**UNIVERSIDADE FEDERAL DE SÃO CARLOS**
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

# On the classification of two dimensional flows

Rodrigo Thomaz

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Dissertation submitted to the Department of Mathematics of Universidade Federal de São Carlos - DM/UFSCar, in partial fulfillment of the requirements for the degree of Master in Mathematics

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Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Matemática

## Folha de Aprovação

Defesa de Dissertação de Mestrado do candidato Rodrigo Thomaz da Silva, realizada em 02/08/2023.

### Comissão Julgadora:

Prof. Dr. Francisco Braun (UFSCar)

Prof. Dr. Luiz Roberto Hartmann Junior (UFSCar)

Prof. Dr. Luis Fernando de Osório Mello (UNIFEI)

O Relatório de Defesa assinado pelos membros da Comissão Julgadora encontra-se arquivado junto ao Programa de Pós-Graduação em Matemática.

*To my family,*
*not restricted to the blood-related one.*

*"Presentemente eu posso me considerar um sujeito de sorte*
*Porque apesar de muito moço me sinto são e salvo e forte*
*E tenho comigo pensado, Deus é brasileiro e anda do meu lado*
*E assim já não posso sofrer no ano passado."*
Sujeito de Sorte - Belchior

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# Resumo

Lawrence Markus, matemático amplamente conhecido, publicou, em 1954, um forte resultado sobre classificação de campos vetoriais planares [10]. Após um certo tempo, Dean Arnold Neumann generalizou tal trabalho para fluxos contínuos [13], e o resultado se tornou conhecido como Teorema de Markus–Neumann. Recentemente, em 2018, José Ginés Espín Buendía and Víctor Jiménez López escreveram um artigo apontando os problemas nos dois trabalhos supracitados [4].

Apresentaremos aqui uma discussão destes resultados clássicos de classificação de campos vetoriais planares e fluxos contínuos, passando pelos três trabalhos citados e detalhando as definições, exemplos e provas presentes.

**Palavras-chave**: Fluxos contínuos; campos vetoriais; sistema diferencial no plano; configuração separatriz; Teorema de Markus–Neumann.

# Abstract

Lawrence Markus, a well-known mathematician, published, in 1954, a strong result about classification of planar vector fields [10]. Afterwards, Dean Arnold Neumann generalized such work for continuous flows [13], and the result became known as the Markus–Neumann Theorem. Recently, in 2018, José Ginés Espín Buendía and Víctor Jiménez López wrote a paper exposing gaps in the two aforementioned articles [4].

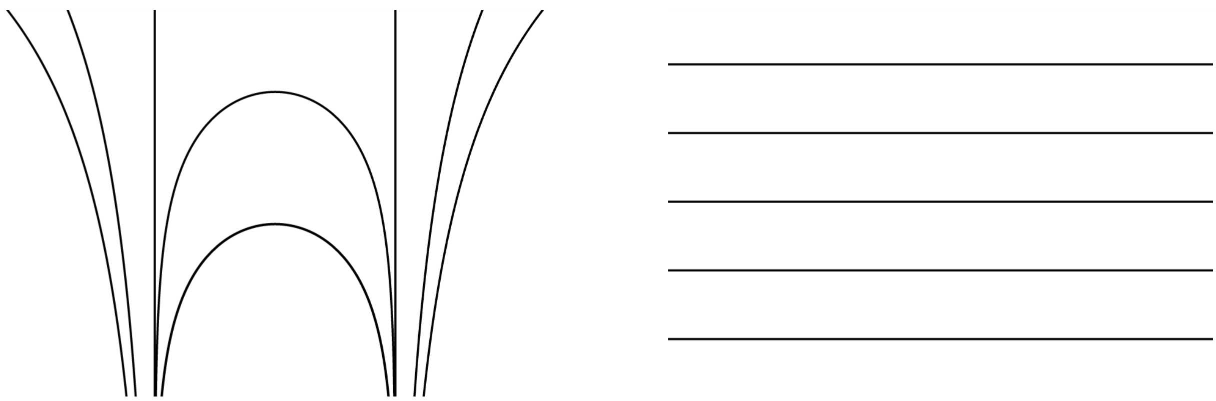
We present here a discussion on these classical results of classification of planar vector fields and continuous flows, going through the three cited works and elaborating on definitions, examples and proofs.

**Keywords**: Continuous flows; vector fields; differential system in the plane; separatrix configuration; Markus–Neumann Theorem.

# Contents

# Introduction

In 1954, Lawrence Markus presented a remarkable result on classification of vector fields in the plane [10], based mainly on the *separatrix configuration*, introduced by him. Several advances on this topic have been achieved after this work; for instance, in [13], Dean Arnold Neumann generalized Markus' result to continuous flows in the plane, and even on two-dimensional manifolds. However, in 2018, Espín Buendía and Jiménez López pointed critical flaws in the former theorem as well as in the generalized version [4], presenting counterexamples to them and also suitable corrections. The main problem was Markus' definition of separatrix: what should be a separatrix was not, since his definition was too restrictive. If we take a look at the flows below, for example, it is quite intuitive that they are, in some sense, different. Roughly speaking, there is no way to "transform" one in a continuous way to get the other: the only way to straighten the bent lines is passing through the two barriers formed by the parallel lines. Even in such simple example the theorem fails, as the definition given by Markus (and carried along the years by many others) cannot identify these barriers (the so-called separatrices).



In this work we aim to elucidate this whole topic. In Chapter 1 we establish notations and present some of the basic results which we shall use throughout the following chapters. In this chapter one can find classical results as the Existence and Uniqueness Theorem of solutions of ODEs, the Jordan Curve Theorem and the Poincaré–Bendixson Theorem. Although most of their proofs will not be presented here, the references for them are properly cited.

In Chapter 2 we present the beginning of the classification problem, going through specific definitions for the topic as *parallel regions* and *complete transversals*, through

Markus' paper which first introduced strong results on the topic (although with several flaws), through Buendía and López article, which redefined the concepts given by Markus and improved the classification theorem and quickly through the general points of Neumann's paper.

The final chapter is dedicated to the updated proof of the classification theorem. Such proof follows the one given by Neumann, with just a few tweaks and more details.

All figures in this text are due to the author.

# Chapter 1

# Preliminaries

There are many topics of great importance for the good understanding of all the content to be presented in this work. We start this text with some of them, as vector fields, flow and transversals. We assume the reader has intermediate knowledge on Ordinary Differential Equations, Analysis in $\mathbb{R}^n$ and Differential Geometry. In this chapter, several proofs will be omitted. For more details about the cited concepts we refer to [15] and to [3].

Though we will usually talk about structures and objects in the plane, some results will be stated and proved for more than two variables. We stress that several definitions and results presented in this chapter have their counterpart in the context of 2-manifolds.

Throughout the text, unless otherwise stated, $I$ will denote an interval of $\mathbb{R}$.

## 1 Vector fields and flows

The following definitions and results are the core of our work.

**Definition 1.1.** By a *differential system* we mean an autonomous ordinary differential equation

$$\dot{x} = f(x), \tag{1.1}$$

where $x = x(t)$ is a differentiable mapping (defined on an interval) to be found and $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^r$ mapping, $r = 1, \ldots, \infty, \omega$; the notation $\mathcal{C}^\omega$ stands for analyticity. In particular, when $n = 2$, our system becomes

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases},$$

with $f, g \in \mathcal{C}^r$. Throughout this work we will often say "a mapping of class $\mathcal{C}^r$", without further mention to the range of $r$. When no confusion is possible, by that we mean $r = 1, \ldots, \infty, \omega$. If in (1.1) we denote $f = (f_1, \ldots, f_n)$, it is possible to talk about the *vector field* associated to the system, which is given by

$$X = f_1(x)\frac{\partial}{\partial x_1} + \cdots + f_n(x)\frac{\partial}{\partial x_n}.$$

To see more details about a vector field as an operator over smooth functions we refer to [19] and to [9]. The solutions of (1.1), i.e., the differentiable mappings $\varphi : I \to \mathbb{R}^n$ such that $\varphi'(t) = f(\varphi(t))$ for all $t \in I$, are called *trajectories* or *integral curves* of $f$ or of the differential equation (1.1). A point $x \in U$ is a *singular point* or a *critical point* of $f$ if $f(x) = 0$ and a *regular point* of $f$ if $f(x) \neq 0$. We will not distinguish between the differential system, its associated vector field and, in some cases, the partition of the plane defined by the *orbits* of the differential system (the *phase portrait*, defined below). This last point is due to the next theorem, which assures a unique orbit through any point $p \in U$.

For the proof of the next theorem, we refer to [15], page 209.

**Theorem 1.2.** *Let $f : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^r$, where $U$ is an open subset of $\mathbb{R}^n$. Then:*

*(a) (Existence and uniqueness of maximal solutions). For each $x \in U$ there exists an open interval $I_x$ on which a unique maximal solution $\varphi_x$ of (1.1) is defined and satisfies the condition $\varphi_x(0) = x$ (here maximal means that if $\psi$ is a solution of (1.1) defined on an interval $I$ such that $\psi(0) = x$, then $I \subset I_x$ and $\varphi_x$ restricted to $I$ equals $\psi$);*

*(b) (Flow properties). If $y = \varphi_x(t)$ for some $t \in I_x$, then $I_y = I_x - t = \{s - t \mid s \in I_x\}$ and $\varphi_y(s) = \varphi_x(t + s)$ for every $s \in I_y$;*

*(c) (Differentiability with respect to initial conditions). Let $D = \{(x, t) \mid x \in U, t \in I_x\}$. Then $D$ is an open set of $\mathbb{R}^{n+1}$ and the mapping $\varphi : D \to \mathbb{R}^n$ given by $\varphi(x, t) = \varphi_x(t)$ is $\mathcal{C}^r$. Moreover, $\varphi$ satisfies*

$$\partial_t \partial_x \varphi(x, t) = df_{\varphi(x,t)} \partial_x \varphi(x, t),$$

*for every $(x, t) \in D$.*

**Definition 1.3.** Let $D = \{(x, t) \mid x \in U, t \in I_x\}$. The mapping $\varphi : D \to U$, $\varphi(x, t) = \varphi_x(t)$ as above, is the *flow generated by $f$*.

**Definition 1.4.** The set $\gamma_p = \{\varphi(p, t) \mid t \in I_p\}$ is the *orbit* of $f$ through $p$. Note that $\gamma_p$ is the image of the integral curve of $f$ through $p$. We can address an orientation to $\gamma_p$ by using $\varphi_p(t)$ orientation (if $p$ is a regular point). It is useful to define the positive (respectively, negative) *semi-orbit* as the subset of $\gamma_p$ where $t \geq 0$ (respectively, $t \leq 0$) and denote it by $\gamma_p^+$ (respectively, $\gamma_p^-$).

**Definition 1.5.** The open set $U$ endowed with the decomposition in oriented orbits of $f$, plus its singular points, is called *phase portrait* of $f$.

The flow generated by a vector field is a particular case of the following.

**Definition 1.6.** A *flow* on a set $N$ is a group action of the additive group $(\mathbb{R}, +)$ on $N$. In other words, a flow $\varphi$ on a set $N$ is a mapping $\varphi : N \times \mathbb{R} \to N$ such that, for all $x \in N$ and $s, t \in \mathbb{R}$, we have

(i) $\varphi(x, 0) = x$;

(ii) $\varphi(\varphi(x, t), s) = \varphi(x, t + s)$.

If, instead of $N \times \mathbb{R}$, we consider the domain

$$D = \{(x, t) \mid x \in N,\ t \in I_x\},$$

where $I_x$ is an interval depending on $x$ and containing 0, then $\varphi : D \to N$ is called a *local flow.* This often is the case when considering flows generated by vector fields.

The next theorem asserts that, in some sense, a solution "escapes" from every compact.

**Theorem 1.7.** *Let $f : U \to \mathbb{R}^n$ be a $\mathcal{C}^r$ vector field. If $x \in U$ and $I_x = (\omega_-(x), \omega_+(x))$ is bounded above (respectively below), i.e., $\omega_+(x) < \infty$ (respectively $\omega_-(x) > -\infty$), then $\varphi_x(t) \to \partial U$ (the boundary of $U$) as $t \to \omega_+(x)$ (respectively $t \to \omega_-(x)$). In other words, for every compact $K \subset U$ there exists $\varepsilon = \varepsilon(K) > 0$ such that if $t \in [\omega_+(x) - \varepsilon, \omega_+(x))$ (respectively $t \in (\omega_-(x), \omega_-(x) + \varepsilon]$), then $\varphi_x(t) \notin K$.*

*Proof.* To get a contradiction, suppose that there exist a compact $K \subset U$ and a sequence $\{t_n\}$ converging to $\omega_+(x)$ such that $\varphi_x(t_n) \in K$ for every $n$. Taking a subsequence if necessary, we can assume $\varphi_x(t_n)$ converges, say, to $x_0 \in K$. From statement (c) of Theorem 1.2, $D$ is an open set. So we can find $b > 0$ and $\alpha > 0$ such that $B_b \times I_\alpha \subset D$, where $B_b = \{x \in \mathbb{R}^n \mid d(x, x_0) < b\}$ and $I_\alpha = \{t \in \mathbb{R} \mid |t| < \alpha\}$. If $n$ is sufficiently large, we have $|t_n - \omega_+(x)| < \frac{\alpha}{2}$ and $y = \varphi_x(t_n) \in B_b \times I_\alpha$. Hence if $\frac{\alpha}{2} < s < \alpha$, $\varphi_y(s)$ is well defined. But from statement (b) of Theorem 1.2 this coincides with $\varphi_x(t_n + s)$, and $t_n + s > \omega_+(x)$, contradiction. □

A direct corollary from the last theorem is that if the orbit $\varphi_x(t)$ stays in some compact set $K$ as $t \to \omega_+(x)$ (respectively $t \to \omega_-(x)$), then it follows that $\omega_+(x) = \infty$ (respectively $\omega_-(x) = -\infty$). Particular cases of this are the *periodic orbits.*

**Definition 1.8.** Let $\varphi_x(t)$ be a integral curve of $f$. It is said to be *periodic* if there exists $\tau > 0$ such that $\varphi_x(t + \tau) = \varphi_x(t)$ for every $t \in \mathbb{R}$.

**Remark.** A periodic orbit is the same of a closed orbit.

**Definition 1.9.** Let $X_1$ and $X_2$ be two vector fields defined in $U_1, U_2 \subset \mathbb{R}^n$, respectively. We say that $X_1$ is *topologically equivalent* (respectively $\mathcal{C}^r$*-equivalent*) *to* $X_2$ when there exists a homeomorphism (respectively a $\mathcal{C}^r$ diffeomorphism) $h : U_1 \to U_2$ which carries an oriented orbit of $X_1$ to an oriented orbit of $X_2$. More precisely, let $p \in U_1$ and $\gamma_p^1$ the oriented orbit of $X_1$ through $p$; then $h(\gamma_p^1)$ is the oriented orbit $\gamma_{h(p)}^2$ of $X_2$ through $h(p)$. The homeomorphism $h$ is called *topological equivalence* (respectively $\mathcal{C}^r$*-equivalence*) between $X_1$ and $X_2$.

**Definition 1.10.** Let $\varphi_1 : D_1 \to \mathbb{R}^n$ and $\varphi_2 : D_2 \to \mathbb{R}^n$ be the flow generated by $X_1 : U_1 \to \mathbb{R}^n$ and $X_2 : U_2 \to \mathbb{R}^n$, respectively. The vector field $X_1$ is said to be *topologically conjugate* (respectively $\mathcal{C}^r$*-conjugate*) *to* $X_2$ when there exists a homeomorphism (respectively a $\mathcal{C}^r$ diffeomorphism) $h : U_1 \to U_2$ such that $h(\varphi_1(x,t)) = \varphi_2(h(x),t)$, for every $(x,t) \in D_1$. The homeomorphism $h$ is called *topological conjugacy* (respectively $\mathcal{C}^r$*-conjugacy*) between $X_1$ and $X_2$. In this case, we necessarily have $I_x = I_{h(x)}$ (on the left the maximal interval for $\varphi_1$ and on the right the maximal interval for $\varphi_2$).

A topological equivalence $h$ defines an equivalence relation between vector fields defined on open sets $U_1$ and $U_2 = h(U_1)$ of $\mathbb{R}^n$. A topological equivalence maps singular points to singular points, and periodic orbits to periodic orbits. If $h$ is a conjugacy, then the period is also preserved.

**Example 1.11.** Let $X$ and $Y$ be two $\mathcal{C}^r$ vector fields on $\mathbb{R}^n$ such that $X(x) = \alpha(x)Y(x)$, $x \in \mathbb{R}^n$, for some function $\alpha \in \mathcal{C}^1$ with $\alpha(x) > 0$ for every $x$. We claim that $X$ and $Y$ are topologically equivalent. Indeed, let $\varphi$ and $\psi$ denote the flows of $X$ and $Y$, respectively. For each $x \in \mathbb{R}^n$, consider the problem

$$\begin{cases} \dot{\beta} = \alpha(\varphi(x,\beta))^{-1} \\ \beta(0) = 0 \end{cases}.$$

Since $\alpha$ does not vanish, the problem is well defined and has a unique solution $\beta(x,t)$ defined on a maximal interval $I_x$. By the continuous dependence on initial conditions and parameters, we see that $\beta$ is continuous as a 2-variables mapping and, moreover, the function $\partial_t\beta(x,t)$ is strictly positive, so $\beta(x,\cdot)$ is a diffeomorphism. Let $\Upsilon(x,t) = \varphi(x,\beta(x,t))$. Then

$$\begin{aligned} \partial_t\Upsilon(x,t) &= \partial_t\beta(x,t)X(\varphi(x,\beta(x,t))) \\ &= \partial_t\beta(x,t)\alpha(\varphi(x,\beta(x,t)))Y(\varphi(x,\beta(x,t))) \\ &= Y(\Upsilon(x,t)). \end{aligned}$$

By uniqueness, $\Upsilon(x,t) = \psi(x,t)$, i.e., $\varphi(x,\beta(x,t)) = \psi(x,t)$ for $t \in I_x$. Denote by $\gamma^1$ and $\gamma^2$ the orbits with respect to $\varphi$ and $\psi$, respectively. Hence, for every $y \in \gamma_x^2$, $y = \psi(x,s)$, it follows that $\varphi(y,\beta(y,t)) = \psi(x,t+s)$, and thus there exists a connected open subset

$\mu_y \subset \gamma_y^1$ (open as a subset of this orbit) such that $y \in \mu_y \subset \gamma_y^2 = \gamma_x^2$. Given $z = \psi(x,s)$, it is possible to cover the compact set $\psi(x,[0,s])$ with finitely many $\mu_i$, whose union is a connected set. Since orbits are connected and disjoint, every $\mu_i$ is contained in the same orbit of $\varphi$. It is possible to choose $\mu_1$ so that it is a subset of $\gamma_x^1$, whence $\psi(x,[0,s]) \subset \gamma_x^1$ and, therefore, $\gamma_x^2 \subset \gamma_x^1$.

Constructing a similar differential equation, one can find a diffeomorphism $\sigma(x,\cdot)$ such that $\psi(x,\sigma(x,t)) = \varphi(x,t)$ for $t$ in the domain of $\sigma(x,\cdot)$. The argument above, then, yields $\gamma_x^1 \subset \gamma_x^2$, and we conclude that the orbits of both flows are the same, possibly with different velocities. The identity provides a $\mathcal{C}^\omega$-equivalence between the vector fields $X$ and $Y$.

For the case where $\alpha(x) < 0$ for every $x \in \mathbb{R}^n$, the above proof shows that the vector fields $X$ and $-Y$ are $\mathcal{C}^\omega$-equivalent. Provided that $Y$ and $-Y$ are equivalent, the transitive property will finish this case. Not every vector field is equivalent to minus itself. A simple example of such case is the vector field $Y(x,y) = (0,y)$. There cannot exist an equivalence between $Y$ and $-Y$, since orbits of $Y$ get closer to the $x$-axis and orbits of $-Y$ get farther from it.

It is important to stress that, even if the vector fields are $\mathcal{C}^\omega$-equivalent, there might exist no conjugacy between them. Take, for instance, $X(x,y) = (ay,-ax)$ and $Y(x,y) = (by,-bx)$, where $a,b > 0$ and $a \neq b$. Then $X = \frac{a}{b}Y$, but there is no conjugacy between them, because every orbit of $X$ has period $2\pi/a$, while every orbit of $Y$ has period $2\pi/b$ (besides the critical point). Since a conjugacy carries periodic orbits onto periodic orbits with the same period, the claim follows.

Next, we have a characterization of $\mathcal{C}^r$-conjugacies between vector fields.

**Lemma 1.12.** *Let $X_1 : U_1 \to \mathbb{R}^n$ and $X_2 : U_2 \to \mathbb{R}^n$ be two $\mathcal{C}^r$ vector fields and $h : U_1 \to U_2$ be a $\mathcal{C}^r$ diffeomorphism. Then $h$ is a conjugacy between $X_1$ and $X_2$ if and only if*

$$dh_p X_1(p) = X_2(h(p)),$$

*for every $p \in U_1$.*

*Proof.* Let $\varphi_1 : D_1 \to \mathbb{R}^n$ and $\varphi_2 : D_2 \to \mathbb{R}^n$ be the flow generated by $X_1$ and $X_2$, respectively. If $h$ is a conjugacy, given $p \in U_1$, we have $h(\varphi_1(p,t)) = \varphi_2(h(p),t)$. Differentiating both sides with respect to $t$ we get

$$dh_{\varphi_1(p,t)} X_1(\varphi_1(p,t)) = dh_{\varphi_1(p,t)} \partial_t \varphi_1(p,t) = \partial_t \varphi_2(h(p),t) = X_2(\varphi_2(h(p),t)),$$

which, when evaluated at $t = 0$, is the desired expression. Conversely, suppose that $h$ satisfies the hypothesis. Given $p \in U_1$, let $Y(t) = h(\varphi_1(p,t))$, where $t \in I_p$ (the maximal interval for $\varphi_1$). Hence $Y$ is a solution for $\dot{x} = X_2(x)$ with $x(0) = h(p)$, since

$$\dot{Y}(t) = dh_{\varphi_1(p,t)} X_1(\varphi_1(p,t)) = X_2(h(\varphi_1(p,t))) = X_2(Y(t)).$$

By uniqueness, it follows that $h(\varphi_1(p,t)) = \varphi_2(h(p),t)$, and the result is proved. □

Another important concept that arises naturally is *transversality*. When dealing with a vector field $X$, one would ask: is there a "simpler" vector field which is equivalent to $X$? Locally, at regular points, the answer is yes; but not every vector field has a global "simplification". We will see next the definition of transversal and the theorem known as Flow Box Theorem, or Straightening-out Theorem.

**Definition 1.13.** Let $X : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^r$, where $U \subset \mathbb{R}^n$ is open, and $A \subset \mathbb{R}^{n-1}$ an open set. A $\mathcal{C}^r$ mapping $f : A \to U$ is a *local transverse section* of $X$ when, for every $a \in A$, the set $\{df_a(\mathbb{R}^{n-1}), X(f(a))\}$ spans $\mathbb{R}^n$. Let $\Sigma = f(A)$ inherit the topology from $\mathbb{R}^n$. If $f : A \to \Sigma$ is a homeomorphism, then $\Sigma$ is a *transverse section* of $X$.

It is always possible to obtain a local transverse section through a regular point $p$. Choosing a sufficiently small neighborhood of $p$, a hyperplane through $p$ with normal vector $X(p)$ will seal the deal.

**Examples 1.14.**

(1) One of the simplest vector fields one can come up with is the following: $X : \mathbb{R}^n \to \mathbb{R}^n$ given by $X(x_1, \ldots, x_n) = (1, 0, \ldots, 0)$. If we let $A = \mathbb{R}^{n-1}$ and $f : A \to \mathbb{R}^n$ be the mapping given by $f(y_1, \ldots, y_{n-1}) = (0, y_1, \ldots, y_{n-1})$, then $f(A)$ is a transverse section of $X$. It is clear that $f$ is a homeomorphism onto its image. Moreover, given $a \in A$, $df_a(\mathbb{R}^{n-1}) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$ and $X(f(a)) = (1, 0, \ldots, 0)$. As every vector $v \in \mathbb{R}^n$ is a linear combination of an element of the first set and $(1, 0, \ldots, 0)$, they span $\mathbb{R}^n$. See Figure 1.15.

(2) Here is an example of a local transverse section which its image is not a transverse section. Take the vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ given by $X(x, y) = (x, y)$. Define $A = \mathbb{R}$ and let $f : A \to \mathbb{R}^2$ be the mapping given by $f(t) = (\cos t, \sin t)$. Given $a \in A$, we see that $df_a(\mathbb{R}) = \{(-\sin a, \cos a)t \mid t \in \mathbb{R}\}$ and $X(f(a)) = (\cos a, \sin a)$. The vectors $(-\sin a, \cos a)$ and $(\cos a, \sin a)$ are linearly independent, so they span $\mathbb{R}^2$. But $f$ is not a homeomorphism, since $f(\mathbb{R})$ is compact.

(3) Let $Y : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field with no closed orbits nor singular points defined on an open connected set. If there exists a transverse section $T$, homeomorphic to an open interval, which intersects every orbit exactly once, then $Y$ is topologically equivalent to $X$, the vector field given in (1) above. To see this, first note that there is a homeomorphism $h$ such that $h(T) = \mathbb{R}$. Also, we can write any $(x, y) \in U$ uniquely as $\varphi(z, t)$, where $\varphi$ is the flow generated by $Y$ and $z \in T$. So we can define $H(\varphi(z, t)) = (t, h(z))$. This mapping is continuous since $h$, $\varphi$ and $t \mapsto t$ are continuous. Moreover, the mapping $G : H(U) \to U$ given by $G(t, h(z)) = \varphi(z, t)$

is its inverse (and is continuous). Hence $H$ is a homeomorphism. It is clearly a topological conjugacy: if $\psi((x,y),t) = (x+t,y)$ is the flow generated by $X$, we see that

$$H(\varphi(z,t)) = (t, h(z)) = \psi((0,h(z)),t) = \psi(H(\varphi(z,0)),t) = \psi(H(z),t).$$

The open set $H(U)$ is connected and contains the $y$-axis. By stretching it, we can assume $H(U) = \mathbb{R}^2$. This process might break conjugacy, but the two vector fields remain equivalent.

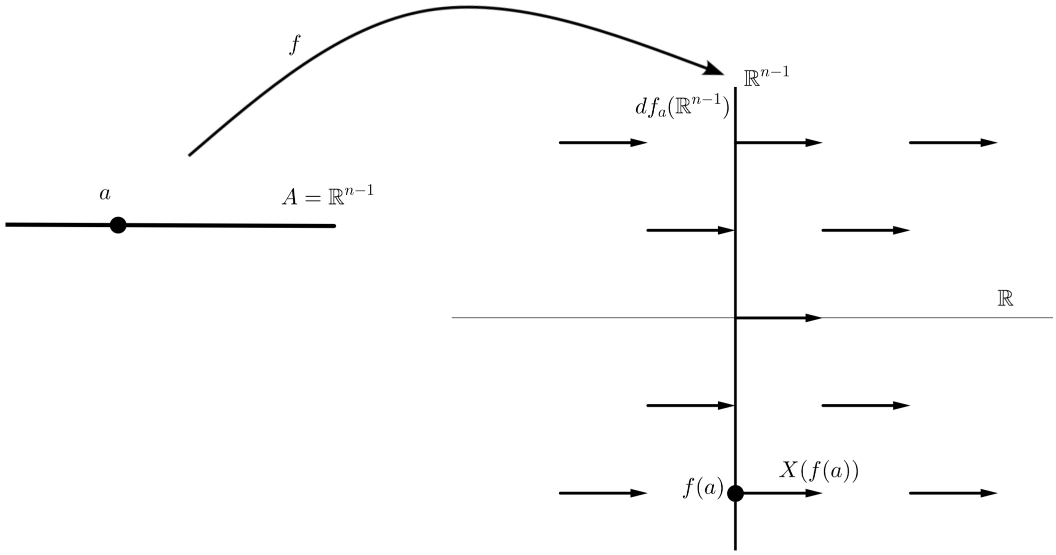



Figure 1.15: Transverse section.

**Theorem 1.16** (Flow Box)**.** *Let $X : U \to \mathbb{R}^n$ be a $\mathcal{C}^r$ vector field, $p$ be a regular point of $X$, $f : A \to U$ be a local transverse section of $X$ of class $\mathcal{C}^r$ with $f(0) = p$, and $\Sigma = f(A)$. Then there exist a neighborhood $V$ of $p$ in $U$ and a diffeomorphism $h : V \to B \times (-\varepsilon, \varepsilon)$ of class $\mathcal{C}^r$, where $\varepsilon > 0$ and $B$ is an open ball in $\mathbb{R}^{n-1}$ with center at the origin $0$ ($= f^{-1}(p)$) such that*

*(a)* $h(\Sigma \cap V) = B \times \{0\}$;

*(b)* *$h$ is a $\mathcal{C}^r$-conjugacy between $X|_V$ and the constant vector field $Y : B \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ given by $Y = (0, \dots, 0, 1)$.*

*Proof.* Let $\varphi : D \to U$ be the flow of $X$. Let $D_A = \{(u,t) \mid (f(u),t) \in D\}$ and define $F : D_A \to U$ by setting $F(u,t) = \varphi(f(u),t)$. Observe that applying $F$ to the straight line $(u_0, t)$ an integral curve of $X$ is produced. We will show that $F$ is a local diffeomorphism at $0 \in \mathbb{R}^{n-1} \times \mathbb{R}$. We will prove that $dF_0$ is an isomorphism, and then apply the Inverse Function Theorem.

We have

$$\partial_t F(0) = \left. \frac{d\varphi(f(0),t)}{dt} \right|_{t=0} = X(\varphi(p,0)) = X(p),$$

and, denoting $u = (u_1, \dots, u_{n-1})$,

$$\partial_{u_i} F(0) = \partial_{u_i} \varphi(f(0), 0) = \partial_{u_i} f(0),$$

for every $i = 1, \dots, n-1$. Since $f$ is a local transverse section, by the above expressions we see that $\{\partial_{u_i} F(0), \partial_t F(0)\}$, $i = 1, \dots, n-1$, spans $\mathbb{R}^n$, i.e., $dF_0$ is an isomorphism.

By the Inverse Function Theorem, there are $\varepsilon > 0$ and an open ball in $\mathbb{R}^{n-1}$ centered at the origin such that $F|_G$, where $G = B \times (-\varepsilon, \varepsilon)$, is a diffeomorphism onto $V = F(G)$. Set $h = (F|_G)^{-1}$. Then

$$h(\Sigma \cap V) = \{(x,t) \mid \varphi(f(x), t) \in \Sigma\} = \{(x, 0) \mid x \in B\} = B \times \{0\},$$

which proves (a). Finally, note that

$$\begin{aligned} dh^{-1}_{(u,t)} Y(u,t) &= dF_{(u,t)}(0, \dots, 0, 1) = \partial_t F(u,t) \\ &= X(\varphi(f(u), t)) = X(F(u,t)) = X(h^{-1}(u,t)), \end{aligned}$$

for every $(u,t) \in B \times (-\varepsilon, \varepsilon)$. By Lemma 1.12, this shows that $h^{-1}$ is a conjugacy between $Y$ and $X$ as desired. □

**Remark.** It is easy to see that the constant vector field given in the preceding theorem is equivalent to the constant vector field in Example 1.14 (1). Since equivalence is transitive, we will usually say that the Flow Box Theorem gives an equivalence between an open neighborhood of a regular point and the constant vector field given in the example without further mentioning, possibly with restricted domain in order to assure conjugacy. Also, any neighborhood $V$ satisfying (a) and (b) is referred to as a *tubular neighborhood* of $p$.

Very similar to the result above, we have the Long Flow Box Theorem.

**Theorem 1.17** (Long Flow Box)**.** *Let $X$, $p$, $f$ and $\Sigma$ be as in the Flow Box Theorem. If $q$ is any point in the orbit of $p$, then there exist an open set $V$ in $U$ containing both $p$ and $q$ and a diffeomorphism $h : V \to B \times (a,b)$ of class $\mathcal{C}^r$, where $B$ is an open ball in $\mathbb{R}^{n-1}$ with center at the origin, such that*

*(a)* $h(\Sigma \cap V) = B \times \{0\}$;

*(b)* *$h$ is a $\mathcal{C}^r$-conjugacy between $X|_V$ and the constant vector field $Y : B \times (a,b) \to \mathbb{R}^n$ given by $Y = (0, \dots, 0, 1)$.*

*Proof.* Let $\varphi : D \to U$ be the flow of $X$ and $t_0 \in I_p$ be the parameter that satisfies $\varphi(p, t_0) = q$. For simplicity's sake, suppose $t_0 > 0$. Since $X$ has no critical points along the orbit of $p$, by using $\varphi$ we can identify $\varphi(p, [0, t_0])$ with an interval of the reals which, by abuse of notation, we shall denote by $[p, q]$. Given $x \in [p, q]$, the Flow Box Theorem guarantees a neighborhood $V_x$ of $x$ such that $X|_{V_x}$ is, through a diffeomorphism

$h_x$, conjugate to $Y_x : B_x \times (-\varepsilon_x, \varepsilon_x) \to \mathbb{R}^n$. The open set $\cup_{x\in[p,q]} V_x$ covers $[p,q]$, so there exist $x_1 = p, x_2, \ldots, x_k = q$ such that

$$[p,q] \subset V_{x_1} \cup \cdots \cup V_{x_k}.$$

If we prove the result for just a pair $V_1$ and $V_2$, where the first is any neighborhood containing $p$ having the properties described in the theorem statement (in particular, $V_{x_1}$ is like this) and $V_2$ is as above, we are done, since, by repetition of the argument for $V_1 \cup V_2$ and, say, $V_3$, it is straightforward to conclude the theorem for any finite set of neighborhoods.

So, suppose that $V_1$ and $V_2$ are two neighborhoods of $p$ and $q$, respectively, as mentioned and that they cover $[p,q]$. Consider the open sets $W = V_1 \cap V_2$ and $V = V_1 \cup V_2$. Using the tubular neighborhoods, we can take a connected transverse section $T$ through a point $r \in [p,q] \cap W$ with the property that every orbit in $V$ intersects $T$ at most once. By reducing the size of $B_1$ or $B_2$ if necessary (and so the size of $V_1$ or $V_2$), we can assume that, given $x \in V$, the orbit $\gamma_x$ intersects $T$. We construct a diffeomorphism $h : V \to B_1 \times (a,b)$ as follows.

Put $k = h_1 \circ h_2^{-1} : h_2(T) \to B_1 \times (-\varepsilon_1, \varepsilon_1)$. Given $x \in V_2$, there is a unique $\overline{x} \in T$ and a unique time $\overline{t}$ such that $x = \varphi(\overline{x}, \overline{t})$. Define $H : B_2 \times (-\varepsilon_2, \varepsilon_2) \to \mathbb{R}^n$ by

$$H(h_2(x)) = H(h_2(\varphi(\overline{x}, \overline{t}))) = (k_1(h_2(\overline{x})), \ldots, k_n(h_2(\overline{x})) + \overline{t}).$$

Such map is a diffeomorphism. Finally, define $h$ by the law

$$h(x) = \begin{cases} h_1(x), & x \in V_1 \\ H \circ h_2(x), & x \in V_2 \end{cases}.$$

Observe that for $x \in W$ we have $(x = \varphi(\overline{x}, \overline{t}))$

$$\begin{aligned} h_1(x) = h_1(\varphi(\overline{x}, \overline{t})) &= (h_1^1(\overline{x}), \ldots, h_1^n(\overline{x}) + \overline{t}) \\ &= (k_1(h_2(\overline{x})), \ldots, k_n(h_2(\overline{x})) + \overline{t}) = H(h_2(\varphi(\overline{x}, \overline{t})) = H \circ h_2(x). \end{aligned}$$

Being a $\mathcal{C}^r$-conjugacy comes from the fact that $h_1$ and $h_2$ are $\mathcal{C}^r$-conjugacies and that $H$ preserves time. This ends the proof. □

# 2 Poincaré–Bendixson theory

We often want to know what happens to an orbit or a group of orbits in the large. In general this is a very hard task. For instance, in $\mathbb{R}^3$ we have the *Lorenz system*, named after the mathematician and meteorologist Edward Lorenz. Despite its simplicity, this system has, for certain parameter values and initial conditions, chaotic solutions. So we cannot, in general, say much about the behavior of orbits.

However, in $\mathbb{R}^2$ and in "planar-like" regions we have the Poincaré–Bendixson Theorem. This result is a very powerful one, as it gives all the possible behaviors for orbits inside a compact set. This will be the main theorem of this section.

**Definition 1.18.** Let $f : U \to \mathbb{R}^n$ be a $C^r$ vector field, $U \subset \mathbb{R}^n$ an open set. Let $\varphi(t) = \varphi(p,t)$ be the integral curve through $p$, defined in its maximal interval $I_p = (\omega_-(p), \omega_+(p))$. If $\omega_+(p) = \infty$, we define

$$\omega(p) = \{x \in U \mid \text{There exists } \{t_n\} \text{ with } t_n \to \infty \text{ and } \varphi(t_n) \to x, \text{ as } n \to \infty\}.$$

Analogously, if $\omega_-(p) = -\infty$, we define

$$\alpha(p) = \{x \in U \mid \text{There exists } \{t_n\} \text{ with } t_n \to -\infty \text{ and } \varphi(t_n) \to x, \text{ as } n \to \infty\}.$$

These sets are called *$\omega$-limit set* (or simply *$\omega$-limit*) and *$\alpha$-limit set* (or simply *$\alpha$-limit*) of $p$.

It is clear that if $x$ is a singular point of a vector field $f$, then $\omega(x) = \alpha(x) = \{x\}$, as $\varphi(x,t) = x$ for every $t$. Also, because of the flow properties, we have $\omega(p) = \omega(q)$ and $\alpha(p) = \alpha(q)$ in case $p$ and $q$ are in the same orbit. Precisely, if $q \in \gamma_p$, we have $\varphi(q,t) = \varphi(p, t+s)$ for some $s \in \mathbb{R}$. Hence, the said property holds by the definition of $\omega$ and $\alpha$-limit sets.

By the preceding paragraph, the following definition can be made.

**Definition 1.19.** The $\omega$-limit set of a given orbit $\gamma$ is the $\omega$-limit set of some $p \in \gamma$. The $\alpha$-limit set of a given orbit $\sigma$ is the $\alpha$-limit set of some $p \in \sigma$. They are denoted, respectively, by $\omega(\gamma)$ and $\alpha(\sigma)$.

**Remark.** Let $\varphi(t) = \varphi(p,t)$ be the integral curve of $f$ through $p$ and $\psi(t) = \psi(p,t)$ be the integral curve of $-f$ through $p$. Hence $\varphi(t) = \psi(-t)$. It follows then that the $\omega$-limit of $\varphi(t)$ is the $\alpha$-limit of $\psi(t)$ and, conversely, the $\omega$-limit of $\psi(t)$ is the $\alpha$-limit of $\varphi(t)$. So, in order to study general properties of $\omega$ and $\alpha$-limit sets, one can restrain their attention to just one of them.

**Theorem 1.20.** *Let $f : U \to \mathbb{R}^n$ be a vector field of class $C^r$ defined on an open set $U \subset \mathbb{R}^n$ and $\gamma_p^+ = \{\varphi(p,t) \mid t \geq 0\}$ (respectively $\gamma_p^- = \{\varphi(p,t) \mid t \leq 0\}$) the positive semi-orbit of $f$ through $p$. If $\gamma_p^+$ (respectively $\gamma_p^-$) is contained in a compact subset $K$ of $U$ (remember that this implies that $\omega_+(p) = \infty$ and, respectively, that $\omega_-(p) = -\infty$), then*

*(a) $\omega(p) \neq \emptyset$ (respectively $\alpha(p) \neq \emptyset$);*

*(b) $\omega(p)$ is compact (respectively $\alpha(p)$);*

*(c) $\omega(p)$ is $f$-invariant (respectively $\alpha(p)$), i.e., if $q \in \omega(p)$, then $\gamma_q$ is contained in $\omega(p)$;*

*(d) $\omega(p)$ is connected (respectively $\alpha(p)$).*

*Proof.* By the previous remark, we only need to prove the theorem for $\omega(p)$.

For (a), as $\varphi(p,t)$ is contained in $K$ for $t \geq 0$, if we consider the sequence $\{\varphi(p,t_n)\}$ for some $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, then, by the compactness of $K$, there exists a subsequence $\{\varphi(p,t_{n_k})\}$ which converges to a point $q \in K$. Since $t_{n_k} \to \infty$ as $k \to \infty$, we have $q \in \omega(p)$.

To prove (b), observe that $\omega(p) \subset \overline{\gamma_p^+} \subset K$; so it suffices to prove that $\omega(p)$ is closed.

Let $\{q_n\}$ be a sequence of points of $\omega(p)$ which converges to $q$. For each $q_n$ there is a sequence $t_k^n$ such that $t_k^n \to \infty$ and $\varphi(p,t_k^n) \to q_n$ as $k \to \infty$. Choose, for each $n$, a number $t_n = t_{k(n)}^n > n$ and such that $d(\varphi(p,t_n),q_n) < \frac{1}{n}$. The triangle inequality yields

$$d(\varphi(p,t_n),q) \leq d(\varphi(p,t_n),q_n) + d(q_n,q) < \frac{1}{n} + d(q_n,q).$$

It follows that $d(\varphi(p,t_n),q)$ goes to 0 as $n \to \infty$, which is to say that $\varphi(p,t_n) \to q$ as $n \to \infty$. Since $t_n \to \infty$ as $n \to \infty$, we conclude that $q \in \omega(p)$.

For (c), let $y$ be a point contained in $\gamma_q$. Then there exists $\tau$ such that $y = \varphi(q,\tau)$. Also, since $q \in \omega(p)$, there exists $\{t_n\}$ with $t_n \to \infty$ and $\varphi(p,t_n) \to q$ as $n \to \infty$. By the continuity of $\varphi$, it follows that

$$y = \varphi(q,\tau) = \varphi(\lim_{n\to\infty} \varphi(p,t_n),\tau) = \lim_{n\to\infty} \varphi(\varphi(p,t_n),\tau)) = \lim_{n\to\infty} \varphi(p,t_n+\tau).$$

Define $s_n = t_n + \tau$. Therefore, $s_n \to \infty$ and $\varphi(p,s_n) \to y$ as $n \to \infty$, i.e., $y \in \omega(p)$, as desired.

In order to prove (d), assume that $\omega(p)$ is disconnected and denote $\varphi(t) = \varphi(p,t)$. Write $\omega(p) = A \cup B$, where $A$ and $B$ are non-empty disjoint closed sets (so, by (b), these sets are compact). Take $a \in A$ and $b \in B$. There exist sequences $\{a_n\}$ and $\{b_n\}$ that go to infinity as $n \to \infty$ such that $\varphi(p,a_n) \to a$ and $\varphi(p,b_n) \to b$ as $n \to \infty$. This shows the existence of a sequence $\{t_n\}$ that goes to infinity as $n \to \infty$ and that has the following property: if we put $\rho = d(A,B) > 0$, the inequalities $d(\varphi(t_n),A) < \rho/2$ and $d(\varphi(t_{n+1}),A) > \rho/2$ hold for every $n$ odd.

Given $n$ odd, the function $g(t) = d(\varphi(t),A)$, $t \in [t_n,t_{n+1}]$, is continuous and $g(t_n) < \rho/2$ and $g(t_{n+1}) > \rho/2$. So, by the Intermediate Value Theorem, there exists a point $x_n \in (t_n,t_{n+1})$ such that $d(\varphi(x_n),A) = g(x_n) = \rho/2$. Now, the set $P = \{x \in K \mid d(x,A) = \rho/2\}$ is compact and contains the sequence $\{\varphi(x_n)\}$. Passing to a convergent subsequence if necessary, we can assume that $\varphi(x_n) \to x \in P$ as $n \to \infty$. Since $x_n$ goes to infinity as $n \to \infty$, the point $x$ is in $\omega(p)$. However, it cannot be that $x \in A$, since $d(\varphi(x),A) = \rho/2 > 0$; but $x$ cannot be in $B$ either, because $d(x,B) \geq d(A,B) - d(x,A) = \rho/2 > 0$. This is a contradiction. Therefore $\omega(p)$ is connected. □

**Corollary.** *With the same assumptions of the last theorem, if $q \in \omega(p)$ (or if $q \in \alpha(p)$), then $\varphi(q,t)$ is defined for every $t \in \mathbb{R}$.*

*Proof.* Since $\omega(p)$ is compact and invariant, the orbit $\gamma_q$ is contained in a compact. So, by Theorem 1.7, $\omega_-(q) = -\infty$ and $\omega_+(q) = \infty$. □

From the proof of the preceding theorem, we see that even if $\gamma_p^+$ (respectively $\gamma_p^-$) is not contained in a compact set, the set $\omega(p)$ (respectively $\alpha(p)$) is $f$-invariant and closed. In this case $\omega(p)$ (respectively $\alpha(p)$) may very well be empty or disconnected.

**Example 1.21.** Consider the vector field $f : [-1, 1] \times \mathbb{R} \to [-1, 1] \times \mathbb{R}$ given by

$$f(x, y) = (f_1(x, y), f_2(x, y)) = (x(1 - x^2 + 2xy) - 2y,\ y(1 - x^2 + 2xy) + 2x).$$

Note that $(0, 0)$ is a critical point of $f$, and thus $\alpha(0, 0) = \omega(0, 0) = \{(0, 0)\}$. Moreover, $f$ has no other critical point. Indeed, $f_1(x, y) = 0$ if, and only if, $|x| = 1$ or $y = x/2$. When $|x| = 1$ we have $f_2(x, y) \neq 0$; when $y = x/2$ we have $f_2(x, y) = 0$ if, and only if, $x = 0$, which proves the claim.

If $|x| = 1$, then $f(x, y) = (0,\ 2x(y^2 + 1))$, and so the orbit $\gamma_1$ through $(1, 0)$ is the straight line $x = 1$ oriented upward and the orbit $\gamma_2$ through $(-1, 0)$ is the straight line $x = -1$ oriented downward. Hence $\alpha(x, y) = \omega(x, y) = \emptyset$.

Fix $(x, y) \in (-1, 1) \times \mathbb{R}$ different from $(0, 0)$. The orbit through $(x, y)$ contains only regular points and does not leave this strip. The inner product between $f(x, y)$ and the vector $(0, 1)$ is equal to

$$\langle f(x, y), (0, 1)\rangle = y(1 - x^2) + 2x(y^2 + 1),$$

while the inner product between $f(x, y)$ and the vector $(1, 0)$ is equal to

$$\langle f(x, y), (1, 0)\rangle = (x - 2y)(1 - x^2).$$

These two identities shows that fixed $y_0 > 0$, along the curve $t \mapsto (-t, y_0)$, $|t| < 1$, when $t$ is close to $-1$ the vector $f(x, y_0)$ points toward the half plane defined by $y > y_0$ and it starts to rotate counterclockwise as $t$ approaches 1. On the other hand, fixing $y_0 < 0$ and using the curve $t \mapsto (t, y_0)$, $|t| < 1$, an analogous conclusion may be obtained. Furthermore, if $x^2 + y^2 > 0$ is sufficiently small, $f$ points outward along this circle. Hence, excluding the singular point, every orbit inside the strip $(-1, 1) \times \mathbb{R}$ spirals counterclockwise, starting at the origin; see Figure 1.22. Therefore, for a point $(x, y)$ in such strip different from the critical point, we have $\alpha(x, y) = \{(0, 0)\}$ and $\omega(x, y) = \gamma_1 \cup \gamma_2$.

In order to prove the Poincaré–Bendixson Theorem, we will state but not prove several lemmas. For their proofs, we refer to [15], pages 248 to 251. Though we will not use all of them explicitly, they have their own importance alone. The theorem immediately after them is the one known as the Poincaré–Bendixson Theorem. Henceforth in this section, $U$ will denote an open set of $\mathbb{R}^2$, $f$ a vector field of class $\mathcal{C}^r$ on $U$ and $\gamma_p^+$ the positive semi-orbit through $p$ ($\gamma_p^+ = \{\varphi(p, t) \mid t \geq 0\}$).

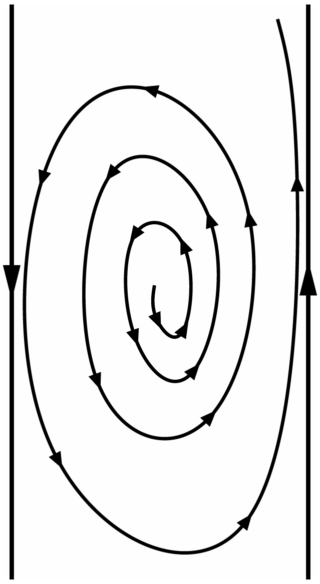


Figure 1.22: Disconnected $\omega$-limit set.

It is important to note that, as we are in $\mathbb{R}^2$, every transverse section $\Sigma$ of a given vector field $f$ has dimension 1. So $\Sigma$ is, locally, the diffeomorphic image of an interval. Therefore, every transverse section $\Sigma$ considered in what comes next will be the diffeomorphic image of an interval. Hence, $\Sigma$ has a total ordering "$\leq$" inherited by the total ordering of the interval. It is possible, then, to talk about monotone sequences in $\Sigma$.

**Lemma 1.23.** *If $p \in \Sigma \cap \omega(\gamma)$, where $\Sigma$ is a transverse section of $f$ and $\gamma = \{\varphi_x(t)\}_{t \in I_x}$ is an orbit of $f$, then $p$ can be written as the limit of a sequence $\{\varphi_x(t_n)\}$, where $\varphi_x(t_n) \in \Sigma$ for every $n$ and $t_n \to \infty$ as $n \to \infty$.*

**Lemma 1.24.** *Let $\Sigma$ be a transverse section of $f$ contained in $U$. If $\gamma$ is an orbit of $f$ and $p \in \Sigma \cap \gamma$, then $\gamma_p^+$ intercepts $\Sigma$ in a monotone sequence $\{p_n\}$.*

**Lemma 1.25.** *If $\Sigma$ is a transverse section of $f$ and $p \in U$, then $\Sigma$ intercepts $\omega(p)$ in at most one point.*

**Lemma 1.26.** *Let $p \in U$, with $\gamma_p^+$ contained in a compact set, and $\gamma$ be an orbit of $f$ contained in $\omega(p)$. If $\omega(\gamma)$ contains regular points, then $\gamma$ is a closed orbit and $\omega(p) = \gamma$.*

**Theorem 1.27** (Poincaré–Bendixson)**.** *Let $\varphi(p,t)$ be an integral curve of $f$, defined for every $t \geq 0$, such that $\gamma_p^+$ is contained in a compact set $K \subset U$. Assume $f$ has a finite number of singularities in $\omega(p)$. Then one of the following assertions holds:*

*(a) If $\omega(p)$ contains only regular points, then $\omega(p)$ is a periodic orbit;*

*(b) If $\omega(p)$ contains both regular and singular points, then $\omega(p)$ is the union of regular orbits and singular points; each orbit tends to one of these singular points when $t \to \pm\infty$;*

*(c) If $\omega(p)$ contains only singular points, then $\omega(p)$ is a unique singular point.*

*Proof.* Suppose $\omega(p)$ has no singular points. If $q \in \omega(p)$, we already know that $\gamma_q \subset \omega(p)$. Since $\omega(p)$ is a compact set, it follows that $\omega(q)$ is non-empty. Lemma 1.26 guarantees that $\omega(p) = \gamma_q$ is a closed orbit.

If $\omega(p)$ has both singular and regular points and $\gamma$ is a regular orbit contained in $\omega(p)$, then, by Lemma 1.26, $\omega(\gamma)$ (and $\alpha(\gamma)$) has no regular points. Since $f$ has only finitely many singularities in $\omega(p)$ and $\omega(\gamma)$ and $\alpha(\gamma)$ are connected, they are, each, a unique singular point.

Finally, assume that $\omega(p)$ has no regular points. Then (c) follows immediately from the fact that $f$ has only finitely many singular points in $\omega(p)$ and that this set is connected. □

An analogous statement holds for $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

**Theorem 1.28.** *Let $f$ be a $\mathcal{C}^r$ vector field on $\mathbb{R}^3$ such that if $x \in S^2$, then $\gamma_x \subset S^2$. If $f$ has finitely many singularities in $S^2$, then one of the same assertions (a), (b) and (c) above holds for the $\omega$-limit set of any $x \in S^2$.*

A very useful corollary of the Poincaré–Bendixson Theorem is that "inside" a closed orbit there must exist a singular point. First, we will formalize the notion of *inside* and *outside* a *Jordan curve* and state the famous *Jordan Curve Theorem*. For the proof and a more complete treatment of this topic, we refer to [12].

**Definition 1.29.** A *Jordan curve* in a topological space $X$ is the image of a continuous injection from $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ to $X$. A *Jordan arc*, or simply an *arc*, in a topological space $X$ is the image of a continuous injection from the unit interval $[0, 1]$ to $X$. The *endpoints* of a Jordan arc $J$ are the two points $p$ and $q$ such that $J - p$ and $J - q$ are connected; the other points are called *interior points* of $J$.

**Definition 1.30.** If $X$ is a connected topological space and $A \subset X$, we say that $A$ *separates* $X$ if $X - A$ is disconnected; if $X - A$ has $n$ components, we say that $A$ *separates $X$ into $n$ components*.

We will state the Jordan Curve Theorem in the sphere $S^2$; however, due to the following lemma, the same result is plainly true to the plane $\mathbb{R}^2$. The proofs of the next three results can be found in [12], pages 377, 389 and 390.

**Lemma 1.31.** *Let $C$ be a compact subset of $S^2$, $p$ be a point of $S^2 - C$ and $h$ be a homeomorphism of $S^2 - p$ with $\mathbb{R}^2$. Suppose $U$ is a component of $S^2 - C$. If $U$ does not contain $p$, then $h(U)$ is a bounded component of $\mathbb{R}^2 - h(C)$. If $U$ contains $p$, then $h(U - p)$ is the unbounded component of $\mathbb{R}^2 - h(C)$. In particular, if $S^2 - C$ has $n$ components, then $\mathbb{R}^2 - h(C)$ has $n$ components.*

**Theorem 1.32.** *Let $A$ be an arc in $S^2$. Then $A$ does not separate $S^2$.*

**Theorem 1.33** (Jordan Curve)**.** *Let $C$ be a Jordan curve in $S^2$. Then $C$ separates $S^2$ into precisely two components $U_1$ and $U_2$. Each of these sets has $C$ as its boundary, i.e., $\overline{U}_i - U_i = C$, for $i = 1, 2$.*

Now it is possible to talk about points inside and outside a closed orbit (a Jordan curve) in a more precise way.

**Definition 1.34.** By *region* we mean an open connected set. Let $\gamma$ be a Jordan curve in $\mathbb{R}^2$. We call the bounded region *interior* of $\gamma$ and denote it by $\operatorname{Int}\gamma$; we call the unbounded region *exterior* of $\gamma$, and denote it by $\operatorname{Ext}\gamma$.

**Theorem 1.35.** *Let $f$ be a vector field of class $\mathcal{C}^r$ on an open set $U \subset \mathbb{R}^2$. If $\gamma$ is a closed orbit of $f$ such that* $\operatorname{Int}\gamma \subset U$*, then there is a singular point of $f$ contained in the interior of $\gamma$.*

*Proof.* To get a contradiction, suppose there are no singular points of $f$ in $\operatorname{Int}\gamma$. Let $\Gamma$ be the set of all closed orbits of $f$ contained in $\overline{\operatorname{Int}\gamma}$, partially ordered by the reverse inclusion: $\gamma_1 \leq \gamma_2$ if and only if $\overline{\operatorname{Int}\gamma_1} \supset \overline{\operatorname{Int}\gamma_2}$. In order to apply the Zorn's Lemma, we will prove that every totally ordered subset of $\Gamma$ has an upper bound.

Given a totally ordered set $\Sigma \subset \Gamma$, let $\mathscr{G} = \cap_{\gamma_i \in \Sigma} \overline{\operatorname{Int}\gamma_i}$. Note that this intersection is non-empty, since the intersection of finitely many of these sets is always non-empty and all of them are compact. Let $p \in \mathscr{G}$. By the Poincaré–Bendixson Theorem, $\omega(p)$ is a closed orbit contained in $\mathscr{G}$, since there are no singular points in $\mathscr{G}$ and it is $f$-invariant ($\mathscr{G}$ is the intersection of $f$-invariant sets). This orbit is an upper bound of $\Sigma$: by the construction there is no $\gamma_i \in \Sigma$ contained in $\overline{\operatorname{Int}\omega(p)}$.

We can now apply the Zorn's Lemma and obtain a maximal element $\mu \in \Gamma$. Hence there is no element of $\Gamma$ contained in $\operatorname{Int}\mu$. But if $q \in \operatorname{Int}\mu$, then again by the Poincaré–Bendixson Theorem the sets $\alpha(q)$ and $\omega(q)$ are both closed orbits. They cannot be both equal $\mu$, otherwise the flow would not be continuous. So one of them is contained in $\operatorname{Int}\mu$, which is the desired contradiction. $\square$

**Example 1.36.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the polynomial vector field given by

$$f(x, y) = (y + x(1 - x^2 - y^2), -x + y(1 - x^2 - y^2))$$

and $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Observe that the origin is the unique singular point of this vector field. Indeed, suppose $(x_0, y_0)$ is a singular point. Let $\alpha = 1 - x_0^2 - y_0^2$. Then $y_0 + \alpha x_0 = 0$, which implies that $y_0 = -\alpha x_0$. On the other hand, we have $-x_0 + \alpha y_0 = 0$. So, using the first identity, it follows that $-x_0(1 + \alpha^2) = 0$. The only possibility is $x_0 = 0$, but then $y_0 = 0$ too, proving the claim. Also, notice that $t \mapsto (\sin t, \cos t)$ is an integral curve of $f$, which is closed. So every trajectory through a point of $B$ is totally contained in $B$; analogously, every trajectory through a point of $\mathbb{R}^2 - \overline{B}$ does not leave this set.

Let us see what happens on the circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$. The vector field takes the form $f(x, y) = (y + (1 - r^2)x, -x + (1 - r^2)y)$; the inner product between this vector and the position is equal to

$$\langle f(x, y), (x, y)\rangle = (1 - r^2)(x^2 + y^2) = r^2(1 - r^2).$$

The above identity says that the vector field $f$ points inward along every circle of radius $r > 1$, is tangent to the circle $S^1$ and points outward along every circle of radius $r < 1$. Hence the closed orbit *attracts* every other orbit (besides, of course, the stationary one). By continuity, we conclude that every orbit spirals toward the periodic orbit; see Figure 1.37. Therefore,

$$\begin{aligned}
&\alpha(0,0) = \omega(0,0) = \{(0,0)\}, \\
&\alpha(p) = \{(0,0)\},\ \omega(p) = S^1,\ \text{if } p \in B - \{(0,0)\}, \\
&\alpha(p) = \omega(p) = S^1,\ \text{if } p \in S^1, \\
&\alpha(p) = S^1,\ \omega(p) = \emptyset,\ \text{if } p \in \mathbb{R}^2 - \overline{B}.
\end{aligned}$$

Note that if it was not obvious that the origin is a singular point of $f$, we could have used Theorem 1.35 to conclude the existence of a singular point inside $S^1$.

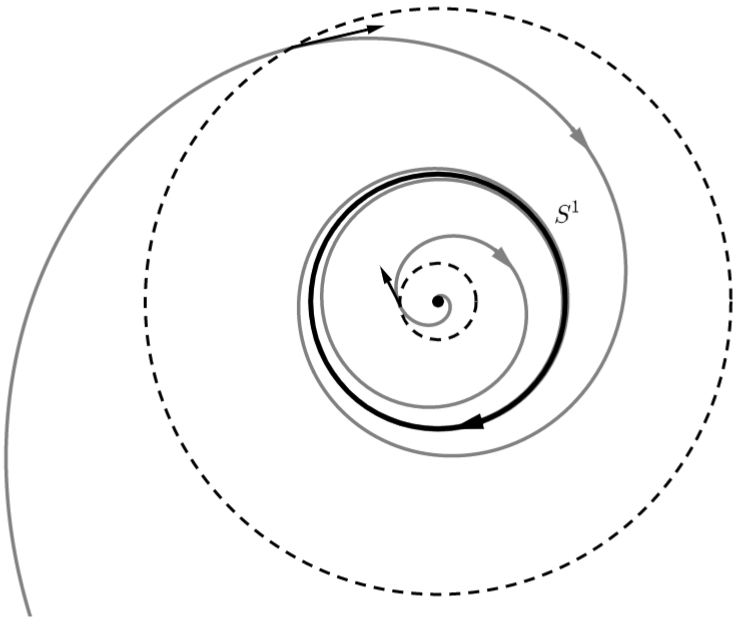



Figure 1.37: Orbits of $f$.

The closed orbit of the vector field above is what we call a *limit cycle*: an orbit that is neighbored by an open set such that it is the only closed orbit of this set.

# 3 Compactification

Several important concepts of vector fields and flows do not depend on the metric of the considered space, as topological equivalence. There are still some that are totally dependent on the metric, such as *stability*. Also, when dealing with vector fields in the plane, for example, one cannot draw their phase portrait in its entirety, as the plane is unbounded. To avoid any problem with the metric or with phase portraits, there is the notion of *compactification of a vector field* or of a continuous flow. We will present the

*Poincaré compactification* and the *one-point compactification* of a continuous flow. The former is widely used when dealing with polynomial vector fields, and it gathers more information about the behavior of the vector field in the large. The latter may be applied to any continuous flow, and will be constructed on topological spaces. The main difference between them is, intuitively, the same difference between seeing the extended real line as a closed interval containing $-\infty$ and $\infty$ and seeing it as the closed circle: in the second one we lose information about the ends. Not every vector field can be compactified, since it may be impossible to restrain its growth near the infinity.

As we will often restrict ourselves to topological manifolds (often called just manifold), we write here what we mean by it.

**Definition 1.38.** Let $M$ be a topological space, $U \subset M$ and $V \subset \mathbb{R}^n$ be open sets and $\psi : U \to V$ a homeomorphism. The pair $(U, \psi)$ is called a *chart* on $M$ and $\psi$ a *coordinate system* on $U$. Sometimes the open set $U$ is referred to as a chart also. An *atlas* on $M$ is a collection of charts $\{(U_\alpha, \psi_\alpha)\}$ such that the collection $\{U_\alpha\}$ covers $M$.

**Definition 1.39.** A second-countable, Hausdorff topological space $M$ is an *$n$-dimensional topological manifold* (or, simply, an $n$-manifold) if it is locally Euclidean, i.e., it admits an atlas $\{(U_\alpha, \psi_\alpha)\}$, $\psi_\alpha : U_\alpha \to \mathbb{R}^n$, $n \in \mathbb{N}$.

In the first half of this section we will work primarily in $\mathbb{R}^2$, based on [3]; in the second, on a topological space $N$.

Let $X = (P, Q)$ be a vector field in $\mathbb{R}^2$, where $P$ and $Q$ are polynomials in two variables, with degree $d_1$ and $d_2$, respectively. Consider, then, the differential system

$$\begin{cases} \dot{x_1} = P(x_1, x_2) \\ \dot{x_2} = Q(x_1, x_2) \end{cases} . \tag{1.2}$$

For the sake of clarity, we will see $\mathbb{R}^2$ as the plane in $\mathbb{R}^3$ given by $\{(x_1, x_2, 1) \in \mathbb{R}^3 \mid x_1, x_2 \in \mathbb{R}\}$. Points in $\mathbb{R}^2$ will be denoted by tuples with subscripted $x$ and points in $\mathbb{R}^3$ by tuples with subscripted $y$. Recall that the sphere $S^2 \subset \mathbb{R}^3$ is given by $\{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1^2 + y_2^2 + y_3^2 = 1\}$. In this context, $S^2$ is called the *Poincaré sphere*. It is tangent to $\mathbb{R}^2$ at the point $(0, 0, 1)$. Define $H_+ = \{y \in S^2 \mid y_3 > 0\}$ (the northern hemisphere), $H_- = \{y \in S^2 \mid y_3 < 0\}$ (the southern hemisphere) and $S^1 = \{y \in S^2 \mid y_3 = 0\}$ (the equator). Our objective is to construct a vector field defined on $S^2$ (a compact space) in such a manner that it is equivalent to $X$. In order to do that, consider the central projections $f^+ : \mathbb{R}^2 \to S^2$ and $f^- : \mathbb{R}^2 \to S^2$, given by

$$f^+(x) = \left( \frac{x_1}{\Delta(x)}, \frac{x_2}{\Delta(x)}, \frac{1}{\Delta(x)} \right)$$

and

$$f^-(x) = \left( \frac{-x_1}{\Delta(x)}, \frac{-x_2}{\Delta(x)}, \frac{-1}{\Delta(x)} \right),$$

where $\Delta(x) = \sqrt{x_1^2 + x_2^2 + 1}$. Geometrically, $f^+(x)$ (respectively, $f^-(x)$) is the intersection of the straight line passing through the origin and $x$ with the northern (respectively, southern) hemisphere of $S^2$. Observe that we produced two copies of $\mathbb{R}^2$ that can be viewed as subsets of $S^2$. Hence, we can obtain two vector fields induced by the two diffeomorphisms. Precisely: for $y = f^+(x)$, we have $\overline{X}(y) = df_x^+ X(x)$ on the northern hemisphere and for $y = f^-(x)$ we have $\overline{X}(y) = df_x^- X(x)$ on the southern hemisphere; so $\overline{X}$ is a vector field on $S^2 - S^1$ (compare with Lemma 1.12).

We would like to extend $\overline{X}$ to the whole sphere. This will be possible once we can guarantee its boundedness when approaching the equator. Luckily this is always possible in the polynomial case after multiplying the vector field by a function that doesn't change its sign when restricted to $H_+$ or $H_-$. Thus the final vector field will not be $\mathcal{C}^\omega$-conjugate to the original one on each hemisphere, but will remain $\mathcal{C}^\omega$-equivalent on $H_+$ and the orbits will be the same on $H_-$ (compare with Example 1.11).

Calculating,

$$df_x^+ = \frac{1}{\Delta(x)} \begin{bmatrix} 1 - \frac{x_1^2}{\Delta(x)^2} & -\frac{x_1 x_2}{\Delta(x)^2} \\ -\frac{x_1 x_2}{\Delta(x)^2} & 1 - \frac{x_2^2}{\Delta(x)^2} \\ -\frac{x_1}{\Delta(x)^2} & -\frac{x_2}{\Delta(x)^2} \end{bmatrix}.$$

Thus, from $(y_1, y_2, y_3) = f^+(x_1, x_2) = \Delta(x)^{-1}(x_1, x_2, 1)$, the above differential can be expressed as

$$df_x^+ = y_3 \begin{bmatrix} 1 - y_1^2 & -y_1 y_2 \\ -y_1 y_2 & 1 - y_2^2 \\ -y_1 y_3 & -y_2 y_3 \end{bmatrix}.$$

Since $(y_1, y_2, y_3) = f^-(x_1, x_2) = -\Delta(x)^{-1}(x_1, x_2, 1)$ in $H_-$, it follows that $df_x^- = df_x^+$. Finally, using the inverse $(x_1, x_2) = (y_1/y_3, y_2/y_3)$, we can write the vector field on $H_+ \cup H_-$ as

$$\overline{X}(y) = y_3 \begin{bmatrix} 1 - y_1^2 & -y_1 y_2 \\ -y_1 y_2 & 1 - y_2^2 \\ -y_1 y_3 & -y_2 y_3 \end{bmatrix} \begin{bmatrix} P\left(\frac{y_1}{y_3}, \frac{y_2}{y_3}\right) \\ Q\left(\frac{y_1}{y_3}, \frac{y_2}{y_3}\right) \end{bmatrix}$$

It lasts to define this vector field on the equator as well. In order to solve the problem of the reciprocals, we define the *degree of* $X$ by $d = \max\{d_1, d_2\}$, the maximum between the degrees of $P$ and $Q$, and multiply the vector field $\overline{X}$ by $y_3^{d-1}$, obtaining $p(X) = y_3^{d-1}\overline{X}$, the *Poincaré compactification* of the vector field $X$. The vector field $p(X)$ is a polynomial defined on $S^2$, and the orbits of $p(X)$ and $\overline{X}$ are the same on $S^2 - S^1$, because $y_3^{d-1}$ is always positive on $H_+$ and either always positive or always negative on $H_-$, as $d$ is odd or even, respectively.

Our work now is to write the expression for this vector field in charts, so calculations and visualization become easier. We take the six charts $U_k = \{y \in S^2 \mid y_k > 0\}$, $V_k = \{y \in S^2 \mid y_k < 0\}$, $k = 1, 2, 3$, of $S^2$. The corresponding coordinate systems

$\phi^k : U_k \to \mathbb{R}^2$ and $\psi^k : V_k \to \mathbb{R}^2$ are given by

$$\phi^k(y) = -\psi^k(y) = \left(\frac{y_m}{y_k}, \frac{y_n}{y_k}\right),$$

for $m < n$ and $m, n \neq k$. For any $k$, the coordinates in the respective chart will be denoted by $(u, v)$, so that $u$ and $v$ play different roles in different situations. It will be clear by the context. For a geometric representation, see Figure 1.40. Note that the points of $S^1$ in any chart are those whose coordinate $v$ vanishes.

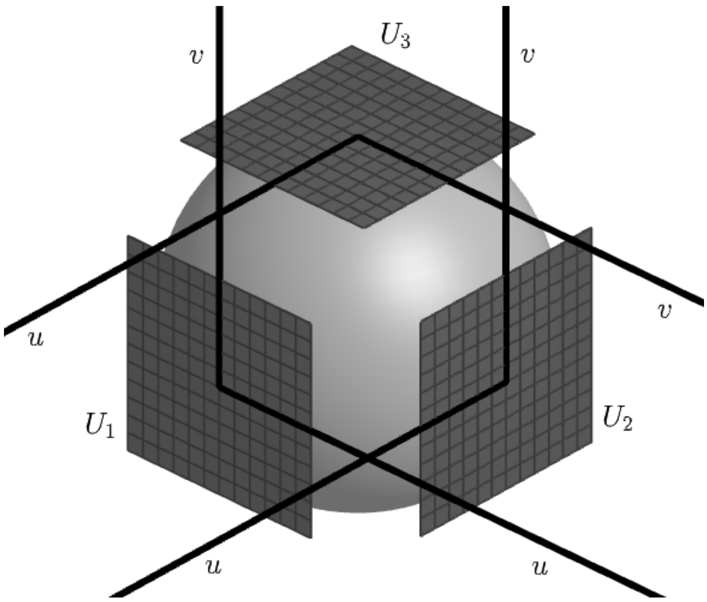



Figure 1.40: Local charts.

For $y = f^+(x) \in H_+$ we have $\overline{X}(y) = df^+_x X(x)$, hence

$$d\phi^1_y p(X)(y) = y_3^{d-1}\, d\phi^1_y\, df^+_x X(x) = y_3^{d-1}\, d(\phi^1 \circ f^+)_x X(x).$$

Let $X^1$ denote the vector field given by $d\phi^1_y p(X)(y)$. This is a vector field in $\mathbb{R}^2$. From

$$(\phi^1 \circ f^+)(x) = \left(\frac{x_2}{x_1}, \frac{1}{x_1}\right) = (u, v),$$

we have

$$\begin{aligned} d(\phi^1 \circ f^+)_x X(x) &= \begin{bmatrix} -\frac{x_2}{x_1^2} & \frac{1}{x_1} \\ -\frac{1}{x_1^2} & 0 \end{bmatrix} \begin{bmatrix} P(x_1, x_2) \\ Q(x_1, x_2) \end{bmatrix} \\ &= \frac{1}{x_1^2}\left(-x_2 P(x_1, x_2) + x_1 Q(x_1, x_2), -P(x_1, x_2)\right) \\ &= v^2 \left(-\frac{u}{v} P\left(\frac{1}{v}, \frac{u}{v}\right) + \frac{1}{v} Q\left(\frac{1}{v}, \frac{u}{v}\right), -P\left(\frac{1}{v}, \frac{u}{v}\right)\right), \end{aligned}$$

so $d(\phi^1 \circ f^+)_x X(x) = v(-uP + Q, -vP)$, where the omitted variables are as above.

Now, observe that $\Delta(u,v) = |x_1|^{-1}\Delta(x) = |y_1|^{-1}$. Recall that in $U_1$ we have $y_1 > 0$. Then $\Delta(u,v) = y_1^{-1}$. Furthermore, from $v = y_3/y_1$ it follows that $y_3 = v\Delta(u,v)^{-1}$. This yields

$$X^1(u,v) = y_3^{d-1}\, d(\phi^1 \circ f^+)_x X(x) = \frac{v^d}{\Delta(u,v)^{d-1}}(-uP+Q, -vP),$$

with the variables as before. When $y = f^-(x)$, since the differentials of $f^-$ and $f^+$ are the same, the result is the same. So the expression holds for every $y \in U_1$. Similar calculations work for $U_2$. Repeating the process above for $V_i$, we see that the expression is the one we got in $U_i$, but multiplied by $(-1)^{d-1}$, because $\Delta(u,v) = |y_i|^{-1} = -y_i^{-1}$, as $y_i < 0$ in $V_i$, $i = 1, 2$.

The case for $U_3$ and $V_3$ is slightly different. Note that $\phi^3(y) = (f^+)^{-1}(y)$, $y \in H_+$, and then the composite is the identity. Moreover, since $y_3 = \Delta(x)^{-1} = \Delta(u,v)^{-1}$, we have

$$d\phi^3_y p(X)(y) = y_3^{d-1} X(x) = \frac{1}{\Delta(u,v)^{d-1}}\left(P(u,v), Q(u,v)\right).$$

The same holds for $V_3$ ($y \in H_-$), but multiplied by $(-1)^{d-1}$, as mentioned before.

We can multiply the vector fields obtained by $(\Delta(u,v))^{d-1} > 0$ in order to obtain a simpler expression for the vector fields and preserve their orbits. In symbols, the expression for the extended differential system on $S^2$ in the chart $(U_1, \phi^1)$ is given by

$$\begin{cases} \dot{u} = v^d\left(-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right) \\ \dot{v} = -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right) \end{cases}, \tag{1.3}$$

in $(U_2, \phi^2)$ is given by

$$\begin{cases} \dot{u} = v^d\left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right) \\ \dot{v} = -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right) \end{cases} \tag{1.4}$$

and in $(U_3, \phi^3)$ is given by

$$\begin{cases} \dot{u} = P(u,v) \\ \dot{v} = Q(u,v) \end{cases}. \tag{1.5}$$

The expression for $p(X)$ in the charts $(V_k, \psi^k)$ is the same as the one for $(U_k, \phi^k)$, but multiplied by $(-1)^{d-1}$, for $k = 1, 2, 3$.

We make a couple of observations.

1. To study $X$ in the complete plane $\mathbb{R}^2$, including its behavior near infinity, it suffices to work on $H_+ \cup S^1$, which is called the *Poincaré disk*. All calculations can be done in the charts $(U_1, \phi^1)$, $(U_2, \phi^2)$ and $(U_3, \phi^3)$. The expressions in each chart are given by (1.3), (1.4) and (1.5), respectively. In the other charts the trajectories are the same, possibly with reversed sense.

2. In order to draw the complete orbits we project the points of $H_+ \cup S^1$ onto the disk

$\{y \in \mathbb{R}^3 \mid y_1^2 + y_2^2 \leq 1, y_3 = 0\}$. This can be done by projecting each point of the sphere along a straight line parallel to the $y_3$-axis.

3. The infinity $S^1$ of $S^2$ is $p(X)$-invariant. Indeed, let $\varphi = (\varphi^1, \varphi^2)$ be the flow of $p(X)$ (in local coordinates) and $x \in S^1 \cap U_1$. Then, in $(U_1, \phi^1)$, we have

$$(\varphi_2)'_x(t) = -(\varphi_2)_x(t)^{d+1} P\left(\frac{1}{(\varphi_2)_x(t)}, \frac{(\varphi_1)_x(t)}{(\varphi_2)_x(t)}\right),$$

where $(\varphi_2)_x(0) = 0$. A solution for such differential equation is $(\varphi_2)_x(t) = 0$ for every $t$ in the maximal interval $I$ of $\varphi_x$. Thus $\varphi_x(I) \subset S^1$. The same reasoning works for $U_2$, $V_1$ and $V_2$.

From now on we deal with a continuous flow $\varphi : N \times \mathbb{R} \to N$ on a topological space $N$ (sometimes referred to as $(N, \varphi)$). We follow [5], Appendix 2, Section III.

**Remark.** As in Chapter 1, the map $\varphi$ defines a group action on $N$, where the group taken is $(\mathbb{R}, +)$. The adjective continuous simply means that this action is continuous ($\varphi$ is continuous).

Let $N$ be noncompact topological space. There is a standard procedure for associating with $N$ a compact topological space, called the *one-point compactification* $N_\infty$ of $N$. Let $\infty$ denote some point not in $N$. We define the set $N_\infty$ to be $N \cup \{\infty\}$. To turn $N_\infty$ into a topological space, we define a subset $U$ of $N_\infty$ to be open in $N_\infty$ if and only if either $U$ is an open subset of $N$ or $N_\infty - U$ is a closed compact subset of $N$. Note that if $U$ is an open set of the second type, then $U \cap N$ is open. We need to verify that this collection is indeed a topology on $N_\infty$.

The empty set is open in $N$, so in $N_\infty$; also, since $N_\infty - N_\infty$ is a closed compact subset of $N$, the whole space $N_\infty$ is open. Checking that the intersection of two open sets is open involves three cases:

(a) The intersection of two open sets $U_1, U_2 \subset N$ is open in $N$, so open in $N_\infty$;

(b) If $N_\infty - U_1$ and $N_\infty - U_2$ are closed compact subsets of $N$, then $N_\infty - (U_1 \cap U_2) = (N_\infty - U_1) \cup (N_\infty - U_2)$ is a closed compact subset of $N$;

(c) If $U_1 \subset N$ is open and $N_\infty - U_2$ is a closed compact subset of $N$, then $U_1 \cap U_2 \subset N$ and

$$N - (U_1 \cap U_2) = (N - U_1) \cup (N_\infty - U_2),$$

which is closed in $N$ since $U_1$ is open. Thus $U_1 \cap U_2$ is open in $N$.

Similarly, for the union of any collection of open sets,

(a) If $\{U_\alpha\}$ are open sets of $N$, then its union is trivially an open set of $N$;

(b) If $\{N_\infty - U_\alpha\}$ are closed compact subsets of $N$, then

$$N_\infty - \bigcup U_\alpha = \bigcap (N_\infty - U_\alpha)$$

is a closed compact subset of $N$;

(c) If $\{U_\alpha\}$ are open sets of $N$ and $\{N_\infty - U_\beta\}$ are closed compact sets of $N$, then

$$\begin{aligned} N_\infty - \left( \bigcup_{\gamma=\alpha,\beta} U_\gamma \right) &= \left[ \bigcap (N_\infty - U_\alpha) \right] \cap \left[ \bigcap (N_\infty - U_\beta) \right] \\ &= \left[ \bigcap (N - U_\alpha) \right] \cap \left[ \bigcap (N_\infty - U_\beta) \right] \end{aligned}$$

is a closed compact set of $N$, since the first intersection is closed and the second is compact.

Moreover, $N$ is a topological subspace of $N_\infty$. This follows directly from the definition of open set in $N_\infty$. To see that $N_\infty$ is compact, let $\mathscr{G}$ be an open covering of $N_\infty$. The collection must contain an open set $U$ such that $N_\infty - U$ is a closed compact subset of $N$. Take all members of $\mathscr{G}$ different from $U$ and intersect them with $N$; they form a collection of open sets of $N$ covering $N_\infty - U$. Because such set is compact, it is possible to extract a finite subcover of $N_\infty - U$. The corresponding finite collection, together with $U$, will cover $N_\infty$. Note that the point $\infty$ is a limit point of $N$ in $N_\infty$, so the closure of $N$ in $N_\infty$ is the whole space $N_\infty$; this is not the case if $N$ is compact.

Suppose, in addition, that $N$ is Hausdorff and locally compact (when the topological space is Hausdorff, locally compactness means: given $x \in N$ and a neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $\overline{V}$ is compact and $\overline{V} \subset U$). In particular, this is the case for an $n$-dimensional manifold. Let $x$ and $y$ be two points in $N_\infty$. If both of them lie in $N$, there are disjoints open sets $U$ and $V$ of $N$ containing $x$ and $y$, respectively. On the other hand, if $x \in N$ and $y = \infty$, we can choose a compact set $C$ of $N$ containing a neighborhood $U$ of $x$. Then $U$ and $N_\infty - C$ are disjoint neighborhoods of $x$ and $y$, respectively, in $N_\infty$. This shows that $N_\infty$ is Hausdorff.

Now let $\varphi$ be a continuous flow on $N$. We define the *one-point compactification* of $\varphi$ to be the map $\varphi_\infty : N_\infty \times \mathbb{R} \to N_\infty$ defined by $\varphi_\infty(x,t) = \varphi(x,t)$ if $x \neq \infty$, and $\varphi_\infty(\infty, t) = \infty$. Plainly $\varphi_\infty$ satisfies the definition of flow. It remains to prove its continuity.

**Lemma 1.41.** *Let $\Phi : N \times \mathbb{R} \to N \times \mathbb{R}$ be defined by $\Phi(x,t) = (\varphi(x,t), t)$. Then $\Phi$ is a homeomorphism.*

*Proof.* The inverse of $\Phi$ is given by $\Phi^{-1}(x,t) = (\varphi(x,-t), t)$. Moreover, if $\pi_1$ and $\pi_2$ denote the projection of $N \times \mathbb{R}$ onto its factors, then $\pi_1 \Phi = \varphi$ and $\pi_2 \Phi = \pi_2$. Since both $\varphi$ and $\pi_2$ are continuous, $\Phi$ is continuous. Similarly $\Phi^{-1}$ is continuous. □

**Theorem 1.42.** *The one-point compactification $\varphi_\infty$ of a continuous flow on $N$ is a continuous flow on $N_\infty$.*

*Proof.* Given $(x, t) \in N_\infty$, and given $V$ a neighborhood of $\varphi_\infty(x, t)$ in $N_\infty$, we must find a neighborhood $U$ of $(x, t)$ in $N_\infty \times \mathbb{R}$ such that $\varphi_\infty(U) \subset V$. There are two cases to consider.

If $x \in N$, then we may assume that $V$ is open in $N$. Thus $U = \varphi_\infty^{-1}(V) = \varphi^{-1}(V)$ is open in $N \times \mathbb{R}$ and hence in $N_\infty \times \mathbb{R}$.

If $x = \infty$, the open set $V$ is a neighborhood of $\infty$ in $N_\infty$, and so $C = N_\infty - V$ is a closed compact subset of $N$. Let

$$K = \varphi^{-1}(C) \cap (N \times [t-1, t+1]) = \Phi^{-1}(C \times [t-1, t+1]),$$

where $\Phi$ is as in Lemma 1.41. Hence $K$ is a closed compact subset of $N \times \mathbb{R}$, and $\pi_1(K)$ is a compact subset of $N$. Moreover, since $\pi_2(K)$ is compact, the restriction $\pi_1|_{N\times\pi_2(K)}$ is a closed map, and so $\pi_1(K)$ is closed. Thus $W = N_\infty - \pi_1(K)$ is an open neighborhood of $\infty$ in $N_\infty$. Finally, taking $U = W \times (t-1, t+1)$, we see that $\varphi_\infty(U)$ does not intersect $C$, so it is contained in $V$. $\square$

# Chapter 2

# Attempts to classify vector fields and tweaks made

The process of classifying things is a pretty tough one. Firstly, one needs to think how the classification will be made. This very first step is already sufficiently hard, and might take years and several brains together to come up with a satisfactory set of classes. Secondly, there must exist a way to determine to which class each object belongs. In this chapter, we shall deal more with the latter, going through the attempts made until we reach the final theorem, which is an easier way to say when two vector fields or continuous flows are "very similar", i.e., they belong to the same class. A very nice discussion on the topic of classification was made by Maurício Matos Peixoto [16], a well-known Brazilian mathematician who have brought great advances in the study of Qualitative Theory of Differential Equations and of Dynamical Systems.

Until now we have worked mainly with vector fields, in particular the planar ones. We focus now on continuous flows on surfaces. The reason for that, besides the fact that the comprehension of vector fields is of utterly importance, is that this train of thought follows the historical one, and is, by our point of view, more natural. So before we start the first section, we extend notations and concepts previously given and establish some new ones; some of them are completely new and others follow [10], [13] and [4], with, possibly, updated writing.

Starting here and during all the text after, $M$ will denote a connected topological 2-manifold without boundary; it is not necessarily compact nor orientable.

**Definition 2.1.** Let $(M, \varphi)$ be a continuous flow. By a *quasi-orbit* of $\varphi$ (or of $\gamma_p$) we mean a connected subset of an orbit (in other words, the image of an interval by $\varphi_p$, for some $p \in M$).

**Definition 2.2.** Two continuous flows $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$ are *locally topologically equivalent* at the points $p_1 \in M_1$ and $p_2 \in M_2$ if there is a homeomorphism $h : U_1 \to U_2$ between open neighborhoods of $p_1$ and $p_2$, with $h(p_1) = p_2$, which takes quasi-orbits of $\varphi_1$ onto quasi-orbits of $\varphi_2$, preserving sense. The homeomorphism $h$ is called a *local* *equivalence* between $\varphi_1$ and $\varphi_2$ at the points $p_1$ and $p_2$. Extending Definition 1.9, when $h$ maps $M_1$ onto $M_2$ taking orbits onto orbits preserving sense, we say that $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$ are *topologically equivalent* and that $h$ is a *topological equivalence* between $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$.

**Definition 2.3.** Let $\varphi$ be a continuous flow on $M$. We say that $A \subset M$ is *$\varphi$-invariant* if it is the union of some orbits. In such a case, $\varphi$ defines a flow on $A$ ($\varphi$ can be restricted to a map from $A \times \mathbb{R}$ to $A$), the so-called *restriction of $\varphi$ to $A$*, and write $(A, \varphi)$ to denote it.

**Definition 2.4.** Let $U \subset M$ be a $\varphi$-invariant region. We call $U$ *parallel* when the restriction $(U, \varphi)$ is equivalent to one of the following:

(i) $\mathbb{R}^2$ with flow defined by $y' = 0$;

(ii) $\mathbb{R}^2 - \{0\}$ with flow defined (in polar coordinates) by $dr/\,dt = 0$, $d\theta/\,dt = 1$;

(iii) $\mathbb{R}^2 - \{0\}$ with flow defined by $dr/\,dt = r$, $d\theta/\,dt = 0$;

(iv) $S^1 \times S^1$ with the flow induced by (i) above, under the covering map which associates $(x, y)$ with $(x + n, y + m)$, where $m, n \in \mathbb{Z}$.

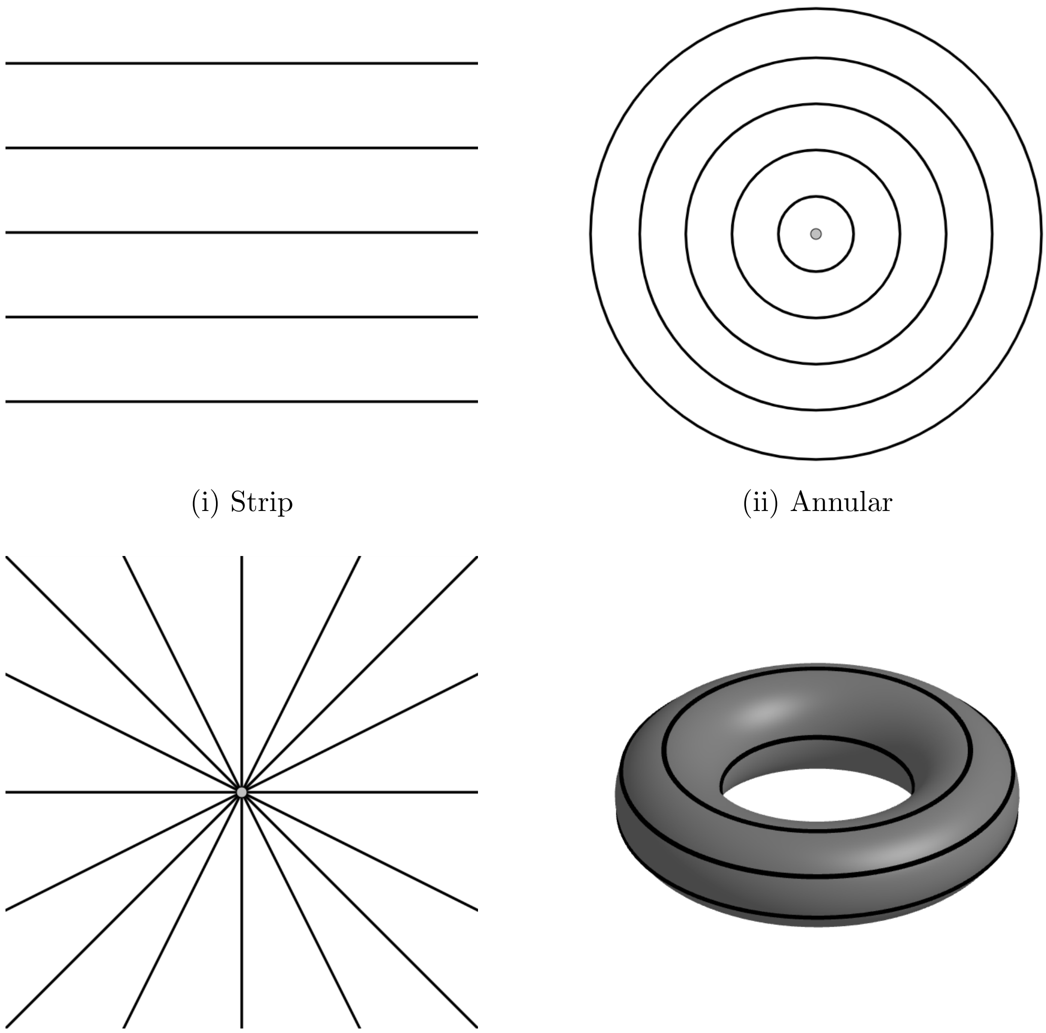
We distinguish these as *strip*, *annular*, *spiral* (or *radial*) and *toral*, respectively (see Figure 2.7 for a representation of the unoriented orbits).

**Remark.** We use the same adjectives given to the region when talking about the flow on each of them.

In the case of continuous flows, since we do not have differentiability, we need to update some notions developed before and the associated results.

**Definition 2.5.** A point $x \in M$ is a *critical point*, or a *rest point*, or an *equilibrium point*, if $\varphi(x, t) = x$ for every $t \in \mathbb{R}$. If a point is not critical, then it is a *regular point*, or a *nonrest point*.

**Example 2.6.** Regard the torus $\mathcal{T}$ as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, with corresponding projection $\pi : \mathbb{R}^2 \to \mathcal{T}$. The image of the line $r$ in $\mathbb{R}^2$ given by $y = \alpha x$ under $\pi$ has two possibilities: if $\alpha$ is rational, say, $\alpha = p/q$, then $y = p$ when $x = q$, and so $\pi(r)$ is a Jordan curve on $\mathcal{T}$; in the irrational case, such image cannot be a closed curve. Moreover, it is dense on the torus. We just need to prove this claim for points $\pi(0, y)$, because any neighborhood $U$ of a point $\pi(x, y)$ may be translated to a neighborhood of the point $\pi(0, y - \alpha x)$. If the curve $\pi(r)$ meets such neighborhood, it will meet $U$ after an "untranslation". So, we will work in $[0, 1)$ and use the notation $[x]$ to indicate $x \mod 1$. First, observe that the set $Z = \{[m\alpha] \mid m \in \mathbb{Z}\}$ is infinite, since $[m\alpha] = [n\alpha]$ with $m \neq n$ would imply $\alpha = \frac{p-q}{m-n}$ for some $p, q \in \mathbb{Z}$, an absurd. Given $n \in \mathbb{Z}$, by the pigeonhole principle there



(i) Strip (ii) Annular

(iii) Spiral (or radial) (iv) Toral

Figure 2.7: Parallel regions.

exist $k \in \{0, \ldots, m-1\}$ and distinct $p, q \in \{1, \ldots, m+1\}$ such that

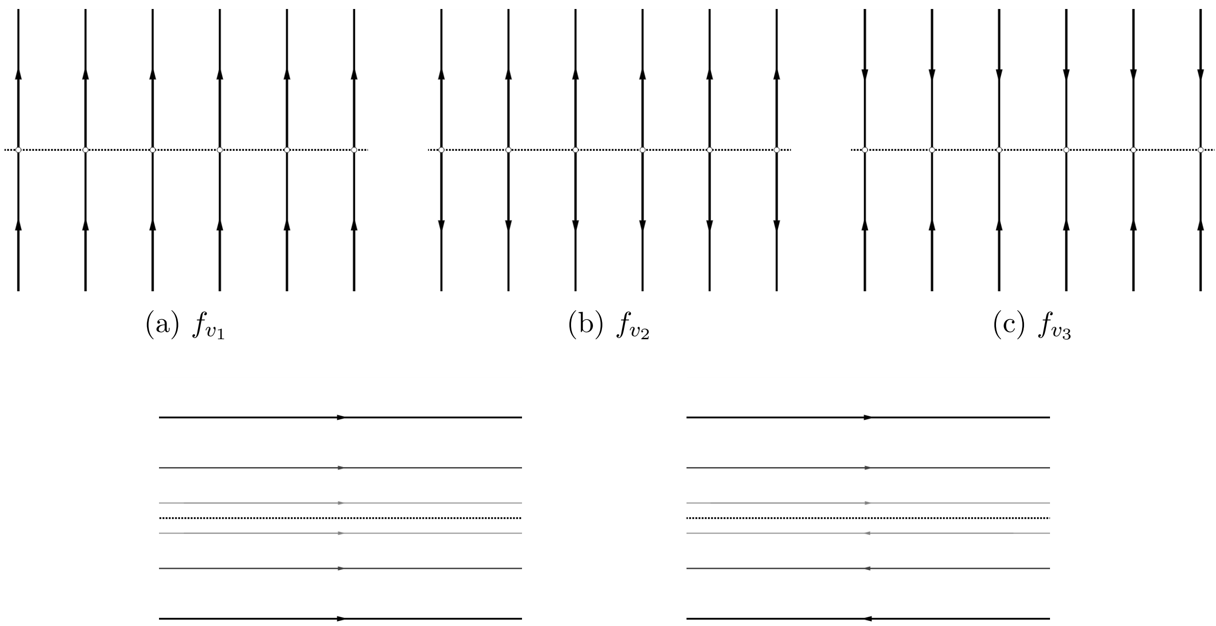
$$\frac{k}{n} \leq [p\alpha] < \frac{k+1}{n} \quad \text{and} \quad \frac{k}{n} \leq [q\alpha] < \frac{k+1}{n}.$$

Hence $|[(p-q)\alpha]| = |[p\alpha] - [q\alpha]| < \frac{1}{n}$. Thus every point of $[0,1)$ is less than $\frac{1}{n}$ distant from the set $\{[m(p-q)\alpha] \mid m \in \mathbb{Z}\}$. This proves the density of $Z$ in $[0,1)$ and, therefore, the density of $\pi(r)$ on $\mathcal{T}$.

Fix $\alpha, \beta \in \mathbb{R}$. Let $\varphi : \mathcal{T} \times \mathbb{R} \to \mathcal{T}$ be the flow defined by $\varphi(\pi(x,y),t) = \pi(x+t\alpha, y+t\beta)$. This is a continuous flow on the torus. Depending on $\beta/\alpha$ rationality or irrationality, this flow is called a *rational rotation* or an *irrational rotation* on the torus, or simply rational flow or irrational flow on the torus. Observe that this quotient gives the slope of a line in $\mathbb{R}^2$, and so by the preceding paragraph, every orbit of the rational rotation is closed, and every orbit of the irrational rotation is dense on $\mathcal{T}$. An interesting fact about these flows on the torus, is that every rational rotation is equivalent ([5], page 36), and so every rational flow is equivalent to the toral flow. The relations between irrational flows are more delicate, but a startling property is that *every* orbit of such flow is a *separatrix* (see Definition 2.25).

Consider the vector fields $f_{v_1}(x,y) = (0,|y|)$, $f_{v_2}(x,y) = (0,y)$, $f_{v_3}(x,y) = (0,-y)$, $f_{h_1}(x,y) = (|y|,0)$ and $f_{h_2}(x,y) = (y,0)$. Let $\varphi_{v_1}$, $\varphi_{v_2}$, $\varphi_{v_3}$, $\varphi_{h_1}$ and $\varphi_{h_2}$ be their respective associated flows (see Figure 2.9).

**Definition 2.8.** Let $p \in M$ be a singular point of the continuous flow $\varphi$. We say that it is *vertical* (respectively, *horizontal*) if there is a local equivalence between $\varphi$ and either $\varphi_{v_1}$, $\varphi_{v_2}$ or $\varphi_{v_3}$ (respectively, either $\varphi_{h_1}$ or $\varphi_{h_2}$) at $p$ and the origin. We call a maximal curve containing only horizontal points a *horizontal orbit*.



(a) $f_{v_1}$ (b) $f_{v_2}$ (c) $f_{v_3}$

(d) $f_{h_1}$ (e) $f_{h_2}$

Figure 2.9: Special vector fields.

**Definition 2.10.** Suppose there is a local equivalence, with homeomorphism $h : U \subset M \to (-\varepsilon,\varepsilon) \times (-\varepsilon,\varepsilon)$, between $\varphi$ and the strip flow at the points $p$ and 0 or between $\varphi$ and one of the horizontal flows at the points $p$ and 0. Then we call $U$ a *tube* of $p$, or simply a tube when we do not need to mention the point taken. The set $T = h^{-1}(\{0\} \times (-\varepsilon,\varepsilon))$ is the *transversal associated to* $U$, or a *local transversal at* $p$ (rigorously, in the horizontal case this is not what we want a transversal to be, as it contains a singular point; but as we will see during the theory, horizontal points are pretty much similar to regular ones). The components that $p$ separates $T$ into are $T$-*transversals*, and the components that $h^{-1}((-\varepsilon,\varepsilon) \times \{0\})$ separates $U$ into are $U$-*tubes*. If $f : \mathbb{R} \to M$ is a continuous injection with the property that, given $s \in \mathbb{R}$, there is $\varepsilon_s > 0$ such that $f((s-\varepsilon_s, s+\varepsilon_s))$ is a local transversal at $f(s)$, then $f(\mathbb{R})$ is a *transversal* to the flow $(M,\varphi)$.

**Definition 2.11.** Let $U$ be a parallel region and $T \subset U$ be a transversal. We say that $T$ is a *complete* transversal to $U$ provided that one of the following conditions holds:

(i) $U$ is either a strip or an annular region, and $T$ intersects each orbit of $U$ at exactly one point. In the strip case, $T$ separates $U$ into two regions, $U_T^-$ and $U_T^+$, corresponding, respectively, to the backward and forward directions of the flow;

(ii) $U$ is a spiral region and, for $p \in T$, each of the two transversals into which $p$ separates $T$ intersects any orbit of $U$ infinitely many times.

A transversal $T \subset U$ is *quasi-complete* when it is a connected subset of a complete transversal.

**Definition 2.12.** Given a continuous flow, let $\gamma$ be a simple curve consisting only of singular points (more than one) which are not horizontal. A point $p \in \gamma$ is called *horizonal* if there is a homeomorphism between $U$ containing $p$ and either $(-\varepsilon, \varepsilon) \times [0, \varepsilon)$ or $(-\varepsilon, \varepsilon)$, carrying $p$ to 0 and quasi-orbits (even horizontal ones) onto horizontal segments. A maximal curve containing only horizonal points is called a *horizon*. When talking about an orbit of a horizontal or horizonal singular point, we mean the horizontal orbit or the horizon containing it, respectively. A singular point which is neither vertical, nor horizontal nor horizonal is called *essential*.

**Remark.** Our definition of essential singular points is slightly less restrictive than the one given by López and Buendía [4].

Horizonal points allow us to talk about a well behaved accumulation of horizontal orbits. There are more horizonal points than these, as any point on the line $y = 0$ in any simple combination between vertical and horizontal flows.

A generalization of Example 1.14 (3) holds.

**Proposition 2.13.** *Suppose there is a transversal $T = f(\mathbb{R})$, $f$ a continuous injection, to the flow $(U, \varphi)$ which intersects each orbit exactly once.*

*(a) If every orbit in $U$ is line homeomorph, then $(U, \varphi)$ is equivalent to $(T \times \mathbb{R}, \psi)$, where the latter flow is given by $\psi((x, s), t) = (x, s + t)$;*

*(b) If every orbit in $U$ is homeomorphic to $S^1$, then $(U, \varphi)$ is equivalent to $(T \times S^1, \Upsilon)$, where the latter flow is given, regarding $S^1$ as the quotient $\mathbb{R}/\mathbb{Z}$, by $\Upsilon((x, [\theta]), t) = (x, [\theta + t])$. The brackets indicate the equivalence class of $\theta \in \mathbb{R}$.*

*Proof.* For (a), suppose every orbit in $U$ is line homeomorph. Since $f$ is continuous, there exists a unique continuous function $\tau : U \to \mathbb{R}$ such that $\varphi(x, \tau(x)) \in T$. We define $H : U \to T \times \mathbb{R}$ by $H(x) = (\varphi(x, \tau(x)), -\tau(x))$. Such map is continuous since $\varphi$ and $\tau$ are continuous. Moreover, it has an inverse $H^{-1}(x, t) = \varphi(x, t)$, which is continuous. So $H$ is a homeomorphism. Finally, given $\varphi(x,t) \in U$, we can write it uniquely as $\varphi(z, -\tau(\varphi(x,t)))$, with $z \in T$; thus $z = \varphi(x, \tau(x))$ and $\tau(\varphi(x,t))) = \tau(x) - t$, and so

$$\begin{aligned} H(\varphi(x,t)) = (z, -\tau(\varphi(x,t))) &= \psi((z, -\tau(\varphi(x,t))), 0) \\ &= \psi((\varphi(x,\tau(x)), -\tau(x)+t), 0) = \psi(H(x), t). \end{aligned}$$

Suppose now that every orbit in $U$ is homeomorphic to $S^1$. For each $z \in T$ (which is the same to take an orbit) there is a period $\xi(z)$ such that $\varphi(z, t+\xi(z)) = \varphi(z,t)$, for every $t \in \mathbb{R}$; in particular, $\varphi_z|_{[0,\xi(z))}$ is injective. Now, for each orbit $\gamma \subset U$, there is a unique $z \in T \cap \gamma$, and so there exists a unique bijection $\tau_z : \gamma \to [0, \xi(z))$ such that $\varphi(x, -\tau_z(x)) = z$ for $x \in \gamma$. If we regard $[0, \xi(z))$ as the quotient $\mathbb{R}/\xi(z)\mathbb{Z}$, the map $\tau_z$ becomes a homeomorphism. Hence, since $f$ is continuous, we may drop the subscript and regard $\tau$ as a continuous map from $U$ onto $S^1$ which is injective along an orbit (we will not distinguish between elements of $[0, \xi(z))$ and of the unit circle). Finally, we define $H : U \to T \times S^1$ by $H(x) = (\varphi(x, -\tau(x)), [\tau(x)])$. It is continuous and has an inverse $H^{-1}(x, [\theta]) = \varphi(x, \theta)$, which is also continuous. So $H$ is a homeomorphism. Given $\varphi(x,t) \in U$, we can write it uniquely as $\varphi(z, \tau(\varphi(x,t)))$ modulo $\xi(z)$, with $z \in T$. Because of the periodicity, we may assume $t \in [0, \xi(z))$, and hence $z = \varphi(x, -\tau(x))$ and $\tau(\varphi(x,t)) = \tau(x) + t$ modulo $\xi(z)$. Therefore

$$\begin{aligned} H(\varphi(x,t)) = (z, [\tau(\varphi(x,t))]) &= \Upsilon((z, [\tau(\varphi(x,t))]), 0) \\ &= \Upsilon((\varphi(x, -\tau(x)), [\tau(x)+t]), 0) = \Upsilon(H(x), t), \end{aligned}$$

and (b) follows. □

Furthermore, a result analogous to the Flow Box Theorem holds. For its proof we refer to [1], page 50. We note that a horizontal singular point always has a tube containing it, by the very definition of horizontal singular point. Similarly, we refer to the set $U$ given in the definition of horizonal points as a tube as well; and in each situation, the preimage of $\{0\} \times (0, \varepsilon)$ or of $\{0\} \times (-\varepsilon, \varepsilon)$ as a local transversal at the horizonal point.

**Theorem 2.14.** *If $p$ is a regular point, then there is a tube containing $p$.*

**Definition 2.15.** Let $\gamma$ be a orbit of $\varphi$ and $p, q \in \gamma$. Let $q = \varphi(p, t_1)$, $t_1 \geq 0$. We define $\gamma[p,q] = \{\varphi(p,t) \mid 0 \leq t \leq t_1\}$. If $T$ is a line homeomorph transversal and $x, y \in T$, using the ordering of the reals, we define $T[x,y]$ as the points of $T$ between $x$ and $y$ (including $x$ and $y$). The sets $\gamma(p,q), \gamma(p,q]$ and $\gamma[p,q)$ are similarly defined, as well as for $T$.

# 1 Lawrence Markus' paper

In 1954, Lawrence Markus published in Transactions of the American Mathematical Society a paper named *Global structure of ordinary differential equations in the plane* [10], whose objective was to develop a topological analysis and classification of real, first order, ordinary differential equations in $\mathbb{R}^2$. There, he introduced the notion of *separatrices*, trying to generalize previous definitions of separatrix. Using the separatrices, he then defined the concepts of parallelism and *canonical regions*.

Markus' paper aimed to give an easier way to see when two vector fields are topologically equivalent. Everything said in the last paragraph was a tool to that purpose. Unfortunately, his definitions and results have several gaps, which we shall point out here.

Our first definition in this section is the one Markus used as *the* definition of separatrix.

**Definition 2.16.** Define $\alpha'(\gamma) = \alpha(\gamma) - \gamma$ and $\omega'(\gamma) = \omega(\gamma) - \gamma$. An orbit $\gamma$ is *m-ordinary* if it is neighbored by a parallel region $N$ such that:

(i) $\alpha'(\mu) = \alpha'(\gamma)$ and $\omega'(\mu) = \omega'(\gamma)$ for every orbit $\mu$ contained in $N$;

(ii) The boundary of $N$ is the union of $\alpha'(\gamma)$, $\omega'(\gamma)$ and exactly two orbits $\gamma_1$ and $\gamma_2$ such that $\alpha'(\gamma_1) = \alpha'(\gamma_2) = \alpha'(\gamma)$ and $\omega'(\gamma_1) = \omega'(\gamma_2) = \omega'(\gamma)$.

An orbit that is not m-ordinary is called *m-separatrix*.

The idea behind the above definition is that some orbits behave differently from others arbitrarily close to them. So, in some sense, they *separate* the flow. We think that, when writing this definition, Markus (and probably others that used this definition later) had some fixed examples and ideas in mind, so when stating and proving his results he would not use precisely the written definition, but would abstract the meaning expressed by it, leading to several logical problems.

**Example 2.17.** Consider the differential system

$$\begin{cases} \dot{x} = x^2 - 1 \\ \dot{y} = x \end{cases} \tag{2.1}$$

and denote the flow generated by this differential system by $\Phi$. Part of its phase portrait is shown in Figure 2.18 below. Intuitively, one would think that the two vertical parallel lines (the orbits defined by $t \mapsto (1, t)$ and $t \mapsto (-1, -t)$) are special: they "separate" two kind of behavior, in a manner that any open $\Phi$-invariant set containing one of them must contain a parabola-like orbit and an orbit similar to a straight line.

However, there are no m-separatrices. To see that any orbit between the parallel lines is m-ordinary, just take an open connected subset of the transversal $x = 0$ which contains the chosen orbit. Then Example 1.14 (3) finishes this case. The case for any orbit in the right side of the right vertical straight line (respectively, left side of the left vertical straight line) is similar; it suffices to take a specific open connected subset of the transversal $y = -x$ (respectively, $y = x$) and then apply the aforementioned example. The case for the parallel lines is a bit trickier. Fix one of them. Take the region bounded

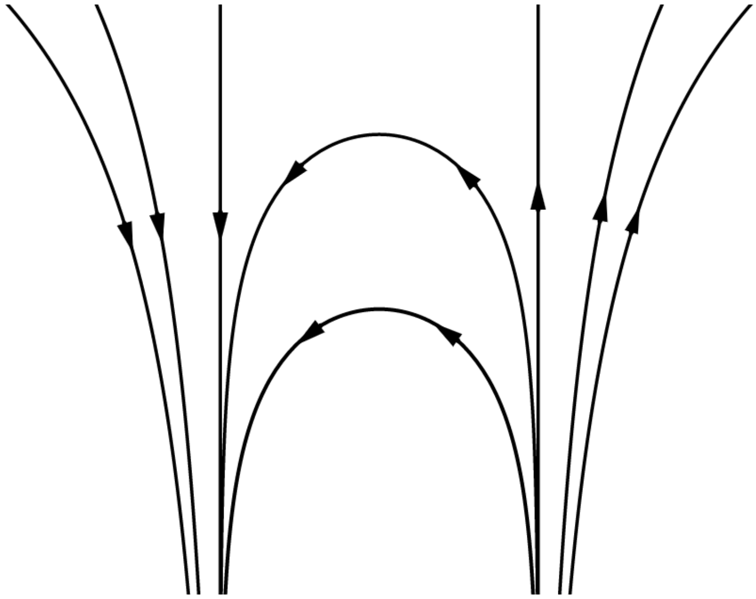


Figure 2.18: Phase portrait.

by the other parallel line and one orbit similar to a straight line close to the fixed orbit; see Figure 2.19. If now we draw the transversal $y = -x^{-1},\ x > 0$ (or $y = x^{-1},\ x < 0$, depending on the chosen vertical line), taking an open connected subset of this transversal containing the fixed orbit finishes this case too, by Example 1.14 (3) again.

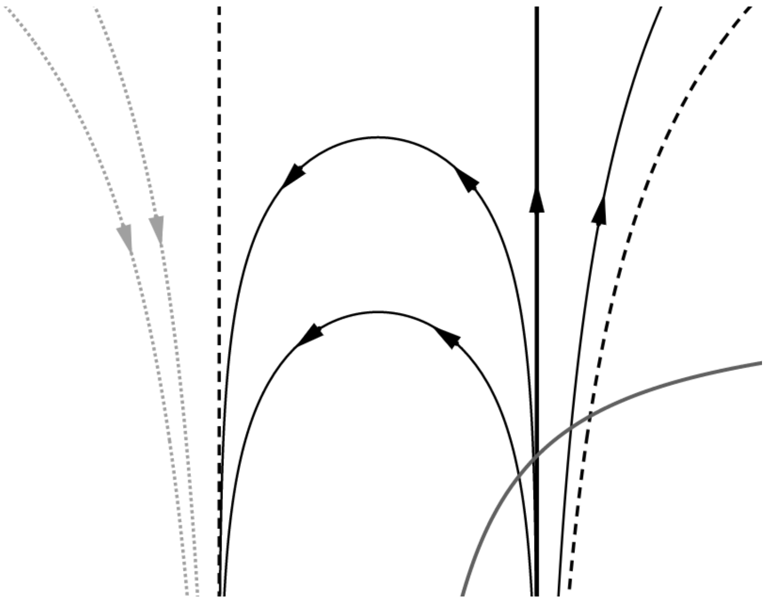


Figure 2.19: Parallel neighborhood.

This shows that every orbit of $\Phi$ is m-ordinary, so the change of behavior is not detected by Markus' definition of separatrix. In future discussions, we will refer to this differential system and its properties as the *classical example*.

Despite the following theorem's simplicity, it plays a crucial role in this theory, as it allows us to state one of the most important definitions.

**Theorem 2.20.** *Let $\mathcal{S}_m$ be the union of all m-separatrices of a differential system. Then $\mathcal{S}_m$ is closed.*

*Proof.* Given an m-ordinary orbit, there is an open set containing it, which is, by definition, formed by m-ordinary orbits. Hence the complement of $\mathcal{S}_m$ is open. □

**Definition 2.21.** Given a differential system or a continuous flow, we denote the union of all its m-separatrices by $\mathcal{S}_m$. Each component of the complement of $\mathcal{S}_m$ is called a *canonical m-region*, or simply an *m-region* of the differential system or of the continuous flow.

It is not always true that an m-separatrix bounds an m-region. Other m-separatrices may cluster about some solution curve.

**Definition 2.22.** If an m-separatrix $\gamma$ belongs to the set $\overline{\mathcal{S}_m - \gamma}$, then it is called *limit m-separatrix.*

Here comes the turning point. Until here — before chapter III in the article —, Markus made the above definitions (with small changes), some not completely true claims and several examples and motivations. Now, he begins to state and prove his basic results. We notice that Markus relies strongly on Kaplan articles, [6] and [7]. Yet, the second lemma of chapter III is wrong as it is, as the classical example shows, since it is noncritical in the entire plane but is not strip parallel. The problems carry over into some of the next theorems. Most importantly, one of his main theorems, stated below with minor modifications, is wrong (again, one can see this using the classical example).

**Important claim.** *Let $X$ and $Y$ be two vector fields in the plane with no limit m-separatrices other than critical points. For each m-region of $X$ take one m-ordinary orbit; denote the union of them by $S_X$. Similarly with $Y$. Then $X$ and $Y$ are equivalent if and only if there is a homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$ carrying the m-separatrices of $X$ onto the m-separatrices of $Y$ and $S_X$ onto $S_Y$.*

The minor modifications mentioned has to do with the definition of equivalence: the homeomorphism must be orientation preserving (the orientation of the plane, that is) and needs not take into account the orientation of orbits.

If the claim were true, the classical example would be equivalent to the strip flow.

A lot of papers published later cited this one by Markus. Most of them used the result for polynomial vector fields. In such setting it is usual practice to apply a definition of separatrix different from the one given by Markus, which, as we shall see in the next section, is equivalent to the "right" definition of separatrix. Perhaps the most notable article that cited Markus' work was published in 1975, by Dean Arnold Neumann. He generalized the theorem to flows with limit m-separatrices on two dimensional manifolds without boundary. Unfortunately, in this case, his results are false.

Only in 2018 the flaws were noticed and suitable corrections were given. We talk about this topic in the next section.

# 2 Buendía and López' paper

José Ginés Espín Buendía and Víctor Jiménez López published an article in 2018 [4] which aimed to solve all problems mentioned above in such a way that the theorem remained true. Though they haven't written the proof of the theorem (which we shall do in Chapter 3, and is basically the proof given by Neumann in [13]), the important definitions were rewritten, examples and counterexamples were given and slight improvements were made in the hypothesis of the theorem. We shall go through all of this, commenting important points and elaborating arguments. We will not, however, do a thorough study on their counterexamples, just minor commentaries.

In their work they start off by giving some counterexamples to the Markus–Neumann Theorem, trying to see where the problem comes from. The intuition gained is then used to reformulate the definitions. The first example is, in some way, very similar to our classical example, and the second one is a vector field on the torus. Both of them share the same characteristic: when choosing parallel regions to show that particular orbits are m-ordinary, all of them cannot be "smaller" than a certain width; for instance, when the chosen parallel region is radial, it is not possible to get "closer" to the desired orbit. To avoid this, they exclude radial regions when defining ordinary orbits and formalize a new kind of region.

**Definition 2.23.** An orbit $\gamma$ is called *recurrent* if it is contained in its $\alpha$ or $\omega$-limit set. Otherwise it is called *nonrecurrent*.

**Remark.** There are orbits which are neither closed orbits nor singular points and yet are recurrent. An example of this case is any orbit of the irrational flow on the torus.

**Definition 2.24.** Let $\varphi$ be a flow on a strip region $U$. We say that $U$ is a *strong strip* region if there are nonrecurrent orbits $\gamma_1$ and $\gamma_2$ such that $(U \cup \gamma_1 \cup \gamma_2, \varphi)$ is equivalent to the strip flow restricted to $\mathbb{R} \times [-1, 1]$. We call $\gamma_1$ and $\gamma_2$ the *border orbits* of $U$ and say that a complete transversal to $U$ is *strong* if it can be extended to a Jordan arc by adding one point from each border orbit. When we talk about the endpoints of a transversal, we mean the endpoints of the Jordan arc which extends the transversal (possibly no extension exists; this is not the case for strong transversals). If $U$ is annular, for convenience, we say it is strong and that a transversal to $U$ is strong if analogous conditions are satisfied.

Surprisingly enough, when trying to redefine ordinary orbits accordingly (and consequently, redefining separatrices), the theorem still fails; the counterexample given by them is a vector field on the torus, in which, when choosing a strong strip neighborhood for a specific orbit, its boundary, in spite of being the one required, has "anomalies" provoked by the $\alpha$-limit and $\omega$-limit sets. So, the final definition is born dealing with all of this.

**Definition 2.25.** Let $\gamma$ be an orbit of $(M, \varphi)$. Consider the following properties about a strong strip $U$ with border orbits $\gamma_1$ and $\gamma_2$:

(i) $\alpha(\mu) = \alpha(\gamma)$ and $\omega(\mu) = \omega(\gamma)$ for every orbit $\mu \subset U \cup \gamma_1 \cup \gamma_2$;

(ii) for every strong transversal $T$ to $U$ with endpoints $p$ and $q$, the boundary of the regions which $T$ separates $U$ into can be written as $\partial U_T^- = T \cup \gamma_p^- \cup \gamma_q^- \cup \alpha(\gamma)$ and $\partial U_T^+ = T \cup \gamma_p^+ \cup \gamma_q^+ \cup \omega(\gamma)$.

We say that $\gamma$ is *ordinary* if it is neighbored by an annular region or a strong strip with properties (i) and (ii). An orbit that is not ordinary is called a *separatrix*. In the case of differential systems, we apply this definition to the flow generated by it.

It is clear that being ordinary is stronger than being m-ordinary. Consequently, being a separatrix is weaker than being an m-separatrix. In other words: every ordinary orbit is m-ordinary, and every m-separatrix is a separatrix.

**Definition 2.26.** Given a differential system or a continuous flow, we denote the union of all its separatrices by $\mathcal{S}$. By the same reasoning of Theorem 2.20, this is a closed set. Each component of the complement of $\mathcal{S}$ is called a *canonical region*. Choose an orbit from each canonical region and call their union $S$. Then the set $\mathcal{S}^+ = \mathcal{S} \cup S$ is called a *separatrix configuration* for the differential system or for the flow. The ordinary orbits of $\mathcal{S}^+$ are the *distinguished orbits* of the separatrix configuration.

**Definition 2.27.** If a separatrix $\gamma$ belongs to the set $\overline{\mathcal{S} - \gamma}$, then it is called a *limit separatrix*.

Since in a canonical region $U$ every orbit has the same $\alpha$-limit and $\omega$-limit sets (except those regions with closed orbits), we can write $\alpha(U) = \alpha(\gamma)$ and $\omega(U) = \omega(\gamma)$, where $\gamma \subset U$ is an ordinary orbit.

**Definition 2.28.** Consider the flows $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$ with separatrix configuration $\mathcal{S}_1^+$ and $\mathcal{S}_2^+$, respectively. We say that $\mathcal{S}_1^+$ and $\mathcal{S}_2^+$ are *equivalent* if there is a homeomorphism $h : M_1 \to M_2$ such that $h(\mathcal{S}_1^+) = \mathcal{S}_2^+$, preserving sense of orbits.

Since there are more separatrices than m-separatrices, every canonical region is contained in some m-region; though more than one canonical region can be within one m-region.

It is not uncommon to see, in the setting of analytic sphere flows (in particular, after a compactification of a polynomial system), a different definition of separatrix. We write it below, following [14], page 293. This is the definition used in the majority of papers citing the Markus–Neumann Theorem, so they are not wrong.

**Definition 2.29.** A *separatrix* of a relatively prime, polynomial system in the plane is a trajectory of such system which is either:

(i) a critical point;

(ii) a limit cycle;

(iii) a trajectory which lies in the boundary of a hyperbolic sector at a critical point of the system on the Poincaré sphere.

It happens that Definitions 2.25 and 2.29 are equivalent in the context described in the latter definition. We will prove this equivalence here, but since our main interest is not polynomial vector fields, much of the terminology and concepts used will neither be defined nor stated here. A more complete study of such topic can be found in [3].

**Proposition 2.30.** *If* $X : \mathbb{R}^2 \to \mathbb{R}^2$ *is a relatively prime, polynomial system, then an orbit* $\gamma$ *is a separatrix of* $X$ *in Definition 2.25 sense if and only if it is one of the three options listed in Definition 2.29.*

*Proof.* We first notice that the assumptions made about $X$ guarantee: a finite number of singular points in $\mathbb{R}^2$, no one-sided isolated periodic orbits, no limit separatrices other than critical points and the finite sectorial decomposition property at every singular point (since there are finitely many, they are isolated).

If $\gamma$ satisfies (i), (ii) or (iii) in Definition 2.29, it is clear that it is a separatrix. On the other hand, suppose $\gamma$ is a separatrix. Assume it is neither a critical point nor a limit cycle; we need to prove that $\gamma$ lies in the boundary of a hyperbolic sector on the Poincaré sphere. It is clear that $\gamma$ cannot be a closed orbit, otherwise it would be within an annular region, contrary to our assumption. Let $p(X)$ be the associated compactified system on the Poincaré sphere. Due to the Poincaré–Bendixson Theorem, the $\omega$-limit set of $\gamma$ has three possibilities. Such set cannot contain any regular point, because it would imply in the existence of limit separatrices, which is not possible. Thus $\omega(\gamma)$ is a unique singular point $p^+$. Similarly, $\alpha(\gamma)$ is a unique singular point $p^-$. Using the finite sectorial decomposition at $p^+$, we conclude that $\gamma$ must border either a hyperbolic sector or an elliptic sector; in any case, using $p^-$ if necessary, such orbit lies in the boundary of a hyperbolic sector at a critical point of the compactified system, as desired. $\square$

**Definition 2.31.** A regular point, or a horizontal or horizonal singular point $p$ in the boundary of a canonical region $R$ is *accessible from* $R$ if there is a transversal contained in $R$ with $p$ as one of its endpoints. The union of all accessible points from $R$ is called *accessible boundary* of $R$, and denoted by $\delta R$.

Sometimes we just say that a point is accessible; from where it is accessible will be clear by the context. We stress that, when saying only accessible, the point is obligatory regular, horizontal or horizonal.

López and Buendía end their paper by giving a proposition which we shall use when proving the main theorem. We change the content of the proposition for the sake of preciseness.

**Proposition 2.32.** *Let $(U, \varphi)$ be a strip or annular canonical region. If $p \in \delta U$, and $L \subset U$ is a transversal with $p$ as one of its endpoints, then $L$ is quasi-complete. If the canonical region is radial, there is a quasi-complete subset of $L$ with $p$ as one of its endpoints.*

*Proof.* We shall consider each case separately. In the first two of them, we must find a complete transversal to $U$ containing $L$. We assume, of course, that $L$ is not complete.

If $U$ is annular, using the homeomorphism which defines the parallelism, $L$ is carried to a transversal in the plane, which we can extend to a complete transversal. Using the homeomorphism again, we just found a complete transversal to $U$ containing $L$. This case is simple because annular regions have compact orbits, with no "strange" alpha or omega limit sets.

Assume $U$ is a strip region and denote by $h$ the homeomorphism which defines the parallelism. There is no loss of generality in assuming that if $T = h^{-1}(\{0\} \times \mathbb{R})$, then $U_T^+ = h^{-1}((0, \infty) \times \mathbb{R})$ and that $q = h^{-1}(0, 0)$ is the other endpoint of $L$.

Let $I = (0, d),\ 0 < d \leq \infty$, be the open interval and $f : I \to \mathbb{R}$ be the continuous map such that $h(L) = \{(f(s), s) \mid s \in I\}$, with $\lim_{s \to d^-} h^{-1}(f(s), s) = p$. We prove that $d = \infty$, so that one can extend $L$ using the canonical region. To get a contradiction, suppose $d < \infty$. We claim that either $\lim_{s \to d^-} f(s) = \infty$ or $\lim_{s \to d^-} f(s) = -\infty$. Otherwise, there would be a sequence $s_n \to d$ with $f(s_n) \to y \in \mathbb{R}$, and thus $h^{-1}(f(s_n), s_n)$ would converge simultaneously to $h^{-1}(y, d)$, a point in $U$, and to $p$, a point in the boundary of $U$, which is impossible.

For the sake of simplicity, suppose $\lim_{s \to d^-} f(s) = \infty$. Furthermore, we can modify $h$ near $q$ if necessary to assume that $f(s) > 0$ for all $s \in I$. Hence $\Sigma = h^{-1}(\{0\} \times (0, \infty))$ does not intersect $L$.

Put $x = h^{-1}(0, d)$. The orbit $\gamma_x$ is ordinary, so there is a strong strip $\Omega \subset U$ neighboring it, with its border orbits also included in $U$, verifying Definition 2.25 (ii). Since the points in $\overline{\Omega_{T \cap \Omega}^+}$ which are not in $U$ belong to $\omega(U) = \omega(x)$, and $p$ is one of such points because $h^{-1}(f(s), s)$ is included in $\Omega$ if $s$ is close enough to $d$, we get $p \in \omega(U)$.

Fix a tube $V$ of $p$, and let $V_1$ and $V_2$ be the two $V$-tubes. We can assume that $V_1 \subset U$ and, moreover, using Proposition 2.13, assume that $L \subset V_1$. Let $A$ be the transversal associated to $V$ and $L'$ the $A$-transversal corresponding to $L$ contained in $V_2$. Since $p \in \omega(U)$, given $z \in T$, the positive semi-orbit $\gamma_z^+$ must intersect $L'$ infinitely many times, because they intersect $L$ at most once. Let $q'$ be the first point where $\gamma_q^+$ intersects $L'$ and consider $L'[p, q']$. Now let $\Sigma_L$ (respectively, $\Sigma_{L'}$) be the set of points $z \in \Sigma$ such that the first intersection point of $\gamma_z^+$ with $L \cup L'[p, q']$ belongs to $L$ (respectively, to $L'[p, q']$). Both sets are disjoint and non-empty ($x \in \Sigma_{L'}$, and all points from $\Sigma \cap V_1$, in particular those close enough to $q$, belong to $\Sigma_L$), their union is the whole $T'$, and they are open in $T'$. This contradicts the connectedness of $T'$.

Finally, we assume that $U$ is a radial region. If we pick a point $q \in L$ and assume that

$L$ is not complete, then at least one of the transversals which $q$ divides $L$ into doesn't intersect all orbits of $U$ infinitely many times. Denote the two components by $L_1$ and $L_2$, where $L_2$ is close to $p$. If $L_2$ intersects all orbits of $U$ infinitely many times, we could extend $L_2$ using the equivalence, and the result would follow; thus, possibly changing the choice of $q$, we suppose $L_2$ doesn't meet every single orbit in $U$, and hence that there is an orbit $\gamma_x$ and a strong strip neighborhood $\Omega \subset U$ of $\gamma_x$ such that $\Omega \cap L_2 = \emptyset$.

We define a new flow $\varphi'$ in the following way: our new flow has the same orbits as $\varphi$ in $M - \gamma_x$, and has $x$ as a singular point; see Figure 2.33. Then the set $\gamma_x$, when seen as a subset of $(M, \varphi')$, consists of three separatrices for $\varphi'$: the singular point $x$ and two regular orbits given by the components which $x$ separates $\gamma_x$ into. Moreover, the region $U' = U - \gamma_x$ is strip and, clearly, a canonical region for $\varphi'$. Due to what we just proved for strip canonical regions, $L_2$ is quasi-complete for $\varphi'$, which is impossible since it doesn't intersect $\Omega$. □

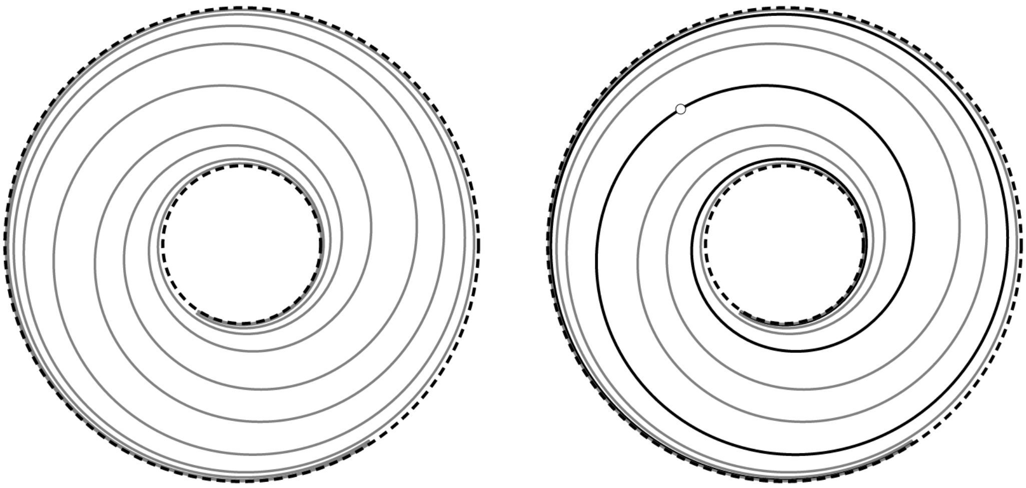


Figure 2.33: Flows $\varphi$ and $\varphi'$, respectively.

**Remarks.**

1. One might think that the previous proposition is obvious, and that it is always possible to extend a transversal to a parallel region using the equivalence. However, being a canonical region is a crucial hypothesis. Take the classical example and consider the strip parallel region shown in the Figure 2.19. All points from the line $x = -1$ are accessible; nevertheless, no transversal ending at a point from this line can be extended to a complete transversal, as it would meet the same orbit before reaching the line $x = 1$. Even being an m-region is not enough, as the first example given by López and Buendía shows.

2. The radial case is treated differently because if the component $L_1$ of $L$ (as in the proof of the proposition) approaches the border of the region without meeting every orbit infinitely many times, it is impossible to extend $L$ itself; see Figure 2.34.

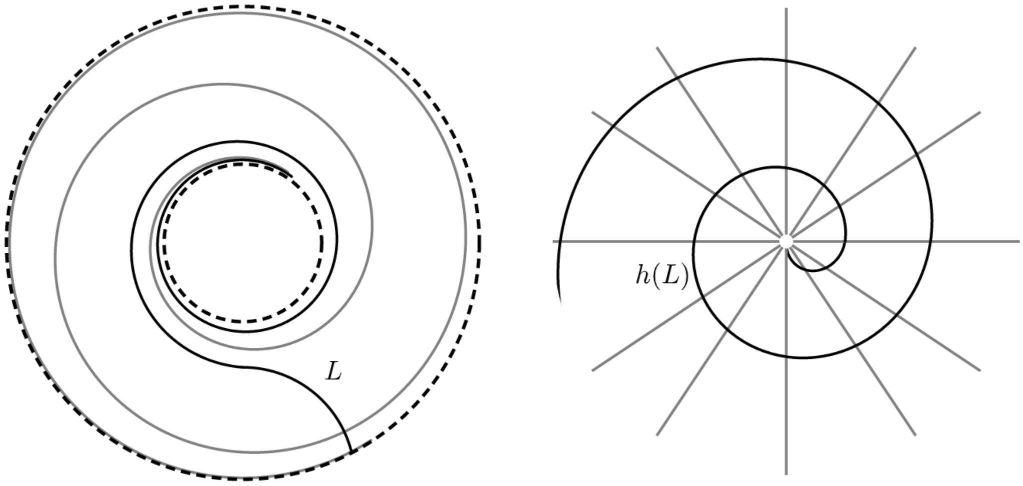



Figure 2.34: Transversal in a radial region.

**Corollary.** *If $U$ is a canonical region which is not toral and $T$ is a transversal to $U$ that can be extended to an arc by adding two points from $\delta U$, then $T$ is complete.*

*Proof.* By the preceding proposition, $T$ is quasi-complete. Since it cannot be further extended, it is complete already. □

# 3 Dean Arnold Neumann's paper

Neumann's paper is, in general, very good to read. However, he stated some strong claims without proof. Since some of them are difficult to verify and others are false, when proving the main theorem we make some adjustments, so that the idea behind the demonstration may still be utilized. One of his greatest mistakes is the adoption of the concept of m-separatrices. For instance, his first lemma (any canonical region is parallel) is false because of it, which implies in the falsehood of the theorem. We split it into two lemmas, with more details and references. The rest of the article, mainly the proof of the theorem, will be treated in the next chapter.

**Lemma 2.35.** *Let $(R, \varphi)$ be a canonical region and consider the equivalence relation $x \sim y$ if and only if $x$ and $y$ are in the same orbit. Then the quotient space $R/\sim$ is a 1-manifold.*

*Proof.* Let $(R, \varphi)$ be a canonical region. Given any orbit, since it is not a separatrix, if it is closed there is an annular region containing it; similarly, if it is not closed, there is a strong strip region containing it. This shows that the set of closed orbit is open, as well as the set of line homeomorph orbits. Hence, by connectedness, $R$ consists either entirely of closed orbits or entirely of line homeomorph orbits.

Now, let $\gamma, \mu \subset R$ be two orbits of $\varphi$. Suppose they cannot be separated by two disjoint parallel neighborhoods. Thus, given $N$ a parallel neighborhood of $\gamma$, we have $\mu \in \overline{N}$; in other words, the orbit $\mu$ is contained in $\bigcap \overline{N}$, where the intersection runs through all parallel neighborhoods of $\gamma$. From the definition of ordinary orbit, such intersection must equal $\gamma \cup \alpha(\gamma) \cup \omega(\gamma)$. The orbit $\mu$ cannot intersect $\gamma$, so it is contained in $\alpha(\gamma) \cup \omega(\gamma)$, which is impossible, since $\mu$ lies in a parallel neighborhood which may be taken to exclude $\gamma$ (this is possible due to our definition of ordinary orbit: now it is always possible to exclude a chosen orbit from the parallel neighborhood).

The map $\pi : R \to R/\sim$, which takes a point to its equivalence class, is a quotient map, so by the above, $R/\sim$ is a Hausdorff topological space. Due to [8], Lemma 3.21, if we prove it is locally Euclidean, it will also be second-countable. So let us define an atlas on $R/\sim$. Given an orbit $\gamma \subset R$ (which is equivalent to take a point $x \in R/\sim$), choose a strong parallel neighborhood $V_\gamma$ containing it, i.e., $V_\gamma$ is strong strip or strong annular. There is a complete strong transversal $\Sigma_\gamma$ to $V_\gamma$, and a homeomorphism $\sigma_\gamma : \Sigma_\gamma \to \mathbb{R}$. The restriction $\pi|_{\Sigma_\gamma}$ is injective, thus bijective; furthermore, it is continuous with a continuous inverse, since $\pi$ is an open map. Set $\iota_\gamma$ its inverse. If we denote $x = \pi(\gamma) \in R/\sim$, the open set $U_x = \pi(V_\gamma) = \pi(\Sigma_\gamma)$ contains $x$ and the induced map $\sigma_x : U_x \to \mathbb{R}$ given by

$$\sigma_x(z) = \sigma_\gamma(\iota_\gamma(z))$$

is a homeomorphism. For convenience, whenever $U_x \cap U_y \neq \emptyset$, with $x = \pi(\gamma)$ and $y = \pi(\mu)$, we assume that $\Sigma_\gamma$ and $\Sigma_\mu$ coincides in $V_\gamma \cap V_\mu$; this is possible since the transversals are strong. Therefore $\{(U_x, \sigma_x)\}$ is an atlas on $R/\sim$, and hence this quotient space is a topological manifold of dimension one. □

For the next lemma we will use the power of *fiber bundles*. As this is a delicate topic, the proper treatment will not be done here. We follow exactly the definition presented in [17], 2.3, page 7. At page 11 one can find the definition of equivalence between coordinate bundles or of fiber bundles; moreover, Lemma 2.7 at page 10 says that equivalent coordinate bundles have homeomorphic bundle spaces. At page 16 it is stated the definition of product bundle and the theorem we want to use.

**Theorem 2.36.** *If the group of a bundle consists of the identity element alone (it is the trivial group), then the bundle is equivalent to a product bundle.*

**Lemma 2.37.** *Any canonical region $R$ of $(M, \varphi)$ is parallel.*

*Proof.* By Lemma 2.35, if we consider the equivalence relation $x \sim y$ if and only if $x$ and $y$ are in the same orbit, the quotient space $R/\sim$ is a 1-manifold. Our aim is to show that $R$ has the structure of product coordinate bundle, so it will be homeomophic to a parallel region. So we put $R$ as the bundle space and analyze what is needed to obtain a product bundle.

Since we already have a natural equivalence relation because of the flow, it is also natural to choose the base space as $R/\sim$ and the map $\pi : R \to R/\sim$ which takes a point to its equivalence class as the projection. The fiber $F$ is chosen as either $\mathbb{R}$ or $S^1$, depending if the orbits are all closed or line homeomorph, since the preimage of a point under $\pi$ is an orbit. The topological transformation group must be the trivial group, otherwise we cannot apply 2.36. The quotient space is naturally equipped with neighborhoods, so the coordinate neighborhoods are $\{U_x\}$, as in Lemma 2.35. To define the coordinate functions $\{\phi_x\}$, we consider the diagram below where all the arrows are homeomorphisms (the right column is due to Proposition 2.13, using that $\gamma = \pi^{-1}(x)$ is homeomorphic to $F$).

$$\begin{tikzcd} U_x \times F \arrow[r, "\phi_x"] \arrow[d] & \pi^{-1}(U_x) \arrow[d] \\ \mathbb{R} \times \gamma \arrow[r] & \Sigma_\gamma \times \gamma \end{tikzcd}$$

By abuse of notation, similarly to what we have done in Proposition 2.13, we write $\phi_x(z,t) = \varphi(z,t)$ (we omit the brackets of the equivalence class if $F = S^1$). Now we verify the conditions the coordinate functions must satisfy. It is clear that $\pi(\phi_x(z,t)) = z$. Put $\phi_{x,z}(t) = \phi_x(z,t)$. Given $z \in U_x \cap U_y$, we have

$$\phi_{x,z}^{-1} \circ \phi_{y,z}(t) = \phi_{x,z}^{-1}(\varphi_z(t)) = t,$$

because the transversals $\Sigma_\gamma$ were chosen so that they coincide whenever there is overlapping. So the homeomorphism $\phi_{x,z}^{-1} \circ \phi_{y,z}$ coincides with the operation of the identity in $F$. Furthermore, the map $g_{y,x} : U_x \cap U_y$ is constant, so continuous. Therefore we have a coordinate bundle as claimed.

By Theorem 2.36, recalling that every connected 1-manifold is homeomorphic to either $\mathbb{R}$ or $S^1$ ([8], Theorem 6.1), we have a product coordinate bundle. Since the flow provides a natural orientation on the fibers, all the possibilities for products yields only four types of regions: the four classes of parallel regions described in Definition 2.4. □

The main problem (using m-separatrices) in the lemma shows up when trying to intersect the closure of all parallel neighborhoods containing a chosen orbit. The result of such intersection is not what we have just written in Lemma 2.35: take our classical example. If one intersects the closure of all parallel regions containing, say, the straight line $x = 1$, the set obtained is the union of the two parallel lines with the region between them. We stress that when dealing with m-separatrices, it is not always possible to get closer to the desired orbit as one might want to.

It should be stressed the power of this lemma. After taking an arbitrary continuous flow and deleting its separatrices, inside each connected component of the open set obtained such flow cannot be arbitrary. Because of the topology, excluding the toral case which is very particular, the flow restricted to each connected component has just three options and nothing more. Also, it follows from Lemma 2.37 that any canonical region admits a complete transversal.

# Chapter 3

# The main theorem, its proof and final considerations

As we said earlier, the purpose of this chapter is to prove the following theorem:

**Theorem 3.1.** *Let $M$ be a 2-manifold and suppose that $\varphi_1$ and $\varphi_2$ are continuous flows on $M$ whose set of essential singular points is discrete. Then $\varphi_1$ and $\varphi_2$ are equivalent if and only if they have equivalent separatrix configurations.*

The proof follows Neumann's paper [13], with modern notation and more details. We divide the proof into two steps. We observe that the first step proves Markus' "Important claim", with "m-objects" replaced by "objects". Since we will use the Pasting Lemma during the proof, we state it here.

**Lemma 3.2.** *Consider two topological spaces $X$ and $Y$. Let $X = A \cup B$, where $A$ and $B$ are closed in $X$. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then the map $h : X \to Y$, the so-called* union *of $f$ and $g$, defined by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$, is continuous.*

*Proof.* Let $C \subset Y$ be a closed set. Now, $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ by elementary set theory. Since $f$ is continuous, $f^{-1}(C)$ is closed in $A$, so closed in $X$. Analogously $g^{-1}(C)$ is closed in $X$. Therefore their union is closed in $X$. □

*Proof of the theorem.* Suppose $\varphi_1$ and $\varphi_2$ have equivalent separatrix configurations and denote them by $\mathcal{S}_1^+$ and $\mathcal{S}_2^+$, respectively. Firstly, we make a simplification. Suppose $k$ is an equivalence between $\mathcal{S}_1^+$ and $\mathcal{S}_2^+$. If $h$ is a homeomorphism of $M$ which is the identity on $\mathcal{S}_2^+$, and an equivalence between the flow induced by $\varphi_1$ under $k$ and $\varphi_2$, then $hk$ is the required equivalence. See the commutative diagram below. Hence we may assume that $\varphi_1$ and $\varphi_2$ have the same separatrix configuration $\mathcal{S}^+$, and construct $h$.

$$\begin{tikzcd} & (M,\Phi) \arrow[d, "h"] \\ (M,\varphi_1) \arrow[ur, "k"] \arrow[r] & (M,\varphi_2) \end{tikzcd}$$

We also note that vertical points are interior to $\mathcal{S}$ (and so to $\mathcal{S}^+$), because all the orbits of the three vertical flows are separatrices, and thus sufficiently small neighborhoods of vertical points contain only separatrices. This shows that vertical points cannot border any canonical region.

Step I: Breaking canonical regions

Let $R$ be a canonical region for the flows. We will describe an equivalence $h$ between $(R, \varphi_1)$ and $(R, \varphi_2)$ which extends by the identity to an equivalence on $R \cup \delta R$. Our strategy is to break $(R, \varphi_1)$ and $(R, \varphi_2)$ into cells in such a way that $h$ is "cellular", being "almost" the identity near $\delta R$. The construction also restricts $h$ in the interior of $R$ in such a manner that the united equivalence (obtained by piecing together the various canonical regions) extends continuously to the limit separatrices. The latter restriction is measured by a positive constant $\varepsilon$, which we assume fixed for the remainder of this step. We may assume that the topology of $M$ is defined by a complete metric $d$ (any second-countable locally compact Hausdorff space is *Polish*; see [18]). Besides the distinguished orbits, which are the same for both flows, the other ordinary orbits of $\varphi_i$ will be denoted by $\gamma^i$, $i = 1, 2$. Let us deal with each type of canonical region.

**Strip:** Suppose $R$ is a strip region and let $\gamma_p \subset \mathcal{S}^+$ be the distinguished orbit. Choose points $p_k \in \gamma_p$, $k \in \mathbb{Z}$, satisfying:

(i) $p_k = \varphi_i(p, t_k^i)$, $i = 1, 2$, where $t_k^i$ strictly increases with $k$ (henceforth we say that $\{p_k\}$ is monotonic) and is unbounded below and above;

(ii) $d(p_k, p_{k+1}) < \varepsilon$ for every $k \in \mathbb{Z}$;

(iii) if $\alpha(p) \neq \emptyset$ (respectively, $\omega(p) \neq \emptyset$), then $\lim_{k \to -\infty} d(p_k, p_{k+1}) = 0$ (respectively, $\lim_{k \to \infty} d(p_k, p_{k+1}) = 0$).

Note that $\gamma_p$ separates $R$ into two half-regions $R^+$ and $R^-$ (both containing $\gamma_p$). If $\delta R \neq \emptyset$, at least one of the sets $\delta R^+$ and $\delta R^-$ is non-empty; without loss of generality, we assume that $\delta R^+ \neq \emptyset$. We construct a subdivision of $R^+$; the case for $R^-$ is completely analogous.

We construct transversals recursively as follows. Let $A_k$ be the set of positive real numbers $a$ for which there exists a transversal to $\varphi_1$ from $p_k$ to $\delta R^+$ of diameter $a$. Because of Proposition 2.32, such set is non-empty, since at each point $x \in \delta R^+$ one can take a transversal contained in $R$ with $x$ as one of its endpoints and extend it, so it meets $\gamma_p$. Using the homeomorphism that defines the parallelism, one can deform the transversal in such a way the point of intersection is $p_k$. Put $a_k = \inf A_k$. Construct a transversal $S_0$ to $\varphi_1$ from $p_0$ to a point $q_0 \in \delta R^+$ with $\operatorname{diam} S_0 < 2a_0$. Let $A \subset \mathbb{Z}$ be the union of 0 and those indices $k$ for which $a_k \leq 1$. For $k \in A$, define $A'_k$ as the set of positive real numbers $a$ for which there exists a transversal to $\varphi_1$ from $p_k$ to $\delta R^+$ disjoint from the previous constructed $S_m$ of diameter $a$ (here "previous" means that $m \in A$ satisfies $|m| < |k|$ and $|k - m| \leq |k - i|$ for every $i \in A$ with $|i| < |k|$). As above, this set is non-empty. Put $a'_k = \inf A'_k$. If it is possible to modify the previous transversal keeping its diameter and disjointness properties so that $a_k = a'_k$, construct a transversal $S_k$ to $\varphi_1$ from $p_k$ to a point $q_k \in \delta R^+$ disjoint from the previous one with $\operatorname{diam} S_k < 2a_k$. Note that, since the region is strip, we can guarantee that every transversal constructed after $S_i$ do not meet any transversal constructed before $S_i$, for any $i \in A - \{0\}$. Redefine $A$ so that it is the union of 0 and those indices $k$ for which we constructed a transversal $S_k$. In a while we will see that, besides the case where the distinguished orbit is sufficiently far from the accessible boundary (so all but a finite number of $a_k$ is greater than 1), we can always construct infinitely many such transversals.

Analogously, for $k \in A$, let $B_k$ be the set of all positive real numbers $b$ for which there exists a transversal to $\varphi_2$ from $p_k$ to $q_k$ of diameter $b$ and define $b_k = \inf B_k$. Construct a transversal $S'_0$ to $\varphi_2$ from $p_0$ to $q_0$ with $\operatorname{diam} S'_0 < 2b_0$. Now, define $B'_k$ as the set of all positive real numbers $b$ for which there exists a transversal to $\varphi_2$ from $p_k$ to $q_k$ disjoint from the previous transversal of diameter $b$. Put $b'_k = \inf B'_k$. If it is possible to modify the previous transversal keeping its diameter and disjointness properties so that $b_k = b'_k$, construct a transversal $S'_k$ to $\varphi_2$ from $p_k$ to $q_k$ disjoint from the previous one with $\operatorname{diam} S'_k < 2b_k$. This process possibly excludes some values of $A$, so we redefine $A$ accordingly (so both $S_k$ and $S'_k$ are constructed for every $k \in A$).

We point out that $A$ is infinite (except when $a_k > 1$ for all but finitely many $k$), because fixed $k_0 \in A$, if $\omega(p)$ ($\alpha(p)$) is empty, then $d(p_{k_0}, p_k) \to \infty$ as $k \to \infty$ ($k \to -\infty$), since $d$ is complete. If $\omega(p)$ ($\alpha(p)$) is non-empty, then there are indices for which $a_k$ is arbitrarily small. Since $S_{k_0}$ is compact, in any case the distance between such transversal and the points $p_k$ eventually becomes sufficiently large, and hence $a_k$ equals $a'_k$ and the transversal $S_k$ can be constructed.

Finally, let $\{d_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\delta R^+$ which is disjoint from $\{q_k\}_{k\in A}$. Construct disjoint transversals $T_n$ to $\varphi_1$ and $T'_n$ to $\varphi_2$, both terminating at $d_n$ and satisfying:

(i) $T_n$ is disjoint from every $S_k$ ($T'_n$ is disjoint from every $S'_k$);

(ii) $\operatorname{diam} T_n \to 0$ ($\operatorname{diam} T'_n \to 0$) as $n \to \infty$;

(iii) if $T_n$ ($T'_n$) has initial point on the orbit $\gamma^1_{r_n}$ ($\gamma^2_{r'_n}$), where $r_n \in S_0$ ($r'_n \in S'_0$), then $r_n$ ($r'_n$) converges monotonically to $q_0$.

The transversals $S_k$, $T_n$ and the orbits $\gamma^1_{r_n}$ separates $R^+$ into a locally finite collection of 2-cells (there are some "open" cells, those homeomorphic to $(0, \infty) \times (0, 1)$), which we refer to as an *$\varepsilon$-subdivision* of $R^+$ with respect to $\varphi_1$; see Figure 3.3. The $S'_k$, $T'_n$ and $\gamma^2_{r'_n}$ provide an $\varepsilon$-subdivision of $R^+$ with respect to $\varphi_2$. We claim that there is an equivalence $h$ between $(R^+, \varphi_1)$ and $(R^+, \varphi_2)$ which takes cells of one subdivision onto the corresponding cells of the other. Indeed, using the transversal $S_0[p_0, q_0)$, one can construct a homeomorphism between the region bounded by $\gamma_p$ and $\delta R^+$, and $\mathbb{R} \times [0, \infty)$

(Proposition 2.13); moreover, we may assume that $S_0[p_0, q_0)$ is mapped onto $\{0\} \times [0, \infty)$. The orbits are mapped onto straight horizontal lines and the image of the sequence $\{r_n\}$ goes to infinity along the $y$-axis. Hence we can adjust the image of each $\gamma^1_{r_n}$, so it is the line $y = n$. The same can be done using $S'_0[p_0, q_0)$. Since all the transversals are disjoint, there is "space" to slide them, so that the second configuration can be obtained from the first by sliding the transversals. The composite map $h$ is, therefore, the required equivalence; see the commutative diagram below, where the flows in the half-plane are induced by the homeomorphisms given in this paragraph.

$$\begin{tikzcd}
(R^+, \varphi_1) \arrow[r, "h"] \arrow[d] & (R^+, \varphi_2) \arrow[d] \\
(\mathbb{R} \times [0, \infty), \Psi_1) \arrow[r] & (\mathbb{R} \times [0, \infty), \Psi_2)
\end{tikzcd}$$

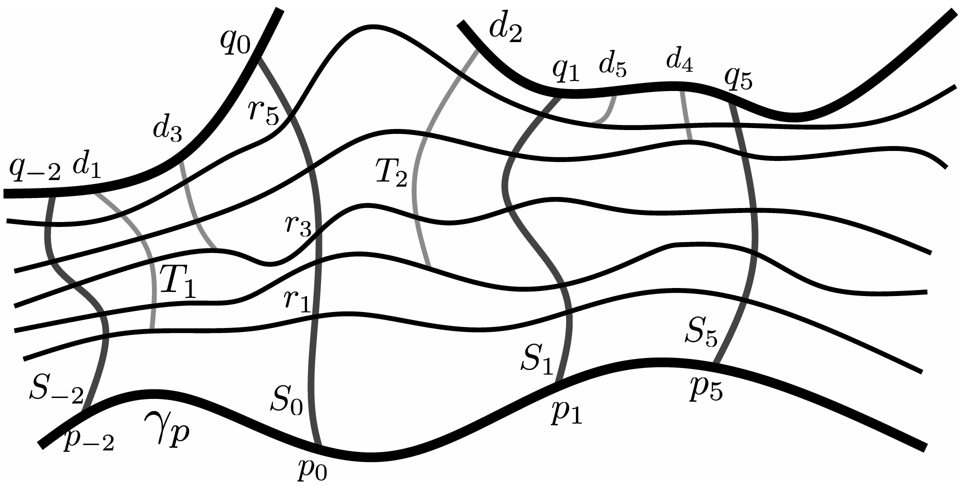



Figure 3.3: Subdivision of a strip canonical region.

We now define $h$ to be the identity on $\delta R^+$. Continuity of the extended mapping is proved as follows. Pick $q \in \delta R^+$ which is not in $\alpha(p) \cup \omega(p)$. Given $U$ a neighborhood of $q$ in $R^+ \cup \delta R^+$, we can choose $i, j$ and $m$ sufficiently large so that the neighborhoods $N$ and $N'$ of $q$ bounded by segments of $T_i, T_j$ and $\gamma^1_{r_m}$ and by segments of $T'_i, T'_j$ and $\gamma^2_{r'_m}$, respectively, both lie in $U$. By construction, $h(N) = N' \subset U$. The case for $q \in \alpha(p) \cup \omega(p)$ is covered by the argument given in Step II, since $\gamma_q$ would then be a limit separatrix. Indeed, take a local transversal at $q$ (if $q$ is horizontal or horizonal, then $\gamma_q$ is a limit separatrix already). Then it intersects $\gamma_p$ infinitely many times. So this local transversal must meet every orbit within $R$ infinitely many times and hence, since it is strip, meet a separatrix. We conclude that there exists a separatrix arbitrarily close to $\gamma_q$.

If $\delta R^+ = \emptyset$, we claim that there are no regular, horizontal or horizonal points in $\partial R^+$. Otherwise we could take a local transversal $T$ at this point, and, since it is not accessible, $T$ would intersect separatrices and the region infinitely many times. But then there would be a separatrix bordering the region directly. A point from such orbit would be accessible, contrary to the hypothesis. Hence $\partial R^+$ is empty (no essential singular point can lie in this boundary, since it would imply in the existence of a separatrix other than such point bordering the region), and we may take $h$ to be any equivalence between $(R^+, \varphi_1)$ and $(R^+, \varphi_2)$ which is the identity on $\gamma_p$.

**Annular:** Here the construction is exactly as above, except that $\{p_k\}$ is a sequence with finitely many values, with distance between consecutive values less than $\varepsilon$ and monotonic along the distinguished orbit $\gamma_p$. Hence, the inductive construction of the transversals $S_k$ and $S'_k$ is unilateral and when trying to construct the next transversal one may modify all previous constructed ones.

**Spiral:** In order to construct the subdivision in this case, we need to prove the existence of an arc which is transversal to both $\varphi_1$ and $\varphi_2$ and has initial and terminal points on the distinguished orbit $\gamma_p$. Let $q$ be a point on the accessible boundary of $R$. Let $N$ denote a closed $V$-tube, for some tube of $q$, which is bounded by transversals for $\varphi_1$ terminating on the distinguished orbit $\gamma_p$, and a quasi-orbit of $\gamma_p$, and let $N' \subset N$ be similarly bounded by $\gamma_p$ and transversals for $\varphi_2$. Let $h_1 : N \to D$ ($h'_2 : N' \to D$) be a homeomorphism, where $D = \{(x, y) \mid |x| \leq 1, 0 \leq y \leq 1\}$, which maps quasi-orbits of $\varphi_1$ ($\varphi_2$) onto horizontal segments and takes $q$ to 0. Put $g = h_1 \circ (h'_2)^{-1} : h'_2(N' \cap \gamma_p) \to D$. Let $k$ be a homeomorphism which extends $g$ to $D$ and takes horizontal segments into horizontal segments (the existence of such extension is proved in a very similar situation in Step II). Thus the mapping $h_2 : N' \to D$ defined by $h_2 = kh'_2$ is such that $h_2 = h_1$ on $N' \cap \gamma_p$. Hence the preimage of a vertical segment provides the desired arc.

We distinguish between two types of spiral regions. Suppose an orbit $\gamma \subset R$ has nonrest or horizontal points in both $\alpha$ and $\omega$-limit sets. We say that $R$ is *orientable* if the orientations on $\delta R$ induced by the flow are compatible with some orientation of $R$; see Figure 3.4. Loosely speaking, the border of an orientable spiral region is made of "two positively oriented Jordan curves or two negatively oriented Jordan curves" (maybe there are no regular orbits on the accessible boundary, but we put an orientation on the curves in such boundary matching the orientation of the flow inside $R$). We say that $R$ is *non-orientable* otherwise.

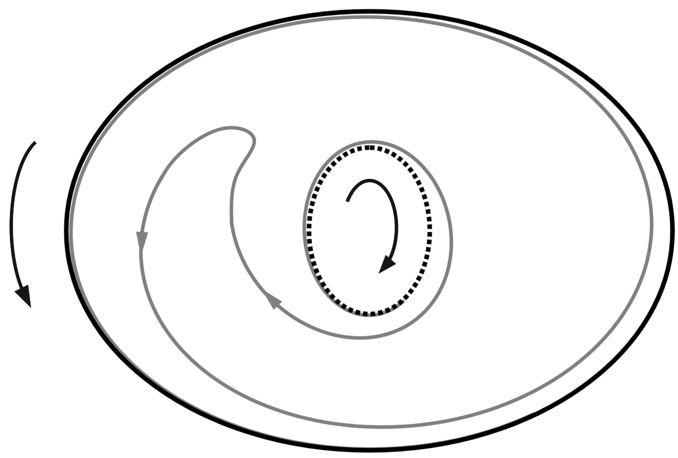


Figure 3.4: Orientable spiral region.

We make a brief comment about such differentiation of spiral regions. Because of the change of orientations inside orientable regions, they cannot accumulate on a limit separatrix (regular or a horizon). So in Step II we need not worry about change of behavior within a canonical region: a sufficiently small neighborhood of a point in a limit separatrix contains only orbits entering the neighborhood from one side and getting out by the other in a "well behaved" way.

Suppose $R$ is a non-orientable spiral region. Let $S$ be an arc which both endpoints lie on $\gamma_p$, no interior point meets $\gamma_p$ and is transversal to both $\varphi_1$ and $\varphi_2$. We may suppose the endpoints of $S$ are $p$ and $q$, with $q \in \gamma_p^+$. Pick $p_0 = p, p_1, \ldots, p_n, p_{n+1} = q$ monotonic along $\gamma_p^+$ and spaced closer together than $\varepsilon$. Put $C = \gamma_p[p_0, p_{n+1}] \cup S$. Thus $C$ is a Jordan curve which separates $R$ into two connected components. Define $R^+$ as the one which corresponds to the future of the flow; similarly define $R^-$.

For $k = 1, \ldots, n+1$, construct disjoint transversals $S_k$ to $\varphi_1$ from $p_k$ to points $q_k \in \delta R^+$ and disjoint transversals $S'_k$ to $\varphi_2$ from $p_k$ to $q_k$ the same way we did in the annular case (remember that the constructed transversals have diameter less than twice the infimum of possible diameters).

If $R$ is orientable (or if $\alpha(p)$ or $\omega(p)$ is empty or contains only essential singular points), we take an arc $S$ transversal to both flows and meeting the distinguished orbit twice, say, at points $p$ and $q$. Put $C = \gamma_p[p, q] \cup S[p, q]$, which is again a Jordan curve which separates $R$ into two regions. The one which corresponds to the future of the flow we define as $R^+$ and the one corresponding to the past of the flow we define as $R^-$.

Finally, for any spiral region $R$ with $\delta R^+ \neq \emptyset$, let $\{d_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\delta R^+$ (disjoint from $\{q_k\}$ in the non-orientable case) and construct local transversals $T_n$ ($T'_n$) to $\varphi_1$ (to $\varphi_2$) pairwise disjoint (and, in the non-orientable case, disjoint from every $S_k$ ($S'_k$)) satisfying:

(i) $T_n$ and $T'_n$ originate at the same point of $\gamma_p$;

(ii) $T_n$ and $T'_n$ terminate at $d_n$;

(iii) both $\operatorname{diam} T_n$ and $\operatorname{diam} T'_n$ goes to zero as $n \to \infty$.

See Figure 3.5.

Similarly to what we have done in the strip case, one can use a subset of a radial region in $\mathbb{R}^2$ to slide transversals and construct a homeomorphism $h$ between $(R^+, \varphi_1)$ and $(R^+, \varphi_2)$ which is the identity on $C$ and takes cells of one partition onto corresponding cells of the other. Again, such equivalence extends to $\delta R^+$ by the identity.

If $\delta R^+ = \emptyset$, take $h$ to be any equivalence which is the identity on $C$. The region $R^-$ is treated similarly.

**Toral:** If $R$ is a toral canonical region, then $R$ is a compact set inside a Hausdorff space, hence closed. Since $M$ is connected, this implies that $R = M$ and $h$ may be

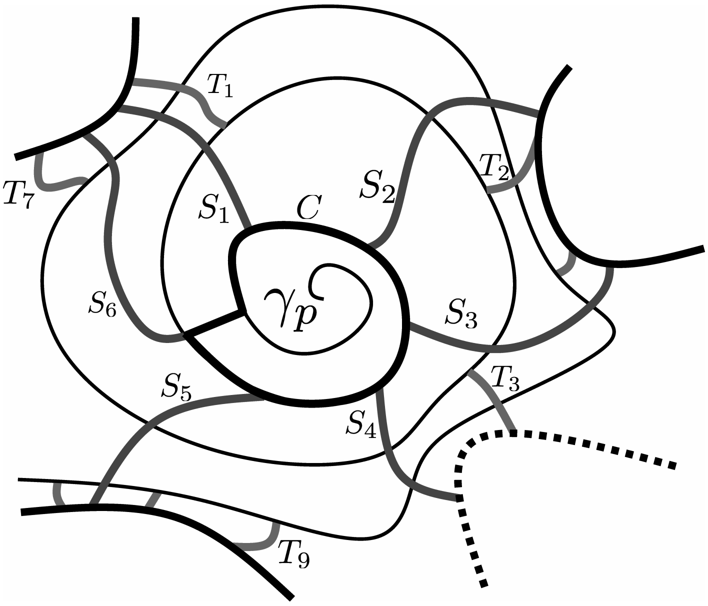



Figure 3.5: Subdivision of a spiral canonical region.

taken as any equivalence between $(M, \varphi_1)$ and $(M, \varphi_2)$ which is the identity on the unique distinguished orbit.

Step II: Dealing with limit separatrices

Throughout this step, when talking about a separatrix, it is possibly a horizontal orbit. It will be clear by the context.

Canonical regions are disjoint open sets. Hence, since $M$ is second-countable, the set of all canonical regions is countable (when there are finitely many canonical regions, we see the continuity of $h$ at limit separatrices the same way we do in the infinite case; the only difference is that when working inside the tubes, the infinitely many separatrices and distinguished orbits are actually a finite number of them, crossing the tube infinitely many times). Denote them by $\{R_n\}_{n\in\mathbb{N}}$. Let $\gamma_{p^n}$ be the distinguished orbit of $R_n$. For each $n$, define $\frac{1}{n}$-subdivisions of $R_n$ with respect to $\varphi_1$ and to $\varphi_2$, as in Step I. Therefore, there is a cellular equivalence $h^n$ between $(R_n, \varphi_1)$ and $(R_n, \varphi_2)$ which extends by the identity to nonlimit separatrices and horizontal points of $\delta R_n$. Define $h$ to be the union of all such $h^n$. Since they agree when there is overlapping, the Pasting Lemma guarantees the continuity of $h$. We extend $h$ by imposing it is the identity on limit separatrices also (and so the restriction $h : \mathcal{S} \to \mathcal{S}$ is the identity). We claim that such mapping is continuous.

Given $\varepsilon > 0$, suppose $p$ is a regular or horizonal point in a limit separatrix. Then, given a neighborhood $U$ of $p$, the orbit $\gamma_p$ separates $U$ into two components; at least one of these, which by simplicity we will denote by $U$ as well, meets separatrices which accumulate at $p$. Let $N$ denote a closed $V$-tube in $\overline{U}$, for some tube $V$ of $p$, which is bounded by transversals for $\varphi_1$ terminating on $\gamma_p$, and a quasi-orbit of a separatrix $\gamma_q$, and let $N' \subset N$ be similarly bounded by $\gamma_q$ and transversals for $\varphi_2$. Let $h_1 : N \to D$ ($h_2' : N' \to D$) be a homeomorphism, where $D = \{(x, y) \mid |x| \leq 1,\ 0 \leq y \leq 1\}$, which maps quasi-orbits of $\varphi_1$ ($\varphi_2$) onto horizontal segments and takes $p$ to 0. Put $g = h_1 \circ (h_2')^{-1} : h_2'(N' \cap \mathcal{S}^+) \to D$.

Let $y_1 < y_2$ be two consecutive heights of segments in $h_2'(N' \cap \mathcal{S}^+)$. Define

$$k(x,y) = \left(\frac{y - y_2}{y_1 - y_2}\right) g(x, y_1) + \left(\frac{y_1 - y}{y_1 - y_2}\right) g(x, y_2);$$

it is an extension of $g$ to the region between such segments which carries horizontal segments into horizontal segments. If we also denote by $k$ the union of all such mappings, then $k$ is continuous by the Pasting Lemma. Furthermore, it is a homeomorphism onto its image. Thus now we can define $h_2 : N' \to D$ by $h_2 = kh_2'$. Note that $h_2 = h_1$ on $N' \cap \mathcal{S}^+$.

By choosing a separatrix closer to $p$ than $\gamma_q$ if necessary (and then redefining the objects above accordingly), we may pick $a > 0$ such that $Q = \{(x, y) \mid |x| \leq a,\ 0 \leq y \leq 1\}$ is contained in both $h_1(N')$ and $h_2(N')$. Set $m = \min\{\varepsilon/6, a/8\}$ and observe that $Q \cap h_i(\mathcal{S}^+)$ includes segments arbitrarily close to 0. The complement of these consists of "strips" which are the intersection of the images of the various half canonical regions with $Q$ (using a vertical segment, the corollary of Proposition 2.32 guarantees a distinguished orbit between two separatrices). Relying on the uniform continuity of both $h_i$ and $h_i^{-1}$, choose $B > 1$ and $C > 1$ such that $\operatorname{diam} h_i(X) < m/2$ whenever $\operatorname{diam} X < m/2B$ and $\operatorname{diam} X < m/4B$ whenever $\operatorname{diam} h_i(X) < m/4C$, for every $X \subset N'$, $i = 1, 2$.

Now choose $\tau > 0$ so that the set $Q_\tau = \{(x, y) \mid |x| \leq a,\ 0 \leq y \leq \tau\}$ has the property that the supremum of widths of strips meeting $Q_\tau$ is less than $m/4C$. Consider a strip $\Omega$ in $Q_\tau$, bounded by segments of $h_i(\gamma_{p^n})$ and $h_i(\gamma_s)$ (so $\gamma_s \subset \delta R_n$ is a separatrix). Let $A^n$ be as $A$ in Step I for the canonical region $R_n$. Take any $h_i(p_k^n)$, $k \in \mathbb{Z}$, lying between $x = -3a/4$ and $x = 3a/4$. There is a strong transversal to $\Omega$, of both $h_1\varphi_1$ and $h_2\varphi_2$, with diameter less than $m/4C$ (a vertical segment will do it). Its preimage, under either $h_i$, has diameter less than $m/4B$; if we suppose it was possible to construct $S_k^n$ and $(S_k')^n$, we would have $\operatorname{diam} S_k^n < m/2B$ and $\operatorname{diam}(S_k')^n < m/2B$, and thus both $\operatorname{diam} h_1 S_k^n$ and $\operatorname{diam} h_2(S_k')^n$ would not reach $m/2$. Taking a $h_i(p_j^n)$ spaced more than $m$ apart from $h_i(p_k^n)$, it is clear that the Euclidean ball of center $h_i(p_j^n)$ and radius $m/2C$ has no point in common with $h_1 S_k^n$ or $h_2(S_k')^n$. Otherwise, picking $\overline{x}$ in such intersection, one would have

$$m < \|h_i(p_k^n) - h_i(p_j^n)\| \leq \|h_i(p_k^n) - \overline{x}\| + \|\overline{x} - h_i(p_j^n)\| < \frac{m}{2} + \frac{m}{2C} < m,$$

a contradiction. This shows that $a_j^n = (a_j')^n$ and $b_j^n = (b_j')^n$. Therefore, taking a smaller $\tau$ if necessary and assuming that the distance between successive $h_i(p_k^n)$ $(k \in A^n)$ along the image of any segment of a distinguished orbit $\gamma_{p^n}$ which meets $Q_\tau$ is less than $2m$, there are at least four $p_k^n$ in $h_i(\gamma_{p^n}) \cap R$, where $R = \{(x, y) \mid |x| < a/2,\ 0 \leq y \leq \delta\}$.

Restricting ourselves to the "cells" bounded by transversals $S_k^n$ $(S_k^n)$, by what we have just done, $R$ is covered by cells (in either subdivision) of diameter less than $3m$, i.e., less than $\varepsilon/2$. Each such cell intersects its image under the map $\overline{h} = h_2 \circ h \circ h_1^{-1} : R \to Q_\tau$, so that if we pick $\delta < \varepsilon/2$, for every $x$ with $\|x\| < \delta$ we have

$$\|\overline{h}(x)\| \leq \|\overline{h}(x) - x\| + \|x\| < \varepsilon,$$

which means that $h$ is continuous at $p$.

We just concluded that $h$ is a homeomorphism on the complement of the discrete set $P$ of essential singular points of $\varphi_i$. Now, given $p \in P$, there exists a sequence of points $\{x_n\}$ disjoint from $P$ with $x_n \to p$ and $h(x_n) \to p$. Indeed, either $p$ can be approached by points on $\mathcal{S}$, what would imply in $h(x_n) = x_n \to p$, or $p$ is the inner boundary of an annular or spiral region, and then, by Step I, we can think that $h$ is any equivalence between $\varphi_1$ and $\varphi_2$ on the unitary closed ball minus the origin, keeping $S^1$ fixed. It is clear that $h(x_n) \to p$ for any $\{x_n\}$ converging to $p$. Therefore, $h$ extends by the identity to $P$.

To prove the converse, we may do a similar simplification and assume that we chose two separatrix configurations $\mathcal{S}_1^+$ and $\mathcal{S}_2^+$ of the same flow $(M, \varphi)$ (hence the separatrices $\mathcal{S}$ are the same); see the commutative diagram below, where $h$ is the homeomorphism we want to construct. Observe that $k$ must map the separatrices of $\varphi_1$ onto the separatrices of $\varphi_2$. Otherwise, by a composition, the ordinariness of a separatrix could be achieved, a clear contradiction.

$$\begin{tikzcd}
& (M, \varphi_2) \arrow[d, "h"] \\
(M, \varphi_1) \arrow[ur, "k"] \arrow[r] & (M, \varphi_2)
\end{tikzcd}$$

Let $R$ be a canonical region, $\gamma^1 \subset R$ be the distinguished orbit of $\mathcal{S}_1^+$ and $\gamma^2 \subset R$ be the distinguished orbit of $\mathcal{S}_2^+$. If they are equal, just define $h$ to be the identity on $R$. If they are different orbits, there exists a strong region $U \subset R$ containing both $\gamma^1$ and $\gamma^2$. Set $\Omega$ as the union of $U$ and its border orbits and fix $\varepsilon > 0$.

**Strip and annular:** Let $\mu$ be the border orbit of $U$ such that $\mu$ and $\gamma^2$ are in different half-regions defined by $\gamma^1$. Define two $\varepsilon$-subdivisions of $R$ taking $\mu$ as the distinguished orbit with the condition that $T_1$ has initial point on $\gamma^1$ and $T_1'$ has initial point on $\gamma^2$. Hence there is an automorphism $h$ of $R$ which takes $\gamma^1$ onto $\gamma^2$ and extends by the identity to the accessible boundary.

**Spiral:** Choose a third orbit $\gamma_x$ and define a new flow $\varphi'$ the same way we did in the proof of Proposition 2.32. The region $R' = R - \gamma_x$ is strip and a canonical region for $\varphi'$. We may apply the argument given in the strip case and obtain an automorphism $h$ which takes $\gamma^1$ onto $\gamma^2$ and, by the first part of the proof, is the identity on $\gamma_x$. Observe that the accessible points for $\varphi$ are no longer accessible for $\varphi'$; instead, they belong to limit separatrices. Due to Step II, $h$ extends by the identity to such points. Therefore we have an automorphism of $R$ which is the identity on $\gamma_x$ and on $\delta R$.

**Toral:** This case is trivial. It suffices to take $h$ as a simply translation.

The mapping $h$ defined as the extension (by the identity) to limit separatrices and essential singular points of the union of all homeomorphisms constructed is an automorphism of $M$ such that $h(\mathcal{S}_1^+) = \mathcal{S}_2^+$, as desired. $\square$

All of our examples of limit separatrices presented in this text are either critical points or a closed orbit approximated by a strip region. In the proof of the above theorem, the main concern is limit separatrices such that infinitely many canonical regions approach them. We give some examples of this case now.

**Examples 3.6.**

(1) Our first example consists of a limit separatrix arbitrarily close to infinitely many strip regions. Let $u : [0, \infty) \to \mathbb{R}$ be the $\mathcal{C}^\infty$ function defined by $u(0) = 0$ and

$$u(x) = e^{-\frac{1}{x}} \sin^2 \frac{\pi}{x}$$

if $x \neq 0$. Define the $\mathcal{C}^\infty$ function $s : \mathbb{R} \to \mathbb{R}$ by $s(x) = u(|x|)$. Finally, consider the function

$$f(x, y) = x + s(x)y.$$

The level curves of this function gives a *foliation* of the plane, which can be viewed as unoriented orbits of a continuous flow in the plane. The line $x = 0$ is a limit separatrix. Details, similar examples and an image picturing the level curves can be found in [2].

(2) We proceed with an example of a closed orbit arbitrarily close to spiral regions. Put $\|z\| = \sqrt{x^2 + y^2}$ and define the vector field $f : \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$ by

$$f(x, y) = \left( -y + \frac{x}{\|z\|}(\|z\| - 1) \sin \frac{\pi}{\|z\| - 1},\ x + \frac{y}{\|z\|}(\|z\| - 1) \sin \frac{\pi}{\|z\| - 1} \right)$$

if $\|z\| \neq 1$, and $f(x, y) = (-y, x)$ if $\|z\| = 1$; in polar coordinates,

$$g(r, \theta) = \left( (r - 1) \sin \frac{\pi}{r - 1},\ 1 \right)$$

if $r \neq 1$, and $g(1, \theta) = (0, 1)$. When $r - 1 = 1/k$, $k \in \mathbb{Z}$, we have a closed orbit going counterclockwise. Between two consecutive closed orbits the radius can grow or shrink, depending mainly on the signal of the sine, so the orbits are line homeomorph, with a spiral behavior. The periodic orbit $r = 1$ is a limit separatrix; see Figure 3.7.

(3) Lastly, we have a continuous flow $\varphi$ on the strip $\mathbb{R} \times (-1, 1)$ induced by a vector field. Consider the *bump function* defined on $\mathbb{R}$ given by $E(x) = \exp\left(-\frac{1}{1-x^2}\right)$ on

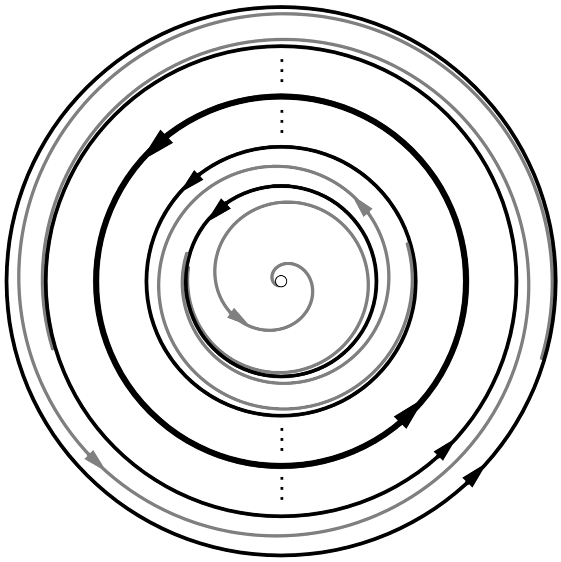


Figure 3.7: Circular limit separatrix.

$(-1, 1)$ and $E(x) = 0$ elsewhere. For $n \in \mathbb{N}$, define $g_n : [\frac{1}{n+1}, \frac{1}{n}] \to [-1, 1]$ by setting

$$g_n(x) = \frac{x+1}{2}\frac{1}{n} + \frac{1-x}{2}\frac{1}{n+1}.$$

Finally, put $h_n = (E \circ g_n)/n$ and set $f_n(x, y) = (-1)^n(h_n(|y|), 0)$. The union of all $f_n$ is continuous due to the Pasting Lemma. Call such union $f$ and extend it to the line $y = 0$ by imposing $f(x, 0) = (0, 0)$. Then $f$ is a continuous vector field on $\mathbb{R} \times (-1, 1)$. If $\varphi$ is the flow generated by $f$, then $\varphi$ is a continuous flow with countable many horizontal orbits. By slightly modifying the construction, we can alternate between orbits going horizontally and orbits going vertically, so the lines $y = \pm\frac{1}{n}$ are horizons and every point on the line $y = 0$ is an essential singular point. Both flows are depicted in Figure 3.8.

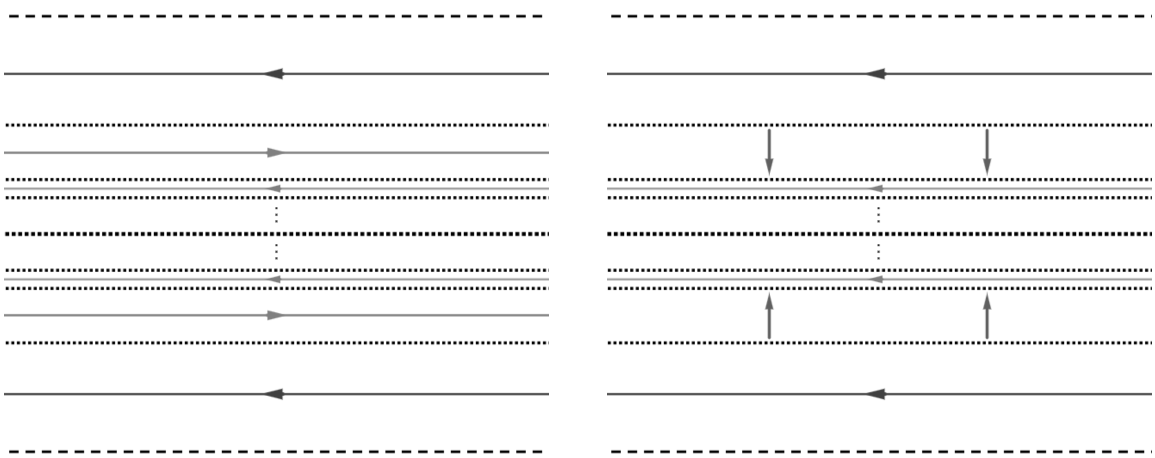


Figure 3.8: Flows with horizons and essential singular points.

The difference between the original theorem stated by Neumann and the new one stated by López and Buendía, is the term *essential*. Neumann proved the theorem for flows with discrete singular points, while the latter authors stated the theorem for flows with discrete essential singular points. As we have pointed, our definition of essential singular point is less restrictive than theirs, so more flows are included. The set of essential singular points of the first flow in Example 3.6 (3) is discrete with our definition (in fact, it is empty), while in theirs *every* singular point on the $x$-axis is essential.

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