UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIENCIAS EXATAS E TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Boa postura analítica e Gevrey da "boa" equação de Boussinesq

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Tese apresentada ao Programa de Pós-Graduação em Matemática da UFSCar como parte dos requisitos necessários para a obtenção do título de Doutor em Matemática.

Orientador: Prof. Dr. Rafael Fernando Barostichi

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Àquele para o qual
a minha vida é dedicada.
Ao meu esposo,
Rafael Figueira.

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Porque baixou os olhos para a humilde condição da sua escrava..." (Lucas 1; 46-48).

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"Talis vita, finis ita."
IMITATIONE CHRISTI

Resumo

Em ambos, círculo e reta, este trabalho tem como finalidade demonstrar que o problema de Cauchy para a "boa" equação de Boussinesq é localmente bem posto em uma classe de funções Gevrey, a qual inclui classes de funções analíticas que podem ser estendidas holomorficamente em uma faixa simétrica no plano complexo em torno do eixo-x. Além disso, informações a respeito da regularidade da solução na variável temporal serão obtidas.

Abstract

In both the line and the circle, we shall to prove that the Cauchy problem for the "good" Boussinesq equation is locally well-posed in a class of Gevrey functions, which includes a class of analytic functions that can be extended holomorphically in a symmetric strip of the complex plane around the x-axis. Additionally, information about the regularity of the solution in the time variable shall be provided.

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Introduction

In 1834, while on horseback along side a narrow barge channel in Edinburgh, John Scott Russell observed a smooth wave that propagated along the channel preserving its shape and speed. Intrigued, the young scientist followed the wave on horseback as it rolled on at about eight or nine miles an hour, but after a chase of one or two miles he lost it. He challenged the mathematical community of the day to explain the phenomenon. In the following, we have his proper words explaining what he saw:

> *"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation. "*
>
> John Scott Russel

Some years of Russel's life was devoted to replicate the Wave of Translation. A 30-foot basin was built by him to test different theories. The discovers of these experiments was presented at a British Science Association meeting in Edinburgh, where was described the waves and the mechanics behind them. Thus began a range of studies and investigations regarding solitary waves. Throughout the next 30 years, the solitons caught attention of the scientific community, these waves were extensively studied appearing in the mathematical activity with application for optics, acoustics, quantum mechanics, oceanography, astrophysics, and others.

In 1872, Joseph Boussinesq proposed the following equation

$$u_{tt} - u_{xx} - u_{xxxx} + (u^2)_{xx} = 0 \tag{1}$$

to describe the propagation of long waves with small amplitude on the surface of water [8]. It possesses soliton traveling wave solutions

$$u(x,t) = \frac{3}{2}\left(c^2-1\right)\operatorname{sech}^2\left[\frac{\sqrt{c^2-1}}{2}(x-ct)\right],$$

where $u(x,t)$ and c are the amplitude and the speed of the wave, respectively, thus providing for the first time mathematical evidence in favor of John Scott Russell's observation of the solitary wave.

In the studies of the Russel's observations others famous equations appeared. We would like to mention the famous KdV equation, which was developed in 1895 by Diederik Korteweg and Gustav de Vries [28], where they expanded Boussinesq's work. The KdV equation is given as follows

$$u_t + u_{xxx} + uu_x = 0.$$

Just KdV alone has a long and celebrated history. In this work, we are going to focus on the Boussinesq equation, more precisely in the "good" one. The equation (1) is nowadays known as "bad" Boussinesq equation. The "badness" of the equation lies in the fact that the corresponding initial value problem is ill-posed.

The mathematical term well-posed problem (or ill-posed problem) stems from a definition given by Jacques Hadamard. He believed that mathematical Cauchy problems of models of physical phenomena should satisfy the following:

1. There is a solution that has the same regularity of the initial data.

2. The solution is unique.

3. The data-to-solution map depends continuously on the initial data.

If an initial value problem satisfies the three items above, then the problem is called well-posed.

The ill-posedness of (1) can be seen, for example, by seeking small amplitude solutions of the form

$$u(x,t) = \varepsilon e^{-ikx-i\omega t},\ \omega^2 = k^2 - k^4,\ k \in \mathbb{R} \text{ and } \varepsilon \ll 1, \tag{2}$$

such that the nonlinear term of (1) is negligible. We observe that even the small amplitude family of solutions (2) grows exponentially with time for $|k| > 1$, since the time frequency ω is imaginary and the wave amplitude is a rate of about $e^{k^2 t}$. In other words, the small initial data $u_0(x) = \varepsilon e^{-ikx}$ immediately evolves to the exponentially large solution, which shows that the third item of the well-posedness definition fails.

One way of solving the issue of ill-posedness is to change the sign of the fourth derivative in (1) from negative to positive resulting in the following equation

$$\partial_t^2 u - \partial_x^2 u + \partial_x^4 u + \partial_x^2(u^2) = 0, \tag{3}$$

which has been suggested by Zakharov [38] as a model of nonlinear vibrations along a string, and also by Turitsyn [37] for describing electromagnetic waves in nonlinear dielectric materials. The dispersion relation is now given by $\omega^2 = k^2 + k^4$, implying that $\omega \in \mathbb{R}$, for all $k \in \mathbb{R}$. Since (3) has results of well-posedness, it is known as the "good" Boussinesq equation. Zakharov [38] has shown that both the "bad" and the "good" Boussinesq equations are integrable. Furthermore, the initial value problem of these equations was analyzed via inverse scattering techniques by Deift, Tomei and Trubowitz [10].

In this work, we consider the initial value problem for the "good" Boussinesq (gB) equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u + \partial_x^2(u^2) = 0, & x \in \mathbb{R} \text{ or } \mathbb{T},\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \\ \partial_t u(x,0) = \partial_x u_1(x), \end{cases} \tag{4}$$

and study its local well-posedness for initial data in Gevrey spaces on both the line and the circle. These include spaces of analytic functions that can be extended holomorphically in a symmetric strip of the complex plane around the x-axis. There is a reason to consider the second initial data as a derivative of a square-integrable function, which is regarding to the Hamiltonian functional related to the equation (for more details see [5] and [29]). Before stating our results precisely we shall recall a few results about this Cauchy problem.

The local well-posedness of the Cauchy problem for the "good" Boussinesq equation (4) in Sobolev spaces has a relatively recent history. Bona and Sachs [5] studied the well-posedness for Boussinesq type equations given by the same expression as in (4), where the nonlinearity u^2 is replaced by a C^∞ function $f(u)$. They showed that the Cauchy problem for such equations is well-posed for initial data $u_0 \in H^{s+2}(\mathbb{R})$ and $u_1 \in H^{s+1}(\mathbb{R})$, with $s > 1/2$, and the solution u satisfies, for some $T > 0$,

$$u \in C([0,T]; H^{s+2}(\mathbb{R})) \cap C^1([0,T]; H^s(\mathbb{R})) \cap C^2([0,T]; H^{s-2}(\mathbb{R})).$$

They also proved that the solitary wave solutions to these equations are nonlinearly stable for a range of their phase speeds, which leads to the conclusion that initial data lying close to a stable solitary wave evolves into a global solution of these equations. These results were improved by Linares [29], who proved the local well-posedness for the Cauchy problem (4) with u^2 replaced by $|u|^\alpha u$, $0 < \alpha < 4$, and with initial data $(u_0, u_1) \in H^0(\mathbb{R}) \times H^{-1}(\mathbb{R})$. He also proved that, for small initial data $(u_0, u_1) \in H^1(\mathbb{R}) \times H^0(\mathbb{R})$, the solution is actually global in time and is in $H^1(\mathbb{R})$ in space variable. We point out that these results hold true for the "good" Boussinesq equation.

Farah [12] improved the local well-posedness results above for the "good" Boussinesq equation by proving that the Cauchy problem (4) is locally well-posed when $(u_0, u_1) \in H^s(\mathbb{R}) \times$

$H^{s-1}(\mathbb{R})$ and $s > -1/4$. One of the main ideas in his proof is to define suitable to the linear part of the equation Bourgain type spaces and using them to derive the appropriate bilinear estimates. More precisely, for $s, b \in \mathbb{R}$ he defines the weighted Sobolev spaces $X_{s,b}$ as the completion of the Schwartz class in $\mathbb{R}^2$ with respect to the norm

$$\|u\|_{X_{s,b}} = \|(1+|\tau-\gamma(\xi)|)^b(1+|\xi|)^s\tilde{u}\|_{L^2_{\tau,\xi}},$$

where $\gamma(\xi) = \sqrt{\xi^2+\xi^4}$ and $\tilde{u}$ denotes the time-space Fourier transform of u. Similar results were proved by Farah and Scialom [13] in the periodic case.

This type of spaces was first introduced by Bourgain [6] and [7] in the studies of nonlinear dispersive wave problems. The Bourgain spaces turn out to be appropriated spaces to establish a fixed point argument. In this work, we will mainly use these spaces in order to prove the well-posedness results.

Let us now define the spaces needed for describing our results. We begin with the spaces of analytic Gevrey functions that our initial data will belong to. For $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, we define the spaces

$$G^{\sigma,\delta,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R});\ \|f\|^2_{G^{\sigma,\delta,s}(\mathbb{R})} = \int \langle\xi\rangle^{2s} e^{2\delta|\xi|^{1/\sigma}} |\hat{f}(\xi)|^2 \, d\xi < \infty \right\}, \tag{5}$$

where $\langle\xi\rangle \doteq \sqrt{1+\xi^2}$. We observe that these spaces satisfy the following

$$G^{\sigma,\delta,s}(\mathbb{R}) \subset H^{s'}(\mathbb{R}), \quad \text{for all } s' \in \mathbb{R}. \tag{6}$$

In addition, if $\varphi \in G^{\sigma,\delta,s}(\mathbb{R})$, then φ belongs to the Gevrey class $\mathbf{G}^\sigma(\mathbb{R})$. In the case when $\sigma = 1$, we denote $G^{\delta,s}(\mathbb{R}) \equiv G^{1,\delta,s}(\mathbb{R})$. Thus, if $\varphi \in G^{\delta,s}(\mathbb{R})$ then φ is analytic on the line and admits a holomorphic extension $\tilde{\varphi}$ on the strip $S_\delta \doteq \{x + iy;\ |y| < \delta\}$.

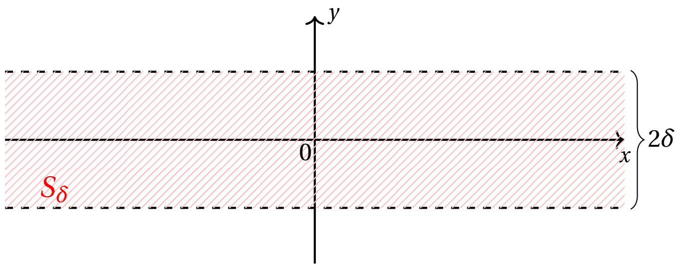

Figure 1: Strip around x-axis.

Hence, in this context, we refer to the parameter $\delta > 0$ as the uniform radius of analyticity of the function φ.

For the periodic case, the norm in the space $G^{\sigma,\delta,s}(\mathbb{T})$ is defined by simply replacing the integral in (5) with a sum as follows

$$G^{\sigma,\delta,s}(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T});\ \|f\|^2_{G^{\sigma,\delta,s}(\mathbb{T})} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} e^{2\delta|k|^{1/\sigma}} |\hat{f}(k)|^2 < \infty \right\}.$$

The above remarks remain true in this case and, moreover, one can easily see that if $\varphi \in G^{\sigma}(\mathbb{T})$, then there exists $\delta > 0$ such that $\varphi \in G^{\sigma,\delta,s}(\mathbb{T})$, for all $s \in \mathbb{R}$.

Next, we introduce the spaces needed for stating our main results precisely. Motivated by [12], we define the analytic Gevrey version of Bourgain spaces that are appropriate for the gB equation. For $s, b \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, we denote by $X_{\sigma,\delta,s,b}(\mathbb{R})$ the completion of the Schwartz class $\mathscr{S}(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{R})} = \left(\iint e^{2\delta|\xi|^{1/\sigma}} \langle |\tau| - \gamma(\xi) \rangle^{2b} \langle \xi \rangle^{2s} |\widehat{u}(\xi,\tau)|^2 \, \mathrm{d}\tau \, \mathrm{d}\xi \right)^{\frac{1}{2}}, \tag{7}$$

where $\gamma(\xi) = \sqrt{\xi^2 + \xi^4}$. In the periodic case, these spaces are defined as the completion of the space of the functions defined on $\mathbb{T} \times \mathbb{R}$ which are in the Schwartz class in time variable and are smooth in space variable, with respect to the norm

$$\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{T})} = \left(\sum_{n \in \mathbb{Z}} \int e^{2\delta|n|^{1/\sigma}} \langle |\tau| - \gamma(n) \rangle^{2b} \langle n \rangle^{2s} |\widehat{u}(n,\tau)|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}}. \tag{8}$$

We often omit $\mathbb{R}$ or $\mathbb{T}$ in the notation of these spaces when it is clear by the context which one is being considered or when the statement holds for both.

Also, for any $T \geq 0$, $X^T_{\sigma,\delta,s,b}$ denotes the localized space endowed with the norm

$$\|u\|_{X^T_{\sigma,\delta,s,b}} = \inf_{\tilde{u} \in X_{\sigma,\delta,s,b}} \left\{ \|\tilde{u}\|_{X_{\sigma,\delta,s,b}} ; \ \tilde{u}(\cdot,t) = u(\cdot,t) \ \text{for all} \ t \in [0,T] \right\}. \tag{9}$$

Furthermore, a very important property of the Bourgain space $X_{\sigma,\delta,s,b}$ is that these spaces are continuously included in the Hadamard space $C\left([0,T], G^{\sigma,\delta,s}\right)$, for every $T > 0$.

Now, we are ready to state our first main result that happens in both the real and the periodic case, which reads as follows.

Theorem 1. *Let $s > -1/4$, $\delta > 0$ and $\sigma \geq 1$. Then, for initial data $(u_0, u_1) \in G^{\sigma,\delta,s} \times G^{\sigma,\delta,s-1}$, there exist a lifespan*

$$T = T(u_0, u_1) = \frac{c_0}{(1 + \|u_0\|_{G^{\sigma,\delta,s}} + \|u_1\|_{G^{\sigma,\delta,s-1}})^{\alpha}}, \tag{10}$$

where $\alpha > 1$ and $0 < c_0 \leq 1$ are constants that depend only on s, and a unique solution u of the Cauchy problem for the "good" Boussinesq equation (4) *such that $u \in C\left([0,T]; G^{\sigma,\delta,s}\right) \cap X^T_{\sigma,\delta,s,b}$. Moreover, the data-to-solution map is locally Lipschitz.*

Our next goal is to study the time regularity of the solution established in Theorem 1, which is motivated by the works [20], [21] and [22] on time regularity of solutions to KdV type equations with analytic Gevrey data. Although the local solution to the Cauchy problem with analytic initial data is analytic in the space variable (see Trubowitz [36] for the periodic case and T. Kato [23], T. Kato and Masuda [24] and K. Kato and Ogawa [25] for the non-periodic case), it may lose regularity in time. However, for initial data in the Gevrey class $G^{\sigma,\delta,s}$, it is proved in [20] that the solution of the periodic higher dispersion KdV equation is Gevrey of order $m\sigma$, where m is the order of the dispersive term.

Looking at the linear part of the gB equation, that is, the equation $\partial_t^2 u - \partial_x^2 u + \partial_x^4 u$ and ignoring the term $\partial_x^2 u$, we see that two time derivatives are equal to four space derivatives, which implies that if the solution is Gevrey of order σ in space variable, then it is going to be Gevrey of order 2σ in time variable. Our second main result says that the same happens for the gB equation.

Theorem 2. *Let $s > -1/4$, $\delta > 0$ and $\sigma \geq 1$. If $(u_0, u_1) \in G^{\sigma,\delta,s} \times G^{\sigma,\delta,s-1}$, then the solution $u \in C([0,T]; G^{\sigma,\delta,s})$ given by Theorem 1 belongs to the Gevrey class $G^{2\sigma}$ in time variable.*

The results presented in this thesis was published in 2019, as the reader can see in [2]. Also, other papers was produced in this period concerning other equations, see in [3], [14] and [15].

This work is organized as follows. In Chapter 1, we introduced some classic spaces and notation that will be useful in what follows. In Chapter 2, we show a detailed proof of the bilinear estimates in Bourgain spaces in the real and in the periodic cases, which are the main results to prove the wel-posedness for gB equation in Sobolev spaces in both $\mathbb{R}$ and $\mathbb{T}$. Chapter 3 is dedicated to present the proof of Theorem 1, while in Chapter 4 the proof of Theorem 2 is provided.

Chapter

1 Preliminaries

This chapter is dedicated to present important spaces that we shall use in the following chapters. Also, it is included some classical results and inequalities about these spaces. Some of the proof are omitted, but the reader can find them in the references that we will cite throughout the chapter.

1.1 The Sobolev Spaces H^s

Natural spaces to measure regularity of the initial data in Cauchy problems are the classical Sobolev spaces H^s, $s \in \mathbb{R}$, which are defined as

$$H^s(\mathbb{R}) = \left\{ f \in \mathscr{S}'(\mathbb{R});\ \|f\|^2_{H^s(\mathbb{R})} = \int_{\mathbb{R}} (1+\xi^2)^s |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi < \infty \right\},$$

in the line, where $\mathscr{S}'(\mathbb{R})$ denotes the class of tempered distributions and $\widehat{f}$ denotes the Fourier transform of f, which is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\mathrm{d}x,$$

if f is an integrable function. We will often use the notation $\langle \xi \rangle = (1+\xi^2)^{\frac{1}{2}}$, then we can write $\|f\|_{H^s} = \|\langle \xi \rangle^s \widehat{f}(\xi)\|_{L^2_\xi}$. In periodic case, we have

$$H^s(\mathbb{T}) = \left\{ f \in \mathscr{S}'(\mathbb{T});\ \|f\|^2_{H^s(\mathbb{T})} = \sum_{n\in\mathbb{Z}} (1+n^2)^s |\widehat{f}(n)|^2 < \infty \right\}.$$

Example 1.1. *Let $f(x) = \chi_{[-1,1]}(x)$. We have that*

$$\widehat{f}(\xi) = \int_{-1}^{1} e^{-ix\xi}\mathrm{d}x = \int_{-1}^{1} \left[\cos(-x\xi) + i\sin(-x\xi)\right]\mathrm{d}x = -2\frac{\sin\xi}{\xi}.$$

Thus, $f \in H^s(\mathbb{R})$ if $s < \frac{1}{2}$.

Example 1.2. *Let $\boldsymbol{\delta}$ be Dirac Delta distribution centered at the origin. Since $\widehat{\boldsymbol{\delta}} = \mathbf{2\pi}$, we have that $\boldsymbol{\delta}$ belongs to H^s for $s < -\frac{1}{2}$.*

Intuitively, a Sobolev space consists of functions with sufficiently many derivatives and equipped with a norm that measures both the size and regularity of a function.

Sobolev spaces are named after the Russian mathematician Sergei Sobolev. Their importance comes from the fact that weak solutions of some important partial differential equations exist in appropriate Sobolev spaces, even when there are no strong solutions in spaces of continuous functions with the derivatives understood in the classical sense.

From the definition of Sobolev spaces we deduce the following properties, which the proofs can be found in [30].

Proposition 1.3 (Proposition 3.1, page 46 in [30])**.**

1. *If $s < s'$, then $H^{s'}(\mathbb{R}) \subset H^s(\mathbb{R})$.*
2. *$H^s(\mathbb{R})$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_s$ defined as follows*

$$\text{If } f, g \in H^s(\mathbb{R}), \text{ then } \langle f, g \rangle_s = \int_{\mathbb{R}} (1+\xi^2)^{\frac{s}{2}} \widehat{f}(\xi) \overline{(1+\xi^2)^{\frac{s}{2}} \widehat{g}(\xi)}\, \mathrm{d}\xi .$$

3. *For any $s \in \mathbb{R}$, the Schwartz space $\mathscr{S}(\mathbb{R})$ is dense in $H^s(\mathbb{R})$.*
4. *If $s_1 \leq s \leq s_2$, with $s = \theta s_1 + (1-\theta)s_2$, $0 \leq \theta \leq 1$, then*

$$\|f\|_{H^s} \leq \|f\|_{H^{s_1}}^{\theta} \|f\|_{H^{s_2}}^{1-\theta} .$$

A really interesting fact is that for positive integer values of s, we can give a description of H^s without using the Fourier transform.

Proposition 1.4 (Theorem 3.1, page 47 in [30])**.** *If k is a positive integer, then $H^k(\mathbb{R})$ coincides with the space of functions $f \in L^2(\mathbb{R})$ whose derivatives (in the distribution sense) $f^{(j)}$ belongs to $L^2(\mathbb{R})$ for every $j \leq k$. In this case, the norms*

$$\|f\|_{H^k} \quad \text{and} \quad \sum_{j=1}^{k} \left\| f^{(j)} \right\|_{L^2}$$

are equivalent.

Furthermore, the following proposition allows us to relate "weak derivatives" with derivatives in the classical sense.

Proposition 1.5 (Embedding - Theorem 3.2, page 47 in [30])**.** *If $s > k + \frac{1}{2}$, then $H^s(\mathbb{R})$ is continuously embedding in $C^k_\infty(\mathbb{R})$, the space of functions with k continuous derivatives vanishing at infinity. In other words, if $f \in H^s(\mathbb{R})$, $s > \frac{1}{2} + k$, then (after a possible modification of f in a set of measure zero) $f \in C^k_\infty(\mathbb{R})$ and*

$$\|f\|_{C^k} \leq \|f\|_{H^s} .$$

From the point of view of nonlinear analysis the next bilinear estimate is essential.

Proposition 1.6 (Theorem 3.4, page 47 in [30]). *If $s > \frac{1}{2}$, then $H^s(\mathbb{R})$ is an algebra with respect to the product of functions. That is, if $f, g \in H^s(\mathbb{R})$, then $fg \in H^s(\mathbb{R})$ with*

$$\|fg\|_{H^s} \le \|f\|_{H^s} \|g\|_{H^s}.$$

The next important property of these spaces will be really useful in Chapter 4.

Proposition 1.7 (Sobolev Lemma). *For $s > \frac{1}{2}$, we have*

$$\|u\|_{L^\infty} \le C\|u\|_{H^s},$$

for some positive constant depending only on s.

Proof. By using the inverse Fourier transform formula, we have

$$2\pi|u(x)| = \left|\int e^{ix\xi}\widehat{u}(\xi)\mathrm{d}\xi\right| \le \int |\widehat{u}(\xi)|\mathrm{d}\xi.$$

Then, it follows from the Cauchy-Schwarz inequality that

$$2\pi|u(x)| \le \int (1+\xi^2)^{-\frac{s}{2}}(1+\xi^2)^{\frac{s}{2}}|\widehat{u}(\xi)|\mathrm{d}\xi \le \left(\int (1+\xi^2)^{-s}\mathrm{d}\xi\right)^{\frac{1}{2}}\|u\|_{H^s}. \tag{1.1}$$

Now, we observe that

$$\int (1+\xi^2)^{-s}\mathrm{d}\xi = 2\int_0^\infty (1+\xi^2)^{-s}\mathrm{d}\xi \le 2\int_0^\infty 2^s(1+\xi)^{-2s}\mathrm{d}\xi = 2^{s+1}\frac{(1+\xi)^{-2s+1}}{-2s+1}\bigg|_0^\infty = \frac{2^{s+1}}{2s-1}, \tag{1.2}$$

for all $s > \frac{1}{2}$. Therefore, we conclude from (1.1) the following bound for the L^∞-norm

$$\|u\|_{L^\infty} \le \frac{1}{\pi(2s-1)}\|u\|_{H^s}, \tag{1.3}$$

which finishes the proof. □

We finish this section with an important inequality, which the proof comes from [18].

Lemma 1.8 (Lemma 3.2, page 170 in [18]). *Let $s \in \mathbb{R}$, $-\frac{1}{2} < b' \le 0 \le b \le b'+1$ and $0 < T \le 1$, then*

$$\left\|\psi_T(t)\int_0^t g(t')\mathrm{d}t'\right\|_{H_t^b} \le CT^{1-(b-b')}\|g\|_{H^{b'}}, \tag{1.4}$$

where ψ is a cut-off function in $C_0^\infty(-2,2)$ with $0 \le \psi \le 1$, $\psi(t) = 1$ on $[-1,1]$ and $\psi_T(t) = \psi\left(\frac{t}{T}\right)$.

Proof. To prove (1.4), we write

$$\psi_T(t)\int_0^t g(t')\,\mathrm{d}t' = \psi_T(t)\frac{1}{2\pi}\int_0^t\int e^{it'\tau}\widehat{g}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t'.$$

By using Fubini's Theorem, we have

$$\psi_T(t)\int_0^t g(t')\,\mathrm{d}t' = \psi_T(t)\frac{1}{2\pi}\int \widehat{g}(\tau)\int_0^t e^{it'\tau}\,\mathrm{d}t'\,\mathrm{d}\tau = \psi_T(t)\frac{1}{2\pi}\int \widehat{g}(\tau)\left(\frac{e^{it\tau}-1}{i\tau}\right)\mathrm{d}\tau\,.$$

Now, we split the integral in two regions $|\tau|T \le 1$ and $|\tau|T > 1$ and obtain

$$\begin{aligned}\psi_T(t)\int_0^t g(t')\,\mathrm{d}t' &= \psi_T(t)\frac{1}{2\pi}\int\limits_{|\tau|T\le 1} \widehat{g}(\tau)\left(\frac{e^{it\tau}-1}{i\tau}\right)\mathrm{d}\tau + \psi_T(t)\frac{1}{2\pi}\int\limits_{|\tau|T>1} \widehat{g}(\tau)\left(\frac{e^{it\tau}-1}{i\tau}\right)\mathrm{d}\tau\\ &= \text{(I)} + \text{(II)} + \text{(III)},\end{aligned}$$

with

$$\begin{aligned}\text{(I)} &= \psi_T(t)\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{t^k}{k!}\int\limits_{|\tau|T\le 1}(i\tau)^{k-1}\widehat{g}(\tau)\mathrm{d}\tau,\\ \text{(II)} &= -\psi_T(t)\frac{1}{2\pi}\int\limits_{|\tau|T>1}\widehat{g}(\tau)(i\tau)^{-1}\mathrm{d}\tau,\\ \text{(III)} &= \psi_T(t)\frac{1}{2\pi}\int\limits_{|\tau|T>1}\widehat{g}(\tau)e^{it\tau}(i\tau)^{-1}\mathrm{d}\tau,\end{aligned}$$

where we used the fact

$$e^{it\tau}-1 = \sum_{k=1}^{\infty}\frac{(it\tau)^k}{k!}.$$

The next step is to estimate the H_t^b-norm of each term above.

Estimation of (I). If $|\tau|T \le 1$, then $|\tau|^{k-1} \le (T^{-1})^{k-1}$ for all $k \ge 1$. Thus, we have

$$\begin{aligned}\|\text{(I)}\|_{H_t^b} &\le \sum_{k=1}^{\infty}\frac{1}{k!}\left\| t^k\psi_T(t)\int\limits_{|\tau|T\le 1}(i\tau)^{k-1}\widehat{g}(\tau)\mathrm{d}\tau\right\|_{H_t^b}\\ &\le \sum_{k=1}^{\infty}\frac{1}{k!}T^{1-k}\left\| t^k\psi_T(t)\int\limits_{|\tau|T\le 1}\widehat{g}(\tau)\mathrm{d}\tau\right\|_{H_t^b}\\ &= \sum_{k=1}^{\infty}\frac{T^{1-k}}{k!}\left|\int\limits_{|\tau|T\le 1}\widehat{g}(\tau)\mathrm{d}\tau\right|\left\| t^k\psi_T(t)\right\|_{H_t^b}.\end{aligned}$$

Using Cauchy-Schwarz inequality, we observe that

$$\left|\int\limits_{|\tau|T\le 1}\widehat{g}(\tau)\mathrm{d}\tau\right| = \left|\int\limits_{|\tau|T\le 1}\widehat{g}(\tau)\langle\tau\rangle^{b'}\langle\tau\rangle^{-b'}\mathrm{d}\tau\right| \le \|g\|_{H^{b'}}\left(\int\limits_{|\tau|T\le 1}\langle\tau\rangle^{-2b'}\mathrm{d}\tau\right)^{\frac{1}{2}},$$

which implies

$$\|\text{(I)}\|_{H_t^b} \le \|g\|_{H^{b'}}\sum_{k=1}^{\infty}\frac{T^{1-k}}{k!}\left\|t^k\psi_T\right\|_{H^b}\left(\int\limits_{|\tau|T\le 1}\langle\tau\rangle^{-2b'}\mathrm{d}\tau\right)^{\frac{1}{2}}.$$

For our goal, it is sufficient to prove

$$\sum_{k=1}^{\infty} \frac{T^{1-k}}{k!} \left\| t^k \psi_T \right\|_{H^b} \left(\int\limits_{|\tau| T \leq 1} \langle \tau \rangle^{-2b'} \, \mathrm{d}\tau \right)^{\frac{1}{2}} \leq C T^{1-(b-b')}. \tag{1.5}$$

We have $\| t^k \psi_T \|_{H^b} = \left\| \langle \tau \rangle^b \, \widehat{t^k \psi_T}(\tau) \right\|_{L^2}$ with

$$\widehat{t^k \psi_T}(\tau) = \int e^{-it\tau} t^k \psi_T(t) \, \mathrm{d}t = \int e^{-it\tau} t^k \psi \left(\frac{t}{T} \right) \mathrm{d}t.$$

By making the change of variables $t = Ts$, we obtain

$$\widehat{t^k \psi_T}(\tau) = \int e^{-i(Ts)\tau} (Ts)^k \psi(s) T \, \mathrm{d}s = T^{k+1} \int e^{-is(T\tau)} s^k \psi(s) \, \mathrm{d}s,$$

that is, $\widehat{t^k \psi_T}(\tau) = T^{k+1} \left[\widehat{t^k \psi}(T\tau) \right]$. Thus,

$$\left\| t^k \psi_T \right\|_{H^b} = T^{k+1} \left(\int \langle \tau \rangle^{2b} \left| \widehat{t^k \psi}(T\tau) \right|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} = T^{k+1} \left(\int \left\langle T^{-1} \rho \right\rangle^{2b} \left| \widehat{t^k \psi}(\rho) \right|^2 T^{-1} \mathrm{d}\rho \right)^{\frac{1}{2}},$$

where we made the change of variables $\rho = T\tau$. Since $T^{-2} \geq T^{-1} \geq 1$, we observe that

$$\left\langle T^{-1} \rho \right\rangle^2 = 1 + T^{-2} \rho^2 \leq T^{-2}(1 + \rho^2) = T^{-2} \left\langle \rho \right\rangle^2,$$

which implies the following

$$\left\| t^k \psi_T \right\|_{H_b} \leq T^{k+1} T^{-\frac{1}{2}} \left(\int T^{-2b} \left\langle \rho \right\rangle^{2b} \left| \widehat{t^k \psi}(\rho) \right|^2 \mathrm{d}\rho \right)^{\frac{1}{2}} = T^{k+\frac{1}{2}} T^{-b} \left\| t^k \psi \right\|_{H^b}. \tag{1.6}$$

Using that $b \leq 1 + b' \leq 1$, we obtain

$$\left\| t^k \psi \right\|_{H^b} \leq \left\| t^k \psi \right\|_{H^1} = \left\| t^k \psi \right\|_{L^2} + \left\| \frac{d}{dt} (t^k \psi) \right\|_{L^2} = \left\| t^k \psi \right\|_{L^2} + \left\| k t^{k-1} \psi \right\|_{L^2} + \left\| t^k \psi' \right\|_{L^2}.$$

Since $\operatorname{supp} \psi \subset [-2,2]$, for all $t \in \mathbb{R}$ and $k \geq 1$ we have the boundness

$$|t^k \psi(t)| \leq 2^k |\psi(t)|, \quad |k t^{k-1} \psi(t)| \leq 2^{k-1} k |\psi(t)| \quad \text{and} \quad |t^k \psi'(t)| \leq 2^k |\psi'(t)|,$$

Thus,

$$\left\| t^k \psi \right\|_{H^b} \leq k 2^k \left(2 \left\| \psi \right\|_{L^2} + \left\| \psi' \right\|_{L^2} \right) = k 2^k \tilde{C}_\psi,$$

for all $k \geq 1$, which implies from (1.6) that

$$\left\| t^k \psi_T \right\|_{H^b} \leq \tilde{C}_\psi T^{-b} T^{k+\frac{1}{2}} k 2^k.$$

Then,

$$\sum_{k=1}^{\infty} \frac{T^{1-k}}{k!} \left\| t^k \psi \right\|_{H^b} \leq \tilde{C}_\psi T^{1-b} T^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{2^k}{(k-1)!} \leq C_\psi T^{1-b} T^{\frac{1}{2}},$$

where we used that

$$\sum_{k=1}^{\infty} \frac{2^k}{(k-1)!} = 2e^2 < \infty.$$

On the other hand, since $b' \leq 0$ and $T^{-2} \geq 1$, we have

$$\left(\int_{|\tau|T\leq 1} \langle\tau\rangle^{-2b'}\mathrm{d}\tau\right)^{\frac{1}{2}} = \left(\int_{|\tau|T\leq 1} (1+\tau^2)^{-b'}\mathrm{d}\tau\right)^{\frac{1}{2}} \leq \left(\int_{|\tau|T\leq 1} (2T^{-2})^{-b'}\mathrm{d}\tau\right)^{\frac{1}{2}}$$

$$\leq 2^{-\frac{b'}{2}} T^{b'}\left(\int_{|\tau|T\leq 1} 1\mathrm{d}\tau\right)^{\frac{1}{2}} = C_{b'}T^{b'}T^{-\frac{1}{2}}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{T^{1-k}}{k!} \left\| t^k \psi_T \right\|_{H^b} \left(\int_{|\tau|T\leq 1} \langle\tau\rangle^{-2b'}\mathrm{d}\tau\right)^{\frac{1}{2}} \leq C_\psi T^{1-b} T^{\frac{1}{2}} C_{b'} T^{b'} T^{-\frac{1}{2}} = C_{\psi,b'} T^{1-(b-b')},$$

which concludes that (1.5) goes true and finishes the desired bound for $\|(\mathrm{I})\|_{H^b_t}$.

Estimation of (II). We observe that

$$\|(\mathrm{II})\|_{H^b_t} \leq \left\| \psi_T(t) \int_{|\tau|T>1} (i\tau)^{-1}\widehat{g}(\tau)\mathrm{d}\tau \right\|_{H^b_t} = \left| \int_{|\tau|T>1} (i\tau)^{-1}\widehat{g}(\tau)\mathrm{d}\tau \right| \left\| \psi_T \right\|_{H^b}. \tag{1.7}$$

Using Cauchy-Schwarz inequality, we obtain

$$\left| \int_{|\tau|T>1} (i\tau)^{-1}\widehat{g}(\tau)\mathrm{d}\tau \right| = \left| \int_{|\tau|T>1} \langle\tau\rangle^{-b'} (i\tau)^{-1} \langle\tau\rangle^{b'} \widehat{g}(\tau)\mathrm{d}\tau \right| \leq \left(\int_{|\tau|T>1} \langle\tau\rangle^{-2b'} |\tau|^{-2}\mathrm{d}\tau \right)^{\frac{1}{2}} \left\| g \right\|_{H^{b'}},$$

with

$$\int_{|\tau|T>1} \langle\tau\rangle^{-2b'} |\tau|^{-2}\mathrm{d}\tau = 2\int_{T^{-1}}^{\infty} (1+\tau^2)^{-b'}\tau^{-2}\mathrm{d}\tau = 2\int_{1}^{\infty} (1+T^{-2}s^2)^{-b'} T^2 s^{-2} T^{-1}\mathrm{d}s$$

$$\leq T^{2b'+1}\int_{1}^{\infty} (1+s^2)^{-b'} s^{-2}\mathrm{d}s,$$

since $T^{-2} \geq 1$ and $-b' \geq 0$. Furthermore,

$$\int_{1}^{\infty} (1+s^2)^{-b'} s^{-2}\mathrm{d}s \leq \int_{1}^{\infty} (2s^2)^{-b'} s^{-2}\mathrm{d}s = 2^{-b'} \int_{1}^{\infty} s^{-2b'-2}\mathrm{d}s < \infty,$$

since $2b'+2 > 1$. Then,

$$\left(\int_{|\tau|T>1} \langle\tau\rangle^{-2b'} |\tau|^{-2}\mathrm{d}\tau \right)^{\frac{1}{2}} \left\| g \right\|_{H^{b'}} \leq C_{b'} T^{b'+\frac{1}{2}} \left\| g \right\|_{H^{b'}}. \tag{1.8}$$

On the other hand, $\left\| \psi_T \right\|_{H^b} = \left\| \langle\tau\rangle^b \widehat{\psi}_T \right\|_{L^2}$ with

$$\widehat{\psi}_T(\tau) = \int e^{-it\tau} \psi\left(\frac{t}{T}\right)\mathrm{d}t = \int e^{-isT\tau} \psi(s) T\mathrm{d}s = T\widehat{\psi}(T\tau).$$

Thus,

$$\begin{aligned}\|\psi_T\|_{H^b} &= \left(\int \langle\tau\rangle^{2b} T^2 |\widehat{\psi}(T\tau)|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} = T\left(\int \left\langle T^{-1}\rho\right\rangle^{2b} |\widehat{\psi}(\rho)|^2 T^{-1}\mathrm{d}\rho\right)^{\frac{1}{2}} \\ &\leq T^{\frac{1}{2}}\left(\int \left\langle T^{-1}\rho\right\rangle^{2b} |\widehat{\psi}(\rho)|^2 \mathrm{d}\rho\right)^{\frac{1}{2}} \leq T^{\frac{1}{2}}\left(\int T^{-2b}\left\langle\rho\right\rangle^{2b} |\widehat{\psi}(\rho)|^2 \mathrm{d}\rho\right)^{\frac{1}{2}} \\ &= T^{\frac{1}{2}-b}\|\psi\|_{H^b},\end{aligned} \tag{1.9}$$

since $\left\langle T^{-1}\rho\right\rangle^{2b} = (1+T^{-2}\rho^2)^b \leq T^{-2b}(1+\rho^2)^b = T^{-2b}\left\langle\rho\right\rangle^{2b}$. Joining inequalities (1.7), (1.8) and (1.9), we conclude that

$$\|(\mathrm{II})\|_{H_t^b} \leq C_{b'} T^{b'+\frac{1}{2}} T^{\frac{1}{2}-b} \|\psi\|_{H^b} \|g\|_{H^{b'}} = C_{\psi,b,b'} T^{1-(b-b')} \|g\|_{H^{b'}},$$

which finishes the desired estimate for $\|(\mathrm{II})\|_{H_t^b}$.

Estimation of (III). We can write $(\mathrm{III}) = (2\pi)^{-1}\psi_T(t)J(t)$, where

$$J(t) = \int_{|\tau|T>1} e^{it\tau}(i\tau)^{-1}\widehat{g}(\tau)\mathrm{d}\tau.$$

Then, we have

$$\|(\mathrm{III})\|_{H_t^b} = (2\pi)^{-1}\left\|\langle\tau\rangle^b \widehat{(\psi_T J)}\right\|_{L^2} = \left\|\langle\tau\rangle^b \widehat{\psi_T} * \widehat{J}(\tau)\right\|_{L^2}. \tag{1.10}$$

Let us consider the following useful inequality

$$\langle\tau\rangle^b \leq C\left(\left\langle\tau-y\right\rangle^b + |y|^b\right), \quad \text{for all } y\in\mathbb{R}. \tag{1.11}$$

Indeed,

$$|\tau|^2 \leq (|\tau-y|+|y|)^2 = |\tau-y|^2 + 2|\tau-y||y| + |y|^2 \leq 2(|\tau-y|^2+|y|^2),$$

since $2|\tau-y||y| \leq |\tau-y|^2+|y|^2$. Thus, using that $b\geq 0$, we obtain

$$\langle\tau\rangle^b = (1+|\tau|^2)^{\frac{b}{2}} \leq \left(1+2(|\tau-y|^2+|y|^2)\right)^{\frac{b}{2}} \leq 2^{\frac{b}{2}}(1+|\tau-y|^2+|y|^2)^{\frac{b}{2}}.$$

To prove (1.11), it is sufficient to guarantee that the following goes true

$$(\alpha+\beta)^{\frac{b}{2}} \leq \alpha^{\frac{b}{2}} + \beta^{\frac{b}{2}}, \quad \text{for all } \alpha,\ \beta\geq 0. \tag{1.12}$$

If $\alpha = 0$, then (1.12) is true as an equality. We assume $\alpha\neq 0$, then (1.12) occurs if, and only if,

$$\left(1+\frac{\beta}{\alpha}\right)^{\frac{b}{2}} \leq 1+\left(\frac{\beta}{\alpha}\right)^{\frac{b}{2}}.$$

Considering the function $f(t) = 1+t^{\frac{b}{2}}-(1+t)^{\frac{b}{2}}$ and noticing that $f(0)=0$ and

$$f'(t) = \frac{b}{2}\left(t^{\frac{b}{2}-1}-(1+t)^{\frac{b}{2}-1}\right) \geq 0, \quad \text{for all } t\geq 0,$$

where we used that $0\leq\frac{b}{2}<1$, we have $f(t)\geq 0$ for all $t\geq 0$. Therefore, (1.12) is proved and also (1.11).

Using (1.11), we obtain

$$\begin{aligned}\langle\tau\rangle^b\widehat{\psi_T}*\widehat{J}(\tau)&=\int\langle\tau\rangle^b\widehat{J}(\tau-y)\widehat{\psi_T}(y)\mathrm{d}y\\&\le C\int\left[\langle\tau-y\rangle^b\widehat{J}(\tau-y)\widehat{\psi_T}(y)+\widehat{J}(\tau-y)|y|^b\widehat{\psi_T}(y)\right]\mathrm{d}y\\&=C\left[\left(\langle\tau\rangle^b\widehat{J}\right)*\widehat{\psi_T}+\widehat{J}*\left(|\tau|^b\widehat{\psi_T}\right)\right].\end{aligned}$$

Hence, returning to (1.10) and applying Young inequality, we have

$$\begin{aligned}\|(\mathrm{III})\|_{H^b}&\le C\left(\left\|\left(\langle\tau\rangle^b\widehat{J}\right)*\widehat{\psi_T}\right\|_{L^2}+\left\|\widehat{J}*\left(|\tau|^b\widehat{\psi_T}\right)\right\|_{L^2}\right)\\&\le C\left(\|\psi_T\|_{L^1}\left\|\langle\tau\rangle^b\widehat{J}\right\|_{L^2}+\left\||\tau|^b\widehat{\psi}_T\right\|_{L^1}\|\widehat{J}\|_{L^2}\right).\end{aligned}$$

Denoting by χ the characteristic function of the set $\{\tau\in\mathbb{R};\ |\tau|T>1\}$, we have

$$J(t)=\int e^{it\tau}\left(\chi(\tau)(i\tau)^{-1}\widehat{g}(\tau)\right)\mathrm{d}\tau=\mathscr{F}_t^{-1}\left(\chi(\tau)(i\tau)^{-1}\widehat{g}(\tau)\right)(t),$$

where $\mathscr{F}_t^{-1}$ denotes the inverse Fourier transform on t variable, which shows that $\widehat{J}(\tau)=\chi(\tau)(i\tau)^{-1}\widehat{g}(\tau)$. Thus,

$$\begin{aligned}\left\|\langle\tau\rangle^b\widehat{J}(\tau)\right\|_{L^2}&=\left\|\langle\tau\rangle^b\chi(\tau)(i\tau)^{-1}\widehat{g}(\tau)\right\|_{L^2}=\left(\int_{|\tau|T>1}|\tau|^{-2}\langle\tau\rangle^{2b}|\widehat{g}(\tau)|^2\mathrm{d}\tau\right)^{\frac12}\\&=\left(\int_{|\tau|T>1}|\tau|^{-2}\langle\tau\rangle^{2(b-b')}\langle\tau\rangle^{2b'}|\widehat{g}(\tau)|^2\mathrm{d}\tau\right)^{\frac12}\le\left(\sup_{|\tau|T>1}|\tau|^{-2}\langle\tau\rangle^{2(b-b')}\right)^{\frac12}\|g\|_{H^{b'}}.\end{aligned}$$

It follows from $0\le b-b'\le1$ and $T^{-1}\ge1$ that

$$\begin{aligned}|\tau|^{-2}\langle\tau\rangle^{2(b-b')}&=|\tau|^{-2}(1+\tau^2)^{b-b'}=\left(|\tau|^{-2(b-b')^{-1}}+|\tau|^{2(1-(b-b')^{-1})}\right)^{b-b'}\\&\le\left(T^{2(b-b')^{-1}}+T^{-2(1-(b-b')^{-1})}\right)^{b-b'}\\&=T^2(1+T^{-2})^{b-b'}\le2^{b-b'}T^{2(1-(b-b'))},\end{aligned}$$

for all $|\tau|>T^{-1}$. Hence, we obtain

$$\left\|\langle\tau\rangle^b\widehat{J}(\tau)\right\|_{L^2}\le C_{b,b'}T^{1-(b-b')}\|g\|_{H^{b'}}.$$

Furthermore, recalling that $\widehat{\psi_T}(t)=T\widehat{\psi}(T\tau)$, we have

$$\|\widehat{\psi_T}\|_{L^1}=\int|\widehat{\psi}(T\tau)|T\mathrm{d}\tau=\int|\widehat{\psi}(s)|\mathrm{d}s=\|\widehat{\psi}\|_{L^1}.$$

Therefore,

$$\|\widehat{\psi_T}\|_{L^1}\left\|\langle\tau\rangle^b\widehat{J}(\tau)\right\|_{L^2}\le C_{\psi,b,b'}T^{1-(b-b')}\|g\|_{H^b}.$$

Now, it remains to estimate $\||\tau|^b\widehat{\psi_T}\|_{L^1}\|\widehat{J}\|_{L^2}$. We notice that

$$\left\||\tau|^b\widehat{\psi_T}\right\|_{L^1}=\int|\tau|^b|\widehat{\psi}(T\tau)|T\mathrm{d}\tau=\int|T^{-1}s|^b|\widehat{\psi}(s)|\mathrm{d}s=T^{-b}\int|s|^b|\widehat{\psi}(s)|\mathrm{d}s=C_{\psi,b}T^{-b},$$

since $\widehat{\psi} \in \mathscr{S}$ and then

$$\int |s|^b |\widehat{\psi}(s)| \mathrm{d}s < \infty, \quad \text{for all } s \in \mathbb{R}.$$

On the other hand, we recall that $\widehat{J}(\tau) = \chi(\tau)(i\tau)^{-1}\widehat{g}(\tau)$, which give us

$$\begin{aligned}\|\widehat{J}\|_{L^2} &= \left(\int\limits_{|\tau|T>1} |\tau|^{-2} |\widehat{g}(\tau)|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} = \left(\int\limits_{|\tau|T>1} |\tau|^{-2} \langle\tau\rangle^{-2b'} \langle\tau\rangle^{2b'} |\widehat{g}(\tau)|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \\ &\le \left(\sup_{|\tau|T>1} |\tau|^{-2} \langle\tau\rangle^{-2b'} \right)^{\frac{1}{2}} \|g\|_{H^{b'}}.\end{aligned}$$

Since $0 \le -b' < 1$ and $T^{-1} \ge 1$, we obtain

$$\begin{aligned}|\tau|^{-2} \langle\tau\rangle^{-2b'} &= |\tau|^{-2}(1+\tau^2)^{-b'} = \left(\tau^{2(b')^{-1}} + \tau^{2(1+(b')^{-1})}\right)^{-b'} \\ &\le \left(T^{-2(b')^{-1}} + T^{-2(1+(b')^{-1})}\right)^{-b'} \\ &= T^2(1+T^{-2})^{-b'} \le 2^{-b'} T^{2(1+b')},\end{aligned}$$

for all $|\tau| > T^{-1}$. Therefore,

$$\left\| |\tau|^b \widehat{\psi_T} \right\|_{L^1} \|\widehat{J}\|_{L^2} \le C_{\psi,b} T^{-b} 2^{-\frac{b'}{2}} T^{1+b'} = C_{\psi,b,b'} T^{1-(b-b')},$$

which finishes the desired bound for (III) and, consequently, concludes the proof of the lemma. □

1.2 The Gevrey Spaces $\mathbf{G}^\sigma$

Let us recall here the definition of the Gevrey classes, which play the role of intermediate spaces between the space of the C^∞ and analytic functions. Given Ω, open set of $\mathbb{R}$, we say that $u \in \mathbf{G}^\sigma(\Omega)$, $\sigma \ge 1$, if $u \in C^\infty(\Omega)$ and for every compact subset K of Ω we have

$$\sup_{x \in K} |u^{(j)}(x)| \le C^{j+1}(j!)^\sigma, \text{ for all } j \in \mathbb{Z}_+,$$

for a constant C depending only on u and K. When $\sigma = 1$, we recapture the analytic case, whereas for $\sigma > 1$ we obtain larger spaces, containing functions with compact support.

Example 1.9. *For given $\sigma > 1$, we define on $\mathbb{R}$ the function*

$$\psi(x) = \begin{cases} e^{-x^{-d}}, & x > 0, \\ 0, & x \le 0, \end{cases} \qquad d = \frac{1}{\sigma - 1}.$$

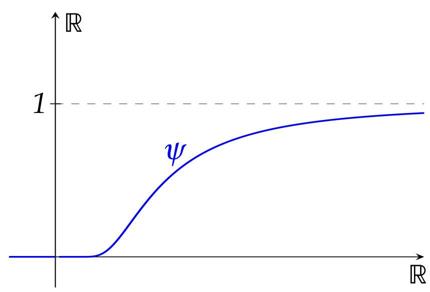

Figure 2: Example of Gevrey function.

One can prove that ψ is a Gevrey function of order σ on $\mathbb{R}$. Furthermore, for $r > 0$, the function φ

defined by $\varphi(x) = \psi(r+x)\psi(r-x)$ *belongs to the Gevrey class* $G^\sigma(\mathbb{R})$ *and is compactly supported on* $[-r, r]$.

It is interesting to observe that if $u \in \mathbf{G}_0^\sigma(\mathbb{R})$, that is, $u \in \mathbf{G}^\sigma(\mathbb{R})$ has compact support, then its Fourier transform $\widehat{u}(\xi)$ satisfies the estimates

$$|\widehat{u}(\xi)| \le Ce^{-\varepsilon|\xi|^{\frac{1}{\sigma}}}, \ \xi \in \mathbb{R}, \tag{1.13}$$

for some positive constants C and ε. Actually, the estimate (1.13) characterize the G^σ-regularity of a Fourier transformable function, or distribution. Observe that (1.13) implies

$$\left\| e^{\delta|\xi|^{\frac{1}{\sigma}}} \widehat{u}(\xi) \right\|_{L^2}^2 = \int e^{2\delta|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi)|^2 \mathrm{d}\xi < \infty,$$

for sufficiently small $\delta > 0$. This fact motivates the definition of the Gevrey-Bourgain space $G^{\sigma,\delta,s}$, which will be considered in the next section.

For more informations and details about these class of functions, we recommend the read of the book [33], where the proof of the next properties can be found.

Proposition 1.10 (Proposition 1.4.5, page 21 in [33])**.** *The space* $\mathbf{G}^\sigma(\Omega)$ *is a vector space and a ring, with respect to the arithmetic product of functions. Moreover,* $\mathbf{G}^\sigma(\Omega)$ *is closed under differentiation.*

Just out of curiosity, the topology considered in the class $\mathbf{G}^\sigma(\mathbb{R})$ is a projective limit topology, which give us the following convergence: A sequence $\varphi_k \in \mathbf{G}^\sigma(\mathbb{R})$, $k = 1, 2, \ldots$, converges to $\varphi \in \mathbf{G}^\sigma(\mathbb{R})$ if, and only if, for any compact $K \subset \mathbb{R}$ there is a constant $C > 0$ such that

$$\sup_{j\in\mathbb{Z}_+} C^{-j}(j!)^{-\sigma}\Big(\sup_{x\in K} |\varphi_k^{(j)}(x) - \varphi^{(j)}(x)|\Big) \longrightarrow 0, \quad \text{as } k \to 0.$$

1.3 The Gevrey-Bourgain Spaces $G^{\sigma,\delta,s}$

In this section, we shall provide some basic properties of the Gevrey-Bourgain spaces, which the definition was given in the introduction and we recall below.

Definition 1.11. *For* $s \in \mathbb{R}$, $\delta > 0$ *and* $\sigma \ge 1$, *we define the spaces*

$$G^{\sigma,\delta,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R});\ \|f\|_{G^{\sigma,\delta,s}(\mathbb{R})}^2 = \int \langle\xi\rangle^{2s} e^{2\delta|\xi|^{1/\sigma}} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi < \infty \right\}, \tag{1.14}$$

where $\langle\xi\rangle \doteq (1+\xi^2)^{1/2}$. *If* $\varphi \in G^{\sigma,\delta,s}(\mathbb{R})$, *then* φ *belongs to the Gevrey class* $G^\sigma(\mathbb{R})$. *In the case when* $\sigma = 1$, *we denote* $G^{\delta,s}(\mathbb{R}) \equiv G^{1,\delta,s}(\mathbb{R})$.

For the periodic case, the norm in the space $G^{\sigma,\delta,s}(\mathbb{T})$ *is defined by simply replacing the integral in* (3.2) *with a sum as follows*

$$G^{\sigma,\delta,s}(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T});\ \|f\|_{G^{\sigma,\delta,s}(\mathbb{T})}^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} e^{2\delta|k|^{1/\sigma}} |\hat{f}(k)|^2 < \infty \right\}.$$

In what follows, we will list some properties of the spaces $G^{\sigma,\delta,s}$ and, just for convenience, we will work only with the space in the line. It is important to point out that all results remains true for the periodic case.

We observe that these spaces satisfy the following embedding

$$G^{\sigma,\delta,s}(\mathbb{R}) \subset G^{\sigma,\delta',s'}(\mathbb{R}), \text{ for all } s', s \in \mathbb{R} \text{ and } 0 < \delta' < \delta. \tag{1.15}$$

In fact,

$$\begin{aligned}
\|\varphi\|^2_{G^{\sigma,\delta',s'}} &= \int e^{2\delta'|\xi|^{1/\sigma}}(1+\xi^2)^{s'}|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi \\
&= \int \left[e^{-2(\delta-\delta')|\xi|^{1/\sigma}}(1+\xi^2)^{s'-s}\right] e^{2\delta|\xi|^{1/\sigma}}(1+\xi^2)^{s}|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi \\
&\le C_{\delta,\delta',s,s'} \int e^{2\delta|\xi|^{1/\sigma}}(1+\xi^2)^{s}|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi \\
&\le C_{\delta,\delta',s,s'}\, \|\varphi\|^2_{G^{\sigma,\delta,s}},
\end{aligned}$$

where we use that $e^{-2(\delta-\delta')|\xi|^{1/\sigma}}(1+\xi^2)^{s'-s}$ is bounded for all $0<\delta'<\delta$ and $s, s' \in \mathbb{R}$.

As a consequence of (1.15), we have the inclusion (6) mentioned in the introduction, that is, $G^{\sigma,\delta,s}(\mathbb{R}) \subset H^{s'}(\mathbb{R})$ for all $s' \in \mathbb{R}$. This embedding will be really useful in the following result.

Proposition 1.12. *Let $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \ge 1$. If $\varphi \in G^{\sigma,\delta,s}(\mathbb{R})$, then there exists a constant $C > 0$ such that, for each $j \in \mathbb{Z}_+$, we have*

$$\sup_{x\in\mathbb{R}} \left|\varphi^{(j)}(x)\right| \le C^{j+1}(j!)^{\sigma}.$$

In particular, $\varphi \in \mathbf{G}^{\sigma}(\mathbb{R})$.

Proof. First, let us prove for the case $s = 0$. If $\varphi \in G^{\sigma,\delta,0}$, using the inverse formula of Fourier Transform, we obtain

$$\varphi^{(j)}(x) = \frac{d^j}{dx^j}\left(\frac{1}{2\pi}\int e^{ix\xi}\widehat{\varphi}(\xi)\,\mathrm{d}\xi\right) = \frac{1}{2\pi}\int (i\xi)^j e^{ix\xi}\widehat{\varphi}(\xi)\,\mathrm{d}\xi.$$

Therefore,

$$\begin{aligned}
|\varphi^{(j)}(x)| &\le \frac{1}{2\pi}\int |\xi|^j|\widehat{\varphi}(\xi)|\,\mathrm{d}\xi = \frac{1}{2\pi}\int |\xi|^j e^{-\delta|\xi|^{1/\sigma}} e^{\delta|\xi|^{1/\sigma}}|\widehat{\varphi}(\xi)|\,\mathrm{d}\xi \\
&\le \frac{1}{2\pi}\left(\int |\xi|^{2j} e^{-2\delta|\xi|^{1/\sigma}}\,\mathrm{d}\xi\right)^{\frac12}\left(\int e^{2\delta|\xi|^{1/\sigma}}|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac12} \\
&= \frac{1}{2\pi}\left(\int |\xi|^{2j} e^{-2\delta|\xi|^{1/\sigma}}\,\mathrm{d}\xi\right)^{\frac12}\|\varphi\|_{G^{\sigma,\delta,0}},
\end{aligned} \tag{1.16}$$

where we used Cauchy-Schwarz inequality. We observe that

$$e^{\frac{\delta}{\sigma}|\xi|^{1/\sigma}} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\delta}{\sigma}|\xi|^{1/\sigma}\right)^k \ge \frac{1}{(2j)!}\left(\frac{\delta}{\sigma}|\xi|^{1/\sigma}\right)^{2j},$$

which implies

$$|\xi|^{\frac{2j}{\sigma}} e^{-\frac{\delta}{\sigma}|\xi|^{1/\sigma}} \le \left(\frac{\sigma}{\delta}\right)^{2j}(2j)!, \text{ that is, } |\xi|^{2j}e^{\delta|\xi|^{1/\sigma}} \le A^{2j}(2j)!^{\sigma},$$

where $A = \left(\frac{\sigma}{\delta}\right)^{\sigma}$, for all $j \in \mathbb{Z}_+$. Then, we have

$$\int |\xi|^{2j} e^{-2\delta|\xi|^{1/\sigma}}\, \mathrm{d}\xi = \int |\xi|^{2j} e^{-\delta|\xi|^{1/\sigma}} e^{-\delta|\xi|^{1/\sigma}}\, \mathrm{d}\xi \le A^{2j} (2j)!^{\sigma} \int e^{-\delta|\xi|^{1/\sigma}}\, \mathrm{d}\xi. \tag{1.17}$$

Denoting by $A_0 = (2\pi)^{-1} \|\varphi\|_{G^{\sigma,\delta,0}}$, it follows from (1.16) and (1.17) that

$$|\varphi^{(j)}(x)| \le A_0 \left(\int |\xi|^{2j} e^{-2\delta|\xi|^{1/\sigma}}\, \mathrm{d}\xi\right)^{\frac{1}{2}} \le A_0 A^j (2j)!^{\frac{\sigma}{2}} \left(\int e^{-\delta|\xi|^{1/\sigma}}\, \mathrm{d}\xi\right)^{\frac{1}{2}}.$$

Also, as we know

$$\sum_{j=0}^{\infty} \frac{(2j)!}{B_0^j j!^2} < \infty,$$

for all $B_0 > 4$, then $(2j)! \le B^{2j} j!^2$ for some $B > 0$, which implies

$$|\varphi^{(j)}(x)| \le A_0 (AB)^j (j!)^{\sigma} \left(\int e^{-\delta|\xi|^{1/\sigma}}\, \mathrm{d}\xi\right)^{\frac{1}{2}}. \tag{1.18}$$

It just remains to prove that the following integral

$$\int e^{-\delta|\xi|^{1/\sigma}}\, \mathrm{d}\xi$$

is finite. In fact, using that $e^{-\delta|\xi|^{1/\sigma}}$ is an even function and making the change of variables $\eta = \delta\xi^{1/\sigma}$, we obtain

$$\begin{aligned}
\int e^{-\delta|\xi|^{1/\sigma}}\, \mathrm{d}\xi &= 2\int_0^{\infty} e^{-\delta\xi^{1/\sigma}}\, \mathrm{d}\xi = 2\sigma\delta^{-\sigma} \int_0^{\infty} e^{-\eta}\eta^{\sigma-1}\, \mathrm{d}\eta \\
&= 2\sigma\delta^{-\sigma} \int_0^{\infty} \left[e^{-\eta/2}\eta^{\sigma-1}\right] e^{-\eta/2}\, \mathrm{d}\eta \\
&\le M \int_0^{\infty} e^{-\eta/2}\, \mathrm{d}\eta = 2M,
\end{aligned}$$

for some positive constant M, since $e^{-\eta/2}\eta^{\sigma-1}$ is bounded for all $\sigma \ge 1$ and

$$\int_0^{\infty} e^{-\eta/2}\, \mathrm{d}\eta = -2e^{-\eta/2}\Big|_0^{\infty} = 2.$$

Therefore, returning to (1.18), we conclude that there is a constant $C > 0$ satisfying

$$|\varphi^{(j)}(x)| \le C^{j+1} (j!)^{\sigma}, \text{ for all } x \in \mathbb{R},$$

which proves that φ belongs to $\mathbf{G}^{\sigma}$.

For the general case $s \in \mathbb{R}$, since we proved the case $s = 0$, we have

$$\varphi \in G^{\sigma,\delta,s}(\mathbb{R}) \subset G^{\sigma,\frac{\delta}{2},0}(\mathbb{R}) \quad \text{implying that} \quad \varphi \in \mathbf{G}^{\sigma}(\mathbb{R}),$$

which finishes the proof. □

One of the most important property of the spaces $G^{\sigma,\delta,s}$, as we mentioned in the introduction, is the following statement:

> If $\varphi \in G^{1,\delta,s} = G^{\delta,s}(\mathbb{R})$, then φ is analytic on the line and can be extended holomorphically in a symmetric strip of the x-axis.

Actually, as we will see in what follows, we have a characterization of the space $G^{\delta,s}$ in terms of functions that admits holomorphic extensions. The final steps in this chapter are devoted to prove this important fact. We begin with a result that give us a motivation to continue in this direction. All the results below was studied in [26].

Proposition 1.13. *If $f:(0,\infty)\to\mathbb{R}$ belongs to $L^2(0,\infty)$, then $\widehat{f}:\mathbb{C}\to\mathbb{C}$ given by*

$$\widehat{f}(z) \doteq \int_0^\infty e^{-izx} f(x)\,\mathrm{d}x, \text{ for all } z\in\mathbb{C},$$

is holomorphic in the half plane $\mathbb{C}^- = \{z_1 + iz_2;\ z_1\in\mathbb{R} \text{ and } z_2<0\}$.

Proof. Given $\tilde{z} = \tilde{z}_1 + i\tilde{z}_2 \in \mathbb{C}^-$, we consider $\delta>0$ such that the closure of the strip $S_{\tilde{z},\delta}$ is included in $\mathbb{C}^-$, where

$$S_{\tilde{z},\delta} = \{z_1 + iz_2;\ z_1\in\mathbb{R} \text{ and } |z_2-\tilde{z}_2|<\delta\}.$$

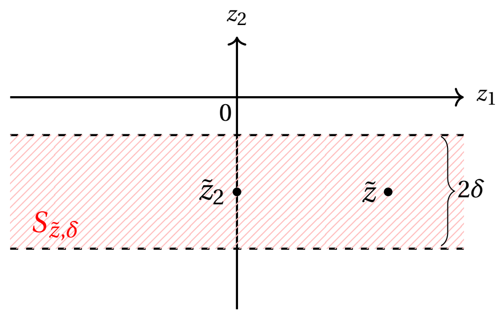

Figure 3: Strip in the half plan $z_2<0$.

Our goal is to prove that $\widehat{f}$ is holomorphic in $S_{\tilde{z},\delta}$.

For all $z\in\mathbb{C}$, we have

$$|\widehat{f}(z)| \le \int_0^\infty \left|e^{-izx} f(x)\right| \mathrm{d}x = \int_0^\infty e^{z_2 x}|f(x)|\,\mathrm{d}x,$$

and by using Cauchy-Schwarz inequality, we obtain

$$|\widehat{f}(z)| \le \left(\int_0^\infty e^{2z_2x}\,\mathrm{d}x\right)^{\frac{1}{2}} \|f\|_{L^2(0,\infty)}. \tag{1.19}$$

Now, if $z\in S_{\tilde{z},\delta}$, then $\tilde{z}_2-\delta<z_2<\tilde{z}_2+\delta<0$ which implies that

$$\int_0^\infty e^{2z_2x}\,\mathrm{d}x \le \int_0^\infty e^{2(\tilde{z}_2+\delta)x}\,\mathrm{d}x < \infty.$$

Therefore, returning to (1.19), we concluded that there is a positive constant C such that

$$|\widehat{f}(z)| \leq C \|f\|_{L^2(0,\infty)}, \text{ for all } z \in S_{\tilde{z},\delta}. \tag{1.20}$$

Since we proved that $|\widehat{f}(z)|$ is bounded in $S_{\tilde{z},\delta}$, it follows from Dominated Convergence Theorem that

$$\left|\frac{\partial \widehat{f}}{\partial z}(z)\right| = \left|\int_0^\infty (-ix)e^{-izx} f(x)\,\mathrm{d}x\right| \leq \int_0^\infty x e^{z_2 x}|f(x)|\,\mathrm{d}x.$$

Using the same arguments above, we obtain

$$\left|\frac{\partial \widehat{f}}{\partial z}(z)\right| \leq \left(\int_0^\infty x^2 e^{2z_2 x}\,\mathrm{d}x\right)^{\frac{1}{2}} \|f\|_{L^2(0,\infty)} \leq C\|f\|_{L^2(0,\infty)}, \text{ for all } z \in S_{\tilde{z},\delta}.$$

Again, since the derivative of $\widehat{f}$ with respect to z is bounded in $S_{\tilde{z},\delta}$, we can take the derivative with respect to $\overline{z}$ under integration to obtain

$$\frac{\partial \widehat{f}}{\partial \overline{z}}(z) = \int_0^\infty \left(\frac{\partial}{\partial \overline{z}} e^{-izx}\right) f(x)\,\mathrm{d}x = 0, \text{ for all } z \in S_{\tilde{z},\delta},$$

where we used the fact that e^{-izx} is a holomorphic function. Therefore, we concluded that $\widehat{f}$ is holomorphic in $S_{\tilde{z},\delta}$, which finishes the proof of the proposition. □

Remark. The key ingredient to prove the result above was the fact that

$$\int_0^\infty e^{\alpha x}\,\mathrm{d}x < \infty, \text{ for all } \alpha < 0.$$

If we put a finite interval (a,b) instead of $(0,\infty)$, we have

$$\int_a^b e^{\beta x}\,\mathrm{d}x < \infty, \text{ for all } \beta \in \mathbb{R}.$$

Therefore, using the same proof, we have the following result

$$\boxed{\text{If } f \in L^2(a,b), \text{ then } \widehat{f}(z) \doteq \int_a^b e^{-izx} f(x)\,\mathrm{d}x \text{ is holomorphic in } \mathbb{C}.}$$

Before we prove another Paley Wiener Theorem, we need to introduce the **F**éjer kernel on $\mathbb{R}$, which we are going to use in the proof. For all $\lambda > 0$, the Féjer kernel is defined by

$$K_\lambda(x) = \lambda K(\lambda x), \tag{1.21}$$

where

$$K(x) = \frac{1}{2\pi}\left(\frac{\sin x/2}{x/2}\right)^2 = \frac{1}{2\pi}\int_{-1}^{1}(1-|\xi|)e^{i\xi x}\,\mathrm{d}\xi. \tag{1.22}$$

The second equality in (1.22) is obtained directly by integration, indeed for all $x \neq 0$

$$\int_{-1}^{1} (1-|\xi|)e^{i\xi x}\,\mathrm{d}\xi = \int_{-1}^{1} e^{i\xi x}\,\mathrm{d}\xi - \int_{-1}^{1} |\xi| e^{i\xi x}\,\mathrm{d}\xi.$$

Now, we observe that

$$\int_{-1}^{1} e^{i\xi x}\,\mathrm{d}\xi = \frac{e^{ix\xi}}{ix}\Big|_{-1}^{1} = \frac{1}{x}\left(\frac{e^{ix}-e^{-ix}}{i}\right) = \frac{2\sin x}{x}$$

and

$$\int_{-1}^{1} |\xi| e^{ix\xi}\,\mathrm{d}\xi = \int_{-1}^{1} |\xi|\cos(\xi x)\,\mathrm{d}\xi + i\int_{-1}^{1} |\xi|\sin(\xi x)\,\mathrm{d}\xi = 2\int_{0}^{1} \xi\cos(\xi x)\,\mathrm{d}\xi,$$

since $|\xi|\cos(\xi x)$ is an even function and $|\xi|\sin(\xi x)$ is an odd function. Then,

$$\int_{-1}^{1} (1-|\xi|)e^{i\xi x}\,\mathrm{d}\xi = \frac{2\sin x}{x} + 2\int_{0}^{1} \xi\cos(\xi x)\,\mathrm{d}\xi.$$

Furthermore, using integration by parts, we obtain

$$\int_{0}^{1} \xi\cos(\xi x)\,\mathrm{d}\xi = \frac{\xi\sin(\xi x)}{x}\Big|_{0}^{1} - \int_{0}^{1} \frac{\sin(\xi x)}{x}\,\mathrm{d}\xi = \frac{\sin x}{x} + \frac{\cos(\xi x)}{x^2}\Big|_{0}^{1} = \frac{\sin x}{x} + \frac{1}{x^2}(\cos x - 1).$$

Thus,

$$\int_{-1}^{1} (1+|\xi|)e^{i\xi x}\,\mathrm{d}\xi = \frac{2\sin x}{x} - \left(\frac{2\sin x}{x} + \frac{2}{x^2}(\cos x - 1)\right) = \frac{2}{x^2}(1-\cos x).$$

Now, using the relations

$$\cos(a+b) = \cos a\cos b - \sin a \sin b \quad \text{and} \quad \sin^2 a + \cos^2 a = 1,$$

which are true for all $a, b \in \mathbb{R}$, we obtain

$$1-\cos x = 1-\cos\left(\frac{x}{2}+\frac{x}{2}\right) = 1-\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) = 2\sin^2\left(\frac{x}{2}\right).$$

Therefore, we concluded that

$$\int_{-1}^{1} (1+|\xi|)e^{i\xi x}\,\mathrm{d}\xi = \frac{2}{x^2}(1-\cos x) = \frac{4}{x^2}\sin^2\left(\frac{x}{2}\right),$$

which proves the second equality in (1.22).

Some important properties of the Féjer Kernel are given in the next lemma, which the proof can be found in [26].

Lemma 1.14 (page 12 in [26]). *The Féjer Kernel K_λ satisfies the following items:*

1. $\int_{\mathbb{R}} K_\lambda(x)\,\mathrm{d}x = \int_{\mathbb{R}} K(x)\,\mathrm{d}x = 1.$

2. $\widehat{K_\lambda}(\xi) = \max\left\{1 - \frac{|\xi|}{\lambda}, 0\right\}.$

3. *Let* $f \in L^1$, *then*

$$f = \lim_{\lambda\to\infty} \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \widehat{K_\lambda}(\xi)\widehat{f}(\xi)e^{ix\xi}\,\mathrm{d}\xi,$$

in the $L^1(\mathbb{R})$ *norm.*

Now we are ready to prove the main result of this section.

Proposition 1.15 (Paley-Wiener - page 174 in [26])**.** *] Let* $\delta > 0$. *For* $f \in L^2(\mathbb{R})$, *the following two conditions are equivalent:*

(a) The function $e^{\delta|\xi|}\widehat{f}$ *belongs to* $L^2(\mathbb{R})$.

(b) f *is the restriction to* $\mathbb{R}$ *of a function* F *holomorphic in the strip*

$$S_\delta = \{z = x + iy;\ x \in \mathbb{R}\, and\ |y| < \delta\},$$

and satisfying

$$\sup_{|y|<\delta} \int |F(x+iy)|^2\,\mathrm{d}x \le C,$$

for some constant $C > 0$.

Proof. (a) implies (b). We define

$$F(x+iy) = F(z) = \frac{1}{2\pi}\int e^{iz\xi}\widehat{f}(\xi)\,\mathrm{d}\xi = \frac{1}{2\pi}\int e^{ix\xi}e^{-y\xi}\widehat{f}(\xi)\,\mathrm{d}\xi.$$

By the inversion formula of Fourier transform we have $F\big|_{\mathbb{R}} = f$. Furthermore, F is well defined in S_δ, indeed, applying Cauchy-Schwartz inequality and using that $e^{\delta|\xi|}\widehat{f} \in L^2(\mathbb{R})$ we obtain

$$\begin{aligned} |F(x+iy)| &\le \int e^{-y\xi}|\widehat{f}(\xi)|\,\mathrm{d}\xi \le \int e^{-(\delta-|y|)|\xi|}e^{\delta|\xi|}|\widehat{f}(\xi)|\,\mathrm{d}\xi \\ &\le \left\|e^{-(\delta-|y|)|\xi|}\right\|_{L^2_\xi}\left\|e^{\delta|\xi|}\widehat{f}\right\|_{L^2_\xi} = C_{y,\delta} < \infty, \end{aligned} \tag{1.23}$$

for all $z = x + iy \in S_\delta$, where $C_{y,\delta}$ is a positive constant that depends on y and δ.

About the L^2_x-norm of F, by Plancherel's formula we have

$$\int |F(x+iy)|^2\,\mathrm{d}x = \int |\widehat{F}^x(\xi+iy)|^2\,\mathrm{d}\xi = \int \left|e^{-y\xi}\widehat{f}(\xi)\right|^2\mathrm{d}\xi \le \int e^{2\delta|\xi|}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi,$$

for all $z = x + iy \in S_\delta$. Then,

$$\sup_{|y|<\delta}\int |F(x+iy)|^2\,\mathrm{d}x \le \left\|e^{\delta|\xi|}\widehat{f}\right\|_{L^2} < \infty.$$

It just remains to prove that F is holomorphic in the strip S_δ, for this we will fix δ_1 such that $0<\delta_1<\delta$. Returning to (1.23) we have

$$|F(x+iy)| \le \left\| e^{-(\delta-|y|)|\xi|}\right\|_{L^2_\xi} \left\| e^{\delta|\xi|}\widehat{f}\right\|_{L^2_\xi} = C_{y,\delta}.$$

Now, if $|y|<\delta_1$, then

$$|F(x+iy)| \le \left\| e^{-(\delta-\delta_1)|\xi|}\right\|_{L^2_\xi} \left\| e^{\delta|\xi|}\widehat{f}\right\|_{L^2_\xi} = C_{\delta_1,\delta}<\infty,$$

which proves that F is bounded in $S_{\delta_1}=\{x+iy;\ |y|<\delta_1\}$. Therefore, it follows from Dominated Convergence Theorem that we can take the derivative with respect to $\overline{z}$ under integration to obtain

$$\frac{\partial \widehat{F}}{\partial \overline{z}}(z) = \int \left(\frac{\partial}{\partial \overline{z}} e^{-iz\xi}\right)\widehat{f}(\xi)\,\mathrm{d}\xi = 0, \text{ for all } z\in S_{\delta_1}.$$

Then, we proved that F is holomorphic in the strip S_{δ_1} for all $0<\delta_1<\delta$, which is sufficient to conclude that F is holomorphic in the hole strip S_δ.

(b) implies (a). We write

$$f_y(x) = F(x+iy), \text{ for all } x,y\in\mathbb{R} \quad (\text{thus } f=f_0).$$

We want to show that

$$\widehat{f_y}(\xi) = \widehat{f}(\xi)e^{-\xi y}, \tag{1.24}$$

since then, by Plancherel's formula we would have

$$\int |\widehat{f}(\xi)|^2 e^{2\xi y}\mathrm{d}\xi = \int \left|\widehat{f}(\xi)e^{-\xi(-y)}\right|^2 \mathrm{d}\xi = \int |\widehat{f_{-y}}(\xi)|^2\,\mathrm{d}\xi = \int |f_{-y}(x)|^2\mathrm{d}x = \int |F(x-iy)|^2\,\mathrm{d}x \le C,$$

for all $|y|<\delta$, which guarantees that $e^{\delta|\xi|}\widehat{f}\in L^2$. Notice that if we assume (a) then, by the first part of the proof, we do have $\widehat{f_y}(\xi)=\widehat{f}(\xi)e^{-\xi y}$.

Thus, our goal is to prove (1.24). For all $\lambda>0$ and $z=x+iy\in S_\delta$, we define

$$G_\lambda(z) = K_\lambda *_x F(x+iy) = \int F(x-x'+iy)K_\lambda(x')\,\mathrm{d}x',$$

where K_λ denotes Féjer Kernel given by (1.21). By using Cauchy Schwarz inequality, Plancherel's formula and item (2) from Lemma 1.14, we obtain that G_λ satisfies

$$\begin{aligned}
|G_\lambda(z)| &\le \int_{\mathbb{R}} |F(x-x'+iy)||K_\lambda(x')|\,\mathrm{d}x' \\
&\le \left(\int |F(x-x'+iy)|^2\,\mathrm{d}x'\right)^{\frac12}\left(\int |K_\lambda(x')|^2\,\mathrm{d}x'\right)^{\frac12} \\
&\le C\left(\int_{-\lambda}^{\lambda} |\widehat{K_\lambda}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac12} \\
&\le C(2\lambda),
\end{aligned}$$

for all $z \in S_\delta$, which is sufficient to prove that G_λ is a holomorphic function in the strip S_δ. Indeed, by using the Dominated Convergence Theorem we can take the derivative with respect to $\overline{z}$ under integration and also using that F is a holomorphic function we obtain

$$\frac{\partial G_\lambda}{\partial \overline{z}}(z) = \int \left(\frac{\partial}{\partial \overline{z}} F(z-x') \right) K_\lambda(x')\, dx' = 0, \text{ for all } z \in S_\delta.$$

Furthermore, we notice that the function $g_{\lambda,y}$ given as follows

$$g_{\lambda,y}(x) \doteq G_\lambda(x+iy) = K_\lambda * f_y(x)$$

satisfies

$$\widehat{g_{\lambda,y}}(\xi) = 2\pi \widehat{K_\lambda}(\xi) \widehat{f}_y(\xi). \tag{1.25}$$

Now, it follows from Lemma 1.14 item (2) that $\widehat{g_{\lambda,y}}$ has a compact support included in $[-\lambda,\lambda]$, thus

$$\widehat{g_{\lambda,y}}(\xi) = \widehat{g_{\lambda,0}}(\xi) e^{-\xi y}. \tag{1.26}$$

Consequently, joining (1.25) and (1.26) we conclude that

$$\widehat{f}_y(\xi) = \widehat{f}(\xi) e^{-\xi y}.$$

Since $\lambda > 0$ is arbitrary, the above holds for all $\xi \in \mathbb{R}$ and the proof is complete. □

1.4 A calculus lemma

This section is devoted to giving a detailed proof of calculus estimates, which is extensively used in the proof of multilinear estimates in Bourgain Spaces.

Before presenting the main result, we would like to introduced the following notation that will be used from now on. We write

$$X \lesssim Y \quad \text{and} \quad X \simeq Y,$$

as a shorthand for $X \le CY$ and $X = CY$, respectively, for some positive constant C.

Also, an inequality that we use several times is

$$\langle a \rangle^s \lesssim \langle a-b \rangle^{|s|} \langle b \rangle^s, \quad \text{for all } a,b,s \in \mathbb{R}, \tag{1.27}$$

which is known as Petri inequality. Indeed, since $|a|^2 \le (|a-b|+|b|)^2 \le 2(|a-b|^2+|b|^2)$, we have

$$\langle a \rangle^2 = 1+|a|^2 \le 2(1+|a-b|^2+|b|^2) \le 2(1+|a-b|^2)(1+|b|^2) = 2\langle a-b \rangle^2 \langle b \rangle^2,$$

which implies $\langle a \rangle \lesssim \langle a-b \rangle \langle b \rangle$. Thus, we have

$$\langle a \rangle^s \lesssim \langle a-b \rangle^s \langle b \rangle^s, \quad \text{for all } s \ge 0. \tag{1.28}$$

Using the same previously calculus, we have

$$\langle b \rangle \lesssim \langle a-b \rangle \langle a \rangle.$$

If $s<0$, then $\langle b\rangle^{-s} \lesssim \langle a-b\rangle^{-s}\langle a\rangle^{-s}$, which implies

$$\langle a\rangle^{s} \lesssim \langle a-b\rangle^{|s|}\langle b\rangle^{s}, \quad \text{for all } s<0. \tag{1.29}$$

The proof of (1.27) is finished by joining (1.28) and (1.29).

Now, we are able to prove the main result of this section, which consists of two inequalities that are used often in this area. For a reference of the following inequalies see, for example, [4] on page 369.

Lemma 1.16. *For $p,q>0$, $p\neq 1$, $q\neq 1$ and $p+q>1$, there exists $C>0$ such that*

$$\int \frac{1}{\langle x-\alpha\rangle^{p}\langle x-\beta\rangle^{q}}\mathrm{d}x \leq \frac{C}{\langle \alpha-\beta\rangle^{r}}, \tag{1.30}$$

where $r=\min\{p,q,p+q-1\}$. Moreover, for $a_0,a_1,a_2\in\mathbb{R}$ and $q>\frac{1}{2}$

$$\int \frac{1}{\langle a_0+a_1x+a_2x^2\rangle^{q}}\mathrm{d}x \leq C. \tag{1.31}$$

Proof. Without loss of generality, we assume that $\alpha\geq\beta$. Taking $z=\frac{\alpha-\beta}{2}\geq 0$ and applying the change of variables $x=y-z+\alpha$, we obtain

$$\int \frac{1}{\langle x-\alpha\rangle^{p}\langle x-\beta\rangle^{q}}\mathrm{d}x = \int \frac{1}{\langle y-z\rangle^{p}\langle y+z\rangle^{q}}\mathrm{d}y.$$

To prove (1.30), it is sufficient to prove that

$$J(z) \doteq \int \frac{1}{\langle y-z\rangle^{p}\langle y+z\rangle^{q}}\mathrm{d}y \leq \frac{C}{\langle z\rangle^{r}}, \quad \text{for all } z\geq 0,$$

since $\langle \alpha-\beta\rangle = \langle 2z\rangle \leq 2\langle z\rangle$ which implies $\langle z\rangle^{-r}\leq 2^{-r}\langle \alpha-\beta\rangle^{-r}$.

First, we observe that $J(z)<\infty$ for all $z\geq 0$. In fact, it follows from (1.27) that

$$\langle y\rangle \lesssim \langle z\rangle\langle y-z\rangle \quad \text{and} \quad \langle y\rangle \lesssim \langle z\rangle\langle y+z\rangle,$$

which implies

$$J(z) \lesssim \int \frac{\langle z\rangle^{p+q}}{\langle y\rangle^{p+q}}\mathrm{d}y = \langle z\rangle^{p+q}\int \frac{1}{(1+y^2)^{\frac{p+q}{2}}}\mathrm{d}y<\infty,$$

since $p+q>1$. Next, we split the region of integration into three subregions, which are

$$A_1=\{y\in\mathbb{R};\ 0\leq y\leq 2z\}, \quad A_2=\{y\in\mathbb{R};\ -2z\leq y<0\} \quad \text{and} \quad A_3=\{y\in\mathbb{R};\ |y|\geq 2z\}.$$

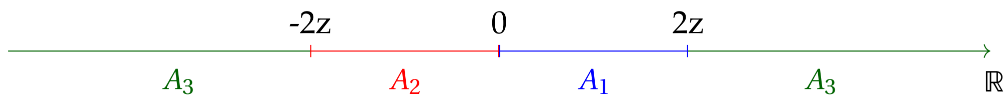

Figure 4: Integration regions A_1, A_2, A_3.

Thus, $J(z) = J_1(z) + J_2(z) + J_3(z)$, where

$$J_i(z) = \int_{A_i} \frac{1}{\langle y-z\rangle^p \langle y+z\rangle^q} \mathrm{d}y, \text{ for } i = 1,2,3.$$

If $y \in A_1$, then $0 \le z \le y+z$. Thus, $\langle z\rangle \le \langle y+z\rangle$, for all $y \in A_1$, which implies

$$J_1(z) \le \frac{1}{\langle z\rangle^q} \int_0^{2z} \frac{1}{\langle y-z\rangle^p} \mathrm{d}y = \frac{1}{\langle z\rangle^q} \int_{-z}^{z} \frac{1}{\langle \mu\rangle^p} \mathrm{d}\mu = \frac{2}{\langle z\rangle^q} \int_0^{z} \frac{1}{\langle \mu\rangle^p} \mathrm{d}\mu \le \frac{C}{\langle z\rangle^q} \int_0^{z} \frac{1}{(1+\mu)^p} \mathrm{d}\mu,$$

since $(1+\mu)^2 \le 2(1+\mu^2) = 2\langle \mu\rangle^2$. Using that $p \neq 1$, we have

$$\int_0^{z} \frac{1}{(1+\mu)^p} \mathrm{d}\mu = \frac{(1+\mu)^{1-p}}{1-p}\bigg|_0^z = \frac{(1+z)^{1-p}}{1-p} - \frac{1}{1-p}.$$

If $1-p > 0$, then

$$\int_0^{z} \frac{1}{(1+\mu)^p} \mathrm{d}\mu \le \frac{(1+z)^{1-p}}{1-p},$$

implying that

$$J_1(z) \lesssim \frac{C}{\langle z\rangle^q} \frac{1}{(1+z)^{p-1}} \le \frac{C}{\langle z\rangle^{q+p-1}},$$

since $\langle z\rangle^2 = 1+z^2 \le (1+z)^2$ for all $z \ge 0$. On the other side, if $1-p<0$, then

$$\int_0^{s} \frac{1}{(1+\mu)^p} \mathrm{d}\mu \le \frac{1}{p-1},$$

which implies

$$J_1(z) \le \frac{C}{\langle z\rangle^q} \left(\frac{1}{p-1}\right) = \frac{C}{\langle z\rangle^q}.$$

Thus, in both cases, we conclude that $J_1(z) \le C\langle z\rangle^{-r}$ with $r = \min\{p, q, p+q-1\}$.

Now, in order to estimate J_2, we observe that

$$J_2(z) = \int_{-2z}^{0} \frac{1}{\langle y-z\rangle^p \langle y+z\rangle^q} \mathrm{d}y = \int_0^{2z} \frac{1}{\langle y+z\rangle^p \langle y-z\rangle^q} \mathrm{d}y.$$

Using the estimate that we proved for J_1, we obtain

$$J_2(z) \le \frac{1}{\langle z\rangle^p} \int_0^{2s} \frac{1}{\langle y-z\rangle^q} \mathrm{d}y \le \frac{C}{\langle z\rangle^r}.$$

Regarding the proof of the estimate for J_3, we have

$$J_3(z) = \int_{|y|\ge 2z} \frac{1}{\langle y-z\rangle^p \langle y+z\rangle^q} \mathrm{d}y = 2\int_{2z}^{+\infty} \frac{1}{\langle y-z\rangle^p \langle y+z\rangle^q} \mathrm{d}y \le 2\int_{2z}^{+\infty} \left\langle \frac{y}{2}\right\rangle^{-(p+q)} \mathrm{d}y,$$

since $y \le y+(y-2z)=2(y-z)$ and $y \le 2(y+z)$, for all $y \ge 2z \ge 0$. Then, by making the change of variables $2u=y$, we obtain

$$\int_{2z}^{+\infty}\left\langle \frac{y}{2}\right\rangle^{-(p+q)}\mathrm{d}y = 2\int_{z}^{+\infty}\langle u\rangle^{-(p+q)}\,\mathrm{d}u \simeq \int_{z}^{+\infty}(1+u)^{-(p+q)}\mathrm{d}u = \frac{(1+u)^{1-(p+q)}}{1-(p+q)}\Bigg|_{z}^{+\infty} \lesssim (1+z)^{1-(p+q)},$$

since $p+q>1$. Thus,

$$J_3(z) \lesssim \frac{1}{(1+z)^{p+q-1}} \lesssim \frac{1}{\langle z\rangle^{p+q-1}} \le \frac{1}{\langle z\rangle^{r}},$$

which finishes the proof of (1.30).

Next, we turn to the proof of (1.31). We write $a_0+a_1x+a_2x^2 = a_2\mathrm{p}(x)$, where $\mathrm{p}(x)=x^2+\frac{a_1}{a_2}x+\frac{a_0}{a_2}$. We observe that

$$\left\langle a_2\mathrm{p}(x)\right\rangle = (1+a_2^2\mathrm{p}(x)^2)^{\frac{1}{2}}\begin{cases} \le \max\{1,|a_2|\}\cdot\left\langle \mathrm{p}(x)\right\rangle \\ \ge \min\{1,|a_2|\}\cdot\left\langle \mathrm{p}(x)\right\rangle. \end{cases}$$

Then, $\left\langle a_0+a_1x+a_2x^2\right\rangle \simeq \left\langle \mathrm{p}(x)\right\rangle$. We consider below three cases concerning the polynomial $\mathrm{p}(x)$.

Case 1. $\mathrm{p}(x)$ has two different real roots.

Let α and β the roots of $\mathrm{p}(x)$. We can write $\mathrm{p}(x)=(x-\alpha)(x-\beta)$. We affirm that

$$\left\langle x-\alpha\right\rangle\left\langle x-\beta\right\rangle \le C\left\langle \mathrm{p}(x)\right\rangle, \quad \text{for all } x\in\mathbb{R}. \tag{1.32}$$

Indeed, we have

$$\langle x-\alpha\rangle^2\left\langle x-\beta\right\rangle^2 = (1+(x-\alpha)^2)(1+(x-\beta)^2) = 1+\mathrm{p}(x)^2+(x-\alpha)^2+(x-\beta)^2.$$

Since

$$\lim_{x\to\infty}\frac{(x-\alpha)^2+(x-\beta)^2}{(x-\alpha)^2(x-\beta)^2} = \lim_{x\to\infty}\frac{1}{(x-\beta)^2}+\frac{1}{(x-\alpha)^2}=0,$$

there exists $M>0$ such that

$$\frac{(x-\alpha)^2+(x-\beta)^2}{(x-\alpha)^2(x-\beta)^2} \le 1, \quad \text{for all } |x|\ge M.$$

Then,

$$\langle x-\alpha\rangle^2\left\langle x-\beta\right\rangle^2 \le 1+2\mathrm{p}(x)^2 \le 2\left\langle \mathrm{p}(x)\right\rangle^2, \quad \text{for all } |x|\ge M. \tag{1.33}$$

Denoting by $\delta=|\alpha-\beta|/2>0$, we observe that

$$|x-\beta| \le |x-\alpha|+|\alpha-\beta| < 3\delta, \quad \text{for all } x\in(\alpha-\delta,\alpha+\delta),$$

which implies

$$0<(x-\alpha)^2+(x-\beta)^2<\delta^2+(3\delta)^2=10\delta^2, \quad \text{for all } x\in(\alpha-\delta,\alpha+\delta).$$

Similarly, we have $0<(x-\alpha)^2+(x-\beta)^2<10\delta^2$ for all $x\in(\beta-\delta,\beta+\delta)$. Thus,

$$\langle x-\alpha\rangle^2\left\langle x-\beta\right\rangle^2 \le 1+\mathrm{p}(x)^2+10\delta^2, \quad \text{for all } x\in(\alpha-\delta,\alpha+\delta)\cup(x-\beta,x+\beta). \tag{1.34}$$

By (1.33) and (1.34), it remains to show that (1.32) occurs in

$$K = [-M, M] \cap \big((\alpha-\delta, \alpha+\delta) \cup (\beta-\delta, \beta+\delta)\big)^c,$$

which is a compact subset. Since $\alpha \notin K$ and $\beta \notin K$, there exists a constant $D > 0$ such that

$$\sup_{x\in K} \frac{(x-\alpha)^2 + (x-\beta)^2}{(x-\alpha)^2(x-\beta)^2} \le D.$$

Then,

$$\langle x-\alpha\rangle^2 \langle x-\beta\rangle^2 \le 1 + \mathrm{p}(x)^2 + D\mathrm{p}(x)^2 \le (1+D)\langle \mathrm{p}(x)\rangle^2, \quad \text{for all } x \in K. \tag{1.35}$$

Putting together (1.33), (1.34) and (1.35), we conclude the proof of inequality (1.32). Thus, it follows from (1.32) and (1.30) that

$$\int \frac{1}{\langle \mathrm{p}(x)\rangle^q} \mathrm{d}x \le C^q \int \frac{1}{\langle x-\alpha\rangle^q \langle x-\beta\rangle^q} \mathrm{d}x \le \frac{C'}{\langle \alpha-\beta\rangle^r} < \infty.$$

Case 2. $\mathrm{p}(x)$ has just one real root.

Let α be the real root of $\mathrm{p}(x)$. In this case, we can write $\mathrm{p}(x) = (x-\alpha)^2$. Then,

$$\int \frac{1}{\langle \mathrm{p}(x)\rangle^q} \mathrm{d}x = \int \frac{1}{(1+(x-\alpha)^4)^{\frac{q}{2}}} \mathrm{d}x = 2\int_0^\infty \frac{1}{(1+y^4)^{\frac{q}{2}}} \mathrm{d}y \le 2^{\frac{3}{2}q+1} \int_0^\infty \frac{1}{(1+y)^{2q}} \mathrm{d}y,$$

where we used that $(1+y)^4 = (1+y)^2(1+y)^2 \le 2^2(1+y^2)^2 \le 2^3(1+y^4)$, for all $y \in \mathbb{R}$. Since $q > \frac{1}{2}$, we have

$$\int_0^\infty \frac{1}{(1+y)^{2q}} \mathrm{d}y = \frac{(1+y)^{1-2q}}{1-2q}\bigg|_0^\infty = \frac{1}{2q-1} < \infty.$$

Therefore, (1.31) holds also in this case.

Case 3. $\mathrm{p}(x)$ has not real root.

In this case, we can write $\mathrm{p}(x) = C + (x-\alpha)^2$ with $C > 0$. In fact,

$$\mathrm{p}(x) = x^2 + bx + c = \left(x - \frac{b}{2}\right)^2 + c - \frac{b^2}{4},$$

where $c - b^2/4 > 0$, since $\Delta = b^2 - 4c < 0$. Thus,

$$\begin{aligned}\int \frac{1}{\langle \mathrm{p}(x)\rangle^q} \mathrm{d}x &= \int \frac{1}{(1+((x-\alpha)^2+C)^2)^{\frac{q}{2}}} \mathrm{d}x \\ &\le \int \frac{1}{(1+(x-\alpha)^4)^{\frac{q}{2}}} \mathrm{d}x \\ &= 2\int_0^\infty \frac{1}{(1+y^4)^{\frac{q}{2}}} \mathrm{d}y,\end{aligned}$$

since $C > 0$. As we did in the case 2, we have

$$2\int_0^\infty \frac{1}{(1+y^4)^{\frac{q}{2}}} \mathrm{d}y \le 2^{\frac{3}{2}q+1} \int_0^\infty \frac{1}{(1+y)^{2q}} \mathrm{d}y = 2^{\frac{3}{2}q+1} \frac{1}{2q-1}.$$

Then,

$$\int \frac{1}{\langle \mathrm{p}(x) \rangle^q} \mathrm{d}x < \infty,$$

which finishes the proof of (1.31). □

Chapter

2 The problem in H^s

We consider the initial value problem for the "good" Boussinesq equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u + \partial_x^2(u^2) = 0, & x \in \mathbb{R} \text{ or } \mathbb{T},\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \\ \partial_t u(x,o) = \partial_x u_1(x), \end{cases} \tag{2.1}$$

with initial data u_0 and u_1 belonging in a Sobolev Space. Our principal aim here is to show a detailed proof of the local well-posedness of (2.1) for low regularity data. The main results presented in this chapter was obtained by Luiz Gustavo Farah in 2009 [12] (real case) and by Luiz Gustavo Farah and Marcia Scialom in 2010 [13] (periodic case), which are the works that motivated the results that will be presented in Chapter 3. More precisely, we will prove here the following theorem.

Theorem 2.1. *Let $s > -1/4$. Then, for every initial data $(u_0, u_1) \in H^s \times H^{s-1}$, there exist a lifespan $T = T\big(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}}\big) > 0$ and a unique solution u of the Cauchy problem for the "good" Boussinesq equation* (2.1) *such that*

$$u \in C([0,T]; H^s) \cap X^T_{s,b}.$$

Moreover, given $T' \in (0,T)$ there exists $R = R(T') > 0$ such that giving the set

$$W = \left\{(\tilde{u}_0, \tilde{u}_1) \in H^s \times H^{s-1};\ \|\tilde{u}_0 - u_0\|_{H^s} + \|\tilde{u}_1 - u_1\|_{H^{s-1}} < R\right\}$$

the map solution

$$\begin{aligned} S : W &\longrightarrow C([0,T]; H^s) \cap X^T_{s,b} \\ (\tilde{u}_0, \tilde{u}_1) &\longmapsto \tilde{u} \end{aligned}$$

is Lipschitz.

2.1 Real Case

In this section we are considering the real version of the Cauchy problem (2.1), that is, the variable x belongs to $\mathbb{R}$.

A formal data-to-solution map

Let us get a formal solution to (2.1). We start by taking the Fourier transform with respect x in (2.1) to obtain

$$\begin{cases} \partial_t^2 \widehat{u}^x + (\xi^2+\xi^4)\widehat{u}^x + \widehat{\partial_x^2(u^2)}^x = 0, \\ \widehat{u}^x(\xi,0) = \widehat{u_0}(\xi), \\ \partial_t \widehat{u}^x(\xi,0) = \widehat{\partial_x u_1}(\xi). \end{cases}$$

Denoting by $U(t) = \widehat{u}^x(\xi,t)$ and $w(x,t) = \partial_x^2(u^2)$, where $\xi \in \mathbb{R}$ is considered as a parameter, we have

$$\begin{cases} U''(t) + (\xi^2+\xi^4)U(t) + \widehat{w}^x(\xi,t) = 0, \\ U(0) = \widehat{u}_0(\xi), \\ U'(0) = \widehat{\partial_x u_1}(\xi). \end{cases} \tag{2.2}$$

Next we use the variation parameters method to solve the above harmonic oscillator initial value problem (IVP) (2.2) with forcing $\widehat{w}^x$. The first step consists in to solve the corresponding homogeneous equation

$$U''(t) + (\xi^2+\xi^4)U(t) = 0. \tag{2.3}$$

A fundamental system of solutions is given by

$$U_1(t) = e^{i\gamma(\xi)t} \quad \text{and} \quad U_2(t) = e^{-i\gamma(\xi)t},$$

where $i\gamma(\xi) = i(\xi^2+\xi^4)^{\frac{1}{2}}$ is a root of the characteristic polynomial $\gamma^2 + (\xi^2+\xi^4)$ associated to equation (2.3). A particular solution of (2.2) is given by

$$U(t) = c_1(t)U_1(t) + c_2(t)U_2(t).$$

For that to hold true, it is sufficient to show that c_1 and c_2 satisfy

$$\begin{cases} c_1'(t)U_1(t) + c_2'(t)U_2(t) = 0, \\ c_1'(t)U_1'(t) + c_2'(t)U_2'(t) = -\widehat{w}^x(\xi,t), \end{cases}$$

that is, from Cramer's rule we must have

$$c_1'(t) = \frac{\det\begin{bmatrix} 0 & U_2(t) \\ -\widehat{w}^x(\xi,t) & U_2'(t) \end{bmatrix}}{W(U_1,U_2)(t)} = \frac{U_2(t)\widehat{w}^x(\xi,t)}{-2i\gamma(\xi)} = -\frac{e^{-i\gamma(\xi)t}}{2i\gamma(\xi)}\widehat{w}^x(\xi,t)$$

and

$$c_2'(t) = \frac{\det\begin{bmatrix} U_1(t) & 0 \\ U_1'(t) & -\widehat{w}^x(\xi,t) \end{bmatrix}}{W(U_1,U_2)(t)} = \frac{-U_1(t)\widehat{w}^x(\xi,t)}{-2i\gamma(\xi)} = \frac{e^{i\gamma(\xi)t}}{2i\gamma(\xi)}\widehat{w}^x(\xi,t),$$

where $W(U_1, U_2)$ denotes the Wronskian of U_1 and U_2. Hence, we obtain

$$c_1(t) = C_1 - \int_0^t \frac{e^{-i\gamma(\xi)t'}}{2i\gamma(\xi)} \widehat{w}^x(\xi, t')\,\mathrm{d}t' \quad \text{and} \quad c_2(t) = C_2 + \int_0^t \frac{e^{i\gamma(\xi)t'}}{2i\gamma(\xi)} \widehat{w}^x(\xi, t')\,\mathrm{d}t',$$

which implies

$$U(t) = C_1 e^{i\gamma(\xi)t} - \int_0^t \frac{e^{i\gamma(\xi)(t-t')}}{2i\gamma(\xi)} \widehat{w}^x(\xi, t')\,\mathrm{d}t' + C_2 e^{-i\gamma(\xi)t} + \int_0^t \frac{e^{-i\gamma(\xi)(t-t')}}{2i\gamma(\xi)} \widehat{w}^x(\xi, t')\,\mathrm{d}t'.$$

Since $U(0) = \widehat{u_0}(\xi)$ and $U'(0) = \widehat{\partial_x u_1}(\xi)$, we have that the constants C_1 and C_2 agree with

$$\begin{cases} C_1 + C_2 = \widehat{u_0}(\xi), \\ C_1 - C_2 = \dfrac{\widehat{\partial_x u_1}(\xi)}{i\gamma(\xi)}, \end{cases}$$

that is,

$$C_1 = \frac{\widehat{u_0}(\xi)}{2} + \frac{\widehat{\partial_x u_1}(\xi)}{2i\gamma(\xi)} \quad \text{and} \quad C_2 = \frac{\widehat{u_0}(\xi)}{2} - \frac{\widehat{\partial_x u_1}(\xi)}{2i\gamma(\xi)}.$$

Therefore,

$$U(t) = \widehat{u_0}(\xi)\left(\frac{e^{i\gamma(\xi)t} + e^{-i\gamma(\xi)t}}{2}\right) + \widehat{\partial_x u_1}(\xi)\left(\frac{e^{i\gamma(\xi)t} - e^{-i\gamma(\xi)t}}{2i\gamma(\xi)}\right) - \int_0^t \left(\frac{e^{i\gamma(\xi)(t-t')} - e^{-i\gamma(\xi)(t-t')}}{2i\gamma(\xi)}\right) \widehat{w}^x(\xi, t')\,\mathrm{d}t'.$$

Thus, we consider the operators W_1 and W_2 defined as follows

$$W_1(t)\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\left(\frac{e^{i\gamma(\xi)t} + e^{-i\gamma(\xi)t}}{2}\right)\widehat{\varphi}(\xi)\mathrm{d}\xi \quad \text{and}$$

$$W_2(t)\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\left(\frac{e^{i\gamma(\xi)t} - e^{-i\gamma(\xi)t}}{2i\gamma(\xi)}\right)\widehat{\varphi}(\xi)\mathrm{d}\xi,$$

which give us

$$\widehat{W_1(t)\varphi}^x(\xi) = \left(\frac{e^{i\gamma(\xi)t} + e^{-i\gamma(\xi)t}}{2}\right)\widehat{\varphi}(\xi) \quad \text{and} \quad \widehat{W_2(t)\varphi}^x(\xi) = \left(\frac{e^{i\gamma(\xi)t} - e^{-i\gamma(\xi)t}}{2i\gamma(\xi)}\right)\widehat{\varphi}(\xi).$$

The solution U of (2.2) can be written by

$$U(t) = \widehat{\left[W_1(t)u_0\right]}^x(\xi) + \widehat{\left[W_2(t)(\partial_x u_1)\right]}^x(\xi) - \int_0^t \widehat{\left[W_2(t-t')w(x,t')\right]}^x(\xi)\mathrm{d}t'.$$

Since $U(t) = \widehat{u}^x(\xi, t)$, it follows that

$$u(x,t) = W_1(t)u_0(x) + W_2(t)\partial_x u_1(x) - \int_0^t W_2(t-t')w(x,t')\,\mathrm{d}t' \tag{2.4}$$

is a formal solution of (2.1).

Next, we localize in the time variable by using a cut-off function $\psi \in C_0^\infty(-2,2)$ with $0 \le \psi \le 1$, $\psi(t) = 1$ on $[-1,1]$ and for $0 < T < 1$ we define $\psi_T(t) = \psi\left(\frac{t}{T}\right)$.

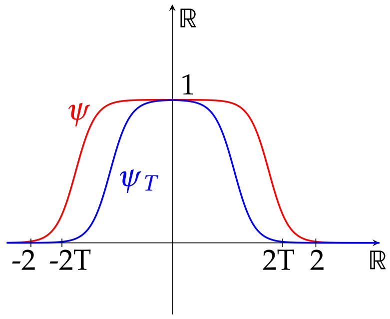

Figure 5: Cut off function ψ_T.

We consider the operator Φ_T given by

$$\boxed{\Phi_T u \doteq \psi(t) W_1(t) u_0(x) + \psi(t) W_2(t) \partial_x u_1(x) - \psi_T(t) \int_0^t W_2(t-t') \partial_x^2 (u^2)(x,t')\, \mathrm{d}t'.} \tag{2.5}$$

Therefore, in order to find a solution of (2.1), our goal is to use a fixed point argument for the map Φ_T, that is, solve the equation $\Phi_T u = u$. A natural question that appears here is: Which is the best space to solve this problem?

To motivate the solution space, we observe that one of the advantages in dealing with integral equation (2.5) is that we can use space-time Fourier transform to express the mapping Φ_T in the phase space (ξ, τ). More precisely, by the inverse Fourier transform formula we can write

$$\widehat{w}^x(\xi, t') = \frac{1}{2\pi} \int e^{i\tau t'} \widehat{w}(\xi, \tau)\, \mathrm{d}\tau,$$

which give us

$$\begin{aligned}\int_0^t W_2(t-t') w(x,t')\, \mathrm{d}t' &= \frac{1}{2\pi} \int_0^t \int e^{ix\xi} \left(\frac{e^{i\gamma(\xi)(t-t')} - e^{-i\gamma(\xi)(t-t')}}{2i\gamma(\xi)} \right) \widehat{w}^x(\xi, t')\, \mathrm{d}\xi\, \mathrm{d}t' \\ &= \frac{1}{4\pi^2} \int_0^t \iint e^{ix\xi} \left(\frac{e^{i\gamma(\xi)(t-t')} - e^{-i\gamma(\xi)(t-t')}}{2i\gamma(\xi)} \right) e^{i\tau t'} \widehat{w}(\xi, \tau)\, \mathrm{d}\tau\, \mathrm{d}\xi\, \mathrm{d}t'.\end{aligned}$$

Performing the t' integration first, that is, using the computation

$$\int_0^t e^{i(\tau \pm \gamma(\xi))t'}\, \mathrm{d}t' = -i \left(\frac{e^{i(\tau \pm \gamma(\xi))t} - 1}{\tau \pm \gamma(\xi)} \right),$$

we have

$$\begin{aligned}\int_0^t W_2(t-t') w(x,t')\, \mathrm{d}t' = &-\frac{i}{4\pi^2} \iint \frac{e^{i(x\xi + \gamma(\xi)t)}}{2i\gamma(\xi)} \left(\frac{e^{i(\tau - \gamma(\xi))t} - 1}{\tau - \gamma(\xi)} \right) \widehat{w}(\xi, \tau)\, \mathrm{d}\xi\, \mathrm{d}\tau \\ &+ \frac{i}{4\pi^2} \iint \frac{e^{i(x\xi - \gamma(\xi)t)}}{2i\gamma(\xi)} \left(\frac{e^{i(\tau + \gamma(\xi))t} - 1}{\tau + \gamma(\xi)} \right) \widehat{w}(\xi, \tau)\, \mathrm{d}\xi\, \mathrm{d}\tau.\end{aligned}$$

The last formula gives us the following decomposition of the map Φ_T

$$\Phi_T u(x,t) = \frac{1}{2\pi}\psi(t)\int e^{ix\xi}\left(\frac{e^{i\gamma(\xi)t}+e^{-i\gamma(\xi)t}}{2}\right)\widehat{u_0}(\xi)\,\mathrm{d}\xi \tag{2.6}$$

$$+\frac{1}{2\pi}\psi(t)\int e^{ix\xi}\left(\frac{e^{i\gamma(\xi)t}-e^{-i\gamma(\xi)t}}{2i\gamma(\xi)}\right)\widehat{\partial_x u_1}(\xi)\,\mathrm{d}\xi \tag{2.7}$$

$$-\frac{i}{4\pi^2}\psi_T(t)\iint \frac{e^{i(x\xi+\gamma(\xi)t)}}{2i\gamma(\xi)}\left(\frac{e^{i(\tau-\gamma(\xi))t}-1}{\tau-\gamma(\xi)}\right)\widehat{w}(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \tag{2.8}$$

$$+\frac{i}{4\pi^2}\psi_T(t)\iint \frac{e^{i(x\xi-\gamma(\xi)t)}}{2i\gamma(\xi)}\left(\frac{e^{i(\tau+\gamma(\xi))t}-1}{\tau+\gamma(\xi)}\right)\widehat{w}(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau\,, \tag{2.9}$$

where $w=\partial_x^2(u^2)$. Now, observing that the space-time Fourier transform of the solution to the homogeneous IVP associates to (2.1) lives on the curve $\tau=\pm\gamma(\xi)$, this motivates the spaces that we are going to introduce in the following subsection.

The Bourgain Spaces $X_{s,b}(\mathbb{R}^2)$

Having as a strategy to make a detailed study of the nonlinearity by using spaces related to the real linear problem of the gB equation, we introduce the Bourgain spaces.

Definition 2.2 (Space $\boldsymbol{X_{s,b}}(\mathbb{R}^2)$.)**.** *Let $s,b\in\mathbb{R}$, $X_{s,b}=X_{s,b}(\mathbb{R}^2)$ denotes the completion of the Schwartz class $\mathscr{S}(\mathbb{R}^2)$ with respect to the norm*

$$\|v\|_{X_{s,b}}=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\big\langle|\tau|-\gamma(\xi)\big\rangle^{2b}\langle\xi\rangle^{2s}|\widehat{v}(\xi,\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}\xi\right)^{\frac{1}{2}},$$

where $\gamma(\xi)=\sqrt{\xi^2+\xi^4}$ and $\langle\xi\rangle=\sqrt{1+\xi^2}$.

Since we are considering local in time well-posedness, we also need the localized $X_{s,b}$ spaces defined as follows.

Definition 2.3 (Space $\boldsymbol{X^T_{s,b}(\mathbb{R}^2)}$.)**.** *Let $s,b\in\mathbb{R}$. For $T\geq 0$, $X^T_{s,b}=X^T_{s,b}(\mathbb{R}^2)$ denotes the space endowed with the norm*

$$\|u\|_{X^T_{s,b}}=\inf_{v\in X_{s,b}}\left\{\|v\|_{X_{s,b}};\ v(x,t)=u(x,t)\ \text{on}\ \mathbb{R}\times[0,T]\right\}.$$

One of the reasons to deal with the Bourgain spaces lies in the next result, which says that for $b>\frac{1}{2}$ the space $X_{s,b}$ is embedding in $C(\mathbb{R},H^s)$.

Lemma 2.4 (Corollary 2.10 in [35] on page 101)**.** *Let $b>\frac{1}{2}$ and $s\in\mathbb{R}$. Then, the space $X_{s,b}$ is included in $C(\mathbb{R},H^s(\mathbb{R}))$ and, furthermore, there exists $C>0$, depending only on b, such that*

$$\sup_{t\in\mathbb{R}}\|u(\cdot,t)\|_{H^s}\leq C\|u\|_{X_{s,b}},\quad \textit{for all } u\in X_{s,b}.$$

Proof. First, we will prove that $X_{s,b}\subset L^\infty(\mathbb{R},H^s)$. Let $u\in X_{s,b}$, we write $u=u_1+u_2$, where

$$\widehat{u_1}=\chi_{\{\tau\leq 0\}}\widehat{u},\ \ \widehat{u_2}=\chi_{\{\tau>0\}}\widehat{u}$$

and χ_A denotes the characteristic function of the set A. Then, since $\left|e^{i\gamma(\xi)t}\right| = 1$, we observe that

$$\|u_1(x,t)\|_{H^s_x} = \left\|\langle\xi\rangle^s \widehat{u_1}^x(\xi,t)\right\|_{L^2_\xi} = \left\|\langle\xi\rangle^s e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right\|_{L^2_\xi} = \left\|\mathscr{F}_x^{-1}\left(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right)\right\|_{H^s_x},$$

for all $t \in \mathbb{R}$, where $\mathscr{F}_x^{-1}$ denotes the inverse Fourier transform with respect to the variable x. Thus, denoting by $\mathscr{F}_t$ the Fourier transform with respect t, we have

$$\begin{aligned}
\|u_1(x,t)\|_{H^s_x} &= \left\|\mathscr{F}_t^{-1}\mathscr{F}_t\left(\mathscr{F}_x^{-1}\left(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right)\right)\right\|_{H^s_x} \\
&= \frac{1}{2\pi}\left\|\int e^{it\tau}\mathscr{F}_t\left(\mathscr{F}_x^{-1}\left(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right)\right)(\tau)\mathrm{d}\tau\right\|_{H^s_x} \\
&= \frac{1}{2\pi}\left(\int \langle\xi\rangle^{2s}\left|\int e^{it\tau}\mathscr{F}_t\left(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right)(\tau)\mathrm{d}\tau\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\
&\le \frac{1}{2\pi}\left(\int\left(\int \langle\xi\rangle^{s}\left|\mathscr{F}_t\left(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right)(\tau)\right|\mathrm{d}\tau\right)^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\
&\le \frac{1}{2\pi}\int\left(\int \langle\xi\rangle^{2s}\left|\mathscr{F}_t\left(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right)(\tau)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}\mathrm{d}\tau,
\end{aligned}$$

where we used in the last step the Minkowski's integral inequality. Therefore,

$$\|u_1(x,t)\|_{H^s_x} \le \frac{1}{2\pi}\int\left\|\mathscr{F}_t\mathscr{F}_x^{-1}\left(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right)(x,\tau)\right\|_{H^s_x}\mathrm{d}\tau,$$

for all $t \in \mathbb{R}$. Using the Cauchy-Schwarz inequality, we obtain

$$\|u_1(x,t)\|_{H^s_x} \le \frac{1}{2\pi}\left(\int \langle\tau\rangle^{-2b}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int \langle\tau\rangle^{2b}\left\|\mathscr{F}_t\mathscr{F}_x^{-1}\left(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right)(x,\tau)\right\|^2_{H^s_x}\mathrm{d}\tau\right)^{\frac{1}{2}}.$$

We observe that

$$\begin{aligned}
\left\|\mathscr{F}_t\mathscr{F}_x^{-1}\left(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right)(x,\tau)\right\|^2_{H^s_x} &= \int \langle\xi\rangle^{2s}\left|\mathscr{F}_t\left(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\right)(\tau)\right|^2\mathrm{d}\xi \\
&= \int \langle\xi\rangle^{2s}\left|\int e^{-it\tau}e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\mathrm{d}t\right|^2\mathrm{d}\xi \\
&= \int \langle\xi\rangle^{2s}\left|\widehat{u}_1(\xi,\tau-\gamma(\xi))\right|^2\mathrm{d}\xi.
\end{aligned}$$

Thus,

$$\|u_1(x,t)\|_{H^s_x} \le \frac{1}{2\pi}\left(\int \langle\tau\rangle^{-2b}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int \langle\tau\rangle^{2b}\int \langle\xi\rangle^{2s}\left|\widehat{u_1}(\xi,\tau-\gamma(\xi))\right|^2\mathrm{d}\xi\mathrm{d}\tau\right)^{\frac{1}{2}},$$

and recalling that $\widehat{u_1} = \chi_{\{\tau\le 0\}}\widehat{u}$, we obtain

$$\begin{aligned}
\|u_1(x,t)\|_{H^s_x} &\le \frac{1}{2\pi}\left(\int \langle\tau\rangle^{-2b}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int \langle\xi\rangle^{2s}\int_{-\infty}^{\gamma(\xi)}\langle\tau\rangle^{2b}\left|\widehat{u}(\xi,\tau-\gamma(\xi)\right|^2\mathrm{d}\tau\mathrm{d}\xi\right)^{\frac{1}{2}} \\
&= \frac{1}{2\pi}\left(\int \langle\tau\rangle^{-2b}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int\int_{-\infty}^{0}\langle\xi\rangle^{2s}\langle\rho+\gamma(\xi)\rangle^{2b}\left|\widehat{u}(\xi,\rho)\right|^2\mathrm{d}\rho\mathrm{d}\xi\right)^{\frac{1}{2}},
\end{aligned}$$

by making the change of variables $\rho = \tau - \gamma(\xi)$. On the other hand, similar arguments imply that

$$\|u_2(x,t)\|_{H^s_x} \le \frac{1}{2\pi}\left(\int \langle\tau\rangle^{-2b}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int\int_0^{+\infty}\langle\xi\rangle^{2s}\left\langle\rho-\gamma(\xi)\right\rangle^{2b}\left|\widehat{u}(\xi,\rho)\right|^2\mathrm{d}\rho\mathrm{d}\xi\right)^{\frac{1}{2}},$$

for all $t\in\mathbb{R}$. Now, by the fact that $b>\frac{1}{2}$ and

$$\left||\rho|-\gamma(\xi)\right| = \begin{cases} |\rho-\gamma(\xi)|, & \text{for } \rho\ge 0,\\ |\rho+\gamma(\xi)|, & \text{for } \rho\le 0,\end{cases}$$

we have

$$\begin{aligned}\|u(x,t)\|_{H^s_x} \le & C_b'\left[\left(\iint_{-\infty}^{0}\langle\xi\rangle^{2s}\left\langle|\rho|-\gamma(\xi)\right\rangle^{2b}\left|\widehat{u}(\xi,\rho)\right|^2\mathrm{d}\rho\mathrm{d}\xi\right)^{\frac{1}{2}}+\left(\iint_{0}^{\infty}\langle\xi\rangle^{2s}\left\langle|\rho|-\gamma(\xi)\right\rangle^{2b}\left|\widehat{u}(\xi,\rho)\right|^2\mathrm{d}\rho\mathrm{d}\xi\right)^{\frac{1}{2}}\right]\\ \le & 2C_b'\,\|u(t,x)\|_{X_{s,b}},\end{aligned}$$

for all $t\in\mathbb{R}$, where $C_b' = (2\pi)^{-1}\left(\int\langle\tau\rangle^{-2b}\mathrm{d}\tau\right)^{\frac{1}{2}}$. Then, we conclude that

$$\|u(x,t)\|_{L^\infty(\mathbb{R},H^s)} \le C_b\,\|u(t,x)\|_{X_{s,b}},$$

that is, $u\in L^\infty(\mathbb{R},H^s)$.

It remains to show continuity to respect the variable t. Let $t\in\mathbb{R}$ and $\{t_n\}\subset\mathbb{R}$ a sequence such that $t_n\to t$. As we did above, we have

$$\|u_1(x,t)-u_1(x,t_n)\|_{H^s_x} = \left\|\int \mathscr{F}_t\mathscr{F}_x^{-1}\big(e^{i\gamma(\xi)t}\widehat{u_1}^x(\xi,t)\big)(x,\tau)\left(e^{it\tau}-e^{it_n\tau}\right)\mathrm{d}\tau\right\|_{H^s_x}. \tag{2.10}$$

Letting $n\to\infty$, two applications of Dominated Convergence Theorem give that the right side of (2.10) goes to zero. Therefore, $u_1\in C(\mathbb{R},H^s)$. Furthermore, the same argument applies to u_2, which concludes the proof. □

Since we would like to prove that Φ_T is a contraction map, we start with the following basic result to estimate its $X_{s,b}$-norm.

Lemma 2.5. *Let $s\in\mathbb{R}$, $u_0\in H^s(\mathbb{R})$, $u_1\in H^{s-1}(\mathbb{R})$ and $0<T<1$. For $f,g\in X_{s,b}$ we define the bilinear operator*

$$\Phi_T(f,g) \doteq \psi(t)\left(W_1(t)u_0(x)+W_2(t)\partial_x u_1(x)\right)-\psi_T(t)\int_0^t W_2(t-t')w_{f,g}(x,t')\mathrm{d}t',$$

where $w_{f,g}=\partial_x^2(fg)$. Let $-\frac{1}{2}<b'\le 0\le b\le b'+1$. Then, there exist a constant $C=C(\psi,b,b')$ such that

$$\left\|\Phi_T(f,g)\right\|_{X_{s,b}} \le C\left(\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}}\right)+CT^{1-(b-b')}\left\|\mathscr{F}^{-1}\left(\frac{\widehat{w_{f,g}}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{s,b'}},$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to both variables x and t.

We split the proof of Lemma 2.5 in the following two lemmas.

Lemma 2.6. *Let $u_0 \in H^s(\mathbb{R})$ and $u_1 \in H^{s-1}(\mathbb{R})$. Then, there exists $C>0$ depending only ψ, s and b such that*

$$\left\|\psi(t)\left(W_1(t)u_0(x)+W_2(t)\partial_x u_1(x)\right)\right\|_{X_{s,b}} \leq C\left(\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}}\right).$$

Proof. Denoting by $v(x,t)=W_1(t)u_0(x)+W_2(t)\partial_x u_1(x)$ and taking the Fourier transform with respect space variable, we have

$$\begin{aligned}
\widehat{\psi v}^{\,x}(\xi,t) &= \psi(t)\,\widehat{\left[W_1(t)u_0\right]}^{x}(\xi,t)+\psi(t)\,\widehat{\left[W_2(t)\partial_x u_1\right]}^{x}(\xi,t)\\
&= \psi(t)\left(\frac{e^{it\gamma(\xi)}+e^{-it\gamma(\xi)}}{2}\right)\widehat{u_0}(\xi)+\psi(t)\left(\frac{e^{it\gamma(\xi)}-e^{-it\gamma(\xi)}}{2i\gamma(\xi)}\right)\widehat{\partial_x u_1}(\xi)\\
&= e^{it\gamma(\xi)}\psi(t)\left(\frac{\widehat{u_0}(\xi)}{2}+\frac{\widehat{\partial_x u_1}(\xi)}{2i\gamma(\xi)}\right)+e^{-it\gamma(\xi)}\psi(t)\left(\frac{\widehat{u_0}(\xi)}{2}-\frac{\widehat{\partial_x u_1}(\xi)}{2i\gamma(\xi)}\right)\\
&= e^{it\gamma(\xi)}\psi(t)h_1(\xi)+e^{-it\gamma(\xi)}\psi(t)h_2(\xi),
\end{aligned}$$

where

$$h_1(\xi)=\frac{\widehat{u_0}(\xi)}{2}+\frac{\widehat{\partial_x u_1}(\xi)}{2i\gamma(\xi)} \quad \text{and} \quad h_2(\xi)=\frac{\widehat{u_0}(\xi)}{2}-\frac{\widehat{\partial_x u_1}(\xi)}{2i\gamma(\xi)}.$$

Thus,

$$\widehat{\psi v}(\xi,\tau)=\int e^{-it\tau}\widehat{\psi v}^{x}(\xi,t)\,\mathrm{d}t = h_1(\xi)\int e^{-it(\tau-\gamma(\xi))}\psi(t)\,\mathrm{d}t+h_2(\xi)\int e^{-it(\tau+\gamma(\xi))}\psi(t)\,\mathrm{d}t,$$

which implies that $\widehat{\psi v}(\xi,\tau)=h_1(\xi)\widehat{\psi}(\tau-\gamma(\xi))+h_2(\xi)\widehat{\psi}(\tau+\gamma(\xi))$. Therefore, we obtain

$$\begin{aligned}
\|\psi v\|_{X_{s,b}} &= \left\|\langle|\tau|-\gamma(\xi)\rangle^b\langle\xi\rangle^s\widehat{\psi v}(\xi,\tau)\right\|_{L^2_{\xi,\tau}}\\
&\leq \left\|\langle|\tau|-\gamma(\xi)\rangle^b\langle\xi\rangle^s h_1(\xi)\widehat{\psi}(\tau-\gamma(\xi))\right\|_{L^2_{\xi,\tau}}+\left\|\langle|\tau|-\gamma(\xi)\rangle^b\langle\xi\rangle^s h_2(\xi)\widehat{\psi}(\tau+\gamma(\xi))\right\|_{L^2_{\xi,\tau}}\\
&\leq \left\|\langle\tau-\gamma(\xi)\rangle^b\langle\xi\rangle^s h_1(\xi)\widehat{\psi}(\tau-\gamma(\xi))\right\|_{L^2_{\xi,\tau}}+\left\|\langle\tau+\gamma(\xi)\rangle^b\langle\xi\rangle^s h_2(\xi)\widehat{\psi}(\tau+\gamma(\xi))\right\|_{L^2_{\xi,\tau}},
\end{aligned}$$

since $\langle|\tau|-\gamma(\xi)\rangle \leq \langle\tau\pm\gamma(\xi)\rangle$ and $b\geq 0$. Now, by making a change of variables in the τ-integral, we observe that

$$\begin{aligned}
\left\|\langle\tau-\gamma(\xi)\rangle^b\langle\xi\rangle^s h_1(\xi)\widehat{\psi}(\tau-\gamma(\xi))\right\|_{L^2_{\xi,\tau}} &= \left(\int\langle\xi\rangle^{2s}|h_1(\xi)|^2\int\langle\tau-\gamma(\xi)\rangle^{2b}|\widehat{\psi}(\tau-\gamma(\xi))|^2\,\mathrm{d}\tau\,\mathrm{d}\xi\right)^{\frac{1}{2}}\\
&= \left(\int\langle\xi\rangle^{2s}|h_1(\xi)|^2\int\langle\rho\rangle^{2b}|\widehat{\psi}(\rho)|^2\,\mathrm{d}\rho\,\mathrm{d}\xi\right)^{\frac{1}{2}}\\
&= \|\psi\|_{H^b}\left\|\langle\xi\rangle^s h_1(\xi)\right\|_{L^2_\xi}.
\end{aligned}$$

Similarly,

$$\left\|\langle\tau+\gamma(\xi)\rangle^b\langle\xi\rangle^s h_2(\xi)\widehat{\psi}(\tau+\gamma(\xi))\right\|_{L^2_{\tau,\xi}} = \|\psi\|_{H^b}\left\|\langle\xi\rangle^s h_2(\xi)\right\|_{L^2_\xi}.$$

On the other hand, we have

$$\left\|\langle\xi\rangle^s h_1(\xi)\right\|_{L^2_\xi} \le \frac{1}{2}\left(\left\|\langle\xi\rangle^s \widehat{u_0}(\xi)\right\|_{L^2_\xi} + \left\|\langle\xi\rangle^s \gamma(\xi)^{-1}\widehat{\partial_x u_1}(\xi)\right\|_{L^2_\xi}\right)$$
$$= \frac{1}{2}\left(\|u_0\|_{H^s} + \left\|\langle\xi\rangle^s \gamma(\xi)^{-1}\xi\widehat{u_1}(\xi)\right\|_{L^2_\xi}\right),$$

and, in the same way,

$$\left\|\langle\xi\rangle^s h_2(\xi)\right\|_{L^2_\xi} \le \frac{1}{2}\left(\left\|\langle\xi\rangle^s \widehat{u_0}(\xi)\right\|_{L^2_\xi} + \left\|\langle\xi\rangle^s \gamma(\xi)^{-1}\widehat{\partial_x u_1}(\xi)\right\|_{L^2_\xi}\right)$$
$$= \frac{1}{2}\left(\|u_0\|_{H^s} + \left\|\langle\xi\rangle^s \gamma(\xi)^{-1}\xi\widehat{u_1}(\xi)\right\|_{L^2_\xi}\right).$$

Then, we conclude that

$$\|\psi v\|_{X_{s,b}} \le \|\psi\|_{H^b}\left(\|u_0\|_{H^s} + \left\|\langle\xi\rangle^s \gamma(\xi)^{-1}\xi\widehat{u_1}(\xi)\right\|_{L^2_\xi}\right)$$
$$= \|\psi\|_{H^b}\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right),$$

since $|\xi|\gamma(\xi)^{-1} = |\xi|(\xi^2+\xi^4)^{-\frac{1}{2}} = (1+\xi^2)^{-\frac{1}{2}} = \langle\xi\rangle^{-1}$. □

The next lemma is concerning the $X_{s,b}$-norm of the nonlinear part of Φ_T, its proof is a consequence of Lemma 1.8.

Lemma 2.7. *Let $s\in\mathbb{R}$, $-\frac{1}{2} < b' \le 0 \le b \le b'+1$ and $0 < T \le 1$, then*

$$\left\|\psi_T(t)\int_0^t W_2(t-t')w(x,t')\,\mathrm{d}t'\right\|_{X_{s,b}} \le CT^{1-(b-b')}\left\|\mathscr{F}^{-1}\left(\frac{\widehat{w}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{s,b'}}. \tag{2.11}$$

Proof. To prove (2.11), we recall that the operator W_2 is given by

$$\widehat{W_2(t)\varphi}^x(\xi) = \left(\frac{e^{i\gamma(\xi)t} - e^{-i\gamma(\xi)t}}{2i\gamma(\xi)}\right)\widehat{\varphi}(\xi).$$

Denoting by $U(x,t) = \psi_T(t)\int_0^t W_2(t-t')w(x,t')\,\mathrm{d}t'$, we have

$$\widehat{U}^x(\xi,t) = \psi_T(t)\int_0^t \left(\frac{e^{i(t-t')\gamma(\xi)} - e^{-i(t-t')\gamma(\xi)}}{2i\gamma(\xi)}\right)\widehat{w}^x(\xi,t')\,\mathrm{d}t'$$
$$= e^{it\gamma(\xi)}\psi_T(t)\int_0^t h_1(\xi,t')\,\mathrm{d}t' - e^{-it\gamma(\xi)}\psi_T(t)\int_0^t h_2(\xi,t')\,\mathrm{d}t',$$

where

$$h_1(\xi,t') = \frac{e^{-it'\gamma(\xi)}\,\widehat{w}^x(\xi,t')}{2i\gamma(\xi)} \quad \text{and} \quad h_2(\xi,t') = \frac{e^{it'\gamma(\xi)}\,\widehat{w}^x(\xi,t')}{2i\gamma(\xi)}.$$

We define v_1 and v_2 by

$$\widehat{v_1}^x(\xi,t) = \psi_T(t)\int_0^t h_1(\xi,t')\,\mathrm{d}t' \quad \text{and} \quad \widehat{v_2}^x(\xi,t) = \psi_T(t)\int_0^t h_2(\xi,t')\,\mathrm{d}t'.$$

Thus, we can write $\widehat{U}^x(\xi,t) = e^{it\gamma(\xi)}\widehat{v_1}^x(\xi,t) - e^{-it\gamma(\xi)}\widehat{v_2}^x(\xi,t)$, which implies that

$$\begin{aligned}\widehat{U}(\xi,\tau) &= \int e^{-it\tau}\widehat{U}^x(\xi,t)\mathrm{d}t\\ &= \int e^{-it(\tau-\gamma(\xi))}\widehat{v_1}^x(\xi,t)\mathrm{d}t - \int e^{-it(\tau+\gamma(\xi))}\widehat{v_2}^x(\xi,t)\mathrm{d}t\\ &= \widehat{v_1}(\xi,\tau-\gamma(\xi)) - \widehat{v_2}(\xi,\tau+\gamma(\xi)).\end{aligned}$$

Now, using the definition of $X_{s,b}$-norm, we obtain

$$\begin{aligned}\|U(x,t)\|_{X_{s,b}} &= \left\|\left\langle|\tau|-\gamma(\xi)\right\rangle^b\langle\xi\rangle^s\,\widehat{U}(\xi,\tau)\right\|_{L^2_{\xi,\tau}}\\ &\le \left\|\left\langle|\tau|-\gamma(\xi)\right\rangle^b\langle\xi\rangle^s\,\widehat{v_1}(\xi,\tau-\gamma(\xi))\right\|_{L^2_{\xi,\tau}} + \left\|\left\langle|\tau|-\gamma(\xi)\right\rangle^b\langle\xi\rangle^s\,\widehat{v_2}(\xi,\tau+\gamma(\xi))\right\|_{L^2_{\xi,\tau}}.\end{aligned}$$

Since $\left\langle|\tau|-\gamma(\xi)\right\rangle \le \left\langle\tau\pm\gamma(\xi)\right\rangle$ and $b\ge 0$, we have

$$\|U(x,t)\|_{X_{s,b}} \le \left\|\left\langle\tau-\gamma(\xi)\right\rangle^b\langle\xi\rangle^s\,\widehat{v_1}(\xi,\tau-\gamma(\xi))\right\|_{L^2_{\xi,\tau}} + \left\|\left\langle\tau+\gamma(\xi)\right\rangle^b\langle\xi\rangle^s\,\widehat{v_2}(\xi,\tau+\gamma(\xi))\right\|_{L^2_{\xi,\tau}}.$$

We observe that

$$\begin{aligned}\left\|\left\langle\tau-\gamma(\xi)\right\rangle^b\langle\xi\rangle^s\,\widehat{v_1}(\xi,\tau-\gamma(\xi))\right\|_{L^2_{\xi,\tau}} &= \left(\iint\left\langle\tau-\gamma(\xi)\right\rangle^{2b}\langle\xi\rangle^{2s}|\widehat{v_1}(\xi,\tau-\gamma(\xi))|^2\mathrm{d}\xi\mathrm{d}\tau\right)^{\frac{1}{2}}\\ &= \left(\int\langle\xi\rangle^{2s}\int\left\langle\tau-\gamma(\xi)\right\rangle^{2b}|\widehat{v_1}(\xi,\tau-\gamma(\xi))|^2\mathrm{d}\tau\mathrm{d}\xi\right)^{\frac{1}{2}}\\ &= \left(\int\langle\xi\rangle^{2s}\int\left\langle\rho\right\rangle^{2b}|\widehat{v_1}(\xi,\rho)|^2\mathrm{d}\rho\mathrm{d}\xi\right)^{\frac{1}{2}}\\ &= \left(\int\langle\xi\rangle^{2s}\,\|\widehat{v_1}^x(\xi,t)\|^2_{H^b_t}\mathrm{d}\xi\right)^{\frac{1}{2}},\end{aligned}$$

where we used the change of variables $\rho=\tau-\gamma(\xi)$ in the τ-integral. Similarly,

$$\left\|\left\langle\tau+\gamma(\xi)\right\rangle^b\langle\xi\rangle^s\,\widehat{v_2}(\xi,\tau+\gamma(\xi))\right\|_{L^2_{\xi,\tau}} = \left(\int\langle\xi\rangle^{2s}\,\|\widehat{v_2}(\xi,t)\|^2_{H^b_t}\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

It follows Lemma 1.8 that

$$\|\widehat{v_1}^x(\xi,t)\|_{H^b_t} = \left\|\psi_T(t)\int_0^t h_1(\xi,t')\,\mathrm{d}t'\right\|_{H^b_t} \le \tilde{C}T^{1-(b-b')}\,\|h_1(\xi,t)\|_{H^{b'}_t}$$

and

$$\|\widehat{v_2}^x(\xi,t)\|_{H^b_t} = \left\|\psi_T(t)\int_0^t h_2(\xi,t')\,\mathrm{d}t'\right\|_{H^b_t} \le \tilde{C}T^{1-(b-b')}\,\|h_2(\xi,t)\|_{H^{b'}_t}.$$

Therefore,

$$\begin{aligned}\|U(t,x)\|_{X_{s,b}} &\le \tilde{C}T^{1-(b-b')}\left(\left(\int \langle\xi\rangle^{2s}\|h_1(\xi,t)\|^2_{H_t^{b'}}\mathrm{d}\xi\right)^{\frac12}+\left(\int \langle\xi\rangle^{2s}\|h_2(\xi,t)\|^2_{H_t^{b'}}\mathrm{d}\xi\right)^{\frac12}\right)\\ &= \tilde{C}T^{1-(b-b')}\left(\left(\iint \langle\xi\rangle^{2s}\langle\tau\rangle^{2b'}\left|\widehat{h_1}^{t}(\xi,\tau)\right|^2\mathrm{d}\tau\mathrm{d}\xi\right)^{\frac12}+\left(\iint \langle\xi\rangle^{2s}\langle\tau\rangle^{2b'}\left|\widehat{h_2}^{t}(\xi,\tau)\right|^2\mathrm{d}\tau\mathrm{d}\xi\right)^{\frac12}\right),\end{aligned}$$

where

$$\widehat{h_1}^{t}(\xi,\tau)=\int e^{-it\tau}h_1(\xi,t)\mathrm{d}t=\frac{1}{2i\gamma(\xi)}\int e^{-it(\tau-\gamma(\xi))}\,\widehat{w}^{x}(\xi,t)\mathrm{d}t=\frac{\widehat{w}(\xi,\tau-\gamma(\xi))}{2i\gamma(\xi)}$$

and

$$\widehat{h_2}^{t}(\xi,\tau)=\int e^{-it\tau}h_2(\xi,t)\mathrm{d}t=\frac{1}{2i\gamma(\xi)}\int e^{-it(\tau+\gamma(\xi))}\,\widehat{w}^{x}(\xi,t)\mathrm{d}t=\frac{\widehat{w}(\xi,\tau+\gamma(\xi))}{2i\gamma(\xi)}.$$

Thus, we obtain

$$\begin{aligned}\|U(x,t)\|_{X_{s,b}} \le& \tilde{C}T^{1-(b-b')}\left(\int \frac{\langle\xi\rangle^{2s}}{|2i\gamma(\xi)|^2}\int \langle\tau\rangle^{2b'}\left|\widehat{w}(\xi,\tau-\gamma(\xi))\right|^2\mathrm{d}\tau\mathrm{d}\xi\right)^{\frac12}\\ &+\tilde{C}T^{1-(b-b')}\left(\int \frac{\langle\xi\rangle^{2s}}{|2i\gamma(\xi)|^2}\int \langle\tau\rangle^{2b'}\left|\widehat{w}(\xi,\tau+\gamma(\xi))\right|^2\mathrm{d}\tau\mathrm{d}\xi\right)^{\frac12}.\end{aligned}$$

By making the change of variables $\rho=\tau-\gamma(\xi)$ in the first integral and $\rho=\tau+\gamma(\xi)$ in the second one, we have

$$\begin{aligned}\|U(x,t)\|_{X_{s,b}} \le& \tilde{C}T^{1-(b-b')}\left(\int \frac{\langle\xi\rangle^{2s}}{|2i\gamma(\xi)|^2}\int \left\langle\rho+\gamma(\xi)\right\rangle^{2b'}\left|\widehat{w}(\xi,\rho)\right|^2\mathrm{d}\rho\,\mathrm{d}\xi\right)^{\frac12}\\ &+\tilde{C}T^{1-(b-b')}\left(\int \frac{\langle\xi\rangle^{2s}}{|2i\gamma(\xi)|^2}\int \left\langle\rho-\gamma(\xi)\right\rangle^{2b'}\left|\widehat{w}(\xi,\rho)\right|^2\mathrm{d}\rho\,\mathrm{d}\xi\right)^{\frac12}\\ \le& 2CT^{1-(b-b')}\left(\iint \langle\xi\rangle^{2s}\left\langle|\rho|-\gamma(\xi)\right\rangle^{2b'}\left|\frac{\widehat{w}(\xi,\rho)}{2i\gamma(\xi)}\right|^2\mathrm{d}\rho\,\mathrm{d}\xi\right)^{\frac12},\end{aligned}$$

where we used that $\left\langle|\rho|-\gamma(\xi)\right\rangle\le\left\langle\rho\pm\gamma(\xi)\right\rangle$ and $b'\le 0$. We conclude that

$$\|U(x,t)\|_{X_{s,b}}\le CT^{1-(b-b')}\left\|\mathscr{F}^{-1}\left(\frac{\widehat{w}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{s,b'}},$$

which finishes the proof of (2.11). □

At this point, we see that we need the bilinear estimate, which is the key ingredient for proving that the map Φ_T is a contraction.

The Bilinear Estimates

We start by showing an elementary inequality that will be useful in the proof of the bilinear estimates.

Lemma 2.8. *There exists $c > 0$ such that*

$$\frac{1}{c} \leq \sup_{x,y\geq 0} \left(\frac{1+|x-y|}{1+\left|x-\sqrt{y^2+y}\right|} \right) \leq c.$$

Proof. For $y \geq 0$, we have $y = \sqrt{y^2} \leq \sqrt{y^2+y}$ and

$$\left(y+\frac{1}{2}\right)^2 = y^2+y+\frac{1}{4} \geq y^2+y,$$

which implies

$$y \leq \sqrt{y^2+y} \leq y+\frac{1}{2},$$

for all $y \geq 0$. Thus,

$$1+\left|x-\sqrt{y^2+y}\right| \geq 1+|x|-\sqrt{y^2+y} \geq 1+x-\frac{1}{2}-y = \frac{1}{2}+(x-y)$$

and

$$1+\left|x-\sqrt{y^2+y}\right| \geq 1+\sqrt{y^2+y}-x \geq 1+(y-x) \geq \frac{1}{2}+(y-x),$$

for all $x, y \geq 0$. Therefore,

$$1+\left|x-\sqrt{y^2+y}\right| \geq \frac{1}{2}+|x-y| \geq \frac{1}{2}\left(1+|x-y|\right),$$

implying that

$$\frac{1+|x-y|}{1+|x-\sqrt{y^2+y}|} \leq 2, \quad \text{for all } x, y \geq 0.$$

On the other hand, we observe that

$$1+|x-y| \geq 1+x-y \geq 1+x-\sqrt{y^2+y} \geq \frac{1}{2}+\left(x-\sqrt{y^2+y}\right)$$

and

$$1+|x-y| \geq 1+y-x \geq 1-\frac{1}{2}+\sqrt{y^2+y}-x = \frac{1}{2}+\left(\sqrt{y^2+y}-x\right).$$

Therefore,

$$1+|x-y| \geq \frac{1}{2}+\left|x-\sqrt{y^2+y}\right| \geq \frac{1}{2}\left(1+\left|x-\sqrt{y^2+y}\right|\right).$$

Then, we conclude that

$$\frac{1+|x-y|}{1+|x-\sqrt{y^2+y}|} \geq \frac{1}{2},$$

which finishes the proof. □

Remark 2.1. In view of the previous lemma, we have an equivalent way to compute the $X_{s,b}$ -norm, that is,

$$\|u\|_{X_{s,b}} \simeq \left\| \left\langle |\tau|-\xi^2 \right\rangle^b \langle\xi\rangle^s \,\widehat{u}(\tau,\xi) \right\|_{L^2_{\xi,\tau}}. \tag{2.12}$$

In fact, we just need to use the result given by Lemma 2.8

$$\frac{1}{c} \le \frac{1+||\tau|-\xi^2|}{1+||\tau|-\gamma(\xi)|} \le c,$$

and the fact $\langle x\rangle \simeq (1+|x|)$, for all $x \in \mathbb{R}$.

In the proof of the next theorem, which is called the bilinear estimates, we are going to use the right side of (2.12) as definition of $X_{s,b}$-norm.

Theorem 2.9 (Sobolev Bilinear Estimates [12])**.** *Let $s > -\frac{1}{4}$ and $u, v \in X_{s,b}(\mathbb{R})$. Then, there exists $C > 0$ such that*

$$\left\| \mathscr{F}^{-1}\left(\frac{|\xi|^2 \widehat{uv}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{s,-a}} \le C \|u\|_{X_{s,b}} \|v\|_{X_{s,b}} \tag{2.13}$$

holds in the following cases:

(i) $s \ge 0$, $b > \frac{1}{2}$ and $\frac{1}{4} < a < \frac{1}{2}$.

(ii) $-\frac{1}{4} < s < 0$, $b > \frac{1}{2}$ and $\frac{1}{4} < a < \frac{1}{2}$ such that $|s| < \frac{a}{2}$.

Moreover, the constant that appears in (2.13) *depends only on a, b and s.*

Proof. We begin by noticing that

$$\begin{aligned}\left\| \mathscr{F}^{-1}\left(\frac{|\xi|^2 \widehat{uv}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{s,-a}} &= \left\| \frac{\langle\xi\rangle^s |\xi|^2}{\big\langle |\tau|-\gamma(\xi)\big\rangle^a 2i\gamma(\xi)} (2\pi)^{-2}\widehat{u} * \widehat{v}(\xi,\tau)\right\|_{L^2_{\xi,\tau}} \\ &\le \left\| \frac{\langle\xi\rangle^s |\xi|^2}{\big\langle |\tau|-\gamma(\xi)\big\rangle^a 2i\gamma(\xi)} (2\pi)^{-2} \iint |\widehat{u}(\xi-\xi_1,\tau-\tau_1)||\widehat{v}(\xi_1,\tau_1)| d\xi_1 d\tau_1 \right\|_{L^2_{\xi,\tau}}.\end{aligned}$$

Then, in order to prove (2.13) it is sufficient to show that

$$\left\| \frac{\langle\xi\rangle^s |\xi|^2}{\big\langle |\tau|-\gamma(\xi)\big\rangle^a 2i\gamma(\xi)} \iint |\widehat{u}(\xi-\xi_1,\tau-\tau_1)||\widehat{v}(\xi_1,\tau_1)| d\xi_1 d\tau_1 \right\|_{L^2_{\xi,\tau}} \le C \|u\|_{X_{s,b}} \|v\|_{X_{s,b}}. \tag{2.14}$$

Let $u, v \in X_{s,b}$ and define

$$f(\xi,\tau) = \big\langle |\tau|-\xi^2\big\rangle^b \langle\xi\rangle^s |\widehat{u}(\xi,\tau)| \quad \text{and} \quad g(\xi,\tau) = \big\langle |\tau|-\xi^2\big\rangle^b \langle\xi\rangle^s |\widehat{v}(\xi,\tau)|,$$

which are functions in $L^2(\mathbb{R}^2)$. We affirm that (2.14) is equivalent to the following inequality

$$|W(f,g,\varphi)| \le C \|f\|_{L^2} \|g\|_{L^2} \|\varphi\|_{L^2}, \quad \text{for all } \varphi \in L^2(\mathbb{R}^2), \tag{2.15}$$

where

$$W(f,g,\varphi) = \int_{\mathbb{R}^4} \frac{|\xi|^2}{2i\gamma(\xi)} \frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^s \langle\xi-\xi_1\rangle^s} \frac{g(\xi_1,\tau_1) f(\xi-\xi_1,\tau-\tau_1)\overline{\varphi}(\xi,\tau)}{\big\langle |\tau|-\xi^2\big\rangle^a \big\langle |\tau_1|-\xi_1^2\big\rangle^b \big\langle |\tau-\tau_1|-(\xi-\xi_1)^2\big\rangle^b} d\xi\, d\tau\, d\xi_1\, d\tau_1.$$

In fact, to prove that (2.14) is equivalent to (2.15), we use a duality argument as follows. We can write

$$W(f,g,\varphi) = \big[h(\xi,\tau), \varphi(\xi,\tau)\big],$$

where $[\,,\,]$ denotes the inner product on $L^2(\mathbb{R}^2)$ and

$$h(\xi,\tau)=\int_{\mathbb{R}^2}\frac{|\xi|^2}{2i\gamma(\xi)}\frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^2\langle\xi-\xi_1\rangle^s}\frac{g(\xi_1,\tau_1)f(\xi-\xi_1,\tau-\tau_1)}{\left\langle|\tau|-\xi^2\right\rangle^a\left\langle|\tau_1|-\xi_1^2\right\rangle^b\left\langle|\tau-\tau_1|-(\xi-\xi_1)^2\right\rangle^b}\,d\xi_1\,d\tau_1$$
$$=\frac{|\xi|^2}{2i\gamma(\xi)}\frac{\langle\xi\rangle^s}{\left\langle|\tau|-\xi^2\right\rangle^a}\int_{\mathbb{R}^2}|\widehat{v}(\xi_1,\tau_1)||\widehat{u}(\xi-\xi_1,\tau-\tau_1)|\,d\xi_1\,d\tau_1\,.$$

If (2.14) holds, using Cauchy-Schwarz inequality we obtain

$$\begin{aligned}|W(f,g,\varphi)|&\le\|h(\xi,\tau)\|_{L^2}\|\varphi\|_{L^2}\\&=\left\|\frac{\langle\xi\rangle^s|\xi|^2}{\left\langle|\tau|-\gamma(\xi)\right\rangle^a2i\gamma(\xi)}\iint|\widehat{u}(\xi-\xi_1,\tau-\tau_1)||\widehat{v}(\xi_1,\tau_1)|d\xi_1d\tau_1\right\|_{L^2}\|\varphi\|_{L^2}\\&\le C\|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\|\varphi\|_{L^2}\\&=C\|f\|_{L^2}\|g\|_{L^2}\|\varphi\|_{L^2},\end{aligned}$$

which proves that (2.15) also holds. On the other hand, if (2.15) holds for all $\varphi\in L^2(\mathbb{R}^2)$, it follows from Riesz Representation Theorem that $h\in L^2(\mathbb{R}^2)$ and $\|h\|_{L^2}\le C\|f\|_{L^2}\|g\|_{L^2}$, which guarantees that inequality (2.14) also occurs.

Then, we observe that to perform the desired estimate we need to analyse all the possible cases for the sign of τ, τ_1 and $\tau-\tau_1$. To do this, we split $\mathbb{R}^4$ into the following regions

$$\begin{aligned}\Gamma_1&=\left\{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4;\ \tau_1<0,\ \tau-\tau_1<0\right\}\\\Gamma_2&=\left\{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4;\ \tau_1\ge0,\ \tau-\tau_1<0,\ \tau\ge0\right\}\\\Gamma_3&=\left\{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4;\tau_1\ge0,\ \tau-\tau_1<0,\ \tau<0\right\}\\\Gamma_4&=\left\{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4;\ \tau_1<0,\ \tau-\tau_1\ge0,\ \tau\ge0\right\}\\\Gamma_5&=\left\{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4;\ \tau_1<0,\ \tau-\tau_1\ge0,\ \tau<0\right\}\\\Gamma_6&=\left\{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4;\ \tau_1\ge0,\ \tau-\tau_1\ge0\right\}.\end{aligned}$$

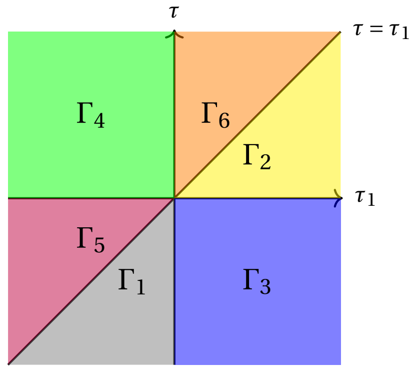

Figure 6: Bilinear estimates regions I.

Thus, it is sufficient to prove inequality (2.15) with $Z(f,g,\varphi)$ instead of $W(f,g,\varphi)$, where

$$Z(f,g,\varphi)=\int_{\mathbb{R}^4}\frac{|\xi|^2}{2i\gamma(\xi)}\frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s}\frac{g(\xi_1,\tau_1)f(\xi_2,\tau_2)\overline{\varphi}(\xi,\tau)}{\langle\sigma\rangle^a\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\,d\xi\,d\tau\,d\xi_1\,d\tau_1\,,$$

with $\xi_2=\xi-\xi_1$, $\tau_2=\tau-\tau_1$ and σ, σ_1, σ_2 belonging to one of the following cases

(I) $\sigma=\tau+\xi^2,\ \sigma_1=\tau_1+\xi_1^2,\ \sigma_2=\tau_2+\xi_2^2.$

(II) $\sigma=\tau-\xi^2,\ \sigma_1=\tau_1-\xi_1^2,\ \sigma_2=\tau_2+\xi_2^2.$

(III) $\sigma=\tau+\xi^2,\ \sigma_1=\tau_1-\xi_1^2,\ \sigma_2=\tau_2+\xi_2^2.$

(IV) $\sigma=\tau-\xi^2,\ \sigma_1=\tau_1+\xi_1^2,\ \sigma_2=\tau_2-\xi_2^2.$

(V) $\sigma=\tau+\xi^2,\ \sigma_1=\tau_1+\xi_1^2,\ \sigma_2=\tau_2-\xi_2^2.$

(VI) $\sigma=\tau-\xi^2,\ \sigma_1=\tau_1-\xi_1^2,\ \sigma_2=\tau_2-\xi_2^2.$

We observe that the cases

$$\sigma=\tau+\xi^2,\ \sigma_1=\tau_1-\xi_1^2,\ \sigma_2=\tau_2-\xi_2^2\quad\text{and}\quad\sigma=\tau-\xi^2,\ \sigma_1=\tau_1+\xi_1^2,\ \sigma_2=\tau_2+\xi_2^2,$$

cannot occur, since $\tau_1 < 0$, $\tau - \tau_1 < 0$ implies $\tau < 0$ and $\tau_1 \geq 0$, $\tau - \tau_1 \geq 0$ implies $\tau \geq 0$.

Applying the change of variables

$$(\xi, \tau, \xi_1, \tau_1) \to -(\xi, \tau, \xi_1, \tau_1)$$

and observing that the L^2−norm is preserved under the reflection operation, the cases (IV), (V) and (VI) can be easily reduced to (III), (II) and (I), respectively. In fact, for example, let us see how the case (IV) is reduced to (III). We assume that (III) has already been proven, then

$$\begin{aligned}
&\left| \int_{\mathbb{R}^4} \frac{|\xi|^2}{2i\gamma(\xi)} \frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s} \frac{g(\xi_1,\tau_1) f(\xi_2,\tau_2)\overline{\varphi}(\xi,\tau)}{\left\langle \tau - \xi^2\right\rangle^a \left\langle \tau_1 + \xi_1^2\right\rangle^b \left\langle \tau_2 - \xi_2^2\right\rangle^b} \, d\xi\, d\tau\, d\xi_1\, d\tau_1 \right| = \\
&\qquad = \left| \int_{\mathbb{R}^4} \frac{|\xi|^2}{2i\gamma(\xi)} \frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s} \frac{g(-\xi_1,-\tau_1) f(\xi_1-\xi,\tau_1-\tau)\overline{\varphi}(-\xi,-\tau)}{\left\langle \tau + \xi^2\right\rangle^a \left\langle \tau_1 - \xi_1^2\right\rangle^b \left\langle \tau_2 + \xi_2^2\right\rangle^b} \, d\xi\, d\tau\, d\xi_1\, d\tau_1 \right| \\
&\qquad \leq C \|f\|_{L^2} \|g(-\xi_1,-\tau_1)\|_{L^2_{\xi_1,\tau_1}} \|\varphi(-\xi,-\tau)\|_{L^2_{\xi,\tau}} \\
&\qquad = C \|f\|_{L^2} \|g\|_{L^2} \|\varphi\|_{L^2}.
\end{aligned}$$

that is, the case (IV) is also proved in this case.

Moreover, in the same way, making the change of variables $\tau_2 = \tau - \tau_1$, $\xi_2 = \xi - \xi_1$ and then $(\xi,\tau,\xi_2,\tau_2) \to -(\xi,\tau,\xi_2,\tau_2)$, the case (II) can be reduced to (III). Therefore, we only need to establish cases (I) and (III).

We first treat the inequality (2.15) with $Z(f,g,\varphi)$ in the case (I). In this situation, we will use the following algebraic relation

$$-(\tau+\xi^2) + (\tau_1+\xi_1^2) + \left(\tau - \tau_1 + (\xi-\xi_1)^2\right) = 2\xi_1(\xi_1 - \xi). \tag{2.16}$$

We can write $\mathbb{R}^4 = A \cup B$, where

$$\begin{aligned}
A &= \left\{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4;\ |\tau - \tau_1 + (\xi-\xi_1)^2| \leq |\tau_1 + \xi_1^2|\right\} \quad \text{and} \\
B &= \left\{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4;\ |\tau - \tau_1 + (\xi-\xi_1)^2| \geq |\tau_1 + \xi_1^2|\right\}.
\end{aligned}$$

Considering $Z(f,g,\varphi)$ in the case (I) and making the change of variables $\xi_2 = \xi - \xi_1$ and $\tau_2 = \tau - \tau_1$, we obtain

$$\begin{aligned}
&\int_B \frac{|\xi|^2}{2i\gamma(\xi)} \frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^s \langle\xi_2\rangle^s} \frac{g(\xi_1,\tau_1) f(\xi_2,\tau_2)\overline{\varphi}(\xi,\tau)}{\left\langle \tau + \xi^2\right\rangle^a \left\langle \tau_1 + \xi_1^2\right\rangle^b \langle \tau_2 + \xi_2\rangle^b} \, d\xi\, d\tau\, d\xi_1\, d\tau_1 \\
&\quad = \int_A \frac{|\xi|^2}{2i\gamma(\xi)} \frac{\langle\xi\rangle^s}{\langle\xi-\xi_2\rangle^s \langle\xi_2\rangle^s} \frac{g(\xi-\xi_2,\tau-\tau_2) f(\xi_2,\tau_2)\overline{\varphi}(\xi,\tau)}{\left\langle \tau + \xi^2\right\rangle^a \left\langle (\tau-\tau_2) + (\xi-\xi_2)^2\right\rangle^b \langle \tau_2 + \xi_2\rangle^b} \, d\xi\, d\tau\, d\xi_2\, d\tau_2.
\end{aligned}$$

Thus, by symmetry we can restrict ourselves to the set A. We divide A into three pieces as following

$$\begin{aligned}
A_1 &= \left\{(\xi,\tau,\xi_1,\tau_1) \in A;\ |\xi_1| \leq 10\right\}, \\
A_2 &= \left\{(\xi,\tau,\xi_1,\tau_1) \in A;\ |\xi_1| \geq 10 \text{ and } |2\xi_1 - \xi| \geq \frac{|\xi_1|}{2}\right\}, \\
A_3 &= \left\{(\xi,\tau,\xi_1,\tau_1) \in A;\ |\xi_1| \geq 10 \text{ and } |\xi_1 - \xi| \geq \frac{|\xi_1|}{2}\right\}.
\end{aligned}$$

We have $A = A_1 \cup A_2 \cup A_3$. Indeed, if $(\xi,\tau,\xi_1,\tau_1) \in A$ we must have

$$\frac{|\xi_1|}{2} \le |2\xi_1 - \xi| \quad \text{or} \quad \frac{|\xi_1|}{2} \le |\xi_1 - \xi|,$$

otherwise we would have

$$|\xi_1| = \frac{|\xi_1|}{2} + \frac{|\xi_1|}{2} > |2\xi_1 - \xi| + |\xi - \xi_1| \ge |2\xi_1 - \xi + \xi - \xi_1| = |\xi_1|,$$

which is an absurd.

Next we split A_3 into two parts

$$A_{3,1} = \left\{(\xi,\tau,\xi_1,\tau_1) \in A_3;\ |\tau_1 + \xi_1^2| \le |\tau + \xi^2|\right\}$$
$$A_{3,2} = \left\{(\xi,\tau,\xi_1,\tau_1) \in A_3;\ |\tau + \xi^2| \le |\tau_1 + \xi_1^2|\right\}.$$

Now, we define the sets R_1 and R_2 as follows

$$R_1 = A_1 \cup A_2 \cup A_{3,1} \quad \text{and} \quad R_2 = A_{3,2}.$$

In what follows, χ_R denotes the characteristic function of the set R. Since $A = R_1 \cup R_2$, then

$$|Z(f,g,\varphi)| \le |\mathscr{R}_1| + |\mathscr{R}_2|,$$

where

$$\mathscr{R}_i = \int_{\mathbb{R}^4} \frac{|\xi|^2}{2\gamma(\xi)} \frac{\langle\xi\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s} \frac{\chi_{R_i}(\xi,\tau,\xi_1,\tau_1) g(\xi_1,\tau_1) f(\xi_2,\tau_2)\overline{\varphi}(\xi,\tau)}{\langle\sigma\rangle^a\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\,\mathrm{d}\xi\,\mathrm{d}\tau\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1.$$

Using Cauchy-Schwarz inequality twice, we have

$$\begin{aligned}
|\mathscr{R}_1| &\le \left[\int_{\mathbb{R}^2} \frac{|\xi|^4\langle\xi\rangle^{2s}}{4\gamma(\xi)^2\langle\sigma\rangle^{2a}} \left(\int_{\mathbb{R}^2} \frac{\chi_{R_1} g(\xi_1,\tau_1) f(\xi_2,\tau_2)}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right)^2 \mathrm{d}\xi\,\mathrm{d}\tau\right]^{\frac{1}{2}} \|\varphi\|_{L^2}\\
&\le \left[\int_{\mathbb{R}^2} \frac{|\xi|^4\langle\xi\rangle^{2s}}{4\gamma(\xi)^2\langle\sigma\rangle^{2a}} \left(\int_{\mathbb{R}_2} \frac{\chi_{R_1}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right)\right.\\
&\quad \times \left.\left(\int_{\mathbb{R}^2} |g(\xi_1,\tau_1)|^2 |f(\xi_2,\tau_2)|^2\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right)\mathrm{d}\xi\,\mathrm{d}\tau\right]^{\frac{1}{2}} \|\varphi\|_{L^2}.
\end{aligned}$$

Now, applying Hölder inequality, we obtain

$$\begin{aligned}
|\mathscr{R}_1| &\le \left\| \frac{|\xi|^4\langle\xi\rangle^{2s}}{4\gamma(\xi)^2\langle\sigma\rangle^{2a}} \int_{\mathbb{R}^2} \frac{\chi_{R_1}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1 \right\|_{L^\infty_{\xi,\tau}}^{\frac{1}{2}}\\
&\quad \times \left[\int_{\mathbb{R}^4} |g(\xi_1,\tau_1)|^2 |f(\xi_2,\tau_2)|^2\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\,\mathrm{d}\xi\,\mathrm{d}\tau\right]^{\frac{1}{2}} \|\varphi\|_{L^2}\\
&= \left\| \frac{|\xi|^4\langle\xi\rangle^{2s}}{4\gamma(\xi)^2\langle\sigma\rangle^{2a}} \int_{\mathbb{R}^2} \frac{\chi_{R_1}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1 \right\|_{L^\infty_{\xi,\tau}}^{\frac{1}{2}} \|f\|_{L^2}\|g\|_{L^2}\|\varphi\|_{L^2}.
\end{aligned}$$

We apply the same steps for $\mathscr{R}_2$,

$$
\begin{aligned}
|\mathscr{R}_2| &\leq \left[\int_{\mathbb{R}^2} \frac{1}{\langle\xi_1\rangle^{2s}\langle\sigma_1\rangle^{2b}} \left(\int_{\mathbb{R}^2} \frac{|\xi|^2\langle\xi\rangle^s f(\xi_2,\tau_2)\overline{\varphi}(\xi,\tau)\chi_{R_2}}{2\gamma(\xi)\langle\xi_2\rangle^s\langle\sigma\rangle^a\langle\sigma_2\rangle^b}\,\mathrm{d}\xi\,\mathrm{d}\tau\right)^2 \mathrm{d}\xi_1\,\mathrm{d}\tau_1\right]^{\frac{1}{2}} \|g\|_{L^2} \\
&\leq \left[\int_{\mathbb{R}^2} \frac{1}{\langle\xi_1\rangle^{2s}\langle\sigma_1\rangle^{2b}} \left(\int_{\mathbb{R}^2} \frac{|\xi|^4}{4\gamma(\xi)^2} \frac{\langle\xi\rangle^{2s}\chi_{R_2}}{\langle\xi_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi\,\mathrm{d}\tau\right)\right. \\
&\quad \left.\times\left(\int_{\mathbb{R}^2} |f(\xi_2,\tau_2)|^2|\overline{\varphi}(\xi,\tau)|^2\,\mathrm{d}\xi\,\mathrm{d}\tau\right)\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right]^{\frac{1}{2}} \|g\|_{L^2} \\
&\leq \left\|\frac{1}{\langle\xi_1\rangle^{2s}\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{|\xi|^4}{4\gamma(\xi)^2} \frac{\langle\xi\rangle^{2s}\chi_{R_2}}{\langle\xi_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi\,\mathrm{d}\tau\right\|_{L^\infty_{\xi_1,\tau_1}} \\
&\quad \times\left(\int_{\mathbb{R}^4} |f(\xi_2,\tau_2)|^2|\overline{\varphi}(\xi,\tau)|^2\,\mathrm{d}\xi\,\mathrm{d}\tau\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right)^{\frac{1}{2}} \|g\|_{L^2} \\
&= \left\|\frac{1}{\langle\xi_1\rangle^{2s}\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{|\xi|^4}{4\gamma(\xi)^2} \frac{\langle\xi\rangle^{2s}\chi_{R_2}}{\langle\xi_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi\,\mathrm{d}\tau\right\|_{L^\infty_{\xi_1,\tau_1}} \|f\|_{L^2}\|g\|_{L^2}\|\varphi\|_{L^2}.
\end{aligned}
$$

Using that $|\xi|^4\gamma(\xi)^{-2} = |\xi|^4\left(|\xi|^2+|\xi|^4\right)^{-1} = (|\xi|^{-2}+1)^{-1} \leq 1$ for all $\xi \in \mathbb{R}$, we conclude that

$$
\begin{aligned}
|Z(f,g,\varphi)| \leq{}& \left\|\frac{\langle\xi\rangle^{2s}}{\langle\sigma\rangle^{2a}} \int_{\mathbb{R}^2} \frac{\chi_{R_1}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right\|_{L^\infty_{\xi,\tau}} \|f\|_{L^2}\|g\|_{L^2}\|\varphi\|_{L^2} \\
&+ \left\|\frac{1}{\langle\xi_1\rangle^{2s}\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{\langle\xi\rangle^{2s}\chi_{R_2}}{\langle\xi_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi\,\mathrm{d}\tau\right\|_{L^\infty_{\xi_1,\tau_1}} \|f\|_{L^2}\|g\|_{L^2}\|\varphi\|_{L^2}.
\end{aligned}
$$

By applying inequality (1.27), we have the following

$$
\frac{\langle\xi\rangle^{2s}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}} \leq \frac{\langle\xi_1\rangle^{2|s|}}{\langle\xi_1\rangle^{2s}} = \langle\xi_1\rangle^{\beta(s)},
$$

where

$$
\beta(s) = \begin{cases} 0, & \text{if } s \geq 0 \\ 4|s|, & \text{if } s < 0. \end{cases}
$$

Therefore,

$$
\left\|\frac{\langle\xi\rangle^{2s}}{\langle\sigma\rangle^{2a}} \int_{\mathbb{R}^2} \frac{\chi_{R_1}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right\|_{L^\infty_{\xi,\tau}} \leq \left\|\frac{1}{\langle\sigma\rangle^{2a}} \int_{\mathbb{R}^2} \frac{\chi_{R_1}\langle\xi_1\rangle^{\beta(s)}}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right\|_{L^\infty_{\xi,\tau}}
$$

and

$$
\left\|\frac{1}{\langle\xi_1\rangle^{2s}\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{\chi_{R_2}\langle\xi\rangle^{2s}}{\langle\xi_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\mathrm{d}\xi\mathrm{d}\tau\right\|_{L^\infty_{\xi_1,\tau_1}} \leq \left\|\frac{\langle\xi_1\rangle^{\beta(s)}}{\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{\chi_{R_2}}{\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\mathrm{d}\xi\mathrm{d}\tau\right\|_{L^\infty_{\xi_1,\tau_1}}.
$$

It follows from Lemma 1.16, more precisely inequality (1.30), that

$$\int_{\mathbb{R}^2} \frac{\chi_{R_1} \langle \xi_1 \rangle^{\beta(s)}}{\langle \tau_1 + \xi_1^2 \rangle^{2b} \langle \tau_2 + \xi_2^2 \rangle^{2b}} \mathrm{d}\xi_1 \mathrm{d}\tau_1 = \int_{\mathbb{R}} \chi_{R_1} \langle \xi_1 \rangle^{\beta(s)} \int_{\mathbb{R}} \frac{1}{\langle \tau_1 - (-\xi_1^2) \rangle^{2b} \langle \tau_1 - (\tau + \xi_2^2) \rangle^{2b}} \mathrm{d}\tau_1 \mathrm{d}\xi_1$$
$$\lesssim \int_{\mathbb{R}} \frac{\chi_{R_1} \langle \xi_1 \rangle^{\beta(s)}}{\langle \tau + \xi_2^2 + \xi_1^2 \rangle^{2b}} \mathrm{d}\xi_1,$$

since $b > \frac{1}{2}$ and $\min\{2b, 4b-1\} = 2b$. Similarly, we have

$$\int_{\mathbb{R}^2} \frac{\chi_{R_2}}{\langle \tau + \xi^2 \rangle^{2a} \langle \tau_2 + \xi_2^2 \rangle^{2b}} \mathrm{d}\xi \mathrm{d}\tau \lesssim \int_{\mathbb{R}} \frac{\chi_{R_2}}{\langle \tau_1 - \xi_2^2 + \xi^2 \rangle^{2a}} \mathrm{d}\xi$$

where we used that $\min\{2b, 2a, 2a+2b-1\} = 2a$ and $a > \frac{1}{4}$.

Since $\tau + \xi_2^2 + \xi_1^2 = \tau + \xi^2 - 2\xi\xi_1 + 2\xi_1^2$ and $\tau_1 - \xi_2^2 + \xi^2 = \tau_1 - \xi_1^2 + 2\xi\xi_1$, it suffices to get bounds for

$$J_1(\xi,\tau) = \frac{1}{\langle \sigma \rangle^{2a}} \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^{\beta(s)}}{\langle \tau + \xi^2 - 2\xi\xi_1 + 2\xi_1^2 \rangle^{2b}} \mathrm{d}\xi_1 \quad \text{on } R_1 \quad \text{and}$$
$$J_2(\xi_1,\tau_1) = \frac{\langle \xi_1 \rangle^{\beta(s)}}{\langle \sigma_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{1}{\langle \tau_1 - \xi_1^2 + 2\xi\xi_1 \rangle^{2a}} \mathrm{d}\xi \quad \text{on } R_2.$$

In region A_1, we have $\langle \xi_1 \rangle^{\beta(s)} \le \langle 10 \rangle^{\beta(s)} \lesssim 1$. Therefore, for $a > 0$ and $b > \frac{1}{2}$ we obtain

$$J_1(\xi,\tau) \lesssim \frac{1}{\langle \sigma \rangle^{2a}} \int_{|\xi_1| \le 10} \frac{1}{\langle \tau + \xi^2 - 2\xi\xi_1 + 2\xi_1^2 \rangle^{2b}} \mathrm{d}\xi_1 \le \int_{|\xi_1| \le 10} 1 \, \mathrm{d}\xi_1 \lesssim 1 \quad \text{on } A_1,$$

since $\langle \eta \rangle \ge 1$ for all $\eta \in \mathbb{R}$. In region A_2, by the change of variables

$$\eta = \tau + \xi^2 + 2\xi_1^2 - 2\xi\xi_1, \; \mathrm{d}\eta = 2|2\xi_1 - \xi| \mathrm{d}\xi_1$$

and the condition $|2\xi_1 - \xi| \ge \frac{|\xi_1|}{2}$, we have

$$J_1(\xi,\tau) = \frac{1}{\langle \sigma \rangle^{2a}} \int_{|\xi_1| \ge 10} \frac{\langle \xi_1 \rangle^{\beta(s)}}{\langle \eta \rangle^{2b}} \frac{\mathrm{d}\eta}{2|2\xi_1 - \xi|} \le \frac{1}{\langle \sigma \rangle^{2a}} \int_{|\xi_1| \ge 10} \frac{\langle \xi_1 \rangle^{\beta(s)}}{\langle \eta \rangle^{2b} |\xi_1|} \mathrm{d}\eta \quad \text{on } A_2.$$

On the other hand, $\langle \xi_1 \rangle = (1 + \xi_1^2)^{\frac{1}{2}} \le (2\xi_1^2)^{\frac{1}{2}} \lesssim |\xi_1|$, which implies that

$$J_1(\xi,\tau) \lesssim \frac{1}{\langle \sigma \rangle^{2a}} \int \frac{\langle \xi_1 \rangle^{\beta(s)-1}}{\langle \eta \rangle^{2b}} \mathrm{d}\eta \quad \text{on } A_2.$$

We observe that $\beta(s) - 1 \le 0$ for all $s > -\frac{1}{4}$, thus

$$J_1(\xi,\tau) \lesssim \frac{1}{\langle \sigma \rangle^{2a}} \int \frac{1}{\langle \eta \rangle^{2b}} \mathrm{d}\eta \lesssim 1 \quad \text{on } A_2,$$

since $b > \frac{1}{2}$. Now, by definition of region $A_{3,1}$ and the relation (2.16), we have

$$\langle \xi_1 \rangle^2 = 1 + |\xi_1|^2 \le 2|\xi_1||\xi_1| \le 4|\xi_1||\xi_1 - \xi| = 2|\sigma_1 - \sigma + \sigma_2| \le 2(|\sigma_1| + |\sigma_2| + |\sigma_3|) \le 6|\sigma| \lesssim \langle \sigma \rangle,$$

since $|\sigma_1|, |\sigma_2| \leq |\sigma|$ in the region $A_{3,1}$. For $a > 0$, we have $\langle\sigma\rangle^{-2a} \lesssim \langle\xi_1\rangle^{-4a}$. Therefore, it follows from Lemma 1.16 (inequality (1.31)) that

$$J_1(\xi,\tau) \lesssim \int \frac{\langle\xi_1\rangle^{\beta(s)-4a}}{\left\langle\tau+\xi^2+2\xi_1^2+2\xi\xi_1\right\rangle^{2b}} \mathrm{d}\xi_1 \lesssim \int \frac{1}{\left\langle\tau+\xi^2+2\xi_1^2+2\xi\xi_1\right\rangle^{2b}} \mathrm{d}\xi_1 \lesssim 1 \quad \text{on } A_{3,1},$$

since $\beta(s) < 4a$.

Next, we are going to estimate $J_2(\xi_1,\tau_1)$ on R_2. Making the change of variables $\eta = \tau_1 - \xi_1^2 + 2\xi\xi_1$ in the region $A_{3,2}$, we obtain

$$J_2(\xi_1,\tau_1) = \frac{\langle\xi_1\rangle^{\beta(s)}}{\langle\sigma_1\rangle^{2b} 2|\xi_1|} \int \frac{1}{\langle\eta\rangle^{2a}} \mathrm{d}\eta.$$

Moreover, in region $A_{3,2}$ we have $|\xi_1 - \xi| \geq \frac{|\xi_1|}{2}$ and $|\sigma|, |\sigma_2| \leq |\sigma_1|$, which implies that

$$|\xi_1|^2 \leq 2|\xi_1||\xi_1 - \xi| = |\sigma_1 - \sigma + \sigma_2| \leq 3|\sigma_1| \lesssim \langle\sigma_1\rangle,$$

by using relation (2.16). Also, for all $(\xi,\tau,\xi_1,\tau_1) \in A_{3,2}$ we have

$$|\eta| = |\tau_1 - \xi_1^2 + 2\xi\xi_1| = |\tau_1 - (\xi-\xi_1)^2 + \xi^2| \leq |\tau_1 - \tau - (\xi-\xi_1)^2| + |\tau+\xi^2| = |\sigma_2| + |\sigma_1| \leq 2|\sigma_1|.$$

Since $|\xi_1| \geq 10$ in $A_{3,2}$, we have

$$J_2(\xi_1,\tau_1) \lesssim \frac{|\xi_1|^{\beta(s)-1}}{\langle\sigma_1\rangle^{2b}} \int\limits_{|\eta|\leq 2|\sigma_1|} \frac{1}{\langle\eta\rangle^{2a}} \mathrm{d}\eta.$$

Now, we observe that

$$\begin{aligned} \int\limits_{|\eta|\leq 2|\sigma_1|} \frac{1}{\langle\eta\rangle^{2a}} \mathrm{d}\eta &\simeq \int\limits_{|\eta|\leq 2|\sigma_1|} \frac{1}{(1+|\eta|)^{2a}} \mathrm{d}\eta = 2\int\limits_0^{2|\sigma_1|} \frac{1}{(1+\eta)^{2a}} = \frac{(1+\eta)^{1-2a}}{1-2a}\Bigg|_0^{2|\sigma_1|} \\ &= \frac{(1+2|\sigma_1|)^{1-2a}}{1-2a} - \frac{1}{1-2a} \leq \frac{2(1+|\sigma_1|)^{1-2a}}{1-2a} \\ &\lesssim \langle\sigma_1\rangle^{1-2a}, \end{aligned}$$

since $a < \frac{1}{2}$. Therefore,

$$J_2(\xi_1,\tau_1) \lesssim \frac{|\xi_1|^{\beta(s)-1}}{\langle\sigma\rangle^{2a+2b-1}} \leq 1, \quad \text{on } R_2,$$

where we used that $\beta(s) - 1 < 0$, for all $s > -\frac{1}{4}$ and $2a + 2b - 1 > 0$ with $a > 0$ and $b > \frac{1}{2}$.

Now, it remains to estimate $|Z(f,g,\varphi)|$ in the conditions of case (III). In what follows, we will make use of the algebraic relation

$$-(\tau+\xi^2) + (\tau_1 - \xi_1^2) + \left(\tau - \tau_1 + (\xi-\xi_1)^2\right) = -2\xi_1\xi. \tag{2.17}$$

First, we split $\mathbb{R}^4$ into four sets

$$\begin{aligned} B_1 &= \left\{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4;\ |\xi_1| \leq 10\right\}, \\ B_2 &= \left\{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4;\ |\xi_1| \geq 10 \text{ and } |\xi| \leq 1\right\}, \\ B_3 &= \left\{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4;\ |\xi_1| \geq 10,\ |\xi| \geq 1 \text{ and } |\xi| \geq |\xi_1|/2\right\}, \\ B_4 &= \left\{(\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4;\ |\xi_1| \geq 10,\ |\xi| \geq 1 \text{ and } |\xi| \leq |\xi_1|/2\right\}. \end{aligned}$$

Next we separate B_4 into three parts, which are

$$B_{4,1}=\{(\xi,\tau,\xi_1,\tau_1)\in B_4;\ |\tau_1-\xi_1^2|,\ |\tau-\tau_1+(\xi-\xi_1)^2|\le|\tau+\xi^2|\},$$
$$B_{4,2}=\{(\xi,\tau,\xi_1,\tau_1)\in B_4;\ |\tau+\xi^2|,\ |\tau-\tau_1+(\xi-\xi_1)^2|\le|\tau_1-\xi_1^2|\},$$
$$B_{4,3}=\{(\xi,\tau,\xi_1,\tau_1)\in B_4;\ |\tau_1-\xi_1^2|,\ |\tau+\xi^2|\le|\tau-\tau_1+(\xi-\xi_1)^2|\}.$$

We can now define the sets S_i, $i=1,2,3$, as follows

$$S_1=B_1\cup B_2\cup B_{4,1},\ \ S_2=B_2\cup B_{4,2}\ \text{and}\ S_3=B_{4,3}.$$

Similarly to the case (I), using Cauchy-Schwarz and Hölder inequalities and duality argument, we can write

$$|Z(f,g,\varphi)|\le\|f\|_{L^2}\|g\|_{L^2}\|\varphi\|_{L^2}\left(\mathscr{S}_1+\mathscr{S}_2+\mathscr{S}_3\right),$$

where

$$\mathscr{S}_1=\left\|\frac{\langle\xi\rangle^{2s}}{\langle\sigma\rangle^{2a}}\int_{\mathbb{R}^2}\frac{\chi_{S_1}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right\|_{L^\infty_{\xi,\tau}}$$

$$\mathscr{S}_2=\left\|\frac{1}{\langle\xi_1\rangle^{2s}\langle\sigma_1\rangle^{2b}}\int_{\mathbb{R}^2}\frac{\chi_{S_2}\langle\xi\rangle^{2s}}{\langle\xi_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi\,\mathrm{d}\tau\right\|_{L^\infty_{\xi_1,\tau_1}}$$

$$\mathscr{S}_3=\left\|\frac{1}{\langle\xi_2\rangle^{2s}\langle\sigma_2\rangle^{2b}}\int_{\mathbb{R}^2}\frac{\chi_{\tilde{S}_3}\langle\xi_1+\xi_2\rangle^{2s}}{\langle\xi_1\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_1\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right\|_{L^\infty_{\xi_2,\tau_2}}$$

with σ, σ_1 and σ_2 are given as in case (III) and

$$\tilde{S}_3\subseteq\left\{(\xi_2,\tau_2,\xi_1,\tau_1)\in\mathbb{R}^4;\ \begin{array}{c}|\xi_1|\ge10,\ |\xi_1+\xi_2|\ge1,\ |\xi_1+\xi_2|\le|\xi_1|/2\\ \text{and}\ |\tau_1-\xi_1^2|,|\tau_1+\tau_2+(\xi_1+\xi_2)^2|\le|\tau_2+\xi_2^2|\end{array}\right\}.$$

Again, by (1.27), we have

$$\frac{\langle\xi\rangle^{2s}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}}\le\langle\xi_1\rangle^{\beta(s)},\ \text{where}\quad \beta(s)=\begin{cases}0, & \text{if } s\ge0\\ 4|s|, & \text{if } s<0.\end{cases}$$

As we did in the case (I), we obtain

$$\mathscr{S}_1\lesssim\left\|\frac{1}{\langle\sigma\rangle^{2a}}\int_{\mathbb{R}^2}\frac{\langle\xi_1\rangle^{\beta(s)}\chi_{S_1}}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right\|_{L^\infty_{\xi,\tau}},$$

$$\mathscr{S}_2\lesssim\left\|\frac{\langle\xi_1\rangle^{\beta(s)}}{\langle\sigma_1\rangle^{2b}}\int_{\mathbb{R}^2}\frac{\chi_{S_2}}{\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi\,\mathrm{d}\tau\right\|_{L^\infty_{\xi_1,\tau_1}},$$

$$\mathscr{S}_3\lesssim\left\|\frac{1}{\langle\sigma_2\rangle^{2b}}\int_{\mathbb{R}^2}\frac{\chi_{\tilde{S}_3}\langle\xi_1\rangle^{\beta(s)}}{\langle\sigma\rangle^{2a}\langle\sigma_1\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right\|_{L^\infty_{\xi_2,\tau_2}}.$$

Applying Lemma 1.16, more precisely, inequality (1.30), we obtain

$$\int_{\mathbb{R}^2} \frac{\langle\xi_1\rangle^{\beta(s)}}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1 = \int_{\mathbb{R}} \langle\xi_1\rangle^{\beta(s)} \int_{\mathbb{R}} \frac{1}{\left\langle\tau_1-\xi_1^2\right\rangle^{2b}\left\langle\tau-\tau_1+(\xi-\xi_1)^2\right\rangle^{2b}}\,\mathrm{d}\tau_1\,\mathrm{d}\xi_1$$
$$\lesssim \int_{\mathbb{R}} \frac{\langle\xi_1\rangle^{\beta(s)}}{\left\langle\tau+\xi^2-2\xi\xi_1\right\rangle^{2b}}\,\mathrm{d}\xi_1,$$
$$\int_{\mathbb{R}^2} \frac{1}{\langle\sigma\rangle^{2a}\langle\sigma_1\rangle^{2b}}\,\mathrm{d}\xi\,\mathrm{d}\tau = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\left\langle\tau+\xi^2\right\rangle^{2a}\left\langle\tau-\tau_1+(\xi-\xi_1)^2\right\rangle^{2b}}\,\mathrm{d}\tau\,\mathrm{d}\xi$$
$$\lesssim \int_{\mathbb{R}} \frac{1}{\left\langle\tau_1-\xi_1^2+2\xi\xi_1\right\rangle^{2a}}\,\mathrm{d}\xi,$$

and

$$\int_{\mathbb{R}^2} \frac{\langle\xi_1\rangle^{\beta(s)}}{\langle\sigma_1\rangle^{2b}\langle\sigma\rangle^{2a}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1 = \int_{\mathbb{R}} \langle\xi_1\rangle^{\beta(s)} \int_{\mathbb{R}} \frac{1}{\left\langle\tau_1-\xi_1^2\right\rangle^{2b}\left\langle\tau_1+\tau_2+(\xi_1+\xi_2)^2\right\rangle^{2a}}\,\mathrm{d}\tau_1\,\mathrm{d}\xi_1$$
$$\lesssim \int_{\mathbb{R}} \frac{\langle\xi_1\rangle^{\beta(s)}}{\left\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_2\right\rangle^{2a}}\,\mathrm{d}\xi_1,$$

since $\min\{2b,\,4b-1\}=2b$ and $\min\{2b,\,2a,\,2b+2a-1\}=2a$. Thus, we conclude that

$$\mathscr{S}_1 \lesssim \left\| \frac{1}{\langle\sigma\rangle^{2a}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{\beta(s)}\,\chi_{S_1}}{\left\langle\tau+\xi^2-2\xi\xi_1\right\rangle^{2b}}\,\mathrm{d}\xi_1 \right\|_{L^\infty_{\tau,\xi}},$$
$$\mathscr{S}_2 \lesssim \left\| \frac{\langle\xi\rangle^{\beta(s)}}{\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}} \frac{\chi_{S_2}}{\left\langle\tau_1-\xi^2+2\xi\xi_1\right\rangle^{2a}}\mathrm{d}\xi \right\|_{L^\infty_{\xi_1,\tau_1}},$$
$$\mathscr{S}_3 \lesssim \left\| \frac{1}{\langle\sigma_2\rangle^{2b}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{\beta(s)}\,\chi_{\tilde{S}_3}}{\left\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_2\xi_1\right\rangle^{2a}}\mathrm{d}\xi_1 \right\|_{L^\infty_{\xi_2,\tau_2}}.$$

Therefore, it is sufficient to get bounds for

$$K_1(\xi,\tau) = \frac{1}{\langle\sigma\rangle^{2a}} \int_{\mathbb{R}} \frac{\langle\xi_1\rangle^{\beta(s)}}{\left\langle\tau+\xi^2-2\xi\xi_1\right\rangle^{2b}}\mathrm{d}\xi_1 \qquad \text{on} \quad S_1$$
$$K_2(\xi_1,\tau_1) = \frac{\langle\xi_1\rangle^{\beta(s)}}{\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}} \frac{1}{\left\langle\tau_1-\xi_1^2+2\xi\xi_1\right\rangle^{2a}}\mathrm{d}\xi \qquad \text{on} \quad S_2$$
$$K_3(\xi_2,\tau_2) = \frac{1}{\langle\sigma_2\rangle^{2b}} \int_{\mathbb{R}} \frac{\langle\xi_1\rangle^{\beta(s)}}{\left\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_2\right\rangle^{2a}}\mathrm{d}\xi_1 \quad \text{on} \quad \tilde{S}_3.$$

In region B_1, we have $|\xi_1|\le 10$ which implies that $\langle\xi_1\rangle^{\beta(s)}\lesssim 1$. Then,

$$K_1(\xi,\tau) \lesssim \frac{1}{\langle\sigma\rangle^{2a}} \int_{|\xi_1|\le 10} \frac{1}{\left\langle\tau+\xi^2-2\xi\xi_1\right\rangle^{2b}}\mathrm{d}\xi_1 \le \int_{|\xi_1|\le 10} 1\,\mathrm{d}\xi_1 \lesssim 1 \quad \text{on} \quad B_1,$$

since $\langle\eta\rangle\ge 1$ for all $\eta\in\mathbb{R}$ and $a,b\ge 0$.

In region B_3, making the change of variables $\eta = \tau + \xi^2 - 2\xi\xi_1$ we obtain

$$K_1(\xi,\tau) = \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{\beta(s)}}{2|\xi|\langle\eta\rangle^{2b}} d\eta \le \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{\beta(s)-1}}{\langle\eta\rangle^{2b}} d\eta \quad \text{on} \quad B_3,$$

since $|\xi_1| \ge 10$, which implies $\langle\xi_1\rangle \le 2|\xi_1|$. Using that $\beta(s) - 1 < 0$ for all $s > -\frac{1}{4}$ and $\langle\eta\rangle \ge 1$ for all $\eta \in \mathbb{R}$, we obtain

$$K_1(\xi_1,\tau_1) \lesssim \int_{\mathbb{R}} \frac{1}{\langle\eta\rangle^{2b}} d\eta \lesssim 1 \quad \text{on} \quad B_3,$$

since $2b > 1$.

Now, by definition of region $B_{4,1}$ and the algebraic relation (2.17) we have

$$\langle\xi_1\rangle \le 2|\xi_1| \le 2|\xi||\xi_1| = |\sigma_1 + \sigma_2 - \sigma| \le 3|\sigma| \lesssim \langle\sigma\rangle,$$

since $|\xi_1| \ge 10$, $|\xi| \ge 1$ and $|\sigma_1|, |\sigma_2| \le |\sigma|$ on $B_{4,1}$. Therefore, the change of variables $\eta = \tau + \xi^2 - 2\xi\xi_1$ and the condition $|\xi| \ge 1$ yield

$$K_1(\xi,\tau) \lesssim \frac{1}{\langle\sigma\rangle^{2a}} \int \frac{\langle\xi_1\rangle^{\beta(s)}}{2|\xi|\langle\eta\rangle^{2b}} d\eta \lesssim \frac{\langle\sigma\rangle^{\beta(s)-2a}}{2|\xi|} \int \frac{1}{\langle\eta\rangle^{2b}} d\eta \lesssim 1,$$

for all $s > -\frac{1}{4}$, $b > \frac{1}{2}$ and $a \in \mathbb{R}$ such that $0 < a < \frac{1}{2}$, if $s \ge 0$ or $2|s| < a < \frac{1}{2}$, if $-\frac{1}{4} < s < 0$, which finishes the desired bound for K_1 on S_1.

Next, we estimate $K_2(\xi_1,\tau_1)$ on S_2. Making the change of variables $\eta = \tau_1 - \xi_1^2 + 2\xi\xi_1$, we obtain

$$K_2(\xi_1,\tau_1) = \frac{\langle\xi_1\rangle^{\beta(s)}}{\langle\sigma_1\rangle^{2b}} \int \frac{1}{2|\xi_1|\langle\eta\rangle^{2a}} d\eta.$$

We observe that in B_2, we have

$$|\eta| = |\tau_1 - \xi_1^2 + 2\xi\xi_1| \le |\sigma_1| + 2|\xi\xi_1| \le 2(|\sigma_1| + |\xi_1|) \quad \text{and} \quad \langle\xi_1\rangle \le 2|\xi_1|.$$

Thus,

$$K_2(\xi_1,\tau_1) \lesssim \frac{|\xi_1|^{\beta(s)-1}}{\langle\sigma_1\rangle^{2b}} \int_{|\eta| \lesssim |\sigma_1|+|\xi_1|} \frac{1}{\langle\eta\rangle^{2a}} d\eta.$$

On the other hand,

$$\int_{|\eta| \lesssim |\sigma_1|+|\xi_1|} \frac{1}{\langle\eta\rangle^{2a}} d\eta \simeq \int_{|\eta| \lesssim |\sigma_1|+|\xi_1|} \frac{1}{(1+|\eta|)^{2a}} d\eta \simeq \int_0^{|\sigma_1|+|\xi_1|} \frac{1}{(1+\eta)^{2a}} d\eta,$$

with

$$\begin{aligned}\int_0^{|\eta_1|+|\xi_1|} \frac{1}{(1+\eta)^{2a}} d\eta &= \frac{(1+\eta)^{1-2a}}{1-2a}\bigg|_0^{|\sigma_1|+|\xi_1|} \\ &= \frac{(1+|\sigma_1|+|\xi_1|)^{1-2a}}{1-2a} - \frac{1}{1-2a} \\ &\lesssim \left(1+|\sigma_1|+|\xi_1|\right)^{1-2a} \\ &\lesssim \left(\langle\sigma_1\rangle + |\xi_1|\right)^{1-2a},\end{aligned}$$

since $1-2a>0$. Therefore,

$$K_2(\xi_1,\tau_1)\lesssim \frac{|\xi_1|^{\beta(s)-1}}{\langle\sigma_1\rangle^{2b}}\big(\langle\sigma_1\rangle+|\xi_1|\big)^{1-2a}\le \frac{|\xi_1|^{\beta(s)-1}}{\langle\sigma_1\rangle^{2b}}\left(\langle\sigma_1\rangle^{1-2a}+|\xi_1|^{1-2a}\right),$$

where we used the fact $\big(|x|+|y|\big)^{\alpha}\le |x|^{\alpha}+|y|^{\alpha}$ for all $0\le\alpha<1$ and for all $x,y\in\mathbb{R}$. Since

$$2b+2a-1>2a>0,\ \langle\eta\rangle\ge 1,\ |\xi_1|\ge 10 \text{ and } \beta(s)-2a<0,$$

we conclude that

$$K_2(\xi_1,\tau_1)\lesssim \frac{|\xi_1|^{\beta(s)-1}}{\langle\sigma_1\rangle^{2b+2a-1}}+\frac{|\xi_1|^{\beta(s)-2a}}{\langle\sigma_1\rangle^{2b}}\le 1,$$

for $s>-\frac{1}{4}$, $b>\frac{1}{2}$ and $0<a<\frac{1}{2}$ such that $\beta(s)\le\min\{1,2a\}$.

In the region $B_{4,2}$, by the algebraic relation (2.17), we have

$$\langle\xi_1\rangle\simeq(1+|\xi_1|)\le 2|\xi_1|\le 2|\xi_1||\xi|=|-\sigma+\sigma_1+\sigma_2|\le 3|\sigma_1|\lesssim\langle\sigma_1\rangle,$$

since $|\xi_1|\ge 10$, $|\xi|\ge 1$ and $|\sigma|$, $|\sigma_2|\le|\sigma_1|$ in B_2. Moreover, the change of variables $\eta=\tau_1-\xi_1^2+2\xi\xi_1$, the restriction in the region B_2 and (2.17) give us

$$|\eta|\le 2|\xi\xi_1|+|\sigma_1|\lesssim\langle\sigma_1\rangle.$$

Therefore,

$$K_2(\xi_1,\tau_1)\lesssim\frac{\langle\xi_1\rangle^{\beta(s)}}{\langle\sigma_1\rangle^{2b}}\int\limits_{|\eta|\lesssim\langle\sigma_1\rangle}\frac{1}{|\xi_1|\langle\eta\rangle^{2a}}\mathrm{d}\eta\lesssim\frac{|\xi_1|^{\beta(s)-1}}{\langle\sigma_1\rangle^{2b}}\int\limits_{|\eta|\lesssim\langle\sigma_1\rangle}\frac{1}{\langle\eta\rangle^{2a}}\mathrm{d}\eta,$$

with

$$\int\limits_{|\eta|\lesssim\langle\sigma_1\rangle}\frac{1}{\langle\eta\rangle^{2a}}\mathrm{d}\eta\simeq\int\limits_0^{\langle\sigma_1\rangle}\frac{1}{(1+\eta)^{2a}}\mathrm{d}\eta=\frac{(1+\eta)^{1-2a}}{1-2a}\Bigg|_0^{\langle\sigma_1\rangle}\lesssim(1+\langle\sigma_1\rangle)^{1-2a}\le 2^{1-2a}\langle\sigma_1\rangle^{1-2a},$$

for all $0<a<\frac{1}{2}$. Thus,

$$K_2(\xi_1,\tau_1)\lesssim\frac{|\xi_1|^{\beta(s)-1}}{\langle\sigma_1\rangle^{2b+2a-1}}\lesssim 1,$$

since $|\xi_1|\ge 10$, $2b+2a-1>0$ and $\beta(s)-1<0$ for all $s>-\frac{1}{4}$, which concludes the bounds for K_2 on S_2.

Finally, we estimate $K_3(\xi_2,\tau_2)$ on $\tilde{S}_3$. By using (2.17), we observe that

$$\langle\xi_1\rangle\le 2|\xi_1|\le 2|\xi\xi_1|=|-\sigma+\sigma_1+\sigma_2|\le 3|\sigma_2|\lesssim\langle\sigma_2\rangle,\quad\text{on } \tilde{S}_3.$$

Therefore, inequality (1.31) from Lemma 1.16 implies

$$\begin{aligned}K_3(\xi_1,\tau_1)&=\frac{1}{\langle\sigma_2\rangle^{2b}}\int\frac{\langle\xi_1\rangle^{\beta(s)}}{\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_1\rangle^{2a}}\mathrm{d}\xi_1\\&\lesssim\int\frac{\langle\xi_1\rangle^{\beta(s)-2b}}{\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_2\rangle^{2a}}\mathrm{d}\xi_1\\&\le\int\frac{1}{\langle\tau_2+\xi_2^2+2\xi_1^2+2\xi_1\xi_2\rangle^{2a}}\mathrm{d}\xi_1<\infty,\end{aligned}$$

since $\beta(s)-2b<0$ for all $s>-\frac{1}{4}$, which finishes the proof of the bilinear estimates. □

Using Lemma 2.5 and Theorem 2.9 we have the following proposition, which will guarantees that Φ_T is a contraction in a ball centered at the origin of $X_{s,b}$.

Proposition 2.10. *Let $s > -\frac{1}{4}$ and $b \in \left(\frac{1}{2}, \frac{3}{4}\right)$, then for all $u_0 \in H^s(\mathbb{R})$, $u_1 \in H^{s-1}(\mathbb{R})$ and $0 < T \leq 1$, there are $d > 0$ and a constant $C_{\psi,b} > 0$ such that*

$$\|\Phi_T u\|_{X_{s,b}} \leq C_{\psi,b}\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + T^d \|u\|^2_{X_{s,b}}\right), \quad \text{for all } u \in X_{s,b}, \tag{2.18}$$

and

$$\|\Phi_T u - \Phi_T v\|_{X_{s,b}} \leq T^d C_{\psi,b} \|u+v\|_{X_{s,b}} \|u-v\|_{X_{s,b}}, \quad \text{for all } u, v \in X_{s,b}. \tag{2.19}$$

Also, from inequalities (2.18) and (2.19) is possible to prove the **uniqueness** of the solution and also the **continuity of the data-to-solution** map Φ_T. Since the proof Proposition 2.10 and the proof of well-posedness in this case is very similar to the one in Gevrey analytic case, we will do this with detail in the next chapter.

2.2 Periodic Case

In this section we are considering the periodic version of the Cauchy problem (2.1), that is, the variable x belongs now to $\mathbb{T}$. The problem approach here are very similar to the way that was showed in the real case, the main change will be in the bilinear estimates, which is the result that we will present in this section with more details.

By doing precisely the same steps that we did in the previous section, we obtain a formal solution map to our problem given by

$$\boxed{\Phi_T u \doteq \psi(t) W_1(t) u_0(x) + \psi(t) W_2(t) \partial_x u_1(x) - \psi_T(t) \int_0^t W_2(t-t') w(x,t')\, \mathrm{d}t',} \tag{2.20}$$

where the operators W_1 and W_2 are now given by

$$W_1(t)\varphi(x) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{inx} \left(\frac{e^{i\gamma(n)t} + e^{-i\gamma(n)t}}{2}\right) \widehat{\varphi}(n) \quad \text{and}$$
$$W_2(t)\varphi(x) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{inx} \left(\frac{e^{i\gamma(n)t} - e^{-i\gamma(n)t}}{2i\gamma(n)}\right) \widehat{\varphi}(n), \quad \gamma(n) = \sqrt{n^2 + n^4},$$

which are the periodic form of the operators W_1 and W_2 presented earlier. Again, our principal aim is to solve the equation $\Phi_T u = u$.

We consider the following decomposition of the map Φ_T

$$\Phi_T u(x,t) = \frac{1}{2\pi}\psi(t)\sum_{n\in\mathbb{Z}} e^{inx}\left(\frac{e^{i\gamma(n)t}+e^{-i\gamma(n)t}}{2}\right)\widehat{u_0}(n) \tag{2.21}$$

$$+\frac{1}{2\pi}\psi(t)\sum_{n\in\mathbb{Z}} e^{inx}\left(\frac{e^{i\gamma(n)t}-e^{-i\gamma(n)t}}{2i\gamma(n)}\right)\widehat{\partial_x u_1}(n) \tag{2.22}$$

$$-\frac{i}{4\pi^2}\psi_T(t)\sum_{n\in\mathbb{Z}}\int \frac{e^{i(nx+\gamma(n)t)}}{2i\gamma(n)}\left(\frac{e^{i(\tau-\gamma(n))t}-1}{\tau-\gamma(n)}\right)\widehat{w}(n,\tau)\,\mathrm{d}\tau \tag{2.23}$$

$$+\frac{i}{4\pi^2}\psi_T(t)\sum_{n\in\mathbb{Z}}\int \frac{e^{i(nx-\gamma(n)t)}}{2i\gamma(n)}\left(\frac{e^{i(\tau+\gamma(n))t}-1}{\tau+\gamma(n)}\right)\widehat{w}(n,\tau)\,\mathrm{d}\tau\,, \tag{2.24}$$

which give us a motivation to define the solution spaces, that is, the periodic Bourgain Spaces.

The Bourgain Spaces $X_{s,b}(\mathbb{T}\times\mathbb{R})$

We introduce the following spaces.

Definition 2.11 (Space $\boldsymbol{X_{s,b}}(\mathbb{T}\times\mathbb{R})$.)**.** *Let $\mathscr{X}$ be the space of functions F such that*

(i) $F:\mathbb{T}\times\mathbb{R}\to\mathbb{C}$.

(ii) $F(x,\cdot)\in\mathscr{S}(\mathbb{R})$ *for each* $x\in\mathbb{T}$.

(iii) $F(\cdot,t)\in C^\infty(\mathbb{T})$ *for each* $t\in\mathbb{R}$.

For $s,b\in\mathbb{R}$, $X_{s,b}=X_{s,b}(\mathbb{T}\times\mathbb{R})$ denotes the completion of $\mathscr{X}$ with respect to the norm

$$\|v\|_{X_{s,b}} = \left(\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\langle|\tau|-\gamma(n)\rangle^{2b}\langle n\rangle^{2s}|\widehat{v}(n,\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}}$$

where $\gamma(n)=\sqrt{n^2+n^4}$ and $\langle n\rangle=(1+n^2)^{\frac{1}{2}}$.

We will also need the localized $X_{s,b}$ spaces defined as follows.

Definition 2.12 (Space $\boldsymbol{X^T_{s,b}}(\mathbb{T}\times\mathbb{R})$.)**.** *For $T\geq 0$, $X^T_{s,b}$ denotes the space endowed with the norm*

$$\|u\|_{X^T_{s,b}} = \inf_{v\in X_{s,b}}\left\{\|v\|_{X_{s,b}};\ v(x,t)=u(x,t)\ \text{on}\ \mathbb{T}\times[0,T]\right\}.$$

It is important to point out that the continuous embedding

$$X_{s,b}(\mathbb{T}\times\mathbb{R})\hookrightarrow C(\mathbb{R};H^s(\mathbb{T}))$$

remains true as we showed in Lemma 2.4 for the real case.

The following lemma give us an elementary bound for the $X_{s,b}$-norm of the map Φ_T and its proof is very similar to the proof of Lemma 2.5.

Lemma 2.13. *Let $s > -\frac{1}{4}$, $u_0 \in H^s(\mathbb{T})$, $u_1 \in H^{s-1}(\mathbb{T})$ and $0 < T < 1$. For $f, g \in X_{s,b}$ we define the bilinear operator*

$$\Phi_T(f,g) \doteq \psi(t)\,(W_1(t)u_0(x) + W_2(t)\partial_x u_1(x)) - \psi_T(t)\int_0^t W_2(t-t')w_{f,g}(x,t')dt$$

where $w_{f,g} = \partial_x^2(fg)$. Let $-\frac{1}{2} < b' \le 0 \le b \le b'+1$. Then, there exist a constant $C = C(\psi, b, b')$ such that

$$\left\|\Phi_T(f,g)\right\|_{X_{s,b}} \le C(\|u_0\|_{H^s} + \|u_1\|_{s-1}) + CT^{1-(b-b')}\left\|\mathscr{F}^{-1}\left(\frac{\widehat{w_{f,g}}(n,\tau)}{2i\gamma(n)}\right)\right\|_{X_{s,b'}}.$$

Now, we are in the crucial part to get the well-posedness for the periodic gB problem, which is to get bound for

$$\left\|\mathscr{F}^{-1}\left(\frac{\widehat{w_{f,g}}(n,\tau)}{2i\gamma(n)}\right)\right\|_{X_{s,b'}}.$$

The next subsection is devoted to show a detailed proof of the bilinear estimates presented by L. G. Farah and M. Scialom in [13].

The Bilinear Estimates

We begin by showing some technical results that will be useful later. For a reference of the following classical result see, for example, Lemma 5.3 in [27] on page 3346.

Lemma 2.14. *If $q > 1/2$, then*

$$\sup_{(n,\tau)\in\mathbb{Z}\times\mathbb{R}} \sum_{n_1\in\mathbb{Z}} \frac{1}{(1+|\tau \pm n_1(n-n_1)|)^q} < \infty. \tag{2.25}$$

Proof. Let $\alpha = \alpha(n,\tau)$ and $\beta = \beta(n,\tau)$ belonging to $\mathbb{C}$ the roots of the polynomial

$$\mathrm{p}(n_1) = \tau \pm n_1(n-n_1),$$

that is,

$$\sum_{n_1\in\mathbb{Z}} \frac{1}{(1+|\tau \pm n_1(n-n_1)|)^q} = \sum_{n_1\in\mathbb{Z}} \frac{1}{(1+|(n_1-\alpha)(n_1-\beta)|)^q}.$$

We write $\mathbb{Z} = A \cup B$, where A and B are given as follows

$$A = \{n_1;\, |n_1-\alpha| \le 2 \text{ or } |n_1-\beta| \le 2\} \quad \text{and} \quad B = \{n_1;\, |n_1-\alpha| > 2 \text{ and } |n_1-\beta| > 2\}.$$

First, we observe that A is finite. In fact, let x be the real part of α and n_1 such that $|n_1-\alpha| \le 2$, then

$$|n_1 - x| \le 2 \Rightarrow -2+x \le n_1 \le 2+x \Rightarrow -2+[x] \le n_1 < 3+[x],$$

where $[x]$ denotes the integer part of x. Therefore, there are only five possibilities for n_1 such that $|n_1-\alpha| \le 2$, which are

$$[x]\pm 2,\ [x]\pm 1 \text{ and } [x].$$

This proves that A has exactly 10 elements and, therefore,

$$\sum_{n_1\in A}\frac{1}{(1+|(n_1-\alpha)(n_1-\beta)|)^q}\le 10,$$

for all $(n,\tau)\in\mathbb{Z}\times\mathbb{R}$.

It just remains to estimate the sum over the set B. If $n_1\in B$, then

$$|n_1-\alpha|\le\frac{|n_1-\alpha||n_1-\beta|}{2}\quad\text{and}\quad|n_1-\beta|\le\frac{|n_1-\alpha||n_1-\beta|}{2},$$

which implies

$$|n_1-\alpha|+|n_1-\beta|\le|n_1-\alpha||n_1-\beta|. \tag{2.26}$$

Using (2.26) we obtain

$$\begin{aligned}\frac{1}{2}\big(1+|n_1-\alpha|\big)\big(1+|n_1-\beta|\big)&=\frac{1}{2}\big(1+|n_1-\alpha|+|n_1-\beta|+|n_1-\alpha||n_1-\beta|\big)\\&\le\frac{1}{2}\big(1+2|n_1-\alpha||n_1-\beta|\big)\\&\le 1+|n_1-\alpha||n_1-\beta|,\end{aligned}$$

that is,

$$\frac{1}{\big(1+|n_1-\alpha||n_1-\beta|\big)}\le\frac{2}{\big(1+|n_1-\alpha|\big)\big(1+|n_1-\beta|\big)}. \tag{2.27}$$

By (2.27) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{n_1\in B}\frac{1}{(1+|(n_1-\alpha)(n_1-\beta)|)^q}&\le 2^q\sum_{n_1\in B}\frac{1}{\big(1+|n_1-\alpha|\big)^q\big(1+|n_1-\beta|\big)^q}\\&\le\left(\sum_{n_1\in B}\frac{1}{(1+|n_1-\alpha|)^{2q}}\right)^{\frac{1}{2}}\left(\sum_{n_1\in B}\frac{1}{(1+|n_1-\beta|)^{2q}}\right)^{\frac{1}{2}}\\&\le\left(\sum_{n_1\in\mathbb{Z}}\frac{1}{(1+|n_1-\alpha|)^{2q}}\right)^{\frac{1}{2}}\left(\sum_{n_1\in\mathbb{Z}}\frac{1}{(1+|n_1-\beta|)^{2q}}\right)^{\frac{1}{2}}\\&\le C_q,\end{aligned}$$

for all $(n,\tau)\in\mathbb{Z}\times\mathbb{R}$, since $q>\frac{1}{2}$. This finishes the proof of the lemma. □

Lemma 2.15. *Let* $0<a<1/2$, $\alpha\in\mathbb{R}$, $\beta\ge 0,\nu>0$ *and* $H=\{h\in\mathbb{R};h=\alpha\pm n,\ n\in\mathbb{Z}\ \text{and}\ |h|\le\beta\}$. *Then*

$$\sum_{h\in H}\frac{1}{(\nu+|h|)^{2a}}\le 2\left(\frac{2}{\nu^{2a}}+\int_0^\beta\frac{1}{(\nu+x)^{2a}}\,\mathrm{d}x\right). \tag{2.28}$$

Proof. Without loss of generality, we assume $\alpha=0$. We are going to prove first the case when $\beta\ge 0$ is a integer number, that is, we would like to start by proving the following inequality

$$\sum_{n=-\beta}^{\beta}\frac{1}{(\nu+n)^{2a}}\le 2\left(\frac{2}{\nu^{2a}}+\int_0^\beta\frac{1}{(\nu+x)^{2a}}\,\mathrm{d}x\right). \tag{2.29}$$

We observe that

$$A_\beta \doteq \sum_{n=-\beta}^{\beta} \frac{1}{(\nu+n)^{2a}} = \frac{1}{\nu^{2a}} + 2\sum_{n=1}^{\beta} \frac{1}{(\nu+n)^{2a}} \tag{2.30}$$

and

$$B_\beta \doteq 2\left(\frac{2}{\nu^{2a}} + \int_0^\beta \frac{1}{(\nu+x)^{2a}}\,\mathrm{d}x\right) = 2\left(\frac{2}{\nu^{2a}} + \frac{1}{1-2a}(\nu+\beta)^{1-2a} - \frac{1}{1-2a}\nu^{1-2a}\right). \tag{2.31}$$

Let us prove the inequality $A_\beta \le B_\beta$ by induction on β. For $\beta = 0$ the desired inequality is trivial, since $A_0 = \frac{1}{\nu^{2a}}$ and $B_0 = \frac{4}{\nu^{2a}}$. For $\beta = 1$, since $1-2a > 0$, we have

$$B_1 = \frac{4}{\nu^{2a}} + \frac{2}{1-2a}(\nu+1)^{1-2a} - \frac{2}{1-2a}\nu^{1-2a} \ge \frac{4}{\nu^{2a}} + \frac{2}{1-2a}(\nu+1)^{1-2a} - \frac{2}{1-2a}(\nu+1)^{1-2a} = \frac{4}{\nu^{2a}},$$

which implies

$$A_1 = \frac{1}{\nu^{2a}} + \frac{2}{(\nu+1)^{2a}} \le \frac{3}{\nu^{2a}} \le B_1.$$

Next, for some natural number $k \ge 1$ we suppose the following

$$A_\beta \le B_\beta, \text{ for all } \beta = 0,1,2,\ldots,k,$$

and we are going to prove that the same happens for $\beta = k+1$. By using the induction hypothesis, we have

$$A_{k+1} = A_k + \frac{2}{(\nu+k+1)^{2a}} \le B_k + \frac{2}{(\nu+k+1)^{2a}},$$

with

$$B_k = 2\left(\frac{2}{\nu^{2a}} + \int_0^k \frac{1}{(\nu+x)^{2a}}\,\mathrm{d}x\right) = B_{k+1} - 2\int_k^{k+1} \frac{1}{(\nu+x)^{2a}}\,\mathrm{d}x.$$

Then, we obtain

$$A_{k+1} \le B_{k+1} + \frac{2}{(\nu+k+1)^{2a}} - 2\int_k^{k+1} \frac{1}{(\nu+x)^{2a}}\,\mathrm{d}x = B_{k+1} - 2\int_k^{k+1} \left[\frac{1}{(\nu+x)^{2a}} - \frac{1}{(\nu+k+1)^{2a}}\right]\mathrm{d}x.$$

For all $x \in [k, k+1]$, we have

$$\frac{1}{(\nu+x)^{2a}} - \frac{1}{(\nu+k+1)^{2a}} \ge 0, \text{ which implies } \int_k^{k+1} \left[\frac{1}{(\nu+x)^{2a}} - \frac{1}{(\nu+k+1)^{2a}}\right]\mathrm{d}x \ge 0.$$

Then, we conclude that $A_{k+1} \le B_{k+1}$ and the proof of (2.29) for integer values of β is done.

Now, let $\beta > 0$, then

$$|n| \le \beta \Rightarrow -[\beta]-1 < -\beta \le n \le \beta < [\beta]+1 \Rightarrow -[\beta] \le n \le [\beta],$$

where $[x]$ denotes the integer part of a number x. Then, using which we proved previously,

$$\sum_{h\in H} \frac{1}{(\nu+|h|)^{2a}} = \sum_{n=-[\beta]}^{[\beta]} \frac{1}{(\nu+n)^{2a}} \le 2\left(\frac{2}{\nu^{2a}} + \int_0^{[\beta]} \frac{1}{(\nu+x)^{2a}}\,\mathrm{d}x\right) \le 2\left(\frac{2}{\nu^{2a}} + \int_0^{\beta} \frac{1}{(\nu+x)^{2a}}\,\mathrm{d}x\right),$$

since $[\beta] \le \beta$. This finishes the proof of the lemma. □

***Remark* 2.2.** In view of Lemma 2.8, we have an equivalent way to compute the $X_{s,b}$ -norm, that is,

$$\|u\|_{X_{s,b}} \simeq \left\| \left\langle |\tau| - n^2 \right\rangle^b \langle n \rangle^s \widehat{u}(\tau,n) \right\|_{l_n^2 L_\tau^2}. \tag{2.32}$$

In fact, we just need to join the result given by Lemma 2.8

$$\frac{1}{c} \leq \frac{1+||\tau|-n^2|}{1+||\tau|-\gamma(n)|} \leq c,$$

with the fact $\langle x \rangle \simeq (1+|x|)$, for all $x \in \mathbb{R}$. In the proof of the next theorem we will use the right side of (2.32) as definition of $X_{s,b}$-norm.

Now we are in position to prove the periodic bilinear estimates.

Theorem 2.16 (Sobolev Bilinear Estimates [13])**.** *Let $s > -\frac{1}{4}$ and $u, v \in X_{s,b}(\mathbb{T} \times \mathbb{R})$. Then, there exists $C > 0$ such that*

$$\left\| \mathscr{F}^{-1}\left(\frac{|n|^2 \widehat{uv}(n,\tau)}{2i\gamma(n)} \right) \right\|_{X_{s,-a}} \leq C \|u\|_{X_{s,b}} \|v\|_{X_{s,b}} \tag{2.33}$$

holds in the following cases:

(i) $s \geq 0$, $b > \frac{1}{2}$ *and* $\frac{1}{4} < a < \frac{1}{2}$.

(ii) $-\frac{1}{4} < s < 0$, $b > \frac{1}{2}$ *and* $\frac{1}{4} < a < \frac{1}{2}$ *such that* $|s| < \frac{a}{2}$.

Moreover, the constant that appears in (2.33) *depends only on* a, b *and* s.

Proof. Let $u, v \in X_{s,b}$ and define

$$f(n,\tau) = \left\langle |\tau| - n^2 \right\rangle^b \langle n \rangle^s \widehat{u}(n,\tau) \quad \text{and} \quad g(n,\tau) = \left\langle |\tau| - n^2 \right\rangle^b \langle n \rangle^s \widehat{v}(n,\tau),$$

which are functions in $l_n^2 L_\tau^2$. By using duality argument, we observe that (2.33) is equivalent to the following inequality

$$|W(f,g,\varphi)| \leq C \|f\|_{l_n^2 L_\tau^2} \|g\|_{l_n^2 L_\tau^2} \|\varphi\|_{l_n^2 L_\tau^2}, \quad \text{for all } \varphi \in l_n^2 L_\tau^2 \tag{2.34}$$

where

$$W(f,g,\varphi) = \sum_{n,n_1 \in \mathbb{Z}} \int_{\mathbb{R}^2} \frac{n^2}{2i\gamma(n)} \frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n-n_1 \rangle^s} \frac{g(n_1,\tau_1) f(n-n_1,\tau-\tau_1) \overline{\varphi}(n,\tau)}{\left\langle |\tau| - n^2 \right\rangle^a \left\langle |\tau_1| - n_1^2 \right\rangle^b \left\langle |\tau-\tau_1| - (n-n_1)^2 \right\rangle^b} \, d\tau \, d\tau_1.$$

Therefore, to perform the desired estimate we need to analyse all the possible cases for sign of τ, τ_1 and $\tau - \tau_1$. To do this, we split $\mathbb{Z}^2 \times \mathbb{R}^2$ into the regions

$$\Gamma_1 = \{(n,n_1,\tau,\tau_1);\ \tau_1 < 0,\ \tau - \tau_1 < 0\},$$
$$\Gamma_2 = \{(n,n_1,\tau,\tau_1);\ \tau_1 \geq 0,\ \tau - \tau_1 < 0,\ \tau \geq 0\},$$
$$\Gamma_3 = \{(n,n_1,\tau,\tau_1);\ \tau_1 \geq 0,\ \tau - \tau_1 < 0,\ \tau < 0\},$$
$$\Gamma_4 = \{(n,n_1,\tau,\tau_1);\ \tau_1 < 0,\ \tau - \tau_1 \geq 0,\ \tau \geq 0\},$$
$$\Gamma_5 = \{(n,n_1,\tau,\tau_1);\ \tau_1 < 0,\ \tau - \tau_1 \geq 0,\ \tau < 0\},$$
$$\Gamma_6 = \{(n,n_1,\tau,\tau_1);\ \tau_1 \geq 0,\ \tau - \tau_1 \geq 0\}.$$

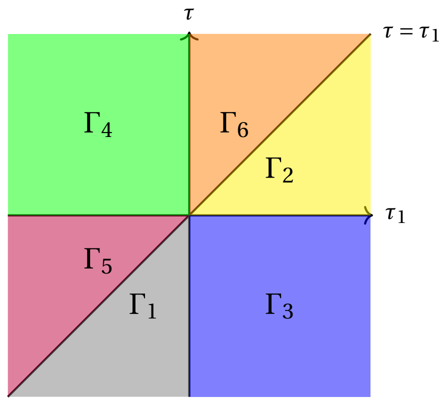

Figure 7: Bilinear estimates regions II.

Thus, it is sufficient to prove inequality (2.34) with $Z(f,g,\varphi)$ instead of $W(f,g,\varphi)$, where

$$Z(f,g,\varphi)=\sum_{n,n_1\in\mathbb{Z}}\int_{\mathbb{R}^2}\frac{|n|^2}{2i\gamma(n)}\frac{\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s}\frac{g(n_1,\tau_1)f(n_2,\tau_2)\overline{\varphi}(n,\tau)}{\langle\sigma\rangle^a\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\,d\tau\,d\tau_1,$$

with $n_2=n-n_1$, $\tau_2=\tau-\tau_1$ and σ, σ_1, σ_2 belonging to one of the following cases

(I) $\sigma=\tau+n^2$, $\sigma_1=\tau_1+n_1^2$, $\sigma_2=\tau_2+n_2^2$.

(II) $\sigma=\tau-n^2$, $\sigma_1=\tau_1-n_1^2$, $\sigma_2=\tau_2+n_2^2$.

(III) $\sigma=\tau+n^2$, $\sigma_1=\tau_1-n_1^2$, $\sigma_2=\tau_2+n_2^2$.

(IV) $\sigma=\tau-n^2$, $\sigma_1=\tau_1+n_1^2$, $\sigma_2=\tau_2-n_2^2$.

(V) $\sigma=\tau+n^2$, $\sigma_1=\tau_1+n_1^2$, $\sigma_2=\tau_2-n_2^2$.

(VI) $\sigma=\tau-n^2$, $\sigma_1=\tau_1-n_1^2$, $\sigma_2=\tau_2-n_2^2$.

We observe that the cases

$$\sigma=\tau+n^2,\ \sigma_1=\tau_1-n_1^2,\ \sigma_2=\tau_2-n_2^2\quad\text{and}\quad \sigma+\tau-n^2,\ \sigma_1=\tau_1+n_1^2,\ \sigma_2=\tau_2+n_2^2,$$

cannot occur, since $\tau_1<0$, $\tau-\tau_1<0$ implies $\tau<0$ and $\tau_1\geq 0$, $\tau-\tau_1\geq 0$ implies $\tau\geq 0$.

Applying the change of variable

$$(n,n_1,\tau,\tau_1)\to-(n,n_1,\tau,\tau_1)$$

and observing that the $l_n^2L_\tau^2$–norm is preserved under the reflection operation, the cases (III), (II) and (I) can be easily reduced to (IV), (V) and (VI) respectively. Moreover, in the same way, making the change of variables $\tau_2=\tau-\tau_1$, $n_2=n-n_1$ and then $(n,n_2,\tau,\tau_2)\to-(n,n_2,\tau,\tau_2)$ the case (V) can be reduced to (IV). Therefore, we need only establish cases (IV) and (VI).

We first treat the inequality (2.34) with $Z(f,g,\varphi)$ in the case (VI), we will use the following algebraic relation

$$-(\tau-n^2)+(\tau_1-n_1^2)+\big((\tau-\tau_1)-(n_1-n)^2\big)=2n_1(n-n_1)\tag{2.35}$$

We can write $\mathbb{Z}^2\times\mathbb{R}^2=A\cup B$, where

$$A=\big\{(n,n_1,\tau,\tau_1);\ |(\tau-\tau_1)-(n-n_1)^2|\leq|\tau_1-n_1^2|\big\}\quad\text{and}$$
$$B=\big\{(n,n_1,\tau,\tau_1);\ |(\tau-\tau_1)-(n-n_1)^2|\geq|\tau_1-n_1^2|\big\}.$$

By symmetry we can restrict ourselves only to the set A. We divide A into three pieces which are

$$A_1=\big\{(n,n_1,\tau,\tau_1)\in A;\ n=0\big\},$$
$$A_2=\big\{(n,n_1,\tau,\tau_1)\in A;\ n_1=0\ \text{or}\ n_1=n\big\},$$
$$A_3=\big\{(n,n_1,\tau,\tau_1)\in A;\ n\neq 0, n_1\neq 0\ \text{and}\ n_1\neq n\big\}.$$

Next we split A_3 into two parts

$$A_{3,1}=\big\{(n,n_1,\tau,\tau_1)\in A_3;\ |\tau_1-n_1^2|\leq|\tau-n^2|\big\}\quad\text{and}$$
$$A_{3,2}=\big\{(n,n_1,\tau,\tau_1)\in A_3;\ |\tau-n^2|\leq|\tau_1-n_1^2|\big\}.$$

Now, we define the sets R_1 and R_2 as follows

$$R_1 = A_1 \cup A_2 \cup A_{3,1} \quad \text{and} \quad R_2 = A_{3,2}.$$

In what follows, χ_R denotes the characteristic function of the set R. Since $A = R_1 \cup R_2$,

$$|Z(f,g,\varphi)| \leq |\mathscr{R}_1| + |\mathscr{R}_2|,$$

where

$$\mathscr{R}_i = \sum_{n,n_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{n^2}{2\gamma(n)} \frac{\langle n\rangle^s}{\langle n_1\rangle^s \langle n_2\rangle^s} \frac{\chi_{R_i}(n,\tau,n_1,\tau_1) g(n_1,\tau_1) f(n_2,\tau_2)\overline{\varphi}(n,\tau)}{\langle\sigma\rangle^a \langle\sigma_1\rangle^b \langle\sigma_2\rangle^b} \,\mathrm{d}\tau\,\mathrm{d}\tau_1 .$$

Using the Cauchy-Schwarz inequality twice, we have

$$\begin{aligned}
|\mathscr{R}_1| &\leq \left[\sum_{n\in\mathbb{Z}} \int \frac{n^4 \langle n\rangle^{2s}}{4\gamma(n)^2 \langle\sigma\rangle^{2a}} \left(\sum_{n_1\in\mathbb{Z}} \int \frac{\chi_{R_1} g(n_1,\tau_1) f(n_2,\tau_2)}{\langle n_1\rangle^s \langle n_2\rangle^s \langle\sigma_1\rangle^b \langle\sigma_2\rangle^b} \,\mathrm{d}\tau_1\right)^2 \mathrm{d}\tau\right]^{\frac{1}{2}} \|\varphi\|_{l_n^2 L_\tau^2} \\
&\leq \left[\sum_{n\in\mathbb{Z}} \int \frac{n^4 \langle n\rangle^{2s}}{4\gamma(n)^2 \langle\sigma\rangle^{2a}} \left(\sum_{n_1\in\mathbb{Z}} \int \frac{\chi_{R_1}}{\langle n_1\rangle^{2s} \langle n_2\rangle^{2s} \langle\sigma_1\rangle^{2b} \langle\sigma_2\rangle^{2b}} \,\mathrm{d}\tau_1\right)\right. \\
&\quad \times \left.\left(\sum_{n_1\in\mathbb{Z}} \int |g(n_1,\tau_1)|^2 |f(n_2,\tau_2)|^2 \,\mathrm{d}\tau_1\right) \mathrm{d}\tau\right]^{\frac{1}{2}} \|\varphi\|_{l_n^2 L_\tau^2} .
\end{aligned}$$

Now applying Hölder inequality, we obtain

$$\begin{aligned}
|\mathscr{R}_1| \leq & \left\| \frac{n^4 \langle n\rangle^{2s}}{4\gamma(n)^2 \langle\sigma\rangle^{2a}} \sum_{n_1\in\mathbb{Z}} \int \frac{\chi_{R_1}}{\langle n_1\rangle^{2s} \langle n_2\rangle^{2s} \langle\sigma_1\rangle^{2b} \langle\sigma_2\rangle^{2b}} \,\mathrm{d}\tau_1 \right\|_{l_n^\infty L_\tau^\infty}^{\frac{1}{2}} \\
& \times \left[\sum_{n,n_1\in\mathbb{Z}} \int_{\mathbb{R}^2} |g(n_1,\tau_1)|^2 |f(n_2,\tau_2)|^2 \,\mathrm{d}\tau_1\,\mathrm{d}\tau\right]^{\frac{1}{2}} \|\varphi\|_{l_n^2 L_\tau^2} \\
= & \left\| \frac{n^4 \langle n\rangle^{2s}}{4\gamma(n)^2 \langle\sigma\rangle^{2a}} \sum_{n_1\in\mathbb{Z}} \int \frac{\chi_{R_1}}{\langle n_1\rangle^{2s} \langle n_2\rangle^{2s} \langle\sigma_1\rangle^{2b} \langle\sigma_2\rangle^{2b}} \,\mathrm{d}\tau_1 \right\|_{l_n^\infty L_\tau^\infty}^{\frac{1}{2}} \|f\|_{l_n^2 L_\tau^2} \|g\|_{l_n^2 L_\tau^2} \|\varphi\|_{l_n^2 L_\tau^2} .
\end{aligned}$$

We apply the same steps for $\mathscr{R}_2$,

$$
\begin{aligned}
|\mathscr{R}_2| &\leq \left[\sum_{n_1\in\mathbb{Z}}\int \frac{1}{\langle n_1\rangle^{2s}\langle\sigma_1\rangle^{2b}}\left(\sum_{n\in\mathbb{Z}}\int \frac{n^2\langle\xi\rangle^s f(n_2,\tau_2)\overline{\varphi}(n,\tau)\chi_{R_2}}{2\gamma(n)\langle n_2\rangle^s\langle\sigma\rangle^a\langle\sigma_2\rangle^b}\,\mathrm{d}\tau\right)^2\mathrm{d}\tau_1\right]^{\frac{1}{2}}\|g\|_{l^2_nL^2_\tau}\\
&\leq \left[\sum_{n_1\in\mathbb{Z}}\int \frac{1}{\langle n_1\rangle^{2s}\langle\sigma_1\rangle^{2b}}\left(\sum_{n\in\mathbb{Z}}\int \frac{n^4}{4\gamma(n)^2}\frac{\langle n\rangle^{2s}\chi_{R_2}}{\langle n_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau\right)\right.\\
&\qquad\times\left.\left(\sum_{n\in\mathbb{Z}}\int |f(n_2,\tau_2)|^2|\overline{\varphi}(n,\tau)|^2\,\mathrm{d}\tau\right)\mathrm{d}\tau_1\right]^{\frac{1}{2}}\|g\|_{l^2_nL^2_\tau}\\
&\leq \left\|\frac{1}{\langle n_1\rangle^{2s}\langle\sigma_1\rangle^{2b}}\sum_{n\in\mathbb{Z}}\int\frac{n^4}{4\gamma(n)^2}\frac{\langle n\rangle^{2s}\chi_{R_2}}{\langle n_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau\right\|_{l^\infty_{n_1}L^\infty_{\tau_1}}\\
&\qquad\times\left(\sum_{n,n_1\in\mathbb{Z}}\int_{\mathbb{R}^2}|f(n_2,\tau_2)|^2|\overline{\varphi}(n,\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}\tau_1\right)^{\frac{1}{2}}\|g\|_{l^2_nL^2_\tau}\\
&\leq \left\|\frac{1}{\langle n_1\rangle^{2s}\langle\sigma_1\rangle^{2b}}\sum_{n\in\mathbb{Z}}\int\frac{n^4}{4\gamma(n)^2}\frac{\langle n\rangle^{2s}\chi_{R_2}}{\langle n_2\rangle^{2s}\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau\right\|_{l^2_{n_1}L^\infty_{\tau_1}}\|f\|_{l^2_nL^2_\tau}\|g\|_{l^2_nL^2_\tau}\|\varphi\|_{l^2_nL^2_\tau},
\end{aligned}
$$

which give us

$$
\begin{aligned}
|Z(f,g,\varphi)| &\leq \left\|\frac{n^4\langle n\rangle^{2s}}{4\gamma(n)^2\langle\sigma\rangle^{2a}}\sum_{n_1\in\mathbb{Z}}\left(\frac{1}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}}\int\frac{\chi_{R_1}}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau_1\right)\right\|^{\frac{1}{2}}_{l^\infty_nL^\infty_\tau}\|f\|_{l^2_nL^2_\tau}\|g\|_{l^2_nL^2_\tau}\|\varphi\|_{l^2_nL^2_\tau}\\
&\quad+\left\|\frac{1}{\langle n_1\rangle^{2s}\langle\sigma_1\rangle^{2b}}\sum_{n\in\mathbb{Z}}\left(\frac{n^4\langle n\rangle^{2s}}{4\gamma(n)^2\langle n_2\rangle^{2s}}\int\frac{\chi_{R_2}}{\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau\right)\right\|_{l^2_{n_1}L^\infty_{\tau_1}}\|f\|_{l^2_nL^2_\tau}\|g\|_{l^2_nL^2_\tau}\|\varphi\|_{l^2_nL^2_\tau}.
\end{aligned}
\tag{2.36}
$$

Since we are considering the case (VI), we have

$$
\begin{aligned}
\int\frac{1}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau_1 &= \int\frac{1}{\langle\tau_1-n_1^2\rangle^{2b}\langle\tau_2-n_2^2\rangle^{2b}}\,\mathrm{d}\tau_1\\
&= \int\frac{1}{\langle\tau_1-n_1^2\rangle^{2b}\langle\tau_1-(\tau-(n-n_1)^2)\rangle^{2b}}\,\mathrm{d}\tau_1.
\end{aligned}
$$

Applying Lemma 1.16, more precisely inequality (1.30) with $p=q=2b$, $\alpha=\tau-(n-n_1)^2$ and $\beta=n_1^2$, we obtain

$$
\int\frac{1}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau_1 \lesssim \frac{1}{\langle\tau-(n-n_1)^2-n_1^2\rangle^{2b}}. \tag{2.37}
$$

Similarly, we have

$$
\begin{aligned}
\int\frac{1}{\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau &= \int\frac{1}{\langle\tau-n^2\rangle^{2a}\langle\tau_2-n_2^2\rangle^{2b}}\,\mathrm{d}\tau\\
&= \int\frac{1}{\langle\tau-n^2\rangle^{2a}\langle\tau-(\tau_1+(n-n_1)^2)\rangle^{2b}}\,\mathrm{d}\tau
\end{aligned}
$$

with

$$
\int\frac{1}{\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau \lesssim \frac{1}{\langle\tau_1+(n-n_1)^2-n^2\rangle^{2a}}, \tag{2.38}
$$

since $\min\{2b, 2a, 2a+2b-1\} = 2a$ and $a > \frac{1}{4}$.

Returning to (2.36), it follows from (2.37) and (2.38) that it is suffices to get bounds for

$$J_1 = \sup_{(n,\tau)\in\mathbb{Z}\times\mathbb{R}} \left(\frac{n^4}{\langle\sigma\rangle^{2a}\gamma(n)^2} \sum_{n_1\in\mathbb{Z}} \frac{\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\langle \tau - n^2 - 2n_1^2 + 2nn_1\rangle^{2b}} \right) \quad \text{on } R_1 \tag{2.39}$$

$$J_2 = \sup_{(n_1,\tau_1)\in\mathbb{Z}\times\mathbb{R}} \left(\frac{1}{\langle\sigma_1\rangle^{2b}} \sum_{n\in\mathbb{Z}} \frac{n^4\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\gamma(n)^2\langle \tau_1 + n_1^2 - 2nn_1\rangle^{2a}} \right) \quad \text{on } R_2. \tag{2.40}$$

Let us start estimating J_1 on $R_1 = A_1 \cup A_2 \cup A_{3,1}$. In region A_1, we have

$$\frac{n^4}{\gamma(n)^2} = \frac{n^4}{n^4+n^2} = \frac{n^2}{n^2+1} = 0,$$

therefore the estimate is trivial. In region A_2, we have

$$\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s} = \langle n\rangle^{2s}\langle n\rangle^{-2s} = 1,$$

and, moreover,

$$\frac{n^4}{\gamma(n)^2} = \frac{n^2}{n^2+1} \le 1.$$

Therefore, since $a, b > 0$, the following happens

$$\sup_{(n,\tau)\in\mathbb{Z}\times\mathbb{R}} \left(\frac{n^4}{\langle\sigma\rangle^{2a}\gamma(n)^2} \sum_{\substack{n_1=0,\\ \text{or } n_1=n}} \frac{\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\langle \tau - n^2 - 2n_1^2 + 2nn_1\rangle^{2b}} \right) \le \sup_{(n,\tau)\in\mathbb{Z}\times\mathbb{R}} \left(\frac{1}{\langle\sigma\rangle^{2a}} \frac{2}{\langle\sigma\rangle^{2b}} \right) \lesssim 1,$$

which proves that J_1 is bounded on A_2.

Now, by inequality (1.27), we have

$$\frac{\langle n\rangle^{2s}}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}} \le \frac{\langle n_1\rangle^{2|s|}}{\langle n_1\rangle^{2s}} = \langle n_1\rangle^{\lambda(s)}, \tag{2.41}$$

for all $n, n_1 \in \mathbb{Z}$, where

$$\lambda(s) = \begin{cases} 0, & \text{if } s \ge 0 \\ 4|s|, & \text{if } s < 0. \end{cases}$$

Using the algebraic relation (2.35), we obtain

$$|2n_1(n-n_1)| = |-\sigma + \sigma_1 + \sigma_2|.$$

In the region $A_{3,1}$ we have $|n - n_1| \ge 1$ and $|\sigma_2| \le |\sigma_1| \le |\sigma|$ which give us

$$|n_1| = \frac{|-\sigma+\sigma_1+\sigma_2|}{2|n-n_1|} \le \frac{3|\sigma|}{2} \lesssim |\sigma|. \tag{2.42}$$

Therefore, using (2.41) and (2.42), we obtain

$$\frac{\langle n\rangle^{2s}}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}} \le \langle\sigma\rangle^{\lambda(s)},$$

on $A_{3,1}$, since $\lambda(s)\geq 0$. Thus, in the region $A_{3,1}$ we have

$$
\begin{aligned}
J_1 &= \sup_{(n,\tau)\in\mathbb{Z}^*\times\mathbb{R}}\left(\frac{n^4}{\langle\sigma\rangle^{2a}\gamma(n)^2}\sum_{\substack{n_1\neq 0,\\ n_1\neq n}}\frac{\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\langle \tau-n^2-2n_1^2+2nn_1\rangle^{2b}}\right)\\
&\leq \sup_{(n,\tau)\in\mathbb{Z}^*\times\mathbb{R}}\left(\frac{1}{\langle\sigma\rangle^{2a-\lambda(s)}}\sum_{\substack{n_1\neq 0,\\ n_1\neq n}}\frac{1}{\langle \tau-n^2-2n_1^2+2nn_1\rangle^{2b}}\right)\\
&\leq \sup_{(n,\tau)\in\mathbb{Z}^*\times\mathbb{R}}\left(\sum_{\substack{n_1\neq 0,\\ n_1\neq n}}\frac{1}{\langle \tau-n^2-2n_1^2+2nn_1\rangle^{2b}}\right),
\end{aligned}
\tag{2.43}
$$

since $2a-\lambda(s)\geq 0$. Applying the change of variables $\tilde{\tau}=\frac{\tau-n^2}{2}$ and Lemma 2.14 on (2.43), we obtain

$$
J_1\leq \sup_{(n,\tilde{\tau})\in\mathbb{Z}^*\times\mathbb{R}}\left(\sum_{\substack{n_1\neq 0,\\ n_1\neq n}}\frac{1}{\langle 2\tilde{\tau}-2n_1^2+2nn_1\rangle^{2b}}\right)\lesssim \sup_{(n,\tilde{\tau})\in\mathbb{Z}^*\times\mathbb{R}}\left(\sum_{\substack{n_1\neq 0,\\ n_1\neq n}}\frac{1}{\langle \tilde{\tau}-n_1^2+nn_1\rangle^{2b}}\right)\lesssim 1,
$$

for all $(n,n_1,\tau,\tau_1)\in A_{3,1}$, which concludes that J_1 is bounded on R_1.

Next, we estimate J_2 on $R_2=A_{3,2}$. Once again, by inequality (1.27), we have

$$
\frac{\langle n\rangle^{2s}}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}}\leq\frac{\langle n_1\rangle^{2|s|}}{\langle n_1\rangle^{2s}}=\langle n_1\rangle^{\lambda(s)},
\tag{2.44}
$$

for all $n,n_1\in\mathbb{Z}$, where

$$
\lambda(s)=\begin{cases}0, & \text{if } s\geq 0\\ 4|s|, & \text{if } s<0.\end{cases}
$$

Using the algebraic relation (2.35), we obtain

$$
|2n_1(n-n_1)|=|-\sigma+\sigma_1+\sigma_2|.
$$

In the region $A_{3,2}$ we have $|n-n_1|\geq 1$, $|\sigma_2|\leq|\sigma_1|$ and $|\sigma|\leq|\sigma_1|$ which give us

$$
|n_1|=\frac{|-\sigma+\sigma_1+\sigma_2|}{2|n-n_1|}\leq\frac{3|\sigma_1|}{2}\leq 2|\sigma_1|.
\tag{2.45}
$$

Therefore, using (2.44) and (2.45), we obtain

$$
\frac{\langle n\rangle^{2s}}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}}\leq\langle\sigma_1\rangle^{\lambda(s)},
$$

on $A_{3,2}$, since $\lambda(s)\geq 0$. Thus, since $n^4/\gamma(n)^2\leq 1$, for all $(n,n_1,\tau,\tau_1)\in A_{3,2}$ we have

$$
\begin{aligned}
J_2 &= \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\left(\frac{1}{\langle\sigma_1\rangle^{2b}}\sum_{\substack{n\neq 0,\\ n\neq n_1}}\frac{n^4\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\gamma(n)^2\langle\tau_1+n_1^2-2nn_1\rangle^{2a}}\right)\\
&\leq \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}}\sum_{\substack{n\neq 0,\\ n\neq n_1}}\frac{1}{(1+|\tau_1+n_1^2-2nn_1|)^{2a}}\right)\\
&= \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}}\frac{1}{|2n_1|^{2a}}\sum_{\substack{n\neq 0,\\ n\neq n_1}}\frac{1}{\left(\frac{1}{|2n_1|}+\left|\frac{\tau_1+n_1^2}{2n_1}-n\right|\right)^{2a}}\right).
\end{aligned}
\tag{2.46}
$$

In the region $A_{3,2}$, we observe that

$$\left|\frac{\tau_1+n_1^2}{2n_1}-n\right| = \frac{1}{|2n_1|}\left|\tau_1+n_1^2-2n_1 n\right| \leq \frac{1}{|2n_1|}\left(|\tau_1-n_1^2|+|2n_1(n-n_1)|\right)$$
$$= \frac{1}{|2n_1|}\left(|\sigma_1|+|-\sigma+\sigma_1+\sigma_2|\right) \leq \frac{2|\sigma_1|}{|n_1|},$$

where we used the algebraic relation (2.35). Applying Lemma 2.15 with

$$h=\frac{\tau_1+n_1^2}{2n_1}-n,\ \nu=\frac{1}{|2n_1|},\ \alpha=\frac{\tau_1+n_1^2}{2n_1} \quad\text{and}\quad \beta=\frac{2|\sigma_1|}{|n_1|},$$

we obtain the following bound in the region $A_{3,2}$

$$\sum_{\substack{n\neq 0,\\ n\neq n_1}} \frac{1}{\left(\frac{1}{|2n_1|}+\left|\frac{\tau_1+n_1^2}{2n_1}-n\right|\right)^{2a}} = \sum_{h\in H}\frac{1}{(\nu+|h|)^{2a}} \leq 2\left(2|2n_1|^{2a}+\int_0^{2|\sigma_1|/|n_1|}\left(\frac{1}{|2n_1|}+x\right)^{-2a}\mathrm{d}x\right).$$

We observe that

$$\int_0^{2|\sigma_1|/|n_1|}\left(\frac{1}{|2n_1|}+x\right)^{-2a}\mathrm{d}x = |2n_1|^{2a}\int_0^{2|\sigma_1|/|n_1|}(1+|2n_1|x)^{-2a}\,\mathrm{d}x = |2n_1|^{2a}\frac{1}{|2n_1|}\int_0^{4|\sigma_1|}(1+y)^{-2a}\,\mathrm{d}y,$$

with

$$\int_0^{4|\sigma_1|}(1+y)^{-2a}\,\mathrm{d}y = \left.\frac{(1+y)^{1-2a}}{1-2a}\right|_0^{4|\sigma_1|} = \frac{(1+4|\sigma_1|)^{1-2a}}{1-2a}-\frac{1}{1-2a} \leq \frac{\langle 4\sigma_1\rangle^{1-2a}}{1-2a} \lesssim \langle\sigma_1\rangle^{1-2a},$$

since $0<2a<1$, which give us

$$\int_0^{2|\sigma_1|/|n_1|}\left(\frac{1}{|2n_1|}+x\right)^{-2a}\mathrm{d}x \lesssim |2n_1|^{2a-1}\langle\sigma_1\rangle^{1-2a}.$$

Therefore,

$$\sum_{\substack{n\neq 0,\\ n\neq n_1}} \frac{1}{\left(\frac{1}{|2n_1|}+\left|\frac{\tau_1+n_1^2}{2n_1}-n\right|\right)^{2a}} \lesssim 2|2n_1|^{2a}+|2n_1|^{2a-1}\langle\sigma_1\rangle^{1-2a}.$$

Returning to (2.46), we have

$$\begin{aligned}
J_2 &\leq \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}}\frac{1}{|2n_1|^{2a}}\sum_{\substack{n\neq 0,\\ n\neq n_1}}\frac{1}{\left(\frac{1}{|2n_1|}+\left|\frac{\tau_1+n_1^2}{2n_1}-n\right|\right)^{2a}}\right)\\
&\leq \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}}\frac{1}{|2n_1|^{2a}}\left(2|2n_1|^{2a}+|2n_1|^{2a-1}\langle\sigma_1\rangle^{1-2a}\right)\right)\\
&= \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\left(\frac{2}{\langle\sigma_1\rangle^{2b-\lambda(s)}}+\frac{1}{|2n_1|\langle\sigma_1\rangle^{2b+2a-1-\lambda(s)}}\right)\\
&\lesssim \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}}+\frac{1}{\langle\sigma_1\rangle^{2b+2a-1-\lambda(s)}}\right),
\end{aligned}$$

for all $(n, n_1, \tau, \tau_1) \in A_{3,2}$. We observe that

$$2b + 2a - 1 - \lambda(s) = (2b - 1) + (2a - \lambda(s)) > 0 \text{ and } 2b - \lambda(s) > 0,$$

since $b > \frac{1}{2}$ and $2b > 2a > \lambda(s)$. Therefore, in the region $A_{3,2}$ we conclude that

$$J_2 \lesssim \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}} + \frac{1}{\langle\sigma_1\rangle^{2b+2a-1-\lambda(s)}} \right) \lesssim 1.$$

Now, we turn to the proof of case (IV). In the following estimates, we will make use of the algebraic relation

$$-(\tau - n^2) + (\tau_1 + n_1^2) + (\tau - \tau_1 - (n - n_1)^2) = 2n_1 n. \tag{2.47}$$

Once again, we split $\mathbb{Z}^2 \times \mathbb{R}^2$ into three regions

$$\begin{aligned} B_1 &= \{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2;\ n = 0\}, \\ B_2 &= \{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2;\ n_1 = 0 \text{ or } n_1 = n\}, \\ B_3 &= \{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2;\ n \neq 0, n_1 \neq 0 \text{ and } n_1 \neq n\}. \end{aligned}$$

Next, we also separate B_3 into three parts

$$\begin{aligned} B_{3,1} &= \left\{(n, n_1, \tau, \tau_1) \in B_3;\ |\tau_1 + n_1^2|, |(\tau - \tau_1) - (n - n_1)^2| \le |\tau - n^2|\right\}, \\ B_{3,2} &= \left\{(n, n_1, \tau, \tau_1) \in B_3;\ |\tau - n^2|, |(\tau - \tau_1) - (n - n_1)^2| \le |\tau_1 + n_1^2|\right\}, \\ B_{3,3} &= \left\{(n, n_1, \tau, \tau_1) \in B_3;\ |\tau_1 + n_1^2|, |\tau - n^2| \le |(\tau - \tau_1) - (n - n_1)^2|\right\}. \end{aligned}$$

We define the sets S_1, S_2 and S_3 as follows

$$S_1 = B_1 \cup B_2 \cup B_{3,1},\ S_2 = B_{3,2} \quad \text{and} \quad S_3 = B_{3,3}.$$

As we did in the previous case, using Cauchy-Schwarz and Hölder inequalities and duality argument, we have

$$|Z(f, g, \varphi)| \le \|f\|_{l_n^2 L_\tau^2} \|g\|_{l_n^2 L_\tau^2} \|\varphi\|_{l_n^2 L_\tau^2} (\mathscr{S}_1 + \mathscr{S}_2 + \mathscr{S}_3),$$

where

$$\begin{aligned} \mathscr{S}_1 &= \left\| \frac{n^4}{\langle\sigma\rangle^{2a}\gamma(n)^2} \sum_{n_1\in\mathbb{Z}} \int \frac{\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}\chi_{S_1}}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\, d\tau_1 \right\|_{l_n^\infty L_\tau^\infty} \\ \mathscr{S}_2 &= \left\| \frac{1}{\langle\sigma_1\rangle^{2b}} \sum_{n\in\mathbb{Z}} \int \frac{n^4}{\gamma(n)^2} \frac{\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}\chi_{S_2}}{\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\, d\tau \right\|_{l_{n_1}^\infty L_{\tau_1}^\infty} \\ \mathscr{S}_3 &= \left\| \frac{1}{\langle\sigma_2\rangle^{2b}} \sum_{n_1\in\mathbb{Z}} \int \frac{|n_1 + n_2|^4}{\gamma(n_1 + n_2)^2} \frac{\langle n_1 + n_2\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}\chi_{\tilde{S}_3}}{\langle\sigma\rangle^{2a}\langle\sigma_1\rangle^{2b}}\, d\tau_1 \right\|_{l_{n_2}^2 L_{\tau_2}^\infty}, \end{aligned}$$

where σ, σ_1 and σ_2 are given in the condition (IV) and

$$\tilde{S}_3 \subseteq \left\{ \begin{array}{r} (n_2,n_1,\tau_2,\tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2;\;\; n_1 \neq 0,\, |n_1+n_2| \neq 0 \text{ and} \\ |\tau_1+n_1^2|, |(\tau_1+\tau_2)-(n_1+n_2)^2| \leq |\tau_2-n_2^2| \end{array} \right\}.$$

Since we are considering the case (IV), we have

$$\begin{aligned}\int \frac{1}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau_1 &= \int \frac{1}{\left\langle\tau_1+n_1^2\right\rangle^{2b}\left\langle\tau_2-n_2^2\right\rangle^{2b}}\,\mathrm{d}\tau_1 \\ &= \int \frac{1}{\left\langle\tau_1-(-n_1^2)\right\rangle^{2b}\left\langle\tau_1-(\tau-(n-n_1)^2)\right\rangle^{2b}}\,\mathrm{d}\tau_1\,.\end{aligned}$$

Applying Lemma 1.16, more precisely (1.30) with $p=q=2b$, $\alpha=\tau-(n-n_1)^2$ and $\beta=-n_1^2$, we obtain

$$\int \frac{1}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau_1 \lesssim \frac{1}{\left\langle\tau-(n-n_1)^2+n_1^2\right\rangle^{2b}} = \frac{1}{\left\langle\tau-n^2+2nn_1\right\rangle^{2b}}. \tag{2.48}$$

Similarly, we have

$$\begin{aligned}\int \frac{1}{\langle\sigma\rangle^{2a}\langle\sigma_2\rangle^{2b}}\,\mathrm{d}\tau &= \int \frac{1}{\left\langle\tau-n^2\right\rangle^{2a}\left\langle\tau_2-n_2^2\right\rangle^{2b}}\,\mathrm{d}\tau \\ &= \int \frac{1}{\left\langle\tau-n^2\right\rangle^{2a}\left\langle\tau-(\tau_1+(n-n_1)^2)\right\rangle^{2b}}\,\mathrm{d}\tau \\ &\lesssim \frac{1}{\left\langle\tau_1+(n-n_1)^2-n^2\right\rangle^{2a}} = \frac{1}{\left\langle\tau_1+n_1^2-2nn_1\right\rangle^{2a}}\end{aligned}$$

and

$$\begin{aligned}\int \frac{1}{\langle\sigma\rangle^{2a}\langle\sigma_1\rangle^{2b}}\,\mathrm{d}\tau &= \int \frac{1}{\left\langle\tau_1+\tau_2-(n_1+n_2)^2\right\rangle^{2a}\left\langle\tau_1+n_1^2\right\rangle^{2b}}\,\mathrm{d}\tau \\ &= \int \frac{1}{\left\langle\tau_1-(-\tau_2+(n_1+n_2)^2)\right\rangle^{2a}\left\langle\tau_1-(-n_1)^2)\right\rangle^{2b}}\,\mathrm{d}\tau \\ &\lesssim \frac{1}{\left\langle\tau_2-(n_1+n_2)^2-n_1^2\right\rangle^{2a}} = \frac{1}{\left\langle\tau_2-n_2^2-2n_1^2-2n_1n_2\right\rangle^{2a}}.\end{aligned}$$

Therefore, it suffices to get bounds for

$$\begin{aligned}K_1 &= \sup_{(n,\tau)\in\mathbb{Z}\times\mathbb{R}}\left(\frac{1}{\langle\sigma\rangle^{2a}}\frac{n^4}{\gamma(n)^2}\sum_{n_1\in\mathbb{Z}}\frac{\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\left\langle\tau-n^2+2nn_1\right\rangle^{2b}}\right) \quad \text{on} \quad S_1, \\ K_2 &= \sup_{(n_1,\tau_1)\in\mathbb{Z}\times\mathbb{R}}\left(\frac{1}{\langle\sigma_1\rangle^{2b}}\sum_{n\in\mathbb{Z}}\frac{n^4}{\gamma(n)^2}\frac{\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\left\langle\tau_1+n_1^2-2nn_1\right\rangle^{2a}}\right) \quad \text{on} \quad S_2, \\ K_3 &= \sup_{(n_2,\tau_2)\in\mathbb{Z}\times\mathbb{R}}\left(\frac{1}{\langle\sigma_2\rangle^{2b}}\sum_{n_1\in\mathbb{Z}}\frac{(n_1+n_2)^4}{\gamma(n_1+n_2)^2}\frac{\langle n_1+n_2\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\left\langle\tau_2-n_2^2-2n_1^2-2n_1n_2\right\rangle^{2a}}\right) \quad \text{on} \quad \tilde{S}_3.\end{aligned}$$

In the region B_1, we have $\frac{n^4}{\gamma(n)^2}=0$, therefore the estimate is trivial.

In the region B_2, we have

$$\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s} = \langle n\rangle^{2s}\langle n\rangle^{-2s} = 1,$$

and, moreover,

$$\frac{n^4}{\gamma(n)^2} = \frac{n^2}{n^2+1} \le 1.$$

Therefore, since $a, b > 0$, the following happens for all $(n, n_1, \tau, \tau_1) \in B_2$

$$K_1 = \sup_{(n,\tau)\in\mathbb{Z}\times\mathbb{R}} \frac{1}{\langle\sigma\rangle^{2a}} \frac{n^4}{\gamma(n)^2} \sum_{\substack{n_1=0,\\ n_1=n}} \frac{\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\langle \tau - n^2 + 2nn_1\rangle^{2b}}$$
$$\le \sup_{(n,\tau)\in\mathbb{Z}\times\mathbb{R}} \left(\frac{1}{\langle\sigma\rangle^{2a}} \frac{2}{\langle\sigma\rangle^{2b}}\right) \lesssim 1,$$

which proves that K_1 is bounded on B_2.

Now, by inequality (1.27), we have

$$\frac{\langle n\rangle^{2s}}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}} \le \frac{\langle n_1\rangle^{2|s|}}{\langle n_1\rangle^{2s}} = \langle n_1\rangle^{\lambda(s)}, \tag{2.49}$$

for all $n, n_1 \in \mathbb{Z}$, where

$$\lambda(s) = \begin{cases} 0, & \text{if } s \ge 0 \\ 4|s|, & \text{if } s < 0. \end{cases}$$

Using the algebraic relation (2.47), we obtain

$$|2n_1 n| = |-\sigma + \sigma_1 + \sigma_2|.$$

In the region $B_{3,1}$ we have $|n| \ge 1$ and $|\sigma_1|, |\sigma_2| \le |\sigma|$ which give us

$$|n_1| = \frac{|-\sigma+\sigma_1+\sigma_2|}{2|n|} \le \frac{3|\sigma|}{2} \lesssim |\sigma|. \tag{2.50}$$

Therefore, using (2.49) and (2.50), we obtain

$$\frac{\langle n\rangle^{2s}}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}} \le \langle\sigma\rangle^{\lambda(s)},$$

on $B_{3,1}$, since $\lambda(s) \ge 0$. Thus, in the region $B_{3,1}$ we have

$$K_1 \le \sup_{(n,\tau)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{1}{\langle\sigma\rangle^{2a}} \sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \frac{\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\langle \tau - n^2 + 2nn_1\rangle^{2b}}\right)$$
$$\le \sup_{(n,\tau)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{1}{\langle\sigma\rangle^{2a-\lambda(s)}} \sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \frac{1}{\langle \tau - n^2 + 2nn_1\rangle^{2b}}\right)$$
$$\le \sup_{(n,\tau)\in\mathbb{Z}^*\times\mathbb{R}} \left(\sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \frac{1}{\langle \tau - n^2 + 2nn_1\rangle^{2b}}\right) \lesssim 1,$$

since $2a - \lambda(s) \ge 0$ and $2b > 1$, which concludes that K_1 is bounded on S_1.

Next we estimate K_2 on $S_2 = B_{3,2}$. In region $B_{3,2}$, by the algebraic relation (2.47), we have

$$|2nn_1| = |-\sigma+\sigma_1+\sigma_2| \le 3|\sigma_1|, \text{ that is, } |n_1| \le \frac{3|\sigma_1|}{2|n|} \le 2|\sigma_1|,$$

since $|n| \geq 1$. Recording inequality (2.49), we obtain

$$\frac{\langle n\rangle^{2s}}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}} \leq \langle n_1\rangle^{\lambda(s)} \lesssim \langle \sigma_1\rangle^{\lambda(s)}.$$

Thus, since $n^4/\gamma(n)^2 \leq 1$, we have

$$\begin{aligned} K_2 &= \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{1}{\langle\sigma_1\rangle^{2b}} \sum_{\substack{n\neq 0,\\ n\neq n_1}} \frac{\langle n\rangle^{2s}\langle n_1\rangle^{-2s}\langle n_2\rangle^{-2s}}{\langle \tau_1+n_1^2-2nn_1\rangle^{2a}} \right) \\ &\leq \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}} \sum_{\substack{n\neq 0,\\ n\neq n_1}} \frac{1}{\langle \tau_1+n_1^2-2nn_1\rangle^{2a}} \right) \\ &= \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}} \frac{1}{|2n_1|^{2a}} \sum_{\substack{n\neq 0,\\ n\neq n_1}} \frac{1}{\left(\frac{1}{|2n_1|}+|\frac{\tau_1+n_1^2}{2n_1}-n|\right)^{2a}} \right), \end{aligned} \tag{2.51}$$

for all $(n,n_1,\tau,\tau_1)\in B_{3,2}$. In the region $B_{3,2}$, we observe that

$$\begin{aligned} \left|\frac{\tau_1+n_1^2}{2n_1}-n\right| &= \frac{1}{|2n_1|}\left|\tau_1+n_1^2-2n_1n\right| \leq \frac{1}{|2n_1|}\left(|\tau_1+n_1^2|+|2nn_1|\right) \\ &= \frac{1}{|2n_1|}\left(|\sigma_1|+|-\sigma+\sigma_1+\sigma_2|\right) \leq \frac{2|\sigma_1|}{|n_1|}, \end{aligned}$$

where we used the algebraic relation (2.47). Applying Lemma 2.15 with

$$h=\frac{\tau_1+n_1^2}{2n_1}-n,\; \nu=\frac{1}{|2n_1|},\; \alpha=\frac{\tau_1+n_1^2}{2n_1} \quad\text{and}\quad \beta=\frac{2|\sigma_1|}{|n_1|},$$

we obtain the following bound in the region $B_{3,1}$

$$\sum_{\substack{n\neq 0,\\ n\neq n_1}} \frac{1}{\left(\frac{1}{|2n_1|}+\left|\frac{\tau_1+n_1^2}{2n_1}-n\right|\right)^{2a}} = \sum_{h\in H}\frac{1}{(\nu+|h|)^{2a}} \leq 2\left(2|2n_1|^{2a}+\int_0^{2|\sigma_1|/|n_1|}\left(\frac{1}{|2n_1|}+x\right)^{-2a}\mathrm{d}x\right).$$

We observe that

$$\int_0^{2|\sigma_1|/|n_1|}\left(\frac{1}{|2n_1|}+x\right)^{-2a}\mathrm{d}x = |2n_1|^{2a}\int_0^{2|\sigma_1|/|n_1|}(1+|2n_1|x)^{-2a}\mathrm{d}x = |2n_1|^{2a}\frac{1}{|2n_1|}\int_0^{4|\sigma_1|}\left(1+y\right)^{-2a}\mathrm{d}y,$$

with

$$\int_0^{4|\sigma_1|}\left(1+y\right)^{-2a}\mathrm{d}y = \left.\frac{(1+y)^{1-2a}}{1-2a}\right|_0^{4|\sigma_1|} = \frac{(1+4|\sigma_1|)^{1-2a}}{1-2a}-\frac{1}{1-2a} \leq \frac{\langle 4\sigma_1\rangle^{1-2a}}{1-2a} \lesssim \langle\sigma_1\rangle^{1-2a},$$

since $2a<1$, which give us

$$\int_0^{2|\sigma_1|/|n_1|}\left(\frac{1}{|2n_1|}+x\right)^{-2a}\mathrm{d}x \lesssim |2n_1|^{2a-1}\langle\sigma_1\rangle^{1-2a}. \tag{2.52}$$

Therefore,

$$\sum_{\substack{n\neq 0,\\ n\neq n_1}} \frac{1}{\left(\frac{1}{|2n_1|}+\left|\frac{\tau_1+n_1^2}{2n_1}-n\right|\right)^{2a}} \lesssim 2|2n_1|^{2a}+|2n_1|^{2a-1}\langle\sigma_1\rangle^{1-2a}.$$

Returning to (2.51), we have

$$\begin{aligned} K_2 &\leq \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}}\frac{1}{|2n_1|^{2a}}\sum_{\substack{n\neq 0,\\ n\neq n_1}}\frac{1}{\left(\frac{1}{|2n_1|}+\left|\frac{\tau_1+n_1^2}{2n_1}-n\right|\right)^{2a}}\right)\\ &\lesssim \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}}\frac{1}{|2n_1|^{2a}}\left(2|2n_1|^{2a}+|2n_1|^{2a-1}\langle\sigma_1\rangle^{1-2a}\right)\right)\\ &= \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{2}{\langle\sigma_1\rangle^{2b-\lambda(s)}}+\frac{1}{|2n_1|\langle\sigma_1\rangle^{2b+2a-1-\lambda(s)}}\right)\\ &\lesssim \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}}+\frac{1}{\langle\sigma_1\rangle^{2b+2a-1-\lambda(s)}}\right), \end{aligned}$$

for all $(n,n_1,\tau,\tau_1)\in B_{3,2}$. We observe that

$$2b+2a-1-\lambda(s)=(2b-1)+(2a-\lambda(s))>0 \text{ and } 2b-\lambda(s)>0,$$

since $b>\frac{1}{2}$ and $2b>2a>\lambda(s)$. Therefore, in the region $B_{3,2}$ we conclude that

$$K_2 \lesssim \sup_{(n_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}} \left(\frac{1}{\langle\sigma_1\rangle^{2b-\lambda(s)}}+\frac{1}{\langle\sigma_1\rangle^{2b+2a-1-\lambda(s)}}\right)\lesssim 1.$$

Finally, we estimate $K_3(n_2,\tau_2)$ on $\tilde{S}_3$. It follows from inequality (1.27) that

$$\frac{\langle n_1+n_2\rangle^{2s}}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}} \leq \frac{\langle n_2\rangle^{2|s|}}{\langle n_2\rangle^{2s}} = \langle n_2\rangle^{\lambda(s)},$$

where

$$\lambda(s)=\begin{cases} 0, & \text{if } s\geq 0\\ 4|s|, & \text{if } s<0, \end{cases}$$

for all $n_1,n_2\in\mathbb{Z}$. Since $n_1\neq 0$ and $n_1+n_2\neq 0$ in the region $\tilde{S}_3$, we have

$$\langle n_2\rangle \lesssim |n_2| \leq |n_1+n_2|+|n_1| \leq |n_1(n_1+n_2)|+|(n_1+n_2)n_1| \leq 2|n_1(n_1+n_2)|.$$

Now, using relation (2.47) with n_1+n_2 instead of n and $\tau_1+\tau_2$ instead of τ, we obtain

$$\langle n_2\rangle \leq 2|n_1(n_1+n_2)| = |-(\tau_1+\tau_2+(n_1+n_2)^2)+(\tau_1+n_1^2)+(\tau_2-n_2^2)| \leq 3|\tau_2-n_2^2| \lesssim \langle\sigma_2\rangle,$$

for all $(n_1,n_2,\tau_1,\tau_2)\in\tilde{S}_3$. Therefore, since $\lambda(s)\geq 0$, we have

$$\frac{\langle n_1+n_2\rangle^{2s}}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}} \leq \langle n_2\rangle^{\lambda(s)} \lesssim \langle\sigma_2\rangle^{\lambda(s)}.$$

Thus, since $(n_1+n_2)^4/\gamma(n_1+n_2)^2\leq 1$, we obtain

$$\begin{aligned} K_3 &\lesssim \sup_{(n_2,\tau_2)\in\mathbb{Z}\times\mathbb{R}} \left(\frac{1}{\langle\sigma_2\rangle^{2b-\lambda(s)}}\sum_{\substack{n_1\neq 0,\\ n_1\neq -n_2}}\frac{1}{\langle\tau_2-n_2^2-2n_1^2-2n_1n_2\rangle^{2a}}\right)\\ &\leq \sup_{(n_2,\tau_2)\in\mathbb{Z}\times\mathbb{R}} \left(\sum_{\substack{n_1\neq 0,\\ n_1\neq -n_2}}\frac{1}{\langle\tau_2-n_2^2-2n_1^2-2n_1n_2\rangle^{2a}}\right), \end{aligned}$$

for all $(n_2, n_1, \tau_2, \tau_1) \in \tilde{S}_3$, since $2a - \lambda(s) \geq 0$. Applying the change of variables $(\tilde{\tau}, \tilde{n}_2) = \left(\frac{\tau_2 - n_2^2}{2}, -n_2\right)$ and Lemma 2.14 for $q = 2a$, we have

$$K_3\Big|_{\tilde{S}_3} \lesssim \sup_{(n_2,\tau_2)\in\mathbb{Z}\times\mathbb{R}} \left(\sum_{n_1\in\mathbb{Z}} \frac{1}{\langle \tau_2 - n_2^2 - 2n_1^2 - 2n_1 n_2 \rangle^{2a}} \right) = \sup_{(\tilde{n}_2,\tilde{\tau})\in\mathbb{Z}\times\mathbb{R}} \left(\sum_{n_1\in\mathbb{Z}} \frac{1}{\langle 2\tilde{\tau} - 2n_1^2 + 2n_1\tilde{n}_2 \rangle^{2a}} \right)$$
$$\lesssim \sup_{(\tilde{n}_2,\tau_2)\in\mathbb{Z}\times\mathbb{R}} \left(\sum_{n_1\in\mathbb{Z}} \frac{1}{\langle \tilde{\tau} - n_1(n_1 - \tilde{n}_2) \rangle^{2a}} \right) \lesssim 1,$$

for all $(n_2, n_1, \tau_2, \tau_1) \in \tilde{S}_3$, which concludes that K_3 is bounded on $\tilde{S}_3$. This finishes the proof of the theorem. □

Therefore, it follows from Lemma 2.13 and Theorem 2.16 the following proposition, which give us Φ_T is a contraction in a ball centered at the origin of $X_{s,b}$.

Proposition 2.17. *Let $s > -\frac{1}{4}$ and $b \in \left(\frac{1}{2}, \frac{3}{4}\right)$, then for all $u_0 \in H^s(\mathbb{T})$, $u_1 \in H^{s-1}(\mathbb{T})$ and $0 < T \leq 1$, there are $d > 0$ and a constant $C_{\psi,b} > 0$ such that*

$$\|\Phi_T u\|_{X_{s,b}} \leq C_{\psi,b}\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + T^d \|u\|_{X_{s,b}}^2\right), \quad \textit{for all } u \in X_{s,b}, \tag{2.53}$$

and

$$\|\Phi_T u - \Phi_T v\|_{X_{s,b}} \leq T^d C_{\psi,b} \|u + v\|_{X_{s,b}} \|u - v\|_{X_{s,b}}, \quad \textit{for all } u, v \in X_{s,b}. \tag{2.54}$$

Once again, it is also possible to show **uniqueness** of the solution and the **continuity of the data-to-solution** map Φ_T, that is, to show that the periodic Cauchy problem (2.1) is well-posed for $s > -\frac{1}{4}$ by using inequalities (2.53) and (2.54). Since the proof Proposition 2.17 and the proof of well-posedness in this case is very similar to the one in Gevrey analytic case, we will do this with detail in the next chapter.

Chapter

3 The problem in $G^{\sigma,\delta,s}$

In this chapter, we consider the initial value problem for the "good" Boussinesq equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u + \partial_x^2(u^2) = 0, & x \in \mathbb{R} \text{ or } \mathbb{T},\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \\ \partial_t u(x,0) = \partial_x u_1(x), \end{cases} \tag{3.1}$$

now with initial data in analytic Gevrey spaces on the line and the circle.

Let us recall the spaces of analytic Gevrey functions that we shall use. For $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, we have the spaces

$$G^{\sigma,\delta,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R});\ \|f\|^2_{G^{\sigma,\delta,s}(\mathbb{R})} = \int \langle\xi\rangle^{2s} e^{2\delta|\xi|^{1/\sigma}} |\widehat{f}(\xi)|^2 \, d\xi < \infty \right\}, \tag{3.2}$$

where $\langle\xi\rangle \doteq (1+\xi^2)^{1/2}$. For the periodic case, the space $G^{\sigma,\delta,s}(\mathbb{T})$ is given by

$$G^{\sigma,\delta,s}(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T});\ \|f\|^2_{G^{\sigma,\delta,s}(\mathbb{R})} = \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} e^{2\delta n^{1/\sigma}} |\widehat{f}(n)|^2 < \infty \right\}. \tag{3.3}$$

We often omit $\mathbb{R}$ or $\mathbb{T}$ in the notation of these spaces when it is clear by the context which one is being considered or when the statement holds for both. If $\varphi \in G^{\sigma,\delta,s}$, then φ belongs to the Gevrey class $\mathbf{G}^\sigma$ (see Proposition 1.12 in Chapter 1.).

In the case when $\sigma = 1$, we denote $G^{\delta,s} \equiv G^{1,\delta,s}$. Thus, if $\varphi \in G^{\delta,s}(\mathbb{R})$ then φ is analytic on the line and admits a holomorphic extension $\tilde{\varphi}$ on the strip $S_\delta \doteq \{x + iy;\ |y| < \delta\}$. Hence, in this context, we refer to the parameter $\delta > 0$ as the radius of analyticity of the function φ (see Proposition 1.15 in Chapter 1.).

By following the same strategy as in Chapter 2 , we consider the following Gevrey-Bourgain spaces:

Definition 3.1. *Let $s, b \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$. We denote by $X_{\sigma,\delta,s,b} = X_{\sigma,\delta,s,b}(\mathbb{R}^2)$ the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm*

$$\|u\|_{X_{\sigma,\delta,s,b}} = \left(\iint e^{2\delta|\xi|^{1/\sigma}} \langle|\tau| - \gamma(\xi)\rangle^{2b} \langle\xi\rangle^{2s} |\widehat{u}(\xi,\tau)|^2 \, d\tau\, d\xi \right)^{\frac{1}{2}},$$

where $\gamma(\xi) = \sqrt{\xi^2 + \xi^4}$ and $\langle\xi\rangle = (1+\xi^2)^{\frac{1}{2}}$.

Since we are considering local in time well-posedness, we shall need the localized Bourgain spaces.

Definition 3.2. *For any $T \geq 0$, $X^T_{\sigma,\delta,s,b} = X^T_{\sigma,\delta,s,b}(\mathbb{R}^2)$ denotes the space endowed with the norm*

$$\|u\|_{X^T_{\sigma,\delta,s,b}} = \inf_{\tilde{u}\in X_{\sigma,\delta,s,b}} \left\{\|\tilde{u}\|_{X_{\sigma,\delta,s,b}};\ \tilde{u}(x,t) = u(x,t)\ on\ \mathbb{R}\times[0,T]\right\}.$$

Let us now give the definitions of the spaces $X_{\sigma,\delta,s,b}$ and $X^T_{\sigma,\delta,s,b}$ on the circle.

Definition 3.3. *Let $\mathscr{X}$ be the space of functions v such that*

(i) $v : \mathbb{T}\times\mathbb{R} \to \mathbb{C}$.

(ii) $v(x,\cdot) \in \mathscr{S}(\mathbb{R})$ *for each* $x \in \mathbb{T}$.

(iii) $v(\cdot,t) \in C^\infty(\mathbb{T})$ *for each* $t \in \mathbb{R}$.

For $\delta > 0$, $s, b \in \mathbb{R}$ and $\sigma \geq 1$, $X_{\sigma,\delta,s,b} = X_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})$ denotes the completion of $\mathscr{X}$ with respect to the norm

$$\|v\|_{X_{\sigma,\delta,s,b}} = \left(\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} \langle|\tau| - \gamma(n)\rangle^{2b}\langle n\rangle^{2s} e^{2\delta|n|^{1/\sigma}}|\widehat{v}(n,\tau)|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}$$

where $\langle n\rangle \doteq \sqrt{1+k^2}$ and $\gamma(n) = \sqrt{n^2+n^4}$. For $T \geq 0$, $X^T_{\sigma,\delta,s,b}$ denotes the space endowed with the norm

$$\|u\|_{X^T_{\sigma,\delta,s,b}} = \inf_{v\in X_{\sigma,\delta,s,b}} \left\{\|v\|_{X_{\sigma,\delta,s,b}};\ v(x,t) = u(x,t)\ on\ \mathbb{T}\times[0,T]\right\}.$$

Our main result in this chapter reads as follows.

Theorem 3.4. *Let $s > -1/4$, $\delta > 0$ and $\sigma \geq 1$. Then, for all initial data $(u_0,u_1) \in G^{\sigma,\delta,s}\times G^{\sigma,\delta,s-1}$, there exist a lifespan*

$$T = T(u_0,u_1) = \frac{c_0}{(1+\|u_0\|_{G^{\sigma,\delta,s}} + \|u_1\|_{G^{\sigma,\delta,s-1}})^\alpha}, \tag{3.4}$$

where $\alpha > 1$ and $c_0 \leq 1$ are positive constants which depend only on s, and a unique solution u of the Cauchy problem for the "good" Boussinesq equation (3.1) *such that*

$$u \in C\big([0,T]; G^{\sigma,\delta,s}\big) \cap X^T_{\sigma,\delta,s,b}.$$

Moreover, the data-to-soltuion map is locally Lipschitz.

In order to prove Theorem 3.4, we are going to follow the same steps as in Chapter 2. Thus, recalling the definition of map Φ_T

$$\Phi_T u \doteq \psi(t)W_1(t)u_0(x) + \psi(t)W_2(t)\partial_x u_1(x) - \psi_T(t)\int_0^t W_2(t-t')w(x,t')\,\mathrm{d}t', \tag{3.5}$$

as we did before, the right side of (3.5) is a formal solution of the IVP (3.1). Our goal again is to solve the equation $\Phi_T u = u$, but now in Gevrey-Bourgain spaces. For this we will use a fixed point argument for the map Φ_T.

In what follows, just to fix notation, we will focus in the proof of Theorem 3.4 for the real case. Since, as the reader will see, all the results have similar proof to the periodic case.

3.1 Estimates in analytic Gevrey-Bourgain spaces

The following two results show the relevance of the space $X_{\sigma,\delta,s,b}$. The first one guarantees that it is continuously embedded in $C\big([0,T], G^{\sigma,\delta,s}\big)$, provided $b > 1/2$, and the second one gives a bilinear estimate which is needed in the proof of the local well-posedness of (3.1).

Lemma 3.5. *Let $b > \frac{1}{2}$, $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$. Then, for all $T > 0$, the inclusion*

$$X_{\sigma,\delta,s,b}(\mathbb{R}^2) \hookrightarrow C\Big([0,T], G^{\sigma,\delta,s}(\mathbb{R})\Big)$$

is continuous, that is,

$$\sup_{t\in[0,T]} \|u(t)\|_{G^{\sigma,\delta,s}} \leq C\,\|u\|_{X_{\sigma,\delta,s,b}}.$$

Proof. First, we observe that the operator A defined by

$$\widehat{Au}^{x}(\xi,t) = e^{\delta|\xi|^{1/\sigma}}\,\widehat{u}^{x}(\xi,t) \tag{3.6}$$

satisfies the relations

$$\|u\|_{G^{\sigma,\delta,s}} = \|Au\|_{H^s} \quad \text{and} \quad \|u\|_{X_{\sigma,\delta,s,b}} = \|Au\|_{X_{s,b}},$$

where $X_{s,b}$ is the space defined in Definition 2.2. Then, by using Lemma 2.4, we have that u belongs to $C\big([0,T]; G^{\sigma,\delta,s}(\mathbb{R})\big)$ and

$$\begin{aligned}
\|u(x,t)\|_{C([0,T],G^{\sigma,\delta,s}(\mathbb{R}))} &= \sup_{0\leq t\leq T}\big(\|u(\cdot,t)\|_{G^{\sigma,\delta,s}}\big) = \sup_{0\leq t\leq T}\big(\|Au(\cdot,t)\|_{H^s}\big)\\
&= \|Au(x,t)\|_{C([0,T],H^s(\mathbb{R}))} \leq C_b\|Au(x,t)\|_{X_{s,b}}\\
&= C_b\|u(x,t)\|_{X_{\sigma,\delta,s,b}},
\end{aligned}$$

which completes the proof of the lemma. □

Next, we show an analytic version of the Bilinear Estimates proved in Chapter 2.

Proposition 3.6 (Gevrey Bilinear Estimates)**.** *If $s > -\frac{1}{4}$ and $u, v \in X_{\sigma,\delta,s,b}$, then there exists a constant $C > 0$ such that the bilinear estimate*

$$\left\|\mathscr{F}^{-1}\left(\frac{|\xi|^2\,\widehat{uv}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{\sigma,\delta,s,-a}} \leq C\,\|u\|_{X_{\sigma,\delta,s,b}}\,\|v\|_{X_{\sigma,\delta,s,b}}$$

holds in the following cases:

(i) $s \geq 0$, $b > \frac{1}{2}$ *and* $\frac{1}{4} < a < \frac{1}{2}$;

(ii) $-\frac{1}{4} < s < 0$, $b > \frac{1}{2}$, $\frac{1}{4} < a < \frac{1}{2}$ *and* $|s| < \frac{a}{2}$.

Proof. We consider again the operator A given by (3.6) and observe that

$$\left\|\mathscr{F}^{-1}\left(\frac{|\xi|^2\,\widehat{uv}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{\sigma,\delta,s,-a}} = \left\|e^{\delta|\xi|^{1/\sigma}}\,\langle\xi\rangle^s\big\langle|\tau|-\gamma(\xi)\big\rangle^{-a}\,\frac{|\xi|^2\,\widehat{uv}(\xi,\tau)}{2i\gamma(\xi)}\right\|_{L^2_{\xi,\tau}}$$

and

$$\begin{aligned}\left|e^{\delta|\xi|^{1/\sigma}}\widehat{uv}(\xi,\tau)\right| &= (2\pi)^{-2}\left|e^{\delta|\xi|^{1/\sigma}}\widehat{u}*\widehat{v}(\xi,\tau)\right| \\ &\le (2\pi)^{-2}\iint e^{\delta|\xi-\xi_1|^{1/\sigma}}|\widehat{u}(\xi-\xi_1,\tau-\tau_1)|\, e^{\delta|\xi_1|^{1/\sigma}}|\widehat{v}(\xi_1,\tau_1)|\,d\tau_1\,d\xi_1 \\ &= (2\pi)^{-2}\iint \left|\widehat{Au}(\xi-\xi_1,\tau-\tau_1)\right|\left|\widehat{Av}(\xi_1,\tau_1)\right| d\xi_1\,d\tau_1,\end{aligned}$$

since

$$|\xi|^{1/\sigma} \le \big(|\xi-\eta|+|\eta|\big)^{1/\sigma} \le |\xi-\eta|^{1/\sigma}+|\eta|^{1/\sigma},$$

for all $\sigma \ge 1$, by using the fact that

$$(a+b)^p \le a^p + b^p, \text{ for all } a,b\ge 0 \text{ and } 0<p\le 1.$$

Thus, we have

$$\left\|\mathscr{F}^{-1}\left(\frac{|\xi|^2\widehat{uv}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{\sigma,\delta,s,-a}} \le \left\|\frac{\langle\xi\rangle^s|\xi|^2}{\big\langle|\tau|-\gamma(\xi)\big\rangle^a 2i\gamma(\xi)}(2\pi)^{-2}\iint \left|\widehat{Au}(\xi-\xi_1,\tau-\tau_1)\right|\left|\widehat{Av}(\xi_1,\tau_1)\right| d\xi_1\,d\tau_1\right\|_{L^2_{\xi,\tau}}$$

Now, by using inequality (2.14) in the proof of Theorem 2.9 (see Theorem 2.16 for the periodic case), we obtain

$$\left\|\mathscr{F}^{-1}\left(\frac{|\xi|^2\widehat{uv}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{\sigma,\delta,s,-a}} \le C\,\|Au\|_{X_{s,b}}\,\|Av\|_{X_{s,b}} = C\|u\|_{X_{\sigma,\delta,s,b}}\|v\|_{X_{\sigma,\delta,s,b}},$$

where the constant $C>0$ is the same one as in Theorem 2.9. □

Remark. The result above has the same proof for the periodic case just by replacing the integral in ξ variable by a sum.

The following two results is concerning the estimate of the map Φ_T in the space $X_{\sigma,\delta,s,b}$.

Lemma 3.7. *If $s\in\mathbb{R}$, $b\ge 0$, $\delta>0$ and $\sigma\ge 1$, then there is a constant $C>0$ depending only on ψ and b such that*

$$\left\|\psi(t)\,[W_1(t)u_0(x)+W_2(t)\partial_x u_1(x)]\right\|_{X_{\sigma,\delta,s,b}} \le C\left(\|u_0\|_{G^{\sigma,\delta,s}}+\|u_1\|_{G^{\sigma,\delta,s-1}}\right),$$

for all $u_0\in G^{\sigma,\delta,s}(\mathbb{R})$ and $u_1\in G^{\sigma,\delta,s-1}(\mathbb{R})$.

Proof. The proof follows the same steps as the proof of Lemma 2.6. Let us denote

$$v(x,t) = W_1(t)u_0(x)+W_2(t)\partial_x u_1(x).$$

Taking the Fourier transform with respect to x, we get

$$\widehat{\psi v}^x(\xi,t) = e^{i\gamma(\xi)t}\psi(t)h_1(\xi)+e^{-i\gamma(\xi)t}\psi(t)h_2(\xi),$$

where

$$h_1(\xi) = \frac{\widehat{u_0}(\xi)}{2}+\frac{\widehat{\partial_x u_1}(\xi)}{2i\gamma(\xi)} \quad\text{and}\quad h_2(\xi) = \frac{\widehat{u_0}(\xi)}{2}-\frac{\widehat{\partial_x u_1}(\xi)}{2i\gamma(\xi)}.$$

Thus,

$$\widehat{\psi v}(\xi,\tau) = h_1(\xi)\widehat{\psi}(\tau-\gamma(\xi)) + h_2(\xi)\widehat{\psi}(\tau+\gamma(\xi)).$$

Since $\langle|\tau|-\gamma(\xi)\rangle \le \langle\tau\pm\gamma(\xi)\rangle$ and $b\ge 0$, we have

$$\begin{aligned}
\|\psi v\|_{X_{\sigma,\delta,s,b}} &\le \left\| e^{\delta|\xi|^{1/\sigma}}\langle\xi\rangle^s\langle|\tau|-\gamma(\xi)\rangle^b h_1(\xi)\widehat{\psi}(\tau-\gamma(\xi))\right\|_{L^2_{\tau,\xi}} \\
&\quad + \left\| e^{\delta|\xi|^{1/\sigma}}\langle\xi\rangle^s\langle|\tau|-\gamma(\xi)\rangle^b h_2(\xi)\widehat{\psi}(\tau+\gamma(\xi))\right\|_{L^2_{\tau,\xi}} \\
&\le \|\psi\|_{H^b}\left\| e^{\delta|\xi|^{1/\sigma}}\langle\xi\rangle^s h_1\right\|_{L^2_\xi} + \|\psi\|_{H^b}\left\| e^{\delta|\xi|^{1/\sigma}}\langle\xi\rangle^s h_2\right\|_{L^2_\xi} \\
&\le \|\psi\|_{H^b}\left(\|u_0\|_{G^{\sigma,\delta,s}} + \left\| e^{\delta|\xi|^{1/\sigma}}\langle\xi\rangle^s\frac{|\xi|}{\gamma(\xi)}\widehat{u_1}\right\|_{L^2_\xi}\right) \\
&\le \|\psi\|_{H^b}\left(\|u_0\|_{G^{\sigma,\delta,s}} + \|u_1\|_{G^{\sigma,\delta,s-1}}\right),
\end{aligned}$$

since $|\xi|/\gamma(\xi) = \langle\xi\rangle^{-1}$, which proves the desired inequality. □

The next result is concerning the $X_{\sigma,\delta,s,b}$-norm of the nonlinear part of Φ_T, its proof is a consequence of Lemma 1.8.

Lemma 3.8. *If* $-\frac{1}{2} < b' \le 0 \le b \le b'+1$ *then*

$$\left\|\psi_T(t)\int_0^t W_2(t-t')w(x,t')\,\mathrm{d}t'\right\|_{X_{\sigma,\delta,s,b}} \le CT^{1-(b-b')}\left\|\mathscr{F}^{-1}\left(\frac{\widehat{w}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{\sigma,\delta,s,b'}},$$

for some constant $C>0$ *which depends on* ψ, b *and* b'.

Proof. We define $U(x,t) = \psi_T(t)\int_0^t W_2(t-t')w(x,t')\,\mathrm{d}t'$. Considering the operator A given by (3.6), we have

$$\begin{aligned}
\widehat{AU}^x(\xi,t) &= e^{\delta|\xi|^{1/\sigma}}\widehat{U}^x(\xi,t) = e^{\delta|\xi|^{1/\sigma}}\psi_T(t)\int_0^t \mathscr{F}_x\left[W_2(t-t')w\right](\xi,t')\,\mathrm{d}t' \\
&= \psi_T(t)\int_0^t\left(\frac{e^{i(t-t')\gamma(\xi)} - e^{-i(t-t')\gamma(\xi)}}{2i\gamma(\xi)}\right)e^{\delta|\xi|^{1/\sigma}}\widehat{w}^x(\xi,t')\,\mathrm{d}t' \\
&= \psi_T(t)\int_0^t \mathscr{F}_x\left[W_2(t-t')(Aw)\right](\xi,t')\,\mathrm{d}t' \\
&= \mathscr{F}_x\left(\psi_T(t)\int_0^t W_2(t-t')(Aw)(x,t')\,\mathrm{d}t'\right)(\xi,t),
\end{aligned}$$

where $\mathscr{F}_x$ denotes the Fourier transform with respect to x variable. Thus,

$$\|U\|_{X_{\sigma,\delta,s,b}} = \|AU\|_{X_{s,b}} = \left\|\psi_T(t)\int_0^t W_2(t-t')(Aw)(x,t')\,\mathrm{d}t'\right\|_{X_{s,b}}.$$

Using Lemma 1.8, we obtain

$$\|U\|_{X_{\sigma,\delta,s,b}} \leq CT^{1-(b-b')} \left\| \mathscr{F}^{-1}\left(\frac{\widehat{Aw}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{s,b'}}.$$

Now, we observe that

$$\begin{aligned}\left\| \mathscr{F}^{-1}\left(\frac{\widehat{Aw}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{s,b'}} &= \left\| \langle\xi\rangle^{s} \big\langle |\tau|-\gamma(\xi)\big\rangle^{b'} \frac{\widehat{Aw}(\xi,\tau)}{2i\gamma(\xi)}\right\|_{L^2_{\xi,\tau}} \\ &= \left\| \langle\xi\rangle^{s} \big\langle |\tau|-\gamma(\xi)\big\rangle^{b'} e^{\delta|\xi|^{1/\sigma}} \frac{\widehat{w}(\xi,\tau)}{2i\gamma(\xi)}\right\|_{L^2_{\xi,\tau}} \\ &= \left\| \mathscr{F}^{-1}\left(\frac{\widehat{w}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{\sigma,\delta,s,b'}},\end{aligned}$$

which finishes the proof. □

Now, we have all the ingredients to prove the well-posedness of the "good" Boussinesq equation in a Gevrey class of funcitons.

3.2 Proof of the well-posedness

In this section we shall prove the well-posedness.

Existence of a solution. For $(u_0, u_1) \in G^{\sigma,\delta,s} \times G^{\sigma,\delta,s-1}$ with $s > -\frac{1}{4}$, and for $0 < T \leq 1$ we recall the definition of the map Φ_T

$$\Phi_T(u)(x,t) = \psi(t)\,(W_1(t)u_0 + W_2(t)\partial_x u_1) + \psi_T(t)\int_0^t W_2(t-t')\partial_x^2\big(u^2\big)(t')\,\mathrm{d}t'.$$

The final step for the existence proof consists in to show that Φ_T is a contraction in

$$X_{\sigma,\delta,s,b}(r) = \left\{u \in X^T_{\sigma,\delta,s,b};\ \|u\|_{X_{\sigma,\delta,s,b}} \leq r\right\}$$

for some $r > 0$ and $0 < T \leq 1$.

Proposition 3.9. *Let $\sigma \geq 1$, $\delta > 0$ and $s > -\frac{1}{4}$, then for all $u_0 \in G^{\sigma,\delta,s}$, $u_1 \in G^{\sigma,\delta,s-1}$ and $0 < T \leq 1$, there are $b \in \left(\frac{1}{2}, \frac{3}{4}\right)$, $d > 0$ and a constant $C_{\psi,b} > 0$ such that*

$$\|\Phi_T u\|_{X_{\sigma,\delta,s,b}} \leq C_{\psi,b}\left(\|u_0\|_{G^{\sigma,\delta,s}} + \|u_1\|_{G^{\sigma,\delta,s-1}} + T^d\,\|u\|^2_{X_{\sigma,\delta,s,b}}\right), \quad \text{for all } u \in X_{\sigma,\delta,s,b}, \tag{3.7}$$

and

$$\|\Phi_T u - \Phi_T v\|_{X_{\sigma,\delta,s,b}} \leq T^d C_{\psi,b}\,\|u+v\|_{X_{\sigma,\delta,s,b}}\,\|u-v\|_{X_{\sigma,\delta,s,b}}, \quad \text{for all } u, v \in X_{\sigma,\delta,s,b}. \tag{3.8}$$

Proof. The first step is to consider $a \in \left(\frac{1}{4}, \frac{1}{2}\right)$ satisfying

(i) If $s \geq 0$, then $d = 1-(b+a) > 0$.

(ii) If $-\frac{1}{4} < s < 0$, then $d = 1-(b+a) > 0$ and $|s| < \frac{a}{2}$.

In this conditions, estimate (3.7) follows from Lemmas 3.7 and 3.8 and Proposition 3.6. In fact, using Lemma 3.7, Lemma 3.8 with $b' = -a$ and Proposition 3.6, we have

$$\begin{aligned}\|\Phi_T u\|_{X_{\sigma,\delta,s,b}} &\le \left\|\psi(t)\left(W_1(t)u_0(x) + W_2(t)\partial_x u_1(x)\right)\right\|_{X_{\sigma,\delta,s,b}} + \left\|\psi_T(t)\int_0^t W_2(t-t')\partial_x^2(u^2)(x,t')\mathrm{d}t'\right\|_{X_{\sigma,\delta,s,b}} \\ &\le \tilde{C}_{\psi,b}\left(\|u_0\|_{G^{\sigma,\delta,s}} + \|u_1\|_{G^{\sigma,\delta,s-1}} + T^d\left\|\mathscr{F}^{-1}\left(\frac{\widehat{\partial_x^2(u^2)}(\xi,\tau)}{2i\gamma(\xi)}\right)\right\|_{X_{\sigma,\delta,s,-a}}\right) \\ &\le C_{\psi,b}\left(\|u_0\|_{G^{\sigma,\delta,s}} + \|u_1\|_{G^{\sigma,\delta,s-1}} + T^d\|u\|^2_{X_{\sigma,\delta,s,b}}\right).\end{aligned}$$

In order to prove estimate (3.8), we observe that

$$\Phi_T u - \Phi_T v = -\psi_T(t)\int_0^t W_2(t-t')w(x,t')\mathrm{d}t',$$

where w now is given by

$$w = \partial_x^2(u^2 - v^2) = \partial_x^2\left[(u+v)(u-v)\right].$$

Thus, applying Lemma 3.8 and Proposition 3.6 we obtain (3.8), which completes the proof. □

The next proposition shows that our map Φ_T is, in fact, a contraction in $X_{\sigma,\delta,s,b}(r)$, for some $T = T(u_0,u_1)$ and $r > 0$.

Proposition 3.10. *Let $\sigma \ge 1$, $\delta > 0$ and $s > -\frac{1}{4}$,. For initial data $u_0 \in G^{\sigma,\delta,s}$ and $u_1 \in G^{\sigma,\delta,s-1}$, there are $b \in \left(\frac{1}{2},\frac{3}{4}\right)$ and $T = T(u_0,u_1) > 0$ such that*

$$\Phi_T : X_{\sigma,\delta,s,b}(r) \longrightarrow X_{\sigma,\delta,s,b}(r)$$

is a contraction, where $X_{\sigma,\delta,s,b}(r)$ is given by

$$X_{\sigma,\delta,s,b}(r) = \left\{u \in X_{\sigma,\delta,s,b};\ \|u\|_{X_{\sigma,\delta,s,b}} \le r\right\} \quad \text{with} \quad r = 2C_{\psi,b}\left(\|u_0\|_{G^{\sigma,\delta,s}} + \|u_1\|_{G^{\sigma,\delta,s-1}}\right),$$

and $C_{\psi,b}$ is the constant that appears in Proposition 3.9.

Proof. In fact, from Proposition 3.9 it follows that

$$\|\Phi_T u\|_{X_{\sigma,\delta,s,b}} \le C_{\psi,b}\left(\|u_0\|_{G^{\sigma,\delta,s}} + \|u_1\|_{G^{\sigma,\delta,s-1}} + T^d\|u\|^2_{X_{\sigma,\delta,s,b}}\right) \le \frac{r}{2} + C_{\psi,b}T^d r^2,$$

for all $u \in X_{\sigma,\delta,s,b}(r)$. Choosing $T = \min\left\{1, (4C_{\psi,b}r)^{-\frac{1}{d}}\right\}$, we obtain

$$\|\Phi_T u\|_{X_{\sigma,\delta,s,b}} \le \frac{r}{2} + \frac{r}{4} < r,$$

for all $u \in X_{\sigma,\delta,s,b}(r)$. Thus, Φ_T maps $X_{\sigma,\delta,s,b}(r)$ into $X_{\sigma,\delta,s,b}(r)$. Also, it is a contraction, since for all u and v belonging to $X_{\sigma,\delta,s,b}(r)$ we have

$$\begin{aligned}\|\Phi_T u - \Phi_T v\|_{X_{\sigma,\delta,s,b}} &\leq T^d C_{\psi,b} \|u+v\|_{X_{\sigma,\delta,s,b}} \|u-v\|_{X_{\sigma,\delta,s,b}} \\ &\leq T^d C_{\psi,b} 2r \|u-v\|_{X_{\sigma,\delta,s,b}} \\ &\leq \frac{1}{2}\|u-v\|_{X_{\sigma,\delta,s,b}},\end{aligned}$$

which finishes the proof. □

By Proposition 3.10, we see that for initial data $u_0 \in G^{\sigma,\delta,s}$ and $u_1 \in G^{\sigma,\delta,s-1}$, there is a $0 < T \leq 1$ such that Φ_T is a contraction on a small ball centered at the origin in $X_{\sigma,\delta,s,b}$. Hence Φ_T has a unique fixed point u in a neighborhood of 0 with respect to the norm $\|\cdot\|_{X_{\sigma,\delta,s,b}}$. Since $\psi(t) = 1$ and $\psi_T(t) = 1$, for all $|t| \leq T$, it follows that u solves the initial value problem (3.1) on $\mathbb{R} \times [0,T]$. Finally, thanks to Lemma 3.5, we have proved the existence of a solution to the Cauchy problem which belongs to the space $C\big([0,T], G^{\sigma,\delta,s}(\mathbb{R})\big)$.

Uniqueness. From the fixed point argument used above, we have uniqueness of the solution of $\Phi_T u = u$ in the set $X_{\sigma,\delta,s,b}(r)$. We will use the same argument due to Bekiranov, Ogawa and Ponce [4] to obtain the uniqueness in the whole space $X^T_{\sigma,\delta,s,b}$.

Let $0 < T \leq 1$, $u \in X_{\sigma,\delta,s,b}$ be the solution of the equation $\Phi_T u = u$ and $\tilde{v} \in X^T_{\sigma,\delta,s,b}$ be a solution of the Cauchy problem (3.1) with the same initial data u_0 and u_1, that is,

$$\tilde{v}(t,x) = W_1(t)u_0(x) + W_2(t)\partial_x u_1(x) - \int_0^t W_2(t-t')\partial_x^2\left(\tilde{v}^2\right)(x,t')\,\mathrm{d}t',$$

for all $(x,t) \in \mathbb{R}\times[0,T]$. Fixing an extension $v \in X_{\sigma,\delta,s,b}$ of $\tilde{v}$, we have

$$v(t,x) = \psi(t)\left(W_1(t)u_0(x) + W_2(t)\partial_x u_1(x)\right) - \psi_T(t)\int_0^t W_2(t-t')\partial_x^2(v^2)(x,t')\,\mathrm{d}t',$$

for all $(x,t) \in \mathbb{R}\times[0,T^*]$ with $0 < T^* \leq T$. Our goal is to show that $u = v$ on $\mathbb{R}\times[0,T]$.

We fix

$$M \geq \max\left\{\|u\|_{X_{\sigma,\delta,s,b}}, \|v\|_{X_{\sigma,\delta,s,b}}\right\}.$$

For any $\varepsilon > 0$, considering the difference $u - v \in X^{T^*}_{\sigma,\delta,s,b}$, there is $\omega \in X_{\sigma,\delta,s,b}$ such that

$$\omega(x,t) = u(x,t) - v(x,t) = \psi_T(t)\int_0^t W_2(t-t')\partial_x^2\left(u^2 - v^2\right)(x,t')\mathrm{d}t', \text{ on } \mathbb{R}\times[0,T^*],$$

and

$$\|\omega\|_{X_{\sigma,\delta,s,b}} \leq \|u-v\|_{X^{T^*}_{\sigma,\delta,s,b}} + \frac{\varepsilon}{2}.$$

We define

$$\tilde{\omega}(t) = \psi_T(t)\int_0^t W_2(t-t')\partial_x^2\big(\omega(u+v)\big)(x,t')\mathrm{d}t'.$$

We have $\tilde{\omega} = u - v$ on $\mathbb{R} \times [0, T^*]$. Therefore, from definition of $\|\cdot\|_{X^{T^*}_{\sigma,\delta,s,b}}$, Lemma 3.8 and Proposition 3.6, it follows that

$$\|u-v\|_{X^{T^*}_{\sigma,\delta,s,b}} \le \|\tilde{\omega}\|_{X_{\sigma,\delta,s,b}} \le C_{\psi,b}\left(T^*\right)^d \|\omega\|_{X_{\sigma,\delta,s,b}} \|u+v\|_{X_{\sigma,\delta,s,b}} \le 2MC_{\psi,b}\left(T^*\right)^d \|\omega\|_{X_{\sigma,\delta,s,b}}.$$

Then, choosing $T^* > 0$ such that $2MC_{\psi,b}(T^*)^d < \frac{1}{2}$, we obtain

$$\|u-v\|_{X^{T^*}_{\sigma,\delta,s,b}} \le \frac{1}{2}\|\omega\|_{X_{\sigma,\delta,s,b}} \le \frac{1}{2}\|u-v\|_{X^{T^*}_{\sigma,\delta,s,b}} + \frac{\varepsilon}{4},$$

which implies

$$\|u-v\|_{X^{T^*}_{\sigma,\delta,s,b}} < \varepsilon.$$

Therefore, $u = v$ on $\mathbb{R} \times [0, T^*]$. Now, since the argument does not depend on the initial data, we can iterate this process a finite number of times to extend the uniqueness result in the whole existence interval $[0, T]$.

Map data-solution is locally Lipschitz. The continuous dependence on the initial data of the solution is given by the following result.

Proposition 3.11. *Let $s > -\frac{1}{4}$, $(u_0, u_1) \in G^{\sigma,\delta,s} \times G^{\sigma,\delta,s-1}$ and $T = T(u_0, u_1)$ satisfying that there are a unique solution $u \in C\left([0,T], G^{\sigma,\delta,s}\right) \cap X^T_{\sigma,\delta,s,b}$ of* (2.1)*. Then, given $T' \in (0,T)$ there exists $R = R(T') > 0$ such that the map solution*

$$\begin{aligned} S : W &\longrightarrow C\big([0,T'], G^{\sigma,\delta,s}\big) \cap X^{T'}_{\sigma,\delta,s,b} \\ (\tilde{u}_0, \tilde{u}_1) &\longmapsto \tilde{u} \end{aligned}$$

is Lipschitz, where W is given by

$$W = \Big\{(\tilde{u}_0, \tilde{u}_1) \in G^{\sigma,\delta,s} \times G^{\sigma,\delta,s-1};\ \|\tilde{u}_0 - u_0\|_{G^{\sigma,\delta,s}} + \|\tilde{u}_1 - u_1\|_{G^{\sigma,\delta,s-1}} < R\Big\}.$$

Proof. First, we observe that for all $(\tilde{u}_0, \tilde{u}_1) \in G^{\sigma,\delta,s} \times G^{\sigma,\delta,s-1}$ there exist $\tilde{T} = \tilde{T}(\tilde{u}_0, \tilde{u}_1) > 0$ and a unique solution $\tilde{u} \in C\left([0,\tilde{T}], G^{\sigma,\delta,s}\right)$ of (3.1). We affirm that given $T' \in (0,T)$, there exists $R > 0$ satisfying

$$T' < \tilde{T}, \quad \text{for all } (\tilde{u}_0, \tilde{u}_1) \in W. \tag{3.9}$$

In fact, since $\tilde{T} = \min\Big\{1, (4C_{\psi,b}\tilde{r})^{-\frac{1}{d}}\Big\}$, where $\tilde{r} = 2C_{\psi,b}\left(\|\tilde{u}_0\|_{G^{\sigma,\delta,s}} + \|\tilde{u}_1\|_{G^{\sigma,\delta,s-1}}\right)$, it is sufficient to show that

$$T' < (4C_{\psi,b}\tilde{r})^{-\frac{1}{d}} = \left(\frac{1}{8C^2_{\psi,b}\left(\|\tilde{u}_0\|_{G^{\sigma,\delta,s}} + \|\tilde{u}_1\|_{G^{\sigma,\delta,s-1}}\right)}\right)^{\frac{1}{d}},$$

which occurs if, and only if,

$$\|\tilde{u}_0\|_{G^{\sigma,\delta,s}} + \|\tilde{u}_1\|_{G^{\sigma,\delta,s-1}} < \frac{1}{8C^2_{\psi,b}T'^d}. \tag{3.10}$$

On the other hand, if $(\tilde{u}_0,\tilde{u}_1)\in W$ then

$$\|\tilde{u}_0\|_{G^{\sigma,\delta,s}}+\|\tilde{u}_1\|_{G^{\sigma,\delta,s-1}}\le\|\tilde{u}_0-u_0\|_{G^{\sigma,\delta,s}}+\|\tilde{u}_1-u_1\|_{G^{\sigma,\delta,s-1}}+\|u_0\|_{G^{\sigma,\delta,s}}+\|u_1\|_{G^{\sigma,\delta,s-1}}$$
$$<R+\|u_0\|_{G^{\sigma,\delta,s}}+\|u_1\|_{G^{\sigma,\delta,s-1}}.$$

Thus, in order to obtain (3.10) it is sufficient to choose R such that

$$0<R<\frac{1}{8C^2_{\psi,b}(T')^d}-\left(\|u_0\|_{G^{\sigma,\delta,s}}+\|u_1\|_{G^{\sigma,\delta,s-1}}\right).$$

This choice of R can be done if the following happens

$$\frac{1}{8C^2_{\psi,b}(T')^d}-\left(\|u_0\|_{G^{\sigma,\delta,s}}+\|u_1\|_{G^{\sigma,\delta,s-1}}\right)>0. \tag{3.11}$$

Since $T'<T$, we have

$$\frac{1}{8C^2_{\psi,b}(T')^d}>\frac{1}{8C^2_{\psi,b}T^d}.$$

Recalling that $T\le(4C_{\psi,b}r)^{-\frac{1}{d}}$, we obtain

$$\frac{1}{8C^2_{\psi,b}(T')^d}>\frac{4C_{\psi,b}r}{8C^2_{\psi,b}}=\frac{r}{2C_{\psi,b}}=\|u_0\|_{G^{\sigma,\delta,s}}+\|u_1\|_{G^{\sigma,\delta,s-1}},$$

which proves that (3.11) goes true. Therefore, we can choose $R=R(T')>0$ satisfying (3.9).

Now, if $(\tilde{u}_0,\tilde{u}_1),(u_0^*,u_1^*)\in W$ with $S(\tilde{u}_0,\tilde{u}_1)=\tilde{u}$ and $S(u_0^*,u_1^*)=u^*$, then

$$\left\|S(\tilde{u}_0,\tilde{u}_1)-S(u_0^*,u_1^*)\right\|_{C([0,T'],G^{\sigma,\delta,s})}=\left\|\tilde{u}-u^*\right\|_{C([0,T'],G^{\sigma,\delta,s})}\le C\left\|\tilde{u}-u^*\right\|_{X_{\sigma,\delta,s,b}}, \tag{3.12}$$

where we used Lemma 3.5. Since $\tilde{u}$ is a fixed point of $\Phi_{\tilde{T}}$ and u^* is a fixed point of Φ_{T^*}, then

$$\tilde{u}(t,x)=\Phi_{\tilde{T}}u(t,x)=\psi(t)\left[W_1(t)\tilde{u}_0(x)+W_2(t)\partial_x\tilde{u}_1(x)\right]-\psi_{\tilde{T}}(t)\int_0^t W_2(t-t')\partial_x^2(\tilde{u}^2)(x,t')\mathrm{d}t'$$

and

$$u^*(t,x)=\Phi_{T^*}u^*(x,t)=\psi(t)\left[W_1(t)u_0^*(x)+W_2(t)\partial_x u_1^*(x)\right]-\psi_{T^*}(t)\int_0^t W_2(t-t')\partial_x^2(u^{*2})(x,t')\mathrm{d}t'.$$

It follows from (3.9) that $\psi_{\tilde{T}}=\psi_{T^*}$ on $[0,T']$, which implies

$$\left\|\tilde{u}-u^*\right\|_{X_{\sigma,\delta,s,b}}\le\left\|\psi(t)\left[W_1(t)(\tilde{u}_0-u_0^*)+W_2(t)\partial_x(\tilde{u}_1-u_1^*)\right]\right\|_{X_{\sigma,\delta,s,b}}$$
$$+\left\|\psi_{T'}(t)\int_0^t W_2(t-t')\partial_x^2(\tilde{u}^2-u^{*2})(x,t')\mathrm{d}t'\right\|_{X_{\sigma,\delta,s,b}}.$$

By Lemma 3.7, we obtain

$$\left\|\psi(t)\left[W_1(t)(\tilde{u}_0-u_0^*)+W_2(t)\partial_x(\tilde{u}_1-u_1^*)\right]\right\|_{X_{\sigma,\delta,s,b}}\le C_{\psi,b}\left(\left\|\tilde{u}_0-u_0^*\right\|_{G^{\sigma,\delta,s}}+\left\|\tilde{u}_1-u_1^*\right\|_{G^{\sigma,\delta,s-1}}\right).$$

Applying Lemma 3.8 and Proposition 3.6, we have

$$\begin{aligned}\left\| \psi_{T'}(t)\int_0^t W_2(t-t')\partial_x^2(\tilde{u}^2-u^{*2})(x,t')\mathrm{d}t' \right\|_{X_{\sigma,\delta,s,b}} &\le C_{\psi,b}T'^d \left\| \tilde{u}+u^* \right\|_{X_{\sigma,\delta,s,b}} \left\| \tilde{u}-u^* \right\|_{X_{\sigma,\delta,s,b}} \\ &\le C_{\psi,b}T'^d \left(\tilde{r}+r^*\right) \left\| \tilde{u}-u^* \right\|_{X_{\sigma,\delta,s,b}},\end{aligned}$$

since $\tilde{u} \in X_{\sigma,\delta,s,b}(\tilde{r})$ and $u^* \in X_{\sigma,\delta,s,b}(r^*)$ with

$$\tilde{r} = 2C_{\psi,b}\left(\|\tilde{u}_0\|_{G^{\sigma,\delta,s}} + \|\tilde{u}_1\|_{G^{\sigma,\delta,s-1}}\right) \quad \text{and} \quad r^* = 2C_{\psi,b}\left(\|u_0^*\|_{G^{\sigma,\delta,s}} + \|u_1^*\|_{G^{\sigma,\delta,s-1}}\right),$$

as we proved on Proposition 3.10. Using that

$$T'^d \le \tilde{T}^d \le (4C_{\psi,b}\tilde{r})^{-1} \quad \text{and} \quad T'^d \le T^{*d} \le (4C_{\psi,b}r^*)^{-1},$$

we obtain $C_{\psi,b}T'^d(\tilde{r}+r^*) \le \frac{1}{2}$. Therefore, we conclude

$$\left\| \tilde{u}-u^* \right\|_{X_{\sigma,\delta,s,b}} \le C_{\psi,b}\left(\left\| \tilde{u}_0-u_0^* \right\|_{G^{\sigma,\delta,s}} + \left\| \tilde{u}_1-u_1^* \right\|_{G^{\sigma,\delta,s-1}}\right) + \frac{1}{2}\left\| \tilde{u}-u^* \right\|_{X_{\sigma,\delta,s,b}},$$

which implies $\|\tilde{u}-u^*\|_{X_{\sigma,\delta,s,b}} \le 2C_{\psi,b}\left(\left\| \tilde{u}_0-u_0^* \right\|_{G^{\sigma,\delta,s}} + \left\| \tilde{u}_1-u_1^* \right\|_{G^{\sigma,\delta,s-1}}\right)$. Finally, returning in (3.12), we establish that

$$\left\| S(\tilde{u}_0,\tilde{u}_1) - S(u_0^*,u_1^*) \right\|_{C([0,T'];G^{\sigma,\delta,s})} \le 2CC_{\psi,b}\left(\left\| \tilde{u}_0-u_0^* \right\|_{G^{\sigma,\delta,s}} + \left\| \tilde{u}_1-u_1^* \right\|_{G^{\sigma,\delta,s-1}}\right),$$

which finishes the proof. □

The proof of Theorem 3.4, that is, local well-posednees of (3.1) in $G^{\sigma,\delta,s}(\mathbb{R})$ is now complete.

CHAPTER

4

Regularity in time variable

In this chapter, we shall prove the following result about regularity in time variable of the solution to the "good" Boussinesq equation with analytic Gevrey initial data, which was inspired by the works [19], [20], [21] and [22] for KdV type equations.

Theorem 4.1. *Let* $s > -\frac{1}{4}$, $\delta > 0$ *and* $\sigma \geq 1$. *The solution* $u(x,t) \in C([0,T]; G^{\sigma,\delta,s})$ *to the Cauchy problem* (3.1) *belongs to* $\mathbf{G}^{2\sigma}(\mathbb{R})$ *in the time variable* t, *for* t *near zero.*

Once again, we will show the proof of Theorem 4.1 in detailed for the real case. The periodic case follows analogously.

4.1 Regularity in space variable

The main result that we show in this section is concerning Gevrey regularity in space variable of the solution to the Cauchy problem (3.1), which is a consequence of the local well-posedness established in Chapter 3.

First, we observe that the solution obtained in Theorem 3.4 satisfies

$$u(\cdot,t) \in G^{\delta,s,b} \;\Rightarrow\; u(\cdot,t) \in H^s, \text{ for all } s \in \mathbb{R},$$

since we have the inclusion (1.15) implying (6). Then, it follows from Proposition 1.5 that u is C^∞ in x variable.

Now, regarding the derivatives on t variable, we already know from [5] that the solution u to the "good" Boussinesq equation belongs to $C^1([0,T]; H^{s-1}) \cap C^2([0,T]; H^{s-2})$ for $s > \frac{5}{2}$, by using Kato's technique. In our case, the solution u belongs to

$$C^1([0,T]; G^{\sigma,\delta,(s+\frac{11}{4})-1}) \cap C^2([0,T]; G^{\sigma,\delta,(s+\frac{11}{4})-2}),$$

for $s > -\frac{1}{4}$, which implies

$$u \in C^2([0,T]; H^s), \text{ for all } s \in \mathbb{R}, \tag{4.1}$$

once again by using (6). We observe that since u is a solution we have

$$u_{tt} = u_{xx} - u_{xxxx} - (u^2)_{xx},$$

with

$$\begin{aligned} u_{xx} &\in C^2([0,T];H^{s-2}), \text{ for all } s \in \mathbb{R}, \\ u_{xxxx} &\in C^2([0,T];H^{s-4}), \text{ for all } s \in \mathbb{R}, \\ (u^2)_{xx} &\in C^2([0,T];H^{s-2}), \text{ for all } s > \frac{1}{2}, \end{aligned}$$

where we used (4.1) and Proposition 1.6. Therefore, we conclude that

$$u_{tt} \in C^2([0,T];H^{s-4}), \text{ for all } s > \frac{1}{2}. \tag{4.2}$$

Then, we just concluded that u is C^4 in the time variable. Now, using again that u is a solution, we have

$$\partial_t^4 u = \partial_t^2(u_{tt}) = (\partial_t^2 u)_{xx} - (\partial_t^2 u)_{xxxx} - (2u_t^2 + 2uu_t)_{xx},$$

with

$$\begin{aligned} (\partial_t^2 u)_{xx} &\in C^2([0,T];H^{(s-4)-2}), \text{ for all } s \in \mathbb{R}, \\ (\partial_t^2 u)_{xxxx} &\in C^2([0,T];H^{(s-4)-4}), \text{ for all } s \in \mathbb{R}, \\ (2u_t^2 + 2uu_t)_{xx} &\in C^2([0,T];H^{(s-4)-2}), \text{ for all } s-4 > \frac{1}{2}, \end{aligned}$$

where we used (4.2) and Proposition 1.6. Therefore, we conclude that

$$\partial_t^4 u \in C^2([0,T];H^{s-8}), \text{ for all } s > \frac{1}{2} + 4. \tag{4.3}$$

By replicating this argument, we conclude the following

$$\partial_t^{2k} u \in C^2([0,T];H^{s-4k}), \text{ for all } s > \frac{1}{2} + 2k \text{ and } k \in \{1,2,\dots\}. \tag{4.4}$$

Therefore, (4.4) allow us to take any time derivatives of the solution u in the classical sense.

Proposition 4.2. *Let $s > -\frac{1}{4}$, $\delta > 0$, $\sigma \geq 1$ and $u \in C([0,T];G^{\sigma,\delta,s})$ be the solution to the Cauchy problem (3.1). Then $u(\cdot,t)$ and $u_t(\cdot,t)$ belong to $\mathbf{G}^\sigma$ for all $t \in [0,T]$, that is, there exists $C > 0$ such that*

$$\left|\partial_x^l u(x,t)\right| \leq C^{l+1}(l!)^\sigma \quad \textit{and} \quad \left|\partial_x^l u_t(x,t)\right| \leq C^{l+1}(l!)^\sigma, \textit{ for all } (x,t) \in \mathbb{R}\times[0,T], \tag{4.5}$$

and for all $l \in \{0,1,2,\dots\}$.

Proof. For any $t \in [0,T]$ and $l \in \{0,1,2,\dots\}$ we have

$$\begin{aligned} \left\|\partial_x^l u(\cdot,t)\right\|_{H^s(\mathbb{R})}^2 &= \int (1+\xi^2)^s \left|\widehat{\partial_x^l u}(\xi,t)\right|^2 \mathrm{d}\xi \\ &= \int (1+\xi^2)^s |\xi|^{2l} |\widehat{u}(\xi,t)|^2 \mathrm{d}\xi \\ &= \int |\xi|^{2l} e^{-2\delta|\xi|^{\frac{1}{\sigma}}} (1+\xi^2)^s e^{2\delta|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi,t)|^2 \mathrm{d}\xi. \end{aligned}$$

We observe that

$$e^{\frac{2\delta}{\sigma}|\xi|^{\frac{1}{\sigma}}}=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{2\delta}{\sigma}|\xi|^{\frac{1}{\sigma}}\right)^{j}\geq\frac{1}{(2l)!}\left(\frac{2\delta}{\sigma}\right)^{2l}|\xi|^{\frac{2l}{\sigma}},$$

for all $l\in\{0,1,2,\ldots\}$ and $\xi\in\mathbb{R}$, which implies $|\xi|^{2l}e^{-2\delta|\xi|^{\frac{1}{\sigma}}}\leq C_{\delta,\sigma}^{2l}(2l)!^{\sigma}$. Thus,

$$\left\|\partial_x^l u(\cdot,t)\right\|_{H^s(\mathbb{R})}^2\leq C_{\delta,\sigma}^{2l}(2l)!^{\sigma}\int(1+\xi^2)^s e^{2\delta|\xi|^{\frac{1}{\sigma}}}|\widehat{u}(\xi,t)|^2\,\mathrm{d}\xi=C_{\delta,\sigma}^{2l}(2l)!^{\sigma}\left\|u(\cdot,t)\right\|_{G^{\sigma,\delta,s}}^2.$$

As we know

$$\sum_{l=0}^{\infty}\frac{(2l)!}{B_0^l(l!)^2}<\infty,\quad\text{for all }B_0>4,$$

then $(2l)!\leq A_1^{2l}(l!)^2$ for some $A_1>0$. Therefore,

$$\left\|\partial_x^l u(\cdot,t)\right\|_{H^s(\mathbb{R})}\leq C_0C_1^l(l!)^{\sigma},\quad\text{for all }t\in[0,T],\tag{4.6}$$

where $C_0=\|u(x,t)\|_{C([0,T];G^{\sigma,\delta,s})}$ and $C_1=A_1^{\sigma}C_{\delta,\sigma}$. Thanks to (4.6) we can prove for $s\geq 0$ that the solution u, in x variable, is Gevrey of order σ. In fact, we split the proof for $s\geq 0$ into two cases.

Case 1. Fix $s>\frac{1}{2}$. By Sobolev Lemma (Proposition 1.7), we have

$$\left\|\partial_x^l u(\cdot,t)\right\|_{L^{\infty}(\mathbb{R})}\leq C_s\left\|\partial_x^l u(\cdot,t)\right\|_{H^s(\mathbb{R})}\leq C_sC_0C_1^l(l!)^{\sigma},$$

which concludes that

$$\left|\partial_x^l u(x,t)\right|\leq C_2C_1^l(l!)^{\sigma},\quad\text{for all }(x,t)\in\mathbb{R}\times[0,T],\tag{4.7}$$

where $C_2=C_sC_0$.

Case 2. Fix $0\leq s\leq\frac{1}{2}$. Applying again Sobolev Lemma (Proposition 1.7) and using the fact that $\|\cdot\|_{L^2}\leq\|\cdot\|_{H^s}$ for all $s\geq 0$, we obtain

$$\begin{aligned}\left\|\partial_x^l u(\cdot,t)\right\|_{L^{\infty}(\mathbb{R})}&\leq C\left\|\partial_x^l u(\cdot,t)\right\|_{H^1(\mathbb{R})}\\&\simeq C\Big(\left\|\partial_x^l u(\cdot,t)\right\|_{L^2(\mathbb{R})}+\left\|\partial_x^{l+1}u(\cdot,t)\right\|_{L^2(\mathbb{R})}\Big)\\&\leq C\Big(\left\|\partial_x^l u(\cdot,t)\right\|_{H^s(\mathbb{R})}+\left\|\partial_x^{l+1}u(\cdot,t)\right\|_{H^s(\mathbb{R})}\Big)\\&\leq CC_0C_1^l(l!)^{\sigma}+CC_0C_1^{l+1}(l+1)!^{\sigma}\\&=CC_0C_1^l(l!)^{\sigma}\left[1+C_1(l+1)^{\sigma}\right]\\&\leq CC_0C_1^l(l!)^{\sigma}\left[1+C_1(l+1)\right]^{\sigma},\end{aligned}$$

since $\sigma\geq 1$. Thanks to the fact $x\leq e^x$ for all $x\geq 0$, we have $1+C_1(l+1)\leq e^{1+C_1}e^{C_1l}$, which give us

$$\left\|\partial_x^l u(\cdot,t)\right\|_{L^{\infty}(\mathbb{R})}\leq e^{\sigma(1+C_1)}CC_0\big(e^{\sigma C_1}C_1\big)^l(l!)^{\sigma}.$$

Therefore,

$$\left|\partial_x^l u(\cdot,t)\right| \le C_3 C_4^l (l!)^\sigma, \quad \text{for all } (x,t)\in\mathbb{R}\times[0,T], \tag{4.8}$$

where $C_3 = e^{\sigma(1+C_1)} C C_0$ and $C_4 = e^{\sigma C_1} C_1$.

Finally, for negative values of s we have Case 3.

Case 3. Let $-\frac{1}{4} < s < 0$. We notice that for $0<\varepsilon<\delta$ there exists a positive constant $B>0$ such that

$$\int e^{2(\delta-\varepsilon)|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi,t)|^2 \,\mathrm{d}\xi \le B\int \frac{e^{2\varepsilon|\xi|^{\frac{1}{\sigma}}}}{(1+\xi^2)^{-s}} e^{2(\delta-\varepsilon)|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi,t)|^2 \,\mathrm{d}\xi = B\int e^{2\delta|\xi|^{\frac{1}{\sigma}}} \langle\xi\rangle^{2s} |\widehat{u}(\xi,t)|^2 \,\mathrm{d}\xi\,,$$

It now follows from this inequality that if $u\in C([0,T];G^{\sigma,\delta,s})$ where $s<0$, then

$$u\in C([0,T];G^{\sigma,\delta-\varepsilon,0})$$

and, therefore, thanks to the second case we can conclude that u satisfies

$$\left|\partial_x^l u(\cdot,t)\right| \le C_5 C_4^l (l!)^\sigma, \quad \text{for all } (x,t)\in\mathbb{R}\times[0,T], \tag{4.9}$$

where C_5 depends now on δ and ε. By inequalities (4.7), (4.8) and (4.9) we have shown that for each fixed $s>-\frac{1}{4}$ there is a constant $C>0$ depending on C_3, δ, σ and s such that

$$\left|\partial_x^l u(\cdot,t)\right| \le C^{l+1} (l!)^\sigma, \quad \text{for all } (x,t)\in\mathbb{R}\times[0,T]. \tag{4.10}$$

It remains to show that the same happens for u_t.

Since u is a solution of (3.1), we have

$$u_{tt} = u_{xx} - u_{xxxx} - (u^2)_{xx},$$

which implies

$$u_t(x,t) - u_t(x,0) = \int_0^t u_{tt}(x,t')\,\mathrm{d}t' = \int_0^t \left[u_{xx}(x,t') - u_{xxxx}(x,t') - (u^2)_{xx}(x,t')\right]\mathrm{d}t',$$

that is,

$$u_t(x,t) = \partial_x u_1(x) + \int_0^t \left[u_{xx}(x,t') - u_{xxxx}(x,t') - (u^2)_{xx}(x,t')\right]\mathrm{d}t'.$$

Thus, for all $l\in\{0,1,2,\dots\}$,

$$\begin{aligned}\left|\partial_x^l u_t(x,t)\right| &\le \left|\partial_x^{l+1} u_1(x)\right| + \int_0^t \left[\left|\partial_x^{l+2} u(x,t')\right| + \left|\partial_x^{l+4} u(x,t')\right| + \left|\partial_x^{l+2}(u^2)(x,t')\right|\right]\mathrm{d}t' \\ &\le \left|\partial_x^{l+1} u_1(x)\right| + T\left(\sup_{0\le t'\le T}\left\{\left|\partial_x^{l+2} u(x,t')\right| + \left|\partial_x^{l+4} u(x,t')\right| + \left|\partial_x^{l+2}(u^2)(x,t')\right|\right\}\right).\end{aligned}$$

Using (4.10) and the fact that u_1 is Gevrey of order σ, we obtain

$$\begin{aligned}
\left|\partial_x^{l+2} u(x,t')\right| &\le C^{l+3}(l+2)!^{\sigma} \\
\left|\partial_x^{l+4} u(x,t')\right| &\le C^{l+5}(l+4)!^{\sigma} \\
\left|\partial_x^{l+1} u_1(x,t')\right| &\le C^{l+2}(l+1)!^{\sigma},
\end{aligned}$$

for all $x \in \mathbb{R}$ and $t' \in [0,T]$. On the other hand, applying Leibniz Rule and (4.10) again, we have

$$\begin{aligned}
\left|\partial_x^{l+2}(u^2)(x,t')\right| &\le \sum_{j=0}^{l+2} \binom{l+2}{j} \left|\partial_x^{l+2-j} u(x,t')\right| \left|\partial_x^{j} u(x,t')\right| \\
&\le \sum_{j=0}^{l+2} \binom{l+2}{j} C^{l+2-j+1}(l+2-j)!^{\sigma} C^{j+1} j!^{\sigma} \\
&\le 2^{l+2} C^{l+4}(l+2)!^{\sigma},
\end{aligned}$$

since $n!m! \le (n+m)!$, for all $n,m \in \mathbb{Z}_+$. Assuming that $C \ge 1$ and remembering that $0 < T \le 1$, we obtain

$$\begin{aligned}
\left|\partial_x^{l} u_t(x,t)\right| &\le C^{l+2}(l+1)!^{\sigma} + C^{l+3}(l+2)!^{\sigma} + C^{l+5}(l+4)!^{\sigma} + 2^{l+2}C^{l+4}(l+2)!^{\sigma} \\
&\le 4(2C)^{l+5}(l+4)!^{\sigma} \\
&\le 4(2C)^{5}(2C)^{l}(l+4)^{4\sigma}(l!)^{\sigma}.
\end{aligned}$$

The fact $x \le e^x$, for all $x \ge 0$, give us $(l+4)^{4\sigma} \le (e^4 e^l)^{4\sigma}$, which implies

$$\left|\partial_x^{l} u_t(x,t)\right| \le 4(2C)^5 e^{16\sigma}(2Ce^{4\sigma})^{l}(l!)^{\sigma} \le A^{l+1}(l!)^{\sigma},$$

for all $(x,t) \in \mathbb{R} \times [0,T]$, where $A = \max\{4(2C)^5 e^{16\sigma}, 2Ce^{\sigma}\}$. Then, the proof of Proposition 4.2 is complete. □

4.2 Bounds for mixed derivatives

We shall follow the strategy adopted in [21]. We start by introducing some notation, for $\varepsilon > 0$ we consider the sequences

$$m_q \doteq \frac{c(q!)^{\sigma}}{(q+1)^2} \ (q = 0,1,2\ldots) \quad \text{and} \quad M_q \doteq \varepsilon^{1-q} m_q \ (q = 1,2,3,\ldots), \tag{4.11}$$

where c is chosen (see [1] page 196) such that the following inequality holds

$$\sum_{l=0}^{k} \binom{k}{l} m_l m_{k-l} \le m_k. \tag{4.12}$$

Removing the ends points 0 and k in the left-hand side of (4.12), we obtain

$$\sum_{l=1}^{k-1} \binom{k}{l} M_l M_{k-l} = \sum_{l=1}^{k-1} \binom{k}{l} \varepsilon^{1-l} m_l \varepsilon^{1-(k-l)} m_{k-l} \le \varepsilon^{2-k} m_k = \varepsilon M_k. \tag{4.13}$$

Next, we observe that for any $\varepsilon > 0$ the sequence M_q satisfies

$$M_j \leq \varepsilon M_{j+1}, \quad \text{for all } j \geq 2. \tag{4.14}$$

In fact,

$$\frac{M_j}{M_{j+1}} = \frac{\varepsilon^{1-j} c (j!)^\sigma}{(j+1)^2} \frac{(j+2)^2}{\varepsilon^{1-(j+1)} c (j+1)!^\sigma} = \varepsilon \frac{(j+2)^2}{(j+1)^{2+\sigma}} \leq \varepsilon,$$

since $(j+2)^2 (j+1)^{-2-\sigma} \leq 1$, for all $j \geq 2$ and $\sigma \geq 1$ (in point of fact, we just need to observe that the function $f(x) = x^{2+\sigma} - (x+1)^2$ satisfies $f'(x) \geq 0$, for all $x \geq 3$ and $f(3) \geq 11 \geq 0$, therefore $f(x) \geq 0$, for all $x \geq 3$).

Also, given $C > 1$ there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, we have

$$C^{j+1} (j!)^\sigma \leq M_j, \quad \text{for all } j \geq 2. \tag{4.15}$$

Indeed, (4.15) happens if, and only if,

$$C^{j+1} (j!)^\sigma \leq \varepsilon^{1-j} \frac{c (j!)^\sigma}{(j+1)^2} \iff \varepsilon^{j-1} \leq \frac{c}{C^{j+1} (j+1)^2} \iff \varepsilon \leq \frac{c^{\frac{1}{j-1}}}{C^{\frac{j+1}{j-1}} (j+1)^{\frac{2}{j-1}}} \doteq a_j.$$

Thus, it is sufficient to choose $\varepsilon_0 > 0$ such that $\varepsilon_0 \leq a_j$ for all $j \geq 2$, which is possible, since $a_j \to 1/C$, when $j \to \infty$.

For $j = 1$, it follows from definition of M_1 and M_2 that

$$M_1 = a \varepsilon M_2, \quad \text{where } a \doteq \frac{9}{4(2!)^\sigma}.$$

We also define the following constants

$$M_0 \doteq \frac{c}{8} \quad \text{and} \quad M \doteq \max\left\{ \sqrt[4]{3}, \frac{8C}{c}, \frac{4C^2}{c} \right\},$$

where c and C are constants as in (4.11) and (4.5), respectively.

Next, we shall prove our main result of this section.

Lemma 4.3. *Let $u(x,t)$ be the solution to the Cauchy problem* (3.1). *If $u(x,t)$ satisfies inequality* (4.5), *then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ we have*

$$\left| \partial_t^j \partial_x^l u(x,t) \right| \leq M^{2j+1} M_{l+2j}, \quad j, l \in \{0, 1, 2, \ldots\}, \tag{4.16}$$

for all $(x,t) \in \mathbb{R} \times [0,T]$, where M_q is defined as in (4.11).

In order to prove Lemma 4.3 we need the following key result.

Lemma 4.4. *Given $k, n \in \{0, 1, 2, \ldots\}$, we have*

$$\sum_{p=0}^{n} \sum_{q=0}^{k} \binom{n}{p} \binom{k}{q} L_{n-p+2(k-q)} L_{p+2q} \leq \sum_{r=0}^{m} \binom{m}{r} L_r L_{m-r}, \tag{4.17}$$

where $\{L_j\}_{j \in \mathbb{Z}_+}$ is any sequence of positive numbers with $m = n + 2k$.

Proof. For $k = n = 0$, inequality (4.17) reads $L_0^2 \le L_0^2$, which is trivially true. Therefore, we assume that either $k \ge 1$ or $n \ge 1$.

Changing the order of the summation and making a change of variables, we obtain

$$\begin{aligned}\sum_{p=0}^{n}\sum_{q=0}^{k}\binom{n}{p}\binom{k}{q}L_{n-p+2(k-q)}L_{p+2q} &= \sum_{q=0}^{k}\sum_{p=0}^{n}\binom{n}{p}\binom{k}{q}L_{n-p+2(k-q)}L_{p+2q}\\ &= \sum_{q=0}^{k}\sum_{r=2q}^{n+2q}\binom{n}{r-2q}\binom{k}{q}L_{m-r}L_r\\ &= \sum_{r=0}^{m}\sum_{q=i_0(r)}^{i_1(r)}\binom{n}{r-2q}\binom{k}{q}L_{m-r}L_r,\end{aligned}$$

with

$$i_0(r) = \max\left\{0, \left[\frac{r-n}{2}\right]\right\} \quad \text{and} \quad i_1(r) = \min\left\{k, \left[\frac{r}{2}\right]\right\},$$

where $[x]$ denotes the integer part of a number x. To complete the proof, we must to show that

$$\sum_{q=i_0(r)}^{i_1(r)}\binom{n}{r-2q}\binom{k}{q} \le \binom{m}{r},$$

which is a consequence of the following result.

Lemma 4.5. *For all $i_0(r) \le \theta \le i_1(r)$, we have*

$$\sum_{q=i_0(r)}^{\theta}\binom{n}{r-2q}\binom{k}{q} \le \binom{m-k+\theta}{r}. \tag{4.18}$$

In fact, using (4.18) with $\theta = i_1(r)$, we obtain

$$\sum_{q=i_0(r)}^{i_1(r)}\binom{n}{r-2q}\binom{k}{p} \le \binom{m-k+i_1(r)}{r}.$$

It suffices to show that

$$\binom{m-k+i_1(r)}{r} \le \binom{m}{r}. \tag{4.19}$$

For $i_1(r) = k$, relation (4.19) holds as an equality. If $0 \le i_1(r) < k$, then $m-k+i_1(r) \le m-1 < m$, which shows that (4.19) is true, since

$$\binom{a'}{b} \le \binom{a}{b}, \quad \text{for all } 0 \le a' \le a. \tag{4.20}$$

Indeed, remembering the elementary property

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}, \quad \text{for all } a, b \in \{1, 2, \ldots\},$$

which implies

$$\binom{a}{b} \ge \binom{a-1}{b}. \tag{4.21}$$

By applying (4.21) sufficiently times, we obtain inequality (4.20).

Therefore, inequalities (4.18) and (4.19) finishes the proof of Lemma 4.4. □

To the proof of Lemma 4.4 be completely done, it just remains the proof of Lemma 4.5, which we will do now.

Proof of Lemma 4.5. We shall prove it by induction on θ. For that, we use the following elementary inequality: If $a,b,c \in \mathbb{Z}_+$, $b \le a$ then

$$\binom{a}{b} \le \binom{a+c}{b+c}. \tag{4.22}$$

Using the definition of m and applying (4.22) with $a=n$, $b=r-2i_0(r)$ and $c=i_0(r)$, we have

$$\binom{n}{r-2i_0(r)} = \binom{m-2k}{r-2i_0(r)} \le \binom{m-2k+i_0(r)}{r-i_0(r)}. \tag{4.23}$$

Now, since the following inequality happens

$$\binom{\beta}{\alpha}\binom{\lambda}{\gamma} \le \binom{\beta+\lambda}{\alpha+\gamma}, \tag{4.24}$$

for $\alpha,\beta,\gamma,\lambda \in \mathbb{Z}_+$ with $\alpha \le \beta$ and $\gamma \le \lambda$, from (4.23) we get

$$\binom{n}{r-2i_0(r)}\binom{k}{i_0(r)} \le \binom{m-2k+i_0(r)}{r-i_0(r)}\binom{k}{i_0(r)} \le \binom{m-k+i_0(r)}{r},$$

which proves (4.18) for $\theta = i_0(r)$.

Next, we assume that (4.18) holds for $i_0(r) \le \theta < i_1(r)$ and we will prove it for $(\theta+1)$. By using the induction hypotheses, we obtain

$$\sum_{q=i_0(r)}^{\theta+1}\binom{n}{r-2q}\binom{k}{q} = \sum_{q=i_0(r)}^{\theta}\binom{n}{r-2q}\binom{k}{q} + \binom{n}{r-2(\theta+1)}\binom{k}{\theta+1} \le \binom{m-k+\theta}{r} + \binom{n}{r-2\theta-2}\binom{k}{\theta+1}.$$

It follows from (4.22) with $a=n$, $b=r-2\theta-2$ and $c=\theta$ that

$$\binom{n}{r-2\theta-2} \le \binom{n+\theta}{r-\theta-2}.$$

Thus, using (4.24) and (4.21) we obtain

$$\begin{aligned}\sum_{q=i_0(r)}^{\theta+1}\binom{n}{r-2q}\binom{k}{q} &\le \binom{m-k+\theta}{r} + \binom{n+\theta}{r-\theta-2}\binom{k}{\theta+1}\\ &\le \binom{m-k+\theta}{r} + \binom{n+\theta+k}{r-1}\\ &= \binom{m-k+\theta}{r} + \binom{m-k+\theta}{r-1}\\ &= \binom{m-k+\theta+1}{r},\end{aligned}$$

which completes the proof of Lemma 4.5. □

Finally, we finish this section with the proof of Lemma 4.3.

Proof of Lemma 4.3. We will prove (4.16) by induction on j. Let $j = 0$. For $l = 0$, it follows from (4.5) that

$$|u(x,t)| \leq C \leq MM_0, \quad \text{for all } (x,t) \in \mathbb{R} \times [0,T],$$

since $M_0 = \frac{c}{8}$ and $M \geq \frac{8C}{c}$. Similarly, for $l = 1$ we have

$$|\partial_x u(x,t)| \leq C^2 \leq MM_1, \quad \text{for all } (x,t) \in \mathbb{R} \times [0,T], \tag{4.25}$$

since $M_1 = m_1 = \frac{c}{4}$ and $M \geq \frac{4C^2}{c}$. For $l \geq 2$, it follows from (4.5) and (4.15) that there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, we have

$$\left|\partial_x^l u(x,t)\right| \leq C^{l+1}(l!)^\sigma \leq M_l \leq MM_l, \text{ for all } (x,t) \in \mathbb{R} \times [0,T],$$

since $M \geq \sqrt[4]{3} \geq 1$. This complete the proof of (4.16) for $j = 0$ and $l \in \{0,1,2,\ldots\}$. Also, since u_t has the same estimates that u in (4.5), similarly we prove that (4.16) holds for $j = 1$ and $l \in \{0,1,2,\ldots\}$.

Next, we will assume that (4.16) is true for $0 \leq q \leq j$ and $l \in \{0,1,2,\ldots\}$ with $j \geq 1$ and we will prove it for $j+1$ and $l \in \{0,1,2,\ldots\}$. We begin by noticing that

$$\left|\partial_t^{j+1}\partial_x^l u\right| = \left|\partial_t^{j-1}\partial_x^l(\partial_t^2 u)\right| \leq \left|\partial_t^{j-1}\partial_x^{l+2}u\right| + \left|\partial_t^{j-1}\partial_x^{l+4}u\right| + \left|\partial_t^{j-1}\partial_x^{l+2}(u^2)\right|,$$

since $j \geq 1$ and u is a solution of (4).

Using the induction hypotheses and (4.14), we obtain that there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$

$$\begin{aligned}\left|\partial_t^{j-1}\partial_x^{l+2}u\right| &\leq M^{2(j-1)+1}M_{l+2+2(j-1)} = M^{2(j+1)+1}M^{-4}M_{l+2(j+1)-2} \\ &\leq \varepsilon^2 M^{-4}M^{2(j+1)+1}M_{l+2(j+1)} \leq \frac{1}{3}M^{2(j+1)+1}M_{l+2(j+1)},\end{aligned} \tag{4.26}$$

since $M \geq \sqrt[4]{3}$ and and we can assume $0 < \varepsilon \leq \varepsilon_0 \leq 1$. Furthermore, in the same way, we have

$$\left|\partial_t^{j-1}\partial_x^{l+4}u\right| \leq M^{2(j-1)+1}M_{l+4+2(j-1)} = M^{-4}M^{2(j+1)+1}M_{l+2(j+1)} \leq \frac{1}{3}M^{2(j+1)+1}M_{l+2(j+1)}. \tag{4.27}$$

About the nonlinear term, applying Leibniz's rule twice and using the induction hypotheses, we have

$$\begin{aligned}\left|\partial_x^{j-1}\partial_x^{l+2}(u^2)\right| &\leq \sum_{p=0}^{l+2}\binom{l+2}{p}\left|\partial_t^{j-1}(\partial_x^{l+2-p}u\partial_x^p u)\right| \\ &\leq \sum_{p=0}^{l+2}\sum_{q=0}^{j-1}\binom{l+2}{p}\binom{j-1}{q}\left|\partial_t^{j-1-q}\partial_x^{l+2-p}u\right|\left|\partial_t^q\partial_x^p u\right| \\ &\leq \sum_{p=0}^{l+2}\sum_{q=0}^{j-1}\binom{l+2}{p}\binom{j-1}{q}M^{2(j-1-q)+1}M_{l+2-p+2(j-1-q)}M^{2q+1}M_{p+2q} \\ &= M^{2j}\sum_{p=0}^{l+2}\sum_{q=0}^{j-1}\binom{l+2}{p}\binom{j-1}{q}M_{l+2-p+2(j-1-q)}M_{p+2q}.\end{aligned}$$

It follows from (4.17) with $n = l+2$, $k = j-1$ and $L_r = M_r$ that

$$\sum_{p=0}^{l+2}\sum_{q=0}^{j-1}\binom{l+2}{p}\binom{j-1}{q}M_{l+2-p+2(j-1-q)}M_{p+2q} \le \sum_{r=0}^{m}\binom{m}{r}M_rM_{m-r},$$

where $m = n+2k = l+2j$. Thus, using (4.13) and (4.14), we obtain for all $0<\varepsilon\le\varepsilon_0$

$$\begin{aligned}\left|\partial_x^{j-1}\partial_x^{l+2}(u^2)\right| &\le M^{2j}\left(2M_mM_0+\sum_{r=1}^{m-1}\binom{m}{r}M_rM_{m-r}\right)\\ &\le M^{2j}(2M_0+\varepsilon)M_{l+2j}\\ &\le M^{2(j+1)+1}M^{-3}(2M_0+\varepsilon)\varepsilon^2M_{l+2(j+1)},\end{aligned}$$

since $l+2j\ge 2$ for all $j\ge1$ and $l\in\{0,1,2,\ldots\}$. Also, we can assume $\varepsilon_0\le(2M_0+1)^{-\frac12}\le 1$, then

$$(2M_0+\varepsilon)\varepsilon^2\le(2M_0+1)\varepsilon_0^2\le 1,$$

which implies that

$$\left|\partial_t^{j-1}\partial_x^{l+2}(u^2)\right|\le\frac{1}{3}M^{2(j+1)+1}M_{l+2(j+1)}, \tag{4.28}$$

since $M^{-3}\le M^{-4}\le\frac13$.

From (4.26), (4.27) and (4.28) we prove (4.16) for $j+1$ and $l\in\{0,1,2,\ldots\}$, which finishes the proof. □

4.3 Proof of regularity in time variable

Finally, in this section we prove our last result.

Proof of Theorem 4.1. Our goal is to prove that there exists a constant $C>0$ such that

$$\left|\partial_t^j u(x,t)\right|\le C^{j+1}(j!)^{2\sigma}, \tag{4.29}$$

for all $j\in\{0,1,2,\ldots\}$ and for all $(x,t)\in\mathbb{R}\times[0,T]$.

Applying (4.16) for $j\in\{1,2,\ldots\}$ and $l=0$, we obtain that there is $\varepsilon>0$ such that

$$\begin{aligned}\left|\partial_t^j u(x,t)\right| &\le M^{2j+1}M_{2j}\\ &= MM^{2j}\varepsilon^{1-2j}\frac{c(2j)!^\sigma}{(2j+1)^2}\\ &\le M\varepsilon c\left(\frac{M}{\varepsilon}\right)^{2j}(2j)!^\sigma\\ &= L_0L^j(2j)!^\sigma,\end{aligned}$$

where $L_0 = M\varepsilon c$ and $L_1 = (M\varepsilon^{-1})^2$. Also, as we know

$$\sum_{j=0}^{\infty}\frac{(2j)!}{A_0^j(j!)^2},\quad \text{for all } A_0>4,$$

then $(2j)! \le A^{2j}(j!)^2$ for some $A > 0$. Therefore,

$$\left|\partial_t^j u(x,t)\right| \le L_0 L^j A^{\sigma j}(j!)^{2\sigma} \le C^{j+1}(j!)^{2\sigma}, \quad \text{for all } (x,t) \in \mathbb{R} \times [0,T],$$

where $C = \max\{L_0, LA^\sigma\}$. This finishes the proof of Theorem 4.1. □

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