UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Attractors for processes on time-dependent spaces

Vitória Soares dos Santos

São Carlos-SP
March 2022

UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Attractors for processes on time-dependent spaces

Vitória Soares dos Santos

Advisor: Prof. Dr. Marcelo José Dias Nascimento

Dissertation submitted to the Departament of Mathematics of Universidade Federal de São Carlos - DM/UFSCar, in partial fulfillment of the requirements for the degree of Master in Mathematics.

São Carlos-SP
March 2022

UNIVERSIDADE FEDERAL DE SÃO CARLOS

Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Matemática

Folha de Aprovação

Assinatura dos membros da comissão examinadora que avaliou e aprovou a Defesa de Dissertação de Mestrado da candidata Vitria Soares dos Santos, realizada em 12/04/2022:

Prof. Dr. Marcelo Jos Dias Nascimento
UFSCar

Severino Horácio da Silva

Prof. Dr. Severino Horácio da Silva
UFCG

Rodiak

Prof. Dr. Rodiak Nicolai Figueroa López
UFSCar

This work is dedicated to
my mom Marlene for her
endless love and support.

Acknowledgments

I would like to thank God Almighty immensely for all my blessings and for the strength to continue pursuing my goals.

I would like to express my sincere gratitude and love...

- to my husband Alexandre, for the love, partnership, patience for all these years and for not letting me give up and always being my support.
- to my sister Amanda and my brother João Vitor, that despite the distance are my comfort and joy;
- to my grandmother Divina, for bringing a lot of love and laughs to my life and being an example of a woman of faith;
- to my stepfather Benjamim, my grandfather José, my cousins, my aunts and my uncles for all the support and moments of joy that we share together;
- to my mother-in-law Bárbara, her mom Anair and her husband Wilson for all the support.

I would also like to express my special thanks...

- to my friends from São Carlos for the company, laughter and games;
- to my Advisor Marcelo for the patience, continuous guidance, and support over these two years;
- to Coordenação de Aperfeiçoamento de Pessoal de Nível Superior Brasil (CAPES) for the financial support;
- to São Paulo Research Foundation (FAPESP) for the financial support under grant 2019/27717-9.

A special and deep gratitude to my grandmother Flora who passed away during my master's degree but was an example in my life, filled me with love, kindness and gentleness.

Resumo

O objetivo desta dissertação de mestrado é estudar o comportamento assintótico de equações diferenciais parciais hiperbólicas não-autônomos com coeficientes dependentes do tempo. Para isso, apresentamos a noção de atratores pullback para processos em espaços dependentes do tempo, onde o operador solução é uma família de aplicações

$$U(t,\tau): X_\tau \to X_t, \quad t \geq \tau \in \mathbb{R},$$

agindo em espaços dependentes do tempo. Além disso, exploramos a minimalidade da propriedade de atração pullback e a existência desses atratores. Finalmente, aplicamos os resultados abstratos de existência de atratores pullback para estudar o comportamento assintótico de equações de onda com velocidade de propagação dependente do tempo.

Palavras-chave: Processos em espaços dependentes do tempo, Atratores pullback, Equações de ondas.

Abstract

The goal of this master's degree dissertation is to study the asymptotic behavior of non-autonomous hyperbolic partial differential equations with time-dependent coefficients. In order to achieve that, we introduce the notion of pullback attractors for processes on time-dependent spaces, where the solution operator is a family of maps

$$U(t,\tau): X_\tau \to X_t, \quad t \geq \tau \in \mathbb{R}$$

acting on a time-dependent family of spaces X_t. Furthermore, we exploiting the minimality with respect to the pullback attraction property and the existence of those attractors. Finally, we applied the abstract results on existence of pullback attractors to study the long term behavior of a damped wave equation with time-dependent speed of propagation.

Keywords: Processes on time-dependent spaces, Pullback attractors, Wave Equations.

Contents

Introduction

Differential Equations are models for several physical, chemical, biological and economic phenomena. If the differential equation involves more than one independent variable, then it is a partial differential equation.

The concept of a dynamical system is of fundamental importance in the understanding of many natural phenomena. Mathematically, a dynamical system is a one-parameter family (generally time) of maps of an abstract space (the set of states) to itself. Many problems in several applied areas, such as physics, chemistry, economics, biology and mechanics, can be classified as dynamical systems. In general, a dynamical system is associated with a differential equation.

Moreover, the evolution of systems arising from mechanics and physics is described in many instances by differential equations of the form

$$\begin{cases} u_t = A(u,t), \quad t > \tau \\ u(\tau) = u_\tau \in X, \end{cases}$$

where X is a normed space and for every fixed t, $A(\cdot,t)$ is a densely operator in X. Assuming the Cauchy problem is well posed and calling $u(t)$ the solution in time t, we can construct the family of solving operators

$$U(t,\tau): X \to X, \quad t \geq \tau \in \mathbb{R},$$

by setting

$$U(t,\tau)u_\tau = u(t).$$

Such family is called a process, characterized by the properties that $U(\tau,\tau) = I_X$ and

$$U(t,s)U(s,\tau) = U(t,\tau), \quad for\ all\ t \geq s \geq \tau \in \mathbb{R}.$$

In order to understand the longtime behavior of solutions to dynamical systems we have to study the dissipative properties of the operators $U(t,\tau)$. A well-established theory of attractors provides a full description of many important autonomous systems from mathematical physics, including non-autonomous models with time-dependent external forces (see [3], [13] and [7] for theoretical background and classical applications).

In evolution problems where the coefficients of the differential operator depend explicitly on time, the standard theory generally fails to capture the dissipation mechanism. For example, consider the following nonlinear damped wave equation in a smooth domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} \varepsilon u_{tt}(x,t) + u_t(x,t) - \Delta u(x,t) + f(u(x,t)) = g(x,t), \\ u(x,t)|_{x\in\partial\Omega} = 0, \end{cases} \tag{1}$$

where $\varepsilon > 0$, f is a nonlinear term and g an external force. If g is independent of time, the system is autonomous and the problem is completely understood within the theory of semigroups. In the nonautonomous case, the dependence of g on time requires further integrability assumptions and the theory of pullback attractors for processes can handle with these situations.

However, if in (1) we assume that ε is not a constant, but rather a positive decreasing function of time $\varepsilon(t)$ vanishing at infinity, the natural energy associated with the system is

$$\mathscr{E}(t) = \int_\Omega |\nabla u(x,t)|^2 dx + \varepsilon(t) \int_\Omega |u_t(x,t)|^2 dx,$$

which exhibits a structural dependence on time. Moreover, the vanishing character of ε at infinity alters the dissipativity of the system and prevents the existence of absorbing sets in the usual sense, that is, bounded sets of the phase space $X = H_0^1(\Omega) \cap L^2(\Omega)$ absorbing all the trajectories after a certain period of time.

In this case, a progress was made in [10]. The authors adopt the point of view of describing the solution operator as a family of maps

$$U(t,\tau) : X_\tau \to X_t, \quad t \geq \tau \in \mathbb{R},$$

acting on a time-dependent family of spaces X_t. For instance, in system (1) all the spaces X_t coincide with the linear space X, that is, are the same linear space. However, the X_t-norm is determinate by the time-dependent energy $\mathscr{E}(t)$ of the solution at time t.

Based on this idea, the paper [10] provides a suitable modification of the classical notion of pullback attractor, establishing a new theory that can handle with time-dependent coefficients of the differential operator in evolution problems. Moreover, the authors apply the framework in a oscillon equation, more specifically, in a Klein-Gordon equation, with nonlinear potential, for a scalar field on a manifold with the Robertson-Walker metric corresponding to an expanding universe with positive Hubble constant. Additionally, the paper [9] based on the abstract theory developed by [10] applied the framework on equation (1) with ε as a time-dependent function, positive decreasing and vanishing at infinity, the same conditions that we refer in the example.

The aim of this master's degree dissertation is to study the long term behavior of a damped wave equation with time-dependent speed of propagation, the same as in [9], and presented the abstract theory for this case. For this, we will study the theoretical background of [10] and [9] exploring the notion of pullback attractor on time-dependent spaces and the minimality with respect to the pullback attraction. Then we apply the abstract result (a result on the existence of pullback attractors) in the non-autonomous case of equation (1).

This dissertation is organized as follows. After this introduction, the subsequent chapter is devoted to recall the notions of process and pullback attractors on fixed metric space X. Also, we will compare these concepts with the semigroup theory. For Chapter 1 the reference used for theoretical background is [5] and [4].

In Chapter 2 we present the abstract theory of pullback attractors on time-dependent spaces based on [10] and [9]. In addition, we will make a comparison in relation to what was done in each article, presenting and explaining the differences between them.

Finally, Chapter 3 is dedicated to apply the abstract results developed in Chapter 2 on the non-autonomous case of (1) with time-dependent speed of propagation, following all the steps presented in [9].

CHAPTER 1

Process on fixed spaces

In this chapter we briefly recall the notions of process and semigroups in a fixed metric space X. Also, we define global attractor for semigroups and pullback attractor for processes presenting a result on the existence of attractors. All discussion is based on Chapters 1 and 2 of [5] (See also [14] and [15]).

1.1 Processes and semigroups

Definition 1.1. A **semigroup** is a one-parameter family of mappings $\{T(t) : X \to X\}_{t\geq 0}$ satisfying the following properties

i) $T(0) = Id_X$;

ii) $T(t)T(s) = T(t+s)$ for $s,t \geq 0$;

iii) $\{t \in \mathbb{R} : t \geq 0\} \ni (t,x) \mapsto T(t)x \in X$ is continuous.

Definition 1.2. A **process** is a two-parameter family of mappings $\{U(t,s) : X \to X\}_{t\geq s}$ with properties

i) $U(t,t) = Id_X$;

ii) $U(t,\tau)U(\tau,s) = U(t,s)$, for all $t \geq \tau \geq s$;

iii) $\{(t,s) \in \mathbb{R}^2 : t \geq s\} \ni (t,s,x) \mapsto U(t,s)x \in X$ is continuous.

The operator $U(t,s)$ takes each state x in X at the initial time s and evolves it to the state $U(t,s)x$ at the time t. Under appropriate assumptions the solutions of the non-autonomous differential equation $\dot{x} = f(t,x)$ will generate a process if one sets $U(t,s)x = x(t,s;x)$, that is, $U(t,s)$ is the solution at time t when $x(s) = x_s$.

Note that, for a fixed τ the operator $U(\tau+s,s)$ will in general be different for each $s \in \mathbb{R}$. Thus, both the initial time s and the elapsed time play an important role in the evolution. A process for which depends only the elapsed time generates an *autonomous process*, which is typically associated with a *semigroup*.

As a convenient shorthand, we will refer to “the process $U(\cdot,\cdot)$” rather than “the process $\{U(t,s) : X \to X\}_{t\geq s}$ in all that follows.

Definition 1.3. Let $\{U(t,s) : X \to X\}_{t\geq s}$ be a process. We say that the process $U(\cdot,\cdot)$ is autonomous if $U(t,s) = U(t-s,0)$, for every $t,s \geq 0$.

Remark 1.4. Given a semigroup $\{T(t) : X \to X\}_{t\geq 0}$ define $U(t,s) = T(t-s)$, $t \geq s$. Then $\{U(t,s) : X \to X\}_{t\geq s}$ is an autonomous process. Conversely, an autonomous process $\{U(t,s) : X \to X\}_{t\geq s}$ defines a *semigroup* given by $T(t) = U(t,0)$, $t \geq 0$.

1.2 Attractors

Ideally, the attractor of a given dynamical system should contain all the assymptotic dynamics. The theory of attractors for autonomous systems is well-established, see [3], [13] and [7] for theoretical background and classical applications. In the non-autonomous case the theory is still under development, however we have solid references, e.g., [5] and [6].

Before defining the concept of *attraction*, we will recall some definitions and properties.

Definition 1.5. For a Banach space X and $A,B \subset X$, we define the **Hausdorff semidistance** as

$$dist_X(A,B) = \sup_{x\in A} d_X(x,B)$$

where

$$d_X(x,B) = \inf_{y\in B} \|y-x\|_X.$$

Remark 1.6. Observe that $dist(A,B) \neq dist(B,A)$. In fact, consider $X = \mathbb{R}$, $A,B \subset X$ such that $A = \{1,2\}$ and $B = \{1\}$. Then,

$$\begin{aligned} dist(B,A) &= \sup_{b\in B} \inf_{a\in A} \|b-a\|_X \\ &= \sup_{b\in B} \inf\{|1-2|,|1-1|\} = \{0\} \end{aligned}$$

and

$$dist(A,B) = \max\{|1-2|,|1-1|\} = 1. \tag{1.1}$$

Lemma 1.7. *For a Banach space X and $A,B \subset X$, it follows that $dist_X(A,B) = 0$ if and only if A is contained in the closure of B.*

Proof. Suppose that $dist_X(A,B) = 0$. For $a \in A$ we have $\inf_{b\in B} \|b-a\|_X = 0$. Hence, there exists a sequence $\{b_n\}_{n\in\mathbb{N}} \subset B$ such that

$$\|b_n - a\|_X \overset{n\to\infty}{\longrightarrow} 0.$$

Thus, $a \in \overline{B}$ and consequently $A \subset \overline{B}$.

Conversely, assume that $A \subset \overline{B}$. Thus, for every $a \in A$ there exists $\{b_n\}_{n\in\mathbb{N}} \subset B$ such that

$$\|b_n - a\|_X \overset{n\to\infty}{\longrightarrow} 0$$

and from the definition of infimum we obtain $\inf_{b\in B} \|b-a\|_X = 0$, for all $a \in A$. Henceforth, the lemma follows by taking the supremum over A.

□

Example 1.8. Consider $X = \mathbb{R}$, $A, B \subset X$ such that $A = (-1,1)$ and $B = \{1\}$. Then,

$$\begin{aligned} dist(B,A) &= \sup_{b\in B} \inf_{a\in A} \|b-a\|_X \\ &= \sup_{b\in B} \inf\{|1-(-1)|, |1-1|\} = \{0\}. \end{aligned}$$

Hence, $B \subset \overline{A} = [-1,1]$.

Definition 1.9. Let X be a normed space. For any given $\varepsilon > 0$, the **ε-neighborhood** of a set $B \subset X$ is defined as

$$\mathscr{O}^{\varepsilon}(B) = \{y \in X : \inf_{x\in B} \|x-y\|_X < \varepsilon\}.$$

Remark 1.10. Notice that if $A, B \subset X$ then

$$dist_X(A,B) < \varepsilon \Leftrightarrow \sup_{x\in A} \inf_{y\in B} \|y-x\|_X < \varepsilon \Leftrightarrow \inf_{y\in B} \|y-x\|_X < \varepsilon, \quad \forall x \in A.$$

Which means that $A \subset \mathscr{O}^{\varepsilon}(B)$.

1.2.1 Global attractors for semigroups

Definition 1.11. Let B and C be subsets of X. We say that B **attracts** C if $dist_X(T(t)C, B) \to 0$, as $t \to +\infty$.

Remark 1.12. The attraction property can be equivalently stated in terms of $\varepsilon-$neighborhood. In fact, if

$$\lim_{t\to\infty} dist_X(T(t)C, B) = 0,$$

i.e., given $\varepsilon > 0$ there exists $s = s(\varepsilon, C) > 0$ such that

$$dist_X(T(t)C, B) < \varepsilon, \quad for\ all\ t \geq s. \tag{1.2}$$

It follows from (1.2) and Remark 1.10 that

$$T(t)C \subset \mathscr{O}^{\varepsilon}(B), \quad for\ all\ t \geq s.$$

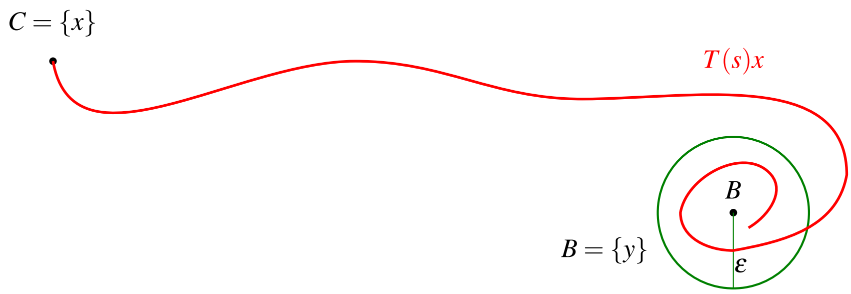

Figure 1.1: B attracts C.

Definition 1.13. A subset A of X is called **invariant** under $\{T(t) : X \to X\}_{t\geq 0}$ if $T(t)A = A$, for all $t \geq 0$.

Definition 1.14. A set $\mathscr{A} \subset X$ is called the **global attractor** for a semigroup $\{T(t) : X \to X\}_{t\geq 0}$ if

i) $\mathscr{A}$ is compact;

ii) $\mathscr{A}$ is invariant;

iii) $\mathscr{A}$ attracts each bounded subset of X.

Lemma 1.15. *The global attractor $\mathscr{A}$ of a semigroup is the minimal closed set that attracts each bounded subset of X. In particular, the global attractor is unique.*

Proof. Let B be a closed set that attracts each bounded subset of X. In particular, B attracts $\mathscr{A}$, and so, since $\mathscr{A} = T(t)\mathscr{A}$ for all $t \geq 0$

$$dist_X(\mathscr{A}, B) = \lim_{t\to+\infty} dist_X(T(t)\mathscr{A}, B) = 0.$$

Then, $\mathscr{A} \subset \overline{B} = B$, since B is closed. □

1.2.2 Pullback attractors

Now, we recall the notion of *pullback attraction*.

Definition 1.16. Given $t \in \mathbb{R}$, a set $K \subset X$ **pullback attracts** a set D at time t under the process $U(\cdot,\cdot)$ if

$$\lim_{s\to-\infty} dist_X(U(t,s)D, K) = 0, \tag{1.3}$$

K pullback attracts bounded sets at time t if (1.3) holds for each bounded set D of X.

A time-dependent family of subsets of X, $K(\cdot)$, pullback attracts bounded subsets of X under $U(\cdot,\cdot)$ if $K(t)$ pullback attracts bounded sets at time t under $U(\cdot,\cdot)$, for each $t \in \mathbb{R}$.

Remark 1.17. The pullback attraction property can be equivalently stated in terms of $\varepsilon-$neighborhood. Indeed, for fixed $t \in \mathbb{R}$, if

$$\lim_{s \to -\infty} dist_X(U(t,s)D, K(t)) = 0,$$

i.e., given $\varepsilon > 0$ there exists $s_0 = s_0(t,D) \leq t$ such that

$$dist_X(U(t,s)D, K(t)) < \varepsilon, \quad for\ all\ s \leq s_0. \tag{1.4}$$

It follows from (1.4) and Remark 1.10 that

$$U(t,s)D \subset \mathscr{O}^{\varepsilon}(K(t)), \quad for\ all\ s \leq s_0.$$

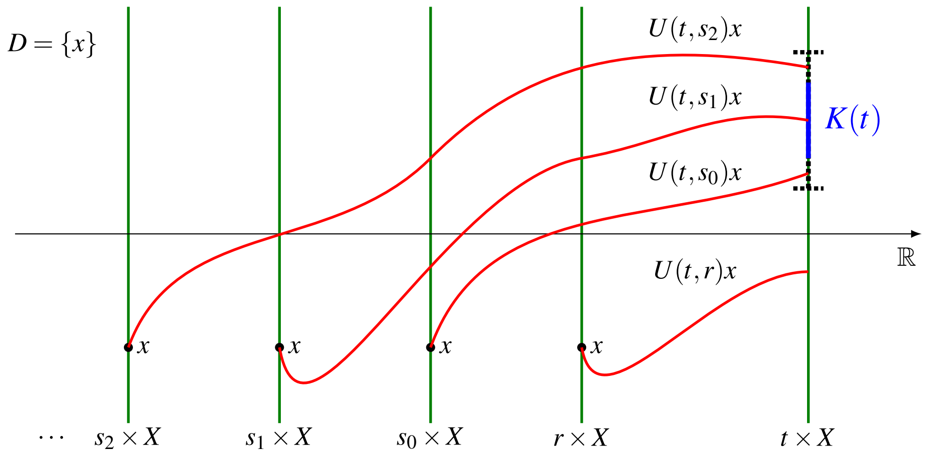

Figure 1.2: $K(t)$ pullback attracts D for every time t.

Definition 1.18. A time-dependent family of sets $\mathscr{D} = \{\mathscr{D}(t)\}_{t\in\mathbb{R}}$ is **invariant** under $U(\cdot,\cdot)$ if

$$U(t,s)\mathscr{D}(s) = \mathscr{D}(t), \quad for\ all\ t,s \in \mathbb{R}\ with\ t \geq s.$$

Definition 1.19. A family $\{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ is the **pullback attractor** for a process $U(\cdot,\cdot)$ if

i) $\mathscr{A}(t)$ is compact for each $t \in \mathbb{R}$;

ii) $\mathscr{A}(\cdot)$ is invariant with respect to $U(\cdot,\cdot)$;

iii) $\mathscr{A}(\cdot)$ pullback attracts bounded sets of X;

iv) $\mathscr{A}(\cdot)$ is the minimal family of closed sets with property (iii).

Example 1.20. Consider the following scalar equation

$$\dot{x} = -\alpha x + f(t) \qquad x(s) = x_0. \tag{1.5}$$

This equation has a explicit solution

$$x(t) = e^{-\alpha(t-s)}x(s) + \int_s^t e^{-\alpha(t-r)} f(r)dr,$$

so if the limit

$$\lim_{s\to-\infty} \int_s^t e^{-\alpha(t-r)} f(r)dr$$

exists, then $\{x^*(t) : t \in \mathbb{R}\}$ is the pullback attractor for (1.5), where

$$x^*(t) := \lim_{s\to-\infty} \int_s^t e^{-\alpha(t-r)} f(r)dr. \tag{1.6}$$

The following result shows that the global attractor for a semigroup $\{T(t) : t \geq 0\}$ and the pullback attractor for $\{U(t,s) = T(t-s) : t \geq s\}$ are essentially the same.

Proposition 1.21. *Let $\{T(t) : t \geq 0\}$ be a semigroup and $\{U(t,s) = T(t-s) : t \geq s\}$ be the corresponding autonomous process. $\{T(t) : t \geq 0\}$ has a global attractor $\mathscr{A}$ if and only if $\{U(t,s) = T(t-s) : t \geq s\}$ has a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$, and then $\mathscr{A}(t) = \mathscr{A}$ for all $t \in \mathbb{R}$.*

Proof. If $T(t)$ has a global attractor $\mathscr{A}$ we claim that $\mathscr{A}(t) = \mathscr{A}$ defines the pullback attractor of $U(\cdot,\cdot)$. Indeed, $U(t,s)\mathscr{A} = T(t-s)\mathscr{A} = \mathscr{A}$ and given a bounded subset B of X we have for each t

$$\lim_{s\to-\infty} dist_X(U(t,s)B, \mathscr{A}) = \lim_{s\to-\infty} dist_X(T(t-s)B, \mathscr{A}) = 0.$$

On the other hand, assume that U has a pullback attractor $\mathscr{A}(\cdot)$. Then given a bounded subset B of X we have

$$\lim_{s\to-\infty} dist_X(T(t-s)B, \mathscr{A}(\tau)) = 0, \ \forall t, \tau \in \mathbb{R}.$$

Thus, given $\tau \in \mathbb{R}$, by minimality $\mathscr{A}(t) = \mathscr{A}(\tau)$, for all $t \in \mathbb{R}$. In this way, it is straightforward to verify that $\mathscr{A} := \mathscr{A}(0)$ is a global attractor for U. □

1.2.3 Existence of pullback attractors

First, we will define the ω-limit set.

Definition 1.22. The **pullback ω-limit** of a subset B of X is the family $\omega_{\mathscr{B}} = \{\omega_B(t) \subset X\}_{t\in\mathbb{R}}$, where $\omega_B(t)$ is defined as

$$\omega_B(t) = \bigcap_{\tau\leq t} \overline{\bigcup_{s\leq\tau} U(t,s)B}^X.$$

An equivalent characterization of the ω-limit set is the following proposition.

Proposition 1.23. *Let B as in the definition above, then*

$$\omega_B(t) = \{z \in X : \text{there are sequences } s_n \to -\infty \text{ as } n \to +\infty \text{ and } z_n \in B \text{ such that } U(t,s_n)z_n \to z \text{ as } n \to +\infty\}.$$

The proof is similar to the proof of Proposition 2.12 and will be omitted.

Remark 1.24. Let $T(\cdot)$ be a semigroup and $U(\cdot,\cdot)$ be the associated evolution process. Then, $\omega_B(t)$ is independent of t and coincides with the definition of the ω-limit set for semigroups (see [13] and [17]):

$$\omega_B(t) = \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s} T(t)B}. \tag{1.7}$$

Definition 1.25. Consider

$$\mathbb{K} = \{K = \{K(t)\}_{t\in\mathbb{R}} : K(t)\subset X \ \ compact,\ K \ \ pullback \ \ attracts\}. \tag{1.8}$$

We say that the process $U(\cdot,\cdot)$ is **asymptotically compact** if $\mathbb{K}\neq\emptyset$.

Lemma 1.26. *If $U(\cdot,\cdot)$ is an asymptotically compact process, then for any bounded sequence $\{x_n\}$, $s_n\to-\infty$, the sequence $\{U(t,s_n)x_n\}$ has a convergent subsequence.*

Proof. Let $\{K(t) : t\in\mathbb{R}\}$ be a family of compact sets that pullback attracts bounded subsets of X. Take sequences $s_k\leq t$ with $s_k\to-\infty$ and $\{x_k\}\in X$ contained in a bounded set B. Since $K(t)$ is compact and $dist_X(U(t,s_k)x_k, K(t))\to 0$ as $s_k\to-\infty$ it is straightforward to show that $\{U(t,s_k)x_k\}$ has a convergent subsequence. □

Remark 1.27. In [5, Definition 2.8] the definition of asymptotically compactness is given by sequences as in the lemma above.

Lemma 1.28. *Let $U(\cdot,\cdot)$ be an asymptotically compact process and B a bounded subset of X. Then*

i) $\omega_B(t)$ *is a non-empty compact set for every* $t\in\mathbb{R}$.

ii) $\omega_B(t)$ *pullback attracts B at time t;*

iii) $\{\omega_B(t) : t\in\mathbb{R}\}$ *is invariant;*

iv) $\omega_B(t)$ *is the minimal family of closed sets that pullback attracts B at time t.*

Proof. i) First, note that there exists a time s_0 such that

$$\overline{\bigcup_{s\leq s_0} U(t,s)B} \tag{1.9}$$

is bounded. If not, there would exist a sequence $s_k\to-\infty$ and a sequence $\{x_k\}\in B$ such that $\{U(t,s_k)x_k\}$ is unbounded, which would contradict the asymptotic compactness.

Now, for any sequences $\{x_k\}\in B$ and $s_k\leq s_0$ with $s_k\to-\infty$ as $k\to+\infty$, it follows from the fact that $U(\cdot,\cdot)$ is asymptotically compact that there exists a convergent subsequence of $\{U(t,s_k)x_k\}$ that converges to some $y\in\omega_B(t)$.

Let us show that $\omega_B(t)$ is compact for every $t \in \mathbb{R}$. Given $\{y_k\} \subset \omega_B(t)$, there are $x_k \in B$ and $s_k \leq \min\{s_0, -k\}$, such that

$$d(U(t,s_k)x_k, y_k) < 1/k.$$

Since $\{U(t,s_k)x_k\}$ has a subsequence that converges to an element y of $\omega_B(t)$, it follows that y_k has a subsequence that converges to y, and hence $\omega_B(t)$ is compact.

ii) We will prove item ii) by contradiction. Indeed, suppose that there exists an $\varepsilon > 0$, a sequence $s_n \to -\infty$ and a sequence $\{x_n\} \in B$ such that

$$dist_X(U(t,s_n)x_n, \omega_B(t)) > \varepsilon, \quad for\ \ all \quad n \in \mathbb{N}. \tag{1.10}$$

On the other hand, since $U(\cdot,\cdot)$ is asymptotically compact, the sequence $\{U(t,s_n)x_n\}$ has a convergent subsequence whose limit belongs to $\omega_B(t)$, contradicting (1.10).

iii) We will prove that $U(t,s)\omega_B(s) = \omega_B(t), t \geq s$. In order to do this, we started with $U(t,s)\omega_B(s) \subset \omega_B(t), t \geq s$. Indeed, if $\omega_B(s) = \emptyset$ there is nothing to show. If $y \in \omega_B(s)$, then there exist $\{x_n\} \in B$ and $s_n \leq s$ such that $s_n \to -\infty$ and $y = \lim_{n\to\infty} U(s,s_n)x_n$. Thus, by continuity, we obtain

$$U(t,s)y = \lim_{n\to\infty} U(t,s_n)x_n \in \omega_B(t).$$

Conversely, let $y \in \omega_B(t)$, then there exist $x_n \in B$ and $s_n \to -\infty$ such that $y = \lim_n U(t,s_n)x_n$. There exists $n_0 \in \mathbb{N}$ such that $s_n \leq s$ for any $n \geq n_0$. Since $U(\cdot,\cdot)$ is asymptotically compact, the sequence $\{U(s,s_n)x_n\}$ has a convergent subsequence with limit $x \in \omega_B(s)$. By continuity, we see that $U(t,s)x = y$ and we conclude that $\omega_B(t) \subset U(t,s)\omega_B(s)$.

iv) Let $\{F(t) : t \in \mathbb{R}\}$ be a family of compact set that pullback attracts B. By contradiction, assume that there exists $y \in \omega_B(t)$ and $y \notin F(t)$. Thus there exists $\delta > 0$ such that

$$d(y, F(t)) > 2\delta.$$

There exist $x_n \in B$ and $s_n \to -\infty$ such that $U(t,s_n)x_n \to y$ as $n \to +\infty$. On the other hand, $F(t)$ attracts B at time t, so there exists s_0 such that

$$U(t,s)B \subset \mathscr{O}^\delta(F(t)),$$

for every $s \leq s_0$. Hence $d(y,F(t)) \leq \delta$ which is a contradiction. Therefore $\omega_B(t) \subset F(t)$ for every $t \in \mathbb{R}$.

□

Theorem 1.29. *Let $U(\cdot,\cdot)$ be an asymptotically compact process. Then there exists a pullback attractor $\{\mathscr{A}(t)\}_{t\in\mathbb{R}}$. In fact, for each $t \in \mathbb{R}$*

$$\mathscr{A}(t) = \overline{\bigcup\{\omega_B(t) : B \subset X,\ B\ bounded\ \}}. \tag{1.11}$$

Proof. Let $\{K(t) : t \in \mathbb{R}\}$ be a family of compact sets that attracts every bounded subset of X. Define for each $t \in \mathbb{R}$, $\mathscr{A}(t)$ as above. Then, from Lemma 1.28 item iv), we have $\omega_B(t) \subset K(t)$ for every bounded set B in X. Thus $\mathscr{A}(t) \subset K(t)$ and so $\mathscr{A}(t)$ is a compact set for each t. Furthermore, from Lemma 1.28 item ii) $\mathscr{A}(t)$ pullback attracts every bounded subset B of X at time t. The minimality also follows from Lemma 1.28 item iv), so we only have to prove the invariance of the family $\{\mathscr{A}(t) : t \in \mathbb{R}\}$.

Since $U(t,s)$ is a continuous map for each $t \geq s$ and $\{\omega_B(t) : t \in \mathbb{R}\}$, is invariant, we have that

$$U(t,s)\mathscr{A}(s) \subset \overline{\bigcup\{U(t,s)\omega_B(s) : B \subset X,\ B \text{ bounded }\}} = \mathscr{A}(t). \tag{1.12}$$

Let $y \in \mathscr{A}(t)$, then there exists $y_n \in \omega_{B_n}(t)$ with $y_n \to y$ as $n \to +\infty$. Thus there exists $x_n \in \omega(B_n,s)$ such that $U(t,s)x_n = y_n$. Since $x_n \in \mathscr{A}(s)$ and $\mathscr{A}(s)$ is compact, there is a subsequence x_{n_j} that converges to some $x_0 \in \mathscr{A}(s)$, for which $U(t,s)x_0 = \lim_{j\to\infty} U(t,s_{n_j})x_{n_j} = \lim_{j\to\infty} y_{n_j} = y$. It follows that $U(t,s)\mathscr{A}(s) \supset \mathscr{A}(t)$. This concludes the proof. □

Corollary 1.30. *Let $T(\cdot)$ be a semigroup in a metric space X and assume that there exists a compact set K that attracts every bounded subset of X. Then $T(\cdot)$ has a global attractor, and in this case $\mathscr{A} = \omega_K$.*

Proof. It follows from Theorem 1.29 and Proposition 1.21 that

$$\mathscr{A} = \overline{\bigcup\{\omega_B : B \subset X,\ B \text{ bounded }\}}. \tag{1.13}$$

is the global attractor for $T(\cdot)$. It is immediate from this that $\omega(K) \subseteq \mathscr{A}$, while, since K attracts bounded subsets of X, we must have $\omega_B \subseteq \omega_K$ for all bounded subsets B of X, which completes the proof. □

CHAPTER 2

Process on time-dependent spaces

The aim of this chapter is to study the abstract theory for processes $U(t,s): X_s \to X_t$ on time-dependent spaces. Henceforth, we will apply the results in the study of the asymptotic behavior of a wave equation with a time-dependent propagation coefficient.

To this end, we will study the abstract theory developed in the papers [10] and [9] and we will make a comparison between them. However, we will use the theory in [9] for the application in Chapter 3.

First, we start defining processes on time-dependent spaces.

Definition 2.1. For $t \in \mathbb{R}$, let X_t be a family of Banach spaces. A **process** is a two-parameter family of mappings $\{U(t,s): X_s \to X_t\}_{t \geq s}$ with the following properties:

(i) $U(t,t) = Id_{X_t}$;

(ii) $U(t,\tau)U(\tau,s) = U(t,s)$ for $t \geq \tau \geq s$.

If, in addition $U(t,s) \in \mathscr{C}(X_s, X_t)$, then the process is called a **continuous process** .

Remark 2.2. Note that the spaces X_t are all the same vector space with the norms $\|\cdot\|_{X_t}$ and $\|\cdot\|_{X_s}$, equivalent for any $t,s \in \mathbb{R}$. Whereas the equivalence blows up as we let $s,t \to +\infty$. However, this is not necessary for most of the theory we develop hereafter in this chapter. We will explicitly use this fact in Chapter 3. This remark will be further discussed in Chapter 3.

2.1 Attractors in time-dependent spaces

This section is devoted to present the abstract theory of [10]. For this, we will consider the process $U(\cdot,\cdot)$ with the continuity condition for all that follows in this section.

2.1.1 Pullback sets

Definition 2.3. A family of subsets $\mathscr{B} = \{\mathscr{B}(t) \subset X_t\}_{t\in\mathbb{R}}$ is **pullback-bounded** if

$$\mathscr{R}(t) = \sup_{s\in(-\infty,t]} \|\mathscr{B}(s)\|_{X_s} < \infty, \qquad for\ all\ t \in \mathbb{R},$$

that is, the function $s \mapsto \|\mathscr{B}(s)\|_{X_s}$ is bounded on $s \in (-\infty,t]$ for every $t \in \mathbb{R}$.

Definition 2.4. A pullback-bounded family $\mathbb{A} = \{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ is **pullback absorber** if for every pullback-bounded family $\mathscr{B}$ and for every $t \in \mathbb{R}$ there exists $s_0 = s_0(t,\mathscr{B}) \le t$ such that

$$U(t,s)\mathscr{B}(s) \subset \mathbb{A}(t), \qquad for\ all\ s \le s_0.$$

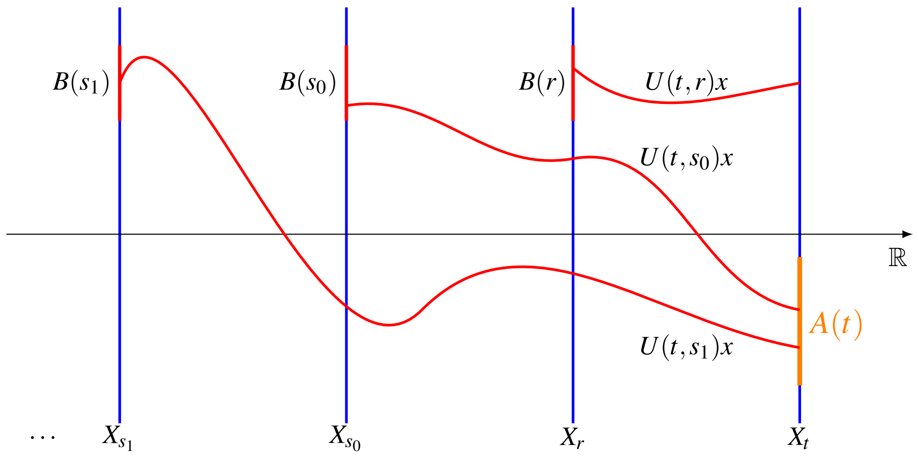

Figure 2.1: $A = \{A(t)\}$ is pullback absorber.

Definition 2.5. A family $\mathscr{D} = \{\mathscr{D}(t)\}_{t\in\mathbb{R}}$ is called **pullback attracting** if

$$\lim_{s\to-\infty} dist_{X_t}(U(t,s)\mathscr{B}(s), \mathscr{D}(t)) = 0, \tag{2.1}$$

for every pullback-bounded family $\mathscr{B} = \{\mathscr{B}(t)\}_{t\in\mathbb{R}}$ and every $t \in \mathbb{R}$.

For any $\varepsilon > 0$, the **ε-neighborhood** of a set $B \subset X_t$ is defined as

$$\mathscr{O}_t^{\varepsilon}(B) = \bigcup_{x\in B} \{y \in X_t : \|x-y\|_{X_t} < \varepsilon\} = \{y \in X_t : \inf_{x\in B} \|x-y\|_{X_t} < \varepsilon\}.$$

Remark 2.6. The pullback attracting property can be equivalently stated in terms of $\varepsilon-$neighborhood. Indeed, if

$$\lim_{s\to-\infty} dist_{X_t}(U(t,s)\mathscr{B}(s), \mathscr{D}(t)) = 0,$$

i.e., given $\varepsilon > 0$ exists $t_0 = t_0(t,\mathscr{B}) \le t$ such that

$$dist_{X_t}(U(t,s)\mathscr{B}(s), \mathscr{D}(t)) < \varepsilon, \quad for\ all\ s \le t_0, \tag{2.2}$$

we obtain from (2.2) and Remark 1.10 that

$$U(t,s)\mathscr{B}(s) \subset \mathscr{O}_t^{\varepsilon}(\mathscr{D}(t)), \quad for\ all\ s \leq t_0.$$

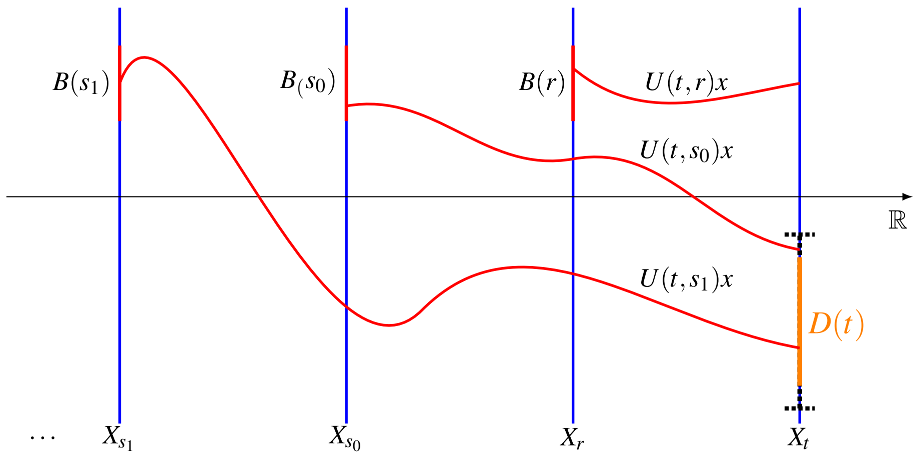

Figure 2.2: $D = \{D(t)\}$ is pullback attracting.

2.1.2 Invariant pullback attractor

In this subsection, our goal is to define the invariant pullback attractor for a process in a family of normed spaces $\{X_t\}_{t\in\mathbb{R}}$.

Definition 2.7. A family of subsets $\mathscr{A} = \{\mathscr{A}(t) \subset X_t\}_{t\in\mathbb{R}}$ is an **invariant pullback attractor** for the process $U(\cdot,\cdot)$ if it fulfills the following properties:

(i) $\mathscr{A}(t)$ is compact for every $t \in \mathbb{R}$;

(ii) $\mathscr{A}(\cdot)$ is invariant;

(iii) $\mathscr{A}(\cdot)$ is pullback attracting;

iv) $\mathscr{A}(\cdot)$ is the minimal family of closed sets with property (iii).

If property iii) holds uniformly with respect to $t \in \mathbb{R}$, $\mathscr{A}$ is a **uniform invariant pullback attractor**.

Remark 2.8. In [5] the pullback attractor is always invariant. However, in this chapter we will differentiate invariant pullback attractors because for non-continuous processes we do not require invariance in the definition of pullback attractors as in Definition 2.30.

Remark 2.9.

1. In general, conditions (i)-(iii) are not sufficient to guarantee the uniqueness of the invariant pullback attractor. For example, consider the ordinary differential equation

$$\dot{y}+y=0,$$

the process it generates is

$$U(t,s)x = xe^{-(t-s)}.$$

Observe that the process $U(\cdot,\cdot)$ has infinitely many invariant pullback attractors in the sense of the definition above; they are given by

$$\mathscr{A}_c = \{\mathscr{A}_c(t) = ce^{-t}\}_{t\in\mathbb{R}},$$

with $c\in\mathbb{R}$. However, only $\mathscr{A}_0$ is also a pullback-bounded family.

2. For uniqueness, instead of (iv) we could require

(iv)' $\mathscr{A}$ is a pullback-bounded family,

then it is possible to prove that there exists at most one family satisfying (i)-(iii), and $(iv)'$, that is, a pullback-bounded invariant pullback attractor is unique in the class of pullback-bounded families. This will be prove in the following proposition.

Proposition 2.10. *If a family of sets $\mathscr{A}$ satisfies conditions (i)-(iii) of Definition 2.7 and $\mathscr{A}$ is a pullback-bounded family, then $\mathscr{A}$ is the unique invariant pullback attractor for the process $U(\cdot,\cdot)$.*

Proof. Indeed, suppose that $\mathscr{A}_1$ and $\mathscr{A}_2$ are two pullback-bounded families satisfying (i)-(iii) of Definition 2.7. Fix $t\in\mathbb{R}$ and notice that from the pullback attraction property of $\mathscr{A}_1$ we obtain $dist_{X_t}(U(t,s)\mathscr{A}_2(s),\mathscr{A}_1(t))\to 0$ as $s\to-\infty$. However, since $\mathscr{A}_2$ is invariant we have $U(t,s)\mathscr{A}_2(s) = \mathscr{A}_2(t)$. Furthermore, since $\mathscr{A}_1(t)$ is closed, it follows from Lemma 1.7 that $\mathscr{A}_2(t)\subset\mathscr{A}_1(t)$. Similarly, exchanging the roles of $\mathscr{A}_1(t)$ and $\mathscr{A}_2(t)$ we obtain the reverse inclusion. Therefore, $\mathscr{A}_1(t)=\mathscr{A}_2(t)$ for each $t\in\mathbb{R}$ the result is proved. □

2.1.3 Time-dependent ω-limit

In the present section, we will introduce the notion of an ω-limit family in time-dependent spaces and prove a characterization for this set.

Definition 2.11. The **time-dependent $\omega-$limit** of a family of sets $\mathscr{B}=\{\mathscr{B}(t)\subset X_t\}_{t\in\mathbb{R}}$ is the family $\omega_{\mathscr{B}}=\{\omega_{\mathscr{B}}(t)\subset X_t\}_{t\in\mathbb{R}}$, where $\omega_{\mathscr{B}}(t)$ is defined as

$$\omega_{\mathscr{B}}(t) = \bigcap_{\tau\le t}\overline{\bigcup_{s\le\tau}U(t,s)\mathscr{B}(s)}^{X_t}.$$

An equivalent characterization of the $\omega-limit$ family is the following proposition.

Proposition 2.12. *Let $\mathscr{B}$ as in Definition 2.11, then*

$$\omega_{\mathscr{B}}(t) = \{z \in X_t : \text{ there are sequences } s_n \to -\infty \text{ as } n \to \infty \text{ and } z_n \in \mathscr{B}(s_n), \text{ such that } \|U(t,s_n)z_n - z\|_{X_t} \to 0 \text{ as } n \to \infty\}.$$

Proof. Fix $t \in \mathbb{R}$ and define

$$W(t) = \{z \in X_t : \text{ there are sequences } s_n \to -\infty \text{ as } n \to \infty \text{ and } z_n \in \mathscr{B}(s_n), \text{ such that } \|U(t,s_n)z_n - z\|_{X_t} \to 0 \text{ as } n \to \infty\}.$$

We want to show that $W(t)$ is equivalent to the Definition 2.11. Indeed, let $z \in \omega_{\mathscr{B}}(t)$ then

$$z \in \overline{\bigcup_{s \leq \tau} U(t,s)\mathscr{B}(s)}^{X_t}, \quad for\ all\ \ \tau \leq t.$$

In particular, for each $n \in \mathbb{N}$, we have

$$z \in \overline{\bigcup_{s \leq t-n} U(t,s)\mathscr{B}(s)}^{X_t}.$$

Consequently, there exists

$$y_n \in \bigcup_{s \leq t-n} U(t,s)\mathscr{B}(s)$$

such that $\|y_n - z\|_{X_t} < \frac{1}{n}$. Furthermore, there are $s_n \leq t-n$ and $z_n \in \mathscr{B}(s_n)$ such that $y_n = U(t,s_n)$. Therefore, $s_n \to -\infty$ as $n \to \infty$ and

$$\|U(t,s_n)z_n - z\|_{X_t} \to 0 \quad as \quad n \to \infty,$$

which implies $z \in W(t)$.

Conversely, suppose that $z \in W(t)$. Hence, there exists $s_n \to -\infty$ as $n \to \infty$ and $z_n \in \mathscr{B}(s_n)$ such that

$$\|U(t,s_n)z_n - z\|_{X_t} \to 0 \quad as \quad n \to \infty.$$

If $\tau \leq t$, then there is $n(\tau) \in \mathbb{N}$ such that $s_{n(\tau)} \leq \tau$. Consequently,

$$U(t,s_n)z_n \in \bigcup_{s \leq \tau} U(t,s)\mathscr{B}(s),\ for\ all\ n \geq n(\tau).$$

Therefore,

$$z \in \overline{\bigcup_{s \leq \tau} U(t,s)\mathscr{B}(s)}^{X_t}.$$

Since $\tau \leq t$ is arbitrary, we obtain $z \in \omega_{\mathscr{B}}(t)$ and this completes the proof. □

Definition 2.13. Consider

$$\mathbb{K} = \{\mathscr{K} = \{\mathscr{K}(t)\}_{t \in \mathbb{R}} : \mathscr{K}(t) \subset X_t\ \ compact,\ \mathscr{K}\ \ pullback\ \ attracting\}. \tag{2.3}$$

We say that the process $U(\cdot,\cdot)$ is **asymptotically compact** if $\mathbb{K} \neq \emptyset$.

Remark 2.14. Observe that this is the same definition as in Chapter 1.

Lemma 2.15. *Let $U(\cdot,\cdot)$ be an asymptotically compact process and $\mathscr{B}$ a pullback-bounded family. Then*

i) $\omega_{\mathscr{B}}(t)$ *is a non-empty compact set for every* $t \in \mathbb{R}$.

ii) $\omega_{\mathscr{B}}(t)$ *pullback attracts* $\mathscr{B}$ *at time* t;

iii) $\{\omega_{\mathscr{B}}(t) : t \in \mathbb{R}\}$ *is invariant;*

iv) $\omega_{\mathscr{B}}(t)$ *is the smallest family of closed sets that pullback attracts* $\mathscr{B}$ *at time* t.

Proof. The proof is the same of Lemma 1.28, with minimal changes. □

2.1.4 Existence of the invariant pullback attractor

This section is devoted to prove a result on the existence of the invariant pullback attractor (Theorem 2.18). However, for do this we will use the Kuratowski measure of noncompactness.

Definition 2.16. For a Banach space X and $D \subset X$, the **Kuratowski measure of noncompactness** is defined by

$$\alpha(D) = \inf\{d > 0 : D \ has \ a \ finite \ cover \ of \ diameter \ less \ than \ d\}.$$

Hereafter we recall some properties of the Kuratowski measure we are going to use, redirecting to [13] for more details and proofs:

(K.1) $\alpha(D) = \alpha(\overline{D})$.

(K.2) If $r_0 \in \mathbb{R}$ and $\{\mathscr{U}_r\}_{r \geq r_0}$ is a family of nonempty closed subsets of X such that $\mathscr{U}_{r_2} \subset \mathscr{U}_{r_1}$ for every $r_2 > r_1 \geq r_0$ and $\lim_{r \to \infty} \alpha(\mathscr{U}_r) = 0$, then $\mathscr{U} = \bigcap_{r \geq r_0} \mathscr{U}_r$ is nonempty and compact.

(K.3) Let $\{\mathscr{U}_r\}_{r \geq r_0}$ be as in (K.2) and let any two sequences $r_n \to \infty$ and $x_n \in \mathscr{U}_{r_n}$ be given. Then, x_n possesses a subsequence that converges to some $x \in \mathscr{U}$.

Remark 2.17. The shorthand α_t stands for the Kuratowski measure in the space X_t. Notice that, for fixed $s, t \in \mathbb{R}$, α_s and α_t are equivalent measures of noncompactness whenever there is a Banach space isomorphism between X_s and X_t.

Theorem 2.18. *Assume that the process $U(\cdot,\cdot)$ possesses an absorber family $\mathbb{A}(\cdot)$ such that*

$$\lim_{s \to -\infty} \alpha_t(U(t,s)\mathbb{A}(s)) = 0, \quad for \ all \ t \in \mathbb{R}. \tag{2.4}$$

Then, $\mathscr{A}(\cdot) = \omega_{\mathbb{A}}(\cdot)$ is an invariant pullback attractor for $U(\cdot,\cdot)$. Furthermore, for each $t \in \mathbb{R}$

$$\mathscr{A}(t) = \bigcup\{\omega_{\mathscr{B}}(t) : \mathscr{B} \ is \ a \ pullback\text{-}bounded \ family \ \}. \tag{2.5}$$

Proof. First, we will prove the compacteness of $\omega_{\mathbb{A}}(t)$ for each $t \in \mathbb{R}$. Indeed, fix $t \in \mathbb{R}$. Given $\varepsilon > 0$, it follows from (2.4) that there exists $t_0 \le t$ such that $\alpha_t(U(t,s)\mathbb{A}(s)) < \varepsilon$, whenever $s \le t_0$. Moreover, from the absorber property of $\mathbb{A}$, we can find $s_0 \le t_0$ satisfying

$$U(t_0,s)\mathbb{A}(s) \subset \mathbb{A}(t_0) \quad for\ \ every\ \ s \le s_0.$$

Hence, for $\tau \le s_0$ we have

$$\bigcup_{s\le\tau} U(t,s)\mathbb{A}(s) = \bigcup_{s\le\tau} U(t,t_0)U(t_0,s)\mathbb{A}(s) \subset \bigcup_{s\le\tau} U(t,t_0)\mathbb{A}(t_0) = U(t,t_0)\mathbb{A}(t_0).$$

Setting

$$\mathscr{U}_\tau = \bigcup_{s\le\tau} U(t,s)\mathbb{A}(s),$$

we obtain $\alpha_t(\mathscr{U}_\tau) < \varepsilon$ whenever $\tau \le s_0$. Observe that the sets $\overline{\mathscr{U}_\tau}$ are closed subsets of X_t and

$$\lim_{\tau\to-\infty} \alpha_t(\overline{\mathscr{U}_\tau}) = 0.$$

Thus, it follows by property (K.2) of the Kuratowski measure that

$$\omega_{\mathbb{A}}(t) = \bigcap_{\tau\le t} \overline{\mathscr{U}_\tau}$$

is nonempty and compact.

Next, we will prove that $\omega_{\mathbb{A}}(t)$ has the pullback attracting property. Suppose not, that is, suppose that there exists $t \in \mathbb{R}$, a pullback-bounded family $\mathscr{B}$, $s_n \to -\infty$, $z_n \in \mathscr{B}(s_n)$ and $\delta > 0$ such that

$$\inf_{z\in\omega_{\mathbb{A}}(t)} \|U(t,s_n)z_n - z\|_{X_t} > \delta. \tag{2.6}$$

Extract a subsequence $\{s_{n_k}\}$ from $\{s_n\}$ as follows: given $s_{n_1},\ldots,s_{n_k}$, choose $s_{n_{k+1}} \le s_{n_k}$ such that $U(s_{n_k},s_{n_{k+1}})\mathscr{B}(s_{n_{k+1}}) \subset \mathbb{A}(s_{n_k})$. Note that

$$U(t,s_{n_{k+1}})z_{n_{k+1}} = U(t,s_{n_k})U(s_{n_k},s_{n_{k+1}})z_{n_{k+1}} = U(t,s_{n_k})w_k,$$

with

$$w_k = U(s_{n_k},s_{n_{k+1}})z_{n_{k+1}} \in \mathbb{A}(s_{n_k}),$$

i.e.,

$$U(t,s_{n_{k+1}})z_{n_{k+1}} \in \mathscr{U}_{s_{n_k}}.$$

By property (K.3) of the Kuratowski measure, the sequence $U(t,s_{n_{k+1}})z_{n_{k+1}}$ has an accumulation point in $\omega_{\mathbb{A}}(t)$, which contradicts (2.6). Therefore, $\omega_{\mathbb{A}}$ is pullback attracting.

Now, we will proof that $\omega_{\mathbb{A}}$ is invariant. In fact, let $s \le t$. We aim to prove that

$$U(t,s)\omega_{\mathbb{A}}(s) = \omega_{\mathbb{A}}(t).$$

First, we will deal with the inclusion

$$U(t,s)\omega_{\mathbb{A}}(s) \subset \omega_{\mathbb{A}}(t).$$

Indeed, let $z \in \omega_{\mathbb{A}}(s)$, then there are sequences $s_n \to -\infty$ and $z_n \in \mathbb{A}(s_n)$ such that

$$\|U(s,s_n)z_n - z\|_{X_s} \to 0 \quad as \quad n \to \infty.$$

Now, extract again a subsequence $\{s_{n_k}\}$ as follows: given $s_{n_1}, \dots, s_{n_k}$, choose $s_{n_{k+1}} \leq s_{n_k}$ such that

$$U(s_{n_k}, s_{n_{k+1}})\mathbb{A}(s_{n_{k+1}}) \subset \mathbb{A}(s_{n_k}).$$

Moreover, set

$$w = U(t,s)z \in U(t,s)\omega_{\mathbb{A}}(s)$$

and

$$w_k = U(s_{n_k}, s_{n_{k+1}})z_{n_{k+1}} \in \mathbb{A}(s_{n_k}).$$

Hence, we have

$$\begin{aligned}\|U(t,s_{n_k})w_k - w\|_{X_t} &= \|U(t,s)U(s,s_{n_k})w_k - U(t,s)z\|_{X_t} \\ &= \|U(t,s)U(s,s_{n_{k+1}})z_{n_{k+1}} - U(t,s)z\|_{X_t} \overset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

This establishes that $w = U(t,s)z \in \omega_{\mathbb{A}}(t)$ and consequently $U(t,s)\omega_{\mathbb{A}}(s) \subset \omega_{\mathbb{A}}(t)$ as claimed.

We now turn to the reverse inclusion

$$U(t,s)\omega_{\mathbb{A}}(s) \supset \omega_{\mathbb{A}}(t).$$

Let $z \in \omega_{\mathbb{A}}(t)$ be arbitrary. Choose sequences $s_n \to -\infty$ and $z_n \in \mathbb{A}(s_n)$ such that $s_n \leq s$ for every n and

$$\|U(t,s_n)z_n - z\|_{X_t} \to 0 \quad as \quad n \to \infty.$$

Using the attraction property of $\omega_{\mathbb{A}}(t)$, proven in the first part, we have

$$\lim_{n\to\infty} \inf_{w\in\omega_{\mathbb{A}}(s)} \|U(s,s_n)z_n - w\|_{X_s} = 0.$$

Thus, there exists a sequence $w_n \in \omega_{\mathbb{A}}(s)$ satisfying

$$\lim_{n\to\infty} \|U(s,s_n)z_n - w_n\|_{X_s} = 0.$$

By compacteness of $\omega_{\mathbb{A}}(s)$, we obtain $w_n \to w \in \omega_{\mathbb{A}}(s)$ in X_s. Thus, $U(s,s_n)z_n \to w$ in X_s. Furthermore, by continuity of $U(t,s)$, we have $U(t,s_n)z_n \to U(t,s)w$ (in X_t) and consequently $z = U(t,s)w$. Therefore, the proof of the second inclusion is complete.

Finally, we only need to show (2.5). Since $\mathbb{A}(\cdot)$ is a pullback-bounded family we see that $\omega_{\mathbb{A}}(t)$ is contained in the union of ω-limit sets. Conversely, the fact that $\mathscr{A}(\cdot)$ is a closed family that pullback attracts all pullback-bounded families and by the same argument of the proof of Lemma 1.28[item iv)] we see that $\omega_{\mathscr{B}}(t) \subset \mathscr{A}(t)$ for all t and every pullback-bounded family $\mathscr{B}$. Thus the prove of (2.5) is concluded and in turn, the proof of Theorem 2.18. □

Corollary 2.19. *If the process $U(\cdot,\cdot)$ with absorber $\mathbb{A}$ possesses a decomposition*

$$U(t,s)\mathbb{A}(s) = P(t,s) + N(t,s)$$

where

$$\lim_{s\to-\infty} \|P(t,s)\|_{X_t} = 0, \quad \forall t \in \mathbb{R}$$

and $N(t,s)$ is a compact subset of X_t for all $t \in \mathbb{R}$ and $s \le t$, then $\mathscr{A}(t) = \omega_{\mathbb{A}}(t)$ is an invariant pullback attractor for the process $U(\cdot,\cdot)$.

First, for $t \in \mathbb{R}$, let X_t be a family of normed spaces, we introduce the **R-ball** of X_t

$$\mathbb{B}_t(R) = \{z \in X_t : \|z\|_{X_t} \le R\}.$$

Proof of Corollary 2.19. Given $\varepsilon > 0$, there exists $s_0 < 0$ such that $\|P(t,s)\|_{X_t} < \varepsilon$ for all $s \le s_0$. Thus

$$P(t,s) \subset \mathbb{B}_t(\varepsilon), \quad for\ all\ \ s \le s_0. \tag{2.7}$$

Now, since $N(t,s)$ is compact in X_t for every $t \ge s$, there exist $\mathbb{B}_t(x_1,\varepsilon),\ldots,\mathbb{B}_t(x_p,\varepsilon)$ such that

$$N(t,s) \subset \bigcup_{i=1}^{p} \mathbb{B}_t(x_i,\varepsilon). \tag{2.8}$$

Hence,

$$P(t,s) + N(t,s) \subset \mathbb{B}_t(\varepsilon) + \bigcup_{i=1}^{p} \mathbb{B}_t(x_i,\varepsilon) \subset \bigcup_{i=1}^{p} \mathbb{B}_t(x_i,2\varepsilon), \tag{2.9}$$

and $\alpha_t(P(t,s)+N(t,s)) \le 2\varepsilon$ for $s \le s_0$. Therefore, (2.4) is satisfy. □

Corollary 2.20. *Let Y_t be a further family of Banach spaces satisfying, for every $t \in \mathbb{R}$,*

i) Y_t is compactly embedded into X_t;

ii) denoting with $\mathscr{I}_t : Y_t \to X_t$ the canonical injection, the maps $\mathscr{I}_s$ are equibounded for $s \le t$, i.e.,

$$C(t) = \sup_{s\le t} \|\mathscr{I}_s\|_{\mathscr{L}(Y_s,X_s)} < \infty;$$

iii) closed balls of Y_t are closed in X_t.

Under the same assumptions as in Corollary 2.19, if in addition

$$h(t) = \sup_{s\in(-\infty,t]} \|N(t,s)\|_{Y_t} < \infty \quad for\ all\ \ t \in \mathbb{R},$$

then the pullback attractor $\mathscr{A}(t)$ is a pullback-bounded family. Furthermore, it satisfies

$$\|\mathscr{A}(t)\|_{Y_t} \le h(t), \quad for\ all\ \ t \in \mathbb{R}.$$

Proof. Fix $t \in \mathbb{R}$ and $z \in \mathscr{A}(t)$. By definition, there exists sequences $s_n \to -\infty$, $z_n \in \mathbb{A}(s_n)$ such that $\|U(t,s_n)z_n - z\|_{X_t} \to 0$ as $n \to \infty$. Using the decomposition of Corollary 2.19,

$$U(t,s_n)z_n = P_{z_n}(t,s_n) + N_{z_n}(t,s_n),$$

with $P_{z_n}(t,s_n) \in P(t,s_n)$ and $N_{z_n}(t,s_n) \in N(t,s_n)$. In particular,

$$\|N_{z_n}(t,s_n)\|_{Y_t} \leq h(t),$$

i.e., the sequence $N_{z_n}(t,s_n)$ is contained in the closed ball of Y_t with radius $h(t)$, which we call B_t. Now, using $\|P(t,s)\|_{X_t} \to 0$ as $s \to -\infty$

$$\|N_{z_n}(t,s_n) - z\|_{X_t} \leq \|U(t,s_n)z_n - z\|_{X_t} + \|P_{z_n}(t,s_n)\|_{X_t} \to 0, \quad n \to \infty.$$

Therefore, z is an accumulation point of B_t (in the topology of X_t). By assumption, B_t is closed in X_t, so that $z \in B_t$ as well.

This establishes that $\mathscr{A}(t) \subset B_t$, i.e.,

$$\|\mathscr{A}(t)\|_{Y_t} \leq h(t) \quad for\ all\ t \in \mathbb{R},$$

which in turn yields that $\mathscr{A}$ is a pullback-bounded family in Y_t. The second assumption yields the existence of $C = C(t) > 0$ such that

$$\|\mathscr{A}(s)\|_{X_s} = \|\mathscr{I}_S(\mathscr{A}(s))\|_{X_s} \leq C(t)h(t), \quad for\ all\ s \leq t.$$

Taking supremum over $s \leq t$, since h is increasing by definition, we obtain

$$\sup_{s\in(-\infty,t]} \|\mathscr{A}(s)\|_{X_s} \leq C(t)h(t),$$

i.e. $\mathscr{A}$ is a pullback-bounded family. □

2.2 Pullback attractor

The aim of this section is to study the abstract theory of [9] which will be used in the application in Chapter 3. Additionally, we will make comparisons with paper [10] presenting the purposes of the changes from one to another. In all that follows for this section the process $U(\cdot,\cdot)$ will not be continuous.

Definition 2.21. A family $\mathscr{C} = \{\mathscr{C}(t) \subset X_t\}_{t\in\mathbb{R}}$ is called **uniformly bounded** if there exists $R > 0$ such that

$$\mathscr{C}(t) \subset \mathbb{B}_t(R), \quad for\ all\ t \in \mathbb{R}.$$

Definition 2.22. A family $\mathbb{D} = \{\mathbb{D}(t)\}_{t\in\mathbb{R}}$ is called **pullback absorbing** if it is uniformly bounded and, for every $R > 0$, there exists $t_0 = t_0(t,R) \le t$ such that

$$U(t,s)\mathbb{B}_s(R) \subset \mathbb{D}(t), \quad for \ all \ s \le t_0. \tag{2.10}$$

The process $U(\cdot,\cdot)$ is called **dissipative** whenever it admits a pullback absorbing family.

Remark 2.23. Although it seems different, we will prove that definitions of pullback absorber (Definition 2.4) and pullback absorbing (Definition 2.22) are equivalent. Indeed, since any family of balls $\{\mathbb{B}_t(R)\}_{t\in\mathbb{R}}$ is pullback-bounded, then (2.10) follows from Definition (2.4). Conversely, if $\mathscr{B}$ is any pullback-bounded family with maximal size $\mathscr{R}$(t) on $(-\infty,t]$, then

$$U(t,s)\mathscr{B}(s) \subset U(t,s)\mathbb{B}_s(\mathscr{R}(t)), \quad for \ all \ s \le t. \tag{2.11}$$

Hence, if $\mathscr{C}$ is a pullback absorbing family in the sense of Definition 2.22 and $t \in \mathbb{R}$ is any fixed time, we have

$$U(t,s)\mathscr{B}(s) \subset U(t,s)\mathbb{B}_s(\mathscr{R}(t)) \subset \mathscr{C}(t) \quad for \ all \ s \le t_0,$$

for some $t \ge t_0 = t_0(t,\mathscr{R}(t))$, where $\mathscr{R}(t)$ depends only on $\mathscr{B}$. However, this is exactly the absorption property (2.11).

Remark 2.24. The notion of pullback attracting can be rephrased in the following way: a uniformly bounded family $\mathscr{D} = \{\mathscr{D}(t)\}_{t\in\mathbb{R}}$ is pullback attracting if and only if

$$\lim_{s\to-\infty} dist_{X_t}(U(t,s)\mathscr{C}(s),\mathscr{D}(t)) = 0, \tag{2.12}$$

for every uniformly bounded family $\mathscr{C} = \{\mathscr{C}(t)\}_t$ and every $t \in \mathbb{R}$.

It was chosen by the authors of [9] to postulate in the definition of absorbing family the stronger property of being uniformly bounded, instead of merely pullback-bounded. Because, such a notion seems to reflect more closely the dissipation mechanism of most equations of mathematical physics, where the dynamics at time are confined in bounded sets $\mathbb{D}(t)$ (the pullback absorbing family) whose size in the phase space X_t remains bounded as $t \to \infty$.

Furthermore, having a pullback-bounded absorbing family does not prevent the possibility of $\mathbb{D}(t)$ becoming larger and larger as time increases, in contrast with the common intuition of dissipation.

Remark 2.25. An interesting question is wheter property (2.12) holds uniformly with respect to intervals of time. This not true in general. In particular, it cannot happen on unbounded intervals. The next result shows that, if the process is sufficiently smooth, then the attraction exerted by any invariant pullback attracting family (such as the time-dependent attractor) is uniform on compact intervals.

Proposition 2.26. *Let $\mathscr{D} = \{\mathscr{D}(t)\}_{t\in\mathbb{R}}$ be an invariant pullback attracting family. Assume that*

$$\|U(t,s)z_1 - U(t,s)z_2\|_{X_t} \le \mathscr{Q}(t-s,r)\|z_1 - z_2\|_{X_s}, \tag{2.13}$$

for all $t \geq s \in \mathbb{R}$ and $\|z_i\|_{X_s} \leq r$, where $\mathscr{Q}$ is a positive function, increasing in each of its arguments. Then, for all $R > 0$,

$$\lim_{s \to -\infty} dist_{X_t}(U(t,s)\mathbb{B}(s)(R), \mathscr{D}(t)) = 0,$$

uniformly for t belonging to a compact set.

Proof. Let $[a,b]$ with $-\infty < a < b < \infty$ be given. Let $R_0 > 0$ be such that

$$\mathscr{O}_a^1(\mathscr{D}(a)) \subset \mathbb{B}_a(R_0).$$

For every q small enough, set

$$\varepsilon = \frac{q}{\mathscr{Q}(b-a,R_0)} < 1.$$

Since $\mathscr{D}$ is pullback attracting, for any given $R > 0$ there exists

$$s_0 = s_0(R,\varepsilon) < a$$

such that

$$U(a,s)\mathbb{B}_s(R) \subset \mathscr{O}^{\varepsilon}\mathscr{D}(a), \quad for \ all \ s < s_0.$$

Now, let $s < s_0$ be fixed, and select any $x \in \mathbb{B}_s(R)$. Calling $z = U(a,s)x$, choose $d \in \mathscr{D}(a)$ for which

$$\|z-d\|_{X_a} < \varepsilon.$$

Then, in light of (2.13), for all $t \in [a,b]$ we have

$$\|U(t,a)z - U(t,a)d\|_{X_t} \leq \mathscr{Q}(t-a,R_0)\|z-d\|_{X_a} \leq \varepsilon\mathscr{Q}(t-a,R_0) = q.$$

Observe that, from the invariance of $\mathscr{D}$

$$U(t,s)x = U(t,a)U(a,s)x = U(t,a)z \quad and \quad U(t,a)d \subset \mathscr{D}(t).$$

Thus,

$$\inf_{d \in D(t)} \|U(t,s)x - d\|_{X_t} \leq \|U(t,a)z - U(t,a)d\|_{X_t} \leq q.$$

In conclusion, we proved that for all $q > 0$ small there exists $s_0 < a$ such that

$$dist_{X_t}(U(t,s)\mathbb{B}_s(R), \mathscr{D}(t)) \leq q, \quad for \ all \ s < s_0.$$

Since s_0 is independent of $t \in [a,b]$, the proof is finished. □

2.2.1 Further lemmas

We can describe the pullback attraction in terms of sequences. To this aim, let Σ_t denote the collection of all possible sequences of the form

$$y_n = U(t, s_n)x_n$$

where $s_n \to -\infty$ and $x_n \in X_{s_n}$ is any uniformly bounded sequence. For any $y_n \in \Sigma_t$, we denote

$$\mathscr{L}_t(y_n) = \{x \in X_t : \|y_{n_i} - x\|_{X_t} \to 0, \ \ for \ \ some \ \ subsequence \ \ n_i \to \infty\}.$$

It is immediately seen from the definitions that a uniformly bounded family $\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}}$ is pullback attracting if and only if

$$d_{X_t}(y_n, \mathscr{K}(t)) \to 0, \quad for \ \ all \ \ y_n \in \Sigma_t, \tag{2.14}$$

for all $t \in \mathbb{R}$. In particular, each element of $\mathscr{L}_t(y_n)$ belongs to the closure of $\mathscr{K}(t)$. Therefore, setting

$$\mathscr{A}^*(t) = \bigcup_{y_n \in \Sigma_t} \mathscr{L}_t(y_n),$$

we have proved the following lemma.

Lemma 2.27. *Assume that there exists a pullback attracting family of closed sets $\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}}$. Then*

$$\mathscr{A}^*(t) \subset \mathscr{K}(t), \quad for \ \ all \ \ t \in \mathbb{R}.$$

Lemma 2.28. *If the process $U(\cdot,\cdot)$ is dissipative, then $\mathscr{A}^* = \{\mathscr{A}^*(t)\}_{t\in\mathbb{R}}$ coincides with the time-dependent ω-limit of any pullback absorbing set $\mathscr{B} = \{\mathscr{B}(t)\}_{t\in\mathbb{R}}$, that is,*

$$\mathscr{A}^*(t) = \bigcap_{s\le t} \overline{\bigcup_{\tau\le s} U(t,\tau)\mathscr{B}(\tau)}^{X_t}. \tag{2.15}$$

In particular, $\mathscr{A}^*(t)$ is closed and contained in $\overline{\mathscr{B}(t)}$ for all $t \in \mathbb{R}$. Therefore, $\mathscr{A}^*$ is uniformly bounded.

Proof. The validity of (2.15) is a direct consequence of the definitions. Moreover, it follows from (2.15) that $\mathscr{A}^*(t)$ is closed for all $t \in \mathbb{R}$. Further, since $\mathscr{B}$ is uniformly bounded, it absorbs itself and

$$U(t,\tau)\mathscr{B}(\tau) \subset \mathscr{B}(t), \quad for \ \ all \ \ \tau \le t_0,$$

for some $t_0 = t_0(t, \mathscr{B}) \le t$, implying the inclusion $\mathscr{A}^*(t) \subset \overline{\mathscr{B}(t)}$. □

Lemma 2.29. *Let $\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}}$ be a uniformly bounded family of compact sets. Then, $\mathscr{K}$ is pullback attracting if and only if for all $t \in \mathbb{R}$*

$$\emptyset \ne \mathscr{L}_t(y_n) \subset \mathscr{K}(t), \quad for \ \ all \ \ y_n \in \Sigma_t.$$

Proof. Let $\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}}$ be a family of compact sets. If $\mathscr{K}$ is pullback attracting, then given $y_n \in \Sigma_t$ we obtain $\mathscr{L}_t(y_n) \subset \mathscr{K}(t)$ and

$$\|\xi_n - y_n\|_{X_t} \to 0, \tag{2.16}$$

for some $\xi_n \in \mathscr{K}(t)$. Since $\mathscr{K}(t)$ is compact, there exists $\xi \in \mathscr{K}(t)$ such that (up to a subsequence)

$$\|\xi_n - \xi\|_{X_t} \to 0. \tag{2.17}$$

It follows from (2.16) and (2.17) that

$$\|y_n - \xi\|_{X_t} \to 0.$$

Thus, $\mathscr{L}_t(y_n) \neq \emptyset$.

Conversely, if $\mathscr{K}$ is not pullback attracting, we deduce from (2.14) that

$$d_{X_t}(y_n, \mathscr{K}(t)) > \varepsilon,$$

for some $t \in \mathbb{R}$, $\varepsilon > 0$ and $y_n \in \Sigma_t$. Therefore, $\mathscr{L}_t(y_n) \cap \mathscr{K}(t) = \emptyset$. □

2.2.2 Existence of pullback attractors

We can deduce from the earlier discussions that a pullback attracting family of compact sets is capable of controlling the regime of the system at any time $t \in \mathbb{R}$. This leads quite naturally to the definition of an attractor as the smallest set possessing such a property. To this aim we consider the collection

$$\mathbb{K} = \{\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}} : \mathscr{K}(t) \subset X_t \ \ compact,\ \mathscr{K} \ \ pullback\ \ attracting\} \tag{2.18}$$

Definition 2.30. We call a **pullback attractor** the minimal element of $\mathbb{K}$, that is, the family $\mathscr{A} = \{\mathscr{A}(t)\}_{t\in\mathbb{R}} \in \mathbb{K}$ such that

$$\mathscr{A}(t) \subset \mathscr{K}(t), \quad for\ \ all\ \ t \in \mathbb{R}, \tag{2.19}$$

for any element $\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}} \in \mathbb{K}$.

The following result is the main result of this abstract theory, because it will be used to prove the existence of attractors for a damped wave equation with time-dependent coefficient in the next chapter. Furthermore, the result tells that the definition is consistent: the minimal element of $\mathbb{K}$ exists (and it is unique) if and only if $\mathbb{K}$ is not empty.

Theorem 2.31. *If the process $U(\cdot,\cdot)$ is asymptotically compact, then the time-dependent attractor $\mathscr{A}$ exists and coincides with $\mathscr{A}^* = \{\mathscr{A}^*(t)\}_{t\in\mathbb{R}}$, where $\mathscr{A}^\star$ is given in Lemma 2.27. In particular, it is unique.*

Proof. Indeed, how the process is $U(\cdot,\cdot)$ is asymptotically compact then $\mathbb{K} \neq \emptyset$. Hence, let $\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}}$ be an element of $\mathbb{K}$. Then, it follows from Lemma 2.27 and Lemma 2.29 that

$$\emptyset \neq \mathscr{A}^*(t) \subset \mathscr{K}(t), \quad for\ all\ t \in \mathbb{R}.$$

Since $U(t,\tau)$ is dissipative, we know by Lemma 2.28 that $\mathscr{A}^*$ is uniformly bounded and $\mathscr{A}^*(t)$ is closed for all $t \in \mathbb{R}$. Moreover, $\mathscr{A}^*(t)$ is contained in (2.14), saying that $\mathscr{A}^*$ is an element of $\mathbb{K}$. Thanks to Lemma 2.27 it is also the minimal element of $\mathbb{K}$, hence it is the (unique) pullback attractor. The uniqueness follows by (2.19), that is, the minimality. □

We now provide a necessary condition for $\mathbb{K}$ to be nonempty, which turns out to be sufficient as well when the spaces X_t are complete. Additionally, we will be able to related the Kuratowski measure of noncompactness with a process asymptotically compact.

Definition 2.32. A process $U(\cdot,\cdot)$ is **ε-dissipative** if for every $t \in \mathbb{R}$ there exists a set $F(t) \subset X_t$ made of a finite number of points such that the family $\{\mathscr{O}_t^\varepsilon(F(t))\}_{t\in\mathbb{R}}$ is pullback absorbing.

The process is called **totally dissipative** whenever it is ε-dissipative for every $\varepsilon > 0$. Note that the sets $F(t)$ need not be the same for all ε.

Theorem 2.33. *Assume that X_t is a Banach space for all $t \in \mathbb{R}$. Then $U(\cdot,\cdot)$ is totally dissipative if and only if $\mathbb{K} \neq \emptyset$.*

Proof. If $\mathbb{K} \neq \emptyset$, then $U(\cdot,\cdot)$ is totally dissipative. Indeed, if $\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}}$ belongs to $\mathbb{K}$, it follows from compactness that any $\mathscr{K}(t)$ can be covered by a finite number of ε-balls, and calling $F(t)$ the union of the centers of those balls, the family $\{\mathscr{O}_t^\varepsilon(F(t))\}_{t\in\mathbb{R}}$ is pullback absorbing.

Conversely, if $U(\cdot,\cdot)$ is totally dissipative, for any arbitrary fixed $\varepsilon > 0$, we can choose a finite set $F^\varepsilon(t)$ such that the family $\{\mathscr{O}_t^\varepsilon(F^\varepsilon(t))\}_{t\in\mathbb{R}}$ is uniformly bounded and absorbing. If we select any $y_n \in \Sigma_t$, then y_n eventually falls into

$$V^\varepsilon(t) = \overline{\mathscr{O}_t^\varepsilon(F^\varepsilon(t))}.$$

Set

$$\mathscr{K}(t) = \bigcap_{\varepsilon>0} V^\varepsilon(t).$$

Accordingly, the family $\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}}$ is uniformly bounded. Furthermore, both $\mathscr{K}(t)$ and $\{y_n\}$ are covered by finitely many balls of arbitrarily small radius, which, in Banach spaces, means precompactness. In particular, $\mathscr{K}(t)$ being closed, it is compact in X_t. Since the sequence y_n is precompact, then $\mathscr{L}_t(y_n)$ is nonempty. Moreover, it is contained in every closed set $V^\varepsilon(t)$ and hence in their intersection $\mathscr{K}(t)$. In other words,

$$d_{X_t}(y_n, \mathscr{K}(t)) \to 0,$$

meaning that $\mathscr{K}$ is pullback attracting. Therefore, $\mathscr{K} \in \mathbb{K}$. □

Now, taking together Theorem 2.31 and Theorem 2.33 we can conclude the following corollary.

Corollary 2.34. *If the process $U(\cdot,\cdot)$ is totally dissipative, then the time-dependent attractor $\mathscr{A}$ exists and coincides with the set $\mathscr{A}^*$. In particular, it is unique and uniformly bounded.*

Remark 2.35. We can characterize a totally dissipative process based on the Kuratowski measure:

The process $U(t,s)$ is totally dissipative if and only if there exists a pullback absorbing set $\mathscr{B} = \{\mathscr{B}(t)\}_{t\in\mathbb{R}}$ for which

$$\lim_{s\to-\infty} \alpha_t(U(t,\tau)\mathscr{B}(\tau)) = 0, \quad for\ \ all\ \ t\in\mathbb{R}.$$

As a result we obtain a relationship between the Kuratowski measure of non-compactness and a process asymptotically compact.

Remark 2.36. Note that Definition 2.30 does no require the invariance as a property. However, this property is a priori postulated in the literature. In particular, in Section 2.1.2 the invariant pullback attractor is by definition a family of compact sets which is at the same pullback attracting and invariant, and it existence is proved by exploiting the continuity of the process $U(\cdot,\cdot)$.

If $\mathscr{K}$ is an invariant pullback attracting family of compact sets, then $\mathscr{K}$ is the smallest element of $\mathbb{K}$, hence it coincides with the invariant pullback attractor $\mathscr{A}$.

To this aim, we will prove by the following results that a pullback attractor is invariant whenever the process $U(\cdot,\cdot)$ is T-closed for some $T>0$.

Proposition 2.37. *Let $\mathscr{A} = \{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ be a pullback attractor. If there exists $T>0$ such that*

$$\mathscr{A}(t) \subset U(t,t-T)\mathscr{A}(t-T), \quad for\ \ all\ \ t\in\mathbb{R},$$

then $\mathscr{A}$ is invariant.

Proof. Fix $t\in\mathbb{R}$. For any $s\geq t$ and any $n\in\mathbb{N}$, we obtain by induction

$$U(s,t)\mathscr{A}(t) \subset U(s,t-T)\mathscr{A}(t-T) \subset ... \subset U(s,t-nT)\mathscr{A}(t-nT). \tag{2.20}$$

Consequently,

$$dist_{X_s}(U(s,t)\mathscr{A}(t),\mathscr{A}(s)) \leq dist_{X_s}(U(s,t-nT)\mathscr{A}(t-nT),\mathscr{A}(s)).$$

Since $\mathscr{A}$ is attracting, , letting $n\to\infty$ we obtain

$$dist_{X_s}(U(s,t)\mathscr{A}(t),\mathscr{A}(s)) = 0,$$

implying in turn, since $\mathscr{A}(s)$ is closed, that

$$U(s,t)\mathscr{A}(t) \subset \mathscr{A}(s), \quad for\ \ all\ \ s\geq t. \tag{2.21}$$

In particular, setting $s = t$ it follows from (2.20) and (2.21) that

$$\mathscr{A}(t) \subset U(t, t-nT)\mathscr{A}(t-nT) \subset \mathscr{A}(t), \tag{2.22}$$

that is,

$$\mathscr{A}(t) = U(t, t-nT)\mathscr{A}(t-nT).$$

Now, consider $\tau \leq t$. Taking n large enough, we infer from (2.21) and (2.22) that

$$\mathscr{A}(t) = U(t, t-nT)\mathscr{A}(t-nT) = U(t, \tau)U(\tau, t-nT)\mathscr{A}(t-nT) \subset U(t, \tau)\mathscr{A}(\tau) \subset \mathscr{A}(t),$$

proving the equality $U(t, \tau)\mathscr{A}(\tau) = \mathscr{A}(t)$. □

In order to establish an invariance criterion, we will need the following definitions.

Definition 2.38. For any pair of fixed times $t \geq \tau$, the map $U(t, \tau) : X_\tau \to X_t$ is said to be **closed** if

$$\begin{cases} x_n \to x \quad in \;\; X_\tau \\ U(t, \tau)x_n \to y \quad in \;\; X_t \end{cases}$$

then $U(t, \tau)x = y$.

Definition 2.39. The process $U(\cdot, \cdot)$ is called

i) **closed** if $U(t, \tau)$ is a closed map for any pair of fixed times $t \geq \tau$;

ii) **T-closed** for some $T > 0$ if $U(t, t-T)$ is a closed map for all t.

Remark 2.40. Observe that if the process $U(\cdot, \cdot)$ is closed it is T-closed, for any $T > 0$. Note also that if the process $U(t, \tau)$ is a continuous map for all $t \geq \tau$, then the process is closed.

Theorem 2.41. *If $U(\cdot, \cdot)$ is a T-closed process for some $T > 0$, which possesses a pullback attractor $\mathscr{A} = \{\mathscr{A}(t)\}_{t\in\mathbb{R}}$, then $\mathscr{A}$ is invariant.*

Proof. In view of Proposition 2.37, it is enough to prove the inclusion

$$\mathscr{A}(t) \subset U(t, t-T)\mathscr{A}_{t-T}, \quad for \;\; all \;\; t \in \mathbb{R}.$$

To this end, select an arbitrary $y \in \mathscr{A}(t)$. By Theorem 2.31,

$$y_n \to y \quad for \;\; some \;\; y_n = U(t, \tau_n)x_n \in \Sigma_t.$$

Now, define the sequence

$$w_n = U(t-T, \tau_n)x_n.$$

On account of Lemma 2.29,

$$w_n \to w \quad for\ \ some\ \ w \in \mathscr{A}(t-T).$$

On the other hand,

$$U(t,t-T)w_n = U(t,\tau_n)x_n,$$

which implies

$$U(t,t-T)w_n \to y.$$

Moreover, since $U(t,t-T)$ is closed we conclude that

$$U(t,t-T)w = y.$$

Therefore,

$$y \in U(t,t-T)\mathscr{A}(t-T),$$

yielding the desired inclusion. □

CHAPTER 3

Wave equations with time-dependent speed of propagation

This chapter is devoted to the study of damped wave equations. More precisely, based on [9], we will applied the abstract results developed in Chapter 2 to study the long time behavior of wave equations with time-dependent speed of propagation.

3.1 Physical Interpretation

First, we will give a physical interpretation for wave equations in general case. Indeed, consider the equation

$$u_{tt} - \Delta u = 0, \tag{3.1}$$

where $t > 0$ and $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is open. The unknown variable is $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$, $u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables $x = (x_1, ..., x_n)$.

The wave equation is a simplified model for a vibrating string ($n = 1$), membrane ($n = 2$), or elastic solid ($n = 3$). In this case $u(x,t)$ represents the displacement in some direction of the point x at time $t \geq 0$. Let U represent any smooth subregion of Ω. The acceleration within U is then

$$\frac{d}{dt}\int_U u dx = \int_U u_{tt} dx$$

and the net contact force is

$$-\int_{\partial U} F \cdot \nu dS,$$

where F denotes the force acting on U through ∂U and the mass density is taken to be unity. Newton's law asserts that the mass times the acceleration equals the net force:

$$\int_U u_{tt} dx = -\int_{\partial U} F \cdot \nu dS.$$

This identity obtains for each subregion U and so

$$u_{tt} = -divF.$$

For elastic bodies, F is a function of the displacement gradient ∇u, whence

$$u_{tt} + divF(\nabla u) = 0.$$

For small ∇u, the linearization $F(\nabla u) \approx -c\nabla u$ is often appropriate. Therefore,

$$u_{tt} - c\Delta u = 0.$$

This is the equation (3.1) if $c = 1$. This physical interpretation strongly suggests it will be mathematically appropriate to specify two initial conditions, on the *displacement* u and the *velocity* u_t, at time $t = 0$.

3.2 Hyperbolic damped wave equation

Now, we will introduce the damped wave equation with time-dependent speed of propagation. Indeed, let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For any $\tau \in \mathbb{R}$, we consider the hyperbolic evolution equation for the unknown variable $u = u(x,t) : \Omega \times [\tau, \infty) \to \mathbb{R}$

$$\varepsilon u_{tt} + \alpha u_t - \Delta u + f(u) = g, \quad t > \tau, \tag{3.2}$$

with Dirichlet boundary condition

$$u_{|\partial\Omega} = 0, \tag{3.3}$$

and to the initial conditions

$$u(x,\tau) = a(x) \quad and \quad u_t(x,\tau) = b(x), \tag{3.4}$$

where $a, b : \Omega \longrightarrow \mathbb{R}$ are assigned data. Moreover, $\varepsilon = \varepsilon(t)$ is a function of t.

Remark 3.1. We can see equation (3.2) as a nonlinear damped wave equation with time-dependent speed of propagation $\frac{1}{\varepsilon(t)}$.

In order to understand the physical interpretation of each term of equation (3.2), we point out which forces are acting in the system in relation to (3.1) and what they represent. In fact, we consider

i) g as an external force;

ii) αu_t as an external force depending on the speed, that is, is the damping force;

iii) $f(u)$ as a restorative force.

3.2.1 Conditions in terms of the equation

We postulate the following assumptions for the terms of equation (3.2).

Conditions on ε:

Suppose $\varepsilon \in \mathscr{C}^1(\mathbb{R})$, a decreasing bounded function satisfying

$$\lim_{t \to +\infty} \varepsilon(t) = 0. \tag{3.5}$$

Proposition 3.2. *With the hypotheses over ε described above, ε' is a bounded function.*

Proof. Suppose not, then

$$\lim_{t \to \infty} \varepsilon'(t) = -\infty \tag{3.6}$$

or

$$\lim_{t \to -\infty} \varepsilon'(t) = -\infty. \tag{3.7}$$

First, note that there exists $N > 0$ such that $0 \leq \varepsilon(t) \leq N$ for every $t \in \mathbb{R}$.

Now, suppose that (3.6) happens. Then, given $M < 0$ there exists $R > 0$ such that

$$\varepsilon'(t) < M, \quad for \;\; all \;\; t > R.$$

In particular, for $M = -1$ there exists $R_M > 0$ such that

$$\varepsilon'(t) < -1, \quad for \;\; all \;\; t > R_M. \tag{3.8}$$

Integrating (3.8) we obtain

$$\int_{R_M}^{t} \varepsilon'(s) ds < -\int_{R_M}^{t} ds.$$

Hence,

$$\varepsilon(t) - \varepsilon(R_M) < R_M - t. \tag{3.9}$$

Furthermore, since $\varepsilon(t) > \varepsilon(R_M)$, we have

$$\varepsilon(t) - \varepsilon(R_M) \geq -N.$$

Then,

$$-N < R_M - t.$$

Setting $t = R_M + 2N$ we have

$$-N < R_M - (R_M + 2N) = -2N,$$

which is a contradiction.

On the other hand, if (3.7) happens, similarly to the arguments above we have for $M < 0$ there exists $T < 0$ such that

$$\varepsilon'(t) < M, \quad for \;\; all \;\; t < T.$$

Then, for $M = -1$ there exists $T_M < 0$ such that

$$\varepsilon'(t) < -1, \quad for\ all\ t < T_M. \tag{3.10}$$

Integrating the above equation we obtain

$$\int_t^{T_M} \varepsilon'(s)ds < -\int_t^{T_M} ds.$$

Thus,

$$-N \leq \varepsilon(T_M) - \varepsilon(t) < t - T_M. \tag{3.11}$$

Then, setting $t = T_M - 2N$ we have

$$-N < (T_M - 2N) - T_M < -2N,$$

which is a contradiction. Therefore, ε' is a bounded function. □

Consequently, taken together there exists $L > 0$ such that

$$\sup_{t\in\mathbb{R}}[|\varepsilon(t)| + |\varepsilon'(t)|] \leq L. \tag{3.12}$$

Conditions on f:

Suppose $f \in \mathscr{C}^2(\mathbb{R})$, with $f(0) = 0$ satisfying, for every $s \in \mathbb{R}$, the growth condition

$$|f''(s)| \leq c(1 + |s|), \quad for\ some\ c \geq 0, \tag{3.13}$$

and the dissipation condition

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{3.14}$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ such that

$$A = -\Delta \quad with \quad domain \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

The operator A is self-adjoint, positive definite and sectorial in $L^2(\Omega)$ (for more details and proofs see [7]). Moreover,

$$H^2(\Omega) \cap H_0^1(\Omega) \Subset L^2(\Omega),$$

that is, $H^2(\Omega) \cap H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ (see [17]).

Note that, in equation (3.2) f takes values in Ω. Thus, was used

$$f^e(u)(x) = f(u(x)), \quad for\ all\ x \in \Omega \tag{3.15}$$

as the Nemytskii operator associated with f.

Conditions on α and g:

The damping coefficient α is a positive constant and the time-independent external source $g = g(x)$ is taken in $L^2(\Omega)$.

3.2.2 The functional setting

We set $H = L^2(\Omega)$, with usual inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Since A is a sectorial operator we will define the fractional power spaces associated with it.

Hence, for $0 \leq \sigma \leq 2$, we define the hierarchy of (compactly) nested Hilbert spaces

$$H_\sigma = D(A^{\frac{\sigma}{2}}), \quad \langle w,v\rangle_\sigma = \langle A^{\frac{\sigma}{2}}w, A^{\frac{\sigma}{2}}v\rangle, \quad \|w\|_\sigma = \|A^{\frac{\sigma}{2}}w\|.$$

Then, for $t \in \mathbb{R}$ and $0 \leq \sigma \leq 2$, we introduce the time-dependent spaces

$$\mathscr{H}_t^\sigma = H_{\sigma+1} \times H_\sigma$$

endowed with the time-dependent product norms

$$\|\{u_1,u_2\}\|^2_{\mathscr{H}_t^\sigma} = \|u_1\|^2_{\sigma+1} + \varepsilon(t)\|u_2\|^2_\sigma.$$

The symbol σ is always omitted whenever is zero. In particular, the time-dependent phase space where we settle the problem is

$$\mathscr{H}_t = H_1 \times H \quad with \quad \|\{u_1,u_2\}\|^2_{\mathscr{H}_t} = \|u_1\|^2_1 + \varepsilon(t)\|u_2\|^2, \tag{3.16}$$

where

$$H_1 = D(A^{\frac{1}{2}}) = H_0^1(\Omega).$$

In addition, how Ω is a bounded domains of $\mathbb{R}^3$, it follows by Poincaré inequality that

$$\|u_1\|^2_1 = \|\nabla u_1\|^2 = \int_\Omega |\nabla u_1|^2 dx \quad for \quad all \quad u_1 \in H_0^1(\Omega). \tag{3.17}$$

Furthermore, we have the compact embeddings

$$\mathscr{H}_t^\sigma \Subset \mathscr{H}_t, \quad 0 < \sigma \leq 2,$$

with injection constants independent of $t \in \mathbb{R}$.

Note that the spaces $\mathscr{H}_t$ are all the same as linear spaces. Moreover, since $\varepsilon(\cdot)$ is a decreasing function of t, for every $z \in H_1 \times H$ and $t \geq \tau \in \mathbb{R}$ there holds

$$\|z\|^2_{\mathscr{H}_t} \leq \|z\|^2_{\mathscr{H}_\tau} \leq \max\left\{1, \frac{\varepsilon(\tau)}{\varepsilon(t)}\right\}\|z\|^2_{\mathscr{H}_t}.$$

Therefore, the norms $\|\cdot\|^2_{\mathscr{H}_t}$ and $\|\cdot\|^2_{\mathscr{H}_\tau}$ are equivalent for any fixed $t,\tau \in \mathbb{R}$. However, the equivalence constant blows up when $t \to +\infty$.

In order to understand the phase space (3.16) where we settle the problem, we look at the energy of the system. For this, multiplying (3.2) by $2u_t$, we obtain

$$\langle 2u_t, \varepsilon u_{tt}\rangle + \langle 2u_t, \alpha u_t\rangle + \langle 2u_t, -\Delta u\rangle + \langle 2u_t, f(u)\rangle = \langle 2u_t, g\rangle, \tag{3.18}$$

that is,

$$2\varepsilon \int_\Omega u_t u_{tt} dx + 2\alpha \int_\Omega |u_t|^2 dx - 2\int_\Omega u_t \Delta u dx + 2\int_\Omega f(u) u_t dx = 2\int_\Omega u_t g dx. \tag{3.19}$$

First, we will solve each term of the left side of the equation (3.19). Indeed,

i)

$$\begin{aligned} 2\varepsilon \int_\Omega u_t u_{tt} dx &= 2\varepsilon(t) \int_\Omega u_t u_{tt} dx \\ &= 2\varepsilon(t)\left(\frac{1}{2}\frac{d}{dt}\langle u_t, u_t\rangle\right) \\ &= \varepsilon(t)\frac{d}{dt}\langle u_t, u_t\rangle \\ &= \frac{d}{dt}\varepsilon(t)\|u_t\|^2 - \varepsilon'(t)\|u_t\|^2. \end{aligned} \tag{3.20}$$

ii)

$$2\alpha \int_\Omega |u_t|^2 dx = 2\alpha \|u_t\|^2. \tag{3.21}$$

iii) It follows from Green's Theorem, (3.3) and (3.17) that

$$\begin{aligned} -2\int_\Omega u_t \Delta u dx &= -2\left(-\int_\Omega \nabla u \nabla u_t dx\right) \\ &= 2\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx \\ &= \frac{d}{dt}\|u\|_1^2. \end{aligned} \tag{3.22}$$

iv)

$$\begin{aligned} 2\int_\Omega f(u) u_t dx &= 2\int_\Omega \frac{d}{dt}\left(\int_0^u f(s) ds\right) dx \\ &= 2\frac{d}{dt}\int_\Omega \left(\int_0^u f(s) ds\right) dx \\ &= 2\frac{d}{dt}\int_\Omega F(u) dx \\ &= 2\frac{d}{dt}\langle F(u), 1\rangle, \end{aligned} \tag{3.23}$$

where, we set

$$F(u) = \int_0^u f(s) ds.$$

Now, for the right side we have

$$\begin{aligned} 2\int_\Omega g u_t dx &= 2\int_\Omega g(x) u_t dx \\ &= 2\frac{d}{dt}\int_\Omega g(x) u dx \\ &= 2\frac{d}{dt}\langle g, u\rangle. \end{aligned} \tag{3.24}$$

Taking all together we obtain

$$\frac{d}{dt}(\|u\|_1^2+\varepsilon(t)\|u_t\|^2+2\langle F(u),1\rangle-2\langle g,u\rangle)=[\varepsilon'(t)-2\alpha]\|u_t\|^2, \tag{3.25}$$

where

$$\varepsilon'(t)-2\alpha<0, \quad for\ all\ \ t\in\mathbb{R},$$

and

$$\mathscr{E}(t)=\|u\|_1^2+\varepsilon(t)\|u_t\|^2+2\langle F(u),1\rangle-2\langle g,u\rangle.$$

Therefore, we can see that the natural energy associated with equation (3.2) posses a structural dependence on time.

3.2.3 Technical lemmas

We shall exploit the following Gronwall-type lemma, that will be used in some results here after.

Lemma 3.3. *Let $\Psi:[\tau,\infty)\to\mathbb{R}^+$ be an absolutely continuous function satisfying the inequality*

$$\frac{d}{dt}\Psi(t)+2\omega\Psi(t)\leq q(t)\Psi(t)+K$$

for some $\omega>0$, $K\geq 0$ and where $q:[\tau,\infty)\to\mathbb{R}^+$ fulfills

$$\int_\tau^\infty q(y)dy\leq m,$$

with $m\geq 0$. Then,

$$\Psi(t)\leq\Psi(\tau)e^m e^{-\omega(t-\tau)}+k\omega^{-1}e^m.$$

Proof. Fix $\tau\in\mathbb{R}$. For $t\in[\tau,\infty)$ we set

$$\eta(t)=\int_\tau^\infty q(y)dy-\omega(t-\tau)\leq m-\omega(t-\tau).$$

Then, by the Gronwall lemma we have

$$\begin{aligned}\psi(t)&\leq\psi(\tau)e^{\eta(\tau)}+\int_\tau^\infty e^{\eta(t)}Kdt\\&\leq\psi(\tau)e^{m-\omega(t-\tau)}+K\int_\tau^\infty e^{m-\omega(t-\tau)}dt\\&=\psi(\tau)e^{m-\omega(t-\tau)}+Ke^m e^{\omega\tau}\int_\tau^\infty e^{-\omega t}dt.\end{aligned}$$

Now, observe that

$$\begin{aligned}\int_\tau^\infty e^{-\omega t}dt&=\lim_{b\to\infty}\frac{-e^{-\omega t}}{\omega}\Big|_\tau^b\\&=\frac{e^{-\omega\tau}}{\omega}+\lim_{b\to\infty}\frac{e^{-\omega b}}{\omega}\\&=\frac{e^{-\omega\tau}}{\omega}.\end{aligned}$$

Therefore,

$$\psi(t) \le \psi(\tau)e^{m-\omega(t-\tau)} + Ke^{m}\omega^{-1},$$

which concludes the proof. □

In light of (3.14) we obtain the following lemma.

Lemma 3.4. *The following inequalities holds for some* $0 < v < 1$ *and* $c_1 \geq 0$*:*

$$2\langle F(u), 1\rangle \geq -(1-v)\|u\|_1^2 - c_1, \tag{3.26}$$

$$\langle f(u), u\rangle \geq -(1-v)\|u\|_1^2 - c_1, \quad for\ all \quad u \in H_1. \tag{3.27}$$

Proof. We will start with (3.26). Indeed, note that

$$-\liminf_{|s|\to\infty} \frac{f(s)}{s} = \limsup_{|s|\to\infty} \frac{-f(s)}{s}.$$

Hence, rewriting (3.14) we obtain

$$\limsup_{|s|\to\infty} \frac{-f(s)}{s} < \lambda_1, \tag{3.28}$$

and setting $h(s) = -f(s)$,

$$\limsup_{|s|\to\infty} \frac{h(s)}{s} < \lambda_1. \tag{3.29}$$

Now, recall that

$$\limsup_{x\to\infty}\{h(x)\} = \inf_{r>0}\left\{\sup_{x>r}\{h(x)\}\right\}.$$

In addition, set

$$\phi(r) := \sup_{x>r}\{h(x)\}.$$

It follows by (3.29) that $\inf_{r>0}\phi(r) < \lambda_1$. Then, there exists $r_1 > 0$ such that

$$\phi(r_1) < \lambda_1.$$

Set

$$\eta := \phi(r_1) = \sup_{|s|>r_1}\left\{\frac{h(s)}{s}\right\} < \lambda_1.$$

Thus,

$$\frac{h(s)}{s} < \lambda_1,$$

for all $|s| > r_1$.

From the definition of supremum, we obtain

$$\frac{h(s)}{s} \leqslant \eta < \lambda_1, \quad \text{for all } |s| > r_1.$$

Set $\eta - \lambda_1 := \gamma < 0$. Suppose that there exists $s_1 > r_1$ such that

$$\frac{h(s_1)}{s_1} - \lambda_1 > \gamma = \eta - \lambda_1,$$

which implies that $\frac{h(s_1)}{s_1} > \eta$. A contradiction. Therefore, there exists $\xi > 0$ (e.g., $\xi = \frac{\eta - \lambda_1}{2}$) such that

$$\frac{h(s)}{s} - \lambda_1 < -\xi, \quad \text{for all} |s| > r_1.$$

Hence,

$$\frac{h(s)}{s} < (\lambda_1 - \xi), \quad for\ all\ \ |s| > r_1 \tag{3.30}$$

Furthermore, how f is a continuous function, there exists $M > 0$ such that

$$|h(s)| \leqslant M, \quad \text{for all } |s| \leqslant r_1.$$

As a result

$$\begin{cases} h(s) < (\lambda_1 - \xi)s, & \text{for all } s > r_1, \\ h(s) > (\lambda_1 - \xi)s, & \text{for all } s < -r_1, \\ |h(s)| \leqslant M, \quad \text{for all } |s| \leqslant r_1. \end{cases}$$

Now, we will analyze the cases in which $t > 0$ and $t < 0$. In fact,

Case 1 ($t > 0$): Firs, suppose that $r_1 < t$. Then,

$$\begin{aligned} \int_0^t h(s)ds = \int_0^{r_1} h(s)ds + \int_{r_1}^t h(s)ds \leqslant \int_0^{r_1} Mds + \int_{r_1}^t (\lambda_1 - \xi)sds \\ = Mr_1 + (\lambda_1 - \xi)\frac{t^2}{2} - (\lambda_1 - \xi)\frac{r_1^2}{2} \\ = (\lambda_1 - \xi)\frac{t^2}{2} + C, \end{aligned}$$

where $C = Mr_1 + |\lambda_1 - \xi|\frac{r_1^2}{2}$.

If $r_1 > t$, then

$$\int_0^t h(s)ds \leqslant \int_0^t Mds = Mt \leqslant Mr_1 \leqslant (\lambda_1 - \xi)\frac{t^2}{2} + C,$$

where $C = Mr_1 + |\lambda_1 - \xi|\frac{r_1^2}{2}$.

Case 2 ($t < 0$): First, suppose $r_1 > 0$ such that $t < -r_1 < 0$. Thus,

$$\begin{aligned} \int_0^t h(s)ds &= \int_0^{-r_1} h(s)ds + \int_{-r_1}^t h(s)ds \\ &= -\int_{-r_1}^0 h(s)ds - \int_t^{-r_1} h(s)ds \\ &\leqslant \int_{-r_1}^0 Mds + \int_t^{-r_1} -(\lambda_1 - \xi)sds \\ &= Mr_1 - (\lambda_1 - \xi)\frac{r_1^2}{2} + (\lambda_1 - \xi)\frac{t^2}{2} \\ &= (\lambda_1 - \xi)\frac{t^2}{2} + C, \end{aligned}$$

where $C = Mr_1 + |\lambda_1 - \xi|\frac{r_1^2}{2}$.

If $-r_1 < t$, then

$$\int_0^t h(s)ds \leqslant \int_0^t Mds = Mt \leqslant Mr_1 \leqslant (\lambda_1 - \xi)\frac{t^2}{2} + C,$$

where $C = Mr_1 + |\lambda_1 - \xi|\frac{r_1^2}{2}$.

Hence,

$$\int_0^t h(s)ds \leqslant (\lambda_1 - \xi)\frac{t^2}{2} + C,$$

that is,

$$2\int_0^t h(s)ds \leqslant (\lambda_1 - \xi)t^2 + 2C.$$

Therefore

$$2\int_0^t f(s)ds \geq -(\lambda_1 - \xi)t^2 - 2C.$$

Taking $\xi \in (\lambda_1 - 1, \lambda_1)$, setting $\nu = 1 + \xi - \lambda_1 \in (0,1)$ and integrating over Ω we obtain the desire inequality (3.26) where $c_1 = 2C|\Omega|$.

Finally, for (3.27), it follows from (3.30) that

$$h(s)s < (\lambda_1 - \xi)s^2, \quad for\ all\ |s| > r_1$$

With similar arguments we can conclude (3.30), see, e.g., [16]. □

Since the aim of the problem is work with time-dependent coefficient $\varepsilon(t)$ in equation 3.2 , in order to avoid technical complications only due to the nonlinear term $f(u)$, we require the additional assumption

$$2\langle f(u), u\rangle \geq 2\langle F(u), 1\rangle - (1-\nu)\|u\|_1^2 - c_1. \tag{3.31}$$

Condition (3.31) is ensured by asking, for instance, that

$$\liminf_{|s|\to\infty} f'(s) > -\lambda_1, \tag{3.32}$$

which is slightly less general than (3.14) but still widely used in the literature, as in [8].

Remark 3.5. Recall that for $x, y, n > 0$, we obtain

$$(x+y)^n \leq 2^n \max\{x^n, y^n\} \leq 2^n(x^n + y^n).$$

We will need the following lemma.

Lemma 3.6. *Let $f \in C^1(\mathbb{R})$ be a function such that there are constants $C > 0$ and $\rho > 1$ such that*

$$|f'(s)| \leq C(1 + |s|^{\rho-1}), \quad for\ all\ s \in \mathbb{R}.$$

Then,

$$|f(s) - f(t)| \leq 2^{\rho-1}C|s-t|(1 + |s|^{\rho-1} + |t|^{\rho-1})$$

for all $s, t \in \mathbb{R}$.

Proof. Consider $s,t \in \mathbb{R}$, it follows from the Mean Value Theorem that there exists $\theta \in (0,1)$ such that

$$\begin{aligned} |f(s)-f(t)| &= |s-t||f'(s(1-\theta)+t\theta)| \\ &\leq C|s-t|(1+|s(1-\theta)+t\theta|^{\rho-1}) \\ &\leq 2^{\rho-1}C|s-t|(1+|s(1-\theta)|^{\rho-1}+|t\theta|^{\rho-1}) \\ &\leq 2^{\rho-1}C|s-t|(1+|s|^{\rho-1}+|t|^{\rho-1}), \end{aligned}$$

which proves the result. □

Remark 3.7. It follows from the growth condition (3.13) of f that

$$|f'(s)| \leq C(1+|s|^2), \quad s \in \mathbb{R}.$$

Consequently, by Lemma 3.6 we obtain

$$|f(s)-f(t)| \leq K|s-t|(1+|s|^2+|t|^2),$$

for all $s,t \in \mathbb{R}$.

3.3 Well-posedness

For any $\tau \in \mathbb{R}$, we rewrite problem (3.2) - (3.4) as

$$\begin{cases} \varepsilon u_{tt} + \alpha u_t + Au + f(u) = g, & t > \tau \\ u(\tau) = a, \\ u_t(\tau) = b. \end{cases} \tag{3.33}$$

Consider the family of maps

$$U(t,\tau) : \mathscr{H}_\tau \to \mathscr{H}_t \quad \textit{with} \quad t \geq \tau,$$

defined by

$$U(t,\tau) = \{u(t), u_t(t)\}.$$

Problem (3.33) generates a continuous process $U(t,\tau) : \mathscr{H}_\tau \to \mathscr{H}_t$, for all $t \geq \tau$, this statement follows by Theorem 3.8.

Global existence of (weak) solutions u of equation (3.33) is classical, and can be obtained by means of a standard Galerkin scheme, as we can see the details and proofs in [17] and [11].

Furthermore, based on Lemma 3.10 described below we can obtain that such solutions satisfy, on any interval $[\tau,t]$ with $t \geq \tau$,

$$u \in \mathscr{C}([\tau,t], H_1)$$

and

$$u_t \in \mathscr{C}([\tau,t], H),$$

see, e.g., [17].

Moreover, the process $U(\cdot,\cdot)$ satisfies the following continuous dependence property.

Theorem 3.8. *Problem (3.33) generates a continuous process $U(t,\tau):\mathscr{H}_\tau\to\mathscr{H}_t$, with $t\geq\tau\in\mathbb{R}$, satisfying the following continuous dependence property: for every pair of initial data $z_i=\{a_i,b_i\}\in\mathscr{H}_\tau$ such that $\|z_i\|_{\mathscr{H}_\tau}\leq R$, i=1,2, the difference of the corresponding solutions satisfies*

$$\|U(t,\tau)z_1-U(t,\tau)z_2\|_{\mathscr{H}_t}\leq e^{K(t-\tau)}\|z_1-z_2\|_{\mathscr{H}_\tau},\quad for\ all\quad t\geq\tau, \tag{3.34}$$

for some constant $K=K(R)\geq 0$.

Remark 3.9. Note that, uniqueness of solutions in problem (3.33) will follow by the continuous dependence estimate (3.34). Indeed, for $\tau\in\mathbb{R}$ and $z=\{a,b\}\in\mathscr{H}_\tau$ such that $\|z_i\|_{\mathscr{H}_\tau}\leq R$. Let $U_1(t,\tau)$ and $U_2(t,\tau)$ be solutions of (3.33), it follows by (3.34) that

$$U_1(t,\tau)z=U_2(t,\tau)z,\quad for\ \ all\ \ t\geq\tau.$$

Therefore, $U_1(t,\tau)=U_2(t,\tau)$ for all $t\geq\tau$.

Taking all the discourses described above together, we can conclude that the family of maps

$$U(t,\tau):\mathscr{H}_\tau\to\mathscr{H}_t\quad with\quad t\geq\tau\in\mathbb{R},$$

defined by

$$U(t,\tau)z=\{u(t),u_t(t)\},$$

where u is the unique solution to (3.33) with initial time τ and initial condition $z=\{a,b\}\in\mathscr{H}_\tau$, defines a continuous process on the family $\{\mathscr{H}_t\}_{t\in\mathbb{R}}$.

For the proof of Theorem 3.8 we will need the following dissipation estimate.

Lemma 3.10. *Let $t\geq\tau$. For $z\in\mathscr{H}_\tau$, let $U(t,\tau)$ be the solution of (3.33) with initial time τ and datum $z=\{a,b\}$. Then, if (3.31) holds, there exist $\omega=\omega(\alpha,\|\varepsilon\|_{L^\infty},\|\varepsilon'\|_{L^\infty})>0$, $K_1\geq 0$ and an increasing positive function ψ such that*

$$\|U(t,\tau)z\|_{\mathscr{H}_t}\leq\psi(\|z\|_{\mathscr{H}_\tau})e^{-\omega(t-\tau)}+K_1,\quad for\ all\quad t\geq\tau.$$

Proof. Let $C\geq 0$ be a generic constant independent of the initial datum z and denote

$$E(t)=\|U(t,\tau)z\|^2_{\mathscr{H}_t}$$

(double) the energy associated with problem (3.33). Due to (3.12), (3.13) and (3.26), the functional

$$\xi=E+\delta\alpha\|u\|^2+2\delta\varepsilon\langle u_t,u\rangle+2\langle F(u),1\rangle-2\langle g,u\rangle$$

fulfills, for $\delta>0$ and some $0<\nu<1$ provided by Lemma 3.4,

$$\nu E(t)-C\leq\xi(t)\leq CE(t)^2+C. \tag{3.35}$$

In light of (3.12), set $\delta > 0$ small enough such that $CL\delta < \frac{\alpha}{2}$, then it follows from Cauchy-Schwarz and Young's inequality that

$$\begin{aligned} 2\delta\varepsilon|\langle u_t, u\rangle| &\leq 2\delta\varepsilon\|u_t\|\|u\| \\ &\leq \frac{\varepsilon}{2}\|u_t\|^2 + \delta\|u\|^2 \\ &\leq \frac{\varepsilon}{2}\|u_t\|^2 + CL\delta^2\|u\|^2 \\ &\leq \frac{\varepsilon}{2}\|u\|^2 + \frac{\delta\alpha}{2}\|u\|^2. \end{aligned}$$

Now, multiplying (3.2) by $2u_t$,

$$\frac{d}{dt}(E(t) + 2\langle F(u), 1\rangle - 2\langle g, u\rangle) + [2\alpha - \varepsilon'(t)]\|u_t\|^2 = 0. \tag{3.36}$$

In addition, multiplying (3.2) by $2\delta u$,

$$2\delta\varepsilon\langle u_{tt}, u\rangle + 2\delta\alpha\langle u_t, u\rangle - 2\delta\langle u, \Delta u\rangle + 2\delta\langle f(u), u\rangle - 2\delta\langle g, u\rangle = 0.$$

Resolving some terms we obtain

$$2\delta\left[\frac{d}{dt}\varepsilon\langle u_t, u\rangle - \varepsilon'\langle u_t, u\rangle - \varepsilon'\|u_t\|^2\right] + 2\delta\alpha\langle u_{tt}, u_t\rangle + 2\delta\|u\|_1^2 + 2\delta\langle f(u), u\rangle - 2\delta\langle g, u\rangle = 0. \tag{3.37}$$

Then, summing (3.36) and (3.37) we obtain

$$\frac{d}{dt}\xi + [2\alpha - \varepsilon' - 2\delta\varepsilon]\|u_t\|^2 + 2\delta\|u\|_1^2 + 2\delta\langle f(u), u\rangle - 2\delta\langle g, u\rangle = 2\delta\varepsilon'\langle u_t, u\rangle.$$

Furthermore, by (3.12), Cauchy-Schwarz and Young's inequality we have

$$\begin{aligned} 2\delta|\varepsilon'\langle u_t, u\rangle| &\leq 2L\delta|\langle u_t, u\rangle| \\ &\leq 2\delta L\|u_t\|\|u\| \\ &\leq \frac{\alpha}{2}\|u_t\|^2 + \frac{\delta\nu}{2}\|u\|_1^2 \end{aligned}$$

for δ small, we arrive at

$$\frac{d}{dt}\xi + \left[\frac{3}{2} - \varepsilon' - 2\delta\varepsilon\right]\|u_t\|^2 + \delta\left[2 - \frac{\nu}{2}\right]\|u\|_1^2 + 2\delta\langle f(u), u\rangle - 2\delta\langle g, u\rangle \leq 0. \tag{3.38}$$

In light of (3.31) we can reconstruct the functional ξ, which provides

$$\frac{d}{dt}\xi + \delta\xi + \alpha\|u_t\|^2 + \Gamma \leq \delta c_1,$$

where

$$\Gamma = \left[\frac{\alpha}{2} - \varepsilon' - 3\delta\varepsilon\right]\|u_t\|^2 + \frac{\delta\nu}{2}\|u\|_1^2 - \delta^2\alpha\|u\|^2 - 2\delta^2\varepsilon\langle u_t, u\rangle.$$

Therefore, setting δ small enough so that $\Gamma \geq 0$, we end up with

$$\frac{d}{dt}\xi + \delta\xi + \alpha\|u_t\|^2 \leq \delta c_1. \tag{3.39}$$

Applying the Gronwall lemma, together with (3.35), we have proved Lemma 3.10. □

Proof of Theorem 3.8. Let $z_1, z_2 \in \mathscr{H}_t$ be such that $\|z_i\|_{\mathscr{H}_t} \le R$, $i = 1,2$, and denote by C a generic positive constant depending on R but independent of z_i. We first observe that the energy estimate in Lemma 3.10 ensures

$$\|U(t,\tau)z_i\|_{\mathscr{H}_t} \le C. \tag{3.40}$$

We call $\{u_i(t), \partial_t u_i(t)\} = U(t,\tau)z_i$ and denote $\bar{z}(t) = \{\bar{u}(t), \bar{u}_t(t)\} = U(t,\tau)z_1 - U(t,\tau)z_2$. Then, the difference between the two solutions satisfies

$$\varepsilon\bar{u}_{tt} + \alpha\bar{u}_t - \Delta\bar{u} + f(u_1) - f(u_2) = 0,$$

with initial datum $z(\tau) = z_1 - z_2$. Multiplying by $2\bar{u}_t$ we obtain

$$\langle 2\bar{u}_t, \varepsilon\bar{u}_{tt}\rangle + \langle 2\bar{u}_t, \alpha\bar{u}_t\rangle + \langle 2\bar{u}_t, -\Delta\bar{u}\rangle = -\langle f(u_1) - f(u_2), \bar{u}_t\rangle,$$

that is,

$$2\varepsilon\int_\Omega \bar{u}_t\bar{u}_{tt}dx + 2\alpha\int_\Omega |\bar{u}_t|^2dx - 2\int_\Omega \bar{u}_t\Delta\bar{u}dx = -\langle f(u_1) - f(u_2), \bar{u}_t\rangle.$$

It follows from (3.20)-(3.22) that

$$\frac{d}{dt}\varepsilon(t)\|\bar{u}_t\|^2 - \varepsilon'(t)\|\bar{u}_t\|^2 + 2\alpha\|\bar{u}_t\|^2 + \frac{d}{dt}\|u\|_1^2 = -\langle f(u_1) - f(u_2), \bar{u}_t\rangle.$$

Thus,

$$\frac{d}{dt}\|\bar{z}\|^2_{\mathscr{H}_t} + [2\alpha - \varepsilon']\|\bar{u}_t\|^2 = -2\langle f(u_1) - f(u_2), \bar{u}_t\rangle. \tag{3.41}$$

Observe that

$$\begin{aligned}-2\langle f(u_1) - f(u_2), \bar{u}_t\rangle &\le 2|\langle f(u_1) - f(u_2), \bar{u}_t\rangle| \\ &\le 2\|f(u_1) - f(u_2)\|\|\bar{u}_t\|.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|f(u_1) - f(u_2)\|^2 &= \int_\Omega |f(u_1) - f(u_2)|^2dx \\ &= \int_\Omega |f(u_1(x)) - f(u_2(x))|^2dx.\end{aligned}$$

Now, it follows by Remark 3.7 that

$$\int_\Omega |f(u_1(x)) - f(u_2(x))|^2dx \le K\int_\Omega |u_1(x) - u_2(x)|^2(1 + |u_1(x)|^2 + |u_2(x)|^2)^2dx.$$

Applying Holder's inequality we obtain

$$\begin{aligned}\int_\Omega |f(u_1(x)) - f(u_2(x))|^2dx &\le K\left(\int_\Omega |u_1(x) - u_2(x)|^6dx\right)^{\frac{1}{3}}\left(\int_\Omega (1 + |u_1(x)|^2 + |u_2(x)|^2)^3dx\right)^{\frac{2}{3}} \\ &= K\left(\|\bar{u}\|^6_{L^6}\right)^{\frac{1}{3}}\left(\int_\Omega (1 + |u_1(x)|^2 + |u_2(x)|^2)^3dx\right)^{\frac{2}{3}}.\end{aligned}$$

Using Remark 3.5 and the embedding $H_0^1(\Omega) \subset L^6(\Omega)$ we have

$$\begin{aligned}\int_\Omega |f(u_1(x)) - f(u_2(x))|^2 dx &\le K\left(\|\overline{u}\|_{L^6}^6\right)^{\frac{1}{3}}\left(\int_\Omega (1+|u_1(x)|^2+|u_2(x)|^2)^3 dx\right)^{\frac{2}{3}}\\ &\le K\left(\|\overline{u}\|_{L^6}^2\right)\left(2^6\int_\Omega (1+|u_1(x)|^6+|u_2(x)|^6)dx\right)^{\frac{2}{3}}\\ &= K\left(\|\overline{u}\|_{L^6}^2\right)\left(2^6\left[|\Omega|+\|u_1\|_{L^6}^6+\|u_2\|_{L^6}^6\right]\right)^{\frac{2}{3}}\\ &\le K\left(K_1\|\overline{u}\|_1^2\right)\left(2^6\left[|\Omega|+K_2\|u_1\|_1^6+K_3\|u_2\|_1^6\right]\right)^{\frac{2}{3}}.\end{aligned}$$

It follows from (3.40) that

$$\int_\Omega |f(u_1(x)) - f(u_2(x))|^2 dx \le L\|\overline{u}\|_1^2.$$

Consequently,

$$-2\langle f(u_1)-f(u_2),\overline{u}_t\rangle \le 2L\|\overline{u}\|_1\|\overline{u}_t\| = 2L\frac{2\alpha}{2\alpha}\|\overline{u}\|_1\|\overline{u}_t\|.$$

Moreover, from Young's inequality we obtain

$$-2\langle f(u_1)-f(u_2),\overline{u}_t\rangle \le M\|\overline{u}\|_1^2+2\alpha\|\overline{u}_t\|^2,$$

where $M = \frac{L}{4\alpha}$. Furthermore, how ε is a decreasing function and satisfies (3.5) we can conclude that

$$\begin{aligned}-2\langle f(u_1)-f(u_2),\overline{u}_t\rangle &\le \varepsilon(t)\|\overline{u}_t\|^2+M\|\overline{u}\|_1^2\\ &\le M\varepsilon(t)\|\overline{u}_t\|^2+M\|\overline{u}\|_1^2.\end{aligned}$$

Therefore, we end up with the differential inequality

$$\frac{d}{dt}\|\overline{z}(t)\|_{\mathscr{H}_t}^2 \le M\|\overline{z}(t)\|_{\mathscr{H}_t}^2, \tag{3.42}$$

and an application of the Gronwall lemma on $[\tau,t]$ completes the proof. □

3.3.1 Absorbing sets

This subsection is devoted to studying the dissipation properties of the process $U(\cdot,\cdot)$ associated with (3.33).

Definition 3.11. A **time-dependent absorbing** set for the process $U(t,\tau)$ is a uniformly bounded family $\mathscr{B} = \{\mathscr{B}(t)\}_{t\in\mathbb{R}}$ with the following property: for every $R \ge 0$ there exists $\theta_e = \theta_e(R) \ge 0$ such that

$$U(t,\tau)\mathbb{B}_\tau(R) \subset \mathscr{B}(t) \quad for\ all \quad \tau \le t-\theta_e,$$

where $\mathbb{B}_\tau(R)$ denotes the R-ball of $\mathscr{H}_t$.

Remark 3.12. The notion of absorption in Definition 3.11 is stronger than the pullback dissipativity of Definition 2.22.

The existence of a time-dependent absorbing set (hence pullback absorbing) for $U(t,\tau)$ it follows by the next result.

Theorem 3.13. *There exists $R_0 > 0$ such that the family $\mathscr{B} = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$ is a time-dependent absorbing set for $U(t,\tau)$. Furthermore,*

$$\sup_{z\in\mathbb{B}_\tau(R_0)} \left[\|U(t,\tau)z\|_{\mathscr{H}_t} + \int_\tau^\infty \|u_t(y)\|^2 dy \right] \le I_0, \quad for\ all \quad \tau \in \mathbb{R}, \tag{3.43}$$

for some $I_0 \ge R_0$.

Remark 3.14. The first statement of the theorem above means: there exists $R_0 > 0$ such that for every $R \ge 0$ there exists $\theta_e = \theta_e(R) \ge 0$ with

$$\tau \le t - \theta_e \quad \Longrightarrow \quad U(t,\tau)\mathbb{B}_\tau(R) \subset \mathbb{B}_t(R_0).$$

This implies that for $z \in \mathbb{B}_\tau(R)$ yields

$$\|U(t,\tau)z\|_{\mathscr{H}_t} \le R_0.$$

Proof of Theorem 3.13. Let $R_0 = 1 + 2K_1$. An application of Lemma 3.10 for $z \in \mathbb{B}_\tau(R)$ yields

$$\|U(t,\tau)z\|_{\mathscr{H}_t} \le \mathscr{Q}(R)e^{-w(t-\tau)} + K_1 \le 1 + 2K_1 = R_0,$$

provided that $t - \tau \ge \theta_e$ where

$$\theta_e = max\left\{0, w^{-1}\log\frac{\mathscr{Q}(R)}{1+K_1}\right\}.$$

This concludes the proof of the existence of the time-dependent absorbing set. In order to prove the integral estimate for $\|u_t\|$, it is enough to integrate (3.39) with $\delta = 0$ on $[\tau, \infty)$ □

Remark 3.15. We can assume that the time-dependent absorbing set $B(t) = \mathbb{B}_t(R_0)$ is positively invariant (namely $U(t,\tau)B(\tau) \subset B(t)$ for all $t \ge \tau$). Indeed, calling θ_e the entering time of $B(t)$ such that

$$U(t,\tau)B(\tau) \subset B(t), \quad for\ all \ \ \tau \le t - \theta_e,$$

we can substitute $B(t)$ with the invariant absorbing family

$$\bigcup_{\tau \le t-\theta_e} U(t,\tau)B(\tau) \subset B(t).$$

3.4 Existence of the pullback attractor

This last section is devoted to prove the main result of this chapter, that is, a result on the existence of the pullback attractor for the problem (3.33). Moreover, this result represents the asymptotic behavior of the problem (3.33).

Theorem 3.16. *The process $U(t,\tau):\mathscr{H}_\tau \longrightarrow \mathscr{H}_t$ generated by problem (3.33) admits an invariant pullback attractor $\mathscr{A}=\{\mathscr{A}(t):t\in\mathbb{R}\}$. Furthermore, $\mathscr{A}(t)$ is bounded in $\mathscr{H}_t^1$, with a bound independent of t.*

The existence of the attractor, according to Definition 2.30, will be proved by a direct application of the abstract Theorem 2.31. Precisely, in order to show that the process is asymptotically compact, we shall exhibit a pullback attracting family of compact sets. To this aim, the strategy classically consists in finding a suitable decomposition of the process in the sum of a decaying part and of a compact one.

3.4.1 The decomposition

We write $f=f_0+f_1$, where $f_0,f_1\in C^2(\mathbb{R})$ fulfill, for some $k\geq 0$,

$$|f_1'(s)|\leq k, \quad for\ all\ \ s\in\mathbb{R}, \tag{3.44}$$

$$|f_0''(s)|\leq k(1+|s|), \quad for\ all\ \ s\in\mathbb{R}, \tag{3.45}$$

$$f_0(0)=f_0'(0)=0, \tag{3.46}$$

$$f_0(s)s\geq 0, \quad for\ all\ \ s\in\mathbb{R}. \tag{3.47}$$

This is possible owing the assumptions (3.13) and (3.14) (as we can see in [2] and [12]).

Let $\mathscr{B}=\{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$ be a time-dependent absorbing set according to Theorem 3.13 and let $\tau\in\mathbb{R}$ be fixed. Then, for any $z\in\mathbb{B}_\tau(R_0)$, we split $U(t,\tau)z$ into the sum

$$U(t,\tau)z=\{u(t),u_t(t)\}=U_0(t,\tau)z+U_1(t,\tau)z,$$

where

$$U_0(t,\tau)z=\{v(t),v_t(t)\}\ \ and\ \ U_1(t,\tau)z=\{w(t),w_t(t)\}$$

solve systems

$$\begin{cases}\varepsilon v_{tt}+\alpha v_t+Av+f_0(v)=0,\\ U_0(t,\tau)=z,\end{cases} \tag{3.48}$$

and

$$\begin{cases}\varepsilon w_{tt}+\alpha w_t+Aw+f(u)+f_0(v)=g,\\ U_1(t,\tau)=0.\end{cases} \tag{3.49}$$

In what follows, the generic constant $C\geq 0$ depends only on $\mathscr{B}$.

Lemma 3.17. *There exists $\delta=\delta(\mathscr{B})>0$ such that*

$$\|U_0(t,\tau)z\|_{\mathscr{H}_t}\leq Ce^{-\delta(t-\tau)},\ \ for\ \ all\ \ t\geq\tau.$$

Proof. Repeating word by word of the proof of Lemma 3.10 with f_0 instead of f we immediately obtain the bound

$$\|U_0(t,\tau)z\|_{\mathscr{H}_t} \le C. \tag{3.50}$$

Then, denoting

$$\mathscr{E}_0 = \|U_0(t,\tau)z\|^2_{\mathscr{H}_t} + \delta\alpha\|v\|^2 + 2\delta\varepsilon\langle v_t, v\rangle + 2\langle F_0(v), 1\rangle, \tag{3.51}$$

with

$$F_0(s) = \int_0^s f_0(y)dy,$$

we multiply (3.48) by $2v_t + 2\delta v$. In view of (3.47) and since $g = 0$, the analogous of the differential inequality (3.38) now reads

$$\frac{d}{dt}\xi_0 + \delta\|U_0(t,\tau)\|^2_{\mathscr{H}_t} \le 0.$$

Exploiting (3.50) and (3.51),

$$\frac{1}{2}\|U_0(t,\tau)z\|^2_{\mathscr{H}_t} \le \xi_0(t) \le C\|U_0(t,\tau)z\|^2_{\mathscr{H}_t}.$$

Moreover, the Gronwall lemma completes the argument. □

Remark 3.18. How $U(\cdot,\cdot)$ is a continuous process and we make a decomposition of this process in the sum of a decaying part and a compact one, summing up, the following uniform bound holds,

$$\sup_{t\ge\tau}[\|U(t,\tau)z\|_{\mathscr{H}_t} + \|U_0(t,\tau)z\|_{\mathscr{H}_t} + \|U_1(t,\tau)z\|_{\mathscr{H}_t}] \le C. \tag{3.52}$$

This fact will be used in the next result.

Lemma 3.19. *There exists $M = M(\mathscr{B}) > 0$ such that*

$$\sup_{t\ge\tau}\|U_1(t,\tau)z\|_{\mathscr{H}_t^{1/3}} \le M.$$

Proof. We choose $\delta > 0$ small and $C > 0$ large enough such that, calling

$$\Lambda = \|U_1(t,\tau)z\|^2_{\mathscr{H}_t^{1/3}} + \delta\alpha\|w\|^2_{1/3} + 2\delta\varepsilon\langle w_t, A^{1/3}w\rangle + 2\langle f(u) - f_0(v) - g, A^{1/3}w\rangle + C,$$

we have

$$\frac{1}{2}\|U_1(t,\tau)z\|^2_{\mathscr{H}_t^{1/3}} \le \Lambda(t) \le 2\|U_1(t,\tau)z\|^2_{\mathscr{H}_t^{1/3}} + 2C. \tag{3.53}$$

Indeed, in view of (3.52) and Remark 3.7,

$$2\langle f(u) - f_0(v), A^{1/3}w\rangle \le 2\|f(u) - f_0(v)\|\|A^{1/3}w\| \le C\|w\|_{2/3} \le \frac{1}{4}\|w\|^2_{4/3} + C.$$

Moreover, by (3.12), for δ small we can estimate

$$2\delta\varepsilon|\langle w_t, A^{1/3}w\rangle| \le \frac{\varepsilon}{2}\|w_t\|_{1/3}^2 + \frac{\delta\alpha}{2}\|w\|_{1/3}^2.$$

By multiplying (3.49) with $2A^{1/3}w_t + 2\delta A^{1/3}w$, we infer that

$$\begin{aligned}\frac{d}{dt}\Lambda + [2\alpha - \varepsilon' - 2\delta\varepsilon]\|w_t\|_{1/3}^2 + 2\delta\|w\|_{4/3}^2 + 2\delta\langle f(u) - f_0(v) - g, A^{1/3}w\rangle \\ = 2\delta\varepsilon'\langle w_t, A^{1/3}w\rangle + I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned} I_1 &= 2\langle [f_0'(u) - f_0'(v)]u_t, A^{1/3}w\rangle, \\ I_2 &= 2\langle f_0'(v)w_t, A^{1/3}w\rangle, \\ I_3 &= 2\langle f_1'(u)u_t, A^{1/3}w\rangle. \end{aligned}$$

Then, for any fixed $\delta > 0$ small enough, we obtain

$$\frac{d}{dt}\Lambda + \delta\Lambda + \alpha\|w_t\|_{1/3}^2 \le I_1 + I_2 + I_3 + \delta C. \tag{3.54}$$

By exploiting conditions (3.45)-(3.46) for f_0 and the embeddings $H_{(3p-6)/2p} \subset L^p(\Omega)$ $(p > 2)$ (see ,e.g., for theoretical background [1] and [11]), we draw from (3.52)-(3.53) the estimates

$$\begin{aligned} I_1 &\le C(1 + \|u\|_{L^6} + \|v\|_{L^6})\|u_t\|\|w\|_{L^{18}}\|A^{1/3}w\|_{L^{18/5}} \\ &\le C\|u_t\|\|w\|_{4/3}^2 \\ &\le \frac{\delta}{2}\Lambda + C\|u_t\|^2\|w\|_{4/3}^2, \\ I_2 &\le C(\|v\|_{L^6} + \|v\|_{L^6}^2)\|w_t\|_{L^{18/7}}\|A^{1/3}w\|_{L^{18/5}} \\ &\le C\|v\|_1\|w_t\|_{1/3}\|w\|_{4/3} \\ &\le \frac{\alpha}{2}\|w_t\|_{1/3}^2 + C\|v\|_1^2\|w\|_{4/3}^2 \end{aligned}$$

Furthermore, in view of (3.44), we have

$$I_3 \le k\|u_t\|\|A^{1/3}w\| \le \|u_t\|^2\|w\|_{4/3}^2 + C.$$

As a consequence, inequality (3.54) improves to

$$\frac{d}{dt}\Lambda + \frac{\delta}{2}\Lambda \le q\Lambda + C,$$

with $q = C\|u_t\|^2 + C\|v\|_1^2$ satisfying

$$\int_\tau^\infty q(y)dy \le C,$$

by virtue of the dissipation integral (3.13) and Lemma 3.17. By Lemma 3.3,

$$\Lambda(t) \le C\Lambda(\tau)e^{-\frac{\delta}{4}(t-\tau)} + C \le C.$$

In turn, (3.53) yields the boundedness of $U_1(t,\tau)z$ in $\mathscr{H}_t^{1/3}$. □

3.4.2 Existence of the pullback attractor

According to Lemma 3.19, we consider the family $\mathscr{K} = \{\mathscr{K}(t)\}_{t\in\mathbb{R}}$ where

$$\mathscr{K}(t) = \{z \in \mathscr{H}_t^{1/3} : \|z\|_{\mathscr{H}_t^{1/3}} \leq M\}.$$

$\mathscr{K}(t)$ is compact by the compact embedding $\mathscr{H}_t^{1/3} \Subset \mathscr{H}_t$. Furthermore, since the injection constants are independent of t, $\mathscr{K}$ is uniformly bounded. Finally, Theorem 3.13, Lemma 3.17 and Lemma 3.19 show that $\mathscr{K}$ is pullback attracting. Indeed,

$$dist_{X_t}(U(t,\tau)\mathbb{B}_\tau(R_0), \mathscr{K}(t)) \leq Ce^{-\delta(t-\tau)}, \ \ for\ all\ \ t \geq \tau.$$

Thus, the process $U(\cdot,\cdot)$ is asymptotically compact, which allows the application of Theorem 2.31 and proves the existence of the unique pullback attractor $\mathscr{A} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$. The invariance of $\mathscr{A}$ follows by the abstract Theorem 2.41 due to the continuity of the process stated in Theorem 3.8. Note that, how the process is continuous, then is T-closed.

Remark 3.20. The attraction exerted by the attractor is uniform on compact intervals of time by virtue of Proposition 2.26, due to the continuous dependence estimate (3.34).

3.4.3 Regularity of the attractor

The minimality of $\mathscr{K}$ in $\mathbb{K}$ establishes that $\mathscr{A}(t) \subset \mathscr{K}(t)$ for all $t \in \mathbb{R}$. Therefore, we immediately have the following regularity result.

Corollary 3.21. *$\mathscr{A}(t)$ is bounded in $\mathscr{H}_t^{1/3}$ (with a bound independent of t).*

To prove that $\mathscr{A}(t)$ is bounded in $\mathscr{H}_t^1$, as claimed in Theorem 3.16, we argue as follows. We fix $\tau \in \mathbb{R}$ and, for $z \in \mathscr{A}(\tau)$, we split the solution $U(t,\tau)z$ into the sum $U_0(t,\tau)z + U_1(t,\tau)z$, where $U_0(t,\tau)z = \{v(t), v_t(t)\}$ and $U_1(t,\tau)z = \{w(t), w_t(t)\}$, instead of (3.48)-(3.49), now solve

$$\begin{cases} \varepsilon v_{tt} + \alpha v_t + Av = 0, \\ U_0(t,\tau) = z, \end{cases}$$

and

$$\begin{cases} \varepsilon w_{tt} + \alpha w_t + Aw + f(u) = g, \\ U_1(t,\tau) = 0. \end{cases}$$

As a particular case of Lemma 3.17, we obtain

$$\|U_0(t,\tau)z\|_{\mathscr{H}_t} \leq Ce^{-\delta(t-\tau)}, \ \ for\ \ all\ \ t \geq \tau. \tag{3.55}$$

Lemma 3.22. *We have the uniform bound*

$$\sup_{t\geq\tau} \|U_1(t,\tau)z\|_{\mathscr{H}_t^1} \leq M_1.$$

for some $M_1 = M_1(\mathscr{K}) > 0$.

Proof. We set

$$\mathscr{E}_1 = \|U_1(t,\tau)z\|^2_{\mathscr{H}^1_t} + \delta\alpha\|w\|^2_1 + 2\delta\varepsilon\langle w_t, Aw\rangle - 2\langle g, Aw\rangle + c,$$

for $\delta > 0$ small and some $c \geq 0$ (depending on $\|g\|$) large enough to ensure

$$\frac{1}{4}\|U_1(t,\tau)z\|^2_{\mathscr{H}^1_t} \leq \mathscr{E}_1(t) \leq 2\|U_1(t,\tau)z\|^2_{\mathscr{H}^1_t} + 2c. \tag{3.56}$$

A multiplication by $2Aw_t + 2\delta Aw$ leads to the equality

$$\begin{aligned}\frac{d}{dt}\mathscr{E}_1 + [2\alpha - \varepsilon' - 2\delta\varepsilon]\|w_t\|^2_1 + 2\delta\|w\|^2_2 - 2g\langle g, Aw\rangle \\ = 2\delta\varepsilon'\langle w_t, Aw\rangle - 2\langle f(u), Aw_t\rangle - 2\delta\langle f(u), Aw\rangle\end{aligned}$$

and after standard computations we get, for δ small enough,

$$\frac{d}{dt}\mathscr{E}_1 + \delta\mathscr{E}_1 \leq -2\langle f(u), Aw_t\rangle - 2\delta\langle f(u), Aw\rangle + \delta c.$$

Denoting by $C > 0$ a generic constant depending on the size of $\mathscr{A}(t)$ in $\mathscr{H}_t^{1/3}$, we find, using the invariance of the attractor,

$$\|U(t,\tau)z\|_{\mathscr{H}_t^{1/3}} \leq C.$$

Hence, exploiting the embeddings $H_{4/3} \subset L^{18}(\Omega)$ and $H_{1/3} \subset L^{18/7}(\Omega)$, we deduce the bound

$$\|f(u)\|_1 \leq \|f'(u)\|_{L^9}\|A^{1/2}u\|_{L^{18/7}} \leq C(1 + \|u\|^2_{L^{18}}) \leq C,$$

yielding

$$-2\langle f(u), Aw_t\rangle - 2\delta\langle f(u), Aw\rangle \leq 2\|f(u)\|_1(\|w_t\|_1 + \|w\|_1) \leq \frac{\delta}{2}\mathscr{E}_1 + C.$$

We finaly end up with

$$\frac{d}{dt}\mathscr{E}_1 + \frac{\delta}{2}\mathscr{E}_1 \leq C,$$

and an application of the standard Gronwall lemma, recalling (3.56), provides the uniform boundedness of $\|U_1(t,\tau)z\|_{\mathscr{H}^1_t}$, as claimed. □

We are now in position to conclude the proof of Theorem 3.16. Indeed, inequality (3.55) and Lemma 3.22 imply that, for all $t \in \mathbb{R}$,

$$\lim_{\tau\to-\infty} dist_{X_t}(U(t,\tau)\mathscr{A}(\tau), \mathscr{K}^1(t)) = 0,$$

having defined

$$\mathscr{K}^1(t) = \{z \in \mathscr{H}^1_t : \|z\|_{\mathscr{H}^1_t} \leq M_1\}.$$

Since $\mathscr{K}$ is invariant, this means

$$dist_{X_t}(\mathscr{A}(t), \mathscr{K}^1(t)) = 0.$$

Therefore, $\mathscr{A}(t) \subset \overline{\mathscr{K}^1(t)} = \mathscr{K}^1(t)$, proving that $\mathscr{A}(t)$ is bounded in $\mathscr{H}_t^1$ with a bound independent of $t \in \mathbb{R}$.

Bibliography

[1] ADAMS, R.A.; FOURNIER, J.F. **Sobolev Spaces.** Oxford: Academic Press, second edition, 2003.

[2] ARRIETA, J.; CARVALHO, A.N.; HALE, J. **A damped hyperbolic equation with critical exponent**. Comm. Partial Differential Equations, vol.17, 1992, pp. 841-866.

[3] BABIN, A.V.; VISHIK, M.I. **Attractors of Evolution Equations**, North-Holland, Amsterdam, 1992.

[4] CARVALHO, A.N. **Sistemas Dinâmicos Não-lineares.** Notas de Aulas, Instituto de Ciências Matemáticas e de Computação, São Carlos, 2012.

[5] CARVALHO, A.N.; LANGA, J.A.; ROBINSON, J.C. **Attractors for Infinite-dimensional Non-autonomous Dynamical Systems.** Springer, New York, 2013, vol. 182.

[6] CHEBAN, D.V. **Global Attractors of Non-autonomous Dissipative Dynamical Systems**, Scientific World, New Jersey, 2004.

[7] CHOLEWA, J.W.; DLOTKO, T. **Global Attractors in Abstract Parabolic Problems.** Cambridge University Press, New York, 2000.

[8] CONTI, M.; MARCHINI, E.M.; PATA, V. **Reaction-diffusion with memory in the minimal state framework**. Trans. Amer. Math. Soc., vol. 366, 2014, pp. 4969-4986.

[9] CONTI, M.; PATA, V.; TEMAM, R. **Attractors for processes on time-dependent spaces. Applications to wave equations.** J. Differential Equations, vol. 255, 2013, pp. 1254 -1277.

[10] DI PLINIO, F.; DUANE, G.S.; TEMAM, R. **Time dependent attractor for the oscillon equation.** Discrete Contin. Dyn. Syst., vol. 29, 2011, pp. 141-167.

[11] EVANS, L.C. **Partial Differential Equations**. American Mathematical Society, vol. 19.

[12] GRASSELI, M.; PATA, V. **Asymptotic behavior of a parabolic-hyperbolic system**. Commun. Pure Appl. Anal., n. 3, 2004, pp. 849-881.

[13] HALE, J. **Asymptotic Behavior of Dissipative Systems.** Mathematical Surveys and Monographs, American Mathematical Society, Providence, 1988.

[14] PAZY, A. **Semigroups of Linear Operators and Applications to Partial Differential Equations**. Springer, New York, 1983.

[15] RENARDY, M.; ROGERS, R.C. **An Introduction to Partial Differential Equations**, Springer, Second edition, 2004.

[16] SANTIAGO, E.B. **Non-autonomous Klein-Gordon-Zakharov system: pullback dynamics in the continuous and impulsive approaches**. Thesis, Doctorate in mathematics, Universidade Federal de São Carlos, São Carlos, 2020.

[17] TEMAM, R. **Infinite-Dimensional Dynamical Systems in Mechanics and Physics.** Springer, New York, 1997, Second edition, vol. 68.