UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Heat equation and the Yamabe flow on manifolds with fibered boundary metric

Bruno Caldeira Carlotti de Souza

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Bruno Caldeira Carlotti de Souza

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Dedicated to my parents
Sandra and Jaime.

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Resumo

Este trabalho é dedicado ao estudo do fluxo de Yamabe em uma classe de variedades Riemannianas (M, g_Φ) não-compactas completas de volume infinito denominadas Φ-variedades. Alguns exemplos dessa classe de variedades são instatons gravitacionais, produtos entre variedades assintoticamente cônicas com variedades fechadas e monopólos magnéticos não-abelianos. Através de suposições sobre a regularidade sobre a curvatura escalar $\mathrm{scal}(g_\Phi)$, verificamos existência e unicidade do fluxo para tempo curto. Além disso, supondo que $\mathrm{scal}(g_\Phi)$ é negativa e limitada tanto superiormente longe do zero quanto inferiomente, provamos que os fluxos de Yamabe normalizados pela curvatura (CYF^+ e CYF^-) existem para todo o tempo e, mais ainda, convergem para métricas Riemannianas sobre M de curvatura escalar constante. Este trabalho estende os resultados obtidos por Serrato-Suárez e Tapie em [SST12].

Para obter estes resultados, provamos: um princípio do máximo no contexto de Φ-variedades, mergulhos compactos entre espaços de funções Hölder contínuas, propriedades de aplicação para o núcleo de calor H, estimativas parabólicas de Schauder e apresentamos uma construção de uma parametrix para uma família de equações do calor. Os argumentos apresentados para o estudo do fluxo de Yamabe global são válidos no contexto mais geral das variedades de geometria limitada. Contudo, o argumento de convergência, bem como as estimativas parabólicas de Schauder consideravelmente mais fortes, são obtidas no contexto de variedades de bordo fibrado munidas de Φ-métricas. Gostaríamos ainda de enfatizar que através de tais estimativas parabólicas de Schauder, conseguimos provar existência em tempo curto do fluxo de Yamabe com um controle preciso do comportamento assintótico das soluções próximo ao bordo. Tal controle não pode ser obtido usando somente estimativas clássicas válidas para variedades de geometria limitada. Finalmente, notamos ainda o fato de Φ-variedades terem volume infinito, o que impede o uso da renormalização usual do fluxo, que garante volume constante em variedades compactas e, eventualmente a convergência do fluxo. Superar esta dificuldade no contexto de variedades não-compactas é uma das principais contribuições desta tese.

Palavras-chave: Fluxo de Yamabe, variedade de bordo fibrado, núcleo do calor, princípio do máximo, estimativas de Schauder, construção de parametrix.

Abstract

This work is dedicated to the study of the Yamabe flow on a class of non-compact complete Riemannian manifolds (M, g_Φ) with fibered boundary and Φ-metrics, called Φ-manifolds. Some examples of this type of manifolds include gravitational instatons, products of an asymptotically conical manifold with a closed manifold and non-abelian magnetic monopoles. Through assumptions on the regularity of the scalar curvature $\mathrm{scal}(g_\Phi)$, we prove both existence and uniqueness of the flow for short-time. Moreover, assuming $\mathrm{scal}(g_\Phi)$ to be negative, bounded and bounded away from zero, we show that the curvature-normalized flows (CYF^+ and CYF^-) exist for all time and, further, that they converge to some Riemannian metric on M with constant scalar curvature. This work extends the results obtained by Serrato-Súarez and Tapie in [SST12].

In order to obtain these results, we proved: a maximum principle in the setting of Φ-manifolds, compact embeddings between Hölder spaces, mapping properties for the heat kernel H, Schauder parabolic estimates and we present a parametrix construction for a family of heat equations. The arguments presented for the study of the Yamabe flow on long-time hold on the more general setting of manifolds of bounded geometry. However, the convergence argument, as well as the considerably stronger parabolic Schauder estimates, are worked out specifically in the setup of manifolds with fibered boundary equipped with Φ-metrics. We also like to emphasize that using these stronger parabolic Schauder estimates, we are able to prove short-time existence of the Yamabe flow with a precise control of the asymptotic of solutions up to the boundary. Such a control is not possible using just the classical estimates on spaces with bounded geometry. Finally, we should also point out that due to the fact that Φ-manifolds have infinite volume, the usual renormalization of the flow, that in the compact setting ensured constant volume and eventually convergence of the flow, does not work here. Overcoming this difficulty in the non-compact setting is one of the main contributions of this thesis.

Keywords: Yamabe flow, fibered boundary manifold, heat kernel, maximum principle, Schauder estimates, parametrix construction.

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Introduction

From its very origin, Riemannian geometry was developed as a way to generalize the theory of compact surfaces to geometric objects of higher dimensions. Naturally, conformal changes of metrics gain importance, given the central role they play in the theory of surfaces. More precisely, it was already known that every compact surface admits a metric of constant Gaussian curvature. This was a groundbreaking result, since it transformed a topological question into a differential geometric one, by classifying all the homeomorphism classes of compact surfaces.

However, generalizations to higher dimension manifolds are not straightforward. In fact, for a Riemannian manifold of dimension m, the Riemann tensor has $m^2(m^2-1)/12$ algebraically independent components. Therefore, to look for a more direct generalization, it would be reasonable to study the "less complex" analogue in manifolds, i.e. the scalar curvature. This would lead to the Yamabe problem, which asks the following question:

"Given a compact Riemannian manifold (M,g) of dimension $m \geq 3$, is it possible to find a Riemannian metric on M conformal to g whose scalar curvature is constant?"

This problem was proposed by Hidehiko Yamabe [Yam60], who attempted to solve this problem in 1960 by using techniques from the theory of Calculus of Variations. Even though his solution was wrong, his attempt to solve the problem paved the way that led to the solution. In fact, his attempt was based on how certain geometric objects – scalar curvature, the Laplace-Beltrami operator, the Riemannian connection operator – transform under the conformal change of metric $\widetilde{g} = u^{4/(m-2)}g$ with $u > 0$ smooth, which allowed him to conclude that $\widetilde{g}$ has constant scalar curvature if and only if

$$(a\Delta_g + \mathrm{scal}(g))u = \lambda u^{(m+2)/(m-2)}, \text{ for some } \lambda \in \mathbb{R}. \tag{1}$$

Yamabe realized that (1) is the Euler-Lagrange equation for the Yamabe energy functional

$$\mathscr{E}(\widetilde{g}) = \frac{\int_M \mathrm{scal}(\widetilde{g})\,\mathrm{dvol}_{\widetilde{g}}}{\left(\int_M \mathrm{dvol}_{\widetilde{g}}\right)^{(m-2)/m}},$$

which means that u is a solution of (1) iff $\widetilde{g} = u^{4/(m-2)}g$ is a critical point of $\mathscr{E}$. Rewriting the expression for $\mathscr{E}$ and employing Hölder's inequality, Yamabe proved that $\mathscr{E}$ is bounded from below, allowing the definition of the Yamabe invariant

$$Y(M,g) = \inf\{\mathscr{E}(\widetilde{g}) \mid \widetilde{g} = u^{4/(m-2)}g,\ u > 0 \text{ smooth}\},$$

whose analysis was key in the argument provided by Yamabe. Through some modifications and under the assumption that $Y(M,g) \leq 0$, Trudinger [Tru68] proved the existence of a conformal metric with constant scalar curvature. In 1976, Aubin [Aub76] extended the work of Trudinger to other possible values of the Yamabe invariant for $m \geq 6$. Finally, Schoen [Sch84] solved the Yamabe problem for dimensions between 3 and 6 by the construction of global test functions.

Parallel to this, Hamilton attempted to solve the Yamabe problem in a slightly different manner, inspired by the work of Eells and Sampson [ES64]. The idea of Hamilton was to find a 1-parameter family of metrics that evolves along a "curvature-diffusion" equation similar to the heat equation, which should lead to the a metric on the manifold with constant scalar curvature. The resulting equation, called the Yamabe flow equation, is an evolution equation conceived by Richard Hamilton [Ham82] as an approach to deal with the Yamabe problem. The flow equation is a heat-type evolution equation whose solution is given as a family of Riemannian metrics $\{g(t)\}_{t\in[0,T)}$ on a fixed underlying smooth manifold M such that the initial metric coincides with a fixed metric g over M. More precisely, such family of metrics is said to be a Yamabe flow on (M,g) if it satisfies

$$\partial_t g(t) = -\operatorname{scal}(g(t))g(t); \;\; g(0) = g. \tag{2}$$

It is noticeable that (2) states that if such flow exists, then the metric must "shrink" along the flow on regions with $\operatorname{scal}(g(t)) > 0$, which means that even in simple cases with positive scalar curvature, it leads to a singularity in finite time due to the collapsing of the volume. For this reason, a normalized version of this flow is also commonly studied. In a classical setting, for (M,g) a compact smooth Riemannian manifold, define the average scalar curvature as

$$\rho(t) = \frac{1}{\operatorname{vol}_{g(t)}(M)} \int_M \operatorname{scal}(g(t))\, \mathrm{dvol}(g(t)), \tag{3}$$

which then allows to consider the normalized Yamabe flow

$$\partial_t g(t) = (\rho(t) - \operatorname{scal}(g(t)))g(t), \;\; g(0) = g. \tag{4}$$

Unlike the original Yamabe flow, a solution to the normalized Yamabe flow is a family of Riemannian metrics on M that preserves the volume of (M,g), keeping the curvature from becoming unbounded. This is interesting because, once the flow is normalized, the curvature evolves along the flow towards the normalizing term.

Both flows are well understood in the setting of compact manifolds. Hamilton [Ham82] himself proved long time existence of the volume normalized flow for any choice of initial metric. Later, Ye [Ye94] proved convergence of the flow for scalar negative, scalar flat and locally conformal flat scalar positive metrics. The case of metrics that are not conformally flat has been studied in a series of papers by Schwetlick and Struwe [SS03] and later by Brendle [Bre05, Bre07]. More recently, Bahuaud and Vertman [BV14, BV19] showed long-time existence and convergence of the normalized flow in the setting of edge manifolds.

In this work, the main concern is the Yamabe problem and, more specifically, the ultimate goal is to study the Yamabe flow on non-compact complete manifolds. In what follows, questions regarding both local and global existence, uniqueness and convergence of the Yamabe flow in the context of so-called Φ-manifolds, which are an example of non-compact manifolds with infinite volume. Note that in the case of infinite volume, the average scalar curvature (3) is ill-defined and only the unnormalized Yamabe flow.

The Yamabe flow in the non-compact setting has been studied on asymptotically conical surfaces by Isenberg, Mazzeo and Sesum [IMS13], who proved locally uniform convergence of a time-rescaled metric to a complete hyperbolic metric with finite area. The flow has also been utilized by Bing-Long Chen and Xi-Ping Zhu [CZ02] to establish a gap theorem for non-compact manifolds with nonnegative Ricci curvature under certain decay conditions at infinity. Ma, Cheng and Zhu [MCZ12] have studied long-time existence of the Yamabe flow, under some L^p conditions on the scalar curvature. Schulz [Sch20] proved global existence of the Yamabe flow on non-compact manifolds with unbounded initial curvature, provided the metric is conformally equivalent to a complete metric with bounded, non-positive scalar curvature and positive Yamabe invariant. A recent work Ma [Ma21] establishes global existence of the Yamabe flow on non-compact manifolds that are asymptotically flat near infinity.

In all of these works, either convergence of the flow is out of reach, since (3) is not defined, or one focuses only on low-dimension geometric objects. Thus, only the unnormalized Yamabe flow has been considered. In this thesis, we study a different type of normalization for the Yamabe flow, that allows to study convergence in the non-compact setting as well. We use the concepts of decreasing and increasing curvature-normalized flows, denoted by CYF^- and CYF^+ respectively, as introduced by Suárez-Serrato and Tapie [SST12] for compact manifolds

$$\begin{aligned} \partial_t g(t) &= (\sup_M \mathrm{scal}(g(t)) - \mathrm{scal}(g(t)))g(t), \;\; g(0) = g, \qquad \left(\mathrm{CYF}^+\right), \\ \partial_t g(t) &= (\inf_M \mathrm{scal}(g(t)) - \mathrm{scal}(g(t)))g(t), \;\; g(0) = g, \qquad \left(\mathrm{CYF}^-\right). \end{aligned} \tag{5}$$

Our interest in these flows lies on the fact that, unlike the standard normalization via average scalar curvature, $\sup_M \mathrm{scal}(g(t))$ and $\inf_M \mathrm{scal}(g(t))$ are well-defined regardless of the volume of the manifold. We study such curvature normalized flows in the setting of fibered boundary manifolds, that generalize the asymptotically flat manifolds considered recently in Ma [Ma21].

Outline of the thesis

Chapter 1 is focused on compiling basic concepts and formulae for the development of the project. First, we present a couple of important formulae in conformal Riemannian geometry in §1.1. In §1.2, we introduce the concept of manifold with corners, which is important for the understanding of polyhomogeneous functions, introduced in §1.3, and of the heat space, which is an extremely useful concept. Finally, we close the chapter with an intuitive explanation of blow-ups and blow-down maps in §1.4.

In Chapter 2, we introduce Φ-manifolds and their geometry-adapted C^∞-structure such as Φ-vector fields and Φ-differential 1-forms. In §2.1, we give an expression for the Φ-volume form, while in §2.2 the expression of the scalar curvature of a Φ-manifold is presented. After this, we discuss maximum principles in Φ-manifolds in §2.3; more precisely, we prove stochastic completeness of Φ-manifolds, which implies the Omori-Yau maximum principle in this setting, and use it to prove the following theorem:

Theorem 0.1. *Let (M, g_Φ) be a Φ-manifold and $u \in C^2_\Phi(M \times [0,T])$ be a function satisfying the following inequalities:*

$$\left|\frac{\partial u}{\partial t}(p,t) - \frac{\partial u}{\partial t}(p,t')\right| \leq C|t-t'|^\gamma, \qquad \left|\frac{\partial u}{\partial t}(p,t)\right| \leq C, \tag{6}$$

for all $(p,t),(p,t') \in M \times [0,T]$, for some constant $C > 0$ and some $\gamma > 0$. Then the Cauchy problem

$$(\partial_t - a\Delta_\Phi)u = 0, \; u|_{t=0} = 0, \tag{7}$$

with the factor "a" being a positive and bounded function, admits only the trivial solution $u = 0$.

Finally, §2.4 discusses the asymptotic expansion of the heat kernel on Φ-manifolds, which can be properly given as a polyhomogenous function defined on a specific manifold with corners (the heat space).

Chapter 3 is dedicated to discussing the Yamabe flow on Φ-manifolds. First, we briefly discuss the transformation of the Yamabe flow into a PDE in terms of the conformal factor. In §3.1, we review the geometry of fibered boundary manifolds equipped with Φ-metrics in their open interior. We also introduce Hölder spaces $C^{k,\alpha}_\Phi(M)$, adapted to this geometry. In §3.2 we study mapping properties of the heat operator with respect to these spaces. Same conclusions follow by classical parabolic Schauder estimates in §3.8. Nevertheless, §3.2 serves as an alternative derivation by microlocal methods and as an exercise for the estimates in §4.2 (which do not follow from classical theory). Based on that, in §3.3 we establish short time existence of the (unnormalized) Yamabe flow (2) within the class of such Φ-manifolds, see Theorem 3.17 for the precise statement.

Theorem 0.2. *Let (M, g_Φ) be a Φ-manifold of dimension $m \geq 3$ such that* $\mathrm{scal}(g_\Phi) \in C^{k+1,\alpha}_\Phi(M)$*, for some $\alpha \in (0,1)$ and any $k \in \mathbb{N}_0$. Then the Yamabe flow $g(t) = u(t)^{4/(m-2)} g_\Phi$ with conformal factor $u \in C^{k+2,\alpha}_\Phi(M \times [0,T])$ solving (3.2), exists for $T > 0$ sufficiently small.*

In §3.5, we turn to the increasing curvature normalized Yamabe flow (CYF^+), introduced in (3.4), whose short-time existence follows from Theorem 0.2 by some time rescaling. The same holds also for the decreasing curvature normalized Yamabe flow CYF^- by a verbatim repetition of the arguments and hence we only write the proofs for CYF^+. In §3.6, we study the evolution of $\mathrm{scal}(g)$ along CYF^+. In §3.7 we derive a priori estimates for solutions of the increasing curvature normalized Yamabe flow.

These a priori estimates allow us to apply the machinery of standard estimates of solutions to parabolic equations, which we review in §3.8. Subsequently, in §3.9 we conclude the global existence of the CYF^+ on Φ-manifolds.

Theorem 0.3. *Let (M, g_Φ) be a Φ-manifold of dimension $m \geq 3$ with $\mathrm{scal}(g_\Phi) \in C^{k,\alpha}_\Phi(M)$ negative, uniformly bounded away from zero and $k \geq 4$. Then the increasing curvature normalized Yamabe flow CYF$^+$ (see Eq. (3.4)) admits a global solution $g = u^{4/(m-2)} g_\Phi$ for some $u \in C^{k,\alpha}_\Phi(M \times \mathbb{R}_+)$.*

Finally, in §3.10 we establish convergence for the CYF^+ and thus settle the Yamabe problem on negatively curved Φ-manifolds. Our result, see Theorem 3.32 for the precise statement, reads as follows:

Theorem 0.4. *Let (M, g_Φ) be a Φ-manifold of dimension $m \geq 3$ such that $\mathrm{scal}(g_\Phi) \in C^{4,\alpha}_\Phi(M)$ is negative and uniformly bounded away from zero. Then, the increasing curvature normalized Yamabe flow CYF$^+$ converges to a Riemannian metric g^* conformal to g_Φ with constant negative scalar curvature.*

In fact, the same arguments apply in the general case of manifolds with bounded geometry, provided the flow exists at least for short time within the corresponding Hölder space. The Φ-geometry is not essential in our arguments. One can view our contribution as an extension of Suárez-Serrato and Tapie [SST12] to a non-compact setting.

In Chapter 4, we study the Yamabe flow under much stronger conditions on the conformal factor. More precisely, we introduce in §4.1 a more restrictive family of Hölder spaces, which forces the functions and their Φ-derivatives to be continuous up to the boundary. This property cannot be inferred from the arguments in Chapter 3. After this, we present in §4.2 some mapping properties of the heat operator acting as a bounded linear operator on functions satisfying these stronger Hölder restrictions. It is interesting to note that these mapping properties prove how the heat kernel acts continuously on functions that are Hölder continuous on manifolds with conic-ends (which are incomplete manifolds, unlike Φ-manifolds themselves). Using these mapping properties, we construct a parametrix in §4.3 for a family of heat-type equations, which leads us to the following theorem:

Theorem 0.5. *Consider a function $a \in C^{k,\beta}_{x^4\Phi}(M \times [0,T])$ which is positive, bounded from below away from zero, for some $0 < \alpha < \beta < 1$. Then both Cauchy problems*

$$(i)\ (\partial_t - a\Delta)u = \ell; u|_{t=0} = 0, \ \ and \ \ (ii)\ (\partial_t - a\Delta)u = 0; u|_{t=0} = u_0 \tag{8}$$

admit solutions $\mathbf{Q}\ell$ and $\mathbf{E}u_0$, respectively, such that

$$\mathbf{Q} : x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T]) \to x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M \times [0,T]),$$
$$\mathbf{E} : x^\gamma C^{k,\alpha}_{x^4\Phi}(M) \to x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M \times [0,T]),$$

are both bounded maps, for any $\gamma \in \mathbb{R}$.

Finally, after completing the parametrix construction, we adapt the contraction argument in the previous chapter to conclude short-time existence and uniqueness of the Yamabe flow on Φ-manifolds for conformal factors that are Hölder continuous in manifolds with conic-ends, leading to our last theorem:

Theorem 0.6. *Let (M, g_Φ) be a Φ-manifold of dimension $m \geq 3$. Assume the scalar curvature $\mathrm{scal}(g_\Phi)$ to lie in $x^\gamma C^{k+1,\alpha}_{x^4\Phi}(M)$ for some $\alpha \in (0,1)$, some $\gamma \geq 0$ and some $k \in \mathbb{N}_0$. Then, there exists some $T > 0$ sufficiently small and some function $u \in C^{k+2,\alpha}_{x^4\Phi}(M \times [0,T])$ such that $g = u^{4/(m-2)} g_\Phi$ is a solution of the Yamabe flow (2) for time $t \in [0,T]$. Moreover, this solution is unique.*

Initially, we intended to prove long-time existence and convergence of the flow in these more restrictive Hölder spaces. That would yield a more detailed understanding of the boundary behavior of the flow. However, we did not manage to carry out certain long-time existence arguments to this setting. Hence, so far, only short-time existence with continuity up to the boundary has been proven. We intend to address this in the future.

Generalizations to manifolds of bounded geometry

We highlight that the arguments presented for the study of the long-time existence of the Yamabe flow (in the weaker Hölder spaces that do not require continuity up to the boundary) can be used in the more general setting of manifolds of bounded geometry, a broader class of Riemannian manifolds which contains Φ-manifolds, provided the flow exists at least for short time within the corresponding Hölder spaces. This happens because the arguments do not rely on any particularities of the Φ-geometry. However, the same cannot be said of the proof of convergence of the curvature-normalized Yamabe flow, since one needs to work with the class of weighted Hölder spaces, which can only be defined once a globally defined boundary defining function is defined. But this is not possible to do on a generic manifold of bounded geometry, meaning that further assumptions are necessary to compensate for this.

Manuscripts

This work compiles results from the two following manuscripts. The first one was developed with Gentile as a part of the Ph.D. projects of each author. This work was supervised by Vertman during the visit of the author of this thesis to Carl von Ossietzky Universität Oldenburg. The second one was developed as a part of the Ph.D. of the author and was supervised by both Hartmann and Vertman.

- [CGct]: Bruno Caldeira and Giuseppe Gentile. "Schauder estimates on manifolds with fibered boundaries", In: (ongoing project).
- [CHV21]: Bruno Caldeira, Luiz Hartmann and Boris Vertman. "Normalized Yamabe flow on some complete manifolds with infinite volume", In: *arXiv preprint* (2021). URL: arxiv.org/abs/2105.14282.

CHAPTER 1

Preliminaries

The present chapter collects the basic notions of b-Calculus used throughout this work. It is assumed that the reader is familiar with the main ideas of the theory of smooth manifolds as in [Lee13], Riemannian geometry as in [O'N83] and basic theory of PDEs as in [Eva10]. A brief recollection of said topics is given in the Appendix.

The main goal here is then to give a quick introduction of some concepts from the theory of b-Calculus, such as the notions of blow-ups and polyhomogeneous conormal functions. The references for this chapter are Melrose's classical book [Mel93], as well as Grieser [Gri01].

Moreover, the ending of the chapter is dedicated to presenting Φ-manifolds , some of its properties and some previous results on them which will be used in this work. The main references for this section are Mazzeo and Melrose [MM98], as well as the recent work of Vertman and Talebi [TV21].

1.1 Conformal Riemannian geometry

In this section, we collect a few formulae from Riemannian geometry which relate geometric information between two conformally equivalent Riemannian metrics. These results will prove themselves useful during our studies of the Yamabe flow in the subsequent sections. In fact, the main technique for the study of the Yamabe flow is to consider a 1-parameter family of Riemannian metrics inside of a conformal class of the initial metric. This is can be justified by the fact that the Yamabe flow preserves the conformal class of a Riemannian metric.

Definition 1.1. Let M be a smooth m-dimensional manifold and g and $\widetilde{g}$ two Riemannian metrics on M. We say that g and $\widetilde{g}$ are conformally equivalent if there is a positive function $u \in C^2(M)$ such that $\widetilde{g} = u \cdot g$, that is,

$$\widetilde{g}(p)(X_p, Y_p) = u(p) \cdot g_p(X_p, Y_p), \tag{1.1}$$

for all $p \in M$ and $X, Y \in \mathscr{V}(M)$. The function u is called the conformal factor.

Throughout this work, we will be exclusively interested in a specific conformal class of Riemannian metrics, which we now define: given a Riemannian metric g on M, with $\dim M = m \geq 3$ and $\eta := (m-2)/4$, define

$$[g] := \left\{ u^{1/\eta} \cdot g \;\middle|\; u \in C^2(M) \right\}. \tag{1.2}$$

Given this conformal class of metrics, we must understand how both the scalar curvature and the Laplace-Beltrami operator transform along the flow. This is important for proving the existence of the flow and for understanding how the scalar curvature evolves as time increases. To do so, consider the following technical lemma, whose proof can be found in detail in [Sch19, Appendix A.1]:

Lemma 1.2. *Let (M,g) be an m-dimensional Riemannian manifold, with $m \geq 2$, and let $\widetilde{g} = u \cdot g$ be a metric on M conformally equivalent to g. Then*

$$\operatorname{scal}(\widetilde{g}) = \frac{\operatorname{scal}(g)}{u} - (m-1)\left(\frac{\Delta_g u}{u^2} + \frac{(m-6)|\nabla u|_g^2}{4u^3}\right). \tag{1.3}$$

Now, as a consequence of the previous result, we prove a small collection of some formulae that will be useful in the following sections.

Proposition 1.3. *Let (M,g) be a m-dimensional Riemannian manifold, with $m \geq 3$, and let $\widetilde{g} \in [g]$. Then*

1. $\operatorname{scal}(\widetilde{g}) = -u^{-(1+1/\eta)}\left[\frac{m-1}{\eta}\Delta_g u - \operatorname{scal}(g)u\right]$;
2. $\Delta_{\widetilde{g}} f = \frac{1}{u^{1/\eta}} \cdot \Delta_g f + \frac{2}{u^{1+1/\eta}} \cdot g(\nabla f, \nabla u)$.

Proof. **Item 1.** Let us first prove the first of the two formulae. To do so, remember that the chain rule of the Laplace operator is the following: given $u \in C^2(M)$ and $f \in C^2(\mathbb{R})$,

$$\Delta_g f(u) = f''(u)|\nabla u|_g^2 + f'(u)\Delta_g u.$$

This can be easily checked by employing the local expression of the Laplace operator. Naturally, this implies that

$$\Delta_g u^{1/\eta} = \frac{1}{\eta}u^{1/\eta-1}\Delta_g u + \frac{1}{\eta}\left(\frac{1}{\eta} - 1\right)u^{1/\eta-2}|\nabla u|_g^2. \tag{1.4}$$

On the other hand, straightforward computations show that the chain rule of the gradient is $\nabla f(u) = f'(u)\nabla u$, which thus gives us

$$\nabla u^{1/\eta} = \frac{1}{\eta}u^{1/\eta-1}\nabla u \Longrightarrow |\nabla u^{1/\eta}|_g^2 = \frac{1}{\eta^2}u^{2/\eta-2}|\nabla u|_g^2. \tag{1.5}$$

Now, note that

$$\frac{1}{\eta}\left(\frac{1}{\eta} - 1\right) = -\frac{1}{\eta^2}\frac{(m-6)}{4}.$$

Therefore, this gives us

$$\begin{aligned}
\frac{\Delta_g u^{1/\eta}}{u^{2/\eta}}+\frac{(m-6)|\nabla u^{1/\eta}|_g^2}{4u^{3/\eta}} =& \frac{1}{u^{2/\eta}}\left(\frac{1}{\eta}u^{1/\eta-1}\Delta_g u-\frac{1}{\eta^2}\frac{(m-6)}{4}u^{1/\eta-2}|\nabla u|_g^2\right)\\
&+\frac{1}{\eta^2}\frac{(m-6)}{4}u^{-1/\eta-2}|\nabla u|_g^2\\
=&\frac{1}{\eta}u^{-1/\eta-1}\Delta_g u,
\end{aligned}$$

which finally leads us to the expression

$$\operatorname{scal}(\widetilde{g})=-u^{-(1+1/\eta)}\left[\frac{m-1}{\eta}\Delta_g u-\operatorname{scal}(g)u\right]. \tag{1.6}$$

Item 2. Now, note that $\sqrt{|\det\widetilde{g}|}=u^{m/2\eta}\cdot\sqrt{|\det g|}$. Thus,

$$\begin{aligned}
\Delta_{\widetilde{g}}f =& \frac{1}{\sqrt{|\det\widetilde{g}|}}\cdot\sum_j\partial_j\left(\sqrt{|\det\widetilde{g}|}\cdot\sum_i\widetilde{g}^{ij}\cdot\partial_i f\right)\\
=&\frac{1}{u^{m/2\eta}\cdot\sqrt{|\det g|}}\sum_j\partial_j\left(u^{(m-2)/2\eta}\cdot\sqrt{|\det g|}\cdot\sum_i g^{ij}\cdot\partial_i f\right)\\
=&\frac{1}{u^{1/\eta}\cdot\sqrt{|\det g|}}\cdot\sum_j\partial_j\left(\sqrt{|\det g|}\cdot\sum_i g^{ij}\cdot\partial_i f\right)\\
&+\frac{2}{u^{1+1/\eta}}\cdot\sum_{i,j}g^{ij}\cdot\partial_j u\cdot\partial_i f\\
=&\frac{1}{u^{1/\eta}}\cdot\Delta_g f+\frac{2}{u^{1+1/\eta}}\cdot g(\nabla f,\nabla u).
\end{aligned}$$

□

1.2 Manifolds with corners

Now, it is necessary to give an introduction to the notion of "manifolds with corners", which are topological spaces whose local smooth structure cannot be properly described locally by neither $\mathbb{R}^m$ or a closed half-space. This concept will prove itself essential along this work, therefore being worth it of a proper presentation.

Recall that a topological m-dimensional manifold is considered to be a paracompact Hausdorff space M with the property that for each point $p\in M$, there exists an open set U_p in M that is homeomorphic to the unitary open ball $\mathbb{B}^n$ in $\mathbb{R}^n$. The algebra of continuous real-valued functions over M is denoted by $C^0(M)$. A subalgebra $F\subseteq C^0(M)$ is said to be a C^∞-subalgebra if for any real-valued $g\in C^\infty(\mathbb{R}^k)$, for every k, and any elements $f_1,...,f_k\in F$ the continuous function $g(f_1,...,f_k)\in F$. The subalgebra is said to be local if it contains each element $g\in C^0(M)$ which has the property that for every set U_α in some covering of M by open sets, there exist $g_\alpha\in F$ with $g=g_\alpha$ on U_α.

Definition 1.4. A *smooth m-manifold M* is a topological n-dimensional manifold with a local C^∞ subalgebra $C^\infty(M) \subseteq C^0(M)$ specified via the following property: M has a covering by open sets $\{U_\alpha\}_{\alpha\in A}$ for which there are m elements $f_1^\alpha,...,f_n^\alpha \in C^\infty(M)$ with $F^\alpha = (f_1^\alpha,...,f_n^\alpha)$ restricted to U_α making it a cooordinate patch and $f \in C^\infty(M)$ if and only if for each $\alpha \in A$ there exists $g_\alpha \in C^\infty(\mathbb{B}^n)$ such that $f = g_\alpha \circ F^\alpha$ on U_α.

In what follows, every time a manifold is mentioned, it is assumed to be a smooth manifold unless otherwise is explicitly stated. It is worthy of note that the definition of smooth manifold presented above is equivalent to the standard definition using smooth charts. A function $f : M \to N$ between two manifolds is said to be a *smooth function* if for each function $g \in C^\infty(N)$, $f^*g = g \circ f \in C^\infty(M)$. Such function is said to be a *diffeomorphism* if is bijective, smooth and its inverse is also a smooth function.

Manifolds with corners of dimension m must be understood as spaces which are locally described as small patches from model spaces of the type $[0,\infty)^k \times \mathbb{R}^{m-k}$, with $0 \le k \le m$. Naturally, this clearly includes the notion of manifolds with boundary, once their model space is only the case $k = 1$. However, though this notion does paint a clear picture of what a manifold with corner can look like, this idea is yet not good enough to give the description desired. In what follows, a precise definition is presented.

Definition 1.5. A smooth *manifold with corners M* is a topological manifold with boundary endowed with a subalgebra C^∞ of $C^0(M)$ satisfying the following property: there is a smooth manifold $\widetilde{M}$ and a map $\iota : M \to \widetilde{M}$ such that

1. $C^\infty(M) = \iota^* C^\infty(\widetilde{M})$,
2. there is a finite collection $\rho_i \in C^\infty(\widetilde{M})$, with $i \in I$, such that

$$\iota(M) = \{p \in \widetilde{M} \mid \rho_i(p) \ge 0 \;\; \forall i \in I\}$$

   and whenever $\rho_i(p) = 0$ for all $i \in J \subset I$, then $\{\rho_i\}_J$ is a subcollection of independent functions at each $p \in \widetilde{M}$ at which they all vanish, that is, for all such $p \in \widetilde{M}$, there is a family of smooth functions $g_1,...,g_{m-|J|}$ such that $(\rho_1,...,\rho_{|J|}, g_1,...,g_{m-|J|})$ restricts to some coordinate chart near p.

The manifold $\widetilde{M}$ is said to be an *extension* of M. Note that for each ρ_i in the definition above, the subset $H_i = \rho_i^{-1}(\{0\})$ is an embedded submanifold of $\widetilde{M}$ (and therefore, of M as well), with $\dim H_i = \dim M - 1$. The submanifolds H_i, for each $i \in I$, are the *boundary hypersurfaces* of M, and their collection is denoted by $\mathscr{M}_1(M)$. Moreover, for every $l \in \mathbb{Z}_+$, let $\mathscr{M}_l(M)$ be the set of boundary faces of codimension l of M (that is, a face that belongs to exactly l distinct boundary hypersurfaces of M) and let $\mathscr{M}(M)$ be the set of all boundary faces of M regardless of dimension .

The advantage of having Definition 1.5 is to be able to claim that $\partial\overline{M}$ is the union of the embedded submanifolds $\rho_i^{-1}\{0\}$ for all $i \in I$. Moreover, from this definition is clear the a function is in $C^\infty(M)$ if admits an extension to a smooth function on a extension $\widetilde{M}$.

Note that the simplest case of a manifold with corners that is not a manifold with boundary (in the smooth sense) is the quadrant $[0,\infty)^2$, since the origin point doesn't admit a neighborhood diffeomorphic to opens subsets in either $\mathbb{R}^2$ or $[0,\infty)\times\mathbb{R}$. In this example, one has $\mathscr{M}_1([0,\infty)^2) = \{O_x, O_y\}$, for O_x and O_y being the x-axis and y-axis respectively, and $\mathscr{M}_2([0,\infty)^2) = \{\{0\}\}$.

Definition 1.6. A map $f : M_1 \to M_2$ between manifolds with corners is said to be *smooth* if $f^*C^\infty(M_2) \subset C^\infty(M_1)$. A smooth function is said to be a *b-map* if, for each $H \in \mathscr{M}_1(M_2)$ one has either

$$f^*\rho_H = 0 \;\text{ or }\; f^*\rho_H = a_H \prod_{G\in\mathscr{M}_1(M_1)} \rho_G^{e_f(G,H)} \text{ with } a_H \in C^\infty(M_1) \;\text{ and }\; e(F,G) \in \mathbb{Z}_{\geq 0}. \tag{1.7}$$

If the first case does not occur, the map is called an *interior b-map*, which is equivalent to demanding in addition that $f(M_1 - \partial M_1) \subseteq M_2 - \partial M_2$ or that $f^{-1}(\partial M_2) \subseteq \partial M_1$.

Any b-map $f : M_1 \to M_2$ can be reduced to an interior b-map in the sense that there is exactly one boundary face of the image space, $F \in M(M_2)$, such that $f(M_1) \subset F$ and $f : M_1 \to F$ is an interior b-map. Hence, one can define the category of manifolds with corners, whose morfisms are exactly the b-maps.

Associated with manifolds with corners, it is reasonable to consider now certain structures that are adapted to its smooth structure. First, note that when at the boundary $\partial\overline{M}$, the tangent direction one might take on M are not as many as one could consider on a regular smooth manifold. Therefore, it makes sense to consider a more restricting class of vector fields on M. Namely, the b-*vector fields* $\mathscr{V}_b(M)$ are defined as

$$\mathscr{V}_b(M) = \left\{X \in \mathscr{V}(M) \;\middle|\; X\big|_{\partial\overline{M}} \in \mathscr{V}(\partial\overline{M})\right\} \tag{1.8}$$

Hence, in a coordinate patch modelled by $[0,\infty)^k \times \mathbb{R}^{m-k}$, it is possible to represent all points as $(x_1,\dots,x_k,y_1,\dots,y_{m-k})$. From this, it follows that, locally,

$$\mathscr{V}_b(M) = \operatorname{span}_{C^\infty(M)} \left\{ x_1\frac{\partial}{\partial x_1}, \dots, x_k\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{m-k}} \right\}. \tag{1.9}$$

Definition 1.7. Let M be a m-dimensional manifold with corners and $H \subset \partial\overline{M}$ one boundary hypersurface. A non-negative function $\rho \in C^\infty(M)$ is a boundary defining function if $\mathrm{d}\rho$ is non-vanishing on H and $\rho^{-1}(\{0\}) = H$.

Clearly, Definition 1.7 implies that each of the functions ρ_i in Definition 1.5 are boundary defining functions for $H_j = \rho_j^{-1}(\{0\})$, respectively. Hence, condition 2 in Definition 1.5 means that every boundary hypersurface of M must admit a smooth boundary defining function and, moreover, this collection of boundary defining functions forms a family of independent functions on the intersection of the surfaces they are defining. Furthermore, it is worth noticing that near each H, one has $[0,\infty)_\rho \times H$ modelling a neighborhood of H in M, implying that one can write points near H as (ρ, z), with ρ identified as a real number and $z \in H$.

1.3 Polyhomogeneous conormal functions

Once understood the concept of boundary defining functions, it is now possible to present a class of functions crucial to this work: the class of polyhomogeneous conormal functions. This is a class of functions that are not smooth on M, but admit an approximation by a power series in terms of the boundary defining functions and with controlled singular behavior. This class of functions is relevant in this work precisely because the heat kernel, which plays a very important role, is not a smooth function; however, when working with “good coordinates” (see §1.4), it is polyhomogenous conormal and hence, admits such approximation. This approximations are key to obtain parabolic Schauder estimates, allowing one to discuss existence of the Yamabe flow.

A set $E \subset \mathbb{C} \times \mathbb{N}_0$ is called an *index set* if it satisfies the following conditions:

1. E is discrete and bounded from below;
2. $E_N := \{(z,p) \in \mathbb{C} \times \mathbb{N}_0 \,|\, \mathrm{Re}(z) < N\}$ is finite, for all N;
3. If $(z,p) \in E$, then $(z+n,p) \in E$ for every $n \in \mathbb{N}$.

A family $\mathscr{E} = (E_1,...,E_k)$ of index sets is called an *index family*.

Definition 1.8. Let M be a manifold with corners, $\{H_1,\dots,H_k\}$ its family of boundary hypersurfaces and $\{\rho_1,\dots,\rho_k\}$ its respective boundary defining functions. A function $f : M \to \mathbb{R}$ is *polyhomogeneous conormal with index family* $\mathscr{E}$ if near each boundary hypersurface $H_j = \rho_j^{-1}(\{0\})$, one can approximate f as follows:

$$f \sim \sum_{(r,n)\in E_j} a_{r,n}\rho_j^r(\log\rho_j)^n, \text{ as } \rho_j \to 0, \tag{1.10}$$

where each $a_{j,r,n}$ is polyhomogeneous conormal on H_j with index family $(E_1,\dots,E_{j-1},E_{j+1},\dots,E_k)$ near the intersections $H_j \cap H_l$ for any $l \neq j$.

The set of polyhomegeneous conormal functions on M with index family $\mathscr{E}$ is denoted by $\mathscr{A}^{\mathscr{E}}_{\mathrm{phg}}(M)$.

From now on, polyhomogeneous conormal functions will be refered to as simply “phg” for short. The expression (1.10) means, in more explicit terms, that for every $N \in \mathbb{N}_0$, by making f_N be partial sums for $r < N$ in (1.10), there exists an uniform constant C_N such that on compact subsets we have

$$|f - f_N| \le C_N\rho_j^N, \text{ as } \rho_j \to 0. \tag{1.11}$$

Moreover, this also implies that similar estimates are true for $\mathscr{V}_{\mathrm{b}} f$ (which is now approximated by $\mathscr{V}_{\mathrm{b}} f_N$), which means that b-derivatives preserve the estimates.

Remark 1.9. Note that, as a consequence of the Taylor series expansion, every smooth function is phg with index set $\{0\} \times \mathbb{N}$. It is also worth noticing that, whenever a function f vanishes to infinite order at some boundary hypersurface H, that is, when $|f| \le \rho_H^n$ for every n as $\rho_H \to 0$, then it is convention that f is phg with index set $\varnothing$ at said face; the simplest (non-trivial) example for this is e^{-1/ρ_H}.

1.4 Blow-up and blow-down maps

The goal of this work is to give results regarding Φ-manifolds (see Chapter 2), which here will be interpreted as a family of singular manifolds. To do so, it is necessary to pass from such singular manifolds to a manifold with corners via resolution of specific submanifolds. This process is called blow-up and consists of replacing these submanifolds $N \subset M$ by its interior-pointing unit-size conormal bundle, creating a new manifold $[M;N]$ (which is a manifold with corners) which relates to M via a blow-down map $\beta : [M;N] \to M$. This is done whenever one needs to study a function near its singular support and the new manifold is a modified version on the original manifold built so that such function becomes a phg function on the new one.

Before presenting a more formal definition, lets discuss an example given by polar coordinates, which can be found in [Gri01], to paint a picture. For such, consider $M = [0,\infty)^2$, which is a manifold with corners, and $N = \{0\}$ a submanifold. Consider then polar coordinates on M, writing

$$x = r\cos\theta,\ y = r\sin\theta, \tag{1.12}$$

with $r \in [0,\infty)$ and $\theta \in [0,\pi/2]$. This means that we are now writing points in M using coordinates coming from the infinite cylinder $[0,\infty) \times [0,\pi/2]$. Note that the two first pictures in Figure 1.1 are,

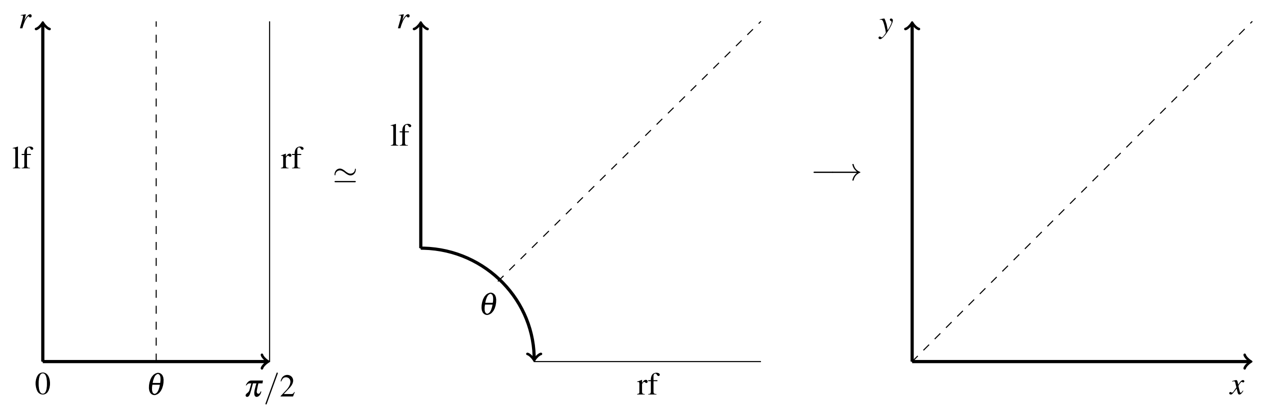

Figure 1.1: Blow-up of $[0,\infty)^2$ on $\{0\}$

in fact, diffeomorphic (denoted by the symbol "≃") , where the second picture represents exactly the description given previously, for the submanifold N was replaced by its interior-pointing unit-size conormal bundle in M (i.e, a quarter of a circle). Then, by writing

$$\mathbb{S}^{m-1}_{+} = \{x = (x_1,\ldots,x_m) \in \mathbb{S}^{m-1} \subset \mathbb{R}^m \mid x_j \geq 0\ \forall j\}, \tag{1.13}$$

one has the blow-up of $[0,\infty)^2$ along $\{0\}$ given by $[[0,\infty)^2;\{0\}] = [0,\infty) \times \mathbb{S}^1_{+}$.

There is, however, a more appropriate choice of coordinates to work on the second picture. Note that, whenever working near its corners, that is $\mathrm{lf} = \{\theta = 0\}$ or $\mathrm{rf} = \{\theta = \pi/2\}$, implies $y \ll x$ or $x \ll y$ respectively. In the first case, represented geometrically by the lower corner in the second picture, admits then reasonable coordinates given by

$$r \approx x,\ \theta \approx \frac{y}{x}, \tag{1.14}$$

where the notation $\approx$ means it is an approximation. This is a reasonable choice of coordinates because, in this scenario, x is "far from zero", meaning that y vanishes faster then x, keeping θ bounded and, moreover, providing a good enough approximation of the real coordinate – at least locally. This coordinates work locally and, in fact, can be used as long as one is "far enough" from the hypersurface $\{x=0\}$. Analogously, is only reasonable to consider coordinates $r \approx y$ and $\theta \approx x/y$ when working near $\{y=0\}$. These types of coordinates are called *projective coordinates* and they will show themselves extremely useful throughout this work.

More generally, the example presented above works exactly the same for higher dimensions, giving

$$[\mathbb{R}^m_+;\{0\}] = \mathbb{R}_+ \times \mathbb{S}^{m-1},\ \ \beta(r,\omega) = r\omega. \tag{1.15}$$

This example is key to understand more general blow-ups, since they are locally modeled by (1.15).

Definition 1.10. Consider a manifold with corners M, with $\dim M = m$ and $N \subset M$ an n-dimensional *p-submanifold* (see [Mel96, §1.7]) , which means that locally N can be written as $\{x_{n+1} = \cdots = x_m = 0\}$, for $(x_1,\dots,x_m)$ local coordinates on M. Under these circumstances, the *blow-up of M along N*, denoted by $[M;N]$, such that $\mathrm{int}([M;N]) \simeq \mathrm{int}(M-N)$.

CHAPTER 2

Φ-manifolds

This section presents the class of manifolds on which this work is based on. The family of Φ-manifolds is a particular case of a Riemannian manifold whose behavior near its "bad region" (that is, its singularities) admits a specific expression. However, before understanding what a Φ-manifold is, one must discuss briefly the idea of manifold with fibered boundaries.

Definition 2.1. A *manifold with fibered boundaries* is a pair $(\overline{M}, \phi)$, with $\overline{M}$ being a compact smooth manifold with boundary $\partial\overline{M}$, M is the open interior and $\phi : \partial\overline{M} \to Y$ a fibration with typical fiber Z, where both Y and Z are closed smooth manifolds.

In order to keep the notation shorter, manifolds with fibered boundaries will be denoted simply as its smooth manifold whenever the map ϕ is implicit. Moreover, from this point on, we fix the notation $b := \dim Y$ and $f := \dim Z$ as the dimensions of the base space and the typical fibers, respectively.

It is worth to take some notes on this definition. Given $\overline{M}$ a manifold with fibered boundary, let x be a choice of a global boundary defining function for $\partial\overline{M}$ (which exists from compactness of $\overline{M}$), that is, x is a non-negative function on M lying in $C^\infty(\overline{M})$, so that $\partial\overline{M} = \{x = 0\}$ and $\mathrm{d}x \neq 0$ on $\partial\overline{M}$. Since $\overline{M}$ is compact, there exists a collar neighborhood U of $\partial\overline{M}$ in $\overline{M}$ such that $U \simeq [0,1) \times \partial\overline{M}$. Hence, it is possible to write every point in U as a pair (x,w), with $x \in [0,1)$ and $w \in \partial\overline{M}$. On the other hand, since $\partial\overline{M}$ is the total space of a fibration with typical fiber Z and base space Y, there is an open subset V of Y such that $\phi^{-1}(V) \simeq V \times Z$, allowing every point in $\partial\overline{M}$ to be written as a pair (y,z) in such open subset. Therefore, locally over the base (that is, for some open subset of U) if is possible to write every $p \in \partial\overline{M}$ as the triple $p = (x,y,z)$.

Once understood these considerations on manifolds with fibered boundary, we are now in a good place to introduce Φ-manifolds.

Definition 2.2. Assume ϕ in Definition 2.1 to be a Riemannian submersion $\phi : (\partial M, g_Z + \phi^* g_Y) \to (Z, g_Z)$. A complete Riemannian metric g_Φ on the interior manifold $M \subset \overline{M}$ is said to be a fibered boundary metric if, near the boundary $\partial\overline{M}$ of $\overline{M}$, it can be expressed as follows:

$$g_\Phi = \frac{\mathrm{d}x^2}{x^4} + \frac{\phi^* g_Y}{x^2} + g_Z + h := g_{\Phi,0} + h, \tag{2.1}$$

where g_Y is a Riemannian metric on the base space Y, g_Z is a symmetric bilinear form defined on $\partial\overline{M}$ which restricts to a Riemannian metric at each fiber Z_y and h corresponds to cross-terms and satisfies $|h|_{g_{\Phi,0}} = O(x)$ as $x \to 0$.

The Riemannian manifold (M, g_Φ) is called a Φ-manifold .

The metric above should be understood in the following manner: the exact part $g_{\Phi,0}$ is constituted precisely of the three diagonal elements specified (2.1), while the remaining possible elements in g_Φ reside within h with the assumption that $|h|_{g_{\Phi,0}} = O(x)$ when $x \to 0$, meaning that whenever near the boundary, its coefficients are of order $O(x)$.

Example 2.3. Consider $M = \mathbb{R}^m$ radially compactified to $\overline{\mathbb{R}^m}$ and endowed with its standard Riemannian metric g_{Euc}. In polar coordinates, one writes $g_{\text{Euc}} = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$, where $\theta \in \mathbb{S}^{m-1}$. Note that, although $\overline{M}$ is compact, M is not and, in fact, its singular region is exactly $\{r = \infty\}$. Consider now a large enough compact $K \subset M$ containing the origin. Then, on $M - K$ one can take $x = r^{-1}$, from where follows that

$$g_{\text{Euc}} = \frac{\mathrm{d}x^2}{x^4} + \frac{\mathrm{d}\theta^2}{x^2},$$

where now the singular region lies in $\{x = 0\}$. It is worth noticing that, in this example, the fiber is $Z = \{\text{pt}\}$. This gives to $\overline{\mathbb{R}^m}$ the structure of a fibered boundary manifold, with base $\mathbb{S}^{m-1}$ and fiber $\{\text{pt}\}$, and it allows us to see $(\mathbb{R}^m, g_{\text{Euc}})$ as a Φ-manifold.

Since ϕ is required to be a Riemannian submersion, the tangent directions on M near $\partial\overline{M}$ are spanned by lifts of vector fields $\{\partial_x, \partial_{y_i}, \partial_{z_j} \mid i = 1, ..., b, j = 1, ..., f\}$ (which, for simplicity, will be denoted omitting the pullback notation). Thus, on this basis of T_pM, one has the matrix representation

$$(g_\Phi) = \begin{pmatrix} x^{-4} & O(x^{-2}) & O(x^{-1}) \\ O(x^{-2}) & O(x^{-2}) & O(1) \\ O(x^{-1}) & O(1) & O(1) \end{pmatrix},$$

where the terms on the diagonal are the exact part of g_Φ, while the remaining terms are in h. Similarly to the case of b-Calculus, when working with Φ-metrics, commonly one chooses to work with a class of vector fields that are adjusted to the singularities in the metric, called Φ-vector fields. This class is given by

$$\mathscr{V}_\Phi(M) = \left\{ V \in \mathscr{V}(M) \;\middle|\; \begin{array}{c} Vx \in x^2 C^\infty(M) \text{ and} \\ V_p \in T_p\phi^{-1}(\phi(p)) \text{ for every } p \in \partial\overline{M} \end{array} \right\}.$$

This is a class of vector fields on M which takes unit size on (M, g_Φ) and are given locally as

$$\mathscr{V}_\Phi(M) = \operatorname{span}\left\{ x^2\frac{\partial}{\partial x}, x\frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_j} \;\middle|\; i = 1, \ldots, b,\ j = 1, ..., f \right\}. \tag{2.2}$$

Unlike the matrix representation of g_Φ on the standard basis, the basis provided by Φ-vector fields makes every entry in the matrix of g_Φ be a bounded function. Once one has the class of Φ-vector fields, one can then consider the class of Φ-forms. In fact, the 1-forms dual to $\mathscr{V}_\Phi(M)$ are

$$\Lambda^1_\Phi(M) = \operatorname{span}\left\{ \frac{\mathrm{d}x}{x^2},\ \frac{\mathrm{d}y_i}{x},\ \mathrm{d}z_j \ \middle|\ i = 1,...,b,\ j = 1,...,f \right\}. \tag{2.3}$$

Naturally, higher order Φ-differential forms are results of exterior products of 1-forms. Since $\mathscr{V}_\Phi(M)$ is a Lie algebra and a $C^\infty(M)$-module, it is natural to consider the enveloping algebra $\mathrm{Diff}^*_\Phi(M)$ given by operators that can be written locally linear combination of elements of $\mathscr{V}_\Phi(M)$. Hence, define the space of Φ-differential operators of order k, denoted by $\mathrm{Diff}^k_\Phi(M)$, as the space of linear operators $P : C^\infty(M) \to C^\infty(M)$ which can be locally expressed by

$$P = \sum_{|\alpha_1|+|\alpha_2|+q\leq k} P_{\alpha_1,\alpha_2,q}(x,y,z)(x^2\partial_x)^q(x\partial_y)^\beta\partial_z^\alpha,$$

where α_1 and α_2 are multi-indices, $\partial_y = \partial_{y_1}...\partial_{y_b}$, $\partial_z = \partial_{z_1}...\partial_{z_f}$ and $P_{\alpha_1,\alpha_2,q}$ is a smooth function. From time to time, we will also refer to $\mathrm{Diff}^k_\Phi(M)$ as $\mathscr{V}^k_\Phi$. Finally, it is possible now to define the class of k-continuously $\mathbb{R}$-valued Φ-differentiable functions

$$C^k_\Phi(M\times[0,T]) := \left\{ u \in C^0(M\times[0,T]) \ \middle|\ \begin{array}{l} (V\circ\partial_t^{l_2})u \in C^\alpha_\Phi(M\times[0,T]), \\ \text{for } V \in \mathrm{Diff}^{l_1}_\Phi(M),\ l_1+2l_2 \leq k \end{array} \right\}. \tag{2.4}$$

It should be pointed out that time-derivatives ∂_t are considered as second-order derivatives. In practical terms, this implies that functions in $C^1_\Phi(M\times[0,T])$ cannot be differentiated in time. This will show itself useful for the study of heat-type equations as we advance. Moreover, we can define $C^k_\Phi(M)$ similarly simply by taking spacial derivatives only.

Manifolds of bounded geometry

Before discussing some further objects associated to Φ-geometries, such as its volume form and scalar curvature, we take the time to discuss briefly a broader class of Riemannian manifolds, which are the manifolds of bounded geometry. For further discussions on this subject, see [Eld13, Chapter 2].

A complete manifold (M,g) is said to have k-th order bounded geometry if the following conditions are met:

1. the global injectivity radius $r_{\mathrm{inj}}(M) = \inf_M r_{\mathrm{inj}}(p)$ is positive, that is, there is a positive number $r_{\mathrm{inj}}(M) = \delta > 0$ such that

$$\exp_p : B(0,\delta) \subset T_pM \to B(p,\delta) \subset M$$

is a diffeomorphism, for all $p \in M$;

2. the curvature tensor R and its covariant derivatives up to order k are uniformly bounded operators (see §A.4).

When no specific order k is given, it is assumed $k = \infty$ and we simply call it a manifold of bounded geometry.

Natural examples of manifolds of bounded geometry are compact manifolds and Euclidean spaces. Moreover, from Appendices A.4 and B, it follows that Φ-manifolds have bounded geometry as well. This means we can employ properties of this class to our work, including the following proposition.

Proposition 2.4. *[Eld13, Proposition 2.5] Let (M,g) be a Riemannian manifold of $k \geq 1$ bounded geometry. For every constant $C > 0$, there is a $\delta > 0$ sufficiently small such that the normal coordinate chart $\varphi_p := \exp_p^{-1}$ is defined on $B(p,\delta) \subset M$ for each $p \in M$ and the Euclidean distance d_{Euc} on the normal coordinates is C-equivalent to the distance d_g induced on M by g, that is,*

$$C^{-1} d_g(p_1,p_2) \leq d_{Euc}(\exp_p(p_1),\exp_p(p_2)) \leq C\, d_g(p_1,p_2), \text{ for all } p_1,p_2 \in B(p,\delta).$$

This means that the distance on a manifold of bounded geometry is locally uniformly equivalent to the Euclidean distance function and, therefore, it allows one to work on this class of manifolds simply by considering the Euclidean distance function. We make use of this in the following chapters.

2.1 Volume form

It is worth noticing that, due to the singularities in the Riemannian metric g_Φ, the volume form on a Φ-manifold has a singularity as well. In fact, consider the exact part $g_{\Phi,0}$ of g_Φ, namely

$$g_{\Phi,0} = \frac{\mathrm{d}x^2}{x^4} + \frac{\phi^* g_Y}{x^2} + g_Z \tag{2.5}$$

locally near boundary $\partial\overline{M}$. Then, it follows that $g_{\Phi,0}$ admits locally the matrix representation

$$(g_{\Phi,0}) = \begin{pmatrix} x^{-4} & & \\ & O(x^{-2}) & \\ & & O(1) \end{pmatrix},$$

where the empty spaces in the matrix above represent null entries. Therefore, it follows from this expression that the volume form for the exact part of the Riemannian metric is then, locally around a point $p = (x,y,z)$, given by

$$\mathrm{dvol}_{g_{\Phi,0}}(p) = x^{-2-b} h_0(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{2.6}$$

with h_0 a bounded smooth function. Now, since the term h in Definition 2.2 is less singular than the exact part of g_Φ, it follows that the volume form for g_Φ is as singular as the volume form for $g_{\Phi,0}$, meaning that for the purposes of the analysis presented here, is good enough to work directly with (2.6). Consequently $\mathrm{vol}(M,g_\Phi) = \infty$, which keeps us from working with the standard definition for the normalized Yamabe flow (for details, see § 3.5).

2.2 Scalar curvature

Obtaining knowledge on the scalar curvature of (M, g_Φ) is crucial to the development of this work. Although the scalar curvature does not impact the proof of the short-time existence of the Yamabe flow, it is important to know its behavior to prove its existence for all time. Hence, we take the time to obtain some information on the scalar curvature of a Φ-manifold.

Just like the analysis conducted in the previous section, it is once again good enough to study the scalar curvature on the exact part of the Riemannian metric. In fact, by taking $\{e_1, \dots, e_b\}$ to be an orthonormal frame on Y and letting $\widetilde{\cdot}$ denote the lift of a vector field to M, we know that there is a family of vector fields $\{\partial_{z_1}, \dots, \partial_{z_f}\}$ on M such that $\{x^2\partial_x, x\widetilde{e_1}, \dots, x\widetilde{e_b}, \partial_{z_1}, \dots, \partial_{z_f}\}$ is an orthonormal frame on M. On this frame, the matrix representation of g_Φ takes the following expression:

$$(g_\Phi) = \begin{pmatrix} 1 & & \\ & \mathbb{1}_b & \\ & & \mathbb{1}_f \end{pmatrix} + \begin{pmatrix} & O(x)_{1\times b} & O(x)_{1\times f} \\ O(x)_{b\times 1} & & O(x)_{b\times f} \\ O(x)_{f\times 1} & O(x)_{f\times b} & \end{pmatrix} \tag{2.7}$$

Hence, the terms out of the exact part $g_{\Phi,0}$ give rise only to higher order terms and, therefore, it will not have any significant impact in our analysis. From §A.6, the Ricci curvature tensor of the exact Φ-metric $g_{\Phi,0}$ is given by

$$(\mathrm{Ric}_{\Phi,0})_{ik} = \begin{cases} (\mathrm{Ric}_Y)_{i-1\,k-1} + (b-1)(g_Y)_{i-1\,k-1}, & \text{if } 2 \le i,k \le b+1 \\ (\mathrm{Ric}_Z)_{i-(b+1)\,k-(b+1)}, & \text{if } b+2 \le i,k \le m \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

Thus, it follows from the definition of the scalar curvature that

$$\begin{aligned} \mathrm{scal}(g_{\Phi,0}) &= \sum_{i,k} (\mathrm{Ric}_{\Phi,0})_{ik} g^{ik}_{\Phi,0} = \sum_{2\le i,k\le b+1} (\mathrm{Ric}_{\Phi,0})_{ik} g^{ik}_{\Phi,0} + \sum_{b+2\le i,k\le m} (\mathrm{Ric}_{\Phi,0})_{ik} g^{ik}_{\Phi,0} \\ &= \sum_{2\le i,k\le b+1} \left((\mathrm{Ric}_Y)_{i-1\,k-1} + (b-1)(g_Y)_{i-1\,k-1}\right) x^2 g_Y^{k-1\,i-1} \\ &\quad + \sum_{b+2\le i,k\le m} (\mathrm{Ric}_Z)_{i-(b+1)\,k-(b+1)} g_Z^{i-(b+1)\,k-(b+1)} \\ &= x^2 \sum_{2\le i,k\le b+1} (\mathrm{Ric}_Y)_{i-1\,k-1} g_Y^{i-1\,k-1} + x^2(b-1) \sum_{2\le i,k\le b+1} (g_Y)_{i-1\,k-1} g_Y^{i-1\,k-1} \\ &\quad + \sum_{b+2\le i,k\le m} (\mathrm{Ric}_Z)_{i-(b+1)\,k-(b+1)} g_Z^{i-(b+1)\,k-(b+1)} \\ &= x^2(\mathrm{scal}(g_Y) + b(b-1)) + \mathrm{scal}(g_Z). \end{aligned} \tag{2.9}$$

In the general case, additional $O(x)$ terms (as $x \to 0$) appear. Far from $x = 0$, we are working in a compact manifold and therefore, scalar curvature is simply a smooth bounded function.

2.3 Maximum principles

For our argument on the parametrix construction (in the following section) to be complete, a maximum principle-type of result is needed in order to say something about uniqueness of solutions of some heat-type equations.

From this point on, we assume the Laplace-Beltrami operator Δ_Φ on (M, g_Φ) to be negatively defined.

First, note that a Φ-manifold, understood as an open Riemannian manifold, is decomposable as the union of a compact region $K \neq \varnothing$ with an open subset U on which the Riemannian metric is given locally by the expression (2.1). Under this assumption, by considering $U \simeq (1,0) \times \partial\overline{M}$, one can identify $K = \{p \in M \mid x \geq 1\}$. Now, by performing the change of coordinates $r = x^{-1}$ on M, one can rewrite the expression for g_Φ as

$$g_\Phi = \mathrm{d}r^2 + r^2\phi^* g_Y + g_Z + h. \tag{2.10}$$

It is worth noticing that, under this expression, the distance between two points near the boundary $\partial\overline{M} = \{p \in \overline{M} \mid r = \infty\}$ is proportional to r. This can be checked by noticing that the distance from the boundary is given by the term $\mathrm{d}r^2$ and, therefore, the distance in this direction is proportional to the Euclidean distance given in polar coordinates.

From [AMR16, Theorem 2.11], a sufficient condition for stochastic completeness of (M, g_Φ) is that

$$\frac{R}{\log \operatorname{vol} B(p_0,R)} \notin L^1(1,+\infty), \text{ i.e. } \int_1^{+\infty} \frac{R}{\log \operatorname{vol} B(p_0,R)} \mathrm{d}R = +\infty, \tag{2.11}$$

for some point $p_0 \in M$ and $B(p_0,R)$ an open disc centered at said point and radius R. Since we can assume, w.l.o.g., to have $K = \{x \geq 1\}$, then naturally we have $K = \{r \leq 1\}$. Consider now the truncated compact subset $M_n = \{r \leq n\}$. This allow us to define a countable family of compact subsets $\{M_i\}_i$, for $i \in \mathbb{N}_{>0}$, satisfying

$$M = \bigcup_{i \in \mathbb{N}_{>0}} M_i \text{ and } M_i \subset M_{i+1} \text{ for all } i. \tag{2.12}$$

Since $K \subset M_1$, it follows from Cantor's intersection theorem that there exists a point $p_0 \in M$ such that $p_0 \in M_i$ for all $i \in \mathbb{N}_{>0}$, satisfying $r(p_0) \leq 1$. Since M is a manifold, it is also a regular topological space and therefore, there is some $0 < \varepsilon < 1/2$ such that $B(p_0, \varepsilon) \subset M_1$. From this, we have the following

Claim: $B(p_0, i) \subset M_{i/\varepsilon}$, for any $i \in \mathbb{N}_{\geq 2}$.

Let us prove this by contradiction, that is, assume the existence of a point $p \in B(p_0, i)$ such that $r(p) > i/\varepsilon$. Thus, from the expression of g_Φ as a function of r and the fact that $r(p_0) < 1$, we have

$$\begin{aligned}\frac{i}{\varepsilon} - 1 &< |r(p) - r(p_0)| \leq d_\Phi((r(p),y,z),(r(p_0),y,z)) \leq d_\Phi(p,p_0) \\ &< i,\end{aligned}$$

implying $i/2 < 1/2$, which does not happen for $i \in \mathbb{N}_{\geq 2}$. Hence, it follows that $B(p_0, i) \subset M_{i/\varepsilon}$.

Hence,

$$\frac{R}{\log \operatorname{vol} B(p_0,R)} \sim \frac{R}{\log \operatorname{vol} M_R},$$

which means that they are the same up to bounded functions, thus the later being integrable if and only if the first one is integrable. Now, the expression for the Φ-metric implies $\mathrm{dvol}_\Phi(p) = h_0 r(p)^b \, dr\, dy\, dz$, where h_0 is a smooth bounded function. From this, it follows that $\mathrm{vol}\, M_R \sim R^{b+1} \leq e^{CR^2}$, for some positive constant C, as R goes to ∞, meaning that

$$\frac{R}{\log \mathrm{vol}\, M_R} \notin L^1(1,+\infty). \tag{2.13}$$

From this, we conclude that (M, g_Φ) is stochastically complete.

According to [AMR16, Theorem 2.8 (i) and (iii)], stochastic completeness and the Omori-Yau maximum principle are equivalent on Riemannian manifolds, from which it follows that for every function $u \in C^2_\Phi(M)$ there is a sequence $\{p_k\} \in M$ satisfying

$$u(p_k) > \sup_M u - \frac{1}{k} \text{ and } \Delta_\Phi u(p_k) < \frac{1}{k}. \tag{2.14}$$

Analogously, for any function $u \in C^2_\Phi(M)$, there exists a sequence of functions $\{p'_k\}$ in M such that

$$u(p'_k) < \inf_M u + \frac{1}{k} \text{ and } \Delta_\Phi u(p'_k) > -\frac{1}{k}. \tag{2.15}$$

Before proving the maximum principle, let us recall the Rademacher's theorem, which gives us a condition for a function to be differentiable almost everywhere. This result is going to be useful in the proof of Proposition 2.6, which is key to proving the maximum principle.

Theorem 2.5. *(Rademacher's theorem, [Hei, Theorem 3.1]) Let $\Omega \subset \mathbb{R}^n$ an open subset and $u : \Omega \to \mathbb{R}^{n'}$ a Lipschitz function. Then u is differentiable almost everywhere.*

Proposition 2.6. *Consider any $u \in C^2_\Phi(M \times [0,T])$ satisfying the following inequalities:*

$$\left| \frac{\partial u}{\partial t}(p,t) - \frac{\partial u}{\partial t}(p,t') \right| \leq C|t-t'|^\gamma, \qquad \left| \frac{\partial u}{\partial t}(p,t) \right| \leq C, \tag{2.16}$$

for some positive constants $C, \gamma > 0$. Then

$$u_{\sup}(t) := \sup_M u(\cdot,t), \quad u_{\inf}(t) := \inf_M u(\cdot,t)$$

are differentiable almost everywhere in $(0,T)$ and at those $t \in (0,T)$ we find in the notation of (2.14) and (2.15)

$$\begin{aligned} \frac{\partial}{\partial t} u_{\sup}(t) &\leq \lim_{\varepsilon \to 0^+} \left(\limsup_{k \to \infty} \frac{\partial u}{\partial t} (p_k(t+\varepsilon), t+\varepsilon) \right), \\ \frac{\partial}{\partial t} u_{\inf}(t) &\geq \lim_{\varepsilon \to 0^+} \left(\liminf_{k \to \infty} \frac{\partial u}{\partial t} (p'_k(t+\varepsilon), t+\varepsilon) \right). \end{aligned} \tag{2.17}$$

Proof. Let $\varepsilon > 0$ and apply (2.14) to $u(t+\varepsilon)$. Then, by the Mean Value Theorem

$$u_{\sup}(t+\varepsilon) \leq u(p_k(t+\varepsilon), t+\varepsilon) + \frac{1}{k}$$

$$= u(p_k(t+\varepsilon),t) + \varepsilon \cdot \frac{\partial u}{\partial t}(p_k(t+\varepsilon),\xi) + \frac{1}{k},$$

for some $\xi \in (t,t+\varepsilon)$. On the other hand, we can write

$$\begin{aligned} u_{\sup}(t+\varepsilon) &= u_{\sup}(t) + \varepsilon \cdot \frac{u_{\sup}(t+\varepsilon) - u_{\sup}(t)}{\varepsilon} \\ &\geq u(p_k(t+\varepsilon),t) + \varepsilon \cdot \frac{u_{\sup}(t+\varepsilon) - u_{\sup}(t)}{\varepsilon}. \end{aligned}$$

Combining these two estimates leads, after cancelling $u(p_k(t+\varepsilon),t)$, to

$$\varepsilon \cdot \frac{u_{\sup}(t+\varepsilon) - u_{\sup}(t)}{\varepsilon} \leq \varepsilon \cdot \frac{\partial u}{\partial t}(p_k(t+\varepsilon),\xi) + \frac{1}{k}.$$

Taking $\limsup$ as $k \to \infty$ on the right hand side, we obtain

$$\varepsilon \cdot \frac{u_{\sup}(t+\varepsilon) - u_{\sup}(t)}{\varepsilon} \leq \varepsilon \cdot \limsup_{k\to\infty} \frac{\partial u}{\partial t}(p_k(t+\varepsilon),\xi).$$

Cancelling ε on both sides, we find

$$\begin{aligned} \frac{u_{\sup}(t+\varepsilon) - u_{\sup}(t)}{\varepsilon} &\leq \limsup_{k\to\infty} \frac{\partial u}{\partial t}(p_k(t+\varepsilon),\xi) \\ &= \limsup_{k\to\infty} \left(\frac{\partial u}{\partial t}(p_k(t+\varepsilon),\xi) - \frac{\partial u}{\partial t}(p_k(t+\varepsilon),t+\varepsilon) \right) \\ &+ \limsup_{k\to\infty} \frac{\partial u}{\partial t}(p_k(t+\varepsilon),t+\varepsilon). \end{aligned} \tag{2.18}$$

We know, from hypothesis, the function u satisfies (2.16). Thus, it follows that

$$\begin{aligned} &\bullet \quad \limsup_{k\to\infty} \left| \frac{\partial u}{\partial t}(p_k(t+\varepsilon),\xi) - \frac{\partial u}{\partial t}(p_k(t+\varepsilon),t+\varepsilon) \right| \leq C\varepsilon^{\gamma}, \\ &\bullet \quad \limsup_{k\to\infty} \left| \frac{\partial u}{\partial t}(p_k(t+\varepsilon),t+\varepsilon) \right| \leq C. \end{aligned} \tag{2.19}$$

Thus the last two summands in (2.18) are bounded uniformly in ε. Repeating the same arguments with the roles of $u(t)$ replaced by $u(t+\varepsilon)$ interchanged, we conclude that $u_{\sup}$ is locally Lipschitz and thus, by the theorem of Rademacher, differentiable almost everywhere. This proves the first statement.

At those $t \in (0,T)$, where $u_{\sup}$ is differentiable, we conclude from (2.18) and the first line in (2.19), taking $\varepsilon \to 0$

$$\frac{\partial}{\partial t} u_{\sup}(t) \leq \lim_{\varepsilon\to 0} \left(\limsup_{k\to\infty} \frac{\partial u}{\partial t}(p_k(t+\varepsilon),t+\varepsilon) \right). \tag{2.20}$$

This proves the first inequality in (2.17). The second inequality follows from the first, using (2.15), with u replaced by $(-u)$. □

Theorem 2.7. *(Maximum Principle) Let (M, g_Φ) be an m-dimensional Φ-manifold and $u \in C^2_\Phi(M \times [0,T])$ be a function satisfying the inequalities in (2.16) and the Cauchy problem*

$$(\partial_t - a\Delta_\Phi)u = 0,\ u|_{t=0} = 0, \tag{2.21}$$

where the function $a = a(p,t)$ is positive, bounded and bounded from below away from zero. Then $u = 0$.

Proof. Combining the first inequality in Proposition 2.6 and (2.14), it follows that

$$\frac{\partial}{\partial t}u_{\sup}(t) \leq \lim_{\varepsilon\to 0}\left(\limsup_{k\to\infty}\frac{a(p_k(t+\varepsilon),t+\varepsilon)}{k}\right) \leq 0.$$

Analogously, combining the second inequality in Proposition 2.6 and (2.15), we get

$$\frac{\partial}{\partial t}u_{\inf}(t) \geq \lim_{\varepsilon\to 0}\left(\liminf_{k\to\infty}\frac{-a(p_k(t+\varepsilon),t+\varepsilon)}{k}\right) \geq 0.$$

This means that the infimum of the function u over M is non-decreasing in time, while the supremum of the function u over M is non-increasing in time; since $u = 0$ at time $t = 0$, follows directly that $u = 0$ on $M \times [0,T]$. □

From this, we conclude Theorem 0.1. Moreover, a straightforward adaptation of the arguments shown in [Eva10, pg.329] allows the following modification of the previous maximum principle:

Corollary 2.8. *Let (M, g_Φ) be an m-dimensional Φ-manifold and $c \in C^0(M \times [0,T])$ a nonnegative function. If $u \in C^2_\Phi(M \times [0,T])$ satisfies the inequalities in (2.16) and the Cauchy problem*

$$(\partial_t - a\Delta_\Phi + c)u = 0,\ u|_{t=0} = 0, \tag{2.22}$$

then $u = 0$.

Proof. Consider the 1-parameter families of open subsets $V_u^+(t), V_u^-(t) \subset M$ defined as follows:

$$V_u^+(t) := \{p \in M \mid u(p,t) > 0\}, \quad V_u^-(t) := \{p \in M \mid u(p,t) < 0\}. \tag{2.23}$$

We want to prove that the function u satisfying (2.22) must be null, which happens if and only if $V_u^+(t) = V_u^-(t) = \varnothing$ for all $t \in [0,T]$.

First, assume $V_u^+(t_0) \neq \varnothing$ for some fixed $t_0 \in [0,T]$. Thus, there is some point $p_0 \in M$ such that $u(p_0,t_0) > 0$, which naturally implies $u_{\sup}(t_0) > 0$. We know that there is a sequence $\{p_k(t_0)\} \subset M$ satisfying (2.14). Hence, it follows directly that there is some $k_0 \in \mathbb{Z}_{>0}$ such that $u(p_k(t_0),t_0) > 0$ for all $k \geq k_0$. This allows us to assume, without loss of generality, that $\{p_k(t_0)\} \subset V_u^+(t_0)$. Moreover, since u is continuous, it is known that there is some $\delta > 0$ such that $u|_{B(p_0,\delta)\times(t_0-\delta,t_0+\delta)} > 0$, thus implying $V_u^+(t) \neq \varnothing$ for all $t \in (t_0-\delta, t_0+\delta)$, while $u|_{t=t_0-\delta} = 0$. From this and (2.22), it follows that

$$\frac{\partial}{\partial t}u(p_k(t_0),t_0) = (a\Delta_\Phi u)(p_k(t_0),t_0) - (cu)(p_k(t_0),t_0) \leq (a\Delta_\Phi u)(p_k(t_0),t_0)$$

$$\leq \|a\|_\infty \frac{1}{k} \xrightarrow{k\to+\infty} 0.$$

Analogously, the same logic can be used for each $t \in (t_0-\delta, t_0+\delta)$. Thus, it follows from Proposition 2.6 that

$$\frac{\partial}{\partial t} u_{\sup}(t) \leq 0, \text{ almost everywhere in } (t_0-\delta, t_0+\delta). \tag{2.24}$$

On the other hand, we know that $u_{\sup}(t_0) \geq u(p_0,t_0) = c > 0$. Hence, (2.24) implies that

$$u_{\sup}(t-\delta+\varepsilon) \geq c, \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Thus, there is some point $\overline{p} \in M$ such that $u(\overline{p}, t-\delta+\varepsilon) > C > 0$, for any $C < c$. Therefore, the continuity of u implies

$$u(\overline{p}, t-\delta) = \lim_{\varepsilon\to 0^+} u(\overline{p}, t-\delta+\varepsilon) \geq C > 0.$$

However, this contradicts the continuity of u, since $u|_{t=t_0-\delta} = 0$. Therefore, we conclude that there cannot exist any $t_0 \in [0,T]$ such that $V_u^+(t_0) \neq \varnothing$ and, therefore, $u \leq 0$.

Now, let us prove that $u \geq 0$. From (2.23), one can easily see that $V_{(-u)}^+(t) = V_u^-(t)$ for all t. The function $-u \in C_\Phi^2(M\times[0,T])$ also satisfies (2.22) and then, from the arguments given in the previous case, it follows directly that there is no $t \in [0,T]$ such that $V_{(-u)}^+(t) \neq \varnothing$, concluding that u cannot be negative anywhere. Therefore, $u = 0$, completing the proof. □

2.4 Heat kernel

This section presents very useful information for the development of this work, being presented here in detail.

The goal of this section is to give a complete presentation of the asymptotic expansion of the heat kernel H near its singularities. This is achieved by building a "modified" version of $\overline{M}^2 \times [0,\infty)_t$ which resolves the singularities of H, expressing them in terms of specific defining functions. This modified manifold is called heat space and it is denoted as M_h^2.

2.4.1 *Review of the heat space M_h^2*

The construction of the heat space is given by 3 iterated blow-ups of $\overline{M}^2 \times [0,\infty)_t$. Such blow-ups are necessary to understand the asymptotic behavior of the heat kernel near its singular points, which lie in the diagonal of $\overline{M}$ at time $t = 0$ and $t = \infty$ (for infinite time). This can be done by replacing the singular regions by new boundary hypersurfaces. We refer to [TV21] for a more detailed discussion on both the construction of the heat space and the properties of the heat kernel given below.

The first blow-up

Consider first the submanifold $S_1 = (\partial\overline{M})^2 \times [0,\infty)_t$ of $\overline{M}^2 \times [0,\infty)_t$. Note that, since $\partial\overline{M}$ is a p-submanifold of $\overline{M}$ (see Definition 1.10), then S_1 is also a p-submanifold of $\overline{M}^2 \times [0,\infty)_t$ and then the blow-up of S_1 in $\overline{M}^2 \times [0,\infty)_t$ is well-defined. By blowing up S_1 in $\overline{M}^2 \times [0,\infty)_t$ we get

$$M^2_{h,1} := \left[\overline{M}^2 \times [0,\infty)_t; S_1\right],$$
$$\beta_1 : M^2_{h,1} \to \overline{M}^2 \times [0,\infty)_t.$$

Note that $M^2_{h,1}$ is now a "new" manifold, built by replacing the codimension 2 submanifold (displayed below as an edge) of $\overline{M}^2 \times [0,\infty)_t$ by a new boundary hypersurface (which is the conormal bundle of S_1 in $\overline{M}^2 \times [0,\infty)_t$). In order to give a proper description of the blow-down map β_1, let us first give the adequate set of projective coordinates to describe this new manifold.

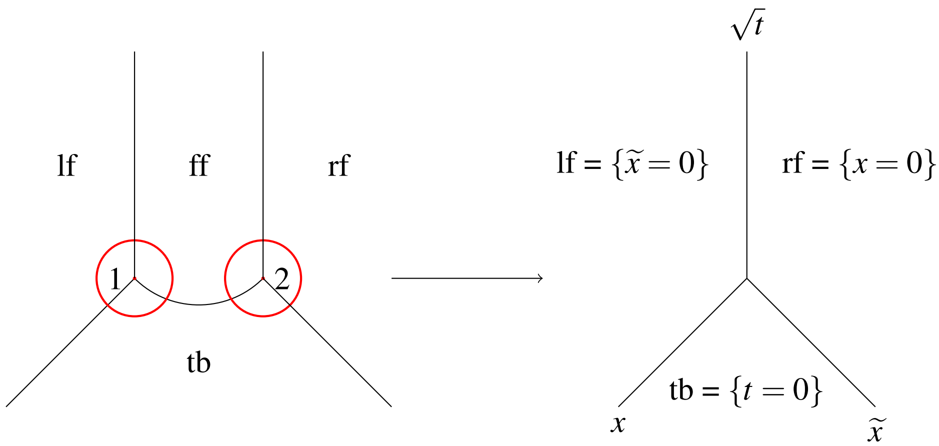

Figure 2.1: First blow-up

Following the steps described in [Gri01], one can describe the projective coordinates for $M^2_{h,1}$ by considering two regions (which from now on, will be called regimes):

• **Regime near the intersection of lf, ff and tb:** This regime is represented in the picture above by "regime 1". Note that such a regime is identified with the region where $\widetilde{x} \ll x$ implying, in particular, that the function $\widetilde{s} = \widetilde{x}^{-1} x$ is bounded. Therefore, by writing $\sqrt{t} =: \tau$, the projective coordinates for the lower-left corner are

$$\left(x, y, z, \frac{\widetilde{x}}{x}, \widetilde{y}, \widetilde{z}, \sqrt{t}\right) = (x, y, z, \widetilde{s}, \widetilde{y}, \widetilde{z}, \tau). \tag{2.25}$$

Hence, on Regime 1 one has $\rho_{\text{ff}} = x$, $\rho_{\text{lf}} = \widetilde{s}$ and $\rho_{\text{tb}} = \tau$.

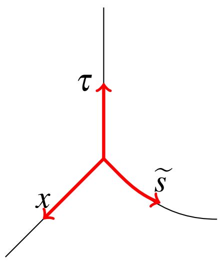

Figure 2.2: Regime near the intersection of lf, ff and tb

• **Regime near the intersection of rf, ff and tb:** This regime is represented in Figure 2.1 by "regime 2", being identified with the case $x \ll \widetilde{x}$. If $x \ll \widetilde{x}$ then $s = x^{-1}\widetilde{x}$ is a bounded function.

Hence, defining τ as above, the projective coordinates for the right-hand corner are

$$\left(\sqrt{t}, \frac{x}{\widetilde{x}}, y, z, \widetilde{x}, \widetilde{y}, \widetilde{z}\right) = (\tau, s, y, z, \widetilde{x}, \widetilde{y}, \widetilde{z}). \tag{2.26}$$

Similarly, on Regime 2 one has $\rho_{\text{ff}} = \widetilde{x}$, $\rho_{\text{rf}} = s$ and $\rho_{\text{tb}} = \tau$.

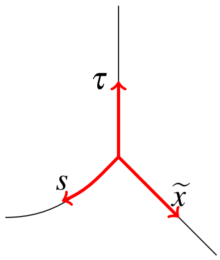

Figure 2.3: Regime near the intersection of rf, ff and tb

Remark 2.9. The projective coordinates defined above for Regimes 1 and 2 are valid in "larger" regions. In fact, one can define both s and $\widetilde{s}$ as long as one stays away from $\{\widetilde{x} = 0\}$ and $\{x = 0\}$ respectively. This information will be useful for computating the parabolic Schauder estimates.

Then, when restricted to the lower-left corner, the blow-down map takes the expression

$$(\beta_1)\big|_1(\tau, x, y, z, \widetilde{s}, \widetilde{y}, \widetilde{z}) = (\tau, x, y, z, x\widetilde{s}, \widetilde{y}, \widetilde{z})$$

and is defined similarly on the lower-right corner.

The second blow-up

Now, we move to the second blow-up, which consists of blowing up the fiber diagonal in time. This means that we want to blow-up the submanifold S_2 of $M^2_{h,1}$ given by

$$\left\{\frac{\widetilde{x}}{x} - 1 = 0 \text{ and } y = \widetilde{y}\right\}.$$

Such submanifold can be seen in the picture above as a line on ff given by its intersection with the plane $\{x = \widetilde{x}\}$ and then, much like in the first blow up, the "new" manifold can be pictured by replacing such path by its conormal bundle on $M^2_{h,1}$ (see picture below). Hence, our "new" manifold has a new hypersurface given by fd = $\{\widetilde{s} - 1 = 0 \text{ and } y = \widetilde{y}\}$ and is now defined

$$M^2_{h,2} := [M^2_{h,1}; S_2],$$
$$\beta_2 : M^2_{h,1} \to M^2_{h,1}$$

and, naturally, one can then consider the iterated blow-down map as the composition $\beta_1 \circ \beta_2 : M^2_{h,2} \to (\overline{M})^2 \times [0, \infty)_\infty$.

Following again the steps as described in [Gri01], is possible to define the projective coordinates on fd by taking

$$\left(\tau, x, y, z, \frac{\widetilde{s} - 1}{x}, \frac{\widetilde{y} - y}{x}, \widetilde{z} - z\right) =: \left(\tau, x, y, z, \widetilde{\mathscr{S}'}, \widetilde{\mathscr{U}'}, \widetilde{\mathscr{Z}'}\right) \tag{2.27}$$

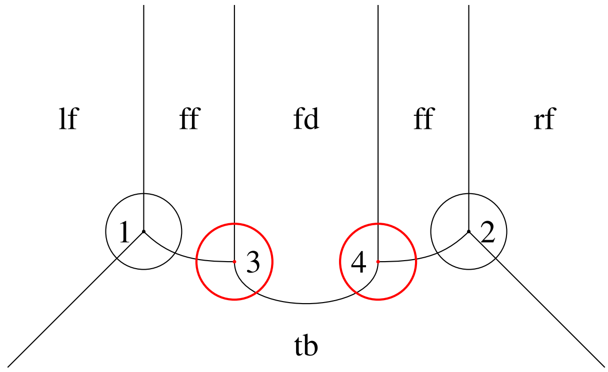

Figure 2.4: Second blow-up

away from $x = 0$ (which corresponds to "regime 3" in Figure 2.4). Similarly, one can consider the projective coordinates on ff away from $\widetilde{x} = 0$ (corresponding to "regime 4" in Figure 2.4) as

$$\left(\tau, \frac{s-1}{\widetilde{x}}, \frac{y-\widetilde{y}}{\widetilde{x}}, z-\widetilde{z}, \widetilde{x}, \widetilde{y}, \widetilde{z}\right) =: \left(\tau, \mathscr{S}', \mathscr{U}', \mathscr{Z}', \widetilde{x}, \widetilde{y}, \widetilde{z}\right).$$

Remark 2.10. Despite the projective coordinates given above for Regimes 3 and 4, one can actually use only one of the sets above to work on both Regimes, since one can understand that approaching ff from fd means that $\|(\mathscr{S}', \mathscr{U}', \mathscr{Z}')\| \to \infty$ (and similarly for $(\widetilde{\mathscr{S}'}, \widetilde{\mathscr{U}'}, \widetilde{\mathscr{Z}'})$). Hence, one can say that on both Regimes 3 and 4, $\rho_{\text{tb}} = \tau$ and $\rho_{\text{fd}} = x$.

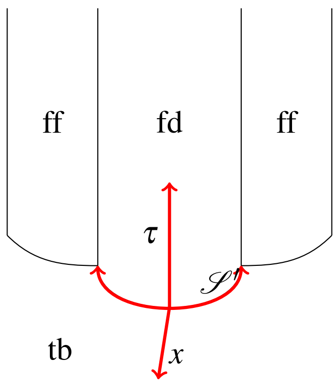

Figure 2.5: Projective coordinates for the second blow-up

The third blow-up

Finally, we move to the third (and last) blow up. This last blow up arises from the singularities of the heat kernel on the spatial diagonal. Therefore, the heat space constructed replacing $\text{diag}(M) \times \{t = 0\}$ by its conormal bundle on $M^2_{h,2}$. Therefore, is defined the blow up

$$M^2_h := \left[M^2_{h,2}; (\beta_1 \circ \beta_2)^{-1}(\text{diag}(M) \times \{t = 0\})\right], \ \beta : M^2_h \to (\overline{M})^2 \times [0, \infty)_t$$

with β being the iterated blow-down map. Note that the heat space has then one more boundary hypersurface td, implying that our heat space has the family of boundary hypersurfaces $\mathscr{M}_1(M^2_h) = \{\text{lf}, \text{rf}, \text{tb}, \text{ff}, \text{fd}, \text{td}\}$.

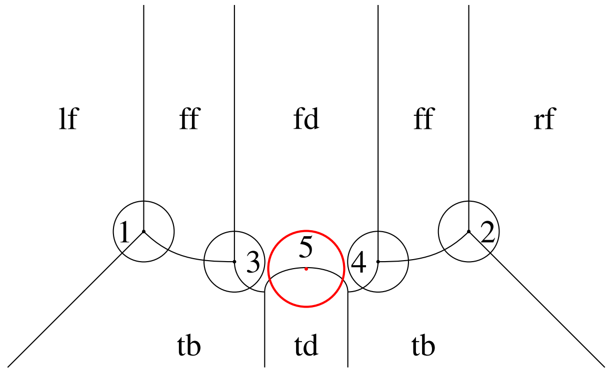

Figure 2.6: Third blow-up

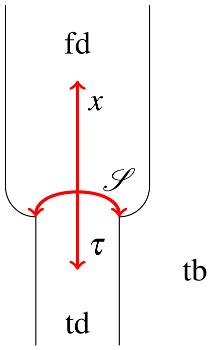

Figure 2.7: Coordinates around the middle regimes

The projective coordinates given then near the intersection of fd and td (which is represented by "regime 5" in Figure 2.6) are

$$\left(\tau, x, y, z, \frac{\mathscr{S}'}{\tau}, \frac{\mathscr{U}'}{\tau}, \frac{\mathscr{Z}'}{\tau}\right) =: (\tau, x, y, z, \mathscr{S}, \mathscr{U}, \mathscr{Z}). \tag{2.28}$$

with $\rho_{\text{fd}} = x$, $\rho_{\text{td}} = \tau$ and $\|(\mathscr{S}, \mathscr{U}, \mathscr{Z})\| \to \infty$ implies leaving the region near the intersection fd$\cap$td.

Remark 2.11. It is still possible to define the projective coordinates near the intersection tb$\cap$td away from fd by taking $(x, y, z, (\widetilde{x}-x)/\tau, (\widetilde{y}-y)/\tau, (\widetilde{z}-z)/\tau, \tau)$.

2.4.2 Asymptotics for the Heat Kernel on M_h^2

First, recall that the heat operator H is the inverse of the differential operator $(\partial_t - \Delta_\Phi)$, that is, $(\partial_t - \Delta_\Phi)\mathbf{H}u = u$. Since the heat operator is an integral operator, it is known that there is a function H smooth function on the open interior of the heat space $M_h^2 = [0, \infty) \times \overline{M}^2$, called the heat kernel of $\mathbf{H}$, such that

$$\mathbf{H}u(t, p) = \int_M H(t, p, \widetilde{p}) u(\widetilde{p})\, \mathrm{dvol}_\Phi(\widetilde{p}), \tag{2.29}$$

for every u. Therefore, understanding H is enough for understanding $\mathbf{H}$.

The point of having a full description of M_h^2 is that, when endowed with the blow-down map β, one can now consider the lift $\beta^* H$ of the heat kernel H to M_h^2. Since β is a diffeomorphism between the interiors of the manifolds, β is then just as a change of coordinates on the interior of $(\overline{M})^2 \times [0, \infty)_t$;

thus, once our work is focused on the interior of the manifolds, it is reasonable to consider the lift in order to do the analysis of the heat kernel.

For every boundary face in M_h^2, as displayed in Figure 2.6, let ρ with a subscript denote the boundary defining function of the face denoted in the subscript.

Theorem 2.12. *[TV21, Theorem 7.2] Let (M, g_Φ) be an m-dimensional complete manifold with fibered boundary endowed with a Φ-metric. Denote by H the heat kernel associated to the Friederichs extension of the Laplacian. The lift β^*H is a polyhomogeneous conormal distribution on M_h^2 with asymptotic behavior described by*

$$\beta^* H \sim \rho_{\mathrm{lf}}^{\infty} \rho_{\mathrm{ff}}^{\infty} \rho_{\mathrm{rf}}^{\infty} \rho_{\mathrm{tb}}^{\infty} \rho_{\mathrm{fd}}^{0} \rho_{rd}^{-m} G \tag{2.30}$$

with G_0 being a bounded function, meaning that β^*H is of leading order $-m$ on td*, bounded on* fd *and vanishes to infinite order on* lf, ff, rf *and* td.

It is worth noting that Theorem 2.12 provides info only on the lower order terms on the asymptotics for β^*H. However, every subsequent term on the asymptotics is less singular then the ones described.

Asymptotics near the intersection $\mathrm{lf} \cap \mathrm{ff} \cap \mathrm{tb}$

Recall that the coordinates valid for this region are $(\tau, x, y, z, \widetilde{s}, \widetilde{y}, \widetilde{z})$, for $\tau = \sqrt{t}$ and $\widetilde{s} = \widetilde{x}/x$. One has

$$\beta^*(x^2\partial_x) = x^2\partial_x - x\widetilde{s}\partial_{\widetilde{s}},\ \beta^*(x\partial_y) = x\partial_y,\ \beta^*\partial_z = \partial_z, \tag{2.31}$$

$$\beta^*\partial_t = \frac{1}{2\tau}\partial_\tau. \tag{2.32}$$

On the other hand, it follows from Theorem 2.12 that the lift of the heat kernel is given by $\beta^*H = (x\widetilde{s}\tau)^\infty G_0$ with G_0 bounded (on it's lower order term). Hence, since $\beta^*\mathcal{V}_\Phi$ above are all in $\mathcal{V}_{\mathrm{b}}$, it follows that

$$\beta^*(VH) = (x\widetilde{s}\tau)^\infty G_0, \tag{2.33}$$

with G_0 bounded, for any $V \in \{\mathrm{id}\} \cup \mathcal{V}_\Phi \cup \mathcal{V}_\Phi^2$. On the other hand, the lift of the volume form is given by

$$\beta^* \mathrm{dvol}_\Phi(\widetilde{x}, \widetilde{y}, \widetilde{z}) = h_0(x\widetilde{s}, \widetilde{y}, \widetilde{z}) x^{-1-b} \widetilde{s}^{-2-b} \,\mathrm{d}\widetilde{s}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{z}. \tag{2.34}$$

Asymptotics near the intersection $\mathrm{rf} \cap \mathrm{ff} \cap \mathrm{tb}$

Near the extreme right corner of M_h^2, coordinates are $(\tau, s, y, z, \widetilde{x}, \widetilde{y}, \widetilde{z})$, for $\tau = \sqrt{t}$ and $s = x/\widetilde{x}$. Thus, follows that

$$\beta^*(x^2\partial_x) = \widetilde{x}s^2\partial_s,\ \beta^*(x\partial_y) = \widetilde{x}s\partial_y,\ \beta^*\partial_z = \partial_z, \tag{2.35}$$

$$\beta^* \partial_t = \frac{1}{2\tau}\partial_\tau. \tag{2.36}$$

Similarly to the previous case, Theorem 2.12 implies that $\beta^* H \sim (\widetilde{x} s \tau)^\infty$. Hence, because $\beta^* \mathcal{V}_\Phi$ lie in $\mathcal{V}_b$, one has

$$\beta^*(VH) = (\widetilde{x} s \tau)^\infty G_0, \tag{2.37}$$

with G_0 bounded, for any $V \in \{\mathrm{id}\} \cup \mathcal{V}_\Phi \cup \mathcal{V}_\Phi^2$. On the other hand, the lift of the volume form is

$$\beta^* \,\mathrm{dvol}_\Phi(\widetilde{x},\widetilde{y},\widetilde{z}) = h_0(\widetilde{x},\widetilde{y},\widetilde{z})\widetilde{x}^{-2-b}\,\mathrm{d}\widetilde{x}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{z}. \tag{2.38}$$

Asymptotics near ff ∩ fd ∩ tb

Around the boundary hypersurface, coordinates are $\left(\tau, x, y, z, \widetilde{\mathscr{S}}', \widetilde{\mathscr{U}}', \widetilde{\mathscr{Z}}'\right)$, with

$$\widetilde{\mathscr{S}}' = \frac{\widetilde{x}-x}{x^2},\ \widetilde{\mathscr{U}}' = \frac{\widetilde{y}-y}{x},\ \widetilde{\mathscr{Z}}' = \widetilde{z}-z,\ \tau = \sqrt{t}. \tag{2.39}$$

Hence, Φ-derivatives lift via β as follow:

$$\beta^*(x^2\partial_x) = x^2\partial_x - (1+2x\widetilde{\mathscr{S}}')\partial_{\widetilde{\mathscr{S}}'} - x\widetilde{\mathscr{U}}'\partial_{\widetilde{\mathscr{U}}'}, \tag{2.40}$$

$$\beta^*(x\partial_y) = x\partial_y - \partial_{\widetilde{\mathscr{U}}'},\ \beta^*\partial_z = \partial_z, \tag{2.41}$$

$$\beta^*\partial_t = \frac{1}{2\tau}\partial_\tau. \tag{2.42}$$

From Theorem 2.12, one knows that $\beta^* H = \tau^\infty x^0 G_0$, where G_0 is a bounded term that vanishes to infinite order whenever $\|(\widetilde{\mathscr{S}}', \widetilde{\mathscr{U}}', \widetilde{\mathscr{Z}}')\| \to \infty$; this can be understood as the behavior of the heat kernel when getting further away from the diagonal of $(\overline{M})^2$ (which meas approaching ff in this regime). Hence, the "worst case scenario" is

$$\beta^*(VH) = \tau^\infty G_0, \tag{2.43}$$

with G_0 still vanishing to infinite order whenever $\|(\widetilde{\mathscr{S}}', \widetilde{\mathscr{U}}', \widetilde{\mathscr{Z}}')\| \to \infty$, for $V \in \{\mathrm{id}\} \cup \mathcal{V}_\Phi \cup \mathcal{V}_\Phi^2$. On the other hand, the lift of the volume form is given by

$$\beta^* \,\mathrm{dvol}_\Phi(\widetilde{x},\widetilde{y},\widetilde{z}) = 2h_0(x+x^2\widetilde{\mathscr{S}}', y+x\widetilde{\mathscr{U}}', z+\widetilde{\mathscr{Z}}')(1+\widetilde{\mathscr{S}}'x)^{-2-b}\,\mathrm{d}\widetilde{\mathscr{S}}'\,\mathrm{d}\widetilde{\mathscr{U}}'\,\mathrm{d}\widetilde{\mathscr{Z}}'. \tag{2.44}$$

Asymptotics near fd ∩ td

Near the middle regimes of M_h^2, the coordinates are $\left(\tau, x, y, z, \widetilde{\mathscr{S}}, \widetilde{\mathscr{U}}, \widetilde{\mathscr{Z}}\right)$, with

$$\widetilde{\mathscr{S}} = \frac{\widetilde{x}-x}{x^2\tau},\ \widetilde{\mathscr{U}} = \frac{\widetilde{y}-y}{x\tau},\ \widetilde{\mathscr{Z}} = \frac{\widetilde{z}-z}{\tau},\ \tau = \sqrt{t}. \tag{2.45}$$

Thus, the lift of Φ-derivatives via β near the middle of M_h^2 are given by

$$\beta^*(x^2\partial_x) = x^2\partial_x - \left(\frac{1}{\tau} + 2x\widetilde{\mathscr{S}}\right)\partial_{\widetilde{\mathscr{S}}} - x\widetilde{\mathscr{U}}\partial_{\widetilde{\mathscr{U}}}, \tag{2.46}$$

$$\beta^*(x\partial_y) = x\partial_y - \frac{1}{\tau}\partial_{\widetilde{\mathscr{U}}}, \tag{2.47}$$

$$\beta^*\partial_z = \partial_z - \frac{1}{\tau}\partial_{\widetilde{\mathscr{Z}}}, \tag{2.48}$$

$$\beta^*\partial_t = \frac{1}{2\tau}\partial_\tau - \frac{1}{2\tau^2}\widetilde{\mathscr{S}}\partial_{\widetilde{\mathscr{S}}} - \frac{1}{2\tau^2}\widetilde{\mathscr{U}}\partial_{\widetilde{\mathscr{U}}} - \frac{1}{2\tau^2}\widetilde{\mathscr{Z}}\partial_{\widetilde{\mathscr{Z}}}. \tag{2.49}$$

If follows from Theorem 2.12 that, when near the middle regimes in M_h^2, $\beta^*H = \tau^{-m}x^0G_0$, with G_0 bounded and vanishing to infinite order whenever $\|(\widetilde{\mathscr{S}}, \widetilde{\mathscr{U}}, \widetilde{\mathscr{Z}})\| \to \infty$, which reflects the behavior of the heat kernel when approaching ff. Hence, the "worst case scenario" is

$$\beta^*(VH) = \tau^{-m-2}G_0, \tag{2.50}$$

with G_0 vanishing to infinite order whenever $\|(\widetilde{\mathscr{S}}, \widetilde{\mathscr{U}}, \widetilde{\mathscr{Z}})\| \to \infty$, for $V \in \{\mathrm{id}\} \cup \mathscr{V}_\Phi \cup \mathscr{V}_\Phi^2$. For the volume form, one has now the expression

$$\beta^*\,\mathrm{dvol}_\Phi(\widetilde{x}, \widetilde{y}, \widetilde{z}) = 2h_0(x + x^2\tau\widetilde{\mathscr{S}}, y + x\tau\widetilde{\mathscr{U}}, z + \tau\widetilde{\mathscr{Z}})(1 + \widetilde{\mathscr{S}}x\tau)^{-2-b}\tau^m\,\mathrm{d}\widetilde{\mathscr{S}}\,\mathrm{d}\widetilde{\mathscr{U}}\,\mathrm{d}\widetilde{\mathscr{Z}}. \tag{2.51}$$

CHAPTER 3

Yamabe flow on Φ-manifolds

The main goal of this chapter is to study the Yamabe flow in the context of Φ-manifolds. Therefore, once defined the conformal class of our interest along this work, let us prove that every Riemannian metric evolving along the flow preserves geometry adapted Hölder-continuity, that is, Hölder spaces defined in terms of Φ-vector fields and the distance function related to g_Φ. This is relevant because it will allow us to use the tools developed in the previous chapter for every metric $g(t)$ along the flow, for any $t > 0$.

Yamabe flow for the conformal factor

The Yamabe flow preserves the conformal class of the metric and can be written as a scalar evolution equation for the conformal factor. More precisely, assume $m := \dim M \geq 3$ and set $\eta := (m-2)/4$. Writing $g(t) = u(t)^{1/\eta} g$, the scalar curvature of $g(t)$ can be computed by (Δ is the negative Laplace Beltrami operator of (M,g))

$$\mathrm{scal}(g(t)) = -u(t)^{-(1+1/\eta)} \left[\frac{m-1}{\eta} \Delta u(t) - \mathrm{scal}(g) u(t) \right]. \tag{3.1}$$

In view of this relation, the Yamabe flow (2) turns into

$$\begin{aligned} \partial_t u(t)^{(m+2)/(m-2)} &= \frac{m+2}{m-2} \Big((m-1)\Delta u(t) - \eta\, \mathrm{scal}(g) u(t) \Big) \\ \Leftrightarrow \quad \partial_t u(t) &= (m-1) u(t)^{-1/\eta} \Delta u(t) - \eta\, \mathrm{scal}(g) u(t)^{1-1/\eta}, \end{aligned} \tag{3.2}$$

with the initial condition $u|_{t=0} = 1$.

Normalized Yamabe flows for the conformal factor

Similar computations as those leading to (3.2) yield the following scalar evolution equation for the conformal factor under the volume normalized Yamabe flow

$$\partial_t u(t) - (m-1) u(t)^{-1/\eta} \Delta u(t) = \eta \Big(\rho(t) u(t) - \mathrm{scal}(g) u(t)^{1-1/\eta} \Big). \tag{3.3}$$

The curvature normalized flows in (5) are similarly given by

$$\begin{aligned} &\partial_t u(t) - (m-1)u(t)^{-1/\eta}\Delta u(t) \\ &\qquad = \eta\left(\sup_M \operatorname{scal}(g(t))\cdot u(t) - \operatorname{scal}(g)u(t)^{1-1/\eta}\right), \qquad (\mathrm{CYF}^+) \\ &\partial_t u(t) - (m-1)u(t)^{-1/\eta}\Delta u(t) \\ &\qquad = \eta\left(\inf_M \operatorname{scal}(g(t))\cdot u(t) - \operatorname{scal}(g)u(t)^{1-1/\eta}\right). \qquad (\mathrm{CYF}^-) \end{aligned} \tag{3.4}$$

3.1 Hölder continuity

In this section, we introduce geometry-adapted Hölder spaces, which form a family of Banach spaces that provide information about both the regularity of a function and its variation proportionally to the distance between points, generalizing the concept of Lipschitz continuity. Moreover, we define these spaces in a way that the natural family of vector fields on (M, g_Φ), i.e. Φ-vector fields, gives the family of differential operators considered for defining regularity of a function.

Definition 3.1. The Hölder space $C^\alpha_\Phi(M\times[0,T])$, for $\alpha\in(0,1)$, is defined as the space of bounded and continuous functions $u\in C^0(M\times[0,T])$ which satisfy

$$[u]_\alpha := \sup_{M_T^2}\left\{\frac{|u(p,t)-u(p',t')|}{d_\Phi(p,p')^\alpha + |t-t'|^{\alpha/2}}\right\} < \infty, \tag{3.5}$$

where the supremum is taken over $M_T^2 := M_T\times M_T$, with $M_T := M\times[0,T]$. The distance function d_Φ is induced by the metric g_Φ and is equivalently given in local coordinates (x,y,z) in U by the following local expression:

$$d_\Phi((x,y,z),(x',y',z')) \approx \sqrt{\left(\frac{|x-x'|}{(x+x')^2}\right)^2 + \left(\frac{\|y-y'\|}{(x+x')}\right)^2 + \|z-z'\|^2}. \tag{3.6}$$

The Hölder norm of any $u\in C^\alpha_\Phi(M\times[0,T])$ is defined by

$$\|u\|_\alpha := \|u\|_\infty + [u]_\alpha. \tag{3.7}$$

The resulting normed vector space $C^\alpha_\Phi(M\times[0,T])$ is a Banach space.

Note that here we do not require the functions to be continuous up to the boundary. Hölder spaces with a stronger control on the boundary behavior will be considered in Chapter 4.

Below, we prove a useful result that proves the Hölder norm $\|\cdot\|_\alpha$ to be equivalent to a slightly more restrictive Hölder norm, which considers only spacial and time differences taken within small local regions.

Lemma 3.2. *The following equation defines an equivalent norm on* $C^\alpha_\Phi(M\times[0,T])$ *:*

$$\|u\|'_\alpha := \|u\|_\infty + [u]'_\alpha, \quad [u]'_\alpha := \sup_{M_{T,\delta}^2}\left\{\frac{|u(p,t)-u(p',t')|}{d_\Phi(p,p')^\alpha + |t-t'|^{\alpha/2}}\right\}, \tag{3.8}$$

where $M^2_{T,\delta} := \{(p,t),(p',t') \in M_T \mid d_\Phi(p,p')^\alpha + |t-t'|^{\alpha/2} \leq \delta\}$. More precisely, we have the following relation between the two norms

$$\|u\|'_\alpha \leq \|u\|_\alpha \leq (1+2\delta^{-1})\|u\|'_\alpha.$$

Proof. It is clear from definition that $\|u\|'_\alpha \leq \|u\|_\alpha$. To prove the second estimate, simply note for any $u \in C^\alpha_\Phi(M\times[0,T])$ and any $(p,t),(p',t') \in M_T$ with

$$d_\Phi(p,p')^\alpha + |t-t'|^{\alpha/2} \geq \delta,$$

we can estimate the Hölder differences as follows

$$\frac{|u(p,t)-u(p',t')|}{d_\Phi(p,p')^\alpha + |t-t'|^{\alpha/2}} \leq \frac{|u(p,t)-u(p',t')|}{\delta} \leq 2\delta^{-1}\|u\|_\infty.$$

From this, we can compute

$$\begin{aligned}\|u\|_\alpha \leq &\|u\|_\infty + [u]'_\alpha + \sup_{M^2_T\setminus M^2_{T,\delta}} \left\{ \frac{|u(p,t)-u(p',t')|}{d_\Phi(p,p')^\alpha + |t-t'|^{\alpha/2}} \right\} \\ \leq &[u]'_\alpha + (1+2\delta^{-1})\|u\|_\infty \\ \leq &(1+2\delta^{-1})\|u\|'_\alpha,\end{aligned}$$

completing the proof. □

From now on we only use the Hölder norm $\|u\|'_\alpha$ defined in (3.8), which we denote from now on without the apostrophe. We also define the higher order Hölder spaces for any given $k \in \mathbb{N}$ by

$$C^{k,\alpha}_\Phi(M\times[0,T]) = \left\{ u \in C^k_\Phi(M\times[0,T]) \;\middle|\; \begin{array}{l} (V\circ\partial_t^{l_2})u \in C^\alpha_\Phi(M\times[0,T]), \\ \text{for } V \in \mathrm{Diff}^{l_1}_\Phi(M),\ \ l_1+2l_2 \leq k \end{array} \right\} \tag{3.9}$$

which is a Banach space (cf. [BV14, Proposition 3.1]) with the norm

$$\|u\|_{k,\alpha} := \|u\|_\alpha + \sum_{l_1+2l_2\leq k} \sum_{V\in\mathscr{V}^{l_1}_\Phi} \|(V\circ\partial_t^{l_2})u\|_\alpha. \tag{3.10}$$

Moreover, we can generalize further the definition above by introducing the weighted Hölder space $x^\gamma C^{k,\alpha}_\Phi(M\times[0,T])$, which is defined as the space of functions $u = x^\gamma v$ with $v \in C^{k,\alpha}_\Phi(M\times[0,T])$ and endowed with the norm $\|u\|_{k,\alpha,\gamma} := \|v\|_{k,\alpha}$.

Remark 3.3. Sometimes, we will also use Hölder spaces for functions depending either only on spacial variables or on time variables, denoted as $C^{k,\alpha}_\Phi(M)$ and $C^{k,\alpha}_\Phi([0,T])$, with Hölder brackets (for $k=0$)

$$[u]_\alpha = \sup_{M^2} \frac{|u(p)-u(p')|}{d_\Phi(p,p')^\alpha} \text{ and } [u]_\alpha = \sup_{[0,T]^2} \frac{|u(t)-u(t')|}{|t-t'|^{\alpha/2}},$$

respectively.

The next proposition shows that every $x^\gamma C^{k,\alpha}_\Phi(M\times[0,T])$ is closed under multiplication of elements in $C^{k,\alpha}_\Phi(M\times[0,T])$.

Proposition 3.4. *Let $\varphi\in C^{k+l,\alpha}_\Phi(M\times[0,T])$. Then the multiplication operator*

$$M_\varphi : x^\gamma C^{k,\alpha}_\Phi(M\times[0,T]) \to x^\gamma C^{k,\alpha}_\Phi(M\times[0,T]) \tag{3.11}$$

acts continuously, for all $l\geq 0$ and $\gamma\in\mathbb{R}$.

Proof. Note that the general case follows as a straightforward generalization of the case $k=l=0$ simply by employing the product rule for derivatives and using the ideas presented below. Therefore, computations are made assuming $k=l=0$.

Let $u=x^\gamma v$, where $v\in C^\alpha_\Phi(M\times[0,T])$. From definition, we know that $\|u\|_{\alpha,\gamma}=\|v\|_\alpha$ and thus, we have the following:

$$\|M_\varphi u\|_{\alpha,\gamma}=\|x^{-\gamma}\varphi x^\gamma v\|_\infty+[x^{-\gamma}\varphi x^\gamma v]_\alpha\leq\|\varphi\|_\infty\|v\|_\infty+[\varphi v]_\alpha.$$

Let us now estimate specifically the second term above. The triangular inequality gives us

$$\begin{aligned}[\varphi v]_\alpha =&\sup\frac{|\varphi u(p,t)-\varphi u(p',t')|}{d(p,p')^\alpha+|t-t'|^{\alpha/2}}\\ \leq&\sup\frac{|v(p,t)(\varphi(p,t)-\varphi(p',t'))|}{d(p,p')^\alpha+|t-t'|^{\alpha/2}}+\sup\frac{|\varphi(p',t')(v(p,t)-v(p',t'))|}{d(p,p')^\alpha+|t-t'|^{\alpha/2}}\\ \leq&\|v\|_\infty[\varphi]_\alpha+\|\varphi\|_\infty[v]_\alpha.\end{aligned}$$

Hence, it follows from the definitions of $\|.\|_\alpha$ and $\|.\|_{\alpha,\gamma}$ that

$$\|M_\varphi u\|_{\alpha,\gamma}\leq\|\varphi\|_\alpha\|u\|_{\alpha,\gamma}.$$

□

Remark 3.5. We point out that the previous result can be altered to functions $\varphi\in x^\gamma C^{k,\alpha}_\Phi(M\times[0,T])$ under the assumption that $\gamma\geq 0$. Moreover, the definition of the distance function is irrelevant to the proof as presented above, which means that this result holds to Hölder spaces defined for different distance functions.

3.1.1 Conformal transformation by Hölder functions

Since the Yamabe flow preserves the conformal class of the metric, we need to look into the effect of conformal transformation by Hölder functions. We first define the conformal class of a Φ-metric g_Φ (we tacitly assume $m\geq 3$)

$$[g_\Phi]=\left\{u^{4/(m-2)}\cdot g_\Phi \;\middle|\; u\in C^2_\Phi(M),\ 0<\inf_M u\leq\sup_M u<+\infty\right\}. \tag{3.12}$$

First, observe that any representative g of the conformal class $[g_\Phi]$ is a complete Riemannian metric on M. In fact, the bounds required on u imply the distance function d_g to be uniformly equivalent to the distance function d_Φ. This means that a sequence $\{p_k\}_k \subset M$ is a Cauchy sequence in (M, d_g) iff is a Cauchy sequence in (M, d_Φ), which is a complete metric space (by the Hopf-Rinow theorem), from which it follows the conclusion.

On the other hand, a generic element of the conformal class $[g_\Phi]$ is *not* necessarily a Φ-metric in the sense of Definition 2.2, since the conformal factor $u^{4/(m-2)}$ cannot in general be expected to admit a partial asymptotic expansion as $x \to 0$. However, if u is bounded away from zero and bounded from above, then it still has $\mathscr{V}_\Phi$ as the space of "bounded vector fields" and thus the distance functions defined with respect to any $g \in [g_\Phi]$ are equivalent. In fact, bounded vector fields on M are understood as to ones which satisfy

$$g_\Phi(V, W) = O(1).$$

It follows directly from the local expressions of g_Φ and $\mathscr{V}_\Phi$ that

$$g_\Phi(\mathscr{V}_\Phi, \mathscr{V}_\Phi) = O(1) \implies u^{1/\eta} \cdot g_\Phi(\mathscr{V}_\Phi, \mathscr{V}_\Phi) = O(1),$$

since u is assumed to be bounded. On the other hand, one knows that

$$d_g(p, p') = \inf\{\text{length}_g(\gamma) \mid \gamma : [0,1] \to M,\ \gamma(0) = p \text{ and } \gamma(1) = p'\}.$$

However, since u is both bounded away from zero (that is, $\inf_M u > 0$) and $\|u\|_\infty < \infty$, we get

$$\text{length}_g(\gamma) = \int_0^t \|\gamma'(t)\|_g \,\mathrm{d}t = \int_0^t u^{1/2\eta} \cdot \|\gamma'(t)\|_\Phi \,\mathrm{d}t \sim \text{length}_\Phi(\gamma).$$

In that sense g still has the same Φ-geometry as g_Φ and we conclude

Proposition 3.6. *The Hölder spaces defined in §3.1 do not depend on the choice of a metric $g \in [g_\Phi]$, if the conformal factor u satisfies $\inf_M u > 0$ and $\|u\|_\infty < \infty$.*

3.1.2 Embedding between Hölder spaces

In this section, we present a few results regarding embedding between Hölder spaces. They will prove themselves useful later on.

Proposition 3.7. *Let (M, g_Φ) be a Φ-manifold and $0 < \beta < \alpha < 1$. Then the inclusion*

$$\iota : C^{k,\alpha}_\Phi(M) \hookrightarrow C^{k',\beta}_\Phi(M) \tag{3.13}$$

is bounded, for all $k' \leq k$.

Proof. Let us consider $u \in C^{k,\alpha}_\Phi(M)$. From definition, it follows directly that $u \in C^k_\Phi(M)$. Thus, it is left for us to check that $[Vu]_\beta < \infty$ for all $V \in \mathscr{V}^l_\Phi$, with $l \leq k'$. To do so, let us assume $l = 0$ for now. Then

$$[u]_\beta = \sup \frac{|u(p) - u(p')|}{d_\Phi(p,p')^\beta} \leq \sup_{d_\Phi(p,p') \leq 1} \frac{|u(p) - u(p')|}{d_\Phi(p,p')^\beta} + \sup_{d_\Phi(p,p') > 1} \frac{|u(p) - u(p')|}{d_\Phi(p,p')^\beta}$$

$$=: A + B.$$

Hence, if A and B are proven finite and, moreover, comparable to $\|u\|_{k,\alpha}$, then our first step will be complete. For estimating the term A, it is enough to note that, since $d(p,p') \le 1$, then

$$A = \sup_{d_\Phi(p,p')\le 1} d_\Phi(p,p')^{\alpha-\beta} \frac{|u(p)-u(p')|}{d_\Phi(p,p')^\alpha} \le [u]_\alpha \le \|u\|_{k,\alpha}.$$

For the term B, we proceed as follows:

$$B \le \sup_{d_\Phi(p,p')>1} \frac{|u(p)|+|u(p')|}{d_\Phi(p,p')^\beta} \le 2\|u\|_\infty \le 2\|u\|_{k,\alpha}.$$

Thus, it follows that $[u]_\beta \le C'\|u\|_{k,\alpha}$ for some constant $C' > 0$. For the case $0 < l < k'$, it is enough to see that the argument provided for $l = 0$ already implies $[Vu]_\beta \le C'\|Vu\|_{k-l,\alpha}$, for every $V \in \mathcal{V}_\Phi^l$. Hence, this implies $\|u\|_{k',\beta} \le C\|u\|_{k,\alpha}$, completing the proof. □

It should be noted that the proof above does not depend on the definition of the distance function d_Φ. Therefore, if one considers a Hölder space defined with a different distance function, the conclusion still holds.

Remark 3.8. First, note that the proof given for Proposition 3.7 can be naturally generalized for weighted Hölder spaces, that is, $\iota : x^\gamma C_\Phi^{k,\alpha}(M) \hookrightarrow x^\gamma C_\Phi^{k',\beta}(M)$. On the other hand, for K a compact manifold, we point out that classical PDE theory implies that $\iota : C^{k,\alpha}(K) \hookrightarrow C^{k',\beta}(K)$ is a compact embedding. Hence, if K is a compact submanifold of M away from the boundary $\partial\overline{M}$, it follows that

$$\iota : C_\Phi^{k,\alpha}(K) \hookrightarrow C_\Phi^{k',\beta}(K)$$

is a compact embedding.

Next, we obtain the following compactness result.

Proposition 3.9. *Consider any* $0 < \beta < \alpha < 1$ *and* $\gamma > 0$. *Then the following inclusion is compact*

$$\iota : C_\Phi^{k,\alpha}(M) \hookrightarrow x^{-\gamma} C_\Phi^{k,\beta}(M). \tag{3.14}$$

Proof. Let $\{u_n\}_n$ be a bounded sequence of functions in $C_\Phi^{k,\alpha}(M)$ and, for any $\delta > 0$, let M_δ be the compact submanifold given by

$$M_\delta = M \setminus \{p \in M \,|\, x(p) < \delta\}. \tag{3.15}$$

We know that $C_\Phi^{k,\alpha}(M_\delta) \hookrightarrow C_\Phi^{k,\beta}(M_\delta)$ compactly for any $\delta > 0$. Therefore, $\{u_n|_{M_\delta}\}_n$ admits a subsequence $\{u_{n_j(\delta)}|_{M_\delta}\}_j$ which converges in $C_\Phi^{k,\beta}(M_\delta)$. Now consider a sequence $\delta_i := 1/i$ for $i \in \mathbb{N}$. We define convergent subsequences in $C_\Phi^{k,\beta}(M_{\delta_i})$ for any i by an iterative procedure: given a convergent subsequence $\{u_{n_j(\delta_i)}|_{M_{\delta_i}}\}_j \subset C_\Phi^{k,\beta}(M_{\delta_i})$, we choose a convergent subsequence $\{u_{n_j(\delta_{i+1})}|_{M_{\delta_{i+1}}}\}_j \subset C_\Phi^{k,\beta}(M_{\delta_{i+1}})$ from $\{u_{n_j(\delta_i)}|_{M_{\delta_{i+1}}}\}_j$. Define the diagonal sequence by

$$\{v_j := u_{n_j(\delta_j)}\}_j. \tag{3.16}$$

We claim that $\{v_j\}_j$ is a Cauchy sequence in $x^{-\gamma}C^{k,\beta}_{\Phi}(M)$. In fact

$$\|v_j\|_{x^{-\gamma}C^{k,\beta}_{\Phi}(M\setminus M_{\delta_j})} = \|x^{\gamma}v_j\|_{C^{k,\beta}_{\Phi}(M\setminus M_{\delta_j})} \le Cj^{-\gamma},$$

where $C>0$ is an upper bound for the norms of $\{u_n\}_n \subset C^{k,\alpha}_{\Phi}(M)$. Now, let $\varepsilon>0$ and choose $j_0\in\mathbb{N}$ sufficient large such that $C\delta_{j_0}^{\gamma}\le\varepsilon/4$. The sequence $\{v_j|_{M_{\delta_{j_0}}}\}\subset C^{k,\beta}_{\Phi}(M_{\delta_{j_0}})$ converges by construction and thus converges also in $x^{-\gamma}C^{k,\beta}_{\Phi}(M_{\delta_{j_0}})$. Hence, there exists some $N_0\in\mathbb{N}$ sufficiently large, such that for every $j,j'\ge N_0$

$$\|v_j - v_{j'}\|_{x^{-\gamma}C^{k,\beta}_{\Phi}(M_{\delta_{j_0}})} \le \varepsilon/2. \tag{3.17}$$

Hence for $J_0=\max\{j_0,N_0\}$, we have for any $j,j'\ge J_0$

$$\begin{aligned}\|v_j - v_s\|_{x^{-\gamma}C^{k,\beta}_{\Phi}(M)} &\le \|v_j - v_s\|_{x^{-\gamma}C^{k,\beta}_{\Phi}(M_{\delta_{j_0}})} + \|v_j - v_s\|_{x^{-\gamma}C^{k,\beta}_{\Phi}(M\setminus M_{\delta_{j_0}})}\\ &< \varepsilon/2+\varepsilon/2=\varepsilon.\end{aligned}$$

Hence, $\{v_j\}$ is a Cauchy sequence in $x^{-\gamma}C^{k,\beta}_{\Phi}(M)$ and by completeness, it admits a convergent subsequence. This proves the statement. □

3.2 Mapping properties of the Heat operator on $C^{\alpha}_{\Phi}(M\times[0,T])$

In this section, we establish parabolic Schauder estimates using the heat operator oh Φ-manifolds. This section is not new, but rather an exercise for the estimates in §4.2. To be more precise, the estimates here are done with respect to the Hölder norms that are equivalent to the usual Hölder norms defined on any manifold with bounded geometry. The parabolic Schauder estimates here then follow by classical estimates in §3.8. Nevertheless, it is interesting to see how on arrives at the same conclusions using microlocal arguments in the Φ-setting.

Consider the homogeneous and inhomogeneous heat equations for some compactly supported smooth functions $v\in C^{\infty}_{\Phi,0}(M\times[0,T])$ and $u_0\in C^{\infty}_{\Phi,0}(M)$

$$\begin{aligned}(\partial_t-\Delta_\Phi)u_{\text{hom}} &= 0, \quad u_{\text{hom}}|_{t=0}=u_0,\\ (\partial_t-\Delta_\Phi)u_{\text{inhom}} &= v, \quad u_{\text{inhom}}|_{t=0}=0.\end{aligned} \tag{3.18}$$

We denote the heat operator corresponding to the unique self-adjoint extension of Δ_Φ in L^2 by $\mathbf{H}$, while its Schwartz kernel is denoted by H. Note that such extension is unique, since (M,g_Φ) is a complete Riemannian manifold (see [Str83, Theorem 2.4]). Then the solutions u_{hom} and u_{inhom} are given by

$$\begin{aligned}u_{\text{hom}}(p,t) &= (\mathbf{H}u_0)(p,t) = \int_M H(t,p,\widetilde{p})u_0(\widetilde{p})\mathrm{dvol}_\Phi(\widetilde{p}),\\ u_{\text{inhom}}(p,t) &= (\mathbf{H}\star v)(p,t) = \int_0^t\int_M H(t-\widetilde{t},p,\widetilde{p})v(\widetilde{p},\widetilde{t})\mathrm{dvol}_\Phi(\widetilde{p})d\widetilde{t},\end{aligned} \tag{3.19}$$

where dvol_Φ denotes the volume form of g_Φ.

Before studying the Yamabe flow itself, we must first prove certain mapping properties for the heat operator acting by convolution in time, that is, the map $\mathbf{H}\star$ above. To do this, we will use the information on the heat kernel provided by Theorem 2.12. This will provide to us important tools for the argument on the short-time existence of the flow.

Theorem 3.10. *The heat operator $\mathbf{H}\star$, acting by convolution in time, defines, for any $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and any $\gamma \in \mathbf{R}$, a bounded linear map*

$$\mathbf{H}\star : x^\gamma C^{k,\alpha}_\Phi(M\times[0,T]) \to x^\gamma\Big(C^{k+2,\alpha}_\Phi \cap \sqrt{t}\,C^{k+1,\alpha}_\Phi\Big)(M\times[0,T]). \tag{3.20}$$

Proof. Both mapping properties can be proven true by going along similar lines. Therefore, only the proof for the first of the two will be presented here, while we argue in the last paragraph why the proof first mapping property implies the latter.

We start the proof with the case $k=0$, that is,

$$\mathbf{H}\star : x^\gamma C^{\alpha}_\Phi(M\times[0,T]) \to x^\gamma C^{2,\alpha}_\Phi(M\times[0,T]).$$

The more general follows directly from the case $k=0$ with the additional use of integration by parts and by the vanishing order of the heat kernel on the boundary $\partial\overline{M}$. In fact, the vanishing order of the heat kernel implies that, for every $\overline{u} \in x^\gamma C^{k,\alpha}_\Phi(M\times[0,T])$, we have

$$V(\mathbf{H}\star\overline{u}) = \mathbf{H}\star(V\overline{u}), \text{ for all } V \in \mathscr{V}^l_\Phi \text{ with } l \le k.$$

Consequently, the case $k>0$ follows from the case $k=0$. Furthermore, for any $\overline{u} \in x^\gamma C^\alpha_\Phi(M\times[0,T])$, there is some function $u \in C^\alpha_\Phi(M\times[0,T])$ such that $\overline{u} = x^\gamma u$. Particularly, it follows that $\mathbf{H}\star\overline{u}$ lies in $x^\gamma C^{2,\alpha}_\Phi(M\times[0,T])$ if and only if $x^{-\gamma}\mathbf{H}\star x^\gamma u$ lies in $C^{2,\alpha}_\Phi(M\times[0,T])$, which is equivalent to prove that

$$\mathbf{H}_\gamma := M_{x^{-\gamma}}\circ\mathbf{H}\star\circ M_{x^\gamma} : C^\alpha_\Phi(M\times[0,T]) \to C^{2,\alpha}_\Phi(M\times[0,T]), \tag{3.21}$$

with M_{x^γ} being the multiplication by x^γ operator. Note that

$$\begin{aligned}\mathbf{H}_\gamma u(t,p) &= \int_0^t\int_M x^{-\gamma}H(t-\widetilde{t},p,\widetilde{p})(x^\gamma u)(\widetilde{t},\widetilde{p})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &= \int_0^t\int_M x^{-\gamma}H(t-\widetilde{t},p,\widetilde{p})\widetilde{x}^\gamma u(\widetilde{t},\widetilde{p})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &=: \int_0^t\int_M H_\gamma(t-\widetilde{t},p,\widetilde{p})u(\widetilde{t},\widetilde{p})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t},\end{aligned}$$

which means that the operator $\mathbf{H}_\gamma$ has a kernel H_γ. Note that β^*H and β^*H_γ have the same asymptotic behavior on M^2_h, from Theorem 2.12, since H vanishes to infinite order on the corners of M^2_h and near the middle of M^2_h one has $x \sim \widetilde{x}$, implying $(\widetilde{x}/x)^\gamma$ to be bounded. In addition to this, from the definition of the Hölder spaces presented previously in (3.9), the statement above is equivalent to prove that the operator $\mathbf{G}$, given by $\mathbf{G} = V\circ\mathbf{H}_\gamma$ with $V \in \{\mathrm{id}\}\cup\mathscr{V}_\Phi\cup\mathscr{V}^2_\Phi$, acts as a bounded operator

$$\mathbf{G} : C^\alpha_\Phi(M\times[0,T]) \to C^\alpha_\Phi(M\times[0,T]).$$

Hence, given a function u lying in $C^{\alpha}_{\Phi}(M\times[0,T])$ and using Lemma 3.2, the goal is to prove

$$\|\mathbf{G}u\|_{\alpha}\leq c\|u\|_{\alpha} \tag{3.22}$$

for some uniform constant $c>0$. The proof is conducted directly by proving the estimate above for $\|\mathbf{G}u\|_{\alpha}$. From the definition of the α-norm in (3.7) we find

$$\|\mathbf{G}u\|_{\alpha}=[\mathbf{G}u]_{\alpha}+\|\mathbf{G}u\|_{\infty}.$$

Furthermore one can see that

$$[\mathbf{G}u]_{\alpha}\leq \sup_{\substack{p,p'\in M\\ p\neq p'}}\frac{|\mathbf{G}u(p,t)-\mathbf{G}u(p',t)|}{d_{\Phi}(p,p')^{\alpha}}+\sup_{\substack{t,t'\geq 0\\ t\neq t'}}\frac{|\mathbf{G}u(p,t)-\mathbf{G}u(p,t')|}{|t-t'|^{\alpha/2}},$$

leading to

$$\|\mathbf{G}u\|_{\alpha}\leq \sup_{\substack{p,p'\in M\\ p\neq p'}}\frac{|\mathbf{G}u(p,t)-\mathbf{G}u(p',t)|}{d_{\Phi}(p,p')^{\alpha}}+\sup_{\substack{t,t'\geq 0\\ t\neq t'}}\frac{|\mathbf{G}u(p,t)-\mathbf{G}u(p,t')|}{|t-t'|^{\alpha/2}}+\|\mathbf{G}u\|_{\infty}.$$

Therefore, the inequalities aforementioned imply that the estimate (3.22) is satisfied if

$$|\mathbf{G}u(p,t)-\mathbf{G}u(p',t)|\leq c\|u\|_{\alpha}d_{\Phi}(p,p')^{\alpha}, \tag{3.23}$$
$$|\mathbf{G}u(p,t)-\mathbf{G}u(p,t')|\leq c\|u\|_{\alpha}|t-t'|^{\alpha/2}, \tag{3.24}$$
$$|\mathbf{G}u(p,t)|\leq c\|u\|_{\alpha}. \tag{3.25}$$

We will therefore proceed in three steps:

i) Uniform estimates of Hölder differences in space, whose proof is presented in §3.2.1,

ii) Uniform estimates of Hölder differences in time, whose proof is presented in §3.2.2,

iii) Uniform estimates of the supremum norm, whose proof is presented in §3.2.3.

The same argumentation as in the previous result leads to an equivalent formulation of the statement, that is, one has to prove that the operator

$$t^{-1/2}\mathbf{H}_{\gamma}:C^{\alpha}_{\Phi}(M\times[0,T])\to C^{1,\alpha}_{\Phi}(M\times[0,T])$$

is bounded. As in the previous theorem one deduces that the above is equivalent to proving that the operator $\mathbf{G}_t$ given by $\mathbf{G}_t u=V(t^{-1/2}\mathbf{H}_{\gamma}u)$, for $V\in\{\mathrm{id}\}\cup\mathscr{V}_{\Phi}$, maps

$$\mathbf{G}_t:C^{\alpha}_{\Phi}(M\times[0,T])\to C^{\alpha}_{\Phi}(M\times[0,T])$$

is bounded. One has

$$(\mathbf{G}_t u)(p,t)=\int_0^t\int_M V((t-\widetilde{t})^{-1/2}H_{\gamma}(t-\widetilde{t},p,\widetilde{p}))u(\widetilde{p},\widetilde{t})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}.$$

The estimates in §3.2.1, §3.2.2 and §3.2.3 will already cover the case $V \in \{\mathrm{id}\} \cup \mathscr{V}_\Phi$. Moreover it is important to note that we are no longer considering elements in $\mathrm{Diff}^2_\Phi(M)$, which will lead to an extra τ term. On the other hand, the term $(t-\widetilde{t})^{-1/2}$ inside the integrand lifts to an extra τ^{-1} in every region of M^2_h. This means that the presence of the term $(t-\widetilde{t})^{-1/2}$ is proportionally compensated by the absence of elements from $\mathrm{Diff}^2_\Phi(M)$. Therefore if one attempts to get these estimates following the same argumentation as in the upcoming sections, then the integrals obtained will have the exact same asymptotics. □

3.2.1 Estimates of Hölder differences in space

Consider $p, p' \in M$ and write

$$M^+ = \{\widetilde{p} \in M \mid d(p,\widetilde{p}) \leq 3d(p,p')\} \text{ and } M^- = \{\widetilde{p} \in M \mid d(p,\widetilde{p}) \geq 3d(p,p')\}.$$

We shall assume that $p = (x,y,z)$ and $p' = (x',y',z')$, with $x' > x$ without loss of generality. The estimates below will be presented only for the regimes near the middle of the heat space, that is, where fd meets td. This is reasonable because the heat kernel vanishes to infinite order near the extreme corners of the heat space, which makes the estimates in said regions to follow straightforwardly.

Now, for $u \in C^\alpha_\Phi(M \times [0,T])$, write

$$\begin{aligned}
\mathbf{G}u(t,p) - \mathbf{G}u(t,p') = &\int_0^t \int_M G(t-\widetilde{t},p,\widetilde{p})u(\widetilde{p},\widetilde{t})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\
&- \int_0^t \int_M G(t-\widetilde{t},p',\widetilde{p})u(\widetilde{p},\widetilde{t})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\
= &\int_0^t \int_{M^-} \left[G(t-\widetilde{t},p,\widetilde{p}) - G(t-\widetilde{t},p',\widetilde{p})\right]\left[u(\widetilde{p},\widetilde{t}) - u(p,\widetilde{t})\right]\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\
&+ \int_0^t \int_{M^+} \left[G(t-\widetilde{t},p,\widetilde{p}) - G(t-\widetilde{t},p',\widetilde{p})\right]\left[u(\widetilde{p},\widetilde{t}) - u(p,\widetilde{t})\right]\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\
&+ \int_0^t \int_M \left[G(t-\widetilde{t},p,\widetilde{p}) - G(t-\widetilde{t},p',\widetilde{p})\right]u(p,\widetilde{t})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\
=: &I_1 + I_2 + I_3.
\end{aligned}$$

Estimates for I_1

Before proving the estimates for I_1, let us prove quickly the following technical lemma:

Lemma 3.11. *Let p'' be a point in M such that $d_\Phi(p',p'') \leq d_\Phi(p,p')$. For every point $\widetilde{p}$ in M^- one has*

$$\frac{1}{3} d_\Phi(p,\widetilde{p}) \leq d_\Phi(p'',\widetilde{p}).$$

Proof. Using triangular inequality, the assumption on p'' and the fact that $\widetilde{p}$ lies in M^- one has

$$\begin{aligned}
d_\Phi(p,\widetilde{p}) &\leq d_\Phi(p,p') + d_\Phi(p',\widetilde{p}) \leq d_\Phi(p,p') + d_\Phi(p',p'') + d_\Phi(p'',\widetilde{p}) \\
&\leq d_\Phi(p,p') + d_\Phi(p,p') + d_\Phi(p'',\widetilde{p}) = 2d_\Phi(p,p') + d_\Phi(p'',\widetilde{p})
\end{aligned}$$

$$\leq \frac{2}{3}d_\Phi(p,\widetilde{p})+d_\Phi(p'',\widetilde{p}).$$

□

Now, let us employ the Mean Value Theorem, where we consider $p_\xi=(\xi,y,z)$, $p_\eta=(x',\eta,z)$ and $p_\zeta=(x',y',\zeta)$ for some intermediate points $\xi\in(x,x')$, $\eta\in(y,y')$ and $\zeta\in(z,z')$. Note that each point $p''\in\{p_\xi,p_\eta,p_\zeta\}$ arising from the Mean Value Theorem satisfies either $d_\Phi(p,p'')\leq d_\Phi(p,p')$ or $d_\Phi(p',p'')\leq d_\Phi(p,p')$, implying that Lemma 3.11 is true for any of them, from where follows that

$$d_\Phi(p,\widetilde{p})\leq d_\Phi(p'',\widetilde{p}),\ \ p''\in\{(\xi,y,z),(x',\eta,z),(x',y',\zeta)\}, \tag{3.26}$$

for any $\widetilde{p}\in M^-$. Thus, we get to write

$$\begin{aligned}
|I_1|\leq&|x-x'|\int_0^t\int_{M^-}\partial_\xi|G(t-\widetilde{t},p_\xi,\widetilde{p})\left[u(\widetilde{t},\widetilde{p})-u(\widetilde{t},p)\right]|\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\
&+\|y-y'\|\int_0^t\int_{M^-}\partial_\eta|G(t-\widetilde{t},p_\eta,\widetilde{p})\left[u(\widetilde{t},\widetilde{p})-u(\widetilde{t},p)\right]|\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\
&+\|z-z'\|\int_0^t\int_{M^-}\partial_\zeta|G(t-\widetilde{t},p_\zeta,\widetilde{p})\left[u(\widetilde{t},\widetilde{p})-u(\widetilde{t},p)\right]|\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\
&\leq C\|u\|_\alpha|x-x'|\int_0^t\int_{M^-}\partial_\xi G(t-\widetilde{t},p_\xi,\widetilde{p})d_\Phi(p_\xi,\widetilde{p})^\alpha\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\
&+C\|u\|_\alpha\|y-y'\|\int_0^t\int_{M^-}\partial_\eta G(t-\widetilde{t},p_\eta,\widetilde{p})d_\Phi(p_\eta,\widetilde{p})^\alpha\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\
&+C\|u\|_\alpha\|z-z'\|\int_0^t\int_{M^-}\partial_\zeta G(t-\widetilde{t},p_\zeta,\widetilde{p})d_\Phi(p_\zeta,\widetilde{p})^\alpha\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\
&=:I_{1,1}+I_{1,2}+I_{1,3}.
\end{aligned}$$

Given the similarities for estimating the terms $I_{1,i}$ above, we present only the estimate for $I_{1,1}$.

Since we estimate in the regime 5 in Figure 2.6, where fd meets td, we use the local projective coordinates $(\tau,\xi,y,z,\mathscr{S}',\mathscr{U}',\mathscr{Z}')$, introduced in (2.27), where

$$\mathscr{S}'=\frac{\widetilde{x}-\xi}{\xi^2},\ \ \mathscr{U}'=\frac{\widetilde{y}-y}{\xi},\ \ \mathscr{Z}'=\widetilde{z}-z\ \text{ and }\ \tau=\sqrt{t-\widetilde{t}}.$$

Then we compute from Theorem 2.12 and $\mathrm{dvol}_\Phi(\widetilde{x},\widetilde{y},\widetilde{z})\sim\widetilde{x}^{-2-b}\,\mathrm{d}\widetilde{x}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{z}$

$$|I_{1,1}|\leq C\frac{|x-x'|}{\xi^2}\cdot\|u\|_\alpha\int_0^{\sqrt{t}}\int_{M^-}\tau^{-m-2}G_0\sqrt{|\mathscr{S}'|^2+\|\mathscr{U}'\|^2+\|\mathscr{Z}'\|^2}^{\,\alpha}\,\mathrm{d}\mathscr{S}'\,\mathrm{d}\mathscr{U}'\,\mathrm{d}\mathscr{Z}'\,\mathrm{d}\tau,$$

with G_0 being bounded and vanishing to infinite order as $\|(\mathscr{S},\mathscr{U},\mathscr{Z})\|\to\infty$, where $(\mathscr{S},\mathscr{U},\mathscr{Z})=(\mathscr{S}'/\tau,\mathscr{U}'/\tau,\mathscr{Z}'/\tau)$. Let us define $r(\mathscr{S}',\mathscr{U}',\mathscr{Z}'):=\sqrt{|\mathscr{S}'|^2+\|\mathscr{U}'\|^2+\|\mathscr{Z}'\|^2}$. Such a function r describes the radial distance in polar coordinates around the origin. Performing a change to polar coordinates, we obtain

$$|I_{1,1}|\leq c\cdot\frac{|x-x'|}{\xi^2}\cdot\|u\|_\alpha\int_0^{\sqrt{t}}\int_{M^-}\tau^{-m-2}r^{m-1+\alpha}G_0\,\mathrm{d}r\,\mathrm{d}(\text{angle})\,\mathrm{d}\tau.$$

Now, setting $\sigma = r^{-1}\tau = \sqrt{|\mathscr{S}|^2 + \|\mathscr{U}\|^2 + \|\mathscr{Z}\|^2}^{-1}$, it follows that G_0 against any negative power of σ is bounded. Hence, integrating out the angular variables, followed by another change of coordinates $\tau \mapsto \sigma$ gives

$$|I_{1,1}| \leq c \cdot \frac{|x-x'|}{\xi^2} \cdot \|u\|_\alpha \int_0^{\sqrt{t}} \int_{M^-} r^{-2+\alpha} \, \mathrm{d}r.$$

Now, it follows from the definition of r that $M^- \subset \{d_\Phi(p,p') \leq cr\}$ for some constant $c > 0$. Thus we can estimate even further

$$\begin{aligned} |I_{1,1}| &\leq c \cdot \frac{|x-x'|}{\xi^2} \cdot \|u\|_\alpha \int_{c\frac{d_\Phi(p,p')}{\xi^2}}^{\infty} r^{-2+\alpha} \, \mathrm{d}r \\ &= c \cdot \frac{|x-x'|}{\xi^2} \cdot d_\Phi(p,p')^{-1+\alpha} \|u\|_\alpha. \end{aligned} \tag{3.27}$$

In order to conclude the desired estimate of I_1, recall from Lemma 3.2, that we may consider only $d_\Phi(p,p') \leq \delta^{1/\alpha} =: \rho$, with any positive $\rho < 1/4$. Then

$$1 - \frac{x}{x'} \leq 2\rho(x+x') \leq 4\rho.$$

Thus $x > (1-4\delta)x'$. Hence we may estimate

$$\begin{aligned} \frac{|x-x'|}{\xi^2} &\leq \frac{|x-x'|}{x^2} \leq (1-4\rho)^{-2} \frac{|x-x'|}{x'^2} \\ &\leq 4(1-4\rho)^{-2} \frac{|x-x'|}{(x+x')^2} \leq 4(1-4\rho)^{-2} d_\Phi(p,p'). \end{aligned}$$

Thus for $\delta > 0$ sufficiently small, we conclude from (3.27) and the last estimate above

$$|I_{1,1}| \leq C d_\Phi(p,p')^\alpha \|u\|_\alpha.$$

Analogously, the remaining terms can be estimated in a similar fashion, which allows us to conclude the same inequalities for both $I_{1,2}$ and $I_{1,3}$ as presented for $I_{1,1}$, leading to the estimate desired for I_1.

Estimates for I_2

Similarly to the computations for the I_2 term, the first thing to do is to give a better expression for the integral I_1. Let us rewrite I_2 as follows:

$$\begin{aligned} I_2 &= \int_0^t \int_{M^+} G(t-\widetilde{t}, p, \widetilde{p})[u(\widetilde{p},\widetilde{t}) - u(p,\widetilde{t})] \, \mathrm{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\widetilde{t} \\ &- \int_0^t \int_{M^+} G(t-\widetilde{t}, p', \widetilde{p})[u(\widetilde{p},\widetilde{t}) - u(p',\widetilde{t})] \, \mathrm{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\widetilde{t} \\ &+ \int_0^t \int_{M^+} G(t-\widetilde{t}, p', \widetilde{p})[u(p,\widetilde{t}) - u(p',\widetilde{t})] \, \mathrm{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\widetilde{t} \\ &=: I_{2,1} - I_{2,2} + I_{2,3}. \end{aligned}$$

Estimates for $I_{2,1}$ and $I_{2,2}$ can be obtained along the same lines. Thus, it is enough for us to present computations only for $I_{2,1}$ and $I_{2,3}$. Now, since we are considering M^+ as the integration region, then points on $\widetilde{p}\in M^+$ are satisfying $d_\Phi(p,\widetilde{p})\leq 3_\Phi d(p,p')$. Moreover, from the triangular inequality follows

$$d_\Phi(p',\widetilde{p})\leq d_\Phi(p',p)+d_\Phi(p,\widetilde{p})\leq 4d_\Phi(p,p').$$

Let us first estimate $I_{2,1}$. Employing the definition of the Hölder norm gives us

$$|I_{2,1}|\leq \|u\|_\alpha\int_0^t\int_{M^+}|G(t-\widetilde{t},p,\widetilde{p})|d_\Phi(p,\widetilde{p})^\alpha\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}.$$

Considering once again coordinates $(\tau,x,y,z,\mathscr{S}',\mathscr{U}',\mathscr{Z}')$ introduced in (2.27), one can obtain constants $c,C>0$ such that

$$d_\Phi(p,\widetilde{p})\leq c\cdot r(\mathscr{S}',\mathscr{U}',\mathscr{Z}'):=c\cdot\sqrt{|\mathscr{S}'|^2+\|\mathscr{U}'\|^2+\|\mathscr{Z}'\|^2}\leq C\cdot d_\Phi(p,\widetilde{p})\leq 3C\cdot d_\Phi(p,p').$$

Performing a change of coordinates gives us

$$\begin{aligned}|I_{2,1}|\leq& c\|u\|_\alpha\int_0^{\sqrt{t}}\int_{M^+}\tau^{-m-1}G_0r^\alpha\,\mathrm{d}\mathscr{S}'\,\mathrm{d}\mathscr{U}'\,\mathrm{d}\mathscr{Z}'\,\mathrm{d}\tau\\ =&c\|u\|_\alpha\int_0^{\sqrt{t}}\int_{\mathbb{S}^{m-1}}\int_0^{3Cd_\Phi(p,p')}\tau^{-m-1}G_0r^{m-1+\alpha}\,\mathrm{d}r\,\mathrm{d}(\mathrm{angle})\,\mathrm{d}\tau.\end{aligned}$$

Now, let $\sigma:=\tau/r$. Performing yet another change of coordiantes, we get to rewrite the integral and, consequently, the estimate as follows:

$$\begin{aligned}|I_{2,1}|\leq& c\|u\|_\alpha\int_0^{\sqrt{t}}\int_{\mathbb{S}^{m-1}}\int_0^{3Cd_\Phi(p,p')}\sigma^{-m-1}G_0r^{-1+\alpha}\,\mathrm{d}r\,\mathrm{d}(\mathrm{angle})\,\mathrm{d}\sigma\\ \leq& C\|u\|_\alpha d_\Phi(p,p')^\alpha.\end{aligned}$$

Estimate for $I_{2,2}$ are obtainable following along the same lines simply by recalling the inequality $d_\Phi(p',\widetilde{p})\leq 4d_\Phi(p,p')$.

For the estimate of $I_{2,3}$, we assume again as before that the heat kernel is supported near fd meeting td, and thus work with local projective coordinates $(\tau,x',y',z',\mathscr{S},\mathscr{U},\mathscr{Z})$ given in (2.28), that is,

$$\mathscr{S}=\frac{\widetilde{x}-x'}{x'^2\tau},\quad \mathscr{U}=\frac{\widetilde{y}-y'}{x'\tau},\quad \mathscr{Z}=\frac{\widetilde{z}-z'}{\tau}\quad\text{and}\quad \tau=\sqrt{t-\widetilde{t}}.$$

We will obtain the estimates using integration by parts. To do so, note that one has (as the "worst case scenario" with $V\in\mathrm{Diff}^2_\Phi(M)$) $G=\tau^{-m-2}(V_1V_2G_0)$ with both $V_1,V_2\in\{\partial_\mathscr{S},\partial_\mathscr{U},\partial_\mathscr{Z}\}$ and G_0 vanishing to infinite order whenever $\|(\mathscr{S},\mathscr{U},\mathscr{Z})\|\to\infty$. For the sake of simplicity, we shall assume $V_1=\partial_\mathscr{S}$. On the other hand, one has by triangle inequality

$$\partial M^+=\{d_\Phi(p,\widetilde{p})=3d_\Phi(p,p')\}\subseteq\{2d_\Phi(p,p')\leq d_\Phi(p',\widetilde{p})\}. \tag{3.28}$$

Moreover we can also write for some smooth function h_0

$$\beta^*(\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t})=h_0\big(x'+\tau x'^2\mathscr{S},y'+\tau x'\mathscr{U},z'+\tau\mathscr{Z}\big)\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau.$$

Since $u(p,\widetilde{t}) - u(p',\widetilde{t}) =: \delta u$ is independent of $\widetilde{p}$, we can integrate by parts

$$\begin{aligned} I_{2,3} &= \int_0^{\sqrt{t}} \delta u \int_{M^+} \tau^{-1} (\partial_{\mathscr{S}} V_2 H) h_0 \, \mathrm{d}\mathscr{S} \, \mathrm{d}\mathscr{U} \, \mathrm{d}\mathscr{Z} \, \mathrm{d}\tau \\ &= \int_0^{\sqrt{t}} \delta u \int_{\partial M^+} \tau^{-1} (V_2 H) h_0 \, \mathrm{d}\mathscr{U} \, \mathrm{d}\mathscr{Z} \, \mathrm{d}\tau \\ &- \int_0^{\sqrt{t}} \delta u \int_{M^+} \tau^{-1} (V_2 H) \partial_{\mathscr{S}} h_0 \, \mathrm{d}\mathscr{S} \, \mathrm{d}\mathscr{U} \, \mathrm{d}\mathscr{Z} \, \mathrm{d}\tau =: I'_{2,3} - I''_{2,3}. \end{aligned}$$

For the $I''_{2,3}$-term, note that h_0 is smooth and therefore $\partial_{\mathscr{S}} h_0 = \tau x'^2 \partial_{\widetilde{x}} h_0$. This cancels the τ^{-1} in the integrand and thus $I''_{2,3}$ can be estimated against $\|u\|_\alpha d_\Phi(p,p')^\alpha$.

For the $I'_{2,3}$-term, by (3.28), we can estimate

$$\begin{aligned} |I'_{2,3}| &\leq \|u\|_\alpha d_\Phi(p,p')^\alpha \int_0^{\sqrt{t}} \int_{\partial M^+} \tau^{-1} \big(V_2 H\big) h_0 \, \mathrm{d}\mathscr{U} \, \mathrm{d}\mathscr{Z} \, \mathrm{d}\tau \\ &\leq \frac{1}{2} \|u\|_\alpha \int_0^{\sqrt{t}} \int_{\partial M^+} \tau^{-1} \big(V_2 H\big) d_\Phi(p',\widetilde{p})^\alpha h_0 \, \mathrm{d}\mathscr{U} \, \mathrm{d}\mathscr{Z} \, \mathrm{d}\tau. \end{aligned}$$

Estimates for I_3

For the estimate of I_3, let us first assume without loss of generality $x < x'$. Moreover, it follows from Lemma 3.2 that it is enough for us to work under the assumption that

$$d_\Phi(p,p')^\alpha + |t-t'|^{\alpha/2} < \delta,$$

for δ sufficiently small. Considering $p_{x'} = (x',y,z) \in M$, write

$$\begin{aligned} I_3 &= \int_0^t \int_M \left[G(t-\widetilde{t},p,\widetilde{p}) - G(t-\widetilde{t},p_{x'},\widetilde{p}) \right] u(p,\widetilde{t}) \, \mathrm{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\widetilde{t} \\ &+ \int_0^t \int_M \left[G(t-\widetilde{t},p_{x'},\widetilde{p}) - G(t-\widetilde{t},p',\widetilde{p}) \right] u(p,\widetilde{t}) \, \mathrm{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\widetilde{t} \\ &= I_{3,1} + I_{3,2}. \end{aligned}$$

Since H is stochastically complete and $\mathbf{G} = V \circ \mathbf{H}_\gamma$, it follows that

$$\begin{aligned} I_{3,2} &= \int_0^t x^\gamma u(p,\widetilde{t}) x'^{-\gamma} \left(\int_M \left[VH(t-\widetilde{t},p_{x'},\widetilde{p}) - VH(t-\widetilde{t},p',\widetilde{p}) \right] \mathrm{dvol}_\Phi(\widetilde{p}) \right) \mathrm{d}\widetilde{t} \\ &= \int_0^t (x^\gamma u(p,\widetilde{t}) x'^{-\gamma} \cdot 0) \, \mathrm{d}\widetilde{t} \\ &= 0. \end{aligned}$$

Therefore, $I_3 = I_{3,1}$. By using the Mean Value Theorem on the x-entry and considering $p_\xi = (\xi,y,z)$ for some $\xi \in (x,x')$, it follows that

$$I_3 = |x-x'| \int_0^t \int_M \partial_\xi G(t-\widetilde{t},p_\xi,\widetilde{p}) u(p,\widetilde{t}) \, \mathrm{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\widetilde{t}$$

Coordinates near td defined in (2.28) give us $\partial_{\xi} G \,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t} = \xi^{-2}\tau^{-1}(VG_0)h_0\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau$, where $h_0(\widetilde{p})$ is a smooth function and G_0 vanishes to infinite order whenever $\|(\mathscr{S},\mathscr{U},\mathscr{Z})\|\to\infty$. Integrating by parts the integral above under the assumption that $V=\partial_{\mathscr{S}}$ (without loss of generality), we obtain

$$\begin{aligned} I_3 =& |x-x'|\int_0^t\int_{\partial M}\xi^{-2}\tau^{-1}G_0h_0(\widetilde{p})u(p,\widetilde{t})\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau \\ &-|x-x'|\int_0^t\int_M\xi^{-2}\tau^{-1}G_0\partial_{\mathscr{S}}(h_0(\widetilde{p})\cdot u(p,\widetilde{t}))\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau \\ =& I_{3,1}-I_{3,2}. \end{aligned}$$

From the vanishing order of the heat kernel near ∂M, it follows that $I_{3,1}=0$. On the other hand, it follows that $\partial_{\mathscr{S}}(h_0(\widetilde{p})\cdot u(p,\widetilde{t})) = u(p,\widetilde{t})\xi^2\tau\partial_{\widetilde{x}}h_0$, which implies

$$\begin{aligned} |I_3| \leq& \|u\|'_{\alpha}|x-x'|\int_0^t\int_M G_0\partial_{\widetilde{x}}h_0\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau \\ \leq& C\|u\|'_{\alpha}\frac{|x-x'|^{\alpha}}{(x+x')^{2\alpha}} \leq C\|u\|'_{\alpha}d_{\Phi}(p,p')^{\alpha}. \end{aligned}$$

This completes the proof of (i).

3.2.2 Estimates for Hölder differences in time

Now, assume $p=p'$ and, without loss of generality, $t<t'$. Suppose first that t and t' satisfy $2t'-t\geqslant 0$ (i.e., $t'<t\leq 2t'$). Hence, we can now define the intervals

$$T_- = [0,2t'-t],\ \ T_+ = [2t'-t,t]\ \text{ and }\ T'_+ = [2t'-t,t'].$$

Once again, writing $\mathbf{G}=V\circ\mathbf{H}_{\gamma}$ for $V\in\{\mathrm{id}\}\cup\mathscr{V}_{\phi}\cup\mathscr{V}_{\phi}^2$, we get

$$\begin{aligned} \mathbf{G}u(p,t)-\mathbf{G}u(p,t') =& |t-t'|\int_{T_-}\int_M\partial_{\theta}G(t-\widetilde{t},p,\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ &+\int_{T_+}\int_M G(t-\widetilde{t},p,\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ &-\int_{T'_+}\int_M G(t'-\widetilde{t},p,\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ &+\int_0^t\int_M G(t-\widetilde{t},p,\widetilde{p})u(p,\widetilde{t})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ &-\int_0^{t'}\int_M G(t'-\widetilde{t},p,\widetilde{p})u(p,\widetilde{t})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ =:& L_1+L_2-L_3+L_4-L_5 \end{aligned}$$

First, let us analyse the terms L_4 and L_5. Since the space-variable is constant at p and (M,g_{Φ}) is stochastically complete, it follows that

$$L_4-L_5 = \int_0^t u(p,\widetilde{t})\,\mathrm{d}\widetilde{t} - \int_0^{t'} u(p,\widetilde{t})\,\mathrm{d}\widetilde{t} \leq C\|u\|_{\infty}|t-t'|^{\alpha/2}.$$

Thus, for us to obtain the estimates, we just need to give estimates for L_1, L_2 and L_3. However, given the similarities between the terms L_2 and L_3, presenting the estimates for one of them gives us the same for the other. In conclusion, we will be presenting the estimates for the terms L_1 and L_2. Moreover, like the estimates presented for the spatial differences, computations for each term will be presented only near the middle regimes of M_h^2, given the fact that estimates near any other regime is straightforward.

Estimate for L_1

The projective coordinates near such intersection are $(\tau, x, y, z, \mathscr{S}, \mathscr{U}, \mathscr{Z})$ presented in (2.28). Hence, from the asymptotics of H_γ near the middle regimes follows that $\beta^* \partial_\theta G \sim \tau^{-m-4} G_0$, with G_0 being polyhomogeneous and vanishing to infinite order when $\|(\mathscr{S}, \mathscr{U}, \mathscr{Z})\| \to \infty$. On the other hand, we have $\beta^*(\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}) \sim \tau^{m+1}\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau$. Moreover, we already know that

$$d_\Phi(p, \widetilde{p}) \leq c\tau\sqrt{|\mathscr{S}|^2 + \|\mathscr{U}\|^2 + \|\mathscr{Z}\|^2} =: c\tau r(\mathscr{S}, \mathscr{U}, \mathscr{Z}) \tag{3.29}$$

where r is bounded whenever its entries are bounded. Thus, it follows that $G_0 r^\alpha$ is bounded everywhere. On the other hand, note that whenever $\widetilde{t} \in T_-$, one has $|\theta - \widetilde{t}| \geq |t - t'|$, from where follows that

$$\begin{aligned}
|L_1| &\leq \|u\|_\alpha |t - t'| \int_{T_-} \int_M \tau^{-3} G_0 d_\Phi(p, \widetilde{p})^\alpha \,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau \\
&\leq c\|u\|_\alpha |t - t'| \int_{\sqrt{t-t'}}^\infty \int_M \tau^{-3+\alpha} x^\alpha G_0 r^\alpha \,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau \\
&\leq C\|u\|_\alpha |t - t'|^{\alpha/2}.
\end{aligned}$$

Estimate for L_2

The estimate near this region of M_h^2 are obtained using the coordinates $(\tau, x, y, z, \mathscr{S}, \mathscr{U}, \mathscr{Z})$ defined in (2.28). With respect to this coordinates one has $\beta^*(G) \sim \tau^{-m-2} G_0$, with G_0 being polyhomogeneous and vanishing to infinite order whenever one has $\|(\mathscr{S}, \mathscr{U}, \mathscr{Z})\| \to \infty$. On the other hand,

$$\beta^*(\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}) \sim \tau^{m+1} h_0 \,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau, \tag{3.30}$$

with h_0 a smooth function of $\widetilde{p}$. From this and from (3.29) follows that

$$\begin{aligned}
|L_2| &\leq \|u\|_\alpha \int_{T_+} \int_M |\tau^{-1} G_0' d(p, \widetilde{p})^\alpha| \,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}\tau \\
&\leq c\|u\|_\alpha \int_{T_+} \int_M |\tau^{-1+\alpha} G_0' r^\alpha| \,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau \\
&\leq c\|u\|_\alpha |t - t'|^{\alpha/2},
\end{aligned}$$

This completes the estimates for time difference with derivatives under the assumption that $2t' - t \geqslant 0$. We will now assume $2t' - t < 0$. Note that, since $t > t'$, we get

$$-3t + 2t' \leq 0 \leq t - 2t' \Rightarrow t \leq 2|t - t'|.$$

Furthermore, since $t' < t$ we conclude that $t' \leq 2|t-t'|$ as well. One then has

$$\begin{aligned}\mathbf{G}u(p,t)-\mathbf{G}u(p,t') &= \int_0^t\int_M G(t-\widetilde{t},p,\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &-\int_0^{t'}\int_M G(t'-\widetilde{t},p,\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &+\int_0^t\int_M [G(t-\widetilde{t},p,\widetilde{p})-G(t'-\widetilde{t},p,\widetilde{p})]u(p,\widetilde{t})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}.\end{aligned}$$

The first two integrals are estimated applying similar argument as for L_2, presented in the previous case, while the third one can be estimated by the same logic applied to L_4.

Remark 3.12. With this estimate, we complete the estimates for time differences with derivatives under the assumption $t' < t$. However, to obtain the estimates for the case $t < t'$, one just need to interchange the roles of t and t'.

From there we conclude immediately the statement (ii).

3.2.3 Estimates for the supremum norm

Finally, we will present the computations for the estimates of the supremum norm of $\mathbf{G}u$. Like the estimates for difference in both space and time, we will only present explicit computations for estimates near fd∪td. Estimates near the corners of M^2_h follow directly. For a given point $(p,t)\in M\times[0,T]$,

$$\begin{aligned}\mathbf{G}u(p,t) &= \int_0^t\int_M G(t-\widetilde{t},p,\widetilde{p})u(\widetilde{p},\widetilde{t})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &= \int_0^t\int_M G(t-\widetilde{t},p,\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &+\int_0^t\int_M G(t-\widetilde{t},p,\widetilde{p})u(p,\widetilde{t})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &= J_1+J_2.\end{aligned}$$

Estimate for J_1

In this regime we will use the projective coordinates $(x,y,z,\mathscr{S},\mathscr{U},\mathscr{Z},\tau)$ defined in (2.28). From the argument employed for the estimate of L_2, one knows that

$$\beta^*(G(t-\widetilde{t},p,\widetilde{p})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}) \sim \tau^{-1}G_0\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau \tag{3.31}$$

with G_0 being polyhomogeneous and vanishing to infinite order whenever $\|(\mathscr{S},\mathscr{U},\mathscr{Z})\|\to\infty$. Thus, by using (3.29) we get

$$\begin{aligned}|J_1| &\leq \|u\|_{\alpha}\int_0^{\sqrt{t}}\int_M \tau^{-1}G_0 d_{\Phi}(p,\widetilde{p})^{\alpha}\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau\\ &= C\|u\|_{\alpha}\int_0^{\sqrt{t}}\int_M \tau^{-1+\alpha}G_0 r^{\alpha}\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau\\ &\leq C\|u\|_{\alpha}.\end{aligned}$$

Estimate for J_2

For the estimates for J_2 near td, one can follow a very similar argumentation to the one presented for L_5. Hence, making use of integration by parts and of the vanishing property of the polyhomogenous function present on the asymptotics of the heat kernel, one can obtain the desired estimates for J_2.

This completes the proof of the statement for $k=0$. For general k, in all of the above integrals we can first pass k Φ-derivatives to the function u using integration by parts in $(\mathscr{S},\mathscr{U},\mathscr{Z})$ and then continue as before in case $k=0$.

3.3 Short-time existence of the Yamabe flow

Consider a Φ-manifold (M,g_Φ) of dimension $m\geq 3$ and set $\eta := (m-2)/4$. In this section we construct a short-time solution to the Yamabe flow equation (3.2) of the conformal factor

$$\partial_t u = (m-1)u^{-1/\eta}\Delta_\Phi u - \eta\,\mathrm{scal}(g_\Phi)u^{1-1/\eta},\;\; u|_{t=0}=1. \tag{3.32}$$

We plan to linearize (3.32), which will provide us a slightly different version for us. This means that the solution for (3.32) will be a translation by a constant of the solution for the linearized equation. After this, we construct a solution as a fixed point of a contraction in $x^\gamma C^{k,\alpha}_\Phi(M\times[0,T])$, for some $\gamma\geq 0$ and some short-time $T>0$. We assume below that $k=0$, since the general case follows the $k=0$ case verbatim. We write $u=1+v$ and obtain from (3.32) an equation for v

$$\partial_t v = (m-1)\Delta_\Phi v(1+v)^{-1/\eta} - \eta\,\mathrm{scal}(g_\Phi)(1+v)^{1-1/\eta};\;\; v|_{t=0}=0. \tag{3.33}$$

Remark 3.13. It is well known that the Binomial series convergence for all $a\in\mathbb{C}$ and $|x|<1$ as follows:

$$(1+x)^a = \sum_{j=0}^{\infty} a_j x^j,\;\; \text{with } a_j := \binom{a}{j} = \frac{a(a-1)\cdots(a-j+1)}{j!}. \tag{3.34}$$

Now, assume $v\in x^\gamma C^{2,\alpha}_\Phi(M\times[0,T])$, for $\gamma\geq 0$, with $\|v\|_{2,\alpha,\gamma}\leq\mu$ for some $\mu<1$. Then the following series converge in the Banach space $C^{2,\alpha}_\Phi(M\times[0,T])$

$$(1+v)^{-1/\eta} = \sum_{j=0}^{\infty} a_j v^j = 1 - \frac{v}{\eta} + \sum_{j=2}^{\infty} a_j v^j =: 1 - \frac{v}{\eta} + v^2 s(v),$$

with $\|(1+v)^{-1/\eta}\|_{2,\alpha}\leq C_\mu$ and $\|s(v)\|_{2,\alpha}\leq C_\mu$, for some $C_\mu>0$, depending only on μ.

Plugging the identity $(1+v)^{-1/\eta}=1-v/\eta+v^2s(v)$ into (3.33) yields after rescaling the time variable by $(m-1)$ the following flow equation

$$\begin{aligned}(\partial_t-\Delta_\Phi)v = &-\frac{1}{\eta}v\Delta_\Phi v + v^2 s(v)\Delta_\Phi v - \frac{\eta}{m-1}\mathrm{scal}(g_\Phi) + \frac{1}{m-1}\mathrm{scal}(g_\Phi)v\\ &+\frac{1}{m-1}\mathrm{scal}(g_\Phi)v^2(1-\eta s(v)-\eta v s(v)).\end{aligned} \tag{3.35}$$

We will simplify the right hand side by introducing two non-linear operators, the first one containing no derivatives of v

$$F_1(v) := -\frac{\eta}{m-1}\operatorname{scal}(g_\Phi) + \frac{1}{m-1}\operatorname{scal}(g_\Phi)v + \frac{1}{m-1}\operatorname{scal}(g_\Phi)v^2(1-\eta s(v) - \eta v s(v)).$$

The second one is in a certain sense quadratic in v and defined by

$$F_2(v) := -\frac{1}{\eta} v\Delta_\Phi v + v^2 s(v)\Delta_\Phi v.$$

In this notation, (3.35) can be written as

$$(\partial_t - \Delta_\Phi)v = (F_1 + F_2)v;\ \ v|_{t=0} = 0. \tag{3.36}$$

Our intention is to prove short-time existence of solution of (3.36) by using the contraction mapping argument (i.e. the Banach fixed-point theorem). We point out that, even though we work on weighted Hölder spaces $x^\gamma C^{k,\alpha}_\Phi(M\times[0,T])$, the proof does not rely on the definition of the distance function chosen on its definition. This means that the proof fits alternative Hölder spaces as well (as we use it in §4.4).

Theorem 3.14. *Consider the Cauchy problem*

$$(\partial_t - a\Delta)v = F(v),\ \ v|_{t=0} = 0, \tag{3.37}$$

where the function a is positive, bounded from below away from zero in $C^{k,\beta}_\Phi(M\times[0,T])$, with $\beta > \alpha$.

Suppose $F : x^\gamma C^{k+2,\alpha}_\Phi(M\times[0,T]) \to x^\gamma C^{k,\alpha}_\Phi(M\times[0,T])$ decomposes as the sum $F = F_1 + F_2$ and that, for $\mu < 1$, there is a constant $C_\mu > 0$ such that

1. *$F_1 : x^\gamma C^{k+2,\alpha}_\Phi(M\times[0,T]) \to x^\gamma C^{k+1,\alpha}_\Phi(M\times[0,T])$ satisfies the estimates*

$$\|F_1(v) - F_1(v')\|_{k+1,\alpha,\gamma} \le C_\mu \|v - v'\|_{k+2,\alpha,\gamma},$$
$$\|F_1(v)\|_{k+1,\alpha,\gamma} \le C_\mu;$$

2. *$F_2 : x^\gamma C^{k+2,\alpha}_\Phi(M\times[0,T]) \to x^\gamma C^{k,\alpha}_\Phi(M\times[0,T])$ satisfies the estimates*

$$\|F_2(v) - F_2(v')\|_{k,\alpha,\gamma} \le C_\mu \max\{\|v\|_{k+2,\alpha,\gamma}, \|v'\|_{k+2,\alpha,\gamma}\}\|v - v'\|_{k+2,\alpha,\gamma},$$
$$\|F_2(v)\|_{k,\alpha,\gamma} \le C_\mu \|v\|^2_{k+2,\alpha,\gamma},$$

whenever $v, v' \in x^\gamma C^{k+2,\alpha}_\Phi(M\times[0,T])$ satisfy $\|v\|_{k+2,\alpha,\gamma}, \|v'\|_{k+2,\alpha,\gamma} \le \mu$.

*Moreover, suppose that the parametrix **Q** for the differential operator $(\partial_t - a\Delta_\Phi)$ maps*

$$\mathbf{Q} : x^\gamma C^{k,\alpha}_\Phi(M\times[0,T]) \to x^\gamma \left(C^{k+2,\alpha}_\Phi \cap \sqrt{t}C^{k+1}_\Phi\right)(M\times[0,T]) \tag{3.38}$$

continuously. Then there are $T' > 0$ and a unique solution $v^* \in x^\gamma C^{k+2,\alpha}_\Phi(M\times[0,T'])$ of (3.37).

Proof. The proof for this statement follows from the Banach fixed-point theorem. For μ and T yet to be specified, define

$$Z_{\mu,T} := \left\{ v \in x^{\gamma} C_{\Phi}^{k+2,\alpha}(M \times [0,T]);\;\; v|_{t=0} = 0, \|v\|_{k,\alpha,\gamma} \leq \mu \right\}.$$

Clearly this is a closed subset in $x^{\gamma} C_{\Phi}^{k+2,\alpha}(M \times [0,T])$ and therefore, since we are working on a Banach space, $Z_{\mu,T}$ is a complete metric space. Consider the map $\Psi(v) := (\mathbf{Q} \circ F)(v)$. From our hypotheses, it follows that Ψ maps $x^{\gamma} C_{\Phi}^{k,\alpha}(M \times [0,T])$ to itself. Therefore, in order to conclude what we desire, we just need to check that the map Ψ is in fact a contraction when restricted to $Z_{\mu,T}$, therefore having a unique fixed point v^* (which is the solution for (3.3)). Now, since Q is a bounded linear map, then one can write

$$\begin{aligned} \Psi(v) &= (\mathbf{Q} \circ F)(v) = (\mathbf{Q} \circ F_1)(v) + (\mathbf{Q} \circ F_2)(v) \\ &=: \Psi_1(v) + \Psi_2(v). \end{aligned}$$

From this point on, denote $\|\cdot\|_{\mathrm{op}}$ as the norm of an operator. For Ψ_1, if one assume to have $\mu < 1$ and $T < (2C_\mu \|\mathbf{Q}\|_{\mathrm{op}})^{-1}\mu$, then for every $v \in Z_{\mu,T}$ one has

$$\begin{aligned} \|\Psi_1(v)\|_{k+2,\alpha,\gamma} &\leq \|\mathbf{Q}\|_{\mathrm{op}} \sqrt{T} \|F_1(v)\|_{k+1,\alpha,\gamma} \leq \|\mathbf{Q}\|_{\mathrm{op}} \sqrt{T} C_\mu \|v\|_{k+2,\alpha,\gamma} \\ &\leq \mu/2. \end{aligned}$$

On the other hand, if we assume $\mu < \min\{(2C_\mu \|\mathbf{Q}\|_{\mathrm{op}})^{-1}, 1\}$ then for every $v \in Z_{\mu,T}$ follows

$$\begin{aligned} \|\Psi_2(v)\|_{k+2,\alpha,\gamma} &\leq \|\mathbf{Q}\|_{\mathrm{op}} \|F_2(v)\|_{k,\alpha,\gamma} \leq \|\mathbf{Q}\|_{\mathrm{op}} C_\mu \|v\|_{k+2,\alpha,\gamma}^2 \\ &\leq \mu/2, \end{aligned}$$

from which is possible to conclude that for μ and T small enough one has $\|\Psi(v)\|_{k+2,\alpha,\gamma} \leq \mu$, therefore implying that Ψ maps $Z_{\mu,T}$ to itself. Moreover, note that in order to prove that Ψ is a contraction on $Z_{\mu,T}$, is enough to show this for Ψ_1 and Ψ_2. But

$$\begin{aligned} \|\Psi_1(v) - \Psi_1(v')\|_{k+2,\alpha,\gamma} &\leq \|\mathbf{Q}\|_{\mathrm{op}} \sqrt{T} C_\mu \|v - v'\|_{k+2,\alpha,\gamma} \\ &\leq \frac{1}{2} \|v - v'\|_{k+2,\alpha,\gamma} \end{aligned}$$

for both μ and T small enough; furthermore,

$$\begin{aligned} \|\Psi_2(v) - \Psi_2(v')\|_{k+2,\alpha,\gamma} &\leq \|\mathbf{Q}\|_{\mathrm{op}} C_\mu \max\{\|v\|_{k+2,\alpha,\gamma}, \|v'\|_{k+2,\alpha,\gamma}\} \|v - v'\|_{k+2,\alpha,\gamma} \\ &\leq \frac{1}{2} \|v - v'\|_{k+2,\alpha,\gamma}, \end{aligned}$$

from where follows that Ψ is, in fact, a contraction on $Z_{\mu,T}$, from where we can conclude our proof. □

Theorem 3.14 gives us some conditions on the operators F_1 and F_2 for the contraction argument to hold. To this end, we prove the following technical lemmas.

Lemma 3.15. *Denote by B the open ball of radius* 1 *in* $x^\gamma C^{2,\alpha}_\Phi(M\times[0,T])$*. Then the map* $F_2 : B \to x^\gamma C^{\alpha}_\Phi(M\times[0,T])$ *is bounded. Moreover, for any two functions* $v, v' \in B \subset x^\gamma C^{2,\alpha}_\Phi(M\times[0,T])$ *satisfying*

$$\|v\|_{2,\alpha,\gamma}, \|v'\|_{2,\alpha,\gamma} \leq \mu < 1,$$

there exists a constant $C_\mu > 0$ *such that*

1. $\|F_2(v) - F_2(v')\|_\alpha \leq C_\mu \max\{\|v\|_{2,\alpha,\gamma}, \|v'\|_{2,\alpha,\gamma}\}\|v - v'\|_{2,\alpha,\gamma}$,
2. $\|F_2(v)\|_{\alpha,\gamma} \leq C_\mu \|v\|^2_{2,\alpha,\gamma}$.

Proof. We shall write $\Delta = \Delta_\Phi$ for simplicity of notation. First, let $v \in B$ with $\|v\|_{2,\alpha,\gamma} \leq \mu < 1$. Then, by the definition of $x^\gamma C^{2,\alpha}_\Phi(M\times[0,T])$ and the fact that $\Delta \in \mathrm{Diff}^2_\Phi(\mathrm{M})$, it follows that $\Delta v \in x^\gamma C^{\alpha}_\Phi(M\times[0,T])$. We can thus estimate

$$\begin{aligned} \|F_2(v)\|_{\alpha,\gamma} &\leq C_\mu \left(\|v\Delta v\|_{\alpha,\gamma} + \|v^2 s(v)\Delta v\|_{\alpha,\gamma}\right) \\ &\leq C_\mu \left(\|v\|_{\alpha,\gamma}\|\Delta v\|_{\alpha,\gamma} + \|v^2 s(v)\|_{\alpha,\gamma}\|\Delta v\|_{\alpha,\gamma}\right) \\ &\leq C_\mu \|v\|^2_{2,\alpha,\gamma}, \end{aligned}$$

for some $C_\mu > 0$ depending only on μ and possibly changing in each estimation step. This proves the second item and in particular boundedness of $F_2 : B \to x^\gamma C^{\alpha}_\Phi(M\times[0,T])$. For the first item we write for any $v, v' \in B$

$$v^2 s(v) - (v')^2 s(v') =: (v - v')O_1(v, v'), \tag{3.39}$$

where $O_1(v,v')$ is a polynomial combination in v and v'. Equation (3.39) implies

$$\begin{aligned} F_2(v) - F_2(v') = &- \frac{1}{\eta}\left(\Delta v(v - v') + v'\Delta(v - v')\right) \\ &+ O_1(v,v')\left((v - v')\Delta v + v'\Delta(v - v')\right), \end{aligned}$$

which then implies

$$\begin{aligned} \|F_2(v) - F_2(v')\|_{\alpha,\gamma} &\leq C_\mu \left(\|\Delta v\|_{\alpha,\gamma}\|v - v'\|_{\alpha,\gamma} + \|v'\|_{\alpha,\gamma}\|\Delta(v - v')\|_{\alpha,\gamma}\right. \\ &\quad \left. + \|\Delta v\|_{\alpha,\gamma}\|v - v'\|_{\alpha,\gamma} + \|v'\|_{\alpha,\gamma}\|v - v'\|_{2,\alpha,\gamma}\right) \\ &\leq C_\mu \max\{\|v\|_{2,\alpha,\gamma}, \|v'\|_{2,\alpha,\gamma}\}\|v - v'\|_{2,\alpha,\gamma}. \end{aligned}$$

□

Lemma 3.16. *Assume that* $\mathrm{scal}(g_\Phi) \in x^\gamma C^{1,\alpha}_\Phi(M)$*. Denote by B the open ball of radius* 1 *in* $x^\gamma C^{2,\alpha}_\Phi(M\times[0,T])$*. Then* F_1 *maps B into* $x^\gamma C^{1,\alpha}_\Phi(M\times[0,T])$*. Furthermore, if* $v, v' \in B$ *with* $\|v\|_{2,\alpha,\gamma}$, $\|v'\|_{2,\alpha,\gamma} \leq \mu < 1$*, there exists a constant* C_μ *such that*

1. $\|F_1(v) - F_1(v')\|_{1,\alpha,\gamma} \leq C_\mu \|v - v'\|_{2,\alpha,\gamma}$,
2. $\|F_1(v)\|_{1,\alpha,\gamma} \leq C_\mu$.

Proof. First, consider $v \in B \subset x^\gamma C^{2,\alpha}_\Phi(M \times [0,T])$. Since by assumption $\mathrm{scal}(g_\Phi) \in x^\gamma C^{1,\alpha}_\Phi(M)$, we find

$$\mathrm{scal}(g_\Phi)v^2(1-\eta s(v) - \eta v s(v)) \in C^{1,\alpha}_\Phi(M \times [0,T]).$$

Now, assume $\|v\|_{2,\alpha,\gamma} \leq \mu < 1$. We can now estimate

$$\|F_1(v)\|_{1,\alpha,\gamma} \leq C_\mu \|\mathrm{scal}(g_\Phi)\|_{1,\alpha,\gamma}\left(1 + \|v\|_{2,\alpha,\gamma} + \|v^2\|_{2,\alpha,\gamma}\right) \leq C_\mu,$$

for some $C_\mu > 0$ depending only on μ and possibly changing in each estimation step. This completes the proof for the second item. In particular, F_1 indeed maps B into $x^\gamma C^{1,\alpha}_\Phi(M \times [0,T])$. For the first item we have for any $v, v' \in B$ with $\|v\|_{2,\alpha,\gamma}, \|v'\|_{2,\alpha,\gamma} \leq \mu < 1$

$$\begin{aligned}\|F_1(v) - F_1(v')\|_{1,\alpha} \leq\ & C_\mu \|\mathrm{scal}(g_\Phi)\|_{1,\alpha,\gamma}\|v - v'\|_{2,\alpha,\gamma} \\ & + C_\mu \|\mathrm{scal}(g_\Phi)\|_{1,\alpha,\gamma}\|v^2 - (v')^2\|_{2,\alpha,\gamma} \\ & + C_\mu \|\mathrm{scal}(g_\Phi)\|_{1,\alpha,\gamma}\|v^2 s(v) - (v')^2 s(v')\|_{2,\alpha,\gamma} \\ & + C_\mu \|\mathrm{scal}(g_\Phi)\|_{1,\alpha,\gamma}\|v^3 s(v) - (v')^3 s(v')\|_{2,\alpha,\gamma} \\ \leq\ & C_\mu \|v - v'\|_{2,\alpha,\gamma},\end{aligned}$$

where in the final estimate we use (3.39) and its analogue for $v^3 s(v)$. This concludes the first item and, naturally, finishes the proof. □

Now, exactly the same argument as in Theorem 3.14 (with $a = 1$) implies directly, that for $\mathrm{scal}(g_\Phi) \in x^\gamma C^{1,\alpha}_\Phi(M)$ with $\gamma \geq 0$, the map $\mathbf{H} \star \circ (F_1 + F_2)$ is a contraction on a closed ball $\overline{B}_\mu \subset x^\gamma C^{2,\alpha}_\Phi(M \times [0,T])$ of radius $\mu > 0$, provided $\mu, T > 0$ are sufficiently small. Thus the flow (3.36) admits a solution $v \in \overline{B}_\mu$ as a fixed point of that contraction. Setting $u = 1 + v$, we obtain a short-time solution for the Yamabe flow (3.32) and thus to (2). The same argument yields a solution in $C^{k+2,\alpha}_\Phi(M \times [0,T])$ for a general $k \in \mathbb{N}_0$, provided $\mathrm{scal}(g_\Phi) \in x^\gamma C^{k+1,\alpha}_\Phi(M)$ for some $\gamma \geq 0$.

Theorem 3.17. *Consider a Φ-manifold (M, g_Φ) of dimension $m \geq 3$. Assume $\mathrm{scal}(g_\Phi) \in x^\gamma C^{k+1,\alpha}_\Phi(M)$ for some $\alpha \in (0,1)$, some $\gamma \geq 0$ and any $k \in \mathbb{N}_0$. Then the Yamabe flow (2) admits a unique solution $g = u^{4/(m-2)} g_\Phi$, where $u \in C^{k+2,\alpha}_\Phi(M \times [0,T])$, for some $T > 0$ sufficiently small.*

This proves Theorem 0.2.

3.4 Uniqueness of solutions

Proposition 3.18. *Consider the Yamabe flow equation as in (3.2)*

$$\partial_t u = (m-1)u^{-1/\eta}\Delta_\Phi u - \eta\, \mathrm{scal}(g_\Phi)u^{1-1/\eta}, \quad u|_{t=0} = u_0, \tag{3.40}$$

for some positive initial data $u_0 \in C^{2,\alpha}_\Phi(M)$. For such a Cauchy problem, a solution in $C^{2,\alpha}_\Phi(M \times [0,T])$ which is positive and bounded from below away from zero is unique for any given $0 < T < \infty$.

Proof. Suppose u and v are two solutions in $C^{2,\alpha}_{\Phi}(M\times[0,T])$ for (3.40). Consider $\omega = u - v \in C^{2,\alpha}_{\Phi}(M\times[0,T])$. Since $u|_{t=0} = v|_{t=0} = u_0$, we find $\omega|_{t=0} = 0$. Moreover, we infer from (3.40)

$$u^{1/\eta}\partial_t u - v^{1/\eta}\partial_t v = (m-1)\Delta_\Phi \omega - \eta\,\mathrm{scal}(g_\Phi)\omega.$$

From the definition of ω, we have

$$\begin{aligned}\partial_t\omega =& u^{-1/\eta}\left(u^{1/\eta}\partial_t u - v^{1/\eta}\partial_t v + (v^{1/\eta}-u^{1/\eta})\partial_t v\right)\\ =& u^{-1/\eta}\left((m-1)\Delta_\Phi\omega - \eta\,\mathrm{scal}(g_\Phi)\omega + (v^{1/\eta}-u^{1/\eta})\partial_t v\right)\\ =& -\left(\eta\,\mathrm{scal}(g_\Phi)u^{-1/\eta} + \frac{\partial_t v}{\eta}\int_0^1 (sv+(1-s)u)^{1/\eta-1}\,\mathrm{d}s\right)\omega\\ &+ (m-1)u^{-1/\eta}\Delta_\Phi\omega,\end{aligned}$$

where the last equality follows from Taylor's Theorem applied for the function $f(s) := (sv + (1-s)u)^{1/\eta}$. This means that ω is a solution of the equation

$$\partial_t\omega = a\Delta_\Phi\omega + b\omega,$$

with $a \in C^{2,\alpha}_{\Phi}(M\times[0,T])$ positive and $b \in C^{\alpha}_{\Phi}(M\times[0,T])$. Since nothing can be said about the sign of the b-term above, we consider any negative constant $c < -\|b\|_\infty$ and apply an integration factor trick by writing $\omega' = e^{ct}\omega$. We obtain an equation for z

$$\partial_t\omega' = a\Delta_\Phi\omega' + (b+c)\omega',$$

with $\omega'|_{t=0} = \omega|_{t=0} = 0$. Now, since $c < -\|b\|_\infty$, we have $(b+c) < 0$. From Corollary 2.8, it follows that $\omega' \equiv 0$ and, consequently, $\omega \equiv 0$. □

3.5 Curvature-normalized Yamabe flow

Consider the increasing curvature normalized Yamabe flow CYF$^+$

$$\partial_t g = (\mathrm{scal}(g)_{\sup} - \mathrm{scal}(g))g, \quad \text{where} \quad \mathrm{scal}(g(t))_{\sup} := \sup_M \mathrm{scal}(g(t)).$$

introduced by Suárez-Serrato and Tapie [SST12] to study entropy rigidity on the Yamabe flow in the compact setting. We are interested in the non-compact setting of a Φ-manifold (M, g_Φ), which is why the usual normalization by (3) does not work and we resort to the CYF$^+$ normalization. We can study the decreasing curvature normalized Yamabe flow CYF$^-$ with $\mathrm{scal}(g)_{\sup}$ replaced by $\mathrm{scal}(g)_{\inf}$ along the same lines.

Short time existence of CYF$^+$ (as well as CYF$^-$) follows by a simple time rescaling. Indeed, let $g(t) = u(t)^{1/\eta} g_\Phi$ be family of Riemannian metrics satisfying the (unnormalized) Yamabe flow (2) with $u \in C^{2,\alpha}_{\Phi}(M \times [0,T])$. Consider the functions

$$\begin{aligned} f(t) &= \exp\left(\int_0^t \eta \operatorname{scal}(g(\theta))_{\sup} \, \mathrm{d}\theta\right), \\ F(t) &= \int_0^t f(\theta)^{1/\eta} \, \mathrm{d}\theta - f(0)^{1/\eta}. \end{aligned} \tag{3.41}$$

Note that f is positive and F is a primitive for f satisfying $F(0) = 0$. Moreover, since $dF/dt > 0$, it follows that F^{-1} is well-defined. Thus, we can define a 1-parameter family of Riemannian metrics by

$$\widetilde{g}(\tau) := \widetilde{u}(\tau)^{1/\eta} g_{\Phi}, \quad \text{where} \quad \widetilde{u}(\tau) := (fu)(F^{-1}(\tau)). \tag{3.42}$$

One can easily check from $u \in C^{2,\alpha}_{\Phi}(M \times [0,T])$ that $\widetilde{u} \in C^{2,\alpha}_{\Phi}(M \times [0,\widetilde{T}])$ with $\widetilde{T} = \max F$.

Claim: The 1-parameter family of Riemannian metrics $\{\widetilde{g}(\tau) := \widetilde{u}(\tau)^{1/\eta} g_{\Phi}\}_{\tau}$ defined above satisfies the following normalized Yamabe flow:

$$\partial_{\tau}\widetilde{g} = \left(\operatorname{scal}(\widetilde{g})_{\sup} - \operatorname{scal}(\widetilde{g})\right)\widetilde{g}, \;\; \widetilde{g}(0) = g_{\Phi}. \tag{3.43}$$

In fact, first note that $\widetilde{u}(0) = 1$, which already proves the claim on the initial condition. Now, computations gives us the following:

$$\begin{aligned} \partial_{\tau}\widetilde{g}(\tau) =& \frac{1}{\eta}\widetilde{u}(\tau)^{1/\eta - 1}\partial_{\tau}\widetilde{u}(\tau) g_{\Phi} \\ =& \frac{1}{\eta}\widetilde{u}(\tau)^{1/\eta-1}\left[(\partial_t f \cdot u + f \cdot \partial_t u)(F^{-1}(\tau)) \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}F^{-1}(\tau)\right] g_{\Phi} \\ =& \frac{1}{\eta}\widetilde{u}(\tau)^{1/\eta-1}\left[\left(\frac{\partial_t f}{f} - \eta \operatorname{scal}(g)\right)(F^{-1}(\tau)) \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}F^{-1}(\tau)\right](fu)(F^{-1}(\tau)) g_{\Phi} \\ =& \frac{1}{\eta}\left[\left(\frac{\partial_t f}{f} - \eta \operatorname{scal}(g)\right)(F^{-1}(\tau)) \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}F^{-1}(\tau)\right]\widetilde{u}(\tau)^{1/\eta} g_{\Phi}. \end{aligned}$$

On the other hand, it follows from the Inverse Function Theorem that $\mathrm{d}F^{-1}/\mathrm{d}\tau = f^{-1/\eta}$. Moreover,

$$\log f(t) = \int_0^t \eta \operatorname{scal}(g(\theta))_{\sup} \, \mathrm{d}\theta \Longrightarrow \frac{\partial_t f}{f}(t) = \eta \operatorname{scal}(g(t))_{\sup}.$$

Finally, after recalling that the scalar curvature transforms under conformal change as in Proposition 1.3, plugging these identities into the above computations gives us

$$\partial_{\tau}\widetilde{g}(\tau) = \left(\operatorname{scal}(\widetilde{g}(\tau))_{\sup} - \operatorname{scal}(\widetilde{g}(\tau))\right)\widetilde{g}(\tau).$$

It is also possible to invert the process and obtain a solution of the standard Yamabe flow from a solution to CYF$^+$, proving said relation. This proves the following corollary of Theorem 0.2.

Corollary 3.19. *Let (M, g_{Φ}) be a m-dimensional Φ-manifold. Assume $\operatorname{scal}(g_{\Phi}) \in x^{\gamma}C^{k+1,\alpha}_{\Phi}(M)$ for some $\alpha \in (0,1)$, some $\gamma \geq 0$ and some $k \in \mathbb{N}_0$. Then both CYF$^+$ and CYF$^-$ admit a unique short-time solution $g = u^{4/(m-2)} g_{\Phi}$, where $u \in C^{k+2,\alpha}_{\Phi}(M \times [0,T])$, for some $T > 0$ sufficiently small.*

3.5.1 Some differential inequalities for solutions to CYF$^+$

First, we point out that all functions in $C^{2,\alpha}_{\Phi}(M\times[0,T])$ satisfy the conditions required in Proposition 2.6. As a direct consequence, we also obtain differential inequalities for solutions to the increasing curvature normalized Yamabe flow CYF$^+$. These will be central later in the derivation of a priori estimates.

Proposition 3.20. *Let $u\in C^{2,\alpha}_{\Phi}(M\times[0,T])$ be a positive (uniformly bounded away from zero) solution to the increasing curvature normalized Yamabe flow CYF$^+$ in (3.4). Then almost everywhere in $(0,T)$*

$$\begin{aligned}\frac{\partial}{\partial t}u_{\sup} &\leq \eta\sup_M \operatorname{scal}(g(t))\cdot u_{\sup}+\eta\sup_M|\operatorname{scal}(g_\Phi)|\cdot u_{\sup}^{1-1/\eta},\\ \frac{\partial}{\partial t}u_{\inf} &\geq \eta\sup_M \operatorname{scal}(g(t))\cdot u_{\inf}+\eta\inf_M|\operatorname{scal}(g_\Phi)|\cdot u_{\inf}^{1-1/\eta}.\end{aligned}\tag{3.44}$$

Proof. First, let us recall the expression for CYF$^+$ (which is satisfied by u by hypothesis):

$$\partial_t u(t)=(m-1)u(t)^{-1/\eta}\Delta u(t)\eta\left(\sup_M \operatorname{scal}(g(t))\cdot u(t)-\operatorname{scal}(g)u(t)^{1-1/\eta}\right)$$

We know, from (2.14), that if $\{p_k(t)\}_k$ is the Omori-Yau sequence for the supremum of u at time t, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}u(p_k(t),t) &\leq \frac{(m-1)}{k}\cdot u^{-1/\eta}(p_k(t),t)+\eta\sup_M\operatorname{scal}(g(t))\cdot u(p_k(t),t)\\ &\quad-\eta\operatorname{scal}(g_\Phi)(p_k(t))\cdot u(p_k(t),t)^{1-1/\eta}\\ &\leq \frac{(m-1)}{k}\cdot u^{-1/\eta}(p_k(t),t)+\eta\sup_M\operatorname{scal}(g(t))\cdot u(p_k(t),t)\\ &\quad+\eta\sup_M|\operatorname{scal}(g_\Phi)|(p_k(t))\cdot u(p_k(t),t)^{1-1/\eta},\end{aligned}\tag{3.45}$$

where the second inequality follows from $\eta>0$, since $m\geq 3$. Since u is positive and uniformly bounded away from zero, we conclude

$$\begin{aligned}\limsup_{k\to\infty}\frac{\partial u}{\partial t}(p_k(t),t) &\leq \eta\sup_M\operatorname{scal}(g(t))\cdot u_{\sup}(t)\\ &\quad+\eta\sup_M|\operatorname{scal}(g_\Phi)|\cdot u_{\sup}(t)^{1-1/\eta}.\end{aligned}\tag{3.46}$$

On the other hand, we know from the first statement of Proposition 2.6 that

$$\begin{aligned}\frac{\partial}{\partial t}u_{\sup}(t) &\leq \lim_{\varepsilon\to 0^+}\left(\limsup_{k\to\infty}\frac{\partial u}{\partial t}(p_k(t+\varepsilon),t+\varepsilon)\right)\\ &\leq \lim_{\varepsilon\to 0^+}\left(\eta\sup_M\operatorname{scal}(g(t+\varepsilon))\cdot u_{\sup}(t+\varepsilon)+\eta\sup_M|\operatorname{scal}(g_\Phi)|\cdot u_{\sup}(t+\varepsilon)^{1-1/\eta}\right).\end{aligned}$$

Finally, since $u|_{t_0+\varepsilon}$ converges uniformly $u|_{t_0}$ as $\varepsilon\to 0^+$ at any $t_0\in[0,T]$, it follows that $u_{\sup}(t+\varepsilon)$ converges to $u_{\sup}(t)$ as $\varepsilon\to 0^+$ and thus, the first statement follows. The second statement follows by (2.15) along the same lines. □

3.6 Evolution of the scalar curvature along CYF$^+$

Note that the increasing curvature normalized Yamabe flow (5) can be rewritten as (recall $\mathrm{scal}(g)_{\sup}$ denotes the supremum of $\mathrm{scal}(g)$)

$$\frac{1}{\eta}\partial_t u = \Big(\mathrm{scal}(g)_{\sup} - \mathrm{scal}(g)\Big)u. \tag{3.47}$$

From here we conclude immediately

$$\begin{aligned}\frac{1}{\eta}\partial_t(u^{-1}\Delta_\Phi u) &= -\frac{1}{\eta}u^{-2}\,\partial_t u\cdot\Delta_\Phi u + \frac{1}{\eta}u^{-1}\Delta_\Phi(\partial_t u)\\ &= -u^{-1}\big(\mathrm{scal}(g)_{\sup} - \mathrm{scal}(g)\big)\cdot\Delta_\Phi u + u^{-1}\Delta_\Phi\Big(\big(\mathrm{scal}(g)_{\sup} - \mathrm{scal}(g)\big)u\Big)\\ &= u^{-1}\Big(\mathrm{scal}(g)\cdot\Delta_\Phi u - \Delta_\Phi\big(\mathrm{scal}(g)u\big)\Big).\end{aligned} \tag{3.48}$$

Moreover, from Lemma 1.3 we obtain

$$\begin{aligned}u^{-1}\Delta_\Phi(\mathrm{scal}(g)u) &= u^{-1}\,\mathrm{scal}(g)\Delta_\Phi u + \Delta_\Phi\,\mathrm{scal}(g) + 2u^{-1}g_\Phi(\nabla u, \nabla\,\mathrm{scal}(g))\\ &= u^{-1}\,\mathrm{scal}(g)\Delta_\Phi u + u^{1/\eta}\Delta_g\,\mathrm{scal}(g),\end{aligned}$$

where Δ_g is the Laplace Beltrami operator of the conformally transformed metric $g = u^{1/\eta}\cdot g_\Phi$. Combined with (3.48) this gives

$$\frac{1}{\eta}\partial_t(u^{-1}\Delta_\Phi u) = -u^{1/\eta}\Delta_g\,\mathrm{scal}(g). \tag{3.49}$$

On the other hand, from (5) is also straightforward that

$$\partial_t u^{-1/\eta} = u^{-1/\eta}(\mathrm{scal}(g) - \mathrm{scal}(g)_{\sup}). \tag{3.50}$$

Finally, combining (3.48) and (3.50) with the transformation formula for the scalar curvature in Proposition 1.3,

$$\mathrm{scal}(g(t)) = \mathrm{scal}(u^{1/\eta}g_\Phi) = -u^{-1/\eta}\left[\frac{m-1}{\eta}u^{-1}\Delta_\Phi u - \mathrm{scal}(g_\Phi)\right]. \tag{3.51}$$

provides us the expression

$$\partial_t\,\mathrm{scal}(g) = (m-1)\Delta_g\,\mathrm{scal}(g) + \mathrm{scal}(g)(\mathrm{scal}(g) - \mathrm{scal}(g)_{\sup}). \tag{3.52}$$

Based on (3.52), we can now prove the following:

Lemma 3.21. *Suppose* $\mathrm{scal}(g_\Phi)\in C^{4,\alpha}_\Phi(M)$ *is negative and bounded away from zero*[1]*, that is, there are constants* $a_1, a_2 > 0$ *such that*

$$-\infty < -a_1 \le \mathrm{scal}(g_\Phi) \le -a_2 < 0. \tag{3.53}$$

Then along CYF$^+$ *with positive solution* $u\in C^{4,\alpha}_\Phi(M\times[0,T])$*, supremum of the the scalar* $\mathrm{scal}(g(t))_{\sup} = \sup_M \mathrm{scal}(g(t))$ *is non-increasing.*

[1] In fact, boundedness away from zero for the scalar curvature will only become important in the next section, but we list it here as a condition for consistency.

Proof. By Corollary 3.19, CYF$^+$ exists for short time in $C^{4,\alpha}_{\Phi}(M \times [0,T])$. From the transformation rule of the scalar curvature (3.51), it follows that $\mathrm{scal}(g) \in C^{2,\alpha}_{\Phi}(M \times [0,T])$. Now, from Proposition 2.6 it follows that $\mathrm{scal}(g)_{\sup}$ is differentiable in time for almost all $t \in [0,T]$. Applying the inequality (2.14) to $\mathrm{scal}(g)$ allows us to conclude that $\Delta_g \mathrm{scal}(g)_{\sup} \leq 0$. Plugging this inequality into (3.52), it follows that we have, for almost all $t \in [0,T]$,

$$\partial_t \mathrm{scal}(g(t))_{\sup} \leq \mathrm{scal}(g(t))_{\sup}(\mathrm{scal}(g(t))_{\sup} - \mathrm{scal}(g(t))_{\sup}) = 0. \tag{3.54}$$

This implies directly that $\mathrm{scal}(g)_{\sup}$ is non-increasing along CYF$^+$. □

Knowing that the supremum of the scalar curvature is non-increasing in time, the next result shows that the scalar curvature approaches its supremum at an exponential rate.

Lemma 3.22. *Suppose* $\mathrm{scal}(g_\Phi) \in C^{4,\alpha}_{\Phi}(M)$ *is negative, bounded away from zero as in Lemma 3.21. Then along CYF$^+$ with positive solution* $u \in C^{4,\alpha}_{\Phi}(M \times [0,T])$*, we have the estimate*

$$\| \mathrm{scal}(g(t))_{\inf} - \mathrm{scal}(g(t))_{\sup} \|_\infty \leq C e^{\mathrm{scal}(g_\Phi)_{\sup} \cdot t},$$

with $C > 0$ *a constant independent of* T*, where* $\mathrm{scal}(g(t))_{\inf} := \inf_M \mathrm{scal}(g(t))$.

Proof. Applying the arguments of §2.3 to $\mathrm{scal}(g)$, we conclude from (3.52) by Proposition 2.6, similar to Corollary 3.20, for almost all $t \in [0,T]$

$$\partial_t \mathrm{scal}(g)_{\inf} \geq \mathrm{scal}(g)_{\inf}(\mathrm{scal}(g)_{\inf} - \mathrm{scal}(g)_{\sup}). \tag{3.55}$$

From here it follows that $\mathrm{scal}(g)_{\inf}$ is non-decreasing along the CYF$^+$. Combining (3.55) with (3.54), we find

$$\begin{aligned}
\partial_t(\mathrm{scal}(g)_{\sup} - \mathrm{scal}(g)_{\inf}) &\leq -\mathrm{scal}(g)_{\inf}(\mathrm{scal}(g)_{\inf} - \mathrm{scal}(g)_{\sup}) \\
&= \mathrm{scal}(g)_{\inf}(\mathrm{scal}(g)_{\sup} - \mathrm{scal}(g)_{\inf}) \\
&\leq \mathrm{scal}(g_\Phi)_{\sup}(\mathrm{scal}(g)_{\sup} - \mathrm{scal}(g)_{\inf}).
\end{aligned}$$

Integrating both sides of the last inequality gives

$$(\mathrm{scal}(g)_{\sup} - \mathrm{scal}(g)_{\inf})(t) \leq C e^{\mathrm{scal}(g_\Phi)_{\sup} t}, \tag{3.56}$$

where C depends only on the initial data. This means that the difference between the supremum and the infimum of the scalar curvature decreases exponentially along the flow. Consequently, the scalar curvature approaches $\mathrm{scal}(g)_{\sup}$ at an exponential rate too, therefore implying the desired outcome. □

3.7 Uniform estimates along CYF$^+$

We start immediately with the central result of the section. If we assume $\mathrm{scal}(g_\Phi) \in C^{4,\alpha}_\Phi(M)$, then the solution $u \in C^{4,\alpha}_\Phi(M \times [0,T'])$ of CYF$^+$ exists by Corollary 3.19 for $T' > 0$ sufficiently small. Assume u in fact lies in $C^{4,\alpha}_\Phi(M \times [0,T))$, for a maximal time $T \geq T'$. Then even in the maximal time interval $[0,T)$ we obtain T-independent a priori estimates.

Theorem 3.23. *Assume* $\mathrm{scal}(g_\Phi) \in C^{4,\alpha}_\Phi(M)$ *is negative and bounded away from zero as in Lemma 3.22. Let* $u \in C^{4,\alpha}_\Phi(M \times [0,T))$ *be the solution of CYF*$^+$ *extended to a maximal time interval* $[0,T)$*. Then there exist constants* $c_1, c_2 > 0$*, depending on* $u(0), \sup|\mathrm{scal}(g_\Phi)|$ *and* $\inf|\mathrm{scal}(g_\Phi)|$*, and independent of* T*, such that*

$$0 < c_1 \leq u(p,t) \leq c_2, \text{ for all } (p,t) \in M \times [0,T).$$

Proof. First, we consider the flow for a short time intervall $[0,T']$, where u is guaranteed to be positive. The estimates below will show that u stay positive, bounded away from zero uniformly on $[0,T']$ and thus all of the arguments hold on the maximal intervall $[0,T)$. By the differential inequalities in Proposition 3.20 we have (a priori almost everywhere on $[0,T']$, however as just explained a posteriori almost everywhere on the full time interval)

$$\begin{aligned} \frac{\partial}{\partial t} u_{\inf} &\geq \eta \sup_M \mathrm{scal}(g(t)) \cdot u_{\inf} + \eta \inf_M |\mathrm{scal}(g_\Phi)| \cdot u_{\inf}^{1-1/\eta}, \\ \frac{\partial}{\partial t} u_{\sup} &\leq \eta \sup_M \mathrm{scal}(g(t)) \cdot u_{\sup} + \eta \sup_M |\mathrm{scal}(g_\Phi)| \cdot u_{\sup}^{1-1/\eta}. \end{aligned} \tag{3.57}$$

Multipying both sides of the first intequality by $\frac{1}{\eta} u_{\inf}^{1/\eta - 1}$, and of the second inequality by $\frac{1}{\eta} u_{\sup}^{1/\eta-1}$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} u_{\inf}^{1/\eta} &\geq \sup_M \mathrm{scal}(g(t)) \cdot u_{\inf}^{1/\eta} + \inf_M |\mathrm{scal}(g_\Phi)|, \\ \frac{\partial}{\partial t} u_{\sup}^{1/\eta} &\leq \sup_M \mathrm{scal}(g(t)) \cdot u_{\sup}^{1/\eta} + \sup_M |\mathrm{scal}(g_\Phi)|. \end{aligned} \tag{3.58}$$

Write $\omega_1 := u_{\inf}^{1/\eta}$ and $\omega_2 := u_{\sup}^{1/\eta}$. We obtain from (3.58)

$$\begin{aligned} \frac{\partial}{\partial t} \omega_1 &\geq \inf_M \mathrm{scal}(g_\Phi) \cdot \omega_1 + \inf_M |\mathrm{scal}(g_\Phi)| =: b\omega_1 + a, \\ \frac{\partial}{\partial t} \omega_2 &\leq \sup_M \mathrm{scal}(g_\Phi) \cdot \omega_2 + \sup_M |\mathrm{scal}(g_\Phi)| =: B\omega_2 + A, \end{aligned} \tag{3.59}$$

where in the first inequality we used the fact that by (3.55) $\mathrm{scal}(g)_{\inf}$ is non-decreasing in time, while the second inequality from Lemma 3.21, since $\mathrm{scal}(g)_{\sup}$ is non-increasing in time.

The first inequality is equivalent to $(e^{-bt}\omega_1)' \geq ae^{-bt}$. Hence, integration on both sides over $[0,t]$ gives the following estimate

$$\omega_1(t) \geq e^{bt}\omega_1(0) + \frac{a}{b}(e^{bt} - 1)$$

$$\iff \quad u_{\inf}^{1/\eta}(t) \geq u_{\inf}^{1/\eta}(0)e^{\inf_M \operatorname{scal}(g_\Phi)\cdot t} + \frac{\inf_M|\operatorname{scal}(g_\Phi)|}{\inf_M \operatorname{scal}(g_\Phi)}\left(e^{\inf_M \operatorname{scal}(g_\Phi)\cdot t} - 1\right)$$

$$\iff \quad u_{\inf}^{1/\eta}(t) \geq u_{\inf}^{1/\eta}(0)e^{\inf_M \operatorname{scal}(g_\Phi)\cdot t} + \frac{\inf_M|\operatorname{scal}(g_\Phi)|}{\sup_M|\operatorname{scal}(g_\Phi)|}\left(1 - e^{\inf_M \operatorname{scal}(g_\Phi)\cdot t}\right)$$

Hence, by setting the right-hand side as a function $f(t)$, it follows that $u_{\inf}^{1/\eta}(t) \geq f(t)$, with

$$f(t) := c\cdot e^{d\cdot t} + c'(1-e^{d\cdot t}), \text{ with } c = u_{\inf}^{1/\eta}(0),\ c' = \frac{\inf_M|\operatorname{scal}(g_\Phi)|}{\sup_M|\operatorname{scal}(g_\Phi)|} \text{ and } d = \operatorname{scal}(g_\Phi)_{\inf}.$$

Direct computations show that $f'(t) = (c-c')d\cdot e^{d\cdot t} \neq 0$ for all t, as long as $c \neq c'$. Hence, if $c \neq c'$, it follows that either $f'(t) < 0$ or $f'(t) > 0$, which means that $f(t)$ is a monotonous function in t. Therefore,

$$u_{\inf}^{1/\eta}(t) \geq f(t) \geq \min\left\{u_{\inf}^{1/\eta}(0), \frac{\inf_M|\operatorname{scal}(g_\Phi)|}{\sup_M|\operatorname{scal}(g_\Phi)|}\right\} > 0.$$

This yields *a priori* positive lower bound for u. On the other hand, if $c = c'$, then it follows straightforwardly that $u_{\inf}^{1/\eta}(0)$ is *a priori* lower bound for $u_{\inf}^{1/\eta}(t)$. Now, let us turn our attention to the second equation in (3.59). This inequality is equivalent $(e^{Bt}\omega_2)' \leq Ae^{-Bt}$, which after integrating over $[0,t]$ implies

$$\omega_2(t) \leq \omega_2(0)e^{Bt} - \frac{A}{B}(1-e^{Bt}) \leq \omega_2(0)\frac{A}{B}(1-e^{Bt})$$

$$\iff \quad u_{\sup}^{1/\eta}(t) \leq u_{\sup}^{1/\eta}(0) + \frac{\sup_M|\operatorname{scal}(g_\Phi)|}{\inf_M|\operatorname{scal}(g_\Phi)|}\left(1 - e^{\sup_M \operatorname{scal}(g_\Phi)\cdot t}\right).$$

Proceeding along the lines of the estimate for $u_{\inf}^{1/\eta}(t)$, consider the right-hand side as a function $F(t)$, where

$$F(t) := C + C'(1-e^{D\cdot t}), \text{ with } C = u_{\sup}^{1/\eta}(0),\ C' = \frac{\sup_M|\operatorname{scal}(g_\Phi)|}{\inf_M|\operatorname{scal}(g_\Phi)|} \text{ and } D = \operatorname{scal}(g_\Phi)_{\sup}.$$

Note that $F'(t) = -C'D\cdot e^{D\cdot t} > 0$ for all t, since $C' > 0$ and $D < 0$. This means $F(t)$ increases in t and, therefore,

$$u_{\sup}^{1/\eta}(t) \leq F(t) \leq \lim_{t\to+\infty} F(t) = u_{\sup}^{1/\eta}(0) + \frac{\sup_M|\operatorname{scal}(g_\Phi)|}{\inf_M|\operatorname{scal}(g_\Phi)|} < +\infty,$$

concluding the proof. □

Proposition 3.24. *Assume* $\operatorname{scal}(g_\Phi) \in C_\Phi^{4,\alpha}(M)$ *is negative and bounded away from zero as in Lemma 3.22. Let* $u \in C_\Phi^{4,\alpha}(M\times[0,T))$ *be the solution of CYF$^+$ extended to a maximal time interval* $[0,T)$*. Then there exists a constant* $C > 0$*, depending on* $u(0), \sup|\operatorname{scal}(g_\Phi)|$ *and* $\inf|\operatorname{scal}(g_\Phi)|$*, and independent of* T*, such that*

$$\|\partial_t u\|_\infty \leq Ce^{\sup_M \operatorname{scal}(g_\Phi)\cdot t}. \tag{3.60}$$

Proof. The CYF$^+$ flow (5) can be rewritten as (cf. (3.47))

$$\frac{1}{\eta}\partial_t u = \left(\operatorname{scal}(g)_{\sup} - \operatorname{scal}(g)\right)u. \tag{3.61}$$

Then, employing Lemma 3.22 and Theorem 3.23, it follows directly that

$$\begin{aligned}\|\partial_t u\|_\infty &\leq |\eta| \| \operatorname{scal}(g)_{\sup} - \operatorname{scal}(g)\|_\infty \|u\|_\infty \\ &\leq C e^{\sup_M \operatorname{scal}(g_\Phi)\cdot t}.\end{aligned}$$

□

3.8 Parabolic Schauder estimates on Φ-manifolds

Consider for any fixed $\delta > 0$ a countable family of points $\{p_i\} \in M$, such that the δ-balls $B_\delta(p_i)$ around these points (with distance measured with respect to g_Φ) cover M. Let $\delta > 0$ be sufficiently small, such that the δ-balls stay inside local coordinate neighborhoods. Obviously we are interested only in those $p_i = (x_i, y_i, z_i) \in U$ in the collar neighborhood of the boundary ∂M. Writing $B_\delta(0) \in \mathbb{R}^m$ for an open ball of radius δ around the origin, we define

$$\begin{aligned}\Psi_i :& B_\delta(0) \times [0,\delta^2] =: Q_\delta \to B_\delta(p_i) \times [0,\delta^2], \\ &(\mathscr{S},\mathscr{U},\mathscr{Z},t) \mapsto \Big(x := x_i + x_i^2 \mathscr{S}, y := y_i + x_i \mathscr{U}, z := z_i + \mathscr{Z}, t\Big).\end{aligned} \tag{3.62}$$

Away from the collar U of the boundary, we may define Ψ_i as usual local coordinate parametrizations. Clearly, the choice of Ψ_i and the notation $(\mathscr{S},\mathscr{U},\mathscr{Z})$ is motivated by the projective coordinates (2.27). We compute the action of Φ-derivatives under the pullback by the transformation Ψ_i:

$$\begin{aligned}\Psi_i^*\big(\partial_x u\big)(\mathscr{S},\mathscr{U},\mathscr{Z},t) &= \partial_{\mathscr{S}}\big(u \circ \Psi_i\big)(\mathscr{S},\mathscr{U},\mathscr{Z},t) \\ &= \partial_x u(\Psi_i(\mathscr{S},\mathscr{U},\mathscr{Z},t)) \cdot \partial_{\mathscr{S}} \Psi_i(\mathscr{S},\mathscr{U},\mathscr{Z},t) \\ &= x_i^2 \partial_x u(\Psi_i(\mathscr{S},\mathscr{U},\mathscr{Z},t)).\end{aligned}$$

From this it follows that

$$\begin{aligned}(1 + x_i \mathscr{S})^2 \partial_{\mathscr{S}} \Psi_i^* u(\mathscr{S},\mathscr{U},\mathscr{Z},t) &= x^2 \partial_x u(\Psi_i(\mathscr{S},\mathscr{U},\mathscr{Z},t)) \\ &= \Psi_i^*\big(x^2 \partial_x u\big)(\mathscr{S},\mathscr{U},\mathscr{Z},t).\end{aligned} \tag{3.63}$$

Analogously, straightforward computations give us

$$\begin{aligned}\Psi_i^*\big(x \partial_y u\big)(\mathscr{S},\mathscr{U},\mathscr{Z},t) &= (1 + x_i \mathscr{S}) \partial_{\mathscr{U}} \Psi_i^* u(\mathscr{S},\mathscr{U},\mathscr{Z},t), \\ \Psi_i^*\big(\partial_z u\big)(\mathscr{S},\mathscr{U},\mathscr{Z},t) &= \partial_Z \Psi_i^* u(\mathscr{S},\mathscr{U},\mathscr{Z},t).\end{aligned} \tag{3.64}$$

On the other hand, one can see explicitly by an easy computation that

$$\Delta_\Phi|_U = x^4 \partial_x^2 + x^2 \Delta_B + \Delta_F + (\text{first order derivatives}). \tag{3.65}$$

Hence we obtain for the heat equation

$$\Psi_i^*\Big((\partial_t - \Delta_\Phi) u\Big) = \Big(\partial_t - \widetilde{\Delta_\Phi}\Big) \Psi_i^* u, \tag{3.66}$$

where $\widetilde{\Delta_\Phi} = \partial_{\mathscr{S}}^2 + \partial_{\mathscr{U}}^2 + \partial_{\mathscr{Z}}^2$ plus first order derivatives in $(\mathscr{S},\mathscr{U},\mathscr{Z})$, up to coefficients that are bounded in Q_δ, uniformly in i. Moreover we observe the following:

Lemma 3.25. *Consider the classical Hölder space $C^{k,\alpha}(Q_\delta)$ with Hölder norm denoted by $\|\cdot\|_{k,\alpha,Q_\delta}$. Then the Hölder norm $\|\cdot\|_{k,\alpha}$ on $C^{k,\alpha}_\Phi(M\times[0,\delta^2])$ defined in terms of (3.8), is equivalent to*

$$\sup_i \|\Psi_i^* u\|_{k,\alpha,Q_\delta}.$$

Proof. The statement follows from (3.63), (3.64) and the fact that, taking the local expression of d_Φ in Definition 3.1, we find in the collar U (we denote the transformation (3.62) without the time variable, again by Ψ_i)

$$d_\Phi\Big(\Psi_i(\mathscr{S},\mathscr{U},\mathscr{Z}),\Psi_i(\mathscr{S}',\mathscr{U}',\mathscr{Z}')\Big)\sim\|\big(\mathscr{S}-\mathscr{S}',\mathscr{U}-\mathscr{U}',\mathscr{Z}-\mathscr{Z}'\big)\|. \tag{3.67}$$

In fact, let us prove this. First, one must understand why (3.67) is enough. The following explanation assumes $k=0$; the general case can be proven analogously. Note that if $u\in C^\alpha_\Phi(M\times[0,\delta^2])$, we have

$$\begin{aligned}|\Psi_i^*u(\mathscr{S},\mathscr{U},\mathscr{Z},t)-\Psi_i^*u(\mathscr{S}',\mathscr{U}',\mathscr{Z}',t')|&\le|u(\Psi_i(\mathscr{S},\mathscr{U},\mathscr{Z},t))-u(\Psi_i(\mathscr{S}',\mathscr{U}',\mathscr{Z}',t))|\\&\quad+|u(\Psi_i(\mathscr{S}',\mathscr{U}',\mathscr{Z}',t))-u(\Psi_i(\mathscr{S}',\mathscr{U}',\mathscr{Z}',t'))|.\end{aligned}$$

Estimates for the second of the two terms above follows directly from the fact that $u\in C^\alpha_\Phi(M\times[0,\delta^2])$. On the other hand, the first can by estimate as

$$|u(\Psi_i(\mathscr{S},\mathscr{U},\mathscr{Z},t))-u(\Psi_i(\mathscr{S}',\mathscr{U}',\mathscr{Z}',t))|\le\|u\|_\alpha d_\Phi\Big(\Psi_i(\mathscr{S},\mathscr{U},\mathscr{Z}),\Psi_i(\mathscr{S}',\mathscr{U}',\mathscr{Z}')\Big)^\alpha,$$

proving our initial claim. Thus, let us know prove (3.67). We have the following:

$$\begin{aligned}d_\Phi\Big(\Psi_i(\mathscr{S},\mathscr{U},\mathscr{Z}),\Psi_i(\mathscr{S}',\mathscr{U}',\mathscr{Z}')\Big)^2&=\frac{|x_i^2(\mathscr{S}-\mathscr{S}')|^2}{x_i^4(2+x_i(\mathscr{S}+\mathscr{S}'))^4}\\&\quad+\frac{\|x_i(\mathscr{U}-\mathscr{U}')\|^2}{x_i^2(2+x_i(\mathscr{S}+\mathscr{S}'))^2}\\&\quad+\|\mathscr{Z}-\mathscr{Z}'\|^2\\&=(2+x_i(\mathscr{S}+\mathscr{S}'))^{-4}|\mathscr{S}-\mathscr{S}'|^2\\&\quad+(2+x_i(\mathscr{S}+\mathscr{S}'))^{-2}\|\mathscr{U}-\mathscr{U}'\|^2\\&\quad+\|\mathscr{Z}-\mathscr{Z}'\|^2.\end{aligned}$$

Now, remember that we are working on a δ-neighborhood of the origin and x_i is sufficiently small, implying $(2+x_i(\mathscr{S}+\mathscr{S}'))$ to be bounded both from above and below away from zero. From this we can conclude (3.67). From this it follows that $\Psi_i^*u\in C^\alpha(Q_\delta)$. The converse follows analogously. □

Now we are ready to convert the a priori estimates in Theorem 3.23 into uniform Hölder regularity on $[0,T]$, where $[0,T)$ is the maximal time intervall, where the CYF$^+$ flow solution u exists in $C^{4,\alpha}_\Phi(M\times[0,T))$. We use the classical Krylov-Safonov estimates, see [KS80] and the exposition in [Pic19, Theorem 12].

Proposition 3.26. *Assume* $\mathrm{scal}(g_\Phi) \in C^{4,\alpha}_\Phi(M)$ *is negative and bounded away from zero. Let* $u \in C^{4,\alpha}_\Phi(M \times [0,T))$ *to be the solution of CYF*$^+$ *extended to a maximal time interval* $[0,T)$*. Then* $u \in C^{\alpha}_\Phi(M \times [0,T])$ *with* T*-independent Hölder norm.*

Proof. Consider the CYF$^+$ flow equation in (3.4)

$$\partial_t u(t) - (m-1)u(t)^{-1/\eta}\Delta_\Phi u(t) = \eta\left(\sup_M \mathrm{scal}(g(t))\cdot u(t) - \mathrm{scal}(g_\Phi)u(t)^{1-1/\eta}\right) =: \ell.$$

Pulling back under Ψ_i we obtain with $a := (m-1)\Psi_i^* u^{-1/\eta}$

$$\left(\partial_t - a\cdot\widetilde{\Delta_\Phi}\right)\Psi_i^* u = \Psi_i^*\ell. \tag{3.68}$$

From Theorem 3.23 we infer that $\Psi_i^*\ell$ and u, u^{-1} (and hence also a, a^{-1}) are bounded in Q_δ, uniformly in i, since u is bounded from below away from zero. Thus by the Krylov-Safonov estimate, see [KS80] and cf. [Pic19, Theorem 12], we find for some uniform constant $C > 0$, depending only on $\delta, \|u\|_\infty$ and $\|u^{-1}\|_\infty$

$$\|\Psi_i^* u\|_{\alpha,Q_\delta} \leq C\left(\|\Psi_i^* u\|_{\infty,Q_\delta} + \|\Psi_i^*\ell\|_{\infty,Q_\delta}\right) \leq C\left(\|u\|_\infty + \|\ell\|_\infty\right).$$

Thus $\Psi_i^* u \in C^\alpha(Q_\delta)$. By Lemma 3.25 we conclude $u \in C^\alpha_\Phi(M \times [0,\delta^2])$. We extend the regularity statement to the whole time interval $[0,T]$ (with constants independent of T) iteratively, by setting $t = \delta^2 + t'$ and obtaining by the argument above $u \in C^\alpha_\Phi(M \times [\delta^2, 2\delta^2])$, and repeating the iteration, until we reach T. □

This first gain in Hölder regularity can now be converted into higher oder regularity by standard parabolic Schauder estimates, see [Kry96] and the exposition in [Pic19, Theorem 6].

Proposition 3.27. *Assume* $\mathrm{scal}(g_\Phi) \in C^{4,\alpha}_\Phi(M)$ *is negative and bounded away from zero. Let* $u \in C^{4,\alpha}_\Phi(M \times [0,T))$ *to be the solution of CYF*$^+$ *extended to a maximal time interval* $[0,T)$*. Then* $u \in C^{4,\alpha}_\Phi(M \times [0,T])$ *with* T*-independent Hölder norm.*

Proof. Consider (3.68). Standard parabolic Schauder estimates, see [Kry96] and cf. [Pic19, Theorem 6], assert that for any $a \in C^{k,\alpha}(Q_\delta)$ positive, uniformly bounded from below away from zero, and for any $\Psi_i^* u \in C^{k,\alpha}(Q_\delta)$, a uniformly bounded solution $\Psi_i^* u$ satisfies

$$\|\Psi_i^* u\|_{k+2,\alpha,Q_\delta} \leq C\left(\|\Psi_i^* u\|_{\infty,Q_\delta} + \|\Psi_i^*\ell\|_{k,\alpha,Q_\delta}\right) \leq C\left(\|u\|_\infty + \|\ell\|_{k,\alpha}\right). \tag{3.69}$$

By Lemma 3.26, $a = (m-1)\Psi_i^* u^{-1/\eta} \in C^\alpha(Q_\delta)$ and $\Psi_i^*\ell \in C^\alpha(Q_\delta)$ uniformly in i. Thus we may apply (3.69) with $k = 0$ and conclude that $\Psi_i^* u \in C^{2,\alpha}(Q_\delta)$, uniformly in i. By Lemma 3.25 we conclude $u \in C^{2,\alpha}_\Phi(M \times [0,\delta^2])$. As before, we may extend the regularity statement to the whole time interval $[0,T]$ (with constants independent of T) by setting $t = \delta^2 + t'$, concluding $u \in C^\alpha_\Phi(M \times [\delta^2, 2\delta^2])$, and repeating the iteration, until we reach T.

Since $\mathrm{scal}(g_\Phi) \in C^{4,\alpha}_\Phi(M)$, this implies that $a \in C^{2,\alpha}(Q_\delta)$ and $\Psi_i^*\ell \in C^{2,\alpha}(Q_\delta)$ uniformly in i. Applying now (3.69) with $k=2$, we conclude exactly as above $u \in C^{4,\alpha}_\Phi(M\times[0,T])$. In fact, in case $\mathrm{scal}(g_\Phi) \in C^{k,\alpha}_\Phi(M)$, we can iterate the arguments until $u \in C^{k+2,\alpha}_\Phi(M\times[0,T])$. This proves the statement. □

Remark 3.28. The arguments above show that in fact, if $\mathrm{scal}(g_\Phi) \in C^{k,\alpha}_\Phi(M)$ with $k\geq 4$ is negative and bounded away from zero, the CYF$^+$ flow solution $u \in C^{2,\alpha}_\Phi(M\times[0,T])$ on any time interval $[0,T]$ is in fact in $C^{k,\alpha}_\Phi(M\times[0,T])$.

3.9 Global existence of the CYF$^+$ on Φ-manifolds

We prove global existence of the flow, i.e. $u \in C^{4,\alpha}_\Phi(M\times[0,\infty))$ by a contradiction. Assume the maximal time $T>0$ is finite. In that case we will now restart the flow at $t=T$, which contradicts maximality of T. Restarting the flow at $t=T$ means constructing a solution u' to the (unnormalized) Yamabe flow equation (3.2) with initial condition $u'|_{t=0} = u|_{t=T}$. A rescaling of the time function, as in §3.5, yields short time existence of the curvature normalized Yamabe flow.

Let us simplify notation by writing $u_0 = u|_{t=T}$ and $\Delta = \Delta_\Phi$. We linearize (3.2) by setting $u' = u_0 + v$ for its solution with initial condition $u'|_{t=0} = u_0$. We obtain from the second equation in (3.2)

$$\left(\partial_t - (m-1)u_0^{-1/\eta}\Delta\right) v = F_1(v) + F_2(v);\ \ v|_{t=0} = 0, \tag{3.70}$$

where we have abbreviated

$$F_1(v) = Q_2(v), \quad F_2(v) = (m-1)u_0^{-1/\eta}\Delta u_0 - \mathrm{scal}(g_\Phi)u_0^{1-1/\eta} + Q_1(v),$$

The terms $Q_1(v)$ include linear combinations of v with coefficients given in terms of u_0 and Δu_0. The terms $Q_2(v)$ include quadratic combinations of v and Δv with coefficients given again in terms of u_0 and Δu_0.

Note that by Proposition 3.27, $u_0 \in C^{4,\alpha}_\Phi(M)$. Thus, F_1 contains quadratic combinations of v and Δv, and F_2 – linear combinations of v; with coefficients being in both cases elements of $C^{2,\alpha}_\Phi(M\times[0,T'])$.

Before we can establish short time existence of v, which we will do by setting up a fixed point as in §3.3, we note a general result from parabolic Schauder theory.

Proposition 3.29. *Consider $a \in C^{k,\alpha}_\Phi(M)$ positive, uniformly bounded away from zero. Then the inhomogeneous heat equation $(\partial_t - a\cdot\Delta_\Phi)v = \ell$, with $v|_{t=0} = 0$ and $\ell \in C^{k,\alpha}_\Phi(M\times[0,T'])$, has a parametrix Q acting as a bounded linear map*

$$\mathbf{Q}: C^{k,\alpha}_\Phi(M\times[0,T']) \to \left(C^{k+2,\alpha}_\Phi \cap t\, C^{k,\alpha}_\Phi\right)(M\times[0,T']). \tag{3.71}$$

Proof. Consider the inhomogeneous heat equation with $\ell \in C^{k,\alpha}_{\Phi}(M\times[0,T'])$ and initial value $v_0 \in C^{k+2,\alpha}_{\Phi}(M)$:

$$(\partial_t - a\cdot\Delta_\Phi)v = \ell, \qquad v|_{t=0} = v_0.$$

Then, by reducing the argument to local δ-balls as in §3.8, we can follow the proof of [LSUc68, Theorem 5.1 on p.320], and conclude for some uniform constant $C>0$ existence of a unique solution $v\in C^{k+2,\alpha}_{\Phi}(M\times[0,T'])$ with

$$\|v\|_{k+2,\alpha} \le C\Big(\|\ell\|_{k,\alpha} + \|v_0\|_{k+2,\alpha}\Big).$$

This proves the first mapping property in (3.71) by setting $v_0 = 0$. For the second mapping property in (3.71), set $\ell = 0$ and obtain a solution $v = \mathbf{R}v_0$ with the solution operator $\mathbf{R}$ acting as a bounded linear map

$$\mathbf{R} : C^{k,\alpha}_{\Phi}(M) \to C^{k,\alpha}_{\Phi}(M\times[0,T']).$$

The solution operator $\mathbf{Q}$ of the inhomogeneous problem is then given by

$$\mathbf{Q}\ell(p,t) = \int_0^t \Big(\mathbf{R}\ell(\widetilde{t})\Big)(p,t-\widetilde{t})d\widetilde{t}. \tag{3.72}$$

Indeed, a direct computation shows

$$\begin{aligned}\Big(\partial_t - a\cdot\Delta_\Phi\Big)\mathbf{Q}\ell(p,t) &= \ell(p,t) + \int_0^t \Big(\partial_t - a\cdot\Delta_\Phi\Big)\Big(\mathbf{R}\ell(\widetilde{t})\Big)(p,t-\widetilde{t})d\widetilde{t} \\ &= \ell(p,t).\end{aligned}$$

This also allow us to conclude the second mapping property. Let us show this for $k=0$; the general case follows analogously. In fact, we have

$$|\mathbf{Q}\ell(p,t)| \le \|\mathbf{R}\ell\|_\infty \int_0^t \mathrm{d}\widetilde{t} = t\,\|\mathbf{R}\ell\|_\infty, \text{ for all } (p,t)\in M\times[0,T'].$$

On the other hand, we can estimate the Hölder brackets. Take a pair of points $(p,t),(p',t)\in M\times[0,T']$, assuming (without loss of generality) $t<t'$. Hence,

$$\begin{aligned}|\mathbf{Q}\ell(p,t) - \mathbf{Q}\ell(p',t')| &\le |\mathbf{Q}\ell(p,t) - \mathbf{Q}\ell(p',t)| + |\mathbf{Q}\ell(p',t) - \mathbf{Q}\ell(p',t')| \\ &= \Big|\int_0^t \Big(\Big(\mathbf{R}\ell(\widetilde{t})\Big)(p,t-\widetilde{t}) - \Big(\mathbf{R}\ell(\widetilde{t})\Big)(p',t-\widetilde{t})\Big)\,\mathrm{d}\widetilde{t}\Big| \\ &+ \Big|\int_t^{t'} \Big(\Big(\mathbf{R}\ell(\widetilde{t})\Big)(p,t-\widetilde{t}) - \Big(\mathbf{R}\ell(\widetilde{t})\Big)(p',t'-\widetilde{t})\Big)\,\mathrm{d}\widetilde{t}\Big| \\ &\le \|\mathbf{R}\ell\|_\alpha d_\Phi(p,p')^\alpha \int_0^t \mathrm{d}\widetilde{t} + \|\mathbf{R}\ell\|_\alpha |t-t'|^{\alpha/2}\int_t^{t'}\mathrm{d}\widetilde{t},\end{aligned}$$

which means $\|\mathbf{Q}\ell\|_\alpha \le Ct$. This implies directly the second mapping property in (3.71) and completes the proof. □

We can now conclude with the proof of Theorem 0.3.

Corollary 3.30. *Assume* $\operatorname{scal}(g_\Phi) \in C^{k,\alpha}_\Phi(M)$ *is negative and bounded away from zero with* $k \geq 4$. *Then the increasing curvature normalized Yamabe flow CYF$^+$ exists for all times with conformal factor* $u \in C^{k,\alpha}_\Phi(M \times [0,\infty))$.

Proof. Using Proposition 3.29, we can construct a solution $v \in C^{2,\alpha}_\Phi(M \times [0,T'])$ to (3.70) for some $T' > 0$ sufficiently small, as a fixed point of

$$\mathbf{Q} \circ (F_1 + F_2) : C^{2,\alpha}_\Phi(M \times [0,T']) \to C^{2,\alpha}_\Phi(M \times [0,T']), \tag{3.73}$$

in the same way as in §3.3. Rescaling time as in §3.5, we obtain a solution $u \in C^{2,\alpha}_\Phi(M \times [0,T+\varepsilon])$ to CYF$^+$, with $\varepsilon > 0$ sufficiently small. Finally, the arguments of Proposition 3.27, cf. Remark 3.28, imply that $u \in C^{k,\alpha}_\Phi(M \times [0,T+\varepsilon])$ with T-independent Hölder norm. This contradicts maximality of $T > 0$ and hence the flow exists for all times. □

Remark 3.31. We point out that the arguments presented for the up to this point hold on the setting of manifolds of bounded geometry as well, as long as the flow does exist for short-time with conformal factor lying in $C^{2,\alpha}_\Phi(M \times [0,T])$. In fact, the Omori-Yau maximum principle holds for manifolds of bounded Ricci curvature, which follows from the definition of manifolds of bounded geometry. Moreover, the Φ-geometry was not used in the proofs given from §3.4.

3.10 Convergence of the CYF$^+$ on Φ-manifolds

This last section presents the convergence of the CYF$^+$. The argument uses a compact embedding of (weighted) Hölder spaces, where the weight is defined in terms of the boundary defining function x is extended to a smooth nowhere vanishing function on M.

Theorem 3.32. *Let* (M, g_Φ) *be an m-dimensional Φ-manifold,* $m \geq 3$, *such that* $\operatorname{scal}(g_\Phi) \in C^{4,\alpha}_\Phi(M)$ *is negative and bounded away from zero. Consider the global solution* $u \in C^{4,\alpha}_\Phi(M \times \mathbb{R}_+)$ *of CYF$^+$. Then the family of metrics* $\{g(t) = u(t)^{1/\eta} g_\Phi\}_{t \geq 0}$ *converges to a metric* $g^* = (u^*)^{1/\eta} g_\Phi$ *with constant negative scalar curvature.*

Proof. By Proposition 3.24, $\|\partial_t u(t)\|_\infty$ decreases exponentially. From the definition of $\partial_t u$ it follows easily that $u(t) \in L^\infty(M)$ is a Cauchy sequence, as $t \to +\infty$, and hence admits a well-defined limit $u^* \in L^\infty(M)$. In fact, we have

$$\frac{\partial u}{\partial t}(t) = \lim_{\varepsilon \to 0} \frac{u(t+\varepsilon) - u(t)}{\varepsilon} \Rightarrow \|u(t+\varepsilon) - u(t)\|_{L^\infty(M)} \leq \varepsilon \cdot C e^{\sup_M \operatorname{scal}(g_\Phi) \cdot t}, \quad \text{as } \varepsilon \to 0.$$

By Proposition 3.9, $u(t) \in C^{4,\alpha}_\Phi(M)$ admits a convergent subsequence in $x^{-\gamma} C^{4,\beta}_\Phi(M)$ for any $\beta < \alpha$ and $\gamma > 0$. Hence $u^* \in x^{-\gamma} C^{4,\beta}_\Phi(M)$ with scalar curvature $\operatorname{scal}^* \in x^{-\gamma} C^{2,\beta}_\Phi(M)$ such that for some divergent sequence $\{t_n\}_n \in \mathbb{R}_+$ going to $+\infty$,

$$\|\operatorname{scal} g(t_n) - \operatorname{scal}^*\|_{x^{-\gamma} C^{2,\beta}_\Phi(M)} \to 0 \ \text{ for } n \to \infty, \tag{3.74}$$

In particular, $\operatorname{scal} g(t_n)$ converges pointwise to scal^*. Note that by Lemma 3.21 the supremum $\sup_M \operatorname{scal} g(t)$ is non-increasing and by (3.55) the infimum $\inf_M \operatorname{scal} g(t)$ is non-decreasing. Thus $\sup_M \operatorname{scal} g(t)$ and $\inf_M \operatorname{scal} g(t)$ are bounded from below and above, respectively, and thus both convergent as $t \to \infty$. By Lemma 3.22

$$\lim_{t\to\infty} \sup_M \operatorname{scal} g(t) = \lim_{t\to\infty} \inf_M \operatorname{scal} g(t) =: \text{const}.$$

We compute from pointwise convergence of $\operatorname{scal} g(t)$ to scal^* at any $p \in M$

$$\begin{aligned} \operatorname{scal}^*(p) &= \lim_{n\to\infty} \operatorname{scal} g(t_n)(p) \leq \lim_{n\to\infty} \sup_M \operatorname{scal} g(t_n) \\ &\Rightarrow \sup_M \operatorname{scal}^* \leq \text{const}. \end{aligned} \tag{3.75}$$

Similar argument applied to the infimum of scal^* yields

$$\begin{aligned} \operatorname{scal}^*(p) &= \lim_{n\to\infty} \operatorname{scal} g(t_n)(p) \geq \lim_{n\to\infty} \inf_M \operatorname{scal} g(t_n) \\ &\Rightarrow \inf_M \operatorname{scal}^* \geq \text{const}. \end{aligned} \tag{3.76}$$

Combining (3.75) and (3.76), proves the statement. □

This proves Theorem 0.4.

Remark 3.33. Unlike the arguments for the long-time existence of the Yamabe flow, the proof of convergence cannot be generalized to manifolds of bounded geometry without further assumptions. This happens because convergence needs the compact embedding between weighted Hölder spaces, which cannot be defined without a globally defined boundary defining function.

CHAPTER 4

Alternative Yamabe flow on Φ-manifolds

In Chapter 3, we proved that the curvature-normalized Yamabe flow on Φ- manifolds exists for all time, is unique and converges to some Riemannian metric within the same conformal class of metrics with constant scalar curvature, under the assumption that the initial scalar curvature is negative and bounded away from zero. In order to do this, we worked on a specific class Hölder spaces $C^{k,\alpha}_{\Phi}$, that requires not only boundedness of the function, but also dictates estimates on the variation of a function accordingly to the distance between the variables.

But one might wonder: is it possible to achieve similar results on the Yamabe flow under the assumption that the conformal factor has a "better behavior"? To be more precise, does there exist a Yamabe flow on Φ-manifolds whose conformal factor is continuous up to the boundary and has an even more controlled variation? This is the question we address in this chapter.

Once again, we proceed in this chapter similarly as we did in Chapter 3. First, we define a family of Hölder spaces that describes the type of behavior we aim for. After this, we prove mapping properties for the heat operator $\mathbf{H}\star$ (as defined in (3.19)) for said family of Hölder spaces. Later, we present a construction of a parametrix $\mathbf{Q}$ for the inhomogeneous Cauchy problem

$$(\partial_t - a\cdot\Delta)u = \ell, \;\; u|_{t=0} = 0, \tag{4.1}$$

where the factor a satisfies analogous conditions as the ones required in Theorem 3.14. Moreover, we extend some of the mapping properties from $\mathbf{H}\star$ to $\mathbf{Q}$. We point out that this construction is based on Bahuaud and Vertman [BV19]. Finally, we extend the contraction argument presented in Theorem 3.14 and used it to conclude the short-time existence of the Yamabe flow with conformal factor in our new family of Hölder spaces.

4.1 Modified Hölder continuity

Now, we present a definition of a considerably more restrictive family of Hölder space. To do so, we introduce a new distance function in terms of local coordinates near the boundary, whose general idea is demonstrated in the following example.

Example 4.1. Let $N = [0,1] \times_x \mathbb{S}^1$ be a Riemannian cone over $\mathbb{S}^1$, that is, consider on N the Riemannian metric given by

$$g_N := \mathrm{d}x^2 + x^2 g_{\mathbb{S}^1}, \tag{4.2}$$

with $g_{\mathbb{S}^1}$ the usual Riemannian metric on $\mathbb{S}^1$; moreover, consider π_1 and π_2 the standard projections from N to $[0,1]$ and $\mathbb{S}^1$ respectively. For two given points $p,q \in N$, consider $\gamma\colon [0,1] \to N$ the geodesic connecting p and q. Thus $\pi_1\gamma$ is a straight line from $\pi_1(p)$ to $\pi_1(q)$ and $\pi_2\gamma$ is an arc on $\mathbb{S}^1$ from $\pi_2(p)$ to $\pi_2(q)$.

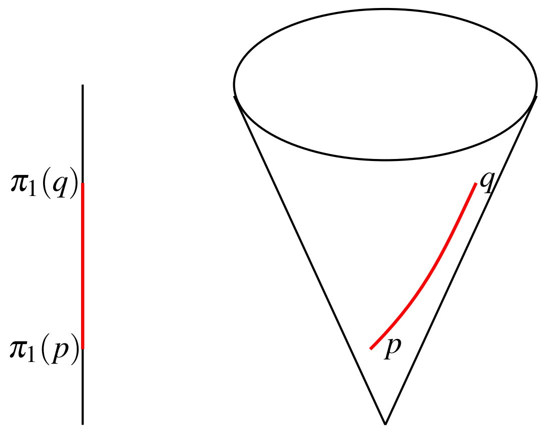

Figure 4.1: Distance on a cone

Hence,

$$\begin{aligned}\operatorname{length}(\gamma) &= \int_0^1 \|\gamma'(t)\|_{g_N}\,\mathrm{d}t \\ &= |\pi_1(p) - \pi_1(q)| + \frac{\pi_1(p)+\pi_1(q)}{2} d_{\mathbb{S}^1}(\pi_2(p),\pi_2(q)).\end{aligned}$$

From this follows that

$$d_N(p,q) \approx |\pi_1(p) - \pi_1(q)| + (\pi_1(p)+\pi_1(q)) d_{\mathbb{S}^1}(\pi_2(p),\pi_2(q)). \tag{4.3}$$

Note that the previous example can be easily generalized to cusps. In fact, if one consider $N = [0,1] \times_{x^k} \mathbb{S}^1$, then the distance function on N is given by the expression

$$d_N(p,q) \approx |\pi_p - \pi_1(q)| + (\pi_1(p)+\pi_1(q))^k d_{\mathbb{S}^1}(\pi_2(p),\pi_2(q)). \tag{4.4}$$

Furthermore, the Riemannian metric for the cone can be written in terms of an exact Φ-metric. In fact, by taking $N=[0,1]\times_x Y$ and $Z=\{\mathrm{pt}\}$, the exact Φ-metric is given by

$$g_{\Phi,0}=\frac{\mathrm{d}x^2}{x^4}+\frac{\phi^* g_Y}{x^2}=\frac{1}{x^4}\left(\mathrm{d}x^2+x^2\phi^* g_Y\right)=\frac{1}{x^4}g_N.$$

Now, let M be a Φ-manifold. Consider a distance function defined on M near $\partial\overline{M}$ as follows: for any two points $p=(x,y,z)$ and $p'=(x',y',z')$, define d locally by the expression

$$d_{x^4\Phi}(p,p')\approx\sqrt{|x-x'|^2+(x+x')^2\|y-y'\|+(x+x')^4\|z-z'\|^2}. \tag{4.5}$$

The function $d_{x^4\Phi}$ is clearly positive, null if and only if $p=p'$, symmetric and, similarly to the argument presented in Example 4.1, it satisfies the triangular inequality. On the other hand, it should be pointed out that $d_{x^4\Phi}$ is not a distance function over $\partial\overline{M}$, since it does vanish altogether on the boundary. However, this is the distance function defined on the interior which captures the type of behavior we are interested in.

Now, recall that we are considering $M_T:=M\times[0,T]$. For $\alpha\in(0,1)$, define α-norm as the map $\|\cdot\|_\alpha^*:C^0(M\times[0,T])\to[0,\infty)$ to be

$$\|u\|_\alpha^*=\|u\|_\infty+\sup_{M_T^2}\left\{\frac{|u(p,t)-u(p',t')|}{d_{x^4\Phi}(p,p')^\alpha+|t-t'|^{\alpha/2}}\right\}=:\|u\|_\infty+[u]_\alpha^*. \tag{4.6}$$

Thus, define the modified Hölder space

$$C^\alpha_{x^4\Phi}(M\times[0,T]):=\{u\in C^0(\overline{M}\times[0,T])\mid \|u\|_\alpha^*<\infty\}.$$

Once endowed with the norm $\|\cdot\|_\alpha^*$, as defined in (4.6), this set turns into a Banach space. Analogously to the standard Hölder spaces, we define the higher-order modified Hölder spaces as

$$C^{k,\alpha}_{x^4\Phi}(M\times[0,T])=\left\{u\in C^k_\Phi(M\times[0,T])\;\middle|\;\begin{array}{l}(V\circ\partial_t^{l_2})u\in C^\alpha_{x^4\Phi}(M\times[0,T]),\\ \text{for } V\in\mathscr{V}_\Phi^{l_1},\;\; l_1+2l_2\le k\end{array}\right\} \tag{4.7}$$

From [BV14, Proposition 3.1] follows that $C^{k,\alpha}_{x^4\Phi}(M\times[0,T])$ is a Banach space as well, when equipped with the norm

$$\|u\|_{k,\alpha}^*=\|u\|_\alpha^*+\sum_{l_1+2l_2\le k}\sum_{V\in\mathscr{V}_\Phi^{l_1}}\|(V\circ\partial_t^{l_2})u\|_\alpha^*.$$

Note that, from the definition, every function lying in an higher-order Hölder spaces also lies in the α-Hölder space (case $l_1=l_2=0$).

Naturally, once defined the basic Hölder space above, it is now easy to generalize to weighted Hölder spaces as follows: for $\gamma\in\mathbb{R}$, define

$$x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T]):=\{x^\gamma u\mid u\in C^{k,\alpha}_{x^4\Phi}(M\times[0,T])\},$$
$$\|u\|_{k,\alpha,\gamma}^*:=\|x^{-\gamma}u\|_{k,\alpha}^*.$$

Remark 4.2. Recall that, as pointed out previously, some of our results achieved for the standard Hölder spaces do not depend on the distance function chosen. In particularl, Proposition 3.4, Proposition 3.6 and Proposition 3.7 are true for the modified Hölder spaces as well. This means that:

1. If we take $\varphi \in C^{k+l,\alpha}_{x^4\Phi}(M \times [0,T])$, with $l \geq 0$, then the multiplication map

$$M_\varphi : x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T]) \to x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T])$$

   is continuous;

2. If we consider a Riemannian metric $g \in \{u^{1/\eta} g_\Phi \mid 0 < \inf_M u \leq \sup_M u < +\infty\}$, then the Hölder spaces are preserved under the change of metrics, i.e. $C^{k,\alpha}_g(M) = C^{k,\alpha}_{x^4\Phi}(M)$.

3. We have the continuous inclusion $C^{k,\alpha}_{x^4\Phi}(M) \hookrightarrow C^{k,\beta}_{x^4\Phi}(M)$, with $0 < \beta < \alpha < 1$.

However, Proposition 3.9 is not true for $C^{k,\alpha}_{x^4\Phi}(M)$. In fact, to obtain a compact embedding between two modified Hölder spaces, stronger conditions on the acquired x-weight are necessary. In what follows, we take the time to discuss this interesting result, in the spirit of curiosity.

Proposition 4.3. *Let (M, g_Φ) be a Φ-manifold and $0 < \beta < \alpha < 1$. Then the inclusion*

$$\iota : C^{k,\alpha}_{x^4\Phi}(M) \hookrightarrow x^{-\gamma} C^{k,\beta}_{x^4\Phi}(M) \tag{4.8}$$

is a compact embedding, for some $\gamma > \beta$.

Proof. Let $\{u_n\}_n$ be a bounded sequence in $C^{k,\alpha}_{x^4\Phi}(M)$. We use the same notation as the one employed in the proof of Proposition 3.9, setting a sequence $\{v_j := u_{n_j(\delta_j)}\}_j$ in the following manner: for the truncated compact manifold $M_{\delta_1} = \{x \geq \delta_1\}$, consider the convergent subsequence $\{u_{n_i(\delta_1)}\}_i$ obtained from the compact embedding $C^{k,\alpha}_{x^4\Phi}(M_{\delta_1}) \hookrightarrow C^{k,\beta}_{x^4\Phi}(M_{\delta_1})$. Repeat the process to said subsequence to the truncated manifold M_{δ_2}. After iterating the process, one obtains the desired sequence $\{v_j\}_j$. This construction only requires the fact that one has a compact embedding $C^{k,\alpha}_{x^4\Phi}(K) \hookrightarrow C^{k,\beta}_{x^4\Phi}(K)$ for any compact subset $K \subset M$ away from the boundary.

Our goal is once again to prove that this sequence is a Cauchy sequence in $x^{-\gamma} C^{k,\beta}_{x^4\Phi}(M)$. Like in the case for the standard Hölder spaces, the proof away from the boundary follows naturally from the compact embeddings for compact subsets away from the boundary. Hence, we only need to prove that this is a Cauchy sequence near the boundary as well. For this, we need the following

• **Claim:** The function $x^\gamma \in C^{\beta}_{x^4\Phi}(M)$ if, and only if, $\gamma \geq \beta$.

Boundedness of x^γ is straightforward, since we assume x to be bounded away from the boundary and x goes to zero near the boundary. Then, we must check that $[x^\gamma]^*_\beta < +\infty$. First, note that

$$|x^\beta - y^\beta| \leq |x - y|^\beta, \quad \text{for all } \beta \in (0,1).$$

In fact, this is equivalent to proving that

$$\left|\left(\frac{x}{y}\right)^{\beta}-1\right|\leq\left|\frac{x}{y}-1\right|^{\beta}.$$

Without loss of generality, assume $x\geq y$. Then, for $t:=x/y$, this is equivalent to showing that

$$t^{\beta}-1\leq(t-1)^{\beta},\quad \text{for } t\geq 1.$$

Set the function $f(t):=(t-1)^{\beta}-(t^{\beta}-1)$. This function is continuously differentiable and we have $f'(t)=\beta(t-1)^{\beta-1}-\beta t^{\beta-1}$. Since $f(1)=0$, the claim will be proven true if $f'(t)\geq 0$ for $t\geq 1$. But

$$f'(t)\geq 0 \Longleftrightarrow \beta(t-1)^{\beta-1}\geq\beta t^{\beta-1}\Longleftrightarrow (t-1)^{1-\beta}\leq t^{1-\beta},$$

which is true for $\beta\in(0,1)$ and $t\geq 1$. Hence, we have $x^{\beta}\in C^{\beta}_{x^4\Phi}(M)$ and, consequently, $x^{\gamma}\in C^{\beta}_{x^4\Phi}(M)$. Moreover, since $Vx^{\gamma}=O(x^{\gamma})$ for $V\in\mathcal{V}_{\Phi}$, we have $x^{\gamma}\in C^{k,\beta}_{x^4\Phi}(M)$ for all $k\in\mathbb{N}$ and $\gamma\geq\beta$.

Now, we prove that the sequence $\{v_j\}_j$ is a Cauchy sequence near the boundary as well. Indeed, from Remark 4.2, item 1., we can conclude

$$\begin{aligned}\|v_j\|^{*}_{x^{-\gamma}C^{k,\beta}_{x^4\Phi}(M\setminus M_{\delta_j})}=\|x^{\gamma}v_j\|^{*}_{C^{k,\beta}_{\Phi}(M\setminus M_{\delta_j})}&\leq C_0 l_j^{\gamma-\beta}\|v_j\|^{*}_{k,\beta}\\&\leq C l_j^{\gamma-\beta}.\end{aligned}$$

Thus, choose $j_0\in\mathbb{N}$ sufficient large such that the inequality $Cl_{j_0}^{\gamma-\beta}\leq\varepsilon/4$ checks (which is possible since $\gamma-\beta>0$). From this it follows that $\{v_j\}$ is a Cauchy sequence in $x^{-\gamma}C^{k,\beta}_{x^4\Phi}(M)$ and, by completeness, admits a convergence subsequence. □

4.2 Mapping properties of the Heat Kernel on $C^{\alpha}_{x^4\Phi}(M\times[0,T])$

Similarly to the proof of existence of the Yamabe flow for standard Hölder spaces, we now study mapping properties of the heat operator $\mathbf{H}\star$ acting on functions in $x^{\gamma}C^{k,\alpha}_{x^4\Phi}(M\times[0,T])$. Recall that

$$\mathbf{H}\star u(t,p):=\int_0^t\int_M H(t-\widetilde{t},p,\widetilde{p})u(\widetilde{t},\widetilde{p})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}.$$

Theorem 4.4. *Let M be a m-dimensional manifold with fibered boundary equipped with a Φ-metric. The operator*

$$\mathbf{H}\star: x^{\gamma}C^{k,\alpha}_{x^4\Phi}(M\times[0,T])\to x^{\gamma}\left(C^{k+2,\alpha}_{x^4\Phi}\cap\sqrt{t}\,C^{k+1,\alpha}_{x^4\Phi}\cap t^{\alpha/2}C^{k+2}_{\Phi}\right)(M\times[0,T])\tag{4.9}$$

is bounded, for any $\alpha\in(0,1)$ and any $\gamma\in\mathbf{R}$.

Proof. Similarly as in the proof of Theorem 3.10, explicit computations will be presented only for the first mapping property, that is, we prove that

$$\mathbf{H}\star: x^{\gamma}C^{k,\alpha}_{x^4\Phi}(M\times[0,T])\to x^{\gamma}C^{k+2,\alpha}_{x^4\Phi}(M\times[0,T]),$$

is a continuous operator. The other two mapping properties can be proven analogously, since the integrals rising during estimate have the same asymptotic behavior. Moreover, we prove only the case for $k = 0$, since the higher order case follows from the case $k = 0$ plus integration by parts.

Once again, for $\mathbf{H}_\gamma := M_{x^{-\gamma}} \mathbf{H} \star M_{x^\gamma}$ and for $\mathbf{G} = V\mathbf{H}_\gamma$, with $V \in \{\mathrm{id}\} \cup \mathscr{V}_\Phi \cup \mathscr{V}_\Phi^2$, we must prove that

$$\mathbf{G} : C^\alpha_{x^4\Phi}(M \times [0,T]) \to C^\alpha_{x^4\Phi}(M \times [0,T]).$$

is a continuous operator. Analogously to the proof of the mapping properties of $\mathbf{H}\star$, we will proceed in three steps:

i) Uniform estimates of Hölder differences in space, whose proof is presented in §4.2.1,

ii) Uniform estimates of Hölder differences in time, whose proof is presented in §4.2.2,

iii) Uniform estimates of the supremum norm, whose proof is presented in §4.2.3.

From this, we conclude the proof of Theorem 4.4. □

4.2.1 Estimates of Hölder differences in space

From this point until the end of §4.2.3, we use freely the notation presented in §2.4.

Let p and p' be points in M and set

$$M^+ = \left\{ \widetilde{p} \in M \,\middle|\, d(p,\widetilde{p}) \leq 3d(p,p') \right\}, \;\; M^- = \left\{ \widetilde{p} \in M \,\middle|\, d(p,\widetilde{p}) \geq 3d(p,p') \right\}.$$

Now, for u function in $C^\alpha_{x^4\Phi}(M \times [0,T])$ and every $V \in \{\mathrm{id}\} \cup \mathscr{V}_\Phi \cup \mathscr{V}_\Phi^2$, write

$$\begin{aligned}
\mathbf{G}u(t,p) - \mathbf{G}u(t,p') = & \int_0^t \int_{M^+} \left[G(t-\widetilde{t},p,\widetilde{p}) - G(t-\widetilde{t},p',\widetilde{p}) \right] \left[u(\widetilde{t},\widetilde{p}) - u(\widetilde{t},p) \right] \mathrm{dvol}_\Phi(\widetilde{p})\, \mathrm{d}\widetilde{t} \\
& + \int_0^t \int_{M^-} \left[G(t-\widetilde{t},p,\widetilde{p}) - G(t-\widetilde{t},p',\widetilde{p}) \right] \left[u(\widetilde{t},\widetilde{p}) - u(\widetilde{t},p) \right] \mathrm{dvol}_\Phi(\widetilde{p})\, \mathrm{d}\widetilde{t} \\
& + \int_0^t \int_{M} \left[G(t-\widetilde{t},p,\widetilde{p}) - G(t-\widetilde{t},p',\widetilde{p}) \right] u(\widetilde{t},p)\, \mathrm{dvol}_\Phi(\widetilde{p})\, \mathrm{d}\widetilde{t} \\
& =: I_1 + I_2 + I_3.
\end{aligned}$$

Again, in order to obtain the desired estimates, it is enough for us to estimate each of the I_j-terms individually. To simplify the notation, we will identify the integration regions M, M^+ and M^- with their lifts.

Furthermore, since the heat kernel vanishes to infinite order near the extreme corners of the heat space, so does the kernel G. This makes estimates near the extreme regimes of M^2_h trivial. Thus, computations for estimates only near the middle regimes of M^2_h (i.e. near $\mathrm{fd} \cup \mathrm{td}$) are presented here.

Estimate for I_2

Going along the lines of the estimate of the term I_1 in §3.2.1, employ the Mean Value Theorem to write

$$\begin{aligned}|I_2| \leq & C\|u\|^*_\alpha |x-x'| \int_0^t \int_{M^-} \partial_\xi G(t-\widetilde{t}, p_\xi, \widetilde{p}) d_\Phi(p_\xi, \widetilde{p})^\alpha \,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ & + C\|u\|^*_\alpha \|y-y'\| \int_0^t \int_{M^-} \partial_\eta G(t-\widetilde{t}, p_\eta, \widetilde{p}) d_\Phi(p_\eta, \widetilde{p})^\alpha \,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ & + C\|u\|^*_\alpha \|z-z'\| \int_0^t \int_{M^-} \partial_\zeta G(t-\widetilde{t}, p_\zeta, \widetilde{p}) d_\Phi(p_\zeta, \widetilde{p})^\alpha \,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ & =: I_{2,1} + I_{2,2} + I_{2,3},\end{aligned}$$

with $p_\xi = (\xi, y, z), p_\eta = (x', \eta, z)$ and $p_\zeta = (x', y', \zeta)$. The previous inequalities above are using the fact that if $p'' \in M$ satisfies $d_{x^4\Phi}(p, p'') \leq d_{x^4\Phi}(p, p')$, then $d_{x^4\Phi}(p, \widetilde{p}) \leq d_{x^4\Phi}(p'', \widetilde{p})$ for every point $\widetilde{p} \in M^-$. The proof of this fact is exactly the same as the one presented for the distance function d_Φ.

Given the similarities of the estimates of each term $I_{2,i}$, we present here only the computations for the term $I_{2,1}$. The two remaining terms can be estimate analogously. For $I_{2,1}$, we will use the projective coordinates $(\tau, \xi, y, z, \widetilde{s}, \widetilde{y}, \widetilde{z})$, with $\widetilde{s} = \widetilde{x}/\xi$ and $\tau = \sqrt{t-\widetilde{t}}$. From the asymptotics of the heat kernel near fd$\cup$td, we have

$$|I_2'| = C\|u\|^*_\alpha |x-x'| \int_0^{\sqrt{t}} \int_{M^-} \tau^{-m-2}\xi^{-3-b} G_0 d_{x^4\Phi}(p_\xi, \widetilde{p}))^\alpha \,\mathrm{d}\widetilde{s}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{z}\,\mathrm{d}\tau, \tag{4.10}$$

with G_0 vanishing to infinite order whenever $\|(\widetilde{s}-1, \widetilde{y}-y, \widetilde{z}-z)\| \to \infty$. On the other hand, $\xi \sim \widetilde{x}$ near the middle regimes, implying $(1+\widetilde{s})$ to be bounded and giving

$$\begin{aligned} d_{x^4\Phi}((\xi, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z})) &= \sqrt{|\xi - \widetilde{x}|^2 + (\xi+\widetilde{x})^2\|y-\widetilde{y}\|^2 + (\xi+\widetilde{x})^4\|z-\widetilde{z}\|^2} \\ &= \sqrt{\xi^2(|1-\widetilde{s}|^2 + (1+\widetilde{s})^2\|y-\widetilde{y}\|^2 + \xi^2(1+\widetilde{s})^4\|z-\widetilde{z}\|^2)} \\ &\sim \sqrt{\xi^2(|1-\widetilde{s}|^2 + \|y-\widetilde{y}\|^2 + \xi^2\|z-\widetilde{z}\|^2)} \\ &=: \xi r(s, y-\widetilde{y}, z-\widetilde{z}),\end{aligned}$$

from where follows that there is a constant c such that $d_{x^4\Phi}(p_\xi, \widetilde{p})^\alpha) \leq c(\xi r)^\alpha$. Such function r describes the radial distance in polar coordinates around the point $(1, y, \xi z)$. Performing a change of coordinates in the integral in (4.10), one has

$$|I_2'| \leq c\|u\|_\alpha |x-x'| \int_0^{\sqrt{t}} \int_{M^-} \tau^{-m-2}\xi^{-m-2+\alpha} r^{m-1+\alpha} G_0 \,\mathrm{d}r\,\mathrm{d}(\text{angle})\,\mathrm{d}\tau.$$

Now, setting $\sigma = r^{-1}\tau\xi$, it follows that the asymptotic behavior of σ^{-1} is

$$\sigma^{-1} \sim \sqrt{|\mathscr{S}|^2 + \|\mathscr{U}\|^2 + \|\mathscr{Z}\|^2}.$$

This implies that the integral of G_0 against any negative power of σ is bounded. Moreover, the definition of r implies, up to some constant, $M^- \subset \{\xi^{-1} d_{x^4\Phi}(p, p') \leq r\}$. Hence, integration on the angular variables followed by another change of coordinates $\tau \mapsto \sigma$ gives

$$\begin{aligned}|I_2'| &\leq C\|u\|_\alpha^* |x-x'| \int_{\xi d_{x^4\Phi}(p,p')}^{\infty} r^{-2+\alpha}\xi^{-1+\alpha}\,\mathrm{d}r \\ &= C\|u\|_\alpha^* |x-x'|\xi^{-1+\alpha}(\xi^{-1}d_{x^4\Phi}(p,p'))^{-1+\alpha} \\ &\leq C\|u\|_\alpha^* d_{x^4\Phi}(p,p')^\alpha.\end{aligned}$$

Estimate for I_1

Proceeding similarly to the term I_2 in §3.2.1, we have

$$\begin{aligned}I_1 = &\int_0^t \int_{M^+} G(t-\widetilde{t},p,\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ &- \int_0^t \int_{M^+} G(t-\widetilde{t},p',\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p',\widetilde{t})]\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ &+ \int_0^t \int_{M^+} G(t-\widetilde{t},p',\widetilde{p})[u(p,\widetilde{t})-u(p',\widetilde{t})]\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t} \\ &=: I_{1,1} - I_{1,2} + I_{1,3}.\end{aligned}$$

Now, since we're considering M^+ as integration region, then points on $\widetilde{p} \in M^+$ satisfy $d_{x^4\Phi}(p,\widetilde{p}) \leq 3d_{x^4\Phi}(p,p')$. Recall that triangular inequality implies

$$d_{x^4\Phi}(p',\widetilde{p}) \leq d_{x^4\Phi}(p',p) + d_{x^4\Phi}(p,\widetilde{p}) \leq 4d_{x^4\Phi}(p,p').$$

For the coordinates $(\tau,x,y,z,\mathscr{S}',\mathscr{U}',\mathscr{Z}')$, with

$$\mathscr{S}' = \frac{\widetilde{x}-x}{x^2},\ \ \mathscr{U}' = \frac{\widetilde{y}-y}{x},\ \ \mathscr{Z}' = \widetilde{z}-z \text{ and } \tau = \sqrt{t-\widetilde{t}},$$

we have $G = \tau^{-m-2}G_0$, with G_0 vanishing to infinite order whenever $\|(\mathscr{S},\mathscr{U},\mathscr{Z})\| \to \infty$. Thus, for these coordinates we have

$$|I_{1,1}| \leq \|u\|_\alpha^* \int_0^{\sqrt{t}} \int_{M^+} \tau^{-m-1}G_0 d_{x^4\Phi}(p,\widetilde{p})^\alpha\,\mathrm{d}\mathscr{S}'\,\mathrm{d}\mathscr{U}'\,\mathrm{d}\mathscr{Z}'\,\mathrm{d}\tau.$$

Now, note that near the middle regimes of the heat space we have $x \sim \widetilde{x}$. Then, if we consider the radius function $r(\mathscr{S}',\mathscr{U}',\mathscr{Z}') := \sqrt{|\mathscr{S}'|^2 + \|\mathscr{U}'\|^2 + \|\mathscr{Z}'\|^2}$, then, near td, we have

$$\begin{aligned}d_{x^4\Phi}((x,y,z),(\widetilde{x},\widetilde{y},\widetilde{z})) &= \sqrt{|x-\widetilde{x}|^2 + (x+\widetilde{x})^2\|y-\widetilde{y}\|^2 + (x+\widetilde{x})^4\|z-\widetilde{z}\|^2} \\ &\sim x^2\sqrt{|\mathscr{S}'|^2 + \|\mathscr{U}'\|^2 + \|\mathscr{Z}'\|^2} \\ &=: cx^2 r(\mathscr{S}',\mathscr{U}',\mathscr{Z}'),\end{aligned}$$

meaning that $M^+ = \{r \leq x^{-2}d_{x^4\Phi}(p,p')\}$ up to some constant. Setting $\sigma := \tau/r$ and taking polar coordinates for $(\mathscr{S}',\mathscr{U}',\mathscr{Z}')$ around $(0,0,0)$ gives the expression

$$|I_{1,1}| \leq c\|u\|_\alpha^* x^{2\alpha} \int_{I(\sigma)} \int_0^{x^{-2}d_{x^4\Phi}(p,p')} \sigma^{-m-1} r^{-1+\alpha} G_0\,\mathrm{d}r\,\mathrm{d}\tau.$$

Since $\sigma^{-m-1}G_0$ is bounded (due to the decay properties of G_0), the estimate follows. The term $I_{1,2}$ can be estimated in the exact same way as $I_{1,1}$, so the computations are not presented here.

For the $I_{1,3}$-term, proceed by integration by parts. First, take the coordinates $(\tau,x,y,z,\mathscr{S},\mathscr{U},\mathscr{Z})$ and note that one has (as the "worst case scenario" for $I_{1,3}$) $G=\tau^{-m-2}(V_1V_2G_0)$, where $V_1,V_2\in\{\partial_{\mathscr{S}},\partial_{\mathscr{U}},\partial_{\mathscr{Z}}\}$. For the sake of simplicity, we shall assume $V_1=\partial_{\mathscr{S}}$; the general case is similar, thus its computations are omitted here. On the other hand, for fixed $(\tau,\mathscr{U},\mathscr{Z})$ one has $M^+=\{|\mathscr{S}|\leq r(\tau,\mathscr{U},\mathscr{Z})\}$. Hence, since

$$\beta^*(\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t})=h_0(x+x^2\tau\mathscr{S},y+x\tau\mathscr{U},z+\tau\mathscr{Z})\tau^{m+1}\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau,$$

with h_0 a smooth function, and $[u(p,\widetilde{t})-u(p',\widetilde{t})]=:\delta u$ is independent of $\widetilde{p}$,

$$\begin{aligned}
I_{1,3}&=\int_0^{\sqrt{t}}\delta u\int_{M^+}\tau^{-1}(\partial_{\mathscr{S}}V_2G_0)h\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau\\
&=\int_0^{\sqrt{t}}\delta u\int_{\partial\overline{M}^+}\tau^{-1}(V_2G_0)\big|_{|\mathscr{S}|=r}h\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau\\
&-\int_0^{\sqrt{t}}\delta u\int_{M^+}\tau^{-1}(V_2G_0)\partial_{\mathscr{S}}h\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau\\
&=:I^1_{1,3}-I^2_{1,3}.
\end{aligned}$$

For the $I^2_{1,3}$-term, since h is a smooth function, then $\partial_{\mathscr{S}}h=x^2\tau h'$, which then cancels the τ^{-1} in the integrand and then, since the rest of the integrand is bounded, the integral is bounded. When restricted to $|\mathscr{S}|=r$ one has $\partial\overline{M}^+=\{\widetilde{p}\in M^+\mid d_{x^4\Phi}(p,\widetilde{p})=3d_{x^4\Phi}(p,p')\}$ and then, from the triangular inequality one gets

$$2d_{x^4\Phi}(p,p')\leq d_{x^4\Phi}(p,\widetilde{p})\leq 4d_{x^4\Phi}(p,p').$$

Considering now the coordinates $(\tau,x,y,z,\mathscr{S}',\mathscr{U}',\mathscr{Z}')$ (which are valid up to fd$\cap$td), one has $\mathscr{S}'=\tau\mathscr{S}$, $\mathscr{U}'=\tau\mathscr{U}$ and $\mathscr{Z}'=\tau\mathscr{Z}$, from where one gets

$$|I^1_{1,3}|\leq\|u\|^*_\alpha\int_0^{\sqrt{t}}\int_{\partial\overline{M}^+}\tau^{-m}(V_2G_0)\big|_{|\mathscr{S}|=r}d_{x^4\Phi}(p,\widetilde{p})^\alpha\,\mathrm{d}U\,\mathrm{d}Z\,\mathrm{d}\tau.$$

Proceed now exactly like in $I_{1,1}$ by taking polar coordinates for $(\mathscr{U}',\mathscr{Z}')$ around $(0,0)$, with radial function R, and once again considering $\sigma=\tau/R$ will then give us

$$\begin{aligned}
|I^1_{1,3}|&\leq\|u\|^*_\alpha\int_0^\infty\int_0^{4d_{x^4\Phi}(p,p')}R^{-1+\alpha}\sigma^{-m}\left(\sqrt{\frac{|\mathscr{S}'|^2+\|\mathscr{U}'\|^2+\|\mathscr{Z}'\|^2}{\|\mathscr{U}'\|^2+\|\mathscr{Z}'\|^2}}\right)^\alpha(V_2G_0)\big|_{|\mathscr{S}|=r}\mathrm{d}R\,\mathrm{d}\tau\\
&\leq C\|u\|^*_\alpha d_{x^4\Phi}(p,p')^\alpha.
\end{aligned}$$

Estimate for I_3

In order to estimate I_3, first let us rewrite it in the following way: by considering $p=(x,y,z)$ and $p'=(x',y',z')$,

$$I_3=\int_0^t\int_M[G(t-\widetilde{t},p,\widetilde{p})-G(t-\widetilde{t},(x',y,z),\widetilde{p})]u(\widetilde{t},p)\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}$$

$$
\begin{aligned}
&+\int_0^t\int_M [G(t-\widetilde{t},(x',y,z),\widetilde{p})-G(t-\widetilde{t},(x',y',z),\widetilde{p})]u(\widetilde{t},p)\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\
&+\int_0^t\int_M [G(t-\widetilde{t},(x',y',z),\widetilde{p})-G(t-\widetilde{t},p',\widetilde{p})]u(\widetilde{t},p)\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\
&=: I_{3,1}+I_{3,2}+I_{3,3}.
\end{aligned}
$$

Remember that Φ-manifolds are stochastically complete. Hence, for $I_{3,2}$ we have

$$
I_{3,2}=\int_0^t x^\gamma u(p,\widetilde{t})\left(\int_M [X(x^{-\gamma}H)(t-\widetilde{t},(x',y,z),\widetilde{p})-X(x^{-\gamma}H)(t-\widetilde{t},(x',y',z),\widetilde{p})]\,\mathrm{dvol}_\Phi(\widetilde{p})\right)\mathrm{d}\widetilde{t},
$$

which then implies

$$
I_{3,2}=\int_0^t ((x')^{-\gamma}-(x')^{-\gamma})x^\gamma u(p,\widetilde{t})\,\mathrm{d}\widetilde{t}=0.
$$

Same argument can be applied to $I_{3,3}$, which means that $I_3=I_{3,1}$. Similar to the estimation for the I_2-term, we once more employ the Mean Value Theorem to obtain

$$
I_3=|x-x'|\int_0^t\int_M \partial_\xi G(t-\widetilde{t},p_\xi,\widetilde{p})u(p,\widetilde{t})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}. \tag{4.11}
$$

Recall that

$$
\beta^*(\partial_\xi)=\partial_\xi-[2\xi^{-1}\mathscr{S}+\xi^{-2}\tau^{-1}]\partial_{\mathscr{S}}-\xi^{-1}\mathscr{U}\partial_{\mathscr{U}},
$$

implying $\beta^*(\partial_\xi G)\sim\xi^{-2}\tau^{-1}\partial_{\mathscr{S}}G_0'$, where G_0' has a similar asymptotic behavior as G_0. Since $u(p,\widetilde{t})$ is constant in the spacial variable, integration by parts gives us

$$
\begin{aligned}
&\int_0^t\int_M \xi^{-2}\tau^{-1}\partial_{\mathscr{S}}G_0'u(p,t-\tau^2)h\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau\\
&\quad=\int_0^t\int_{\partial M}\xi^{-2}\tau^{-1}G_0'\big|_{|\mathscr{S}|=\infty}u(p,t-\tau^2)h_0\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau\\
&\quad-\int_0^t\int_M \xi^{-2}\tau^{-1}G_0'u(p,t-\tau^2)\partial_{\mathscr{S}}h_0\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau.
\end{aligned}
$$

Due to the decay properties of the heat kernel near $\partial\overline{M}$, the first of the two integrals above vanishes. On the other hand, since h is a smooth, then similarly to the $I_{1,3}^2$-estimation one has $\partial_{\mathscr{S}}h_0=\xi^2\tau h_0'$, then giving us

$$
\begin{aligned}
&\int_0^t\int_M \xi^{-2}\tau^{-1}\partial_{\mathscr{S}}G_0'u(p,t-\tau^2)h_0\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau\\
&\qquad=-\int_0^t\int_M G_0'u(p,t-\tau^2)h_0'\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau.
\end{aligned}
$$

Then one can now follow the same procedure as presented for the $I_{1,3}^2$, only keeping in mind that (unlike for $I_{1,3}$) the boundary term will once again vanish. Therefore, the estimate follows.

This completes the proof of the estimates for Hölder differences in space.

4.2.2 Estimations for Hölder differences in time

Let us now give estimates for the Hölder differences only in the time variable. Assume $p=p'$ and, without loss of generality, $t<t'$. Suppose first that t and t' satisfy $2t'-t\geqslant 0$ (i.e., $t'<t\leq 2t'$). Recall that we can define the intervals

$$T_-=[0,2t'-t],\ \ T_+=[2t'-t,t]\ \text{ and }\ T'_+=[2t'-t,t'].$$

Thus, we have

$$\begin{aligned}\mathbf{G}u(p,t)-\mathbf{G}u(p,t')=|t-t'|&\int_{T_-}\int_M \partial_\theta G\big|_{(t-\widetilde{t},p,\widetilde{p})}[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &+\int_{T_+}\int_M G(t-\widetilde{t},p,\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &-\int_{T'_+}\int_M G(t'-\widetilde{t},p,\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &+\int_0^t\int_M G(t-\widetilde{t},p,\widetilde{p})u(p,\widetilde{t})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &-\int_0^{t'}\int_M G(t'-\widetilde{t},p,\widetilde{p})u(p,\widetilde{t})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t}\\ &=:L_1+L_2-L_3+L_4-L_5.\end{aligned}$$

Similarly to the estimates in §3.2.2, we can estimate L_4 and L_5 straightforwardly, since Φ-manifolds are stochastically complete, which implies

$$L_4-L_5=\int_0^t u(p,\widetilde{t})\,\mathrm{d}\widetilde{t}-\int_0^{t'}u(p,\widetilde{t})\,\mathrm{d}\widetilde{t}\leq C\|u\|_\infty|t-t'|^{\alpha/2}.$$

Given the similarities between L_2 and L_3, it is enough to present computations for just one of them. Thus, for us to obtain the estimates, we just need to give estimates for L_1 and L_3. Moreover, since the heat kernel vanishes to infinite order away from $\mathrm{fd}\cup\mathrm{td}$, estimates are straightforward. Hence, we present only computations near $\mathrm{fd}\cup\mathrm{td}$.

Estimates for L_1

Consider projective coordinates near such intersection given by $(\tau,x,y,z,\mathscr{S},\mathscr{U},\mathscr{Z})$. In this case, we have $\beta^*\partial_\theta G\sim\tau^{-m-4}G_0$, with G_0 being polyhomogeneous and vanishing to infinite order when $\|(\mathscr{S},\mathscr{U},\mathscr{Z})\|\to\infty$, and $\beta^*(\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\widetilde{t})\sim\tau^{m+1}\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau$. Moreover, since $x\sim\widetilde{x}$ when near $\mathrm{fd}\cup\mathrm{td}$, we get to write

$$d_{x^4\Phi}(p,\widetilde{p})\leq c\tau\rho_{\mathrm{fd}}\sqrt{|\mathscr{S}|^2+\|\mathscr{U}\|^2+\|\mathscr{Z}\|^2}=:c\tau x r(\mathscr{S},\mathscr{U},\mathscr{Z}),\tag{4.12}$$

where r is bounded whenever its entries are bounded. Consequently, G_0r^α is bounded everywhere. On the other hand, note that whenever $\widetilde{t}\in T_-$, one has $|\theta-\widetilde{t}|\geq|t-t'|$, from where follows that

$$|L_1|\leq\|u\|^*_\alpha|t-t'|\int_{T_-}\int_M|\tau^{-3}G_0d(p,\widetilde{p})^\alpha|\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau$$

$$\leq C\|u\|_{\alpha}^{*}|t-t'|\int_{\sqrt{t-t'}}^{\infty}\int_{M}|\tau^{-3+\alpha}x^{\alpha}G_{0}'r^{\alpha}|\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau$$
$$\leq C\|u\|_{\alpha}^{*}|t-t'|^{\alpha/2}.$$

This completes the estimates for the L_1-term.

Estimates for L_2

Take the projective coordinates $(\tau,x,y,z,\mathscr{S},\mathscr{U},\mathscr{Z})$. It is known that $\beta^*G\sim\tau^{-m-2}G_0$, with G_0 polyhomogeneous and vanishing to infinite order if $\|(\mathscr{S},\mathscr{U},\mathscr{Z})\|\to\infty$. On the other hand,

$$\beta^*(\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t})\sim\tau^{m+1}h_0\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau, \tag{4.13}$$

with h_0 smooth on $\widetilde{p}=(x+x^2\mathscr{S}\tau,y+x\mathscr{U}\tau,z+\tau\mathscr{Z})$. From this and from (4.12) follows that

$$|L_2|\leq\|u\|_{\alpha}^{*}\int_{T_+}\int_{M}|\tau^{-1}G_0d_{x^4\Phi}(p,\widetilde{p})^{\alpha}|\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}\tau$$
$$\leq C\|u\|_{\alpha}^{*}\int_{T_+}\int_{M}|\tau^{-1+\alpha}G_0r^{\alpha}|\,\mathrm{d}\mathscr{S}\,\mathrm{d}\mathscr{U}\,\mathrm{d}\mathscr{Z}\,\mathrm{d}\tau$$
$$\leq C\|u\|_{\alpha}^{*}|t-t'|^{\alpha/2},$$

concluding the estimates for the L_2-term.

This completes the estimates for time difference with derivatives under the assumption that $2t'-t\geqslant 0$. Computations under the assumption $2t'-t<0$ follow analogously to estimates for time difference in §3.2.2.

4.2.3 Estimates for the supremum norm

Computations for the estimates of the supremum norm follow *ipsis literis* the estimates presented in §3.2.3. In fact, for a given point $(p,t)\in M\times[0,T]$, write

$$\begin{aligned}\mathbf{G}u(p,t)&=\int_0^t\int_M G(t-\widetilde{t},p,\widetilde{p})u(\widetilde{p},\widetilde{t})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}\\&=\int_0^t\int_M G(t-\widetilde{t},p,\widetilde{p})[u(\widetilde{p},\widetilde{t})-u(p,\widetilde{t})]\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}\\&+\int_0^t\int_M G(t-\widetilde{t},p,\widetilde{p})u(p,\widetilde{t})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\widetilde{t}\\&=J_1+J_2.\end{aligned}$$

Once again, estimates away from fd∪td are straightforward, meaning that we must focus near fd∪td. Now, note that the estimates in §3.2.3 uses the inequality $d_\Phi\leq C\tau r$ near the middle regimes, where r is some radial function and C is some constant. This property is true for $d_{x^4\Phi}$ as well, meaning that the argument can be employed here too.

From this, we conclude the estimate for the supremum norm.

4.3 Parametrix construction for heat-type equations

Now that we have proven some mapping properties for the heat kernel, we are in a good place to construct a parametrix, i.e., an approximate inverse, for a slightly more general heat-type equation. This is important for this work because this parametrix provides a way to find solutions for a modified version of the Yamabe flow, which is our ultimate goal.

Let M be a manifold with fibered boundary equipped with a Φ-metric. We consider an heat-type operator P that is

$$P = \partial_t - a\Delta$$

where Δ is the self-adjoint extension of the negative Laplace-Beltrami operator on functions and $a : M \times [0,T] \to \mathbb{R}$. Remember that such extension is unique, given that (M, g_Φ) is a complete Riemannian manifold. It is clear that we can not expect a to be as generic as possible. Indeed, for instance, we need a restrictive enough to preserve parabolicity of the heat operator. On the other hand, we need a generic enough so the parametrix constructed for P can be used to study the Yamabe flow.

This analysis will provide a tool to discuss, in an appropriate function space, the short-time solvability of Cauchy problems on M of the form

$$(\partial_t - a\Delta)u = \ell, \;\; u|_{t=0} = u_0. \tag{4.14}$$

We will approach this problem following the techniques presented in [BV19] and [EM13]. The idea is to construct an appropriate boundary parametrix, giving us an approximate inverse near the boundary, and an interior parametrix, which will be obtained via classical parabolic PDE theory on compact manifolds. A combination of those will lead to an operator that will be used to prove short time existence of equation (4.14).

Lemma 4.5. *Let $\varphi, \psi \in C^\alpha_\Phi(M)$ be compactly supported smooth functions so that ψ is supported away from the boundary. The operator*

$$\mathbf{R^0} := M_\psi \mathbf{H} \star M_\varphi : x^\gamma C^{k,\alpha}_\Phi(M \times [0,T]) \to \sqrt{t} x^\gamma C^{k+1,\alpha}_\Phi(M \times [0,T])$$

has operator norm $\|\mathbf{R^0}\|_{\mathrm{op}}$ converging to 0 as T goes to 0.

Proof. Since ψ is supported away from the boundary of M, hence away from the singularities of H, the lift of $\psi H \varphi$ to M^2_h is compactly supported away from ff, fd, lf and rf. Using the formulas described in §2.4.2 one finds that the asymptotics of $\psi H \varphi$ are given by

$$\beta^*(\psi H \varphi) \sim \tau^{-m} G_0$$

where G_0 is a bounded function vanishing to infinite order as $\|(\mathscr{S}, \mathscr{U}, \mathscr{Z})\| \to \infty$. This implies Theorem 4.4 to hold also for $M_\psi \mathbf{H} \star M_\varphi$ by straightforward estimates. Hence, the operator norm of

$M_\psi \mathbf{H} \star M_\varphi$ can be obtained by

$$\|M_\psi \mathbf{H} \star M_\varphi\|_{\text{op}} = \sup_{\|u\|^*_{k,\alpha,\gamma}=1} \|M_\psi \mathbf{H} \star (M_\varphi u)\|^*_{k+1,\alpha,\gamma} \leq \sup_{\|u\|^*_{k,\alpha,\gamma}=1} c\sqrt{t}\|u\|^*_{k,\alpha,\gamma} = c\sqrt{t},$$

which clearly converges to 0 as t goes to 0. Note that this holds in particular for T going to 0. □

4.3.1 Boundary parametrix

From now on, let $U_r = \{x \leq r\}$ be a fixed collar neighborhood of $\partial\overline{M}$. In order to localize our argument, we need to fix a specific covering of M. Given the family of half-cubes

$$B(d) = [0,d) \times (-d,d)^b \times (-d,d)^f \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^b \times \mathbb{R}^f,$$

for every point $p \in \partial\overline{M}$ there exists a coordinate chart A_p around p so that the half-cube $B(1)$ is diffeomorphic to A_p. Since by assumption the boundary is compact, $\partial\overline{M}$ can be covered with finitely many charts $\{A_i, p_i, \phi_i\}_{i=1,\dots,n}$ where $\phi_i : B(1) \to A_i$. Clearly, for r sufficiently small, these open sets will cover the fixed collar neighborhood U_r. Such a covering of $\partial\overline{M}$ can be extended to a cover of M by adding an open subset $A_0 = M \setminus \{x \leq r/2\}$.

We can now proceed with the construction of the boundary parametrix. First of all let us fix a smooth compactly supported function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}$ so that $\sigma(s) = 1$ for $s \leq 1/2$ and $\sigma(s) = 0$ for $s \geq 1$. Smoothness implies σ to lie in $C^{k,\alpha}(\mathbb{R}_{\geq 0})$ for every $k \geq 0$. Note that here $\sigma \in C^{k,\alpha}(\mathbb{R}_{\geq 0})$ means that σ is α-Hölder in the classical sense, as well as all its derivatives up to order k. Let $\widetilde{\varphi}, \widetilde{\psi} : \mathbb{R}_{\geq 0} \times \mathbb{R}^b \times \mathbb{R}^f \to \mathbb{R}$ be defined, for any $m = 1, \dots, n$, by

$$\begin{aligned} \widetilde{\varphi}(x,y,z) &= \sigma(x)\sigma(\|y\|)\sigma(\|z\|) \\ \widetilde{\psi}(x,y,z) &= \sigma\left(\frac{x}{2}\right)\sigma\left(\frac{\|y\|}{2}\right)\sigma\left(\frac{\|z\|}{2}\right). \end{aligned}$$

Since σ lies in $C^{k,\alpha}(\mathbb{R}_{\geq 0})$ for every $k \geq 0$, it follows that $\widetilde{\varphi}, \widetilde{\psi} \in C^{k,\alpha}(\mathbb{R}_{\geq 0} \times \mathbb{R}^b \times \mathbb{R}^f)$. Furthermore, note that $\widetilde{\psi} \equiv 1$ on $\operatorname{supp}(\widetilde{\varphi})$.

For any point $\bar{p} \in \partial\overline{M}$ there exists a coordinate patch A_i so that $\bar{p}$ lies in A_i and has coordinate $(0, \bar{y}, \bar{z})$. For any $\varepsilon \in (0,1)$, one can consider the functions $\bar{\varphi}_{i,\bar{p}}, \bar{\psi}_{i,\bar{p}} : A_i \to \mathbb{R}$ defined by

$$\begin{aligned} \bar{\varphi}_{i,\bar{p}}(p) &= \widetilde{\varphi}\left(\frac{x}{\varepsilon}, y-\bar{y}, \varepsilon(z-\bar{z})\right) \\ \bar{\psi}_{i,\bar{p}}(p) &= \widetilde{\psi}\left(\frac{x}{\varepsilon}, y-\bar{y}, \varepsilon(z-\bar{z})\right), \end{aligned} \tag{4.15}$$

where the local coordinates used are given as follows:

$$\begin{aligned} &x = x(\phi_i^{-1}(p)), \\ &y = y(\phi_i^{-1}(p)) \text{ and } \bar{y} = y(\phi_i^{-1}(\bar{p})), \\ &z = z(\phi_i^{-1}(p)) \text{ and } \bar{z} = z(\phi_i^{-1}(\bar{p})). \end{aligned}$$

From the definition of $\widetilde{\varphi}$ and $\widetilde{\psi}$, it is clear that $\bar{\psi}_{i,\bar{p}} \equiv 1$ on $\operatorname{supp}(\bar{\varphi}_{i,\bar{p}})$. Furthermore, it is clear from the definition that both $\bar{\varphi}_{i,\bar{p}}$ and $\bar{\psi}_{i,\bar{p}}$ equal 1 near $\bar{p}$. Note that the bump-functions $\bar{\varphi}_{i,\bar{p}}$ and $\bar{\psi}_{i,\bar{p}}$ are chosen precisely such that the following estimates hold:

$$\bullet\, [\bar{\varphi}_{i,\bar{p}}]^*_\alpha = \sup_{M^2} \frac{|\bar{\varphi}_{i,\bar{p}}(p) - \bar{\varphi}_{i,\bar{p}}(p')|}{d_{x^4\Phi}(p,p')^\alpha} \leq C\varepsilon^{-\alpha}, \tag{4.16}$$

$$\bullet \text{ if } p,p' \in \operatorname{supp} \bar{\varphi}_{i,\bar{p}} \text{ then } d_{x^4\Phi}(p,p') \leq C\varepsilon^\alpha, \tag{4.17}$$

with the same statement holding for $\bar{\psi}_{i,\bar{p}}$ for some uniform constant $C > 0$. Note also that Φ-derivatives do not worsen, but even improve the ε-estimate, since they carry extra x-powers, implying

$$[V\bar{\varphi}_{i,\bar{p}}]^*_\alpha, [V\bar{\psi}_{i,\bar{p}}]^*_\alpha \leq C\varepsilon^{-\alpha+k}, \text{ for every } V \in \mathscr{V}^k_\Phi(M). \tag{4.18}$$

The bump-functions defined above can be extended smoothly to the whole manifold M by letting them be 0 everywhere outside the support, which is contained in A_i. With abuse of notation we will call these extensions again $\bar{\varphi}$ and $\bar{\psi}$. We want to construct partitions of unity near the boundary. To this end, for every $i = 1,\dots,n$ and for every $\varepsilon \in (0,1)$, we can consider the set

$$E_{i,\varepsilon} = A_m \cap \left\{\phi_i(0,\varepsilon z) \;\middle|\; z \in \mathbb{Z}^{b+f}\right\}.$$

Due to the diffeomorphism $\phi_i : B(1) \to A_i$, the set $\#E_{i,\varepsilon}$ is finite, hence the set

$$\{\bar{\psi}_{i,p} \,|\, i = 1,\dots,n \;\; p \in E_{i,\varepsilon}\} \tag{4.19}$$

is also finite for any $\varepsilon \in (0,1)$. Hence every point $q \in \partial\overline{M}$ is contained in the support of at most a finite number of the functions in the above set. Further, setting

$$\varphi_{i,p} = \frac{\bar{\varphi}_{i,p}}{\sum_{\ell=1}^n \sum_{\bar{p}\in E_{\ell,\varepsilon}} \bar{\varphi}_{\ell,\bar{p}}}, \qquad \psi_{i,p} = \frac{\bar{\psi}_{i,p}}{\sum_{\ell=1}^n \sum_{\bar{p}\in E_{\ell,\varepsilon}} \bar{\psi}_{\ell,\bar{p}}},$$

one has that, for every $\varepsilon \in (0,1)$, the above functions are partitions of unity and the sum

$$\phi = \sum_{i=1}^n \sum_{p\in E_{i,\varepsilon}} \varphi_{i,p} \tag{4.20}$$

is identically one on an open neighborhood of the boundary $\partial\overline{M}$. The above functions still satisfy (4.16) and will allow us to localize problem (4.14) in a neighborhood of a point lying on the boundary of the manifold M. In §3.2 , parabolic Schauder estimates for the heat operator of the Laplace-Beltrami operator have been established. The idea is to use those estimates, upon an appropriate rescaling, to a localized version of problem (4.14). This will be accomplished employing the technique of frozen coefficients.

Before we proceed, we assume from this point on that the factor a in (4.14) lies in $C^{k,\beta}_{x^4\Phi}(M \times [0,T])$ for some $0 < \alpha < \beta < 1$. Under this assumption, we have

$$\|a - a(\overline{p},0)\|_{\infty,\operatorname{supp}\psi_{m,\bar{p}}} \leq C(T^{\beta/2} + \varepsilon^\beta), \tag{4.21}$$

for some uniform constant $C > 0$. Moreover, additional Φ-derivatives, once again, only add positive ε-powers, leading to similar estimates as the ones shown in (4.18).

Fix a point $\bar{p}$ on the boundary $\partial\overline{M}$. Note that, by construction, any point q on the boundary, lies on the support of at most a fixed number of functions in the set (4.19). Hence, without loss of generality, we can consider $\bar{p}$ lying in $E_{i,\varepsilon}$ for some $i = 1,\dots,n$. Problem (4.14) can be localize, near $\bar{p}$, as follows. Let us freeze the factor $a(\bar{p},t)$ in front of the Laplacian at $(\bar{p},0)$ and consider, for a given $\ell \in x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T])$, the Cauchy problem

$$P(\bar{p},0)\bar{u}_{\bar{p}} := (\partial_t - a(\bar{p},0)\Delta)\bar{u}_{\bar{p},0} = \varphi_{i,\bar{p}}\ell, \;\; \bar{u}_{\bar{p}}|_{t=0} = 0. \tag{4.22}$$

Assuming that a is positive and denoting the solution of the solution operator of (4.22) by $\mathbf{H}_{\gamma,\bar{p}}$ (which, up to rescaling, is the heat operator), a solution to (4.22) is given by:

$$\bar{u}_{\bar{p}} = \mathbf{H}_{\gamma,\bar{p}}(\varphi_{m,\bar{p}}\ell) \in x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M\times[0,T]).$$

Let us define the function

$$u_{\bar{p}} = \psi_{i,\bar{p}}\mathbf{H}_{\gamma,\bar{p}}(\varphi_{i,\bar{p}}\ell). \tag{4.23}$$

Lemma 4.6. *Let $a \in C^{k,\beta}_{x^4\Phi}(M\times[0,T])$, with $0<\alpha<\beta<1$, be positive, bounded from below away from zero. Then the function $u_{\bar{p}}$ defined in (4.23) satisfies*

$$Pu_{\bar{p}} := (\partial_t - a\Delta)u_{\bar{p}} = \varphi_{i,\bar{p}}\ell + R^1_{i,\bar{p}}\ell + R^2_{i,\bar{p}}\ell, \tag{4.24}$$

where

a) $R^1_{i,\bar{p}} : x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T]) \to x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T])$ is a bounded operator with a uniform constant such that

$$\|R^1_{i,\bar{p}}\ell\|^*_{k,\alpha,\gamma} \le C\left(T\varepsilon^{-\alpha} + T^{\alpha/2}\varepsilon^{-2\alpha} + T^{\alpha/2}\varepsilon^{\beta-\alpha} + \varepsilon^{\beta-\alpha}\right)\|\ell\|^*_{k,\alpha,\gamma}.$$

b) $R^2_{i,\bar{p}} : x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T]) \to x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T])$ is a bounded operator and its operator norm goes to 0 as $T\to 0^+$ i.e.

$$\lim_{T\to 0^+}\|R^2_{i,\bar{p}}\|_{\mathrm{op}} = 0.$$

Proof. In order to avoid the plethora of indices we will suppress all the indices on φ,ψ and the error terms R^0 and R^1. Computing $Pu_{\bar{p}}$ gives us the following:

$$\begin{aligned} Pu_{\overline{p}} =& (\partial_t - a\Delta)\psi\mathbf{H}_{\overline{p}}(\varphi\ell) \\ =& \psi\partial_t\mathbf{H}_{\overline{p}}(\varphi\ell) - a\Delta(\psi H_{\overline{p}}(\varphi\ell)) \\ =& \psi\partial_t\mathbf{H}_{\overline{p}}(\varphi\ell) - a\mathbf{H}_{\overline{p}}(\varphi\ell)\Delta\psi - 2a\langle\nabla\psi,\nabla\mathbf{H}_{\overline{p}}(\varphi\ell)\rangle - a\psi\Delta\mathbf{H}_{\overline{p}}(\varphi\ell). \end{aligned}$$

On the other hand, for every function v sufficiently regular we have

$$\begin{aligned}[\psi, a\Delta]v &:= a\psi\Delta v - a\Delta(\psi v) = a\psi\Delta v - a(v\Delta\psi + 2\langle\nabla\psi, \nabla v\rangle + \psi\Delta v) \\ &= -2a\langle\nabla\psi, \nabla v\rangle - av\Delta\psi.\end{aligned} \tag{4.25}$$

Consequently, we get to rewrite $Pu_{\overline{p}}$ as

$$\begin{aligned}Pu_{\overline{p}} &= \psi(\partial_t - a\Delta)\mathbf{H}_{\overline{p}}[\varphi\ell] + [\psi, a\Delta](\mathbf{H}_{\overline{p}}(\varphi\ell)) \\ &= \psi(\partial_t - a(\overline{p},0))\Delta)\mathbf{H}_{\overline{p}}(\varphi\ell) + \psi((a(\overline{p},0) - a)\Delta)\mathbf{H}_{\overline{p}}(\varphi\ell) \\ &+ [\psi, a\Delta]\mathbf{H}_{\overline{p}}(\varphi\ell) \\ &= \psi\varphi\ell + \psi(a(\overline{p},0) - a)\Delta\mathbf{H}_{\overline{p}}(\varphi\ell) + [\psi, a\Delta]\mathbf{H}_{\overline{p}}(\varphi\ell) \\ &=: \psi\varphi\ell + R^1\ell + R^2\ell.\end{aligned} \tag{4.26}$$

Note that the first term in equation (4.26) is obtained from the fact that $\mathbf{H}_{\overline{p}}(\varphi\ell)$ is a solution of the localized Cauchy problem. Moreover, since $\psi = 1$ on the support of φ we can conclude that $\psi\varphi\ell = \varphi\ell$.

Let us estimate $R^1\ell$ and $R^2\ell$ for $\gamma = 0$ and $k = 0$. The cases $\gamma \in \mathbb{R}$ and $k \in \mathbb{N}$ can be proven analogously with minor adjustments.

For the estimate of $R^1\ell$, let us first note by Theorem 4.4,

$$\Delta\mathbf{H}_{\overline{p}} : C^{\alpha}_{x^4\Phi}(M \times [0,T]) \to t^{\alpha/2}C^0(M \times [0,T]). \tag{4.27}$$

Note that in the estimate of $R^1\ell$, the suprema in the definition of Hölder norm can be taken over $\operatorname{supp}\psi \equiv \operatorname{supp}\psi_{i,\bar{p}}$. We note then from (4.16)

- $[\Delta\mathbf{H}_{\overline{p}}(\varphi\ell)]^*_\alpha \leq C\varepsilon^{-\alpha}\|\ell\|^*_\alpha$,
- $[\psi]^*_\alpha \leq C\varepsilon^{-\alpha}$.

For the supremum-norm, taken again over $\operatorname{supp}\psi$, we find from (4.16) and (4.27):

- $\|a(\bar{p},0) - a\|_\infty \leq C(T^{\beta/2} + \varepsilon^\beta)$,
- $\|\Delta\mathbf{H}_{\overline{p}}(\varphi\ell)\|_\infty \leq CT^{\alpha/2}\|\ell\|_\infty$.

From there we conclude

$$\begin{aligned}\|R^1\ell\|^*_\alpha &= \|\psi\|_\infty\|a - a(\bar{p},0)\|_\infty\|\Delta\mathbf{H}_{\overline{p}}(\varphi\ell)\|_\infty \\ &+ [\psi]^*_\alpha\|a - a(\bar{p},0)\|_\infty\|\Delta\mathbf{H}_{\overline{p}}(\varphi\ell)\|_\infty \\ &+ \|\psi\|_\infty[a - a(\bar{p},0)]^*_\alpha\|\Delta\mathbf{H}_{\overline{p}}(\varphi\ell)\|_\infty \\ &+ \|\psi\|_\infty\|a - a(\bar{p},0)\|_\infty[\Delta\mathbf{H}_{\overline{p}}(\varphi\ell)]^*_\alpha \\ &\leq C\|\ell\|^*_\alpha\left((T^{\beta/2} + \varepsilon^\beta)T^{\alpha/2} + T^{\alpha/2}\varepsilon^{-\alpha}(T^{\beta/2} + \varepsilon^\beta) + T^{\alpha/2} + \varepsilon^{-\alpha}(T^{\beta/2} + \varepsilon^\beta)\right).\end{aligned}$$

Altogether, we obtain

$$\|R^1\ell\|^*_\alpha \leq C\|\ell\|^*_\alpha \left(T^{(\alpha+\beta)/2}\varepsilon^{-\alpha} + T^{\alpha/2} + T^{\alpha/2}\varepsilon^{\beta-\alpha} + \varepsilon^{\beta-\alpha}\right).$$

Let us now prove the second part of the statement, that is proving R^2 to be a bounded operator with operator norm converging to 0 as T goes to 0. From (4.25), we get to rewrite R^2 as

$$R^2\ell = -2a\langle\nabla\psi, \nabla\mathbf{H}_{\overline{p}}(\varphi\ell)\rangle - a\mathbf{H}_{\overline{p}}(\varphi\ell)\Delta\psi.$$

Since ψ is constant on a neighborhood of the boundary $\partial\overline{M}$, Lemma 4.5 guarantees that the operator $R^2 : C^{k,\alpha}_{x^4\Phi}(M\times[0,T]) \to C^{k,\alpha}_{x^4\Phi}(M\times[0,T])$ is a bounded operator with operator norm converging to 0 as T goes to 0.

Now, let us prove the case $k>0$. First, note that the proof presented of the estimate for R^2 does not rely on the value of k, hence requiring no further explanations. Therefore, we must prove that

$$\|VR^1\ell\|^*_\alpha \leq C\|\ell\|^*_\alpha \left(T^{(\alpha+\beta)/2}\varepsilon^{-\alpha} + T^{\alpha/2} + T^{\alpha/2}\varepsilon^{\beta-\alpha} + \varepsilon^{\beta-\alpha}\right),$$

for all $V \in \mathscr{V}^l_\Phi$, with $l \leq k$. Naturally, it is reasonable to decompose $V = V_1V_2V_3$, where $V_j \in \mathscr{V}^{l_j}_\Phi(M)$ and $l_1+l_2+l_3 = l$. Thus, we get

$$VR^1\ell = \sum_{l_1+l_2+l_3\leq k} V_1\psi \cdot V_2(a(\overline{p},0)-a)\cdot V_3\Delta\mathbf{H}_{\overline{p}}(\varphi\ell).$$

Assume $l_3 \leq k-1$. Then $V_{l_3}\Delta \in \mathrm{Diff}^{k+1}_\Phi(M)$ and thus, from Theorem 4.4, it follows that

$$\|VR^1\ell\|^*_\alpha \leq C\sum_{l_1+l_2+l_3\leq k} T^{\alpha/2}\|V_1\psi\|^*_\alpha\|V_2(a(\overline{p},0)-a)\|^*_\alpha\|\ell\|^*_\alpha,$$

which, combined with (4.18) and (4.21) allows us to conclude the estimate for $l_3\leq k-1$. Now, assume $l_3 = k$, which means $V_{l_3}\Delta \in \mathrm{Diff}^{k+2}_\Phi(M)$. From Theorem 3.10, it follows that $V_{l_3}\Delta\mathbf{H}_{\overline{p}}(\varphi\ell) \in C^\alpha_{x^4\Phi}(M\times[0,T])$ and thus, employing once again (4.16) and (4.21), the estimate follows. □

Remark 4.7. We point out that the condition $C^{k,\beta}_{x^4\Phi}(M\times[0,T])$ on the factor a cannot be improved for this construction of the parametrix – we cannot ask for a to lie in any Hölder space with an exponent that does not exceed α – because we must be able to choose ε sufficiently small such that $\varepsilon^{\beta-\alpha} < \delta/4C$, which cannot be true if $\beta\leq\alpha$.

We can now construct a boundary parametrix. Set

$$\mathscr{Q}_B\ell = \sum_{i=1}^n \sum_{\overline{p}\in E_{i,\varepsilon}} \psi_{i,\overline{p}}\mathbf{H}_{\overline{p}}(\varphi_{i,\overline{p}}\ell). \tag{4.28}$$

Proposition 4.8. *For every $\delta>0$ one can find $\varepsilon>0$ and $T_0>0$ small enough so that*

$$\mathscr{Q}_B : x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T_0]) \to x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M\times[0,T_0]),$$

$$\mathscr{Q}_B : x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T_0]) \to x^\gamma\sqrt{t}C^{k+1,\alpha}_{x^4\Phi}(M\times[0,T_0])$$

are bounded operators satisfying, for ϕ defined as in (4.20),

$$(\partial_t - a\Delta)(\mathscr{Q}_B\ell) = \phi\ell + \mathbf{R}^1\ell + \mathbf{R}^2\ell$$

with $\|\mathbf{R}^1\ell\|^*_{k,\alpha,\gamma} \le \delta$ and $\|\mathbf{R}^2\ell\|^*_{k,\alpha,\gamma}$ converging to 0 as T goes to 0.

Proof. First of all let us note that the sums in (4.28) are finite. Boundedness of $\mathscr{Q}_B$ follows directly from Theorem 3.10. Indeed, both multiplication operators $M_{\psi_{i,\overline{p}}}$ and $M_{\varphi_{i,\overline{p}}}$ are bounded operators preserving the regularity (from Proposition 3.4). Thus, we have

$$\begin{aligned}\mathscr{Q}_B : &x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T]) \xrightarrow{M_{\psi_{i,\overline{p}}}} x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T]) \xrightarrow{\mathbf{H}_{\overline{p}}} \\ &\xrightarrow{\mathbf{H}_{\overline{p}}} x^\gamma\sqrt{t}C^{k+1,\alpha}_{x^4\Phi}(M\times[0,T])) \xrightarrow{M_{\varphi_{i,\overline{p}}}} x^\gamma\sqrt{t}C^{k+1,\alpha}_{x^4\Phi}(M\times[0,T]).\end{aligned}$$

Following along the same lines one can see that $\mathscr{Q}_B$ is also bounded when taken from $x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T])$ to $x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M\times[0,T])$.

Computing explicitly $(\partial_t - a\Delta)(\mathscr{Q}_B\ell)$ and applying Lemma 4.6 one has

$$\begin{aligned}(\partial_t - a\Delta)(\mathscr{Q}_B\ell) &= \sum_{i=1}^n \sum_{\overline{p}\in E_{i,\varepsilon}} (\partial_t - a\Delta)(\psi_{i,\overline{p}}\mathbf{H}_{\overline{p}}(\varphi_{i,\overline{p}}\ell)) \\ &= \phi\ell + \sum_{i=1}^n \sum_{\overline{p}\in E_{i,\varepsilon}} R^1_{i,\overline{p}}\ell + \sum_{i=1}^n \sum_{\overline{p}\in E_{i,\varepsilon}} R^2_{i,\overline{p}}\ell.\end{aligned}$$

For simplicity let us denote, $\mathbf{R}^j\ell = \sum_{i=1}^n \sum_{\overline{p}\in E_{i,\varepsilon}} R^j_{i,\overline{p}}\ell$ for $j=1,2$. We will denote all the uniform positive constants arising from the estimates by $C>0$. From Lemma 4.6 follows that

$$\|R^1_{i,\overline{p}}\ell\|^*_{k,\alpha} \le C\left(T^{(\alpha+\beta)/2}\varepsilon^{-\alpha} + T^{\alpha/2} + T^{\alpha/2}\varepsilon^{\beta-\alpha} + \varepsilon^{\beta-\alpha}\right)\|\ell\|^*_{k,\alpha}.$$

Hence we estimate $\|R^1_{i,\overline{p}}\|_{\text{op}}$ as follows

$$\|R^1_{i,\overline{p}}\|_{\text{op}} = \sup_{\|\ell\|^*_{k,\alpha}=1} \|R^1_{m,\overline{p}}\ell\|^*_{k,\alpha} \le C\left(T^{(\alpha+\beta)/2}\varepsilon^{-\alpha} + T^{\alpha/2} + T^{\alpha/2}\varepsilon^{\beta-\alpha} + \varepsilon^{\beta-\alpha}\right).$$

Given any $\delta>0$, choose both $\varepsilon>0$ and $T>0$ small enough in a way that

$$T^{(\alpha+\beta)/2}\varepsilon^{-\alpha}, T^{\alpha/2}\varepsilon^{\beta-\alpha}, T^{\alpha/2}, \varepsilon^{\beta-\alpha} < \delta/4C,$$

and $x=\varepsilon$ is a smooth hypersurface. In such a way we get $\|R^1\|_{\text{op}} < \delta$.

The statement about R^2 follows automatically from Lemma 4.6. □

Remark 4.9. Choosing ε small enough so that $x=\varepsilon$ is a smooth hypersurface will be useful in the next subsection.

4.3.2 Construction of the Parametrix

In the previous subsection, through localization, we used mapping properties of $\mathbf{H}\star$ to construct an approximate boundary parametrix. In this subsection we will construct first an approximate interior parametrix and we will conclude with the construction of a right inverse for the Cauchy problem (4.14).

First of all note that on compact subspaces of M, the construction of a parametrix is a mere application of classical PDE theory on compact manifolds. With respect to ε as in the previous section, an ε-neighborhood of $\partial\overline{M}$ is fixed and the function ϕ is identically 1 on such a neighborhood. The idea is to cut off a neighborhood of the boundary from M. Let $Y_\varepsilon = \{p \in M \mid x = x(p) \geq \varepsilon/2\}$; it is clear that Y_ε is a manifold with boundary. Denote by $\overline{Y}$ the double space of Y_ε, consisting in two copies of Y_ε glued together along the boundary, which is a compact manifold without boundary. Note that the double space construction does not lead to a smooth metric on $\overline{Y}$. In order to smooth it up we consider a smoothing of such a metric so that the metric on $\overline{Y}$ and the one on M coincide on $Y_{2\varepsilon}$. Moreover on $\overline{Y}$ we are working away from the boundary hence the α-Hölder norms are the classical ones.

We can extend the function $(1-\phi)$ to a function, still denoted by $(1-\phi)$, on $\overline{Y}$ by setting it to be 0 on the second copy of Y_ε. In particular such a function $(1-\phi)$ defines a smooth cut off function over Y_ε in $\overline{Y}$. Similarly, let $\overline{P}$ denote the uniform parabolic extension of $P|_{Y_\varepsilon}$ to $\overline{Y}$.

It is well known, from classical parabolic PDE theory, that there exists a parametrix $\overline{Q}_I$ for the heat operator $\overline{P}$ so that the maps

$$\overline{Q}_I : C^{k,\alpha}(\overline{Y}\times[0,T]) \to \left(C^{k+2,\alpha}\cap\sqrt{t}C^{k+1,\alpha}\right)(\overline{Y}\times[0,T]) \tag{4.29}$$

is bounded. The idea is to use such a parametrix $\overline{Q}_I$ and the boundary parametrix constructed above to construct a parametrix $\mathscr{Q}$ for the Cauchy problem (4.14).

Note that, for a given function $\widehat{u}\in C^{k,\alpha}(\overline{Y}\times[0,T])$, one has $\overline{Q}_I\widehat{u}\in\sqrt{t}C^{k+1,\alpha}(\overline{Y}\times[0,T])$. In order to turn $\overline{Q}_I\widehat{u}$ into a function in $C^{k,\alpha}_{x^4\Phi}(M\times[0,T])$, let us consider a cut off function $\overline{\Psi}$ on $\overline{Y}$ so that $\overline{\Psi}=1$ on $\mathrm{supp}(1-\phi)$. We can now define the operator

$$\mathscr{Q}_I := M_{\overline{\Psi}}\overline{Q}_I M_{(1-\phi)}.$$

As pointed out in Remark 4.2, $M_{\overline{\Psi}}$ and $M_{(1-\phi)}$ preserve the regularity and are bounded operators. Hence, it follows that

$$\begin{aligned}\mathscr{Q}_I : x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T]) &\xrightarrow{M_{(1-\phi)}} C^{k,\alpha}(\overline{Y}\times[0,T]) \xrightarrow{\overline{Q}_I}\\ &\xrightarrow{\overline{Q}_I} \sqrt{t}C^{k+1,\alpha}(\overline{Y}\times[0,T]) \xrightarrow{M_{\overline{\Psi}}} \sqrt{t}C^{k+1,\alpha}(Y_\varepsilon\times[0,T])\end{aligned}$$

acts continuously. Moreover, since we are working away from the boundary of M, the spaces $C^{k+1,\alpha}(Y_\varepsilon\times[0,T])$ can be identified with the space $x^\gamma C^{k+1,\alpha}_\Phi(Y_\varepsilon\times[0,T])$. We can hence conclude that the operator

$\mathcal{Q}_I$ mapping

$$\mathcal{Q}_I : x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T]) \to x^\gamma \sqrt{t} C^{k+1,\alpha}_{x^4\Phi}(M \times [0,T])$$

is bounded. It is then clear that a parametrix for the Cauchy problem in (4.14) is given by

$$\mathcal{Q}\ell = \mathcal{Q}_B\ell + \mathcal{Q}_I\ell$$

with the map

$$\mathcal{Q} : x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T]) \to x^\gamma \sqrt{t} C^{k+1,\alpha}_{x^4\Phi}(M \times [0,T])$$

being bounded.

Proposition 4.10. *Let $a \in C^{k,\beta}_{x^4\Phi}(M \times [0,T])$, with $0 < \alpha < \beta < 1$, be positive, bounded from below away from zero, and consider the operator $P = \partial_t - a\Delta$. For $T_0 > 0$ sufficiently small there exists an operator $\mathbf{Q}$ so that the map*

$$\mathbf{Q} : x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T_0]) \to x^\gamma \left(C^{k+2,\alpha}_{x^4\Phi} \cap \sqrt{t} C^{k+1,\alpha}_{x^4\Phi} \right)(M \times [0,T_0])$$

is bounded. Moreover, for every function ℓ in $x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T])$, $\mathbf{Q}\ell$ is a solution of the inhomogeneous Cauchy problem

$$(\partial_t - a\Delta)u = \ell, \ \ u|_{t=0} = u_0. \tag{4.30}$$

Proof. Let ℓ be a function in $x^\gamma C^{k,\alpha}_{\Phi}(M \times [0,T])$. Applying Proposition 4.8 and the construction above, we get

$$(\partial_t - a\Delta)(\mathcal{Q}\ell) = \phi\ell + \mathbf{R}^1\ell + \mathbf{R}^2\ell + (1-\phi)\ell + \mathbf{R}^3\ell$$

where $\mathbf{R}^1$ and $\mathbf{R}^2$ are the ones arising from Proposition 4.8 while $\mathbf{R}^3$ is given by

$$\mathbf{R}^3\ell = [\overline{\psi}, a\Delta](\overline{Q}_I(1-\phi)\ell).$$

It is clear from the definition that $\mathbf{R}^3 : x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T]) \to x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T])$. Further, the operator norm of $\mathbf{R}^3$ can be estimated in the same way as we have already estimated $\mathbf{R}^2$ in Lemma 4.6. In particular it follows that both $\|\mathbf{R}^2\|_{\text{op}}$ and $\|\mathbf{R}^3\|_{\text{op}}$ converge to 0 as T goes to 0 while $\|\mathbf{R}^1\|_{\text{op}} < \delta$. We can now choose T_0 small enough so that, denoting by $\mathbf{R} := \mathbf{R}^1 + \mathbf{R}^2 + \mathbf{R}^3$,

$$\|\mathbf{R}\|_{\text{op}} \leq \|\mathbf{R}^1\|_{\text{op}} + \|\mathbf{R}^2\|_{\text{op}} + \|\mathbf{R}^3\|_{\text{op}} < 1.$$

It is now clear that $\text{id} + \mathbf{R}$ is invertible, with inverse obtained via the von Neumann series of $\mathbf{R}$. The claimed right parametrix of P will then be

$$\mathbf{Q} = \mathcal{Q}(\text{id} + \mathbf{R})^{-1}.$$

□

Corollary 4.11. *Let $a \in C^{k,\beta}_{x^4\Phi}(M \times [0,T])$, with $0 < \alpha < \beta < 1$, be positive, bounded from below away from zero. For T_0 sufficiently small there exists an operator* $\mathbf{E}$

$$\mathbf{E} : x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M) \to x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M \times [0,T_0])$$

so that $\mathbf{E}$ *is bounded and, if u_0 is a function in $x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M)$, $u = \mathbf{E}u_0$ is a solution of the homogeneous Cauchy problem*

$$(\partial_t - a\Delta)u = 0, \;\; u|_{t=0} = u_0. \tag{4.31}$$

Proof. Since $u_0 \in x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M)$ then $a\Delta u_0$ lies in $x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T])$. Using the right inverse for the inhomogeneous Cauchy problem constructed in Proposition 4.10, set

$$\mathbf{E}u_0 = u_0 + \mathbf{Q}(a\Delta u_0).$$

An easy computation shows that $\mathbf{E}u_0$ indeed solves the homogeneous Cauchy problem. □

Even though Proposition 4.10 and Corollary 4.11 together do provide solutions for the inhomogeneous Cauchy problem we wish to solve, they do this at the cost of possibly shrinking the time interval on which a is defined on. To fix this issue, we now improve upon Proposition 4.10 by employing Theorem 2.7 (the maximum principle). To be more precise, we need to know that the homogeneous Cauchy problem

$$(\partial_t - a\Delta)u = 0, \;\; u|_{t=0} = 0$$

has only the trivial solution $u = 0$ in $x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T])$. But this is a consequence of Theorem 2.7, since the functions in $x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T])$ satisfy the conditions imposed in Proposition 2.6. Therefore, we have the following theorem.

Theorem 4.12. *Consider a function $a \in C^{k,\beta}_{x^4\Phi}(M \times [0,T])$, with $0 < \alpha < \beta < 1$, be positive, bounded from below away from zero. Then the equations*

$$(\partial_t - a\Delta)u = \ell; u|_{t=0} = 0, \tag{4.32}$$

$$(\partial_t - a\Delta)u = 0; u|_{t=0} = u_0 \tag{4.33}$$

have solutions $\mathbf{Q}\ell$ and $\mathbf{E}u_0$, respectively, such that

$$\mathbf{Q} : x^\gamma C^{k,\alpha}_{x^4\Phi}(M \times [0,T]) \to x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M \times [0,T]),$$
$$\mathbf{E} : x^\gamma C^{k,\alpha}_{x^4\Phi}(M) \to x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M \times [0,T]),$$

are bounded maps, for any $\gamma \in \mathbb{R}$.

Proof. First, let us focus on proving the existence of a solution for the first of the two Cauchy problems above.

From Proposition 4.10, we know that the Cauchy problem above, in fact, does have a solution, say $u \in x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M \times [0,T_0])$. If $T_0 \geq T$, clearly there is nothing to do. Thus, let us assume $T_0 < T$.

Our goal is to prove that u can actually be extended to a solution on the entire interval $[0,T]$. Consider $0<\varepsilon<T_0$ and the Cauchy problem

$$(\partial_t - a\Delta)v_1 = 0;\ \ v_1|_{t=0} = u|_{t=T_0-\varepsilon}. \tag{4.34}$$

Once again, the parametrix construction (Corollary 4.11) ensures the existence of a solution v_1 in $x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M\times[0,T_0])$. Observe that a simple change of variables given by the translation $t\mapsto t+(T_0-\varepsilon)$ allows us to define v_1 as a function defined on $[T_0-\varepsilon, 2T_0-\varepsilon]$. Moreover, consider now the inhomogeneous Cauchy problem

$$(\partial_t - a\Delta)u_1 = \ell;\ \ u_1|_{t=0} = 0. \tag{4.35}$$

Similarly, we do have a solution u_1 in $x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M\times[0,T_0])$ and, following the same logic on the previous paragraph, we are allowed to define u_1 as a function on $[T_0-\varepsilon, 2T_0-\varepsilon]$. Now, note that on the interval $[T_0-\varepsilon, T_0]$ we have

$$(\partial_t - a\Delta)(u_1+v_1) = \ell;\ \ (u_1+v_1)|_{t=T_0-\varepsilon} = u|_{t=T_0-\varepsilon}, \tag{4.36}$$

which is a Cauchy problem that $(u_1+v_1)\in x^\gamma C^{k+2,\alpha}_{x^4\Phi}(M\times[T_0-\varepsilon,T_0])$ satisfies. However, the function u also satisfies the same Cauchy problem and, therefore, from Theorem 2.7 it follows that $u=u_1+v_1$ on $[T_0-\varepsilon,T_0]$. Hence we can now extend u past T_0 by defining

$$\widetilde{u}(p,t) = \begin{cases} u(p,t), & \text{if } 0\le t\le T_0,\\ (u_1+v_1)(p,t), & \text{if } T_0<t\le 2T_0-\varepsilon. \end{cases}$$

If $2T_0-\varepsilon\ge T$, then we have already an extension of u on the entire desired interval. Otherwise, repeat the process with $\widetilde{u}$ until $nT_0-n\varepsilon\ge T$ (which is possible in a finite number of repetitions since $[0,T]$ is compact). Thus we have an extension of u defined on $M\times[0,T]$. Note that this extension was obtained employing the parametrix construction, namely the maps **Q** and **E**, which are bounded, from where follows that the extended map **Q** given by $\ell\mapsto\widetilde{u}$ is bounded too, therefore enabling us to now extend **E** as well, completing the proof. □

This concludes the proof of Theorem 0.5.

4.4 Short-time existence and regularity of solutions

Now that we finished constructing a parametrix in the previous section, we turn our attention once again to the Yamabe flow. The ultimate goal of this chapter is to discuss whether or not there exists a Yamabe flow on Φ-manifolds with conformal factor in $C^{2,\alpha}_{x^4\Phi}(M\times[0,T])$, which is a stronger assumption than the one imposed in the previous chapter.

To do this, write once again the flow equation in terms of the conformal factor and linearize it near the time $t = 0$ (as in §3.3), obtaining

$$\begin{aligned}(\partial_t - \Delta_\Phi)v = &-\frac{1}{\eta}v\Delta_\Phi v + v^2 s(v)\Delta_\Phi v - \frac{\eta}{m-1}\mathrm{scal}(g_\Phi) + \frac{1}{m-1}\mathrm{scal}(g_\Phi)v \\ &+ \frac{1}{m-1}\mathrm{scal}(g_\Phi)v^2(1-\eta s(v) - \eta v s(v)),\end{aligned} \tag{4.37}$$

where $u = 1+v$ is the linearized expression of the conformal factor near $t = 0$. Simplifying the notation on the right hand by taking

$$\begin{aligned}F_1'(v) &:= -\frac{\eta}{m-1}\mathrm{scal}(g_\Phi) + \frac{1}{m-1}\mathrm{scal}(g_\Phi)v \\ &\quad + \frac{1}{m-1}\mathrm{scal}(g_\Phi)v^2(1-\eta s(v) - \eta v s(v)), \\ F_2'(v) &:= -\frac{1}{\eta}v\Delta_\Phi v + v^2 s(v)\Delta_\Phi v.\end{aligned}$$

Hence, we once again must look for a function that satisfies

$$(\partial_t - \Delta_\Phi)v = (F_1' + F_2')v, \quad v|_{t=0} = 0. \tag{4.38}$$

First, note that the constant function 1 lies in any $C^{k,\beta}_{x^4\Phi}(M\times[0,T])$. Moreover, Proposition 4.10 guarantees that the conditions imposed on the parametrix by Theorem 3.14 are met. Furthermore, the proof of Theorem 3.14 does not depend on the definition of the Hölder brackets. Consequently, Theorem 3.14 holds for functions on the modified Hölder spaces $x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T])$ as well. Thus, we must only check that each F_1' and F_2' satisfy the conditions required. However, the proofs of Lemma 3.15 and Lemma 3.16 are also independent on the definition of the Hölder brackets, which means that

1. Lemma 3.15 holds for F_2' with the exact same formulation, simply exchanging $x^\gamma C^{k,\alpha}_{\Phi}(M\times[0,T])$ by $x^\gamma C^{k,\alpha}_{x^4\Phi}(M\times[0,T])$;

2. Lemma 3.16 holds for F_1' analogously as in the changes described in (i), with the only difference that we must require $\mathrm{scal}(g_\Phi)\in x^\gamma C^{k-1,\alpha}_{x^4\Phi}(M)$.

Therefore, we just proved the following theorem.

Theorem 4.13. *Let (M, g_Φ) be a Φ-manifold of dimension $m \geq 3$. Assume $\mathrm{scal}(g_\Phi)\in x^\gamma C^{k+1,\alpha}_{x^4\Phi}(M)$ for some $\alpha\in(0,1)$, some $\gamma\geq 0$ and any $k\in\mathbb{N}_0$. Then the Yamabe flow (2) admits a unique solution $g = u^{4/(m-2)}g_\Phi$, where $u\in C^{k+2,\alpha}_{x^4\Phi}(M\times[0,T])$, for some time $T>0$ sufficiently small.*

Naturally, this proves Theorem 0.6

4.4.1 The problem of the global Yamabe flow in $C^{\alpha}_{x^4\Phi}(M \times [0,+\infty))$

The results we obtained for the Yamabe flow for conformal factor in $C^{\alpha}_{x^4\Phi}(M \times [0,T])$ as similar to the ones we got when the conformal factor was a function in $C^{\alpha}_{\Phi}(M \times [0,T])$. However, from this point on, this is no longer true. In fact, although we managed to show that the Yamabe flow does exist and is unique for some short-time T when $u \in C^{\alpha}_{x^4\Phi}(M \times [0,T])$, we could not extend the short-time solution to a global solution of the flow which remains continuous up to the boundary.

First, recall that $\eta = (m-2)/4$. Let $u \in C^{2,\alpha}_{x^4\Phi}(M \times [0,T])$ be the conformal factor such that $g = u^{1/\eta} g_\Phi$ is the solution of the Yamabe flow. To prove long-time existence of the flow, we could proceed as in §3.9 and linearize u' at time T by setting $u' = u_0 + v$, with $u_0 = u|_{t=T}$, with initial condition $u'|_{t=0} = u_0$. We obtain from the second equation in (3.2)

$$\left(\partial_t - (m-1)u_0^{-1/\eta}\Delta\right) v = F_1'(v) + F_2'(v), \;\; v|_{t=0} = 0, \tag{4.39}$$

where we have abbreviated

$$F_1'(v) = Q_2(v), \quad F_2'(v) = (m-1)u_0^{-1/\eta}\Delta u_0 - \mathrm{scal}(g_\Phi)u_0^{1-1/\eta} + Q_1(v),$$

where $Q_1(v)$ are linear combinations of v with coefficients given in terms of u_0 and Δu_0 and the terms in $Q_2(v)$ include quadratic combinations of v and Δv with coefficients in terms of u_0 and Δu_0.

For us to be able to use the parametrix construction in §4.3, the function $u_0 = u|_{t=T} \in C^{2,\alpha}_{x^4\Phi}(M)$ must lie in $C^{2,\beta}_{x^4\Phi}(M)$ for some $\alpha < \beta < 1$, which is generally not true! Moreover, as pointed out previously in Remark 4.7, this condition cannot be improved in the construction provided in §4.3.

APPENDIX A

Curvature on Φ-manifolds

This appendix presents explicit computations of the Riemann curvature tensor on an open manifold M endowed with an exact Φ-metric. The general case behaves similarly in terms of its lower order terms, preserving the overall behavior we are interested in. Before discussing how to proceed, let us present the concept of warped product of Riemannian manifolds.

Definition A.1. Consider two Riemannian manifolds (N_1, g_{N_1}) and (N_2, g_{N_2}) and let $\psi \in C^\infty(N_2)$ be a positive function. On the product manifold $N_1 \times N_2$, let π_* denote the standard projection map from $N_1 \times N_2$ to each individual manifold. The warped product $N_1 \; {}_\psi\!\times N_2$ is the product manifold $N_1 \times N_2$ furnished with the Riemannian metric

$$g_\psi := (\psi \circ \pi_{N_2})^2 \pi_{N_1}^* g_{N_1} + \pi_{N_2}^* g_{N_2}. \tag{A.1}$$

N_1 is called the fiber, N_2 is called the base and the function ψ is called the warping function. Often, the metric g_ψ is written omitting the projections π_*, as we do in this present work.

Naturally, product metrics are example of warped metrics, with $\psi = 1$.

From the previous definition, it is clear that the exact Φ-metric $g_{\Phi,0}$ on M, which is given near the boundary by the expression

$$g_{\Phi,0} := \frac{\mathrm{d}x^2}{x^4} + \frac{\phi^* g_Y}{x^2} + g_Z,$$

is a product metric between a warped metric on $(0,1) \times Y$ and a metric on Z. Now, the curvature tensor R on a generic manifold is defined by the expression

$$\mathrm{R}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{A.2}$$

for X, Y, Z vector fields. In O'Neill's book on Semi-Riemannian geometry [O'N83], we have the following

Proposition A.2. *[O'N83, pg. 210] Let $N_1 \,{}_\psi\!\times N_2$ be a warped product manifold. If $U,V,W \in \mathscr{V}(N_1)$ and $X,Y,W \in \mathscr{V}(N_2)$ are vector fields, then*

1. $\mathrm{R}_\psi(\widetilde{X},\widetilde{Y})\widetilde{Z} = \mathrm{R}_{N_2}\widetilde{(X,Y)Z}$;
2. $\mathrm{R}_\psi(\widetilde{V},\widetilde{X})\widetilde{Y} = -\dfrac{\operatorname{Hess}\psi(X,Y)}{\psi}\widetilde{V}$;
3. $\mathrm{R}_\psi(\widetilde{X},\widetilde{Y})\widetilde{V} = \mathrm{R}_\psi(\widetilde{V},\widetilde{W})\widetilde{X} = 0$;
4. $\mathrm{R}_\psi(\widetilde{X},\widetilde{V})\widetilde{W} = -\dfrac{g_\psi(\widetilde{V},\widetilde{W})}{\psi}\widetilde{\nabla_X \operatorname{grad}}\,\psi$;
5. $\mathrm{R}_\psi(\widetilde{V},\widetilde{W})\widetilde{U} = \mathrm{R}_{N_1}\widetilde{(V,W)}U - \dfrac{g_{N_2}(\operatorname{grad}\psi,\operatorname{grad}\psi)}{\psi^2}\left(g_\psi(\widetilde{W},\widetilde{U})\widetilde{V} - g_\psi(\widetilde{V},\widetilde{U})\widetilde{W}\right)$,

where $\widetilde{\cdot}$ represents the lift of a vector field to $N_1 \,{}_\psi\!\times N_2$.

Since $g_{\Phi,0}$ is a product metric between a warped metric on $(0,1)\times Y$ and a metric in Z, and since $\operatorname{Hess}1 = \operatorname{grad}1 = 0$, it follows from the previous proposition that the curvature of Z contributes only with bounded terms to the curvature of M (since Z is closed). Thus, to understand the curvature of M, it suffices to study the curvature of $(0,1)\times_{x^{-1}} Y$. Note that this coincides with the case $\partial\overline{M} = Y$, which means that ϕ can be omitted.

Consider M an open manifold endowed with a metric

$$g_{\Phi,0} = \frac{\mathrm{d}x^2}{x^4} + \frac{g_Y}{x^2}, \tag{A.3}$$

where Y is a b-dimensional closed manifold. Considering a local frame $\{\partial_x, \partial_{y_{i-1}} \mid i = 1,...,b\}$, we obtain (locally) a matrix representation

$$(g_{\Phi,0}) =: (g_{ij}) = \begin{pmatrix} x^{-4} & \\ & x^{-2}((g_Y)_{ij}) \end{pmatrix} \Longrightarrow (g_{\Phi,0})^{-1} =: (g^{ij}) = \begin{pmatrix} x^4 & \\ & x^2((g_Y)^{ij}) \end{pmatrix}.$$

A.1 Christoffel symbols

It is well known that, locally, the Christoffel symbols are given by

$$\Gamma^m_{ij} = \frac{1}{2}\sum_k \left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right) g^{km}. \tag{A.4}$$

Naturally, this means $\Gamma^m_{ij} = \Gamma^m_{ji}$ for all i,j.

- **Case $m = 1$:** In this case, $g^{k1} \neq 0$ iff $k = 1$ and, therefore,

$$(\Gamma_{\Phi,0})^1_{ij} = \frac{1}{2}\left(\partial_i g_{j1} + \partial_j g_{1i} - \partial_x g_{ij}\right) x^4.$$

Then,

$$(\Gamma_{\Phi,0})^1_{11} = \frac{1}{2}\partial_x x^{-4}\cdot x^4 = -2x^{-1},$$

$$(\Gamma_{\Phi,0})^1_{1j} = \frac{1}{2}(\partial_x g_{j1} + \partial_{y_{j-1}} g_{11} - \partial_x g_{1j})x^4 = 0, \text{ for all } j \geq 2,$$
$$(\Gamma_{\Phi,0})^1_{ij} = \frac{1}{2}(\partial_{y_{i-1}} g_{j1} + \partial_{y_{j-1}} g_{1i} - \partial_x g_{ij}) = -\frac{1}{2}\partial_x x^{-2}(g_Y)_{i-1\,j-1} \cdot x^4 = x(g_Y)_{i-1\,j-1}, \text{ for all } i,j \geq 2.$$

• **Case** $m \geq 2$**:** In this case, $g^{km} \neq 0$ iff $k \geq 2$ and, therefore,

$$(\Gamma_{\Phi,0})^m_{ij} = \frac{1}{2}\sum_{k\geq 2}\left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_{y_{k-1}} g_{ij}\right) x^2 (g_Y)^{k-1\,m-1}.$$

Therefore,

$$(\Gamma_{\Phi,0})^m_{11} = \frac{1}{2}\sum_{k\geq 2}(\partial_x g_{1k} + \partial_x g_{k1} - \partial_{y_{k-1}} x^{-4})x^2 g_Y^{k-1\,m-1} = 0,$$
$$(\Gamma_{\Phi,0})^m_{1j} = \frac{1}{2}\sum_{k\geq 2}(\partial_x x^{-2}(g_Y)_{j-1\,k-1} + \partial_{y_{j-1}} g_{k1} - \partial_{y_{k-1}} g_{1j})x^2 g_Y^{k-1\,m-1} = -x^{-1}\delta^m_j,$$
$$(\Gamma_{\Phi,0})^m_{ij} = \frac{1}{2}\sum_{k\geq 2} x^{-2}(\partial_{y_{i-1}}(g_Y)_{j-1\,k-1} + \partial_{y_{j-1}}(g_Y)_{k-1\,i-1} - \partial_{y_{k-1}}(g_Y)_{i-1\,j-1})x^2 g_Y^{k-1\,m-1} = (\Gamma_Y)^{m-1}_{i-1\,j-1},$$

for all $i,j \geq 2$.

A.2 Riemann curvature

The Riemann curvature can be expressed, locally, by the expression:

$$\mathrm{R}^s_{ijk} = \sum_l \Gamma^l_{jk}\Gamma^s_{il} - \sum_l \Gamma^l_{ik}\Gamma^s_{jl} + \partial_i\Gamma^s_{jk} - \partial_j\Gamma^s_{ik}. \tag{A.5}$$

• **Case** $s = 1$**:** If we assume $s = 1$, then

$$(\mathrm{R}_{\Phi,0})^1_{ijk} = \sum_l \Gamma^l_{jk}\Gamma^1_{il} - \sum_l \Gamma^l_{ik}\Gamma^1_{jl} + \partial_i\Gamma^1_{jk} - \partial_j\Gamma^1_{ik}.$$

This means that if $k \neq i$ and $k \neq j$, then $(\mathrm{R}_{\Phi,0})^1_{ijk} = 0$. Thus, the interesting cases are either $k = i$ or $k = j$.

If $k = i$, then

$$(\mathrm{R}_{\Phi,0})^1_{iji} = \sum_l \Gamma^l_{ji}\Gamma^1_{il} - \sum_l \Gamma^l_{ii}\Gamma^1_{jl} + \partial_i\Gamma^1_{ji} - \partial_j\Gamma^1_{ii}.$$

Therefore, if $i = 1$,

$$(\mathrm{R}_{\Phi,0})^1_{111} = \Gamma^1_{11}\Gamma^1_{11} - \Gamma^1_{11}\Gamma^1_{11} + \partial_x\Gamma^1_{11} - \partial_x\Gamma^1_{11} = 0,$$
$$(\mathrm{R}_{\Phi,0})^1_{1j1} = \Gamma^1_{j1}\Gamma^1_{11} - \Gamma^1_{11}\Gamma^1_{j1} + \partial_x\Gamma^1_{j1} - \partial_{y_{j-1}}\Gamma^1_{11} = 0, \text{ for all } j \geq 2.$$

On the other hand, if $i \geq 2$,

$$(\mathrm{R}_{\Phi,0})^1_{iji} = \sum_{l\geq 2} \Gamma^l_{ji}\Gamma^1_{il} - \sum_l \Gamma^l_{ii}\Gamma^1_{jl} + \partial_{y_{i-1}}\Gamma^1_{ji} - \partial_j\Gamma^1_{ii}.$$

Thus,

$$
\begin{aligned}
(\mathrm{R}_{\Phi,0})^1_{i1i} &= \sum_{l\geq 2}(-x^{-1}\delta^l_i)x(g_Y)_{i-1\,l-1} - (-2x^{-1})x(g_Y)_{i-1\,i-1} - \partial_x x(g_Y)_{i-1\,i-1} = 0,\\
(\mathrm{R}_{\Phi,0})^1_{iji} &= x\left(\sum_{l\geq 2}(\Gamma_Y)^{l-1}_{j-1\,i-1}(g_Y)_{i-1\,l-1} - \sum_{l\geq 2}(\Gamma_Y)^{l-1}_{i-1\,i-1}(g_Y)_{j-1\,l-1}\right.\\
&\quad \left. +\partial_{y_{i-1}}(g_Y)_{j-1\,i-1} - \partial_{y_{j-1}}(g_Y)_{i-1\,i-1}\right),
\end{aligned}
$$

for all $j \geq 2$, which completes computations of the case $i = k$. On the other hand, if we assume $j = k$, similar computations give the following outcomes:

$$
\begin{aligned}
(\mathrm{R}_{\Phi,0})^1_{i11} &= \mathrm{R}^1_{1jj} = 0, \text{ for all } i,j \geq 2,\\
(\mathrm{R}_{\Phi,0})^1_{ijj} &= x\left(\sum_{l\geq 2}(\Gamma_Y)^{l-1}_{j-1\,j-1}(g_Y)_{i-1\,l-1} - \sum_{l\geq 2}(\Gamma_Y)^{l-1}_{i-1\,j-1}(g_Y)_{j-1\,l-1}\right.\\
&\quad +\partial_{y_{i-1}}(g_Y)_{j-1\,j-1} - \partial_{y_{j-1}}(g_Y)_{i-1\,j-1},
\end{aligned}
$$

for all $i \geq 2$.

• **Case** $s \geq 2$**:** Unlike the previous case, assuming $s \geq 2$ does not provide any further information on $(\mathrm{R}_{\Phi,0})^s_{ijk}$ *a priori*. Then, to obtain information on these terms, we need to split computations into several cases. We obtain the following:

$$
\begin{aligned}
(\mathrm{R}_{\Phi,0})^s_{11k} &= \sum_l \Gamma^l_{1k}\Gamma^s_{1l} - \sum_l \Gamma^l_{1k}\Gamma^s_{1l} + \partial_x\Gamma^s_{1k} - \partial_x\Gamma^s_{1k} = 0, \text{ for all } k,\\
(\mathrm{R}_{\Phi,0})^s_{1j1} &= \sum_{l\geq 2}(-x^{-1})\delta^l_j(-x^{-1})\delta^s_l - (-2x^{-1})(-x^{-1})\delta^s_j + \partial_x(-x^{-1}\delta^s_j) = 0, \text{ for all} j \geq 2,\\
(\mathrm{R}_{\Phi,0})^s_{1jk} &= \sum_{l\geq 2}(\Gamma_Y)^{l-1}_{j-1\,k-1}(-x^{-1}\delta^s_l) - \sum_{l\geq 2}(-x^{-1}\delta^l_k)(\Gamma_Y)^{s-1}_{j-1\,l-1} = 0, \text{ for all} j,k \geq 2,\\
(\mathrm{R}_{\Phi,0})^s_{i11} &= (-2x^{-1})(-x^{-1}\delta^s_i) - \sum_{l\geq 2}(-x^{-1}\delta^l_i)(-x^{-1}\delta^s_l) - \partial_x(-x^{-1}\delta^s_i) = 0, \text{ for all} i \geq 2,\\
(\mathrm{R}_{\Phi,0})^s_{i1k} &= \sum_{l\geq 2}(-x^{-1}\delta^l_k)(\Gamma_Y)^{s-1}_{i-1\,l-1} - \sum_{l\geq 2}(\Gamma_Y)^{s-1}_{i-1\,k-1}(-x^{-1}\delta^s_l) = 0, \text{ for all} i,k \geq 2,\\
(\mathrm{R}_{\Phi,0})^s_{ij1} &= \sum_{l\geq 2}(-x^{-1}\delta^l_j)(\Gamma_Y)^{s-1}_{i-1\,l-1} - \sum_{l\geq 2}(-x^{-1}\delta^l_i)(\Gamma_Y)^{s-1}_{j-1\,l-1} = 0, \text{ for all} i,j \geq 2.
\end{aligned}
$$

Finally, for $i,j,k \geq 2$, we get

$$
\begin{aligned}
(\mathrm{R}_{\Phi,0})^s_{ijk} &= x(g_Y)_{j-1\,k-1}(-x^{-1}\delta^s_i) + \sum_{l\geq 2}(\Gamma_Y)^{l-1}_{j-1\,k-1}(\Gamma_Y)^{s-1}_{i-1\,l-1} - x(g_Y)_{i-1\,k-1}(-x^{-1}\delta^s_j)\\
&\quad - \sum_{l\geq 2}(\Gamma_Y)^{l-1}_{i-1\,k-1}(\Gamma_Y)^{s-1}_{j-1\,l-1} + \partial_{y_{i-1}}(\Gamma_Y)^{s-1}_{j-1\,k-1} - \partial_{y_{j-1}}(\Gamma_Y)^{s-1}_{i-1\,k-1}\\
&= (\mathrm{R}_Y)^{s-1}_{i-1\,j-1\,k-1} + \delta^s_j(g_Y)_{i-1\,k-1} - \delta^s_i(g_Y)_{j-1\,k-1}.
\end{aligned}
$$

Since Y is a closed manifold, it follows that R^s_{ijk} is bounded.

A.3 Curvature tensor

From definition, the curvature tensor of $(M, g_{\Phi,0})$ is given by

$$\mathrm{R}_{\Phi,0}(X,Y,Z,W) := g_{\Phi,0}(\mathrm{R}(X,Y)Z,W),$$

for any given vector fields $X,Y,Z,W \in \mathscr{V}(M)$. Thus, considering the local frame $\{\partial_x, \partial_{y_{i-1}} \mid i = 1,\ldots,b\}$, one can write

$$\begin{aligned}\mathrm{R}_{\Phi,0}(\partial_i,\partial_j,\partial_k,\partial_s) = g_{\Phi,0}(\mathrm{R}_{\Phi,0}(\partial_i,\partial_j)\partial_k,\partial_s) &= g_{\Phi,0}\left(\sum_l (\mathrm{R}_{\Phi,0})^l_{ijk}\partial_l, \partial_s\right)\\ &= \sum_l (\mathrm{R}_{\Phi,0})^l_{ijk} g_{\Phi,0}(\partial_l,\partial_s) = \sum_l (\mathrm{R}_{\Phi,0})^l_{ijk} g_{ls} =: (\mathrm{R}_{\Phi,0})_{ijks}.\end{aligned}$$

Hence, determining each $(\mathrm{R}_{\Phi,0})_{ijks}$ is enough to fully determine $\mathrm{R}_{\Phi,0}$. However, we have the following identities:

$$\mathrm{R}_{jiks} = -\mathrm{R}_{ijks}, \;\; \mathrm{R}_{ijsk} = -\mathrm{R}_{ijks} \;\text{ and }\; \mathrm{R}_{ksij} = \mathrm{R}_{ijks}\,. \tag{A.6}$$

This means that we need to computate only a few cases. In fact, if we assume $i = 1$, we obtain

$$(\mathrm{R}_{\Phi,0})_{1jks} = (\mathrm{R}_{\Phi,0})^1_{1jk} g_{1s} + \sum_{m\geq 2} (\mathrm{R}_{\Phi,0})^m_{1jk} g_{ms}$$

As proven above, $(\mathrm{R}_{\Phi,0})^1_{ijk} = 0$ if $k \neq i$ and $k \neq j$. Moreover, $(\mathrm{R}_{\Phi,0})^m_{ijk} = 0$ if $m \geq 2$ and either $i = 1$ or $j = 1$ of $k = 1$. If $k = 1$, then $(\mathrm{R}_{\Phi,0})^1_{1j1} = (\mathrm{R}_{\Phi,0})^m_{1j1} = 0$ for all j and $m \geq 2$. On the other hand, if $k = j$ implies $(\mathrm{R}_{\Phi,0})^1_{1jj} = 0$. Hence, $(\mathrm{R}_{\Phi,0})_{1jks} = 0$. Thus, from (A.6) it follows that

$$\begin{aligned}&(\mathrm{R}_{\Phi,0})_{i1ks} = (\mathrm{R}_{\Phi,0})_{ij1s} = (\mathrm{R}_{\Phi,0})_{ijk1} = 0\\ \Longrightarrow\; &(\mathrm{R}_{\Phi,0})_{ijks} = 0, \;\text{ whenever }\; i = 1 \;\text{ or }\; j = 1 \;\text{ or }\; k = 1 \;\text{ or }\; s = 1.\end{aligned}$$

From this, we conclude that the only nontrivial terms $(\mathrm{R}_{\Phi,0})_{ijks}$ are the ones with all its indices greater or equal than 2, in which case

$$\begin{aligned}(\mathrm{R}_{\Phi,0})_{ijks} &= \sum_{m\geq 2} \Big((\mathrm{R}_Y)^{m-1}_{i-1\,j-1\,k-1} + \delta^m_j (g_Y)_{i-1\,k-1} - \delta^m_i (g_Y)_{j-1\,k-1} \Big) x^{-2} (g_Y)_{m-1\,s-1}\\ &= x^{-2}\left((\mathrm{R}_Y)_{i-1\,j-1\,k-1\,s-1} + (g_Y)_{i-1\,k-1}(g_Y)_{j-1\,s-1} - (g_Y)_{j-1\,k-1}(g_Y)_{i-1\,s-1} \right)\\ &= x^{-2}(F_1)_{ijks},\end{aligned}$$

where each $(F_1)_{ijks}$ is a smooth function on Y. Similarly to the argument for $(\mathrm{R}_{\Phi,0})^s_{ijks}$, it follows that each $(F_1)_{ijks}$ is also a bounded function.

A.4 Norm of the curvature tensor

From definition, the norm of the curvature tensor R is locally given by

$$\|\,\mathrm{R}_{\Phi,0}\,\|_{g_{\Phi,0}} := \sqrt{\sum_{i,j,k,s} (\mathrm{R}_{\Phi,0})_{ijks} (\mathrm{R}_{\Phi,0})^{ijks}}, \;\text{ with }\; (\mathrm{R}_{\Phi,0})^{sijk} = \sum_{a,b,c} g^{ia} g^{jb} g^{kc} (\mathrm{R}_{\Phi,0})^s_{ijk}. \tag{A.7}$$

From §A.3, we know that $(\mathrm{R}_{\Phi,0})_{ijks} \neq 0$ iff all its indices are strictly greater than 1, in which case g^{ia}, g^{jb} and g^{kc} are nonvanishing iff $a,b,c \geq 2$ as well. Therefore, if $i,j,k,s \geq 2$,

$$(\mathrm{R}_{\Phi,0})^{sijk} = x^6 \sum_{a,b,c\geq 2} (g_Y)^{i-1\, a-1}(g_Y)^{j-1\, b-1}(g_Y)^{k-1\, c-1}(\mathrm{R}_{\Phi,0})^{s}_{ijk} = x^6 (F_2)_{ijks},$$

where each $(F_2)_{ijks}$ is a smooth function on Y. Once again, the fact that Y is a closed manifold implies that each $(F_2)_{ijks}$ is bounded. Therefore,

$$\|\,\mathrm{R}_{\Phi,0}\,\|_{g_{\Phi,0}} = \sqrt{\sum_{i,j,k,s} (\mathrm{R}_{\Phi,0})_{ijks}(\mathrm{R}_{\Phi,0})^{ijks}} = \sqrt{\sum_{i,j,k,s\geq 2} (\mathrm{R}_{\Phi,0})_{ijks}(\mathrm{R}_{\Phi,0})^{ijks}} = O(x^2).$$

Furthermore, one can obtain similar estimates to the derivatives of the curvature tensor. In fact, the derivative of the curvature tensor is given by

$$\begin{aligned}\nabla \mathrm{R}_{\Phi,0}(X,Y,Z,W,U) &:= \nabla_U \mathrm{R}_{\Phi,0}(X,Y,Z,W) = \nabla_U g_{\Phi,0}(\mathrm{R}_{\Phi,0}(X,Y)Z,W) \\ &= g_{\Phi,0}(\nabla_U \mathrm{R}_{\Phi,0}(X,Y)Z,W) + g_{\Phi,0}(\mathrm{R}_{\Phi,0}(X,Y)Z,\nabla_U W).\end{aligned} \tag{A.8}$$

One can prove, via computations on local frames, that each derivative of the curvature tensor worsens the singularity by two powers of x, that is, its lowest order term is of the order of x^{-4}. However, when "raising the indices" as before, one must introduce yet another term from $(g_{\Phi,0})^{-1}$, whose lower order term is of the order of x^2. Thus, the increase in the singular term is compensated by the extra term from $(g_{\Phi,0})^{-1}$. Then,

$$\|\nabla^i \mathrm{R}_{\Phi,0}\,\|_{g_{\Phi,0}} = O(x^2), \quad \text{for all } \ i \geq 1.$$

A.5 Sectional curvature

Another important information (that proves itself useful in Chapter B) is the sectional curvature of $(M, g_{\Phi,0})$. From definition,

$$K_{\Phi,0}(X\wedge Y) := \frac{\mathrm{R}_{\Phi,0}(X,Y,X,Y)}{g_{\Phi,0}(X,X)g_{\Phi,0}(Y,Y) - g_{\Phi,0}(X,Y)^2}, \quad \text{with } \ X\wedge Y = \mathrm{span}\{X,Y\}, \tag{A.9}$$

where $X,Y \in \mathscr{V}(M)$. It follows from the definition that it is enough to determine the sectional curvature on a local frame $\{\partial_x, \partial_{y_{i-1}} \mid i = 1,\ldots,b\}$, since any plane on the tangent space can be spanned by vector fields in such local frame.

For $X = \partial_x$ and $Y = \partial_{y_{i-1}}$,

$$K_{\Phi,0}(\partial_x \wedge \partial_{y_{i-1}}) = \frac{(\mathrm{R}_{\Phi,0})_{1i1i}}{g_{11}g_{ii} - g_{1i}^2} = 0.$$

On the other hand, for $X = \partial_{y_{i-1}}$ and $Y = \partial_{y_{j-1}}$, we have

$$K_{\Phi,0}(\partial_{y_{i-1}} \wedge \partial_{y_{j-1}}) = \frac{(\mathrm{R}_{\Phi,0})_{ijij}}{g_{ii}g_{jj} - g_{ij}^2}$$

$$
\begin{aligned}
&= x^4 \frac{g_{\Phi,0}\left(\sum_{l\geq 2}(\mathrm{R}_Y)^{l-1}_{i-1\ j-1\ i-1}\partial_{y_{l-1}}, \partial_{y_{j-1}}\right)}{(g_Y)_{i-1\,i-1}(g_Y)_{j-1\ j-1} - (g_Y)^2_{i-1\ j-1}} \\
&+ x^4 \frac{g_{\Phi,0}\left(\sum_{l\geq 2}(\delta^l_j (g_Y)_{i-1\ j-1} - \delta^l_i (g_Y)_{j-1\ i-1})\partial_{y_{l-1}}, \partial_{y_{j-1}}\right)}{(g_Y)_{i-1\,i-1}(g_Y)_{j-1\ j-1} - (g_Y)^2_{i-1\ j-1}} \\
&= x^2(K_Y(\partial_{y_{i-1}} \wedge \partial_{y_{j-1}}) + 1).
\end{aligned}
$$

Therefore, this means the sectional curvature K_M is bounded on M.

A.6 Ricci curvature

One of the main ingredients of this work is the study of the scalar curvature and, to do this, it is necessary to obtain information on the scalar curvature of a Φ-manifold. However, to obtain such information, one needs to determine the Ricci curvature first.

From definition, the Ricci curvature is localy given by

$$
(\mathrm{Ric}_{\Phi,0})_{ik} := \sum_l (\mathrm{R}_{\Phi,0})^l_{ilk} \tag{A.10}
$$

Directly from the computations in §A.2, it follows that

$$
(\mathrm{Ric}_{\Phi,0})_{ik} = 0, \text{ if either } i = 1 \text{ or } k = 1.
$$

On the other hand, for $i, k \geq 2$, we get

$$
\begin{aligned}
(\mathrm{Ric}_{\Phi,0})_{ik} = (\mathrm{R}_{\Phi,0})^1_{i1k} + \sum_{l\geq 2}(\mathrm{R}_{\Phi,0})^l_{ilk} &= \sum_{l\geq 2}\Big((\mathrm{R}_Y)^{l-1}_{i-1\ l-1\ k-1} + \delta^l_l (g_Y)_{i-1\ k-1} - \delta^l_i (g_Y)_{l-1\ i-1}\Big) \\
&= (\mathrm{Ric}_Y)_{i-1\ k-1} + \sum_{l\geq 2}\Big((g_Y)_{i-1\ k-1} - \delta^l_i (g_Y)_{l-1\ i-1}\Big) \\
&= (\mathrm{Ric}_Y)_{i-1\ k-1} + (b-1)(g_Y)_{i-1\ k-1}.
\end{aligned}
$$

Similarly to the argument employed for the sectional curvature, one can conclude from this that the Ricci curvature is bounded as well, since Y is a closed manifold.

APPENDIX B

Injectivity radius on Φ-manifolds

This appendix discusses the injectivity radius of a Φ-manifold, which is a relevant information to guarantee that such class of spaces has bounded geometry. First of all, let us present the definition of injectivity radius.

Definition B.1. Let (M,g) be a Riemannian manifold. Given a point $p \in M$ and denoting $B(p,R)$ as an open ball in M centered at p and with radius R, the injectivity radius at p is defined as

$$r_{\text{inj}}(p) := \sup\{\, R > 0 \mid \exp_p : B(p,R) \to M \text{ is a diffeomorphism } \}. \tag{B.1}$$

Then, the injectivity radius of M is given by

$$r_{\text{inj}}(M) := \inf\{\, r_{\text{inj}}(p) \mid p \in M \,\}. \tag{B.2}$$

Therefore, the injectivity radius of a manifold is the biggest R that defines an open covering of M of normal coordinated charts of uniform radii. Naturally, the first problem to overcome in order to estimate $r_{\text{inj}}(M)$ is to understand when is the exponential map even well-defined at a generic point. Since Φ-manifolds are a class of complete Riemannian manifolds, this can be easily answered by the Hopf-Rinow theorem, see [GHL90, Theorem 2.103, pg. 94], which states that a Riemannian manifold is complete if, and only, if, the exponential map at each point is defined on the entire tangent space at the same point, that is,

$$(M,g) \text{ is complete} \iff \exp_p : T_pM \to M, \text{ for all } p \in M. \tag{B.3}$$

Hence, the exponential map at each point is always well-defined for all $t \geq 0$, which means where are now left with the task of estimating the injectivity radius from below. For compact manifolds, it is a well known fact (proven by Klingenberg) that the injectivity radius is bounded from below away from zero. Thus, it follows that $r_{\text{inj}}(M_n) \geq c(n) > 0$, with $M_n := \{x \leq 1/n\}$. This means we must look for estimates of $r_{\text{inj}}(p)$ for points near the boundary. On the other hand, from §A.5, it is known that Φ-manifolds have bounded sectional curvature. Thus, we are allowed to employ the following result due to Cheeger, Gromov and Taylor:

Theorem B.2. *[CGT82, Theorem 4.7] Let (M,g) be a m-dimensional complete, connected Riemannian manifold such that there exist constants λ,Λ satisfying $\lambda \le K_M \le \Lambda$, and let $p \in M$. Furthermore, let $r > 0$ and assume $r < \pi/(4\sqrt{\Lambda})$ if $\Lambda > 0$. Then the injectivity radius at p can be estimated from below as follows:*

$$r_{\text{inj}}(p) \ge r\frac{\operatorname{vol} B(p,r)}{\operatorname{vol} B(p,r) + V_\lambda^m(2r)}, \tag{B.4}$$

where $V_\lambda^m(\rho)$ denotes the volume of a ball of radius ρ in the m-dimensional model space M_λ^m with constant sectional curvature λ.

This means that, in order to provide an estimate to the injectivity radius, one must look for estimates for the volume of an open ball on a Φ-manifold. First, set $r := x^{-1}$, which implies

$$g_{\Phi,0} := \mathrm{d}r^2 + r^2\phi^* g_Y + g_Z.$$

This means that the distance function on Φ-manifolds to be, locally, equivalent to

$$d_\Phi(p,p') \approx |r - r'| + (r + r')d_Y(y,y') + d_Z(z,z'), \tag{B.5}$$

where $p = (r,y,z)$ and $p' = (r',y',z')$. Let $p, p_0 \in M$ and assume

$$|r - r_0| < R/3,\ (r + r_0)d_Y(y,y_0) < R/3 \ \text{ and }\ d_Z(z,z_0) < R/3 \Longrightarrow d_\Phi(p,p_0) < R.$$

By using Fubini's theorem and the above arguments, one can check that

$$\begin{aligned} \operatorname{vol} B(p_0,R) &\ge \frac{R^{m-1}}{3^b 6^f}\frac{\operatorname{vol} B_Y(y_0,1)\operatorname{vol} B_Z(z_0,1)}{b+1}\frac{1}{(r_0+R/6)^b}\left((r_0+R/3)^{b+1} - (r_0-R/3)^{b+1}\right) \\ &\sim R^{m-1}\frac{1}{(r_0+R/6)^b}\left((r_0+R/3)^{b+1} - (r_0-R/3)^{b+1}\right). \end{aligned}$$

Similar arguments can be used to estimate the volume of such ball from above, in which the expression differs only by scaling to some bounded factor, which means the overall behavior to be the same. This means that we can estimate the volume of a ball centered at p_0 on a Φ-manifold in terms of its distance between p_0 and the boundary. Moreover, note that

$$\lim_{r_0\to+\infty}\frac{1}{(r_0+R/6)^b}\left((r_0+R/3)^{b+1} - (r_0-R/3)^{b+1}\right) \ge c > 0. \tag{B.6}$$

Hence, it is possible to estimate such volume from below by a constant. On the other hand, a quick analysis to the estimate provided in Theorem B.2 reveals that the left-hand side term of the inequality (that is, $r_{\text{inj}}(p_0)$) does not depend on R and, therefore, because of the estimate obtained for $\operatorname{vol} B(p_0,R)$, it is possible to see that $r_{\text{inj}}(M) \ge c_0 > 0$, as we desired.

APPENDIX C

On the distance function of a Φ-manifold

This appendix presents further explanations on the local expression of the distance function d_Φ of a Φ-manifold. This appendix relies on the fact that Φ-manifolds have bounded geometry. Thus, to fully understand this appendix, see the subsection on manifolds of bounded geometry in Chapter 2.

First, lets us look at the distance function on the Riemannian manifold $((0,+\infty), g := x^{-4}\,\mathrm{d}x^2)$. For any two given points $x, x' \in (0,+\infty)$, we know that the path connecting x and x' is given by $\gamma(t) = x + t(x'-x)$. Thus,

$$\begin{aligned} d_g(x,x') = \text{lenght}(\gamma) &= \int_0^1 \|\gamma'(t)\|_g\,\mathrm{d}t = |x-x'| \int_0^1 \frac{1}{(x+t(x'-x))^2}\,\mathrm{d}t \\ &= \frac{1}{x} - \frac{1}{x'} = \frac{|x-x'|}{xx'}. \end{aligned}$$

Now, remember that Φ-manifolds have bounded geometry and, therefore, there is an open covering of M given by balls $B(p,\delta)$ centered at each $p \in M$ and of uniform radii δ such that the distance function d_Φ is uniformly equivalent to the Euclidean distance on normal coordinated charts. Hence,

$$d_\Phi(p,p') \approx \sqrt{\frac{|x-x'|^2}{(xx')^2} + \frac{\|y-y'\|^2}{(x+x')^2} + \|z-z'\|^2}. \tag{C.1}$$

Note that the expression in (C.1) is not the same as in (3.6). However, it is enough for us to prove that both expressions are equivalent locally, since we use such expressions only restricted to the open subsets given by the bounded geometry property of (M, g_Φ). Given $p_0 \in M$, for any two points $p, p' \in B(p_0,\delta)$, one has

$$p,p' \in B(p_0,\delta) \Longrightarrow \frac{|x-x'|}{xx'} \leq 2\delta \Longrightarrow |x-x'| \leq 2\delta xx'.$$

Without loss of generality, assume $x \geq x'$. Thus,

$$x - x' \leq 2\delta xx' \Longrightarrow x \leq (2\delta x + 1)x'.$$

Since both $x, x' \leq 1$ and δ can be taken sufficiently small, then

$$2\delta x + 1 \leq 2 \Longrightarrow x \leq 2x'.$$

Analogously, $x' \leq 2x$. Then, it follows that, in $B(p_0, \delta)$, we have $x \sim x'$, that is

$$x \sim x' \Longleftrightarrow \frac{x}{x'}, \frac{x'}{x} \leq C + \infty. \tag{C.2}$$

Claim 1: If $x \sim x'$, then $\frac{|x-x'|}{xx'} \leq K \frac{|x-x'|}{(x+x')^2}$, for some constant $K > 0$.

In fact, from (C.2) it follows that

$$\begin{aligned}
&(x+x')^2 = (x+x')(x+x') \leq (x+Cx)(x'+Cx') \leq K_0 xx' \\
&\Longrightarrow \frac{1}{xx'} \leq K_0^{-1} \frac{1}{(x+x')^2} \Longrightarrow \frac{|x-x'|}{xx'} \leq K_0^{-1} \frac{|x-x'|}{(x+x')^2}.
\end{aligned}$$

On the other hand, we have $x + x' \geq 2\sqrt{x}\sqrt{x'}$ and, therefore, $(x+x')^2 \geq 4xx'$, which implies

$$\frac{1}{(x+x')^2} \leq \frac{1}{4}\frac{1}{xx'} \Longrightarrow \frac{|x-x'|}{(x+x')^2} \leq \frac{1}{4}\frac{|x-x'|}{xx'}.$$

Therefore, it follows from this that

$$d_\Phi(p, p') \approx \sqrt{\left(\frac{|x-x'|}{(x+x')^2}\right)^2 + \left(\frac{\|y-y'\|}{(x+x')}\right)^2 + \|z-z'\|^2}$$

on each $B(p_0, \delta)$. Hence, this equivalence holds on each open ball given by the bounded geometry property of a Φ-manifold.

Remark C.1. It should be noted that the expression in (3.6) not a globally defined distance function on Φ-manifolds. In fact, the triangular inequality does not hold, in general, for this local expression. However, the local expression (3.6), is a distance function when restricted to each $B(p_0, \delta)$, which is where we are interested in working with such local expression. In fact, this is a consequence of a more general fact, which follows:

Claim 2: If $d' \geq 0$ is uniformly equivalent to d, and d is a distance function, then d' is a distance function as well.

First, note that $d' \sim d$ means that there is a constant $C > 0$ such that $C^{-1}d \leq d' \leq Cd$. Thus, it follows directly from this that $d'(p,q) = 0$ iff $p = q$ and, moreover, $d'(p,q) = d'(q,p)$, for any p, q. Furthermore,

$$\begin{aligned}
d'(p,q) &\leq Cd(p,q) \leq C(d(p,r) + d(r,q)) \leq CC^{-1}(d'(p,r) + d'(r,q)) \\
&= d'(p,r) + d'(r,q),
\end{aligned}$$

concluding the proof that d' is a distance function as well.

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