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Modified Control Chart For Variance Monitoring

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ABSTRACT

Control charts are powerful tools used by many industries to monitor the quality of their processes and detect special causes of variation. They are often used to monitor the mean of some process quality characteristic with the well known $\bar{X}$, while the process variability can be monitored with either a control chart for the range, called R control chart, for the standard deviation, called S control chart or for the variance, called S^2 control chart. This work will focus on the S^2 control chart. In their original formulation, if the actual process mean (μ) or variance (σ^2) are different or larger from their specified in-control values (μ_0 and σ_0^2, respectively), the process is declared out-of-control. However, in many practical situations, even though the process may be declared out-of-control, it might be still capable from a practical point of view in terms of the proportion of nonconforming items produced. Thus, it may not be necessary to stop the process and start looking for assignable causes, which can save time and resources. The Modified Control Charts were designed to monitor the process mean in such a capable situation. However, regarding the monitoring the process variance, there is still no control chart presented in the Statistical Process Control (SPC) literature that considers the capability of the process while monitoring the process variance. According to Juran and Godfrey (1998), capability is the sense of a competence, based on tested performance, to produce quality products. With this background as motivation, in this work, we derive a Modified Control Chart developed to monitor the mean, for monitoring the process variance, considering the capability of the process.

Keywords: Modified Control Chart. S^2 Control Chart. Chart Performance.

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LIST OF SYMBOLS

σ	:	True process standard deviation
σ_0	:	Known standard deviation
σ^2	:	True process variance
σ_1^2	:	Actual process variance
σ_0^2	:	Known or target variance
σ_U^2	:	Largest permissible value for the variance
σ_L^2	:	Smallest permissible value for the variance
$\hat{\sigma}$	:	Estimated standard deviation
$\hat{\sigma}_0^2$	:	Estimator of σ_0^2
$\hat{\sigma}^2$	:	Estimated variance
σ_{MAX}^2	:	The maximum allowed value for process variance
S^2	:	Sample variance
S_p	:	Pooled standard deviation
μ	:	True process mean
μ_1	:	Actual process mean
μ_0	:	Known or target mean
μ_U	:	Largest permissible value for the mean
μ_L	:	Smallest permissible value for the mean
$\bar{x}$	:	Arithmetic sample mean
α	:	Type I error probability
β	:	Type II error probability
δ	:	Maximum rate acceptable of nonconforming
γ	:	Rate of undesirable nonconforming units
n	:	Sample size
m	:	Size of individual samples
ω	:	Ratio between variance from Phases I and II
τ	:	Error factor of the estimate
Δ	:	Variance shift
k	:	Constant in terms of (1-α)-quantile of a chi-square distribution
C_p	:	Potential Process Capability Index
C_{pk}	:	Actual Process Capability Index
C_{pm}	:	Capability index in terms of specification limits, mean of the process and a provided target.
C_{pU}	:	Upper C_p
C_{pL}	:	Lower C_p
T	:	Specification nominal value

Y	:	Random variable that follows a chi-squared distribution with m(n-1) degrees of freedom
F_{CFAR}	:	Cumulative Distribution Function of CFAR
F_{CARL}	:	Cumulative Distribution Function of CARL
f_{CFAR}	:	Probability Density Function of CFAR
f_{CARL}	:	Probability Density Function of CARL
$\Phi(*)$	:	Cumulative distribution function ($p.d.f.$) of a standard normal random variable
$f(u)$	:	Associated function to the $p.d.f$ of Chi-Square distribution
f_{X^2}	:	Probability density function of the Chi-Square distribution
F_{X^2}	:	Cumulative density function of the Chi-Square distribution

ABBREVIATIONS

ARL	:	Average run length
ARL_0	:	In Control ARL
ARL_1	:	Out-of-Control ARL
C.D.F.	:	Cumulative distribution function
CARL	:	Conditional average run length
CFAR	:	Conditional false alarm rate
FAR	:	False alarm rate
FAR_{MAX}	:	Maximum false alarm rate
IC	:	In-control (process)
LCL	:	Lower control limit
LSL	:	Lower specification limits
OOC	:	Out-of-control (process)
PCI	:	Process Capability Indices
PCR	:	Process Capability Ratio
P.D.F.	:	Probability density function
RL	:	Run length
SPC	:	Statistical Process Control
UCL	:	Upper control limit
USL	:	Upper specification limit

1 INTRODUCTION

Control charts are powerful tools used by many industries to monitor the quality of processes and detect special cause of variations on them. The Shewhart S^2 Control Chart is one of the most used tools to monitor if the variance of some quality characteristic (X) that is assumed to be normally distributed and may change from an in-control (IC) to an out-of-control (OCC) situation. According to Montgomery (2009), an in-control process is subject only to natural variability under the presence of random, common, and inevitable causes. In other hand, an out-of-control process presents special, or unexpected causes, which move the process away from the statistical stability and indicate that there are problems that must be identified and corrected. In this way, Shewhart's Control Charts suggest that whenever special causes are detected, the process should be put to a stop for interventions to eliminate such causes and regain their stability. These graphs are based on the stability of the control variable, so that any change of this variable in relation to the nominal value (target) should be considered an out-of-control situation (OCC). Therefore, the main objective of this chart is to detect increases of any magnitude in the process variance as soon as possible. In this context, if the actual process variance is larger than an in-control single level point, the process is considered being in an out-of-control state.

The basic procedure of the Shewhart S^2 Control Chart is samples of size n (of some quality characteristic, X, of the product being produced) are collected at regular intervals, so the sample variance (S^2) can be computed. This sample variance is compared with a control limit and if S^2 is above the control limit, chances should be high that the process is out-of-control, or in other words, chances should be high that the actual process variance is larger than the nominal in-control value.

However, in some situations, even if a process is declared out-of-control, it might still be capable from a practical point of view in the sense that it still produces an acceptable low proportion of nonconforming items and hence the process does not need to be stopped in order to look for assignable causes. This can save valuable time and resources. In other words, if the process variance is allowed to be a bit larger than the in-control variance value and yet the rate of nonconforming items being produced is small enough, this may be a tolerable situation from a practical point of view. This relates to Process Capability Indices such C_p and C_{pk} (see 3.3 for detailed information), which have been proposed in the manufacturing industry to provide numerical measures of whether a process is capable of producing items within the preset specification limits (OPRIME *et al*., 2019). A major reason for quantifying the process capability, for instance the process variation, is to be able to compute the ability of the process to hold product specifications (JURAN and GODFREY, 1998). The process capability relates to total amount of products that fall outside specification limits and therefore are claimed non-conforming units.

In summary, it is of interest to monitor the process mean and variance with control charts with a broader definition of “in-control” together with the capability of the process. Unfortunately, the original Shewhart $\bar{X}$ and S^2 control charts are not designed for this type of monitoring. Instead, in this situation, the Modified and the Acceptance charts (that are Shewhart-type charts) and introduced respectively by Hill (1956) and Freund (1957), are more appropriate tools, since they allow the process mean to vary between two specified/tolerated limits (MONTGOMERY, 2009). These charts also aim to ensure that only a small proportion of nonconforming items are produced, so there is no need to declare the process out-of-control and start a search for assignable causes.

According to Montgomery (2009), modified charts use limits that are generally used in situations where the natural variability or “spread” of the process is considerably smaller than the spread in the specification limits, that is, C_p and C_{pk} is far greater than 1. In other hand, the Acceptance control charts approach monitors the fraction of nonconforming units, or the fraction of units exceeding specifications.

The Modified and Acceptance charts are also powerful tools to avoid many false alarms, which is very important nowadays where several systems with many control charts can be used simultaneously, as emphasized recently by Woodall and Faltin (2019). Modified (and Acceptance) control charts generate less false alarms (compared with the Shewhart $\bar{X}$ chart) because, as explained above, they are designed to detected only genuinely important changes in the process mean (changes that generate a rate of nonconforming items larger than what is specified). So, even though these charts were created a long time ago (in the 50’s), they may be still of great value in practice today. We can find more applications of these types of charts in Mohammadian and Amiri (2012), Oliveira *et al*., (2018) and Wu (1998).

Unfortunately, the Modified and Acceptance Control charts were designed only focusing on monitoring the process mean and as emphasized by several authors, see, for example, Hill (1956), monitoring the process variance is also important to avoid the production of an undesirable number of nonconforming units. Given this background as motivation, this work extent the idea of the Modified Control Charts by focusing on monitoring the process dispersion.

In this research, it is also discussed the effect of variance estimation, considering that usually the process parameters are unknown by practitioners and need to be estimated, which is done through sampling data, being expected a variation between the collected data and true process value. According to Chakraborti (2006), this variability negatively affects some properties of chart, such False Alarm Rate (FAR), which increases significantly. Since more false alarm are generated, the average length until a signal decreases. This also negatively affect the chart performance, which is usually measured by Average Run Length (ARL). Based on this, it is important to assess the effect of parameter estimation on the construction and use of the modified charts.

The practical implication of this work is to develop an S^2 Modified Control Chart that detects only genuinely increases in the process variance, which significantly increase the rate of nonconforming items being produced, preventing unnecessary process stop and assessment for assignable causes if only a small increase in the process variance occurs, contributing for higher process efficiency and reduce costs.

The chart in scope of this work is to monitor process variance, as Shewhart S^2 chart, considering that the variable to be monitored follows a normal distribution.

1.1 RESEARCH QUESTION

As described at Introduction, the modified chart is an important tool for controlling highly capable processes, since it allows certain shifts of mean, and in the scope of this work also variance, without considering it out-of-control. Industrial processes can greatly benefit from this approach, where costs can be reduced by improper process stops because of signaling events that do not actually affect nonconforming production rates. However, despite the importance of the subject, no such development has been done for variance monitoring and its performance during monitoring phase (Phase II) when variance is estimated.

This literature gap motivated the development of the present study, which aims to propose and evaluate a statistical model for expanded control limits (Modified chart) for variance, taking into consideration the ability of the process produce conforming parts, contributing to reduce false alarms, where the process practitioner may claim that the process is out-of-control when actually the process compliance to the specification limits still meets the required quality performance avoiding losses due to process over-control.

Based on these theoretical foundations, the following research question is proposed:

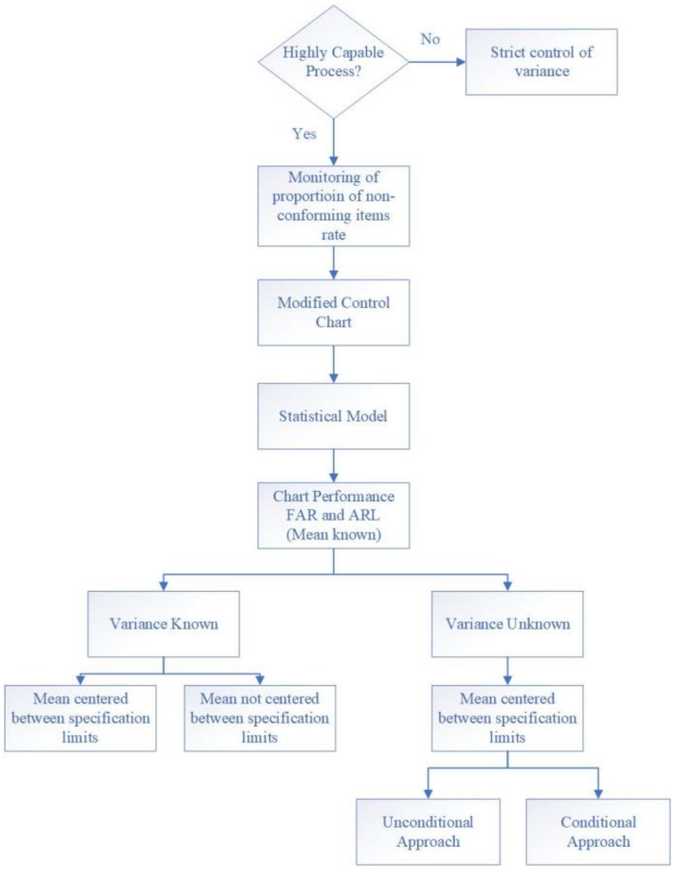
The use of Modified Control Chart for process variance can contribute to avoid unnecessary interventions in the process contributing to improve its efficiency?

Some questions that motivated the research project are:

- Why to control rigorously the process variance with traditional Shewart control chart when a certain variation may be allowed without cause harm to quality performance?
- How modified chart could avoid process over-control and improve management by reducing false alarms in highly capable processes?
- Due to the modified control limits are calculated with known parameters, can the modified control chart minimize the impact of False Alarm Rate and the control limits are estimated?

To answer these questions, this work will propose statistical models to provide the exact calculation of modified control limits for variance and also the False Alarm Rate (FAR) and the average run length (ARL) when the process variance is known (Case K) and assess the chart performance measures when the process variance is unknown (Case U) and therefore, estimated. The Figure 1.1 summarizes the research questions in scope of this work.

Figure 1.1 - Research Problem



Source: The author (2021)

1.2 RESEARCH OBJECTIVE

The objective of this research is to propose the chart named S^2 modified control chart where the process variance (σ^2) is allowed to be larger than the in-control variance value (σ_0^2) until a maximum value (σ_{MAX}^2), as long as the process remains capable, in the sense that it produces a specified (tolerated) small fraction of nonconforming items.

In addition to the objective listed above, the measures of the False Alarm Rate (FAR) and the average run length (ARL) when the process variance is known (Case K) and process variance is unknown (Case U), will be assessed. Data simulation was used for Phase II analysis for all three illustrative examples provided in this work to enhance the understanding of how the S^2 control limit for both known and unknown variances, compares with the S^2 modified control limit, in order to support process practitioner decisions.

The main variable inputs for the model proposed are process variance (σ^2), sample size (m), number of elements in each sample (n), upper and lower specification limits (LSL and USL) process specification limits and allowed a fraction of nonconforming items (γ).

1.3 RESEARCH JUSTIFICATION

A situation in which some slack is often allowed in the process occurs when the process is highly capable (WOODALL and FALTIN, 2019), however the traditional Shewhart control chart (SHEWHART, 1931, 1941) does not consider if the specification limits are so wide relative to the process variation that attention should be directed to more pressing issues, recommending to stop the process when there is an indication of special causes, in order to keep the production process at a stable variation level.

Despite the relevance of the Modified Control Chart for industrial processes, that contributes to detection of important process shifts that genuinely shows an impact of the capacity of process manufacture compliant products and preventing unnecessary actions, little was done regarding process variance monitoring. Up to this date, no studies have been found on modeling the modified charts for S^2 charts and the effects of estimating the process variance. This gap in the literature has opened an opportunity for the present work.

The contribution of such chart is that there are cases in which it is not financially appropriate to intervene, even in the presence of special causes. Starting also from the premise that the purpose of a process control system is decision making that result in economic gains over the process, it is possible to balance the consequences of these decisions even considering two situations: (a) take action when not needed (over control), versus (b) does not take action when it is necessary (lack of control). For this reason, when the benefits of stopping the process in the presence of special causes are lower than the costs, it may be considered an over-control. Therefore, it is worth having an alternative method to achieve the benefit of both the control charts for variables and the process capability analysis (OPRIME *et al*., 2019). Therefore, have an allowed process variance (σ^2) that can be larger as much a maximum value (σ^2_{MAX}), meeting the process acceptance criteria, based on the tolerated small fraction of nonconforming items, may facilitate the management by preventing process over-control and improve the decision-making process.

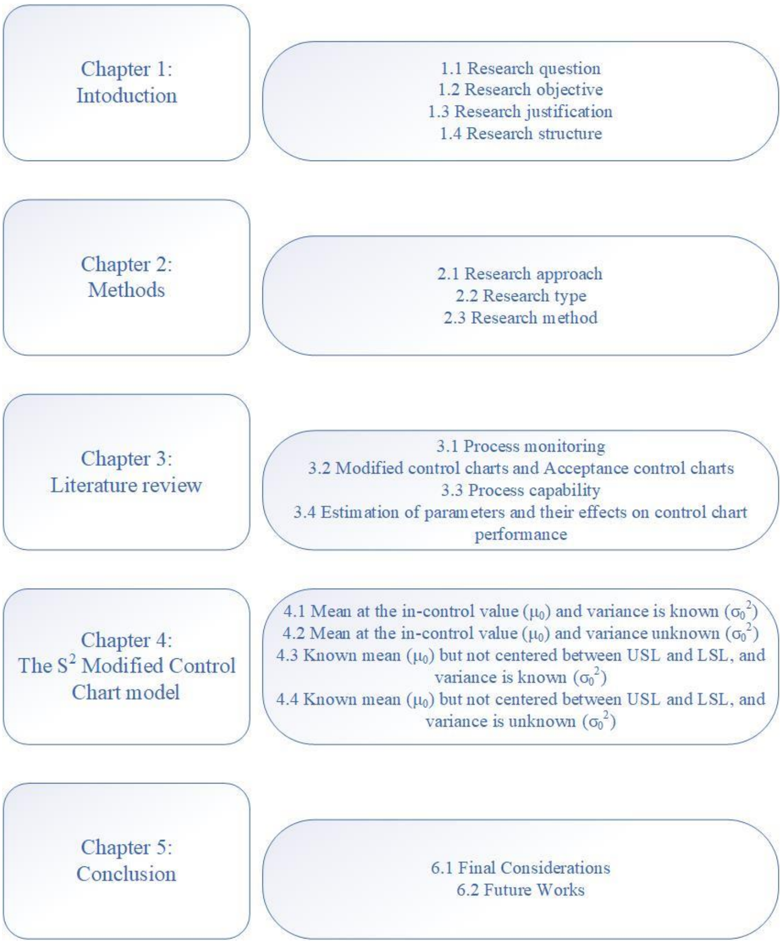
1.4 RESEARCH STRUCTURE

The present research will be divided into three major steps:

1. Bibliographic review of the modified chart, taking into consideration existing knowledge about modified control charts, fit it into the application proposed in this present research.
2. Development of statistical models based on the theoretical framework that evaluates the performance of the chart using both known and unknown variance. The unconditional approach (does not consider the practitioner-to-practitioner variability) will be used for development of this research. A more detailed insight about the differences between these two perspectives is provided in section 4.2. In addition is presented the statistical models for in-control mean not centered between the specification limits and variance known.
3. Validation of the models created by providing examples that show the usefulness of the proposed work.

The division of this work is presented in Figure 1.2, where chapters 1 and 2 were elaborated for the presentation of the theme, research problem, research method, objective, and justification of this dissertation. chapter 3 regards to bibliographic review to substantiate the concepts of Statistical Process Control (SPC) and Modified Charts and chapter 4 develops the statistical models for the S^2 Modified Control Chart, for three different cases, mean at the in-control value and variance known, mean at the in-control value and variance unknown and known mean but not centered between specification limits with variance. At last there is the conclusion chapter 5, with final considerations and suggestion of future works.

Figure 1.42 - Research Structure



Source: The author (2021)

2 METHODS

It is seen by many researchers as the validity of the results of a research (MIGUEL *et al*., 2012), that is, research is considered scientific as long as it meets some methodological criterion.

The basic procedures proposed by the study of the methods can be divided into five major stages, that are: determination of the research approach, determination of the type of research, method chosen for the research, research project and research preparation. In this work, all these procedures were used as a path, which will be better clarified in the next topics.

2.1 RESEARCH APPROACH

This dissertation has a quantitative approach, where the results presented will be expressed numerically, that is, in a quantified way. According to Bryman (2011), the major concerns of the quantitative approach are measurement, causality, generalization, and replication. The definition of each of these terms is described in Table 2.1 (MIGUEL *et al*., 2012).

Table 2.1 - Characteristics of quantitative research

Characteristics	Descriptions
Measurement	It refers to the ability to measure the variables in order to test the hypotheses.
Causality	It is related to the ability to establish relationships Refers to the causal relationship between the dependent variable (effect) and the independent variables (causes). The research seeks, therefore, to prove the existence of such a relationship between the variables.
Generalization	It deals with the possibility that the results obtained are generalized beyond the limits of the research.
Replication	It deals with the possibility for one researcher to repeat one search of another and find its results.

Source: The author (2021)

This dissertation fits into the description of the Measurement characteristic, where the hypotheses raised are going to be tested through a set of variables that can be measured. The present work is also classified as normative quantitative axiomatic research and will use the simulation method of generation of data to represent an actual manufacturing process. According to Miguel *et al*., (2012), quantitative axiomatic research produces knowledge about the behavior of certain variables of the model, based on assumptions about the behavior of certain variables of the model, and normative axiomatic research develops norms, policies, strategies and actions, in order to improve or compare the performance of strategies that deal with the same problem.

The present work aims to build a statistical model, which is the Modified S^2 Monitoring Chart for variance taking into consideration a permitted rate of undesirable nonconforming units (γ) and a highly capable process. Few illustrative examples will be provided to show its application for actual situations. See section 1.4 for further details about the research structure.

2.2 RESEARCH TYPE

The research that intends to analyze quantitative models, with the main objective of understanding the modeled process or explaining its characteristics, is considered descriptive axiomatic in nature. In view of this, from an idealized problem (not observed) theories are created, supported by mathematical, statistical and computational methods in order to obtain a better understanding of the problem under study (MIGUEL *et al.*, 2012).

This work focus on understanding the theoretical statistical model existing in the literature on the modified graph/acceptance and, going from it, develop a new statistical model that use the variables of interest of the research question (process stability and capability), aiming that one day it may be used by practitioners in real processes, for monitoring and decision-making. For this reason, the nature of this work can be considered as descriptive axiomatic research.

2.3 RESEARCH METHOD

The method used in this dissertation is divided into two steps: the first, focused on the bibliographic review, in which the concepts related to process monitoring, modified and acceptance control charts, process capability and the effect of parameter estimation on control chart performance were assessed. The second step consists of the development of the statistical model for S^2 Modified Control Chart, which was done for in-control mean centered between specification limits, for both known and unknown variance, and mean not centered between the specification limits, for known variance only. As informed in section 1.2, data simulation was used for Phase II analysis for all illustrative examples provided in this research.

The bibliographic review provides insights about how the traditional Shewhart charts may be enhanced in order to accommodate a certain shift of the chart parameters to avoid over-control and still meet the quality requirements in place. The effects caused by parameter estimation was also studied and additional chart design steps are proposed in a practical and user driven manner.

According to Miguel *et al.* (2012) the act of measure a research variable is to measure research variables is the most outstanding characteristic of the quantitative approach. In this context, the researcher shall capture the research evidences by measuring the variables, thus, no subjectivism will be influencing the understanding of facts in the use of induction for the generation of knowledge. The researcher also does not interfere or interfere with little in the research variables.

In the context, to conduct the quantitative research proposed in this work, the research methods used is modeling and simulation, which according to Chung (2004), this is the process of creating and experimenting with a computerized mathematical model of a physical system. A system is defined as a collection of interacting components that receive input and provide output for some purpose. The simulation modeling and analysis of different types of systems are conducted for the purposes of:

- Gaining insight into the operation of a system;
- Developing operating or resource policies to improve system performance;
- Testing new concepts and/or systems before implementation;
- Gaining information without disturbing the actual system.

In order to simulate a production process, in which it is possible to obtain the results of performance, this work used Microsoft Excel software to simulate Phase I (when applicable) and Phase II for all illustrative examples provided. The use of simulation to evaluate the performance of control charts, when parameters are estimated, and the design chart phases, is more realistic and gets closer to the process manager/user needs. This research is divided in three major steps, that are the literature review, the development of statistical models and validation of the models created by examples that show the usefulness of the proposed work.

A illustrative example is provided for each of the models developed in chapter 4, with the objective to show its applicability in a practical and reasonable manner, to readers that are inserted in process manufacturing.

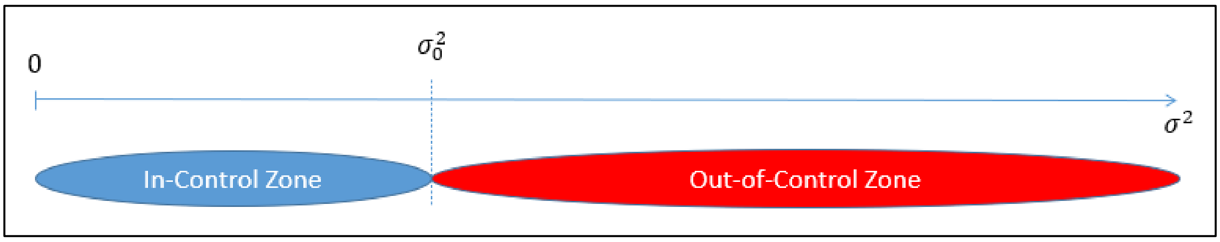
3 LITERATURE REVIEW

3.1 PROCESS MONITORING

The control chart is one of the fundamental techniques used in SPC, given his alleged operational simplicity. According to Montgomery (2009), it contains a center line that represents the average value of the quality characteristic corresponding to the in-control state and one, or two horizontal lines called the upper control limit (UCL) and lower control limit (LCL). These control limits are chosen so that if the process is in control, nearly all the sample points will fall between them. When we work with a quality characteristic that is a variable, it is necessary to monitor both the mean value of this characteristic and its variability. As previously stated, the major focus of this research is on the variability. Now, concerning the chart design, when the parameters of a certain characteristic quality are unknown, the control chart is usually built in two phases. In Phase I (pre-stage prospective) statistical control limits are estimated. When we use the traditional chart of Shewhart, it's common the extract of 25 samples of size five ($n = 5$) to estimate parameters of the process and the limits of statistical control. In Phase II, with the chart already set, new samples are taken, and it is considered that the process is stable when the result of the observed characteristic is plotted between the control limits. Otherwise, it follows that the process has lost its stability condition and is subject to the action of special causes (JENSEN *et al*., 2006; MONTGOMERY, 2009).

As presented in the Introduction, the main objective of the S^2 Control Chart is to detect increases (of any magnitude) in the process variance (σ^2), as soon as possible. According to Chakraborti and Graham (2019), a process is considered in a state of statistical control or in-control (IC) if it is operating according to what is targeted or expected in the presence of common causes. In contrast, the out-of-control (OOC) state is when there are reasons to believe that the of some special causes influences the process and this needs to be identified and if possible, eliminated, so the process is restored to the IC state. In this context, if the actual process variance (σ^2) is larger (by any magnitude) than an in-control single level point (σ_0^2), the process is considered being in an out-of-control state, otherwise the process is declared in-control. Figure 3.1 illustrates this situation.

Figure 3.1 - The In-Control and Out-of-Control Zones of the $\boldsymbol{S^2}$ Control Chart



Source: The author (2021)

When the process parameters are unknown, the control chart creation goes through the two phases explained earlier in this section, which are Phase I, with estimation of the unknown parameter(s) and setup of control limits, and Phase II, the monitoring phase.

To monitor the process variance (σ^2) with the S^2 Control Chart, samples of size n of the quality characteristic (X) are collected at regular intervals so the sample variance (S^2) can be computed. S^2 is also known as the plotting statistic of the chart and it is given by

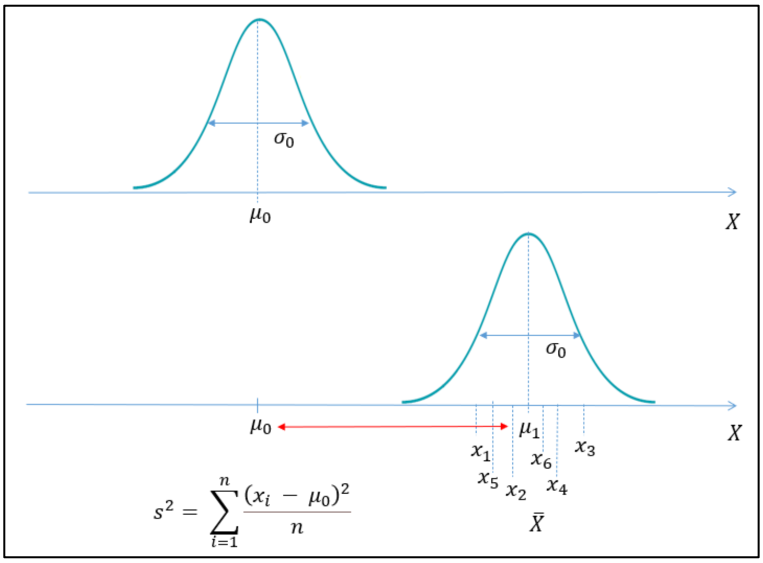
$$S^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_j - \bar{X}\right)^2, \quad (1)$$

where X_j is the j-th observation of the quality characteristic of each sample being collected at regular intervals ($j = 1,2, \dots, n$). X_j is considered normally distributed with mean μ_0 and variance σ^2, n is the size of each sample being collected at regular intervals and $\bar{X}$ is the sample mean of each sample given by

$$\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j. \quad (2)$$

By using $\bar{X}$, one degree of freedom is lost and $n - 1$ shall be used. The $\bar{X}$ is used in equation 1 instead of μ_0 $\left(S^2 = \sum_{i=1}^{n}\frac{(x_i - \mu_0)^2}{n}\right)$, because in actual situations the actual process mean (μ) can change along the different sample subgroups collecting. If we use μ_0 (known or target mean) for calculation of subgroup variance, in case of the mean changes from μ_0 to μ_1, the S^2 would notice this change and incorrectly would provide a signal of variance change when in reality it was the mean that changed. The Figure 3.2 below shows this event.

Figure 3.2 - Demonstration of the use of $\bar{\boldsymbol{X}}$ instead of $\boldsymbol{\mu_0}$ for S^2 calculation



Source: The author (2021)

3.2 MODIFIED CONTROL CHARTS AND ACCEPTANCE CONTROL CHARTS

As stated by Montgomery (2009) the modified control limits are generally used in situations where the natural variability or "spread" of the process is considerably smaller than the spread in the specification limits; that is, C_p or C_{pk} (see section 3.3 for further details about these indices) is much greater than 1, occurring occasionally in practice. In situations where six sigma (6σ) is much smaller than the spread in the specifications ($USL- LSL$), the process mean or variance can sometimes be allowed to vary up to a maximum value without having an important effect of the overall performance of the process. This chart was first introduced by Hill (1956). When this situation occurs (MONTGOMERY, 2009) we can use a modified control chart instead of the usual control chart for both mean, and as developed by this present work, for variance. The modified control chart is concerned only with detecting whether the true process mean (μ) or variance (σ^2) is located such that the process is producing a fraction nonconforming over some specified value, allowing therefore, these the mean and variance to vary over a determined interval, as follows:

$\mu_L \leq \mu \leq \mu_U$, where μ_L and μ_U are chosen as the smallest and largest permissible values of μ (mean), and
$\sigma_L^2 \leq \sigma^2 \leq \sigma_U^2$, where σ_L^2 and σ_U^2 are chosen as the smallest and largest permissible values of σ^2 (variance).

Considering the fact that the major concern regarding variance, is the chart user detect the increase of process dispersion (the lowest dispersion the better), this work will restrict to the case of the upper one-sided charts and largest permissible value of σ^2 (without a lower control limit and smallest permissible value). Therefore, this work considers the charts with just one upper control limit and the maximum value ($\sigma_U^2 = \sigma_{MAX}^2$) allowed for process variance.

Now, regarding the origin of the acceptance chart, it was created by Freund in 1957 with the aim of monitor the fraction of nonconforming units, or the fraction of units exceeding specifications (MONTGOMERY, 2009).

The common sense between Hill (1956) and Freund (1957) is that highly capable process have a natural deviation that is much lower than the limits imposed by specifications, and for this reason there is no need to maintain the rigid control on mean or variance at a nominal value, since a certain variation would not cause harm to the quality of the process in terms of the production of noncompliant items (OPRIME and MENDES, 2017).

Some process parameters need to be defined for modified and acceptance chart design, where the main are: α (probability of Type I error), β (probability of Type II error), δ (maximum rate acceptable of nonconforming), γ (a specific rate of undesirable nonconforming units). The process means (μ_0) and standard deviation (σ_0) can be known or estimated. The process specification limits, established by the project, manager or consumer, and the sample size for calculating the sample mean ($\bar{X}$) must also be defined. At Phase I, for SPC chart creation, it is required to know or estimate these parameters, since the control limits definition requires them.

Therefore, the modified and acceptance charts are designed to detect only genuinely important in the process mean and/or variance, that generate a fraction of nonconforming no larger than is tolerable, being able to avoid false alarms, which is very important as emphasized by Woodall and Faltin (2019).

The main difference between Modified and Acceptance Charts is the while the first controls the maximum probability of Type I error, the second controls a specified probability of the Type II error (see section 3.2.1 for literature review about Type I and Type II error). Usually, the practitioners are concerned with controlling the probability of rejecting the process when, in fact, it should be approved (Type I error - α), however, if the practitioner is interested in monitoring the probability of accepting the process when it should be rejected (β), the option is the second chart, that is based on Type II error.

In the SPC context, the Type I error is also known as the false alarm rate (FAR), so the Modified Control Chart is constructed for a maximum FAR, whereas the traditional Shewhart Control Charts, with a strict definition of in control and out-of-control process, are constructed for a specified FAR, being designed to provide a signal every time the monitored parameter, mean or dispersion, fall out-of-control limits (CHAKRABORTI, 2000). However, for the Modified Control Chart, there are many possible false alarm rates, since the process variance can vary within a range of values without being considered out-of-control. In such cases, a true alarm occurs every time the chart alerts out-of-control condition for a process whose variance has shifted out of the tolerable threshold. Hence, a false alarm is obtained every time a signal indicates an out-of-control condition, and the variance is still equal or below σ_{MAX}^2. As there are numerous permissible locations for the variance within the range of 0 to σ_{MAX}^2, for each location assumed, there is a new probability of a false alarm. The False Alarm Rate (FAR) is an important measure because it occurs when the process is declared OOC when in fact it is IC, leading to unnecessary investigations and implementation of corrective action.

3.2.1 Type I and Type II error

According to Montgomery (2009) two kinds of errors may be committed when testing hypothesis. If the null hypothesis is rejected when it is true, a Type I error has occurred, while if we fail to reject the null hypothesis when it is false, then we have a Type II error. The probabilities of these two types of errors can be denoted as,

$\alpha = P\{type\ I\ error\} = P\{reject\ H_0|H_0\ is\ true\}$, and
$\beta = P\{type\ II\ error\} = P\{fail\ to\ reject\ H_0|H_0\ is\ false\}$.

Still, according to Montgomery (2009), sometimes it is more convenient to work with the power of a statistical test, where $Power = 1 - \beta = \{reject\ H_0|H_0\ is\ false\}$.

Being the power, therefore, the probability of correctly reject H_0. In quality control work, α is sometimes called the producer's risk because it denotes the probability of a good lot be rejected, or the probability that a process producing acceptable values of a particular quality characteristic will be rejected as performing unsatisfactorily. The β, in other hand, is sometimes called the consumer's risk because it denotes de probability of accepting a lot of poor quality or allowing a process that is operating unsatisfactorily regarding a specific quality characteristic to continue in operation.

The process owner can directly control or chose the α risk and the β risk, which is usually a function of sample size and how different the true value of a parameter is from the hypothesized value. The larger is the sample size used, the smaller is the β risk.

3.3 PROCESS CAPABILITY

Process capability refers to the reproducibility over a long period of time with normal changes in workers, materials, and other process conditions (JURAN and GODFREY, 1998). In accordance with Montgomery (2009), process capability studies have a considerable impact on many management decision problems that occur during the product cycle, including production purchase decisions and process improvements that reduce process variability and contractual agreements with customers or suppliers regarding product quality.

According to Wu, Pearn and Kotz (2009), the Process Capability Indices (PCIs) have been developed in certain manufacturing industry as capability measures based on various criteria, including process consistency, process departure from a target, process yield, and process loss. The (PCI)s, are a popular numerical instrument for measuring process performance and providing information to the consumer and producer to assess whether a product complies with process specification requirements. Process capability indices such C_p and C_{pk}, for example, have been proposed in the manufacturing industry and the service industry, providing numerical measurements on whether a process is capable of produce items within factory pre-set specification limits (PEARN and LIN, 2004).

The C_p, C_{pk} and C_{pm} indices, according to Montgomery (2009), are defined by the quality characteristic with upper and lower limits of USL and LSL specification, respectively, as:

$$C_p = \frac{\text{USL}-\text{LSL}}{6\sigma} \mathrm{C_p} = \frac{\text{LSE-LIE}}{6\sigma} \quad (3)$$

Where the C_p does consider where the process mean is in relation to the specifications, measuring only the dispersion of the specifications in relation to the Six Sigma dispersion of the Process. In this sense, the C_{pk} index was proposed by Kane (1986), which considers both the deviation in the process and the displacement of the process in relation to the midpoint of the specification interval, and is defined as follows (Montgomery, 2009):

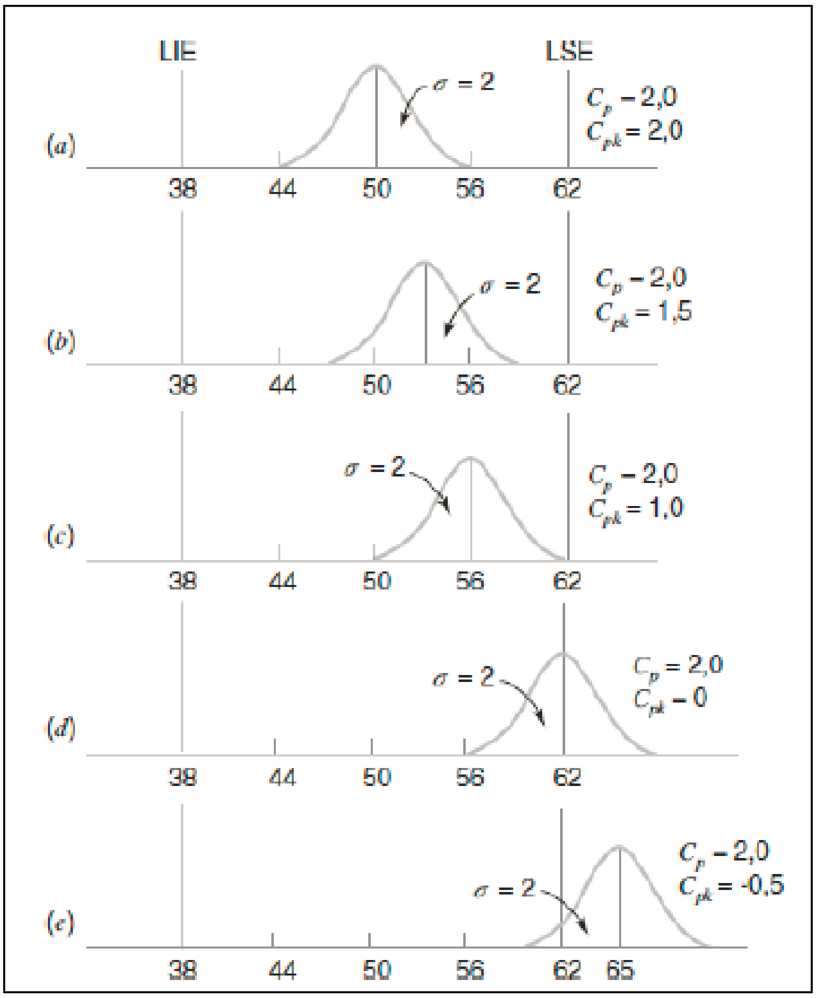
$$C_{pk} = \text{Min}\left\{\frac{\text{LSE}-\mu}{3\sigma}, \frac{\mu-\text{LIE}}{3\sigma}\right\} \quad (4)$$

Where μ is the process mean, LSL is the lower specification limit, USL is the upper specification limit, and σ is the standard deviation of the process.

In general, if $C_p = C_{pk}$, the process is centered on the midpoint of the specifications, and when $C_p < C_{pk}$, the process is decentralized. Figure 3.3 shows the relationship between C_p and C_{pk}.

The number of standard deviations used in equations 3 and 4 refers to a six-sigma process, the most common case studied in the literature, since it guarantees with a 99,73% chance that the sample data will fall within the specification limits. The natural limits of a six-sigma process fluctuate within 3 standard deviations to the right and left, regarding its nominal mean. As can be seen in equations 3 and 4, capacity indices are calculated based on process specification limits.

Figure 3.2.13 - Relation between C_p and C_{pk}



Source: Montgomery (2009)

C_p and C_{pk} indices can be applied to products with higher and lower specifications. However, they cannot be used when the product has only a one-sided specification limit, so Kane (1986) has further proposed two one-sided capacity indices, C_{pU} (upper C_p) and C_{pL} (lower C_p). The first is used to measure the process performance of the "the higher the better" quality characteristic with an LSL (Lower Specification Limit), and the second is for the "the lower the better" type quality characteristic with an USL (Upper Specification Limit). They are defined as follows (MONTGOMERY, 2009):

$$C_{pL} = \frac{\mu - \text{LSL}}{3\sigma} \quad (5)$$

$$C_{pU} = \frac{\text{LSE}- \mu}{3\sigma} \quad (6)$$

Where μ is the process mean, LSL is the lower specification limit, USL is the upper specification limit, and σ is the standard deviation of the process.

According to Montgomery (2009), the reason the C_{pk} index was initially created was because the C_p index did not adequately address the case of a process with the mean not centered between specification limits, however, the C_{pk} itself is still an inadequate measure of centralization of a process, where a large C_{pk} value can say nothing about the location of the mean, in the range from LSL to USL.

Where USL is the upper specification limit, LSL is the lower specification limit, σ is the process standard deviation, μ is the process mean, and T is the specification nominal value. However, usually the process standard deviation (σ) is unknown and should be replaced by an estimated parameter.

The modified chart takes into consideration a specific rate of undesirable nonconforming units, here denoted as γ, that has a direct relation with process capability ratio C_p, that measures the ability of the process to manufacture a product that meet the specification. The Table 3.1 presents several values of C_p along with the associated values of process fallout, expressed in defective parts or nonconforming units of product per million (ppm). These values were calculated following the assumptions that the quality characteristic has a normal distribution, the process is in statistical control and for the case of two-sided specifications, the process mean is centered between the lower and upper specification limits.

For illustration, the table above shows that a process normally distributed with $C_p = 0{,}90$ expresses a fallout rate of 6.934 products per million (ppm) for two-sided specifications, whereas a $C_p = 1{,}30$ for this process implies a fallout rate of only 96 products per million (ppm) for two-sided specifications. The interpretation for one-sided specification is analogous for the two-sided specification. In this dissertation, the C_p index is used for development of modified control chart when the process mean is centered between the specification limits (Section 4.1), while the C_{pk} is used when the process mean is no longer centered between specification limits (see section 4.3).

Table 3.2.11 - Values of the Process Capability Ratio ($\boldsymbol{PCR}$) and associated Fallout for a Normally Distributed Process (in Defective ppm) that is in statistical control.

	Process Fallout (in Defective ppm)	
PCR	One-Sided Specifications	Two-Sided Specifications
0,25	226.628	453.255
0,50	66.807	133.614
0,60	35.931	71.861
0,70	17.865	35.729
0,80	8.198	16.395
0,90	3.467	6.934
1,00	1.350	2.700
1,10	484	967
1,20	159	318
1,30	48	96
1,40	14	27
1,50	4	7
1,60	1	2
1,70	0,17	0,34
1,80	0,03	0,06
2,00	0,0009	0,0018

Source: Montgomery (2009)

3.4 ESTIMATION OF PARAMETERS AND THEIR EFFECTS ON CONTROL CHART PERFORMANCE

A control chart is a process monitoring tool used to detect the presence of any assignable causes of variation, such as process shifts, so that any necessary corrective actions can be taken. However, in many control charting applications, the process mean and/or the standard deviations are unknown parameters, needing to be estimated and used instead of known parameter values (CHAKRABORTI, 2006).

According to Chakraborti and Graham (2019) the SPC usually comprises of two phases, namely Phase I and Phase II, in which monitoring objectives are different so that the control charts are constructed under different performance criteria. The first phase, known as Phase I or the retrospective phase, is mainly exploratory and is typically used to establish control or stability of a process based on an analysis of historical or retrospective data. Then the unknown parameters are estimated, distributional assumptions are checked, and then control limits are calculated. The next part of SPC, which is the future monitoring of the process, is referred to Phase II, that is a prospective analysis. The phase II uses the parameter estimates and the control limits developed in the course of the Phase I analysis. When estimates are used in place of known parameters, the variability of the estimators can result in chart performance that differs from that of charts designed with known parameters (JENSEN *et al*., 2006).

Understand the statistical performance of a control chart is important, because when a charting statistic plots or falls on or outside the control limits, it signals the possibility of the presence of assignable causes and at that point the process may be declared OOC (CHAKRABORTI and GRAHAM, 2019). However, the process can be declared OOC when in fact it is IC, being this event called a false alarm and the probability of it occur referred to as the False Alarm Rate (FAR), and this understand contributes for a more appropriate chart design across the two Phases in order to prevent excessive false alarms, that may require unnecessary corrective actions being applied to the process. Still according to Chakraborti and Graham (2019), the performance of a control chart is studied via its run-length distribution, and the most popular performance measure is the expected value of the run-length distribution, the so-called average run-length (ARL). The ARL of a control chart is the expected number of charting statistics that must be plotted (subgroups that must be collected) before the control chart signals for the first time.

According to Chakraborti (2006), some of the effects of parameter estimation in the calculation of FAR is the increase, sometimes substantial, of the number of false alarms that, as mentioned previously, results in a loss of time and money. In addition, both FAR and ARL become random variables, which are represented by probability distributions and have their own parameters (μ, σ). This scenario may compromise the quality of these measurements, since the FAR and ARL means can significantly get itself away from the target value (got when the parameters are known).

Still according to Chakraborti (2006), the interpretation and implementation of FAR is easy when the parameters are specified or known, but when the need to be estimated from data, the events when there is a signal become statistically dependent due to multiple uses of these estimates and because of the signaling events are dependent, the run-length distribution of the chart is no longer geometric and must be obtained by taking the effect of estimation and the resulting dependence into account. In section 4.2 we develop the modified control chart when the variance is estimated and present an additional Phase for chart design that we denote as Phase 0.

4 THE S² MODIFIED CONTROL CHART MODEL

4.1 MEAN AT THE IN-CONTROL VALUE ($\boldsymbol{\mu_0}$) AND VARIANCE IS KNOWN ($\boldsymbol{\sigma_0^2}$)

Here, we assume the process mean at the in-control value (μ_0) and at the exact middle point between the specification limits, consistently with the purpose of detecting relevant increases in the process variance only. As described previously, because of the major concern for variance is the chart user detects the increase of process dispersion, this work will restrict to the case of the upper one-sided charts and largest permissible value of σ^2.

Suppose that $\left(X_{i,1}, X_{i,2}, \ldots, X_{m,n}\right)$ are random variables, and $i = 1,2, \ldots m$ are independent samples extracted from a process in Phase II, known variance σ_0^2, where i which identifies the subgroup. In Phase II, the samples of size n are extracted, and s_i^2 are calculated from $\left\{X_{i,1}, X_{i,2}, \ldots, X_{i,n}\right\}$, where the type I error is $\alpha = Pr(s_i^2 \notin (0, UCL))$ or $\alpha = 1 - Pr(s_i^2 \in (0, UCL))$, and is represented by α_{nom}. The UCL is the Upper Control Limit.

When the process is under control (IC), with mean centered between USL and LSL, and known variance, using $\bar{X}$ as stated in section 3.1, can write the following equation 7:

$$Pr\left(s_i^2 \leq UCL_{S^2}\right) = Pr\left(n - 1 \,.\, \frac{s_i^2}{\sigma_0^2} \leq \chi^2_{n-1,1-\alpha}\right) = Pr\left(s_i^2 \leq \sigma_0^2 \,.\, \frac{\chi^2_{n-1,1-\alpha}}{n-1}\right) \qquad (7)$$

Therefore, we can get the Upper Control Limit for S^2 by the following equation 8:

$$UCL_{S^2} = \sigma_0^2 \, \frac{\chi^2_{n\text{-}1,\, 1\text{-}\alpha}}{n\text{-}1}, \qquad (8)$$

The plotting statistic (S^2) given by equation 1, should be compared with the Upper Control limit (UCL_{S^2}) of the S^2 Control Chart, which is given by equation 8, where σ_0^2 is the nominal in-control process variance, $\chi^2_{n\text{-}1,\, 1\text{-}\alpha}$ is the ($1\text{-}\alpha$)-quantile of a chi-square distribution with $n\text{-}1$ degrees of freedom and α is the nominal false alarm rate (or in other words, the false alarm probability) chosen by the practitioner (usually, $\alpha_{nom} = 0{,}0027$).

A false alarm is defined as a signal (alarm) when the process is in control. The maximum false alarm rate happens when $\sigma^2 = \sigma_0^2$. So, note that the Control Limits given by equation 8 is derived in order to provide a maximum false alarm rate equal to α_{nom}, as shown in Equations 9 and 10 below.

$$FAR_{MAX} = 1 - P(S^2 < UCL_{S^2} \mid \sigma^2 = \sigma_0^2)$$
$$= 1 - P\left(S^2 < \sigma_0^2 \frac{\chi^2_{n\text{-}1,1\text{-}\alpha}}{n\text{-}1} \middle| \sigma^2 = \sigma_0^2\right)$$
$$= 1 - P\left(\frac{(n\text{-}1)S^2}{\sigma_0^2} < \sigma_0^2 \frac{(n\text{-}1)\,\chi^2_{n\text{-}1,1\text{-}\alpha}}{\sigma_0^2(n\text{-}1)}\right), \tag{9}$$

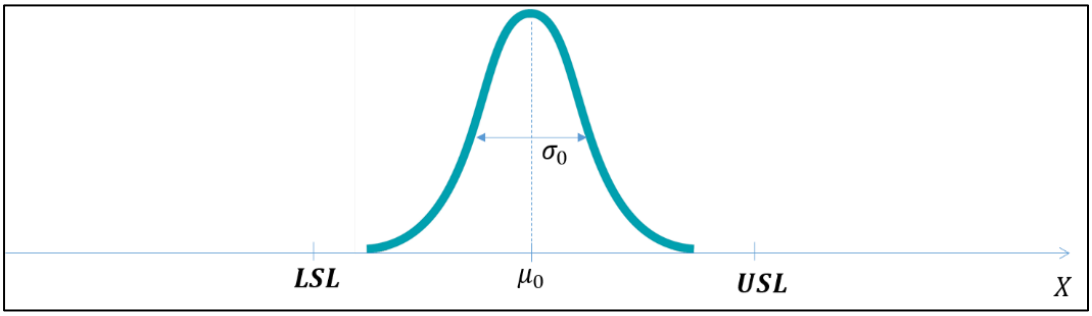
where $\frac{(n\text{-}1)S^2}{\sigma_0^2} = \chi^2_{n\text{-}1}$ is a random variable that follows a chi-squared distribution with $n\text{-}1$ degrees of freedom, so

$$FAR_{MAX} = 1 - P(\chi^2_{n-1} < \chi^2_{n-1,1-\alpha}) = \alpha_{nom}. \tag{10}$$

For highly capable processes, in situations where the natural variability is considerably smaller than the extension of specification limits, i.e., the C_p and C_{pk} indices are much larger than 1,0 (OPRIME *et al.*, 2019). When the actual variance of the process (σ^2) is exactly at the specified or known in-control process variance (σ_0^2) value, that meets the process requirements in terms of failures, the proportion of nonconforming units being produced should be small. In other words, the probability of the quality characteristic (X) be smaller than the lower specification limits (LSL) or larger than the upper specification limits (USL), should also be small. Figure 4.1 illustrates this situation. Note that these specification limits are provided by the project/manager.

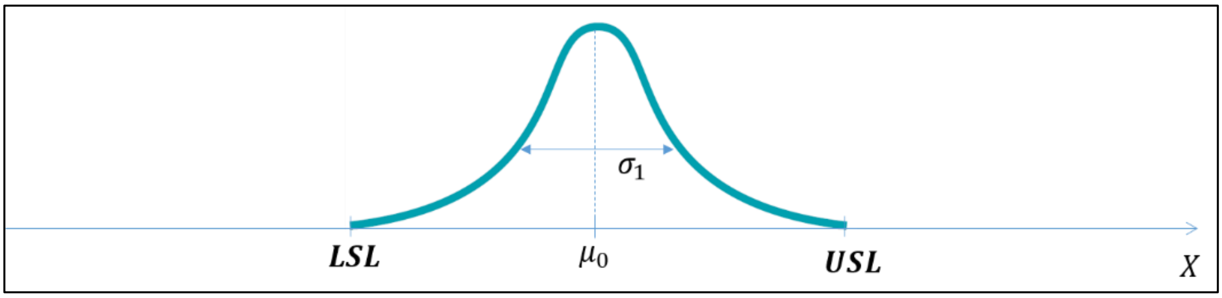
The S^2 Control Chart is designed to detect increases (larger than σ_0^2) of any magnitude in the actual process variance (σ^2), even increases that do not affect the rate of nonconforming items being produced and these increases will tend to produce a signal (alarm) on the control chart. Consider the illustration provided by Figure 4.2 where the actual process variance is larger than σ_0^2 ($\sigma^2 = \sigma_1^2 > \sigma_0^2$), but yet the rate of nonconforming items is still small.

Figure 4.1 - Process running with the nominal in-control variance ($\sigma^2 = \sigma_0^2$) with all the items being produced within the specification limits



Source: The author (2021)

Figure 4.2 - Process running with a variance ($\boldsymbol{\sigma_1^2}$) larger than $\boldsymbol{\sigma_0^2}$, but still with all the item being produced within the specification limits



Source: The author (2021)

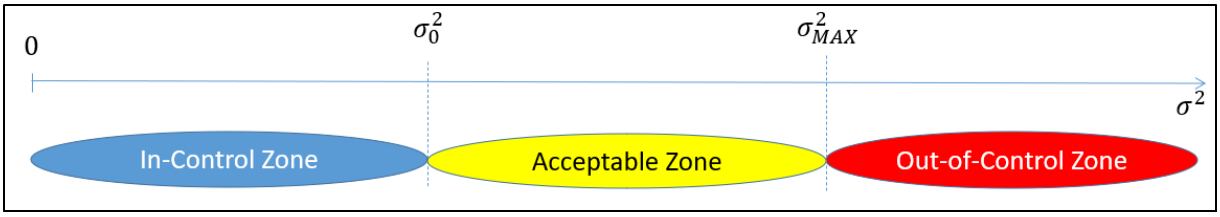
Note that since σ_1^2 is larger than σ_0^2, from the perspective of the traditional S^2 Control Chart, the process should be declared out-of-control. In this case, the chart will tend to signal an alarm. However, this may be a problem because, as can be seen in Figure 4.2, the process is still not producing numerous nonconforming items (almost all the items being produced are still within the specification limits, even though $\sigma^2 = \sigma_1^2 > \sigma_0^2$. So, trying to fix this increase on the variance may be a waste of time and money, since in most of the cases, the process would have to be paused. Thus, it is of interest to monitor the process variance with a control chart with a broader definition of "in-control" which considers the specification limits. Here in chapter 4, we develop such kind of Control Chart for variance. We named this chart as the S^2 Modified Control Chart, in consonance with the Modified Control chart for monitoring the process mean introduced by Hill (1956).

Thus, it is possible to use the concept of practical significance to make economically viable decisions on the process condition. According to Woodall (1985) it is necessary to react in the presence of special causes. However, this decision is taken only when this special cause has a sufficient impact to be economically feasible for its removal, in order to improve quality indicators.

From an economic point of view (it is recommended to consult the work of Magalhães *et al.*, 2001, which deals with economical design of control charts) excessive intervention on the process because of false alarms is harmful to productivity. Adopting the same understanding, it is not convenient to act on small deviations in the parameter statistically monitored, even though the chart shows an alarm identification of a special cause.

As discussed in the Introduction, the main idea of the chart proposed in this work is that the process variance (σ^2) is allowed to be larger than the in-control variance value (σ_0^2) until a maximum value (σ_{MAX}^2), as long as the process remains capable, in the sense that it produces a specified (tolerated) small fraction of nonconforming items (γ). In the situation we are concerned with, instead of the in-control situation be represented by $\sigma^2 \le \sigma_0^2$ (where σ_0^2 represents the specified in-control target value for the process variance), we allow the process to be "roughly in-control" or acceptable when $\sigma^2 \le \sigma_{MAX}^2$ (where $\sigma_0^2 \le \sigma_{MAX}^2$). If σ^2 assumes a value larger than σ_{MAX}^2, the process is deemed out-of-control (OOC). Figure 4.3 illustrates this situation.

Figure 4.3 - In-Control Zone, Acceptable Zone and Out-of-Control Zone of the Modified Control Chart



Source: The author (2021)

The σ_{MAX}^2 value must be chosen with care, depending on the lower and upper specification limits, LSL and USL, respectively, and the maximum rate (probability) of nonconforming units produced (denoted here by γ) that may be tolerated (or allowed). LSL, USL and γ are specified by the management/project and have the following relationship:

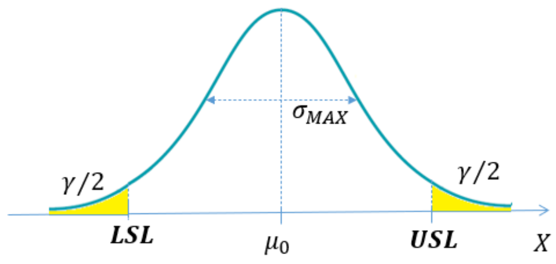
$$\gamma = P[(X < LSL) + (X > USL) | \sigma^2 = \sigma_{MAX}^2]$$

$$\gamma = 1 - P[(LSL < X < USL) | \sigma^2 = \sigma_{MAX}^2], \qquad (11)$$

where X is the quality characteristic of the process and follows a normal distribution with mean μ_0 and variance σ^2. As considered in the traditional S^2 Control Chart, it is assumed that the process mean is in-control value (μ_0).

So γ is the maximum tolerated probability of X being smaller than the LSL or greater than the USL that can be tolerated in a specific application. Figure 4.4 illustrates this situation where the process is running at the maximum allowed tolerated rate of nonconforming units (γ), which happens when $\sigma^2 = \sigma_{MAX}^2$. Note that if $\sigma^2 > \sigma_{MAX}^2$ the rate of nonconforming units produced will be larger than the specified γ, and hence, the process will be declared OOC.

Figure 4.4 - Process running at the maximum rate of nonconforming units being produced ($\boldsymbol{\sigma^2 = \sigma^2_{MAX}}$)



Source: The author (2021)

The maximum tolerated variance (σ^2_{MAX}) can be calculated given the specification limits, and the maximum tolerated rate of nonconforming units (γ). From Figure 4.4, one can write

$$\frac{\gamma}{2} = 1 - P(X < USL | \sigma^2 = \sigma^2_{MAX}) = 1 - P\left(\frac{X - \mu_0}{\sigma_{MAX}} < \frac{USL - \mu_0}{\sigma_{MAX}}\right)$$

$$\frac{\gamma}{2} = 1 - P\left(Z < \frac{USL - \mu_0}{\sigma_{MAX}}\right) = 1 - \Phi\left(\frac{USL - \mu_0}{\sigma_{MAX}}\right), \quad (12)$$

where Z is a random variable that follows a standard normal distribution and $\Phi(*)$ is the cumulative distribution function (c.d.f.) of a standard normal random variable. From Equation 12, one has:

$$\frac{USL - \mu_0}{\sigma_{MAX}} = \Phi^{-1}\left(1 - \frac{\gamma}{2}\right) = z_{1-\gamma/2}, \quad (13)$$

where $\Phi^{-1}\left(1 - \frac{\gamma}{2}\right) = z_{1-\gamma/2}$ is the $\left(\frac{\gamma}{2}\right)$-quantile of a standard normal distribution. Since the normal distribution is symmetric around μ_0, one can write $USL - \mu_0 = \frac{USL-LSL}{2}$, so, σ_{MAX} can be calculated as:

$$\sigma_{MAX} = \frac{USL-LSL}{2\, z_{1-\gamma/2}}. \quad (14)$$

The equation 14 is useful because in practice, what is usually defined by the project/manager is the specification limits (USL and LSL) and the maximum allowed rate of nonconforming units (γ).

To calculate the upper control limit (UCL_{Mod}) of the S^2 Modified Control Chart, one just need to replace σ_0^2 in the original control limit equation of the S^2 Control Chart (see equation 8 by σ_{MAX}^2, as shown below:

$$UCL_{Mod} = \sigma_{MAX}^2 \frac{\chi^2_{n-1,\,1-\alpha}}{n-1} = \frac{(USL-LSL)^2\,\chi^2_{n-1,1-\alpha}}{4(n-1)\,(z_{1-\gamma/2})^2}. \quad (15)$$

The plotting statistic of the S^2 Modified Control Chart is equal to the plotting statistic of the well-known S^2 Control Chart, which is given by Equation 1. Therefore, the control limit showed in equation 15 is designed so that the maximum false alarm rate (which now happens when $\sigma^2 = \sigma_{MAX}^2$) is actually α_{nom}. This can be shown replacing σ_0^2 by σ_{MAX}^2 in Equations 8 and 9, as presented below.

$$\begin{aligned} FAR_{MAX} &= 1 - P(S^2 < UCL_{Mod} \mid \sigma^2 = \sigma_{MAX}^2) \\ &= 1 - P\left(S^2 < \sigma_{MAX}^2 \frac{\chi^2_{n-1,\,1-\alpha}}{n-1} \,\middle|\, \sigma^2 = \sigma_{MAX}^2\right) \\ &= 1 - P\left(\frac{(n-1)S^2}{\sigma_{MAX}^2} < \sigma_{MAX}^2 \frac{\chi^2_{n-1,\,1-\alpha}}{(n-1)} \frac{(n-1)}{\sigma_{MAX}^2}\right), \end{aligned} \quad (16)$$

where $\frac{(n-1)S^2}{\sigma_{MAX}^2} = \chi^2_{n-1}$ is a random variable that follows a chi-squared distribution with $n-1$ degrees of freedom, so

$$FAR_{MAX} = 1 - P\left(\chi^2_{n-1} < \chi^2_{n-1,\,1-\alpha}\right) = \alpha_{nom}. \quad (17)$$

Note that σ_1^2 illustrated in Figure 4.2 is exactly in the Acceptable Zone showed in Figure 4.3 (i.e., $\sigma_0^2 \le \sigma_1^2 \le \sigma_{\mathrm{MAX}}^2$). So, differently from the well-known S^2 Control Chart, the S^2 Modified Control Chart will not tend to signal an alarm when $\sigma^2 = \sigma_1^2$. This is desirable, since when $\sigma^2 = \sigma_1^2$, the process is still not producing an unacceptable rate of nonconforming units. In the next section, we provide an illustrative example showing the advantages of the proposed S^2 Control Chart in the case illustrated in Figure 4.2 (i.e., in the case of $\sigma^2 = \sigma_1^2$).

Since the False Alarm Rate assumes numerous values within the acceptable zone, is possible to find a curve that characterizes the behavior of this rate for the Modified Chart. However, this curve does not indicate that the FAR is a random variable, it just points out that for each variance assumed by the process, there is a fixed-value for FAR. The FAR becomes a random variable when it depends on another random variable, but in this section, the variance is known. The equation 18 below considers several variances $\sigma_1^2 \in [0; \sigma_{MAX}^2)$ as follow:

$$FAR = 1 - P\left(S^2 < UCL_{Mod} \middle| \sigma_1^2 \in (0; \sigma_{MAX}^2]\right)$$

$$=1 - P\left(S^2 < \sigma_{MAX}^2 \frac{\chi^2_{n\text{-}1,\ 1\text{-}\alpha}}{n\text{-}1} \middle| \sigma_1^2 \in (0; \sigma_{MAX}^2]\right)$$

$$=1 - P\left(\frac{(n\text{-}1)S^2}{\sigma_1^2} < \sigma_{MAX}^2 \frac{\chi^2_{n\text{-}1,\ 1\text{-}\alpha}}{(n\text{-}1)} \frac{(n\text{-}1)}{\sigma_1^2}\right)$$

$$=1 - P\left(\frac{(n\text{-}1)S^2}{\sigma_1^2} < \frac{\sigma_{MAX}^2}{\sigma_1^2} \chi^2_{n\text{-}1,\ 1\text{-}\alpha}\right),$$

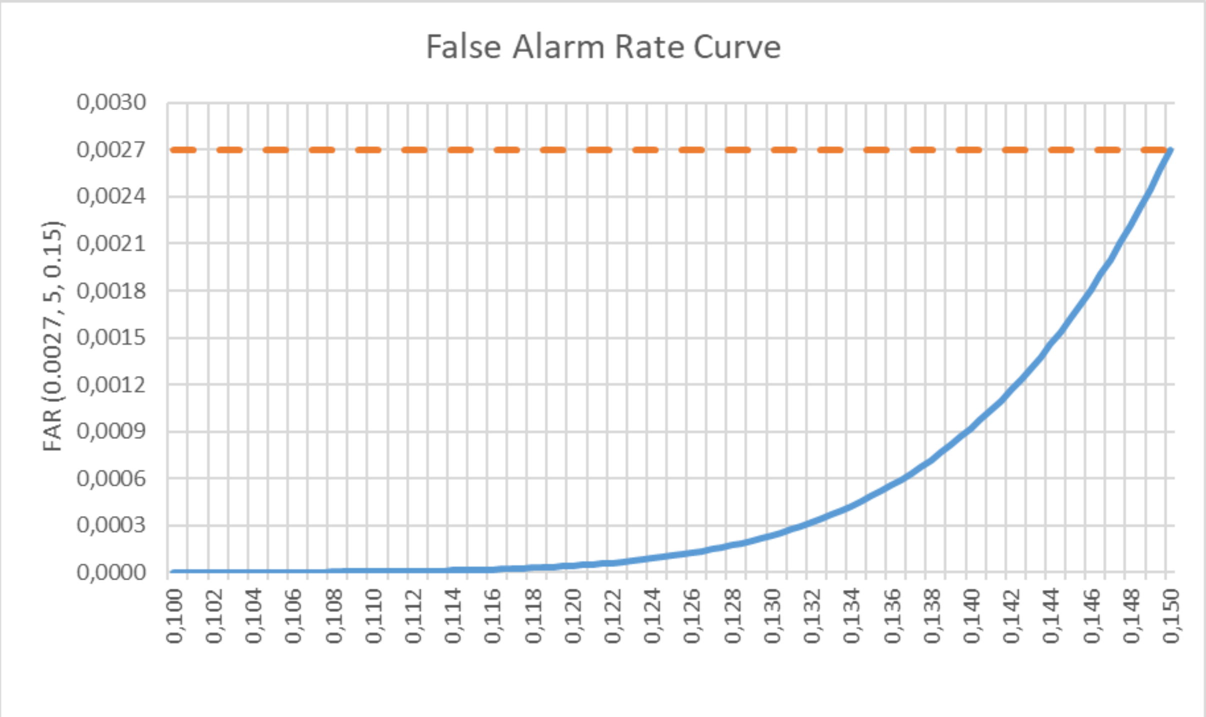
where $\frac{(n\text{-}1)S^2}{\sigma_1^2} = \chi^2_{n\text{-}1}$ is a random variable that follows a chi-squared distribution with $n\text{-}1$ degrees of freedom, so

$$FAR = 1 - P\left(\chi^2_{n\text{-}1} < \frac{\sigma_{MAX}^2}{\sigma_1^2} \chi^2_{n\text{-}1,\ 1\text{-}\alpha}\right) \quad (18)$$

Figure 4.5 below shows an example of this curve, using the following parameters and variables, α_{nom} = 0,0027, $n = 5$, $\sigma_{MAX}^2 = 0{,}0225$, and σ_1^2 varying from 0,100 to σ_{MAX}^2.

The numeric data of Figure 4.5 is listed in Table 4.1 below.

Figure 4.5 - False Alarm Rate curve for Modified Chart (Known Parameters)



Source: The author (2021)

Table 4 Erro! Nenhum texto com o estilo especificado foi encontrado no documento.**.1** – Numeric data of False Alarm Rate for Modified Chart

$\boldsymbol{\alpha_{nom}} = \mathbf{0{,}0027}, \boldsymbol{n} = \mathbf{5},$ $\boldsymbol{\sigma^2_{MAX}} = \mathbf{0,0225}$	
$\boldsymbol{\sigma_1^2}$	**FAR**
0,100	0,000000
0,105	0,000001
0,110	0,000004
0,115	0,000015
0,120	0,000042
0,125	0,000105
0,130	0,000237
0,135	0,000485
0,140	0,000918
0,145	0,001622
0,150	0,002700

Source: The author (2021)

The process user is more interested in the maximum occurrence probability of a false alarm (FAR_{MAX}) and not the minimum, since it is when the process is about to move to out-of-control state, with σ_1^2 coinciding with σ^2_{MAX}. In the next section 4.1.1, the ARL when variance is known will be discussed.

4.1.1 Average Run Length (ARL) when variance is known ($\boldsymbol{\sigma_0^2}$)

The Average Run Length (ARL) is the average number of points that must be plotted before a point indicates an out-of-control condition and can be used to evaluate the performance of the control chart (MONTGOMERY, 2009) at monitoring stage (Phase II). According to Chakraborti (2000) for an "efficient" control chart, one would like to have the in-control ARL to be "large" and the out-of-control ARL to be "small". According to Montgomery (2009) the in-control ARL is denoted ARL_0 and the out-of-control, denoted as ARL_1.

The Run-length (RL) is a random variable whose experiment involves repeat the event of taking a sample from the process until the chart plotted data indicates an out-of-control state and when the process parameters are known (μ_0 and σ_0^2), we can state that these trials are independent and have the same probability of occurrence. Since the probability p of obtaining a success (an alarm) is constant, for this case, RL_0 follows a geometric distribution, with parameter p (OLIVEIRA, 2020).

For the modified chart, there are numerous possible False Alarm Rate (FAR), because for each value assumed by FAR there is a new parameter p for distribution of RL_0, hence, the smaller the process dispersion, here denoted by σ_1^2, is, smaller the FAR will be, and the probability p will be equal to FAR_{MAX} when the process variation moves towards to σ_{MAX}^2.

According to Chakraborti (2000), considering an IC process with known parameters, RL_0 follows a geometric distribution, and its mean (ARL_0) is equal to the reciprocal of false alarm rate (FAR), hence, for S^2 Shewhart chart $ARL_0 = 1/FAR = 1/\alpha$. Thus, we can say that the reciprocal of the maximum false alarm rate of the modified chart is the $ARL_{0,FAR_{MAX}}$, where the FAR_{MAX} is the worst scenario for chart FAR, which is when it is about to move from IC to OOC state. For $\sigma_1^2 = \sigma_{MAX}^2$, it is expected the average number of samples required to detect the first alarm will be the lowest possible. This is when the ARL_0 reaches its minimum value, as shown by equation 19.

$$ARL_{0,FAR_{MAX}} = \frac{1}{FAR_{MAX}} = \frac{1}{1-P\left(\chi_{n-1}^2 < \chi_{n-1,\,1-\alpha}^2\right)} \tag{19}$$

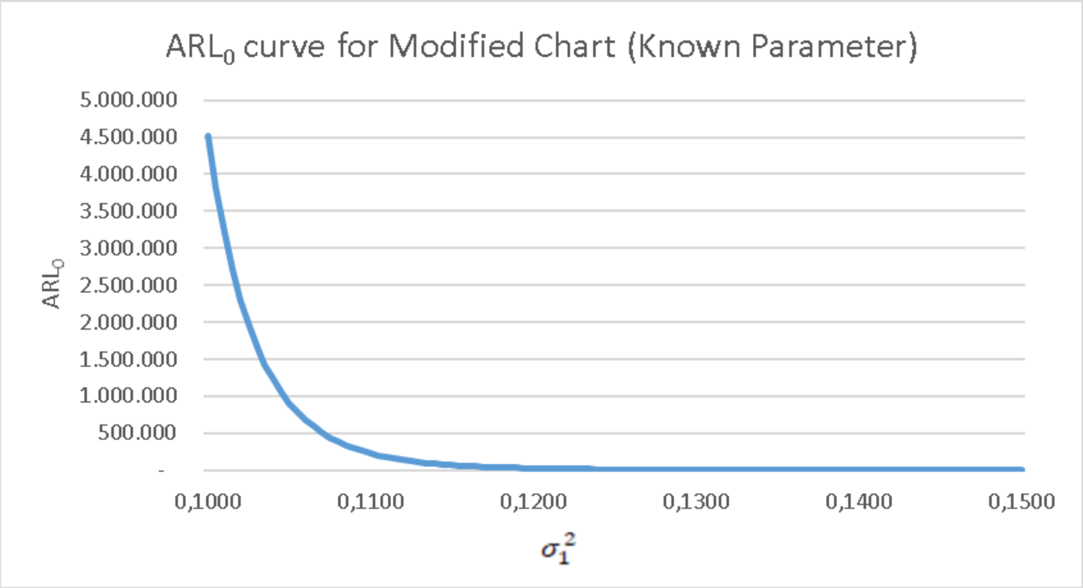
When the process parameters are known, the ARL_0 is not a random variable, and it is possible to describe graphically its behavior for Modified Chart for many variances within the acceptable range.

Derived from equation 19, the equation 20 below considers several variances $\sigma_1^2 \in (0; \sigma_{MAX}^2]$ as follows:

$$ARL_{0,FAR} = \frac{1}{FAR} = \frac{1}{1-P\left(\chi_{n-1}^2 < \frac{\sigma_{MAX}^2}{\sigma_1^2}\chi_{n-1,\,1-\alpha}^2\right)} \tag{20}$$

It is worth to notice that $ARL_{0,FAR} = ARL_{0,FAR_{MAX}}$ when $\sigma_1^2 = \sigma_{MAX}^2$. The Figure 4.6 shows the curve that describes the behavior of ARL_0 for each shift of placement of σ_1^2, $\alpha_{nom} = FAR_{MAX} = 0{,}0027$, $n = 5$, $\sigma_{MAX}^2 = 0{,}0225$, and σ_1^2 varying from 0,100 to σ_{MAX}^2.

Figure 4.6 - ARL_0 curve for Modified Chart (Known Parameters)



Source: The author (2021)

The numeric data in Figure 4.6 is listed in Table 4.2 below.

Table 4.2 – Numeric data of Average Run Length for Modified Chart

α_{nom} = 0,0027, n = 5, σ^2_{MAX} = 4	
σ_1^2	**ARL**
0,100	4.517.034
0,105	905.194
0,110	226.420
0,115	68.049
0,120	23.840
0,125	9.501
0,130	4.224
0,135	2.061
0,140	1.089
0,145	616
0,150	370

Source: The author (2021)

When σ_1^2 is substantially lower than σ^2_{MAX}, the probability of a false alarm is too low, thus, the average number of samples to be collected before the control chart signals for the first time is extremely high. As example, when $\sigma_1^2 = 0{,}100$, the ARL is 4.517.034.

4.1.2 An Illustrative Example for mean (μ_0)variance (σ_0^2) known

We illustrate the ideas of the S^2 Modified Control Chart for known variance in an automobile engine manufacturing process that uses a forging process to make piston rings. A more detailed description of this example is given in Montgomery (2009, p. 251). The quality characteristic variable (X) is the internal diameter of the piston rings, which has a two-sided specification limits of 74,000 $\pm$ 0,050 mm. It is assumed that the piston rings diameter (X) follows a normal distribution. Different from the book, here we assume that the in-control mean (μ_0) and the in-control standard deviation (σ_0) of the piston rings diameter are known, being respectively 74,000 mm and 0,0100 mm. The process leadership defined as acceptable up to 96 nonconforming parts per million (ppm) of units produced, which represents a potential capacity (Cp) of 1,30, according to Table 3.1.

In other words, the maximum allowed rate of nonconforming items (γ) is $96/1000000 = 0{,}000096$, which provides $z_{1-\gamma/2} = 3{,}9$. So, the maximum standard-deviation allowed (σ_{MAX}) for the piston ring diameter can be calculated using Equation 14, as shown below.

$$\sigma_{MAX} = \frac{USL - LSL}{2\, z_{1-\gamma/2}} = \frac{74{,}05 - 73{,}95}{2 \times 3{,}9} = \frac{0{,}1}{7{,}8} = 0{,}0128$$

All parameters given by this example are summarized in Table 4.3.

Table 4.3 - Parameters provided by the Example

μ_0	σ_0	σ_{MAX}	USL	LSL	γ
74,000	0,0100	0,0128	74,050	73,950	0,000096

Source: The author (2021)

The first analysis was done on the probability of nonconform units with $\sigma_0 = 0{,}0100$, which provided a C_p equal to 1,67, sigma level of 5 and a defective ppm of 0,57, which is significantly better than the process requirement. Therefore, the use of a modified acceptance chart has become appropriate.

By using the data presented in Table 4.3, the upper control limits for the traditional S^2 chart and the modified chart for the process were defined, considering $\sigma_0 = 0{,}01$, which are shown in Table 4.4.

Table 4.4 - Upper Control Limits for variance and potential capacity

	Upper Limit for S_i^2	Equation
UCL_{S^2}	0,000406	$s_i^2 \leq \frac{\sigma_0^2}{n-1} \cdot \chi^2_{n-1,1-\alpha}$
UCL_{Mod}	0,00067	$s_i^2 \leq \frac{(USL-LSL)^2 \cdot \chi^2_{n-1,1-\alpha}}{4(n-1)\, z_{\gamma/2}^2}$

Source: The Author (2021)

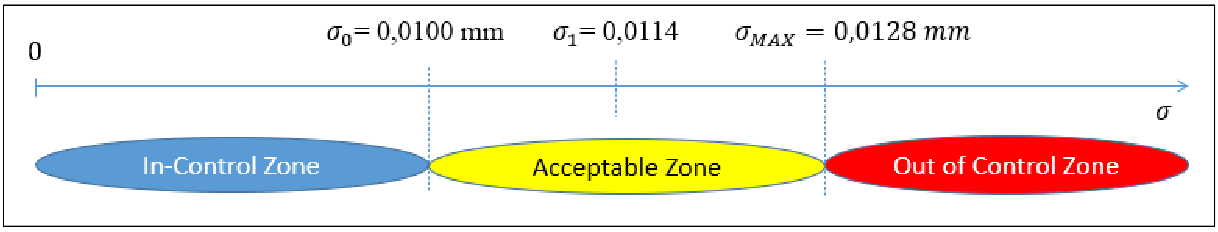
Suppose that the practitioner decides to monitor the process variance (σ^2) with samples of size $n = 5$ of the piston's rings diameter collected at regular intervals. To this end, the practitioner can use the well-known S^2 Control Chart or the S^2 Modified Control Chart proposed here. Considering a maximum false alarm rate (α) of 0,0027 for each chart, the control limits of both charts (UCL_{S^2} and UCL_{Mod}) can be calculated as shown below.

$$UCL_{S^2} = \sigma_0^2 \frac{\chi^2_{n-1,\, 1-\alpha}}{n-1} = 0{,}0100^2 \frac{16{,}25}{5-1} = 0{,}000406$$

$$UCL_{Mod} = \sigma_{MAX}^2 \frac{\chi^2_{n-1,\, 1-\alpha}}{n-1} = 0{,}0128^2 \frac{16{,}25}{5-1} = 0{,}00067$$

Now, let's suppose that the process standard deviation (σ) moved from the in-control value $\sigma = \sigma_0 = 0{,}0100$ to $\sigma = \sigma_1 = 0{,}0114$. Note that, since $\sigma_0 < \sigma_1 < \sigma_{MAX}$, even though the process standard-deviation increased, it is still in the Acceptable Zone (see Figures 4.3, 4.4 and 4.7), so the process is still producing an acceptable rate of nonconforming items (i.e., a rate smaller than $\gamma = 0{,}000096$).

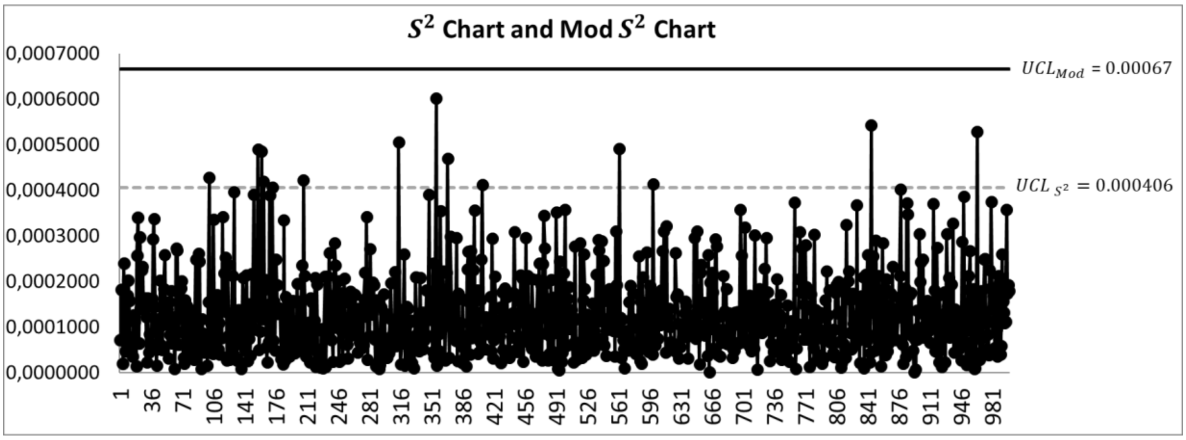
Figure 4.7 - Illustration showing the actual process standard deviation (σ) in the Acceptable Zone ($\sigma = \sigma_1 = 0{,}0114$)



Source: The author (2021)

To further examine the behavior of both charts (the well-known S^2 Control Chart and the S^2 Modified Control Chart proposed here), we simulated one thousand independent samples of the piston ring diameter (each sample with size 5), from a normal distribution with mean $\mu_0 = 74{,}000$ and standard deviation $\sigma = \sigma_1 = 0{,}0114$. With the simulated data, we calculated the sample variances (S^2) using equation 1, which is the plotting statistic for both charts, and plotted these against the control limits values shown in Table 4.4. The simulated sample variances (S^2) and the control limits (UCL_{S^2} and UCL_{Mod}) are shown in Figure 4.8. UCL_{S^2} is shown in a dashed grey line and UCL_{Mod} in a solid black line.

Figure 4.8 - The S^2 Control Chart and the S^2 Modified Control Chart for monitoring the variance of a process $X \sim N(\mu_0 = 74{,}000, \sigma = \sigma_1 = 0{,}0114)$ given the in-control parameters in Table 4.3



Source: The author (2021)

As it can be seen in Figure 4.8, signals above the UCL_{S^2} dashed line are frequent. If the user were using just the well-known S^2 Control Chart, he would typically suspect that an assignable cause has occurred and that the process variance (σ^2) is larger than the in-control target ($\sigma_0^2 = 0{,}01^2$) increasing the production of nonconforming items. In this case, the user would stop the process and start looking for assignable causes, wasting time and decreasing production (what is also a waste of money). However, even though the process variance has indeed increased to $\sigma^2 = \sigma_1^2 = 0{,}0114^2$, it is still smaller than the maximum variance allowed ($\sigma_{\text{MAX}}^2 = 0{,}0128^2$), this means that the proportion of nonconforming items being produced is still acceptable according to the specification of the project. Therefore, the production does not really need to be stopped to search for assignable causes. So, it is clear that the use of the well-known S^2 Control Chart alone can mislead the user.

Now, considering the S^2 Modified Control Chart proposed in this work, since the process is still capable in the sense it is still producing an acceptable rate of nonconforming units, there is no signal above the UCL_{Mod} black solid line, what is indeed expected since the probability of a false alarm when $\sigma^2 = \sigma_1^2 = 0{,}0114^2$ is smaller than $\alpha_{nom} = 0{,}0027$, as shown below.

$$False\ Alarm\ Rate = 1 - P(S^2 < UCL_{Mod} | \sigma^2 = \sigma_1^2 = 0{,}0114^2)$$
$$= 1 - P\left(S^2 < \sigma_{MAX}^2 \frac{\chi_{n\text{-}1,\ 1\text{-}\alpha}^2}{n\text{-}1} \middle| \sigma^2 = \sigma_1^2 = 0{,}0114^2\right)$$
$$= 1 - P\left(\frac{(n\text{-}1)S^2}{\sigma_1^2} < \frac{\sigma_{MAX}^2}{\sigma_1^2} \chi_{n\text{-}1,\ 1\text{-}\alpha}^2\right),$$

where $\frac{(n\text{-}1)S^2}{\sigma_1^2} = \chi_{n\text{-}1}^2$ is a random variable that follows a chi-squared distribution with $n\text{-}1$ degrees of freedom, so

$$False\ Alarm\ Rate = 1 - P\left(\chi_{n-1}^2 < \frac{0.0128^2}{0.0114^2}\ 16{,}25\right) = 1 - P(\chi_{n-1}^2 < 20{,}486)$$
$$False\ Alarm\ Rate = 0{,}0004 < \alpha_{nom} = 0{,}0027$$

The large number of signals between UCL_{S^2} and UCL_{Mod} control limits when the actual process variance (σ^2) is in between σ_0^2 and σ_{MAX}^2 (as shown here with $\sigma^2 = \sigma_1^2$) and the smaller frequency of signal above UCL_{Mod} given the small value of the false alarm rate in this case ($FAR = 0{,}0004$), motivates the use of the S^2 Modified Control Chart proposed in the present work.

In summary, as σ^2 moves from σ_0 towards σ_{MAX}, the chart tends to more quickly signal points higher than UCL_{S^2} and not higher than UCL_{Mod}, since the process is still capable. If the process had been monitored only by using the S^2 control chart (i.e., only by using the UCL_{S^2} control limit), it would have signaled several alarms, however, that do not compromise the process in meet its quality requirements, which could keep running in order to fulfill the process expectancies in terms of efficiency and quality.

4.2 MEAN AT THE IN-CONTROL VALUE (μ_0) AND VARIANCE UNKNOWN (σ^2)

Here, we also assume that the process mean is in-control value (μ_0) and in the exact middle point between the specification limits, consistently with the purpose of detecting relevant increases in the process variance only. The estimation of the process variance is traditionally done by collecting m samples with size n elements from an in-control (IC) process during Phase I. In this dissertation we also present the execution of what we call Phase 0 and a proposed change on how Phase I is executed, that are properly presented in section 4.2.2. Considering that the estimation is done from sampling data, there is a data variability when compared to the entire process data which is known as "practitioner-to-practitioner variability" (SALEH, *et al*., 2015) since each chart practitioner can get a different sample, which influences the parameter estimation, where the control limits become conditioned to these estimates and according to Chakraborti (2006), some operational properties of the control chart, such as FAR and ARL are compromised. Some of the parameter estimation effects on FAR calculation is the increase of false alarms.

The analysis of the effects of parameter estimation on the performance of the control graph can be done based on two perspectives and according to Jardim, Chakraborti and Epprecht (2019), these are: conditional and unconditional perspectives.

The conditional perspective considers the practitioner-to-practitioner variability, examining the performance of each control chart individually (assessing the graph based on the estimate of the parameter found). That is, the performance of the graph is conditioned to the value of the estimator, so FAR conditioning is called $CFAR$, and ARL conditioning as $CARL$ (JARDIM; CHAKRABORTI; EPPRECHT, 2019).

The unconditional perspective does not take into account the practitioner-to-practitioner variability, since it analyzes the average performance of a many different control charts, each corresponding to a set of parameters estimated from samples of the same process (CHAKRABORTI, 2000).

These both perspectives (conditional and unconditional) are going to be considered in this work, with only the process variance (S^2 chart) being estimated, with known process mean. However, the unconditional perspective developed in this present chapter will be further assessed in terms of chart performance and presentation of an illustrative example, whilst the conditional perspective will have only first developments and analysis, as a kick-off for future research.

After the design phase of the control chart (Phase I), when the reference samples were collected and the control limits have been calculated from the estimated parameters (recalling that in this study, the process mean is known), the Phase II starts, which is when the process monitoring actually takes place, assessing process samples and if there are special causes that move the process from an in-control state (IC) to an out-of-control state (OOC). Here, the chart provides information to process user to act on the process, so it moves back to a control state.

Because the process monitoring is done based on sampling, a particular sample collected may show that the process is in control when actually it is not, and vice versa, it is important that the process manager assess the control chart performance in order to minimize or prevent unnecessary process stop, which contributes for lower process efficiency and increased costs.

According to Epprecht, Loureiro and Chakraborti (2015), due to the skewness of the distribution of S^2, this study only considers the case of probability control limits, which aims to provide a pre-specified False Alarm Rate and upper one-sided charts (without a lower control limit), being the major concern of the chart user is to detect increases in process variance.

For modified charts, because the standard deviation or variance shifts within the acceptable range, there are several possible values for the False Alarm Rate (FAR). When the parameters are estimated, the control limits are affected by this estimation and, hence, the probability of a false alarm occurrence depends on this estimate. Thus, the false alarm rate becomes a random variable to the estimator (OLIVEIRA, 2020).

Here we have the subgroups of Phase I and Phase II with same size (n) and all the observations also follow a normal distribution, with known mean μ_0 and unknown variance σ^2. Thus, the process variance in phases I and II may be different. Based on this, we may state that:

$$\omega = \frac{\sigma}{\sigma_0} \tag{21}$$

Where σ^2 is the unknown variance from Phase II and σ_0^2 the variance from Phase I.

When the process is In Control (IC), with $\sigma^2 = \sigma_0^2$, and hence $\omega = 1$. If at a given moment during Phase II the process variance increases, then $\sigma^2 = \sigma_1^2 > \sigma_0^2$, and as a consequence, the process is deemed out-of-control (OOC), with $\omega > 1$.

Now, we denote $\hat{\sigma}_0^2$ an estimator of σ_0^2 from the reference sample from Phase I and we have the error factor of the estimator (EPRECHT; LOUREIRO; CHAKRABORTI, 2015), that is defined as:

$$\tau = \frac{\hat{\sigma}_0}{\sigma_0} \tag{22}$$

Once the estimator $\hat{\sigma}_0^2$ is calculated, the UCL for the one sided S^2 chart to be used during Phase II monitoring can be defined, for a specified or nominal False Alarm Rate (FAR) and α_{NOM}, equation 8 can be rewritten as:

$$\widehat{UCL}_{S^2} = \hat{\sigma}_0^2 \, \frac{\chi^2_{n\text{-}1,\, 1\text{-}\alpha}}{n\text{-}1} \tag{23}$$

In the next section we will discuss some properties of variance estimation.

4.2.1 Probability of a Signal

As previously described, a false alarm is defined as a signal (alarm) when the process is actually in-control. The maximum false alarm rate happens when $\sigma^2 = \sigma_0^2$, in this case, where the variance parameter is estimated, $\hat{\sigma}_0^2$, and is not affected by change of the mean to a not centered location between the USL and LSL. The probability of a signal for the upper one sided S^2 from Phase II is defined by:

$$\begin{aligned} FAR_{MAX} &= P\left(S^2 > UCL_{S^2} \mid \sigma^2 = \hat{\sigma}_0^2\right) \\ &= P\left(S^2 > \hat{\sigma}_0^2 \frac{\chi^2_{n\text{-}1,1\text{-}\alpha}}{n\text{-}1} \,\middle|\, \sigma^2 = \hat{\sigma}_0^2\right) \\ &= 1 - P\left(\frac{(n\text{-}1)S^2}{\sigma^2} < \frac{\hat{\sigma}_0^2}{\sigma_0^2}\frac{\sigma_0^2}{\sigma^2}\chi^2_{n\text{-}1,1\text{-}\alpha}\right), \end{aligned}$$

Which can be rewritten as

$$FAR_{MAX} = 1 - P\left(\frac{(n\text{-}1)S^2}{\sigma^2} < \frac{\tau^2}{\omega^2}\chi^2_{n\text{-}1,1\text{-}\alpha}\right).$$

Where $\frac{(n\text{-}1)S^2}{\sigma^2} = \chi^2_{n\text{-}1}$ is a random variable that follows a chi-squared distribution with $n\text{-}1$ degrees of freedom, so

$$FAR_{MAX} = 1 - P\left(\chi^2_{n\text{-}1} < \frac{\tau^2}{\omega^2}\chi^2_{n\text{-}1,1\text{-}\alpha}\right). \tag{24}$$

According to Epprecht, Loureiro and Chakraborti (2015), the probability of a signal in Phase II can be calculated conditionally on the Phase I estimator $\hat{\sigma}_0^2$ or, unconditionally, averaging over the distribution $\hat{\sigma}_0^2$, being this probability a function of the random variable τ and given one particular value of τ and a given value of ω it assumes a single value, and the distribution of the random variable "probability of signal" is parameterized by the number of retrospective sample m and the size of the individual sample, n.

Being the False Alarm Rate an important characteristic in the design and implementation phases and the probability of a signal, the occurrence of a signaling event when the process is actually in-control (IC), and in this case $\omega = 1$. Hence, the actual FAR attained by a particular S^2 chart, with an estimated parameter, here denominated FAR_{MAX} may be obtained from equation 24 as:

$$FAR_{MAX} = P(S^2 > UCL_{S^2} \mid \sigma^2 = \hat{\sigma}_0^2)$$

$$= P\left(S^2 > \frac{\hat{\sigma}_0^2}{\sigma_0^2}\chi^2_{n\text{-}1,1\text{-}\alpha} \middle| \sigma^2 = \hat{\sigma}_0^2\right)$$

$$= P\left(S^2 > \frac{\hat{\sigma}_0^2}{\sigma_0^2}\chi^2_{n\text{-}1,1\text{-}\alpha} \middle| \sigma^2 = \hat{\sigma}_0^2\right)$$

$$FAR_{MAX} = 1\text{-}\ F_{\chi^2_{n-1}}\left(\frac{\hat{\sigma}_0^2}{\sigma_0^2}\ \chi^2_{n\text{-}1,1\text{-}\alpha} \mid \sigma^2 = \hat{\sigma}_0^2\right) \qquad (25)$$

Where the $F_{\chi^2_{n-1}}$ represents the c.d.f of the chi-square distribution with $(n-1)$ degrees of freedom. Thus, FAR_{MAX} is a random variable, being a monotonically decreasing function of the Phase I variance estimator $\hat{\sigma}_0^2$.

According to Epprecht, Loureiro and Chakraborti (2015), the attained FAR_{MAX} in equation 24 is in general different to α even when the process is in-control (IC), as it depends on the ratio $\tau^2 = \frac{\hat{\sigma}_0^2}{\sigma_0^2}$, the error factor of the estimate and due to this error factor ω is a random variable, the attained FAR_{MAX} is also a random variable. The FAR_{MAX} distribution is parametrized by m and n and when $m \to \infty$, $FAR_{MAX} \to \alpha_{nom}$, as in that case, $\hat{\sigma}_0^2 \to \sigma_0^2$ in probability, provided that σ_0^2 is a consistent estimator. Thus, when $m \to \infty$, that is, when there is a very large sample, the attained FAR_{MAX} converges to the nominal FAR.

In this dissertation, the estimator chosen and recommended is the pooled estimator $\left(S_p^2\right)$ based on the variances of the Phase I samples, that according to Mahmoud (2010), this is the best estimators for σ_0^2. The pooled variance $\left(S_p^2\right)$ is calculated by the sample variance means of the samples collected in Phase I, as shown:

$$S_p^2 = \frac{1}{m}\sum_{i=1}^{m} S_i^2, \text{ where } S_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{i,j} - \bar{X}_i\right)^2 \qquad (26)$$

4.2.2 Unconditional Average-Run-Length (Traditional Shewhart's S^2 chart)

Suppose that $(X_{i,1}, X_{i,2}, \dots, X_{m,n})$ are random variables, and $i = 1,2, \dots m$ are independent samples extracted from a process in Phase I, with variance σ_0^2 unknown, where i which identifies the subgroup. When the process is in control (IC), for $\sigma_1 = \Delta\sigma_0$ and traditional Shewhart's S^2 chart, we have that $\Delta = 1$, and when the process is out-of-control (OOC), $\Delta > 1$. For σ_0^2 unknown, the IC state occurs when $s_i^2 \in \left(0, \widehat{UCL}_{S^2}\right)$.

In Phase II, the samples of size n are extracted, and s_i^2 are calculated from $\{X_{i,1}, X_{i,2}, \dots, X_{i,n}\}$, where the Type I error is $\alpha = Pr(s_i^2 \notin (0, \widehat{UCL}_{S^2}))$ or $\alpha = 1 - Pr(s_i^2 \in (0, \widehat{UCL}_{S^2}))$, and is represented by α_{NOM}. As stated previously, for a OOC process for traditional Shewhart's S^2 chart, $\sigma_1 = \Delta\sigma_0$, for $\Delta > 1$, and the estimation of Upper Control Limit is given by Equation 23, listed in section 4.2:

$$\widehat{UCL}_{S2} = \hat{\sigma}_0^2 \, \frac{\chi^2_{n\text{-}1,\, 1\text{-}\alpha}}{n-1} \qquad (23)$$

For IC state, the equation 27 can be written:

$$Pr\left(s_i^2 \leq \widehat{USC}\right) = Pr\left(\frac{s_i^2(n-1)}{\sigma_1^2} \leq \frac{\widehat{UCL}\,(n-1)}{\sigma_1^2}\right) \qquad (27)$$

Developing the Equation 27 we have the following:

$$Pr\left(s_i^2 \leq \widehat{USC}\right) = Pr\left(\frac{s_i^2(n-1)}{\sigma_1^2} \leq \frac{\chi^2_{n-1,1-\alpha_{nom}}\hat{\sigma}_0^2\,(n-1)}{(n-1)\,\sigma_1^2}\right),$$

where $\frac{s_i^2\,(n\text{-}1)}{\sigma_1^2} = \chi^2_{n\text{-}1}$ is a random variable that follows a chi-squared distribution with $n\text{-}1$ degrees of freedom, so

$$Pr\left(s_i^2 \leq \widehat{USC}\right) = Pr\left(\chi^2_{n-1} \leq \frac{\hat{\sigma}_0^2}{\sigma_1^2}\chi^2_{n-1,1-\alpha_{nom}}\right). \qquad (28)$$

Being $\sigma_1^2 = \Delta^2\sigma_0^2$ and $\tau^2 = \frac{\hat{\sigma}_0^2}{\sigma_0^2}$, we have the following Equation 29:

$$Pr(s_i^2 \leq \widehat{USC}) = Pr\left(X_{n-1}^2 \leq \chi^2_{n\text{-}1,\ 1\text{-}\alpha}\frac{\tau^2}{\Delta^2}\right) \tag{29}$$

Being Type I error $\alpha = 1 - Pr\left(s_i^2 \in \left(0, \widehat{LSC}\right)\right)$ for, when $\Delta = 1$ (In-Control Process), hence, $\alpha = 1 - Pr\left(X^2 \leq \chi^2_{n\text{-}1,\ 1\text{-}\alpha}\tau^2\right)$, resulting in the following equation:

$$\alpha = 1 - F_{X^2}\left(\chi^2_{n\text{-}1,\ 1\text{-}\alpha}\tau^2\right) \tag{30}$$

The probability function of U is $f(u) = m(n-1)\, f_{X^2}(m(n-1)u)$ and the unconditional ARL is given by the following equation:

$ARL = \int_0^\infty \frac{1}{\alpha} f(u)\, du$, where $\alpha = 1 - F_{X^2}\left(\chi^2_{n\text{-}1,\ 1\text{-}\alpha}\,\tau^2\right)$, thus

$$E(ARL) = \int_0^\infty \frac{1}{1 - Pr\left(X_{n-1}^2 \leq \chi^2_{n\text{-}1,\ 1\text{-}\alpha}\frac{\tau^2}{\Delta^2}\right)} f(u)\, du$$

$$E(ARL) = \int_0^\infty \frac{1}{1 - Pr\left(X_{n-1}^2 \leq \chi^2_{n\text{-}1,\ 1\text{-}\alpha}\frac{\tau^2}{\Delta^2}\right)} m(n-1)\, f_{X^2}(m(n-1)u\, du$$

$$AARL(m,n,\delta) = \int_0^\infty \frac{1}{1 - F_{X^2}\left(\frac{\chi^2_{n\text{-}1,\ 1\text{-}\alpha}\tau^2}{\Delta^2}\right)} m(n-1) f_{X^2}(m(n-1)u\, du).$$

Being $\chi^2_{n\text{-}1,\ 1\text{-}\alpha}$ the constant k in terms of (*1-α*)-quantile of a chi-square distribution with *n-1* degrees of freedom and *α* is the nominal false alarm rate, we have:

$$AARL(m,n,k,\delta) = \int_0^\infty \frac{1}{1 - F_{X^2}\left(\frac{k\tau^2}{\Delta^2}\right)} m(n-1)\, f_{X^2}(m(n-1)u\, du) \tag{31}$$

Where f_{X^2} is the Probability Density Function of a chi-square distribution with $m(n-1)$ degrees of freedom.

The analysis of the performance of the Shewhart variance charts with estimated parameter was performed using the Maple Software, version 18. The numerical solution of the

ARL mathematical function is given by equation 31. This equation was obtained by the construction of the program according to Figure 4.9 shown below.

Figure 4.9 - Mathematical function in software Maple version 18

$$RR2 := \textbf{proc}(m, n, \Delta, k)\ \textbf{local}\ fu, pl, ARL; pl := CDF\left(ChiSquare(n-1), \frac{k \cdot u}{\Delta^2}, numeric\right); fu := m \cdot (n-1) \cdot PDF(ChiSquare(m \cdot (n-1)), m \cdot (n-1) \cdot u, numeric); ARL := evalf\left(int\left(\frac{fu}{1-pl}, u = 0..100, numeric, digits = 5\right)\right); print(ARL); \textbf{end proc}$$

Source: The author (2021)

Where m is the number of samples taken in Phase I of size n; k is a constant; and Δ is the deviation of population variance. The following table presents the results obtained by numerical methods for $\alpha_{nom} = 0{,}0027$ and $\Delta = \{1{,}00; 1{,}05; 1{,}10; 1{,}15; 1{,}20; 1{,}25; 1{,}30; 1{,}35\}$

Table 4.5 - Values of ARL form $\alpha_{nom} = 0{,}0027$ and Δ={1,00; 1,05; 1,10; 1,15; 1,20; 1,25; 1,30; 1,35}

M	n	α_{nom}	0,0027	Δ							
		k	Performance	1	1,05	1,10	1,15	1,20	1,25	1,30	1,35
20	3	11,83	AARL	1110,4	516,2	271,9	158,0	99,3	66,5	46,9	34,5
	5	16,25	AARL	802,4	350,7	175,2	97,4	59,0	38,3	26,4	19,1
	9	23,57	AARL	667,0	254,5	114,5	58,7	33,5	20,8	13,9	9,8
50	3	11,83	AARL	541,9	291,6	171,4	108,4	72,7	51,2	37,7	28,7
	5	16,25	AARL	490,5	237,7	128,4	75,7	48,1	32,4	23,0	17,0
	9	23,57	AARL	460,9	191,3	91,6	42,7	29,1	18,6	12,7	9,2
200	3	11,83	AARL	405,2	230,1	141,1	92,2	63,5	45,7	34,2	26,4
	5	16,25	AARL	396,0	200,2	111,71	67,6	43,8	30,0	21,5	16,1
	9	23,57	AARL	390,0	168,0	82,7	45,4	27,3	17,7	12,2	8,9
500	3	11,83	AARL	383,9	220,1	136,0	89,4	61,8	44,7	33,5	25,9
	5	16,25	AARL	380,2	193,8	108,8	66,1	43,0	29,5	21,3	15,9
	9	23,57	AARL	377,7	163,8	81,1	44,7	27,0	17,5	12,1	8,8
1000	3	11,83	AARL	377,1	216,9	134,4	88,5	61,3	44,4	33,3	25,8
	5	16,25	AARL	375,1	191,7	107,8	65,7	42,7	29,4	21,2	15,9
	9	23,57	AARL	373,7	162,5	80,6	44,5	26,8	17,4	12,1	8,8

Source: The author (2021)

The Table 4.5 shows that the ARL converges to $1/\alpha_{NOM} = 370{,}4$ as m increases (see $m = 1000$) and the ARL decreases as Δ increases, meaning that the FAR gets higher, which is expected, as $\Delta > 1$. If we increase from $n = 3$ to $n = 5$ and $n = 9$, for same m, the ARL converges slowly to $1/\alpha_{NOM}$, and faster as m increases.

The Modified control limit however, is not a random variable because it is not estimated, being calculated by given parameters such upper and lower specification limits that are defined by process manager or consumers and acceptance rate of nonconforming units, that are also defined by process manager in order to fulfil customer requirements. This being said, the Modified Chart for variance performance has the same behavior found for Shewhart'd S^2 Chart when parameters are known.

In order to evaluate how the UCL_{MOD} and $\widehat{UCL}_{S^2}$ correlates, we propose a additional Phase 0, before well known traditional Phase I. In addition, to provide to process practitioners guidance to define if the chart developed fits to real use, we also propose the assessment of the ratio between σ^2_{MAX} and σ_0^2 as detailed in the following paragraphs.

When we need to estimate σ_0^2, it means that the $\widehat{UCL}_{S^2}$ may vary because the variance now is a random variable. However, the UCL_{MOD}, considering that the mean is between the USL and LSL, does not shift. Thus, if the $\widehat{UCL}_{S^2}$ is located below UCL_{MOD} most of the time, it is possible to conclude that the issues caused by the estimations and its inherent errors could be less severe. Hence, we calculate the probability of $\widehat{UCL}_{S^2}$ is less than UCL_{MOD}, that is $P(\widehat{UCL}_{S^2} < UCL_{MOD})$, recalling that the UCL_{MOD} is calculated by using given parameters (see equation 15).

The estimation of Upper Control Limit is given by Equation 23, already presented in section 4.2.

$$\widehat{UCL}_{S^2} = \hat{\sigma}_0^2 \ \frac{\chi^2_{n-1,\ 1-\alpha}}{n-1} \tag{23}$$

As stated previously in section 4.2, according to Mahmoud (2010), the pooled variance $\left(S_p^2\right)$ is the best estimators for σ_0^2 and is calculated by equation 26. Replacing $\hat{\sigma}_0^2$ by S_p^2 we have the equation (32).

$$\widehat{UCL}_{S^2} = S_p^2 \ \frac{\chi^2_{n-1,\ 1-\alpha}}{n-1} \tag{32}$$

Therefore, $P(\widehat{UCL}_{S^2} < UCL_{MOD}) = P\left(S_p^2 \frac{\chi^2_{n\text{-}1,\,1\text{-}\alpha}}{n\text{-}1} < \sigma^2_{MAX} \frac{\chi^2_{n\text{-}1,\,1\text{-}\alpha}}{n\text{-}1}\right)$

$$= P\left(S_p^2 < \sigma^2_{MAX}\right) \quad (33)$$

Developing the Equation 33 we have the following:

$$P(\widehat{UCL}_{S^2} < UCL_{MOD}) = P\left(\frac{m(n-1)S_p^2}{\sigma_0^2} < \frac{m(n-1)\,\sigma^2_{MAX}}{\sigma_0^2}\right)$$

where $\frac{m(n-1)S_p^2}{\sigma_0^2} = \chi^2_{m(n-1)}$, here denoted as Y, is a random variable that follows a chi-squared distribution with $m(n-1)$ degrees of freedom, so we have the following Equation 34. Is important to notice that here we are still in Phase I, thus we have the in-control variance σ_0^2.

$$P(\widehat{UCL}_{S^2} < UCL_{MOD}) = P\left(Y < \frac{m(n-1)\,\sigma^2_{MAX}}{\sigma_0^2}\right) \quad (34)$$

Recalling that UCL_{MOD} has all parameters known and is fixed, after proper derivations, we found that the equation 34 depends on the in-control variance σ_0^2, which is unkwown. Assuming that the process practitioner does not known the actual value of σ_0^2, note that $\sigma^2_{MAX}/\sigma_0^2$, is the ratio between σ^2_{MAX} (the maximum value that the process variance is allowed to be compared with the in-control process value, in the sense that the process produces a tolerated small fraction of nonconforming items) and the in-control variance σ_0^2. By dividing σ^2_{MAX} by σ_0^2, we may define how much larger the maximum allowed variance is required to be when compared with the the target variance.

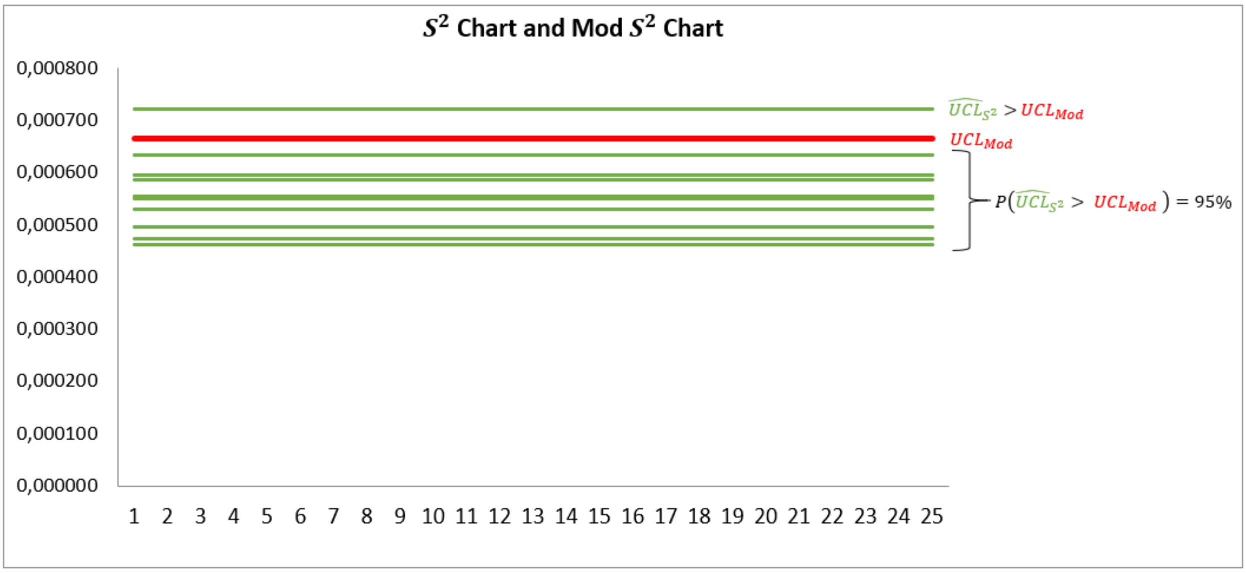
Hence, what should this ratio be so that the probability of the traditional S^2 control limit of the estimated parameter is less than the modified control limit, at a given probability of 95%, 90% or 99%? That is, 95% of all possible $\widehat{UCL}_{S^2}$, due to S_p^2 is a random variable, will be smaller than UCL_{MOD}. The Figure 4.10 below shows graphically this behavior, for Phase I, where the green lines representing $\widehat{UCL}_{S^2}$ were estimated for a $m = 25, n = 5$ and the parameters listed in Table 4.6, and the red line representing UCL_{MOD}, which is fixed and calculated in terms of σ_{MAX}, specification limits, degrees of freedom and α_{nom}, all given parameters.

Table 4.6 - Parameters used for calculation of control limits of Figure 4.10

μ_0	σ_0	σ_{MAX}	USL	LSL	α
74,000	0,0117	0,0128	74,050	73,950	0,27%

Source: The author (2021)

Figure 4.10 - Graphical demonstration of how the $\widehat{UCL}_{S^2}$ locates around UCL_{MOD}



Source: The author (2021)

Is also important to notice that we can find different required ratios for $\frac{\sigma_{MAX}^2}{\sigma_0^2}$ by varying m and n. Based on these informations, we can raise the following question:

What should be the value of $\frac{\sigma_{MAX}^2}{\sigma_0^2}$ so that $P\left(Y < \frac{m(n-1)\,\sigma_{MAX}^2}{\sigma_0^2}\right) = 95\%$ (or any other desired probability)?

The chart user defines this probability, recalling that the ideal scenario would be that σ_0^2 is known, however it is an unknown process parameter, that can be either given for calculation matters or replaced by the pooled variance (S_p^2).

For $m = 25$, $n = 5$ and $P = 95\%$, we have that $\sigma_{MAX}^2/\sigma_0^2 = 1{,}2434$, that means if the practitioner collects 25 samples of size 5 during Phase I, the σ_{MAX}^2 shall be at least 24% larger than the in-control variance, or in this case, the pooled variance (S_p^2) from Phase I, given by equation 26. If the ratio is 1,2434, we have that 95% possible estimated S^2 control limits $(\widehat{UCL}_{S^2})$, will locate below the modified control limit (UCL_{MOD}). In a new scenario we raise $m = 100$ and keep $n = 5$ and $P = 95\%$, and now we have that now $\sigma_{MAX}^2/\sigma_0^2 = 1{,}1191$, that means, if the process practitioner wants to ensure that all possible 95% possible estimated S^2 control limits ($\widehat{UCL}_{S^2}$), will locate below the modified control limit (UCL_{MOD}).

In Table 4.7 below, there are several ratios to be considered varying m, n and the given probability of $\widehat{UCL}_{S^2}$ being less than UCL_{MOD}. Is possible to notice that as m and n get larger, the required ratio gets smaller, that means that σ^2_{MAX} is required to be less in magnitude than σ^2_0, which can contribute to achieve the required ratio easier.

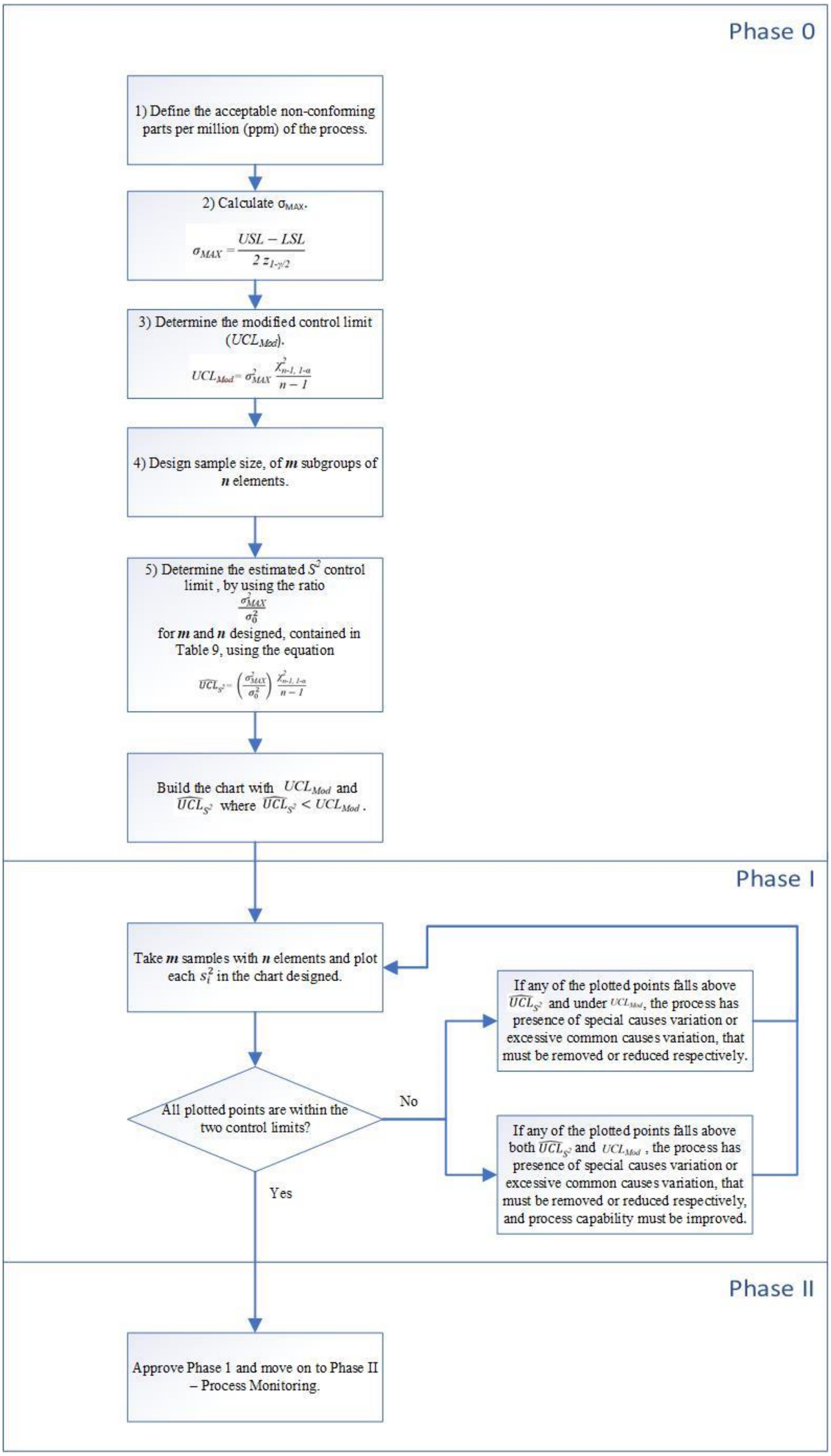
Table 4.7 - Ratio $\sigma^2_{MAX}/\sigma^2_0$ calculation varying m, n and the given probability of $\widehat{UCL}_{S^2}$ being less than UCL_{MOD}

m	n	Degrees of Freedom $m(n-1)$	Ratio (90%)	Ratio (95%)	Ratio (99%)
25	3	50	1,2633	1,3501	1,5231
	5	100	1,1850	1,2434	1,3581
	9	200	1,1301	1,1700	1,2472
50	3	100	1,1850	1,2434	1,3581
	5	200	1,1301	1,1700	1,2472
	9	400	1,0916	1,1191	1,1718
100	3	200	1,1301	1,1700	1,2472
	5	400	1,0916	1,1191	1,1718
	9	800	1,0646	1,0836	1,1200
200	3	400	1,0916	1,1191	1,1718
	5	800	1,0646	1,0836	1,1200
	9	1600	1,0456	1,0589	1,0841
500	3	1000	1,0577	1,0747	1,1070
	5	2000	1,0407	1,0526	1,0750
	9	4000	1,0288	1,0371	1,0528
1000	3	2000	1,0407	1,0526	1,0750
	5	4000	1,0288	1,0371	1,0528
	9	8000	1,0203	1,0261	1,0372
2000	3	4000	1,0288	1,0371	1,0528
	5	8000	1,0203	1,0261	1,0372
	9	16000	1,0144	1,0185	1,0262

Source: The author (2021)

Here, as stated previously, we propose one additional phase for chart design, when compared to what is traditionally accepted. In Phase 0, we determine UCL_{MOD} and $\widehat{UCL}_{S^2}$, following by the control limits and chart build. Then we have Phase I, that comprises to take $\boldsymbol{m}$ samples with $\boldsymbol{n}$ elements and plot each s_i^2. If all points are within the two control limits, we approve the process to move on to monitoring phase (Phase II).

Figure 4.11 – Proposed steps to build the Modified Chart for unknown variance



Source: The author (2021)

However, if any of the plotted points falls above $\widehat{UCL}_{S^2}$ and under UCL_{Mod}, the process has presence of special causes of variation or excessive common causes of variation, that must be removed or reduced, respectively. In addition, if any of the plotted points falls above both $\widehat{UCL}_{S^2}$ and UCL_{Mod}, the special causes of variation, if present, must be removed as low as possible, the excessive common causes of variation must be reduced, and process capability must be improved. After all corrections required are completed, take another $\boldsymbol{m}$ samples with $\boldsymbol{n}$ elements and plot each s_i^2, and verify if all of them fall under both control limits. The Figure 4.11 shows the steps to be followed to build the Modified Chart for unknown variance.

In the next section an illustrative example is provided to show the use of the proposed Modified Chart and its building steps when the in-control mean is centered between LSL and USL and the variance is unknown.

It is at Phase II that the process control takes place, where through samples assessment, it is evaluated whether there is the presence of special causes that may lead it to an out-of-control condition. In this case, the chart alerts the process practitioner that some intervention may be needed in order to return the process to its control state. Some attributes of run-length distribution and False Alarm Rate (Type I error probability) are used to measure and describe chart performance. This work does not provide any new insights about Phase II and due to this stage be well known and developed in terms of literature, no further guidance will be provided in here. Some suggestions of literature regarding Phase II are Montgomery (2009), Chakraborti and Graham (2019) and Epprecht, Loureiro and Chakraborti (2015).

4.2.3 An Illustrative Example for unknown variance ($\boldsymbol{\sigma_0^2}$) and in control mean $\boldsymbol{\mu_0}$ centered between LSL and USL

In this section, we illustrate the use S^2 Modified Control Chart for unknown variance using the same example from previous sections 4.1.2 and 4.2.1. Again, the quality characteristic variable (X) is the internal diameter of the piston rings, which has a two-sided specification limits of 74.000 $\pm$ 0.050 mm. It is assumed that the piston rings diameter (X) follows a normal distribution and the in-control mean (μ_0) is known, centered in the middle of specification limits. However, the in-control standard deviation (σ_0) of the piston rings diameter is unknown and therefore it needs to be estimated, being the pooled variance $\left(S_p^2\right)$ used. Different from previous illustrative examples, the Phase I assessment needs to be executed, where according to

Montgomery (2009), typically $m = 20$ or 25 subgroups are used in this phase. Here, the Phase I will change to the proposed method explained in the previous section.

The process leadership still defined as acceptable up to 96 nonconforming parts per million (ppm) of units produced, which represents a potential capacity (Cp) of 1.30. Here we start the Phase 0, when the chart control limits (UCL_{Mod} and $\widehat{UCL}_{S^2}$) will be defined. Starting with UCL_{Mod}, being USL, LSL and the acceptable nonconforming parts rate known, σ_{MAX} and UCL_{Mod} are calculated as follows (considering a maximum false alarm rate (α) of 0,0027):

$$\sigma_{MAX} = \frac{USL - LSL}{2\, z_{1-\gamma/2}} = \frac{74{,}05 - 73{,}95}{2 \times 3{,}9} = \frac{0{,}1}{7{,}8} = 0{,}0128$$

$$UCL_{Mod} = \sigma^2_{MAX} \frac{\chi^2_{n-1,\,1-\alpha}}{n-1} = 0{,}0128^2 \frac{16{,}25}{5-1} = 0{,}00067$$

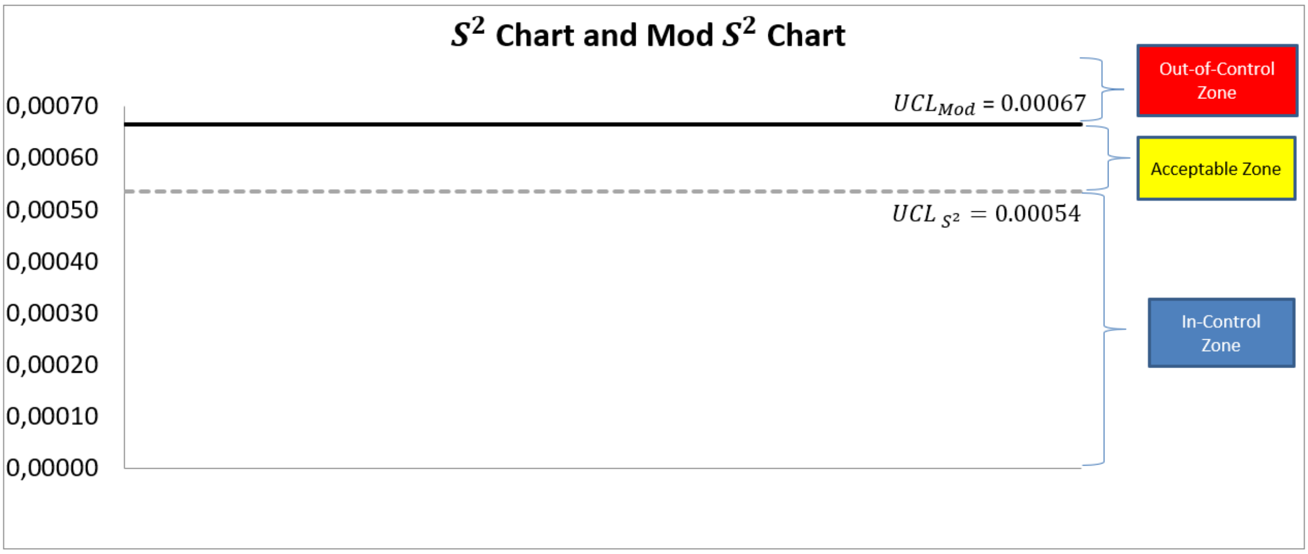
During Phase I, the process manager decided to take $m = 25$ subgroups with $n = 5$ elements each. In addition, it is defined the 95% probability of S^2 estimated upper control limit ($\widehat{UCL}_{S^2}$) is less of UCL_{Mod}. Thus, based on Table 4.7, the ratio $\sigma^2_{MAX}/\sigma^2_0$ shall be at least 1,2434 and hence, considering that σ^2_{MAX} is 0,000164 ($\sigma_{MAX} = 0{,}0128$), the maximum in-control process tolerable variance σ^2_0 is 0,000132. Therefore, the $\widehat{UCL}_{S^2}$ is calculated as follows (considering a maximum false alarm rate (α) of 0,0027):

$$\widehat{UCL}_{S^2} = \left(\frac{\sigma^2_{MAX}}{1{,}2434}\right) \frac{\chi^2_{n-1,\,1-\alpha}}{n-1} = 0{,}000132 \frac{16{,}25}{5-1} = 0{,}00054$$

Hence, after Phase 0, we have the following designed control chart shown in Figure 4.12 and in the next phase, we will take m samples with n elements and plot each Phase I s_i^2, in the same chart designed shown in Figure 4.12.

To examine the behavior of Phase I, we simulated 25 independent samples of the piston ring diameter (each sample with size 5), from a normal distribution with mean $\mu_0 = 74{,}000$ and a standard deviation equals to $\sigma^2_{MAX}/1{,}2434$, based on Table 4.7 for $m = 25$, $n = 5$ and $P = 95\%$. The simulated sample variances (S^2) and the control limits (UCL_{S^2} and UCL_{Mod}) are shown in Figure 4.13. UCL_{S^2} is shown in a dashed grey line and UCL_{Mod} in a solid black line.

Figure 4.12 - Graphical demonstration of how the $\widehat{UCL}_{S^2}$ locates around UCL_{MOD}

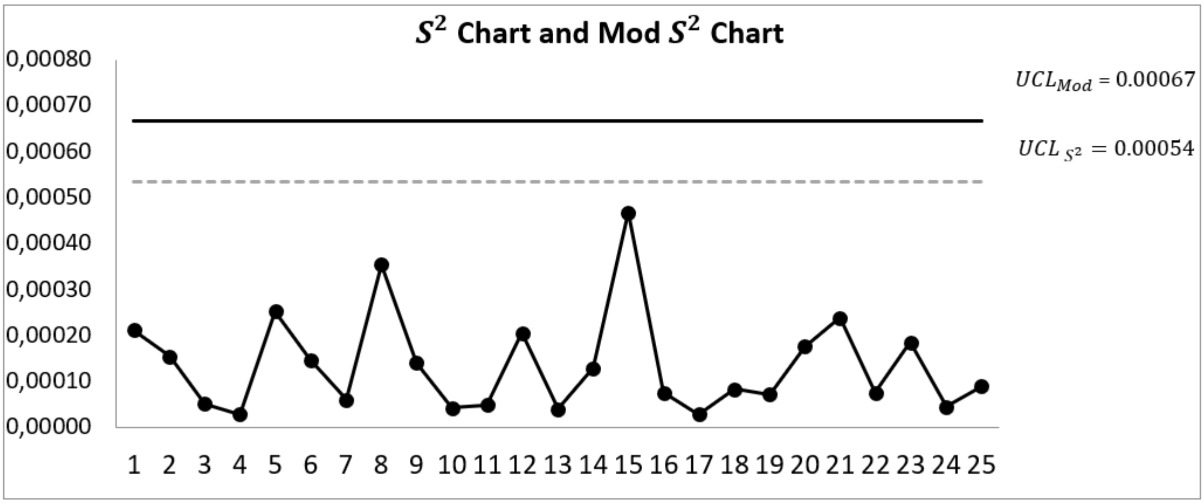


Source: The author (2021)

UCL_{S^2} is shown in a dashed grey line and UCL_{Mod} in a solid black line. Supposing that the unknown process variance (unknown) is $\sigma^2 = 0{,}000132$, that equals to $\sigma^2_{MAX}/1{,}2434$, we have the chart shown in Figure 4.13.

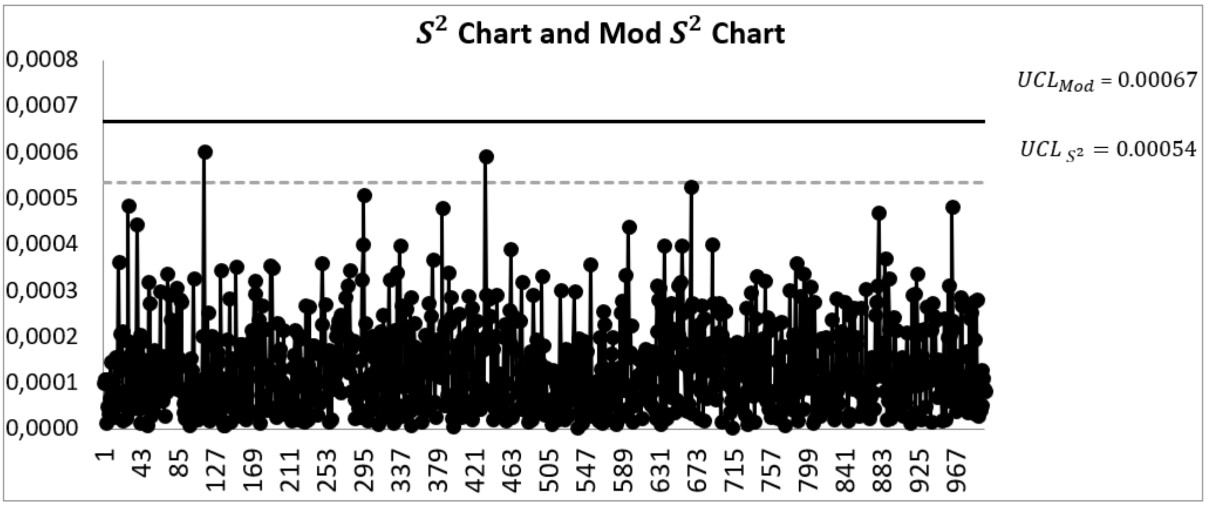
As we can see in Figure 4.13, signals above the $\widehat{UCL}_{S^2}$ dashed line were not found and therefore, the practitioner can move to Phase II, process monitoring. Now in Figure 4.14, we simulated 1,000 Phase II samples variances with size 5.

Figure 4.13 - The S^2 Control Chart and the S^2 Modified Control Chart for Phase I execution of a process $X \sim N(\mu_0 = 74{,}000, \sigma^2 = 0{,}000132)$ given the in-control parameters in the example



Source: The author (2021)

Figure 4.14 - The S^2 Control Chart and the S^2 Modified Control Chart for monitoring the variance of a process $X \sim N(\mu_0 = 74, \sigma^2 = 0{,}000132)$



Source: The author (2021)

We can also see in Figure 4.14 signals above the $\widehat{UCL}_{S^2}$ dashed line, and again, instead of process practitioner suspect that an assignable cause has occurred, by using UCL_{Mod}, the process variance is still smaller than the maximum variance allowed ($\sigma^2_{\mathrm{MAX}} = 0{,}0128^2$), so the proportion of nonconforming items being produced is still acceptable according to the specification of the project.

The S_p^2 for the data shown in Figure 4.14 is $0{,}0001322$, so the probability of a false alarm is smaller than $\alpha_{nom} = 0{,}0027$, as shown below.

$$False\ Alarm\ Rate = 1 - P\left(S^2 < UCL_{Mod} \middle| S_p^2 = 0{,}0001322\right)$$

$$= 1 - P\left(S^2 < \sigma^2_{MAX}\,\frac{\chi^2_{n\text{-}1,\,1\text{-}\alpha}}{n\text{-}1} \middle| S_p^2 = 0{,}0001322\right)$$

$$= 1 - P\left(\frac{(n\text{-}1)S^2}{S_p^2} < \frac{\sigma^2_{MAX}}{S_p^2}\,\chi^2_{n\text{-}1,\,1\text{-}\alpha}\right),$$

where $\frac{(n\text{-}1)S^2}{\sigma_1^2} = \chi^2_{n\text{-}1}$ is a random variable that follows a chi-squared distribution with $n\text{-}1$ degrees of freedom, so

$$False\ Alarm\ Rate = 1 - P\left(\chi^2_{n-1} < \frac{0{,}000164}{0{,}0001322}\,16{,}25\right) = 1 - P(\chi^2_{n-1} < 20{,}1365)$$

$$False\ Alarm\ Rate = 0{,}00047 < \alpha_{nom} = 0{,}0027$$

The False Alarm Rate less than the Type I error (α_{nom}) determined by the process practitioner, means that it is expected less false signals that the process shifted when it is still actually, meeting the manufacturing requirements. This may contribute for even less unnecessary process stop.

4.3 IN-CONTROL MEAN (μ_0) BUT NOT CENTERED BETWEEN USL AND LSL, AND VARIANCE IS KNOWN (σ_0^2)

Here, we assume that the process mean is known and in-control (μ_0) but not centered between the specification limits, and variance also known (σ_0^2). The chart is still constructed with the purpose of detecting relevant increases in the process variance only.

Suppose that $\left(X_{i,1}, X_{i,2}, \dots, X_{m,n}\right)$ are random variables, and $i = 1,2, \dots m$ are independent samples extracted from a process in Phase II, known variance σ_0^2, where i which identifies the subgroup. In Phase II, the samples of size n are extracted, and s_i^2 are calculated from $\{X_{i,1}, X_{i,2}, \dots, X_{i,n}\}$, where the type I error is $\alpha = Pr(s_i^2 \notin (0, UCL))$ or $\alpha = 1 - Pr(s_i^2 \in (0, UCL))$, and is represented by α_1. The UCL is the Upper Control Limit.

When the process is in-control (IC), with mean not centered between USL and LSL, and known variance, using $\bar{X}$ as stated in section 3.1, we can write the same equation 7, already written before:

$$Pr\left(s_i^2 \leq UCL_{S^2}\right) = Pr\left(n-1\,.\frac{s_i^2}{\sigma_0^2} \leq \chi^2_{n-1,1-\alpha}\right) = Pr\left(s_i^2 \leq \sigma_0^2\,.\frac{\chi^2_{n-1,1-\alpha}}{n-1}\right) \tag{7}$$

Therefore, we can get the Upper Control Limit for S^2 by the same equation 8, already shown before. The shift of in-control value (μ_0) does not affect the position of Upper Control Limit.

$$UCL_{S^2} = \sigma_0^2\ \frac{\chi^2_{n-1,\ 1-\alpha}}{n-1}, \tag{8}$$

The plotting statistic (S^2) given by equation 1, should be compared with the Upper Control limit (UCL_{S^2}) of the S^2 Control Chart which is given by equation 8, where σ_0^2 is the nominal in-control process variance, $\chi^2_{n-1,\ 1-\alpha}$ is the (1-α)-quantile of a chi-square distribution with *n-1* degrees of freedom and α is the nominal false alarm rate (or the false alarm probability) chosen by the practitioner (usually, $\alpha = 0{,}0027$).

A false alarm is defined as a signal (alarm) when the process is in control. The maximum false alarm rate happens when $\sigma^2 = \sigma_0^2$ and is not affected by change of the mean to a not centered location between the USL and LSL, being the equation 10 already shown in section 4.1.

$$FAR_{MAX} = 1 - P\left(\chi^2_{n-1} < \chi^2_{n-1,1-\alpha}\right) = \alpha. \tag{10}$$

When the actual variance of the process (σ^2) is exactly at the in-control process variance (σ_0^2) value, the proportion of nonconforming units being produced should be small, for slight changes in mean (μ_1). In other words, the probability of the quality characteristic (X) be smaller than the lower specification limits (LSL) or larger than the upper specification limits (USL), should be small, but higher when compared with the scenario where the mean in the middle point between the USL and LSL. Figure 4.15 illustrates this situation. Note that these specification limits are provided by the project/manager.

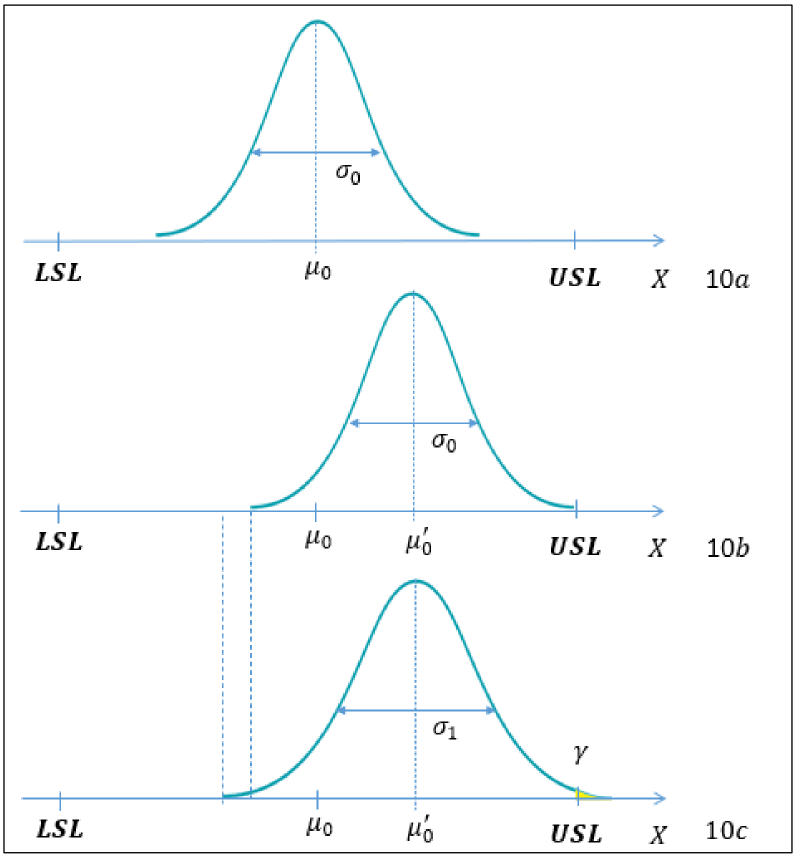
The S^2 Control Chart is designed to detect increases (larger than σ_0^2) of any magnitude in the actual process variance (σ^2), even increases that do not affect the rate of nonconforming items being produced. These increases will tend to produce a signal (alarm) on the control chart. Considering now that the mean changed from μ_0 to μ_0', by how large the actual variance σ_1^2 could be compared to σ_0^2 to still provide a nonconforming rate that meets the company or consumer acceptance criteria (see Figures 4.15a and 4.15b)?

Note that since σ_1^2 is larger than σ_0^2, with mean no more in the middle of the specification limits, from the perspective of the S^2 Control Chart, the process should be declared out-of-control, for this new scenario, the process may be still not producing many nonconforming items (almost all the items being produced are still within the specification limits even though $\sigma^2 = \sigma_1^2 > \sigma_0^2$ and $\mu = \mu_0 > \mu_0'$ or $\mu_0 < \mu_0'$. So, in the same way stated in section 4.1, trying to fix this increase on the variance may be a waste of time and money, since in most of the cases, the process would have to be paused.

Starting from the premise of Section 3.1, that the purpose of a process control system is decisions making that result in economic gains over the process, it is possible to balance the consequences of these decisions even considering two situations: (a) take action when not needed (over control), versus (b) does not take action when it is necessary (lack of control).

Thus, it is possible to use the concept of practical significance to make economically viable decisions on the process condition.

Figure 4.15 – 4.15a. Process running with the nominal in-control variance ($\sigma^2 = \sigma_0^2$) with all the items being produced within the specification limits. 4.15b. Process running with the nominal in control variance ($\sigma^2 = \sigma_0^2$) and mean changed from μ_0 to μ_0'. 4.15c. Process running with variance changed from σ_0^2 to σ_1^2 and mean from μ_0 to μ_0', where γ maximum tolerated rate of nonconforming units



Source: The author (2021)

The σ_{MAX}^2 value still must be chosen with care, depending on the lower and upper specification limits, LSL and USL, the magnitude of the change for the actual mean ($\mu = \mu_0'$), and the maximum rate (probability) of nonconforming units produced (denoted here by γ) that may be tolerated (or allowed).

Let us consider the figure 4.15c, where μ_0' is larger than the middle point:

$$\gamma = 1 - P(X < USL | \sigma^2 = \sigma_{MAX}^2) = 1 - P\left(\frac{X - \mu_0'}{\sigma_{MAX}} < \frac{USL - \mu_0'}{\sigma_{MAX}}\right)$$

$$\gamma = 1 - P\left(Z < \frac{USL - \mu_0'}{\sigma_{MAX}}\right) = 1 - \Phi\left(\frac{USL - \mu_0'}{\sigma_{MAX}}\right) \quad (35)$$

where Z is a random variable that follows a standard normal distribution and $\Phi(*)$ is the cumulative distribution function (c.d.f.) of a standard normal random variable. From equation 35, one has:

$$\frac{USL - \mu_0'}{\sigma_{MAX}} = \Phi^{-1}(1-\gamma) = z_{1-\gamma}$$

$$\sigma_{MAX} = \frac{USL - \mu_0'}{z_{1-\gamma}} \quad (36)$$

Is important to note that in the case presented, $USL - \mu_0'$ and $z_{1-\gamma}$ are positive values.

Now, let us consider the case where μ_0' is smaller than the middle point.

$$\gamma = P(X < LSL |\, \sigma^2 = \sigma_{MAX}^2) = P\left(\frac{X - \mu_0'}{\sigma_{MAX}} < \frac{LSL - \mu_0'}{\sigma_{MAX}}\right)$$

$$\gamma = P\left(Z < \frac{LSL - \mu_0'}{\sigma_{MAX}}\right) = \Phi\left(\frac{LSL - \mu_0'}{\sigma_{MAX}}\right) = \frac{LSL - \mu_0'}{\sigma_{MAX}} = \Phi^{-1}(\gamma) = z_\gamma$$

$$\sigma_{MAX} = \frac{LSL - \mu_0'}{z_\gamma} \quad (37)$$

Note that in this case presented $\mu_0' - LSL$ and z_γ are negative values.

Given that the shift in the first case (where μ_0' is larger than the middle point) has the same length of the dislocation in the second case (where μ_0' is smaller than the middle point). We have:

$$\sigma_{MAX} = \frac{LSL - \mu_0'}{z_\gamma} = \frac{USL - \mu_0'}{z_{1-\gamma}} \quad (38)$$

To calculate the upper control limit (UCL_{Mod}) of the S^2 Modified Control Chart, one just need to replace σ_0^2 in the original control limit equation of the S^2 Control Chart (see equation 8 by σ_{MAX}^2, as shown below.

For μ_0' larger than the middle point:

$$UCL_{Mod} = \sigma_{MAX}^2 \frac{\chi^2_{n-1,\, 1-\alpha}}{n-1} = \frac{(USL - \mu_0')^2\, \chi^2_{n-1, 1-\alpha}}{(n-1)\,(z_{1-\gamma})^2}. \quad (39)$$

For μ'_0 smaller than the middle point:

$$UCL_{Mod} = \sigma^2_{MAX} \frac{\chi^2_{n-1,\ 1-\alpha}}{n-1} = \frac{(LSL-\mu'_0)^2\ \chi^2_{n-1,1-\alpha}}{(n-1)\left(z_{\gamma}\right)^2}. \quad (40)$$

The FAR and ARL assessment executed in section 4.1, for in-control mean centered between USL and LSL and variance is known, extends to this section 4.2, where the only difference is the in-control mean no longer centered between USL and LSL. The reason is that the FAR and ARL for process variance are not affected if the in-control mean is centered between the specification limits or not.

4.3.1 An Illustrative Example for variance known (σ_0^2) and in control mean μ'_0 not centered between LSL and USL

Returning to the same example shown in Section 4.1.1, to illustrate the ideas of the S^2 Modified Control Chart for known variance and mean changing from μ_0 to μ_1, in an automobile engine manufacturing process that uses a forging process to make piston rings. The quality characteristic variable (X) is the internal diameter of the piston rings, which has a two-sided specification limits of 74,000 $\pm$ 0,050 mm. It is assumed that the piston rings diameter (X) follows a normal distribution. Now, we assume that the in-control mean (μ_0) 74,000 mm shifted to μ'_0, being now the in-control mean equals to 74,007 and the in-control standard deviation (σ_0) of the piston rings diameter are known, still being 0,010 mm. The process leadership defined as acceptable, still up to 96 nonconforming parts per million (ppm) of units produced, which represents a potential capacity (*Cp*) of 1,30, according to Table 3.1.

In other words, the maximum allowed rate of nonconforming items (γ) is $96/1000000 = 0{,}000096$, which provides $z_{1-\gamma} = 3{,}729$. So, the maximum standard-deviation allowed (σ_{MAX}) for the piston ring diameter can be calculated using Equation 21, as shown below.

$$\sigma_{MAX} = \frac{USL-\mu'_0}{z_{1-\gamma}} = \frac{74{,}050-74{,}008}{3{,}729} = \frac{0{,}042}{3{,}729} = 0{,}0113$$

As expected, by having the in-control shifted from μ_0 to μ_0', no longer in the center point of USL and LSL, the σ_{MAX} is less than the one found in section 4.1.1, for centered in-control mean. All parameters given by this example are summarized in Table 4.8.

Table 4.8 - Parameters provided by the Example

μ_0'	σ_0	σ_{MAX}	USL	LSL	γ
74,008	0,0100	0,0113	74,050	73,950	0,000096

Source: The author

The first analysis was done on the probability of nonconform units with $\sigma_0 = 0{,}0100$, which provided a C_p equal to 1,67 and C_{pk} = min $\left(C_{PL} = \frac{\mu - \text{LSL}}{3\sigma} = 1{,}93; C_{PU} = \frac{LSE - \mu}{3\sigma} = 1{,}40\right)$ sigma level of 4.2 and a defective ppm of 13.3, which is still better than the process requirement (ppm of 96). Therefore, the use of a modified acceptance chart has become appropriate.

By using the data presented in Table 11, the upper control limits for the traditional S^2 chart and the modified chart for the process were defined, considering $\sigma_0 = 0{,}01$, which are shown in Table 6.

Table 4.9 - Upper Control Limits for variance and potential capacity

	Upper Limit for S_i^2	Equation
UCL_{S^2}	0,000406	$s_i^2 \leq \frac{\sigma_0^2}{n-1} \cdot \chi^2_{n-1,1-\alpha}$
UCL_{Mod}	0,00052	$s_i^2 \leq \frac{(LSL - \mu_0')^2 \chi^2_{n-1,1-\alpha}}{(n-1)\left(z_\gamma\right)^2}$

Source: The author (2021)

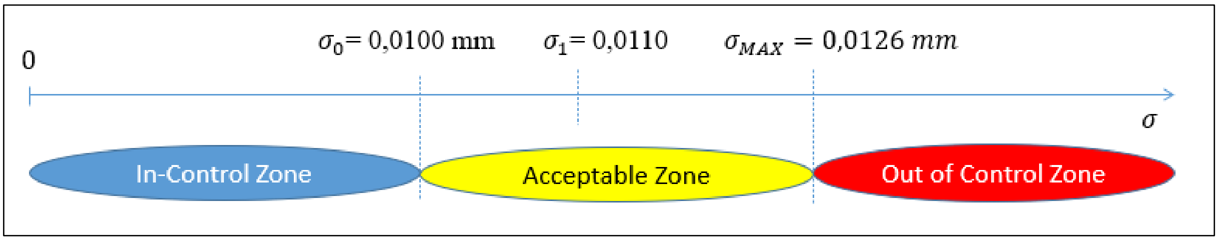
Suppose that the practitioner decides to monitor the process variance (σ^2) with samples of size $n = 5$ of the piston's rings diameter collected at regular intervals. To this end, the practitioner can use the well-known S^2 Control Chart or the S^2 Modified Control Chart proposed here. Considering a maximum false alarm rate (α) of 0,0027 for each chart, the control limits of both charts (UCL_{S^2} and UCL_{Mod}) can be calculated as shown below.

$$UCL_{S^2} = \sigma_0^2 \frac{\chi^2_{n-1,\ 1-\alpha}}{n-1} = 0{,}0100^2 \frac{16{,}25}{5-1} = 0{,}000406$$

$$UCL_{Mod} = \sigma_{MAX}^2 \frac{\chi^2_{n-1,\ 1-\alpha}}{n-1} = 0{,}0113^2 \frac{16{,}25}{5-1} = 0{,}00052$$

Now, let's suppose that the process standard deviation (σ) moved from the in-control value $\sigma = \sigma_0 = 0{,}0100$ to $\sigma = \sigma_1 = 0{,}0110$. Note that, since $\sigma_0 < \sigma_1 < \sigma_{MAX}$, even though the process standard-deviation increased, it is still in the Acceptable Zone (see Figure 4.16), so the process is still producing an acceptable rate of nonconforming items (i.e., a rate smaller than $\gamma = 0{,}000096$).

Figure 4.16 - Illustration showing the actual process standard deviation (σ) in the Acceptable Zone ($\sigma = \sigma_1 = 0{,}0110$)



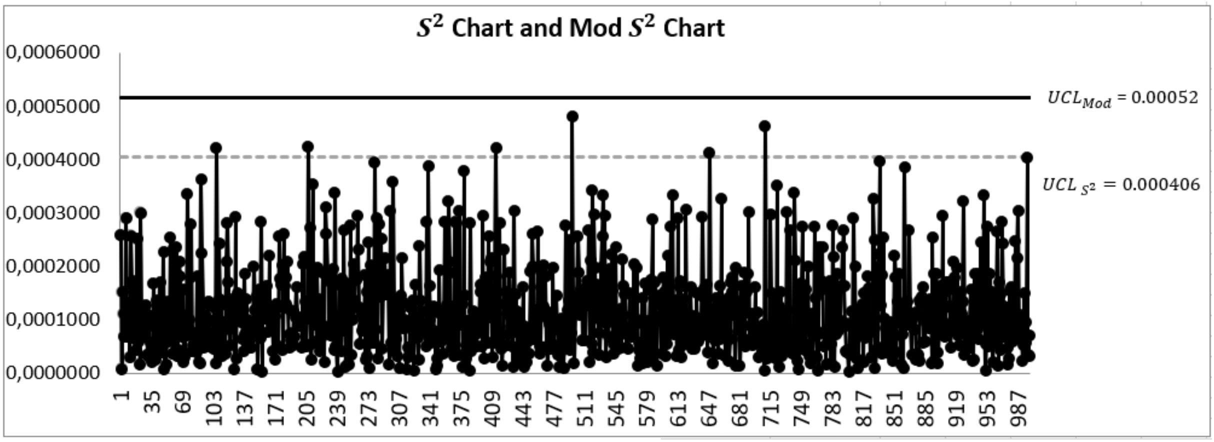
Source: The author (2021)

To further examine the behavior of both charts (the well-known S^2 Control Chart and the S^2 Modified Control Chart proposed here), we simulated one thousand independent samples of the piston ring diameter (each sample with size 5), from a normal distribution with mean $\mu_0' = 74{,}008$ and standard deviation $\sigma = \sigma_1 = 0{,}0110$. With the simulated data, we calculated the sample variances (S^2) using Equation 1, which is the plotting statistic for both charts, and plotted these against the control limits values shown in Equations 22 and 23. The simulated sample variances (S^2) and the control limits (UCL_{S^2} and UCL_{Mod}) are shown in Figure 4.17. UCL_{S^2} is shown in a dashed grey line and UCL_{Mod} in a solid black line.

As it can be seen in Figure 4.17, signals above the UCL_{S^2} dashed line are frequent. If the user were using just the well-known S^2 Control Chart, he would still typically suspect that an assignable cause has occurred and that the process variance (σ^2) is larger than the in-control target ($\sigma_0^2 = 0{,}010^2$) increasing the production of nonconforming items. In this case, the user would stop the process and start looking for assignable causes, wasting time and decreasing production (what is also a waste of money). However, even though the process variance has indeed increased to $\sigma^2 = \sigma_1^2 = 0{,}0110^2$, it is still smaller than the maximum variance allowed

($\sigma^2_{\text{MAX}} = 0{,}0113^2$), this means that the proportion of nonconforming items being produced is still acceptable according to the specification of the project. Therefore, the production does not really need to be stopped to search for assignable and again, the use of the well-known S^2 Control Chart alone can mislead the user.

Figure 4.17 - The S^2 Control Chart and the S^2 Modified Control Chart for monitoring the variance of a process $X \sim N(\mu'_0 = 74{,}008, \sigma = \sigma_1 = 0{,}0110)$ given the in-control parameters in Table 4.8



Source: The author (2021)

Since the process is still capable in the sense it is still producing an acceptable rate of nonconforming units, there is no signal above the UCL_{Mod} black solid line, what is indeed expected since the probability of a false alarm when $\sigma^2 = \sigma^2_1 = 0{,}0110^2$ is smaller than $\alpha = 0{,}0027$, as shown below.

$$False\ Alarm\ Rate = 1 - P(S^2 < UCL_{Mod} | \sigma^2 = \sigma^2_1 = 0{,}0110^2)$$

$$= 1 - P\left(S^2 < \sigma^2_{MAX} \frac{\chi^2_{n\text{-}1,\ 1\text{-}\alpha}}{n\text{-}1} \middle| \sigma^2 = \sigma^2_1 = 0{,}0110^2\right)$$

$$= 1 - P\left(\frac{(n\text{-}1)S^2}{\sigma^2_1} < \frac{\sigma^2_{MAX}}{\sigma^2_1} \chi^2_{n\text{-}1,\ 1\text{-}\alpha}\right),$$

where $\frac{(n\text{-}1)S^2}{\sigma^2_1} = \chi^2_{n\text{-}1}$ is a random variable that follows a chi-squared distribution with $n\text{-}1$ degrees of freedom, so

$$False\ Alarm\ Rate = 1 - P\left(\chi^2_{n-1} < \frac{0.0113^2}{0.0110^2}\ 16{,}25\right) = 1 - P(\chi^2_{n-1} < 17{,}034)$$

$$False\ Alarm\ Rate = 0{,}0019 < \alpha_{nom} = 0{,}0027$$

As already stated previously, the False Alarm Rate less than the Type I error (α_{nom}) determined by the process practitioner may contribute for even less unnecessary process stop.

4.4 IN-CONTROL MEAN (μ_0) BUT NOT CENTERED BETWEEN USL AND LSL, AND VARIANCE IS UNKNOWN (σ^2)

As stated previously in section 4.2, the estimation of the process variance is traditionally done by collecting m samples with size n elements from an in-control (IC) process during Phase I. In addition, in this dissertation we also present the execution of what we call Phase 0 and a proposed change on how Phase I is executed, that is explained in detail in section 4.2.2 and remains valid to this section, due to the development previously done does not take in consideration the location of the in-control mean. Once the estimator $\hat{\sigma}_0^2$ is calculated, the UCL for the one sided S^2 chart to be used during Phase II monitoring can be defined, for a specified or nominal False Alarm Rate (FAR) and α_{NOM}, through the equation 23 presented in section 4.2:

$$\widehat{UCL}_{S^2} = \hat{\sigma}_0^2 \ \frac{\chi^2_{n-1,\ 1-\alpha}}{n-1} \qquad (23)$$

Therefore, the estimation of $\widehat{UCL}_{S^2}$ is analogous to the in-control mean centered between the specification limits. A s stated previously in section 4.2.1, according to Mahmoud (2010), the pooled variance $\left(S_p^2\right)$ is the best estimators for σ_0^2 and is calculated by equation (33). Replacing $\hat{\sigma}_0^2$ by S_p^2 we have the equation 32 of section 4.2.2.

$$\widehat{UCL}_{S^2} = S_p^2 \ \frac{\chi^2_{n-1,\ 1-\alpha}}{n-1} \qquad (32)$$

The upper control limit (UCL_{Mod}) of the S^2 Modified Control Chart however, for not centered in-control mean, is calculated by equations 39 and 40, from section 4.3:

For μ_0' larger than the middle point:

$$UCL_{Mod} = \sigma_{MAX}^2 \frac{\chi^2_{n\text{-}1,\ 1\text{-}\alpha}}{n\text{-}1} = \frac{(USL-\mu_0')^2 \chi^2_{n\text{-}1,1\text{-}\alpha}}{(n\text{-}1)\left(z_{1-\gamma}\right)^2} \tag{39}$$

For μ_0' smaller than the middle point:

$$UCL_{Mod} = \sigma_{MAX}^2 \frac{\chi^2_{n\text{-}1,\ 1\text{-}\alpha}}{n\text{-}1} = \frac{(LSL-\mu_0')^2 \chi^2_{n\text{-}1,1\text{-}\alpha}}{(n\text{-}1)\left(z_{\gamma}\right)^2} \tag{40}$$

Following the same steps proposed in section 4.2.2, here we also include the additional Phase 0, before the well-known Phase 1. Here we also need to estimate σ_0^2, it means that the $\widehat{UCL}_{S^2}$ may vary because the variance now is a random variable and the UCL_{Mod}, being larger or smaller than the middle point between USL and LSL. Thus, if the $\widehat{UCL}_{S^2}$ is located below UCL_{MOD} most of the time, it is possible to conclude that the issues caused by the estimations and its inherent errors could be less severe. Hence, we calculate the probability of $\widehat{UCL}_{S^2}$ is less than UCL_{MOD}, that is $P(\widehat{UCL}_{S^2} < UCL_{MOD})$, recalling that the UCL_{MOD} is calculated by using given parameters.

Therefore, analogous to what is shown in section 4.2.2, equation 33,

$$P(\widehat{UCL}_{S^2} < UCL_{MOD}) = P\left(S_p^2 \frac{\chi^2_{n\text{-}1,\ 1\text{-}\alpha}}{n\text{-}1} < \sigma_{MAX}^2 \frac{\chi^2_{n\text{-}1,\ 1\text{-}\alpha}}{n\text{-}1}\right)$$
$$= P\left(S_p^2 < \sigma_{MAX}^2\right) \tag{33}$$

Developing the Equation 33 we have the following:

$$P(\widehat{UCL}_{S^2} < UCL_{MOD}) = P\left(\frac{m(n-1)S_p^2}{\sigma_0^2} < \frac{m(n-1)\,\sigma_{MAX}^2}{\sigma_0^2}\right)$$

where $\frac{m(n-1)S_p^2}{\sigma_0^2} = \chi^2_{m(n-1)}$, here denoted as Y, is a random variable that follows a chi-squared distribution with $m(n-1)$ degrees of freedom, so we have the same Equation 34 first provided in section 4.2.2. Is important to notice that here we are still in Phase I, thus we have the in-control variance σ_0^2.

$$P(\widehat{UCL}_{S^2} < UCL_{MOD}) = P\left(Y < \frac{m(n-1)\,\sigma^2_{MAX}}{\sigma^2_0} \right) \tag{34}$$

Recalling from section 4.2.2, we have that UCL_{MOD} has all parameters known and is fixed and the equation 34 depends on the in-control variance σ_0^2, which is unkwown. The ratio $\sigma^2_{MAX}/\sigma^2_0$, between σ^2_{MAX} (the maximum value that the process variance is allowed to be compared with the in-control process value, in the sense that the process produces a tolerated small fraction of nonconforming items) and the in-control variance σ_0^2, developed in details in section 4.2.2 for in-control mean centered between USL and LSL, including table 4.7 is still valid, being the only difference when compared with the in-control mean not centered between USL and LSL, the way that σ^2_{MAX} is calculeted. Therefore, dividing σ^2_{MAX} by σ_0^2, we may define how much larger the maximum allowed variance is required to be when compared with the the target variance.

Here we also propose one additional phase for chart design, when compared to what is traditionally accepted. In Phase 0, we determine UCL_{MOD} and $\widehat{UCL}_{S^2}$, following by the control limits and chart build. Then we have Phase I, that comprises to take $\boldsymbol{m}$ samples with $\boldsymbol{n}$ elements and plot each s_i^2. If all points are within the two control limits, we approve the process to move on to monitoring phase (Phase II). However, if any of the plotted points falls above $\widehat{UCL}_{S^2}$ and under UCL_{Mod}, the process has presence of special causes of variation or excessive common causes of variation, that must be removed or reduced, respectively. In addition, if any of the plotted points falls above both $\widehat{UCL}_{S^2}$ and UCL_{Mod}, the special causes of variation, if present, must be removed as low as possible, the excessive common causes of variation must be reduced, and process capability must be improved. After all corrections required are completed, take another $\boldsymbol{m}$ samples with $\boldsymbol{n}$ elements and plot each s_i^2, and verify if all them fall under both control limits. The Figure 4.11 shows the steps to be followed to build the Modified Chart for unknown variance.

In the next section an illustrative example is provided to show the use of the proposed Modified Chart and its building steps when the in-control mean is not centered between LSL and USL and the variance is unknown.

4.4.1 An Illustrative Example for unknown variance (σ_0^2) and in control mean μ_0 not centered between LSL and USL

In this section, we illustrate the use S^2 Modified Control Chart for unknown variance and mean changing from μ_0 to μ_1, not centered between LSL and USL, using the same example from previous section 4.3.1. The quality characteristic variable (X) is still the internal diameter of the piston rings, which has a two-sided specification limits of 74.000 ± 0.050 mm. It is assumed that the piston rings diameter (X) follows a normal distribution and the in-control mean μ_0' is known and is equal to 74,008, not centered in the middle of specification limits. However, the in-control standard deviation (σ_0) of the piston rings diameter is unknown and therefore it needs to be estimated, being the pooled variance $\left(S_p^2\right)$ used. Different from illustrative example in section 4.3.1, the Phase I assessment needs to be executed, where according to Montgomery (2009), typically $m = 20$ or 25 subgroups are used in this phase. Here, the Phase I will change to the proposed method explained in the previous section.

The process leadership still defined as acceptable up to 96 nonconforming parts per million (ppm) of units produced, which represents a potential capacity (Cp) of 1.30. First we execute the Phase 0, that is when the chart control limits (UCL_{Mod} and $\widehat{UCL}_{S^2}$) will be defined. Starting with UCL_{Mod}, being USL, LSL and the acceptable nonconforming parts rate known, σ_{MAX} and UCL_{Mod} are calculated as follows (considering a maximum false alarm rate (α) of 0,0027):

$$\sigma_{MAX} = \frac{USL-\mu_0'}{z_{1-\gamma}} = \frac{74{,}050-74{,}008}{3{,}729} = \frac{0{,}042}{3{,}729} = 0{,}0113$$

The maximum allowed rate of nonconforming items (γ) is $96/1000000 = 0{,}000096$, which provides $z_{1-\gamma} = 3{,}729$.

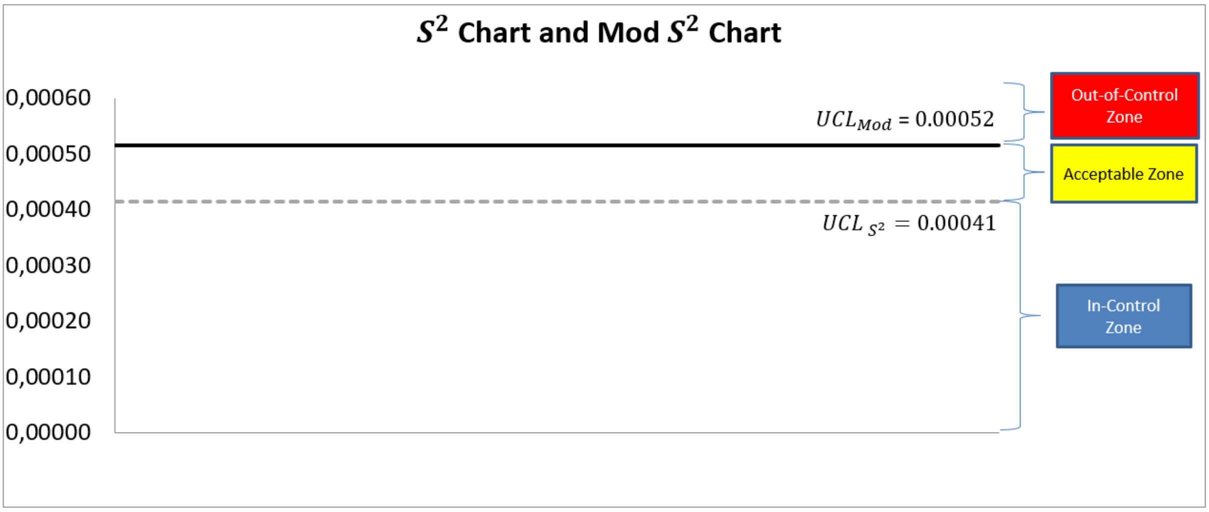
$$UCL_{Mod} = \sigma_{MAX}^2 \frac{\chi^2_{n-1,\,1-\alpha}}{n-1} = 0{,}0113^2 \frac{16{,}25}{5-1} = 0{,}00052$$

During Phase I, the process manager decided to take $m = 25$ subgroups with $n = 5$ elements each. In addition, it is defined the 95% probability of S^2 estimated upper control limit ($\widehat{UCL}_{S^2}$) is less of UCL_{Mod}. Thus, based on Table 4.7, the ratio $\sigma_{MAX}^2/\sigma_0^2$ shall be at least 1,2434 and hence, considering that σ_{MAX}^2 is 0,000127 ($\sigma_{MAX} = 0{,}0113$), the maximum in-control process tolerable variance σ_0^2 is 0,000102. Therefore, the $\widehat{UCL}_{S^2}$ is calculated as follows (considering a maximum false alarm rate (α) of 0,0027):

$$\widehat{UCL}_{S^2} = \left(\frac{\sigma_{MAX}^2}{1{,}2434}\right)\frac{\chi^2_{n-1,\,1-\alpha}}{n-1} = 0{,}000102\frac{16{,}25}{5-1} = 0{,}00041$$

Hence, after Phase 0, we have the following designed control chart:

Figure 4.18 - Graphical demonstration of how the $\widehat{UCL}_{S^2}$ locates around UCL_{MOD}



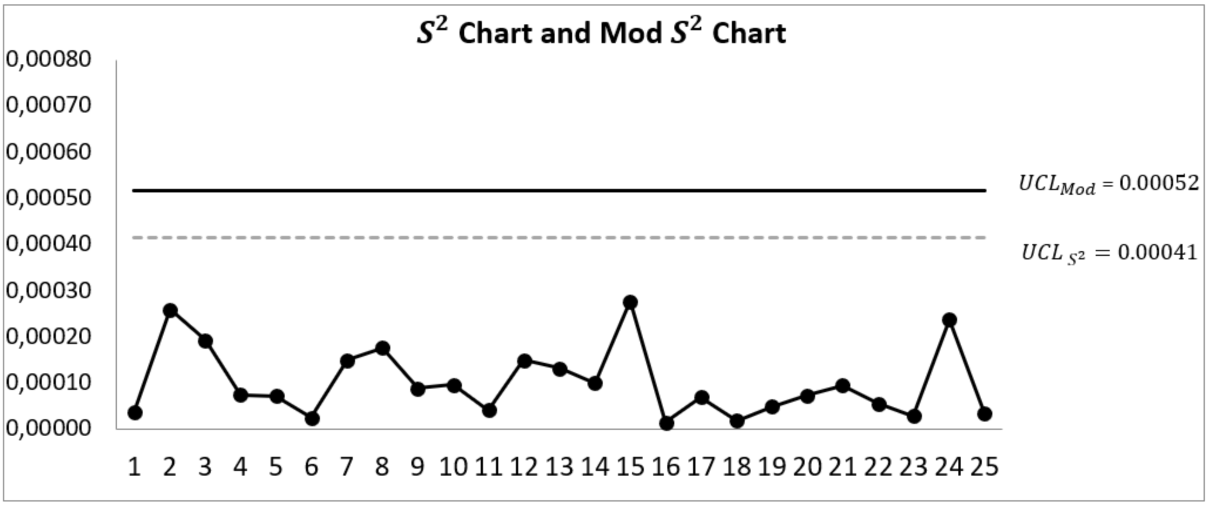
Source: The author (2021)

In the next phase, we will take m samples with n elements and plot each Phase I s_i^2, in the chart designed in Figure 4.18.

To examine the behavior of Phase I, we simulated 25 independent samples of the piston ring diameter (each sample with size 5), from a normal distribution with mean $\mu_0 = 74{,}008$ and a standard deviation equals to $\sigma_{MAX}^2/1{,}2434$, based on Table 4.7 for $m = 25$, $n = 5$ and $P = 95\%$. The simulated sample variances (S^2) and the control limits (UCL_{S^2} and UCL_{Mod}) are shown in Figure 4.19. UCL_{S^2} is shown in a dashed grey line and UCL_{Mod} in a solid black line. Supposing that the unknown process variance (unknown) is $\sigma^2 = 0{,}000102$, that equals to $\sigma_{MAX}^2/1{,}2434$, we have the chart shown in Figure 4.19.

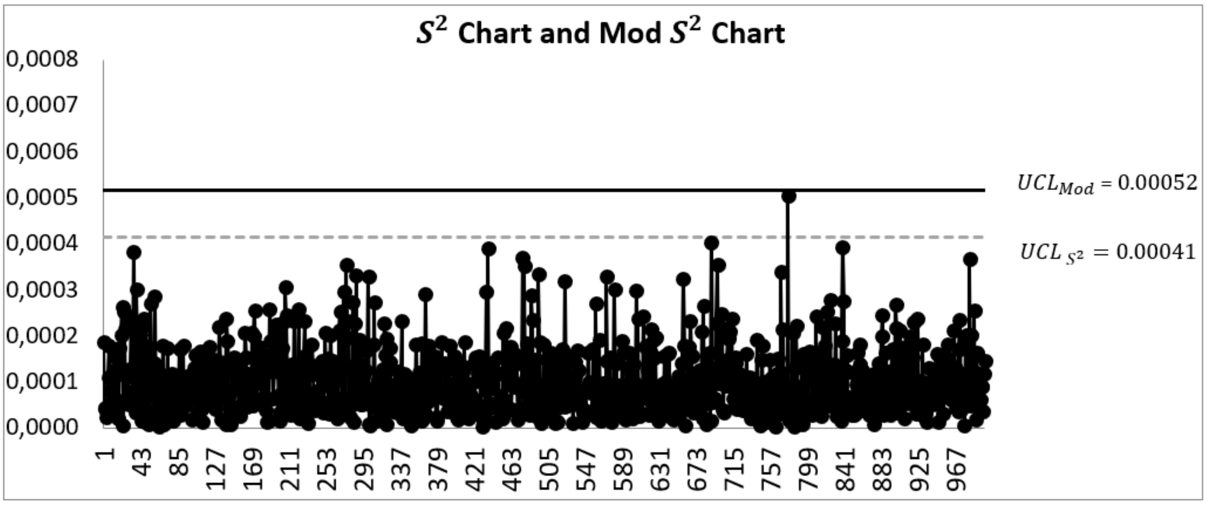
As we can see in Figure 4.19, signals above the $\widehat{UCL}_{S^2}$ dashed line were not found and therefore, the practitioner can move to Phase II, process monitoring. Now in Figure 4.20, we simulated 1,000 Phase II samples variances with size 5.

Figure 4.19 - The S^2 Control Chart and the S^2 Modified Control Chart for Phase I execution of a process $X \sim N(\mu_0' = 74{,}008, \sigma^2 = 0{,}000102)$ given the in-control parameters in the example



Source: The author (2021)

Figure 4.20 - The S^2 Control Chart and the S^2 Modified Control Chart for monitoring the variance of a process $X \sim N(\mu_0 = 74{,}008, \sigma^2 = 0{,}000102)$



Source: The author (2021)

We can also see in Figure 4.20 signals above the $\widehat{UCL}_{S^2}$ dashed line, and again, instead of process practitioner suspect that an assignable cause has occurred, by using UCL_{Mod}, the process variance is still smaller than the maximum variance allowed ($\sigma^2_{\text{MAX}} = 0{,}0113^2$), so the proportion of nonconforming items being produced is still acceptable according to the specification of the project.

The S_p^2 for the data shown in Figure 4.20 is 0,000104, so the probability of a false alarm is smaller than $\alpha_{nom} = 0{,}0027$, as shown below.

$$False\ Alarm\ Rate = 1 - P\left(S^2 < UCL_{Mod} \middle| S_p^2 = 0{,}000104\right)$$
$$= 1 - P\left(S^2 < \sigma_{MAX}^2 \frac{\chi^2_{n\text{-}1,\ 1\text{-}\alpha}}{n\text{-}1} \middle| S_p^2 = 0{,}000104\right)$$
$$= 1 - P\left(\frac{(n\text{-}1)S^2}{S_p^2} < \frac{\sigma_{MAX}^2}{S_p^2} \chi^2_{n\text{-}1,\ 1\text{-}\alpha}\right),$$

where $\frac{(n\text{-}1)S^2}{\sigma_1^2} = \chi^2_{n\text{-}1}$ is a random variable that follows a chi-squared distribution with $n\text{-}1$ degrees of freedom, so

$$False\ Alarm\ Rate = 1 - P\left(\chi^2_{n-1} < \frac{0{,}000127}{0{,}000104}\ 16{,}25\right) = 1 - P(\chi^2_{n-1} < 19{,}818)$$
$$False\ Alarm\ Rate = 0{,}00054 < \alpha_{nom} = 0{,}0027$$

As already stated previously, the False Alarm Rate less than the Type I error (α_{nom}) determined by the process practitioner may contribute for even less unnecessary process stop.

5 CONCLUSION

5.1 FINAL CONSIDERATIONS

This work shows the reasons why the modified chart is useful when running high capable processes this is because, differently from the well-known S^2 Control Chart, this new tool considers, in its formulation, the process specifications limits provided by the project/manager and not only the nominal in-control process variance. The dissertation author named this new chart as the S^2 Modified Control Chart, since it is a natural extension of the Modified Control Chart for monitoring the process mean presented in Montgomery (2009) and introduced by Hill (1956).

This allowed range for variation shift is defined based on the acceptable nonconforming index, which protects the process from produce a rate of defective items larger than what is accepted. The practical implication is that the S^2 Modified Control Chart, designed to detect only genuinely increases in the process variance, may preventing unnecessary process stop and assessment for assignable causes and contribute to higher process efficiency. This is desirable in the sense that small increases in the variance may not affect much the rate of not-conforming items being produced and pausing the process generate extra costs.

In this present dissertation, it was developed 4 different scenarios involving variance monitoring, differing in if the mean in centered or not between the specification limits and if the variance is known or unknown. The σ^2_{MAX} and modified control limit expressions have been performed for both mean centered and not centered between the specification limits, and its values calculated in illustrative examples. When mean is not centered, the σ^2_{MAX} as consequence the modified control limit are lower than the scenario of centered mean. Now regarding when the variance is unknown, the main contribution of this work is to provide an additional work phase where the $\widehat{UCL}_{S^2}$ is determined based on a desired probability of S^2 control limit of the estimated variance parameter is less than the modified control limit.

Despite being a relevant tool for process monitoring, none other work was found related to provide a range where the process variance may shift without compromising its ability to fulfil quality requirements. Hence, the present work aimed to define the variables and parameters necessary for the construction of the S^2 Modified Control Chart for known and unknown variance (unconditional approach) and to expand the knowledge about the performance of this chart, which is important for the conduct of Phase II. This work also presented an additional step before Phase I, when the process variance is unknown, providing a practical approach on how to define the monitoring control limits, supporting the process manager decision.

The performance measures false alarm rate (FAR) and the average run length (ARL) were considered in this work, for known and unknown variance, however taking the process mean as a known parameter, which is unlikely and is one limitation of the research. The chart performance when variance is unknown has been assessed in order to understand the effects of parameter estimation in these measurements, however this study focused mainly on the unconditional approach, proposing a new way to define the estimated control limit for variance, to work combined with modified control chart in a practical way that can be used for actual processes. The chart performance analysis for the conditional approach has been presented but further studies are suggested as future works, such compare its measures performance when compared to unconditional and provide guidance for more fit charts for the actual scenario of manufacturing process, where time and samples are important constraints when designing a statistical process control.

It is possible to notice that the variance estimation compromises the performance of the graph, mainly due to the fact that FAR and ARL become random variables conditioned to the estimates. Once they become random variables, you can define their behavior from their probability distributions, p.d.f. (probability distribution function) and c.d.f. (cumulative density function).

Thus, the major contributions of this work were provided evidence that for highly capable processes, a certain room for variance shift may be allowed without harm quality performance and show how the modified chart can contribute to avoid process over-control and improve management by reducing false alarms. In addition, the use of the modified control chart may minimize the impact of the False Alarm Rate when the control limits are estimated due to estimation due to the modified control limit be defined based on known or target parameters.

The S^2 modified chart was briefly assessed when process mean is no longer centered between specification limits, with variance known and unknown, showing how the mean shift affect the maximum allowed variance and hence, the modified control limit. The effect of parameter estimation needs to be further assessed and is a suggestion for future research.

Finally, it was possible to verify that the acceptance chart is actually indicated for highly capable processes, because, even if the variance slightly shifts from the nominal value within a specific range of values) and its capacity decreases moderately, the performance of the graph remains almost unchanged, with probability of a false alarm very close to zero. The analysis of the simulated data showed that by using the proposed S^2 Modified Control Chart the number of unnecessary interventions in the process could be decreased, contributing to improve its efficiency. However, if the variance moves to a value close to the acceptance limit, the risks of a false alarm increase. And if the variance goes out of the specification limits, the process is then considered uncapable and hence, the chart would signal an out-of-control condition.

5.2 FUTURE WORKS

It is recommended as future studies aligned to what was presented in this research, the analysis of modified chart performance when both mean and variance are estimated, and the design of a modified chart providing combining the possibility of both mean and variance be able to shift within a tolerable range. As future work, also the assessment of what would be the most appropriate sample size to achieve a reasonable, practical, and economic significance for actual processes in order to provide both confidence and efficiency to monitoring of highly capable manufacturing processes.

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