p(x,x)$ para todo $x \in \mathbb{R}^N$ com $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfazendo:

$$\begin{aligned} &p \text{ é simétrico, isto é, } p(x,y) = p(y,x), \\ &1 < p^- := \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) \leqslant \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) := p^+ < \frac{N}{s}, \ \ s \in (0,1), \end{aligned} \tag{p_1}$$

e consideramos o expoente variável crítico fracionário relacionado a $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ definido por $p_s^\star(x) = \dfrac{Np(x)}{N - sp(x)}$.

O operador integrodiferencial não-local geral $\mathscr{L}_{\mathcal{A}K}$ é definido em espaços de Sobolev fracionário adequados (consulte Capítulo 1, Subseção 1.1) por

$$\mathscr{L}_{\mathcal{A}K}u(x) = P.V. \int_{\mathbb{R}^N} \mathcal{A}(u(x) - u(y))K(x,y)\, dy, \ \ x \in \mathbb{R}^N, \tag{2}$$

onde $P.V.$ é o valor principal.

O aplicação $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ é uma função mensurável que satisfaz as seguintes condições:

(a_1) $\mathcal{A}$ é contínua, ímpar e a aplicação $\mathscr{A} : \mathbb{R} \to \mathbb{R}$ definida por

$$\mathscr{A}(t) := \int_0^{|t|} \mathcal{A}(\tau) d\tau$$

é estritamente convexa;

(a_2) Existem constantes positivas $c_{\mathcal{A}}$ e $C_{\mathcal{A}}$, tal que para todo $t \in \mathbb{R}$ e para todo $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$

$$\mathcal{A}(t)t \geqslant c_{\mathcal{A}}|t|^{p(x,y)} \quad \text{e} \quad |\mathcal{A}(t)| \leqslant C_{\mathcal{A}}|t|^{p(x,y)-1};$$

(a_3) $\mathcal{A}(t)t \leqslant p^+\mathscr{A}(t)$ para todo $t \in \mathbb{R}$.

O kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^+$ é uma função mensurável que satisfaz a seguinte propriedade:

$(\mathcal{K})$ Existem constantes b_0 e b_1, tal que $0 < b_0 \leqslant b_1$,

$$b_0 \leqslant K(x,y)|x-y|^{N+sp(x,y)} \leqslant b_1 \text{ para todo } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ e } x \neq y.$$

Vale ressaltar que as hipóteses (a_1)-(a_3) e $(\mathcal{K})$ foram similarmente introduzidas em [3, 12, 15, 24, 30, 37, 43, 44, 52, 53, 59, 69, 75] e uma generalização matemática muito especial para $\mathcal{A}$ e K satisfazendo (a_1)-(a_3) e $(\mathcal{K})$ é quando $\mathcal{A}(t) = |t|^{p(x,y)-2}t$ e $K(x,y) = |x-y|^{-(N+sp(x,y))}$. Assim o operador $\mathscr{L}_{\mathcal{A}K}$ torna-se o operador $p(\cdot)$-Laplaciano fracionário $(-\Delta)^s_{p(\cdot)}$, o qual é definido por

$$(-\Delta)^s_{p(\cdot)}u(x) = P.V. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\,dy \text{ para todo } x \in \mathbb{R}^N.$$

Para mais detalhes do operador $p(\cdot)$-Laplaciano fracionário veja [7, 8, 41, 50].

Na literatura vários trabalhos têm sido desenvolvidos sobre existência e também multiplicidade de soluções não-triviais para problemas envolvendo o operadores locais e não-locais com não-linearidades de crescimento subcrítico, dentre eles destacamos alguns de nosso interesse. Em [21] os autores estudaram um problema do tipo (1) reduzido ao operador p-Laplaciano, funções peso que podem mudar de sinal e não-linearidades do tipo côncavo-convexo. Usando o método da aplicação fibração, introduzido e desenvolvido por Pohozaev em [28, 65, 66, 67] e a minimização restrita nas variedades Nehari, eles obtiveram resultados significativos para determinar a existência de duas soluções distintas para o problema. Em [4, 7, 8, 13, 41, 50] o problema do tipo (1) foi estudado quando o operador $\mathscr{L}_{\mathcal{A}K}$ é um caso particular, o operador $p(\cdot)$-Laplaciano fracionário ($(-\Delta)^s_{p(\cdot)}$ definido acima), o parâmetro $\beta = 1$ ou $\beta = 0$, a função peso $\mathfrak{a}$ positiva, e as não-linearidades associada a expoentes variáveis do tipo superlinear e sublinear. Nestes trabalhos sob apropriadas suposições os autores provaram resultados de existência de soluções e também de existência de autovalores para problemas do tipo (1) via o métodos variacionais adequados.

Portanto, motivados pelos trabalhos citados anteriormente, ao abordar o problema (1), nossa proposta foi estudar novos resultados de existência e multiplicidade de soluções para uma ampla classe de equações elípticas envolvendo um novo operador integrodiferenciail não-local, funções peso que podem mudar de sinal e não-linearidades adequadas. Ou seja, o estudo do problema (1) foi concentrado em três situações distintas, a saber. Na primeira situação, reduzimos o problema (1) para um problema com não-linearidade do tipo côncavo-convexo na estrutura de espaços de Sobolev fracionário com expoentes constantes e as funções peso mudando de sinal. Usando o método de minimização em conjuntos de Nehari provamos a existência de duas soluções ground state não-triviais distintas para o problema (1). Já na segunda e terceira situação reduzimos o problema (1), separadamente, para um problema com não-linearidade do tipo sublinear e superlinear envolvendo também uma função peso que pode mudar de sinal, na estrutura de espaços de Sobolev fracionário com expoentes variáveis. Para tais situações provamos a multiplicidade de solução através da existência de uma família contínua de autovalores utilizando o princípio variacional de Ekeland e o método direto do cálculo variacional, respectivamente.

Durante o estudo enfrentamos uma série de dificuldades as quais precisaram ser contornadas para obtermos os resuldados desejados. Primeiramente observamos que o operador $\mathscr{L}_{\mathcal{A}K}$ é não-homogêneo e não é apenas uma mera extensão do operador $p(\cdot)-$Laplaciano fracionário, pois pelas condições (a_1)-(a_4) e $(\mathcal{K})$, a aplicação $\mathcal{A}$ e o kernel K são bastante gerais e K incluem kernels singulares. Logo surgem de forma natural inúmeras dificuldades de ordem bastante técnicas e cuidadosas ao trabalharmos com este tipo de operador. Em seguida é importante notar que o espaço para trabalharmos com o operador $\mathscr{L}_{\mathcal{A}K}$ deve ser um espaço de Sobolev fracionário com expoentes variáveis conforme podemos ver em [7, 8, 41, 50]. Mas, devido a caracterização do problema (1) foi necessário definirmos um novo espaço de Sobolev fracionário associado ao estudo. Assim, inspirados no espaço $W_0^{s,p}(\Omega)$ (definido em [46]) e nos espaços de Sobolev fracionário com expoentes variáveis citados acima, definimos o seguinte espaço:

$$\mathscr{W} = W_0^{s,p(\cdot,\cdot)} := \{u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) : u = 0 \text{ q.t.p. em } \mathbb{R}^N \setminus \Omega\},$$

o qual é um espaço de Banach separável e reflexivo. Para este espaço provamos um resultado de equivalência de normas e que um importante resultado de mergulho compacto e contínuo permanece válido neste contexto. Para mais detalhes consulte o Apêndice, Lema 3.1 e Lema 3.2. Além disso, como problemas envolvendo operadores com expoentes variáveis possuem uma grande dificuldade em aplicar métodos variacionais, no nosso caso para suprir parte desta dificuldade, no Lemma 3.3, exploramos um novo resultado relacionado a uma propriedade intrínseca do operador que governa o nosso problema. Ou seja, foi necessário mostrarmos que tal operador satisfaz a propriedade (S_+), que é uma propriedade de compacidade do operador a qual é geralmente essencial para obter outras propriedades, como a condição de compacidade de Palais–Smale ou a condição de Cerami em uma estrutura variacional.

Para finalizar, destacamos que a mudança de sinal dos pesos $\mathfrak{a}$ e $\mathfrak{b}$ geram algumas dificuldades, dentre elas, ao analisarmos o funcional de Euler Lagrange associado ao problema (1), não podemos aplicar diretamente métodos variacionais e a análise para a existência de solução do problema para cada tipo de não-linearidade passa a ser bastante delicada. Por exemplo, no caso da não-linearidade do tipo côncavo-convexo, conforme detalhado na subseção 1.2.1, foi essencial dividirmos a variedade de Nehari em duas partes: $\mathcal{N}_{\lambda,1} = \mathcal{N}^+_{\lambda,1} \cup \mathcal{N}^-_{\lambda,1}$, usar a aplicação fibração para obtermos uma projeção única em cada parte $\mathcal{N}^\pm_{\lambda,1}$ e usando o procedimento de minimização padrão obtermos ao menos uma solução para cada conjunto $\mathcal{N}^\pm_{\lambda,1}$. O mesmo ocorreu para as não-linearidades do tipo sublinear e superlinear, em cada caso foi essencial considerarmos conjuntos solução apropriados, conforme veremos nas Subseções 1.3.1 e 1.3.2, para obtermos os resultados esperados.

Os principais resultados do Capítulo 1 serão enunciados abaixo.

Em nossos primeiros resultados, consideraremos o problema (1) quando os expoentes são constantes. Para isso, suponhamos que

$$\begin{cases} 1 < \mathfrak{m}_1 < l \leqslant p \leqslant m < \mathfrak{m}_2 < p^\star_s = \frac{Np}{N-sp}, s \in (0,1); \\ \\ 1 < l \leqslant p \leqslant m < \frac{N}{s}; \\ \\ (\mathfrak{m}_2 - 1)(m-l) < (\mathfrak{m}_2 - l)(m - \mathfrak{m}_1). \end{cases} \tag{$\mathcal{H}$}$$

Além disso, suponhamos (a_1)-(a_2) e a condição:

(a_3') $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ é uma função de classe $C^2(\mathbb{R}, \mathbb{R})$ e para todo $t \in \mathbb{R}$ são válidos:

$$(i)\ \ l\mathscr{A}(t) \leqslant \mathcal{A}(t)t \leqslant p\mathscr{A}(t);$$
$$(ii)\ \ (l-1)\mathcal{A}(t)t \leqslant t^2\mathcal{A}'(t) \leqslant (m-1)\mathcal{A}(t)t;$$
$$(iii)\ \ (l-2)\mathcal{A}'(t) \leqslant \mathcal{A}''(t)t \leqslant (m-2)\mathcal{A}'(t).$$

Portanto, obtemos os seguintes resultados envolvendo não-linearidade do tipo côncavo-convexo.

Teorema 0.1 (Theorem 1.1). *Suponha as condições (a_1), (a_2), (a_3'), $(\mathcal{K})$, $(\mathcal{H})$ válidas, e que as funções peso $\mathfrak{a}, \mathfrak{b} \in L^\infty(\Omega)$ são tais que $a^+, b^+ \not\equiv 0$, isto é, podem mudar de sinal em Ω. Então existe $\tilde{\lambda} > 0$ tal que o problema (1), com $\beta = 1$, admite ao menos uma solução ground state u em $\mathcal{N}^+_{\lambda,1}$ satisfazendo $\mathcal{J}_{\lambda,1}(u) < 0$ para todo $0 < \lambda < \tilde{\lambda}$. ($\mathcal{N}^+_{\lambda,1}$ é definido em (1.11) e $\mathcal{J}_{\lambda,1}$ é definido em (1.2))*

Teorema 0.2 (Theorem 1.2). *Sob as mesmas condições do Teorema 0.1 existe* $\tilde{\lambda} > 0$ *tal que o problema* (1), *com* $\beta = 1$, *admite ao menos uma solução ground state* u *em* $\mathcal{N}^{-}_{\lambda,1}$ *satisfazendo* $\mathcal{J}_{\lambda,1}(u) > 0$ *para todo* $0 < \lambda < \tilde{\lambda}$. *(*$\mathcal{N}^{-}_{\lambda,1}$ *é definido em* (1.11)*)*

Nossos próximos resultados são para expoentes variáveis e envolvem não-linearidades do tipo sublinear e superlinear.

Teorema 0.3 (Theorem 1.3). *Suponha as condições* (a_1)-(a_3) *e* $(\mathcal{K})$ *válidas. Seja* $q \in C^{+}(\overline{\Omega})$ *e* $\underline{\mathfrak{m}}_1^{-} \leqslant \underline{\mathfrak{m}}_1^{+} < p^{-} \leqslant p^{+} < \frac{N}{s} < \underline{q}^{-} \leqslant \underline{q}^{+}$. *Além disso, suponha* $\mathfrak{a} \in L^{q(\cdot)}(\Omega)$ *e que existe* $\Omega_0 \subset \Omega$ *um conjunto mensurável de interior não vazio e de medida positiva tal que* $\mathfrak{a}(x) > 0$ *para todo* $x \in \overline{\Omega}_0$. *Então existe* $\lambda^{\star} > 0$ *tal que qualquer* $\lambda \in (0, \lambda^{\star})$ *é um autovalor do problema* (1) *em* $\mathscr{W}$ *para* $\beta = 0$.

Teorema 0.4 (Theorem 1.4). *Suponha as condições* (a_1)-(a_3), $(\mathcal{K})$ *válidas. Seja* $q \in C^{+}(\overline{\Omega})$, $p^{-} \leqslant p^{+} < \underline{\mathfrak{m}}_1^{-} \leqslant \underline{\mathfrak{m}}_1^{+}$ *e* $\mathfrak{m}_1(x) < p_s^{\star}(x)$ *para todo* $x \in \overline{\Omega}$. *Além disso, seja* $\mathfrak{a} \in L^{q(\cdot)}(\Omega)$ *e* $q(x) > \max\left\{1, \frac{Np(x)}{Np(x)+sp(x)\mathfrak{m}_1(x)-N\mathfrak{m}_1(x)}\right\}$ *para todo* $x \in \overline{\Omega}$.
Então obtemos que:

1) *Existe* $\lambda^{\star\star}$ *e* $\mu^{\star\star}$, *autovalor positivo e negativo do problema* (1), *respectivamente, satisfazendo* $\mu^{\star\star} \leqslant \mu_{\star} < 0 < \lambda_{\star} \leqslant \lambda^{\star\star}$ *em* $\mathscr{W}$ *para* $\beta = 0$. *(*$\lambda^{\star\star}$, $\mu^{\star\star}$, $\lambda_{\star}$ *e* $\mu_{\star}$ *são definidos em* (1.56)*)*

2) $\lambda \in (-\infty, \mu^{\star\star}) \cup (\lambda^{\star\star}, +\infty)$ *é um autovalor do problema* (1), *enquanto que todo* $\lambda \in (\mu_{\star}, \lambda_{\star})$ *não é um autovalor em* $\mathscr{W}$ *para* $\beta = 0$.

O estudo destes resultados de existência de solução e existência autovalores visa estender e complementar os principais resultados obtidos em [4, 7, 8, 21, 32, 33, 34, 39, 50, 61] no sentido que os operadores considerados estão incluídos em nossa classe de operadores e nossas funções peso mudam de sinal. Ou seja, nossos resultados, Teoremas 0.1 e 0.2, estendem e complementam os resultados recentes de [21] sobre a existência de soluções para problemas locais com não-linearidade do tipo côncavo-convexo. Já, o Teorema 0.3, no caso sublinear, estende e complementa os principais resultados de [4, 7, 8, 39, 33, 61] que tratam de problemas locais, não-locais com expoente variáveis e função peso que não muda de sinal. Finalmente o Teorema 0.4, no caso superlinear, estende e complementa os principais resultados de [32, 34, 39, 50] que tratam também de problemas locais, não-locais com expoente variáveis e função peso que não muda de sinal.

No **Capítulo 2**, estudamos a existência, a multiplicidade e o comportamento assintótico de soluções fracas para equações elípticas envolvendo o mesmo operador integrodiferencial não-local geral com expoentes variáveis $\mathscr{L}_{\mathcal{A}K}$ do Capítulo 1. Mais precisamente consideremos o seguinte problema

$$\begin{cases} \mathscr{L}_{\mathcal{A}K} u &= \lambda f(x,u) \quad \text{em } \Omega, \\ u &= 0 \quad\quad\quad \text{em } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3}$$

onde $\lambda > 0$ é um parâmetro real, $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ é um domínio suave e limitado e o operador integrodiferencial não-local geral $\mathscr{L}_{\mathcal{A}K}$ é o definido em (2) onde a aplicação $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ satisfaz as condições (a_1)-(a_3) e o kernel K satizfaz a condição $(\mathcal{K})$.

A não-linearidade $f : \Omega \times \mathbb{R} \to \mathbb{R}$ é uma função Carathéodory e satisfaz:

(f_0) Existe uma constante positiva c_1 tal que f satisfaz a condição de crescimento subcrítico
$$|f(x,t)| \leqslant c_1(1 + |t|^{\vartheta(x)-1})$$
para todo $(x,t) \in \Omega \times \mathbb{R}$, onde $\vartheta \in C(\overline{\Omega})$, $1 < p^+ < \underline{\vartheta}^- \leqslant \vartheta(x) \leqslant \underline{\vartheta}^+ < p_s^\star(x)$ para $x \in \overline{\Omega}$, e $\underline{\vartheta}^- := \inf\limits_{x\in\overline{\Omega}} \vartheta(x)$, $\underline{\vartheta}^+ := \sup\limits_{x\in\overline{\Omega}} \vartheta(x)$;

(f_1) $\lim\limits_{|t|\to+\infty} \dfrac{F(x,t)}{|t|^{p^+}} = +\infty$ uniformemente para quase todo ponto (q.t.p.) $x \in \Omega$, isto é, f é p^+-superlinear no infinito, a função F é a primitiva de f com respeito a segunda variável, isto é, $F(x,t) := \displaystyle\int_\Omega f(x,\tau)\, d\tau$;

(f_2) $f(x,t) = o(|t|^{p^+-1})$, quando $t \to 0$, uniformemente em quase todo ponto $x \in \Omega$;

(f_3) Existe constante positiva $\mathsf{c}_\star > 0$ tal que
$$\mathcal{G}(x,t) \leqslant \mathcal{G}(x,\tau) + \mathsf{c}_\star$$
para todo $x \in \Omega,\ 0 < t < \tau$ ou $\tau < t < 0$, onde $\mathcal{G}(x,t) := tf(x,t) - p^+F(x,t)$.

Com a intenção de encontrar infinitas soluções, é natural impor certas condições de simetria à não-linearidade. Na sequência, assumiremos a seguinte suposição em f :

(f_4) f é ímpar em t, isto é, $f(x,-t) = -f(x,t)$ para todo $x \in \Omega$ e $t \in \mathbb{R}$.

Além disso, para provar que o funcional Euler Lagrange associado ao problema (3) verifica a condição Cerami $(C)_c$, assumimos que as funções $\mathscr{A}$ e $\mathcal{A}$ satisfazem a seguinte condição:

(a_4) $\mathcal{H}(at) \leqslant \mathcal{H}(t)$ para todo $t \in \mathbb{R}$ e $a \in [0,1]$ onde $\mathcal{H}(t) = p^+\mathscr{A}(t) - \mathcal{A}(t)t$.

Na literatura, o problema (3) é investigado principalmente no contexto de equações conduzidas pelo Laplaciano (problemas semilineares), ou seja, a seguinte classe de problemas

$$\begin{cases} -\Delta u &= \lambda f(x,u) \quad \text{em } \Omega, \\ u &= 0 \quad \text{em } \partial\Omega. \end{cases} \tag{4}$$

Ambrosetti e Rabinowitz, em [2], foram os primeiros a usar o Teorema do Passo da Montanha para provar que os problemas do tipo (4) admitem uma solução quando a seguinte condição de não-linearidade para $f(x,\cdot)$, bem conhecida na literatura como condição de Ambrosetti-Rabinowitz (condição (AR), para abreviar), é empregada: existem $\mu > 2$ e $M > 0$ tais que

$$0 < \mu F(x,t) \leqslant f(x,t)t \text{ para todo } x \in \Omega \text{ e para todo } |t| \geqslant M \tag{5}$$

onde $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ é uma função contínua e $F(x,t) = \displaystyle\int_0^t f(t,z)\,dz$.

A condição (AR) é uma ferramenta para obter soluções fracas de problemas superlineares e seu principal papel é garantir a compacidade, mais especificamente, a limitação da sequência Palais-Smale exigida pelos argumentos de minimax. Porém, é uma condição um pouco restrita e elimina algumas não-linearidades de $F(x,\cdot)$. Uma integração direta de (5) nos permite obtermos a seguinte condição mais fraca para a função potencial $F(x,t)$,

$$F(x,t) \geqslant \alpha_1|t|^\mu - \alpha_2 \text{ para todo } x \in \overline{\Omega} \text{ e para todo } t \in \mathbb{R} \text{ com constantes } \alpha_1, \alpha_2 > 0. \tag{6}$$

No entanto, ainda existem muitas funções que são superlineares no infinito e que não satisfazem a condição (AR). De fato, a condição (6) implica que a função $F(x,\cdot)$ exiba pelo menos um crescimento polinomial μ próximo a $\pm\infty$ e, desde que $\mu > p$

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^p} = +\infty. \tag{7}$$

A função $f(x,t) = t\ln(1+t)$ é um exemplo de função que satisfaz a condição (7), mas não satisfaz a condição (6) e, portanto, também não satisfaz a condição (5).

Nesse sentido, o estudo de problemas envolvendo não apenas o operador Laplaciano, mas também o operador $p(\cdot)-$Laplaciano sem a condição (AR) tornou-se o objeto de estudo de muitos pesquisadores e as referências na literatura aumentaram constantemente. No artigos [10, 43, 48, 56, 57, 58, 60] os autores estabeleceram a existência de pelo menos soluções não-triviais para os problemas do tipo (4) sem a condição (AR) e em geral, a principal ferramenta usada para obter os resultados é o Teorema do Passo da Montanha com a condição de Cerami.

Além desse avanço significativo para operadores locais sem a condição (AR), recentemente, alguns pesquisadores começaram a estudar problemas do tipo (4) não-locais sem a condição (AR). Mais especificamente, no artigo [77] considerando $\mathcal{A}(t) = t$, $K(x, y) = |x - y|^{N+2s}$ e $\lambda = 1$, os autores provaram que o problema (3) possui infinitas soluções usando o Teorema de Fountain. No artigo [64], o autor, para o caso $p-$Laplaciano fracionário e $\lambda = 1$ estuda a existência de uma solução fraca para o problema (3) utilizando o Teorema do Passo da Montanha combinado com a desigualdade de Moser-Trudinger fracionária. Já em [79], para $\mathcal{A}(t) = t$ e algumas suposições semelhantes para o kernel K, usando um novo resultado de pontos críticos apresentado por [54, Teorema 2.6], o autor prova que existe $\lambda_0 > 0$ tal que problema do tipo (3) tem duas soluções fracas distintas para cada $\lambda \in (0, \lambda_0)$

Portanto, motivados pelas referências acima, principalmente pelos artigos [43, 60, 77, 79], buscando novos avanços nesta teoria, o Capítulo 2, mostra a existência, a multiplicidade e o comportamento assintótico de soluções para o problema (3) sem a condição (AR). Para obtermos os resultados desejados, usamos diferentes versões do Teorema do Passo da Montana, bem como o Teorema de Fountain, Dual Fountain com condição de Cerami e uma nova versão do Teorema do Passo da Montanha simétrico introduzido por Kajikya [49].

Enunciaremos agora os principais resultados do Capítulo 2.

Teorema 0.5 (Theorem 2.1). *Suponha* (a_1)-(a_4), $(\mathcal{K})$ *e que* f *satisfaz* (f_0)-(f_3). *Então o problema* (3) *possui ao menos uma solução fraca em* $\mathscr{W}$ *para todo* $\lambda > 0$.

Teorema 0.6 (Theorem 2.2). *Suponha* (a_1)-(a_4), $(\mathcal{K})$, *e que* f *satisfaz* (f_0)-(f_4). *Então o problema* (3) *tem infinitas soluções para todo* $\lambda > 0$.

Teorema 0.7 (Theorem 2.3). *Suponha* (a_1)-(a_4), $(\mathcal{K})$, *e que* f *satisfaz* (f_0), (f_1), (f_3) *e* (f_4). *Então para cada* $\lambda \in \left(0, \frac{c_{\mathcal{A}} b_0}{p^+}\right)$, *o problema* (3) *tem infinitas soluções fracas* $u_k \in \mathscr{W}$, $k \in \mathbb{N}$ *tal que* $\Psi_\lambda(u_k) \to +\infty$, *quando* $k \to +\infty$. *(*$\Psi_\lambda(\cdot)$ *é definido em* (2.1)*)*

Teorema 0.8 (Theorem 2.4). *Suponha* (a_1)-(a_4), $(\mathcal{K})$, *e que* f *satisfaz* (f_0), (f_1), (f_3) *e* (f_4). *Então para cada* $\lambda \in \left(0, \frac{c_{\mathcal{A}} b_0}{p^+}\right)$, *o problema* (3) *tem uma sequência de soluções fracas* $v_k \in \mathscr{W}$, $k \in \mathbb{N}$ *tal que* $\Psi_\lambda(v_k) < 0$, $\Psi_\lambda(v_k) \to 0$ *quando* $k \to +\infty$.

Teorema 0.9 (Theorem 2.5). *Suponha* (a_1)-(a_4) *e* $(\mathcal{K})$. *Se* f *satisfaz* (f_0), (f_1) *e* (f_3). *Além disso,* $f(x,0) = 0$, $f(x,t) \geqslant 0$ *q.t.p.* $x \in \Omega$ *e para todo* $t \geqslant 0$. *Então existe uma constante positiva* $\overline{\lambda}$ *tal que o problema* (3) *possui ao menos uma solução para todo* $\lambda \in (0, \overline{\lambda})$, *tal que*

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{\mathscr{W}} = +\infty.$$

($\|\cdot\|_{\mathscr{W}}$ *é definido na Subseção*(3.0.2)*)*

Teorema 0.10 (Theorem 2.6). *Suponha* (a_1)-(a_3), $(\mathcal{K})$, *e que* f *satisfaz* (f_4). *Além disso, suponhamos a seguinte condição:*

(f_5) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *é uma função contínua e existem constantes positivas* C_0, C_1 *tal que*

$$C_0|t|^{\mathfrak{m}(x)-1} \leqslant f(x,t) \leqslant C_1|t|^{\mathfrak{m}(x)-1},$$

para todo $x \in \overline{\Omega}$ *and* $t \geqslant 0$, *onde* $\mathfrak{m} \in C(\overline{\Omega})$ *tal que* $1 < \mathfrak{m}(x) < p^\star(x)$ *para todo* $x \in \overline{\Omega}$, *com* $\underline{\mathfrak{m}}^+ < p^-$. *Então o problema* (3) *tem infinitas soluções* $u_k \in \mathscr{W}$, $k \in \mathbb{N}$ *tal que*

$$\lim_{k \to +\infty} \|u_k\|_{\mathscr{W}} = 0$$

para todo $\lambda > 0$.

Os resultados mencionados acima, visam estender e completar os principais resultados de [20, 34, 38, 43, 55, 48, 56, 60, 64, 72, 74, 79, 80] de tal forma que os mesmos permaneçam válidos para uma classe mais ampla de operadores não-locais que envolvem expoentes variáveis sem satsifazer a condição (AR). Ou seja, os Teoremas 0.5 e 0.6 estendem e completam os principais resultados obtidos em [20, 34, 43, 60, 64, 74, 80]. Os Teoremas 0.7 e 0.8 estendem e completam os principais resultados de [38, 43, 55, 48, 56, 72, 79, 80]. Finalmente, os Teoremas 0.9 e 0.10 estendem e completam alguns dos principais resultados de [43]. Além disso, gostaríamos de destacar que até onde sabemos, não há resultados nessa abordagem, mesmo envolvendo os problemas do p-Laplaciano fracionário, bem como os problemas envolvendo o $p(\cdot)$-Laplaciano fracionário, embora consideremos algumas das técnicas conhecidas.

Para finalizar, no Apêndice, apresentamos alguns resultados e definições importantes que utilizamos no decorrer deste trabalho. Provamos resultados importantes como o Lema 3.1 e o Lema 3.2 com respeito ao novo espaço definido $\mathscr{W}$ e o Lema 3.3 que fornece características importantes para o funcional associado ao operador $\mathscr{L}_{AK}$.

Gostaríamos de esclarecer que a formatação dos capítulos desta tese é baseada nos artigos que escrevemos e que foram submetidos para publicação. Os Capítulos 1 e 2 estão distribuídos da seguinte forma:

Capítulo 1 *A class of elliptic equations involving a general nonlocal integrodifferential operators with sign-changing weight functions*, submetido para publicação.

Capítulo 2 *Multiplicity results for elliptic problems involving nonlocal integrodifferential operators without Ambrosetti-Rabinowitz condition,* submetido para publicação.

Por esse motivo, a identificação das hipóteses em cada um dos capítulos é descrita no início do respectivo capítulo e os resultados referentes ao espaço e ao operador envolvido nos problemas (1) e (3) são enunciados e demonstrados no Apêndice.

Chapter

1

A class of elliptic equations involving a general nonlocal integrodifferential operators with sign-changing weight functions

The deal of this chapter is to show results concerning the existence of weak solutions for a class of elliptic equations involving a general nonlocal integrodifferential operator with two real parameters, two weight functions which can be sign-changing and different subcritical nonlinearities. More precisely, in a smooth bounded domain Ω of $\mathbb{R}^N$ $(N \geqslant 2)$, we consider the following problem

$$\begin{cases} \mathscr{L}_{AK} u = \lambda \mathfrak{a}(x)|u|^{\mathfrak{m}_1(x)-2}u + \beta \mathfrak{b}(x)|u|^{\mathfrak{m}_2(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{$\mathscr{P}$}$$

where λ and β are real parameters, the weight functions $\mathfrak{a}, \mathfrak{b} : \overline{\Omega} \to \mathbb{R}$ can be sign-changing in Ω, $\mathfrak{m}_1$ and $\mathfrak{m}_2 \in C^+(\overline{\Omega})$, and to define the general nonlocal integrodifferential operator $\mathscr{L}_{AK}$ we will consider the variable exponent $p(x) := p(x, x)$ for all $x \in \mathbb{R}^N$ with $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying:

$$\begin{aligned} &p \text{ is symmetric, that is, } p(x,y) = p(y,x), \\ &1 < p^- := \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) \leqslant \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) := p^+ < \frac{N}{s}, \ \ s \in (0,1), \end{aligned} \tag{p_1}$$

and we consider the fractional critical variable exponent related to $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$

defined by $p_s^\star(x) = \dfrac{Np(x)}{N - sp(x)}$.

The general nonlocal integrodifferential operator $\mathscr{L}_{\mathcal{A}K}$ is defined on suitable fractional Sobolev spaces (see Subsection 1.1) by

$$\mathscr{L}_{\mathcal{A}K}u(x) = P.V. \int_{\mathbb{R}^N} \mathcal{A}(u(x) - u(y))K(x,y)\,dy, \;\; x \in \mathbb{R}^N,$$

where $P.V.$ is the principal value.

The map $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying the next assumptions:

(a_1) $\mathcal{A}$ is continuous, odd, and the map $\mathscr{A} : \mathbb{R} \to \mathbb{R}$ given by

$$\mathscr{A}(t) := \int_0^{|t|} \mathcal{A}(\tau)d\tau$$

is strictly convex;

(a_2) There exist positive constants $c_{\mathcal{A}}$ and $C_{\mathcal{A}}$, such that for all $t \in \mathbb{R}$ and for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$

$$\mathcal{A}(t)t \geqslant c_{\mathcal{A}}|t|^{p(x,y)} \quad \text{and} \quad |\mathcal{A}(t)| \leqslant C_{\mathcal{A}}|t|^{p(x,y)-1};$$

(a_3) $\mathcal{A}(t)t \leqslant p^+\mathscr{A}(t)$ for all $t \in \mathbb{R}$.

The kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^+$ is a measurable function satisfying the following property:

$(\mathcal{K})$ There exist constants b_0 and b_1, such that $0 < b_0 \leqslant b_1$,

$$b_0 \leqslant K(x,y)|x-y|^{N+sp(x,y)} \leqslant b_1 \text{ for all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ and } x \neq y.$$

In our first results we will consider the $(\mathscr{P})$ when the exponents are constant. For this, we assume that

$$\begin{cases} 1 < \mathfrak{m}_1 < l \leqslant p \leqslant m < \mathfrak{m}_2 < p_s^\star = \frac{Np}{N-sp}, s \in (0,1); \\[2ex] 1 < l \leqslant p \leqslant m < \frac{N}{s}; \\[2ex] (\mathfrak{m}_2 - 1)(m - l) < (\mathfrak{m}_2 - l)(m - \mathfrak{m}_1). \end{cases} \tag{$\mathcal{H}$}$$

We assume that map $\mathcal{A}:\mathbb{R}\to\mathbb{R}$ satisfies the conditions (a_1)-(a_2) and additionally, the conditions:

(a_3') $\mathcal{A}:\mathbb{R}\to\mathbb{R}$ is a map of class $C^2(\mathbb{R},\mathbb{R})$ and all $t\in\mathbb{R}$ is hold:

$$
\begin{aligned}
&(i)\ \ l\mathscr{A}(t)\leqslant \mathcal{A}(t)t\leqslant p\mathscr{A}(t);\\
&(ii)\ \ (l-1)\mathcal{A}(t)t\leqslant t^2\mathcal{A}'(t)\leqslant (m-1)\mathcal{A}(t)t;\\
&(iii)\ \ (l-2)\mathcal{A}'(t)\leqslant \mathcal{A}''(t)t\leqslant (m-2)\mathcal{A}'(t).
\end{aligned}
$$

Therefore, we obtain the following result involving concave-convex nonlinearities.

Theorem 1.1. *Suppose that assumptions (a_1), (a_2), (a_3'), $(\mathcal{K})$, $(\mathcal{H})$ hold, and that weight functions $\mathfrak{a},\mathfrak{b}\in L^\infty(\Omega)$ are such that $a^+,b^+\not\equiv 0$, i.e., can be sign-changing in Ω. Then there exists $\tilde{\lambda}>0$ such that problem $(\mathscr{P})$, with $\beta=1$, admits at least one ground state solution u in $\mathcal{N}^+_{\lambda,1}$ satisfying $\mathcal{J}_{\lambda,1}(u)<0$ for all $0<\lambda<\tilde{\lambda}$. ($\mathcal{N}^+_{\lambda,1}$ is defined in (1.11) and $\mathcal{J}_{\lambda,1}$ is defined in (1.2))*

Theorem 1.2. *Under the same conditions of Theorem 1.1 there exists $\tilde{\lambda}>0$ such that problem $(\mathscr{P})$, with $\beta=1$, admits at least one ground state solution u in $\mathcal{N}^-_{\lambda,1}$ satisfying $\mathcal{J}_{\lambda,1}(u)>0$ for all $0<\lambda<\tilde{\lambda}$. ($\mathcal{N}^-_{\lambda,1}$ is defined in (1.11))*

Our last results are for variable exponent.

Theorem 1.3. *Suppose that assumptions (a_1)-(a_3), $(\mathcal{K})$ hold. Let $q\in C^+(\overline{\Omega})$ and $\underline{\mathfrak{m}_1^-}\leqslant\underline{\mathfrak{m}_1^+}<p^-\leqslant p^+<\frac{N}{s}<\underline{q}^-\leqslant\underline{q}^+$. Moreover, assume $\mathfrak{a}\in L^{q(\cdot)}(\Omega)$ and that there exists $\Omega_0\subset\Omega$ a measurable set with nonempty interior and measure positive such that $\mathfrak{a}(x)>0$ for all $x\in\overline{\Omega}_0$. Then there exists $\lambda^\star>0$ such that any $\lambda\in(0,\lambda^\star)$ is an eigenvalue of problem $(\mathscr{P})$ in $\mathscr{W}$ whenever $\beta=0$. ($\mathscr{W}$ is defined in subsection 1.1)*

Theorem 1.4. *Suppose that assumptions (a_1)-(a_3), $(\mathcal{K})$ hold. Let $q\in C^+(\overline{\Omega})$, $p^-\leqslant p^+<\underline{\mathfrak{m}_1^-}\leqslant\underline{\mathfrak{m}_1^+}$, and $\mathfrak{m}_1(x)<p_s^\star(x)$ for all $x\in\overline{\Omega}$. Moreover, let $\mathfrak{a}\in L^{q(\cdot)}(\Omega)$ and $q(x)>\max\left\{1,\frac{Np(x)}{Np(x)+sp(x)\mathfrak{m}_1(x)-N\mathfrak{m}_1(x)}\right\}$ for all $x\in\overline{\Omega}$.*
Then we have:

1) *There are $\lambda^{\star\star}$ and $\mu^{\star\star}$, positive and negative eigenvalue of problem $(\mathscr{P})$, respectively, satisfying $\mu^{\star\star}\leqslant\mu_\star<0<\lambda_\star\leqslant\lambda^{\star\star}$ in $\mathscr{W}$ whenever $\beta=0$. ($\lambda^{\star\star}$, $\mu^{\star\star}$, $\lambda_\star$ and $\mu_\star$ are defined in (1.56))*

2) *$\lambda\in(-\infty,\mu^{\star\star})\cup(\lambda^{\star\star},+\infty)$ is an eigenvalue of problem $(\mathscr{P})$, while every $\lambda\in(\mu_\star,\lambda_\star)$ is not an eigenvalue in $\mathscr{W}$ whenever $\beta=0$.*

1.1 Variational framework

We start with the definition of the Lebesgue and Sobolev spaces with variable exponent and some properties of them, for more details see [1, 27, 29, 32, 34, 71] and references therein. Throughout the work, $\Omega \subset \mathbb{R}^N$, denotes a smooth bounded domain. Put

$$C^+(\overline{\Omega}) = \left\{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega}\right\}$$

and for all $h \in C^+(\overline{\Omega})$, we define $\underline{h}^- := \inf_{x\in\overline{\Omega}} h(x)$ and $\underline{h}^+ := \sup_{x\in\overline{\Omega}} h(x)$.

For $h \in C^+(\overline{\Omega})$, the variable exponent Lebesgue space $L^{h(\cdot)}(\Omega)$ is defined by

$$L^{h(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \exists\, \zeta > 0 : \int_\Omega \left|\frac{u(x)}{\zeta}\right|^{h(x)} dx < +\infty \right\}. \tag{1.1}$$

We consider this space endowed with the so-called Luxemburg norm

$$\|u\|_{L^{h(\cdot)}(\Omega)} := \inf\left\{ \zeta > 0 : \int_\Omega \left|\frac{u(x)}{\zeta}\right|^{h(x)} dx \leqslant 1 \right\}.$$

When p is constant, the Luxemburg norm $\|\cdot\|_{L^{h(\cdot)}(\Omega)}$ coincide with the standard norm $\|\cdot\|_{L^h(\Omega)}$ of the Lebesgue space $L^h(\Omega)$. From Proposition 3.1, the spaces $L^{h(\cdot)}(\Omega)$ is separable and reflexive Banach spaces

Let $s \in (0,1)$. We consider two variable exponents $q : \overline{\Omega} \to \mathbb{R}$ and $p : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$, where both $q(\cdot)$ and $p(\cdot,\cdot)$ are continuous function. We assume that:

$$\begin{gathered} p \text{ is symmetric, this is, } p(x,y) = p(y,x), \\ 1 < \underline{p}^- := \inf_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \leqslant p(x,y) \leqslant \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) := \underline{p}^+ < +\infty, \end{gathered} \tag{q_1}$$

and

$$1 < \underline{q}^- := \inf_{x\in\overline{\Omega}} q(x) \leqslant q(x) \leqslant \sup_{x\in\overline{\Omega}} q(x) := \underline{q}^+ < +\infty. \tag{q_2}$$

The fractional Sobolev space with variable exponents $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ is defined by

$$\left\{ u \in L^{q(\cdot)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\zeta^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dx\, dy < +\infty, \text{ for some } \zeta > 0 \right\}$$

and we set

$$[u]_{\Omega}^{s,p(\cdot,\cdot)} = \inf\left\{\zeta > 0; \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\zeta^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy \leqslant 1\right\}$$

the variable exponent Gagliardo seminorm. It is already known that $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ is a separable and reflexive Banach space with the norm

$$\|u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)} := \|u\|_{L^{q(\cdot)}(\Omega)} + [u]_{\Omega}^{s,p(\cdot,\cdot)},$$

see [4, 8, 50].

Remark 1.1. *Throughout this chapter, when $q(x) = p(x,x)$ we denote $p(x)$ instead of $p(x,x)$ and we will write $W^{s,p(\cdot,\cdot)}(\Omega)$ instead of $W^{s,p(\cdot),p(\cdot,\cdot)}(\Omega)$.*

Now, we consider the space $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ as following

$$\left\{u \in L^{p(\cdot)}(\mathbb{R}^N) : \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{\zeta^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < +\infty, \text{ for some } \zeta > 0\right\}$$

where the space $L^{p(\cdot)}(\mathbb{R}^N)$ is defined analogous the space $L^{p(\cdot)}(\Omega)$. The corresponding norm for this space is

$$\|u\| := \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} + [u]_{\mathbb{R}^N}^{s,p(\cdot,\cdot)}.$$

The space $(W^{s,p(\cdot,\cdot)}(\mathbb{R}^N), \|\cdot\|)$ has the same properties that $(W^{s,p(\cdot,\cdot)}(\Omega), \|\cdot\|_{W^{s,p(\cdot,\cdot)}(\Omega)})$, this is, it is a reflexive and separable Banach space.

We define the space were will study problem $(\mathscr{P}_\lambda)$. Let we will consider the variable exponents $p(x) := p(x,x)$ for all $x \in \mathbb{R}^N$ with $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying (p_1) and we denote by

$$\mathscr{W} = W_0^{s,p(\cdot,\cdot)} := \{u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

Note that $\mathscr{W}$ is a closed subspace of $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$, thus $\mathscr{W}$ is a reflexive and separable Banach space with the norm

$$\|u\| := \|u\|_{L^{p(\cdot)}(\Omega)} + [u]_{\mathbb{R}^N}^{s,p(\cdot,\cdot)},$$

once the norms $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^N)}$ and $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ coincide in $\mathscr{W}$.

In order to work with a simpler norm in the defined space we prove in the Lemma 3.1 that the space $(\mathscr{W}, \|\cdot\|)$ is equivalently defined with respect to the Gagliardo seminorm $[\cdot]_{\mathbb{R}^N}^{s,p(\cdot,\cdot)}$. Moreover, we proof in Lemma 3.2 an important result of compact and continuous embedding of space $\mathscr{W}$ as consequence of Corollary 3.1 and Lemma 3.1. Therefore, along this work we will consider the space $\mathscr{W}$ with norm $\|u\|_{\mathscr{W}} = [u]_{\mathbb{R}^N}^{s,p(\cdot,\cdot)}$ and we denote $(\mathscr{W}, \|\cdot\|_{\mathscr{W}})$.

For each λ and β real parameters, we introduce the Euler Lagrange functional $\mathcal{J}_{\lambda,\beta} : \mathscr{W} \to \mathbb{R}$ associated with problem $(\mathscr{P})$ defined by

$$\mathcal{J}_{\lambda,\beta}(u) = \Phi(u) - \lambda \mathcal{I}_{\mathfrak{a}}(u) - \beta \mathcal{I}_{\mathfrak{b}}(u) \text{ for all } u \in \mathscr{W} \tag{1.2}$$

where Φ is defined in the Lemma 3.3,

$$\mathcal{I}_{\mathfrak{a}}(u) = \int_\Omega \frac{\mathfrak{a}(x)|u|^{\mathfrak{m}_1(x)}}{\mathfrak{m}_1(x)}\, dx \text{ and } \mathcal{I}_{\mathfrak{b}}(u) = \int_\Omega \frac{\mathfrak{b}(x)|u|^{\mathfrak{m}_2(x)}}{\mathfrak{m}_2(x)}\, dx.$$

Definition 1.1. *We say that $u \in \mathscr{W}$ is a weak solution of the problem $(\mathscr{P})$ if and only if*

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x)-u(y))(v(x)-v(y))K(x,y)\, dx\, dy = &\int_\Omega \lambda \mathfrak{a}(x)|u|^{\mathfrak{m}_1(x)-2} u\, v\, dx \\ &+ \int_\Omega \beta \mathfrak{b}(x)|u|^{\mathfrak{m}_2(x)-2} u\, v\, dx\end{aligned} \tag{1.3}$$

for all $v \in \mathscr{W}$. When $\beta = 0$, we say that λ is an eigenvalue of problem $(\mathscr{P})$, if there exists $u \in \mathscr{W} \setminus \{0\}$ satisfying (1.3)*, that is, u is the corresponding eigenfunction to λ.*

1.2 Proof of Theorems 1.1 and 1.2

Now will show the existence of solution to problem $(\mathscr{P})$ for constants exponents with concave-convex nonlinearities and weight functions $\mathfrak{a}, \mathfrak{b} : \overline{\Omega} \to \mathbb{R}$ that are sign-changing in Ω. In this case, the space $\mathscr{W}$ coincide with the space $W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$ defined in [46], then $\mathscr{W} = W_0^{s,p}(\Omega)$. We consider $\mathcal{J}_{\lambda,1}$ the Euler Lagrange functional associated to problem $(\mathscr{P})$. To proof Theorem 1.1 and Theorem 1.2 we will consider the Nehari manifold $\mathcal{N}_{\lambda,1}$ introduced in [62], the fibering map and the different "sign-subsets" of the Nehari set that will be used to find critical points of the

Euler Lagrange functional $\mathcal{J}_{\lambda,1}$.

1.2.1 The Nehari Manifold

The Nehari manifold associated to the functional $\mathcal{J}_{\lambda,1}$ is given by

$$\begin{aligned} \mathcal{N}_{\lambda,1} =& \left\{ u \in W_0^{s,p}(\Omega) \setminus \{0\} : \langle \mathcal{J}'_{\lambda,1}(u), u \rangle = 0 \right\} \\ =& \left\{ u \in W_0^{s,p}(\Omega) \setminus \{0\} : \lambda \int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\, dx + \int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\, dx \right. \\ & \left. = \int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\, dx\, dy \right\}. \end{aligned} \tag{1.4}$$

Note that when $u \in \mathcal{N}_{\lambda,1}$, by (1.4) we obtain

$$\begin{aligned} \mathcal{J}_{\lambda,1}(u) =& \Phi(u) - \frac{1}{\mathfrak{m}_1} \int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\, dx\, dy \\ & + \left(\frac{1}{\mathfrak{m}_1} - \frac{1}{\mathfrak{m}_2} \right) \int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\, dx, \end{aligned} \tag{1.5}$$

or it can be rewritten as

$$\begin{aligned} \mathcal{J}_{\lambda,1}(u) =& \Phi(u) - \frac{1}{\mathfrak{m}_2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\, dx\, dy \\ & + \lambda \left(\frac{1}{\mathfrak{m}_2} - \frac{1}{\mathfrak{m}_1} \right) \int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\, dx. \end{aligned} \tag{1.6}$$

The characterization above for the functional $\mathcal{J}_{\lambda,1}$ is relevant to results we will study the following.

As first step, we shall prove that $\mathcal{J}_{\lambda,1}$ is coercive and bounded below on $\mathcal{N}_{\lambda,1} \subset W_0^{s,p}(\Omega)$ which allows us to find a ground state solution that is a critical point for $\mathcal{J}_{\lambda,1}$.

Proposition 1.1. *The functional $\mathcal{J}_{\lambda,1}$ is coercive and bounded below on $\mathcal{N}_{\lambda,1}$.*

Proof. For $u \in \mathcal{N}_{\lambda,1}$ using (1.6), (a_2), (a_3')-(i), $(\mathcal{K})$, and $(\mathcal{H})$, we obtain

$$\mathcal{J}_{\lambda,1}(u) \geqslant \left(\frac{1}{p} - \frac{1}{\mathfrak{m}_2}\right) c_{\mathcal{A}} b_0 \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy + \lambda \left(\frac{1}{\mathfrak{m}_2} - \frac{1}{\mathfrak{m}_1}\right) \int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\, dx. \tag{1.7}$$

Now, from continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{\mathfrak{m}_1}(\Omega)$, $\mathfrak{a} \in L^\infty(\Omega)$, $\mathfrak{a}^+ \not\equiv 0$, and $(\mathcal{H})$, it follows that

$$\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\, dx \leqslant \|\mathfrak{a}^+\|_\infty \|u\|_{L^{\mathfrak{m}_1}}^{\mathfrak{m}_1} \leqslant \|\mathfrak{a}^+\|_\infty C_{\mathfrak{m}_1}^{\mathfrak{m}_1(\Omega)} \|u\|_{W_0^{s,p}(\Omega)}^{\mathfrak{m}_1}. \tag{1.8}$$

Then by (1.7) and (1.8), we infer that

$$\mathcal{J}_{\lambda,1}(u) \geqslant \left(\frac{1}{p} - \frac{1}{\mathfrak{m}_2}\right) c_{\mathcal{A}} b_0 \|u\|_{W_0^{s,p}(\Omega)}^p + \lambda \left(\frac{1}{\mathfrak{m}_2} - \frac{1}{\mathfrak{m}_1}\right) \|\mathfrak{a}^+\|_\infty C_{\mathfrak{m}_1}^{\mathfrak{m}_1} \|u\|_{W_0^{s,p}(\Omega)}^{\mathfrak{m}_1}.$$

Therefore, since $p > \mathfrak{m}_1$ $\mathcal{J}_{\lambda,1}$ is coercive and consequently $\mathcal{J}_{\lambda,1}$ is bounded below on $\mathcal{N}_{\lambda,1}$. □

Let us introduce the fibering maps associated to the functional $\mathcal{J}_{\lambda,1}$. For every fixed $u \in W_0^{s,p}(\Omega) \setminus \{0\}$, we will define the fibering map $\wp_u : (0, +\infty) \to \mathbb{R}$ by

$$\wp_u(t) := \mathcal{J}_{\lambda,1}(tu) = \Phi(tu) - \frac{\lambda t^{\mathfrak{m}_1}}{\mathfrak{m}_1} \int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\, dx - \frac{t^{\mathfrak{m}_2}}{\mathfrak{m}_2} \int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\, dx \text{ for all } t \in (0, +\infty).$$

Our objective is we will analyze the behavior the fibering maps and show its relation with the Nehari manifold. More specifically as fibering maps are considered together with the Nehari manifold in order to ensure the existence of critical points for $\mathcal{J}_{\lambda,1}$. In particular, for concave-convex nonlinearities, knowledge the geometry for $\wp_u$ is important, see for instance [17].

Furthermore, using again arguing as in the Lemma 3.3 and standard arguments, we conclude that $\wp_u$ is of class $C^1(\mathbb{R}^+, \mathbb{R})$. Then differentiating $\wp_u(t)$ with respect to t, we obtain

$$\begin{aligned} \wp_u'(t) = &\int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{A}(tu(x) - tu(y))(u(x) - u(y)) K(x,y)\, dx\, dy \\ &- \lambda^{\mathfrak{m}_1 - 1} \int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\, dx - \lambda^{\mathfrak{m}_2 - 1} \int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\, dx. \end{aligned} \tag{1.9}$$

Therefore, $tu \in \mathcal{N}_{\lambda,1}$ if and only if $\wp_u'(t) = 0$. In particular, $u \in \mathcal{N}_{\lambda,1}$ if and only if $\wp_u'(1) = 0$. In other words, it is sufficient to find stationary points of fibering maps in order to get critical points for $\mathcal{J}_{\lambda,1}$ on $\mathcal{N}_{\lambda,1}$.

Furthermore, again arguing as in the Lemma 3.3 and standard arguments, we have that $\wp_u$ is of class $C^2(\mathbb{R}^+, \mathbb{R})$ with second derivative given by

$$\begin{aligned}\wp_u''(t) = &\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}'(tu(x) - tu(y))(u(x) - u(y))^2 K(x, y)\, dx\, dy \\ &- \lambda t^{\mathfrak{m}_1-2}(\mathfrak{m}_1 - 1)\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\, dx - t^{\mathfrak{m}_2-2}(\mathfrak{m}_2 - 1)\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\, dx.\end{aligned} \tag{1.10}$$

Thus, as $\wp_u'' \in C^2(\mathbb{R}^+, \mathbb{R})$ it is natural to divide $\mathcal{N}_{\lambda,1}$ into three sets as was pointed by [16, 17]:

$$\begin{aligned}\mathcal{N}_{\lambda,1}^+ &= \{u \in \mathcal{N}_{\lambda,1}; \wp_u''(1) > 0\};\\ \mathcal{N}_{\lambda,1}^- &= \{u \in \mathcal{N}_{\lambda,1}; \wp_u''(1) < 0\};\\ \mathcal{N}_{\lambda,1}^0 &= \{u \in \mathcal{N}_{\lambda,1}; \wp_u''(1) = 0\}.\end{aligned} \tag{1.11}$$

Here we mention that $\mathcal{N}_{\lambda,1}^+, \mathcal{N}_{\lambda,1}^-$, and $\mathcal{N}_{\lambda,1}^0$ correspond to critical points of minimum, maximum and inflexions points, respectively of $\wp_u$.

Remark 1.2. *Note that if $u \in \mathcal{N}_{\lambda,1}$, then by* (1.9) *and* (1.10)*, we obtain*

$$\begin{aligned}\wp_u''(1) =&(\mathfrak{m}_1 - \mathfrak{m}_2)\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\, dx + \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}'(u(x) - u(y))(u(x) - u(y))^2 K(x, y)\, dx\, dy\\ &+ (1 - \mathfrak{m}_1)\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x) - u(y))(u(x) - u(y))K(x, y)\, dx\, dy\\ =&\lambda(\mathfrak{m}_2 - \mathfrak{m}_1)\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\, dx + \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}'(u(x) - u(y))(u(x) - u(y))^2 K(x, y)\, dx\, dy\\ &+ (1 - \mathfrak{m}_2)\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x) - u(y))(u(x) - u(y))K(x, y)\, dx\, dy.\end{aligned}$$

Lemma 1.1. *For each $\lambda > 0$ sufficiently small, we have that*

(1) $\mathcal{N}_{\lambda,1}^0 = \emptyset$;

(2) $\mathcal{N}_{\lambda,1} = \mathcal{N}_{\lambda,1}^+ \cup \mathcal{N}_{\lambda,1}^-$ *is a C^1-manifold.*

Proof. (1) We suppose that $\mathcal{N}_{\lambda,1}^0 \neq \emptyset$. Let $u \in \mathcal{N}_{\lambda,1}^0$ be a fixed function. Thus,

$\wp_u''(1) = \wp_u'(1) = 0$. Using Remark 1.2, (a_2), (a_3')-(ii), $(\mathcal{K})$, and $(\mathcal{H})$, we obtain

$$\begin{aligned}(\mathfrak{m}_2 - \mathfrak{m}_1)\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx =& \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}'(u(x)-u(y))(u(x)-u(y))^2K(x,y)\,dx\,dy \\ &+ (1-\mathfrak{m}_1)\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\,dx\,dy \\ \geqslant& (l-\mathfrak{m}_1)c_{\mathcal{A}}b_0\|u\|^p_{W_0^{s,p}(\Omega)}.\end{aligned} \tag{1.12}$$

Now, from continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{\mathfrak{m}_2}(\Omega)$, $\mathfrak{b} \in L^\infty(\Omega)$, $\mathfrak{b}^+ \not\equiv 0$, and $(\mathcal{H})$, it follows that

$$\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx \leqslant \|\mathfrak{b}^+\|_\infty\|u\|^{\mathfrak{m}_2}_{L^{\mathfrak{m}_2}} \leqslant \|\mathfrak{b}^+\|_\infty C_{\mathfrak{m}_2}^{\mathfrak{m}_2(\Omega)}\|u\|^{\mathfrak{m}_2}_{W_0^{s,p}(\Omega)}. \tag{1.13}$$

Thus by (1.12), (1.13), and $(\mathcal{H})$, we achieve

$$\|u\|^{\mathfrak{m}_2-p}_{W_0^{s,p}(\Omega)} \geqslant \left(\frac{l-\mathfrak{m}_1}{\mathfrak{m}_2-\mathfrak{m}_1}\right)\frac{c_{\mathcal{A}}b_0}{\|\mathfrak{b}^+\|_\infty C_{\mathfrak{m}_2}^{\mathfrak{m}_2}} := C_1. \tag{1.14}$$

Now, using again Remark 1.2, (a_2), (a_3')-(ii), and $(\mathcal{H})$, we infer that

$$\begin{aligned}\lambda(\mathfrak{m}_2 - \mathfrak{m}_1)\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx =& \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}'(u(x)-u(y))(u(x)-u(y))^2K(x,y)\,dx\,dy \\ &+ \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\,dx\,dy \\ &- \mathfrak{m}_2\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\,dx\,dy \\ \geqslant& (\mathfrak{m}_2-m)c_{\mathcal{A}}b_0\|u\|^p_{W_0^{s,p}(\Omega)}.\end{aligned} \tag{1.15}$$

Therefore, by (1.8), (1.14), (1.15), and $(\mathcal{H})$, we obtain that

$$\lambda \geqslant \left(\frac{\mathfrak{m}_2-m}{\mathfrak{m}_2-\mathfrak{m}_1}\right)\frac{c_{\mathcal{A}}b_0}{C_{\mathfrak{m}_1}^{\mathfrak{m}_1}\|\mathfrak{a}^+\|_\infty}C_1^{\frac{p-\mathfrak{m}_1}{\mathfrak{m}_2-p}} > 0.$$

Which is a contradiction for each $\lambda > 0$ small enough. Hence, the proof of item (1) it is complete.

(2) Without loss of generality suppose that $u \in \mathcal{N}^+_{\lambda,1}$. Define the function $\mathsf{G}_\lambda : \mathcal{N}^+_{\lambda,1} \to \mathbb{R}$

$$\begin{aligned}\mathsf{G}_\lambda(u) =& \langle \mathcal{J}'_{\lambda,1}(u), u\rangle \\ =& \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\,dx\,dy - \lambda\int_\Omega \mathfrak{a}(x)|u|^{\mathtt{m}_1}\,dx \\ &- \int_\Omega \mathfrak{b}(x)|u|^{\mathtt{m}_2}\,dx \text{ for all } u \in \mathcal{N}^+_{\lambda,1}.\end{aligned}$$

Note that

$$\langle \mathsf{G}'_\lambda(u), u\rangle = \langle \mathcal{J}''_{\lambda,1}(u)(u,u), u\rangle + \langle \mathcal{J}'_{\lambda,1}(u), u\rangle = \wp''(1) \text{ for all } u \in \mathcal{N}^+_{\lambda,1}. \tag{1.16}$$

Hence $\mathcal{N}^+_{\lambda,1} = \mathsf{G}^{-1}_\lambda(\{0\})$ is a C^1-manifold. Indeed, for $u \in \mathcal{N}^+_{\lambda,1}$, using (1.16), we get $\langle \mathsf{G}'_\lambda(u), u\rangle > 0$. Therefore, $\langle \mathsf{G}'_\lambda(u), u\rangle \neq 0$. As $\mathsf{G}_\lambda(u)$ is class $C^1(W^{s,p}_0(\Omega), \mathbb{R})$ it follows that $\mathcal{N}^+_{\lambda,1} = \mathsf{G}^{-1}_\lambda(\{0\})$ is a C^1-manifold. Similarly, we may show that $\mathcal{N}^-_{\lambda,1}$ is a C^1-manifold. Consequently, as $\mathcal{N}^0_{\lambda,1} = \emptyset$ for all $\lambda > 0$ small enough, it follows that $\mathcal{N}_{\lambda,1} = \mathcal{N}^+_{\lambda,1} \cup \mathcal{N}^-_{\lambda,1}$ is a C^1-manifold. □

Lemma 1.2. *Let u_0 be a local minimum (or local maximum) of $\mathcal{J}_{\lambda,1}$ in such a way that $u_0 \notin \mathcal{N}^0_{\lambda,1}$. Then u_0 is a critical point for $\mathcal{J}_{\lambda,1}$.*

Proof. Without loss of generality, we suppose that u_0 is a local minimum of $\mathcal{J}_{\lambda,1}$. Define the function $\mathsf{H} : W^{s,p}_0(\Omega) \to \mathbb{R}$ by

$$\mathsf{H}(u) = \langle \mathcal{J}'_{\lambda,1}(u), u\rangle \text{ for all } u \in W^{s,p}_0(\Omega).$$

We observe that u_0 is a solution for the minimization problem

$$\begin{cases} \min \mathcal{J}_{\lambda,1}(u) \\ \mathsf{H}(u) = 0. \end{cases} \tag{1.17}$$

Now note that

$$\langle \mathsf{H}'(u), v\rangle = \langle \mathcal{J}''_{\lambda,1}(u)(u,u), v\rangle + \langle \mathcal{J}'_{\lambda,1}(u), v\rangle \text{ for all } u, v \in W^{s,p}_0(\Omega). \tag{1.18}$$

Then taking $u = v = u_0$ in (1.18), we infer that

$$\langle \mathsf{H}'(u_0), u_0\rangle = \langle \mathcal{J}''_{\lambda,1}(u_0)(u_0,u_0), u_0\rangle + \langle \mathcal{J}'_{\lambda,1}(u_0), u_0\rangle = \wp''_{u_0}(1) > 0. \tag{1.19}$$

Thus $u_0 \notin \mathcal{N}^0_{\lambda,1}$ and by Lemma 1.1 we conclude that problem (1.17) has a solution in the following form

$$\mathcal{J}'_{\lambda,1}(u_0) = \mu \mathsf{H}'(u_0),$$

where $\mu \in \mathbb{R}$ which is given by Lagrange Multipliers Theorem. Since $u_0 \in \mathcal{N}_{\lambda,1}$ we obtain that

$$\mu \langle \mathsf{H}'(u_0), u_0 \rangle = \langle \mathcal{J}'_{\lambda,1}(u_0), u_0 \rangle = 0. \tag{1.20}$$

However by (1.19) we infer that $\langle \mathsf{H}'(u_0), u_0 \rangle \neq 0$. Thus, by (1.20) we conclude that $\mu = 0$. Therefore, $\mathcal{J}'_{\lambda,1}(u_0) = 0$ and u_0 is a critical point for $\mathcal{J}_{\lambda,1}$ on $W^{s,p}_0(\Omega)$. □

1.2.2 The fibering map

In this subsection, we will do a complete analysis of the fibering map associated with problem ($\mathscr{P}$). The essential nature for the fibering maps is determined by the signs of $\displaystyle\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx$ and $\displaystyle\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx$.

Throughout this subsection, fixed $u \in W^{s,p}_0(\Omega) \setminus \{0\}$ it is useful to consider the auxiliary function $\mathsf{M}_u : \mathbb{R} \to \mathbb{R}$ by

$$\mathsf{M}_u(t) = t^{-\mathfrak{m}_1} \int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{A}(tu(x) - tu(y))(tu(x) - tu(y))K(x,y)\,dx\,dy - t^{-\mathfrak{m}_1} \int_\Omega \mathfrak{b}(x)|tu|^{\mathfrak{m}_2}\,dx$$

for all $t \in \mathbb{R}$.

Note that M_u has possible forms when $\displaystyle\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx \leqslant 0$ and $\displaystyle\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx > 0$, respectively

Figure 1.1: Sketches of M_u

Lemma 1.3. *Let $t > 0$ be fixed. Then $tu \in \mathcal{N}_{\lambda,1}$ if and only if is a solution of* $\mathsf{M}_u(t) = \lambda \displaystyle\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx$.

Proof. Fix $t > 0$ such that $tu \in \mathcal{N}_{\lambda,1}$. Then,

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(tu(x)-tu(y))(tu(x)-tu(y))K(x,y)\,dx\,dy =&\lambda t^{\mathfrak{m}_1}\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx\\ &+t^{\mathfrak{m}_2}\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx.\end{aligned} \tag{1.21}$$

Thus, multiplying (1.21) by $t^{-\mathfrak{m}_1}$, we have that

$$\begin{aligned}\lambda\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx =&t^{-\mathfrak{m}_1}\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(tu(x)-tu(y))(tu(x)-tu(y))K(x,y)\,dx\,dy\\ &-t^{\mathfrak{m}_2-\mathfrak{m}_1}\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx.\end{aligned}$$

□

Lemma 1.4.

(*a*) *Assume that* $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx \leqslant 0$. *Then we obtain*

$\mathsf{M}_u(0) := \lim_{t\to 0^+}\mathsf{M}_u(t) = 0, \mathsf{M}_u(\infty) := \lim_{t\to+\infty}\mathsf{M}_u(t) = +\infty,$ *and* $\mathsf{M}_u'(t) > 0$ *for all* $t > 0$;

(*b*) *Assume that* $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx > 0$ *and* $(\mathfrak{m}_2-1)(m-l) < (\mathfrak{m}_2-l)(m-\mathfrak{m}_1)$. *Then there exists a unique critical point for* M_u, *i.e., there is a unique point* $\tilde{t} > 0$ *in such a way that* $\mathsf{M}_u(\tilde{t}) = 0$. *Furthermore, we know that* $\tilde{t} > 0$ *is a global maximum point for* M_u *and* $\mathsf{M}_u(\infty) = -\infty$.

Proof. (*a*) Note that using (a_2), (a_3')-(ii),$(\mathcal{K})$, and $(\mathcal{H})$, we obtain that

$$\begin{aligned}\mathsf{M}_u'(t) \geqslant&(l-\mathfrak{m}_1)t^{-\mathfrak{m}_1-1}\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(tu(x)-tu(y))(tu(x)-tu(y))K(x,y)\,dx\,dy\\ &-(\mathfrak{m}_2-\mathfrak{m}_1)t^{\mathfrak{m}_2-\mathfrak{m}_1-1}\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx.\end{aligned} \tag{1.22}$$

Once $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx \leqslant 0$, from (1.22), we obtain $\mathsf{M}_u'(t) > 0$ for all $t > 0$.

Now, we shall prove that $\mathsf{M}_u(0) = 0$. Indeed, using (a_2) and $(\mathcal{K})$, we deduce that

$$\mathsf{M}_u(t) \geqslant c_{\mathcal{A}} b_0 t^{p-\mathfrak{m}_1}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - t^{\mathfrak{m}_2-\mathfrak{m}_1}\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx. \tag{1.23}$$

On the other hand, using (a_1), (a_2), (a_3')-(i), and $(\mathcal{K})$, we infer that

$$\mathsf{M}_u(t) \leqslant t^{p-\mathfrak{m}_1} C_{\mathcal{A}} b_1 \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - t^{\mathfrak{m}_2-\mathfrak{m}_1}\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx. \tag{1.24}$$

Using (1.23), (1.24), and $(\mathcal{H})$, we conclude that $\lim\limits_{t\to 0^+}\mathsf{M}_u(t)=0$. Moreover, by (1.23) and $(\mathcal{H})$ also we observe that $\mathsf{M}_u(\infty)=\lim\limits_{t\to+\infty}\mathsf{M}_u(t)=+\infty$.

(b) As a first step we note that $\lim\limits_{t\to 0^+}\mathsf{M}_u(t)=0$, M_u is increasing for $t>0$ small enough and $\lim\limits_{t\to+\infty}\mathsf{M}_u(t)=-\infty$. More specifically, for $0<t<1$ we observe that using (1.22) and the fact that $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx>0$ we obtain that $\mathsf{M}_u'(t)>0$, i.e., $\mathsf{M}_u(t)$ is increasing for $t\in(0,1)$. Moreover, from (1.22) and (1.23), we obtain $\lim\limits_{t\to 0^+}\mathsf{M}_u(t)=0$. Finally using (1.24) and $(\mathcal{H})$, it follows that $\lim\limits_{t\to+\infty}\mathsf{M}_u(t)=-\infty$.

Now the main goal in this proof is to show that M_u has a unique critical point $\tilde{t}>0$. Note that $\mathsf{M}_u'(t)=0$ if and only if, we have

$$\begin{aligned}(\mathfrak{m}_2-\mathfrak{m}_1)\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx =& t^{-\mathfrak{m}_2}\Bigg[\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathcal{A}(tu(x)-tu(y))(tu(x)-tu(y))K(x,y)\,dx\,dy\\ &-\mathfrak{m}_1\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathcal{A}(tu(x)-tu(y))(tu(x)-tu(y))K(x,y)\,dx\,dy\\ &+\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathcal{A}'(tu(x)-tu(y))(tu(x)-tu(y))^2K(x,y)\,dx\,dy\Bigg].\end{aligned}$$

Define the auxiliary function $\xi_u:\mathbb{R}\to\mathbb{R}$ given by

$$\begin{aligned}\xi_u(t) =&(1-\mathfrak{m}_1)t^{-\mathfrak{m}_2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathcal{A}(tu(x)-tu(y))(tu(x)-tu(y))K(x,y)\,dx\,dy\\ &+t^{-\mathfrak{m}_2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathcal{A}'(tu(x)-tu(y))(tu(x)-tu(y))^2K(x,y)\,dx\,dy.\end{aligned}$$

Note that using (a_2), (a_3')-(ii), $(\mathcal{K})$, and $(\mathcal{H})$, we infer that

$$\xi_u(t) \geqslant (l-\mathfrak{m}_1)t^{p-\mathfrak{m}_2}c_{\mathcal{A}}b_0\|u\|^p_{W_0^{s,p}(\Omega)}. \tag{1.25}$$

Then using (1.25) and $(\mathcal{H})$ for $0<t<1$, we obtain $\lim\limits_{t\to 0^+}\xi_u(t)=+\infty$. Moreover, $\lim\limits_{t\to+\infty}\xi_u(t)=0$ and ξ_u is a decreasing function. Indeed, using (a_1), (a_2), (a_3')-(i), (a_3')-(ii), $(\mathcal{K})$, and $(\mathcal{H})$, we have that

$$\xi_u(t) \leqslant (m-\mathfrak{m}_1)t^{p-\mathfrak{m}_2}C_{\mathcal{A}}b_1\|u\|^p_{W_0^{s,p}(\Omega)}. \tag{1.26}$$

Therefore, for all $t>1$ using (1.25) and (1.26), we obtain that $\lim_{t\to+\infty}\xi_u(t)=0$.

Now using (a_2), (a_3'), $(\mathcal{K})$, and $(\mathcal{H})$, we infer that

$$\begin{aligned}\xi_u'(t) \leqslant &(\mathfrak{m}_1\mathfrak{m}_2-\mathfrak{m}_1 l-\mathfrak{m}_2 l+ml)t^{-\mathfrak{m}_2-1}\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathcal{A}(tu(x)-tu(y))(tu(x)-tu(y))K(x,y)\,dx\,dy\\ &+(l-m)t^{-\mathfrak{m}_2-1}\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathcal{A}(tu(x)-tu(y))(tu(x)-tu(y))K(x,y)\,dx\,dy\\ <&0.\end{aligned}$$

Therefore, ξ_u is decreasing function proving that M_u has a unique critical point which is a maximum critical point for M_u. □

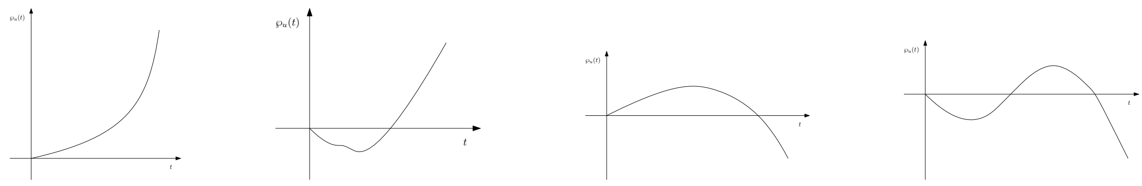
Lemma 1.5. *Let $u\in W_0^{s,p}(\Omega)\setminus\{0\}$ be a fixed function. Then we shall consider the following assertions:*

(a) *Assume that $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx\leqslant 0$. Then $\wp_u'(t)\neq 0$ for all $t>0$ and $\lambda>0$ whenever $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx\leqslant 0$. Moreover, there exists a unique $t_1=t_1(u,\lambda)$ such that $\wp_u'(t_1)=0$ and $t_1u\in\mathcal{N}^+_{\lambda,1}$ whenever $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx>0$;*

(b) *Assume that $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx>0$. Then exists a unique $t_1=t_1(u,\lambda)>\tilde{t}$ such that $\wp_u'(t_1)=0$ and $t_1u\in\mathcal{N}^-_{\lambda,1}$ whenever $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx\leqslant 0$;*

(c) *For each $\lambda>0$ small enough there exists unique $0<t_1=t_1(u,\lambda)<\tilde{t}<t_2=t_2(u,\lambda)$ such that $\wp_u'(t_1)=\wp_u'(t_2)=0$, $t_1u\in\mathcal{N}^+_{\lambda,1}$, and $t_2u\in\mathcal{N}^+_{\lambda,1}$ whenever $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx>0$, $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx>0$, and $(\mathfrak{m}_2-1)(m-l)<(\mathfrak{m}_2-l)(m-\mathfrak{m}_1)$.*

Remark 1.3. *We observe that for $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx\leqslant 0$ and $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx\leqslant 0$, $\wp_u$ has a graph as the Figure 1.2(a). For $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx>0$ and $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx\leqslant 0$, $\wp_u$ has a graph as the Figure 1.2(b). For $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx\leqslant 0$ and $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx>0$, $\wp_u$ has a graph as the Figure 1.2(c). For $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx>0$ and $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx>0$, $\wp_u$ has a graph as the Figure 1.2(d).*



(a) (b) (c) (d)

Figure 1.2: Sketches of $\wp_u$

Proof. (a) We shall consider the proof for the case $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx \leqslant 0$ and $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx \leqslant 0$. Using the Lemma 1.4$-(a)$ it follows that

$$\mathsf{M}_u(0)=0, \mathsf{M}_u(\infty) := \lim_{t\to+\infty}\mathsf{M}_u(t) = +\infty, \text{ and } \mathsf{M}_u'(t) > 0 \text{ for all } t>0. \tag{1.27}$$

Thus, we achieve that $\mathsf{M}_u(t) \neq \lambda \int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx$ for all $t>0$ and $\lambda>0$. Then by Lemma 1.3 we conclude that $tu \notin \mathcal{N}_{\lambda,1}$ for all $t>0$. In particular, $\wp_u'(t) \neq 0$ for each $t>0$, that is, from (1.9), $\wp_u'(t)>0$ for each $t>0$. Now we shall consider the case $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx \leqslant 0$ and $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx > 0$. From Lemma 1.4$-(a)$ is valid (1.27). In particular, the equation $\mathsf{M}_u(t) = \lambda\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx$ admits exactly one solution $t_1 = t_1(u,\lambda) > 0$. Consequently by Lemma 1.3, we conclude that $t_1u \in \mathcal{N}_{\lambda,1}$. Therefore, $\wp_u'(t_1)=0$.
Moreover, since that $\mathsf{M}_u(t) = t^{1-\mathfrak{m}_1}\wp_u'(t) + \lambda\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx$, we have that $\mathsf{M}_u'(t) = t^{1-\mathfrak{m}_1}\wp_u''(t) + (1-\mathfrak{m}_1)t^{-\mathfrak{m}_1}\wp_u'(t)$. Thus we conclude that $t_1^{1-\mathfrak{m}_1}\wp_u''(t_1) = \mathsf{M}_u'(t_1) > 0$ and consequently $t_1u \in \mathcal{N}_{\lambda,1}^+$.
(b) Now consider the case $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx > 0$ and $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx \leqslant 0$. Using Lemma 1.4$-(b)$ the function M_u admits a unique critical point $\tilde{t}>0$, i.e., $\mathsf{M}_u'(t)=0$, $t>0$ if and only if $t=\tilde{t}$. Moreover, $\tilde{t}$ a global maximum point for M_u such that $\mathsf{M}_u(\tilde{t})>0$ and $\mathsf{M}_u(\infty) = -\infty$. Hence, once M_u is a decreasing function in $t \in (\tilde{t},+\infty)$, there exists a unique $t_1 > \tilde{t}$ such that $\mathsf{M}_u(t_1) = \lambda\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx$ and $\mathsf{M}_u'(t_1)<0$. Therefore, $t^{1-\mathfrak{m}_1}\wp_u''(t_1) = \mathsf{M}_u'(t_1) < 0$ and we conclude that $t_1u \in \mathcal{N}_{\lambda,1}^-$.
(c) From $\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx > 0$ we can consider $\lambda>0$ small enough in such a way that $\mathsf{M}_u(\tilde{t}) > \lambda\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx$. Moreover, due to the proof of Lemma 1.4$-(b)$, M_u is

increasing in $(0,\tilde{t})$ and decreasing in $(\tilde{t},+\infty)$. In this sense, there is exactly two points $0 < t_1 = t_1(u,\lambda) < \tilde{t} < t_2 = t_2(u,\lambda)$ such that $\mathsf{M}_u(t_i) = \lambda \int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx$ for $i = 1,2$. Additionally, we have that $\mathsf{M}'_u(t_1) > 0$ and $\mathsf{M}'_u(t_2) < 0$. So with the same argument as before we get $t^{1-\mathfrak{m}_1}\wp''_u(t_2) = \mathsf{M}'_u(t_2) < 0$ and $t^{1-\mathfrak{m}_1}\wp''_u(t_1) = \mathsf{M}'_u(t_1) > 0$. Then $t_2u \in \mathcal{N}^-_{\lambda,1}$ and $t_1u \in \mathcal{N}^+_{\lambda,1}$. □

Lemma 1.6. *There exists $\tilde{\lambda} > 0$ such that $\wp_u$ takes positive values for all $u \in W^{s,p}_0(\Omega)\backslash\{0\}$ whenever $0 < \lambda < \tilde{\lambda}$.*

Proof. We shall split the proof into two cases.
Case 1: $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx \leqslant 0$.
From (a_2), (a'_3)-(i), and $(\mathcal{K})$, we get

$$\wp_u(t) \geqslant \frac{c_{\mathcal{A}} b_0 t^p}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \lambda\frac{t^{\mathfrak{m}_1}}{\mathfrak{m}_1}\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\,dx - \frac{t^{\mathfrak{m}_2}}{\mathfrak{m}_2}\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx.$$

Since $\mathfrak{m}_1 < p < \mathfrak{m}_2$ and $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx \leqslant 0$, it follows that $\lim_{t\to+\infty}\wp_u(t) = +\infty$. In particular, there is $\bar{t} > 0$ such that $\wp_u(t) > 0$ for each $t > \bar{t}$.
Case 2: $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx > 0$.
First, let $\mathsf{h}_u : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathsf{h}_u(t) := \int_{\mathbb{R}^N\times\mathbb{R}^N}\mathscr{A}(tu(x)-tu(y))K(x,y)\,dx\,dy - \frac{t^{\mathfrak{m}_2}}{\mathfrak{m}_2}\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx \text{ for all } t \in \mathbb{R}$$

an function of class $C^1(\mathbb{R},\mathbb{R})$. Note that

$$\mathsf{h}'_u(t) = \int_{\mathbb{R}^N\times\mathbb{R}^N}\mathcal{A}(tu(x)-tu(y))(u(x)-u(y))K(x,y)\,dx\,dy - t^{\mathfrak{m}_2-1}\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx.$$

Then h_u admits a critical point $t > 0$, this is, $\mathsf{h}'_u(t) = 0$ for some point $t > 0$ which is a local maximum point for h_u (see Lemma 1.5 items (b) and (c)). Moreover, for all $t > 0$, we observe that $\mathsf{h}'_u(t) = 0$ if and only if

$$\frac{t}{\mathfrak{m}_2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathcal{A}(tu(x)-tu(y))(u(x)-u(y))K(x,y)\,dx\,dy = \frac{t^{\mathfrak{m}_2}}{\mathfrak{m}_2}\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\,dx. \quad (1.28)$$

Thus using (a_2), (a_3')-(i), $(\mathcal{K})$, $(\mathcal{H})$, and (1.28), we obtain

$$\mathsf{h}_u(t) \geqslant \left(\frac{1}{p} - \frac{1}{\mathfrak{m}_2}\right) c_{\mathcal{A}} b_0 \|tu\|^p_{W_0^{s,p}(\Omega)} > 0. \tag{1.29}$$

Now, using (a_2), $(\mathcal{K})$, the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{\mathfrak{m}_2}(\Omega)$, (1.8) (1.13), and (1.28), it follows that

$$\|tu\|_{W_0^{s,p}(\Omega)} \geqslant \left(\frac{c_{\mathcal{A}} b_0}{C_{\mathfrak{m}_2}^{\mathfrak{m}_2} \|\mathfrak{b}^+\|_\infty}\right)^{\frac{1}{\mathfrak{m}_2 - p}}. \tag{1.30}$$

Therefore, by (1.29) and (1.30), we obtain that

$$\mathsf{h}_u(t) \geqslant \left(\frac{1}{p} - \frac{1}{\mathfrak{m}_2}\right) c_{\mathcal{A}} b_0 \left(\frac{c_{\mathcal{A}} b_0}{C_{\mathfrak{m}_2}^{\mathfrak{m}_2} \|\mathfrak{b}^+\|_\infty}\right)^{\frac{p}{\mathfrak{m}_2 - p}} := \delta > 0 \text{ for all } u \in W_0^{s,p}(\Omega) \setminus \{0\} \text{ and } t \in \mathbb{R}.$$

Now, by the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{\mathfrak{m}_1}(\Omega)$ using (1.8) and (1.29), we infer that

$$\begin{aligned} \int_\Omega \frac{\mathfrak{a}(x)|tu|^{\mathfrak{m}_1}}{\mathfrak{m}_1}\, dx &\leqslant \frac{\|\mathfrak{a}^+\|_\infty C_{\mathfrak{m}_1}^{\mathfrak{m}_1} \|tu\|^{\mathfrak{m}_1}_{W_0^{s,p}(\Omega)}}{\mathfrak{m}_1} \\ &\leqslant \frac{\|\mathfrak{a}^+\|_\infty C_{\mathfrak{m}_1}^{\mathfrak{m}_1}}{\mathfrak{m}_1 \left(\left(\frac{1}{p} - \frac{1}{\mathfrak{m}_2}\right) c_{\mathcal{A}} b_0\right)^{\frac{\mathfrak{m}_1}{p}}} \big(\mathsf{h}_u(t)\big)^{\frac{\mathfrak{m}_1}{p}} := \tilde{D} \big(\mathsf{h}_u(t)\big)^{\frac{\mathfrak{m}_1}{p}}, \end{aligned}$$

where $\tilde{D} = \frac{\|\mathfrak{a}^+\|_\infty C_{\mathfrak{m}_1}^{\mathfrak{m}_1}}{\mathfrak{m}_1 \left(\left(\frac{1}{p} - \frac{1}{\mathfrak{m}_2}\right) c_{\mathcal{A}} b_0\right)^{\frac{\mathfrak{m}_1}{p}}} > 0.$
Therefore,

$$\wp_u(t) = \mathsf{h}_u(t) - \frac{\lambda t^{\mathfrak{m}_1}}{\mathfrak{m}_1} \int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1}\, dx \geqslant \big(\mathsf{h}_u(t)\big)^{\frac{\mathfrak{m}_1}{p}} \left[\big(\mathsf{h}_u(t)\big)^{1 - \frac{\mathfrak{m}_1}{p}} - \lambda \tilde{D}\right].$$

Since $\mathsf{h}_u(t) > \delta$, taking $\tilde{\lambda} = \frac{\delta^{1 - \frac{\mathfrak{m}_1}{p}}}{2\tilde{D}} > \lambda$ we obtain that $\wp_u(t) > \frac{1}{2}\delta > 0$. This conclude the proof of Lemma. □

Lemma 1.7. *There exist $\tilde{\delta} > 0$ and $\tilde{\lambda} > 0$ in such a way that $\mathcal{J}_{\lambda,1}(u) \geqslant \tilde{\delta}$ for all $u \in \mathcal{N}^-_{\lambda,1}$ where $0 < \lambda < \tilde{\lambda}$.*

Proof. Fix $u \in \mathcal{N}^-_{\lambda,1}$, consequently $\mathcal{J}_{\lambda,1}$ admits a global maximum in $t = 1$ and $\int_\Omega \mathfrak{b}(x)|u|^{\mathfrak{m}_2}\, dx > 0$. Indeed, for $u \in \mathcal{N}^-_{\lambda,1}$ the fibering map has a behavior described in

Lemma 1.5. On the other hand, using Lemma 1.6 there is $t_0 > 0$ such that $\mathsf{h}_u(t_0) > \mathsf{h}_u(t)$ for each $t > 0$. Hence

$$\mathcal{J}_{\lambda,1}(u) = \wp_u(1) \geqslant \wp_u(t) \geqslant \mathsf{h}_u(t_0)^{\frac{\mathtt{m}_1}{p}}(\mathsf{h}_u(t_0)^{1-\frac{\mathtt{m}_1}{p}} - \lambda\tilde{D}) \geqslant \delta^{\frac{\mathtt{m}_1}{p}}(\delta^{1-\frac{\mathtt{m}_1}{p}} - \lambda\tilde{D})$$

for $\delta > 0$ obtained in the Lemma 1.6. Thus, taking $\tilde{\delta} = \delta^{\frac{\mathtt{m}_1}{p}}(\delta^{1-\frac{\mathtt{m}_1}{p}} - \lambda\tilde{D})$, the proof for this lemma is completed by choosing $0 < \lambda < \tilde{\lambda}$ small enough. □

1.2.3 Proof of Theorem 1.1

Proof of Theorem 1.1. From Proposition 1.1 the functional $\mathcal{J}_{\lambda,1}$ is bounded below on $\mathcal{N}_{\lambda,1}$ and so on $\mathcal{N}^+_{\lambda,1}$, then there exists a minimizing sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{N}^+_{\lambda,1}$ such that

$$\lim_{k\to+\infty} \mathcal{J}_{\lambda,1}(u_k) = \inf_{u\in\mathcal{N}^+_{\lambda,1}} \mathcal{J}_{\lambda,1}(u) := \mathbb{J}^+. \tag{1.31}$$

Again by Proposition 1.1 the functional $\mathcal{J}_{\lambda,1}$ is coercive on $\mathcal{N}^+_{\lambda,1}$, consequently $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{s,p}_0(\Omega)$. Since $W^{s,p}_0(\Omega)$ is a reflexive Banach space, there exists $u_0 \in W^{s,p}_0(\Omega)$ such that, up to a subsequence $u_k \rightharpoonup u_0$ in $W^{s,p}_0(\Omega)$,

$$u_k(x) \to u_0(x) \text{ a.e. } x \in \Omega, \;\; u_k \to u_0 \text{ in } L^{\mathtt{m}_1}(\Omega) \quad\text{and}\quad u_k \to u_0 \text{ in } L^{\mathtt{m}_2}(\Omega) \text{ as } k \to +\infty.$$

Consequently as $\mathfrak{a}, \mathfrak{b} \in L^\infty(\Omega)$, we conclude that

$$\int_\Omega \mathfrak{a}(x)|u_k|^{\mathtt{m}_1}\,dx \to \int_\Omega \mathfrak{a}(x)|u_0|^{\mathtt{m}_1}\,dx, \quad \int_\Omega \mathfrak{b}(x)|u_k|^{\mathtt{m}_2}\,dx \to \int_\Omega \mathfrak{b}(x)|u_0|^{\mathtt{m}_2}\,dx \text{ as } k \to +\infty. \tag{1.32}$$

Now suppose by contradiction that $u_k \nrightarrow u_0$ in $W^{s,p}_0(\Omega)$ as $k \to +\infty$. Hence from Lemma 3.3

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u_0(x) - u_0(y))(u_0(x) - u_0(y))K(x,y)\,dx\,dy < \\ \liminf_{k\to+\infty} &\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u_k(x) - u_k(y))(u_k(x) - u_k(y))K(x,y)\,dx\,dy.\end{aligned} \tag{1.33}$$

Now, fix $t_0 > 0$ such that $t_0u_0 \in \mathcal{N}^+_{\lambda,1}$ and using (1.32) and (1.33), we get

$$0 = \wp'_{u_0}(t_0) < \liminf_{k\to+\infty} \wp'_{u_k}(t_0).$$

Consequently $\wp'_{u_k}(t_0) > 0$ for all $k \geqslant k_0$ where k_0 is big enough.
Now, note that for all $u_k \in \mathcal{N}^+_{\lambda,1}$, using Lemma 1.5, we obtain that $\wp'_{u_k}(t) < 0$ for all $t \in (0,1)$ and $\wp'_{u_k}(1) = 0$. Indeed, as $u_k \in \mathcal{N}^+_{\lambda,1}$ then $\wp''_{u_k}(1) > 0$ and $\wp'_{u_k}(1) = 0$ for all $k \in \mathbb{N}$. Moreover using (a'_3)-(i) and $(\mathcal{H})$, we infer that

$$\begin{aligned}\wp''_{u_k}(1)+&(\mathfrak{m}_2 - m)\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u_k(x)-u_k(y))(u_k(x)-u_k(y))K(x,y)\,dx\,dy\\ &\leqslant\lambda(\mathfrak{m}_2-\mathfrak{m}_1)\int_\Omega \mathfrak{a}(x)|tu_k|^{\mathfrak{m}_1}\,dx \text{ for all } k\in\mathbb{N}.\end{aligned} \tag{1.34}$$

By (1.34), $(\mathcal{H})$, and the fact that $\wp''_{u_k}(1) > 0$, we conclude that $\int_\Omega \mathfrak{a}(x)|tu_k|^{\mathfrak{m}_1}\,dx > 0$. Therefore, by Lemma 1.5 items (b) and (c) we obtain that $\wp''_{u_k}(t) < 0$ for $t \in (0,1)$ and $\wp'_{u_k}(1) = 0$ for all $k \in \mathbb{N}$. Consequently, $t_0 > 1$.
On the other hand, as $t_0u_0 \in \mathcal{N}^+_{\lambda,1}$, $\wp_{u_0}(t)$ is decreasing for $t\in(0,t_0)$. Indeed, note that if exists $u \in W^{s,p}_0(\Omega)$ and $\int_\Omega \mathfrak{a}(x)|tu|^{\mathfrak{m}_1}\,dx > 0$, by Lemma 1.5, should exist $t_1(u,\lambda)\in\mathcal{N}^+_{\lambda,1}$ such that $\wp_u(t_1(u,\lambda)) = \mathcal{J}_{\lambda,1}(t_1(u,\lambda)u) < 0$. Therefore,

$$\inf_{u\in\mathcal{N}^+_{\lambda,1}} \mathcal{J}_{\lambda,1}(u) < 0. \tag{1.35}$$

Now observe that

$$\begin{aligned}\lambda\left(\frac{1}{\mathfrak{m}_2}-\frac{1}{\mathfrak{m}_1}\right)\int_\Omega \mathfrak{a}(x)|u_k|^{\mathfrak{m}_1}\,dx =& \int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\mathcal{A}(u_k(x)-u_k(y))(u_k(x)-u_k(y))K(x,y)}{\mathfrak{m}_2}\,dx\,dy\\ &-\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathscr{A}(u_k(x)-u_k(y))K(x,y)\,dx\,dy + \mathcal{J}_{\lambda,1}(u_k)\end{aligned} \tag{1.36}$$

for all $k\in\mathbb{N}$.
Then taking $\liminf$ in (1.36) and using (a_3)-(i), (1.31), (1.32), (1.33), and (1.35), we obtain that

$$\begin{aligned}\lambda\left(\frac{1}{\mathfrak{m}_1}-\frac{1}{\mathfrak{m}_2}\right)\int_\Omega \mathfrak{a}(x)|u_0|^{\mathfrak{m}_1}\,dx \geqslant& \int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\mathcal{A}(u_0(x)-u_0(y))(u_0(x)-u_0(y))K(x,y)}{p}\,dx\,dy\\ &-\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\mathcal{A}(u_0(x)-u_0(y))(u_0(x)-u_0(y))K(x,y)}{\mathfrak{m}_2}\,dx\,dy\\ &-\inf_{u\in\mathcal{N}^+_{\lambda,1}}\mathcal{J}_{\lambda,1}(u).\end{aligned}$$

Thus $\displaystyle\int_\Omega \mathfrak{a}(x)|u_0|^{\mathtt{m}_1}\,dx > 0$. Consequently, as $t_0u_0 \in \mathcal{N}^+_{\lambda,1}$, from Lemma 1.5 we obtain that $\wp_{u_0}$ is decreasing for $t \in (0, t_0)$. Therefore,

$$\mathcal{J}_{\lambda,1}(t_0u_0) < \mathcal{J}_{\lambda,1}(u_0) < \liminf_{k\to+\infty} \mathcal{J}_{\lambda,1}(u_k) := \mathbb{J}^+$$

which contradicts (1.31). Then $u_k \to u_0$ in $W^{s,p}_0(\Omega)$ as $k \to +\infty$. Hence, from Lemma 3.3 and (1.32), we get

$$\mathcal{J}_{\lambda,1}(u_0) = \lim_{k\to+\infty} \mathcal{J}_{\lambda,1}(u_k),$$

that is, u_0 is a minimizer for $\mathcal{J}_{\lambda,1}$ in $\mathcal{N}^+_{\lambda,1}$. Moreover using (1.35), we get $\mathcal{J}_{\lambda,1}(u_0) = \inf_{u\in\mathcal{N}^+_{\lambda,1}} \mathcal{J}_{\lambda,1}(u) < 0$. This finishes the proof. □

1.2.4 Proof Theorem 1.2

Proof Theorem 1.2. From Lemma 1.7 there exists $\tilde{\delta} > 0$ such that $\mathcal{J}_{\lambda,1}(u) \geqslant \tilde{\delta}$ for all $u \in \mathcal{N}^-_{\lambda,1}$. Thus

$$\mathbb{J}^- := \inf_{u\in\mathcal{N}^-_{\lambda,1}} \mathcal{J}_{\lambda,1}(u) \geqslant \tilde{\delta} > 0. \tag{1.37}$$

Let a minimizer sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{N}^-_{\lambda,1}$, this is, $\lim_{k\to+\infty} \mathcal{J}_{\lambda,1}(u_k) = \mathbb{J}^-$. From Proposition 1.1, $\mathcal{J}_{\lambda,1}$ is coercive on $\mathcal{N}_{\lambda,1}$ and also on $\mathcal{N}^-_{\lambda,1}$, then $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{s,p}_0(\Omega)$. Since $W^{s,p}_0(\Omega)$ is a reflexive Banach space, there exists $\tilde{u}_0 \in W^{s,p}_0(\Omega)$ such that, up to a subsequence $u_k \rightharpoonup \tilde{u}_0$ in $W^{s,p}_0(\Omega)$,

$$u_k(x) \to \tilde{u}_0(x) \text{ a.e. } x \in \Omega,\ \ u_k \to \tilde{u}_0 \text{ in } L^{\mathtt{m}_1}(\Omega) \ \text{ and } \ u_k \to \tilde{u}_0 \text{ in } L^{\mathtt{m}_2}(\Omega) \text{ as } k \to +\infty.$$

Consequently as $\mathfrak{a}, \mathfrak{b} \in L^\infty(\Omega)$ we conclude that is valid (1.32). We want to prove that $u_k \to \tilde{u}_0$ in $W^{s,p}_0(\Omega)$ as $k \to +\infty$ and conclude that

$$\mathcal{J}_{\lambda,1}(\tilde{u}_0) = \lim_{k\to+\infty} \mathcal{J}_{\lambda,1}(u_k) = \inf_{u\in\mathcal{N}^-_{\lambda,1}} \mathcal{J}_{\lambda,1}(u).$$

Now, using (1.5), (a_3')-(i), $(\mathcal{K})$, and $(\mathcal{H})$ note that

$$\begin{aligned}\mathcal{J}_{\lambda,1}(u_k) \leqslant & \left(1-\frac{l}{\mathfrak{m}_1}\right)\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathscr{A}(u_k(x)-u_k(y))K(x,y)\,dx\,dy \\ & +\left(\frac{1}{\mathfrak{m}_1}-\frac{1}{\mathfrak{m}_2}\right)\int_\Omega \mathfrak{b}(x)|u_k|^{\mathfrak{m}_2}\,dx \text{ for all } k\in\mathbb{N}.\end{aligned} \tag{1.38}$$

Hence, from (1.38), (a_3')-(i), $(\mathcal{K})$, and $(\mathcal{H})$, we obtain that

$$\begin{aligned}\left(\frac{1}{\mathfrak{m}_1}-\frac{1}{\mathfrak{m}_2}\right)\int_\Omega \mathfrak{b}(x)|u_k|^{\mathfrak{m}_2}\,dx \geqslant & \int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\mathcal{A}(u_k(x)-u_k(y))(u_k(x)-u_k(y))K(x,y)}{\mathfrak{m}_1 p}\,dx\,dy \\ & -\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\mathcal{A}(u_k(x)-u_k(y))(u_k(x)-u_k(y))K(x,y)}{p}\,dx\,dy \\ & +\mathcal{J}_{\lambda,1}(u_k) \text{ for all } k\in\mathbb{N}.\end{aligned} \tag{1.39}$$

Consequently using (1.32), (1.37), (a_2), $(\mathcal{K})$, and $(\mathcal{H})$ in (1.39), we conclude that $\int_\Omega \mathfrak{b}(x)|\tilde{u}_0|^{\mathfrak{m}_2}\,dx > 0$. Thus from Lemma 1.5, the fibering map $\wp_{\tilde{u}_0}$ admits a unique critical point $t_1>0$ in such a way that $\wp'_{\tilde{u}_0}(t_1)=0$ and $t_1\tilde{u}_0\in\mathcal{N}_\lambda^-$.
Now we suppose by contradiction that $u_k \nrightarrow \tilde{u}_0$ in $W_0^{s,p}(\Omega)$. Using (a_1), (a_2), (a_3'), $(\mathcal{K})$, and the Brezis–Lieb Lemma (see [14]), we infer that

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathscr{A}(\tilde{u}_0(x)-\tilde{u}_0(y))K(x,y)\,dx\,dy < \liminf_{k\to+\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathscr{A}(u_k(x)-u_k(y))K(x,y)\,dx\,dy. \tag{1.40}$$

Since $(u_k)_{k\in\mathbb{N}}\subset\mathcal{N}_{\lambda,1}^-$, we obtain $\wp_{u_k}(1)\geqslant\wp_{u_k}(t)$ for all $t>0$, consequently

$$\mathcal{J}_{\lambda,1}(u_k)\geqslant\mathcal{J}_{\lambda,1}(tu_k) \text{ for all } t>0 \text{ and } k\in\mathbb{N}. \tag{1.41}$$

Thence, from (1.32), (1.40), and (1.41), we conclude that

$$\begin{aligned}\mathcal{J}_{\lambda,1}(t_1\tilde{u}_0) < & \liminf_{k\to+\infty}\Bigg(\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathscr{A}(u_k(x)-u_k(y))K(x,y)\,dx\,dy \\ & -\frac{\lambda}{\mathfrak{m}_1}\int_\Omega \mathfrak{a}(x)|u_k|^{\mathfrak{m}_1}\,dx-\frac{1}{\mathfrak{m}_2}\int_\Omega \mathfrak{b}(x)|u_k|^{\mathfrak{m}_2}\,dx\Bigg) \\ < & \liminf_{k\to+\infty}\mathcal{J}_{\lambda,1}(u_k)=\mathbb{J}^-.\end{aligned}$$

Therefore, $t_1\tilde{u}_0 \in \mathcal{N}_\lambda^-$ and $\mathcal{J}_{\lambda,1}(t_1\tilde{u}_0) < \mathbb{J}^-$ which is a contradiction due the fact that $(u_k)_{k\in\mathbb{N}}$ is minimizer sequence. Hence $u_k \to \tilde{u}_0$ in $W_0^{s,p}(\Omega)$ as $k \to +\infty$. Thus from Lemma 3.3 and by equation (1.32) we can conclude that $\mathcal{J}_{\lambda,1}(u_k) \to \mathcal{J}_{\lambda,1}(\tilde{u}_0)$ as $k \to +\infty$ and $\tilde{u}_0$ is point minimum of $\mathcal{J}_{\lambda,1}$ in $\mathcal{N}_\lambda^-$, then a critical point of the functional $\mathcal{J}_{\lambda,1}$. □

1.3 Proof of Theorems 1.3 and 1.4

1.3.1 Proof of Theorem 1.3

To prove Theorem 1.3 we will apply Ekeland's Variational Principle, Theorem 3.2 combined with the Lemmas that we will prove in the sequence.

First we denote by $q'(x)$ the conjugate exponent of the function $q(x)$ and put $\alpha(x) := \dfrac{q(x)\mathfrak{m}_1(x)}{q(x)-\mathfrak{m}_1(x)}$ for all $x \in \overline{\Omega}$. Since $\underline{\mathfrak{m}_1^-} \leqslant \underline{\mathfrak{m}_1^+} < p^- \leqslant p^+ < \dfrac{N}{s} < \underline{q}^- \leqslant \underline{q}^+$, we have $q'(x)\mathfrak{m}_1(x) < \alpha(x)$ and $\alpha(x) < p_s^\star(x)$, for all x in $\overline{\Omega}$. Thus, by Lemma 3.2, the embeddings $\mathscr{W} \hookrightarrow L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)$ and $\mathscr{W} \hookrightarrow L^{\alpha(\cdot)}(\Omega)$ are compact and continuous.

The proof of Lemma 1.8 follows by standards arguments.

Lemma 1.8. *Assume the hypotheses of Theorem 1.3 are fulfilled. Then the functional* $\mathcal{J}_{\lambda,0} \in C^1(\mathscr{W},\mathbb{R})$.

Lemma 1.9. *Assume the hypotheses of Theorem 1.3 are fulfilled. Then there exists* $\lambda^\star > 0$ *such that, for all* $\lambda \in (0,\lambda^\star)$*, there exist* $\overline{\mathcal{R}}$*,* $R > 0$ *where* $\mathcal{J}_{\lambda,0}(u) \geqslant \overline{\mathcal{R}} > 0$ *for all* $u \in \mathscr{W}$ *with* $\|u\|_{\mathscr{W}} = R$.

Proof. Indeed, since embedding $\mathscr{W} \hookrightarrow L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)$ is continuous there exists a constant $c_4 > 0$ such that

$$\|u\|_{L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)} \leqslant c_4\|u\|_{\mathscr{W}} \text{ for all } u \in \mathscr{W}. \tag{1.42}$$

Fix $R \in (0,1)$ such that $R < \frac{1}{c_4}$. From (1.42), we obtain

$$\|u\|_{L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)} \leqslant 1 \text{ for all } u \in \mathscr{W} \text{ with } R = \|u\|_{\mathscr{W}}.$$

Thus, by Proposition 3.1 and Proposition 3.3, we have

$$\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1(x)}dx \leqslant \|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)}\|u\|^{\mathfrak{m}_1^-}_{L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)} \text{ for all } u \in \mathscr{W}. \tag{1.43}$$

Since $\|u\|_{\mathscr{W}} = R < 1$, using (a_2), (a_3), $(\mathcal{K})$, Proposition 3.4, and (1.43), we have that

$$\mathcal{J}_{\lambda,0}(u) \geqslant R^{\underline{\mathfrak{m}_1}^-}\left(\frac{c_{\mathcal{A}}b_0}{p^+}R^{p^+-\underline{\mathfrak{m}_1}^-} - \frac{\lambda c_4^{\underline{\mathfrak{m}_1}^-}\|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)}}{\underline{\mathfrak{m}_1}^-}\right). \tag{1.44}$$

Thus, by (1.44), we can choose $\lambda^\star$ in order to $\dfrac{c_{\mathcal{A}}b_0}{p^+}R^{p^+-\underline{\mathfrak{m}_1}^-} - \dfrac{\lambda c_4^{\underline{\mathfrak{m}_1}^-}\|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)}}{\underline{\mathfrak{m}_1}^-} > 0$. Therefore, for all $\lambda \in (0, \lambda^\star)$ with

$$\lambda^\star = \frac{c_{\mathcal{A}}b_0R^{p^+-\underline{\mathfrak{m}_1}^-}}{2p^+}\cdot\frac{\underline{\mathfrak{m}_1}^-}{c_4^{\underline{\mathfrak{m}_1}^-}\|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)}} \tag{1.45}$$

and for all $u \in \mathscr{W}$ with $\|u\|_{\mathscr{W}} = R$, there exists $\overline{\mathcal{R}} = \dfrac{c_{\mathcal{A}}b_0R^{p^+-\underline{\mathfrak{m}_1}^-}}{2p^+} > 0$ such that $\mathcal{J}_{\lambda,0}(u) \geqslant \overline{\mathcal{R}} > 0$. □

Lemma 1.10. *Assume the hypotheses of Theorem 1.3 are fulfilled. Then there exists $\underline{v} \in \mathscr{W}$ such that, $\underline{v} \neq 0$ and $\mathcal{J}_{\lambda,0}(\gamma_1\underline{v}) < 0$ for all γ_1 small enough.*

Proof. Indeed, since $\mathfrak{m}_1(x) < p(x,y)$ for all $x, y \in \overline{\Omega}_0$. In the sequence we will use the following notation, $\underline{\mathfrak{m}_0}^- := \inf_{x\in\overline{\Omega}_0}\mathfrak{m}_1(x)$ and $\underline{\mathfrak{m}_0}^+ := \sup_{x\in\overline{\Omega}_0}\mathfrak{m}_1(x)$. Thus, there exists $\varepsilon_0 > 0$ such that $\underline{\mathfrak{m}_0}^- + \varepsilon_0 \leqslant p^-$. Since $\mathfrak{m}_1 \in C^+(\overline{\Omega})$ there exists an open set $\Omega_1 \subset \Omega_0$ such that

$$|\mathfrak{m}_1(x) - \underline{\mathfrak{m}_0}^-| \leqslant \varepsilon \text{ for all } x \in \Omega_1. \tag{1.46}$$

Consequently, we can conclude that $\mathfrak{m}_1(x) \leqslant \underline{\mathfrak{m}_0}^- + \varepsilon_0 \leqslant p^-$ for all $x \in \Omega_1$.
Let $\underline{v} \in C_0^\infty(\Omega_0)$ such that $\overline{\Omega}_1 \subset supp\,\underline{v}$, $\underline{v}(x) = 1$ for all $x \in \overline{\Omega}_1$ and for $0 < \underline{v} < 1$ in Ω_0. Without loss of generality, we way assume $\|\underline{v}\|_{\mathscr{W}} = 1$, by Proposition 3.4, it follows

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\underline{v}(x) - \underline{v}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy = 1. \tag{1.47}$$

Then, by (a_2), (a_3), $(\mathcal{K})$, (1.46), (1.47), Proposition 3.4, and $\gamma_1 \in (0,1)$, we have

$$\mathcal{J}_{\lambda,0}(\gamma_1\underline{v}) \leqslant \frac{\gamma_1^{p^-}}{p^-}C_{\mathcal{A}}b_1 - \frac{\lambda\gamma_1^{\underline{\mathfrak{m}_0}^-+\varepsilon}}{\underline{\mathfrak{m}_0}^+}\int_{\Omega_1}\mathfrak{a}(x)|\underline{v}|^{\mathfrak{m}_1(x)}\,dx.$$

Thus $\mathcal{J}_{\lambda,0}(\gamma_1\underline{v}) < 0$ for all $\gamma_1 < \beta^{\overline{\frac{1}{p^- - \underline{\mathfrak{m}_0}^- - \varepsilon}}}$ with

$$0 < \beta < \min\left\{1, \frac{\lambda p^-}{\underline{\mathfrak{m}}_0^+ C_{\mathcal{A}} b_1}\int_{\Omega_1} \mathfrak{a}(x)|\underline{v}|^{\mathfrak{m}_1(x)}\,dx\right\}.$$

Therefore, the proof this Lemma is completed. □

Proof of Theorem 1.3. Consider $\lambda^\star > 0$ be defined as in (1.45) and let $\lambda \in (0, \lambda^\star)$. By Lemma 1.8, $\mathcal{J}_{\lambda,0} \in C^1(\mathscr{W}, \mathbb{R})$, and by Lemma 1.9 it follows that

$$\inf_{v \in \partial B_R(0)} \mathcal{J}_{\lambda,0}(v) > 0, \tag{1.48}$$

where $\partial B_R(0) = \{u \in B_R(0) : \|u\|_{\mathscr{W}} = R\}$ and $B_R(0)$ is the ball centred at the origin in $\mathscr{W}$.

Besides, by Lemma 1.10, there exists $\underline{v} \in \mathscr{W}$ such that $\mathcal{J}_{\lambda,0}(\gamma_1\underline{v}) < 0$ for all small enough $\gamma_1 > 0$. Moreover, by (1.44), for all $u \in B_R(0)$, we have

$$\mathcal{J}_{\lambda,0}(u) \geqslant \frac{c_{\mathcal{A}} b_0}{p^+}\|u\|_{\mathscr{W}}^{p^+} - \frac{\lambda c_1^{\underline{\mathfrak{m}}_1^-}}{\underline{\mathfrak{m}}_1^-}\|u\|_{\mathscr{W}}^{\underline{\mathfrak{m}}_1^-}. \tag{1.49}$$

Hence,

$$-\infty < \overline{d} = \inf_{v \in \overline{B_R(0)}} \mathcal{J}_{\lambda,0}(v) < 0. \tag{1.50}$$

Consequently, by (1.48) and (1.50) let $\varepsilon > 0$ such that

$$0 < \varepsilon < \inf_{v \in \partial B_R(0)} \mathcal{J}_{\lambda,0}(v) - \inf_{v \in \overline{B_R(0)}} \mathcal{J}_{\lambda,0}(v).$$

Applying the Ekeland's Variational Principle, Theorem 3.2, to the functional $\mathcal{J}_{\lambda,0} : \overline{B_R(0)} \to \mathbb{R}$, there exists $u_\varepsilon \in \overline{B_R(0)}$ such that

$$\begin{cases} \mathcal{J}_{\lambda,0}(u_\varepsilon) < \inf_{v \in \overline{B_R(0)}} \mathcal{J}_{\lambda,0}(v) + \varepsilon, \\ \mathcal{J}_{\lambda,0}(u_\varepsilon) < \mathcal{J}_{\lambda,0}(u) + \varepsilon\|u - u_\varepsilon\|_{\mathscr{W}} \text{ for all } u \neq u_\varepsilon. \end{cases} \tag{1.51}$$

Since,

$$\mathcal{J}_{\lambda,0}(u_\varepsilon) \leqslant \inf_{v \in \overline{B_R(0)}} \mathcal{J}_{\lambda,0}(v) + \varepsilon < \inf_{v \in \partial B_R(0)} \mathcal{J}_{\lambda,0}(v)$$

we deduce that $u_\varepsilon \in B_R(0)$.

Now, we define $\mathbb{T}^\varepsilon_{\lambda,\mathfrak{a}} : \overline{B_R(0)} \to \mathbb{R}$ by $\mathbb{T}^\varepsilon_{\lambda,\mathfrak{a}}(u) = \mathcal{J}_{\lambda,0}(u) + \varepsilon\|u - u_\varepsilon\|_{\mathscr{W}}$. Note that by (1.51), we get $\mathbb{T}^{\varepsilon}_{\lambda,\mathfrak{a}}(u_\varepsilon) = \mathcal{J}_{\lambda,0}(u_\varepsilon) < \mathbb{T}^{\varepsilon}_{\lambda,\mathfrak{a}}(u)$ for all $u \neq u_\varepsilon$. Thus u_ε is a minimum point of $\mathbb{T}^{\varepsilon}_{\lambda,\mathfrak{a}}$ on $\overline{B_R(0)}$. Therefore,

$$\frac{\mathbb{T}^{\varepsilon}_{\lambda,\mathfrak{a}}(u_\varepsilon + tv) - \mathbb{T}^{\varepsilon}_{\lambda,\mathfrak{a}}(u_\varepsilon)}{t} \geqslant 0 \text{ for all small enough } t > 0 \text{ and } v \in B_R(0).$$

By this fact, we obtain

$$\frac{\mathcal{J}_{\lambda,0}(u_\varepsilon + tv) - \mathcal{J}_{\lambda,0}(u_\varepsilon)}{t} + \varepsilon\|v\|_{\mathscr{W}} \geqslant 0.$$

Taking $t \to 0^+$, it follows that $\langle \mathcal{J}'_{\lambda,0}(u_\varepsilon), v\rangle + \varepsilon\|v\|_{\mathscr{W}} \geqslant 0$ and we infer that

$$\|\mathcal{J}'_{\lambda,0}(u_\varepsilon)\|_{\mathscr{W}'} \leqslant \varepsilon. \tag{1.52}$$

Then, by (1.50) and (1.52) we deduce that there exists a sequence $(w_k)_{k\in\mathbb{N}} \subset B_R(0)$ such that

$$\mathcal{J}_{\lambda,0}(w_k) \to \overline{d} \quad \text{and} \quad \mathcal{J}'_{\lambda,0}(w_k) \to 0 \quad \text{as } k \to +\infty. \tag{1.53}$$

From (1.49) and (1.53), we have that sequence $(w_k)_{k\in\mathbb{N}}$ is bounded in $\mathscr{W}$. Indeed, if $\|w_k\|_{\mathscr{W}} \to +\infty$, by (1.49) and since $\underline{\mathfrak{m}_1^-} < p^+$ we get $\mathcal{J}_{\lambda,0}(w_k) \to +\infty$, which is a contradiction with (1.53). Therefore the sequence $(w_k)_{k\in\mathbb{N}}$ is bounded in $\mathscr{W}$. From Lemma 3.2, there exists $w \in \mathscr{W}$ such that $w_k \rightharpoonup w$ in $\mathscr{W}$, $w_k \to w$ in $L^{\mathfrak{m}_1(\cdot)}(\Omega)$, and $w_k(x) \to w(x)$ a.e. $x \in \Omega$ as $k \to +\infty$.
To finalize the proof we will show that $w_k \to w$ in $\mathscr{W}$ as $k \to +\infty$.
Claim c1.

$$\lim_{k\to+\infty} \int_\Omega \mathfrak{a}(x)|w_k|^{\mathfrak{m}_1(x)-2} w_k(w_k - w)\, dx = 0.$$

Indeed, from Proposition 3.1 and Proposition 3.3, we have

$$\int_\Omega \mathfrak{a}(x)|w_k|^{\mathfrak{m}_1(x)-2} w_k(w_k - w)\, dx \leqslant \|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)}(1 + \|w_k\|^{\mathfrak{m}_1^+-1}_{L^{\mathfrak{m}_1(\cdot)}(\Omega)})\|w_k - w\|_{L^{\alpha(\cdot)}(\Omega)}. \tag{1.54}$$

Since $\mathscr{W}$ is continuously embedded in $L^{\mathfrak{m}_1(\cdot)}(\Omega)$ and $(w_k)_{k\in\mathbb{N}}$ is bounded in $\mathscr{W}$, so $(w_k)_{k\in\mathbb{N}}$ is bounded in $L^{\mathfrak{m}_1(\cdot)}(\Omega)$. From Lemma 3.2, the embedding $\mathscr{W} \hookrightarrow L^{\alpha(\cdot)}(\Omega)$ is compact, we deduce $\|w_k - w\|_{L^{\alpha(\cdot)}(\Omega)} \to 0$ as $k \to +\infty$. Therefore, using (1.54) the proof of **Claim c1** is complete.
On the other hand, by (1.53), we infer that

$$\lim_{k\to+\infty}\langle \mathcal{J}'_{\lambda,0}(w_k), w_k - w\rangle = 0. \tag{1.55}$$

Consequently by **Claim c1** and (1.55), we get

$$\lim_{k\to+\infty}\langle \Phi'(w_k), w_k - w\rangle = \lim_{k\to+\infty}\langle \mathcal{J}'_{\lambda,0}(w_k), w_k - w\rangle = 0.$$

Thus, by Lemma 3.3 it follows that $w_k \to w$ in $\mathscr{W}$ as $k \to +\infty$. Since $\mathcal{J}_{\lambda,0} \in C^1(\mathscr{W}, \mathbb{R})$, using (1.53), we obtain

$$\mathcal{J}_{\lambda,0}(w) \leqslant \lim_{k\to+\infty} \mathcal{J}_{\lambda,0}(w_k) = \overline{d} < 0 \quad \text{and} \quad \mathcal{J}'_{\lambda,0}(w) = 0.$$

Therefore, w is a nontrivial weak solution to problem $(\mathscr{P})$ and thus any $\lambda \in (0, \lambda^\star)$ is an eigenvalue of problem $(\mathscr{P})$. □

1.3.2 Proof of Theorem 1.4

Before we prove Theorem 1.4 we define

$$\mathscr{W}^+ = \left\{ u \in \mathscr{W} : \int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1(x)}dx > 0 \right\}, \mathscr{W}^- = \left\{ u \in \mathscr{W} : \int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1(x)}dx < 0 \right\},$$

$$\lambda^{\star\star} = \inf_{u\in\mathscr{W}^+} \frac{\Phi(u)}{\mathcal{I}_{\mathfrak{a}}(u)}, \quad \lambda_\star = \inf_{u\in\mathscr{W}^+} \frac{\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\,dx\,dy}{\displaystyle\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1(x)}\,dx},$$

$$\mu^{\star\star} = \sup_{u\in\mathscr{W}^-} \frac{\Phi(u)}{\mathcal{I}_{\mathfrak{a}}(u)}, \quad \mu_\star = \inf_{u\in\mathscr{W}^-} \frac{\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\,dx\,dy}{\displaystyle\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1(x)}\,dx}. \tag{1.56}$$

Proof of Theorem 1.4. Note that if λ is an eigenvalue of problem $(\mathscr{P})$ with weight function $\mathfrak{a}$, then $-\lambda$ is an eigenvalue of problem $(\mathscr{P})$ with weight $-\mathfrak{a}$. Hence, it is enough to show Theorem 1.4 only for $\lambda > 0$. So problem $(\mathscr{P})$ has only to be considered in $\mathscr{W}^+$. For this case, the proof is divided into the following four steps.
Step 1. $\lambda_\star > 0$.

First we observe that since $\mathscr{A}$ is strictly convex (see (a_1)) and by [51, Lemma 15.4], we get

$$\mathscr{A}(t) \leqslant \mathcal{A}(t)t \text{ for all } t \in \mathbb{R}. \tag{1.57}$$

Then, by (1.57) it follows that

$$\frac{\Phi(u)}{\mathcal{I}_{\mathfrak{a}}(u)} \leqslant \underline{\mathfrak{m}}_1^+ \frac{\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\,dx\,dy}{\displaystyle\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1(x)}\,dx} \text{ for all } u \in \mathscr{W}^+. \tag{1.58}$$

On the other hand, from (a_3)

$$\frac{\Phi(u)}{\mathcal{I}_{\mathfrak{a}}(u)} \geqslant \frac{\underline{\mathfrak{m}}_1^-}{p^+} \frac{\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x)-u(y))(u(x)-u(y))K(x,y)\,dx\,dy}{\displaystyle\int_\Omega \mathfrak{a}(x)|u|^{\mathfrak{m}_1(x)}\,dx} \text{ for all } u \in \mathscr{W}^+. \tag{1.59}$$

Then by (1.56), (1.58), and (1.59), we get

$$\frac{\underline{\mathfrak{m}}_1^-}{p^+}\lambda_\star \leqslant \lambda^{\star\star} \leqslant \underline{\mathfrak{m}}_1^+\lambda_\star. \tag{1.60}$$

Since $p^+ < \underline{\mathfrak{m}}_1^-$, it follows that $\lambda^{\star\star} \geqslant \lambda_\star \geqslant 0$.

Claim:

$$(a) \lim_{\|u\|_{\mathscr{W}}\to 0, u\in\mathscr{W}^+} \frac{\Phi(u)}{\mathcal{I}_{\mathfrak{a}}(u)} = +\infty; \quad (b) \lim_{\|u\|_{\mathscr{W}}\to +\infty, u\in\mathscr{W}^+} \frac{\Phi(u)}{\mathcal{I}_{\mathfrak{a}}(u)} = +\infty.$$

Indeed, using Proposition 3.1 and Proposition 3.3, it follows that

$$|\mathcal{I}_{\mathfrak{a}}(u)| \leqslant \frac{2}{\underline{\mathfrak{m}}_1^-}\|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)}\|u\|^{\underline{\mathfrak{m}}_1^l}_{L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)}, \tag{1.61}$$

where $l = -$, if $\|u\|_{L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)} < 1$ and $l = +$, if $\|u\|_{L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)} \geqslant 1$.

Since $q(x) > \max\left\{1, \frac{Np(x)}{Np(x)+sp(x)\mathfrak{m}_1(x)-N\mathfrak{m}_1(x)}\right\}$ for all $x \in \overline{\Omega}$, we have $1 < q'(x)\mathfrak{m}_1(x) < p_s^\star(x)$ for all $x \in \overline{\Omega}$, then by Lemma 3.2 there exist a constant $c_5 > 0$ such that

$$\|u\|_{L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)} \leqslant c_5\|u\|_{\mathscr{W}}. \tag{1.62}$$

Thus by (1.61) and (1.62), we get

$$|\mathcal{I}_{\mathfrak{a}}(u)| \leqslant \frac{2C_5^{\underline{\mathfrak{m}}_1^l}}{\underline{\mathfrak{m}}_1^-}\|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)}\|u\|_{\mathscr{W}}^{\underline{\mathfrak{m}}_1^l}. \tag{1.63}$$

On the other hand, by (a_2), (a_3), and $(\mathcal{K})$, we infer that

$$\Phi(u) \geqslant \frac{c_{\mathcal{A}} b_0}{p^+}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy. \tag{1.64}$$

Then for $u \in \mathscr{W}^+$ with $\|u\|_{\mathscr{W}} \leqslant 1$ by (1.62), (1.63), (1.64), and Proposition 3.4, we have

$$\frac{\Phi(u)}{\mathcal{I}_{\mathfrak{a}}(u)} \geqslant \frac{c_{\mathcal{A}} b_0 \underline{\mathfrak{m}}_1^-}{2C_5^{\underline{\mathfrak{m}}_1^-} p^+}\frac{\|u\|_{\mathscr{W}}^{p^+-\underline{\mathfrak{m}}_1^-}}{\|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)}}. \tag{1.65}$$

Since $\underline{\mathfrak{m}}_1^+ > p^+$, using (1.65) we conclude that

$$\frac{\Phi(u)}{\mathcal{I}_{\mathfrak{a}}(u)} \to +\infty \quad \text{as} \quad \|u\|_{\mathscr{W}} \to 0,\ \ u \in \mathscr{W}^+.$$

Therefore, the relation (a) holds.

Now, since $\underline{\mathfrak{m}}_1^+ - \frac{1}{2} < \underline{\mathfrak{m}}_1^-$ it follows that there exists $\eta > 0$ such that $\underline{\mathfrak{m}}_1^+ - \frac{1}{2} < \eta < \underline{\mathfrak{m}}_1^-$, which implies that

$$\underline{\mathfrak{m}}_1^+ - 1 < \underline{\mathfrak{m}}_1^+ - \frac{1}{2} < \eta \Rightarrow 1 + \eta - \underline{\mathfrak{m}}_1^+ > 0 \text{ and } 2(\underline{\mathfrak{m}}_1^- - \eta) \leqslant 2(\underline{\mathfrak{m}}_1^+ - \eta) < 1. \tag{1.66}$$

Taking $r(x)$ be any measurable function satisfying

$$\begin{gathered}\max\left\{\frac{q(x)}{1+\eta q(x)}, \frac{p_s^\star(x)}{p_s^\star(x)+\eta-\mathfrak{m}_1(x)}\right\} \leqslant r(x) \leqslant \min\left\{\frac{p_s^\star(x)}{p_s^\star(x)+\eta q(x)}, \frac{1}{1+\eta-\mathfrak{m}_1(x)}\right\}\\ \text{for all } x\in\overline{\Omega},\ \text{and}\\ \eta\left(\frac{r^+}{r^-}+1\right) < \underline{\mathfrak{m}}_1^-.\end{gathered} \tag{1.67}$$

Thus by (1.66) and (1.67) $r \in L^\infty(\Omega)$ and $1 < r(x) < q(x)$ for all $x \in \overline{\Omega}$. Then, for $u \in \mathscr{W}^+$ using Proposition 3.1, we obtain

$$|\mathcal{I}_{\mathfrak{a}}(u)| \leqslant \frac{1}{\underline{\mathfrak{m}}_1^-}\left(\frac{1}{\underline{r}^-} + \frac{1}{\underline{r}'^-}\right)\|\mathfrak{a}|u|^{\eta}\|_{L^{r(\cdot)}(\Omega)}\||u|^{\mathfrak{m}_1(x)-\eta}\|_{L^{r'(\cdot)}(\Omega)}. \tag{1.68}$$

Now, note that by Proposition 3.2

$$\|\mathfrak{a}|u|^{\eta}\|_{L^{r(\cdot)}(\Omega)} \leqslant \begin{cases} 1, \text{ if } \|\mathfrak{a}|u|^{\eta}\|_{L^{r(\cdot)}(\Omega)} \leqslant 1; \\ \left[\displaystyle\int_{\Omega} |\mathfrak{a}(x)|^{r(x)}|u|^{\eta r(x)}\,dx\right]^{\frac{1}{\underline{r}^-}} \leqslant 2\||\mathfrak{a}|^{r(x)}\|^{\frac{1}{\underline{r}^-}}_{L^{\frac{q(\cdot)}{r(\cdot)}}(\Omega)}\||u|^{\eta r(x)}\|^{\frac{1}{\underline{r}^-}}_{L^{\left(\frac{q(\cdot)}{r(\cdot)}\right)'}(\Omega)}, \\ \text{if } \|\mathfrak{a}|u|^{\eta}\|_{L^{r(\cdot)}(\Omega)} > 1. \end{cases}$$

From Proposition 3.3, we have

$$\|\mathfrak{a}|u|^{\eta}\|_{L^{r(\cdot)}(\Omega)} \leqslant 1 + 2\left(1 + \|\mathfrak{a}\|^{\frac{\underline{r}^+}{\underline{r}^-}}_{L^{q(\cdot)}(\Omega)}\right)\left(1 + \|u\|^{\eta\frac{\underline{r}^+}{\underline{r}^-}}_{L^{\eta r(\cdot)\left(\frac{q(\cdot)}{r(\cdot)}\right)'}(\Omega)}\right). \tag{1.69}$$

Similarly using the same arguments as above, we obtain

$$\||u|^{\mathfrak{m}_1(x)-\eta}\|_{L^{r'(\cdot)}(\Omega)} \leqslant \begin{cases} 1, \text{ if } \|u\|_{L^{(\mathfrak{m}_1(\cdot)-\eta)r'(\cdot)}(\Omega)} \leqslant 1; \\ \|u\|^{\underline{\mathfrak{m}}_1^+-\eta}_{L^{(\mathfrak{m}_1(\cdot)-\eta)r'(\cdot)}(\Omega)}, \text{ if } \|u\|_{L^{(\mathfrak{m}_1(\cdot)-\eta)r'(\cdot)}(\Omega)} > 1. \end{cases}$$

Thus,

$$\||u|^{\mathfrak{m}_1(x)-\eta}\|_{L^{r'(\cdot)}(\Omega)} \leqslant 1 + \|u\|^{\underline{\mathfrak{m}}_1^+-\eta}_{L^{(\mathfrak{m}_1(\cdot)-\eta)r'(\cdot)}(\Omega)}. \tag{1.70}$$

Therefore, using (1.68), (1.69), and (1.70), we have that

$$\begin{aligned} |\mathcal{I}_{\mathfrak{a}}(u)| \leqslant & \frac{1}{\underline{\mathfrak{m}}_1^-}\left(\frac{1}{\underline{r}^-} + \frac{1}{\underline{r}'^-}\right)\Bigg[1 + \|u\|^{\underline{\mathfrak{m}}_1^+-\eta}_{L^{(\mathfrak{m}_1(\cdot)-\eta)r'(\cdot)}(\Omega)} + 2(1 + \|\mathfrak{a}\|^{\frac{\underline{r}^+}{\underline{r}^-}}_{L^{q(\cdot)}(\Omega)})\|u\|^{\underline{\mathfrak{m}}_1^+-\eta}_{L^{(\mathfrak{m}_1(\cdot)-\eta)r'(\cdot)}(\Omega)} \\ & + 2\left(1 + \|\mathfrak{a}\|^{\frac{\underline{r}^+}{\underline{r}^-}}_{L^{q(\cdot)}(\Omega)}\right) + 2\left(1 + \|\mathfrak{a}\|^{\frac{\underline{r}^+}{\underline{r}^-}}_{L^{q(\cdot)}(\Omega)}\right)\|u\|^{\eta\frac{\underline{r}^+}{\underline{r}^-}}_{L^{\eta r(\cdot)\left(\frac{q(\cdot)}{r(\cdot)}\right)'}(\Omega)} \\ & + \left(1 + \|\mathfrak{a}\|^{\frac{\underline{r}^+}{\underline{r}^-}}_{L^{q(\cdot)}(\Omega)}\right)\left(\|u\|^{2(\underline{\mathfrak{m}}_1^+-\eta)}_{L^{(\mathfrak{m}_1(\cdot)-\eta)r'(\cdot)}(\Omega)} + \|u\|^{2\eta\frac{\underline{r}^+}{\underline{r}^-}}_{L^{\eta r(\cdot)\left(\frac{q(\cdot)}{r(\cdot)}\right)'}(\Omega)}\right)\Bigg]. \end{aligned} \tag{1.71}$$

Now, since $r(x)$ is chosen such that (1.67) is fulfilled, then

$$1 \leqslant \eta r(x)\left(\frac{q(x)}{r(x)}\right)', \text{ and } (\mathfrak{m}_1(x)-\eta)r'(x) \leqslant p_s^\star(x) \text{ a.e. } x \in \overline{\Omega}.$$

Thus, we obtain continuous embedding $\mathscr{W} \hookrightarrow L^{\eta r(\cdot)\left(\frac{q(\cdot)}{r(\cdot)}\right)'}(\Omega)$ and $\mathscr{W} \hookrightarrow L^{(\mathfrak{m}_1(\cdot)-\eta)r'(\cdot)}(\Omega)$. Consequently, there exist positive constants c_6, c_7, M_1, and M_2 such that by (1.71), follows that

$$|\mathcal{I}_{\mathfrak{a}}(u)| \leqslant M_1 + M_2\left(\|u\|_{\mathscr{W}}^{2(\underline{\mathfrak{m}}_1^+-\eta)} + \|u\|_{\mathscr{W}}^{2\eta\frac{r^+}{\underline{r}^-}}\right). \tag{1.72}$$

Since $p^- > 1 > 2(\underline{\mathfrak{m}}_1^+ - \eta) \geqslant 2(\underline{\mathfrak{m}}_1^- - \eta) \geqslant 2\eta\frac{r^+}{\underline{r}^-} > 2\eta$ by (1.66) and (1.67), using (1.64), (1.72), and Proposition 3.4 for all $u \in \mathscr{W}^+$ with $\|u\|_{\mathscr{W}} > 1$, we have that

$$\frac{\Phi(u)}{\mathcal{I}_{\mathfrak{a}}(u)} \geqslant \frac{1}{p^+}\frac{\|u\|_{\mathscr{W}}^{p^-}}{M_1 + M_2\left(\|u\|_{\mathscr{W}}^{2(\underline{\mathfrak{m}}_1^+-\eta)} + \|u\|_{\mathscr{W}}^{2\eta\frac{r^+}{\underline{r}^-}}\right)} \to +\infty \text{ as } \|u\|_{\mathscr{W}} \to +\infty,$$

Therefore, the relation (b) is holds.

Now, we proving that $\lambda_\star > 0$. Let us suppose by contradiction that $\lambda_\star = 0$, then by (1.60) it follows that $\lambda^{\star\star} = 0$. Let $(u_k)_{k\in\mathbb{N}} \subset \mathscr{W}^+ \setminus \{0\}$ be such that

$$\lim_{k\to+\infty} \frac{\Phi(u_k)}{\mathcal{I}_{\mathfrak{a}}(u_k)} = 0. \tag{1.73}$$

Note that by (1.65), we have

$$\frac{\Phi(u_k)}{\mathcal{I}_{\mathfrak{a}}(u_k)} \geqslant \begin{cases} \dfrac{c_{\mathcal{A}} b_0 \underline{\mathfrak{m}}_1^-}{2c_5^{\underline{\mathfrak{m}}_1^+} p^+} \dfrac{\|u\|_{\mathscr{W}}^{p^+-\underline{\mathfrak{m}}_1^-}}{\|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)}}, \text{ if } \|u\|_{\mathscr{W}} \leqslant 1; \\ \dfrac{c_{\mathcal{A}} b_0 \underline{\mathfrak{m}}_1^-}{2c_5^{\underline{\mathfrak{m}}_1^-} p^+} \dfrac{\|u\|_{\mathscr{W}}^{p^--\underline{\mathfrak{m}}_1^+}}{\|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)}}, \text{ if } \|u\|_{\mathscr{W}} > 1. \end{cases} \tag{1.74}$$

By hypothesis, we know that $p^+ - \underline{\mathfrak{m}}_1^- < 0$ and $p^- - \underline{\mathfrak{m}}_1^+ < 0$. Thus (1.74) implies that $\|u_k\|_{\mathscr{W}} \to +\infty$ as $k \to +\infty$ and using (b), we conclude

$$\lim_{k\to+\infty} \frac{\Phi(u_k)}{\mathcal{I}_{\mathfrak{a}}(u_k)} = +\infty.$$

However, this contradicts (1.73). Consequently, we have $\lambda_\star > 0$.

Step 2. $\lambda^{\star\star}$ is an eigenvalue of problem $(\mathscr{P})$.

Indeed, let $(u_k)_{k\in\mathbb{N}} \subset \mathscr{W}^+ \setminus \{0\}$ be a minizing sequence for $\lambda^{\star\star}$, that is

$$\lim_{k\to+\infty} \frac{\Phi(u_k)}{\mathcal{I}_\mathfrak{a}(u_k)} = \lambda^{\star\star} > 0. \tag{1.75}$$

Note that by (a) and (1.75), we have $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathscr{W}$. Since $\mathscr{W}$ is reflexive, it follows that there exists $u_0 \in \mathscr{W}$ such that $u_k \rightharpoonup u_0$ in $\mathscr{W}$, thus by Lemma 3.3, we have that

$$\lim_{k\to+\infty} \Phi(u_k) \geqslant \Phi(u_0). \tag{1.76}$$

On the other hand, by Lemma 3.2 we get $u_k \to u_0$ in $L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)$. Then, we infer that $\|u_k\|_{L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)} \to \|u_0\|_{L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)}$, $\||u_k|^{\mathfrak{m}_1(\cdot)}\|_{L^{q'(\cdot)}(\Omega)} \to \||u_0|^{\mathfrak{m}_1(\cdot)}\|_{L^{q'(\cdot)}(\Omega)}$, $\||u_k|^{\mathfrak{m}_1(\cdot)}\|_{L^{q'(\cdot)}(\Omega)}$ is bounded and $|u_k|^{\mathfrak{m}(\cdot)} \rightharpoonup |u_0|^{\mathfrak{m}(\cdot)}$ in $L^{q'(\cdot)}(\Omega)$. Thus $|u_k|^{\mathfrak{m}(\cdot)} \to |u_0|^{\mathfrak{m}(\cdot)}$ and from (1.63), we conclude that

$$|\mathcal{I}_\mathfrak{a}(u_k) - \mathcal{I}_\mathfrak{a}(u_0)| \leqslant \frac{2c_5}{\underline{\mathfrak{m}}_1^-} \|\mathfrak{a}\|_{L^{q(\cdot)}(\Omega)} \||u_k|^{\mathfrak{m}_1(x)} - |u_0|^{\mathfrak{m}_1(x)}\|_{L^{q'(\cdot)}(\Omega)} \to 0 \text{ as } k \to +\infty.$$

Therefore,

$$\lim_{k\to+\infty} \mathcal{I}_\mathfrak{a}(u_k) = \mathcal{I}_\mathfrak{a}(u_0). \tag{1.77}$$

In view of (1.76) and (1.77), we get

$$\frac{\Phi(u_0)}{\mathcal{I}_\mathfrak{a}(u_0)} = \lambda^{\star\star} \text{ if } u_0 \neq 0.$$

It remains to be shown that u_0 is nontrivial. We suppose by contradiction that $u_0 = 0$. Then $u_k \rightharpoonup 0$ in $\mathscr{W}$ and $u_k \to 0$ in $L^{\mathfrak{m}_1(\cdot)q'(\cdot)}(\Omega)$. Therefore,

$$\lim_{k\to+\infty} \mathcal{I}_\mathfrak{a}(u_k) = 0. \tag{1.78}$$

Now using (1.75), give $\varepsilon \in (0, \lambda^{\star\star})$ fixed there exists k_0 such that

$$|\mathcal{J}_{\lambda,0}(u_k) - \lambda^{\star\star}\mathcal{I}_\mathfrak{a}(u_k)| < \varepsilon\mathcal{I}_\mathfrak{a}(u_k) \text{ for all } k \geqslant k_0,$$

that is,

$$(\lambda^{\star\star} - \varepsilon)\mathcal{I}_\mathfrak{a}(u_k) < \mathcal{J}_{\lambda,0}(u_k) < (\lambda^{\star\star} + \varepsilon)\mathcal{I}_\mathfrak{a}(u_k) \text{ for all } k \geqslant k_0.$$

Passing to the limit in the above inequalities as $k \to +\infty$ and using (1.78), we get

$$\lim_{k\to+\infty} \mathcal{J}_{\lambda,0}(u_k) = 0. \tag{1.79}$$

Consequently by (1.2), (1.78), and (1.79), we obtain

$$\lim_{k\to+\infty} \Phi(u_k) = 0. \tag{1.80}$$

In contrast, by (a_2), (a_3), and $(\mathcal{K})$, it follows that

$$\Phi(u_k) \geqslant \frac{c_{\mathcal{A}} b_0}{p^+} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_k(x)-u_k(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy \geqslant 0. \tag{1.81}$$

Thus, by (1.80) and (1.81) we deduce

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_k(x)-u_k(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy \to 0 \text{ as } k \to +\infty \text{ for all } k \in \mathbb{N}.$$

Then by Proposition 3.4, we conclude

$$u_k \to 0 \text{ in } \mathscr{W} \text{ as } k \to +\infty. \tag{1.82}$$

Hence by (1.82) and Claim (a), we have

$$\lim_{\|u_k\|_{\mathscr{W}}\to 0, u\in\mathscr{W}^+} \frac{\Phi(u_k)}{\mathcal{I}_{\mathfrak{a}}(u_k)} = +\infty$$

which is a contradiction with (1.75). Thus $u_0 \neq 0$. Therefore, u_0 is an eigenfunction and Step 2 is proved.

Step 3. Given any $\lambda \in (\lambda^{\star\star}, +\infty)$, λ is an eigenvalue of problem $(\mathscr{P})$.

Let $\lambda \in (\lambda^{\star\star}, +\infty)$ fixed. Then λ is an eigenvalue of problem $(\mathscr{P})$ if and only if there exists $u_\lambda \in \mathscr{W}^+ \setminus \{0\}$ a critical point of $\mathcal{J}_{\lambda,0}$. Note that $\mathcal{J}_{\lambda,0}$ is coercive. Indeed, using (1.66) and (1.67), we infer that $p^- > 1 > 2(\underline{\mathfrak{m}}_1^+ - \eta) \geqslant 2(\underline{\mathfrak{m}}_1^- - \eta) \geqslant 2\eta\frac{r^+}{\underline{r}^-} > 2\eta$, thus using (1.64) and (1.72), for all $u \in \mathscr{W}^+$ with $\|u\|_{\mathscr{W}} \geqslant 1$, we have

$$\mathcal{J}_{\lambda,0}(u) \geqslant \frac{c_{\mathcal{A}} b_0}{p^+}\|u\|_{\mathscr{W}}^{p^-} - \lambda\left[M_1 + M_2\left(\|u\|_{\mathscr{W}}^{2(\underline{\mathfrak{m}}_1^+-\eta)} + \|u\|_{\mathscr{W}}^{2\eta\frac{r^+}{\underline{r}^-}}\right)\right] \to +\infty \text{ as } \|u\|_{\mathscr{W}} \to +\infty.$$

In addition, by proof of Step 2 enable us to affirm that $\mathcal{I}_{\mathfrak{a}}$ is weakly-strongly continuous, namely $u_k \rightharpoonup u$ implies $\mathcal{I}_{\mathfrak{a}}(u_k) \to \mathcal{I}_{\mathfrak{a}}(u)$. Thus by Lemma 3.3, the functional Φ is weakly lower semicontinuous, then $\mathcal{J}_{\lambda,0}$ is weakly lower semicontinuous. Therefore, by Theorem 3.1 there is a global minimum point $u_\lambda \in \mathscr{W}^+$ of $\mathcal{J}_{\lambda,0}$, hence u_λ is a critical point of $\mathcal{J}_{\lambda,0}$. To complete Lemma proof we will show that u_λ is nontrivial. Indeed, since

$$\lambda^{\star\star} = \inf_{u \in \mathscr{W}^+} \frac{\Phi(u)}{\mathcal{I}_{\mathfrak{a}}(u)}$$

and $\lambda > \lambda^{\star\star}$, we deduce that there is v_λ in $\mathscr{W}^+$ such that $\dfrac{\Phi(v_\lambda)}{\mathcal{I}_{\mathfrak{a}}(v_\lambda)} < \lambda$, this is, $\mathcal{J}_{\lambda,0}(v_\lambda) < 0$. Thus $\inf_{v \in \mathscr{W}^+} \mathcal{J}_{\lambda,0}(v) < 0$. Consequently, we conclude that u_λ is a nontrivial critical point of $\mathcal{J}_{\lambda,0}$. Therefore, λ is an eigenvalue of problem $(\mathscr{P})$.

Step 4. Given any $\lambda \in (0, \lambda_\star)$, λ is not an eigenvalue of problem $(\mathscr{P})$.

Indeed, suppose by contradiction that there exists an eigenvalue $\lambda \in (0, \lambda_\star)$ of problem $(\mathscr{P})$. Then there exists $u_\lambda \in \mathscr{W}^+$, such that

$$\langle \Phi'(u_\lambda), v \rangle = \lambda \langle \mathcal{I}'_{\mathfrak{a}}(u_\lambda), v \rangle \text{ for all } \ v \in \mathscr{W}^+.$$

Then taking $v = u_\lambda$, since $\lambda \in (0, \lambda_\star)$ and by definition of $\lambda_\star$, we have that

$$\langle \Phi'(u_\lambda), u_\lambda \rangle = \lambda \langle \mathcal{I}'_{\mathfrak{a}}(u_\lambda), u_\lambda \rangle < \lambda_\star \langle \mathcal{I}'_{\mathfrak{a}}(u_\lambda), u_\lambda \rangle \leqslant \langle \Phi'(u_\lambda), u_\lambda \rangle$$

which is a contradiction. Hence, there does not exist $\lambda \in (0, \lambda_\star)$ eigenvalue of problem $(\mathscr{P})$. Thus the claim is verified.

Therefore, the proof of Theorem 1.4 is complete. □

Chapter

2

Multiplicity results for elliptic problems involving nonlocal integrodifferential operators without Ambrosetti-Rabinowitz condition

The aim of this chapter is to prove the existence and multiplicity of weak solutions for elliptic equations involving the nonlocal integrodifferential operators with variable exponents. We consider the nonlinear elliptic problem:

$$\begin{cases} \mathscr{L}_{\mathcal{A}K} u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{$\mathscr{P}_\lambda$}$$

where $\lambda > 0$ is a real parameter, $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ is a smooth bounded domain, and to define the nonlocal integrodifferential operator $\mathscr{L}_{\mathcal{A}K}$ we will need the variable exponents $p(x) := p(x,x)$ for all $x \in \mathbb{R}^N$ with $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying:

$$\begin{aligned} & p \text{ is symmetric, that is, } p(x,y) = p(y,x), \\ & 1 < p^- := \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) \leqslant \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) := p^+ < \frac{N}{s}, \;\; s \in (0,1), \end{aligned} \tag{p_1}$$

and we consider the fractional critical variable exponent related to $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ defined by $p_s^\star(x) = \dfrac{Np(x)}{N - sp(x)}$.

The nonlocal integrodifferential operator $\mathscr{L}_{\mathcal{A}K}$ is defined on suitable fractional Sobolev spaces (see Subsection 2.1) by

$$\mathscr{L}_{\mathcal{A}K}u(x) = P.V. \int_{\mathbb{R}^N} \mathcal{A}(u(x)-u(y))K(x,y)\,dy, \ \ x \in \mathbb{R}^N,$$

where $P.V.$ is the principal value.

The map $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying the next assumptions:

(a_1) $\mathcal{A}$ is continuous, odd, and the map $\mathscr{A} : \mathbb{R} \to \mathbb{R}$ given by

$$\mathscr{A}(t) := \int_0^{|t|} \mathcal{A}(\tau)d\tau$$

is strictly convex;

(a_2) There exist positive constants $c_{\mathcal{A}}$ and $C_{\mathcal{A}}$, such that for all $t \in \mathbb{R}$ and for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$

$$\mathcal{A}(t)t \geqslant c_{\mathcal{A}}|t|^{p(x,y)} \quad \text{and} \quad |\mathcal{A}(t)| \leqslant C_{\mathcal{A}}|t|^{p(x,y)-1};$$

(a_3) $\mathcal{A}(t)t \leqslant p^+\mathscr{A}(t)$ for all $t \in \mathbb{R}$.

The kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^+$ be a measurable function satisfying the following property:

$(\mathcal{K})$ There exist constants b_0 and b_1, such that $0 < b_0 \leqslant b_1$,

$$b_0 \leqslant K(x,y)|x-y|^{N+sp(x,y)} \leqslant b_1 \text{ for all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ and } x \neq y.$$

We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $F(x,t) := \int_0^t f(x,\tau)\,d\tau$, that is, the function F is the primitive of f with respect to the second variable, such that satisfied:

(f_0) There exists a positive constant c_1 such that f satisfies the subcritical growth condition

$$|f(x,t)| \leqslant c_1(1+|t|^{\vartheta(x)-1})$$

for all $(x,t) \in \Omega \times \mathbb{R}$, where $\vartheta \in C(\overline{\Omega})$, $1 < p^+ < \underline{\vartheta}^- \leqslant \vartheta(x) \leqslant \underline{\vartheta}^+ < p_s^\star(x)$ for $x \in \overline{\Omega}$, and $\underline{\vartheta}^- := \inf_{x\in\overline{\Omega}} \vartheta(x)$, $\underline{\vartheta}^+ := \sup_{x\in\overline{\Omega}} \vartheta(x)$;

(f_1) $\lim\limits_{|t|\to+\infty} \dfrac{F(x,t)}{|t|^{p^+}} = +\infty$ uniformly for almost everywhere a.e. $x \in \Omega$, that is, f is p^+-superlinear at infinity;

(f_2) $f(x,t) = o(|t|^{p^+-1})$, as $t \to 0$, uniformly a.e. $x \in \Omega$;

(f_3) There exists a positive constant $\mathsf{c}_\star > 0$ such that

$$\mathcal{G}(x,t) \leqslant \mathcal{G}(x,\tau) + \mathsf{c}_\star$$

for all $x \in \Omega,\ 0 < t < \tau$ or $\tau < t < 0$, where $\mathcal{G}(x,t) := tf(x,t) - p^+F(x,t)$.

With intending to find infinite solutions is natural to impose certain symmetry condition on the nonlinearity. In the sequel, we will assume the following assumption on f :

(f_4) f is odd in t, that is, $f(x,-t) = -f(x,t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$.

In addition, to prove that the Euler Lagrange functional associated to problem $(\mathscr{P}_\lambda)$ verifies the Cerami condition $(C)_c$, we assume that the functions $\mathscr{A}$ and $\mathcal{A}$ satisfy the following condition

(a_4) $\mathcal{H}(at) \leqslant \mathcal{H}(t)$ for all $t \in \mathbb{R}$ and $a \in [0,1]$ where $\mathcal{H}(t) = p^+\mathscr{A}(t) - \mathcal{A}(t)t$.

Our main results are the following Theorems.

Theorem 2.1. *Assume* (a_1)-(a_4), $(\mathcal{K})$, *and* f *satisfies* (f_0)-(f_3). *Then problem* $(\mathscr{P}_\lambda)$ *has at least one nontrivial weak solution in* $\mathscr{W}$ *for all* $\lambda > 0$. *($\mathscr{W}$ is defined in Subsection 2.1)*

Theorem 2.2. *Assume* (a_1)-(a_4), $(\mathcal{K})$, *and* f *satisfies* (f_0)-(f_4). *Then problem* $(\mathscr{P}_\lambda)$ *has infinitely many solutions in* $\mathscr{W}$ *for all* $\lambda > 0$.

Theorem 2.3. *Assume* (a_1)-(a_4), $(\mathcal{K})$, *and* f *satisfies* (f_0), (f_1), (f_3) *and* (f_4). *Then, for each* $\lambda \in \left(0, \frac{c_{\mathcal{A}} b_0}{p^+}\right)$, *the problem* $(\mathscr{P}_\lambda)$ *has infinitely many weak solutions* $u_k \in \mathscr{W}$, $k \in \mathbb{N}$ *such that* $\Psi_\lambda(u_k) \to +\infty$, *as* $k \to +\infty$. *(Ψ_λ is defined (2.1))*

Theorem 2.4. *Assume* (a_1)-(a_4), $(\mathcal{K})$, *and* f *satisfies* f *satisfies* (f_0), (f_1), (f_3) *and* (f_4). *Then, for each* $\lambda \in \left(0, \frac{c_{\mathcal{A}} b_0}{p^+}\right)$, *the problem* $(\mathscr{P}_\lambda)$ *has a sequence of weak solutions* $v_k \in \mathscr{W}$, $k \in \mathbb{N}$ *such that* $\Psi_\lambda(v_k) < 0$, $\Psi_\lambda(v_k) \to 0$ *as* $k \to +\infty$.

Theorem 2.5. *Assume* (a_1)-(a_4) *and* $(\mathcal{K})$. *If* f *satisfies* (f_0), (f_1), *and* (f_3), *moreover* $f(x,0)=0$, $f(x,t)\geqslant 0$ *a.e.* $x\in\Omega$ *and for all* $t\geqslant 0$. *Then there exists a positive constant* $\overline{\lambda}$ *such that problem* $(\mathscr{P}_\lambda)$ *possesses at least one solution for all* $\lambda\in(0,\overline{\lambda})$. *Moreover*

$$\lim_{\lambda\to 0^+}\|u_\lambda\|_{\mathscr{W}}=+\infty.$$

($\|\cdot\|_{\mathscr{W}}$ is defined in Subsection 2.1)

Theorem 2.6. *Assume* (a_1)-(a_3), $(\mathcal{K})$, *and* f *satisfies* (f_4). *In addition we will assume the following condition:*

(f_5) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ *is a continuous function and there exist positive constants* C_0, C_1 *such that*

$$C_0|t|^{\mathfrak{m}(x)-1}\leqslant f(x,t)\leqslant C_1|t|^{\mathfrak{m}(x)-1} \text{ for all } x\in\overline{\Omega} \text{ and } t\geqslant 0,$$

where $\mathfrak{m}\in C(\overline{\Omega})$ *such that* $1<\mathfrak{m}(x)<p^\star(x)$ *for all* $x\in\overline{\Omega}$, *with* $\underline{\mathfrak{m}}^+<p^-$. *Then problem* $(\mathscr{P}_\lambda)$ *it has a sequence of on trivial solutions* $u_k\in\mathscr{W}$, $k\in\mathbb{N}$ *such that*

$$\lim_{k\to+\infty}\|u_k\|_{\mathscr{W}}=0$$

for all $\lambda>0$.

2.1 Variational framework

The problem $(\mathscr{P}_\lambda)$ has a variational structure and the natural space to look for solutions are variable exponent fractional Sobolev spaces. Let us consider the separable and reflexive Banach space $\mathscr{W}$, defined in Subsection 1.1 of following form

$$\mathscr{W}=W_0^{s,p(\cdot,\cdot)}:=\{u\in W^{s,p(\cdot,\cdot)}(\mathbb{R}^N): u=0 \text{ a.e. in } \mathbb{R}^N\setminus\Omega\}.$$

endowed with the norm

$$\|u\|_{\mathscr{W}}:=[u]_{\mathbb{R}^N}^{s,p(\cdot,\cdot)}.$$

Throughout the present section c_i and C_i for $i=1,2,\ldots$ will denote generic positive constants which may vary from line to line, but are independent of the terms which take part in any limit process.

Definition 2.1. *We say that $u \in \mathscr{W}$ is a weak solution to problem $(\mathscr{P}_\lambda)$ if, and only if,*

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{A}(u(x)-u(y))(v(x)-v(y))K(x,y)\,dx\,dy = \lambda \int_\Omega f(x,u)v\,dx \text{ for all } v \in \mathscr{W}.$$

The weak solutions to problem $(\mathscr{P}_\lambda)$ coincide with the critical points of the Euler Lagrange functional $\Psi_\lambda : \mathscr{W} \to \mathbb{R}$ given by

$$\Psi_\lambda(u) = \Phi(u) - \lambda \int_\Omega F(x,u)\,dx \text{ for all } u \in \mathscr{W}, \tag{2.1}$$

where Φ is defined in the Lemma 3.3.
Moreover, by Lemma 3.3 and standard arguments, the functional Ψ_λ is differentiable in $u \in \mathscr{W}$ and

$$\begin{aligned}\langle \Psi'_\lambda(u), v\rangle = &\int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{A}(u(x)-u(y))(v(x)-v(y))K(x,y)\,dx\,dy \\ &- \lambda \int_\Omega f(x,u)v\,dx \text{ for all } v \in \mathscr{W}.\end{aligned}$$

Definition 2.2. *We say that Ψ_λ satisfies the $(C)_c$ condition if for every sequence $(u_k)_{k\in\mathbb{N}} \subset \mathscr{W}$ such that $\Psi_\lambda(u_k) \to c$ and $\|\Psi'_\lambda(u_k)\|_{\mathscr{W}'}(1+\|u_k\|_{\mathscr{W}}) \to 0$, as $k \to +\infty$, has a convergent subsequence.*

2.2 Proof of Theorems 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6

2.2.1 Proof of Theorem 2.1

To prove Theorem 2.1 we need of Lemmas below and Theorem 3.3.

Lemma 2.1. *Assume (a_1)-(a_3), $(\mathcal{K})$, and (f_0)-(f_2) are holds. Then we have the following assertions:*

(i) There exists $v \in \mathscr{W}$, $v > 0$, such that $\Psi_\lambda(tv) \to -\infty$ as $t \to +\infty$;

(ii) There exist $r > 0$ and $\mathcal{R} > 0$ such that $\Psi_\lambda(u) \geqslant \mathcal{R}$ for any $u \in \mathscr{W}$ with $\|u\|_{\mathscr{W}} = r$.

Proof. (i) From (f_1), it follows that for any $\mathscr{C} > 0$ there exists a constant $c_{\mathscr{C}} > 0$ such that

$$F(x,t) \geqslant \mathscr{C}|t|^{p^+} - c_{\mathscr{C}} \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}. \tag{2.2}$$

Take $v \in \mathscr{W}$ with $v > 0$, for $t > 1$, by (2.2), (a_1), (a_2), and $(\mathcal{K})$, we obtain

$$\Psi_\lambda(tv) \leqslant t^{p^+}\left[\frac{C_{\mathcal{A}}b_1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy - \lambda\mathscr{C}\int_\Omega v^{p^+}\,dx\right] + \lambda c_{\mathscr{C}}|\Omega| \tag{2.3}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Hence, from (2.3) and taking $\mathscr{C}$ large enough such that

$$\frac{C_{\mathcal{A}}b_1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy - \lambda\mathscr{C}\int_\Omega v^{p^+}\,dx < 0,$$

we have

$$\lim_{t\to+\infty}\Psi_\lambda(tv) = -\infty,$$

which completes the proof of (i).
(ii)First, since the embeddings $\mathscr{W} \hookrightarrow L^{p^+}(\Omega)$ and $\mathscr{W} \hookrightarrow L^{\vartheta(x)}(\Omega)$ are continuous (Lemma 3.2), there exist positive constants c_2, c_3, such that

$$\|u\|_{L^{p^+}(\Omega)} \leqslant c_2\|u\|_{\mathscr{W}}, \ \|u\|_{L^{\vartheta(\cdot)}(\Omega)} \leqslant c_3\|u\|_{\mathscr{W}} \quad \text{for all} \quad u \in \mathscr{W}. \tag{2.4}$$

Now, let $0 < \varepsilon < \dfrac{c_{\mathcal{A}}b_0}{\lambda c_2^{p^+}}$. From (f_0) and (f_2), it follows that for all given $\varepsilon > 0$, there exists $c_\varepsilon > 0$, such that

$$F(x,t) \leqslant \frac{\varepsilon}{p^+}|t|^{p^+} + c_\varepsilon|t|^{\vartheta(x)} \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}. \tag{2.5}$$

Now, Thus, for $u \in \mathscr{W}$ with $\|u\|_{\mathscr{W}} < 1$ sufficiently small, from (a_2), (a_3), $(\mathcal{K})$, (2.4), and (2.5), we obtain

$$\Psi_\lambda(u) \geqslant \frac{\|u\|_{\mathscr{W}}^{p^+}}{p^+}\left(c_{\mathcal{A}}b_0 - \lambda\varepsilon c_2^{p^+}\right) - \lambda c_\varepsilon c_3^{\vartheta^-}\|u\|_{\mathscr{W}}^{\vartheta^-}. \tag{2.6}$$

Therefore, since $\underline{\vartheta}^- > p^+$ from (2.6) we can choose $\mathcal{R} > 0$ and $r > 0$ such that $\Psi_\lambda(u) \geqslant \mathcal{R} > 0$ for every $u \in \mathscr{W}$ and $\|u\|_{\mathscr{W}} = r$. This completes the proof of (ii). □

Lemma 2.2. *Assume that the condition* (a_1)*-*(a_4)*,* $(\mathcal{K})$*, and* f *satisfies* (f_0)*,* (f_1) *and*

(f_3). Then the functional Ψ_λ satisfies the $(C)_c$ condition at any level $c > 0$.

Proof. Let $c \in \mathbb{R}$ and $(u_k)_{k\in\mathbb{N}} \subset \mathscr{W}$ be a $(C)_c$ sequence for Ψ_λ, that is,

$$\Psi_\lambda(u_k) \to c > 0 \text{ and } \|\Psi'_\lambda(u_k)\|_{\mathscr{W}'}(1 + \|u_k\|_{\mathscr{W}}) \to 0 \text{ as } k \to +\infty. \tag{2.7}$$

Initially, we prove that the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathscr{W}$. Indeed, arguing by contradiction, up to a subsequence, still denoted by $(u_k)_{k\in\mathbb{N}}$, we suppose that $(u_k)_{k\in\mathbb{N}}$ is unbounded in $\mathscr{W}$. Define $\omega_k := \dfrac{u_k}{\|u_k\|_{\mathscr{W}}}$ for all $k \in \mathbb{N}$, then $(\omega_k)_{k\in\mathbb{N}} \subset \mathscr{W}$ and $\|\omega_k\|_{\mathscr{W}} = 1$. Thus, we can extract a subsequence, still denoted by $(\omega_k)_{k\in\mathbb{N}}$ and $\omega \in \mathscr{W}$ such that $\omega_k \rightharpoonup \omega$ in $\mathscr{W}$ as $k \to +\infty$. From Lemma 3.2, it follows that

$$\omega_k(x) \to \omega(x) \text{ a.e. } x \in \Omega,\ \omega_k \to \omega \text{ in } L^{p^+}(\Omega), \text{ and } \omega_k \to \omega \text{ in } L^{\vartheta(\cdot)}(\Omega) \text{ as } k \to +\infty. \tag{2.8}$$

We consider $\Omega_\star := \{x \in \Omega : \omega(x) \neq 0\}$. If $x \in \Omega_\star$, by (2.8), we have

$$|u_k(x)| = |\omega_k(x)|\|u_k\|_{\mathscr{W}} \to +\infty \text{ a.e. } x \in \Omega_\star \text{ as } k \to +\infty.$$

Therefore, by (f_1), we obtain for each $x \in \Omega_\star$

$$\lim_{k\to+\infty} \frac{F(x,u_k)}{|u_k|^{p^+}} \frac{|u_k|^{p^+}}{\|u_k\|_{\mathscr{W}}^{p^+}} = \lim_{k\to+\infty} \frac{F(x,u_k)}{|u_k|^{p^+}} |\omega_k|^{p^+} = +\infty. \tag{2.9}$$

Also, by (f_1) there exists $D > 0$ such that

$$\frac{F(x,t)}{|t|^{p^+}} > 1 \text{ for all } (x,t) \in \Omega \times \mathbb{R} \text{ with } |t| \geq D. \tag{2.10}$$

Since $F(x,t)$ is continuous on $\overline{\Omega} \times [-D, D]$, there exists a positive constant c_5 such that

$$|F(x,t)| \leqslant c_5 \text{ for all } (x,t) \in \overline{\Omega} \times [-D, D]. \tag{2.11}$$

Hence, by (2.10) and (2.11), we conclude that there is a constant c_6 such that

$$F(x,t) \geqslant c_6 \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R},$$

which shows that
$$\frac{F(x,u_k)-c_6}{\|u_k\|_{\mathscr{W}}^{p^+}} \geqslant 0 \text{ for all } x\in\Omega \text{ and } k\in\mathbb{N},$$
that is,
$$\frac{F(x,u_k)}{|u_k|^{p^+}}|\omega_k|^{p^+} - \frac{c_6}{\|u_k\|_{\mathscr{W}}^{p^+}} \geqslant 0 \text{ for all } x\in\Omega \text{ and } k\in\mathbb{N}. \tag{2.12}$$
Now, by (2.7), (a_2), (a_3), and $(\mathcal{K})$, we have
$$c \geqslant \frac{c_{\mathcal{A}}b_0}{p^+}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u_k(x)-u_k(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy - \lambda\int_\Omega F(x,u_k)\,dx + o_k(1).$$
Then,
$$\int_\Omega F(x,u_k)\,dx \geqslant \frac{c_{\mathcal{A}}b_0}{\lambda p^+}\|u_k\|_{\mathscr{W}}^{p^-} - \frac{c}{\lambda} + \frac{o_k(1)}{\lambda} \to +\infty \text{ as } k\to+\infty. \tag{2.13}$$
On the other hand, from (a_1), (a_2), and $(\mathcal{K})$, we also that
$$c \leqslant \frac{C_{\mathcal{A}}b_1}{p^-}\|u_k\|_{\mathscr{W}}^{p^+} - \lambda\int_\Omega F(x,u_k)\,dx + o_k(1) \text{ for all } k\in\mathbb{R}.$$
Consequently, by (2.13), we achieve
$$\|u_k\|_{\mathscr{W}}^{p^+} \geqslant \frac{cp^-}{C_{\mathcal{A}}b_1} + \frac{\lambda p^-}{C_{\mathcal{A}}b_1}\int_\Omega F(x,u_k)\,dx - \frac{p^-}{C_{\mathcal{A}}b_1}o_k(1) > 0, \tag{2.14}$$
for k large enough.

We claim that $|\Omega_\star|=0$. Indeed, if $|\Omega_\star|\neq 0$, then by (2.9), (2.12), (2.14), and Fatou's Lemma, we have
$$\begin{aligned}+\infty &= \int_{\Omega_\star}\liminf_{k\to+\infty}\frac{F(x,u_k)}{|u_k|^{p^+}}|\omega_k(x)|^{p^+}\,dx - \int_{\Omega_\star}\limsup_{k\to+\infty}\frac{c_6}{\|u_k\|_{\mathscr{W}}^{p^+}}\,dx \\ &\leqslant \liminf_{k\to+\infty}\frac{\displaystyle\int_\Omega F(x,u_k)\,dx}{\frac{\lambda p^-}{C_{\mathcal{A}}b_1}\displaystyle\int_\Omega F(x,u_k)\,dx - o_k(1)}.\end{aligned} \tag{2.15}$$

Therefore, from (2.14) and (2.15), we obtain that $+\infty \leqslant \frac{C_{\mathcal{A}}b_1}{\lambda p^-}$, which is a contradiction. This proves that $|\Omega_\star|=0$ and thus $\omega(x)=0$ a.e. $x\in\Omega$.

Now as in [47], we define the continuous function $B_k:[0,1]\longrightarrow\mathbb{R}$ by $B_k(t):=\Psi_\lambda(tu_k)$. Since $B_k(t):=\Psi_\lambda(tu_k)$ is continuous in $[0,1]$, we can say that for each $k\in\mathbb{N}$ there exists

$t_k \in [0,1]$ such that

$$\Psi_\lambda(t_k u_k) := \max_{t\in[0,1]} B_k(t). \tag{2.16}$$

(If for $k \in \mathbb{N}$ is not unique we choose the smaller possible value). Note that $t_k > 0$ for all $k \in \mathbb{N}$. Indeed, passing to a subsequence if necessary, we have $\Psi_\lambda(u_k) \geq \frac{c}{2}$ for all $k \in \mathbb{N}$. So, if $t_k = 0$ for all $k \in \mathbb{N}$ it follows that

$$\Psi_\lambda(t_k u_k) = \Psi_\lambda(0) = 0, \tag{2.17}$$

however,

$$0 < \frac{c}{2} \leqslant \Psi_\lambda(u_k) \leqslant \max_{t\in[0,1]} \Psi_\lambda(tu_k) = \Psi_\lambda(t_k u_k). \tag{2.18}$$

Thus, from (2.17) and (2.18), we obtain a contradiction.
If $t_k \in (0,1)$, by (2.16), we infer that

$$\frac{d}{dt}_{|t=t_k} \Psi_\lambda(tu_k) = 0 \quad \text{for all} \quad k \in \mathbb{N}.$$

Moreover, if $t_k = 1$, by (2.7) we have $\langle \Psi'_\lambda(u_k), u_k\rangle = o_k(1)$. So we always have

$$\langle \Psi'_\lambda(t_k u_k), t_k u_k\rangle = t_k \frac{d}{dt}_{|t=t_k} \Psi_\lambda(tu_k) = o_k(1).$$

Let $(r_j)_{j\in\mathbb{N}}$ be a positive sequence of real numbers such that $r_j > 1$ and $\lim_{j\to+\infty} r_j = +\infty$. Then $\|r_j\omega_k\|_{\mathscr{W}} = r_j > 1$ for all j and $k \in \mathbb{N}$. Fix $j \in \mathbb{N}$, since $\omega_k \to 0$ in $L^{\vartheta(\cdot)}(\Omega)$ and $\omega_k(x) \to 0$ a.e. $x \in \Omega$, as $k \to +\infty$, using the condition (f_0), there exists a positive constant c_7 such that

$$\Big|F(x, r_j\omega_k)\Big| \leqslant c_7 \left(r_j|\omega_k(x)| + r_j^{\vartheta(x)}|\omega_k(x)|^{\vartheta(x)}\right) \tag{2.19}$$

and by continuity of the function F, we achieve

$$F(x, r_j\omega_k) \to F(x, r_j\omega) = 0 \text{ a.e. } x \in \Omega \text{ as } k \to +\infty, \tag{2.20}$$

for each $j \in \mathbb{N}$. Consequently, from (2.19), (2.20), and the Dominated Convergence Theorem, we obtain

$$\lim_{k\to+\infty} \int_\Omega F(x, r_j\omega_k)\, dx = 0 \tag{2.21}$$

for all $j \in \mathbb{N}$.

Since $\|u_k\|_{\mathscr{W}} \to +\infty$ as $k \to +\infty$, we also have either $\|u_k\|_{\mathscr{W}} > r_j$ or $\frac{r_j}{\|u_k\|_{\mathscr{W}}} \in (0,1)$ for k large enough. Thus, using (2.16), (2.21), (a_2), (a_3), $(\mathcal{K})$, and Proposition 3.4, we deduce that

$$\Psi_\lambda(t_k u_k) \geqslant \Psi_\lambda\left(\frac{r_j}{\|u_k\|_{\mathscr{W}}} u_k\right) = \Psi_\lambda(r_j \omega_k) \geqslant \frac{c_{\mathcal{A}} b_0 r_j^{p^-}}{p^+} - \lambda \int_\Omega F(x, r_j \omega_k)\, dx \tag{2.22}$$

for all k large enough .

Therefore, by (2.22) letting $k, j \to +\infty$, we conclude

$$\limsup_{k \to +\infty} \Psi_\lambda(t_k u_k) = +\infty. \tag{2.23}$$

Now, we affirm that $\limsup\limits_{k \to +\infty} \Psi_\lambda(t_k u_k) \leqslant \delta$, for a suitable positive constant δ. Indeed, from (a_4), (f_3), (2.7), and for all k large enough, we have

$$\begin{aligned}
\Psi_\lambda(t_k u_k) =& \Psi_\lambda(t_k u_k) - \frac{1}{p^+}\langle \Psi'_\lambda((t_k u_k)), t_k u_k\rangle + o_k(1) \\
=& \frac{1}{p^+}\mathcal{H}(t_k u_k) + \frac{\lambda}{p^+}\int_\Omega \mathcal{G}(x, t_k u_k)\, dx + o_k(1) \\
\leqslant& \frac{1}{p^+}\int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{H}(u_k) K(x,y)\, dx\, dy + \frac{\lambda}{p^+}\int_\Omega (\mathcal{G}(x, u_k) + c_\star)\, dx + o_k(1) \\
=& \Psi(u_k) - \frac{1}{p^+}\langle \Psi'_\lambda(u_k), u_k\rangle + \frac{\lambda c_\star |\Omega|}{p^+} + o_k(1).
\end{aligned}$$

Then,

$$\Psi_\lambda(t_k u_k) \longrightarrow c + \frac{\lambda \mathsf{c}_\star}{p^+}|\Omega| \text{ as } k \longrightarrow +\infty. \tag{2.24}$$

Consequently, from (2.23) and (2.24) we obtain a contradiction. Therefore, the sequence $(u_k)_{k \in \mathbb{N}}$ is bounded in $\mathscr{W}$.

Now, with standard arguments, we prove that any $(C)_c$ sequence has a convergent subsequence. Since $\mathscr{W}$ is a reflexive Banach space there exists $u \in \mathscr{W}$ such that, up to a subsequence still denoted by $(u_k)_{k \in \mathbb{N}}$ we obtain that $u_k \rightharpoonup u$ in $\mathscr{W}$ and by Lemma 3.2, we achieve

$$u_k(x) \to u(x) \text{ a.e. } x \in \Omega, \ \ u_k \to u \text{ in } L^{\vartheta(\cdot)}(\Omega), \text{ and } u_k \to u \text{ in } L^{\mathsf{m}(\cdot)}(\Omega) \text{ as } k \to +\infty.$$

Hence, using the Hölder's inequality, we have

$$\Big|\int_\Omega f(x,u_k)(u_k-u)\,dx\Big| \leqslant C\|1+|u_k|^{\vartheta(x)-1}\|_{L^{\vartheta'(\cdot)}(\Omega)}\|u_k-u\|_{L^{\vartheta(\cdot)}(\Omega)} \to 0 \text{ as } k\to+\infty. \tag{2.25}$$

and from (2.7), it follows that

$$\langle \Psi'_\lambda(u_k), u_k-u\rangle \to 0 \text{ as } k\to+\infty. \tag{2.26}$$

Thus, using (2.25) and (2.26), we get

$$\langle \Phi'(u_k), u_k-u\rangle = \lambda\int_\Omega f(x,u_k)(u_k-u)\,dx + \langle \Psi'_\lambda(u_k), u_k-u\rangle \to 0 \;as\; k\to+\infty.$$

Thus, since the operator Φ' is of type (S_+) (see Lemma 3.3), we conclude that $u_k \to u$ in $\mathscr{W}$. Therefore, this proves that Ψ_λ satisfies the $(C)_c$ condition on $\mathscr{W}$ and we finish the proof of Lemma. □

Proof of Theorem 2.1. From Lemma 2.1 and Lemma 2.2, the Euler Lagrange functional Ψ_λ satisfies the geometric conditions the Mountain Pass Theorem. Moreover $\Psi_\lambda(0) = 0$. Therefore, by Theorem 3.3, the functional Ψ_λ has a critical value $c \geqslant \mathcal{R} > 0$, that is, exists $u \in \mathscr{W}$ such that problem $(\mathscr{P}_\lambda)$ has at least one nontrivial weak solution in $\mathscr{W}$. □

2.2.2 Proof of Theorem 2.2

To prove Theorem 2.2 we use some Lemmas presented below and Theorem 3.4 what is "$\mathbb{Z}_2 - symmetric$" version (for even functionals) of the Mountain Pass.

Lemma 2.3. *Assume (a_1)-(a_3), $(\mathcal{K})$, and (f_0)-(f_2) are fulfilled. Then for each $\lambda > 0$, there exist $\mathcal{R}_1 > 0$ and $r_1 > 0$ such that $\Psi_\lambda(u) \geqslant \mathcal{R}_1$ for all $u \in \mathscr{W}$ with $\|u\|_{\mathscr{W}} = r_1$.*

Proof. The proof is as in the Lemma 2.1. □

Lemma 2.4. *Assume (a_1)-(a_3), $(\mathcal{K})$, and (f_1) are fulfilled. For every finite dimensional subspace $\widehat{\mathscr{W}} \subset \mathscr{W}$ there exists $\mathcal{R}_2 = \mathcal{R}_2(\widehat{\mathscr{W}})$ such that*

$$\Psi_\lambda(u) \leqslant 0 \text{ for all } u \in \widehat{\mathscr{W}} \setminus B_{\mathcal{R}_2}(\mathscr{W}),$$

where $B_{\mathcal{R}_2}(\mathscr{W}) = \{u \in \widehat{\mathscr{W}} : \|u\|_{\mathscr{W}} < \mathcal{R}_2\}$.

Proof. Consider $\widehat{\mathscr{W}}$ be a finite dimensional subspace of $\mathscr{W}$ and let $u \in \widehat{\mathscr{W}}$ with $\|u\|_{\mathscr{W}} = 1$ fixed. Thus, for all $t \geqslant 1$ using (a_1), (a_2), $(\mathcal{K})$, (f_1), (2.2), and Proposition 3.4, we get

$$\Psi_\lambda(tu) \leqslant \frac{C_{\mathcal{A}} b_1}{p^-} t^{p^+} - \lambda \mathscr{C} c_{p^+} t^{p^+} + \lambda c_{\mathscr{C}} |\Omega|.$$

Taking $\Pi(t) = \left[\frac{C_{\mathcal{A}} b_1}{p^-} - \lambda \mathscr{C} c_{p^+}\right] t^{p^+} + \lambda c_{\mathscr{C}} |\Omega|$, if $\mathscr{C}$ is large enough, then $\Pi(t) \to -\infty$ as $t \to +\infty$.

Therefore, we have

$$\sup\{\Psi_\lambda(u) : u \in \widehat{\mathscr{W}}, \|u\|_{\mathscr{W}} = \mathcal{R}_3\} = \sup\{\Psi_\lambda(\mathcal{R}_3 u) : u \in \widehat{\mathscr{W}}, \|u\|_{\mathscr{W}} = 1\} \to -\infty$$

as $\mathcal{R}_3 \to +\infty$.

Hence, there exists $\mathcal{R}_2 > 0$ sufficiently large such that $\Psi_\lambda(u) \leqslant 0$ for all $u \in \widehat{\mathscr{W}}$ with $\|u\|_{\mathscr{W}} = \mathcal{R}_3$ and $\mathcal{R}_3 \geqslant \mathcal{R}_2$. □

Proof of Theorem 2.2. From Lemma 2.1, the Euler Lagrange functional Ψ_λ satisfies the $(C)_c$ condition. Moreover, $\Psi_\lambda(0) = 0$ and Ψ_λ is even functional by conditions (a_1) and (f_4). Therefore, using the Lemma 2.3, Lemma 2.4, and Theorem 3.4 we conclude the existence of an unbounded sequence of weak solutions to problem $(\mathscr{P}_\lambda)$ and this completes the proof. □

2.2.3 Proof of Theorem 2.3

To prove the Theorem 2.3 we verify the hypotheses of Theorem 3.5 through the Lemmas presented below.

Since that $\mathscr{W}$ is a separable and reflexive Banach space, using ([31, Chapter 4], or [78, Section 17]) or [36], there exist sequence $(e_l)_{l\in\mathbb{N}} \subset \mathscr{W}$ and $(e_l^\star)_{\mathbb{N}} \subset \mathscr{W}'$ such that

$$\mathscr{W} = \overline{span\{e_l : l = 1, 2, \ldots\}},\ \mathscr{W}' = \overline{span\{e_l^\star : l = 1, 2, \ldots\}}^{\omega^\star},$$

and

$$\langle e_i^\star, e_l \rangle = \begin{cases} 1 & \text{se} \quad i = l, \\ 0 & \text{se} \quad i \neq l. \end{cases}$$

We denote

$$\mathscr{W}_l = span\{e_l\},\ Y_j = \bigoplus_{l=1}^{j} \mathscr{W}_l = span\{e_1, \ldots, e_j\},\ \text{and}\ Z_j = \overline{\bigoplus_{l=j}^{\infty} \mathscr{W}_l} = \overline{span\{e_j, e_{j+1} \ldots\}}.$$

Lemma 2.5. *If $\vartheta \in C^+(\overline{\Omega}),\ p^+ < \vartheta(x) < p_s^{\star -}$ for all $x \in \overline{\Omega}$, denote*

$$\beta_j := \sup\{\|u\|_{L^{\vartheta(\cdot)}(\Omega)} : \|u\|_{\mathscr{W}} = 1,\ u \in Z_j\},$$

then $\lim\limits_{j \to +\infty} \beta_j = 0.$

Proof. It is clear that, $0 \leqslant \beta_{j+1} \leqslant \beta_j$, thus $\beta_j \to \beta \geqslant 0$ as $j \to +\infty$. Let $u_j \in Z_j$ satisfy

$$\|u_j\|_{\mathscr{W}} = 1,\ 0 \leqslant \beta_j - \|u_j\|_{L^{\vartheta(\cdot)}(\Omega)} < \frac{1}{j} \text{ for } j \in \mathbb{N}.$$

Since $\mathscr{W}$ is reflexive there exist a subsequence of $(u_j)_{j\in\mathbb{N}}$ such that $u_j \rightharpoonup u$ as $j \to +\infty$. We claim $u = 0$. Indeed, for all $e_m^\star \in \{e_l^\star : l = 1, 2, \ldots\}$, we obtain $\langle e_m^\star, u_j\rangle = 0,\ j > m$. So $\langle e_m^\star, u_j\rangle \to 0$ *as* $j \to \infty$, this concludes $\langle e_m^\star, u\rangle = 0$ for all $e_m^\star \in \{e_l^\star : l = 1, 2, \ldots.\}$. Therefore, $u = 0$ and so $u_j \to 0$ as $j \to +\infty$. Since the embedding from $\mathscr{W} \hookrightarrow L^{\vartheta(x)}(\Omega)$ is compact, then $u_j \to 0$ in $L^{\vartheta(x)}(\Omega)$ as $j \to +\infty$. Hence we get $\beta_j \to 0$ as $j \to +\infty$. □

Lemma 2.6. *Assume that $\Xi : \mathscr{W} \to \mathbb{R}$ is weakly-strongly continuous, namely, $u_k \rightharpoonup u$ implies $\Xi(u_k) \to \Xi(u)$, and $\Xi(0) = 0$. Then for each $\tau > 0$ and $j \in \mathbb{N}$ there exists*

$$\alpha_j := \sup\{|\Xi(u)| : u \in Z_j,\ \|u\|_{\mathscr{W}} < \tau\} < \infty.$$

Moreover, $\lim\limits_{j \to +\infty} \alpha_j = 0.$

Proof. It is clear that, $0 \leqslant \alpha_{j+1} \leqslant \alpha_j$, thus $\alpha_j \to \alpha \geqslant 0\ j \to +\infty$. Let $u_j \in Z_j,\ \|u\|_{\mathscr{W}} \leqslant \tau$ such that

$$0 \leqslant \alpha_j - |\Xi(u_j)| < \frac{1}{j}.$$

Since $\mathscr{W}$ is reflexive there exist a subsequence of $(u_j)_{j\in\mathbb{N}}$ such that $u_j \rightharpoonup u$. Proceeding as in the previous lemma, we obtain $u = 0$. The weak- strong continuity of Ξ, guarantees $\Xi(u_j) \to \Xi(0) = 0$. Therefore, $\alpha_j \to 0$ as $j \to +\infty$. □

Lemma 2.7. *Assume (a_1)-(a_3), $(\mathcal{K})$, (f_0), and (f_2) are satisfied. Moreover, let $\lambda \in \left(0, \dfrac{c_{\mathcal{A}} b_0}{p^+}\right)$. Then there exist $\rho_j > r_j > 0$ such that:*

(h_2) $b_j := \inf\{\Psi_\lambda(u) : u \in Z_j,\ \|u\|_{\mathscr{W}} = r_j\} \to +\infty$ *as* $j \to +\infty$;

(h_3) $a_j := \max\{\Psi_\lambda(u) : u \in Y_j,\ \|u\|_{\mathscr{W}} = \rho_j\} \leqslant 0$.

Proof. (h_2) First, notice that for all $u \in Z_j$ with $\|u\|_{\mathscr{W}} > 1$, using (a_2), (a_3), $(\mathcal{K})$, and (f_0), we infer

$$\begin{aligned}\Psi_\lambda(u) \geqslant & \frac{c_{\mathcal{A}} b_0}{p^+} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy - \lambda c_1 \int_\Omega \left(|u| + \frac{|u|^{\vartheta(x)}}{\vartheta(x)} \right) dx \\ \geqslant & \frac{c_{\mathcal{A}} b_0}{p^+} \|u\|_{\mathscr{W}}^{p^-} - \lambda \frac{c_1 \beta_j^{\vartheta^+}}{\underline{\vartheta}^-} \|u\|_{\mathscr{W}}^{\vartheta^+} - \lambda c_8 \|u\|_{\mathscr{W}},\end{aligned} \tag{2.27}$$

for a constant $c_8 > 0$ and $\beta_j := \sup\{\|u\|_{L^{\vartheta(x)}(\Omega)} : \|u\|_{\mathscr{W}} = 1,\ u \in Z_j\}$.
Now, since $p^- \leqslant p^+ < \underline{\vartheta}^+$, by the Lemma 2.5, it is easy see that $r_j := (c_1 \beta_j^{\vartheta^+})^{\frac{1}{p^- - \underline{\vartheta}^+}} \to +\infty$ as $j \to +\infty$. Then, for j sufficiently large, $u \in Z_j$ with $\|u\|_{\mathscr{W}} = r_j > 1$, and by (2.27), we conclude

$$\Psi_\lambda(u) \geqslant \left(\frac{c_{\mathcal{A}} b_0}{p^+} - \lambda \right) r_j^{p^-} - \lambda c_8 r_j.$$

Therefore, since $p^- > 1$ and $\dfrac{c_{\mathcal{A}} b_0}{p^+} > \lambda$, we obtain that $b_j := \inf\{\Psi_\lambda(u) : u \in Z_j,\ \|u\|_{\mathscr{W}} = r_j\} \to +\infty$ as $j \to +\infty$.

(h_3) Note that for all $v \in Y_j$ with $\|v\|_{\mathscr{W}} = 1$, using (a_2), (a_3), $(\mathcal{K})$, (f_0) and (2.2) for $t > 1$, we have

$$\Psi_\lambda(tv) \leqslant t^{p^+} \left(\frac{C_{\mathcal{A}} b_1}{p^-} \|v\|_{\mathscr{W}}^{p^+} - \lambda \mathscr{C} \int_\Omega |v|^{p^+}\, dx \right) + \lambda c_{\mathscr{C}} |\Omega|. \tag{2.28}$$

It is clear that we can choose $\mathscr{C} > 0$ large enough such that

$$\frac{C_{\mathcal{A}} b_1}{p^-} \|v\|_{\mathscr{W}}^{p^+} - \lambda \mathscr{C} \int_\Omega |v|^{p^+}\, dx < 0.$$

From (2.28), it follows that

$$\lim_{t \to +\infty} \Psi_\lambda(tv) = -\infty.$$

Therefore, there exists $t_0 > r_j > 1$ large enough such that $\Psi_\lambda(t_0 v) \leqslant 0$ and thus, if set

$\rho_j = t_0$, we conclude that

$$a_j := \max\{\Psi_\lambda(u) : u \in Y_j;\ \|u\|_{\mathscr{W}} = \rho_j\} \leqslant 0.$$

□

Proof of Theorem 2.3. From conditions (a_1) and (f_4) the Euler Lagrange functional Ψ_λ is even functional and by Lemma 2.2, Ψ_λ satisfies the $(C)_c$ condition for every $c > 0$. Then, with that and by Lemma 2.7 all conditions of the Theorem 3.5 are satisfied. Therefore, we obtain a sequence of critical points $(u_k)_{k\in\mathbb{N}}$ in $\mathscr{W}$ such that $\Psi_\lambda(u_k) \to +\infty$ as $k \to +\infty$. □

2.2.4 Proof of Theorem 2.4

To prove Theorem 2.4 we use the Lemma presented below and we verify hypotheses Theorem 3.6.

Lemma 2.8. *Suppose that the hypotheses in Theorem 2.4 hold, then Ψ_λ satisfied the $(C)_c^\star$ condition.*

Proof. Let $c \in \mathbb{R}$ and the sequence $(u_k)_{k\in\mathbb{N}} \subset \mathscr{W}$ be such that $u_k \in Y_k$ for all $k \in \mathbb{N}$, $\Psi_\lambda(u_k) \to c$ and $\left\|\Psi'_{\lambda|_{Y_{k_j}}}(u_k)\right\|_{\mathscr{W}'}(1 + \|u_k\|_{\mathscr{W}}) \to 0$, as $k \to +\infty$. Therefore, we have

$$c = \Psi_\lambda(u_k) + o_k(1) \text{ and } \langle \Psi'_\lambda(u_k), u_k\rangle = o_k(1),$$

where $o_k(1) \to 0$ as $k \to +\infty$. Analogously to the proof of Lemma 2.2, we can prove that the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathscr{W}$. Since $\mathscr{W}$ is reflexive, we can extract a subsequence of $(u_k)_{k\in\mathbb{N}}$, denoted for $(u_{k_j})_{j\in\mathbb{N}}$, and $u \in \mathscr{W}$ such that $u_{k_j} \rightharpoonup u$ in $\mathscr{W}$ as $j \to +\infty$.
On the other hand, as $\mathscr{W} = \overline{\cup_k Y_k} = \overline{span\{e_k : k \geqslant 1\}}$, we can choose $v_k \in Y_k$ such that $v_k \to u$ in $\mathscr{W}$ as $k \to +\infty$. Hence, we conclude that

$$\langle \Psi'_\lambda(u_{k_j}), u_{k_j} - u\rangle = \langle \Psi'_\lambda(u_{k_j}), u_{k_j} - v_{k_j}\rangle + \langle \Psi'_\lambda(u_{k_j}), v_{k_j} - u\rangle.$$

Since $\Psi'_{\lambda|_{Y_{k_j}}}(u_{k_j}) \to 0$ and $u_{k_j} - v_{k_j} \rightharpoonup 0$ in Y_{k_j}, as $j \to +\infty$, we achieve

$$\lim_{j\to+\infty} \langle \Psi'_\lambda(u_{k_j}), u_{k_j} - u\rangle = 0.$$

Furthermore, using Hölder's inequality, we obtain that

$$\int_\Omega f(x,u_{k_j})(u_{k_j}-u)\,dx \to 0 \text{ as } j\to+\infty.$$

Therefore,

$$\langle \Phi'(u_{k_j}), u_{k_j}-u\rangle = \lambda\int_\Omega f(x,u_{k_j})(u_{k_j}-u)\,dx + \langle \Psi'_\lambda(u_{k_j}), u_{k_j}-u\rangle \to 0 \ as\ j\to+\infty.$$

Consequently, since Φ' is of type (S_+), it follows that $u_{k_j}\to u$ in $\mathscr{W}$ as $j\to+\infty$. Then, we conclude that Ψ_λ satisfies the $(C)^\star_c$ condition. Thus, we obtain that $\Psi'_\lambda(u_{k_j})\to\Psi'_\lambda(u)$ as $j\to+\infty$.

Let us prove $\Psi'_\lambda(u)=0$. Indeed, taking $\omega_l\in Y_l$, notice that when $k_j\geqslant l$, we achieve

$$\begin{aligned}\langle \Psi'_\lambda(u),\omega_l\rangle &= \langle \Psi'_\lambda(u)-\Psi'_\lambda(u_{k_j}),\omega_l\rangle + \langle \Psi'_\lambda(u_{k_j}),\omega_l\rangle\\ &= \langle \Psi'_\lambda(u)-\Psi'_\lambda(u_{k_j}),\omega_l\rangle + \langle \Psi'_{\lambda|Y_{k_j}}(u_{k_j}),\omega_l\rangle,\end{aligned}$$

so, passing the limit on the right side of the equation above as $j\to+\infty$, we conclude

$$\langle \Psi'_\lambda(u), w_l\rangle = 0 \text{ for all } \omega_l\in Y_l.$$

Therefore, $\Psi'_\lambda(u)=0$ in $\mathscr{W}'$ and this show that Ψ_λ satisfies the $(C)^\star_c$ condition for every $c\in\mathbb{R}$. □

Proof of Theorem 2.4. First we observe that from (a_1), (f_3), and Lemma 2.8 the Euler Lagrange functional Ψ_λ is even functional and satisfies the $(C)^\star_c$ condition for all $c\in\mathbb{R}$. Now we will show that the conditions (g_1), (g_2), and (g_3) of the Dual Fountain Theorem are satisfied:

(g_1) First we note that using (f_0) and Young's inequality, we have

$$|F(x,t)|\leqslant c_9(1+|t|^{\vartheta(x)}) \tag{2.29}$$

for a positive constant c_9. Moreover, as $\lambda<\dfrac{c_{\mathcal{A}}b_0}{p^+}$, we have

$$\lim_{j\to+\infty}\left(\frac{c_{\mathcal{A}}b_0}{p^+}-\lambda\right)(c_9\beta_j^{\vartheta^+})^{\frac{p^-}{p^--\vartheta^+}} = +\infty.$$

Then, there exists $j_0 \in \mathbb{N}$ such that

$$\left(\frac{c_{\mathcal{A}}b_0}{p^+} - \lambda\right)(c_9\beta_j^{\vartheta^+})^{\frac{p^-}{p^- - \vartheta^+}} - \lambda c_9|\Omega| \geqslant 0 \quad \text{for all } j \geqslant j_0.$$

Taking $\rho_j = (c_9\beta_j^{\vartheta^+})^{\frac{1}{p^- - \vartheta^+}}$ for $j \geqslant j_0$. It is clear that $\rho_j > 1$ for all $j \in \mathbb{N}$, $j \geqslant j_0$, since $\lim_{j\to+\infty} \rho_j = +\infty$. Using same the arguments of Theorem 2.3, for all $u \in Z_j$, with $\|u\|_{\mathscr{W}} = \rho_j$, and using (a_2), (a_3), $(\mathcal{K})$, and (2.29), we conclude that

$$\Psi_\lambda(u) \geqslant \left(\frac{c_{\mathcal{A}}b_0}{p^+} - \lambda\right)(c_9\beta_j^{\vartheta^+})^{\frac{p^-}{p^- - \vartheta^+}} - \lambda c_9|\Omega| \geqslant 0.$$

So, we obtain

$$a_j := \inf\{\Psi_\lambda(u) : u \in Z_j,\ \|u\|_{\mathscr{W}} = \rho_j\} \geqslant 0.$$

(g_2) Since Y_j is finite dimensional all the norms are equivalent, there exists a constant $c_{10} > 0$ such that $\|u\|_{L^{p^+}(\Omega)} \geqslant c_{10}\|u\|_{\mathscr{W}}$ for all $u \in Y_j$. Then, from (a_1), (a_2), $(\mathcal{K})$, (f_1), (2.2), and Proposition 3.4, we infer

$$\Psi_\lambda(u) \leqslant \frac{C_{\mathcal{A}}b_1}{p^-}\|u\|_{\mathscr{W}}^{p^+} - \lambda c_{10}\mathscr{C}^{p^+}\|u\|_{\mathscr{W}}^{p^+} + \lambda c_{\mathscr{C}}|\Omega| \text{ for all } u \in Y_j \text{ with } \|u\|_{\mathscr{W}} \geqslant 1.$$

Let $\mathscr{B}(t) = \dfrac{C_{\mathcal{A}}b_1}{p^-}t^{p^+} - \lambda c_{10}\mathscr{C}^{p^+}t^{p^+} + \lambda c_{\mathscr{C}}|\Omega|$. However, we can choose $\mathscr{C} > 0$ large enough, such that

$$\lim_{t\to+\infty} \mathscr{B}(t) = -\infty.$$

Therefore, there exists $\bar{t} \in (1, +\infty)$ such that

$$\mathscr{B}(t) < 0 \text{ for all } t \in [\bar{t}, +\infty).$$

Consequently, $\Psi_\lambda(u) < 0$ for all $u \in Y_j$ with $\|u\|_{\mathscr{W}} = \bar{t}$. Then, choosing $r_j = \bar{t}$ for all $j \in \mathbb{N}$, we have

$$b_j := \max\{\Psi_\lambda(u) : u \in Y_j,\ \|u\|_{\mathscr{W}} = r_j\} < 0.$$

We observe that we can change j_0 on other more large, if necessary, so that $\rho_j > r_j > 0$ for all $j \geqslant j_0$.

(g_3) Since $Y_j \cap Z_j \neq \emptyset$ and $0 < r_j < \rho_j$, so, we have

$$d_j := \inf\{\Psi_\lambda(u) : u \in Z_j,\ \|u\|_{\mathscr{W}} \leqslant \rho_j\} \leqslant b_j := \max\{\Psi_\lambda(u) : u \in Y_j,\ \|u\|_{\mathscr{W}} = r_j\} < 0.$$

From (f_0), we obtain $|F(x,u)| \leqslant c_{11}(|t| + |t|^{\vartheta(x)})$, for a constant positive c_{11}, and for all $(x,t) \in \Omega \times \mathbb{R}$. Consider, $\Sigma_1 : \mathscr{W} \to \mathbb{R}$ and $\Sigma_2 : \mathscr{W} \to \mathbb{R}$ defined by

$$\Sigma_1(u) = \int_\Omega \lambda c_{11} |u|^{\vartheta(x)}\, dx \text{ and } \Sigma_2(u) = \int_\Omega \lambda c_{11} |u|\, dx. \tag{2.30}$$

We have $\Sigma_i(0) = 0$, $i = 1, 2$, and they are weakly-strongly continuous. Let us denote

$$\eta_j = \sup\{|\Sigma_1(u)| : u \in Z_j,\ \|u\|_{\mathscr{W}} \leqslant 1\}, \quad \xi_j = \sup\{|\Sigma_2(u)| : u \in Z_j,\ \|u\|_{\mathscr{W}} \leqslant 1\}.$$

Thus, since the embedding $\mathscr{W} \hookrightarrow L^{\vartheta(\cdot)}(\Omega)$ is compact it follows that by Lemma 2.6 that $\lim_{j\to+\infty} \eta_j = \lim_{j\to+\infty} \xi_j = 0$.
Now, consider $v \in Z_j$ with $\|v\|_{\mathscr{W}} = 1$ and $0 < t < \rho_j$. Then, from (a_1), (a_2), (a_3), $(\mathcal{K})$, and (2.30), we obtain

$$\Psi_\lambda(tv) \geqslant -\lambda \int_\Omega F(x, tv)\, dx \geqslant -\Sigma_1(tv) - \Sigma_2(tv),$$

and since

$$\Sigma_1(tv) \leqslant t^{\vartheta^+_-} \Sigma_1(v) \text{ and } \Sigma_2(tv) = t\Sigma_2(v),$$

we achieve

$$\Psi_\lambda(tv) \geqslant -\rho_j^{\vartheta^+_-} \Sigma_1(v) - \rho_j \Sigma_2(v) \geqslant -\rho_j^{\vartheta^+_-} \eta_j - \rho_j \xi_j$$

for all $t \in (0, \rho_j)$ and $v \in Z_j$ with $\|v\|_{\mathscr{W}} = 1$. Thus, $d_j \geqslant -\rho_j^{\vartheta^+_-} \eta_j - \rho_j \xi_j$ and as $d_j < 0$ for all $j \geqslant j_0$, we conclude that $\lim_{j\to+\infty} d_j = 0$.
Therefore, the conditions of Theorem 3.6 are satisfied. Consequently, there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathscr{W}$ of weak solutions of problem such that $\Psi_\lambda(u_k) < 0$ and $\Psi_\lambda(u_k) \to 0$ as $k \to +\infty$ for $\lambda \in \left(0, \frac{c_{\mathcal{A}} b_0}{p^+}\right)$. □

2.2.5 Proof of Theorem 2.5

To prove Theorem 2.1 we need of Lemmas below and Theorem 3.3.

Lemma 2.9. *Suppose* (a_1)-(a_3) *and* (f_1) *holds. There is* $v \in \mathscr{W} \setminus \{0\}$, *such that* $\lim_{t \to +\infty} \Psi_\lambda(tv) = -\infty$.

Proof. The proof is as in the Lemma 2.1-(i). □

Lemma 2.10. *Suppose* (a_1)-(a_3), $(\mathcal{K})$, (f_0), $f(x,0)=0$, *and* $f(x,t) \geqslant 0$ *a.e.* $x \in \Omega$ *and for all* $t \geqslant 0$ *holds. Then, there exist* $\overline{\lambda} > 0$, *positive constants* C_λ *and* ρ_λ *for* $\lambda \in (0, \overline{\lambda})$ *such that* $\lim_{\lambda \to 0^+} \mathsf{C}_\lambda = +\infty$ *and* $\Psi_\lambda(u) > \mathsf{C}_\lambda > 0$ *whenever* $\|u\|_{\mathscr{W}} = \rho_\lambda$.

Proof. We consider $u \in \mathscr{W}$ with $\|u\|_{\mathscr{W}} > 1$. Then, using (2.29), Young's inequality, and Lemma 3.4, we conclude that

$$\Psi_\lambda(u) \geqslant \frac{c_{\mathcal{A}} b_0}{p^+} \|u\|^{p^-} - \lambda c_{13} \|u\|_{\mathscr{W}}^{\underline{\vartheta}^+} - \lambda c_9 |\Omega| \tag{2.31}$$

for constant positive c_{13}.
Let $\gamma \in \left(0, \frac{1}{\underline{\vartheta}^+ - p^-}\right)$ and $u \in \mathscr{W}$ such that $\|u\|_{\mathscr{W}} = \lambda^{-\gamma}$. We define $\rho_\lambda := \lambda^{-\gamma}$ and we observe that $\rho_\lambda > 1$ for λ small enough. Hence, from (2.31), we conclude

$$\Psi_\lambda(u) \geqslant \frac{c_{\mathcal{A}} b_0}{p^+} \lambda^{-\gamma p^-} - c_{13} \lambda^{1-\gamma \underline{\vartheta}^+} - \lambda c_9 |\Omega|.$$

Since $\gamma < \frac{1}{\underline{\vartheta}^+ - p^-}$ it follows that $-\gamma p^- < 1 - \gamma \underline{\vartheta}^+$. Thus $\mathsf{C}_\lambda := \frac{c_{\mathcal{A}} b_0}{p^+} \lambda^{-\gamma p^-} - c_{13} \lambda^{1-\gamma \underline{\vartheta}^+} - \lambda c_9 |\Omega| \to +\infty$ as $\lambda \to 0^+$. Hence, there exists $\overline{\lambda} > 0$ small enough such that $\mathsf{C}_\lambda > 0$ for all $\lambda \in (0, \overline{\lambda})$. Then we obtain that

$$\Psi_\lambda(u) \geqslant \mathsf{C}_\lambda > 0 = \Psi_\lambda(0)$$

for all $u \in \mathscr{W}$ with $\|u\|_{\mathscr{W}} = \rho_\lambda = \lambda^{-\gamma}$ and $\lambda \in (0, \overline{\lambda})$. Therefore, C_λ verifies the assertion of the Lemma. □

Proof of Theorem 2.5. From Lemma 2.2 the functional Ψ_λ satisfy the $(C)_c$ condition. Moreover, $\Psi_\lambda(0) = 0$. Then, by Lemma 2.9, Lemma 2.10, and Theorem 3.3 we obtain that there are constant $\overline{\lambda}$ and a nontrivial critical point u_λ for Ψ_λ with $\lambda \in (0, \overline{\lambda})$ such that

$$c = \Psi_\lambda(u_\lambda) \geqslant \mathsf{C}_\lambda.$$

Then, from (a_1), (a_2), $(\mathcal{K})$, (f_0), (2.29), Proposition 3.4, and Lemma 3.2, we achieve

$$\mathsf{C}_\lambda \leqslant \Psi_\lambda(u_\lambda) \leqslant \frac{C_{\mathcal{A}} b_1}{p^-} \max\{\|u_\lambda\|_{\mathscr{W}}^{p^+}, \|u_\lambda\|_{\mathscr{W}}^{p^-}\} + \lambda c_{13} \max\{\|u_\lambda\|_{\mathscr{W}}^{\vartheta^+}, \|u_\lambda\|_{\mathscr{W}}^{\vartheta^-}\} + \lambda c_9 |\Omega|. \tag{2.32}$$

Hence, taking $\lambda \to 0^+$ in (2.32) as $\mathsf{C}_\lambda \to +\infty$, we get

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{\mathscr{W}} = +\infty.$$

Therefore, we conclude the proof of Theorem 2.5. □

2.2.6 Proof of Theorem 2.6

To proof Theorem 2.6 it is enough to verify that Ψ_λ satisfies the hypotheses of Theorem 3.7.

Lemma 2.11. *Suppose (a_1)- (a_3), $(\mathcal{K})$, and (f_5) holds. Then the functional Ψ_λ is bounded from below and satisfies the (PS) condition.*

Proof. Let $u \in \mathscr{W}$ and suppose $\|u\|_{\mathscr{W}} > 1$. Then, from (a_2), (a_3), $(\mathcal{K})$, (f_5), Lemma 3.2, and Lemma3.3 for each $\lambda > 0$, we infer that

$$\Psi_\lambda(u) \geqslant \frac{c_{\mathcal{A}} b_0}{p^-} \|u\|_{\mathscr{W}}^{p^-} - \frac{\lambda c_{14}}{\underline{\mathfrak{m}}^-} \|u\|_{\mathscr{W}}^{\underline{\mathfrak{m}}^+}.$$

Since $\underline{\mathfrak{m}}^+ < p^-$, follows that Ψ_λ is coercive. Therefore, Ψ_λ is bounded from below.
Now, let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathscr{W}$ such that

$$\Psi_\lambda(u_k) \to c \text{ and } \Psi_\lambda'(u_k) \to 0 \text{ in } \mathscr{W}' \text{ as } k \to +\infty.$$

By coercivity of Ψ_λ, we have $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathscr{W}$. Hence up to a subsequence, still denoted by $(u_k)_{k\in\mathbb{N}}$, we have $u_0 \in \mathscr{W}$ such that $u_k \rightharpoonup u_0$ in $\mathscr{W}$ and from Lemma 3.4 we infer that $u_k \to u_0$ in $L^{\mathfrak{q}(\cdot)}(\Omega)$ and $u_k(x) \to u_0(x)$ a.e. $x \in \Omega$, as $k \to +\infty$.

Since $\Psi_\lambda'(u_k) \to 0$ in $\mathscr{W}$ as $k \to +\infty$ and the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathscr{W}$, we have that

$$\langle \Psi_\lambda'(u_k), u_k - u_0 \rangle \to 0 \textit{ as } k \to +\infty. \tag{2.33}$$

Now, using (f_5) and Hölder's inequality, we obtain that

$$\Big|\int_\Omega f(x,u_k)(u_k-u_0)\,dx\Big| \leqslant C_2\Big\||u_k|^{\mathtt{m}(x)-1}\Big\|_{L^{\frac{\mathtt{m}(\cdot)}{\mathtt{m}(\cdot)-1}}(\Omega)}\|u_k-u_0\|_{L^{\mathtt{m}(\cdot)}(\Omega)}.$$

Thus, taking into account that $u_k \to u_0$ in $L^{\mathtt{m}(\cdot)}(\Omega)$ as , $k \to +\infty$, we achieve

$$\int_\Omega f(x,u_k)(u_k-u_0)\,dx \to 0 \ as \ k \to +\infty. \tag{2.34}$$

Therefore, from (2.33) and (2.34), we conclude that

$$\langle \Phi'(u_k), u_k-u_0\rangle = \langle \Psi'_\lambda(u_k), u_k-u_0\rangle - \lambda\int_\Omega f(x,u_k)(u_k-u_0)\,dx \longrightarrow 0 \text{ as } k \to +\infty.$$

Since Φ' is of type (S_+) (see Lemma 3.3), we obtain $u_k \to u_0$ in $\mathscr{W}$ as $k \to +\infty$, concluding the proof of the Palais-Smale condition. □

Proof of Theorem 2.6. From (a_1) and (f_4) the Euler Lagrange functional Ψ_λ is even functional. Furthermore, by Lemma 2.11 Ψ_λ is bounded from below and satisfies the Palais-Smale condition. Hence the item $(I1)$ of Theorem 3.7 is verified. Let us show that Ψ_λ satisfies $(I2)$. Since $\mathscr{W}$ is a reflexive and separable Banach space, for each $j \in \mathbb{N}$, consider a j-dimensional linear subspace $\mathscr{W}_j \subset C_0^\infty(\Omega)$ of $\mathscr{W}$. We define

$$\mathbb{S}^j_{\mathcal{R}_4} = \{u \in \mathscr{W}_j : \|u\|_{\mathscr{W}} = \mathcal{R}_4\}$$

where $\mathcal{R}_4 > 0$ will be determined later on. Since $\mathscr{W}_j$ and $\mathbb{R}^j$ are isomorphic and $\mathbb{S}^j_{\mathcal{R}_4}$ is homeomorphic to the $(j-1)$-dimensional sphere $\mathbb{S}^{j-1}$ by an odd mapping, it has genus j, i.e., $\gamma(\mathbb{S}^j_{\mathcal{R}_4}) = j$.
Now, from (a_1), (a_2), $(\mathcal{K})$, and (f_5), we obtain

$$\Psi_\lambda(u) \leqslant \frac{C_{\mathcal{A}} b_1}{p^-}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy - \lambda C_0\int_\Omega |u|^{\mathtt{m}(x)}\,dx. \tag{2.35}$$

By Proposition 3.4, if $\|u\|_{\mathscr{W}} < 1$, we have $\rho_{\mathscr{W}}(u) \leqslant \|u\|^{p^-}_{\mathscr{W}}$, and $\rho_{\mathtt{m}(\cdot)}(u) \geqslant \|u\|^{\mathtt{m}^+}_{L^{\mathtt{m}(\cdot)}(\Omega)}$ for every $u \in \mathscr{W}$. Moreover, since $\mathscr{W}_j$ is a finite dimensional space, any norm in $\mathscr{W}_j$ is equivalent to each other. Thus, there exist a constant $C(j) > 0$ such that $C(j)\|u\|^{\mathtt{m}^+}_{\mathscr{W}} \leqslant \int_\Omega |u|^{\mathtt{m}(x)}\,dx$

for every $u \in \mathscr{W}_j$. Consequently, by (2.35), we get

$$\Psi_\lambda(u) \leqslant \|u\|_{\mathscr{W}}^{\underline{\mathfrak{m}}^+}\left(\frac{C_{\mathcal{A}} b_1}{p^-}\|u\|_{\mathscr{W}}^{p^- - \underline{\mathfrak{m}}^+} - \lambda C(j) C_0\right),$$

for every $u \in \mathscr{W}_j$ with $\|u\|_{\mathscr{W}} < 1$. Let $\mathcal{R}_5 \in (0,1)$ such that

$$\frac{C_{\mathcal{A}} b_1}{p^-}\mathcal{R}_5^{p^- - \underline{\mathfrak{m}}^+} < \lambda C(j) C_0.$$

Thus, for all $0 < \mathcal{R}_4 < \mathcal{R}_5$ and $u \in \mathbb{S}^j_{\mathcal{R}_4}$, we have that

$$\Psi_\lambda(u) \leqslant \mathcal{R}_4^{\underline{\mathfrak{m}}^+}\left(\frac{C_{\mathcal{A}} b_1}{p^-}\mathcal{R}_4^{p^- - \underline{\mathfrak{m}}^+} - \lambda C(j) C_0\right) \leqslant \mathcal{R}_5^{\underline{\mathfrak{m}}^+}\left(\frac{C_{\mathcal{A}} b_1}{p^-}\mathcal{R}_5^{p^- - \underline{\mathfrak{m}}^+} - C(j) C_0\right) < 0 = \Psi_\lambda(0).$$

Therefore, we conclude that

$$\sup_{u \in \mathbb{S}^j_{\mathcal{R}_4}} \Psi_\lambda(u) < 0 = \Psi_\lambda(0).$$

Thus the hypotheses of Theorem 3.7 are satisfied and we conclude that there exists a sequence nontrivial weak solutions u_j in $\mathscr{W}$ such that

$$\Psi_\lambda(u_j) \leqslant 0, \;\; \Psi'_\lambda(u_j) = 0 \text{ and } \|u_j\|_{\mathscr{W}} \to 0 \text{ as } j \to +\infty.$$

□

Chapter 3

Appendix

3.0.1 Lebesgue spaces with variable exponent

Proposition 3.1. *(a) The space $(L^{h(\cdot)}(\Omega), \|\cdot\|_{L^{h(\cdot)}(\Omega)})$ is a separable and reflexive Banach space;*

(b) Let $h_i \in C^+(\overline{\Omega})$ for $i = 1, \dots m$ with $\sum_{i=1}^{m} \frac{1}{h_i(x)} = 1$. If $u_i \in L^{h_i(\cdot)}(\Omega)$, then

$$\int_\Omega |u_1(x) \cdots u_m(x)|\, dx \leqslant C_H \|u_1\|_{L^{h_1(\cdot)}(\Omega)} \cdots \|u_m\|_{L^{h_m(\cdot)}(\Omega)}$$

where $C_H = \frac{1}{\underline{h}_1^-} + \frac{1}{\underline{h}_2^-} + \cdots + \frac{1}{\underline{h}_m^-}$.

Let h be a function in $C^+(\overline{\Omega})$. An important role in manipulating the generalized Lebesgue – Sobolev spaces is played by the $h(\cdot)$-modular of the space $L^{h(\cdot)}(\Omega)$, which is the convex function $\rho_{h(\cdot)} : L^{h(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{h(\cdot)}(u) = \int_\Omega |u(x)|^{h(x)} dx,$$

along any function u in $L^{h(\cdot)}(\Omega)$.
The function $\rho_{h(\cdot)}(\cdot)$ verifies the following properties:

(a) $\rho_{h(\cdot)}(u) \geqslant 0$ for every $u \in L^{h(\cdot)}(\Omega)$;

(b) $\rho_{h(\cdot)}(u) = 0$ if and only if $u = 0$;

(c) $\rho_{h(\cdot)}(u) = \rho_{h(\cdot)}(-u)$ for every $u \in L^{h(\cdot)}(\Omega)$;

(*d*) $\rho_{h(\cdot)}(u)$ is convex.

Every functional that satisfies the properties (*a*)-(*d*) is called *convex modular*.

The following result show relations between the norm $\|\cdot\|_{L^{h(\cdot)}(\Omega)}$ and modular $\rho_{h(\cdot)}(\cdot)$.

Proposition 3.2. *For $u \in L^{h(\cdot)}(\Omega)$ and $(u_k)_{k\in\mathbb{N}} \subset L^{h(\cdot)}(\Omega)$, we have:*

(*a*) *For $u \in L^{h(\cdot)}(\Omega) \setminus \{0\}$, $\zeta = \|u\|_{L^{h(\cdot)}(\Omega)}$ if and only if $\rho_{h(\cdot)}\left(\frac{u}{\zeta}\right) = 1$;*

(*b*) $\|u\|_{L^{h(\cdot)}(\Omega)} \geqslant 1 \Rightarrow \|u\|^{h^-}_{L^{h(\cdot)}(\Omega)} \leqslant \rho_{h(\cdot)}(u) \leqslant \|u\|^{h^+}_{L^{h(\cdot)}(\Omega)}$;

(*c*) $\|u\|_{L^{h(\cdot)}(\Omega)} \leqslant 1 \Rightarrow \|u\|^{h^+}_{L^{h(\cdot)}(\Omega)} \leqslant \rho_{h(\cdot)}(u) \leqslant \|u\|^{h^-}_{L^{h(\cdot)}(\Omega)}$;

(*d*) $\lim\limits_{k\to+\infty} \|u_k\|_{L^{h(\cdot)}(\Omega)} = 0 \Leftrightarrow \lim\limits_{k\to+\infty} \rho_{h(\cdot)}(u_k) = 0$;

(*e*) $\lim\limits_{k\to+\infty} \|u_k\|_{L^{h(\cdot)}(\Omega)} = +\infty \Leftrightarrow \lim\limits_{k\to+\infty} \rho_{h(\cdot)}(u_k) = +\infty$.

Proposition 3.3. *Let $h_1 \in L^\infty(\Omega)$ such that $1 \leqslant h_1(x)h_2(x) \leqslant +\infty$ for a.e. $x \in \Omega$. Let $u \in L^{h_2(\cdot)}(\Omega)$ and $u \neq 0$. Then*

$$\|u\|_{L^{h_1(\cdot)h_2(\cdot)}(\Omega)} \leqslant 1 \Rightarrow \|u\|^{h_1^+}_{L^{h_1(\cdot)h_2(\cdot)}(\Omega)} \leqslant \||u|^{h_1(x)}\|_{L^{h_2(\cdot)}(\Omega)} \leqslant \|u\|^{h_1^-}_{L^{h_1(\cdot)h_2(\cdot)}(\Omega)};$$
$$\|u\|_{L^{h_1(\cdot)h_2(\cdot)}(\Omega)} \geqslant 1 \Rightarrow \|u\|^{h_1^-}_{L^{h_1(\cdot)h_2(\cdot)}(\Omega)} \leqslant \||u|^{h_1(x)}\|_{L^{h_2(\cdot)}(\Omega)} \leqslant \|u\|^{h_1^+}_{L^{h_1(\cdot)h_2(\cdot)}(\Omega)}.$$

3.0.2 The functional space $\mathscr{W}$ and their properties

The following result is an consequence of [41, Theorem 3.2].

Corolllary 3.1. *Let $\Omega \subset \mathbb{R}^N$ a smooth bounded domain, $s \in (0,1)$, $p(x,y)$ and $p(x)$ be continuous variable exponents such that (q_1)-(q_2) be satisfied and that $s\underline{p}^+ < N$. Then, for all $r : \overline{\Omega} \to (1,+\infty)$ a continuous function such that $p_s^\star(x) > r(x)$ for all $x \in \overline{\Omega}$, the space $W^{s,p(\cdot,\cdot)}(\Omega)$ is continuously and compactly embedding in $L^{r(\cdot)}(\Omega)$.*

Lemma 3.1. *Assume Ω be a smooth bounded domain in $\mathbb{R}^N$. Let $p(x) := p(x,x)$ for all $x \in \mathbb{R}^N$ with $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying (p_1) and $p_s^\star(x) > p(x)$ for $x \in \mathbb{R}^N$. Then there exists $\zeta_1 > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leqslant \frac{1}{\zeta_1}[u]^{s,p(\cdot,\cdot)}_{\mathbb{R}^N} \text{ for all } u \in \mathscr{W}.$$

Proof. We consider the set $\mathbb{M} = \{u \in \mathscr{W}; \|u\|_{L^{p(\cdot)}(\Omega)} = 1\}$. Then to prove this Lemma it suffices to prove that

$$\inf_{u\in\mathbb{M}} [u]_{\mathbb{R}^N}^{s,p(\cdot,\cdot)} = \zeta_1 > 0.$$

Initially, we observe that $\zeta_1 \geqslant 0$ and we prove ζ_1 is attained in $\mathbb{M}$. Let $(u_k)_{k\in\mathbb{N}} \subset \mathbb{M}$ be a minimizing sequence, that is, $[u_k]_{\mathbb{R}^N}^{s,p(\cdot,\cdot)} \to \zeta_1$ as $k \to +\infty$. This implies that $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathscr{W}$ and $L^{p(\cdot)}(\Omega)$, therefore in $W^{s,p(\cdot,\cdot)}(\Omega)$. Consequently up to a subsequence $u_k \rightharpoonup u_0$ in $W^{s,p(\cdot,\cdot)}(\Omega)$ as $k \to +\infty$. Thus, from Corollary 3.1, it follows that $u_k \to u_0$ in $L^{p(\cdot)}(\Omega)$ as $k \to +\infty$. We extend u_0 to $\mathbb{R}^N$ by setting $u_0(x) = 0$ on $x \in \mathbb{R}^N \setminus \Omega$. This implies $u_k(x) \to u_0(x)$ a.e. $x \in \mathbb{R}^N$ as $k \to +\infty$. Hence by using Fatou's Lemma, we have

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \leqslant \liminf_{k\to+\infty} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_k(x) - u_k(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy$$

which implies that

$$[u_0]_{\mathbb{R}^N}^{s,p(\cdot,\cdot)} \leqslant \liminf_{k\to+\infty} [u_k]_{\mathbb{R}^N}^{s,p(\cdot,\cdot)} = \zeta_1,$$

and thus $u_0 \in \mathscr{W}$. Moreover, $\|u_0\|_{L^{p(\cdot)}(\Omega)} = 1$ and then $u_0 \in \mathbb{M}$. In particular, $u_0 \neq 0$ and $[u_0]_{\mathbb{R}^N}^{s,p(\cdot,\cdot)} = \zeta_1 > 0$. □

Lemma 3.2. *Assume Ω be a smooth bounded domain in $\mathbb{R}^N$. Let $p(x) := p(x,x)$ for all $x \in \mathbb{R}^N$ with $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying (p_1) and $p_s^\star(x) > p(x)$ for $x \in \mathbb{R}^N$. Assume that $r : \overline{\Omega} \to (1,+\infty)$ is a continuous function. Thus, the space $(\mathscr{W}, \|\cdot\|_{\mathscr{W}})$ is continuously and compactly embedding in $L^{r(\cdot)}(\Omega)$ for all $r(x) \in (1, p_s^\star(x))$ for all $x \in \overline{\Omega}$.*

Proof. First we observe that by Lemma 3.1, for all $u \in \mathscr{W}$, we get

$$\|u\|_{W^{s,p(\cdot,\cdot)}(\Omega)} \leqslant \|u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{\mathscr{W}} \leqslant \left(\frac{1}{\zeta_1} + 1\right)\|u\|_{\mathscr{W}}, \tag{3.1}$$

that is, $\mathscr{W}$ is continuously embedded in $W^{s,p(\cdot,\cdot)}(\Omega)$, and by Corollary 3.1 we conclude that $\mathscr{W}$ is continuously embedded in $L^{r(\cdot)}(\Omega)$. To prove that the embedding above is compact we consider $(u_k)_{k\in\mathbb{N}}$ a bounded sequence in $\mathscr{W}$. Using (3.1), follows that $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{s,p(\cdot,\cdot)}(\Omega)$. Hence by Corollary 3.1, we infer that there exists $u_0 \in L^{r(\cdot)}(\Omega)$ such that up to a subsequence $u_k \to u_0$ in $L^{r(\cdot)}(\Omega)$ as $k \to +\infty$. Since that $u_k = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ for all $k \in \mathbb{N}$, so we define $u_0 = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ and obtain that the convergence occurs in $L^{r(\cdot)}(\Omega)$. This completes the proof this Lemma. □

An important role in manipulating the fractional Sobolev spaces with variable exponent is played by the $(s,p(\cdot,\cdot))$-convex modular function $\rho_{\mathscr{W}}:\mathscr{W}\to\mathbb{R}$ defined by

$$\rho_{\mathscr{W}}(u)=\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy.$$

The following proposition show the relationship between the norm $\|\cdot\|_{\mathscr{W}}$ and the $\rho_{\mathscr{W}}$ convex modular function.

Proposition 3.4. *For $u\in\mathscr{W}$ and $(u_k)_{k\in\mathbb{N}}\subset\mathscr{W}$, we have:*

(a) *For $u\in\mathscr{W}\setminus\{0\}$, $\zeta=\|u\|_{\mathscr{W}}$ if and only if $\rho_{\mathscr{W}}\big(\frac{u}{\zeta}\big)=1$;*

(b) $\|u\|_{\mathscr{W}}\geqslant 1\Rightarrow\|u\|_{\mathscr{W}}^{p^-}\leqslant\rho_{\mathscr{W}}(u)\leqslant\|u\|_{\mathscr{W}}^{p^+}$;

(c) $\|u\|_{\mathscr{W}}\leqslant 1\Rightarrow\|u\|_{\mathscr{W}}^{p^+}\leqslant\rho_{\mathscr{W}}(u)\leqslant\|u\|_{\mathscr{W}}^{p^-}$;

(d) $\lim\limits_{k\to+\infty}\|u_k-u\|_{\mathscr{W}}=0\Leftrightarrow\lim\limits_{k\to+\infty}\rho_{\mathscr{W}}(u_k-u)=0$;

(e) $\lim\limits_{k\to+\infty}\|u_k\|_{\mathscr{W}}=+\infty\Leftrightarrow\lim\limits_{k\to+\infty}\rho_{\mathscr{W}}(u_k)=+\infty$.

3.0.3 Variational theorems

The results below can be see in [26, Theorem 1.2] and [76, Theorem 2.4] respectively.

Theorem 3.1. *Let X a reflexive Banach space and suppose that a functional $\Psi:X\to\mathbb{R}$ is weakly lower-semicontinuous (weakly l.s.c.) and coercive. Then Ψ is bounded from below and there exists $u_0\in X$ such that*

$$\Psi(u_0)=\inf_{u\in X}\Psi(u).$$

Theorem 3.2. *Let X be a Banach space, $\Psi\in C^1(X,\mathbb{R})$ bounded below, $v\in X$ and ε, $\delta>0$. If*

$$\Psi(v)\leqslant\inf_X\Psi+\varepsilon$$

then there is $u\in X$ such that

$$\Psi(u)\leqslant\inf_X\Psi+2\varepsilon,\ \ \|\Psi'(u)\|_{X'}<\frac{8\varepsilon}{\delta},\ \ \|u-v\|\leqslant 2\delta.$$

The condition $(C)_c$, introduced by Cerami in [22, 23], is a little more weak version of the Palais–Smale $(PS)_c$ condition, a condition more common that we find in the literature. Thus, since the Deformation Theorem is still valid under the Cerami a condition it follows that the Mountain Pass Theorem, Fountain Theorem, and Dual Fountain Theorem under Palais–Smale $(PS)_c$ and $(PS)_c^\star$ condition holds true also under this compactness.

The results below Mountain Pass Theorem, "$\mathbb{Z}_2$ − *symmetric*" version (for even functionals) Mountain Pass Theorem, Fountain Theorem and Dual Fountain Theorem can be seen respectively in [25, Theorem I], [70, Theorem 9.12], [58, Theorem 2.9], and [9, Theorem 2].

Theorem 3.3. *Let X be a real Banach space, let $\Psi : X \to \mathbb{R}$ be a functional of class $C^1(X, \mathbb{R})$ that satisfies the $(C)_c$ condition for any $c > 0$, $\Psi(0) = 0$, and the following conditions hold:*

(i) There exist positive constants ρ and $\mathcal{R}$ such that $\Psi(u) \geqslant \mathcal{R}$ for any $u \in X$ with $\|u\|_X = \rho$;

(ii) There exists a function $e \in X$ such that $\|e\|_X > \rho$ and $\Psi(e) < 0$.

Then, the functional Ψ has a critical value $c \geqslant \mathcal{R}$, that is, there exists $u \in X$ such that $\Psi(u) = c$ and $\Psi'(u) = 0$ in X'.

Theorem 3.4. *Assume that X has infinite dimension and let $\Psi \in C^1(X, \mathbb{R})$ be a functional satisfying the $(C)_c$ condition as well as the following properties*

i) $\Psi(0) = 0$, and there exist two constants $r, \rho > 0$ such that $\Psi_{|\partial B_r} \geqslant \rho$;

ii) Ψ is even;

iii) For all finite dimensional subspace $\widehat{X} \subset X$ there exists $\mathcal{R} = \mathcal{R}(\widehat{X}) > 0$ such that

$$\Psi(u) \leqslant 0 \text{ for all } u \in \widehat{X} \setminus B_{\mathcal{R}}(\widehat{X})$$

where $B_{\mathcal{R}}(\widehat{X}) = \{u \in \widehat{X} : \|u\|_X < \mathcal{R}\}$.

Then Ψ possesses an unbounded sequence of critical values.

Let X be a real, reflexive, and separable Banach space, it is known ([31, Chapter 4], or [78, Section 17]) or [36] that for a separable and reflexive Banach space there exist sequence $(e_l)_{l\in\mathbb{N}} \subset X$ and $(e_l^\star)_{\mathbb{N}} \subset X'$ such that

$$X = \overline{span\{e_l : l = 1, 2, \dots\}},\ X' = \overline{span\{e_l^\star : l = 1, 2, \dots\}}^{\omega^\star},$$

and

$$\langle e_i^\star, e_l \rangle = \begin{cases} 1 & \text{se} \quad i = l, \\ 0 & \text{se} \quad i \neq l. \end{cases}$$

We denote

$$X_l = span\{e_l\},\ Y_j = \bigoplus_{l=1}^{j} X_l = span\{e_1, \ldots, e_j\},\ \text{and}\ Z_j = \overline{\bigoplus_{l=j}^{\infty}} X_l = \overline{span\{e_j, e_{j+1} \ldots\}}.$$

Theorem 3.5. *Assume*

(h_1) *X is a Banach space, $\Psi \in C^1(X, \mathbb{R})$ is an even functional.*
If for every $j \in \mathbb{N}$ there exist $\rho_j > r_j > 0$ such that:

(h_2) $b_j := \inf\{\Psi(u) : u \in Z_j, \|u\|_X = r_j\} \to +\infty$ *as* $j \to +\infty$;

(h_3) $a_j := \max\{\Psi(u) : u \in Y_j, \|u\|_X = \rho_j\} \leqslant 0$;

(h_4) *Ψ satisfies the $(C)_c$ condition for every $c > 0$.*

Then Ψ has a sequence of critical points $(u_j)_{j \in \mathbb{N}}$ such that $\Psi(u_j) \to +\infty$.

Definition 3.1. *Let X be a separable and reflexive Banach space, $\Psi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We say that Ψ satisfies the $(C)_c^\star$ condition (with respect to Y_k), if any sequence $(u_k)_{k \in \mathbb{N}} \subset X$ for which $u_k \in Y_k$, for any $k \in \mathbb{N}$, $\Psi(u_k) \to c$ and $\|\Psi'_{|Y_k}(u_k)\|_{X'}(1 + \|u_k\|_X) \to 0$, as $k \to +\infty$, contain a subsequence converging to a critical point of Ψ.*

Theorem 3.6. *Suppose (h_1). If for each $j \geqslant j_0$ there exist $\rho_j > r_j > 0$ such that*

(g_1) $a_j = \inf\{\Psi(u) : u \in Z_j,\ \|u\|_X = \rho_j\} \geqslant 0$;

(g_2) $b_j = \max\{\Psi(u) : u \in Y_j,\ \|u\|_X = r_j\} < 0$;

(g_3) $d_j = \inf\{\Psi(u) : u \in Z_j,\ \|u\|_X \leqslant \rho_j\} \to 0$, *as* $j \to +\infty$;

(g_4) *Ψ satisfies the $(C)_c^\star$ condition for every $c \in [d_{j_0}, 0)$.*

Then Ψ has a sequence of negative critical values converging to 0.

3.0.4 Krasnoselskii's genus

We will present some basic notions on Krasnoselskii's genus and introducing a critical point theorem related to the new version of the symmetric Mountain Pass Theorem studied by Kajikiya, see [49, Theorem 1].

Definition 3.2. *Let X be a real Banach space and B a subset of X. B is said to be symmetric if $u \in B$ implies $-u \in B$. For a closed symmetric set B which does not contain the origin, we define a Krasnoselskii genus $\gamma(B)$ of B by the smallest integer j such that there exists an odd continuous mapping from B to $\mathbb{R}^j\backslash\{0\}$. If there does not exist such a j, we define $\gamma(B) = +\infty$. Moreover, we set $\gamma(\emptyset) = 0$.*

Let us consider the following set,

$$\Gamma_j = \{B_j \subset X : B_j \text{ is closed, symmetric and } 0 \notin B_j \text{ such that the genus } \gamma(B_j) \geqslant j\}$$

Theorem 3.7. *Let X be an infinite-dimensional space, $B \in \Gamma_j$, and $\Psi \in C^1(X, \mathbb{R})$ satisfying the following conditions:*

$(I1)$ $\Psi(u)$ is even, bounded from below, $\Psi(0) = 0$ and $\Psi(u)$ satisfies the Palais-Smale condition;

$(I2)$ For each $j \in \mathbb{N}$, there exists an $B_j \in \Gamma_j$ such that $\sup_{u \in B_j} \Psi(u) < 0$.

Then either $(R1)$ or $(R2)$ below holds

$(R1)$ There exists a sequence $(u_j)_{j\in\mathbb{N}}$ such that $\Psi'(u_j) = 0$, $\Psi(u_j) < 0$ and $(u_j)_{j\in\mathbb{N}}$ converges to zero;

$(R2)$ There exist two sequences $(u_j)_{j\in\mathbb{N}}$ and $(v_j)_{j\in\mathbb{N}}$ such that $\Psi'(u_j) = 0$, $\Psi(u_j) < 0$, $u_j \neq 0$, $\lim_{j\to+\infty} u_j = 0$, $\Psi'(v_j) = 0$, $\Psi(v_j) < 0$, $\lim_{j\to+\infty} v_j = 0$, and $(v_j)_{j\in\mathbb{N}}$ converges to a non-zero limit.

Remark 3.1. *From Theorem 3.7 we have a sequence $(u_j)_{j\in\mathbb{N}}$ of critical points such that $\Psi(u_j) \leqslant 0$, $u_j \neq 0$, and $\lim_{j\to+\infty} u_j = 0$.*

3.0.5 The (S_+) condition

Lemma 3.3. *Assume that (a_1)-(a_3), and $(\mathcal{K})$ is hold. We consider the functional $\Phi : \mathscr{W} \to \mathbb{R}$ defined by*

$$\Phi(u) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \mathscr{A}(u(x) - u(y))K(x, y)\, dx\, dy \text{ for all } u \in \mathscr{W},$$

has the following properties:

(i) The functional Φ is well defined on $\mathscr{W}$, is of class $C^1(\mathscr{W}, \mathbb{R})$, and its Gâteaux derivative is given by

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{A}(u(x) - u(y))(v(x) - v(y))K(x, y)\, dx\, dy \text{ for all } u, v \in \mathscr{W};$$

(ii) The functional Φ is weakly lower semicontinuous, that is, $u_k \rightharpoonup u$ in $\mathscr{W}$ as $k \to +\infty$ implies that $\Phi(u) \leqslant \liminf_{k\to+\infty} \Phi(u_k)$;

(iii) The functional $\Phi' : \mathscr{W} \to \mathscr{W}'$ is an operator of type (S_+) on $\mathscr{W}$, that is, if

$$u_k \rightharpoonup u \text{ in } \mathscr{W} \text{ and } \limsup_{k\to+\infty} \langle \Phi'(u_k), u_k - u\rangle \leqslant 0, \tag{3.2}$$

then $u_k \to u$ in $\mathscr{W}$ as $k \to +\infty$.

Proof. (i) Using standard arguments proof this item.
(ii) From (i) the functional Φ is of class $C^1(\mathscr{W}, \mathbb{R})$, and by hypothesis (a_1), the functional Φ' is monotone. Thus, by [51, Lemma 15.4] we conclude that $\langle \Phi'(u), u_k - u\rangle + \Phi(u) \leqslant \Phi(u_k)$ for all $k \in \mathbb{N}$.
Thus, since $u_k \rightharpoonup u$ in $\mathscr{W}$, as $k \to +\infty$ we obtain $\Phi(u) \leqslant \liminf_{k\to+\infty} \Phi(u_k)$. That is, the functional Φ is weakly lower semicontinuous.

(iii) Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathscr{W}$ as in the statement. We have that by (i), Φ' is a continuous functional. Therefore,

$$\lim_{k\to+\infty} \langle \Phi'(u), u_k - u\rangle = 0. \tag{3.3}$$

Now, we observe that

$$\langle \Phi'(u_k) - \Phi'(u), u_k - u\rangle = \langle \Phi'(u_k), u_k - u\rangle - \langle \Phi'(u), u_k - u\rangle \text{ for all } k \in \mathbb{R}. \tag{3.4}$$

Thus by (3.2), (3.3), and (3.4), we infer

$$\limsup_{k\to+\infty} \langle \Phi'(u_k) - \Phi'(u), u_k - u\rangle \leqslant 0. \tag{3.5}$$

Furthermore, since Φ is strictly convex by hypothesis (a_1), Φ' is monotone (see [51, Lemma 15.4]), we have

$$\langle \Phi'(u_k) - \Phi'(u), u_k - u\rangle \geqslant 0 \quad \text{for all } k \in \mathbb{N}. \tag{3.6}$$

Therefore, by (3.5) and (3.6), we infer that

$$\lim_{k\to+\infty} \langle \Phi'(u_k) - \Phi'(u), u_k - u\rangle = 0. \tag{3.7}$$

Consequently, by (3.3), (3.4), and (3.7), we conclude

$$\lim_{k\to+\infty} \langle \Phi'(u_k), u_k - u\rangle = 0. \tag{3.8}$$

Since Φ is strictly convex, we get

$$\Phi(u) + \langle \Phi'(u_k), u_k - u\rangle \geqslant \Phi(u_k) \text{ for all } k \in \mathbb{N}. \tag{3.9}$$

Thus, by (3.8) and (3.9), we have

$$\Phi(u) \geqslant \lim_{k\to+\infty} \Phi(u_k). \tag{3.10}$$

Since Φ is weakly lower semicontinuous (see (ii)) and by (3.10), we conclude that

$$\Phi(u) = \lim_{k\to+\infty} \Phi(u_k). \tag{3.11}$$

On the other hand, by (3.7) the sequence $(\mathcal{U}_k(x,y))_{k\in\mathbb{N}}$ converge to 0 in $L^1(\mathbb{R}^N \times \mathbb{R}^N)$ as $k \to +\infty$, where

$$\mathcal{U}_k(x,y) := [\mathcal{A}(u_k(x) - u_k(y)) - \mathcal{A}(u(x) - u(y))[(u_k(x) - u_k(y)) - (u(x) - u(y)]K(x,y) \geqslant 0.$$

Hence there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ of $(u_k)_{k\in\mathbb{N}}$ such that

$$\mathcal{U}_{k_j}(x,y) \to 0 \text{ a.e. } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ as } j \to +\infty. \tag{3.12}$$

We denoted $\mu_j(x,y) = u_{k_j}(x) - u_{k_j}(y)$ and $\mu(x,y) = u(x) - u(y)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.
Claim a. If $\mathcal{U}_{k_j}(x,y) \to 0$ a.e. $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, then $\mu_j(x,y) \to \mu(x,y)$ as $j \to +\infty$ for almost all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.
Indeed, fixed $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $x \neq y$ we suppose by contradiction that the sequence $(\mu_j(x,y))_{j\in\mathbb{N}}$ is unbounded for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ fixed. Using (3.12) we get $\mathcal{U}_{k_j}(x,y) \to 0$ as $j \to +\infty$ in $\mathbb{R}$, consequently there is $M > 0$ such that for all $j \in \mathbb{N}$

$$\Big|[\mathcal{A}(\mu_j(x,y)) - \mathcal{A}(\mu(x,y))](\mu_j(x,y) - \mu(x,y))K(x,y)\Big| \leqslant M. \tag{3.13}$$

Thus, denoting

$$V_{\mathcal{A}} := [\mathcal{A}(\mu_j(x,y))\mu_j(x,y) + \mathcal{A}(\mu(x,y))\mu(x,y)]K(x,y)\text{for all } j \in \mathbb{R},$$

we get from (3.13) that

$$V_{\mathcal{A}} \leqslant M + \mathcal{A}(\mu(x,y))\mu_j(x,y)K(x,y) + \mathcal{A}(\mu_j(x,y))\mu(x,y)K(x,y)\text{for all } j \in \mathbb{R}.$$

So using (a_2) and $(\mathcal{K})$ in inequality above, we have for all $j \in \mathbb{R}$ that,

$$\begin{aligned} c_{\mathcal{A}} b_0 \frac{|\mu_j(x,y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + c_{\mathcal{A}} b_0 \frac{|\mu(x,y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \leqslant & M + C_{\mathcal{A}} b_1 \frac{|\mu(x,y)|^{p(x,y)-1}|\mu_j(x,y)|}{|x-y|^{N+sp(x,y)}} \\ & + C_{\mathcal{A}} b_1 \frac{|\mu_j(x,y)|^{p(x,y)-1}|\mu(x,y)|}{|x-y|^{N+sp(x,y)}}. \end{aligned} \tag{3.14}$$

Dividing (3.14) by $|\mu_j(x,y)|^{p(x,y)}$, we achieve

$$\begin{aligned} \frac{c_{\mathcal{A}} b_0}{|x-y|^{N+sp(x,y)}} + \frac{c_{\mathcal{A}} b_0 |\mu(x,y)|^{p(x,y)}}{|\mu_j(x,y)|^{p(x,y)}|x-y|^{N+sp(x,y)}} \leqslant & \frac{M}{|\mu_j(x,y)|^{p(x,y)}} \\ & + \frac{C_{\mathcal{A}} b_1 |\mu(x,y)|^{p(x,y)-1}}{|\mu_j(x,y)|^{p(x,y)-1}|x-y|^{N+sp(x,y)}} \\ & + \frac{C_{\mathcal{A}} b_1 |\mu(x,y)|}{|\mu_j(x,y)||x-y|^{N+sp(x,y)}} \end{aligned} \tag{3.15}$$

for all $j \in \mathbb{R}$. Since we are supposing that the sequence $(\mu_j(x,y))_{j\in\mathbb{N}}$ is unbounded, we can assume that $|\mu_j(x,y)| \to +\infty$ as $j \to +\infty$, then by (3.15) we obtain $c_{\mathcal{A}} b_0 \leqslant 0$ which is an contradiction.

Therefore, the sequence $(\mu_j(x,y))_{j\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and up to a subsequence $(\mu_j(x,y))_{j\in\mathbb{N}}$ converges to some $\nu \in \mathbb{R}$. Thus we obtain $\mu_j(x,y) \to \nu$ as $j \to +\infty$. Thence denoting

$$\mathcal{U}(x,y) := [\mathcal{A}(\nu) - \mathcal{A}(\mu(x,y))](\nu - (\mu(x,y))K(x,y)$$

and using (a_1) we conclude that

$$\mathcal{U}_{k_j}(x,y) \to \mathcal{U}(x,y) \text{ as } j \to +\infty. \tag{3.16}$$

Consequently, by (3.12) and (3.16), we get

$$\mathcal{U}(x,y) = [\mathcal{A}(\nu) - \mathcal{A}(\mu(x,y))](\nu - (\mu(x,y))K(x,y) = 0. \tag{3.17}$$

In this way, by strictly convexity of $\mathscr{A}$, (3.17) this occurs only if $\nu = \mu(x,y) = u(x) - u(y)$. Therefore, by uniqueness of limit

$$u_{k_j}(x) - u_{k_j}(y) = \mu_j(x,y) \to \mu(x,y) = u(x) - u(y) \text{ in } \mathbb{R} \text{ as } j \to +\infty \tag{3.18}$$

for almost all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

Now we consider the sequence $(g_{k_j})_{j\in\mathbb{N}}$ in $L^1(\mathbb{R}^N \times \mathbb{R}^N)$ defined pointwise for all $j \in \mathbb{N}$ by

$$g_{k_j}(x,y) := \left[\frac{1}{2}\bigg(\mathscr{A}(\mu_j(x,y)) + \mathscr{A}(\mu(x,y))\bigg) - \mathscr{A}\left(\frac{\mu_j(x,y) - \mu(x,y)}{2}\right)\right]K(x,y).$$

By convexity the map $\mathscr{A}$ (see (a_1)), $g_{k_j}(x,y) \geqslant 0$ for almost all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Furthermore, by continuity of map $\mathscr{A}$ (see (a_1)) and (3.18), we have

$$g_{k_j}(x,y) \to \mathscr{A}(\mu(x,y))K(x,y) \text{ as } j \to +\infty \quad \text{for all } \ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Therefore, using this above information, (3.11), and Fatou's Lemma, we get

$$\Phi(u) \leqslant \liminf_{j\to+\infty} g_{k_j}(x,y) = \Phi(u) - \limsup_{j\to+\infty} \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathscr{A}\left(\frac{\mu_j(x,y) - \mu(x,y)}{2}\right)K(x,y)\,dx\,dy.$$

Then

$$\limsup_{j\to+\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathscr{A}\left(\frac{\mu_j(x,y)-\mu(x,y)}{2}\right)K(x,y)\,dx\,dy\leqslant 0. \tag{3.19}$$

On the other hand, by (a_2), (a_3), $(\mathcal{K})$, and Proposition 3.4, we infer that

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathscr{A}\left(\frac{\mu_j(x,y)-\mu(x,y)}{2}\right)K(x,y)\,dx\,dy \geqslant&\frac{c_{\mathcal{A}}b_0}{2^{p^-}p^+}\min\left\{\|u_{k_j}-u\|_{\mathscr{W}}^{p^-},\|u_{k_j}-u\|_{\mathscr{W}}^{p^+}\right\}\\ \geqslant&0 \text{ for all } j\in\mathbb{N}.\end{aligned} \tag{3.20}$$

Consequently, by (3.19) and (3.20), we achieve

$$\lim_{j\to+\infty}\|u_{k_j}-u\|_{\mathscr{W}}=0.$$

Therefore, we can conclude that $u_{k_j}\to u$ in $\mathscr{W}$ as $j\to+\infty$. Since $(u_{k_j})_{j\in\mathbb{N}}$ is an arbitrary subsequence of $(u_k)_{k\in\mathbb{N}}$, this shows that $u_k\to u$ as $k\to+\infty$ in $\mathscr{W}$, as required. □

Bibliography

[1] C. O. Alves, M. C. Ferreira, *Nonlinear perturbations of a $p(x)$-Laplacian equation with critical growth in* $\mathbb{R}^N$, Math. Nachr. 287 (2014), 849–868.

[2] A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and apllications,* J. Functional Analysis. 14 (1973), 349-381.

[3] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, J. J. Toledo-Melero, *Nonlocal diffusion problems,* volume 165 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid (2010).

[4] E. Azroul, A. Benkirane, M. Shimi, *Eigenvalue problems involving the fractional $p(x)$-Laplacian fractional operator,* Adv. Oper. Theory 4 (2019), 539-555.

[5] G. Autuori, P. Pucci, *Existence of entire solutions for a class of quasilinear elliptic equations,* Nonlinear Differential Equations and Appl. 20 (2013), 977–1009.

[6] A. Bahrouni, *Trudinger–Moser type inequality and existence of solution for perturbed nonlocal elliptic operators with exponential nonlinearity,* Comm. Pure Appl. Anal. 16 (2017), 243–252.

[7] A. Bahrouni, *Comparasion and sub-supersolution principles for the fractional $p(x)$-Laplacian,* J. Math. Anal. Appl. 458 (2018), 1363-1372.

[8] A. Bahrouni, V. D. Rădulescu, *On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent,* Discrete Contin. Dyn. Syst. 11 (2018), 379-389.

[9] T. Bartsch, M. Willem, *On an elliptic equation with concave and convex nonlinearities,* Proc. Amer. Math. Soc. (11) 123 (1995), 3555–3561.

[10] G. Bin, *On superlinear $p(x)$-Laplacian like problem without Ambrsetti and Rabinowitz condition,* Bull. Korean Math. Soc. 51 (2014), 409-421.

[11] G. M. Bisci, V. D. Rădulescu, *Ground state solutions of scalar field fractional Schrödinger equations,* Calc. Var. 54 (2015), 2985–3008.

[12] G. M. Bisci, D. D. Repovš, *Multiple solutions for elliptic equations involving a general operator in divergence form,* Ann. Acad. Sci. Fenn. Math. 39 (2014), 259-273.

[13] L. Brasco, E. Parini, M. Squassina, *Stability of variational eigenvalues for the fractional p-Laplacian,* Discrete Cont. Dyn. Sys. **36** (2016), 1813–1845.

[14] L. Brasco, S. Squassina, Y. Yang, *Global compactness results for nonlocal problems,* Discrete and Continuous Dynamical Systems **11** (2018), 391-424.

[15] H. Brezis, H-M. Nguyen, *Non-local functionals related to the total variation and applications,* Image Processing. Ann. PDE 4 (2018), 77 pp.

[16] K. J. Brown, T. F. Wu, *A fibering map approach to a semilinear elliptic boundary value problem,* Electron. J. Differential Equations **69** (2007), 1–9.

[17] K. J. Brown, Y. Zhang, *The Nehari manifold for semilinear elliptic equation with a sign-changing weight function,* J. Differential Equation **193** (2003), 481–499.

[18] C. Bucur, E. Valdinoci, *Nonlocal diffusion and applications,* Lect. Notes Unione Mat. Ital., vol.20, Springer/Unione Matematica Italiana, Cham/Bologna (2016).

[19] L. Caffarelli, *Nonlocal equations, drifts and games,* Nonlinear Partial Differential Equations 7 (2012), 37–52.

[20] M. L. M. Carvalho, J. V. A. Gonçalves, E.D. da Silva, *On quasilinear elliptic problems without the Ambrosetti-Rabinowitz condition,* J. Math. Anal. Appl. 426 (2015), 466-483.

[21] M. L. M. Carvalho, D. E. da Silva, C. Goulart, *Quasilinear elliptic problems with concave-convex nonlinearities,* In Commun. Contemp. Math. **19** (2017), 1650050.

[22] G. Cerami, *An existence criterion for the critical points on unbounded manifolds,* Istit. Lombardo Accad. Sci. Lett. Rend. A 112 (1978), 332–336.

[23] G. Cerami, *On the existence of eigenvalues for a nonlinear boundary value problem,* Ann. Mat. Pura Appl. (4) 124 (1980), 161–179.

[24] F. Colasuonno, P. Pucci, C. Varga, *Multiple solutions for an eigenvalue problem involving p–Laplacian type operators,* Nonlinear Anal 75 (2012), 4496–4512.

[25] D. G. Costa, C. A. Magalhães, *Existence results for perturbations of the p-Laplacian,* Nonlinear Anal. 24 (1995), 409-418.

[26] D. G. Costa, *An Invitation to Variational Methods in Differential Equations,* Boston, Basel, Berlin : Birkhäuser, cop. 2007.

[27] L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents,* Lecture Notes in Mathematics, Vol. 2017, Springer-Verlag, Heidelberg (2011).

[28] P. Drábek, S. I. Pohozaev, *Positive solutions for the p−Laplacian: application of the fibering method,* Proc. Roy. Soc. Edinburgh **127A** (1997), 703–726.

[29] D. E. Edmunds, J. Ráskosník, *Sobolev embeddings with variable exponent,* Studia Math. 143 (2000), 267-293.

[30] E. Emmrich, D. Puhst, *Measure-valued and weak solutions to the nonlinear peridynamic model in nonlocal elastodynamics,* Nonlinearity 28 (2014), 285–307.

[31] M. Fabian, P. Habala, P Hajék, V. Montesinos, V. Zizler, *Banach Space Theory: The Basis for Linear and Nonlinear Analysis,* New York Springer (2011).

[32] X. Fan, X. Han, *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$*, Nonlinear Anal. 59 (2004), 173-188.

[33] X. L. Fan, *Remarks on eigenvalue problems involving the $p(x)$-Laplacian,* J. Math. Anal. Appl. 352 (2009), 85-98.

[34] X. Fan, Q. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem,* Nonlinear Anal. 52 (2003), 1843-1852.

[35] A. Fiscella, G. M. Bisci, R. Servadei, *Multiplicity results for fractional Laplace problems with critical growth,* Manuscripta Math. 155 (2018), 369-388.

[36] S. Fučík, O. John, J. Nečas, *On the existence of Schauder bases in Sobolev spaces,* Commentationes Mathematicae Universitatis Carolinae, 1, Vol. 13 (1972), 163-175.

[37] C. Gal, M. Warma, *Bounded solutions for nonlocal boundary value problems on Lipschitz manifolds with boundary,* Adv. Nonlinear Stud. 16 (2016), 529–550.

[38] B. Ge, *On superlinear $p(x)$-Laplacian like problem without Ambrosetti and Rabinowitz condition,* Bull. Korean Math. Soc. 51 (2014), 409-421.

[39] B. Ge, Q. M. Zhou, Y. H. Wu, *Eigenvalues of the $p(x)$-biharmonic operator with indefinite weight,* Z. Angew. Math Phys. 66 (2015), 1007-1021.

[40] T. C. Halsey, *Electrorheological fluids,* Science 258 (1992), 761–766.

[41] K. Ho, Y. H. Kim, *A-priori bounds and multiplicity of solutions for nonlinear elliptic problems involving the fractional $p(\cdot)$-Laplacian,* Nonlinear Analysis 188 (2019), 179–201.

[42] E. J. Hurtado, *Nonlocal diffusion equations involving the fractional $p(\cdot)$-Laplacian,* Journal of Dynamics and Differential Equations (2019). doi.org/10.1007/s10884-019-09745-2.

[43] E. J. Hurtado, O. H. Miyagaki, R. S. Rodrigues, *Existence and multiplicity of solutions for a class of elliptic equations without Ambrosetti-Rabinowitz type conditions,* Journal of Dynamics and Differential Equations 30 (2018), 405-432.

[44] E. J. Hurtado, O. H. Miyagaki, R. S. Rodrigues, *Existence and asymptotic behaviour for a Kirchhoff type equation with variable critical growth exponent,* Milan J. Math. 85 (2017), 71–102.

[45] A. Iannizzotto, M. Squassina, *$\frac{1}{2}$-Laplacian problems with exponential nonlinearity,* J. Math. Anal. Appl. 414 (2014), 372–385.

[46] A. Iannizzotto, M. Squassina, *Weyl-type laws for fractional p-eigenvalue problems,* Asymptot. Anal. 4 (2014), 233–245.

[47] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$*, Proc. Roy. Soc. Edinburgh A. 129 (1999), 787-809.

[48] C. Ji, *On the superlinear problem involving $p(x)$-laplacian,* Electron. J. Qual. Theory Differ. Equ. 40 (2011), 1-9.

[49] R. Kajikiya, *A critical point theorem related to the symmetric mountain-pass lemma and its applications to elliptic equations,* J. Funct. Anal. 225 (2005), 352–370.

[50] U. Kaufmann, J. D. Rossi, R. Vidal, *Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacian,* Electron. J. Qual. Theory Differ. Equ. 76 (2017), 1-10.

[51] O. Kavian, *Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques,* Paris; New York Springer-Verlag (1993).

[52] I. H. Kim, Y. H. Kim, it Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents, Manuscripta Math. 147 (2015), 169–191.

[53] T. Kussi, G. Mingione, Y. Sire, *Nonlocal Equations with Measure Data,* Commun. Math. Phys. 337 (2015), 1317-1368.

[54] J. Lee, Y. H. Kim, *Multiplicity results for nonlinear Neumann boundary value problems involving p-Laplace type operators,* Bound. Value Probl. 95 (2016), 1–25.

[55] J. I. Lee, J. M. Kim, Y.H. Kim, J. Lee, *Multiplicity of weak solutions to non-local elliptic equations involving the fractional $p(x)$-Laplacian,* Journal of Mathematical Physics, 61(1) (2020), 011505.

[56] G. Li , C. Yang, *The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition,* Nonlinear Anal. 72 (2010), 4602-4613.

[57] Z. Liu, Z. Q. Wang, *On the Ambrosetti-Rabinowitz superlinear condition,* Adv. Nonlinear Stud. 4 (2004), 563-574.

[58] S. B. Liu, S. J. Li, *On superlinear problems without the Ambrosetti and Rabinowitz condition,* Nonlinear Anal. 73 (2010), 788–795.

[59] Mazón, J. M., Rossi, J. D., Toledo J. J., *Nonlocal perimeter, curvature and minimal surfaces for measurable sets*, Frontiers in Mathematics, Birkäuser (2019).

[60] O. H. Miyagaki, M. A. S Souto, *Superlinear problems without Ambrosetti and Rabinowitz growth condition,* J. Differential Equations. 245 (2008), 3628-3638.

[61] M. Mihăilescu, V. D. Rădulescu, *On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent,* Proc. Amer. Math. Soc. 135 (2007), 2929-2937.

[62] Z. Nehari, *On a class of nonlinear second-order equations,* Trans. Amer. Math. Soc. 95 (1960), 101–123.

[63] G. Palatucci, T. Kuusi, *Recent developments in the Nonlocal Theory, Book Series on Measure Theory,* De Gruyter Berlin (2016).

[64] R. Pei, *Fractional p-laplacian equations with subcritical and critical exponential growth without the Ambrosett-Rabinowitz condition,* Mediterranean Journal of Mathematics (2) 15 (2018).

[65] S. I. Pohozaev, *On one approach to nonlinear equations,* Dokl. Akad. Nauk 247 (1979) 1327-1331 (in Russian) (20, 912-916 (1979) (in English)).

[66] S. I. Pohozaev, *On a constructive method in calculus of variations,* Dokl. Akad. Nauk 298, 1330-1333 (1988)(in Russian) (37, 274-277 (1988) (in English)).

[67] S. I. Pohozaev, *On fibering method for the solutions of nonlinear boundary value problems,* Trudy Mat. Inst. Steklov 192 (1990), 146–163 (in Russian).

[68] P. Pucci, X. Mingqi, Z. Binlin, *Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations,* Adv. Nonlinear Anal. 5 (2016), 27–55.

[69] P. Pucci, S. Saldi, *Multiple solutions for an eigenvalue problem involving nonlocal elliptic p-Laplacian operators,* Geometric Methods in PDE's - Springer INdAM Series - Vol. 13, G. Citti, M. Manfredini, D. Morbidelli, S. Polidoro, F. Uguzzoni Eds. (2015), pages 16.

[70] P. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations,* CBMS Regional Conf. Ser. inMath. 65. American Mathematical Society, Providence, RI, (1986).

[71] V. D. Rădulescu, D. Repovš, *Partial differential equations with variable exponents: Variational Methods and Qualitative Analysis,* CRC Press, Taylor & Francis Group, Boca Raton, FL, (2015).

[72] M. M. Rodrigues, *Multiplicity of solutions on a nonlinear eigenvalue problem for $p(x)$-laplacian-like operators,* Mediterr. J. Math. 9 (2012), 211–223.

[73] R. Servadei, E. Valdinoci, *Mountain pass solutions for nonlocal elliptic operators,* J. Math. Anal. Appl. 389 (2012), 887–898.

[74] M. Z. Sun,*Multiple solutions of a superlinearp-Laplacian equation without AR-condition,* Applicable Analysis, 89(3) (2010), 325-336.

[75] J. L. Vázquez, *The mathematical theories of diffusion: nonlinear and fractional diffusion,* In: Bonforte, M., Grillo, G. (eds.) Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions. Lecture Notes in Mathematics 2186 (2017), 205–278.

[76] M. Willem, *Minimax Theorems. In: Progress in NLDE and their Applications,* vol 24, Birkhäuser Boston Inc., Boston, MA, (1996).

[77] B. L. Zhang, G. Molica Bisci, R. Servadei, *Superlinear nonlocal fractional problems with infinitely many solutions,* Nonlinearity 28 No. 7 (2015), 2247-2264.

[78] J. F. Zhao, *Structure Theory of Banach Spaces,* Wuhan University Press, Wuhan, (1991) (in Chinese).

[79] Q. M. Zhou, *On a class of superlinear nonlocal fractional problems without Ambrosetti-Rabinowitz type conditions,* Electronic Journal of Qualitative Theory of Differential Equations 17 (2019), 1-12.

[80] Q. M. Zhou, *On the superlinear problems involving $p(x)$-Laplacian-like operators without AR-condition,* Nonlinear Analysis: Real World Applications. 21 (2015), 161-169.