UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

ON THE EXISTENCE OF FREE ACTIONS OF THE GROUPS $\mathbb{Z}_2$, S^1 AND S^3 ON SOME FINITISTIC SPACES AND COHOMOLOGY OF ORBIT SPACES

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Call to me and I will answer you, and will tell you great and hidden things that you have not known.

Jeremiah 33.3

For from him and through him and to him are all things. To him be glory forever. Amen.

Romans 11.36

Resumo

Sejam G um grupo de Lie compacto e X um espaço finitístico. Quando G atua de forma contínua em X podemos construir a fibração

$$X \hookrightarrow X_G \longrightarrow B_G, \tag{1}$$

chamada fibração de Borel, onde $G \hookrightarrow E_G \to B_G$ denota o $G-$fibrado univeral e X_G é o espaço de órbitas $(E_G \times X)/G$, também chamado de espaço de Borel.

Se a ação G em X é livre, então existe uma equivalência de homotopia entre o espaço de órbitas X/G e o espaço X_G. Portanto, podemos usar a sequência espectral de Leray-Serre $\{E_r^{*,*}, d_r\}$, associada à fibração (1), que converge para a cohomologia do espaço total X_G, para obter o anel de cohomologia do espaço de órbitas X/G.

Nessa tese, utilizamos estas ferramentas para investigar a existência de ações livres dos grupos de Lie compactos $\mathbb{Z}_2$, S^1 e S^3 em alguns espaços finitísticos. Precisamente, estudamos a existência de ações livres em espaços finitísticos que possuem cohomologia mod 2 de uma variedade de Dold $P(m,n)$, variedade de Wall $Q(m,n)$, variedade de Milnor $H(m,n)$, um produto de esferas, espaços projetivos (reais, complexos ou quaterniônicos) e espaços do tipo (a,b). Quando o espaço X em questão admite essa estrutura, computamos a cohomologia dos respectivos espaços de órbitas X/G.

Palavras chave e frases: Ações livres, Espaços de Órbitas, Fibração de Borel, Sequência espectral de Leray-Serre, Cohomologia.

Abstract

Let G be a compact Lie group and X be a finitistic space. If G acts continuously on X, we can construct the fibration

$$X \hookrightarrow X_G \longrightarrow B_G, \tag{2}$$

called Borel fibration, where $G \hookrightarrow E_G \to B_G$ denotes the universal $G-$bundle and X_G is the orbit space $(E_G \times X)/G$, also known as the Borel space.

When the action on G on X is free, there is a homotopy equivalence between the orbit space X/G and the space X_G. Therefore, we can use the Leray-Serre spectral sequence $\{E_r^{*,*}, d_r\}$, associated to the fibration (2), which converges to the cohomology of the total space X_G, to get the cohomology ring of the orbit space X/G.

In this thesis, we use these tools to investigate the existence of free actions of the compact Lie groups $\mathbb{Z}_2$, S^1 and S^3 on some finitistic spaces. Precisely, we study the existence of free action on finitistic spaces with mod 2 cohomology of a Dold manifold $P(m, n)$, a Wall manifold $Q(m, n)$, a Milnor manifold $H(m, n)$, a product of spheres, the (real, complex or quaternionic) projective spaces and spaces of type (a, b). When the space X admit such such structure, we compute the mod 2 cohomology of the respective orbit space X/G.

Key words and phrases: Free Actions, Orbit Spaces, Borel Fibration, Leray-Serre spectral Sequence, Cohomology.

Contents

SOME NOTATIONS

S^n : $n-$dimensional sphere.

$\mathbb{R}P^n$: $n-$dimensional real projective space.

$\mathbb{C}P^n$: $2n-$dimensional complex projective space.

$\mathbb{H}P^n$: $4n-$dimensional quaternionic projective space.

$E_\infty^{*,*}$: Limit term of the spectral sequence $\{E_r^{p,q}, d_r\}$.

$H^q(X; R)$: Singular cohomology of the topological space X with coefficients in the ring R.

$H^q(X) = \check{H}^q(X; \mathbb{Z}_2)$: Čech cohomology of the topological space X with coefficients in the ring $\mathbb{Z}_2$.

$\mathcal{H}^*(\,-\,)$: Local System of coefficients.

B_G : Classifying space for the group G.

$\mu(g, x) = g * x$: Action of the element $g \in G$ on $x \in X$.

G_x : Isotropy subgroup relative to $x \in X$.

X/G : Orbit space of X by an action of the group G.

Fix(T) : Fixed point set of the function T.

$X \star Y$: *Join* of the topological spaces X and Y.

T^* : Homomorphism induced by T on cohomology ring.

$a \smile b = a \cdot b = ab$: Cup product between the elements a and b.

$\deg x$: Cohomological degree of the element x, that is, $\deg x \in \mathbb{N}$ such that $x \in H^{\deg x}(X; R)$.

$dimX$: Dimension of the manifold X.

$\omega_G = (E_G, p_G, B_G, G)$: A universal $G-$bundle with total space E_G.

X_G : Borel space relative to the topological space X.

$H^*_G(X; R)$: $G-$equivariant cohomology of the topological space X with coefficients in the ring R.

(X_G, p, B_G, X) : Borel fibration associated to the $G-$space X.

$G\langle x_1, x_2, \cdots, x_n\rangle$: A group isomorphic to G with generators $x_1, \cdots, x_n$.

$X \cong_2 Y$: Means that the topological spaces X and Y has the same cohomology with coefficients in $\mathbb{Z}_2$.

$X \amalg Y$: Disjoint union of the topological spaces X and Y.

$X \vee Y$: One point union between the topological spaces X and Y.

$P(m, n)$: Dold manifold of dimension $m + 2n$.

$Q(m, n)$: Wall manifold of dimension $m + 2n + 1$.

$H(m, n) = \mathbb{R}H_{m,n}$: Milnor manifold of dimension $m + n - 1$.

ind(X, T) : Index of the free involution T on X.

co-ind$_{\mathbb{Z}_2}(X)$: Co-index of the free involution T on X.

$\mathfrak{N}_*$: Unoriented cobordism ring.

INTRODUCTION AND STATEMENT OF RESULTS

Let X be a topological space, G a topological group and $* : G \times X \to X$ be a continuous action of G on X. A typical transformation group problem associated to the pair (X, G) is to determine under what conditions the action $*$ is free. Recall that the action of G on X is called free if for any point x of X the isotropy group

$$G_x = \{g \in G\,;\, g * x = x\}. \tag{3}$$

contains only the trivial element 1 of G.

The interest in this type of problem, particularly when G is a finite group, has become greater since the publication of work [15], by H. Hopf in 1926, in which one formalizes the purpose of classification of all manifolds whose universal covering is homeomorphic to a sphere S^n. This problem, as we know, is equivalent to the classification of all finite groups that can act freely on S^n, and J. Milnor [26] gives an answer to this question by showing that the symmetrical group $\mathbb{S}_3$ cannot act freely on S^n.

A possible generalization for this kind of problem can be proposed by considering the topological space X as a product of spheres. For example, L. W. Cusick [4] shows that if a finite group G act freely on a product of spheres of even dimensions, $S^{2n_1} \times \cdots \times S^{2n_k}$, then G must be isomorphic to a group of the type $(\mathbb{Z}_2)^r$, for some $r \leq k$. Dotzel et al. [12] considered the problem for free actions of the groups $\mathbb{Z}_p$, p prime, and S^1, on a product of spheres $S^m \times S^n$. Jahren and Kwasik [17] provide some answers to the problem of the existence of free involutions on products of the type $S^1 \times S^n$.

When the group G acts freely on a topological space X, we can study certain aspects of the orbit space X/G in the same way as it happens on the projective spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$, and $\mathbb{H}P^n$, that are orbit spaces of certain free actions of $\mathbb{Z}_2$, S^1 and S^3 on spheres S^n, S^{2n+1} and S^{4n+3}, respectively. We also realized by these cases that the knowledge of the cohomology of the orbit spaces plays an important role in a large number of classic results. Thus, we can say that this specific type of situation motivates a more general approach when we consider free actions of these groups on other relevant spaces, which is what we proposed to do in this thesis.

However, to compute the cohomology of the orbit space X/G can be a difficult task, particularly when the space X it is not a sphere or, more generally, when X has nontrivial cohomology on several levels. On the other hand, many results in that direction have been obtained by using some tools of equivariant cohomology theory. This is due to the fact that as long as G is a compact Lie group acting freely on a space X, there is a homotopy equivalence between the orbit space X/G and the Borel space X_G, so that we can use the so-called Leray-Serre spectral sequence associated to the Borel fibration

$$X \hookrightarrow X_G \longrightarrow B_G, \tag{4}$$

where B_G denotes the classifying space for the group G, to investigate the cohomology ring $H^*(X_G; R) \cong H^*(X/G; R)$.

For example, Pergher et al. [33] got conditions to guarantee the existence of free actions of the Lie groups $\mathbb{Z}_2$ and S^1 on spaces of type (a, b), which are spaces with have nontrivial cohomology only on the levels $0, n, 2n$ and $3n$, for certain natural number n, and also computed the cohomology ring of the respective orbit spaces. Singh [36] deals with the existence of free involutions on the lens space $L_p^{2m-1}(q)$, which are the manifolds defined as the orbit space of free actions of $\mathbb{Z}_p$ on a sphere S^{2m-1}, and gives a list of all possible structures for the cohomology of the orbit spaces obtained by an arbitrary free action.

In this work, we are interested in investigating the existence of free actions of the compact Lie groups $\mathbb{Z}_2$, S^1 and S^3, on some finitistic spaces which have mod 2 cohomology of certain known spaces, as the Wall manifolds of the type $Q(m, n)$, the Dold manifolds $P(m, n)$, product of spheres, the projective spaces $\mathbb{C}P^n$ and $\mathbb{H}P^n$ and spaces of type (a, b).

Recall that a finitistic space is a paracompact Hausdorff space whose every open covering has a finite dimensional open refinement, where the dimension of a covering is one less than the maximum number of members of the covering which intersect non-trivially. In this way, since X is finitistic if and only if the orbit space X/G is finitistic, for G a compact Lie group, we can consider the problem of cohomology classification of the orbit spaces up to finitistic spaces of cohomology isomorphic to the initial space X.

The Dold manifolds, as they came to be known, were defined by A. Dold, in 1956, in his work [11], as orbit spaces of free actions of $\mathbb{Z}_2$ or, equivalently, free involutions on a product of the form $S^m \times \mathbb{C}P^n$. Dold showed that such constructions provide generators in odd dimensions to the unoriented cobordism ring[1] $\mathfrak{N}_*$. Precisely, for each $i = 2^r(2s+1)$, we have $x_i = [P(2^r - 1, s2^r)]$, and then

$$\Big\{[\mathbb{R}P^{2i}], [P(2^r - 1, s2^r)]\,;\, i, r, s \geq 1\Big\}, \tag{5}$$

[1] See appendix A.

is a generator set for the unoriented cobordism ring.

The Wall manifolds $Q(m,n)$ were introduced in [42] by C. T. C. Wall in 1960. These spaces are defined as the orbit spaces of a certain involution on $P(m,n)\times[0,1]$ or, equivalently, $Q(m,n)$ is the differentiable manifold obtained by the mapping torus constructed from a specific involution S on $P(m,n)$, that is,

$$Q(m,n)=\frac{P(m,n)\times[0,1]}{([x,z],0)\sim(S[x,z],1)}, \tag{6}$$

and, in similar way to the result presented by Dold, Wall showed that the manifolds $Q(m,n)$ also represent generators for $\mathfrak{N}_*$, by demonstrating that

$$\Big\{[\mathbb{R}P^{2i}],[Q(2^r-2,s2^r)]\,;\,i,r,s\geq 1\Big\}, \tag{7}$$

is a generator set of $\mathfrak{N}_*$.

In 1965, J. Milnor [27] constructed a new generator set for $\mathfrak{N}_*$, defining the spaces that later came to be called Milnor manifolds. In that work Milnor defined the set

$$\Big\{[\mathbb{R}P^{2i}],[H(2^k,2t2^k)]\,;\,i,k,t\geq 1\Big\}, \tag{8}$$

which is another generator set for $\mathfrak{N}_*$, where $H(m,n)$ denotes a Milnor manifold of dimension $m+n-1$.

For those reasons, the manifolds $P(m,n),Q(m,n)$ and $H(m,n)$ have been extensively studied, with the greatest aim being to investigate the existence of certain algebraic and geometric invariants, as it is done on the works [30, 31, 32] of Mukerjee. We also emphasize that the difficulty of making an investigation of the existence of free actions by cohomological criteria over the Dold, Wall and Milnor manifold and computing the cohomology of the resulting orbit space is found in the fact that these spaces have nontrivial cohomology ring at all possible levels.

When investigating the existence of free actions of $\mathbb{Z}_2$ on Dold manifolds, Morita et. [29], in 2018, partially solved the problem by considering free involutions on $P(1,n)$, for $n\geq 1$ an odd integer. Later, in 2019, the problem was completely solved by Dey in [8].

In this thesis we contribute with this question by computing the cohomology of the orbit spaces of free actions of the circle group S^1 on Dold manifolds of the type $P(m,n)$, for $n\geq 1$ an odd integer and $m>0$ an integer, according with Theorem 3.1.2, in which we conclude that:

$$H^*(P(m,n)/S^1;\mathbb{Z}_2)\cong\begin{cases}\mathbb{Z}_2[x,y]/\langle x^{(m+1)/2},y^{n+1}\rangle, & \text{if } m \text{ is odd},\\ \mathbb{Z}_2[x,y]/\langle x^{(m+2)/2},y^{n+1}\rangle, & \text{if } m \text{ is even},\end{cases} \tag{9}$$

where $\deg x=\deg y=2$. As an application of Theorem 3.1.2 we get a Borsuk-Ulam type result in which we showed the non-existence of S^1-equivariant maps of S^{2j-1} on $P(m,n)$, for all $n \geq 1$ odd and all $j \geq 1$, according to Theorem 3.2.1.

We observe that the Dold manifolds of type $P(m, n)$, for n even, do not admit the existence of free actions of the groups $\mathbb{Z}_2$ and S^1, due to the Khare's results presented in his work [20].

Regarding the Milnor manifolds, the problem was solved in 2019 by Dey and Singh in [9], both for actions of $\mathbb{Z}_2$ and S^1. We contribute with this problem by showing that the Milnor manifolds do not admit any free action of the group S^3, according to Theorem 4.1.3. Since in [9] the authors showed that there are free actions of $\mathbb{Z}_2$ and S^1 on the Milnor manifolds, the result of Theorem 4.1.3 can be seen, in a way, as an obstruction to the extension of these actions to the entire group S^3.

When considering the Wall manifolds $Q(m, n)$ we were able to show, according to the construction of Proposition 2.2.2, that such spaces admit the existence of free involutions, as long as n be an odd number, and we got the complete description of the cohomology of the orbit spaces $Q(m, n)/\mathbb{Z}_2$, both for the cases when $m = 1$ or when $m > 1$ is an even integer, according to Theorems 2.5.1 and 2.6.1, respectively. Precisely, for $m = 1$ we showed that $H^*(Q(m, n)/\mathbb{Z}_2; \mathbb{Z}_2)$ is isomorphic to the graded polynomial algebra:

$$\mathbb{Z}_2[\alpha, \beta, \gamma, \delta]/\langle \alpha^3, \beta^3, \gamma^2, \beta^2 + \beta\gamma, \delta^{\frac{n+1}{2}} \rangle, \tag{10}$$

where $\deg \alpha = \deg \beta = \deg \gamma = 1$ and $\deg \delta = 4$.

If $m > 1$ is even, we showed that $H^*(Q(m, n)/\mathbb{Z}_2; \mathbb{Z}_2)$ must be isomorphic to one of the following graded polynomial algebras:

$$\mathbb{Z}_2[\alpha, \beta, \gamma, \delta]/\langle \alpha^3, \beta^2, \gamma^{m+1} + \gamma^m\beta, \delta^{\frac{n+1}{2}} \rangle, \tag{11}$$

where $\deg \alpha = \deg \beta = \deg \gamma = 1$ and $\deg \delta = 4$, or

$$\mathbb{Z}_2[\alpha, \beta, \gamma, \delta]/\langle \alpha^2, \beta^{m+1}, \beta^2 + \gamma, \delta^{n+1} \rangle, \tag{12}$$

where $\deg \alpha = \deg \beta = 1$ and $\deg \gamma = \deg \delta = 2$.

As an application of Theorems 2.5.1 and 2.6.1, we derive a Borsuk-Ulam type result for $Q(m, n)$, both for $m = 1$ or m an even number, according to the Theorems 2.7.1 and 2.7.2, who claimed that there are no $\mathbb{Z}_2-$equivariant maps of S^k on $Q(m, n)$, n odd, for all $k \geq 2$ and for $m = 1$ or m even.

With respect to the existence of free actions of S^1 on Wall manifolds $Q(m, n)$, we showed on Theorem 3.3.1 that these spaces do not admit such structures, for every positive integers m and n.

We observe that the choice of the cases $Q(1, n)$ and $Q(\text{even}, n)$, in the context of Theorems 2.5.1 and 2.6.1, were motivated by the work of Khare [21], where he showed that $Q(m, n)$ bounds if and only if, n is odd or $n = 0$ and m is odd. Therefore, these spaces not only admit the existence of free involutions, according to Proposition 2.2.2, but also the particular choices of m are convenient in reason of the relations between the generators of the cohomology ring, which naturally limit the possible structures for the cohomology ring $H^*(Q(m,n)/\mathbb{Z}_2;\mathbb{Z}_2)$.

The main tool that we used was the Leray-Serre spectral sequence $\{E_r^{*,*}, d_r\}$, associated to the Borel fibration $X_G \to B_G$, and for that purpose an additional difficult is to find conditions to guarantee that the action of the fundamental group of the classifying space B_G is trivial on the cohomology of the space X, which implies that the E_2-term of the sequence is given in terms of the tensor product

$$E_2^{p,q} \cong H^p(B_G;R) \otimes_R H^q(X;R), \tag{13}$$

instead of the traditional way as the cohomology ring of B_G with local coefficients $\mathcal{H}^*(X;R)$. When X is a Wall manifold, we emphasize that the use of Theorem 7.4 of [1] was fundamental to guarantee that this happens, similarly as it is done in [29] and [33], for example. On the other side, when G is the group S^1 or the group S^3, it is already well known that the fundamental group of the classifying space B_G is simply connected, therefore the action must be trivial.

In [19], the authors investigated the existence of free actions of the group S^3 on spheres, real projective spaces and lens spaces, while Pergher et al. [33] considered free actions of the groups $\mathbb{Z}_2$ and S^1 on spaces of the type (a,b). We extended these results by considering free actions of S^3 over certain product of spheres, complex and quaternionic projective spaces and spaces of type (a,b). In particular, we showed on Theorem 4.3.1 that the complex and quaternionic projective spaces do not admit any free actions of S^3.

When considering free actions of S^3 on a cohomology sphere S^m, Singh et al. [19] showed that m must be of the form $4k-1$, for some integer k. Assuming that S^3 acts on a product of the type $S^m \times S^n$, for $m \leq n$, we showed on Theorem 4.2.2 that m or n must be an odd of the form $4k-1$, for some integer k. We also got the mod 2 cohomology of the orbit space $(S^m \times S^n)/S^3$, which is isomorphic to the following graded polynomial ring:

$$\mathbb{Z}_2[x,y]/\langle x^k, y^2\rangle, \tag{14}$$

where $\deg x = 4$ and $\deg y = n$ or $\deg y = m$.

On Theorem 4.2.3 we showed that if the group S^3 acts on a finitistic space that have mod 2 cohomology of a $n-$torus $S^1 \times \cdots \times S^1$, then this action cannot be free.

By assuming that the group S^3 acts freely over a space X of the type (a,b), characterized by some integer $n > 1$, we showed on Theorem 4.4.3 that we must have $a = 0$, b must be an odd number and n an odd of the type $4k-1$, and in that case the orbit space X/S^3 has the mod 2 cohomology isomorphic to the following graded polynomial ring:

$$\mathbb{Z}_2[x,y]/\langle x^k, y^2\rangle, \tag{15}$$

where $\deg x = 4$ and $\deg y = 2n$.

The similarly between the cohomology rings presented in (14) and (15) is due to the fact that $S^m \times S^{2m}$ is a space of type $(0,1)$, according to Example 4.4.1. That is, the same result of Theorem 4.2.2, when applied on the product space $S^m \times S^{2m}$, can be derived of Theorem 4.4.3, which shows the consistency of the results.

CHAPTER 1

PRELIMINARIES

1.1 Leray-Serre spectral sequence

The main tool we used in this work - both to calculate the cohomology ring of orbit spaces and to obtain obstructions for the existence of free actions - was the Leray-Serre spectral sequence, in which, given a fibration $F \hookrightarrow E \xrightarrow{\pi} B$, the Theorem associates to it a spectral sequence converging to the cohomology of the total space E. In this Section we briefly summarize the main results regarding the Leray-Serre spectral sequence we used during the work, based on McCleary [24].

Definition 1.1.1. A (cohomological) *spectral sequence* is a collection of bigraded differential $R-$modules $\{E_r^{*,*}, d_r\}$, with $r = 1, 2, \cdots$, where all the differentials have bidegree $(r, 1-r)$, that is,

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}, \tag{1.1}$$

such that

$$E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r) = \ker d_r^{p,q} / \mathrm{im}\ d_r^{p-r,q-1+r}, \tag{1.2}$$

for all $p, q, r \in \mathbb{Z}$. In this case, the bigraded module $E_r^{*,*}$ is called the E_r-*term*, or the E_r-*page* of the spectral sequence.

We say that the sequence is of the *first quadrant* if $E^{p,q} = \{0\}$ for $p < 0$ or $q < 0$. When the spectral sequence *collapses* on the N^{th} term, that is, $d_r = 0$, for all $r \geq N$, we have $E_{r+1}^{p,q} \cong E_r^{p,q}$, for all $r \geq N$.

Remark 1.1.1. Using the notations

$$Z_2^{p,q} = \ker(d_2 : E_2^{p,q} \to E_2^{p+2,q-1}) \tag{1.3}$$

and

$$B_2^{p,q} = \mathrm{im}(d_2 : E_2^{p-2,q+1} \to E_2^{p,q}), \tag{1.4}$$

we have $B_2^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}$, since $d \circ d = 0$. On the other hand, by definition,

$$E_3^{p,q} \cong H^{p,q}(E_2^{*,*}, d_2) = \frac{Z_2^{p,q}}{B_2^{p,q}}. \tag{1.5}$$

Let us consider

$$\overline{Z}_3^{p,q} = \ker(d_3 : E_3^{p,q} \to E_3^{p+3,q-2}) \tag{1.6}$$

and

$$\overline{B}_3^{p,q} = \mathrm{im}(d_3 : E_3^{p-3,q+2} \to E_3^{p,q}). \tag{1.7}$$

We can see that $\overline{Z}_3^{p,q}$ and $\overline{B}_3^{p,q}$ are submodules of $E_3^{p,q}$, so it follows by (1.5) that there are submodules $Z_3^{p,q}$ and $B_3^{p,q}$ of $Z_2^{p,q}$, containing $B_2^{p,q}$, such that $\overline{Z}_3^{p,q} \cong Z_3^{p,q}/B_2^{p,q}$ and $\overline{B}_3^{p,q} \cong B_3^{p,q}/B_2^{p,q}$. Thus,

$$E_4^{p,q} \cong H^{p,q}(E_3^{*,*}, d_3) = \frac{\overline{Z}_3^{p,q}}{\overline{B}_3^{p,q}} = \frac{Z_3^{p,q}/B_2^{p,q}}{B_3^{p,q}/B_2^{p,q}} \cong \frac{Z_3^{p,q}}{B_3^{p,q}} \tag{1.8}$$

and we get the chain of submodules $B_2^{p,q} \subseteq B_3^{p,q} \subseteq Z_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}$. By continuing this process, we get an infinite sequence of submodules of $E_2^{p,q}$,

$$B_2^{p,q} \subseteq B_3^{p,q} \subseteq \cdots \subseteq B_n^{p,q} \subseteq \cdots \subseteq Z_n^{p,q} \subseteq \cdots \subseteq Z_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}, \tag{1.9}$$

which has the property $E_{n+1}^{p,q} \cong Z_n^{p,q}/B_n^{p,q}$ for all $n \geq 2$. Then, with these notations, we can define the following submodules of $E_2^{p,q}$:

$$Z_\infty^{p,q} = \bigcap_{n=2}^{\infty} Z_n^{p,q} \quad \text{and} \quad B_\infty^{p,q} = \bigcup_{n=2}^{\infty} B_n^{p,q}. \tag{1.10}$$

Thus, the elements of $Z_\infty^{p,q}$ are precisely the elements of $E_2^{p,q}$ that survive in all the E_r-pages, that is, which are cocycles at all stages. The submodule $B_\infty^{p,q}$ is formed by elements that are eventually image of some element at some stage of the spectral sequence.

Definition 1.1.2. It follows from the chain of submodules displayed in (1.9) that $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$, so it makes sense to define the bigraded module

$$E_\infty^{p,q} := Z_\infty^{p,q}/B_\infty^{p,q}, \tag{1.11}$$

called the *limit term* (or simply the $E_\infty-$term) of the spectral sequence $\{E_r^{*,*}, d_r\}$.

Definition 1.1.3. A (decreasing) *filtration* F over an $R-$module A is a family of submodules $\{F^pA\,;\, p \in \mathbb{Z}\}$, such that $\cdots \subseteq F^{p+1}A \subseteq F^pA \subseteq \cdots A$.

When $A = \{A^n\,;\, n \in \mathbb{N}\}$ is a graded module, then $F^pA^n = F^pA \cap A^n$, for each $p, n \geq 0$, and in this case we define the bigraded module $\{E_0^{p,q}(A, F)\,;\, p, q \geq 0\}$, where

$$E_0^{p,q}(A, F) = \frac{F^pA^{p+q}}{F^{p+1}A^{p+q}}. \tag{1.12}$$

Definition 1.1.4. We say that an $R-$module A is a *differential graded module* when

$$A = \bigoplus_{n=0}^{\infty} A^n \tag{1.13}$$

and there is an $R-$linear homomorphism $d : A \to A$, of degree ± 1, satisfying $d \circ d = 0$. If A has a filtration F and d respects this filtration, that is, $d(F^p) \subseteq F^pA$, then we say that A is a *filtered differential graded module*.

When, associated with the differential graded module (A, d), there is a morphism of graded module $\psi : A \otimes_R A \to A$, also known as a product, where

$$\psi(a \otimes b) = a \cdot b \in A^{p+q} \tag{1.14}$$

for each $a \otimes b \in A^p \otimes_R A^q$, we say that A is a *differential graded algebra* (over R). Furthermore, the morphism ψ satisfies the commutative diagram

$$\begin{array}{ccc} A \otimes_R A \otimes_R A & \xrightarrow{\psi\otimes 1} & A \otimes_R A \\ \Big\downarrow{\scriptstyle 1\otimes\psi} & & \Big\downarrow{\scriptstyle \psi} \\ A \otimes_R A & \xrightarrow{\quad\psi\quad} & A \end{array} \tag{1.15}$$

which expresses the associativity of the product in A and d satisfies Leibniz's rule

$$d(a \cdot b) = d(a) \cdot b + (-1)^{\deg a} a \cdot d(b). \tag{1.16}$$

Definition 1.1.5. A *differential bigraded algebra over* R is a differential bigraded $R-$module (E, d), associated with a morphism $\psi : E \otimes_R E \to E$, called the product, such that

$$\begin{array}{rlcl} \psi : & E^{p,q} \otimes_R E^{r,s} & \to & E^{p+r,q+s} \\ & e \otimes e' & \mapsto & \psi(e \otimes e') = e \otimes e' \end{array} \tag{1.17}$$

which is associative and such that d satisfies Leibniz's rule

$$d(e \otimes e') = d(e) \otimes e' + (-1)^{p+q} e \otimes d(e'), \tag{1.18}$$

for each $e \in E^{p,q}$ and each $e' \in E^{r,s}$. If $R = \mathbb{Z}_2$ then the above Leibniz's rule reduces to

$$d(e \otimes e') = d(e) \otimes e' + e \otimes d(e'). \tag{1.19}$$

Definition 1.1.6. A *spectral sequence of algebras over a ring* R is a spectral sequence $\{E_r^{*,*}, d_r\}$ along with algebra structures $\psi_r : E_r \otimes_R E_r \to E_r$, for each r, so that ψ_{r+1} can be written as the composition described in the commutative diagram below

$$\begin{tikzcd} E_{r+1} \otimes_R E_{r+1} \arrow[r, "\psi_{r+1}"] \arrow[d, "\cong"] & E_{r+1} \\ H(E_r) \otimes_R H(E_r) \arrow[d, "p"] & \\ H(E_r \otimes_R E_r) \arrow[r, "H(\psi_{r+1})"] & H(E_r) \arrow[uu, "\cong"'] \end{tikzcd} \tag{1.20}$$

where p is the homomorphism defined by $p([u] \otimes [v]) = [u \otimes v]$.

Definition 1.1.7. We say that a spectral sequence of algebras $\{E_r^{*,*}, d_r\}$ *converge* to $H = H^*$, as a graded algebra, if there is a filtration F of H, stable with respect to the product ψ, that is,

$$\psi(F^r H \otimes_R F^s H) \subseteq F^{r+s} H, \tag{1.21}$$

and such that $E_\infty^{p,q} \cong E_0^{p,q}(H, F)$.

Remark 1.1.2. Let $\{E_r^{*,*}, d_r\}$ be a (cohomological) spectral sequence of algebras that converges to a graded algebra $H = H^*$. If the $E_\infty-$term of the sequence is a free bigraded algebra, then H^* is isomorphic to Tot(E_∞), according to [24], p. 25, Example 1.K, where Tot is the functor

$$\begin{array}{rccc} \text{Tot}: & \textit{BigradedAlg} & \to & \textit{GradedAlg} \\ & E & \mapsto & \text{Tot}(E) \end{array} \tag{1.22}$$

with Tot$(E)^n = \bigoplus_{n=p+q} E^{p,q}$, as in [24], p. 24.

Theorem 1.1.1 (Leray-Serre spectral sequence, [24], p. 135, Theorem 5.2)**.** *Let R be a commutative ring with identity. If $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration with B path-connected and F connected, then there is a first quadrant spectral sequence of algebras $\{E_r^{*,*}, d_r\}$ converging to $H^*(E; R)$ as an algebra, such that*

$$E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F; R)), \tag{1.23}$$

the cohomology of B with local coefficients[1] *on the cohomology of the fiber of π. In addition,*

[1] [24], section 5.3.

the cup product and the product $\cdot$ in $E_2^{*,*}$ are related by

$$u \cdot v = (-1)^{qr} u \smile v \tag{1.24}$$

for all $u \in E_2^{p,q}$ and all $v \in E_2^{r,s}$.

Remark 1.1.3. Given a fibration $F \hookrightarrow E \xrightarrow{\pi} B$, there is an action of $\pi_1(B)$ on the cohomology of the fiber F, as shown in [5], p. 120, Corollary 6.13. If, under the same hypotheses as in Theorem (1.1.1), the induced action is trivial, then the local system of coefficient system $\mathcal{H}^q(F; R)$ is simple. Therefore, the E_2-term of the spectral sequence as in (1.23) gets the form

$$E_2^{p,q} \cong H^p(B; R) \otimes_R H^q(F; R), \tag{1.25}$$

according to [24], p. 140, Proposition 5.6. In addition, by the isomorphism (1.25), if $u \otimes v \in E_2^{p,q} \cong H^p(B; R) \otimes_R H^q(F; R)$, then

$$\begin{aligned} u \otimes v &= (u \smile 1) \otimes (1 \smile v) \\ &= (u \otimes 1) \smile (1 \otimes v) \\ &= (u \otimes 1) \cdot (1 \otimes v), \end{aligned} \tag{1.26}$$

which implies that $d_r^{p,q} = 0$, whenever $d^{0,q} = 0$, for all $r \geq 2$, since

$$\begin{aligned} d_r^{p,q}(u \otimes v) &= d_r^{p,q}((u \otimes 1) \cdot (1 \otimes v)) \\ &= d_r^{p,0}(u \otimes 1) \cdot (1 \otimes v) + (-1)^p (u \otimes 1) \cdot d_r^{0,q}(1 \otimes v). \end{aligned} \tag{1.27}$$

Theorem 1.1.2 ([24], p. 147, Theorem 5.9)**.** *Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibration, with B path-connected, F connected and such that the local system of coefficients in B is simple. Then the composed homomorphisms*

$$H^q(B; R) = E_2^{q,0} \twoheadrightarrow E_3^{q,0} \twoheadrightarrow \cdots \twoheadrightarrow E_q^{q,0} \twoheadrightarrow E_{q+1}^{q,0} = E_\infty^{q,0} \subseteq H^q(E; R) \tag{1.28}$$

and

$$H^q(E; R) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subseteq E_q^{0,q} \subseteq \cdots \subseteq E_2^{0,q} = H^q(F; R) \tag{1.29}$$

coincide with the induced homomorphisms $\pi^* : H^q(B; R) \to H^q(E; R)$ and $i^* : H^q(E; R) \to H^q(F; R)$, respectively.

1.2 Group actions

Since the goal of this work is to investigate the existence of free actions of the groups $\mathbb{Z}_2$, S^1 and S^3 on some known spaces and, if that happens, to compute the cohomology of the orbit spaces, we made use of some results about actions of topological groups. In this Section we briefly recall these results, using the text [1] as the main reference.

Definition 1.2.1. Let G be a topological group and X be a topological Hausdorff space. An *action* of G on X is a continuous map $\mu : G \times X \to X$, such that

(i) $\mu(1, x) = x$, for any $x \in X$, where 1 denotes the neutral element in G;

(ii) $\mu(g, \mu(h, x)) = \mu(gh, x)$, for all $g, h \in G$ and all $x \in X$.

We say in this case that the space X is a $G-$*space*. Most of the time we will denote the action μ simply by $\mu(g, x) = *(g, x) = g * x$. For each $x \in X$, we call the *orbit* of x the set

$$\overline{x} = \{g * x\,;\, g \in G\}, \tag{1.30}$$

and denote by X/G the set consisting of the union of all orbits of X by the action of G. The map $\pi : X \to X/G$, which associates to each x the orbit $\overline{x}$, is called *orbit map* (or quotient map) and X/G, with the quotient topology induced by π, is called the *orbit space*.

Definition 1.2.2. Let X be a $G-$space. For each $x \in X$ the subgroup

$$G_x = \{g \in G\,;\, g * x = x\}, \tag{1.31}$$

is called the *isotropy subgroup* in x. We say that the action of the group G on X is *free* if $G_x = \{1\}$, for all $x \in X$ and, in this case, we call X a *free* $G-$*space*.

Definition 1.2.3. Let X and Y be $G-$spaces. We say that a continuous map $f : X \to Y$ is $G-$*equivariant* if

$$f(g * x) = g * f(x),\ \forall x \in X, \tag{1.32}$$

and, in this case, the map f passes to the quotient as $\overline{f} : X/G \to Y/G$, according to the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ \dfrac{X}{G} & \xrightarrow{\overline{f}} & \dfrac{Y}{G}. \end{array} \tag{1.33}$$

Definition 1.2.4. Let X be a manifold. We say that a continuous map $T : X \to X$ is an *involution* if $T \circ T = 1$, which also implies that $T^{-1} = T$. When $T(x) \neq x$, for all $x \in X$, that is,

$$\text{Fix}(T) = \{x \in X\,;\, T(x) = x\} = \emptyset, \tag{1.34}$$

we say that T is a *free involution*.

Remark 1.2.1. Let X be a $\mathbb{Z}_2-$free space. Being $\mathbb{Z}_2 = \langle g \rangle$, We can consider the free involution $T : X \to X$ defined by $T(x) = g * x$, that is, saying that X is a $\mathbb{Z}_2-$free space is equivalent to the statement that X admits a free involution.

Note also that T induces a homeomorphism $T^* : H^*(X;R) \to H^*(X;R)$, which in turn can be seen as an action of $\mathbb{Z}_2$ on the cohomology ring of X.

Proposition 1.2.1. *Let $S, T : X \to X$ be free involutions. If S and T commute, that is, if $S \circ T = T \circ S$, then they induce involutions $\overline{S} : X/T \to X/T$ and $\overline{T} : X/S \to X/S$. If, in addition, the coincidence set*

$$Coin(S,T) = \{x \in X\,;\, T(x) = S(x)\} \tag{1.35}$$

is empty, then the involutions $\overline{S}$ and $\overline{T}$ are free.

Proof. For each element $x \in X$, let us consider $[x]$ the correspondent class in X/S, that is, $[x] = \{x, s(x)\}$, and let us define the map

$$\overline{T}[x] = [T(x)] = \{T(x), S \circ T(x)\}. \tag{1.36}$$

Since the involutions S and T commute, we can conclude that the map $\overline{T}$ is well defined. In fact, if $y \in [x]$, where $y \neq x$, then $y = S(x)$ and consequently

$$\overline{T}[y] = [T(y)] = \{T \circ S(x), S \circ S(x)\} = \{S \circ T(x), x\} = \overline{T}[x]. \tag{1.37}$$

The conclusion that $\overline{T}$ is an involution is trivial.

Let $F = \{[x] \in X/S\,;\, \overline{T}[x] = [x]\}$ be the fixed point set of the involution $\overline{T}$ and note that $[x] \in F$ is equivalent to

$$\{T(x), T \circ S(x)\} = \{x, S(x)\} \Leftrightarrow T(x) = S(x) \Leftrightarrow x \in Coin(S,T). \tag{1.38}$$

Therefore, if the coincidence set is empty then F is also empty. □

Corollary 1.2.1. *Let $T : X \to X$ be a free involution and $\sim$ be an equivalence relation in X. If T preserves the relation $\sim$, that is,*

$$x \sim y \Rightarrow T(x) \sim T(y), \tag{1.39}$$

then T induces an involution $\overline{T}$ on the quotient space $X/\!\sim$. If, in addition, the classes $[x]$ and $[T(x)]$ are disjoint for any $x \in X$, then $\overline{T}$ is free.

Definition 1.2.5. A paracompact Hausdorff space X is said to be *finitistic* if every open covering has a finite dimensional open refinament, where the dimension of a covering is one less than the maximum number of members of the covering which intersect non-trivially.

Remark 1.2.2. It is known [6, 7] that if G is a compact Lie group acting continuously on X, then X is finitistic if and only if the orbit space X/G is finitistic. Therefore we can consider the problem of cohomology classification of the orbit spaces up to finitistic spaces of isomorphic cohomology of the initial space X.

Theorem 1.2.1 ([1], p. 407, Theorem 7.4)**.** *Let T be a free involution on a finitistic space X. Suppose that $\check{H}^i(X;\mathbb{Z}_2) = \{0\}$, for $i > 2l$ and that $T^* = 1$ in $\check{H}^{2l}(X;\mathbb{Z}_2)$. If there is an element $a \in \check{H}^l(X;\mathbb{Z}_2)$, such that*

$$a \smile T^*(a) = a \cdot T^*(a) \neq 0, \tag{1.40}$$

then the fixed point set of T is not empty.

Definition 1.2.6. We say that a group $(G, \cdot)$ is a *Lie group* when G is also a differentiable manifold such that the operation

$$G \times G \to G, \;\; (g, h) \mapsto g^{-1} \cdot h, \tag{1.41}$$

is differentiable. It naturally follows that the well known groups $\mathbb{Z}_2, S^1$ and S^3 are examples of compact Lie groups. We say that H is a (Lie) *subgroup* of G if H is also a Lie group and a submanifold of G, so that the immersion $\iota : H \to G$ is a group homomorphism. For example, we have the chain of Lie subgroups

$$\mathbb{Z}_2 \subset S^1 \subset S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2. \tag{1.42}$$

Remark 1.2.3. Let G be a compact Lie group acting in a differentiable way on a differentiable manifold X. In general, we can consider the quotient space X/G only as an orbit space. However, if the action is free and proper,[2] the quotient X/G has a differentiable structure, according with Theorem 1.2.2, known as the Quotient Manifold Theorem.

Theorem 1.2.2 ([23], p. 544, Theorem 21.10)**.** *Suppose that G is a Lie group acting smoothly, freely, and properly on a differentiable manifold X. Then the orbit space X/G is a topological manifold of dimension dim$X-$dim$G$, and has a unique differentiable structure with the property that the quotient map $\pi : X \to X/G$ is a differentiable submersion.*

Without imposing many conditions regarding the action of G on a space X, Deo and Tripathi in [6], showed the following result:

Theorem 1.2.3 ([6], p. 398, Proposition 3.7)**.** *Let G be a compact Lie group. If X is a $G-$space of finite dimension, paracompact and Hausdorff, then dim$X/G \leq$ dim$X$.*

More specifically, we have the following versions for Theorem 1.2.3, considering free actions of the groups $\mathbb{Z}_2, S^1$ and S^3 on a finitistic space.

Theorem 1.2.4 ([1], p. 374, Theorem 1.5)**.** *Let X be a finitistic space and $n \in N$ such that $\check{H}^j(X;\mathbb{Z}_2) = \{0\}$, for all $j > n$. If X is a $\mathbb{Z}_2-$free space then $\check{H}^j(X/\mathbb{Z}_2;\mathbb{Z}_2) = \{0\}$, for all $j > n$.*

[2] Means that the map $\sigma : G \times X \to X \times X$, $\sigma(g, x) = (g * x, x)$ is proper.

Theorem 1.2.5 ([35], p. 05)**.** *Let X be a finitistic space and $n \in N$ such that $\check{H}^j(X;\mathbb{Z}_2) = \{0\}$, for all $j > n$. If X is a S^1-free space then $\check{H}^j(X/S^1;\mathbb{Z}_2) = \{0\}$, for all $j \geq n$.*

Theorem 1.2.6 ([19], p. 06, Lemma 3.1)**.** *Let X be a finitistic space and $n \in N$ such that $\check{H}^j(X;\mathbb{Z}_2) = \{0\}$, for all $j > n$. If X is a S^3-free space then $\check{H}^j(X/S^3;\mathbb{Z}_2) = \{0\}$, for all $j > n$.*

1.3 Universal bundles and equivariant cohomology

To use the Leray-Serre spectral sequence, according to Theorem 1.1.1, we need to define a specific fibration, known as the Borel fibration, where the total space E will be homotopically equivalent to the obit space X/G. In this Section, we briefly set up this stage, based on the texts [10, 14, 16, 25].

Definition 1.3.1. Let G be a topological group and let E, B be topological spaces. A (left) *principal $G-$bundle* consists of a map $p : E \to B$, continuous and surjective, such that E is a free $G-$space, satisfying the following conditions:

(i) $p(g * x) = p(x)$, for all $x \in E$ and all $g \in G$;

(ii) Each $b \in B$ has a open neighborhood $V \subseteq B$ and a $G-$homeomorphism $\varphi : p^{-1}(V) \to G \times V$ such that the diagram below is commutative, where π_2 is the projection on the second coordinate.

$$\begin{tikzcd} p^{-1}(V) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & G \times V \arrow[dl, "\pi_2"] \\ & V & \end{tikzcd} \tag{1.43}$$

We say in this case that E is the *total space*, B is the *base space*, p is the *projection*, G is the *fiber* and φ is a *trivialization* of p over V. We will use the notation $\xi = (E, p, B, G)$ to indicate a $G-$bundle as this. We say the the $G-$bundle is *trivial* if there is a trivialization over all the space B.

Definition 1.3.2. Let $\xi = \{E, p, B, G\}$ be a principal $G-$bundle. Given a continuous map $f : B' \to B$, we can construct the space

$$f^*(E) := \{(b', e) \in B' \times E\,;\, f(b') = p(e)\} \subseteq B' \times E, \tag{1.44}$$

which is called the *pullback* of ξ by f

Proposition 1.3.1 ([16], p. 44)**.** *Let $\xi = (E, p, B, G)$ be a principal $G-$bundle and $f : B' \to B$ a continuous map. The map p^*, defined by*

$$\begin{array}{rrcl} p^* : & f^*(E) & \to & B' \\ & (b', e) & \mapsto & b', \end{array} \tag{1.45}$$

gives rise to a principal $G-$bundle, named as the principal $G-$bundle induced of ξ by f and indicated by the symbol $f^*(\xi) = (f^*(E), p^*, B', G)$.

Proposition 1.3.2 ([1], p. 88)**.** *Let G a compact Lie group and X be a paracompact and free $G-$space. The bundle $(X, \pi, X/G, G)$ is a principal $G-$bundle.*

Definition 1.3.3. Let $\xi = (E, p, B, G)$ and $\xi' = (E', p', B, G)$ be two principal $G-$bundles over the same base space B. A *principal $B-$morphism* $u : \xi \to \xi'$ is a $G-$equivariant map $u : E \to E'$ such that $p = p' \circ u$. When u is a homeomorphism, we will say that u is a *$B-$isomorphism.*

Proposition 1.3.3 ([16] p. 52)**.** *Let $\xi = (E, p, B, G)$ be a principal $G-$bundle over a paracompact space B and $u, v : B' \to B$ be continuous and homotopic maps. Then, the induced $G-$bundles $u^*(\xi)$ and $v^*(\xi)$ are $B'-$isomorphic.*

$$\begin{tikzcd} v^*(\xi) \arrow[rr, "\cong"] \arrow[dr, "p_v^*"'] & & u^*(\xi) \arrow[dl, "p_u^*"] & E \arrow[d, "p"] \\ & B^* \arrow[rr, bend left, "u"] \arrow[rr, bend right, "v"'] & & B \end{tikzcd} \tag{1.46}$$

Remark 1.3.1. On the set $F_G(B)$, of all principal $G-$bundles ξ over a paracompact space B, let us define the equivalence relation:

$$\xi \sim \eta \Leftrightarrow \xi \text{ and } \eta \text{ are } B-\text{isomorphic}. \tag{1.47}$$

We indicate by $K_G(B)$ the quotient of the space $F_G(B)$ by this relation, and we denote by $\{\xi\}$ the class of the $G-$bundle ξ in this quotient.

Let us consider the $G-$bundle $\omega = (E_0, p_0, B_0, G) \in F_G(B_0)$. For each paracompact space B, we can conclude by 1.3.3 that the map

$$\begin{array}{rcll} \phi_\omega^B: & [B, B_0] & \to & K_G(B), \\ & [u] & \mapsto & \phi_\omega^B[u] = \{u^*(\omega)\} = \{(u^*(E), p^*, B, G)\} \end{array} \tag{1.48}$$

is well defined.

Definition 1.3.4. Let $\omega = (E_0, p_0, B_0, G)$ be a principal $G-$bundle over a paracompact space B_0. We say that ω is a *universal $G-$bundle* if the map ϕ_ω^B is bijective, for each paracompact space B. In this case we will use the notations

$$E_0 = E_G,\ B_0 = B_G,\ p_0 = p_G \text{ and } \omega = \omega_G, \tag{1.49}$$

and we establish a bijection between the sets $K_G(B)$ and $[B, B_G]$.

Proposition 1.3.4 ([16] p. 57)**.** *Let $\omega_G = (E_G, p_G, B_G, G)$ be a universal $G-$bundle. For each principal $G-$bundle $\xi = (E, p, B, G)$ over the paracompact space B, there is a continuous map $q : B \to B_G$ such that ξ and $q^*(\omega_G)$ are $B-$isomorphic. The map q is called a* classifying map *for ξ.*

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[d, "p"'] & q^*(E) \arrow[d, "q^*"] & E_G \arrow[d, "p_G"] \\ B \arrow[r, equal] & B \arrow[r, "q"'] & B_G \end{tikzcd} \tag{1.50}$$

Remark 1.3.2 (Milnor's construction [25])**.** For any compact Lie group G we can construct the universal $G-$bundle

$$G \hookrightarrow E_G \xrightarrow{p_G} B_G, \tag{1.51}$$

where the space E_G is the $G-$space defined as the join infinite copies of G, (see [16], Section 4.11), the quotient space $B_G = E_G/G$ is the classifying space and p_G is the projection.

Remark 1.3.3 (Classifying spaces for $\mathbb{Z}_2, S^1$ and S^3)**.** Throughout the text we will use these constructions for $G = \mathbb{Z}_2, S^1$ or $G = S^3$, and for these three cases we have $E_G = S^\infty$.

For $G = \mathbb{Z}_2$, according to [16], p. 55, Example 11.3, we have $B_G = E_G/G \cong \mathbb{R}P^\infty$. Consequently, the mod 2 cohomology of the classifying space B_G is given by

$$H^*(B_G; \mathbb{Z}_2) = \mathbb{Z}_2[t], \tag{1.52}$$

where $\deg t = 1$, as we can see in [14], p. 212, Theorem 3.12.

For $G = S^1$, since $B_G = E_G/G \cong \mathbb{C}P^\infty$, as we can see in [10], p. 183, Proposition 2.1, then $\pi_1(B_G) = 1$ and the mod 2 cohomology is given by

$$H^*(B_G; \mathbb{Z}_2) = \mathbb{Z}_2[\tau], \tag{1.53}$$

where $\deg \tau = 2$, according to [14], p. 212, Theorem 3.12.

Considering the group $G = S^3$, we have $B_G = E_G/G \cong \mathbb{H}P^\infty$. We can see in [41], p. 189, Corollary 8.13, that $\pi_i(B_G) \cong \pi_{i-1}(G)$, therefore $\pi_1(B_G) \cong 1$ and the mod 2 cohomology of B_G is given by

$$H^*(B_G; \mathbb{Z}_2) = \mathbb{Z}_2[\tau], \tag{1.54}$$

where $\deg \tau = 4$, according to [14], p. 214.

Definition 1.3.5. As the diagonal action of G on $E_G \times X$, given by $g(y, x) = (gy, gx)$, is itself free (since E_G is a free $G-$space), we can define the orbit space

$$X_G = \frac{E_G \times X}{G}, \tag{1.55}$$

which is called the *Borel space* associated to the space X. We call the cohomology

$$H_G^*(X;R) = H^*(X_G;R) \tag{1.56}$$

by the *equivariant cohomology ring* of the $G-$space X, with coefficients in the ring R.

Remark 1.3.4. If G is the trivial group, then $H_G^*(X;R)$ coincides with the usual singular cohomology of X with coefficients in R, so that such a cohomological theory can be thought of as a kind of generalization to the singular theory, in a category of $G-$spaces. If the space X is contractible, we can note that

$$H_G^*(X_G;R) \cong H^*\left(E_G/G;R\right) = H^*(B_G;R), \tag{1.57}$$

which is known as the cohomology of G, when the group G is finite.

The reason why this cohomological theory is used in the context of this work is the fact that, when X is a free $G-$space, then the equivalent cohomology ring coincides with the orbit space cohomology ring, as we will see in the Remark 1.3.5.

Definition 1.3.6. By considering the commutative diagram below

$$\begin{tikzcd} X \times E_G \arrow[r, "\pi_2"] \arrow[d] & E_G \arrow[d, "p_G"] \\ X_G \arrow[r, "\pi"] & B_G \end{tikzcd} \tag{1.58}$$

where $\pi : X_G \to B_G$ is the map induced by the projection $\pi_2 : X \times E_G \to E_G$, we get the fibration

$$X \hookrightarrow X_G \xrightarrow{\pi} B_G, \tag{1.59}$$

called the *Borel fibration* associated to the $G-$space X.

Remark 1.3.5. Note that the other projection $\pi_1 : X \times E_G \to X$ is $G-$equivariant. Thus, it induces a principal $G-$bundle

$$E_G \hookrightarrow X_G \xrightarrow{p} X/G. \tag{1.60}$$

When X is a free $G-$space, then p is a homotopy equivalence, which induces a natural isomorphism

$$p^* : H^*(X/G;R) \to H_G^*(X;R) = H^*(X_G;R) \tag{1.61}$$

for any commutative ring with unity R, according to [10], p. 180. Therefore, in this context it makes sense to use Theorem 1.1.1 applied to a Borel fibration as in (1.59) to obtain the cohomology ring of a space of X/G orbits through isomorphism (1.61).

Theorem 1.3.1. *Let X be a finitistic space equiped with a free $G-$action, for $G = \mathbb{Z}_2, S^1$ or $G = S^3$ and $n \in \mathbb{N}$ such that $\check{H}(X;\mathbb{Z}_2) = \{0\}$, for all $j > n$. If $\pi_1(B_G)$ acts trivially on $\check{H}^*(X;\mathbb{Z}_2)$ and $\{E_r^{*,*}, d_r\}$ is the Leray-Serre spectral sequence associate to Borel Fibration $X \hookrightarrow X_G \to B_G$, then the sequence does not collapse on it E_2-page.*

Proof. Under the above hypothesis, by Theorem 1.1.1 and Remark 1.1.3, the spectral sequence converges to the cohomology of the Borel space X_G and

$$E_2^{p,q} = \check{H}^p(B_G;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \check{H}^q(X;\mathbb{Z}_2).$$

Therefore if the sequence collapsed on it E_2-page, we will have

$$\check{H}^j(X/G;\mathbb{Z}_2) \cong \check{H}^j(X_G;\mathbb{Z}_2) = \mathrm{Tot}^j(E_\infty^{*,*}) = \mathrm{Tot}^j(E_2^{*,*}),$$

for all $j \in \mathbb{N}$.

Let us suppose that $G = S^p$, for $p = 0, 1$ or $p = 3$ (where we are considering $S^0 \cong \mathbb{Z}_2$) and let t be the generator of $\check{H}^*(B_G;\mathbb{Z}_2)$. By Theorem 1.1.2 it follows that $\pi^*(t) = t \otimes 1$ and in general $(t \otimes 1)^j = t^j \otimes 1 = \pi^*(t^j)$, so that

$$0 \neq t^j \otimes 1 \in E_2^{j,0} = E_\infty^{j,0},$$

for all $j \equiv 0(\mathrm{mod}\ (p+1))$.

Thus, $t^j \otimes 1$ it would give rise to a nonzero element $\alpha \in \check{H}^j(X/G;\mathbb{Z}_2)$, which is a contradiction with Theorems 1.2.4, 1.2.5 and 1.2.6, for $p = 0, 1$ or $p = 3$, respectively, for all $j > n$. □

Remark 1.3.6. During the text we will use the symbol $X \cong_2 Y$ to indicate that the two spaces X and Y have the same modulo 2 cohomology, that is, when there is an isomorphism

$$H^*(X;\mathbb{Z}_2) \cong H^*(Y;\mathbb{Z}_2). \tag{1.62}$$

CHAPTER 2

FREE INVOLUTIONS ON WALL MANIFOLDS

In this Chapter we define the Dold manifolds $P(m,n)$ and the Wall manifolds of the form $Q(m,n)$ and we investigate the existence of free involutions or, equivalently, the existence of free $\mathbb{Z}_2-$actions on $Q(m,n)$. We show that such spaces admit these structures when n is an arbitrary odd number. Then, we calculate the cohomology ring of the orbit spaces $Q(m,n)/\mathbb{Z}_2$, for $m = 1$ and m an even number and, as an application of these results, we exhibit Borsuk-Ulam Theorems for finitistic Spaces $X \cong_2 Q(m,n)$, for certain values of m and n.

In order to simplify the notations, all cohomologies considered will be the $\check{C}$ech cohomology with coefficients in $\mathbb{Z}_2$, according to [1], Section 3.6, that is, the symbol $H^*(X)$ will indicate the graded ring $\check{H}^*(X;\mathbb{Z}_2)$. The cup product, usually denoted by $a \smile b$, will be indicated simply by $a \cdot b$ or by ab.

2.1 The Dold and Wall manifolds

This Section is dedicated to the definition of the Dold and Wall manifolds as well as the presentation of their respective cohomology rings structures along with some elementary facts about them.

Definition 2.1.1. Let m and n be non-negative integers. The *Dold manifold* $P(m,n)$, of dimension $m+2n$, is the orbit space $(S^m \times \mathbb{C}P^n)/T$, where T is the free involution

$$\begin{array}{rccc} T: & S^m \times \mathbb{C}P^n & \to & S^m \times \mathbb{C}P^n \\ & (x,[z]) & \mapsto & (-x,[\overline{z}]). \end{array} \tag{2.1}$$

Proposition 2.1.1 ([11])**.** *The mod 2 cohomology ring of $P(m,n)$ is a graded polynomial ring in the variables c and d, of degree 1 and 2, respectively, truncated by the relations $c^{m+1} =$*

$d^{n+1} = 0$. *More precisely, one has*

$$H^*(P(m,n)) = \mathbb{Z}_2[c,d]/\langle c^{m+1}, d^{n+1}\rangle. \tag{2.2}$$

In 1969 Wall [42] constructed a new set of generators of $\mathfrak{R}_*$, using the Wall manifolds $Q(m,n)$. We will reproduce the construction of these spaces in the next results.

Proposition 2.1.2. *Let M be a closed differentiable manifold of dimension n and $f : M \to M$ a diffeomorphism. On the product $M \times [0,1]$ we define the equivalence relation:*

$$(x,0) \sim_f (y,1) \Leftrightarrow y = f(x), \tag{2.3}$$

which identifies only the boundary points $\partial(M \times [0,1])$ and fixes the others. Then the quotient space $(M \times [0,1]) \sim_f$ is a closed differentiable manifold of dimension $n+1$, which we denote by $\mathcal{W}(M,f)$.

Proof. Let us consider a boundary point $(x,0) \in \partial(M \times [0,1])$. Let $\varphi : U \to \mathbb{R}^{n+1}$ be a (boundary) local chart of $(x,0)$ and consider the diffeomorphism

$$\begin{array}{rccc} h: & M \times [0,1] & \to & M \times [0,1] \\ & (x,t) & \mapsto & (f(x), 1-t). \end{array} \tag{2.4}$$

Then we can choose an open subset $U \subset M \times [0,1]$ such that $h(U) \cap U = \emptyset$. Thus, $(U \cup h(U))/\sim_f$ is a domain of a (interior) local chart of $(M \times I)/\sim_f$. $\square$

Remark 2.1.1. Let us consider the involution

$$R \times 1 : S^m \times \mathbb{C}P^n \to S^m \times \mathbb{C}P^n, \tag{2.5}$$

where $R : S^m \to S^m$ denote the reflection of the last coordinate

$$R(x_0, \cdots, x_m) = (x_0, \cdots, x_{m-1}, -x_m), \tag{2.6}$$

and $1 : \mathbb{C}P^n \to \mathbb{C}P^n$ indicates the identity map. Since $R \times 1$ commutes with the involution T in 2.1, i. e.,

$$T \circ (R \times 1)(x,w) = (R \times 1) \circ T(x,w), \tag{2.7}$$

for all $(x,w) \in S^m \times \mathbb{C}P^n$, it follows by Proposition 1.2.1 that $R \times 1$ induces an involution $\overline{R \times 1} : P(m,n) \to P(m,n)$, which we will indicate by S, according to the commutative diagram below.

$$\begin{array}{ccc} S^m \times \mathbb{C}P^n & \xrightarrow{R\times 1} & S^m \times \mathbb{C}P^n \\ \downarrow & & \downarrow \\ \dfrac{S^m \times \mathbb{C}P^n}{T} & \xrightarrow{S} & \dfrac{S^m \times \mathbb{C}P^n}{T} \end{array} \tag{2.8}$$

Definition 2.1.2. For each pair of non-negative integers m, n, the *Wall manifold* of dimension $m + 2n + 1$ is the closed differentiable manifold

$$\mathcal{W}(P(m,n),S) = \frac{P(m,n) \times [0,1]}{\sim_S}, \tag{2.9}$$

which we will denote by $Q(m, n)$.

Remark 2.1.2. Similarly to Definition 2.1.2, we can construct the Wall manifold $Q(m, n)$ as the *mapping torus*

$$Q(m,n) = \frac{P(m,n) \times [0,1]}{([x,z],0) \sim (S[x,z],1)}. \tag{2.10}$$

Proposition 2.1.3 ([42], Lema 4)**.** *The mod 2 cohomology ring of $Q(m, n)$ is a graded polynomial ring in the variables x and c of degree 1, and d of degree 2, truncated by the relations $x^2 = 0$, $d^{n+1} = 0$ and $c^{m+1} = c^m \cdot$. More precisely, one has*

$$H^*(Q(m,n)) = \mathbb{Z}_2[x,c,d]/\langle x^2, c^{m+1} + c^m \cdot x, d^{n+1}\rangle. \tag{2.11}$$

Remark 2.1.3 ([42] and [32])**.** The projection $\pi_3 : S^m \times \mathbb{C}P^n \times [0,1]$, $\pi_3(x,z,t) = t$, induces a fibration

$$P(m,n) \hookrightarrow Q(m,n) \xrightarrow{\beta} S^1, \tag{2.12}$$

according to the commutative diagram (2.13) below, where the vertical arrows represent the respective quotient maps (identifications).

$$\begin{tikzcd} S^m \times \mathbb{C}P^n \arrow[r, hook] \arrow[d] & S^m \times \mathbb{C}P^n \times [0,1] \arrow[r, "\pi_3"] \arrow[d] & {[0,1]} \arrow[d] \\ P(m,n) \arrow[r, hook] & Q(m,n) \arrow[r, "\beta"] & S^1 \end{tikzcd} \tag{2.13}$$

The projection $\pi_{13} : S^m \times \mathbb{C}P^n \times [0,1] \to S^m \times [0,1]$, $\pi_{13}(x,z,t) = (x,t)$, induces another fibration $\gamma : Q(m,n) \to Q(m,0)$ with fiber $\mathbb{C}P^n$ and group structure $\mathbb{Z}_2$, for which there is a classifying map $\theta : Q(m,0) \to \mathbb{R}P^{m+1}$, according to the commutative diagram (2.14),

$$\begin{tikzcd} Q(m,n) \arrow[r, "\Theta"] \arrow[d, "\gamma"] & P(m+1,n) \arrow[d] \\ Q(m,0) \arrow[r, "\theta"] & \mathbb{R}P^{m+1} \end{tikzcd} \tag{2.14}$$

where $\Theta(x,z,t) = (x_0, x_1, \cdots, x_m \cos \pi t, x_m \sin \pi t, z)$.

In order to prove Proposition 2.1.3, Wall used the fibrations β and γ. Moreover he found out that $x = \beta^*(\iota)$, for ι the generator of $H^1(S^1)$, satisfies the relation $x^2 = 0$ and the generators c and d are obtained from Θ^*, applied to the respective generators also denoted by $c, d \in H^*(P(m,n))$, from where it follows the relations $c^{m+1} = c^m \cdot x$ and $d^{n+1} = 0$.

Remark 2.1.4 (Fundamental groups of Wall manifolds)**.** Since we know the fundamental groups of S^1 and the fundamental groups of the Dold manifolds[1], it follows by the homotopy exact sequence associated to β that

$$\pi_1(Q(m,n)) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z} & \text{if} \quad m > 1, \\ \mathbb{Z} \times \mathbb{Z} & \text{if} \quad m = 1. \end{cases} \tag{2.15}$$

Definition 2.1.3. A *Wall structure* on a differentiable manifold M is a homotopy class of maps $\varphi : M \to S^1$, such that $\varphi^*(\iota) = w_1(M)$, where $\iota \in H^1(S^1)$ is the generator element and $w_1(M)$ denotes the first Stiefel-Whitney class[2] of M.

The fibration β, as presented in Remark 2.1.3, defines a Wall structure on $Q(m,n)$, according to Definition 2.1.3. In 1960, C. T. C. Wall showed by these constructions that

$$\left\{ [\mathbb{R}p^{2i}], [Q(2^r - 2, s2^r)] \,;\, i, r, s \geq 1 \right\} \tag{2.16}$$

is a generator set of $\mathfrak{R}_*$. Before that, A. Dold [11] had already shown that the set

$$\left\{ [\mathbb{R}P^{2i}], [P(2^r - 1, s2^r)] \,;\, i, r, s \geq 1 \right\} \tag{2.17}$$

is a generator set to the unoriented cobordism ring. In 1965, J. Milnor [27] defined another generator set for $\mathfrak{R}_*$, defining what came to be known as Milnor manifolds[3]. Precisely, Milnor defined the following generator set of $\mathfrak{R}_*$,

$$\left\{ [\mathbb{R}P^{2i}], [H(2^k, 2t2^k)] \,;\, i, k, t \geq 1 \right\}, \tag{2.18}$$

where $H(m,n)$ denotes a Milnor manifold of dimension $m + n - 1$.

For these reasons, the analysis of certain structures and algebraic invariants related to the Dold, Milnor and Wall manifolds is a relevant research topic, as the interest of investigating the existence of free actions of compact Lie groups on these spaces and also the cohomology classification of the respective orbit spaces.

In fact, this problem stems from the more general question of homotopic classification. However, as we know, generally the homotopic classification problem can be quite difficult to answer, so we can opt for a weaker approach when considering the cohomological classification.

[1] Just use the exact homotopy sequence associated with the bundle $\mathbb{C}P^n \hookrightarrow P(m,n) \to \mathbb{R}P^m$.

[2] Definition A.1.2.

[3] Definition 4.1.1.

2.2 Existence of free involutions on Wall manifolds

From the moment a given space admits the existence of a free involution we can consider the problem of cohomological classification of resulting orbit spaces. In this Section show that we can construct involution on Wall manifolds $Q(m,n)$, for certain values of m and n.

Remark 2.2.1. It is known from cobordism theory ([2], Theorem 24.2) that, when a differentiable manifold X does not bound, there cannot exist any free involution T on X. However, when X does bound, then there may or may not exist a free involution $T : X \to X$.

Regarding the existence of free involutions on Wall manifolds, Khare [21] provides an answer in that direction proving that the following holds.

Proposition 2.2.1 ([21]). *A Wall manifold $Q(m,n)$ bounds if, and only if, n is odd or $n = 0$ and m is even.*

We can conclude by Proposition 2.2.1 that we do not need to consider the cases $Q(m,n)$, for n even, since such spaces do not admit free involutions. We also know that this is not enough to guarantee the existence of free involutions for the other cases, nor to obstruct their existence. However, for such cases, we have a means of displaying them explicitly, according to Proposition 2.2.2.

Proposition 2.2.2. *The Wall manifold $Q(m,n)$ admits a free involution, for any odd integer $n \geq 1$.*

Proof. Let $\varphi : P(m,n) \times [0,1] \to P(m,n) \times [0,1]$ be the free involution given by

$$\varphi\left([x,[z_0 : \cdots : z_n]], s\right) = \left([-x, [-\overline{z}_1 : \overline{z}_0 : \cdots : -\overline{z}_n : \overline{z}_{n-1}]], s\right).$$

If $a = ([x,[z]], s) \sim_S ([y,[w]], t) = b$, that is, $s = 0$, $t = 1$ and

$$[y,[w]] = S\,[x,[z]] = [(x_0, \cdots, x_{m-1}, -x_m), [\overline{z}]]\,,$$

then we see that $\varphi(a) \sim_S \varphi(b)$. therefore, by Corollary 1.2.1, the free involution φ induces an involution $F : Q(m,n) \to Q(m,n)$. Since a is not related with $\varphi(a)$ (relative to the equivalence relation $\sim_S$), for all $a \in P(m,n) \times [0,1]$, it follows that the involution F is free. $\square$

Proposition 2.2.2 ensures that it makes sense to ask about the nature of the orbit space $Q(m, \text{odd})/T$, for T an arbitrary free involution. Our aim will be to give a complete description of the cohomology of these orbit spaces, considering the cases when $m = 1$ or m even. We will see that the choice of such values of m, particularly the case $m = 1$, was convenient due to existing relations between the generators of the cohomology ring of $Q(m,n)$, which made it possible to obtain conditions for guaranteeing that the induced $\mathbb{Z}_2-$action on $H^*(Q(m,n))$ is trivial.

This means that to use the Leray-Serre spectral sequence whose E_2-term is given more conveniently in the form

$$E_2^{p,q} \cong H^p(B_G; G) \otimes_G H^q(X; G), \tag{2.19}$$

as in (1.25), for $G = \mathbb{Z}_2$ and $X = Q(m,n)$, according to the Remark 1.1.3, it is necessary to ensure that the induced action of $\pi_1(B_G)$ on $H^*(Q(m,n))$ is trivial, and we will deal with this condition in the following Section 2.3.

2.3 Induced action on cohomology ring

Let T be a free involution on $Q(m,n)$. Supposing that $m = 1$, by Proposition 2.1.3 the generators $x, c, d \in H^*(X)$ satisfy the relations

$$x^2 = 0,\ d^{n+1} = 0 \text{ and } c^2 = c \cdot x, \tag{2.20}$$

from where we get the following general description of the cohomology groups of $X = Q(1,n)$:

$$\begin{array}{rcccl} H^{2k+1}(X) & \cong & \langle c \cdot d^k \rangle \oplus \langle x \cdot d^k \rangle & \cong & \mathbb{Z}_2^2;\ \ 0 \leq k \leq n; \\ H^{2k}(X) & \cong & \langle d^k \rangle \oplus \langle d^{k-1} \cdot c^2 \rangle & \cong & \mathbb{Z}_2^2;\ \ 1 \leq k \leq n, \end{array} \tag{2.21}$$

and $H^{2n+2}(X) \cong \mathbb{Z}_2\langle d^n \cdot c^2 \rangle$, where $2n+2$ is the dimension of X. Therefore, even when $m = 1$, it is not immediate the conclusion that the induced $\mathbb{Z}_2-$action T^* on $H^*(Q(m,n); \mathbb{Z}_2)$ must be trivial.

In general, we have

$$H^j(Q(m,n)) \cong \mathbb{Z}_2^{\psi(j,m)}, \tag{2.22}$$

where $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a function such that $\psi(j,m) \geq 2$, for all $2 \leq j \leq m + 2n$.

However, we will show in the next results that, under some conditions, the induced $\mathbb{Z}_2-$action $T^* : H^*(Q(m,n)) \to H^*(Q(m,n))$ is indeed trivial.

Remark 2.3.1. Suppose that there is an action of the group $\mathbb{Z}_2 = \langle g \rangle$ on $H^*(Q(m,n))$. Since the cohomology ring $H^*(Q(m,n))$ is generated by the elements x, c and d, such action is entirely determined by $g * x$, $g * c$ and $g * d$. This means that it is enough to analyze the actions

$$\begin{array}{rccc} g^* : & H^j(Q(m,n)) & \to & H^j(Q(m,n)),, \\ & \alpha & \mapsto & g * \alpha, \end{array} \tag{2.23}$$

for $j = 1, 2$. In order to investigate this, we remember that by the naturality of the cup product ([14], p. 2010, Proposition 3.10), we have the following equality, for any integers $r, s \geq 0$,

$$g * (\alpha^r \cdot \beta^s) = (g * \alpha)^r \cdot (g * \beta)^s, \tag{2.24}$$

for all $\alpha, \beta \in H^*(Q(m,n))$.

Lemma 2.3.1. *If $\mathbb{Z}_2 = \langle g \rangle$ acts on $H^*(Q(m,n))$, then $g * x = x$ and $g * d \neq c \cdot x$. Moreover, if $m = 1$, then $g * d = d$ or $g * d = c^2 + d$.*

Proof. Let us suppose that $g * x \neq x$. Then, note that $g * x = c$ or $g * x = c + x$, and in both cases we can conclude, using the equality in (2.24), that $0 = g * x^2 = (g * x)^2 = c^2$, which is a contradiction.

With respect to the generator d, we have in general that

$$g * d \in \{d, c^2, c \cdot x, d + c^2, d + c \cdot x, d + c^2 + c \cdot x\}.$$

However, if $m = 1$, By relation $c^2 = c \cdot x$ we can conclude that $g * d \in \{d, c^2, d + c^2\}$. If $g * d = c^2 = c \cdot x$, then $g * d^2 = (g * d)^2 = (c \cdot x)^2 = 0$, which is a contradiction, since $d^2 \neq 0$. □

Lemma 2.3.2. *Let T be a free involution on $Q(1,n)$ and T^* be the induced $\mathbb{Z}_2-$action on $H^*(Q(1,n))$. If $n \equiv 1(\mathrm{mod}\, 4)$, then $T^*(d) = d$.*

Proof. Let $\mathbb{Z}_2 = \langle g \rangle$. According to Remark 1.2.1, we have $T^*(\alpha) = g * \alpha$, for all $\alpha \in H^*(Q(1,n))$. If $T^*(d) = g * d \neq d$, then $g * d = cx + d = c^2 + d$. Let us consider the integer $s = (n+1)/2$ such that $d^s \in H^{n+1}(Q(1,n))$. Due to hypothesis $n \equiv 1(\mathrm{mod}\, 4)$ it follows that $s = \binom{s}{s-1}$ is odd, consequently

$$\begin{aligned} d^s \cdot T^*(d^s) &= d^s \cdot (c^2 + d)^s \\ &= d^s \cdot \textstyle\sum_{k=0}^{s} \binom{s}{k} (c^2)^{s-k} d^k \\ &= d^s \cdot (c^2 \cdot d^{s-1} + d^s) \\ &= d^n \cdot c^2 \\ &\neq 0 \end{aligned}$$

since $d^n \cdot c^2$ is the generator of $H^{2n+2}(Q(1,n))$. Therefore, taking $a = d^s$ and $l = n+1$, it follows by Theorem 1.2.1 that the fixed point set of T is non-empty, which is not true. □

Theorem 2.3.1. *Let T be a free involution on $Q(1,n)$ and let T^* be the induced $\mathbb{Z}_2-$action on $H^*(Q(1,n))$. If $T^*(c) = c$ and $n \equiv 1(\mathrm{mod}\, 4)$, then T^* is trivial.*

Proof. It follows from Lemmas 2.3.1 and 2.3.2. □

Remark 2.3.2. Assuming that m is an even number then any action of $\mathbb{Z}_2$ on the cohomology ring $H^*(Q(m,n))$ must be trivial on the generators x and c. In fact, since $m+1$ is odd, if $T^*(c) = g * c = c + x$, then

$$g * c^{m+1} = (c + x)^{m+1} = c^{m+1} + c^m \cdot x = 0, \tag{2.25}$$

which is a contradiction. In turn this implies that $g * d \neq c^2$ and $g * d \neq c^2 + c \cdot x$, since any action of $\mathbb{Z}_2$ induces an automorphism

$$\varphi_j : H^j(Q(m,n)) \to H^j(Q(m,n)), \text{ for all } j \geq 0. \tag{2.26}$$

Therefore, if $g * d = c^2$, then $g * (d + c^2) = 0$, that is, $d + c^2 \in \ker \varphi_2 = \{0\}$, which is a contradiction. Similarly, if $g * d = c^2 + c \cdot x$ then $d + c^2 + c \cdot x \in \ker \varphi_2 = \{0\}$, which is also a contradiction.

Then, for every m even, we will have in general:

$$T^*(d) \in \{d, cx + d, c^2 + d, cx + c^2 + d\}. \tag{2.27}$$

Theorem 2.3.2. *Let T be an involution on $Q(m,n)$ and T^* be the induced $\mathbb{Z}_2-$action on $H^*(Q(m,n))$. Suppose that $n > 1$, m is even and $n + 1 \leq m/2$. Then T^* is trivial.*

Proof. By Remark 2.3.2, it follows that $T^*(c) = c$ and $T^*(d) \in \{d, cx+d, c^2+d, cx+c^2+d\}$. In this way, it only remains to show that $T^*(d) = d$, and in order to do this, we will show that all the other alternatives in (2.27) lead to some contradiction.

Let us suppose that $T^*(d) = c^2+d$. By hypothesis, we have $m/2 \geq n+1$, so that $T^*(d^{m/2}) = 0$. However, on the other hand we have

$$\begin{array}{rcl} T^*(d^{m/2}) & = & (c^2 + d)^{m/2} \\ & = & \sum_{k=0}^{m/2} \binom{m/2}{k} (c^2)^{(m/2)-k} d^k \\ & = & c^m + \binom{m/2}{1} c^{m-2} d + \cdots + \binom{m/2}{(m/2)-1} c^2 d^{(m/2)-1} \\ & \neq & 0, \end{array}$$

since c^m is nonzero and there are no relations between the other elements appearing in the binomial expansion that can result on a zero sum.

Observe that the relation $(cx)^2 = 0$ guarantees that the same argument works to show that $T^*(d) \neq c^2 + cx + d$.

If $T^*(d) = cx + d$, then $T^*(d^2) = (cx + d)^2 = d^2$, therefore T^* is trivial on $H^j(Q(m,n))$, for all $j \geq 4$.

Let us consider the element $a = (c^2+d) \in H^2(Q(m,n))$ and note that $T^*(a) = c^2+cx+d$. Since $a^3 \in H^6(Q(m,n))$, so we should have $T^*(a^3) = a^3$. However, we see that

$$T^*(a^3) = (c^2 + cx + d)^3 = a^3 + a^2cx \neq a^3,$$

which is a contradiction. □

2.4 Uniqueness of cohomology ring of orbit space of free involutions on $Q(1, \mathbf{odd})$

In this Section we will show that, under some hypothesis, there is only one cohomology ring structure for the orbit space $Q(1,n)/\mathbb{Z}_2$.

For that we will use the Leray-Serre spectral sequence associated to the appropriate Borel fibration and we will show that there is only one possible structure for the cohomology ring $H^*(Q(1,n)/\mathbb{Z}_2)$, considering an arbitrary free $\mathbb{Z}_2-$action.

Let us consider the free $G-$space $Q(1,n)$, $n > 0$ odd, where $G = \mathbb{Z}_2$, and the associated Borel fibration

$$Q(1,n) \hookrightarrow Q(1,n)_G \xrightarrow{\pi} B_G. \tag{2.28}$$

Under the hypothesis of Theorem 2.3.1 the fundamental group $\pi_1(B_G) = \mathbb{Z}_2$ acts trivially on the cohomology of the fiber $Q(1,n)$. So, by Theorem 1.1.1 and Remark 1.1.3, we can conlude that there is a first quadrant spectral sequence $\{E_r^{*,*}, d_r\}$, whose E_2-term is given by

$$E_2^{p,q} \cong H^p(B_G) \otimes_G H^q(Q(1,n)), \tag{2.29}$$

converging (as an algebra) to $H^*(Q(1,n)_G)$. Since the $\mathbb{Z}_2-$action on $Q(1,n)$ is free, it follows from Remark 1.3.5 that $H^*(Q(1,n)) \cong H^*(Q(1,n)/G)$. Therefore we can use the Leray-Serre spectral sequence to get the possible structures for the cohomology ring of the orbit space $Q(1,n)/G$.

Remark 2.4.1. By Theorem 1.3.1, the spetral sequence $\{E_r^{*,*}, d_r\}$ cannot collapse on E_2-page, then there must be integers $r_i \geq 2$ such that some differential

$$d_{r_i}^{p,q} : E_{r_i}^{p,q} \to E_{r_i}^{p+r_i, q+1-r_i} \tag{2.30}$$

is nontrivial. If $r = \min\{r_i\} \geq 2$, then $E_2^{p,q} = \cdots = E_r^{p,q}$, for every $p, q \geq 0$, and since d_s is trivial for all $2 \leq s \leq r-1$, it follows that

$$E_{s+1}^{p,q} = \frac{\ker(d_s^{p,q})}{\operatorname{im}(d_s^{p-s,q+s-1})} = \frac{E_s^{p,q}}{\{0\}} = E_s^{p,q}, \ \forall p, q \geq 0. \tag{2.31}$$

We will analyze the possible cases where the differential d_r can be nontrivial. Since x, c and d are the generators of $H^*(Q(1,n))$, it is enough to look at the possible values of

$$d_r^{0,1}(1 \otimes x), \ d_r^{0,1}(1 \otimes c) \text{ and } d_r^{0,2}(1 \otimes d). \tag{2.32}$$

We also know that the generators x, c, d of $H^*(Q(1,n))$ have dimension 1, 1 and 2, respectively, and that $\operatorname{im}(d_r^{0,1}) \subseteq E_r^{r,2-r}$, $\operatorname{im}(d_r^{0,2}) \subseteq E_r^{r,3-r}$. Therefore, the differentials $d_r^{*,*}$ must be nontrivial for $r = 2$ or $r = 3$.

However, looking at the E_3-page of the sequence, it is easy to see that

$$d_3^{0,1}(1 \otimes x) = d_3^{0,1}(1 \otimes c) \in E_3^{3,-1} = \{0\}, \tag{2.33}$$

that is, for $r = 3$ the only possible nonzero element is $d_3^{0,2}(1 \otimes d)$. By these facts, therefore, we must at first analyze the following cases when the differential is potentially nontrivial:

(A) $d_3^{0,1}(1 \otimes x) = 0$, $d_3^{0,1}(1 \otimes c) = 0$ and $d_3^{0,2}(1 \otimes d) \neq 0$;

(B) $d_2^{0,1}(1 \otimes x) = 0$, $d_2^{0,1}(1 \otimes c) = 0$ and $d_2^{0,2}(1 \otimes d) \neq 0$;

(C) $d_2^{0,1}(1 \otimes x) \neq 0$, $d_2^{0,1}(1 \otimes c) \neq 0$ and $d_2^{0,2}(1 \otimes d) = 0$;

(D) $d_2^{0,1}(1 \otimes x) \neq 0$, $d_2^{0,1}(1 \otimes c) \neq 0$ and $d_2^{0,2}(1 \otimes d) \neq 0$;

(E) $d_2^{0,1}(1 \otimes x) \neq 0$, $d_2^{0,1}(1 \otimes c) = 0$ and $d_2^{0,2}(1 \otimes d) = 0$;

(F) $d_2^{0,1}(1 \otimes x) \neq 0$, $d_2^{0,1}(1 \otimes c) = 0$ and $d_2^{0,2}(1 \otimes d) \neq 0$;

(G) $d_2^{0,1}(1 \otimes x) = 0$, $d_2^{0,1}(1 \otimes c) \neq 0$ and $d_2^{0,2}(1 \otimes d) \neq 0$;

(H) $d_2^{0,1}(1 \otimes x) = 0$, $d_2^{0,1}(1 \otimes c) \neq 0$ and $d_2^{0,2}(1 \otimes d) = 0$.

Lemma 2.4.1. *If k is even, then $d_i^{l,k}(t^l \otimes c^k) = 0$, $d_i^{l,2k}(t^l \otimes d^k) = 0$ and $d_i^{l,2k+1}(t^l \otimes \nu d^k) = 0$, for ν equal to c or x and for all $i \geq 2$.*

Proof. We will only show the first statement, since the others are analogous. Let us consider $k = 2j$ and note that $d_i^{0,k}(1 \otimes c^k) = 2(1 \otimes c^j)d_i^{0,j}(1 \otimes c^j) = 0$, so

$$d_i^{l,k}(t^l \otimes c^k) = d_i^{l,k}((t^l \otimes 1)(1 \otimes c^k)) = 2(t^l \otimes 1)(1 \otimes c^j)d_i^{0,j}(1 \otimes c^j) = 0,$$

for all $i \geq 2$. □

Proposition 2.4.1. *Cases C, D, E, F, G and H are not possible.*

Proof. We claim that we cannot have simultaneously $d_2^{0,1}(1 \otimes x) \neq 0$ and $d_2^{0,1}(1 \otimes c) \neq 0$. In fact, if this happened then both would be equal to $t^2 \otimes 1$ and, using the relation $c^2 = cx$ (along with the multiplicative properties of the differentials) we would have

$$\begin{aligned} d_2^{0,2}(1 \otimes cx) &= d_2^{0,2}((1 \otimes c)(1 \otimes x)) \\ &= d_2^{0,1}(1 \otimes c)(1 \otimes x) + (1 \otimes c)d_2^{0,1}(1 \otimes x) \\ &= (t^2 \otimes 1)(1 \otimes x) + (1 \otimes c)(t^2 \otimes 1) \\ &= t^2 \otimes (c + x), \end{aligned}$$

while (using the Lemma 2.4.1 in its simplest form) $d_2^{0,2}(1 \otimes c^2) = 0$, which is a contradiction. Therefore, cases C and D cannot occur.

To show the obstructions to the cases E and F, note that it is impossible to happen simultaneously $d_2^{0,1}(1\otimes x)\neq 0$ and $d_2^{0,1}(1\otimes c)=0$, since in this case we would have

$$\begin{aligned} d_2^{0,2}(1\otimes c\cdot x) &= d_2^{0,1}\left((1\otimes c)\cdot(1\otimes x)\right)\\ &= d_2^{0,1}(1\otimes c)\cdot(1\otimes x)+(1\otimes c)\cdot d_2^{0,1}(1\otimes x)\\ &= (1\otimes c)\cdot(t^2\otimes 1)\\ &= t^2\otimes c, \end{aligned}$$

which would imply that $0=d_2^{0,2}(1\otimes c^2)=d_2^{0,2}(1\otimes cx)=t^2\otimes c$, a contradiction. The obstructions for the cases G and H are obtained in a similar way. □

Remark 2.4.2. For D we still have another obstruction. Suppose that $d_2^{0,1}(1\otimes x)=d_2^{0,1}(1\otimes c)=t^2\otimes 1\neq 0$. If

$$d_2^{0,2}(1\otimes d)\in E_2^{2,1}\cong H^2(B_{\mathbb{Z}_2})\otimes H^1(X)$$

is nonzero, then it can only be equal to $t^2\otimes(x+c)$.

In fact, as $d_2^{2,1}(t^2\otimes x)=d_2^{2,1}(t^2\otimes c)=t^4\otimes 1$, then this is the only way to

$$\operatorname{im}(d_2^{0,2})\subseteq\ker(d_2^{2,1})=\langle t^2\otimes(x+c)\rangle.$$

Now, Let us observe that $d_2^{0,2}(1\otimes cx)=t^2\otimes(x+c)$, that is, $d_2^{0,2}(1\otimes d)=d_2^{0,2}(1\otimes cx)$. On the other hand,

$$t^2\otimes(x+c)=d_2^{0,2}(1\otimes d)=d_2^{0,2}(1\otimes cx)=d_2^{0,2}(1\otimes c^2)=0,$$

which is a contradiction.

Proposition 2.4.2. *Case B is not possible.*

Proof. In this case, it is clear that $d_2^{0,2}(1\otimes d)$ needs to be equal to some element of the group

$$E_2^{2,1}\cong H^2(B_{\mathbb{Z}_2})\otimes H^1(Q(1,n))=\{0,t^2\otimes x,t^2\otimes c,t^2\otimes(x+c)\}.$$

Therefore, there are the following possibilities:

(B1) $d_2^{0,2}(1\otimes d)=t^2\otimes x$;

(B2) $d_2^{0,2}(1\otimes d)=t^2\otimes c$;

(B3) $d_2^{0,2}(1\otimes d)=t^2\otimes(x+c)$.

We will show that all these possibilities lead to the same limit of convergence for the spectral sequence in question and, furthermore, we will see that none of them can occur due to Theorem 1.3.1.

Let us consider t^l the generator element of $H^l(B_{\mathbb{Z}_2})$. For $B1$ note that $d_2^{0,3}(1 \otimes xd) = t^2 \otimes x^2 = 0$, $d_2^{0,3}(1 \otimes cd) = t^2 \otimes cx$, and in general we have

$$d_2^{l,2k}(t^l \otimes d^k) = (t^l \otimes 1)d_2^{0,2k}(1 \otimes d^k) = \begin{cases} 0 & \text{if} \quad k \equiv 0 (\text{mod } 2), \\ t^{l+2} \otimes d^{k-1}x & \text{if} \quad k \equiv 1 (\text{mod } 2), \end{cases}$$

$$d_2^{l,2k+1}(t^l \otimes cd^k) = \begin{cases} 0 & \text{if} \quad k \equiv 0 (\text{mod } 2), \\ t^{l+2} \otimes d^{k-1}cx & \text{if} \quad k \equiv 1 (\text{mod } 2), \end{cases}$$

$d_2^{l,2k+1}(t^l \otimes xd^k) = 0$ and $d_2^{l,2k+2}(t^l \otimes c^2 d^k) = 0$ for all $l \geq 0$ and for all $k \in \{0, \cdots, n\}$. With these descriptions of the differentials we can conclude that the E_3-page of the spectral sequence has the following form, from where we can see the module 4 pattern that is repeated:

q					
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\cdots$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	$\cdots$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\cdots$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	$\cdots$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
					p

Figure 2.1: Pattern for E_3-page

Equivalently, we have the following pattern for $q \leq 2n$:

$$E_3^{p,q} = \begin{cases} \mathbb{Z}_2, & \text{if } q = 0 \text{ or } q \equiv 3(\text{mod } 4), \\ \mathbb{Z}_2, & \text{if } q \equiv 1(\text{mod } 4) \text{ and } p \geq 2, \\ \mathbb{Z}_2^2, & \text{if } q \neq 0 \text{ and } q \equiv 0(\text{mod } 4), \\ \mathbb{Z}_2^2, & \text{if } q \equiv 1(\text{mod } 4) \text{ and } p = 0, 1, \\ \{0\}, & \text{if } q \equiv 2(\text{mod } 4) \text{ and } p \geq 2. \end{cases} \tag{2.34}$$

We claim that all differentials $d_3^{p,q}$ are trivial.

If $q \equiv 0(\text{mod } 4)$, then $q - 2 \equiv 2(\text{mod } 4)$ and $\text{im} d_3^{p,q} \subseteq E_3^{p+3,q-2} = \{0\}$. Therefore, $d_3^{p,q}$ is trivial.

If $q \equiv 1(\text{mod } 4)$, note that

$$E_3^{p,q} = \langle t^p \otimes cd^{(q-1)/2} \rangle \oplus \langle t^p \otimes xd^{(q-1)/2} \rangle, \text{ for } p = 0, 1 \tag{2.35}$$

and

$$E_3^{p,q} = \langle t^p \otimes cd^{(q-1)/2}, \rangle \text{ for } p \geq 2. \tag{2.36}$$

Since $(q-1)/2$ is even, it follows by Lemma 2.4.1 that $d_3^{p,q}(t^p \otimes \nu d^{(q-1)/2}) = 0$, both for $\nu = x$ or for $\nu = c$. Therefore, $d_3^{p,q}$ is trivial.

If $q \equiv 2 (\text{mod } 4)$ then $E_3^{p,q} = \langle t^p \otimes c^2 d^{(q-2)/2} \rangle$ for $p = 0, 1$ and it is trivial for $p \geq 2$. Again, by the parity of $(q-2)/2$, it follows that $d_3^{p,q} = 0$.

If $q \equiv 3 (\text{mod } 4)$ then $E_3^{p,q} = \langle t^p \otimes x d^{(q-1)/2} \rangle$ and the analysis of the differential $d_3^{p,q}$ depends only of $d_3^{0,q-1}$, since

$$d_3^{p,q} \left(t^p \otimes x d^{(q-1)/2} \right) = (t^p \otimes x) \cdot d_3^{0,q-1} \left(1 \otimes d^{(q-1)/2} \right).$$

We claim that $d_3^{0,q-1}(1 \otimes d^{(q-1)/2}) = 0$. In fact, since $1 \otimes d$ represents the null element of $E_3^{0,2}$, due to the isomorphism

$$E_3^{0,2} = (\ker d_2^{0,2} / \text{im} d_2^{-2,2}) = \langle 1 \otimes c^2 \rangle \cong (\langle 1 \otimes c^2 \rangle \oplus \langle 1 \otimes d \rangle) / \langle 1 \otimes d \rangle,$$

it follows immediately that $d_3^{0,2}(1 \otimes d) = 0$ and, using the fact that $q - 3 \equiv 0 (\text{mod } 4)$, we can conclude that $d_3^{0,q-3}$ is trivial. Therefore, it follows from these statements that

$$\begin{aligned} d_3^{0,q-1}(1 \otimes d^{(q-1)/2}) &= d_3^{0,q-1}((1 \otimes d) \cdot (1 \otimes d^{(q-3)/2})) \\ &= \underbrace{d_3^{0,2}(1 \otimes d)}_{0} (1 \otimes d^{(q-3)/2}) + (1 \otimes d) \underbrace{d_3^{0,q-3}(1 \otimes d^{(q-3)/2})}_{0} \\ &= 0. \end{aligned}$$

For $B2$ we note that all differentials can be derived by the following rules:

$$d_2^{l,2k}(t^l \otimes d^k) = \begin{cases} 0 & \text{if} \quad k \equiv 0 (\text{mod } 2), \\ t^{l+2} \otimes d^{k-1} c & \text{if} \quad k \equiv 1 (\text{mod } 2), \end{cases}$$

$$d_2^{l,2k+1}(t^l \otimes cd^k) = d_2^{l,2k+1}(t^l \otimes xd^k) = \begin{cases} 0 & \text{if} \quad k \equiv 0 (\text{mod } 2), \\ t^{l+2} \otimes d^{k-1} c^2 & \text{if} \quad k \equiv 1 (\text{mod } 2), \end{cases}$$

and $d_2^{l,2k+2}(t^l \otimes c^2 d^k) = 0$, for all $k \in \{0, \cdots, n\}$. So we can conclude that the E_3-page of the spectral sequence has the same pattern of the one in case $B1$ in (2.34). Therefore, by a similar argument we can conclude that all differentials are trivial.

For $B3$, all the differentials can be derived by the rules

$$d_2^{l,2k}(t^l \otimes d^k) = \begin{cases} 0 & \text{if} \quad k \equiv 0 (\text{mod } 2), \\ t^{l+2} \otimes (x + c) d^{k-1} & \text{if} \quad k \equiv 1 (\text{mod } 2), \end{cases}$$

$$d_2^{l,2k+1}(t^l \otimes xd^k) = d_2^{l,2k+1}(t^l \otimes cd^k) = \begin{cases} 0 & \text{if} \quad k \equiv 0(\text{mod } 2), \\ t^{l+2} \otimes c^2 d^{k-1} & \text{if} \quad k \equiv 1(\text{mod } 2), \end{cases}$$

and $d_2^{l,2k+2}(t^l \otimes c^2 d^k) = 0$ for all $k \in \{0, \cdots, n\}$. From this, we can conclude that the E_3-page of the spectral sequence has the same pattern of the presented in (2.34). Therefore, all the differentials are trivial.

Since we showed that all differentials are trivial, it follows that the sequence collapses on it E_3-page, so

$$H^j(X/G) \cong \text{Tot}(E_\infty)^j = \text{Tot}(E_3)^j,$$

for all $j \geq 0$, which means that the cohomology of the orbit space $Q(1, n)/\mathbb{Z}_2$ has nontrivial elements in all dimensions, contradicting Theorem 1.3.1 (or directly the Theorem 1.2.4). Thus, we conclude that case B cannot occur. □

2.5 Cohomology of orbit spaces of free involutions on $Q(1, \text{odd})$

For us to obtain the cohomology of the orbit spaces $Q(1, n)/\mathbb{Z}_2$, it follows from Propositions 2.4.1 and 2.4.2 that we just need to use the spectral sequence in which the differentials $d_2^{p,q}$ act on the generators of the form

$$d_3^{0,1}(1 \otimes x) = 0,\ d_3^{0,1}(1 \otimes c) = 0 \text{ and } d_3^{0,2}(1 \otimes d) \neq 0 \tag{2.37}$$

which we called case A.

In order to do this, by considering t^l the generator of $H^l(B_G)$ for $k \in \{0, \cdots, n\}$ and by using (2.37), it follows that the differentials $d_3^{p,q}$ act generally on the elements of the E_3-page in the following way:

$$d_3^{l,2k}(t^l \otimes d^k) = (t^l \otimes 1)d_3^{0,k}(1 \otimes d^k) = \begin{cases} t^{l+3} \otimes d^{k-1} & \text{if} \quad k \equiv 1(\text{mod } 2), \\ 0 & \text{if} \quad k \equiv 0(\text{mod } 2), \end{cases}$$

$$d_3^{l,2k+1}(t^l \otimes xd^k) = (1 \otimes x)d_3^{l,2k}(t^l \otimes d^k) = \begin{cases} t^{l+3} \otimes xd^{k-1} & \text{if} \quad k \equiv 1(\text{mod } 2), \\ 0 & \text{if} \quad k \equiv 0(\text{mod } 2), \end{cases},$$

$$d_3^{l,2k+i}(t^l \otimes c^i d^k) = \begin{cases} t^{l+3} \otimes c^i d^{k-1} & \text{if} \quad k \equiv 1(\text{mod } 2), \\ 0 & \text{if} \quad k \equiv 0(\text{mod } 2), \end{cases} \quad \text{for } i = 1, 2,$$

so we have a general expression for all the differentials $d_3^{p,q}$, which allows us to get the E_4-page of the spectral sequence.

By making the necessary calculations, recalling that $E_4^{*,*} \cong H(E_3^{*,*}, d_3)$, we get the following pattern for the E_4-page:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	0	$\cdots$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	$\cdots$
0	0	0	0	0	$\cdots$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	$\cdots$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	0	$\cdots$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	$\cdots$
0	0	0	0	0	$\cdots$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	$\cdots$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	0	$\cdots$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	$\cdots$

q p

Figure 2.2: Pattern for E_4-page

Equivalently, we have the following description for $q \leq 2n$:

$$E_4^{p,q} = \begin{cases} \mathbb{Z}_2, & \text{if } q \equiv 0(\text{mod } 4) \text{ or } q \equiv 2(\text{mod } 4),\ p = 0,1,2, \\ \mathbb{Z}_2^2, & \text{if } q \equiv 1(\text{mod } 4),\ p = 0,1,2, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Therefore, since $\text{im} d_4^{p,q} \subseteq E_4^{p+4,q-3} = \{0\}$, it follows that the sequence collapses at the E_4-term, that is, $E_\infty = E_4$. Thus,

$$H^j(Q(1,n)/G) \cong \text{Tot}(E_\infty)^j = \bigoplus_{p+q=j} E_\infty^{p,q} = E_\infty^{0,j} \oplus E_\infty^{1,j-1} \oplus E_\infty^{2,j-2} \tag{2.38}$$

To be more specific, we have the following additive structure for $H^*(Q(1,n)/G)$:

$$H^j(Q(1,n)/G) = \begin{cases} \mathbb{Z}_2, & \text{if} \quad j = 0 \text{ or } j = 2n+2, \\ \mathbb{Z}_2^2, & \text{if} \quad j \equiv 0(\text{mod } 4),\ 0 < j < 2n+2, \\ \mathbb{Z}_2^3, & \text{if} \quad j \equiv 1(\text{mod } 4),\ 0 < j < 2n+2, \\ \mathbb{Z}_2^3, & \text{if} \quad j \equiv 3(\text{mod } 4),\ 0 < j < 2n+2, \\ \mathbb{Z}_2^4, & \text{if} \quad j \equiv 2(\text{mod } 4),\ 0 < j < 2n+2, \\ \{0\}, & \text{if} \quad j > 2n+2. \end{cases}$$

Now we will obtain the ring structure for the cohomology of the orbit spaces, which is possible due to the fact that the Leray-Serre spectral sequence provides a convergence of algebras. To do this, we will use homomorphisms π^* and i^*, in the way they are shown in the Theorem 1.1.2.

For that, let us consider $\alpha = t \otimes 1 \in E_\infty^{1,0}$. By (1.28) we have $\pi^*(t) = \alpha \in H^1(X_G)$ and $\alpha^l = \pi^*(t^l) \in E_\infty^{l,0}$, that is, $\alpha^l = 0$ for $l \geq 3$ and $\alpha^l \neq 0$ for $l = 1,2$.

The elements $1 \otimes c$, $1 \otimes x \in E_2^{0,1}$ and $1 \otimes d^2 \in E_2^{0,4}$ are permanent cocycles and so they determine nonzero elements $y, z \in E_\infty^{0,1}$ and $w \in E_\infty^{0,4}$, respectively, such that $y^l = z^l = 0$ for $l \geq 3$ and $w^k = 0$ for $k \geq (n+1)/2$.

By (1.29) we can conclude that there are elements $\beta, \gamma \in H^1(X_G)$ that represent y and z, respectively, such that $i^*(\beta) = c$ and $i^*(\gamma) = x$,

$$H^1(X_G) \twoheadrightarrow E_\infty^{0,1} \cong E_4^{0,1} \cong E_3^{0,1} \cong E_2^{0,1} \cong H^1(X). \tag{2.39}$$

Since $H^1(X_G) \cong \mathbb{Z}_2^3$ and $i^*(\alpha) = i^* \circ \pi^*(t) = 0$, then $\alpha \neq \beta \neq \gamma$ and

$$H^1(X_G) \cong \langle \alpha \rangle \oplus \langle \beta \rangle \oplus \langle \gamma \rangle. \tag{2.40}$$

In E_∞ we have $z^2 = 0$, $y^2 = zy$ and $y^2 z \in E_\infty^{0,3} = \{0\}$, so $\gamma^2 = 0$ and $\beta^2 = \beta\gamma$. Therefore,

$$H^2(X_G) \cong \langle \alpha^2 \rangle \oplus \langle \alpha\beta \rangle \oplus \langle \alpha\gamma \rangle \oplus \langle \beta\gamma = \beta^2 \rangle. \tag{2.41}$$

Similarly, we have $H^3(X_G) \cong \langle \alpha\beta^2 = \alpha\beta\gamma \rangle \oplus \langle \alpha^2\beta \rangle \oplus \langle \alpha^2\gamma \rangle$, where $\beta^2\gamma = 0$.

By using (1.29) again, we can conclude that there is an element $\delta \in H^4(X_G) \cong \mathbb{Z}_2^2$ which represents w and such that $i^*(\delta) = d^2$, because we know that $1 \otimes dc^2 \in E_3^{0,4}$ is not a permanent cocycle and that i^* factors through inclusions as in (2.42) below:

$$H^4(X_G) \twoheadrightarrow E_\infty^{0,4} \cong E_4^{0,4} \subseteq E_3^{0,4} \cong E_2^{0,4} \cong H^4(X). \tag{2.42}$$

This way, we see that $H^4(X_G) \cong \langle \alpha^2\beta^2 = \alpha^2\beta\gamma \rangle \oplus \langle \delta \rangle$ and $\delta^{\frac{n+1}{2}} = 0$.

Note also that $w^{\frac{n-1}{2}} x^2 y^2 = w^{\frac{n-1}{2}} x^2 yz \in E_\infty^{2,2n} \neq \{0\}$ and $E_\infty^{l,m} = \{0\}$ for all $l \geq 0$ and $m \geq 2n+1$, that is, in general we have $\delta^i \alpha^j \beta\gamma \neq 0$ for all $0 \leq i \leq (n-1)/2$ and $j = 1, 2$, with the element $\delta^{\frac{n-1}{2}} \alpha^2 \beta^2$ being the generator in the top dimension. Therefore, $H^*(X/G)$ is isomorphic to the following graded polynomial algebra:

$$\frac{\mathbb{Z}_2[\alpha, \beta, \gamma, \delta]}{\langle \alpha^3, \beta^3, \gamma^2, \beta^2 + \beta\gamma, \delta^{\frac{n+1}{2}} \rangle}, \tag{2.43}$$

which completes the proof of Theorem 2.5.1.

Theorem 2.5.1. *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \cong_2 Q(1,n)$, with $n \geq 3$ odd and such that the induced action on the cohomology of X is trivial. Then*

$$H^*(X/G) \cong \frac{\mathbb{Z}_2[\alpha, \beta, \gamma, \delta]}{\langle \alpha^3, \beta^3, \gamma^2, \beta^2 + \beta\gamma, \delta^{\frac{n+1}{2}} \rangle},$$

where $\deg \alpha = \deg \beta = \deg \gamma = 1$ *and* $\deg \delta = 4$.

Corollary 2.5.1. *Let $F : Q(1,n) \to Q(1,n)$ be the involution defined in Proposition 2.2.2. Then $H^*(Q(1,n)/F)$ is isomorphic to the graded polynomial algebra presented in (2.43).*

2.6 Cohomology of orbit spaces of free involutions on $Q(\mathbf{even}, \mathbf{odd})$

Let X be a Wall manifold of the type $Q(m, n)$. Since dim$X = m + 2n + 1$, the dimension of X is odd if and only if m is an even number. By proposition 2.2.2 we know that, if n is odd, then X admits the existence of a free involution $T : X \to X$ or, equivalently, the existence of a free action of the group $G = \mathbb{Z}_2$.

In this section we will compute all the possible cohomology structures of the orbit spaces X/T, for X a Wall manifold of odd dimension, that is, when m is even.

Let $\{E_r^{*,*}, d_r\}$ be the Leray-Serre spectral sequence associated to the Borel fibration $X_G \to B_G$, with fiber X. Under the hypothesis of Theorem 2.3.2, the induced action of G on $H^*(X)$ is trivial. Therefore, we have

$$E_2^{p,q} \cong H^p(B_G) \otimes_{\mathbb{Z}_2} H^q(X). \tag{2.44}$$

By Theorem 1.3.1, this spectral sequence does not collapse on E_2-term. Therefore, there must exist some nontrivial differential d_r, for some $r \geq 2$, such that

$$E_r^{p,q} \cong E_{r-1}^{p,q} \cong \cdots \cong E_2^{p,q}, \tag{2.45}$$

for all $p, q \geq 0$. Since the generators x, c and d of $H^*(X)$ have degrees 1 and 2, respectively, this is only possible if either $r = 2$ or $r = 3$. Therefore, at first, there are the following possibilities for the actions of the differentials on generators of $E_r^{0,q}$:

(1) $d_3^{0,1}(1 \otimes x) = 0$, $d_3^{0,1}(1 \otimes c) = 0$ and $d_3^{0,2}(1 \otimes d) \neq 0$.

(2) $d_2^{0,1}(1 \otimes x) \neq 0$, $d_2^{0,1}(1 \otimes c) = 0$ and $d_2^{0,2}(1 \otimes d) = 0$.

(3) $d_2^{0,1}(1 \otimes x) = 0$, $d_2^{0,1}(1 \otimes c) \neq 0$ and $d_2^{0,2}(1 \otimes d) = 0$.

(4) $d_2^{0,1}(1 \otimes x) = 0$, $d_2^{0,1}(1 \otimes c) = 0$ and $d_2^{0,2}(1 \otimes d) \neq 0$.

(5) $d_2^{0,1}(1 \otimes x) \neq 0$, $d_2^{0,1}(1 \otimes c) \neq 0$ and $d_2^{0,2}(1 \otimes d) = 0$.

(6) $d_2^{0,1}(1 \otimes x) \neq 0$, $d_2^{0,1}(1 \otimes c) = 0$ and $d_2^{0,2}(1 \otimes d) \neq 0$.

(7) $d_2^{0,1}(1 \otimes x) = 0$, $d_2^{0,1}(1 \otimes c) \neq 0$ and $d_2^{0,2}(1 \otimes d) \neq 0$.

(8) $d_2^{0,1}(1 \otimes x) \neq 0$, $d_2^{0,1}(1 \otimes c) \neq 0$ and $d_2^{0,2}(1 \otimes d) \neq 0$.

Lemma 2.6.1. *Cases* $(2), (3), (6), (7)$ *and* (8) *are not possible.*

Proof. Let us suppose that $d_2^{0,1}(1 \otimes x) \neq 0$ and $d_2^{0,1}(1 \otimes c) = 0$. Then, by relation $c^{m+1} = c^m x$, we have

$$0 = d_2^{0,m+1}(1 \otimes c^{m+1}) = d_2^{0,m+1}(1 \otimes c^m x) = t^2 \otimes c^m,$$

which is a contradiction. Therefore, we can eliminate cases (2) and (6).

If $d_2^{0,1}(1 \otimes x) = 0$ and $d_2^{0,1}(1 \otimes c) \neq 0$, then, by relation $c^{m+1} = c^m x$ again, it follows that

$$\begin{aligned} t^2 \otimes c^m &= d_2^{0,m+1}(1 \otimes c^{m+1}) \\ &= (1 \otimes c^m) d_2^{0,1}(1 \otimes x) + (1 \otimes x) d_2^{0,m}(1 \otimes c^m) \\ &= 0, \end{aligned}$$

which produce the same contradiction as the previous one. This in turn allows us to eliminate cases (3) and (7).

Let us suppose now that case (8) occurs. If $d_2^{0,2}(1 \otimes d) = t^2 \otimes x$ or $d_2^{0,2}(1 \otimes d) = t^2 \otimes c$, then

$$\operatorname{im} d_2^{0,3} = \langle t^2 \otimes cx \rangle \oplus \langle t^2 \otimes c^2 \rangle \nsubseteq \langle t^2 \otimes c^2 \rangle = \ker d_2^{2,2},$$

which is a contradiction. □

Lemma 2.6.2. *Case* (4) *is not possible.*

Proof. Let us suppose that $d_2^{0,1}(1 \otimes x) = d_2^{0,1}(1 \otimes c) = 0$ and $d_2^{0,2}(1 \otimes d) \neq 0$. Since

$$d_2^{0,2}(1 \otimes d) \in E_2^{2,1} = H^2(B_G) \otimes H^1(Q(m,n)),$$

there are the following possibilities:

(a) $d_2(1 \otimes d) = t^2 \otimes x$,

(b) $d_2(1 \otimes d) = t^2 \otimes c$,

(c) $d_2(1 \otimes d) = t^2 \otimes (c + x)$.

We will show that none of these cases can occur, due to Theorem 1.3.1 (or directly by Theorem 1.2.4).

Suppose case (a) is true. Then the differential is given by the rules:

$$d_2^{l,2k}(t^l \otimes d^k) = \begin{cases} 0, & \text{if } k \equiv 0 (\text{mod } 2), \\ t^{l+2} \otimes x d^{k-1}, & \text{if } k \equiv 1 (\text{mod } 2), \end{cases}$$

$$d_2^{l,2k+s}(t^l \otimes c^s d^k) = \begin{cases} 0, & \text{if } k \equiv 0 (\text{mod } 2) \text{ or } s = m+1, \\ t^{l+2} \otimes c^s x d^{k-1}, & \text{if } k \equiv 1 (\text{mod } 2) \text{ and } s < m+1, \end{cases}$$

$d_2^{l,2k+1}(t^l \otimes x d^k) = 0$ and $d_2^{2k+s+1}(t^l \otimes c^s x d^k) = 0$, for all $l \geq 0$, $s \in \{0, \cdots, m+1\}$ and $k \in \{0, \cdots, n\}$.

Therefore, the E_3-term of the spectral sequence has infinite nonzero elements on every line $E_3^{*,q}$, for $q \leq m + 2n + 1$. Precisely, since for $p = 0, 1$ we have

$$E_3^{p,q} = \ker d_2^{p,q} / \operatorname{im} d_2^{p-2,q+1} = \ker d_2^{p,q},$$

then, for $q \equiv 0(\bmod 4)$ and $q \neq 0$, it follows that

$$E_3^{1,q} = \langle t \otimes d^{\frac{q}{2}}, t \otimes cxd^{\frac{q-2}{2}}, t \otimes c^4 d^{\frac{q-4}{2}}, t \otimes c^3 xd^{\frac{q-4}{2}}, \cdots, t \otimes c^{q-3}xd, t \otimes c^{q-1}x, t \otimes c^q \rangle.$$

If $q \equiv 1(\bmod 4)$, then

$$E_3^{1,q} = \langle t \otimes cd^{\frac{q-1}{2}}, t \otimes xd^{\frac{q-1}{2}}, t \otimes c^2 xd^{\frac{q-3}{2}}, t \otimes c^5 d^{\frac{q-5}{2}}, \cdots, t \otimes c^{q-3}xd, t \otimes c^{q-1}x, t \otimes c^q \rangle.$$

If $q \equiv 2(\bmod 4)$, then

$$E_3^{1,q} = \langle t \otimes c^2 d^{\frac{q-2}{2}}, t \otimes cxd^{\frac{q-2}{2}}, t \otimes c^3 xd^{\frac{q-4}{2}}, t \otimes c^6 d^{\frac{q-6}{2}}, \cdots, t \otimes c^{q-3}xd, c^{q-1}x, t \otimes c^q \rangle.$$

Since $\mathrm{Im} d_2^{p-2,q+1} \neq \{0\}$, for all $p \geq 2$, we can see that there are less elements on $E_3^{p,q}$, for $p \geq 2$, than for the other cases $p = 0, 1$. Thus, if $d_3^{1,q}$ is trivial, this implies that $d_3^{p,q}$ is trivial, for all $p \geq 2$.

Since $1 \otimes d$ represents the zero element in

$$E_3^{0,2} \cong \langle 1 \otimes c^2, 1 \otimes cx, 1 \otimes d \rangle / \langle 1 \otimes d \rangle \cong \langle 1 \otimes c^2, 1 \otimes cx \rangle$$

we have $d_3^{p,2k}(t^p \otimes d^k) = 0$, for all $k \geq 1$. For dimensional reasons, we have $d_3^{0,1}(1 \otimes c) = d_3^{0,1}(1 \otimes x) = 0$ and, consequently, $d_3^{0,s+1}(1 \otimes c^s x) = 0$, for all $s \geq 0$. These facts together ensure that

$$d_3^{p,2k+s}(t^p \otimes c^s d^k) = d_3^{p,2k+s}((1 \otimes c^s)(t^p \otimes d^k)) = 0,$$

$$d_3^{p,2k+1}(t^p \otimes xd^k) = d_3^{p,2k+1}((1 \otimes x)(t^p \otimes d^k)) = 0$$

and

$$d_3^{p,2k+s+1}(t^p \otimes c^s xd^k) = d_3^{p,2k+s+1}((1 \otimes c^s x)(t^p \otimes d^k)) = 0,$$

for all $s, k \geq 1$.

Therefore, it follows that $d_3^{p,q}(\nu) = 0$, for any $\nu \in E_3^{p,q}$, which, in turn, means that all differentials $d_3^{p,q}$ are trivial. This implies that the sequence collapses on it E_3-term, which contradicts Theorem 1.3.1.

Let us suppose now that case (b) occurs. Then, the differentials are given by the following rules:

$$d_2^{l,2k}(t^l \otimes d^k) = \begin{cases} 0, & \text{if } k \equiv 0(\bmod 2), \\ t^{l+2} \otimes cd^{k-1}, & \text{if } k \equiv 1(\bmod 2), \end{cases}$$

$$d_2^{l,2k+1}(t^l \otimes xd^k) = \begin{cases} 0, & \text{if } k \equiv 0(\bmod 2), \\ t^{l+2} \otimes cxd^{k-1}, & \text{if } k \equiv 1(\bmod 2), \end{cases}$$

$$d_2^{l,2k+s}(t^l \otimes c^s d^k) = \begin{cases} 0, & \text{if } k \equiv 0(\bmod 2) \text{ or } s = m+1, \\ t^{l+2} \otimes c^{s+1} d^{k-1}, & \text{if } k \equiv 1(\bmod 2) \text{ and } s < m+1, \end{cases}$$

$$d_2^{l,2k+s+1}(t^l \otimes c^s x d^k) = \begin{cases} 0, & \text{if } k \equiv 0 (\text{mod } 2) \text{ or } s = m, \\ t^{l+2} \otimes c^{s+1} x d^{k-1}, & \text{if } k \equiv 1 (\text{mod } 2) \text{ and } s < m, \end{cases}$$

for all $l \geq 0$, $s \in \{0, \cdots, m+1\}$ and $k \in \{0, \cdots, n\}$.

Then, for $q \leq m + 2n - 1$, we have

$$E_3^{p,q} = \begin{cases} \ker d_2^{p,q} \neq \{0\}, & \text{if } p = 0, 1, \\ \langle t^p \otimes d^{\frac{q}{2}} \rangle, & \text{if } p \geq 2 \text{ and } q \equiv 0 (\text{mod } 4), \\ \{0\}, & \text{if } p \geq 2 \text{ and } q \not\equiv 0 (\text{mod } 4). \end{cases}$$

This implies that the sequence collapses on the E_3-term. But all the lines $E_3^{*,q}$, such that $q \equiv 0 (\text{mod } 4)$, have infinite nonzero elements, which again contradicts Theorem 1.3.1.

For case (c), the differentials are given by the rules:

$$d_2^{l,2k}(t^l \otimes d^k) = \begin{cases} 0, & \text{if } k \equiv 0 (\text{mod } 2), \\ t^{l+2} \otimes (c + x) d^{k-1}, & \text{if } k \equiv 1 (\text{mod } 2), a \end{cases}$$

$$d_2^{l,2k+1}(t^l \otimes x d^k) = \begin{cases} 0, & \text{if } k \equiv 0 (\text{mod } 2), \\ t^{l+2} \otimes c x d^{k-1}, & \text{if } k \equiv 1 (\text{mod } 2), \end{cases}$$

$$d_2^{l,2k+s}(t^l \otimes c^s d^k) = \begin{cases} 0, & \text{if } k \equiv 0 (\text{mod } 2) \text{ or } s \geq m, \\ t^{l+2} \otimes c^s (c + x) d^{k-1}, & \text{if } k \equiv 1 (\text{mod } 2) \text{ and } s < m, \end{cases}$$

$$d_2^{l,2k+s+1}(t^l \otimes c^s x d^k) = \begin{cases} 0, & \text{if } k \equiv 0 (\text{mod } 2) \text{ or } s \geq m, \\ t^{l+2} \otimes c^{s+1} x d^{k-1}, & \text{if } k \equiv 1 (\text{mod } 2) \text{ and } s < m, \end{cases}$$

for all $l \geq 0$, $s \in \{0, \cdots, m+1\}$ and $k \in \{0, \cdots, n\}$. Similarly to the previous case, the sequence collapses on the E_3-page, with the lines $E_3^{*,q}$ containing infinite nonzero elements, for all $q \leq m + 2n - 1$. This, again, contradicts Theorem 1.3.1. □

Theorem 2.6.1. *Let T be a free involution on a finitistic space $X \simeq_2 Q(m, n)$, with m even and n odd, such that the induced $\mathbb{Z}_2-$action T^* on $H^*(X)$ is trivial. Then $H^*(X/\mathbb{Z}_2)$ is isomorphic to one of the following graded polynomial algebras:*

- (i) $\mathbb{Z}_2[\alpha, \beta, \gamma, \delta]/\langle \alpha^3, \beta^2, \gamma^{m+1} + \gamma^m \beta, \delta^{\frac{n+1}{2}} \rangle$, *where* $\deg \alpha = \deg \beta = \deg \gamma = 1$ *and* $\deg \delta = 4$.
- (ii) $\mathbb{Z}_2[\alpha, \beta, \gamma, \delta]/\langle \alpha^2, \beta^{m+1}, \beta^2 + \gamma, \delta^{n+1} \rangle$, *where* $\deg \alpha = \deg \beta = 1$ *and* $\deg \gamma = \deg \delta = 2$.

Proof. By Lemmas 2.6.1 and 2.6.2, we only need to consider the cases (1) and (5).

Let us suppose that (1) is true. Then, all the differentials are determined by the rule:

$$d_3^{l,2k+\deg \nu}(t^l \otimes \nu d^k) = \begin{cases} 0, & \text{if } k \equiv 0 (\text{mod } 2), \\ t^{l+3} \otimes \nu d^{k-1}, & \text{if } k \equiv 1 (\text{mod } 2), \end{cases}$$

for all $l \geq 0$ and $k \in \{0, \cdots, n\}$, where ν is any element of $H^*(Q(m,n))$.

Then, for $q \leq m+2n-1$ and $p = 0, 1, 2$, the group $E_4^{p,q}$ is equal to:

$\langle t^p \otimes 1 \rangle,$ if $q = 0$,

$\langle t^p \otimes d^{\frac{q}{2}}, t^p \otimes c^4 d^{\frac{q-4}{2}}, t^p \otimes c^3 x d^{\frac{q-4}{2}}, \cdots, t^p \otimes c^q, t^p \otimes c^{q-1}x \rangle,$ if $q \equiv 0 (\text{mod } 4)$ and $q \neq 0$,

$\langle t^p \otimes c d^{\frac{q-1}{2}}, t^p \otimes x d^{\frac{q-1}{2}}, t^p \otimes c^5 d^{\frac{q-5}{2}}, t^p \otimes c^4 x d^{\frac{q-5}{2}}, \cdots, t^p \otimes c^q, t^p \otimes c^{q-1}x \rangle,$ if $q \equiv 1 (\text{mod } 4)$,

$\langle t^p \otimes c^2 d^{\frac{q-2}{2}}, t^p \otimes c x d^{\frac{q-2}{2}}, t^p \otimes c^6 d^{\frac{q-6}{2}}, t^p \otimes c^5 x d^{\frac{q-6}{2}}, \cdots, t^p \otimes c^q, t^p \otimes c^{q-1} \rangle,$ if $q \equiv 2 (\text{mod } 2)$,

$\langle t^p \otimes c^3 d^{\frac{q-3}{2}}, t^p \otimes c^2 x d^{\frac{q-3}{2}}, t^p \otimes c^7 d^{\frac{q-7}{2}}, t^p \otimes c^6 x d^{\frac{q-7}{2}}, \cdots, t^p \otimes c^q, t^p \otimes c^{q-1}x \rangle,$ if $q \equiv 3 (\text{mod } 4)$,

and $E_4^{p,q} = \{0\}$, for all $p \geq 3$.

Therefore, it is clear that the sequence collapses on E_4-term, so

$$H^j(X/G) \cong \text{Tot}(E_\infty)^j = \bigoplus_{p+q=j} E_\infty^{p,q} = E_\infty^{0,j} \oplus E_\infty^{1,j-1} \oplus E_\infty^{2,j-2},$$

for all $j \geq 0$. In particularly, if $m \geq 4$, then

$$H^j(X/G) \cong \begin{cases} \mathbb{Z}_2^3, & \text{if } j = 1, \\ \mathbb{Z}_2^5, & \text{if } j = 2, \\ \mathbb{Z}_2^6, & \text{if } j = 3, \\ \mathbb{Z}_2^7, & \text{if } j = 4. \end{cases}$$

Let us consider $\alpha = t \otimes 1 \in E_\infty^{1,0}$. By (1.28), it follows that $\pi^*(t) = \alpha \in H^1(X_G)$, and $\alpha^l = \pi^*(t^l) \in E_\infty^{l,0}$. Then, $\alpha^l = 0$ for all $l \geq 3$ and $\alpha^l \neq 0$ if $l = 1, 2$.

The elements $1 \otimes x, 1 \otimes c \in E_4^{0,1}$ and $1 \otimes d^2 \in E_4^{0,4}$ are the only permanent co-cycles, therefore they determine nontrivial elements $y, z \in E_\infty^{0,1}$ and $w \in E_\infty^{0,4}$, respectively, such that $y^2 = 0$, $z^{m+1} = z^m y$ and $w^{\frac{n+1}{2}} = 0$.

By (1.29), there are elements $\beta, \gamma \in H^1(X_G)$ and $\delta \in H^4(X_G)$ which represent y, z and w, respectively, such that $i^*(\beta) = x$, $i^*(\gamma) = c$ and $i^*(\delta) = d^2$. Therefore, it follows that $\beta^2 = 0$, $\gamma^{m+1} = \gamma^m \beta$ and $\delta^{\frac{n+1}{2}} = 0$. Then, we have

$$H^1(X/G) \cong \langle \alpha \rangle \oplus \langle \beta \rangle \oplus \langle \gamma \rangle$$

and, similarly,

$$H^j(X/G) \cong \begin{cases} \langle \alpha^2 \rangle \oplus \langle \alpha\beta \rangle \oplus \langle \alpha\gamma \rangle \oplus \langle \beta\gamma \rangle \oplus \langle \gamma^2 \rangle, & \text{if } j = 2, \\ \langle \alpha^2\beta \rangle \oplus \langle \alpha^2\gamma \rangle \oplus \langle \alpha\beta\gamma \rangle \oplus \langle \alpha\gamma^2 \rangle \oplus \langle \beta\gamma^2 \rangle \oplus \langle \gamma^3 \rangle, & \text{if } j = 3, \\ \langle \delta \rangle \oplus \langle \beta\gamma^3 \rangle \oplus \langle \gamma^4 \rangle \oplus \langle \alpha\beta\gamma^2 \rangle \oplus \langle \alpha\gamma^3 \rangle \oplus \langle \alpha^2\beta\gamma \rangle \oplus \langle \alpha^2\gamma^2 \rangle, & \text{if } j = 4, \end{cases}$$

and $H^{m+2n+1}(X/G) \cong \langle \alpha^2 \beta \gamma^m \delta^{\frac{n-1}{2}} \rangle$.

Therefore, we can conclude that $H^*(X/G)$ is isomorphic to the following polynomial graded algebra:

$$\frac{\mathbb{Z}_2[\alpha,\beta\gamma,\delta]}{\langle \alpha^3,\beta^2,\gamma^{m+1}+\gamma^m\beta,\delta^{\frac{n+1}{2}}\rangle},$$

where $\deg\alpha=\deg\beta=\deg\gamma=1$ and $\deg\delta=4$. This corresponds to item (i) of our Theorem.

Let us suppose now that case (5) is true. Then, the differentials are given by the following rules, for all $l\geq 0$, $k\in\{1\cdots,n\}$ and $s\in\{1,\cdots,m+1\}$:

$$d_2^{l,2k}(t^l\otimes d^k)=0,$$

$$d_2^{l,2k+1}(t^l\otimes xd^k)=t^{l+2}\otimes d^k,$$

$$d_2^{l,s}(t^l\otimes c^s)=\begin{cases} 0, & \text{if } s\equiv 0(\text{mod } 2),\\ t^{l+2}\otimes c^{s-1}, & \text{if } s\equiv 1(\text{mod } 2),\end{cases}$$

$$d_2^{l,s+1}(t^l\otimes c^s x)=\begin{cases} t^{l+2}\otimes c^s, & \text{if } s\equiv 0(\text{mod } 2),\\ t^{l+2}\otimes c^{s-1}(x+c), & \text{if } s\equiv 1(\text{mod } 2),\end{cases}$$

$$d_2^{l,s+1+2k}(t^l\otimes c^s xd^k)=\begin{cases} t^{l+2}\otimes c^s d^k, & \text{if } s\equiv 0(\text{mod } 2),\\ t^{l+2}\otimes c^{s-1}d^k(x+c), & \text{if } s\equiv 1(\text{mod } 2),\end{cases}$$

$$d_2^{l,s+2k}(t^l\otimes c^s d^k)=\begin{cases} 0, & \text{if } s\equiv 0(\text{mod } 2),\\ t^{l+2}\otimes c^{s-1}d^k, & \text{if } s\equiv 1(\text{mod } 2),\end{cases}$$

Then, for $p=0,1$ and $q\leq m+2n-1$, $E_3^{p,q}$ is equal to:

$\langle t^p\otimes 1\rangle,$ if $q=0$,

$\langle t^p\otimes d^{\frac{q}{2}}, t^p\otimes c^2d^{\frac{q-2}{2}}, t^p\otimes c^4d^{\frac{q-4}{2}},\cdots,t^p\otimes c^{q-2}d, t^p\otimes c^q\rangle,$ if q even,

$\langle t^p\otimes d^{\frac{q-1}{2}}(c+x), t^p\otimes c^2d^{\frac{q-3}{2}}(c+x),\cdots,t^p\otimes c^{q-3}d(c+x), t^p\otimes c^{q-1}(c+x)\rangle,$ if q is odd,

and $E_3^{p,q}=\{0\}$, for all $p\geq 2$.

Therefore, the sequence collapses on E_3-term and we have

$$H^j(X/G)\cong \text{Tot}(E_\infty)^j=\bigoplus_{p+q=j}E_\infty^{p,q}=E_\infty^{0,j}\oplus E_\infty^{1,j-1},$$

for all $j\geq 0$.

Similarly to the previous case, let us consider $\alpha=1\otimes t\in E_\infty^{1,0}$ such that $\pi^*(t)=\alpha$. Then $\alpha^l=0$ for all $l\geq 2$.

The elements $1\otimes(x+c)\in E_3^{0,1}$ and $1\otimes c^2, 1\otimes d\in E_3^{0,2}$ are the only permanent co-cycles, so they determine non-trivial elements $y\in E_\infty^{0,1}$ and $z,w\in E_\infty^{0,2}$, respectively, such that $y^{m+1}=0$, $y^2=z$ and $w^{n+1}=0$. By (1.29), there are elements $\beta\in H^1(X_G)$ and $\gamma,\delta\in H^2(X_G)$ which represent y,z and w, respectively, such that $i^*(\beta)=c+x$, $i^*(\gamma)=c^2$ and $i^*(\delta)=d$.

This implies that

$$H^j(X/G) \cong \begin{cases} \langle\alpha\rangle \oplus \langle\beta\rangle, & \text{if } j=1, \\ \langle\alpha\beta\rangle \oplus \langle\gamma\rangle \oplus \langle\delta\rangle, & \text{if } j=2, \\ \langle\alpha\gamma\rangle \oplus \langle\alpha\delta\rangle \oplus \langle\beta\delta\rangle \oplus \langle\beta\gamma\rangle, & \text{if } j=3, \\ \langle\alpha\beta\gamma\rangle \oplus \langle\alpha\beta\delta\rangle \oplus \langle\gamma^2\rangle \oplus \langle\gamma\delta\rangle \oplus \langle\delta^2\rangle, & \text{if } j=4, \end{cases}$$

and $H^{m+2n+1}(X/G) \cong \langle\alpha\beta^m\delta^n\rangle = \langle\alpha\gamma^{\frac{m}{2}}\delta^n\rangle$.

Therefore, we can conclude that $H^*(X/G)$ must be isomorphic to the following graded polynomial algebra:

$$\frac{\mathbb{Z}_2[\alpha,\beta,\gamma,\delta]}{\langle\alpha^2,\beta^{m+1},\beta^2+\gamma,\delta^{n+1}\rangle},$$

where $\deg\alpha = \deg\beta = 1$ and $\deg\gamma = \deg\delta = 2$. This corresponds to our item (ii). $\square$

Example 2.6.1. If $X \cong_2 Q(2,1)$ then, supposing that the induced $\mathbb{Z}_2-$action on $H^*(X)$ is trivial, it follows that $H^*(X/\mathbb{Z}_2)$ must be isomorphic to one of the following graded polynomial algebras:

(i) $\mathbb{Z}_2[\alpha,\beta,\gamma]/\langle\alpha^2,\beta^2,\gamma^3+\gamma^2\beta\rangle$, where $\deg\alpha = \deg\beta = \deg\gamma = 1$.

(ii) $\mathbb{Z}_2[\alpha,\beta,\gamma,\delta]/\langle\alpha^2,\beta^3,\beta^2+\gamma,\delta^2\rangle$, where $\deg\alpha = \deg\beta = 1$ and $\deg\gamma = \deg\delta = 2$.

2.7 Existence of equivariant maps

Let X be a paracompact and Hausdorff space equipped with a free involution $T : X \to X$ or, equivalently, a free $\mathbb{Z}_2-$action. Let us consider S^n the $n-$dimensional sphere and $A : S^n \to S^n$ the antipodal map.

In [3], Conner and Floyd proposed the following question: For which integer n is there a $\mathbb{Z}_2-$equivariant map from S^n to X but in a way that it doesn't exist such map from S^{n+1} to X? This question motivated the definition of an index, which depends on the involution T :

Definition 2.7.1. Let T an free involution on X. We call of *index* of T in X to the number

$$\text{ind}(X,T) = \max\{n\,;\ \text{there is a } \mathbb{Z}_2-\text{equivariant map } f : S^n \to X\}. \tag{2.46}$$

For $X = \emptyset$, by convention we have $\text{ind}(X,T) = -1$ and, when there is no such maximum number n, one defines $\text{ind}(X,T) = \infty$.

Remark 2.7.1. An equivalent formulation of the Borsuk-Ulam Theorem states that there is no equivariant map from S^{n+1} to S^n. Hence, it follows in particular that $\text{ind}(S^n, A) = n$.

Lemma 2.7.1 ([3], (3.2))**.** *Let X and Y be topological spaces and T, S be free involutions on X and Y, respectively. If $f : X \to Y$ is an equivariant map, then ind$(X,T) \leq$ ind(Y,S).*

By using characteristic classes with coefficients in $\mathbb{Z}_2$ (which are the Stiefel-Whitney classes[4]), the authors in [3] obtained a cohomological criterion to study the initial question of the existence of $\mathbb{Z}_2-$equivariant maps, by defining the following cohomological index:

Definition 2.7.2. Let T be a free involution on X and $w_1 = w_1(\xi)$ be the Stiefel-Whitney class associated to the principal $\mathbb{Z}_2-$bundle $\xi = (X, \pi, X/\mathbb{Z}_2, \mathbb{Z}_2)$. We call the *co-index* of T in X the number

$$\text{co-ind}_{\mathbb{Z}_2}(X,T) = \max\{n\,;\, w_1^n \neq 0\}. \tag{2.47}$$

Lemma 2.7.2 ([3], (4.5))**.** *ind*$(X,T) \leq$ *co-ind*$_{\mathbb{Z}_2}(X,T)$

Remark 2.7.2 (Sufficient condition for the existence of equivariant maps, [3], p. 426)**.** Suppose that there is a $G-$equivariant map $f : X \to Y$ and let $\sigma = w_1(\xi)$ and $\omega = w_1(\eta)$ be the respective Stiefel-Whitney classes associated to the principal $G-$bundles $\xi = (X, \pi, X/G, G)$ and $\eta = (Y, \pi, Y/G, G)$.

$$\begin{array}{ccc} H^*(Y/G) & \xrightarrow{F^*} & H^*(X/G) \\ \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} \\ H^*(Y) & \xrightarrow{f^*} & H^*(X) \end{array} \tag{2.48}$$

If $F : X/\mathbb{Z}_2 \to Y/\mathbb{Z}_2$ is the map induced by f, then $F^*(\omega) = \sigma$, so ω^j is nonzero whenever σ^j is nonzero or, equivalently, we have the condition $\omega^j = 0 \Rightarrow \sigma^j = 0$. By this condition, together with Lemmas 2.7.1 and 2.7.2 and the result of Theorems 2.5.1 and 2.6.1, we obtain the Theorems 2.7.1 and 2.7.2 below:

Theorem 2.7.1. *If $X \cong_2 Q(1, odd)$, then there is no $\mathbb{Z}_2-$equivariant map of S^k to X, for all $k \geq 2$.*

Proof. Let us consider $X \cong_2 Q(1, \text{odd})$, $G = \mathbb{Z}_2$ and suppose that there is an equivariant map of S^k to X. By considering $q : X/G \to B_G$ a classifying map relative to the principal $G-$bundle

$$G \hookrightarrow X \longrightarrow X/G, \tag{2.49}$$

let $s : X/G \to X_G$ be an inverse to the homotopy equivalence p, given in Remark 1.3.5. Then, by Proposition 1.3.4, the composition $\pi \circ s : X/G \to B_G$ is a classifying map, where $\pi : X_G \to B_G$ is the Borel fibration. Therefore, $\pi \circ s$ is homotopic to the map q.

$$\begin{tikzcd} X \arrow[d] & & X_G \arrow[d, "\pi"] \arrow[dll, bend right, "p"'] \\ X/G \arrow[urr, bend right, "s"'] \arrow[rr, "q"'] & & B_G \end{tikzcd} \tag{2.50}$$

[4]Definition A.1.2.

Thus, to obtain the Stiefel-Whitney class w_1 associated with the $G-$bundle $X \to X/G$, we just have to analyze the map

$$\pi^* : H^1(B_G; G) \to H^1(X_G; G) \cong H^1(X/G; G),$$

according to Theorem 1.1.2. It follows from the proof of Theorem 2.5.1 that $w_1(X/G) = \pi^*(t) = \alpha$, with $\alpha^j = 0$ for $j \geq 3$. Then we have

$$\text{co-ind}_G(X) = 2 \geq \text{ind}(X)$$

and consequently such a $G-$equivariant map cannot exist for all $k \geq 2$. $\square$

Theorem 2.7.2. *If $X \cong_2 Q(m, odd)$, with m even, then there is no $\mathbb{Z}_2-$equivariant map of S^k to X, for all $k \geq 2$.*

Proof. Analogous to the proof of Theorem 2.7.1. Just use the proof of Theorem 2.6.1 to get $\text{Ind}(X) \leq \text{co-ind}_G(X) \leq 2$. $\square$

CHAPTER 3

FREE ACTIONS OF S^1 ON DOLD AND WALL MANIFOLDS

In this Chapter we investigate the existence of free actions of the group S^1 on the Dold and Wall manifolds. In particular we calculate the cohomological structure of the orbit spaces $P(m,n)/S^1$ and we show that none of the Wall manifolds admit the existence of free actions of S^1.

In order to simplify the notations, all cohomologies considered will be the $\check{C}$ech cohomology with coefficients in $\mathbb{Z}_2$, according to [1], Section 3.6. That is, the symbol $H^*(X)$ will indicate the graded ring $\check{H}^*(X;\mathbb{Z}_2)$. The cup product, usually denoted by $a \smile b$, will be simply indicated by $a \cdot b$ or by ab.

3.1 Free actions of S^1 on Dold manifolds and cohomology of orbit spaces

As we know, the Dold manifolds are defined by the quotient

$$P(m,n) = \frac{S^m \times \mathbb{C}P^n}{T}, \tag{3.1}$$

where T is the free involution defined in 2.1. In [11], A. Dold described the mod 2 cohomology ring structure of $P(m,n)$, according to Proposition 2.1.1, which is given by

$$H^*(P(m,n)) \cong \frac{\mathbb{Z}_2[c,d]}{\langle c^{m+1}, d^{n+1}\rangle}, \tag{3.2}$$

where $\deg c = 1$ and $\deg d = 2$.

In [20], Khare investigates when Dold manifolds bounds proving that the following holds.

Theorem 3.1.1 ([20])**.** *Let $P(m,n)$ be a Dold manifold.*

(i) *If n is odd, then $P(m,n)$ bounds for every n;*

(ii) *If m and n are both even, then $P(m,n)$ does not bound;*

(iii) *If m is odd and n is even, then $P(m,n)$ bounds if, and only if, $m > n$ and 2^α divides $m-(n+1)$, for some α such that $2^\alpha > n$.*

Remark 3.1.1. The previous result allow us to restrict any free S^1-action on a Dold manifold $P(m,n)$ to a free action of $\mathbb{Z}_p$, for any p prime. In particular, for $p = 2$, we obtain free involutions, which cannot happen according to Theorem 3.1.1, for m and n both even or n even and $m \leq n$.

However, if n is odd, we can ask about the existence and nature of free S^1-actions on Dold manifolds $P(m,n)$, for every $m \geq 1$. Moreover, if there exists a such an action, we can compute the cohomology algebra of orbit space $P(m,n)/S^1$.

Remark 3.1.2. If $G = S^1$, we know that[1] $B_G = \mathbb{C}P^\infty$, thus $\pi_1(B_G) = 1$ acts trivially on $H^*(P(m,n))$. Therefore, considering the spectral sequence associated to the Borel fibration

$$P(m,n) \hookrightarrow P(m,n)_G \xrightarrow{\pi} B_G, \tag{3.3}$$

it follows from Remark 1.1.3 that the E_2-term of the sequence is given by the tensor product:

$$E_2^{p,q} \cong H^p(B_G) \otimes H^q(P(m,n)). \tag{3.4}$$

Lemma 3.1.1. *Let $\{E_r^{*,*}, d_r\}$ be a spectral sequence, where $E_2^{p,q} \cong \{0\}$ for all p odd. If this sequence does not collapses on the E_2-page, i.e. there is some differential $d_2^{p,q} \neq 0$, then the sequence collapses on the E_3-page.*

Proof. Let us consider a non-zero element α in $E_3^{p,q}$. By hypothesis, $p = 2s$ for some $s \in \mathbb{N}$. Therefore, $d_3^{2s,q}(\alpha) \in E_3^{2s+3,q-2} = \{0\}$ since $2s+3$ is odd. $\square$

Theorem 3.1.2. *Let X be a finitistic space, such that $X \cong_2 P(m,n)$ for n odd, equipped with a free action of $G = S^1$.*

(i) *If m is odd, then*

$$H^*(X/G) = \frac{\mathbb{Z}_2[x,y]}{\langle x^{(m+1)/2}, y^{n+1}\rangle},$$

where $\deg x = \deg y = 2$. *In particular,*

$$H^q(X/G) \cong H^q(P(m,n))$$

---

[1] See Remark 1.3.3.

if q is even and $q < m + 2n$, and $H^q(X/G) = \{0\}$ otherwise;

(ii) *If m is even, then*

$$H^*(X/G) = \frac{\mathbb{Z}_2[x, y]}{\langle x^{(m+2)/2}, y^{n+1} \rangle},$$

where $\deg x = \deg y = 2$. *In particular,*

$$H^q(X/G) \cong H^q(P(m, n))$$

if q is even and $q \leq m + 2n$, and $H^q(X/G) = \{0\}$ otherwise.

Proof. Let $\{E_r^{*,*}, d_r\}$ be the spectral sequence associated to the Borel fibration

$$X \hookrightarrow X_G \xrightarrow{\pi} B_G, \tag{3.5}$$

converging, as an algebra, to the cohomology of the Borel space X_G, which has the same homotopy type of the orbit space X/G.

The fundamental group of the classifying space $B_G \cong \mathbb{C}P^\infty$ is trivial, so the E_2-page of the spectral sequence has the suitable form

$$E_2^{p,q} = H^p(B_G) \otimes_{\mathbb{Z}_2} H^q(X). \tag{3.6}$$

By Theorem 1.3.1, the spectral sequence does not collapse on the E_2-page; therefore, some differential must be nontrivial. As the generators of $H^*(X)$ are c and d, of degree 1 and 2, respectively, we only need to analyze the following possibilities:

(I) $d_2^{0,1}(1 \otimes c) = \tau \otimes 1$ and $d_2^{0,2}(1 \otimes d) = \tau \otimes c$,

(II) $d_2^{0,1}(1 \otimes c) = 0$ and $d_2^{0,2}(1 \otimes d) = \tau \otimes c$,

(III) $d_2^{0,1}(1 \otimes c) = \tau \otimes 1$ and $d_2^{0,2}(1 \otimes d) = 0$.

We will show that the cases (I) and (II) do not occur.

Suppose that case (I) occurs. Then, note that the differential $d_2^{2,1}$ is an isomorphism, so $\ker d_2^{2,1} = \{0\}$, while $\mathrm{im} d_2^{0,2} = \langle \tau \otimes c \rangle$, that is, $\mathrm{im} d_2^{0,2} \nsubseteq \ker d_2^{2,1}$, which is a contradiction with the structure of the spectral sequence.

If case (II) is true, then the general expressions of the differentials are given by:

$$d_2^{r,k}(\tau^r \otimes c^k) = 0,$$

$$d_2^{r,2l}(\tau^r \otimes d^l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ \tau^{r+1} \otimes cd^{l-1}, & \text{if } l \text{ is odd,} \end{cases}$$

$$d_2^{r,2l+k}(\tau^r \otimes c^k d^l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ \tau^{r+1} \otimes c^{k+1} d^{l-1}, & \text{if } l \text{ is odd,} \end{cases}$$

for all $r \geq 0$, $k \in \{1, \cdots, m\}$ and $l \in \{1, \cdots, n\}$. Then, the E_3-page has the following pattern, for $q \leq m+2n$:

$$E_3^{p,q} \cong \begin{cases} \mathbb{Z}_2, & \text{if } q \equiv 0(\text{mod } 4) \text{ and } p \equiv 0(\text{mod } 2), \\ \mathbb{Z}_2^{\phi(q)}, & \text{if } p = 0, \text{ where } \phi(q) > 1 \text{ depends of } q. \end{cases}$$

By Lemma 3.1.1, the sequence collapses on the E_3-page. But note that, for all $q \equiv 0(\text{mod } 4)$, the lines $E_3^{*,q}$ have infinite elements, which contradicts Theorem 1.2.5.

Suppose now that case (III) holds. Then, the general expressions of the differentials are given by:

$$d_2^{r,k}(\tau^r \otimes c^k) = \begin{cases} 0, & \text{if } k \text{ is even }, \\ \tau^{r+1} \otimes c^{k-1}, & \text{if } k \text{ is odd}, \end{cases}$$

$$d_2^{r,2l+k}(\tau^r \otimes c^k d^l) = \begin{cases} 0, & \text{if } k \text{ is even }, \\ \tau^{r+1} \otimes c^{k-1} d^l, & \text{if } k \text{ is odd}, \end{cases}$$

and $d_2^{r,2l}(\tau^r \otimes d^l) = 0$, for all $r \geq 0$, $k \in \{1, \cdots, m\}$ and for all $l \in \{1, \cdots, n\}$.

If m is an odd number, then the differentials

$$d_2^{p,m+2n} : E_2^{p,m+2n} \to E_2^{p+2,m+2n-1}$$

are isomorphisms, where $E_2^{p,m+2n} = \langle \tau^p \otimes c^m d^n \rangle$ and $E_2^{p,m+2n-1} = \langle \tau^p \otimes c^{m-1} d^n \rangle$. Therefore, we have the following pattern for the E_3-page, $q < m+2n$:

$$E_3^{p,q} = \begin{cases} H^0(B_G) \otimes H^q(P(m,n)), & \text{for } q \text{ even and } p = 0, \\ \{0\}, & \text{for } q \text{ odd or any } p \geq 1. \end{cases}$$

As the sequence collapses on the E_3-page, where only the column $E_3^{0,q}$ is nontrivial, then

$$H^q(X/G) \cong H^0(B_G) \otimes_{\mathbb{Z}_2} H^q(P(m,n)) \cong H^q(P(m,n)),$$

for every q even and $q < m+2n$.

The elements $1 \otimes c^2$ and $1 \otimes d$ are permanent co-cycles and, therefore, determine nontrivial elements $\overline{x}, \overline{y} \in E_\infty^{0,2}$. By (1.28), there are $x, y \in H^2(X_G)$, such that $i^*(x) = c^2$ and $i^*(y) = d$. In particular, the homomorphisms

$$i^* : H^q(X/G) \cong H^q(X_G) \longrightarrow H^q(X)$$

are isomorphisms for all q even and it follows by the structure of the $E_\infty-$page that $x^{(m+1)/2} = 0$ and $y^{n+1} = 0$, from where we get

$$H^*(P(m,n)/G) = \frac{\mathbb{Z}_2[x,y]}{\langle x^{(m+1)/2}, y^{n+1} \rangle},$$

which corresponds to item (i).

If m is even, then $x^{m/2} \neq 0$, since $1 \otimes c^m d^n \neq 0$; therefore,

$$H^*(P(m,n)) = \mathbb{Z}_2[x,y]/\langle x^{(m+2)/2}, y^{n+1}\rangle,$$

which corresponds to item (ii). $\square$

Example 3.1.1. Let G be the group S^1 and m, n be odd integers, where $m = 2k-1$, for some $k \geq 1$. Then, we can consider $S^m \subseteq \mathbb{C}^k$ and the action of G on $S^m \times \mathbb{C}P^n$. We define the free action

$$z * (w, v) \mapsto \Big((zw_1, \cdots, zw_k), [zv_0 : \cdots : zv_n]\Big), \tag{3.7}$$

where $w = (w_1, \cdots, w_k) \in S^m \subseteq \mathbb{C}^k$ and $v = [v_0 : \cdots : v_n] \in \mathbb{C}P^n$.

Let us consider an arbitrary element

$$[w, v] \in P(m,n) = (S^m \times \mathbb{C}P^n)/T.$$

We claim that the isotropy subgroup $G_{[w,v]}$ is trivial. In fact, suppose that $z \in G_{[w,v]}$. Then, we see that $z * [w, v] = [w, v]$ is equivalent to the following equality on the elements of $P(m, n) = (S^k \times \mathbb{C}P^n)/T$:

$$\Big[(zw_1, \cdots, zw_k), [zv_0 : \cdots : zv_n]\Big] = [w, v]. \tag{3.8}$$

Therefore, z must be equal to $1 \in G$, that is, $G_{[w,v]} = \{1\}$, so the induced action on Dold manifold $P(m, n)$ is free. By Theorem 2.5.1, it follows that the cohomology of the quotient space of $P(m, n)$, by this action, is isomorphic to the graded polynomial algebra according to case (i) of Theorem 3.1.2.

If n is odd and m is an even number of the form $2k$, we can consider

$$S^m = \{(w,t) \in \mathbb{C}^k \times \mathbb{R}\,;\ \|w\| + |t| = 1\},$$

and the analogous free action of G on $S^m \times \mathbb{C}P^n$ is defined by

$$z * ((w,t), v) \mapsto \Big((zw_1, \cdots, zw_k, t), [zv_0 : \cdots : zv_n]\Big). \tag{3.9}$$

Since this action is free, it induces a free action of G on $P(m, n)$, as in the previous case. Therefore, the cohomology of the orbit space $P(m, n)/G$ is isomorphic to the one given in case (ii) of Theorem 3.1.2

3.2 Existence of equivariant maps

By following the same strategies as the ones in the proof of Theorem 2.7.1, we obtain Theorem 3.2.1 below.

Theorem 3.2.1. *Let us consider $G = S^1$ and $X \cong_2 P(m,n)$, with $n > 0$ odd, regarded as a free $G-$space. Let S^j, for j odd, be a $j-$sphere equipped with the free S^1-action induced by the complex multiplication. There is no equivariant map of S^j to X, for any $j > 0$.*

Proof. Suppose that there is such equivariant map $f : S^j \to X$ and let

$$\mathbb{C}P^{\frac{j-1}{2}} \cong \frac{S^j}{G} \xrightarrow{F} \frac{X}{G}$$

be the map induced by f. As in the proof of Theorem 2.7.1, we just analyze the homomorphism

$$\pi^* : H^*(B_G) \to H^*(X_G) \cong H^*(X/G)$$

to obtain a Stiefel-Whitney class. From the proof of Theorem 3.1.2 it follows that $\pi^*(\tau) = 0$. But, on the other hand, we have $0 = F^*(0) = \tau$, which is a contradiction. □

3.3 Non existence of free actions of S^1 on Wall manifolds

Khare showed[2] in [21] that a Wall manifold $Q(m,n)$ bounds if, and only if, either n is odd or $m = 0$ and n is even; therefore, by an argument similar to the one in Remark 3.1.1, it follows that there cannot exist any free actions of S^1 on a Wall manifold of the type $Q(m,n)$, for n even. In this Section, we extend this result showing that the same occurs for the other cases.

Theorem 3.3.1. *There is no free action of the group $G = S^1$ on any finitistic space $X \cong_2 Q(m,n)$, for all $m,n > 0$.*

Proof. Suppose that there is a free action of $G = S^1$ on $X \cong_2 Q(m,n)$, where n is odd, and let us consider

$$X \hookrightarrow X_G \xrightarrow{\pi} B_G \tag{3.10}$$

the associated Borel fibration.

Since $\pi_1(B_G) = \pi_1(\mathbb{C}P^\infty) = 1$, there is a spectral sequence $\{E_r^{*,*}, d_r\}$, where

$$E_2^{p,q} \cong H^p(B_G) \otimes_{\mathbb{Z}_2} H^q(X), \tag{3.11}$$

converging, as an algebra, to $H^*(X_G) \cong H^*(X/G)$.

[2]Proposition 2.2.1.

By Theorem 1.3.1, this sequence does not collapse on the E_2-page; therefore, it must have some nontrivial differential $d_2^{*,*}$. We will analyze the cases where these differentials can be possibly nontrivial and show that all of them produce contradictions. To this purpose, we will divide the proof in two cases:

Case m odd: In this case we have $d_2^{0,m+1}(1 \otimes c^{m+1}) = 0$. In fact, since $m+1 = 2r$ for some $r > 0$, then

$$\begin{aligned} d_2^{0,2r}(1 \otimes c^{2r}) &= d_2^{0,2r}((1 \otimes c^r)(1 \otimes c^r)) \\ &= 2(1 \otimes c^r) d_2^{0,r}(1 \otimes c^r) \\ &= 0. \end{aligned}$$

However, by the relation $c^{m+1} = c^m x$, it follows that

$$\begin{aligned} d_2^{0,m+1}(1 \otimes c^{m+1}) &= d_2^{0,m+1}((1 \otimes c^m x)) \\ &= (1 \otimes c^m) d_2^{0,1}(1 \otimes x) + (1 \otimes x) d_2^{0,m}(1 \otimes c^m). \end{aligned}$$

Thus, there cannot occur simultaneously $d_2^{0,1}(1 \otimes x) \neq 0$ and $d_2^{0,1}(1 \otimes c) \neq 0$, because in this case we would have $0 = \tau \otimes c^{m-1}(c+x)$, which is a contradiction .

Similarly, there cannot occur $d_2^{0,1}(1 \otimes x) = 0$ and $d_2^{0,1}(1 \otimes c) \neq 0$, simultaneosly. If this happened, we would have $0 = \tau \otimes x c^{m-1}$. Also, if $d_2^{0,1}(1 \otimes x) \neq 0$ and $d_2^{0,1}(1 \otimes c) = 0$, we would have

$$0 = \tau \otimes c^m x = \tau \otimes c^{m+1},$$

which is a contradiction. Therefore, it follows that $d_2^{0,1}(1 \otimes x) = d_2^{0,1}(1 \otimes c) = 0$, and this implies that there are only the following possibilities:

(1) $d_2^{0,1}(1 \otimes c) = d_2^{0,1}(1 \otimes x) = 0$ and $d_2^{0,2}(1 \otimes d) = \tau \otimes c$,

(2) $d_2^{0,1}(1 \otimes c) = d_2^{0,1}(1 \otimes x) = 0$ and $d_2^{0,2}(1 \otimes d) = \tau \otimes x$.

We claim that (1) and (2) cannot occur.

Suppose (1) occurs; then, for any $j \geq 0$, $k \in \{1 \cdots, m\}$ and $l \in \{1 \cdots, n\}$, it follows that

$$d_2^{j,2l}(\tau^j \otimes d^l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ \tau^{j+1} \otimes c d^{l-1}, & \text{if } l \text{ is odd,} \end{cases}$$

$$d_2^{j,k+2l}(\tau^j \otimes c^k d^l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ \tau^{j+1} \otimes c^{k+1} d^{l-1}, & \text{if } l \text{ is odd,} \end{cases}$$

$$d_2^{j,2l+1}(\tau^j \otimes x d^l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ \tau^{j+1} \otimes c x d^{l-1}, & \text{if } l \text{ is odd,} \end{cases}$$

$$d_2^{j,2l+k+1}(\tau^j \otimes x c^k d^l) = \begin{cases} 0, & \text{if } l \text{ is even ,} \\ \tau^{j+1} \otimes x c^{k+1} d^{l-1}, & \text{if } l \text{ is odd,} \end{cases}$$

therefore, we have $E_3^{p,q} \cong \{0\}$, for all p odd or $q \equiv 2 (\text{mod } 4)$ and $q > 0$. By Lemma 3.1.1, the sequence collapses on the E_3-page. However, $E_3^{2r,q} \neq \{0\}$ for all $q \equiv s(\text{mod } 4)$, $s = 0, 2, 3$ and $r \geq 0$, which contradicts Theorem 1.2.5.

Suppose case (2) occurs; then, for all $j \geq 0$, $k \in \{1 \cdots, m\}$ and $l \in \{1 \cdots, n\}$, we have

$$d_2^{j,2l}(\tau^j \otimes d^l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ \tau^{j+1} \otimes x d^{l-1}, & \text{if } l \text{ is odd,} \end{cases}$$

$$d_2^{j,2l+k}(\tau^j \otimes c^k d^l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ \tau^{j+1} \otimes x c^k d^{l-1}, & \text{if } l \text{ is odd,} \end{cases}$$

while $d_2^{j,2l+1}(\tau^j \otimes x d^l) = 0$, since $x^2 = 0$. Therefore, similarly to case (1), we conclude that this case is not possible either.

<u>**Case m even**</u>: In this case we have $d_2^{0,m}(1 \otimes c^m) = 0$, so

$$d_2^{0,m+1}(1 \otimes c^m x) = (1 \otimes c^m) d_2^{0,1}(1 \otimes x),$$

while, by the relation $c^{m+1} = c^m x$,

$$d_2^{0,m+1}(1 \otimes c^m x) = d_2^{0,m+1}(1 \otimes c^{m+1}) = (1 \otimes c^m) d_2^{0,1}(1 \otimes c).$$

Therefore, we should necessarily have $d_2^{0,1}(1 \otimes c) = d_2^{0,1}(1 \otimes x)$.

If $d_2^{0,1}(1 \otimes c) = d_2^{0,1}(1 \otimes x) = \tau \otimes 1$, then $d_2^{0,2}(1 \otimes d) = 0$; otherwise, we would have

$$\text{im} d_2^{0,2} = \langle d_2^{0,2}(1 \otimes d) \rangle \nsubseteq \ker d_2^{2,1} \langle \tau \otimes (c + x) \rangle,$$

which is a contradiction.

Let us suppose now that $d_2^{0,1}(1 \otimes c)$ and $d_2^{0,1}(1 \otimes x)$ are nonzero and that $d_2^{0,2}(1 \otimes d) = \tau \otimes c$. Then we have, for example,

$$\text{im } d_2^{0,4} = \langle \tau \otimes cd \rangle \oplus \langle \tau \otimes c^2 x \rangle \nsubseteq \ker d_2^{2,3} = \langle \tau \otimes x c^2 \rangle \oplus \langle \tau \otimes c^3 \rangle,$$

which is a contradiction.

Because of this, for m even, we only need to consider the following cases:

(i) $d_2^{0,1}(1 \otimes c) = d_2^{0,1}(1 \otimes x) = \tau \otimes 1$ and $d_2^{0,2}(1 \otimes d) = 0$;

(ii) $d_2^{0,1}(1 \otimes c) = d_2^{0,1}(1 \otimes x) = 0$ and $d_2^{0,2}(1 \otimes d) = \tau \otimes x$.

We will show that both cases (i) and (ii) cannot occur.

Suppose (i) is true; then, for all $j \geq 0$, $k \in \{1 \cdots, m\}$ and all $l \in \{1 \cdots, n\}$, we have

$$d_2^{j,k}(\tau^j \otimes c^k) = \begin{cases} 0, & \text{if } k \text{ is even,} \\ \tau^{j+1} \otimes c^{k-1}, & \text{if } k \text{ is odd,} \end{cases}$$

$$d_2^{j,k+1}(\tau^j \otimes c^k x) = \begin{cases} \tau^{j+1} \otimes c^k, & \text{if } k \text{ is even,} \\ \tau^{j+1} \otimes (c+x)c^{k-1}, & \text{if } k \text{ is odd,} \end{cases}$$

$$d_2^{j,2l+k}(\tau^j \otimes c^k d^l) = \begin{cases} 0, & \text{if } k \text{ is even,} \\ \tau^{j+1} \otimes c^{k-1} d^l, & \text{if } k \text{ is odd,} \end{cases}$$

and $d_2^{j,2l+1}(\tau^j \otimes x d^l) = \tau^{j+1} \otimes d^l$. Therefore, $E_3^{p,q} \cong \{0\}$, for all $q \not\equiv 1 (\text{mod } 4)$, and $p > 0$. However, for $q \equiv 1 (\text{mod } 4)$, $q \geq 5$, we have

$$\begin{aligned} E_3^{2j,q} &\cong \frac{\langle \tau^j \otimes (c+x) d^{(q-1)/2} \rangle \oplus \langle \tau^j \otimes (c+x) c^{q-1} \rangle}{\langle \tau^j \otimes (c+x) c^{q-1} \rangle} \\ &\cong \langle \tau^j \otimes (c+x) d^{(q-1)/2} \rangle \\ &\neq \{0\}, \end{aligned}$$

which contradicts Theorem 1.2.5.

For (ii), note that it would result in a similar pattern to the one in case (2); so, by the same arguments, it cannot hold either. □

CHAPTER 4

FREE ACTIONS OF S^3 ON SOME FINITISTIC SPACES

In this Chapter we will investigate the existence of free actions of the group S^3 on some spaces and also describe the cohomology ring of the orbit spaces in the cases which admit such structures.

In order to simplify the notations, all cohomologies considered will be the $\check{C}$ech cohomology with coefficients in $\mathbb{Z}_2$, according to [1], Section 3.6. That is, the symbol $H^*(X)$ will indicate the graded ring $\check{H}^*(X;\mathbb{Z}_2)$. The cup product, usually denoted by $a \smile b$, will be simply indicated by $a \cdot b$ or by ab.

4.1 Non existence of free actions of S^3 on Dold, Wall and Milnor manifolds

As we know[1], the Dold manifolds $P(m,n)$ cannot admit the existence of free involutions if n is even, so they cannot admit free actions of S^1 and S^3 either. Let us assume now that there is a free action of the group $G = S^3$ on $P(m,n)$, for n odd. Let $\{E_r^{*,*}, d_r\}$ be the Leray-Serre spectral sequence associated to the Borel fibration

$$P(m,n) \hookrightarrow P(m,n)_G \longrightarrow B_G. \tag{4.1}$$

Since $\pi_1(B_G) = 1$, it follows that

$$E_2^{p,q} = H^p(B_G) \otimes_{\mathbb{Z}_2} H^q(P(m,n)), \quad \forall p,q \geq 0. \tag{4.2}$$

By Theorem 1.3.1 (or directly by Theorem 1.2.6) we can conclude that the sequence cannot

[1]See Theorem 3.1.1.

collapse on E_2-term; that is, there must exist some nontrivial differential d_r, for some $r \geq 2$. By the cohomological structure of the classifying space B_G, this is only possible when $r \equiv 0(\text{mod } 4)$. This way, by considering $r = 4$ we must have $E_2^{p,q} \cong E_3^{p,q} \cong E_4^{p,q}$.

Therefore, there must be some nontrivial differential d_4 and, since c and d are the generators of $H^*(P(m,n))$ and $H^3(P(m,n)) = \langle c^3\rangle \oplus \langle cd\rangle$, there are only the possibilities $d_4^{0,3}(1 \otimes cd) \neq 0$ and/or $d_4^{0,3}(1 \otimes c^3) \neq 0$. However, if $d_4^{0,3}(1 \otimes cd) = \tau \otimes 1$, where τ denotes the generator of $H^*(B_G)$, then

$$\tau \otimes 1 = d_4^{0,3}(1 \otimes cd) = d_4^{0,3}((1 \otimes c) \cdot (1 \otimes d)) = 0, \tag{4.3}$$

since $\operatorname{im} d_4^{0,i} \subseteq E_4^{4,i-3} = \{0\}$ for $i = 0, 1, 2$, which is an absurd. Similarly, if $d_4^{0,3}(1 \otimes c^3) \neq 0$, we must have $\tau \otimes 1 = 0$. Therefore, we can conclude that no space $X \cong_2 P(m,n)$ can admit the existence of a free action of the group S^3.

Since $\deg c = 1$ and $\deg d = 2$ it follows that the same contraductions can be found supposing that $r > 4$, with $r \equiv 0(\text{mod } 4)$.

About the Wall manifolds, it follows from Theorem 3.3.1 that the manifolds $Q(m,n)$ cannot admit the existence of any free S^3-action.

These considerations demonstrate Theorem 4.1.1 below.

Theorem 4.1.1. *The group S^3 does not act freely on any finitistic space $X \cong_2 P(m,n)$ or $X \cong_2 Q(m,n)$, for any values of m and n.*

Definition 4.1.1. Let m, n be integers such that $0 \leq n \leq m$. We call by a (real) *Milnor manifold* of dimension $n+m-1$ the differentiable closed submanifold of codimension 1 in $\mathbb{R}P^m \times \mathbb{R}P^n$, described in homogeneous coordinates as

$$\mathbb{R}H_{m,n} = \Big\{([x_0, \cdots, x_m], [y_0, \cdots, y_n]) \in \mathbb{R}P^m \times \mathbb{R}P^n \,|\, x_0y_0 + \cdots + x_ny_n = 0\Big\},$$

also denoted by $H(m,n)$. Equivalently, $\mathbb{R}H_{m,n}$ is the total space of the bundle

$$\mathbb{R}P^{m-1} \hookrightarrow \mathbb{R}H_{m,n} \xrightarrow{\pi} \mathbb{R}P^n.$$

Theorem 4.1.2 ([31], p. 161)**.** *The mod 2 cohomology ring of $H(m,n)$ is isomorphic to the following graded polynomial ring*

$$\mathbb{Z}_2[a,b]/\langle a^{n+1}, b^m + ab^{m-1} + \cdots + a^n b^{m-n}\rangle,$$

where $\deg a = \deg b = 1$.

The Milnor manifold was introduced in 1965 by J. Milnor [27] and provides representatives for the generators in odd dimensions for the unoriented cobordism ring $\mathfrak{N}_*$, as the Dold and Wall manifolds do. This is one of the reasons why these spaces have been extensively studied in the literature, such as in [8, 31, 34].

In [8], the authors showed that S^1 acts freely on a Milnor manifold $\mathbb{R}H_{m,n} = H(m,n)$ if, and only if, m and n are odd numbers, and in this case they also obtained the cohomology ring of the orbit spaces relative to these actions. With respect to the group $G = S^3$, we show in Theorem 4.1.3 below that the Milnor manifolds do not admit free actions of G, for any values of m and n.

Theorem 4.1.3. *The group S^3 does not act freely in any finitistic space $X \cong_2 H(m,n)$, for any values of m and n.*

Proof. Let us suppose that $G = S^3$ acts freely on X and let $\{E_r^{*,*}, d_r\}$ be the spectral sequence associated to the Borel fibration $X_G \to B_G$, with fiber X, converging to the cohomology of the Borel space X_G. Since $\pi_1(B_G) = 1$, then

$$E_2^{p,q} \cong H^p(B_G) \otimes_{\mathbb{Z}_2} H^q(X),$$

and, by the cohomological structure of $B_G \cong \mathbb{H}P^\infty$, it follows that

$$E_2^{p,q} \cong \begin{cases} \mathbb{Z}_2, & \text{if } p = 0, 4, 8, 12, \cdots \\ \{0\}, & \text{otherwise.} \end{cases}$$

This way, note that $\mathrm{Im} d_2^{p,q} = \mathrm{Im} d_3^{p,q} = \{0\}$, for all $p, q \geq 0$, that is, d_r is trivial for $r = 2, 3$. Consequently, we have $E_2^{p,q} \cong E_3^{p,q} \cong E_4^{p,q}$, for all $p, q \geq 0$.

Since the action is free, X_G has the same homotopy type of the orbit space X/G; therefore, it follows by Theorem 1.3.1 (or directly by Theorem 1.2.6) that this spectral sequence cannot collapse in the E_r−term, for $r = 2, 3, 4$. This means that there must be at least one nontrivial differential d_r, for some $r \geq 4$, such that $d_s \equiv 0$ for all $s < r$. If we suppose that

$$4 = \min\{r \geq 4\,;\, d_r \text{ is nontrivial}\},$$

then there must be an element $\alpha \in E_4^{0,3} \cong H^0(B_G) \otimes H^3(X)$ such that $d_4^{0,3}(\alpha) = \tau \otimes 1 \neq 0$.

Since

$$E_4^{0,3} \cong \langle 1 \otimes a^3 \rangle \oplus \langle 1 \otimes b^3 \rangle \oplus \langle 1 \otimes a^2 b \rangle \oplus \langle 1 \otimes ab^2 \rangle \cong \mathbb{Z}_2^4,$$

if $\alpha = 1 \otimes a^3$, then we have

$$\tau \otimes 1 = d_4^{0,3}(\alpha) = (1 \otimes a) \cdot d_4^{0,2}(1 \otimes a^2) + (1 \otimes a^2) \cdot d_4^{0,1}(1 \otimes a) = 0,$$

which is a contradiction. Since the same happens for any other element in $E_4^{0,3}$ and the argument is based on the fact that $\mathrm{im} d_4^{0,1} \subseteq E_4^{4,-2} = \{0\}$, this will happen to any differential d_r, for $r \geq 4$. Therefore, we can conclude that G cannot act freely on X. $\square$

Remark 4.1.1. Note that $H(m,0) \cong \mathbb{R}P^{m-1}$; therefore, it follows by Theorem 4.1.3 that S^3 cannot act freely on the real projective spaces. We note that this result is already known and was directly obtained by Singh et al. in [19].

Remark 4.1.2. Similarly to the definition of the real Milnor manifolds, we can define a complex Milnor manifold of dimension $2(n+m-1)$ by

$$\mathbb{C}H_{m,n} = \Big\{([z_0,\cdots,z_m],[w_0,\cdots,w_n]) \in \mathbb{C}P^m \times \mathbb{C}P^n \,|\, z_0\overline{w}_0 + \cdots + z_n\overline{w}_n = 0\Big\}.$$

It has already been demonstrated by Dey and Singh [9] any finitistic space $X \cong_2 \mathbb{C}H_{m,n}$ does not admit any free action of S^1; therefore, such spaces do not admit any free actions of S^3 either.

4.2 Free actions of S^3 on product of spheres

Let G be the Lie group S^3, seen as a subgroup of $\mathbb{R}^4 \cong \mathbb{H}$, where $\mathbb{H}$ denotes the quaternion algebra

$$\mathbb{H} = \left\{ \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} ;\, \alpha,\beta \in \mathbb{C} \right\}. \tag{4.4}$$

We can identify G with the unitary quaternions group $SU(2) = \{w \in \mathbb{H};\, \|w\| = 1\}$. Therefore, G acts freely, by multiplication, on every $(4n-1)-$sphere, seen as a subset of $\mathbb{R}^{4n} \cong \mathbb{H}^n$. From this, we get the principal $G-$bundle

$$G \hookrightarrow S^{4n-1} \longrightarrow \frac{S^{4n-1}}{G} \cong \mathbb{H}P^{n-1}, \tag{4.5}$$

over the quaternionic projective space.

In [19], the authors investigated the existence of free actions of S^3 on spheres and showed that the only ones that admit such structures must have a dimension of the form $4k-1$, according to Theorem 4.2.1 below.

Theorem 4.2.1 ([19], p. 06, Theorem 3.2)**.** *If $G = S^3$ acts freely on a finitistic space $X \cong_2 S^m$, then $m = 4k-1$, $k \geq 1$ and, in this case, we have*

$$H^*(X/G) \cong \mathbb{Z}_2[x]/\langle x^k \rangle,$$

where $\deg x = 4$.

As a consequence of Theorem 4.2.1, there is only one cohomology algebra $H^*(Y)$ whenever Y is an orbit space of $X \cong_2 S^m$ by any free action of S^3. In addiction, this result implies that any free action of $G = S^3$ on a sphere S^{4k-1} produces a principal $G-$bundle as in (4.5).

By considering X as a finitistic space with mod 2 cohomology of a product of spheres, we get Theorems 4.2.2 and 4.2.3 below.

Theorem 4.2.2. *Let G be the group S^3, acting freely on a finitistic space $X \cong_2 S^m \times S^n$, where $m \leq n$. Then m or n must be an odd number of the form $4k-1$, for some $k \geq 1$, and the cohomology ring $H^*(X/G)$ must be isomorphic to the graded polynomial algebra*

$$\mathbb{Z}_2[x,y]/\langle x^k, y^2\rangle,$$

where $\deg x = 4$ *and* $\deg y = m$ *or* $\deg y = n$.

Proof. Let $\{E_r^{*,*}, d_r\}$ be the spectral sequence associated to the Borel fibration $X_G \to B_G$, so that

$$E_2^{p,q} = H^p(B_G) \otimes_{\mathbb{Z}_2} H^q(X),$$

converging, as an algebra, to $H^*(X_G)$.

By Theorem 1.3.1 (or directly by Theorem 1.2.6), it follows that this sequence does not collapse on the E_2-page; therefore, there must exist some nontrivial differential d_{r_i}, for some $r_i \geq 2$. If $r = \min\{r_i\}$, then

$$E_r^{p,q} \cong E_{r-1}^{p,q} \cong \cdots \cong E_2^{p,q},$$

and note that a differential d_r is only nontrivial when $r = 4k$, for some $k \geq 1$ such that $m = 4k-1$ or $n = 4k-1$; otherwise, there would always be a nonzero line that would survive until the limit page of the sequence, according to the graphic below. This would contradict Theorem 1.3.1.

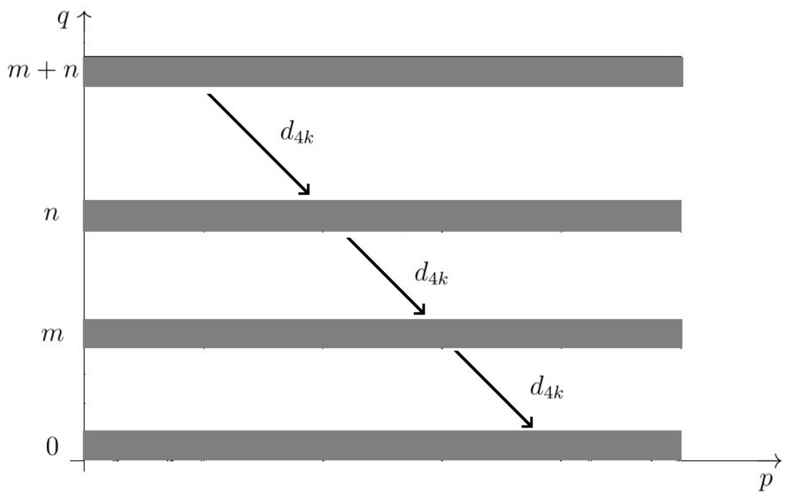

Figure 4.1: Pattern for the action of the differentials on $E_{4k}-$page

We will suppose that $m = 4k-1$. Let u be the generator of $H^m(S^m)$ and v be the generator of $H^n(S^n)$. By a Kunneth theorem ([14], p. 219, Theorem 3.16) there is an isomorphism of graded algebras

$$H^*(X) \cong H^*(S^m \times S^n) \xrightarrow{\cong} H^*(S^m) \otimes H^*(S^m),$$

so that we can identify the element $u \otimes v \in H^m(S^m) \otimes H^n(S^n)$ with $uv \in H^{m+n}(X)$.

Let us initially assume that $m < n$. Then, by the equality

$$d_r(1 \otimes uv) = (1 \otimes v)d_r(1 \otimes u) + (1 \otimes u)d_r(1 \otimes v),$$

we only need to analyze the possible values for $d_r(1 \otimes u)$ and $d_r(1 \otimes v)$.

If $d_r(1\otimes u) = \tau^k \otimes 1$ and $d_r(1\otimes v) = 0$, then we have $d_r(1\otimes uv) = \tau^k \otimes v$ and, consequently, the $E_{4k+1}-$page of the spectral sequence has the following pattern:

$$E_{4k+1}^{p,q} \cong \begin{cases} \mathbb{Z}_2, & \text{if } p = 0, 4, \cdots, 4(k-1) \text{ and } q = 0, n, \\ \{0\}, & \text{otherwise.} \end{cases}$$

With that we can see that the sequence collapses at the $E_{4k+1}-$term, with $E_\infty^{p,q} \cong E_{4k+1}^{p,q}$, for all $p, q \geq 0$; therefore, we have the following additive structure of $H^*(X/G)$:

$$H^j(X/G) \cong \text{Tot}^j(E_\infty).$$

As $\tau \otimes 1$ and $1 \otimes v$ are the only permanent co-cycles, they determine nonzero elements x and y in $E_\infty^{4,0}$ and $E_\infty^{0,n}$, respectively. By (1.28) we have $\pi^*(\tau) = x$; then, $0 = \pi^*(\tau^j) = x^j$, for all $j \geq k$. It follows from the structure of the $E_\infty-$term that $y^2 = 0$; therefore, $H^*(X/G)$ is isomorphic to the graded polynomial algebra

$$\mathbb{Z}_2[x, y]/\langle x^k, y^2\rangle = \mathbb{Z}_2[x, y]/\langle x^{\frac{m+1}{4}}, y^2\rangle,$$

where $\deg x = 4$ and $\deg y = n$.

If $d_r(1 \otimes u) = \tau^k \otimes 1$ and $d_r(1 \otimes v) = \tau^k \otimes u$, then all the differentials $d_{4k}^{4l,m}, d_{4k}^{4l,n}$ and $d_{4k}^{4l,m+n}$ would be isomorphisms, which would imply that

$$\text{im}\, d_{4k}^{4l,m+n} \cong \mathbb{Z}_2 \not\subseteq \{0\} = \ker d_{4k}^{4(l+1),n},$$

a contradiction.

If $d_r(1 \otimes u) = 0$ and $d_r(1 \otimes v) = \tau^k \otimes u$, then $d_r(1 \otimes uv) = 0$, which implies that all the homomorphisms $d_{4k}^{4l,m+n}$ are trivial. Then, if the sequence collapse on $E_{4k+1}-$page, then we will have

$$E_\infty^{4l,m+n} \cong E_{4k+1}^{4l,m+n} = \ker d_{4k}^{4l,m+n} = \langle \tau^l \otimes uv\rangle,$$

for all $l \geq 1$. Since this cannot occur, due to Theorem 1.2.6, we have a contradiction.

However, if we suppose that the sequence doesn't collapse on $E_{4k+1}-$page, then the only possibility for the existence of a nontrivial differential d_r is for $r = m + n + 1$, so that

$$d_r : E_r^{p,m+n} \to E_r^{p+r,0} \cong H^{p+r}(B_G) \otimes H^0(X) \neq \{0\},$$

for $p + r \equiv 0(\text{mod } 4)$.

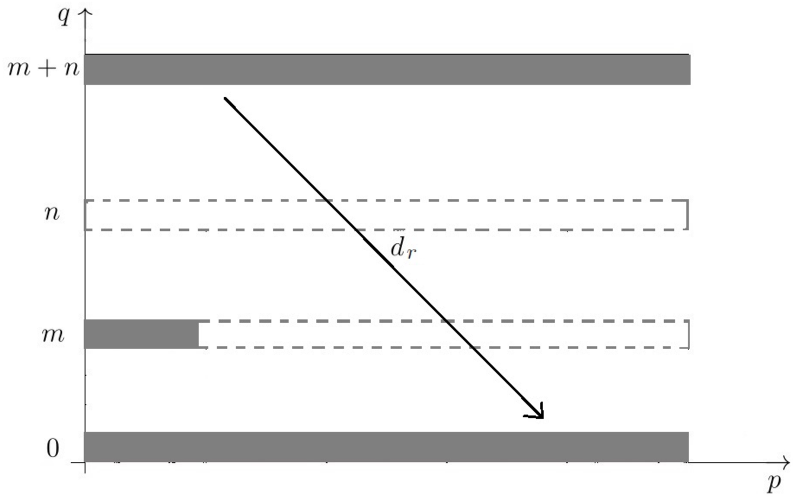

Figure 4.2: Pattern for the action of the differentials on $E_{m+n+1}-$page

But this way, since $r > m$ and $r > n$, then $d_r(1 \otimes u) = 0$ and $d_r(1 \otimes v) = 0$, therefore $d_r(1 \otimes uv) = 0$, and again we will have

$$E_\infty^{4l,m+n} \cong E_r^{4l,m+n} = \ker d_r^{4l,m+n} = \langle \tau^l \otimes uv \rangle.$$

Moreover, this shows that

$$E_{4k+1}^{*,*} \cong \cdots \cong E_{m+n+1}^{*,*} \cong E_\infty^{*,*},$$

which is a contradiction with Theorem 1.2.6.

If $m < n = 4k - 1$ then we must have the only possibility $d_{4k}(1 \otimes u) = 0$ and $d_{4k}(1 \otimes v) = \tau^k \otimes 1$, so that the differentials $d_{4k}^{p,m+n}$ are isomorphisms, for all $p \geq 0$; thus we can conclude in a very similar way that

$$H^*(X/G) \cong \mathbb{Z}_2[x, y]/\langle x^k, y^2 \rangle,$$

where $\deg x = 4$ and $\deg y = m$.

If $m = 4k_1 - 1$ and $n = 4k_2 - 1$, for certain $k_1 < k_2$, then it follows that r must be equal to $4k_1$ or $4k_2$, for which it will produce the same previous structures for $H^*(X/G) \cong \mathbb{Z}_2[x, y]/\langle x^{k_i}, y^2 \rangle$, where $\deg x = 4$ and $\deg y = m$ or $\deg y = n$.

If $m = n$, then both must be equal to $4k - 1$, for some $k \geq 1$. In fact, this is the only way there can exist a nontrivial differential d_r, where $r = 4k$, and such that

$$E_2^{p,q} \cong \cdots \cong E_{4k}^{p,q},$$

for all $p, q \geq 0$.

In this case, similarly to the previous one, we have the following possibilities for the action of the differentials d_r on the generators $1 \otimes u$ and $1 \otimes v$:

(a) $d_r(1 \otimes u) = \tau^k \otimes 1$ and $d_r(1 \otimes v) = 0,$

(b) $d_r(1 \otimes u) = 0$ and $d_r(1 \otimes v) = \tau^k \otimes 1,$

(c) $d_r(1 \otimes u) = \tau^k \otimes 1$ and $d_r(1 \otimes v) = \tau^k \otimes 1.$

We will see that all these cases produce the same limit for the spectral sequence.

Suppose that case (a) occurs. Then,

$$d_{4k}^{4l,2m}(\tau^l \otimes uv) = \tau^l \otimes v$$

and the $E_{4k+1}-$term has the pattern:

$$E_{4k+1}^{p,q} \cong \begin{cases} \mathbb{Z}_2, & \text{if } p = 0, 4, 8, \cdots, 4(k-1) \text{ and } q = 0, m, \\ \{0\}, & \text{otherwise,} \end{cases}$$

for all $q \leq m.$

Therefore, it follows that the sequence collapses on $E_{4k+1}-$term, that is, $E_{4k+1}^{p,q} \cong E_\infty^{p,q}$, for all $p, q \geq 0$, and

$$H^*(X/G; \mathbb{Z}_2) \cong \mathrm{Tot}(E_\infty).$$

The elements $\tau \otimes 1$ and $1 \otimes v$ are the only permanent co-cycles, so they determine nonzero elements x and y in $E_\infty^{4,0}$ and $E_\infty^{0,m}$, respectively. By (1.28), we have $\pi^*(\tau) = x$, so $0 = \pi^*(\tau^j) = x^j$, for all $j \geq k$. It follows from the structure of the $E_{4k+1}-$page that $y^2 = 0$; therefore,

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/\langle x^k, y^2 \rangle,$$

where $\deg x = 4$ and $\deg y = m = n.$

By considering case (b), note that it determines the same limit page E_∞, whose only difference lies in the fact that y represents the permanent co-cycle $1 \otimes u$. Similarly, case (c) determines the same limit page, where y represents, in this case, the permanent co-cycle $1 \otimes (u + v)$. □

Example 4.2.1. Let X and Y be spaces equipped with an action of the group G. We can consider the diagonal action of G on the product space $X \times Y$, in which we have

$$g * (x, y) \mapsto (g * x, g * y),$$

for any $(x, y) \in X \times Y$ and for any $g \in G$. Moreover, if the action is free in some coordinate then the diagonal action will be free.

Let $X = S^7 \times S^{11}$. Since 7 and 11 are of the type $4k - 1$, for $k = 2, 3$, respectively, then we can consider two types of free actions of S^3 on X :

(i) If the action is free on the first coordinate and trivial on the second, then

$$X/S^3 \cong_2 \mathbb{H}P^1 \times S^{11},$$

so that $H^*(X/S^2) \cong \mathbb{Z}_2[x,y]/\langle x^2, y^2\rangle$, where $\deg x = 4$ and $\deg y = 11$

(ii) If the action is free only on the second coordinate, then

$$X/S^3 \cong_2 S^7 \times \mathbb{H}P^2,$$

and $H^*(X/S^2) \cong \mathbb{Z}_2[x,y]/\langle x^3, y^2\rangle$, where $\deg x = 4$ and $\deg y = 7$.

Therefore, cases (i) and (ii) are being covered by the Theorem 4.2.2.

A significant consequence of Theorem 4.2.2 is that, even when $m, n \equiv 3(\text{mod } 4)$,i. e., there are $k_1, k_2 \geq 1$ such that $m = 4k_1 + 3$ and $n = 4k_2 + 3$, and the action of S^3 on X is free at both coordinates, then we will have the same cohomology structure for the resulting orbit spaces, whose only difference is in the element that gives rise to the generator y (and consequently the cohomological degree), according to the proof of Theorem.

In other words, the conclusion of the Theorem 4.2.2 tells us that any free action of S^3 on $S^m \times S^n$ behaves like a diagonal action in which it acts trivially on some of the coordinates.

Corollary 4.2.1. *Let $X \cong_2 S^m \times S^n$, with $m \leq n$, equipped with an action of S^3. If $m, n \not\equiv 3(\textit{mod } 4)$ then the action is not free.*

Theorem 4.2.3. *The group S^3 cannot act freely on any finitistic space $X \cong_2 S^1 \times \cdots \times S^1 = (S^1)^n$.*

Proof. For $n < 3$ the result is trivial; then, we will suppose that $n \geq 3$ and that $G = S^3$ acts freely on X.

Let $x_1, \cdots, x_n \in H^1(X)$ be the generators. By Theorem 1.2.6, the spectral sequence $\{E_r^{*,*}, d_r\}$ associated to the Borel fibration $X_G \to B_G$ does not collapse on the E_2-term. Therefore, there must exist some nontrivial differential $d_r^{p,q}$, for a certain $r \geq 2$, such that

$$E_r^{p,q} \cong E_{r-1}^{p,q} \cong \cdots \cong E_2^{p,q} = H^p(B_G) \otimes_{\mathbb{Z}_2} H^q(X),$$

and it is clear that this is only possible when $r \geq 4k$, for $k \in \mathbb{N}$.

Let us suppose that $r = 4$ and let $y = x_{i_1}x_{i_2}x_{i_3} \in H^3(X)$ be an element for which $d_4^{0,3}(1 \otimes y) = \tau \otimes 1$. By dimensional reasons, $d_4^{0,1}(1 \otimes x_i) = 0$ for all $1 \leq i \leq n$; therefore, it follows that

$$\tau \otimes 1 = d_4^{0,3}(1 \otimes y) = (1 \otimes x_{i_1})(1 \otimes x_{i_2})d_4^{0,1}(1 \otimes x_{i_3}) = 0,$$

which is a contradiction.

Since this argument works for any $r \geq 4$ and for any $y \in H^j(X)$, $j \geq 3$, it follows that G cannot act freely on X. $\square$

4.3 Non existence of free actions of S^3 on projective spaces

In [19], Theorem 3.7, the authors showed that the group $G = S^3$ cannot act freely on any finitistic space $X \cong_2 \mathbb{R}P^n$, for all $n \in \mathbb{N}$. In this section, we will show that the same occurs with the complex and quaternionic projective spaces, according to Theorem 4.3.1.

Theorem 4.3.1. *The group $G = S^3$ cannot act freely on any finitistic space $X \cong_2 \mathbb{C}P^n$ or $X \cong_2 \mathbb{H}P^n$.*

Proof. Let us suppose that the group $G = S^3$ acts freely on $X \cong_2 \mathbb{C}P^n$. Then, the spectral sequence $\{E_r^{*,*}, d_r\}$ associated to the Borel fibration $X \hookrightarrow X_G \to B_G$, which has the E_2-term given by

$$E_2^{p,q} = H^p(B_G) \otimes_{\mathbb{Z}_2} H^q(X),$$

converges to $H^*(X_G) \cong H^*(X/G)$, as an algebra. By the cohomology structures of $B_G \cong \mathbb{H}P^\infty$ and $X \cong_2 \mathbb{C}P^n$, it follows that

$$E_2^{p,q} \cong \begin{cases} \mathbb{Z}_2, & \text{if } p = 4i \text{ and } q = 2j, \text{ for all } i, j \geq 0, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Therefore, a differential

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1-r}$$

with bidegree $(r, 1-r)$, is nontrivial only if $p = 4i$ and $q = 2j \leq 2n$, for some integers i and j. In this case, we have the following equality involving the bidegrees:

$$(4i + r, 2j + 1 - r) = (4k, 2l),$$

for certain integers $k, l > 0$, that is, these numbers must satisfy the linear system

$$\begin{cases} 4i + r = 4k, \\ 2j + 1 - r = 2l, \end{cases}$$

that clearly has no integer solution.

Therefore, we conclude that all differentials $d_r^{*,*}$ are trivial, for all $r \geq 2$. This implies that the sequence collapses on its $E_r \cong E_2-$term and contradicts Theorem 1.3.1.

Similarly, let us suppose that the group S^1 acts freely on $X \cong_2 \mathbb{H}P^n$, and let us consider $\{E_r^{*,*}, d_r\}$ the spectral sequence associated to the Borel fibration $X_{S^1} \to B_{S^1}$, whose E_2-term is given by

$$E_2^{p,q} \cong H^p(B_{S^1}) \otimes_{\mathbb{Z}_2} H^q(X).$$

Let t be the generator of $H^*(\mathbb{C}P^\infty) \cong H^*(B_{S^1})$ and τ be the generator of $H^*(\mathbb{H}P^n)$. Then,

$$E_2^{p,q} \cong \begin{cases} \mathbb{Z}_2, & \text{if } p = 2i \text{ and } q = 4j,\ i, j \geq 0, \\ \{0\}, & \text{otherwise.} \end{cases}$$

By Theorem 1.3.1, the spectral sequence does not collapse on its E_2-term; therefore, there must exist some nontrivial differential $d_r^{*,*}$. If $r \geq 2$ is the smallest integer for which this happens, so that

$$E_r^{p,q} \cong E_{r-1}^{p,q} \cong \cdots \cong E_2^{p,q},$$

for all $p, q \geq 0$, we see that this is only possible when the integers r, i, j and k (which are obtained from the equality between the bidegrees involved) satisfy the linear system

$$\begin{cases} r = 2i, \\ 4j + 1 - r = 4k. \end{cases}$$

But this system has no integer solution. Therefore, the group S^1 cannot acts freely on X. Since S^1 is a subgroup of $G = S^3$, then X does not admit any free action of G. $\square$

Remark 4.3.1. Note that on the proof of Theorem 4.3.1 we actually showed that the group S^1 cannot act freely on any finitistic space $X \cong_2 \mathbb{H}P^n$, which in turn implies that S^3 cannot act freely on X either.

4.4 Free actions of S^3 on spaces of type (a, b)

This Section is dedicated to investigating the existence of free action of S^3 on the so-called spaces of type (a, b), where a and b are integers associated with its cohomological structure, which we will define.

The relevance of these spaces lies in the fact that they appear as quite elementary constructions involving projective spaces and spheres, for example, as we shall see.

Definition 4.4.1. Let X be a finite CW complex. We say that X is a space of type (a, b), characterized by an integer $n > 0$, if

$$H^j(X; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } j = 0, n, 2n, 3n, \\ \{0\}, & \text{otherwise,} \end{cases} \tag{4.6}$$

whose generators $u_i \in H^{in}(X; \mathbb{Z})$ satisfy the relations

$$\begin{cases} au_2 = u_1^2, \\ bu_3 = u_1 u_2, \end{cases} \tag{4.7}$$

for certain integers a and b.

Remark 4.4.1. Let X be a space of type (a, b). By the universal coefficient Theorem ([5], p. 47, Theorem 2.33), the mod 2 cohomology of X is given by

$$H^{in}(X; \mathbb{Z}_2) \cong H^{in}(X; \mathbb{Z}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2, \tag{4.8}$$

for $n = 0, 1, 2, 3$, and the relations come to depend only on the parity of the numbers a and b. In this case, we will use the same symbols to denote the generators, i. e.,

$$\mathbb{Z}_2\langle u_i \rangle \cong H^{in}(X) = H^{in}(X; \mathbb{Z}_2). \tag{4.9}$$

Example 4.4.1. The spaces of type (a, b) were first studied by James [18] and Toda [40], and we can construct examples of these spaces by considering products or unions between certain known spaces, as spheres and projective spaces. For example:

(i) The projective spaces $\mathbb{R}P^3$, $\mathbb{C}P^3$ e $\mathbb{H}P^3$ are examples of spaces of type $(1, 1)$.

(ii) The cartesian product $S^n \times S^{2n}$ is a space of type $(0, 1)$.

(iii) Let us consider the construction $X = S^n \vee S^{2n} \vee S^{3n}$, where the symbol $A \vee B$ denotes the wedge sum[2] of the spaces A and B. This means that, from the disjoint union $A \amalg B$, we fix the points $a_0 \in X$ and $b_0 \in Y$ and define

$$A \vee B = \frac{A \amalg B}{a_0 \sim b_0}. \tag{4.10}$$

The space X, constructed in this way, is an example of a type $(0, 0)$ space.

(iv) The space $Y = S^6 \vee \mathbb{C}P^2$ is of type $(1, 0)$.

Theorem 4.4.1 ([40])**.** *If there exists a space of type (a, b), then there exist also spaces of types (ka, hb), for arbitrary integers k and h.*

Remark 4.4.2. A relevant consequence of Theorem 4.4.1 together with the example 4.4.1, is that it is possible to obtain a space of type (a, b) for any integers a and b, by choosing an appropriate integer $n > 0$.

In 2010, Pergher et al. [33] investigated the existence of free actions of the groups $\mathbb{Z}_2$ and S^1 on spaces of type (a, b), whose conclusions are summarized on Theorem 4.4.2 below.

Theorem 4.4.2 ([33])**.** *Let X be a space of type (a, b), characterized by $n > 1$.*

(i) *If a is odd and b is even, then $\mathbb{Z}_2$ cannot act freely on X.*

(ii) *If $a \neq 0$, then S^1 cannot act freely on X.*

[2]also known as the *one point union space*.

About free actions of the group S^3 on spaces of type (a, b), we can conclude that:

Theorem 4.4.3. *Let X be a space of type (a, b), characterized by $n > 1$. If $G = S^3$ acts freely on X, then n is an odd number of the form $4k - 1$, for some $k \geq 1$, $a = 0$ and b is odd. In this case, the cohomology algebra of the orbit space X/G is isomorphic to the graded polynomial algebra*

$$\mathbb{Z}_2[x, y]/\langle x^k, y^2\rangle,$$

where $\deg x = 4$ *and* $\deg y = 2n$.

Proof. By Theorem 4.4.2, if $a \neq 0$ then G cannot act freely on X; therefore, we can suppose $a = 0$.

Let $\{E_r^{*,*}, d_r\}$ be the spectral sequence associated to the Borel fibration $X_G \to B_G$, with fiber X, such that

$$E_2^{p,q} = H^p(B_G) \otimes_{\mathbb{Z}_2} H^q(X),$$

which converges to $H^*(X_G) \cong H^*(X/G)$.

By Theorem 1.3.1 (or directly by Theorem 1.2.6), it follows that this sequence does not collapse on its E_2-term. Then, there must exist some nontrivial differential d_{r_i}, for some $r_i \geq 2$. If $r = \min\{r_i\}$, then

$$E_r^{p,q} \cong E_{r-1}^{p,q} \cong \cdots \cong E_2^{p,q},$$

and this is possible only if $r = 4k$ and $n = 4k - 1$, for some $k \geq 1$. This provides the following possibilities for the action of the differentials $d_{4k}^{4l,q}$, for $q = n, 2n, 3n$:

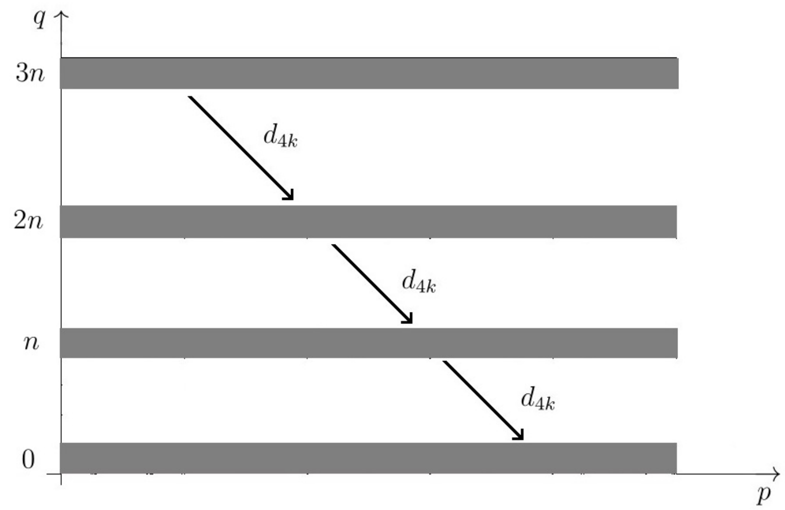

Figure 4.3: Pattern for the actions of the differentials on $E_{4k}-$page

(a) $d_r(1 \otimes u_1) = 0$, $d_r(1 \otimes u_2) = \tau^k \otimes u_1$ and $d_r(1 \otimes u_3) = \tau^k \otimes u_2$,

(b) $d_r(1 \otimes u_1) = \tau^k \otimes 1$, $d_r(1 \otimes u_2) = \tau^k \otimes u_1$ and $d_r(1 \otimes u_3) = 0$,

(c) $d_r(1\otimes u_1)=\tau^k\otimes 1$, $d_r(1\otimes u_2)=0$ and $d_r(1\otimes u_3)=\tau^k\otimes u_2$,

(d) $d_r(1\otimes u_1)=0$, $d_r(1\otimes u_2)=\tau^k\otimes u_1$ and $d_r(1\otimes u_3)=\tau^k\otimes u_2$,

(e) $d_r(1\otimes u_1)=0$, $d_r(1\otimes u_2)=0$ and $d_r(1\otimes u_3)=\tau^k\otimes u_2$,

(f) $d_r(1\otimes u_1)=0$, $d_r(1\otimes u_2)=\tau^k\otimes u_1$ and $d_r(1\otimes u_3)=0$,

(g) $d_r(1\otimes u_1)=\tau^k\otimes 1$, $d_r(1\otimes u_2)=0$ and $d_r(1\otimes u_3)=0$.

We will divide the analysis of these cases according to the parity of b.

Case b odd: In this case, we have the relation $u_1u_2=u_3$ and, by the multiplicative properties of the differentials, we have

$$d_r(1\otimes u_3)=(1\otimes u_1)d_r(1\otimes u_2)+(1\otimes u_2)d_r(1\otimes u_1).$$

So, if one of the cases $(b),(d),(e)$ or (g) occurred, it would lead to the contradiction $0=\tau^k\otimes u_2$.

If case (a) occurred, then the differentials $d_{4k}^{4i,3n}$ and $d_{4k}^{4i,2n}$ would be isomorphisms, whence it would follow that

$$\mathrm{im} d_{4k}^{4i,3n}\cong\mathbb{Z}_2\not\subseteq\{0\}=\ker d_{4k}^{4(i+k),2n},$$

which is a contradiction.

If case (f) occurred, then the sequence would collapse on its $E_{4k+1}-$term, with the lines $E_{4k+1}^{*,0}$ and $E_{4k+1}^{*,3n}$ containing an infinite number of nonzero elements. This would contradict Theorem 1.2.6.

Therefore, (c) is the only possible case, and it produces the following pattern:

$$E_{4k+1}^{p,q}=\begin{cases}\mathbb{Z}_2, & \text{if } p=0,4,\cdots,4(k-1) \text{ and } q=2n,\\ \{0\}, & \text{otherwise.}\end{cases}$$

Then, the sequence collapses on its $E_{4k+1}-$term, and $E_\infty^{p,q}\cong E_{4k+1}^{p,q}$, for all $p,q\geq 0$. So,

$$H^j(X/G)\cong \mathrm{Tot}^j(E_\infty).$$

The elements $\tau\otimes 1$ and $1\otimes u_2$ are the only permanent co-cycles, so they determine the nonzero elements x and y in $E_\infty^{4,0}$ and $E_\infty^{0,2n}$, respectively. By (1.28), we have $\pi^*(\tau)=x$; then, $0=\pi^*(\tau^j)=x^j$ for all $j\geq k$. By the structure of the $E_\infty-$term, it follows that $y^2=0$; therefore, $H^*(X/G)$ is isomorphic to the graded polynomial algebra

$$\mathbb{Z}_2[x,y]/\langle x^k,y^2\rangle,$$

where $\deg x=4$ and $\deg y=2n$.

Case b even: We will show that if b is even, then none of the cases can occur. By the relation $u_1u_2 = 0$, we have

$$0 = (1 \otimes u_1)d_r(1 \otimes u_2) + (1 \otimes u_2)d_r(1 \otimes u_1),$$

and this allows us to eliminate the cases $(b), (c)$ and (g), since they produce the contradiction $0 = \tau^k \otimes u_2$.

By the same reason of the previous case (b odd) we can eliminate case (a); that is, it implies that

$$\mathrm{im} d_{4k}^{4i,3n} \not\subseteq \ker d_{4k}^{4(i+k),2n}.$$

By a similar reason we can eliminate (d), since it implies that the differentials $d_{4k}^{4i,2n}$ and $d_{4k}^{4j,3n}$ are isomorphisms.

For case (e), the sequence would collapse on its $E_{4k+1}-$term, with the lines $E_{4k}^{*,0}$ and $E_{4k}^{*,n}$ containing infinite nonzero elements, which would contradict Theorem 1.2.6. Finally, by the same reason of the previous case, we can eliminate (f); therefore, when b is even, the space X does not admit any free action of G. □

Example 4.4.2. As we saw in the Example 4.4.1, $X = S^n \vee S^{2n} \vee S^{3n}$ is a space of type $(0, 0)$ and $Y = S^6 \vee \mathbb{C}P^2$ is a space of type $(1, 0)$. Therefore, by Theorem 4.4.3, the group S^3 cannot act freely, in both cases. In general, we can affirm that S^3 cannot act freely on every finitistic space $W \cong_2 X$ or $W \cong_2 Y$.

Example 4.4.3. In order to investigate the existence of free actions of the finite groups $\mathbb{Z}_p$ on spaces of type $(a, 0)$, in [13] the author constructed the space $Z = (S^{n-1} \star \mathbb{R}P^2) \vee S^n$, for $n > 0$ even, which is an example of a type $(0, 0)$ space. By Theorem 4.4.3, we can conclude that S^3 cannot act freely on any finitistic space with mod 2 cohomology of Z.

Remark 4.4.3. It follows by Theorem 4.4.2 that if S^3 acts freely on a space of type (a, b) characterized by n, then $a = 0$, and the statement that n must be odd can be obtained by Theorem 1 in [40]. However, we improved this result by showing that it is necessary that $n \equiv 3(\mod 4)$, according to Theorem 4.4.3.

Remark 4.4.4. The same result of Theorem 4.2.2, when applied on a finitistic space $X \cong_2 S^m \times S^{2m}$, can be obtained by using Theorem 4.4.3, since in this case X is a space of type $(0, 1)$. This shows the consistency of the results.

Remark 4.4.5. We can directly conclude by Theorem 4.4.3 that S^3 cannot act freely on every finitistic space $X \cong_2 \mathbb{C}P^3$ or $\mathbb{H}P^3$, since these spaces are of type $(1, 1)$, even though it has already been demonstrated, more generally, on Theorem 4.3.1.

APPENDIX A

STIEFEL-WHITNEY CLASSES AND COBORDISM

Since the Dold, Wall and Milnor manifolds (which are fundamental in most of the results we present) emerged in the context of cobordism theory, we added this appendix to the text in order to show the motivation and the way in which these spaces appear in the literature.

A.1 Stiefel-Whitney classes

The definitions and the results that we present in this Section is based on the text of Milnor and Stasheff [28].

Definition A.1.1. Let R be a ring and $\xi = (E, p, B)$ be a (vector) $n-$bundle over a paracompact base space B. A *characteristic class* c of degree q of ξ is an association $c(\xi) \in H^q(B; R)$, such that, for any $n-$bundles $\xi_1 = (E_1, p_1, B_1)$ and $\xi_2 = (E_2, p_2, B_2)$ and any bundle map $(F, f) : \xi_1 \to \xi_2$, we have $f^*c(\xi_2) = c(\xi_1)$.

Remark A.1.1. If ξ and η are isomorphic $n-$bundles over the same paracompact space B, then $c(\xi) = c(\eta) \in H^q(B; R)$, for all $q \geq 0$.

Definition A.1.2. Let Λ_q be the set of all characteristic classes of degree q associated to a $n-$bundle $\xi = (E, p, B)$. Then Λ_a has the structure of an abelian group, given by the natural operation

$$c_1 + c_2 = c_1(\xi) + c_2(\xi) \in H^1(B; R). \tag{A.1}$$

if $\deg c_1 = r$ and $\deg c_2 = s$, we can define the product

$$c_1 \cdot c_2 = c_1(\xi) \smile c_2(\xi) \in H^{r+s}(B; R), \tag{A.2}$$

which makes the sum

$$\Lambda = \bigoplus_{q \geq 0} \Lambda_q \tag{A.3}$$

a graded ring, called the *characteristic classes ring* of ξ. For $R = \mathbb{Z}_2$, we have the classes $w_i(\xi) \in H^i(B; \mathbb{Z}_2)$, which are called *Stiefel-Whitney classes*, and are uniquely determined by the following axioms:

(a) $w_0(\xi) = 1$ and $w_i(\xi) = 0$ if $i > \dim \xi$;

(b) $w_i(\xi \oplus \eta) = \sum_{j=0}^{i} w_j(\xi) \smile w_{i-j}(\eta)$;

(c) $w_1(\gamma_1) \neq 0$, where $\gamma = (E, \pi, \mathbb{R}P^\infty)$ is the universal line bundle over $G_1(\mathbb{R}^\infty) = \mathbb{R}P^\infty$.

Remark A.1.2 ([28], p. 38-39)**.** The following results can be directly obtained from the axioms above.

(i) If ξ and η are isomorphic $n-$bundles, then $w_i(\xi) = w_i(\eta)$, for all $i \geq 0$.

(ii) If ξ is a trivial $n-$bundle, then $w_i(\xi) = 0$, for all $i > 0$.

(iii) If ε is trivial, then $w_i(\xi \oplus \varepsilon) = w_i(\xi)$, for all $n-$bundle ξ.

Definition A.1.3. Let M be a closed manifold of dimension n. By excision, we know that

$$H_i(M, M - \{x\}; \mathbb{Z}_2) = \begin{cases} \{0\}, & \text{if } i \neq n, \\ \mathbb{Z}_2, & \text{if } i = n, \end{cases} \tag{A.4}$$

for each $x \in M$.

We say that the generator μ_x of $H_n(M, M - \{x\}; \mathbb{Z}_2)$ is a $\mathbb{Z}_2-$*orientation* of M in x. We also know that there is a class

$$\mu_M \in H_n(M; \mathbb{Z}_2) \cong \mathbb{Z}_2\langle \mu_M \rangle, \tag{A.5}$$

such that $(i_x)_*(\mu_M) = \mu_x$, for all $x \in M$, where $i_x : (M, \emptyset) \to (M, M - \{x\})$ is the inclusion map. This class μ_M is called the *fundamental homology class* of M.

Remark A.1.3. Let M be a closed differentiable manifold and let τ_M be the tangent bundle of M. If $w_i(\tau_M) \in H^i(M; \mathbb{Z}_2)$, then for any $r_i \in \mathbb{Z}$ we have $w_i^{r_i}(\tau_M) \in H^{ir_i}(M; \mathbb{Z}_2)$. Therefore, it makes sense to consider the index

$$\langle w_i^{r_i}(\tau_M)\,,\, \sigma \rangle, \text{ for any element } \sigma \in H_{ir_i}(M; \mathbb{Z}_2). \tag{A.6}$$

Definition A.1.4. Let M be a closed manifold and $r_1, \cdots, r_n$ be non-negative integers so that $1 \cdot r_1 + 2 \cdot r_2 + \cdots + n \cdot r_n = \dim M$. The *Stiefel-Whitney number* associated to the $n-$uple

$(r_1, \cdots, r_n)$ is defined as the number

$$\langle w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n} \, , \, \mu_M \rangle \in \mathbb{Z}_2. \tag{A.7}$$

We say that two manifolds M and N, of same dimensions n, have the same Stiefel-Whitney number when

$$\langle w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n} \, , \, \mu_M \rangle = \langle w_1(\tau_N)^{r_1} \cdots w_n(\tau_N)^{r_n} \, , \, \mu_N \rangle \tag{A.8}$$

for all $n-$uple $(r_1, \cdots, r_n)$.

A.2 The unoriented cobordism ring $\mathfrak{R}_*$

In this Section we define the cobordism ring and we present the initial motivation for the construction of the Dold, Wall and Milnor manifolds.

Remark A.2.1. Throughout this Section we note that the term *manifold* will always mean a *differentiable manifold.*

Definition A.2.1. Two closed manifolds M and N are called *cobordant* when there is a compact manifold W, with boundary, such that $\partial W \cong M \amalg N$. Here, the symbol $\amalg$ means the disjoint union. In particular, we say that a manifold M *bounds* when it is the boundary of a compact manifold.

Proposition A.2.1 ([2], p. 08)**.** *On the category of all closed manifolds, the relation*

$$M \sim N \Leftrightarrow M \textit{ and } N \textit{ are cobordant} \tag{A.9}$$

is an equivalence relation; we denote by $[M]$ the class represented by M.

Definition A.2.2. For each natural number n, we call the $n-$*th cobordism group* the set

$$\mathfrak{R}_n = \{[M] \, ; \, \dim M = n\}, \tag{A.10}$$

where the operation $+$ is induced by the disjoint union operation $\amalg$. Formally, for each class $[M]$ and $[N]$ in $\mathfrak{R}_n$, we have

$$[M] + [N] = [M \amalg N]. \tag{A.11}$$

The *unoriented cobordism ring* is defined as the sum

$$\mathfrak{R}_* = \bigoplus_{n=0}^{\infty} \mathfrak{R}_n, \tag{A.12}$$

where the product operation is induced by the Cartesian product, i. e.,

$$[M] \cdot [N] = [M \times N]. \tag{A.13}$$

Remark A.2.2. Note that $\mathfrak{N}_*$ has the structure of a graded module over $\mathbb{Z}_2$, since we have

$$[M] + [M] = [M \amalg M] = [\partial(M \times [0,1])] = [\emptyset] = 0_{\mathfrak{N}_*}, \tag{A.14}$$

for any closed manifold M.

Theorem A.2.1. *Let M be a closed manifold of dimension n. Then $[M] = 0 \in \mathfrak{N}_*$ if, and only if,*

$$\langle w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n} \, , \, \mu_M \rangle = 0, \tag{A.15}$$

for any $n-$uple $(r_1, \cdots, r_n)$, such that $1 \cdot r_1 + 2 \cdot r_2 + \cdots + n \cdot r_n = n$. That is, a closed manifold is the boundary of another manifold if, and only if, all the Stiefel-Whitney numbers are equal to zero.

The proof of the first part of the Theorem A.2.1, that is, the statement that the existence of a manifold W, of dimension $n+1$, such that $\partial W = M$ implies that all the Stiefel-Whitney number associated to M are zero, is due to L. S. Pontrjagin, and can be found in [28], Theorem 4.9. The demonstration of the second statement is due to R. Thom, whose proof (which is considerably complex) can be found in [39].

In 1954, R. Thom [39] showed the existence of an isomorphism of algebras

$$\mathfrak{N}_* = \mathbb{F}_2[x_2, x_4, x_5, \cdots] = \mathbb{F}_2[x_i \, ; \, i \neq 2^j - 1], \tag{A.16}$$

where $x_{2i} = [\mathbb{C}P^{2i}]$, for each i. In 1956, A. Dold [11] constructed representatives in odd dimensions of the generators x_i, who came to be known as *Dold manifolds.* Precisely, Dold showed that for each $i = 2^r(2s+1)$, we have $x_i = [P(2^r - 1, s2^r)]$, where $P(m,n)$ is the orbit space defined by the involution T as in 2.1. That is, Dold showed that

$$\left\{[\mathbb{R}P^{2i}], [P(2^r - 1, s2^r)] \, ; \, i, r, s \geq 1\right\} \tag{A.17}$$

is a generator set for the unoriented cobordism ring.

In his work [42], C. T. C. Wall defined a new class of manifolds by the orbit spaces of certain involutions defined on cartesian products of the type $P(m,n) \times [0,1]$, according to Definition 2.1.2, that were called *Wall manifolds* and are denoted by $Q(m,n)$. Similarly to the work of Dold, Wall showed that the manifolds $Q(m,n)$ also represent generators for $\mathfrak{N}_*$, so that

$$\left\{[\mathbb{R}p^{2i}], [Q(2^r - 2, s2^r)] \, ; \, i, r, s \geq 1\right\} \tag{A.18}$$

is a new generator set of $\mathfrak{N}_*$.

Finally, in 1965, J. Milnor [27] constructed a generator set of $\mathfrak{N}_*$ defining what came to be known as *Milnor manifolds*. Moreover, Milnor defined the following generator set of $\mathfrak{N}_*$:

$$\Big\{[\mathbb{R}P^{2i}], [H(2^k, 2t2^k)] \,;\, i, k, t \geq 1\Big\}, \tag{A.19}$$

where $H(m, n)$ denotes a Milnor manifold of dimension $m + n - 1$.

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