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Polynomial Weingarten Surfaces of Tubular Type

São Carlos - SP

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Contents

Abstract

This work seeks to contribute to the classification of Weingarten surfaces. More precisely, it fully classifies three families of surfaces (named tubular, cyclic and canal surfaces) in a tridimensional space form (Euclidean, Lorentzian and Hyperbolic spaces) that verify an arbitrary polynomial relation among its Gaussian and mean curvatures. The results obtained provide geometric features of the surface as well as algebraic conditions over the polynomial that defines a surface as Weingarten. Furthermore, results that allow us to investigate Weingarten surfaces only by the polynomial analysis are presented.

Keywords:

Weingarten surfaces, Weingarten tubular surfaces, Weingarten cyclic surfaces, Weingarten canal surfaces, polynomial Weingarten surface.

Resumo

Esse trabalho busca contribuir com a classificação de superfícies de Weingarten. Mais precisamente, esse trabalho classifica três famílias de superfícies (a saber, as superfícies: tubular, cíclica e canal) em um espaço tridimensional com curvatura seccional constante (os espaços Euclidiano, Lorentziano e Hiperbólico) que verificam uma relação arbitrária polinomial entre suas curvaturas Gaussiana e média. Os resultados obtidos fornecem características geométricas da superfície bem como condições algébricas sobre o polinômio que a define como superfície de Weingarten. Além disso, são apresentados resultados que nos permitem investigar superfícies de Weingarten exclusivamente através da análise polinomial.

Keywords:
Superfície de Weingarten, Superfície tubular de Weingarten, Superfície cíclica de Weingarten, superfície canal de Weingarten, Superfície polinomial de Weingarten.

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Introduction

An important research subject in classical differential geometry is to discover which global properties of a manifold one can obtain from hypotheses on its curvatures. The relevance of this topic initially throwbacks to *Theorema Egregium* of Gauss until it culminates in Riemann's groundbreaking achievements, named after him, Riemannian geometry.

In this direction, a topic that has received much attention over the years is that of Weingarten surfaces in a space form, that is, two dimensional manifold whose Gaussian curvature K, and mean curvature H, satisfy a smooth non trivial relation in a space with constant sectional curvature:

$$\Phi\,(K, H) \equiv 0.$$

One of the many reasons that motivates the investigation of Weingarten surfaces is that it includes relevant and well studied families of surfaces like CMC (constant mean curvature), CGC (constant Gaussian curvature) and CCC (constant Casorati curvature). It is also worth to remark that Weingarten surfaces have several applications in computer aided geometric design.

Despite being an ancient topic that attracts so much interest, there is still much to be discovered about Weingarten surfaces, once in general, the results in the literature classify a particular familiy of surfaces in a classical environment and verifying a specific relation. For instance, the class of non-developable ruled surfaces verifying a non-trivial relation in the Euclidean space (denoted by $\mathbb{E}^3$) was presented by Beltrami and Dini, while in [5] Khünel and Dillen discussed Weingarten ruled surfaces in Lorentzian 3-space (denoted by $\mathbb{L}^3$). Returning to $\mathbb{E}^3$, Khünel in [10] revisited this class of surfaces presenting another proof by the already known result and expanded the classification considering other types of curvatures (more precisely, the second Gaussian curvature K_{II}). Following the investigation of classifying ruled surfaces under other types of curvatures, F. Dillen and W. Sodsiri in [6], [7], described ruled surfaces that verify a relation among the K_{II} and H_{II} (where H_{II} denotes the second mean curvature).

The classification of translational surfaces was presented by Dillen, Goemans and Woestyne in [4] that studied these surfaces in Euclidean and Lorentzian spaces. Other examples can be found in [3] where Do Carmo and M. Dajczer studied CMC helicoidal surfaces and Rosenberg and Sá Earp in [21] research embedded surfaces in $\mathbb{E}^3$ that verify a linear relation. A rare analysis of a non linear relation is given by A. Barreto, F. Fontenele and L. Hartmann that classifies CCC rotational surfaces in [1]. Finally, more recently, in the work [24] López and Pámpano provided a classification of linear Weingarten rotational surfaces among its principal curvatures.

Concerning tubular surfaces, in [22] Sorour provides a classification for the linear relation among its gaussian and mean curvatures, besides K_{II} and H_{II}. In Lorentzian space, Karacan, Yoon and Tuncer presented in [13] a classification for linear relations among every curvatures.

The CMC surfaces foliated by circles was fully classified by López in Euclidean, Hyperbolic and Lorentzian spaces in [28], who also provided a complete classification of Linear Weingarten (*i.e.*, surface that verify a relation given by the linear polynomial $ax + by - c$) in [16]. Then, the later relation was also studied, now in $\mathbb{L}^3$, by Kallan, López and Saglam in [29].

For canal surfaces in the Euclidean 3-space, Kim, Liu and Qian in [30] classified the linear relation among the Gaussian, mean and second Gaussian curvatures. While Tunçer and Yoon studied in [31] the relation $ax + by + cz + d$ among the three mentioned curvatures. In the Lorentzian 3-space, the Linear Weingarten canal surface was classified by J. Qian, M. Su, X. Fu and S. D. Jung in [20] and [19].

The main challenge of classifying a surface locally parametrized by $\psi(s,t)$ that verify a non trivial relation $\Phi(x,y)$ is that usually the approach is based on considering the following composition

$$(s,t) \longrightarrow (K(s,t), H(s,t)) \longrightarrow \Phi(K(s,t), H(s,t)),$$

then the Weingarten hypothesis implies that

$$\Phi(K(s,t), H(s,t)) \equiv 0$$

hence, the derivative of above equation gives us

$$\det(J\Phi) = \det \begin{pmatrix} K_s & H_s \\ K_t & H_t \end{pmatrix} = 0$$

which implies to classificate surfaces which derivative in parameter s and t of Gaussian and mean curvatures verifies

$$K_s H_t - K_t H_s = 0. \tag{1}$$

In general, expressing each of the previous terms is already a difficult task. Therefore, to compute (1) has several obstacles as computational limitations and polynomial analysis. This process concluded, it is obtained a list of all possible surfaces (of fixed family of surfaces) that may be Weingarten, however, there is no information about the relation itself. In other words, for each particular relation Φ, this procedure basically must be repetead (now using the explicit expression of Φ) in order to classify which of the listed surfaces verifies the specific given relation.

To exemplify, in [22] it is presented that every tubular surface in $\mathbb{E}^3$ is Weingarten (for some unknown relation) and it is given a classification of linear Weingarten tubular surfaces. Nevertheless for arbitrary relations this classification was open untill this work. It occurs especially because even for polynomial relation of degree 2, besides the early mentioned problems, we now also have that the techniques of differential equations are not applicable anymore.

Moreover, we must highlight that the previous technique does not always apply, once for several families of surfaces the equation (1) is trivially satisfyied. In other words, the differential equation presented in (1) vanishes identically, therefore there is no equation to be analysed (hence, no geometric description of the surface is obtained).

Cases where the equation (1) does not provide information include, but do not resume to, surfaces whose curvatures are in one parameter only, as the rotational surface.

The latter statement can be illustrated in the work [1] that classifies CCC rotational surfaces by solving the following equation:

$$\left(\theta'\left(t\right)\right)^2 + \frac{\cos^2\theta\left(t\right)}{z^2\left(t\right)} = 1. \tag{2}$$

We observe that the Theory of Differential Equations does not ensure the existence and uniqueness of the solution for the previous equation, therefore, the analysis becomes very specific for each case. That is a relevant cause for the difficulties of more general classifications of Weingarten surfaces and what makes this topic so captivating.

Motivated by the aforementioned results, our work seeks to contribute to the investigation of Weingarten surfaces. More precisely, we research Weingarten surfaces whose relation verified by its curvatures is a polynomial relation. In other words, we suggest the following definition:

Definition 1 *A Polynomial Weingarten surface is a (Weingarten) surface whose Gaussian and mean curvatures verify*

$$Q\left(K, H\right) \equiv 0$$

where $Q\left(x, y\right)$ *is a polynomial in* $\mathbb{R}\left[x, y\right]$.

The relevance of this particular class (the Polynomial Weingarten surfaces class) of Weingarten surfaces, lies on the fact that the most famous investigated relations can be written as polynomials. Therefore, Polynomial Weingarten surfaces provides classification of several relations as CGC, CMC and Linear Weingarten surfaces (which are given by a linear polynomial), as well as CCC surface (which is given by a non linear relation $4H^2 - 2K - c \equiv 0$ or, equivalently, in terms of principal curvatures $k_1^2 + k_2^2 - c \equiv 0$).

Furthermore, we list all surfaces that verify a given polynomial relation. More precisely, let us stablish an important (original) concept of this work:

Given a polynomial $Q(x,y) \in \mathbb{R}[x,y]$, we define $\mathcal{S}(Q)$ the set of all regular surfaces (in this work we will study this set for tubular, cyclic and canal surfaces) in a space form (we will study in the tridimensional spaces: Euclidean, Lorentzian and Hyperbolic) whose Gaussian and mean curvatures verify $Q(K,H) \equiv 0$. So, fixed a polynomial $Q(x,y)$ we will present geometric features and also conditions over the surface whose Gaussian and mean curvatures vanishes Q.

Conversely, for a given surface S, we define $\mathcal{Q}(S)$ as the set of all polynomials $Q(x,y) \in \mathbb{R}[x,y]$ verifying $Q(K,H) \equiv 0$. For tubular surfaces (in $\mathbb{E}^3$, $\mathbb{L}^3$ and $\mathbb{H}^3$) we are able to present a complete characterization of the set $\mathcal{Q}(S)$. For cyclic and canal surfaces we obtain an important discriminant that provides aspects of the elements (polynomials) of $\mathcal{Q}(S)$.

We point out that a Polynomial Weingarten surface is equivalent to the $Q(S)$ not being empty.

In the Chapter 2, we investigate tubular surfaces which are the surfaces obtained by the moviment of a circle of constant radius $r > 0$ along a central curve. Then, our discussion starts with the study of the linear polynomial relations. So let us recall the definition:

Definition 2 *A Polynomial Weingarten surface is called **linear** when a linear polynomial among its Gaussian and mean curvatures are verified, that is,*

$$aK + bH - c \equiv 0$$

where a, b, $c \in \mathbb{R}$ with $(a,b) \neq (0,0)$.

Endowed with the above presented nomenclature, the previous definition can be expressed as: A Polynomial Weingarten surface S is said linear when there exists a polynomial of degree 1 in $\mathcal{Q}(S)$. Associated with the above discussion, we display the following result.

Theorem 3 *Every Polynomial Weingarten tubular surface is linear. More precisely, every tubular surface of radius $r > 0$ verify the relation $xr^2 - 2ry + 1$.*

Proceeding with our analysis, we started the research of relations that are not linear. However, we must observe that, once a surface verifies a linear polynomial $ax + by - c$, it is clearly that it will verify $(ax + by - c)^n R(x,y)$ for every $n \in \mathbb{N}$ and $R(x,y) \in \mathbb{R}[x,y]$ not identically null, as well. Hence, we do not consider the latter relation as a true non linear relation. Roughly speaking, we study non linear polynomial relations that cannot be written as before.

In this direction, we want to understand which type of tubular surfaces verify a fixed polynomial relation $Q(K,H) \equiv 0$ (*i.e.,* we would like to know more information about $\mathcal{S}(Q)$) and which type of polynomial vanishes at the curvatures of a fixed tubular surface S (*i.e.,* we would like to know more information about $\mathcal{Q}(S)$). Before to present this answers, we suggest some new definitions that will help us to improve our analysis.

Definition 4 *The degree of a surface S is defined by*

$$\partial S = \min\{\partial Q\ ;\ Q \in \mathcal{Q}(S)\}.$$

Where the symbol ∂ associated with a polynomial represents the degree of the polynomial. Hence, a Polynomial Weingarten surface S is linear if and only if $\partial S = 1$.

The degree of a surface measures the minimal required degree of a polynomial to be an element of $\mathcal{Q}(S)$. Besides, the degree of a surface S also provides a first discriminant that indicates which factor of a polynomial we should investigate.

For example, let S be a surface such that $\partial S = 2$, and consider the polynomial $Q(x,y) = Q_1(x,y)\,Q_2(x,y) \in \mathcal{Q}(S)$ given by

$$Q_1(x,y) = ax + by - c \qquad \text{and} \qquad Q_2(x,y) = x^2 - y^2 \tag{3}$$

where a, b, $c \in \mathbb{R}$ with $(a,b,c) \neq (0,0,0)$.

Then by the degree of the surface, we already know that $Q_1 \notin \mathcal{Q}(S)$, since $\partial Q_1 < 2 = \partial S$. So, we wonder which information we can gather about $Q_2(x,y)$.

In this direction, we focus our attention to the study of the polynomials. Once we set the definitions that are sensible to conditions over the surface, we must understand the polynomials. More precisely, our goal is to obtain a quality from the polynomials that indicates which cases the polynomial can be factorized in a smaller polynomial that vanishes the curvatures of the surface.

Returning to the last example, where we have a surface S with $\partial S = 2$ and a polynomial $Q(x,y) = Q_1(x,y)\,Q_2(x,y)$ given by (3) where $\partial Q = 3$. Here, we may have that the polynomial $Q_2(x,y)$ of degree 2 is the responsible to vanish the polynomial relation $Q(K,H) \equiv 0$ or we may have that the polynomial $Q(x,y)$ as a whole is needed to verify $Q(K,H) \equiv 0$.

In view of the above discussion, we present our next definition:

Definition 5 *Consider a Polynomial Weingarten surface S, and let Q be a polynomial in $\mathcal{Q}(S)$. We define the degree of Q relative to S by:*

$$\partial_S Q = \{\partial R\ ;\ R \in \mathcal{Q}(S) \ \text{and } Q \text{ belongs to the ideal in } \mathbb{R}[x,y] \text{ generated by } R\}.$$

We would like to remark that $\partial_S Q \leq \partial Q$. Moreover, if Q is a irreducible polynomial the equality is achieved. We also observe that the condition of irreducibility of the polynomial is not a necessary condition for the equality to be reached.

Endowed with the above terminology, we notice that, in the previous example, the degree of polynomial $Q(x,y)$ relative to S may be $\partial_S Q = 2$ (if $Q_2(K,H) \equiv 0$ everywhere) or $\partial_S Q = 3$ (otherwise).

The interest behind the previous definition lies on the fact that we are looking for "true" relations, that is, we seek to investigate the essential factor of the polynomial needed in order to the surface to verify the polynomial relation. *In suma,* the definition of degree of the polynomial relative to the surface captures this quality. Then, we are able to investigate only the relevant part of the polynomial.

Finally, in the aim to study "true" non linear relations, we suggest the next definition:

Definition 6 *We said that* $Q(K,H) \equiv 0$ *is a* ***true nonlinear relation*** *when* $\partial_S Q > 1$.

In the view of above discussion we have the following theorem:

Theorem 7 *The cylinders are the only tubular surfaces that verify a true nonlinear relation* $Q(K,H) \equiv 0$.

All the previous results are consequence of our main theorem of the Chapter 2. In the aim to present it in a more suitable way, we will first consider the following definition:

Definition 8 *The radius of a polynomial* $Q(x,y) \in \mathbb{R}[x,y]$ *is defined as the set*

$$Rad(Q) = \left\{ r \in (0,+\infty) \ ; \ Q\left(0, \frac{1}{2r}\right) = 0 \right\}.$$

We say that Q *is* ***tubular*** *or* ***non tubular*** *according to the* $Rad(Q)$ *being either non empty or empty, respectively.*

The following result is what motivates the above definition. Furthermore, our result provides a necessary and sufficient condition to the existence of Polynomial Weingarten tubular surface:

Proposition 9 *There is a Polynomial Weingarten tubular surface in Euclidean (respect. Lorentzian or Hyperbolic) 3-spaces (of radius* $r > 0$*) verifying* $Q(K,H) = 0$ *if and only if* Q *is tubular (and* $r \in Rad(Q)$*).*

Besides presenting a characterization of Polynomial Weingarten tubular surfaces, the previous theorem provides a list of all possible radius. The theorem that determines the set of all polynomials whose set of zeros contains the Gaussian and mean curvatures of a given regular tubular surface is read as follows:

Theorem 10 *Consider a regular tubular surface* S *of radius* $r > 0$ *in Euclidean (respect. Lorentzian or Hyperbolic) 3-space and let* K, H *be its Gaussian and mean curvatures. Denote by* $\mathcal{Q}(S)$ *the set of all polynomials* $Q \in \mathbb{R}[x,y]$ *verifying* $Q(K,H) \equiv 0$.

i. If S *is a cylinder, then* $\mathcal{Q}(S) = \{Q \in \mathbb{R}[x,y] \ ; \ r \in Rad(Q)\}$;

ii. If S *is not a cylinder, then* $\mathcal{Q}(S)$ *is the ideal in* $\mathbb{R}[x,y]$ *generated by* $xr^2 - 2ry + 1$.

In particular, every tubular surface is Polynomial Weingarten.

As we indicated before, we were also able to describe the set of all tubular surfaces whose Gaussian and mean curvatures vanishes at a given polynomial.

Theorem 11 *Given a tubular polynomial* $Q(x, y) \in \mathbb{R}[x, y]$*, denote by* $\mathcal{S}(Q)$ *the set of all regular tubular surfaces in Euclidean (respect. Lorentzian or Hyperbolic) 3-space whose Gaussian and mean curvatures* K*,* H *verify* $Q(K, H) \equiv 0$*. Then, the elements of* $\mathcal{S}(Q)$ *are:*

i. *The cylinders whose radius* r *belongs to* $Rad(Q)$*;*

ii. *The tubular surfaces of radius* $r \in Rad(Q)$ *such that* Q *is in the ideal of* $\mathbb{R}[x, y]$ *generated by* $xr^2 - 2ry + 1$*.*

Many particular results can be obtained as consequence of our main theorems of the Chapter 2. We feature here the classification of tubular surfaces verifying a linear relation and the classification of tubular surfaces with second fundamental form of constant length (or constant Casorati curvature):

Corollary 12 *Let* a, b, c *be real numbers such that* $(a, b, c) \neq (0, 0, 0)$*, and define* $\Delta = b^2 + 4ac$*. Consider the polynomial*

$$Q(x, y) = ax + by - c.$$

Then, $\mathcal{S}(Q) \neq \emptyset$ *if and only if* $b = c = 0$ *or* $bc > 0$*. Moreover:*

i. *If* $b = c = 0$*, then* $\mathcal{S}(Q)$ *contains all right cylinders of any radius* r*;*

ii. *If* $bc > 0$ *and* $\Delta = 0$*, then* $\mathcal{S}(Q)$ *contains all tubular surfaces of radius* $\frac{b}{2c}$*;*

iii. *If* $bc > 0$ *and* $\Delta \neq 0$*, then* $\mathcal{S}(Q)$ *contains all right cylinders of radius* $\frac{b}{2c}$*.*

Corollary 13 *The cylinders are the unique regular tubular Weingarten surfaces with second fundamental form of constant length (or constant Casorati curvature).*

The next natural step is to investigate cyclic surfaces that, roughly speaking, may be seen as tubular surfaces such that the radius r is a smooth function that assumes only positive values. Hence, in Chapter 3 we discuss cyclic surfaces that verify a polynomial relation among its Gaussian and mean curvatures. As a matter of fact, we furnish a classification of Polynomial Weingarten cyclic surfaces in $\mathbb{E}^3$and $\mathbb{L}^3$.

Before presenting the statement of our theorems, we would like to remark that this analysis yields the necessity to suggest another definition that is related to the concept of tubular polynomial. Therefore, we define:

Definition 14 *The radius star of a polynomial* $Q(x, y) \in \mathbb{R}[x, y]$ *is defined as the set*

$$Rad^*(Q) = \left\{ r \in Rad(Q) \ ; \ Q(x, y) \in \left\langle xr^2 - 2ry + 1 \right\rangle \right\}.$$

We recall that the symbol $\langle I(x,y)\rangle$ denotes the ideal in $\mathbb{R}[x,y]$ generated by $I(x,y)$.

In the view of the above definition, our main theorem of the Chapter 3 can be stated as:

Theorem 15 *Consider the polynomial $Q(x,y) \in \mathbb{R}[x,y]$, and let $\mathcal{S}(Q)$ be the set of all regular cyclic surfaces in Euclidean (respect. Lorentzian) 3-space whose Gaussian and mean curvatures K, H verify $Q(K,H) \equiv 0$. Then, the elements of $\mathcal{S}(Q)$ are (smooth) combinations of Rotational surfaces and Tubular surfaces of radius $r \in Rad^*(Q)$.*

The importance of our theorem lies on the fact that we provide geometric features of cyclic surfaces. More precisely, a cyclic surface whose Gaussian and mean curvatures vanish a polynomial relation has (at least) one of the following properties: Locally, either the radius is constant (hence locally is a tubular surface) or either the curvature of the central curve is identically null (hence locally is a rotational surface). This characterizations have profund impact in the curvatures of the surface and, in addition, we already have a complete classification of tubular surfaces.

When we articulate the previous theorem with the concept of polynomials $Q(x,y) \in \mathbb{R}[x,y]$ such that $Rad^*(Q) = \emptyset$, we obtain a characterization that relates the set $\mathcal{S}(Q)$ with conditions over the polynomials $Q(x,y)$ belonging to the set $\mathcal{Q}(S)$.

Corollary 16 *Let $Q(x,y) \in \mathbb{R}[x,y]$ be a polynomial. $Rad^*(Q) = \emptyset$ if and only if the unique elements of $\mathcal{S}(Q)$ are the globally rotational surfaces.*

Many particular results can be obtained as consequence of our main theorems of the Chapter 3. We feature here the next result that provides that, for cyclic surfaces verifying a linear relation, there is no combination of tubular surfaces with rotational surfaces in $\mathcal{S}(Q)$. In other words, a LW-cyclic surface is globally a tubular surface of radius r in the radius star or a globally rotational surface:

Corollary 17 *Let a, b, c be real numbers such that $(a,b) \neq (0,0)$, and define $\Delta = b^2 + 4ac$. Consider the polynomial*

$$Q(x,y) = ax + by - c.$$

Besides, the set $\mathcal{S}(Q)$ contains only globally tubular surfaces or globally rotational surfaces. More precisely:

i. *If $bc > 0$ and $\Delta = 0$, then $\mathcal{S}(Q)$ contains all tubular surfaces of radius $\frac{b}{2c}$;*

ii. *Otherwise, we have that $\mathcal{S}(Q)$ contains rotational surfaces.*

For the class of irreducible polynomials, we also have that the set $\mathcal{S}(Q)$ only accepts one of the (mentioned) subclasses of cyclic surfaces. More precisely, we have the following corollary:

Corollary 18 *If $Q(x,y) \in \mathbb{R}[x,y]$ is an irreducible polynomial. Then, the elements of $\mathcal{S}(Q)$ are globally rotational surfaces or globally tubular surfaces.*

Finally, in Chapter 4, we study surfaces that are obtained by the sweeping of 1-parameter family of spheres with variable radius along a central curve. This class of surfaces is named canal surfaces.

We start this chapter with the investigation of canal surfaces that verify a polynomial relation among its principal curvatures. In the end, our analysis leads us to a classification of canal surfaces whose Gaussian and mean curvatures verify $Q(K,H) \equiv 0$. Thus, we obtain as the main theorem of this chapter:

Theorem 19 *Consider the polynomial $Q(x,y) \in \mathbb{R}[x,y]$ and let $\mathcal{S}(Q)$ be the set of all regular canal surfaces in Euclidean (respect. Lorentzian) 3-space whose Gaussian and mean curvatures K, H verify $Q(K,H) \equiv 0$. Then, the elements of $\mathcal{S}(Q)$ are (smooth) combinations of Rotational surfaces and Tubular surfaces of radius $r \in Rad^*(Q)$.*

As a consequence of our main theorem of Chapter 4, we have the following result:

Corollary 20 *Let $Q(x,y) \in \mathbb{R}[x,y]$ be a polynomial. $Rad^*(Q) = \emptyset$ if and only if the unique elements of $\mathcal{S}(Q)$ are the globally rotational surfaces.*

Chapter 1

Semi-Riemannian Geometry, an overview

1.1 Semi-Riemannian manifold

A **semi-Riemannian manifold** is a smooth manifold M endowed with a metric tensor

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^{\infty}(M, \mathbb{R})$$

which induces in tangent plane of each $p \in M$ a scalar product

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

of constant index (*i.e.*, the largest natural that is the dimension of a subspace Σ of T_pM on which $g_p|\Sigma$ is negative definite). In particular, when the index of g is 0, we obtain a Riemannian manifold.

An **isometry** between two semi-Riemannian manifolds (M, g) and (N, h) is a diffeomorphism $f : M \longrightarrow N$ such that

$$g(V, W) = h(df(V), df(W))$$

for every V, $W \in \mathfrak{X}(M)$.

If $x^1, ..., x^n$ is a coordinate system on a neighborhood $\Omega \subset M$ the componentes of the metric tensor g on Ω are

$$g_{i,j} = g(\partial_i, \partial_j) \in C^{\infty}(\Omega, \mathbb{R}) \qquad 1 \leq i \leq j \leq n.$$

Thus, for vector fields $V = \sum V^i \partial_i$ and $W = \sum W^i \partial_i$ in $\mathfrak{X}(\Omega)$, we have

$$g(V, W) = \sum g_{i,j} V^i W^j.$$

A relevant example for our purposes is the **Lorentzian n-space** ($n \geq 2$), denoted by $\mathbb{L}^n$, which is the semi-Riemannian manifold obtained by endowing $\mathbb{R}^n$ with the scalar product of index 1, given by

$$g\left(V, W\right) = \sum_{i=1}^{n-1} V^i W^i - V^n W^n$$

where $V = \sum V^i e_i$ and $W = \sum W^i e_i$ are vector fields in $\mathfrak{X}\left(\mathbb{L}^n\right)$.

Let (M, g) be a semi-Riemannian manifold of index $m > 0$. A trichotomy of vectors and subspaces classes named as **causality** arises because of index m. The causality of a vector $v \in T_pM$ is said

i. Spacelike, when $g_p\left(v, v\right) > 0$ or $v = 0$;

ii. Timelike, when $g_p\left(v, v\right) < 0$;

iii. Lightlike, when $g_p\left(v, v\right) = 0$ and $v \neq 0$.

A vector subspace Σ of T_pM will be denominated

i. Spacelike when the induced metric on Σ is positive definite, *i.e.*, when all vectors in Σ are spacelike;

ii. Timelike when the induce metric on Σ is nondegenerated and has index m, *i.e.*, there is no lightlike vector on Σ and the dimension of subspace of T_pM spanned by the timelike vectors is m;

iii. Lightlike when the induced metric is degenerated, *i.e.*, Σ contains a lightlike vector.

An immersion $\varphi : S \hookrightarrow M$ of a differential manifold S on (M, g) is said to be **nondegenerated** if $d_x\varphi\left(T_xS\right)$ is not lightlike for every $x \in S$. A nondegenerated immersion is named **spacelike** (**timelike** respect.) if $d_x\varphi\left(T_xS\right)$ is a spacelike (timelike respect.) subspace of $T_{\varphi(x)}M$, for every $x \in S$. Recall that S always can be endowed with natural semi-Riemannian metric $\varphi^* g$ induced by the immersion φ and the semi-Riemannian metric g. More precisely

$$\left(\varphi^* g\right)_x\left(u, v\right) = g_{\varphi(x)}\left(d_x\varphi\left(u\right), d_x\varphi\left(v\right)\right)$$

for every $x \in S$, $u, v \in T_xS$. Note that, for spacelike immersions, $(S, \varphi^* g)$ is a Riemannian manifold.

An important example of spacelike immersion on semi-Riemannian manifold is the **Hyperbolic n-space**

$$\mathbb{H}^n = \left\{x \in \mathbb{L}^{n+1} \ ; \ g\left(x, x\right) = -1 \text{ where } x_{n+1} > 0\right\}$$

endowed with the metric induced by the canonical inclusion. It is possible to show that $\mathbb{H}^n$ is isometric to

$$H^n = \{(x_1, ..., x_n) \in \mathbb{R}^n \ ; \ x_n > 0\}$$

endowed with the Riemannian metric

$$h_x(V, W) = \frac{\sum V_i W_i}{(x_n)^2}$$

where $x = (x_1, ..., x_n) \in H^n$ and $V, W \in T_x H^n$.

The previous discussion on immersions has impact on Local Curve Theory, specifically in the Frenet Frame of the curve. More precisely, let γ be a smooth regular ($\|\gamma'\|_{\mathbb{L}} \neq 0$) curve in $\mathbb{L}^3$ parametrized by arc lenght. The curvature of γ is defined by $\kappa = \|\gamma''\|_{\mathbb{L}}$. When γ is biregular ($\|\gamma'\|_{\mathbb{L}} \neq 0$ and $\kappa \neq 0$), the tangent vector, the principal normal vector and binormal vector are given, respectively, by

$$T = \gamma', \qquad N = \tfrac{\gamma''}{\|\gamma''\|_{\mathbb{L}}} \qquad \text{and } B = T \times_{\mathbb{L}} N,$$

in that case, the causality of γ is $g_p(T, T)$.

A regular curve in $\mathbb{L}^3$, parametrized by arc lenght, is called a **Frenet curve** if it is non degenated and all of its principal normals, where it is defined, are spacelike or timelike. For Frenet curves, the torsion is defined by $\tau = g_p(N', B)$ and the Frenet relations of the curve are given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\varepsilon_T \varepsilon_N \kappa & 0 & \tau \\ 0 & -\varepsilon_N \varepsilon_B \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \tag{1.1}$$

where $\varepsilon_T = g_p(T, T)$, $\varepsilon_N = g_p(N, N)$, and $\varepsilon_B = g_p(B, B)$. Notice that the previous coefficients are related by

$$\varepsilon_T = -\varepsilon_N \varepsilon_B \ , \qquad \varepsilon_N = -\varepsilon_T \varepsilon_B \qquad \text{and} \qquad \varepsilon_B = -\varepsilon_T \varepsilon_N. \tag{1.2}$$

Since $\varepsilon_T = \varepsilon_N = \varepsilon_B = 1$ in the Euclidean case, observe that the usual Euclidean Frenet relations can also be retrieved by (1.1).

1.2 The Levi-Civita connection

An **affine connection** on a differential manifold M is an application

$$(V, W) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \nabla_V W \in \mathfrak{X}(M)$$

such that

C1. $\nabla_{fV_1 + gV_2}(W) = f \nabla_{V_1}(W) + g \nabla_{V_2}(W)$;

C2. $\nabla_V (W_1 + W_2) = \nabla_V (W_1) + \nabla_V (W_2)$;

C3. $\nabla_V (fW) = (Vf) W + f\nabla_V (W)$

where $V_1, V_2, W_1, W_2 \in \mathfrak{X}(M)$ and $f,\ g \in C^\infty (M, \mathbb{R})$.

Its a well-know result the existence of a unique connection (called the **Levi-Civita**) on a semi-Riemannian manifold M verifying:

C4. $[V, W] = \nabla_V W - \nabla_W V$;

C5. $U \langle V, W\rangle = \langle \nabla_U V, W\rangle + \langle V, \nabla_U W\rangle$,

for all $U, V, W \in \mathfrak{X}(M)$.

The affine connection of $\mathbb{L}^n$ and $\mathbb{R}^n$ is

$$\nabla_V W = dW(V)$$

for every $V, W \in \mathfrak{X}(M)$. And the affine conection of $\left(H^3, h_x\right)$ is given by

$$\nabla_V W(p) = dW(V) - \frac{g(V(p), e_3)}{x_3} W(p) - \frac{g(W(p), e_3)}{x_3} V(p) + \frac{g(V(p), W(p))}{x_3} e_3$$

where $x = (x_1, ..., x_3) \in H^3$, $\{e_i\}$ is the canonical basis of $\mathbb{R}^3$ and $V, W \in T_x H^n$. To obtain the connection of $\mathbb{H}^n$ it is necessary to revisit the following result:

If $\varphi : S \hookrightarrow M$ is an immersion of differential manifold S on a semi-Riemannian manifold (M, g) then the Levi-Civita connection of $(S, \varphi^* g)$ is given by

$$d\varphi(\nabla_V W) = \left(\overline{\nabla}_{\overline{V}} \overline{W}\right)^{\intercal} \circ \varphi$$

where $V, W \in \mathfrak{X}(S)$, $\overline{V}, \overline{W} \in \overline{\mathfrak{X}}(M)$ are extensions of V and W (*i.e.*, $\overline{V} \circ \varphi = d\varphi(V)$ and $\overline{W} \circ \varphi = d\varphi(W)$), $\overline{\nabla}$ is the Levi-Civita connection of M and $(.)^{\intercal}$ is the projection on $d\varphi\left(T_{\varphi(.)} S\right)$.

Curvature

Let (M, g) be a semi-Riemannian manifold with Levi-Civita connection. The tensor

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

given by

$$R(U, V) W = \nabla_{[U,V]} W - \nabla_U \nabla_V W + \nabla_V \nabla_U W$$

is called **Riemannian curvature tensor** of M.

For every nondegenerated plane Π of $T_p M$, we define the **sectional curvature of** Π by

$$K_M(\Pi) = \frac{g(R(v, w) v, w)}{g(v, v) g(w, w) - g(v, w)^2}, \tag{1.3}$$

where v, w is a basis of Π. It is possible to prove that $K_M(\Pi)$ is well defined and it does not depend on the choice of the basis.

The sectional curvature of Euclidean n-space and Lorentzian n-space are null for every Π and the sectional curvature of Hyperbolic n-space is -1 for every Π.

1.3 Semi-Riemannian immersions and Hypersurfaces

Let $\varphi : S \hookrightarrow M$ be a nondegenerated immersion of a differential manifold S on a semi-Riemannian manifold (M, g). The **second fundamental form of** φ is the symmetric tensor given by

$$\sigma : (V, W) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \left(\overline{\nabla}_V W\right)^{\perp} \in \mathfrak{X}(M)$$

where $(.)^{\perp}$ is the projection on the orthogonal complement of $d\varphi\left(T_{\varphi(.)}S\right)$ in $T_{\varphi(.)}M$.

A particular relevant case of semi-Riemannian immersion is

$$\dim M = \dim S + 1.$$

In this case, S is said to be a semi-Riemannian hypersurface of M.

Consider a local unit normal vector field $\mathcal{N}$ on a semi-Riemannian hypersurface S of M. Using the canonical identification between $T_{(.)}S$ and $d\varphi\left(T_{\varphi(.)}S\right)$ we can define the self-adjoint operator (named **shape operator**)

$$\mathcal{A} : V \in \mathfrak{X}(S) \longrightarrow -\overline{\nabla}_V N \in \mathfrak{X}(S). \qquad \text{(Weingarten Formula)}$$

It is possible to show that

$$\langle \mathcal{A}(V), W \rangle = \langle \sigma(V, W), N \rangle$$

for all $V, W \in \mathfrak{X}(S)$.

When the hypersurface is spacelike, the shape operator is diagonalizable (this conclusion is guaranteed by Real Spectral Theorem). In this case, its eingenvalues are called **principal curvatures**.

An important result relating the shape operator and sectional curvatures is the Gauss equation:

$$K_S(v, w) = K_M(d\varphi(v), d\varphi(w)) + \varepsilon \frac{g(d\varphi(\mathcal{A}v), d\varphi(v))g(d\varphi(\mathcal{A}w), d\varphi(w)) - g(d\varphi(\mathcal{A}v), d\varphi(w))^2}{g(d\varphi(v), d\varphi(v))g(d\varphi(w), d\varphi(w)) - g(d\varphi(v), d\varphi(w))^2} \qquad (1.4)$$

where $v, w \in T_x S$ and $g(d\varphi\mathcal{N}, d\varphi\mathcal{N}) = \varepsilon$.

Now let us concentrate our atention on the particular case where $\dim S = 2$ and $\dim M = 3$ (this type of hypersurface is simply called **surface**). For a coordinate system u, v in S the components of the metric tensor are traditionally denoted by

$$E = g\left(d\varphi\partial_u, d\varphi\partial_u\right) \qquad F = g\left(d\varphi\partial_u, d\varphi\partial_v\right) \qquad G = g\left(d\varphi\partial_v, d\varphi\partial_v\right)$$

where for indexing purposes $u = u^1$ and $v = u^2$.

The coefficients of the second fundamental form σ with respect to basis $\mathcal{B} = \{\partial_u, \partial_v\}$ is expressed as

$$\begin{aligned} e &= \varphi^* g\left(\mathcal{A}\partial_u, \partial_u\right) = -g\left(\mathcal{N}, \left(\nabla_{\partial_u}\partial_u\right)^{\perp}\right); \\ f &= \varphi^* g\left(\mathcal{A}\partial_u, \partial_v\right) = -g\left(\mathcal{N}, \left(\nabla_{\partial_u}\partial_v\right)^{\perp}\right); \\ g &= \varphi^* g\left(\mathcal{A}\partial_v, \partial_v\right) = -g\left(\mathcal{N}, \left(\nabla_{\partial_v}\partial_v\right)^{\perp}\right), \end{aligned}$$

therefore the Gaussian and mean curvatures of a non degenerated surface locally parametrized by an immersion φ are respectively given by

$$K = \varepsilon \frac{eg - f^2}{EG - F^2} \qquad \text{and} \qquad H = \varepsilon \frac{eG - 2fF + gE}{2EG - F^2}, \tag{1.5}$$

where $\varepsilon = g\left(\mathcal{N}, \mathcal{N}\right)$.

The previous formula can be rewritten as follows: Denote $\omega = EG - F^2$, where $\omega > 0$ if the surface is spacelike and $\omega < 0$ if the surface is timelike. Thus, consider the normal vector

$$\mathcal{N} = \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|_{\mathbb{L}}},$$

and we remark that $\|\varphi_u \times \varphi_v\|_{\mathbb{L}} = \sqrt[2]{|EG - F^2|} = -\varepsilon\omega$. Therefore, the formula (1.5) is expressed by

$$K = -\frac{eg - f^2}{\omega^2} \qquad \text{and} \qquad H = -\frac{eG - 2fF + gE}{2\left(-\varepsilon\omega\right)^{\frac{3}{2}}}. \tag{1.6}$$

As mentioned before, the diagonalization of the Weingarten map is guaranteed only for spacelike surfaces. On the other hand, Gaussian and mean curvatures can be computed for every causality. Henceforth, we will call **principal curvatures** every continuous solution of the system

$$k_1 k_2 = \varepsilon K \qquad \text{and} \qquad k_1 + k_2 = 2\varepsilon H. \tag{1.7}$$

Note that, in the Riemannian case and for spacelike surfaces, the above definition of principal curvatures agrees with the usual ones.

Chapter 2

Weingarten Tubular Surfaces

In this chapter we will classify Polynomial Weingarten tubular surfaces. In the aim to accomplish that, in the Section 2.1 we will introduce the notion of tubular surfaces and discuss the nuances whithin such geometrical objects immersed in different environments ($\mathbb{E}^3$, $\mathbb{L}^3$ and $\mathbb{H}^3$).

Once the definition and characterization of tubular surface is well stablished, we will change our focus and present the Section 2.2 which is an investigation of polynomial results. More precisely, for a given polynomial $P(x,y) \in \mathbb{R}[x,y]$, we will study conditions of its coefficients in order to determine when $P(x,y)$ belongs to the ideal in $\mathbb{R}[x,y]$ generated by $xr^2 - 2yr + 1$, for some $r > 0$.

Proceeding through this chapter, in the Section 2.3 we fully classificate Polynomial Weingarten tubular surface and present several applications. Moreover, in Theorem 54 we describe every tubular surface whose Gaussian and mean curvatures verify a given polynomial. Conversely, in Theorem 61, fixed a tubular surface, we provide families of polynomials that verify $P(K,H) \equiv 0$.

2.1 Tubular Surfaces

This section was elaborated to introduce tubular surfaces and discuss the nuances whithin such geometrical objects immersed in different environments.

In the section's first part, we briefly discuss the well-known class of tubular surface in Euclidean space, it is also exhibited the condition of regularity and it is presented the Gaussian and mean curvatures. In contrast, the second and third part, we investigate more profoundly the particularities of each space.

In the second part that correspondes to the Lorentzian 3-space, we analyze the notion of "circle" and give a more wide conception of circle that is allowed by the non definite positive scalar product. Indeed, the index 1 in metric of the $\mathbb{L}^3$ generates several types of tubular surfaces, possibly with no circular orthogonal sections that are not isometric between them (since isometries preserves causality).

Besides that, the Lorentzian Weingarten map is not always diagonalizable, therefore the existence of principal curvatures is no longer guaranteed. We overcome this difficulty by providing natural functions with similar properties (see (1.5) and (1.7) for more details).

Finally, we present the regularity condition for tubular surfaces in the Lorentzian 3-space, also the Gaussian and mean curvatures are computed and the principal curvatures are presented.

The final part of the chapter concerns about the Hyperbolic 3-space. We study the conditions to a tubular surface belong to $\mathbb{H}^n$ and how it influences on its parametrization. Therefore, we present the regularity condition and present the Gaussian and mean curvature. In this space form, we also have the Sectional curvature.

2.1.1 Euclidean Tubular Surfaces

In the Euclidean 3-space, a **tubular surface** of radius $r > 0$ around a regular curve $\gamma : (a, b) \to \mathbb{R}^3$, called **central curve**, is the set obtained by the union of all circles $S_r(\gamma(s))$ of radius r and center $\gamma(s)$ contained in the normal planes $T_s\gamma^{\perp}$ of γ.

In intervals where γ is birregular, a tubular surface can be parametrized by the application:

$$\psi : (s, t) \in (a, b) \times \mathbb{R} \longmapsto \gamma(s) + r\cos(t)\, N(s) + r\sin(t)\, B(s) \in \mathbb{R}^3. \qquad (2.1)$$

In particular, if the central curve is a straight line, the Frenet Frame can be regarded as a trivial orthogonal frame.

Proposition 21 *Given a tubular surface in Euclidean 3-space of radius $r > 0$, consider an interval I where the central curve γ is birregular. The parametrization (2.1) is an immersion if and only if*

$$\xi_{\mathbb{E}}(s, t) = 1 - r\kappa(s)\cos t \neq 0, \qquad \textit{for every } (s, t) \in (a, b) \times \mathbb{R}.$$

Proof. Using (1.1) and (2.1) we obtain

$$\psi_t = -r\sin(t)\, N(s) + r\cos(t)\, B(s) \qquad \text{and} \qquad \psi_s = \xi_{\mathbb{E}} T + \tau\psi_t.$$

Therefore, the vectors ψ_s and ψ_t are linearly independent if and only if $\xi_{\mathbb{E}} \neq 0$, for every $(s, t) \in I \times \mathbb{R}$. ■

The above regularity condition provides that $\xi_{\mathbb{E}} > 0$ everywhere, since $\xi_{\mathbb{E}}\left(., \frac{\pi}{2}\right) = 1$.

Definition 22 *A tubular surface is called* ***regular*** *if γ is parametrized by arc length and*

$$\xi_{\mathbb{E}}(s, t) > 0, \qquad \textit{for every } (s, t) \in (a, b) \times \mathbb{R}.$$

Proposition 23 *Consider a regular tubular surface of radius $r > 0$ in Euclidean 3-space and let $\gamma : (a, b) \to \mathbb{R}^3$ be its central curve. The principal curvatures of the tubular surface are*

$$k_1 = -\frac{\kappa \cos t}{\xi_{\mathbb{E}}} \qquad \textit{and} \qquad k_2 = \frac{1}{r}.$$

Hence the Gaussian and Mean curvatures are respectively:

$$K = -\frac{\kappa \cos t}{r\xi_{\mathbb{E}}} \qquad \textit{and} \qquad H = -\frac{2r\kappa \cos t - 1}{2r\xi_{\mathbb{E}}}$$

Proof. In intervals where the central curve γ is birregular, we can use the parametrization ψ in (2.1) to obtain the coefficients of the first and second fundamental forms which are

$$E = \xi_{\mathbb{E}}^2 + (r\tau)^2 \qquad F = \tau r^2 \qquad G = r^2$$

$$e = r\tau^2 - \kappa\xi_{\mathbb{E}} \cos t \qquad f = r\tau \qquad g = r.$$

Thus, on the image of ψ, the Gaussian and mean curvature of the tubular surface are respectively indicate as

$$K = -\frac{\kappa \cos t}{r\xi_{\mathbb{E}}} \qquad \text{and} \qquad H = -\frac{2r\kappa \cos t - 1}{2r\xi_{\mathbb{E}}}$$

and the principal curvatures are

$$k_1 = -\frac{\kappa \cos t}{\xi_{\mathbb{E}}} \qquad \text{and} \qquad k_2 = \frac{1}{r}.$$

■

Note that above curvatures are also valid in intervals where the curvature κ of γ is null (*i.e.* where the surface is a right cylinder). By continuity, it is concluded that expressions are true on the entire tubular surface.

Finally, observe that $k_1 \leq k_2$ once $\xi_{\mathbb{E}} > 0$ and

$$r\kappa \cos t > r\kappa \cos t - 1 = -\xi_{\mathbb{E}}$$

which implies

$$\frac{r\kappa \cos t}{-\xi_{\mathbb{E}}} < 1$$

and the statement is concluded.

2.1.2 Lorentzian Tubular Surfaces

Remember from Section 1.1 that Lorentzian 3-space, denoted by $\mathbb{L}^3$, is the semi-Riemannian manifold obtained endowing $\mathbb{R}^3$ with the bilinear form of index 1

$$g_{\mathbb{L}}\left((x_1, x_2, x_3), (y_1, y_2, y_3)\right) = x_1y_1 + x_2y_2 - x_3y_3,$$

where $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$.

As a means to parametrize tubular surface in $\mathbb{L}^3$, it is necessary discuss the notion of a tubular surface as the set obtained by moving a circle along a central curve (see 2.1.1).

First, a circle (*e.g.* in Euclidean 3-space) is the set of all points contained in a plane whose distance (called radius) from a fixed point (called center) is constant. This definition is usually stablished with a norm $\|.\|$ which is positive definite (recall the Lorentzian norm is $\|x\|_{\mathbb{L}} = \sqrt{|g_{\mathbb{L}}(x,x)|}$, hence always positive). However this approach not fully utilizes the possibilities provided by a scalar product with index (as the notion of negative distance). It motivates a generalized idea of circle, named:

i. A **Lorentzian circle** is the set

$$S_r(c) = \left\{x \in \mathbb{L}^3 \ ; \ g_{\mathbb{L}}(x-c, x-c) = r^2\right\};$$

ii. A **Lorentzian hyperboles** is the set

$$S_r(c) = \left\{x \in \mathbb{L}^3 \ ; \ g_{\mathbb{L}}(x-c, x-c) = -r^2\right\}.$$

Remark 24 *We point out some authors may define the same previous sets but add the prefix 'pseudo' in the name.*

In Lorentzian 3-space the Lorentzian circle and Lorentzian hyperboles plays the same role as Euclidean spheres in $\mathbb{E}^3$, once the Gauss map has these as codomain.

Finally note that Lorentzian hyperboles are always Euclidean hyperboles, but Lorentzian circles can be Euclidean circles or Euclidean hyperboles. With the new realization of circles, we are able to define a tubular surface in Lorentzian 3-space.

In the Lorentzian 3-space, a **tubular surface** of radius $r > 0$ around a regular curve $\gamma : (a,b) \to \mathbb{R}^3$, called **central curve**, is the set obtained by the union of all Lorentzian circles or Lorentzian hyperboles $S_r(\gamma(s))$ of radius r and center $\gamma(s)$ contained in the normal planes $T_s\gamma^{\perp}$ of γ.

It is important to emphasize that condition of γ be regular is needed to existence of well-defined normal subspace $T_s\gamma^{\perp}$.

In intervals where the central curve $\gamma \subset \mathbb{L}^3$ is biregular, the tubular surface admits a local parameterization that depends on the normal section, also depends on the causality of the central curve and causality of the principal normal, once each of these qualities impacts on Frenet formula. More precisely:

Let (a,b) be an interval where γ is birregular and consider the abstract parametrization of a tubular surface as the envelope of circles (*i.e.* set obtained by moving a circle along a central curve)

$$\psi(s,t) = \gamma(s) + f(s,t)\,T(s) + g(s,t)\,N(s) + h(s,t)\,B(s)$$

where $\{T(s), N(s), B(s)\}$ is the Frenet frame of γ and $f(s,t)$, $g(s,t)$ and $h(s,t)$ are smooth functions defined in (a,b). Our objective is to explicitly express those functions.

In order to do that, first notice that for each $s_0 \in (a,b)$ we want to describe a circle of radius εr^2 and center $\gamma(s_0)$ contained in the normal plane $T_{s_0}\gamma^{\perp}$, that is

$$\varepsilon_T f^2 + \varepsilon_N g^2 - \varepsilon_T \varepsilon_N h^2 = g_{\mathbb{L}}\left(\psi(s_{0,}t) - \gamma(s_0), \psi(s_{0,}t) - \gamma(s_0)\right) = \varepsilon r^2 \quad (2.2)$$

where $\varepsilon \in \{-1, 1\}$ fixed constant. The derivative in parameter s of these previous paramatrization and equation gives us

$$\varepsilon_T f f_s + \varepsilon_N g g_s - \varepsilon_T \varepsilon_N h h_s = 0 \quad (2.3)$$

and

$$\psi_s = (1 + f_s - \varepsilon_N \varepsilon_T g\kappa) T + (g_s + f\kappa + \varepsilon_T h\tau) N + (h_s + g\tau) B. \quad (2.4)$$

Then, observe that $\psi - \gamma$ is a normal vector to the tubular surface, thus we have

$$0 = g_{\mathbb{L}}(\psi - \gamma, \psi_s) = \varepsilon_T f. \quad (2.5)$$

Applying the condition (2.5) in Equation (2.3) it is obtained the following system

$$\varepsilon_N g^2 - \varepsilon_T \varepsilon_N h^2 = \varepsilon r^2,$$

whose solutions are given by chosening the causality of central curve (ε_T), of principal normal (ε_N) and the normal section (ε).

The table below contain all possibilities:

Curva	Normal	Section	Parametrization
spacelike	spacelike	Lorentzian circles	$\gamma \pm r\cosh(t)N + r\sinh(t)B$
spacelike	timelike	Lorentzian circles	$\gamma + r\sinh(t)N \pm r\cosh(t)B$
timelike	spacelike	Lorentzian circles	$\gamma + r\cos(t)N + r\sin(t)B$
spacelike	spacelike	Lorentzian hyperboles	$\gamma + r\sinh(t)N \pm \cosh(t)B$
spacelike	timelike	Lorentzian hyperboles	$\gamma \pm \cosh(t)N + r\sinh(t)B$
timelike	spacelike	Lorentzian hyperboles	Does not exist

Table 2.1: Table of Parametrizations

In the aim to encompass and deal with all possible local parametrizations simultaneosly, we will use a pair of functions $(\mu(t), \eta(t))$ to represent one of the followings pairs

$$(\delta\cos t, \sin t),\ (\delta\cosh t, \sinh t),\ (\sinh t, \delta\cosh t). \quad (2.6)$$

where $\delta = \pm 1$. Then, in intervals where the central curve $\gamma \subset \mathbb{L}^3$ is biregular, the parametrizations presented in Table 2.1 can always be rewritten as

$$\psi(s,t) = \gamma(s) + r\mu(t) N(s) + r\eta(t) B(s). \quad (2.7)$$

For calculations purposes, it is observed that derivatives of $\mu(t)$, $\eta(t)$ behaves as follow

$$\mu'(t) = \delta\varepsilon_T \eta(t) \qquad \text{and} \qquad \eta'(t) = \delta\mu(t).$$

Definition 25 *A Lorentzian tubular surface is said* ***regular*** *when γ is birregular and*

$$\xi_{\mathbb{L}}(s,t) = 1 + \varepsilon_B r\kappa(s)\mu(t) \neq 0, \qquad \textit{for every } (s,t) \in (a,b). \tag{2.8}$$

Despite the above condition be very similar to the Euclidean case, it is not possible determine the signal of an arbitrary $\xi_{\mathbb{L}}$, since $\mu(t)$ can be choosen as any function in (2.6).

Proposition 26 *Given a tubular surface in Lorentzian 3-space of radius $r > 0$, consider an interval I where the central curve γ is birregular. The parametrization (2.7) is an immersion if and only if $\xi_{\mathbb{L}} \neq 0$, for every $(s,t) \in I \times \mathbb{R}$.*

Proof. Using (1.1), (2.7) and (1.2) we obtain

$$\psi_t = r\eta\varepsilon_T N + r\mu B \qquad \text{and} \qquad \psi_s = \xi_{\mathbb{L}} T + \delta\tau\psi_t.$$

Therefore, the vectors ψ_s and ψ_t are linearly independent if and only if $\xi_{\mathbb{L}} \neq 0$, for every $(s,t) \in I \times \mathbb{R}$. ■

As mentioned before, self-adjoint endomorphisms are only diagonalizable with the additional condition that the scalar product is positive definite. Therefore, in the Lorentzian 3-space the Lorentzian Weingarten map is not always diagonalizable, hence the existence of (usual) principal curvatures is no longer guaranteed.

We overcome this difficulty by providing natural functions with similar properties (see (1.7)), which for convenience of the reader, it will be re-stated:

The continuous functions k_1 and k_2 are called principal curvatures if they verify the system

$$k_1 k_2 = \varepsilon K \qquad \text{and} \qquad k_1 + k_2 = 2\varepsilon H.$$

where K and H are, respectively, Gaussian and mean curvatures, $\mathcal{N}$ is the normal vector to the surface and $\varepsilon = g_{\mathbb{L}}(\mathcal{N}, \mathcal{N})$.

For us, the extended notion of principal curvatures are important because allows us to demonstrate our main theorem, as we will see in Chapter 2.3.

So the next proposition provides the Gaussian and mean curvatures for a regular tubular surface in Lorentzian 3-space (which always can be computed) and exhibits their principal curvatures as well.

Proposition 27 *Consider a regular tubular surface of radius $r > 0$ in Lorentzian 3-space an let $\gamma : (a,b) \in \mathbb{L}^3$ be its central curve. The Gaussian and Mean curvature of S are respectively:*

$$K = \varepsilon \frac{\kappa\mu\varepsilon_B}{r\xi_{\mathbb{L}}} \qquad \textit{and} \qquad H = \varepsilon \frac{\xi_{\mathbb{L}} + r\kappa\mu\varepsilon_B}{r\xi_{\mathbb{L}}}. \tag{2.9}$$

In particular, the principal curvatures of the tubular surface are given by

$$k_1 = \frac{\kappa\mu\varepsilon_B}{\xi_{\mathbb{L}}} \qquad \text{and} \qquad k_2 = \frac{1}{r}. \tag{2.10}$$

Proof. In intervals where the central curve γ is birregular, consider the generic parametrization in (2.7). In this case, the normal vector of the tubular surface is

$$\mathcal{N} = -\mu N - \eta B$$

and the coefficients of first and the second fundamental forms are

$$E = \varepsilon_T \xi_{\mathbb{L}}^2 - \varepsilon\varepsilon_T (r\tau)^2, \qquad F = -\varepsilon\varepsilon_T \delta\tau r^2, \qquad G = -\varepsilon\varepsilon_T r^2$$

$$e = \varepsilon_T \varepsilon_B \kappa\mu\xi_{\mathbb{L}} - \varepsilon\varepsilon_T r\tau^2, \qquad f = -\varepsilon\varepsilon_T \delta r\tau, \qquad g = -\varepsilon\varepsilon_T r,$$

where $\varepsilon = g_{\mathbb{L}}(\mathcal{N}, \mathcal{N})$.

Thus, on the image of ψ, the Gaussian and mean curvature of the tubular surface are respectively indicate as

$$K = \varepsilon \frac{\kappa\mu\varepsilon_B}{r\xi_{\mathbb{L}}} \qquad \text{and} \qquad H = \varepsilon \frac{\xi_{\mathbb{L}} + r\kappa\mu\varepsilon_B}{r\xi_{\mathbb{L}}}$$

and the principal curvatures are

$$k_1 = \frac{\kappa\mu\varepsilon_B}{\xi_{\mathbb{L}}} \qquad \text{and} \qquad k_2 = \frac{1}{r}.$$

■

Note that above curvatures are also valid in intervals where the curvature κ of γ is null (*i.e.* where the surface is a right cylinder). By continuity, it is concluded that expressions are true on the entire tubular surface.

Remark 28 *When $\mu(t) = \delta \cos t$ or $\mu(t) = \sinh h(t)$ we must have $\xi_{\mathbb{L}} > 0$. As in Euclidean case, this implies that $k_1 < k_2$. However, when $\mu(t) = \delta \cosh t$, the sign of $\xi_{\mathbb{L}}$ (and the order of the principal curvtures) depends on the sign of δ. Note that, when $\xi_{\mathbb{L}} < 0$, the central curve cannot have points where the curvature is zero.*

To avoid ambiguity, from now on the principal curvatures k_1 and k_2 will always correspond to the expressions in (2.10).

2.1.3 Hyperbolic Tubular Surfaces

The hyperbolic space is extensively studied once it was the first sample of non-Euclidean geometry (*i.e.* a space obtained by revoking the famous fifth axiom of Euclides). Moreover, this space has several model that are isometric between them, for instance: Poincaré half plane model, Poincaré ball model, Beltrami–Klein model.

For the purpose of explore tubular surfaces, a suitable model is the spacelike immersion on semi-Riemannian manifold $\mathbb{L}^4$:

$$\mathbb{H}^3 = \left\{x \in \mathbb{L}^4 \ ; \ g_{\mathbb{L}}(x,x) = -1 \text{ where } x_4 > 0\right\}$$

endowed with the metric induced by the canonical inclusion.

In the Hyperbolic 3-space, a **tubular surface** of radius $r > 0$ around a regular curve $\gamma : (a,b) \to \mathbb{H}^3$, called **central curve**, is the set obtained by the union of all circles $S_r(\gamma(s))$ of radius r and center $\gamma(s)$ contained in the normal planes $T_s\gamma^{\perp}$ of γ.

It is important to emphasize that condition of γ be regular is needed to existence of well-defined normal subspace $T_s\gamma^{\perp}$.

Let (a,b) be an interval where γ is birregular and consider the abstract parametrization of a tubular surface as the envelope of circles (*i.e.* set obtained by moving a circle along a central curve)

$$\psi(s,t) = \gamma + f(s,t)\,T(s) + g(s,t)\,N(s) + h(s,t)\,B(s) \tag{2.11}$$

where $\{T(s), N(s), B(s)\}$ is the Frenet frame of γ and $f(s,t)$, $g(s,t)$ and $h(s,t)$ are smooth functions defined in (a,b). Our objective is to explicitly express those functions.

In order to do that, first notice that for each $s_0 \in (a,b)$ we want to describe a circle of radius r^2 and center $\gamma(s_0)$ contained in the normal plane $T_{s_0}\gamma^{\perp}$, that is,

$$f^2 + g^2 + h^2 = g_{\mathbb{L}}\left(\psi(s_{0,}t) - \gamma(s_0), \psi(s_{0,}t) - \gamma(s_0)\right) = r^2$$

The derivative in parameter s of these previous paramatrization and equation gives us

$$ff_s + gg_s + hh_s = 0 \tag{2.12}$$

and

$$\psi_s = (f_s - g\kappa + 1)\,T + (g_s + f\kappa - h\tau)\,N + (h_s + g\tau)\,B. \tag{2.13}$$

Then, observe that $\psi - \gamma$ is a normal vector to the tubular surface, thus we have

$$0 = g_{\mathbb{L}}(\psi - \gamma, \psi_s) = f(s,t). \tag{2.14}$$

Applying the condition (2.14) in Equation (2.12) it is obtained the following system

$$g^2 + h^2 = r^2$$

whose solutions are given by

$$g(s,t) = r\cos t \qquad \text{and} \qquad h(s,t) = r\sin t.$$

Therefore the parametrization is

$$\psi(s,t) = \gamma + r\cos t N(s) + r\sin t B(s), \tag{2.15}$$

however, notice that $\psi(s,t)$ must belong to H^3 for every $(s,t) \in I \times \mathbb{R}$, which yields

$$-1 = g_{\mathbb{L}}(\psi(s,t), \psi(s,t)) = -1 + r^2$$

since $r > 0$, we do not achieved the desired (*i.e.* $\psi \notin H^3$). To correct this problem, we add (*a priori*) functions λ_1 and λ_2 in (2.11) (*a posteriori*, it will be proved they are constant) such that

$$-1 = g_{\mathbb{L}}(\psi(s,t), \psi(s,t)) = -\lambda_1 + \lambda_2 r^2.$$

Thus, the solutions are

$$\lambda_1 = -\cosh r \qquad \text{and} \qquad \lambda_2 = \frac{1}{r}\sinh r. \tag{2.16}$$

In the view of above discussion, the parametrization of a tubular surface of radius r in Hyperbolic 3-space is given by

$$\psi(s,t) = (\cosh r)\,\gamma + \sinh r\,(\cos tN + \sin tB)\,. \tag{2.17}$$

Proposition 29 *Given a tubular surface in Hyperbolic* 3*-space of radius* $r > 0$*, consider an interval* I *where the central curve* γ *is birregular. The parametrization (2.17) is an immersion if and only if*

$$\xi_{\mathbb{H}}(s,t) = \cosh r - \kappa\cos t\sinh r \neq 0, \qquad \textit{for every } (s,t) \in (a,b) \times \mathbb{R}.$$

Proof. Using (1.1) and (2.17) we obtain

$$\psi_t = -\sinh r\sin tN(s) + \sinh r\cos tB(s) \qquad \text{and} \qquad \psi_s = \xi_{\mathbb{H}}T + \tau\psi_t.$$

Therefore, the vectors ψ_s and ψ_t are linearly independent if and only if $\xi_{\mathbb{H}} \neq 0$, for every $(s,t) \in I \times \mathbb{R}$. ∎

The above regularity condition provides that $\xi_{\mathbb{H}} > 0$ everywhere, since

$$\cosh r - \kappa\cos t\sinh r = 0 \Longleftrightarrow 1 - \kappa\cos t\tanh r = 0$$

hence $\xi_{\mathbb{H}}\left(., \frac{\pi}{2}\right) = 1$.

Definition 30 *A tubular surface is called* ***regular*** *if* γ *is parametrized by arc length and*

$$\xi_{\mathbb{H}}(s,t) > 0, \qquad \textit{for every } (s,t) \in (a,b) \times \mathbb{R}.$$

Proposition 31 *Consider a regular tubular surface of radius* $r > 0$ *in Euclidean* 3*-space an let* $\gamma : (a,b) \in \mathbb{L}^3$ *be its central curve. The principal curvatures of the tubular surface are*

$$k_1 = -\frac{\kappa\cos t}{\xi_{\mathbb{H}}} \qquad \textit{and} \qquad k_2 = \frac{1}{\sinh r}.$$

Hence the Gaussian and Mean curvatures are respectively:

$$K = -\frac{\kappa\cos t}{\xi_{\mathbb{H}}\sinh r} \qquad \textit{and} \qquad H = \frac{\cosh r - 2\kappa\cos t\sinh r}{2\xi_{\mathbb{H}}\sinh r}$$

Proof. In intervals where the central curve γ is birregular, we can use the parametrization ψ in (2.1) to obtain the coefficients of the first and second fundamental forms which are

$$E = \xi_{\mathbb{H}}^2 + (\tau \sinh r)^2 \qquad F = \tau \sinh^2 r \qquad G = \sinh^2 r$$
$$e = \tau^2 \sinh r - \kappa \xi_{\mathbb{H}} \cos t \qquad f = \tau \sinh r \qquad g = \sinh r.$$

Thus, on the image of ψ, the Gaussian and mean curvature of the tubular surface are respectively indicate as

$$K = -\frac{\kappa \cos t}{\xi_{\mathbb{H}} \sinh r} \qquad \text{and} \qquad H = \frac{\cosh r - 2\kappa \cos t \sinh r}{2\xi_{\mathbb{H}} \sinh r}$$

and the principal curvatures are

$$k_1 = -\frac{\kappa \cos t}{\xi_{\mathbb{H}}} \qquad \text{and} \qquad k_2 = \frac{1}{\sinh r}.$$

■

Corollary 32 *The Sectional curvature of a tubular surface in Hyperbolic* 3-*space is*

$$K_S = 1 - \frac{\kappa \cos t}{\xi_{\mathbb{H}} \sinh r}.$$

Note that above curvatures are also valid in intervals where the curvature κ of γ is null (*i.e.* where the surface is a right cylinder). By continuity, it is concluded that expressions are true on the entire tubular surface.

Finally, observe that $k_1 < k_2$. Indeed, since $r > 0$ it implies that $\cosh r > 0$ and $\sinh r > 0$, and by the remark in the beginning of section, we have

$$0 < \xi_{\mathbb{H}} = \cosh r - \kappa \cos t \sinh r$$

therefore, notice that $\frac{\kappa \cos t}{\xi_H}$ is the product of positive functions, so

$$-\frac{\kappa \cos t}{\xi_{\mathbb{H}}} = k_1 < 0$$

while

$$0 < k_2 = \frac{1}{\sinh r}$$

and the statement is concluded.

2.2 Polynomials results for Tubular Surfaces

This section is devoted for analysis and characterization of polynomials of the form $Q(X, Y) \in \mathbb{R}[x]$ which is obtained by the composition of a polynomial $Q(x, y) \in \mathbb{R}[x, y]$ with the polynomials X, $Y \in \mathbb{R}[x]$ such that $X(x) = \frac{x}{r}$ and $Y(x) = \frac{xr+1}{2r}$.

The interest to study this specific type of polynomial arises narutally in the investigation of the geometric problem that is the classification of Weingarten surfaces whose polynomial relation its among the Gaussian curvature and mean curvature. More precisely, in the aim to fully describe the set of all polynomials Q such that $Q(K,H)=0$ it leads us to an algebraic problem that is finding (a reasonably simple to check) condition to determine when $Q(x,y)$ can be factorized as $\left(xr^2-2yr+1\right)R(x,y)$, where $R(x,y)\in\mathbb{R}[x,y]$. As a matter of fact, we obtain the following theorem.

Theorem 33 *Consider a polynomial $Q(x,y)\in\mathbb{R}[x,y]$ and let r be nonzero constant. Then Q belongs to the ideal generated by $xr^2-2yr+1$ if and only if $Q\left(\frac{x}{r},\frac{xr+1}{2r}\right)\in\mathbb{R}[x]$ is identically null.*

In addition to the initial incentive of this result, the above theorem is an alternative to the usual method to decide under which conditions one polynomial can be rewritten as another two polynomials, where (at least) one of them is linear.

A numerical example will show one first manner to apply the theorem.

Example 34 *Consider the polynomial*

$$\begin{aligned} Q(x,y) &= 4x^4+8x^2y^2-12xy^3 \\ &\quad +9x^3+9x^2y-9xy^2-4y^3 \\ &\quad +22x^2-8xy-7y^2 \\ &\quad -91x+98y \\ &\quad -24 \end{aligned}$$

and notice that $Q\left(\frac{x}{2},\frac{2x+1}{4}\right)\in\mathbb{R}[x]$ is identically null, it implies that

$$Q(x,y)=(4x-4y+1)R(x,y)$$

where $R(x,y)\in\mathbb{R}[x,y]$. Once the Theorem 44 already gives us one factor, the another one is "easier" to obtain through the equality of polynomials. In fact, we have that

$$R(x,y)=x^3+x^2y+3xy^2+2x^2+4xy+y^2+5x+2y-24.$$

Remark 35 *The first presented propostion of this Chapter will furnish us a proper way to express a polynomial of the form $Q(X,Y)$ in order to obtain more information about $R(x,y)$.*

Another functionality of our result is that the Theorem 44 allows us to answer in which cases a tubular surface of radius $r>0$ verifies a relation $Q(K,H)\equiv 0$ (see Chapter 2.3 for more details), named:

Theorem 36 *Consider a polynomial $Q(x,y)\in\mathbb{R}[x,y]$. If there is a non cylindrical tubular surface S whose its Gaussian and mean curvatures verify $Q(K,H)\equiv 0$, then Q has a unique factorization*

$$Q(x,y)=\left(xr^2-2yr+1\right)^m R(x,y),$$

where r is the radius of S, $m\in\mathbb{N}$ and $R(x,y)\in\mathbb{R}[x,y]$ such that $R(K,H)\not\equiv 0$.

In other words, if we have a tubular surface verifying a polynomial relation Q of degree $n \geq 1$, then Q is a multiple of a linear polynomial and the zero of $Q(K, H) = 0$ comes from the linear relation. So, we obtain the theorem stated in Introduction:

Corollary 37 *Cylinder are the unique non linear Polynomial Weingarten surfaces.*

Therefore the importance of this chapter lies on the connection established by Theorem 44 between conditions on polynomials coefficients and ideals in the polynomial ring that Q belongs to. Furthermore, this result was the crucial problem to be solved as a means to fully describe the set of all polynomials whose zeros are the Gaussian and mean curvatures of a tubular surface.

The first difficulty we face when we are trying to classify polynomials is finding a favorable expression that allows us to examine them. In this direction, a polynomial $Q \in \mathbb{R}[x, y]$ evaluated in $X(x) = \frac{x}{r}$ and $Y(x) = \frac{xr+1}{2r}$ furnishes a polynomial $Q\left(\frac{x}{r}, \frac{xr+1}{2r}\right) \in \mathbb{R}[x]$ whose expression is $\sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{i,j}\left(\frac{x}{r}\right)^i\left(\frac{xr+1}{2r}\right)^j$. Then, the following Propostion provides a reorganization of coefficients that permits us to write in a more suitable (and canonical) way $Q\left(\frac{x}{r}, \frac{xr+1}{2r}\right) = \sum_{k=0}^{n}\Gamma_k(r)x^k$ where each $\Gamma_k(r)$ are the constant coefficents.

Proposition 38 *Consider a polynomial* $Q(x, y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{i,j}x^i y^j \in \mathbb{R}[x, y]$ *and a nonzero number* r. *Then*

$$Q\left(\frac{x}{r}, \frac{xr+1}{2r}\right) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{i,j}\left(\tfrac{x}{r}\right)^i\left(\tfrac{xr+1}{2r}\right)^j = \sum_{k=0}^{n}\Gamma_k(r)x^k,$$

where

$$\Gamma_k(r) = \sum_{i=0}^{k}\left(\sum_{j=0}^{n-k}\tbinom{k-i+j}{j}\tfrac{a_{i,k-i+j}}{2^{k-i+j}r^{j+i}}\right). \tag{2.18}$$

Proof. Going along the lines of the proof, we will demonstrate

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{i,j}\left(\tfrac{x}{r}\right)^i\left(\tfrac{xr+1}{2r}\right)^j - \sum_{k=0}^{n}\Gamma_k(r)x^k = 0, \tag{2.19}$$

by induction on n. Notice that the above relation is verified for $n = 1$. Indeed, a straightforward calculation provides

$$\sum_{i=0}^{1}\sum_{j=0}^{1-i} a_{i,j}\left(\tfrac{x}{r}\right)^i\left(\tfrac{xr+1}{2r}\right)^j = a_{0,0} + \tfrac{1}{2r}a_{0,1} + \tfrac{1}{2}a_{0,1}x + \tfrac{1}{r}a_{1,0}x = \sum_{k=0}^{1}\Gamma_k(r)x^k$$

which concludes this case. So assume that the relation (2.19) is true for some $n \in \mathbb{N}$, *i.e.*,

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{i,j}\left(\frac{x}{r}\right)^{i}\left(\tfrac{xr+1}{2r}\right)^{j} - \sum_{k=0}^{n}\Gamma_k(r)\,x^k = 0,$$

and consider the case $n+1$:

$$\sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i} a_{i,j}\left(\tfrac{x}{r}\right)^{i}\left(\tfrac{xr+1}{2r}\right)^{j} - \sum_{k=0}^{n+1}\Gamma_k(r)\,x^k.$$

Replacing the definition of $\Gamma_k(r)$ as in (2.18) and by Proposition 127 & Proposition 128 the previous equation can be rewritten as

$$\sum_{i=0}^{n+1} a_{i,n+1-i}\left(\tfrac{x}{r}\right)^{i}\left(\tfrac{xr+1}{2r}\right)^{n+1-i} - \sum_{k=0}^{n+1}\sum_{i=0}^{k}\left(\binom{n+1-i}{n+1-k}\tfrac{a_{i,n+1-i}}{2^{n+1-i}r^{n+1-k+i}}\right)x^k$$
$$+\sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{i,j}^{i}\left(\tfrac{x}{r}\right)^{i}\left(\tfrac{xr+1}{2r}\right)^{j} - \sum_{k=0}^{n}\sum_{i=0}^{k}\left(\sum_{j=0}^{n-k}\binom{k-i+j}{j}\tfrac{a_{i,k-i+j}}{2^{k-i+j}r^{j+i}}\right)x^k$$

so the induction step implies

$$\sum_{i=0}^{n+1} a_{i,n+1-i}\left(\frac{x}{r}\right)^{i}\left(\tfrac{xr+1}{2r}\right)^{n+1-i} - \sum_{k=0}^{n+1}\sum_{i=0}^{k}\binom{n+1-i}{n+1-k}\tfrac{a_{i,n+1-i}}{2^{n+1-i}r^{n+1-k+i}}x^k \quad (2.20)$$
$$= \sum_{i=0}^{n+1}\sum_{k=0}^{n+1-i}\binom{n+1-i}{k}\tfrac{a_{i,n+1-i}}{2^{n+1-i}r^{i+k}}x^{n+1-k} - \sum_{k=0}^{n+1}\sum_{i=0}^{k}\binom{n+1-i}{n+1-k}\tfrac{a_{i,n+1-i}}{2^{n+1-i}r^{n+1-k+i}}x^k.$$

Before proceeding with the calculus of (2.20), let us focus our attention in each of the terms above separately and rewrite them in a suitable way as follows:

$$\sum_{i=0}^{n+1}\sum_{k=0}^{n+1-i}\binom{n+1-i}{k}\tfrac{a_{i,n+1-i}}{2^{n+1-i}r^{i+k}}x^{n+1-k}$$
$$= \sum_{i=0}^{n}\sum_{k=1}^{n+1-i}\binom{n+1-i}{k}\tfrac{a_{i,n+1-i}}{2^{n+1-i}r^{i+k}}x^{n+1-k} + \sum_{i=0}^{n}\tfrac{a_{i,n+1-i}}{2^{n+1-i}r^{i}}x^{n+1} + \tfrac{a_{n+1,0}}{r^{n+1}}x^{n+1}$$

therefore, we have

$$\sum_{i=0}^{n+1}\sum_{k=0}^{n+1-i}\binom{n+1-i}{k}\tfrac{a_{i,n+1-i}}{2^{n+1-i}r^{i+k}}x^{n+1-k}$$
$$= \sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n+1-i}{k+1}\tfrac{a_{i,n+1-i}}{2^{n+1-i}r^{i+k+1}}x^{n-k} + \sum_{i=0}^{n+1}\tfrac{a_{i,n+1-i}}{2^{n+1-i}r^{i}}x^{n+1}$$

by Proposition 125 it is obtained

$$\begin{aligned}
&\sum_{i=0}^{n+1}\sum_{k=0}^{n+1-i}\binom{n+1-i}{k}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{i+k}}x^{n+1-k} \\
= &\sum_{i=0}^{n}\sum_{k=0}^{i}\binom{i+1}{k+1}\frac{a_{n-i,i+1}}{2^{i+1}r^{n-i+k+1}}x^{n-k}+\sum_{i=0}^{n+1}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{i}}x^{n+1}
\end{aligned}$$

and the Proposition 126 gives us

$$\begin{aligned}
&\sum_{i=0}^{n+1}\sum_{k=0}^{n+1-i}\binom{n+1-i}{k}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{i+k}}x^{n+1-k} \\
= &\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{n+1-i}{n+1-k}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{n+1-k+i}}x^{k}+\sum_{i=0}^{n+1}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{i}}x^{n+1}.
\end{aligned} \tag{2.21}$$

Now, consider the other term that can be expressed as

$$\begin{aligned}
&\sum_{k=0}^{n+1}\sum_{i=0}^{k}\binom{n+1-i}{n+1-k}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{n+1-k+i}}x^{k} \\
= &\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{n+1-i}{n+1-k}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{n+1-k+i}}x^{k}+\sum_{i=0}^{n+1}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{i}}x^{n+1}.
\end{aligned} \tag{2.22}$$

Given the further explored expressions (2.21) and (2.22) we are able to return to the Equation (2.20) and obtain the desired conclusion:

$$\begin{aligned}
&\sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i}a_{i,j}\left(\frac{x}{r}\right)^{i}\left(\frac{xr+1}{2r}\right)^{j}-\sum_{k=0}^{n+1}\Gamma_{k}\left(r\right)x^{k} \\
= &\sum_{i=0}^{n}\sum_{k=0}^{k}\binom{(n-i)+1}{(n-k)+1}\frac{a_{i,n-i+1}}{2^{n-i+1}r^{i+n-k+1}}x^{k}+\sum_{i=0}^{n+1}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{i}}x^{n+1} \\
&-\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{n+1-i}{n+1-k}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{n+1-k+i}}x^{k}-\sum_{i=0}^{n+1}\frac{a_{i,n+1-i}}{2^{n+1-i}r^{i}}x^{n+1} \\
= &\ 0
\end{aligned}$$

■

As mentioned before, we will revisit the Example 34 to present how the above Proposition 38 associated with the Theorem 44 allows us to gather more information about a polynomial Q and explain how to choose r to test whether $Q\left(\frac{x}{r},\frac{xr+1}{2r}\right)$ is identically null or not.

Example 39 *Consider the polynomial*

$$\begin{aligned} Q(x,y) &= 4x^4+8x^2y^2-12xy^3 \\ &\quad +9x^3+9x^2y-9xy^2-4y^3 \\ &\quad +22x^2-8xy-7y^2 \\ &\quad -91x+98y \\ &\quad -24 \end{aligned}$$

presented in the Example 34. Without further explanation we took $r=2$ and verify that $Q\left(\frac{x}{2},\frac{2x+1}{4}\right)\equiv 0$.

The first matter in question is: What did we base ourselves on for choosing the r and what were the possible choices for that?

To answer that, we need to consider the polynomial $Q\left(\frac{x}{r},\frac{xr+1}{2r}\right)\in\mathbb{R}[x]$ which by Proposition 38 can be rewritten as

$$Q\left(\tfrac{x}{r},\tfrac{xr+1}{2r}\right)=\Gamma_4(r)\,x^4+\Gamma_3(r)\,x^3+\Gamma_2(r)\,x^2+\Gamma_1(r)\,x+\Gamma_0(r)\,,$$

where $\Gamma_k(r)$ are as in 2.18. To obtain the possible candidates r to verify if the above expression is identically null or not , we compute just the 0-th coefficient

$$\Gamma_0(r)=\frac{49}{r}-\frac{7}{4r^2}-\frac{1}{2r^3}-24,$$

and notice that it admits the following factorization

$$\Gamma_0(r)=-\frac{1}{4r^3}(8r-1)(r-2)(12r+1)\,.$$

Therefore, we test $Q\left(\frac{x}{r},\frac{xr+1}{2r}\right)$ for $r\in\left\{-\frac{1}{12},\frac{1}{8},2\right\}$ to obtain that $Q\left(-12x,\frac{1}{2}x-6\right)$ and $Q\left(8x,\frac{1}{2}x+4\right)$ are not identically null, while $Q\left(\frac{x}{2},\frac{2x+1}{4}\right)$ is. By Theorem 44 we conclude that $Q(x,y)$ belongs to the ideal generated by $4x-4y+1$.

Remark 40 *The last propostion on this chapter will provide the explication to our require that $\Gamma_0(r)$ be zero.*

Remark 41 *In the case of Weingarten surfaces, we assume only $r>0$ (once there is a geometrical meaning of r be the radius), so in the above example we just have to test $r\in\left\{\frac{1}{8},2\right\}$.*

The next proposition is a technical result that presents us a characterization of polynomials Q that belongs to the ideal generated by $xr^2-2yr+1$, in other words, we describe conditions on coefficients of Q in order to it be divisible by $xr^2-2yr+1$.

Proposition 42 *A polynomial $Q(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n-i}a_{i,j}x^iy^j\in\mathbb{R}[x,y]$ is divisible by $xr^2-2yr+1$ if and only if there exists a family*

$$\mathcal{C}=\{c_{i,j}\in\mathbb{R}\ ;\ i,j\in\mathbb{Z}\}$$

such that:

1. $c_{i,j} = 0$ *whenever* $i < 0$ *or* $j < 0$ *or* $i + j \geq n$;

2. $a_{i,j} = r^2 c_{i-1,j} - 2rc_{i,j-1} + c_{i,j}$, *for every* $i, j \in \mathbb{Z}$.

Proof. Suppose that

$$Q(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} a_{i,j} x^i y^j \in \mathbb{R}[x,y] \tag{2.23}$$

is divisible by $Xr^2 - 2Yr + 1$, then exists a family $\mathcal{C} = \{c_{i,j} \in \mathbb{R} \; ; \; i, \, j \in \mathbb{N} \text{ and } i + j \leq n-1\}$ such that

$$Q(X,Y) = \left(xr^2 - 2yr + 1\right) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} c_{i,j} X^i Y^j. \tag{2.24}$$

Therefore, by (2.23) and (2.24) it is obtained the following relation

$$a_{i,j} = r^2 c_{i-1,j} - 2rc_{i,j-1} + c_{i,j}$$

for every $i = 1, \ldots, n$ and for every $j = 1, \ldots, n-i$.

If we set

$$c_{i,j} = 0$$

whenever $i < 0$ or $j < 0$ or $i + j \geq n$, hence we constructed a family $\mathcal{C} = \{c_{i,j} \in \mathbb{R} \; ; \; i, j \in \mathbb{Z}\}$ verifying items 1 and 2.

Conversely, given a family

$$\mathcal{C} = \{c_{i,j} \in \mathbb{R} \; ; \; i, j \in \mathbb{Z}\}$$

we define the polynomial $R(x,y) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} c_{i,j} X^i Y^j$. By hypothesis we have

$$\begin{aligned} Q(x,y) &= \sum_{i=0}^{n} \sum_{j=0}^{n-i} a_{i,j} x^i y^j \\ &= \left(Xr^2 - 2Yr + 1\right) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} c_{i,j} X^i Y^j \\ &= \left(Xr^2 - 2Yr + 1\right) R(x,y), \end{aligned}$$

so Q is divisible by $\left(Xr^2 - 2Yr + 1\right)$. ■

The following Proposition is an important result, once it articulates both of the previous propositions: In one hand, it is considered the characterization of polynomials in the ideal generated by $xr^2 - 2yr + 1$; In the other hand, we study the consequences of $Q\left(\frac{x}{r}, \frac{xr+1}{2r}\right)$ be identically null, that is, when $\Gamma_k(r) \equiv 0$ for every (possible) k. Furthermore, in the aim to prove it, several results concerning binomials identities and summatories identities are obtained (see Appendix A for more details).

Proposition 43 *Given $n \in \mathbb{N}$, consider a family*

$$\mathcal{A} = \{a_{i,j} \in \mathbb{R} \; ; \; i, j \in \mathbb{N}\}$$

such that $a_{i,j} = 0$, whenever either $i < 0$ or $j < 0$ or $i + j > n$.

Then there is a family

$$\mathcal{C} = \{c_{i,j} \in \mathbb{R} \; ; \; i, j \in \mathbb{N}\}$$

verifying

1. *$c_{i,j} = 0$ whenever $i < 0$ or $j < 0$ or $i + j \geq n$;*
2. *$a_{i,j} = r^2 c_{i-1,j} - 2rc_{i,j-1} + c_{i,j}$ equivalently $c_{i,j} = a_{i,j} - r^2 c_{i-1,j} + 2rc_{i,j-1}$, for every $i, j \in \mathbb{N}$*

if and only if the coefficients $a_{i,j}$ verify

$$\Gamma_k(r) = \sum_{i=0}^{k} \left(\sum_{j=0}^{n-k} \binom{k-i+j}{j} \frac{1}{2^{k-i+j} r^{j+i}} a_{i,k-i+j} \right) = 0 \qquad \forall k = 0, \ldots, n. \tag{2.25}$$

Proof. Assume the existence of $\mathcal{C} = \{c_{i,j} \in \mathbb{R} \; ; \; i, j \in \mathbb{N}\}$ and we will prove that $\Gamma_k(r) = 0$ for every k. To prove this first part, we will consider three cases for k, named:

If $k = 0$, then

$$\begin{aligned} \Gamma_0(r) &= \sum_{j=0}^{n} \tfrac{1}{2^j r^j} a_{0,j} = \sum_{j=0}^{n} \tfrac{r^2 c_{-1,j} - 2rc_{0,j-1} + c_{0,j}}{2^j r^j} \\ &= -\sum_{j=1}^{n} \tfrac{c_{0,j-1}}{2^{j-1} r^{j-1}} - \tfrac{c_{0,-1}}{2^{-1} r^{-1}} + \sum_{j=0}^{n-1} \tfrac{c_{0,j}}{2^j r^j} + \tfrac{c_{0,n}}{2^n r^n} \end{aligned}$$

since $c_{0,n} = c_{0,-1} = 0$ by hypothesis, it implies

$$\sum_{j=0}^{n-1} \tfrac{c_{0,j}}{2^j r^j} - \sum_{j=1}^{n} \tfrac{c_{0,j-1}}{2^{j-1} r^{j-1}} = \sum_{j=0}^{n-1} \tfrac{c_{0,j}}{2^j r^j} - \sum_{j=0}^{n-1} \tfrac{c_{0,j}}{2^j r^j} = 0,$$

which concludes this case.

If $k = n$, then

$$\Gamma_n(r) = \sum_{i=0}^{n} \tfrac{a_{i,n-i}}{2^{n-i} r^i} = \sum_{i=0}^{n} \tfrac{r^2 c_{i-1,n-i} - 2rc_{i,n-i-1} + c_{i,n-i}}{2^{n-i} r^i}$$

once $c_{i,n-i} = 0$ for every i (because $i + n - i = n$) and by definition $c_{-1,n} = c_{n,-1} = 0$, thus

$$\sum_{i=1}^{n} \tfrac{c_{i-1,n-i}}{2^{n-i} r^{i-2}} - \sum_{i=0}^{n-1} \tfrac{c_{i,n-i-1}}{2^{n-i-1} r^{i-1}} = \sum_{i=0}^{n-1} \tfrac{c_{i,n-i-1}}{2^{n-i-1} r^{i-1}} - \sum_{i=0}^{n-1} \tfrac{c_{i,n-i-1}}{2^{n-i-1} r^{i-1}} = 0,$$

which concludes this case.

If $k = 1, \ldots, n-1$, then

$$\begin{aligned}
\Gamma_k(r) &= \sum_{i=0}^{k} \sum_{j=0}^{n-k} \binom{k-i+j}{j} \frac{a_{i,k-i+j}}{2^{k-i+j} r^{j+i}} \\
&= \sum_{i=0}^{k} \sum_{j=0}^{n-k} \binom{k-i+j}{j} \frac{r^2 c_{i-1,k-i+j} - 2r c_{i,k-i+j-1} + c_{i,k-i+j}}{2^{k-i+j} r^{j+i}}
\end{aligned}$$

expanding the last term (which is possible since $k \geq 1$) it gives us the following expression

$$\begin{aligned}
&\sum_{j=0}^{n-k} \binom{k+j}{j} \frac{c_{-1,k+j}}{2^{k+j} r^{j-2}} + \sum_{i=1}^{k} \sum_{j=0}^{n-k} \binom{k-i+j}{j} \frac{c_{i-1,k-i+j}}{2^{k-i+j} r^{j+i-2}} \\
&- \sum_{i=0}^{k-1} \sum_{j=0}^{n-k} \binom{k-i+j}{j} \frac{c_{i,k-i+j-1}}{2^{k-i+j-1} r^{j+i-1}} - \sum_{j=0}^{n-k} \frac{c_{k,j-1}}{2^{j-1} r^{j+k-1}} \\
&+ \sum_{i=0}^{k-1} \sum_{j=0}^{n-k} \binom{k-i+j}{j} \frac{c_{i,k-i+j}}{2^{k-i+j} r^{j+i}} + \sum_{j=0}^{n-k} \frac{c_{k,j}}{2^{j} r^{j+k}}
\end{aligned} \tag{2.26}$$

Notice that

$$\sum_{j=0}^{n-k} \binom{k+j}{j} \frac{c_{-1,k+j}}{2^{k+j} r^{j-2}} = 0, \tag{2.27}$$

once $c_{-1,m} = 0$, for every $m \in \mathbb{N}$. Also, observe that next terms can be simplified as

$$\sum_{j=0}^{n-k} \frac{c_{k,j-1}}{2^{j-1} r^{j+k-1}} = \sum_{j=1}^{n-k} \frac{c_{k,j-1}}{2^{j-1} r^{j+k-1}} + \frac{c_{k,-1}}{2^{-1} r^{k-1}} = \sum_{j=1}^{n-k} \frac{c_{k,j-1}}{2^{j-1} r^{j+k-1}} = \sum_{j=0}^{n-k-1} \frac{c_{k,j}}{2^{j} r^{j+k}} \tag{2.28}$$

and

$$\sum_{j=0}^{n-k} \frac{c_{k,j}}{2^{j} r^{j+k}} = \sum_{j=0}^{n-k-1} \frac{c_{k,j}}{2^{j} r^{j+k}} + \frac{c_{k,n-k}}{2^{n-k} r^{n}} = \sum_{j=0}^{n-k-1} \frac{c_{k,j}}{2^{j} r^{j+k}} \tag{2.29}$$

since $c_{k,n-k} = 0$ and $c_{m,-1} = 0$ for every $m \in \mathbb{N}$. Therefore by (2.27), (2.28) &

(2.29) the expression (2.26) resumes to

$$\begin{aligned}
&\sum_{i=0}^{k-1}\sum_{j=0}^{n-k}\tbinom{k-i+j-1}{j}\tfrac{c_{i,k-i+j-1}}{2^{k-i+j-1}r^{j+i-1}}-\sum_{i=0}^{k-1}\sum_{j=0}^{n-k}\tbinom{k-i+j}{j}\tfrac{c_{i,k-i+j-1}}{2^{k-i+j-1}r^{j+i-1}}+\sum_{i=0}^{k-1}\sum_{j=0}^{n-k}\tbinom{k-i+j}{j}\tfrac{c_{i,k-i+j}}{2^{k-i+j}r^{j+i}}\\
=\;&\sum_{i=0}^{k-1}\tfrac{c_{i,k-i-1}}{2^{k-i-1}r^{i-1}}+\sum_{i=0}^{k-1}\sum_{j=1}^{n-k}\tbinom{k-i+j-1}{j}\tfrac{c_{i,k-i+j-1}}{2^{k-i+j-1}r^{j+i-1}}\\
&-\sum_{i=0}^{k-1}\tfrac{c_{i,k-i-1}}{2^{k-i-1}r^{i-1}}-\sum_{i=0}^{k-1}\sum_{j=1}^{n-k}\tbinom{k-i+j}{j}\tfrac{c_{i,k-i+j-1}}{2^{k-i+j-1}r^{j+i-1}}\\
&+\sum_{i=0}^{k-1}\sum_{j=1}^{n-k}\tbinom{k-i+j-1}{j-1}\tfrac{c_{i,k-i+j-1}}{2^{k-i+j-1}r^{j+i-1}}+\sum_{i=0}^{k-1}\tbinom{n-i}{n-k}\tfrac{c_{i,n-i}}{2^{n-i}r^{n-k+i}}
\end{aligned}$$

again, $c_{i,n-i}=0$ (because of $i+n-i=n$), follows that

$$\begin{aligned}
\Gamma_k(r) &= \sum_{i=0}^{k}\left(\sum_{j=0}^{n-k}\tbinom{k-i+j}{j}\tfrac{a_{i,k-i+j}}{2^{k-i+j}r^{j+i}}\right)\\
&= \sum_{i=0}^{k-1}\sum_{j=1}^{n-k}\left(\tbinom{k-i+j-1}{j}-\tbinom{k-i+j}{j}+\tbinom{k-i+j-1}{j-1}\right)\tfrac{c_{i,k-i+j-1}}{2^{k-i+j-1}r^{j+i-1}}
\end{aligned}$$

finally, by Pascal's rule we have

$$\tbinom{k-i+j}{j}=\tbinom{k-i+j-1}{j}+\tbinom{k-i+j-1}{j-1}$$

hence it is concluded

$$\Gamma_k(r)=\sum_{i=0}^{k}\left(\sum_{j=0}^{n-k}\tbinom{k-i+j}{j}\tfrac{a_{i,k-i+j}}{2^{k-i+j}r^{j+i}}\right)=0,$$

for every $0\leq k\leq n$.

Reciprocally, consider the following definition

$$c_{i,j}=\begin{cases}\sum\limits_{k=0}^{i}\left(\sum\limits_{l=0}^{j}(-1)^k\tbinom{l+k}{l}2^l r^{l+2k}a_{i-k,j-l}\right) & \text{if } i,j\in\mathbb{N}\\ 0 & \text{if } i<0 \text{ or if } j<0\end{cases},\quad (2.30)$$

we proceed with this proof beginning by item 2. Our goal is to show

$$r^2c_{i-1,j}-2rc_{i,j-1}+c_{i,j}=a_{i,j},$$

for every $i,\ j\in\mathbb{N}$. Once again, we will consider several cases for indeces i and j.

Choose $i,\ j\in\mathbb{N}$, if $i=0=j$, follows from (2.30) that

$$c_{0,0}=a_{0,0}\qquad\text{and}\qquad c_{-1,0}=0=c_{0,-1}$$

consequently

$$r^2 c_{-1,0} - 2r c_{0,-1} + c_{0,0} = c_{0,0} = a_{0,0},$$

and the desired is achieved.

If $j = 0$ and $i \geq 1$, we have

$$\begin{aligned} c_{i,0} &= \sum_{k=0}^{i} (-1)^k r^{2k} a_{i-k,0} \\ c_{i,0-1} &= 0 \\ c_{i-1,0} &= \sum_{k=0}^{i-1} (-1)^k r^{2k} a_{i-1-k,0} \end{aligned}$$

hence

$$\begin{aligned} & r^2 c_{i-1,0} - 2r c_{i,0-1} + c_{i,0} \\ = \; & r^2 \sum_{k=0}^{i-1} (-1)^k r^{2k} a_{i-1-k,0} + \sum_{k=0}^{i} (-1)^k r^{2k} a_{i-k,0} \\ = \; & \sum_{k=0}^{i-1} (-1)^k r^{2(k+1)} a_{i-1-k,0} + \sum_{k=0}^{i} (-1)^k r^{2k} a_{i-k,0} \\ = \; & \sum_{k=1}^{i} (-1)^{(k-1)} r^{2((k-1)+1)} a_{i-1-(k-1),0} + \sum_{k=0}^{i} (-1)^k r^{2k} a_{i-k,0} \\ = \; & -\sum_{k=1}^{i} (-1)^k r^{2k} a_{i-k,0} + \sum_{k=1}^{i} (-1)^k r^{2k} a_{i-k,0} + a_{i,0} \\ = \; & a_{i,0} \end{aligned}$$

and the desired is achieved.

If $i = 0$ and $j \geq 1$ we have

$$\begin{aligned} c_{0,j} &= \left(\sum_{l=0}^{j} 2^l r^l a_{0,j-l} \right) \\ c_{0,j-1} &= \left(\sum_{l=0}^{j-1} 2^l r^l a_{0,j-1-l} \right) \\ c_{0-1,j} &= 0 \end{aligned}$$

hence

$$\begin{aligned}
& r^2 c_{0-1,j} - 2rc_{0,j-1} + c_{0,j} \\
= \quad & -2r\sum_{l=0}^{j-1} 2^l r^l a_{0,j-1-l} + \sum_{l=0}^{j} 2^l r^l a_{0,j-l} \\
= \quad & -\sum_{l=0}^{j-1} 2^{l+1} r^{l+1} a_{0,j-1-l} + \sum_{l=0}^{j} 2^l r^l a_{0,j-l} \\
= \quad & -\sum_{l=1}^{j} 2^{(l-1)+1} r^{(l-1)+1} a_{0,j-1-(l-1)} + \sum_{l=0}^{j} 2^l r^l a_{0,j-l} \\
= \quad & -\sum_{l=1}^{j} 2^l r^l a_{0,j-l} + \sum_{l=1}^{j} 2^l r^l a_{0,j-l} + a_{0,j} \\
= \quad & a_{0,j}
\end{aligned}$$

and the desired is achieved.

If i, $j \geq 1$, by definition (2.30) each $c_{(i-1),j}$, $c_{i,(j-1)}$ and $c_{i,j}$ has the following expression

$$\begin{aligned}
c_{(i-1),j} &= \sum_{k=0}^{(i-1)} \left(\sum_{l=0}^{j} (-1)^k \binom{l+k}{l} 2^l r^{l+2k} a_{(i-1)-k,j-l} \right) \\
c_{i,(j-1)} &= \sum_{k=0}^{i} \left(\sum_{l=0}^{(j-1)} (-1)^k \binom{l+k}{l} 2^l r^{l+2k} a_{i-k,(j-1)-l} \right) \\
c_{i,j} &= \sum_{k=0}^{i} \left(\sum_{l=0}^{j} (-1)^k \binom{l+k}{l} 2^l r^{l+2k} a_{i-k,j-l} \right)
\end{aligned}$$

then expanding each one in a suitable way, we have

$$\begin{aligned}
& r^2 \sum_{k=1}^{i} \sum_{l=0}^{j} (-1)^{(k-1)} \binom{l+(k-1)}{l} 2^l r^{l+2(k-1)} a_{i-k,j-l} \qquad (2.31) \\
= \quad & -\sum_{k=1}^{i} (-1)^k r^{2k} a_{i-k,j} \\
& + \sum_{k=1}^{i} \sum_{l=1}^{j} (-1)^{(k-1)} \binom{l+(k-1)}{l} 2^l r^{l+2k} a_{i-k,j-l}
\end{aligned}$$

$$\begin{aligned} & -2r\sum_{k=0}^{i}\sum_{l=1}^{j}(-1)^k\tbinom{(l-1)+k}{(l-1)}2^{(l-1)}r^{(l-1)+2k}a_{i-k,j-l} \\ = \quad & -\sum_{l=1}^{j}2^l r^l a_{i,j-l} \\ & -\sum_{k=1}^{i}\sum_{l=1}^{j}(-1)^k\tbinom{(l-1)+k}{(l-1)}2^l r^{l+2k}a_{i-k,j-l} \end{aligned} \tag{2.32}$$

$$\begin{aligned} & \sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^k\tbinom{l+k}{l}2^l r^{l+2k}a_{i-k,j-l} \\ = \quad & a_{i,j}+\sum_{l=1}^{j}2^l r^l a_{i,j-l}+\sum_{k=1}^{i}(-1)^k r^{2k}a_{i-k,j} \\ & +\sum_{k=1}^{i}\sum_{l=1}^{j}(-1)^k\tbinom{l+k}{l}2^l r^{l+2k}a_{i-k,j-l} \end{aligned} \tag{2.33}$$

replacing the above equalities in $r^2c_{i-1,j}-2rc_{i,j-1}+c_{i,j}$ it is obtained

$$\sum_{k=1}^{i}\sum_{l=1}^{j}(-1)^{(k)}\,2^l r^{l+2k}\left(-\tbinom{l+(k-1)}{l}-\tbinom{(l-1)+k}{(l-1)}+\tbinom{l+k}{l}\right)a_{i-k,j-l}+a_{i,j}$$

finally, the Pascal rule provides the identity

$$\tbinom{l+k}{l}=\tbinom{l+k-1}{l}+\tbinom{l+k-1}{l-1}$$

which implies

$$r^2c_{i-1,j}-2rc_{i,j-1}+c_{i,j}=a_{i,j}$$

and the proof of item 2 is complete.

To prove item 1, by definition (2.30) remains to verify

$$c_{i,j}=0$$

whenever $i+j\geq n$, since the case $c_{i,j}=0$ for every $i+j=n$ is a rather difficulty, we will assume *a priori* $c_{k,n-k}=0$ for every k, then we will use it to prove that

$$c_{i,j}=0,$$

for every $i,j,l\in\mathbb{N}$ with $l\geq 1$ such that $i+j=n+l$. This proof is given by induction on l.

If $l = 1$, so

$$\begin{aligned} c_{i,j} &= c_{i,n+1-i} \\ &= a_{i,n+1-i} - r^2 c_{i-1,n+1-i} + 2rc_{i,n-i} \\ &= -r^2 c_{i-1,n+1-i} + 2rc_{i,n-i} \\ &= 2rc_{i,n-i} \end{aligned}$$

note that $i + j = n + 1$ hence $i \leq n + 1$ implies $i - 1 \leq n$, so in case $i = n + 1$, it follows $c_{n+1,n-(n+1)} = c_{n+1,-1} = 0$. In case of $i \leq n$, then $c_{i,n-i} = 0$ by hypothesis. Then it is concluded that $c_{i,j}$ is zero.
Proof. Assume that $c_{i,n+l-i} = 0$ for some l and we will verify the equality for $l + 1$. In fact,

$$\begin{aligned} c_{i,n+l+1-i} &= a_{i,n+l+1-i} - r^2 c_{i-1,n+l+1-i} + 2rc_{i,n+l-i} \\ &= -r^2 c_{i-1,n+l+1-i} + 2rc_{i,n+l-i} \\ &= 0 \end{aligned}$$

and the desired is achieved. To finish the demonstration of item 1, consequently the Theorem, still necessary to prove that

$$c_{k,n-k} = 0,$$

for every $k \in \mathbb{N}$, in the presence of the hypothesis (2.25), which for convenience of the reader, it will be restated in a slightly different but very helpful way:

$$\Gamma_{i-k}(r) = \sum_{k=0}^{(i-m)} \left(\sum_{l=0}^{n-(i-m)} \binom{(i-m)-k+l}{l} \frac{1}{2^{(i-m)-k+l} r^{l+k}} a_{k,(i-m)-k+l} \right) = 0, \tag{2.34}$$

so we will show that each $c_{i,n-i}$ is a linear combination of certains $\Gamma_{i-k}(r)$. More precisely,

$$c_{i,n-i} = \sum_{k=0}^{i} (-1)^k \binom{n-i+k}{n-i} \frac{(2r)^n r^k \Gamma_{i-k}}{2^i}, \tag{2.35}$$

where Γ_{i-k} denotes $\Gamma_{i-k}(r)$ for practical reasons. In the aim to provide the proof, it will be presented that the difference of above equality is null. First of all, is important do a treatment of expressions to express them in a proper manner: Applying the Prop (Inversão de contagem) to $c_{i,n-i}$ gives us

$$c_{i,n-i} = \sum_{k=0}^{i} \sum_{l=0}^{n-i} (-1)^{i-k} \binom{l+i-k}{l} 2^l r^{l+2i-2k} a_{k,n-i-l} \tag{2.36}$$

and consider the explicity expression of right side of Equation (2.35)

$$\begin{aligned} &\sum_{k=0}^{i} (-1)^k \binom{n-i+k}{n-i} \frac{(2r)^n r^k \Gamma_{i-k}}{2^i} \\ =\ &\sum_{m=0}^{i} \sum_{l=0}^{n-(i-m)} \sum_{k=0}^{i-m} (-1)^m 2^{k-l+m+n-2i} r^{m-l-k+n} \binom{n-i+m}{n-i} \binom{i-m-k+l}{l} a_{k,(i-m)-k+l} \end{aligned}$$

then apply Proposition 130 to obtain

$$\sum_{k=0}^{i}(-1)^k\binom{n-i+k}{n-i}\frac{(2r)^n r^k\Gamma_{i-k}}{2^i} \tag{2.37}$$
$$= \sum_{k=0}^{i}\sum_{m=0}^{i-k}\sum_{l=0}^{n-i+m}(-1)^m 2^{k-l+m+n-2i}r^{m-l-k+n}\binom{n-i+m}{n-i}\binom{i-m-k+l}{l}a_{k,(i-m)-k+l}$$

so we define

$$H_n = c_{i,n-i} - \sum_{k=0}^{i}(-1)^k\binom{n-i+k}{n-i}\frac{(2r)^n r^k\Gamma_{i-k}}{2^i}$$

by (2.36) and (2.37) we have

$$H_n =$$
$$\sum_{k=0}^{i}\sum_{l=0}^{n-i}(-1)^{i-k}\binom{l+i-k}{l}2^l r^{l+2i-2k}a_{k,n-i-l}$$
$$-\sum_{k=0}^{i}\sum_{m=0}^{i-k}\sum_{l=0}^{n-i+m}(-1)^m 2^{k-l+m+n-2i}r^{m-l-k+n}\binom{n-i+m}{n-i}\binom{i-m-k+l}{l}a_{k,i-m-k+l}$$

and our goal resumes to verify $H_n = 0$, so it is sufficienty verify

$$\sum_{l=0}^{n-i}(-1)^{i-k}\binom{l+i-k}{l}2^l r^{l+2i-2k}a_{k,n-i-l} \tag{2.38}$$
$$-\sum_{m=0}^{i-k}\sum_{l=0}^{n-i+m}(-1)^m 2^{k-l+m+n-2i}r^{m-l-k+n}\binom{n-i+m}{n-i}\binom{i-m-k+l}{l}a_{k,(i-m)-k+l}$$
$$= 0$$

for each $k = 0, ..., i$. Observe that Equation (2.38) allows us to do further simplifications. Redefine

$$p = i - k, \qquad q = n - i \qquad \text{and} \qquad b_j = a_{k,j}$$

hence the previous equation can be rewritten as

$$\sum_{l=0}^{q}(-1)^p\binom{l+p}{l}2^l r^{l+2p}b_{q-l}$$
$$-\sum_{m=0}^{p}\sum_{l=0}^{q+m}(-1)^m 2^{q+m-l-p}r^{m-l+p+q}\binom{q+m}{q}\binom{p-m+l}{l}b_{p-m+l}$$

also notice that Proposition 125 furnishes the following equality

$$\sum_{l=0}^{q}(-1)^p\binom{l+p}{l}2^l r^{l+2p}b_{q-l} = \sum_{l=0}^{q}(-1)^p\binom{q-l+p}{q-l}2^{q-l}r^{q-l+2p}b_l$$

and the Proposition 126 gives us

$$\begin{aligned} & \sum_{m=0}^{p}\sum_{l=0}^{q+m}(-1)^m\, 2^{q+m-l-p} r^{m-l+p+q}\binom{q+m}{q}\binom{p-m+l}{l} b_{p-m+l} \\ = & \sum_{m=0}^{p}\sum_{l=0}^{q+(p-m)}(-1)^{(p-m)}\, 2^{q-m-l} r^{2p-m-l+q}\binom{q+(p-m)}{q}\binom{m+l}{l} b_{m+l}. \end{aligned}$$

Finally, we set

$$\begin{aligned} J_{p,q} = & \sum_{l=0}^{q}(-1)^p\binom{q-l+p}{q-l}2^{q-l}r^{q-l+2p}b_l \\ & -\sum_{m=0}^{p}\sum_{l=0}^{q+(p-m)}(-1)^{(p-m)}\, 2^{q-m-l} r^{2p-m-l+q}\binom{q+(p-m)}{q}\binom{m+l}{l} b_{m+l} \end{aligned} \tag{2.39}$$

and our goal is to verify that $J_{p,q}=0$ which is equivalent to $H_n=0$ (also equivalent to (2.38) be null). ■

We have several cases to consider in order to prove that $J_{p,q}=0$.

Case $p=q=0$. A straighfoward computation provides

$$J_{0,0}=b_0-b_0=0$$

which concludes this case.

Case $p=q\geq 1$, it implies that Equation (2.39) has the following expression

$$J_{q,q}=\sum_{l=0}^{q}(-1)^q\binom{2q-l}{q-l}2^{q-l}r^{3q-l}b_l-\sum_{m=0}^{q}\sum_{l=0}^{2q-m}(-1)^{q-m}\, 2^{q-m-l} r^{3q-m-l}\binom{2q-m}{q}\binom{m+l}{l} b_{m+l} \tag{2.40}$$

and once again we need to do some simplifications in each of terms above.

In the aim to apply Theorem 129, we set

$$\lambda_{m,l}=(-1)^{q-m}\, 2^{q-m-l} r^{3q-m-l}\binom{2q-m}{q}\binom{m+l}{l}$$

therefore, we have

$$\begin{aligned} \lambda_{m,l-m} &= (-1)^{q-m}\, 2^{q-l} r^{3q-l}\binom{2q-m}{q}\binom{l}{l-m} \\ \lambda_{m,q+l-m} &= (-1)^{q-m}\, 2^{-l} r^{2q-l}\binom{2q-m}{q}\binom{q+l}{q+l-m} \end{aligned}$$

hence

$$\sum_{m=0}^{q}\sum_{l=0}^{2q-m}(-1)^{q-m}\,2^{q-m-l}r^{3q-m-l}\binom{2q-m}{q}\binom{m+l}{l}b_{m+l} \tag{2.41}$$

$$= \sum_{l=0}^{q}\sum_{m=0}^{l}(-1)^{q-m}\,2^{q-l}r^{3q-l}\binom{2q-m}{q}\binom{l}{l-m}b_l$$

$$+\sum_{l=1}^{q}\sum_{m=0}^{q}(-1)^{q-m}\,2^{-l}r^{2q-l}\binom{2q-m}{q}\binom{q+l}{q+l-m}b_{q+l} \tag{2.42}$$

$$= \sum_{l=0}^{q}\sum_{m=0}^{l}(-1)^{q-m}\,2^{q-l}r^{3q-l}\binom{2q-m}{q}\binom{l}{l-m}b_l$$

$$+\sum_{l=q+1}^{2q}\sum_{m=0}^{q}(-1)^{q-m}\,2^{-l+q}r^{3q-l}\binom{2q-m}{q}\binom{l}{l-m}b_l \tag{2.43}$$

where the last equality is furnished by rearranging the indices of last summatory.

Then the Equation (2.40) become by (2.41)

$$J_{q,q} = \sum_{l=0}^{q}2^{q-l}r^{3q-l}(-1)^q\left(\binom{2q-l}{q-l}-\sum_{m=0}^{l}(-1)^m\binom{2q-m}{q}\binom{l}{l-m}\right)b_l$$
$$-\sum_{l=q+1}^{2q}\left(\sum_{m=0}^{q}(-1)^{q-m}\,2^{-l+q}r^{3q-l}\binom{2q-m}{q}\binom{l}{l-m}\right)b_l$$

and once verified the following statements

$$\sum_{m=0}^{l}(-1)^m\binom{2q-m}{q}\binom{l}{l-m}=\binom{2q-l}{q-l}$$

and

$$\sum_{m=0}^{q}\binom{2q-m}{q}\binom{l}{l-m}=0$$

we will achived the desired.

CLAIM 1:

$$\sum_{m=0}^{l}(-1)^m\binom{2q-m}{q}\binom{l}{m}=\binom{2q-l}{q},$$

for $l < q$ and $q \geq 1$.

Indeed, the Propostion 133 states

$$\sum_{m=0}^{q}(-1)^m\binom{2q-m}{q}\binom{l}{m}=\binom{2q-l}{q}$$

since $l < q$, it yields

$$\begin{aligned}
& \sum_{m=0}^{l} (-1)^m \binom{2q-m}{q}\binom{l}{m} + \sum_{m=l+1}^{q} (-1)^m \binom{2q-m}{q}\binom{l}{m} \\
= & \sum_{m=0}^{q} (-1)^m \binom{2q-m}{q}\binom{l}{m} \\
= & \binom{2q-l}{q}
\end{aligned}$$

however $\sum_{m=l+1}^{q} (-1)^m \binom{2q-m}{q}\binom{l}{m} = 0$ once $\binom{z}{k} = 0$ whenever $k \geq z$, thus

$$\sum_{m=0}^{l} (-1)^m \binom{2q-m}{q}\binom{l}{m} = \binom{2q-l}{q}$$

so the Claim 1 is proved.

CLAIM 2:

$$\sum_{m=0}^{q} \binom{2q-m}{q}\binom{l}{l-m} = 0,$$

for every $l \in \{q+1, \dots, 2q\}$.

In order to apply Proposition 132 we set

$$y = 2q, \qquad x = -1, \qquad r = q \qquad \text{and } n = l,$$

and we obtain

$$\sum_{m=0}^{l} (-1)^m \binom{2q-m}{q}\binom{l}{m} = 0.$$

Observe that $2q-m \geq 2q-l \geq 0$, for every $l \leq 2q$, and note that $2q-m-q \geq 0$ whenever $q \geq m$ which implies

$$\binom{2q-m}{q} = 0,$$

for every $m \in \{q+1, \dots, l\}$, thus

$$\sum_{m=0}^{q} (-1)^m \binom{2q-m}{q}\binom{l}{m} = 0$$

and the Claim 2 is proved.

Then $J_{q,q} = 0$ and the case $p = q \geq 1$ is finalized.

Case $p < q$; In this case, we can consider the following expression of first summatory in (2.39):

$$\begin{aligned} & \sum_{l=0}^{q} (-1)^p \binom{q-l+p}{q-l} 2^{q-l} r^{q-l+2p} b_l \\ = & \sum_{l=0}^{p} (-1)^p \binom{q-l+p}{q-l} 2^{q-l} r^{q-l+2p} b_l \\ & + \sum_{l=p+1}^{q} (-1)^p \binom{q-l+p}{q-l} 2^{q-l} r^{q-l+2p} b_l . \end{aligned} \tag{2.44}$$

For the second summatory of (2.39), we define

$$\lambda_{m,l} = (-1)^{p-m}\, 2^{q-m-l} r^{2p-m-l+q} \binom{q+p-m}{q} \binom{m+l}{l}$$

the we have

$$\begin{aligned} \lambda_{m,l-m} &= (-1)^{p-m}\, 2^{q-l} r^{2p-l+q} \binom{q+p-m}{q} \binom{l}{l-m} \\ \lambda_{m,p+l-m} &= (-1)^{p-m}\, 2^{q-p-l} r^{p-l+q} \binom{q+p-m}{q} \binom{l+p}{l-m+p} \end{aligned}$$

and once again we apply the Theorem 129, so it is obtained

$$\begin{aligned} & \sum_{m=0}^{p} \sum_{l=0}^{q+(p-m)} (-1)^{(p-m)}\, 2^{q-m-l} r^{2p-m-l+q} \binom{q+(p-m)}{q} \binom{m+l}{l} b_{m+l} \\ & = \sum_{l=0}^{p} \sum_{m=0}^{l} (-1)^{p-m}\, 2^{q-l} r^{2p-l+q} \binom{q+p-m}{q} \binom{l}{l-m} b_l \\ & + \sum_{l=1}^{q} \sum_{m=0}^{p} (-1)^{p-m}\, 2^{q-p-l} r^{p-l+q} \binom{q+p-m}{q} \binom{l+p}{l-m+p} b_{p+l} \end{aligned} \tag{2.45}$$

then replacing (2.44) and (2.45) in (2.39) we have

$$
\begin{aligned}
J_{p,q} &= \sum_{l=0}^{p} (-1)^p \, 2^{q-l} r^{q-l+2p} \left(\tbinom{q-l+p}{q-l} - \sum_{m=0}^{l} (-1)^m \tbinom{q+p-m}{q} \tbinom{l}{l-m} \right) b_l \\
&\quad + \sum_{l=1}^{q-p} (-1)^p \, 2^{q-l-p} r^{q-l+p} \tbinom{q-l}{q-l-p} b_{p+l} \\
&\quad - \sum_{l=1}^{q} \sum_{m=0}^{p} (-1)^{p-m} \, 2^{q-p-l} r^{p-l+q} \tbinom{q+p-m}{q} \tbinom{l+p}{l-m+p} b_{p+l} \\
&= \sum_{l=0}^{p} (-1)^p \, 2^{q-l} r^{q-l+2p} \left(\tbinom{q-l+p}{q-l} - \sum_{m=0}^{l} (-1)^m \tbinom{q+p-m}{q} \tbinom{l}{l-m} \right) b_l \\
&\quad + \sum_{l=1}^{q-p} (-1)^p \, 2^{q-l-p} r^{q-l+p} \left(\tbinom{q-l}{q-l-p} - \sum_{m=0}^{p} (-1)^m \tbinom{q+p-m}{q} \tbinom{l+p}{l-m+p} \right) b_{p+l} \\
&\quad - \sum_{l=q-p+1}^{q} \sum_{m=0}^{p} (-1)^{p-m} \, 2^{q-p-l} r^{p-l+q} \tbinom{q+p-m}{q} \tbinom{l+p}{l-m+p} b_{p+l}
\end{aligned}
$$

thus, it will be show the following equalities

$$\sum_{m=0}^{l} (-1)^m \tbinom{q+p-m}{q} \binom{l}{m} = \tbinom{q-l+p}{p} \text{ for every } l = 0, \ldots, p \text{ with } p < q;$$

$$\sum_{m=0}^{p} (-1)^m \tbinom{q+p-m}{q} \tbinom{l+p}{m} = \tbinom{q-l}{p} \text{ for every } l = 1, \ldots, q-p \text{ with } p < q;$$

$$\sum_{m=0}^{p} (-1)^m \tbinom{q+p-m}{q} \tbinom{l+p}{m} = 0 \text{ for every } l = q-p+1, \ldots, q \text{with } p < q.$$

CLAIM 3:

$$\sum_{m=0}^{l} (-1)^m \tbinom{q+p-m}{q} \tbinom{l}{m} = \tbinom{q+p-l}{p},$$

for every $l = 0, \ldots, p$ with $p < q$. This claim follows immediatly from Propostion 133.

CLAIM 4:

$$\sum_{m=0}^{p} (-1)^m \tbinom{q+p-m}{q} \tbinom{l+p}{m} = \tbinom{q-l}{p}$$

for every $l = 1, \ldots, q-p$ with $p < q$. By Proposition 134, it is given the equality

$$\sum_{m=0}^{n} (-1)^m \tbinom{x-m}{j} \tbinom{n}{m} = \tbinom{x-n}{j-n}$$

then we choose $x = q + p$, $j = q$ and $n = l + p$, hence

$$\sum_{m=0}^{n} (-1)^m \binom{(q+p)-m}{(q)}\binom{(l+p)}{m} = \binom{q-l}{q-l-p} = \binom{q-l}{q}$$

and the Claim 4 is conluded.

CLAIM 5:

$$\sum_{m=0}^{p} (-1)^m \binom{q+p-m}{q}\binom{l+p}{m} = 0$$

for every $l = q-p+1, \ldots, q$ with $p < q$. By Proposition 132 is given

$$\sum_{m=0}^{n} (-1)^m \binom{y+xm}{r}\binom{n}{m} = 0$$

for every $x, y, r \in \mathbb{Z}$ such that $r < n$. So, we set $n = l + p$, $r = q$, $y = p + q$ and $x = -1$, therefore

$$\sum_{m=0}^{l+p} (-1)^m \binom{p+q-m}{q}\binom{l+p}{m} = 0$$

and notice that $(p + q - m) - q = p - m \geq 0 \iff m \leq p$ which implies $\binom{p+q-m}{q} = 0$ for every $m \in \{p+1, ..., p+l\}$, then

$$0 = \sum_{m=0}^{l+p} (-1)^m \binom{p+q-m}{q}\binom{l+p}{m} = \sum_{m=0}^{p} (-1)^m \binom{p+q-m}{q}\binom{l+p}{m}$$

and the Claim 5 is proved.

So the case $p < q$ is demonstrated.

For the last case, consider $q < p$ which is very similar to the previous one. In Equation (2.39) once we set

$$\lambda_{m,l} = (-1)^{p-m}\, 2^{q-m-l} r^{2p-m-l+q} \binom{q+p-m}{q}\binom{m+l}{l}$$

we obtain the following terms

$$\begin{aligned} \lambda_{m,l-m} &= (-1)^{p-m}\, 2^{q-l} r^{2p-l+q} \binom{q+p-m}{q}\binom{l}{l-m} \\ \lambda_{m,l} &= (-1)^{p-m}\, 2^{q-p-l} r^{p-l+q} \binom{q+p-m}{q}\binom{p+l}{p+l-m} \end{aligned}$$

then using Theorem 129 in the second summatory in Equation (2.39), is rewritten as

$$\begin{aligned}
&\sum_{m=0}^{p}\sum_{l=0}^{q+p-m}(-1)^{p-m}\,2^{q-m-l}r^{2p-m-l+q}\binom{q+p-m}{q}\binom{m+l}{l}b_{m+l}\\
=\;&\sum_{l=0}^{p}\sum_{m=0}^{l}(-1)^{p-m}\,2^{q-l}r^{2p-l+q}\binom{q+p-m}{q}\binom{l}{l-m}b_l\\
&+\sum_{l=1}^{q}\sum_{m=0}^{p}(-1)^{p-m}\,2^{q-p-l}r^{p-l+q}\binom{q+p-m}{q}\binom{p+l}{p+l-m}b_{p+l}\\
=\;&\sum_{l=0}^{q}\sum_{m=0}^{l}(-1)^{p-m}\,2^{q-l}r^{2p-l+q}\binom{q+p-m}{q}\binom{l}{l-m}b_l\\
&+\sum_{l=q+1}^{p}\sum_{m=0}^{l}(-1)^{p-m}\,2^{q-l}r^{2p-l+q}\binom{q+p-m}{q}\binom{l}{l-m}b_l\\
&+\sum_{l=1}^{q}\sum_{m=0}^{p}(-1)^{p-m}\,2^{q-p-l}r^{p-l+q}\binom{q+p-m}{q}\binom{p+l}{p+l-m}b_{p+l}
\end{aligned}$$

thus the Equation (2.39) is

$$\begin{aligned}
J_{p,q}\;=\;&\sum_{l=0}^{q}(-1)^{p}\,2^{q-l}r^{q-l+2p}\left(\binom{q-l+p}{q-l}-\sum_{m=0}^{l}(-1)^{m}\binom{q+p-m}{q}\binom{l}{l-m}\right)b_l\\
&-\sum_{l=q+1}^{p}\sum_{m=0}^{l}(-1)^{p-m}\,2^{q-l}r^{2p-l+q}\binom{q+p-m}{q}\binom{l}{l-m}b_l\\
&-\sum_{l=1}^{q}\sum_{m=0}^{p}(-1)^{p-m}\,2^{q-p-l}r^{p-l+q}\binom{q+p-m}{q}\binom{p+l}{p+l-m}b_{p+l}
\end{aligned}$$

and we will prove the following claims:

$$\sum_{m=0}^{l}(-1)^{m}\binom{q+p-m}{q}\binom{l}{m}=\binom{q-l+p}{p}\text{ for every } q<p \text{ and } l=0,...,q;$$

$$\sum_{m=0}^{l}(-1)^{m}\binom{q+p-m}{q}\binom{l}{m}=0\text{ for every } q<p \text{ and } l=q+1,...,p;$$

$$\sum_{m=0}^{p}(-1)^{m}\binom{q+p-m}{q}\binom{p+l}{m}=0\text{ for every } q<p \text{ and } l=1,...,q.$$

CLAIM 6:

$$\sum_{m=0}^{l}(-1)^{m}\binom{q+p-m}{q}\binom{l}{m}=\binom{q-l+p}{p}$$

for every $q < p$ and $l = 0, ..., q$. In Proposition 132 it is only necessary set $r = p - q$ and the result follows.

CLAIM 7:

$$\sum_{m=0}^{l} (-1)^m \binom{q+p-m}{q}\binom{l}{m} = 0$$

for every $q < p$ and $l = q+1, ..., p$. We define

$$y = q + p, \qquad x = -1, \qquad r = q, \qquad n = l$$

and notice $q < l$, since $l = q+1, ..., p$, so we are in hypothesis to apply Proposition 132 and we obtain the desired.

CLAIM 8:

$$\sum_{m=0}^{p} (-1)^m \binom{q+p-m}{q}\binom{p+l}{m} = 0$$

for every $q < p$ and $l = 1, ..., q$. We define

$$y = q + p, \qquad x = -1, \qquad r = q, \qquad n = p + l$$

and notice that $q < p + l$, since $q < p$ and $l \in \{1, ..., q\}$. By Proposition 132 we have

$$\begin{aligned} 0 &= \sum_{m=0}^{n} (-1)^m \binom{y+xm}{r}\binom{n}{m} \\ &= \sum_{m=0}^{p} (-1)^m \binom{q+p-m}{q}\binom{p+l}{m} \\ &\quad + \sum_{m=p+1}^{p+l} (-1)^m \binom{q+p-m}{q}\binom{p+l}{m} \\ &= \sum_{m=0}^{p} (-1)^m \binom{q+p-m}{q}\binom{p+l}{m} \\ &\quad + \sum_{m=0}^{p+l-(p+1)} (-1)^{(m+p+1)} \binom{q+p-(m+p+1)}{q}\binom{p+l}{(m+p+1)} \\ &= \sum_{m=0}^{p} (-1)^m \binom{q+p-m}{q}\binom{p+l}{m} \end{aligned}$$

and the desired is obtained.

So we demonstrated that

$$J_{p,q} = 0$$

for every p and q. ∎

In view of the foregoing, we are able to proof the Theorem 44 which for convenience of the reader, will be restated. We remark that theorem provides the technical tool to demonstrate our main results in next chapter.

Theorem 44 *Consider a polynomial* $Q(x,y) \in \mathbb{R}[x,y]$ *and let* r *be a strictly positive constant. Then* Q *belongs to the ideal generated by* $xr^2 - 2yr + 1$ *if and only if* $Q\left(\frac{x}{r}, \frac{xr+1}{2r}\right) \in \mathbb{R}[x]$ *is identically null.*

Proof. If $Q\left(\frac{x}{r}, \frac{xr+1}{2r}\right) \in \mathbb{R}[x]$ is identically null, by Proposition 38 we have that $\Gamma_k(r) \equiv 0$ for each k, which is precisely the needed condition in Proposition 43 to conclude the existence of the family $\mathcal{C} = \{c_{i,j} \in \mathbb{R}\ ;\ i,j \in \mathbb{N}\}$. Finally, it implies, by Proposition 42, that Q is divisible by $xr^2 - 2yr + 1$, *i.e.*, Q belongs to ideal generated by $xr^2 - 2yr + 1$.

The other direction is straightforward. ■

2.3 Main result and applications for Tubular Surfaces

In this section we present our main theorems that fully classify Polynomial Weingarten tubular surfaces in Euclidean, Lorentzian and Hyperbolic 3-space.

In the aim to exemplify the type of problem and which challenges are presented in the classification of Polynomial Weingarten tubular surfaces, let us consider a polynomial $Q(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} a_{i,j}x^i y^j \in \mathbb{R}[x,y]$ of degree 2 (one of the most simple cases to be considered) and assume that Gaussian and mean curvatures are roots of Q, then

$$0 \equiv Q(K,H) = Q_2(s)\cos^2 t + Q_1(s)\cos t + Q_0(s), \qquad (2.46)$$

where

$$\begin{aligned} Q_2(s) &= \kappa^2\left(ra_{1,1} + r^2a_{1,0} + r^2a_{0,2} + r^3a_{0,1} + r^4a_{0,0} + a_{2,0}\right); \\ Q_1(s) &= -\tfrac{1}{2}\kappa\left(2ra_{1,0} + 2ra_{0,2} + 3r^2a_{0,1} + 4r^3a_{0,0} + a_{1,1}\right); \\ Q_0(s) &= a_{0,0}r^2 + \tfrac{1}{2}a_{0,1}r + \tfrac{1}{4}a_{0,2}, \end{aligned}$$

since the Equation (2.46) is valid for every $t \in \mathbb{R}$, it follows that each $Q_i(s)$ must be null, for every i. This study leads us to the necessity to write Q in a suitable way and also to investigate under which conditions the coefficients $a_{i,j}$ provides that $Q_i(s) \equiv 0$.

In the end, our researches gaves us that analogous conditions were requeried for arbitrary polynomials of degree n. Hence, this is clearly the reason that motivates the Chapter 2.2.

So, once detected the pattern, we can discuss the polynomials rather than analyze enormous equations. Thus, we dodge the difficulty of studying particular polynomials and its associated equations to furnish a fully description of tubular surfaces whose curvatures vanishes a polynomial.

Throughout this chapter we will represent the second principal curvature of a tubular surface in Hyperbolic 3-space by $k_2 = \frac{1}{r}$ instead of $\frac{1}{\sinh r}$. Since $\sinh r$

is constant, we named it by constant r. The only reason is to present the results with simple statements and deal with all spaces at once.

We start this chapter recalling the definiton exhibited in Introduction.

Definition 45 *The radius of a polynomial $Q(x,y) \in \mathbb{R}[x,y]$ is defined as the set*

$$Rad(Q) = \left\{ r \in (0,+\infty) \ ; \ Q\left(0, \tfrac{1}{2r}\right) = 0 \right\}.$$

*We say that Q is **tubular** or **non tubular** according to the $Rad(Q)$ being either non empty or empty, respectively.*

Definition 46 *The radius star of a polynomial $Q(x,y) \in \mathbb{R}[x,y]$, denoted by $Rad^*(Q)$, is defined as the set of all $r \in Rad(Q)$ such that $Q(x,y)$ belongs to the ideal in $\mathbb{R}[x,y]$ generated by $xr^2 - 2ry + 1$.*

Remark 47 *Given a polynomial $Q(x,y) \in \mathbb{R}[x,y]$, by Proposition 38 we have*

$$Q\left(\tfrac{x}{r}, \tfrac{xr+1}{2r}\right) = \sum_{k=0}^{n} \Gamma_k(r)\, x^k.$$

where $\Gamma_k(r)$ are as in (2.18). Notice that $Q\left(0, \frac{1}{2r}\right)$ is precisely $\Gamma_0(r)$.

The terminology used in above definition is motivated, as we will see, by the fact that exists a Weingarten tubular surface verifying a polynomial relation if and only if the polynomial is tubular.

The next example illustrates the class of tubular polynomials of degree 2:

Example 48 *Consider a polynomial of degree* 2

$$Q(x,y) = a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0} \in \mathbb{R}[x,y].$$

Follows from Theorem 38 that Q is tubular if and only if exists a strictly positive real number r such that

$$\Gamma_0(r) = \frac{1}{2r}a_{0,1} + \frac{1}{4r^2}a_{0,2} + a_{0,0} = 0.$$

Before to present the main theorem, let us re-state the Theorem 44 that incorporates all results in Chapter 2.2 to show that the condition of a polynomial Q to belong to the ideal generated by $xr^2 - 2yr + 1$ can be replaced by another one that is more technical however, easier to check. Moreover, the next proposition is an important argument in our main theorems.

Theorem 49 *Consider a polynomial $Q(x,y) \in \mathbb{R}[x,y]$ and let r be a nonzero constant. Then Q belongs to the ideal generated by $xr^2 - 2yr + 1$ if and only if $Q\left(\frac{x}{r}, \frac{xr+1}{2r}\right) \in \mathbb{R}[x]$ is identically null.*

Hence, the above Proposition associated with previous Definition 45, furnishes that tubular polynomials contains the polynomials that are multiples of $xr^2 - 2yr + 1 \in \mathbb{R}[x, y]$, for which $r \in Rad(Q)$.

The below Lemma is the first acquired result such that an algebraic hypothesis (on the polynomial) delivers a geometric feature on the tubular surface.

Lemma 50 *Let $Q(x, y) \in \mathbb{R}[x, y]$ be a polynomial that is not in the ideal generated by $xr^2 - 2ry + 1$. If a tubular surface of radius r verify $Q(K, H) = 0$, then*

$$K \equiv 0 \qquad \text{and} \qquad H \equiv \frac{1}{2r}.$$

Proof. Since $Q(x, y) \notin \left\langle xr^2 - 2ry + 1 \right\rangle$ it follows by Theorem 44 that

$$Q\left(\tfrac{x}{r}, \tfrac{xr+1}{2r}\right) \neq 0$$

and considering the additional assumption that exists a tubular surface which verifies $Q(K, H) = 0$, it is concluded that k_1 is constant equal to $\lambda \in \mathbb{R}$, that is,

$$k_1(s, t) = \frac{\kappa(s)\,\mu(t)\,\varepsilon_B}{1 + r\kappa(s)\,\mu(t)\,\varepsilon_B} = \lambda.$$

If $\lambda \neq 0$, the above equation gives us

$$\varepsilon_B \kappa(s)(1 - r\lambda)\,\mu(t) = \lambda$$

which implies that $\kappa(s)(1 - r\lambda) \neq 0$, for every $s \in (a, b)$. Then, for a fixed $s_0 \in (a, b)$, we have

$$\mu(t) = \frac{\lambda}{\varepsilon_B \kappa(s_0)(1 - r\lambda)} \qquad \text{for every } t \in \mathbb{R}.$$

This yields that μ is a constant function, which is an absurd because $\mu(t)$ represents $\delta \cos t$, $\sin t$, $\sinh t$ or $\delta \cosh t$. Consequently it is obtained

$$k_1(s, t) = 0 \qquad \therefore K \equiv 0 \text{ and } H \equiv \frac{1}{2r},$$

which concludes this proof. ■

Remark 51 *Going along the lines of previous proof, we remark that the same argument is valid for the first principal curvature of a tubular surface in Euclidean or in Hyperbolic 3-space which provides the same conclusion, that is, in every case it is obtained that $K \equiv 0$ and $H \equiv \frac{1}{2r}$.*

In view of the above discussion, now we are able to state one of our main theorems. The first Theorem presents the necessary and sufficient condition to existence of Polynomial Weingarten tubular surface.

Theorem 52 *Consider a polynomial $Q(x,y) \in \mathbb{R}[x,y]$. There is a Weingarten tubular surface in Euclidean (respect. Lorentzian or Hyperbolic) 3-spaces (of radius $r > 0$) verifying $Q(K,H) = 0$ if and only if Q is tubular (and $r \in Rad(Q)$).*

Proof. If Q is tubular (and $r \in Rad(Q)$) it is easy to see that every right cylinder of radius r verifies $Q(K,H) = 0$. Conversely, suppose there is a tubular surface S verifying the polynomial relation $Q(x,y)$ among its Gaussian and mean curvatures. If Q belongs to the ideal generated by $xr^2 - 2yr + 1$, follows from Theorem 44 that is equivalent to $Q\left(\frac{x}{r}, \frac{xr+1}{2r}\right)$ be identically null, hence we set $x = 0$ and the desired is concluded. Otherwise (that is, $Q \notin \langle xr^2 - 2yr + 1 \rangle$), since $Q(K,H) = 0$, the Lemma 50 provides that

$$Q(K,H) = Q\left(0, \tfrac{1}{2r}\right) = 0$$

which is the required condition to Q be a tubular polynomial. ■

As a consequence, the previous theorem establishes a relation between tubular polynomials and Polynomial Weingarten tubular surfaces that permits us to study only the relation in the aim to classify the surfaces. We also remark that theorem is what motivates the Definition 45.

Now let us revisit the Example 48 to further explore.

Example 53 *It is already known that a polynomial Q of degree 2 is tubular if and only if there is $r > 0$ such that*

$$\frac{1}{2r}a_{0,1} + \frac{1}{4r^2}a_{0,2} + a_{0,0} = 0.$$

In the case that $a_{0,1}$, $a_{0,2}$, $a_{0,0}$ are all null, we have that

$$Rad(Q) = (0, +\infty)$$

therefore, for each fixed $r \in Rad(Q)$, the previous Theorem ensures the existence of a tubular surface of radius r.

As illustrated by above example, the Theorem 52 guarantees the existence of a Polynomial Weingarten tubular surface, however it still lacks in more information as in which cases the tubular surface is a cylinder or an arbitrary tube. In this sense, we are leaded to our next main theorem to fill this gaps.

For a given fixed (tubular) polynomial $Q(x,y) \in \mathbb{R}[x,y]$, the below Theorem describes the set $\mathcal{S}(Q)$ of all regular tubular surfaces whose Gaussian and mean curvatures are roots of Q. Moreover, the result gives geometric features of the tubular surface. In other words, it is exhibit information about the required radius r and condition on the curvature of central curve (see Chapter 2) in order to the tubular surface be a Weingarten surface for the relation Q.

Theorem 54 *Given a tubular polynomial $Q(x,y) \in \mathbb{R}[x,y]$, denote by $\mathcal{S}(Q)$ the set of all regular tubular surfaces in Euclidean (respect. Lorentzian or Hyperbolic) 3-space whose Gaussian and mean curvatures K, H verify $Q(K,H) \equiv 0$. Then, the elements of $\mathcal{S}(Q)$ are:*

i. The cylinders whose radius r belongs to $Rad(Q)$.

ii. The tubular surfaces of radius $r \in Rad^*(Q)$.

In particular, if $Rad(Q) = \emptyset$ then $\mathcal{S}(Q)$ is empty.

Proof. First, we will check that cylinders and tubular surfaces are, indeed, elements of $\mathcal{S}(Q)$. It is immediate that cylinders of radius $r \in Rad(Q)$ belongs to $S(Q)$, once

$$Q(K,H) = Q\left(0, \tfrac{1}{2r}\right) = 0.$$

For a tubular surface of radius $r \in Rad(Q)$ such that Q is in the ideal of $\mathbb{R}[x,y]$ generated by $xr^2 - 2ry + 1$, it is obtained that Q admits the following factorization

$$Q(x,y) = \left(xr^2 - 2ry + 1\right) R(x,y) \qquad \text{where } R(x,y) \in \mathbb{R}[x,y],$$

then, evaluating Q in (K,H), it yields

$$Q(K,H) = \left(\tfrac{k_1}{r} r^2 - 2r\left(\tfrac{k_1 r + 1}{2r}\right) + 1\right) R(K,H) = 0,$$

and the desired is achieved. So it is verified that tubular surfaces of type i and type ii (as in the statement) are in $\mathcal{S}(Q)$ and the first part is finished.

Now, let S be a tubular surface of radius r in $S(Q)$ (which means that $r \in Rad(Q)$) and consider K and H its Gaussian and mean curvatures. Our objective is describe the types of surfaces in $S(Q)$. Hence consider the relation

$$0 = Q(K,H).$$

If Q is in the ideal of $\mathbb{R}[x,y]$ generated by $xr^2 - 2ry + 1$, as discussed above, we have that S is of type ii. If $Q \notin \left\langle xr^2 - 2ry + 1\right\rangle$, the Lemma 50 ensures that S is of type i. ■

The example below elucidates the usefulness of the above theorem with a numerical case:

Example 55 *Consider the polynomial $Q(x,y) = 14y - 25x + 100xy - 40y^2 - 1 \in \mathbb{R}[x,y]$. An easy calculation shows that*

$$Rad(Q) = \{2, 5\}.$$

Note that $Q\left(\frac{x}{5}, \frac{5x+1}{10}\right)$ vanishes identically and $Q\left(\frac{x}{2}, \frac{2x+1}{4}\right)$ not. So, the tubular surfaces that belongs to $\mathcal{S}(Q)$ are arbitrary regular tubular surfaces of radius 5 and right cylinders of radius 2. Ultimately, because of our Theorem 44 this problem also can be solved by polynomial analysis, once we observe that

$$Q(x,y) = (25x - 10y + 1)(4y - 1)$$

and apply the Theorem 54.

As a immediatly corollary of the Theorem 52 with the notation of Theorem 54 we have

Corollary 56 *A polynomial* $Q(x,y) \in \mathbb{R}[x,y]$ *is tubular if and only if* $\mathcal{S}(Q) \neq \emptyset$*. Consequently, we are always able to choose at least one regular tubular surface whose Gaussian and mean curvatures vanishes a tubular polynomial.*

Observe that the Theorem 54 allows us to classify tubular Polynomial Weingarten surfaces through the study of polynomials (exclusively). To portray the funcionality, let us show how some well-known results can be obtained from it:

Example 57 *The cylinders are the only tubular surfaces with constant mean curvature. In fact, consider the polynomial*

$$Q(x,y) = y - c \in \mathbb{R}[x,y].$$

Note that Q *is tubular if and only if* $c > 0$*. Therefore for each* $c \in (0,+\infty)$*, we have that* $\frac{1}{2c}$ *belongs to* $Rad(Q)$ *in Euclidean or Lorentzian 3-space. Moreover, in this case, follows immediately that* $Q(x,y)$ *is not in the ideal of* $\mathbb{R}[x,y]$ *generated by* $xr^2 - 2ry + 1$*. So* $\mathcal{S}(Q)$ *only contains cylinders of radius* $\frac{1}{2c}$*.*

Example 58 *The cylinders are the only tubular surfaces with constant (zero) Gaussian curvature.*

Indeed, given $c \in \mathbb{R}$*, the polynomial relation is*

$$Q(x,y) = x - c \in \mathbb{R}[x,y].$$

Just for $c = 0$ *we have that* Q *is tubular and, in this case,* $Rad(Q) = (0,+\infty)$*. Since* Q *is not in the ideal of* $\mathbb{R}[x,y]$ *generated by* $xr^2 - 2ry + 1$*, it implies that* $\mathcal{S}(Q)$ *contains all the right cylinders of radius* r*.*

As mentioned in the introduction, a special and very studied polynomial relation among the Gaussian and the mean curvatures, is the linear one. This type of relation is relevant because it permits us to see each of curvatures as function of another (*i.e.* $K = f(H)$ or $H = f(K)$).

Corollary 59 *Let* a, b, c *be real numbers such that* $(a,b,c) \neq (0,0,0)$ *and define* $\Delta = b^2 + 4ac$*. Consider the polynomial relation*

$$Q(x,y) = ax + by - c.$$

Then, $\mathcal{S}(Q) \neq \emptyset$ *if and only if* $b = c = 0$ *or* $bc > 0$*. Moreover:*

i. If $b = c = 0$*, then* $\mathcal{S}(Q)$ *contains all right cylinders of any radius* $r > 0$*;*

ii. If $bc > 0$ *and* $\Delta = 0$*, then* $\mathcal{S}(Q)$ *contains all tubular surfaces of radius* $\frac{b}{2c}$*;*

iii. If $bc > 0$ *and* $\Delta \neq 0$*, then* $\mathcal{S}(Q)$ *contains all right cylinders of radius* $\frac{b}{2c}$*.*

Proof. To prove that Q is tubular, it is sufficient to observe that $Q\left(0,\frac{1}{2r}\right)$ has positive roots if and only if $b=c=0$ or $bc>0$. Then, by Corollary 56 we obtain our first assertion. Moreover, we have

$$Rad\,(Q)=(0,+\infty) \qquad \text{or} \qquad Rad\,(Q)=\left\{\frac{b}{2c}\right\}$$

in these cases.

When $b=c=0$, we must have $a\neq 0$ by hypothesis. Hence $Q\,(x,y)$ is not in the ideal of $\mathbb{R}\,[x,y]$ generated by $xr^2-2ry+1$. Suppose now $bc>0$. So $Q\left(\frac{2cx}{b},\frac{2c+bx}{2b}\right)=\frac{1}{2b}\Delta x$ vanishes identically if and only if $\Delta=0$. ∎

In the investigation of important relations, we also have the length of second fundamental form and the Casorati curvature that are defined, respectively, by the non-linear relations:

$$|\mathcal{A}|=\sqrt{4H^2-2K} \qquad \text{and} \qquad K_C=\frac{|\mathcal{A}|}{2}.$$

It follows from Example 48 that the polynomial $Q\,(x,y)=-2x+4y^2-c$ is tubular whenever $c>0$ and, in this case, $Rad\,(Q)=\left\{\frac{1}{\sqrt[2]{c}}\right\}$. Finally, notice that $Q\left(\sqrt{c}x,\frac{x+\sqrt{c}}{2}\right)=x^2\in\mathbb{R}\,[x]$, so combining the Theorem 54 with this remark we obtain the next corollary.

Corollary 60 *The cylinders are the unique regular tubular Polynomial Weingarten surfaces with second fundamental form of constant length (or constant Casorati curvature).*

The remaining question will be answered by the our last main Theorem, once it acts in the another direction of the previous one. More precisely, now we consider a (fixed) regular tubular surface S and we present the set $\mathcal{Q}\,(S)$ of all polynomials that vanishes in its Gaussian and mean curvatures. Furthermore, an algebraic characterizations of the polynomials are expressed as a discriminant to analyze tubular Polynomial Weingarten surfaces.

Theorem 61 *Consider a regular tubular surface S of radius $r>0$ in Euclidean (respect. Lorentzian or Hyperbolic) 3-space and let K, H be its Gaussian and mean curvatures. Denote by $\mathcal{Q}\,(S)$ the set of all polynomials $Q\in\mathbb{R}\,[x,y]$ verifying $Q\,(K,H)\equiv 0$.*

i. If S is a cylinder, then $\mathcal{Q}\,(S)=\{Q\in\mathbb{R}\,[x,y]\ ;\ r\in Rad\,(Q)\}$;

ii. If S is not a cylinder, then $\mathcal{Q}\,(S)=\{Q\in\mathbb{R}\,[x,y]\ ;\ r\in Rad^*\,(Q)\}$.

In particular, every tubular surface is Weingarten.

Proof. Let S be a cylinder and consider $Q_0 \in \mathcal{Q}(S)$, it implies that

$$Q_0(K,H) = 0$$

since the Gaussian and mean curvatures of S are precisely

$$K \equiv 0 \qquad \text{and} \qquad H = \tfrac{1}{2r}, \tag{2.47}$$

we have that $0 = Q_0(K,H) = Q\left(0, \frac{1}{2r}\right)$. Therefore $r \in Rad(Q)$ and consequently $Q_0 \in \{Q \in \mathbb{R}[x,y] \ ; r \in Rad(Q)\}$. So

$$\mathcal{Q}(S) \subset \{Q \in \mathbb{R}[x,y] \ ; r \in Rad(Q)\}.$$

Similarly, for a given $Q_0 \in \{Q \in \mathbb{R}[x,y] \ ; \ r \in Rad(Q)\}$ follows immediately from (2.47) that $0 = Q_0\left(0, \frac{1}{2r}\right) = Q_0(K,H)$ then $Q_0 \in \mathcal{Q}(S)$ and the equality of the sets is obtained

$$\mathcal{Q}(S) = \{Q \in \mathbb{R}[x,y] \ ; r \in Rad(Q)\}.$$

If S is not a cylinder and consider K and H its Gaussian and mean curvatures, in a given order. If $Q_0 \in \mathcal{Q}(S)$, it yields

$$0 = Q_0(K,H).$$

In the aim to prove that Q_0 is the ideal in $\mathbb{R}[x,y]$ generated by $xr^2 - 2ry + 1$, by Theorem 44 it is only necessary verify

$$Q_0\left(\tfrac{x}{r}, \tfrac{xr+1}{2r}\right) \equiv 0 \ \forall x.$$

Suppose that $Q_0\left(\frac{x}{r}, \frac{xr+1}{2r}\right)$ is not null, then by Lema 50 we have that S is a cylinder and we arrive at a contradiction. Then

$$\mathcal{Q}(S) \subset \left\langle xr^2 - 2yr + 1 \right\rangle.$$

Conversely, consider $Q_0 \in \left\langle xr^2 - 2yr + 1 \right\rangle$, the Theorem 44 provides that

$$0 = Q_0\left(\tfrac{k_1}{r}, \tfrac{k_1 r+1}{2r}\right) = Q_0(K,H)$$

thus $Q_0 \in \mathcal{Q}(S)$. Hence it gives

$$\mathcal{Q}(S) = \left\langle xr^2 - 2yr + 1 \right\rangle.$$

∎

An outcome of our last main theorem is presented in the next example:

Example 62 *Denote by* $\mathbf{C}$ *the set of all cylinders in Euclidean (respect. Lorentzian or Hyperbolic)* 3*-space and by* $C_{r,\gamma}$ *the cylinder of radius* r *around the straight line* γ*. Then, the ideal generated by* x *in* $\mathbb{R}[x,y]$ *verify*

$$\langle x \rangle = \bigcap_{S\in\mathbf{C}} \mathcal{Q}(\mathbf{C}) = \bigcap_{r\in\mathbb{R}_+^*} \mathcal{Q}(C_{r,\gamma}),$$

for every fixed straight line γ.

In fact, if we consider $Q(x,y) \in \bigcap_{S\in\mathbf{C}} \mathcal{Q}(S)$ it follows that

$$Q\left(0, \tfrac{1}{2r}\right) = 0$$

for every radius r of a cylinder. Therefore, $Q(0,y) \in \mathbb{R}[y]$ is identically null and because $Q(x,y)$ is a non trivial relation, it implies

$$Q(x,y) = xR(x,y)$$

where $R(x,y) \in \mathbb{R}[x,y]$ not null. The conversely is straightforward.

The interesting in the second equality lies on the fact that is not necessary consider every cylinder, it is only needed consider every cylinder around a fixed straight line γ.

Finally, Theorem 61 promotes the answer (and proof) for the question about non linear Weingarten surfaces for the case of tubular surfaces.

Theorem 63 *The cylinders are the only tubular surfaces that verify a true nonlinear relation $Q(K,H) \equiv 0$.*

We conclude this work achieving a complete classification of Polynomial Weingarten tubular surfaces for any polynomial relation. Furthermore, we presented the necessary and sufficient condition for existence of Polynomial Weingarten surfaces and we also proved that cylinders are the only non linear Weingarten tubular surfaces.

The fully description of the sets $\mathcal{S}(Q)$ and $\mathcal{Q}(S)$ allows us to construct and analyse families of tubular surfaces that vanishes a relation and define but also study polynomials whose roots are the Gaussian and mean curvatures of a tubular surface, respectively. Therefore, the investigation of Polynomial Weingarten tubular surfaces is completed.

2.3.1 (k_1, k_2)-Weingarten Tubular Surfaces

Motivated by the observation that every polynomial relation $P(K,H) = 0$, where K and H denotes the Gaussian and mean curvatures, can always be rewritten as a polynomial relation between the principal curvatures, but the conversely is not (necessarily) true. We propose to ourselves to investigate the class of tubular surfaces whose principal curvatures verify an arbitrary polynomial relation. Hence, these surfaces, sometimes called (k_1, k_2)-Weingarten surfaces are the focus of our study in this section.

Inspired by the Theorem 61 and Theorem 54, this section fully classificates Weingarten regular tubular surfaces in Euclidean, Lorentzian and Hyperbolic 3-space whose principal curvatures vanishes an arbitrary polynomial relation. Furthermore, our results determines the set of all polynomials vanished by the principal curvatures of a given regular tubular surface. More precisely, in Euclidean case, we have the Theorem below:

Theorem 64 *Consider a regular tubular surface S and let $k_1 \leq k_2$ be its principal curvatures. Denote by $\mathcal{P}(S)$ the set of all polynomials $P \in \mathbb{R}[x,y]$ verifying $P(k_1,k_2) \equiv 0$.*

i. If S is a cylinder, then $\mathcal{P}(S) = \{P \in \mathbb{R}[x,y]\ ;\ P(0,k_2) \equiv 0\}$

ii. If S is not a cylinder, then $\mathcal{P}(S)$ is the ideal in $\mathbb{R}[x,y]$ generated by $y - k_2$.

In particular, S is a Weingarten surface.

Theorem 65 *Consider a polynomial $P(x,y) \in \mathbb{R}[x,y]$. The set $\mathcal{S}(P)$ of all regular tubular surfaces whose principal curvatures $k_1 \leq k_2$ verify $P(k_1,k_2) \equiv 0$ is composed by:*

i. Cylinders of radius $r > 0$, where $P\left(0, \frac{1}{r}\right) \equiv 0$

ii. Tubular surfaces of radius $r > 0$, where $P\left(0, \frac{1}{r}\right) \equiv 0$ and P is in the ideal of $\mathbb{R}[x,y]$ generated by $y - \frac{1}{r}$.

In order to present the main theorem in a more suitable way, we will consider the following definition:

Definition 66 *A polynomial $P(x,y) = x^n A_n(y) + ... + xA_1(y) + A_0(y)$ is called (k_1,k_2)-**tubular**, if $A_0(y)$ has strictly positive roots. In this case, the set*

$$R_P = \left\{ r \in \mathbb{R}\ ;\ \frac{1}{r} \text{ is strictly positive roots of } A_0(y) \right\}$$

*is called the radius of P; Otherwise,it will be called **non** (k_1,k_2)-**tubular**.*

The terminology used in above definition is motivated, as we will see, by the fact that exists a Weingarten tubular surface verifying a polynomial relation if and only if the polynomial is tubular.

Before reformulating the statement of Theorems mentioned in the beginning of the section, let us present a proposition showing that the condition of a polynomial belong to the ideal generated by $y - \frac{1}{r}$ can be replaced by another one that is easier to check.

Proposition 67 *Consider a (k_1,k_2)-tubular polynomial $P(x,y) \in \mathbb{R}[x,y]$ and $r \in Rad_P$. Then P is divided by $y - \frac{1}{r}$ if and only if $P\left(x, \frac{1}{r}\right) \in \mathbb{R}[x]$ is identically null.*

Proof. Consider polynomials $A_i(y) \in \mathbb{R}[y]$ as in Definition 45. and write

$$P\left(x, \frac{1}{r}\right) = x^n A_n\left(\frac{1}{r}\right) + ... + xA_1\left(\frac{1}{r}\right) + A_0\left(\frac{1}{r}\right),$$

If $P\left(x,\frac{1}{r}\right)$ is identically null, we have $A_i\left(\frac{1}{r}\right)=0$, for every $i=1,\ldots,n$. Then each polynomial $A_i(y)$ is factorized as $A_i(y)=\left(y-\frac{1}{r}\right)B_i(y)$ and this implies that

$$P(x,y)=\left(y-\frac{1}{r}\right)\left(x^nB_n(y)+...+xB_1(y)+B_0(y)\right).$$

The other direction is straightforward. ■

To illustrate the previous definition and propositon, let us present below the full classification of (k_1,k_2)-tubular polynomial of degree 2.

Example 68 *A polynomials of degree 2 is (k_1,k_2)-tubular if and only if is written as*

$$\begin{aligned} P_1(x,y) &= ax^2+bxy+cy^2+dx-c\left(\frac{1}{r_1}+\frac{1}{r_2}\right)y+c\frac{1}{r_1r_2}; \\ P_2(x,y) &= ax^2+bxy+dx+ey-\frac{e}{r_1}; \\ P_3(x,y) &= ax^2+bxy+dx. \end{aligned}$$

where a, b, d, $e\in\mathbb{R}$, $c\neq0$, $e\neq0$, $r_1>0$ and $r_2\in\mathbb{R}$. In these cases

$$Rad_{P_1}=\begin{cases}\{r_1,r_2\} & \text{if } r_2>0\\ \{r_1\} & \text{if } r_2\leq0\end{cases},\qquad Rad_{P_2}=\{r_1\},\qquad Rad_{P_3}=(0,+\infty).$$

Moreover, $P_i\left(x,\frac{1}{r_j}\right)$ is identically null if and only if

$$a=0 \qquad \text{and} \qquad b+dr_j=0,$$

for $1\leq i\leq 3$ and $1\leq j\leq 2$.

In view of the above discussion, Theorems 65 & 64 stated in the beginning of this section are consequences of the following theorem:

Theorem 69 *Let $P(x,y)$ be a polynomial in $\mathbb{R}[x,y]$. Then:*

1. *There is a Weingarten tubular surface in Euclidean, Lorentzian or Hyperbolic 3-spaces of radius $r>0$ verifying $P(k_1,k_2)=0$ if and only if P is (k_1,k_2)-tubular and $r\in R_p$.*

Moreover, assuming P is (k_1,k_2)-tubular and $r\in R_p$, we have

2. *$P\left(x,\frac{1}{r}\right)\in\mathbb{R}[x]$ is identically null if and only if every tubular surface of radius $r\in R_p$ verify $P(k_1,k_2)=0$.*

3. *$P\left(x,\frac{1}{r}\right)\in\mathbb{R}[x]$ is not null if and only if cylinders of radius $r\in R_p$ are the only tubular surfaces verifying $P(k_1,k_2)=0$.*

Proof. First of all, let $A_i(y) \in \mathbb{R}[y]$ be polynomials as in Definition (66).

If $P(x,y)$ is a (k_1,k_2)-tubular polynomial and $r \in R_P$, it is easy to see that right cylinders of radius r verify $P(k_1,k_2)=0$. Conversely, suppose there is a tubular surface S verifying

$$0 = P(k_1,k_2) = (k_1)^n A_n(k_2) + \ldots + k_1 A_1(k_2) + A_0(k_2) \tag{2.48}$$

and consider the polynomial

$$P\left(x,\frac{1}{r}\right) = x^n A_n\left(\frac{1}{r}\right) + \ldots + x A_1\left(\frac{1}{r}\right) + A_0\left(\frac{1}{r}\right). \tag{2.49}$$

By Definition (66), we only need to prove that $A_0\left(\frac{1}{r}\right)=0$.

Suppose that $A_0\left(\frac{1}{r}\right)$ is not null. Follows from (2.48) and (2.49) that $k_1(s,t)$ is a non zero constant function. More precisely, there is $\lambda \neq 0$ such that

$$k_1(s,t) = \frac{\kappa(s)\,\mu(t)\,\varepsilon_B}{1 + r\kappa(s)\,\mu(t)\,\varepsilon_B} = \lambda. \tag{2.50}$$

The above equation provides

$$\varepsilon_B \kappa(s)(1-r\lambda)\,\mu(t) = \lambda \tag{2.51}$$

which implies that $\kappa(s)(1-r\lambda) \neq 0$, for every $s \in (a,b)$. Then,

$$\mu(t) = \frac{\lambda}{\varepsilon_B \kappa(s)(1-r\lambda)}, \tag{2.52}$$

for every $(s,t) \in (a,b)\times\mathbb{R}$. This implies that μ is a constant function, which is an absurd because $\mu(t)$ represents $\delta\cos t$, $\sin t$, $\sinh t$ or $\delta\cosh t$. This concludes the proof of item (1).

Assume that P is (k_1,k_2)-tubular and fix $r \in R_p$. In this case

$$P\left(x,\frac{1}{r}\right) = x^n A_n\left(\frac{1}{r}\right) + \ldots + x A_1\left(\frac{1}{r}\right).$$

If $P\left(x,\frac{1}{r}\right)$ is identically null, follows from (2.10) and Propostion 67 that every tubular surface of radius $r \in R_P$ verifies $P(k_1,k_2)=0$.

Suppose that $P\left(x,\frac{1}{r}\right)$ is not identically null and consider a tubular surface S verifying $P(k_1,k_2)$. In this case, the principal curvature $k_1(s,t)$ of S must be a constant function. Moreover, $k_1(s,t)$ must be identically null because, as proved in Item (1), we arrive in a contradiction when $k_1(s,t)$ is a non zero constant function. Then

$$k_1(s,t) = \frac{\kappa(s)\,\mu(t)\,\varepsilon_B}{1 + r\kappa(s)\,\mu(t)\,\varepsilon_B} = 0 \qquad \therefore\ \kappa(s)\,\mu(t) = 0,$$

for every $(s,t) \in (a,b)\times\mathbb{R}$. Since $\mu(t)$ represents $\cos t$, $\sin t$, $\sinh t$ or $\cosh t$, the above equality implies that the curvature κ of the central curve of S is identically null. As a consequence, the cylinders are the only tubular surfaces of radius r verifying $P(k_1,k_2)=0$. With this assertion we finish the proof of item (2) and (3). ■

Remark 70 *Notice that if exists a Weingarten tubular surface of radius $r > 0$ of some causality verifying $P(k_1, k_2) = 0$, in fact, exists a Weingarten tubular surface of radius $r > 0$ verifying $P(k_1, k_2) = 0$, for every possible causality.*

A special and very studied polynomial relation among the principal curvatures, as well as among the Gaussian and the mean curvatures, is the linear one.

Corollary 71 *Consider $a, b, c \in \mathbb{R}$, not all null.*

1. *There is a tubular surface in Euclidean Lorentzian or Hyperbolic 3-space verifying the relation*
$$ak_1 + bk_2 = c \tag{2.53}$$
if and only if $b = c = 0$ or $bc > 0$.

More precisely, we have:

2. *$b = c = 0$ if and only if every right cylinder verify Relation (2.53);*
3. *$a = 0$ and $bc > 0$ if and only if every tubular surfaces of radius $\frac{b}{c}$ verify Relation (2.53);*
4. *$a \neq 0$ and $bc > 0$ if and only if right cylinders of radius $\frac{b}{c}$ verify Relation (2.53).*

Proof of the Euclidean case:. Consider the polynomial $P(x, y) = ax + by - c$. To prove that P is tubular, it is sufficient to observe that $A_0(y) = by - c$ has positive roots if and only if $b = c = 0$ or $bc > 0$. Moreover, we have

$$R_P = (0, +\infty) \qquad \text{or} \qquad R_P = \left\{ \frac{b}{c} \right\}$$

is these cases.

When $b = c = 0$, we must have $a \neq 0$ by hypothesis. Hence $P(x, y) = ax$ is clearly not null. Suppose now $bc > 0$. So $P\left(x, \frac{c}{b}\right) = ax$ vanishes identically if and only if $a = 0$. ■

Chapter 3

Weingarten Cyclic Surfaces

In this chapter we will classify Polynomial Weingarten cyclic surfaces. We start our discussion in Section 3.1 with the definition of cyclic surface in the Euclidean 3-space, thus we proceed to the investigation of cyclic surfaces in the Lorentzian 3-space, where for each combination of causality of central curve, causality of principal normal and, finally, foliation of cyclic surface (*i.e.* if the normal sections can be Lorentzian circles or Lorentzian hyperboles), there is a specific type of cyclic surface.

In the Section 3.2, motivated by the challenges that we face in the study of Polynomial Weingarten cyclic surface, we obtain several results concerning polynomial characterization, degree of composition of polynomials and, also, several summatory identities. A sample of the work presented in the Section 3.2 is that we are primarily interested in conditions to decide if either a polynomial $P(x,y)$ may be factorized as $\left(-x+y^2\right)^n R(x,y)$, where $R(x,y)\in\mathbb{R}[x,y]$ and $n\in\mathbb{N}$ or not. This is relevant, once we remark that the algebraic condition $-x+y^2$ provides a geometric feature of the surface. More precisely, in $\mathbb{E}^3$, a surface whose curvatures verify $-K+H^2$ is a totally umbilic surface.

Once, we achieve several polynomial results, they are applied in the Section 3.3 where we display our main theorem that fully describe Polynomial Weingarten cyclic surfaces. The theorem 102 furnishes that a Weingarten cyclic surface must be a (smooth) combination of tubular surface and rotational surface. Moreover, we present conditions over the polynomial that allows us to conclude when the cyclic surface is, indeed, a globally rotational surface. Finally, we give several applications of our main result.

3.1 Cyclic Surfaces

In this section we will discuss cyclic surfaces in the Euclidean and in the Lorentzian 3-space. We start in section's first part that we study the class of cyclic surface in Euclidean space where we will exhibit the condition of regularity and it is presented the Gaussian and mean curvatures.

In the second part that correspondes to the Lorentzian 3-space, once again, we will focus in the several possibilities of foliation for a cyclic surface. In other words, cyclic surfaces may have Lorentzian circles or Lorentzian hyperboles as foliation (which is a consequence of the index of the metric).

Moreover, we will use a different moving frame than the Frenet one to parametrize our cyclic surface. Specifically, here we will use the Monge frame that is a moving frame that better suits our goals. The reason of this choice lies on the necessity to have the expression of Gaussian and mean curvatures in a form that we can easily apply our main theorem (see Section 3.3 for more details). Then, we will present the Gaussian and mean curvatures of the cyclic surfaces in $\mathbb{L}^3$.

3.1.1 Euclidean Cyclic Surfaces

In the Euclidean 3-space, a **cyclic surface** around a regular curve $\gamma : (a,b) \rightarrow \mathbb{R}^3$, called **central curve**, with radius $r : (a,b) \rightarrow \mathbb{R}$ such that $r(s) > 0$ for every $s \in (a,b)$, is the set obtained by the union of all circles $S_{r(s)}(\gamma(s))$ of radius $r(s) > 0$ and center $\gamma(s)$ contained in the normal planes $T_s\gamma^{\perp}$ of γ.

In intervals where γ is birregular, a cyclic surface can be parametrized by the application:

$$\psi(s,t) = \gamma(s) + (r(s)\cos t)\, N(s) + (r(s)\sin t)\, B(s), \tag{3.1}$$

where $\{T, N, B\}$ is the Frenet frame.

To avoid undesirable cases, we will include in the definition an additional condition for the regularity of curvature of central curve. We will say that a cyclic surface is regular if there is an interval $\Sigma \subset \mathbb{N}$ (possibly infinite) and there is a strictly increasing sequence $(\xi_n)_{n\in\Sigma}$ in (a,b) of isolated points such that

i. The curvature κ of γ verify $\kappa(\xi_n) = 0$ for every $n \in \Sigma$

ii. For every $n \in \Sigma$, consider the set

$$\Sigma_n = \left\{t \in \left(\xi_n, \xi_{n+1}\right) \;;\; \kappa(t) = 0\right\}$$

then

$$\operatorname{int}\Sigma_n = \emptyset \qquad \text{or} \qquad \operatorname{int}\Sigma_n = \left(\xi_n, \xi_{n+1}\right).$$

Proposition 72 *Given a cyclic surface with central curve γ and radius $r(s) > 0$ in Euclidean 3-space. Consider an interval I where γ is birregular. The parametrization (3.1) is an immersion if and only if*

$$\beta(s,t) = r'(s)^2 + (1 - \kappa(s)\, r(s)\cos t)^2 > 0,$$

for every $(s,t) \in I \times \mathbb{R}$.

Proof. Using (1.1) and (3.1) we obtain

$$\psi_s = (1 - r\kappa\cos t)\, T + (r'\cos t - r\tau\sin t)\, N + (r'\sin t + r\tau\cos t)\, B \tag{3.2}$$

and

$$\psi_t = -\left(r\sin t\right)N + \left(r\cos t\right)B \tag{3.3}$$

therefore

$$\left\|\psi_s \times \psi_t\right\|_{\mathbb{E}}^2 = r^2\left(s\right)\beta\left(s,t\right)$$

since $r\left(s\right) > 0$ everywhere, we achieved the desired. ∎

Definition 73 *A cyclic surface with central curve γ and radius $r\left(s\right) > 0$ in Euclidean 3-space is called* ***regular*** *if γ is parametrized by arc length and*

$$\beta\left(s,t\right) = r'\left(s\right)^2 + \left(1-\kappa\left(s\right)r\left(s\right)\cos t\right)^2 > 0, \tag{3.4}$$

for every $(s,t) \in I \times \mathbb{R}$.

Proposition 74 *Consider a regular cyclic surface with central curve γ and radius $r\left(s\right) > 0$ in Euclidean 3-space. The Gaussian and mean curvature are respectively given by*

$$\begin{aligned} K &= \frac{\left(r^4\kappa^4\right)\cos^4 t - 3r^3\kappa^3\cos^3 t + r^2\kappa\left(3\kappa(r')^2 + r\kappa' r' + 3\kappa - rr''\kappa\right)\cos^2 t + r^3 r'\kappa^2\tau\cos t\sin t}{r^2\left((r')^2 + (1-\kappa(s)r(s)\cos t)^2\right)^2} \\ &\quad + \frac{-r\left(2\kappa(r')^2 + r\kappa' r' + \kappa - 2rr''\kappa\right)\cos t - r^2 r'\kappa\tau\sin t - \left(r(r')^2\kappa^2 + r''\right)}{r^2\left((r')^2 + (1-\kappa(s)r(s)\cos t)^2\right)^2} \end{aligned}$$

and

$$H = \frac{-2r^4\kappa^3\cos^3 t + 5r^3\kappa^2\cos^2 t - r^2\left(3\kappa(r')^2 + r\kappa' r' + 4\kappa - rr''\kappa\right)\cos t - r^3 r'\kappa\tau\sin t + r^2\left((r')^2 - rr'' + 1\right)}{2r^2\left((r')^2 + (1-\kappa(s)r(s)\cos t)^2\right)^{\frac{3}{2}}}.$$

Proof. The coefficients of first fundamental form are

$$E = \left(1 - r\kappa\cos t\right)^2 + r^2\tau^2 + \left(r'\right)^2, \qquad F = r^2\tau, \qquad G = r^2$$

and the coefficients of second fundamental form are

$$\begin{aligned} e &= \frac{r'\left(-r\kappa'\cos t - 2r'\kappa\cos t + r\kappa\tau\sin t\right) + (r\kappa\cos t - 1)\left(-r\kappa^2\cos^2 t + \kappa\cos t - r\tau^2 + r''\right)}{\sqrt[2]{\beta}} \\ f &= \frac{r\tau(1-\kappa(s)r(s)\cos t) + \left(r'\right)r\kappa\sin t}{\sqrt[2]{\beta}} \\ g &= \frac{r(1-\kappa(s)r(s)\cos t)}{\sqrt[2]{\beta}} \end{aligned}$$

By the Formula 1.5 and by the above coefficients, a straightforward calculus provide the desired. ∎

Notation 75 *For practical purpposes, we will write the Gaussian and mean curvatures of a regular cyclic surface as*

$$K = \frac{\Delta}{r^2\beta^2} \qquad \text{and} \qquad H = \frac{\alpha}{2r^2\beta^{\frac{3}{2}}} \tag{3.5}$$

where

$$\begin{aligned}
\Delta &= r^4\kappa^4\cos^4 t - 3r^3\kappa^3\cos^3 t + r^2\kappa\left(3\kappa(r')^2 + r\kappa' r' + 3\kappa - rr''\kappa\right)\cos^2 t \\
&\quad + r^3 r'\kappa^2\tau\cos t\sin t - r\left(2\kappa(r')^2 + r\kappa' r' + \kappa - 2rr''\kappa\right)\cos t \\
&\quad - r^2 r'\kappa\tau\sin t - r\left(r(r')^2\kappa^2 + r''\right) \\
\alpha &= -2r^4\kappa^3\cos^3 t + 5r^3\kappa^2\cos^2 t - r^2\left(3\kappa(r')^2 + r\kappa' r' + 4\kappa - rr''\kappa\right)\cos t \\
&\quad - r^3 r'\kappa\tau\sin t + r\left((r')^2 - rr'' + 1\right) \\
\beta &= (r')^2 + (1 - r\kappa\cos t)^2
\end{aligned} \tag{3.6}$$

3.1.2 Lorentzian Cyclic Surfaces

In the Lorentzian 3-space, a **cyclic surface** around a regular curve $\gamma : (a,b) \to \mathbb{L}^3$, called **central curve**, with radius $r : (a,b) \to \mathbb{R}$ such that $r(s) > 0$ for every $s \in (a,b)$, is the set obtained by the union of all Lorentzian circles or Lorentzian hyperboles, $S_{r(s)}(\gamma(s))$, of radius $r(s) > 0$ and center $\gamma(s)$ contained in the normal planes $T_s\gamma^{\perp}$ of γ.

In intervals where γ is birregular, a cyclic surface can be parametrized by the application:

$$\psi(s,t) = \gamma(s) + r(s)\,b(t)\,N(s) + r(s)\,c(t)\,B(s), \tag{3.7}$$

where $\{T, N, B\}$ is the Frenet frame and $b(t)$, $c(t)$ are smooth function that we will study and explicit it later.

To avoid undesirable cases, we will include in the definition an additional condition for the regularity of curvature of central curve. We will say that a cyclic surface is regular if there is an interval $\Sigma \subset \mathbb{N}$ (possibly infinite) and there is a strictly increasing sequence $(\xi_n)_{n\in\Sigma}$ in (a,b) of isolated points such that

i. The curvature κ of γ verify $\kappa(\xi_n) = 0$ for every $n \in \Sigma$

ii. For every $n \in \Sigma$, consider the set

$$\Sigma_n = \left\{t \in \left(\xi_n, \xi_{n+1}\right)\ ;\ \kappa(t) = 0\right\}$$

then

$$\operatorname{int}\Sigma_n = \emptyset \qquad \text{or} \qquad \operatorname{int}\Sigma_n = \left(\xi_n, \xi_{n+1}\right).$$

Now, let us comeback to the investigation of the functions $b(t)$ and $c(t)$, we decide express (3.7) with a generic parametrization because for each fixed $s_0 \in (a,b)$, we have a Lorentzian circle or Lorentzian hyperbole in $\mathbb{L}^3$, then the following system must be verified

$$\varepsilon r^2 = g_{\mathbb{L}}\left(\psi(s_0,t) - \gamma(s_0), \psi(s_0,t) - \gamma(s_0)\right) = \varepsilon_N r^2 b^2 + \varepsilon_B r^2 c^2,$$

where $\varepsilon \in \{-1, 1\}$ is fixed, depending on if $S_{r(s)}(\gamma(s))$ is a **Lorentzian circle** ($\varepsilon = 1$) which is given by the set

$$S_{r(s)}(\gamma(s)) = \left\{ x \in \mathbb{L}^3 \ ; \ g_{\mathbb{L}}(x - \gamma(s), x - \gamma(s)) = r(s)^2 \right\};$$

or if $S_{r(s)}(\gamma(s))$ is a **Lorentzian hyperboles** ($\varepsilon = -1$) which is given by the set

$$S_{r(s)}(\gamma(s)) = \left\{ x \in \mathbb{L}^3 \ ; \ g_{\mathbb{L}}(x - \gamma(s), x - \gamma(s)) = -r(s)^2 \right\}.$$

Therefore, the solutions of the system

$$\varepsilon_N b(t)^2 + \varepsilon_B c(t)^2 = \varepsilon.$$

are given by chosening the causality of central curve (ε_T), of principal normal (ε_N) and the normal section (ε). Hence, we have the below cases to consider:

If $\gamma' = T$ is spacelike, $\varepsilon_T = 1$, it yields that $\varepsilon_N \varepsilon_B = -1$, so we have the following cases to study:

1. If $\varepsilon = 1$, then $0 < r = \varepsilon_N b^2 + \varepsilon_B c^2$, in the case that $\varepsilon_N = -1$, it implies that
$$b = \sinh t \qquad \text{and} \qquad c = \cosh t \ ;$$

2. If $\varepsilon = 1$, then $0 < r = \varepsilon_N b^2 + \varepsilon_B c^2$, in the case that $\varepsilon_N = 1$, it implies that
$$b = \cosh t \qquad \text{and} \qquad c = \sinh t \ ;$$

3. If $\varepsilon = -1$, then $r = \varepsilon_N b^2 + \varepsilon_B c^2 < 0$, in the case that $\varepsilon_N = -1$, it implies that
$$b = \cosh t \qquad \text{and} \qquad c = \sinh t \ ;$$

4. If $\varepsilon = -1$, then $r = \varepsilon_N b^2 + \varepsilon_B c^2 < 0$, in the case that $\varepsilon_N = 1$, it implies that
$$b = \sinh t \qquad \text{and} \qquad c = \cosh t \ .$$

On the other hand, if we have $\gamma' = T$ timelike, $\varepsilon_T = -1$, it yields that $\varepsilon_N = 1 = \varepsilon_B$, thus we have the following cases to study:

5. If $\varepsilon = 1$, then $0 < r = b^2 + c^2$ therefore

$$b = \cos t \qquad \text{and} \qquad c = \sin t \ ;$$

6. If $\varepsilon = -1$, then $r = b^2 + c^2 < 0$ follows the non existence of this case.

In the next calculus we will present each possible conception of cyclic surface, that is, we will rewrite (3.7) for each option of $b(t)$ and $c(t)$. In other words, we will parametrize the cyclic surface for each of the above cases. Moreover, for each of them, we will exhibit the Gaussian and mean curvature.

The applied technique, however, is a little different than the one used in the rest of the text, once we employ the Monge frame (instead of the Frenet frame), but before to proceed to the discussion of moving frames, let us explain the reason to make that choice.

In the aim to accomplish a theorem that holds for cyclic surfaces in the Euclidean and in the Lorentzian 3-space, it is convenient that Gaussian and mean curvatures were similar between the different spaces. So, the Monge frame is the more suitable frame for our purpposes.

As you may known, the moving frame is a classical differential geometry method that transcribes (extrinsic) geometrical features trhough the study of the notion of a local ordered basis of a vector space.

The Monge frame has the particular property that its undo the torsion presented in the surface, however, for cyclic surfaces, undo the torsion preserves the geometric shape of the surface, since for circles we just consider a rotated basis. Thus, the cyclic surface maintain the geometric form, but the calculus become more convenient for us.

If the surface is of the type 1, the parametrization is

$$\phi\left(s,t\right)=\alpha+r\left(N_{\psi}\sinh t+B_{\psi}\cosh t\right)$$

where

$$N_{\psi}=\cosh\psi N+\sinh\psi B \qquad \text{and} \qquad B_{\psi}=\sinh\psi N+\cosh\psi B$$

with $\{T,N,B\}$ is the Frenet frame and $\psi=-\tau$.

As a matter of fact, the derivative of N_{ψ} provides

$$\left(N_{\psi}\right)'=\left(N_{\psi}\right)^{'}=\kappa\cosh\psi T+\left(\tau+\psi'\right)\sinh\psi N+\left(\tau+\psi'\right)\cosh\psi B$$

and since $\psi=-\tau$ it follows

$$\left(N_{\psi}\right)'=\kappa\cosh\psi T.$$

Similarly, we have that

$$\left(B_{\psi}\right)'=\kappa\sinh\psi T+\left(\tau+\psi'\right)\cosh\psi N+\left(\tau+\psi'\right)\sinh\psi B$$

therefore we conclude that

$$\left(B_{\psi}\right)'=\kappa\sinh\psi T.$$

Then, the Monge Frame in this case is

$$\begin{pmatrix} T' \\ N_{\psi}' \\ B_{\psi}' \end{pmatrix}=\begin{pmatrix} 0 & \kappa & 0 \\ \kappa\cosh\psi & 0 & 0 \\ \kappa\sinh\psi & 0 & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We remark that causality of vectors are perserved, once we have

$$\langle N_{\psi},N_{\psi}\rangle=-1 \qquad \text{and} \qquad \langle B_{\psi},B_{\psi}\rangle=1.$$

Moreover, $\{T,N_{\psi},B_{\psi}\}$ is a orthonormal basis.

Proposition 76 *For cyclic surface with central curve $\gamma(s)$ and radius $r(s) > 0$ of the type 1 in the Lorentzian 3-space, the Gaussian and mean curvatures are as follows:*

$$K = \frac{\left(r^4\kappa^4\right)\sinh^4(t+\psi)+\left(3r^3\kappa^3\right)\sinh^3(t+\psi)+r^2\kappa\left(3\kappa(r')^2+r\kappa'r'+3\kappa-rr''\kappa\right)\sinh^2(t+\psi)+r^3r'\kappa^2\psi'\sinh(t+\psi)\cosh(t+\psi)}{r^2\left((r\kappa\sinh(t+\psi)+1)^2+(r')^2\right)^2}$$
$$+\frac{r\left(2\kappa(r')^2+r\kappa'r'+\kappa-2rr''\kappa\right)\sinh(t+\psi)+r^2r'\kappa\psi'\cosh(t+\psi)-r\left(r''-r(r')^2\kappa^2\right)}{r^2\left((r\kappa\sinh(t+\psi)+1)^2+(r')^2\right)^2}$$

$$H = \frac{\left(2r^4\kappa^3\right)\sinh^3(t+\psi)+\left(5r^3\kappa^2\right)\sinh^2(t+\psi)+r^2\left(3\kappa(r')^2+r\kappa'r'+4\kappa-rr''\kappa\right)\sinh(t+\psi)}{2r^2(\varepsilon)^{\frac{3}{2}}\left((r\kappa\sinh(t+\psi)+1)^2+(r')^2\right)^{\frac{3}{2}}}$$
$$+\frac{r^3r'\kappa\psi'\cosh(t+\psi)+r\left((r')^2-rr''+1\right)}{2r^2(\varepsilon)^{\frac{3}{2}}\left((r\kappa\sinh(t+\psi)+1)^2+(r')^2\right)^{\frac{3}{2}}}$$

Proof. The coefficients of first fundamental form are

$$E = (r\kappa\cosh(t+\psi)+1)^2 + (r')^2, \qquad F = 0, \qquad G = -r^2$$

and the coefficients of second fundamental form are

$$e = \frac{\left(-r^2\kappa^3\right)\sinh^3(t+\psi)+\left(-2r\kappa^2\right)\cosh^2(t+\psi)+\left(-rr'\kappa\psi'\right)\cosh(t+\psi)}{\sqrt[2]{-\varepsilon\left(-\left((r\kappa\sinh(t+\psi)+1)^2+(r')^2\right)\right)}}$$
$$+\frac{+\left(rr''\kappa-r'\left(r\kappa'+2r'\kappa\right)-\kappa\right)\sinh(t+\psi)+\left(2r\kappa^2+r''\right)}{\sqrt[2]{-\varepsilon\left(-\left((r\kappa\sinh(t+\psi)+1)^2+(r')^2\right)\right)}}$$
$$f = \frac{-rr'\kappa\cosh(t+\psi)}{\sqrt[2]{-\varepsilon\left(-\left((r\kappa\sinh(t+\psi)+1)^2+(r')^2\right)\right)}}$$
$$g = \frac{r(r\kappa\sinh(t+\psi)+1)}{\sqrt[2]{-\varepsilon\left(-\left((r\kappa\sinh(t+\psi)+1)^2+(r')^2\right)\right)}}$$

By the Formula 1.6 and by the above coefficients, a straightforward calculus provide the desired. ■

If the surface is of the type 2, the parametrization is

$$\phi(s,t) = \alpha + r\left(N_\psi\cosh t + B_\psi\sinh t\right)$$

where

$$N_\psi = \cosh\psi N + \sinh\psi B \qquad \text{and} \qquad B_\psi = \sinh\psi N + \cosh\psi B$$

with $\{T, N, B\}$ is the Frenet frame and $\psi = -\tau$.

As a matter of fact, the derivative of N_ψ provides

$$(N_\psi)' = -\kappa\cosh\psi T + \left(\tau+\psi'\right)N\sinh\psi + \left(\tau+\psi'\right)B\cosh\psi$$

and since $\psi = -\tau$ it follows

$$(N_\psi)' = -\kappa\cosh\psi T.$$

Similarly, we have that

$$(B_\psi)' = -\kappa \sinh\psi T + \left(\tau + \psi'\right) N \cosh\psi + \left(\tau + \psi'\right) B \sinh\psi$$

therefore we conclude that

$$(B_\psi)' = -\kappa \sinh\psi T.$$

Then, the Monge Frame in this case is

$$\begin{pmatrix} T' \\ N'_\psi \\ B'_\psi \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa\cosh\psi & 0 & 0 \\ -\kappa\sinh\psi & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We remark that causality of vectors are perserved, once we have

$$\langle N_\psi, N_\psi \rangle = 1 \qquad \text{and} \qquad \langle B_\psi, B_\psi \rangle = -1.$$

Moreover, $\{T, N_\psi, B_\psi\}$ is a orthonormal basis.

Proposition 77 *For cyclic surface with central curve $\gamma(s)$ and radius $r(s) > 0$ of the type 2 in the Lorentzian 3-space, the Gaussian and mean curvatures are as follows:*

$$K = \frac{\left(r^4\kappa^4\right)\cosh^4(t+\psi) - 3r^3\kappa^3\cosh^3(t+\psi) + r^2\kappa\left(3\kappa(r')^2 + r\kappa' r' + 3\kappa - rr''\kappa\right)\cosh^2(t+\psi) + r^3 r'\kappa^2\psi'\cosh(t+\psi)\sinh(t+\psi)}{(r^2)\left((1-r\kappa\cosh(t+\psi))^2 + (r')^2\right)^2}$$
$$+ \frac{-r\left(2\kappa(r')^2 + r\kappa' r' + \kappa - 2rr''\kappa\right)\cosh(t+\psi) + r^2 r'\kappa\psi'\sinh(t+\psi) - \left(r^2(r')^2\kappa^2 + r'' r\right)}{(r^2)\left((1-r\kappa\cosh(t+\psi))^2 + (r')^2\right)^2}$$

$$H = \frac{-2r^4\kappa^3\cosh^3(t+\psi) + \left(5r^3\kappa^2\right)\cosh^2(t+\psi) + r^2\left(3\kappa(r')^2 + r\kappa' r' + 4\kappa - rr''\kappa\right)\cosh(t+\psi)}{2r^2(-\varepsilon)^{\frac{1}{2}}\left((1-r\kappa\cosh(t+\psi))^2 + (r')^2\right)^{\frac{3}{2}}}$$
$$+ \frac{r^3 r'\kappa\psi'\sinh(t+\psi) + \left(r^2 r'' - r\left((r')^2 + 1\right)\right)}{2r^2(-\varepsilon)^{\frac{1}{2}}\left((1-r\kappa\cosh(t+\psi))^2 + (r')^2\right)^{\frac{3}{2}}}$$

Proof. The coefficients of first fundamental form are

$$E = (1 - r\kappa\cosh(t+\psi))^2 + (r')^2, \qquad F = 0, \qquad G = -r^2$$

and the coefficients of second fundamental form are

$$e = \frac{\left(-r^2\kappa^3\right)\cosh^3(t+\psi) + \left(2r\kappa^2\right)\cosh^2(t+\psi) + \left(-2\kappa(r')^2 - r\kappa' r' - \kappa\right)\cosh(t+\psi)}{\sqrt[2]{-\varepsilon\left((1-r\kappa\cosh(t+\psi))^2 + (r')^2\right)}}$$
$$+ \frac{-rr'\kappa\psi'\sinh(t+\psi) + rr''\kappa\cosh(t+\psi) - r''}{\sqrt[2]{-\varepsilon\left((1-r\kappa\cosh(t+\psi))^2 + (r')^2\right)}}$$

$$f = \frac{-r' r\kappa\sinh(t+\psi)}{\sqrt[2]{-\varepsilon\left((1-r\kappa\cosh(t+\psi))^2 + (r')^2\right)}}$$

$$g = \frac{r(r\kappa\cosh(t+\psi) - 1)}{\sqrt[2]{-\varepsilon\left((1-r\kappa\cosh(t+\psi))^2 + (r')^2\right)}}$$

By the Formula 1.6 and by the above coefficients, a straightforward calculus provide the desired. ■

If the surface is of the type 3, the parametrization is

$$\phi(s,t) = \alpha + r(N\cosh t + B\sinh t)$$

where

$$N_\psi = \cosh\psi N + \sinh\psi B \qquad \text{and} \qquad B_\psi = \sinh\psi N + \cosh\psi B$$

with $\{T, N, B\}$ is the Frenet frame and $\psi = -\tau$.

As a matter of fact, the derivative of N_ψ provides

$$(N_\psi)' = \kappa\cosh\psi T + (\tau + \psi')\sinh\psi N + (\tau + \psi')\cosh\psi B$$

and since $\psi = -\tau$ it follows

$$(N_\psi)' = \kappa\cosh\psi T.$$

Similarly, we have that

$$(B_\psi)' = (\kappa\sinh\psi)T + (\tau + \psi')\cosh\psi N + (\tau + \psi')\sinh\psi B$$

therefore we conclude that

$$(B_\psi)' = \kappa\sinh\psi T.$$

Then, the Monge Frame in this case is

$$\begin{pmatrix} T' \\ N'_\psi \\ B'_\psi \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa\cosh\psi & 0 & 0 \\ \kappa\sinh\psi & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We remark that causality of vectors are perserved, once we have

$$\langle N_\psi, N_\psi\rangle = -1 \qquad \text{and} \qquad \langle B_\psi, B_\psi\rangle = 1.$$

Moreover, $\{T, N_\psi, B_\psi\}$ is a orthonormal basis.

Proposition 78 *For cyclic surface with central curve $\gamma(s)$ and radius $r(s) > 0$ of the type 3 in the Lorentzian 3-space, the Gaussian and mean curvatures are as follows:*

$$K = -\frac{(r^4\kappa^4)\cosh^4(t+\psi) + (3r^3\kappa^3)\cosh^3(t+\psi) + r^2\kappa\left(-3\kappa(r')^2 - r\kappa' r' + 3\kappa + rr''\kappa\right)\cosh^2(t+\psi) - r^3 r'\kappa^2\psi'\cosh(t+\psi)\sinh(t+\psi)}{r^2\left((r\kappa\cosh(t+\psi(s))+1)^2 - (r'(s))^2\right)^2}$$
$$-\frac{r\left(\kappa - r'\left(r\kappa' + 2r'\kappa\right) + 2rr''\kappa\right)\cosh(t+\psi) - r^2 r'\kappa\psi'\sinh(t+\psi) + \left(r^2(r')^2\kappa^2 + r''r\right)}{r^2\left((r\kappa\cosh(t+\psi(s))+1)^2 - (r'(s))^2\right)^2}$$

$$H = \frac{-2r^4\kappa^3\cosh^3(t+\psi) - 5r^3\kappa^2\cosh^2(t+\psi) - r^2\left(-3\kappa(r')^2 - r\kappa' r' + 4\kappa + rr''\kappa\right)\cosh(t+\psi)}{2r^2(-\varepsilon)^{\frac{3}{2}}\left((r')^2 - (r\kappa\cosh(t+\psi)+1)^2\right)^{\frac{3}{2}}}$$
$$+\frac{r^3 r'\kappa\psi'\sinh(t+\psi) + \left(r\left((r')^2 - 1\right) - r^2 r''\right)}{2r^2(-\varepsilon)^{\frac{3}{2}}\left((r')^2 - (r\kappa\cosh(t+\psi)+1)^2\right)^{\frac{3}{2}}}$$

Proof. The coefficients of first fundamental form are

$$E = (r\kappa \cosh(t+\psi)+1)^2 - (r')^2, \qquad F = 0, \qquad G = r^2$$

and the coefficients of second fundamental form are

$$\begin{aligned} e &= \frac{\left(-r^2\kappa^3\right)\cosh^3(t+\psi)+\left(-2r\kappa^2\right)\cosh^2(t+\psi)}{\sqrt[2]{-\varepsilon\left((r\kappa\cosh(t+\psi(s))+1)^2-(r'(s))^2\right)}} \\ &\quad + \frac{\left(r'\left(r\kappa'+2r'\kappa\right)-\kappa-rr''\kappa\right)\cosh(t+\psi)+rr'\kappa\psi'\sinh(t+\psi)-r''}{\sqrt[2]{-\varepsilon\left((r\kappa\cosh(t+\psi(s))+1)^2-(r'(s))^2\right)}} \\ f &= \frac{rr'\kappa\sinh(t+\psi)}{\sqrt[2]{-\varepsilon\left((r\kappa\cosh(t+\psi(s))+1)^2-(r'(s))^2\right)}} \\ g &= \frac{-r(r\kappa\cosh(t+\psi)+1)}{\sqrt[2]{-\varepsilon\left((r\kappa\cosh(t+\psi(s))+1)^2-(r'(s))^2\right)}} \end{aligned}$$

By the Formula 1.6 and by the above coefficients, a straightforward calculus provide the desired. ■

If the surface is of the type 4, the parametrization is

$$\phi(s,t) = \alpha + r\left(N_\psi \sinh t + B_\psi \cosh t\right)$$

where

$$N_\psi = \cosh\psi N + \sinh\psi B \qquad \text{and} \qquad B_\psi = \sinh\psi N + \cosh\psi B$$

with $\{T, N, B\}$ is the Frenet frame and $\psi = -\tau$.

As a matter of fact, the derivative of N_ψ provides

$$(N_\psi)' = (-\kappa\cosh\psi)\,T + \left(\tau+\psi'\right)N\sinh\psi + \left(\tau+\psi'\right)B\cosh\psi$$

and since $\psi = -\tau$ it follows

$$(N_\psi)' = -\kappa\cosh\psi T.$$

Similarly, we have that

$$(B_\psi)' = (-\kappa\sinh\psi)\,T + \left(\tau+\psi'\right)N\cosh\psi + \left(\tau+\psi'\right)B\sinh\psi$$

therefore we conclude that

$$(B_\psi)' = -\kappa\sinh\psi T.$$

Then, the Monge Frame in this case is

$$\begin{pmatrix} T' \\ N'_\psi \\ B'_\psi \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa\cosh\psi & 0 & 0 \\ -\kappa\sinh\psi & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We remark that causality of vectors is perserved, once we have

$$\langle N_\psi, N_\psi\rangle = 1 \qquad \text{and} \qquad \langle B_\psi, B_\psi\rangle = -1.$$

Moreover, $\{T, N_\psi, B_\psi\}$ is a orthonormal basis.

Proposition 79 *For cyclic surface with central curve $\gamma(s)$ and radius $r(s) > 0$ of the type 4 in the Lorentzian 3-space, the Gaussian and mean curvatures are as follows:*

$$
\begin{aligned}
K &= -\frac{\left(r^4\kappa^4\right)\sinh^4(t+\psi)+\left(-3r^3\kappa^3\right)\sinh^3(t+\psi)+\left(r\left(2r\kappa^2+r\kappa\left(\kappa-r'\left(r\kappa'+2r'\kappa\right)+rr''\kappa\right)\right)-r^2(r')^2\kappa^2\right)\sinh^2(t+\psi)}{r^2\left(\left((1-r\kappa\sinh(t+\psi))^2-(r')^2\right)\right)^2} \\
&\quad -\frac{\left(-r^3r'\kappa^2\psi'\right)\sinh(t+\psi)\cosh(t+\psi)+\left(-r\left(\kappa-r'\left(r\kappa'+2r'\kappa\right)+2rr''\kappa\right)\right)\sinh(t+\psi)+r^2r'\kappa\psi'\cosh(t+\psi)+\left(rr''-r^2(r')^2\kappa^2\right)}{r^2\left(\left((1-r\kappa\sinh(t+\psi))^2-(r')^2\right)\right)^2} \\
H &= \frac{\left(2r^4\kappa^3\right)\sinh^3(t+\psi)+\left(-5r^3\kappa^2\right)\sinh^2(t+\psi)+\left(r^2\left(\kappa-r'\left(r\kappa'+2r'\kappa\right)+rr''\kappa\right)+r\left(2r\kappa-r\kappa\left((r')^2-1\right)\right)\right)\sinh(t+\psi)}{r^2(-\varepsilon)^{\frac{3}{2}}\left((1-r\kappa\sinh(t+\psi))^2-(r')^2\right)^{\frac{3}{2}}} \\
&\quad +\frac{-r^3r'\kappa\psi'\cosh(t+\psi)+\left(r\left((r')^2-1\right)-r^2r''\right)}{r^2(-\varepsilon)^{\frac{3}{2}}\left((1-r\kappa\sinh(t+\psi))^2-(r')^2\right)^{\frac{3}{2}}}
\end{aligned}
$$

Proof. The coefficients of first fundamental form are

$$E = (1 - r\kappa\sinh(t+\psi))^2 - (r')^2, \qquad F = 0, \qquad G = r^2$$

and the coefficients of second fundamental form are

$$
\begin{aligned}
e &= \frac{\left(r^2\kappa^3\right)\sinh^3(t+\psi)+\left(-2r\kappa^2\right)\sinh^2(t+\psi)}{\sqrt[2]{-\varepsilon\left((1-r\kappa\sinh(t+\psi))^2-(r')^2\right)}} \\
&\quad +\frac{\left(\kappa-r'\left(r\kappa'+2r'\kappa\right)+rr''\kappa\right)\sinh(t+\psi)+\left(-rr'\kappa\psi'\right)\cosh(t+\psi)-r''}{\sqrt[2]{-\varepsilon\left((1-r\kappa\sinh(t+\psi))^2-(r')^2\right)}} \\
f &= \frac{-r'r\kappa\cosh(t+\psi)}{\sqrt[2]{-\varepsilon\left((1-r\kappa\sinh(t+\psi))^2-(r')^2\right)}} \\
g &= \frac{r(r\kappa\sinh(t+\psi)-1)}{\sqrt[2]{-\varepsilon\left((1-r\kappa\sinh(t+\psi))^2-(r')^2\right)}}
\end{aligned}
$$

By the Formula 1.6 and by the above coefficients, a straightforward calculus provide the desired. ■

If the surface is of the type 5, the parametrization is

$$\phi(s,t) = \alpha + r\left(N_\psi \cos t + B_\psi \sin t\right)$$

where

$$N_\psi = \sin\psi N + \cos\psi B \qquad \textit{and} \qquad B_\psi = -\cos\psi N + \sin\psi B$$

with $\{T, N, B\}$ is the Frenet frame and $\psi = -\tau$.

As a matter of fact, the derivative of N_ψ provides

$$(N_\psi)' = (\kappa\sin\psi)\,T + \left(\psi' - \tau\right)\cos\psi N + \left(\tau - \psi'\right)\sin\psi$$

and since $\psi = \tau$ it follows

$$(N_\psi)' = \kappa\sin\psi T.$$

Similarly, we have that

$$(B_\psi)' = (-\kappa\cos\psi)\,T + \left(\psi' - \tau\right)\sin\psi N + \left(\psi' - \tau\right)\cos\psi B$$

therefore we conclude that

$$(B_\psi)' = -\kappa \cos \psi T.$$

Then, the Monge Frame in this case is

$$\begin{pmatrix} T' \\ N'_\psi \\ B'_\psi \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa \sin \psi & 0 & 0 \\ -\kappa \cos \psi & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We remark that causality of vectors are perserved, once we have

$$\langle N_\psi, N_\psi \rangle = 1 \qquad \text{and} \qquad \langle B_\psi, B_\psi \rangle = 1.$$

Moreover, $\{T, N_\psi, B_\psi\}$ is a orthonormal basis.

Proposition 80 *For cyclic surface with central curve $\gamma(s)$ and radius $r(s) > 0$ of the type 5 in the Lorentzian 3-space, the Gaussian and mean curvatures are as follows:*

$$\begin{aligned} K = {} & \frac{r^4\kappa^4 \sin^4(\psi-t) + 3r^3\kappa^3 \sin^3(\psi-t) + r^2\kappa\left(-3\kappa(r')^2 - r\kappa' r' + 3\kappa + rr''\kappa\right)\sin^2(\psi-t) + r^3 r'\kappa^2\psi' \sin(\psi-t)\cos(\psi-t)}{r^2\left((r')^2 - (r\kappa\sin(\psi-t)+1)^2\right)^2} \\ & + \frac{-\left(\kappa' r^2 r' - 2r''\kappa r^2 + 2\kappa r(r')^2 - \kappa r\right)\sin(\psi-t) - r^2 r'\kappa\psi'\cos(\psi-t) + \left(r^2(r')^2\kappa^2 + r'' r\right)}{r^2\left((r')^2 - (r\kappa\sin(\psi-t)+1)^2\right)^2} \\ H = {} & \frac{2r^4\kappa^3\sin^3(\psi-t) + 5r^3\kappa^2\sin^2(\psi-t) + \left(r''\kappa r^3 - \kappa' r^3 r' - 3\kappa r^2 (r')^2 + 4\kappa r^2\right)\sin(\psi-t)}{2r^2(-\varepsilon)^{\frac{3}{2}}\left(-(r\kappa\sin(\psi-t)+1)^2 + (r')^2\right)^{\frac{3}{2}}} \\ & + \frac{-r^3 r'\kappa\psi'\cos(\psi-t) - r(r')^2 + r^2 r'' + r}{2r^2(-\varepsilon)^{\frac{3}{2}}\left(-(r\kappa\sin(\psi-t)+1)^2 + (r')^2\right)^{\frac{3}{2}}} \end{aligned}$$

Proof. The coefficients of first fundamental form are

$$E = -\left(r\kappa\sin(\psi - t) + 1\right)^2 + (r')^2, \qquad F = 0, \qquad G = r^2$$

and the coefficients of second fundamental form are

$$\begin{aligned} e &= \frac{\left(r^2\kappa^3\right)\sin^3(\psi-t) + \left(2r\kappa^2\right)\sin^2(\psi-t) + \left(\kappa - r'\left(r\kappa' + 2r'\kappa\right) + rr''\kappa\right)\sin(\psi-t)}{\sqrt[2]{-\varepsilon\left(-(r\kappa\sin(\psi-t)+1)^2 + (r')^2\right)}} \\ &\quad + \frac{-rr'\kappa\psi'\cos(\psi-t) + r''}{\sqrt[2]{-\varepsilon\left(-(r\kappa\sin(\psi-t)+1)^2 + (r')^2\right)}} \\ f &= \frac{r' r\kappa\cos(\psi-t)}{\sqrt[2]{-\varepsilon\left(-(r\kappa\sin(\psi-t)+1)^2 + (r')^2\right)}} \\ g &= \frac{-r(r\kappa(\sin(\psi-t)) - 1)}{\sqrt[2]{-\varepsilon\left(-(r\kappa\sin(\psi-t)+1)^2 + (r')^2\right)}} \end{aligned}$$

By the Formula 1.6 and by the above coefficients, a straightforward calculus provide the desired. ■

3.2 Polynomial results for Cyclic Surfaces

This section is dedicated to the investigation of several parallel topics whose necessity for understanding emerged from the fact that they were crucial problems in the geometric description of Polynomial Weingarten cyclic surfaces. Moreover, as consequence of this studies, we obtain several results that allows us to achieve a complete classification of such surfaces.

Briefly, we can summarize the previous mentioned topics as the characterization of polynomials belonging to the ideal generated by $-x+y^2 \in \mathbb{R}[x,y]$, properties of degree of composition of polynomials in n-variables and, finally, we will present several results concerning summatories identities for polynomial series. It is also relevant to observe that each of our results in this section has it own value, besides potencial applications in other problems.

Throughout this section we use the symbol ∂ to represent the degree of a polynomial, and we will assume without loss of generality that a polynomial $P(x,y) \in \mathbb{R}[x,y]$ always has the monomial of degree $2n$, for some $n \in \mathbb{N}$. In fact, in the case that $\partial P = 2n-1$, we have $P(x,y) = \sum_{i=0}^{2n-1} \sum_{j=0}^{2n-1-i} a_{i,j} x^i y^j$, hence we just consider the monomial $\sum_{i=0}^{2n} a_{i,2n-i} x^i y^{2n-i}$ such that $a_{i,2n-i} = 0$ for every $0 \le i \le 2n$, therefore we may write

$$P(x,y) = \sum_{i=0}^{2n} \sum_{j=0}^{2n-i} a_{i,j} x^i y^j. \tag{3.8}$$

Of course, this procedure does not affect the degree of P.

We also recall that we may write the previous polynomial as

$$P(x,y) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{i,j} x^i y^j$$

just setting $a_{i,j} = 0$ whenever $i < 0$ or $j < 0$ or $i + j > 2n$. So, for a more suitable expression of (3.8) we define the following set

$$\Omega_k = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,;\; 0 \le i \le k \text{ and } 0 \le j \le k-i\}. \tag{3.9}$$

Hence, the polynomial $P(x,y)$ can be indicate as

$$P(x,y) = \sum_{(i,j) \in \Omega_{2n}} a_{i,j} x^i y^j. \tag{3.10}$$

Finally, we remark that even when $P(x,y)$ is written as in (3.8) we still understanding it as an infinite polynomial as in (3.10) since the coefficients $a_{i,j} \notin \Omega_{2n}$ are necessarily zero. This technicalities are important because it permits us to achieve our results especially those concerning summatories identities.

In the analysis of cyclic surfaces whose Gaussian and mean curvatures verify $P(K,H) \equiv 0$, it was needed to investigate under which conditions the following polynomial

$$\mathcal{P}(x) = \sum_{i=0}^{2n-1} \sum_{j=0}^{1} \left(\sum_{k=0}^{2n-i-j} a_{i-k,j+2k} \right) x^{4n-2i-j} \in \mathbb{R}[x]. \tag{3.11}$$

(which is obtained by rearranging the coefficients of $P(x,y)$ as in (3.8)) could vanish identically. As a matter of fact, the investigation of above polynomial leads us to our theorem that states that the hypothesis

$$\sum_{k=0}^{2n-i-j} a_{i-k,j+2k} = 0 \tag{3.12}$$

for every $(i,j) \in \Delta = \{0,\ldots,2n-1\} \times \{0,1\} \cup \{(2n,0)\}$ is precisely the sufficient and necessary condition to the polynomial $P(x,y)$ belong to the ideal in $\mathbb{R}[x,y]$ generated by $-x+y^2$. A simple illustration of above remark can be checked in the next example:

Example 81 *Given the polynomial*

$$P(x,y) = a_{1,1}xy + a_{1,0}x + a_{0,2}y^2 + a_{0,1}y + a_{0,0} \in \mathbb{R}[x,y]$$

consider the associated polynomial

$$\mathcal{P}(x) = (a_{0,0} + a_{-1,2} + a_{-2,4})x^4 + (a_{0,1} + a_{-1,3})x^3 + (a_{1,0} + a_{0,2})x^2 + a_{1,1}x$$

since every coeficiente verify $a_{i,j} = 0$ whenever $i < 0$, or $j < 0$, we have

$$\mathcal{P}(x) = a_{0,0}x^4 + a_{0,1}x^3 + (a_{1,0} + a_{0,2})x^2 + a_{1,1}x.$$

Notice that $\mathcal{P}(x)$ may vanish identically. In fact, in order to obtain that it is necessary choose $a_{1,0} = -a_{0,2}$ and set all the others coefficients as zero, hence we obtain

$$\mathcal{P}(x) \equiv 0(x).$$

Then our theorem allows us to conclude information about $P(x,y) \in \mathbb{R}[x,y]$ by studying $\mathcal{P}(x) \in \mathbb{R}[x]$. More precisely, we have that $P(x,y) = y^2a_{0,2} - xa_{0,2}$.

Remark 82 *Usually and differently from above example, we do not know (a priori) the polynomial $P(x,y)$, however from information gathered from $\mathcal{P}(x)$ our theorem provides substantial details of $P(x,y)$.*

In other words, our theorem allow us to characterize polynomials of the form

$$P(x,y) = \left(-x+y^2\right)^n R(x,y) \in \mathbb{R}[x,y]$$

for some $n \in \mathbb{N}$ (possibly zero) and for some $R(x,y) \in \mathbb{R}[x,y]$ through the study of an associated polynomial $\mathcal{P}(x) \in \mathbb{R}[x]$ as in (3.11) which is significantly easier for computational calculus. The statement of our result is read as follows:

Theorem 83 *Given a positive integer $n \geq 2$, the polynomial*

$$P(x,y) = \sum_{i=0}^{2n} \sum_{j=0}^{2n-j} a_{i,j} x^i y^j = \sum_{(i,j)\in\Omega_{2n}} a_{i,j} x^i y^j \qquad (3.13)$$

belongs to the ideal in $\mathbb{R}[x,y]$ generated by $-x+y^2$ if and only if its coefficients a_{ij} verify

$$\sum_{k=0}^{2n-i-j} a_{i-k,j+2k} = 0 \qquad (3.14)$$

for every $(i,j) \in \Delta = \{0,\ldots,2n-1\} \times \{0,1\} \cup \{(2n,0)\}$.

Proof. Assume that $P(x,y)$ belongs to the ideal in $\mathbb{R}[x,y]$ generated by $-x+y^2$, hence it yields that

$$P(x,y) = \left(-x+y^2\right) Q(x,y) \qquad (3.15)$$

where $Q(x,y) \in \mathbb{R}[x,y]$ is a polynomial of degree $2n-2$ that may be written as

$$Q(x,y) = \sum_{i=0}^{2n-2} \sum_{j=0}^{2n-2-i} b_{i,j} x^i y^j = \sum_{(i,j)\in\Omega_{2n-2}} b_{i,j} x^i y^j.$$

Moreover, if we define

$$b_{i,j} = 0 \qquad \text{whenever } (i,j) \in \mathbb{Z}\times\mathbb{Z}-\Omega_{2n-2}$$

it implies that

$$Q(x,y) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} b_{i,j} x^i y^j = \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} b_{i,j} x^i y^j.$$

Thus, by (3.13), (3.15) and the above remark, we have

$$a_{i,j} = -b_{i-1,j} + b_{i,j-2}$$

for every $i = 0,\ldots,2n$ and $j = 0,\ldots,2n-i$. Given $(i,j) \in \Delta$, it is obtained

$$\begin{aligned}
\sum_{k=0}^{2n-i-j} a_{i-k,j+2k} &= \sum_{k=0}^{2n-i-j} \left(-b_{i-k-1,j+2k} + b_{i-k,j+2k-2}\right) \\
&= -\sum_{k=0}^{2n-i-j-1} b_{i-k-1,j+2k} - b_{i-(2n-i-j)-1,j+2(2n-i-j)} \\
&\quad +b_{i,j-2} + \sum_{k=1}^{2n-i-j} b_{i-k,j+2k-2} \\
&= -b_{j-2n-1+2i,4n-j-2i} + b_{i,j-2} \\
&\quad -\sum_{k=0}^{2n-i-j-1} b_{i-k-1,j+2k} + \sum_{k=0}^{2n-i-j-1} b_{i-(k+1),j+2(k+1)-2} \\
&= -b_{j-2n-1+2i,4n-j-2i} + b_{i,j-2}.
\end{aligned}$$

Notice that $(i,j)\in\Delta$, therefore $j<2$ and

$$(j-2n-1+2i)+(4n-j-2i)=2n-1>2n-2$$

which implies that $(j-2n-1+2i,4n-j-2i)\notin\Omega_{2n-2}$, hence

$$\sum_{k=0}^{2n-i-j} a_{i-k,j+2k}=-b_{j-2n-1+2i,4n-j-2i}+b_{i,j-2}=0-0=0.$$

Reciprocally, assume that

$$\sum_{k=0}^{2n-i-j} a_{i-k,j+2k}=0$$

for every $(i,j)\in\Delta=\{0,\dots,2n-1\}\times\{0,1\}\cup\{(2n,0)\}$.

If we set

$$a_{i,j}=0 \qquad \text{for every } (i,j)\notin\Omega_{2n}$$

it follows that we can write

$$P(x,y)=\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_{i,j}x^iy^j=\sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} a_{i,j}x^iy^j.$$

Then, we define

$$b_{i,j}=\begin{cases}\sum\limits_{k=0}^{i} a_{i-k,j+2k+2} & \text{se } j\geq 0\\ 0 & \text{se } j<0\end{cases}\quad,\qquad \forall\,(i,j)\in\mathbb{Z}\times\mathbb{Z}, \tag{3.16}$$

also we consider the following polynomial

$$Q(x,y)=\sum_{i=0}^{2n-2}\sum_{j=0}^{2n-2-i} b_{i,j}x^iy^j.$$

We define

$$\widetilde{P_{2n}}(x,y)=\sum_{i=0}^{2n}\sum_{j=0}^{2n-i}\widetilde{a_{i,j}}x^iy^j=\left(-x+y^2\right)Q(x,y)$$

and we point out that

$$\widetilde{a_{i,j}}=b_{i,j-2}-b_{i-1,j}$$

for every $(i,j)\in\Omega_{2n}$. Our objective is to show $P_{2n}=\widetilde{P_{2n}}$, so it is sufficient to verify

$$a_{i,j}=\widetilde{a_{i,j}}=b_{i,j-2}-b_{i-1,j},$$

for every $(i,j)\in\Omega_{2n}$.

For every $(i,j) \in \Omega_{2n}$ such that $j \geq 2$, we have

$$\begin{aligned}
\widetilde{a_{i,j}} &= b_{i,j-2} - b_{i-1,j} \\
&= \sum_{k=0}^{2n-i-j} a_{i-k,j+2k} - \sum_{k=1}^{2n-i-j} a_{i-k,j+2k} \\
&= a_{i,j} + \sum_{k=1}^{2n-i-j} a_{i-k,j+2k} - \sum_{k=1}^{2n-i-j} a_{i-k,j+2k} \\
&= a_{i,j}.
\end{aligned}$$

thus, in this case, the desired is achieved.

Still remaining three cases to be considered, named $(0,0)$, $(i,0)$ and $(i,1)$. First, let $(i,j) = (0,0)$ so in this case

$$\widetilde{a_{0,0}} = b_{0,0-2} - b_{0-1,0} = b_{0,-2} - b_{-1,0} = 0 - 0 = 0. \tag{3.17}$$

On the other hand, the hypothesis applied to $(i,j) = (0,0)$ provide to us

$$0 = \sum_{k=0}^{2n} a_{-k,2k} = a_{0,0} + \sum_{k=1}^{2n} a_{-k,2k} = a_{0,0}. \tag{3.18}$$

By Remarks 3.17 and 3.18, we conclude this case.

Proceeding to the case $(i,0)$ for every $i \in \{1,\ldots,2n\}$. Notice that, in on hand we have

$$\widetilde{a_{i,0}} = b_{i,-2} - b_{i-1,0} = -\sum_{k=0}^{i-1} a_{i-(k+1),2(k+1)} = -\sum_{k=1}^{i} a_{i-k,2k}.$$

On the other hand, the hypothesis applied for $(i,0)$ gives

$$0 = \sum_{k=0}^{2n-i-0} a_{i-k,0+2k} = a_{i,0} + \sum_{k=1}^{2n-i} a_{i-k,2k}, \tag{3.19}$$

then

$$\widetilde{a_{i,0}} = 0 - \sum_{k=1}^{i} a_{i-k,2k} = \left(a_{i,0} + \sum_{k=1}^{2n-i} a_{i-k,2k} \right) - \sum_{k=1}^{i} a_{i-k,2k}. \tag{3.20}$$

i. If $2n - i = i$, the Equality 3.20 is rewritten as

$$\widetilde{a_{i,0}} = a_{i,0} + \sum_{k=1}^{i} a_{i-k,2k} - \sum_{k=1}^{i} a_{i-k,2k} = a_{i,0}.$$

■

Proof.

ii. If $2n-i>i$, the Equality 3.20 is rewritten as

$$\begin{aligned}\widetilde{a_{i,0}} &= a_{i,0}+\sum_{k=i+1}^{2n-i}a_{i-k,2k}+\sum_{k=1}^{i}a_{i-k,2k}-\sum_{k=1}^{i}a_{i-k,2k}\\ &= a_{i,0}+\sum_{k=i+1}^{2n-i}a_{i-k,2k}\end{aligned}$$

therefore, notice that

$$i-k\leq i-(2n-i)=-(2n-2i)<0$$

since $2n-2i>0$, so

$$\widetilde{a_{i,0}}=a_{i,0}+\sum_{k=i+1}^{2n-i}a_{i-k,2k}=a_{i,0}.$$

iii. If $2n-i<i$, the Equality 3.20 is rewritten as

$$\begin{aligned}\widetilde{a_{i,0}} &= a_{i,0}+\sum_{k=1}^{2n-i}a_{i-k,2k}-\sum_{k=1}^{2n-i}a_{i-k,2k}-\sum_{k=2n-i+1}^{i}a_{i-k,2k}\\ &= a_{i,0}-\sum_{k=2n-i+1}^{i}a_{i-k,2k},\end{aligned}$$

now note that

$$(i-k)+2k=i+k\geq i+(2n-i+1)=2n+1$$

therefore

$$a_{i-k,2k}=0$$

for every $k\in\{2n-i+1,...,i\}$. Hence this case is concluded.

In the case $(i,j)=(0,1)$, we have

$$\widetilde{a_{0,1}}=b_{0,1-2}-b_{0-1,1}=b_{0,-1}-b_{-1,1}=0-0=0.$$

While the hypothesis applied to $(i,j)=(0,1)$ gives us

$$0=\sum_{k=0}^{2n-1}a_{-k,1+2k}=a_{0,1}+\sum_{k=1}^{2n-1}a_{-k,1+2k}=a_{0,1}$$

so

$$\widetilde{a_{0,1}}=0=a_{0,1}.$$

Finally, the case that $(i,1)$ for every $i\in\{1,\ldots,2n-1\}$. In one hand, we have

$$\widetilde{a_{i,1}}=b_{i,1-2}-b_{i-1,1}=-\sum_{k=0}^{i-1}a_{i-(k+1),1+2(k+1)}=-\sum_{k=1}^{i}a_{i-k,1+2k}.$$

On the other hand, the hypothesis applied for $(i,1)$ provides

$$0 = \sum_{k=0}^{2n-i-1} a_{i-k,1+2k} = a_{i,1} + \sum_{k=1}^{2n-i-1} a_{i-k,1+2k},$$

so

$$\widetilde{a_{i,1}} = 0 - \sum_{k=1}^{i} a_{i-k,1+2k} = a_{i,1} + \sum_{k=1}^{2n-i-1} a_{i-k,1+2k} - \sum_{k=1}^{i} a_{i-k,1+2k}. \quad (3.21)$$

i. If $2n-i-1=i$, the Equality 3.21 is rewritten as

$$\widetilde{a_{i,1}} = a_{i,1} + \sum_{k=1}^{i} a_{i-k,1+2k} - \sum_{k=1}^{i} a_{i-k,1+2k} = a_{i,1}.$$

ii. If $2n-i-1>i$, the Equality 3.21 is rewritten as

$$\begin{aligned} \widetilde{a_{i,1}} &= a_{i,1} + \sum_{k=i+1}^{2n-i-1} a_{i-k,1+2k} + \sum_{k=1}^{i} a_{i-k,1+2k} - \sum_{k=1}^{i} a_{i-k,1+2k} \\ &= a_{i,1} + \sum_{k=i+1}^{2n-i-1} a_{i-k,1+2k} \\ &= a_{i,1} \end{aligned}$$

iii. If $2n-i-1<i$, the Equality 3.21 is rewritten as

$$\begin{aligned} \widetilde{a_{i,1}} &= a_{i,1} + \sum_{k=1}^{2n-i-1} a_{i-k,1+2k} - \sum_{k=1}^{2n-i-1} a_{i-k,1+2k} - \sum_{k=2n-i}^{i} a_{i-k,1+2k} \\ &= a_{i,1} - \sum_{k=2n-i}^{i} a_{i-k,1+2k} \end{aligned}$$

therefore

$$(i-k)+(1+2k) = k+i+1 \geq (2n-i)+i+1 = 2n+1$$

which yields

$$a_{i-k,1+2k} = 0.$$

■

Proof. Then, we conclude that

$$P_{2n} = \widetilde{P_{2n}},$$

and the proof is finished. ■

Now we proceed to the investigation of degree of composition of polynomials in n-variables. This analysis is motivated by the challenge of to study the next function

$$\begin{aligned} 0 &= \left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j} 2^{2n-2j}a_{i,2j}r^{4n-2(i+2j)}\beta^{4n-(2i+3j)}\Delta^{i}\alpha^{2j}\right)^{2} \\ &\quad -\beta\left(\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)} 2^{2n-(2j-1)}a_{i,(2j-1)}r^{4n-2(i+(2j-1))}\beta^{4n-2i-3j+1}\Delta^{i}\alpha^{(2j-1)}\right)^{2} \end{aligned}$$

which is obtained from the assumption that $0 \equiv P(K,H)$, where the Gaussian and mean curvatures are expressed as in (3.5), but for convenience of the reader, it will be restate:

$$K = \frac{\Delta}{r^2\beta^2} \qquad \text{and} \qquad H = \frac{\alpha}{2r^2\beta^{\frac{3}{2}}},$$

where we recall that α, β and Δ are smooth functions in parameter (s,t). So we translate the above problem to the research of the next polynomial

$$\begin{aligned} \mathfrak{P}(x_1,x_2,x_3) &= \left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j} 2^{2n-2j}a_{i,2j}r^{4n-2(i+2j)}x_3^{4n-(2i+3j)}x_1^{i}x_2^{2j}\right)^{2} \\ -x_3 &\left(\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)} 2^{2n-(2j-1)}a_{i,(2j-1)}r^{4n-2(i+(2j-1))}x_2^{4n-2i-3j+1}x_1^{i}x_2^{(2j-1)}\right)^{2} \end{aligned}$$

that verify $\mathfrak{P}(\Delta,\alpha,\beta) = 0$. Although, with this change we must decide either we have a monomial of maximum degree, that is, when the monomial provides the degree of the polynomial. Hence, we stated the discussion of degree of composition of polynomials.

Before to present our results, we recall that every polynomial $P(x,y) \in \mathbb{R}[x,y]$ of degree n, always admits the following expression

$$P(x,y) = H(x,y) + Q(x,y) \tag{3.22}$$

where $H(x,y)$ is a monomial of degree n and $Q(x,y) \in \mathbb{R}[x,y]$ a polynomial such that $\partial Q(x,y) < n$. It is easy to check that expression (3.22) is unique.

In the aim to present the result that we apply in our analysis, it is necessary other two propositions. In this direction, our first result studies a threshold for the degree of $Q(x,y)$ as in (3.22) in the case that $P(x,y)$ is obtained by the produtory of k-polynomials. More precisely:

Proposition 84 *Consider the polynomials $P_1(x,y),\ldots,P_k(x,y) \in \mathbb{R}[x,y]$ such that the degree of $P_i(x,y)$ is $n_i \in \mathbb{N}$, for every $1 \le i \le k$. For each i, we write*

$$P_i(x,y) = H_i(x,y) + Q_i(x,y)$$

where $H_i(x,y)$ is an homogeneous monomial of degree n_i and $Q_i(x,y) \in \mathbb{R}[x,y]$ is a polynomial such that $\partial Q_i < n_i$. Then

$$\prod_{i=1}^{k} P_i(x,y) = \left(\prod_{i=1}^{k} H_i(x,y)\right) + Q(x,y)$$

where $\partial Q < \sum_{i=1}^{k} n_i = \partial\left(\prod_{i=1}^{k} H_i(x,y)\right)$.

Proof. The proof is given by induction on k. In the case that $k=1$, the result follows immediately. Assume that statement is valid for some $k \in \mathbb{N}$ and we will show that still true for $k+1$.

In the case that we have $P_1(x,y), \ldots, P_k(x,y), P_{k+1}(x,y) \in \mathbb{R}[x,y]$, follows from the hypothesis that

$$\prod_{i=1}^{k} P_i(x,y) = \left(\prod_{i=1}^{k} H_i(x,y)\right) + Q(x,y)$$

where $\partial Q < \sum_{i=1}^{k} n_i = \partial\left(\prod_{i=1}^{k} H_i(x,y)\right)$. Therefore,

$$\begin{aligned}
\prod_{i=1}^{k+1} P_i(x,y) &= P_{k+1}(x,y) \prod_{i=1}^{k} P_i(x,y) \\
&= \left(H_{k+1}(x,y) + Q_{k+1}(x,y)\right)\left(\left(\prod_{i=1}^{k} H_i(x,y)\right) + Q(x,y)\right) \\
&= \left(\prod_{i=1}^{k+1} H_i(x,y)\right) + \overline{Q}(x,y)
\end{aligned}$$

where

$$\overline{Q}(x,y) = H_{k+1}(x,y)\, Q(x,y) + Q_{k+1}(x,y)\left(\prod_{i=1}^{k} H_i(x,y)\right) + Q_{k+1}(x,y)\, Q(x,y) \tag{3.23a}$$

Then, notice that:

1. $\partial\left(\prod_{i=1}^{k+1} H_i(x,y)\right) = \partial H_{k+1}(x,y) + \partial\left(\prod_{i=1}^{k} H_i(x,y)\right) = n_{k+1} + \sum_{i=1}^{k} n_i = \sum_{i=1}^{k+1} n_i$;
2. $\partial\left(H_{k+1}(x,y)\, Q(x,y)\right) = \partial H_{k+1}(x,y) + \partial Q(x,y) = n_{k+1} + \partial Q(x,y) < n_{k+1} + \sum_{i=1}^{k} n_i = \sum_{i=1}^{k+1} n_i$;

3. $\partial\left(Q_{k+1}(x,y)\left(\prod_{i=1}^{k}H_i(x,y)\right)\right)=\partial Q_{k+1}(x,y)+\sum_{i=1}^{k}n_i<n_{k+1}+\sum_{i=1}^{k}n_i=\sum_{i=1}^{k+1}n_i$;

4. $\partial(Q_{k+1}(x,y)\,Q(x,y))=\partial Q_{k+1}(x,y)+\partial Q(x,y)<\partial Q_{k+1}(x,y)+\sum_{i=1}^{k}n_i<n_{k+1}+\sum_{i=1}^{k}n_i=\sum_{i=1}^{k+1}n_i$,

Then, follows from items 2, 3 and 4 that

$$\begin{aligned}&\partial\overline{Q}(x,y)\\ \leq\ &\max\left\{\partial(H_{k+1}(x,y)\,Q(x,y)),\partial\left(Q_{k+1}(x,y)\prod_{i=1}^{k}H_i(x,y)\right),\partial(Q_{k+1}(x,y)\,Q(x,y))\right\}\\ <\ &\sum_{i=1}^{k+1}n_i\end{aligned}$$

Thus, by (3.23a) and above remarks, it yields the conclusion of proof. ■

The other needed result provides a threshold for $Q(x,y)$ when it is obtained by the evaluation of k polynomials in a monomial of k-variables:

Proposition 85 *For given $k\in\mathbb{N}$, Consider the monomial*

$$M(X_1,...,X_k)=\mu(X_1)^{m_1}\ldots(X_k)^{m_k}$$

where $\mu\in\mathbb{R}$. Let $P_1(x,y),\ldots,P_k(x,y)\in\mathbb{R}[x,y]$ be polynomials whose degree are $n_1,\ldots,n_k\in\mathbb{N}$, respectively. For each $1\leq i\leq k$, we write

$$P_i(x,y)=H_i(x,y)+Q_i(x,y),$$

where $H_i(x,y)$ is homogenous with degree n_i and $\partial Q_i<n_i$. Then, exists a polynomial $Q(x,y)\in\mathbb{R}[x,y]$ verifying

$$M(P_1(x,y),\ldots,P_k(x,y))=M(H_1(x,y),\ldots,H_k(x,y))+Q(x,y)$$

such that $\partial Q<\sum_{i=1}^{k}m_in_i=\partial(M(H_1(x,y),\ldots,H_k(x,y)))$.

Proof. We remark that

$$(P_1(x,y))^{m_1}\ldots(P_k(x,y))^{m_k}=\prod_{i=1}^{m_1}P_1(x,y)\ldots\prod_{i=1}^{m_k}P_k(x,y)$$

therefore, follows from Proposition 84 that above equality is rewritten as

$$(H_1(x,y))^{m_1}\ldots(H_k(x,y))^{m_k}+Q(x,y)$$

where

$$\begin{aligned}\partial Q &< n_1 + \ldots + n_1 + \ldots + n_k + \ldots + n_k = \sum_{i=1}^{k} m_i n_i \\ &= \partial \left(H_1(x,y)\right)^{m_1} \ldots \left(H_k(x,y)\right)^{m_k}.\end{aligned}$$

Thus

$$\begin{aligned}M\left(P_1(x,y), \ldots, P_k(x,y)\right) &= \mu \left(P_1(x,y)\right)^{m_1} \ldots \left(P_k(x,y)\right)^{m_k} \\ &= \mu \left(H_1(x,y)\right)^{m_1} \ldots \left(H_k(x,y)\right)^{m_k} + \mu Q(x,y) \\ &= M\left(H_1(x,y), \ldots, H_k(x,y)\right) + \overline{Q}(x,y)\end{aligned}$$

where $\overline{Q}(x,y) = \mu Q(x,y)$. Finally, we observe that

$$\partial \overline{Q}(x,y) = \partial Q(x,y) < \sum_{i=1}^{k} m_i n_i$$

since

$$\partial M\left(H_1(x,y), \ldots, H_k(x,y)\right) = \partial \left(H_1(x,y)\right)^{m_1} \ldots \left(H_k(x,y)\right)^{m_k} = \sum_{i=1}^{k} m_i n_i$$

so the desired is achieved. ∎

Endowed with both previous results we are able to state the corollary that is a natural next question about the threshold of $Q(x,y)$ when we evaluete n polynomials in a given polynomial of n variables.

Corollary 86 *Consider the polynomial* $\mathfrak{P}(x_1, \ldots, x_k) \in \mathbb{R}[x_1, \ldots, x_k]$ *such that*

$$\mathfrak{P}(x_1, \ldots, x_k) = \sum_{l=0}^{n} M_l(x_1, \ldots, x_k)$$

where for every l, $M_l(x_1, \ldots, x_k) = \mu_l (x_1)^{m_{l1}} \ldots (x_k)^{m_{lk}}$ *is a monomial (with* $\mu_l \in \mathbb{R}$*). Let* $P_1(x,y), \ldots, P_k(x,y) \in \mathbb{R}[x,y]$ *be polynomials. For each* $0 \leq i \leq k$*, assume that*

$$P_i(x,y) = a_i x^{n_i} + Q_i(x,y)$$

where $a_i \in \mathbb{R}$ *nonzero and* $\partial Q_i(x,y) < n_i \in \mathbb{N}$*. Then*

$$\mathfrak{P}\left(P_1(x,y), \ldots, P_k(x,y)\right) = \mathfrak{P}\left(a_1 x^{n_1}, ..., a_k x^{n_k}\right) + Q(x,y)$$

where $\partial Q(x,y) < \max\limits_{l} \sum_{i=1}^{k} m_{li} n_i$.

Proof. The Proposition 85 ensures that for each $0 \leq l \leq n$, we may express the monomials as follows

$$M_l\left(P_1(x,y),\ldots,P_k(x,y)\right) = M_l\left(a_1x^{n_1},\ldots,a_kx^{n_k}\right) + Q_l(x,y)$$

where $\partial Q_l(x,y) < \sum_{i=1}^{k} m_{li}n_i$. Therefore,

$$\begin{aligned}\mathfrak{P}\left(P_1(x,y),\ldots,P_k(x,y)\right) &= \sum_{l=0}^{n} M_l\left(P_1(x,y),\ldots,P_k(x,y)\right)\\ &= \sum_{l=0}^{n} M_l\left(a_1x^{n_1},\ldots,a_kx^{n_k}\right) + \sum_{l=0}^{n} Q_l(x,y)\\ &= \mathfrak{P}\left(a_1x^{n_1},\ldots,a_kx^{n_k}\right) + Q(x,y)\end{aligned}$$

where $Q(x,y) = \sum_{l=0}^{n} Q_l(x,y)$. Then, notice that

$$\partial Q(x,y) \leq \max_l \partial Q_l(x,y) < \max_l \sum_{i=1}^{k} m_{li}n_i.$$

■

This result concludes the study of degree of composition of polynomials.

Finally, the last topic that we will show in this section is about polynomial identities. More precisely, our theorem permits us to retrieve and articulate the condition presented in the Theorem 83 with the writing of a polynomial $\mathcal{P}(x)$ presented in (3.11). Moreover, given a polynomial

$$P(x) = \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i} a_{i,j}x^{4n-j-2i} \in \mathbb{R}[x],$$

our theorem presents a suitable expression that hightlights the condition (3.12), that is

$$\begin{aligned}P(x) = \sum_{i=0}^{2n-1}\sum_{j=0}^{1}\left(\sum_{k=0}^{2n-i-j} a_{i-k,j+2k}\right)x^{4n-2i-j}\\ -\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-(i+1)} a_{-(i+1),2n+(i+1)-j}x^{2n+j+(i+1)} \in \mathbb{R}[x].\end{aligned}$$

So the statement of our theorem is read as follows:

Theorem 87 *For given $n \in \mathbb{N}$, the next polynomial equality holds*

$$\begin{aligned}&\sum_{i=0}^{2n-1}\sum_{j=0}^{1}\left(\sum_{k=0}^{2n-i-j} a_{i-k,j+2k}\right)x^{4n-2i-j}\\ =&\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i} a_{i,j}x^{4n-j-2i} + \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-(i+1)} a_{-(i+1),2n+(i+1)-j}x^{2n+j+(i+1)}.\end{aligned}$$

In order to transcribe the previous equality we will exhibit three auxiliary results. The Proposition 88 deserves a special attention once its proof is different from the usual induction proves, because in this case we have to add and substract several terms to reach our goal.

Proposition 88 *For given $n \in \mathbb{N}$, we have*

$$\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i-1} A_{i,j} = \sum_{i=0}^{n-1}\sum_{j=0}^{2n-2i-1} A_{j,2n-2i-j-1} + \sum_{i=0}^{n-1}\sum_{j=0}^{2n-2i-2} A_{j,2n-2i-j-2} \quad (3.24)$$

Proof. We will prove this statement by induction on n. It is easy to see that above equality is valid for $n=1$:

$$\sum_{i=0}^{2(1)-1}\sum_{j=0}^{2(1)-i-1} A_{i,j} = A_{0,0} + A_{0,1} + A_{1,0} = \sum_{j=0}^{1} A_{j,1-j} + A_{0,0}.$$

Suppose that Equality 3.24 its true for some $n \in \mathbb{N}$, we will show that the equality for $n+1$. More precisely, we will verify that

$$\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} = \sum_{i=0}^{n}\sum_{j=0}^{2n-2i+1} A_{j,2n-2i-j+1} + \sum_{i=0}^{n}\sum_{j=0}^{2n-2i} A_{j,2n-2i-j}.$$

The left-hand side of the above equality gives

$$\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} = \sum_{i=2n}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} + \sum_{i=0}^{2n-1}\sum_{j=2n-i}^{2n-i+1} A_{i,j} + \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i-1} A_{i,j}$$

thus, the induction step implies

$$\begin{aligned}\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} &= \sum_{i=2n}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} + \sum_{i=0}^{2n-1}\sum_{j=2n-i}^{2n-i+1} A_{i,j} \\ &+ \left(\sum_{i=0}^{n-1}\sum_{j=0}^{2n-2i-1} A_{j,2n-2i-j-1} + \sum_{i=0}^{n-1}\sum_{j=0}^{2n-2i-2} A_{j,2n-2i-j-2}\right).\end{aligned}$$

Rearranging the indices of previous terms, it is obtained

$$\begin{aligned}\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} &= \sum_{i=2n}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} + \sum_{i=0}^{2n-1}\sum_{j=2n-i}^{2n-i+1} A_{i,j} \\ &+ \left(\sum_{i=1}^{n}\sum_{j=0}^{2n-2i+1} A_{j,2n-2i-j+1}\right) + \left(\sum_{i=1}^{n}\sum_{j=0}^{2n-2i} A_{j,2n-2i-j}\right),\end{aligned}$$

so if we add and subtract $\sum_{j=0}^{2n+1} A_{j,2n-j+1}$ and $\sum_{j=0}^{2n} A_{j,2n-j}$, it follows

$$\begin{aligned}\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} &= \sum_{i=2n}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} + \sum_{i=0}^{2n-1}\sum_{j=2n-i}^{2n-i+1} A_{i,j} \\ &\quad + \left(\sum_{i=1}^{n}\sum_{j=0}^{2n-2i+1} A_{j,2n-2i-j+1} + \sum_{j=0}^{2n+1} A_{j,2n-j+1} - \sum_{j=0}^{2n+1} A_{j,2n-j+1}\right) \\ &\quad + \left(\sum_{i=1}^{n}\sum_{j=0}^{2n-2i} A_{j,2n-2i-j} + \sum_{j=0}^{2n} A_{j,2n-j} - \sum_{j=0}^{2n} A_{j,2n-j}\right),\end{aligned}$$

hence

$$\begin{aligned}&\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} \qquad (3.25)\\ =\ &\sum_{i=2n}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} + \sum_{i=0}^{2n-1}\sum_{j=2n-i}^{2n-i+1} A_{i,j} \\ &+ \left(\sum_{i=0}^{n}\sum_{j=0}^{2n-2i+1} A_{j,2n-2i-j+1} - \sum_{j=0}^{2n+1} A_{j,2n-j+1}\right) \\ &+ \left(\sum_{i=0}^{n}\sum_{j=0}^{2n-2i} A_{j,2n-2i-j} - \sum_{j=0}^{2n} A_{j,2n-j}\right).\end{aligned}$$

Finally, notice that

$$\sum_{i=2n}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} = A_{2n,0} + A_{2n,1} + A_{2n+1,0} \qquad (3.26)$$

and

$$\sum_{i=0}^{2n-1}\sum_{j=2n-i}^{2n-i+1} A_{i,j} = \sum_{i=0}^{2n-1} A_{i,2n-i} + \sum_{i=0}^{2n-1} A_{i,2n-i+1} \qquad (3.27)$$

therefore, by the observations (3.26) and (3.27) we have

$$\sum_{i=2n}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} + \sum_{i=0}^{2n-1}\sum_{j=2n-i}^{2n-i+1} A_{i,j} = \sum_{i=0}^{2n} A_{i,2n-i} + \sum_{i=0}^{2n+1} A_{i,2n-i+1} \qquad (3.28)$$

So, replacing (3.28) into (3.25) it is obtained

$$\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i,j} = \sum_{i=0}^{n}\sum_{j=0}^{2n-2i+1} A_{j,2n-2i-j+1} + \sum_{i=0}^{n}\sum_{j=0}^{2n-2i} A_{j,2n-2i-j}$$

which finishes the demonstration. ■

As immediate consequence we have the next result which is a direction application of the previous result to our case of interest.

Proposition 89 *For any* $n \in \mathbb{N}$*, we have*

$$\sum_{i=0}^{2n-1} \sum_{j=0}^{2n-(i+1)} a_{-(i+1),2n+(i+1)-j} x^{2n+j+(i+1)} \tag{3.29}$$
$$= \sum_{i=0}^{n-1} \sum_{j=0}^{2n-2i-1} a_{-j-1,2j+2i+2} x^{4n-2i} + \sum_{i=0}^{n-1} \sum_{j=0}^{2n-2i-2} a_{-j-1,2j+2i+3} x^{4n-2i-1}.$$

Proof. Consider

$$\sum_{i=0}^{2n-1} \sum_{j=0}^{2n-(i+1)} a_{-(i+1),2n+(i+1)-j} x^{2n+j+(i+1)}$$

hence notice that if we set

$$A_{i,j} = a_{-(i+1),2n+(i+1)-j} x^{2n+j+(i+1)}$$

it yields

$$A_{j,2n-2i-j-1} = a_{-j-1,2j+2i+2} x^{4n-2i} \qquad \text{and} \qquad A_{j,2n-2i-j-2} = a_{-j-1,2j+2i+3} x^{4n-2i-1},$$

then, applying the Proposition 88, we achieve the desired. ■

The third and last technical result necessary to demonstrate the Theorem 87 is also a relevant identity of summations whose proof lies deeply in the detailed analysis of each of the terms, as well as in the need to add and subtract terms in order to reach the desired expression. As mentioned earlier, the following proposition also has potential applications in problems where rewriting the coefficients in a more appropriate way is relevant.

Theorem 90 *For any* $n \in \mathbb{N}$*, we have*

$$\sum_{i=0}^{2n-1} \sum_{j=0}^{2n-i} A_{i,j} = \sum_{i=0}^{n-1} \sum_{j=0}^{i} A_{i-j,2j} + \sum_{i=n}^{2n-1} \sum_{j=0}^{2n-i} A_{i-j,2j} + \sum_{i=0}^{n-1} \sum_{j=0}^{i} A_{i-j,1+2j} + \sum_{i=n}^{2n-1} \sum_{j=0}^{2n-i-1} A_{i-j,1+2j}. \tag{3.30}$$

Proof. This prove is given by induction on n. First, we observe that in the case $n = 1$, the above equality is verified:

$$\begin{aligned} & \sum_{i=0}^{1} \sum_{j=0}^{2-i} A_{i,j} \\ = \quad & A_{0,0} + A_{0,1} + A_{1,0} + A_{0,2} + A_{1,1} \\ = \quad & A_{0,0} + A_{0,1} + \sum_{j=0}^{1} A_{1-j,2j} + A_{1,1}. \end{aligned}$$

Suppose that Equality (3.30) holds for some $n \in \mathbb{N}$. We will demonstrate that equality is true also for $n+1$. More precisely, our goal is to prove:

$$\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i,j} = \sum_{i=0}^{n}\sum_{j=0}^{i} A_{i-j,2j} + \sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i-j,2j} + \sum_{i=0}^{n}\sum_{j=0}^{i} A_{i-j,1+2j} + \sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i-j,1+2j}. \tag{3.31}$$

The left-hand side of the previous equality gives to us

$$\begin{aligned}
&\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i,j}\\
= \; & \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i+2} A_{i,j} + \sum_{i=2n}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i,j}\\
= \; & \sum_{i=0}^{2n-1}\left(\sum_{j=0}^{2n-i} A_{i,j} + A_{i,2n-i+1} + A_{i,2n-i+2}\right)\\
& + \sum_{j=0}^{1} A_{2n+1,j} + \sum_{j=0}^{2} A_{2n,j}\\
= \; & \left(\sum_{i=0}^{2n-1} A_{i,2n-i+2} + \sum_{i=0}^{2n-1} A_{i,2n-i+1} + \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i} A_{i,j}\right)\\
& + \left(A_{2n+1,1} + A_{2n+1,0} + A_{2n,2} + A_{2n,1} + A_{2n,0}\right)\\
= \; & \left(A_{2n+1,1} + A_{2n,2} + \sum_{i=0}^{2n-1} A_{i,2n-i+2}\right) + \left(A_{2n+1,0} + A_{2n,1} + \sum_{i=0}^{2n-1} A_{i,2n-i+1}\right)\\
& + A_{2n,0} + \left(\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i} A_{i,j}\right)\\
= \; & \left(\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i} A_{i,j}\right) + \left(\sum_{i=0}^{2n+1} A_{i,2n-i+2}\right) + \left(\sum_{i=0}^{2n+1} A_{i,2n-i+1}\right) + A_{2n,0},
\end{aligned}$$

hence, follows from the induction step that

$$\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i,j} \tag{3.32}$$
$$= \left(\sum_{i=0}^{n-1}\sum_{j=0}^{i} A_{i-j,2j} + \sum_{i=n}^{2n-1}\sum_{j=0}^{2n-i} A_{i-j,2j} + \sum_{i=0}^{n-1}\sum_{j=0}^{i} A_{i-j,1+2j} + \sum_{i=n}^{2n-1}\sum_{j=0}^{2n-i-1} A_{i-j,1+2j}\right)$$
$$+ \left(\sum_{i=0}^{2n+1} A_{i,2n-i+2}\right) + \left(\sum_{i=0}^{2n+1} A_{i,2n-i+1}\right) + A_{2n,0}.$$

Now we will examine and study several of above terms. First, we notice that

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i} A_{i-j,2j} = \sum_{i=0}^{n}\sum_{j=0}^{i} A_{i-j,2j} - \sum_{j=0}^{n} A_{n-j,2j} \tag{3.33}$$

moreover,

$$\sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i-j,2j} = \sum_{i=n+1}^{2n-1}\left(\sum_{j=2n-i+1}^{2n-i+2} A_{i-j,2j} + \sum_{j=0}^{2n-i} A_{i-j,2j}\right)$$
$$+\sum_{j=0}^{2} A_{2n-j,2j} + \sum_{j=0}^{1} A_{(2n+1)-j,2j}$$

if we expand and rearrange the terms in a proper way, we get

$$\sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i-j,2j} = \sum_{i=n+1}^{2n-1} A_{-2n+2i-2,4n-2i+4} + A_{2n,2} + A_{2n-2,4}$$
$$+\sum_{i=n+1}^{2n-1} A_{-2n+2i-1,4n-2i+2} + A_{2n+1,0} + A_{2n-1,2}$$
$$+\sum_{i=n+1}^{2n-1}\sum_{j=0}^{2n-i} A_{i-j,2j} + A_{2n,0},$$

therefore, if we add $0 = \sum_{j=0}^{n} A_{n-j,2j} - \sum_{j=0}^{n} A_{n-j,2j}$ it is obtained

$$\sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i-j,2j} \tag{3.34}$$
$$= \sum_{i=n+1}^{2n+1} A_{-2n+2i-2,4n-2i+4} + \sum_{i=n+1}^{2n+1} A_{-2n+2i-1,4n-2i+2}$$
$$-\sum_{j=0}^{n} A_{n-j,2j} + A_{2n,0} + \sum_{i=n+1}^{2n-1}\sum_{j=0}^{2n-i} A_{i-j,2j} + \sum_{j=0}^{n} A_{n-j,2j}$$

then

$$\begin{aligned}
&\sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i-j,2j} \qquad (3.35)\\
= &\sum_{i=n+1}^{2n+1} A_{-2n+2i-2,4n-2i+4} + \sum_{i=n+1}^{2n+1} A_{-2n+2i-1,4n-2i+2}\\
&-\sum_{j=0}^{n} A_{n-j,2j} + \sum_{i=n}^{2n-1}\sum_{j=0}^{2n-i} A_{i-j,2j} + A_{2n,0}.
\end{aligned}$$

Also observe that

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i} A_{i-j,1+2j} = \sum_{i=0}^{n}\sum_{j=0}^{i} A_{i-j,1+2j} - \sum_{j=0}^{n} A_{n-j,1+2j} \qquad (3.36)$$

and note that

$$\begin{aligned}
&\sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i-j,1+2j}\\
= \;& A_{2n+1,1} + A_{2n-1,3} + A_{2n,1} + \sum_{i=n+1}^{2n-1}\sum_{j=0}^{2n-i+1} A_{i-j,1+2j}
\end{aligned}$$

if we expand and rearrange the terms in a proper way, we get

$$\begin{aligned}
&\sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i-j,1+2j}\\
= &\sum_{i=n+1}^{2n-1}\sum_{j=0}^{2n-i-1} A_{i-j,1+2j}\\
&+\sum_{i=n+1}^{2n-1} A_{2i-2n,4n-2i+1} + A_{2n,1}\\
&+\sum_{i=n+1}^{2n-1} A_{2i-2n-1,4n-2i+3} + A_{2n+1,1} + A_{2n-1,3}
\end{aligned}$$

moreover, simplifying the summations and adding $0 = \sum_{j=0}^{n-1} A_{n-j,1+2j} - \sum_{j=0}^{n-1} A_{n-j,1+2j}$

we have

$$
\begin{aligned}
&\sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i-j,1+2j}\\
=\;& \sum_{i=n+1}^{2n+1} A_{-2n+2i-1,4n-2i+3} + \sum_{i=n+1}^{2n} A_{-2n+2i,4n-2i+1}\\
&+\sum_{i=n+1}^{2n-1}\sum_{j=0}^{2n-i-1} A_{i-j,1+2j} + \left(\sum_{j=0}^{n-1} A_{n-j,1+2j} - \sum_{j=0}^{n-1} A_{n-j,1+2j}\right)
\end{aligned}
$$

as consequence,

$$
\begin{aligned}
&\sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i-j,1+2j}\\
=\;& \sum_{i=n+1}^{2n+1} A_{-2n+2i-1,4n-2i+3} + \sum_{i=n+1}^{2n} A_{-2n+2i,4n-2i+1}\\
&-\sum_{j=0}^{n-1} A_{n-j,1+2j} + \sum_{i=n}^{2n-1}\sum_{j=0}^{2n-i-1} A_{i-j,1+2j}.
\end{aligned}
\tag{3.37}
$$

Thus, by considereing (3.33),(3.35),(3.36) and (3.37) the Equality (3.32) is rewritten as follows

$$
\begin{aligned}
&\sum_{i=0}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i,j}\\
=\;& \sum_{i=0}^{n}\sum_{j=0}^{i} A_{i-j,2j} + \sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+2} A_{i-j,2j}\\
&+\sum_{i=0}^{n}\sum_{j=0}^{i} A_{i-j,1+2j} + \sum_{i=n+1}^{2n+1}\sum_{j=0}^{2n-i+1} A_{i-j,1+2j}\\
&+\sum_{i=0}^{2n+1} A_{i,2n-i+2} + \sum_{i=0}^{2n+1} A_{i,2n-i+1}\\
&-\sum_{i=n+1}^{2n+1} A_{-2n+2i-2,4n-2i+4} - \sum_{i=n+1}^{2n+1} A_{-2n+2i-1,4n-2i+2}\\
&-\sum_{i=n+1}^{2n+1} A_{-2n+2i-1,4n-2i+3} - \sum_{i=n}^{2n} A_{-2n+2i,4n-2i+1}.
\end{aligned}
$$

Then, in the aim to prove the Equality (3.31) it is equivalent to show

$$\begin{aligned} 0 = & \sum_{i=0}^{2n+1} A_{i,2n-i+2} + \sum_{i=0}^{2n+1} A_{i,2n-i+1} \\ & - \sum_{i=n+1}^{2n+1} A_{-2n+2i-2,4n-2i+4} - \sum_{i=n+1}^{2n+1} A_{-2n+2i-1,4n-2i+3} \\ & - \sum_{i=n+1}^{2n+1} A_{-2n+2i-1,4n-2i+2} - \sum_{i=n}^{2n} A_{-2n+2i,4n-2i+1}. \end{aligned} \tag{3.38}$$

Again we will examine the previous terms individually. So, we point out that

$$\begin{aligned} \sum_{i=n+1}^{2n+1} A_{-2n+2i-2,4n-2i+4} &= \sum_{i=0}^{n} A_{-2n+2(i+n+1)-2,4n-2(i+n+1)+4} = \sum_{i=0}^{n} A_{2i,2n-2i+2} \\ \sum_{i=n+1}^{2n+1} A_{-2n+2i-1,4n-2i+3} &= \sum_{i=0}^{n} A_{-2n+2(i+n+1)-1,4n-2(i+n+1)+3} = \sum_{i=0}^{n} A_{2i+1,2n-2i+1} \\ \sum_{i=n+1}^{2n+1} A_{-2n+2i-1,4n-2i+2} &= \sum_{i=0}^{n} A_{-2n+2(i+n+1)-1,4n-2(i+n+1)+2} = \sum_{i=0}^{n} A_{2i+1,2n-2i} \\ \sum_{i=n}^{2n} A_{-2n+2i,4n-2i+1} &= \sum_{i=0}^{n} A_{-2n+2(i+n),4n-2(i+n)+1} = \sum_{i=0}^{n} A_{2i,2n-2i+1} \end{aligned}$$

hence, applying the above remark in (3.38) it implies

$$\begin{aligned} 0 = & \sum_{i=0}^{2n+1} A_{i,2n-i+2} + \sum_{i=0}^{2n+1} A_{i,2n-i+1} \\ & - \sum_{i=0}^{n} A_{2i,2n-2i+2} - \sum_{i=0}^{n} A_{2i+1,2n-2i+1} \\ & - \sum_{i=0}^{n} A_{2i+1,2n-2i} - \sum_{i=0}^{n} A_{2i,2n-2i+1} \end{aligned}$$

thus, we must verify the next equality:

$$\begin{aligned} & \sum_{i=0}^{2n+1} A_{i,2n-i+2} + \sum_{i=0}^{2n+1} A_{i,2n-i+1} \\ = & \left(\sum_{i=0}^{n} A_{2i,2n-2i+2} + \sum_{i=0}^{n} A_{2i+1,2n-2i+1} \right) + \left(\sum_{i=0}^{n} A_{2i,2n-2i+1} + \sum_{i=0}^{n} A_{2i+1,2n-2i} \right). \end{aligned}$$

By the Lemma 138, if we consider

$$k = n + 1$$

and we set

$$B_i = A_{i,2n-i+2}$$

which follows

$$\sum_{i=0}^{2n+1} A_{i,2n-i+2} = \sum_{i=0}^{n} A_{2i,2n-2i+2} + \sum_{i=0}^{n} A_{2i+1,2n-2i+1} \tag{3.39}$$

and by the Lemma 139, consider

$$k = n+1$$

and we set

$$B_i = A_{i,2n-i+1}$$

thus it is obtained

$$\sum_{i=0}^{2n+1} A_{i,2n-i+1} = \sum_{i=0}^{n} A_{2i,2n-2i+1} + \sum_{i=0}^{n} A_{2i+1,2n-2i}, \tag{3.40}$$

so by (3.39) and (3.40) we conclude the desired. ■

We are in the position to prove the theorem mentioned in the beginning of this topic.

Theorem 91 *For given $n \in \mathbb{N}$, the next polynomial equality holds*

$$\begin{aligned} &\sum_{i=0}^{2n-1}\sum_{j=0}^{1}\left(\sum_{k=0}^{2n-i-j} a_{i-k,j+2k}\right)x^{4n-2i-j} \\ =\ &\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i} a_{i,j}x^{4n-j-2i} + \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-(i+1)} a_{-(i+1),2n+(i+1)-j}x^{2n+j+(i+1)} \end{aligned}$$

Proof. Notice that the left-hand side of the previous equality can be expressed as

$$\begin{aligned} &\sum_{i=0}^{2n-1}\sum_{j=0}^{1}\left(\sum_{k=0}^{2n-i-j} a_{i-k,j+2k}\right)x^{4n-2i-j} \\ =\ &\sum_{i=0}^{2n-1}\left(\sum_{k=0}^{2n-i} a_{i-k,2k}\right)x^{4n-2i} + \sum_{i=0}^{2n-1}\left(\sum_{k=0}^{2n-i-1} a_{i-k,1+2k}\right)x^{4n-2i-1}. \end{aligned} \tag{3.41}$$

Let us focus on the analysis of each of the above terms separately. The first one gives us

$$\sum_{i=0}^{2n-1}\left(\sum_{j=0}^{2n-i} a_{i-j,2j}\right) x^{4n-2i} = \sum_{i=0}^{n-1}\left(\sum_{j=0}^{2n-i} a_{i-j,2j}\right) x^{4n-2i} + \sum_{i=n}^{2n-1}\left(\sum_{j=0}^{2n-i} a_{i-j,2j}\right) x^{4n-2i} \tag{3.42}$$

$$= \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i-j,2j}\right) x^{4n-2i} + \sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{2n-i} a_{i-j,2j}\right) x^{4n-2i} + \sum_{i=n}^{2n-1}\left(\sum_{j=0}^{2n-i} a_{i-j,2j}\right) x^{4n-2i}$$

$$= \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i-j,2j}\right) x^{4n-2i} + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{2n-i-(i+1)} a_{i-(j+i+1),2(j+i+1)}\right) x^{4n-2i} + \sum_{i=n}^{2n-1}\left(\sum_{j=0}^{2n-i} a_{i-j,2j}\right) x^{4n-2i}$$

$$= \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i-j,2j}\right) x^{4n-2i} + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{2n-2i-1} a_{-j-1,2j+2i+2}\right) x^{4n-2i} + \sum_{i=n}^{2n-1}\left(\sum_{j=0}^{2n-i} a_{i-j,2j}\right) x^{4n-2i}$$

while the second term can be express as

$$\sum_{i=0}^{2n-1}\left(\sum_{j=0}^{2n-i-1} a_{i-j,1+2j}\right) x^{4n-2i-1} \tag{3.43}$$

$$= \sum_{i=0}^{n-1}\left(\sum_{j=0}^{2n-i-1} a_{i-j,1+2j}\right) x^{4n-2i-1} + \sum_{i=n}^{2n-1}\left(\sum_{j=0}^{2n-i-1} a_{i-j,1+2j}\right) x^{4n-2i-1}$$

$$= \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i-j,1+2j}\right) x^{4n-2i-1} + \sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{2n-i-1} a_{i-j,1+2j}\right) x^{4n-2i-1} + \sum_{i=n}^{2n-1}\left(\sum_{j=0}^{2n-i-1} a_{i-j,1+2j}\right) x^{4n-2i-1}$$

$$= \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i-j,1+2j}\right) x^{4n-2i-1} + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{2n-i-1-(i+1)} a_{i-(j+i+1),1+2(j+i+1)}\right) x^{4n-2i-1} + \sum_{i=n}^{2n-1}\left(\sum_{j=0}^{2n-i-1} a_{i-j,1+2j}\right) x^{4n-2i-1}$$

$$= \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i-j,1+2j}\right) x^{4n-2i-1} + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{2n-2i-2} a_{-j-1,2j+2i+3}\right) x^{4n-2i-1} + \sum_{i=n}^{2n-1}\left(\sum_{j=0}^{2n-i-1} a_{i-j,1+2j}\right) x^{4n-2i-1} \tag{3.44}$$

Hence, by the previous studies in (3.42) and (3.43), we obtain that the

Equality 3.41 is rewritten as

$$\sum_{i=0}^{2n-1}\sum_{j=0}^{1}\left(\sum_{k=0}^{2n-i-j} a_{i-k,j+2k}\right)x^{4n-2i-j} = \sum_{i=0}^{n-1}\left(\sum_{j=0}^{2n-2i-1} a_{-j-1,2j+2i+2}\right)x^{4n-2i}$$
$$+\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i-j,2j}\right)x^{4n-2i} + \sum_{i=n}^{2n-1}\left(\sum_{j=0}^{2n-i} a_{i-j,2j}\right)x^{4n-2i}$$
$$+\sum_{i=0}^{n-1}\left(\sum_{j=0}^{2n-2i-2} a_{-j-1,2j+2i+3}\right)x^{4n-2i-1}$$
$$+\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i-j,1+2j}\right)x^{4n-2i-1} + \sum_{i=n}^{2n-1}\left(\sum_{j=0}^{2n-i-1} a_{i-j,1+2j}\right)x^{4n-2i-1}.$$

The Theorem 89 provide to us

$$\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-(i+1)} a_{-(i+1),2n+(i+1)-j}x^{2n+j+(i+1)}$$
$$= \sum_{i=0}^{n-1}\sum_{j=0}^{2n-2i-1} a_{-j-1,2j+2i+2}x^{4n-2i} + \sum_{i=0}^{n-1}\sum_{j=0}^{2n-2i-2} a_{-j-1,2j+2i+3}x^{4n-2i-1},$$

finally, applying the Theorem 90 to $\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i} a_{i,j}x^{4n-j-2i}$, we obtain the following equality

$$\begin{aligned}\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-i} a_{i,j}x^{4n-j-2i} &= \sum_{i=0}^{n-1}\sum_{j=0}^{i} a_{i-j,2j}x^{4n-2i} + \sum_{i=n}^{2n-1}\sum_{j=0}^{2n-i} a_{i-j,2j}x^{4n-2i}\\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{i} a_{i-j,1+2j}x^{4n-2i-1} + \sum_{i=n}^{2n-1}\sum_{j=0}^{2n-i-1} a_{i-j,1+2j}x^{4n-2i-1},\end{aligned}$$

then the desired is achieved. ■

3.3 Main result and applications for Cyclic Surfaces

In this section we present our main theorems that fully classify Polynomial Weingarten cyclic surfaces in the Euclidean and in the Lorentzian 3-space.

The cyclic surface is the first and most natural generalization of tubular surface, in the sense that we may obtain a cyclic surface as a tubular surface that we permit the radius r to be a (smooth) function instead of being a constant number. For mathematical purposes, we have to demand that the radius satisfies $r(s) > 0$ for every $s \in (a, b)$.

This simple and almost innocent change has a massive impact in the curvatures of the surface that (in the Euclidean case) are expressed as

$$\begin{aligned} K &= \frac{\left(r^3\kappa^4\right)\cos^4 t - 3r^2\kappa^3\cos^3 t + r\kappa\left(3\kappa(r')^2 + r\kappa' r' + 3\kappa - rr''\kappa\right)\cos^2 t}{r\left((r')^2 + (1-\kappa(s)r(s)\cos t)^2\right)^2} \\ &+ \frac{+r^2 r'\kappa^2\tau\cos t\sin t - \left(2\kappa(r')^2 + r\kappa' r' + \kappa - 2rr''\kappa\right)\cos t - rr'\kappa\tau\sin t - \left(r(r')^2\kappa^2 + r''\right)}{r\left((r')^2 + (1-\kappa(s)r(s)\cos t)^2\right)^2} \end{aligned}$$

and

$$H = \frac{-2r^3\kappa^3\cos^3 t + 5r^2\kappa^2\cos^2 t - r\left(3\kappa(r')^2 + r\kappa' r' + 4\kappa - rr''\kappa\right)\cos t - r^2 r'\kappa\tau\sin t + r\left((r')^2 - rr'' + 1\right)}{2r\left((r')^2 + (1-\kappa(s)r(s)\cos t)^2\right)^{\frac{3}{2}}}.$$

Moreover, the assumption that the above curvatures verify a polynomial relation is a difficult challenge since it leads us to arduous differential equations that cannot be solved with the techniques that are commonly used.

In order to exemplify the previous claim, let us consider the simplest polynomial relation to be studied that is the linear one,

$$P(x, y) = a_{1,0}x + a_{0,1}y + a_{0,0} \in \mathbb{R}[x, y],$$

then the assumption that the cyclic surface is Polynomial Weingarten provides that

$$0 \equiv P(K, H) = \sum_{i=0}^{8}\sum_{j=0}^{1} P_{i,j}(s)\cos^i t\sin^j t \tag{3.45}$$

where $P_{i,j}(s)$ are smooth functions. The reason to omit the explicit expression of the coefficients $P_{i,j}(s)$ is because of the size of the equations would make the reading exaustive and because of the page formatting. Althoug to illustrate the type of problem that we face, we will present some of $P_{i,j}(s)$. For instance,

$$\begin{aligned} P_{8,0}(s) &= 4r^8\kappa^8 r^2 a_{0,0}^2 - r^2 a_{0,1}^2 + 2ra_{0,0}a_{1,0} + a_{1,0}^2 \\ P_{5,1}(s) &= 8r'\tau a_{0,0}r^8\kappa^6 a_{1,0} - 4r'\tau r^9\kappa^6 a_{0,1}^2 + 8r'\tau r^7\kappa^6 a_{1,0}^2 \end{aligned}$$

and the coefficient $P_{3,1} = -2r^5 r'\kappa^3\tau\widetilde{Q}_{3,1}$ where

$$\begin{aligned} \widetilde{Q}_{3,1} &= -24\kappa a_{1,0}^2 + 16r^2\kappa a_{0,1}^2 - 12(r')^2\kappa a_{1,0}^2 + 5r^2(r')^2\kappa a_{0,1}^2 \\ &\quad + 4rr''\kappa a_{1,0}^2 - 4rr'\kappa' a_{1,0}^2 - r^3 r''\kappa a_{0,1}^2 + r^3 r'\kappa' a_{0,1}^2 \\ &\quad - 40r\kappa a_{0,0}a_{1,0} - 8r(r')^2\kappa a_{0,0}a_{1,0}. \end{aligned}$$

Hence, our approach to deal with this type of problem, essentially, consists in to consider an arbitrary polynomial $P(x, y) \in \mathbb{R}[x, y]$ and investigate the associated function

$$0 \equiv P(K, H) = \sum_{i,j} f_{i,j}(s)\cos^i t\sin^j t$$

which is obtained by the composition of $P(x,y)$ with $K(s,t)$ and $H(s,t)$ (as we presented in (3.45)). So, for each fixed s_0 we define $\mathfrak{P}(s_0,x,y) \in \mathbb{R}[x,y]$ the polynomial that verify the following property

$$\mathfrak{P}(s_0, \cos t, \sin t) = P(K,H) \equiv 0$$

for every $t \in \mathbb{R}$. Then, we analyse the polynomial $\mathfrak{P}(s_0,x,y)$ under the algebraic geometry point of view, which ensures us that we can analyse each $f_{i,j}(s) \equiv 0$ individually. This method allowed us to obtain our main result that full classificates cyclic surface whose Gaussian and mean curvatures verify an arbitrary polynomial relation. The statement of our result read as follows:

Theorem 92 *A Polynomial Weingarten cyclic surface is a (smooth) combination of tubular surface with rotational surface.*

A first impact of the previous theorem is a charactarization of geometric features of the Polynomial Weingarten cyclic surface. More precisely, the above classification describes conditions in the curvature of the central curve of the cyclic surface and also provides conditions on the radius functions.

As a consequence of our theorem, we obtain a classification for Linear Weingarten cyclic surfaces:

Corollary 93 *A Linear Weingarten cyclic surface is either a globally tubular surface or globally a rotational surface.*

The previous corollary provides an improvement in the complete classification, in the sense that we proof that some particular relations does not accept combinations between the forementioned surfaces. Indeed, we have obtained the following result that ensures our claim.

Corollary 94 *Let $Q(x,y) \in \mathbb{R}[x,y]$ be a polynomial. Then $Rad^*(Q) = \emptyset$ if and only if the unique elements of $\mathcal{S}(Q)$ are the globally rotational surfaces.*

Here we would like to recall that the set $Rad^*(Q)$ is the collection of the $r_0 \in Rad(Q)$ such that $Q(x,y)$ belongs to the ideal in $\mathbb{R}[x,y]$ generated by $xr_0^2 - 2r_0 y + 1$.

In other words, throughout the study of the polynomials, we can provide a first discriminant that guarantees either the surface may or may not be a combination of tubular and rotational surfaces.

Before we start, we would like to (briefly) discuss about rotational surfaces. A rotational surface (also known as surface of revolution) is a surface obtained by the rotation of a plane curve $g(s)$ around an axis in its plane. The parametrization of a rotational surface is given by

$$\psi(s,t) = (g(s)\cos t, g(s)\sin t, t)$$

and its Gaussian and mean curvatures are

$$K = \left(\frac{-g''}{g\left(1+(g')^2\right)^2} \right) \qquad \text{and} \qquad H = \left(\frac{(g')^2 - gg'' + 1}{2g\left((g')^2+1\right)^{\frac{3}{2}}} \right).$$

As mentioned before, in this section we will use a classical result of Algebraic Geometry, but in order to proper state it, let us define the following set of all points $(a,b) \in \mathbb{R}^2$ such that $P(a,b) = 0$, for a given polynomial $P(x,y) \in \mathbb{R}[x,y]$. We denote this set as

$$V(P) = \left\{ (a,b) \in \mathbb{R}^2 \ ; \ P(a,b) = 0 \right\}.$$

For example, in the case that $P(x,y) \in \mathbb{R}[x,y]$ is given by $P(x,y) = x - y$, we have that $V(P)$ is the principal diagonal of $\mathbb{R}^2$. Other example is if $P(x,y) = x^2 + y^2 - 1$, then $V(P)$ has the geometric shape of a circle of radius 1 centered in the origin.

With this newly notation, we are able to state the theorem of Algebraic Geometry that we will made use:

Theorem 95 *Let $F(x,y)$ and $G(x,y)$ be polynomials in $\mathbb{R}[x,y]$ with no common factors. Then $V(F,G) = V(F) \bigcap V(G)$ is a finite set of points.*

The proof can be found in several classical books of algebraic geometry, we suggest for instance [8].

In the direction to apply the forementioned theorem, we first must remark that the polynomial $x^2 + y^2 - 1 \in \mathbb{R}[x,y]$ is irreducible, once it will play an important role in the next result. In fact, asume by absurd the existence of two polynomials of degree 1, named,

$$P_1(x,y) = ax + by + c \qquad \text{and} \qquad P_2(x,y) = dx + ey + f \in \mathbb{R}[x,y] \tag{3.46}$$

such that

$$x^2 + y^2 - 1 = P_1(x,y)\, P_2(x,y).$$

The polynomial equality provides that the following system must be verified:

$$\begin{aligned} ad &= 1, \qquad be = 1, \qquad cf = 1 \\ ae + bd &= 0, \qquad af + cd = 0, \qquad ce + bf = 0. \end{aligned}$$

Therefore, in on hand we have a, b, d and e must be all nonzero, moreover the coefficients satisfy

$$a = \frac{1}{d} \qquad \text{and} \qquad b = \frac{1}{e}.$$

On the other hand, we observe that above conditions implies that

$$0 = ae + bd = \frac{e^2 + d^2}{de},$$

hence

$$e = d = 0.$$

which is an absurd. Thus, we conclude that $x^2 + y^2 - 1$ is an irreducible polynomial.

So the following result is a consequence of the Theorem 95 along with previous observation and is a fundamental argument in the demonstration of our main theorem of classification of Polynomial Weingarten cyclic surfaces. The statement read as:

Lemma 96 *Consider the polynomial*

$$F(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{1} a_{i,j}x^i y^j = \left(\sum_{i=0}^{n} a_{i,0}x^i\right) + \left(\sum_{i=0}^{n} a_{i,1}x^i\right) y \in \mathbb{R}[x,y]$$

such that $F(\cos t, \sin t) \equiv 0$ *for every* t *in an interval. Then* $F(x,y)$ *is the null polynomial.*

Proof. Consider the irreducible polynomial $Q(x,y) = x^2 + y^2 - 1$ and notice that $Q(\cos t, \sin t) \equiv 0$ for every $t \in \mathbb{R}$. Therefore, we remark that $V(Q)$ is the circle $\mathbb{S}^1$ of radius 1 and center in the origin. Furthermore, the hypothesis that $F(\cos t, \sin t) \equiv 0$ furnishes that $V(F)$ contains $\mathbb{S}^1$, thus

$$V(F) \bigcap V(Q) \neq 0.$$

By Theorem 95 we have that Q and F has a factor in common and since Q is a irreducible polynomial, it implies that

$$F(x,y) = Q(x,y)R(x,y) = \left(x^2 - 1\right)R(x,y) + y^2 R(x,y)$$

for some $R(x,y) \in \mathbb{R}[x,y]$.

Notice that the power of variable y of polynomial $F(x,y)$ is up to 1, by hypothesis. This yields that $R(x,y)$ must be identically null, thus F is the null polynomial. ■

We remark that the above result naturally still holds in the case that the polynomials is given by

$$F(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{1} a_{i,j}y^i x^j = \left(\sum_{i=0}^{n} a_{i,0}y^i\right) + \left(\sum_{i=0}^{n} a_{i,1}y^i\right) x \in \mathbb{R}[x,y],$$

which we will state, but the proof will be omitted since is completely analogous.

Lemma 97 *Consider the polynomial*

$$F(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{1} a_{i,j}y^i x^j = \left(\sum_{i=0}^{n} a_{i,0}y^i\right) + \left(\sum_{i=0}^{n} a_{i,1}y^i\right) x \in \mathbb{R}[x,y]$$

such that $F(\cos t, \sin t) \equiv 0$ *for every* t *in an interval. Then* $F(x,y)$ *is the null polynomial.*

For hyperbolic trigonometric functions (*i.e.* for the functions $\cosh t$ and $\sinh t$), there is a complete analogous result to the Lemma 96. But in order to obtain that, first we have to notice that the polynomial $x^2 - y^2 - 1 \in \mathbb{R}[x, y]$ is also irreducible. Therefore, we obtain the following:

Lemma 98 *Consider the polynomial*

$$F(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{1} a_{i,j} x^i y^j = \left(\sum_{i=0}^{n} a_{i,0} x^i \right) + \left(\sum_{i=0}^{n} a_{i,1} x^i \right) y \in \mathbb{R}[x, y]$$

such that $F(\cosh t, \sinh t) \equiv 0$ for every t in an interval. Then $F(x,y)$ is the null polynomial.

Going along the lines to prove the above statement, it is only necessary remark that $Q(x,y) = x^2 - y^2 - 1$ is irreducible and verify

$$0 \equiv Q(\cosh t, \sinh t) = \cosh^2 t - \sinh^2 t - 1$$

for every $t \in \mathbb{R}$. Then, the result follows similarly as the prove of Lemma 96.

A relevant proposition is obtained when we articulate the previous lemma along with the assuption that the Gaussian and mean curvatures of a cyclic surface with central curve γ and radius r vanishes the polynomial $Q(x,y) = x - y^2$. More precisely, the below proposition is the first acquired result that describes geometric features of a cyclic surface that verify $Q(K, H) \equiv 0$.

Moreover, our result garantees that under the previous conditions, the curvature κ of γ vanishes everywhere which implies that cyclic surface is defined on a straight line (hence, it belongs specifically to the class of rotational surfaces). The precisely statement of our proposition follows:

Proposition 99 *Let $Q(x,y) \in \mathbb{R}[x, y]$ be the polynomial given by $Q(x,y) = x - y^2$. If a cyclic surface in the Euclidean 3-space is such that its Gaussian and mean curvatures verify $Q(K, H) \equiv 0$, then the surface is, in fact, a rotational surface.*

Proof. Assume the existence of a cyclic surface S with central curve γ and radius r such that its Gaussian and mean curvatures verify

$$0 = Q(K, H) = -K + H^2.$$

The above equation implies that S must be a totally umbilical cyclic surface. In fact,

$$0 = -K + H^2 = \frac{1}{4}(k_1 - k_2)^2,$$

where k_1 and k_2 are the principal curvatures of S. Since the class of totally umbilical are well-known, our work resumes to analyse if S can be either (part of) a sphere or (part of) a plane. First, we assume the existence of a point $s_0 \in I$ where the curvature κ of central curve γ is

$$\kappa(s_0) \neq 0,$$

therefore, there is an open interval J containing s_0 such that $\kappa(s) \neq 0$ for every $s \in J$. In this interval, we are able to parametrized the surface S_J which is obtained by restricting S to J, more precisely:

$$\psi(s,t) = \gamma(s) + (r(s)\cos t)\, N(s) + (r(s)\sin t)\, B(s)$$

where $s \in J$ and $t \in \mathbb{R}$.

Now we have two cases to examine, named, if S_J is contained in a sphere or if S_J is contained in a plane.

Suppose that exists an open set Ω of S_J which is entirely contained in the sphere of center ζ and radius r, named $S(\zeta, r)$. Thus, there is an open set $U = (a,b) \times (c,d) \subset J \times \mathbb{R}$, such that

$$g(\psi(s,t) - \zeta, \psi(s,t) - \zeta) = r^2,$$

for every $(s,t) \in U$.

So, the following equality is obtained by derivation of previous equation:

$$\begin{aligned} 0 &= g(\psi_t, \psi - \zeta) \\ &= rg(N(s), \zeta - \gamma(s))\sin t - rg(B(s), \zeta - \gamma(s))\cos t. \end{aligned}$$

Therefore, for each fixed $s_1 \in (a,b)$ we can apply the Lemma 96 that provides $g(N(s_1), \zeta - \gamma(s_1))$ and $g(B(s_1), \zeta - \gamma(s_1))$ must vanish identically, since s_1 is arbitrary, it implies

$$g(N(s), \zeta - \gamma(s)) \equiv 0 \qquad \text{and} \qquad g(B(s), \zeta - \gamma(s)) \equiv 0$$

for every $s \in (a,b)$. Finally, notice that

$$\begin{aligned} 0 &= \frac{d}{ds} g(N(s), \zeta - \gamma(s)) \\ &= g(-\kappa T, \zeta - \gamma(s)) + \tau g(B, \zeta - \gamma(s)) \\ &= -\kappa g(T, \zeta - \gamma(s)). \end{aligned}$$

Then $g(T, \zeta - \gamma(s))$ must be null for every $s \in (a,b)$. As consequence, we have

$$\gamma - \zeta = g(T, \gamma - \zeta)\, T + g(N, \gamma - \zeta)\, N + g(B, \gamma - \zeta)\, B = 0,$$

that is, $\gamma(s)$ is constant (equal to ζ) which is an absurd with the regularity of γ. We conclude that S_J cannot be part of sphere.

It still remains prove that S_J is not (part of) a plane. So suppose that exists an open set Ω of S_J which is entirely contained in a plane $\pi(v,d) =$

$\{u \in \mathbb{R}^3 \; ; \; \langle u, v \rangle = d\}$ where $v \in S^2 \subset \mathbb{R}^3$ and $d \in \mathbb{R}$ constant. Thus, there is an open set $U = (a, b) \times (c, d) \subset J \times \mathbb{R}$, such that

$$g(\psi(s,t), v) = d$$

for every $(s, t) \in U$.

Thus, the following equalities are obtained

$$g(\psi_s, v) = 0 \qquad \text{and} \qquad g(\psi_t, v) = 0,$$

we also remark that

$$0 = g(\psi_s, v) = -g(N(s), v)\sin t + g(B(s), v)\cos t.$$

For each $s_1 \in (a, b)$, the Lemma 96 provides that $g(N(s_1), v)$ and $g(B(s_1), v)$ must vanish identically, since s_1 is arbitrary, it yields

$$g(N(s), v) \equiv 0 \qquad \text{and} \qquad g(B(s), v) \equiv 0$$

for every $s \in (a, b)$. As consequence, we have

$$T(s) = v$$

for every $s \in (a, b)$, which is an absurd because the above equality gives

$$k(s) = \|T'\| = 0,$$

hence, S_J cannot be part of a plane.

Then, we conclude that curvature of central curve must be null. Moreover, a circle of radius $r(s)$ along a straight line is a rotational surface. ■

We obtain the same conclusions of the above statement also for cyclic surfaces in the Lorentzian 3-space, that is, a cyclic surface with central curve γ and radius r that vanishes the polynomial $Q(x, y) = x - y^2$ is, as a matter of fact, a rotational surface. More precisely, the cyclic surface is defined on a straight line.

However, we must to highlight that the arguments presented in the proof of previous proposition are not valid for the Lorentzian 3-space. Since the class of totally umbilical surfaces in $\mathbb{L}^3$ is, naturally, different than the ones in $\mathbb{E}^3$. Moreover, the notion of totally umbilical surfaces requires that the Weingarten map be diagonalizable.

In the end, we present the following result for a cyclic surface in $\mathbb{L}^3$ whithout any additional hypothesis.

Proposition 100 *Let $Q(x, y) \in \mathbb{R}[x, y]$ be the polynomial given by $Q(x, y) = x - y^2$. If a cyclic surface in the Lorentzian 3-space is such that its Gaussian and mean curvatures verify $Q(K, H) \equiv 0$, then the surface is, in fact, a rotational surface.*

Proof. Assume the existence of a cyclic surface S of type 1 (see Section 3.1.2 for more details) with central curve γ and radius r such that its Gaussian and mean curvatures verify

$$0 = Q\,(K, H) = -K + H^2. \tag{3.47}$$

We assume the existence of a point $s_0 \in I$ where the curvature κ of central curve γ is $\kappa\,(s_0) \neq 0$,therefore, there is an open interval J containing s_0 such that $\kappa\,(s) \neq 0$ for every $s \in J$. In this interval, we are able to parametrized the surface S_J which is obtained by restricting S to J, more precisely:

$$\psi\,(s, t) = \alpha + (r\,(s) \sinh t)\, N_\psi + (r\,(s) \cosh t)\, B_\psi$$

where $s \in J$ and $t \in \mathbb{R}$.

Then, by the Proposition 76 and by (3.47) we obtain the following differential equation:

$$\begin{aligned}
0 \;\equiv\;& 4r^8\kappa^6\,(\varepsilon-1)\sinh^6(t+\psi) + 20r^7\kappa^5\,(\varepsilon-1)\sinh^5(t+\psi)\\
&+r^6\kappa^3\begin{pmatrix} -40\kappa+41\kappa\varepsilon-16(r')^2\kappa+12(r')^2\kappa\varepsilon \\ +4rr''\kappa-4rr'\kappa'-4rr''\kappa\varepsilon+4rr'\varepsilon\kappa' \end{pmatrix}\sinh^4(t+\psi)\\
&+\left(4r^7r'\kappa^4\varepsilon\psi'-4r^7r'\kappa^4\psi'\right)\sinh^3(t+\psi)\cosh(t+\psi)\\
&+2r^5\kappa^2\begin{pmatrix} -20\kappa+22\kappa\varepsilon-22(r')^2\kappa+17(r')^2\kappa\varepsilon \\ +8rr''\kappa-6rr'\kappa'-7rr''\kappa\varepsilon+5rr'\varepsilon\kappa' \end{pmatrix}\sinh^3(t+\psi)\\
&+2r^6r'\kappa^3\psi'\,(5\varepsilon-6)\sinh^2(t+\psi)\cosh(t+\psi)\\
&-r^4\begin{pmatrix} 20\kappa^2+40(r')^2\kappa^2+12(r')^4\kappa^2-26\kappa^2\varepsilon-34(r')^2\kappa^2\varepsilon-9(r')^4\kappa^2\varepsilon+4r^2(r')^2\kappa^4 \\ -24rr''\kappa^2+18rr''\kappa^2\varepsilon+4r(r')^3\kappa\kappa'-4r(r')^2r''\kappa^2+12rr'\kappa\kappa'-r^2(r'')^2\kappa^2\varepsilon \\ -r^2(r')^2\varepsilon(\kappa')^2-6r(r')^3\kappa\varepsilon\kappa'+6r(r')^2r''\kappa^2\varepsilon-8rr'\kappa\varepsilon\kappa'+2r^2r'r''\kappa\varepsilon\kappa' \end{pmatrix}\sinh^2(t+\psi)\\
&+2r^5r'\kappa\psi'\begin{pmatrix} -6\kappa+4\kappa\varepsilon-2(r')^2\kappa \\ +3(r')^2\kappa\varepsilon-rr''\kappa\varepsilon+rr'\varepsilon\kappa' \end{pmatrix}\sinh(t+\psi)\\
&+\left(r^6(r')^2\kappa^2\varepsilon(\psi')^2\right)\cosh^2(t+\psi)\\
&+\left(2r^4r'\kappa\varepsilon\psi'\left((r')^2-rr''+1\right)-4r^4r'\kappa\psi'\left((r')^2+1\right)\right)\cosh(t+\psi)\\
&+\left(r^2\varepsilon\left((r')^2-rr''+1\right)^2+4r^3\left((r')^2+1\right)\left(r''-r(r')^2\kappa^2\right)\right)
\end{aligned}$$

Then, we notice that, in order to apply the Lemma 96, we must replace every

$$\cosh^2 t = \sinh^2 t + 1$$

in above equation. Hence, it follows

$$
\begin{aligned}
0 \equiv{} & 4r^8\kappa^6(\varepsilon-1)\sinh^6(t+\psi)+20r^7\kappa^5(\varepsilon-1)\sinh^5(t+\psi)\\
&+r^6\kappa^3\begin{pmatrix} -40\kappa+41\kappa\varepsilon-16(r')^2\kappa+12(r')^2\kappa\varepsilon \\ +4rr''\kappa-4rr'\kappa'-4rr''\kappa\varepsilon+4rr'\varepsilon\kappa' \end{pmatrix}\sinh^4(t+\psi)\\
&+4r^7r'\kappa^4\psi'(\varepsilon-1)\sinh^3(t+\psi)\cosh(t+\psi)\\
&+2r^5\kappa^2\begin{pmatrix} -20\kappa+22\kappa\varepsilon-22(r')^2\kappa+17(r')^2\kappa\varepsilon \\ +8rr''\kappa-6rr'\kappa'-7rr''\kappa\varepsilon+5rr'\varepsilon\kappa' \end{pmatrix}\sinh^3(t+\psi)\\
&+2r^6r'\kappa^3\psi'(5\varepsilon-6)\sinh^2(t+\psi)\cosh(t+\psi)\\
&-r^4\begin{pmatrix} 20\kappa^2+40(r')^2\kappa^2+12(r')^4\kappa^2-26\kappa^2\varepsilon-34(r')^2\kappa^2\varepsilon \\ -9(r')^4\kappa^2\varepsilon+4r^2(r')^2\kappa^4-24rr''\kappa^2+18rr''\kappa^2\varepsilon+4r(r')^3\kappa\kappa' \\ -4r(r')^2r''\kappa^2+12rr'\kappa\kappa'-r^2(r'')^2\kappa^2\varepsilon \\ -r^2(r')^2\varepsilon(\kappa')^2-6r(r')^3\kappa\varepsilon\kappa'+6r(r')^2r''\kappa^2\varepsilon \\ -8rr'\kappa\varepsilon\kappa'+2r^2r'r''\kappa\varepsilon\kappa' \end{pmatrix}\sinh^2(t+\psi)\\
&+2r^5r'\kappa\psi'\begin{pmatrix} -6\kappa+4\kappa\varepsilon-2(r')^2\kappa+3(r')^2\kappa\varepsilon \\ -rr''\kappa\varepsilon+rr'\varepsilon\kappa' \end{pmatrix}\sinh(t+\psi)\cosh(t+\psi)\\
&-2r^3\begin{pmatrix} 2\kappa-4\kappa\varepsilon+6(r')^2\kappa+4(r')^4\kappa \\ -7(r')^2\kappa\varepsilon-3(r')^4\kappa\varepsilon \\ -8rr''\kappa+2rr'\kappa'+4r^2(r')^2\kappa^3+2r(r')^3\kappa' \\ -r(r')^3\varepsilon\kappa'-r^2(r'')^2\kappa\varepsilon \\ +5rr''\kappa\varepsilon-rr'\varepsilon\kappa'-4r(r')^2r''\kappa \\ +4r(r')^2r''\kappa\varepsilon+r^2r'r''\varepsilon\kappa' \end{pmatrix}\sinh(t+\psi)\\
&-2r^4r'\kappa\psi'\left(-\varepsilon+2(r')^2-(r')^2\varepsilon+rr''\varepsilon+2\right)\cosh(t+\psi)\\
&+r^2\varepsilon\left((r')^2-rr''+1\right)^2+4r^3\left((r')^2+1\right)\left(r''-r(r')^2\kappa^2\right)+r^6(r')^2\kappa^2\varepsilon(\psi')^2
\end{aligned}
$$

Now, we have that each term of above differential equation must be null.

In the case that $\varepsilon=-1$, we have

$$-8r^8\kappa^6 \equiv 0$$

which is an absurd since

$$r(s)>0 \qquad \text{and} \qquad \kappa(s)\neq 0$$

for every $s\in J$, therefore, in this case we conclude that there is no point s_0 such that $\kappa(s_0)\neq 0$.

In the case that $\varepsilon = 1$, the previous differential equations becomes

$$
\begin{aligned}
0 \quad \equiv \quad & -r^6\kappa^4\left(2r'-1\right)\left(2r'+1\right)\sinh^4\left(t+\psi\right) \\
& -2r^5\kappa^2\left(-2\kappa+5(r')^2\kappa-rr''\kappa+rr'\kappa'\right)\sinh^3\left(t+\psi\right) \\
& -2r^6r'\kappa^3\psi'\sinh^2\left(t+\psi\right)\cosh\left(t+\psi\right) \\
& +r^4\left(\begin{array}{c} 6\kappa^2-6(r')^2\kappa^2-3(r')^4\kappa^2+r^2(r'')^2\kappa^2 \\ -4r^2(r')^2\kappa^4+r^2(r')^2(\kappa')^2+6rr''\kappa^2 \\ +2r(r')^3\kappa\kappa'+r^2(r')^2\kappa^2(\psi')^2 \\ -2r(r')^2r''\kappa^2-4rr'\kappa\kappa'-2r^2r'r''\kappa\kappa' \end{array}\right)\sinh^2\left(t+\psi\right) \\
& +2r^5r'\kappa\psi'\left(-2\kappa+(r')^2\kappa-rr''\kappa+rr'\kappa'\right)\sinh\left(t+\psi\right)\cosh\left(t+\psi\right) \\
& -2r^3\left(\begin{array}{c} -2\kappa-(r')^2\kappa+(r')^4\kappa-r^2(r'')^2\kappa \\ -3rr''\kappa+rr'\kappa'+4r^2(r')^2\kappa^3 \\ +r(r')^3\kappa'+r^2r'r''\kappa' \end{array}\right)\sinh\left(t+\psi\right) \\
& -2r^4r'\kappa\psi'\left(rr''+(r')^2+1\right)\cosh\left(t+\psi\right) \\
& +r^2\left(\begin{array}{c} r^2(r'')^2+2rr''+2(r')^2+(r')^4 \\ -4r^2(r')^2\kappa^2-4r^2(r')^4\kappa^2+2r(r')^2r'' \\ +r^4(r')^2\kappa^2(\psi')^2+1 \end{array}\right)
\end{aligned}
$$

From the first line, we have that

$$\left(2r'-1\right)\left(2r'+1\right)=0$$

which yields that $r'(s) \equiv \pm\frac{1}{2}$ for every $s \in J$, consequently, $r'' \equiv 0$ in J. The third line provides that

$$-2r^6r'\kappa^3\psi' \equiv 0$$

so $\psi'(s) \equiv 0$ for every $s \in J$. The second equation gives

$$-2r^5\kappa^2\left(-2\kappa+5(r')^2\kappa-rr''\kappa+rr'\kappa'\right) \equiv 0$$

thus, in the case that $r'(s) \equiv \frac{1}{2}$ the previous equation gives

$$\frac{1}{2}r\kappa'-\frac{3}{4}\kappa \equiv 0$$

since $r \neq 0$ and $\kappa \neq 0$ we can express κ' in terms of forementioned functions, named

$$\kappa' = \frac{3\kappa}{2r}.$$

Now, notice that in on hand, the last equality provides

$$\left(\begin{array}{c} r^2(r'')^2+2rr''+2(r')^2+(r')^4-4r^2(r')^2\kappa^2 \\ -4r^2(r')^4\kappa^2+2r(r')^2r''+r^4(r')^2\kappa^2(\psi')^2+1 \end{array}\right) \equiv 0$$

since $r'' = 0 = \psi'$ and $r' = \frac{1}{2}$ it follows

$$\frac{5}{4} = r^2\kappa^2. \tag{3.48}$$

On the other hand, the fourth equation is

$$\begin{pmatrix} 6\kappa^2 - 6(r')^2\kappa^2 - 3(r')^4\kappa^2 + r^2(r'')^2\kappa^2 \\ -4r^2(r')^2\kappa^4 + r^2(r')^2(\kappa')^2 + 6rr''\kappa^2 \\ +2r(r')^3\kappa\kappa' + r^2(r')^2\kappa^2(\psi')^2 \\ -2r(r')^2r''\kappa^2 - 4rr'\kappa\kappa' - 2r^2r'r''\kappa\kappa' \end{pmatrix} \equiv 0,$$

then applying the already known information, we have

$$-\frac{1}{4}\kappa^2\left(2r\kappa - 3\right)\left(2r\kappa + 3\right) \equiv 0$$

therefore

$$2r\kappa - 3 = 0 \qquad \text{or} \qquad 2r\kappa + 3 = 0.$$

Hence

$$r\kappa = \frac{3}{2} \qquad \text{or} \qquad r\kappa = -\frac{3}{2},$$

either way, we have $(r\kappa)^2 = \left(\pm\frac{3}{2}\right)^2 = \frac{9}{4}$ which contradicts to the required condition in (3.48). So, in the case that $r' \equiv \frac{1}{2}$ we conclude that there is no $s_0 \in I$ such that $\kappa\left(s_0\right) \neq 0$. Then, the cyclic surface that verify $-K + H^2 \equiv 0$ is, indeed, a rotational surface.

In the case that $r' \equiv -\frac{1}{2}$, we already have that $\psi' \equiv 0$ and $r'' \equiv 0$ in J. The second equation gives

$$-2r^5\kappa^2\left(-2\kappa + 5(r')^2\kappa - rr''\kappa + rr'\kappa'\right) \equiv 0$$

thus, the previous equation gives

$$\kappa' = \frac{-3\kappa}{2r}.$$

Now, notice that in on hand, the last equality provides

$$\begin{pmatrix} r^2(r'')^2 + 2rr'' + 2(r')^2 + (r')^4 - 4r^2(r')^2\kappa^2 \\ -4r^2(r')^4\kappa^2 + 2r(r')^2r'' + r^4(r')^2\kappa^2(\psi')^2 + 1 \end{pmatrix} \equiv 0$$

since $r'' = 0 = \psi'$ and $r' = -\frac{1}{2}$ it follows

$$r^2\kappa^2 = \frac{5}{4}.$$

On the other hand, the fourth equation is

$$\begin{pmatrix} 6\kappa^2 - 6(r')^2\kappa^2 - 3(r')^4\kappa^2 + r^2(r'')^2\kappa^2 \\ -4r^2(r')^2\kappa^4 + r^2(r')^2(\kappa')^2 + 6rr''\kappa^2 \\ +2r(r')^3\kappa\kappa' + r^2(r')^2\kappa^2(\psi')^2 \\ -2r(r')^2r''\kappa^2 - 4rr'\kappa\kappa' - 2r^2r'r''\kappa\kappa' \end{pmatrix} \equiv 0,$$

then applying the already known information, we have

$$-\frac{1}{4}\kappa^2\left(2r\kappa - 3\right)\left(2r\kappa + 3\right) \equiv 0$$

therefore

$$2r\kappa - 3 = 0 \qquad \text{or} \qquad 2r\kappa + 3 = 0.$$

Hence

$$r\kappa = \frac{3}{2} \qquad \text{or} \qquad r\kappa = -\frac{3}{2},$$

either way, we have $(r\kappa)^2 = \left(\pm\frac{3}{2}\right)^2 = \frac{9}{4}$ which contradicts to the required condition in (3.48). So, in the case that $r' \equiv -\frac{1}{2}$ we conclude that there is no $s_0 \in I$ such that $\kappa(s_0) \neq 0$. Then, the cyclic surface that verify $-K + H^2 \equiv 0$ is, indeed, a rotational surface. ■

Remark 101 *Despite the proof of the Proposition 100 be made by the cyclic surface of type 1 in Lorentzian* 3*-space, it is immediate to conclude that the statement holds for cyclic surface of type 2, 3, 4 and 5, once the Gaussian and mean curvatures of these surfaces has the only difference in the* $\cos t, \sin t, \cosh t$ *and* $\sinh t$*. So we conclude that the result is valid for a cyclic surface (of any type) in Lorentzian* 3*-space.*

In view of the above discussion, we present our main theorem that provides a geometric description of cyclic surfaces that verify a polynomial $Q(x,y) \in \mathbb{R}[x,y]$. Furthermore, the result achieves that in the aim to classify cyclic surfaces such that its curvatures verify $Q(K,H) \equiv 0$, it is only needed investigate (smooth) combinations of tubular surfaces and rotational surfaces whose curvatures vanishes the polynomial.

This is a relevant simplification since locally either the radius is constant or either the curvature of central curve is null. Finally, our theorem shows conditions over the polynomial to guarantee existence of Polynomial Weingarten cyclic surfaces.

The result exhibited in the beginning of this section is a simplified version of our main theorem whose precisely statement will be presented below.

Theorem 102 *Consider the polynomial* $Q(x,y) \in \mathbb{R}[x,y]$ *and let* $\mathcal{S}(Q)$ *be the set of all regular cyclic surfaces in Euclidean* 3*-space whose Gaussian and mean curvatures* K*,* H *verify* $Q(K,H) \equiv 0$*. Then, the elements of* $\mathcal{S}(Q)$ *are (smooth) combinations of Rotational surfaces and Tubular surfaces of radius* $r \in Rad^*(Q)$*.*

Proof. Consider the following expression (which is always possible) of the given polynomial

$$Q(x,y) = \left(-x + y^2\right)^n P(x,y),$$

where $n \in \mathbb{N}$ (possibly zero) and $P(x,y) \in \mathbb{R}[x,y]$ is a polynomial that does not belong to the ideal generated by $-x + y^2$.

For an arbitraty element $s_0 \in (a,b)$ such that the curvature κ of the central curve γ is not null, that is $\kappa(s_0) \neq 0$, we have the existence of a neighborhood J of s_0 such that $\kappa(s) \neq 0$ for every $s \in J$. In this interval, we are able to parametrized the surface S_J which is obtained by the restriction of S to J, more precisely:

$$\psi(s,t) = \gamma(s) + (r(s)\cos t)\, N(s) + (r(s)\sin t)\, B(s) \tag{3.49}$$

where $s \in J$ and $t \in \mathbb{R}$. By the Proposition 99 we already have that the Gaussian and mean curvatures of S_J does not vanish the polynomial $-x+y^2$, therefore the hypothesis that the Gaussian and mean curvatures of S verify $Q(K,H) \equiv 0$ implies that $P(K,H)$ must be null, that is

$$0 \equiv P(K,H) = \sum_{j=0}^{2n} \sum_{i=0}^{2n-j} a_{i,j} (K)^i (H)^j ,$$

and by the Notation 75 the previous equation is rewritten as

$$0 \equiv \sum_{j=0}^{2n} \sum_{i=0}^{2n-j} a_{i,j} \frac{\Delta(s,t)^i \alpha(s,t)^j}{2^j r(s)^{2(i+j)} \beta(s,t)^{\frac{4i+3j}{2}}}. \tag{3.50}$$

So it is necessary express the above equation in a more suitable way. In the aim to accomplish that, we first need to write every term with the same denominator, then we have to analyze the exponent of β which is

$$\frac{4i+3j}{2} = 2i + j + \frac{j}{2} < 2i + 2j \leq 4n.$$

So mulitplying the Equation 3.50 both sides by $\left(2r^2\beta^2\right)^{2n}$ it is obtained

$$0 = \sum_{j=0}^{2n} \sum_{i=0}^{2n-j} 2^{2n-j} a_{i,j} r^{2n-2(i+j)} \beta^{4n-\frac{4i+3j}{2}} \Delta^i \alpha^j. \tag{3.51}$$

Now the next step to reach our proper expression is to separate between the terms whose β is to the power of odds and even numbers because our objective is express the above equation as a polynomial, therefore we will study the nth roots of β.

In this direction, it is easy to see that the parity of $\frac{4i+3j}{2}$ is given exclusively by j, once $4i$ is even for every $i \in \mathbb{N}$. Thus, applying the Lemma 135 to the Equation 3.51, it can be expressed as two summatories

$$\sum_{j=0}^{n} \sum_{i=0}^{2n-2j} 2^{2n-2j} a_{i,2j} r^{4n-2(i+2j)} \beta^{4n-(2i+3j)} \Delta^i \alpha^{2j}$$

$$+\beta^{\frac{1}{2}} \sum_{j=1}^{n} \sum_{i=0}^{2n-(2j-1)} 2^{2n-(2j-1)} a_{i,(2j-1)} r^{4n-2(i+(2j-1))} \beta^{4n-2i-3j+1} \Delta^i \alpha^{(2j-1)} = 0,$$

which implies

$$\left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j} 2^{2n-2j} a_{i,2j} r^{4n-2(i+2j)} \beta^{4n-(2i+3j)} \Delta^i \alpha^{2j}\right)^2$$
$$= \left(-\beta^{\frac{1}{2}}\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)} 2^{2n-(2j-1)} a_{i,(2j-1)} r^{4n-2(i+(2j-1))} \beta^{4n-2i-3j+1} \Delta^i \alpha^{(2j-1)}\right)^2$$

Finally, we achieve the following expression

$$\left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j} 2^{2n-2j} a_{i,2j} r^{4n-2(i+2j)} \beta^{4n-(2i+3j)} \Delta^i \alpha^{2j}\right)^2 \tag{3.52}$$
$$-\beta\left(\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)} 2^{2n-(2j-1)} a_{i,(2j-1)} r^{4n-2(i+(2j-1))} \beta^{4n-2i-3j+1} \Delta^i \alpha^{(2j-1)}\right)^2 = 0.$$

Before to proceed, let us define, for each $s \in J$, the following polynomials

$$\begin{aligned}
\Delta_s(x,y) &= r^4\kappa^4x^4 - 3r^3\kappa^3x^3 + r^2\kappa\left(3\kappa(r')^2 + r\kappa'r' + 3\kappa - rr''\kappa\right)x^2 \\
&\quad + r^3r'\kappa^2\tau xy - r\left(2\kappa(r')^2 + r\kappa'r' + \kappa - 2rr''\kappa\right)x \\
&\quad - r^2r'\kappa\tau y - r\left(r(r')^2\kappa^2 + r''\right) \\
\alpha_s(x,y) &= -2r^4\kappa^3x^3 + 5r^3\kappa^2x^2 - r^2\left(3\kappa(r')^2 + r\kappa'r' + 4\kappa - rr''\kappa\right)x \\
&\quad - r^3r'\kappa\tau y + r\left((r')^2 - rr'' + 1\right) \\
\beta_s(x,y) &= \left(r^2\kappa^2\right)x^2 + (-2r\kappa)x + \left((r')^2 + 1\right)
\end{aligned}$$

where the functions r, r', r'', κ, τ are evalueted in s (therefore they are scalars).

Notice that the above polynomials has the property that $\Delta_s(\cos t, \sin t) = \Delta(s,t)$, $\alpha_s(\cos t, \sin t) = \alpha(s,t)$ and $\beta_s(\cos t, \sin t) = \beta(s,t)$ for every $t \in \mathbb{R}$. Before to proceed, let us define the polynomial $\mathfrak{P}(x,y,z) \in \mathbb{R}[x,y,z]$, given by

$$\mathfrak{P}(x,y,z) = \left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j} 2^{2n-2j} a_{i,2j} r^{4n-2(i+2j)} z^{4n-(2i+3j)} x^i y^{2j}\right)^2 \tag{3.53}$$
$$-z\left(\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)} 2^{2n-(2j-1)} a_{i,(2j-1)} r^{4n-2(i+(2j-1))} z^{4n-2i-3j+1} x^i y^{(2j-1)}\right)^2.$$

Now we consider the following map $\mathfrak{P}(\Delta_s(x,y), \alpha_s(x,y), \beta_s(x,y)) \in \mathbb{R}[x,y]$. Moreover, we remark that $\mathfrak{P}(\Delta_s(x,y), \alpha_s(x,y), \beta_s(x,y))$ has degree constant equal to $16n$. In fact, applying the Lemma 140 in (3.53) we are able to express

$\mathfrak{P}(x,y,z)$ as

$$\mathfrak{P}(x,y,z)=\sum_{j=0}^{n}\sum_{i=0}^{2n-2j}\sum_{k=0}^{n}\sum_{l=0}^{2n-2k}2^{4n-2k-2j}r^{4n-2j-2k-l-i}a_{i,2j}a_{l,2k}x^{i+l}y^{2j+2k}z^{8n-3j-3k-2l-2i}$$
$$-\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)}\sum_{k=1}^{n}\sum_{l=0}^{2n-(2k-1)}2^{4n-2k-2j+2}r^{4n-2j-2k-l-i+2}a_{i,2j-1}a_{l,2k-1}x^{i+l}y^{2j+2k-2}z^{8n-3j-3k-2l-2i+3}.$$

then, we have that the degree of $\mathfrak{P}(\Delta_s(x,y),\alpha_s(x,y),\beta_s(x,y))$ is calculated as

$$4(i+l)+3(2j+2k)+2(8n-3j-3k-2l-2i)=16n$$

and also

$$4(i+l)+3(2j+2k-2)+2(8n-3j-3k-2l-2i+3)=16n$$

Therefore, the Corollary 86 provides to us the existence of a polynomial $Q_s(x,y)\in\mathbb{R}[x,y]$ such that

$$\mathfrak{P}(\Delta_s(x,y),\alpha_s(x,y),\beta_s(x,y))=\mathfrak{P}\left(r^4\kappa^4x^4,-2r^4\kappa^3x^3,r^2\kappa^2x^2\right)+Q_s(x,y)$$

where $\partial Q_s<16n$.

Then, notice that

$$\mathfrak{P}(\Delta_s(\cos t,\sin t),\alpha_s(\cos t,\sin t),\beta_s(\cos t,\sin t))\equiv 0$$

for every $t\in\mathbb{R}$. So replacing every y^2 by $1-x^2$ in the next polynomial

$$\mathfrak{P}\left(\Delta_{s_0}(x,y),\alpha_{s_0}(x,y),\beta_{s_0}(x,y)\right)=\mathfrak{P}\left(r^4\kappa^4x^4,-2r^4\kappa^3x^3,r^2\kappa^2x^2\right)+Q_{s_0}(x,y)$$

we obtain a polynomial $\widetilde{\mathfrak{P}_{s_0}}(x,y)$ verifying

$$\widetilde{\mathfrak{P}_{s_0}}(\cos t,\sin t)=0$$

for every t. Follows from the Lemma 96 that $\widetilde{\mathfrak{P}_{s_0}}$ must be the null polynomial. Furthermore, we remark that

$$0(x,y)=\widetilde{\mathfrak{P}_{s_0}}(x,y)=\mathfrak{P}\left(r^4\kappa^4x^4,-2r^4\kappa^3x^3,r^2\kappa^2x^2\right)+\widetilde{Q_s}(x,y)$$

where $\partial\widetilde{Q_s}\le\partial Q_s<16n$. It is important to observe that procedure of changing y^2 by $1-x^2$ does not affect the monomial $\mathfrak{P}\left(r^4\kappa^4x^4,-2r^4\kappa^3x^3,r^2\kappa^2x^2\right)$ once it is a polynomial in $\mathbb{R}[x]$. Indeed, a straightfoward computation provides that

$$\begin{aligned}&\mathfrak{P}\left(r^4\kappa^4x^4,-2r^4\kappa^3x^3,r^2\kappa^2x^2\right) \qquad (3.54)\\ =\;&2^{4n}\kappa^{16n}\left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j}r^{2n}r^{10n-2j-2i}a_{i,2j}\right)^2x^{16n}\\ &-2^{4n}\kappa^{16n}r^2\left(\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)}r^{2n}r^{10n-2j-2i}a_{i,2j-1}\right)^2x^{16n},\end{aligned}$$

therefore we achieve this claim.

Furthermore, let us come back to the monomial $\mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right)$ for further analysis. For practical purposes, we recall the next well-known property

$$A^2 - B^2 = (A - B)(A + B)$$

it provides that the Equation 3.54 can be expressed as

$$\mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right) = 2^{4n}\kappa^{16n}r^{4n}\mathcal{A}(r)\,\mathcal{B}(r)\,x^{16n}$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined as polynomials evalueted in r. More precisely, we define the next polynomials

$$\mathcal{A}(w) = \left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j}\left(w^{10n-2j-2i}a_{i,2j}\right) - w\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)}\left(w^{10n-2j-2i}a_{i,2j-1}\right)\right) \tag{3.55}$$

and

$$\mathcal{B}(w) = \left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j}\left(w^{10n-2j-2i}a_{i,2j}\right) + w\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)}\left(w^{10n-2j-2i}a_{i,2j-1}\right)\right) \tag{3.56}$$

Since, by hypothesis, $P(x, y)$ does not belong to the ideal in $\mathbb{R}[x, y]$ generated by $-x + y^2$ and it implies that

$$\mathcal{A}(w)\,\mathcal{B}(w) \not\equiv 0\,(w),$$

in other words, $\mathcal{A}(w)\,\mathcal{B}(w)$ is not the null polynomial.

In fact, we will examine each of the polynomials $\mathcal{A}(w)$ and $\mathcal{B}(w)$ individually. Notice that, the Lemma 135, allows us to rewrite (3.55) and (3.56) respectively as

$$\mathcal{A}(w) = \left(\sum_{j=0}^{2n}\sum_{i=0}^{2n-j}(-1)^j\,a_{i,j}w^{4n-j-2i}\right)$$

and

$$\mathcal{B}(w) = \left(\sum_{j=0}^{2n}\sum_{i=0}^{2n-j}a_{i,j}w^{4n-j-2i}\right).$$

Thus, the Theorem 91 provides the following expression of above polynomials

$$\mathcal{A}(w) = \sum_{i=0}^{2n-1}\sum_{j=0}^{1}(-1)^j\left(\sum_{k=0}^{2n-i-j}a_{i-k,j+2k}\right)w^{4n-2i-j} + a_{2n,0}$$

and

$$\mathcal{B}(w) = \sum_{i=0}^{2n-1}\sum_{j=0}^{1}\left(\sum_{k=0}^{2n-i-j}a_{i-k,j+2k}\right)w^{4n-2i-j} + a_{2n,0}.$$

It is relevant to notice that in previous polynomials each power of w appears only one time. In fact,

$$\mathcal{A}(w) = \sum_{i=0}^{2n-1}\left(\sum_{k=0}^{2n-i} a_{i-k,2k}\right) w^{4n-2i} - \sum_{i=0}^{2n-1}\left(\sum_{k=0}^{2n-i-1} a_{i-k,1+2k}\right) w^{4n-2i-1} + a_{2n,0}$$

and

$$\mathcal{B}(w) = \sum_{i=0}^{2n-1}\left(\sum_{k=0}^{2n-i} a_{i-k,2k}\right) w^{4n-2i} + \sum_{i=0}^{2n-1}\left(\sum_{k=0}^{2n-i-1} a_{i-k,1+2k}\right) w^{4n-2i-1} + a_{2n,0},$$

therefore, the polynomials will be null if and only if

$$\left(\sum_{k=0}^{2n-i} a_{i-k,2k}\right) \equiv 0 \tag{3.57}$$

for every $(i,j) \in \Delta = \{0,\ldots,2n-1\} \times \{0,1\} \cup \{(2n,0)\}$ which is precisely de condition to apply our main Theorem 83 that gives to us that 3.57 vanishes identically if and only if $P(x,y)$ belongs to the ideal in $\mathbb{R}[x,y]$ generated by $-x+y^2$ (which is not the case).

Then we conclude that $\mathcal{A}(w)$ and $\mathcal{B}(w)$ are non null, hence

$$\mathcal{A}(w)\,\mathcal{B}(w) \not\equiv 0\,(w).$$

However, we recall that

$$\mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right) = 2^{4n}\kappa^{16n}r^{4n}\mathcal{A}(r)\,\mathcal{B}(r)\,x^{16n} \tag{3.58}$$

whose degree is equal to $16n$ and since $\partial\widetilde{Q_s} < 16n$ we conclude that (3.58) is, in fact, the term of highest degree. On the other hand, it must be null, so it yields

$$\mathcal{A}(r)\,\mathcal{B}(r) = 0$$

since $r \neq 0$ for every $s \in (a,b)$, in particular $r \neq 0$ for every $s \in J$, and r is a smooth function whose vanishes a polynomial, then we conclude that r must be constant, that is,

$$r(s) = r_0$$

for every $s \in J$.

If exists a $r_0 \in Rad^*(P)$, we conclude by Theorem 54 that the elements of $\mathcal{S}(P)$ are the tubular surfaces of radius r_0.

So it remains to investigate the points $s_0 \in (a,b)$ such that $\kappa(s) = 0$. If s_0 is an isolated point, we notice that there is a neighborhood J of s_0 such that for every $s \in J - \{s_0\}$ we have $\kappa(s) \neq 0$, then the same argument as before provides that we have a combination of tubular surfaces and since, in this case, the radius is constant, it implies that the surface is a tubular surface in J.

Finally, in elements $s_0 \in (a,b)$ such that $\kappa(s_0) = 0$, where we have an open neighborhood L such that $\kappa(s) = 0$ for every $s \in L$. So, if we consider S_L which is defined analagous to S_J, it follows that S_L is a rotational surface. Then we conclude that S is a smooth combination of tubular surfaces and rotational surfaces. ■

Remark 103 *It is important to notice that the set $\mathcal{S}(Q)$ includes (but do not resume to) surfaces that are combination of Tubular surface and Rotational surfaces, that is, the combination may be empty. In other words,we admit globally tubular surface and globally rotational surfaces as combinations of the forementioned surfaces.*

We also achieve a complete classification of cyclic surfaces in Lorentzian 3-space whose Gaussian and mean curvatures verify $Q(K,H) \equiv 0$, for $Q(x,y) \in \mathbb{R}[x,y]$ an arbitrary polynomial relation. As a matter of fact, the following demonstration is completly analogous to the previous theorem. The only distinct point between the proofs is the presence of the signal ε from the curvatures in $\mathbb{L}^3$.

Theorem 104 *Consider the polynomial $Q(x,y) \in \mathbb{R}[x,y]$ and let $\mathcal{S}(Q)$ be the set of all regular cyclic surfaces in Lorentzian 3-space whose Gaussian and mean curvatures K, H verify $Q(K,H) \equiv 0$. Then, the elements of $\mathcal{S}(Q)$ are (smooth) combinations of timelike Rotational surfaces and timelike Tubular surfaces of radius $r \in Rad^*(Q)$.*

Proof. Consider the following expression (which is always possible) of the given polynomial

$$Q(x,y) = \left(-x+y^2\right)^n P(x,y),$$

where $n \in \mathbb{N}$ (possibly zero) and $P(x,y) \in \mathbb{R}[x,y]$ is a polynomial that does not belong to the ideal generated by $-x+y^2$.

Assume the existence of a cyclic surface S of type 1 (see Section 3.1.2 for more details) with central curve γ and radius r. For an arbitraty element $s_0 \in (a,b)$ such that the curvature κ of the central curve γ is not null, that is $\kappa(s_0) \neq 0$, we have the existence of a neighborhood J of s_0 such that $\kappa(s) \neq 0$ for every $s \in J$. In this interval, we are able to parametrized the surface S_J which is obtained by the restriction of S to J, more precisely:

$$\psi(s,t) = \alpha + (r(s)\sinh t) N_\psi + (r(s)\cosh t) B_\psi \tag{3.59}$$

where $s \in J$ and $t \in \mathbb{R}$. By the Proposition 100 we already have that the Gaussian and mean curvatures of S_J does not vanish the polynomial $-x+y^2$, therefore the hypothesis that the Gaussian and mean curvatures of S verify $Q(K,H) \equiv 0$ implies that $P(K,H)$ must be null, that is

$$0 \equiv P(K,H) = \sum_{j=0}^{2n} \sum_{i=0}^{2n-j} a_{i,j} (K)^i (H)^j,$$

the previous equation is rewritten as

$$0 \equiv \sum_{j=0}^{2n} \sum_{i=0}^{2n-j} a_{i,j} \frac{\Delta(s,t)^i \alpha(s,t)^j}{2^j (-\varepsilon)^{\frac{3}{2}j} r(s)^{2(i+j)} \beta(s,t)^{\frac{4i+3j}{2}}}, \tag{3.60}$$

where $\varepsilon = -1$ if the surface is spacelike and $\varepsilon = 1$ if the surface is timelike.

So it is necessary express the above equation in a more suitable way. In the aim to accomplish that, we first need to write every term with the same denominator, then we have to analyze the exponent of β which is

$$\frac{4i+3j}{2} = 2i + j + \frac{j}{2} < 2i + 2j \leq 4n.$$

So mulitplying the Equation 3.60 both sides by $\left(2R^2\beta^2\right)^{2n}$ it is obtained

$$0 = \sum_{j=0}^{2n} \sum_{i=0}^{2n-j} 2^{2n-j} \left(-\varepsilon\right)^{\frac{3}{2}j} a_{i,j} r^{4n-2(i+j)} \beta^{4n-\frac{4i+3j}{2}} \Delta^i \alpha^j. \tag{3.61}$$

Now the next step to reach our proper expression is to separate between the terms whose β is to the power of odds and even numbers because our objective is express the above equation as a polynomial, therefore we will study the nth roots of β.

In this direction, it is easy to see that the parity of $\frac{4i+3j}{2}$ is given exclusively by j, once $4i$ is even for every $i \in \mathbb{N}$. Thus, applying the Lemma 135 to the Equation 3.61, it can be expressed as two summatories

$$\sum_{j=0}^{n} \sum_{i=0}^{2n-2j} 2^{2n-2j} \left(-\varepsilon\right)^{2j} a_{i,2j} r^{4n-2(i+2j)} \beta^{4n-(2i+3j)} \Delta^i \alpha^{2j}$$
$$+\beta^{\frac{1}{2}} \sum_{j=1}^{n} \sum_{i=0}^{2n-(2j-1)} 2^{2n-(2j-1)} \left(-\varepsilon\right)^{2j-1} a_{i,(2j-1)} r^{4n-2(i+(2j-1))} \beta^{4n-2i-3j+1} \Delta^i \alpha^{(2j-1)} = 0,$$

which implies

$$\left(\sum_{j=0}^{n} \sum_{i=0}^{2n-2j} 2^{2n-2j} \left(-\varepsilon\right)^{2j} a_{i,2j} r^{4n-2(i+2j)} \beta^{4n-(2i+3j)} \Delta^i \alpha^{2j}\right)^2$$
$$= \left(-\beta^{\frac{1}{2}} \sum_{j=1}^{n} \sum_{i=0}^{2n-(2j-1)} 2^{2n-(2j-1)} \left(-\varepsilon\right)^{2j-1} a_{i,(2j-1)} r^{4n-2(i+(2j-1))} \beta^{4n-2i-3j+1} \Delta^i \alpha^{(2j-1)}\right)^2$$

Finally, we achieve the following expression

$$\left(\sum_{j=0}^{n} \sum_{i=0}^{2n-2j} 2^{2n-2j} \left(-\varepsilon\right)^{i+2j} a_{i,2j} r^{2n-(i+2j)} \beta^{4n-(2i+3j)} \Delta^i \alpha^{2j}\right)^2 \tag{3.62}$$
$$-\beta \left(\sum_{j=1}^{n} \sum_{i=0}^{2n-(2j-1)} 2^{2n-(2j-1)} \left(-\varepsilon\right)^{i+2j-1} a_{i,(2j-1)} r^{2n-(i+(2j-1))} \beta^{4n-2i-3j+1} \Delta^i \alpha^{(2j-1)}\right)^2 = 0.$$

For each $s \in J$, we define the following polynomials

$$\begin{aligned} \Delta_s(x,y) &= r^4\kappa^4x^4 - 3r^3\kappa^3x^3 + r^2\kappa\left(3\kappa(r')^2 + r\kappa'r' + 3\kappa - rr''\kappa\right)x^2 \\ &\quad + r^3r'\kappa^2\tau xy - r\left(2\kappa(r')^2 + r\kappa'r' + \kappa - 2rr''\kappa\right)x \\ &\quad - r^2r'\kappa\tau y - r\left(r(r')^2\kappa^2 + r''\right) \\ \alpha_s(x,y) &= -2r^4\kappa^3x^3 + 5r^3\kappa^2x^2 - r^2\left(3\kappa(r')^2 + r\kappa'r' + 4\kappa - rr''\kappa\right)x \\ &\quad - r^3r'\kappa\tau y + r\left((r')^2 - rr'' + 1\right) \\ \beta_s(x,y) &= \left(r^2\kappa^2\right)x^2 + (-2r\kappa)x + \left((r')^2 + 1\right) \end{aligned}$$

where the functions r, r', r'', κ, τ are evalueted in s (therefore they are scalars).

Notice that the above polynomials has the property that $\Delta_s(\cosh t, \sinh t) = \Delta(s,t)$, $\alpha_s(\sinh t, \cosh t) = \alpha(s,t)$ and $\beta_s(\sinh t, \cosh t) = \beta(s,t)$ for every $t \in \mathbb{R}$. Before to proceed, let us define the polynomial $\mathfrak{P}(x,y,z) \in \mathbb{R}[x,y,z]$, given by

$$\mathfrak{P}(x,y,z) = \left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j} 2^{2n-2j}(-\varepsilon)^{i+2j}\, a_{i,2j}r^{4n-2(i+2j)}z^{4n-(2i+3j)}x^iy^{2j}\right)^2 \tag{3.63}$$

$$-z\left(\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)} 2^{2n-(2j-1)}(-\varepsilon)^{i+2j-1}\, a_{i,(2j-1)}r^{4n-2(i+(2j-1))}z^{4n-2i-3j+1}x^iy^{(2j-1)}\right)^2.$$

Now we consider the following map $\mathfrak{P}(\Delta_s(x,y), \alpha_s(x,y), \beta_s(x,y)) \in \mathbb{R}[x,y]$. Moreover, we remark that $\mathfrak{P}(\Delta_s(x,y), \alpha_s(x,y), \beta_s(x,y))$ has degree constant equal to $16n$. In fact, applying the Lemma 140 in (3.63) we are able to express $\mathfrak{P}(x,y,z)$ as

$$\mathfrak{P}(x,y,z) = \sum_{j=0}^{n}\sum_{i=0}^{2n-2j}\sum_{k=0}^{n}\sum_{l=0}^{2n-2k} 2^{4n-2k-2j}R^{4n-2j-2k-l-i}a_{i,2j}a_{l,2k}x^{i+l}y^{2j+2k}z^{8n-3j-3k-2l-2i}$$
$$-\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)}\sum_{k=1}^{n}\sum_{l=0}^{2n-(2k-1)} 2^{4n-2k-2j+2}R^{4n-2j-2k-l-i+2}a_{i,2j-1}a_{l,2k-1}x^{i+l}y^{2j+2k-2}z^{8n-3j-3k-2l-2i+3}.$$

then, we have that the degree of $\mathfrak{P}(\Delta_s(x,y), \alpha_s(x,y), \beta_s(x,y))$ is calculated as

$$4(i+l) + 3(2j+2k) + 2(8n-3j-3k-2l-2i) = 16n$$

and also

$$4(i+l) + 3(2j+2k-2) + 2(8n-3j-3k-2l-2i+3) = 16n$$

Therefore, the Corollary 86 provides to us the existence of a polynomial $Q_s(x,y) \in \mathbb{R}[x,y]$ such that

$$\mathfrak{P}(\Delta_s(x,y), \alpha_s(x,y), \beta_s(x,y)) = \mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right) + Q_s(x,y)$$

where $\partial Q_s < 16n$.

Then, notice that

$$\mathfrak{P}\left(\Delta_s\left(\cos t, \sin t\right), \alpha_s\left(\cos t, \sin t\right), \beta_s\left(\cos t, \sin t\right)\right) \equiv 0$$

for every $t \in \mathbb{R}$. So replacing every y^2 by $1 - x^2$ in the next polynomial

$$\mathfrak{P}\left(\Delta_{s_0}\left(x, y\right), \alpha_{s_0}\left(x, y\right), \beta_{s_0}\left(x, y\right)\right) = \mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right) + Q_{s_0}\left(x, y\right)$$

we obtain a polynomial $\widetilde{\mathfrak{P}_{s_0}}\left(x, y\right)$ verifying

$$\widetilde{\mathfrak{P}_{s_0}}\left(\sinh t, \cosh t\right) = 0$$

for every t. Follows from the Lemma 98 that $\widetilde{\mathfrak{P}_{s_0}}$ must be the null polynomial. Furthermore, we remark that

$$0\left(x, y\right) = \widetilde{\mathfrak{P}_{s_0}}\left(x, y\right) = \mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right) + \widetilde{Q_s}\left(x, y\right)$$

where $\partial\widetilde{Q_s} \leq \partial Q_s < 16n$. It is important to observe that procedure of changing y^2 by $1 - x^2$ does not affect the monomial $\mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right)$ once it is a polynomial in $\mathbb{R}\left[x\right]$. Indeed, a straightfoward computation provides that

$$\begin{aligned} & \mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right) \\ = \; & 2^{4n}\kappa^{16n}\left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j}\left(-\varepsilon\right)^{2j} r^{2n} r^{10n-2j-2i} a_{i,2j}\right)^2 x^{16n} \\ & -2^{4n}\kappa^{16n}\left(r^2\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)}\left(-\varepsilon\right)^{2j-1} r^{2n} r^{10n-2j-2i} a_{i,2j-1}\right)^2 x^{16n}, \end{aligned} \tag{3.64}$$

therefore we achieve this claim.

Furthermore, let us come back to the monomial $\mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right)$ for further analysis. For practical purposes, we recall the next well-known property

$$A^2 - B^2 = \left(A - B\right)\left(A + B\right)$$

it provides that the Equation 3.64 can be expressed as

$$\mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right) = 2^{4n}\kappa^{16n}r^{4n}\mathcal{A}\left(r\right)\mathcal{B}\left(r\right)x^{16n}$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined as polynomials evalueted in r. More precisely, we define the next polynomials

$$\mathcal{A}\left(w\right) = \left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j}\left(\left(-\varepsilon\right)^{2j} w^{10n-2j-2i} a_{i,2j}\right) - w\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)}\left(\left(-\varepsilon\right)^{2j-1} w^{10n-2j-2i} a_{i,2j-1}\right)\right), \tag{3.65}$$

and

$$\mathcal{B}(w)=\left(\sum_{j=0}^{n}\sum_{i=0}^{2n-2j}\left((-\varepsilon)^{2j}w^{10n-2j-2i}a_{i,2j}\right)+w\sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)}\left((-\varepsilon)^{2j-1}w^{10n-2j-2i}a_{i,2j-1}\right)\right). \tag{3.66a}$$

Since, by hypothesis, $P(x,y)$ does not belong to the ideal in $\mathbb{R}[x,y]$ generated by $-x+y^2$ and it implies that

$$\mathcal{A}(w)\,\mathcal{B}(w)\not\equiv 0\,(w),$$

in other words, $\mathcal{A}(w)\,\mathcal{B}(w)$ is not the null polynomial.

In fact, we will examine each of the polynomials $\mathcal{A}(w)$ and $\mathcal{B}(w)$ individually. Notice that, the Lemma 135, allows us to rewrite (3.65) and (3.66a) respectively as

$$\mathcal{A}(w)=\left(\sum_{j=0}^{2n}\sum_{i=0}^{2n-j}(-1)^j(-\varepsilon)^j a_{i,j}w^{4n-j-2i}\right)$$

and

$$\mathcal{B}(w)=\left(\sum_{j=0}^{2n}\sum_{i=0}^{2n-j}(-\varepsilon)^j a_{i,j}w^{4n-j-2i}\right).$$

Thus, the Theorem 91 provides the following expression of above polynomials

$$\mathcal{A}(w)=\sum_{i=0}^{2n-1}\sum_{j=0}^{1}(-1)^j(-\varepsilon)^j\left(\sum_{k=0}^{2n-i-j}a_{i-k,j+2k}\right)w^{4n-2i-j}+a_{2n,0}$$

and

$$\mathcal{B}(w)=\sum_{i=0}^{2n-1}\sum_{j=0}^{1}(-\varepsilon)^j\left(\sum_{k=0}^{2n-i-j}a_{i-k,j+2k}\right)w^{4n-2i-j}+a_{2n,0}.$$

It is relevant to notice that in previous polynomials each power of w appears only one time. In fact,

$$\mathcal{A}(w)=\sum_{i=0}^{2n-1}\left(\sum_{k=0}^{2n-i}a_{i-k,2k}\right)w^{4n-2i}-\sum_{i=0}^{2n-1}(-\varepsilon)\left(\sum_{k=0}^{2n-i-1}a_{i-k,1+2k}\right)w^{4n-2i-1}+a_{2n,0}$$

and

$$\mathcal{B}(w)=\sum_{i=0}^{2n-1}\left(\sum_{k=0}^{2n-i}a_{i-k,2k}\right)w^{4n-2i}+\sum_{i=0}^{2n-1}(-\varepsilon)\left(\sum_{k=0}^{2n-i-1}a_{i-k,1+2k}\right)w^{4n-2i-1}+a_{2n,0},$$

therefore, the polynomials will be null if and only if

$$\left(\sum_{k=0}^{2n-i}a_{i-k,2k}\right)\equiv 0 \tag{3.67}$$

for every $(i,j) \in \Delta = \{0,\ldots,2n-1\} \times \{0,1\} \cup \{(2n,0)\}$ which is precisely de condition to apply our main Theorem 83 that gives to us that 3.57 vanishes identically if and only if $P(x,y)$ belongs to the ideal in $\mathbb{R}[x,y]$ generated by $-x+y^2$ (which is not the case).

Then we conclude that $\mathcal{A}(w)$ and $\mathcal{B}(w)$ are non null, hence

$$\mathcal{A}(w)\,\mathcal{B}(w) \not\equiv 0\,(w)\,.$$

However, we recall that

$$\mathfrak{P}\left(r^4\kappa^4x^4, -2r^4\kappa^3x^3, r^2\kappa^2x^2\right) = 2^{4n}\kappa^{16n}r^{4n}\mathcal{A}(r)\,\mathcal{B}(r)\,x^{16n} \qquad (3.68)$$

whose degree is equal to $16n$ and since $\partial\widetilde{Q_s} < 16n$ we conclude that (3.68) is, in fact, the term of highest degree. On the other hand, it must be null, so it yields

$$\mathcal{A}(r)\,\mathcal{B}(r) = 0$$

since $r \neq 0$ for every $s \in (a,b)$, in particular $r \neq 0$ for every $s \in J$, and r is a smooth function whose vanishes a polynomial, then we conclude that r must be constant, that is,

$$r(s) = r_0$$

for every $s \in J$.

If exists a $r_0 \in Rad^*(P)$, we conclude by Theorem 54 that the elements of $\mathcal{S}(P)$ are the tubular surfaces of radius r_0.

So it remains to investigate the points $s_0 \in (a,b)$ such that $\kappa(s) = 0$. If s_0 is an isolated point, we notice that there is a neighborhood J of s_0 such that for every $s \in J - \{s_0\}$ we have $\kappa(s) \neq 0$, then the same argument as before provides that we have a combination of tubular surfaces and since, in this case, the radius is constant, it implies that the surface is a tubular surface in J.

Finally, in elements $s_0 \in (a,b)$ such that $\kappa(s_0) = 0$, where we have an open neighborhood L such that $\kappa(s) = 0$ for every $s \in L$. So, if we consider S_L which is defined analagous to S_J, it follows that S_L is a rotational surface. Then we conclude that S is a smooth combination of tubular surfaces and rotational surfaces. ■

Remark 105 *Once again, we remark that the prove of the above theorem still valid for cyclic surfaces of any type, since the only difference in surfaces of type 2, 3, 4 and 5 lies on the* $\cos t, \sin t, \cosh t$ *and* $\sinh t$ *of Gaussian and mean curvatures of these surfaces and it does not affect the conclusion. Then, we conclude that the theorem is for a cyclic surface of any type.*

As discussed in the beginning of this section, one first consequence of our main theorem is that, in some sense, there is no Polynomial Weingarten cyclic surface. In other words, the cylic surfaces whose Gaussian and mean curvatures verify a polynomial relation belong, in fact, to subclasses (more specifically to tubular surfaces and rotational surfaces). Since our Theorems 54 and 61 provide a complete classification of tubular surfaces, it only remains to investigate the rotational surfaces (which is being done by important authors as in [24], [25], [1], [26], [27], among others).

Theorem 106 *Let $Q(x,y) \in \mathbb{R}[x,y]$ be a polynomial. $Rad^*(Q) = \emptyset$ if and only if the unique elements of $\mathcal{S}(Q)$ are the globally rotational surfaces.*

Proof. If we have that

$$Rad^*(Q) = \emptyset,$$

it implies that for every $r \in Rad(Q)$, the polynomial $Q(x,y)$ does not belong to the ideal in $\mathbb{R}[x,y]$ generated by $xr^2 - 2ry + 1$. Then by (the proof of) the Theorem 102 we conclude that every cyclic surface $S \in \mathcal{S}(Q)$ must be define on a straight line which concludes this direction.

Conversely, assume by the contraposition the existence of a cyclic surface $S \in \mathcal{S}(P)$ with central curve γ and radius r such that there is an element $s_0 \in (a,b)$ where the curvature $\kappa(s_0) \neq 0$. Therefore, there is a neighborhood J of s_0 where $\kappa(s) \neq 0$ for every $s \in J$. Hence we consider the surface S_J which is obtained by restricting the surface S on the interval $J \times \mathbb{R} \subset (a,b) \times \mathbb{R}$ (as in (3.49) in the Theorem 102). Then, we have that S_J is a tubular surface and the Theorem 61 provides that the polynomial Q must verify $Rad^*(Q) \neq \emptyset$. ■

A famous and intensively investigate particular relation is the linear one, that is, the relation given by $a_{1,0}x + a_{0,1}y + a_{0,0}$, where $(a_{1,0}, a_{0,1}, a_{0,0}) \neq (0,0,0)$, whose the main motivation to understand it, is because it allows us to relate the Gaussian and mean curvature through a affine function which provides several applications in various fields (for instance, in engineer and architecture).

As consequence of our Theorem 102 we classify Linear Weingarten cyclic surfaces and with a study of this particular relation we obtain an improvement in the description of the surface. In the following sense, as mentioned in the beginning we assets that the globally tubular surface and the globally rotational surfaces are the cyclic surfaces that verify the linear relation.

The precisely statement of our corollary read as follows:

Theorem 107 *Let $a_{1,0}$, $a_{0,1}$, $a_{0,0}$ be real numbers such that $(a_{1,0}, a_{0,1}) \neq (0,0)$ and define $\Delta = a_{0,1}^2 + 4a_{1,0}a_{0,0}$. Consider the polynomial*

$$Q(x,y) = a_{1,0}x + a_{0,1}y - a_{0,0}.$$

Then $\mathcal{S}(P)$ contains only globally tubular surfaces or globally rotational surfaces. More precisely:

i. *If $a_{0,1}a_{0,0} > 0$ and $\Delta = 0$, then $\mathcal{S}(Q)$ contains all tubular surfaces of radius $\frac{a_{0,1}}{2a_{0,0}}$;*

ii. *Otherwise we have that $\mathcal{S}(Q)$ contains rotational surfaces.*

Proof. For the particular polynomial

$$P(x,y) = a_{1,0}x + a_{0,1}y - a_{0,0} = 0 \tag{3.69}$$

we may consider two cases in order to fully understand the above relation, named $a_{0,0} \neq 0$ and $a_{0,0} = 0$.

If $a_{0,0} \neq 0$, we assume without loss of generality that $a_{0,0} = 1$, therefore the relation becomes $P(x,y) = a_{1,0}x + a_{0,1}y - 1$. Hence, the Theorem 52 provides that a tubular surface verify $P(K,H) \equiv 0$ if and only if P is a tubular polynomial. Thus, we must have $Rad(P) \neq \emptyset$, more precisely it is obtained

$$Rad(P) = \left\{ \frac{a_{0,1}}{2} \right\}$$

as consequence, we conclude that $a_{0,1} > 0$.

We remark that the combination of cylinder with rotational surface is simply a rotational surface. Then, assume the existence of tubular surfaces besides the right cylinder in $\mathcal{S}(Q)$. By our Theorem 61 it implies that belongs to the ideal in $\mathbb{R}[x,y]$ generated by $x\left(\frac{a_{0,1}}{2}\right)^2 - 2\left(\frac{a_{0,1}}{2}\right)y + 1$. Thus, the previous condition furnishes that the coefficients of $P(x,y)$ must satisfy

$$a_{1,0} = -\frac{a_{0,1}^2}{4} \qquad \text{and} \qquad a_{0,1} > 0 \tag{3.70}$$

Let $c \in (a,b)$ and suppose the existence of a cyclic surface S with central curve γ defined in (a,b) and radius r. Also suppose the following

1. In $[a,c]$ the surface S is a rotational surface;

2. In $[c,b]$ the surface S is a tubular surface of radius $\frac{a_{0,1}}{2}$ different of a cylinder.

For rotational surface the associated differential equation to the linear case is expressed as

$$a_{1,0}\left(\frac{-r''}{r\left(1+(r')^2\right)^2}\right) + a_{0,1}\left(\frac{(r')^2 - rr'' + 1}{2r\left((r')^2+1\right)^{\frac{3}{2}}}\right) - 1 = 0,$$

consequently, we have

$$2r\left(1+(r')^2\right)^2 = a_{0,1}\left(1+(r')^2\right)^{\frac{3}{2}} - \left(2a_{1,0} + ra_{0,1}\left(1+(r')^2\right)^{\frac{1}{2}}\right)r''. \tag{3.71}$$

CLAIM 1. Exists a number ε with $0 < \varepsilon < c - a$ such that

$$2a_{1,0} + ra_{0,1}\left(1+(r')^2\right)^{\frac{1}{2}} \equiv 0 \tag{3.72}$$

for every $s \in (c-\varepsilon, c)$.

Indeed, suppose that claim is false, therefore it implies the existence of a sequence $(s_n) \in (a, c)$ such that $s_n \to c$ and

$$\left(2a_{1,0} + ra_{0,1}\left(1 + (r')^2\right)^{\frac{1}{2}}\right)(s_n) \neq 0$$

for every $n \in \mathbb{N}$. So, for every s_n, by (3.70) and (3.71) we may write

$$r'' = -\frac{2\left((r')^2 + 1\right)^{\frac{3}{2}}}{a_{0,1}} < 0.$$

hence

$$\lim_{n\to\infty} r''(s_n) = \lim_{s\to b} -\frac{2\left(1 + (r'(s_n))^2\right)^{\frac{3}{2}}}{a_{0,1}} = -\frac{2}{a_{0,1}} \neq 0.$$

Then, we have $s_n \to c$ but $r''(s_n) \not\to r''(c) = 0$ which is an absurd. Hence we conclude the proof of our first claim.

Let us consider the number ε given by the above statement. By the remark that $a_{1,0} = -\frac{a_{0,1}^2}{4}$, we have that Equation 3.72 becomes

$$\frac{1}{2}a_{0,1}\left(2r\sqrt{(r')^2 + 1} - a_{0,1}\right) = 0$$

for all $s \in (c - \varepsilon, c)$, which implies that

$$\left(2r\sqrt{(r')^2 + 1} - a_{0,1}\right) \equiv 0$$

since $a_{0,1} > 0$ (by (3.70)), hence

$$\sqrt{(r')^2 + 1} = \frac{a_{0,1}}{2r}$$

therefore

$$(r')^2 = \frac{a_{0,1}^2 - 4r^2}{4r^2} = \frac{a_{0,1}^2}{4r^2} - 1$$

for every $s \in (c - \varepsilon, c)$. The derivative of above equality gives

$$2r'r'' = -8a_{0,1}^2 rr' \tag{3.73}$$

CLAIM 2. Exists a number δ with $0 < \delta < \varepsilon$ such that $r' \equiv 0$ in $[c - \delta, c]$.

We observe that (3.73) is rewritten as

$$2\left(4ra_{0,1}^2 + r''\right)r' = 0$$

and notice that

$$\left(4ra_{0,1}^2 + r''\right)(c) = 2a_{0,1}^3 > 0$$

so by continuity there is δ with $0 < \delta < \varepsilon$ where $\left(4ra_{0,1}^2 + r''\right) \neq 0$ for every $[c - \delta, c)$. Hence r' must be null in $[c - \delta, c)$. Since $r'(c) = 0$ the statement holds.

By the above claims we conclude that S is a tubular surface in $[c-\delta, c)$.

We consider the following set

$$\mathcal{L} = \{s \in (a, c) \ ; \ S \text{ is a tubular surface in } [s, b)\}$$

and we define $\lambda = \inf \mathcal{L}$. By construction, we already have that $\lambda \leq c$ and by previous analysis we have $\lambda \leq c - \delta$.

Note that $\lambda \geq a$, then we assume by absurd that $\lambda > a$. Then, in this case, we have that S is a tubular surface in $[\lambda, b)$. Applying the previous argument as before, it is obtained the existence of $\delta' > 0$ such that S is a tubular surface in $\left[\lambda - \delta', b\right)$, which is an absurd with minimality of λ. So we conclude that

$$a = \inf \mathcal{L}$$

therefore S is a tubular surface in (a, b).

In the case that $a_{0,0} = 0$, the polynomial in (3.69) is expressed as

$$P(x, y) = a_{1,0}x + a_{0,1}y,$$

so by the Definition 45, it follows that P is a tubular polynomial if and only if $a_{0,1} = 0$. Moreover, the Theorem 52 provides that exists a tubular surface if and only if the polynomial is tubular. So it is necessary study only the case

$$P(x, y) = a_{1,0}x$$

which lies on the classification of developable surfaces, since we have $a_{1,0}K \equiv 0$. Hence we conclude that is a cylinder (a particular case of tubular surface) or part of a cone. ■

Several interesting and relevant results can be obtained when we add hypothesis on the radius function or in the curvature of central curve. In order to present particular results from our main theorem, let us discourse about gluing points (presents in surfaces obtained by the combination of others surfaces).

We recall that the definition of regular cylcic surfaces requires the existence of an interval $\Sigma \subset \mathbb{N}$ (possibly infinite) and the existence of a strictly increasing sequence $(\xi_n)_{n\in\Sigma}$ in (a, b) of isolated points such that $\kappa(\xi_n) = 0$ for every $n \in \Sigma$. Moreover, for each $n \in \Sigma$ we consider the following set $\Sigma_n = \left\{t \in \left(\xi_n, \xi_{n+1}\right) \ ; \ \kappa(t) = 0\right\}$ which has the following property:

$$\operatorname{int}\Sigma_n = \emptyset \qquad \text{or} \qquad \operatorname{int}\Sigma_n = \left(\xi_n, \xi_{n+1}\right).$$

Then, we point out that for the regular cyclic surface which are obtained by the (smooth) combination of tubular surface and rotational surface, the gluing points are necessarily elements of $(\xi_n)_{n\in\Sigma}$. Furthermore, for a gluing point ξ_i (for some $i \in \Sigma$), the sets Σ_i and Σ_{i+1} verify

$$\operatorname{int}\Sigma_i = \emptyset \qquad \text{and} \qquad \operatorname{int}\Sigma_{i+1} = \left(\xi_i, \xi_{i+1}\right).$$

So our corollaries read as follows:

Corollary 108 *Given a polynomial* $Q(x,y) \in \mathbb{R}[x,y]$*, consider* $\mathcal{S}_r(Q)$ *the set of all regular cyclic surfaces in Euclidean (respect. Lorentzian) 3-space around a central curve* γ *with analytic radius* $r > 0$ *whose Gaussian and mean curvatures* K*,* H *verify* $Q(K,H) \equiv 0$*. Then, the elements of* $\mathcal{S}(Q)$ *are:*

i. *Globally Tubular surfaces of radius* $r \in Rad^*(Q)$*;*

ii. *Globally rotational surface.*

Proof. By the Theorem 102 it is only necessary to investigate the item i. More precisely, we will show that there is globally tubular surface and globally rotational surfaces as element of $\mathcal{S}(Q)$. Then, assume by absurd the existence of a surface $S \in \mathcal{S}(Q)$ such that S is a combination of tubular surface and rotational surfaces.

So we consider the gluing point $\xi \in (a,b)$, that is, the point where the tubular surface and the rotational surface are glued. Without loss of generality, assume that (a,ξ) the surface is tubular and in (ξ,b) the surface is rotational.

Notice that in ξ the Gaussian and mean curvatures of tubular and of rotational surfaces agrees and since r is a analytic function in (a,b) there is an open neighborhood $V_\xi = (\xi-\delta, \xi+\delta)$, for some $\delta > 0$, of ξ such that

$$r(\xi+s) = \sum_{n=0}^{\infty} \frac{r^{(n)}(\xi)}{n!}(s)^n$$

for every $s \in V_\xi$. On the other hand, r is constant (once is the radius of the tubular surface), then the above equality is rewritten as

$$r(\xi+s) = r(\xi) \equiv r_0$$

for every $s \in V_\xi$ which is an absurd, since ξ is the gluing point. In other words, the radius r is constant in (ξ,b), which implies that the surface is tubular in this interval, which is an absurd since we assumed that the surface is rotational in (ξ,b). Then we conclude the desired. ■

A particular class of polynomials that attracts much interest and is very studied is the class of irreducible polynomials. In the same direction of previous corollaries, we have an improviment of the classification of the set $\mathcal{S}(Q)$ when $Q(x,y) \in \mathbb{R}[x,y]$ is an irreducible polynomial.

Corollary 109 *Given an irreducible polynomial* $Q(x,y) \in \mathbb{R}[x,y]$ *and let* $\mathcal{S}(Q)$ *be the set of all regular cyclic surfaces in Euclidean (respect. Lorentzian) 3-space whose Gaussian and mean curvatures* K*,* H *verify* $Q(K,H) \equiv 0$*. Then, the elements of* $\mathcal{S}(Q)$ *are globally rotational surfaces or globally tubular surfaces of radius* $r \in Rad^*(Q)$*.*

Corollary 110 *Given an irreducible polynomial* $Q(x,y) \in \mathbb{R}[x,y]$ *with* $\partial Q \geq 2$ *and let* $\mathcal{S}(Q)$ *be the set of all regular cyclic surfaces in Euclidean (respect. Lorentzian) 3-space whose Gaussian and mean curvatures* K*,* H *verify* $Q(K,H) \equiv 0$*. Then, the elements of* $\mathcal{S}(Q)$ *are globally rotational surfaces.*

Another consequence of our main theorem can be obtained if we assume that the curvature of the central curve never vanishes:

Corollary 111 *Given a polynomial $Q(x,y) \in \mathbb{R}[x,y]$, consider $\mathcal{S}_\kappa(Q)$ the set of all regular cyclic surfaces in Euclidean (respect. Lorentzian) 3-space around a central curve γ such that $\kappa \neq 0$ everywhere and with radius $r > 0$ whose Gaussian and mean curvatures K, H verify $Q(K,H) \equiv 0$. Then, the elements of $\mathcal{S}_\kappa(Q)$ are globally Tubular surfaces of radius $r \in Rad^*(Q)$.*

Besides the well-knonw CMC (Constant mean curvature) and CGC (constant Gaussian curvature) surfaces, another important class of surfaces is the CCC (Constant Casorati curvature) surfaces that relates the principal curvatures as follows

$$\frac{k_1^2 + k_2^2}{2} \equiv c$$

where $c \in \mathbb{R}$. The above relation can be obtained in terms of the Gaussian and mean curvatures by the polynomial

$$-2K + 4H^2 \equiv 2c.$$

For CCC Cyclic surfaces we have the next result:

Corollary 112 *The cylinders in $\mathbb{E}^3$ are the unique complete regular cyclic Weingarten surfaces with second fundamental form of constant length (or constant Casorati curvature).*

Proof. By the Theorem 102 and Theorem 106 it follows that the cyclic surface must be a rotational surface and by Corollary 1.2 in [1] we achieve the desired. ■

Chapter 4

Weingarten Canal Surfaces

In this chapter we will classify Polynomial Weingarten cyclic surfaces. Therefore, to achieve our goal, we first study the parametrization of canal surface in $\mathbb{E}^3$ and $\mathbb{L}^3$, which is presented in Section 4.1. For canal surfaces in the Euclidean and in the Lorentzian 3-space, we present the principal curvatures whenever they exist.

Finally, in the Section 4.2 we present a technical results that provides a summatory identity that plays an important role in the classification of Polynomial Weingarten canal surfaces. Proceeding in this section, we display our main theorem that fully characterizes Polynomial Weingarten canal surfaces among the Gaussian and mean curvature and another theorem that characterizes Polynomial Weingarten canal surfaces among its principal curvatures. *In suma,* our main theorem provides that a canal surface that verify a polynomial relation is, in fact, a (smooth) combination of tubular surface and rotational surface.

4.1 Canal Surfaces

This section was elaborated to introduce canal surface in the Euclidean and in the Lorentzian 3-space. In the section's first part, we discuss canal surface where we present the parametrization and also exhibit the Gaussian and mean curvatures.

Proceeding to the second part, we will study canal surface as the envelope of Lorentzian spheres or Lorentzina hyperboles, so we obtain the parametrization for each type of canal surface in $\mathbb{L}^3$. Finally, we presented the Gaussian and mean curvatures of the canal surface.

Furthermore, we will present the principal curvatures of canal surface, for those the principal curvatures exists, since in the Lorentzian 3-space this is not always ensured. For the surface those the principal curvature does not exist, we will present a pair of functions whose behavior is similar enough to the principal curvatures. Here it is important to mention that pair of function, despite not being the principal curvatures, they permit us to classify Polynomial Weingarten canal surfaces (see Section 4.2 for more details).

4.1.1 Euclidean Canal Surfaces

Given a smooth curve $\gamma(s) \subset \mathbb{R}^3$ such that for each $\gamma(s)$ we define a circle of radius $r > 0$ (constant), then in the process of to unite every circle, it is obtained the tubular surfaces. If it is permited that radius $r > 0$ be a smooth function that never vanishes, the same procedure gives the cyclic surfaces. The final next natural generalization is in the case that we consider spheres insted of circles.

In this direction, we obtain the canal surface which is formed by the movement of spheres of variable or constant radius. More precisely, we have:

In Euclidean 3-space, canal surface is defined as the envelope of a 1-parameter family of spheres. The central curve is the name given for the curve whose centers of each sphere lies on. The radius function $r : (a, b) \longrightarrow \mathbb{R}$ such that $r(s) > 0$ for every $s \in (a, b)$ verify the condition that $r(s)$ is the radius of sphere $S^2(s)$.

In case of central curve (denoted by) $\gamma : (a, b) \longrightarrow \mathbb{R}^3$ is parametrized by arc lenght and biregular ($\|\gamma\| = 1$, $\gamma'' \neq 0$) the parametrization of canal surface is expressed by

$$\psi(s,t) = \gamma(s) + r(s)\left(-r'(s)T(s) + \sqrt{1-(r')^2}\cos tN + \sqrt{1-(r')^2}\sin tB\right) \tag{4.1}$$

where $\{T, N, B\}$ is the Frenet frame of γ. When γ is a straigh line $\{T, N, B\}$ can be regarded as constant vectors.

To avoid undesirable cases, we will include in the definition an additional condition for the regularity of curvature of central curve. We will say that a cyclic surface is regular if there is an interval $\Sigma \subset \mathbb{N}$ (possibly infinite) and there is a strictly increasing sequence $(\xi_n)_{n\in\Sigma}$ in (a, b) of isolated points such that

i. The curvature κ of γ verify $\kappa(\xi_n) = 0$ for every $n \in \Sigma$

ii. For every $n \in \Sigma$, consider the set

$$\Sigma_n = \left\{t \in \left(\xi_n, \xi_{n+1}\right)\ ;\ \kappa(t) = 0\right\}$$

then

$$\operatorname{int}\Sigma_n = \emptyset \qquad \text{or} \qquad \operatorname{int}\Sigma_n = \left(\xi_n, \xi_{n+1}\right).$$

For practical purposes, we notice that an inherent conditions over the derivative of radius functions emerge from the parametrization (4.1) which is

$$\sqrt{1-(r'(s))^2} \geq 0$$

for every $s \in (a, b)$. Then, we may describe the function $r' : (a, b) \longrightarrow \mathbb{R}$ by an another smooth function $\phi(s)$ that verifies

$$-r'(s) = \cos\phi(s).$$

where we have that $\phi(s) \in [0, \pi)$. Therefore, the parametrization (4.1) becomes

$$\psi(s,t) = \gamma(s) + r(s)\left(-r'(s)T(s) + \sin\phi\cos tN + \sin\phi\sin tB\right) \tag{4.2}$$

Proposition 113 *The Gaussian and mean curvatures of a canal surface in the Euclidean 3-space are given by*

$$\begin{aligned} K &= \frac{r'' + k\sin\phi\cos t}{r^2\left(r'' + k\sin\phi\cos t\right) - r\sin^2\phi}, \\ H &= \frac{1}{2}\frac{2rr'' + 2kr\cos t\sin\phi - \sin^2 y}{r\left(rr'' - \sin^2\phi + kr\cos t\sin\phi\right)} \end{aligned}$$

Hence the principal curvatures are respectively:

$$k_1 = \frac{r'' + k\sin\phi\cos t}{r\left(r'' + k\sin\phi\cos t\right) - \sin^2\phi} \qquad \text{and} \qquad k_2 = \frac{1}{r}.$$

Proof. Using (1.1) and (4.2) we obtain the coefficients of the first fundamental form

$$\begin{aligned} E &= r\left(-4(r')^2\phi' + r\kappa^2\sin\phi\right)\sin\phi\cos^2 t + 4r^2r'\tau\phi'\sin\phi\cos t\sin t \\ &\quad +2r\kappa\left(-\sin^3\phi - (r')^2\sin\phi + rr''\sin\phi + r(r')^2\phi'\right)\cos t \\ &\quad +\left(2r^2r'\kappa\tau\sin\phi\right)\sin t \\ &\quad +\left(\left(r'\sin\phi + rr'\phi'\right)^2 + \left(rr'' - \sin^2\phi\right)^2 + r^2(r')^2\kappa^2 + r^2\tau^2\sin^2\phi\right) \end{aligned}$$

$$f = \tau r^2\sin^2\phi + \kappa r^2 r'\sin\phi\sin t \qquad \text{and} \qquad G = r^2\sin^2\phi.$$

And the coefficients of the second fundamental form are

$$\begin{aligned} e &= -r\kappa^2\sin^2\phi\cos^2 t + \left(\kappa\sin\phi - 2r\kappa\phi'\right)\cos t \\ &\quad -2rr'\kappa\tau\sin\phi\sin t - r\left(r'\right)^2\kappa^2\sin^2\phi - r\left(\phi'\right)^2 + r'' \end{aligned}$$

$$f = -r\tau\sin^2\phi - rr'\kappa\sin\phi\sin t \qquad \text{and} \qquad g = -r\sin^2\phi$$

By the Formula 1.5 and by the above coefficients, a straightforward calculus provide the desired. ■

4.1.2 Lorentzian Canal Surfaces

Motivated by the (abstract) definition of Canal surface in $\mathbb{E}^3$, the natural generalization of Canal surface in Lorentzian 3-space is to permit the envelope of Lorentzian spheres or Lorentzian hyperboles.

So, let $\gamma : (a,b) \longrightarrow \mathbb{L}^3$ a smooth curve and consider the abstract parametrization of a canal surface as the envelope of Lorentzian spheres or Lorentzian hyperboles (*i.e.* set obtained by moving a Lorentzian spheres or Lorentzian hyperboles, respectively, along a central curve)

$$\psi(s,t) = \gamma(s) + f(s,t)T(s) + g(s,t)N(s) + h(s,t)B(s)$$

where $\{T(s), N(s), B(s)\}$ is the Frenet frame of γ and $f(s,t)$, $g(s,t)$ and $h(s,t)$ are smooth functions defined in (a,b) whose the explicitly expression will be given later on. First, we would like to add one hypothesis to our curve.

To avoid undesirable cases, we will include in the definition an additional condition for the regularity of curvature of central curve. We will say that a cyclic surface is regular if there is an interval $\Sigma \subset \mathbb{N}$ (possibly infinite) and there is a strictly increasing sequence $(\xi_n)_{n\in\Sigma}$ in (a,b) of isolated points such that

i. The curvature κ of γ verify $\kappa(\xi_n) = 0$ for every $n \in \Sigma$

ii. For every $n \in \Sigma$, consider the set

$$\Sigma_n = \left\{t \in \left(\xi_n, \xi_{n+1}\right) \ ; \ \kappa(t) = 0\right\}$$

then

$$\operatorname{int}\Sigma_n = \emptyset \qquad \text{or} \qquad \operatorname{int}\Sigma_n = \left(\xi_n, \xi_{n+1}\right).$$

Now, let us comeback to explicity writing of the functions $f(s,t)$, $g(s,t)$ and $h(s,t)$. In order to do that, first notice that for each $s_0 \in (a,b)$ we want to describe a Lorentzian spheres ($\varepsilon = 1$) or Lorentzina hyperbole ($\varepsilon = -1$) of radius εr^2,

$$\varepsilon_T f^2 + \varepsilon_N g^2 + \varepsilon_B h^2 = g_{\mathbb{L}}\left(\psi(s_{0,}t) - \gamma(s_0), \psi(s_{0,}t) - \gamma(s_0)\right) = \varepsilon r^2 \qquad (4.3)$$

where $\varepsilon \in \{-1, 1\}$ fixed constant. The derivative in parameter s of these previous paramatrization and equation gives us

$$\varepsilon_T f f_s + \varepsilon_N g g_s + \varepsilon_B h h_s = \varepsilon r r' \qquad (4.4)$$

and

$$\psi_s = (f_s - g\kappa\varepsilon_N\varepsilon_T + 1)\,T + (g_s + f\kappa - h\tau\varepsilon_B\varepsilon_N)\,N + (h_s + g\tau)\,B \qquad (4.5)$$

Then, observe that $\psi - \gamma$ is a normal vector to the Canal surface, thus we have

$$0 = g_{\mathbb{L}}(\psi - \gamma, \psi_s) = \varepsilon_T f + \varepsilon r r', \qquad (4.6)$$

which yields

$$f = -\varepsilon_T \varepsilon r r'$$

Applying the condition (4.6) in Equation (4.4) it is obtained the following system

$$\varepsilon_N g^2 + \varepsilon_B h^2 = \varepsilon r^2 - \varepsilon_T (r r')^2, \qquad (4.7)$$

whose solutions are given by chosening the causality of central curve (ε_T), of principal normal (ε_N) and the normal section (ε). Hence, we have the below cases to consider:

In the case that we foliate the surface by Lorentzian circles, it yields that $\varepsilon = 1$, so we must consider the next options:

If we have that $\gamma' = T$ is spacelike, $\varepsilon_T = 1$, it yields that $\varepsilon_N\varepsilon_B = -1$, so we have the following cases to study:

1. If $\varepsilon_N = -1$, then (4.7) is rewritten as

$$-g^2 + h^2 = r^2 - (rr')^2$$

then,

$$g = r\sqrt{1-(r')^2}\sinh t \qquad \text{and} \qquad h = \sqrt{1-(r')^2}\cosh t$$

2. If $\varepsilon_N = 1$, then (4.7) is rewritten as

$$g^2 - h^2 = r^2 - (rr')^2$$

then,

$$g = r\sqrt{1-(r')^2}\cosh t \qquad \text{and} \qquad h = \sqrt{1-(r')^2}\sinh t$$

If we have that $\gamma' = T$ is timelike, $\varepsilon_T = -1$, it implies that $\varepsilon_N = 1 = \varepsilon_B$, so we have that:

3. In this case (4.7) is rewritten as

$$g^2 + h^2 = r^2 + (rr')^2$$

then,

$$g = r\sqrt{1+(r')^2}\cos t \qquad \text{and} \qquad h = \sqrt{1+(r')^2}\sin t$$

In the case that we foliate the surface by Lorentzian hyperboles, it yields that $\varepsilon = -1$, so we must consider the next options:

If we have that $\gamma' = T$ is spacelike, $\varepsilon_T = 1$, it yields that $\varepsilon_N\varepsilon_B = -1$, so we have the following cases to study:

4. If $\varepsilon_N = -1$, then (4.7) is rewritten as

$$-g^2 + h^2 = -r^2 + (rr')^2$$

then,

$$g = r\sqrt{1+(r')^2}\cosh t \qquad \text{and} \qquad h = \sqrt{1+(r')^2}\sinh t$$

5. If $\varepsilon_N = 1$, then (4.7) is rewritten as

$$g^2 - h^2 = -r^2 + (rr')^2$$

then,

$$g = r\sqrt{1+(r')^2}\sinh t \qquad \text{and} \qquad h = r\sqrt{1+(r')^2}\cosh t$$

If we have that $\gamma' = T$ is timelike, $\varepsilon_T = -1$, it implies that $\varepsilon_N = 1 = \varepsilon_B$, so we have that:

6. In this case (4.7) is rewritten as

$$g^2 + h^2 = -r^2 + (rr')^2,$$

which follows the non existence of solutions in this case.

Then, we conclude the existence of five distinct types of Canal surface in $\mathbb{L}^3$.

In the following results, more specifically, in the Propositions 114, 115, 116, 117 and 119 where the Gaussian and mean curvatures of each type of canal surface are exhibited, the detailed calculus will be omited, once this computations can be found in the works [19] and [20] where the authors presented a profund and precise study of the Gaussian and mean curvatures of each type of canal surfaces in Lorentzian 3-space.

Canal surface of type 1. The parametrization is

$$\psi(s,t) = \gamma(s) + r(s)\left(-r'(s)T(s) + \sqrt{1-(r')^2}\sinh tN + \sqrt{1-(r')^2}\cosh tB\right)$$

hence, once we consider the smooth function $\phi(s)$ such that

$$-r'(s) = \cos\phi$$

the above parametrization is rewritten as

$$\psi(s,t) = \gamma(s) + r(s)(-r'(s)T(s) + \sin\phi\sinh tN + \sin\phi\cosh tB)$$

Proposition 114 *The Gaussian and mean curvatures of a canal surface in Lorentzian 3-space of type 1 are given by*

$$\begin{aligned} K &= \frac{r'' - \kappa\sin\phi\sinh t}{r\left(rr'' - r\kappa\sin\phi\sinh t - \sin^2\phi\right)}, \\ H &= \frac{2rr'' - 2r\kappa\sinh t\sin\phi - \sin^2\phi}{2r\left(rr'' - r\kappa\sin\phi\sinh t - \sin^2\phi\right)} \end{aligned}$$

Proof. See in [19] and [20]. ■

Based on the previous proposition, we noticed that we could express the curvatures as product and average of the following functions (principal curvatures)

$$k_1 = \frac{r'' - \kappa\sin\phi\sinh t}{r(r'' - \kappa\sin\phi\sinh t) - \sin^2\phi} \qquad \text{and} \qquad k_2 = \frac{1}{r} \tag{4.8}$$

and, therefore, they verify the condition (1.7).

Canal surface of type 2. The parametrization is

$$\psi(s,t)=\gamma(s)+r(s)\left(-r'(s)T(s)+\sqrt{1-(r')^2}\cosh tN+\sqrt{1-(r')^2}\sinh tB\right)$$

hence, once we consider the smooth function $\phi(s)$ such that

$$-r'(s)=\cos\phi$$

the above parametrization is rewritten as

$$\psi(s,t)=\gamma(s)+r(s)\left(-r'(s)T(s)+\sin\phi\cosh tN+\sin\phi\sinh tB\right)$$

Proposition 115 *The Gaussian and mean curvatures of a canal surface in Lorentzian* 3*-space of type 2 are given by*

$$\begin{aligned} K &= \frac{r''+\kappa\sin\phi\cosh t}{r\left(rr''+r\kappa\sin\phi\cosh t-\sin^2\phi\right)}, \\ H &= \frac{2rr''-\sin^2\phi+2r\kappa\sin\phi\cosh t}{2r\left(rr''-\sin^2\phi+r\kappa\sin\phi\cosh t\right)} \end{aligned}$$

Proof. See in [19] and [20]. ■

Based on the previous proposition, we noticed that we could express the curvatures as product and average of the following functions (principal curvatures)

$$k_1=\frac{r''+\kappa\sin\phi\cosh t}{r\left(r''+\kappa\sin\phi\cosh t\right)-\sin^2\phi}\qquad\text{and}\qquad k_2=\frac{1}{r}$$

and, therefore, they verify the condition (1.7).

Canal surface of type 3. The parametrization is

$$\psi(s,t)=\gamma(s)+r(s)\left(-r'(s)T(s)+\sqrt{1+r_s^2}\cos tN+\sqrt{1+r_s^2}\sin tB\right)$$

hence, once we consider the smooth function $\phi(s)$ such that

$$r'(s)=\tan\phi$$

the above parametrization is rewritten as

$$\psi(s,t)=\gamma(s)+r(s)\left(-r'(s)T(s)r+\sec\phi\cos tN+\sec\phi\sin tB\right)$$

Proposition 116 *The Gaussian and mean curvatures of a canal surface in Lorentzian* 3*-space of type 3 are given by*

$$\begin{aligned} K &= \frac{r''+\kappa\sec\phi\cos t}{r\left(rr''+r\kappa\sec\phi\cos t+\sec^2\phi\right)}, \\ H &= \frac{2rr''+2r\kappa\sec\phi\cos t+\sec^2\phi}{r\left(rr''+r\kappa\sec\phi\cos t+\sec^2\phi\right)} \end{aligned}$$

Proof. See in [19] and [20]. ■

Based on the previous proposition, we noticed that we could express the curvatures as product and average of the following functions (principal curvatures)

$$k_1 = \frac{r'' + \kappa \sec\phi \cos t}{r\left(r'' + \kappa \sec\phi \cos t\right) + \sec^2\phi} \qquad \text{and} \qquad k_2 = \frac{1}{r}$$

and, therefore, they verify the condition (1.7).

Canal surface of type 4. The parametrization is

$$\psi(s,t) = \gamma(s) + r(s)\left(r'(s)\,T(s) + \sqrt{1+r_s^2}\cosh tN + \sqrt{1+r_s^2}\sinh tB\right)$$

hence, once we consider the smooth function $\phi(s)$ such that

$$r'(s) = \sinh\phi$$

the above parametrization is rewritten as

$$\psi(s,t) = \gamma(s) + r(s)\left(r'(s)\,T(s) + \sinh\phi\cosh tN + \sinh\phi\sinh tB\right)$$

Proposition 117 *The Gaussian and mean curvatures of a canal surface in Lorentzian* 3*-space of type 4 are given by*

$$\begin{aligned} K &= -\frac{\left(r'' + \kappa\cosh\phi\cosh t\right)}{r\left(rr'' + r\kappa\cosh\phi\cosh t + \cosh^2\phi\right)}, \\ H &= \frac{2rr'' + 2r\kappa\cosh\phi\cosh t + \cosh^2\phi}{2r\left(rr'' + r\kappa\cosh\phi\cosh t + \cosh^2\phi\right)} \end{aligned}$$

Proof. See in [19] and [20]. ■

Based on the previous proposition, we noticed that we could express the curvatures as the following functions

$$\widetilde{k_1} = \frac{\left(r'' + \kappa\cosh\phi\cosh t\right)}{r\left(r'' + \kappa\cosh\phi\cosh t\right) + \cosh^2\phi} \qquad \text{and} \qquad \widetilde{k_2} = \frac{1}{r}$$

whose verifies the condition

$$\widetilde{k_1}\widetilde{k_2} = -K \qquad \text{and} \qquad \frac{\widetilde{k_1} + \widetilde{k_2}}{2} = H.$$

Remark 118 *It is clear that functions are not the principal curvatures, however,for our purposes (see Section 4.2 for more details), we just need two functions that behaves closes enough as principal curvatures, i.e. functions that verify the above relation except by a sign.*

Canal surface of type 5. The parametrization is

$$\psi(s,t) = \gamma(s) + r(s)\left(r'(s)T(s) + \sqrt{1+r_s^2}\sinh tN + \sqrt{1+r_s^2}\cosh tB\right)$$

hence, once we consider the smooth function $\phi(s)$ such that

$$r'(s) = \sinh\phi$$

the above parametrization is rewritten as

$$\psi(s,t) = \gamma(s) + r(s)(r'(s)T(s) + \cosh\phi\sinh tN + \cosh\phi\cosh tB)$$

Proposition 119 *The Gaussian and mean curvatures of a canal surface in Lorentzian* 3*-space of type 5 are given by*

$$\begin{aligned} K &= -\frac{r'' - \kappa\cosh\phi\sinh t}{r\left(rr'' - r\kappa\cosh\phi\sinh t + \cosh^2\phi\right)}, \\ H &= \frac{2rr'' - 2r\kappa\cosh\phi\sinh t + \cosh^2\phi}{2r\left(rr'' - r\kappa\cosh\phi\sinh t + \cosh^2\phi\right)} \end{aligned}$$

Proof. See in [19] and [20]. ■

Based on the previous proposition, we noticed that we could express the curvatures as the following functions

$$\widetilde{k_1} = \frac{r'' - \kappa\cosh\phi\sinh t}{r(r'' - \kappa\cosh\phi\sinh t) + \cosh^2\phi} \qquad \text{and} \qquad \widetilde{k_2} = \frac{1}{r}$$

whose verifies the condition

$$\widetilde{k_1}\widetilde{k_2} = -K \qquad \text{and} \qquad \frac{\widetilde{k_1} + \widetilde{k_2}}{2} = H$$

Remark 120 *It is clear that functions are not the principal curvatures, however,for our purposes (see Section 4.2 for more details), we just need two functions that behaves closes enough as principal curvatures, i.e. functions that verify the above relation except by a sign.*

4.2 Main result and applications for Canal Surfaces

In this section we present our main theorems that fully classify Polynomial Weingarten canal surfaces in the Euclidean and in the Lorentzian 3-space.

Discovered and named by Monge in 1850, Canal surfaces (originaly created in the Euclidean 3-space) are defined by the envelope of spheres whose center belongs to a smooth central curve (see Sections 4.1.1 & 4.1.2 for details). This is a very interesting family of surfaces once they contains the class of tubular surfaces and have a large intersection with other important families like rotational surfaces and cyclic surfaces just to name a few.

In applied context, Canal surfaces has an important part since they have several utilities, as in the CAGD (computer aided geometric design), where they are used to model infrastructures, buildings and biological structures.

Inspired by the importance and wide class of applications of the Canal surface, we start to investigate those surfaces whose curvatures are roots of a polynomial relation and, in the end, we fully classificates canal surfaces such that $P(K,H) \equiv 0$.

But, before to present our results, let us to illustrate the general technique contained in our theorems, by presenting a particular case of a polynomial of degree 2. It is relevant to mention that precisely particular case is what permits us to achieve the complete classification of Polynomial Weingarten canal surface.

Let $P(x,y) \in \mathbb{R}[x,y]$ be a polynomial given by

$$P(x,y) = a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0}$$

then, we express the previous polynomial as

$$P(x,y) = x^2 A_2(y) + xA_1(y) + A_0(y)$$

where, each $A_i(y) \in \mathbb{R}[y]$ whose explicity expression is given by

$$\begin{aligned} A_2(y) &= a_{2,0}; \\ A_1(y) &= a_{1,1}y + a_{1,0}; \\ A_0(y) &= a_{0,2}y^2 + a_{0,1}y + a_{0,0}. \end{aligned}$$

Follows from the assumption that the principal curvatures of a canal surface vanishes $P(x,y)$ that

$$P(k_1,k_2) \equiv 0$$

and since

$$k_1 = \frac{r'' + k\sin\phi\cos t}{r(r'' + k\sin\phi\cos t) - \sin^2 y} \qquad \text{and} \qquad k_2 = \frac{1}{r},$$

we obtain the next differential equation:

$$\left(\frac{r'' + k\sin y\cos t}{r(r'' + k\sin y\cos t) - \sin^2 y}\right)^2 A_2\left(\frac{1}{r}\right) + \left(\frac{r'' + k\sin y\cos t}{r(r'' + k\sin y\cos t) - \sin^2 y}\right) A_1\left(\frac{1}{r}\right) + A_0(k_2) \equiv 0. \tag{4.9}$$

The previous differential equation is what leads us to the investigation of ours Lemmas 121 and 122. More precisely, in the aim to express (4.9) in a more suitable way, we obtain the results that allows us express the above differential equation as

$$\sum_{i=0}^{2}\left(\sum_{j=i}^{2}\binom{j}{i}(r'')^{j-i}\left(k^2\sin^2\phi\right)^{2-j} R_j(r)\right)(k\sin y)^i\cos^i t \equiv 0 \tag{4.10}$$

where

$$R_j(r) = \sum_{l=0}^{j} \binom{n-l}{n-j} r^{2+j-l} A_i \left(\frac{1}{r}\right).$$

The writing in (4.10) is relevant since it assosicated with Proposition 96 we can conclude that for each $0 \le i \le n$, the coefficients

$$\sum_{j=i}^{2} \binom{j}{i} (r'')^{j-i} \left(k^2 \sin^2 \phi\right)^{2-j} R_j(r) \tag{4.11}$$

must vanish identically. Therefore, the following system must be verified for every $s \in (a, b)$:

$$\begin{aligned} 0 &= A_2 + rA_1 + r^2 A_0 \\ 0 &= 2\left(A_2 + rA_1 + r^2 A_0\right) r'' + (k \sin y)^2 (A_1 + 2rA_0) \\ 0 &= \left(A_2 + A_1 r + A_0 r^2\right) (r'')^2 + (\kappa \sin y)^2 (A_1 + 2rA_0) r'' + (k \sin y)^4 A_0 \end{aligned}$$

Let us remark that even more interesting is to notice the pattern present in each line of the system. For instance, if the first line is identically null, we obtain information in the second line and so on.

In the end, we replace the problem to solve the differential equation (4.9) to find a solution of the system (4.11).

The presented technique is the core to the analysis of a polynomial of degree n.

Hence, in order to deal and encompass the investigation of Polynomial Weingarten Canal surfaces in the Euclidean and also in the Lorentzian 3-spaces, evenly and simultaneously, we will introduce an abstract expression of the (generic) principal curvatures presented in the previous section. More precisely, we write

$$k_1(s,t) = \frac{\alpha(s,t)}{\beta(s,t)} = \frac{a + b\mu}{r(a + b\mu) + c} \tag{4.12}$$

where the function $\mu(t)$ represents the trigonometric function (possibly hyperbolic trigonometric function) that depends on the parameter t, the function $\eta(s)$ stands for the trigonometric function (possibly hyperbolic trigonometric function) that models the r' function. Finally, the symbols stands for $a = r''$, $b = \varepsilon_b \kappa \eta$ and $c = \varepsilon_c \eta^2$ with ε_b, $\varepsilon_c \in \{-1, 1\}$.

For instance, a Canal surface in Euclidean 3-space, the above functions becomes

$$\mu(t) = \cos t \qquad \text{and} \qquad \eta(s) = \sin\phi(s)$$

hence, we obtain

$$k_1(s,t) = \frac{r'' + \kappa \sin\phi \cos t}{r(r'' + \kappa \sin\phi \cos t) - \sin^2\phi}$$

which agrees with the one exhibit in the Proposition 113.

Another sample can be seen, when we choose

$$\mu(t) = \sinh t \qquad \text{and} \qquad \eta(s) = \sin\phi(s)$$

then, it is obtained

$$k_1(s,t) = \frac{r'' - \kappa \sin\phi \sinh t}{r(r'' - \kappa \sin\phi \sinh t) - \sin^2\phi}$$

which agrees with the one exhibit in (4.8).

Although the expression (4.12) seems to add unnecessary dificulty to write each curvatures presented in the Sections 4.1.1 & 4.1.2, we point out that the generic expression (4.12) allows us to analyse and study Canal surfaces in $\mathbb{R}^3$ and in $\mathbb{L}^3$ all at once.

The first obtained result is a summatory identity whose application to the next result is very important.

Lemma 121 *For given $n \in \mathbb{N}$, we have the following equality*

$$\sum_{l=0}^{n}\sum_{j=0}^{n-l}\sum_{i=0}^{n-j} a_{i,j,l} = \sum_{i=0}^{n}\sum_{j=i}^{n}\sum_{l=0}^{j} a_{i,n-j,l}.$$

Proof. We consider the summatory

$$\sum_{l=0}^{n}\sum_{j=0}^{n-l}\sum_{i=0}^{n-j} a_{i,j,l}$$

for a given $n \in \mathbb{N}$. Then, we change the indices l and i, which implies

$$\sum_{l=0}^{n}\sum_{j=0}^{n-l}\sum_{i=0}^{n-j} a_{i,j,l} = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{l=0}^{n-j} a_{i,j,l}$$

(roughly speaking we rename the first and third indices). From the above equality we have the next expression

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{l=0}^{n-j} a_{i,j,l} = \sum_{i=0}^{n}\sum_{j=i}^{(n-i)+i}\sum_{l=0}^{n-(j-i)} a_{i,(j-i),l} = \sum_{i=0}^{n}\sum_{j=i}^{n}\sum_{l=0}^{n-(j-i)} a_{i,j-i,l}$$

then, we redefine the indice j as follows

$$\sum_{i=0}^{n}\sum_{j=i}^{n}\sum_{l=0}^{n-(j-i)} a_{i,j-i,l} = \sum_{i=0}^{n}\sum_{j=i}^{n}\sum_{l=0}^{n-((n-j+i)-i)} a_{i,(n-j+i)-i,l} = \sum_{i=0}^{n}\sum_{j=i}^{n}\sum_{l=0}^{j} a_{i,n-j,l}$$

which concludes the demonstration. ■

The next lemma is purely technical, however it plays a relevant role in our main theorem. The reason lies on the fact that our main theorem is based on a suitable expression of an arbitrary polynomial relation and it is achieved thanks to the following lemma.

We also observe that the lemma is what allows us to transfer the problem of solve a differential equation to solve a system of differential equations.

Lemma 122 *For every $a, b, c, x \in \mathbb{R}$ and for any $n \in \mathbb{N}$, the following equality holds*

$$\sum_{i=0}^{n} (a+bx)^{i} \left(r\left(a+bx\right)+c\right)^{n-i} \mathcal{A}_{i}\left(r\right) = \sum_{i=0}^{n} \left(\sum_{j=i}^{n} \binom{j}{i} a^{j-i} c^{n-j} R_{j}\left(r\right) \right) b^{i} x^{i}$$

where $R_j\left(z\right) = \sum_{l=0}^{j} \binom{n-l}{n-j} z^{j-l} \mathcal{A}_l\left(z\right)$.

Proof. In fact, notice that

$$\sum_{l=0}^{n} (a+bx)^{l} \left(r\left(a+bx\right)+c\right)^{n-l} \mathcal{A}_{l}\left(r\right) \tag{4.13}$$

where each of above terms that can be expanded as follows

$$\left(r\left(a+bx\right)+c\right)^{n-l} = \sum_{j=0}^{n-l} \binom{n-l}{j} r^{n-l-j} \left(a+bx\right)^{n-l-j} c^{j}$$

and

$$\left(a+bx\right)^{n-j} = \sum_{i=0}^{n-j} \binom{n-j}{i} a^{n-j-i} b^{i} x^{i}$$

therefore (4.13) is rewritten as

$$\sum_{l=0}^{n} \sum_{j=0}^{n-l} \sum_{i=0}^{n-j} \binom{n-l}{j} \binom{n-j}{i} r^{n-l-j} a^{n-j-i} b^{i} x^{i} c^{j} \mathcal{A}_{l}\left(r\right).$$

Applying the Lemma 121 in the previous equation, it is obtained the following expression

$$\sum_{i=0}^{n} \sum_{j=i}^{n} \sum_{l=0}^{j} \binom{j}{i} \binom{n-l}{n-j} r^{j-l} a^{j-i} b^{i} x^{i} c^{n-j} \mathcal{A}_{l}\left(r\right)$$

and rearranging the terms, we have

$$\sum_{i=0}^{n} \left(\sum_{j=i}^{n} \binom{j}{i} a^{j-i} c^{n-j} \left(\sum_{l=0}^{j} \binom{n-l}{n-j} r^{j-l} \mathcal{A}_{l}\left(r\right) \right) \right) b^{i} x^{i}.$$

Finally, if we set

$$R_j(r) = \sum_{l=0}^{j} \binom{n-l}{n-j} r^{j-l} \mathcal{A}_l(r)$$

the desired is achieved. ■

We start the study of canal surface whose principal curvatures verify $P(k_1, k_2) \equiv 0$ (called (k_1, k_2)-Weingarten surfaces). As a consequence, we classificate the canal surface whose Gaussian and mean curvatures verify $Q(K, H) \equiv 0$ (where $Q\left(xy, \frac{x+y}{2}\right)$ is the polynomial associated to $P(x, y)$), once this curvatures always can be rewritten as

$$K = k_1 k_2 \qquad \text{and} \qquad H = \frac{k_1 + k_2}{2}.$$

Moreover, we noticed that we could investigate the classification of surfaces over two (or more) functions whose behavior is similar to the principal curvatures and still, the result will follows for Gaussian and mean curvatures. More precisely, for canal surfaces in $\mathbb{L}^3$ of type 4 or type 5, we consider smooth functions $\widetilde{k_1}$ and $\widetilde{k_2}$ that verify

$$-K = \widetilde{k_1}\widetilde{k_2} \qquad \text{and} \qquad H = \frac{\widetilde{k_1} + \widetilde{k_2}}{2}.$$

Then, we remark that the sign does not affect our analysis, because we study an arbitrary polinomial, therefore, we may incorporate the minus sign in the coefficients of the polynomial.

We would like to highlight that the following theorem is, indeed, a classification of Canal surface whose principal curvatures verify an arbitrary polynomial, for canal surfaces in the Euclidean 3-space and for canal surfaces in the Lorentzian 3-space of the type 1, 2 and 3. However, for canal surface in $\mathbb{L}^3$ of type 4 or type 5, we classificate for functions $\widetilde{k_1}$ and $\widetilde{k_2}$.

Finally, the next theorem is the first (and fundamental) step to us to achieve the classification of (k_1, k_2)-Weingarten canal surface.

Theorem 123 *A (k_1, k_2)-Weingarten Canal surface is a (smooth) combination of tubular surface and rotational surface.*

Proof. Given an arbitrary polynomial relation $P(x, y) \in \mathbb{R}[x, y]$ we consider the following generic representation (which always exists)

$$P(x, y) = x^n A_n(y) + x^{n-1} A_{n-1}(y) + \ldots + x A_1(y) + A_0(y), \tag{4.14}$$

where for each $0 \le i \le n$, we have that

$$A_i(y) = \sum_{j=0}^{m_i} a_{i,j} y^j \in \mathbb{R}[y]. \tag{4.15}$$

Let S be a canal surface with central curve γ defined in (a,b). Suppose that $s_0 \in (a,b)$ is such that $\kappa(s_0) \neq 0$, it yields the existence of an open neighborhood J of $\gamma(s)$, with $\kappa(s) \neq 0$, for every $s \in J$. Consider the closed set $\overline{J}$. Then, in $\overline{J} \times \mathbb{R}$, the abstract principal curvatures are

$$k_1 = \frac{\alpha(s,t)}{\beta(s,t)} = \frac{a + b\cos t}{r(a + b\cos t) + c} \qquad \text{and} \qquad k_2 = \frac{1}{r}. \tag{4.16}$$

Therefore, we consider the following relation

$$\widetilde{P}(x,y) = \beta^n r^m P(x,y), \tag{4.17}$$

where $m = \max\{m_1, \dots, m_n\}$. Since $P(k_1,k_2) \equiv 0$, it implies that $\widetilde{P}(k_1,k_2) \equiv 0$.

Notice that (4.17) can be expressed as

$$\widetilde{P}(k_1,k_2) = \sum_{i=0}^{n} \beta(s,t)^{n-i} \alpha(s,t)^i r^m A_i\left(\frac{1}{r}\right) = 0.$$

Then we define the map

$$z \in \mathbb{R}^* \longmapsto \mathcal{A}_i(z) = z^m A_i\left(\frac{1}{z}\right)$$

which yields

$$0 \equiv \widetilde{P}(k_1,k_2) = \sum_{i=0}^{n} \alpha(s,t)^i \beta(s,t)^{n-i} \mathcal{A}_i(r)$$

thus, by (4.16) the above writing becomes

$$0 \equiv \widetilde{P}(k_1,k_2) = \sum_{i=0}^{n} (a + b\cos t)^i (r(a + b\cos t) + c)^{n-i} \mathcal{A}_i(r).$$

Follows from Lemma 122 that

$$\widetilde{P}(k_1,k_2) = \sum_{i=0}^{n} \left(\sum_{j=i}^{n} \binom{j}{i} a^{j-i} c^{n-j} R_j(r) \right) b^i \cos^i t \equiv 0$$

where

$$R_j(z) = \sum_{l=0}^{j} \binom{n-l}{n-j} z^{j-l} \mathcal{A}_l(z) \in \mathbb{R}[z]. \tag{4.18}$$

Before to proceed, let us define, for each $s \in \overline{J}$, the following polynomials

$$\mathfrak{P}_s(x) = \sum_{i=0}^{n} \left(\sum_{j=i}^{n} \binom{j}{i} a^{j-i} c^{n-j} R_j(r) \right) b^i x^i \in \mathbb{R}[x] \tag{4.19}$$

then we notice that

$$\mathfrak{P}_s(\cos t) \equiv 0$$

for every $t \in \mathbb{R}$. By the Lemma 96 it follows that

$$\sum_{j=i}^{n} \binom{j}{i} a^{j-i} c^{n-j} R_j(r)\, b^i \equiv 0$$

for each $i \in \{0, ..., n\}$. We would like to recall that

$$b(s) = \kappa(s) \sin \phi(s) \neq 0$$

where $\kappa(s) \neq 0$ on $\overline{J}$. Furthermore, $\sin \phi(s) \neq 0$ almost everywhere, otherwise it will contradict the regularity of the surface. Finally, since $c = \sin^2 \phi(s)$ we have that

$$c(s) \neq 0.$$

Then, we conclude that

$$\sum_{j=i}^{n} \binom{j}{i} a^{j-i} c^{n-j} R_j(r) \equiv 0, \tag{4.20}$$

for each $i \in \{0, ..., n\}$, hence we must consider two cases, named the case that exists $R_i(z)$ not null (for some $0 \leq i \leq n$) and the case that $R_i(z)$ are all null.

CASE i. Assume the existence of i_0 such that $R_{i_0}(z)$ is not null and $R_i(z)$ is null for every $i > i_0$. Follows from the (4.18), (4.19) and (4.20) that

$$\sum_{j=i_0}^{n} \binom{j}{i_0} a^{j-i_0} c^{n-j} R_j(r) = \binom{i_0}{i_0} a^{i_0-i_0} c^{n-i_0} R_j(r) + \sum_{j=i_0+1}^{n} \binom{j}{i_0} a^{j-i_0} c^{n-j} R_j(r)$$

it follows that

$$R_{i_0}(r(s)) \equiv 0$$

for every $s \in J$. Then we conclude that $r(s)$ must be constant, as consequence, the canal surface on $\overline{J} \times \mathbb{R}$ is, as a matter of fact, a tubular surface.

CASE ii. Every $R_i(z)$ is null.

We will prove the following statement:

CLAIM: Every $\mathcal{A}_i(z) \equiv 0$.

Assume, by absurd, that statement is false. Consider i_0 such that $\mathcal{A}_{i_0}(x)$ is not null and $\mathcal{A}_i(x) = 0$ for every $i < i_0$. Then, we remark that

$$R_{i_0}(r) = \binom{n-l}{n-i_0} r^{i_0-l} \mathcal{A}_l(r) = \binom{n-i_0}{n-i_0} r^{i_0-i_0} \mathcal{A}_{i_0}(r) = \mathcal{A}_{i_0}(r),$$

so we reach that R_{i_0} is not null, which constradicts this case. Then, we have that $\mathcal{A}_i(z) \equiv 0$ for every $0 \leq i \leq n$.

Let $\widetilde{A_i}(z)$ be the associated polynomial function to $A_i(z)$. Since $\mathcal{A}_i$ is the null polynomial, it implies that $\widetilde{A_i}(z)$ must vanish everywhere. Then, consider the following relation

$$\widetilde{A_i}(z) = z^m \widetilde{\mathcal{A}_i}\left(\frac{1}{z}\right) = 0$$

which implies that $\widetilde{A_i}(z)$ is null, therefore $A_i(z)$ is the null polynomial. However, by the definition of $A_i(z)$ in (4.15) and by (4.14) the previous conclusions provides that the polynomial $P(x,y)$ is the null polynomial, which is an absurd!

Therefore we conclude that, in $\overline{J} \times \mathbb{R}$ the canal surface is, as a matter of fact, a tubular surface.

So it remains to investigate the points $s_0 \in (a,b)$ such that $\kappa(s_0) = 0$. If s_0 is an isolated point, we notice that there is a neighborhood $(s_0 - \delta, s_0 + \delta)$ of s_0 such that for every $s \in (s_0 - \delta, s_0 + \delta) - \{s_0\}$ we have $\kappa(s) \neq 0$.

Then the same argument as before provides that we have a combination of two tubular surfaces, one defined in $(s_0 - \delta, s_0)$ and another defined in $(s_0, s_0 + \delta)$. Since, in this case, the radius is constant, it implies that the surface is a tubular surface in $(s_0 - \delta, s_0 + \delta)$.

Finally, in elements $s_0 \in (a,b)$ such that $\kappa(s_0) = 0$, where we have an open neighborhood L such that $\kappa(s) = 0$ for every $s \in L$, we conclude that, in $L \times \mathbb{R}$ the canal surface is, as a matter of fact, a rotational surface. ■

Once we classificated (k_1, k_2)-Weingarten Canal surface, we are able to the next step which is investigate Weingarten Canal surfaces, that is, Canal surfaces whose Gaussian and mean curvatures verify $Q(K,H) \equiv 0$.

In the aim to accomplish that, we remark that it is only needed to notice that in Section 4.1.1 and in Section 4.1.2, either we presented the principal curvatures of the Canal surface or either we introduce a pair of functions whose average provides the mean curvature of the Canal surface, while the product gives minus the Gaussian curvature of the Canal surface.

Then, we always can to rewrite the Gaussian and mean curvatures in terms of the principal curvatures k_1 and k_2 (or in terms of two functions that behaves the same).

For a given a polynomial $Q(x,y) = \sum_{i=0}^{p} \sum_{j=0}^{q} a_{ij} x^i y^j \in \mathbb{R}[x,y]$, the assumption that the Canal surface verify $P(K,H) \equiv 0$, provides that

$$0 \equiv Q(K,H) = \sum_{i=0}^{p} \sum_{j=0}^{q} a_{ij} (K)^i (H)^j , \tag{4.21}$$

so, we may express

$$K = \varepsilon_* k_1 k_2 \qquad \text{and} \qquad H = \frac{k_1 + k_2}{2}$$

where $\varepsilon_* \in \{-1, 1\}$ depending if k_1 and k_2 are the principal curvatures (in this case $\varepsilon_* = 1$); Otherwise, we have $\varepsilon_* = -1$. Hence, (4.21) is rewritten as

$$0 \equiv Q(K,H) = \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{j} \frac{(\varepsilon_*)^i a_{ij}}{2^j} \binom{j}{k} k_1^{j-k+i} k_2^{k+i}.$$

Then, we define

$$\widetilde{Q}(x,y)=\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{j}\widetilde{a_{i,j}}x^{j-k+i}y^{k+i}\in\mathbb{R}[x,y]$$

where

$$\widetilde{a_{i,j}}=\frac{(\varepsilon_*)^i a_{ij}}{2^j}\binom{j}{k}$$

and apply The Theorem 123 to the polynomial $\widetilde{Q}(x,y)\in\mathbb{R}[x,y]$ which yields the following result:

Theorem 124 *Consider the polynomial* $Q(x,y)\in\mathbb{R}[x,y]$ *and let* $\mathcal{S}(Q)$ *be the set of all regular canal surfaces in Euclidean* 3*-space (Lorentzian* 3*-space) whose Gaussian and mean curvatures* K*,* H *verify* $Q(K,H)\equiv 0$*. Then, the elements of* $\mathcal{S}(Q)$ *are (smooth) combinations of Rotational surfaces and Tubular surfaces of radius* $r\in Rad^*(Q)$*.*

Appendix A

Appendix for Tubular Surfaces

A.1 Summatories Identities

Proposition 125 *Let $n \in \mathbb{N}$, then*

$$\sum_{i=0}^{n}\sum_{k=0}^{n-i} A_{i,k} = \sum_{i=0}^{n}\sum_{k=0}^{i} A_{n-i,k}$$

Proposition 126 *Let $n \in \mathbb{N}$, then*

$$\sum_{i=0}^{n}\sum_{k=0}^{i} A_{i,k} = \sum_{k=0}^{n}\sum_{i=0}^{k} A_{n-i,n-k}. \tag{A.1}$$

Proof. This prove is given by induction on n. Notice that (A.1) is valid for $n = 1$,

$$\sum_{i=0}^{(1)}\sum_{k=0}^{i} A_{i,k} = A_{0,0} + A_{1,0} + A_{1,1} = \sum_{k=0}^{(1)}\sum_{i=0}^{k} A_{(1)-i,(1)-k}.$$

Thus, we assume that equality holds for some $n \in \mathbb{N}$:

$$\sum_{i=0}^{n}\sum_{k=0}^{i} A_{i,k} = \sum_{k=0}^{n}\sum_{i=0}^{k} A_{n-i,n-k}$$

and we will verify that (A.1) is valid for $n+1$. In fact,

$$\begin{aligned}
& \sum_{i=0}^{(n+1)} \sum_{k=0}^{i} A_{i,k} - \sum_{k=0}^{(n+1)} \sum_{i=0}^{k} A_{(n+1)-i,(n+1)-k} && \text{(A.2)} \\
= & \sum_{i=0}^{n}\sum_{k=0}^{i} A_{i,k} + \sum_{k=0}^{(n+1)} A_{n+1,k} && \text{(A.3)} \\
& -\sum_{k=0}^{n}\sum_{i=0}^{k} A_{(n+1)-i,(n+1)-k} - \sum_{i=0}^{(n+1)} A_{(n+1)-i,0}. && \text{(A.4)}
\end{aligned}$$

Before to proceed with previous equation, we remark that

$$\begin{aligned}
& \sum_{k=0}^{n}\sum_{i=0}^{k} A_{(n+1)-i,(n+1)-k} \\
= \quad & A_{n+1,n+1} + \sum_{k=0}^{n-1}\sum_{i=0}^{k} A_{n-i,n-k} + \sum_{k=1}^{n} A_{(n+1),(n+1)-k}
\end{aligned}$$

therefore, applying the above equality to (A.2) it is obtained

$$\begin{aligned}
& \sum_{i=0}^{n}\sum_{k=0}^{i} A_{i,k} + \sum_{k=0}^{n} A_{n+1,k} - \sum_{k=0}^{n-1}\sum_{i=0}^{k} A_{n-i,n-k} \\
& -\sum_{k=1}^{n} A_{(n+1),(n+1)-k} - A_{(n+1),0} - \sum_{i=1}^{(n+1)} A_{n+1-i,0} \\
= \quad & \sum_{i=0}^{n}\sum_{k=0}^{i} A_{i,k} + \sum_{k=0}^{n} A_{n+1,k} - \sum_{k=0}^{n-1}\sum_{i=0}^{k} A_{n-i,n-k} \\
& -\sum_{k=1}^{n} A_{(n+1),(n+1)-k} - A_{(n+1),0} - \sum_{i=0}^{n} A_{n-i,0} \\
= \quad & \sum_{i=0}^{n}\sum_{k=0}^{i} A_{i,k} - \sum_{k=0}^{n}\sum_{i=0}^{k} A_{n-i,n-k} \\
& +\sum_{k=0}^{n} A_{n+1,k} - \sum_{k=1}^{n} A_{(n+1),(n+1)-k} - A_{(n+1),0}.
\end{aligned}$$

The induction step gives us

$$\begin{aligned}
& \sum_{k=0}^{n} A_{n+1,k} - \sum_{k=1}^{n} A_{(n+1),(n+1)-k} - A_{(n+1),0} \\
= \quad & \sum_{k=0}^{n} A_{n+1,k} - \sum_{k=0}^{n} A_{n+1,n-k},
\end{aligned}$$

finally, through the Propostion 125 we achieve the desired. ■

Proposition 127

$$\sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i} M_{i,j} = \sum_{i=0}^{n+1} M_{i,n+1-i} + \sum_{i=0}^{n}\sum_{j=0}^{n-i} M_{i,j}$$

Proof. The proof is by induction on n. First, notice the equality holds for $n = 1$

$$\begin{aligned} & \sum_{i=0}^{2}\sum_{j=0}^{2-i} M_{i,j} \\ = \; & M_{0,0} + M_{0,1} + M_{1,0} + M_{0,2} + M_{1,1} + M_{2,0} \\ = \; & \sum_{i=0}^{2} M_{i,2-i} + \sum_{i=0}^{1}\sum_{j=0}^{1-i} M_{i,j} \end{aligned}$$

and suppose that is also true for $n = k$:

$$\sum_{i=0}^{k+1}\sum_{j=0}^{k+1-i} M_{i,j} = \sum_{i=0}^{k+1} M_{i,k+1-i} + \sum_{i=0}^{k}\sum_{j=0}^{k-i} M_{i,j},$$

we will prove for $n = k + 1$.

In one hand, we have

$$\sum_{i=0}^{(k+1)+1}\sum_{j=0}^{(k+1)+1-i} M_{i,j} = \sum_{i=0}^{k+1}\sum_{j=0}^{k+1-i} M_{i,j} + \sum_{i=0}^{k+1} M_{i,k+2-i} + M_{k+2,0}$$

and the induction step gives

$$\sum_{i=0}^{k+1} M_{i,k+1-i} + \sum_{i=0}^{k}\sum_{j=0}^{k-i} M_{i,j} + \sum_{i=0}^{k+1} M_{i,k+2-i} + M_{k+2,0}.$$

On the other hand,

$$\begin{aligned} & \sum_{i=0}^{k+2} M_{i,k-i+2} + \sum_{i=0}^{k+1}\sum_{j=0}^{k+1-i} M_{i,j} \\ = \; & \sum_{i=0}^{k+1} M_{i,k+2-i} + M_{k+2,0} + \sum_{i=0}^{k}\sum_{j=0}^{k-i+1} M_{i,j} + M_{k+1,0} \end{aligned}$$

and the desired is obtained. ■

Proposition 128

$$\sum_{k=0}^{n+1}\sum_{i=0}^{k}\sum_{j=0}^{n+1-k} M_{i,j,k} = \sum_{i=0}^{n+1} M_{i,0,n+1} + \sum_{k=0}^{n}\sum_{i=0}^{k}\sum_{j=0}^{n+1-k} M_{i,j,k}$$

Proof. This demonstration is given by induction on n. When $n = 1$ is easy to see that both sides agrees. Assume the equality is valid for $n = l$:

$$\sum_{k=0}^{(l)+1}\sum_{i=0}^{k}\sum_{j=0}^{(l)+1-k} M_{i,j,k} = \sum_{i=0}^{(l)+1} M_{i,0,(l)+1} + \sum_{k=0}^{(l)}\sum_{i=0}^{k}\sum_{j=0}^{(l)+1-k} M_{i,j,k}$$

we will show the result holds for $n = l + 1$.

Indeed,

$$\begin{aligned}
&\sum_{k=0}^{(l+1)+1}\sum_{i=0}^{k}\sum_{j=0}^{(l+1)+1-k} M_{i,j,k} \\
= \; &\sum_{k=0}^{l+1}\sum_{i=0}^{k}\sum_{j=0}^{l+2-k} M_{i,j,k} + \sum_{i=0}^{l+2} M_{i,0,l+2} \\
= \; &\sum_{k=0}^{l+1}\sum_{i=0}^{k}\sum_{j=0}^{l+1-k} M_{i,j,k} + \sum_{k=0}^{l+1}\sum_{i=0}^{k} M_{i,l+2-k,k} + \sum_{i=0}^{l+2} M_{i,0,l+2}
\end{aligned}$$

and the induction step provides

$$\begin{aligned}
\therefore \quad &\left(\sum_{i=0}^{l+1} M_{i,0,l+1} + \sum_{k=0}^{l}\sum_{i=0}^{k}\sum_{j=0}^{l+1-k} M_{i,j,k}\right) \\
&+ \sum_{k=0}^{l+1}\sum_{i=0}^{k} M_{i,l+2-k,k} + \sum_{i=0}^{l+2} M_{i,0,l+2} \\
= \quad &\sum_{i=0}^{l+2} M_{i,0,l+2} + \sum_{i=0}^{l+1} M_{i,0,l+1} \\
&+ \sum_{k=0}^{l}\sum_{i=0}^{k}\sum_{j=0}^{l+1-k} M_{i,j,k} + \sum_{k=0}^{l}\sum_{i=0}^{k} M_{i,l+2-k,k} + \sum_{i=0}^{l+1} M_{i,1,l+1}.
\end{aligned}$$

Then observe that

$$\begin{aligned}
&\sum_{k=0}^{(l+1)}\sum_{i=0}^{k}\sum_{j=0}^{(l+1)+1-k} M_{i,j,k} \\
= \; &\sum_{k=0}^{l}\sum_{i=0}^{k}\sum_{j=0}^{l+1-k} M_{i,j,k} + \sum_{i=0}^{l+1} M_{i,0,l+1} \\
&+ \sum_{k=0}^{l}\sum_{i=0}^{k} M_{i,l+2-k,k} + \sum_{i=0}^{l+1} M_{i,1,(l+1)}
\end{aligned}$$

which concludes the proof. ∎

Proposition 129

$$\sum_{m=0}^{p}\sum_{l=0}^{q+p-m}\lambda_{m,l}b_{m+l}=\sum_{l=0}^{p}\sum_{m=0}^{l}\lambda_{m,l-m}b_l+\sum_{l=1}^{q}\sum_{m=0}^{p}\lambda_{m,p+l-m}b_{p+l}, \tag{A.5}$$

for every p and $q \geq 0$.

Proof. This demonstration is given by induction on $p \in \mathbb{N}$. First, in the case of $p = 0$, the left-hand side of the equation (A.5) provides

$$\sum_{m=0}^{0}\sum_{l=0}^{q-m}\lambda_{m,l}b_{m+l}=\sum_{l=0}^{q}\lambda_{0,l}b_l.$$

while the right-hand side of the equation (A.5) gives us

$$\sum_{l=0}^{(0)}\sum_{m=0}^{l}\lambda_{m,l-m}b_l+\sum_{l=1}^{q}\lambda_{0,l}b_l=\lambda_{0,0}b_0+\sum_{l=1}^{q}\lambda_{0,l}b_l=\sum_{l=0}^{q}\lambda_{0,l}b_l$$

therefore, the equality is obtained.

Assume that Equation (A.5) is true for some p and for every $q \geq 1$, that is,

$$\sum_{m=0}^{p}\sum_{l=0}^{q+p-m}\lambda_{m,l}b_{m+l}=\sum_{l=0}^{p}\sum_{m=0}^{l}\lambda_{m,l-m}b_l+\sum_{l=1}^{q}\sum_{m=0}^{p}\lambda_{m,p+l-m}b_{p+l}.$$

Hence, we will verify that equality holds for $p + 1$.

In on hand, we have

$$\begin{aligned}
&\sum_{m=0}^{(p+1)}\sum_{l=0}^{q+(p+1)-m}\lambda_{m,l}b_{m+l}\\
= &\sum_{m=0}^{p}\sum_{l=0}^{q+p-m}\lambda_{m,l}b_{m+l}+\sum_{m=0}^{p}\lambda_{m,q+(p+1)-m}b_{q+(p+1)}\\
&+\sum_{l=0}^{q}\lambda_{p+1,l}b_{p+1+l}
\end{aligned}$$

therefore, the induction step furnishes

$$\begin{aligned}
& \sum_{l=0}^{p}\sum_{m=0}^{l}\lambda_{m,l-m}b_l + \sum_{l=1}^{q}\sum_{m=0}^{p}\lambda_{m,p+l-m}b_{p+l} \\
& + \sum_{m=0}^{p}\lambda_{m,q+(p+1)-m}b_{q+(p+1)} + \sum_{l=0}^{q}\lambda_{p+1,l}b_{p+1+l} \\
= & \sum_{l=0}^{p}\sum_{m=0}^{l}\lambda_{m,l-m}b_l + \sum_{l=0}^{q-1}\sum_{m=0}^{p}\lambda_{m,p+(l+1)-m}b_{p+(l+1)} \\
& + \sum_{m=0}^{p}\lambda_{m,q+(p+1)-m}b_{q+(p+1)} + \sum_{l=0}^{q}\lambda_{p+1,l}b_{p+1+l} \\
= & \sum_{l=0}^{p}\sum_{m=0}^{l}\lambda_{m,l-m}b_l + \sum_{l=0}^{q}\sum_{m=0}^{p}\lambda_{m,p+(l+1)-m}b_{p+(l+1)} + \sum_{l=0}^{q}\lambda_{p+1,l}b_{p+1+l} \\
= & \sum_{l=0}^{p}\sum_{m=0}^{l}\lambda_{m,l-m}b_l + \sum_{l=1}^{q}\sum_{m=0}^{p}\lambda_{m,p+(l+1)-m}b_{p+(l+1)} \\
& + \sum_{m=0}^{p+1}\lambda_{m,p+1-m}b_{p+1} + \sum_{l=1}^{q}\lambda_{p+1,l}b_{p+1+l}.
\end{aligned}$$

On the other hand, we consider

$$\begin{aligned}
& \sum_{l=0}^{(p+1)}\sum_{m=0}^{l}\lambda_{m,l-m}b_l + \sum_{l=1}^{q}\sum_{m=0}^{(p+1)}\lambda_{m,(p+1)+l-m}b_{(p+1)+l} \\
= & \sum_{l=0}^{p}\sum_{m=0}^{l}\lambda_{m,l-m}b_l + \sum_{m=0}^{(p+1)}\lambda_{m,(p+1)-m}b_{(p+1)} \\
& + \sum_{l=1}^{q}\sum_{m=0}^{p}\lambda_{m,(p+1)+l-m}b_{(p+1)+l} + \sum_{l=1}^{q}\lambda_{(p+1),(p+1)+l-(p+1)}b_{(p+1)+l} \\
= & \sum_{l=0}^{p}\sum_{m=0}^{l}\lambda_{m,l-m}b_l + \sum_{l=1}^{q}\sum_{m=0}^{p}\lambda_{m,(p+1)+l-m}b_{(p+1)+l} \\
& + \sum_{m=0}^{(p+1)}\lambda_{m,(p+1)-m}b_{(p+1)} + \sum_{l=1}^{q}\lambda_{(p+1),l}b_{(p+1)+l}
\end{aligned}$$

so we conclude that

$$\sum_{m=0}^{(p+1)}\sum_{l=0}^{q+(p+1)-m}\lambda_{m,l}b_{m+l} = \sum_{l=0}^{(p+1)}\sum_{m=0}^{l}\lambda_{m,l-m}b_l + \sum_{l=1}^{q}\sum_{m=0}^{(p+1)}\lambda_{m,(p+1)+l-m}b_{(p+1)+l}$$

■

Proposition 130 *Let $n \in \mathbb{N}$, then the below equality is true*

$$\sum_{m=0}^{i} \sum_{l=0}^{n-(i-m)} \sum_{k=0}^{i-m} A_{k,m,l} = \sum_{k=0}^{i} \sum_{m=0}^{i-k} \sum_{l=0}^{n-i+m} A_{k,m,l}, \tag{A.6}$$

for every $i \in \{0, ..., n\}$.

Proof. Before to actually present the prove, it is relevant to note that for a given $n \in \mathbb{N}$, it follows that Equality A.6 is valid for every $i \in \{0, ..., n\}$. Indeed, otherwise (*i.e.* $i > n$) we can express

$$i = n + \varepsilon,$$

where $\varepsilon > 0$. Then, the second summand of the left-hand side of the Equation A.6 becomes

$$n - (i - m) = n - ((n + \varepsilon) - m) = m - \varepsilon$$

and notice that $0 \leq m \leq i = n + \varepsilon$, hence

$$m - \varepsilon \leq 0$$

which is not well-defined in the sum.

In the view of above discussion, now we will prove the Equation A.6 by induction on n. In the case of $n = 1$, there is two possibilities for i, since $i \in \{0, 1\}$. For $i = 0$, a straightforward calculation provides

$$\sum_{l=0}^{1} A_{0,0,l} = A_{0,0,0} + A_{0,0,1} = \sum_{l=0}^{1} A_{0,0,l}$$

which concludes this case. For $i = 1$, it follows

$$\begin{aligned} & \sum_{m=0}^{1} \sum_{l=0}^{m} \sum_{k=0}^{1-m} A_{k,m,l} \\ = \quad & A_{0,0,0} + A_{0,1,0} + A_{1,0,0} + A_{0,1,1} \\ = \quad & \sum_{k=0}^{1} \sum_{m=0}^{1-k} \sum_{l=0}^{m} A_{k,m,l} \end{aligned}$$

and the desired is achieved.

Assume that Equality A.6 is valid for some $n \in \mathbb{N}$ and for every $i \in \{0, ..., n\}$. More precisely:

$$\sum_{m=0}^{i} \sum_{l=0}^{n-(i-m)} \sum_{k=0}^{i-m} A_{k,m,l} = \sum_{k=0}^{i} \sum_{m=0}^{i-k} \sum_{l=0}^{n-i+m} A_{k,m,l}. \tag{A.7}$$

Hence, it is necessary to verify that equality is still true for some $n + 1$ and for every $i \in \{0, ..., n + 1\}$.

To prove that part, we will consider two cases, named: $0 \leq i \leq n$ and $i = n+1$, it is important to remark that induction step does not stand for $n+1$, so we have to separate the sums up to n and the sum $n+1$.

First, we consider $i \in \{0, ..., n\}$, then

$$\sum_{k=0}^{i}\sum_{m=0}^{i-k}\sum_{l=0}^{(n+1)-i+m} A_{k,m,l} = \sum_{k=0}^{i}\sum_{m=0}^{i-k}\sum_{l=0}^{n-i+m} A_{k,m,l} + \sum_{k=0}^{i}\sum_{m=0}^{i-k} A_{k,m,n+1-i+m}$$

so, the induction step presented in (A.7) yields

$$\sum_{m=0}^{i}\sum_{l=0}^{n-(i-m)}\sum_{k=0}^{i-m} A_{k,m,l} + \sum_{k=0}^{i}\sum_{m=0}^{i-k} A_{k,m,n+1-i+m}.$$

On the other hand,

$$\begin{aligned} & \sum_{m=0}^{i}\sum_{l=0}^{n-(i-m)}\sum_{k=0}^{i-m} A_{k,m,l} + \sum_{k=0}^{i}\sum_{m=0}^{i-k} A_{k,m,n+1-i+m} \\ = & \sum_{m=0}^{i}\sum_{l=0}^{n-(i-m)}\sum_{k=0}^{i-m} A_{k,m,l} + \sum_{m=0}^{i}\sum_{k=0}^{i-m} A_{k,m,(n+1)-(i-m)}, \end{aligned}$$

which finishes this case.

For the remaining case where $i = n+1$, we consider

$$\begin{aligned} & \sum_{k=0}^{n+1}\sum_{m=0}^{n+1-k}\sum_{l=0}^{m} A_{k,m,l} \qquad \text{(A.8)} \\ = & \sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{l=0}^{m} A_{k,m,l} + \sum_{k=0}^{n}\sum_{l=0}^{n+1-k} A_{k,n+1-k,l} + A_{(n+1),0,0} \\ = & \sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{l=0}^{m} A_{k,m,l} + \sum_{k=0}^{n+1}\sum_{l=0}^{n+1-k} A_{k,n+1-k,l} \end{aligned}$$

consequently, the induction step for $i = n$ gives us the following equality

$$\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{k=0}^{n-m} A_{k,m,l} = \sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{l=0}^{n-n+m} A_{k,m,l},$$

for every $i \in \{0, ..., n\}$. Thus, applying the above remark in Equation A.8 it is obtained

$$\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{k=0}^{n-m} A_{k,m,l} + \sum_{k=0}^{n+1}\sum_{l=0}^{n+1-k} A_{k,n+1-k,l}. \qquad \text{(A.9)}$$

On the other hand,

$$\sum_{m=0}^{(n+1)} \sum_{l=0}^{m} \sum_{k=0}^{(n+1)-m} A_{k,m,l} \tag{A.10}$$

$$= \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{k=0}^{n-m} A_{k,m,l} + \sum_{m=0}^{n+1} \sum_{l=0}^{m} A_{n+1-m,m,l}. \tag{A.11}$$

Then, Equation by A.9 and by Equation A.10 we just have to prove:

$$\sum_{k=0}^{n+1} \sum_{l=0}^{n+1-k} A_{k,n+1-k,l} = \sum_{m=0}^{n+1} \sum_{l=0}^{m} A_{n+1-m,m,l}. \tag{A.12}$$

which is precisely the result contained in Proposition **??**, because the Equality A.12 resumes to

$$\sum_{k=0}^{n+1} A_{k,n+1-k} = \sum_{m=0}^{n+1} A_{n+1-m,m},$$

so we conclude this proof. ■

A.2 Binomial Identities

Proposition 131 (Pascal rule) *Let n, $k \in \mathbb{Z}$, then the following identity is true*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proposition 132 *Given $n \in \mathbb{N}^*$, the equality is true*

$$\sum_{m=0}^{n} (-1)^m \binom{y+xm}{r} \binom{n}{m} = 0 \tag{A.13}$$

for every $x, y, r \in \mathbb{Z}$ such that $r < n$.

Proof. To prove the equality in (A.13) we proceed by induction on n. In case $n = 1$, choose x, y, $z \in \mathbb{Z}$ as in the statement and notice that $r \leq 0$. If $r < 0$, we have

$$\binom{y+xm}{r} = 0\ , \qquad \forall m \in \mathbb{Z}$$

and the result follows immediately. Assume $r = 0$, then

$$\binom{y+xm}{0} = \frac{(y+xm)_0}{0!} = 1\ , \qquad \forall m \in \mathbb{Z}$$

which implies

$$\sum_{m=0}^{n} (-1)^m \tbinom{y+xm}{0} \tbinom{n}{m} = \sum_{m=0}^{n} (-1)^m \tbinom{n}{m} = \delta_{n,0} = 0.$$

Suppose the equality (A.13) is true for some $n \geq 1$:

$$\sum_{m=0}^{n} (-1)^m \tbinom{y+xm}{r}\tbinom{n}{m} = 0 \tag{A.14}$$

and for every $x, y, r \in \mathbb{Z}$ such that $r < n$. We will show the result is valid for $n+1$. Since $r < n+1$ there is two distinct cases to consider, named $r = n$ and $r < n$.

So given $x, y, r \in \mathbb{Z}$ such that $r < n+1$. First consider the case $r = n$. Thus

$$\sum_{m=0}^{n+1} (-1)^m \tbinom{y+xm}{n}\tbinom{n+1}{m} = \tbinom{y}{n} + \sum_{m=1}^{n} (-1)^m \tbinom{y+xm}{n}\tbinom{n+1}{m} + (-1)^{n+1} \tbinom{y+x(n+1)}{n}$$

applying Pascal's formula in $\binom{n+1}{m}$ it is obtained

$$\begin{aligned}
& \tbinom{y}{n} + \sum_{m=1}^{n} (-1)^m \tbinom{y+xm}{n}\tbinom{n}{m} + \sum_{m=1}^{n} (-1)^m \tbinom{y+xm}{n}\tbinom{n}{m-1} + (-1)^{n+1} \tbinom{y+x(n+1)}{n} \\
= & \sum_{m=0}^{n} (-1)^m \tbinom{y+xm}{n}\tbinom{n}{m} + \sum_{m=1}^{n+1} (-1)^m \tbinom{y+xm}{n}\tbinom{n}{m-1} \\
= & \sum_{m=0}^{n} (-1)^m \tbinom{y+xm}{n}\tbinom{n}{m} - \sum_{m=0}^{n} (-1)^m \tbinom{(y+x)+xm}{n}\tbinom{n}{m}
\end{aligned}$$

Notice that, the reverse Pascal's formula provide

$$\tbinom{y+xm}{n} = \tbinom{y+xm-1}{n} + \tbinom{(y-1)+xm}{n-1}$$

and

$$\tbinom{(y+x)+xm}{n} = \tbinom{(y+x)+xm-1}{n} + \tbinom{(y+x-1)+xm}{n-1}$$

therefore, the above observation yields

$$\begin{aligned}
& \sum_{m=0}^{n} (-1)^m \tbinom{y+xm-1}{n}\binom{n}{m} + \sum_{m=0}^{n} (-1)^m \tbinom{(y-1)+xm}{n-1}\binom{n}{m} \\
& - \sum_{m=0}^{n} (-1)^m \tbinom{(y+x)+xm-1}{n}\binom{n}{m} - \sum_{m=0}^{n} (-1)^m \tbinom{(y+x-1)+xm}{n-1}\binom{n}{m}.
\end{aligned}$$

Again, the Pascal's formula gives

$$\begin{aligned}
& \sum_{m=0}^{n} (-1)^m \tbinom{y+xm-2}{n}\tbinom{n}{m} + \sum_{m=0}^{n} (-1)^m \tbinom{(y-2)+xm}{n-1}\tbinom{n}{m} + \sum_{m=0}^{n} (-1)^m \tbinom{(y-1)+xm}{n-1}\tbinom{n}{m} \\
& - \sum_{m=0}^{n} (-1)^m \tbinom{(y+x)+xm-2}{n}\tbinom{n}{m} - \sum_{m=0}^{n} (-1)^m \tbinom{(y+x)+xm-2}{n-1}\tbinom{n}{m} - \sum_{m=0}^{n} (-1)^m \tbinom{(y+x-1)+xm}{n-1}\tbinom{n}{m}
\end{aligned}$$

however, now we are able to use the induction step in above equation, which provides to us

$$\sum_{m=0}^{n}(-1)^m\binom{y+xm-2}{n}\binom{n}{m}-\sum_{m=0}^{n}(-1)^m\binom{(y+x)+xm-2}{n}\binom{n}{m},$$

repeating the above process yields

$$\sum_{m=0}^{n}(-1)^m\binom{y+xm-3}{n}\binom{n}{m}-\sum_{m=0}^{n}(-1)^m\binom{(y+x)+xm-3}{n}\binom{n}{m}$$

therefore,

$$\sum_{m=0}^{n}(-1)^m\binom{y+xm-4}{n}\binom{n}{m}-\sum_{m=0}^{n}(-1)^m\binom{(y+x)+xm-4}{n}\binom{n}{m}$$

proceeding in this way, consequently, we get

$$\sum_{m=0}^{n}(-1)^m\binom{y+xm-\mu}{n}\binom{n}{m}-\sum_{m=0}^{n}(-1)^m\binom{(y+x)+xm-\eta}{n}\binom{n}{m}$$

for some $\mu,\eta\geq 0$. We remark that μ and η does not necessarily to be equal, once we can apply the hypothesis only in one of the terms.

So take μ, $\eta\geq 0$ such that

$$y-\mu=y+x-\eta$$

(which is always possible). In this case,

$$\begin{aligned}
&\sum_{m=0}^{n}(-1)^m\binom{y+xm-\mu}{n}\binom{n}{m}-\sum_{m=0}^{n}(-1)^m\binom{(y+x)+xm-\eta}{n}\binom{n}{m}\\
=\;&\sum_{m=0}^{n}(-1)^m\binom{(y-\mu)+xm}{n}\binom{n}{m}-\sum_{m=0}^{n}(-1)^m\binom{(y+x-\eta)+xm}{n}\binom{n}{m}\\
=\;&\sum_{m=0}^{n}(-1)^m\binom{(y-\mu)+xm}{n}\binom{n}{m}-\sum_{m=0}^{n}(-1)^m\binom{(y-\mu)+xm}{n}\binom{n}{m}\\
=\;&0.
\end{aligned}$$

Finally, assume that $r<n$, it follows

$$\begin{aligned}
&\sum_{m=0}^{n+1}(-1)^m\binom{y+xm}{r}\binom{n+1}{m}\\
=\;&(-1)^{n+1}\binom{y+x(n+1)}{r}+\sum_{m=1}^{n}(-1)^m\binom{y+xm}{r}\binom{n+1}{m}+\binom{y}{r}\\
=\;&(-1)^{n+1}\binom{y+x(n+1)}{r}+\sum_{m=1}^{n}(-1)^m\binom{y+xm}{r}\binom{n}{m}+\sum_{m=1}^{n}(-1)^m\binom{y+xm}{r}\binom{n}{m-1}+\binom{y}{r}\\
=\;&(-1)^{n+1}\binom{y+x(n+1)}{r}+\sum_{m=0}^{n}(-1)^m\binom{y+xm}{r}\binom{n}{m}+\sum_{m=1}^{n}(-1)^m\binom{y+xm}{r}\binom{n}{m-1},
\end{aligned}$$

then, the induction step implies

$$
\begin{aligned}
& (-1)^{n+1}\tbinom{y+x(n+1)}{r} + \sum_{m=1}^{n} (-1)^m \tbinom{y+xm}{r}\tbinom{n}{m-1} \\
= & \sum_{m=1}^{n+1} (-1)^m \tbinom{y+xm}{r}\tbinom{n}{m-1} \\
= & -\sum_{m=0}^{n} (-1)^m \tbinom{y+x(m+1)}{r}\tbinom{n}{m} \\
= & -\sum_{m=0}^{n} (-1)^m \tbinom{(y+x)+xm}{r}\tbinom{n}{m}
\end{aligned}
$$

and applying again the induction step we achieved the desired and then concludes the proof. ∎

Proposition 133 *Given $p \leq q \in \mathbb{N}$, we have*

$$\sum_{m=0}^{l} (-1)^m \binom{q+p-m}{q-r}\binom{l}{m} = \binom{q+p-l}{p+r} \tag{A.15}$$

for every $l \in \{0, \dots, p\}$ and $r \in \{0, \dots, q-p\}$.

Proof. The prove of equality (A.15) is given by induction on p. For $p = 1$, chose q, l, r as in the statement and in this case $l \in \{0, 1\}$. If $l = 0$, the conclusion is immediate. If $l = 1$, we have

$$\sum_{m=0}^{1} (-1)^m \tbinom{q+1-m}{q-r}\tbinom{1}{m} = \tbinom{q+1}{q-r} - \tbinom{q}{q-r},$$

applying Pascal rule in above equation, it is obtained the desired.

Assume the equality (A.15) is true for some p:

$$\sum_{m=0}^{l} (-1)^m \tbinom{q+p-m}{q-r}\tbinom{l}{m} = \tbinom{q+p-l}{p+r} \tag{A.16}$$

and it will be shown that still valid for $p+1$. Consider $q \geq p+1$, $l \in \{0, \dots, p+1\}$ and $r \leq q-(p+1)$. If $l \leq p$, then

$$\sum_{m=0}^{l} (-1)^m \tbinom{q+(p+1)-m}{q-r}\tbinom{l}{m} = \sum_{m=0}^{l} (-1)^m \tbinom{(q+1)+p-m}{(q+1)-(r+1)}\tbinom{l}{m},$$

the induction step and rearranging the coefficients finishes this case:

$$\tbinom{(q+1)+p-l}{p+(r+1)} = \tbinom{q+(p+1)-l}{(p+1)+r}.$$

If $l = p + 1$,

$$\sum_{m=0}^{p+1} (-1)^m \binom{q+(p+1)-m}{q-r}\binom{p+1}{m}$$
$$= \binom{q+p+1}{q-r} + \sum_{m=1}^{p} (-1)^m \binom{(q+1)+p-m}{q-r}\binom{p+1}{m} + (-1)^{p+1}\binom{q}{q-r}$$

applying the Pascal rule to $\binom{p+1}{m}$ in above equation provides

$$\binom{q+p+1}{q-r} + \sum_{m=1}^{p} (-1)^m \binom{(q+1)+p-m}{(q+1)-(r+1)}\binom{p}{m} \tag{A.17}$$
$$+ \sum_{m=1}^{p} (-1)^m \binom{(q+1)+p-m}{q-r}\binom{p}{m-1} + (-1)^{p+1}\binom{q}{q-r}. \tag{A.18}$$

Notice that induction step (A.16) gives

$$\sum_{m=1}^{p} (-1)^m \binom{(q+1)+p-m}{(q+1)-(r+1)}\binom{p}{m} = \binom{q+1}{p+r+1} - \binom{q+1+p}{q-r}$$

replacing it on (A.17) we have

$$\binom{q+p+1}{q-r} + \binom{q+1}{p+r+1} - \binom{q+1+p}{q-r} + \sum_{m=1}^{p} (-1)^m \binom{(q+1)+p-m}{q-r}\binom{p}{m-1} + (-1)^{p+1}\binom{q}{q-r}$$

and rearranging the coefficients of the sum and observing

$$-(-1)^{(p+1)-1}\binom{q}{q-r} = -(-1)^{(p+1)-1}\binom{q+p-((p+1)-1)}{q-r}\binom{p}{(p+1)-1},$$

follows

$$\binom{q+1}{p+r+1} - \sum_{m=1}^{p+1} (-1)^{m-1}\binom{q+p-(m-1)}{q-r}\binom{p}{m-1}$$
$$= \binom{q+1}{p+r+1} - \sum_{m=0}^{p} (-1)^m \binom{q+p-(m)}{q-r}\binom{p}{(m)}$$

the induction step furnishes

$$\binom{q+1}{p+r+1} - \binom{q+p-p}{p+r} = \binom{q+1}{p+r+1} - \binom{q}{p+r}$$

finally, the Pascal rule implies

$$\binom{q}{p+r+1} = \binom{q+(p+1)-l}{(p+1)+r}$$

since $l = p + 1$. ■

Proposition 134 *Let $n \in \mathbb{N}$ and consider $x, j \in \mathbb{R}$, then*

$$\sum_{m=0}^{n} (-1)^m \binom{x-m}{j}\binom{n}{m} = \binom{x-n}{j-n}.$$

Proof. The proof is by induction on n. Indeed, for $n = 1$ we have

$$\sum_{m=0}^{1} (-1)^m \binom{x-m}{j}\binom{1}{m} = \binom{x}{j} - \binom{x-1}{j} = \binom{x-1}{j-1}$$

and the last equality is given by the Pascal rule.

Assume the equality is true for n:

$$\sum_{m=0}^{n} (-1)^m \binom{x-m}{j}\binom{n}{m} = \binom{x-n}{j-n}$$

and we will prove for $n+1$,

$$\begin{aligned} &\sum_{m=0}^{(n+1)} (-1)^m \binom{x-m}{j}\binom{(n+1)}{m} \\ = \quad &\binom{x}{j} + \sum_{m=1}^{(n+1)} (-1)^m \binom{x-m}{j}\binom{(n+1)}{m} \end{aligned}$$

the Pascal rule implies

$$\begin{aligned} &\binom{x}{j} + \sum_{m=1}^{(n+1)} (-1)^m \binom{x-m}{j}\binom{n}{m} + \sum_{m=1}^{(n+1)} (-1)^m \binom{x-m}{j}\binom{n}{m-1} \\ = \quad &\sum_{m=0}^{n} (-1)^m \binom{x-m}{j}\binom{n}{m} + (-1)^{n+1}\binom{x-(n+1)}{j}\binom{n}{(n+1)} - \sum_{m=0}^{n} (-1)^m \binom{(x-1)-m}{j}\binom{n}{m} \end{aligned}$$

notice, by definition, $\binom{n}{n+1}$ is null, then

$$\sum_{m=0}^{n} (-1)^m \binom{x-m}{j}\binom{n}{m} - \sum_{m=0}^{n} (-1)^m \binom{(x-1)-m}{j}\binom{n}{m}$$

applying the induction step for each term yields

$$\binom{x-n}{j-n} - \binom{(x-1)-n}{j-n}.$$

Now, we have several cases to consider, since Pascal rule is not always applicable.

Case 1. If $j - n < 0$, follows $j - n - 1 < 0$, then

$$\binom{x-n}{j-n} = 0, \qquad \binom{(x-1)-n}{j-n} = 0 \qquad \text{and} \qquad \binom{x-n-1}{j-n-1} = 0$$

and the equality holds.

Case 2. If $j-n=0$, we have

$$\binom{x-n}{0}-\binom{(x-1)-n}{0}=\frac{1}{1!}-\frac{1}{1!}=0=\binom{x-n-1}{j-n-1}$$

and the equality holds.

Case 3. If $j-n\geq 1$, there is some subcases to consider, named

- Case $x\geq j+1$, the result follows immediately from Pascal rule.
- Case $x=j$, the equations are

$$\binom{j-n}{j-n}=1,\qquad \binom{(j-1)-n}{j-n}=0\qquad \text{and}\qquad \binom{j-n-1}{j-n-1}=1$$

  and the result is valid.
- Case $x<j$, notice that

$$\begin{aligned}
&\binom{x-n}{j-n}-\binom{(x-1)-n}{j-n}\\
=\;&\tfrac{(x-n)(x-n-1)(x-n-2)\ldots(x-n-(j-n)+1)}{(j-n)!}-\tfrac{(x-1-n)((x-1-n)-1)((x-1-n)-2)\ldots((x-1-n)-(j-n)+1)}{(j-n)!}\\
=\;&\tfrac{(x-n)(x-n-1)(x-n-2)\ldots(x-j+1)-(x-n-1)(x-n-2)(x-n-3)\ldots(x-j)}{(j-n)!}\\
=\;&\tfrac{((x-n-1)(x-n-2)\ldots(x-j+1))}{(j-n)!}(j-n)\\
=\;&\tfrac{((x-n-1)(x-n-2)\ldots(x-j+1))}{(j-n-1)!}\\
=\;&\binom{x-n-1}{j-n-1}
\end{aligned}$$

Since every possible case verify the equation, it is concluded

$$\sum_{m=0}^{(n+1)}(-1)^m\binom{x-m}{j}\binom{(n+1)}{m}=\binom{x-(n+1)}{j-(n+1)}$$

■

Appendix B

Appendix for Cyclic Surfaces

Lemma 135

$$\sum_{j=0}^{2n}\sum_{i=0}^{2n-j} A_{i,j} = \sum_{j=0}^{n}\sum_{i=0}^{2n-2j} A_{i,2j} + \sum_{j=1}^{n}\sum_{i=0}^{2n-(2j-1)} A_{i,2j-1}$$

Proposition 136 *For any $n \in \mathbb{N}$, we have*

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i} A_{i,2j} = \sum_{i=0}^{n-1}\sum_{j=0}^{i} A_{i-j,2j}. \tag{B.1}$$

The proof of this result is by induction on n. Notice that the Equality (B.1) is valid for $n = 1$, in fact, the left-hand side of the equality provide to us

$$\sum_{i=0}^{1-1}\sum_{j=0}^{1-1-i} A_{i,2j} = \sum_{i=0}^{0}\sum_{j=0}^{0-i} A_{i,2j} = A_{0,0},$$

which is precisely the same obtained by the right-hand side of the equality

$$\sum_{i=0}^{1-1}\sum_{j=0}^{i} A_{i-j,2j} = \sum_{i=0}^{0}\sum_{j=0}^{i} A_{i-j,2j} = A_{0,0},$$

therefore, we conclude this case.

Assume that Equality (B.1) is true for some $n \in \mathbb{N}$. Our goal is to show that equality still holds for $n + 1$:

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i} A_{i,2j} = \sum_{i=0}^{n+1-1}\sum_{j=0}^{n+1-1-i} A_{i,2j} = \sum_{i=0}^{n+1-1}\sum_{j=0}^{i} A_{i-j,2j} = \sum_{i=0}^{n}\sum_{j=0}^{i} A_{i-j,2j}. \tag{B.2}$$

In one hand, we have

$$\sum_{i=0}^{n}\sum_{j=0}^{i} A_{i-j,2j} = \sum_{j=0}^{n} A_{n-j,2j} + \sum_{i=0}^{n-1}\sum_{j=0}^{i} A_{i-j,2j}$$

by induction step it is obtained

$$\sum_{i=0}^{n}\sum_{j=0}^{i} A_{i-j,2j} = \sum_{j=0}^{n} A_{n-j,2j} + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i} A_{i,2j}. \tag{B.3}$$

On the other hand, when we consider

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i} A_{i,2j} = A_{n,0} + \sum_{i=0}^{n-1}\sum_{j=0}^{n-i} A_{i,2j}$$

which can be rewritten as

$$\begin{aligned} & \sum_{i=0}^{n}\sum_{j=0}^{n-i} A_{i,2j} \qquad \text{(B.4)} \\ = \; & A_{n,0} + \sum_{i=0}^{n-1}\left(A_{i,2(n-i)} + \sum_{j=0}^{n-1-i} A_{i,2j}\right) \\ = \; & A_{n,0} + \sum_{i=0}^{n-1} A_{i,2(n-i)} + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i} A_{i,2j}. \end{aligned}$$

Then, comparing (B.3) and (B.4), follows that it is sufficient verify:

$$\sum_{i=0}^{n} A_{n-i,2i} = \sum_{j=0}^{n} A_{n-j,2j} = A_{n,0} + \sum_{i=0}^{n-1} A_{i,2(n-i)} = \sum_{i=0}^{n} A_{i,2(n-i)}. \tag{B.5}$$

If we set

$$B_i = A_{i,2(n-i)} \qquad \text{and} \qquad B_{n-i} = A_{n-i,2(n-(n-i))} = A_{n-i,2i}$$

the Equality (B.5) is expressed as

$$\sum_{i=0}^{n} B_{n-i} = \sum_{i=0}^{n} B_i$$

which is trivial. So, we achieve the desired.

Proposition 137 *For any $n \in \mathbb{N}$, we have*

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i} A_{i,2j+1} = \sum_{i=0}^{n-1}\sum_{j=0}^{i} A_{i-j,1+2j}. \tag{B.6}$$

The demonstration will be by induction on n. We remark that Equality (B.6) is true for $n = 2$. Indeed,

$$\sum_{i=0}^{(2)-1} \sum_{j=0}^{(2)-1-i} A_{i,2j+1} = A_{0,1} + A_{1,1} + A_{0,3} = \sum_{i=0}^{(2)-1} \sum_{j=0}^{i} A_{i-j,1+2j},$$

so, we will assume that Equality (B.6) holds for some $n \in \mathbb{N}$, then we must proof that it still valid for $n + 1$, *i.e.*,

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} A_{i,2j+1} = \sum_{i=0}^{(n+1)-1} \sum_{j=0}^{(n+1)-1-i} A_{i,2j+1} = \sum_{i=0}^{(n+1)-1} \sum_{j=0}^{i} A_{i-j,1+2j} = \sum_{i=0}^{n} \sum_{j=0}^{i} A_{i-j,1+2j}.$$

In one hand, the right-hand side of the above equality can be expressed as

$$\sum_{i=0}^{n} \sum_{j=0}^{i} A_{i-j,1+2j} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} A_{i,2j+1} + \sum_{j=0}^{n} A_{n-j,1+2j}. \tag{B.7}$$

On the other hand, by the left-hand side of the equality we obtain

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} A_{i,2j+1} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} A_{i,2j+1} + A_{n,1},$$

which can be write as

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} A_{i,2j+1} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} A_{i,2j+1} + \sum_{i=0}^{n-1} A_{i,2(n-i)+1} + A_{n,1}$$

therefore

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} A_{i,2j+1} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} A_{i,2j+1} + \sum_{i=0}^{n} A_{i,2(n-i)+1}. \tag{B.8}$$

Hence, by (B.7) and (B.8), still remains to prove that

$$\sum_{i=0}^{n} A_{n-i,1+2i} = \sum_{j=0}^{n} A_{n-j,1+2j} = \sum_{i=0}^{n} A_{i,2(n-i)+1}$$

so, we notice that when we define

$$B_i = A_{i,2(n-i)+1} \qquad \text{and} \qquad B_{n-i} = A_{(n-i),2(n-(n-i))+1} = A_{n-i,2i+1}$$

it follows that

$$\sum_{i=0}^{n} B_{n-i} = \sum_{i=0}^{n} B_i$$

which is trivial. Then we conclude this demonstration.

Lemma 138

$$\sum_{i=0}^{2k} B_i = \sum_{i=0}^{k} B_{2i} + \sum_{i=1}^{k} B_{2i-1} \tag{B.9}$$

This demonstration it will be given by induction on k. Note that the Equality B.9 is true for $k = 1$, in fact

$$\sum_{i=0}^{2} B_i = B_0 + B_1 + B_2 = \sum_{i=0}^{1} B_{2i} + \sum_{i=1}^{1} B_{2i-1}.$$

Assume the Equality B.9 holds for some $k \in \mathbb{N}$. We will show that its still valid for $k+1$. Thus, consider

$$\sum_{i=0}^{2k+2} B_i = \sum_{i=0}^{2k} B_i + \sum_{i=2k+1}^{2k+2} B_i,$$

follows from induction step that

$$\begin{aligned} \sum_{i=0}^{2k+2} B_i &= \sum_{i=0}^{k} B_{2i} + \sum_{i=1}^{k} B_{2i-1} + B_{2k+1} + B_{2k+2} \\ &= \sum_{i=0}^{k+1} B_{2i} + \sum_{i=1}^{k+1} B_{2i-1}. \end{aligned}$$

Lemma 139

$$\sum_{i=0}^{2k-1} B_i = \sum_{i=0}^{k-1} B_{2i} + \sum_{i=0}^{k-1} B_{2i+1}$$

Lemma 140 *For every $n \in \mathbb{N}$, we have the following equality*

$$\left(\sum_{j=0}^{(n)} \sum_{i=0}^{2(n)-2j} A_{i,j} \right)^2 = \sum_{j=0}^{(n)} \sum_{i=0}^{2(n)-2j} \left(\sum_{k=0}^{(n)} \sum_{l=0}^{2(n)-2k} A_{i,j} A_{l,k} \right)$$

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