UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

The period function for some planar piecewise vector fields

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I dedicate this work
to everyone who contributed
in my moral and intellectual
education

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Resumo

Nesta tese, estudamos a função período para famílias de campos vetoriais diferenciais suaves por partes com uma reta de descontinuidade. Tais sistemas, chamados indistintamente de descontínuos ou não suaves, aparecem em diversas aplicações, incluindo, entre outras, controle ótimo, mecânica descontínua e manipulação robótica. Para uma família, utilizando um método baseado em equações de Picard–Fuchs para curvas algébricas, caracterizamos o comportamento global da função período. Ou seja, determinamos regiões no espaço de parâmetros para as quais a função período correspondente é monótona ou possui períodos críticos. Além disso, em outra família estudamos a bifurcação de períodos críticos no interior do anel de período do centro fraco e do centro isócrono usando o cálculo do desenvolvimento de Taylor das constantes de períodos próximas ao centro. Adicionalmente, apresentamos o início do estudo do comportamento global da função período para os sistemas suaves por partes do tipo linear-linear que contém um anel de período no infinito.

Palavras-chave: sistemas suaves por partes, função período, monotonicidade, períodos críticos, bifurcação de períodos críticos.

Abstract

In this work, we study the period function for fixed families of piecewise differential vector fields with a line of discontinuities. These systems, indistinctly called piecewise or nonsmooth, appear in several applications, including among others optimal control, nonsmooth mechanics, and robotic manipulation. For one family, by using a method based upon Picard–Fuchs equations for algebraic curves, we characterize the global behavior of the period function. That is, we determine regions in the parameter space for which the corresponding period function is monotonous or it has critical periods. Furthermore, in one of these families we study the bifurcation of critical periods in the interior of the period annulus from the weak center and from the isochronous center by using the calculation of the Taylor developments of the periods constants near the center. We further present the beginning of the study of the global behavior of the period function for the planar piecewise linear system that contains a period annulus at infinity.

Keywords: piecewise systems, period function, monotonicity, critical periods, bifurcation of critical periods.

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Introduction

The measurement of time has been an enormous incentive for the development of physics and mathematics. Galileo discovered in 1583 that pendulums of the same length have the same oscillation time, i.e. all periods are the same independently of the initial angle, this phenomenon became known as the Law of Isochrony. Inspired by this fact, and realizing the need for an accurate measurement of time, C. Huygens in 1656 created a pendulum clock which for a long time was the most accurate (see for instance [41]). The work with period functions goes back at least to 1673 when Huygens observed that the pendulum clock has a monotone period function, and therefore, oscillates with a shorter period when the energy is decreasing, that is the clock spring unwinds.

In recent decades, there has been an increasing interest in studying the isochronicity of vector fields in the plane. We find an overview of the results obtained in the survey of Chavarriga and Sabatini [15]. However, there are few families of polynomial vector fields in which a complete classification has been found. Some of the classes of isochronous systems that have already been studied are the potential Hamiltonian systems ([90, 91]), quadratic systems ([66]), cubic systems with homogeneous nonlinearities ([83]), Kukles systems ([25]), and isochronous centers of a linear center perturbed by third, fourth, and fifth degree homogeneous polynomials ([12, 13, 14]).

Consider a family of planar vector field $(x, y) \mapsto \mathbf{F}(x, y, \lambda)$, parametrized by $\lambda \in \mathbb{R}^m$, where for all λ we have a nondegenerate center at the origin, that is, a center where the linear part has two pure imaginary eigenvalues (see [11]). Let $\gamma(\rho)$, $\rho \in I$ where I is a real open interval, be a smooth parametrized continua of periodic orbits. The period function $T(\rho, \lambda)$ associated to the parameter λ, is the function that assigns to each ρ the minimal period of the periodic trajectory through $\rho \in I$. By the Implicit Function Theorem, T it is analytic. The zeros of its derivative with respect to ρ, denoted by $T'(\rho, \lambda)$, are called *critical periods* or *oscillations* and determining them is a key point to know the behavior of T.

In the last years, the function T has been extensively studied by many authors with different methods, as in [23, 36, 66, 95]. Mostly, with the interest of determining its qualitative behavior ([10, 50, 53, 69, 88]). In [6] the authors state that "the period function is important to study theoretical properties of planar ordinary differential equation and their perturbations, see [22]; to understand some mathematical models in physics or ecology, see [42, 85, 95] and the references therein; in the description of the dynamics of some discrete dynamical systems, see [5, 27, 28]; or for counting the solutions of some boundary value problems, see [19, 20]." There are also many researchers in the area dedicated to find conditions for the period function to be monotonous, since monotonicity implies existence and uniqueness for certain boundary value problems, and provides a nondegeneracy condition for the bifurcation of subharmonic solutions of periodically forced Hamiltonian systems ([21]).

Another approach considered is finding the maximum number of critical periods that a period function can have for a n-th degree planar polynomial system by using similar techniques developed for dealing with the problem of determining the number and location of their limit cycles (isolated closed orbits of a vector field), known as a particular case of the second part of Hilbert's 16th problem ([61]). In other words, there are many works that intend to establish these values in terms of the degree n, but it remains unsolved if these numbers are finite, for all n, which would guarantee their existence. However, since it is believed to be finite, the number of limit cycles is denoted by $\mathcal{H}(n)$ (Hilbert number) and the number of critical periods by $\mathcal{C}(n)$. Due to the difficulty of determining exactly $\mathcal{H}(n)$ and $\mathcal{C}(n)$, several researchers try to obtain the highest possible lower bound for them. The best result concerning any configuration of limit cycles provides $\mathcal{H}(n) \geq kn^2 \log(n)$, for some $k > 0$, that is, it grows at least as rapidly as $n^2 \log(n)$ ([26]). In [30] the author provides the following lower bound: $\mathcal{C}(n) \geq 2[(n-2)/2]$, where $[\cdot]$ denotes the integer part, and from the result of [50] it is known that $\mathcal{C}(n) \geq n^2/4$, that is, it grows as $n^2/4$. In [10] the lower bound of $\mathcal{C}(n)$ becomes $n^2/2 + n - 5/2$ when n is odd, and $n^2/2 - 2$ when n is even. Recently, in [37] this bound has been proved to be $n^2 - 2$ when n is odd, and $n^2 - 2n - 1$ when n is even. A natural question established by Gasull in [47] is whether there is a similar result to what has been proved for limit cycles, that is: "Is it true that $\mathcal{C}(n) \geq cn^2 \log(n)$, for some $c > 0$?"

There are only a few families where the global qualitative behavior of T is found, that is $\mathcal{C}(n)$ is found, among them some families of smooth polynomial potential systems. These families are given by Hamiltonian systems with total energy $H(x, y) = y^2/2 + V(x)$, where $V(x)$ is the potential energy. It is well known that for smooth potential systems the set of all periodic orbits can be parametrized by the energy h and, therefore in this case, we can introduce the period function $T(h)$, which gives the period of the periodic orbit with energy h. It can be found in Chow and Sanders ([23]) and Gavrilov ([53]) the study of $T(h)$ for the case in which the potential energy is given by $V(x) = (1/2)x^2 + (a/3)x^2 + (b/4)x^4$, and in Mañosas and Villadelprat ([69]) the case where $V(x) = (1/2)x^2 + (a/4)x^4 + (b/6)x^6$, with $a, b \in \mathbb{R}$ and $b \neq 0$.

In general, the qualitative behavior of $T(h)$ for smooth potential systems is found by using that it satisfies a second order Picard–Fuchs equation for algebraic curves and the ellipticity of the level curves of H allows to see that $x(h) = T'(h)/T(h)$ satisfies a Riccati equation. Then, instead of this equation, we consider the equivalent polynomial system on the plane and study its global phase portrait. It is worth mentioning that Picard–Fuchs equations was already used by many authors to study the number of zeros of Abelian integrals, because these zeros correspond to limit cycles appearing in non-conservative perturbations of Hamiltonian, or, more general, integrable systems (see [80, 101] for instance).

Even the global study of the period function for the reversible quadratic center is not complete. It is well known that this system can be brought to the Loud normal form

$$\begin{aligned} \dot{x} &= -y + Bxy, \\ \dot{y} &= x + Dx^2 + Fy^2, \end{aligned} \tag{1}$$

where $B, D, F \in \mathbb{R}$. It is also called *homogeneous Loud system* (see [21]). It is proved in [49] that if $B = 0$ then the period function of the center at the origin is monotonous increasing. Thus the case in which $B \neq 0$ is the most interesting one and, by a rescaling, we can only consider family (1) with $B = 1$, i.e.

$$\begin{aligned} \dot{x} &= -y + xy, \\ \dot{y} &= x + Dx^2 + Fy^2, \end{aligned} \tag{2}$$

called *dehomogenized Loud family.* One of the most famous open problems about critical periods conjectured by Chicone in 1994 and also listed by Gasull in [47] is the following question: Is 2 the maximum number of critical periods for family (2)?

Another problem, less intricate, is describing the local behavior of the function T, in a neighborhood of an equilibrium point. To this end, there are several papers whose interest is to provide the maximum number of zeros of T' which can bifurcate near a center for planar autonomous polynomial vector fields of degree n, denoted by $\mathcal{C}_0(n)$. The problem of determining $\mathcal{C}_0(n)$ is called *problem of bifurcations of critical periods* or *criticality problem.* Chicone and Jacobs in [21] solved this problem for analytical quadratic systems, and they obtained that $\mathcal{C}_0(2) = 2$. In particular, they proved that at most one local critical period can bifurcate from a nonlinear isochronous center and at most two critical periods bifurcate from the linear isochronous center for the homogeneous Loud system (1), which is a family with three parameters. The approach for determining $\mathcal{C}_0(n)$ in terms of the techniques is similar to that taken to provide the maximum number of small amplitude limit cycles bifurcating from an elementary center or an elementary focus of a planar polynomial system of degree n, denoted by $\mathcal{H}_0(n)$ ([81]). In other words, determining $\mathcal{C}_0(n)$ (resp. $\mathcal{H}_0(n)$) is equivalent to solve the *local criticality problem* (resp. *local cyclicity problem*). Clearly, $\mathcal{C}_0(n) < \mathcal{C}(n)$ and $\mathcal{H}_0(n) < \mathcal{H}(n)$.

There are two useful results to get good lower bounds for $\mathcal{H}_0(n)$, whose statements and proofs can be found in Christopher ([24]). The first result shows how we can use the first-order Taylor approximation of the Lyapunov constants. The second result shows how sometimes we can obtain more limit cycles using high-order Taylor developments of the Lyapunov constants. In these results, if we replace the concept of Lyapunov constants by period constants and initially consider an isochronous center, we can use the same method to get the local criticality of isochronous centers, and this is what was made in [88] for reversible holomorphic (isochronous) centers. In general, if the number of parameters is large, the computations can be very hard, with a high computational demand. One method that can be useful to compute the linear parts of the period constants is the so called *parallelization*, which was developed in [63] and allows decreasing both the total computation time and the memory requirements.

It is worth mentioning that there are also works that study other types of bifurcation phenomena. For example, there exist several papers which aim to develop tools for the study of bifurcations of critical points of the period function at the outer boundary (i.e., the polycycle) for family (2), see more details in [72, 73, 74, 75, 77, 78]. Other studies aims to determine the regions of monotonicity by using that (2) can be brought by means of a coordinate transformation to a potential system. Then the authors can apply the monotonicity criteria for potential smooth systems, for example the Schaaf's criterium ([89]), to study the period function of its center ([70, 92, 93]) or even Picard–Fuchs approach ([99, 100]).

Although there is still much to be done regarding the period function for smooth systems, our goal is to contribute to the study of the period function for families of *Filippov vector fields*. Since many significant dynamical systems, that arise in practice, for modeling problems raised from mechanics, electrical, engineering, or automation control contain terms that are non-smooth functions of their arguments, then an unavoidable issue is the use of *non-smooth systems* for describing them, see [38]. Many works have been made to get a characterization of the dynamical behavior that such systems can exhibit by proceeding in a case-by-case approach, but this study is from being completely finished. Up to now, most of the results concerning about piecewise differential systems aimed to either identify conditions for the origin to be a center and an isochronous center (see [31, 33, 52, 62, 68]), or investigate the *cyclicity problem* (see [16, 32, 59, 64, 65, 98]). There exist only few studies in the sense of determining the number of critical periods as in [17, 18, 79, 97, 102].

In Chapter 1 we bring some of the definitions and results needed for the reader who is unfamiliar with the tools we use in this text. There, we make an introduction of the Filippov system, also called *piecewise system*. Furthermore, we present definitions for the study of the period function of a center and a first approach of its bifurcation theory, including the results presented in [88] for increasing the criticality for families of isochronous centers.

Chapter 2 is devoted to determine the complete bifurcation diagram of the period function of the center at the origin for a piecewise continuous planar Hamiltonian system of ordinary equations with two parameters. This family is the aggregation of two different polynomial potential systems with a straight line of separation. In order to describe the period function of the family studied, we use the fact that the period function can be written as a linear combination of the period functions for the smooth potential systems that define it. Then, it is enough to determine the behavior of the period in the smooth case by using Picard–Fuchs approach. The results are presented in the paper: A. C. Rezende, M. A. S. Santos, J. Torregrosa. *Period function for a family of planar piecewise Hamiltonian systems. Preprint*, 2023. In particular, we highlight Theorem 2.13 which we may say that is the most relevant result of this thesis.

In Chapter 3 we estimate lower bounds of the local criticality for the family of piecewise quadratic reversible centers. The two systems that define this family are smooth reversible quadratic systems of the form (1) after a simple coordinate trasformation. We notice that the results we have obtained are very similar to those ones found in [21], since the quadratic family studied has six parameters and we have found that at least four local critical periods can bifurcate from a nonlinear isochronous center, and at most five critical periods bifurcate from the linear one. This problem can be solved by calculating the Taylor series of the period function in the neighborhood of the center, and by determining the order of its first non-constant term. This computation is purely algorithmic and its advantage is that it provides information about the behavior of the period function in a neighborhood of the origin. The results we obtained are presented in the manuscript: A. C. Rezende, M. A. S. Santos, J. Torregrosa. *Period function of planar piecewise reversible quadratic systems. Preprint*, 2023.

Chapter 4 is a work in progress that aims to study the period function of a center at infinity for the lowest degree family for discontinuous system, the one which has the aggregation of two different linear systems with a straight line of separation, the planar piecewise linear system, which is also called a linear-linear system. It comes in line with the large amount of work that has been made in recent years with the aim to understand its dynamic behavior, we refer the reader to [7, 8, 9, 43, 45, 82]. Under the hypothesis of focus-focus dynamic, a reduced canonical form with only five parameters is obtained and the authors of [44] characterize the centers at infinity, called center-center case and focus-focus case together with the limit cycles bifurcating from them. Here we try to determine the global behavior of the period function for these cases, and we believe that the center-center case has at most one critical period. In the focus-focus case, we found that there are no oscillations and we classify when the period function is monotonous decreasing, constant, or monotonous increasing. This study is also presented in: A. C. Rezende, M. A. S. Santos, J. Torregrosa. *Period function for piecewise linear centers at infinity. Preprint*, 2023.

The last chapter contains a brief description of what can still be done in the future, and also some concluding remarks.

For all the needed calculations we have used the computer algebra system Maple [71].

CHAPTER 1

Preliminaries

1.1 Piecewise differential systems

In this section we introduce the basic concepts of a *planar Filippov system*, which we shall often refer as a *planar piecewise differential system*, necessary for the development of the next chapters. Basically, we follow the approach introduced in [57].

Let X and Y be smooth vector fields defined in an open set $U \subset \mathbb{R}^2$ containing the origin and $h : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function having 0 as a regular value and $\Sigma = h^{-1}(0) \cap U$ a smooth codimension-one submanifold. Denote by $\Sigma^- = \{(x, y) \in U \subset \mathbb{R}^2 : h(x, y) < 0\}$ and $\Sigma^+ = \{(x, y) \in U \subset \mathbb{R}^2 : h(x, y) > 0\}$ the regions having Σ as a separating boundary. A *planar piecewise differential system* on $\mathbb{R}^2$ is a pair of C^r (with $r \geq 1$) differential systems in $\mathbb{R}^2$ separated by the separation manifold Σ, that is

$$Z(x, y) = \begin{cases} X(x, y), & \text{if } (x, y) \in \Sigma^- \cup \Sigma, \\ Y(x, y), & \text{if } (x, y) \in \Sigma^+. \end{cases} \tag{1.1}$$

Clearly, an orbit is well defined while it evolves without touching the separation set. However, we shall assume the Filippov's convention [40] for the definition of trajectories arriving at Σ.

The contact between the vector field X (or Y) and the line of separation Σ is characterized by the Lie derivative of h in the direction of the vector field X and Y, that is $Xh(p) = \langle \nabla h(p), X(p) \rangle$ and $Yh(p) = \langle \nabla h(p), Y(p) \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^2$. According to the terminology established by Filippov, we can split the line of separation Σ into the following sets:

(a) *Crossing set*: $\Sigma^c := \{p \in \Sigma \colon Xh(p) \cdot Yh(p) > 0\}$;

(b) *Sliding set*: $\Sigma^s := \{p \in \Sigma \colon Xh(p) < 0 \text{ and } Yh(p) > 0\}$;

(c) *Escaping set*: $\Sigma^e := \{p \in \Sigma\colon Xh(p) > 0 \text{ and } Yh(p) < 0\}$.

The *escaping* Σ^e or *sliding* Σ^s regions are respectively defined on points in Σ, where both vector fields X and Y simultaneously point outwards or inwards from Σ, while the interior of its complement in Σ defines the *crossing region* Σ^c (see Figure 1.1). The complementary of $\Sigma^c \cup \Sigma^e \cup \Sigma^s$ is the set formed by *tangency points*, that is, points where one of the two vector fields is tangent to Σ, which are the points $p \in \Sigma$ where $Xh(p) = 0$ or $Yh(p) = 0$. These points are on the boundary of Σ^c, Σ^e, and Σ^s, which we denote by $\partial\Sigma^c$, $\partial\Sigma^e$, and $\partial\Sigma^s$, respectively. This union also contains the equilibrium points of X and Y at Σ.

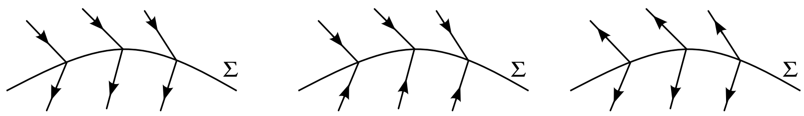

Figure 1.1: Crossing, sliding and escaping regions, respectively

In this work, we will assume that the tangency points are isolated in Σ. This happens when we study low-codimension bifurcations in planar Filippov systems. For simplicity, the definition of orbit that is established here applies only to Filippov systems with isolated equilibrium points.

Next, we define a trajectory passing through a point p at Σ^c, Σ^e, and Σ^s following the Filippov convention. For a point $p \in \Sigma^c$, both vector fields X and Y simultaneously point towards Σ^+ and Σ^-, since the transversal components of X and Y have the same sign in p, then it is enough to connect the solutions of X and Y at p. In points $p \in \Sigma^s \cup \Sigma^e$ the transversal components of X and Y have opposite signs, i.e. the two vector fields are pushing in opposite directions, then the state of the system is forced to remain on the boundary and slide on it. The local orbit of system (1.1) on Σ is given by the vector field Z^s, which is a convex linear combination of $X(p)$ and $Y(p)$, so that $Z^s(p)$ is tangent to the separation manifold Σ, as we can see in Figure 1.2. Such vector field is defined as the *sliding vector field*, and it is given by

$$Z^s(p) = \frac{1}{Yh(p) - Xh(p)}\Big(Yh(p)X(p) - Xh(p)Y(p)\Big). \tag{1.2}$$

This way to define the motion on Σ is the most natural and it is called *Filippov convex method* or *Filippov convention*.

In general, we can formulate the following definition of orbits to Filippov systems with isolated equilibrium points, where $\varphi_V(t,p)$ denotes the flow of the vector field V defined

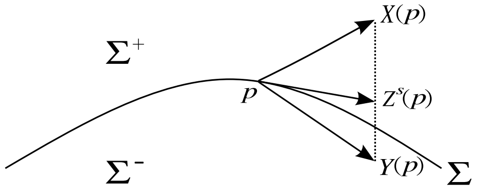

Figure 1.2: Sliding vector field

in time $t \in I_p \subset \mathbb{R}$, that is

$$\begin{cases} \dfrac{d}{dt}\varphi_V(t,p) = V(\varphi_V(t,p)), \\ \varphi_V(0,p) = p, \end{cases}$$

and I_p is a real interval which depends on the point p and on the vector field V.

Definition 1.1. The *local trajectories* of the planar Filippov system (1.1) are defined as:

(i) For $p \in \Sigma^-$ (resp. $p \in \Sigma^+$) such that $X(p) \neq 0$ (resp. $Y(p) \neq 0$), the trajectories are given by $\varphi_Z(t,p) = \varphi_X(t,p)$ (resp. $\varphi_Z(t,p) = \varphi_Y(t,p)$), for $t \in I_p \subset \mathbb{R}$.

(ii) For $p \in \Sigma^c$ such that $Xh(p)$, $Yh(p) > 0$ and considering the orbit starting at p we have that $\varphi_Z(t,p) = \varphi_Y(t,p)$, for $t \in I_p \cap \{t \leq 0\}$, and $\varphi_Z(t,p) = \varphi_X(t,p)$, for $t \in I_p \cup \{t \geq 0\}$. For $Xh(p)$, $Yh(p) < 0$, the definition is the same, but reversing time.

(iii) For $p \in \Sigma^s \cup \Sigma^e$ such that $Z^s(p) \neq 0$, $\varphi_Z(t,p) = \varphi_{Z^s}(t,p)$, for $t \in I_p \subset \mathbb{R}$, where Z^s is the sliding vector field defined in (1.2).

(iv) For $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$ such that the definitions of trajectories for points in Σ in both sides of p can be extended to p and coincide, then the orbit through p is this trajectory. We call these points *regular tangency points*.

(v) For any other point, $\varphi_Z(t,p) = p$, for all $t \in \mathbb{R}$. This is the case of the tangency points in Σ which are not regular, called *singular tangency points*, and the equilibrium points of X in Σ^+, Y in Σ^-, and Z^s in $\Sigma^s \cup \Sigma^e$.

Definition 1.2. The *local orbit* of a point $p \in U$ is the set $\gamma(p) = \{\varphi_Z(t,p) : t \in I\}$.

Since we are dealing with autonomous systems, from now on we will use the terms *trajectory* and *orbit* indistinctly.

Thus, the *phase portrait* of a Filippov system is the union of all orbits in $\mathbb{R}^2$ composed by the sliding phase portrait on the boundary Σ and of the standard phase portraits in each region. We note that if the orbit has sliding motion, then it can overlap.

Definition 1.3. The points $p \in \Sigma^s \cup \Sigma^e$ that satisfy $Z^s(p) = 0$, i.e. the equilibrium points of the sliding vector field are called *pseudoequilibria* of Z.

The equilibrium points of the Filippov system are characterized in the following definition.

Definition 1.4. *Equilibrium points* of the planar Filippov system (1.1) are:

(i) $p \in \Sigma^-$ (resp. $p \in \Sigma^+$) such that $X(p) = 0$ or $Y(p) = 0$, that is, p is an equilibrium point of X or Y;

(ii) $p \in \Sigma^s \cup \Sigma^e$ such that p is a pseudoequilibrium, that is, $Z^s(p) = 0$;

(iii) $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$, that are regular or singular tangencies, i.e. the points p such that $Xh(p) = 0$ or $Yh(p) = 0$.

The remaining points are called *regular points*.

In Filippov systems there exist equilibrium points (regular tangency points) which have an orbit such that $\gamma(p) \neq \{p\}$. For this reason they can be classified as *distinguished equilibrium points*, which are points p such that $\gamma(p) = \{p\}$, and *non-distinguished equilibrium points*, which are points $p \in \Sigma$ which are regular tangency points and then, even if they are not regular points, their local orbit is homeomorphic to $\mathbb{R}$.

Definition 1.5. A *distinguished equilibrium point* of the planar Filippov system (1.1) is a point p such that $\gamma(p) = \{p\}$ and it can be classified as:

(i) $p \in \Sigma^-$ (resp. $p \in \Sigma^+$) such that $X(p) = 0$ or $Y(p) = 0$, that is, p is an equilibrium point of X or Y;

(ii) $p \in \Sigma^s \cup \Sigma^e$ such that p is a pseudoequilibrium, that is, $Z^s(p) = 0$;

(iii) $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$, such that p is a singular tangency point.

The vector field X (resp. Y) can have equilibrium points that do not belong to $\overline{\Sigma^-}$ (resp. $\overline{\Sigma^+}$). We call these points as *virtual equilibrium points* or *non-admissible equilibrium points.* The equilibrium points of X (resp. Y) that are in Σ^- (resp. Σ^+) are called *admissible or real equilibrium points.* A special case of pseudoequilibrium points which are the solutions of $X(p) = 0$ or $Y(p) = 0$ and $h(p) = 0$ are named *boundary equilibrium points.*

Analogously, invariant objects (stable and unstable manifolds, periodic orbits) of the smooth vector fields X and Y not belonging to $\overline{\Sigma^-}$ and $\overline{\Sigma^+}$, respectively, are also referred to as *non-admissible.*

Even if the chosen definition of orbit leads to the uniqueness property, a point $p \in \Sigma$ may belong to the closure of several other orbits. Taking into account this fact, we can use the following definition.

Definition 1.6. Given a trajectory $\varphi_Z(t,p) \in \Sigma^+ \cup \Sigma^-$ and a point $p \in \Sigma$, we say that p is a starting point of $\varphi_Z(t,p)$, if there exists $t_0 < 0$ such that $\lim_{t \to t_0^+} \varphi_Z(t,q) = p$, and that it is an arrival point of $\varphi_Z(t,q)$, if there exists $t_0 > 0$ such that $\lim_{t \to t_0^-} \varphi_Z(t,q) = p$.

According to Definition 1.1, if $p \in \Sigma^c$, p is a starting point of $\varphi_Z(t,p)$ for any q belonging to the forward orbit

$$\gamma^+(p) = \{\varphi_Z(t,p) : t \in I_p \cap \{t \geq 0\}\},$$

and is an arrival point of $\varphi_Z(t,p)$ for any q belonging to the backward orbit

$$\gamma^-(p) = \{\varphi_Z(t,p) : t \in I_p \cap \{t \leq 0\}\}.$$

Namely, the orbit through a point $p \in \Sigma^c$ is the union of the point and its starting and arrival orbits, that is, $\gamma(p) = \{p\} \cup \gamma^+(p) \cup \gamma^-(p)$.

Once we have defined the local orbit through a point, we can state rigorously the definition of maximal orbit. Depending on the point, it can be either a regular orbit, or a sliding orbit, or a distinguished equilibrium point.

Definition 1.7. A *maximal regular orbit* of Z is a piecewise smooth curve γ such that:

(i) $\gamma \cap \Sigma^-$ and $\gamma \cap \Sigma^+$ are a union of orbits of the smooth vector fields X and Y, respectively;

(ii) The intersection $\gamma \cap \Sigma$ consists only of crossing points and regular tangency points in $\partial\Sigma^c$;

(iii) γ is maximal with respect to these conditions.

Definition 1.8. A *maximal sliding orbit* of Z is a smooth curve $\gamma \subset \overline{\Sigma^s} \cup \overline{\Sigma^e}$ such that it is a maximal orbit of the smooth vector field Z^s.

The previous definitions lead to two relevant results: first, the uniqueness of solutions, that is, any point p belongs to only one orbit, and second, any neighborhood U of p is decomposed into a disjoint union of orbits.

We can genaralize the concept of separatrix for planar Filippov systems.

Definition 1.9. An *unstable separatrix* is either:

(i) A regular orbit Γ which is the unstable invariant manifold of a regular saddle point $p \in \overline{\Sigma^-}$ of X or $p \in \overline{\Sigma^+}$ of Y, that is,

$$\Gamma = \left\{ q \in U \text{ such that } \varphi_Z(t,q) \text{ is defined for } t \in (-\infty, 0) \text{ and } \lim_{t \to -\infty} \varphi_Z(t,q) = p \right\}.$$

We denote it by $W^u(p)$;

(ii) A regular orbit which has a distinguished equilibrium point $p \in \Sigma$ as a starting point. We denote it by $W^u_\pm(p)$, where the subscript $\pm$ means that it leaves p from $\Sigma^\pm$.

In the first case, as it is well known in smooth systems, the trajectory lying on the separatrix reaches p in infinite time whereas in the second case, it may reach the singularity in finite time.

Stable separatrices $W^s(p)$ and $W^s_\pm(p)$ are defined analogously. If a separatrix is simultaneously stable and unstable it is a *separatrix connection.*

In Filippov system, beyond periodic orbits of X in Σ^- and of Y in Σ^+, there exist other regular trajectories of Σ^- and Σ^+ that have p as an arrival or starting point. Remember that these orbits reach p in a finite time. The next definition generalizes the concept of periodic orbit in this context.

Since our goal is to study nonlinear phenomena around Σ, we look for possible periodic orbits not totally contained in Σ^+ and Σ^-. These orbits must be of one of the following two types, depending on the nature of their points on the separation manifold Σ: If the periodic orbit has sliding points, then it is called a *sliding periodic orbit*, otherwise we have a *crossing periodic orbit.*

Definition 1.10. A *regular periodic orbit* is a regular orbit $\gamma = \{\varphi_Z(t,p) : t \in \mathbb{R}\}$, which therefore belongs to $\Sigma^+ \cup \Sigma^- \cup \overline{\Sigma^c}$ ($\overline{\Sigma^c}$ denotes the closure of Σ^c) and satisfies $\varphi_Z(t+T,p) = \varphi_Z(t,p)$, for some $T > 0$, called the *period.*

The regular periodic orbits are called *standard periodic orbits*, if they stay in $\Sigma^- \cup \Sigma^+$, and *crossing periodic orbits*, if they intersect $\overline{\Sigma^c}$.

The sliding periodic orbits are the so-called cycles and they are presented in the next definition.

Definition 1.11. A *period cycle* is the closure of a finite set of pieces of orbits $\gamma_1, \dots, \gamma_n$ such that γ_{2k} is a piece of sliding orbit, γ_{2k+1} is a maximal regular orbit and the starting and arrival points of γ_{2k+1} belong to $\overline{\gamma_{2k}}$ and $\overline{\gamma_{2k+2}}$, respectively. The *period of the cycle* is the sum of the periods of time that are spent in each of the pieces of orbit γ_i, $i = 1, \dots, n$.

Definition 1.12. The planar piecewise vector field (1.1) is called *continuous* if

$$X(p) = Y(p), \quad \text{for all} \quad p \in \Sigma^c.$$

Otherwise, it is called *discontinuous.*

In particular, if $\Sigma^c = \Sigma$, i.e Σ^e and Σ^s are empty sets, the piecewise vector field (1.1) is continuous if $X(p) = Y(p)$ for all $p \in \Sigma$.

Example 1.13. Consider the planar piecewise vector field given by

$$(\dot{x}, \dot{y}) = \begin{cases} \left(\dfrac{\partial H^-(x,y)}{\partial y}, -\dfrac{\partial H^-(x,y)}{\partial x}\right), & \text{if } (x,y) \in \Sigma^-, \\ \left(\dfrac{\partial H^+(x,y)}{\partial y}, -\dfrac{\partial H^+(x,y)}{\partial x}\right), & \text{if } (x,y) \in \Sigma^+. \end{cases}$$

This vector field is called *planar piecewise Hamiltonian vector field* with *Hamiltonian function* given by

$$H(x,y) = \begin{cases} H^-(x,y), & \text{if } (x,y) \in \Sigma^-, \\ H^+(x,y), & \text{if } (x,y) \in \Sigma^+. \end{cases}$$

In the case of $H^-(x,y) = H^+(x,y)$, if $(x,y) \in \Sigma$, the system is a *continuous piecewise Hamiltonian system.*

1.2 Centers and their period functions

The same notion of center and period function for planar vector field are valid for the planar piecewise differential equation (1.1).

Definition 1.14. An isolated equilibrium point p of (1.1) is a *center* if and only if there exists a punctured neighborhood $\mathcal{V}$ of p, $\mathcal{V} \subset \mathbb{R}^2$, such that every point in $\mathcal{V}$ belongs to a periodic orbit surrounding p.

The largest connected set $\mathcal{V}_0$ covered with periodic orbits surrounding the center p is called *central region.*

Definition 1.15. For any center p of a planar differential system, the largest neighborhood of p which is entirely covered by periodic orbits is called the *period annulus.* It is denoted by $\mathcal{P}$, and its boundary is denoted by $\partial\mathcal{P}$.

Definition 1.16. A center is a *global center* when its period annulus is the whole plane, that is, if every solution of (1.1) is periodic. Otherwise, this equilibrium is called a *local center.*

We consider that the center is at the origin, which we denote it by O.

In what follows the period function is defined over the set of periodic orbits inside its corresponding period annulus by the following definition.

Definition 1.17. Let O be a center and $\mathcal{P}$ its period annulus. The *period function* is the map which associates to any periodic orbit $\gamma \in \mathcal{P}$ its period.

Remark 1.18. Given a smooth system $(\dot{x}, \dot{y}) = (P(x, y), Q(x, y))$ with a nondegenerate center at the origin, that is, a center where the linear part has two pure imaginary eigenvalues. Let $\gamma(\rho)$, $\rho \in I$ where I is a real open interval, be a smooth parametrized continua of periodic orbits. The period function T is the function that assigns to each $\rho \in I$ the minimal period of the periodic trajectory through ρ. And can be computed by the expression

$$T(\rho) = \int_0^T dt = \int_{\gamma(\rho)} \frac{dx}{\dot{x}} = \int_{\gamma(\rho)} \frac{dx}{P(x, y)},$$

or even by

$$T(\rho) = \int_0^T dt = \int_{\gamma(\rho)} \frac{dy}{\dot{y}} = \int_{\gamma(\rho)} \frac{dy}{Q(x, y)}.$$

Note that a vector field can have several annular regions foliated by periodic orbits. In this case, for each one of these period annuli, we can consider the corresponding period function. That is, each period function is defined over the set of periodic orbits inside its corresponding period annulus.

Definition 1.19. The period function of a center is *monotonous increasing* (resp. *decreasing*) if for any pair of periodic orbits inside $\mathcal{P}$, say γ_1 and γ_2 with $\gamma_1 \subset \mathrm{Int}(\gamma_2)$, we have that the period of γ_2 is greater (resp. smaller) than the one of γ_1.

A very special case is the one in which we have a constant period function, according to the next definition.

Definition 1.20. A center is called *isochronous center* when all periodic solutions in a neighborhood of it have a constant period.

The problem of isochronicity is investigated in the same way that the well-known center problem. Both problems are completely solved only for quadratic vector fields. For the most interested readers, an excellent text for a first reading about the theory of centers of planar polynomial systems can be found in [35], and to better understand many problems about isochronicity for analytic systems we recommend the survey of Chavarriga and Sabatini [15], already mentioned in the introduction. It is important to note that the isochronicity problem appears only for nondegenerate centers, that is centers whose linear part has nonzero imaginary eigenvalues (see [11]), since the period function goes to infinity near a degenerate center, as in the next example, which can be found in [94].

Example 1.21. Consider the cubic system given by

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -2x^3, \end{cases}$$

with a degenerate center at $(0,0)$. The first integral of this system is given by $H(x,y) = x^4 + y^2$, hence, the integrating curve passing throungh $(\rho, 0)$ can be expressed as $y^2 + x^4 - \rho^4 = 0$. Thus, $y = \pm\sqrt{\rho^4 - x^4}$ and, by the Remark 1.18, the period function T, which assigns to the periodic solution $\gamma(\rho)$, that intersects the x-axis in ρ and $-\rho$, its minimum period can be expressed by

$$T(\rho) = \int_{\gamma(\rho)} \frac{dx}{\dot{x}} = 2\int_{\rho}^{-\rho} \frac{dx}{y} = 2\int_{\rho}^{-\rho} \frac{dx}{\sqrt{\rho^4 - x^4}} = \frac{2}{\rho}\int_{-1}^{1} \frac{1}{1 - s^4} ds,$$

where a change of variables $x = \rho s$ is applied on the last equality. Therefore,

$$\lim_{\rho \to 0} T(\rho) = \infty.$$

A special case of isochronicity that has been studied in the literature is given in the next definition, see [34, 51].

Definition 1.22. A center is a *uniform isochronous center* if the system, in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, takes the form $\dot{r} = G(\theta, r)$, $\dot{\theta} = k$ or, equivalently, the equality $x\dot{y} - y\dot{x} = k(x^2 + y^2)$ holds, for some $k \in \mathbb{R} \setminus \{0\}$.

In other words, the angular velocity is the same for all orbits. A system which possesses this property is also called a *rigid system*.

As indicated in [31, 87], inside the class of smooth isochronous system there are some special cases, that are presented in the next definition.

Definition 1.23. Consider the analytic system

$$\begin{cases} \dot{x} = -y + p(x,y), \\ \dot{y} = x + q(x,y), \end{cases}$$

where p and q are analytic functions in a neighborhood of the origin starting with terms at least of degree two and with an isochronous center at the origin. We say that the isochronous center is a *k-strong isochronous center* ($k \in \mathbb{N}^*$) if there exist exactly k half-straight lines (radials) given by $L(\theta_i) := \{a(\cos\theta_i, \sin\theta_i) : a > 0\}$ (called, for convenience, *isochronicity radials*), $\theta_i \in (0, 2\pi)$, $i = 1, \ldots, k$ such that all periodic solutions spend the time θ_i to go from the positive x-axis to $L(\theta_i)$, for every $i = 1, \ldots, k$.

The 0-strong isochronous center is referred to the case of no isochronicity radials and ∞-strong isochronous systems to the case of infinite ones. Obviously, a uniformly (or rigid) isochronous center is the ∞-strong isochronous center.

1.3 Piecewise systems with a line of discontinuities

Consider the following class of piecewise planar systems of ordinary differential equations with $\Sigma = \{(x, y) : x = 0\}$ given by

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + P^-(x, y, \lambda^-),\ x + Q^-(x, y, \lambda^-)), & \text{if } x \leq 0, \\ (-y + P^+(x, y, \lambda^+),\ x + Q^+(x, y, \lambda^+)), & \text{if } x > 0, \end{cases} \tag{1.3}$$

where $\lambda = (\lambda^-, \lambda^+) \in \mathbb{R}^m$ are parameters, $P^\pm(x, y, \lambda^\pm)$ and $Q^\pm(x, y, \lambda^\pm)$ are convergent real series which start at least with quadratic monomials in the variables x and y. We call the system defined in the left half-plane ($x \leq 0$) by the *left system*, and the system defined in the right half-plane ($x > 0$) by the *right system*. This system has the origin as a *monodromic equilibrium point*, that is, there are no orbits tending or leaving the point with a given direction.

We highlight that we only consider *crossing periodic orbits* for the piecewise linear systems (1.3), i.e. the two sides of the periodic orbit crosses transversally the separation line Σ and, at these crossing points, both vector fields points towards the same half-plane. Then, these periodic orbits have only isolated points of intersection with the curve of separation. In particular, when we consider limit cycles for piecewise system we refer to them as *crossing limit cycles*.

With the results presented here one may notice that piecewise planar differential equations have richer dynamics than smooth dynamical systems.

There exist many problems in science where their mathematical models are given by planar piecewise systems whose phase plane is composed by two "uncoupled" smooth systems matched by a straight line, as in the following example where $h(x, y) = x$.

Example 1.24 (Formulation of a Mechanical Problem)**.** Consider the movement without friction of a ball of mass m on a curve under the action of the gravitational potential.

Assume that this curve is given by the function

$$y(x) = \begin{cases} F(x), & \text{if } x \leq 0, \\ G(x), & \text{if } x > 0, \end{cases}$$

where $G'(x) = x + O(2)$, $F'(x) = x + O(2)$, and F and G are analytic at zero, see Figure 1.3.

Let $s \geq 0$ (resp. $s \leq 0$) be the arc length starting at $(0, 0)$ of the curve $y(x) = G(x)$ (resp. $y(x) = F(x)$) (see [31, 55]), where the potential energy is the gravitational potential $V(y) = mgy$ (where m represents the mass and g stands for the gravitational constant). Then, for the left half-plane (resp. right half-plane) the differential equation governing the

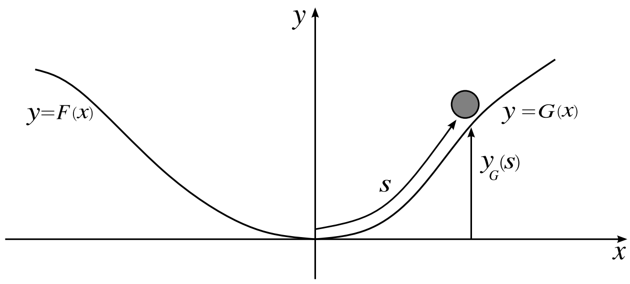

Figure 1.3: Movement of a ball on a curve

movement of the ball is given by a conservative second order scalar differential equation with total energy given by the Hamiltonian function

$$H(s,w)=\frac{m}{2}s^2+V(w), \text{ with } w=\dot{s}.$$

Then, the piecewise differential equation governing the movement of the ball is

$$(\dot{s},\dot{w})=\begin{cases}\left(w,-\frac{1}{m}\frac{dV(y_F(s))}{ds}\right), & \text{if } s\leq 0,\\ \left(w,-\frac{1}{m}\frac{dV(y_G(s))}{ds}\right), & \text{if } s>0.\end{cases}$$

It is well-known that the two smooth vector fields $(\dot{x},\dot{y})=(-y+P^-(x,y,\lambda^-),x+Q^-(x,y,\lambda^-))$ and $(\dot{x},\dot{y})=(-y+P^+(x,y,\lambda^+),x+Q^+(x,y,\lambda^+))$ that determine (1.3) have the origin as a nondegerenate linear center. In this case, there exists a simple way to determine if the origin of the planar piecewise system (1.3) is a center by using reversibility with respect to the separation line.

Definition 1.25. A system (1.3) is *reversible with respect to a straight line* l through O if it is invariant with respect to reflection about l and a reversion of the time t.

From Definition 1.25, it follows that if the system is invariant under the change $(x,y,t)\mapsto(x,-y,-t)$ or $(x,y,t)\mapsto(-x,y,-t)$ we say that it is *time-reversible* with respect to the x-axis or y-axis, respectively. Then, the next proposition is valid.

Proposition 1.26. *The planar piecewise system* (1.3) *has a center at the origin if it is a reversible system, that is invariant with respect to the change of variables* $(x,y,t)\mapsto(x,-y,-t)$.

Suppose that a planar smooth system has a conserved quantity E and the origin is an isolated equilibrium. It is well known that, if the origin is an isolated fixed point and a minimum local of E, then it is a center. However, for a piecewise system we must also require that the energy function of the left and right systems coincide on the separation line as we can see in Proposition 2.1 of [18]. For convenience, we state it below:

Proposition 1.27 ([18]). *If the left system and the right one in* (1.3) *have first integrals* $H^-(x,y)$ *and* $H^+(x,y)$ *near* O*, respectively, and either they are even functions in* y *or they verify* $H^-(0,y) \equiv H^+(0,y)$*, then the origin* O *of* (1.3) *is a center.*

Example 1.28. Consider the class of piecewise Hamiltonian systems

$$(\dot{x}, \dot{y}) = \begin{cases} (y, -x + ax^2), & \text{if } x \leq 0, \\ (y, -x + bx^2), & \text{if } x > 0, \end{cases}$$

where $(a,b) \in \mathbb{R}^2$. For all $(a,b) \in \mathbb{R}^2$, the Hamiltonian function is given by

$$H(x,y) == \begin{cases} H^-(x,y) = \dfrac{y^2 + x^2}{2} - a\dfrac{x^3}{3}, & \text{if } x \leq 0, \\ H^+(x,y) = \dfrac{y^2 + x^2}{2} - b\dfrac{x^3}{3}, & \text{if } x > 0. \end{cases}$$

Note that O is a nondegenerate center for the left and right systems. Furthermore, $H^\pm(0,y) = H^\pm(0,y) = y^2/2$, for every $(0,y) \in \Sigma$, then by Proposition 1.27 the origin is a center. In addition, from Definition 1.12, it is a continuous piecewise Hamiltonian system.

Note that it is not so easy, for any type of planar piecewise system, to find a solution for the center problem, that is, determining conditions for O to be a center is quite different from the case of analytic systems. For example, in the next system, the origin is a center even if the systems that define it do not have a center at the origin.

Example 1.29. Consider the following piecewise system

$$(\dot{x}, \dot{y}) = \begin{cases} (2y + 4y^3, 3x^2), & \text{if } x \leq 0, \\ (2y + 4y^3, -3x^2), & \text{if } x > 0. \end{cases} \tag{1.4}$$

The origin of (1.4) is a center, by Proposition 1.26, because there is invariant by the change of variables $(x,y,t) \to (x,-y,-t)$. However, the phase portrait near the origin of the left and the right systems are not of center type, as indicated in Figure 1.4 (*i*) and (*ii*), respectively. In fact, the origin is the only singular point of cusp type for both systems (it can be proved by using the blow-up thecnique). But the origin is a center of (1.4), as we see in Figure 1.4 (*iii*).

On the other hand, if the origin is a center for both systems, we cannot ensure that the piecewise system has a center at the origin. The next two examples (given as propositions in [68]) reinforce this statement.

Example 1.30 ([68]). Consider the piecewise potential Hamiltonian system given by

$$(\dot{x}, \dot{y}) = \begin{cases} (-y, V_1'(x)), & \text{if } y \geq 0, \\ (-y, V_2'(x)), & \text{if } y < 0, \end{cases} \tag{1.5}$$

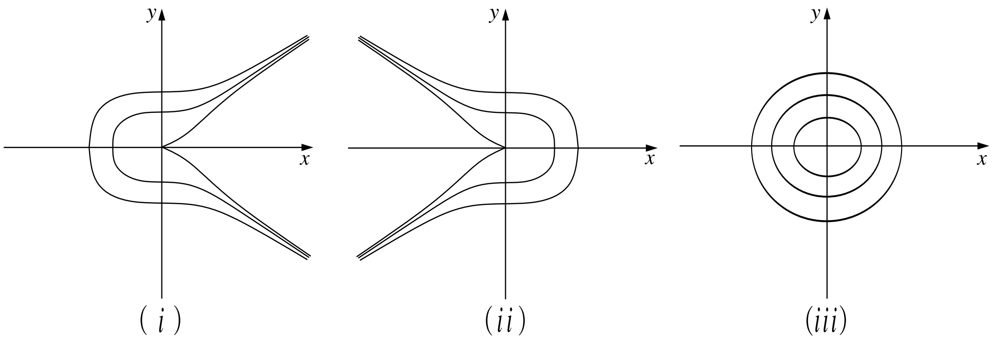

Figure 1.4: (*i*) phase portrait of the left system of (1.4), (*ii*) phase portrait of the right system of (1.4), and (*iii*) phase portrait of (1.4)

where the analytic potentials are $V_1 = ax^2 + O(x^2)$ and $V_2 = bx^2 + O(x^2)$, with $a > 0$ and $b > 0$. This system has a center at the origin if and only if there exists an analytic diffeomorphism g at the origin such that $V_2 = g(V_1)$.

Example 1.31 ([68])**.** Let F be an analytic function at 0 with $F(0) = F'(0) = 0$ and $|F''(0)| < 1$. The piecewise potential Hamiltonian system given by

$$(\dot{x}, \dot{y}) = \begin{cases} (-y, x + F'(x)), & \text{if } y \geq 0, \\ (-y, x - F'(x)), & \text{if } y < 0, \end{cases} \tag{1.6}$$

is a center if and only if F is even.

In fact, for the piecewise systems (1.5) and (1.6) we need additional properties to ensure that the origin is a center when considering the union of two systems with a center.

1.4 Return map and period function

Some efforts were made to the center problem for planar piecewise system with a separation line (see [16, 31, 33, 52, 62]). This study is usually done, as follows, by finding the two half-return maps associated with the two smooth differential equations that define them and the whole return map is the composition of these two maps.

Remark 1.32. In what follows we consider $\lambda^- \in \mathbb{R}^j$ and $\lambda^+ \in \mathbb{R}^k$, so that $j + k = m$ is the dimension of the parameter space.

Definition 1.33 ([31])**.** Consider the planar piecewise differential system (1.3).

(i) The function $\Pi^- : (0, \alpha) \times \mathbb{R}^j \to \mathbb{R}^-$, where $(0, \alpha) \subset \mathbb{R}^+$ is an interval on the semi-axis OY^+, which gives, for each point $(0, y) \in \mathbb{R}^2$ with $0 < y < \alpha$ and fixed parameter $\lambda^- \in \mathbb{R}^j$, the first intersection, in positive time, of the orbit that passes through $(0, y)$ at $t = 0$ with the semi-axis OY^-, is called *left half-return map* associated to (1.3).

(ii) The function $\Pi^+ : (\beta, 0) \times \mathbb{R}^k \to \mathbb{R}^+$, where $(\beta, 0) \subset \mathbb{R}^-$ is an interval on the semi-axis OY^-, which gives, for each point $(0, y) \in \mathbb{R}^2$ with $\beta < y < 0$ fixed parameter and $\lambda^+ \in \mathbb{R}^k$, the first intersection, in positive time, of the orbit that passes through $(0, y)$ at $t = 0$ with the semi-axis OY^+, is called *right half-return map* associated to (1.3).

Hence, the *return map* is defined as $(\rho, \lambda) \mapsto \Pi(\rho, \lambda)$, where

$$\Pi(\rho, \lambda) = \Pi^+(\Pi^-(\rho, \lambda^-), \lambda^+), \text{ see Figure 1.5.} \tag{1.7}$$

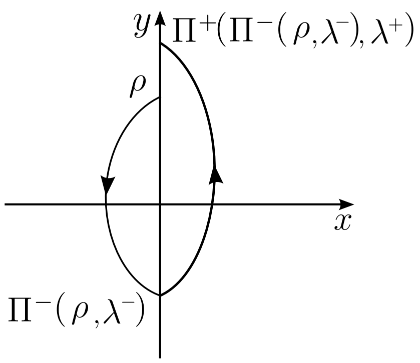

Figure 1.5: Return map $\Pi(\rho, \lambda)$ for system (1.3)

The *displacement function* is given by

$$d(\rho, \lambda) = \Pi(\rho, \lambda) - \rho, \tag{1.8}$$

where $d(0, \lambda) = 0$, for all $\lambda \in \mathbb{R}^m$, since both vector fields that define (1.3) have a equilibrium at the origin and we are avoiding sliding motion. Therefore, the origin is a fixed point of d.

It is easy to conclude that the origin is a center of system (1.3) if and only if $d(\rho, \lambda) = 0$, for $0 < \rho \ll 1$. The isolated zeros of $d(\rho, \lambda) = 0$ near $\rho = 0$ correspond to limit cycles around the origin. The analyticity of $P^\pm$ and $Q^\pm$ implies that the half-return maps $\Pi^\pm$ are analytic, for sufficiently small $|\rho|$, and then the displacement function is analytic and so it can be expanded as

$$d(\rho, \lambda) = V_1\rho + V_2\rho^2 + \cdots, \tag{1.9}$$

where V_i is called the *i-th Lyapunov constant* of the piecewise system (1.3). For $k \geq 1$, the Lyapunov constant V_k belongs to the ideal $\langle V_1, V_2, V_3, \ldots, V_{k-1}\rangle$ over the ring $\mathbb{R}\{\lambda_1, \lambda_2, \ldots, \lambda_m\}_{\lambda_0}$, for each $\lambda_0 \in \mathbb{R}^N$. Therefore, when an expression for V_k is given, it makes sense only when $V_1 = V_2 = \cdots = V_{k-1} = 0$, and the origin is a center if and only if all the Lyapunov constants in (1.9) vanish.

Note that as we consider just half-returns for determining the displacement function (1.8) for nondegenerate equilibrium points of planar piecewise systems, the expressions found for the Lyapunov constants are even larger than the ones obtained in smooth vector fields. In fact, in the smooth case there appear some cancellations related to considering the flow that gives a complete turn around the origin.

It is worth remembering that in planar smooth differential systems the stability of a nondegenerate weak focus (equilibrium point with complex eigenvalues) is given by the sign of the trace, if it is different from zero. In planar piecewise systems this stability is reduced to the computation of the Lyapunov constants (see [3]). Furthermore the first nonzero constant allows us to find bounds for the maximum number of small periodic orbits which appear from this equilibrium point via a degenerate Hopf bifurcation.

For smooth systems the first nonzero V_j occurs for odd j, and the order of a weak focus O is defined as k if the first nonzero Lyapunov constant is V_{2k+1} (see [2] for instance). But this is not true for piecewise systems, because this first nonzero constant can be any natural number. In order to investigate the center-focus and cyclicity problems for (1.3) we have also to consider the even constants, and the order of a weak focus is defined as in the following definition.

Definition 1.34. The origin is a *weak focus of order k* of (1.3) if $V_1 = V_2 = \cdots = V_k = 0$ and $V_{k+1} \neq 0$.

Thus, if we have a weak focus of order k, the displacement function can be expanded as

$$d(\rho, \lambda) = V_{k+1}\rho^{k+1} + V_{k+2}\rho^{k+2} + \cdots .$$

Remark 1.35. The center problem can be investigated by computing the Lyapunov constants and finding algebraic equalities satisfied by these coefficients, see more details in [2]. As indicated in [31, 52], at most k limit cycles bifurcate from a focus of order k of (1.3) and this number can be attained. A difference between (1.3) and a smooth system is that it is possible to generate k limit cycles only from $V_1, V_2, \ldots, V_{k+1}$ while for a smooth system, $V_1, V_3, \ldots, V_{2k+1}$ must be used.

According to Remark 1.18, the *period function* can be defined as $T : (0, \alpha) \to \mathbb{R}^+$, where $(0, \alpha)$ is an interval of the semi-axis OY^+ and for each point $(0, \rho) \in \mathbb{R}^2$ with $0 < \rho < \alpha$, $T(\rho)$ gives the time required for the flow of (1.3) to intercept again the semi-axis OY^+. Then, in order to explicitly determine the period function for planar piecewise systems with a separation line given by (1.3), we need to find the two half-period functions, as follows, associated with the two smooth differential equations that define them and then consider their sum.

Definition 1.36 ([31])**.** Consider the planar differential system (1.3) with a fixed parameter $\lambda = (\lambda^-, \lambda^+) \in \mathbb{R}^m$.

(i) The function $T^- : (0, \alpha) \times \mathbb{R}^j \to \mathbb{R}^-$, where $(0, \alpha) \subset \mathbb{R}^+$ is an interval on the semi-axis OY^+, which gives, for each point $(0, y) \in \mathbb{R}^2$ with $0 < y < \alpha$ and fixed parameter $\lambda^- \in \mathbb{R}^j$, the smallest positive time required to the orbit that passes through $(0, y)$ at $t = 0$ to reach the semi-axis OY^-, is called *left half-period function* associated to (1.3).

(ii) The function $T^+ : (\beta, 0) \times \mathbb{R}^k \to \mathbb{R}^+$, where $(\beta, 0) \subset \mathbb{R}^-$ is an interval on the semi-axis OY^-, which gives, for each point $(0, y) \in \mathbb{R}^2$ with $\beta < y < 0$ and fixed parameter $\lambda^+ \in \mathbb{R}^k$, the smallest positive time required to the orbit that passes through $(0, y)$ at $t = 0$ to reach the semi-axis OY^+, is called *right half-period function* associated to (1.3).

Hence the *period function* is defined as $(\rho, \lambda) \mapsto T(\rho, \lambda)$, where

$$T(\rho, \lambda) = T^-(\rho, \lambda^-) + T^+(\Pi^-(\rho, \lambda^-), \lambda^+). \tag{1.10}$$

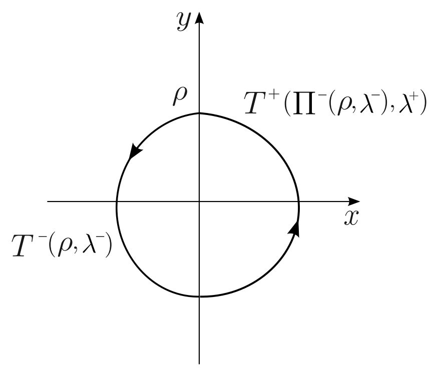

Figure 1.6: Period function $T(\rho, \lambda)$

Remark 1.37. Using the fact that the solution of a differential equation is smoothly dependent on its initial value and by using the Implicit Function Theorem, it can be proved that Π^-, Π^+, T^-, and T^+ are smooth functions, then Π and T are also smooth.

As in the center problem, the discussion on isochronicity for piecewise systems is more complicated than those for smooth systems. For example, both systems defined in the two half-planes can have an isochronous center at the origin, but O may not be an isochronous center of the piecewise system. The next example can be found in [18].

Example 1.38. The planar piecewise system given below is non-isochronous

$$(\dot{x}, \dot{y}) = \begin{cases} (-y - x^2, x - 4xy), & \text{if } y \geq 0, \\ (-y, x), & \text{if } y < 0, \end{cases} \tag{1.11}$$

but the two systems that define it have an isochronous center. In fact, one is the linear isochronous center and the other is equivalent to the quadratic isochronous center classified in [15]. And the period function is either equal to π for the system in the lower half-plane and $4\arctan(\rho + \sqrt{1+\rho^2})$ (to be proved in the last equality of equation (3.27) of Section 3.4) for the system in the upper half-plane. Then, the period function is given by $\pi + 4\arctan(\rho + \sqrt{1+\rho^2})$, for all ρ in the domain, then the origin is not an isochronous center of (1.11).

In what follows, we use the symbol $f'(\rho, \lambda)$ in order to denote the derivative of a given function f with respect to the first variable ρ.

Definition 1.39. A ρ_0 for which $T'(\rho_0, \lambda) = 0$ is called a *critical period.* In addition, if ρ_0 is a simple zero of T', that is $T''(\rho_0, \lambda) \neq 0$, it is called a *simple critical period* or *hyperbolic critical period.*

For each fixed parameter $\lambda_0 = (\lambda_0^-, \lambda_0^+)$, the *period function* (1.10) can be indicated as a power series

$$T(\rho, \lambda) = 2\pi + \sum_{j=1}^{\infty} T_j(\lambda)\rho^j, \tag{1.12}$$

for $|\rho|$ and $|\lambda - \lambda_0|$ sufficiently small, where $T_j(\lambda) \in \mathbb{R}\{\lambda_1, \lambda_2, \ldots, \lambda_m\}_{\lambda_0}$, the ring of convergent power series at λ_0, and $T_j(\lambda)$ are known as the *period constants* of the center. These period constants are homogeneous polynomials in the parameters of the systems ([29]), and they can be found by the procedure we shall describe and use in Section 3.1, together with a computer algebra system.

From [21] one can conclude that the period constants for planar quadratic differential systems are polynomials in the components of the bifurcation parameter λ. This is also true for any planar polynomial system for both period and Lyapunov constants, see for example [29] or [84]. Furthermore, it is well known that for analytic systems the first nonzero T_j occurs always for even j and the order of a weak center O is defined as $k \geq 0$ if the first nonzero period constant is T_{2k+2}. For the piecewise case we will consider the order of a weak center given by the next definition:

Definition 1.40. The origin is a *weak center of finite order k* of the system (1.3) for the parameter $\lambda = \lambda_0$ if

$$T'(0, \lambda_0) = T''(0, \lambda_0) = \cdots = T^{(k)}(0, \lambda_0) = 0, \text{ but } T^{(k+1)}(0, \lambda_0) \neq 0,$$

where the derivatives are taken with respect to the first variable of T. If $T^{(k+1)}(0, \lambda_0) = 0$, for all k, the origin is a *weak center of infinite order*.

Note that, if the origin is a weak center of infinite order, we have that $T(\rho, \lambda_0)$ is constant for all ρ, so the origin is an isochronous center. Actually, a consequence of (1.12) is the following characterization for isochronicity: the origin is an isochronous center if and only if $T_j = 0$, for all $j \in \mathbb{N}$. Then, the period constants play the same role when studying isochronicity as Lyapunov constants when characterizing centers.

1.5 Bifurcation of critical points of the period function

In this section we intend to formalize some definitions and results for the general piecewise system (1.3) and show how we study the bifurcation of critical points of the period function $\rho \mapsto T(\rho, \lambda)$ for that piecewise system. That is, roughly speaking, we deal with the solutions of the equation $T'(\rho, \lambda) = 0$, near $\rho = 0$, as the parameter λ varies. We basically present here what was done for smooth vector fields in [21] with some adaptations.

Before the study of the bifurcation of critical points of T, it is worth remembering here the next two theorems, which are proved in many textbooks, see for instance [96].

Theorem 1.41 (Implicit Function Theorem)**.** *Let $f_j(x_1, \ldots, x_m, y_1, \ldots, y_n)$ $(1 \leq j \leq n)$ be smooth functions of the variables indicated, such that at $O = (0, \ldots, 0)$ we have each $f_j(O) = 0$ and the matrix $J_y f := \left(\frac{\partial f_j}{\partial y_k}(O)\right)$ is nonsingular. Then, in some neighborhood of O, there exist unique functions $h_j(x_1, \ldots, x_m)$, with $h_j(0, \ldots, 0) = 0$, such that $f_j(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0$, for each j if and only if $y_j = h_j(x)$, for each j.*

In particular, if $f(x, y)$ is such that $f(0, 0) = 0$ and $\frac{\partial f}{\partial y}(0, 0) \neq 0$, there exists a unique function $h(x)$ with $h(0) = 0$ such that, in some neighborhood of $(0, 0)$, $f(x, y) = 0$ if and only if $y = h(x)$.

Theorem 1.42 (Weierstrass Preparation Theorem)**.** *Let $F(x, \lambda)$ be an analytic function with $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}^n$ in a neighborhood of the origin. Let k be the smallest natural number such that*

$$F(0,0) = 0, \quad \frac{\partial F(0,0)}{\partial x} = 0, \quad \ldots, \quad \frac{\partial^{k-1} F(0,0)}{\partial^{k-1} x} = 0, \quad \frac{\partial^k F(0,0)}{\partial^k x} \neq 0.$$

Then in some neighborhood $|x| < \varepsilon$, $|\lambda| < \delta$ of the point $(0, 0)$ the function $F(x, \lambda)$ can be represented as

$$F(x, \lambda) = \left(x^k + A_{k-1}(\lambda)x^{k-1} + \cdots + A_1(\lambda)x + A_0(\lambda)\right)\phi(x, \lambda),$$

where $\phi(x, \lambda)$ is an analytic function not equal to zero in the chosen neighborhood and $A_1(\lambda), \ldots, A_k(\lambda)$ are analytic functions for $|\lambda| < \delta$.

From Theorem 1.42, it follows that the equation $F(x,\lambda) = 0$, in a sufficiently small neighborhood of the point $(0,0)$, is equivalent to the equation

$$x^k + A_{k-1}(\lambda)x^{k-1} + \cdots + A_1(\lambda)x + A_0(\lambda) = 0,$$

whose left-hand side is a polynomial with respect to x. Thus, the Weirstrass Preparation Theorem reduces the study of zeros of an analytic function $F(x,\lambda)$ to the study of the number of zeros of a polynomial of degree k.

Remark 1.43. By Remark 1.37, the period function of (1.3) is an analytic function, and so is $T'(\rho,\lambda)$.

For the analytic function $(\rho,\lambda) \mapsto T'(\rho,\lambda)$, we write its expansion in series using the Taylor series of T in (1.12). Then, near $\rho = 0$, $T'(\rho,\lambda) = T_1(\lambda)+2T_2(\lambda)\rho+3T_3(\rho,\lambda)\rho^2+\cdots$, where each function $\lambda \mapsto T_i(\lambda)$ is analytic and, for each λ, the series is convergent in some neighborhood of $\rho = 0$. Given a point λ_0 where $T'(0,\lambda_0) = 0$ $(T_1(\lambda_0) = 0)$, we wish to know how many zeros the function $\rho \mapsto T'(\rho,\lambda)$ has near $\rho = 0$ for perturbations λ of λ_0.

The bifurcation parameter λ_0 has *finite order* k if the origin is a weak center of order k for $\lambda = \lambda_0$ and, consequently, from Definition 1.40, we have $T_1(\lambda_0) = \cdots = T_k(\lambda_0) = 0$ and $T_{k+1}(\lambda_0) \neq 0$, and the parameter λ_0 has *infinite order* if the origin is a weak center of infinite order for $\lambda = \lambda_0$, and then $T_i(\lambda_0) = 0$ for all $i \geq 1$.

Definition 1.44. Let (ρ_0,λ_0) be a critical period which arises from a bifurcation of a weak center. The period $T(\rho_0,\lambda_0)$ is called a *local critical period.*

It remains, therefore, to single out what it means that there exist k local critical periods bifurcating from a weak center of finite order at the origin.

Definition 1.45. Consider the vector field (1.3) for which the center at the origin corresponding to the parameter value λ_0 is a weak center of order k. We say that k local critical periods bifurcate from the weak center if, for every $\varepsilon > 0$, sufficiently small, there exists a neighborhood W of λ_0 such that, for any $\lambda \in W$, $T(\rho,\lambda)$ has at most k critical points in $\mathcal{U} := (0,\varepsilon)$. Moreover, any neighborhood W of λ_0 contains a point $\lambda_1 \in W$ such that $T'(\rho,\lambda_1) = 0$ has k solutions in $\mathcal{U}$.

Remark 1.46. It is clear that we are counting only the positive solutions of $T'(\rho,\lambda) = 0$, because the nontrivial zeros of T' are the positive and negative intersection of the critical periodic orbit with the y-axis.

Note that the next lemma is an immediate consequence of the Weierstrass Preparation Theorem (Theorem 1.42).

Lemma 1.47. *Consider the planar piecewise differential system* (1.3) *with a weak center at the origin O, corresponding to a parameter value λ_0. If the weak center has order k, then no more than k local critical periods bifurcate from this weak center at the parameter value λ_0.*

Proof. If λ_0 is a finite parameter of order k, i.e. the origin is a weak center of order k, then at most k zeros bifurcate from $(0, \lambda_0)$. This is equivalent to the proof for the cyclicity of limit cycles of finite codimension (see [86]). In this case the period function becomes $T'(\rho, \lambda_0) = (k+1)T_{k+1}(\lambda_0)\rho^k + O(\rho^{k+1})$, and it follows from Weierstrass Preparation Theorem that there exist functions $U(\rho, \lambda)$, with $U(0, \lambda_0) \neq 0$ and $\alpha_0(\lambda), \dots, \alpha_{k-1}(\lambda)$ in neighborhoods of $(0, \lambda_0)$ and λ_0, respectively, such that

$$T'(\rho, \lambda) = U(\rho, \lambda)\left(\rho^k + \sum_{j=0}^{k-1} \alpha_j(\lambda)\rho^j\right).$$

It follows that the study of zeros of $T'(\rho, \lambda) = 0$ has been changed into the study of the number of zeros of a polynomial equation of degree k

$$\rho^k + \sum_{j=1}^{k-1} \alpha_j(\lambda)\rho^j = 0,$$

and, consequently, it has at most k zeros. Then, the number of critical periods bifurcating from a weak center of order k is k. □

Remark 1.48. It is worth mentioned that in a piecewise system, we can use the Weierstrass Preparation Theorem to determine the highest lower bound for criticality, but in the analytical case the proof has to be performed by using another method, since the ideal generated by all period constants is equals to the ideal generated only by those period constants of even orders ([29]).

The case of a weak center of infinite order, i.e. an isochronous center, is much more delicate and the following remark can be useful for solving the bifurcation problem in a neighborhood of the isochronous center corresponding to a parameter value λ_0.

Remark 1.49. As the coefficients $T_k(\lambda)$ of the period function (1.12) are in the Noetherian ring $\mathbb{R}\{\lambda_1, \lambda_2, \dots, \lambda_m\}_{\lambda_0}$ of convergent power series at λ_0, the ideal generated by all the Taylor coefficients, which we will denote by I, is finitely generated. That is,

$$I = \langle T_1, \dots, T_k \rangle = \left\{ \sum_{j=1}^{k} h_j T_j : h_1, \dots, h_k \in \mathbb{R}[\lambda_1, \lambda_2, \dots, \lambda_m] \right\},$$

for some k. Then, I is called the *ideal generated by the polynomials* $T_1, \dots, T_k$, which are called *generators* or *basis*. For finding such k generators, these coefficients are tested successively in order to detect if all element of the sequence is already in the ideal generated by its predecessors. This can be made, for example, by using mathematical algorithms for checking ideal membership which rely on the computation of a Gröbner basis for the ideal, see more details in [84]. In this case the origin has an isochronous center if $T_1 = \cdots = T_k = 0$.

Lemma 1.50. *Consider the planar piecewise differential system* (1.3) *with an isochronous center at the origin O, corresponding to a parameter value λ_0. If all the Taylor coefficients of the function $T'(\rho, \lambda_0)$ are in the ideal $\langle T_1, \ldots, T_k \rangle$ over $\mathbb{R}\{\lambda_1, \lambda_2, \ldots, \lambda_m\}_{\lambda_0}$, then there are at most $k-1$ local critical periods which bifurcate from the isochronous center at λ_0.*

Proof. If all the coefficients are generated by $\langle T_1, T_2, \ldots, T_k \rangle$, the functions T_i, for $i > k$, are written in terms of the functions in these initial segments, i.e. $T_i = \alpha_{i1} T_1 + \alpha_{i2} T_2 + \cdots + \alpha_{ik} T_k$, for $i > k$, and then

$$T'(\rho, \lambda) = \sum_{i=1}^{k} T_i(\lambda) \rho^{i-1} \big(1 + \phi(\rho, \lambda)\big),$$

where $\phi(0, \lambda) = 0$, for $i = 0, 1, 2, \ldots, k$. Thus, the bifurcation function behaves as a polynomial of degree $k-1$ near λ_0 and at most $k-1$ zeros near $\rho = 0$ bifurcate for values of λ near λ_0. □

Then, the problem of bifurcation from a zero is reduced to obtaining the smallest value k such that the corresponding initial segment $\langle T_1, T_2, \ldots, T_k \rangle$ is a basis for the ideal of all Taylor coefficients for the expansion of the period function (see Remark 1.49).

Remark 1.51. Note that it is possible to generate k critical periods bifurcating from a weak center of order k for (1.3) only by using $T_1, T_2, \ldots, T_{k+1}$. Now, for smooth systems, $T_2, T_4, \ldots, T_{2k+2}$ must be used. This is analogous to the statement made for limit cycles in Remark 1.35. For example, consider a smooth planar quadratic system. Given a point λ_0 for which $T'(0, \lambda_0) = 0$, the maximum number of critical periods that can bifurcate from $(0, \lambda_0)$ is 2 using $T_2(\lambda_0), T_4(\lambda_0)$, and $T_6(\lambda_0)$, see [21]. While for the piecewise system (1.3), it is enough to use $T_1(\lambda_0), T_2(\lambda_0)$, and $T_3(\lambda_0)$ in order to obtain the same number of critical periods bifurcating from the origin. Then, it is expected that the piecewise system can have the double of the number of local critical periods than in the smooth case.

1.6 A result on the criticality of isochronous centers

Since it is not always possible to determine k such that $\langle T_1, T_2, \ldots, T_k \rangle$ is the basis of all period constants, other techniques for determining the criticality of isochronous centers have been developed. In general, these results are based on the results used for solving the cyclicity problem. That is, fix a class of systems of type (1.3) and determine the maximum number of limit cycles which bifurcate from the origin under the variation of the parameters inside this class of systems.

In this section we present the technique that can be used to increase the number of critical periods with respect to the bounds obtained by linear developments. These results are introduced in [58] and they are better developed in [54, 56] for studying cyclicity in families of centers. Such results were applied in [88] to find the highest number of critical periods in a class of planar smooth systems of polynomial differential equations for fixed degree having a center.The authors stated and proved the next results. We judge convenient to write their proofs here for a better understanding of the proofs of the results presented in Sections 3.3 and 3.4.

We consider a family of isochronous centers with some parameters, and add a perturbation that keeps the center property. The following proposition shows the structure of the first order terms of the period constants for a perturbed family of isochronous centers.

Proposition 1.52 ([88]). *Consider a polynomial family of isochronous centers parametrized by $A \in \mathbb{R}^P$, for some $P \in \mathbb{N}$, and add a polynomial perturbation with coefficients $\lambda \in \mathbb{R}^N$, for some $N \in \mathbb{N}$, which does not break the center property.*

(i) The k-th period constant T_k of the perturbed system is a polynomial on the perturbative parameters of the form

$$T_k = \sum_{j=1}^{N} g_k^{(j)}(A)\lambda_j + O_2(\lambda_1, \lambda_2, \dots, \lambda_N), \tag{1.13}$$

where $g_k^{(j)}(A)$ are polynomials in A which are the coefficients of the linear part of T_k with respect to λ and $O_2(\lambda_1, \lambda_2, \dots, \lambda_N)$ denotes a sum of monomials of degree at least 2 on the parameters.

(ii) The matrix of coefficients of linear parts of the first m period constant is the $m \times m$ matrix $G_m(A)$, whose element in position (i, j) is $g_i^{(j)}(A)$ in (1.13). If $\det G_N(A)=0$ and $\det G_{N-1}(A) \neq 0$, there exists a linear change of variables such that the first $N-1$ first period constants take the form

$$T_k = u_k + O_2(u_1, u_2, \dots, u_N), \tag{1.14}$$

for $k = 1, \dots, N-1$, where the linear part of T_k is u_k, $u_N := \lambda_N$, and we denote the higher order terms by $O_2(u_1, u_2, \dots, u_N)$.

(iii) Under the assumptions of (ii), the first $N+M$ period constants for some $M \in \mathbb{N}$ can be written as

$$T_k = \begin{cases} v_k, & \text{if } k = 1,\ldots,N-1, \\ \sum_{j=1}^{N-1} \tilde{g}_k^{(j)}(A)v_j + f_{k-N}(A)u_N + O_2(v,u_N), & \text{if } k = N,\ldots,N+M, \end{cases} \tag{1.15}$$

where $v = (v_1,\ldots,v_{N-1})$ are new variables, $f_{k-N}(A)$ and $\tilde{g}_k^{(j)}(A)$ are the corresponding coefficients of $v_1, v_2, \ldots, v_{N-1}, u_N$, which are rational functions in $A \in \mathbb{R}^P$, and $O_2(v,u_N)$ are analytical functions of order two in $v_1,\ldots,v_{N-1},u_N$.

Proof. (i) Recall that the period constants are polynomials in the parameters of the system. Because the parameter A does not break the isochronicity of the system, they cannot appear isolated, then when considering the power series expansion of the period constants T_k, its linear part must be a linear combination of the perturbative parameters λ with the coefficients being polynomials in A.

(ii) Consider the system of N equations $\sum_{j=1}^{N} g_k^{(j)}\lambda_j = u_k$. Then, we can write it as a system with unknowns $\lambda_1,\ldots.\lambda_{N-1}$ of the form

$$\begin{array}{ccccccccl} g_1^{(1)}(A)\lambda_1 & + & g_1^{(2)}(A)\lambda_2 & + & \cdots & + & g_1^{(N-1)}(A)\lambda_{N-1} & = u_1 - g_1^{(N)}(A)\lambda_N, \\ g_2^{(1)}(A)\lambda_1 & + & g_1^{(2)}(A)\lambda_2 & + & \cdots & + & g_2^{(N-1)}(A)\lambda_{N-1} & = u_2 - g_2^{(N)}(A)\lambda_N, \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ g_{N-1}^{(1)}(A)\lambda_1 & + & g_{N-1}^{(2)}(A)\lambda_2 & + & \cdots & + & g_{N-1}^{(N-1)}(A)\lambda_{N-1} & = u_N - g_{N-1}^{(N)}(A)\lambda_N. \end{array} \tag{1.16}$$

By applying the Cramer's rule to this system, since $\det G_{N-1}(A) \neq 0$, we have that the system of equations (1.16) has a unique solution which determines a linear change of variables that proves (1.14). By using this method, it is clear that the coefficients which define the change of variables are rational functions in A.

(iii) By using (1.14) define the new variables $v_1, v_2, \ldots, v_{N-1}, v_N$, where

$$v_k = h_k(\mathbf{u}) := u_k + O_2(u_1,\ldots,u_N), \qquad \text{for } k = 1\ldots,N-1,$$

$h_N(\mathbf{u}) := u_N$ (then $v_N = u_N$) and $\mathbf{u} = (u_1,\ldots,u_N)$ (note that $h_k(\mathbf{u}) = T_k$). We observe that these new variables are related to the coordinates $u_1,\ldots,u_{N-1},u_N$ by the equation $F = 0$, where

$$F(v_1,v_2,\ldots,v_{N-1},v_N,u_1,u_2,\ldots,u_{N-1},u_N) = (h_1(\mathbf{u})-v_1,\ldots,h_{N-1}(\mathbf{u})-v_{N-1},h_N(\mathbf{u})-v_N).$$

The Jacobian matrix of F in every $(\hat{\mathbf{v}},\hat{\mathbf{u}}) = (\hat{v}_1,\hat{v}_2,\ldots,\hat{v}_{N-1},\hat{v}_N,\hat{u}_1,\hat{u}_2,\ldots,\hat{u}_{N-1},\hat{u}_N)$ is given by

$$DF(\hat{\mathbf{v}},\hat{\mathbf{u}}) = \left[\begin{array}{cccccccc} -1 & \ldots & 0 & 0 & \frac{\partial h_1}{\partial u_1}(\hat{\mathbf{u}}) & \ldots & \frac{\partial h_1}{\partial u_{N-1}}(\hat{\mathbf{u}}) & \frac{\partial h_1}{\partial u_N}(\hat{\mathbf{u}}) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & -1 & 0 & \frac{\partial h_{N-1}}{\partial u_1}(\hat{\mathbf{u}}) & \ldots & \frac{\partial h_{N-1}}{\partial u_{N-1}}(\hat{\mathbf{u}}) & \frac{\partial h_{N-1}}{\partial u_N}(\hat{\mathbf{u}}) \\ 0 & \ldots & 0 & -1 & \frac{\partial h_N}{\partial u_1}(\hat{\mathbf{u}}) & \ldots & \frac{\partial h_N}{\partial u_{N-1}}(\hat{\mathbf{u}}) & \frac{\partial h_N}{\partial u_N}(\hat{\mathbf{u}}) \end{array}\right] = [-I_N|J],$$

where I_N denotes the identity matrix $N \times N$ and J is the matrix $N \times N$ of the partial derivatives of the $h_k's$. Since $u_1, \ldots, u_{N-1}$ are independent, then the matrix J has rank N and the Implicit Function Theorem 1.41 can be applied to write $u_1, \ldots, u_{N-1}$ as functions of $v_1, \ldots, v_{N-1}, v_N = v_1, \ldots, v_{N-1}, u_N$. That is,

$$u_k = F_k(v_1, \ldots, v_{N-1}, u_N), \text{ for } k = 1, \ldots, N-1, \tag{1.17}$$

for some real function F_k and $F_N(v_1, \ldots, v_{N-1}, u_N) \equiv u_N$ from the uniqueness. Then, by applying (1.13) from part (i) and (1.14) from part (ii), together with the change given by (1.17), the period constants take the form (1.15), where $\tilde{g}_{N+d}^{(j)}(A)$ and $f_d(A)$, for $d = 0, \ldots, M$ and $j = 1, \ldots, N-1$, are the corresponding coefficients of $v_1, v_2, \ldots, v_{N-1}, u_N$, respectively, and they are functions of $A \in \mathbb{R}^M$, and each $O_2(v, u_N)$ is an analytical function of order at least two in $v_1, v_2, \ldots, v_{N-1}, u_N$ due to the application of the Implicit Function Theorem. Then, the statement follows. □

Now we can present the aforementioned results which are useful to obtain a better lower bound for the local criticality on a parametrized family of isochronous centers.

Theorem 1.53 ([88]). *We consider a polynomial family of isochronous centers parametrized by $A \in \mathbb{R}^P$, for some $P \in \mathbb{N}$, and a polynomial perturbation with coefficients $\lambda \in \mathbb{N}$, for some $N \in \mathbb{N}$, which does not break the center property. We denote by $G_m(A)$ the $m \times m$ matrix as defined in Proposition 1.52.*

(i) If there exists $A^* \in \mathbb{R}^P$ such that $\det G_N(A^*) \neq 0$, then the linear parts of the first period constants have rank N and at least $N-1$ simple critical periods can bifurcate.

(ii) If there exists $A^* \in \mathbb{R}^P$ such that $\det G_N(A^*) = 0$, $\det G_{N-1}(A) \neq 0$, $f_i(A^*) = 0$, for $i = 0, \ldots, M-1$, $f_M(A^*) \neq 0$ (where $f_0, \ldots, f_M$ are those defined in (1.15)) and the Jacobian determinant satisfies $J(A^*) := \det \mathrm{Jac}_{(f_0, \ldots, f_{M-1})}(A^*) \neq 0$, then M extra critical periods can bifurcate, which leads to a total of at least $N + M - 1$ critical periods.

Proof. (i) By the hypothesis the matrix of the coefficients of the linear parts of the first N period constants $G_N(A^*)$ has determinant nonzero, and we can apply the results obtained in Proposition 1.52 (ii) to obtain a change of variables of N new independent variables $u_1, \ldots, u_N$ such that the first period constants are written as

$$T_i = u_i + O_2(u_1, \ldots, u_N), \quad i = 1, \ldots, N.$$

Using the Implicit Function Theorem 1.41, it is clear that we can write $T_i = v_i$, for $i = 1, \ldots, N$. Then, the first N coefficients of the period function are independent and this implies the existence of a curve, in the parameter space, of weak centers of order $N-1$, or the ideal of all Taylor coefficients for the expansion of the period function has at least N generators (see Remark 1.49). By applying Weierstrass Preparation Theorem 1.42, as described in Section 1.5, at least $N-1$ critical periods can bifurcate near such a curve.

(ii) About the condition $\det G_{N-1}(A^*) \neq 0$, for some $A^* \in \mathbb{R}^P$, we can apply Proposition 1.52 (iii) and write the first $N+M$ period constants as (1.15). If we consider the problem in the manifold $\{v_1 = v_2 = \cdots = v_{N-1} = 0\}$, the structure becomes

$$T_k = \begin{cases} 0, & \text{if } k = 1, \ldots, N-1, \\ \sum_{j=1}^{N-1} u_N \left(f_{k-N}(A) + \sum_{l=1}^{\infty} f_{k-N}^{(l)}(A) u_N^l \right), & \text{if } k = N, \ldots, N+M, \end{cases}$$

for some functions $f_d^{(l)}(A)$ with $d = 0, \ldots, M$. As by assumption there exists $A^* \in \mathbb{R}^P$ such that the Jacobian determinant $J(A^*) \neq 0$, the Implicit Function Theorem guarantees that in a neighborhood of $A = A^*$ and $u_N = 0$, the following change of variables can be performed in $T_N, \ldots, T_{N+M-1}$:

$$v_{N+k} = f_k(A) + \sum_{l=1}^{\infty} f_k^{(l)}(A) u_N^l, \text{ if } k = 0, \ldots, M-1.$$

As we suppose that $f_i(A^*) = 0$, for $i = 0, \ldots, M-1$, but $f_M(A^*) \neq 0$, we rewrite

$$T_{N+k} = \begin{cases} u_N v_{N+k}, & \text{if } k = 0, ..., M-1, \\ u_N \left(f_M(A^*) + \sum_{l=1}^{\infty} f_M^{(l)}(A^*) u_N^l \right) =: u_N v_{N+M}, & \text{if } k = M. \end{cases}$$

Finally, again by the Implicit Function Theorem, since we have obtained M new independent variables, we get the existence of M extra critical periods. □

The next corollaries tell us that the same conclusion is also valid when the number of parameters is bigger than or equal to N.

Corollary 1.54 ([88]). *We consider a polynomial family of isochronous centers parametrized by $A \in \mathbb{R}^P$, for some $P \in \mathbb{N}$, and a polynomial perturbation with coefficients $\lambda \in \mathbb{R}^m$, for some $m \in \mathbb{N}$ where $m \geq N$, which does not break the center property. For each $k \leq m$, we denote by $G_k(A)$ the $k \times k$ matrix as defined in Proposition 1.52. If there exists $A^* \in \mathbb{R}^P$ such that $\det G_k(A^*) \neq 0$, then generically at least $k-1$ simple critical periods bifurcate from the origin.*

Proof. The proof is straightforward by following the ideas of the proof of Theorem 1.53. If $\det G_k(A)$ is not identically zero, then as it is a polynomial and we have that $\det G_k(A) \neq 0$, except for a set of zero Lebesgue measure, which implies that the rank of $G_N(A)$ is N and, therefore, $N-1$ critical periods unfold. □

Corollary 1.55 ([88]). *Under the same conditions of Corollary 1.54, if there exists* $A^* \in \mathbb{R}^P$ *such that* $\det G_k(A^*) = 0$, $\det G_{k-1}(A^*) \neq 0$, $f_i(A^*) = 0$, *for* $i = 0, \ldots, l-1$, $f_l(A^*) \neq 0$ *(where* $f_0, \ldots, f_l$ *are those defined in an analogous way as in Proposition 1.52(iii)) and the Jacobian determinant satisfies* $J(A^*) := \det \mathrm{Jac}_{(f_0,\ldots,f_{l-1})}(A^*) \neq 0$, *then* l *extra critical periods can bifurcate, which leads to a total of* $k + l - 1$ *critical periods.*

Remark 1.56. We can restrict the analysis to a representative system of family (1.3).

CHAPTER 2

Period function for a family of planar piecewise Hamiltonian systems

In this chapter we obtain the period function of the center at the origin for the family of planar piecewise continuous Hamiltonian systems of ordinary differential equations given by

$$(\dot{x}, \dot{y}) = \begin{cases} (y, -x - ax^3), & \text{if } x \leq 0, \\ (y, -x - bx^3), & \text{if } x \geq 0, \end{cases} \tag{2.1}$$

associated to the Hamiltonian function

$$H(x, y) = \begin{cases} \dfrac{1}{2}y^2 + \dfrac{1}{2}x^2 + \dfrac{a}{4}x^4, & \text{if } y \in \mathbb{R},\ x \leq 0, \\ \dfrac{1}{2}y^2 + \dfrac{1}{2}x^2 + \dfrac{b}{4}x^4, & \text{if } y \in \mathbb{R},\ x \geq 0, \end{cases} \tag{2.2}$$

with potential energy

$$V(x) = \begin{cases} \dfrac{1}{2}x^2 + \dfrac{a}{4}x^4, & \text{if } y \in \mathbb{R},\ x \leq 0, \\ \dfrac{1}{2}x^2 + \dfrac{b}{4}x^4, & \text{if } y \in \mathbb{R},\ x \geq 0, \end{cases} \tag{2.3}$$

where a and b are real numbers. The main tool used is the Picard–Fuchs equations for algebraic curves that has been used in the study of the period function for smooth vector field by many authors, see, for example, Chow and Sanders [23].

Remark 2.1. There exists functions that can be written as the integrals of a form over a basis of cycles, called period integrals. Such functions satisfy a differential equation known as *Picard–Fuchs equation.* In the simplest cases such equations has degree two and the quotient of the derivative of these functions and themselves satisfies a first-order ordinary differential equation that is quadratic in the unknown function, i.e. an equation of the form $y'(x) = q_0(x) + q_1(x)y(x) + q_2(x)y^2(x)$, where q_2 is nonzero, called *Riccati equation*.

It is clear that system (2.1) is integrable with the piecewise Hamiltonian function (2.2) that is known as its *piecewise first integral*. In other words, the solutions of system (2.1) are contained in the level curves of (2.2). Furthermore, according to Example 1.13, $H^+(x, y) = y^2/2 + x^2/2 + ax^4/4$, $H^-(x, y) = y^2/2 + x^2/2 + bx^4/4$, and then $H^+(0, y) = H^-(0, y)$, i.e. the level curves of H^+ and H^- intersect on the y-axis at the same points. Therefore, from Proposition 1.27, the system (2.1) has a center at the origin.

Because the vector fields are continuous, it is not necessary to consider the classical definition of Filippov vector fields (see Section 1.1). We remark that there is no sliding or escaping segments, only crossing points.

Our strategy is to use the symmetries of the piecewise Hamiltonian system (2.1) to define its period function as a linear combination of the period functions for the right-system and the left-system. Then, to carry out a study of the period function of (2.1), first of all we need to know the behavior of the period function for the smooth Hamiltonian systems that define such a family.

2.1 Analysis of the smooth planar Hamiltonian system

In this section we make an analysis of the potential system associated to the Hamiltonian function

$$H(x, y) = \frac{y^2}{2} + \frac{x^2}{2} + \frac{kx^4}{4} \tag{2.4}$$

which has potential energy

$$V(x) = \frac{x^2}{2} + \frac{kx^4}{4}, \tag{2.5}$$

given by

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - kx^3. \end{cases} \tag{2.6}$$

It is clear that due to a rescaling and a linear change of variables we can consider that $k = 1$, if $k > 0$ or $k = -1$, if $k < 0$. In the case $k = 1$, the origin is a global center, and if $k = -1$, the origin is a local center and there exist two saddle points located at $(-1, 0)$ and $(1, 0)$ with a heteroclinic connection between them, as it can be seeing in Figure 2.1.

In both cases the origin is a center and there exists only one annular region foliated by periodic orbits called *period annulus*, denoted by $\mathcal{P}$, and its boundary by $\partial\mathcal{P}$ (see Definition 1.15).

It is clear that system (2.6) is integrable with the Hamiltonian function (2.4) known as its first integral. Note that $H(0, 0) = 0$ and that the system has a nondegenerate

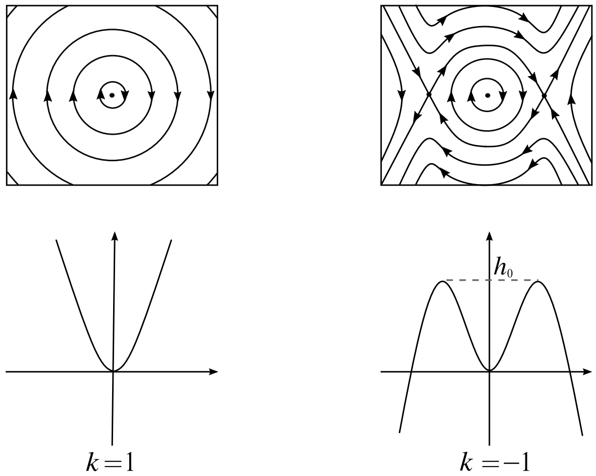

Figure 2.1: Phase portraits for family (2.6) and graphic of V in (2.5) for $k = 1, -1$

center at the origin. We can show that $H(z) \neq 0$ for every point $z \in \mathcal{P}$ different from the origin ([48]). Thus we shall assume, without loss of generality, that $H(z) > 0$ for all $z \in \mathcal{P} \setminus \{(0,0)\}$. In this case $H(\mathcal{P}) = [0, h_0)$, where $h_0 \in \mathbb{R}^+ \cup \{\infty\}$ is the level curve of the Hamiltonian that contains $\partial\mathcal{P}$. Moreover, if $h_0 < \infty$ then $\mathcal{P}$ is bounded, otherwise $\mathcal{P}$ is the whole plane and the center is global.

In addition one can prove that the set of all the periodic orbits in the period annulus can be parametrized by the energy (see [30], for instance). More generally, if the system is Hamiltonian, the parameter h that parametrize the continuum of periodic orbits of the system is taken to be its energy. Thus, for each $h \in (0, h_0)$, we denote the periodic orbit in $\mathcal{P}$ of energy level h by

$$\gamma_h := \{(x, y) \in \mathbb{R}^2 : H(x, y) = h\}, \tag{2.7}$$

where H is the Hamiltonian (2.4) and the corresponding period function which assigns to each value of the parameter $h \in (0, h_0)$ the period of the corresponding periodic orbit γ_h by $T(h)$.

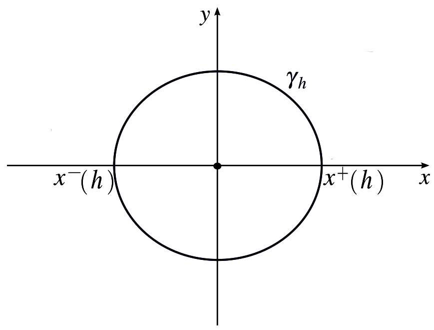

Figure 2.2: Periodic orbit γ_h

The *period function* $T:[0,h_0)\to\mathbb{R}^+$ *associated to the Hamiltonian system* can be computed by means of

$$T(h):=\int_{\gamma_h}\frac{dx}{y}=2\int_{x^-(h)}^{x^+(h)}\frac{dx}{\sqrt{2h-x^2-kx^4/2}}=4\int_0^{x^+(h)}\frac{dx}{\sqrt{2h-x^2-kx^4/2}}, \tag{2.8}$$

for each $h\in(0,h_0)$, where $x^-(h)$ and $x^+(h)$ are the intersection points of γ_h with the x-axis. Obviously, by the symmetry, we get $x^+(h)=-x^-(h)$ and the last equality holds. Clearly, $T(h)$ is an analytic function on $(0,h_0)$ and, since the center is nondegenerate, it is well known that it can be extended analytically to $h=0$ (see [94]). Therefore, $T(h)$ is defined in $[0,h_0)$.

Remark 2.2. Due to the form of the period function (2.8) we can say that it depends on k and it would be natural to denote it by $T_k(h)$. However, for simplicity we only write $T(h)$.

Our goal is to study the monotonicity of the period function $T(h)$ defined in (2.8). For this, we use that it satisfies a second order Picard–Fuchs equation that will be given in Lemma 2.3, and $x(h)=T'(h)/T(h)$ satisfies a Riccati equation, as it will be shown in Corollary 2.4 (see Remark 2.1).

Lemma 2.3. *If $T(h)$ is the period function defined in* (2.8) *then it satisfies the following homogeneous second order differential equation*

$$4h(4kh+1)T''(h)+4(8kh+1)T'(h)+3kT(h)=0, \tag{2.9}$$

for all $h\in[0,h_0)$.

Proof. In this case the Hamiltonian function is (2.4) and γ_h is as in (2.7). First we define the expression $I_j(h)=\int_{\gamma_h}x^jydx$, for $j=0,1,2,\ldots$, and we proceed closely as the procedure developed in [39].

Since $I_j'(h)=\int_{\gamma_h}\frac{x^j}{y}dx$, we get

$$\begin{aligned}I_j(h)&=\int_{\gamma_h}\frac{x^jy^2}{y}dx=\int_{\gamma_h}\frac{x^j\left(2h-x^2-\frac{k}{2}x^4\right)}{y}dx\\&=2hI_j'(h)-I_{j+2}'(h)-\frac{k}{2}I_{j+4}'(h).\end{aligned} \tag{2.10}$$

Differentiating the expression $y^2/2+x^2/2+kx^4/4=h$ with respect to the variable x we have $ydy+(x+kx^3)dx=0$. Using such an expression and integrating by parts, we have

$$
\begin{aligned}
I_j(h) &= \int_{\gamma_h} x^j y dx = \int_{\gamma_h} \frac{y}{j+1} dx^{j+1} = -\frac{1}{j+1}\int_{\gamma_h} x^{j+1} dy \\
&= \frac{1}{j+1}\int_{\gamma_h} x^{j+1}\left(\frac{x+kx^3}{y}\right) dx \\
&= \frac{1}{j+1}\int_{\gamma_h} \frac{x^{j+2}+kx^{j+4}}{y} dx \\
&= \frac{1}{j+1}[I'_{j+2}(h)+kI'_{j+4}(h)].
\end{aligned} \tag{2.11}
$$

By using equation (2.11) we remove $I'_{j+4}(h)$ of (2.10) and we obtain

$$(j+3)I_j(h) = 4hI'_j(h) - I'_{j+2}(h). \tag{2.12}$$

Taking $j = 0, 1, 2$ in expression (2.12), we have:

$$
\begin{aligned}
3I_0(h) &= 4hI'_0(h) - I'_2(h), \\
4I_1(h) &= 4hI'_1(h) - I'_3(h), \\
5I_2(h) &= 4hI'_2(h) - I'_4(h).
\end{aligned} \tag{2.13}
$$

Note that along γ_h we have $y^2dy + (x + kx^3)ydx = 0$, hence

$$0 \equiv \int_{\gamma_h} (x+kx^3)ydx = I_1(h) + kI_3(h). \tag{2.14}$$

Using the derivative of (2.14) and (2.11) with $j = 0$ we remove I'_3 and I'_4, from (2.13), and we finally obtain

$$3I_0(h) = 4hI'_0(h) - I'_2(h), \tag{2.15}$$

$$4I_1(h) = 4hI'_1(h) + (1/k)I'_1(h), \tag{2.16}$$

$$(1/k)I_0(h) + 5I_2(h) = 4hI'_2(h) + (1/k)I'_2(h). \tag{2.17}$$

We note that I_1 and I'_1 do not appear in (2.15) or (2.17). By eliminating (2.16) we consider the equation

$$\begin{pmatrix} 3 & 0 \\ \frac{1}{k} & 5 \end{pmatrix}\begin{pmatrix} I_0 \\ I_2 \end{pmatrix} = \begin{pmatrix} 4h & -1 \\ 0 & 4h+\frac{1}{k} \end{pmatrix}\begin{pmatrix} I'_0 \\ I'_2 \end{pmatrix}. \tag{2.18}$$

Differentiating (2.18), we get

$$\begin{pmatrix} -4h & 1 \\ 0 & 4h+\frac{1}{k} \end{pmatrix}\begin{pmatrix} I''_0 \\ I''_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{k} & 1 \end{pmatrix}\begin{pmatrix} I'_0 \\ I'_2 \end{pmatrix}.$$

Then, we obtain the Picard–Fuchs equation for the algebraic curve $y^2 = 2h-x^2-kx^4/2$ given by

$$I''_0(h) = \frac{1}{\delta}\left(4hI'_0(h) - I'_2(h)\right), \tag{2.19}$$

$$I''_2(h) = \frac{1}{\delta}\left(-\frac{4h}{k}I'_0(h) - 4hI'_2(h)\right). \tag{2.20}$$

where $\delta = -4h(4kh+1)/k$.

By using (2.19) and (2.20) we have $I_2'(h)$ and $I_2''(h)$ as functions of $I_0(h)$ and its derivatives. Differentiating again (2.19) and replacing $I_2'(h)$ and $I_2''(h)$ by the expressions previously found, after some simplifications, we obtain

$$4h(4kh+1)I_0^{(3)}(h) + 4(8kh+1)I_0''(h) + 3kI_0'(h) = 0.$$

Note that this proves (2.9) since $I_0'(h) = T(h)$ (see equation (2.8)). Thus $I_0''(h) = T'(h)$ and $I_0^{(3)}(h) = T''(h)$, and the lemma is proved. □

Following the ideas presented in [53] we have the next corollary.

Corollary 2.4. *Let $T(h)$ be the function that satisfies* (2.9). *Then the function $x(h)$ defined by $T'(h)/T(h)$ verifies the Riccati equation*

$$4h(4kh+1)x'(h) + 4h(4kh+1)x^2(h) + 4(8kh+1)x(h) + 3k = 0, \tag{2.21}$$

for all $h \in [0, h_0)$.

Proof. As $T(h) > 0$, for all $h \in [0, h_0)$, the function $x(h)$ takes only finite values and through direct calculations replacing $x(h) = T'(h)/T(h)$ and using equation (2.9) we can prove equation (2.21). □

Lemma 2.5. *Consider the period function $T(h)$ given in* (2.8), *for $h \in [0, h_0)$. Then, $T(0) = 2\pi$, $T'(0) = -3k\pi/2$, and*

$$T(h) = 2\pi - \frac{3k\pi}{2}h + \frac{105k^2\pi}{32}h^2 - \frac{1155k^3\pi}{128}h^3 + O(h^4) \tag{2.22}$$

is the Taylor series of T around $h = 0$.

Proof. We write system (2.6) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and we get

$$\dot{r} = kr^3 \sin\theta \cos^3\theta =: R(r,\theta), \tag{2.23}$$
$$\dot{\theta} = 1 + kr^2\cos^4\theta =: \Theta(r,\theta). \tag{2.24}$$

Moreover, the ratio $r_h(\theta)$ of γ_h given by (2.7) is the unique solution of

$$\frac{dr}{d\theta} = \frac{R(r,\theta)}{\Theta(r,\theta)},$$

with initial condition $(r,\theta) = (x^+(h), 0) := (r_0, 0)$, where r_0 is the smaller positive root of $x^2/2 + kx^4/4 = h$. Then, $r_h(\theta) = r_0 + \sum_{j=2}^{\infty} U_j(\theta) r_0^j$. Substituting this last expression into equation (2.24) we obtain an equation of the form $d\theta/dt = 1+\sum_{k=1}^{\infty} V_j(\theta)r_0^j = \Theta(r_h(\theta),\theta)$ and then $dt = d\theta/\Theta(r_h(\theta),\theta)$ and we can compute the period of γ_h by

$$T(h) := T(r_h(\theta)) = \int_0^{2\pi} \frac{1}{\Theta(r_h(\theta),\theta)} d\theta = 2\pi + \sum_{j=1}^{\infty} S_j(\theta)r_0^j, \tag{2.25}$$

therefore $T(0) = 2\pi$. Every series converges for $0 < \theta \leq 2\pi$ and sufficiently small $r_0 \geq 0$.

If $k = 0$, then

$$T(h) = \int_0^{2\pi} d\theta = 2\pi,$$

for all h from the above expression and, hence, the center is isochronous.

Now, for $k \neq 0$ we can determine the values of $T^{(j)}(h)$ by using the Picard–Fuchs equation (2.9). In fact, considering $h = 0$ in (2.9) we obtain $T'(0) = -3k\pi/2$. Taking now the derivative of (2.9):

$$(16kh^2 + 4h)T^{(3)}(h) + (64kh + 8)T''(h) + 35kT'(h) = 0, \tag{2.26}$$

and by replacing $h = 0$ and $T'(0)$ we can show that $T''(0) = 105k^2\pi/16$. Differentiating equation (2.26) we have

$$(16kh^2 + 4h)T^{(4)}(h) + (96kh + 12)T^{(3)}(h) + 99kT''(h) = 0, \tag{2.27}$$

and then by replacing $h = 0$ and $T''(0)$ in equation (2.27), we obtain $T^{(3)}(0) = -3465k^3\pi/64$. We could continue this inductive procedure to determine all the higher order derivatives.

Therefore, we can write the Taylor series by $T(h) = 2\pi + \sum_{j=1}^{\infty} c_j h^j$ where $c_j = T^{(j)}(0)/j!$ and we obtain (2.22). □

Lemma 2.6. *Assume that $k > 0$ in system* (2.6) *and its period function $T(h)$ is given by* (2.8)*. Then, $T(h)$ goes to 0 as h tends to $+\infty$.*

Proof. We can assume, without loss of generality, that $k = 1$ and that (2.6) can be written in polar coordinates, as the system given by (2.23) and (2.24). From the expression (2.24) we have that

$$\Theta(r,\theta) > 1, \text{ for any } (r,\theta), \text{ consequently } \frac{1}{\Theta(r,\theta)} < 1. \tag{2.28}$$

Note also that the distance of γ_h to the origin tends to infinity as $h \to \infty$. The proof of Lemma 2.5 gives us a parametrization of the radius with respect to the angle for any h, denoted by $r_h(\theta)$, and that we can compute the period of γ_h, for each h, by means of (2.25). On the other hand, note that, since $r(\theta,h) \to \infty$ as $h \to \infty$, for each θ,

$$\lim_{h\to\infty} \frac{1}{\Theta(r(\theta,h),\theta)} = \begin{cases} 1, & \text{if } \theta = \pm\dfrac{\pi}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.29}$$

Therefore, by using (2.28), we can apply the Dominate Convergence Theorem and assert that $\lim_{h\to\infty} T(h) = 0$. □

Aiming to make the complete study on the monotonicity of the period function $T(h)$, instead of the Riccati equation (2.21), we study the global phase portrait of the equivalent autonomous differential system on the plane. For this system, the phase curve $(h, x(h)) = (h, T'(h)/T(h))$ has the following fundamental geometric property: suppose that for $h = \hat{h}$ the periodic solution $\gamma_{\hat{h}}$ vanishes. Then, $\lim_{h\to\hat{h}} T'(h)/T(h) = \hat{x} \neq \pm\infty$, the equilibrium point $(\hat{h}, \hat{x})$ is a saddle and the curve $(h, x(h))$ is a separatrix solution of $(\hat{h}, \hat{x})$. Consequently, after determining this phase curve and the isoclines we can obtain the behavior of $T'(h)$ and $T''(h)$. This is done in a similar way to the one carried out in [53, 69].

Lemma 2.7. *Consider the period function $T(h)$ given in (2.8). Thus, if $k = 0$, then $T(h)$ is constant, if $k < 0$, then it is monotonous increasing, and if $k > 0$, then it is monotonous decreasing. Moreover, for $k \neq 0$, we have that $T'(h)$ is monotonous increasing, i.e. $T''(h) > 0$, for all $h \in [0, h_0)$.*

Proof. If $k = 0$, by the proof of Lemma 2.5, $T(h) = 2\pi$ for all h.

For the other cases we can define $x(h) = T'(h)/T(h)$ which verifies the Riccati equation (2.21), for all $h \in [0, h_0)$, by Corollary 2.4. Instead of this equation we study an equivalent autonomous differential system on the plane, namely

$$\begin{cases} \dot{h} = -4h(4kh + 1), \\ \dot{x} = 4h(4kh + 1)x^2 + 4(8kh + 1)x + 3k, \end{cases} \tag{2.30}$$

where the dot means the derivative with respect to time (i.e. d/dt).

The roots of $-4h(4kh + 1) = 0$ are $h = 0$ and $h = -1/(4k)$, and they correspond to invariant vertical straight lines on the (h, x)-plane. Moreover, $p_0 = (0, -3k/4)$ and $p_1 = (-1/(4k), 3k/4)$ are equilibrium points of system (2.30) for each k. The linear parts of the vector fields at p_0 and at p_1 are

$$\begin{pmatrix} -4 & 0 \\ -\frac{87}{4}k^2 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 & 0 \\ \frac{87}{4}k^2 & -4 \end{pmatrix},$$

respectively. Therefore, they are hyperbolic saddles whose eigenvectors are $(1, 87k^2/32)$ and $(0, 1)$, respectively.

Remember that, for the analytical case, it is sufficient to consider $k = -1$, for $k < 0$, and $k = 1$, for $k > 0$. We reproduce here all the computations for both cases, although the proof follows closely.

Now we describe the idea of the proof. First we show that the graphic of $x(h)$ coincides with a separatrix of the saddle p_0. We denote such a separatrix by Γ and we represent it in Figures 2.3 and 2.4 by the dotted line. Then, we identify the horizontal isoclines P_j and Q_j of (2.30) and we analyze the behavior of the vector field in the regions bordered by them. First, for $k = -1$, due to the fact that Γ reaches the saddle point with a slope less than the one of Q_1 and due to the behavior of the vector field in the regions determined by P_1 and Q_1, it follows that Γ is as on the right in Figure 2.3. Performing a similar analysis for $k = 1$, now for P_2 and Q_2, we can only guarantee that Γ is below P_2. So, to prove that Γ is in the region between P_2 and Q_2, as represented on the right of Figure 2.4, we must find a special curve. Then, it is possible to determine the sign of the derivative of the period function in both cases.

Finally, for the proof of the second statement, it is enough to prove that $x'(h) > 0$, for all $h \in [0, h_0)$, since from Lemma 2.5, $T''(0) = 105k^2\pi/16 > 0$, and we have the following equivalences

$$x'(h) > 0 \Leftrightarrow \frac{T''(h)T(h) - (T'(h))^2}{T(h)^2} > 0 \Leftrightarrow T''(h) > \frac{(T'(h))^2}{T(h)} > 0,$$

that is $x'(h) > 0$ if and only if $T''(h) > 0$.

Case $k = -1$: In this case the Hamiltonian differential system associated to the Hamiltonian function (2.4) has only the period annulus of the center at the origin, which is bounded (see Figure 2.1). Moreover, $T(h)$ is defined for $h \in [0, h_0)$, where $h_0 = -1/(4k) = 1/4$. Then, $T(h) \to \infty$, as $h \to 1/4$, and this implies that $T'(h)$ assumes positive values as $h \to 1/4$.

Since $k = -1$, the invariant vertical straight lines of (2.30) are $h = 0$ and $h = 1/4$, and the equilibrium points are saddles at $p_0 = (0, 3/4)$ and $p_1 = (1/4, -3/4)$.

Consider the vertical strip $\mathcal{U} = \{(h, x) : 0 \leq h \leq 1/4,\ x \geq 0\}$ and notice that there exists a unique orbit, here denoted by Γ, lying in $\mathrm{Int}(\mathcal{U})$ and having the saddle p_0 as an ω-limit point, that is Γ is the stable separatrix of p_0 and it is drawn on the right in Figure 2.3 with dashed line.

We claim that the graphic of $x(h)$ coincides with the stable separatrix Γ. In fact, first by using the Taylor series obtained in Lemma 2.5 we have that the curve $x(h) = T'(h)/T(h)$, for $h \in (0, h_0) = (0, 1/4)$, is an integral curve of system (2.30) that tends to $3/4$ as $h \to 0$ and this value is the x-component of p_0. Then, the statement follows from the uniqueness of solutions.

Now we can show that $T'(h) > 0$ and $x'(h) > 0$, for all $h \in [0, 1/4)$.

For $h = 0$, we have that $x'(0) = 87/32$ (this is simply obtained by computing the slope of the eigenspace at the saddle p_0). Now, consider the second equation in (2.30) equals to 0, then we obtain the points where the vector field is horizontal. This equation defines

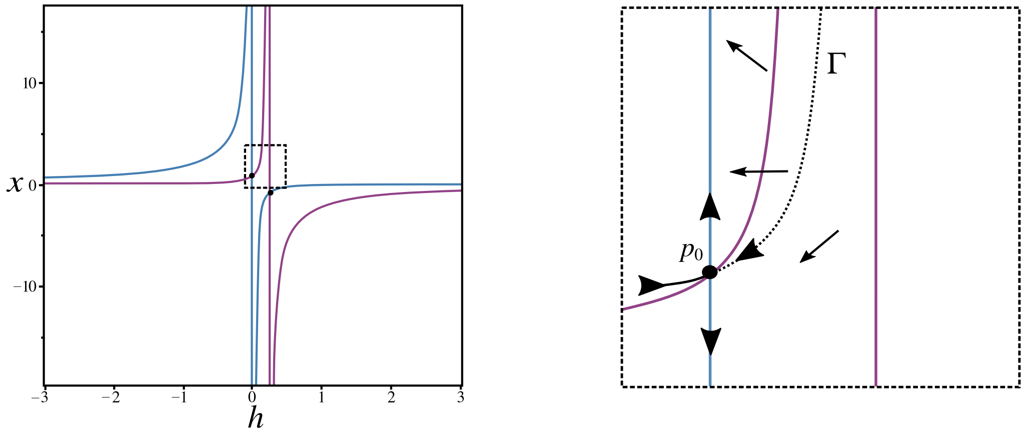

Figure 2.3: Curves P_1 (blue) and Q_1 (purple) and relative position of Γ

two curves given by

$$P_1(h) = \frac{-8h + 1 + \sqrt{52h^2 - 13h + 1}}{h(4h-1)} \quad \text{and} \quad Q_1(h) = \frac{-8h + 1 - \sqrt{52h^2 - 13h + 1}}{h(4h-1)},$$

whose connected components are graphics of functions in h. The curve P_1 is drawn in blue and Q_1 is drawn in purple in Figure 2.3. The vector field is transversal to P_1 and Q_1.

We now study the position of P_1 and Q_1 with respect to Γ. The curves P_1 and Q_1 divide the vertical strip $\mathcal{U}$ into two regions where the vector field points upwards on the top and points downwards on the bottom region (see Figure 2.3).

At p_0, the slope of the tangent line to Q_1 is equal to $87/16$, which is bigger than the slope $x'(0) = 87/32$ of Γ at the same point. Then, in a neighborhood of p_0, the separatrix Γ is below Q_1. Since the vector field is transversal to Q_1 and directed to the left, the orbit Γ is not allowed to intersect Q_1 for $t \to -\infty$ and the orbit Γ is always below Q_1. Moreover, Γ is above the h-axis since the vector field points downwards on $(0, 1/4) \times \{0\}$, because the second component of the vector field is constant and it is equal to -3. Therefore, the graphic of $x(h)$, which is the orbit Γ, is entirely located in $\mathcal{U}$. This implies that $x(h) = T'(h)/T(h) > 0$, then $T'(h) > 0$, for all $h \in [0, 1/4)$. We also have that $x'(h) = (dx/dt)/(dh/dt) > 0$, since $dx/dt < 0$ in the region below Q_1 in the vertical strip $\mathcal{U}$ and $dh/dt < 0$ for all $h \in (0, 1/4)$.

Case $k = 1$: We already know that in this case the planar Hamiltonian differential system (2.6) associated to the Hamiltonian function (2.4) has only the period annulus of the center at the origin, which is global (see Figure 2.1). Moreover, the corresponding period function is defined for $h \in [0, \infty)$.

Since $k = 1$, the invariant vertical straight lines of (2.30) are $h = 0$ and $h = -1/4$, and the equilibrium points are saddles at $p_0 = (0, -3/4)$ and $p_1 = (-1/4, 3/4)$.

Consider the half-plane $\mathcal{U} = \{(h, x) : h \geq 0\}$ and notice that there exists a unique orbit, here denoted by Γ, lying in $\mathrm{Int}(\mathcal{U})$ and having the saddle p_0 as an ω-limit point, that is Γ is the stable separatrix of p_0 and is drawn on the right in Figure 2.4 with dashed line.

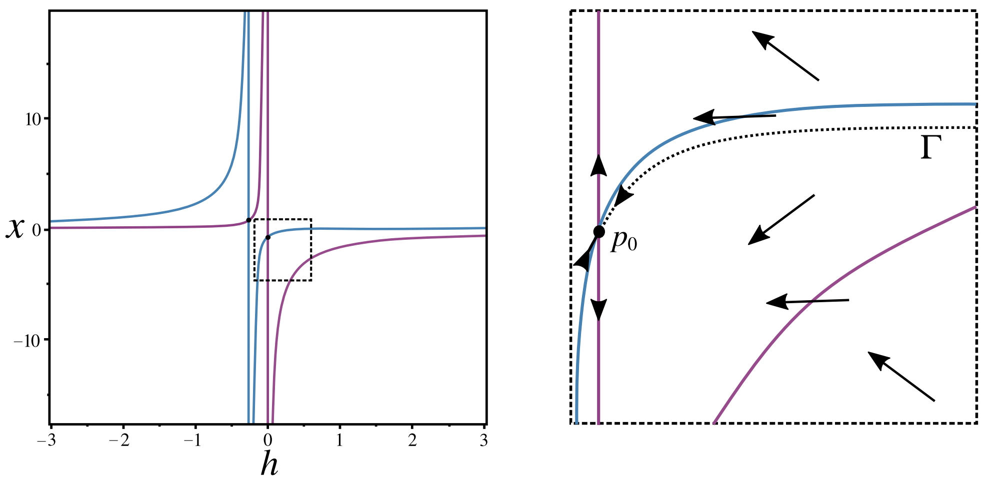

Figure 2.4: Curves P_2 (blue) and Q_2 (purple) and relative position of Γ

We claim that the graphic of $x(h)$ coincides with stable separatrix Γ. In fact, first by using the Taylor series obtained in Lemma 2.5, we have that the curve $x(h) = T'(h)/T(h)$, for $h \in (0, \infty)$, is a integral curve of system (2.30) that tends to $-3/4$ as $h \to 0$ and this value is the x-component of p_0. Then, the statement follows from the uniqueness of solutions.

Now we can show that $T'(h) < 0$ and $x'(h) > 0$, for all $h \in [0, \infty)$.

For $h = 0$, we have that $x'(0) = 87/32$ (this is simply obtained by computing the slope of the eigenspace at the saddle p_0). Now, consider the second equation in (2.30) equals to 0, then we obtain the points where the vector field is horizontal. This equation defines two curves given by

$$P_2(h) = \frac{-8h - 1 + \sqrt{52h^2 + 13h + 1}}{2h(1 + 4h)} \quad \text{and} \quad Q_2(h) = \frac{-8h - 1 - \sqrt{52h^2 + 13h + 1}}{2h(1 + 4h)},$$

whose connected components are graphics of functions in h. The curve P_2 is drawn in blue and Q_2 is drawn in purple in Figure 2.4. The vector field is transversal to P_2 and Q_2 and directed to the left.

We now study the position of P_2 and Q_2 with respect to Γ. The curves P_2 and Q_2 divide the half-plane $\mathcal{U}$ into three regions where the vector field points upwards on the top and on the bottom regions and it points downwards on the middle region (see Figure 2.4). At p_0, the slope of the tangent line to P_2 is equal to $87/16$, which is bigger than the slope

$x'(0) = 87/32$ of Γ at the same point. Then, in a neighborhood of p_0, the separatrix Γ is below P_2. Since the vector field is transversal to P_2 and directed to the left, the orbit Γ is not allowed to intersect P_2 again for $t \to -\infty$. The orbit Γ is then entirely located in $\mathcal{U}$, below P_2.

Proving that Γ is above the curve Q_2 requires a more accurate analysis. Some calculations show that

$$\lim_{h\to 0^+} P_2(h) = -3/4, \ \lim_{h\to\infty} P_2(h) = 0, \ \lim_{h\to 0^+} Q_2(h) = -\infty, \text{ and } \lim_{h\to\infty} Q_2(h) = 0.$$

Now notice that any convex combination

$$R_s(h) = sP_2(h) + (1-s)Q_2(h),$$

with $s \in [0,1]$, is a curve that has the following properties: $R_s(h)$ is monotonous increasing, $Q_2(h) \le R_s(h) \le P_2(h)$ for all $h \in [0,\infty)$,

$$\lim_{h\to\infty} R_s(h) = 0, \quad \text{and} \quad \lim_{h\to 0^+} R_s(h) = -\infty.$$

There exists a value $s^* \in [0,1]$ such that the curve Γ is in the region between the curves R_{s^*} and P_2, so that we can conclude that Γ is above the curve Q_2. In fact, taking $s^* = 7/8$, then $R_{s^*}(h) = 7P_2(h)/8 + Q_2(h)/8$. Therefore, the graphic of R_{s^*} is the zero level curve of the function

$$F(h,x) = 2xh(4h+1) + 8h + 1 - 3\sqrt{52h^2+13h+1}/4,$$

i.e. $(h, R_{s^*}(h)) := \{(h,x) \colon F(h,x) = 0\}$.

In the region near the saddle point p_0, the curve R_{s^*} is below Γ since $x(0) = -3/4$, $\lim_{h\to 0+} R_{s^*}(h) = -\infty$, and R_{s^*} is monotonous increasing. Moreover, by solving the equation $R_{s^*}(h) = -3/4$, it follows that the curve R_{s^*} intersects the line $x = -3/4$ at the point $(1/3, -3/4)$.

Note that the gradient of F is given by

$$\nabla F = \Big(2x\big(4h+1\big) + 8xh + 8 - 3\big(104h+13\big)/\big(8\sqrt{52h^2+13h+1}\big), 2h\big(4h+1\big)\Big),$$

and it is always orthogonal to R_{s^*} and it is directed upwards. Since

$$\langle \nabla F, (\dot{h}, \dot{x})\rangle\big|_{F=0} = \frac{5}{8}(132h^2+33h+5) - \frac{3(8h+1)(52h^2+13h+2)}{2\sqrt{52h^2+13h+1}}$$

is always negative in $h \in (1/3,\infty)$, because evaluating this scalar product at $h = 1$ we get $\sqrt{66}(425\sqrt{66}/8 - 1809/4)/33 < 0$, and since the polynomial that is obtained by eliminating the root by squaring has no roots in the interval $(1/3,\infty)$, then the angle

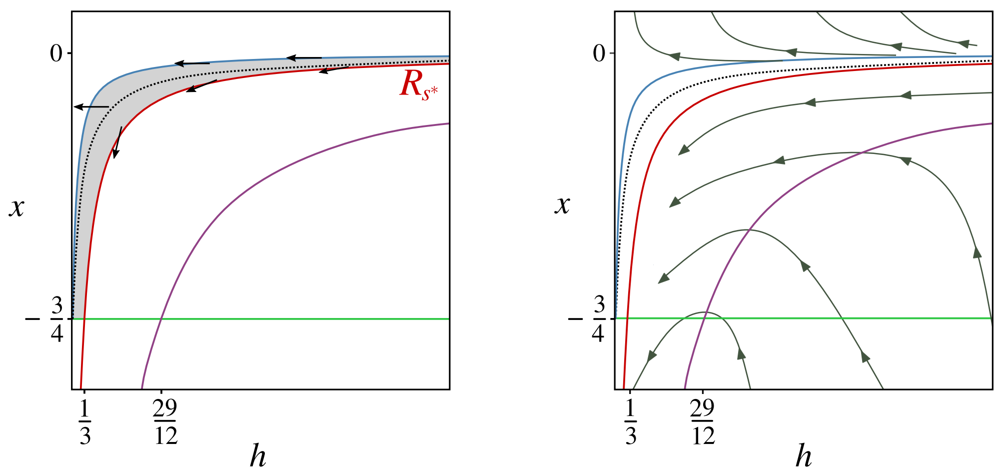

Figure 2.5: Region negatively invariant in gray and curve R_{s^*} in red

formed by ∇F and $(\dot{h}, \dot{x})$ is greater than $\pi/2$, for all $h \in (1/3, \infty)$. So the behavior of the vector field (2.30) along the curve R_{s^*} is as presented in Figure 2.5.

Then, the region in gray on the left in Figure 2.5 is a trapping region (negatively invariant set) for the system (2.30), i.e. solutions that start inside this region (including on its boundary) stay inside of it, for all negative times. Thus, the curve Γ remains above the curve R_{s^*} for all $h \in [0, \infty)$, implying that Γ is above Q_2.

Therefore, the graphic of x is located below the h-axis, between P_2 and Q_2. Since it is below P_2, the second component $T'(h)/T(h) < 0$, for all h, so $T'(h) < 0$, for all $h \in [0, \infty)$. Moreover, this implies that $x'(h) = (dx/dt)/(dh/dt) > 0$, since $dx/dt < 0$ in the region between P_2 and Q_2 and $dh/dt < 0$ for all $h \in [0, \infty)$. □

2.2 Phase portraits

This section is devoted to determine all topologically different phase portraits for system (2.1). The key point of this section is the well-known behavior of the continuous case studied in Section 2.1.

Remember that $\mathcal{P}$ and $\partial\mathcal{P}$ denote the period annulus and its boundary, respectively, as in Definition 1.15. In this case, we can also assume that $H(z) > 0$ for all $z \in \mathcal{P} \setminus \{(0, 0)\}$ and $H(\mathcal{P}) = [0, h_0)$, where h_0 is the level curve of the Hamiltonian (2.2) that contains $\partial\mathcal{P}$.

The next result states that there exist four topologically different phase portraits for system (2.1), but only three different period annuli, and they are called, respectively, global center, saddle loop, and two saddle cycle, see Figure 2.6.

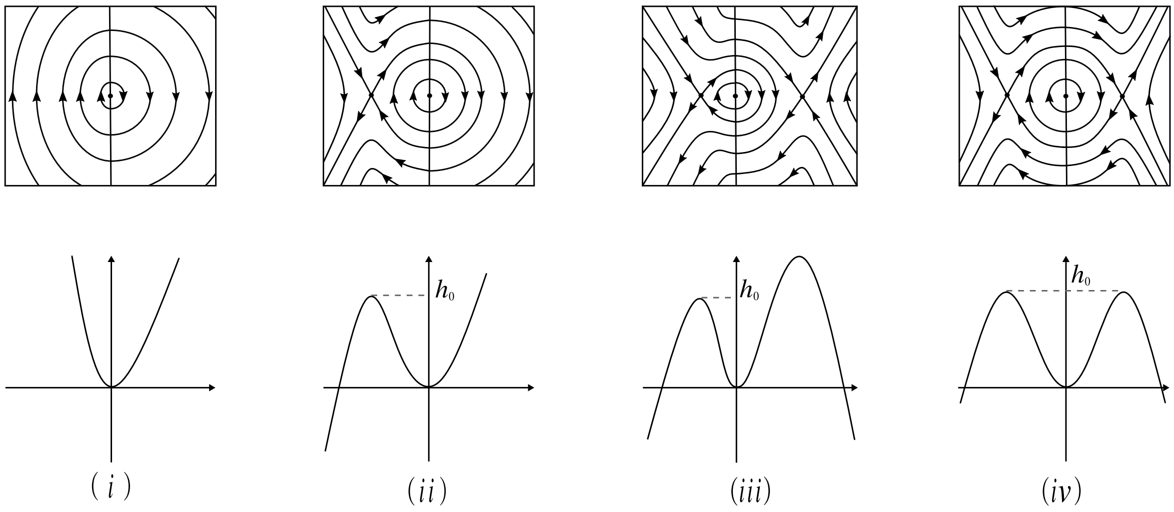

Figure 2.6: Phase portraits for family (2.1) and graphic of V in (2.3)

Proposition 2.8. *System* (2.1) *has a center at the origin and the following statements hold:*

(i) For $a \geq 0$ and $b \geq 0$, it has only one equilibrium point at the origin which is a global center (i.e. $h_0 = \infty$).

(ii) For $a \geq 0$ and $b < 0$ or $a < 0$ and $b \geq 0$, it has two equilibrium points, which are a center and a saddle, and $\partial\mathcal{P}$ is a finite homoclinic connection in $h_0 = \max\{-1/(4a), -1/(4b)\}$.

(iii) For $a < 0$, $b < 0$ and $a \neq b$, it has three equilibrium points, which are a center and two saddles, and $\partial\mathcal{P}$ is a finite homoclinic connection in $h_0 = \min\{-1/(4a), -1/(4b)\}$.

(iv) For $a < 0$, $b < 0$ and $a = b$, it has three equilibrium points, which are a center and two saddle, and $\partial\mathcal{P}$ is a finite heteroclinic connection in $h_0 = -1/(4a) = -1/(4b)$.

The phase portraits for each item (i) to (iv) are topologically equivalent, respectively, to those ones presented in Figure 2.6.

Proof. We have that family (2.1) has a center at the origin from Proposition 1.26 or even by Proposition 1.27, since their solutions are invariant with respect to the change of variables $(x, y, t) \to (-x, y, -t)$ and $H^-(0, y) = H^+(0, y)$.

Before we consider the case-by-case study, we describe the critical levels of the Hamiltonian function (2.2).

In order to determine h_0 we study the graphic of $V^-(x) = x^2/2 + ax^4/4$ for $x \leq 0$ and $V^+(x) = x^2/2 + bx^4/4$ for $x \geq 0$, as it was done in [69] and from equation (2.3) we write

$$V(x) = \begin{cases} V^-(x), & \text{if } x \leq 0, \\ V^+(x), & \text{if } x \geq 0. \end{cases} \tag{2.31}$$

We have that the critical levels of (2.2) for $x \leq 0$ (resp. $x \geq 0$) are the positive zeros of the discriminant of $V^-(x) - h$ (resp. $V^+(x) - h$) with respect to x, that is, the common solutions of $(V^-)'(x) = 0$ and $V^-(x) = h$ (resp. $(V^+)'(x) = 0$ and $V^+(x) = h$). As we have that such discriminants are equal to $-ah(1+4ah)^2/4$ for V^- and $-bh(1+4bh)^2/4$ for V^+, the union of the zeros of the two discriminants is $\{0\}$ if a and b are zeros, $\{0, h^-, h^+\}$, where $h^- = -1/(4a)$ and $h^+ = -1/(4b)$, if a and b are nonzero, $\{0, h^+\}$ if $a = 0$ or $\{0, h^-\}$ if $b = 0$.

To completely analyze family (2.1), we study five different cases in terms of the signs of the parameters a and b. The first three cases are those ones when the two parameters have the same signs and the last two cases correspond to the cases when the two parameters have different signs.

Case 1. For $a \geq 0$ and $b \geq 0$: Note first that in this case both systems of the form (2.6) associated to the Hamiltonian function given by (2.4) with $k = a$ for the left half-plane and with $k = b$ for the right half-plane have a global center at the origin. Then the piecewise system also has a global center at the origin and the graphic of V and the phase portrait are represented by (i) in Figure 2.6.

Case 2. For $a < 0, b < 0$ and $|a| \neq |b|$: In this case the Hamiltonian function given by (2.4) with $k = a$ (resp. $k = b$) is a first integral of system (2.6) in the left (resp. right) half-plane, then there exists one saddle for $x \leq 0$ (resp. $x \geq 0$) in the level h^- (resp. h^+) and a center at the origin. Since $|a| \neq |b|$, $h^- \neq h^+$ are two positive values and then $h_0 = \min\{h^-, h^+\}$. Therefore $\partial\mathcal{P}$ which is contained in h_0 has one saddle with a finite homoclinic connection. The graphic of V and the phase portrait are represented by (iii) in Figure 2.6.

Case 3. For $a < 0, b < 0$ and $|a| = |b|$: In this case the Hamiltonian function given by (2.4) with $k = a$ (resp. $k = b$) is a first integral of system (2.6) in the left (resp. right) half-plane, then there exists one saddle for $x \leq 0$ (resp. $x \geq 0$) at the level h^- (resp. h^+) and a center at the origin. Since $|a| = |b|$ then $h_0 = h^- = h^+$. Therefore system (2.1) has two saddles with a finite heteroclinic connection between them at level h_0. The graphic of V and the phase portrait are represented by (iv) in Figure 2.6.

Case 4. For $a < 0$ and $b \geq 0$: In this case the Hamiltonian function given by (2.4) with $k = a$ (resp. $k = b$) is a first integral of system (2.6) in the left (resp. right) half-plane. Then there exist a center at the origin and a saddle at the level h^- for $x \leq 0$ and only a center at the origin for $x \geq 0$. Therefore, there exists a positive value $h_0 = \max\{h^-, h^+\}$. The graphic of V and the phase portrait are represented by (ii) in Figure 2.6.

Case 5. For $a \geq 0$ and $b < 0$: This case can be obtained by the previous one by using the change of variables $(x, y, t) \mapsto (-x, y, -t)$.

□

It is clear that Case 1 corresponds to item (i) in Proposition 2.8, Case 2 corresponds to item (iii), Case 3 to item (iv), and Cases 4 and 5 to item (ii). Moreover, for each case we can define only one period function, since in all of them there exists just one region which is entirely covered by periodic orbits.

2.3 Period function and its oscillations

Similarly as what is done in the continuous case, in piecewise Hamiltonian systems the periodic orbits in the period annulus can be parametrized by the energy. For each $h \in [0, h_0)$, we denote by γ_h the set of points in $\mathbb{R}^2$ that verifies $H(x, y) = h$, where H is defined by (2.2) (see Figure 2.7). The period function that associates each γ_h to its minimum period is also denoted by $T(h)$.

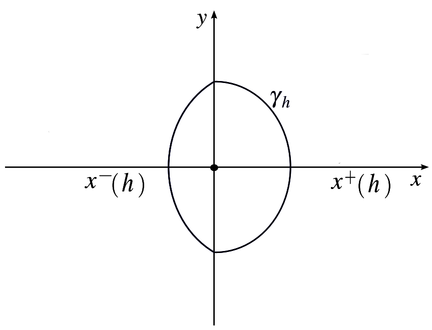

Figure 2.7: Orbit γ_h

Note that system (2.1) is reversible with respect to both x-axis and y-axis (double symmetry). In fact, using the transformation $(x, y, t) \mapsto (-x, -y, t)$ we obtain this double symmetry and, therefore, from the expression of the period function of the continuous case in (2.8) we can conclude that the period function of system (2.1) is described as in the next lemma.

Lemma 2.9. *The period function of system* (2.1) *defined in* $[0, h_0)$ *is given by*

$$T(h) = \frac{1}{2}\left(T_a(h) + T_b(h)\right), \tag{2.32}$$

where $T_k(h)$ *is given by* (2.8) *(see Remark 2.2).*

As we have that $T'(h) = \big(T_a'(h) + T_b'(h)\big)/2$ and $T''(h) = \big(T_a''(h) + T_b''(h)\big)/2$, from Lemmas 2.5 and 2.7 we obtain the following result.

Lemma 2.10. *Consider the period function* $T(h)$ *given in* (2.32). *Then* $T(0) = 2\pi$, $T'(0) = -3\pi(a+b)/4$, *and* $T''(h) > 0$ *for all* $h \in [0, h_0)$.

Although we do not use this fact, in what follows we want to highlight that if we have two functions such that each of them verifies a Picard–Fuchs equation, then the sum of these two functions also verifies a Picard–Fuchs equation, but of a higher order.

Lemma 2.11. *Consider two functions* δ *and* σ *that satisfy the following Picard–Fuchs equations, respectively,*

$$P_2(h)\delta''(h) + P_1(h)\delta'(h) + P_0(h)\delta(h) = 0, \tag{2.33}$$

$$Q_2(h)\sigma''(h) + Q_1(h)\sigma'(h) + Q_0(h)\sigma(h) = 0. \tag{2.34}$$

Then we have that the sum $\tau = \delta + \sigma$ *verifies a fourth order Picard–Fuchs equation.*

Proof. By the identity $\tau = \delta + \sigma$ we have

$$\sigma = \tau - \delta, \qquad \sigma' = \tau' - \delta', \qquad \sigma'' = \tau'' - \delta''. \tag{2.35}$$

We replace the values of σ, σ', and σ'' obtained in (2.35) in (2.34). Then we replace δ'' by the expression $-\big(P_1(h)\delta'(h) + P_0(h)\delta(h)\big)/P_2(h)$ found by means of (2.33) and we have

$$\begin{aligned}&\Big(Q_2(h)P_1(h) - Q_1(h)P_2(h)\Big)\delta'(h) + \Big(Q_2(h)P_0(h) - Q_0(h)P_2(h)\Big)\delta(h)\\ &+Q_2(h)P_2(h)\tau''(h) + Q_1(h)P_2(h)\tau'(h) + Q_0(h)P_2(h)\tau(h) = 0.\end{aligned} \tag{2.36}$$

We differentiate expression (2.36) and replace δ'' by the expression previously found and we obtain the next combination of τ (and its derivatives up to order three), δ, and δ'

$$\begin{aligned}&\Big(-Q_1'(h)P_2(h) - Q_1(h)P_2'(h) + Q_1(h)P_1(h) - Q_0(h)P_2(h)\\ &+P_1'(h)Q_2(h) - P_1(h)^2Q_2(h)/P_2(h) + P_1(h)Q_2'(h) + P_0(h)Q_2(h)\Big)\delta'(h)\\ &+\Big(Q_1(h)P_0(h) - Q_0'(h)P_2(h) - Q_0(h)P_2'(h) - P_1(h)P_0(h)Q_2(h)/P_2(h)\\ &+P_0'(h)Q_2(h) + P_0(h)Q_2'(h)\Big)\delta(h) + P_2(h)Q_2(h)\tau^{(3)}(h)\\ &+\Big(Q_1(h)P_2(h) + P_2(h)Q_2'(h) + Q_2(h)P_2'(h)\Big)\tau''(h)\\ &+\Big(Q_0(h)P_2(h) + Q_1'(h)P_2(h) + Q_1(h)P_2'(h)\Big)\tau'(h)\\ &+\Big(Q_0'(h)P_2(h) + Q_0(h)P_2'(h)\Big)\tau(h) = 0.\end{aligned} \tag{2.37}$$

With equations (2.36) and (2.37) by solving as if they were a system of linear equations on the variables δ and δ', we can determine δ, δ', and δ'' only in function of τ and its derivatives. Differentiating again equation (2.37) and replacing δ'', δ', and δ by these expressions we obtain a new equation as a combination of τ (and its derivatives up to order four). Then we have that the sum τ verifies a fourth order Picard–Fuchs equation. We do not show the mentioned equation due to its size. □

By Lemmas 2.3 and 2.11 the period function $T(h)$ defined in (2.32) verifies the following result.

Corollary 2.12. *If $T(h)$ is the period function of* (2.1) *then $P_4(h)T^{(4)}(h)+P_3(h)T^{(3)}(h)+P_2(h)T''(h)+P_1(h)T'(h)+P_0T(h)=0$ for all $h\in[0,h_0)$, where*

$$\begin{aligned}
P_4(h) &= 16h^2\big(256a^3bh^4+64a^2(a+3b)h^3+48a(a+b)h^2+4(3a+b)h+1\big),\\
P_3(h) &= 8h\big(3584a^3bh^4+32(19a+75b)a^2h^3+48(8a+11b)ah^2+2(39a+19b)h+5\big),\\
P_2(h) &= 41472a^3bh^4+64(61a+385b)a^2h^3+16(127a+284b)ah^2+4(71a+61b)h+5,\\
P_1(h) &= 8448a^3bh^3+4(71a+61b)a^2h^2+88(a+7b)ah+11(a+b),\\
P_0(h) &= 528a^3bh^2+264a^2bh+33ab.
\end{aligned}$$

As Proposition 2.8 made clear how the domain of the period function is, the following result gives us information about the monotonicity and the existence of oscillations of such a function.

Theorem 2.13. *The period function of system* (2.1) *defined in* $[0,h_0)$ *satisfies the following conditions:*

(i) For $a=b=0$ it is constant.

(ii) For $a\geq 0$ and $b\geq 0$, not simultaneously zero, it is monotonous decreasing.

(iii) For $a<0$ and $b<0$, or $ab<0$ with $|\min\{a,b\}|\geq\max\{a,b\}$, it is monotonous increasing.

(iv) For $ab<0$ with $|\min\{a,b\}|<\max\{a,b\}$, has one simple critical period, which is a minimum point.

Proof. First, if $a=0$ and $b=0$ then the center is isochronous, T is constant and it is equal to 2π.

As a matter of fact, we can further reduce the number of parameters by one. In other words, we can assume that there is only one free parameter. To achieve this, as we have that $b\neq 0$, we use the following rescaling in system (2.1):

$$x\to\lambda x,\ y\to\lambda y,$$

for $\lambda \neq 0$, to obtain a new family given by

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y, -x - \dfrac{a}{\lambda^2} x^3\right), & \text{if } x \leq 0, \\ \left(y, -x - \dfrac{b}{\lambda^2} x^3\right), & \text{if } x \geq 0. \end{cases}$$

Hence, when $b > 0$, we consider $\lambda = \sqrt{b}$ and we get the following equivalent system

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y, -x - \dfrac{a}{b} x^3\right), & \text{if } x \leq 0, \\ (y, -x - x^3), & \text{if } x \geq 0, \end{cases}$$

and when $b < 0$ we consider $\lambda = \sqrt{-b}$ and the family is topologically equivalent to

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y, -x + \dfrac{a}{b} x^3\right), & \text{if } x \leq 0, \\ (y, -x + x^3), & \text{if } x \geq 0. \end{cases}$$

Therefore, we can assume that in the right hand side we have $b = 1$ or $b = -1$.

For the case $a \geq 0$ and $b > 0$, with a rescaling we obtain $a > 0$ and $b = 1$. By (i) in Proposition 2.8 the origin is a global center and $h_0 = \infty$, then the period function is defined in $[0, \infty)$ and by Lemma 2.10 $T'(0) = -3\pi(a+1)/4 < 0$. Moreover, by Lemma 2.7, $T_a'(h) < 0$ and $T_b'(h) < 0$ for all $h \in (0, \infty)$. Thus, $T'(h) = T_a'(h) + T_b'(h) < 0$, $T''(h) > 0$ from Lemma 2.10, and $\lim_{h\to\infty} T(h) = 0$ from Lemma 2.6, hence T is monotonous decreasing, convex and its graphic is as represented in the blue region in Figure 2.8.

In the case that $a < 0$ and $b < 0$ we can assume $b = -1$. The period function is defined in $[0, h_0)$ with $h_0 = \min\{-1/(4a), 1/4\}$ by (iii) and (iv) in Proposition 2.8. By Lemma 2.10, $T'(0) = -3\pi(a - 1)/4 > 0$ and Lemma 2.7 implies that $T_a'(h) > 0$ and $T_b'(h) > 0$ for all $h \in [0, h_0)$. Thus, $T'(h) = T_a'(h) + T_b'(h) > 0$ and T is monotonous increasing, convex as $T''(h) > 0$ from Lemma 2.10 with a horizontal asymptote in $h = h_0$. Therefore, the graphic of T is as represented in the green region in Figure 2.8.

Now, if $ab < 0$ we can assume that we are in the case where $a < 0$ and $b > 0$, since the other case is symmetric. We can rescale and consider $b = 1$ and $a < 0$ as our free parameter. Moreover, by (ii) in Proposition 2.8 the period function is defined in $[0, h_0)$, where $h_0 = \max\{-1/(4a), -1/4\} = -1/(4a)$ and by Lemma 2.10

$$T'(0) = \frac{-3\pi(a+1)}{4}, \tag{2.38}$$

and $T''(h) > 0$ for all $h \in [0, h_0)$.

First suppose that $|a| > 1$, then we have $T'(0) > 0$ by equation (2.38), since $T''(h) > 0$, then $T'(h) > 0$ for all $h \in [0, -1/(4a))$ and the period function is monotonous increasing and the graphic of T is as represented in the green region in Figure 2.8. When $a = -1$

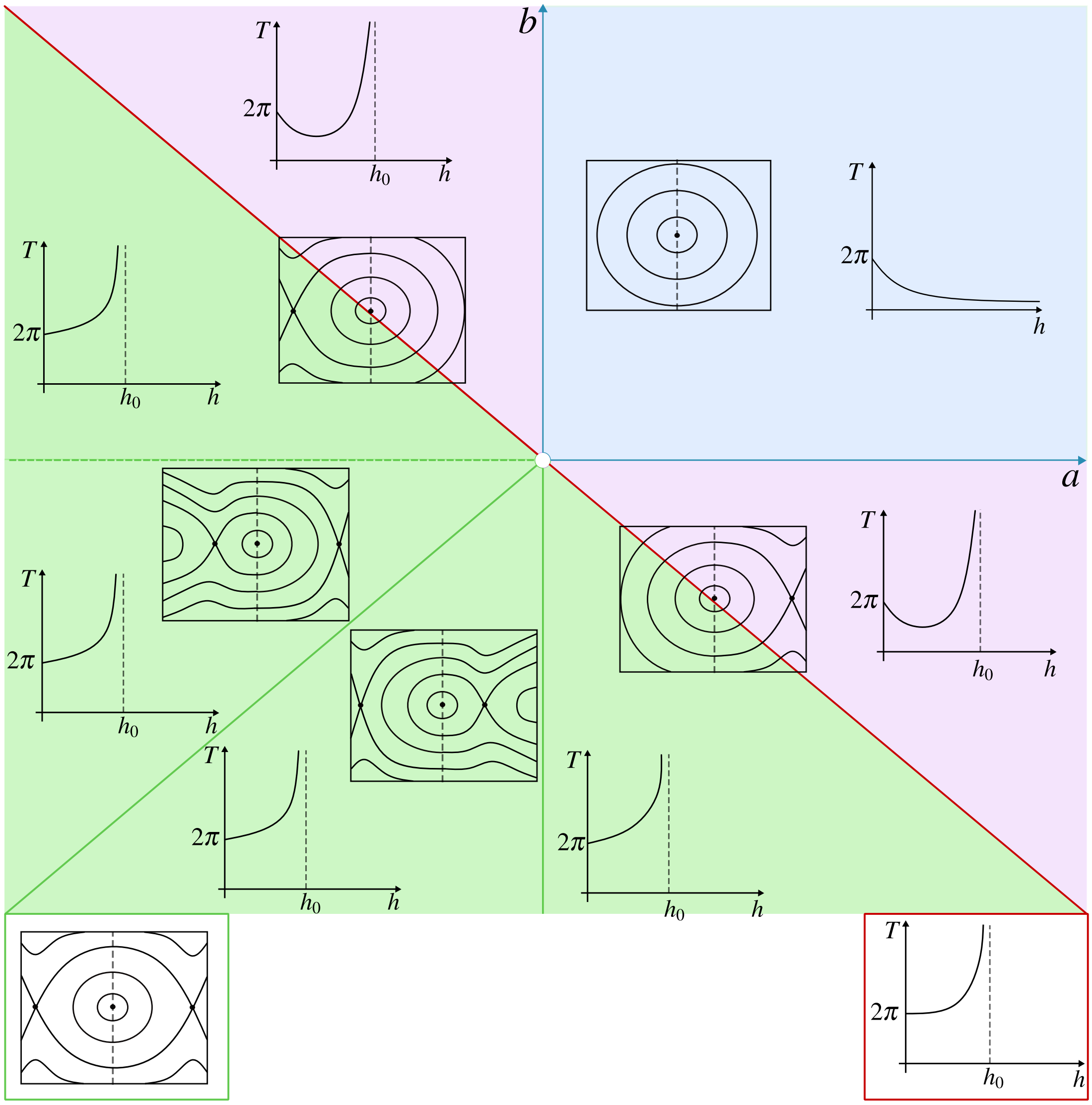

Figure 2.8: Bifurcation diagram of the period function of the center at the origin of system (2.1)

we have $T'(0) = 0$ by equation (2.38), $T'(h) > 0$ and $T''(h) > 0$ for all $h \in (0, 1/4)$. Consequently the graphic of T is as represented in the red square in Figure 2.8.

Now take $|a| < 1$, thus we have $T'(0) < 0$ by equation (2.38), and since $T''(h) > 0$ for all $h \in [0, -1/(4a))$, the period function has one simple critical period, which is a minimum point. Furthermore, the derivative T' cannot have a horizontal asymptote at 0. In fact, as $a < 0$, it is well known that $\lim_{h \to h_0} T_a(h) = \infty$, where $h_0 = -1/(4a)$. Then $\lim_{h \to h_0} T(h) = \lim_{h \to h_0}(T_a(h) + T_b(h))/2 = \infty$ and therefore $T'(h) > 0$ when $h \to h_0$. The later properties proves that the graphic of T is as represented in the purple region in Figure 2.8. □

From Theorem 2.13 we see that the period functions can have at most one simple critical period for family (2.1).

Remark 2.14. The piecewise system (2.1) is a special case of asymmetric oscillator, which is well-known and frequently encountered in physical problems ([60]). As a first work about the period function of piecewise system, we decide to do an analysis of this family that has a double symmetry, that is with respect to both axis, which allows us to find an expression for T in terms of the energy h. Therefore, we established its bifurcation diagram determining the behavior of $T(h)$. Moreover, it is important to mention that the period function of the analytic potential vector field that determines the left and right system of (2.1) have already been studied in Chicone [19] and Gasull, Guillamon, Mañosa and Mañosas [48]. We find convenient to present in Section 2.1 the study of the monotonous behavior of the period function for them to establish one of the methods that can be used.

CHAPTER 3

Period function of planar piecewise reversible quadratic systems

In this chapter we study the bifurcation of local critical periods near the origin in the class of planar piecewise quadratic systems of the form (1.3), which are invariant by the transformation $(x, y, t) \mapsto (-x, y, -t)$, and so they are reversible with respect to the y-axis. In this way, we intend to present an estimative for the maximum number of zeros of the derivative T' which can bifurcate from the origin in the class of *planar piecewise reversible quadratic system* given by

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + a_1x^2 + a_2y^2, x + a_3xy), & \text{if } y \geq 0, \\ (-y + b_1x^2 + b_2y^2, x + b_3xy), & \text{if } y < 0, \end{cases} \tag{3.1}$$

where a_i, b_i $(i = 1, 2, 3)$ are real numbers. It is clear that the origin is a center of system (3.1) because of the reversibility.

Note that system (3.1) belongs to the class of planar piecewise systems of ordinary differential equations with the x-axis as its separation line given by

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + P^+(x, y, \lambda^+), x + Q^+(x, y, \lambda^+)), & \text{if } y \geq 0, \\ (-y + P^-(x, y, \lambda^-), x + Q^-(x, y, \lambda^-)), & \text{if } y < 0, \end{cases} \tag{3.2}$$

where $\lambda = (\lambda^+, \lambda^-) \in \mathbb{R}^m$ are parameters, $P^\pm(x, y, \lambda^\pm)$ and $Q^\pm(x, y, \lambda^\pm)$ are homogeneous polynomials of degree 2 in the variables x and y. The system defined in the upper half-plane ($y \geq 0$) is called the *upper system* and the system defined in the lower half-plane ($y \leq 0$) is called the *lower system*.

In the planar piecewise system (3.2), where the separation line is the x-axis, the half-return maps are defined as in Definition 1.33, only changing the y-axis into the x-axis, and they are called, respectively, *positive and negative half-return maps*. Also in an analogous way as in Definition 1.36 we define the *positive half-period function* $T^+(\rho, \lambda^+)$ (resp. the *negative half-period function* $T^-(\rho, \lambda^-)$) as the least period of the trajectory of (3.2) passing through $(x, y) = (\rho, 0)$ on the positive x-axis (resp. on the negative x-axis) to reach the negative x-axis (resp. the positive x-axis). Hence, for (ρ, λ) where $\lambda = (\lambda^+, \lambda^-)$ the *Poincaré return map* and the *period function* are defined as $(\rho, \lambda) \mapsto \Pi(\rho, \lambda)$ and $(\rho, \lambda) \mapsto T(\rho, \lambda)$, where

$$\Pi(\rho, \lambda) = \Pi^-(\Pi^+(\rho, \lambda^+), \lambda^-) \text{ and } T(\rho, \lambda) = T^+(\rho, \lambda^+) + T^-(\Pi^+(\rho, \lambda^+), \lambda^-),$$

as represented in Figure 3.1.

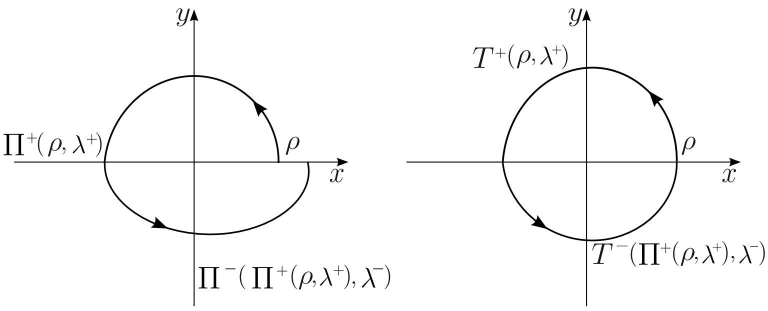

Figure 3.1: Return map $\Pi(\rho, \lambda)$ and period function $T(\rho, \lambda)$ for system (3.2)

For proving the results of this chapter we use the results of Sections 1.5 and 1.6.

3.1 Lyapunov and period constants

As discussed in the introduction, two interesting problems are determining $\mathcal{H}_0(n)$ and $\mathcal{C}_0(n)$, i.e to solve the *local cyclicity problem* and *local criticality problem*. In general, we can find lower bounds for these numbers using the Taylor coefficients of the expansion of the Poincaré return map and period function, i.e. determining the *Lyapunov constants* and the *period constants*. Such definitions have already been introduced in Section 1.4.

First, we will present a method developed in [52] that can easily be implemented in a computer algebraic system for the calculation of the Lyapunov constants for the piecewise system (3.2). After we will describe a method to find the period constants described in [18] for (3.2).

Consider the following form of a planar piecewise system

$$(\dot{x}, \dot{y}) = \begin{cases} (\delta^+ x - y + P^+(x, y, \lambda^+), x + \delta^+ y + Q^+(x, y, \lambda^+)), & \text{if } y \geq 0, \\ (\delta^- x - y + P^-(x, y, \lambda^-), x + \delta^- y + Q^-(x, y, \lambda^-)), & \text{if } y < 0, \end{cases} \tag{3.3}$$

where $\lambda = (\lambda^+, \lambda^-) \in \mathbb{R}^m$ are the parameters, $P^\pm(x, y, \lambda^+)$ and $Q^\pm(x, y, \lambda^-)$ are analytic functions in x and y starting from at least the second-order terms. Under the polar coordinates transformation, $x = r\cos\theta$, and $y = r\sin\theta$, system (3.3) can be transformed into

$$(\dot r, \dot\theta) = \begin{cases} \Big(r\big(\delta^+ + R^+(r,\theta,\lambda^+)\big), 1 + \Theta^+(r,\theta,\lambda^+)\Big), & \text{for } \theta \in (0,\pi], \\ \Big(r\big(\delta^- + R^-(r,\theta,\lambda^-)\big), 1 + \Theta^-(r,\theta,\lambda^-)\Big), & \text{for } \theta \in (\pi,2\pi), \end{cases} \tag{3.4}$$

where $R^\pm$ (resp. $\Theta^\pm$) are analytic functions in $r > 0$, $\sin\theta$ and $\cos\theta$ starting at least with first (resp. second) order terms in $r > 0$, and third order terms in $\sin\theta$ and $\cos\theta$ given by

$$\begin{aligned} R^\pm(r,\theta,\lambda^\pm) &= \cos\theta P^\pm(r\cos\theta, r\sin\theta, \lambda^\pm) + \sin\theta Q^\pm(r\cos\theta, r\sin\theta, \lambda^\pm), \\ \Theta^\pm(r,\theta,\lambda^\pm) &= \frac{1}{r}\big(\cos\theta Q^\pm(r\cos\theta, r\sin\theta, \lambda^\pm) - \sin\theta P^\pm(r\cos\theta, r\sin\theta, \lambda^\pm)\big). \end{aligned}$$

Eliminating t we obtain the equation of orbits on the phase plane

$$\frac{dr}{d\theta} = \begin{cases} r\big(\delta^+ + R^+(r,\theta,\lambda^+)\big)/\big(1 + \Theta^+(r,\theta,\lambda^+)\big), & \text{for } \theta \in (0,\pi], \\ r\big(\delta^- + R^-(r,\theta,\lambda^-)\big)/\big(1 + \Theta^-(r,\theta,\lambda^-)\big), & \text{for } \theta \in (\pi,2\pi). \end{cases} \tag{3.5}$$

Let $r^+(\rho,\theta,\delta^+,\lambda^+)$ and $r^-(\rho,\theta,\delta^-,\lambda^-)$ denote the solutions of (3.5) for $\theta \in (0,\pi)$ associated with the initial condition $r^+(\rho,0,\delta^+,\lambda^+) = \rho$, and for $\theta \in (\pi,2\pi)$ associated with the initial condition $r^-(\rho,\pi,\delta^-,\lambda^-) = \rho$, respectively. The positive half-return map $\Pi^+ : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^-$ and the negative half-return map $\Pi^- : \mathbb{R}^- \times \mathbb{R}^2 \to \mathbb{R}^+$ can be defined respectively by

$$\Pi^+(\rho,\delta^+,\lambda^+) = r^+(\rho,\pi,\delta^+,\lambda^+) \text{ and } \Pi^-(\rho,\delta^-,\lambda^-) = r^-(\rho,2\pi,\delta^-,\lambda^-).$$

Then, the return map $\Pi : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^+$ for the piecewise system (3.3) can be constructed by the composition $\Pi(\rho,\delta,\lambda) := \Pi^-\big(\Pi^+(\rho,\delta^+,\lambda^+),\delta^-,\lambda^-\big)$, where $\delta = (\delta^+,\delta^-)$ and $\lambda = (\lambda^+,\lambda^-)$. As $\Pi^\pm$ are analytic for $|\rho|$ small enough, $\Pi^\pm$ can be expanded, respectively, as

$$\Pi^+(\rho,\delta^+,\lambda^+) = e^{\pi\delta^+}\rho + \sum_{j\ge2} v_j^+(\delta^+,\lambda^+)\rho^j \text{ and } \Pi^-(\rho,\delta^-,\lambda^-) = e^{\pi\delta^-}\rho + \sum_{j\ge2} v_j^-(\delta^-,\lambda^-)\rho^j,$$

where v_j^+'s and v_j^-'s are the Taylor's coefficients. Thus,

$$\Pi(\rho,\delta,\lambda) = \Pi^-(\Pi^+(\rho,\delta^+,\lambda^+),\delta^-,\lambda^-) = e^{\pi\, v_1(\delta,\lambda)}\rho + \sum_{j\ge2} v_j(\delta,\lambda)\rho^j, \tag{3.6}$$

for sufficiently small $|\rho|$, where $v_1(\delta,\lambda) = \delta^+ + \delta^-$. This last expression implies that 0 is a center of system (3.3) if and only if $\Pi(\rho,\delta,\lambda) = \rho$, for all small ρ, i.e. $\delta^+ + \delta^- = 0$ and $v_j(\delta,\lambda) = 0$, for all $j \ge 2$, by (3.6). The j-th Lyapunov constants are defined by $V_j := v_j(\delta,\lambda)$. Hence, the displacement function can be expanded as

$$d(\rho,\delta,\lambda) = \Pi(\rho,\delta,\lambda) - \rho = (e^{2\pi(\delta^+ + \delta^-)} - 1)\rho + \sum_{j\ge2} V_j\rho^j. \tag{3.7}$$

Since it is not easy to compute the Lyapunov constants by using (3.6), then Gasull and Torregrosa in [52] used the following method for the calculation of Lyapunov constants for the piecewise system (3.2), given by the piecewise system (3.3) with $\delta = (\delta^+, \delta^-) = (0, 0)$. Consider the expansion of (3.4) and of equation (3.5) given by

$$(\dot{r}, \dot{\theta}) = \begin{cases} \left(\sum_{j=1}^{\infty} \varphi_j^+(\theta) r^{j+1}, 1 + \sum_{j=1}^{\infty} \phi_j^+(\theta) r^{j+1}\right), & \text{for } \theta \in (0, \pi], \\ \left(\sum_{j=1}^{\infty} \varphi_j^-(\theta) r^{j+1}, 1 + \sum_{j=1}^{\infty} \phi_j^-(\theta) r^{j+1}\right), & \text{for } \theta \in (\pi, 2\pi), \end{cases} \tag{3.8}$$

and

$$\frac{dr}{d\theta} = \begin{cases} \sum_{j=1}^{\infty} R_j^+(\theta, \lambda^+) r^j, & \text{for } \theta \in (0, \pi], \\ \sum_{j=1}^{\infty} R_j^-(\theta, \lambda^-) r^j, & \text{for } \theta \in (\pi, 2\pi), \end{cases} \tag{3.9}$$

respectively, where $\varphi_j^\pm$ and $\phi_j^\pm$ are homogeneous polynomials in $\sin\theta$ and $\cos\theta$ of degree $j+2$, $R_j^\pm(\theta, \lambda^\pm)$ are 2π-periodic functions of θ and the series are convergent for all θ and for all sufficiently small $r > 0$. The curve of solutions with $r^+(\rho, 0, \lambda^+) = r^-(\rho, \pi, \lambda^-) = \rho$, of the upper and lower equations of (3.9) can also be expanded by

$$\begin{cases} r^+(\rho, \theta, \lambda^+) = \rho + \sum_{j=2}^{\infty} u_j^+(\theta, \lambda^+) \rho^j, & \text{for } \theta \in (0, \pi], \\ r^-(\rho, \theta, \lambda^-) = \rho + \sum_{j=2}^{\infty} u_j^-(\theta, \lambda^-) \rho^j, & \text{for } \theta \in (\pi, 2\pi). \end{cases} \tag{3.10}$$

These series are convergent for all θ and all $\rho < r^*$, for some sufficiently small $r^* > 0$.

Then, the positive half-return map is $\Pi^+(\rho, \lambda^+) = r^+(\rho, \pi, \lambda^+)$, the negative half-return map is $\Pi^-(\rho, \lambda^-) = r^-(\rho, 2\pi, \lambda^-)$, and the *complete return map* Π can be expanded as (3.6), where $V_1 = 0$ and $V_j := v_j((0,0), \lambda) := v_j(\lambda)$, that is

$$\Pi(\rho, \lambda) = \rho + \sum_{j \geq 2} v_j(\lambda) \rho^j = \rho + \sum_{j \geq 2} V_j \rho^j.$$

In this case the *displacement function* $d(\rho, \lambda) = \Pi(\rho, \lambda) - \rho$, given in (3.7), becomes

$$d(\rho, \lambda) = V_2 \rho^2 + V_3 \rho^3 + \cdots . \tag{3.11}$$

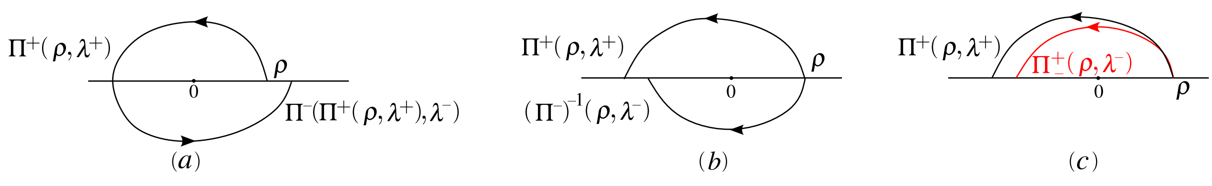

Figure 3.2: Return maps

From (3.6), in order to obtain the constants V_j's, we need a method to compute $\Pi^+(\rho, \lambda^+)$ and $\Pi^-(\rho, \lambda^-)$ and afterward to compose them. But, in [52] the authors present a simpler way to compute the Lyapunov constants by using the difference function,

$$\Pi^+(\rho, \lambda^+) - (\Pi^-)^{-1}(\rho, \lambda^-) = W_2 \rho^2 + W_3 \rho^3 + \cdots , \tag{3.12}$$

where $(\Pi^-)^{-1}(\rho, \lambda^-)$ is the inverse of the negative half-return map $\Pi^-(\rho, \lambda)$, see Figure 3.2(b). We have that $(\Pi^-)^{-1}(\rho, \lambda^-) =: \Pi_-^+(\rho, \lambda^+)$, where $\Pi_-^+(\rho, \lambda^-)$ is the positive half-return map of the following piecewise system

$$(\dot{x}, \dot{y}) = \begin{cases} (-y - P^-(x, -y, \lambda^-), x + Q^-(x, -y, \lambda^-)), & \text{if } y \geq 0, \\ (-y - P^+(x, -y, \lambda^+), x + Q^+(x, -y, \lambda^+)), & \text{if } y < 0, \end{cases} \tag{3.13}$$

obtained by the substitution $(x, y, t) \to (x, -y, -t)$ on the system (3.2) (see Figure 3.2(c)). Thus, to get (3.12) we only need to compute the two positive half-return maps $\Pi^+(\rho, \lambda^+)$ and $\Pi_-^+(\rho, \lambda^-)$. In this case, for $|\rho|$ small enough, we have the following expansion for $\Pi_-^+(\rho, \lambda^-)$

$$\Pi_-^+(\rho, \lambda) = \rho + \sum_{j \geq 2} u_j^+(\lambda) \rho^j,$$

where u_j^+'s are the Taylor coefficients. Hence, the expansion of the displacement function becomes

$$d(\rho, \lambda) = \sum_{j \geq 2} (v_j^+(\lambda) - u_j^+(\lambda)) \rho^j = \sum_{j \geq 2} W_j \rho^j,$$

Thus, to compute higher order $W_j := v_j^+(\lambda) - u_j^+(\lambda)$ for piecewise system (3.2), we only need to compute $v_j^+(\lambda)$ and $u_j^+(\lambda)$ for two positive half-return maps.

The conditions $V_j \neq 0$, $V_i = 0$, $2 \leq i \leq j-1$ of (3.11) are equivalent to $W_j \neq 0$, $W_i = 0$, $2 \leq i \leq j-1$ of (3.12), from [52]. Therefore, we can use these new constants to find the center conditions. However, if the sign of the first nonzero coefficient of the return map (3.6) is positive (resp. negative) the equilibrium point is repulsive (resp. attractor). On the other hand, the way we have determined the difference (3.12), the sign of the first W_j nonzero is the inverse, then we cannot use it to find the stability.

Remark 3.1. Although the calculations of the Lyapunov constants are straightforward, it is very difficult to solve the center-focus and the cyclicity problems for planar piecewise systems because it is not easy to find the common zeros of the Lyapunov constants. Then, some techniques that reduce the computational complexity have been developed ([63]). Furthermore, it is expected that in the piecewise system it can appear the double of the number of small amplitude limit cycles bifurcating from an elementary center or an elementary focus rather than in the smooth case. For instance, while for quadratic systems the maximum number of limit cycles that bifurcate from the origin is three, by using V_1, V_3, V_5, and V_7, since the Lyapunov constants of even order vanish when the previous Lyapunov constants of odd orders are zeros (see [4]), for piecewise quadratic systems it is enough to use V_1, V_2, V_3, and V_4 in order to generate also three limit cycles. And six limit cycles can be generated from the origin by using $V_1, \ldots, V_7$.

In the following, we describe the method used in [18] to find the period constants $T_j(\lambda)$ that determine the expansion of the period function associated with the center at the origin of the piecewise system (3.2) given by

$$T(\rho, \lambda) = 2\pi + \sum_{j=1}^{\infty} T_j(\lambda)\rho^j, \tag{3.14}$$

for $|\rho|$ and $|\lambda - \lambda_0|$ sufficiently small, by using the fact that $T(\rho, \lambda) = T^+(\rho, \lambda^+) + T^-(\Pi^+(\rho, \lambda^+), \lambda^-)$.

Substituting (3.10) into the right hand side of the upper system and the lower system of (3.4), we have two equations (one with the superscript + and the other with the superscript −):

$$\dot{\theta} = 1 + \sum_{j=1}^{\infty} \xi_j^{\pm}(\theta, \lambda^{\pm})\rho^j.$$

Rewriting those equations as

$$dt = \frac{d\theta}{1 + \sum_{j=1}^{\infty} \xi_j^{\pm}(\theta, \lambda^{\pm})\rho^j} = \left(1 + \sum_{j=1}^{\infty} \phi_j^{\pm}(\theta, \lambda^{\pm})\rho^j\right) d\theta,$$

and integrating, we get

$$t - \theta = \sum_{j=1}^{\infty} \psi_j^{\pm}(\theta)\rho^j, \tag{3.15}$$

where

$$\psi_j^+(\theta) = \int_0^{\theta} \phi_j^+(s)ds, \quad \psi_j^-(\theta) = \int_{\pi}^{\theta} \phi_j^-(s)ds,$$

and the series in (3.15) converges for all θ and sufficiently small $\rho \geq 0$.

From (3.15) it follows that the *positive half-period* of (3.2) passing through $(x, y) = (\rho, 0)$ on the positive x-axis for $\rho \neq 0$ that reaches the negative x-axis is given by

$$T^+(\rho, \lambda^+) = \pi + \sum_{k\geq 1} T_k^+(\lambda^+)\rho^k,$$

where $T_k^+(\lambda^+) = \psi_k(\pi) = \int_0^{\pi} \phi_k^+(s)ds$.

The *negative half-period* from the point $(x, y) = (\Pi^+(\rho, \lambda^+), 0)$ on the negative x-axis for $\rho \neq 0$ is given by

$$\begin{aligned} T^-(\Pi^+(\rho, \lambda^+), \lambda^-) &= \pi + \textstyle\sum_{j\geq 1} T_j^-(\lambda^-)(\Pi^+(\rho, \lambda^+))^j \\ &= \pi + \textstyle\sum_{j\geq 1} T_j^-(\lambda^-)\big(r^+(\rho, \pi, \lambda^+)\big)^j \\ &= \pi + \textstyle\sum_{j\geq 1} T_j^-(\lambda^-)\big(\rho + \sum_{i\geq 2} u_i^+(\pi, \lambda^+)\rho^i\big)^j \\ &= \pi + \textstyle\sum_{j\geq 1} \hat{T}_j^-(\lambda^+, \lambda^-)\rho^j, \end{aligned}$$

where $T_j^-(\lambda^-) = \psi_j(2\pi) = \int_{\pi}^{2\pi} \phi_j^-(s)ds$, for the second equality we use the expression obtained in (3.10) and $\hat{T}_j^-(\lambda^-, \lambda^+)$'s are the Taylor coefficients of the before expression.

Thus, the least period of the trajectory of (3.2) passing through $(x, y) = (\rho, 0)$, for $\rho \neq 0$, is given by (3.14), where $T_j(\lambda) = T_j^+(\lambda^+) + \hat{T}_j(\lambda^+, \lambda^-)$ is the *j-th period constant*. Obviously, the center O of (3.2) is isochronous if and only if all period constants vanish.

Remark 3.2. Note that if the piecewise system (3.2) is reversible with respect to the y-axis then we have $\Pi^+(\rho, \lambda^+) = -\rho$ and the negative half-period of (3.2) passing through $(-\rho, 0)$ is simply given by

$$T^-(\rho, \lambda^-) = \pi + \sum_{j\geq 1} T_j^-(\lambda^-)\rho^j,$$

where $T_j^-(\lambda^-) = \psi_j(2\pi) = \int_\pi^{2\pi} \phi_j^-(s)ds$ and, therefore, the period constants of (3.14) are of the form $T_j(\lambda) = T_j^+(\lambda^+) + T_j^-(\lambda^-)$.

But, in general, it is not easy to find the coefficients $\hat{T}_j(\lambda^+, \lambda^-)$ since we need to know the expression of $\Pi^+(\rho, \lambda^+)$. In order to avoid this, we can use the transformation $(x, y, t) \to (x, -y, -t)$ in the lower system of (3.2), then the negative half-period function becomes the positive half-period function of (3.13) which is denoted by $\tilde{T}^-(\rho, \lambda^-)$ found in analogous way of $T^+(\rho, \lambda^+)$ with expansion in form of series

$$\tilde{T}^-(\rho, \lambda^-) = \pi + \sum_{j\geq 1} \tilde{T}_j(\lambda^-)\rho^j.$$

Hence, the expansion of the period function becomes

$$\begin{aligned} T(\rho, \lambda^+, \lambda^-) &= T^+(\rho, \lambda^+) + \tilde{T}^-(\rho, \lambda^-) \\ &= 2\pi + \textstyle\sum_{j\geq 1}(T_j^+(\lambda^+) + \tilde{T}_j^-(\lambda^-))\rho^j, \end{aligned}$$

and the period constants are $T_j(\lambda) = T_j^+(\lambda^+) + \tilde{T}_j^-(\lambda^-)$.

Remark 3.3. As for $j \geq 1$ the period constant T_j belongs to the ideal $\langle T_1, T_2, T_3, \ldots, T_{j-1}\rangle$ over the ring $\mathbb{R}\{\lambda_1, \lambda_2, \ldots, \lambda_m\}_\lambda$ for each $\lambda \in \mathbb{R}^m$, the expression of a period constant $T_j(\lambda)$ has only meaning when $T_1(\lambda) = \cdots = T_{j-1}(\lambda) = 0$.

The next lemma provides the first six period constants found by the method before presented for family (3.1) and the other statements follows by using the computer algebra system Maple. We will not give more details because of the size of their expressions. Note that in the case of system (3.1) the corresponding period constants can be thought as $T_j(\mathbf{a}, \mathbf{b}) = T_j^+(\mathbf{a}) + T_j^-(\mathbf{b})$, where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

Lemma 3.4. *The period constants T_j for family* (3.1) *are homogeneous polynomials of degree j in the parameters. Moreover, they write as $T_j = \overline{T}_j(a_1, a_2, a_3) - \overline{T}_j(b_1, b_2, b_3)$ for*

$j=1,3,5$ and $T_j=\overline{T}_j(a_1,a_2,a_3)+\overline{T}_j(b_1,b_2,b_3)$ for $j=2,4,6$ where:

$$\begin{aligned}
\overline{T}_1(z_1,z_2,z_3) &= \frac{1}{3}\big(2z_1+4z_2-2z_3\big),\\
\overline{T}_2(z_1,z_2,z_3) &= \frac{\pi}{24}\big(4z_1^2+10z_1z_2-5z_1z_3+10z_2^2-z_2z_3+z_3^2\big),\\
\overline{T}_3(z_1,z_2,z_3) &= \frac{1}{405}\big(206z_1^3+624z_1^2z_2-312z_1^2z_3+960z_1z_2^2-204z_1z_2z_3+114z_1z_3^2\\
&\quad +640z_2^3+120z_2^2z_3+12z_2z_3^2-8z_3^3\big),\\
\overline{T}_4(z_1,z_2,z_3) &= \frac{\pi}{2304}\big(400z_1^4+1424z_1^3z_2-712z_1^3z_3+2772z_1^2z_2^2-828z_1^2z_2z_3+369z_1^2z_3^2\\
&\quad +3080z_1z_2^3+168z_1z_2^2z_3+126z_1z_2z_3^2-58z_1z_3^3+1540z_2^4+700z_2^3z_3\\
&\quad +21z_2^2z_3^2-2z_2z_3^3+z_3^4\big),
\end{aligned}$$

$$\begin{aligned}
\overline{T}_5(z_1,z_2,z_3) &= \frac{1}{42525}\big(26342z_1^5+107708z_1^4z_2-53854z_1^4z_3+247520z_1^3z_2^2-90956z_1^3z_2z_3\\
&\quad +35786z_1^3z_3^2+360640z_1^2z_2^3-20832z_1^2z_2^2z_3+23772z_1^2z_2z_3^2-8882z_1^2z_3^3\\
&\quad +313600z_1z_2^4+98560z_1z_2^3z_3+3192z_1z_2^2z_3^2-1604z_1z_2z_3^3+592z_1z_3^4\\
&\quad +125440z_2^5+89600z_2^4z_3+9520z_2^3z_3^2-56z_2^2z_3^3-40z_2z_3^4+16z_3^5\big),\\
\overline{T}_6(z_1,z_2,z_3) &= \frac{\pi}{2488320}\big(578624z_1^6+2670432z_1^5z_2-1335216z_1^5z_3+6994704z_1^4z_2^2\\
&\quad -2964576z_1^4z_2z_3+1076988z_1^4z_3^2+12236840z_1^3z_2^3-1891308z_1^3z_2^2z_3\\
&\quad +1121838z_1^3z_2z_3^2-366193z_1^3z_3^3+14294280z_1^2z_2^4+2721180z_1^2z_2^3z_3\\
&\quad +262458z_1^2z_2^2z_3^2-149091z_1^2z_2z_3^3+46227z_1^2z_3^4+10210200z_1z_2^5\\
&\quad +5825820z_1z_2^4z_3+404250z_1z_2^3z_3^2-11379z_1z_2^2z_3^3+1380z_1z_2z_3^4\\
&\quad -291z_1z_3^5+3403400z_2^6+3303300z_2^5z_3+690690z_2^4z_3^2+11935z_2^3z_3^3\\
&\quad -699z_2^2z_3^4+417z_2z_3^5-139z_3^6\big).
\end{aligned}$$

Additionally, the ideal $\langle T_1,\ldots,T_6\rangle$ is not radical, $T_7^2,T_8^2,T_9^2\in\langle T_1,\ldots,T_6\rangle$ but $T_7,T_8,T_9\notin\langle T_1,\ldots,T_6\rangle$.

Remark 3.5. We decided to present the expressions of the period constants obtained directly without the simplification commented in Remark 3.3 because they are short.

3.2 Strong isochronicity and discontinuous isochronous centers

Chen and Zhang in [18] studied the isochronicity of a center in a piecewise Bautin system and they proved Theorems 3.7 and 3.12 (we present in the following) that give us conditions for the origin to be a regular isochronous center of (3.2). In order to do this, they used a modification of two techniques applied for analytic systems, namely linearizations and finding the associated commuting systems (see [1, 76, 87]), and also the fact that the separation line plays the role of the isochronicity radial (remember

Definition 1.23). These results determines how to prove that the conditions found in Theorem 3.15 of Section 3.4 are sufficient.

Definition 3.6 ([18]). Consider the piecewise system (3.2). We say that the center is a *regular isochronous center* if all orbits of the upper system (resp. the lower system) spend the same time to go from the positive (resp. negative) x-axis to the negative (resp. positive) x-axis. The other cases are refereed as *irregular isochronous centers.*

The irregular isochronous centers are more complicated than the regular ones because we have to compute the period functions for the upper system and the lower system separately and prove that their sum is constant near O. In the next theorem we introduce a technique to find regular isochronous centers.

Theorem 3.7 ([18]). *Let O be a center of system* (3.2). *Suppose that one of the following two conditions holds for the upper system:*

(A_+) $\dot{\theta} \equiv 1$ *for* $\theta \in (0, \pi)$ *in* (3.4)*;*

(B_+) O is an isochronous center of the upper system and $L(\pi)$ is an isochronicity radial (see Definition 1.23);

and that one of the following two conditions holds for the lower system:

(A_-) $\dot{\theta} \equiv 1$ *for* $\theta \in (\pi, 2\pi)$ *in* (3.4)*;*

(B_-) O is an isochronous center of the lower system and $L(\pi)$ is an isochronicity radial.

Then, the center O of system (3.2) *is isochronous and regular.*

Proof. It follows directly from Definition 1.23 and that the positive half-period function is $T^+(\rho, \lambda^+) = \int_0^\pi 1/\dot{\theta} d\theta$ and the negative half-period function is $T^-(\Pi^+(\rho, \lambda^+), \lambda^-) = \int_\pi^{2\pi} 1/\dot{\theta} d\theta$. □

Note that, if the condition (A_+) (resp. (A_-)) holds, we have that the upper (resp. lower) system of (3.2) has a uniform isochronous center at the origin.

Before we present the next result, we briefly present what it means to say that two vector fields commute and the existence of a linearization for smooth planar vector fields.

Given two smooth planar vector fields $F(x, y)$ and $G(x, y)$ given by

$$(\dot{x}, \dot{y}) = (F_1(x, y), F_2(x, y)), \tag{3.16}$$

and

$$(\dot{x}, \dot{y}) = (G_1(x, y), G_2(x, y)), \tag{3.17}$$

respectively, we denote by $\phi(t,p)$ (resp. $\psi(s,p)$) the solution of (3.16) (resp. (3.17)), such that $\phi(0,p)=p$ (resp. $\psi(0,p)=p$).

Let A and B be positive real numbers, and let $R=[0,A]\times[0,B]$ be the parametric rectangle.

Definition 3.8. The local flows $\phi(t,p)$ and $\psi(t,p)$ *commute* if, for every parametric rectangle R such that both $\phi(t,\psi(s,p))$ and $\psi(s,\phi(t,p))$ exist whenever $(t,p)\in R$, then

$$\phi(t,\psi(s,p))=\psi(s,\phi(t,p)).$$

By a classical result (see [15]), two local flows commute if and only if their Lie bracket is equal to 0, i.e. $[F,G]=0$. Then, F and G *commute*, or that G is a *commutator* of F, if they satisfy the next equations:

$$\begin{aligned}\left(F_1\frac{\partial G_1}{\partial x}-G_1\frac{\partial F_1}{\partial x}\right)+\left(F_2\frac{\partial G_1}{\partial y}-G_2\frac{\partial F_1}{\partial y}\right)&\equiv 0,\\ \left(F_1\frac{\partial G_2}{\partial x}-G_1\frac{\partial F_2}{\partial x}\right)+\left(F_2\frac{\partial G_2}{\partial y}-G_2\frac{\partial F_2}{\partial y}\right)&\equiv 0.\end{aligned}$$

If $G_1=F_2$ and $G_2=-F_1$ we say that such systems are *orthogonal* to each other.

We recall that F and G, of degrees n and m, are said to be *transversal* to each other at a point (x,y) if

$$F_1(x,y)G_2(x,y)-F_2(x,y)G_1(x,y)\neq 0.$$

Theorem 3.9 ([1])**.** *The smooth system*

$$\begin{cases}\dot{x}=-y+p(x,y),\\ \dot{y}=x+q(x,y),\end{cases}\tag{3.18}$$

where p and q are analytic functions in a neighborhood of the origin starting with terms at least of degree two, has an isochronous center at the origin O if and only if there exists a smooth vector field $(\dot{x},\dot{y})=(x+O_2(|x,y|^2),y+O_2(|x,y|^2))$, defined in a neighborhood of O such that it commutes with the vector field defined by (3.18) *and it is transversal at nonsingular points.*

Also in [76], the authors introduced the linearization criterion for isochronicity of a center given by the next result.

Theorem 3.10 ([76])**.** *A center of the analytic system* (3.18) *is isochronous if and only if there exists an analytic change of coordinates of the form*

$$\begin{aligned}u&=x+O(|(x,y)|),\\ v&=y+O(|(x,y)|),\end{aligned}$$

reducing the system to the linear isochronous system $(\dot{u},\dot{v})=(-v,u)$.

The next definition is important in this context.

Definition 3.11 ([18]). A *consistent linearizing transformation* of the piecewise system (3.2) is a transformation

$$(u,v)=\begin{cases}(x+V^{+}(x,y),y+yW^{+}(x,y)), & \text{if } y\geq 0,\\ (x+V^{-}(x,y),y+yW^{-}(x,y)), & \text{if } y<0,\end{cases} \tag{3.19}$$

where $V^{\pm}(x,y)$ and $yW^{\pm}(x,y)$ are analytic functions starting at least with degree 2 and $V^{+}(x,0)\equiv V^{-}(x,0)$, if the upper transformation and the lower one in (3.19) reduce the upper system and the lower one in (3.2), respectively, to $(\dot{u},\dot{v})=(-v,u)$.

By using the two techniques for analytic systems, linearizations by Theorem 3.10 and finding commuting systems by Theorem 3.9, the authors of [18] obtained isochronicity conditions for piecewise systems as follows in Theorem 3.12.

Theorem 3.12 ([18]). *The equilibrium point at the origin O for the piecewise system* (3.2) *is a regular isochronous center if one of the following conditions is satisfied:*

(i) O is a center of system (3.2)*. Moreover, the upper system (resp. the lower system) in* (3.2) *has either a linearizing transformation of the form $(u,v)=(x+V^{+}(x,y),y+yW^{+}(x,y))$ (resp. $(u,v)=(x+V^{-}(x,y),y+yW^{-}(x,y))$), or a transversal commuting system of the form $(\dot{x},\dot{y})=(x+F^{+}(x,y),y+yG^{+}(x,y))$ (resp. $(\dot{x},\dot{y})=(x+F^{-}(x,y),y+yG^{-}(x,y))$), where $V^{\pm}(x,y)$, $yW^{\pm}(x,y)$, $F^{\pm}(x,y)$ and $yG^{\pm}(x,y)$ are analytic functions starting at least with degree* 2*.*

(ii) System (3.2) *has a consistent linearizing transformation of the form* (3.19)*.*

3.3 Bifurcation of local critical periods

In this section, we shall prove a result that provides lower bounds for the criticallity problem that is a limitation of the number of local critical periods that bifurcates from a weak center of finite order (see Definition 1.44).

Henceforth, we will denote by $T_j^{[k]}$ the k-th order terms of the period constant T_j.

In order to prove the next theorem, it is convenient to do a change in the parameter space given by

$$\begin{aligned} a_1&=(w_1+z_1)/2, & a_2&=(w_2+z_2)/2, & a_3&=(w_3+z_3)/2,\\ b_1&=(w_1-z_1)/2, & b_2&=(w_2-z_2)/2, & b_3&=(w_3-z_3)/2. \end{aligned} \tag{3.20}$$

This change is natural, because as the upper system is equal to the lower one, after the change $(a_1,a_2,a_3)\to(b_1,b_2,b_3)$, it is reasonable to think about the case where both systems are coincident, then the difference is equal to zero, and the other cases are sums. Also, note that in Lemma 3.4 the period constants are differences or sums of the same expression in the three variables depending on whether it is odd or even.

Again we use the period constants found in Lemma 3.4 and consider p_1 as T_1 after the change (3.20). Next, for even $j \geq 2$, p_j is T_j/π after the change of variables reduced with respect to the ideal $\langle p_1, \ldots, p_{j-1} \rangle$, and for odd $j > 2$, p_j is T_j after the change of variables reduced with respect to the ideal $\langle p_1, \ldots, p_{j-1} \rangle$.

Now we state and prove our first result of this section.

Theorem 3.13. *For system* (3.1) *the following families have a weak center of order* 5 *and the number of local critical periods bifurcating from the origin when perturbing inside the class of reversible quadratic system is at most* 5*:*

(i) $a_1 = b_1 = -5a_2/3,\ a_3 = b_3 = -2a_2/3,\ b_2 = a_2$*;*

(ii) $a_1 = \alpha a_2,\ a_3 = -5(3559\alpha + 4361)a_2/(401\alpha + 827),\ b_2 = a_2$*, where* α *is one of the two roots of the polynomial* $p(\alpha) = 2\alpha^2 + 15\alpha + 15$.

Proof. Using the computer algebra system Maple, we solve $p_1 = p_2 = \cdots = p_5 = 0$ and we consider the solutions such that $p_6 \neq 0$. Then, those ones represent weak centers of order 5 and are given by:

(1) $w_1 + 5w_2/3 = w_3 + 2w_2/3 = z_i = 0,\ i = 1, 2, 3$;

(2) $w_1 - \alpha w_2 = w_3 + 5\,(3559\alpha + 4361)\,w_2/2\,(401\alpha + 827) = z_i = 0\ i = 1, 2, 3$, where α is one of the two roots of the polynomial $p(\alpha) = 2\alpha^2 + 15\alpha + 15$;

(3) $w_1 - \alpha w_2 = w_3 - p_1(\alpha)w_2/p_2(\alpha) = z_1 - \big(q(\alpha) + k_1\beta^2\big)p_3(\alpha)w_2/p_4(\alpha) = z_2 - k_2\big(q(\alpha) + k_1\beta^2\big)w_2 = z_3 - \big(q(\alpha) + k_1\beta^2\big)p_5(\alpha)w_2/p_6(\alpha)$, where α is a real root of a polynomial $r(\alpha)$ of degree 31, $q(\alpha)$ and $p_i(\alpha)$, for $i = 1, \ldots, 6$, are polynomials of degree 30, and k_i is a constant value, for $i = 1, 2$.

The polynomials in item (3) are not shown here because of their large size: $r(\alpha)$ has degree 31 in which the coefficients possess at least 34 digits; $q(\alpha)$ has coefficients with at least 708 digits; $p_1(\alpha)$ has coefficients with at least 330 digits; $p_2(\alpha)$ has coefficients with at least 328 digits; $p_3(\alpha)$ has coefficients with at least 2189 digits; $p_4(\alpha)$ has coefficients with at least 2177 digits; $p_5(\alpha)$ has coefficients with at least 1455 digits; $p_6(\alpha)$ has coefficients with at least 1466 digits; k_1 has 682 digits; and $k_2 = 1/580608000$.

The conditions (1) and (2) are, respectively, equivalent to conditions (i) and (ii) of Theorem (3.13). Now we analyze the bifurcation problem in these cases.

Under the condition (i), system (3.1) can be written as

$$(\dot{x},\dot{y}) = \begin{cases} (-y - 5a_2x^2/3 + a_2y^2, x - 2a_2xy/3), & \text{if } y \geq 0, \\ (-y - 5a_2x^2/3 + a_2y^2, x - 2a_2xy/3), & \text{if } y < 0. \end{cases} \tag{3.21}$$

It is clear that we only have to consider the case $a_2 \neq 0$. Additionally, with the map $(x,y) \mapsto (3a_2^{-1}x, 3a_2^{-1}y)$ we transform (3.21) into

$$(\dot{x},\dot{y}) = \begin{cases} (-y - 5x^2 + 3y^2, x - 2xy), & \text{if } y \geq 0, \\ (-y - 5x^2 + 3y^2, x - 2xy), & \text{if } y < 0, \end{cases} \tag{3.22}$$

and for this case, by using the period constants found in Lemma 3.4, we have $T_1 = \cdots = T_5 = 0$ and $T_6 = 63\pi/5$. We consider the time-reversible perturbation in terms of the parameters $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ in system (3.22):

$$(\dot{x},\dot{y}) = \begin{cases} (-y + (-5+e_1)x^2 + (3+e_2)y^2, x + (-2+e_3)xy), & \text{if } y \geq 0, \\ (-y + (-5+e_4)x^2 + (3+e_5)y^2, x + (-2+e_6)xy), & \text{if } y < 0. \end{cases}$$

Then, the linear development in power series with respect to the perturbative parameters of the first five period constants is given by

$$\begin{aligned} T_1^{[1]} &= \frac{1}{3}\big(2e_1 + 4e_2 - 2e_3 - 2e_4 - 4e_5 + 2e_6\big), \\ T_2^{[1]} &= \frac{\pi}{2}e_2 + \frac{3\pi}{4}e_3 + \frac{\pi}{2}e_5 + \frac{3\pi}{4}e_6, \\ T_3^{[1]} &= 2e_1 + \frac{8}{5}e_2 - 4e_3 - 2e_4 - \tfrac{8}{5}e_5 + 4e_6, \\ T_4^{[1]} &= -2\pi e_1 + 5\pi e_3 - 2\pi e_4 + 5\pi e_6, \\ T_5^{[1]} &= \frac{2966}{105}e_1 + \frac{524}{75}e_2 - \frac{27898}{525}e_3 - \frac{2966}{105}e_4 - \frac{524}{75}e_5 + \frac{27898}{525}e_6, \end{aligned}$$

and the rank of these linear developments with respect to the parameters $\{e_1, e_2, e_3, e_4, e_5\}$ is 5. Then, by using the Implicit Function Theorem, there exists a linear change of variables in the parameter space, well defined, in a neighborhood of the origin, such that $T_k = v_k$, for $k = 1, \ldots, 5$ and at least 4 simple critical periods can bifurcate. Furthermore, as the sixth period constant is not equal to zero, we can apply the Weierstrass Preparation Theorem to obtain that at most 5 local critical periods bifurcating from this center.

The proof is analogous for the condition (ii) of Theorem 3.15. □

Remark 3.14. It can be proved that Theorem 3.13 is not classificatory because there is another family given by polynomials in which the coefficients are integer numbers between 34 and 2226 digits long. So there would be a lot of numerical work to reach the same conclusion that have already been obtained through (i) and (ii).

3.4 Perturbing piecewise isochronous quadratic systems

In this section we prove a theorem that provides all the conditions in the parameters for family (3.1) to have an isochronous center at the origin and we study lower bounds for the criticality of such isochronous centers. From Remark 1.56, we can apply the technique developed in Section 1.6 for the study of the criticallity of the families of isochronous centers classified. We shall see that at least 4 local critical periods can unfold in the class of piecewise reversible quadratic systems using only first order developments. In all the studied cases, no more critical periods can be found using higher order developments up to order 4.

It is important to highlight that the problem of isochronicity for analytical quadratic systems was studied by Loud [66], where he proved that the origin is an isochronous center if and only if through a linear change of coordinates and a time rescaling the quadratic system can be written as one of the following systems, denoted by S_1, S_2, S_3, and S_4 in [15]:

$$S_1 : \begin{cases} \dot{x} = -y + x^2 - y^2, \\ \dot{y} = x + 2xy, \end{cases} \qquad S_2 : \begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy, \end{cases}$$
$$S_3 : \begin{cases} \dot{x} = -y - 4x^2/3, \\ \dot{y} = x - 16xy/3, \end{cases} \qquad S_4 : \begin{cases} \dot{x} = -y + 16x^2/3 - 4y^2/3, \\ \dot{y} = x + 8xy/3. \end{cases} \tag{3.23}$$

Note that after applying the change of variables $(x, y, t) \mapsto (x, -y, -t)$ in family (3.1), which we shall denote by η, the upper and the lower systems exchange their equations. Note that what actually happens is an interchange of parameters

$$(a_1, a_2, a_3, b_1, b_2, b_3) \leftrightarrow (-b_1, -b_2, -b_3, -a_1, -a_2, -a_3).$$

As in the previous section, we denote by p_1 the first period constant T_1 after the change presented in (3.20), and denote by p_j the constant T_j/π after the change of variables (3.20) reduced with respect to the ideal $\langle p_1, \ldots, p_{j-1} \rangle$ for even $j \geq 2$, and for odd $j > 2$, the period constant T_j after the change of variables (3.20) reduced with respect to the ideal $\langle p_1, \ldots, p_{j-1} \rangle$.

Theorem 3.15. *Up to a linear change of coordinates and a time rescaling including the change η, the origin is an isochronous center of the piecewise quadratic system* (3.1) *if and only if either the system is the isochronous linear center or one of the following conditions holds:*

(i) $a_1 = b_1 = 1$, $a_2 = b_2 = -1$, $a_3 = b_3 = 2$;

(ii) $a_2 = b_2 = a_3 - a_1 = b_3 - b_1 = 0$;

(iii) $a_1 = b_1 = -1$, $a_2 = b_2 = 0, a_3 = b_3 = -4$;

(iv) $a_1 + 4a_2 = a_3 + 2a_2 = b_1 + 4b_2 = b_3 + 2b_2 = 0$;

(v) $a_1 = -1$, $a_2 = 1$, $a_3 = -2$, $b_1 = -1$, $b_2 = 0, b_3 = -4$;

(vi) $a_2 = a_3 - a_1 = b_1 + 4b_2 = b_3 + 2b_2 = 0$.

Proof. By solving $p_1 = p_2 = \cdots = p_6 = 0$, we get 8 necessary conditions to obtain isochronous centers which are expressed and separated into their respective types in Table 3.1.

Table 3.1: Isochronous conditions

Type	Isochronous conditions found by solving $p_1 = p_2 = \cdots = p_6 = 0$
I	$w_1 + w_2 = 2w_2 + w_3 = z_i = 0,\ i = 1, 2, 3$
II	$w_1 - w_3 = w_2 = z_1 - z_3 = z_2 = 0$
III	$w_2 = w_3 - 4w_1 = z_i = 0,\ i = 1, 2, 3$
IV	$w_1 + 4w_2 = 2w_2 + w_3 = z_1 + 4z_2 = 2z_2 + z_3 = 0$
V	(1) $w_1 + 2w_2 = 6w_2 + w_3 = w_2 - z_2 = 2w_2 - z_3 = z_1 = 0$ (2) $w_1 + 2w_2 = 6w_2 + w_3 = w_2 + z_2 = 2w_2 + z_3 = z_1 = 0$
VI	(1)$w_1 + 2w_2 - w_3 = w_1 + 8w_2 + z_1 = w_2 - z_2 = w_1 + 6w_2 + z_3 = 0$ (2) $w_1 + 2w_2 - w_3 = w_1 + 8w_2 - z_1 = w_2 + z_2 = w_1 + 6w_2 - z_3 = 0$

The conditions in type I, type II, type IV, type V, and type VI in Table 3.1 are respectively equivalent to conditions (i), (ii), (iii), (iv), (v), and (vi), and now we shall prove that they are sufficient.

Related to list (3.23), systems (i) and (iii) are equivalent to the isochronous quadratic systems S_1 and S_3, (ii) is equivalent to S_2 if $a_1 = b_1$, and (iv) is equivalent to S_4, if $a_2 = b_2$. In [18] we find the proof of the conditions for systems (3.1) to have isochronous centers at the origin in each case: (ii) with $a_1 \neq b_1$, (iv) with $a_2 \neq b_2$, (v) and (vi). By completeness, we reproduce it here by using the definitions and results of Section 3.2.

When condition (ii) holds with $a_1 \neq b_1$, system (3.1) takes the form

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + a_1x^2, x + a_1xy), & \text{if } y \geq 0, \\ (-y + b_1x^2, x + b_1xy), & \text{if } y < 0. \end{cases} \tag{3.24}$$

After the change of variables $(x, y) \mapsto (r\cos\theta, r\sin\theta)$, we can transform the upper and lower systems one in (3.24) into the polar coordinates form and obtain that $\dot{\theta} \equiv 1$ for any θ and, by Theorem 3.7, it follows that O is an isochronous center.

When condition (iv) holds with $a_2 \neq b_2$, system (3.1) takes the form

$$(\dot{x}, \dot{y}) = \begin{cases} (-y - 4a_2x^2 + a_2y^2, x - 2a_2xy), & \text{if } y \geq 0, \\ (-y - 4b_2x^2 + b_2y^2, x - 2b_2xy), & \text{if } y < 0, \end{cases} \tag{3.25}$$

which has a transversal commuting system

$$(\dot{x}, \dot{y}) = \begin{cases} (x - 4a_2xy + 4a_2^2xy^2, y - 3a_2y^2 + 2a_2^2y^3), & \text{if } y \geq 0, \\ (x - 4b_2xy + 4b_2^2xy^2, y - 3b_2y^2 + 2b_2^2y^3), & \text{if } y < 0. \end{cases}$$

Thus, by Theorem 3.12, the origin O is an isochronous center of (3.25).

About the condition (v), we have

$$(\dot{x}, \dot{y}) = \begin{cases} (-y - x^2, x - 4xy), & \text{if } y \geq 0, \\ (-y - x^2 + y^2, x - 2xy), & \text{if } y < 0. \end{cases} \tag{3.26}$$

For the upper system of (3.26), it is possible to prove that it has an integrating factor $\mu(y) = (1 - 4y)^{-3/2}$, from which we obtain a first integral $H^+(x, y) = (1 + 2x^2 - 2y)/\sqrt{1 - 4y}$, hence, the integrating curve passing though $(\rho, 0)$ can be expressed as $1 + 2x^2 - 2y - (1 + 2\rho^2)\sqrt{1 - 4y} = 0$. Thus, $y = (1 + 2\rho^2)\sqrt{\rho^2 + \rho^4 - x^2} - 2\rho^2 - 2\rho^4 + x^2$. Therefore, the positive half-period function can be computed as

$$\begin{aligned} T^+(\rho) &= \int_{\rho}^{-\Pi^+(\rho)} \frac{dx}{\dot{x}} = \int_{\rho}^{-\rho} \frac{dx}{-y - x^2} \\ &= \int_{\rho}^{-\rho} \frac{dx}{2\rho^2 + 2\rho^4 - 2x^2 - (1 + 2\rho^2)\sqrt{\rho^2 + \rho^4 - x^2}} \\ &= 2\int_{\rho}^{0} \frac{dx}{2\rho^2 + 2\rho^4 - 2x^2 - (1 + 2\rho^2)\sqrt{\rho^2 + \rho^4 - x^2}} \\ &= 2\int_{\arcsin\left(\frac{1}{\sqrt{1+\rho^2}}\right)}^{0} \frac{d\alpha}{2\rho\sqrt{1 + \rho^2}\cos\alpha - (1 + 2\rho^2)} \\ &= 4\arctan(\rho + \sqrt{1 + \rho^2}), \end{aligned} \tag{3.27}$$

where a change of variables $x = \rho\sqrt{1 + \rho^2}\sin\alpha$ is applied in the fourth line. For the lower system, we apply the change of variables $x = r\cos\theta$ and $y = r\sin\theta$ to transform it into the polar coordinate form $\dot{r} = -r^2\cos\theta$, $\dot{\theta} = 1 - r\sin\theta$. Solving the first order differential equation

$$\frac{dr}{d\theta} = \frac{-r^2\cos\theta}{1 - r\sin\theta},$$

associated to the initial condition $r(\pi) = \Pi^+(\rho) = \rho$, we get that

$$r(\theta) = -\rho^2\sin\theta + \rho\sqrt{\rho^2\sin^2\theta + 1}.$$

Thus, the negative half-period function can be computed as

$$
\begin{aligned}
T^-(\rho) &= \int_\pi^{2\pi} \frac{d\theta}{\dot\theta} = \int_\pi^{2\pi} \frac{d\theta}{1+\rho^2\sin^2\theta - \rho\sin\theta\sqrt{\rho^2\sin^2\theta+1}} \\
&= \int_\pi^{2\pi} \frac{\sqrt{\rho^2\sin^2\theta+1}+\rho\sin\theta}{\sqrt{\rho^2\sin^2\theta+1}} d\theta \\
&= \pi - \rho\int_\pi^{2\pi} \frac{d\cos\theta}{\rho^2+1-\rho^2\cos^2\theta} \\
&= \pi - \int_{-1}^{1} \frac{ds}{\sqrt{\frac{\rho^1+1}{\rho^2}-s^2}} = \pi - 2\arcsin\left(\frac{\rho}{\sqrt{1+\rho^2}}\right).
\end{aligned} \tag{3.28}
$$

Note that $\beta := 2\arctan\left(\rho+\sqrt{1+\rho^2}\right) - \arcsin\left(\frac{\rho}{\sqrt{1+\rho^2}}\right) \in [-\pi/2, \pi]$, and

$$
\sin\beta = \frac{-\rho}{\sqrt{1+\rho^2}}\frac{1-(\rho+\sqrt{1+\rho^2})^2}{1+(\rho+\sqrt{1+\rho^2})^2} + \frac{2(\rho+\sqrt{1+\rho^2})}{1+(\rho+\sqrt{1+\rho^2})^2}\frac{1}{\sqrt{1+\rho^2}} \equiv 1,
$$

which implies that $\beta = \pi/2$. It follows from (3.27), (3.28), and $\beta = \pi/2$ that $T(\rho) = T^+(\rho) + T^-(\rho) = \pi + 2\beta \equiv 2\pi$. Therefore, O is an isochronous center of (3.26).

When the condition (vi) holds, system (3.1) takes the form

$$
(\dot x, \dot y) = \begin{cases} (-y + a_1x^2, x + a_1xy), & \text{if } y \geq 0, \\ (-y - 4b_2x^2 + b_2y^2, x - 2b_2xy), & \text{if } y < 0, \end{cases} \tag{3.29}
$$

which has a transversal commuting system

$$
(\dot x, \dot y) = \begin{cases} (x + a_1xy, y + a_1y^2), & \text{if } y \geq 0, \\ (x - 4b_2xy + 4b_2xy^2, y - 3b_2y^2 + 2b_2^2y^3), & \text{if } y < 0. \end{cases}
$$

Thus, by Theorem 3.12, the origin O is an isochronous center of (3.29).

Then, $p_1 = \cdots = p_6 = 0$ implies that the origin is an isochronous center, since for each case we do not need the period constant p_7 to get the isochronous center condition.

□

The last results of this chapter aim to investigate lower bounds for the criticality by adding a reversible perturbation on some isochronous centers obtained in Theorem 3.15 by applying the results presented in Section 1.6.

Proposition 3.16. *Consider the isochronous quadratic systems given by condition (i) in Theorem 3.15. The number of critical periods bifurcating from the origin when perturbing into the class of piecewise reversible quadratic system is at most* 3.

Proof. We consider the time-reversible quadratic perturbation

$$(\dot{x},\dot{y}) = \begin{cases} (-y+(1+e_1)x^2+(-1+e_2)y^2, x+(-2+e_3)xy), & \text{if } y \geq 0,\\ (-y+(1+e_4)x^2+(-1+e_5)y^2, x+(-2+e_6)xy), & \text{if } y < 0. \end{cases}$$

The matrix of $T_1^{[1]}$, $T_2^{[1]}$, $T_3^{[1]}$, and $T_5^{[1]}$ with respect to the parameters $\{e_1,e_2,e_3,e_4\}$ is given by

$$\begin{pmatrix} \frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3}\\ -\frac{\pi}{2} & -\frac{\pi}{2} & 0 & -\frac{\pi}{2}\\ -\frac{2}{15} & -\frac{8}{15} & \frac{4}{5} & \frac{2}{15}\\ \frac{2}{35} & \frac{12}{35} & -\frac{6}{7} & -\frac{2}{35} \end{pmatrix} \tag{3.30}$$

and it has rank four since, its determinant is equal to $-64\pi/1575$. Then, there exists a linear change of variables given by

$$\begin{aligned} e_1 &= \frac{1}{32\pi}\big(99\pi u_1 - 32u_2 + 570\pi u_3 + 455\pi u_4 - 32\pi u_5\big),\\ e_2 &= -\frac{27}{16}u_1 - \frac{105}{8}u_3 - \frac{175}{16}u_4 + u_5,\\ e_3 &= -\frac{3}{8}u_1 - \frac{15}{4}u_3 - \frac{35}{8}u_4 + u_6,\\ e_4 &= -\frac{1}{32\pi}\big(45\pi u_1 + 32u_2 + 150\pi u_3 + 105\pi u_4 + 32\pi u_5\big), \end{aligned} \tag{3.31}$$

where $u_5 = e_5$ and $u_6 = e_6$, such that $T_1^{[1]} = u_1$, $T_2^{[1]} = u_2$, $T_3^{[1]} = u_3$, $T_5^{[1]} = u_4$. Using these expressions we can rewrite $T_1,\ldots,T_5$ in terms of these new variables. By the properties presented in Proposition 1.52 we can simplify the period constant T_4 and then we get $T_4^{[1]} = 0$. Using the simplification $u_1 = u_2 = u_3 = 0$, we obtain the second order terms of T_j, for $j = 4,5$, which depend on the remaining parameters (u_4,u_5,u_6):

$$\begin{aligned} T_4^{[2]} &= \frac{\pi}{24576}\big(385875u_4^2 - 53760u_4u_5 - 26880u_4u_6 + 4096u_5^2 + 4096u_5u_6 + 1024u_6^2\big),\\ T_5^{[2]} &= \frac{u_4}{480}\big(480 + 82215u_4 - 12848u_5 - 5464u_6\big). \end{aligned}$$

As the solutions of $T_4^{[2]} = 0$ are complex, the only way the fourth period constant to be zero is $u_4 = u_5 = u_6 = 0$, so that all the next period constants vanish and, therefore, we do not obtain more than 3 oscillations. □

Proposition 3.17. *Consider the isochronous quadratic systems given by conditions (iii), and (v) presented in Theorem 3.15. The number of critical periods bifurcating from the origin when perturbing in the class of piecewise reversible quadratic system is at least* 4*, using developments up to order* 4.

Proof. We present the proof for the condition (iii) and for the other condition the analysis is carried out in a similar way.

We consider the time-reversible quadratic perturbation

$$(\dot{x},\dot{y}) = \begin{cases} (-y+(-1+e_1)x^2+e_2y^2, x+(-4+e_3)xy), & \text{if } y \geq 0,\\ (-y+(-1+e_4)x^2+e_5y^2, x+(-4+e_6)xy), & \text{if } y < 0. \end{cases}$$

Initially, we compute the first eight period constants and we obtain that the rank of the linear developments of the first five period constants with respect to the parameters $\{e_1, e_2, e_3, e_4, e_5\}$ is 5. Then, up to a linear change of variables in the parameter space as in (3.31), we can write $T_j^{[1]} = u_j$, for $j = 1, \ldots, 5$. Using Proposition 1.52, we can simplify the next period constants to get $T_j^{[1]} = 0$, for $j = 6, 7, 8$. Applying item (i) of Theorem 1.53 up to order 1, we get that only four critical periods bifurcate from the origin. Computing the higher developments up to order 4, and using the simplification mechanism described in item (ii) of Theorem 1.53, we get that $T_6^{[k]} = T_7^{[k]} = T_8^{[k]} = 0$, for $k = 2, 3, 4$. Then, the result follows up to order 4. □

For the case of conditions (ii), (iv), and (vi) presented in Theorem 3.15, we have a polynomial family of isochronous centers parametrized by two parameters. In the parameter plane there are some curves for which the rank of the linear parts of the period constants is not maximal along them. In these more degenerate cases, it would be possible to obtain a greater number of oscillations by using Theorem 1.53 (ii). Thus, it is possible to split into cases and some of them are presented in Table 3.2.

Table 3.2: Regions in parameters space

Type	Condition
II^*	$a_1b_1(a_1+b_1) \neq 0$
II^{**}	$a_1(a_1+b_1)(a_1+2b_1) \neq 0$
IV^*	$a_2b_2(a_2+b_2) \neq 0$
IV^{**}	$a_2(a_2+b_2)(a_2+2b_2) \neq 0$
IV^{***}	$a_2(4a_2+5b_2)(a_2+2b_2) \neq 0$
VI^*	$a_1b_2(a_1-4b_2) \neq 0$
VI^{**}	$a_1 \neq 0$

Proposition 3.18. *Consider the isochronous quadratic systems given by conditions* II^*, IV^*, *and* VI^*. *The number of critical periods bifurcating from the origin when perturbing in the class of piecewise reversible quadratic system is at least* 3, *using developments up to order* 4.

Proof. We shall see why we consider the restriction presented in Table 3.2 for the isochronous center under condition (ii) in Theorem 3.15, for the other cases the study is analogous.

Consider the time-reversible quadratic perturbation for the isochronous center presented in condition (ii) of Theorem 3.15

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + (a_1 + e_1)x^2 + e_2 y^2, x + (a_1 + e_3)xy), & \text{if } y \geq 0, \\ (-y + (b_1 + e_4)x^2 + e_5 y^2, x + (b_1 + e_6)xy), & \text{if } y < 0. \end{cases}$$

Doing a first order analysis, we see that the matrix, as the one presented in (3.30), corresponding to the linear terms of the first four period constants, with respect to the parameters $\{e_1, e_2, e_4, e_5\}$, has determinant equals to $-\pi^2 a_1 b_1 (a_1 + b_1)^4/720$. Then, there exists a linear change, as the one at (3.31), if $a_1 b_1 (a_1 + b_1) \neq 0$. Therefore, we obtain the type II*.

The remaining of the proof is carried out in a similar way as it was done in the proofs of Propositions 3.16 and 3.17. □

Remark 3.19. Under the remaining conditions in Table 3.2 we were unable to obtain a greater number of oscillations than those ones obtained in the last results and, therefore, we shall not detail them here.

CHAPTER 4

Period function for piecewise linear centers at infinity

The analysis of the piecewise linear systems with only two half-planes separated by a straight line started some years ago. The first and most relevant question about periodicity in continuous piecewise linear differential systems was given in 1990 ([67]), when Lum and Chua conjectured that such systems had at most one limit cycle. But, the first proof of this maximality, together with a more complete study of the dynamics of these piecewise systems, was given only in 1998 by Freire, Ponce, Rodrigo, and Torres ([43]). In [45] the authors introduced a Liénard-like canonical form for general piecewise planar systems with two linear systems separated by a straight line. Recently, in [44] the authors classified the *centers at infinity*, that is, when there exists a period annulus at infinity and then the infinity behaves like a center. Moreover, they studied the limit cycles bifurcating from these centers when the infinity has dynamics of focus type in both regions called of *monodromic type*. Additionally, in [7] it was constructed an integral characterization of its Poincaré half-maps. For fundamental properties of these Poincaré half-maps and related studies, we recommend [8, 9].

In this chapter, we present the beginning of the study that we are performing for the period function for the class of piecewise linear systems with a center at infinity that are characterized in [44]. Such systems have two equilibrium points, symmetric with respect to the separation line (y-axis), that can be invisible, visible, or coincident and belong to this line. For the case where both the left and the right systems have a center, we refer as the center-center case and we believe that the period function will have at most one oscillation. The other cases where both systems have a focus, we refer as the focus-focus case. We will prove that there are no oscillations of the period function and we will determine the cases in which the period function is monotonous decreasing, constant, and monotonous increasing. We point out that in this class we have no systems of center-focus type. We chose to carry out the study only for the centers classified in [44], because the solution for the center problem for planar piecewise linear system with two zones is not finished yet.

In this chapter, the period function will be given by the difference $T = T^- - T^+$, where T^- is the left flight time with respect to y-axis, obtained going through the orbit of the left system in the direction of its flow, and T^+ is the right flight time with respect to y-axis obtained considering the opposite direction of the flow of the right system (see Figure 4.1). In the focus-focus cases classified in [44], we will not get critical periods because both T^- and $-T^+$ have the same monotonic behavior, then T is monotonous increasing or monotonous decreasing, if T^- and $-T^+$ are monotonous increasing or monotonous decreasing. Then, we can expect a more interesting and rich behavior for T only in the center-center case, since in this case T^- and $-T^+$ may have different monotonic behaviors and it is possible that oscillations arise.

4.1 Preliminaries

Without loss of generality, we assume that the separation line is $\Sigma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and the system that we consider is given by

$$\dot{\mathbf{x}} = \mathbf{F}(x) = \begin{cases} \mathbf{F}^-(\mathbf{x}) = \left(F_1^-(\mathbf{x}), F_2^-(\mathbf{x})\right)^T = A^-\mathbf{x} + \mathbf{b}^-, & \text{if } \mathbf{x} \in \Sigma^- \cup \Sigma, \\ \mathbf{F}^+(\mathbf{x}) = \left(F_1^+(\mathbf{x}), F_2^+(\mathbf{x})\right)^T = A^+\mathbf{x} + \mathbf{b}^+, & \text{if } \mathbf{x} \in \Sigma^+, \end{cases} \tag{4.1}$$

where $\mathbf{x} = (x, y)^T$, $A^- = (a_{ij}^-)$, and $A^+ = (a_{ij}^+)$ are 2×2 constant matrices with real coefficients, and $\mathbf{b}^- = (b_1^-, b_2^-)^T$, $\mathbf{b}^+ = (b_1^+, b_2^+)^T$ are constant vectors in $\mathbb{R}^2$. This system with $F^-(0, y) = F^+(0, y)$ has been studied in [43] and the one with no continuity assumption was investigated in [45].

In this case, the differential function $h\colon \mathbb{R}^2 \to \mathbb{R}$, having 0 as regular value that defines Σ, is given by $h(x) = x$. The Lie derivative of h in the direction of the vector fields that compose (4.1), $\mathbf{F}^-$ and $\mathbf{F}^+$, on points $(0, y) \in \Sigma$, are given by $\mathbf{F}^-h(0, y) = \big\langle \big(F_1^-(0, y), F_2^-(0, y)\big), \big(1, 0\big) \big\rangle = F_1^-(0, y)$ and $\mathbf{F}^+h(0, y) = \big\langle \big(F_1^+(0, y), F_2^+(0, y)\big), \big(1, 0\big) \big\rangle = F_1^+(0, y)$.

Then, from the definitions presented in Section 1.1, we say that $(0, y) \in \Sigma$ is a crossing point if $F_1^-(0, y)F_1^+(0, y) > 0$, so the crossing set Σ^c is defined as follows:

$$\Sigma^c = \left\{(0, y) : \left(a_{12}^- y + b_1^-\right)\left(a_{12}^+ y + b_1^+\right) > 0\right\}. \tag{4.2}$$

We are interested in the family of such differential systems in which orbits are sufficiently far from the origin and cross the separation line Σ transversally, allowing the existence of periodic orbits lying in both half-planes. These orbits are called crossing periodic orbits and they exist if points $(0, y)$, where y is big enough, are of crossing type. The condition in (4.2) is equivalent to

$$a_{12}^{-}a_{12}^{+} > 0, \tag{4.3}$$

and we will refer to it as the *generic condition.* In fact, when $a_{12}^{-}a_{12}^{+} \leq 0$, it is easy to see that the crossing set, if it exists, is an open interval of the y-axis, bounded for $a_{12}^{-}a_{12}^{+} < 0$ and unbounded for $a_{12}^{-}a_{12}^{+} = 0$. Due to condition (4.3), the points in Σ that cannot be part of a crossing orbit, i.e. sliding or escaping points, form a bounded set.

The linear-linear system (4.1) has twelve parameters. However, there exists a canonical form with only seven parameters, called Liénard-canonical form obtained in Proposition 3.1 of [45] by making a continuous piecewise linear change of variables such that the resulting transformation is a homeomorphism, keeping invariant the separation line Σ and the half-planes Σ^{-} and Σ^{+}. This canonical form is topologically equivalent to the original system, if one is interested in the dynamics not involving sliding orbits.

Proposition 4.1 ([45])**.** *Under the condition* (4.3) *the piecewise linear system* (4.1) *is reduced to the following Liénard canonical form:*

$$\begin{aligned}
\dot{\mathbf{x}} = \boldsymbol{G}^{-}(\mathbf{x}) &= \begin{pmatrix} \mathsf{T}_L & -1 \\ \mathsf{D}_L & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 0 \\ a_L \end{pmatrix}, \quad \textit{if} \quad \mathbf{x} \in \Sigma^{-} \cup \Sigma, \\
\dot{\mathbf{x}} = \boldsymbol{G}^{+}(\mathbf{x}) &= \begin{pmatrix} \mathsf{T}_R & -1 \\ \mathsf{D}_R & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} -b \\ a_R \end{pmatrix}, \quad \textit{if} \quad \mathbf{x} \in \Sigma^{+},
\end{aligned} \tag{4.4}$$

where T_L *(resp.* T_R*) and* D_L *(resp.* D_R*) are the trace and the determinant of the matrix* A^{-} *(resp.* A^{+}*), and*

$$a_L = a_{12}^{-}b_2^{-} - a_{22}^{-}b_1^{-}, \quad b = \frac{a_{12}^{-}}{a_{12}^{+}}b_1^{+} - b_1^{-}, \quad a^{+} = \frac{a_{12}^{-}}{a_{12}^{-}}\left(a_{12}^{+}b_2^{+} - a_{22}^{+}b_1^{+}\right).$$

For the canonical form (4.4), it is enough to consider $b \geq 0$, since it is invariant under the change of variables $(x, y, t) \to (x, -y, -t)$ and the change of parameters

$$(\mathsf{D}_R, \mathsf{D}_L, \mathsf{T}_R, \mathsf{T}_L, a_R, a_L, b) \to (\mathsf{D}_R, \mathsf{D}_L, -\mathsf{T}_R, -\mathsf{T}_L, a_R, a_L, -b),$$

simultaneously. Therefore, the sliding set of (4.4) is determined by

$$\Sigma^{s} = \{(0, y) : y(y - b) \leq 0\} = \{(x, y) : x = 0, 0 \leq y \leq b\},$$

that is the segment joining two invisible tangencies at its endpoints: the origin $(0, 0)$ and the point $(0, b)$. This set becomes repulsive, if $b > 0$ (the normal component of both vector fields points outward from the sliding set), and it shrinks to the origin, if $b = 0$ (the origin is the single sliding point and it is always a topological focus).

The parameters a_L and a_R are related to the position of equilibria and the visibility of the tangencies, and when some of them vanish, we have a boundary equilibrium point. In fact, by computing the sign of $\ddot{x}$ at the tangency points, we obtain

$$\ddot{x}_{|(x,y)=(0,0)} = a_L, \quad \ddot{x}_{|(x,y)=(0,b)} = a_R,$$

so that the left (resp. right) tangency is called visible or real, if $a_L < 0$ (resp. $a_R > 0$), and it is invisible or virtual, if $a_L > 0$ (resp. $a_R < 0$), see [43] for details.

From now on, we suppose that $a_L^2 + \mathsf{D}_L^2 \neq 0$ and $a_R^2 + \mathsf{D}_R^2 \neq 0$, otherwise, there is no possible return to section Σ. Also assume that $b = 0$, i.e. $\Sigma^s = \{(0,0)\}$, then the first equation of system (4.4) evaluated on section Σ is reduced to $\dot{x}|_\Sigma = -y$. Therefore, the flow of the system crosses Σ from the right half-plane Σ^+ to the left half-plane Σ^- when $y > 0$, from Σ^- to Σ^+ when $y < 0$, and it is tangent to Σ at the origin. Then, the origin is the unique tangency point of the flow of the piecewise vector field in the separation line. Under these conditions, system (4.4) becomes

$$\dot{\mathbf{x}} = \begin{cases} (\mathsf{T}_L x - y, \mathsf{D}_L x - a_L), & \text{if } \mathbf{x} \in \Sigma^-, \\ (\mathsf{T}_R x - y, \mathsf{D}_R x - a_R), & \text{if } \mathbf{x} \in \Sigma^+. \end{cases} \tag{4.5}$$

Now we shall define, in the usual way, the Poincaré half-maps and the respective half-flight times of system (4.5) corresponding to the section Σ. Namely, the *forward Poincaré half-map* (resp. *backward Poincaré half-map*), denoted by Π^- (resp. Π^+), and the *left flight time* (resp. *right flight time*), denoted by T^- (resp. T^+).

Consider a point $(0, y_0) \in \Sigma$ with $y_0 \geq 0$ and let $\psi(t, y_0) = (\psi_1(t, y_0), \psi_2(t, y_0))$ be the orbit of the left system of (4.4) that satisfies $\psi(0, y_0) = (0, y_0)$. If there exists a value $T^-(y_0) > 0$ such that $\psi_1(T^-(y_0), y_0) = 0$ and $\psi_1(t, y_0) < 0$, for every $t \in (0, T^-(y_0))$, we say that $y_1 = \psi_2(T^-(y_0), \rho_0) \leq 0$ is the image of y_0 by the *forward Poincaré half-map*, then $y_1 = \Pi^-(y_0)$. The value $T^-(y_0)$ is the corresponding *left flight time* for the solution of (4.4) starting at $(0, y_0)$ and ending at $(0, y_1)$. Considering the solution starting at $(0, y_0)$ we can similarly define the *backward Poincaré half-map* ($\Pi^+(y_0)$) and *right flight time* ($T^+(y_0)$), now backward on time, for the right system of (4.4). See Figure 4.1.

As we consider the right flight time in the opposite direction to the flow of the vector field, then $T^+(y_0) < 0$, for all $y_0 > 0$. Then, the complete flight time, denoted by T, is the left flight time minus the right flight time, i.e. we have the expression

$$T(y_0) = T^-(y_0) - T^+(y_0). \tag{4.6}$$

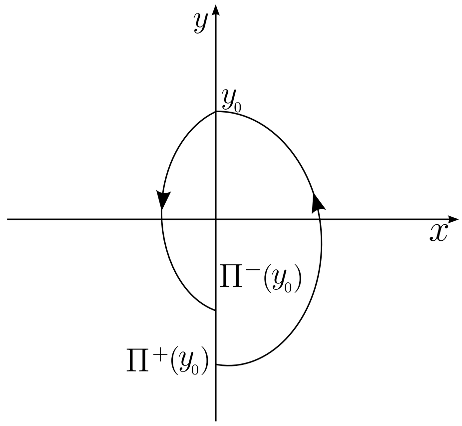

Figure 4.1: Forward and backward Poincaré half-maps associated to $(0, y_0)$, with $y_0 > 0$

Remark 4.2. In the case in which $\Pi^-(0)$ (resp. $\Pi^+(0)$) cannot be defined but for every $\varepsilon > 0$ there exist $y_0 \in (0, \varepsilon)$ and $y_1 \in (-\varepsilon, 0)$ (resp. $y_2 \in (-\varepsilon, 0)$) such that $\Pi^-(y_0) = y_1$ (resp. $\Pi^+(y_0) = y_2$), the left (resp. right) Poincaré half-map can be extended with $\Pi^-(0) = 0$ (resp. $\Pi^+(0) = 0$). That is, having an equilibrium point at the origin or an invisible tangency in the half-plane $\{x < 0\}$ (resp. $\{x > 0\}$).

In Theorem 19 of [7], the authors determine the following integral characterization of the forward and backward Poincaré half-maps and they use it to obtain the half-flight times. The *left Poincaré half-map or forward Poincaré half-map* is the unique function $\Pi^- : I_L \subset [0, \infty) \to (-\infty, 0]$ that, for every $y_0 \in I_L$, satisfies

$$\mathrm{PV}\left\{\int_{\Pi^-(y_0)}^{y_0} \frac{-y}{W_L(y)}\right\} = c_L \mathsf{T}_L, \tag{4.7}$$

where $W_L(y) = \mathsf{D}_L y^2 - a_L \mathsf{T}_L y + a_L^2$ and c_L is given, in terms of the parameters, as follows:

(i) $c_L = 0$, if $a_L > 0$,

(ii) $c_L = \pi(\mathsf{D}_L\sqrt{4\mathsf{D}_L - \mathsf{T}_L^2})^{-1} \in \mathbb{R}$, if $a_L = 0$,

(iii) $c_L = 2\pi(\mathsf{D}_L\sqrt{4\mathsf{D}_L - \mathsf{T}_L^2})^{-1} \in \mathbb{R}$, if $a_L < 0$.

When $a_L = 0$ the integral given in (4.7) diverges and the Cauchy principal value at the origin, which is defined as

$$\mathrm{PV}\left\{\int_{y_1}^{y_0} \frac{-y}{W_L(y)}\right\} = \lim_{\varepsilon \to 0}\left(\int_{y_1}^{-\varepsilon} \frac{-y}{W_L(y)}dy + \int_{\varepsilon}^{y_0} \frac{-y}{W_L(y)}dy\right),$$

for $y_1 < 0 < y_0$ must be used. On the other hand, if $a_L \neq 0$, the integrating function is continuous and we can just take the value of the integral. Moreover, the corresponding left flight time is

$$T^-(y_0) = 2\mathsf{D}_L c_L + \int_{\Pi^-(y_0)}^{y_0} \frac{a_L}{W_L(y)}dy. \tag{4.8}$$

In an analogous way, the *right Poincaré half-map or backward Poincaré half-map* is the unique function $\Pi^{+} : I_R \subset [0,\infty) \to (-\infty, 0]$ that, for every $y_0 \in I_R$, satisfies

$$\mathrm{PV}\left\{\int_{\Pi^{+}(y_0)}^{y_0} \frac{-y}{W_R(y)}\right\} = -c_R \mathsf{T}_R,$$

where $W_R(y) = \mathsf{D}_R y^2 - a_R \mathsf{T}_R y + a_R^2$ and c_R is given, in terms of the parameters, as follows:

(i) $c_R = 0$, if $a_R < 0$,

(ii) $c_R = \pi(\mathsf{D}_R\sqrt{4\mathsf{D}_R - \mathsf{T}_R^2})^{-1} \in \mathbb{R}$, if $a_R = 0$,

(iii) $c_R = 2\pi(\mathsf{D}_R\sqrt{4\mathsf{D}_R - \mathsf{T}_R^2})^{-1} \in \mathbb{R}$, if $a_R > 0$.

Moreover, the corresponding right flight time is

$$T^{+}(y_0) = -2\mathsf{D}_R c_R + \int_{\Pi^{+}(y_0)}^{y_0} \frac{a_R}{W_R(y)} dy. \tag{4.9}$$

Remark 4.3. The smallest positive root of W_L, if it exists, is the right endpoint of I_L, and the greatest negative root of W_L, if exists, is the left endpoint of $\Pi^{-}(I_L)$. If $\mathsf{T}_L^2 - 4\mathsf{D}_L < 0$, then the equilibrium point of the left system of (4.4) is either a focus or a center, and the domain I_L and the range $\Pi^{-}(I_L)$ of the Poincaré half-map Π^{-} are unbounded, and $\Pi^{-}(y_0) \to -\infty$ as $y_0 \to \infty$. In this case, the intervals are $I_L = [0,\infty)$ and $\Pi^{-}(I_L) = (-\infty, 0]$, except when the equilibrium point is a focus (i.e. $\mathsf{T}_L \neq 0$) and it is located in the left half-plane Σ^{-} (i.e. $a_L < 0$). In fact, when $\mathsf{T}_L > 0$, i.e. in the unstable focus case, there exists a value $\hat{y}_1$ such that $\Pi^{-}(0) = \hat{y}_1$ and the interval $\Pi^{-}(I_L)$ is reduced to $(-\infty, \hat{y}_1]$ (see Figure 4.2 (a)). Analogously, for $\mathsf{T}_L < 0$, i.e. in the stable focus case, there exists a value $\hat{y}_0$ such that $(\Pi^{-})^{-1}(0) = \hat{y}_0$ and the interval $I_L = [\hat{y}_0, \infty)$ (see Figure 4.2 (b)).

Remark 4.4. An analogous observation as Remark 4.3 is valid for the backward Poincaré half-map, only with the following change: if the equilibrium point is a focus of the right system of (4.4), that is $\mathsf{T}_R^2 - 4\mathsf{D}_R < 0$, the range $\Pi^{+}(I_R)$ is $(-\infty, \hat{y}_1]$, where $\hat{y}_1 = \Pi^{+}(0)$, if it is a stable focus, that is $\mathsf{T}_R < 0$. The domain I_R is the interval $[\hat{y}_0, \infty)$, with $\hat{y}_0 = (\Pi^{+})^{-1}(0)$, if it is an unstable focus, that is $\mathsf{T}_R > 0$ (see Figure 4.3).

In [44] the authors introduce a symmetric canonical form obtained under the hypotheses $\mathsf{T}_\Lambda^2 - 4\mathsf{D}_\Lambda < 0$, for $\Lambda \in \{L, R\}$, in which both dynamics are of focus type named *Liénard reduced form.* In the following we assume, for shortness, that $\Lambda \in \{L, R\}$.

Proposition 4.5 ([44])**.** *Assume that* $\mathsf{T}_\Lambda = 2\alpha_\Lambda$*,* $\mathsf{D}_\Lambda = (\alpha_\Lambda)^2 + (\omega_\Lambda)^2$ *with* $\omega_\Lambda > 0$ *in the canonical form* (4.4)*, and introduce the parameters* $\gamma_R = \alpha_R/\omega_R$*,* $\gamma_L = \alpha_L/\omega_L$*,*

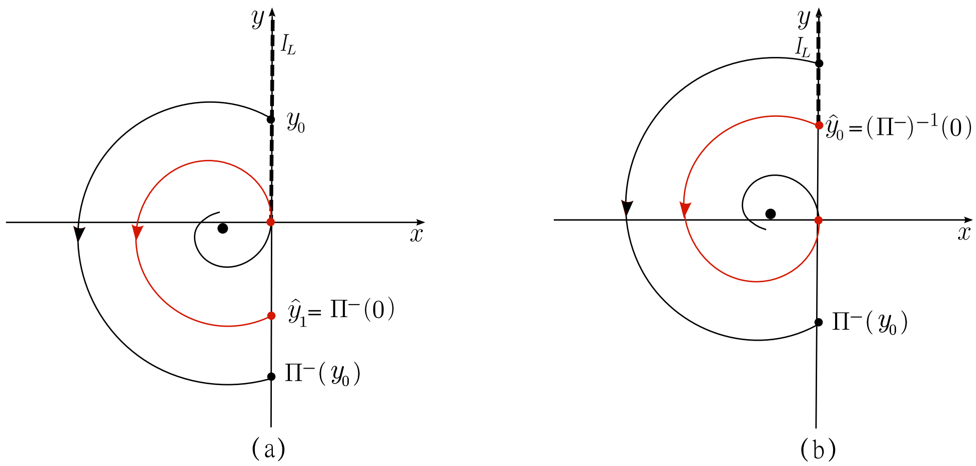

Figure 4.2: The left Poincaré half-map Π^- and its interval of definition I_L for the cases: (a) unstable focus and (b) stable focus

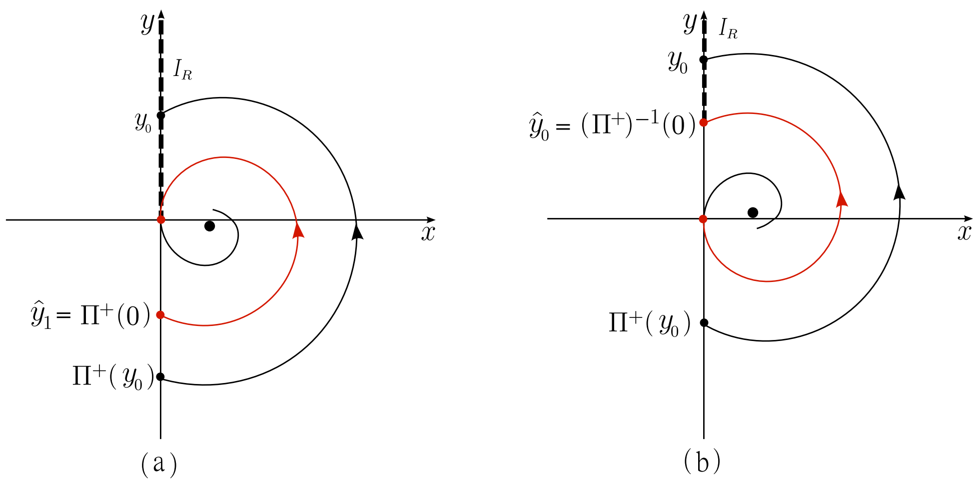

Figure 4.3: The right Poincaré half-map Π^+ and its interval of definition I_R for the cases: (a) stable focus and (b) unstable focus

$\alpha_R = a_R/\omega_R$, *and* $\alpha_L = a_L/\omega_L$. *Then, there exists a change of variable that transforms the canonical form of system* (4.4) *into the form*

$$(\dot{x}, \dot{y}) = \begin{cases} (2\gamma_L x - y, (1+\gamma_L^2)x - \alpha_L), & \text{if } x \leq 0, \\ (2\gamma_R x - y + b, (1+\gamma_R^2)x - \alpha_R), & \text{if } x > 0. \end{cases} \tag{4.10}$$

Proof. Since we have $\omega_\Lambda > 0$ such that $(\omega_\Lambda)^2 = \mathsf{D}_\Lambda - \mathsf{T}_\Lambda^2/4$ and $\sigma_\Lambda = \mathsf{T}_\Lambda/2$, the eigenvalues of the matrix ruling the dynamics on the half-plane $\Sigma^\mp$ in (4.4) is $\sigma_\Lambda \pm i\omega_\Lambda$. The canonical form is obtained by making the change of variables $(x, y, t) \to (X, Y, \tau) := (\omega_\Lambda x, y, \omega_\Lambda t)$ in each half-plane of (4.4) and after introducing the new parameters given by γ_L, γ_R, α_L, and α_R. $\square$

There are two equilibrium points of (4.10) of focus type

$$(x_L, y_L) = (x_L, 2\gamma_L x_L) = \left(\frac{\alpha_L}{1+\gamma_L^2}, \frac{2\gamma_L\alpha_L}{1+\gamma_L^2}\right)$$

and

$$(x_R, y_R) = (x_R, 2\gamma_R x_R + b) = \left(\frac{\alpha_R}{1+\gamma_R^2}, \frac{2\gamma_R\alpha_R}{1+\gamma_R^2} + b\right).$$

The equilibrium points (x_L, y_L) and (x_R, y_R) are stable for $\gamma_L < 0$ and $\gamma_R < 0$, unstable for $\gamma_L > 0$ and $\gamma_R > 0$, and a center if $\gamma_L = \gamma_R = 0$. Such equilibria will be real when $\alpha_L < 0$ or $\alpha_R > 0$, boundary equilibria for $\alpha_L = \alpha_R = 0$, and virtual ones when $\alpha_L > 0$ or $\alpha_R < 0$.

Now, in terms of the coordinates of the equilibrium points (x_L, y_L) and (x_R, y_R) of system (4.10), we can rewrite it as follows:

$$(\dot{x}, \dot{y}) = \begin{cases} \big(2\gamma_L(x - x_L) - (y - y_L), (1+\gamma_L^2)(x - x_L)\big), & \text{if } x \le 0, \\ \big(2\gamma_R(x - x_R) - (y - y_R), (1+\gamma_R^2)(x - x_R)\big), & \text{if } x > 0, \end{cases}$$

or equivalently

$$(\dot{x}, \dot{y}) = \begin{cases} \big(2\gamma_L x - y, (1+\gamma_L^2)(x - x_L)\big), & \text{if } x \le 0, \\ \big(2\gamma_R x - y + b, (1+\gamma_R^2)(x - x_R)\big), & \text{if } x > 0. \end{cases} \tag{4.11}$$

Such systems are invariant under the transformations

$$\begin{aligned} (x, y, \tau, \gamma_L, x_L, b, \gamma_R, x_R) &\mapsto (-x, y, -\tau, -\gamma_R, -x_R, -b, -\gamma_L, -x_L), \\ (x, y, \tau, \gamma_L, x_L, b, \gamma_R, x_R) &\mapsto (x, -y, -\tau, -\gamma_L, x_L, -b, -\gamma_R, x_R), \end{aligned}$$

and their composition

$$(x, y, \tau, \gamma_L, x_L, b, \gamma_R, x_R) \mapsto (-x, -y, \tau, \gamma_R, x_R, b, \gamma_L, x_L).$$

Note that the time τ is the one introduced in the proof of Proposition 4.5, that is, $\tau = \omega_L t$, for the left system, and $\tau = \omega_R t$, for the right system.

Remark 4.6. By Remarks 4.3 and 4.4, if $x_L < 0$ (resp. $x_R > 0$) when $\gamma_L > 0$ (resp. $\gamma_R < 0$), the interval $\Pi^-(I_L)$ is reduced to $(-\infty, \hat{y}_1]$, where $\hat{y}_1 = \Pi^-(0)$ (resp. $\hat{y}_1 = (\Pi^+)^{-1}(0)$), and for $\gamma_L < 0$ (resp. $\gamma_R > 0$), $I_L = [\hat{y}_0, \infty)$ with $\hat{y}_0 = (\Pi^-)^{-1}(0)$ (resp. $\hat{y}_0 = \Pi^+(0)$), see Figures 4.2 and 4.3. In fact, the change $(x, y, t) \to (\omega_L x, y, \omega_L t)$ keeps the origin unchanged.

The next result, obtained in [44], says that we can restrict ourselves to studying the family (4.11) with $b = 0$.

Theorem 4.7 ([44])**.** *System (4.11) has a center at infinity if and only if it is time-reversible with respect to x-axis or y-axis. The centers time-reversible with respect to $y = 0$ if and only if $b = 0$ and $\gamma_L = \gamma_R = 0$. The centers time-reversible with respect to $x = 0$ if and only if $b = 0$, $\gamma_L = -\gamma_R \neq 0$ and either $\alpha_L = \alpha_R = 0$ or $\alpha_L = -\alpha_R \neq 0$.*

The authors of [44] classify the centers at infinity for the family

$$(\dot{x}, \dot{y}) = \begin{cases} \big(2\gamma_L x - y, (1+\gamma_L^2)(x - x_L)\big), & \text{if } x \leq 0, \\ \big(2\gamma_R x - y, (1+\gamma_R^2)(x - x_R)\big), & \text{if } x > 0. \end{cases} \tag{4.12}$$

Consequently, we can use the left Poincaré half-map (4.7) and the left flight time (4.8) and do the following changes

$$\begin{aligned} \mathsf{T}_L &= 2\gamma_L\omega_L, \;\; \mathsf{D}_L = (1+\gamma_L^2)(\omega_L)^2, \;\; \text{and} \;\; a_L = (1+\gamma_L^2)x_L\omega_L, \\ \mathsf{T}_R &= 2\gamma_R\omega_R, \;\; \mathsf{D}_R = (1+\gamma_R^2)(\omega_R)^2, \;\; \text{and} \;\; a_R = (1+\gamma_R^2)x_R\omega_R, \end{aligned} \tag{4.13}$$

on them.

4.2 Study of the flight time

A special case of smooth system is the so called *rigid system* or *uniformly isochronous system*, that is, by using a homeomorphism $\mathbf{X} = h(\mathbf{x})$, such that the new system may be brought into the form $\dot{\theta} \equiv 1$, for any θ after the polar coordinates transformation $X = r\cos\theta$ and $Y = r\sin\theta$. By using that the study of the period function is equivalent to the one in which the system is in its normal form, roughly speaking, we can say that the half-flight time is the angle with respect to the equilibrium point for rigid systems. In particular, we have that each linear system which defines equation (4.12) has this property by the following proposition.

Proposition 4.8. *Consider the differential linear system*

$$\begin{cases} \dot{x} = 2\gamma_\Lambda x - y, \\ \dot{y} = (1+\gamma_\Lambda^2)(x - x_\Lambda), \end{cases} \tag{4.14}$$

where γ_Λ and x_Λ are real. The half-period function for the solution of system (4.14) passing through $(0, y_0)$ and $(0, y_1)$ is the angle of the sector centered at the equilibrium and defined by those points (see Figure 4.4).

Proof. By using the translation $(x, y) \mapsto (x - x_\Lambda, y - 2\gamma_\Lambda x_\Lambda)$ the system becomes

$$\begin{cases} \dot{x} = 2\gamma_\Lambda x - y, \\ \dot{y} = (1+\gamma_\Lambda^2)x. \end{cases}$$

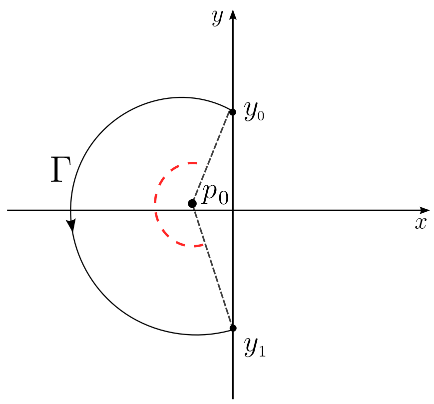

Figure 4.4: Sector centered at the equilibrium p_0 and defined by the points $(0, y_0)$ and $(0, y_1)$

We can apply a linear transformation to the normal form given by

$$(x, y) \to (y, -(1 + \gamma_\Lambda^2)x + \gamma_\Lambda y) := (X, Y)$$

that transforms this system in the Jordan canonical form

$$\begin{cases} \dot{X} = \gamma_\Lambda X - Y, \\ \dot{Y} = X + \gamma_\Lambda Y, \end{cases}$$

where after the polar coordinates transformation, $X = r\cos\theta$, and $Y = r\sin\theta$ we obtain that $\dot{\theta} \equiv 1$ for any θ. □

This result tells us that the behavior of the half-period function of (4.12) is equal to the corresponding flying time which in this case is the angle we go along the solution. Then, in order to study the complete period function, it is enough to determine the behavior of the angles defined by the equilibrium points of the left and right systems of (4.12).

Now we show a simple and important technical result which will be useful later for studying the behavior of the period function for the considered cases.

Definition 4.9. Consider a planar differential system with a monodromic equilibrium point p_0 and a straight line ℓ. Then, ℓ splits the phase plane into two *connected components*, called Σ^- and Σ^+. Fixed the connected component Σ^- (resp. Σ^+), when $p_0 \in \Sigma^-$ (resp. $p_0 \in \Sigma^+$) it is called a *visible equilibrium*, if $p_0 \in \ell$ is a *boundary equilibria*, and when $p_0 \notin \Sigma^-$ (resp. $p_0 \notin \Sigma^+$), it is an *invisible equilibrium*.

In the next result, $\mathrm{d}(\cdot,\cdot)$ denotes the Euclidean distance in $\mathbb{R}^2$.

Proposition 4.10. *Consider a planar differential system with a monodromic equilibrium point p_0 and a straight line ℓ, then $p_0 \in \ell$, or there exists a tangency point p_1 of the flow of the differential system with the straight line ℓ. Fixed the connected component Σ^-, if p_0*

is visible, we take $k_0 = \mathrm{d}(p_1, p_2)$*, where* p_2 *is the first intersection of the orbit that passes through* p_1 *at* $t = 0$ *with the straight line* ℓ*, and* $k_0 = 0$*, if* $p_0 \in \ell$*, or* p_0 *is an invisible point. Since* p_0 *is a monodromic point, for each* $k \in (k_0, \infty) \subset \mathbb{R}^+$*, there exist two points* $q_0^k, q_1^k \in \ell$ *with a piece of orbit connecting them such that* $\mathrm{d}(q_0^k, q_1^k) = k$*. The function which associates each* $k \in [k_0, \infty) \subset \mathbb{R}^+$ *to the angle* $\mathcal{T}_k$ *of the sector centered at its focus and defined by those points* q_0^k *and* q_1^k *in* ℓ *contained in* Σ^- *(see Figure 4.5) satisfies the following conditions:*

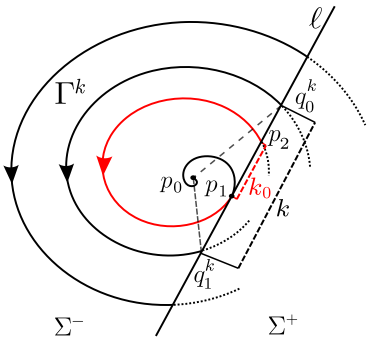

Figure 4.5: Sector centered at its focus p_0 and defined by the points q_0^k and q_1^k in ℓ

(*i*) *is monotonous decreasing, if* p_0 *is visible;*

(*ii*) *is constant, if* $p_0 \in \ell$*;*

(*iii*) *is monotonous increasing, if* p_0 *is invisible.*

Proof. When $p_0 \in \ell$, the angle is constant and equals to π. For the other cases, we consider $k_1 < k_2$ and we fix the connected component Σ^-. For Σ^+ the proof is analogous. Denote by Γ^{k_1} (resp. Γ^{k_2}) the piece of an orbit that starts at $q_0^{k_1}$ (resp. $q_0^{k_2}$) and ends at $q_1^{k_1}$ (resp. $q_1^{k_2}$), where the points $q_0^{k_1}$, $q_1^{k_1}$, $q_0^{k_2}$, and $q_1^{k_2}$ are considered as in the statement, according to Figure 4.6. Note that, if the focus $p_0 \in \Sigma^-$ (resp. $p_0 \notin \Sigma^-$), then the angle determined by it and Γ^{k_1} is smaller (resp. greater) than the angle determined by the equilibrium and Γ^{k_2} and the statement follows. □

Remark 4.11. If we fix the other connected component Σ^+, we have the same conclusions.

The following lemma is a direct consequence of Proposition 4.10.

Lemma 4.12. *Suppose we are under the conditions of Proposition 4.10. Then,*

(*i*) *if* p_0 *is visible,* $\mathcal{T}_{k_0} \in (\pi, 2\pi]$*,*

(*ii*) *if* $p_0 \in \ell$*,* $k_0 = 0$ *and* $\mathcal{T}_0 = \pi$*,*

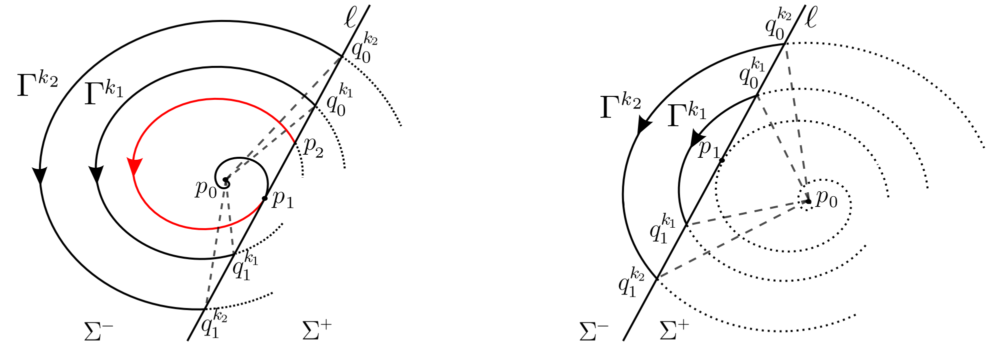

Figure 4.6: The flow, illustration when the equilibrium of focus type $p_0 \in \Sigma^-$ and $p_0 \notin \Sigma^-$, respectively

(iii) if p_0 is invisible, $k_0 = 0$ and $\mathcal{T}_{k_0} = 0$.

Furthermore, if p_0 is of center type and visible, $k_0 = 0$ and $\mathcal{T}_0 = 2\pi$ (see Figure 4.7).

The last statement of this lemma follows by using that, if a line intersects a circle in exactly one point, the line is said to be tangent to the circle, as represented in Figure 4.7.

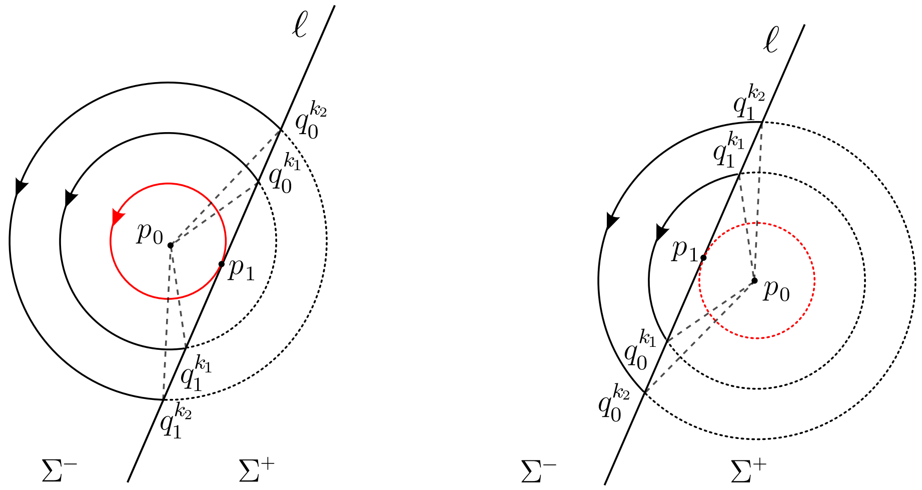

Figure 4.7: The flow, illustration when the equilibrium of center type $p_0 \in \Sigma^-$ and $p_0 \notin \Sigma^-$, respectively

Note that we can do the study of the angle defined by the equilibrium points for the left and right planar differential systems that define system (4.12) taking ℓ being the y-axis. Then, from Proposition 4.8, this is sufficient to determine the behavior of the left (resp. right) flight time that defines T in (4.6).

Remark 4.13. From Proposition 4.8 for the left (resp. right) system, the angle function defined by $k \mapsto \mathcal{T}_k$, obtained in Proposition 4.10, verifies that, for each $k \in \mathbb{R}^+$, $\mathcal{T}_k$ is $T^-(y_0)$ (resp. $-T^+(y_0)$), where $y_0 > 0$ and $(0, y_0)$ is the point in the y-axis such that $\mathrm{d}((0,y_0),(0,\Pi^-(y_0))) = k$ (resp. $\mathrm{d}((0,y_0),(0,\Pi^+(y_0))) = k$).

Proposition 4.14. *Consider the system* (4.12)*, where the left system (resp. right system) has a monodromic equilibrium point at* (x_L, y_L) *(resp.* (x_R, y_R)*). The left (resp. minus the right) flight time* $T^-(y_0)$ *(resp.* $-T^+(y_0)$*) with respect to y-axis associated to the period annulus at infinity, satisfies the following conditions:*

(*i*) *if* $x_L < 0$ *(resp.* $x_R > 0$*), it is monotonous decreasing,*

(*ii*) *if* $x_L = 0$ *(resp.* $x_R = 0$*), it is constant,*

(*iii*) *if* $x_L > 0$ *(resp.* $x_R < 0$*), it is monotonous increasing.*

Proof. Under the condition of statement (*i*), we have a real monodromic visible equilibrium in $(x_L, 2\gamma_L x_L)$ (resp. $(x_R, 2\gamma_R x_R)$). From Proposition 4.10(i), the left (resp. right) flight time is monotonous decreasing.

Due to the condition (*ii*) the origin is a boundary equilibrium and the left (resp. right) flight time is constant from Proposition 4.10(ii).

Regarding statement (*iii*), the system has a monodromic invisible equilibrium point $(x_L, 2\gamma_L x_L)$ (resp. $(x_R, 2\gamma_R x_R)$). From Proposition 4.10(iii), the flight time is monotonous increasing. □

Remark 4.15. In the following, we are considering the angle of the sector centered at the equilibrium and defined by two points in the separation line when we refer to the left and right flight times.

From Lemma 4.12, we have the following conditions for the initial values of the left flight time T^- and minus the right flight time $-T^+$.

Lemma 4.16. *Suppose we are under the conditions of Proposition 4.14 and the domain of the left (resp. right) flight time* $T^-(y_0)$ *(resp.* $-T^+(y_0)$*) is* $I_L = [\hat{y}_0, \infty)$ *(resp.* $I_R = [\hat{y}_0, \infty)$*). Then,*

(*i*) *if* $x_L < 0$ *(resp.* $x_R > 0$*),* $\hat{y}_0 = (\Pi^-)^{-1}(0)$ *(resp.* $\hat{y}_0 = (\Pi^+)^{-1}(0)$*) and* $T^-(\hat{y}_0) \in (\pi, 2\pi]$ *(resp.* $-T^+(\hat{y}_0) \in (\pi, 2\pi]$*),*

(*ii*) *if* $x_L = 0$*,* $\hat{y}_0 = 0$ *and* $T^-(0) = \pi$ *(resp.* $-T^+(0) = \pi$*),*

(*iii*) *if* $x_L > 0$*,* $\hat{y}_0 = 0$ *and* $T^-(0) = 0$ *(resp.* $-T^+(0) = 0$*).*

Furthermore, if (x_L, y_L) *(resp.* (x_R, y_R)*) is of center type and visible,* $\hat{y}_0 = 0$ *and* $T^-(0) = 2\pi$ *(resp.* $-T^+(0) = 2\pi$*).*

We can also directly study the algebraic expressions in equations (4.8) and (4.9) obtained in [7] for the half-flight times, after the substitutions given in (4.13), to determine the behavior of T.

Lemma 4.17. *For the systems that define (4.12), with $\gamma_\Lambda \neq 0$, the half flight times $T^{\mp}$ are given by:*

(i) $T^{\mp}(y_0) = \dfrac{\xi}{\omega_\Lambda}\left(2\pi + \arctan\left(\dfrac{y_0-\gamma_\Lambda x_\Lambda}{x_\Lambda}\right) - \arctan\left(\dfrac{\Pi^{\mp}(y_0)-\gamma_\Lambda x_\Lambda}{x_\Lambda}\right)\right)$, *if* $\xi x_\Lambda < 0$;

(ii) $T^{\mp}(y_0) = \dfrac{\xi\pi}{\omega_\Lambda}$, *if* $x_\Lambda = 0$;

(iii) $T^{\mp}(y_0) = \dfrac{\xi}{\omega_\Lambda}\left(\arctan\left(\dfrac{y_0-\gamma_\Lambda x_\Lambda}{x_\Lambda}\right) - \arctan\left(\dfrac{\Pi^{\mp}(y_0)-\gamma_\Lambda x_\Lambda}{x_\Lambda}\right)\right)$, *if* $\xi x_\Lambda > 0$,

and its first and second derivatives, for all x_Λ, *are:*

$$\begin{aligned}(T^{\mp})'(y_0) &= \frac{\xi x_\Lambda(\Pi^{\mp}(y_0)-y_0)}{\omega_\Lambda((\gamma_\Lambda x_\Lambda - y_0)^2 + x_\Lambda^2)\Pi^{\mp}(y_0)},\\ (T^{\mp})''(y_0) &= -\frac{\xi x_\Lambda(\Pi^{\mp}(y_0)-y_0)Q(y_0)}{\omega_\Lambda((\gamma_\Lambda x_\Lambda - y_0)^2 + x_\Lambda^2)\Pi^{\mp}(y_0)^3},\end{aligned} \tag{4.15}$$

where $Q(y_0) = \big((\gamma_\Lambda x_\Lambda - \Pi^{\mp}(y_0))^2 + x_\Lambda^2\big)\big(\Pi^{\mp}(y_0)+y_0\big) - \Pi^{\mp}(y_0)^2\big(\Pi^{\mp}(y_0)-y_0\big)$, $\Lambda = L$ *and* $\xi = 1$ *for the left flight time* T^-, $\Lambda = R$ *and* $\xi = -1$ *for right flight time* T^+.

Proof. It follows by using the expression given in (4.8) for the left and right flight times and the substitutions given in equation (4.13). □

Lemma 4.18. *For the right system of* (4.12) *with* $x_R = -x_L$ *and* $\gamma_R = -\gamma_L \neq 0$, *the right flight time is given by* $T^+(y_0) = -(\omega_L/\omega_R)\,T^-(y_0)$. *Then, the complete time is* $T(y_0) = (1+\omega_L/\omega_R)\,T^-(y_0)$.

To study the sign of the second derivative of T^-, it is important to determine the sign of the difference $y_0 - (-\Pi^-(y_0))$ that is given in [9] by the next proposition.

Proposition 4.19 ([9])**.** *The following statements hold.*

(i) The left Poincaré half-map Π^- *satisfies* $\operatorname{sign}\big(y_0 + \Pi^-(y_0)\big) = -\operatorname{sign}\big(\mathsf{T}_L\big)$, *for* $y_0 \in I_L \setminus \{0\}$. *In addition, when* $0 \in I_L$ *and* $\Pi^-(0) \neq 0$, *or when* $\mathsf{T}_L = 0$, *the identity also holds for* $y_0 = 0$.

(ii) The right Poincaré half-map Π^+ *satisfies* $\operatorname{sign}\big(y_0 + \Pi^+(y_0)\big) = \operatorname{sign}\big(\mathsf{T}_R\big)$, *for* $y_0 \in I_R \setminus \{0\}$. *In addition, when* $0 \in I_R$ *and* $\Pi^+(0) \neq 0$, *or when* $\mathsf{T}_R = 0$, *the identity also holds for* $y_0 = 0$.

From the last proposition and equation (4.13), we have that

$$\operatorname{sign}\big(y_0 + \Pi^-(y_0)\big) = -\operatorname{sign}\big(\gamma_L\big) \quad \text{and} \quad \operatorname{sign}\big(y_0 + \Pi^+(y_0)\big) = \operatorname{sign}\big(\gamma_R\big). \tag{4.16}$$

4.3 Series expansions of the Poincaré half-map and the flight time

In the next propositions the Taylor or Newton–Puiseux series expansion of the forward Poincaré half-map Π^- at the tangency point is presented, as obtained in [8], for the piecewise system in the Liénard canonical form (4.4), in the different scenarios after the substitutions given in (4.13).

Proposition 4.20 ([8]). *Let $x_L \neq 0$ and $0 \in I_L$. If $\Pi^-(0) = 0$, then the Taylor expansion of Π^- around the origin is written as*

$$\Pi^-(y_0) = \quad -y_0 - \frac{4\gamma_L}{3x_L(1+\gamma_L^2)}y_0^2 - \frac{16\gamma_L^2}{9x_L^2(1+\gamma_L^2)^2}y_0^3 + O(y_0^4). \tag{4.17}$$

Lemma 4.21. *Under the same conditions of Proposition 4.20, the Taylor expansion of the left flight time T^- around the origin for system* (4.12) *writes as*

$$T^-(y_0) = \frac{1}{\omega_L}\left(\frac{2}{x_L(\gamma_L^2+1)}y_0 + \frac{4\gamma_L}{3x_L^2(\gamma_L^2+1)^2}y_0^2 + \frac{2(5\gamma_L^2-3)}{9x_L^3(\gamma_L^2+1)^3}y_0^3 + O(y_0^4)\right). \tag{4.18}$$

Proof. Note that $0 \in I_L$ if and only if $x_L > 0$. In order to get (4.18), it is enough to replace the expansion of $\Pi^-(y_0)$ given in (4.17) on the left flight time $T^-(y_0)$ determined in Lemma 4.17 (iii), and then consider the Taylor expansion around the origin for the obtained expression. □

Remark 4.22. Note that $0 \in I_L$ (resp. $0 \in I_R$) and $\Pi^-(0) = 0$ if and only if $x_L > 0$ (resp. $x_R < 0$), from Remark 4.3 and equation (4.13).

Proposition 4.23 ([8]). *Assume that $0 \in I_L$. If $\Pi^-(0) = \hat{y}_1 < 0$, then $x_L < 0$, $\gamma_L > 0$, $\hat{y}_1$ is the right endpoint of the interval $\Pi^-(I_L)$ (see Figure 4.2(a)). The left Poincaré half-map Π^- is a real analytic function in I_L and its Taylor expansion around the origin writes as*

$$\Pi^-(y_0) = \quad \hat{y}_1 + \frac{(\gamma_L x_L - \hat{y}_1)^2 + x_L^2}{2x_L^2\hat{y}_1(1+\gamma_L^2)}y_0^2 + \frac{2\gamma_L((\gamma_L x_L - \hat{y}_1)^2 + x_L^2)}{3x_L^3\hat{y}_1(1+\gamma_L^2)^2}y_0^3 + O(y_0^4). \tag{4.19}$$

Lemma 4.24. *Under the same conditions of Proposition 4.23 the Taylor expansion of the left flight time T^- around the origin for system* (4.12) *writes as*

$$\begin{aligned} T^-(y_0) = \quad & \frac{1}{\omega_L}\Bigg(T^-(0) + \frac{1}{x_L(1+\gamma_L^2)}y_0 - \frac{(\gamma_L x_L - \hat{y}_1)^2 + x_L^2}{2x_L^2\hat{y}_1(1+\gamma_L^2)^2}y_0^2 \\ & -\frac{(\gamma_L^3 x_L^2 + 4\gamma_L^2 x_L\hat{y}_1 + \gamma_L x_L^2 + \gamma_L\hat{y}_1^2 - 2x_L\hat{y}_1)}{3x_L^3\hat{y}_1(1+\gamma_L^2)^3}y_0^3 + O(y_0^4)\Bigg), \end{aligned} \tag{4.20}$$

where $T^-(0) = 2\pi - \arctan(\gamma_L) - \arctan((\hat{y}_1 - \gamma_L x_L)/x_L) \in (\pi, 2\pi)$.

Proof. In order to get (4.20), it is enough to replace the expansion of $\Pi^-(y_0)$ given in (4.19) on the left flight time $T^-(y_0)$ given in Lemma 4.17 (i), and then consider the Taylor expansion around the origin for the obtained expression. □

Remark 4.25. If the equilibrium point is a visible unstable focus, see the left side of Figure 4.6, it is not easy to determine the extremes of the interval $\Pi^-(I_L)$, i.e. $\hat{y}_1$ such that $\Pi^-(I_L) = (-\infty, \hat{y}_1]$, as we would have to find, by equation (4.7), the solution of the equation

$$\gamma_L \arctan\left(\frac{\gamma_L x_L - \hat{y}_1}{x_L}\right) - \gamma_L \arctan(\gamma_L) + \frac{\ln(\gamma_L^2 x_L^2 + x_L^2)}{2} - \frac{\ln((\gamma_L x_L + \hat{y}_1)^2 + x_L^2)}{2} = 2\gamma_L \pi.$$

When there exists a point $\hat{y}_0$ such that $\Pi^-(\hat{y}_0) = 0$, as illustrated in Figure 4.2(b), the left Poincaré Π^- is a non-analytic function at $\hat{y}_0$ and so the authors of [8] use that the inverse function $(\Pi^-)^{-1}$ is analytic at the origin to get a Newton–Puiseux series expansion for the left Poincaré half-map Π^- around $\hat{y}_0$ by means of an inversion.

Proposition 4.26 ([8])**.** *Assume that there exists a value $\hat{y}_0 > 0$ with $\Pi^-(\hat{y}_0) = 0$. Then, $x_L < 0$, $\gamma_L < 0$, $\hat{y}_0$ is the left endpoint of the interval I_L, the inverse function $(\Pi^-)^{-1}$ is a real analytic function, and the left Poincaré half-map Π^- admits the Newton–Puiseux expansion around the point $\hat{y}_0$ given by*

$$\begin{aligned}\Pi^-(y_0) = & \frac{\sqrt{2\hat{y}_0(1+\gamma_L^2)((\gamma_L x_L - \hat{y}_0)^2 + x_L^2)}}{(\gamma_L x_L - \hat{y}_0)^2 + x_L^2}(y_0 - \hat{y}_0)^{1/2} \\ & - \frac{4\gamma_L x_L \hat{y}_0}{3((\gamma_L x_L - \hat{y}_0)^2 + x_L^2)}(y_0 - \hat{y}_0) + O((y_0 - \hat{y}_0)^{3/2}),\end{aligned} \tag{4.21}$$

which is valid for $y_0 \geq \hat{y}_0$.

Lemma 4.27. *Under the same conditions of Proposition 4.26, the Newton–Puiseux series expansion around the point $\hat{y}_0$ for system* (4.12) *writes as*

$$\begin{aligned}T^-(y_0) = & \frac{1}{\omega_L}\Bigg(T^-(\hat{y}_0) + \left(\sqrt{\frac{2\hat{y}_0}{(1+\gamma^2)((\gamma x_L - \hat{y}_0)^2 + x_L^2)}} + \frac{x_L}{(\gamma x_L - \hat{y}_0)^2 + x_L^2}\right)(y_0 - \hat{y}_0)^{1/2} \\ & + \left(\frac{(\gamma_L x_L - \hat{y}_0)x_L}{((\gamma x_L - \hat{y}_0)^2 + x_L^2)^2} - \frac{2}{3}\frac{\gamma_L \hat{y}_0}{(1+\gamma^2)((\gamma x_L - \hat{y}_0)^2 + x_L^2)}\right)(y_0 - \hat{y}_0) \\ & + O((y_0 - \hat{y}_0)^{3/2}\Bigg),\end{aligned}$$

which is valid for $y_0 \geq \hat{y}_0$, where $T^-(\hat{y}_0) = 2\pi + \arctan(\gamma_L) - \arctan((\gamma_L x_L - \hat{y}_0)/x_L) \in (\pi, 2\pi)$.

Proof. In order to get (4.27), it is enough to replace the expansion of $\Pi^-(y_0)$ given in (4.21) on the left flight time $T^-(y_0)$ given in Lemma 4.17 (i), and then consider the Newton–Puiseux series expansion around the point $\hat{y}_0$ for the obtained expression. □

Remark 4.28. If the equilibrium point is a visible stable focus, see the right side of Figure 4.6, it is not easy to determine the extremes of interval of I_L, i.e. $\hat{y}_0$ such that $I_L = [\hat{y}_0, 0)$, as we would have to find, by equation (4.7), the solution of the equation

$$\gamma_L \arctan(\gamma_L) - \gamma_L \arctan\left(\frac{\gamma_L x_L - \hat{y}_0}{x_L}\right) + \frac{\ln((\gamma_L x_L - \hat{y}_0)^2 + x_L^2)}{2} - \frac{\ln(\gamma_L^2 x_L^2 + x_L^2)}{2} = 2\gamma_L \pi.$$

4.4 Centers at infinity and its period functions

The hypotheses of monodromy at infinity implies that the dynamics is of center-focus type in both regions which is characterized by $\mathsf{T}_\Lambda^2 - 4\mathsf{D}_\Lambda < 0$. This allows the existence of a periodic orbit at infinity. Next proposition provides the classification of the centers at infinity for the canonical form (4.12).

Proposition 4.29 ([44])**.** *There exists a period annulus at infinity for system* (4.12) *if and only if we are in one of the following four cases:*

(i) The conditions $\gamma_L = \gamma_R = 0$ hold, which imply $y_L = y_R = 0$ and the phase plane is the result of the matching of two linear centers $(x_L, 0)$ and $(x_R, 0)$, both symmetric with respect to the x-axis, which can be real or virtual equilibrium. Moreover, the system is reversible and, if at least one of such equilibrium points is virtual, then the center is global.

(ii) The conditions $\gamma_L = -\gamma_R \neq 0$, and $x_L = -x_R \neq 0$, with $x_L < 0 < x_R$ hold, which imply $y_L = y_R \neq 0$. So, we have two real equilibrium points on $(x_L, 2\gamma_L x_L)$ and $(-x_L, 2\gamma_L x_L)$. The phase plane exhibits a reversible nonlinear center at infinity. Such a center is not global ending in a heart-shaped homoclinic orbit to a pseudo-saddle at the origin, which contains the two foci in its interior.

(iii) The conditions $\gamma_L = -\gamma_R \neq 0$, and $x_L = x_R = 0$ hold, which imply $y_L = y_R = 0$, and the origin is a boundary focus from both sides, constituting a reversible global nonlinear center.

(iv) The conditions $\gamma_L = -\gamma_R \neq 0$, and $x_L = -x_R \neq 0$, with $x_R < 0 < x_L$ hold (hence $y_L = y_R \neq 0$), then we have two virtual equilibria at $(x_L, 2\gamma_L x_L)$ and $(-x_L, 2\gamma_L x_L)$. The phase plane exhibits a reversible global nonlinear center at infinity.

The possible phase portraits and locations of the equilibrium points can be visualized in [44] and in Figure 4.8.

Remark 4.30. Let $\Lambda = L$ for the left flight time T^{-}, and $\Lambda = R$ for the right flight time T^{+}. When the origin is either a boundary equilibrium point of the center or a focus type for both systems that define (4.4), i.e. $4\mathsf{D}_\Lambda - \mathsf{T}_\Lambda^2 > 0$, it is known that $T^{\mp}(y_0) = 2\pi/\sqrt{4\mathsf{D}_\Lambda - \mathsf{T}_\Lambda^2}$ (see [43]), for any $y_0 > 0$, then the half-period functions can be extended to the origin with $T^{\mp}(0) = 2\pi/\sqrt{4\mathsf{D}_\Lambda - \mathsf{T}_\Lambda^2}$.

By using that for the piecewise system (4.1), the period function $T(y_0)$ behaves as the difference of the left flight time and the right flight time, we have the next theorems that determine the bifurcation diagram of the period function for the planar piecewise system (4.12) for the centers at infinity presented in the last proposition. Theorem 4.31 presents the behavior of the period function for the center-center case and Theorem 4.32 deals with the focus-focus case.

In the proof of the next result, the two argument function $\arctan(y, x)$, for real arguments x, y, computes the principal value of the argument of the complex number $x + iy$, so $-\pi < \arctan(y, x) \leq \pi$.

Theorem 4.31. *The period function* (4.6) *of the planar piecewise system* (4.12) *with* $\gamma_L = \gamma_R = 0$ *(condition (i) of Proposition 4.29), where* ω_L *and* ω_R *are given in the proof of Proposition 4.5 coincide, satisfies the following conditions*

(i) if $x_L = x_R$*, it is constant,*

(ii) if $x_L \leq 0$ *and* $x_R \geq 0$*, not simultaneously zero, it is monotonous decreasing with* $T(0) = 4\pi$*,*

(iii) if $x_L \geq 0$ *and* $x_R \leq 0$*, not simultaneously zero, it is monotonous increasing with* $T(0) = 0$*,*

(iv) if $x_L x_R > 0$ *and* $x_L > x_R$*, it has one simple critical period which is a minimum point with* $T(0) = 2\pi$*,*

(v) if $x_L x_R > 0$ *and* $x_L < x_R$*, it has one simple critical period which is a maximum point with* $T(0) = 2\pi$*.*

Furthermore, for all the cases, $\lim_{y_0 \to \infty} T(y_0) = 2\pi$.

Proof. Under these conditions, the planar piecewise system reduces to

$$(\dot{x}, \dot{y}) = \begin{cases} (-y, x - x_L), & \text{if } x \leq 0, \\ (-y, x - x_R), & \text{if } x > 0, \end{cases} \tag{4.22}$$

which are defined by systems that are time-reversible with respect to x-axis, and the half-return maps satisfy $\Pi^-(y_0) = \Pi^+(y_0) = -y_0$. First, note that, if $x_L = x_R$, the system in (4.22) becomes the analytical system with a linear center and, then, it has an isochronous center, and (i) follows.

For the other cases, by Proposition 4.8 we need to find the corresponding left and right flight times in order to obtain the behavior of the period function $T(y_0)$.

Consider the parametrized solution of the left planar piecewise system of (4.12)

$$(x(t), y(t)) = (x_L + r\cos t, r\sin t)\,.$$

Let τ_0 and $\tau_0 + \tau_1$ be the values of the time such that the solution starts at $(0, y_0)$ and ends at $(0, y_1)$. Substituting the values of $\cos\tau_0$ and $\sin\tau_0$, from the equations $x(\tau_0) = 0$ and $y(\tau_0) = y_0$ into the equation $x(\tau_0 + \tau_1) = 0$, we obtain that τ_1 satisfies $f_{y_0}(\tau_1) = 0$, with $f_{y_0}(\tau) = x_L - x_L\cos\tau - y_0\sin\tau$. Solving the equation $f_{y_0}(\tau_1) = 0$, we obtain

$$\tau_1 = \arctan\left(\frac{2x_L y_0}{x_L^2 + y_0^2}, \frac{x_L^2 - y_0^2}{x_L^2 + y_0^2}\right).$$

Note that τ_1 is the left flight time $T^-(y_0)$.

By using the parametrized solution of the equation in Σ^+ given by

$$(x(t), y(t)) = (x_R + r\cos t, r\sin t)\,,$$

with analogous calculations, now for the orbit starting at point $(0, -y_0)$, entering the zone Σ^+ until it reaches Σ at the point $(0, y_0)$, the time τ_2 satisfies $g_{y_0}(\tau_2) = 0$, for $g_{y_0}(\tau) = x_R - x_R\cos\tau + y_0\sin\tau$. Solving the equation $g_{y_0}(\tau_2) = 0$, we obtain

$$\tau_2 = \arctan\left(-\frac{2x_R y_0}{x_R^2 + y_0^2}, \frac{x_R^2 - y_0^2}{x_R^2 + y_0^2}\right).$$

Note that $T^+(y_0)$ coincides with $-\tau_2$.

Then, the complete period function is:

$$\begin{aligned} T(y_0) &= T^-(y_0) - T^+(y_0) = \tau_1 - (-\tau_2) = \tau_1 + \tau_2 \\ &= \arctan\left(\frac{2x_L y_0}{x_L^2 + y_0^2}, \frac{x_L^2 - y_0^2}{x_L^2 + y_0^2}\right) + \arctan\left(-\frac{2x_R y_0}{x_R^2 + y_0^2}, \frac{x_R^2 - y_0^2}{x_R^2 + y_0^2}\right). \end{aligned}$$

By using the first and second derivative of T given by

$$\begin{aligned} T'(y_0) &= -\frac{2(x_L - x_R)(x_L x_R - y_0^2)}{(x_L^2 + y_0^2)(x_R^2 + y_0^2)}, \\ T''(y_0) &= \frac{4y_0(x_L - x_R)(x_L^3 x_R + x_L^2 x_R^2 + x_L x_R^3 + 2x_L^2 x_R^2 y_0^2 - y_0^4)}{(x_L^2 + y_0^2)^2(x_L^2 + y_0^2)^2}, \end{aligned} \tag{4.23}$$

we determine the bifurcation diagram for these considered cases.

Note that, regarding conditions (ii) and (iii), that is if $x_L x_R \leq 0$, in the case where $x_L \leq 0$ and $x_R \geq 0$, not simultaneously zero, we have $T'(y_0) < 0$ and $T''(y_0) > 0$, for all y_0, and if $x_L \geq 0$ and $x_R \leq 0$, not simultaneously zero, $T'(y_0) > 0$ and $T''(y_0) < 0$, for all y_0. In fact, in these cases, $\operatorname{sign}(T') = \operatorname{sign}(x_L - x_R)$ and $\operatorname{sign}(T'') = -\operatorname{sign}(x_L - x_R)$, as $Y_1(y_0) = x_L x_R - y_0^2 < 0$ and $Y_2(y_0) = x_L^3 x_R + x_L^2 x_R^2 + x_L x_R^3 + 2x_L^2 x_R^2 y_0^2 - y_0^4 < 0$, for all y_0, coming from the fact that $Y_1(0) < 0$, $Y_2(0) < 0$ and both has no real roots. Then, T has no critical periods, since the derivative T' has no real root, and T is concave under condition (ii) and convex for (iii).

Therefore, for (ii) the half-flight times T^- and $-T^+$ are monotonous decreasing, from Proposition 4.14(i), and $T^-(0) = -T^+(0) = 2\pi$, from Lemma 4.16, $T'(y_0) > 0$, and $T''(y_0) > 0$, for all y_0, from the previous discussion. Hence, the sum T is monotonous decreasing with $T(0) = 4\pi$ and its graphic is as in region I of Figure 4.8.

On the other hand, with a similar approach, regarding statement (iii), T is monotonous increasing, as sum of T^- and T^+ which has this monotonic behavior from Proposition 4.14(iii). Furthermore, the period function T satisfies that $T(0) = 0$ given that $T^-(0) = -T^+(0) = 0$, from Lemma 4.16(iii), $T'(y_0) > 0$, $T''(y_0) < 0$, for all y_0, and its graphic is as in region II of Figure 4.8.

For the cases (iv) and (v), i.e $x_L x_R > 0$, we have from Proposition 4.14 that the behavior of one of the half-flight times is monotonous increasing and the other is monotonous decreasing, then we need to perform a more detailed analysis of these cases and see if there are period oscillations in this case. From Lemma 4.16, we can already conclude that $T(0) = 2\pi$. In this case we have the aggregation of a system with a visible center and another with an invisible center.

From the expression of the derivative $T'(y_0)$ in (4.23), if $x_L \neq x_R$ with $x_L x_R > 0$, we have one critical period given by the positive zero $p_0 := \sqrt{x_L x_R}$ of $Y_1(y_0)$, i.e. one oscillation. Furthermore, by using the second derivative in (4.23), we have that the number of inflection points of T is at most 1. In fact, this number coincides with the number of positive zeros of $Y_2(y_0) = x_L^3 x_R + x_L^2 x_R^2 + x_L x_R^3 + 2x_L^2 x_R^2 y_0^2 - y_0^4$. Hence, as for $x_L x_R > 0$ the discriminant of Y_2 with respect to y_0 is given by

$$-256 x_L^3 x_R^3 \left(x_L^2 + x_L x_R + x_R^2\right) \left(x_L^3 x_R^3 + x_L^2 + x_L x_R + x_R^2\right)^2,$$

which is negative in every domain of T, then we have two real roots, one positive and the other negative, and two complex conjugate roots. So, we have only a positive real root,

$$q_0 := \sqrt{x_L x_R + \sqrt{x_L^3 x_R + 2x_L^2 x_R^2 + x_L x_R^3}},$$

obtained by solving $Y_2(y_0) = 0$. Note that $p_0 < q_0$.

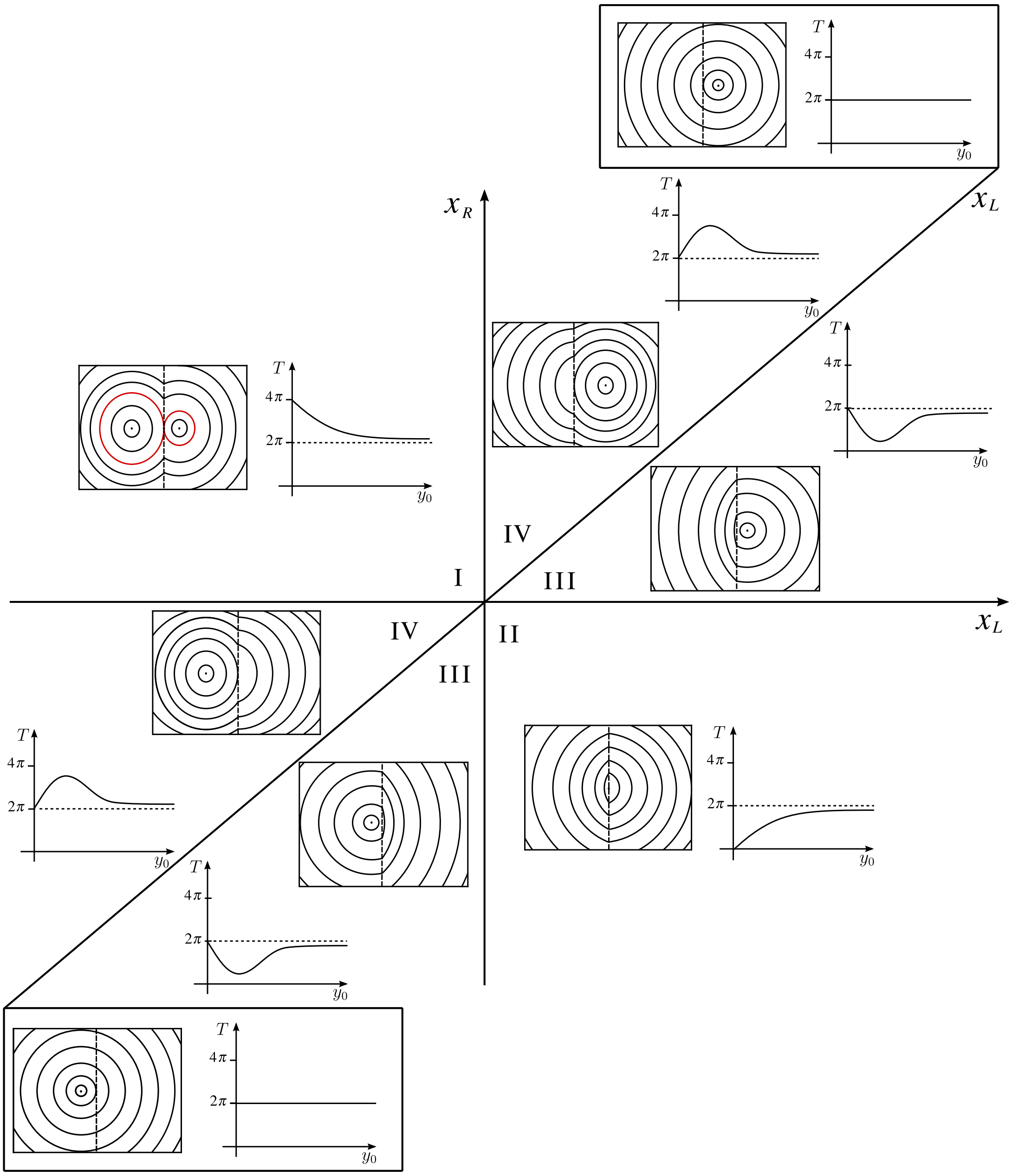

Figure 4.8: Bifurcation diagram of the period function of system (4.22)

Doing a simple analysis of the expressions of the first and second derivative of T given in (4.23), we have:

$$\begin{aligned} \operatorname{sign}(T') &= -\operatorname{sign}(x_L - x_R), \text{ for } y_0 \in (0, p_0);\\ \operatorname{sign}(T') &= \operatorname{sign}(x_L - x_R), \text{ for } y_0 \in (p_0, \infty);\\ \operatorname{sign}(T'') &= \operatorname{sign}(x_L - x_R), \text{ for } y_0 \in (0, q_0);\\ \operatorname{sign}(T'') &= -\operatorname{sign}(x_L - x_R), \text{ for } y_0 \in (q_0, \infty). \end{aligned} \tag{4.24}$$

So, we can conclude that, under the condition (iv), i.e. $x_L > x_R$, we have $T(0) = 2\pi$, $T'(y_0) < 0$, for $0 \le y_0 < p_0$, $T'(p_0) = 0$, $T'(y_0) > 0$, for $y_0 > p_0$, $T''(q_0) > 0$, for

$0 \leq y_0 < q_0$, $T''(q_0) = 0$, and $T''(y_0) < 0$, for $y_0 > q_0$, as $\text{sign}(x_L - x_R) = +1$, and we have (4.24). Hence, p_0 and q_0 are a minimum point and inflection point of T, respectively. Then, the behavior of $T(y_0)$ is as in region III of Figure 4.8. Analogously, under the condition (v), i.e. $x_L < x_R$, we have $T(0) = 2\pi$, $T'(y_0) > 0$, for $0 \leq y_0 < p_0$, $T'(p_0) = 0$, $T'(y_0) < 0$, for $y_0 > p_0$, $T''(q_0) < 0$, for $0 \leq y_0 < q_0$, $T''(q_0) = 0$, and $T''(y_0) > 0$, for $y_0 > q_0$, as $\text{sign}(x_L - x_R) = -1$, and we have (4.24). Thus, p_0 is a maximum point of T and q_0 is its inflection point. Then the behavior of $T(y_0)$ is as in region IV of Figure 4.8.

The last statement follows by using that we can parametrize the radios of the solution of the left and right system of (4.22), with respect to the angle, for any y_0. In fact, fixing the left system, it writes in polar coordinates as

$$\begin{cases} \dot{r} = -x_L \sin\theta =: R(r,\theta), \\ \dot{\theta} = 1 + \dfrac{x_L \cos\theta}{r} =: \Theta(r,\theta). \end{cases} \tag{4.25}$$

Note first that $\Theta(r, \pm\pi/2) = 1$, for all r, and $\Theta(r,\theta) \to 1$, as $r \to \infty$, then

$$\Theta(r,\theta) > 1, \text{ for } r \text{ big enough and any } \theta \Rightarrow \frac{1}{\Theta(r,\theta)} < 1. \tag{4.26}$$

Note also that the distance of the solution that starts in $(0, y_0)$ to the origin tends to infinity, as $y_0 \to \infty$. Hence, we can parametrize the radius of this solution with respect to the angle, for any y_0. Indeed, it is given by the solution, here denoted by $r(\theta, y_0)$, of

$$\frac{dr}{d\theta} = \frac{R(r,\theta)}{\Theta(r,\theta)},$$

that verifies $r(0, y_0) = y_0$. Using this parametrization, we can compute the left flight time, for each y_0, by means of

$$T^-(y_0) = \int_{\pi/2}^{3\pi/2} \frac{1}{\Theta(r(\theta, y_0), \theta)} d\theta. \tag{4.27}$$

Note, on the other hand, that, since $r(\theta, y_0) \to \infty$ as $y_0 \to \infty$, for each θ,

$$\lim_{y_0 \to \infty} \frac{1}{\Theta(r(\theta, h), \theta)} = 1. \tag{4.28}$$

Therefore, by using (4.26), we can apply the Dominate Convergence Theorem and assert that $\lim_{y_0 \to \infty} T^-(y_0) = \pi$.

For the right system, $\lim_{y_0 \to \infty} -T^+(y_0) = \pi$. The proof is analogous as $\dot{\theta} = 1 - x_R \cos(\theta)/r =: \Theta(r(\theta, y_0), \theta)$ and $-T^+(y_0) = \int_{-\pi/2}^{\pi/2} 1/\Theta(r(\theta, y_0), \theta) d\theta$. It follows that $T(y_0) = T^-(y_0) - T^+(y_0) \to 2\pi$, as $y_0 \to \infty$.

□

Theorem 4.32. *The period function of the planar piecewise system* (4.12)*, with* $\gamma_R = -\gamma_L \neq 0$ *and* $x_R = -x_L$ *(conditions (ii), (iii), and (iv) of Proposition 4.29), satisfies the following conditions:*

(i) if $x_L < 0$*, it is monotonous decreasing,*

(ii) if $x_L = 0$*, it is constant,*

(iii) if $x_L > 0$*, it is monotonous increasing.*

Furthermore, for all the cases, $\lim_{y_0\to\infty} T(y_0) = 2\pi$.

Proof. First, note that assuming the conditions $\gamma_L = -\gamma_R$ and $x_L = -x_R$ given in (i), (ii), and (iii), from equations in (4.13), we have $\omega_L = \omega_R$. Then, after a reparameterization, we can consider $\omega_L = \omega_R = 1$ and $T(y_0) = (1 + \omega_L/\omega_R)T^-(y_0) = 2T^-(y_0)$ (see Lemma 4.18). In cases (i) and (iii) the planar piecewise system becomes

$$(\dot{x}, \dot{y}) = \begin{cases} (2\gamma_L x - y, (1+\gamma_L^2)(x - x_L)), & \text{if } x \leq 0, \\ (-2\gamma_L x - y, (1+\gamma_L^2)(x + x_L)), & \text{if } x > 0, \end{cases} \tag{4.29}$$

and we have to consider only two cases, $\gamma_L < 0$ and $\gamma_L > 0$ (see, Proposition 4.29 and Remark 4.6). For condition (ii), we have

$$(\dot{x}, \dot{y}) = \begin{cases} (2\gamma_L x - y, (1+\gamma_L^2)x), & \text{if } x \leq 0, \\ (-2\gamma_L x - y, (1+\gamma_L^2)x), & \text{if } x > 0, \end{cases} \tag{4.30}$$

whose phase portrait is presented in I of Figure 4.9.

For condition (i) with $\gamma_L < 0$, the domain of T is given by $[\hat{y}_0, \infty)$, where $\hat{y}_0 > 0$ satisfies $\Pi^-(\hat{y}_0) = 0$. Furthermore, we know that T^- and T^+ are monotonous decreasing, from Proposition 4.14(i), and $T^-(\hat{y}_0), -T^+(\hat{y}_0) \in (\pi, 2\pi)$, from Lemma 4.16(i), then T is monotonous decreasing with $T(\hat{y}_0) \in (2\pi, 4\pi)$. Moreover, we have $T'(y_0) < 0$, and $T''(y_0) > 0$, for all $y_0 \in (\hat{y}_0, \infty)$. In fact, since $T(y_0) = 2T^-(y_0)$, we can analyse the sign of T' and T'' by using equations (4.15) and (4.16), for the left flight time T^-. Hence, the period annulus corresponding to $\gamma_L < 0$ and the period function T are depicted on region I of Figure 4.9.

Now, if we consider $\gamma_L > 0$, the domain of T becomes $[0, \infty)$, and it behaviors as the above case. In fact, we can obtain the previous case with a composition of a symmetry and a reversibility of the time given by $(x, y, \tau) \to (x, -y, -\tau)$. Hence, the period function T (see Remark 4.15) and its phase portrait are depicted in region II of Figure 4.9.

Under the condition (iii) for both cases $\gamma_L < 0$ and $\gamma_L > 0$, the period annulus is unbounded and the domain of T is given by $(0, \infty)$. The situation is analogous to the condition (ii) and it is enough to consider $\gamma_L < 0$. In this case, T is monotonous increasing

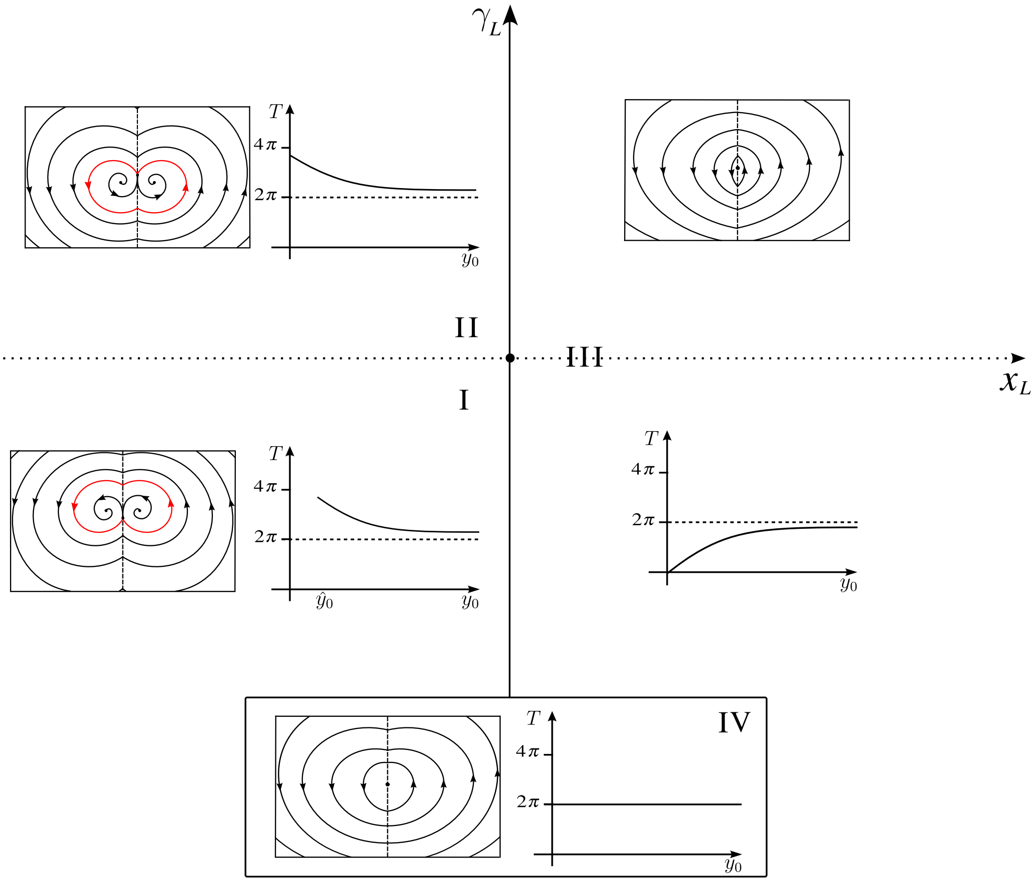

Figure 4.9: Bifurcation diagram of the period functions for the focus-focus case

by Proposition 4.14(iii), $T(0) = 0$ from Lemma 4.16(c), and $T''(y_0) < 0$ for all $y_0 \in (0, \infty)$ from equations (4.15) and (4.16). Hence, the period annulus and the period function T are depicted in region III of Figure 4.9.

Finally, under the condition (ii), that is system (4.30) has a global center at the origin, by Proposition 4.14(ii), the period is constant. Furthermore, by Lemma 4.16(ii), $T(0) = 2\pi$ and, therefore, $T(y_0) = 2\pi$, for all $y_0 > 0$, and the origin is an isochronous center and the period function is as depicted in region IV.

The last statement follows by using that the complete time is $T(y_0) = 2T^-(y_0)$, where $T^-(y_0)$ has the expressions given in Lemma 4.17 with $\omega_L = 1$. Then, T is given by

(i) $T(y_0) = 2\left(2\pi + \arctan\left(\dfrac{y_0 - \gamma_L x_L}{x_L}\right) - \arctan\left(\dfrac{\Pi^-(y_0) - \gamma_L x_L}{x_L}\right)\right)$, if $x_L < 0$;

(ii) $T(y_0) = 2\pi$, if $x_L = 0$;

(iii) $T(y_0) = 2\left(\arctan\left(\dfrac{y_0 - \gamma_L x_L}{x_L}\right) - \arctan\left(\dfrac{\Pi^-(y_0) - \gamma_L x_L}{x_L}\right)\right)$, if $x_L > 0$.

Therefore, as $\lim_{y_0\to\infty} \Pi^-(y_0) = -\infty$, we simply calculate the limit of the above expressions (i), (ii), and (iii), and we obtain $\lim_{y_0\to\infty} T(y_0) = 2\pi$, for all values of x_L.

□

Remark 4.33. About the conditions (i), (ii), and (iii) of Theorem 4.32 we have the focus-focus case invisible, the origin is a boundary equilibrium point of focus type for both systems, and focus-focus case visible, respectively.

Remark 4.34. About the focus-focus case with $x_L < 0$, $|x_L|$ small, we have that the initial value approaches the value 4π and tends to 2π slower. On the other hand, for $|x_L|$ big, it starts with a smaller and closer value to 2π and tends to 2π faster. If $|x_L| = 0$, then the period function is constant. For $x_L > 0$ with $|x_L|$ small, the period function takes its first value at 0 with a bigger slope and tends to 2π faster. On the other hand, for $|x_L|$ big, the left flight time T^- starts in 0 with a smaller slope and tends to 2π slower.

Remark 4.35. For the center-center case we have only considered the case where $\omega_L = \omega_R$. We are studying the remaining case where $\omega_L \neq \omega_R$.

CHAPTER 5

Final considerations

It is important to mention that we are studying how we can determine the total number of period oscillations for some families of piecewise systems and not just the local problem, that is, to establish a lower bound for the number of period oscillations around the center. The advantage of choosing this approach to the problem is that it allowed us to study different problems and techniques. Furthermore, as this study takes a different direction from those that before we decided to study the period function, there are still many possibilities for future works. Although it is an interesting approach it is difficult.

For the first work in Chapter 2, we studied a special case of piecewise potential system, also called asymmetric oscillator, already well-known and frequently encountered in physical problems ([60]). Such a family has a certain symmetry that facilitates to find an expression for the period function and so we can use the study carried out for the analytical system that determines our family to obtain its bifurcation diagram. The interesting thing is that we could try to carry out a similar study for piecewise potentials of any degree as in Chicone [19] and Gasull, Guillamon, Mañosa, and Mañosas [48] for the analytical case, at first those that have the same symmetry of system (2.1), and later without this symmetry.

As we have studied the local problem for the planar piecewise reversible quadratic system in Chapter 3, we could have used the same technique and applied it to study different types of piecewise systems. For example, try to find a good limitation of the number of local critical periods that bifurcates from a planar piecewise vector field with degree n.

Furthermore, as it was seen in the thesis, we could have chosen to work with the Poincaré map for piecewise systems and establishing the number of limit cycles, since the tools coincide with those we used for the period function. The preference for studying the period function in piecewise systems is precisely because there are still not many works in this direction.

In this context, we can specify some open problems:

1. We point out that, as we studied in Chapter 2, the period function and its oscillations for the cubic planar Hamiltonian system based on Picard–Fuchs equations for algebraic curves, we expect that we could do the same approach for other families of piecewise systems without symmetries, and with higher degree.

2. Given a planar analytic system $(\dot{x}, \dot{y}) = (f(x, y), g(x, y))$, we can write it in complex coordinates as $\dot{z} = F(z, \overline{z})$, where $z = x + \mathrm{i}\, y$. Moreover, when the origin is a weak focus, after a constant rescaling of time, it writes as $\dot{z} = \mathrm{i}\, z + Z(z, \overline{z})$, where Z is a convergent series which starts with at least quadratic terms. Since holomorphic systems are isochronous (see [46]), it is interesting to work with the maximum number of local critical periods that can bifurcate from holomorphic centers. In this way the authors of [88] studied perturbations of reversible holomorphic isochronous centers by adding nonholomorphic perturbations which keeps the center property, more precisely the family of n-th degree reversible system

$$\dot{z} = \mathrm{i}\, z \prod_{j=1}^{n-1} (1 - a_j z),$$

where $n > 1$ and $a_j \in \mathbb{R} \setminus \{0\}$ are real parameters such that $a_j \neq a_i$ for every $i, j \in \{1, \ldots, n-1\}$, $i \neq j$. This system is reversible with respect to the horizontal axis, then we could consider the piecewise systems which are the aggregation of two of these systems with the y-axis as straight line of separation, namely

$$\dot{z} = \mathrm{i}\, z \prod_{j=1}^{n-1} (1 - a_j z), \quad \text{if} \;\; \mathrm{Re}(z) < 0, \qquad \dot{z} = \mathrm{i}\, z \prod_{j=1}^{n-1} (1 - b_j z), \quad \text{if} \;\; \mathrm{Re}(z) > 0,$$

where $n > 1$ and $a_j, b_j \in \mathbb{R} \setminus \{0\}$ are real parameters such that $a_j \neq a_i$ and $b_j \neq b_i$, for every $i, j \in \{1, \ldots, n-1\}$, $i \neq j$, which has a center because of the reversibility and try to find the maximum number of local critical periods which can bifurcate from the origin. Or even, we could study the global problem for these lower degree systems.

3. One can also consider the cases of centers for the planar piecewise linear system (4.1) and try to describe the behavior of the flight time for the remaining cases: saddle-saddle, node-node, etc.

4. There are still many families of planar piecewise polynomial systems for studying the criticality problem. By using the same methods that we have presented in

Chapter 3, we could perform a case-by-case study by starting with lower degree and, then, increasing the degree.

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