**UNIVERSIDADE FEDERAL DE SÃO CARLOS**
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

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São Carlos - SP, Brazil
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Advisor: Prof. Dr. Marcelo José Dias Nascimento
Co-Advisor: Prof. Dr. Everaldo de Mello Bonotto

Thesis submitted to the Department of Mathematics at *Universidade Federal de São Carlos*, DM-UFSCar, along with the Postgraduate Program in Mathematics, in partial fulfillment of the requirements for the degree of Doctor in Science - Concentration Area: Mathematics.

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# Sistema Klein-Gordon-Zakharov não autônomo: dinâmica pullback nas abordagens contínua e impulsiva

Eric Busatto Santiago

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Coorientador: Prof. Dr. Everaldo de Mello Bonotto

Tese apresentada ao Departamento de Matemática da Universidade Federal de São Carlos, DM-UFSCar, junto ao Programa de Pós-Graduação em Matemática, como parte dos requisitos para obtenção do título de Doutor em Ciências - Área de Concentração: Matemática.

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Dedicated to all mathematicians that may find themselves between analysis and dynamics.

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*"It is through science that we prove, but through intuition that we discover."*
— Henri Poincaré

# *Abstract*

This work is dedicated to study a non-autonomous formulation of the Klein-Gordon-Zakharov system, which is a coupled system consisting of two non-autonomous evolution equations, where each one is of second order in time. This model is closely related to the interaction of waves and it appears frequently in thermoelasticity, mechanics, plasma physics, and other areas alike.

The present work is divided into two main parts. In a first moment, using the uniform sectorial operators theory, we will show that our formulation has parabolic structure and then, making use of the natural energy associated to the system, we will obtain its global well-posedness. With the global solution in hands, we can define a nonlinear evolution process. Thus, in order to study the long-time dynamics of solutions, we shall use the abstract evolution processes theory to prove existence, regularity and upper semicontinuity of pullback attractors.

In the second main moment of this work, we are going to investigate the asymptotic dynamics of solutions of the non-autonomous Klein-Gordon-Zakharov system when they are subject to the action of impulses. To do that, we will study the qualitative properties of evolution processes under conditions of impulses and present sufficient conditions for the existence of pullback attractors for evolution processes in the impulsive scenario. Finally, we apply the abstract results in order to ensure the existence of an impulsive pullback attractor for the impulsive evolution process associated with the non-autonomous Klein-Gordon-Zakharov system with impulsive action.

**Keywords:** Klein-Gordon-Zakharov system, Global well-posedness, Pullback attractor, Upper semicontinuity, Impulses.

# *Resumo*

Este trabalho é dedicado ao estudo de uma formulação não autônoma do sistema de Klein-Gordon-Zakharov, o qual é um sistema acoplado composto por duas equações de evolução não autônomas, onde cada uma é de segunda ordem no tempo. Este modelo está intimamente relacionado a interação de ondas e ele aparece com frequência em termoelasticidade, mecânica, física de plasma, e outras áreas semelhantes.

O presente trabalho é dividido em duas partes principais. Em um primeiro momento, usando a teoria de operadores uniformemente setoriais, iremos mostrar que nossa formulação possui estrutura parabólica e então, fazendo uso da energia natural associada ao sistema, iremos obter a sua boa postura global. Com a solução global em mãos, podemos definir um processo de evolução não linear. Assim, a fim de estudar a dinâmica a longo prazo das soluções, deveremos usar a teoria abstrata dos processos de evolução para provar a existência, regularidade e semicontinuidade superior dos atratores pullback.

No segundo momento principal deste trabalho, vamos investigar a dinâmica assintótica das soluções do sistema de Klein-Gordon-Zakharov não autônomo quando elas estão sob ação de impulsos. Para fazer isto, iremos estudar as propriedades qualitativas de processos de evolução sob condições de impulsos e apresentar condições suficientes para a existência de atratores pullback para processos de evolução no cenário impulsivo. Finalmente, aplicaremos os resultados abstratos para garantir a existência de um atrator pullback impulsivo para o processo de evolução impulsivo associado ao sistema de Klein-Gordon-Zakharov não autônomo com ação impulsiva.

**Palavras-chave:** Sistema de Klein-Gordon-Zakharov, Boa colocação global. Atrator pullback, Semicontinuidade superior, Impulsos.

# *Contents*

# *Introduction*

The theory and applications of infinite dimensional dynamical systems have called the attention of many mathematicians over the past decades. Several real world phenomena can be described using dynamical aspects: the turbulence of fluids, celestial mechanics, climate changes, economic models, chemical and biological reactions, and so forth.

In particular, one of the main challenges of this area is to understand and predict the asymptotic behaviour of solutions associated with non-autonomous ordinary and partial differential equations, because for a large amount of problems it is not possible to obtain an explicit expression for its solutions (when they exist). In order to overcome this obstacle, the most notorious attempt is to show the existence of a specific object that attracts all the trajectories of the dynamical system generated by these solutions. To this object it is given the name of attractor and the main idea behind this purpose is to reduce the complexity of the system and study what happens to the solutions inside this object.

The concept of attractor is closely related to some kind of dissipation of energy over the time, which is one of the main ingredients used to ensure its existence. The investigation of this theory involves elements of classical analysis and some geometric viewpoint of the qualitative theory of differential equations.

In this work, we study a non-autonomous version of the well known Klein-Gordon-Zakharov system, given by the following initial-boundary value problem

$$\begin{cases} u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}} u_t + a_\epsilon(t)(-\Delta)^{\frac{1}{2}} v_t = f(u), & (x,t) \in \Omega \times (\tau, \infty), \\ v_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}} v_t - a_\epsilon(t)(-\Delta)^{\frac{1}{2}} u_t = 0, & (x,t) \in \Omega \times (\tau, \infty), \end{cases} \tag{1}$$

where $\eta$ is a positive constant, subject to boundary conditions

$$u = v = 0, \ (x,t) \in \partial\Omega \times (\tau, \infty),$$

and initial conditions

$$u(\tau, x) = u_0(x), \ u_t(\tau, x) = u_1(x), \ v(\tau, x) = v_0(x), \ v_t(\tau, x) = v_1(x), \ x \in \Omega, \ \tau \in \mathbb{R},$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, with the boundary $\partial\Omega$ assumed to be regular enough, $\eta > 0$ is constant, $a_\epsilon$ is a Hölder continuous function and $f \in C^1(\mathbb{R})$ is a dissipative nonlinearity.

In the case that $a_\epsilon(t) \equiv a$, the system (1) represents the autonomous version of the Klein-Gordon-Zakharov system. Within the autonomous case, if $n = 3$ then the Klein-Gordon-Zakharov system arises to describe the interaction of a Langmuir wave and an ion acoustic wave in a plasma, see [6, 31, 43] and references therein.

These types of systems have been considered by many researchers in recent years. In what follows, we recall some related results for these kinds of systems. In [43], the authors considered the following system (in dimension 2 and 3)

$$\begin{cases} u_{tt} - \Delta u + u + vu = 0, \\ v_{tt} - c_0^2 \Delta v = \Delta(|u|^2), \end{cases}$$

and they proved instability of solutions in the sense that small perturbations of the initial data can make the perturbed solution blow up in finite time.

In [2], it is considered the following coupled system of wave equations:

$$\begin{cases} u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s)ds + \alpha v = 0, \\ v_{tt} - \Delta v + \alpha u = 0, \end{cases}$$

where the authors showed the dissipativeness of this system, and, moreover, they proved that the associated semigroup is not exponentially stable. Later in [37], the authors studied a more general and abstract version of the previous system presented in [2]. In fact, they obtained existence of solutions and an optimal energy decay estimate for the following coupled system of second order abstract evolution equations:

$$\begin{cases} u_{tt}(t) + A_1 u(t) - \int_0^{+\infty} g(s)Au(t-s)ds + Bv(t) = 0, \\ v_{tt}(t) + A_2 v(t) + Bu(t) = 0, \end{cases}$$

where $A$, $A_1$ and $A_2$ are positive self-adjoint linear operators in a Hilbert space $H$, $B$ is a positive self-adjoint bounded linear operator in $H$, and $g$ is a non-increasing function satisfying some properties. With this formulation, this system covers the well-known Timoshenko system, which appears in mechanics and thermoelasticity, and models the transverse vibrations of a beam.

For a deeper and more detailed discussion about systems consisting of wave equations and other types of physical models, we refer to [32], [33],[41], [42] and [44].

This work is divided into two main parts. In the first one, the main purpose is to show the global well-posedness and to study the long-time dynamics of solutions of the evolution system

(1). In order to do that, we shall use the uniform sectorial operators theory, following Amann [3] and Henry [35], to show the local and global well-posedness of system (1), and we will use the abstract evolution processes theory to prove existence, regularity and upper semicontinuity of pullback attractors.

It is appropriate to observe that, in the literature, it is common to obtain the existence of pullback attractors by using a decomposition of the nonlinear evolution process to show that it has the property called *pullback asymptotic compactness.* This is done by writing this decomposition as two maps, where one decays to zero and the another one is compact. See [7], [8], [22], [23] and [25] for more details. However, in this work, it is established the compactness of the nonlinear process associated with the system (1) in a direct way. See Proposition 2.6.

The regularity of pullback attractors for the system (1) will be obtained using a combination of energy estimates in fractional power spaces and the so called "bootstrapping argument", which is an idea used very often in the theory of elliptic partial differential equations to increase regularity of solutions. See Theorem 2.8. Finally, after improving the regularity, we will apply this result to obtain the upper semicontinuity of pullback attractors for the system (1), which is an important achievement from the stability viewpoint, because this means that the attractor does not explode when we make a small perturbation in it. See Theorem 2.9.

The results that were mentioned previously are contained in the paper [17], which was already submitted for publication.

The second main part of this work is concerned with the long-time dynamics of solutions of the system (1) when they are subject to impulsive effects at variable times. That is, the problem to be studied now has the form

$$\begin{cases} u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}} u_t + a_\epsilon(t)(-\Delta)^{\frac{1}{2}} v_t = f(u), & (x,t) \in \Omega \times (\tau, \infty), \\ v_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}} v_t - a_\epsilon(t)(-\Delta)^{\frac{1}{2}} u_t = 0, & (x,t) \in \Omega \times (\tau, \infty), \\ u = v = 0, & (x,t) \in \partial\Omega \times (\tau, \infty), \\ I : M \subset Y_0 \to Y_0. \end{cases} \tag{2}$$

Here, the set $M$, called the *impulsive set*, is a nonempty closed subset of the phase space $Y_0$, and it will satisfy some transversality properties in relation with the continuous evolution process generated by the global solution of the system (1). The function $I : M \subset Y_0 \to Y_0$, called the *impulse function*, is assumed to be continuous and will be responsible by the occurrence of discontinuities in the trajectory when the solution hits $M$.

The theory of impulsive dynamical systems is used to comprehend the structure of systems where the continuity of the flow is interrupted by an abrupt change of state. Briefly speaking, when a dynamical system is subject to impulsive effects, the continuous flow is interrupted when it hits the impulsive set and then, at the moment of this impact, the continuity of the dynamics is broken and the flow will restart its evolution from another point of the phase space. This process we just described can eventually come to an end, if the flow hits the impulsive set only a finite number of times, or this process can generate an infinite number of discontinuities if the flow keeps hitting the impulsive set indefinitely.

The states of several real world problems can change in a gap of time so small that is expected for this phenomena to happen almost instantaneously. Therefore, it is natural to think that these abrupt changes may occur in the form of impulses. It makes more sense when we think in the real applications: optimal control theory in economics, a health treatment involving the periodic ingestion of medicines and electric systems are some concrete examples where the action of impulses are observable.

Another example that we may cite is a billiard system where there are balls colliding to each other. In this system, when the balls are hitting one another, their positions do not change at the moments of impact, that is, when the impulses happen. But we can see that the velocities of the balls will gain finite increments. Thus, in this example, the impulses are acting on the velocity according to the position of the ball.

In particular, the study of systems of coupled wave equations with impulses is motivated by the fact that, when modeling the interaction between different fluids, discontinuities may appear naturally in the state variables, which are influenced by several physical aspects, such as density, viscosity, and molecular cohesion. For instance, the discontinuities in the molecular cohesion are responsible by the phenomenon of surface tension, which is modeled using jump conditions in the pressure field. Moreover, the discontinuity in the density variable can change the shape of air bubbles in the water. A survey on fluids with discontinuity conditions can be found in [36].

The present work is divided into four chapters and it is organized as follows. The Chapter 1 is dedicated to give a collection of preliminary facts that are useful for the understanding of the forthcoming chapters of this work. In Section 1.1, we present a brief summary on the theory of semigroups of bounded linear operators, including basic properties and the main theorems on generation of semigroups, and we also present results concerning sectorial operators and their fractional powers. In Section 1.2 we reunite the main concepts and existence results involving the theory of pullback attractors for nonlinear evolution processes, and we also include a quick overview of the theory of abstract parabolic problems.

The Chapter 2 is devoted to study the non-autonomous version of the Klein-Gordon-Zakharov system, given by (1), and it is organized in five parts. In Section 2.1, we present the conditions and assumptions that will ensure the local and global well-posedness of the problem (1), which are two topics that are going to be discussed in Section 2.2. The other three remaining sections of Chapter 2 are dedicated to investigate the long-time dynamics of solutions of the evolution system (1) using the nonlinear evolution processes framework and the theory of pullback attractors. Thus, in Section 2.3, we obtain the existence of the pullback attractor for the non-autonomous problem (1). Section 2.4 deals with the regularity of the pullback attractor and, finally, in Section 2.5, we prove the upper semicontinuity of pullback attractors.

The Chapter 3 is dedicated to present the theory of evolution processes under conditions of impulses. In Section 3.1, we exhibit the construction of an impulsive evolution process, we give the conditions at which the existence of an impulsive flow is guaranteed for all time, and we also define the concepts of invariance, asymptotic compactness and dissipativeness in the framework of evolution processes with impulses. In Section 3.2, we discuss the continuity of the function that represents the smallest time for which the trajectory of a point meets the impulsive set. In Section 3.3, we present qualitative properties concerning the convergence of the impulsive flow that are crucial to establish the invariance of the impulsive omega-limit set, which is the main goal of Section 3.4. In Section 3.5, we prove an abstract result on existence of pullback attractors for evolution processes subject to impulses. Finally, Section 3.6 is devoted to obtain a result on upper semicontinuity of a family of impulsive pullback attractors.

The Chapter 4 is reserved to study the asymptotic dynamics of the impulsive non-autonomous problem (2). Our main goal in Section 4.1 is to show that the impulsive problem (2) possesses an impulsive pullback attractor. To do that, we will construct a compact absorbing set for the impulsive evolution process associated with this problem, see Theorem 4.1, and then we will assure the existence of such attractor by applying the abstract existence result presented in Chapter 3. Finally, in Section 4.2 we study the robustness of the family of impulsive pullback attractors associated with the impulsive problem (2).

Chapter

1

# Preliminaries

This chapter is dedicated to provide a quick look on the main tools that are necessary for a better comprehension of this work.

## 1.1 Semigroups of linear operators

The purpose of this section is to give a review on standard facts about semigroups of bounded linear operators and their infinitesimal generators, with focus in presenting the main results on generation of $C_0$-semigroups, namely the Hille-Yosida Theorem and the Lumer-Phillips Theorem, and also some of its consequences. Properties of sectorial operators and fractional powers of operators are also listed. These notions are used in the modern theory of differential equations, mostly in evolution problems, to study existence and uniqueness of solutions, continuous dependence with respect to the initial data, and even the existence of global attractors. The content related to linear operators and semigroups can be found in [20], [38] and [45]. Meanwhile, [3], [30] and [35] are good references for sectorial operators and fractional power spaces.

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed vector spaces. We denote by $\mathcal{L}(E,F)$ the space of bounded linear operators $T\colon E \to F$ endowed with the norm

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{\|x\|_E \leq 1} \|Tx\|_F.$$

As usual, we write $\mathcal{L}(E)$ to denote the space $\mathcal{L}(E,E)$.

### 1.1.1 Semigroups and their generators

Throughout the following definitions, $(X, \|\cdot\|_X)$ will be a Banach space over a field $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

**Definition 1.1.** A *semigroup of bounded linear operators on* $X$, or simply *semigroup*, is a one parameter family $\{T(t)\colon t \geq 0\} \subset \mathcal{L}(X)$ satisfying:

(*a*) $T(0) = I$, with $I$ being the identity operator on $X$;

(*b*) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$.

The semigroups of bounded linear operators are classified as follows.

**Definition 1.2.** A semigroup $\{T(t)\colon t \geq 0\} \subset \mathcal{L}(X)$ is called:

(*a*) *uniformly continuous*, if $\lim_{t\downarrow 0} \|T(t) - I\|_{\mathcal{L}(X)} = 0$;

(*b*) *strongly continuous*, if $\lim_{t\downarrow 0} T(t)x = x$ for all $x \in X$ and, in this case, we say that the semigroup is of class $C_0$, or we simply call it a $C_0$-semigroup.

**Definition 1.3.** For a semigroup $\{T(t)\colon t \geq 0\} \subset \mathcal{L}(X)$, we define its *infinitesimal generator* as the linear operator $A\colon D(A) \subset X \to X$ given by

$$D(A) = \left\{ x \in X\colon \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t}, \text{ for } x \in D(A).$$

**Theorem 1.1.** [45, Theorem 1.2] *A linear operator $A\colon D(A) \subset X \to X$ is the infinitesimal generator of a uniformly continuous semigroup $\{T(t)\colon t \geq 0\} \subset \mathcal{L}(X)$ if, and only if $A$ is a bounded linear operator.*

The next result says that every bounded linear operator is the infinitesimal generator of a unique uniformly continuous semigroup.

**Theorem 1.2.** [45, Theorem 1.3] *Let $\{T(t)\colon t \geq 0\}$ and $\{S(t)\colon t \geq 0\}$ be two uniformly continuous semigroups. If*

$$\lim_{t\downarrow 0} \frac{T(t) - I}{t} = A = \lim_{t\downarrow 0} \frac{S(t) - I}{t},$$

*then $T(t) = S(t)$ for all $t \geq 0$.*

Every $C_0$-semigroup is exponentially dominated, as it is established in the next result.

**Theorem 1.3.** [45, Theorem 2.2] *Let $\{T(t)\colon t \geq 0\}$ be a $C_0$-semigroup. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \text{ for all } t \geq 0.$$

**Remark 1.1.** In Theorem 1.3, if $\omega = 0$, then the $C_0$-semigroup $\{T(t)\colon t \geq 0\}$ is called *uniformly bounded* and, if $\omega = 1$, then $\{T(t)\colon t \geq 0\}$ is called a $C_0$-*semigroup of contractions.*

The following result collects some useful facts about $C_0$-semigroups.

**Theorem 1.4.** [45, Corollary 2.3, Theorem 2.4, Corollary 2.5, Theorem 2.7] *Let $\{T(t)\colon t \geq 0\}$ be a $C_0$-semigroup and let $A$ be its infinitesimal generator.*

(*i*) *For all $x \in X$, the map $\mathbb{R}_+ \ni t \mapsto T(t)x \in X$ is continuous.*

(*ii*) *$D(A)$ is dense in $X$ and $A$ is a closed operator.*

(*iii*) *For $x \in D(A)$, $T(t)x \in D(A)$, the map $\mathbb{R}_+ \ni t \mapsto T(t)x \in X$ is continuously differentiable, and*
$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \quad t > 0.$$

(*iv*) *If $D(A^n)$ is the domain of $A^n$, for $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} D(A^n)$ is dense in $X$.*

### 1.1.2 The Hille-Yosida and the Lumer-Phillips Theorems

When dealing with applications of the semigroup theory of linear operators to partial differential equations, one needs to know how to determine conditions that ensure when a given operator on a Banach space is the generator of a $C_0$-semigroup.

There are two mainly results in this direction. One is the Hille-Yosida Theorem, which provides necessary and sufficient conditions for a linear operator to be the infinitesimal generator of a $C_0$-semigroup of contractions, but these conditions can be difficult to verify. The other one is the Lumer-Phillips Theorem, which gives necessary and sufficient conditions for generation of $C_0$-semigroups in terms of dissipativity, and this result is quite useful in a Hilbert space setting.

Recall that if $A\colon D(A) \subset X \to X$ is a linear operator, not necessarily bounded, then the *resolvent set* of $A$, denoted by $\rho(A)$, is defined by

$$\rho(A) = \{\lambda \in \mathbb{C}\colon (\lambda I - A)^{-1}\colon X \to X \text{ is a bounded linear operator}\}.$$

For $\lambda \in \rho(A)$, the operator

$$R(\lambda\colon A) = (\lambda I - A)^{-1}\colon X \to X$$

is called *resolvent operator.*

**Theorem 1.5** (Hille-Yosida)**.** [45, Theorem 3.1] *A linear operator $A\colon D(A) \subset X \to X$ is the infinitesimal generator of a $C_0$-semigroup of contractions $\{T(t)\colon t \geq 0\}$ in $X$ if and only if the following conditions hold:*

(*i*) *$A$ is closed and $\overline{D(A)} = X$;*

(*ii*) *the resolvent set* $\rho(A)$ *contains* $\mathbb{R}_+$ *and, for all* $\lambda > 0$,

$$\|R(\lambda\colon A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

Some consequences of the Hille-Yosida Theorem are given next.

**Corollary 1.1.** [45, Corollary 3.6] *Let* $A\colon D(A) \subset X \to X$ *be the infinitesimal generator of a* $C_0$*-semigroup of contractions* $\{T(t)\colon t \geq 0\}$ *in* $X$*. Then*

$$\rho(A) \supseteq \{\lambda \in \mathbb{C}\colon Re(\lambda) > 0\}$$

*and, for such* $\lambda$*, it holds*

$$\|R(\lambda\colon A)\|_{\mathcal{L}(X)} \leq \frac{1}{Re(\lambda)}.$$

**Corollary 1.2.** [45, Corollary 3.8] *Let* $A\colon D(A) \subset X \to X$ *be a linear operator. The following statements are equivalent:*

(*i*) $A$ *is the infinitesimal generator of a* $C_0$*-semigroup of contractions* $\{T(t)\colon t \geq 0\}$ *in* $X$ *satisfying*

$$\|T(t)\|_{\mathcal{L}(X)} \leq e^{\omega t}$$

*for some* $\omega \geq 0$ *and for all* $t \geq 0$*;*

(*ii*) $A$ *is closed, densely defined, its resolvent set* $\rho(A)$ *contains* $(\omega, \infty)$ *and*

$$\|R(\lambda\colon A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda - \omega}$$

*for all* $\lambda > \omega$*.*

Let $X^*$ be the *topological dual space* of $X$, that is, $X^*$ is the space of all continuous linear functionals defined in $X$ and taking values in $\mathbb{K}$. The value of $f \in X^*$ in a point $x \in X$ will be denoted by $\langle f, x\rangle$. The usual norm on $X^*$ is defined by

$$\|f\|_{X^*} = \sup_{\|x\|_X \leq 1} |\langle f, x\rangle|.$$

It is well known that $(X^*, \|\cdot\|_{X^*})$ is a Banach space (this fact is true even when $X$ is not complete). For $x \in X$, the *duality set* $F(x) \subseteq X^*$ is defined by

$$F(x) = \{f \in X^*\colon \mathrm{Re}(\langle f, x\rangle) = \|x\|_X^2 \text{ and } \|f\|_{X^*} = \|x\|_X\}.$$

From the Hahn-Banach Theorem, it follows that $F(x) \neq \emptyset$ for all $x \in X$.

**Definition 1.4.** A linear operator $A\colon D(A) \subset X \to X$ is called *dissipative* if, for each $x \in D(A)$, there is $f \in F(x)$ such that $\mathrm{Re}\,(\langle f, Ax\rangle) \leq 0$.

**Definition 1.5.** If a linear operator $A\colon D(A) \subset X \to X$ is such that $-A$ is dissipative, then $A$ is called *accretive*. If, additionally, one has $R(I + A) = X$, then $A$ is called *maximal accretive*.

A general characterization of dissipativity for linear operators is given next.

**Theorem 1.6.** [45, Theorem 4.2] *A linear operator $A\colon D(A) \subset X \to X$ is dissipative if, and only if*

$$\|(\lambda I - A)x\|_X \geq \lambda \|x\|_X$$

*for all $x \in D(A)$ and $\lambda > 0$.*

Now, with the concept of dissipativity in mind, we are able to present the Lumer-Phillips Theorem.

**Theorem 1.7** (Lumer-Phillips)**.** [45, Theorem 4.3] *Let $A\colon D(A) \subset X \to X$ be a densely defined linear operator.*

*(i) If $A$ is dissipative and there exists $\lambda_0 > 0$ such that $R(\lambda_0 I - A) = X$, then $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions in $X$.*

*(ii) If $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions in $X$, then $R(\lambda I - A) = X$ for all $\lambda > 0$, and $A$ is dissipative. Moreover,*

$$Re\,(\langle f, Ax\rangle) \leq 0$$

*for all $x \in D(A)$ and all $f \in F(x)$.*

Recall that the *adjoint operator* $A^*\colon D(A^*) \subset Y^* \to X^*$ of a densely defined unbounded linear operator $A\colon D(A) \subset X \to Y$ is defined in the following way. Its domain is given by

$$D(A^*) = \{f \in Y^*\colon \text{there is } c_0 \geq 0 \text{ such that } |\langle f, Ax\rangle| \leq c_0\|x\|_X \text{ for all } x \in D(A)\},$$

which is a dense subspace of $Y^*$. Now, for $f \in D(A^*)$, consider a map $h\colon D(A) \to \mathbb{R}$ defined by

$$h(x) = \langle f, Ax\rangle,\ \ x \in D(A),$$

and note that

$$|h(x)| \leq c_0\|x\|_X \quad \text{for all} \quad x \in D(A).$$

Then, by the analytic form of the Hahn-Banach Theorem, there exists a linear map $\tilde{h}\colon X \to \mathbb{R}$ that extends $h$ and satisfy

$$|\tilde{h}(x)| \leq c_0 \|x\|_X \quad \text{for all} \quad x \in X.$$

It follows that $\tilde{h} \in X^*$. Moreover, note that the extension of $h$ is unique, since $D(A)$ is dense in $X$. Now, set $A^* f = \tilde{h}$. The unbounded linear operator $A^*\colon D(A^*) \subset Y^* \to X^*$, defined in this way, is called the adjoint of $A$.

The fundamental relation between $A$ and $A^*$ is given by the following rule:

$$\langle f, Ax \rangle = \langle A^* f, x \rangle \quad \text{for all } x \in D(A) \text{ and all } f \in D(A^*).$$

Another important property states that: if $A$ is a bounded linear operator from $X$ into $Y$, then $A^*$ is a bounded operator from $Y^*$ into $X^*$ and it holds that

$$\|A^*\|_{\mathcal{L}(Y^*,X^*)} = \|A\|_{\mathcal{L}(X,Y)}.$$

Furthermore, the adjoint $A^*$ is always a closed operator, that is, the graph $G(A^*)$ is closed in $Y^* \times X^*$.

Now, the corollary below is a consequence of the Lumer-Phillips Theorem.

**Corollary 1.3.** [45, Corollary 4.4] *Let $A\colon D(A) \subset X \to X$ be a densely defined closed linear operator. If both $A$ and $A^*$ are dissipative, then $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions in $X$.*

### 1.1.3 Sectorial operators and analytic semigroups

For $a \in \mathbb{R}$ and $\phi \in \left(0, \frac{\pi}{2}\right)$, a *sector* of the complex plane, denoted by $\mathcal{S}_{a,\phi}$, is a subset of $\mathbb{C}$ given by

$$\mathcal{S}_{a,\phi} = \{\lambda \in \mathbb{C}\colon \phi \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\}.$$

**Definition 1.6.** A densely defined closed linear operator $A\colon D(A) \subset X \to X$ is called a *sectorial operator* if there exist constants $a \in \mathbb{R}$, $\phi \in \left(0, \frac{\pi}{2}\right)$ and $M > 0$ such that the resolvent set $\rho(A)$ contains the sector $\mathcal{S}_{a,\phi}$ and the estimate

$$\|R(\lambda\colon A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - a|}$$

holds for each $\lambda \in \mathcal{S}_{a,\phi}$.

**Example 1.1.** The following list shows some examples of sectorial operators.

($a$) Every bounded linear operator defined on a Banach space is a sectorial operator.

($b$) If $A\colon D(A) \subset X \to X$ and $B\colon D(B) \subset Y \to Y$ are sectorial operators, where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, then the operator $(A, B)\colon D(A) \times D(B) \subset X \times Y \to X \times Y$, defined by $(A, B)(x, y) = (Ax, By)$, for each $(x, y) \in D(A) \times D(B)$, is sectorial in $X \times Y$.

($c$) Let $A\colon D(A) \subset H \to H$ be a self-adjoint and densely defined linear operator in a Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$. If there is $c \in \mathbb{R}$ such that $\langle Ax, x\rangle_H \geq c\|x\|_H^2$ for all $x \in D(A)$, then $A$ is a sectorial operator in $H$.

($d$) Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, where the boundary $\partial\Omega$ is assumed to be of class $C^2$, $X = L^2(\Omega)$, and $A\colon D(A) \subset X \to X$ is the linear operator defined by $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $Au = (-\Delta)u$ for all $u \in D(A)$. Then $A$ is sectorial and positive definite in $X$.

($e$) The bi-Laplacian operator $\Delta^2\colon H^4(\Omega) \cap H_0^2(\Omega) \to L^2(\Omega)$, where $\partial\Omega \in C^4$, is sectorial and positive definite in $L^2(\Omega)$.

Items ($a$), ($b$), ($d$) and ($e$) were taken, respectively, from examples 1.3.1, 1.3.2, 1.3.6 and 1.3.7 contained in [28], where the reader can also find details about the proofs. The assertion on item ($c$) is proved in [28, Proposition 1.3.3]. Moreover, [30, Chapter 5] also contains the statements of items ($a$), ($c$) and ($d$).

The following result gives equivalent conditions for sectoriality.

**Proposition 1.1.** [28, Proposition 1.3.1] *Let $A\colon D(A) \subset X \to X$ be a densely defined closed linear operator and, for $\omega \in \mathbb{R}$, consider the operator $A_\omega = A + \omega I$. Then the following conditions are equivalent:*

($i$) *$A_\omega$ is sectorial in $X$ for some $\omega \in \mathbb{R}$;*

($ii$) *$A_\omega$ is sectorial in $X$ for each $\omega \in \mathbb{R}$;*

($iii$) *there exist $k, \omega \in \mathbb{R}$ such that the resolvent set $\rho(A_\omega)$ contains the half-plane $\{\lambda \in \mathbb{C}\colon Re(\lambda) \leq k\}$ and, for such $\lambda$, it holds the estimate*

$$\|\lambda(\lambda I - A_\omega)^{-1}\|_{\mathcal{L}(X)} \leq M,$$

*where $M$ is a positive constant.*

**Definition 1.7.** A $C_0$-semigroup $\{T(t)\colon t \geq 0\}$ in $X$ is called an *analytic semigroup* if there exist a sector of the complex plane

$$\mathcal{S} = \{z \in \mathbb{C}\colon \phi_1 < \arg z < \phi_2\},$$

with $\phi_1 < 0 < \phi_2$, and a family of bounded linear operators $T(z)\colon X \to X$, $z \in \mathcal{S}$, which coincides with $T(t)$ for $t \in [0, \infty)$, such that

(*a*) the map $z \mapsto T(z)x$ is analytic in $\mathcal{S}$ for each $x \in X$;

(*b*) for $z \in \mathcal{S}$, $T(z)x \to x$ as $z \to 0$, for all $x \in X$;

(*c*) $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \mathcal{S}$.

In the next result, the notation $\mathrm{Re}(\sigma(A)) > a$ means that $\mathrm{Re}(\lambda) > a$ whenever $\lambda \in \sigma(A)$.

**Theorem 1.8.** [35, Theorem 1.3.4] *If an operator $A\colon D(A) \subset X \to X$ is sectorial, then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-At}\colon t \geq 0\}$, where*

$$e^{-At} = \frac{1}{2\pi i} \int_\Gamma (\lambda + A)^{-1} e^{\lambda t} d\lambda, \quad t \geq 0,$$

*where $\Gamma$ is a contour in $\rho(-A)$, with $\arg \lambda \to \pm\theta$ as $|\lambda| \to \infty$, for some $\theta \in \left(\frac{\pi}{2}, \pi\right)$. Furthermore, $\{e^{-At}\colon t \geq 0\}$ can be extended analytically in a sector $\{t \neq 0\colon |\arg t| < \varepsilon\}$ that contains the positive real axis and, if $Re(\sigma(A)) > a > 0$, then*

$$\|e^{-At}\|_{\mathcal{L}(X)} \leq Ce^{-at}, \ \|Ae^{-At}\|_{\mathcal{L}(X)} \leq \frac{C}{t} e^{-at}, \tag{1.1}$$

*for $t > 0$ and some positive constant $C$. Moreover, it holds that*

$$\frac{d}{dt} e^{-At} = -Ae^{-At} \quad \text{for} \quad t > 0.$$

### 1.1.4 Fractional powers of operators

Now, let $A\colon D(A) \subset X \to X$ be a sectorial operator in $X$ with $\mathrm{Re}(\sigma(A)) > 0$. The boundedness in (1.1) allows to define, for $\alpha \in (0, \infty)$, the *fractional powers* $A^{-\alpha}\colon X \to X$, associated with $A$, by the following integral formula

$$A^{-\alpha} v = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha - 1} e^{-At} v dt.$$

**Theorem 1.9.** [35, Theorem 1.4.2] *For each $\alpha \in (0, \infty)$, $A^{-\alpha}\colon X \to X$ is a well defined* *bounded linear operator satisfying*

$$A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)} \quad \textit{for} \quad \alpha, \beta \in (0, \infty).$$

*Furthermore, for* $0 < \alpha < 1$,

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1} d\lambda.$$

As we can see above, $A^{-\alpha}$ is invertible for each $\alpha \in (0, \infty)$ and its inverse operator is denoted by $A^\alpha$.

**Definition 1.8.** If $A\colon D(A) \subset X \to X$ is a sectorial operator in a Banach space $X$, then we define, for each $\alpha \in [0, \infty)$, $X^\alpha = D(A_1^\alpha)$ with the graph norm

$$\|x\|_{X^\alpha} = \|A_1^\alpha x\|_X, \; x \in X^\alpha,$$

where $A_1 = A + \omega I$ and $\omega$ is chosen so that $\mathrm{Re}(\sigma(A_1)) > 0$. We observe that different choices of $\omega$ leads to equivalent norms on $X^\alpha$, so we will omit the dependence of $\omega$. See [35] for more details

In the particular case when $\alpha = 0$, it is a convention to denote $A^0 = I$ (identity operator) and $X^0 = X$.

For the next result, recall that a linear operator $A\colon D(A) \subset X \to X$ has *compact resolvent* if the operator $(\lambda I - A)^{-1}\colon X \to X$ is a compact map for each $\lambda \in \rho(A)$.

**Proposition 1.2.** [28, Proposition 1.3.5] *For each* $\alpha \in [0, \infty)$, $X^\alpha$ *is a Banach space when equipped with the norm* $\| \cdot \|_{X^\alpha} = \|A^\alpha \cdot \|_X$, *and* $A^\alpha\colon X^\alpha \to X$ *is a densely defined and closed linear operator, satisfying*

$$A^\alpha A^\beta = A^\beta A^\alpha = A^{\alpha+\beta},$$

*for any* $\alpha, \beta \in [0, \infty)$. *Furthermore,* $X^\alpha$ *is a dense subset of* $X^\beta$ *for* $\alpha \geq \beta \geq 0$, *and the following inclusions are dense and continuous:*

$$X^\alpha \subset X^\beta, \; \alpha > \beta \geq 0,$$

*and, if A has compact resolvent, then they are also compact.*

## 1.2 Evolution processes and pullback attractors

The aim of this section is to collect the definitions and results that together are the basis of the theory of pullback attractors for nonlinear evolution processes. This content will be applied in the forthcoming chapters of this work. For further details, it is worthwhile to consult the references [23], [25] and [27].

### 1.2.1 Existence results

Let $(Z, d)$ be a metric space. An *evolution process* in $Z$ is a two-parameter family $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ of maps from $Z$ into itself such that:

(*a*) $S(t,t) = I$ for all $t \in \mathbb{R}$, ($I$ is the identity operator in $Z$),

(*b*) $S(t,\tau) = S(t,s)S(s,\tau)$ for all $t \geq s \geq \tau$, and

(*c*) the map $\{(t,\tau) \in \mathbb{R}^2 \colon t \geq \tau\} \times Z \ni (t,\tau,x) \mapsto S(t,\tau)x \in Z$ is continuous.

**Remark 1.2.** In the particular case when $X$ is a Banach space, an evolution process arises naturally as a non-autonomous dynamical system associated with a non-autonomous differential equation. More precisely, if $f\colon \mathbb{R} \times B \subset \mathbb{R} \times X \to X$ is a suitable function and we have the global well-posedness of the following Cauchy problem

$$\begin{cases} \dot{u} = f(t,u), & t > \tau, \\ u(\tau) = u_0 \in X, & \tau \in \mathbb{R}, \end{cases}$$

then one can define

$$S_f(t,\tau)u_0 = u(t,\tau,f,u_0), \quad \text{for all} \quad t \geq \tau,$$

where $u(\cdot,\tau,f,u_0)$, $t \geq \tau$, is the global solution of the above problem. Therefore, $\{S_f(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ is an evolution process in $X$.

**Remark 1.3.** In the case when $X$ is a Banach space, if $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\} \subset \mathcal{L}(X)$, then we will refer to this process as a *linear evolution process.*

Recall that the Hausdorff semidistance between two nonempty subsets $A$ and $B$ of $Z$ is defined by

$$d_H(A,B) = \sup_{a\in A} \inf_{b \in B} d(a,b).$$

The Hausdorff semidistance measures how far $A$ is from being inside the closure of $B$. It is important to emphasize that $d_H$ is not a metric, since $d_H(A,B) = 0$ implies only that $\overline{A} \subseteq \overline{B}$.

**Definition 1.9.** Let $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ be an evolution process in $Z$. Given $t \in \mathbb{R}$ and $A, B$ subsets of $Z$, we say that $A$ *pullback attracts* $B$ at time $t$ if

$$\lim_{\tau \to -\infty} d_H(S(t,\tau)B, A) = 0, \tag{1.2}$$

where $S(t,\tau)B = \{S(t,\tau)x\colon x \in B\}$ is the image of $B$ under $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$.

The set $A$ pullback attracts bounded sets at time $t$, if (1.2) holds for every bounded subset $B$ of $Z$.

Moreover, we say that a time-dependent family $\{A(t)\colon t \in \mathbb{R}\}$ of subsets of $Z$ pullback attracts bounded subsets of $Z$, if $A(t)$ pullback attracts bounded sets at time $t$, for each $t \in \mathbb{R}$.

**Remark 1.4.** It is worthwhile to emphasize that the term "pullback" refers to make the initial time of the evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ goes to minus infinity, that is, $\tau \to -\infty$, which is not the same thing as going back in time. The evolution will always be forward in time as $t \geq \tau$.

Now, we are in position to define the concept of pullback attractor.

**Definition 1.10.** A family of compact subsets $\{\mathbb{A}(t)\colon t \in \mathbb{R}\}$ of $Z$ is a *pullback attractor* for the evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ if

(*i*) $\{\mathbb{A}(t)\colon t \in \mathbb{R}\}$ is invariant; that is, $S(t,\tau)\mathbb{A}(\tau) = \mathbb{A}(t)$ for all $t \geq \tau$,

(*ii*) $\{\mathbb{A}(t)\colon t \in \mathbb{R}\}$ pullback attracts bounded subsets of $Z$, in the sense of Definition 1.9, and

(*iii*) $\{\mathbb{A}(t)\colon t \in \mathbb{R}\}$ is the minimal family of closed sets satisfying property (*ii*).

The "pullback version" of the $\omega$-limit set in the evolution processes framework can be stated as follows.

**Definition 1.11.** Let $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ be an evolution process in $Z$. We define the *pullback $\omega$-limit set* of a subset $B$ of $Z$ at time $t$ as the set

$$\omega(B,t) = \bigcap_{\sigma \leq t} \overline{\bigcup_{\tau \leq \sigma} S(t,\tau)B}.$$

Equivalently,

$$\omega(B,t) = \{y \in Z\colon \text{there are sequences } \{\tau_k\}_{k\in\mathbb{N}} \subset (-\infty,t] \text{ and } \{x_k\}_{k\in\mathbb{N}} \subset B,$$

$$\text{with } \tau_k \xrightarrow{k\to\infty} -\infty, \text{such that } y = \lim_{k\to\infty} S(t,\tau_k)x_k\}.$$

Next, we define what it means the concept of compactness for evolution processes from the asymptotic viewpoint in the pullback sense.

**Definition 1.12.** An evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ in $Z$ is said to be *pullback asymptotically compact* if, for each $t \in \mathbb{R}$, each sequence $\{\tau_k\}_{k\in\mathbb{N}}$ with $\tau_k \leq t$ for all $k \in \mathbb{N}$ and $\tau_k \xrightarrow{k\to\infty} -\infty$, and each bounded sequence $\{x_k\}_{k\in\mathbb{N}} \subset Z$, then the sequence $\{S(t,\tau_k)x_k\}_{k\in\mathbb{N}}$ has a convergent subsequence.

**Definition 1.13.** We say that a set $B \subset Z$ *pullback absorbs* bounded sets at time $t \in \mathbb{R}$ if, for each bounded subset $D$ of $Z$, there exists a time $T = T(t, D) \leq t$ such that $S(t, \tau)D \subset B$ for all $\tau \leq T$. Moreover, we say that a time-dependent family $\{B(t) \colon t \in \mathbb{R}\}$ of subsets of $Z$ pullback absorbs bounded subsets of $Z$, if $B(t)$ pullback absorbs bounded sets at time $t$, for each $t \in \mathbb{R}$.

**Remark 1.5.** If a set pullback absorbs bounded sets at time $t \in \mathbb{R}$, then it pullback attracts bounded sets at time $t \in \mathbb{R}$.

**Definition 1.14.** We say that an evolution process $\{S(t, \tau) \colon t \geq \tau \in \mathbb{R}\}$ in $Z$ is:

(i) *pullback strongly bounded* if, for each bounded subset $B$ of $Z$ and each $t \in \mathbb{R}$, then the set $\bigcup_{s \leq t} \gamma_p(B, s)$ is bounded, where $\gamma_p(B, t) = \bigcup_{\tau \leq t} S(t, \tau)B$ is the *pullback orbit* of $B \subset Z$ at time $t \in \mathbb{R}$.

(ii) *pullback strongly bounded dissipative* if, for each $t \in \mathbb{R}$, then there is a bounded subset $B(t)$ of $Z$ which pullback absorbs bounded subsets of $Z$ at time $s$ for each $s \leq t$; that is, given a bounded subset $D$ of $Z$ and $s \leq t$, there exists $\tau_0(s, D)$ such that $S(s, \tau)D \subset B(t)$ for all $\tau \leq \tau_0(s, D)$.

The theorem (existence result) below characterizes the class of evolution processes which have pullback attractors.

**Theorem 1.10.** [25, Theorem 2.23] *If an evolution process $\{S(t, \tau) \colon t \geq \tau \in \mathbb{R}\}$ in $Z$ is pullback strongly bounded dissipative and pullback asymptotically compact, then $\{S(t, \tau) \colon t \geq \tau \in \mathbb{R}\}$ has a pullback attractor $\{\mathbb{A}(t) \colon t \in \mathbb{R}\}$, with the property that $\bigcup_{\tau \leq t} \mathbb{A}(\tau)$ is bounded in $Z$ for each $t \in \mathbb{R}$.*

**Definition 1.15.** A *global solution* for an evolution process $\{S(t, \tau) \colon t \geq \tau \in \mathbb{R}\}$ in $Z$ is a function $\xi \colon \mathbb{R} \to Z$ such that $S(t, \tau)\xi(\tau) = \xi(t)$ for all $t \geq \tau$.

**Definition 1.16.** A global solution $\xi \colon \mathbb{R} \to Z$ for a process $\{S(t, \tau) \colon t \geq \tau \in \mathbb{R}\}$ in $Z$ is said to be *backwards-bounded* (respectively, *forwards-bounded*), or bounded in the past (respectively, bounded in the future), if there exists $s \in \mathbb{R}$ such that the set $\{\xi(t) \colon t \leq s\}$ (respectively, $\{\xi(t) \colon t \geq s\}$) is a bounded subset of $Z$.

From the previous definition, it follows that, if a process $\{S(t, \tau) \colon t \geq \tau \in \mathbb{R}\}$ has a pullback attractor $\{\mathbb{A}(t) \colon t \in \mathbb{R}\}$ and $\xi \colon \mathbb{R} \to Z$ is a backwards-bounded global solution, then $\xi(t) \in \mathbb{A}(t)$ for all $t \in \mathbb{R}$, because $\{\mathbb{A}(t) \colon t \in \mathbb{R}\}$ pullback attracts the bounded subset $\{\xi(t) \colon t \leq \tau\}$.

It is well-known that if a semigroup has a global attractor, then it is characterized as the union of all bounded global solutions. In the non-autonomous case, an equivalent characterization is given by the following result.

**Theorem 1.11.** [25, Theorem 1.17] *If an evolution process* $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ *has a pullback attractor* $\{\mathbb{A}(t)\colon t \in \mathbb{R}\}$ *which is bounded in the past, that is, the union* $\bigcup_{s\leq t} \mathbb{A}(s)$ *is bounded for each* $t \in \mathbb{R}$*, then* $\mathbb{A}(t)$ *is given by*

$$\mathbb{A}(t) = \{\xi(t)\colon\ \xi\colon \mathbb{R} \to Z \text{ is a backwards-bounded global solution}\},$$

*for all* $t \in \mathbb{R}$*.*

### 1.2.2 Continuity of attractors

The principal reason in studying the continuity of attractors lies in the fact that it ensures the robustness of these objects when the dynamical system is under small perturbations. In the literature, this part of the theory is usually divided into two main concepts: the *upper semicontinuity* and the *lower semicontinuity*.

Roughly speaking, the upper semicontinuity ensures that the original attractor does not explode when the perturbation is well behaved, while the lower semicontinuity shows that no implosion can happen, meaning that the original attractor cannot degenerate or collapse. The first property is expected to hold simply using some consequences of the existence of attractors, but the second one requires the development of a more sophisticated part of the theory about pullback attractors and also structural assumptions on the dynamics inside the attractor. For this last reason, the problems concerning the lower semicontinuity of pullback attractors will not be explored in this work.

**Definition 1.17.** Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of subsets of $Z$ indexed on a metric space $\Lambda$. We say that the family $\{A_\lambda\}_{\lambda\in\Lambda}$ is

($a$) *upper semicontinuous as* $\lambda \to \lambda_0$ (or at $\lambda_0$) if $\lim_{\lambda\to\lambda_0} d_H(A_\lambda, A_{\lambda_0}) = 0$;

($b$) *lower semicontinuous as* $\lambda \to \lambda_0$ (or at $\lambda_0$) if $\lim_{\lambda\to\lambda_0} d_H(A_{\lambda_0}, A_\lambda) = 0$;

($c$) *continuous as* $\lambda \to \lambda_0$ (or at $\lambda_0$) if it is both upper and lower semicontinuous as $\lambda \to \lambda_0$.

The following characterization is frequently used in the study of upper and lower semicontinuities.

**Lemma 1.1.** [25, Lemma 3.2] *Let* $\Lambda$ *be a metric space and let* $\{A_\lambda\}_{\lambda\in\Lambda}$ *be a family of compact subsets of* $Z$*. Then it holds that*

($i$) *the family* $\{A_\lambda\}_{\lambda\in\Lambda}$ *is upper semicontinuous as* $\lambda \to \lambda_0$ *if and only if every sequence* $x_n \in A_{\lambda_n}$ *has a convergent subsequence whose limit lies in* $A_{\lambda_0}$*, whenever* $\lambda_n \xrightarrow{n\to\infty} \lambda_0$*;*

(*ii*) *the family* $\{A_\lambda\}_{\lambda\in\Lambda}$ *is lower semicontinuous as* $\lambda \to \lambda_0$ *if and only if for any point* $x_0 \in A_{\lambda_0}$ *and any sequence* $\lambda_n \xrightarrow{n\to\infty} \lambda_0$, *there exists a sequence* $x_n \in A_{\lambda_n}$ *such that* $x_n \xrightarrow{n\to\infty} x_0$.

The next result presents sufficient conditions for a family of pullback attractors to be upper semicontinuous as a small parameter approaches to zero.

**Proposition 1.3.** [25, Proposition 1.20] *For each* $\epsilon \in [0,1]$ *let* $\{S_{(\epsilon)}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ *be an evolution process in* $Z$*. Moreover, assume that:*

(*i*) $\{S_{(\epsilon)}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ *has a pullback attractor* $\{\mathbb{A}_{(\epsilon)}(t)\colon t \in \mathbb{R}\}$ *for all* $\epsilon \in [0,1]$*;*

(*ii*) *given* $t \in \mathbb{R}$, $T \geq 0$ *and a bounded set* $D \subset Z$,

$$\sup_{s\in[0,T],u_0\in D} d(S_{(\epsilon)}(t+s,t)u_0, S_{(0)}(t+s,t)u_0) \to 0 \quad \text{as} \quad \epsilon \to 0^+;$$

(*iii*) *there exist* $\delta_0 > 0$ *and* $t_0 \in \mathbb{R}$ *such that*

$$\bigcup_{\epsilon\in(0,\delta_0)} \bigcup_{s\leq t_0} \mathbb{A}_{(\epsilon)}(s)$$

*is bounded.*

*Then, the family* $\{\mathbb{A}_{(\epsilon)}(t)\colon t \in \mathbb{R}\}$ *of pullback attractors is upper semicontinuous as* $\epsilon \to 0^+$*, that is, for each* $t \in \mathbb{R}$,

$$d_H(\mathbb{A}_{(\epsilon)}(t), \mathbb{A}_{(0)}(t)) \to 0 \quad \text{as} \quad \epsilon \to 0^+.$$

### 1.2.3 Parabolic structure

The main purpose of this subsection is to briefly introduce the reader to some terminology and facts about the theory of abstract parabolic problems. Let $X$ be a Banach space and $\{\mathcal{B}(t)\colon t \in \mathbb{R}\}$ be a family of unbounded closed linear operators, where each $\mathcal{B}(t)$ has the same dense subspace $D$ of $X$ as domain.

Consider the singularly non-autonomous abstract linear parabolic problem of the form

$$\begin{cases} \dfrac{du}{dt} = -\mathcal{B}(t)u, \ t > \tau, \\ u(\tau) = u_0 \in D. \end{cases} \tag{1.3}$$

The term *singularly non-autonomous* is used to evidence the fact that the unbounded operator $\mathcal{B}(t)$ has explicit dependence with the time. When it comes to the parabolic structure of the above problem, we assume the following conditions:

$(A1)$ The family of operators $\mathcal{B}(t)\colon D \subset X \to X$ is *uniformly sectorial* in $X$; that is, $\mathcal{B}(t)$ is closed and densely defined for every $t \in \mathbb{R}$, with domain $D$ fixed, and for all $T \in \mathbb{R}$ there exists a constant $C_1 > 0$, independent of $T$, such that

$$\|(\mathcal{B}(t) + \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_1}{|\lambda| + 1}$$

for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ and for all $t \in [-T, T]$.

$(A2)$ The map $\mathbb{R} \ni t \mapsto \mathcal{B}(t)$ is *uniformly Hölder continuous* in $X$; that is, for all $T \in \mathbb{R}$ there are constants $C_2 > 0$ and $0 < \epsilon_0 \leq 1$, both independent of $T$, such that

$$\|[\mathcal{B}(t) - \mathcal{B}(s)]\mathcal{B}^{-1}(\tau)\|_{\mathcal{L}(X)} \leq C_2|t - s|^{\epsilon_0}$$

for every $t, s, \tau \in [-T, T]$.

Denote by $\mathcal{B}_0$ the operator $\mathcal{B}(t_0)$ for some $t_0 \in \mathbb{R}$ fixed. If $X^\alpha$ denotes the domain of $\mathcal{B}_0^\alpha$, $\alpha > 0$, with the graph norm, and $X^0 = X$, then $\{X^\alpha\colon \alpha \geq 0\}$ is the fractional power scale associated with $\mathcal{B}_0$. For more details about fractional powers of operators, see Henry [35].

From $(A1)$, it follows that $-\mathcal{B}(t)$ is the generator of an analytic semigroup

$$\{e^{-\tau\mathcal{B}(t)}\colon \tau \geq 0\} \subset \mathcal{L}(X).$$

Using this and the fact that $0 \in \rho(\mathcal{B}(t))$, one can obtain a constant $C > 0$ such that the following estimates hold:

$$\|e^{-\tau\mathcal{B}(t)}\|_{\mathcal{L}(X)} \leq C,\ \ \tau \geq 0,\ \ t \in \mathbb{R},$$

and

$$\|\mathcal{B}(t)e^{-\tau\mathcal{B}(t)}\|_{\mathcal{L}(X)} \leq C\tau^{-1},\ \ \tau > 0,\ \ t \in \mathbb{R}.$$

For a given bounded set $I \subset \mathbb{R}^2$, it follows from $(A2)$, that there exists a constant $K = K(I) > 0$ such that

$$\|\mathcal{B}(t)\mathcal{B}^{-1}(\tau)\|_{\mathcal{L}(X)} \leq K,$$

for all $(t, \tau) \in I$.

Also, the semigroup $\{e^{-\tau\mathcal{B}(t)}\colon \tau \geq 0\}$ satisfies

$$\|e^{-\tau\mathcal{B}(t)}\|_{\mathcal{L}(X^\beta, X^\alpha)} \leq C(\alpha, \beta)\tau^{\beta-\alpha},\ \ \tau > 0,\ \ t \in \mathbb{R},$$

where $0 \leq \beta \leq \alpha < 1 + \epsilon_0$ (see [50]).

**Remark 1.6.** If the operator $\mathcal{B}(t)\colon D \subset X \to X$ of equation (1.3) is uniformly sectorial and uniformly Hölder continuous, then there exists a linear evolution process $\{L(t, \tau)\colon t \geq \tau \in \mathbb{R}\}$

associated with $\mathcal{B}(t)$, which is given by

$$L(t,\tau) = e^{-(t-\tau)\mathcal{B}(\tau)} + \int_\tau^t L(t,s)[\mathcal{B}(\tau) - \mathcal{B}(s)]e^{-(s-\tau)\mathcal{B}(\tau)}ds, \quad t \geq \tau.$$

Furthermore, the process $\{L(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ satisfies the following condition:

$$\|L(t,\tau)\|_{\mathcal{L}(X^\beta,X^\alpha)} \leq C(\alpha,\beta)(t-\tau)^{\beta-\alpha},$$

where $0 \leq \beta \leq \alpha < 1 + \epsilon_0$. The reader may consult [26] and [50] for more details.

Now, let us consider the following singularly non-autonomous abstract parabolic problem

$$\begin{cases} \dfrac{du}{dt} = -\mathcal{B}(t)u + g(u), \ t > \tau, \\ u(\tau) = u_0 \in D, \end{cases} \tag{1.4}$$

where the operator $\mathcal{B}(t)\colon D \subset X \to X$ is uniformly sectorial and uniformly Hölder continuous, and the nonlinearity $g$ satisfies some suitable conditions that will be specified later.

The nonlinear evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ associated with $\mathcal{B}(t)$ is given by

$$S(t,\tau) = L(t,\tau) + \int_\tau^t L(t,s)g(S(s,\tau))ds, \quad t \geq \tau.$$

**Definition 1.18.** Let $g\colon X^\alpha \to X^\beta$, $\alpha \in [\beta, \beta+1)$, be a continuous function. A continuous function $u\colon [\tau, \tau+t_0] \to X^\alpha$ is said to be a *local solution* of the problem (1.4), starting at $u_0 \in X^\alpha$, if the following conditions hold:

(*a*) $u \in C([\tau,\tau+t_0], X^\alpha) \cap C^1((\tau,\tau+t_0], X^\alpha)$;

(*b*) $u(\tau) = u_0$;

(*c*) $u(t) \in D(\mathcal{B}(t))$ for all $t \in (\tau, \tau+t_0]$;

(*d*) $u(t)$ satisfies (1.4) for all $t \in (\tau, \tau+t_0]$.

Now we state the following abstract local well-posedness result. The reader may consult [26] for a more general version that includes the critical growth case.

**Theorem 1.12.** [22, Theorem 2.3] *Assume that the family of operators $\{\mathcal{B}(t)\colon t \in \mathbb{R}\}$ is uniformly sectorial and uniformly Hölder continuous in $X^\beta$. If $g\colon X^\alpha \to X^\beta$, $\alpha \in [\beta, \beta+1)$, is a Lipschitz continuous map in bounded subsets of $X^\alpha$, then given $r > 0$ there exists a time $t_0 > 0$ such that for all $u_0 \in B_{X^\alpha}(0,r)$ there exists a unique solution of the problem (1.4) starting in $u_0$ and defined on $[\tau, \tau+t_0]$. Moreover, such solutions are continuous with respect to the initial data in $B_{X^\alpha}(0,r)$.*

Chapter

2

# A non-autonomous Klein-Gordon-Zakharov system

In this chapter, we study a non-autonomous version of the well known Klein-Gordon-Zakharov system. We consider the following initial-boundary value problem

$$\begin{cases} u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}} u_t + a_\epsilon(t)(-\Delta)^{\frac{1}{2}} v_t = f(u), & (x,t) \in \Omega \times (\tau, \infty), \\ v_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}} v_t - a_\epsilon(t)(-\Delta)^{\frac{1}{2}} u_t = 0, & (x,t) \in \Omega \times (\tau, \infty), \end{cases} \tag{2.1}$$

where $\eta$ is a positive constant, subject to boundary conditions

$$u = v = 0, \ (x,t) \in \partial\Omega \times (\tau, \infty), \tag{2.2}$$

and initial conditions

$$u(\tau, x) = u_0(x), \ u_t(\tau, x) = u_1(x), \ v(\tau, x) = v_0(x), \ v_t(\tau, x) = v_1(x), \ x \in \Omega, \ \tau \in \mathbb{R}, \tag{2.3}$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$ with $n \geq 3$, and the boundary $\partial\Omega$ is assumed to be regular enough.

All the results of this chapter that have no references are presented in the article [17].

## 2.1 Setup of the problem and general assumptions

In this section, we present the general conditions to obtain the local and global well-posedness of the problem (2.1) − (2.3) in some appropriate space which will be specified later. Assume that the function $a_\epsilon \colon \mathbb{R} \to (0, \infty)$ is continuously differentiable in $\mathbb{R}$ and satisfies the following condition:

$$0 < a_0 \leq a_\epsilon(t) \leq a_1, \tag{2.4}$$

for all $\epsilon \in [0,1]$ and $t \in \mathbb{R}$, with positive constants $a_0$ and $a_1$, and we also assume that the first derivative of $a_\epsilon$ is uniformly bounded in $t$ and $\epsilon$, that is, there exists a constant $b_0 > 0$ such that

$$|a_\epsilon'(t)| \leq b_0 \quad \text{for all} \quad t \in \mathbb{R},\ \epsilon \in [0,1]. \tag{2.5}$$

Furthermore, we assume that $a_\epsilon$ is $(\beta, C)$-Hölder continuous, for each $\epsilon \in [0,1]$; that is,

$$|a_\epsilon(t) - a_\epsilon(s)| \leq C|t-s|^\beta \tag{2.6}$$

for all $t, s \in \mathbb{R}$ and $\epsilon \in [0,1]$. Concerning the nonlinearity $f$, we assume that $f \in C^1(\mathbb{R})$ and it satisfies the dissipativeness condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} \leq 0, \tag{2.7}$$

and also satisfies the subcritical growth condition given by

$$|f'(s)| \leq c(1 + |s|^{\rho-1}), \tag{2.8}$$

for all $s \in \mathbb{R}$, where $1 < \rho < \frac{n}{n-2}$, with $n \geq 3$, and $c > 0$ is a constant.

In order to formulate the non-autonomous problem (2.1) – (2.3) in a nonlinear evolution process setting, we introduce some notations. Let $X = L^2(\Omega)$ and denote by $A\colon D(A) \subset X \to X$ the negative Laplacian operator, that is, $Au = (-\Delta)u$ for all $u \in D(A)$, where $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Thus $A$ is a positive self-adjoint operator in $X$ with compact resolvent and, therefore, $-A$ generates a compact analytic semigroup on $X$. Following Henry [35], $A$ is a sectorial operator in $X$. Now, denote by $X^\alpha$, $\alpha > 0$, the fractional power spaces associated with the operator $A$; that is, $X^\alpha = D(A^\alpha)$ endowed with the graph norm. With this notation, we have $X^{-\alpha} = (X^\alpha)'$ for all $\alpha > 0$, see [3].

In this framework, the non-autonomous problem (2.1) – (2.3) can be rewritten as an ordinary differential equation in the following abstract form

$$\begin{cases} W_t + \mathcal{A}(t)W = F(W), & t > \tau, \\ W(\tau) = W_0, & \tau \in \mathbb{R}, \end{cases} \tag{2.9}$$

where $W = W(t)$, for all $t \in \mathbb{R}$, and $W_0 = W(\tau)$ are respectively given by

$$W = \begin{bmatrix} u \\ u_t \\ v \\ v_t \end{bmatrix} \quad \text{and} \quad W_0 = \begin{bmatrix} u_0 \\ u_1 \\ v_0 \\ v_1 \end{bmatrix},$$

and, for each $t \in \mathbb{R}$, the unbounded linear operator $\mathcal{A}(t)\colon D(\mathcal{A}(t)) \subset Y \to Y$ is defined by

$$\mathcal{A}(t)\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 0 & -I & 0 & 0 \\ A+I & \eta A^{\frac{1}{2}} & 0 & a_\epsilon(t)A^{\frac{1}{2}} \\ 0 & 0 & 0 & -I \\ 0 & -a_\epsilon(t)A^{\frac{1}{2}} & A & \eta A^{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} -v \\ (A+I)u + \eta A^{\frac{1}{2}}v + a_\epsilon(t)A^{\frac{1}{2}}z \\ -z \\ -a_\epsilon(t)A^{\frac{1}{2}}v + Aw + \eta A^{\frac{1}{2}}z \end{bmatrix} \tag{2.10}$$

for each $\begin{bmatrix} u & v & w & z \end{bmatrix}^T$ in the domain $D(\mathcal{A}(t))$ defined by the space

$$D(\mathcal{A}(t)) = X^1 \times X^{\frac{1}{2}} \times X^1 \times X^{\frac{1}{2}}, \tag{2.11}$$

where

$$Y = Y_0 = X^{\frac{1}{2}} \times X \times X^{\frac{1}{2}} \times X$$

is the phase space of the problem (2.1) – (2.3). The nonlinearity $F$ is given by

$$F(W) = \begin{bmatrix} 0 \\ f^e(u) \\ 0 \\ 0 \end{bmatrix}, \tag{2.12}$$

where $f^e(u)$ is the Nemitskiĭ operator associated with $f(u)$; that is,

$$f^e(u)(x) = f(u(x)), \quad \text{for all} \quad x \in \Omega.$$

Now, we observe that the norms

$$\|(x,y,z,w)\|_1 = \|x\|_{X^{\frac{1}{2}}} + \|y\|_X + \|z\|_{X^{\frac{1}{2}}} + \|w\|_X$$

and

$$\|(x,y,z,w)\|_2 = (\|x\|^2_{X^{\frac{1}{2}}} + \|y\|^2_X + \|z\|^2_{X^{\frac{1}{2}}} + \|w\|^2_X)^{\frac{1}{2}}$$

are equivalent in $Y_0$. In this way, we shall use the same notation $\|(x,y,z,w)\|_{Y_0}$ for both norms and the choice will be as convenient.

## 2.2 Local and global well-posedness

This section concerns the investigation of the existence of the global solution for (2.9). We start by obtaining some spectral properties for the unbounded linear operator $\mathcal{A}(t)$ given in (2.10) and (2.11).

It is not difficult to see that $\det(\mathcal{A}(t)) = A(A+I)$, and therefore that $0 \in \rho(\mathcal{A}(t))$, for all $t \in \mathbb{R}$. Moreover, for each $t \in \mathbb{R}$, the operator $\mathcal{A}^{-1}(t)\colon Y_0 \to Y_0$ is defined by

$$\mathcal{A}^{-1}(t) \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} \eta A^{\frac{1}{2}}(A+I)^{-1} & (A+I)^{-1} & a_\epsilon(t) A^{\frac{1}{2}}(A+I)^{-1} & 0 \\ -I & 0 & 0 & 0 \\ -a_\epsilon(t) A^{-\frac{1}{2}} & 0 & \eta A^{-\frac{1}{2}} & A^{-1} \\ 0 & 0 & -I & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}. \tag{2.13}$$

**Proposition 2.1.** *For each fixed $t \in \mathbb{R}$, the operator $\mathcal{A}(t)$ defined in (2.10) − (2.11) is maximal accretive.*

*Proof.* The proof is analogous to the proof of [8, Proposition 4.3]. We include here only the proof of accretivity of $\mathcal{A}(t)$. Let $t \in \mathbb{R}$ be fixed and arbitrary, and let $x = \begin{bmatrix} u & v & w & z \end{bmatrix}^T \in D(\mathcal{A}(t))$. At first, we note that $\langle v, u \rangle_{X^{\frac{1}{2}}} = \left\langle (A+I)^{\frac{1}{2}} v, (A+I)^{\frac{1}{2}} u \right\rangle_X$, because from [28, Corollary 1.3.5], we have $D((A+I)^{\frac{1}{2}}) = D(A^{\frac{1}{2}})$. Thus,

$$\begin{aligned} \langle \mathcal{A}(t)x, x \rangle_{Y_0} &= \langle -v, u \rangle_{X^{\frac{1}{2}}} + \langle (A+I)u + \eta A^{\frac{1}{2}} v + a_\epsilon(t) A^{\frac{1}{2}} z, v \rangle_X + \langle -z, w \rangle_{X^{\frac{1}{2}}} \\ &\quad + \langle -a_\epsilon(t) A^{\frac{1}{2}} v + Aw + \eta A^{\frac{1}{2}} z, z \rangle_X \\ &= \left\langle (A+I)^{\frac{1}{2}} u, (A+I)^{\frac{1}{2}} v \right\rangle_X - \left\langle (A+I)^{\frac{1}{2}} v, (A+I)^{\frac{1}{2}} u \right\rangle_X \\ &\quad + a_\epsilon(t) \left( \langle A^{\frac{1}{2}} z, v \rangle_X - \langle v, A^{\frac{1}{2}} z \rangle_X \right) \\ &\quad + \langle Aw, z \rangle_X - \langle z, Aw \rangle_X + \eta \| A^{\frac{1}{4}} v \|_X^2 + \eta \| A^{\frac{1}{4}} z \|_X^2. \end{aligned}$$

Hence,

$$\mathrm{Re}(\langle \mathcal{A}(t)x, x \rangle_{Y_0}) = \eta \| A^{\frac{1}{4}} v \|_X^2 + \eta \| A^{\frac{1}{4}} z \|_X^2 \geq 0, \tag{2.14}$$

which proves the accretivity of $\mathcal{A}(t)$. □

**Remark 2.1.** By the Lumer-Phillips Theorem, we have $-\mathcal{A}(t)$ is the infinitesimal generator of a $C_0$-semigroup of contractions in $Y_0$, which we denote as $\{e^{-\tau \mathcal{A}(t)}\colon \tau \geq 0\}$, for each $t \in \mathbb{R}$.

**Proposition 2.2.** *If $Y_{-1}$ denotes the extrapolation space of $Y_0 = X^{\frac{1}{2}} \times X \times X^{\frac{1}{2}} \times X$ generated by the operator $\mathcal{A}^{-1}(t)$, then*

$$Y_{-1} = X \times X^{-\frac{1}{2}} \times X \times X^{-\frac{1}{2}}.$$

*Proof.* Recall that the extrapolation space $Y_{-1}$ is the completion of the normed space $(Y_0, \|\mathcal{A}^{-1}(t) \cdot \|_{Y_0})$. Let $x = \begin{bmatrix} u & v & w & z \end{bmatrix}^T \in X \times X^{-\frac{1}{2}} \times X \times X^{-\frac{1}{2}}$, and note that

$$\begin{aligned}\|\mathcal{A}^{-1}(t)x\|_{Y_0} &= \|\eta A^{\frac{1}{2}}(A+I)^{-1}u + (A+I)^{-1}v + a_\epsilon(t)A^{\frac{1}{2}}(A+I)^{-1}w\|_{X^{\frac{1}{2}}} + \|-u\|_X \\ &+ \|-a_\epsilon(t)A^{-\frac{1}{2}}u + \eta A^{-\frac{1}{2}}w + A^{-1}z\|_{X^{\frac{1}{2}}} + \|-w\|_X \\ &\leq \eta\|A^{\frac{1}{2}}(A+I)^{-1}u\|_{X^{\frac{1}{2}}} + \|(A+I)^{-1}v\|_{X^{\frac{1}{2}}} + a_1\|A^{\frac{1}{2}}(A+I)^{-1}w\|_{X^{\frac{1}{2}}} \\ &+ (1+a_1)\|u\|_X + (1+\eta)\|w\|_X + \|z\|_{X^{-\frac{1}{2}}}.\end{aligned} \tag{2.15}$$

Now, since $(A+I)(A+I)^{-1} = I$ and $A$ is uniformly sectorial, we have

$$\|A(A+I)^{-1}\|_{\mathcal{L}(X)} \leq 1 + \|(A+I)^{-1}\|_{\mathcal{L}(X)} \leq 1 + M, \tag{2.16}$$

for some constant $M > 0$.

Then, we can estimate the first two terms that appears in (2.15) as follows:

$$\eta\|A^{\frac{1}{2}}(A+I)^{-1}u\|_{X^{\frac{1}{2}}} = \eta\|A(A+I)^{-1}u\|_X \leq \eta(1+M)\|u\|_X, \tag{2.17}$$

and

$$\begin{aligned}\|(A+I)^{-1}v\|_{X^{\frac{1}{2}}} &= \|A^{\frac{1}{2}}(A+I)^{-1}AA^{-1}v\|_X = \|A(A+I)^{-1}A^{-\frac{1}{2}}v\|_X \\ &\leq \|A(A+I)^{-1}\|_{\mathcal{L}(X)}\|A^{-\frac{1}{2}}v\|_X \leq (1+M)\|v\|_{X^{-\frac{1}{2}}}.\end{aligned} \tag{2.18}$$

Similarly, for the third term in (2.15) we have

$$a_1\|A^{\frac{1}{2}}(A+I)^{-1}w\|_{X^{\frac{1}{2}}} \leq a_1(1+M)\|w\|_X. \tag{2.19}$$

Finally, combining (2.15), (2.17), (2.18) and (2.19) we obtain

$$\|\mathcal{A}^{-1}(t)x\|_{Y_0} \leq C_1\Big(\|u\|_X + \|v\|_{X^{-\frac{1}{2}}} + \|w\|_X + \|z\|_{X^{-\frac{1}{2}}}\Big) = C_1\|x\|_{X\times X^{-\frac{1}{2}}\times X\times X^{-\frac{1}{2}}}, \tag{2.20}$$

where $C_1$ is a positive constant.

On the other hand, taking $x = \begin{bmatrix} u & v & w & z\end{bmatrix}^T \in X \times X^{-\frac{1}{2}} \times X \times X^{-\frac{1}{2}}$, since

$$\|x\|_{X\times X^{-\frac{1}{2}}\times X\times X^{-\frac{1}{2}}} = \|u\|_X + \|v\|_{X^{-\frac{1}{2}}} + \|w\|_X + \|z\|_{X^{-\frac{1}{2}}}, \tag{2.21}$$

note that the term $\|z\|_{X^{-\frac{1}{2}}}$ can be estimated as follows:

$$\begin{aligned}\|z\|_{X^{-\frac{1}{2}}} &= \|A^{-1}z\|_{X^{\frac{1}{2}}} \\ &\leq \|-a_\epsilon(t)A^{-\frac{1}{2}}u + \eta A^{-\frac{1}{2}}w + A^{-1}z\|_{X^{\frac{1}{2}}} + \|a_\epsilon(t)A^{-\frac{1}{2}}u\|_{X^{\frac{1}{2}}} + \|\eta A^{-\frac{1}{2}}w\|_{X^{\frac{1}{2}}} \\ &\leq \|-a_\epsilon(t)A^{-\frac{1}{2}}u + \eta A^{-\frac{1}{2}}w + A^{-1}z\|_{X^{\frac{1}{2}}} + a_1\|u\|_X + \eta\|w\|_X.\end{aligned} \tag{2.22}$$

Now, to deal with the term $\|v\|_{X^{-\frac{1}{2}}}$, we use the boundedness

$$\|A^{-1}(A+I)\|_{\mathcal{L}(X)} \leq N$$

to obtain

$$\begin{aligned}
\|v\|_{X^{-\frac{1}{2}}} &= \|A^{-\frac{1}{2}}(A+I)(A+I)^{-1}v\|_X = \|A^{-1}(A+I)A^{\frac{1}{2}}(A+I)^{-1}v\|_X \\
&\leq \|A^{-1}(A+I)\|_{\mathcal{L}(X)}\|A^{\frac{1}{2}}(A+I)^{-1}v\|_X \\
&\leq N\|(A+I)^{-1}v\|_{X^{\frac{1}{2}}} \\
&\leq N\|\eta A^{\frac{1}{2}}(A+I)^{-1}u + (A+I)^{-1}v + a_\epsilon(t)A^{\frac{1}{2}}(A+I)^{-1}w\|_{X^{\frac{1}{2}}} \\
&+ N\|\eta A^{\frac{1}{2}}(A+I)^{-1}u\|_{X^{\frac{1}{2}}} + N\|a_\epsilon(t)A^{\frac{1}{2}}(A+I)^{-1}w\|_{X^{\frac{1}{2}}}
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&\leq N\|\eta A^{\frac{1}{2}}(A+I)^{-1}u + (A+I)^{-1}v + a_\epsilon(t)A^{\frac{1}{2}}(A+I)^{-1}w\|_{X^{\frac{1}{2}}} \\
&+ N\eta(1+M)\|u\|_X + Na_1(1+M)\|w\|_X.
\end{aligned}$$

Combining (2.21) with the estimates obtained in (2.22) and (2.23), it follows that

$$\begin{aligned}
\|x\|_{X\times X^{-\frac{1}{2}}\times X\times X^{-\frac{1}{2}}} &\leq N\|\eta A^{\frac{1}{2}}(A+I)^{-1}u + (A+I)^{-1}v + a_\epsilon(t)A^{\frac{1}{2}}(A+I)^{-1}w\|_{X^{\frac{1}{2}}} \\
&+ [1 + a_1 + N\eta(1+M)]\|u\|_X \\
&+ \| - a_\epsilon(t)A^{-\frac{1}{2}}u + \eta A^{-\frac{1}{2}}w + A^{-1}z\|_{X^{\frac{1}{2}}} \\
&+ [1 + \eta + Na_1(1+M)]\|w\|_X \\
&\leq C_2\|\mathcal{A}^{-1}(t)x\|_{Y_0},
\end{aligned} \tag{2.24}$$

where $C_2$ is a positive constant.

That is, from (2.20) and (2.24), we have shown that there are positive constants $C_1$ and $C_2$ such that

$$\|\mathcal{A}^{-1}(t)x\|_{Y_0} \leq C_1\|x\|_{X\times X^{-\frac{1}{2}}\times X\times X^{-\frac{1}{2}}}$$

and

$$\|x\|_{X\times X^{-\frac{1}{2}}\times X\times X^{-\frac{1}{2}}} \leq C_2\|\mathcal{A}^{-1}(t)x\|_{Y_0}$$

for all $x = \begin{bmatrix} u & v & w & z \end{bmatrix}^T \in X \times X^{-\frac{1}{2}} \times X \times X^{-\frac{1}{2}}$, which proves the desired result. □

**Proposition 2.3.** *The operator $\mathcal{A}^{-1}(t)$ given in* (2.13) *is a compact map for each $t \in \mathbb{R}$.*

*Proof.* Let $B \subset Y_0$ be a bounded set and denote $Y_1 = X^1 \times X^{\frac{1}{2}} \times X^1 \times X^{\frac{1}{2}}$. Let $x = \begin{bmatrix} u & v & w & z \end{bmatrix}^T \in B$. Thus, using the boundedness (2.16), we have

$$
\begin{aligned}
&\|\mathcal{A}^{-1}(t)x\|_{Y_1}\\
&= \|\eta A^{\frac{1}{2}}(A+I)^{-1}u + (A+I)^{-1}v + a_\epsilon(t)A^{\frac{1}{2}}(A+I)^{-1}w\|_{X^1} + \|-u\|_{X^{\frac{1}{2}}}\\
&+ \|-a_\epsilon(t)A^{-\frac{1}{2}}u + \eta A^{-\frac{1}{2}}w + A^{-1}z\|_{X^1} + \|-w\|_{X^{\frac{1}{2}}}\\
&\le \eta\|AA^{\frac{1}{2}}(A+I)^{-1}u\|_X + \|A(A+I)^{-1}v\|_X + a_1\|AA^{\frac{1}{2}}(A+I)^{-1}w\|_X + \|u\|_{X^{\frac{1}{2}}}\\
&+ a_1\|u\|_{X^{\frac{1}{2}}} + \eta\|w\|_{X^{\frac{1}{2}}} + \|z\|_X + \|w\|_{X^{\frac{1}{2}}}\\
&\le \eta\|A(A+I)^{-1}\|_{\mathcal{L}(X)}\|A^{\frac{1}{2}}u\|_X + \|A(A+I)^{-1}\|_{\mathcal{L}(X)}\|v\|_X\\
&+ a_1\|A(A+I)^{-1}\|_{\mathcal{L}(X)}\|A^{\frac{1}{2}}w\|_X + (1+a_1)\|u\|_{X^{\frac{1}{2}}} + (1+\eta)\|w\|_{X^{\frac{1}{2}}} + \|z\|_X\\
&\le [\eta(1+M)+1+a_1]\|u\|_{X^{\frac{1}{2}}} + (1+M)\|v\|_X + [a_1(1+M)+1+\eta]\|w\|_{X^{\frac{1}{2}}} + \|z\|_X\\
&\le C\left(\|u\|_{X^{\frac{1}{2}}} + \|v\|_X + \|w\|_{X^{\frac{1}{2}}} + \|z\|_X\right),
\end{aligned}
$$

where $C$ is a positive constant, that is,

$$\|\mathcal{A}^{-1}(t)x\|_{Y_1} \le C\|x\|_{Y_0}.$$

Thus, $\mathcal{A}^{-1}(t)B$ is bounded in $Y_1$. Using the compact embedding $Y_1 \hookrightarrow Y_0$, we conclude that the operator $\mathcal{A}^{-1}(t)$ is compact. □

**Proposition 2.4.** *The family of operators* $\{\mathcal{A}(t)\colon t \in \mathbb{R}\}$, *defined in* (2.10)−(2.11), *is uniformly Hölder continuous in* $Y_{-1}$.

*Proof.* Firstly, note that

$$\mathcal{A}(t) - \mathcal{A}(s) = [a_\epsilon(t) - a_\epsilon(s)]\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & A^{\frac{1}{2}}\\ 0 & 0 & 0 & 0\\ 0 & -A^{\frac{1}{2}} & 0 & 0 \end{bmatrix}$$

for all $t, s \in \mathbb{R}$. Consequently,

$$[\mathcal{A}(t) - \mathcal{A}(s)]\mathcal{A}^{-1}(\tau) = [a_\epsilon(t) - a_\epsilon(s)]\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -A^{\frac{1}{2}} & 0\\ 0 & 0 & 0 & 0\\ A^{\frac{1}{2}} & 0 & 0 & 0 \end{bmatrix}$$

for all $t, s, \tau \in \mathbb{R}$. Now, let $T \in \mathbb{R}$ be fixed. Given $x = \begin{bmatrix} u & v & w & z \end{bmatrix}^T \in Y_{-1}$, if $t, s, \tau \in [-T, T]$, then

$$\begin{aligned}\|[\mathcal{A}(t)-\mathcal{A}(s)]\mathcal{A}^{-1}(\tau)x\|_{Y_{-1}} &= |a_\epsilon(t)-a_\epsilon(s)| \left\| \begin{bmatrix} 0 & -A^{\frac{1}{2}}w & 0 & A^{\frac{1}{2}}u \end{bmatrix}^T \right\|_{Y_{-1}} \\ &= |a_\epsilon(t)-a_\epsilon(s)| \left( \|-A^{\frac{1}{2}}w\|_{X^{-\frac{1}{2}}} + \|A^{\frac{1}{2}}u\|_{X^{-\frac{1}{2}}} \right) \\ &\le C|t-s|^\beta (\|u\|_X + \|w\|_X) \\ &\le C|t-s|^\beta \left( \|u\|_X + \|v\|_{X^{-\frac{1}{2}}} + \|w\|_X + \|z\|_{X^{-\frac{1}{2}}} \right) \\ &= C|t-s|^\beta \|x\|_{Y_{-1}}.\end{aligned}$$

From this we obtain

$$\|[\mathcal{A}(t)-\mathcal{A}(s)]\mathcal{A}^{-1}(\tau)\|_{\mathcal{L}(Y_{-1})} \le C|t-s|^\beta$$

for all $t, s, \tau \in [-T, T]$. Since $T \in \mathbb{R}$ is arbitrary, this ends the proof of the result. □

The next step is to show the analyticity of the semigroup $\{e^{-\tau\mathcal{A}(t)} : \tau \ge 0\}$. For that, we will make use of the following auxiliary result.

**Theorem 2.1.** [40, Theorem 1.3.3] *Let $\{T(\tau) : \tau \ge 0\}$ be a $C_0$-semigroup of contractions in a Hilbert space $H$ with infinitesimal generator $\mathcal{B}$. Suppose that $i\mathbb{R} \subset \rho(\mathcal{B})$. Then $\{T(\tau) : \tau \ge 0\}$ is analytic if, and only if*

$$\limsup_{|\beta|\to\infty} \|\beta(i\beta I - \mathcal{B})^{-1}\|_{\mathcal{L}(H)} < \infty.$$

The next lemma shows that $i\mathbb{R} \subset \rho(-\mathcal{A}(t))$ for all $t \in \mathbb{R}$.

**Lemma 2.1.** *The semigroup $\{e^{-\tau\mathcal{A}(t)} : \tau \ge 0\}$, generated by $-\mathcal{A}(t)$, satisfies*

$$i\mathbb{R} \subset \rho(-\mathcal{A}(t))$$

*for all $t \in \mathbb{R}$.*

*Proof.* Arguing by contradiction, suppose that there exists $0 \ne \beta \in \mathbb{R}$ such that $i\beta$ is in the spectrum of $-\mathcal{A}(t)$ for some $t \in \mathbb{R}$. Then $i\beta$ must be an eigenvalue of $-\mathcal{A}(t)$, since the operator $\mathcal{A}^{-1}(t)$ is compact. Consequently, there exists

$$U = \begin{bmatrix} u & v & w & z \end{bmatrix}^T \in D(\mathcal{A}(t)), \ \|U\|_{Y_0} = 1,$$

such that $i\beta U - (-\mathcal{A}(t))U = 0$ or, equivalently,

$$
\begin{aligned}
i\beta u - v &= 0,\\
i\beta v + Au + u + \eta A^{\frac{1}{2}} v + a_\epsilon(t) A^{\frac{1}{2}} z &= 0,\\
i\beta w - z &= 0,\\
i\beta z - a_\epsilon(t) A^{\frac{1}{2}} v + Aw + \eta A^{\frac{1}{2}} z &= 0.
\end{aligned}
$$

Now, taking the real part of the inner product of $i\beta U + \mathcal{A}(t)U$ with $U$ in $Y_0$, we have

$$
\begin{aligned}
\langle i\beta U + \mathcal{A}(t)U, U\rangle_{Y_0} = \langle 0, U\rangle_{Y_0} = 0 &\implies i\beta \|U\|_{Y_0}^2 + \langle \mathcal{A}(t)U, U\rangle_{Y_0} = 0\\
&\implies \mathrm{Re}(\langle \mathcal{A}(t)U, U\rangle_{Y_0}) = 0\\
&\implies \eta \|A^{\frac{1}{4}} v\|_X^2 + \eta \|A^{\frac{1}{4}} z\|_X^2 = 0\\
&\implies \|A^{\frac{1}{4}} v\|_X^2 = \|A^{\frac{1}{4}} z\|_X^2 = 0\\
&\implies v = z = 0.
\end{aligned}
$$

Consequently, $u = w = 0$. Therefore, $U = 0$, which is a contradiction. This proves our claim. □

Now, we are in position to prove that the semigroup generated by $-\mathcal{A}(t)$ is analytic.

**Theorem 2.2.** *The semigroup $\{e^{-\tau \mathcal{A}(t)} \colon \tau \geq 0\}$, generated by $-\mathcal{A}(t)$, is analytic for each $t \in \mathbb{R}$.*

*Proof.* We are going to use Theorem 2.1. Let $t \in \mathbb{R}$. In view of Lemma 2.1, it is enough to prove that there exists a positive constant $C$ such that

$$
|\beta| \|U\|_{Y_0} \leq C \|F\|_{Y_0},
$$

for all $F \in Y_0$ and all $\beta \in \mathbb{R}$, where

$$
U = (i\beta I + \mathcal{A}(t))^{-1} F \in D(\mathcal{A}(t)).
$$

In fact, denoting $U = \begin{bmatrix} u & v & w & z \end{bmatrix}^T$ and $F = \begin{bmatrix} f & g & h & k \end{bmatrix}^T$, we can write the resolvent equation

$$
(i\beta I + \mathcal{A}(t))U = F \tag{2.25}
$$

in $Y_0$ in terms of its components, obtaining the following scalar equations

$$
i\beta u - v = f,
$$

$$
Au + u + i\beta v + \eta A^{\frac{1}{2}} v + a_\epsilon(t) A^{\frac{1}{2}} z = g, \tag{2.26}
$$

$$
i\beta w - z = h, \tag{2.27}
$$

$$-a_\epsilon(t)A^{\frac{1}{2}}v + Aw + i\beta z + \eta A^{\frac{1}{2}}z = k.$$

Taking the inner product of (2.25) with $U$ in $Y_0$, we obtain

$$i\beta\|U\|_{Y_0}^2 + \langle \mathcal{A}(t)U, U\rangle_{Y_0} = \langle F, U\rangle_{Y_0}. \tag{2.28}$$

By the proof of Proposition 2.1, see (2.14), we get

$$\mathrm{Re}(\langle \mathcal{A}(t)U, U\rangle_{Y_0}) = \eta\|A^{\frac{1}{4}}v\|_X^2 + \eta\|A^{\frac{1}{4}}z\|_X^2 \geq 0.$$

It follows by the Cauchy-Schwartz inequality that

$$\eta\|A^{\frac{1}{4}}v\|_X^2 + \eta\|A^{\frac{1}{4}}z\|_X^2 = |\mathrm{Re}(\langle \mathcal{A}(t)U, U\rangle_{Y_0})| = |\mathrm{Re}(\langle F, U\rangle_{Y_0})| \leq |\langle F, U\rangle_{Y_0}| \leq \|F\|_{Y_0}\|U\|_{Y_0}$$

and, therefore, we obtain

$$\|A^{\frac{1}{4}}v\|_X^2 \leq \frac{1}{\eta}\|F\|_{Y_0}\|U\|_{Y_0} \quad \text{and} \quad \|A^{\frac{1}{4}}z\|_X^2 \leq \frac{1}{\eta}\|F\|_{Y_0}\|U\|_{Y_0}. \tag{2.29}$$

Now, taking the inner product of (2.25) with $x_1 = \begin{bmatrix} A^{-\frac{1}{2}}v & 0 & 0 & 0\end{bmatrix}^T$ in $Y_0$, it leads to

$$\begin{aligned}
\langle (i\beta I + \mathcal{A}(t))U, x_1\rangle_{Y_0} = \langle F, x_1\rangle_{Y_0} &\iff \langle i\beta u - v, A^{-\frac{1}{2}}v\rangle_{X^{\frac{1}{2}}} = \langle f, A^{-\frac{1}{2}}v\rangle_{X^{\frac{1}{2}}} \\
&\iff i\beta\langle A^{\frac{1}{2}}u, v\rangle_X - \langle A^{\frac{1}{2}}v, v\rangle_X = \langle A^{\frac{1}{2}}f, v\rangle_X \\
&\iff \langle A^{\frac{1}{2}}u, -i\beta v\rangle_X - \|A^{\frac{1}{4}}v\|_X^2 = \langle A^{\frac{1}{2}}f, v\rangle_X
\end{aligned}$$

and then, using (2.26), we conclude that

$$\langle A^{\frac{1}{2}}u, Au + u + \eta A^{\frac{1}{2}}v + a_\epsilon(t)A^{\frac{1}{2}}z - g\rangle_X - \|A^{\frac{1}{4}}v\|_X^2 = \langle A^{\frac{1}{2}}f, v\rangle_X.$$

Thus, from Cauchy-Schwartz and Young inequalities and (2.29), we obtain

$$\begin{aligned}
&\|A^{\frac{3}{4}}u\|_X^2 \\
&= -\|A^{\frac{1}{4}}u\|_X^2 - \eta\langle A^{\frac{3}{4}}u, A^{\frac{1}{4}}v\rangle_X - a_\epsilon(t)\langle A^{\frac{3}{4}}u, A^{\frac{1}{4}}z\rangle_X + \langle A^{\frac{1}{2}}u, g\rangle_X + \langle A^{\frac{1}{2}}f, v\rangle_X + \|A^{\frac{1}{4}}v\|_X^2 \\
&\leq \eta\|A^{\frac{3}{4}}u\|_X\|A^{\frac{1}{4}}v\|_X + a_1\|A^{\frac{3}{4}}u\|_X\|A^{\frac{1}{4}}z\|_X + \|A^{\frac{1}{2}}u\|_X\|g\|_X + \|A^{\frac{1}{2}}f\|_X\|v\|_X + \|A^{\frac{1}{4}}v\|_X^2 \\
&\leq \frac{\epsilon_1}{2}\eta^2\|A^{\frac{3}{4}}u\|_X^2 + \frac{1}{2\epsilon_1}\|A^{\frac{1}{4}}v\|_X^2 + \frac{\epsilon_2}{2}a_1^2\|A^{\frac{3}{4}}u\|_X^2 + \frac{1}{2\epsilon_2}\|A^{\frac{1}{4}}z\|_X^2 + \left(\frac{1}{\eta} + 2\right)\|F\|_{Y_0}\|U\|_{Y_0} \\
&\leq \left(\frac{\epsilon_1}{2}\eta^2 + \frac{\epsilon_2}{2}a_1^2\right)\|A^{\frac{3}{4}}u\|_X^2 + \left(\frac{1}{2\eta\epsilon_1} + \frac{1}{2\eta\epsilon_2} + \frac{1}{\eta} + 2\right)\|F\|_{Y_0}\|U\|_{Y_0},
\end{aligned}$$

for all $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Now, it is enough to choose $\epsilon_1 = \frac{1}{2\eta^2}$ and $\epsilon_2 = \frac{1}{2a_1^2}$, and so we get

$$\|A^{\frac{3}{4}}u\|_X^2 \leq \left(2\eta + \frac{2(a_1^2+1)}{\eta} + 4\right)\|F\|_{Y_0}\|U\|_{Y_0}. \tag{2.30}$$

Next, taking the inner product of (2.25) with $x_2 = \begin{bmatrix} 0 & 0 & 0 & A^{\frac{1}{2}}w \end{bmatrix}^T$, we have

$$\langle (i\beta I + \mathcal{A}(t))U, x_2\rangle_{Y_0} = \langle F, x_2\rangle_{Y_0}$$

$$\langle -a_\epsilon(t)A^{\frac{1}{2}}v + Aw + i\beta z + \eta A^{\frac{1}{2}}z, A^{\frac{1}{2}}w\rangle_X = \langle k, A^{\frac{1}{2}}w\rangle_X$$

that is,

$$-a_\epsilon(t)\langle A^{\frac{1}{2}}v, A^{\frac{1}{2}}w\rangle_X + \|A^{\frac{3}{4}}w\|_X^2 + \langle A^{\frac{1}{2}}z, -i\beta w\rangle_X + \eta\langle A^{\frac{1}{2}}z, A^{\frac{1}{2}}w\rangle_X = \langle k, A^{\frac{1}{2}}w\rangle_X$$

and then, using (2.27), we have

$$\|A^{\frac{3}{4}}w\|_X^2 = a_\epsilon(t)\langle A^{\frac{1}{4}}v, A^{\frac{3}{4}}w\rangle_X - \eta\langle A^{\frac{1}{4}}z, A^{\frac{3}{4}}w\rangle_X + \|A^{\frac{1}{4}}z\|_X^2 + \langle A^{\frac{1}{2}}z, h\rangle_X + \langle k, A^{\frac{1}{2}}w\rangle_X.$$

Using again the Cauchy-Schwartz and Young inequalities, and (2.29), we obtain

$$\begin{aligned}
&\|A^{\frac{3}{4}}w\|_X^2 \\
&\leq a_1\|A^{\frac{1}{4}}v\|_X\|A^{\frac{3}{4}}w\|_X + \eta\|A^{\frac{1}{4}}z\|_X\|A^{\frac{3}{4}}w\|_X + \|A^{\frac{1}{4}}z\|_X^2 + \|z\|_X\|A^{\frac{1}{2}}h\|_X + \|k\|_X\|A^{\frac{1}{2}}w\|_X \\
&\leq \frac{\epsilon_3}{2}a_1^2\|A^{\frac{3}{4}}w\|_X^2 + \frac{1}{2\epsilon_3}\|A^{\frac{1}{4}}v\|_X^2 + \frac{\epsilon_4}{2}\eta^2\|A^{\frac{3}{4}}w\|_X^2 + \frac{1}{2\epsilon_4}\|A^{\frac{1}{4}}z\|_X^2 + \left(\frac{1}{\eta}+2\right)\|F\|_{Y_0}\|U\|_{Y_0} \\
&\leq \left(\frac{\epsilon_3}{2}a_1^2 + \frac{\epsilon_4}{2}\eta^2\right)\|A^{\frac{3}{4}}w\|_X^2 + \left(\frac{1}{2\eta\epsilon_3} + \frac{1}{2\eta\epsilon_4} + \frac{1}{\eta} + 2\right)\|F\|_{Y_0}\|U\|_{Y_0},
\end{aligned}$$

for all $\epsilon_3 > 0$ and $\epsilon_4 > 0$. Choosing $\epsilon_3 = \frac{1}{2a_1^2}$ and $\epsilon_4 = \frac{1}{2\eta^2}$, we get

$$\|A^{\frac{3}{4}}w\|_X^2 \leq \left(2\eta + \frac{2(a_1^2+1)}{\eta} + 4\right)\|F\|_{Y_0}\|U\|_{Y_0}. \tag{2.31}$$

By [28, Corollary 1.3.5], we have $D((A+I)^{\frac{1}{2}}) = D(A^{\frac{1}{2}})$, consequently,

$$\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\rangle_X = \langle u, v\rangle_{X^{\frac{1}{2}}} = \left\langle (A+I)^{\frac{1}{2}}u, (A+I)^{\frac{1}{2}}v\right\rangle_X.$$

Using this fact and the proof of Proposition 2.1, we obtain

$$\begin{aligned}
\langle \mathcal{A}(t)U, U\rangle_{Y_0} &= \left\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\right\rangle_X - \left\langle A^{\frac{1}{2}}v, A^{\frac{1}{2}}u\right\rangle_X \\
&\quad + a_\epsilon(t)\left(\langle A^{\frac{1}{2}}z, v\rangle_X - \langle v, A^{\frac{1}{2}}z\rangle_X\right) \\
&\quad + \langle Aw, z\rangle_X - \langle z, Aw\rangle_X + \eta\|A^{\frac{1}{4}}v\|_X^2 + \eta\|A^{\frac{1}{4}}z\|_X^2,
\end{aligned}$$

and, taking the imaginary part, we have

$$\begin{aligned}\mathrm{Im}(\langle \mathcal{A}(t)U, U\rangle_{Y_0}) &= 2\mathrm{Im}(\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\rangle_X) + 2a_\epsilon(t)\mathrm{Im}(\langle A^{\frac{1}{4}}z, A^{\frac{1}{4}}v\rangle_X)\\ &+ 2\mathrm{Im}(\langle A^{\frac{3}{4}}w, A^{\frac{1}{4}}z\rangle_X)\\ &= 2\mathrm{Im}(\langle A^{\frac{3}{4}}u, A^{\frac{1}{4}}v\rangle_X) + 2a_\epsilon(t)\mathrm{Im}(\langle A^{\frac{1}{4}}z, A^{\frac{1}{4}}v\rangle_X)\\ &+ 2\mathrm{Im}(\langle A^{\frac{3}{4}}w, A^{\frac{1}{4}}z\rangle_X).\end{aligned}$$

With this last equality and taking the imaginary part in (2.28), it follows by the Cauchy-Schwartz and Young inequalities that

$$\begin{aligned}&\beta\|U\|^2_{Y_0} = \mathrm{Im}(\langle F, U\rangle_{Y_0}) - \mathrm{Im}(\langle \mathcal{A}(t)U, U\rangle_{Y_0})\\ &\leq \|F\|_{Y_0}\|U\|_{Y_0} + 2\|A^{\frac{3}{4}}u\|_X\|A^{\frac{1}{4}}v\|_X + 2a_1\|A^{\frac{1}{4}}z\|_X\|A^{\frac{1}{4}}v\|_X + 2\|A^{\frac{3}{4}}w\|_X\|A^{\frac{1}{4}}z\|_X\\ &\leq \|F\|_{Y_0}\|U\|_{Y_0} + \|A^{\frac{3}{4}}u\|^2_X + (1+a_1)\|A^{\frac{1}{4}}v\|^2_X + \|A^{\frac{3}{4}}w\|^2_X + (a_1+1)\|A^{\frac{1}{4}}z\|^2_X\end{aligned}$$

and, using the estimates obtained in (2.29), (2.30) and (2.31), we get

$$\beta\|U\|^2_{Y_0} \leq \left(1 + 2\left(2\eta + \frac{2(a_1^2+1)}{\eta} + 4\right) + \frac{2a_1+2}{\eta}\right)\|F\|_{Y_0}\|U\|_{Y_0},$$

that is, there exists a positive constant $C$, independent of $\beta$, such that

$$\beta\|(i\beta I + \mathcal{A}(t))^{-1}F\|_{Y_0} \leq C\|F\|_{Y_0}$$

for all $F \in Y_0$ and all $\beta \in \mathbb{R}$. Since this holds for $\beta \in \mathbb{R}$ arbitrary,

$$|\beta|\|(i\beta I + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(Y_0)} \leq C, \quad \text{for all} \quad \beta \in \mathbb{R},$$

and, therefore, we conclude that

$$\limsup_{|\beta|\to\infty} \|\beta(i\beta I + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(Y_0)} < \infty.$$

By Theorem 2.1, the semigroup $\{e^{-\tau\mathcal{A}(t)} : \tau \geq 0\}$ is analytic. □

Next, we present an auxiliary result about the class of nonlinearities that are being considered in this work.

**Lemma 2.2.** [21, Lemma 2.4] *Let $f \in C^1(\mathbb{R})$ be a function such that there are constants $c > 0$ and $\rho > 1$ such that $|f'(s)| \leq c(1 + |s|^{\rho-1})$ for all $s \in \mathbb{R}$. Then*

$$|f(s) - f(t)| \leq 2^{\rho-1}c|s-t|(1 + |s|^{\rho-1} + |t|^{\rho-1})$$

*for all* $s, t \in \mathbb{R}$.

*Proof.* Recall that for $a, b, s > 0$, one has $(a+b)^s \leq 2^s \max\{a^s, b^s\} \leq 2^s(a^s + b^s)$. Now, given $s, t \in \mathbb{R}$, it follows from the Mean Value Theorem that there exists $\theta \in (0,1)$ such that

$$\begin{aligned}
|f(s) - f(t)| &= |s-t||f'(s(1-\theta) + t\theta)| \\
&\leq c|s-t|(1 + |s(1-\theta) + t\theta|^{\rho-1}) \\
&\leq 2^{\rho-1}c|s-t|(1 + |s(1-\theta)|^{\rho-1} + |t\theta|^{\rho-1}) \\
&\leq 2^{\rho-1}c|s-t|(1 + |s|^{\rho-1} + |t|^{\rho-1}),
\end{aligned}$$

which proves the result. □

**Remark 2.2.** We have the following description of the fractional power scale for the operator $\mathcal{A}(t)$, given as follows

$$Y_0 \hookrightarrow Y_{\alpha-1} \hookrightarrow Y_{-1}, \quad \text{for all} \quad 0 < \alpha < 1,$$

where

$$\begin{aligned}
Y_{\alpha-1} = [Y_{-1}, Y_0]_\alpha &= [X \times X^{-\frac{1}{2}} \times X \times X^{-\frac{1}{2}}, X^{\frac{1}{2}} \times X \times X^{\frac{1}{2}} \times X]_\alpha \\
&= [X, X^{\frac{1}{2}}]_\alpha \times [X^{-\frac{1}{2}}, X]_\alpha \times [X, X^{\frac{1}{2}}]_\alpha \times [X^{-\frac{1}{2}}, X]_\alpha \\
&= X^{\frac{\alpha}{2}} \times X^{\frac{\alpha-1}{2}} \times X^{\frac{\alpha}{2}} \times X^{\frac{\alpha-1}{2}},
\end{aligned}$$

where $[\cdot,\cdot]_\alpha$ denotes the complex interpolation functor, see [51]. The first equality follows from Proposition 2.1 (recall that $0 \in \rho(\mathcal{A}(t))$), see [3, Example 4.7.3 (b)] and the others equalities follow from [24, Proposition 2].

In what follows, Proposition 2.5 below provides sufficient conditions for the nonlinearity $F\colon Y_0 \to Y_{\alpha-1}$ to be Lipschitz continuous in bounded subsets of $Y_0$.

**Proposition 2.5.** *Assume that* $1 < \rho < \frac{n+2(1-\alpha)}{n-2}$, *with* $\alpha \in (0,1)$. *Then the map* $F\colon Y_0 \to Y_{\alpha-1}$, *defined in* (2.12), *is Lipschitz continuous in bounded subsets of* $Y_0$.

*Proof.* Let $x_i = \begin{bmatrix} u_i & v_i & w_i & z_i \end{bmatrix}^T \in Y_0$ for $i = 1, 2$. Then, from Lemma 2.2, and using Hölder's inequality, we have

$$\begin{aligned}
&\|F(x_1) - F(x_2)\|_{Y_{\alpha-1}} = \|f^e(u_1) - f^e(u_2)\|_{X^{\frac{\alpha-1}{2}}} \\
&\leq c_1 \|f^e(u_1) - f^e(u_2)\|_{L^{\frac{2n}{n+2(1-\alpha)}}(\Omega)} \\
&= c_1 \left( \int_\Omega |f^e(u_1)(x) - f^e(u_2)(x)|^{\frac{2n}{n+2(1-\alpha)}} dx \right)^{\frac{n+2(1-\alpha)}{2n}} \\
&= c_1 \left( \int_\Omega |f(u_1(x)) - f(u_2(x))|^{\frac{2n}{n+2(1-\alpha)}} dx \right)^{\frac{n+2(1-\alpha)}{2n}}
\end{aligned}$$

$$\begin{aligned}
&\le c_2\left(\int_\Omega \left[|u_1(x)-u_2(x)|\left(1+|u_1(x)|^{\rho-1}+|u_2(x)|^{\rho-1}\right)\right]^{\frac{2n}{n+2(1-\alpha)}}dx\right)^{\frac{n+2(1-\alpha)}{2n}}\\
&\le c_2\left(\int_\Omega |u_1(x)-u_2(x)|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{2n}}\left(\int_\Omega \left|1+|u_1(x)|^{\rho-1}+|u_2(x)|^{\rho-1}\right|^{\frac{n}{2-\alpha}}dx\right)^{\frac{2-\alpha}{n}}\\
&\le c_2\|u_1-u_2\|_{L^{\frac{2n}{n-2}}(\Omega)}\|1+|u_1|^{\rho-1}+|u_2|^{\rho-1}\|_{L^{\frac{n}{2-\alpha}}(\Omega)}\\
&\le c_3\|u_1-u_2\|_{L^{\frac{2n}{n-2}}(\Omega)}\left(1+\|u_1\|^{\rho-1}_{L^{\frac{n(\rho-1)}{2-\alpha}}(\Omega)}+\|u_2\|^{\rho-1}_{L^{\frac{n(\rho-1)}{2-\alpha}}(\Omega)}\right),
\end{aligned}$$

where $c_1 > 0$ is the embedding constant of $L^{\frac{2n}{n+2(1-\alpha)}}(\Omega) \hookrightarrow X^{\frac{\alpha-1}{2}}$. Moreover, we have the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ holds if and only if $p \le \frac{2n}{n-2}$. Since $\rho < \frac{n+2(1-\alpha)}{n-2}$ if and only if $\frac{n(\rho-1)}{2-\alpha} < \frac{2n}{n-2}$, then we obtain

$$H^1(\Omega) \hookrightarrow L^{\frac{n(\rho-1)}{2-\alpha}}(\Omega).$$

Therefore,

$$\begin{aligned}
\|F(x_1)-F(x_2)\|_{Y_{\alpha-1}} &\le c_4\|u_1-u_2\|_{X^{\frac12}}\left(1+\|u_1\|^{\rho-1}_{X^{\frac12}}+\|u_2\|^{\rho-1}_{X^{\frac12}}\right)\\
&\le c_4\|x_1-x_2\|_{Y_0}\left(1+\|x_1\|^{\rho-1}_{Y_0}+\|x_2\|^{\rho-1}_{Y_0}\right),
\end{aligned}$$

for some constant $c_4 > 0$, which concludes the proof. □

Proposition 2.5 and Theorem 1.12 ensure the local well-posedness of (2.9) in the phase space $Y_0$, and this allows us to establish the following existence result.

**Corollary 2.1.** *Let $1 < \rho < \frac{n+2(1-\alpha)}{n-2}$, with $\alpha \in (0,1)$, and let $f \in C^1(\mathbb{R})$ be a function satisfying* (2.7)-(2.8). *Assume that conditions* (2.4)-(2.6) *hold and let $F\colon Y_0 \to Y_{\alpha-1}$ be defined as in* (2.12). *Then given $r > 0$, there exists a time $t_0 = t_0(r) > 0$ such that for all $W_0 \in B_{Y_0}(0,r)$, there exists a unique solution $W\colon [\tau,\tau+t_0] \to Y_0$ of the problem* (2.9) *starting in $W_0$. Moreover, such solutions are continuous with respect to the initial data in $B_{Y_0}(0,r)$.*

In order to obtain the global well-posedness of solutions, we give an auxiliary result.

**Lemma 2.3.** [4, Proposition 4.1] [28, Observation 6.2.1] *Let $f \in C^1(\mathbb{R})$ be a function satisfying* (2.7)-(2.8). *The following conditions hold:*

(*i*) *There exists a constant $c > 0$ such that $|f(s)| \le c(1+|s|^\rho)$ for all $s \in \mathbb{R}$.*

(*ii*) *Given $\delta > 0$, there exists a constant $C_\delta > 0$ such that*

$$\int_\Omega f(u)u dx \le C_\delta + \delta\|u\|_X^2 \quad \text{and} \quad \int_\Omega\int_0^u f(s)dsdx \le C_\delta + \delta\|u\|_X^2,$$

*for all $u \in X$.*

(*iii*) *Given $r > 0$, there exist constants $C_r > 0$ and $C > 0$ (which does not depend on $r$) such that*

$$\left|\int_\Omega f(u)udx\right| \le C_r\|u\|^2_{X^{\frac12}} \quad \textit{and} \quad \left|\int_\Omega \int_0^u f(s)dsdx\right| \le C_r\|u\|^2_{X^{\frac12}} + C$$

*for all $u \in X^{\frac12}$ with $\|u\|_{X^{\frac12}} \le r$.*

*Proof.* (*i*) In view of Lemma 2.2, there are constants $c > 0$ and $\rho > 1$ such that

$$|f(s)| - |f(t)| \le |f(s) - f(t)| \le 2^{\rho-1}c|s-t|(1 + |s|^{\rho-1} + |t|^{\rho-1})$$

for all $s, t \in \mathbb{R}$. On the other hand, using the Young inequality $ab \le \epsilon\frac{a^p}{p} + \frac{1}{\epsilon^{q/p}}\frac{b^q}{q}$ with $\epsilon = 1, a = |s|, b = 1, p = \rho$ and $q = \frac{\rho}{\rho-1}$, we obtain

$$|s| \le \frac{|s|^\rho}{\rho} + \frac{\rho-1}{\rho} \quad \text{for all} \quad s \in \mathbb{R}.$$

Hence,

$$\begin{aligned}
|f(s)| &\le |f(0)| + 2^{\rho-1}c|s|(1 + |s|^{\rho-1}) = |f(0)| + 2^{\rho-1}c(|s| + |s|^\rho) \\
&\le |f(0)| + 2^{\rho-1}c\left(\frac{|s|^\rho}{\rho} + \frac{\rho-1}{\rho} + |s|^\rho\right) \\
&\le |f(0)| + 2^{\rho-1}c(1 + |s|^\rho)\max\left\{\frac{\rho-1}{\rho}, \frac{1+\rho}{\rho}\right\} \\
&\le |f(0)| + 2^{\rho-1}c(1 + |s|^\rho)\left(\frac{\rho-1}{\rho} + \frac{1+\rho}{\rho}\right) \\
&= |f(0)| + 2^\rho c(1 + |s|^\rho) \le (|f(0)| + 2^\rho c)(1 + |s|^\rho).
\end{aligned}$$

Therefore, $f$ satisfies $|f(s)| \le \tilde{c}(1 + |s|^\rho)$ for all $s \in \mathbb{R}$ and some constant $\tilde{c} > 0$.
(*ii*) We shall make use of the dissipativeness condition (2.7). In fact, to deal with the first assertion, note that, by (2.7), for all $\delta > 0$ given, there exists $R_\delta > 0$ such that, for $|s| > R_\delta$, one has $\frac{f(s)}{s} \le \delta$ and, therefore, $f(s)s \le \delta s^2$. Moreover, since the function $\mathbb{R} \ni s \mapsto f(s)s$ is bounded on the interval $[-R_\delta, R_\delta]$, there exists $M_\delta > 0$ such that

$$f(s)s \le M_\delta + \delta s^2 \quad \text{for all} \quad s \in \mathbb{R}.$$

Thus, given $u \in X$, we have $f(u(x))u(x) \le M_\delta + \delta u^2(x)$ for all $x \in \Omega$ and then, integrating this inequality on $\Omega$, we obtain

$$\int_\Omega f(u)udx \le M_\delta|\Omega| + \delta\|u\|^2_X. \tag{2.32}$$

Now, let us show the second assertion of this item. For a given $\delta > 0$, let $R_\delta > 0$ be as above. Since $\mathbb{R} \ni s \mapsto f(s)$ is bounded on the interval $[-R_\delta, R_\delta]$, there exists $m_\delta > 0$ such that $|f(s)| \leq m_\delta$ for all $s \in [-R_\delta, R_\delta]$. At first, we claim that

$$\int_0^s f(\theta)d\theta \leq m_\delta R_\delta + \delta s^2$$

for all $s \in \mathbb{R}$. In fact, we have some cases to consider. If $s \in [0, R_\delta]$, then

$$\begin{aligned}\int_0^s f(\theta)d\theta &\leq \int_0^s |f(\theta)|d\theta \leq \int_0^{R_\delta} |f(\theta)|d\theta \\ &\leq \int_0^{R_\delta} m_\delta d\theta = m_\delta R_\delta \leq m_\delta R_\delta + \delta s^2.\end{aligned}$$

If $s > R_\delta$, then we have

$$\int_0^s f(\theta)d\theta = \int_0^{R_\delta} f(\theta)d\theta + \int_{R_\delta}^s f(\theta)d\theta \leq \int_0^{R_\delta} |f(\theta)|d\theta + \int_{R_\delta}^s \delta\theta d\theta$$

$$\leq \int_0^{R_\delta} m_\delta d\theta + \delta\left(\frac{s^2}{2} - \frac{R_\delta^2}{2}\right) \leq m_\delta R_\delta + \frac{\delta}{2}s^2 < m_\delta R_\delta + \delta s^2.$$

If $s \in [-R_\delta, 0]$, then

$$\begin{aligned}\int_0^s f(\theta)d\theta &= -\int_s^0 f(\theta)d\theta \leq \int_s^0 |f(\theta)|d\theta \leq \int_{-R_\delta}^0 |f(\theta)|d\theta \\ &\leq \int_{-R_\delta}^0 m_\delta d\theta = m_\delta R_\delta \leq m_\delta R_\delta + \delta s^2.\end{aligned}$$

Lastly, if $s < -R_\delta < 0$, then

$$\begin{aligned}\int_0^s f(\theta)d\theta &= -\int_s^0 f(\theta)d\theta = -\int_s^{-R_\delta} f(\theta)d\theta - \int_{-R_\delta}^0 f(\theta)d\theta \\ &\leq -\int_s^{-R_\delta} \delta\theta d\theta + \int_{-R_\delta}^0 |f(\theta)|d\theta \leq -\delta\left(\frac{R_\delta^2}{2} - \frac{s^2}{2}\right) + \int_{-R_\delta}^0 m_\delta d\theta \\ &\leq m_\delta R_\delta + \frac{\delta}{2}s^2 < m_\delta R_\delta + \delta s^2.\end{aligned}$$

This ends the proof of our claim.

Hence, given $u \in X$, we have

$$\int_0^u f(s)ds \leq K_\delta + \delta u^2$$

with $K_\delta = m_\delta R_\delta > 0$ being a constant and then, integrating this inequality on $\Omega$, we get

$$\int_\Omega \int_0^u f(s)dsdx \leq K_\delta|\Omega| + \delta\|u\|_X^2. \tag{2.33}$$

In conclusion, by (2.32) and (2.33), and taking $C_\delta = |\Omega| \max\{M_\delta, K_\delta\} > 0$, we obtain

$$\int_\Omega f(u)udx \leq C_\delta + \delta\|u\|_X^2 \quad \text{and} \quad \int_\Omega \int_0^u f(s)dsdx \leq C_\delta + \delta\|u\|_X^2$$

for all $u \in X$, which proves the statement of this item.
$(iii)$ Let $u \in X^{\frac{1}{2}}$. Using the Hölder's inequality, the Poincaré inequality $\|u\|_X^2 \leq \lambda_1^{-1}\|u\|_{X^{\frac{1}{2}}}^2$ and also item $(i)$ above, we have

$$\begin{aligned}
\left|\int_\Omega f(u)udx\right| &\leq \int_\Omega |f(u)||u|dx \leq \left(\int_\Omega |u|^2dx\right)^{\frac{1}{2}} \left(\int_\Omega |f(u)|^2dx\right)^{\frac{1}{2}} \\
&\leq \|u\|_X^2 \left(\int_\Omega [c(1+|u|^\rho)]^2dx\right)^{\frac{1}{2}} \leq c\lambda_1^{-1}\|u\|_{X^{\frac{1}{2}}}^2 \left(\int_\Omega (1+|u|^\rho)^2dx\right)^{\frac{1}{2}} \\
&\leq c_1\|u\|_{X^{\frac{1}{2}}}^2 \left(|\Omega| + \int_\Omega |u|^{2\rho}dx\right)^{\frac{1}{2}} \leq c_2\|u\|_{X^{\frac{1}{2}}}^2 \left(|\Omega|^{\frac{1}{2}} + \left(\int_\Omega |u|^{2\rho}dx\right)^{\frac{1}{2}}\right) \\
&\leq c_3\|u\|_{X^{\frac{1}{2}}}^2 \left(1 + \|u\|_{L^{2\rho}(\Omega)}^\rho\right)
\end{aligned}$$

with $c_3 > 0$ being a constant. Thanks to our assumption on the exponent $\rho$, we have $2\rho < \frac{2n}{n-2}$ and, moreover, since we know that the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ holds if and only if $p \leq \frac{2n}{n-2}$, it follows that $H^1(\Omega) \hookrightarrow L^{2\rho}(\Omega)$. Thus, there exists $\kappa > 0$ such that $\|u\|_{L^{2\rho}(\Omega)} \leq \kappa\|u\|_{X^{\frac{1}{2}}}$ and, hence,

$$\left|\int_\Omega f(u)udx\right| \leq c_3\|u\|_{X^{\frac{1}{2}}}^2 \left(1 + \kappa^\rho\|u\|_{X^{\frac{1}{2}}}^\rho\right).$$

Now, given $r > 0$, if $\|u\|_{X^{\frac{1}{2}}} \leq r$, then we get

$$\left|\int_\Omega f(u)udx\right| \leq c_3(1+\kappa^\rho r^\rho)\|u\|_{X^{\frac{1}{2}}}^2. \tag{2.34}$$

In what follows, we show the other inequality. At first, we claim that

$$\left|\int_0^s f(\theta)d\theta\right| \leq c\left(|s| + \frac{1}{\rho+1}|s|^{\rho+1}\right)$$

for all $s \in \mathbb{R}$. Indeed, using item $(i)$, if $s \geq 0$ then

$$\left| \int_0^s f(\theta) d\theta \right| \leq \int_0^s |f(\theta)| d\theta \leq \int_0^s c(1 + |\theta|^\rho) d\theta = c \left( |s| + \frac{1}{\rho+1} |s|^{\rho+1} \right).$$

If $s < 0$, then

$$\begin{aligned} \left| \int_0^s f(\theta) d\theta \right| &\leq \int_s^0 |f(\theta)| d\theta \leq \int_s^0 c(1 + |\theta|^\rho) d\theta = c \left( -s + \frac{1}{\rho+1} (-s)^{\rho+1} \right) \\ &= c \left( |s| + \frac{1}{\rho+1} |s|^{\rho+1} \right). \end{aligned}$$

Thus, the claim is proved.

Now, let $u \in X^{\frac{1}{2}}$. Using the Poincaré inequality $\|u\|_X^2 \leq \lambda_1^{-1} \|u\|_{X^{\frac{1}{2}}}^2$, we obtain

$$\begin{aligned} \left| \int_\Omega \int_0^u f(s) ds dx \right| &\leq \int_\Omega \left| \int_0^u f(s) ds \right| dx \leq \int_\Omega c \left( |u| + \frac{1}{\rho+1} |u|^{\rho+1} \right) dx \\ &\leq \int_\Omega c \left( \frac{1}{2} + \frac{1}{2} |u|^2 + \frac{1}{\rho+1} |u|^{\rho+1} \right) dx \\ &= \frac{c}{2} |\Omega| + \frac{c}{2} \int_\Omega |u|^2 dx + \frac{c}{\rho+1} \int_\Omega |u|^{\rho+1} dx \end{aligned}$$

$$\begin{aligned} &= \frac{c}{2} |\Omega| + \frac{c}{2} \|u\|_X^2 + \frac{c}{\rho+1} \|u\|_{L^{\rho+1}(\Omega)}^{\rho+1} \\ &\leq \frac{c}{2} |\Omega| + \frac{c\lambda_1^{-1}}{2} \|u\|_{X^{\frac{1}{2}}}^2 + \frac{c}{\rho+1} \|u\|_{L^{\rho+1}(\Omega)}^{\rho+1}. \end{aligned}$$

Since $1 < \rho < \frac{n}{n-2}$, with $n \geq 3$, we have

$$2 < \rho + 1 < \frac{2n-2}{n-2} < \frac{2n}{n-2},$$

which ensures that $H^1(\Omega) \hookrightarrow L^{\rho+1}(\Omega)$. Thus, there exists a constant $\kappa_0 > 0$ such that

$$\|u\|_{L^{\rho+1}(\Omega)} \leq \kappa_0 \|u\|_{X^{\frac{1}{2}}}$$

and, hence,

$$\begin{aligned} \left| \int_\Omega \int_0^u f(s) ds dx \right| &\leq \frac{c}{2} |\Omega| + \frac{c\lambda_1^{-1}}{2} \|u\|_{X^{\frac{1}{2}}}^2 + \frac{c}{\rho+1} \kappa_0^{\rho+1} \|u\|_{X^{\frac{1}{2}}}^{\rho+1} \\ &= \frac{c}{2} |\Omega| + \left( \frac{c\lambda_1^{-1}}{2} + \frac{c\kappa_0^{\rho+1}}{\rho+1} \|u\|_{X^{\frac{1}{2}}}^{\rho-1} \right) \|u\|_{X^{\frac{1}{2}}}^2. \end{aligned}$$

Now, given $r > 0$, if $\|u\|_{X^{\frac{1}{2}}} \leq r$, then we get

$$\left|\int_\Omega \int_0^u f(s)dsdx\right| \leq \frac{c}{2}|\Omega| + \left(\frac{c\lambda_1^{-1}}{2} + \frac{c\kappa_0^{\rho+1} r^{\rho-1}}{\rho+1}\right) \|u\|^2_{X^{\frac{1}{2}}}. \tag{2.35}$$

Therefore, we conclude that, for all $r > 0$ given, and for all $u \in X^{\frac{1}{2}}$ with $\|u\|_{X^{\frac{1}{2}}} \leq r$, taking

$$C_r = \max\left\{c_3(1+\kappa^\rho r^\rho), \frac{c\lambda_1^{-1}}{2} + \frac{c\kappa_0^{\rho+1} r^{\rho-1}}{\rho+1}\right\} > 0$$

and

$$C = \frac{c}{2}|\Omega|,$$

it follows by (2.34) and (2.35) that

$$\left|\int_\Omega f(u)udx\right| \leq C_r \|u\|^2_{X^{\frac{1}{2}}} \quad \text{and} \quad \left|\int_\Omega \int_0^u f(s)dsdx\right| \leq C_r \|u\|^2_{X^{\frac{1}{2}}} + C,$$

which concludes the proof of item $(iii)$. The proof of the result is now complete. □

**Remark 2.3.** According to the proof of [4, Proposition 4.1] if $f(0) = 0$, then there exists a constant $c > 0$ such that $|f(s)| \leq c(|s| + |s|^\rho)$ for all $s \in \mathbb{R}$. In addition, given $r > 0$, there exists a constant $C_r > 0$ such that

$$\left|\int_\Omega f(u)udx\right| \leq C_r \|u\|^2_{X^{\frac{1}{2}}} \quad \text{and} \quad \left|\int_\Omega \int_0^u f(s)dsdx\right| \leq C_r \|u\|^2_{X^{\frac{1}{2}}}$$

for all $u \in X^{\frac{1}{2}}$ with $\|u\|_{X^{\frac{1}{2}}} \leq r$.

**Theorem 2.3. [Global Well-Posedness]** *Let $f \in C^1(\mathbb{R})$ be a function satisfying* (2.7)-(2.8), *assume that conditions* (2.4)-(2.6) *hold and let $F\colon Y_0 \to Y_{\alpha-1}$ be defined as in* (2.12). *Then for any initial data $W_0 \in Y_0$ the problem* (2.9) *has a unique global solution $W(t)$ such that*

$$W(t) \in C([\tau, \infty), Y_0).$$

*Moreover, such solutions are continuous with respect to the initial data on $Y_0$.*

*Proof.* By Corollary 2.1, the problem (2.1)-(2.3) has a local solution $(u(t), u_t(t), v(t), v_t(t))$ in $Y_0$ defined on some interval $[\tau, \tau + t_0]$. Consider the original system (2.1). Multiplying the first equation in (2.1) by $u_t$, and the second by $v_t$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \eta\|(-\Delta)^{\frac{1}{4}} u_t\|^2_X \\ &+ a_\epsilon(t)\langle(-\Delta)^{\frac{1}{2}} v_t, u_t\rangle_X = \frac{d}{dt}\int_\Omega\int_0^u f(s)dsdx, \end{aligned} \tag{2.36}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v_t|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2dx + \eta\|(-\Delta)^{\frac{1}{4}}v_t\|_X^2 - a_\epsilon(t)\langle(-\Delta)^{\frac{1}{2}}u_t, v_t\rangle_X = 0, \tag{2.37}$$

for all $\tau < t \leq \tau + t_0$. Combining (2.36) and (2.37), we get

$$\frac{d}{dt}\mathcal{E}(t) = -\eta\|(-\Delta)^{\frac{1}{4}}u_t\|_X^2 - \eta\|(-\Delta)^{\frac{1}{4}}v_t\|_X^2 \tag{2.38}$$

for all $\tau < t \leq \tau + t_0$, where

$$\begin{aligned}\mathcal{E}(t) = {} & \frac{1}{2}\|u(t)\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|u(t)\|_X^2 + \frac{1}{2}\|u_t(t)\|_X^2 + \frac{1}{2}\|v(t)\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|v_t(t)\|_X^2 \\ & - \int_\Omega\int_0^u f(s)dsdx\end{aligned} \tag{2.39}$$

is the total energy associated with the solution $(u(t), u_t(t), v(t), v_t(t))$ of the problem (2.1)-(2.3) in $Y_0$. The identity (2.38) means that the map $t \mapsto \mathcal{E}(t)$ is monotone decreasing along solutions. Moreover, using the property $\mathcal{E}(t) \leq \mathcal{E}(\tau)$ for all $\tau \leq t \leq \tau + t_0$, we can obtain a priori estimate of the solution $(u(t), u_t(t), v(t), v_t(t))$ in $Y_0$. In fact, given $\delta > 0$, it follows by Lemma 2.3, item $(ii)$, that there is $C_\delta > 0$ such that

$$\int_\Omega\int_0^u f(s)dsdx \leq C_\delta + \delta\|u\|_X^2.$$

Thus, for all $\tau < t \leq \tau + t_0$, we have

$$\begin{aligned}&\|u\|^2_{X^{\frac{1}{2}}} + \|u_t\|_X^2 + \|v\|^2_{X^{\frac{1}{2}}} + \|v_t\|_X^2 \leq \|u\|^2_{X^{\frac{1}{2}}} + \|u\|_X^2 + \|u_t\|_X^2 + \|v\|^2_{X^{\frac{1}{2}}} + \|v_t\|_X^2 \\ &= 2\mathcal{E}(t) + 2\int_\Omega\int_0^u f(s)dsdx \leq 2\mathcal{E}(\tau) + 2(\delta\|u\|_X^2 + C_\delta) \\ &\leq 2(\mathcal{E}(\tau) + C_\delta) + 2\delta\lambda_1^{-1}\|u\|^2_{X^{\frac{1}{2}}} \\ &\leq 2(\mathcal{E}(\tau) + C_\delta) + 2\delta\lambda_1^{-1}(\|u\|^2_{X^{\frac{1}{2}}} + \|u_t\|_X^2 + \|v\|^2_{X^{\frac{1}{2}}} + \|v_t\|_X^2),\end{aligned}$$

where we have used the Poincaré inequality, and $\lambda_1 > 0$ denotes the first eigenvalue of the negative Laplacian operator with homogeneous Dirichlet boundary condition.

Now, choosing $\delta = \frac{\lambda_1}{4}$, we get

$$\|u\|^2_{X^{\frac{1}{2}}} + \|u_t\|_X^2 + \|v\|^2_{X^{\frac{1}{2}}} + \|v_t\|_X^2 \leq 4\left(\mathcal{E}(\tau) + C_{\frac{\lambda_1}{4}}\right),$$

that is,

$$\|(u(t), u_t(t), v(t), v_t(t))\|_{Y_0}^2 \leq 4\left(\mathcal{E}(\tau) + C_{\frac{\lambda_1}{4}}\right).$$

This ensures that the problem $(2.1) - (2.3)$ has a global solution $W(t)$ in $Y_0$, which proves the result. □

Since the problem $(2.1)-(2.3)$ has a global solution $W(t)$ in $Y_0$, we can define an evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ in $Y_0$ by

$$S(t,\tau)W_0 = W(t), \quad t \geq \tau \in \mathbb{R}. \tag{2.40}$$

By [26], we have

$$S(t,\tau)W_0 = L(t,\tau)W_0 + U(t,\tau)W_0, \quad t \geq \tau \in \mathbb{R}, \tag{2.41}$$

where $\{L(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ is the linear evolution process in $Y_0$ associated with the homogeneous problem

$$\begin{cases} W_t + \mathcal{A}(t)W = 0, \ t > \tau, \\ W(\tau) = W_0, \ \tau \in \mathbb{R}, \end{cases} \tag{2.42}$$

and

$$U(t,\tau)W_0 = \int_\tau^t L(t,s)F(S(s,\tau)W_0)ds. \tag{2.43}$$

## 2.3 Existence of the pullback attractor

In this section, we prove the existence of the pullback attractor of the problem (2.1)-(2.3). To this end, we need to make a modification on the energy functional. More precisely, for $\gamma_1, \gamma_2 \in \mathbb{R}_+$, let us define $L_{\gamma_1,\gamma_2}\colon Y_0 \to \mathbb{R}$ by the map

$$\begin{aligned} L_{\gamma_1,\gamma_2}(\phi,\varphi,\psi,\Phi) = \frac{1}{2}\|\phi\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|\phi\|^2_X + \frac{1}{2}\|\varphi\|^2_X + \frac{1}{2}\|\psi\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|\Phi\|^2_X \\ + \gamma_1\langle\phi,\varphi\rangle_X + \gamma_2\langle\psi,\Phi\rangle_X - \int_\Omega\int_0^\phi f(s)dsdx. \end{aligned} \tag{2.44}$$

We start by noting that if

$$\gamma_i < \frac{1}{2} \ \text{ and } \ \frac{\gamma_i}{2}\lambda_1^{-1} < \frac{1}{4}, \ i = 1,2,$$

then

$$\begin{aligned} \frac{1}{4}\|(\phi,\varphi,\psi,\Phi)\|^2_{Y_0} &\leq L_{\gamma_1,\gamma_2}(\phi,\varphi,\psi,\Phi) + \int_\Omega\int_0^\phi f(s)dsdx \\ &\leq \frac{3}{4}(1+\lambda_1^{-1})\|(\phi,\varphi,\psi,\Phi)\|^2_{Y_0}. \end{aligned} \tag{2.45}$$

Indeed, using the Cauchy-Schwartz and Young inequalities, we obtain

$$
\begin{aligned}
|\gamma_1\langle\phi,\varphi\rangle_X+\gamma_2\langle\psi,\Phi\rangle_X| &\leq \gamma_1\|\phi\|_X\|\varphi\|_X+\gamma_2\|\psi\|_X\|\Phi\|_X\\
&\leq \frac{\gamma_1}{2}(\|\phi\|_X^2+\|\varphi\|_X^2)+\frac{\gamma_2}{2}(\|\psi\|_X^2+\|\Phi\|_X^2)\\
&\leq \frac{\gamma_1}{2}\lambda_1^{-1}\|\phi\|^2_{X^{\frac{1}{2}}}+\frac{\gamma_1}{2}\|\varphi\|_X^2+\frac{\gamma_2}{2}\lambda_1^{-1}\|\psi\|^2_{X^{\frac{1}{2}}}+\frac{\gamma_2}{2}\|\Phi\|_X^2\\
&\leq \frac{1}{4}\|(\phi,\varphi,\psi,\Phi)\|^2_{Y_0},
\end{aligned}
$$

which leads to

$$
\frac{1}{4}\|(\phi,\varphi,\psi,\Phi)\|^2_{Y_0}\leq\frac{1}{2}\|(\phi,\varphi,\psi,\Phi)\|^2_{Y_0}+\gamma_1\langle\phi,\varphi\rangle_X+\gamma_2\langle\psi,\Phi\rangle_X\leq\frac{3}{4}\|(\phi,\varphi,\psi,\Phi)\|^2_{Y_0}. \tag{2.46}
$$

Consequently,

$$
\frac{1}{4}\|(\phi,\varphi,\psi,\Phi)\|^2_{Y_0}\leq L_{\gamma_1,\gamma_2}(\phi,\varphi,\psi,\Phi)+\int_\Omega\int_0^\phi f(s)dsdx\leq\frac{3}{4}\|(\phi,\varphi,\psi,\Phi)\|^2_{Y_0}+\frac{1}{2}\|\phi\|^2_X.
$$

But since $\|\phi\|_X^2\leq\lambda_1^{-1}\|\phi\|^2_{X^{\frac{1}{2}}}$, we have

$$
\frac{3}{4}\|(\phi,\varphi,\psi,\Phi)\|^2_{Y_0}+\frac{1}{2}\|\phi\|^2_X\leq\frac{3(1+\lambda_1^{-1})}{4}\|(\phi,\varphi,\psi,\Phi)\|^2_{Y_0}, \tag{2.47}
$$

and the claim is proved.

**Theorem 2.4.** *There exists $R>0$ such that for any bounded subset $B\subset Y_0$ one can find $t_0(B)>0$ satisfying*

$$
\|(u,u_t,v,v_t)\|^2_{Y_0}\leq R\quad \text{for all}\quad t\geq\tau+t_0(B).
$$

*In particular, the evolution process $\{S(t,\tau)\colon t\geq\tau\in\mathbb{R}\}$ defined in (2.40) is pullback strongly bounded dissipative.*

*Proof.* At first, note that we can differentiate the expression (2.44) along the solution $W(t)=(u(t),u_t(t),v(t),v_t(t))$ and, using (2.38) and (2.39), we get

$$
\begin{aligned}
&\frac{d}{dt}L_{\gamma_1,\gamma_2}(u,u_t,v,v_t)=\frac{d}{dt}\mathcal{E}(t)+\gamma_1\langle u_t,u_t\rangle_X+\gamma_1\langle u,u_{tt}\rangle_X+\gamma_2\langle v_t,v_t\rangle_X+\gamma_2\langle v,v_{tt}\rangle_X\\
&=-\eta\|A^{\frac{1}{4}}u_t\|^2_X-\eta\|A^{\frac{1}{4}}v_t\|^2_X+\gamma_1\|u_t\|^2_X+\gamma_1\langle u,-Au-u-\eta A^{\frac{1}{2}}u_t-a_\epsilon(t)A^{\frac{1}{2}}v_t+f(u)\rangle_X\\
&+\gamma_2\|v_t\|^2_X+\gamma_2\langle v,-Av-\eta A^{\frac{1}{2}}v_t+a_\epsilon(t)A^{\frac{1}{2}}u_t\rangle_X\\
&=-\eta\|u_t\|^2_{X^{\frac{1}{4}}}-\eta\|v_t\|^2_{X^{\frac{1}{4}}}+\gamma_1\|u_t\|^2_X-\gamma_1(\|u\|^2_{X^{\frac{1}{2}}}+\|u\|^2_X)-\gamma_1\eta\langle A^{\frac{1}{2}}u,u_t\rangle_X\\
&-\gamma_1a_\epsilon(t)\langle A^{\frac{1}{2}}u,v_t\rangle_X+\gamma_1\langle u,f(u)\rangle_X+\gamma_2\|v_t\|^2_X-\gamma_2\|v\|^2_{X^{\frac{1}{2}}}\\
&-\gamma_2\eta\langle A^{\frac{1}{2}}v,v_t\rangle_X+\gamma_2a_\epsilon(t)\langle A^{\frac{1}{2}}v,u_t\rangle_X.
\end{aligned}
$$

Now, if $c > 0$ is the embedding constant of $X^{\frac{1}{4}} \hookrightarrow X$, then one has

$$-\eta\|\cdot\|^2_{X^{\frac{1}{4}}} \leq -\eta\frac{1}{c^2}\|\cdot\|^2_X. \tag{2.48}$$

Moreover, by Lemma 2.3, item $(ii)$, for each $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\int_\Omega f(u)udx \leq \delta\|u\|^2_X + C_\delta,$$

which implies

$$\gamma_1\langle u, f(u)\rangle_X \leq \gamma_1\delta\|u\|^2_X + \gamma_1 C_\delta \leq \gamma_1\delta\lambda_1^{-1}\|u\|^2_{X^{\frac{1}{2}}} + \gamma_1 C_\delta. \tag{2.49}$$

Thus, using (2.48), (2.49) and the Cauchy-Schwartz and Young inequalities, we have

$$\begin{aligned}
&\frac{d}{dt}L_{\gamma_1,\gamma_2}(u,u_t,v,v_t)\\
&\leq -\gamma_1(1-\delta\lambda_1^{-1})\|u\|^2_{X^{\frac{1}{2}}} - \left(\eta\frac{1}{c^2}-\gamma_1\right)\|u_t\|^2_X - \gamma_2\|v\|^2_{X^{\frac{1}{2}}} - \left(\eta\frac{1}{c^2}-\gamma_2\right)\|v_t\|^2_X\\
&+\gamma_1 C_\delta + \gamma_1\eta\left(\frac{\epsilon_1}{2}\|u\|^2_{X^{\frac{1}{2}}} + \frac{1}{2\epsilon_1}\|u_t\|^2_X\right) + \gamma_1 a_1\left(\frac{1}{2\epsilon_2}\|u\|^2_{X^{\frac{1}{2}}} + \frac{\epsilon_2}{2}\|v_t\|^2_X\right)\\
&+\gamma_2 a_1\left(\frac{1}{2\epsilon_3}\|v\|^2_{X^{\frac{1}{2}}} + \frac{\epsilon_3}{2}\|u_t\|^2_X\right) + \gamma_2\eta\left(\frac{\epsilon_4}{2}\|v\|^2_{X^{\frac{1}{2}}} + \frac{1}{2\epsilon_4}\|v_t\|^2_X\right)\\
&= -\gamma_1\left(1-\delta\lambda_1^{-1}-\eta\frac{\epsilon_1}{2}-a_1\frac{1}{2\epsilon_2}\right)\|u\|^2_{X^{\frac{1}{2}}} - \left(\eta\frac{1}{c^2}-\gamma_1-\gamma_1\eta\frac{1}{2\epsilon_1}-\gamma_2 a_1\frac{\epsilon_3}{2}\right)\|u_t\|^2_X\\
&-\gamma_2\left(1-a_1\frac{1}{2\epsilon_3}-\eta\frac{\epsilon_4}{2}\right)\|v\|^2_{X^{\frac{1}{2}}} - \left(\eta\frac{1}{c^2}-\gamma_2-\gamma_1 a_1\frac{\epsilon_2}{2}-\gamma_2\eta\frac{1}{2\epsilon_4}\right)\|v_t\|^2_X + \gamma_1 C_\delta
\end{aligned}$$

for all $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$. Choosing $\delta = \frac{\lambda_1}{8}$, $\epsilon_1 = \epsilon_4 = \frac{1}{\eta}$ and $\epsilon_2 = \epsilon_3 = 2a_1$, we obtain

$$\begin{aligned}
\frac{d}{dt}L_{\gamma_1,\gamma_2}(u,u_t,v,v_t) \leq &-\frac{1}{8}\gamma_1\|u\|^2_{X^{\frac{1}{2}}} - \left(\eta\frac{1}{c^2}-\gamma_1\left(1+\frac{\eta^2}{2}\right)-\gamma_2 a_1^2\right)\|u_t\|^2_X - \frac{1}{4}\gamma_2\|v\|^2_{X^{\frac{1}{2}}}\\
&-\left(\eta\frac{1}{c^2}-\gamma_1 a_1^2-\gamma_2\left(1+\frac{\eta^2}{2}\right)\right)\|v_t\|^2_X + \gamma_1 C_{\frac{\lambda_1}{8}}.
\end{aligned}$$

We may choose $\gamma_i > 0, i = 1, 2$, sufficiently small such that

$$\gamma_i < \frac{\eta}{4c^2}\min\left\{\frac{1}{a_1^2},\ \left(1+\frac{\eta^2}{2}\right)^{-1}\right\},\ i = 1, 2.$$

Now, taking

$$C_1 = \min\left\{\frac{1}{8}\gamma_1,\ \eta\frac{1}{c^2}-\gamma_1\left(1+\frac{\eta^2}{2}\right)-\gamma_2 a_1^2,\ \frac{1}{4}\gamma_2,\ \eta\frac{1}{c^2}-\gamma_1 a_1^2-\gamma_2\left(1+\frac{\eta^2}{2}\right)\right\} > 0,$$

and $C_2 = \gamma_1 C_{\frac{\lambda_1}{8}} > 0$, we obtain

$$\frac{d}{dt} L_{\gamma_1,\gamma_2}(u, u_t, v, v_t) \leq -C_1 \|(u, u_t, v, v_t)\|_{Y_0}^2 + C_2. \tag{2.50}$$

Note that $C_1$ and $C_2$ are independent of $B$.

We claim that there exists $K > 0$ such that $L_{\gamma_1,\gamma_2}(u, u_t, v, v_t) \geq \frac{1}{8}\|(u, u_t, v, v_t)\|_{Y_0}^2 - K$.

In fact, by Lemma 2.3, item $(ii)$, given $\tilde{\delta} > 0$, there exists a constant $C_{\tilde{\delta}} > 0$ such that

$$\int_\Omega \int_0^u f(s) ds dx \leq \tilde{\delta} \|u\|_X^2 + C_{\tilde{\delta}},$$

which, together with $\|u\|_X^2 \leq \lambda_1^{-1} \|u\|_{X^{\frac{1}{2}}}^2$, implies

$$\begin{aligned}
L_{\gamma_1,\gamma_2}(u, u_t, v, v_t) &\geq \frac{1}{4}\|(u, u_t, v, v_t)\|_{Y_0}^2 - \int_\Omega \int_0^u f(s) ds dx \\
&\geq \frac{1}{4}\|(u, u_t, v, v_t)\|_{Y_0}^2 - \tilde{\delta}\lambda_1^{-1} \|u\|_{X^{\frac{1}{2}}}^2 - C_{\tilde{\delta}} \\
&\geq \left(\frac{1}{4} - \tilde{\delta}\lambda_1^{-1}\right) \|(u, u_t, v, v_t)\|_{Y_0}^2 - C_{\tilde{\delta}}.
\end{aligned}$$

Choosing $\tilde{\delta} = \frac{\lambda_1}{8}$, we get

$$L_{\gamma_1,\gamma_2}(u, u_t, v, v_t) \geq \frac{1}{8}\|(u, u_t, v, v_t)\|_{Y_0}^2 - K, \tag{2.51}$$

where $K = C_{\frac{\lambda_1}{8}} > 0$, which proves the claim.

Now, define the set

$$\ell_r = \sup\{\|(u, u_t, v, v_t)\|_{Y_0}^2 \colon t \geq \tau, \|(u(\tau), u_t(\tau), v(\tau), v_t(\tau))\|_{Y_0}^2 \leq r\}.$$

Note that $\ell_r < \infty$ for each $r > 0$. In fact, by the proof of Theorem 2.3, we have

$$\|W(t)\|_{Y_0}^2 = \|(u(t), u_t(t), v(t), v_t(t))\|_{Y_0}^2 \leq 4\left(\mathcal{E}(\tau) + C_{\frac{\lambda_1}{4}}\right), \quad t \geq \tau,$$

where

$$\begin{aligned}
\mathcal{E}(\tau) &= \frac{1}{2}\|W(\tau)\|_{Y_0}^2 + \frac{1}{2}\|u(\tau)\|_X^2 - \int_\Omega \int_0^{u(\tau)} f(s) ds dx \\
&\leq \frac{1}{2}\|W(\tau)\|_{Y_0}^2 + \lambda_1^{-1}\|u(\tau)\|_{X^{\frac{1}{2}}}^2 + \left| \int_\Omega \int_0^{u(\tau)} f(s) ds dx \right| \\
&\leq \left(\frac{1}{2} + \lambda_1^{-1}\right) \|W(\tau)\|_{Y_0}^2 + C_r \|u(\tau)\|_{X^{\frac{1}{2}}}^2 + C
\end{aligned}$$

$$\leq \left(\frac{1}{2} + \lambda_1^{-1}\right) r + C_r r + C.$$

This shows that $\ell_r < \infty$.

Now, we claim that given a bounded set $B \subset Y_0$ there exists $t_0(B) > 0$ such that

$$\|(u, u_t, v, v_t)\|_{Y_0}^2 \leq \max\left\{8K, \ell_{\frac{C_2+1}{C_1}}\right\} \quad \text{for all} \quad t \geq \tau + t_0(B).$$

In fact, let $B \subset Y_0$ be a bounded set. Let $r_0 > 0$ be such that $B \subset B_{Y_0}(0, r_0)$. By (2.45) and Lemma 2.3, we obtain

$$L_{\gamma_1,\gamma_2}(u(\tau), u_t(\tau), v(\tau), v_t(\tau)) \leq \frac{3}{4}(1 + \lambda_1^{-1}) r_0 + r_0 C_{r_0} + C = T_{r_0},$$

for all $(u(\tau), u_t(\tau), v(\tau), v_t(\tau)) \in B$.

Let $(u(\tau), u_t(\tau), v(\tau), v_t(\tau)) \in B$ be arbitrary. If $\|(u, u_t, v, v_t)\|_{Y_0}^2 > \frac{C_2+1}{C_1}$ for all $t \geq \tau$ then

$$\frac{d}{dt} L_{\gamma_1,\gamma_2}(u, u_t, v, v_t) \leq -C_1 \|(u, u_t, v, v_t)\|_{Y_0}^2 + C_2 \leq -1 \quad \text{for all} \quad t \geq \tau,$$

which implies

$$L_{\gamma_1,\gamma_2}(u, u_t, v, v_t) \leq L_{\gamma_1,\gamma_2}(u(\tau), u_t(\tau), v(\tau), v_t(\tau)) - (t - \tau) \quad \text{for all} \quad t \geq \tau.$$

Thus, $L_{\gamma_1,\gamma_2}(u, u_t, v, v_t) \leq 0$ for all $t \geq \tau + T_{r_0}$. Consequently, using (2.51), we have

$$\|(u, u_t, v, v_t)\|_{Y_0}^2 \leq 8K \quad \text{for all} \quad t \geq \tau + T_{r_0}.$$

On the other hand, if there exists $t_u \geq \tau$ such that $\|(u(t_u), u_t(t_u), v(t_u), v_t(t_u))\|_{Y_0}^2 \leq \frac{C_2+1}{C_1}$ (take the smallest $t_u$ with this property) then

$$\|(u, u_t, v, v_t)\|_{Y_0}^2 \leq \ell_{\frac{C_2+1}{C_1}} \quad \text{for all} \quad t \geq t_u.$$

Set

$$B^u = \Bigg\{ w_0 \in B\colon \text{there exists } t_u^{w_0} > \tau \text{ such that } \|W(t_u^{w_0}) w_0\|_{Y_0}^2 = \frac{C_2 + 1}{C_1} \text{ and}$$
$$\|W(t) w_0\|_{Y_0}^2 > \frac{C_2 + 1}{C_1} \text{ for all } \tau \leq t < t_u^{w_0} \Bigg\}.$$

We claim that $T_u(B) = \sup\{t_u^{w_0}\colon w_0 \in B^u\} < \infty$. In fact, suppose to the contrary that there exists a sequence $\{w_0^n\}_{n\in\mathbb{N}} \subset B^u$ such that $t_u^{w_0^n} \to \infty$ as $n \to \infty$. Since $\|W(t) w_0^n\|_{Y_0}^2 \geq \frac{C_2+1}{C_1}$ for all $\tau \leq t \leq t_u^{w_0^n}$, we conclude that

$$L_{\gamma_1,\gamma_2}(W(t)w_0^n) \leq L_{\gamma_1,\gamma_2}(w_0^n) - (t-\tau) \leq T_{r_0} - t + \tau \quad \text{for all} \quad \tau \leq t \leq t_u^{w_0^n}.$$

This implies that $\lim\limits_{n\to\infty} L_{\gamma_1,\gamma_2}(W(t_u^{w_0^n})w_0^n) = -\infty$. But, using (2.45), we obtain

$$\begin{aligned}
\frac{1}{4}\|W(t_u^{w_0^n})w_0^n\|_{Y_0}^2 &\leq L_{\gamma_1,\gamma_2}(W(t_u^{w_0^n})w_0^n) + \int_\Omega \int_0^{u(t_u^{w_0^n})} f(s)dsdx \\
&\leq L_{\gamma_1,\gamma_2}(W(t_u^{w_0^n})w_0^n) + \left| \int_\Omega \int_0^{u(t_u^{w_0^n})} f(s)dsdx \right| \\
&\leq L_{\gamma_1,\gamma_2}(W(t_u^{w_0^n})w_0^n) + C_{\frac{C_2+1}{C_1}} \|u(t_u^{w_0^n})\|_{X^{\frac{1}{2}}}^2 + C \\
&\leq L_{\gamma_1,\gamma_2}(W(t_u^{w_0^n})w_0^n) + C_{\frac{C_2+1}{C_1}} \|W(t_u^{w_0^n})w_0^n\|_{Y_0}^2 + C \\
&= L_{\gamma_1,\gamma_2}(W(t_u^{w_0^n})w_0^n) + C_{\frac{C_2+1}{C_1}} \frac{C_2+1}{C_1} + C
\end{aligned}$$

which contradicts the fact that $\lim\limits_{n\to\infty} L_{\gamma_1,\gamma_2}(W(t_u^{w_0^n})w_0^n) = -\infty$.

Taking $t_0(B) = \max\{T_u(B), T_{r_0}\}$, we conclude that

$$\|(u, u_t, v, v_t)\|_{Y_0}^2 \leq \max\left\{8K, \ell_{\frac{C_2+1}{C_1}}\right\} \quad \text{for all} \quad t \geq \tau + t_0(B).$$

This shows that, if $s \leq t$ and $B \subset Y_0$ is a bounded set then

$$S(s,\tau)B \subset \overline{B_{Y_0}(0,R)} \quad \text{for all} \quad \tau \leq \tau_0(s,B),$$

where $\tau_0(s,B) = s - t_0(B)$ and $R = \max\left\{8K, \ell_{\frac{C_2+1}{C_1}}\right\}$. Therefore, the process given by (2.40) is pullback strongly bounded dissipative. $\square$

Next, we prove that the solutions of problem (2.9) are uniformly exponentially dominated when the initial data are in bounded subsets of $Y_0$.

**Theorem 2.5.** *Let $B \subset Y_0$ be a bounded set. If $W\colon [\tau,\infty) \to Y_0$ is the global solution of* (2.9) *starting at $W_0 \in B$, then there are positive constants $\sigma = \sigma(B)$, $K_1 = K_1(B)$ and $K_2 = K_2(B)$ such that*

$$\|W(t)\|_{Y_0}^2 \leq K_1 e^{-\sigma(t-\tau)} + K_2, \quad t \geq \tau.$$

*Proof.* Let $r > 0$ be such that $B \subset B_{Y_0}(0,r)$. We claim that there is $M_r > 0$ and $C > 0$ such that $L_{\gamma_1,\gamma_2}(u,u_t,v,v_t) \leq M_r\|(u,u_t,v,v_t)\|_{Y_0}^2 + C$ for all $t \geq \tau$. In fact, by (2.45), we have

$$L_{\gamma_1,\gamma_2}(u,u_t,v,v_t) + \int_\Omega \int_0^u f(s)dsdx \leq \frac{3}{4}(1+\lambda_1^{-1})\|(u,u_t,v,v_t)\|_{Y_0}^2.$$

By the proof of Theorem 2.4, the set

$$\ell_r = \sup\{\|(u,u_t,v,v_t)\|_{Y_0}^2 \colon t \geq \tau, \|(u(\tau),u_t(\tau),v(\tau),v_t(\tau))\|_{Y_0}^2 \leq r\} < \infty.$$

Now, using Lemma 2.3, condition $(iii)$, there are constants $C_{K_r} > 0$ and $C > 0$ such that

$$\left|\int_\Omega \int_0^u f(s)dsdx\right| \leq C_{\ell_r}\|u\|_{X^{\frac{1}{2}}}^2 + C$$

whenever $\|(u(\tau),u_t(\tau),v(\tau),v_t(\tau))\|_{Y_0}^2 \leq r$. Hence, if $\|(u(\tau),u_t(\tau),v(\tau),v_t(\tau))\|_{Y_0}^2 \leq r$, then

$$\begin{aligned}L_{\gamma_1,\gamma_2}(u,u_t,v,v_t) &\leq \frac{3}{4}(1+\lambda_1^{-1})\|(u,u_t,v,v_t)\|_{Y_0}^2 - \int_\Omega\int_0^u f(s)dsdx \\ &\leq \frac{3}{4}(1+\lambda_1^{-1})\|(u,u_t,v,v_t)\|_{Y_0}^2 + C_{\ell_r}\|u\|_{X^{\frac{1}{2}}}^2 + C \\ &\leq M_r\|(u,u_t,v,v_t)\|_{Y_0}^2 + C,\end{aligned}$$

where $M_r = \frac{3}{4}(1+\lambda_1^{-1}) + C_{\ell_r} > 0$, which proves the claim.

Using the proof of Theorem 2.4, it follows by (2.50) that

$$\frac{d}{dt}L_{\gamma_1,\gamma_2}(W(t)) \leq -\frac{C_1}{M_r}L_{\gamma_1,\gamma_2}(W(t)) + \frac{CC_1}{M_r} + C_2, \quad t \geq \tau,$$

which implies

$$L_{\gamma_1,\gamma_2}(W(t)) \leq L_{\gamma_1,\gamma_2}(W(\tau))e^{-\frac{C_1}{M_r}(t-\tau)} + \left(C_2 + \frac{CC_1}{M_r}\right)\frac{M_r}{C_1}, \quad t \geq \tau,$$

and, using the fact that $\frac{1}{8}\|W(t)\|_{Y_0}^2 - K \leq L_{\gamma_1,\gamma_2}(W(t))$ (see (2.51)), we conclude that

$$\|W(t)\|_{Y_0}^2 \leq 8L_{\gamma_1,\gamma_2}(W(\tau))e^{-\frac{C_1}{M_r}(t-\tau)} + 8\left(C_2\frac{M_r}{C_1} + C + K\right), \quad t \geq \tau.$$

Since $L_{\gamma_1,\gamma_2}(W(\tau)) \leq K_rM_r + C$, we get

$$\|W(t)\|_{Y_0}^2 \leq 8(K_rM_r + C)e^{-\frac{C_1}{M_r}(t-\tau)} + 8\left(C_2\frac{M_r}{C_1} + C + K\right), \quad t \geq \tau,$$

and the result follows by taking $\sigma = \frac{C_1}{M_r}$, $K_1 = 8(K_rM_r + C)$ and $K_2 = 8\left(C_2\frac{M_r}{C_1} + C + K\right)$. □

**Theorem 2.6.** *Let $B \subset Y_0$ be a bounded set and denote by $L\colon [\tau,\infty) \to Y_0$ the solution of the homogeneous problem* (2.42) *starting in $W_0 \in B$. Then there exist positive constants $K = K(B)$ and $\zeta$ such that*

$$\|L(t)\|_{Y_0}^2 \leq Ke^{-\zeta(t-\tau)}, \quad t \geq \tau.$$

*Proof.* Denoting by $L = (u, u_t, v, v_t)\colon [\tau, \infty) \to Y_0$ the solution of the problem (2.42) starting in $W_0 = (u_0, u_1, v_0, v_1) \in B$, we define the map $\tilde{L}_{\gamma_1,\gamma_2}\colon Y_0 \to \mathbb{R}$ by

$$\begin{aligned}\tilde{L}_{\gamma_1,\gamma_2}(u, u_t, v, v_t) = \frac{1}{2}\|u\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|u\|^2_X + \frac{1}{2}\|u_t\|^2_X + \frac{1}{2}\|v\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|v_t\|^2_X \\ + \gamma_1\langle u, u_t\rangle_X + \gamma_2\langle v, v_t\rangle_X\end{aligned} \tag{2.52}$$

with $\gamma_1, \gamma_2 \in \mathbb{R}_+$.

Then, thanks to the regularity of solutions, established in Corollary 2.1, and using the Cauchy-Schwartz and Young inequalities, we have

$$\begin{aligned}&\frac{d}{dt}\tilde{L}_{\gamma_1,\gamma_2}(u, u_t, v, v_t)\\
&= -\eta\|u_t\|^2_{X^{\frac{1}{4}}} - \eta\|v_t\|^2_{X^{\frac{1}{4}}} + \gamma_1\|u_t\|^2_X - \gamma_1(\|u\|^2_{X^{\frac{1}{2}}} + \|u\|^2_X) - \gamma_1\eta\langle A^{\frac{1}{2}}u, u_t\rangle_X\\
&- \gamma_1 a_\epsilon(t)\langle A^{\frac{1}{2}}u, v_t\rangle_X + \gamma_2\|v_t\|^2_X - \gamma_2\|v\|^2_{X^{\frac{1}{2}}} - \gamma_2\eta\langle A^{\frac{1}{2}}v, v_t\rangle_X + \gamma_2 a_\epsilon(t)\langle A^{\frac{1}{2}}v, u_t\rangle_X\\
&\le -\eta\frac{1}{c^2}\|u_t\|^2_X - \eta\frac{1}{c^2}\|v_t\|^2_X + \gamma_1\|u_t\|^2_X - \gamma_1\|u\|^2_{X^{\frac{1}{2}}} + \gamma_1\eta\left(\frac{\epsilon_1}{2}\|u\|^2_{X^{\frac{1}{2}}} + \frac{1}{2\epsilon_1}\|u_t\|^2_X\right)\\
&+ \gamma_1 a_1\left(\frac{1}{2\epsilon_2}\|u\|^2_{X^{\frac{1}{2}}} + \frac{\epsilon_2}{2}\|v_t\|^2_X\right) + \gamma_2\|v_t\|^2_X - \gamma_2\|v\|^2_{X^{\frac{1}{2}}}\\
&+ \gamma_2 a_1\left(\frac{1}{2\epsilon_3}\|v\|^2_{X^{\frac{1}{2}}} + \frac{\epsilon_3}{2}\|u_t\|^2_X\right) + \gamma_2\eta\left(\frac{\epsilon_4}{2}\|v\|^2_{X^{\frac{1}{2}}} + \frac{1}{2\epsilon_4}\|v_t\|^2_X\right)\\
&= -\gamma_1\left(1 - \eta\frac{\epsilon_1}{2} - a_1\frac{1}{2\epsilon_2}\right)\|u\|^2_{X^{\frac{1}{2}}} - \left(\eta\frac{1}{c^2} - \gamma_1 - \gamma_1\eta\frac{1}{2\epsilon_1} - \gamma_2 a_1\frac{\epsilon_3}{2}\right)\|u_t\|^2_X\\
&- \gamma_2\left(1 - a_1\frac{1}{2\epsilon_3} - \eta\frac{\epsilon_4}{2}\right)\|v\|^2_{X^{\frac{1}{2}}} - \left(\eta\frac{1}{c^2} - \gamma_2 - \gamma_1 a_1\frac{\epsilon_2}{2} - \gamma_2\eta\frac{1}{2\epsilon_4}\right)\|v_t\|^2_X\end{aligned}$$

for all $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$, where $c > 0$ is the embedding constant of $X^{\frac{1}{4}} \hookrightarrow X$. Now, it is enough to choose

$$\epsilon_1 = \frac{1}{\eta},\ \epsilon_2 = 2a_1,\ \epsilon_3 = 2a_1,\ \epsilon_4 = \frac{1}{\eta},$$

so that we get

$$\begin{aligned}\frac{d}{dt}\tilde{L}_{\gamma_1,\gamma_2}(u, u_t, v, v_t) \le -\frac{1}{4}\gamma_1\|u\|^2_{X^{\frac{1}{2}}} - \left(\eta\frac{1}{c^2} - \gamma_1\left(1 + \frac{\eta^2}{2}\right) - \gamma_2 a_1^2\right)\|u_t\|^2_X - \frac{1}{4}\gamma_2\|v\|^2_{X^{\frac{1}{2}}}\\
- \left(\eta\frac{1}{c^2} - \gamma_1 a_1^2 - \gamma_2\left(1 + \frac{\eta^2}{2}\right)\right)\|v_t\|^2_X.\end{aligned}$$

Taking $\gamma_i > 0, i = 1, 2$, sufficiently small such that

$$\gamma_i < \frac{\eta}{4c^2}\min\left\{\frac{1}{a_1^2},\ \left(1 + \frac{\eta^2}{2}\right)^{-1}\right\},\ i = 1, 2,$$

and then taking

$$C_0 = \min\left\{\frac{1}{4}\gamma_1,\ \eta\frac{1}{c^2} - \gamma_1\left(1+\frac{\eta^2}{2}\right) - \gamma_2 a_1^2,\ \frac{1}{4}\gamma_2,\ \eta\frac{1}{c^2} - \gamma_1 a_1^2 - \gamma_2\left(1+\frac{\eta^2}{2}\right)\right\} > 0,$$

we obtain

$$\frac{d}{dt}\tilde{L}_{\gamma_1,\gamma_2}(u,u_t,v,v_t) \leq -C_0\|(u,u_t,v,v_t)\|_{Y_0}^2. \tag{2.53}$$

Observe that, from (2.45) and (2.52), we have

$$\frac{1}{4}\|(u,u_t,v,v_t)\|_{Y_0}^2 \leq \tilde{L}_{\gamma_1,\gamma_2}(u,u_t,v,v_t) \leq \frac{3}{4}(1+\lambda_1^{-1})\|(u,u_t,v,v_t)\|_{Y_0}^2, \tag{2.54}$$

and, combining this with (2.53), we get

$$\frac{d}{dt}\tilde{L}_{\gamma_1,\gamma_2}(L(t)) \leq -\zeta\tilde{L}_{\gamma_1,\gamma_2}(L(t)), \quad \text{for all} \quad t \geq \tau,$$

where $\zeta = \frac{4C_0}{3(1+\lambda_1^{-1})} > 0$, which yields

$$\tilde{L}_{\gamma_1,\gamma_2}(L(t)) \leq \tilde{L}_{\gamma_1,\gamma_2}(L(\tau))e^{-\zeta(t-\tau)}, \quad \text{for all} \quad t \geq \tau.$$

Finally, it follows from (2.54) that

$$\|L(t)\|_{Y_0}^2 \leq 4\tilde{L}_{\gamma_1,\gamma_2}(L(\tau))e^{-\zeta(t-\tau)} \leq 3(1+\lambda_1^{-1})\|L(\tau)\|_{Y_0}^2 e^{-\zeta(t-\tau)},$$

for all $t \geq \tau$, and the result is proved. □

Our intention is to apply Theorem 1.10 in order to conclude that the problem (2.1) − (2.3) has a pullback attractor in the phase space $Y_0$. However, instead of proving that the evolution process defined in (2.40) is pullback asymptotically compact (see Definition 1.12), in the next result we will establish, in a direct way, its compactness as a map from the phase space $Y_0$ into itself.

**Proposition 2.6.** *For each $t > \tau \in \mathbb{R}$, the evolution process $S(t,\tau)\colon Y_0 \to Y_0$ given in* (2.40) *is a compact map.*

*Proof.* Using the identity (2.38), the energy functional (2.39) and the Cauchy-Schwartz and Young inequalities, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{X^{\frac{1}{2}}}^2 + \|u\|_X^2 + \|u_t\|_X^2 + \|v\|_{X^{\frac{1}{2}}}^2 + \|v_t\|_X^2\right) + \eta\|u_t\|_{X^{\frac{1}{4}}}^2 + \eta\|v_t\|_{X^{\frac{1}{4}}}^2 \\
&= \langle f(u), u_t\rangle_X \leq \|f(u)\|_X\|u_t\|_X \leq \tilde{c}\|f(u)\|_X\|u_t\|_{X^{\frac{1}{4}}} \leq \frac{1}{2\epsilon}\|f(u)\|_X^2 + \frac{\epsilon}{2}\tilde{c}^2\|u_t\|_{X^{\frac{1}{4}}}^2,
\end{aligned}$$

for all $\epsilon > 0$, where $\tilde{c} > 0$ is the embedding constant of $X^{\frac{1}{4}} \hookrightarrow X$. Choosing $\epsilon = \frac{\eta}{\tilde{c}^2}$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{X^{\frac{1}{2}}}+\|u\|^2_X+\|u_t\|^2_X+\|v\|^2_{X^{\frac{1}{2}}}+\|v_t\|^2_X\right)+\frac{\eta}{2}\|u_t\|^2_{X^{\frac{1}{4}}}+\eta\|v_t\|^2_{X^{\frac{1}{4}}}\\&\leq\frac{\tilde{c}^2}{2\eta}\|f(u)\|^2_X.\end{aligned}\tag{2.55}$$

Now, knowing that the embedding $X^{\frac{1}{2}}\hookrightarrow L^{2\rho}(\Omega)$ holds for $1<\rho\leq\frac{n}{n-2}$, and using Lemma 2.3, condition $(i)$, we get

$$\begin{aligned}\|f(u)\|^2_X&\leq\int_\Omega[c(1+|u|^\rho)]^2dx\leq c_1\int_\Omega(1+|u|^{2\rho})dx\\&=c_1|\Omega|+c_1\|u\|^{2\rho}_{L^{2\rho}(\Omega)}\leq c_1|\Omega|+c_2\|u\|^{2\rho}_{X^{\frac{1}{2}}}\\&\leq c_1|\Omega|+c_2\|W\|^{2\rho}_{Y_0},\end{aligned}\tag{2.56}$$

where $c_1,c_2$ are positive constants and $W(t)=(u(t),u_t(t),v(t),v_t(t))$. Thus, combining (2.55) and (2.56), we obtain

$$\frac{d}{dt}\left(\|W\|^2_{Y_0}+\|u\|^2_X\right)+\eta\|u_t\|^2_{X^{\frac{1}{4}}}+2\eta\|v_t\|^2_{X^{\frac{1}{4}}}\leq\frac{\tilde{c}^2c_1|\Omega|}{\eta}+\frac{\tilde{c}^2c_2}{\eta}\|W\|^{2\rho}_{Y_0}.$$

Integrating the previous inequality from $\tau$ to $t$, we obtain

$$\begin{aligned}&\|W(t)\|^2_{Y_0}+\|u(t)\|^2_X+\eta\int_\tau^t\|u_t(r)\|^2_{X^{\frac{1}{4}}}dr+2\eta\int_\tau^t\|v_t(r)\|^2_{X^{\frac{1}{4}}}dr\\&\leq\frac{\tilde{c}^2c_1|\Omega|}{\eta}(t-\tau)+\frac{\tilde{c}^2c_2}{\eta}\int_\tau^t\|W(r)\|^{2\rho}_{Y_0}dr+\|W(\tau)\|^2_{Y_0}+\|u(\tau)\|^2_X\\&\leq\frac{\tilde{c}^2c_1|\Omega|}{\eta}(t-\tau)+\frac{\tilde{c}^2c_2}{\eta}\int_\tau^t\|W(r)\|^{2\rho}_{Y_0}dr+(1+\lambda_1^{-1})\|W(\tau)\|^2_{Y_0},\end{aligned}\tag{2.57}$$

where we have used the Poincaré Inequality $\|u(\tau)\|^2_X\leq\lambda_1^{-1}\|u(\tau)\|^2_{X^{\frac{1}{2}}}$. Also, note that inequality (2.57) implies

$$\begin{aligned}&\int_\tau^t\|u_t(r)\|^2_{X^{\frac{1}{4}}}dr+\int_\tau^t\|v_t(r)\|^2_{X^{\frac{1}{4}}}dr\\&\leq\frac{\tilde{c}^2c_1|\Omega|}{\eta^2}(t-\tau)+\frac{\tilde{c}^2c_2}{\eta^2}\int_\tau^t\|W(r)\|^{2\rho}_{Y_0}dr+\frac{1+\lambda_1^{-1}}{\eta}\|W(\tau)\|^2_{Y_0}.\end{aligned}\tag{2.58}$$

Now, consider the original system (2.1). By taking the inner product of the first equation in (2.1) with $A^{\frac{1}{2}}u$, and also the inner product of the second equation in (2.1) with $A^{\frac{1}{2}}v$, and noticing the identity

$$\langle u_{tt},A^{\frac{1}{2}}u\rangle_X=\frac{d}{dt}\langle u_t,A^{\frac{1}{2}}u\rangle_X-\|u_t\|^2_{X^{\frac{1}{4}}},$$

we obtain,

$$\begin{aligned}
&\frac{d}{dt}\langle u_t, A^{\frac{1}{2}}u\rangle_X - \|u_t\|^2_{X^{\frac{1}{4}}} + \|u\|^2_{X^{\frac{3}{4}}} + \|u\|^2_{X^{\frac{1}{4}}} + \frac{\eta}{2}\frac{d}{dt}\|u\|^2_{X^{\frac{1}{2}}} + a_\epsilon(t)\langle A^{\frac{1}{2}}v_t, A^{\frac{1}{2}}u\rangle_X \\
&+ \frac{d}{dt}\langle v_t, A^{\frac{1}{2}}v\rangle_X - \|v_t\|^2_{X^{\frac{1}{4}}} + \|v\|^2_{X^{\frac{3}{4}}} + \frac{\eta}{2}\frac{d}{dt}\|v\|^2_{X^{\frac{1}{2}}} - a_\epsilon(t)\langle A^{\frac{1}{2}}u_t, A^{\frac{1}{2}}v\rangle_X \\
&= \langle f(u), A^{\frac{1}{2}}u\rangle_X.
\end{aligned}$$

Once again, using the Cauchy-Schwartz and Young inequalities, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\langle u_t, A^{\frac{1}{2}}u\rangle_X + \langle v_t, A^{\frac{1}{2}}v\rangle_X\right) + \frac{\eta}{2}\frac{d}{dt}\left(\|u\|^2_{X^{\frac{1}{2}}} + \|v\|^2_{X^{\frac{1}{2}}}\right) + \|u\|^2_{X^{\frac{3}{4}}} + \|u\|^2_{X^{\frac{1}{4}}} + \|v\|^2_{X^{\frac{3}{4}}} \\
&\leq \|u_t\|^2_{X^{\frac{1}{4}}} + \|v_t\|^2_{X^{\frac{1}{4}}} + a_1\|v_t\|_{X^{\frac{1}{4}}}\|u\|_{X^{\frac{3}{4}}} + a_1\|u_t\|_{X^{\frac{1}{4}}}\|v\|_{X^{\frac{3}{4}}} + \|f(u)\|_X\|u\|_{X^{\frac{1}{2}}} \\
&\leq \left(1 + \frac{1}{2\epsilon_2}\right)\|u_t\|^2_{X^{\frac{1}{4}}} + \left(1 + \frac{1}{2\epsilon_1}\right)\|v_t\|^2_{X^{\frac{1}{4}}} + \frac{\epsilon_1}{2}a_1^2\|u\|^2_{X^{\frac{3}{4}}} + \frac{\epsilon_2}{2}a_1^2\|v\|^2_{X^{\frac{3}{4}}} \\
&+ \frac{1}{2}\|f(u)\|^2_X + \frac{1}{2}\|u\|^2_{X^{\frac{1}{2}}},
\end{aligned}$$

for all $\epsilon_1, \epsilon_2 > 0$. Choosing $\epsilon_1 = \epsilon_2 = \frac{1}{a_1^2}$, and using (2.56), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\langle u_t, A^{\frac{1}{2}}u\rangle_X + \langle v_t, A^{\frac{1}{2}}v\rangle_X\right) + \frac{\eta}{2}\frac{d}{dt}\left(\|u\|^2_{X^{\frac{1}{2}}} + \|v\|^2_{X^{\frac{1}{2}}}\right) + \frac{1}{2}\|u\|^2_{X^{\frac{3}{4}}} + \frac{1}{2}\|v\|^2_{X^{\frac{3}{4}}} \\
&\leq \frac{2 + a_1^2}{2}\|u_t\|^2_{X^{\frac{1}{4}}} + \frac{2 + a_1^2}{2}\|v_t\|^2_{X^{\frac{1}{4}}} + \frac{c_1|\Omega|}{2} + \frac{c_2}{2}\|W\|^{2\rho}_{Y_0} + \frac{1}{2}\|W\|^2_{Y_0}.
\end{aligned}$$

Integrating the previous inequality from $\tau$ to $t$, and using (2.58), we obtain

$$\begin{aligned}
&\frac{\eta}{2}\left(\|u(t)\|^2_{X^{\frac{1}{2}}} + \|v(t)\|^2_{X^{\frac{1}{2}}}\right) + \frac{1}{2}\int_\tau^t \|u(r)\|^2_{X^{\frac{3}{4}}}dr + \frac{1}{2}\int_\tau^t \|v(r)\|^2_{X^{\frac{3}{4}}}dr \\
&\leq \frac{2 + a_1^2}{2}\left(\int_\tau^t \|u_t(r)\|^2_{X^{\frac{1}{4}}}dr + \int_\tau^t \|v_t(r)\|^2_{X^{\frac{1}{4}}}dr\right) + \frac{c_1|\Omega|}{2}(t-\tau) + \frac{c_2}{2}\int_\tau^t \|W(r)\|^{2\rho}_{Y_0}dr \\
&+ \frac{1}{2}\int_\tau^t \|W(r)\|^2_{Y_0}dr - \langle u_t(t), A^{\frac{1}{2}}u(t)\rangle_X - \langle v_t(t), A^{\frac{1}{2}}v(t)\rangle_X \\
&+ \langle u_t(\tau), A^{\frac{1}{2}}u(\tau)\rangle_X + \langle v_t(\tau), A^{\frac{1}{2}}v(\tau)\rangle_X + \frac{\eta}{2}\left(\|u(\tau)\|^2_{X^{\frac{1}{2}}} + \|v(\tau)\|^2_{X^{\frac{1}{2}}}\right) \\
&\leq \frac{2 + a_1^2}{2}\left(\frac{\tilde{c}^2c_1|\Omega|}{\eta^2}(t-\tau) + \frac{\tilde{c}^2c_2}{\eta^2}\int_\tau^t \|W(r)\|^{2\rho}_{Y_0}dr + \frac{1+\lambda_1^{-1}}{\eta}\|W(\tau)\|^2_{Y_0}\right) + \frac{c_1|\Omega|}{2}(t-\tau) \\
&+ \frac{c_2}{2}\int_\tau^t \|W(r)\|^{2\rho}_{Y_0}dr + \frac{1}{2}\int_\tau^t \|W(r)\|^2_{Y_0}dr + \frac{1}{2}\|W(t)\|^2_{Y_0} + \frac{1+\eta}{2}\|W(\tau)\|^2_{Y_0},
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_\tau^t \|u(r)\|^2_{X^{\frac{3}{4}}}dr + \int_\tau^t \|v(r)\|^2_{X^{\frac{3}{4}}}dr \\
&\leq (2 + a_1^2)\left(\frac{\tilde{c}^2c_1|\Omega|}{\eta^2}(t-\tau) + \frac{\tilde{c}^2c_2}{\eta^2}\int_\tau^t \|W(r)\|^{2\rho}_{Y_0}dr + \frac{1+\lambda_1^{-1}}{\eta}\|W(\tau)\|^2_{Y_0}\right)
\end{aligned} \tag{2.59}$$

$$+ c_1|\Omega|(t-\tau) + c_2 \int_\tau^t \|W(r)\|_{Y_0}^{2\rho} dr + \int_\tau^t \|W(r)\|_{Y_0}^2 dr$$
$$+ \|W(t)\|_{Y_0}^2 + (1+\eta)\|W(\tau)\|_{Y_0}^2.$$

On the other hand, taking the inner product of the first equation in (2.1) with $A^{\frac{1}{2}}u_t$, and also the inner product of the second equation in (2.1) with $A^{\frac{1}{2}}v_t$, and using (2.56), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2_{X^{\frac{1}{4}}} + \|u\|^2_{X^{\frac{3}{4}}} + \|u\|^2_{X^{\frac{1}{4}}} + \|v_t\|^2_{X^{\frac{1}{4}}} + \|v\|^2_{X^{\frac{3}{4}}}\right) + \eta\|u_t\|^2_{X^{\frac{1}{2}}} + \eta\|v_t\|^2_{X^{\frac{1}{2}}}\\
&= \langle f(u), A^{\frac{1}{2}}u_t\rangle_X \le \|f(u)\|_X\|u_t\|_{X^{\frac{1}{2}}} \le \frac{1}{2\eta}\|f(u)\|_X^2 + \frac{\eta}{2}\|u_t\|^2_{X^{\frac{1}{2}}}\\
&\le \frac{c_1|\Omega|}{2\eta} + \frac{c_2}{2\eta}\|W\|_{Y_0}^{2\rho} + \frac{\eta}{2}\|u_t\|^2_{X^{\frac{1}{2}}},
\end{aligned}$$

which yields

$$\frac{d}{dt}\left(\|u\|^2_{X^{\frac{3}{4}}} + \|u\|^2_{X^{\frac{1}{4}}} + \|u_t\|^2_{X^{\frac{1}{4}}} + \|v\|^2_{X^{\frac{3}{4}}} + \|v_t\|^2_{X^{\frac{1}{4}}}\right) \le \frac{c_1|\Omega|}{\eta} + \frac{c_2}{\eta}\|W\|_{Y_0}^{2\rho}.$$

Integrating the previous inequality from $r$ to $t$, for $\tau < r < t$, we have

$$\begin{aligned}
&\|u(t)\|^2_{X^{\frac{3}{4}}} + \|u(t)\|^2_{X^{\frac{1}{4}}} + \|u_t(t)\|^2_{X^{\frac{1}{4}}} + \|v(t)\|^2_{X^{\frac{3}{4}}} + \|v_t(t)\|^2_{X^{\frac{1}{4}}}\\
&\le \frac{c_1|\Omega|}{\eta}(t-r) + \frac{c_2}{\eta}\int_r^t \|W(s)\|_{Y_0}^{2\rho} ds + \|u(r)\|^2_{X^{\frac{3}{4}}} + \|u(r)\|^2_{X^{\frac{1}{4}}} + \|u_t(r)\|^2_{X^{\frac{1}{4}}}\\
&+ \|v(r)\|^2_{X^{\frac{3}{4}}} + \|v_t(r)\|^2_{X^{\frac{1}{4}}},
\end{aligned}$$

consequently,

$$\begin{aligned}
&\|u(t)\|^2_{X^{\frac{3}{4}}} + \|u_t(t)\|^2_{X^{\frac{1}{4}}} + \|v(t)\|^2_{X^{\frac{3}{4}}} + \|v_t(t)\|^2_{X^{\frac{1}{4}}}\\
&\le \frac{c_1|\Omega|}{\eta}(t-r) + \frac{c_2}{\eta}\int_r^t \|W(s)\|_{Y_0}^{2\rho} ds + \tilde{k}\|W(r)\|_{Y_0}^2 + \|u(r)\|^2_{X^{\frac{3}{4}}}\\
&+ \|u_t(r)\|^2_{X^{\frac{1}{4}}} + \|v(r)\|^2_{X^{\frac{3}{4}}} + \|v_t(r)\|^2_{X^{\frac{1}{4}}},
\end{aligned} \tag{2.60}$$

where we have used the embedding $X^{\frac{1}{2}} \hookrightarrow X^{\frac{1}{4}}$, i.e., $\|u(r)\|^2_{X^{\frac{1}{4}}} \le \tilde{k}\|u(r)\|^2_{X^{\frac{1}{2}}}$.

Now, by integrating inequality (2.60), with respect to $r$, from $\tau$ to $t$, we obtain

$$\begin{aligned}
&(t-\tau)\left(\|u(t)\|^2_{X^{\frac{3}{4}}} + \|u_t(t)\|^2_{X^{\frac{1}{4}}} + \|v(t)\|^2_{X^{\frac{3}{4}}} + \|v_t(t)\|^2_{X^{\frac{1}{4}}}\right)\\
&\le \frac{c_1|\Omega|}{2\eta}(t-\tau)^2 + \frac{c_2}{\eta}\int_\tau^t\int_r^t \|W(s)\|_{Y_0}^{2\rho} ds dr + \tilde{k}\int_\tau^t \|W(r)\|_{Y_0}^2 dr\\
&+ \int_\tau^t \|u(r)\|^2_{X^{\frac{3}{4}}} dr + \int_\tau^t \|u_t(r)\|^2_{X^{\frac{1}{4}}} dr
\end{aligned} \tag{2.61}$$

$$+\int_\tau^t \|v(r)\|^2_{X^{\frac{3}{4}}} dr + \int_\tau^t \|v_t(r)\|^2_{X^{\frac{1}{4}}} dr.$$

Combining the inequalities obtained in (2.58), (2.59) and (2.61), we get

$$\begin{aligned}
&\|u(t)\|^2_{X^{\frac{3}{4}}} + \|u_t(t)\|^2_{X^{\frac{1}{4}}} + \|v(t)\|^2_{X^{\frac{3}{4}}} + \|v_t(t)\|^2_{X^{\frac{1}{4}}} \\
&\leq \frac{\tilde{c}^2 c_1 |\Omega|(3+a_1^2)}{\eta^2} + c_1|\Omega| + \frac{c_1|\Omega|}{2\eta}(t-\tau) + \frac{c_2}{\eta(t-\tau)} \int_\tau^t \int_r^t \|W(s)\|^{2\rho}_{Y_0} ds dr \\
&+ \frac{1}{t-\tau}\left(\frac{\tilde{c}^2 c_2(3+a_1^2)}{\eta^2} + c_2\right)\int_\tau^t \|W(r)\|^{2\rho}_{Y_0} dr + \frac{\tilde{k}+1}{t-\tau}\int_\tau^t \|W(r)\|^2_{Y_0} dr \\
&+ \frac{1}{t-\tau}\|W(t)\|^2_{Y_0} + \frac{(1+\lambda_1^{-1})(3+a_1^2)+\eta(1+\eta)}{\eta(t-\tau)}\|W(\tau)\|^2_{Y_0}.
\end{aligned} \tag{2.62}$$

Now, if the global solution $W(t) = (u(t), u_t(t), v(t), v_t(t))$ of the problem (2.1)$-$(2.3) starts in a bounded subset $B$ of $Y_0$, then $\|W(\tau)\|_{Y_0} \leq M$ for some positive constant $M$. Moreover, remember that from Theorem 2.4 there exist positive constants $\sigma$, $K_1 = K_1(B)$ and $K_2$ such that

$$\|W(t)\|^2_{Y_0} \leq K_1 e^{-\sigma(t-\tau)} + K_2, \quad t \geq \tau.$$

With this, we can handle with the three integrals that appear on the right hand side of inequality (2.62). In fact, first note that

$$\int_\tau^t \|W(r)\|^2_{Y_0} dr \leq \int_\tau^t [K_1 e^{-\sigma(r-\tau)} + K_2] dr \leq \frac{K_1}{\sigma} + K_2(t-\tau)$$

and

$$\int_\tau^t \|W(r)\|^{2\rho}_{Y_0} dr \leq \int_\tau^t [\tilde{K_1} e^{-\rho\sigma(r-\tau)} + \tilde{K_2}] dr \leq \frac{\tilde{K_1}}{\rho\sigma} + \tilde{K_2}(t-\tau),$$

where $\tilde{K_1}, \tilde{K_2}$ are positive constants. For the last integral remaining, note that

$$\int_r^t \|W(s)\|^{2\rho}_{Y_0} ds \leq \int_r^t \left[\tilde{\tilde{K}}_1 e^{-\rho\sigma(s-\tau)} + \tilde{\tilde{K}}_2\right] ds \leq \frac{\tilde{\tilde{K}}_1}{\rho\sigma} e^{-\rho\sigma(r-\tau)} + \tilde{\tilde{K}}_2(t-r),$$

for positive constants $\tilde{\tilde{K}}_1$ and $\tilde{\tilde{K}}_2$, and then it follows that

$$\int_\tau^t \int_r^t \|W(s)\|^{2\rho}_{Y_0} ds dr \leq \int_\tau^t \left[\frac{\tilde{\tilde{K}}_1}{\rho\sigma} e^{-\rho\sigma(r-\tau)} + \tilde{\tilde{K}}_2(t-r)\right] dr \leq \frac{\tilde{\tilde{K}}_1}{(\rho\sigma)^2} + \frac{\tilde{\tilde{K}}_2}{2}(t-\tau)^2. \tag{2.63}$$

Finally, combining all the estimates in (2.62)−(2.63), we conclude that there exist positive constants $k_1, k_2, k_3, k_4, k_5$ such that

$$\|u(t)\|^2_{X^{\frac{3}{4}}} + \|u_t(t)\|^2_{X^{\frac{1}{4}}} + \|v(t)\|^2_{X^{\frac{3}{4}}} + \|v_t(t)\|^2_{X^{\frac{1}{4}}} \le k_1 + k_2(t-\tau) + \frac{1}{t-\tau}[k_3 e^{-k_4(t-\tau)} + k_5].$$

Hence, $S(t,\tau)B$ is bounded in $X^{\frac{3}{4}} \times X^{\frac{1}{4}} \times X^{\frac{3}{4}} \times X^{\frac{1}{4}}$. Since $X^{\frac{3}{4}} \times X^{\frac{1}{4}} \times X^{\frac{3}{4}} \times X^{\frac{1}{4}} \hookrightarrow Y_0$, and this embedding is compact, we conclude that $S(t,\tau)\colon Y_0 \to Y_0$, given in (2.40), is compact for each $t > \tau$. □

**Theorem 2.7. [Pullback Attractors]** *Under the conditions of Theorem 2.3, the problem* (2.1) − (2.3) *has a pullback attractor* $\{\mathbb{A}(t)\colon t \in \mathbb{R}\}$ *in* $Y_0$ *and*

$$\bigcup_{t\in\mathbb{R}} \mathbb{A}(t) \subset Y_0$$

*is bounded.*

*Proof.* Theorem 2.4 assures that the evolution process $S(t,\tau)\colon Y_0 \to Y_0$ given by (2.40) is pullback strongly bounded dissipative. Additionally, it follows by Proposition 2.6 that $S(t,\tau)\colon Y_0 \to Y_0$ is compact, and, consequently, it is pullback asymptotically compact. Now the result is a simple consequence of Theorem 1.10. □

## 2.4 Regularity of the pullback attractor

The purpose of this section is to show that the regularity of the pullback attractor can be improved, using energy estimates and progressive increases of regularity.

**Theorem 2.8. [Regularity of Pullback Attractors]** *Assume that* $\frac{n-1}{n-2} \le \rho < \frac{n}{n-2}$. *The pullback attractor* $\{\mathbb{A}(t)\colon t \in \mathbb{R}\}$ *for the problem* (2.1) − (2.3)*, obtained in Theorem 2.7, lies in a more regular space than* $Y_0$*. More precisely,*

$$\bigcup_{t\in\mathbb{R}} \mathbb{A}(t)$$

*is a bounded subset of* $X^1 \times X^{\frac{1}{2}} \times X^1 \times X^{\frac{1}{2}}$.

*Proof.* Let $\xi\colon \mathbb{R} \to Y_0$ be a bounded global solution for the system (2.1). Since $\bigcup_{t\in\mathbb{R}} \mathbb{A}(t)$ is bounded in $Y_0$ (see Theorem 2.7), we have $\{\xi(t)\colon t \in \mathbb{R}\}$ is a bounded subset of $Y_0$ by Theorem 1.11. Moreover, $\xi(\cdot) = (\mu(\cdot), \mu_t(\cdot), \nu(\cdot), \nu_t(\cdot))\colon \mathbb{R} \to Y_0$ is such that $\xi(t) \in \mathbb{A}(t)$ for all $t \in \mathbb{R}$, and by (2.41),

$$\xi(t) = L(t,\tau)\xi(\tau) + \int_\tau^t L(t,s)F(\xi(s))ds, \quad t \ge \tau.$$

Using the decay of $L(\cdot,\cdot)$, which was established in Theorem 2.6, and letting $\tau \to -\infty$, we get

$$\xi(t) = \int_{-\infty}^{t} L(t,s)F(\xi(s))ds, \quad t \in \mathbb{R}.$$

Now, for $\tau \in \mathbb{R}$ fixed, we write $W_0 = (u_0, u_1, v_0, v_1) = \xi(\tau)$ and consider

$$(u(t), u_t(t), v(t), v_t(t)) = U(t,\tau)W_0 = \int_{\tau}^{t} L(t,s)F(S(s,\tau)W_0)ds,$$

where $U(\cdot,\cdot)$ is defined as in (2.43). Note that $(u(\cdot), v(\cdot))$ solves the system

$$\begin{cases} u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}}u_t + a_\epsilon(t)(-\Delta)^{\frac{1}{2}}v_t = f(u(t,\tau;u_0)), & (x,t) \in \Omega \times (\tau,\infty), \\ v_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}}v_t - a_\epsilon(t)(-\Delta)^{\frac{1}{2}}u_t = 0, & (x,t) \in \Omega \times (\tau,\infty), \end{cases} \tag{2.64}$$

with

$$u(\tau,x) = 0, \ v(\tau,x) = 0, \ x \in \Omega. \tag{2.65}$$

To estimate the solution of (2.64) – (2.65) for $(u_0, u_1, v_0, v_1)$ in a bounded subset $B \subset Y_0$, we again consider the maps

$$\mathcal{E}(t) = \frac{1}{2}\|u(t)\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|u(t)\|^2_X + \frac{1}{2}\|u_t(t)\|^2_X + \frac{1}{2}\|v(t)\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|v_t(t)\|^2_X - \int_\Omega \int_0^{u(t)} f(s)dsdx,$$

and

$$\begin{aligned}\mathcal{L}(t) = &\frac{1}{2}\|u(t)\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|u(t)\|^2_X + \frac{1}{2}\|u_t(t)\|^2_X + \frac{1}{2}\|v(t)\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|v_t(t)\|^2_X \\ &+ \gamma_1\langle u(t), u_t(t)\rangle_X + \gamma_2\langle v(t), v_t(t)\rangle_X\end{aligned}$$

with $\gamma_1, \gamma_2 \in \mathbb{R}^+$. Using (2.64), we can write

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &= \frac{d}{dt}\left(\mathcal{E}(t) + \gamma_1\langle u, u_t\rangle_X + \gamma_2\langle v, v_t\rangle_X + \int_\Omega\int_0^u f(s)dsdx\right) \\ &= -\eta\|u_t\|^2_{X^{\frac{1}{4}}} - \eta\|v_t\|^2_{X^{\frac{1}{4}}} + \gamma_1\|u_t\|^2_X - \gamma_1(\|u\|^2_{X^{\frac{1}{2}}} + \|u\|^2_X) - \gamma_1\eta\langle A^{\frac{1}{2}}u, u_t\rangle_X \\ &\quad - \gamma_1 a_\epsilon(t)\langle A^{\frac{1}{2}}u, v_t\rangle_X + \gamma_1\langle u, f(u)\rangle_X + \gamma_2\|v_t\|^2_X - \gamma_2\|v\|^2_{X^{\frac{1}{2}}} - \gamma_2\eta\langle A^{\frac{1}{2}}v, v_t\rangle_X \\ &\quad + \gamma_2 a_\epsilon(t)\langle A^{\frac{1}{2}}v, u_t\rangle_X + \langle f(u), u_t\rangle_X.\end{aligned}$$

In the first place, let's deal with the nonlinearity $f$. By Lemma 2.3, it follows that for each $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\int_\Omega f(u)udx \le \delta\|u\|^2_X + C_\delta.$$

Further, once the condition $1 < \frac{n-1}{n-2} \leq \rho < \frac{n}{n-2}$ implies $X^{\frac{1}{2}} \hookrightarrow L^{2\rho}(\Omega)$, and using again Lemma 2.3, condition $(i)$, we have

$$\begin{aligned}\|f(u)\|_X &\leq \left(\int_\Omega [c(1+|u|^\rho)]^2 dx\right)^{\frac{1}{2}} \leq \tilde{c}\left(|\Omega| + \int_\Omega |u|^{2\rho} dx\right)^{\frac{1}{2}} \\ &\leq \tilde{\tilde{c}}\left(|\Omega|^{\frac{1}{2}} + \|u\|^\rho_{L^{2\rho}(\Omega)}\right) \leq \overline{c}\|u\|^\rho_{X^{\frac{1}{2}}} + \tilde{\tilde{c}}|\Omega|^{\frac{1}{2}} \leq \overline{c}r^\rho + \tilde{\tilde{c}}|\Omega|^{\frac{1}{2}} = \overline{C},\end{aligned}$$

whenever $\|u\|_{X^{\frac{1}{2}}} \leq r$.

Hence, using the Poincaré and Young Inequalities, one can obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\leq -\eta\frac{1}{c^2}\|u_t\|^2_X - \eta\frac{1}{c^2}\|v_t\|^2_X + \gamma_1\|u_t\|^2_X - \gamma_1\|u\|^2_{X^{\frac{1}{2}}} + \gamma_1\eta\left(\frac{\epsilon_1}{2}\|u\|^2_{X^{\frac{1}{2}}} + \frac{1}{2\epsilon_1}\|u_t\|^2_X\right) \\ &+ \gamma_1 a_1\left(\frac{1}{2\epsilon_2}\|u\|^2_{X^{\frac{1}{2}}} + \frac{\epsilon_2}{2}\|v_t\|^2_X\right) + \gamma_1\delta\lambda_1^{-1}\|u\|^2_{X^{\frac{1}{2}}} + \gamma_1 C_\delta + \gamma_2\|v_t\|^2_X - \gamma_2\|v\|^2_{X^{\frac{1}{2}}} \\ &+ \gamma_2 a_1\left(\frac{1}{2\epsilon_3}\|v\|^2_{X^{\frac{1}{2}}} + \frac{\epsilon_3}{2}\|u_t\|^2_X\right) + \gamma_2\eta\left(\frac{\epsilon_4}{2}\|v\|^2_{X^{\frac{1}{2}}} + \frac{1}{2\epsilon_4}\|v_t\|^2_X\right) + \frac{1}{2\epsilon_5}\|f(u)\|^2_X + \frac{\epsilon_5}{2}\|u_t\|^2_X \\ &\leq -\gamma_1\left(1 - \delta\lambda_1^{-1} - \eta\frac{\epsilon_1}{2} - a_1\frac{1}{2\epsilon_2}\right)\|u\|^2_{X^{\frac{1}{2}}} - \left(\eta\frac{1}{c^2} - \gamma_1 - \gamma_1\eta\frac{1}{2\epsilon_1} - \gamma_2 a_1\frac{\epsilon_3}{2} - \frac{\epsilon_5}{2}\right)\|u_t\|^2_X \\ &- \gamma_2\left(1 - a_1\frac{1}{2\epsilon_3} - \eta\frac{\epsilon_4}{2}\right)\|v\|^2_{X^{\frac{1}{2}}} - \left(\eta\frac{1}{c^2} - \gamma_2 - \gamma_1 a_1\frac{\epsilon_2}{2} - \gamma_2\eta\frac{1}{2\epsilon_4}\right)\|v_t\|^2_X + \frac{1}{2\epsilon_5}\overline{C}^2 + \gamma_1 C_\delta\end{aligned}$$

for all $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5 > 0$, where $c > 0$ is the embedding constant of $X^{\frac{1}{4}} \hookrightarrow X$. Choosing

$$\delta = \frac{\lambda_1}{8},\ \epsilon_1 = \frac{1}{\eta},\ \epsilon_2 = 2a_1,\ \epsilon_3 = 2a_1,\ \epsilon_4 = \frac{1}{\eta},\ \epsilon_5 = \frac{\eta}{c^2},$$

it follows that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq &-\frac{1}{8}\gamma_1\|u\|^2_{X^{\frac{1}{2}}} - \left(\frac{\eta}{2c^2} - \gamma_1\left(1 + \frac{\eta^2}{2}\right) - \gamma_2 a_1^2\right)\|u_t\|^2_X - \frac{1}{4}\gamma_2\|v\|^2_{X^{\frac{1}{2}}} \\ &- \left(\frac{\eta}{2c^2} - \gamma_1 a_1^2 - \gamma_2\left(1 + \frac{\eta^2}{2}\right)\right)\|v_t\|^2_X + \frac{c^2}{2\eta}\overline{C}^2 + \gamma_1 C_{\frac{\lambda_1}{8}}.\end{aligned}$$

Now, note that one can take $\gamma_i > 0, i = 1, 2,$ sufficiently small such that

$$\gamma_i < \frac{\eta}{8c^2}\min\left\{\frac{1}{a_1^2},\ \left(1 + \frac{\eta^2}{2}\right)^{-1}\right\},\ i = 1, 2.$$

Setting

$$C_1 = \min\left\{\frac{1}{8}\gamma_1, \frac{\eta}{2c^2} - \gamma_1\left(1 + \frac{\eta^2}{2}\right) - \gamma_2 a_1^2,\ \frac{1}{4}\gamma_2,\ \frac{\eta}{2c^2} - \gamma_1 a_1^2 - \gamma_2\left(1 + \frac{\eta^2}{2}\right)\right\} > 0$$

and $C_2 = \frac{c^2}{2\eta}\overline{C}^2 + \gamma_1 C_{\frac{\lambda_1}{8}} > 0$, we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -C_1\|(u,u_t,v,v_t)\|_{Y_0}^2 + C_2.$$

Using (2.46) and (2.47), we get

$$\frac{1}{4}\|(u,u_t,v,v_t)\|_{Y_0}^2 \leq \mathcal{L}(t) \leq \frac{3(1+\lambda_1^{-1})}{4}\|(u,u_t,v,v_t)\|_{Y_0}^2,$$

and putting $C_3 = C_1\left(\frac{3(1+\lambda_1^{-1})}{4}\right)^{-1}$, one has

$$\frac{1}{4}\|(u,u_t,v,v_t)\|_{Y_0}^2 \leq \mathcal{L}(t) \leq \mathcal{L}(\tau)e^{-C_3(t-\tau)} + \frac{C_2}{C_3}, \quad t \geq \tau.$$

From this, we obtain

$$\bigcup_{\tau\leq s\leq t} U(s,\tau)B \text{ is a bounded subset of } Y_0.$$

On the other hand, note that $(\phi,\varphi) = (u_t, v_t)$ solves the system

$$\begin{cases} \phi_{tt} - \Delta\phi + \phi + \eta(-\Delta)^{\frac{1}{2}}\phi_t + a_\epsilon(t)(-\Delta)^{\frac{1}{2}}\varphi_t + a_\epsilon'(t)(-\Delta)^{\frac{1}{2}}\varphi = f'(u)\phi, \\ \varphi_{tt} - \Delta\varphi + \eta(-\Delta)^{\frac{1}{2}}\varphi_t - a_\epsilon(t)(-\Delta)^{\frac{1}{2}}\phi_t - a_\epsilon'(t)(-\Delta)^{\frac{1}{2}}\phi = 0. \end{cases} \tag{2.66}$$

We want to estimate $(\phi,\phi_t,\varphi,\varphi_t)$ in $Y_0$, but our solutions are not regular enough for this to be done in a direct way. Thus, instead, the process will be done by progressive increases of regularity. For $\alpha > 0$, let us consider the fractional power spaces $X^\alpha = D(A^\alpha)$ endowed with the graph norm, and let $X^{-\alpha} = (X^\alpha)'$. For

$$(\phi,\phi_t,\varphi,\varphi_t) \in X^{\frac{1-\alpha}{2}} \times X^{-\frac{\alpha}{2}} \times X^{\frac{1-\alpha}{2}} \times X^{-\frac{\alpha}{2}},$$

let us define

$$\begin{aligned} \mathcal{L}_\alpha(t) = \frac{1}{2}\Big(&\|\phi(t)\|_{X^{\frac{1-\alpha}{2}}}^2 + \|\phi(t)\|_{X^{-\frac{\alpha}{2}}}^2 + \|\phi_t(t)\|_{X^{-\frac{\alpha}{2}}}^2 + \|\varphi(t)\|_{X^{\frac{1-\alpha}{2}}}^2 + \|\varphi_t(t)\|_{X^{-\frac{\alpha}{2}}}^2\Big) \\ &+ \gamma_1\langle\phi(t),\phi_t(t)\rangle_{X^{-\frac{\alpha}{2}}} + \gamma_2\langle\varphi(t),\varphi_t(t)\rangle_{X^{-\frac{\alpha}{2}}}, \end{aligned}$$

with $\gamma_1, \gamma_2 \in \mathbb{R}^+$. Using (2.66), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_\alpha(t) = &\langle\phi_t,\phi\rangle_{X^{\frac{1-\alpha}{2}}} + \langle\phi_t,\phi\rangle_{X^{-\frac{\alpha}{2}}} + \langle\phi_{tt},\phi_t\rangle_{X^{-\frac{\alpha}{2}}} + \langle\varphi_t,\varphi\rangle_{X^{\frac{1-\alpha}{2}}} + \langle\varphi_{tt},\varphi_t\rangle_{X^{-\frac{\alpha}{2}}} \\ &+ \gamma_1\langle\phi_t,\phi_t\rangle_{X^{-\frac{\alpha}{2}}} + \gamma_1\langle\phi,\phi_{tt}\rangle_{X^{-\frac{\alpha}{2}}} + \gamma_2\langle\varphi_t,\varphi_t\rangle_{X^{-\frac{\alpha}{2}}} + \gamma_2\langle\varphi,\varphi_{tt}\rangle_{X^{-\frac{\alpha}{2}}} \end{aligned}$$

$$
\begin{aligned}
&= \gamma_1\|\phi_t\|^2_{X^{-\frac{\alpha}{2}}} + \gamma_2\|\varphi_t\|^2_{X^{-\frac{\alpha}{2}}} - \gamma_1\|\phi\|^2_{X^{-\frac{\alpha}{2}}} - \gamma_1\|\phi\|^2_{X^{\frac{1-\alpha}{2}}} - \gamma_2\|\varphi\|^2_{X^{\frac{1-\alpha}{2}}} - \eta\|\phi_t\|^2_{X^{\frac{1-2\alpha}{4}}} \\
&- \eta\|\varphi_t\|^2_{X^{\frac{1-2\alpha}{4}}} - \eta\gamma_1\langle\phi, A^{\frac12}\phi_t\rangle_{X^{-\frac{\alpha}{2}}} - a_\epsilon(t)\gamma_1\langle\phi, A^{\frac12}\varphi_t\rangle_{X^{-\frac{\alpha}{2}}} - \eta\gamma_2\langle\varphi, A^{\frac12}\varphi_t\rangle_{X^{-\frac{\alpha}{2}}} \\
&+ a_\epsilon(t)\gamma_2\langle\varphi, A^{\frac12}\phi_t\rangle_{X^{-\frac{\alpha}{2}}} - a'_\epsilon(t)\langle A^{\frac12}\varphi, \phi_t\rangle_{X^{-\frac{\alpha}{2}}} + a'_\epsilon(t)\langle A^{\frac12}\phi, \varphi_t\rangle_{X^{-\frac{\alpha}{2}}} - a'_\epsilon(t)\gamma_1\langle\phi, A^{\frac12}\varphi\rangle_{X^{-\frac{\alpha}{2}}} \\
&+ a'_\epsilon(t)\gamma_2\langle\varphi, A^{\frac12}\phi\rangle_{X^{-\frac{\alpha}{2}}} + \gamma_1\langle\phi, f'(u)\phi\rangle_{X^{-\frac{\alpha}{2}}} + \langle f'(u)\phi, \phi_t\rangle_{X^{-\frac{\alpha}{2}}}.
\end{aligned}
$$

Next, we shall estimate the terms that appear on the right hand side of the above expression, beginning with those in which the nonlinearity $f'$ is explicit. To do this, consider

$$\alpha_1 = \frac{(\rho-1)(n-2)}{2}.$$

Since $\frac{n-1}{n-2} \le \rho < \frac{n}{n-2}$, we have $\frac12 \le \alpha_1 < 1$.

Noticing that

$$\langle f'(u)\phi, \phi_t\rangle_{X^{-\frac{\alpha}{2}}} \le \|f'(u)\phi\|_{X^{-\frac{\alpha}{2}}}\|\phi_t\|_{X^{-\frac{\alpha}{2}}} \tag{2.67}$$

and that the embedding $X^{\frac{\alpha}{2}} = H^\alpha(\Omega) \hookrightarrow L^p(\Omega)$ or, equivalently, $L^{\frac{p}{p-1}}(\Omega) \hookrightarrow X^{-\frac{\alpha}{2}}$, holds for any $2 \le p \le \frac{2n}{n-2\alpha}$, one can obtain an estimate for the term $\|f'(u)\phi\|_{X^{-\frac{\alpha}{2}}}$ using Hölder's Inequality and the growth condition, in the following way:

$$
\begin{aligned}
\|f'(u)\phi\|_{X^{-\frac{\alpha}{2}}} &\le c_1\|f'(u)\phi\|_{L^{\frac{2n}{n+2\alpha}}(\Omega)} \le c_1\|\phi\|_{L^2(\Omega)}\|f'(u)\|_{L^{\frac{n}{\alpha}}(\Omega)} \\
&\le c_1\|\phi\|_X\left(\int_\Omega [c(1+|u|^{\rho-1})]^{\frac{n}{\alpha}}dx\right)^{\frac{\alpha}{n}} \le c_2\|\phi\|_X\left(|\Omega| + \int_\Omega |u|^{\frac{(\rho-1)n}{\alpha}}dx\right)^{\frac{\alpha}{n}} \\
&\le c_3\|\phi\|_X\left(1 + \|u\|^{\rho-1}_{L^{\frac{(\rho-1)n}{\alpha}}(\Omega)}\right).
\end{aligned}
$$

Now, once the embedding $H^1(\Omega) \hookrightarrow L^{\frac{(\rho-1)n}{\alpha}}(\Omega)$ holds, if and only if $\alpha \ge \frac{(\rho-1)(n-2)}{2}$ and $\frac{(\rho-1)n}{\alpha} \ge 2$, that is, $\frac{(\rho-1)(n-2)}{2} \le \alpha \le \frac{(\rho-1)n}{2}$, then for $\alpha = \alpha_1$ we have

$$\|f'(u)\phi\|_{X^{-\frac{\alpha_1}{2}}} \le c_3\|\phi\|_X\left(1 + \|u\|^{\rho-1}_{L^{\frac{(\rho-1)n}{\alpha_1}}(\Omega)}\right) \le c_5\|\phi\|_X\left(1 + \|u\|^{\rho-1}_{X^{\frac12}}\right) \le c_6, \tag{2.68}$$

since $u$ and $\phi$ remain in bounded subsets of $X^{\frac12}$ and $X$, respectively. Hence, from Young's inequality, and using (2.67) and (2.68), we get

$$
\begin{aligned}
\langle f'(u)\phi, \phi_t\rangle_{X^{-\frac{\alpha_1}{2}}} &\le \frac{1}{2\epsilon_0}\|f'(u)\phi\|^2_{X^{-\frac{\alpha_1}{2}}} + \frac{\epsilon_0}{2}\|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}} \\
&\le \frac{1}{2\epsilon_0}c_6^2 + \frac{\epsilon_0}{2}\|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}}
\end{aligned}
$$

for all $\epsilon_0 > 0$. With this in mind, it is possible to obtain an estimate for the other term that has the nonlinearity $f'$, that is,

$$\gamma_1\langle\phi, f'(u)\phi\rangle_{X^{-\frac{\alpha_1}{2}}} \leq \frac{\epsilon_1}{2}\|\phi\|^2_{X^{-\frac{\alpha_1}{2}}} + \frac{1}{2\epsilon_1}\gamma_1^2\|f'(u)\phi\|^2_{X^{-\frac{\alpha_1}{2}}} \leq \frac{\epsilon_1}{2}\|\phi\|^2_{X^{-\frac{\alpha_1}{2}}} + \frac{1}{2\epsilon_1}\gamma_1^2 c_6^2$$

for all $\epsilon_1 > 0$.

Next, from Cauchy-Schwartz and Young inequalities, we have

$$-\eta\gamma_1\langle\phi, A^{\frac{1}{2}}\phi_t\rangle_{X^{-\frac{\alpha_1}{2}}} \leq \eta\gamma_1\frac{\epsilon_2}{2}\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \eta\gamma_1\frac{1}{2\epsilon_2}\|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}},$$

$$-a_\epsilon(t)\gamma_1\langle\phi, A^{\frac{1}{2}}\varphi_t\rangle_{X^{-\frac{\alpha_1}{2}}} \leq a_1\gamma_1\frac{\epsilon_3}{2}\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}} + a_1\gamma_1\frac{1}{2\epsilon_3}\|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}},$$

$$-\eta\gamma_2\langle\varphi, A^{\frac{1}{2}}\varphi_t\rangle_{X^{-\frac{\alpha_1}{2}}} \leq \eta\gamma_2\frac{\epsilon_4}{2}\|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \eta\gamma_2\frac{1}{2\epsilon_4}\|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}}$$

and

$$a_\epsilon(t)\gamma_2\langle\varphi, A^{\frac{1}{2}}\phi_t\rangle_{X^{-\frac{\alpha_1}{2}}} \leq a_1\gamma_2\frac{\epsilon_5}{2}\|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}} + a_1\gamma_2\frac{1}{2\epsilon_5}\|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}},$$

for all $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon_4 > 0$ and $\epsilon_5 > 0$.

Since $\frac{1}{2} \leq \alpha_1 < 1$, we have the embedding $X \hookrightarrow X^{\frac{1-2\alpha_1}{4}}$, that is,

$$\|\cdot\|_{X^{\frac{1-2\alpha_1}{4}}} \leq \tilde{c}\|\cdot\|_X$$

for some constant $\tilde{c} > 0$. From this, and by condition (2.5), and also using the fact that $\varphi$ remains in a bounded subset of $X$, we get

$$-a'_\epsilon(t)\langle A^{\frac{1}{2}}\varphi, \phi_t\rangle_{X^{-\frac{\alpha_1}{2}}} \leq b_0\|\varphi\|_{X^{\frac{1-2\alpha_1}{4}}}\|\phi_t\|_{X^{\frac{1-2\alpha_1}{4}}} \leq b_0\frac{1}{2\epsilon_6}\|\varphi\|^2_{X^{\frac{1-2\alpha_1}{4}}} + b_0\frac{\epsilon_6}{2}\|\phi_t\|^2_{X^{\frac{1-2\alpha_1}{4}}}$$
$$\leq b_0\frac{1}{2\epsilon_6}\tilde{c}^2\|\varphi\|^2_X + b_0\frac{\epsilon_6}{2}\|\phi_t\|^2_{X^{\frac{1-2\alpha_1}{4}}} \leq \frac{1}{2\epsilon_6}b_0 c_7 + b_0\frac{\epsilon_6}{2}\|\phi_t\|^2_{X^{\frac{1-2\alpha_1}{4}}}$$

for all $\epsilon_6 > 0$,

$$a'_\epsilon(t)\langle A^{\frac{1}{2}}\phi, \varphi_t\rangle_{X^{-\frac{\alpha_1}{2}}} \leq b_0\frac{1}{2\epsilon_7}\|\phi\|^2_{X^{\frac{1-2\alpha_1}{4}}} + b_0\frac{\epsilon_7}{2}\|\varphi_t\|^2_{X^{\frac{1-2\alpha_1}{4}}}$$
$$\leq b_0\frac{1}{2\epsilon_7}\tilde{c}^2\|\phi\|^2_X + b_0\frac{\epsilon_7}{2}\|\varphi_t\|^2_{X^{\frac{1-2\alpha_1}{4}}} \leq \frac{1}{2\epsilon_7}b_0 c_8 + b_0\frac{\epsilon_7}{2}\|\varphi_t\|^2_{X^{\frac{1-2\alpha_1}{4}}}$$

for all $\epsilon_7 > 0$,

$$-a'_\epsilon(t)\gamma_1\langle\phi, A^{\frac{1}{2}}\varphi\rangle_{X^{-\frac{\alpha_1}{2}}} \leq b_0\gamma_1\frac{\epsilon_8}{2}\|\phi\|^2_{X^{\frac{1-2\alpha_1}{4}}} + b_0\gamma_1\frac{1}{2\epsilon_8}\|\varphi\|^2_{X^{\frac{1-2\alpha_1}{4}}}$$
$$\leq \frac{b_0\gamma_1\epsilon_8\tilde{c}^2}{2}\|\phi\|^2_X + \frac{b_0\gamma_1\tilde{c}^2}{2\epsilon_8}\|\varphi\|^2_X \leq c_9$$

and

$$a'_\epsilon(t)\gamma_2\langle\varphi, A^{\frac{1}{2}}\phi\rangle_{X^{-\frac{\alpha_1}{2}}} \leq b_0\gamma_2\frac{\epsilon_9}{2}\|\varphi\|^2_{X^{\frac{1-2\alpha_1}{4}}} + b_0\gamma_2\frac{1}{2\epsilon_9}\|\phi\|^2_{X^{\frac{1-2\alpha_1}{4}}}$$
$$\leq \frac{b_0\gamma_2\epsilon_9\tilde{c}^2}{2}\|\varphi\|^2_X + \frac{b_0\gamma_2\tilde{c}^2}{2\epsilon_9}\|\phi\|^2_X \leq c_{10}$$

for some constants $c_9 > 0$ and $c_{10} > 0$.

Finally, combining all the estimates obtained before, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_{\alpha_1}(t) \leq &-\left(\gamma_1 - \eta\gamma_1\frac{\epsilon_2}{2} - a_1\gamma_1\frac{\epsilon_3}{2}\right)\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}}\\ &-\left(-\gamma_1 - \eta\gamma_1\frac{1}{2\epsilon_2} - a_1\gamma_2\frac{1}{2\epsilon_5} - \frac{\epsilon_0}{2}\right)\|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}}\\ &-\left(\gamma_2 - \eta\gamma_2\frac{\epsilon_4}{2} - a_1\gamma_2\frac{\epsilon_5}{2}\right)\|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}}\\ &-\left(-\gamma_2 - a_1\gamma_1\frac{1}{2\epsilon_3} - \eta\gamma_2\frac{1}{2\epsilon_4}\right)\|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}}\\ &-\left(\gamma_1 - \frac{\epsilon_1}{2}\right)\|\phi\|^2_{X^{-\frac{\alpha_1}{2}}} + \left(b_0\frac{\epsilon_6}{2} - \eta\right)\|\phi_t\|^2_{X^{\frac{1-2\alpha_1}{4}}} + \left(b_0\frac{\epsilon_7}{2} - \eta\right)\|\varphi_t\|^2_{X^{\frac{1-2\alpha_1}{4}}}\\ &+\frac{1}{2\epsilon_0}c_6^2 + \frac{1}{2\epsilon_1}\gamma_1^2c_6^2 + \frac{1}{2\epsilon_6}b_0c_7 + \frac{1}{2\epsilon_7}b_0c_8 + c_9 + c_{10}.\end{aligned}$$

Now, by choosing $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon_4 > 0$, $\epsilon_5 > 0$, $\epsilon_6 > 0$ and $\epsilon_7 > 0$, respectively, such that

$$\epsilon_1 = 2\gamma_1,\ \epsilon_2 = \frac{1}{2\eta},\ \epsilon_3 = \frac{1}{2a_1},\ \epsilon_4 = \frac{1}{2\eta},\ \epsilon_5 = \frac{1}{2a_1},\ \epsilon_6 = \frac{\eta}{b_0} \text{ and } \epsilon_7 = \frac{3\eta}{2b_0},$$

we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_{\alpha_1}(t) \leq &-\frac{1}{2}\gamma_1\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}} - \left(-\gamma_1 - \eta^2\gamma_1 - a_1^2\gamma_2 - \frac{\epsilon_0}{2}\right)\|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}} - \frac{1}{2}\gamma_2\|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}}\\ &-\left(-\gamma_2 - a_1^2\gamma_1 - \eta^2\gamma_2\right)\|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}} - \frac{\eta}{2}\|\phi_t\|^2_{X^{\frac{1-2\alpha_1}{4}}} - \frac{\eta}{4}\|\varphi_t\|^2_{X^{\frac{1-2\alpha_1}{4}}}\\ &+\frac{1}{2\epsilon_0}c_6^2 + \frac{1}{4}\gamma_1c_6^2 + \frac{b_0^2c_7}{2\eta} + \frac{b_0^2c_8}{3\eta} + c_9 + c_{10}.\end{aligned} \tag{2.69}$$

As $\frac{1-2\alpha_1}{4} = -\frac{\alpha_1}{2} + \frac{1}{4} > -\frac{\alpha_1}{2}$, we have the embedding $X^{\frac{1-2\alpha_1}{4}} \hookrightarrow X^{-\frac{\alpha_1}{2}}$, and so

$$\|\cdot\|_{X^{-\frac{\alpha_1}{2}}} \leq \tilde{\tilde{c}}\|\cdot\|_{X^{\frac{1-2\alpha_1}{4}}}$$

for some constant $\tilde{\tilde{c}} > 0$, which implies

$$-\|\cdot\|^2_{X^{\frac{1-2\alpha_1}{4}}} \leq -\frac{1}{\tilde{\tilde{c}}^2}\|\cdot\|^2_{X^{-\frac{\alpha_1}{2}}}. \tag{2.70}$$

Hence, combining (2.69) and (2.70), we get

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_{\alpha_1}(t) \leq & -\frac{1}{2}\gamma_1\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}} - \left(\frac{\eta}{2\tilde{\tilde{c}}^2} - \gamma_1 - \eta^2\gamma_1 - a_1^2\gamma_2 - \frac{\epsilon_0}{2}\right)\|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}} \\ & - \frac{1}{2}\gamma_2\|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}} - \left(\frac{\eta}{4\tilde{\tilde{c}}^2} - \gamma_2 - a_1^2\gamma_1 - \eta^2\gamma_2\right)\|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}} \\ & + \frac{1}{2\epsilon_0}c_6^2 + \frac{1}{4}\gamma_1 c_6^2 + \frac{b_0^2 c_7}{2\eta} + \frac{b_0^2 c_8}{3\eta} + c_9 + c_{10}. \end{aligned} \tag{2.71}$$

At last, choosing $\epsilon_0 > 0$ such that $\epsilon_0 = \frac{\eta}{2\tilde{\tilde{c}}^2}$, expression (2.71) turns into

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_{\alpha_1}(t) \leq & -\frac{1}{2}\gamma_1\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}} - \left(\frac{\eta}{4\tilde{\tilde{c}}^2} - (1+\eta^2)\gamma_1 - a_1^2\gamma_2\right)\|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}} \\ & - \frac{1}{2}\gamma_2\|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}} - \left(\frac{\eta}{4\tilde{\tilde{c}}^2} - a_1^2\gamma_1 - (1+\eta^2)\gamma_2\right)\|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}} \\ & + \frac{\tilde{\tilde{c}}^2}{\eta}c_6^2 + \frac{1}{4}\gamma_1 c_6^2 + \frac{b_0^2 c_7}{2\eta} + \frac{b_0^2 c_8}{3\eta} + c_9 + c_{10}. \end{aligned}$$

Now, taking $\gamma_i > 0, i = 1,2$, sufficiently small such that

$$\gamma_i < \min\left\{\frac{1}{2\tilde{k}}, \frac{\eta}{16\tilde{\tilde{c}}^2}\frac{1}{1+\eta^2}, \frac{\eta}{16\tilde{\tilde{c}}^2}\frac{1}{a_1^2}\right\}, \; i = 1,2,$$

where $\tilde{k} > 0$ is the embedding constant of $X^{\frac{1-\alpha_1}{2}} \hookrightarrow X^{-\frac{\alpha_1}{2}}$ and taking

$$M_1 = \min\left\{\frac{1}{2}\gamma_1, \; \frac{\eta}{4\tilde{\tilde{c}}^2} - (1+\eta^2)\gamma_1 - a_1^2\gamma_2, \; \frac{1}{2}\gamma_2, \; \frac{\eta}{4\tilde{\tilde{c}}^2} - a_1^2\gamma_1 - (1+\eta^2)\gamma_2\right\} > 0$$

and $M_2 = \frac{\tilde{\tilde{c}}^2}{\eta}c_6^2 + \frac{1}{4}\gamma_1 c_6^2 + \frac{b_0^2 c_7}{2\eta} + \frac{b_0^2 c_8}{3\eta} + c_9 + c_{10} > 0$, it follows that

$$\frac{d}{dt}\mathcal{L}_{\alpha_1}(t) \leq -M_1\left(\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}} + \|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}}\right) + M_2. \tag{2.72}$$

Observe that

$$|\gamma_1\langle\phi,\phi_t\rangle_{X^{-\frac{\alpha_1}{2}}} + \gamma_2\langle\varphi,\varphi_t\rangle_{X^{-\frac{\alpha_1}{2}}}| \leq \frac{1}{4}\left(\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}} + \|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}}\right).$$

In this way, using a similar argument as in (2.46) and (2.47), we get

$$\begin{aligned} & \frac{1}{4}\left(\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}} + \|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}}\right) \\ & \leq \mathcal{L}_{\alpha_1}(t) \leq \frac{3(1+\tilde{k}^2)}{4}\left(\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}} + \|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}}\right). \end{aligned}$$

This estimate together with (2.72) implies that

$$\|\phi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \|\phi_t\|^2_{X^{-\frac{\alpha_1}{2}}} + \|\varphi\|^2_{X^{\frac{1-\alpha_1}{2}}} + \|\varphi_t\|^2_{X^{-\frac{\alpha_1}{2}}} \leq 4\mathcal{L}_{\alpha_1}(\tau)e^{-M_3(t-\tau)} + M_4,$$

with positive constants $M_3$ and $M_4$. This assures that $(\phi, \phi_t, \varphi, \varphi_t)$ is bounded in the space $X^{\frac{1-\alpha_1}{2}} \times X^{-\frac{\alpha_1}{2}} \times X^{\frac{1-\alpha_1}{2}} \times X^{-\frac{\alpha_1}{2}}$. But we want to conclude that $\bigcup_{t\in\mathbb{R}} \mathbb{A}(t)$ is bounded in $X^{\frac{2-\alpha_1}{2}} \times X^{\frac{1-\alpha_1}{2}} \times X^{\frac{2-\alpha_1}{2}} \times X^{\frac{1-\alpha_1}{2}}$. We already know that $u_t$ and $v_t$ are bounded in $X^{\frac{1-\alpha_1}{2}}$. Now, to show that $u \in X^{\frac{2-\alpha_1}{2}}$ and it is bounded, it is enough to show that $\|Au\|_{X^{-\frac{\alpha_1}{2}}} \leq C_1$ for some constant $C_1 > 0$, since

$$\|Au\|_{X^{-\frac{\alpha_1}{2}}} = \|A^{\frac{2-\alpha_1}{2}}u\|_X = \|u\|_{X^{\frac{2-\alpha_1}{2}}}.$$

Indeed, note that

$$\begin{aligned}
&\|-Au\|_{X^{-\frac{\alpha_1}{2}}} - \|u + \eta A^{\frac{1}{2}}u_t + a_\epsilon(t)A^{\frac{1}{2}}v_t - f(u)\|_{X^{-\frac{\alpha_1}{2}}} \\
&\leq \|-Au - u - \eta A^{\frac{1}{2}}u_t - a_\epsilon(t)A^{\frac{1}{2}}v_t + f(u)\|_{X^{-\frac{\alpha_1}{2}}} \\
&= \|u_{tt}\|_{X^{-\frac{\alpha_1}{2}}} = \|\phi_t\|_{X^{-\frac{\alpha_1}{2}}} \leq k_1,
\end{aligned}$$

which yields

$$\|Au\|_{X^{-\frac{\alpha_1}{2}}} \leq k_1 + \|u\|_{X^{-\frac{\alpha_1}{2}}} + \eta\|A^{\frac{1}{2}}u_t\|_{X^{-\frac{\alpha_1}{2}}} + a_1\|A^{\frac{1}{2}}v_t\|_{X^{-\frac{\alpha_1}{2}}} + \|f(u)\|_{X^{-\frac{\alpha_1}{2}}}.$$

Thus, we need to obtain estimates for the terms that are on the right hand side of the above inequality. Using the embedding $L^{\frac{2n}{n+2\alpha_1}}(\Omega) \hookrightarrow X^{-\frac{\alpha_1}{2}}$ and Lemma 2.3, condition $(i)$, we have

$$\begin{aligned}
\|f(u)\|_{X^{-\frac{\alpha_1}{2}}} &\leq c_1\|f(u)\|_{L^{\frac{2n}{n+2\alpha_1}}(\Omega)} \leq c_1\left(\int_\Omega [c(1+|u|^\rho)]^{\frac{2n}{n+2\alpha_1}}dx\right)^{\frac{n+2\alpha_1}{2n}} \\
&\leq c_2\left(|\Omega| + \int_\Omega |u|^{\frac{2n\rho}{n+2\alpha_1}}dx\right)^{\frac{n+2\alpha_1}{2n}} \leq c_3\left(1 + \|u\|^\rho_{L^{\frac{2n\rho}{(n-2)\rho+2}}(\Omega)}\right).
\end{aligned}$$

Since the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ holds, if and only if $p \leq \frac{2n}{n-2}$, and

$$(n-2)\rho + 2 > (n-2)\rho \implies \frac{2n\rho}{(n-2)\rho+2} < \frac{2n\rho}{(n-2)\rho} = \frac{2n}{n-2},$$

it follows that

$$H^1(\Omega) \hookrightarrow L^{\frac{2n\rho}{(n-2)\rho+2}}(\Omega)$$

and, therefore,

$$\|f(u)\|_{X^{-\frac{\alpha_1}{2}}} \leq c_3 \left(1 + \|u\|^{\rho}_{L^{\frac{2n\rho}{(n-2)\rho+2}}(\Omega)}\right) \leq c_5 \left(1 + \|u\|^{\rho}_{X^{\frac{1}{2}}}\right) \leq k_2.$$

For the remaining terms, note that

$$\|u\|_{X^{-\frac{\alpha_1}{2}}} \leq \tilde{c}\|u\|_{X^{\frac{1}{2}}} \leq k_3,$$

since $X^{\frac{1}{2}} \hookrightarrow X^{-\frac{\alpha_1}{2}}$, and, moreover,

$$\eta\|A^{\frac{1}{2}}u_t\|_{X^{-\frac{\alpha_1}{2}}} = \eta\|\phi\|_{X^{\frac{1-\alpha_1}{2}}} \leq k_4,$$

and

$$a_1\|A^{\frac{1}{2}}v_t\|_{X^{-\frac{\alpha_1}{2}}} = a_1\|\varphi\|_{X^{\frac{1-\alpha_1}{2}}} \leq k_5.$$

Therefore, we conclude that

$$\|Au\|_{X^{-\frac{\alpha_1}{2}}} \leq k_1 + k_2 + k_3 + k_4 + k_5 = C_1,$$

as desired.

Now, to show that $v \in X^{\frac{2-\alpha_1}{2}}$ and it is bounded, the idea is similar, because

$$\begin{aligned}
&\| - Av\|_{X^{-\frac{\alpha_1}{2}}} - \|\eta A^{\frac{1}{2}}v_t - a_\epsilon(t)A^{\frac{1}{2}}u_t\|_{X^{-\frac{\alpha_1}{2}}} \\
&\leq \| - Av - \eta A^{\frac{1}{2}}v_t + a_\epsilon(t)A^{\frac{1}{2}}u_t\|_{X^{-\frac{\alpha_1}{2}}} \\
&= \|v_{tt}\|_{X^{-\frac{\alpha_1}{2}}} = \|\varphi_t\|_{X^{-\frac{\alpha_1}{2}}} \leq k_6,
\end{aligned}$$

which implies

$$\begin{aligned}
\|v\|_{X^{\frac{2-\alpha_1}{2}}} &= \|Av\|_{X^{-\frac{\alpha_1}{2}}} \leq k_6 + \eta\|A^{\frac{1}{2}}v_t\|_{X^{-\frac{\alpha_1}{2}}} + a_1\|A^{\frac{1}{2}}u_t\|_{X^{-\frac{\alpha_1}{2}}} \\
&= k_6 + \eta\|\varphi\|_{X^{\frac{1-\alpha_1}{2}}} + a_1\|\phi\|_{X^{\frac{1-\alpha_1}{2}}} \leq C_2,
\end{aligned}$$

with $C_2 > 0$ being constant.

From the previous observations and from the fact that

$$\mathbb{A}(t) = \{\xi(t)\colon \xi(t) \text{ is a bounded global solution}\},$$

we conclude that

$$\bigcup_{t\in\mathbb{R}} \mathbb{A}(t) \text{ is bounded in } X^{\frac{2-\alpha_1}{2}} \times X^{\frac{1-\alpha_1}{2}} \times X^{\frac{2-\alpha_1}{2}} \times X^{\frac{1-\alpha_1}{2}}. \tag{2.73}$$

Now, we turn our attention once again to the term $\|f'(u)\phi\|_{X^{-\frac{\alpha}{2}}}$ that appears in (2.67). Note that the embedding $X^{\frac{1-\alpha_1}{2}} = H^{1-\alpha_1}(\Omega) \hookrightarrow L^p(\Omega)$ holds, if and only if $p \leq \frac{2n}{n-2(1-\alpha_1)}$. Hence, using (2.73), the Hölder's inequality and the growth condition (2.8), we have

$$\begin{aligned}
\|f'(u)\phi\|_{X^{-\frac{\alpha}{2}}} &\leq c_1\|f'(u)\phi\|_{L^{\frac{2n}{n+2\alpha}}(\Omega)} \leq c_1\|u_t\|_{L^{\frac{2n}{n-2(1-\alpha_1)}}(\Omega)}\|f'(u)\|_{L^{\frac{n}{1-\alpha_1+\alpha}}(\Omega)} \\
&\leq c_2\|u_t\|_{X^{\frac{1-\alpha_1}{2}}}\left(\int_\Omega [c(1+|u|^{\rho-1})]^{\frac{n}{1-\alpha_1+\alpha}}dx\right)^{\frac{1-\alpha_1+\alpha}{n}} \\
&\leq c_3\|u_t\|_{X^{\frac{1-\alpha_1}{2}}}\left(|\Omega| + \int_\Omega |u|^{\frac{(\rho-1)n}{1-\alpha_1+\alpha}}dx\right)^{\frac{1-\alpha_1+\alpha}{n}} \\
&\leq c_4\|u_t\|_{X^{\frac{1-\alpha_1}{2}}}\left(|\Omega|^{\frac{1-\alpha_1+\alpha}{n}} + \|u\|^{\rho-1}_{L^{\frac{(\rho-1)n}{1-\alpha_1+\alpha}}(\Omega)}\right) \\
&\leq c_5\|u_t\|_{X^{\frac{1-\alpha_1}{2}}}\left(1 + \|u\|^{\rho-1}_{L^{\frac{(\rho-1)n}{1-\alpha_1+\alpha}}(\Omega)}\right).
\end{aligned}$$

Now, note that the embedding $X^{\frac{2-\alpha_1}{2}} = H^{2-\alpha_1}(\Omega) \hookrightarrow L^{\frac{(\rho-1)n}{1-\alpha_1+\alpha}}(\Omega)$ holds, if and only if $(2-\alpha_1) - \frac{n}{2} \geq -\frac{1-\alpha_1+\alpha}{(\rho-1)}$ and $\frac{(\rho-1)n}{1-\alpha_1+\alpha} \geq 2$, that is

$$\frac{(\rho-1)(n-2)}{2} + \rho(\alpha_1 - 1) \leq \alpha \leq \frac{(\rho-1)n}{2} + \alpha_1 - 1.$$

If $\frac{(\rho-1)(n-2)}{2} + \rho(\alpha_1 - 1) = \alpha_1 + \rho(\alpha_1 - 1) \geq 0$, then using (2.73) and restarting the whole process from (2.67) with $\alpha_2 = \alpha_1 + \rho(\alpha_1 - 1)$, we will get

$$\bigcup_{t\in\mathbb{R}} \mathbb{A}(t) \text{ is bounded in } X^{\frac{2-\alpha_2}{2}} \times X^{\frac{1-\alpha_2}{2}} \times X^{\frac{2-\alpha_2}{2}} \times X^{\frac{1-\alpha_2}{2}}.$$

We continue with this iterative process getting $\alpha_{k+1} = \alpha_1 + \rho(\alpha_k - 1)$ for $k \geq 1$ while $\alpha_k \geq 0$.

There will be an integer $k_0 \geq 1$ such that $\alpha_{k_0} \geq 0$ and $\alpha_{k_0+1} < 0$. Thus, we obtain

$$\bigcup_{t\in\mathbb{R}} \mathbb{A}(t) \text{ is bounded in } X^{\frac{2-\alpha_{k_0}}{2}} \times X^{\frac{1-\alpha_{k_0}}{2}} \times X^{\frac{2-\alpha_{k_0}}{2}} \times X^{\frac{1-\alpha_{k_0}}{2}},$$

but we cannot assure the boundedness in $X^{\frac{2-\alpha_{k_0+1}}{2}} \times X^{\frac{1-\alpha_{k_0+1}}{2}} \times X^{\frac{2-\alpha_{k_0+1}}{2}} \times X^{\frac{1-\alpha_{k_0+1}}{2}}$ because of the embeddings. Here, we set $\alpha = 0$ and we restart the whole process from (2.67), with the obvious adaptations using the boundedness in $X^{\frac{2-\alpha_{k_0}}{2}} \times X^{\frac{1-\alpha_{k_0}}{2}} \times X^{\frac{2-\alpha_{k_0}}{2}} \times X^{\frac{1-\alpha_{k_0}}{2}}$, and we conclude that

$$\bigcup_{t\in\mathbb{R}} \mathbb{A}(t) \text{ is bounded in } X^1 \times X^{\frac{1}{2}} \times X^1 \times X^{\frac{1}{2}},$$

and the proof is complete. $\square$

## 2.5 Upper semicontinuity of pullback attractors

This last section is devoted to study the upper semicontinuity of pullback attractors with respect to the functional parameter $a_\epsilon$. To this end, we will use the regularity result obtained in the previous section. Let $\{a_\epsilon \colon \epsilon \in [0,1]\}$ be a family of real valued functions of one real variable satisfying (2.4). For each $\epsilon \in [0,1]$ denote by

$$\{S_{(\epsilon)}(t,\tau)\colon t \geq \tau \in \mathbb{R}\} \quad \text{and} \quad \{\mathbb{A}_{(\epsilon)}(t)\colon t \in \mathbb{R}\},$$

respectively, the evolution process and its pullback attractor associated with the problem (2.1)-(2.3). Moreover, we will assume that

$$\|a_\epsilon - a_0\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad \epsilon \to 0^+.$$

Our main goal is to prove that, for each $t \in \mathbb{R}$, $d_H(\mathbb{A}_{(\epsilon)}(t), \mathbb{A}_{(0)}(t)) \to 0$ as $\epsilon \to 0^+$.

**Theorem 2.9. [Upper Semicontinuity]** *For each $\eta > 0$ and $\epsilon \in [0,1]$, let $W^{(\epsilon)}(\cdot) = S_{(\epsilon)}(\cdot,\tau)W_0$ be the solution of* (2.1) *in $Y_0$. Assume that $\|a_\epsilon - a_0\|_{L^\infty(\mathbb{R})} \to 0$ as $\epsilon \to 0^+$. Then, for each $T > 0$, $W^{(\epsilon)}$ converges to $W^{(0)}$ in $C([0,T], Y_0)$ as $\epsilon \to 0^+$. Moreover, the family of pullback attractors $\{\mathbb{A}_{(\epsilon)}(t)\colon t \in \mathbb{R}\}$ is upper semicontinuous at $\epsilon = 0$.*

*Proof.* Let $W = W^{(\epsilon)} - W^{(0)}$, where

$$W^{(\epsilon)} = (u^{(\epsilon)}, u_t^{(\epsilon)}, v^{(\epsilon)}, v_t^{(\epsilon)}) \ \text{ and } \ W^{(0)} = (u^{(0)}, u_t^{(0)}, v^{(0)}, v_t^{(0)}),$$

with $u = u^{(\epsilon)} - u^{(0)}$ and $v = v^{(\epsilon)} - v^{(0)}$. From this, we have

$$\begin{cases} u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}}u_t + a_\epsilon(t)(-\Delta)^{\frac{1}{2}}v_t^{(\epsilon)} - a_0(t)(-\Delta)^{\frac{1}{2}}v_t^{(0)} = f(u^{(\epsilon)}) - f(u^{(0)}), \\ v_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}}v_t - a_\epsilon(t)(-\Delta)^{\frac{1}{2}}u_t^{(\epsilon)} + a_0(t)(-\Delta)^{\frac{1}{2}}u_t^{(0)} = 0, \end{cases}$$

for all $t > \tau$ and $x \in \Omega$. Taking the inner product of the first equation with $u_t$, and also the inner product of the second equation with $v_t$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \eta\|(-\Delta)^{\frac{1}{4}}u_t\|_X^2 \\ &+ a_\epsilon(t)\langle(-\Delta)^{\frac{1}{2}}v_t^{(\epsilon)}, u_t^{(\epsilon)}\rangle_X - a_\epsilon(t)\langle(-\Delta)^{\frac{1}{2}}v_t^{(\epsilon)}, u_t^{(0)}\rangle_X \\ &- a_0(t)\langle(-\Delta)^{\frac{1}{2}}v_t^{(0)}, u_t^{(\epsilon)}\rangle_X + a_0(t)\langle(-\Delta)^{\frac{1}{2}}v_t^{(0)}, u_t^{(0)}\rangle_X \\ &= \int_\Omega [f(u^{(\epsilon)}) - f(u^{(0)})]u_t dx, \end{aligned}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v_t|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2dx + \eta\|(-\Delta)^{\frac{1}{4}}v_t\|_X^2 - a_\epsilon(t)\langle(-\Delta)^{\frac{1}{2}}u_t^{(\epsilon)}, v_t^{(\epsilon)}\rangle_X$$
$$+ a_\epsilon(t)\langle(-\Delta)^{\frac{1}{2}}u_t^{(\epsilon)}, v_t^{(0)}\rangle_X + a_0(t)\langle(-\Delta)^{\frac{1}{2}}u_t^{(0)}, v_t^{(\epsilon)}\rangle_X - a_0(t)\langle(-\Delta)^{\frac{1}{2}}u_t^{(0)}, v_t^{(0)}\rangle_X = 0,$$

and combining these two last equations, it follows that

$$\frac{d}{dt}\frac{1}{2}\left(\int_\Omega |\nabla u|^2dx + \int_\Omega |u|^2dx + \int_\Omega |u_t|^2dx + \int_\Omega |\nabla v|^2dx + \int_\Omega |v_t|^2dx\right)$$
$$+ \eta\|(-\Delta)^{\frac{1}{4}}u_t\|_X^2 + \eta\|(-\Delta)^{\frac{1}{4}}v_t\|_X^2 + (a_0 - a_\epsilon)(t)\langle(-\Delta)^{\frac{1}{4}}v_t^{(\epsilon)}, (-\Delta)^{\frac{1}{4}}u_t^{(0)}\rangle_X$$
$$+ (a_\epsilon - a_0)(t)\langle(-\Delta)^{\frac{1}{4}}u_t^{(\epsilon)}, (-\Delta)^{\frac{1}{4}}v_t^{(0)}\rangle_X$$
$$= \int_\Omega [f(u^{(\epsilon)}) - f(u^{(0)})]u_t dx.$$

Now, using the Young's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|^2_{X^{\frac{1}{2}}} + \|u\|_X^2 + \|u_t\|_X^2 + \|v\|^2_{X^{\frac{1}{2}}} + \|v_t\|_X^2\right)\\
&= -2\eta\|A^{\frac{1}{4}}u_t\|_X^2 - 2\eta\|A^{\frac{1}{4}}v_t\|_X^2 + 2(a_\epsilon - a_0)(t)\langle A^{\frac{1}{4}}v_t^{(\epsilon)}, A^{\frac{1}{4}}u_t^{(0)}\rangle_X\\
&+ 2(a_0 - a_\epsilon)(t)\langle A^{\frac{1}{4}}u_t^{(\epsilon)}, A^{\frac{1}{4}}v_t^{(0)}\rangle_X + 2\int_\Omega [f(u^{(\epsilon)}) - f(u^{(0)})]u_t dx\\
&\leq 2|(a_\epsilon - a_0)(t)|\left(\frac{1}{2}\|v_t^{(\epsilon)}\|^2_{X^{\frac{1}{4}}} + \frac{1}{2}\|u_t^{(0)}\|^2_{X^{\frac{1}{4}}}\right)\\
&+ 2|(a_\epsilon - a_0)(t)|\left(\frac{1}{2}\|u_t^{(\epsilon)}\|^2_{X^{\frac{1}{4}}} + \frac{1}{2}\|v_t^{(0)}\|^2_{X^{\frac{1}{4}}}\right)\\
&+ 2\int_\Omega |[f(u^{(\epsilon)}) - f(u^{(0)})]u_t| dx\\
&\leq \|a_\epsilon - a_0\|_{L^\infty(\mathbb{R})}\left(\|u_t^{(\epsilon)}\|^2_{X^{\frac{1}{4}}} + \|u_t^{(0)}\|^2_{X^{\frac{1}{4}}} + \|v_t^{(\epsilon)}\|^2_{X^{\frac{1}{4}}} + \|v_t^{(0)}\|^2_{X^{\frac{1}{4}}}\right)\\
&+ 2\int_\Omega |[f(u^{(\epsilon)}) - f(u^{(0)})]u_t| dx.
\end{aligned} \tag{2.74}$$

On the other hand, from Theorem 2.8, we know that $W^{(\epsilon)}$ and $W^{(0)}$ are bounded in $X^1 \times X^{\frac{1}{2}} \times X^1 \times X^{\frac{1}{2}}$. In particular, there exists a constant $C > 0$, independent of $\epsilon$, such that

$$\|u_t^{(\epsilon)}\|_{X^{\frac{1}{2}}}, \|u_t^{(0)}\|_{X^{\frac{1}{2}}}, \|v_t^{(\epsilon)}\|_{X^{\frac{1}{2}}}, \|v_t^{(0)}\|_{X^{\frac{1}{2}}} \leq C. \tag{2.75}$$

Therefore, from (2.74), (2.75), and the embedding $X^{\frac{1}{2}} \hookrightarrow X^{\frac{1}{4}}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|^2_{X^{\frac{1}{2}}} + \|u\|_X^2 + \|u_t\|_X^2 + \|v\|^2_{X^{\frac{1}{2}}} + \|v_t\|_X^2\right)\\
&\leq C'\|a_\epsilon - a_0\|_{L^\infty(\mathbb{R})} + 2\int_\Omega |[f(u^{(\epsilon)}) - f(u^{(0)})]u_t| dx,
\end{aligned} \tag{2.76}$$

where $C' > 0$ is independent of $\epsilon$.

By the Mean Value Theorem, there exists $\sigma \in (0,1)$ such that

$$|f(u^{(\epsilon)}) - f(u^{(0)})| = |f'(\sigma u^{(\epsilon)} + (1-\sigma)u^{(0)})||u^{(\epsilon)} - u^{(0)}| = |f'(\sigma u^{(\epsilon)} + (1-\sigma)u^{(0)})||u|,$$

and so

$$\int_\Omega |[f(u^{(\epsilon)}) - f(u^{(0)})]u_t|dx = \int_\Omega |f'(\sigma u^{(\epsilon)} + (1-\sigma)u^{(0)})||u||u_t|dx.$$

As in the proof of Theorem 2.8, the condition $1 < \rho \leq \frac{n}{n-2}$ implies $X^{\frac{1}{2}} \hookrightarrow L^{2\rho}(\Omega)$. Since $\frac{(\rho-1)}{2\rho} + \frac{1}{2\rho} + \frac{1}{2} = 1$, then Hölder's inequality gives us

$$\int_\Omega |[f(u^{(\epsilon)}) - f(u^{(0)})]u_t|dx \leq \|f'(\sigma u^{(\epsilon)} + (1-\sigma)u^{(0)})\|_{L^{\frac{2\rho}{\rho-1}}(\Omega)} \|u\|_{L^{2\rho}(\Omega)} \|u_t\|_{L^2(\Omega)}; \tag{2.77}$$

but note that

$$\begin{aligned}
&\|f'(\sigma u^{(\epsilon)} + (1-\sigma)u^{(0)})\|_{L^{\frac{2\rho}{\rho-1}}(\Omega)} \leq \left(\int_\Omega [c(1 + |\sigma u^{(\epsilon)} + (1-\sigma)u^{(0)}|^{\rho-1})]^{\frac{2\rho}{\rho-1}} dx\right)^{\frac{\rho-1}{2\rho}} \\
&\leq \tilde{c}\left(|\Omega| + \int_\Omega |\sigma u^{(\epsilon)} + (1-\sigma)u^{(0)}|^{2\rho} dx\right)^{\frac{\rho-1}{2\rho}} \\
&\leq \tilde{\tilde{c}}\left(|\Omega|^{\frac{\rho-1}{2\rho}} + \|\sigma u^{(\epsilon)} + (1-\sigma)u^{(0)}\|^{\rho-1}_{L^{2\rho}(\Omega)}\right) \\
&\leq \tilde{\tilde{\tilde{c}}}\left[1 + \left(\|\sigma u^{(\epsilon)}\|_{L^{2\rho}(\Omega)} + \|(1-\sigma)u^{(0)}\|_{L^{2\rho}(\Omega)}\right)^{\rho-1}\right] \\
&\leq \tilde{\tilde{\tilde{\tilde{c}}}}\left(1 + \|u^{(\epsilon)}\|^{\rho-1}_{X^{\frac{1}{2}}} + \|u^{(0)}\|^{\rho-1}_{X^{\frac{1}{2}}}\right) \leq C_0,
\end{aligned} \tag{2.78}$$

where $C_0 > 0$ is independent of $\epsilon$. Thus, combining (2.77), (2.78) and the Young's inequality, we obtain

$$\begin{aligned}
&\int_\Omega |[f(u^{(\epsilon)}) - f(u^{(0)})]u_t|dx \leq C_0 \|u\|_{L^{2\rho}(\Omega)} \|u_t\|_{L^2(\Omega)} \leq \hat{c}\|u\|_{X^{\frac{1}{2}}} \|u_t\|_X \\
&\leq \frac{\hat{c}}{2}\left(\|u\|^2_{X^{\frac{1}{2}}} + \|u_t\|^2_X\right) \leq \frac{\hat{c}}{2}\left(\|u\|^2_{X^{\frac{1}{2}}} + \|u\|^2_X + \|u_t\|^2_X + \|v\|^2_{X^{\frac{1}{2}}} + \|v_t\|^2_X\right).
\end{aligned} \tag{2.79}$$

Now, denoting $G(t) = \|u(t)\|^2_{X^{\frac{1}{2}}} + \|u(t)\|^2_X + \|u_t(t)\|^2_X + \|v(t)\|^2_{X^{\frac{1}{2}}} + \|v_t(t)\|^2_X$, from (2.76) and (2.79), it follows that

$$\frac{d}{dt}G(t) \leq C'\|a_\epsilon - a_0\|_{L^\infty(\mathbb{R})} + \hat{c}G(t) \leq \overline{C}\|a_\epsilon - a_0\|_{L^\infty(\mathbb{R})} + \overline{C}G(t),$$

where $\overline{C} = \max\{C', \hat{c}\}$. Since this holds for all $t > \tau$, and noticing that $G(\tau) = 0$, we get

$$G(t)e^{-\overline{C}(t-\tau)} \leq -\|a_\epsilon - a_0\|_{L^\infty(\mathbb{R})} e^{-\overline{C}(t-\tau)} + \|a_\epsilon - a_0\|_{L^\infty(\mathbb{R})}, \ t > \tau,$$

that is,

$$\|u\|^2_{X^{\frac{1}{2}}} + \|u\|^2_X + \|u_t\|^2_X + \|v\|^2_{X^{\frac{1}{2}}} + \|v_t\|^2_X \leq e^{\overline{C}(t-\tau)}\|a_\epsilon - a_0\|_{L^\infty(\mathbb{R})} \to 0$$

as $\epsilon \to 0^+$ with $t, \tau$ in compact subsets of $\mathbb{R}$, and uniformly for $W_0$ in bounded subsets of $Y_0$. This proves the first part of the result.

In order to show the upper semicontinuity of the family of pullback attractors $\{\mathbb{A}_{(\epsilon)}(t) \colon t \in \mathbb{R}\}$ at $\epsilon = 0$, let $\delta > 0$ be given. Let $t \in \mathbb{R}$ be fixed but arbitrary and

$$B \supset \bigcup_{s \leq t} \mathbb{A}_{(\epsilon)}(s)$$

be a bounded set in $Y_0$, whose existence is guaranteed by Theorem 1.10. Now, let $\tau \in \mathbb{R}$, $\tau < t$, be such that

$$d_H(S_{(0)}(t,\tau)B, \mathbb{A}_{(0)}(t)) < \frac{\delta}{2}.$$

Using the convergence obtained in the first part of this proof, there exists $\epsilon_0 > 0$ such that

$$\sup_{u_\epsilon \in \mathbb{A}_{(\epsilon)}(\tau)} \|S_{(\epsilon)}(t,\tau)u_\epsilon - S_{(0)}(t,\tau)u_\epsilon\|_{Y_0} < \frac{\delta}{2}$$

for all $\epsilon < \epsilon_0$. Finally, for $\epsilon < \epsilon_0$, we have

$$\begin{aligned}
& d_H(\mathbb{A}_{(\epsilon)}(t), \mathbb{A}_{(0)}(t)) \\
& \leq d_H(S_{(\epsilon)}(t,\tau)\mathbb{A}_{(\epsilon)}(\tau), S_{(0)}(t,\tau)\mathbb{A}_{(\epsilon)}(\tau)) + d_H(S_{(0)}(t,\tau)\mathbb{A}_{(\epsilon)}(\tau), \mathbb{A}_{(0)}(t)) \\
& = \sup_{u_\epsilon \in \mathbb{A}_{(\epsilon)}(\tau)} \|S_{(\epsilon)}(t,\tau)u_\epsilon - S_{(0)}(t,\tau)u_\epsilon\|_{Y_0} + d_H(S_{(0)}(t,\tau)\mathbb{A}_{(\epsilon)}(\tau), \mathbb{A}_{(0)}(t)) \\
& < \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\end{aligned}$$

which proves the upper semicontinuity of the family of pullback attractors. □

Chapter

3

# Impulsive evolution processes

The theory of impulsive differential equations describes the evolution of processes whose continuous dynamics are interrupted by abrupt changes of state. While a differential equation describes the period of continuous variation of state, an external condition describes the discontinuities of the solutions (the impulses). These equations give us an effective tool to model real problems whose evolution are not continuous.

One of the reasons in the study of the theory of differential equations with impulses is because they are examples of dynamical systems of infinite dimension, presenting complex dynamics. The theory of impulsive dynamical systems started by Rozko in the papers [47, 48]. In the framework of autonomous systems, the theory of impulsive dynamical systems has been studied intensively in [9, 10, 11, 12, 29, 39]. On the other hand, in recent works, the non-autonomous case started to be explored in [13, 14, 16, 19].

In this chapter, we present the theory of evolution processes under conditions of impulses. Results on convergence are established in Section 3.3. Section 3.4 deals with the impulsive pullback $\omega$-limit set. In Section 3.5, we present a result on the existence of an impulsive pullback attractor. Lastly, in Section 3.6, we study the upper semicontinuity of the impulsive pullback attractor.

All the results of this chapter that have no references are presented in the article [18].

## 3.1 Fundamental properties

Let $Z$ be a metric space and $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ be an evolution process acting in $Z$.

**Definition 3.1.** Given $D \subseteq Z$ and an interval $J \subseteq [\tau, \infty)$, we define

$$F(D, J, \tau) = \{x \in Z\colon S(t,\tau)x \in D \text{ for some } t \in J\}.$$

A point $x \in Z$ is called an *initial point* at fiber $\tau$ if $F(x, t, \tau) = \emptyset$ for all $t > \tau$.

In the next definition, we exhibit the concept of an impulsive evolution process.

**Definition 3.2.** An *impulsive evolution process* $(Z, S, M, I)$ consists of an evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ in $Z$, a nonempty closed subset $M \subset Z$ satisfying the property: for each $x \in M$ and $\tau \in \mathbb{R}$, there exists $\epsilon = \epsilon(x,\tau) > 0$ such that

$$\bigcup_{t\in(0,\epsilon)} F(x,\tau,\tau-t) \cap M = \emptyset \quad \text{and} \quad \{S(t,\tau)x\colon 0 < t-\tau < \epsilon\} \cap M = \emptyset,$$

and a continuous function $I\colon M \to Z$, whose role will be specified in a forthcoming definition. The set $M$ is called an *impulsive set*, and the function $I$ is called an *impulse function*.

For $x \in Z$, we also define the auxiliary set

$$M^+(x,\tau) = \{S(t,\tau)x\colon t > \tau\} \cap M.$$

**Lemma 3.1.** *Let $(Z, S, M, I)$ be an impulsive evolution process, $\tau \in \mathbb{R}$ and $x \in Z$. If $M^+(x,\tau) \neq \emptyset$, then there exists $\bar{t} = \bar{t}(x,\tau) > \tau$ such that $S(\bar{t},\tau)x \in M$ and $S(r,\tau)x \notin M$ for $\tau < r < \bar{t}$.*

*Proof.* Since $M^+(x,\tau) \neq \emptyset$, there exists $t_1 > \tau$ such that $S(t_1,\tau)x \in M$. Now, suppose to the contradiction that there exists a sequence $\{s_n\}_{n\in\mathbb{N}} \subset (\tau,\infty)$ such that $s_n \xrightarrow{n\to\infty} \tau$ and

$$S(s_n,\tau)x \in M \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.1}$$

Using the continuity of $S$, we obtain $S(s_n,\tau)x \xrightarrow{n\to\infty} S(\tau,\tau)x = x$. Since $M$ is closed and $\{S(s_n,\tau)x\}_{n\in\mathbb{N}} \subset M$, it follows that $x \in M$. On the other hand, from Definition 3.2, there exists $\epsilon = \epsilon(x,\tau) > 0$ such that $S(s,\tau)x \notin M$ for all $s$ with $0 < s-\tau < \epsilon$. But this contradicts (3.1). Hence, the proof is completed. □

According to Lemma 3.1, we are able to define, for each fixed fiber $\tau \in \mathbb{R}$, a function $\phi(\cdot,\tau)\colon Z \to (0,\infty]$, called the *impact time map*, given by

$$\phi(x,\tau) = \begin{cases} s-\tau, & \text{if } S(s,\tau)x \in M \text{ and } S(r,\tau)x \notin M \text{ for } \tau < r < s, \\ \infty, & \text{if } S(t,\tau)x \notin M \text{ for all } t > \tau, \end{cases} \tag{3.2}$$

for each $x \in Z$. When $M^+(x,\tau) \neq \emptyset$, the value $s = \phi(x,\tau) + \tau$ represents the smallest time for which the trajectory of the point $x$ starting at time $\tau$ meets $M$. In this case, we say that the point $S(\phi(x,\tau)+\tau,\tau)x$ is the *impulsive point* of $x$. With this, we can now describe the trajectory of a point under the action of an impulsive evolution process.

**Definition 3.3** (Impulsive trajectory)**.** Given $\tau \in \mathbb{R}$, the *impulsive trajectory* of a point $x \in Z$, starting at time $\tau$, under the action of the impulsive evolution process $(Z, S, M, I)$, is a map

$\tilde{S}(\cdot,\tau)x$ defined on some interval $J_{(x,\tau)} \subseteq [\tau,\infty)$, which contains $\tau$, taking values in $Z$ given inductively by the following rule: if $M^+(x,\tau)=\emptyset$, then $\phi(x,\tau)=\infty$ and in this case we define

$$\tilde{S}(t,\tau)x = S(t,\tau)x \quad \text{for all} \quad t \geq \tau.$$

However, if $M^+(x,\tau) \neq \emptyset$, then $\phi(x_0,\tau) < \infty$. Denoting $x = x_0^+$, we define $\tilde{S}(\cdot,\tau)x$ in $[\tau, \phi(x_0^+,\tau)+\tau]$ by setting

$$\tilde{S}(t,\tau)x = \begin{cases} S(t,\tau)x_0^+, & \text{if } \tau \leq t < \phi(x_0^+,\tau)+\tau, \\ I(S(\phi(x_0^+,\tau)+\tau,\tau)x_0^+), & \text{if } t = \phi(x_0^+,\tau)+\tau. \end{cases}$$

Now, let $\tau_1 = \phi(x_0^+,\tau)+\tau$, $x_1 = S(\tau_1,\tau)x_0^+$ and $x_1^+ = I(x_1)$. In this case, since $\tau_1 < \infty$, this process can go on, but now starting at the point $x_1^+$ and at time $\tau_1$. Thus, if $M^+(x_1^+,\tau_1)=\emptyset$, then $\phi(x_1^+,\tau_1)=\infty$. Consequently, we define

$$\tilde{S}(t,\tau)x = S(t,\tau_1)x_1^+ \quad \text{for all} \quad t \geq \tau_1.$$

However, if $M^+(x_1^+,\tau_1) \neq \emptyset$, then we define $\tilde{S}(\cdot,\tau)x$ in $[\tau_1, \tau_1+\phi(x_1^+,\tau_1)]$ by setting

$$\tilde{S}(t,\tau)x = \begin{cases} S(t,\tau_1)x_1^+, & \text{if } \tau_1 \leq t < \tau_1+\phi(x_1^+,\tau_1), \\ I(S(\tau_1+\phi(x_1^+,\tau_1),\tau_1)x_1^+), & \text{if } t = \tau_1+\phi(x_1^+,\tau_1). \end{cases}$$

As before, let $\tau_2 = \tau_1+\phi(x_1^+,\tau_1)$, $x_2 = S(\tau_2,\tau_1)x_1^+$ and $x_2^+ = I(x_2)$. In this last case, since $\tau_2 < \infty$, this process can go on, but now starting at the point $x_2^+$ and at time $\tau_2$.

Assume now that $\tilde{S}(\cdot,\tau)x$ is defined on the interval $[\tau_{n-1},\tau_n]$, $n \geq 1$, where

$$\tau_0 = \tau, \quad \tau_n = \tau_{n-1}+\phi(x_{n-1}^+,\tau_{n-1}),\ n \geq 1, \quad \text{and} \quad x_0^+ = x.$$

In addition, assume that $\tilde{S}(\tau_n,\tau)x = x_n^+ = I(x_n)$. If $M^+(x_n^+,\tau_n)=\emptyset$, we define

$$\tilde{S}(t,\tau)x = S(t,\tau_n)x_n^+$$

for $\tau_n \leq t < \infty$ (note in this case that $\phi(x_n^+,\tau_n)=\infty$). However, if $M^+(x_n^+,\tau_n) \neq \emptyset$, then we set $\tau_{n+1} = \tau_n+\phi(x_n^+,\tau_n)$, $x_{n+1}^+ = I(x_{n+1}) = I(S(\tau_{n+1},\tau_n)x_n^+)$ and we define $\tilde{S}(\cdot,\tau)x$ in $[\tau_n,\tau_{n+1}]$ by

$$\tilde{S}(t,\tau)x = \begin{cases} S(t,\tau_n)x_n^+, & \text{if } \tau_n \leq t < \tau_{n+1}, \\ x_{n+1}^+, & \text{if } t = \tau_{n+1}. \end{cases}$$

The process described above can either ends after a finite number of steps (when

$M^+(x_n^+, \tau_n) = \emptyset$ for some $n \in \mathbb{N} \cup \{0\}$) or it may proceed indefinitely (when $M^+(x_n^+, \tau_n) \neq \emptyset$ for all $n \in \mathbb{N} \cup \{0\}$) and, in the second possibility, the impulsive trajectory $\tilde{S}(\cdot, \tau)x$ is defined on the interval $[\tau, T(x, \tau))$, where

$$T(x, \tau) = \tau + \sum_{i=0}^{\infty} \phi(x_i^+, \tau_i). \tag{3.3}$$

Note that $T(x, \tau)$ can be finite for some $x \in Z$ and $\tau \in \mathbb{R}$. However, since we shall investigate the asymptotic dynamics of an impulsive evolution process $(Z, S, M, I)$, we will assume throughout this work that

$$T(x, \tau) = \infty \quad \text{for all} \quad x \in Z \ \text{ and } \ \tau \in \mathbb{R}. \tag{H0}$$

Moreover, throughout this work, we shall denote the impulsive evolution process $(Z, S, M, I)$ simply by $\{\tilde{S}(t, \tau) \colon t \geq \tau \in \mathbb{R}\}$.

**Remark 3.1.** If there exists $\xi > 0$ such that $\phi(x, s) \geq \xi$ for all $x \in I(M)$ and $s \in \mathbb{R}$, then condition (H0) holds. It follows directly from the expression (3.3). The existence of $\xi > 0$ means that there is a minimum time at which the evolution process takes to reach the impulsive set $M$ when it leaves the set $I(M)$. This condition is satisfied, for instance, when $I(M)$ is compact and $I(M) \cap M = \emptyset$.

**Remark 3.2.** If $I(M) \cap M = \emptyset$, then no point $x \in M$ belongs to any impulsive trajectory, except if the trajectory starts at $x$. This fact is a consequence of the definition of impulsive trajectories.

Although an impulsive evolution process is not continuous, it satisfies the following properties which are also valid for continuous evolution processes. A proof of the next lemma can be found in [19] with the obvious adaptations.

**Lemma 3.2.** [19, Proposition 2.14] *Suppose condition* (H0) *is satisfied. Then the following properties hold:*

*(i)* $\tilde{S}(t, t)x = x$ *for all* $x \in Z$ *and all* $t \in \mathbb{R}$*;*

*(ii)* $\tilde{S}(t, \tau) = \tilde{S}(t, s)\tilde{S}(s, \tau)$ *for all* $t \geq s \geq \tau \in \mathbb{R}$*.*

In the sequel, we present the concepts of invariance, pullback attraction, pullback asymptotic compactness, and pullback strongly bounded dissipativeness in the context of impulsive evolution processes.

**Definition 3.4.** Let $\{\tilde{S}(t, \tau) \colon t \geq \tau \in \mathbb{R}\}$ be an impulsive evolution process on $Z$ and $\hat{B} = \{B(t) \colon t \in \mathbb{R}\}$ be a family of nonempty subsets of $Z$. We say that $\hat{B}$ is $\tilde{S}$*-invariant* if

$$\tilde{S}(t,\tau)B(\tau) = B(t) \quad \text{for all} \quad t \geq \tau \in \mathbb{R}.$$

Moreover, we say that $\hat{B}$ is *positively* (*negatively*, respectively) $\tilde{S}$-*invariant* if

$$\tilde{S}(t,\tau)B(\tau) \subseteq B(t) \quad \Big(\tilde{S}(t,\tau)B(\tau) \supseteq B(t), \text{ respectively}\Big) \quad \text{for all} \quad t \geq \tau \in \mathbb{R}.$$

**Definition 3.5.** Let $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ be an impulsive evolution process in $Z$. Given $t \in \mathbb{R}$ and $A, B$ nonempty subsets of $Z$, we say that $A$ *pullback* $\tilde{S}$-*attracts* $B$ at time $t$ if

$$\lim_{\tau\to-\infty} d_H(\tilde{S}(t,\tau)B, A) = 0. \tag{3.4}$$

The set $A$ pullback $\tilde{S}$-attracts bounded sets at time $t$, if (3.4) holds for every bounded subset $B$ of $Z$. In addition, a time-dependent family $\{A(t)\colon t \in \mathbb{R}\}$ of subsets of $Z$ pullback $\tilde{S}$-attracts bounded subsets of $Z$, if $A(t)$ pullback $\tilde{S}$-attracts bounded sets at time $t$, for each $t \in \mathbb{R}$.

**Definition 3.6.** An impulsive evolution process $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ in $Z$ is said to be *pullback* $\tilde{S}$-*asymptotically compact* if, for each $t \in \mathbb{R}$, each sequence $\{\tau_k\}_{k\in\mathbb{N}}$ with $\tau_k \leq t$ for all $k \in \mathbb{N}$ and $\tau_k \xrightarrow{k\to\infty} -\infty$, and each bounded sequence $\{x_k\}_{k\in\mathbb{N}} \subset Z$, then the sequence $\{\tilde{S}(t,\tau_k)x_k\}_{k\in\mathbb{N}}$ has a convergent subsequence.

The next definition is not a generalization of Definition 1.14. We present a new version of pullback strongly bounded dissipativeness for impulsive evolution processes.

**Definition 3.7.** An impulsive evolution process $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ in $Z$ is said to be *pullback* $\tilde{S}$-*strongly bounded dissipative* if, for each $t \in \mathbb{R}$, then there is a bounded subset $B(t)$ of $Z$ which pullback $\tilde{S}$-absorbs bounded subsets of $Z$ at time $t$, that is, there exists $\epsilon_0 > 0$ such that, for each bounded subset $D$ of $Z$, one can find a time $T = T(t,D) \leq t$ such that

$$\tilde{S}(t+\epsilon,\tau)D \subset B(t) \quad \text{for all} \quad \tau \leq T \quad \text{and} \quad \epsilon \in [0,\epsilon_0].$$

In this case, the family $\hat{B} = \{B(t)\colon t \in \mathbb{R}\}$ is called an *absorbing set* for the impulsive evolution process $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$. If the absorbing set $\hat{B} = \{B(t)\colon t \in \mathbb{R}\}$ is compact and there exists $t_0 \in \mathbb{R}$ such that $\bigcup_{t\leq t_0} B(t)$ is bounded in $Z$, then we say that the impulsive evolution process $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ is *pullback* $\tilde{S}$-*strongly compact dissipative*.

## 3.2 Continuity of the impact time map $\phi$

As presented in [13, 14, 19], the function $\phi$ defined in (3.2) is not continuous in general. The continuity of $\phi$ in $Z\backslash M$ depends on extra conditions on the impulsive set $M$. In [13, 14], the authors studied the continuity of $\phi$ using tube conditions. In [19], the authors were inspired by the work [15] to consider a special condition to study the continuity of $\phi$. This condition, namely condition (T), was first considered for impulsive autonomous multivalued dynamical systems, since it is easier to verify in applications than the tube conditions. Here, we also consider condition (T) in order to obtain qualitative properties of impulsive evolution processes. In other words, we say that an impulsive evolution process $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ satisfies condition (T) if

$$\begin{cases} \text{Given } \tau \in \mathbb{R},\ t > \tau,\ x \in M, \text{ and a convergent sequence } \{x_n\}_{n\in\mathbb{N}} \text{ in } Z \text{ such} \\ \text{that } S(t,\tau)x_n \xrightarrow{n\to\infty} x, \text{ there exist a subsequence } \{x_{n_k}\}_{k\in\mathbb{N}} \text{ of } \{x_n\}_{n\in\mathbb{N}} \text{ and} \\ \text{a sequence } \{\alpha_k\}_{k\in\mathbb{N}} \subset \mathbb{R}, \text{ with } t+\alpha_k \geq \tau \text{ for all } k \in \mathbb{N} \text{ and } \alpha_k \xrightarrow{k\to\infty} 0, \text{ such} \\ \text{that } S(t+\alpha_k,\tau)x_{n_k} \in M \text{ for all } k \in \mathbb{N}. \end{cases} \tag{T}$$

Now, using condition (T), we present sufficient conditions for the impact time map $\phi$ to be continuous on $Z \setminus M$. This result is not new and the reader can find a more general version in [19, Proposition 2.33].

**Proposition 3.1** (Continuity of the impact time map)**.** *Assume that condition* (T) *holds. Then, for each* $\tau \in \mathbb{R}$ *fixed, the impact time map* $\phi(\cdot,\tau)\colon Z \to (0,\infty]$*, given by* (3.2)*, is continuous on* $Z \setminus M$*.*

*Proof.* Let $\tau \in \mathbb{R}$ be fixed and arbitrary. Now, let $x \in Z\backslash M$ and $\{x_n\}_{n\in\mathbb{N}} \subset Z$ be a sequence such that $x_n \xrightarrow{n\to\infty} x$. We may assume without loss of generality that $x_n \notin M$ for all $n \in \mathbb{N}$, because $M$ is a closed subset of $Z$. We will split the proof into two steps: the lower semicontinuity and the upper semicontinuity.

At first, let us prove the lower semicontinuity of $\phi(\cdot,\tau)$ at $x$. We may assume that $\liminf_{n\to\infty} \phi(x_n,\tau) = c \in [0,\infty)$, because if $\liminf_{n\to\infty} \phi(x_n,\tau) = \infty$, then there is nothing to prove. Thus, one can obtain a subsequence of $\{x_n\}_{n\in\mathbb{N}}$, which we still denote by the same, such that $\phi(x_n,\tau) \xrightarrow{n\to\infty} c$. Using the continuity of $S$, we get

$$S(\phi(x_n,\tau)+\tau,\tau)x_n \xrightarrow{n\to\infty} S(c+\tau,\tau)x.$$

Since $S(\phi(x_n,\tau)+\tau,\tau)x_n \in M$, for all $n \in \mathbb{N}$, and $M$ is closed in $Z$, it follows that $S(c+\tau,\tau)x \in M$. By the definition of the impact time map, we obtain

$$\tau < \phi(x,\tau) + \tau \leq c + \tau,$$

that is, $0 < \phi(x,\tau) \leq c$, which proves our first claim.

Now, let us prove the upper semicontinuity of $\phi(\cdot,\tau)$ at $x$. Indeed, if $\phi(x,\tau) = \infty$, then the result is obvious. Thus, we may assume that $\phi(x,\tau) < \infty$. Moreover, let

$c_0 = \limsup_{n\to\infty} \phi(x_n,\tau)$ and consider a subsequence of $\{x_n\}_{n\in\mathbb{N}}$, which we still denote by the same, such that $\phi(x_n,\tau) \xrightarrow{n\to\infty} c_0$. Note that $\phi(x,\tau)+\tau > \tau$, $S(\phi(x,\tau)+\tau,\tau)x \in M$, and $S(\phi(x,\tau)+\tau,\tau)x_n \xrightarrow{n\to\infty} S(\phi(x,\tau)+\tau,\tau)x$. Using condition (T), there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ and a sequence $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$, with $\alpha_k \xrightarrow{k\to\infty} 0$, such that

$$\phi(x,\tau)+\tau+\alpha_k \geq \tau \quad \text{and} \quad S(\phi(x,\tau)+\tau+\alpha_k,\tau)x_{n_k} \in M \quad \text{for all} \quad k \in \mathbb{N}.$$

By the definition of the impact time map, we obtain

$$\phi(x_{n_k},\tau) \leq \phi(x,\tau)+\alpha_k, \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence, as $k\to\infty$, it follows that $c_0 \leq \phi(x,\tau)$, which proves our second claim.

In conclusion, $\phi(\cdot,\tau)$ is continuous at $x \notin M$. □

## 3.3 Convergence properties

In this section, we present some convergence properties for an impulsive evolution process $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$. Theorem 3.1, Theorem 3.2, Lemma 3.3, and Corollary 3.1 are presented in [19], however, the authors in [19] used the tube conditions to prove them. Here, we use condition (T), which is more general, to get these results.

**Theorem 3.1.** *Assume that condition* (T) *holds. Let $\tau \in \mathbb{R}$, $x \in Z$, $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{x_n\}_{n\in\mathbb{N}} \subset Z$ be sequences such that $x_n \xrightarrow{n\to\infty} x$ and $\tau_n \xrightarrow{n\to\infty} \tau$. If $x \notin M$, then $\phi(x_n,\tau_n) \xrightarrow{n\to\infty} \phi(x,\tau)$.*

*Proof.* This result is a particular case of Theorem 3.4. □

**Theorem 3.2.** *Assume that condition* (T) *holds. Let $t,\tau \in \mathbb{R}$, $x \in Z\setminus M$, $\{x_n\}_{n\in\mathbb{N}} \subset Z$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset (-\infty,t]$ be sequences such that $x_n \xrightarrow{n\to\infty} x$ and $\tau_n \xrightarrow{n\to\infty} \tau$. Then, there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, with $t+\eta_n \geq \tau_n$ and $\eta_n \xrightarrow{n\to\infty} 0$, such that $\tilde{S}(t+\eta_n,\tau_n)x_n \xrightarrow{n\to\infty} \tilde{S}(t,\tau)x$.*

*Proof.* See Theorem 3.5. □

**Lemma 3.3.** *Assume that condition* (T) *holds. Let $t \in \mathbb{R}$ and $x \in Z\setminus M$ be given, and let $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}, \{\beta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}, \{x_n\}_{n\in\mathbb{N}} \subset Z$ be sequences such that $\alpha_n \xrightarrow{n\to\infty} 0, \beta_n \xrightarrow{n\to\infty} 0$, and $x_n \xrightarrow{n\to\infty} x$, with $\beta_n \leq \alpha_n$ and $x_n \notin M$ for all $n \in \mathbb{N}$. Then, $\tilde{S}(t+\alpha_n,t+\beta_n)x_n \xrightarrow{n\to\infty} x$.*

*Proof.* See Lemma 3.10. □

**Corollary 3.1.** *Assume that $I(M)\cap M = \emptyset$. Under the assumptions of Theorem 3.2, we can assume that $\eta_n \geq 0$ for all $n \in \mathbb{N}$.*

*Proof.* See Corollary 3.2. □

We end this section with a convergence result which leads with points that belong to the impulsive set $M$.

**Lemma 3.4.** *Assume that condition* (T) *holds and* $I(M) \cap M = \emptyset$. *Let* $t \in \mathbb{R}$, $x \in M$, $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $\{\beta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, *and* $\{x_n\}_{n\in\mathbb{N}} \subset Z$ *be sequences such that* $\alpha_n \geq \beta_n$ *for all* $n \in \mathbb{N}$, $\alpha_n \xrightarrow{n\to\infty} 0$, $\beta_n \xrightarrow{n\to\infty} 0$, *and* $x_n \xrightarrow{n\to\infty} x$. *Then, there exists a subsequence* $\{\phi(x_{n_k}, t+\beta_{n_k})\}_{k\in\mathbb{N}}$ *of* $\{\phi(x_n, t+\beta_n)\}_{n\in\mathbb{N}}$ *such that* $\phi(x_{n_k}, t+\beta_{n_k}) \xrightarrow{k\to\infty} 0$. *Moreover,*

*(i) if* $\alpha_{n_k} - \beta_{n_k} < \phi(x_{n_k}, t+\beta_{n_k})$ *for all* $k \in \mathbb{N}$, *then* $\tilde{S}(t+\alpha_{n_k}, t+\beta_{n_k})x_{n_k} \xrightarrow{k\to\infty} x$;

*(ii) if* $\alpha_{n_k} - \beta_{n_k} \geq \phi(x_{n_k}, t+\beta_{n_k})$ *for all* $k \in \mathbb{N}$, *then* $\tilde{S}(t+\alpha_{n_k}, t+\beta_{n_k})x_{n_k} \xrightarrow{k\to\infty} I(x)$.

*Proof.* See Lemma 3.11. □

## 3.4 The impulsive pullback $\omega$-limit set

In this section, we present the generalization of the $\omega$-limit set in the framework of impulsive evolution processes.

**Definition 3.8.** Let $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ be an impulsive evolution process in $Z$. The *impulsive pullback $\omega$-limit set* of a subset $B$ of $Z$ at time $t \in \mathbb{R}$ is defined as the set

$$\tilde{\omega}(B,t) = \bigcap_{\sigma \leq t} \overline{\bigcup_{\tau \leq \sigma} \bigcup_{\epsilon \geq 0} \tilde{S}(t+\epsilon, \tau)B}.$$

The impulsive pullback $\omega$-limit set can be characterized in the following way.

**Lemma 3.5.** *For each* $t \in \mathbb{R}$ *and* $B \subset Z$, *we have*

$$\tilde{\omega}(B,t) = \{x \in Z\colon \text{ there are sequences } \{\tau_n\}_{n\in\mathbb{N}} \subset (-\infty, t], \{\epsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+ \text{ and } \{x_n\}_{n\in\mathbb{N}} \subseteq B \\ \text{such that } \tau_n \xrightarrow{n\to\infty} -\infty,\ \epsilon_n \xrightarrow{n\to\infty} 0 \text{ and } \tilde{S}(t+\epsilon_n, \tau_n)x_n \xrightarrow{n\to\infty} x\}$$

*and* $\tilde{\omega}(B,t)$ *is closed in* $Z$.

**Lemma 3.6.** *Let* $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ *be a pullback* $\tilde{S}$*-asymptotically compact impulsive evolution process in* $Z$. *Assume that condition* (T) *holds,* $I(M) \cap M = \emptyset$ *and* $B$ *is a nonempty bounded subset of* $Z$. *Then, for each* $t \in \mathbb{R}$, *the impulsive pullback* $\omega$*-limit set* $\tilde{\omega}(B,t)$ *is nonempty, compact and pullback* $\tilde{S}$*-attracts* $B$ *at time* $t$.

*Proof.* Let $t \in \mathbb{R}$ be fixed and arbitrary. Now, let $\{\tau_k\}_{k\in\mathbb{N}} \subset (-\infty, t]$ be a sequence such that $\tau_k \xrightarrow{k\to\infty} -\infty$, and $\{x_k\}_{k\in\mathbb{N}} \subset B$. Since $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ is pullback $\tilde{S}$-asymptotically compact, the sequence $\{\tilde{S}(t,\tau_k)x_k\}_{k\in\mathbb{N}}$ admits a convergent subsequence, which converges to some point $x \in Z$. Hence, $x \in \tilde{\omega}(B,t)$ and $\tilde{\omega}(B,t)$ is nonempty.

Now, let us show that $\tilde{\omega}(B,t)$ is compact. Let $\{z_n\}_{n\in\mathbb{N}} \subset \tilde{\omega}(B,t)$ be a sequence. For each $n \in \mathbb{N}$, there exist sequences $\{x_k^n\}_{k\in\mathbb{N}} \subset B$, $\{\tau_k^n\}_{k\in\mathbb{N}} \subset (-\infty,t]$ and $\{\epsilon_k^n\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\tau_k^n \xrightarrow{k\to\infty} -\infty$, $\epsilon_k^n \xrightarrow{k\to\infty} 0$, and

$$\tilde{S}(t+\epsilon_k^n,\tau_k^n)x_k^n \xrightarrow{k\to\infty} z_n.$$

For each $n \in \mathbb{N}$, there exists a natural number $k_n \geq n$ such that

$$d(\tilde{S}(t+\epsilon_{k_n}^n,\tau_{k_n}^n)x_{k_n}^n, z_n) < \frac{1}{n}, \quad n \in \mathbb{N}. \tag{3.5}$$

Since $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ is pullback $\tilde{S}$-asymptotically compact, we may assume without loss of generality that there exists $y \in Z$ such that

$$\tilde{S}(t,\tau_{k_n}^n)x_{k_n}^n \xrightarrow{n\to\infty} y.$$

By Lemma 3.4, up to a subsequence, there exists $z \in Z$ (either $z = y$ or $z = I(y)$) such that

$$\tilde{S}(t+\epsilon_{k_n}^n,\tau_{k_n}^n)x_{k_n}^n = \tilde{S}(t+\epsilon_{k_n}^n,t)\tilde{S}(t,\tau_{k_n}^n)x_{k_n}^n \xrightarrow{n\to\infty} z. \tag{3.6}$$

Note that $z \in \tilde{\omega}(B,t)$. Using (3.5) and (3.6), as $n \to \infty$, we obtain $z_n \xrightarrow{n\to\infty} z$. Since $\tilde{\omega}(B,t)$ is a closed subset of $Z$, we conclude that $\tilde{\omega}(B,t)$ is compact.

At last, let us prove that $\tilde{\omega}(B,t)$ pullback $\tilde{S}$-attracts $B$ at time $t$. Indeed, suppose to the contrary that there are $\epsilon_0 > 0$ and sequences $\tau_n \xrightarrow{n\to\infty} -\infty$ and $\{x_n\}_{n\in\mathbb{N}} \subset B$ such that

$$d_H(\tilde{S}(t,\tau_n)x_n, \tilde{\omega}(B,t)) \geq \epsilon_0, \quad n \in \mathbb{N}. \tag{3.7}$$

Since $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ is pullback $\tilde{S}$-asymptotically compact, we may assume without loss of generality that there exists $z \in Z$ such that

$$\tilde{S}(t,\tau_n)x_n \xrightarrow{n\to\infty} z.$$

But this shows that $z \in \tilde{\omega}(B,t)$, which contradicts (3.7) as $n \to \infty$. □

Next, we show that the family $\{\tilde{\omega}(B,t) \setminus M\colon t \in \mathbb{R}\}$ is positively $\tilde{S}$-invariant.

**Lemma 3.7.** *Assume that condition* (T) *holds and* $I(M) \cap M = \emptyset$. *Let* $B$ *be a nonempty subset of* $Z$. *Then,* $\{\tilde{\omega}(B,t) \setminus M\colon t \in \mathbb{R}\}$ *is positively* $\tilde{S}$*-invariant.*

*Proof.* Let $t,\tau \in \mathbb{R}$ with $\tau \leq t$. We are going to show that $\tilde{S}(t,\tau)[\tilde{\omega}(B,\tau) \setminus M] \subset \tilde{\omega}(B,t) \setminus M$. If $t = \tau$, then the result is done. Assume that $\tau < t$. For a given $x \in \tilde{\omega}(B,\tau) \setminus M$, there are sequences $\{x_n\}_{n\in\mathbb{N}} \subset B$, $\{\epsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset (-\infty, \tau]$ such that $\epsilon_n \xrightarrow{n\to\infty} 0$, $\tau_n \xrightarrow{n\to\infty} -\infty$ and

$$w_n = \tilde{S}(\tau + \epsilon_n, \tau_n)x_n \xrightarrow{n\to\infty} x.$$

Since $x \notin M$, it follows by Corollary 3.1 that there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, with $t + \eta_n \geq \tau + \epsilon_n$ and $\eta_n \xrightarrow{n\to\infty} 0$, such that $\tilde{S}(t + \eta_n, \tau + \epsilon_n)w_n \xrightarrow{n\to\infty} \tilde{S}(t, \tau)x$, that is,

$$\tilde{S}(t + \eta_n, \tau_n)x_n \xrightarrow{n\to\infty} \tilde{S}(t, \tau)x.$$

Thus, $\tilde{S}(t, \tau)x \in \tilde{\omega}(B, t)$. Since $\tau < t$ and $I(M) \cap M = \emptyset$, we have $\tilde{S}(t, \tau)x \in \tilde{\omega}(B, t) \setminus M$. Hence, the result is complete. □

In Lemma 3.8, we establish the negative $\tilde{S}$-invariance of the family $\{\tilde{\omega}(B, t) \setminus M : t \in \mathbb{R}\}$.

**Lemma 3.8.** *Assume that condition* (T) *holds and $I(M) \cap M = \emptyset$. Let $B$ be a nonempty subset of $Z$. Moreover, assume that the family $\{\tilde{\omega}(B, t) : t \in \mathbb{R}\}$ is compact and pullback $\tilde{S}$-attracts $B$. The following properties hold:*

*(i) if $y \in \tilde{\omega}(B, t) \cap M$, then $I(y) \in \tilde{\omega}(B, t) \setminus M$;*

*(ii) $\{\tilde{\omega}(B, t) \setminus M : t \in \mathbb{R}\}$ is negatively $\tilde{S}$-invariant.*

*Proof.* (*i*) Given $y \in \tilde{\omega}(B, t) \cap M$, there are sequences $\{x_n\}_{n\in\mathbb{N}} \subset B$, $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset (-\infty, t]$ such that $\gamma_n \xrightarrow{n\to\infty} 0$, $\tau_n \xrightarrow{n\to\infty} -\infty$ and

$$y_n = \tilde{S}(t + \gamma_n, \tau_n)x_n \xrightarrow{n\to\infty} y.$$

Since $y \in M$, it follows by Lemma 3.4, along some subsequence, that $\sigma_k = \phi(y_{n_k}, t + \gamma_{n_k}) \xrightarrow{k\to\infty} 0$ and

$$\tilde{S}(t + \gamma_{n_k} + \sigma_k, t + \gamma_{n_k})y_{n_k} \xrightarrow{k\to\infty} I(y).$$

Hence,

$$\tilde{S}(t + \gamma_{n_k} + \sigma_k, \tau_{n_k})x_{n_k} \xrightarrow{k\to\infty} I(y),$$

which assures that $I(y) \in \tilde{\omega}(B, t)$. As $I(M) \cap M = \emptyset$, we conclude that $I(y) \in \tilde{\omega}(B, t) \setminus M$.

(*ii*) Let $t, \tau \in \mathbb{R}$ with $\tau \leq t$. We are going to show that $\tilde{\omega}(B, t) \setminus M \subset \tilde{S}(t, \tau)[\tilde{\omega}(B, \tau) \setminus M]$. Indeed, given $x \in \tilde{\omega}(B, t) \setminus M$, there exist sequences $\{x_n\}_{n\in\mathbb{N}} \subset B$, $\{\epsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset (-\infty, t]$ such that $\epsilon_n \xrightarrow{n\to\infty} 0$, $\tau_n \xrightarrow{n\to\infty} -\infty$ and

$$\tilde{S}(t + \epsilon_n, \tau_n)x_n \xrightarrow{n\to\infty} x. \tag{3.8}$$

Since $\tilde{\omega}(B,\tau)$ pullback $\tilde{S}$-attracts $B$, we get

$$d_H(\tilde{S}(\tau,\tau_n)x_n,\tilde{\omega}(B,\tau)) \xrightarrow{n\to\infty} 0.$$

Moreover, by the compactness of $\tilde{\omega}(B,\tau)$, we may assume, up to a subsequence, that there exists $y\in\tilde{\omega}(B,\tau)$ such that

$$\tilde{S}(\tau,\tau_n)x_n \xrightarrow{n\to\infty} y.$$

<u>Case 1:</u> $y\notin M$. In this case, using Corollary 3.1, we can obtain a sequence $\{\eta_n\}_{n\in\mathbb{N}}\subseteq\mathbb{R}_+$ such that $\eta_n \xrightarrow{n\to\infty} 0$ and

$$\tilde{S}(t+\eta_n,\tau_n)x_n=\tilde{S}(t+\eta_n,\tau)\tilde{S}(\tau,\tau_n)x_n \xrightarrow{n\to\infty} \tilde{S}(t,\tau)y\notin M. \tag{3.9}$$

Now, since $x\notin M$, applying Lemma 3.3 in (3.8) and in (3.9), respectively, we obtain

$$\tilde{S}(t+\epsilon_n+\eta_n,\tau_n)x_n=\tilde{S}(t+\epsilon_n+\eta_n,t+\epsilon_n)\tilde{S}(t+\epsilon_n,\tau_n)x_n \xrightarrow{n\to\infty} x$$

and

$$\tilde{S}(t+\eta_n+\epsilon_n,\tau_n)x_n=\tilde{S}(t+\eta_n+\epsilon_n,t+\eta_n)\tilde{S}(t+\eta_n,\tau_n)x_n \xrightarrow{n\to\infty} \tilde{S}(t,\tau)y,$$

which yields $x=\tilde{S}(t,\tau)y\in\tilde{S}(t,\tau)[\tilde{\omega}(B,\tau)\setminus M]$.

<u>Case 2:</u> $y\in M$. Let $z_n=\tilde{S}(\tau,\tau_n)x_n$, $n\in\mathbb{N}$. In this case, by Lemma 3.4, there exists a subsequence $\{\phi(z_{n_k},\tau)\}_{k\in\mathbb{N}}$ such that $\phi(z_{n_k},\tau) \xrightarrow{k\to\infty} 0$ and

$$\tilde{S}(\tau+\phi(z_{n_k},\tau),\tau)z_{n_k} \xrightarrow{k\to\infty} I(y).$$

Using Corollary 3.1, we can obtain a sequence $\{\gamma_k\}_{k\in\mathbb{N}}\subseteq\mathbb{R}_+$ such that $\gamma_k \xrightarrow{k\to\infty} 0$ and

$$\tilde{S}(t+\gamma_k,\tau+\phi(z_{n_k},\tau))\tilde{S}(\tau+\phi(z_{n_k},\tau),\tau)z_{n_k} \xrightarrow{k\to\infty} \tilde{S}(t,\tau)I(y),$$

that is,

$$\tilde{S}(t+\gamma_k,\tau_{n_k})x_{n_k} \xrightarrow{k\to\infty} \tilde{S}(t,\tau)I(y). \tag{3.10}$$

Now, since $x\notin M$, applying Lemma 3.3 in (3.8) and in (3.10), respectively, we get

$$\tilde{S}(t+\epsilon_{n_k}+\gamma_k,\tau_{n_k})x_{n_k}=\tilde{S}(t+\epsilon_{n_k}+\gamma_k,t+\epsilon_{n_k})\tilde{S}(t+\epsilon_{n_k},\tau_{n_k})x_{n_k} \xrightarrow{k\to\infty} x$$

and

$$\tilde{S}(t+\gamma_k+\epsilon_{n_k},\tau_{n_k})x_{n_k}=\tilde{S}(t+\gamma_k+\epsilon_{n_k},t+\gamma_k)\tilde{S}(t+\gamma_k,\tau_{n_k})x_{n_k} \xrightarrow{k\to\infty} \tilde{S}(t,\tau)I(y),$$

which implies $x = \tilde{S}(t,\tau)I(y)$. By item $(i)$, $I(y) \in \tilde{\omega}(B,\tau) \setminus M$. Thus, $x \in \tilde{S}(t,\tau)[\tilde{\omega}(B,\tau) \setminus M]$.

Therefore, $\tilde{\omega}(B,t) \setminus M \subset \tilde{S}(t,\tau)[\tilde{\omega}(B,\tau) \setminus M]$ for every $t \geq \tau$. □

## 3.5 Pullback attractors for impulsive evolution processes

This section concerns sufficient conditions for the existence of pullback attractors in the context of evolution processes with impulses.

**Definition 3.9.** A family $\{\tilde{\mathbb{A}}(t) \colon t \in \mathbb{R}\}$ of subsets of $Z$ is called an *impulsive pullback attractor* for the impulsive evolution process $\{\tilde{S}(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$ if:

$(i)$ $\{\tilde{\mathbb{A}}(t) \colon t \in \mathbb{R}\}$ is compact;

$(ii)$ $\{\tilde{\mathbb{A}}(t) \setminus M \colon t \in \mathbb{R}\}$ is $\tilde{S}$-invariant;

$(iii)$ $\{\tilde{\mathbb{A}}(t) \colon t \in \mathbb{R}\}$ pullback $\tilde{S}$-attracts bounded subsets of $Z$;

$(iv)$ $\{\tilde{\mathbb{A}}(t) \colon t \in \mathbb{R}\}$ is the minimal family of closed sets satisfying property $(iii)$.

Condition $(iv)$ of Definition 3.9 says that, if $\{\tilde{\mathbb{A}}_1(t) \colon t \in \mathbb{R}\}$ and $\{\tilde{\mathbb{A}}_2(t) \colon t \in \mathbb{R}\}$ are two impulsive pullback attractors for the impulsive evolution process $\{\tilde{S}(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$, then

$$\tilde{\mathbb{A}}_1(t) = \tilde{\mathbb{A}}_2(t) \quad \text{for every} \quad t \in \mathbb{R}.$$

**Theorem 3.3.** *Let $\{\tilde{S}(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$ be an impulsive evolution process in $Z$. Assume that condition* (T) *holds, $I(M) \cap M = \emptyset$ and $\{\tilde{S}(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$ is pullback $\tilde{S}$-strongly compact dissipative. Then $\{\tilde{S}(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$ has an impulsive pullback attractor $\{\tilde{\mathbb{A}}(t) \colon t \in \mathbb{R}\}$ such that there exists $t_0 \in \mathbb{R}$ satisfying $\bigcup_{t \leq t_0} \tilde{\mathbb{A}}(t)$ is bounded in $Z$.*

*Proof.* Since $\{\tilde{S}(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$ is pullback $\tilde{S}$-strongly compact dissipative, it admits a compact absorbing set $\hat{K} = \{K(t) \colon t \in \mathbb{R}\}$ such that $\bigcup_{t \leq t_0} K(t)$ is bounded in $Z$ for some $t_0 \in \mathbb{R}$. Thus, for each $t \in \mathbb{R}$, there exists $\epsilon_0 > 0$ such that, for every bounded set $D \subset Z$, one can find a time $T = T(t,D) \leq t$ such that

$$\tilde{S}(t+\epsilon,\tau)D \subset K(t) \quad \text{for all} \quad \tau \leq T \quad \text{and} \quad \epsilon \in [0,\epsilon_0]. \tag{3.11}$$

Note, in particular, that $\{\tilde{S}(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$ is pullback $\tilde{S}$-asymptotically compact.

Now, for each $t \in \mathbb{R}$, define

$$\tilde{\mathbb{A}}(t) = \bigcup \{\tilde{\omega}(B,t) \colon B \subset Z,\ B \text{ bounded}\}.$$

By Lemmas 3.6, 3.7 and 3.8, the family $\{\tilde{\mathbb{A}}(t) \colon t \in \mathbb{R}\}$ is nonempty, $I$-invariant and pullback $\tilde{S}$-attracts bounded subsets of $Z$, and the family $\{\tilde{\mathbb{A}}(t) \setminus M \colon t \in \mathbb{R}\}$ is $\tilde{S}$-invariant.

Let us show that $\tilde{\mathbb{A}}(t)$ is compact, for each $t \in \mathbb{R}$. At first, we claim that $\tilde{\omega}(B,t) \subset K(t)$ for all bounded set $B \subset Z$ and $t \in \mathbb{R}$. Indeed, let $B \subset Z$ be a bounded set and $t \in \mathbb{R}$. Given $x \in \tilde{\omega}(B,t)$, there are sequences $\{x_n\}_{n\in\mathbb{N}} \subset B$, $\{\epsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset (-\infty, t]$ such that $\epsilon_n \xrightarrow{n\to\infty} 0$, $\tau_n \xrightarrow{n\to\infty} -\infty$ and

$$\tilde{S}(t+\epsilon_n, \tau_n)x_n \xrightarrow{n\to\infty} x.$$

According to (3.11), we have $\tilde{S}(t+\epsilon_n,\tau_n)x_n \in K(t)$ for $n$ sufficiently large and, consequently, $x \in K(t)$. Hence, the claim is proved, and it yields $\tilde{\mathbb{A}}(t) \subset K(t)$ for all $t \in \mathbb{R}$. Since $\hat{K}$ is compact, it follows that the family $\{\tilde{\mathbb{A}}(t) \colon t \in \mathbb{R}\}$ is compact.

Since $\bigcup_{t\leq t_0} \tilde{\mathbb{A}}(t) \subset \bigcup_{t\leq t_0} K(t)$, we have $\bigcup_{t\leq t_0} \tilde{\mathbb{A}}(t)$ is bounded in $Z$.

At last, let $\hat{C} = \{C(t) \colon t \in \mathbb{R}\}$ be a family of closed sets that pullback $\tilde{S}$-attracts bounded subsets of $Z$. We claim that $\tilde{\omega}(B,t) \subset C(t)$ for all bounded set $B \subset Z$ and $t \in \mathbb{R}$. Indeed, since $\bigcup_{t\leq t_0} \tilde{\mathbb{A}}(t)$ is bounded in $Z$, we obtain

$$d_H(\tilde{\omega}(B,t) \setminus M, C(t)) = \lim_{\tau\to-\infty} d_H(\tilde{S}(t,\tau)[\tilde{\omega}(B,\tau) \setminus M], C(t)) = 0.$$

Thus, $\tilde{\omega}(B,t) \setminus M \subset C(t)$.

Now, let $x \in \tilde{\omega}(B,t) \cap M$. Then there are sequences $\{x_n\}_{n\in\mathbb{N}} \subset B$, $\{\epsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset (-\infty, t]$ such that $\epsilon_n \xrightarrow{n\to\infty} 0$, $\tau_n \xrightarrow{n\to\infty} -\infty$ and

$$\tilde{S}(t+\epsilon_n, \tau_n)x_n \xrightarrow{n\to\infty} x.$$

Since $\{\tilde{S}(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$ is pullback $\tilde{S}$-asymptotically compact, we may assume, up to a subsequence, that

$$\tilde{S}(t,\tau_n)x_n \xrightarrow{n\to\infty} b \in \tilde{\omega}(B,t).$$

Note that $b \in C(t)$, since $C(t)$ pullback $\tilde{S}$-attracts bounded subsets of $Z$ at time $t$. Now, according to Lemma 3.4, either

$$\tilde{S}(t+\epsilon_n, t)\tilde{S}(t,\tau_n)x_n \xrightarrow{n\to\infty} b \quad \text{or} \quad \tilde{S}(t+\epsilon_n, t)\tilde{S}(t,\tau_n)x_n \xrightarrow{n\to\infty} I(b).$$

In the first case, we obtain $x = b \in C(t)$. In the second case, using Lemma 3.8, we get $x = I(b) \in \tilde{\omega}(B,t) \setminus M \subset C(t)$. Consequently, $\tilde{\omega}(B,t) \cap M \subset C(t)$.

Thus, we conclude that $\tilde{\omega}(B,t) \subset C(t)$ for all $B \subset Z$ bounded and $t \in \mathbb{R}$, which implies $\tilde{\mathbb{A}}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Therefore, it follows that the family $\{\tilde{\mathbb{A}}(t) \colon t \in \mathbb{R}\}$ is an impulsive pullback attractor for the impulsive evolution process $\{\tilde{S}(t,\tau) \colon t \geq \tau \in \mathbb{R}\}$, and the proof of the result is complete. □

## 3.6 Upper semicontinuity of impulsive pullback attractors

In this section, we deal with the upper semicontinuity at zero for a family of impulsive pullback attractors $\{\tilde{\mathbb{A}}_{(\epsilon)}(t)\colon t \in \mathbb{R}\}$ associated with a family of impulsive evolution processes $\{\tilde{S}_{(\epsilon)}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$, $\epsilon \in [0,1]$, that is, we will prove that for each $t \in \mathbb{R}$ we have

$$\lim_{\epsilon\to 0^+} \mathrm{d_H}(\tilde{\mathbb{A}}_{(\epsilon)}(t), \tilde{\mathbb{A}}_{(0)}(t)) = 0. \tag{3.12}$$

In order to achieve our goal, we will make use of the following assumptions:

($A1$) for each $\epsilon \in [0,1]$, the impulsive evolution process $\{\tilde{S}_{(\epsilon)}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ admits an impulsive pullback attractor $\{\tilde{\mathbb{A}}_{(\epsilon)}(t)\colon t \in \mathbb{R}\}$;

($A2$) $S_{(\epsilon)}(t,\tau)x \xrightarrow{\epsilon\to 0^+} S_{(0)}(t,\tau)x$ uniformly in bounded subsets of $\{(t,\tau) \in \mathbb{R}^2\colon t \geq \tau\} \times Z$;

($A3$) $I_{(0)}(M_{(0)})$ is bounded and $I_{(0)}(M_{(0)}) \cap M_{(0)} = \emptyset$;

($A4$) if $\epsilon_n \xrightarrow{n\to\infty} 0$, then $\mathrm{d_H}(M_{(\epsilon_n)}, M_{(0)}) + \mathrm{d_H}(M_{(0)}, M_{(\epsilon_n)}) \xrightarrow{n\to\infty} 0$;

($A5$) if $\epsilon_n \xrightarrow{n\to\infty} 0$, then $\mathrm{d_H}(I_{(\epsilon_n)}(M_{(\epsilon_n)}), I_{(0)}(M_{(0)})) \xrightarrow{n\to\infty} 0$;

($A6$) there exists $\xi > 0$ such that $\phi_{(0)}(I_{(0)}(x), t) \geq 2\xi$ for all $(x,t) \in M_{(0)} \times \mathbb{R}$;

($A7$) the impulse function $I_{(0)}\colon M_{(0)} \to Z$ is an injective map;

($A8$) given $x \in I_{(0)}(M_{(0)})$ and $t \in \mathbb{R}$, there exist $y \in I_{(0)}(M_{(0)})$ and $\tau \geq 2\xi > 0$ such that

$$S_{(0)}(t, t-\tau)y \in M_{(0)} \quad \text{and} \quad \tilde{S}_{(0)}(t, t-\tau)y = x.$$

($A9$) given $\tau \in \mathbb{R}$ and $t > \tau$, the evolution process $S_{(0)}(t,\tau)\colon Z \to Z$ is a compact map.

The family $\{\tilde{S}_{(\epsilon)}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$, $\epsilon \in [0,1]$, satisfies the *collective condition* (T) if given $\tau \in \mathbb{R}$, $t > \tau$, $x \in M_{(0)}$, $\{\epsilon_n\}_{n\in\mathbb{N}} \subset [0,1]$ with $\epsilon_n \xrightarrow{n\to\infty} 0$, and a convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in $Z$ such that $S_{(\epsilon_n)}(t,\tau)x_n \xrightarrow{n\to\infty} x$, then there exist a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ and a sequence $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$, with $t + \alpha_k \geq \tau$ for all $k \in \mathbb{N}$ and $\alpha_k \xrightarrow{k\to\infty} 0$, such that $S_{(\epsilon_{n_k})}(t+\alpha_k, \tau)x_{n_k} \in M_{(\epsilon_{n_k})}$ for all $k \in \mathbb{N}$.

### 3.6.1 Some convergence results

This subsection concerns some convergence results for a family of impulsive evolution processes $\{\tilde{S}_{(\epsilon)}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$, $\epsilon \in [0,1]$.

**Lemma 3.9.** [12, Lemma 3.1] *Assume condition* ($A4$) *holds. If* $\{\epsilon_n\}_{n\in\mathbb{N}} \subset [0,1]$ *and* $\{x_n\}_{n\in\mathbb{N}} \subset Z$ *are sequences such that* $\epsilon_n \xrightarrow{n\to\infty} 0$, $x_n \in M_{(\epsilon_n)}$ *for* $n \in \mathbb{N}$ *and* $x_n \xrightarrow{n\to\infty} x$, *then* $x \in M_{(0)}$.

**Theorem 3.4.** *Assume that conditions* ($A2$), ($A4$) *and the collective condition* (T) *hold. Let* $\tau \in \mathbb{R}$, $x \in Z$, $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *and* $\{x_n\}_{n\in\mathbb{N}} \subset Z$ *be sequences such that* $x_n \xrightarrow{n\to\infty} x$ *and* $\tau_n \xrightarrow{n\to\infty} \tau$. *If* $x \notin M_{(0)}$, *then* $\phi_{(\epsilon_n)}(x_n,\tau_n) \xrightarrow{n\to\infty} \phi_{(0)}(x,\tau)$.

*Proof.* Suppose, at first, that $\phi_{(0)}(x,\tau) = \infty$. If there exists a subsequence $\{\phi_{(\epsilon_{n_k})}(x_{n_k},\tau_{n_k})\}_{k\in\mathbb{N}}$ of $\{\phi_{(\epsilon_n)}(x_n,\tau_n)\}_{n\in\mathbb{N}}$ such that

$$\phi_{(\epsilon_{n_k})}(x_{n_k},\tau_{n_k}) \xrightarrow{k\to\infty} \lambda \in (0,\infty),$$

then, by condition ($A2$) and Lemma 3.9,

$$\lim_{k\to\infty} S_{(\epsilon_{n_k})}(\tau_{n_k} + \phi_{(\epsilon_{n_k})}(x_{n_k},\tau_{n_k}),\tau_{n_k})x_{n_k} = S_{(0)}(\tau+\lambda,\tau)x \in M_{(0)},$$

which yields the contradiction $\phi_{(0)}(x,\tau) \leq \lambda$. Hence, $\lim\limits_{n\to\infty} \phi_{(\epsilon_n)}(x_n,\tau_n) = \infty = \phi_{(0)}(x,\tau)$.

Now, suppose $\phi_{(0)}(x,\tau) < \infty$. We claim that $\{\phi_{(\epsilon_n)}(x_n,\tau_n)\}_{n\in\mathbb{N}}$ is bounded. Indeed, suppose there exists a subsequence $\{\phi_{(\epsilon_{n_k})}(x_{n_k},\tau_{n_k})\}_{k\in\mathbb{N}}$ of $\{\phi_{(\epsilon_n)}(x_n,\tau_n)\}_{n\in\mathbb{N}}$ such that $\phi_{(\epsilon_{n_k})}(x_{n_k},\tau_{n_k}) \xrightarrow{k\to\infty} \infty$. Since

$$S_{(\epsilon_{n_k})}(\tau_{n_k} + \phi_{(0)}(x,\tau),\tau_{n_k})x_{n_k} \xrightarrow{k\to\infty} S_{(0)}(\tau+\phi_{(0)}(x,\tau),\tau)x \in M_{(0)},$$

it follows by the collective condition (T), up to a subsequence, that there exists a sequence $\alpha_{n_k} \xrightarrow{k\to\infty} 0$, such that $\tau_{n_k} + \phi_{(0)}(x,\tau) + \alpha_{n_k} \geq \tau_{n_k}$ and $S_{(\epsilon_{n_k})}(\tau_{n_k} + \phi_{(0)}(x,\tau) + \alpha_{n_k},\tau_{n_k})x_{n_k} \in M_{(\epsilon_{n_k})}$ for all $k \in \mathbb{N}$. Consequently,

$$\phi_{(\epsilon_{n_k})}(x_{n_k},\tau_{n_k}) \leq \phi_{(0)}(x,\tau) + \alpha_{n_k}, \quad k \in \mathbb{N},$$

which is a contradiction as $\phi_{(\epsilon_{n_k})}(x_{n_k},\tau_{n_k}) \xrightarrow{k\to\infty} \infty$. Hence, $\{\phi_{(\epsilon_n)}(x_n,\tau_n)\}_{n\in\mathbb{N}}$ is bounded.

Now, let $\{\phi_{(\epsilon_{n_m})}(x_{n_m},\tau_{n_m})\}_{m\in\mathbb{N}}$ be an arbitrary subsequence of $\{\phi_{(\epsilon_n)}(x_n,\tau_n)\}_{n\in\mathbb{N}}$. This subsequence admits another subsequence, which we denote by the same, which converges to some $\lambda \in (0,\infty)$. We claim that $\lambda = \phi_{(0)}(x,\tau)$. In fact, suppose at first that $\lambda < \phi_{(0)}(x,\tau)$. Since $\{S_{(\epsilon_{n_m})}(\tau_{n_m} + \phi_{(\epsilon_{n_m})}(x_{n_m},\tau_{n_m}),\tau_{n_m})x_{n_m}\}_{m\in\mathbb{N}} \subset M_{(\epsilon_{n_m})}$, condition ($A4$) holds and

$$S_{(\epsilon_{n_m})}(\tau_{n_m} + \phi_{(\epsilon_{n_m})}(x_{n_m},\tau_{n_m}),\tau_{n_m})x_{n_m} \xrightarrow{m\to\infty} S_{(0)}(\tau+\lambda,\tau)x,$$

we have $S_{(0)}(\tau+\lambda,\tau)x \in M_{(0)}$, which yields the contradiction $\phi_{(0)}(x,\tau) \leq \lambda < \phi_{(0)}(x,\tau)$. Hence, $\phi_{(0)}(x,\tau) \leq \lambda$.

On the other hand, since

$$S_{(\epsilon_{n_m})}(\tau_{n_m} + \phi_{(0)}(x,\tau), \tau_{n_m})x_{n_m} \xrightarrow{m\to\infty} S_{(0)}(\tau + \phi_{(0)}(x,\tau), \tau)x \in M_{(0)},$$

we can use the collective condition (T) again, up to a subsequence, to obtain a sequence $\beta_{n_m} \xrightarrow{m\to\infty} 0$, such that $\tau_{n_m} + \phi_{(0)}(x,\tau) + \beta_{n_m} \geq \tau_{n_m}$ and $S_{(\epsilon_{n_m})}(\tau_{n_m} + \phi_{(0)}(x,\tau) + \beta_{n_m}, \tau_{n_m})x_{n_m} \in M_{(\epsilon_{n_m})}$ for all $m \in \mathbb{N}$. It implies

$$\phi_{(\epsilon_{n_m})}(x_{n_m}, \tau_{n_m}) \leq \phi_{(0)}(x,\tau) + \beta_{n_m}, \quad m \in \mathbb{N},$$

and, as $m \to \infty$, we obtain

$$\lambda \leq \phi_{(0)}(x,\tau).$$

Therefore, $\lambda = \phi_{(0)}(x,\tau)$. This shows that $\phi_{(\epsilon_n)}(x_n,\tau_n) \xrightarrow{n\to\infty} \phi_{(0)}(x,\tau)$. □

**Theorem 3.5.** *Assume that conditions (A2), (A4), (A5), and the collective condition* (T) *hold. Let $t, \tau \in \mathbb{R}$, $x \in Z \setminus M_{(0)}$, $\{x_n\}_{n\in\mathbb{N}} \subset Z$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset (-\infty, t]$ be sequences such that $x_n \xrightarrow{n\to\infty} x$ and $\tau_n \xrightarrow{n\to\infty} \tau$. Then, there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, with $t + \eta_n \geq \tau_n$ and $\eta_n \xrightarrow{n\to\infty} 0$, such that $\tilde{S}_{(\epsilon_n)}(t+\eta_n, \tau_n)x_n \xrightarrow{n\to\infty} \tilde{S}_{(0)}(t,\tau)x$.*

*Proof.* Since $x \notin M_{(0)}$ and $M_{(0)}$ is a closed subset of $Z$, we may assume that $x_n \notin M_{(\epsilon_n)}$ for all $n \in \mathbb{N}$. By Theorem 3.4, we have $\phi_{(\epsilon_n)}(x_n,\tau_n) \xrightarrow{n\to\infty} \phi_{(0)}(x,\tau)$. If $\phi_{(0)}(x,\tau) = \infty$, then $\phi_{(\epsilon_n)}(x_n,\tau_n) > t - \tau_n$ for $n$ sufficiently large and, therefore,

$$\tilde{S}_{(\epsilon_n)}(t,\tau_n)x_n = S_{(\epsilon_n)}(t,\tau_n)x_n \xrightarrow{n\to\infty} S_{(0)}(t,\tau)x = \tilde{S}_{(0)}(t,\tau)x,$$

and the assertion follows by taking $\eta_n = 0$ for all $n \in \mathbb{N}$.

Now, suppose $\phi_{(0)}(x,\tau) < \infty$. In this way, we may assume that $\phi_{(\epsilon_n)}(x_n,\tau_n) < \infty$ for all $n \in \mathbb{N}$. Set $\tau^1 = \phi_{(0)}(x,\tau) + \tau$. We will consider the following cases:

<u>Case 1:</u> $\tau \leq t < \tau^1$.

Given $\epsilon \in (0, \tau^1 - t)$, there exists $n_0 \in \mathbb{N}$ such that, if $n \geq n_0$, then

$$|\phi_{(\epsilon_n)}(x_n,\tau_n) - \phi_{(0)}(x,\tau)| < \frac{\epsilon}{2} \quad \text{and} \quad |\tau_n - \tau| < \frac{\epsilon}{2}$$

and, therefore,

$$t < \tau^1 - \epsilon = \phi_{(0)}(x,\tau) + \tau - \epsilon < \frac{\epsilon}{2} + \phi_{(\epsilon_n)}(x_n,\tau_n) + \frac{\epsilon}{2} + \tau_n - \epsilon = \phi_{(\epsilon_n)}(x_n,\tau_n) + \tau_n,$$

which yields

$$\tilde{S}_{(\epsilon_n)}(t,\tau_n)x_n = S_{(\epsilon_n)}(t,\tau_n)x_n \xrightarrow{n\to\infty} S_{(0)}(t,\tau)x = \tilde{S}_{(0)}(t,\tau)x,$$

and the assertion follows by taking once again $\eta_n = 0$ for all $n \in \mathbb{N}$.

Case 2: $t = \tau^1$.

Note that

$$\tilde{S}_{(0)}(t,\tau)x = \tilde{S}_{(0)}(\tau^1,\tau)x = I_{(0)}(S_{(0)}(\tau^1,\tau)x)$$

and, using conditions $(A2)$ and $(A5)$,

$$\begin{aligned}&\tilde{S}_{(\epsilon_n)}(\phi_{(\epsilon_n)}(x_n,\tau_n)+\tau_n,\tau_n)x_n\\&= I_{(\epsilon_n)}(S_{(\epsilon_n)}(\phi_{(\epsilon_n)}(x_n,\tau_n)+\tau_n,\tau_n)x_n) \xrightarrow{n\to\infty} I_{(0)}(S_{(0)}(\tau^1,\tau)x) = \tilde{S}_{(0)}(t,\tau)x.\end{aligned}$$

Now, taking $\eta_n = \phi_{(\epsilon_n)}(x_n,\tau_n) - \phi_{(0)}(x,\tau) + \tau_n - \tau$ for all $n \in \mathbb{N}$, we have $\eta_n \xrightarrow{n\to\infty} 0$ and

$$\tilde{S}_{(\epsilon_n)}(t+\eta_n,\tau_n)x_n = \tilde{S}_{(\epsilon_n)}(\phi_{(\epsilon_n)}(x_n,\tau_n)+\tau_n,\tau_n)x_n \xrightarrow{n\to\infty} \tilde{S}_{(0)}(t,\tau)x,$$

which proves the assertion.

Case 3: $t > \tau^1$.

Let $\tau^0 = \tau$ and $(\tau_n)_0 = \tau_n$ for $n \in \mathbb{N}$. Using the notation of the Definition 3.3, we have

$$(\tau_n)_1 = \phi_{(\epsilon_n)}(x_n,(\tau_n)_0) + (\tau_n)_0 \xrightarrow{n\to\infty} \phi_{(0)}(x,\tau^0) + \tau^0 = \tau^1$$

and, consequently,

$$(x_n)_1 = S_{(\epsilon_n)}((\tau_n)_1,(\tau_n)_0)x_n \xrightarrow{n\to\infty} S_{(0)}(\tau^1,\tau^0)x = x_1.$$

Hence, $(x_n)_1^+ = I_{(\epsilon_n)}((x_n)_1) \xrightarrow{n\to\infty} I_{(0)}(x_1) = x_1^+$.

Proceeding with this argument, we conclude, for every $j \geq 1$, that

$$(\tau_n)_{j+1} = (\tau_n)_j + \phi_{(\epsilon_n)}((x_n)_j^+,(\tau_n)_j) \xrightarrow{n\to\infty} \tau^j + \phi_{(0)}(x_j^+,\tau^j) = \tau^{j+1},$$

$$(x_n)_{j+1} = S_{(\epsilon_n)}((\tau_n)_{j+1},(\tau_n)_j)(x_n)_j^+ \xrightarrow{n\to\infty} S_{(0)}(\tau^{j+1},\tau^j)x_j^+ = x_{j+1},$$

and, finally, $(x_n)_{j+1}^+ = I_{(\epsilon_n)}((x_n)_{j+1}) \xrightarrow{n\to\infty} I_{(0)}(x_{j+1}) = x_{j+1}^+$.

Since $t > \tau^1$, there exists $k = k(x,\tau) \in \mathbb{N}$ such that

$$t = \tau + \sum_{j=0}^{k-1} \phi_{(0)}(x_j^+,\tau^j) + \tau' \quad \text{with} \quad \tau' \in [0,\phi_{(0)}(x_k^+,\tau^k)).$$

Moreover, $\tilde{S}_{(0)}(t,\tau)x = S_{(0)}(t,\tau^k)x_k^+$. Now, let us define

$$\eta_n = \sum_{j=0}^{k-1} \phi_{(\epsilon_n)}((x_n)_j^+, (\tau_n)_j) - \sum_{j=0}^{k-1} \phi_{(0)}(x_j^+, \tau^j) + \tau_n - \tau, \quad n \in \mathbb{N}.$$

Note that $\eta_n \xrightarrow{n\to\infty} 0$. Furthermore, we have $\tau' < \phi_{(\epsilon_n)}((x_n)_k^+, (\tau_n)_k)$ for $n$ sufficiently large. Thus,

$$\begin{aligned}
&\lim_{n\to\infty} \tilde{S}_{(\epsilon_n)}(t+\eta_n, \tau_n)x_n \\
&= \lim_{n\to\infty} \tilde{S}_{(\epsilon_n)}\left(\tau_n + \sum_{j=0}^{k-1} \phi_{(\epsilon_n)}((x_n)_j^+, (\tau_n)_j) + \tau', \tau_n\right) x_n \\
&= \lim_{n\to\infty} S_{(\epsilon_n)}\left(\tau_n + \sum_{j=0}^{k-1} \phi_{(\epsilon_n)}((x_n)_j^+, (\tau_n)_j) + \tau', (\tau_n)_k\right) (x_n)_k^+ \\
&= S_{(0)}\left(\tau + \sum_{j=0}^{k-1} \phi_{(0)}(x_j^+, \tau^j) + \tau', \tau^k\right) x_k^+.
\end{aligned}$$

Since $\tilde{S}_{(0)}(t,\tau)x = S_{(0)}(t,\tau^k)x_k^+ = S_{(0)}\left(\tau + \sum_{j=0}^{k-1} \phi_{(0)}(x_j^+, \tau^j) + \tau', \tau^k\right) x_k^+$, we have

$$\tilde{S}_{(\epsilon_n)}(t+\eta_n, \tau_n)x_n \xrightarrow{n\to\infty} \tilde{S}_{(0)}(t,\tau)x,$$

which proves the assertion. The proof of the result is complete. □

**Lemma 3.10.** *Assume that conditions* (*A*2), (*A*4)*, and the collective condition* (T) *hold. Let* $t \in \mathbb{R}$ *and* $x \in Z \setminus M_{(0)}$ *be given, and let* $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}, \{\beta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}, \{x_n\}_{n\in\mathbb{N}} \subset Z$ *be sequences such that* $\alpha_n \xrightarrow{n\to\infty} 0, \beta_n \xrightarrow{n\to\infty} 0,$ *and* $x_n \xrightarrow{n\to\infty} x$*, with* $\beta_n \leq \alpha_n$ *for all* $n \in \mathbb{N}$*. Then,* $\tilde{S}_{(\epsilon_n)}(t+\alpha_n, t+\beta_n)x_n \xrightarrow{n\to\infty} x$*.*

*Proof.* Since $\beta_n \leq \alpha_n$ for all $n \in \mathbb{N}$, $\alpha_n - \beta_n \xrightarrow{n\to\infty} 0$, and $\phi_{(\epsilon_n)}(x_n, t+\beta_n) \xrightarrow{n\to\infty} \phi_{(0)}(x,t)$ (see Theorem 3.4), there exists $n_0 \in \mathbb{N}$ such that

$$0 \leq \alpha_n - \beta_n < \frac{\phi_{(0)}(x,t)}{2} \quad \text{and} \quad |\phi_{(\epsilon_n)}(x_n, t+\beta_n) - \phi_{(0)}(x,t)| < \frac{\phi_{(0)}(x,t)}{2}$$

for all $n \geq n_0$. Consequently, for $n \geq n_0$, we have

$$0 \leq \alpha_n - \beta_n < \frac{\phi_{(0)}(x,t)}{2} < \phi_{(\epsilon_n)}(x_n, t+\beta_n)$$

and, therefore,

$$\begin{aligned}&\lim_{n\to\infty}\tilde{S}_{(\epsilon_n)}(t+\alpha_n,t+\beta_n)x_n\\&=\lim_{n\to\infty}\tilde{S}_{(\epsilon_n)}(t+\beta_n+(\alpha_n-\beta_n),t+\beta_n)x_n\\&=\lim_{n\to\infty}S_{(\epsilon_n)}(t+\beta_n+(\alpha_n-\beta_n),t+\beta_n)x_n\\&=S_{(0)}(t,t)x=x,\end{aligned}$$

which proves the result. □

**Corollary 3.2.** *Under the assumptions of Theorem 3.5, assume that $I_{(0)}(M_{(0)})\cap M_{(0)}=\emptyset$. Then we also can assume that $\eta_n\geq 0$ for all $n\in\mathbb{N}$.*

*Proof.* We already know that, by Theorem 3.5, given $t,\tau\in\mathbb{R}$, $x\in Z\setminus M_{(0)}$, $\{x_n\}_{n\in\mathbb{N}}\subset Z$ and $\{\tau_n\}_{n\in\mathbb{N}}\subset(-\infty,t]$ sequences such that $x_n\xrightarrow{n\to\infty}x$ and $\tau_n\xrightarrow{n\to\infty}\tau$, we can find a sequence $\eta_n\xrightarrow{n\to\infty}0$ such that $t+\eta_n\geq\tau_n$ and

$$\tilde{S}_{(\epsilon_n)}(t+\eta_n,\tau_n)x_n\xrightarrow{n\to\infty}\tilde{S}_{(0)}(t,\tau)x.$$

Now, choose the sequences $\alpha_n=\eta_n+|\eta_n|$ and $\beta_n=\eta_n$ for all $n\in\mathbb{N}$. Since $I_{(0)}(M_{(0)})\cap M_{(0)}=\emptyset$, we have $\tilde{S}_{(0)}(t,\tau)x\notin M_{(0)}$. By Lemma 3.10,

$$\tilde{S}_{(\epsilon_n)}(t+\eta_n+|\eta_n|,t+\eta_n)\tilde{S}_{(\epsilon_n)}(t+\eta_n,\tau_n)x_n\xrightarrow{n\to\infty}\tilde{S}_{(0)}(t,\tau)x,$$

that is, $\tilde{S}_{(\epsilon_n)}(t+\alpha_n,\tau_n)x_n\xrightarrow{n\to\infty}\tilde{S}_{(0)}(t,\tau)x$ with $\alpha_n=\eta_n+|\eta_n|\geq 0$ for all $n\in\mathbb{N}$. □

We end this section with a convergence result which leads with points that belong to the impulsive set $M$.

**Lemma 3.11.** *Assume that conditions (A2), (A5), and the collective condition (T) hold. Also, assume that $I_{(0)}(M_{(0)})\cap M_{(0)}=\emptyset$. Let $t\in\mathbb{R}$, $x\in M_{(0)}$, $\{\alpha_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_+$, $\{\beta_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_+$, and $\{x_n\}_{n\in\mathbb{N}}\subset Z$ be sequences such that $\alpha_n\geq\beta_n$ for all $n\in\mathbb{N}$, $\alpha_n\xrightarrow{n\to\infty}0$, $\beta_n\xrightarrow{n\to\infty}0$, and $x_n\xrightarrow{n\to\infty}x$. Then there exists a subsequence $\{\phi_{(\epsilon_{n_k})}(x_{n_k},t+\beta_{n_k})\}_{k\in\mathbb{N}}$ of $\{\phi_{(\epsilon_n)}(x_n,t+\beta_n)\}_{n\in\mathbb{N}}$ such that $\phi_{(\epsilon_{n_k})}(x_{n_k},t+\beta_{n_k})\xrightarrow{k\to\infty}0$. Moreover,*

- *(i) if $\alpha_{n_k}-\beta_{n_k}<\phi_{(\epsilon_{n_k})}(x_{n_k},t+\beta_{n_k})$ for all $k\in\mathbb{N}$, then $\tilde{S}_{(\epsilon_{n_k})}(t+\alpha_{n_k},t+\beta_{n_k})x_{n_k}\xrightarrow{k\to\infty}x$;*
- *(ii) if $\alpha_{n_k}-\beta_{n_k}\geq\phi_{(\epsilon_{n_k})}(x_{n_k},t+\beta_{n_k})$ for all $k\in\mathbb{N}$, then $\tilde{S}_{(\epsilon_{n_k})}(t+\alpha_{n_k},t+\beta_{n_k})x_{n_k}\xrightarrow{k\to\infty}I_{(0)}(x)$.*

*Proof.* Note that $S_{(\epsilon_n)}\left(t+\alpha_n,t+\beta_n\right)x_n\xrightarrow{n\to\infty}x\in M_{(0)}$. By the collective condition (T), there exist a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ and a sequence $\gamma_k\xrightarrow{k\to\infty}0$, such that $t+\gamma_k+\alpha_{n_k}\geq t+\beta_{n_k}$ and

$$S_{(\epsilon_{n_k})}\left(t+\gamma_k+\alpha_{n_k},t+\beta_{n_k}\right)x_{n_k}\in M_{(\epsilon_{n_k})}$$

for all $k \in \mathbb{N}$. Consequently,

$$\phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k}) \leq \gamma_k + \alpha_{n_k} - \beta_{n_k} \xrightarrow{k\to\infty} 0.$$

Hence, $\phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k}) \xrightarrow{k\to\infty} 0$.

$(i)$ Note that $\tilde{S}_{(\epsilon_{n_k})}(t + \tau, t + \beta_{n_k})x_{n_k} = S_{(\epsilon_{n_k})}(t + \tau, t + \beta_{n_k})x_{n_k}$ for all $\tau \in [\beta_{n_k}, \beta_{n_k} + \phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k}))$, $k \in \mathbb{N}$. Thus, if $\alpha_{n_k} - \beta_{n_k} < \phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k})$ for all $k \in \mathbb{N}$, then

$$\tilde{S}_{(\epsilon_{n_k})}(t + \alpha_{n_k}, t + \beta_{n_k})x_{n_k} = S_{(\epsilon_{n_k})}(t + \alpha_{n_k}, t + \beta_{n_k})x_{n_k} \xrightarrow{k\to\infty} S_{(0)}(t, t)x = x.$$

$(ii)$ Since $S_{(\epsilon_{n_k})}(t + \beta_{n_k} + \phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k}), t + \beta_{n_k})x_{n_k} \in M_{(\epsilon_{n_k})}$ for all $k \in \mathbb{N}$, and

$$S_{(\epsilon_{n_k})}(t + \beta_{n_k} + \phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k}), t + \beta_{n_k})x_{n_k} \xrightarrow{k\to\infty} x,$$

it follows by the continuity of $I_{(\epsilon_{n_k})}$ that

$$\begin{aligned}
&\lim_{k\to\infty} \tilde{S}_{(\epsilon_{n_k})}(t + \beta_{n_k} + \phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k}), t + \beta_{n_k})x_{n_k} \\
&= \lim_{k\to\infty} I_{(\epsilon_{n_k})}(S_{(\epsilon_{n_k})}(t + \beta_{n_k} + \phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k}), t + \beta_{n_k})x_{n_k}) \\
&= I_{(0)}(x) \notin M_{(0)}.
\end{aligned}$$

Let $w_k = \tilde{S}_{(\epsilon_{n_k})}(t + \beta_{n_k} + \phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k}), t + \beta_{n_k})x_{n_k}$, $k \in \mathbb{N}$. Thus, if $\alpha_{n_k} - \beta_{n_k} \geq \phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k})$ for all $k \in \mathbb{N}$, then by Lemma 3.10,

$$\tilde{S}_{(\epsilon_{n_k})}(t + \alpha_{n_k}, t + \beta_{n_k} + \phi_{(\epsilon_{n_k})}(x_{n_k}, t + \beta_{n_k}))w_k \xrightarrow{k\to\infty} I_{(0)}(x),$$

that is,

$$\tilde{S}_{(\epsilon_{n_k})}(t + \alpha_{n_k}, t + \beta_{n_k})x_{n_k} \xrightarrow{k\to\infty} I_{(0)}(x).$$

Hence, the proof is complete. □

### 3.6.2 Upper semicontinuity

At first place, we need to obtain informations about the impulsive dynamics that happens inside the attractor $\{\tilde{\mathbb{A}}_{(0)}(t) \colon t \in \mathbb{R}\}$. In fact, we prove that all the points that undergo impulse must enter in the attractor $\{\tilde{\mathbb{A}}_{(0)}(t) \colon t \in \mathbb{R}\}$.

**Proposition 3.2.** *Assume that conditions* $(A1)$, $(A3)$ *and* $(A8)$ *hold. The inclusion* $I_{(0)}(M_{(0)}) \subset \tilde{\mathbb{A}}_{(0)}(t) \setminus M_{(0)}$ *holds for all* $t \in \mathbb{R}$.

*Proof.* Let $t \in \mathbb{R}$ be fixed and arbitrary, and let $x \in I_{(0)}(M_{(0)})$. By condition $(A8)$, there exist

$\tau_1 \geq 2\xi$ and $y_1 \in I_{(0)}(M_{(0)})$ such that

$$\tilde{S}_{(0)}(t, t-\tau_1)y_1 = x.$$

Since $y_1 \in I_{(0)}(M_{(0)})$, we may use condition $(A8)$ again to obtain the existence of $\tau_2 \geq 2\xi$ and $y_2 \in I_{(0)}(M_{(0)})$ such that

$$\tilde{S}_{(0)}(t-\tau_1, t-\tau_1-\tau_2)y_2 = y_1.$$

Hence,

$$\tilde{S}_{(0)}(t, t-\tau_1)\tilde{S}_{(0)}(t-\tau_1, t-\tau_1-\tau_2)y_2 = \tilde{S}_{(0)}(t, t-\tau_1)y_1 = x,$$

that is,

$$\tilde{S}_{(0)}(t, t-\tau_1-\tau_2)y_2 = x.$$

Proceeding with this recursive process, we get the existence of sequences $\{\tau_k\}_{k\in\mathbb{N}} \subset [2\xi, \infty)$ and $\{y_k\}_{k\in\mathbb{N}} \subset I_{(0)}(M_{(0)})$ satisfying

$$\tilde{S}_{(0)}(t, t-s_k)y_k = x \quad \text{for all} \quad k \in \mathbb{N},$$

where $s_k = \sum_{i=1}^{k} \tau_i$. Note that $\{s_k\}_{k\in\mathbb{N}}$ is increasing because $\tau_i \geq 2\xi$ for all $i \in \mathbb{N}$. Thus, $s_k \xrightarrow{k\to\infty} \infty$. Now, since

$$\begin{aligned} d(x, \tilde{\mathbb{A}}_{(0)}(t)) &= d(\tilde{S}_{(0)}(t, t-s_k)y_k, \tilde{\mathbb{A}}_{(0)}(t)) \\ &\leq d_H(\tilde{S}_{(0)}(t, t-s_k)I_{(0)}(M_{(0)}), \tilde{\mathbb{A}}_{(0)}(t)), \quad k \in \mathbb{N}, \end{aligned}$$

and $I_{(0)}(M_{(0)})$ is bounded by condition $(A3)$, it follows that

$$d(x, \tilde{\mathbb{A}}_{(0)}(t)) \leq \lim_{k\to\infty} d_H(\tilde{S}_{(0)}(t, t-s_k)I_{(0)}(M_{(0)}), \tilde{\mathbb{A}}_{(0)}(t)) = 0,$$

that is, $d(x, \tilde{\mathbb{A}}_{(0)}(t)) = 0$. Therefore, $x \in \tilde{\mathbb{A}}_{(0)}(t) \setminus M_{(0)}$ as we have condition $(A3)$. □

**Proposition 3.3.** *Assume that conditions* $(A1)$, $(A3)$, $(A7)$ *and* $(A8)$ *hold. The inclusion* $M_{(0)} \subset \tilde{\mathbb{A}}_{(0)}(t)$ *holds for all* $t \in \mathbb{R}$. *In particular,* $M_{(0)}$ *is compact in* $Z$.

*Proof.* Let $t \in \mathbb{R}$ be fixed and arbitrary, and let $x \in M_{(0)}$ be given. By condition $(A8)$, there exist $y \in I_{(0)}(M_{(0)})$ and $\tau \geq 2\xi$ such that

$$S_{(0)}(t, t-\tau)y = z \in M_{(0)} \quad \text{and} \quad \tilde{S}_{(0)}(t, t-\tau)y = I_{(0)}(x) \in I_{(0)}(M_{(0)}).$$

Hence,

$$I_{(0)}(x) = \tilde{S}_{(0)}(t, t-\tau)y = I_{(0)}(z)$$

and, since $I_{(0)}$ is injective (see condition $(A7)$), we obtain $x = z$.

On the other hand, since $S_{(0)}(t, t-\tau)y = z \in M_{(0)}$, we get $\tau = \phi_{(0)}(y, t-\tau)$. Then

$$x = z = S_{(0)}(t, t-\tau)y = \lim_{s \to \tau^-} S_{(0)}(t, t-s)y = \lim_{s \to \tau^-} \tilde{S}_{(0)}(t, t-s)y.$$

Let $\{s_k\}_{k\in\mathbb{N}} \subset [0, \tau)$ be a sequence such that $s_k \xrightarrow{k\to\infty} \tau$. Consequently, $\lim_{k\to\infty} \tilde{S}_{(0)}(t, t-s_k)y = x$.

According to Proposition 3.2, we have

$$I_{(0)}(M_{(0)}) \subset \tilde{\mathbb{A}}_{(0)}(t - s_k) \backslash M_{(0)}, \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence,

$$\tilde{S}_{(0)}(t, t-s_k)y \in \tilde{S}_{(0)}(t, t-s_k)[\tilde{\mathbb{A}}_{(0)}(t-s_k) \backslash M_{(0)}] = \tilde{\mathbb{A}}_{(0)}(t) \setminus M_{(0)}, \quad \text{for all} \quad k \in \mathbb{N},$$

that is, $\{\tilde{S}_{(0)}(t, t-s_k)y\}_{k\in\mathbb{N}} \subset \tilde{\mathbb{A}}_{(0)}(t)$. Since $\tilde{S}_{(0)}(t, t-s_k)y \xrightarrow{k\to\infty} x$ and $\tilde{\mathbb{A}}_{(0)}(t)$ is closed, we obtain $x \in \tilde{\mathbb{A}}_{(0)}(t)$, which proves the result. □

**Proposition 3.4.** *Assume that conditions $(A1) - (A9)$ hold. Assume that $\bigcup_{\epsilon\in[0,1]} \tilde{\mathbb{A}}_{(\epsilon)}(t)$ is bounded in $Z$ for each $t \in \mathbb{R}$. Then, the set*

$$\overline{\bigcup_{\epsilon\in[0,1]} \tilde{\mathbb{A}}_{(\epsilon)}(t)}$$

*is compact, for each $t \in \mathbb{R}$.*

*Proof.* Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\bigcup_{\epsilon\in[0,1]} \tilde{\mathbb{A}}_{(\epsilon)}(t)$. For each $n \in \mathbb{N}$, there exists $\epsilon_n \in [0, 1]$ such that $x_n \in \tilde{\mathbb{A}}_{(\epsilon_n)}(t)$. We may assume without loss of generality that $\epsilon_n \xrightarrow{n\to\infty} \epsilon_0 \in [0, 1]$.

Suppose $x_n \in M_{(\epsilon_n)}$ for every $n \in \mathbb{N}$ (up to a subsequence). Using the fact that $d_H(M_{(\epsilon_n)}, M_{(\epsilon_0)}) \xrightarrow{n\to\infty} 0$ and $M_{(\epsilon_0)}$ is compact by Proposition 3.3, then $\{x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence.

Now, suppose up to a subsequence that $x_n \notin M_{(\epsilon_n)}$ for every $n \in \mathbb{N}$. Then

$$x_n = \tilde{S}_{(\epsilon_n)}(t, t-\xi)b^n$$

for some $b^n \in \tilde{\mathbb{A}}_{(\epsilon_n)}(t-\xi) \setminus M_{(\epsilon_n)}$, $n \in \mathbb{N}$. By hypothesis, there exists a bounded set $B_0 \subset Z$ such that $\bigcup_{\epsilon\in[0,1]} \tilde{\mathbb{A}}_{(\epsilon)}(t) \subset B_0$.

<u>Case 1:</u> $\phi_{(\epsilon_n)}(b^n, t-\xi) > \xi$, up to a subsequence.

In this case,

$$x_n = \tilde{S}_{(\epsilon_n)}(t, t-\xi)b^n = S_{(\epsilon_n)}(t, t-\xi)b^n.$$

Since $S_{(0)}(t,t-\xi)B_0$ is relatively compact (see $(A9)$), condition $(A2)$ holds and

$$d_H(x_n, S_{(0)}(t,t-\xi)B_0) \leq d_H(x_n, S_{(0)}(t,t-\xi)b^n) + d_H(S_{(0)}(t,t-\xi)b^n, S_{(0)}(t,t-\xi)B_0),$$

then $\{x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence.
Case 2: $\phi_{(\epsilon_n)}(b^n, t-\xi) \leq \xi$, up to a subsequence.

In this case,

$$x_n = \tilde{S}_{(\epsilon_n)}(t,t-\xi)b^n = S_{(\epsilon_n)}(t, t-\xi+\phi_{(\epsilon_n)}(b^n,t-\xi))(b^n)_1^+,$$

as we have condition $(A6)$. Note that $\{\phi_{(\epsilon_n)}(b^n,t-\xi)\}_{n\in\mathbb{N}}$ and $\{(b^n)_1^+\}_{n\in\mathbb{N}}$ admit convergent subsequences. Using condition $(A2)$, we conclude that $\{x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence. Hence, the proof is complete. □

Next, we exhibit the upper semicontinuity of a family of impulsive pullback attractors.

**Theorem 3.6.** *Assume that conditions* $(A1)-(A9)$ *and the collective condition* (T) *hold. Assume that* $\bigcup_{\epsilon\in[0,1]}\bigcup_{t\in\mathbb{R}}\tilde{\mathbb{A}}_{(\epsilon)}(t)$ *is bounded in* $Z$*. The family* $\{\tilde{\mathbb{A}}_{(\epsilon)}(t)\colon t\in\mathbb{R}\}$ *of impulsive pullback attractors is upper semicontinuous at* $\epsilon=0$*, that is,*

$$d_H(\tilde{\mathbb{A}}_{(\epsilon)}(t), \tilde{\mathbb{A}}_{(0)}(t)) \xrightarrow{\epsilon\to 0^+} 0,$$

*for all* $t\in\mathbb{R}$*.*

*Proof.* Suppose, arguing by contradiction, that there exist $t\in\mathbb{R}$, $\delta_0>0$, a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}\subset(0,1]$ with $\epsilon_n \xrightarrow{n\to\infty} 0$, and a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$x_n\in\tilde{\mathbb{A}}_{(\epsilon_n)}(t) \quad \text{and} \quad d(x_n,\tilde{\mathbb{A}}_{(0)}(t))\geq\delta_0 \quad \text{for all} \quad n\in\mathbb{N}. \tag{3.13}$$

Suppose $x_n\in M_{(\epsilon_n)}$ for every $n\in\mathbb{N}$ (up to a subsequence). Using the fact that $d_H(M_{(\epsilon_n)}, M_{(0)}) \xrightarrow{n\to\infty} 0$ and $M_{(0)}$ is compact by Proposition 3.3, then $\{x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence to some point $x_0\in M_{(0)}\subset\tilde{\mathbb{A}}_{(0)}(t)$ which contradicts (3.13) as $n\to\infty$.

Now, let us assume that $x_n\notin M_{(\epsilon_n)}$ for all $n\in\mathbb{N}$. Using the invariance of the impulsive pullback attractor, there exists $b^n_{-k}\in\tilde{\mathbb{A}}_{(\epsilon_n)}(-k)\setminus M_{(\epsilon_n)}$ such that

$$x_n = \tilde{S}_{(\epsilon_n)}(t,-k)b^n_{-k},$$

for each natural $k$ such that $t+k\geq 1$. By Proposition 3.4, we may assume that

$$x_n\xrightarrow{n\to\infty} x_0 \quad \text{and} \quad b^n_{-k}\xrightarrow{n\to\infty} b_{-k}.$$

Since $M_{(0)} \subset \tilde{\mathbb{A}}_{(0)}(t)$ (Proposition 3.3), we may assume that $x_0 \notin M_{(0)}$.

According to the hypothesis, there exists a bounded set $C_0 \subset Z$ such that $\{b^n_{-k}\}_{n\in\mathbb{N}} \subset C_0$ and $b_{-k} \in C_0$ for all integer $k \geq -t+1$.

Case 1: $b_{-k} \notin M_{(0)}$ for all natural $k \geq -t+1$, up to a subsequence.

By Corollary 3.2, there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, with $t+\eta_n \geq -k$ and $\eta_n \xrightarrow{n\to\infty} 0$, such that $\tilde{S}_{(\epsilon_n)}(t+\eta_n, -k)b^n_{-k} \xrightarrow{n\to\infty} \tilde{S}_{(0)}(t,-k)b_{-k}$. Since $x_0 \notin M_{(0)}$, it follows by Lemma 3.10 that

$$\tilde{S}_{(\epsilon_n)}(t+\eta_n, t)\tilde{S}_{(\epsilon_n)}(t,-k)b^n_{-k} \xrightarrow{n\to\infty} x_0.$$

Hence, $x_0 = \tilde{S}_{(0)}(t,-k)b_{-k}$. As $n \to \infty$ in (3.13), we obtain

$$d_H(\tilde{S}_{(0)}(t,-k)b_{-k}, \tilde{\mathbb{A}}_{(0)}(t)) \geq \delta_0$$

for all $k \geq -t+1$. But $d_H(\tilde{S}_{(0)}(t,-k)b_{-k}, \tilde{\mathbb{A}}_{(0)}(t)) \xrightarrow{k\to\infty} 0$, which is a contradiction.

Case 2: $b_{-k} \in M_{(0)}$ for all natural $k \geq -t+1$, up to a subsequence.

By Lemma 3.11, we may assume that $\phi_{(\epsilon_n)}(b^n_{-k}, -k) \xrightarrow{n\to\infty} 0$ and

$$\tilde{S}_{(\epsilon_n)}(\phi_{(\epsilon_n)}(b^n_{-k}, -k) - k, -k)b^n_{-k} \xrightarrow{n\to\infty} I_{(0)}(b_{-k}) \in I_{(0)}(M_{(0)}).$$

Since $k \geq -t+1$, we conclude, using Corollary 3.2, that there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, with $\eta_n \xrightarrow{n\to\infty} 0$, such that $t+\eta_n > \phi_{(\epsilon_n)}(b^n_{-k}, -k) - k$ for $k$ sufficiently large and

$$\tilde{S}_{(\epsilon_n)}(t+\eta_n, -k)b^n_{-k} \xrightarrow{n\to\infty} \tilde{S}_{(0)}(t,-k)I_{(0)}(b_{-k}).$$

Since $x_0 \notin M_{(0)}$, it follows by Lemma 3.10 that

$$\tilde{S}_{(\epsilon_n)}(t+\eta_n, t)\tilde{S}_{(\epsilon_n)}(t,-k)b^n_{-k} \xrightarrow{n\to\infty} x_0.$$

Hence, $x_0 = \tilde{S}_{(0)}(t,-k)I_{(0)}(b_{-k})$. As $n \to \infty$ in (3.13), we obtain

$$d_H(\tilde{S}_{(0)}(t,-k)I_{(0)}(b_{-k}), \tilde{\mathbb{A}}_{(0)}(t)) \geq \delta_0$$

for all $k \geq -t+1$. But, as $I_{(0)}(M_{(0)})$ is bounded by condition $(A3)$, it follows that $d_H(\tilde{S}_{(0)}(t,-k)I_{(0)}(b_{-k}), \tilde{\mathbb{A}}_{(0)}(t)) \xrightarrow{k\to\infty} 0$, which is a contradiction.

In conclusion, we obtain that the family $\{\tilde{\mathbb{A}}_{(\epsilon)}(t)\colon t \in \mathbb{R}\}$ is upper semicontinuous at $\epsilon = 0$, and this ends the result. □

Chapter

4

# Non-autonomous Klein-Gordon-Zakharov system with impulsive action

This chapter is devoted to the study of the non-autonomous Klein-Gordon-Zakharov system presented in Chapter 2, given by (2.1) – (2.3), subject to impulsive effects at variable times. We are interested in the asymptotic dynamics of the solutions of the following impulsive non-autonomous problem

$$\begin{cases} u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}} u_t + a_\epsilon(t)(-\Delta)^{\frac{1}{2}} v_t = f(u), & (x,t) \in \Omega \times (\tau, \infty), \\ v_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}} v_t - a_\epsilon(t)(-\Delta)^{\frac{1}{2}} u_t = 0, & (x,t) \in \Omega \times (\tau, \infty), \\ u = v = 0, & (x,t) \in \partial\Omega \times (\tau, \infty), \\ I \colon M \subset Y_0 \to Y_0, \end{cases} \tag{4.1}$$

with initial conditions

$$u(\tau, x) = u_0(x),\ u_t(\tau, x) = u_1(x),\ v(\tau, x) = v_0(x),\ v_t(\tau, x) = v_1(x),\ x \in \Omega,\ \tau \in \mathbb{R}, \tag{4.2}$$

where, as presented in Chapter 2, $\eta > 0$ is constant, $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, with the boundary $\partial\Omega$ assumed to be regular enough, the function $a_\epsilon \colon \mathbb{R} \to (0, \infty)$ is continuously differentiable in $\mathbb{R}$ and it is $(\beta, C)$-Hölder continuous for each $\epsilon \in [0,1]$ (see conditions (2.4) and (2.6)), and the nonlinearity $f \in C^1(\mathbb{R})$ satisfies the dissipativeness condition given by (2.7) and also the subcritical growth condition given by (2.8).

The set $M$, called the *impulsive set*, is a nonempty closed subset of the phase space

$$Y_0 = X^{\frac{1}{2}} \times X \times X^{\frac{1}{2}} \times X,$$

where $X = L^2(\Omega)$, which satisfies the conditions presented in Definition 3.2. The *impulse function* $I \colon M \subset Y_0 \to Y_0$ is assumed to be continuous and will be responsible by the occurrence of impulses at variable times.

All the results of this chapter are presented in the article [18].

## 4.1 Existence of the impulsive pullback attractor

We shall assume that the impulsive set $M \subset Y_0$ satisfies condition (T) with respect to the evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ given in (2.40), and the impulse function $I\colon M \to Y_0$ satisfies the following conditions:

$(H1)$ $I(M) \cap M = \emptyset$;

$(H2)$ there exists $\mu > 0$ such that $\|I(w)\|_{Y_0}^2 \leq \mu$ for all $w \in M$;

$(H3)$ there exists $\xi > 0$ such that $\phi(w,\tau) \geq 2\xi$ for all $w \in I(M)$ and $\tau \in \mathbb{R}$.

Let $\tilde{W}(t) = \tilde{S}(t,\tau)W_0$, $t \geq \tau$, be the impulsive solution of the impulsive non-autonomous problem

$$\begin{cases} W_t + \mathcal{A}(t)W = F(W), \ t > \tau, \\ W(\tau) = W_0 \in Y_0, \ \tau \in \mathbb{R}, \\ I\colon M \to Y_0, \end{cases} \tag{4.3}$$

where $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ is its associated impulsive evolution process.

The method chosen to show the existence of the impulsive pullback attractor is to construct a family of compact absorbing sets. To achieve this goal, first we need some technical lemmas. In what follows, we prove that it is possible to obtain some kind of control over the impulsive trajectories.

**Lemma 4.1.** *Given $r \geq \mu$, there exists $\ell_r > 0$ such that $\|\tilde{S}(t,\tau)W_0\|_{Y_0}^2 \leq \ell_r$ for all $W_0 \in Y_0$ with $\|W_0\|_{Y_0}^2 \leq r$ and $t \geq \tau \in \mathbb{R}$.*

*Proof.* In fact, let $\tau \in \mathbb{R}$, $W_0 \in Y_0$ with $\|W_0\|_{Y_0}^2 \leq r$ and $t \geq \tau$. By the proof of Theorem 2.4,

$$\ell_r = \sup\{\|S(t,\tau)W_0\|_{Y_0}^2\colon t \geq \tau, \|W_0\|_{Y_0}^2 \leq r\} < \infty.$$

Since $I(M) \subset \overline{B_{Y_0}(0,\mu)} \subset \overline{B_{Y_0}(0,r)}$ due to condition $(H2)$, it follows that $\|\tilde{S}(t,\tau)W_0\|_{Y_0}^2 \leq \ell_r$ for all $t \geq \tau$. □

**Lemma 4.2.** *There exists $R \geq \mu$ such that for any bounded subset $B$ of $Y_0$, one can find $t_0(B) \geq 0$ such that*

$$\|\tilde{S}(t,\tau)W_0\|_{Y_0}^2 \leq R,$$

*for all $W_0 \in B$ and $t \geq \tau + t_0(B)$.*

*Proof.* By Theorem 2.4, there exist $r_0 > 0$ (independent of $B$) and $t_0 = t_0(B) > 0$ such that

$$S(t,\tau)B \subset \overline{B_{Y_0}(0,r_0)} \quad \text{for all} \quad t \geq \tau + t_0(B).$$

According to Lemma 4.1, taking $R = \max\{\ell_\mu, r_0\}$, we obtain

$$\|\tilde{S}(t,\tau)W_0\|_{Y_0}^2 \leq R,$$

for all $W_0 \in B$ and $t \geq \tau + t_0(B)$. □

In the following result, we show that the impulsive process $\{\tilde{S}(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ is a map that takes precompact sets into precompact sets.

**Lemma 4.3.** *If $G$ is a precompact subset of $Y_0$ and $t \geq \tau \in \mathbb{R}$ satisfies $0 \leq t - \tau < \xi$, then $\tilde{S}(t,\tau)G$ is precompact in $Y_0$.*

*Proof.* Let $\tau \in \mathbb{R}$ be fixed, and observe that for $t = \tau$, it follows that the set

$$\tilde{S}(\tau,\tau)G = G$$

is precompact in $Y_0$. Thus, let us assume that $0 < t - \tau < \xi$. Note that we may consider only the case in which we have $\phi(u,\tau) \leq t - \tau$ for all $u \in G$. In fact, otherwise we could write $G = G_1 \cup G_2$, where

$$\phi(\cdot,\tau)|_{G_1} \leq t - \tau \quad \text{and} \quad \phi(\cdot,\tau)|_{G_2} > t - \tau$$

and so, in this last one, we have $\tau < t < \phi(u,\tau) + \tau$ for all $u \in G_2$, that is, $\tilde{S}(t,\tau)G_2 = S(t,\tau)G_2$, and then the precompactness of $\tilde{S}(t,\tau)G_2$ follows from the compactness of the evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$, see Proposition 2.6. Thereby, we assume $\phi(u,\tau) \leq t - \tau$ for all $u \in G$.

Now, define the auxiliary set

$$B = \bigcup_{r \in [\tau,t]} S(r,\tau)G.$$

Note that $B$ is precompact. In fact, take a sequence $\{x_n\}_{n\in\mathbb{N}} \subset B$. Then $x_n = S(r_n,\tau)z_n$ for some $r_n \in [\tau,t]$ and $z_n \in G$, for each $n \in \mathbb{N}$. Since $\{z_n\}_{n\in\mathbb{N}} \subset \overline{G}$, and $\overline{G}$ is compact, we can assume, up to subsequences, that $z_n \xrightarrow{n\to\infty} z \in \overline{G}$ and $r_n \xrightarrow{n\to\infty} r \in [\tau,t]$. Hence, by continuity, we get

$$x_n = S(r_n,\tau)z_n \xrightarrow{n\to\infty} S(r,\tau)z = x$$

and, therefore, $x \in \overline{B}$. This shows the claim.

Since $I$ is a continuous map and $B \cap M$ is precompact, we may use the previous argument to conclude that

$$\bigcup_{r\in[\tau,t]} S(t,r)(I(B\cap M))$$

is also precompact.

In order to conclude this result, we are going to show that

$$\tilde{S}(t,\tau)G \subseteq \bigcup_{r\in[\tau,t]} S(t,r)(I(B\cap M)).$$

Indeed, given $u \in G$ and taking into account that $\phi(u,\tau) \leq t-\tau$ for all $u \in G$, we have

$$\tilde{S}(t,\tau)u = \tilde{S}(t,\phi(u,\tau)+\tau)u_1^+,$$

where

$$u_1^+ = \tilde{S}(\phi(u,\tau)+\tau,\tau)u = I(S(\phi(u,\tau)+\tau,\tau)u) \in I(B\cap M).$$

On the other hand, setting $\tau_1 = \phi(u,\tau)+\tau$ and using $(H3)$, we obtain

$$0 \leq t-\tau-\phi(u,\tau) < t-\tau < \xi < \phi(u_1^+,\tau_1),$$

that is, $\tau_1 \leq t < \phi(u_1^+,\tau_1)+\tau_1$ and this condition ensures that

$$\tilde{S}(t,\tau)u = \tilde{S}(t,\tau_1)u_1^+ = S(t,\tau_1)u_1^+ \in S(t,\tau_1)(I(B\cap M)).$$

Therefore,

$$\tilde{S}(t,\tau)G \subseteq \bigcup_{r\in[\tau,t]} S(t,r)(I(B\cap M)),$$

which ensures the precompactness of $\tilde{S}(t,\tau)G$ in $Y_0$. □

Finally, in the next result we will construct a family of compact sets that pullback absorbs bounded subsets of the phase space.

**Theorem 4.1.** *The impulsive evolution process $\{\tilde{S}(t,s)\colon t \geq s \in \mathbb{R}\}$ is pullback $\tilde{S}$-strongly compact dissipative.*

*Proof.* Let $B_0 = \{w \in Y_0\colon \|w\|_{Y_0}^2 \leq R\}$, where $R \geq \mu$ comes from Lemma 4.2, and let $\tau \in (\xi, 2\xi)$ be fixed. We claim that the set $G(t) = \tilde{S}(t,t-\tau)B_0$ is precompact in $Y_0$ for each $t \in \mathbb{R}$. In fact, we can write $B_0 = C_1(t) \cup C_2(t)$, where

$$C_1(t) = \{w \in B_0\colon \phi(w,t-\tau) > \xi\} \quad \text{and} \quad C_2(t) = \{w \in B_0\colon \phi(w,t-\tau) \leq \xi\}.$$

Let $y \in G(t)$ be given. Then $y = \tilde{S}(t,t-\tau)w$ for some $w \in B_0$. If $w \in C_1(t)$, then $\phi(w,t-\tau) > \xi$, that is,

$$y = \tilde{S}(t, t-\tau+\xi)\tilde{S}(t-\tau+\xi, t-\tau)w = \tilde{S}(t, t-\tau+\xi)S(t-\tau+\xi, t-\tau)w,$$

that is, $y \in \tilde{S}(t, t-\tau+\xi)S(t-\tau+\xi, t-\tau)C_1(t)$.

Now, if $w \in C_2(t)$, then $\phi(w, t-\tau) \leq \xi$. We may write

$$y = \tilde{S}(t, t-\tau)w = \tilde{S}(t, t-\tau+\phi(w, t-\tau))\tilde{S}(t-\tau+\phi(w, t-\tau), t-\tau)w.$$

Since $z = \tilde{S}(t-\tau+\phi(w, t-\tau), t-\tau)w \in I(M)$, it follows from $(H3)$ that

$$\phi(z, t-\tau+\phi(w, t-\tau)) \geq 2\xi > \tau > \tau - \phi(w, t-\tau).$$

Thus, $y = S(t, t-\tau+\phi(w, t-\tau))\tilde{S}(t-\tau+\phi(w, t-\tau), t-\tau)w$, consequently,

$$y \in S(t, t-\tau+\phi(w, t-\tau))\tilde{S}(t-\tau+\phi(w, t-\tau), t-\tau)C_2(t).$$

In this way, we conclude that

$$G(t) = \tilde{S}(t, t-\tau+\xi)S(t-\tau+\xi, t-\tau)C_1(t)$$

$$\bigcup S(t, t-\tau+\phi(w, t-\tau))\tilde{S}(t-\tau+\phi(w, t-\tau), t-\tau)C_2(t).$$

Since $\tilde{S}(t-\tau+\phi(w, t-\tau), t-\tau)C_2(t)$ is bounded, because of Lemma 4.1, and using the fact that $S(t,s)\colon Y_0 \to Y_0$ is a compact map for each $t > s$ (see Proposition 2.6), it follows that the set

$$S(t, t-\tau+\phi(w, t-\tau))\tilde{S}(t-\tau+\phi(w, t-\tau), t-\tau)C_2(t)$$

is precompact in $Y_0$. Furthermore, since $S(t-\tau+\xi, t-\tau)C_1(t)$ is precompact in $Y_0$ by Proposition 2.6, and $0 < t-(t-\tau+\xi) = \tau-\xi < \xi$, we can apply Lemma 4.3 to guarantee that the set $\tilde{S}(t, t-\tau+\xi)S(t-\tau+\xi, t-\tau)C_1(t)$ is precompact in $Y_0$. Hence, $\overline{G(t)}$ is compact in $Y_0$, for each $t \in \mathbb{R}$, which proves our claim.

Let $\epsilon_0 > 0$ and define $K(t) = \overline{\bigcup_{0 \leq \epsilon \leq \epsilon_0} \tilde{S}(t+\epsilon, t)G(t)}$ which is compact in $Y_0$.

Now, it remains to prove that the family $\{K(t)\colon t \in \mathbb{R}\}$ pullback $\tilde{S}$-absorbs bounded subsets of $Y_0$ under the action of $\{\tilde{S}(t,s)\colon t \geq s \in \mathbb{R}\}$. To this end, let a bounded subset $B$ of $Y_0$ be given. In Lemma 4.2, we have shown that there exists $t_0 = t_0(B) \geq 0$ such that $\|\tilde{S}(t,s)W_0\|^2_{Y_0} \leq R$ for all $W_0 \in B$ and $t-s \geq t_0$. Thus,

$$\tilde{S}(t-\tau, t-\tau-r)B \subset B_0 \ \text{ for all } \ r \geq t_0.$$

Hence,

$$\tilde{S}(t+\epsilon, t-\tau-r)B = \tilde{S}(t+\epsilon, t-\tau)\tilde{S}(t-\tau, t-\tau-r)B$$
$$\subset \tilde{S}(t+\epsilon, t-\tau)B_0 = \tilde{S}(t+\epsilon, t)G(t) \subset K(t)$$

for all $r \geq t_0$ and $0 \leq \epsilon \leq \epsilon_0$. Therefore,

$$\tilde{S}(t+\epsilon, s)B \subset K(t) \quad \text{whenever} \quad t-s > t_0 + \xi.$$

In addition, since $(t+\epsilon) - (t-\tau) = \tau + \epsilon > 0$, and $R \geq \mu$, it follows by Lemma 4.1 that

$$\|\tilde{S}(t+\epsilon, t-\tau)W_0\|_{Y_0}^2 \leq \ell_R, \quad \text{for all} \quad t \in \mathbb{R} \quad \text{and all} \quad W_0 \in B_0.$$

Thus, $K(t) \subset \overline{B_{Y_0}(0, \ell_R)}$ for all $t \in \mathbb{R}$. Hence, $\bigcup_{t\in\mathbb{R}} K(t) \subset \overline{B_{Y_0}(0, \ell_R)}$ and the impulsive evolution process $\{\tilde{S}(t,s) \colon t \geq s \in \mathbb{R}\}$ is pullback $\tilde{S}$-strongly compact dissipative. □

By Theorems 4.1 and 3.3, we have the following straightforward result.

**Theorem 4.2.** *The impulsive evolution process* $\{\tilde{S}(t,s) \colon t \geq s \in \mathbb{R}\}$ *associated with the impulsive non-autonomous problem* (4.3) *has an impulsive pullback attractor* $\{\tilde{\mathbb{A}}(t) \colon t \in \mathbb{R}\}$ *such that* $\bigcup_{t \leq t_0} \tilde{\mathbb{A}}(t)$ *is bounded in* $Y_0$ *for all* $t_0 \in \mathbb{R}$.

## 4.2 Upper semicontinuity

The aim of this section is to study the robustness of the family $\{\tilde{\mathbb{A}}_{(\epsilon)}(t) \colon t \in \mathbb{R}\}$ of impulsive pullback attractors, associated with the impulsive non-autonomous problem (4.1), as the parameter $\epsilon \in [0,1]$ approaches zero.

Let us assume that conditions $(A1), (A3), (A4), (A5), (A6), (A7)$ and $(A8)$ from Section 3.6 hold. In addition, we also assume that the collective condition (T) holds.

By Theorem 2.9, condition $(A2)$ is already verified. Moreover, by Proposition 2.6, the condition $(A9)$ also holds. According to the proof of Theorem 4.1, we can conclude that $\bigcup_{t\in\mathbb{R}} \tilde{\mathbb{A}}_{(\epsilon)}(t) \subset \bigcup_{t\in\mathbb{R}} K_{(\epsilon)}(t) \subset \overline{B_{Y_0}(0, \ell_R)}$, for all $\epsilon \in [0,1]$, with $R$ independent of $\epsilon$. Thus, we may assume that $\bigcup\limits_{\epsilon\in[0,1]} \bigcup\limits_{t\in\mathbb{R}} \tilde{\mathbb{A}}_{(\epsilon)}(t)$ is bounded in $Y_0$.

Under these previous conditions and according to Theorem 3.6, we can state the following upper semicontinuity result.

**Theorem 4.3.** *The family* $\{\tilde{\mathbb{A}}_{(\epsilon)}(t) \colon t \in \mathbb{R}\}$ *of impulsive pullback attractors associated with the impulsive non-autonomous problem* (4.1) *is upper semicontinuous at* $\epsilon = 0$.

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