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Differential Cohomology on Maps of Pairs and Relative-Parallel Product

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“El Eterno te guardará de todo mal, él guardará tu vida.
Guardará tu salida y tu entrada, ahora y siempre”.
(Salmos 121: 7,8)

Resumo

Nesta tese estudamos três questões importantes. A primeira consiste em fornecer um arcabouço axiomático para a cohomologia diferencial no contexto dos mapas de pares de espaços topológicos, introduzindo o produto entre classes relativas e classes paralelas. A segunda consiste em mostrar alguns modelos concretos de cohomologia diferencial com produto relativo-paralelo, para que os axiomas enunciados anteriormente não fiquem vazios. Enfim, a terceira questão consiste na construção e na axiomatização de algumas versões do morfismo de Thom e do mapa de integração, em que o produto paralelo-relativo é essencial. Estas definições constituem a aplicação principal do produto introduzido, sendo ao mesmo tempo a motivação principal que nos levou a considerá-lo.

Palavras-chaves: Cohomologia diferencial, mapas de pares, produto relativo-paralelo, K-teoria diferencial, integração relativa e com suporte (verticalmente) compacto.

Abstract

In this thesis, we study three important topics. The first one provides an axiomatic framework to differential cohomology on maps of pairs of topological spaces, introducing the product between relative and parallel classes. The second one describes some concrete models of differential cohomology endowed with the relative-parallel product, showing that the axioms stated above are not empty. The third topic deals with the construction and the axiomatization of some meaningful versions of the Thom morphism and of the integration map, in which the parallel-relative product is essential. These definitions constitute the main application of this product, and they are at the same time the fundamental motivation that led us to consider it.

Keywords: Differential cohomology, maps of pairs, relative-parallel product, differential $K-$theory, relative and (vertically)-compactly-supported integration.

Contents

Introduction

We begin by recalling the idea of differential cohomology. Intuitively, differential cohomology theories are refinements of topological cohomology theories, that encode geometric as well as topological information.

A historically well-know example is provided by ordinary differential cohomology with integral coefficients. In degree 2, the singular cohomology group $H^2(X;\mathbb{Z})$ classifies complex line bundles on X up to isomorphism. If we endow a line bundle with a connection (compatible with a fixed Hermitian metric), we add a geometric piece of information, partially described by the corresponding curvature. We call $\hat{H}^2(X)$ the group of (Hermitian) line bundles with connection, up to isomorphism. Since every line bundle admits a connection, we have the forgetful surjective morphism $\hat{H}^2(X) \to H^2(X;\mathbb{Z})$, showing that $\hat{H}^2(X)$ is a refinement of $H^2(X;\mathbb{Z})$. This is a clear and motivating example of differential cohomology. Considering any degree (not only 2), there are various concrete realizations of ordinary differential cohomology, e.g., smooth Deligne cohomology (see (BRYLINSKI, 2007)). In this case the differential refinement is implemented by considering the set of transition functions of a line bundle or of an abelian gerbe, thought of as a Čech cocyle, and adding the local potentials that describe a connection. Differential ordinary cohomology of degree n can also be modeled by Cheeger-Simons differential characters (see (CHEEGER; SIMONS, 1985) and (BäR; BECKER, 2014)). Here a differential class is formed by a pair (χ, ω), where χ is an homomorphism defined on $(n-1)$-smooth cycles with values in $\mathbb{R}/\mathbb{Z}$ (in the case of line bundles, it is the holonomy of the corresponding connection), which satisfies a Stokes-type formula over boundaries; i.e., the value of χ on a boundary $\partial\sigma$ is equal to the integral of ω on σ modulo $\mathbb{Z}$. Here ω is the curvature.

Another important example of differential refinement of a cohomology theory is provided by differential $K-$theory. Here we quote two models. In (FREED; LOTT, 2010), Freed and Lott proposed a model, where each topological K-theory class is refined by adding a connection and a real odd-dimensional differential form. In (SIMONS; SULLIVAN, 2008), Simons and Sullivan proposed a model where differential K-theory classes are are represented by a structured bundle (by definition, it is a pair given by a complex vector bundle equipped with a suitable equivalence class of connections). The importance of differential $K-$theory is not only theoretical, but it is also due to its applications in physics and mathematical physics (see (FREED, 2002) among many other sources).

Perhaps the most important example, from a theoretical point of view, is the model of Hopkins and Singer (see (HOPKINS; SINGER, 2005)). This model refines any cohomology theory represented by a spectrum and, in (UPMEIER, 2012), it has been completed describing in detail the S^1-integration the multiplicative structure. Moreover, in (BUNKE; SCHICK, 2010), Bunke and Schick axiomatized differential cohomology theories, and in particular they showed that a differential extension of a fixed topological theory is essentially unique under rather mild hypotheses.

The models summarized up to now concern the *absolute* case. It is natural to wonder if in the *relative* version it is possible to recover some of those facts; indeed, it was studied by Ruffino and Rocha in (RUFFINO; BARRIGA, 2018). They proposed an axiomatic framework for the relative differential cohomology groups, generalizing the one developed for the absolute case. They show the existence of a family of long exact sequences that combine the differential and topological groups. Using an adaptation of the Hopkins-Singer model, they showed the existence of the relative differential refinement for any cohomology theory and its uniqueness; moreover, given any rationally-even cohomology theory, the relative theory is multiplicative in the sense that there exists a module structure on relative classes over the absolute ones, i.e, given a smooth map $\rho : A \to X$ we have the module structure $\hat{h}^\bullet(\rho) \otimes \hat{h}^\bullet(X) \to \hat{h}^\bullet(\rho)$, equivalently, we have the exterior product of the following form

$$\cdot : \hat{h}^\bullet(\rho) \otimes_{\mathbf{Z}} \hat{h}^\bullet(Y) \to \hat{h}^\bullet(\rho \times \mathrm{id}_Y : A \times Y \to X \times Y). \tag{1}$$

As an application of this product (1), in (RUFFINO, 2015a), it was also reviewed in (RUFFINO; BARRIGA, 2018), they consider compact integration that requires the Thom morphism as an intermediate step. Such morphism, with respect to a real vector bundle $\pi : E \to X$ of rank n, is of the form

$$\begin{aligned} T : \hat{h}^\bullet(X) &\to \hat{h}^{\bullet+n}_{\mathrm{cpt}}(E) \\ \hat{\alpha} &\to \hat{u} \cdot \pi^* \hat{\alpha}, \end{aligned}$$

where $\hat{u}$ is a differential Thom class of E (by definition, it is represented by a parallel class $\hat{u}_0 \in \hat{h}^\bullet_{\mathrm{par}}(E, E \setminus H)$, where $H \subset E$ is compact). Hence, given a class $\hat{\alpha} \in \hat{h}^\bullet(X)$, it is natural to define the product $\hat{u} \cdot \pi^* \hat{\alpha}$ as the class represented by $\hat{u}_0 \cdot \pi^* \hat{\alpha}$, this last product is the one defined on (1).

We point out that an important hypothesis in the problem above is that the base X is compact. It is natural to inquire if we may achieve the same result when X is not compact. In this case, the Thom class of a vector bundle is *vertically*-compactly supported and we can define two versions of the Thom morphism: the ordinary one and the compactly-supported one. In the latter case, the class $\hat{\alpha}$ is compactly-supported, therefore it is represented by a parallel class $\hat{\alpha}_0 \in \hat{h}^\bullet_{\mathrm{par}}(X, X \setminus K)$. Following the same ideas as above, it seems natural to define the product $\hat{u} \cdot \pi^* \hat{\alpha}$ as the class represented by $\hat{u}_0 \cdot \pi^* \hat{\alpha}_0$, but the latter is a product of two parallel classes. Since both terms are relative, such a product does not fit in the relative-absolute one, described in (1).

Actually, we encounter the same problem when dealing with *relative* integration, even in the compact setting. In fact, in this case we have to multiply a relative class by a representative of the Thom class, the latter being parallel. Hence, we need again to multiply two relative classes, but one of the two keeps on being parallel.

Tentative of solution. One possible and natural solution, that we could think, is to "refine" the topological well know product for two cofibrations (X, A) and (Y, B).

$$\cdot : h^\bullet(X, A) \otimes_{\mathbf{Z}} h^\bullet(Y, B) \to h^\bullet(X \times Y, (X \times B) \cup (A \times Y)), \tag{2}$$

i.e., we would have the relative-relative "differential" product for cofibrations

$$\cdot : \hat{h}^\bullet(X, A) \otimes_{\mathbf{Z}} \hat{h}^\bullet(Y, B) \to \hat{h}^\bullet(X \times Y, (X \times B) \cup (A \times Y)). \tag{3}$$

The solution would be as follow: we have that $\hat{u}_0 \in \hat{h}_{\mathrm{par}}(E, H')$ and $\pi^*\hat{\alpha}_0 \in \hat{h}_{\mathrm{par}}(E, K')$, where we set $H' := E \setminus H$ and $K' := E \setminus \pi^{-1}(K)$; therefore, using the product (3), we get the class $\hat{u}_0 \times \pi^*\hat{\alpha}_0 \in \hat{h}^{\bullet+n}_{\mathrm{par}}(E \times E, (E \times K') \cup (H' \times E))$. Then, applying the pull-back through the diagonal embedding $\Delta : (E, K' \cup H') \to (E \times E, (E \times K') \cup (H' \times E))$, we get $\hat{u}_0 \cdot \pi^*\hat{\alpha}_0 \in \hat{h}^{\bullet+n}_{\mathrm{par}}(E, K' \cup H')$, where $K' \cup H' = E \cap (\pi^{-1}(K) \cap H)$. The subset $H \subset E$ is vertically compact, while the subset $\pi^{-1}(K) \subset E$ is "horizontally compact"; thus, it easily follows that $\pi^{-1}(K) \cap H$ is compact. This means that $\hat{u}_0 \cdot \pi^*\hat{\alpha}_0$ represents a compactly-supported class on E, that we call $\hat{u} \cdot \pi^*\hat{\alpha}$ by definition, hence we get the Thom morphism $T(\hat{\alpha}) := \hat{u} \cdot \pi^*\hat{\alpha}$.

Unlike the first case, this tentative of solution could be meaningless as we point out now. Notice that, there are two important conditions that make the first case works well;

- the smoothness of the map $\rho \times \mathrm{id}_Y$ when ρ and Y are also smooth;
- given two differential classes, $\hat{\alpha} \in \hat{h}^\bullet(\rho)$ and $\hat{\beta} \in \hat{h}^\bullet(Y)$, with associated curvatures (that are closed forms), $R(\hat{\alpha}) := (\omega, \eta) \in \Omega^\bullet_{\mathrm{cl}}(\rho)$ and $R(\hat{\beta}) := \omega' \in \Omega^\bullet_{\mathrm{cl}}(Y)$, the associate curvature of the class $\hat{\alpha} \cdot \hat{\beta} \in \hat{h}^\bullet(\rho \times \mathrm{id}_Y)$ is the well define wedge product

$$(\omega, \eta) \wedge \omega' := (\omega \wedge \omega', \eta \wedge \omega') \in \Omega^\bullet_{\mathrm{cl}}(\rho \times \mathrm{id}_Y). \tag{4}$$

This two conditions, as we may notice, could not hold for the problem that we are facing. In fact, first, when (X, A) and (Y, B) are smooth manifold pairs, $(X \times B) \cup (A \times Y)$ is not a sub-manifold of $X \times Y$ in general, and second, given two relative differential classes $\hat{\alpha} \in \hat{h}^\bullet(\rho : A \to X)$ and $\hat{\beta} \in \hat{h}^\bullet(\gamma : B \to Y)$ with associated curvatures $R(\hat{\alpha}) := (\omega, \eta) \in \Omega^\bullet_{\mathrm{cl}}(\rho)$ and $R(\hat{\beta}) := (\omega', \eta') \in \Omega^\bullet_{\mathrm{cl}}(\gamma)$, the curvature associated of the class $\hat{\alpha} \cdot \hat{\beta}$ would be the "wedged product" $(\omega, \eta) \wedge (\omega', \eta')$. Here is the problem, there is no natural definition of the wedge product of two relative forms.

Nevertheless, this second situation can be modified, since in the problem that we are facing, we do not need to multiply two generic relative differential classes, instead we multiply a relative class $\hat{\alpha} \in \hat{h}^\bullet(\rho)$ by a *parallel* one $\hat{\beta} \in \hat{h}_{\text{par}}(Y, B)$ (by definition, the curvature of a parallel class is a closed form that looks like an absolute form, i.e., $R(\beta) := \omega' \in \Omega^\bullet_{\text{cl}}(Y, B)$), then, it could be possible to define this two forms similar to the one on (4).

Two resolve the first situation, we must look for a suitable candidate "?" on the codomain of the product below

$$\cdot : \hat{h}^\bullet(\rho) \otimes_{\mathbf{Z}} \hat{h}^\bullet_{\text{par}}(Y, B) \to \hat{h}^\bullet(?), \tag{5}$$

in such a way that when ρ is a smooth map and (Y, B) is a smooth manifold pair, our missing candidate "?" should be smooth as well; moreover, when we consider $B = \varnothing$, in (5), we should recover the particular product (1) and in the topological framework, when $\rho : A \hookrightarrow X$ is a cofibration, the product

$$\cdot : h^\bullet(\rho) \otimes_{\mathbf{Z}} h^\bullet(Y, B) \to h^\bullet(?),$$

should be equivalent to (2).

Then, motivated by the problem already explained, we look for this suitable candidate "?" that surely could not fit on the category of smooth manifold, smooth par of manifold or a smooth map. Moreover, as in the relative case, we propose an axiomatic framework for the differential cohomology groups of this new object, we establish a suitable set of axioms to be satisfied by (5), in order to define axiomatically a multiplicative cohomology theory in this extended sense. To complete the theory, we construct a natural product of the form (5) in some relevant models of differential cohomology, i.e. differential character for ordinary cohomology, the Freed-Lott model for K-theory and the Hopkins-Singer model for any rationally even cohomology theory.

The work is organized as follows:

- In chapter 1, we give a brief summary of the axioms of relative cohomology theory. The last section lays the topological foundations of the product between two relative cohomology classes, i.e., classes that belong to cohomology groups of any map that is not necessarily an embedding. As we will see, the intermediate case, i.e., the product of two classes when one of them belongs to a cohomology group of an embedding, will be useful for the construction of the relative-parallel product in chapter [2].

- Chapter 2 begins with a brief summary of the axioms of relative differential cohomology. In a similar way we give the axioms of cohomolgy theory on maps of pairs, i.e., on the category whose objects are morphisms between pairs. In the next section, we axiomatize the differential product. As we will see, it will be necessary to consider cohomology on a higher category. We will be dealing with the category whose objects are maps of $n-$tuples, which we shall call maps of finite sequences.

- In chapters 3–5, we present three classic models of differential extension cohomology on maps of pairs, and we construct the relative-parallel differential product in each model.
    - In chapter 3, the particular cohomology theory that we consider is the ordinary cohomology, i.e., the singular cohomology with integral coefficients, and the model that we choose for the construction of its differential extension is the one of differential characters.
    - In Chapter 4, the cohomology theory that we fix is the $K-$theory. Its differential extension is given through the Freed-Lott model. We explain in detail the relative version of this model, to which we paid particular attention in this thesis, both in the topological and in the differential framework.
    - In Chapter 5 we adapt the Hopkins-Singer model to the case of maps of pairs, i.e, we prove that any cohomology theory on maps of pairs has a differential extension. Moreover, we construct the relative-parallel product for rationally-even cohomology theory.
- In chapter 6, we describe three important parts. First, in the topological (differential) framework, we describe two versions of Thom isomorphisms (morphism) with the differential compactly-supported Thom morphism being the most important for us since its construction shows the necessity of using a more general product (namely, the differential relative-parallel product) of the usual one. In the second part, we define two versions of integration, namely, compactly-supported and vertically-compactly-supported integration. As we will see, this is a more general case of the one given on (RUFFINO; BARRIGA, 2018) since we do not ask that the compromised smooth manifolds be compact. Finally, in the last part, we axiomatize the integration.
- In appendix A, we briefly recall the definitions and properties of push-outs and homotopy push-outs that are useful for the construction of natural maps that will lead us to define the product for two relative cohomology classes. We also adapt the splittings of the Künneth sequence on the level of cycles on the context of maps of pairs, which is crucial in defining the relative-parallel product of differential character described in chapter 3.

1 Preliminaries

In this chapter, we briefly review the axioms of relative cohomology in the topological framework (see (RUFFINO; BARRIGA, 2018)), keeping in mind that the expression "relative cohomology" refers to the cohomology groups of any continuous function, not necessarily an embedding. In the last section, we construct the topological product in this context as a preliminary step in order to define the parallel-relative product in the next chapter.

We use the following notation:

$\mathcal{C}$: category of spaces with the homotopy type of a CW-complex or of a finite CW-complex (depending on the cohomology theory we are considering);

$\mathcal{C}_+$: category whose objects are the ones of $\mathcal{C}$ with a marked point, and whose morphisms are the continuous functions that respect the marked points;

$\mathcal{C}_2$: the category of morphism of $\mathcal{C}$;

taking the quotient of the morphisms of $\mathcal{C}$, $\mathcal{C}_+$ and $\mathcal{C}_2$ up to homotopy (relative to the marked point in $\mathcal{C}_+$), we get the categories $\mathcal{HC}$, $\mathcal{HC}_+$ and $\mathcal{HC}_2$ respectively.

$\mathcal{A}_{\mathbb{Z}}$: category of $\mathbb{Z}$-graded abelian groups;

$\mathcal{R}_{\mathbb{Z}}$: category of $\mathbb{Z}$-graded commutative rings.

We define the natural functors $\Pi : \mathcal{C}_2 \to \mathcal{C}$ and $\Pi : \mathcal{HC}_2 \to \mathcal{HC}$ in the following way: to each object $\rho : A \to X$ and morphism $(f, g) : \rho \to \eta$, we have $\Pi(\rho) = A$ and $\Pi(f, g) = g$.

1.1 Relative cohomology.

A *cohomology theory* on $\mathcal{C}_2$ is defined by a functor $h^\bullet : \mathcal{HC}_2^{\mathrm{op}} \to \mathcal{A}_{\mathbb{Z}}$ and a morphism of functors $\beta : h^\bullet \circ \Pi \to h^{\bullet+1}$, satisfying the axioms of long exact sequence, excision and multiplicativity on path-connected components (the latter being redundant in the case of finite CW-complexes), i.e.,

1) *Long exact sequence:* the functor $h^\bullet$ and the morphism of functors $\beta^\bullet$ define a functor from $\mathcal{HC}_2$ to the category of long exact sequences of abelian groups that assigns to an object $\rho : A \to X$ the sequence:

$$\cdots \to h^n(\rho) \xrightarrow{\pi^*} h^n(X) \xrightarrow{\rho^*} h^n(A) \xrightarrow{\beta} h^{n+1}(\rho) \to \cdots,$$

where π is the natural morphism from $\varnothing \to X$ to $\rho : A \to X$, and to morphism $(f, g) : \rho \to \eta$ the morphism of exact sequences: the corresponding morphism of exact sequences.

2) *Excision:* if $Z \subset A$ and $A \to X$ is a embedding such that the closure of $\rho(Z)$ is contained in the interior of $\rho(A)$, then the morphism

$$\begin{array}{ccc} A \setminus Z & \xrightarrow{\rho'} & X \setminus \rho(Z) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} \\ A & \xrightarrow{\quad\rho\quad} & X. \end{array}$$

induces an isomorphism between $h^\bullet(\rho)$ and $h^\bullet(\rho')$.

Remark 1. Such a definition of cohomology theory is equivalent to the usual one on pairs of spaces or on spaces with a marked point. In fact, starting from a reduced cohomology theory on $\mathcal{HC}_+$, the cohomology groups of a morphism $\rho : A \to X$ are defined as the reduced ones of the cone $C(\rho) := X \sqcup_A CA$, and the axioms are satisfied. Conversely, if we start from the axioms on the category $\mathcal{HC}_2$, then $h^\bullet(\rho)$ is naturally isomorphic to $\tilde{h}^\bullet(C(\rho))$, hence the theory is the unique possible extension to $\mathcal{HC}_2$ of a reduced cohomology theory on $\mathcal{HC}_+$.

1.1.1 Relative S^1−integration

We also recall the definition of the topological S^1-integration map on relative cohomology,

$$\int_{S^1} : h^{\bullet+1}(\mathrm{id}_{S^1} \times \rho) \to h^\bullet(\rho). \tag{1.1}$$

Let $\rho : A \to X$ be a morphism and fix a marked point on S^1. Consider the natural maps $i_1 : \rho \to \mathrm{id}_{S^1} \times \rho$ and $\pi_1 : \mathrm{id}_{S^1} \times \rho \to \rho$ with its induced maps $C(i_1) : C_\rho \to C_{\mathrm{id}_{S^1} \times \rho}$ and $C(\pi_1) : C_{\mathrm{id}_{S^1} \times \rho} \to C_\rho$. Since $C(\pi_1) \circ C(i_1) = \mathrm{id}_{C_\rho}$, we have the following split exact sequence:

$$0 \longrightarrow \tilde{h}^\bullet \left(C_{\mathrm{id}_{S^1} \times \rho} / C_\rho \right) \xrightarrow{C(\pi)^*} \tilde{h}^\bullet(C_{\mathrm{id}_{S^1} \times \rho}) \xrightarrow{C(i_1)^*} \tilde{h}^\bullet(C_\rho) \longrightarrow 0,$$

where $C(\pi) : C_{\mathrm{id}_{S^1} \times \rho} \to C_{\mathrm{id}_{S^1} \times \rho} / C_\rho$. Defining $h^\bullet(i_1) := h^\bullet(C(i_1)) \simeq \tilde{h}^\bullet \left(C_{\mathrm{id}_{S^1} \times \rho} / C_\rho \right)$, we get the split exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & h^\bullet(i_1) \arrow[r, "\pi^*"] & h^\bullet(\mathrm{id}_{S^1} \times \rho) \arrow[r, "i_1^*"] \arrow[l, bend left, "\xi"] & h^\bullet(\rho) \arrow[r] \arrow[l, bend left, "\pi_1^*"] & 0, \end{tikzcd} \tag{1.2}$$

were $\xi(\alpha) = (\pi^*)^{-1}(\alpha - \pi_1^* i_1^* \alpha)$. Using the natural isomorphism

$$s : h^\bullet(i_1) \simeq \tilde{h}^\bullet \left(C_{\mathrm{id}_{S^1} \times \rho} / C_\rho \right) \simeq \tilde{h}^\bullet(\Sigma(C_\rho)) \simeq \tilde{h}^{\bullet-1}(C_\rho) \simeq h^{\bullet-1}(\rho);$$

we get from (1.2)

$$h^\bullet(\mathrm{id}_{S^1} \times \rho) \simeq h^{\bullet-1}(\rho) \otimes h^\bullet(\rho). \tag{1.3}$$

The S^1-integration map is defined by the composition $\int_{S^1} = s \circ \xi$.

To finish this first part, lets us recall that given a morphism $\rho : A \to X$, the group $h^\bullet(\rho)$ has a natural external product over $h^\bullet(X)$ as follow:

$$h^\bullet(\rho) \otimes_{\mathbb{Z}} h^\bullet(X) \simeq \tilde{h}^\bullet(C_\rho) \otimes_{\mathbb{Z}} \tilde{h}^\bullet(X_+) \to \tilde{h}^\bullet(C_\rho \wedge X_+) \simeq \tilde{h}^\bullet(C_{\rho\times \mathrm{id}_X}) \simeq h^\bullet(\rho \times \mathrm{id}_X). \tag{1.4}$$

1.1.2 Product in relative cohomology.

In appendix (A.1), we briefly recall the definition of push-out and homotopy push-out and give some results that are useful to introduce the product for two classes in relative cohomology.

Given two maps $\rho : A \to X$ and $\eta : B \to Y$, denote their cartesian product by $\rho\times\eta : A\times B \to X\times Y$. By definition, a multiplicative cohomology theory on $\mathcal{C}_2$ is endowed with an internal product in $h^\bullet$, and we can easily define the exterior product

$$\begin{aligned} h^\bullet(\rho) \otimes h^\bullet(\eta) &\to h^\bullet(\rho \times \eta) \\ (\alpha, \beta) \quad &\to \pi_1^*\alpha \cdot \pi_2^*\beta, \end{aligned}$$

where $\pi_1 : \rho \times \eta \to \rho$ and $\pi_2 : \rho \times \eta \to \eta$ are the natural projections. When ρ and η are embeddings, we get the exterior product $h^\bullet(X, A) \times h^\bullet(Y, B) \to h^\bullet(X \times Y, A \times B)$. Actually, for cofibrations, we can also define the following more general product:

$$\cdot : h^\bullet(X, A) \times h^\bullet(Y, B) \to h^\bullet(X \times Y, (X \times B) \cup (A \times Y)) \tag{1.5}$$

as follows: since $X \times Y/((X \times B) \cup (A \times Y)) \simeq (X/A) \wedge (Y/B)$, we have

$$\begin{aligned} h^\bullet(X, A) \times h^\bullet(Y, B) &\simeq \tilde{h}^\bullet(X/A) \times \tilde{h}^\bullet(Y/B) \\ &\to \tilde{h}^\bullet(X/A \wedge Y/B) \\ &\simeq \tilde{h}^\bullet\left(\frac{X \times Y}{(X \times B) \cup (A \times Y)}\right) \\ &\simeq h^\bullet(X \times Y, (X \times B) \cup (A \times Y)). \end{aligned}$$

The product (1.5) can be generalized for any generic maps $\rho : A \to X$ and $\eta : B \to Y$. For this purpose, as an intermediate passage, the idea is to try to construct a natural map through homotopy push-out such that, its induced morphism in cohomology, will help to define such product.

In fact, consider the homotopy push-out of the maps $\rho \times \mathrm{id}_B : A \times B \to X \times B$ and $\mathrm{id}_A \times \eta : A \times B \to A \times Y$, as we can see in example (4) of the appendix, it is a quadruple

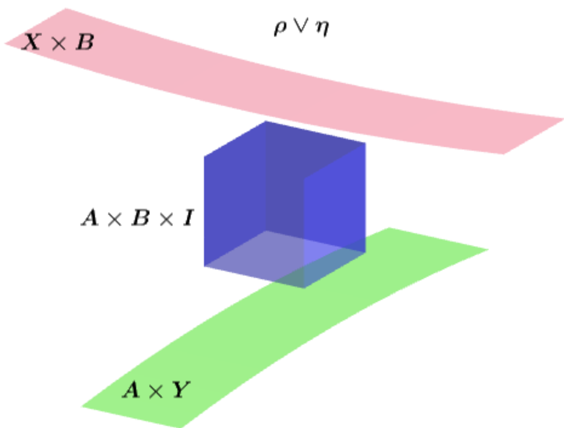
$(\rho \vee \eta, j_{(X\times B)}, j_{(A\times Y)}, F)$, where

$$\rho \vee \eta := (X \times B) \coprod_{A\times B} (A \times Y) = \frac{(X \times B) \sqcup (A \times B \times I) \sqcup (A \times Y)}{\sim}$$

is the space defined through the identifications (see the picture below):

$$(a, b, 1) \in A \times B \times I \sim (a, \eta(b)) \in A \times Y \text{ and } (a, b, 0) \in A \times B \times I \sim (\rho(a), b) \in X \times B.$$

The homotopy $F : j_{(A\times Y)} \circ (\mathrm{id}_A \times \eta) \simeq j_{(X\times B)} \circ (\rho \times \mathrm{id}_B)$ is defined by $F(a, b, t) = [(a, b, t)]$.



We now consider the homotopy push-out of the maps $\rho \times \mathrm{id}_B$ and $\mathrm{id}_A \times \eta$ defined by the quadruple $(X \times Y, \mathrm{id}_X \times \eta, \rho \times \mathrm{id}_Y, G)$, see diagram (1.6), with $G : (\rho \times \mathrm{id}_Y) \circ (\mathrm{id}_A \times \eta) \simeq (\mathrm{id}_X \times \eta) \circ (\rho \times \mathrm{id}_B)$ being the trivial homotopy, by definition of homotopy push-out 48, there is a function $\rho \wedge \eta : \rho \vee \eta \to X \times Y$ and two homotopies, $H : \mathrm{id}_X \times \eta \simeq (\rho \wedge \eta) \circ j_{(X\times B)}$ and $K : \rho \times \mathrm{id}_Y \simeq (\rho \wedge \eta) \circ j_{(A\times Y)}$, such that the following diagram is homotopy commutative

$$\begin{tikzcd}
A \times B \arrow[r, "\rho\times \mathrm{id}_B"] \arrow[d, "\mathrm{id}_A\times\eta"] & X \times B \arrow[d, "j_{(X\times B)}"] \arrow[ddr, bend left, "\mathrm{id}_X\times\eta"] & \\
A \times Y \arrow[r, "j_{(A\times Y)}"] \arrow[drr, bend right, "\rho\times\mathrm{id}_Y"'] & \rho \vee \eta \arrow[dr, dashed, "\rho\wedge\eta"] & \\
 & & X \times Y,
\end{tikzcd} \tag{1.6}$$

we can easily see that the map $\rho \wedge \eta$ is defined by

$$\begin{aligned}
\rho \wedge \eta : \rho \vee \eta &\to X \times Y \\
[(x, b)] &\mapsto (x, \eta(b)) \\
[(a, y)] &\mapsto (\rho(a), y) \\
[(a, b, t)] &\mapsto (\rho(a), \eta(b)).
\end{aligned}$$

Lemma 1. *Considering the mapping cones of ρ, η and $\rho\wedge\eta$. We have a canonical(pointed) homeomorphism $C_{\rho\wedge\eta} \simeq C_\rho \wedge C_\eta$ uniquely determined up to homotopy equivalence.*

Proof. We fix a function $\Theta = (\Theta_1, \Theta_2) : I \times I \to I \times I$ with the following properties:

- $\Theta(I \times \{0\}) = \{(0,0)\}$;
- the restriction $\Theta_| : I \times I \setminus I \times \{0\} \to I \times I \setminus \{(0,0)\}$ is an orientation-preserving homeomorphism;
- $\Theta(\{0\} \times I) = \{0\} \times I$;
- $\Theta(I \times \{1\}) = I \times \{1\} \cup \{1\} \times I$;
- $\Theta(\{1\} \times I) = I \times \{0\}$.

Such a function is unique up to homotopy. A fixed choice may be

$$\Theta(t,s) := \begin{cases} s(2t, 1) & \text{for } 0 \leq t \leq \frac{1}{2} \\ s(1, 2-2t) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now, we define the following map

$$\begin{aligned} \Psi_\Theta : C_{\rho\wedge\eta} &\to C_\rho \wedge C_\eta \\ [(x,y)] &\to [[x],[y]] \\ [(x,b,s)] &\to [[x],[(b,s)]] \\ [(a,y,t)] &\to [[(a,t)],[b]] \\ [(a,b,t,s)] &\to [[(a,\Theta_1(t,s))],[(b,\Theta_2(t,s)]]. \end{aligned}$$

It is not difficult to see that Ψ_Θ is a well defined homeomorphism.

Informally, keeping in mind the picture above, we could think that Ψ_Θ identifies $X \times Y$ with $X \times Y$, the cone $C(X \times B)$ with the space $X \times C(B)/\sim$, the cone $C(A \times Y)$ with the space $C(A) \times Y/\sim$ and the cone $C(A \times B \times I)$ with the space $C(A) \times C(B)/\sim$.

Given any other function $\Theta' = (\Theta'_1, \Theta'_2) : I \times I \to I \times I$ satisfying the conditions above, it is easy to prove that Ψ_Θ and $\Psi_{\Theta'}$ are homotopics. Moreover, any such Ψ_Θ is canonical, since Θ is fixed a priori, independently of ρ and η. $\square$

Corollary 1. *There exists an isomorphism between the two functors from $\mathcal{C}_2 \times \mathcal{C}_2$ to $\mathcal{C}_+$, defined on objects by $(\rho,\eta) \mapsto C_{\rho\wedge\eta}$ and $(\rho,\eta) \mapsto C_\rho \wedge C_\eta$, that can be determined in an essentially unique way.*

We now define the exterior product

$$\times : h^\bullet(\rho) \otimes_{\mathbb{Z}} h^\bullet(\eta) \to h^\bullet(\rho \wedge \eta) \tag{1.7}$$

through the composition

$$\tilde{h}^\bullet(C_\rho) \times \tilde{h}^\bullet(C_\eta) \to \tilde{h}^\bullet(C_\rho \wedge C_\eta) \xrightarrow{\Psi_\Theta^*} \tilde{h}^\bullet(C_{\rho\wedge\eta}).$$

This product does not depend on the homotopy class of Ψ_Θ (hence on the choice of Θ in the construction).

1.1.2.1 Product and cofibrations

In the particular case when one of the two maps is a cofibration, e.g. η, we can apply the isomorphism (A.6) and replace the homotopy push-out. In particular, we set

$$\rho \veebar \eta := (X \times B) \sqcup_{A\times B} (A \times Y)$$

and we define the function

$$\rho \bar{\wedge} \eta : \rho \veebar \eta \to X \times Y \tag{1.8}$$

through the following homotopy commutative diagram:

$$\begin{tikzcd}
A \times B \arrow[r, "\rho\times \mathrm{id}_B"] \arrow[d, "\mathrm{id}_A\times\eta"] & X \times B \arrow[d, "j_{X\times B}"] \arrow[ddr, bend left, "\mathrm{id}_X\times\eta"] & \\
A \times Y \arrow[r, "j_{A\times Y}"] \arrow[drr, bend right, "\rho\times\mathrm{id}_Y"'] & \rho \veebar \eta \arrow[dr, dashed, "\rho\bar{\wedge}\eta"] & \\
& & X \times Y.
\end{tikzcd}$$

It follows that $\rho\bar{\wedge}\eta$ is defined by $(x, b) \to (x, \eta(b))$ and $(a, y) \to (\rho(a), y)$. We get diagram (A.5) with the spaces we are considering now:

$$\begin{tikzcd}
\rho \vee \eta \arrow[r, "\rho\wedge\eta"] \arrow[d, "p"] & X \times Y \arrow[d, "\mathrm{id}"] \\
\rho \veebar \eta \arrow[r, "\rho\bar{\wedge}\eta"] & X \times Y,
\end{tikzcd} \tag{1.9}$$

hence the isomorphism (A.6) takes the form

$$(\mathrm{id}, p)^* : h^\bullet(\rho\bar{\wedge}\eta) \to h^\bullet(\rho \wedge \eta). \tag{1.10}$$

Composing the product (1.7) with the inverse of the isomorphism (1.10), we get the product

$$\times : h^\bullet(\rho) \otimes_{\mathbb{Z}} h^\bullet(\eta) \to h^\bullet(\rho\bar{\wedge}\eta). \tag{1.11}$$

If both ρ and η are cofibration, then $\rho\veebar\eta = (A\times Y) \cup (X \times B)$ and $\rho\bar{\wedge}\eta$ is the corresponding inclusion in $X \times Y$, hence we recover (1.5).

In particular, taking $\eta : \varnothing \to Y$ on (1.11), we get the exterior relative-absolute product $h^\bullet(\rho) \otimes_{\mathbb{Z}} h^\bullet(Y) \to h^\bullet(\rho \times \mathrm{id}_Y)$, given on (1.4).

1.2 Cohomology theory on maps of pairs

A cohomology theory can be defined on a single space X, i.e., as a functor from the opposite homotopy category of $\mathcal{C}$ to $\mathcal{A}_{\mathbb{Z}}$ or $\mathcal{R}_{\mathbb{Z}}$. As described in section 1.1, we can extend the definition to generic maps $\rho : A \to X$, i.e., we can replace $\mathcal{C}$ by the category $\mathcal{C}_2$, the objects of the latter being the morphisms of $\mathcal{C}$. In principle, we can further generalize this construction, replacing $\mathcal{C}_2$ by the category $(\mathcal{C}_2)_2$, whose objects are the morphisms of $\mathcal{C}_2$. This means that we define the cohomology groups $h^\bullet(\rho, \eta)$, where $(\rho, \eta) : f \to g$ is the following morphism between the objects f and g of $\mathcal{C}_2$:

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "\eta"] & A \arrow[d, "\rho"] \\ X' \arrow[r, "g"] & X. \end{tikzcd} \tag{1.12}$$

The definition can be given axiomatically (requiring excision, long exact sequence and multiplicativity on path-components), or through the mapping cone. In fact, the morphism (ρ, η) induces the (pointed) map $C(\rho, \eta) : C_f \to C_g$, hence we set $h^\bullet(\rho, \eta) := \tilde{h}^\bullet(\tilde{C}_{C(\rho,\eta)})$, where $\tilde{C}$ denotes the reduced cone. Actually, since C_f and C_g are well-pointed, the reduced cone $\tilde{C}_{C(\rho,\eta)}$ has the same homotopy type of the unreduced one; hence, we have the canonical isomorphism

$$h^\bullet(\rho, \eta) \simeq h^\bullet(C(\rho, \eta)), \tag{1.13}$$

that might also be used as the definition. We could keep go on in this way, defining a functor on $((\mathcal{C}_2)_2)_2$, and so on.

Of course we are not going to deal with such a generalization at any level, but only with the following picture. Between the definitions on $\mathcal{C}$ and on $\mathcal{C}_2$ (i.e., on single spaces and on maps), we have the intermediate step of pairs of spaces that is the usual definition of relative cohomology. We denote by $\mathcal{C}_2'$ the full subcategory of $\mathcal{C}_2$ whose objects are pairs. Consequently, between the definitions on $\mathcal{C}_2$ and $(\mathcal{C}_2)_2$, we get the intermediate step $(\mathcal{C}_2')_2$, i.e. we define the cohomology theory on the category of morphisms of pairs. This means that we suppose that f and g are inclusions of a subspace in diagram (1.12). This implies that η is necessarily the restriction of ρ, hence we are dealing with objects (not morphisms) of the following form:

$$(\rho, \rho') : (A, A') \to (X, X'),$$

i.e., $\rho : A \to X$ is a map such that $\rho|_{A'} = \rho'$.

Remark 2. Sometimes when there is no risk of confusion, we will just write $\rho := (\rho, \rho')$.

We review more in detail the axioms of cohomology theory on $(\mathcal{C}_2')_2$ as sketched above.

- An object of $(\mathcal{C}_2')_2$ is a map of the form $(\rho, \rho') : (A, A') \to (X, X')$;

- a morphism between the objects $(\eta, \eta') : (B, B') \to (Y, Y')$ and $(\rho.\rho') : (A, A') \to (X, X')$ is a pair of maps, $(f, f') : (Y, Y') \to (X, X')$ and $(g, g') : (B, B') \to (A, A')$, making the following diagram commutative:

$$\begin{array}{ccc} (B, B') & \xrightarrow{(\eta,\eta')} & (Y, Y') \\ \downarrow{\scriptstyle (g,g')} & & \downarrow{\scriptstyle (f,f')} \\ (A, A') & \xrightarrow{(\rho,\rho')} & (X, X'). \end{array} \tag{1.14}$$

A *homotopy* between two morphisms $((f_0, f_0'), (g_0, g_0')), ((f_1, f_1'), (g_1, g_1')) : (\eta, \eta') \to (\rho, \rho')$ is a morphism $((F, F'), (G, G')) : (\eta, \eta') \times \mathrm{id}_I \to (\rho, \rho')$, where id_I is the identity of the interval and $(\eta, \eta) \times \mathrm{id}_I : (B \times I, B' \times I) \to (Y \times I, Y' \times I)$, such that, for $i = 0, 1$, we have $(F, F')|_{(Y\times\{i\},Y'\times\{i\})} = (f_i, f_i')$ and $(G, G')|_{(B\times\{i\},B'\times\{i\})} = (g_i, g_i')$.

Taking the quotient of the morphisms of $(\mathcal{C}_2')_2$ up to homotopy, we define the category $\mathcal{H}(\mathcal{C}_2')_2$. We have the natural functors $\Pi : (\mathcal{C}_2')_2 \to \mathcal{C}_2'$ and $\Pi : \mathcal{H}(\mathcal{C}_2')_2 \to \mathcal{H}\mathcal{C}_2$, defined in the following way: if (ρ, ρ') is the object, then $\Pi(\rho, \rho') := (A, A')$; if $((f, f'), (g, g'))$ is the morphism (1.14), then $\Pi((f, f'), (g, g')) := (g, g')$.

A *cohomology theory* on $(\mathcal{C}_2')_2$ is defined by a functor $h^\bullet : \mathcal{H}(\mathcal{C}_2')_2^{\mathrm{op}} \to \mathcal{A}_\mathbb{Z}$ and a morphism of functors $\beta^\bullet : h^\bullet \circ \Pi \to h^{\bullet+1}$, satisfying the following axioms:

- *Long exact sequence:* the functor $h^\bullet$ and the morphism of functors $\beta^\bullet$ define a functor from $\mathcal{H}(\mathcal{C}_2')_2$ to the category of long exact sequences of abelian groups that assigns to the object $(\rho, \rho') : (A; A') \to (X, X')$ the sequence

$$\cdots \longrightarrow h^n(\rho, \rho') \xrightarrow{\pi^*} h^n(X, X') \xrightarrow{(\rho,\rho')^*} h^n(A, A') \xrightarrow{\beta} h^{n+1}(\rho, \rho') \longrightarrow \cdots,$$

  (π being the natural morphism from $(\varnothing, \varnothing) \to (X, X')$ to (ρ, ρ')) and to a morphism the corresponding morphism of exact sequences,

- if $Z \subset A'$ and $W \subset X'$ are subspaces whose closure is contained in the interior of the bigger space, in such a way that $\rho(Z) \subset W$, then the morphism

$$\begin{array}{ccc} (A \setminus Z, A' \setminus Z) & \xrightarrow{(\bar\rho,\bar\rho')} & (X \setminus W, X' \setminus W) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ (A, A') & \xrightarrow{(\rho,\rho')} & (X, X') \end{array}$$

  induces an isomorphism between $h^\bullet(\rho, \rho')$ and $h^\bullet(\bar\rho, \bar{\rho}')$, where the morphism $\bar\rho$ is the restriction of ρ to $A \setminus Z$.

Remark 3. If necessary, we must add the multiplicativity axiom (MILNOR,) that is enough to state in its absolute version. Such a definition of cohomology theory is equivalent to the usual one on pairs of spaces or on spaces with a marked point. In fact, starting from a reduced cohomology theory on $\mathcal{HC}_+$, the cohomology groups of a morphism $(\rho, \rho') : (A, A') \to (X, X')$ are defined as the reduced ones of the reduced mapping cone of the induced map $C(\rho, \rho') : C(X, X') \to C(A, A')$ between the corresponding relative cones, and the axioms are satisfied. Conversely, if we start from the axioms on the category $\mathcal{H}(\mathcal{C}_2')_2$, we can prove that $h^\bullet(\rho, \rho')$ is naturally isomorphic to $\tilde{h}^\bullet\left(\tilde{C}_{C(\rho,\rho')}\right)$; hence, the theory is the unique possible extension to $\mathcal{H}(\mathcal{C}_2')_2$ of a reduced cohomology theory on $\mathcal{HC}_+$.

The cohomology theory $h^\bullet$ is called multiplicative if it can be refined to a functor $h^\bullet : \mathcal{H}(\mathcal{C}_2')_2 \to \mathcal{R}_\mathbb{Z}$, in such a way that the product satisfies a suitable compatibility condition with the morphisms $\beta^\bullet$. The isomorphism $h^\bullet(\rho, \rho') \simeq \tilde{h}^\bullet(\tilde{C}_{C(\rho,\rho')})$ is a ring isomorphism; hence, the product in relative cohomology is canonically induced by the one on the corresponding reduced cohomology theory.

1.2.1 Some preliminary results.

We now apply the previous construction, given on section 1.1.2, to products, we state some lemmas that will be useful in the following.

Lemma 2. *If (A, A') and (X, X') are cofibrations, then the cohomology groups of the map $(\rho, \rho') : (A, A') \to (X, X')$ are canonically isomorphic to the ones of the projection $\bar{\rho} : A/A' \to X/X'$.*

Proof. Considering the pointed map $(\bar{\rho}, \bar{\rho}') : (A/A', A'/A') \to (X/X', X'/X')$, the natural projection morphism $(\rho, \rho') \to (\bar{\rho}, \bar{\rho}')$ induces the canonical isomorphism $h^\bullet(\rho, \rho') \simeq h^\bullet(\bar{\rho}, \bar{\rho}')$. In fact, it is enough to apply the five lemma to the corresponding long exact sequences:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & h^{\bullet-1}(A/A') & \longrightarrow & h^\bullet(\bar{\rho}, \bar{\rho}') & \longrightarrow & h^\bullet(X/X') & \longrightarrow & h^\bullet(A/A') & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & h^{\bullet-1}(A, A') & \longrightarrow & h^\bullet(\rho, \rho') & \longrightarrow & h^\bullet(X, X') & \longrightarrow & h^\bullet(A, A'). & \longrightarrow & \cdots
\end{array}$$

The natural morphism $\bar{\rho} \to (\bar{\rho}, \bar{\rho}')$ induces the following morphism of long exact sequences:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \tilde{h}^{\bullet-1}(A/A') & \longrightarrow & h^\bullet(\bar{\rho}, \bar{\rho}') & \longrightarrow & \tilde{h}^\bullet(X/X', *) & \longrightarrow & \tilde{h}^\bullet(A/A', *) & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & h^{\bullet-1}(A/A') & \longrightarrow & h^\bullet(\bar{\rho}) & \longrightarrow & h^\bullet(X/X') & \longrightarrow & h^\bullet(A/A'), & \longrightarrow & \cdots
\end{array}$$

since $h^\bullet(X/X') \simeq \tilde{h}^\bullet(X/X') \oplus h^\bullet(pt)$ and $h^\bullet(A/A') \simeq \tilde{h}^\bullet(A/A') \oplus h^\bullet(pt)$ canonically, cutting $h^\bullet(pt)$ on both groups we obtain reduced cohomology on both lines of the previous diagram. Therefore, because of the five lemma, we get the canonical isomorphism $h^\bullet(\bar{\rho}) \simeq h^\bullet(\bar{\rho}, \bar{\rho}')$. $\square$

Lemma 3. *Let us consider a map $(\rho, \rho') : (A, A') \to (X.X')$. The corresponding mapping cones form the pair $(C\rho, C\rho')$. We have the canonical isomorphism $h^\bullet(\rho, \rho') \simeq h^\bullet(C\rho, C\rho')$.*

Proof. It follows from (1.13) that $h^\bullet(\rho, \rho') \simeq h^\bullet(C(\rho, \rho'))$, where $C(\rho, \rho') : C(A, A') \to C(X, X')$ is the induced map between the relative mapping cones. We have that $h^\bullet(\rho, \rho') \simeq \tilde{h}^\bullet(C_{C(\rho,\rho')})$ and we have the canonical homeomorphism $C_{C(\rho,\rho')} \simeq C(C\rho, C\rho')$. It follows that $\tilde{h}^\bullet(C_{C(\rho,\rho')}) \simeq \tilde{h}^\bullet(C(C\rho, C\rho')) \simeq h^\bullet(C\rho, C\rho')$. □

Corollary 2. *If (A, A') and (X, X') are cofibrations and $\rho' : A' \to X'$ is a homeomorphism, then the cohomology groups of $(\rho, \rho') : (A, A') \to (X, X')$ are canonically isomorphic to the ones of $\rho : A \to X$.*

Proof. It follows from lemma 3 that $h^\bullet(\rho, \rho') \simeq h^\bullet(C\rho, C\rho')$. Since ρ' is a homeomorphism, $C\rho' \simeq CA'$, hence it is contractible. Since (A, A') and (X, X') are cofibration, the embedding $C\rho' \hookrightarrow C\rho$ is a cofibration too, hence $C\rho$ and $C\rho/C\rho'$ have the same homotopy type. Hence, $h^\bullet(C\rho, C\rho') \simeq \tilde{h}^\bullet(C\rho/C\rho') \simeq \tilde{h}(C\rho) \simeq h^\bullet(\rho)$. □

Lemma 4. *Let us consider the following data:*

- *an adjunction space $A \sqcup_{A'} B$, defined through a cofibration $A \hookrightarrow A'$ and a map $f : A' \to B$;*
- *a map $\rho_A : A \to X$ and a cofibration $\rho_B : B \hookrightarrow X$, such that $\rho_B \circ f = (\rho_A)|_{A'}$;*
- *the map $\rho : A \sqcup_{A'} B \to X$ induced by ρ_A and ρ_B.*

We have the canonical isomorphism

$$h^\bullet(\rho : A \sqcup_{A'} B \to X) \simeq h^\bullet(\rho_A : (A, A') \to (X, B)).$$

Proof. It follows from the hypotheses that $B \hookrightarrow A \sqcup_{A'} B$ is a cofibration too. Identifying B with its image through ρ_B, it follows from corollary 2 that $h^\bullet(\rho : A \sqcup_{A'} B \to X)$ is canonically isomorphic to $h^\bullet(\rho : (A \sqcup_{A'} B, B) \to (X, B))$. We consider the natural morphism $(\mathrm{id}_X, j) : \rho_A \to \rho$, defined as follows:

$$\begin{tikzcd}
(A, A') \arrow[r, "\rho_A"] \arrow[d, "\pi"] & (X, B) \arrow[d, equal, "\mathrm{id}_X"] \\
(A \sqcup_{A'} B, B) \arrow[r, "\rho"] & (X, B).
\end{tikzcd}$$

Since both the domain and the codomain of π are cofibrations and, since we have the canonical homeomorphism $A/A' \simeq A \sqcup_{A'} B/B$, the pull-back π^* is an isomorphism. Hence, it follows from the five lemma that $(\mathrm{id}_X, \pi)^*$ is an isomorphism too. □

Corollary 3. *Let us consider a map $\rho : A \cup B \to X$ such that $A \cap B \hookrightarrow A$ and $\rho|_B : B \hookrightarrow X$ are cofibrations. We have the canonical isomorphism*

$$h^\bullet(\rho : A \cup B \to X) \simeq h^\bullet(\rho|_A : (A, A \cap B) \to (X, B)).$$

Proof. It is enough to choose $A' = A \cap B$ in the statement of lemma 4, the function f being the embedding $A \cap B \hookrightarrow B$ and ρ_A and ρ_B being the restrictions of ρ. $\square$

1.2.2 S^1-Integration on cohomology of maps of pairs

We now define the S^1-integration map on relative cohomology on maps of pairs,

$$\int_{S^1} : h^{\bullet+1}(\mathrm{id}_{S^1} \times (\rho, \rho')) \to h^\bullet(\rho, \rho').$$

Let $(\rho, \rho') : (A, A') \to (X, X')$ be a morphism and fix a marked point on S^1, consider the natural maps $i_1 : (\rho, \rho') \to \mathrm{id}_{S^1} \times (\rho, \rho')$ and $\pi_1 : \mathrm{id}_{S^1} \times (\rho, \rho') \to (\rho, \rho')$ with its induced maps

$$C(i_1) : \frac{C_\rho}{C_{\rho'}} \hookrightarrow \frac{C_{\mathrm{id}_{S^1} \times \rho}}{C_{\mathrm{id}_{S^1} \times \rho'}} \quad \text{and} \quad C(\pi_1) : \frac{C_{\mathrm{id}_{S^1} \times \rho}}{C_{\mathrm{id}_{S^1} \times \rho'}} \to \frac{C_\rho}{C_{\rho'}}.$$

Since $C(\pi_1) \circ C(i_1) = \mathrm{id}_{C_\rho / C_{\rho'}}$, we have the following split exact sequence:

$$0 \longrightarrow \tilde{h}^\bullet\left(\frac{\dfrac{C_{\mathrm{id}_{S^1} \times \rho}}{C_{\mathrm{id}_{S^1} \times \rho'}}}{\dfrac{C_\rho}{C_{\rho'}}}\right) \xrightarrow{C(\pi)^*} \tilde{h}^\bullet\left(\frac{C_{\mathrm{id}_{S^1} \times \rho}}{C_{\mathrm{id}_{S^1} \times \rho'}}\right) \xrightarrow{C(i_1)^*} \tilde{h}^\bullet\left(\frac{C_\rho}{C_{\rho'}}\right) \longrightarrow 0,$$

or equivalent, the split exact sequence:

$$0 \longrightarrow h^\bullet\left(\frac{C_{\mathrm{id}_{S^1} \times \rho}}{C_{\mathrm{id}_{S^1} \times \rho'}}, \frac{C_\rho}{C_{\rho'}}\right) \xrightarrow{C(\pi)^*} h^\bullet\left(C_{\mathrm{id}_{S^1} \times \rho}, C_{\mathrm{id}_{S^1} \times \rho'}\right) \xrightarrow{C(i_1)^*} h^\bullet\left(C_\rho, C_{\rho'}\right) \longrightarrow 0,$$

Defining

$$h^\bullet(i_1) := h^\bullet\left(\frac{C_{\mathrm{id}_{S^1} \times \rho}}{C_{\mathrm{id}_{S^1} \times \rho'}}, \frac{C_\rho}{C_{\rho'}}\right),$$

and using lemma 3, we get the split exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & h^\bullet(i_1) \arrow[r, "\pi^*"] & h^\bullet(\mathrm{id}_{S^1} \times (\rho, \rho')) \arrow[l, bend left, "\xi"] \arrow[r, "i_1^*"] & h^\bullet(\rho, \rho') \arrow[l, bend left, "\pi_1^*"] \arrow[r] & 0, \end{tikzcd}$$

were $\xi(\alpha) = (\pi^*)^{-1}(\alpha - \pi_1^* i_1^* \alpha)$. Using the natural isomorphism

$$s : h^\bullet(i_1) := h^\bullet\left(\frac{C_{\mathrm{id}_{S^1}\times\rho}}{C_{\mathrm{id}_{S^1}\times\rho'}}, \frac{C_\rho}{C_{\rho'}}\right) \simeq \tilde{h}^\bullet\left(\frac{\dfrac{C_{\mathrm{id}_{S^1}\times\rho}}{C_{\mathrm{id}_{S^1}\times\rho'}}}{\dfrac{C_\rho}{C_{\rho'}}}\right) \simeq \tilde{h}^\bullet\left(\frac{\dfrac{C_{\mathrm{id}_{S^1}\times\rho}}{C_\rho}}{\dfrac{C_{\mathrm{id}_{S^1}\times\rho'}}{C_{\rho'}}}\right)$$
$$\simeq \tilde{h}^\bullet\left(\Sigma C_\rho / \Sigma C_{\rho'}\right) \simeq \tilde{h}^\bullet\left(\Sigma (C_\rho / C_{\rho'})\right) \simeq \tilde{h}^{\bullet-1}\left(C_\rho / C_{\rho'}\right) \simeq h^{\bullet-1}(\rho, \rho')$$

we define the S^1-integration map as the composition $\int_{S^1} = s \circ \xi$.

1.2.3 Maps of pairs and product

Let us consider the function $\rho\bar{\wedge}\eta : \rho \veebar \eta \to X \times Y$ defined in (1.8), and notice that, replacing the data A, B, A', f, ρ_A and ρ_B respectively by $A \times Y$, $X \times B$, $A \times B$, $\rho \times \mathrm{id}_B$, $\rho \times \mathrm{id}_Y$ and $\mathrm{id}_X \times \eta$ on the statements of lemma 4, we get the isomorphism

$$h^\bullet(\rho\bar{\wedge}\eta) \simeq h^\bullet\left(\rho \times \mathrm{id}_{(Y,B)} : (A \times Y, A \times B) \to (X \times Y, X \times B)\right) := h^\bullet(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_B);$$

thus, denoting the cofibration η as the pair (Y, B), we can formulate the product (1.11) as follows:

$$\times : h^\bullet(\rho : A \to X) \otimes_{\mathbb{Z}} h^\bullet(Y, B) \to h^\bullet(\rho \times \mathrm{id}_{(Y,B)}) = h^\bullet(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_B). \tag{1.15}$$

We could equivalently obtain (1.15) directly through the mapping cones. In fact, using lemma 3 and the canonical homeomorphism $C_\rho \wedge Y/B \simeq C_{\rho\times\mathrm{id}_Y}/C_{\rho\times\mathrm{id}_B}$, we have

$$\begin{aligned}\tilde{h}^\bullet(C_\rho) \otimes \tilde{h}^\bullet(Y/B) \to \tilde{h}^\bullet(C_\rho \wedge Y/B)& \\ \simeq \tilde{h}^\bullet(C_{\rho\times\mathrm{id}_Y}/C_{\rho\times\mathrm{id}_B})& \\ \simeq \tilde{h}^\bullet(C_{\rho\times\mathrm{id}_Y}, C_{\rho\times\mathrm{id}_B})& \\ \overset{3}{\simeq} h^\bullet(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_B).&\end{aligned}$$

This is the most satisfactory formulation for our purposes, since it can be restricted to pairs and maps of smooth manifolds without any extension of the notion of smoothness.

1.3 Cohomology theory on finite sequences of space

In order to axiomatize the differential relative-parallel product that we will explain in the next chapter, we are compelled to generalize the product (1.15) in the following way.

We generalize a cofibration (X, X') to a finite sequence $(X, X_1, \dots, X_n)$ such that $X_i \subset X$, no necessarily $X_i \subset X_{i+1}$, and (X, X_i) is a cofibration for all $i = 1, \dots, k$, and, a map of pairs $(\rho, \rho') : (A, A') \to (X, X')$ to a *map of finite sequences* that we denote by $(\rho, \rho_1, \dots, \rho_n) : (A, A_1, \dots, A_n) \to (X, X_1, \dots, X_n)$, i.e., a map $\rho : A \to X$ such that $\rho|_{Ai} = \rho_i$ for every i. We denote this map by $(\rho, \vec{\rho}_n) : (A, \vec{A}_n) \to (X, \vec{X}_n)$.

Denote by C'_ω the category whose objects are finite sequences of this form (of any length) and whose morphisms are the corresponding maps. Morphisms between two sequences of distinct lengths are well defined, completing the shortest one with empty spaces (equivalently, we may think of an object $(X, \vec{X}_n)$ as an infinite sequence whose elements are eventually empty). We denote by $(C'_\omega)_2$ the category of morphisms of C'_ω. It follows that an object of $(C'_\omega)_2$ is a map $(\rho, \vec{\rho}_n) : (A, \vec{A}_n) \to (X, \vec{X}_n)$ and a morphism from $(\eta, \vec{\eta}_n) : (B, \vec{B}_n) \to (Y, \vec{Y}_n)$ to $(\rho, \vec{\rho}_n)$ is pair of maps $((f, \vec{f}_n), (g, \vec{g}_n))$ making the following diagram commutative:

$$\begin{array}{ccc} (B, \vec{B}_n) & \xrightarrow{(\eta, \vec{\eta}_n)} & (Y, \vec{Y}_n) \\ \Big\downarrow{\scriptstyle (g, \vec{g}_n)} & & \Big\downarrow{\scriptstyle (f, \vec{f}_n)} \\ (A, \vec{A}_n) & \xrightarrow{(\rho, \vec{\rho}_n)} & (X, \vec{X}_n). \end{array}$$

We have a natural functor $\mathcal{C}'_\omega \to \mathcal{C}_2$ defined on objects by

$$(X, X_1, \dots, X_n) \mapsto (X, X_1 \cup \ldots \cup X_n)$$

and on morphisms by

$$((\rho, \vec{\rho}_n) : (A, A_1, \dots, A_n) \to (X, X_1, \dots, X_n)) \mapsto (\rho : (A, A_1 \cup \ldots \cup A_n) \to (X, X_1 \cup \ldots \cup X_n)) .$$

This functor naturally extends to $(C'_\omega)_2 \to (C'_2)_2$; therefore, we define the topological cohomology theory $h^\bullet$ on $(C'_\omega)_2$ by composition. Basically, this means that

$$h^\bullet(X, X_1, \dots, X_n) := h^\bullet(X, X_1 \cup \ldots \cup X_n), \tag{1.16}$$

with the corresponding definition of pull-back. Equivalently, we can state the axiom of cohomology theory on $(C'_\omega)_2$ as we did in section 1.2, simply replacing a pair of the form (X, X') or (A, A') by a finite sequence of the form $(X, \vec{X}_n)$ or $(A, \vec{A}_n)$. Such a theory is completely determined up to equivalence by its restriction to $(C'_2)_2$, since it satisfies (1.16), the latter becoming a canonical isomorphism instead of a definition. Now, using the results of subsection 1.2.1 appropriately, it is possible to generalize the product (1.15), i.e., given a map of finite sequences $(\rho, \vec{\rho}_n) : (A, \vec{A}_n) \to (X, \vec{X}_n)$ we have

$$\times : h^\bullet(\rho, \vec{\rho}_n) \otimes_{\mathbb{Z}} h^\bullet(Y, \vec{B}_m) \to h^\bullet((\rho, \vec{\rho}_n) \times \mathrm{id}_{(Y, \vec{B}_m)}) = h^\bullet(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_{\vec{B}_m}, \vec{\rho}_n \times \mathrm{id}_Y), \tag{1.17}$$

explicitly, the group on the right $h^\bullet(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_{\vec{B}_m}, \vec{\rho}_n \times \mathrm{id}_Y)$ is given by

$$\begin{aligned} h^\bullet((&A \times Y, A \times B_1, \dots, A \times B_m, A_1 \times Y, \dots, A_n \times Y) \\ &\to (X \times Y, X \times B_1, \dots, X \times B_m, X_1 \times Y, \dots, X_n \times Y)). \end{aligned}$$

2 DIFFERENTIAL COHOMOLOGY ON MAPS OF PAIRS

In this chapter, we briefly review the axioms of relative cohomology in the differential framework. In a similar way, we state the axioms of relative cohomology on maps of pairs. Finally, in the last section, we describe the differential relative-parallel product, tools that we have been announcing in the introduction and will be useful for the applications the we describe on chapter 6.

2.1 Summary on relative differential cohomology.

Let $\mathcal{M}$ be the category of smooth manifolds or of smooth compact manifolds (even with boundary). We consider a cohomology theory $h^\bullet$, defined on a category including M. We use the following notation:

$$\mathfrak{h}^\bullet := h^\bullet(\{pt\}), \qquad \mathfrak{h}^\bullet_{\mathbb{R}} := \mathfrak{h}^\bullet \otimes_{\mathbb{Z}} \mathbb{R}.$$

We consider the category $\mathcal{M}_2$ of morphisms of $\mathcal{M}$. For any object $\rho : A \to X$ of $\mathcal{M}_2$, we call $\mathrm{ch} : h^\bullet(\rho) \to H^\bullet_{\mathrm{dR}}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}})$ the generalized Chern character.

Definition 1. A *Relative differential extension* of $h^\bullet$ is a quadruple $(\hat{h}^\bullet, R, I, a)$, where $\hat{h}^\bullet : \mathcal{M}_2^{\mathrm{op}} \to \mathcal{A}_{\mathbb{Z}}$ is a contravariant functor together with natural transformations of $\mathcal{A}_{\mathbb{Z}}-$valued functors: $R : \hat{h}^\bullet(\rho) \to \Omega^\bullet_{\mathrm{cl}}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}})$ called curvature, $I : \hat{h}^\bullet(\rho) \to h^\bullet(\rho)$ and $a : \Omega^{\bullet-1}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \to \hat{h}^\bullet(\rho)$, such that the following two diagrams are commutative:

$$\begin{tikzcd} h^{\bullet-1}(\rho) \arrow[r, "\mathrm{ch}"] & \Omega^{\bullet-1}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "a"] \arrow[dr, "d"] & \hat{h}^\bullet(\rho) \arrow[r, "I"] \arrow[d, "R"] & h^\bullet(\rho) \arrow[r] \arrow[d, "\mathrm{ch}"] & 0 \\ & & \Omega^\bullet_{\mathrm{cl}}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[r, "\mathrm{dR}"] & H^\bullet_{\mathrm{dR}}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}}), & \end{tikzcd} \tag{2.1}$$

with an exact upper horizontal line.

$$\begin{tikzcd} \hat{h}^\bullet(\rho) \arrow[r, "\pi^*"] \arrow[d, "\mathrm{cov}"] & \hat{h}^\bullet(X) \arrow[d, "\rho^*"] \\ \Omega^{\bullet-1}(A) \arrow[r, "a"] & \hat{h}^\bullet(A), \end{tikzcd} \tag{2.2}$$

where π is the natural morphism from $\varnothing \to X$ to $\rho : A \to X$ and $\mathrm{cov}(\rho)$ is the second component of the curvature $R(\rho)$.

Definition 2. A class $\hat{\alpha} \in \hat{h}^\bullet(\rho)$ is called flat if $R(\hat{\alpha}) = 0$, and it is called *parallel* if $\mathrm{cov}(\hat{\alpha}) = 0$. We denote by $\hat{h}^\bullet_{\mathrm{fl}}(\rho)$ and $\hat{h}^\bullet_{\mathrm{par}}(\rho)$ the subgroups respectively of flat and parallel classes.

Given a functor $\mathcal{F} : \mathcal{M}_2 \to \mathcal{C}$, it induce a new functor $S\mathcal{F} : \mathcal{M}_2 \to \mathcal{C}$ defined by $S\mathcal{F}\rho := \mathcal{F}(\mathrm{id}_{S^1} \times \rho)$ and $S\mathcal{F}(f,g) := \mathcal{F}(\mathrm{id}_{S^1} \times f, \mathrm{id}_{S^1} \times g)$. Moreover, given a morphism $t : S^1 \to S^1$, we consider the morphism $(t \times \mathrm{id}_X, t \times \mathrm{id}_A) : \mathrm{id}_{S^1} \times \rho \to \mathrm{id}_{S^1} \times \rho$.

Definition 3. A *relative differential extension with integration* of $h^\bullet$ is a relative differential extension $(\hat{h}^\bullet, R, I, a)$ together with a natural transformation:

$$\int_{S^1} : S\hat{h}^{\bullet+1} \to \hat{h}^\bullet$$

such that:

*I*1. for $t : S^1 \to S^1$ defined by $t(e^{i\theta}) := e^{-i\theta}$, then $\int_{S^1} \circ (t \times \mathrm{id}_X, t \times \mathrm{id}_A)^* = -\int_{S^1}$;

*I*2. $\int_{S^1} \circ \pi_1^* = 0$, where $\pi_1 : \mathrm{id}_{S^1} \times \rho \to \rho$ is the projection;

*I*3. the follows diagram is commutative

$$\begin{tikzcd}
\Omega^\bullet(\mathrm{id}_{S^1} \times \rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "a"] \arrow[d, "\int_{S^1}"] & \hat{h}^{\bullet+1}(\mathrm{id}_{S^1} \times \rho) \arrow[r, "I"] \arrow[d, "\int_{S^1}"] \arrow[rr, bend left, "R"] & h^{\bullet+1}(\mathrm{id}_{S^1} \times \rho) \arrow[d, "\int_{S^1}"] & \Omega^{\bullet+1}_{\mathrm{cl}}(\mathrm{id}_{S^1} \times \rho; \mathfrak{h}^\bullet_R) \arrow[d, "\int_{S^1}"] \\
\Omega^{\bullet-1}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "a^{-1}"] & \hat{h}^\bullet(\rho) \arrow[r, "I^{-1}"] \arrow[rr, bend right, "R^{-1}"'] & h^\bullet(\rho) & \Omega^\bullet_{\mathrm{cl}}(\rho; \mathfrak{h}^\bullet_R),
\end{tikzcd} \tag{2.3}$$

where the first and last vertical arrows are defined by $\int_{S^1}(\omega, \eta) := (\int_{S^1} \omega, \int_{S^1} \eta)$, and, the third one is the topological S^1-integration defined in the previous section .

Assume now that $h^\bullet$ is a multiplicative cohomology theory, in particular $\Omega^\bullet(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})$ and $H^\bullet(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})$ are $\mathcal{A}_{\mathbb{Z}}$-graded rings.

Definition 4. A *multiplicative relative differential extension* of $h^\bullet$ is a relative differential extension $(\hat{h}^\bullet, R, I, a)$ such that, there is a natural right $\hat{h}^\bullet(X)$-module structure on $\hat{h}^\bullet(\rho)$, the transformations I, R are multiplicative and for every $\hat{\alpha} \in \hat{h}^\bullet(\rho)$, $\hat{\beta} \in \hat{h}^\bullet(X)$, $(\omega, \eta) \in \Omega^\bullet(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d)$ and $\omega' \in \Omega^\bullet(X; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d)$, the identities hold:

- $\hat{\alpha} \cdot a(\omega') = a(R(\hat{\alpha}) \wedge \omega')$;
- $a(\omega, \eta) \cdot \hat{\beta} = a((\omega, \eta) \wedge R(\hat{\beta}))$.

Remark 4. When $\rho : A \hookrightarrow X$ is a closed embedding, we denote $\hat{h}^\bullet(\rho)$ also by $\hat{h}^\bullet(X, A)$. Restricting to the case $A = \varnothing$, we obtain an absolute differential extension of $h^\bullet$ given in (BUNKE; SCHICK, 2010).

2.2 Differential cohomology on maps of pairs

We call $\mathcal{M}$ the subcategory of $\mathcal{C}$ whose objects are smooth manifolds or smooth compact manifolds (even with boundary) and whose morphisms are smooth maps. The categories $\mathcal{M}_2$, $(\mathcal{M}_2)_2$ and so on are defined from $\mathcal{M}$ similar to the corresponding ones from $\mathcal{C}$. The full subcategory $\mathcal{M}'_2$ of $\mathcal{M}_2$ is the one whose objects are closed embeddings (equivalently, manifold pairs), that are in particular cofibrations. It follows that $\mathcal{M}'_2$ is a (non-full) subcategory of $\mathcal{C}_2$ The category $(\mathcal{M}'_2)_2$ is defined similarly to $(\mathcal{C}'_2)_2$ and it is a subcategory of the latter.

2.2.1 Axioms of differential cohomology.

Since we are dealing with cohomology theories on $(\mathcal{C}'_2)_2$, it is natural (and necessary for us) to consider the corresponding differential extensions on $(\mathcal{M}'_2)_2$; hence, we briefly state the corresponding axioms following (RUFFINO; BARRIGA, 2018) [section2.3].

We begin defining the de-Rham complex in this framework. Given a manifold pair (X, X'), we call $\Omega^\bullet(X, X')$ the complex of smooth differential forms on X that vanish on X'. Given a smooth map of manifold pairs $(\rho, \rho') : (A, A') \to (X, X')$, we call $\Omega^\bullet(\rho, \rho')$ the cone complex of $(\rho, \rho')^* : \Omega^\bullet(X, X') \to \Omega^{\bullet-1}(A, A')$, i.e., the cochain complex $\Omega^\bullet(X, X') \oplus \Omega^{\bullet-1}(A, A')$ with coboundary $d(\omega, \eta) = (d\omega, \rho^*\omega - d\eta)$. We get the following short exact sequence of chain complexes:

$$0 \longrightarrow (\Omega^{\bullet-1}(A, A'); -d^{\bullet-1}) \xrightarrow{i} (\Omega^\bullet(\rho, \rho'); d^\bullet) \xrightarrow{\pi} (\Omega^\bullet(X, X'); d^\bullet) \longrightarrow 0,$$

where $i(\eta) = (0, \eta)$ and $\pi(\omega, \eta) = \omega$. We denote by $H^\bullet_{\mathrm{dR}}(\rho, \rho')$ the cohomology of $\Omega^\bullet(\rho, \rho')$, i.e., the de-Rham cohomology of (ρ, ρ').
For any object $(\rho, \rho') : (A, A') \to (X, X')$ of $(\mathcal{M}'_2)_2$, we call $\mathrm{ch} : h^\bullet(\rho, \rho') \to H^\bullet((\rho, \rho'); \mathfrak{h}^\bullet_\mathbb{R})$ the generalized Chern character (HOPKINS; SINGER, 2005) [sec. 4.8 p. 383].

Definition 5. A *differential extension on maps of pairs* of $h^\bullet$ is a functor $h^\bullet : (\mathcal{M}'_2)_2^{\mathrm{op}} \to \mathcal{A}_\mathbb{Z}$ together with the following natural transformation of $\mathcal{A}_\mathbb{Z}-$valued functor:

- $I : \hat{h}^\bullet(\rho, \rho') \to h^\bullet(\rho, \rho')$;
- $R : \hat{h}^\bullet(\rho, \rho') \to \Omega^\bullet_{\mathrm{cl}}((\rho, \rho'); \mathfrak{h}^\bullet_\mathbb{R})$ called curvatura;
- $a : \Omega^{\bullet-1}((\rho, \rho'); \mathfrak{h}^\bullet_\mathbb{R})/\mathrm{Im}(d) \to \hat{h}^\bullet(\rho, \rho')$,

such that:

*A*1. $R \circ a = d$;

*A*2. the following diagram is commutative:

$$\begin{tikzcd}
\hat{h}^\bullet(\rho,\rho') \arrow[r, "I"] \arrow[d, "R"] & h^\bullet(\rho,\rho') \arrow[d, "\mathrm{ch}"] \\
\Omega^\bullet_{\mathrm{cl}}((\rho,\rho');\mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[r, "dR"] & H^\bullet_{\mathrm{dR}}((\rho,\rho');\mathfrak{h}^\bullet_{\mathbb{R}});
\end{tikzcd}$$

*A*3. the following sequence is exact:

$$h^{\bullet-1}(\rho,\rho') \xrightarrow{\mathrm{ch}} \Omega^{\bullet-1}((\rho,\rho');\mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \xrightarrow{a} \hat{h}^\bullet(\rho,\rho') \xrightarrow{I} h^\bullet(\rho,\rho') \to 0;$$

*A*4. calling $\mathrm{cov}(\rho,\rho')$ the second component of the curvature $R(\rho,\rho')$ and π is the natural morphism from $(\varnothing,\varnothing) \to (X,X')$ to $(A,A') \to (X,X')$, the following diagram is commutative:

$$\begin{tikzcd}
\hat{h}^\bullet(\rho,\rho') \arrow[r, "\pi^*"] \arrow[d, "\mathrm{cov}"] & \hat{h}^\bullet(X,X') \arrow[d, "\rho^*"] \\
\Omega^{\bullet-1}(X,X') \arrow[r, "a"] & \hat{h}^\bullet(A,A'),
\end{tikzcd}$$

where $\hat{h}^\bullet(X,X')$ denotes the cohomology of $(\varnothing,\varnothing) \to (X,X')$.

We also call $\hat{h}^\bullet$ differential cohomology theory on maps of pairs.

When A' and X' are empty, we recover the usual notion of relative differential cohomology on maps of spaces (RUFFINO; BARRIGA, 2018). If A is empty too, then we get differential cohomology on absolute spaces (BUNKE; SCHICK, 2010).

Definition 6. A class $\hat{\alpha} \in \hat{h}^\bullet(\rho,\rho')$ is called flat if $R(\hat{\alpha}) = 0$ and it is called *parallel* if $\mathrm{cov}(\hat{\alpha}) = 0$. We denote by $\hat{h}^\bullet_{\mathrm{fl}}(\rho,\rho')$ and $\hat{h}^\bullet_{\mathrm{par}}(\rho,\rho')$ the subgroups respectively of flat and parallel classes.

2.2.2 Parallel classes.

When dealing with parallel classes, we use the following notation:

- $\Omega^\bullet_0(\rho,\rho')$ is the sub-group of $\Omega^\bullet(X,X')$ containing the forms ω such that $\rho^*\omega = 0$;
- $\Omega^\bullet_{\mathrm{cl},0}(\rho,\rho')$ is the intersection between $\Omega^\bullet_0(\rho,\rho')$ and $\Omega^\bullet_{\mathrm{cl}}(X,X')$;
- $\Omega^\bullet_{\mathrm{ch},0}(\rho,\rho')$ is the subgroup of $\Omega^\bullet_{\mathrm{cl},0}(\rho,\rho')$ containing the forms ω such that the relative cohomology class $[(\omega,0)] \in H^\bullet_{\mathrm{dR}}(\rho,\rho')$ belongs to the image of the Chern character.

If $(\omega, 0)$ is the curvature of a parallel class, then $\omega \in \Omega^{\bullet}_{\mathrm{ch},0}(\rho, \rho')$. We get the functor $\hat{h}^{\bullet}_{\mathrm{par}} : (\mathcal{M}'_2)_2^{\mathrm{op}} \to \mathcal{A}_{\mathbb{Z}}$, together with the following natural transformations of $\mathcal{A}_{\mathbb{Z}}$-valued functors:

- $I' : \hat{h}^{\bullet}_{\mathrm{par}}(\rho, \rho') \to h^{\bullet}(\rho, \rho')$, which is the restriction of the functor I;
- $R' : \hat{h}^{\bullet}_{\mathrm{par}}(\rho, \rho') \to \Omega^{\bullet}_{\mathrm{cl},0}((\rho, \rho'); \mathfrak{h}^{\bullet}_{\mathbb{R}})$, which is the first component of the curvature R;
- $a' : \Omega_0^{\bullet-1}((\rho, \rho'); \mathfrak{h}^{\bullet}_{\mathbb{R}})/\mathrm{Im}(d) \to \hat{h}^{\bullet}_{\mathrm{par}}(\rho, \rho')$, defined by $a(\omega) := a(\omega, 0)$.

Parallel classes are well-behaved when ρ is a closed embedding. In this case they satisfy four properties analogous to axioms $(A1) - (A4)$ in definition 1 or 5. We remark that, in the case of a closed embedding, parallel classes on $(A, A') \hookrightarrow (X, X')$ are equivalent to the ones on $(X, A \cup X')$, but $(X, A \cup X')$ is not a submanifold in general. Actually, we will see that this is not a problem since we can deal with a finite union of submanifolds as well, but we can avoid this little generalization at this stage, since in the following sections, we will only need to deal with parallel classes on pairs, not on maps of pairs.

All of the main properties and results about relative differential cohomology stated in chapters $2 - 5$ of (RUFFINO; BARRIGA, 2018), can be reproduced in the framework of maps of pairs adapting the proofs in the straightforward way, as an example, due to the relevance in this thesis, we state two results.

Proposition 1. *For any smooth map of manifold pairs $(\rho, \rho') : (A, A') \to (X, X')$, we have the following short exact sequence:*

$$0 \longrightarrow \hat{h}_{fl}(\rho, \rho') \longrightarrow \hat{h}_{par}(\rho, \rho') \xrightarrow{R'} \Omega^{\bullet}_{ch,0}(\rho, \rho') \longrightarrow 0.$$

Proof. It is enough to prove that R' is surjective. Let $\omega \in \Omega^{\bullet}_{\mathrm{ch},0}(\rho, \rho')$, by definition of $\Omega^{\bullet}_{\mathrm{ch},0}(\rho, \rho')$ there is class $\alpha \in h^{\bullet}(\rho, \rho')$ such that $\mathrm{ch}(\alpha) = [(\omega, 0)]$, from the sequence of axiom $A3$, the morphism I is surjective, then there is a class $\hat{\alpha} \in \hat{h}^{\bullet}(\rho, \rho')$ such that $I(\hat{\alpha}) = \alpha$, axiom $A2$ shows that $[R(\hat{\alpha})] = \mathrm{ch}(I(\hat{\alpha})) = [(\omega, 0)]$, so that there is a form $(\omega', \eta') \in \Omega^{\bullet}(\rho, \rho')$ such that $R(\hat{\alpha}) = (\omega, 0) + d(\omega', \eta')$, by axiom $A1$ we have $R(\hat{\alpha} - a(\omega', \eta')) = (\omega, 0)$. Thus, there is a parallel class $\hat{\alpha} - a(\omega', \eta')$ such that $R'(\hat{\alpha} - a(\omega', \eta')) = \omega$. ☐

Proposition 2. *If $\hat{h}^{\bullet}$ is differential cohomology theory on maps of pairs with S^1- integration, then the flat theory $\hat{h}^{\bullet}_{fl}$ is a cohomology theory on $(\mathcal{M}_2)_2$.*

2.3 Relative-parallel product.

Assuming that the theory $h^{\bullet}$ is multiplicative, we require that the topological product (1.15) be refined in the differential framework to the following product between relative and parallel classes:

$$\times : \hat{h}^\bullet(\rho) \otimes_{\mathbb{Z}} \hat{h}^\bullet_{\mathrm{par}}(Y, B) \to \hat{h}^\bullet(\rho \times \mathrm{id}_{(Y,B)}) := \hat{h}^\bullet(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_B), \tag{2.4}$$

in such a way that:

- $I(\alpha \times \beta) = I(\alpha) \times I'(\beta)$;
- $R(\alpha \times \beta) = R(\alpha) \wedge R'(\beta)$;
- $\alpha \times a'(\omega) = a(R(\alpha) \wedge \omega)$;
- $a(\omega, \eta) \times \beta = a((\omega, \eta) \wedge R'(\beta))$.

This is the product that we have been stating in the introduction. With this formulation we can see that given a smooth map $\rho : A \to X$ and smooth manifold pair (Y, B), the map of pair $\rho \times \mathrm{id}_{(Y,B)}$ is a smooth map of pair; moreover, given a relative class $\hat{\alpha} \in \hat{h}^\bullet(\rho)$ and a parallel class $\hat{\beta} \in \hat{h}^\bullet_{\mathrm{par}}(Y, B)$ their curvatures $R(\hat{\alpha})$ and $R'(\hat{\beta})$ can be multiplied through the wedged product

$$\wedge : \Omega^\bullet(\rho) \otimes_{\mathbb{Z}} \Omega^\bullet(Y, B) \to \Omega^\bullet(\rho \times \mathrm{id}_{(Y,B)}) := \Omega^\bullet(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_B)$$
$$(\omega, \eta) \wedge \omega' := (\omega \wedge \omega', \eta \wedge \omega').$$

This is the most relevant formulation for the applications that we are going to show; however, we cannot achieve a complete axiomatization since the condition of associativity could not hold, e.g., let $\hat{\alpha} \in \hat{h}^\bullet(\rho)$, $\hat{\beta} \in \hat{h}^\bullet_{\mathrm{par}}(Y, B) \simeq \hat{h}^\bullet_{\mathrm{par}}(\iota : Y \hookrightarrow B)$ and $\hat{\gamma} \in \hat{h}^\bullet_{\mathrm{par}}(Z, C)$, we have $\hat{\beta} \times \hat{\gamma} \in \hat{h}_{\mathrm{par}}(\iota \times \mathrm{id}_{(Y,B)})$ but $\hat{\alpha} \times (\hat{\beta} \times \hat{\gamma}) \in \hat{h}(?)$.

Actually, in order to find a suitable axiomatization, as we anticipated in the previous section, we will need to further generalize the product (2.4). In fact, it can be generalized refining the topological product (1.17) as follows

$$\times : \hat{h}^\bullet((\rho, \vec{\rho}_n)) \otimes_{\mathbb{Z}} \hat{h}^\bullet_{\mathrm{par}}(Y, \vec{B}_m) \to \hat{h}^\bullet((\rho, \vec{\rho}_n) \times \mathrm{id}_{(Y, \vec{B}_m)}) = \hat{h}^\bullet(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_{\vec{B}_m}, \vec{\rho}_n \times \mathrm{id}_Y). \tag{2.5}$$

Notice that, in particular, for $n = 0$ and $m = 1$ we get (2.4). Now, we can state the axioms.

Definition 7. A *multiplicative relative differential extension* of $h^\bullet$ is a relative differential extension $(\hat{h}^\bullet, R, I, a)$ such that, there is an external product on $\hat{h}^\bullet(\rho, \vec{\rho}_n)$ over $\hat{h}_{\mathrm{par}}(Y, \vec{B}_m)$, of the form (2.5), in such a way that

(M_1) *Mixed associativity* : for every $\hat{\alpha} \in h^\bullet(\rho, \vec{\rho}_n)$, $\hat{\beta} \in \hat{h}^\bullet_{\mathrm{par}}(Y, B)$ and $\hat{\gamma} \in h^\bullet_{\mathrm{par}}(Z, C)$ we have that

$$(\hat{\alpha} \times \hat{\beta}) \times \hat{\gamma} = \hat{\alpha} \times (\hat{\beta} \times \hat{\gamma}).$$

The product $\hat{\beta} \times \hat{\gamma}$ is given on the cohomology of $(Y \times Z, Y \times C, B \times Z)$.

(M_2) *Graded-commutativity on parallel classes*: for every $\hat{\alpha} \in \hat{h}^{\bullet}_{\text{par}}(X, A)$ and $\hat{\beta} \in \hat{h}^{\bullet}_{\text{par}}(Y, B)$ we have

$$\hat{\beta} \times \hat{\alpha} = (-1)^{nm} \hat{\alpha} \times \hat{\beta},$$

up to the canonical identification $(X \times Y, X \times B, A \times Y)$, $(Y \times X, Y \times A, B \times X)$, the two terms $Y \times A$ and $B \times X$ being interchangeable.

(M_3) *Distributivity* : $(\hat{\alpha} + \hat{\beta}) \times \hat{\gamma} = (\hat{\alpha} \times \hat{\gamma}) + (\hat{\beta} \times \hat{\gamma})$ and $\hat{\alpha} \times (\hat{\beta} + \hat{\gamma}) = (\hat{\alpha} \times \hat{\beta}) + (\hat{\alpha} \times \hat{\gamma})$ whenever these operations make sense.

(M_4) *Unitarity* : There exists the unit element $1 \in \hat{h}^0(pto)$.

(M_5) *Compatibility with the natural transformations of* $\hat{h}^{\bullet}$: The following identities hold:

- $I(\alpha \times \beta) = I(\alpha) \times I'(\beta)$;
- $R(\alpha \times \beta) = R(\alpha) \wedge R'(\beta)$;
- $\alpha \times a'(\omega) = a(R(\alpha) \wedge \omega)$;
- $a(\omega, \eta) \times \beta = a((\omega, \eta) \wedge R'(\beta))$,

Notice that, given two manifold pairs (X, A) and (Y, B), we could have the following products

$$\hat{h}^{\bullet}_{\text{par}}(X, A) \otimes_{\mathbb{Z}} \hat{h}^{\bullet}_{\text{par}}(Y, B) = \hat{h}^{\bullet}_{\text{par}}(A \hookrightarrow B) \otimes_{\mathbb{Z}} \hat{h}^{\bullet}_{\text{par}}(Y, B) \to \hat{h}_{\text{par}}((A \times Y, A \times B) \to (X \times Y, X \times B))$$

$$\hat{h}^{\bullet}_{\text{par}}(X, A) \otimes_{\mathbb{Z}} \hat{h}^{\bullet}_{\text{par}}(Y, B) = \hat{h}^{\bullet}_{\text{par}}((\varnothing, \varnothing) \to (X, A)) \otimes_{\mathbb{Z}} \hat{h}^{\bullet}_{\text{par}}(Y, B) \to \hat{h}_{\text{par}}(X \times Y, X \times B, A \times Y);$$

and apparently associativity would not make sense; however, it is not a problem since it is possible to prove that

$$\hat{h}_{\text{par}}((A \times Y, A \times B) \to (X \times Y, X \times B)) \simeq \hat{h}_{\text{par}}(X \times Y, X \times B, A \times Y),$$

so, axiom M_1 makes sense. Moreover, given a manifold triple (X, A, B), the following isomorphism holds:

$$\hat{h}_{\text{par}}(X, A, B) \simeq \hat{h}_{\text{par}}(X, B, A),$$

thus, axiom M_2 makes sense.

3 Differential characters

In this chapter the fixed cohomology theory $h^\bullet$ will be ordinary cohomology, i.e., singular cohomology with integral coefficients that we denote by $H^\bullet$. We consider its differential extension $\hat{H}^\bullet$, realized through the model of differential characters.

3.1 Review on relative differential characters

We begin by briefly reviewing the definition of relative differential characters given in (BäR; BECKER, 2014), (BRIGHTWELL; TURNER, 2004) and (ZUCCHINI, 2003). The definition and properties of differential characters on the absolute case is given in (CHEEGER; SIMONS, 1985), the original source.

Given a smooth map of manifolds $\rho : A \to X$, we call $C^{\mathrm{sm}}_\bullet(\rho)$ the corresponding graded group of smooth singular chains, defined through the mapping cone. This means that $C^{\mathrm{sm}}_\bullet(\rho) = C^{\mathrm{sm}}_\bullet(X) \oplus C^{\mathrm{sm}}_{\bullet-1}(A)$ as a group, with boundary $\partial(\alpha, \beta) := (\partial\alpha + \rho_*\beta, -\partial\beta)$. We denote by $Z^{\mathrm{sm}}_\bullet(\rho)$ and $B^{\mathrm{sm}}_\bullet(\rho)$ the corresponding subgroups of cycles and boundaries. We use a similar notation about singular cohomology and we call $\Omega^\bullet(\rho)$ the graded group of relative differential forms, already defined above.
Given a form $(\omega, \eta) \in \Omega^n(\rho)$ and a chain $(\alpha, \beta) \in C^{\mathrm{sm}}_n(\rho)$, we define the relative integration $\int_{(\alpha,\beta)}(\omega, \eta) := \int_\alpha \omega + \int_\beta \eta$. We say that (ω, η) has *integral relative periods* if its integral over any cycle $(\alpha, \beta) \in Z^{\mathrm{sm}}_\bullet(\rho)$ is an integral number (this implies that (ω, η) is closed). The set of all forms $(\omega, \eta) \in \Omega^\bullet(\rho)$ with integral relative periods is denote by $\Omega^\bullet_0(\rho)$.

Definition 8. A *differential character* of degree n on ρ is a pair $(\chi, (\omega, \eta))$ such that:

i) $\chi : Z^{\mathrm{sm}}_{n-1}(\rho) \to \mathbb{R}/\mathbb{Z}$ is a group morphism;

ii) $(\omega, \eta) \in \Omega^n(\rho)$ has integral relative periods;

iii) on a boundary $\partial(\alpha, \beta) \in B^{\mathrm{sm}}_{n-1}(\rho)$, we have that

$$\chi(\partial(\alpha, \beta)) = \int_{(\alpha,\beta)} (\omega, \eta) \bmod \mathbb{Z}.$$

We denote by $\hat{H}^\bullet(\rho)$ the corresponding graded group. The natural transformations I, R and a are constructed as follows:

- Since $Z^{\mathrm{sm}}_{\bullet-1}(\rho)$ is a projective module, we can lift χ to $\tilde{\chi} : Z^{\mathrm{sm}}_{\bullet-1}(\rho) \to \mathbb{R}$. We define $\psi : C^{\mathrm{sm}}_\bullet(\rho) \to \mathbb{Z}$ by $(\alpha, \beta) \to \int_{(\alpha,\beta)}(\omega, \eta) - \tilde{\chi}(\partial(\alpha, \beta))$. It is easy to verify that ψ is a cocycle, hence it defines a cohomology class $[\psi] \in H^\bullet(\rho, \mathbb{Z})$. We set $I(\chi, (\omega, \eta)) := [\psi]$;

- $R(\chi(\omega,\eta)) := (\omega,\eta)$;
- Given a form $(\xi,\nu) \in \Omega^{\bullet-1}(\rho)$, we set $a(\xi,\nu) := (\chi_{(\xi,\nu)}, d(\xi,\nu))$, where we set $\chi_{(\xi,\nu)}(\alpha,\beta) := \int_{(\alpha,\beta)}(\xi,\nu) \bmod \mathbb{Z}$.

If ρ is a closed embedding and we deal with parallel characters, then $Z^{\text{sm}}_{\bullet}(\rho)$ and $\Omega^{\bullet}(\rho)$ can be equivalently replaced, respectively, by $Z^{\text{sm}}_{\bullet}(X,A)$, defined (as usual) as the graded group of cycles of the complex $C^{\text{sm}}_{\bullet}(X,A) := C^{\text{sm}}_{\bullet}(X)/C^{\text{sm}}_{\bullet}(A)$, and by $\Omega^{\bullet}(X,A) := \{\omega \in \Omega^{\bullet}(X) : \omega|_A = 0\}$.

3.2 Relative differential characters on maps of pairs

Given a smooth map of manifolds pairs $(\rho,\rho') : (A;A') \to (X,X')$, we set $C^{\text{sm}}_{\bullet}(\rho,\rho') := C^{\text{sm}}_{\bullet}(X,X') \oplus C^{\text{sm}}_{\bullet-1}(A,A')$ the graded group of smooth singular chains of pairs with boundary, defined naturally by $\partial_\rho(\alpha,\beta) := (\partial\alpha + \rho^*\beta, -\partial\beta)$. We denote by $Z^{\text{sm}}_{\bullet}(\rho,\rho')$ and $B^{\text{sm}}_{\bullet}(\rho,\rho')$ the corresponding subgroups of cycles and boundaries. We use a similar notation about singular cohomology, and we call $\Omega^{\bullet}(\rho,\rho')$ the graded group of relative differential forms defined as $\Omega^{\bullet}(\rho,\rho') := \Omega^{\bullet}(X,X') \oplus \Omega^{\bullet}(A,A')$ with coboundary $d(\omega,\eta) := (d\omega, \rho^*\omega - d\eta)$.

The definition of relative integration and integral periods is similar to the case of maps of manifolds.

Definition 9. A *differential character* of degree n on (ρ,ρ') is a pair $(\chi,(\omega,\eta))$ such that:

i) $\chi : Z^{\text{sm}}_{n-1}(\rho,\rho') \to \mathbb{R}/\mathbb{Z}$ is a group morphism;

ii) $(\omega,\eta) \in \Omega^n(\rho,\rho')$ has integral relative periods;

iii) on a boundary $\partial(\alpha,\beta) \in B^{\text{sm}}_{n-1}(\rho,\rho')$, we have that

$$\chi(\partial(\alpha,\beta)) = \int_{(\alpha,\beta)}(\omega,\eta) \bmod \mathbb{Z}. \tag{3.1}$$

We denote by $\hat{H}^n(\rho,\rho')$ the corresponding Abelian group of differential characters.

The natural transformations I, R and a are defined similarly. For the definition of I it is enough to notice that $Z^{\text{sm}}_{\bullet-1}(\rho,\rho') \subset C^{\text{sm}}_{\bullet}(X,X') \oplus C^{\text{sm}}_{\bullet-1}(A,A')$ is a projective module.

Definition 10. The natural transformations are:

$$I : \hat{H}^n(\rho,\rho') \to H^n(\rho,\rho')$$
$$(\chi,(\omega,\eta)) \mapsto \left[\psi^{\tilde{\chi}}\right];$$

$$\begin{aligned} R : \hat{H}^n(\rho, \rho') &\to \Omega^n_{\mathrm{cl}}((\rho, \rho'); \mathfrak{h}^\bullet_{\mathbb{R}}) \\ (\chi, (\omega, \eta)) &\mapsto (\omega, \eta); \end{aligned}$$

$$\begin{aligned} a : \Omega^{-1}((\rho, \rho'); \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) &\to \hat{H}^n(\rho, \rho') \\ (\omega, \eta) &\mapsto (\chi_{(\omega,\eta)}, d(\omega, \eta)), \end{aligned}$$

where $\chi_{(\omega,\eta)}$ is defined by $\chi_{(\omega,\eta)}(\alpha, \beta) := \int_{(\alpha,\beta)}(\omega, \eta) \bmod \mathbb{Z}$.

Notice that since (ω, η) is closed, by the Stokes theorem we can easily verify that $\psi^{\tilde{\chi}}$ is a cocycle, thus $[\psi^{\tilde{\chi}}] \in H^n((\rho, \rho'); \mathbb{Z})$. Also this class does not depend on the choice of the lift $\tilde{\chi}$; in fact, suppose that $\tilde{\chi}$ $\tilde{\chi}'$ are two liftings of χ, and $\psi^{\tilde{\chi}}$, $\psi^{\tilde{\chi}'}$ are the associated morphisms respectively, we easily see that $\psi^{\tilde{\chi}} - \psi^{\tilde{\chi}'} = \delta(\tilde{\chi}' - \tilde{\chi})$, so $[\psi^{\tilde{\chi}}] = [\psi^{\tilde{\chi}'}]$.

We now prove that the relative singular cohomology of map of pairs $H^\bullet$ has a differential extension.

Theorem 1. *The morphisms I, R and a hold*

i) $R \circ a = d$;

ii) the following diagram

$$\begin{tikzcd} \hat{H}^\bullet(\rho, \rho') \arrow[r, "I"] \arrow[d, "R"] & H^\bullet(\rho, \rho') \arrow[d, "ch"] \\ \Omega^\bullet_0((\rho, \rho'); \mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[r, "dR"] & H^\bullet_{dR}((\rho, \rho'); \mathfrak{h}^\bullet_{\mathbb{R}}), \end{tikzcd} \tag{3.2}$$

is commutative;

iii) the sequence

$$0 \to \Omega^{\bullet-1}((\rho, \rho'); \mathfrak{h}^\bullet_{\mathbb{R}})/\Omega^{\bullet-1}_0 \xrightarrow{a} \hat{H}^\bullet(\rho, \rho') \xrightarrow{I} H^\bullet(\rho, \rho') \to 0$$

is exact;

iv) the following diagram is commutative:

$$\begin{tikzcd} \hat{H}^\bullet(\rho, \rho') \arrow[r, "\pi^*"] \arrow[d, "cov"] & \hat{H}^\bullet(X, X') \arrow[d, "\rho^*"] \\ \Omega^{\bullet-1}_{cl}(A, A') \arrow[r, "a"] & \hat{H}^\bullet(A, A') \end{tikzcd}$$

Proof.

- The first part of the theorem is easy to see. In fact, given $(\omega, \eta) \in \Omega^{\bullet-1}(\rho, \rho')$ and $(\alpha, \beta) \in C_\bullet((\rho, \rho'); \mathbb{Z})$, we have

$$a(\omega, \eta)(\partial(\alpha, \beta)) = \int_{\partial(\alpha,\beta)} (\omega, \eta) \bmod \mathbb{Z} = \int_{(\alpha,\beta)} d(\omega, \eta) \bmod \mathbb{Z},$$

thus $R \circ a(\omega, \eta) = d(\omega, \eta)$.

- Given $(\chi, (\omega, \eta)) \in \hat{H}^\bullet(\rho, \rho')$, we have:

$$\mathrm{ch} \circ I((\chi, (\omega, \eta))) = \mathrm{ch}[(\omega, \eta) - \delta\tilde{\chi}] = [(\omega, \eta)]_{\mathrm{dR}} = \mathrm{dR} \circ R((\chi, (\omega, \eta))).$$

- Exactness at $H(\rho, \rho')$. Let $[\alpha] \in H^\bullet((\rho, \rho'); \mathbb{Z})$, and consider the class $i([\alpha]) = [\alpha'] \in H^\bullet((\rho, \rho'); \mathbb{R})$ induced by the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{R}$, by the De Rham's theorem there is a form (ω, η) such that $[\int(\omega, \eta)] = [\alpha']$; thus, for some $\alpha'' : C^{\mathrm{sm}}_{\bullet-1}(\rho, \rho') \to \mathbb{R}$ we have $\int(\omega, \eta) - \alpha' = \delta\alpha''$. Denote by $\tilde{\chi} := \alpha''|_{Z^{\mathrm{sm}}_{\bullet-1}(\rho,\rho')} : Z^{\mathrm{sm}}_{\bullet-1}(\rho, \rho') \to \mathbb{R}$ the restriction of α'' to the cycles, clearly it holds $\tilde{\chi}(\partial c) = \int_c(\omega, \eta) - \alpha(c)$. Let $\chi : Z^{\mathrm{sm}}_{\bullet-1}(\rho) \to \mathbb{R}/\mathbb{Z}$ be the composing of $\tilde{\chi}$ with the projection $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, we have that $I(\chi, (\omega, \eta)) = [\alpha]$. In fact, first, notice that, since α takes values in $\mathbb{Z}$, then

$$\chi(\partial c) = \tilde{\chi}(\partial c) \bmod \mathbb{Z} = \int_c (\omega, \eta) \bmod \mathbb{Z},$$

i.e, $(\chi, (\omega, \eta))$ is a differential character on (ρ, ρ'); moreover,

$$I(\chi)(c) = \int_c (\omega, \eta) - \left(\int_c (\omega, \eta) - \alpha(c)\right) = \alpha(c);$$

thus, I is surjective.
Exactness at $\hat{h}^\bullet(\rho, \rho')$. Given $(\omega, \eta) \in \Omega^{\bullet-1}((\rho, \rho'); \mathfrak{h}^\bullet_{\mathbb{R}})/\Omega^{\bullet-1}_0$, it is easy to see that $I \circ a(\omega, \eta) = 0$. On the other hand, let $(\chi, (\omega, \eta)) \in \hat{H}^\bullet(\rho, \rho')$ be a differential character such that $I(\chi, (\omega, \eta)) = 0$, then we have that $\int(\omega, \eta) = \delta\tilde{\gamma}$ for some $\tilde{\gamma} : C^{\mathrm{sm}}_{\bullet-1}(\rho) \to \mathbb{R}$, by the isomorphism of De Rham there is a form $(\xi, \nu) \in \Omega^{\bullet-1}(\rho)$ such that $\int(\xi, \nu) = \tilde{\gamma}$, thus

$$a(\xi, \nu)(\partial(\alpha, \beta)) = \int_{\partial(\alpha,\beta)} (\xi, \nu) = \tilde{\gamma}(\partial(\alpha, \beta)) = \int_{(\alpha,\beta)} (\omega) = \chi(\partial(\alpha, \beta)).$$

-
$$(a \circ \mathrm{cov})(\chi)(\beta) = \int_{(0,\beta)} (0, \eta) \bmod \mathbb{Z} = \chi(\partial(0, \beta)) = \chi(\rho_*(\beta), 0)$$
$$(\rho^* \circ \pi^*)(\chi)(\beta) = \rho^*(\pi^*(\chi))(\beta) = \pi^*(\chi)(\rho_*(\beta)) = \chi(\rho_*(\beta), 0)$$

□

3.2.1 Relative-parallel product

In order to construct the product (2.4), in appendix *A.2*, we generalize the splitting of the Künneth sequence, described in (BäR; BECKER, 2014) [section 4.2.1], that we need here.

Given $(\chi, (\omega, \eta)) \in \hat{H}^n(\rho)$ and $(\chi', \omega') \in \hat{H}^m_{\mathrm{par}}(Y, B)$, we define a group morphism

$$\chi \times \chi' : Z^{\mathrm{sm}}_{n+m-1}\left(\rho \times \mathrm{id}_{(Y,B)}\right) \to \mathbb{R}/\mathbb{Z}$$

in the following way. Thanks to (A.11), a cycle $(\alpha_{n+m-1}, \beta_{n+m-2}) \in Z_{n+m-1}(\rho \times \mathrm{id}_{(Y,B)})$ can be decomposed as $(\alpha_{n+m-1}, \beta_{n+m-2}) = x + (\gamma_{n+m-1}, \delta_{n+m-2})$, in such a way that:

- x is a sum of cycles of the form $(\mu_k, \nu_{k-1}) \otimes \mu'_h$, where $(\mu_k, \nu_{k-1}) \in Z_k(\rho)$, $\mu'_h \in Z_h(Y, B)$ and $k + h = n + m - 1$;

- $(\gamma_{n+m-1}, \delta_{n+m-2})$ represents a torsion homology class.

For this reason, it is enough to evaluate the product $\chi \times \chi'$ on cycles of the form $(\mu_k, \nu_{k-1}) \otimes \mu'_h$, such that $k + h = n + m - 1$, and on cycles of degree $n + m - 1$ representing a torsion class. Formally, we set:

$$(\chi \times \chi')((\mu_k, \nu_{k-1}) \otimes \mu'_h) := \begin{cases} \chi(\mu_{n-1}, \nu_{n-2}) \cdot I(\chi')(\mu'_m) & \text{if } k = n-1,\ h = m \\ I(\chi)\left((-1)^n(\mu_n, \nu_{n-1})\right) \cdot \chi'(\mu'_{m-1}) & \text{if } k = n,\ h = m-1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

In the latter case, supposing that $N \in \mathbb{N}$ satisfies $N(\gamma_{n+m-1}, \delta_{n+m-2}) = \partial(A_{n+m}, B_{n+m-1})$, we choose a cocycle $\psi \in Z^{n+m}(\rho \times \mathrm{id}_{(Y,B)})$ representing $I(\chi) \times I(\chi')$ and we set

$$\chi \times \chi'(\gamma_{n+m-1}, \delta_{n+m-2}) := \left(\frac{1}{N}\left(\int_{(A_{n+m}, B_{n+m-1})} (\omega, \eta) \wedge \omega' - \psi(A_{n+m}, B_{n+m-1})\right)\right) \bmod \mathbb{Z}. \tag{3.4}$$

Notice that the value of $\chi \times \chi'$ on a torsion cycle $(\gamma_{n+m-1}, \delta_{n+m-2})$ is independent of the choice of cocycle ψ, the chain (A_{n+m}, B_{n+m-1}) and of $N \in \mathbb{N}$.
In fact, suppose that $\psi' \in Z^{n+m}(\rho \times \mathrm{id}_{(Y,B)})$ is another cocycle that represents $I(\chi) \times I(\chi')$, then $\psi - \psi' = \delta l$ for some $l \in C^{n+m-1}(\rho \times \mathrm{id}_{(Y,B)}, \mathbb{Z})$, now notice that

$$\frac{1}{N} \delta l(A_{n+m}, B_{n+m-1}) = l(\gamma_{n+m-1}, \delta_{n+m-2}) \in \mathbb{Z}.$$

Now suppose that $N(\gamma_{n+m-1}, \delta_{n+m-2}) = \partial(A'_{n+m}, B'_{n+m-1})$, since $(A'', B'') = (A_{n+m}, B_{n+m-1}) - (A'_{n+m}, B'_{n+m-1}) \in Z_{n+m}(\rho \times \mathrm{id}_{(Y,B)})$ we have

$$\left(\int_{(A'', B'')} (\omega, \eta) \wedge \omega'\right) - \psi(A'', B'') = 0.$$

With the notations above we have

Proposition 3. *The pair $(\chi \times \chi', (\omega, \eta) \wedge \omega')$ is a differential character of degree $n + m$ on $\rho \times id_{(Y,B)}$, i. e. $(\chi \times \chi', (\omega, \eta) \wedge \omega') \in \hat{H}^{n+m}(\rho \times id_{(Y,B)})$.*

Proof. We easily to see that $(\omega, \eta) \wedge \omega'$ has integral relative periods. Let us see that condition (3.1) holds. Let $(\mu_k, \nu_{k-1}) \otimes \mu'_h \in Z_k(\rho) \otimes Z_h(Y, B)$ be a cycle with $k + h = n + m - 1$, since a cross product of cycles is a boundary if and only if one of the factors is a boundary, we may suppose that

- for $k = n - 1$ and $h = m$,

– if $(\mu_k, \nu_{k-1}) = \partial(\alpha_{k+1}, \beta_k) \in Z_{n-1}(\rho)$, then we have

$$\begin{aligned}
\chi \times \chi'(\partial((\alpha_{k+1}, \beta_k) \otimes \mu'_h)) &= \chi \times \chi'(\partial(\alpha_{k+1}, \beta_k) \otimes \mu'_h) \\
&= \chi(\partial(\alpha_n, \beta_{n-1})) \cdot I(\chi')(\mu'_m) \\
&= \int_{(\alpha_n, \beta_{n-1})} (\omega, \eta) \bmod \mathbb{Z} \cdot \int_{\mu'_m} \omega' \\
&= \int_{(\alpha_{k+1}, \beta_k) \otimes \mu'_h} (\omega, \eta) \wedge \omega' \bmod \mathbb{Z},
\end{aligned}$$

– if $\mu'_h = \partial\gamma_{h+1} \in Z_m(Y, B)$, we get

$$\begin{aligned}
\chi \times \chi'(\partial((\mu_k, \nu_k) \otimes \gamma_{h+1})) &= \chi \times \chi' \left((-1)^{h-1}(\mu_k, \nu_{k-1}) \otimes \partial\gamma_{h+1}\right) \\
&= \chi\left((-1)^{n-1}(\mu_{n-1}, \nu_{n-2})\right) \cdot I(\chi')(\partial\gamma_{m+1}) \\
&= 0 \\
&= \int_{(\mu_k, \nu_{k-1}) \otimes \gamma_{h+1}} (\omega, \eta) \wedge \omega' \bmod \mathbb{Z},
\end{aligned}$$

notice that since ω' is closed, $I(\chi')(\partial\gamma_{h+1}) = 0$ and, due to the degree of γ_{h+1} we have $\int_{\gamma_{h+1}} \omega' = 0$, thus the last term is 0.

Similarly for $k = n$, $h = m - 1$ and otherwise.

- Now, let $(\gamma_{n+m-1}, \delta_{n+m-2}) \in T_{n+m-1}(\rho \times \mathrm{id}_{(Y,B)})$, supposing that $N \in \mathbb{N}$ satisfies $N(\gamma_{n+m-1}, \delta_{n+m-2}) = \partial(A_{n+m}, B_{n+m-1})$, from (3.4) we easily have that

$$\chi \times \chi'(\partial(A_{n+m}, B_{n+m-1})) = \int_{(A_{n+m}, B_{n+m-1})} (\omega, \eta) \wedge \omega' \bmod \mathbb{Z}$$

□

Theorem 2. *The exterior product between the relative and parallel differential characters groups*

$$\times : \hat{H}^n(\rho) \otimes_{\mathbb{Z}} \hat{H}^m_{par}(Y, B) \to \hat{H}^{n+m}(\rho \times id_{(Y,B)})$$

holds: the natural morphisms I, R and a are multiplicative, i.e.,

i) $I(\chi \times \chi', (\omega, \eta) \wedge \omega') = I(\chi, (\omega, \eta)) \times I(\chi', \omega')$;

ii) $R(\chi \times \chi', (\omega, \eta) \wedge \omega') = R(\chi, (\omega, \eta)) \times R(\chi', \omega')$;

iii) $a(\omega, \eta) \times (\chi', \omega') = a((\omega, \eta) \wedge R(\chi', \omega'))$;

iv) $(\chi, (\omega, \eta)) \times a(\omega') = a(R(\chi, (\omega, \eta)) \wedge \omega')$.

3.2.2 Product of parallel characters

When ρ is a closed embedding, we can construct in a more direct way the product of parallel characters, leading to a ring structure on them. In this setting, we can generalize manifold pairs to pairs of the form $(X, A_1 \cup \cdots \cup A_n)$, where X is a smooth manifolds and $A_1, \cdots, A_n$ are smooth embedded sub-manifolds. In fact, we define the relative de-Rham complex $\Omega^\bullet(X, A_1 \cup \cdots \cup A_n)$ as the one made by forms $\omega \in \Omega^\bullet(X)$ such that $\omega|_{A_i} = 0$ for every i, and this is enough not to have troubles with the differential structure.

Thanks to (A.13), a cycle $\alpha_n \in Z_n(X \times Y, (X \times B) \cup (A \times Y))$ can be decomposed as $\alpha_n = x + \gamma_n$, similarly to the general case, where x is a sum of cycles of the form $\mu_k \otimes \mu'_h$. Hence, given $(\chi, \omega) \in H^n_{\mathrm{par}}(X, A)$ and $(\chi', \omega') \in H^m_{\mathrm{par}}(Y, B))$, it is enough to evaluate the product $\chi \times \chi'$ on cycles of the form $\mu_k \otimes \mu'_h$, such that $k + h = n + m - 1$, and on cycles of degree $n + m - 1$ representing a torsion class. In the former case, we set:

$$(\chi \times \chi')(\mu_k \otimes \mu'_h) := \begin{cases} \chi(\mu_{n-1}) \cdot I(\chi')(\mu'_m) & \text{if } k = n-1,\ h = m \\ I(\chi)(\mu_n) \cdot \chi'(\mu'_{m-1}) & \text{if } k = n,\ h = m-1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

In the latter case, supposing that $N \in \mathbb{N}$ satisfies $N\gamma_{n+m-1} = \partial A_{n+m}$, we choose a cocycle $\psi \in Z^{n+m}(X \times Y$ representing $I(\chi) \times I(\chi')$ and we set

$$\chi \times \chi'(\gamma_n) := \frac{1}{N}\left(\left(\int_{(A_{n+m}} \omega \wedge \omega'\right) - \psi(A_{n+m})\right). \tag{3.6}$$

Applying the pull-back through the diagonal embedding $\triangle\colon (X, A), \hookrightarrow (X \times X, (X \times A) \cup (A \times X))$, we get a ring structure on $\hat{H}^\bullet_{\mathrm{par}}(X, A)$.

4 Differential K-Theory

This part of the thesis contains three important parts. We first describe the relative Freed-Lott model, both inn the topological and differential framework. On the second one, we adapt this model to describe parallel $K-$theory. Moreover, we state a theorem that shows that parallel $K-$theory is a particular case of the relative one. In the last section we describe the Freed-Lott model for maps of pairs and the relative-parallel product through this model. We also give a sketch of the refinement of K-theory of sequences.

4.1 Relative topological K-theory

Definition 11. Let $\rho : A \to X$ be a map, a relative vector bundle on ρ is defined as the triple (E, F, ψ), where:

- E and F are complex vector bundle on X;
- $\psi : \rho^*E \xrightarrow{\simeq} \rho^*F$ is an isomorphism.

A triple of the form (E, E, id) is called elementary.

Definition 12.
- A morphism between relative vector bundles (E, F, ψ) and (E', F', ψ') on ρ, is a pair $(f, g) : (E, F, \psi) \to (E', F', \psi')$, where $f : E \to E'$ and $g : F \to F'$ are morphism of vector bundles such that, the following diagram commutes.

$$\begin{tikzcd} \rho^*E \arrow[r, "\psi"] \arrow[d, "\rho^*f"] & \rho^*F \arrow[d, "\rho^*g"] \\ \rho^*E' \arrow[r, "\psi'"] & \rho^*F', \end{tikzcd}$$

we say that (f, g) is an isomophism of relative fiber bundles if f and g are isomorphism of vector bundles;

- Let $(f, g) : \xi \to \rho$ be a morphism between smooth maps, and let $(E, F.\psi)$ a relative vector buble on ρ. The relative pull-back $(f, g)^*(E, F, \psi)$ is the relative vector bundle on ξ defined by

$$(f, g)^*(E, F, \psi) := (f^*E, f^*F, g^*\psi).$$

4.1.1 Relative K-theory groups

We now introduce the basic definitions and properties of Relative K-theory, referring mainly to (KAROUBI, 1978), (ATIYAH, 1994) and (HUSEMöLLER, 1994).

In the topological framework, given a map $\rho : A \to X$ between spaces with the homotopy type of a finite CW-complex, We call $\Gamma(\rho)$ the semi-group of isomorphism classes of relative vector bundle with the operation of direct sum. Moreover, we introduce the following equivalence relation on $\Gamma(\rho)$: we set $[(E, F, \psi)] \approx [(E', F', \psi')]$ if and only if there exist two elementary triples (G, G, id) and (G', G', id) such that

$$[(E, F, \psi) \oplus (G, G, \mathrm{id})] = [(E', F', \psi') \oplus (G', G', \mathrm{id})]. \tag{4.1}$$

The quotient by this equivalence relation is denoted by $K(\rho) = \Gamma(\rho)/\approx$. We get an abelian group, whose zero element is $[[(E, E, \mathrm{id})]]$, for any E, and such that $-[[(E, F, \psi)]] = [[(F, E, \psi^{-1})]]$. In the following paragraphs, we denote $[[(E, F, \psi)]]$ simply by $[(E, F, \psi)]$.

The relative $K-$theory groups of higher degree are defined as follow: let $\mathbb{T}^n$ be the $n-$dimensional torus, that is, $\mathbb{T}^n := S^1 \times \cdots \times S^1$ and let us consider the embeddings

$$i_j = (i_{Aj}, i_{Xj}) : \mathrm{id}_{\mathbb{T}^{n-1}} \times \rho \hookrightarrow \mathrm{id}_{\mathbb{T}^n} \times \rho$$

$$\begin{tikzcd}
\mathbb{T}^{n-1} \times A \arrow[r, "\mathrm{id}_{\mathbb{T}^{n-1}} \times \rho"] \arrow[d, hook, "i_{Aj}"] & \mathbb{T}^{n-1} \times X \arrow[d, hook, "i_{Xj}"] \\
\mathbb{T}^n \times A \arrow[r, "\mathrm{id}_{\mathbb{T}^n} \times \rho"] & \mathbb{T}^n \times X,
\end{tikzcd} \tag{4.2}$$

where $j = 1, \cdots, n$ and

$$\begin{aligned}
&i_{Xj} : \mathbb{T}^{n-1} \times X \to \mathbb{T}^n \times X \\
&i_{Xj}(t_1, \cdots, t_{n-1}, x) = (t_1, \cdots, , t_{j-1}, 1, t_j, \cdots, t_{n-1}, x),
\end{aligned}$$

$$\begin{aligned}
&i_{Aj} : \mathbb{T}^{n-1} \times A \to \mathbb{T}^n \times A \\
&i_{Xj}(t_1, \cdots, t_{n-1}, a) = (t_1, \cdots, , t_{j-1}, 1, t_j, \cdots, t_{n-1}, a).
\end{aligned}$$

Definition 13. The relative K-theory groups in negative degree is given by

$$K^{-n}(\rho) := \bigcap_j \mathrm{Ker}\left(i_j^* : K(\mathrm{id}_{\mathbb{T}^n} \times \rho) \to K(\mathrm{id}_{\mathbb{T}^{n-1}} \times \rho)\right).$$

We now prove that the relative $K-$theory is, in fact, a relative cohomology theory. Instead of checking all the axioms of a cohomology theory, according to remark 1, we need to prove that:

$$K^\bullet(\rho) \simeq \tilde{K}^\bullet(C_\rho). \tag{4.3}$$

First, we define the natural function $\Pi : \mathrm{Cyl}(\rho) \to X$ through the following push-out diagram, see definition 47:

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[d, "i_0"] & X \arrow[d, "j_X"] \arrow[ddr, bend left, equal, "\mathrm{id}_X"] & \\
I \times A \arrow[r, "j_{A\times I}"] \arrow[drr, bend right, "\rho\circ\pi_A"'] & \mathrm{Cyl}(\rho) \arrow[dr, dashed, "\Pi"] & \\
& & X.
\end{tikzcd} \tag{4.4}$$

We now consider the natural inclusion $\iota_1 : j_{I\times A}(\{1\} \times A) \hookrightarrow \mathrm{Cyl}(\rho)$ and the following natural maps:

$$\pi_1 : \{1\} \times A \to j_{A\times I}(\{1\} \times A), \quad i_0 : \{0\} \times A \hookrightarrow I \times A, \quad i_1 : \{1\} \times A \hookrightarrow I \times A$$

$$i_{0,1} : \{0\} \times A \to \{1\} \times A, \quad \bar{i}_0 : A \to \{0\} \times A, \quad \bar{i}_1 : A \to \{1\} \times A,$$

$$P : \{0\} \times A \to X, \text{ defined by } P(a, 0) = \rho(a).$$

We can easily verify the identities

$$j_{A\times I} \circ i_1 \circ \pi_1^{-1} = \iota_1, \quad j_{A\times I} \circ i_0 = j_X \circ P, \quad P \circ \bar{i}_0 = \rho, \tag{4.5}$$

and the homotopies

$$i_0 \circ i_{0,1}^{-1} \simeq i_1, \quad j_X \circ \Pi \simeq \mathrm{id}_{\mathrm{Cyl}(\rho)}, \quad \text{and} \quad \Pi \circ j_X = \mathrm{id}_{\mathrm{Cyl}(\rho)}. \tag{4.6}$$

Remark 5. Since A, X, and thus $\mathrm{Cyl}(\rho)$, are compact Hausdorf spaces, by (HATCHER, 2017)(Theorem 1.6) and (4.6) we get the following isomorphisms:

$$(i_{0,1}^{-1})^* i_0^* E \cong i_1^* E \quad \text{and} \quad \Pi^* j_X^* E' \cong E'$$

for any bundles $E \to I \times A$ and $E' \to \mathrm{Cyl}(\rho)$.

Theorem 3. *Let $\rho : A \to X$ be a continuous map between compact topological spaces. There exists a natural isomorphism*

$$\Phi : K(\rho) \to K(\iota_1 : j_{I\times A}(\{1\} \times A) \hookrightarrow Cyl(\rho)) = K(Cyl(\rho), \{1\} \times A).$$

Proof. Given a vector bundle E on X, we naturally get a vector bundle $\Pi^* E$ on $\mathrm{Cyl}(\rho)$, see diagram (4.4), similarly on the other direction. Considering diagram (4.4) and the natural maps, we define the following homomorphisms

$$\begin{aligned}
&\Phi : K(\rho) \to K(\iota_1) \\
&[E, F, \psi] \mapsto [\Pi^* E, \Pi^* F, (\pi_1^{-1})^* (i_{0,1}^{-1})^* (\bar{i}_0^{-1})^* \psi],
\end{aligned}$$

$$\begin{aligned}\Phi' : K(\iota_1) &\to K(\rho)\\ [E, F, \psi] &\mapsto \left[j_X^* E, j_X^* F, \bar{i}_0^* i_{0,1}^* \pi_1^* \psi\right].\end{aligned}$$

Diagram (4.7), proves that Φ' is a well-defined homomorphism, similarly for Φ. By remark 5 and the homotopies (4.6) we can easily see that Φ and Φ' are inverses of each other.

Let us see that Φ' is well defined, again, considering the natural maps and the identities (4.5) we get the following diagram:

$$\begin{tikzcd}
\iota_1^* E \arrow[r, "\psi"] \arrow[d, equal] & \iota_1^* F \arrow[r] \arrow[d, equal] & j_{A\times I}(\{1\}\times A) \arrow[d, equal] \\
(\pi_1^{-1})^* i_1^* j_{A\times I}^* E \arrow[r, "\psi\ 4.5"] \arrow[d] & (\pi_1^{-1})^* i_1^* j_{A\times I}^* F \arrow[r] \arrow[d] & j_{A\times I}(\{1\}\times A) \arrow[d, "(\pi_1)^{-1}"] \\
i_1^* j_{A\times I}^* E \arrow[r, "\pi_1^*\psi"] \arrow[d] & i_1^* j_{A\times I}^* F \arrow[r] \arrow[d] & \{1\}\times A \arrow[d, "i_{0,1}^{-1}"] \\
i_0^* j_{A\times I}^* E \arrow[r, "i_{0,1}^*\pi_1^*\psi"] \arrow[d, equal, "4.5"] & i_0^* j_{A\times I}^* F \arrow[r] \arrow[d, equal] & \{0\}\times A \arrow[d, equal] \\
P^* j_X^* E \arrow[r, "i_{0,1}^*\pi_1^*\psi"] \arrow[d] & P^* j_X^* F \arrow[r] \arrow[d] & \{0\}\times A \arrow[d, "\bar{i}_0^{-1}"] \\
\bar{i}_0^* P^* j_X^* E \arrow[r, "\bar{i}_0^* i_{0,1}^*\pi_1^*\psi"] \arrow[d, equal, "4.5"] & \bar{i}_0^* P^* j_X^* F \arrow[r] \arrow[d, equal] & A \arrow[d, equal] \\
\rho^* j_X^* E \arrow[r, "\bar{i}_0^* i_{0,1}^*\pi_1^*\psi"] & \rho^* j_X^* F \arrow[r] & A,
\end{tikzcd} \tag{4.7}$$

where all the arrows between bundles are isomorphisms and the arrows of the third column are bijectives. Thus, we have that Φ' is a well-defined homomorphism. $\square$

Corollary 4. *Let $\rho : A \to X$ be a continuous map between compact topological spaces, then*

$$K(\rho) \cong \tilde{K}(C_\rho).$$

Proof. Since $j_{I\times A}(\{1\}\times A) \subset \mathrm{Cyl}(\rho)$ is a compact subspace of $\mathrm{Cyl}(\rho)$ we know that $K(\iota_1) \cong \tilde{K}(\mathrm{Cyl}(\rho)/j_{I\times A}(\{1\}\times A)) \cong \tilde{K}(C_\rho)$, the result immediately follows from theorem 3. $\square$

Consider the inclusions $i'_j : \mathrm{id}_{\mathbb{T}^{n-1}} \times \iota_1 \hookrightarrow \mathrm{id}_{\mathbb{T}^n} \times \iota_1$, where $i'_j = (i'_{Aj}, i'_{\rho j})$

$$\begin{tikzcd}
\mathbb{T}^{n-1}\times(\{1\}\times A) \arrow[r, "\mathrm{id}_{\mathbb{T}^{n-1}}\times\iota_1"] \arrow[d, "i'_{Aj}"] & \mathbb{T}^{n-1}\times \mathrm{Cyl}(\rho) = \mathrm{Cyl}(\mathrm{id}_{\mathbb{T}^{n-1}}\times\rho) \arrow[d, "i'_{\rho j}"] \\
\mathbb{T}^{n}\times(\{1\}\times A) \arrow[r, "\mathrm{id}_{\mathbb{T}^{n}}\times\iota_1"] & \mathbb{T}^{n}\times \mathrm{Cyl}(\rho) = \mathrm{Cyl}(\mathrm{id}_{\mathbb{T}^{n}}\times\rho),
\end{tikzcd} \tag{4.8}$$

with i'_{Aj} defined similar to i_{Aj} on (4.2) and $i'_{\rho j}$ defined naturally such that the diagram

$$\begin{tikzcd}
\mathrm{Cyl}(\mathrm{id}_{\mathbb{T}^{n-1}} \times \rho) \arrow[r, "i'_{\rho j}"] \arrow[d, "\Pi"] & \mathrm{Cyl}(\mathrm{id}_{\mathbb{T}^{n}} \times \rho) \arrow[d, "\Pi"] \\
\mathbb{T}^{n-1} \times X \arrow[r, "i_{Xj}"] & \mathbb{T}^{n} \times X
\end{tikzcd}$$

commutes for $j = 1, \cdots, n$.

Proposition 4. *With the notation above, for every $j = 1, \cdots, n$ the isomorphism holds*

$$Ker(i_j^*) = Ker(i_j'^*).$$

Proof. Using proposition 3 and the morphism (4.8), we can easily see that the following diagram is also commutative

$$\begin{tikzcd}
K(\mathrm{id}_{\mathbb{T}^n} \times \rho) \arrow[r, "i_j^*"] \arrow[d, "\simeq 3"] & K(\mathrm{id}_{\mathbb{T}^{n-1}} \times \rho) \arrow[d, "\simeq 3"] \\
K(\mathrm{Cyl}(\mathrm{id}_{\mathbb{T}^n} \times \rho), \{1\} \times (\mathbb{T}^n \times A)) \arrow[r, "i_j'^*"] \arrow[d, equal] & K(\mathrm{Cyl}(\mathrm{id}_{\mathbb{T}^{n-1}} \times \rho), \{1\} \times (\mathbb{T}^{n-1} \times A)) \arrow[d, equal] \\
K(\mathbb{T}^n \times \mathrm{Cyl}(\rho), \mathbb{T}^n \times (\{1\} \times A)) \arrow[r, "i_j'^*"] & K(\mathbb{T}^{n-1} \times \mathrm{Cyl}(\rho), \mathbb{T}^{n-1} \times (\{1\} \times A)),
\end{tikzcd}$$

thus $\mathrm{Ker}(i_j^*) \cong \mathrm{Ker}(i_j'^*)$ for every $j = 1, \cdots, n$. □

Corollary 5. *For $n \geq 0$, we have the isomorphism*

$$K^{-n}(\rho) \cong K^{-n}(Cyl(\rho), \{1\} \times A).$$

Proof. By definition we have

$$K^{-n}(\mathrm{Cyl}(\rho), \{1\} \times A) \cong \bigcap_j \mathrm{Ker}\left(i_j'^* : K(\mathrm{id}_{\mathbb{T}^n} \times \iota_1) \to K(\mathrm{id}_{\mathbb{T}^{n-1}} \times \iota_1)\right),$$

then from the proposition above, we have

$$K^{-n}(\rho) :\overset{13}{=} \bigcap_j \mathrm{Ker}(i_j^*) \overset{4}{\cong} \bigcap_j \mathrm{Ker}(i_j'^*) = \bigcap_j \mathrm{Ker}\left(i_j'^* : K(\mathrm{id}_{\mathbb{T}^n} \times \iota_1) \to K(\mathrm{id}_{\mathbb{T}^{n-1}} \times \iota_1)\right).$$

□

Proposition 5. ***(Relative Bott periodicity)*** *For any map $\rho : A \to X$ between compact spaces, we have the following isomorphism*

$$K^{-n}(\rho) \simeq K^{-n-2}(\rho).$$

Proof. Since A and X are compact spaces, the cone C_ρ is also compact, so applying the Bott periodicity theorem on the absolute case and corollary (5), we get

$$\begin{aligned} K^{-n}(\rho) &\simeq \tilde{K}^{-n}(C_\rho) \\ &\simeq \tilde{K}^{-n-2}(C_\rho) \\ &\simeq K^{-n-2}(\rho). \end{aligned}$$

□

Theorem 4. *Relative K-theory is a generalized relative cohomology theory.*

Proof. From remark 1, we just need to prove the isomorphism

$$K^\bullet(\rho) \simeq \tilde{K}^\bullet(C_\rho), \tag{4.9}$$

By corollary 5 we have that for every $n \geq 0$, $K^{-n}(\rho) \simeq K^{-n}(\mathrm{Cyl}(\rho), \{1\} \times A)$, and since $\tilde{K}^{-n}(C_\rho) \simeq K^{-n}(\mathrm{Cyl}(\rho), \{1\} \times A)$, we get

$$K^{-n}(\rho) \simeq \tilde{K}^{-n}(C_\rho),$$

using the relative Bott periodicity, we obtain the isomorphism (4.9). □

4.1.2 Topological relative-absolute product

We define the relative-absolute product similar to the case when ρ is a cofibration, see (KAROUBI, 1978) 5.14, i.e., given two classes $[E, F, \psi] \in K(\rho)$ and $[E'] - [F'] \in K(Y)$, define

$$\cdot : K(\rho) \otimes_{\mathbb{Z}} K(Y) \to K(\rho \times \mathrm{id}_Y)$$

$$[E, F, \psi] \cdot ([E'] - [F']) = [E \otimes E', F \otimes E', \psi \otimes \mathrm{id}_{E'}] - [E \otimes F', F \otimes F', \psi \otimes \mathrm{id}_{F'}] \tag{4.10}$$

More generally, we can define products

$$K^{-m}(\rho) \boxtimes K^{-n}(Y) \to K^{-m-n}(\rho \times \mathrm{id}_Y) \tag{4.11}$$

for $m, n \leq 0$ as follows:
Given $\alpha \in K^{-m}(\rho) \subset K(\mathrm{id}_{\mathbb{T}^m} \times \rho)$ and $\beta \in K^{-m}(Y) \subset K(\mathbb{T}^n \times Y)$, we consider $\alpha \cdot \beta \in K(\mathrm{id}_{\mathbb{T}^m} \times \rho \times \mathrm{id}_{\mathbb{T}^n \times Y})$, where $\cdot_0$ is the exterior product in degree 0 defined in (4.10). Then considering the natural isomorphism

$$\varphi_{m,n} : \mathrm{id}_{\mathbb{T}^{m+n}} \times \rho \times \mathrm{id}_Y \to \mathrm{id}_{\mathbb{T}^m} \times \rho \times \mathrm{id}_{\mathbb{T}^n \times Y}$$

we define:

$$\alpha \cdot \beta := \varphi^*_{n,m}(\alpha \cdot_0 \beta).$$

Using both, the absolute and relative Bott periodicity, we can extend this product to any degree.

4.2 Relative Chern-Simons class

Let $E \to X$ be complex vector bundles over a smooth manifold X with connections ∇_0^E and ∇_1^E. For $I = [0,1]$, let $\pi_X : I \times X \to X$ the natural projection and consider the vector bundle $\pi_X^* E \to I \times X$ on $I \times X$.

Definition 14. We say that the connection $\nabla_{0,1}^{\pi_X^* E}$ (called path connection) interpolates ∇_0^E and ∇_1^E if its restriction to the edges $\{0\} \times X$ and $\{1\} \times X$ hold: $\nabla_{0,1}^{\pi_X^* E}|_{\{0\}\times X} = \nabla_0^E$ and $\tilde{\nabla}_{0,1}^{\pi_X^* E}|_{\{1\}\times X} = \nabla_1^E$.

Example 1. There is a canonical path connection defined by

$$\left(\nabla_{0,1}^{\pi_X^* E}{}_{(a,X)} V\right)_{(t,p)} := (1-t)(\nabla_{0\,X_p}^E V)_p + t(\nabla_{1\,X_p}^E V)_p + a\partial_t V. \tag{4.12}$$

where $(a, X) \in \mathfrak{X}(I \times X)$, $V \in \Gamma(\pi_X^* E)$ and $(t,p) \in I \times X$.

Remark 6. The example above can be generalized in the following way. Let ∇_0^E, ∇_1^E and ∇_2^E connection on E and consider the standard simplex $\Delta^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1 \text{ and } x_i \geq 0\}$ and the interval $I_{0,1} := \{(x_1.x_2, x_3) \in \Delta^2 : x_3 = 0\}$, similarly define $I_{1,2}$ and $I_{2,0}$. Let $\nabla_{0,1}^{\pi_{I_{0,1},X}^* E}$ be a path connection between ∇_0^E and ∇_1^E, where $\pi_{I_{0,1},X} : I_{0.1} \times X \to X$ is the natural projection, similarly for $\nabla_{0,1}^{\pi_{I_{1,2},X}^* E}$ and $\nabla_{0,1}^{\pi_{I_{2,0},X}^* E}$. A connection that interpolates these three connections is a connection $\hat{\nabla}$ $\pi_{\Delta^2,X}^* E$, where $\pi_{\Delta^2,X} : \Delta^2 \times X \to X$ is the natural projection, in such a way that its restriction to the edges are $\hat{\nabla}|_{I_{1,2}\times X} = \nabla_{1,2}^{\pi_X^* E}$, $\hat{\nabla}|_{I_{2,0}\times X} = \nabla_{2,0}^{\pi_X^* E}$ and $\hat{\nabla}|_{I_{0,1}\times X} = \nabla_{0,1}^{\pi_X^* E}$.

Definition 15. The *Chern character form* associated to ∇^E, is the form on $\Omega^{\mathrm{ev}}(X)$ given by

$$\mathrm{ch}\nabla^E := \mathrm{Tr}\exp\left(\frac{-1}{2\pi i} R^E\right) \in \Omega^{\mathrm{ev}}(X),$$

where R^E is the curvature of the connection ∇^E.

Proposition 6. *The Chern character form holds:*

- *$ch\nabla^E$ is a closed form;*
- *Let ∇_0^E and ∇_1^E any two connections over E, then there is a differential form $\omega \in \Omega^\bullet(X)$ such that $ch\nabla_1^E - ch\nabla_0^E = d\omega$; moreover, ω can be chosen to be the form $\int_I ch\nabla_{0,1}^{\pi_X^* E}$;*
- *$ch\nabla^{E\oplus F} = ch\nabla^E + ch\nabla^F$;*
- *$ch\nabla^{E\otimes F} = ch\nabla^E \wedge ch\nabla^F$.*

Definition 16. The Chern-Simons transgression form between two connections ∇_0^E and ∇_1^E is given and denoted by

$$\mathrm{CS}(\nabla_0^E, \nabla_1^E) := \int_I \mathrm{ch}\nabla_{0,1}^{\pi_X^* E}.$$

Its class $[\mathrm{CS}(\nabla_0^E, \nabla_1^E)] \in \Omega^{\mathrm{odd}}(X)/\mathrm{Im}(d)$ is called the *Chern-Simons class.*

The second item of the proposition above says that, the class $[\mathrm{CS}(\nabla_0^E, \nabla_1^E)] \in \Omega^{\mathrm{odd}}(X)/\mathrm{Im}(d)$ is independent of the choice of the connections ∇_0^E and ∇_1^E.

Let $\rho : A \to X$ be a smooth map and consider two vector bundles E and F over X,

Proposition 7. *The Chern-Simons transgression and its class, holds:*

- $\rho^* CS(\nabla_0^E, \nabla_1^E) = CS(\nabla_0^{\rho^* E}, \nabla_1^{\rho^* E})$;
- $[CS(\nabla_0^E, \nabla_1^E)] = -[CS(\nabla_1^E, \nabla_0^E)]$;
- $[CS(\nabla_1^E, \nabla_2^E)] + [CS(\nabla_2^E, \nabla_3^E)] = [CS(\nabla_1^E, \nabla_3^E)]$;
- *let* $\Phi : F \to E$ *be an isomorphism of vector bundles, then*
$$[CS(\Phi^*\nabla_0^E, \Phi^*\nabla_1^E)] = [CS(\nabla_0^E, \nabla_1^E)];$$
- *given the connections* ∇_0^E, ∇_1^E *and* ∇_0^F, ∇_1^F, *then the connections* $\nabla_0^E \oplus \nabla_0^F$ *and* $\nabla_1^E \oplus \nabla_1^F$ *over* $E \oplus F$, *holds*
$$[CS(\nabla_0^E \oplus \nabla_0^F, \nabla_1^E \oplus \nabla_1^F)] = [CS(\nabla_0^E, \nabla_1^E)] + [CS(\nabla_0^F, \nabla_1^F)].$$

Proof. The first one is clear, if follows due to the naturality of ch and the $I-$integration, the rest of the items is similar to the third one that will be proved. It is enough to consider a connection as definned in remark 6, then, by the fiberwise Stokesâ€™ theorem, see (GREUB, 1972), p. 311, we have

$$\begin{aligned} d\int_{\Delta^2} \mathrm{ch}\hat{\nabla} &= \int_{\Delta^2} d\mathrm{ch}\hat{\nabla} + \int_{\partial\Delta^2} \mathrm{ch}\hat{\nabla} \\ &= \mathrm{CS}(\nabla_1^E, \nabla_2^E) + \mathrm{CS}(\nabla_2^E, \nabla_3^E) - \mathrm{CS}(\nabla_1^E, \nabla_3^E). \end{aligned}$$

□

We now extend the definitions of the Chern Character form and the Chern-Simons class to the relative case. On the previous section we defined what we mean by a relative vector bundle on any map $\rho : A \to X$, which is a triple where the first two entries are vector bundle and the third one linking these bundles, following the same pattern, we define.

Definition 17. A connection on (E, F, ψ) is a triple $\nabla^{(E,F,\psi)} := (\nabla^E, \nabla^F, \tilde{\nabla}^{\pi_A^*\rho^* E})$, where:

- ∇^E and ∇^F are connections on E and F respectively;
- $\tilde{\nabla}^{\pi_A^*\rho^* E}$ is a path connection on $\pi_A^*\rho^* E$ between $\nabla^{\rho^* E}$ and $\nabla^{\psi^*\rho^* F}$.

Definition 18. The relative *Chern character form* associated to $\nabla^{(E,F,\psi)}$, is the form on $\Omega^{\text{even}}(\rho)$ defined by

$$\mathrm{ch}\nabla^{(E,F,\psi)} := \left(\mathrm{ch}\nabla^E - \mathrm{ch}\nabla^F, \int_I \mathrm{ch}\tilde{\nabla}^{\pi_A^*\rho^*E}\right) \in \Omega^{\text{ev}}(\rho),$$

where $\mathrm{ch}\tilde{\nabla}^{\pi_A^*\rho^*E}$ is a path connection between ∇^E and $\psi^*\nabla^F$.

The relative version of the proposition 6 holds here and takes the form.

Proposition 8. *The relative Chern Character form holds:*

- *The form $ch\nabla^{(E,F,\psi)}$ is closed;*
- *Given $\nabla_0^{(E,F,\psi)}$ and $\nabla_1^{(E,F,\psi)}$, any two relative connection on (E,F,ψ), then there is a relative differential form $(\omega,\eta) \in \Omega^{odd}(\rho)$ such that*

$$ch\nabla_1^{(E,F,\psi)} - ch\nabla_0^{(E,F,\psi)} = d(\omega,\eta).$$

Proof. Let us prove the second item. By definition we have

$$\begin{aligned}
\mathrm{ch}\nabla_1^{(E,F,\psi)} - \mathrm{ch}\nabla_0^{(E,F,\psi)} &= \left((\mathrm{ch}\nabla_1^E - \mathrm{ch}\nabla_0^E) - (\mathrm{ch}\nabla_1^F - \mathrm{ch}\nabla_0^F), \int_I \left(\mathrm{ch}\tilde{\nabla}_1^{\pi_A^*\rho^*E} - \mathrm{ch}\tilde{\nabla}_0^{\pi_A^*\rho^*E}\right)\right) \\
&= \left(d\int_I \left(\mathrm{ch}\nabla_{0,1}^{\pi_X^*E} - \mathrm{ch}\nabla_{0,1}^{\pi_X^*F}\right), \int_I \left(\mathrm{ch}\tilde{\nabla}_1^{\pi_A^*\rho^*E} - \mathrm{ch}\tilde{\nabla}_0^{\pi_A^*\rho^*E}\right)\right),
\end{aligned}$$

it is enough to prove that the form

$$\rho^*\int_I \left(\mathrm{ch}\nabla_{0,1}^{\pi_X^*E} - \mathrm{ch}\nabla_{0,1}^{\pi_X^*F}\right) - \int_I \left(\mathrm{ch}\tilde{\nabla}_1^{\pi_A^*\rho^*E} - \mathrm{ch}\tilde{\nabla}_0^{\pi_A^*\rho^*E}\right)$$

is exact in A. In fact,

$$\begin{aligned}
&\rho^*\int_I \left(\mathrm{ch}\nabla_{0,1}^{\pi_X^*E} - \mathrm{ch}\nabla_{0,1}^{\pi_X^*F}\right) - \int_I \left(\mathrm{ch}\tilde{\nabla}_1^{\pi_A^*\rho^*E} - \mathrm{ch}\tilde{\nabla}_0^{\pi_A^*\rho^*E}\right) \\
&\quad= \int_I (\rho\times\mathrm{id}_I)^*\left(\mathrm{ch}\nabla_{0,1}^{\pi_X^*E} - \mathrm{ch}\nabla_{0,1}^{\pi_X^*F}\right) - \int_I \left(\mathrm{ch}\tilde{\nabla}_1^{\pi_A^*\rho^*E} - \mathrm{ch}\tilde{\nabla}_0^{\pi_A^*\rho^*E}\right) \\
&\quad= \int_I \left(\mathrm{ch}\nabla_{0,1}^{\pi_A^*\rho^*E} - \mathrm{ch}\nabla_{0,1}^{\pi_A^*\rho^*F}\right) - \int_I \left(\mathrm{ch}\tilde{\nabla}_1^{\pi_A^*\rho^*E} - \mathrm{ch}\tilde{\nabla}_0^{\pi_A^*\rho^*E}\right) \\
&\quad= \int_I d\int_I \mathrm{ch}\tilde{\nabla}_{0,1}^{\pi_{A\times I}^*\pi_A^*\rho^*E} - \int_{\partial I}\int_I \mathrm{ch}\tilde{\nabla}_{0,1}^{\pi_{A\times I}^*\pi_A^*\rho^*E} \\
&\quad= d\int_I\int_I \mathrm{ch}\tilde{\nabla}_{0,1}^{\pi_{A\times I}^*\pi_A^*\rho^*E},
\end{aligned}$$

where the last identity is due to the fiberwise Stokes theorem and $\tilde{\nabla}_{0,1}^{\pi_{A\times I}^*\pi_A^*\rho^*E}$ is a path connection between the four connections $\nabla_{0,1}^{\pi_A^*\rho^*E}$, $\nabla_{0,1}^{(\psi')^*\pi_A^*\rho^*F}$, $\tilde{\nabla}_0^{\pi_A^*\rho^*E}$ and $\tilde{\nabla}_1^{\pi_A^*\rho^*E}$, with $\psi' : \pi_A^*\rho^*E \to \pi_A^*\rho^*F$ the isomorphism iduced by ψ. □

Definition 19. The relative Chern-Simons transgression form between two relative connections $\nabla_0^{(E,F,\psi)}$ and $\nabla_1^{(E,F,\psi)}$ is given and denoted by

$$\mathrm{CS}\left(\nabla_0^{(E,F,\psi)}, \nabla_1^{(E,F,\psi)}\right) := \int_I \left(\mathrm{ch}\nabla_{0,1}^{\pi_X^*E} - \mathrm{ch}\nabla_{0,1}^{\pi_X^*F}, \int_I \mathrm{ch}\tilde{\nabla}_{0,1}^{\pi_{A\times I}^*\pi_A^*\rho^*E}\right),$$

Its class $\left[\mathrm{CS}(\nabla_0^{(E,F,\psi)}, \nabla_1^{(E,F,\psi)})\right] \in \Omega^{\text{odd}}(\rho)/\mathrm{Im}(d)$ is called the relative *Chern-Simons class.*

Explicitly, the relative Chern-Simons of $\nabla_0^{(E,F,\psi)}$ and $\nabla_1^{(E,F,\psi)}$ is given by the $I-$integration of the form (ω,η) that appears in the proposition 8.

We can easily see that the equality holds

$$\mathrm{CS}\left(\nabla_0^{(E,F,\psi)},\nabla_1^{(E,F,\psi)}\right)=\left(\mathrm{CS}(\nabla_0^E,\nabla_1^E)-\mathrm{CS}(\nabla_0^F,\nabla_1^F);\int_I \mathrm{CS}(\tilde{\nabla}_0^{\pi_A^*\rho^*E},\tilde{\nabla}_1^{\pi_A^*\rho^*E})\right). \quad (4.13)$$

The statements and proofs of the proposition 7, can be adapted to the relative version as follows.

Proposition 9. *The relative Chern-Simons class holds:*

- $\left[CS\left(\nabla_1^{(E,F,\psi)},\nabla_2^{(E,F,\psi)}\right)\right]=-\left[CS\left(\nabla_2^{(E,F,\psi)},\nabla_1^{(E,F,\psi)}\right)\right]$;
- $\left[CS\left(\nabla_1^{(E,F,\psi)},\nabla_2^{(E,F,\psi)}\right)\right]+\left[CS\left(\nabla_2^{(E,F,\psi)},\nabla_3^{(E,F,\psi)}\right)\right]=\left[CS\left(\nabla_1^{(E,F,\psi)},\nabla_3^{(E,F,\psi)}\right)\right]$;
- *Let $(f,g):(E',F',\psi')\to(E,F,\psi)$ be an isomorphism, then*
$$\left[CS\left((f,g)^*\nabla_1^{(E,F,\psi)},(f,g)^*\nabla_2^{(E,F,\psi)}\right)\right]=\left[CS\left(\nabla_1^{(E,F,\psi)},\nabla_2^{(E,F,\psi)}\right)\right].$$

Proof. For the proof of the second item, we just apply proposition 7 in each entries. In fact, there exist connections $\hat{\nabla}_E$ and $\hat{\nabla}_F$ on $\pi^*_{\Delta^2,X}E$ and $\pi^*_{\Delta^2,X}F$ respectively, such that:

$$\mathrm{CS}(\nabla_1^E,\nabla_2^E)+\mathrm{CS}(\nabla_2^E,\nabla_3^E)-\mathrm{CS}(\nabla_1^E,\nabla_3^E)=d\int_{\Delta^2}\mathrm{ch}\hat{\nabla}_E,$$
$$\mathrm{CS}(\nabla_1^F,\nabla_2^F)+\mathrm{CS}(\nabla_2^F,\nabla_3^F)-\mathrm{CS}(\nabla_1^F,\nabla_3^F)=d\int_{\Delta^2}\mathrm{ch}\hat{\nabla}_F;$$

moreover, there is a connection $\hat{\nabla}_{\pi_A^*\rho^*E}$ on $\pi^*_{\Delta^2,I\times A}E$, where $\pi_{\Delta^2,I\times A}:\Delta\times I\times A\to I\times A$ is the natural projection, such that

$$\mathrm{CS}(\nabla_1^{\pi_A^*\rho^*E},\nabla_2^{(\psi')^*\pi_A^*\rho^*F})+\mathrm{CS}(\nabla_2^{\pi_A^*\rho^*E},\nabla_3^{(\psi')^*\pi_A^*\rho^*F})-\mathrm{CS}(\nabla_1^{\pi_A^*\rho^*E},\nabla_3^{(\psi')^*\pi_A^*\rho^*F})=d\int_{\Delta^2}\mathrm{ch}\hat{\nabla}_{\pi_A^*\rho^*E},$$

thus, from (4.13) we get

$$\begin{aligned}\mathrm{CS}\left(\nabla_1^{(E,F,\psi)},\nabla_2^{(E,F,\psi)}\right)+\mathrm{CS}\left(\nabla_2^{(E,F,\psi)},\nabla_3^{(E,F,\psi)}\right)-\mathrm{CS}\left(\nabla_1^{(E,F,\psi)},\nabla_3^{(E,F,\psi)}\right)\\=\left(d\int_{\Delta^2}(\mathrm{ch}\hat{\nabla}_E-\mathrm{ch}\hat{\nabla}_F),\int_I d\int_{\Delta^2}\mathrm{ch}\hat{\nabla}_{\pi_A^*\rho^*E}\right),\end{aligned}$$

then as in the proof of proposition 9, using the fiberwise Stokes theorem we have that $\rho^*\int_{\Delta^2}(\mathrm{ch}\hat{\nabla}_E-\mathrm{ch}\hat{\nabla}_F)-\int_I d\int_{\Delta^2}\mathrm{ch}\hat{\nabla}_{\pi_A^*\rho^*E}$ is exact in A.

Definition 20. Let $(f,g):(E',F',\psi')\to(E,F,\psi)$ be a morphism, and let $\nabla^{(E,F,\psi)}$ be a connection on (E,F,ψ), the pull-back of this connection is defined by

$$(f,g)^*(\nabla^E,\nabla^F,\tilde{\nabla}^{\pi_A^*\rho^*E}):=(f^*\nabla^{E'},g^*\nabla^{F'},\hat{f}^*\tilde{\nabla}^{\pi_A^*\rho^*E'}),$$

where $\hat{f}:\pi_A^*\rho^*E\to\pi_A^*\rho^*E'$ is the natural morphism between vector bundles on $I\times A$ induced by f.

□

4.3 Relative Freed-Lott model

We now consider the differential extension of $K(\rho)$, assuming that $\rho : A \to X$ is a smooth map between compact manifolds. From now on, we assume that every vector bundle is endowed with an Hermitian metric, every isomorphism is unitary, and every connection is compatible with the corresponding metric.

Definition 21. (**Relative Differential vector bundles**)

A *relative differential vector bundle* over ρ, is a triple $\Big((E,F,\psi),\nabla^{(E,F,\psi)},(\omega,\eta)\Big)$, such that:

- (E,F,ψ) is a relative complex vector bundle on ρ;
- $\nabla^{(E,F,\psi)}$ is a connection on (E,F,ψ);
- $(\omega,\eta)\in\Omega^{\mathrm{odd}}(\rho)$.

A triple of the form $\Big((G,G,\mathrm{id}),\nabla^{(G,G,\mathrm{id})},(0,0)\Big)$ is called elementary.

We say that two relative differential vector bundles are isomorphic, denoted by

$$\Big((E,F,\psi),(\nabla^E,\nabla^F,\tilde{\nabla}^{\pi_A^*\rho^*E}),(\omega,\eta)\Big)\simeq\Big((E',F',\psi'),(\nabla^{E'},\nabla^{F'},\tilde{\nabla}^{\pi_A^*\rho^*E'}),(\omega',\eta')\Big)$$

if and only if there is an isomorphism $(f,g):(E,F,\psi)\to(E',F',\psi')$ such that

$$(\omega,\eta)-(\omega',\eta')\in\Big[\mathrm{CS}\left(\nabla^{(E,F,\psi)},(f,g)^*\nabla^{(E',F',\psi')}\right)\Big]. \tag{4.14}$$

Notice that, from (4.13) and definition 20, the relative Chern-Simons class above is

$$\left[\left(\mathrm{CS}(\nabla^E,f^*\nabla^{E'})-\mathrm{CS}(\nabla^F,g^*\nabla^{F'});\int_I\mathrm{CS}(\tilde{\nabla}^{\pi_A^*\rho^*E},\hat{f}^*\tilde{\nabla}^{\pi_A^*\rho^*E'})\right)\right],$$

where $\hat{f}:\pi_A^*\rho^*E\to\pi_A^*\rho^*E'$ is the isomorphism between vector bundles on $I\times A$ induced by f.

By proposition 9, we can easily prove that $\simeq$ is an equivalent relation, for example, suppose that $\Big((E,F,\psi),\nabla^{(E,F,\psi)},(\omega,\eta)\Big)\simeq\Big((E',F',\psi'),\nabla^{(E',F',\psi')},(\omega',\eta')\Big)$ and $\Big((E',F',\psi'),\nabla^{(E',F',\psi')},(\omega',\eta')\Big)\simeq\Big((E'',F'',\psi''),\nabla^{(E'',F'',\psi'')},(\omega'',\eta'')\Big)$ through isomorphisms $(f,g):(E,F,\psi)\to(E',F',\psi')$ and $(f',g'):(E',F',\psi')\to(E'',F'',\psi'')$, we have

$$\begin{aligned}
(\omega,\eta)-(\omega'',\eta'') &= ((\omega,\eta)-(\omega',\eta'))+((\omega',\eta')-(\omega'',\eta''))\\
&\in\Big[\mathrm{CS}\left(\nabla^{(E,F,\psi)},(f,g)^*\nabla^{(E',F',\psi')}\right)\Big]\\
&+\Big[\mathrm{CS}\left(\nabla^{(E',F',\psi')},(f',g')^*\nabla^{(E'',F'',\psi'')}\right)\Big]\\
&=\Big[\mathrm{CS}\left(\nabla^{(E',F',\psi')},(f,g)^*\nabla^{(E'',F'',\psi'')}\right)\Big]\\
&+\Big[\mathrm{CS}\left((f,g)^*\nabla^{(E,F,\psi)},(f,g)^*(f',g')^*\nabla^{(E',F',\psi')}\right)\Big]\\
&=\Big[\mathrm{CS}\left(\nabla^{(E,F,\psi)},(f'\circ f,g'\circ g)^*\nabla^{(E'',F'',\psi'')}\right)\Big],
\end{aligned}$$

thus, $\simeq$ is transitive.

4.3.1 Relative differential K-Theory Groups

We call $\hat{\Gamma}(\rho)$ the semi-group of isomorphism classes of differential vector bundle with the operation of direct sum, i.e.,

$$\begin{aligned}\Big[(E,F,\psi),\nabla^{(E,F,\psi)},(\omega,\eta)\Big]\oplus\Big[(E',F',\psi'),\nabla^{(E',F',\psi')},(\omega',\eta')\Big]:=\\ \Big[(E,F,\psi)\oplus(E',F',\psi'),\nabla^{(E,F,\psi)}\oplus\nabla^{(E',F',\psi')},(\omega,\eta)+(\omega',\eta')\Big],\end{aligned}$$

we now define an equivalence relation on $\hat{\Gamma}(\rho)$ as follow:

$$\Big[(E,F,\psi),\nabla^{(E,F,\psi)},(\omega,\eta)\Big]\approx\Big[(E',F',\psi'),\nabla^{(E',F',\psi')},(\omega',\eta')\Big]$$

iff there exist two elementaries $\Big((G,G,\mathrm{id}),\nabla^{(G,G,\mathrm{id})},0\Big)$ and $\Big((G',G',\mathrm{id}),\nabla^{(G',G',\mathrm{id})},0\Big)$ such that

$$\begin{aligned}&\Big((E,F,\psi)\oplus(G,G,\mathrm{id}),\nabla^{(E,F,\psi)}\oplus\nabla^{(G,G,\mathrm{id})},(\omega,\eta)\Big)\\ &\qquad\simeq\Big((E',F',\psi')\oplus(G',G',\mathrm{id}),\nabla^{(E',F',\psi')}\oplus\nabla^{(G',G',\mathrm{id})},(\omega',\eta')\Big).\end{aligned}$$

Remark 7. We denote by $\hat{K}(\rho):=\hat{\Gamma}(\rho)/\approx$ the set of equivalence classes

$$\Big[\Big[(E,F,\psi),\nabla^{(E,F,\psi)},(\omega,\eta)\Big]\Big].$$

Notice that $\hat{K}(\rho)$ is an abelian group where the identity element is $\Big[\Big[(G,G,\mathrm{id}_{\rho^*G}),\nabla^{(G,G,\mathrm{id}_{\rho^*G})},0\Big]\Big]$ and the inverse element of $\Big[\Big[(E,F,\psi),\nabla^{(E,F,\psi)},(\omega,\eta)\Big]\Big]$ is $\Big[\Big[(F,E,\psi^{-1}),\nabla^{(F,E,\psi^{-1})},-(\omega,\eta)\Big]\Big]$. In the following paragraphs, we denote $\Big[\Big[(E,F,\psi),\nabla^{(E,F,\psi)},(\omega,\eta)\Big]\Big]$ simply by

$$\Big[(E,F,\psi),\nabla^{(E,F,\psi)},(\omega,\eta)\Big].$$

Definition 22. The relative differential K-theory group of $\rho:A\to X$ is the group

$$\hat{K}(\rho):=\hat{\Gamma}(\rho)/\approx.$$

With the notations and definitions introduced on the previous subsections, let us show that we get a differential extension of $K^\bullet$. First, we need to define the natural transformations I,R and a of $\hat{K}(\rho)$. In fact, we have the following.

Definition 23. Let $\hat{\alpha}=\Big[(E,F,\psi),\nabla^{(E,F,\psi)},(\omega,\eta)\Big]\in\hat{K}(\rho)$, the natural transformations of degree 0 are:

$$\begin{aligned}I:\hat{K}(\rho)&\to K(\rho)\\ \hat{\alpha}&\mapsto[E,F,\psi];\\ R:\hat{K}(\rho)&\to\Omega^{\mathrm{even}}_{\mathrm{cl}}(\rho,\mathfrak{h}^\bullet_{\mathbb{R}})\\ \hat{\alpha}&\mapsto \mathrm{ch}\nabla^{(E,F,\psi)}-d(\omega,\eta);\\ a:\Omega^{-1}(\rho,\mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d)&\to\hat{K}(\rho)\\ (\omega,\eta)&\mapsto[0,0,-(\omega,\eta)];\end{aligned}$$

R is called curvature.

We recall that ch^{-1} is the homomorphism given by

$$\begin{aligned} \mathrm{ch}^{-1} : K^{-1}(\rho) &\to \Omega^{-1}(\rho, \mathfrak{h}^{\bullet}_{\mathbb{R}})/\mathrm{Im}(d) \\ [(E, F, \psi)] &\mapsto \left[\int_{S^1} \mathrm{ch}\nabla^{(E,F,\psi)}\right]. \end{aligned}$$

which, from propositions 8, it is a well defined homomorphism.

The transformations defined above make diagrams (2.1) and (2.2) commutative, moreover we have the following theorem.

Theorem 5. *The morphisms I, R and a hold*

A1. $R \circ a = d$;

A2. $dR \circ R = ch \circ I$, *where dR is the morphism that takes a closed form to its de Rham class and ch is the Chern character in K-theory*

$$\begin{tikzcd} \hat{K}(\rho) \arrow[r, "I"] \arrow[d, "R"] & K(\rho) \arrow[d, "ch"] \\ \Omega_{cl}(\rho; \mathfrak{h}^{\bullet}_{\mathbb{R}}) \arrow[r, "dR"] & H_{dR}(\rho; \mathfrak{h}^{\bullet}_{\mathbb{R}}); \end{tikzcd} \tag{4.15}$$

A3. *the sequence*

$$K^{-1}(\rho) \xrightarrow{ch^{-1}} \Omega^{-1}(\rho; \mathfrak{h}^{\bullet}_{\mathbb{R}})/Im(d) \xrightarrow{a} \hat{K}(\rho) \xrightarrow{I} K(\rho) \longrightarrow 0 \tag{4.16}$$

is exact;

A4. *the diagram below*

$$\begin{tikzcd} \hat{K}^{\bullet}(\rho) \arrow[r, "\pi^*"] \arrow[d, "cov"] & \hat{K}^{\bullet}(X) \arrow[d, "\rho^*"] \\ \Omega^{\bullet-1}(A) \arrow[r, "a"] & \hat{K}^{\bullet}(A) \end{tikzcd} \tag{4.17}$$

is commutative, where π is the natural morphism from $\varnothing \to X$ to $\rho^*A \to X$ and $cov(\rho)$ is the second component of the curvature $R(\rho)$.

The proof of items $A1$, $A2$, $A4$, and, the exactness at $K(\rho)$ and $\hat{K}(\rho)$ are not difficult. For the proof of the exactness at $\Omega^{-1}(\rho; \mathfrak{h}^{\bullet}_{\mathbb{R}})/\mathrm{Im}(d)$ we will give a previous result.

Let $\left((E, F, \psi), \nabla^{(E,F,\psi)}\right)$ be a relative vector bundle on ρ, and denote by $\mathrm{Aut}(E, F, \psi)$ the family of all isomorphism from (E, F, ψ) to itself, i.e.,

$$\mathrm{Aut}(E, F, \psi) := \{(f, g) : (E, F, \psi) \to (E, F, \psi) : (f, g) \text{ is an isomorphism}\}.$$

For each $(f, g) \in \mathrm{Aut}(E, F, \psi)$ we obtain an element

$$\left[\mathrm{CS}\left(\nabla^{(E,F,\psi)}, (f, g)^*\nabla^{(E,F,\psi)}\right)\right] \in \Omega^{\mathrm{odd}}(\rho)/\mathrm{Im}(d).$$

We now define the following function:

$$\begin{aligned}\Theta_{(E,F,\psi)}:\mathrm{Aut}(E,F,\psi)&\to\Omega^{\mathrm{odd}}(\rho,\mathfrak{h}^{\bullet}_{\mathbb{R}})/\mathrm{Im}(d)\\ (f,g)&\mapsto\left[\mathrm{CS}\left(\nabla^{(E,F,\psi)},(f,g)^*\nabla^{(E,F,\psi)}\right)\right];\end{aligned}$$

from proposition 9, we can easily see that this function is well defined, i.e., it does not depend on the connection. In fact, denote $\mathcal{E}:=(E,F,\psi)$, then

$$\begin{aligned}&\left[\mathrm{CS}\left(\nabla^{\mathcal{E}},(f,g)^*\nabla^{\mathcal{E}}\right)\right]-\left[\mathrm{CS}\left(\nabla'^{\mathcal{E}},(f,g)^*\nabla'^{\mathcal{E}}\right)\right]\\ &\quad=\left(\left[\mathrm{CS}\left(\nabla^{\mathcal{E}},(f,g)^*\nabla^{\mathcal{E}}\right)\right]+\left[\mathrm{CS}\left((f,g)^*\nabla^{\mathcal{E}},(f,g)^*\nabla'^{\mathcal{E}}\right)\right]\right)\\ &\quad-\left(\left[\mathrm{CS}\left((f,g)^*\nabla^{\mathcal{E}},(f,g)^*\nabla'^{\mathcal{E}}\right)\right]+\left[\mathrm{CS}\left(\nabla'^{\mathcal{E}},(f,g)^*\nabla'^{\mathcal{E}}\right)\right]\right)\\ &\quad=\left[\mathrm{CS}\left(\nabla^{\mathcal{E}},(f,g)^*\nabla'^{\mathcal{E}}\right)\right]-\left[\mathrm{CS}\left(\nabla^{\mathcal{E}},(f,g)^*\nabla'^{\mathcal{E}}\right)\right]=0.\end{aligned}$$

Lemma 5. *Let Θ be the function defined by*

$$\begin{aligned}\Theta:\bigcup_{(E,F,\psi)}Aut(E,F,\psi)&\to\Omega^{odd}(\rho;\mathfrak{h}^{\bullet}_{\mathbb{R}})/Im(d)\\ (f,g)&\mapsto\Theta_{(E,F,\psi)}(f,g),\quad \textit{if } (f,g)\in Aut(E,F,\psi).\end{aligned}$$

Then $Im(\Theta)=Im(ch^{-1})$.

Proof. • $\mathrm{Im}(\Theta)\subset\mathrm{Im}(\mathrm{ch}^{-1})$. Suppose that $y\in\mathrm{Im}(\Theta)$, then for some $(f,g)\in\mathrm{Aut}(E,F,\psi)$,

$$y=\Theta(f,g)=\Theta_{(E,F,\psi)}(f,g)=\left[\mathrm{CS}\left(\nabla_1^{(E,F;G)},(f,g)^*\nabla_1^{(E,F;G)}\right)\right].$$

Let $(E_f,F_g,\psi_{(f,g)})$ be the relative vector bundle on $\mathrm{id}_{S^1}\times\rho$, where E_f is defined as

$$\begin{aligned}E_f=\pi_X^*E/\sim&\to S^1\times X\\ [(e_x,(t,x))]&\mapsto(e^{2\pi it},x),\ \text{for } (e_x,(0,x))\sim(f(e_x),(1,x)).\end{aligned}$$

F_g is defined similarly and $\psi_{(f,g)}:(\mathrm{id}_{S^1}\times\rho)^*E_f\to(\mathrm{id}_{S^1}\times\rho)^*F_g$ is the isomorphism induced by ψ.

Let $\nabla_{1,2}^{(E,F,\psi)}$ be the path connection between $\nabla_1^{(E,F,\psi)}$ and $\nabla_2^{(E,F,\psi)}=(f,g)^*\nabla_1^{(E,F,\psi)}$ and let $\nabla^{(E_f,F_g,\psi_{(f,g)})}$ be the connection induced by $\nabla_{1,2}^{(E,F,\psi)}$.

Since $\nabla_{1,2}^{\pi_X^*E}\simeq\pi_{I,S^1}^*\nabla^{E_f}$, where $\pi_{I,S^1}:I\times X\to S^1\times X$ is the natural projection, we have $\mathrm{ch}\nabla_{1,2}^{\pi_X^*E}=\pi_{I,S^1}^*\mathrm{ch}\nabla^{E_f}$, therefore:

$$\mathrm{CS}(\nabla_1^E,f^*\nabla_1^E)=\int_I\mathrm{ch}\nabla_{1,2}^{\pi_X^*E}=\int_I\pi_{I,S^1}^*\mathrm{ch}\nabla^{E_f}=\int_{S^1}\mathrm{ch}\nabla^{E_f},$$

similarly we have

$$\mathrm{CS}(\nabla_1^F,g^*\nabla_1^F)=\int_{S^1}\mathrm{ch}\nabla^{E_g}\ \text{ and }\ \mathrm{CS}(\tilde\nabla_1^{\pi_A^*\rho^*E},\tilde\nabla_2^{\pi_A^*\rho^*E})=\int_{S^1}\mathrm{ch}\tilde\nabla^{\pi_{S^1\times A}^*(\mathrm{id}_{S^1}\times\rho)^*E_f}.$$

The same steps show that, for $(E_{\mathrm{id}_E}, F_{\mathrm{id}_F}, \psi_{(\mathrm{id}_E,\mathrm{id}_F)})$ relative bundle on $\mathrm{id}_{S^1} \times \rho$ induced by (E, F, ψ), we get

$$\int_{S^1} \mathrm{ch}\nabla^{E_{\mathrm{id}_E}} = 0, \int_{S^1} \mathrm{ch}\nabla^{F_{\mathrm{id}_F}} = 0 \text{ and } \int_{S^1} \mathrm{ch}\nabla^{E_f} = 0$$

Finally, let $\alpha = [(E_f, F_g, \psi_{(f,g)})] - [(E_{\mathrm{id}_E}, F_{\mathrm{id}_F}, \psi_{(\mathrm{id}_E,\mathrm{id}_F)})] \in K(\mathrm{id}_{S^1} \times \rho)$. It is easy to see that $i^*\alpha = 0$, i. e., $\alpha \in K^{-1}(\rho)$ and

$$\begin{aligned}\mathrm{ch}^{-1}(\alpha) &= \left[\int_{S^1}\left(\mathrm{ch}\nabla^{E_f} - \mathrm{ch}\nabla^{F_g}, \int_I \mathrm{ch}\tilde{\nabla}^{\pi^*_{S^1\times A}(\mathrm{id}_{S^1}\times\rho)^*E_f}\right)\right] \\ &= \left[\left(\mathrm{CS}(\nabla^E_1, f^*\nabla^E_1) - \mathrm{CS}(\nabla^F_1, g^*\nabla^F_1), \int_I \mathrm{CS}(\tilde{\nabla}_1^{\pi^*_A\rho^*E}, \hat{f}^*\tilde{\nabla}_1^{\pi^*_A\rho^*E})\right)\right] \\ &= \left[\mathrm{CS}\left(\nabla_1^{(E,F;G)}, (f,g)^*\nabla_1^{(E,F;G)}\right)\right] = y\end{aligned}$$

- $\mathrm{Im}(\mathrm{ch}^{-1}) \subset \mathrm{Im}(\Theta)$. Let $y \in \mathrm{Im}(\mathrm{ch}^{-1})$ and suppose that $y = \mathrm{ch}^{-1}([(\tilde{E}, \tilde{F}, \tilde{\psi})])$, by definition (13), $[(\tilde{E}, \tilde{F}, \tilde{\psi})] \in K(\mathrm{id}_{S^1} \times \rho)$ with $i_1^*[(\tilde{E}, \tilde{F}, \tilde{\psi})] = 0$, then there exist two elementary vector bundles on ρ, (H, H, id) and (G, G, id), such that

$$i_1^*(\tilde{E}, \tilde{F}, \tilde{\psi}) \oplus (H, H, \mathrm{id}) \simeq (G, G, \mathrm{id}).$$

applying $j_1 = (j_{X1}, j_{A1}) : \mathrm{id}_{S^1} \times \rho \to \rho$, the left homotopy inverse of i_1^*, we get the relative vector bundle

$$j_1^*(G, G, \mathrm{id}) \simeq (\tilde{E}, \tilde{F}, \tilde{\psi}) \oplus j_1^*(H, H, \mathrm{id}).$$

By remark 5, there is an isomorphism $f : G \simeq (\pi_X^*G)|_{\{0\}\times X} \to (\pi_X^*G)|_{\{1\}\times X} \simeq G$ that identifies the boundaries of π_X^*G, since $\pi^*_{I,S^1}j^*_{X1}G = \pi^*_XG$, thus $j^*_{X1}G \simeq G_f$. Then

$$\mathrm{ch}^{-1}([\tilde{E}, \tilde{F}, \tilde{\psi}]) = \mathrm{ch}^{-1}(j_1^*[G, G, \mathrm{id}]) = \mathrm{ch}^{-1}([G_f, G_f, \mathrm{id}]),$$

computing $\mathrm{ch}^{-1}([G_f, G_f, \mathrm{id}])$ as in the first part of the proposition, we get

$$\mathrm{ch}^{-1}([\tilde{E}, \tilde{F}, \tilde{\psi}]) = \left[\mathrm{CS}(\nabla^{(G,G,\mathrm{id})}, (f,f)^*\nabla^{(G,G,\mathrm{id})})\right] = \Theta(f, f).$$

□

We now prove theorem 5.

Proof. Let $\hat{\alpha} = \left[(E, F, \psi), \nabla^{(E,F,\psi)}, (\omega, \eta)\right] \in \hat{K}(\rho)$, then

- $R \circ a(\omega, \eta) = R([0, 0, (-\omega, -\eta)]) = d(\omega, \eta)$;
- $dR \circ R(\hat{\alpha}) = [\mathrm{ch}\nabla^{(E,F,\psi)} - d(\omega, \eta)]_{dR} = [\mathrm{ch}\nabla^{(E,F,\psi)}]_{dR} = \mathrm{ch}[E, F, \psi] = \mathrm{ch} \circ I(\hat{\alpha})$;
- sequence (4.16) is exact:

- The surjectivity of I is immediate;
- Exactness at $\hat{K}(\rho)$. Clearly $I \circ a = 0$. Now, suppose that $I(\hat{\alpha}) = [E, F, \psi] = 0$, then there exist relative bundles (G, G, id) and (H, H, id) such that

$$(E \oplus H, F \oplus H, \psi \oplus \mathrm{id}) \simeq (G, G, \mathrm{id}),$$

hence the class

$$\begin{aligned}\hat{\alpha} &= [(E \oplus H, F \oplus H, \psi \oplus \mathrm{id}), \nabla^{(E\oplus H, F\oplus H, \psi\oplus\mathrm{id})}, (\omega, \eta)] \\ &= [(G, G, \mathrm{id}), \nabla^{(G,G,\mathrm{id})}, (\omega, \eta)] \\ &= a(-\omega, -\eta).\end{aligned}$$

So $\mathrm{Ker}(I) \subset \mathrm{Im}(a)$.

- Exactness at $\Omega^{-1}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d)$. Using Lemma 5, it is enough to prove that $\mathrm{Im}(\Theta) = \mathrm{Ker}(a)$. Let $y \in \mathrm{Im}(\Theta)$, then for some $(f, g) \in \bigcup_{(E,F,\psi)} \mathrm{Aut}(E, F, \psi)$,

$$y = \Theta(f, g) = \left[\mathrm{CS}\left(\nabla^{(E,F,\psi)}, (f, g)^* \nabla^{(E,F,\psi)}\right)\right],$$

condition (4.14) says that $a(y) = 0$. To see the other inclusion, take $(\omega, \eta) \in \Omega^{-1}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d)$ such that

$$a(\omega, \eta) = [(0, 0, -(\omega, \eta))] = [(0, 0, 0)],$$

adding the zero class $\left[(G, G, \mathrm{id}), \nabla^{(G,G,\mathrm{id})}, 0\right]$ to both sides, we have that

$$\left((G, G, \mathrm{id}), \nabla^{(G,G,\mathrm{id})}, 0\right) \approx \left((G, G, \mathrm{id}), \nabla^{(G,G,\mathrm{id}_{\rho^* G})}, -(\omega, \eta)\right);$$

then, there is an isomorphism $(f, g) : (G, G, \mathrm{id}) \to (G, G, \mathrm{id})$ such that

$$(\omega, \eta) = \left[\mathrm{CS}\left(\nabla^{(G,G,\mathrm{id})}, (f, g)^* \nabla^{(G,G,\mathrm{id})}\right)\right];$$

thus, $(\omega, \eta) = \Theta(f, g) \in \mathrm{Im}(\Theta)$.

□

4.3.2 Relative differential K-Theory Groups: Higher Degree

In this subsection, our objective is to define the relative differential $K-$theory groups $\hat{K}^{-n}(\rho)$, as well as its natural morphisms, for all $n \in \mathbb{Z}$ and prove the axioms of differential extension.

We call $\mathfrak{h}^\bullet_{\mathbb{R}}$, the K-theory ring of a point with real coefficients, in particular, $\mathfrak{h}^{2n}_{\mathbb{R}} \cong \mathbb{R}$ and $\mathfrak{h}^{2n+1}_{\mathbb{R}} = 0$. It follows that $\Omega^{2n}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}) = \Omega^{\mathrm{ev}}(\rho)$ and $\Omega^{2n+1}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}) = \Omega^{\mathrm{odd}}(\rho)$.

According to definition 13, on the relative topological case the following relation holds: for every $n > 0$

$$K^{-n}(\rho) \cong \bigcap_j \mathrm{Ker}\left(i_j^* : K(\mathrm{id}_{\mathbb{T}^n} \times \rho) \to K(\mathrm{id}_{\mathbb{T}^{n-1}} \times \rho)\right) \subset K(\mathrm{id}_{\mathbb{T}^n} \times \rho).$$

Definition 24. We define $\hat{K}^{-n}(\rho)$ as the subgroup of $\hat{K}(\mathrm{id}_{\mathbb{T}^n} \times \rho)$ whose elements are the classes $\hat{\alpha}$ such that:

- for every $j = 1, \cdots, n$:
$$\hat{\alpha} \in \bigcap_j \mathrm{Ker}\left(i_j^* : \hat{K}(\mathrm{id}_{\mathbb{T}^n} \times \rho) \to \hat{K}(\mathrm{id}_{\mathbb{T}^{n-1}} \times \rho)\right); \tag{4.18}$$
- there exists $\Theta \in \Omega^{-n}(\rho; \mathfrak{h}_{\mathbb{R}}^{\bullet})$ such that the curvature satisfies:
$$R(\hat{\alpha}) = dt_1 \wedge \cdots \wedge dt_n \wedge \pi_{\mathbb{T}^n}^* \Theta; \tag{4.19}$$
where $\pi_{\mathbb{T}^n} : \mathrm{id}_{\mathbb{T}^n} \times \rho \to \rho$ is the projection.

The extension of the natural morphisms I, R and a to any negative degrees are defined as follows. Let $\hat{\alpha} = \left[(E, F, \psi), \nabla^{(E,F,\psi)}, \Theta\right] \in \hat{K}^{-n}(\rho)$, due to the conditions (4.18) and (4.19) we define:

$$\begin{aligned}
I^{-n} : \hat{K}^{-n}(\rho) &\to K^{-n}(\rho) \\
\hat{\alpha} &\mapsto I(\hat{\alpha}); \\
R^{-n} : \hat{K}^{-n}(\rho) &\to \Omega^{-n}(\rho; \mathfrak{t}_{\mathbb{R}}^{\bullet}) \\
\hat{\alpha} &\mapsto \int_{\mathbb{T}^n} R(\hat{\alpha}); \\
a^{-n} : \Omega^{-n-1}(\rho; \mathfrak{t}_{\mathbb{R}}^{\bullet})/\mathrm{Im}(d) &\to \hat{K}^{-n}(\rho) \\
\Theta &\mapsto [0, 0, (-1)^{n+1} dt_1 \wedge \cdots \wedge dt_n \wedge \pi_{\mathbb{T}^n}^* \Theta].
\end{aligned}$$

As in the 0-degree, these morphisms preserve the properties established in proposition 5.

Proposition 10. *The following diagram*

$$\begin{tikzcd}
K^{-n-1}(\rho) \arrow[r, "ch"] & \Omega^{-n-1}(\rho; \mathfrak{h}_{\mathbb{R}}^{\bullet})/Im(d) \arrow[r, "a"] \arrow[dr, "d"] & \hat{K}^{-n}(\rho) \arrow[r, "I"] \arrow[d, "R"] & K^{-n}(\rho) \arrow[r] \arrow[d, "ch"] & 0 \\
 & & \Omega_{cl}^{-n}(\rho; \mathfrak{h}_{\mathbb{R}}^{\bullet}) \arrow[r, "dR"] & H_{dR}^{-n}(\rho; \mathfrak{h}_{\mathbb{R}}^{\bullet}) &
\end{tikzcd}$$

is commutative with an exact upper horizontal line. Moreover, $a \circ cov(\rho) = \rho^* \circ \pi^*$.

Proof. Using theorem 5, we can easily prove the proposition above, for example, the commutativity of the triangle. Given $\Theta \in \Omega^{-n-1}(\rho; \mathfrak{h}_{\mathbb{R}})/\mathrm{Im}(d)$, we have

$$\begin{aligned}
R^{-n} \circ a^{-n}(\Theta) &= R^{-n}\left([0, 0, (-1)^{n+1} dt_1 \wedge \cdots \wedge dt_n \wedge \pi_{\mathbb{T}^n}^* \Theta]\right) \\
&= \int_{\mathbb{T}^n} R\left([0, 0, (-1)^{n+1} dt_1 \wedge \cdots \wedge dt_n \wedge \pi_{\mathbb{T}^n}^* \Theta]\right) \\
&= \int_{\mathbb{T}^n} -d\left((-1)^{n+1} dt_1 \wedge \cdots \wedge dt_n \wedge \pi_{\mathbb{T}^n}^* \Theta\right) \\
&= \int_{\mathbb{T}^n} dt_1 \wedge \cdots \wedge dt_n \wedge \pi_{\mathbb{T}^n}^* d\Theta \\
&= d\Theta.
\end{aligned}$$

The commutativity of the square, given $\hat{\alpha} \in \hat{K}^{-n}(\rho)$,

$$dR \circ R^{-n}(\hat{\alpha}) = \int_{\mathbb{T}^n} dR(R(\hat{\alpha})) = \int_{\mathbb{T}^n} \mathrm{ch}(I(\hat{\alpha})) = \mathrm{ch}^{-n}\left(\int_{\mathbb{T}^n} I(\hat{\alpha})\right) = \mathrm{ch}^{-n} \circ I^{-n}(\hat{\alpha}).$$

The exactness at $\hat{K}^{-n}(\rho)$. In fact, it is clear that $I^{-n} \circ a^{-n} = 0$. Now, suppose that $\hat{\alpha} \in \hat{K}^{-n}(\rho)$ such that $I^{-n}(\hat{\alpha}) = 0$; then, as a class of $\hat{K}(\mathrm{id}_{\mathbb{T}^n} \times \rho)$ we have that $I(\hat{\alpha}) = 0$, by the exactness at $\hat{K}(\mathrm{id}_{\mathbb{T}^n} \times \rho)$, item *A3* of theorem 5, it follows that $a(\upsilon) = \hat{\alpha}$ for some $\upsilon \in \Omega^{-1}(\mathrm{id}_{\mathbb{T}^n} \times \rho)$, thus by condition (4.19), we have

$$d\upsilon = dt_1 \wedge \cdots \wedge dt_n \wedge \pi^*_{\mathbb{T}^n} d\mu;$$

hence, $\upsilon = \xi + (-1)^n dt_1 \wedge \cdots \wedge dt_n \wedge \pi^*_{\mathbb{T}^n}\mu$, with $d\xi = 0$. Since $\hat{\alpha} \in \ker(i^*_j)$ for $j = 1, \ldots, n$ then $a(i^*_n\xi) = i^*_n a(\xi) = i^*_n\hat{\alpha} = 0$. Now, consider

$$\xi^{(1)} = \xi - \pi^*_n i^*_n \xi \quad \text{and} \quad \upsilon^{(1)} = \upsilon - \pi^*_n i^*_n \xi,$$

we have $a(\upsilon^{(1)}) = a(\upsilon) = \hat{\alpha}$ with $i^*_n\xi^{(1)} = 0$; hence, we can suppose that $\xi^{(1)} = dt_n \wedge \pi^*_n \tilde{\xi}^{(1)}$; thus, we obtain $\upsilon^{(1)} = -dt_n \wedge \pi^*_n \tilde{\upsilon}^{(1)}$. Repeating successively the argument above for i^*_j with $j = n-1, n-2, \cdots, 1$ we get a form $\upsilon^{(n)} = (-1)^n dt_1 \wedge \cdots \wedge dt_n \wedge \pi^*_{\mathbb{T}^n}\tilde{\upsilon}^{(n)}$ such that $a^{-n}(\upsilon) = a(\upsilon^{(n)}) = \hat{\alpha}$. The exactness at $K^{-n}(\rho)$ and $\Omega^{-n-1}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d)$ can be proved similarly. □

4.3.3 Relative-absolute product

Now we have all the tools to refine the relative-absolute product (1.4),

$$\cdot : \hat{K}(\rho) \otimes_{\mathbb{Z}} \hat{K}(Y) \to \hat{K}(\rho \times \mathrm{id}_Y),$$

through the Freed-Lott model. Given a relative class $\hat{\alpha} = \left[(E, F, \psi), \nabla^{(E,F,\psi)}, (\omega, \eta)\right] \in \hat{K}(\rho)$ and an absolute class $\hat{\beta} = [E, \nabla^{E'}, \omega'] \in \hat{K}(Y)$, we define $\hat{\alpha} \cdot \hat{\beta} \in \hat{K}(\rho \times \mathrm{id}_Y)$ by

$$\hat{\alpha} \cdot \hat{\beta} := [(E \otimes E', F \otimes E', \psi \otimes \mathrm{id}_{E'}), (\nabla^E \otimes \nabla^{E'}, \nabla^F \otimes \nabla^{E'}, \tilde{\nabla}^{\pi^*_I \rho^* E} \otimes \nabla^{E'}), (\tilde{\omega}, \tilde{\eta})],$$

where:

$$(\tilde{\omega}, \tilde{\eta}) = (\omega, \eta) \wedge R(\hat{\beta}) + R(\hat{\alpha}) \wedge \omega' + (\omega, \eta) \wedge d\omega'$$

Theorem 6. *The exterior product between the relative and absolute differential $K-$theory groups*

$$\cdot : \hat{K}(\rho) \otimes_{\mathbb{Z}} \hat{K}(Y) \to \hat{K}(\rho \times id_Y)$$

holds: the natural morphisms I, R and a are multiplicative, i.e., for every $\alpha \in \hat{K}(\rho)$ and $\beta \in \hat{K}(Y)$

- $I(\hat{\alpha} \cdot \hat{\beta}) = I(\hat{\alpha}) \cdot I(\hat{\beta})$;

- $R(\hat{\alpha} \cdot \hat{\beta}) = R(\hat{\alpha}) \wedge R(\hat{\beta})$;

and for every $(\omega, \eta) \in \Omega(\rho)/Im(d)$ *and* $\omega' \in \Omega(Y)/Im(d)$

- $a(\omega, \eta) \cdot \hat{\beta} = a((\omega, \eta) \wedge R(\hat{\beta}))$;
- $\hat{\alpha} \cdot a(\omega') = a(R(\hat{\alpha}) \wedge \omega')$.

Proof. The first item is simple, we prove the second and third items, and the last one can be proved similarly. let $\hat{\alpha} = \left[(E, F, \psi), \nabla^{(E,F,\psi)}, (\omega, \eta)\right] \in \hat{K}(\rho)$ and $\hat{\beta} = [E', \nabla^{E'}, \omega'] \in \hat{K}(Y)$, then

$$\begin{aligned}
R(\hat{\alpha} \cdot \hat{\beta}) = \mathrm{ch}\left(\nabla^{(E,F,\psi)} \otimes \nabla^{E'}\right) - d(\tilde{\omega}, \tilde{\eta}) \\
&= \mathrm{ch}\nabla^{(E,F,\psi)} \wedge \mathrm{ch}\nabla^{E'} - d(\omega, \eta) \wedge R(\hat{\beta}) - R(\hat{\alpha}) \wedge d\omega' - d(\omega, \eta) \wedge d\omega' \\
&= \mathrm{ch}\nabla^{(E,F,\psi)} \wedge \mathrm{ch}\nabla^{E'} - d(\omega, \eta) \wedge \mathrm{ch}\nabla^{E'} - \mathrm{ch}\nabla^{(E,F,\psi)} \wedge d\omega' + d(\omega, \eta) \wedge d\omega' \\
&= R(\hat{\alpha}) \wedge R(\hat{\beta});
\end{aligned}$$

$$\begin{aligned}
a(\omega, \eta) \cdot \hat{\beta} &= [0, 0, 0, -(\omega, \eta)] \cdot [E', \nabla^{E'}, \omega'] \\
&= [0, 0, 0, -(\omega, \eta) \wedge R(\hat{\beta}) + R(\hat{\alpha}) \wedge \omega' - (\omega, \eta) \wedge d\omega'] \\
&= [0, 0, 0, -(\omega, \eta) \wedge R(\hat{\beta}) + d(\omega, \eta) \wedge \omega' - (\omega, \eta) \wedge d\omega'] \\
&= [0, 0, 0, -(\omega, \eta) \wedge R(\hat{\beta})] \\
&= a((\omega, \eta) \wedge R(\hat{\beta}).
\end{aligned}$$

□

We now extend the product defined above to all non-positive degrees,

$$\cdot : \hat{K}^{-n}(\rho) \otimes_{\mathbb{Z}} \hat{K}^{-m}(Y) \to \hat{K}^{-n-m}(\rho \times \mathrm{id}_Y),$$

as follows. Let $\hat{\alpha} \in \hat{K}^{-n}(\rho)$ and $\hat{\beta} \in \hat{K}^{-m}(Y)$, then as elements of $\hat{K}(\mathrm{id}_{\mathbb{T}^n} \times \rho)$ and $\hat{K}(\mathbb{T}^m \times Y)$ respectively, we have

$$\hat{\alpha} \cdot_0 \hat{\beta} \in \hat{K}(\mathrm{id}_{\mathbb{T}^m} \times \rho \times \mathrm{id}_{\mathbb{T}^n} \times \mathrm{id}_Y),$$

where $\cdot_0$ is the exterior product defined in degree 0. Then we consider the natural diffeomorphism $\varphi_{m,n} : \mathrm{id}_{\mathbb{T}^{m+n}} \times \rho \times \mathrm{id}_Y \to (\mathrm{id}_{\mathbb{T}^m} \times \rho \times \mathrm{id}_{\mathbb{T}^n} \times \mathrm{id}_Y)$ and we define:

$$\hat{\alpha} \cdot \hat{\beta} := (-1)^{nm} \varphi^*_{m,n}(\hat{\alpha} \cdot_0 \hat{\beta}),$$

since $\hat{\alpha}$ and $\hat{\beta}$ both hold condition of definition 24, also $\hat{\alpha} \cdot \hat{\beta}$ satisfies, so $\hat{\alpha} \cdot \hat{\beta} \in \hat{K}^{-n-m}(\rho \times \mathrm{id}_Y)$. It remains to verify the axioms of multiplicativity.

Theorem 7. *The cross product between relative and absolute differential $K-$ Theory*

$$\cdot : \hat{K}^{-n}(\rho) \otimes_{\mathbb{Z}} \hat{K}^{-m}(Y) \to \hat{K}^{-n-m}(\rho \times id_Y)$$

hold: for every $\hat{\alpha} \in \hat{K}^{-m}(\rho)$ *and* $\hat{\beta} \in \hat{K}^{-n}(Y)$

- $I^{-n-m}(\hat{\alpha}\cdot\hat{\beta}) = I^{-m}(\hat{\alpha})\cdot I^{-n}(\hat{\beta})$;
- $R^{-n-m}(\hat{\alpha}\cdot\hat{\beta}) = R^{-m}(\hat{\alpha})\wedge R^{-n}(\hat{\beta})$;

and for every $(\omega,\eta)\in\Omega(\rho)^{-n}/Im(d)$ *and* $\omega'\in\Omega(Y)^{-m}/Im(d)$

- $\hat{\alpha}\cdot a^{-n}(\omega') = a^{-n-m}(R(\hat{\alpha})\wedge\omega')$;
- $a^{-m}(\omega,\eta)\cdot\hat{\beta} = a^{-n-m}((\omega,\eta)\wedge R(\hat{\beta}))$.

Proof. Let us prove the second and the last item.

$$\begin{aligned} R(\hat{\alpha}\cdot_0\hat{\beta}) &= R(\hat{\alpha})\wedge R(\hat{\beta}) \\ &= dt_1\wedge\cdots\wedge dt_n\wedge\pi^*_{\mathbb{T}^n}R^{-m}(\hat{\alpha})\wedge dt_{n+1}\wedge\cdots\wedge dt_{n+m}\wedge\pi^*_{\mathbb{T}^m}R^{-n}(\hat{\beta}) \\ &= (-1)^{nm}dt_1\wedge\cdots\wedge dt_{n+m}\wedge\pi^*_{\mathbb{T}^{n+m}}(\pi^*_\rho R^{-n}(\hat{\alpha}))\wedge\pi^*_Y R^{-m}(\hat{\beta})) \end{aligned}$$

then multiplying for $(-1)^{mn}$ we get

$$\begin{aligned} R^{-n-m}(\hat{\alpha}\cdot\hat{\beta}) &= \pi^*_\rho R^{-m}(\hat{\alpha})\wedge\pi^*_Y R^{-m}(\hat{\beta}) \\ &= R^{-m}(\hat{\alpha})\wedge R^{-n}(\hat{\beta}); \end{aligned}$$

$$\begin{aligned} a^{-n}(\omega,\eta)\cdot\hat{\beta} &= [0,0,0,(-1)^{n+1}dt_1\wedge\cdots\wedge dt_n\wedge\pi^*_{\mathbb{T}^n}(\omega,\eta)]\cdot\hat{\beta} \\ &= [(0,0,0,(-1)^{n+1}dt_1\wedge\cdots\wedge dt_n\wedge\pi^*_{\mathbb{T}^n}(\omega,\eta)\wedge R(\hat{\beta})] \\ &= [(0,0,0,(-1)^{nm+n+1}dt_1\wedge\cdots\wedge dt_n\wedge\pi^*_{\mathbb{T}^n}(\omega,\eta) \\ &\qquad\qquad \wedge dt_{n+1}\wedge\cdots\wedge dt_{n+m}\wedge\pi^*_{\mathbb{T}^m}R^{-m}(\hat{\beta}))] \\ &= [(0,0,0,(-1)^{n+m+1}dt_1\wedge\cdots\wedge dt_{n+m}\wedge\pi^*_{\mathbb{T}^{n+m}}((\omega,\eta)\wedge R^{-m}(\hat{\beta})))] \\ &= a^{-n-m}((\omega,\eta)\wedge R^{-m}(\hat{\beta})). \end{aligned}$$

□

In order to extend all the results above for any integer, we extend the Bott periodicity $K^{-n}(\rho)\simeq K^{-n-2}(\rho)$, given in proposition 5, to the differential framework as follows. Consider a generator $\kappa-1\in\tilde{K}(\mathbb{T}^2)\simeq\mathbb{Z}$ and the topological isomorphism

$$\begin{aligned} B^{-n}: K^{-n}(\rho) &\to K^{-n-2}(\rho) \\ \alpha &\mapsto (\kappa-1)\boxtimes\alpha, \end{aligned} \tag{4.20}$$

this map can be refined defining

$$\begin{aligned} \hat{B}^{-n}: \hat{K}^{-n}(\rho) &\to \hat{K}^{-n-2}(\rho) \\ \hat{\alpha} &\mapsto (\hat{\kappa}-1)\boxtimes\hat{\alpha} \end{aligned}$$

where $\hat{\kappa}\in\hat{K}(\mathbb{T}^2)$ with $I(\hat{\kappa}-1)=\kappa-1$, $R(\hat{\kappa}-1)=dt_1\wedge dt_2$ and $i_1^*(\hat{\kappa}-1)=i_2^*(\hat{\kappa}-1)=0$, for $i_1,i_2:\mathbb{T}\to\mathbb{T}^2$. Thus, such a class $\hat{\kappa}-1\in\hat{K}^{-2}(pto)$ with curvature 1 and first Chern class 1.

Proposition 11. *(**Differential Bott periodicity**) The map $\hat{B}^{-n}$ is an isomorphism.*

Proof. • Surjective. Let $\hat{\beta} \in \hat{K}^{-n-2}(\rho)$ and suppose that $I^{-n-2}(\hat{\beta}) = \beta \in K^{-n-2}(\rho)$, due to the surjectivity of B^{-n} there is a class $\alpha \in K^{-n}(\rho)$ such that $(\kappa - 1) \boxtimes \alpha = \beta$. Condition (4.19) says that $R(\hat{\beta}) = dt_1 \wedge \cdots \wedge dt_{n+2} \wedge \pi^*_{\mathbb{T}^{n+2}} R^{-n-2}(\hat{\beta})$. Choose a class $\hat{\alpha} \in \hat{K}^{-n}(\rho)$ such that $I^{-n}(\hat{\alpha}) = \alpha$; moreover, it is possible to chose $\hat{\alpha}$ in such a way that $R^{-n}(\hat{\alpha}) = R^{-n-2}(\hat{\beta})$, i.e., $R(\hat{\alpha}) = dt_1 \wedge \cdots \wedge dt_n \wedge \pi^*_{\mathbb{T}^n} R^{-n-2}(\hat{\beta})$, proposition 7

$$I^{-n-2}((\hat{\kappa} - 1) \boxtimes \hat{\alpha}) = (\kappa - 1) \boxtimes \alpha = \beta = I^{-n-2}(\hat{\beta});$$

then, $\hat{\beta} = (\hat{\kappa} - 1) \boxtimes \hat{\alpha} + a^{-n-2}(\mu)$ with $d\mu = 0$. Since the classes $\hat{\beta}$ and $(\hat{\kappa} - 1) \boxtimes \hat{\alpha}$ belong to $\mathrm{Ker}(i_j^*)$ for $j = 1, \cdots, n+2$, we have $\mu = (-1)^n dt_1 \wedge \cdots \wedge dt_{n+2} \wedge \nu + d\gamma$ and

$$\begin{aligned} \hat{\beta} = (\hat{\kappa} - 1) \boxtimes \hat{\alpha} + a^{-n-2}(\mu) &= (\hat{\kappa} - 1) \boxtimes \hat{\alpha} + a^{-n}(dt_1 \wedge dt_2 \wedge \nu) \\ &= (\hat{\kappa} - 1) \boxtimes \hat{\alpha} + (\hat{k} - 1) \boxtimes a^{-n}(\nu) \\ &= (\hat{\kappa} - 1) \boxtimes (\hat{\alpha} + a^{-n}(\mu)) = \hat{B}^{-n}(\hat{\alpha} + a^{-n}(\nu)), \end{aligned}$$

this proves that $\hat{B}^{-n}$ is surjective.

• Injective. Suppose that $(\hat{\kappa} - 1) \boxtimes \hat{\alpha} = 0$, then, by proposition 7

$$(\kappa - 1) \boxtimes I^{-n}(\hat{\alpha}) = I^{-n-2}((\hat{\kappa} - 1) \boxtimes \hat{\alpha}) = 0,$$

and by the topological Bott isomorphism, we have that $I^{-n}(\hat{\alpha}) = 0$. By proposition 10, $\hat{\alpha} = a^{-n}(\mu)$ and since the curvature is multiplicative, 7, we have

$$R^{-n}(\hat{\alpha}) = 1 \cdot R^{-n}(\hat{\alpha}) = R^{-2}(\hat{\kappa} - 1) \boxtimes R^{-n}(\hat{\alpha}) = R^{-n-2}((\hat{\kappa} - 1) \boxtimes \hat{\alpha}) = 0,$$

then $d\mu = R^{-n} \circ a^{-n}(\mu) = R^{-n}(\hat{\alpha}) = 0$ and

$$a^{-n-2}(\mu) = a^{-n}(dt_1 \wedge dt_2 \wedge \mu) = (\hat{k} - 1) \boxtimes \hat{\alpha} = 0,$$

thus, by (10), μ is in the image of the Chern character, i.e., $\hat{\alpha} = a^{-n}(\mu) = 0$.

∎

Thanks to relative differential Bott periodicity, we can define the relative differential groups for all $n > 0$,

$$\hat{K}^n(\rho) := \hat{K}^{-n}(\rho).$$

Using the relative topological Bott periodicity 5, and the canonical isomorphism $\Omega^n(\rho, \mathfrak{h}^\bullet_{\mathbb{R}}) = \Omega^{-n}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}})$ the natural morphism with positive degree are define by

$$I^n : \hat{K}^n(\rho) \to K^n(\rho), \quad \alpha \mapsto I^{-n}(\alpha);$$

$$R^n : \hat{K}^n(\rho) \to \Omega^n_{\mathrm{cl}}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}}), \quad \alpha \mapsto R^{-n}(\alpha);$$

$$a^n : \Omega^{n-1}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \to \hat{K}^n(\rho), \quad (\omega, \eta) \mapsto a^{-n}(\omega, \eta).$$

Clearly, proposition 10 also holds for positive degree. Thus, we conclude that $(\hat{K}^\bullet, R, I, a)$ is a Relative Differential Extension of $K^\bullet$.

4.3.4 S^1-Integration

From 1.1, we know that $K^\bullet$ has a topological S^1- integration map. In this section, we will see that its relative differential extension $(\hat{K}^\bullet, R, I, a)$ has an integration.

According to definition 3, for the S^1-integration, we need to define a map

$$\int_{S^1} : \hat{K}^n(\mathrm{id}_{S^1} \times \rho) \to \hat{K}^{n-1}(\rho) \tag{4.21}$$

satisfying axioms $I1$, $I2$ and $I3$. Let us see the case when $n = 0$, the case for $n < 0$ can be constructed similarly and using Bott periodicity we recover all the degrees.

In fact, given any differential class $\hat{\alpha} \in \hat{K}(\mathrm{id}_{S^1} \times \rho)$, from the isomorphism (1.3) the class $I(\hat{\alpha}) \in K(\mathrm{id}_{S^1} \times \rho)$ can we written as

$$I(\hat{\alpha}) = (I(\hat{\alpha}) - \pi_1^* i_1^* I(\hat{\alpha})) + \pi_1^* i_1^* I(\hat{\alpha}) := \beta + \pi_1^* \gamma \in K^{-1}(\rho) \oplus K(\rho),$$

due to the properties of diagram 2.1, there exists classes $\hat{\beta} \in \hat{h}^{-1}(\rho)$ and $\hat{\alpha} \in \hat{h}(\rho)$ such that

$$\hat{\alpha} = \hat{\beta} + \pi_1^* \hat{\gamma} + a^{-1}(\Theta), \tag{4.22}$$

for some form $\Theta \in \Omega^{-1}(\mathrm{id}_{S^1} \times \rho)$. We now define $\int_{S^1} \hat{\beta} := \hat{\beta}$, i.e., the S^1-integration restricts to the identity on $\hat{K}^{-1}(\rho)$, since we demand that axioms $I2$ holds, we define $\int_{S^1} \pi_1^* \hat{\gamma} := 0$ and since axioms $I3$ must hold, we set $\int_{S^1} a^{-1}(\Theta) := a^{-1}(\int_{S^1} \Theta)$; thus, from (4.22) we define

$$\int_{S^1} \hat{\alpha} := \hat{\beta} + a^{-1} \left(\int_{S^1} \Theta \right). \tag{4.23}$$

Let us see that this integral does not depend of the expression (4.22). First, notice that if $\hat{\alpha} = \hat{\beta}' + \pi_1^* \hat{\gamma}' + a^{-1}(\Theta')$, then there exists Θ'' such that

$$\begin{aligned} \hat{\alpha} = \hat{\beta} + \pi_1^* \hat{\gamma} + ((\hat{\beta}' + \pi_1^* \hat{\gamma}') - (\hat{\beta} + \pi_1^* \hat{\gamma})) + a^{-1}(\Theta') \\ = \hat{\beta} + \pi_1^* \hat{\gamma} + a^{-1}(\Theta'') + a^{-1}(\Theta') = \hat{\beta} + \pi_1^* \hat{\gamma} + a^{-1}(\Theta'''), \end{aligned}$$

this happens due to $I(\hat{\beta}' + \pi_1^* \hat{\gamma}') = I(\hat{\beta} + \pi_1^* \hat{\gamma})$. Thus, it is enough to see that the integral does not depend of Θ. In fact, suppose that $a^{-1}(\Theta) = a^{-1}(\Theta')$, then we may write $\Theta - \Theta' = dt_1 \wedge \pi_1^* d\mu$, thus $dt_1 \wedge (\pi_1^* \int_{S^1}(\Theta - \Theta')) = d\mu$, i.e., $a^{-1}(\int_{S^1} \Theta) = a^{-1}(\int_{S^1} \Theta')$.

It is not difficult to see that this definition does not depend of the expression (4.22) and satisfies axioms $I1, I2$ and $I3$, i.e., we have the following proposition.

Proposition 12. *The morphism of S^1-integration has the following properties:*

- *Let $t : S^1 \to S^1$, $z \mapsto \bar{z}$, and consider the map $(t \times id_X, t \times id_A) : id_{S^1} \times \rho \to id_{S^1} \times \rho$, we have*

$$\int_{S^1} \circ (t \times id_X, t \times id_A)^* = - \int_{S^1};$$

- $\int_{S^1} \circ \pi_1^* = 0$, *where* $\pi_1 : id_{S^1} \times \rho \to \rho$ *is the projection;*
- *the following diagram is commutative*

$$\begin{tikzcd}
\Omega^{-1}(id_{S^1} \times \rho; \mathfrak{h}^\bullet_R)/Im(d) \arrow[r, "a"] \arrow[d, "\int_{S^1}"] & \hat{K}(id_{S^1} \times \rho) \arrow[r, "I"] \arrow[d, "\int_{S^1}"] \arrow[rr, bend left, "R"] & K(id_{S^1} \times \rho) \arrow[d, "\int_{S^1}"] & \Omega_{cl}(id_{S^1} \times \rho; \mathfrak{h}_{\mathbb{R}}) \arrow[d, "\int_{S^1}"] \\
\Omega^{-2}(\rho; \mathfrak{h}^\bullet_R)/Im(d) \arrow[r, "a^{-1}"] & \hat{K}^{-1}(\rho) \arrow[r, "I^{-1}"] \arrow[rr, bend right, "R^{-1}"'] & K^{-1}(\rho) & \Omega^{-1}_{cl}(\rho; \mathfrak{h}^\bullet_{\mathbb{R}}).
\end{tikzcd}$$

Proof. The second item and the commutativity with a and I follows by construction. Let us see that the integral commutes with the curvature. In fact, from (4.23) we have

$$R^{-1}\left(\int_{S^1} \hat{\alpha}\right) = R^{-1}(\hat{\beta}) + d\left(\int_{S^1} \Theta\right),$$

applying R to (4.22), we get

$$R(\hat{\alpha}) = dt_1 \wedge \pi_1^* R^{-1}(\hat{\beta}) + \pi_1^* R(\hat{\gamma}) + d\left(dt_1 \wedge \pi_1^*\left(\int_{S^1} \Theta\right)\right),$$

thus, since the integral kills π_1^*, second item, we have

$$\int_{S^1} R(\hat{\alpha}) = R^{-1}(\hat{\beta}) + d\left(\int_{S^1} \Theta\right).$$

Let us prove the first item. We denote $t_\sharp := (t \times \mathrm{id}_X, t \times \mathrm{id}_A)$, it is not difficult to see that for topological classes and differential forms, we have $t_\sharp^* = -\mathrm{id}^*$. Thus, applying $t_\sharp^*$ to the expression (4.22), we get

$$t_\sharp^* \hat{\alpha} = t_\sharp^* \hat{\beta} + t_\sharp^* \pi_1^* \hat{\gamma} + t_\sharp^* a^{-1}(\Theta) = t_\sharp^* \hat{\beta} + \pi_1^* t_\sharp^* \hat{\gamma} + a^{-1}(t_\sharp^* \Theta),$$

since $t_\sharp^* \Theta = -\Theta$ we have

$$\int_{S^1} t_\sharp^* \hat{\alpha} = t_\sharp^* \hat{\beta} - \int_{S^1} a^{-1}(\Theta),$$

thus, it is enough to prove that $t_\sharp^* \hat{\beta} = -\hat{\beta}$. Let us call $\hat{\lambda} := t_\sharp^* \hat{\beta} + \hat{\beta}$, then $t_\sharp^* \hat{\lambda} = \hat{\lambda}$. Since $I^{-1}(\hat{\lambda}) = t_\sharp^* I^{-1}(\hat{\beta}) + I^{-1}(\hat{\beta}) = -I^{-1}(\hat{\beta}) + I^{-1}(\hat{\beta}) = 0$, we have $\hat{\lambda} = a^{-1}(\xi) = -a(dt \wedge \pi_1^* \xi)$; therefore, $t_\sharp^* \hat{\lambda} = a(dt \wedge \pi_1^* \xi) = -\hat{\lambda}$. Hence $\hat{\lambda} = 0$, i.e., $t_\sharp^* \hat{\beta} = -\hat{\beta}$. □

4.4 Freed-Lott model on cofibrations

Parallel classes on cofibrations can be described through a simpler model. Let $\rho : A \hookrightarrow X$ be a closed embedding, we denote this map by the pair (X, A).

Definition 25. A relative vector bundle on (X, A) is a tripla (E, F, ψ), where:

- E and F are complex vector bundle on X;
- $\psi : E|_A \xrightarrow{\simeq} F|_A$ is an isomorphism.

A parallel connection on (E, F, ψ) is a pair $\nabla^{(E,F,\psi)} := (\nabla^E, \nabla^F)$, where:

- ∇^E and ∇^F are connections on E and F respectively;
- $\psi : (E, \nabla^E) \to (F, \nabla^F)$ is an isomorphism, i.e., $\psi^*(\nabla^F|_A) = \nabla^E|_A$.

Definition 26. Let (∇^E, ∇^F) be a parallel connection on (E, F, ψ), the *parallel Chern Character* form associated to this connection is defined by

$$\mathrm{ch}(\nabla^E, \nabla^F) := \mathrm{ch}\nabla^E - \mathrm{ch}\nabla^F \in \Omega^{\mathrm{ev}}_{\mathrm{cl}}(X, A; \mathbb{R});$$

if (∇'^E, ∇'^F) is another parallel connections on (E, F, ψ), the relative Chern-Simons class associated to both connections is given by

$$\left[\mathrm{CS}\left((\nabla^E, \nabla^F); (\nabla'^E, \nabla'^F)\right)\right] := \left[\int_I (\mathrm{ch}\tilde{\nabla}^{\pi_X^* E} - \mathrm{ch}\tilde{\nabla}^{\pi_X^* F})\right] \in \Omega^{\mathrm{odd}}(X, A)/\mathrm{Im}(d),$$

where $\tilde{\nabla}^{\pi_X^* E}$ interpolates ∇^E and ∇'^E, and, $\tilde{\nabla}^{\pi_X^* F}$ interpolates ∇^F and ∇'^F; Moreover, $\tilde{\nabla}^{\pi_X^* E}$ and $\tilde{\nabla}^{\pi_X^* F}$ are defined in such a way that, extending ψ to $A \times I$, we have $\psi^*(\tilde{\nabla}^{\pi_X^* F}|_{A\times I}) = \tilde{\nabla}^{\pi_X^* E}$.

Definition 27. A *differential parallel vector bundle* on (X, A) is a triple $((E, F, \psi), (\nabla^E, \nabla^F), \omega)$ such that:

- (E, F, ψ) is a vector bundle on (X, A), where ψ is a geometric isomorphism;
- (∇^E, ∇^F) is a parallel connection on (E, F, ψ);
- $\omega \in \Omega^{\mathrm{odd}}(X, A)$.

We say that two parallel differential vector bundles, $((E, F, \psi), (\nabla^E, \nabla^F), \omega)$ and $((E', F', \psi'), (\nabla^{E'}, \nabla^{F'}), \omega')$, are equivalent if there is an isomorphism $(f, g) : (E, F, \psi) \to (E', F', \psi')$ such that

$$\omega - \omega' \in \left[\mathrm{CS}\left((\nabla^E, \nabla^F), (f, g)^*(\nabla^{E'}, \nabla^{F'})\right)\right] = \left[\mathrm{CS}(\nabla^E, f^*\nabla^{E'}) - \mathrm{CS}(\nabla^F, g^*\nabla^{F'})\right]. \tag{4.24}$$

We call $\hat{\Gamma}_{\mathrm{par}}(\rho)$ the semi-group of isomorphism classes of parallel triples with the operation of direct sum and we introduce the equivalence relation analogous to the one defined on the previous subsection. We set

$$\hat{K}_{\mathrm{par}}(X, A) := \hat{\Gamma}_{\mathrm{par}}(\rho)/\approx . \tag{4.25}$$

The natural transformations, that characterize the parallel sub-theory of a differential cohomology theory, are defined as follows:

$$I : \hat{K}_{\mathrm{par}}(X, A) \to K(X, A), \quad \alpha \mapsto [(E, F, \psi)];$$

$$R : \hat{K}_{\mathrm{par}}(X, A) \to \Omega^{\mathrm{ev}}_{\mathrm{cl}}(X, A; \mathfrak{h}^\bullet_{\mathbb{R}}), \quad \alpha \mapsto \mathrm{ch}(\nabla^E, \nabla^F) - d\omega;$$

$$a : \Omega^{\mathrm{odd}}(X, A; \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \to \hat{K}_{\mathrm{par}}(X, A), \quad \omega \mapsto [(0, 0, -\omega)].$$

After adapting Lemma 5 to the context of a pair (X, A), we immediately get the following commutative diagram

$$\begin{tikzcd}
h^{\bullet-1}(X,A) \arrow[r, "\mathrm{ch}"] & \Omega^{\bullet-1}(X,A;\mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \arrow[r, "a"] \arrow[dr, "d"] & \hat{K}^\bullet_{\mathrm{par}}(X,A) \arrow[r, "I"] \arrow[d, "R"] & K^\bullet(X,A) \arrow[r] \arrow[d, "\mathrm{ch}"] & 0 \\
 & & \Omega^\bullet_{\mathrm{cl}}(X,A;\mathfrak{h}^\bullet_{\mathbb{R}}) \arrow[r, "\mathrm{dR}"] & H^\bullet_{\mathrm{cl}}(X,A;\mathfrak{h}^\bullet_{\mathbb{R}}) &
\end{tikzcd}$$

with an exact upper horizontal line and $\rho^* \circ \pi^* = 0$. Here, $\pi : (X, \varnothing) \to (X, A)$ is the natural morphism.

The construction of the differential groups of higher degree $\hat{K}^n_{\mathrm{par}}(X, A)$, definitions of the natural transformations; I^{-n}, R^{-n} and a^{-n} and the verification of the axioms of higher degree are given similarly to the preceding sections with the straightforward adaptations. Thus, in this context of cofibrations, $(\hat{K}^\bullet, R, I, a)$ is a relative differential extension of $K^\bullet$; moreover, $(\hat{K}^\bullet, R, I, a)$ is a relative differential extension with integration defined as in (4.23).

4.4.1 Comparison of the two Notions of Relative Differential $K-$Theory

In definition 22 , let us consider, the particular case when $\rho = i_A : A \hookrightarrow X$ is a close embedding, and we want to see the relation of the two relative differential $K-$theory groups $\hat{K}_{\mathrm{par}}(i_A)$ and $\hat{K}_{\mathrm{par}}(X, A)$, the last one is defined on (4.25). Define the following morphism

$$\bar{\iota} : \hat{K}_{\mathrm{par}}(X, A) \hookrightarrow \hat{K}_{\mathrm{par}}(i_A) \tag{4.26}$$

$$[[(E, F, \psi), (\nabla^E, \nabla^F), \omega]] \mapsto [[(E, F, \psi), (\nabla^E, \nabla^F, \tilde{\nabla}), (\omega, 0)]], \tag{4.27}$$

where $\tilde{\nabla}$ is the trivial path connection between $\nabla^{E|_A}$ and $\nabla^{F|_A}$. Notice that we are using double $[[.]]$ in order to be more specific we the classes, see remark 7. Calling

$$\hat{\alpha} := ((E, F, \psi), (\nabla^E, \nabla^F), \omega) \quad \text{and} \quad \iota(\hat{\alpha}) := ((E, F, \psi), (\nabla^E, \nabla^F, \tilde{\nabla}), (\omega, 0))$$

we can easily see the following facts:

- $[[\iota(\hat{\alpha})]]$ is a parallel class. In fact, we have $\mathrm{cov}([[\iota(\hat{\alpha})]]) = \int_I \mathrm{ch}(\tilde{\nabla}^{\pi_A^* E|_A}) - \omega|_A = \int_I \pi_A^* \mathrm{ch}(\nabla^{E|_A}) + 0 = 0$, where we used the fact that $\tilde{\nabla}^{\pi_A^* E|_A} = \nabla^{\pi_A^* E|_A} = \pi_A^* \nabla^{E|_A}$, since $\tilde{\nabla}^{\pi_A^* E|_A}$ is the trivial interpolation, and $\int_I \pi_A^* = 0$.

- ι is well defined. It is an immediate consequence of the fallowing fact: $\hat{\alpha} \approx \hat{\alpha}'$ if and only if $\iota(\hat{\alpha}) \approx \iota(\hat{\alpha}')$, with $\approx$ defined as in (4.24) and (4.14). Using the same argument as above we can easily see that the second entry of (4.14), $\int_I \mathrm{CS}(\tilde{\nabla}, \tilde{\nabla}') = 0$.

- Every elementary relative differential vector bundle belongs to the image of ι (actually, we can identify an elementary $\hat{\gamma} := ((G, G, \mathrm{id}), (\nabla^G, \nabla^G), 0)$ with $\iota(\hat{\gamma})$.

- $\bar{\iota}$ is injective. Moreover, if $\bar{\iota}([[\hat{\alpha}]]) = \bar{\iota}([[\hat{\alpha}']])$, then, there exists an elementaries of the form $\iota(\hat{\gamma})$ and $\iota(\hat{\gamma}')$ such that $[\iota(\hat{\alpha})] \oplus [\iota(\hat{\gamma})] \approx [\iota(\hat{\alpha}')] \oplus [\iota(\hat{\gamma}')]$, then $[\iota(\hat{\alpha} \oplus \hat{\gamma})] \approx [\iota(\hat{\alpha}' \oplus \hat{\gamma}')]$ if and only if $[\hat{\alpha} \oplus \hat{\gamma}] \approx [\hat{\alpha}' \oplus \hat{\gamma}']$, thus $[[\hat{\alpha}]] = [[\hat{\alpha}']]$.

With the notation above, we have the following theorem.

Theorem 8. *The morphism defined on* (4.26) *is an isomorphism.*

Proof. We already saw that $\bar{\iota}$ is injective morphism. Let us see that it is surjective. Let $\hat{\beta} := ((E, F, \psi), (\nabla^E, \nabla^F, \tilde{\nabla}), (\omega, \eta))$ be a relative differential vector bundle on i_A such that $[[\hat{\beta}]] \in \hat{h}_{\mathrm{par}}(i_A)$, we must find a parallel differential class $[[\hat{\alpha}]] \in \hat{h}_{\mathrm{par}}(X, A)$ such that $\bar{\iota}([[\hat{\alpha}]]) = [[\hat{\beta}]]$.

Let us consider the two connections $\nabla^E|_A$ and $\psi^*(\nabla^F|_A)$ on $E|_A$. Since we are dealing with two connections on the same vector bundle, there exists a $1-$form $\Omega : TA \to \mathrm{End}(E|_A)$ such that $\psi^*(\nabla^F|_A) = \nabla^E|_A + \Omega$, hence $\nabla^F|_A = \psi_*(\nabla^E|_A) + \psi_*\Omega$, where $\psi_*\Omega : TA \to \mathrm{End}(F|_A)$. We extend $\psi_*\Omega$ to $\Omega' : TX|_A \to \mathrm{End}(F|_A)$ by composing with any projection $TX|_A \to TA$, defined through any metric on TX. Then, since A is a closed sub-manifold of X, we extend Ω' to $\Omega'' : TX \to \mathrm{End}(F)$ and we set $\overline{\nabla}^F := \nabla^F - \Omega''$. In this way, $\bar{\nabla}^F|_A = \psi_*(\nabla^E|_A)$, hence

$$\psi^*(\overline{\nabla}^F|_A) = \nabla^E|_A. \tag{4.28}$$

We now define the following relative differential vector bundle

$$\hat{\beta}' := ((E, F, \psi), (\nabla^E, \overline{\nabla}^F, \tilde{\nabla}^{\mathrm{cte}}), (\omega', \eta')),$$

where (ω', η') is any form(up to exact ones) such that

$$(\omega', \eta') - (\omega, \eta) \in [\mathrm{CS}((\nabla^E, \overline{\nabla}^F, \tilde{\nabla}^{\mathrm{cte}}), (\nabla^E, \nabla^F, \tilde{\nabla}))].$$

Clearly, by construction, the differential vector bundle $\hat{\beta}$ is isomorphic to $\hat{\beta}'$, thus $[[\hat{\beta}']] = [[\hat{\beta}]] \in \hat{K}_{\mathrm{par}}(i_A)$.

Finally, consider the differential vector bundle $\hat{\alpha} := ((E, F, \psi), (\nabla^E, \overline{\nabla}^F), \omega' - d\tilde{\eta}')$, where $\tilde{\eta}'$ is the extension of η' to X through a partition of unity. Notice that, by (4.28) and since $0 = \mathrm{cov}([[\hat{\beta}']]) = \omega|_A - d\eta' = (\omega' - d\tilde{\eta}')|_A$, we have that $[[\hat{\alpha}]] \in \hat{K}_{\mathrm{par}}(X, A)$; moreover, by construction $\iota(\hat{\alpha}) \approx \hat{\beta}$. Thus, we get $\bar{\iota}([[\hat{\alpha}]]) = [[\iota(\hat{\alpha})]] = [[\hat{\beta}']] = [[\hat{\beta}]]$.

□

4.5 Freed-Lott model on maps of pairs

In this section, we briefly describe the Freed-Lott model on maps of pairs. In the topological framework, given a map of pairs $(\rho, \rho') : (A, A') \to (X, X')$ such that all the spaces involved have the homotopy type of a finite CW-complex, we define.

Definition 28. A vector bundle on a map of pairs (ρ, ρ') is a triple $(\mathcal{E}, \mathcal{F}, \Theta)$, where:

- $\mathcal{E} = (E, \tilde{E}, \alpha)$ and $\mathcal{F} = (F, \tilde{F}, \beta)$ are relative vector bundle on (X, X');
- $\Theta = (\theta, \tilde{\theta}) : (\rho, \rho')^*\mathcal{E} \xrightarrow{\simeq} (\rho, \rho')^*\mathcal{F}$ is an isomorphism of relative vector bundles on (A, A').

Explicitly, the second item states that $\theta : \rho^*E \to \rho^*F$ and $\tilde{\theta} : \rho^*\tilde{E} \to \rho^*\tilde{F}$ are isomorphisms such that the following diagram commutes:

$$\begin{tikzcd}
(\rho^*E)|_{A'} \arrow[r, "\rho^*\alpha"] \arrow[d, "\theta|_{A'}"] & (\rho^*\tilde{E})|_{A'} \arrow[d, "\tilde{\theta}|_{A'}"] \\
(\rho^*F)|_{A'} \arrow[r, "\rho^*\beta"] & (\rho^*\tilde{F})|_{A'}
\end{tikzcd} \tag{4.29}$$

A triple of the form $(\mathcal{E}, \mathcal{E}, id)$ is called elementary.

Definition 29. An isomophism from $(\mathcal{E}, \mathcal{F}, \Theta)$ to $(\mathcal{E}', \mathcal{F}', \Theta')$ is a pair (F, G), where

- $F = (f_1, f_2) : \mathcal{E} \xrightarrow{\simeq} \mathcal{E}'$ and $G = (g_1, g_2) : \mathcal{F} \xrightarrow{\simeq} \mathcal{F}'$ are isomorphisms of relative vector bundles;
- the following diagram commutes:

$$\begin{tikzcd}
(\rho, \rho')^*\mathcal{E} \arrow[r, "\Theta"] \arrow[d, "(\rho,\rho')^*F"] & (\rho, \rho')^*\mathcal{F} \arrow[d, "(\rho,\rho')^*G"] \\
(\rho, \rho')^*\mathcal{E}' \arrow[r, "\Theta'"] & (\rho, \rho')^*\mathcal{F}'.
\end{tikzcd}$$

A triple of the form $(\mathcal{E}, \mathcal{E}, id)$ is called elementary.

Explicitly, from (4.29), the diagram above is commutative means that the following cube commutes

$$\begin{tikzcd}
 & (\rho^*E)|_{A'} \arrow[rr, "(\rho')^*\alpha"] \arrow[dl, "(\rho^*f_1)|_{A'}"'] \arrow[dd, dotted, "\theta|_{A'}"] & & (\rho^*\tilde{E})|_{A'} \arrow[dl, "(\rho^*f_2)|_{A'}"'] \arrow[dd, "\tilde{\theta}|_{A'}"] \\
(\rho^*E')|_{A'} \arrow[rr, "(\rho')^*\alpha'"] \arrow[dd, "\theta'|_{A'}"'] & & (\rho^*F')|_{A'} \arrow[dd, "\tilde{\theta}'|_{A'}"] & \\
 & (\rho^*F)|_{A'} \arrow[rr, dotted, "(\rho')^*\beta"] \arrow[dl, dotted, "(\rho^*g_1)|_{A'}"] & & (\rho^*\tilde{F})|_{A'} \arrow[dl, "(\rho^*g_2)|_{A'}"] \\
(\rho^*\tilde{E}')|_{A'} \arrow[rr, "(\rho')^*\beta'"] & & (\rho^*\tilde{F}')|_{A'}. &
\end{tikzcd} \tag{4.30}$$

4.5.1 K-theory on maps of pairs

Let $(\rho, \rho') : (A, A') \to (X, X')$ be a map of pair, such that all the spaces involved have the homotopy type of a finite CW-complex. The construction of the $K-$theory groups on maps of pairs follows similar to section 4.1.1. In fact, denote by $\Gamma(\rho, \rho')$ the semi-group of isomorphism classes of vector bundle on maps of pairs $(\mathcal{E}, \mathcal{F}, \Theta)$, where the operation of direct sum of vector bundle (on map or pairs) is defined component wise, the next step is to define in $\Gamma(\rho, \rho')$ a equivalence relation $\approx$ analogous to the one defined on (4.1), i.e., $[(\mathcal{E}, \mathcal{F}, \Theta)] \approx [(\mathcal{E}', \mathcal{F}', \Theta')]$ if and only if there exist two elementary triples $(\mathcal{G}, \mathcal{G}, \mathrm{id})$ and $(\mathcal{G}', \mathcal{G}', \mathrm{id})$ such that

$$[(\mathcal{E}, \mathcal{F}, \Theta) \oplus (\mathcal{G}, \mathcal{G}, \mathrm{id})] = [(\mathcal{E}', \mathcal{F}', \Theta') \oplus (\mathcal{G}', \mathcal{G}', \mathrm{id})],$$

then, the abelian group of $K-$theory on maps of pairs is given by the quotient

$$K(\rho, \rho') = \Gamma(\rho, \rho')/\approx .$$

The definition of K-theory groups on maps of pairs $K^{-n}(\rho, \rho')$, of non-positive degree, is defined similar to definition 13 with the obvious adaptations.

Let $(\rho, \rho') : (A, A') \to (X, X')$ be a morphism of pair, let us see that the $K-$theory groups on maps of pairs satisfy the axioms of cohomology theory on maps of pairs. In fact, according to remark 3, we just need to see that

$$K^{\bullet}(\rho, \rho') \cong \tilde{K}^{\bullet}(C_{C(\rho,\rho')}),$$

where $C(\rho, \rho') : C(A, A') \to C(X, X')$ is the cone of (ρ, ρ'). Consider the map of pairs

$$(\bar{\rho}, 1 \times \rho') : (\mathrm{Cyl}(A, A'), \{1\} \times A') \to (\mathrm{Cyl}(X, X'), \{1\} \times X'),$$

where $\bar{\rho} : \mathrm{Cyl}(A, A') \to \mathrm{Cyl}(X, X')$ is the natural morphism that makes the diagram

$$\begin{tikzcd} \mathrm{Cyl}(A, A') \arrow[r, "\bar{\rho}"] \arrow[d, "\Pi"] & \mathrm{Cyl}(X, X') \arrow[d, "\Pi"] \\ A \arrow[r, "\rho"] & X \end{tikzcd}$$

commutative. Since the pairs $(\mathrm{Cyl}(A, A'), \{1\} \times A')$ and $(\mathrm{Cyl}(X, X'), \{1\} \times X')$ are cofibrations, using lemma 2, we get the isomorphism

$$K(\bar{\rho}, 1 \times \rho') \cong K\left(\frac{\mathrm{Cyl}(A, A')}{\{1\} \times A'} \to \frac{\mathrm{Cyl}(X, X')}{\{1\} \times X'}\right) \cong K(C(\rho, \rho')).$$

Theorem 9. *Let $(\rho, \rho') : (A, A') \to (X, X')$ and map of pair with (A, A') and (X, X') cofibrations, then we have the isomorphism:*

$$K(\rho, \rho') \cong K(\bar{\rho}, 1 \times \rho').$$

Proof. Consider the notations previous to theorem 3. We define

$$\begin{aligned}
&\Phi : K(\rho,\rho') \to K(\bar{\rho}, 1\times\rho' : (\mathrm{Cyl}(A,A'),\{1\}\times A') \to (\mathrm{Cyl}(X,X'),\{1\}\times X'))\\
&[(E,F,\psi),(\tilde{E},\tilde{F},\tilde{\psi}),(\phi,\tilde{\phi}))] \mapsto\\
&\qquad [(\Pi^*E,\Pi^*F,(\pi_1^{-1})^*(i_{0,1}^{-1})^*(\bar{i}_0^{-1})^*\psi),(\Pi^*\tilde{E},\Pi^*\tilde{F},(\pi_1^{-1})^*(i_{0,1}^{-1})^*(\bar{i}_0^{-1})^*\tilde{\psi}),(\Pi^*\phi,\Pi^*\tilde{\phi})].
\end{aligned}$$

Notice that, $(\Pi^*E,\Pi^*F,(\pi_1^{-1})^*(i_{0,1}^{-1})^*(\bar{i}_0^{-1})^*\psi)$ is a vector bundle on $(\mathrm{Cyl}(X,X'),1\times X')$ induced by (E,F,ψ), similarly for the other entries. Since the following diagram

$$\begin{tikzcd}
(\rho^*E)|_{A'} \arrow[r, "(\rho')^*\psi"] \arrow[d, "\phi|_{A'}"] & (\rho^*F)|_{A'} \arrow[d, "\tilde{\phi}|_{A'}"] \\
(\rho^*\tilde{E})|_{A'} \arrow[r, "(\rho')^*\tilde{\psi}"] & (\rho^*\tilde{F})|_{A'}
\end{tikzcd}$$

commutes, we can easily see that the next one

$$\begin{tikzcd}
(\bar{\rho}^*\Pi^*E)|_{1\times A'} \arrow[r, "(1\times\rho')^*(\pi_1^{-1})^*(i_{0,1}^{-1})^*(\bar{i}_0^{-1})^*\psi"] \arrow[d, "\Pi^*\phi|1\times A'"] & (\bar{\rho}^*\Pi^*F)|_{1\times A'} \arrow[d, "\Pi^*\tilde{\phi}|_{1\times A'}"] \\
(\bar{\rho}^*\Pi^*\tilde{E})|_{1\times A'} \arrow[r, "(1\times\rho')^*(\pi_1^{-1})^*(i_{0,1}^{-1})^*(\bar{i}_0^{-1})^*\tilde{\Psi}"] & (\bar{\rho}^*\Pi^*\tilde{F})|_{1\times A'}
\end{tikzcd}$$

is also commutative. This proves that Φ is well-defined.

Following the same ideas, we define the inverse morphism Φ' by

$$\begin{aligned}
&\Phi' : K(\bar{\rho},1\times\rho') \to (\mathrm{Cyl}(X,X'),\{1\}\times X')) \to K(\rho,\rho')\\
&[(E,F,\psi),(\tilde{E},\tilde{F},\tilde{\psi}),(\phi,\tilde{\phi}))] \mapsto [j_X^*E, j_X^*F, \bar{i}_0^* i_{0,1}^*\pi_1^*\psi, j_X^*\tilde{E}, j_X^*\tilde{F}, \bar{i}_0^* i_{0,1}^*\pi_1^*\psi, (j_X^*\phi, j_X^*\tilde{\phi})].
\end{aligned}$$

The relations

$$\Pi\circ j_X = \mathrm{id}_{\mathrm{Cyl}(X,X')} \text{ and } j_X\circ\Pi \simeq \mathrm{id}_{\mathrm{Cyl}(X,X')}$$

proves that Φ and Φ' are inverses of each other. □

Corollary 6. *Let $(\rho,\rho') : (A,A') \to (X,X')$ and map of pair with (A,A') and (X,X') cofibrations, then*

$$K(\rho,\rho') \cong K(C(\rho,\rho')).$$

Corollary 5 also holds when ρ is a map of pairs and the proof is similar.

Corollary 7. *For $n\geq 0$, we have the isomorphism*

$$K^{-n}(\rho,\rho') \cong K^{-n}(C(\rho,\rho')).$$

Proposition 13. ***(Relative Bott periodicity of pairs)*** *Let $(\rho,\rho') : (A,A') \to (X,X')$ a morphism of pair, then*

$$K^{-n}(\rho,\rho') = K^{-n-2}(\rho,\rho').$$

Proof. Using (4.5.1), we have

$$\begin{aligned} K^{-n}(\rho,\rho') &\simeq \tilde{K}^{-n}(C_{C(\rho,\rho')}) \\ &\simeq \tilde{K}^{-n-2}(C_{C(\rho,\rho')}) \\ &\simeq K^{-n-2}(\rho,\rho') \end{aligned}$$

□

Theorem 10. *Relative K-theory of map of pairs is a generalized relative cohomology theory of maps of pairs.*

4.5.1.1 Topological relative-absolute product

We now construct the relative-relative product (1.15) on $K-$theory. First, we define the exterior product in degree 0, i.e., given two classes $\alpha = [(E, F, \psi)] \in K(\rho)$ and $\beta = [(E', F', \psi')] \in K(Y, B)$, we set

$$\cdot : K(\rho) \otimes_{\mathbb{Z}} K(Y, B) \to K(\rho \times \mathrm{id}_{(Y,B)}),$$

$$\alpha \cdot \beta = [(E \otimes E', E \otimes F', \mathrm{id}_{\rho^* E} \times \psi'), (F \otimes E', F \otimes F', \mathrm{id}_{\rho^* F} \times \psi'), (\psi \times \mathrm{id}_{E'}, \psi \times \mathrm{id}_{F'})] .$$

The exterior product in any non-positive degree:

$$\cdot : K^{-n}(\rho) \otimes_{\mathbb{Z}} K^{-n}(Y, B) \to K^{-n-m}(\rho \times \mathrm{id}_{(Y,B)}),$$

is defined similar to the relative-absolute product (4.11). In fact, given $\alpha \in K^{-n}(\rho) \subset K(\mathrm{id}_{\mathbb{T}^n} \times \rho)$ and $\beta \in K^{-m}(Y, B) \subset K(\mathbb{T}^n \times (Y, B)$, we consider $\alpha \cdot_0 \beta \in K(\mathrm{id}_{\mathbb{T}^m} \times \rho \times \mathrm{id}_{\mathbb{T}^n \times (Y,B)})$, where $\cdot_0$ is the exterior product in degree 0 defined in (4.10). Then considering the natural isomorphism

$$\varphi_{m,n} : \mathrm{id}_{\mathbb{T}^{m+n}} \times \rho \times \mathrm{id}_{(Y,B)} \to \mathrm{id}_{\mathbb{T}^m} \times \rho \times \mathrm{id}_{\mathbb{T}^n \times (Y,B)}$$

we define:

$$\alpha \otimes \beta := \varphi^*_{n,m}(\alpha \otimes_0 \beta).$$

4.5.2 Differential K-theory on maps of pairs

In this section, we consider the differential extension of $K(\rho, \rho')$, assuming that $(\rho, \rho') : (A, A') \to (X, X')$ is a smooth map between compact manifold pairs. The definition becomes similar to the definition given in 21.

Definition 30. A connection on $(\mathcal{E}, \mathcal{F}, \Theta)$ is a tripla $\nabla^{(\mathcal{E},\mathcal{F},\Theta)} := (\nabla^{\mathcal{E}}, \nabla^{\mathcal{F}}, \tilde{\nabla})$ such that:

- $\nabla^{\mathcal{E}} = (\nabla^E, \nabla^{\tilde{E}})$ and $\nabla^{\mathcal{F}} = (\nabla^F, \nabla^{\tilde{F}})$ are parallel connections on $\mathcal{E}$ and $\mathcal{F}$ respectively;

- $\tilde{\nabla} = (\tilde{\nabla}^{\pi_A^*\rho^* E}, \tilde{\nabla}^{\pi_A^*\rho^*\tilde{E}})$ is a path connection between $(\rho,\rho')^*\nabla^{\mathcal{E}}$ and $\Theta^*(\rho,\rho')^*\nabla^{\mathcal{F}}$, i.e.:
  - $\tilde{\nabla}^{\pi_A^*\rho^* E}$ is a path connection between $\nabla^{\rho^* E}$ and $\nabla^{\theta^*\rho^* F}$;
  - $\tilde{\nabla}^{\pi_A^*\rho^*\tilde{E}}$ is a path connection between $\nabla^{\rho^*\tilde{E}}$ and $\nabla^{\tilde{\theta}^*\rho^*\tilde{F}}$.

In this context, the definitions of Chern character form $\mathrm{ch}\nabla^{(\mathcal{E},\mathcal{F},\Theta)} \in \Omega^{\mathrm{ev}}_{\mathrm{cl}}(\rho,\rho')$ and Cheeger-Simons class $\left[\mathrm{CS}\left(\nabla_1^{(\mathcal{E},\mathcal{F},\Theta)}, \nabla_2^{(\mathcal{E},\mathcal{F},\Theta)}\right)\right] \in \Omega^{\mathrm{odd}}(\rho,\rho')/\mathrm{Im}(d)$ are similarly to definitions 18 and 19, respectively.

Definition 31. A *differential vector bundle on maps of pairs* (ρ,ρ') is a triple

$$\left((\mathcal{E},\mathcal{F},\Theta), \nabla^{(\mathcal{E},\mathcal{F},\Theta)}, (\omega,\eta)\right),$$

such that:

- $(\mathcal{E},\mathcal{F},\Theta)$ is a complex vector bundle (ρ,ρ');
- $\nabla^{(\mathcal{E},\mathcal{F},\Theta)}$ is a connection over $(\mathcal{E},\mathcal{F},\Theta)$;
- $(\omega,\eta) \in \Omega^{\mathrm{odd}}(\rho,\rho')$.

Notice that $\left((\mathcal{E},\mathcal{F},\Theta), \nabla^{(\mathcal{E},\mathcal{F},\Theta)}, (\omega,\eta)\right)$ is the reduced notation of

$$\left(((E,\tilde{E},\alpha),(F,\tilde{F},\beta),(\theta,\tilde{\theta})),((\nabla^E,\nabla^{\tilde{E}}),(\nabla^F,\nabla^{\tilde{F}}),(\tilde{\nabla}^{\pi_A^*\rho^* E},\tilde{\nabla}^{\pi_A^*\rho^*\tilde{E}})),(\omega,\eta)\right).$$

The construction of the relative differential K-Theory Groups on maps of pairs follows similar to the one given on subsection 4.3.1. We denote this group by $\hat{K}(\rho,\rho')$ and its elements are the classes of the form $\left[(\mathcal{E},\mathcal{F},\Theta), \nabla^{(\mathcal{E},\mathcal{F},\Theta)}, (\omega,\eta)\right]$.

The natural transformations $I : \hat{K}(\rho,\rho') \to K(\rho,\rho')$, $a : \dfrac{\Omega^{\mathrm{odd}}(\rho,\rho')}{\mathrm{Im}(d)} \to \hat{K}(\rho,\rho')$ and the curvature $R : \hat{K}(\rho,\rho') \to \Omega^{\mathrm{even}}_{\mathrm{cl}}(\rho,\rho')$ are also defined similarly as definition (23). The extension to any degree and the definition of S^1-integration follow as we have seen above.

4.5.3 Relative-Parallel Product

Now we have all the tools to construct the relative-parallel product (2.4) through the Freed-Lott model. In fact, given a relative class $\hat{\alpha} = \left[(E,F,\psi), \nabla^{(E,F,\psi)}, (\omega,\eta)\right] \in \hat{K}(\rho)$ and a parallel class $\hat{\beta} = [(E',F',\psi'),(\nabla^{E'},\nabla^{F'}),\omega'] \in \hat{K}_{\mathrm{par}}(Y,B)$, we have to construct the corresponding product

$$\cdot : \hat{K}(\rho) \otimes_{\mathbb{Z}} \hat{K}_{\mathrm{par}}(Y,B) \to \hat{K}\left(\rho \times \mathrm{id}_{(Y,B)}\right) = \hat{K}(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_B)$$

we set: $\hat{\alpha}\cdot\hat{\beta} = \left[(E,F,\psi)\otimes(E',F',\psi'), \nabla^{(E,F,\psi)}\otimes(\nabla^{E'},\nabla^{F'}), (\tilde{\omega},\tilde{\eta})\right]$, where

$$(\tilde{\omega},\tilde{\eta}) = (\omega,\eta)\wedge R(\hat{\beta}) + R(\hat{\alpha})\wedge\omega' + (\omega,\eta)\wedge d\omega'.$$

Explicitly, the first and the second entries are the triples given by

$$\begin{aligned}&(\mathcal{E},\mathcal{F},\Theta) := \\ &\qquad ((E\otimes E', E\otimes F', \mathrm{id}_E\otimes\psi'), (F\otimes E', F\otimes F', \mathrm{id}_F\otimes\psi'), (\psi\otimes\mathrm{id}_{E'}, \psi\otimes\mathrm{id}_{F'}))\end{aligned}$$

$$\begin{aligned}&(\nabla^{\mathcal{E}},\nabla^{\mathcal{F}},\tilde{\nabla}) := \\ &\qquad \left((\nabla^{E}\otimes\nabla^{E'}, \nabla^{E}\otimes\nabla^{F'}), (\nabla^{F}\otimes\nabla^{E'}, \nabla^{F}\otimes\nabla^{F'}), (\tilde{\nabla}\otimes\nabla^{E'}, \tilde{\nabla}\otimes\nabla^{F'})\right).\end{aligned}$$

As we can easily see, from definition 28, the triple $(\mathcal{E},\mathcal{F},\Theta)$ is a vector bundle on $(\rho\times\mathrm{id}_Y, \rho\times\mathrm{id}_B) : (A\times Y, A\times B)\to(X\times Y, X\times B)$. In fact, $(E\otimes E', E\otimes F', \mathrm{id}_E\otimes\psi')$ and $(F\otimes E', F\otimes F', \mathrm{id}_F\otimes\psi')$ are vector relative bundles on $(X\times Y, X\times B)$; moreover, the following diagram

$$\begin{array}{ccc} ((\rho\times\mathrm{id}_Y)^*(E\otimes E'))|_{A\times B} & \xrightarrow{(\rho\times\mathrm{id}_Y)^*(\mathrm{id}_E\otimes\psi')} & ((\rho\times\mathrm{id}_Y)^*(E\otimes F'))|_{A\times B} \\ \Big\downarrow{\scriptstyle (\psi\otimes\mathrm{id}_{E'})|_{A\times B}} & & \Big\downarrow{\scriptstyle (\psi\times\mathrm{id}_{F'})|_{A\times B}} \\ ((\rho\times\mathrm{id}_Y)^*(F\otimes E'))|_{A\times B} & \xrightarrow{(\rho\times\mathrm{id}_Y)^*(\mathrm{id}_F\otimes\psi')} & ((\rho\times\mathrm{id}_Y)^*(F\otimes F'))|_{A\times B} \end{array}$$

$$= \begin{array}{ccc} \rho^*E\otimes E'|_B & \xrightarrow{\mathrm{id}_{\rho^*E}\otimes\psi'} & \rho^*E\otimes F'|_B \\ \Big\downarrow{\scriptstyle \psi\otimes\mathrm{id}_{E'|_B}} & & \Big\downarrow{\scriptstyle \psi\otimes\mathrm{id}_{F'|_B}} \\ \rho^*F\otimes E'|_B & \xrightarrow{\mathrm{id}_{\rho^*F}\otimes\psi'} & \rho^*F\otimes F'|_B \end{array}$$

is commutative. Similarly we can see that the long expression $(\nabla^{\mathcal{E}},\nabla^{\mathcal{F}},\tilde{\nabla})$ is a connection over $(\mathcal{E},\mathcal{F},\Theta)$

Theorem 11. *The cross product between relative and parallel differential $K-$theory,*

$$\cdot : \hat{K}(\rho)\otimes_{\mathbb{Z}}\hat{K}_{par}(Y,B)\to\hat{K}\left(\rho\times id_{(Y,B)}\right),$$

holds: the natural morphisms I, R and a are multiplicative.

Proof. The demonstration is similar to the proof of theorem 6 with the straightforward adaptations. Let us prove that the curvature is multiplicative. In fact, for $\hat{\alpha} = \left[(E,F,\psi), \nabla^{(E,F,\psi)}, (\omega,\eta)\right]\in\hat{K}(\rho)$ and $\hat{\beta} = [(E',F',\psi'), (\nabla^{E'},\nabla^{F'}), \omega']\in\hat{K}(Y,B)$, we have

$$\begin{aligned} R(\hat{\alpha}\cdot\hat{\beta}) &= \mathrm{ch}\left(\nabla^{(E,F,\psi)}\otimes(\nabla^{E'},\nabla^{F'})\right) - d(\tilde{\omega},\tilde{\eta}) \\ &= \mathrm{ch}\nabla^{(E,F,\psi)}\wedge\mathrm{ch}(\nabla^{E'},\nabla^{F'}) - d(\omega,\eta)\wedge R(\hat{\beta}) - R(\hat{\alpha})\wedge d\omega' - d(\omega,\eta)\wedge d\omega' \\ &= \mathrm{ch}\nabla^{(E,F,\psi)}\wedge\mathrm{ch}(\nabla^{E'},\nabla^{F'}) - d(\omega,\eta)\wedge\mathrm{ch}(\nabla^{E'},\nabla^{F'}) - \mathrm{ch}\nabla^{(E,F,\psi)}\wedge d\omega' \\ &+ d(\omega,\eta)\wedge d\omega' \\ &= R(\hat{\alpha})\wedge R(\hat{\beta}). \end{aligned}$$

□

In order to construct the relative-parallel product (2.5), the Freed Lott model can be further generalized, and as expected, it can be reproduced in the framework of sequence of maps of sequences in a straightforward way. For example, let $(\rho, \vec{\rho}_n) : (A\vec{A}_n) \to (X, \vec{X}_n)$ be a map of sequences, we define

Definition 32. A relative vector bundle on $(X, \vec{X}_n)$ is a tripla $(E, F, \vec{\psi}_n)$, where:

- E and F are complex vector bundle on X;
- $\psi_i : E|_{X_i} \xrightarrow{\simeq} F|_{X_i}$ is an isomorphism for every $i = 1, \cdots, n$.

A parallel connection on $(E, F, \vec{\psi}_n)$ is a pair $\nabla^{(E,F,\vec{\psi}_n)} := (\nabla^E, \nabla^F)$, where:

- ∇^E and ∇^F are connections on E and F respectively;
- $\psi_i : (E, \nabla^E) \to (F, \nabla^F)$ is an isomorphism, i.e., $\psi^*(\nabla^F|_{X_i}) = \nabla^E|_{X_i}$.

Definition 33. A vector bundle on a map of sequences $(\rho, \vec{\rho}_n)$ is a triple $(\mathcal{E}, \mathcal{F}, \vec{\Theta}_n)$, where:

- $\mathcal{E} = (E, \tilde{E}, \vec{\alpha}_n)$ and $\mathcal{F} = (F, \tilde{F}, \vec{\beta}_n)$ are relative vector bundle on $(X, \vec{X}_n)$;
- $\vec{\Theta}_n = (\theta, \vec{\tilde{\theta}}_n) : (\rho, \vec{\rho}_n)^*\mathcal{E} \xrightarrow{\simeq} (\rho, \vec{\rho}_n)^*\mathcal{F}$ is an isomorphism of relative vector bundles on $(A, \vec{A}_n)$.

Explicitly, the second item states that $\theta : \rho^*E \to \rho^*F$ and $\tilde{\theta} : \rho^*\tilde{E} \to \rho^*\tilde{F}$ are isomorphisms such that, for every $i = 1, \cdots, n$, the following diagram commutes:

$$\begin{array}{ccc} (\rho^*E)|_{A_i} & \xrightarrow{\rho^*\alpha_i} & (\rho^*\tilde{E})|_{A_i} \\ \downarrow{\scriptstyle \theta|_{A_i}} & & \downarrow{\scriptstyle \tilde{\theta}|_{A_i}} \\ (\rho^*F)|_{A'} & \xrightarrow{\rho^*\beta_i} & (\rho^*\tilde{F})|_{A_i} \end{array} \tag{4.31}$$

With this definitions we can easily construct the topological $K-$theory groups, $K^\bullet_{\text{par}}(X, \vec{X}_n)$ and $K^\bullet(\rho, \vec{\rho}_n)$, and, differential $K-$theory groups $\hat{K}^\bullet_{\text{par}}(X, \vec{X}_n)$ and $\hat{K}^\bullet(\rho, \vec{\rho}_n)$ with their respective natural morphisms. The topological product for $K-$theory of sequences or map of sequences has the picture

$$\times : K^\bullet(X, \vec{A}_n) \otimes_{\mathbb{Z}} K^\bullet(Y, \vec{B}_m) \to K^\bullet(X \times Y, X \times \vec{B}_m, \vec{A}_n \times Y).$$

$$[(E, F, \vec{\psi}_n)] \cdot [(E', F', \vec{\psi}'_m)] :=$$
$$[(E \otimes E' + F \otimes F', E \otimes F' + F \otimes E', \mathrm{id}_E \otimes \vec{\psi'}_m + \mathrm{id}_F \otimes \vec{\psi'}^{-1}_m, \vec{\psi}_n \otimes \mathrm{id}_{E'} + \vec{\psi}^{-1}_n \otimes \mathrm{id}_{F'})].$$

$$\times : K^\bullet(\rho, \vec{\rho}_n) \otimes_{\mathbb{Z}} K^\bullet_{\text{par}}(Y, \vec{B}_m) \to K^\bullet((\rho, \vec{\rho}_n) \times \mathrm{id}_{(Y, \vec{B}_m)}) = K^\bullet(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_{\vec{B}_m}, \vec{\rho}_n \times \mathrm{id}_Y).$$
$$[(\mathcal{E}, \mathcal{F}, \vec{\Theta}_n)] \cdot [(E', F', \vec{\psi}'_m)] := [\mathcal{E} \otimes (E', F', \vec{\psi}'_m), \mathcal{F} \otimes (E', F', \vec{\psi}'_m), \vec{\Theta}_n \otimes \mathrm{id}_{(E', F', \vec{\psi'}_m)})],$$

where $\mathcal{E}\otimes(E',F',\vec{\psi}'_m)$ is a vector bundle on $(X,\vec{X}_n)\times(Y,\vec{B}_m) := (X\times Y, X\times\vec{B}_m, \vec{X}_n\times Y)$ defined similar as the previous product, the same for $\mathcal{F}\otimes(E',F',\vec{\psi}'_m)$. Finally, we define the generalized relative-parallel product

$$\times : \hat{K}^\bullet(\rho,\vec{\rho}_n)\otimes_{\mathbb{Z}}\hat{K}^\bullet_{\text{par}}(Y,\vec{B}_m)\to K^\bullet((\rho,\vec{\rho}_n)\times\mathrm{id}_{(Y,\vec{B}_m)}) = \hat{K}^\bullet(\rho\times\mathrm{id}_Y, \rho\times\mathrm{id}_{\vec{B}_m}, \vec{\rho}_n\times\mathrm{id}_Y).$$

For $\hat{\alpha} = \left[(\mathcal{E},\mathcal{F},\vec{\Theta}_n), \nabla^{(\mathcal{E},\mathcal{F},\vec{\Theta}_n)}, (\omega,\eta)\right]\in\hat{K}(\rho,\vec{\rho}_n)$ and $\hat{\beta} = [(E',F',\vec{\psi'}_m), (\nabla^{E'},\nabla^{F'}), \omega']\in\hat{K}(Y,\vec{B}_m)$, set

$$\hat{\alpha}\cdot\hat{\beta} = \left[(\mathcal{E},\mathcal{F},\vec{\Theta}_n)\otimes(E',F',\vec{\psi'}_m), \nabla^{(\mathcal{E},\mathcal{F},\vec{\Theta}_n)}\otimes(\nabla^{E'},\nabla^{F'}), (\tilde{\omega},\tilde{\eta})\right],$$

where

$$(\tilde{\omega},\tilde{\eta}) = (\omega,\eta)\wedge R(\hat{\beta}) + R(\hat{\alpha})\wedge\omega' + (\omega,\eta)\wedge d\omega'.$$

With these definitions, we can easily prove the axioms $M1-M5$ of definition 7.

5 Existence

Given a rationally-even cohomology theory $h^\bullet$, the Hopkins-Singer model, in (HOPKINS; SINGER, 2005), provides a multiplicative differential extension $\hat{h}^\bullet$, see (UPMEIER, 2012), (RUFFINO, 2015b). We briefly review its definition on maps between spaces. Afterwards, we show how to adapt the construction to maps of pairs and we define the parallel-relative product in this framework.

5.1 Differential Functions

Given a smooth function $\rho : A \to X$, consider the corresponding cylinder $\mathrm{Cyl}(\rho)$ and the natural map $\iota_\rho : \mathrm{Cyl}(\rho) \to X \times I$, defined by $\iota_\rho(x) := (x, 0)$, for every $x \in X$, and $\iota_\rho[(a,t)] := (\rho(a), t)$, for every $(a,t) \in A \times I$.

Definition 34.
- Let Y be a manifold, we say that a continuous function $f : Y \to \mathrm{Cyl}(\rho)$ is smooth if and only if $\iota_\rho \circ f$ is;
- let $\nu : B \to Y$ a smooth map, a continuous function $f : \mathrm{Cyl}(\nu) \to \mathrm{Cyl}(\rho)$ is smooth if and only if, for any manifold Z and any smooth map $\xi : Z \to \mathrm{Cyl}(\nu)$, the composition $f \circ \xi$ is smooth.

With these preliminaries, we can define smooth singular chains and cochains on $\mathrm{Cyl}(\rho)$ as usually on manifolds. Moreover, a differential form $\omega \in \Omega^n(X, V^\bullet)$ induces a smooth singular cochain $\varphi_\omega \in C^n_{sm}(\mathrm{Cyl}(\rho); V^\bullet)$ as follows. Denote by φ^0_ω the singular cochain induced by ω on X, calling $\pi_\rho : \mathrm{Cyl}(\rho) \to X$ the natural projection, we set $\varphi_\omega := \pi^*_\rho \varphi^0_\omega$, thus, we get a natural morphism

$$\chi_\rho : \Omega^\bullet(X; V^\bullet) \to C^\bullet_{sm}(\mathrm{Cyl}(\rho); V^\bullet), \tag{5.1}$$

where $V^\bullet$ is a graded real vector space.

Let $\rho : A \to X$ be a smooth function, Y a topological space and $\kappa_n \in C^n(Y, V^\bullet)$ a real singular cocycle. We now give the following

Definition 35. A *differential function* $(f, h, \omega) : \rho \to (Y, \kappa_n)$, from ρ to (Y, κ_n), is a triple (f, h, ω) such that:

- $f : \mathrm{Cyl}(\rho) \to Y$ is a continuous function;
- $h \in C^{n-1}_{sm}(\mathrm{Cyl}(\rho); V^\bullet)$;
- $\omega \in \Omega^n_{\mathrm{cl}}(X; V^\bullet)$;

satisfying the following condition

$$\delta^{n-1}h = \chi_\rho(\omega) - f^*\kappa_n. \tag{5.2}$$

Consider the natural map $\rho \times \mathrm{id}_I : A \times I \to X \times I$ and the natural projection $\pi_X : X \times I \to X$.

Definition 36. A homotopy between two differential functions $(f_0, h_0, \omega_0), (f_1, h_1, \omega_1) : \rho \to (Y, \kappa_n)$, is a differential function $(F, H, \pi_X^*\omega) : \rho \times \mathrm{id}_I \to (Y, \kappa_n)$, such that:

- $\omega_0 = \omega_1$
- $f_0 \simeq_F f_1$;
- $H|_{(\mathrm{Cyl}(\rho)\times\{i\}, A\times\{i\})} = h_i$, for $i = 0, 1$.

We naturally denote the homotopy by $(F, H, \pi_X^*\omega) : (f_0, h_0, \omega_0) \simeq (f_1, h_1, \omega_1)$.

In the previous definition it is possible that $A = \varnothing$. In this case we get a differential function from the manifold X to (Y, κ_n).

Definition 37. Let X be manifold. Given:

- a differential function $(f, h, \omega) : X \to (Y, \kappa_n)$;
- a marked point $y_0 \in Y$ and the constant function $c_{y_0} : X \to Y$;
- $\eta \in \Omega^{n-1}(X, V^\bullet)$;

a *strong topological trivialization* of (f, h, ω), induced by η, is a homotopy $(F, H, \pi_X^*\omega) : X \times I \to (Y, \kappa_n)$ between (f, h, ω) and $(c_{y_0}, \chi_X(\eta), d\eta)$, where $\chi_X : \Omega^\bullet(X, V^\bullet) \to C^\bullet_{sm}(X, V^\bullet)$ is the natural homomorphism, i.e.,

$$(F, H, \pi_X^*\omega) : (f, h, \omega) \simeq (c_{y_0}, \chi_X(\eta), d\eta).$$

5.2 Relative Hopkins-Singer Model.

Given a smooth function $\rho : A \to X$ between manifolds, denote by:

- $\iota_{\mathrm{Cyl}(A)} : \mathrm{Cyl}(A) \to \mathrm{Cyl}(\rho)$ the natural map defined by $(a, t) \mapsto [(a, t)]$;
- $\pi_A : \mathrm{Cyl}(A) \to A$ the projection;
- $A \times \{1\}$ the upper base of the cylinders $\mathrm{Cyl}(A)$ and $\mathrm{Cyl}(\rho)$.

Remark 8. We fix a rationally-even cohomology theory $h^\bullet$, represented by an $\Omega-$spectrum (E_n, e_n, ϵ_n), where e_n is the marked point of E_n and $\epsilon_n : (\sum E_n, \sum e_n) \to (E_{n+1}, e_{n+1})$ is the structure map, whose adjoint $\tilde{\epsilon}_n : E_n \to \Omega_{e_{n+1}} E_{n+1}$ is a homeomorphism. We set $\mathfrak{h}^\bullet_{\mathbb{R}} = h^\bullet(\{pt\}) \otimes_{\mathbb{Z}} \mathbb{R}$ and we fix real singular cocycles $\iota_n \in C^n(E_n, e_n; \mathfrak{h}^\bullet_{\mathbb{R}})$, representing the Chern character of $h^\bullet$, such that $\iota_{n-1} = \int_{S^1} \epsilon_n^* \iota_n$, the integration of cochains being defined through the prisma map, see (UPMEIER, 2012).

Let $(f, h, \omega) : \rho \to (E_n, \iota_n)$ be a differential function and consider the "pull-backs" differential functions:

$$\rho^*(f, h, \omega) := (f|_X \circ \rho, \rho^*(h|_X), \rho^*\omega) : A \to (E_n, \iota_n),$$

$$\iota^*_{\mathrm{Cyl}(A)}(f, h, \omega) := (f \circ \iota_{\mathrm{Cyl}(A)}, \iota^*_{\mathrm{Cyl}(A)} h, \pi_A^* \rho^* \omega) : A \times I \to (E_n, \iota_n).$$

Definition 38. We denote by $\hat{h}^n(\rho)$ the set of equivalence classes $[(f, h, (\omega, \eta))]$, where:

- $(f, h, \omega) : \rho \to (E_n, \iota_n)$ is a differential function such that:

$$\iota^*_{\mathrm{Cyl}(A)}(f, h, \omega) : \rho^*(f, h, \omega) \simeq (c_{e_n}, \chi_A(\eta), d\eta);$$

- two representatives $(f_0, h_0, (\omega, \eta))$ and $(f_1, h_1, (\omega, \eta))$ are equivalent if there exists a homotopy $(F, H, \pi_X^*\omega) : (f_0, h_0, \omega) \simeq (f_1, h_1, \omega)$, that is constant on the upper base of the cylinder, i.e., such that $(F, H, \pi_X^*\omega)|_{A\times\{1\}\times I} = (c_{e_n}, \chi_{A\times I}(\pi_A^*\eta), \pi_A^* d\eta)$.

Notice that, if we restrict the homotopy $(F, H, \pi_X^*\omega)$ to $A\times\{1\}\times\{0\}$ or $A\times\{1\}\times\{1\}$ we have that $\rho^*\omega = d\eta$, i.e., $(\omega, \eta) \in \Omega^n_{\mathrm{cl}}(\rho, \mathfrak{h}^\bullet_{\mathbb{R}})$.

Moreover, given two maps, $\rho : A \to X$ and $\nu : B \to Y$, and a morphism $(\varphi, \psi) : \nu \to \rho$, there is a natural map $(\varphi, \psi)_\# : \mathrm{Cyl}(\nu) \to \mathrm{Cyl}(\rho)$, $y \mapsto \varphi(y)$, $[(b, t)] \mapsto [(\psi(b), t)]$ that induces the following pull-back

$$\begin{aligned} &(\varphi, \psi)^* : \hat{h}^n(\rho) \to \hat{h}^n(\nu) \\ &[(f, h, (\omega, \eta))] \mapsto [(f \circ (\varphi, \psi)_\#, (\varphi, \psi)^*_\# h, \varphi^*\omega, \psi^*\eta)]. \end{aligned}$$

We now describe the abelian group structure on $\hat{h}^n(\rho)$. First, we fix the following data (UPMEIER, 2012):

- Let $\alpha_n : E_n \times E_n \to E_n$ representing the addition in cohomology, i.e., given a topological space X and two maps $f, g : X \to E_n$, representing the cohomology classes $[f], [g] \in h^n(X)$(resp.), the cohomology class $[f] + [g] \in h^n(X)$ is represented by the following composition:

$$X \xrightarrow{(f,g)} E_n \times E_n \xrightarrow{\alpha_n} E_n,$$

we require that, calling $\varphi_n : \Sigma(E_n \times E_n) \to E_{n+1} \times E_{n+1}$ the structure maps of the spectrum $E_n \times E_n$ (defined via the factorization $\Sigma(E_n \times E_n) \to \Sigma E_n \times \Sigma E_n \to E_{n+1} \times E_{n+1}$), see remark 8, one has $\epsilon_{n-1} \circ \Sigma\alpha_{n-1} = \alpha_n \circ \varphi_{n-1}$.

- Let $\pi_{1,n}, \pi_{2,n} : E_n \times E_n \to E_n$ be the two projections, representatives of the two classes $[\pi_{1,n}], [\pi_{2,n}] \in h^n(E_n \times E_n)$ (resp.), the sum of these classes is represented by $\alpha_n \circ (\pi_{1,n}, \pi_{2,n})$. Since $(\pi_{1,n}, \pi_{2,n}) = \mathrm{id}_{E_n \times E_n}$, we have that $\alpha_n \circ (\pi_{1,n}, \pi_{2,n}) = \alpha_n$, thus

$$\pi_{1,n}^*[\iota_n] + \pi_{2,n}^*[\iota_n] = \mathrm{ch}([\pi_{1,n}]) + \mathrm{ch}([\pi_{2,n}]) = \mathrm{ch}([\pi_{1,n}] + [\pi_{2,n}]) = \mathrm{ch}[\alpha_n] = \alpha_n^*[\iota_n],$$

hence there exists $A_{n-1} \in C^{n-1}(E_n \times E_n, e_n \times e_n, \mathfrak{h}_{\mathbb{R}}^\bullet)$ such that:

$$\pi_{1,n}^*(\iota_n) + \pi_{2,n}^*(\iota_n) - \alpha_n^*(\iota_n) = \delta^{n-1} A_{n-1}. \tag{5.3}$$

Since we are assuming that $\mathfrak{h}_{\mathbb{R}}^{\mathrm{odd}} = 0$, it follows that A_{n-1} is unique up to a coboundary for n even (UPMEIER, 2012). We set $A_{n-2} := -\int_{S^1} \varphi_{n-1}^* A_{n-1}$, where φ_{n-1} is the structure map of the spectrum $E_n \times E_n$ defined above, see remark 8. In this way A_{n-1} is uniquely defined up to a coboundary for every n.

We define the sum in $\hat{h}^n(\rho)$ as follows:

$$[(f_0, h_0, (\omega_0, \eta_0))] + [(f_1, h_1, (\omega_1, \eta_1))] = \\ [(\alpha_n \circ (f_0, f_1), h_0 + h_1 + (f_0, f_1)^* A_{n-1}, (\omega_0, \eta_0) + (\omega_1, \eta_1)], \tag{5.4}$$

where $(f_0, f_1) : \mathrm{Cyl}(\rho) \to E_n \times E_n$ denote the map induced by f_0 and f_1.

Let us show that we get a differential extension of $h^\bullet$, constructing the corresponding natural transformations of $\hat{h}^\bullet(\rho)$. We define

$$I : \hat{h}^n(\rho) \to h^n(\rho), \quad [(f, h, (\omega, \eta))] \mapsto [f];$$

$$R : \hat{h}(\rho) \to \Omega_{\mathrm{cl}}^{\mathrm{even}}(\rho; \mathfrak{h}_{\mathbb{R}}), \quad [(f, h, (\omega, \eta))] \mapsto (\omega, \eta);$$

$$a : \Omega^{\mathrm{odd}}(\rho; \mathfrak{h}_{\mathbb{R}}^\bullet)/\mathrm{Im}(d) \to \hat{h}(\rho), \quad (\omega, \eta) \mapsto [(c_{e_n}, \chi_\rho(\omega, \eta), d(\omega, \eta)],$$

where the morphism $\chi_\rho^n : \Omega^n(\rho; \mathfrak{h}_{\mathbb{R}}^\bullet) \to C_{sm}^n(\mathrm{Cyl}(\rho); \mathfrak{h}_{\mathbb{R}}^\bullet)$ is a map that induces a form $(\omega, \eta) \in \Omega^n(\rho; \mathfrak{h}_{\mathbb{R}}^\bullet)$ to a smooth cochain on $\mathrm{Cyl}(\rho)$ up to coboundaries, whose restriction to the upper base is $\chi_A(\rho^*\omega - d\eta)$. We should not confuse this with map (5.1), also denoted by χ_ρ, that induces a form $\omega \in \Omega^n(X; \mathfrak{h}_{\mathbb{R}}^\bullet)$ to smooth cochain on $\mathrm{Cyl}(\rho)$.

Given $(\omega, \eta) \in \Omega^n(\rho; \mathfrak{h}_{\mathbb{R}}^\bullet)$, we define the smooth singular cochain $\chi_\rho^n(\omega, \eta)$ as follows. Fix a real number $\epsilon \in (0, 1)$ and take a smooth non-decreasing function $\theta : I \to I$ such that $\theta(t) = 0$ for $t \leq \epsilon$ and $\theta(1) = 1$. We also fix an open cover $\{U, W\}$ of $\mathrm{Cyl}(\rho)$ defined by $U := A \times (\frac{\epsilon}{3}, 1]$ and $W := A \times [0, \frac{\epsilon}{2}) \sqcup_\rho X$. Now, for each smooth chain $\sigma : \Delta^n \to \mathrm{Cyl}(\rho)$, take the iterated barycentric subdivision, so that the image of each sub-chain is contained in U or in W; then, for each small chain σ', we set

$$\chi^n_\rho(\omega,\eta)(\sigma') := \begin{cases} \chi^n_{A\times I}(\pi_A^*\rho^*\omega - d(\theta\pi_A^*\eta))(\sigma') & \text{if } \sigma' \subset U \\ \chi^n_X(\omega)(\pi_X \circ \sigma') & \text{if } \sigma' \subset W, \end{cases}$$

where $\pi_X : W \to X$ is the natural projection defined by $[a,t] \mapsto \rho(a)$ and $[x] \mapsto x$. Note that the morphism is well defined for $\sigma' \subset U \cap W$, since $\theta(t) = 0$ for $t \leq \epsilon$. The cochain $\chi^n_\rho(\omega,\eta)$ depends on the choice of the function θ up to coboundaries.

5.2.1 S^1-integration and product

Considering the topological S^1-integration defined on (1.1), we define the differential S^1-integration $\int_{S^1} : \hat{h}^{n+1}(\rho \times \mathrm{id}_{S^1}) \to \hat{h}^n(\rho)$, as follow:

Given $\hat{\alpha} \in \hat{h}^{n+1}(\rho \times \mathrm{id}_{S^1})$, we set $\hat{\alpha}' := \hat{\alpha} - \pi_1^* i_1^* \hat{\alpha}$ and we represent it as $\hat{\alpha}' = [(f, h, (\omega, \eta))]$. Since $i_1^*\hat{\alpha}' = 0$ by construction, in particular $i_1^* I(\hat{\alpha}') = 0$, therefore the function $f : (\mathrm{Cyl}(\rho) \times S^1, A \times S^1) \to (E_{n+1}, e_{n+1})$ can be chosen in such a way that $f(i_1(\mathrm{Cyl}(\rho), A)) = e_{n+1}$. Hence, it induces $f : (\mathrm{Cyl}(\rho), A) \to (\Omega_{e_{n+1}} E_{n+1}, c_{e_{n+1}})$, where $c_{e_{n+1}}$ is the constant loop. Composing with the inverse of the adjoint of the structure map, see remark 8, we get $\int_{S^1} f := \tilde{\epsilon}_n^{-1} \circ f : (\mathrm{Cyl}(\rho), A) \to (E_n, e_n)$, hence we set $\int_{S^1} \hat{\alpha} := [(\int_{S^1} f, \int_{S^1} h, \int_{S^1}(\omega,\eta))]$.

We now describe the abelian group structure on $\hat{h}^n(\rho)$. First, We fix the following data (UPMEIER, 2012):

We now describe the exterior product between relative and absolute classes, we call $\mu_{n,m} : E_n \wedge E_m \to E_{n+m}$ the maps making E a ring spectrum and we fix the following data:

- Let $\mu_{n,m} : E_n \wedge E_m \to E_{n+m}$ be the maps making E a ring spectrum, i.e., given two topological spaces with marked point (X, x_0) and (Y, y_0) and two maps $f : (X, x_0) \to (E_n, e_n)$ and $g : (Y, y_0) \to (E_m, e_m)$, representing the reduced cohomology classes $[f] \in \tilde{h}^n(X)$ and $[g] \in \tilde{h}^m(Y)$, the cohomology class $[f] \times [g] \in \tilde{h}^{n+m}(X \wedge Y)$ is represented by the following composition:

  $$X \wedge Y \xrightarrow{f\wedge g} E_n \wedge E_m \xrightarrow{\mu_{n,m}} E_{n+m}.$$

  Therefore, one has $\mathrm{ch}[f] \times \mathrm{ch}[g] = \mathrm{ch}([f] \times [g]) = \mathrm{ch}([\mu_{n,m} \circ (f \wedge g)])$. Choosing $f = \mathrm{id}_{E_n}$ and $g = \mathrm{id}_{E_m}$, we get $\mathrm{ch}[\mathrm{id}_{E_n}] \times \mathrm{ch}[\mathrm{id}_{E_m}] = \mathrm{ch}[\mu_{n,m}]$. Hence, there exists $M_{n,m} \in C^{n+m-1}(E_n \wedge E_m, e_n \wedge e_m, \mathfrak{h}^\bullet_\mathbb{R})$ such that:

  $$\delta^{n+m-1} M_{n,m} = \iota_n \times \iota_m - \mu^*_{n,m} \iota_{n+m}.$$

  Since we are assuming that $\mathfrak{h}^{\mathrm{odd}}_\mathbb{R} = 0$, it follows that $M_{n,m}$ is unique up to a coboundary for n and m even (UPMEIER, 2012).

- We fix a chain homotopy between the wedge product of differential forms and the cup product of the associated singular cochains. In particular, given two manifolds X and Y, we consider the two maps

$$P, Q : \Omega^n(X; \mathfrak{h}^\bullet_{\mathbb{R}}) \otimes \Omega^m(Y; \mathfrak{h}^\bullet_{\mathbb{R}}) \to C^{n+m}(X \times Y; \mathfrak{h}^\bullet_{\mathbb{R}})$$

defined by $P(\omega_0 \otimes \omega_1) := \chi_{X\times Y}(\omega_0 \wedge \omega_1)$ and $Q(\omega_0 \otimes \omega_1) := \chi_X(\omega_0) \cup \chi_Y(\omega_1)$, where $\chi_* : \Omega^\bullet(*; \mathfrak{h}^\bullet_{\mathbb{R}}) \to C^\bullet(*; \mathfrak{h}^\bullet_{\mathbb{R}})$ is the natural homomorphism. The coboundary of $\Omega^n(X; \mathfrak{h}^\bullet_{\mathbb{R}}) \otimes \Omega^m(Y; \mathfrak{h}^\bullet_{\mathbb{R}})$ is defined as $d(\omega_0 \otimes \omega_1) := d\omega_0 \otimes \omega_1 + (-1)^n \omega_0 \otimes d\omega_1$. There exists a chain homotopy

$$B : \Omega^n(X; \mathfrak{h}^\bullet_{\mathbb{R}}) \otimes \Omega^m(Y; \mathfrak{h}^\bullet_{\mathbb{R}}) \to C^{n+m-1}(X \times Y; \mathfrak{h}^\bullet_{\mathbb{R}})$$

between P and Q, which by definition satisfies

$$\chi_{X\times Y}(\omega_0 \wedge \omega_1) - \chi_X(\omega_0) \cup \chi_Y(\omega_1) = \delta B(\omega_0 \otimes \omega_1) + Bd(\omega_0 \otimes \omega_1).$$

Given $\hat{\alpha} = [(f_0, h_0, (\omega_0, \eta_0))] \in \hat{h}^n(\rho)$, where $\rho : A \to X$ is a smooth map, and $\hat{\beta} = [(f_1, h_1, \omega)] \in \hat{h}^m(Y)$, with n and m even, the class $\hat{\alpha} \times \hat{\beta} \in \hat{h}^{n+m}(\rho \times \mathrm{id}_Y)$ is defined by

$$\begin{aligned}[(f_0, h_0, (\omega_0, \eta_0)] \times [(f_1, h_1, \omega_1)] := [(&\mu_{n,m} \circ (f_0 \times f_1), h_0 \cup \chi_Y(\omega_1) + \chi_\rho(\omega_0, \eta_0) \cup h_1 \\ &+ (\pi_\rho \times \mathrm{id}_Y)^* B(\omega_0 \otimes \omega_1) - h_0 \cup \delta h_1 + (f_0, f_1)^* M_{n,m}, \omega_0 \wedge \omega_1, \eta_0 \wedge \omega_1)].\end{aligned} \quad (5.5)$$

In the first entry $\mu_{n,m} \circ (f_0 \times f_1)$, we actually considered the following composition:

$$\mathrm{Cyl}(\rho \times \mathrm{id}_Y) \approx \ \mathrm{Cyl}(\rho) \times Y \xrightarrow{f_0 \wedge f_1} E_n \times E_m \to E_n \wedge E_m \xrightarrow{\mu_{n,m}} E_{n+m},$$

thinking of it as $\mu_{n,m} \circ (f_0 \times f_1) : (\mathrm{Cyl}(\rho) \times Y, A \times \{1\} \times Y) \to (E_{n+m}, e_{n+m})$.

For any $\hat{\alpha} \in \hat{h}^n(\rho)$ (without restrictions on n), there exists a unique class $\hat{\alpha}' \in h^{n+1}(\rho \times \mathrm{id}_{S^1})$ such that $\int_{S^1} \hat{\alpha}' = \hat{\alpha}$ and $R(\hat{\alpha}') = dt \wedge \pi^*_{1,\rho} R(\hat{\alpha})$, where $\pi_{1,\rho} : \rho \times \mathrm{id}_{S^1} \to \rho$ is the projection. The same statement holds replacing ρ by Y. Hence, still supposing that n and m are even, we define:

- for $\hat{\alpha} \in \hat{h}^{n-1}(\rho)$ and $\hat{\beta} \in h^m(Y)$, $\hat{\alpha} \times \hat{\beta} := \int_{S^1} \pi^*_{1,\rho} \hat{\alpha} \times \hat{\beta}$;
- $\hat{\alpha} \in \hat{h}^n(\rho)$ and $\hat{\beta} \in h^{m-1}(Y)$, $\hat{\alpha} \times \hat{\beta} := \int_{S^1} \hat{\alpha} \times \pi^*_Y \hat{\beta}$;
- $\hat{\alpha} \in \hat{h}^{n-1}(\rho)$ and $\hat{\beta} \in h^{m-1}(Y)$, $\hat{\alpha} \times \hat{\beta} := -\int_{S^1} \int_{S^1} \pi_{1,\rho} \hat{\alpha}' \times \pi^*_Y \hat{\beta}'$.

5.3 Parallel classes

When $\rho : A \hookrightarrow X$ is a closed embedding, we can simplify the model described above without considering the cylinder, see (UPMEIER, 2012). With the same data of definition 35 we have the following

Definition 39. A *differential function* $(f,h,\omega):(X,A)\to(Y,y_0,\kappa_n)$, from (X,A) to (Y,y_0,κ_n), is a triple (f,h,ω) such that:

- $f:(X,A)\to(Y,y_0)$ is a continuous function;
- $h\in C^{n-1}_{sm}(X,A;V^\bullet)$;
- $\omega\in\Omega^n_{\mathrm{cl}}(X,A;V^\bullet)$;

satisfying the condition that $\delta^{n-1}h=\chi_{(X,A)}(\omega)-f^*\kappa_n$, where $\chi_{(X,A)}:\Omega^\bullet(X,A;V^\bullet)\to C^\bullet_{sm}(X,A;V^\bullet)$ is the natural morphism defined similarly to the case of maps (5.1).

In other words, a differential function from (X,A) to (Y,y_0,κ_n) is a differential function $(f,h,\omega):X\to(Y,\kappa_n)$ such that $(f,h,\omega)|_A=(c_{y_0},0,0)$.

A homotopy between two differential functions $(f_0,h_0,\omega),(f_1,h_1,\omega):(X,A)\to(Y,y_0,\kappa_n)$ is a differential function $(F,H,\pi_X^*\omega):(X\times I,A\times I)\to(Y,y_0,\kappa_n)$ defined similarly to 36.

Definition 40. We denote by $\hat{h}^n_{\mathrm{par}}(X,A)$ the set of equivalence classes, denoted by $[(f,h,\omega)]$, of differential functions $(f,h,\omega):(X,A)\to(E_n,e_n,\iota_n)$ modulo the equivalence relation described above.

We define the sum in $\hat{h}^n(X,A)$ similarly to (5.4), i.e.,

$$[(f_0,h_0,\omega_0]+[(f_1,h_1,\omega_1]:=[(\alpha_n\circ(f_0,f_1),h_0+h_1+(f_0,f_1)^*A_{n-1},\omega_0+\omega_1)],$$

where $(f_0,f_1):(X,A)\to(E_n\times E_n,e_n\times e_n)$.

The natural transformations that characterize the parallel theory are defined as

$$I:\hat{h}^n(X,A)\to h^n(X,A),\quad [(f,h,\omega)]\mapsto[f];$$

$$R:\hat{h}(X,A)\to\Omega^{\mathrm{even}}_{\mathrm{cl}}(X,A;\mathfrak{h}_{\mathbb{R}}),\quad [(f,h,\omega)]\mapsto\omega;$$

$$a:\Omega^{\mathrm{odd}}(X,A;\mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d)\to\hat{h}(X,A),\quad \omega\mapsto[(0,0,\omega].$$

We define S^1-integration like in section 5.2.1, replacing the pair $(\mathrm{Cyl}(\rho),A)$ by (X,A).

Given a closed embedding $\rho:A\hookrightarrow X$, we denote by $\hat{h}^n_{\mathrm{par}}(\rho)$ the parallel subgroup of the model defined above. We have the natural morphism

$$\begin{aligned} h^n_{\mathrm{par}}(X,A)&\to\hat{h}^n_{\mathrm{par}}(\rho)\\ [(f,h,\omega)]&\mapsto[(f\circ\rho,\pi_\rho^*h,\omega,0)], \end{aligned}$$

where $\pi_\rho:\mathrm{Cyl}(\rho)\to X$ is the projection. In (RUFFINO, 2015b), it is proven that it is an isomorphism.

5.4 Hopkins-Singer model on maps of pairs

Given a map of pairs $(\rho, \rho') : (A, A') \to (X, X')$, we get the pair $(\mathrm{Cyl}(\rho), \mathrm{Cyl}(\rho'))$, which is a cofibration. Moreover, given a topological space with marked point (Y, y_0), a graded real vector space $V^\bullet$ and a real singular cocycle $\kappa_n \in C^n(Y, y_0, V^\bullet)$, we have the following

Definition 41. A *differential function* $(f, h, \omega) : (\rho, \rho') \to (Y, y_0, \kappa_n)$, from (ρ, ρ') to (Y, y_0, κ_n), is a triple (f, h, ω), where

- $f : (\mathrm{Cyl}(\rho), \mathrm{Cyl}(\rho')) \to (Y, y_0)$ is a continuous function;
- $h \in C^{n-1}_{sm}(\mathrm{Cyl}(\rho), \mathrm{Cyl}(\rho'); V^\bullet)$;
- $\omega \in \Omega^n_{\mathrm{cl}}(X, X'; V^\bullet)$;

such that $\delta^{n-1} h = \chi_{(\rho,\rho')}(\omega) - f^* \kappa_n$, with $\chi_{(\rho,\rho')} : \Omega^\bullet(X, X'; V^\bullet) \to C^\bullet_{sm}(Cyl(\rho), Cyl(\rho'); V^\bullet)$ being the natural morphism defined similarly to the case of maps (5.1).

In other words, we are defining a differential function from (ρ, ρ') to (Y, y_0, κ_n) as in definition 35, but requiring that $(f, h, \omega)|_{Cyl(\rho')} = (y_0, 0, 0)$.

A homotopy between two differential functions $(f_0, h_0, \omega), (f_1, h_1, \omega) : (\rho, \rho') \to (Y, y_0, \kappa_n)$ is a differential function $(F, H, \pi_X^* \omega) : (\rho, \rho') \times \mathrm{id}_I \to (Y, y_0, \kappa_n)$, defined as 36, with the additional condition that it is constant on $\mathrm{Cyl}(\rho') \times I$, i.e.,

$$(F, H, \pi_X^* \omega)|_{\mathrm{Cyl}(\rho') \times I} = (c_{y_0}, 0, 0).$$

Basically we are joining definitions 35 and 40, since we are considering the cylinder of (ρ, ρ'), as in the former, and we are requiring triviality on a suitable subspace, as in the latter. In Keeping with this principle, given a differential function $(f, h, \omega) : (X, X') \to (Y, y_0, \kappa_n)$, we define a strong topological trivialization of (f, h, ω), induced by $\eta \in \Omega^{n-1}(X, X'; V^\bullet)$, like in definition 37, but requiring that the homotopy is constant on $X' \times I$, i.e. that $(F, H, \pi_X^* \omega)|_{X' \times I} = (c_{y_0}, 0, 0)$.

With these preliminaries, we apply definition 38 to a map of pairs without variations. We observe in particular that a homotopy between two representatives must be constant both on $\mathrm{Cyl}(\rho') \times I$ (by definition of homotopy in this setting) and on $A \times \{1\} \times I$ (by definition 38). The sum is defined by formula (5.4), where $(f_0, f_1) : (\mathrm{Cyl}(\rho), \mathrm{Cyl}(\rho') \cup (A \times \{1\})) \to (E_n \times E_n, e_n \times e_n)$. The natural transformations I, R and a are defined as in the relative framework, simply checking that they respect triviality on X', A' and $\mathrm{Cyl}(\rho')$ in the case of maps of pairs. The same consideration holds about S^1-integration.

5.4.1 Relative-parallel product

Using the tools developed up to now, it is straightforward to define the relative-absolute product (2.4) in the Hopkins-Singer model.

In fact, given $\hat{\alpha} = [(f_0, h_0, \omega_0, \eta_0)] \in \hat{h}^n(\rho)$ and $\hat{\beta} = [(f_1, h_1, \omega_1)] \in \hat{h}^m_{\text{par}}(Y, B)$, with n and m even, the class $\hat{\alpha} \times \hat{\beta} \in \hat{h}^{n+m}(\rho \times \mathrm{id}_{(Y,B)})$ is defined similarly to the formula (5.5) , i.e.,

$$[(f_0, h_0, (\omega_0, \eta_0)] \times [(f_1, h_1, \omega_1)] := [(\mu_{n,m} \circ (f_0 \times f_1), h_0 \cup \chi_Y(\omega_1) + \chi_\rho(\omega_0, \eta_0) \cup h_1 \\ + \pi^*_\rho(\omega_0 \times \omega_1) - h_0 \cup \delta h_1 + (f_0, f_1)^* M_{n,m}, \omega_0 \wedge \omega_1, \eta_0 \wedge \omega_1)].$$

we think of $\mu_{n,m} \circ (f_0 \times f_1) : (\mathrm{Cyl}(\rho) \times Y, (\mathrm{Cyl}(\rho) \times B) \cup (A \times \{1\} \times Y)) \to (E_{n+m}, e_{n+m})$. The other components vanish on $\mathrm{Cyl}(\rho) \times B$ too, since each term defined on Y vanishes on B by hypothesis. The extension to n and m not necessarily even is defined trough S^1-integration, as at the end of section 5.2.1.

6 Integration

In this chapter, we construct compactly supported and vertically-compactly-supported integration in differential cohomology. For it, we briefly recall the definitions of compactly-supported cohomology and vertically compactly-supported cohomology, both, topological and differential. In the second section, since the integration is defined as a composition of Thom isomorphism with some natural maps, we describe some versions of the Thom isomorphisn. Here we use the relative-parallel product described in the section 2.3. Finally, in the last section, we characterize axiomatically this integration .

6.1 Vertically-compactly-supported cohomology

Let X be a locally compact space, and let $\langle \mathcal{K}_X, \subseteq \rangle$ be the directed partially ordered set, where $\mathcal{K}_X$ denote the set formed by the compact subsets of X, and $\subseteq$ the inclusion of set. Let us consider a family of morphisms $i_{K,K'} : (X, X \setminus K') \to (X, X \setminus K)$ for all $K \subseteq K'$ satisfying $i_{K,K} = \mathrm{id}_{(X,X\setminus K)}$ and $i_{K,K''} = i_{K,K'} \circ i_{K',K''}$ for all $K \subseteq K' \subseteq K''$.

Definition 42. Given a cohomology theory $h^\bullet$, the corresponding *compactly-supported cohomology* $h^\bullet_{\mathrm{cpt}}(X)$ is by definition the direct limit of the system $\left\langle (h^\bullet(X, X \setminus K))_{K \in \mathcal{K}_X}, i^*_{K,K'} \right\rangle$ over $\mathcal{K}_X$, i. e.,

$$h^\bullet_{\mathrm{cpt}}(X) := \varinjlim_{K \subset X} h^\bullet(X, X \setminus K).$$

If we consider a differential extension $\hat{h}^\bullet : \mathcal{M}_2^{\mathrm{op}} \to \mathcal{A}_{\mathbb{Z}}$ of $h^\bullet$, for X a manifold, the corresponding differential extension $\hat{h}^\bullet_{\mathrm{cpt}}(X)$ is given by

$$\hat{h}^\bullet_{\mathrm{cpt}}(X) := \varinjlim_{K \subset X} \hat{h}^\bullet_{\mathrm{par}}(X, X \setminus K).$$

Notice that an element $\hat{\alpha} \in \hat{h}^\bullet_{\mathrm{cpt}}(X)$ is an equivalence class $\hat{\alpha} = [\hat{\alpha}_0]$ represented by a parallel class $\hat{\alpha}_0 \in \hat{h}^\bullet_{\mathrm{par}}(X, X \setminus K)$, with K being a compact subset of X. If X is compact, we get a canonical isomorphism $\hat{h}^\bullet_{\mathrm{cpt}}(X) = \hat{h}^\bullet(X)$.

Similarly, we define the following direct limits $\Omega^\bullet_{\mathrm{cpt}}(X) := \varinjlim_{K \subset X} \Omega^\bullet(X, X \setminus K)$, $\Omega^\bullet_{\mathrm{cl,cpt}}(X) := \varinjlim_{K \subset X} \Omega^\bullet_{\mathrm{cl}}(X, X \setminus K)$, etc.

In this context, we also get the natural transformations of $\mathcal{A}_{\mathbb{Z}}-$value functors $I_{\mathrm{cpt}} : \hat{h}^\bullet_{\mathrm{cpt}}(X) \to h^\bullet_{\mathrm{cpt}}(X)$, $R_{\mathrm{cpt}} : \hat{h}^\bullet_{\mathrm{cpt}}(X) \to \Omega^\bullet_{\mathrm{cl,cpt}}(X, \mathfrak{h}^\bullet_{\mathbb{R}})$ and $a_{\mathrm{cpt}} : \Omega^{\bullet-1}_{\mathrm{cpt}}(X, \mathfrak{h}^\bullet_{\mathbb{R}})/\mathrm{Im}(d) \to \hat{h}^\bullet_{\mathrm{cpt}}(X)$ that make diagram (2.1) commutative after adapting to the compactly supported version.

Remark 9. Notice that, in general, given any smooth map $f : Y \to X$, it does not always induce the natural maps $f^* : h^\bullet_{\mathrm{cpt}}(X) \to h^\bullet_{\mathrm{cpt}}(Y)$ and $f^* : \hat{h}^\bullet_{\mathrm{cpt}}(X) \to \hat{h}^\bullet_{\mathrm{cpt}}(Y)$. Nevertheless, an open embedding $\iota : Y \hookrightarrow X$ induces the push-forward $\iota_* : h^\bullet_{\mathrm{cpt}}(Y) \to h^\bullet_{\mathrm{cpt}}(X)$ and $\iota_* : \hat{h}^\bullet_{\mathrm{cpt}}(Y) \to \hat{h}^\bullet_{\mathrm{cpt}}(X)$, see (RUFFINO; BARRIGA, 2018).

Definition 43. Let $\pi : E \to X$ be a fibre bundle. A subset $H \subset E$ is called *vertically compact* if H is closed in E and $H \cap \pi^{-1}(K)$ is compact for every $K \subset X$ compact.

Definition 44. Let $\pi : E \to X$ be a fibre bundle with E and X locally compact, let $\mathcal{K}_{E\to X}$ be the set formed by the vertically compact subsets of E. Following the same construction above, the *vertically compactly-supported cohomology* $h^\bullet_{\mathrm{vcpt}}(h)$ is defined by

$$h^\bullet_{\mathrm{vcpt}}(E) := \varinjlim_{H \subset E} h^\bullet(E, E \setminus H),$$

similarly we define

$$\hat{h}^\bullet_{\mathrm{vcpt}}(E) := \varinjlim_{H \subset E} \hat{h}^\bullet_{\mathrm{par}}(E, E \setminus H).$$

Remark 10. Similar to the previous remark, if $\pi : E \to X$ is a fibre bundle and $\iota : E \hookrightarrow F$ is an open embedding of fiber bundles over X, then, ι induces the well-defined push-forwards $\iota_* : h^\bullet_{\mathrm{vcpt}}(E) \to h^\bullet_{\mathrm{vcpt}}(F)$ and $\iota_* : \hat{h}^\bullet_{\mathrm{vcpt}}(E) \to \hat{h}^\bullet_{\mathrm{vcpt}}(F)$, i.e., both the topological and the differential groups are functorial with respect to open embeddings of fibre bundles over the base X.

6.2 Thom morphism

Let X be a manifold and $\pi : E \to X$ a real vector bundle of rank n.

Definition 45. The bundle E is orientable with respect to a multiplicative cohomology theory $h^\bullet$ if there exists a topological *Thom class* $u \in h^\bullet_{\mathrm{vcpt}}(E)$.

Definition 46. Let $\hat{h}^\bullet$ be a multiplicative differential extension of $h^\bullet$. A *differential Thom class* of E is a class $\hat{u} \in \hat{h}^\bullet_{\mathrm{vcpt}}(E)$ such that $I(\hat{u}) \in h^\bullet_{\mathrm{vcpt}}(E)$ is a topological Thom class.

Let us suppose that the Thom class $u \in h^\bullet_{\mathrm{vcpt}}(E)$ is represented by $u_0 \in h^\bullet(E, E\setminus H)$, with $H \subset E$ vertically compact, given $\alpha \in h^\bullet(X)$, then $\pi^*\alpha \in h^\bullet(E)$, the relative-absolute product

$$\cdot : h^\bullet(E, E \setminus H) \otimes_{\mathbb{Z}} h^\bullet(E) \to h^\bullet(E, E \setminus H)$$

allows us to define a version of the Thom isomorphism

$$\begin{aligned} T : h^\bullet(X) &\to h^{\bullet+n}_{\mathrm{vcpt}}(E) \\ \alpha &\mapsto u \cdot \pi^*\alpha, \end{aligned} \tag{6.1}$$

where the class $u \cdot \pi^*\alpha$ is represented by $u_0 \cdot \pi^*\alpha \in h^\bullet(E, E \setminus H)$.

Now, let $\hat{u} \in \hat{h}^{\bullet}_{\mathrm{vcpt}}(E)$ be a differential Thom class represented by $\hat{u}_0 \in \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H)$, for $\hat{\alpha} \in \hat{h}^{\bullet}(X)$, then $\pi^*\hat{\alpha} \in \hat{h}^{\bullet}(E)$, the refinement of the product above

$$\cdot : \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H) \otimes_{\mathbb{Z}} \hat{h}^{\bullet}(E) \to \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H), \tag{6.2}$$

allows to define the differential Thom morphism

$$T : \hat{h}^{\bullet}(X) \to \hat{h}^{\bullet+n}_{\mathrm{vcpt}}(E),\ \hat{\alpha} \mapsto \hat{u} \cdot \pi^*\hat{\alpha} \tag{6.3}$$

that refines the corresponding topological Thom isomorphism (6.1), where the class $\hat{u} \cdot \pi^*\hat{\alpha}$ is represented by $\hat{u}_0 \cdot \pi^*\hat{\alpha} \in \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H)$.

Another version of the Thom isomorphism is the **Compactly-supported Thom morphism**, both topological and differential. In order to define the differential one, the parallel-absolute product (6.2) is not enough, we will use the parallel-relative product (2.4) as we will see.

Let $\pi : E \to X$ a real vector bundle of rank n. Fix a Thom class $u \in h^{\bullet}_{\mathrm{vcpt}}(E)$ represented by $u_0 \in h^{\bullet}(E, E \setminus H)$ with $H \subset E$ vertically compact, suppose that $\alpha \in h^{\bullet}_{\mathrm{cpt}}(X)$ is represented by $\alpha_0 \in h^{\bullet}(X, X \setminus K)$ with $K \subseteq X$ compact, notice that $\pi^*\alpha_0 \in h^{\bullet}\left(E, E \setminus \pi^{-1}(K)\right)$. By remark 11, $h^{\bullet}(E, E \setminus H) \simeq h^{\bullet}(E, E \setminus H^{\circ})$ similarly $h^{\bullet}(E, E \setminus \pi^{-1}(K)) \simeq h^{\bullet}(E, E \setminus \pi^{-1}(K)^{\circ})$, then since the pairs $(E, E \setminus H^{\circ})$ and $(E, E \setminus \pi^{-1}(K)^{\circ})$ are cofibrations, we have the well-define product

$$\begin{aligned}
&\cdot : h^{\bullet}(E, E \setminus H) \otimes_{\mathbb{Z}} h^{\bullet}(E, E \setminus \pi^{-1}(K)) \simeq h^{\bullet}(E, E \setminus H^{\circ}) \otimes_{\mathbb{Z}} h^{\bullet}(E, E \setminus \pi^{-1}(K)^{\circ}) \\
&\xrightarrow{1.5} h^{\bullet}\left(E \times E, E \times (E \setminus \pi^{-1}(K)^{\circ}) \cup (E \setminus H^{\circ}) \times E\right) \xrightarrow{\Delta^* 6.5} h^{\bullet}\left(E, (E \setminus \pi^{-1}(K)^{\circ}) \cup (E \setminus H^{\circ})\right) \\
&\qquad = h^{\bullet}\left(E, E \setminus (\pi^{-1}(K) \cap H)^{\circ}\right) \simeq h^{\bullet}\left(E, E \setminus (\pi^{-1}(K) \cap H)\right),
\end{aligned}$$

due to the compactness of $\pi^{-1}(K) \cap H$, we define the topological *compactly supported* Thom isomorphism

$$\begin{aligned} T : h^{\bullet}_{\mathrm{cpt}}(X) &\to h^{\bullet+n}_{\mathrm{cpt}}(E) \\ \alpha &\mapsto u \cdot \pi^*\alpha, \end{aligned} \tag{6.4}$$

where $u \cdot \pi^*\alpha$ is represented by $\Delta^*(u_0 \cdot \pi^*\alpha_0) \in h^{\bullet}\left(E, E \setminus (\pi^{-1}(K) \cap H)\right)$ with Δ^* being the pull-back of the diagonal morphism

$$\left(E, (E \setminus \pi^{-1}(K)^{\circ}) \cup (E \setminus H^{\circ})\right) \xrightarrow{\Delta} \left(E \times E, E \times (E \setminus \pi^{-1}(K)^{\circ}) \cup (E \setminus H^{\circ}) \times E\right). \tag{6.5}$$

Remark 11. On definition 44, it is possible to assume that H and $E \setminus H^{\circ}$ are non-empty submanifolds of E of dimension n, moreover, using the collar neighborhood theorem, we have that H and H° are homotopy equivalent, thus $h^{\bullet}(E, E \setminus H) \simeq h^{\bullet}(E, E \setminus H^{\circ})$. This result can be refined to the case of parallel class, i.e.,

$$\hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H) \simeq \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H^{\circ}).$$

In fact, let us consider the natural inclusion of pairs $\nu : (E, E \setminus H) \to (E, E \setminus H^{\circ})$ and the corresponding pull-back. By proposition 1 adapted to the relative case we get the following morphism of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \hat{h}^{\bullet}_{\mathrm{fl}}(E, E \setminus H^{\circ}) \arrow[r] \arrow[d, "\nu^*_{\mathrm{fl}}"] & \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H^{\circ}) \arrow[r] \arrow[d, "\nu^*_{\mathrm{par}}"] & \Omega^{\bullet}_{\mathrm{ch},0}(E, E \setminus H^{\circ}) \arrow[r] \arrow[d, "\nu^*_{\Omega}"] & 0 \\
0 \arrow[r] & \hat{h}^{\bullet}_{\mathrm{fl}}(E, E \setminus H) \arrow[r] & \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H) \arrow[r] & \Omega^{\bullet}_{\mathrm{ch},0}(E, E \setminus H) \arrow[r] & 0,
\end{tikzcd}$$

where $\Omega^{\bullet}_{\mathrm{ch},0}$ contains the possible curvatures of parallel classes. By proposition 2, the flat groups are topological, hence ν^*_{fl} is an isomorphism. The curvature is a form on E that vanishes respectively on $E \setminus H^{\circ}$ or on $E \setminus H$, that is the same by continuity, and it represents a class in the image of the Chern character, that coincides in the two cases because of the equivalence up to homotopy, therefore ν^*_{Ω} is an isomorphism. It follows from the five lemma that ν^*_{par} is an isomorphism too.

For the refinement of the isomorphism (6.4), we fix a differential Thom class $\hat{u} \in \hat{h}^{\bullet}_{\mathrm{vcpt}}(E)$ represented by $\hat{u}_0 \in \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H)$. Let $\hat{\alpha} \in \hat{h}^{\bullet}_{\mathrm{cpt}}(X)$ a class represented by $\hat{\alpha}_0 \in \hat{h}^{\bullet}_{\mathrm{par}}(X, X \setminus K)$, notice that $\pi^*\alpha_0 \in \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus \pi^{-1}(K)) = \hat{h}^{\bullet}_{\mathrm{par}}(\iota : (E \setminus \pi^{-1}(K)) \hookrightarrow E)$, by remark 11 we have that $\hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H) \simeq \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H^{\circ})$, then, the parallel-relative product

$$\begin{aligned}
&\cdot : \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus H^{\circ}) \otimes_{\mathbb{Z}} \hat{h}^{\bullet}_{\mathrm{par}}(\iota) \xrightarrow{2.4} \hat{h}^{\bullet}_{\mathrm{par}}(\mathrm{id}_{(E, E \setminus H^{\circ})} \times \iota) \\
&= \hat{h}^{\bullet}_{\mathrm{par}} \left((E \times (E \setminus \pi^{-1}(K)), (E \setminus H^{\circ}) \times (E \setminus \pi^{-1}(K))) \to (E \times E, (E \setminus H^{\circ}) \times E) \right) \\
&\xrightarrow{\Delta^*(6.8)} \hat{h}^{\bullet}_{\mathrm{par}} \left((E \setminus \pi^{-1}(K), (E \setminus H^{\circ}) \cap (E \setminus \pi^{-1}(K))) \to (E, E \setminus H^{\circ}) \right) \\
&\overset{\text{remark } 12}{\simeq} \hat{h}^{\bullet}_{\mathrm{par}} \left(((E \setminus H^{\circ})^{\circ} \cup (E \setminus \pi^{-1}(K))) \hookrightarrow E \right) \simeq \hat{h}^{\bullet}_{\mathrm{par}} \left(E, E \setminus (H \cap \pi^{-1}(K)) \right),
\end{aligned} \tag{6.6}$$

allows us to define the compactly-supported Thom morphism

$$\begin{aligned}
T : \hat{h}^{\bullet}_{\mathrm{cpt}}(X) &\to \hat{h}^{\bullet+n}_{\mathrm{cpt}}(E) \\
\alpha &\mapsto \hat{u} \cdot \pi^*\hat{\alpha},
\end{aligned} \tag{6.7}$$

where $\hat{u} \cdot \pi^*\hat{\alpha}$ is represented by $\Delta^*(\hat{u}_0 \cdot \pi^*\hat{\alpha}_0) \in \hat{h}^{\bullet}_{\mathrm{par}}(E, E \setminus (\pi^{-1}(K) \cap H))$ with Δ^* being the pull-back of the diagonal morphism

$$\begin{tikzcd}
(E \setminus \pi^{-1}(K), (E \setminus H^{\circ}) \cap (E \setminus \pi^{-1}(K))) \arrow[r, hook] \arrow[d, "\Delta"] & (E, E \setminus H^{\circ}) \arrow[d, "\Delta"] \\
(E \times (E \setminus \pi^{-1}(K)), (E \setminus H^{\circ}) \times (E \setminus \pi^{-1}(K))) \arrow[r, hook] & (E \times E, (E \setminus H^{\circ}) \times E)
\end{tikzcd} \tag{6.8}$$

Remark 12. Let us see that the first isomorphism of (6.6). We set $K' := E \setminus \pi^{-1}(K)$, $H' := E \setminus H^{\mathrm{o}}$, then we need to prove that

$$\hat{h}^{\bullet}_{\mathrm{par}}\left((K', H' \cap K') \xrightarrow{(\iota_1, \iota'_1)} (E, H')\right) \simeq \hat{h}^{\bullet}_{\mathrm{par}}\left((H'^{\mathrm{o}} \cup K', \varnothing) \xrightarrow{(\iota_3, \iota'_3)} (E, \varnothing)\right).$$

In fact, using the intermediary map of pairs $(H'^{\mathrm{o}} \cup K', (H'^{\mathrm{o}} \cup K') \cap H') \xrightarrow{(\iota_2, \iota'_2)} (E, H')$, we prove that the two groups above are isomorphic to $\hat{h}^{\bullet}_{\mathrm{par}}(\iota_2, \iota'_2)$. Let us consider the natural inclusion of map of pairs $(j, \mathrm{id}) : (\iota_1, \iota'_1) \to (\iota_2, \iota'_2)$

$$\begin{tikzcd}
(K', H' \cap K') \arrow[r, "(\iota_1, \iota'_1)"] \arrow[d, hook, "j"] & (E, H') \arrow[d, equal, "\mathrm{id}"] \\
(H'^{\mathrm{o}} \cup K', (H'^{\mathrm{o}} \cup K') \cap H') \arrow[r, "(\iota_2, \iota'_2)"] & (E, H'),
\end{tikzcd} \tag{6.9}$$

and the corresponding pull-back. By proposition 1, we get the following morphism of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \hat{h}^{\bullet}_{\mathrm{fl}}(\iota_1, \iota'_1) \arrow[r] \arrow[d, "(j,\mathrm{id})^*_{\mathrm{fl}}"] & \hat{h}^{\bullet}_{\mathrm{par}}(\iota_1, \iota'_1) \arrow[r] \arrow[d, "(j,\mathrm{id})^*_{\mathrm{par}}"] & \Omega^{\bullet}_{\mathrm{ch},0}(\iota_1, \iota'_1) \arrow[r] \arrow[d, "(j,\mathrm{id})^*_{\Omega}"] & 0 \\
0 \arrow[r] & \hat{h}^{\bullet}_{\mathrm{fl}}(\iota_2, \iota'_2) \arrow[r] & \hat{h}^{\bullet}_{\mathrm{par}}(\iota_2, \iota'_2) \arrow[r] & \Omega^{\bullet}_{\mathrm{ch},0}(\iota_2, \iota'_2) \arrow[r] & 0,
\end{tikzcd}$$

since H' and H'^{o} have the same homotopy type, then applying corollary 3 we get that $(j, \mathrm{id})^*_{\mathrm{fl}}$ is an isomorphism; and since in the two rows of diagram (6.9) we get forms in E that vanish on H' and on K', the image of the Chern character being the same again because of corollary 3, we get that $(j, \mathrm{id})^*_{\Omega}$ is an isomorphism. Hence, applying the five lemma to this diagram we get that $(j, \mathrm{id})^*_{\mathrm{par}}$ is an isomorphism. Similarly we prove that $\hat{h}^{\bullet}_{\mathrm{par}}(\iota_2, \iota'_2) \simeq \hat{h}^{\bullet}_{\mathrm{par}}(\iota_3, \iota'_3)$.

6.3 Integration

Once the topological Thom isomorphisms (6.1) is given, we define the Gysin map $f_{(\iota,u,\varphi)!} : h^{\bullet}(Y) \to h^{\bullet - n}(X)$ associated to a smooth fibre bundle $f : Y \to X$ with compact fibres without boundary $f : Y \to X$, where the triple (ι, u, φ) is called a *representative* of an $h^{\bullet}$–orientation of f. Similar to the case when Y and X are compact, see (RUFFINO; BARRIGA, 2018), the triple is defined as follows:

- $\iota : Y \hookrightarrow X \times \mathbb{R}^N$ is a neat fibre-bundle embedding over X, for any $N \in \mathbb{N}$;
- u is a Thom class of the normal bundle $N_{\iota(Y)}\left(X \times \mathbb{R}^N\right)$;
- $\varphi : N_{\iota(Y)}(X \times \mathbb{R}^N) \to U$ is a diffeomorphism, for U a neat tubular neighborhood of $\iota(Y)$ in $X \times \mathbb{R}^N$, such that the image of the fibre of $y \in Y$ is contained in $\{f(y)\} \times \mathbb{R}^N$.

An orientation is an equivalence class $[(\iota, u, \varphi)]$ of representatives up to homotopy and stabilization, as defined as in (RUFFINO; BARRIGA, 2018) with the natural adaptations without the compactness hypothesis.

With this ingredients, the Gysin map $f_! : h^\bullet(Y) \to h^{\bullet-n}(X)$, for $n = \dim Y - \dim X$, is given by the composition

$$h^\bullet(Y) \xrightarrow{T} h^{\bullet+N-n}_{\text{vcpt}}\left(N_{\iota(Y)}(X\times\mathbb{R}^N)\right) \xrightarrow{\varphi_*} h^{\bullet+N-n}_{\text{vcpt}}(U) \xrightarrow{i_*} h^{\bullet+N-n}_{\text{vcpt}}(X\times\mathbb{R}^N) \xrightarrow{\int_{\mathbb{R}^N}} h^{\bullet-n}(X),$$

i.e.,

$$f_!(\alpha) = \int_{\mathbb{R}^N} i_*\varphi_*(u\cdot\pi^*\alpha), \tag{6.10}$$

where :

- T is the topological Thom isomorphism defined as in (6.1);
- i_* is the push-forward in vertically-compactly supported cohomology induced by the inclusion $i : U \hookrightarrow X\times\mathbb{R}^N$, see remark 10;
- φ_* is the induced isomorphism by the diffeomorphism φ;
- the integration map $\int_{\mathbb{R}^N} : h^{\bullet+N}_{\text{vcpt}}(X\times\mathbb{R}^N) \to h^\bullet(X)$ is defined as follows: the open embedding $j : R^N \hookrightarrow (R^+)^N \simeq (S^1)^N$ induces the pushforward $(id\times j)_* : h^\bullet_{\text{vcpt}}(X\times R^N) \to h^\bullet(X\times(S^1)^N)$, hence we set
$$\int_{\mathbb{R}^N} := \int_{S^1}\cdots\int_{S^1}\circ(\mathrm{id}\times j)_*.$$

Similarly, using the differential Thom morphism (6.3) we refine the Gysin (6.10), we get $f_! : \hat{h}^\bullet(Y) \to \hat{h}^{\bullet-n}(X)$ defined by

$$f_!(\hat\alpha) = \int_{\mathbb{R}^N} i_*\varphi_*(\hat u\cdot\pi^*\hat\alpha),$$

where the triple $(\iota, \hat u, \varphi)$ is called a *representative* of an $\hat{h}^\bullet-$orientation of f and is defined similar to the topological case just considering a differential Thom class $\hat u$.

6.3.1 Compactly-supported integration

Given a neat submersion $f : Y \to X$ between manifolds (not necessarily a fibre bundle), we define a representative of an orientation of f as a triple $(\iota, \hat u, \varphi)$, where:

- $\iota : Y \hookrightarrow X\times\mathbb{R}^N$ is a neat embedding, for any $N\in\mathbb{N}$, with $\pi_X\circ\iota = f$;
- $\hat u$ is a Thom class of the normal bundle $N_{\iota(Y)}\left(X\times\mathbb{R}^N\right)$;
- $\varphi : N_{\iota(Y)}(X\times\mathbb{R}^N) \to U$ is a diffeomorphism, for U a neat tubular neighborhood of $\iota(Y)$ in $X\times\mathbb{R}^N$, such that the image of the fibre of $y\in Y$ is contained in $\{f(y)\}\times\mathbb{R}^N$.

Now, as in the previous subsection, we define the Gysin map in compactly-supported cohomology $f_! : h^\bullet_{\text{cpt}}(Y) \to h^{\bullet-n}_{\text{cpt}}(X)$ associated to f, as the composition

$$\hat{h}^\bullet_{\text{cpt}}(Y) \xrightarrow{T} \hat{h}^{\bullet+N-n}_{\text{cpt}}\left(N_{\iota(Y)}(X \times \mathbb{R}^N)\right) \xrightarrow{\varphi_*} \hat{h}^{\bullet+N-n}_{\text{cpt}}(U) \xrightarrow{i_*} \hat{h}^{\bullet+N-n}_{\text{cpt}}(X \times \mathbb{R}^N) \xrightarrow{\int_{\mathbb{R}^N}} \hat{h}^{\bullet-n}_{\text{cpt}}(X),$$

i.e.,

$$f_!(\hat{\alpha}) = \int_{\mathbb{R}^N} i_*\varphi_*(\hat{u} \cdot \pi^*\hat{\alpha}), \text{ for } n = \text{dim}Y - \text{dim}X, \tag{6.11}$$

where T is the topological Thom isomorphism defined as in (6.7), i_* is the push-forward induced by the inclusion $i : U \hookrightarrow X \times \mathbb{R}^N$, φ_* is the induced isomorphism by the diffeomorphism φ, and, the integration map $\int_{\mathbb{R}^N} : \hat{h}^{\bullet+N}_{\text{cpt}}(X \times \mathbb{R}^N) \to \hat{h}^\bullet(X)$ is defined as follows:

The open embedding $j : \mathbb{R} \to \mathbb{R}^+ \simeq S^1$ induces the push-forward $(\text{id} \times j)^* : \hat{h}^\bullet_{\text{cpt}}(X \times \mathbb{R}) \to \hat{h}^\bullet_{\text{cpt}}(X \times S^1)$. A class $\alpha \in \hat{h}^\bullet_{\text{cpt}}(X \times S^1)$ can be represented without loss of generality by a class $\alpha_0 \in \hat{h}^\bullet(X \times S^1, (X \setminus K) \times S^1)$, with $K \subset X$ compact. We set $\int_{S^1} \alpha := [\int_{S^1} \alpha_0]$, so that

$$\int_{\mathbb{R}} := \int_{S^1} \circ (\text{id} \times j)_*. \tag{6.12}$$

6.4 Axioms

Theorem 12. *Given a multiplicative differential cohomology theory $\hat{h}^\bullet$ with S^1-integration, the integration map (6.11), defined for any $\hat{h}^\bullet-$oriented neat submersion $f : Y \to X$, satisfies:*

(I_1) *Naturality with respect to the natural transformations of $\hat{h}^\bullet$, i.e. the following diagram commutes:*

$$\begin{tikzcd}
\Omega^\bullet_{cpt}(Y; \mathfrak{h}^\bullet_{\mathbb{R}})/Im(d) \arrow[r, "a_{cpt}"] \arrow[d, "R_{(\iota,\hat{u},\varphi)}"] & \hat{h}^\bullet_{cpt}(Y) \arrow[r, "I_{cpt}"] \arrow[d, "f_!"] \arrow[rr, bend left, "R_{cpt}"] & h^\bullet_{cpt}(Y) \arrow[d, "f_!"] & \Omega^\bullet_{cl,cpt}(Y, \mathfrak{h}^\bullet_R) \arrow[d, "R_{(\iota,\hat{u},\varphi)}"] \\
\Omega^{\bullet-n}_{cpt}(X; \mathfrak{h}^\bullet_{\mathbb{R}})/Im(d) \arrow[r, "a_{cpt}"] & \hat{h}^{\bullet-n}_{cpt}(X) \arrow[r, "I_{cpt}"] \arrow[rr, bend right, "R_{cpt}"'] & h^{\bullet-n}_{cpt}(X) & \Omega^{\bullet-n}_{cl,cpt}(X, \mathfrak{h}^\bullet_R),
\end{tikzcd}$$

where $R(\iota, \hat{u}, \varphi)$ is the curvature map on differential forms, defined as follows:

$$\begin{aligned} R_{(\iota,\hat{u},\varphi)} : \Omega^\bullet_{cpt}(Y; \mathfrak{h}^\bullet_{\mathbb{R}}) &\to \Omega^\bullet_{cpt}(X; \mathfrak{h}^\bullet_{\mathbb{R}}) \\ \omega &\mapsto \int_{X\times\mathbb{R}^N/X} i_*\varphi_*(R(\hat{u} \wedge \pi^*\omega)). \end{aligned}$$

(I_2) *Naturality with respect to composition, i.e., given two neat* $\hat{h}^\bullet-$*oriented submersions* $f : Y \to X$ *and* $g : Z \to Y$ *, if we endow* $f \circ g : Z \to X$ *of the natural induced orientation, see lemma 6.29 of (RUFFINO; BARRIGA, 2018), then* $(f \circ g)_! = f_! \circ g_!$.

We now consider the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d, "f"] & Y' \arrow[d, "f'"] \\ X \arrow[r, hook, "i"] & X', \end{tikzcd} \tag{6.13}$$

where f and f' are neat submersions and i and j are open embeddings.

Lemma 6. *A* $\hat{h}^\bullet-$*orientation* $[(\iota', \hat{u}', \varphi')]$ *of* f' *naturally induces an orientation of* f.

Proof. We define a representative of an orientation of f, $(\iota, \hat{u}, \varphi)$, as follows

- Define $\iota : Y \hookrightarrow X \times \mathbb{R}^N$ and $\varphi : N_{\iota(Y)}(X \times \mathbb{R}^N) \to U$ in such a way that the following diagrams commute (respectively)

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d, hook, "\iota"] & Y' \arrow[d, hook, "\iota'"] \\ X \times \mathbb{R}^N \arrow[r, hook, "i \times \mathrm{id}"] & X' \times \mathbb{R}^N, \end{tikzcd} \qquad \begin{tikzcd} N_{\iota(Y)}(X \times \mathbb{R}^N) \arrow[r, hook, "k"] \arrow[d, "\varphi"] & N_{\iota'(Y')}(X' \times \mathbb{R}^N) \arrow[d, "\varphi'"] \\ U \subset X \times \mathbb{R}^N \arrow[r, hook, "i \times \mathrm{id}"] & X' \times \mathbb{R}^N \supset U', \end{tikzcd}$$

  i.e., $\iota := (i^{-1} \times \mathrm{id}) \circ \iota' \circ j : Y \hookrightarrow X \times \mathbb{R}^N$, clearly $\pi_X \circ \iota = f$, and $\varphi := (i \times \mathrm{id})^{-1} \circ \varphi' \circ k$, where k is essentially the differential of $i \times \mathrm{id}$.

- Now consider the following diagram

$$\begin{tikzcd} N_{\iota(Y)}(X \times \mathbb{R}^N) \arrow[r, hook, "k"] \arrow[d, "\pi_N"] & N_{\iota'(Y')}(X' \times \mathbb{R}^N) \arrow[d, "\pi_{N'}"] \\ Y \arrow[r, hook, "j"] & Y'. \end{tikzcd}$$

  Suppose that $\hat{u}' \in \hat{h}^\bullet_{\mathrm{vcpt}}(N_{\iota(Y)}(X \times \mathbb{R}^N))$ is a differential Thom class of $N_{\iota'(Y')}(X' \times \mathbb{R}^N)$ represented by $\hat{u}'_0 \in \hat{h}^\bullet_{\mathrm{par}}(N_{\iota'(Y')}(X' \times \mathbb{R}^N), N_{\iota'(Y')}(X' \times \mathbb{R}^N) \setminus H')$, then we have that the class $\hat{u} = k^*\hat{u}' \in h^\bullet_{\mathrm{vcpt}}(N_{\iota(Y)}(X \times \mathbb{R}^N))$ is represented by $\hat{u}_0 = k^*\hat{u}'_0 \in h^\bullet(N_{\iota(Y)}(X \times \mathbb{R}^N), N_{\iota(Y)}(X \times \mathbb{R}^N) \setminus H)$, with $H := k^{-1}(H')$.

Thus we get the a $\hat{h}^\bullet-$orientation $[(\iota, \hat{u}, \varphi)]$ of f. ∎

Theorem 13. *Given a multiplicative differential cohomology theory* $\hat{h}^\bullet$ *with* S^1-*integration, the integration map (6.11), defined for any* $\hat{h}^\bullet-$*oriented neat submersion* $f : Y \to X$*, satisfies:*

(I_3) *Naturality with respect to open embeddings, i.e., given a commutative diagram of the form (6.13) and assuming that the $\hat{h}^\bullet-$orientation of f is induced by the one of f', lemma 6, we have that $f'_! j_* \hat{\alpha} = i_* f_! \hat{\alpha}$ for any class $\hat{\alpha} \in \hat{h}^\bullet_{cpt}(Y)$.*

Proof. We need to prove that the following diagram

$$\begin{tikzcd}
\hat{\alpha} \in \hat{h}_{\mathrm{cpt}}(Y) \arrow[r, "j_*"] \arrow[d, "T"] & \hat{h}_{\mathrm{cpt}}(Y') \ni j_*\hat{\alpha} \arrow[d, "T'"] \\
\hat{u} \cdot \pi^*\hat{\alpha} \in \hat{h}_{\mathrm{cpt}}(N) \arrow[r, "k_*"] \arrow[d, "\iota_*\varphi_*"] & \hat{h}_{\mathrm{cpt}}(N') \ni \hat{u}' \cdot \pi'^* j_*\hat{\alpha} \arrow[d, "\iota'_*\varphi'_*"] \\
\hat{h}_{\mathrm{cpt}}(X \times \mathbb{R}^N) \arrow[r, "(i \times \mathrm{id})_*"] \arrow[d, "\int_{\mathbb{R}^N}"] & \hat{h}_{\mathrm{cpt}}(X' \times \mathbb{R}^N) \arrow[d, "\int_{\mathbb{R}^N}"] \\
\hat{h}_{\mathrm{cpt}}(X) \arrow[r, "i_*"] & \hat{h}_{\mathrm{cpt}}(X')
\end{tikzcd}$$

is commutative, i.e., $\int_{\mathbb{R}^N} \iota_*\varphi_*(\hat{u} \cdot \pi^*\hat{\alpha}) = i_* \int_{\mathbb{R}^N} \iota'_*\varphi'_*(\hat{u}' \cdot \pi'^* j_*\hat{\alpha})$. In fact, let $\hat{\alpha} \in \hat{h}^\bullet_{\mathrm{cpt}}(Y)$ and suppose that this class is represented by $\hat{\alpha}_0 \in \hat{h}^\bullet_{\mathrm{par}}(Y, Y \setminus K)$ and $j_*\hat{\alpha} \in \hat{h}^\bullet_{\mathrm{cpt}}(Y')$ is represented by $\hat{\alpha}_1 \in \hat{h}^\bullet_{\mathrm{par}}(Y', Y' \setminus j(H))$ in such a way that $\hat{\alpha}_0 = j^*\hat{\alpha}_1$.
Considering the notations an result of lemma 6 and the product 6.6, it follows that $T(\hat{\alpha})$ is represented by

$$\hat{\beta}_0 := \Delta^*(\hat{u}_0 \times \pi^*\hat{\alpha}_0) \in \hat{h}^\bullet_{\mathrm{par}}\left(N_{\iota(Y)}(X \times \mathbb{R}^N), N_{\iota(Y)}(X \times \mathbb{R}^N) \setminus (\pi^{-1}(K) \cap H)\right),$$

where $\hat{u}_0 \in \hat{h}^\bullet_{\mathrm{par}}(N, N \setminus H)$ represents the Thom class, and, $T'(j_!\hat{\alpha}) \in \hat{h}^\bullet_{\mathrm{cpt}}(Y')$ is represented by

$$\hat{\beta}_1 := \Delta^*(\hat{u}'_0 \times \pi'^*\hat{\alpha}_1) \in \hat{h}^\bullet_{\mathrm{par}}\left(N'_{\iota'(Y')}(X' \times \mathbb{R}^N)), N'_{\iota'(Y')}(X' \times \mathbb{R}^N)) \setminus (\pi'^{-1}(j(K)) \cap H')\right).$$

Since $\hat{u}' = k^*\hat{u}$ by hypothesis, we suppose that $H = k^{-1}(H')$ and $\hat{u}_0 = k^*\hat{u}'_0$, it follows that $\pi^*\hat{\alpha}_0 = \pi^* j\hat{\alpha}_1 = k^*\pi'^*\hat{\alpha}_1$, then

$$k^*\beta_1 = k^*\Delta^*(\hat{u}'_0 \times \pi^*\hat{\alpha}_1) = \Delta^*(k,k)^*(\hat{u}'_0 \times \pi'^*\hat{\alpha}_1) = \Delta^*(\hat{u}_0 \times \pi^*\hat{\alpha}_0) = \beta_0.$$

Applying $\iota_*\varphi_*$ and $\iota'_*\varphi'_*$ respectively to $[\hat{\beta}_0] \in \hat{h}^\bullet_{\mathrm{cpt}}(N_{\iota(Y)}(X \times \mathbb{R}^N))$ and $[\hat{\beta}_1] \in \hat{h}^\bullet_{\mathrm{cpt}}(N'_{\iota'(Y')}(X' \times \mathbb{R}^N))$, we get $\iota_*\varphi_*[\hat{\beta}_0] \in \hat{h}^\bullet_{\mathrm{par}}(X \times \mathbb{R}^n)$ and $\iota'_*\varphi'_*[\hat{\beta}_1] \in \hat{h}^\bullet_{\mathrm{par}}(X' \times \mathbb{R}^n)$ such that

$$(i \times id)^*\iota'_*\varphi'_*[\hat{\beta}_1] = \iota_*\varphi_*[\hat{\beta}_0].$$

Applying $\int_{\mathbb{R}^N}$, since S^1-integration is a natural transformation, we have that

$$i^* \int_{\mathbb{R}^n} \iota'_*\varphi'_*[\hat{\beta}_1] = \int_{\mathbb{R}^n} \iota_*\varphi_*[\hat{\beta}_0], \text{ thus } \left[\int_{\mathbb{R}^n} \iota'_*\varphi'_*[\hat{\beta}_1]\right] = i_* \left[\int_{\mathbb{R}^n} \iota_*\varphi_*[\hat{\beta}_0]\right],$$

i.e., $f'_! j_*\hat{\alpha} = i_* f_!\hat{\alpha}$. □

Remark 13. In a differential cohomology theory with S^1-integration, the product line bundle $\pi : X \times \mathbb{R} \to X$ has a natural orientation defined as follows:

- Thinking of $S^1 \subset \mathbb{C}$, we fix 1 as a marked point. There exists a unique class $\hat{v} \in \hat{h}^1_{\text{par}}(S^1, 1)$ such that $\int_{S^1} \hat{v} = 1$ and $R(\hat{v}) = dt$, see (RUFFINO; BARRIGA, 2018) Lemma 2.16;

- We fix an open interval $U := \exp(-\varepsilon, \varepsilon)$ around 1, where $\exp(t) := e^{2\pi i t}$, and a smooth increasing function $\phi : I \to I$ such that $\phi[0, \varepsilon) = 0$ and $\phi(1-\varepsilon, 1] = 1$. We get the smooth map of pairs $\varphi : (S^1, U) \to (S^1, 1)$, $\exp(t) \to \exp(\phi(t))$;

- We set $D_{\mathbb{R}} := [-1, 1]$ and $D' := \mathbb{R} \setminus D_{\mathbb{R}}$. We fix a diffeomorphism $\psi : (\mathbb{R}, D') \to (S^1 \setminus 1, U \setminus 1)$, that preserves the orientation in the usual sense, and we consider the embedding $\iota : (S^1 \setminus 1, U \setminus 1) \to (S^1, U)$ that induces an excision isomorphism. The class $\hat{u}_0 := \psi^* \iota_* \varphi_* \hat{v} \in \hat{h}^1_{\text{par}}(\mathbb{R}, D')$ represents a Thom class $\hat{u}$ of $X \times \mathbb{R}$, that depends on the choice of ϕ and ψ only up to homotopy, see (RUFFINO; BARRIGA, 2018) Def. 6.19, hence it represents a well-defined orientation of $X \times \mathbb{R}$.

It follows that the product bundle $\pi : X \times \mathbb{R}^n \to X$ of any rank has a natural orientation too.

Remark 14. In a multiplicative differential cohomology theory with S^1-integration, a $\hat{h}^\bullet-$orientation of a real vector bundle $\pi : E \to X$, defined as a differential Thom class up to homotopy, naturally induces a $\hat{h}^\bullet-$orientation of π as a smooth map. In fact, since X is a manifold, there exists a vector bundle $\pi' : F \to X$ such that $E \oplus F \simeq X \times \mathbb{R}^N$, therefore we can fix an embedding $\iota : E \hookrightarrow X \times \mathbb{R}^N$ such that $\iota(E)$ is a vector sub-bundle of $X \times \mathbb{R}^N$. In this case the role of F is performed by $\iota(E)^\perp$ with respect to the standard metric of $\mathbb{R}^n$. We identify $N_\iota(E)(X \times \mathbb{R}^N)$ with $\pi^*_{\iota(E)}(\iota(E)^\perp)$, whose total space is $X \times \mathbb{R}^N$ (i.e. the same of $\iota(E) \oplus \iota(E)^\perp$, so that the map φ is the identity. Since $X \times \mathbb{R}^N$ is canonically oriented by remark 13 , it follows by the 2×3 rule that an orientation of E induces an orientation of $\iota(E)^\perp$, therefore of $N_{\iota(E)}(X \times \mathbb{R}^N)$ by pull-back. This completes the orientation of π as a smooth map.

Theorem 14. *Given a multiplicative differential cohomology theory $\hat{h}^\bullet$ with S^1-integration, the integration map (6.11), defined for any $\hat{h}^\bullet-$oriented neat submersion $f : Y \to X$, satisfies:*

(I_4) *If $\pi : E \to X$ is a $\hat{h}^\bullet-$oriented real vector bundle, see remark 14, then $\pi_!$ is left-inverse to the Thom morphism, i.e. $\pi_! T(\hat{\alpha}) = \hat{\alpha}$ for every $\hat{\alpha} \in \hat{h}_{cpt}(X)$.*

Proof. We call $\hat{u}_E$ the Thom class of E, and orienting π as in remark 14, $\hat{u}_N$ the Thom class of $N_{\iota(E)}(X \times \mathbb{R}^n)$. Similarly, we call π_E and π_N the corresponding projections and T_E

and T_N the corresponding Thom morphisms, identifying E with $\iota(E)$. Given $\hat{\alpha} \in \hat{h}^{\bullet}_{\mathrm{cpt}}(X)$, we have to prove that $\pi_! T_E(\hat{\alpha}) = \hat{\alpha}$. In fact, the first step of $\pi_!$ consists of the application of T_N; therefore, we get

$$\begin{aligned} T_N T_E(\hat{\alpha}) = \hat{u}_N \cdot \pi_N^*(\hat{u}_E \cdot \pi_E^* \hat{\alpha}) &= (\hat{u}_N \cdot \pi_N^* \hat{u}_E) \cdot \pi_N^* \pi_E^* \hat{\alpha} \\ &= (\pi_E^* \hat{u}_{E^\perp} \cdot \pi_N^* \hat{u}_E) \cdot \pi^*_{X \times \mathbb{R}^N} \hat{\alpha} \\ &= \hat{u}_{X \times \mathbb{R}^N} \cdot \pi^*_{(X \times \mathbb{R}^N)} \hat{\alpha} = T_{X \times \mathbb{R}^N}(\hat{\alpha}). \end{aligned}$$

By definition, $\hat{u}_{X \times \mathbb{R}^N} = \hat{u}_{X \times \mathbb{R}} \times \cdots \times \hat{u}_{X \times \mathbb{R}}$. Using the notation of remark 13 and of definition 6.12,

$$\int_{\mathbb{R}} \hat{u}_{X \times \mathbb{R}} \cdot \pi^*_{X \times \mathbb{R}} \hat{\alpha} = \int_{S^1} ((\mathrm{id} \times j)^* \hat{u}_{X \times \mathbb{R}}) \cdot \pi^*_{X \times S^1} \hat{\alpha} = \left(\int_{S^1} \hat{v} \right) \cdot \hat{\alpha} = \hat{\alpha}.$$

Applying iteratively this identity, it easily follows from definition 6.12 that $\int_{\mathbb{R}^N} \hat{u}_{X \times \mathbb{R}^N} \cdot \pi^*_{X \times \mathbb{R}^N} \hat{\alpha} = \hat{\alpha}$, therefore $\pi_! T(\hat{\alpha}) = \hat{\alpha}$ as required. □

Theorem 15. *Given a multiplicative differential cohomology theory $h^\bullet$ with S^1-integration, the integration map (6.11), defined for any $\hat{h}^\bullet$−oriented neat submersion $f : Y \to X$, is the unique one satisfying axioms I_1, I_2, I_3 and I_4.*

Proof. We first observe that, given a $\hat{h}^\bullet$−oriented real vector bundle $\pi : E \to X$, axioms (I_1) and (I_4) completely determine $\pi_!$. In fact, given $\hat{\alpha} \in \hat{h}^{\bullet}_{\mathrm{cpt}}(E)$, we set $\alpha := I(\hat{\alpha})$. Since the Thom morphism is topologically an isomorphism, we set $\beta := T^{-1}(\alpha)$. Refining β to any differential class $\hat{\beta}$, we get $\hat{\alpha} = T(\hat{\beta}) + a(\omega)$, for a suitable $\omega \in \Omega^{\bullet-1}_{\mathrm{cpt}}(E; \mathfrak{h}^\bullet_{\mathbb{R}})$. Therefore, applying (I_1) and (I_4), we get

$$\pi_!(\hat{\alpha}) = \pi_! T(\hat{\beta}) + \pi_! a(R_{(i,u,\hat{\varphi})}(\omega)) = \hat{\beta} + a(R_{(i,u,\hat{\varphi})}(\omega)).$$

In particular, if $\pi : X \times \mathbb{R}^N \to X$ is the product bundle endowed with its natural orientation, as in remark 13, then $\pi_! = \int_{\mathbb{R}^N}$, since $\int_{\mathbb{R}^N}$ verifies (I_1) and (I_4), as it is easy to verify applying the axiom of S^1-integration. Another particular case is the zero-bundle $\mathrm{id} : X \to X$, with the trivial orientation 1. In this case, we have $(\mathrm{id})_! = \mathrm{id}$.

Now, let us suppose that $f_!$ is any integration map satisfying $(I_1) - (I_4)$. Given a $\hat{h}^\bullet$−oriented neat submersion $f : Y \to X$, we consider the following diagram:

$$\begin{tikzcd} N_{\iota(Y)}(X \times \mathbb{R}^N) \arrow[r, hook, "i \circ \varphi"] \arrow[d, "f \circ \pi_N"] & X \times \mathbb{R}^N \arrow[d, "\pi_{X \times \mathbb{R}^N}"] \\ X \arrow[r, equal] & X, \end{tikzcd}$$

and its induced diagram in compact-supported differential cohomology:

$$\begin{tikzcd} \hat{h}^{\bullet}_{\mathrm{cpt}}(N_{\iota(Y)}(X \times \mathbb{R}^N)) \arrow[r, hook, "i_* \varphi_*"] \arrow[d, "(f \circ \pi_N)_!"] & \hat{h}^{\bullet}_{\mathrm{cpt}}(X \times \mathbb{R}^N) \arrow[d, "(\pi_{X \times \mathbb{R}^N})_!"] \\ \hat{h}^{\bullet}_{\mathrm{cpt}}(X) \arrow[r, equal] & \hat{h}^{\bullet}_{\mathrm{cpt}}(X). \end{tikzcd}$$

Given $\hat{\beta} \in \hat{h}^{\bullet}_{\mathrm{cpt}}(N_{\iota(Y)}(X \times \mathbb{R}^N))$, applying axioms (I_2) and (I_3) we have that

$$f_!(\pi_N)_!\hat{\beta} = (f \circ \pi_N)_!\hat{\beta} = (\pi_{X\times\mathbb{R}^N})_! i_*\varphi_*\hat{\beta} = \int_{\mathbb{R}^N} i_*\varphi_*\hat{\beta}.$$

Given $\hat{\alpha} \in \hat{h}^{\bullet}_{\mathrm{cpt}}(Y)$ and setting $\hat{\beta} := T(\hat{\alpha})$, we get $f_!(\pi_N)_!T(\hat{\alpha}) = \int_{\mathbb{R}^N} i_*\varphi_*T(\hat{\alpha})$. Applying axiom (I_4) we get $f_!\hat{\alpha} = \int_{\mathbb{R}^N} i_*\varphi_*T(\hat{\alpha})$, i.e. $f_!$ coincides with (6.10). This proves the uniqueness of the integration map. $\square$

A Some topological notions

A.1 Push-outs and and homotopy push-outs

Definition 47. (**Push-out**). The pushout of the morphisms $f : Z \to X$ and $g : Z \to Y$ consists of an object P and two morphisms $i_X : X \to P$ and $i_Y : Y \to P$ such that the diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "i_X"] \\ Y \arrow[r, "i_Y"] & P \end{tikzcd}$$

commutes and such that (P, i_X, i_Y) is universal with respect to this diagram. That is, for any other such tripla (W, h, k) for which the following diagram commutes, there must exist a unique $\varphi : P \to W$ also making the diagram commute:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "i_X"] \arrow[ddr, bend left, "h"] & \\ Y \arrow[r, "i_Y"] \arrow[drr, bend right, "k"] & P \arrow[dr, dashed, "\varphi"] & \\ & & W. \end{tikzcd} \tag{A.1}$$

Example 2. The standard construction of the push-out of any two maps $f : Z \to X$ and $g : Z \to Y$ is the tripla $(X \sqcup_Z Y, i_X, i_Y)$, where

$$X \sqcup_Z Y = \frac{X \sqcup Y}{\sim}, \quad f(z) \sim g(z)\, \forall z \in Z$$
$$i_X(x) = [x], \qquad i_Y(y) = [y].$$

The space $X \sqcup_Z Y$ is unique up to canonical homeomorphism. In particular, the push-out of the two maps $\rho : A \to X$ and $i_0 : A \to A \times I$, with $i_0(a) = (a, 0)$, is the mapping cylinder $(\mathrm{Cyl}(\rho), j_X, j_{A\times I})$.

Definition 48. (**homotopy push-out**). The homotopy push-out of the morphisms $f : Z \to X$ and $g : Z \to Y$ consists of an object P, two morphisms, $j_X : X \to P$ and $j_Y : Y \to P$, and an homotopy $F : f \circ j_X \simeq j_Y \circ g$ such that the diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "j_X"] \\ Y \arrow[r, "j_Y"] & P \end{tikzcd}$$

is homotopy commutative and such that (P, j_X, j_Y, F) is universal with respect to this diagram. That is to say that, for any other such quadruple (W, h, k, G) for which its diagram is homotopy commutative, it holds:

- there exists a map $\psi : P \to W$ and homotopies $H : h \simeq \psi \circ j_X$ and $K : k \simeq \psi \circ j_Y$ such that the whole diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "j_X"] \arrow[ddr, bend left, "h"] & \\ Y \arrow[r, "j_Y"] \arrow[drr, bend right, "k"] & P \arrow[dr, dashed, "\psi"] & \\ & & W \end{tikzcd} \tag{A.2}$$

  with all maps and homotopies above is homotopy commutative;

- if there exists another map $\psi' : P \to W$ and homotopies $H' : h \simeq \psi' \circ j_X$ and $K' : k \simeq \psi' \circ j_Y$ such that the whole diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "j_X"] \arrow[ddr, bend left, "h"] & \\ Y \arrow[r, "j_Y"] \arrow[drr, bend right, "k"] & P \arrow[dr, dashed, "\psi'"] & \\ & & W \end{tikzcd}$$

  is homotopy commutative, then there exists a homotopy $M : \psi \simeq \psi'$ such that the whole diagram with all maps and homotopies above is homotopy commutative.

Example 3. The standard construction of the homotopy push-out of any two maps $f : Z \to X$ and $g : Z \to Y$ is the quadruple $(X \coprod_Z Y, j_X, j_Y, F)$, where

$$X \coprod_Z Y = \frac{X \sqcup (Z \times I) \sqcup Y}{\sim}, \quad (z, 0) \sim f(z)\,, (z, 1) \sim g(z)\ \forall z \in Z \tag{A.3}$$

$$j_X(x) = [x], \qquad j_Y(y) = [y], \qquad F(z,t) = [(z,t)].$$

The space $X \coprod_Z Y$ is unique up to an essentially unique homotopy equivalence.

Example 4. Considering the push-out $(X \sqcup_Z Y, i_X, i_Y)$ of the example above, we can easily get a homotopy push-out $(X \sqcup_Z Y, i_X, i_Y, G)$, where $G : i_X \circ f \simeq i_Y \circ g$ is the trivial homotopy. Thus, there is a natural map $p : X \coprod_Z Y \to X \sqcup_Z Y$ defined by the following diagram:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "j_X"] \arrow[ddr, bend left, "i_X"] & \\ Y \arrow[r, "j_Y"] \arrow[drr, bend right, "i_Y"] & X \coprod_Z Y \arrow[dr, dashed, "p"] & \\ & & X \sqcup_Z Y. \end{tikzcd} \tag{A.4}$$

The map p is defined by $[x] \mapsto [x]$, $[y] \mapsto [y]$ and $[(z,t)] \mapsto [f(z)] = [g(z)]$. The homotopies $H : i_X \simeq p \circ j_X$ and $K : i_Y \simeq p \circ j_Y$ are trivial too.

Lemma 7. *If one of two maps f and g in diagram (A.4) is a cofibration, then the map $p : X \coprod_Z Y \to X \sqcup_Z Y$ is a homotopy equivalence.*

Proof. By (DIECK, 2008) [Prop. 5.3.2 p. 112] the push-out $X \sqcup_Z Y$ is a homotopy push-out too; hence, $X \sqcup_Z Y$ and $X \coprod_Z Y$ have the same homotopy type. In particular, we consider the following two diagrams:

$$
\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "j_X"] \arrow[ddr, bend left, "i_X"] \arrow[dddrr, bend left, "j_X"] & & \\
Y \arrow[r, "j_Y"] \arrow[drr, bend right, "i_Y"] \arrow[ddrrr, bend right, "j_Y"] & X \coprod_Z Y \arrow[dr, dashed, "p"] & & \\
 & & X \sqcup_Z Y \arrow[dr, dashed, "q"] & \\
 & & & X \coprod_Z Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "i_X"] \arrow[ddr, bend left, "j_X"] \arrow[dddrr, bend left, "i_X"] & & \\
Y \arrow[r, "i_Y"] \arrow[drr, bend right, "j_Y"] \arrow[ddrrr, bend right, "i_Y"] & X \sqcup_Z Y \arrow[dr, dashed, "q"] & & \\
 & & X \coprod_Z Y \arrow[dr, dashed, "p"] & \\
 & & & X \sqcup_Z Y.
\end{tikzcd}
$$

Because of the uniqueness up to homotopy of the induced map, $q \circ p$ and $p \circ q$ are homotopic to the corresponding identities, hence p and q are homotopy equivalence. $\square$

Let us consider two maps $h : X \to W$ and $k : Y \to W$ such that $h \circ f = k \circ g$. In the diagram (A.1), the map φ is uniquely defined. If in diagram (A.2), we choose G to be the constant homotopy, then the induced map ψ can be chosen to be $\varphi \circ p$. We get the following diagram, whose horizontal arrows are objects of $\mathcal{C}_2$ and whose vertical arrows define a morphism in $\mathcal{C}_2$:

$$
\begin{tikzcd}
X \coprod_Z Y \arrow[r, "\psi"] \arrow[d, "p"] & W \arrow[d, "\mathrm{id}"] \\
X \sqcup_Z Y \arrow[r, "\varphi"] & W.
\end{tikzcd}
\tag{A.5}
$$

Corollary 8. *With the hypotheses stated above diagram (A.5), if one of the two maps f and g is a cofibration; then the morphism*

$$(id, p)^* : h^\bullet(\varphi) \to h^\bullet(\psi) \tag{A.6}$$

is a canonical isomorphism.

Proof. Applying the five lemma to the morphism of long exact sequences induced by (id, p), the result immediately follows from lemma 7. $\square$

A.2 Splitting of the Kunneth sequence on cycles

An important result for the construction of the product (2.4) through the model of differential character, on subsection 3.2.1, was the Splitting of the Künneth sequence on cycles. We now describe this result, basically as a generalization of the one given on (BäR; BECKER, 2014).

A.2.1 Summary on the absolute case

In the absolute setting, given two spaces X and Y, we have the Künneth sequence

$$0 \longrightarrow [H_\bullet(X) \otimes H_\bullet(Y)]_n \longrightarrow H_n(X \times Y) \longrightarrow \mathrm{Tor}(H_\bullet(X), H_\bullet(Y))_{n-1} \longrightarrow 0. \tag{A.7}$$

Such a sequence splits at the level of cycles as follows. Let us consider the Alexander-Whitney and Eilenberg-Zilber maps

$$C_\bullet(X \times Y) \underset{EZ_{X,Y}}{\overset{AW_{X,Y}}{\rightleftarrows}} C_\bullet(X) \otimes C_\bullet(Y). \tag{A.8}$$

We have that $EZ_{X,Y} \circ AW_{X,Y} \simeq \mathrm{id}_{C_\bullet(X \times Y)}$ and $AW_{X,Y} \circ EZ_{X,Y} = \mathrm{id}_{C_\bullet(X) \otimes C_\bullet(Y)}$. Now we consider the following sequences, and we choose two splittings s and t

$$\begin{tikzcd} 0 \arrow[r] & Z_\bullet(X) \arrow[r, hook, "i"] & C_\bullet(X) \arrow[r, "\partial_X"] \arrow[l, dashed, bend left, "s"] & B_{\bullet-1}(X) \arrow[r] & 0, \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & Z_\bullet(Y) \arrow[r, hook, "j"] & C_\bullet(Y) \arrow[r, "\partial_Y"] \arrow[l, dashed, bend left, "t"] & B_{\bullet-1}(Y) \arrow[r] & 0, \end{tikzcd}$$

then, we get the following splitting of (A.7) on cycles

$$\begin{tikzcd} 0 \arrow[r] & (Z_\bullet(X) \otimes Z_\bullet(Y))_n \arrow[r, shift left, "i\otimes j"] \arrow[dr, shift left, "K"] & Z(C_\bullet(X) \otimes C_\bullet(Y))_n \arrow[r] \arrow[l, dashed, shift left, "s\otimes t"] \arrow[d, shift left, "EZ_{X,Y}"] & \cdots \\ & & Z_n(X \times Y). \arrow[u, dashed, shift left, "AW_{X,Y}"] \arrow[ul, dashed, shift left, "S"] & \end{tikzcd}$$

A.2.2 The relative Künneth Splitting

We now consider the relative setting. Given a map $\rho : A \to X$ and a pair (Y, B), we consider the map of pairs $\rho \times \mathrm{id}_{(Y,B)} : (A \times Y, A \times B) \to (X \times Y, X \times B)$, and we get the Künneth sequence

$$0 \longrightarrow [H_\bullet(\rho) \otimes H_\bullet(Y, B)]_n \longrightarrow H_n(\rho \times (Y, B)) \longrightarrow \mathrm{Tor}(H_\bullet(\rho), H_\bullet(Y, B))_{n-1} \longrightarrow 0. \tag{A.9}$$

Such a sequence splits at the level of cycles as follows. Given a subspace $B \subset Y$, the maps in (A.8) restrict to $C_\bullet(X \times B)$ and $C_\bullet(X) \otimes C_\bullet(B)$; hence, they project to

$$C_\bullet(X \times Y, X \times B) \underset{EZ_{X,(Y,B)}}{\overset{AW_{X,(Y,B)}}{\rightleftarrows}} C_\bullet(X) \otimes C_\bullet(Y, B); \tag{A.10}$$

therefore, replacing X by the map $\rho : A \to X$, we get the following Alexander-Whitney and Eilenberg-Zilber maps (we denote the mapping cone by â€~Coneâ€™, since C denotes the singular chain complex):

$$\begin{tikzcd} C_\bullet(\rho) \otimes C_\bullet(Y, B) \arrow[d, shift left, "EZ_{\rho,(Y,B)}"] & = & \mathrm{Cone}_\bullet(\rho_* : C_\bullet(A) \to C_\bullet(B)) \otimes C_\bullet(Y, B) & \simeq & \mathrm{Cone}_\bullet(\rho_* \otimes \mathrm{id} : C_\bullet(A) \otimes C_\bullet(Y, B) \to C_\bullet(X) \otimes C_\bullet(Y, B)) \arrow[d, shift left, "EZ_{\rho\times \mathrm{id}_B}"] \\ C_*(\rho, \times \mathrm{id}_{(Y,B)}) \arrow[u, shift left, "AW_{\rho,(Y,B)}"] & & = & & \mathrm{Cone}_\bullet((\rho \otimes \mathrm{id})_* : C_\bullet(A \otimes Y, A \times B) \to C_\bullet(X \otimes Y, X \times B)). \arrow[u, shift left, "AW_{\rho\times \mathrm{id}_B}"] \end{tikzcd}$$

We have that $EZ_{\rho,(Y,B)} \circ AW_{\rho,(Y,B)} \simeq \mathrm{id}_{C_\bullet(\rho\times \mathrm{id}_{(Y,B)})}$ and $AW_{\rho,(Y,B)} \circ EZ_{\rho,(Y,B)} = \mathrm{id}_{C_\bullet(\rho)\otimes C_\bullet(Y,B)}$. Now we consider the following sequences, and we choose two splittings s and t

$$\begin{tikzcd} 0 \arrow[r] & Z_\bullet(\rho) \arrow[r, hook, "i"] & C_\bullet(\rho) \arrow[r, "\partial_\rho"] \arrow[l, dashed, bend left, "s"] & B_{\bullet-1}(\rho) \arrow[r] & 0; \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & Z_\bullet(Y,B) \arrow[r, hook, "j"] & C_\bullet(Y,B) \arrow[r, "\partial_{(Y,B)}"] \arrow[l, dashed, bend left, "t"] & B_{\bullet-1}(Y,B) \arrow[r] & 0. \end{tikzcd}$$

We get the following splitting of (A.9) on cycles

$$\begin{tikzcd} 0 \arrow[r] & (Z_\bullet(\rho)\otimes Z_\bullet(Y,B))_n \arrow[r, shift left, "i\otimes j"] \arrow[dr, "K"] & Z(C_\bullet(\rho)\otimes C_\bullet(Y,B))_n \arrow[r] \arrow[l, shift left, dashed, "s\otimes t"] \arrow[d, shift left, "EZ_{\rho,(Y,B)}"] & \cdots \\ & & Z_n(\rho\times \mathrm{id}_{(Y,B)}), \arrow[u, shift left, dashed, "AW_{\rho,(Y,B)}"] \arrow[ul, dashed, shift left, "S"] & \end{tikzcd}$$

i.e., we have

$$Z_n(\rho\times \mathrm{id}_{(Y,B)}) \simeq (Z_\bullet(\rho)\otimes Z_\bullet(Y,B))_n \oplus T_n(\rho\times \mathrm{id}_{(Y,B)}). \tag{A.11}$$

From (A.11), a cycle $(\alpha_n, \beta_{n-1}) \in Z_n(\rho\times \mathrm{id}_{(Y,B)})$ can be decomposed as $(\alpha_n,\beta_{n-1}) = x + (\gamma_n, \delta_{n-1})$, in such a way that:

- x is a sum of cycles of the form $(\mu_k, \nu_{k-1})\otimes \mu'_h$, where $(\mu_k,\nu_{k-1}) \in Z_k(\rho)$, $\mu'_h \in Z_h(Y,B)$ and $k+h=n$;
- (γ_n, δ_{n-1}) represents a torsion homology class.

In particular, when $\rho : A \hookrightarrow X$ is an inclusion, we get the following K*ü*nneth sequence:

$$0 \longrightarrow [H_\bullet(x,A)\otimes H_\bullet(Y,B)]_n \longrightarrow H_n(X\times Y, (X\times B)\cup(A\times Y)) \longrightarrow \mathrm{Tor}(H_\bullet(X,A), H_\bullet(Y,B))_{n-1} \longrightarrow 0. \tag{A.12}$$

Such a sequence splits at the level of cycles as follows. The maps in (A.10) restrict to $C_\bullet(A\times Y, A\times B)$ and $C_\bullet(A)\otimes C_\bullet(Y,B)$, hence they project to

$$\begin{tikzcd} C_\bullet(X\times Y, (X\times B)\cup(A\times Y)) \arrow[r, shift left, "AW_{(X,A),(Y,B)}"] & C_\bullet(X,A)\otimes C_\bullet(Y,B). \arrow[l, shift left, "EZ_{(X,A),(Y,B)}"] \end{tikzcd}$$

Now we consider the following sequences and we choose two splittings s and t

$$\begin{tikzcd} 0 \arrow[r] & Z_\bullet(X,A) \arrow[r, hook, "i"] & C_\bullet(X,A) \arrow[r, "\partial_{(X,A)}"] \arrow[l, dashed, bend left, "s"] & B_{\bullet-1}(X,A) \arrow[r] & 0; \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & Z_\bullet(Y,B) \arrow[r, hook, "j"] & C_\bullet(Y,B) \arrow[r, "\partial_{(Y,B)}"] \arrow[l, dashed, bend left, "t"] & B_{\bullet-1}(Y,B) \arrow[r] & 0. \end{tikzcd}$$

We get the following splitting of (A.12) on cycles

$$\begin{tikzcd} 0 \arrow[r] & (Z_\bullet(X,A)\otimes Z_\bullet(Y,B))_n \arrow[r, shift left, "i\otimes j"] \arrow[dr, "K"] & Z(C_\bullet(X,A)\otimes C_\bullet(Y,B))_n \arrow[r] \arrow[l, shift left, dashed, "s\otimes t"] \arrow[d, shift left, "EZ_{X,Y}"] & \cdots \\ & & Z_n(X\times Y, (X\times B)\cup(A\times Y)). \arrow[u, shift left, dashed, "AW_{X,Y}"] \arrow[ul, dashed, shift left, "S"] & \end{tikzcd} \tag{A.13}$$

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