UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Campos de Vetores Suaves por Partes: Preservação de Medida, Pressão Topológica e Dinâmica Simbólica

Marco Aurélio do Carmo Florentino

São Carlos-SP
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Dedico este trabalho
a Zita Rosa do Carmo,
minha vó.

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“ Pode ser difícil agora, mas você deve silenciar esses pensamentos!
Pare de contar apenas as coisas que você perdeu! O que se foi, se foi!”
(Eiichiro Oda- Mangá One Piece.)

Resumo

O estudo de campos de vetores suaves por partes (CVSPs) tem se consolidado nos últimos anos não apenas pela beleza dos resultados teóricos, mas também pela proximidade dessa área com as ciências aplicadas como mecânica, engenharia, eletrônica e biologia, além das ciências sociais e econômicas. A principal diferença entre CVSPs e campos de vetores suaves é o fato de que pode não haver unicidade de trajetória para todo ponto de um CVSP. Com a existência de caos, podemos buscar maneiras de calcular a entropia topológica, uma vez que entropia estima o quão caótico é o sistema.

Neste trabalho seguimos esta linha de investigação e obtemos um conjunto de trajetórias dos campos de vetores suaves por partes onde a aplicação de tempo um está bem definida. Deste modo, obtemos uma conjugação entre o itinerário de uma trajetória contida neste conjunto e de sequências sobre um conjunto finito de símbolos. Assim, estudamos alguns aspectos do formalismo termodinâmico, mais especificamente pressão topológica e, consequentemente, entropia topológica para campos de vetores suaves por partes, usando conjugação topológica com shifts unilaterais e o Operador de Ruelle-Perron-Frobenius. Algumas relações entre entropia, dimensão de Hausdorff e dimensão de Minkowski também são apresentadas. Neste sentido, quando a pressão é zero, podemos usar a teoria da cadeias de Markov juntamente ao operador de Ruelle-Perron-Frobenius, para calcular o tempo de relaxação e estimar o tempo de mistura para CVSPs.

Por fim, introduzimos o conceito de conexão deslize-escape para CVSPs e estabelecemos condições para obter um conjunto de trajetórias que preserva a medida mesmo no caso em que o movimento de deslize é permitido. Como consequência, resultados clássicos da teoria ergódica de sistemas dinâmicos podem ser adaptados para o contexto de CVSPs com uma conexão deslize-escape, a saber, o Teorema de Recorrência de Poincaré e o Teorema Ergódico de Birkhoff.

Palavras-chave: Campos de vetores suaves por partes, Conexão deslize-escape, Pressão topológica, Shifts unilaterais.

Abstract

The study of piecewise smooth vector fields (PSVFs) has been consolidated in recent years not only because of the beauty of the theoretical results, but also because of the proximity of this area to applied sciences such as mechanics, engineering, electronics and biology, in addition to social sciences and economical. The main difference between PSVFs and smooth vector fields is the fact that there may not be unique the trajectory passing through each point a PSVF. With the existence of chaos, we can look for ways to calculate the topological entropy, since entropy estimates how chaotic the environment is system.

In this work we follow this line of investigation and obtain a set of piecewise smooth vector field trajectories where the application of time one is well defined. In this way, we obtain a conjugacy between the itinerary of a trajectory contained in this set and sequences over a finite set of symbols. Thus, we study some aspects of thermodynamic formalism, more specifically topological pressure and, consequently, topological entropy for piecewise smooth vector fields, using topological conjugacy with one-sided shifts and the Ruelle-Perron-Frobenius Operator. Some relations between entropy, Hausdorff dimension and Minkowski dimension are also presented. In this sense, when the pressure is zero, we can use the Markov chain theory together with the Ruelle-Perron-Frobenius operator to calculate the relaxation time and estimate the mixing time for PSVFs.

Finally, we introduce the concept of sliding-escaping connection for piecewise smooth vector fields and establish conditions in order to obtain a set of trajectories that preserves measure even in the case where sliding motion is allowed. As consequence, classical results from the ergodic theory of dynamical systems can be adapted for the context of piecewise smooth vector fields with a sliding-escaping connection, namely, the Poincaré's Recurrence Theorem and the Birkhoff's Ergodic Theorem.

Keywords: Piecewise smooth vector fields, Sliding-escaping connection, Topological pressure, One-sided shifts.

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CHAPTER 1

Introduction

The theory of piecewise smooth vector fields (PSVFs) admits the existence of switching manifolds separating the phase portrait into a finite number of disjoint regions. Then it is defined a vector field (not necessarily the same) in each of these regions. It means, among other things, that at the boundary of each region it is defined at least two vector fields in such a way that the global trajectory may not be smooth. Here it is clear the useful of PSVFs to model applied problems where a kind of *"on-off"* situation is considered. This is the case of intermittent treatments of cancer and HIV (see [14, 30]), intermittent protocols of containment for COVID-19 (see [13]) among others. The terminology that presents the behavior of trajectories in the switching manifold was presented by Filippov (see [24]). Some landmarks in the theory of PSVFs are the works [8, 52] and references therein.

At the moment, a strongly explored line of research in PSVFs theory is to verify which results from the theory of smooth vector fields are still valid for the piecewise smooth scenario. In this direction, some particular results have been obtained over the last few years. It is quite clear that the Existence and Uniqueness Theorem is not true in the context of PSVFs. On the other hand, under adequate hypotheses, we already know that Poincaré Index Theorem, Poincaré-Bendixson Theorem and Peixoto's Theorem have versions for PSVFs (see [9, 12, 51]). Other works introduce the concepts of invariance, minimality and chaoticity for PSVFs, and they achieve amazing characteristics in relation to these objects, such as the existence of non-trivial minimal sets and chaotic vector fields in dimension 2 (see [11, 15, 32]), which does not happens for smooth vector fields.

In Chapter 2, we provide the main ideas and general definitions concerning PSVFs, topological pressure, topological entropy, Bernoulli shift spaces, random variables, Markov chains, Perron-Frobenius theory, Hausdorff measure and Minkowski dimension that we use along this text.

There are several concepts of entropy, among them, Shannon entropy which first appeared in the 1940s for the need of information theory, Kolmogorov-Sinai entropy which arose to solve a fundamental problem of ergodic theory in the 1950s and topological entropy for the study of topological dynamical systems in the 1960s. All these concepts are mathematical measures of uncertainty, whose origins come from the classic Boltzmann entropy in thermodynamics. In 1948 Claude Shannon in his article *"A mathematical theory of communication"* [47] proposed the notion of entropy to measure how the information inside a signal can be quantified with absolute precision as the amount of unexpected data contained in the message. In 1958, Kolmogorov [34] introduced the concept of entropy in the dynamical system as a measure-preserving map and studied the concomitant property of completely positive entropy. Soon after in 1959, his student Sinai [49] formulated the Kolmogorov-Sinai entropy which is suitable for automorphisms of Lebesgue spaces. The Kolmogorov-Sinai entropy is equivalent to a generalized version of the Shannon entropy under certain plausible assumptions. Topological entropy was introduced by R. Adler, A. Konheim and M. McAndrew [1] to describe the complexity of a single map acting on a compact metric space, we have a metric space. In this way, the topological entropy is a measure of the "disorder" of the system, that is, it can be thought of as a quantitative measure of sensitive dependence on initial conditions. So entropy quantitatively measures how chaotic the system is. In addition, Bowen extended the definition to non-compact spaces, which is also very useful in applications.

Topological pressure is a weighted version of topological entropy, where the "weights" are determined by a continuous function called the potential. The idea of pressure was brought from Statistical Mechanics to Ergodic Theory by the mathematician and theoretical physicist David Ruelle [44], one of the originators of the differentiable ergodic theory, and was later extended by the British mathematician Peter Walters [55].

In [3], the authors proposed a new way to approach PSVFs through the construction of a metric space of all possible trajectories. Using this, they defined the topological entropy of a planar PSVFs, proved the existence of planar PSVFs with positive entropy (finite and infinite) and provided suficient conditions for a planar PSVF to have infinite entropy, in addition to showing examples of PSVFs where the entropy is always $\log r$, for a **positive integer** r. Furthermore, based on this metric space, in [2] it is proposed a way to combine the dynamics of a planar PSVF with the two-sided shift map in sequence spaces. This approach is absolutely new in the literature and estates tools of discrete dynamics that can be used in order to prove results concerning PSVFs.

The fact that the Existence and Uniqueness Theorem of solutions does not apply to PSVFs adds an extra difficulty when studying invariant measures in this scenario. However, there are ways to deal with this problem and a necessary condition for this is given by [38]. Our focus will be on natural invariant measures referring to PSVFs that are absolutely continuous with respect to the Lebesgue measure and its corresponding density invariant, where density can be considered as a set of initial conditions. The dynamics action in this case is described by the Perron-Frobenius operator $\mathscr{L}$. Invariant densities are the sets fixed under the linear operator $\mathscr{L}$, that is, they are eigenfunctions with eigenvalues 1.

In the present work, we construct the Ruelle-Perron Frobenius operator restricted to a subset of global trajectories of a PSVF for an expanding and topologically mixing map. Firstly, on this subset of global trajectories, we construct a metric making this subset a metric space. The Ruelle-Perron-Frobenius operator will be obtained through the geometric potential for the time-one map. However, the time-one map is not well defined in this context, since, after a time $t = 1$, the same starting point can be associated with arcs of different trajectories. To deal with this problem, which is the biggest hurdle to overcome, we consider a quotient space, for which two trajectories are equal if and only if they have the same itinerary. In addition, the quotient set will also be a metric space with the induced metric. Thus, time-one map is well defined for the considered PSVFs, and as we will see it will be an expanding and topologically mixing map.

In Chapter 3 we present our first contributions. Using the time-one map induced in this quotient space, we build conditions for a subset of trajectories of a PSVF to be associated with a finite-type subshift through a topological conjugacy. Once this topological conjugacy is established, we actually show that under certain conditions, there is an ergodic equivalence between the space of shifts with the Bernoulli measure and the space of the global trajectories of a PSVF with the Lebesgue measure. Since entropy and topological pressure are preserved via topological conjugacy, we can use such properties of shift spaces to obtain the entropy and topological pressure for the PSVF. Furthermore, we present examples of PSVFs whose entropy for a subset contained in the quotient space of all trajectories of these PSVFs is r, for a positive **real number** r. Finally, we use the relationship between topological entropy, Hausdorff dimension and Minkowski dimensions (box dimension) for shifts of finite type given by Simpson [48] to calculate such quantities for PSVFs.

In stochastic processes we say that $\pi = (\pi(i))_{i\in\Xi}$ is a probability distribution over a state space Ξ if $\pi(i) \geq 0$ for all $i \in \Xi$ and the sum of $\pi(i)$ is equal to one. Given a Markov chain, let us denote by π_t the probability distribution at time t. Suppose that the Markov Chain starts with an initial distribution $\pi_0 = \pi$, and that $\pi_t = \pi$ for all t (discrete or continuous). When this occurs, π is called a *stationary distribution*. Now, consider $\varepsilon > 0$, a stationary distribution $\tilde{\pi}_t$ and another distribution $\tilde{\tilde{\pi}}_t$. The *mixing time* t_{mix} is the first positive number such that the distance between $\tilde{\pi}_t$ and $\tilde{\tilde{\pi}}_t$ is smaller that ε. In other words mixing time helps to define a method of measuring how long it takes Markov chains to converge to their stationary distributions. In the classical theory of *discrete dynamical systems*, the *relaxation time* t_{rel} is the rate at which a chaotic system "mixes" the state space and it is related to the second largest eigenvalue of the *Ruelle-Perron-Frobenius operator*.

Similar to case of the one-sided shifts, still in Chapter 3, we construct a Markov chain associated with PSVFs. Therefore, we use Markov chain theory in order to calculate the relaxation time and estimate the mixing time for PSVFs. In Example 8 and Example 9, we display stationary distributions associated with the respective PSVFs.

Finally, in Chapter 4 we establish conditions in order to obtain a set of trajectories of PSVFs that preserves measure even in the case where sliding motion is allowed. In fact, for planar PSVFs, a sliding region (see the precise definition in Chapter 2) transforms a small open set $A \subset \mathbb{R}^n$ (with positive measure) into a line segment (with null measure). However, the escaping region has the power to transform this same line segment into a positive measure set. Along this work we deal with these transformations and we will be particularly interested in the case where the escaping region produces a set having the same measure of the original set A. In order to do this, we construct an appropriated set of trajectories and introduce a measure on it.

In addition to the results for entropy and topological pressure, Hausdorff and Minkowski dimensions, obtained in this thesis, the importance of this work lies in studying PSVFs that *preserve measure*, which allows us to take an ergodic approach to this class of PSVFs. As far as we know, this approach is completely new in the literature and constitutes a *very powerful tool* in the study of PSVFs that opens a new horizon in the theory.

CHAPTER 2

Preliminary

2.1 Preliminary

2.1.1 Basic Notations on PSVFs

Definition 1. *A piecewise-smooth vector field is a triple* (M, Σ, Z) *where*

(i) *M is a suitable manifold;*

(ii) *Σ is formed by a finite union of simple curves $\Sigma = \Sigma_1 \dot{\cup} \cdots \dot{\cup} \Sigma_n$ splitting M into $n+1$ connected components regions R_i, where $\Sigma_i = f_i^{-1}(0)$ and $f_i : M \to \mathbb{R}$ are smooth functions having 0 as regular value, $i = 1, \cdots, n$, (that is, $\nabla f_i(p) \neq 0$, for $p \in f_i^{-1}(0)$);*

(ii) *Z is a collection of $n+1$ vector fields of class C^r defined on M, say $Z = (X^1, \cdots, X^{n+1})$, being each X^i defined on the closure of R_i.*

We shall denote a PSVF by Z in stead of the triple (M, Σ, Z) unless there is some confusion on M or Σ. We call Σ the **switching manifold** and we notice that Z is multi-valuated on Σ. In particular, every component X^i of Z is a vector field defined on whole M which has been restricted to R_i. Because Z is multi-valuated on each connected component of Σ, it is necessary to establish some rule describing how trajectories interact to Σ, switching to one side of Σ to another or even remaining on it. In this thesis, we adopted the Filippov convention that we will describe below. We will do the case where Σ is formed by a single curve that separates an open set $V \subset M$ into two regions. In addition, we will fix the notations that we will use from now on. For the general case of Filippov's convention, in which Σ is formed by the union of n simple curves see [5, 22].

Let V be an open set of $\mathbb{R}^n$. Consider a manifold $\Sigma \subset V$ of codimension 1 in $\mathbb{R}^n$ given by $\Sigma = f^{-1}(0) = \{q \in V : f(q) = 0\}$, where $f : V \to \mathbb{R}$ is smooth having $0 \in \mathbb{R}$ as a regular value. Consider Σ a switching manifold whose boundary separates the regions $\Sigma^+ = \{q \in V : f(q) \geq 0\}$ and $\Sigma^- = \{q \in V : f(q) \leq 0\}$.

Call $\mathfrak{X}^r$ the space of the C^r-vector fields in $V \subset \mathbb{R}^n$ endowed with the C^r-topology, with $r \geq 1$ large enough depending on the need. Call $\mathscr{Z}^r$ the space of PSVFs $Z : V \to \mathbb{R}^n$ such that

$$Z(q) = \begin{cases} X^+(q)\ if\ q \in \Sigma^+ \\ X^-(q)\ if\ q \in \Sigma^- \end{cases}, \tag{2.1}$$

where $X^+ = \left(X_1^+, X_2^+, \cdots, X_n^+\right)$, $X^- = \left(X_1^-, X_2^-, \cdots, X_n^-\right) \in \mathfrak{X}^r$.

We denote (2.1) simply by $Z = (X^+, X^-)$ when there is no confusion about the switching manifold. We equip $\mathscr{Z}^r$ with the product topology, i.e.,

$$\| Z \|_{C^r} = max\left\{| X^+ |_{C^r}, | X^- |_{C^r}\right\},$$

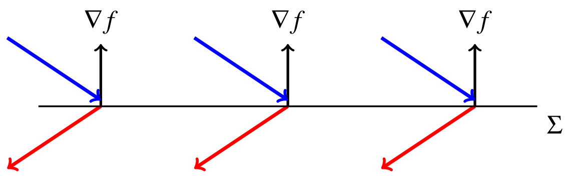
where $| \cdot |_{C^r}$ denotes the classical C^r-norm of the smooth vector fields X^+ and X^- restricted to Σ^+ and Σ^-, respectively.

In order to define rigorously the flow of Z passing through a point $p \in V$, we distinguish whether this point is at $\Sigma^{\pm} \setminus \Sigma$ or Σ. For the first two regions, the local trajectory is defined as being the one gives by X^+ and X^- respectively, as usual, but for Σ we rely on the contact between the vector fields X^+, X^- and Σ characterized by the Lie derivative $X^{\pm}f(q) = \langle \nabla f(q), X^{\pm}(q)\rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product. We also use higher order derivatives given by $(X^{\pm})^k f = (X^{\pm})((X^{\pm})^{k-1} f) = \left\langle \nabla (X^{\pm})^{k-1} f, X^{\pm}\right\rangle$, with $k > 1$ a positive integer. Using the Lie derivatives, it appears the following generic regions on Σ:

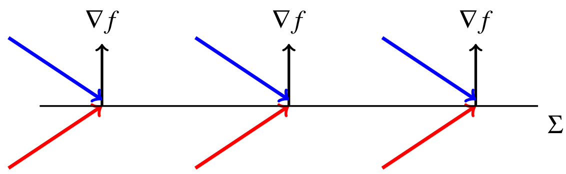
- **Crossing Region** is defined by $\Sigma^c = \{p \in \Sigma \mid X^+f(q)X^-f(q) > 0\}$; In addition we denote $\Sigma^{c^+} = \{p \in \Sigma \mid X^+f(q) > 0, X^-f(q) > 0\}$ and $\Sigma^{c^-} = \{p \in \Sigma \mid X^+f(q) < 0, X^-f(q) < 0\}$;

- **Sliding Region :** $\Sigma^s = \{p \in \Sigma \mid X^+f(q) < 0, X^-f(q) > 0\}$;

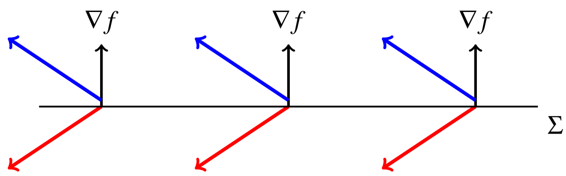
- **Escaping Region :** $\Sigma^e = \{p \in \Sigma \mid X^+f(q) > 0, X^-f(q) < 0\}$.



(a) Cossing Region



(b) Sliding Region



(c) Escaping Region

Any $q \in \Sigma$ such that $X^+f(q)X^-f(q) = 0$ is called a boundary singularity. The boundary singularities can be of two types: (i) an equilibrium of X^+ or X^- over Σ or (ii) a point where a trajectory of X^+ or X^- is tangent to Σ (and it is not an equilibrium of X^+ or X^-). In the second case, we call $q \in \Sigma$ a tangential singularity (or tangency point) and we denote the set of these points by Σ^t. If there exists an orbit of the vector field $X^+ |_{\Sigma^+}$ (respectively $X^- |_{\Sigma^-}$) reaching $q \in \Sigma^t$ in a finite time such that the trajectory continues in Σ^+ (respectively Σ^-), then such tangency is called a visible tangency for X^+ (respectively X^-), otherwise we call q an invisible tangency for X^+ (respectively X^-).

In the case $X^+f(p) = 0$, the trajectories of X^+ are tangent to Σ in p and we say that p is a tangential singularity of X^+. A tangential singularity $p \in \Sigma$ is a fold point of X^+ if $X^+f(p) = 0$, but $(X^+)^2f(p) \neq 0$. Moreover, $p \in \Sigma$ is a visible (respectively, invisible) fold point of X^+ if $X^+f(p) = 0$ and $(X^+)^2f(p) > 0$ (respectively, $(X^+)^2f(p) < 0$). Analogously for X^- reversing the last two inequalities. When p is a fold point for both X^+ and X^-, we say that p is a fold–fold singularity or two-fold singularity. A two-fold is called

1. visible-visible, if it is a visible tangency for both X^+ and X^-;

2. invisible-invisible, if it is an invisible tangency for X^+ and X^-;

3. visible-invisible, whether it is a visible tangency for X^+ and an invisible tangency for X^- or vice versa.

The trajectories of a PSVF passing through a crossing point are defined as the concatenation of the trajectories of X^+ and X^- by that point since the vector fields X^+ and X^- point in the same direction. However, in the sliding and escaping regions, we need to define an auxiliary vector field. So, we consider the Filippov's convention [24, 35] to define a new vector field on $\Sigma^s \cup \Sigma^e$. This new vector field, called sliding vector field, is a convex linear combination of $X^+(p)$ and $X^-(p)$ in such a way that Z^s is tangent to Σ in the cone generated by $X^+(p)$ and $X^-(p)$. See Figure 2.2

Definition 2. *The* ***sliding vector field*** $Z^s : \Sigma^s \cup \Sigma^e \to \mathbb{R}^n$ *is defined as*

$$Z^s(p) = (1-\delta)X^+(p) + \delta X^-(p),$$

where for each $p \in \Sigma^s \cup \Sigma^e$, the value of δ is chosen such that $\langle \nabla f(p), Z^s(p) \rangle = 0$, i.e.,

$$\delta = \delta(p) = \frac{X^+ f(p)}{X^+ f(p) - X^- f(p)}$$

provided that the denominator of the previous expression does not vanish.

When $\Sigma^s \cup \Sigma^e \neq \emptyset$, the sliding vector field can be extend to $\overline{\Sigma^s} \cup \overline{\Sigma^e}$. Note that, $\delta \in (0,1)$ for all $p \in \Sigma^s \cup \Sigma^e$, while $\delta = 0$ implies that $X^- f(p) = 0$, i.e., p is a tangency point of the vector field X^- with the boundary Σ, and $\delta = 1$ implies that $X^+ f(p) = 0$, i.e., p is a tangency point of the vector field X^+ with the boundary Σ. A point $p \in \Sigma^s \cup \Sigma^e$ such that $Z^s(p) = 0$ is called a **pseudo equilibrium** of Z.

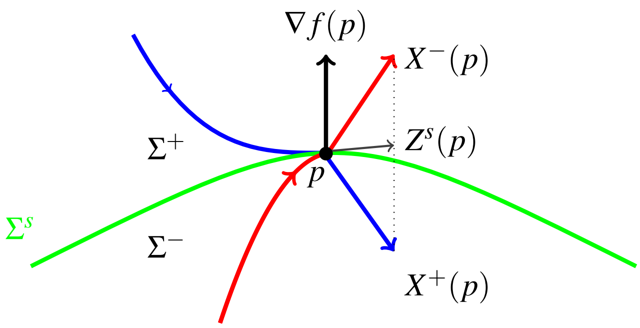


Figure 2.2: Sliding Vector

Note that that if $p \in \Sigma^s$ then $p \in \Sigma^e$ for $(-Z)$. So we can define the *escaping vector field Z^e* on Σ^e associated to Z by $Z^e = -(-Z)^s$. We will use the notation Z^T to both, Z^s and Z^e.

The following definition establishes the classical convention about the trajectories of a PSVF:

Definition 3. *The local trajectory (orbit) $\phi_Z(t,p)$ of a PSVF $Z = (X^+, X^-)$ through a small neighborhood of $p \in U$ is defined as follows:*

(i) For $p \in \Sigma^+ \setminus \Sigma$ and $p \in \Sigma^- \setminus \Sigma$ the trajectory is given by $\phi_Z(t,p) = \phi_{X^+}(t,p)$ and $\phi_Z(t,p) = \phi_{X^-}(t,p)$ respectively.

(ii) For $p \in \Sigma^{c^+}$ and taking the origin of time at p the trajectory is defined as $\phi_Z(t,p) = \phi_{X^-}(t,p)$ for $t \leq 0$ and $\phi_Z(t,p) = \phi_{X^+}(t,p)$ for $t \geq 0$. If $p \in \Sigma^{c^-}$ the definition is the same reversing the time;

(iii) For $p \in \Sigma^e$ and taking the origin of time at p the trajectory is defined as $\phi_Z(t,p) = \phi_{Z^s}(t,p)$ for $t \leq 0$ and $\phi_Z(t,p)$ is either $\phi_{X^+}(t,p)$ or $\phi_{X^-}(t,p)$ or $\phi_{Z^s}(t,p)$ for $t \geq 0$. For $p \in \Sigma^s$ the definition is the same reversing the time;

(iv) For p being a tangential singularity and taking the origin of time at p the trajectory is defined as $\phi_Z(t,p) = \phi_1(t,p)$ for $t \leq 0$ and $\phi_Z(t,p) = \phi_2(t,p)$ for $t \geq 0$, where each ϕ_1, ϕ_2 is either ϕ_{X^+} or ϕ_{X^-} or ϕ_{Z^T} ;

(v) *For $p \in V \subset \mathbb{R}^2$ a singular tangency point, $\phi_Z(t,p) = p$ for all $t \in \mathbb{R}$.*

Definition 4. *A global trajectory $\Gamma_Z(t,p_0)$ is a concatenation of local trajectories. Moreover, a maximal trajectory is a global trajectory that can not be extended to any other global trajectories by joining local ones, that is, if $\widetilde{\Gamma_Z}$ is a global trajectory containing Γ_z then $\Gamma_z = \widetilde{\Gamma_Z}$. In this case, we call $I_{max} = (\tau^-(p_0), \tau^+(p_0))$ the maximal interval of the solution Γ_Z. A global trajectory is a **positive** (respectively, **negative**) global trajectory if $t > 0$ (respectively, $t < 0$) and $t_0 = 0$. **We will denote by Λ is the set of all global trajectories of Z.***

Remark 1. *The maximal interval of the solution may not cover the interval $(-\infty,\infty)$, that is, $\tau^{\pm}(\Gamma_Z, p_0)$ could be finite values. When there is no danger of confusion, we will prefer use the notation $I_{max} = (\tau^-(p_0), \tau^+(p_0))$ instead of $I_{max}^{\Gamma_Z} = (\tau^-(\Gamma_Z, p_0), \tau^+(\Gamma_Z, p_0))$.*

In Chapter 3, we will only consider planar PSVFs. In this case, we say that a tangency point $p \in V$ is singular if p is an invisible-invisible tangency for both X^+ and X^-. On the other hand, a tangency point $p \in V$ is regular if it is not singular.

Definition 5. *Consider an n-dimensional PSVF $Z \in \mathscr{Z}^r$. The set*

$$Sat(A) = \bigcup_{\phi_Z \in \Lambda} \bigcup_{p \in A} \phi_Z(t,p) \text{ and } t \in I_{max} = (\tau^-(p), \tau^+(p))$$

*will be called the **saturation** of the set $A \subset \mathbb{R}^n$.*

Definition 6. *A set A is Z-invariant if for each $p \in A$ and any global trajectory $\Gamma_Z(t,p)$ passing through p it holds $\Gamma_Z(t,p) \subset A$.*

Definition 7. *A set $B \subset \mathbb{R}^n$ is minimal for a PSVF Z if*

(i) *$B \neq \emptyset$;*

(ii) *B is compact;*

(iii) *B is Z-invariant;*

(iv) *B does not contain proper subset satisfying $(i), (ii)$, and (iii).*

Definition 8. *A PSVF Z is topologically transitive if given two arbitrary open sets $\mathscr{U}$ and $\mathscr{V}$ of $A \subseteq \mathbb{R}^n$, there exist a global trajectory γ connecting these sets.*

In our discussion, it is crucial that the volume measure is preserved in PSVFs, which is not very simple to obtain, however there is a class of piecewise smooth systems satisfying $X^+ f(p) = X^- f(p)$, $p \in \Sigma$, this class constitutes a well-known class of Filippov systems called **refractive systems** (see Liouville's Lemma in [10]). In chapter 3, we work with refractive systems which, in addition, satisfy $div(X^{\pm}) = 0$ in $\Sigma^{\pm}$, in other words, preserves the volume measure in $\Sigma^{\pm}$ (see [53]). Therefore, by Corollary A in [38], the PSVFs preserves the volume measure, that is, the Lebesgue measure, here denoted by *med*.

Definition 9. *Let $Z = (X^+, X^-)$ a PSVF defined over a compact 2-dimensional surface M and $\Lambda = \{\gamma : \text{global trajectory of } Z\}$. Define $\rho : \Lambda \times \Lambda \to \mathbb{R}$ by:*

$$\rho(\gamma_1, \gamma_2) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \int_i^{i+1} \mid \gamma_1(t) - \gamma_2(t) \mid dt,$$

where $| \cdot |$ denotes the distance between the points $\gamma_1(t)$ and $\gamma_2(t)$.

Remark 2. *Note that if p is a pseudo-equilibrium point, the time stop, since $\gamma(t) = \phi^t_{Z^s}(p) = p$, for all $t \in I_{max}$.*

Proposition 1. *The space (Λ, ρ) is a metric space.*

Proof. Let $\gamma_1, \gamma_2 \in \Lambda$. Observe that M being compact, implies $|\gamma(t) - \gamma_2(t)|$ isuniformly bounded for all $t \in \mathbb{R}$, thus the series above converges for any γ_1, γ_2. If $\rho(\gamma_1, \gamma_2) = 0$ then $\int_i^{i+1} |\gamma_1(t) - \gamma_2(t)| dt = 0$ for all $i \in \mathbb{N}$ which implies $\gamma_1(t) = \gamma_2(t)$ for all $t \in \mathbb{R}$ and therefore by continuity $\gamma_1 = \gamma_2$. The fact $\rho(\gamma_1, \gamma_2) = \rho(\gamma_2, \gamma_1)$ follows immediately from $|\gamma_1(t) - \gamma_2(t)| = |\gamma_2(t) - \gamma_1(t)|$. And, finally, for the triangle inequality part it is enough to notice that $|\gamma_1(t) - \gamma_3(t)| \leq |\gamma_1(t) - \gamma_2(t)| + |\gamma_2(t) - \gamma_3(t)|$ for all $t \in \mathbb{R}$ gives the inequality $\rho(\gamma_1, \gamma_3) \leq \rho(\gamma_1, \gamma_3) + \rho(\gamma_2, \gamma_3)$. □

Let $\Omega = \{\text{positive global trajectories of } Z\}$. Consider the map:

$$\begin{aligned} T : \mathbb{R}^+ \times \Omega &\to \Omega \\ (t, \gamma) &\mapsto T(t, \gamma)(\cdot) = \gamma(\cdot + t). \end{aligned} \tag{2.2}$$

Then we have the time one map $T_1(\gamma) = T(1, \gamma)(.) = \gamma(\cdot + 1)$.

Remark 3. *Note that (Ω, ρ), with $i \in \mathbb{N}$ is also a metric space. Furthermore, $\mathbb{R}^+ = \{t \in \mathbb{R} | t \geq 0\}$.*

Proposition 2. *The map $T_1 : \Omega \to \Omega$ defined above is continuous.*

Proof. Note that

$$\int_i^{i+1} \mid T_1(\gamma_1)(t) - T_1(\gamma_2)(t) \mid dt = \int_i^{i+1} \mid \gamma_1(t+1) - \gamma_2(t+1) \mid dt = \int_{i+1}^{i+2} \mid \gamma_1(t) - \gamma_2(t) \mid dt.$$

Using the relation above, we obtain:

$$\rho\left(T_1(\gamma_1)(t), T_1(\gamma_2)(t)\right) = \sum_{i \in \mathbb{N}} \frac{1}{2^i} \int_i^{i+1} \mid T_1(\gamma_1)(t) - T_1(\gamma_2)(t) \mid dt =$$

$$\sum_{i \in \mathbb{N}} \frac{1}{2^i} \int_{i+1}^{i+2} \mid \gamma_1(t) - \gamma_2(t) \mid dt = \lim_{n \to +\infty} \left(\sum_{i=0}^{n} \frac{1}{2^i} \int_{i+1}^{i+2} \mid \gamma_1(t) - \gamma_2(t) \mid dt \right) =$$

$$\lim_{n \to +\infty} \left(2 \sum_{i=0}^{n} \frac{1}{2^{i+1}} \int_{i+1}^{i+2} \mid \gamma_1(t) - \gamma_2(t) \mid dt \right) = \lim_{n \to +\infty} \left(2 \sum_{j=1}^{n+1} \frac{1}{2^j} \int_j^{j+1} \mid \gamma_1(t) - \gamma_2(t) \mid dt \right)$$

$$\leq \lim_{n \to +\infty} \left(2 \sum_{j=1}^{n+1} \frac{1}{2^j} \int_j^{j+1} \mid \gamma_1(t) - \gamma_2(t) \mid dt \right) + \int_0^1 \mid \gamma_1(t) - \gamma_2(t) \mid dt =$$

$$2 \lim_{n \to +\infty} \left(\sum_{j=1}^{n+1} \frac{1}{2^j} \int_j^{j+1} \mid \gamma_1(t) - \gamma_2(t) \mid dt + \frac{1}{2} \int_0^1 \mid \gamma_1(t) - \gamma_2(t) \mid dt \right) = 2\rho(\gamma_1, \gamma_2).$$

Hence T_1 is continuous. □

2.1.2 Random Variables and Markov Chains

Definition 10. *A **probability space** is a set Ξ, together with a family of subsets of Ξ whose elements are called events. Events satisfy the following closure properties:*

(i) *Ξ is an event;*

(ii) *If $B_1, B_2, \cdots$ are all events, then the union $\bigcup_{i=1}^{\infty} B_i$ is also an event;*

(iii) *If B is an event, so is $\Xi \setminus B$.*

Given a probability space, a probability measure is a non-negative function *Prob* defined on events and satisfaying the probability axioms:

(i) $Prob(\Xi) = 1$;

(ii) For any sequence of events $B_1, B_2, \cdots$ which are mutually disjoint, meaning $B_i \cap B_j = \emptyset$ for i, j,

$$Prob\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} Prob(B_i).$$

If Ξ is a countable set, a **probability distribution** on Ξ is a function $\pi : \Xi \to [0,1]$ so that $\sum_{\zeta \in \Xi} \pi(\zeta) = 1$. For any subset $B \subset \Xi$

$$\pi(B) = \sum_{\zeta \in B} \pi(\zeta).$$

The set function $B \mapsto \pi(B)$ is a probability measure.

Given a set Ξ (not necessarily enumerable) with a σ-algebra $\mathscr{F}$, a function $g : \Xi \to \mathbb{R}$ is called **measurable** if $g^{-1}(B)$ is an element of $\mathscr{F}$ for all open sets B. If $\Xi = D$ is an open subset of $\mathbb{R}^n$ and $g : D \to [0, \infty)$ is a measurable function satisfying $\int_D g(x)\, dx = 1$, then g is called a *density function*. Given a density function, the set function defined for Borel sets B by

$$\mu_g(B) = \int_B g(x)\, dx$$

is a probability measure (here, the integral is the Lebesgue integral and it agrees with the usual Riemann integral wherever the Riemann integral is defined).

Given a probability space $(\Xi, \mathscr{F}, Prob)$, a **random variable** $\mathscr{X}$ is a measurable function defined on Ξ. We will use the notation $\{\mathscr{X} \in A\}$ as an abbreviation for the set

$$\{w \in \Xi; \mathscr{X}(w) \in A\} = \mathscr{X}^{-1}(A).$$

The distribution of a random variable $\mathscr{X}$ is the probability measure $\mu_{\mathscr{X}}$ on $\mathbb{R}$ defined for Borel sets B by

$$\mu_{\mathscr{X}}(B) := Prob(\mathscr{X} \in B) := Prob(\{x \in B\}).$$

Definition 11. *We call a random variable $\mathscr{X}$* **discrete** *if there is a finite or countable set S, called the* ***support*** *of $\mathscr{X}$, such that $\mu_{\mathscr{X}}(S) = 1$. In this case, the function*

$$\pi_{\mathscr{X}}(a) = Prob(\mathscr{X} = a)$$

is a probability distribution on S.

A set of random variables $(\mathscr{X}_m)_{m\in M}$ indexed by a set M, defined in a probability space $(\Xi, \mathscr{F}, Prob)$ and taking values in a set S, called the state space, is a **stochastic process**.

A random variable $\mathscr{X}$ is called absolutely continuous if there is a density function g on $\mathbb{R}$ such that

$$\mu_{\mathscr{X}}(A) = \int_A g(x)\,dx.$$

For $A \subset \Xi$, we define the function $\mathbf{1}_A : \Xi \to \mathbb{R}$ by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{2.3}$$

which is called the indicator function of A. $\mathbf{1}_A$ is a random variable on $(\Xi, \mathscr{F})$ if and only if $A \in \mathscr{F}$. We say that a random variable $\mathscr{X}$ on $(\Xi, \mathscr{F})$ is simple if there are $A_1, \cdots, A_n \in \mathscr{F}$, $A_i \cap A_j = \emptyset$, $i \neq j$, such that $\mathscr{X} = \sum\limits_{i=1}^{n} a_i \mathbf{1}_{A_i}$ with $a_1, \cdots, a_n \in \mathbb{R}$.

We define

$$\mathbb{E}[\mathscr{X}] = \sum_{i=1}^{n} a_i Prob(A_i).$$

For a discrete random variable $\mathscr{X}$, the expectation $\mathbb{E}(\mathscr{X})$ can be computed by the formula

$$\mathbb{E}[\mathscr{X}] = \sum_{x\in\mathbb{R}} x Prob(\mathscr{X} = x).$$

For an absolutely continuous random variable $\mathscr{X}$, the expectation is computed by the formula

$$\mathbb{E}[\mathscr{X}] = \int_{\mathbb{R}} x \cdot g_{\mathscr{X}}(x)\,dx.$$

Definition 12. *Let $(M, \mathscr{M}, Prob)$ be a probability space and let $F : M \to M$ be a measurable map. The map F is said to be measure-preserving if for every $A \in \mathscr{M}$,*

$$Prob(F^{-1}(A)) = Prob(A).$$

The triple $(M, \mathscr{M}, Prob, F)$ is then said to be a measure-preserving system.

Definition 13. *A stochastic process $(\mathscr{X}_m)_{m\in\mathbb{N}}$ have the Markovian property if*

$$Prob(\mathscr{X}_{m+1} = j \mid \mathscr{X}_0 = i_0, \cdots, \mathscr{X}_{m-1} = i_{m-1}, \mathscr{X}_m = i) = \tag{2.4}$$

$$Prob(\mathscr{X}_{m+1} = j \mid \mathscr{X}_m = i),$$

for $m \in \mathbb{N}$ and $i, j, i_0, \cdots i_{m-1} \in \mathscr{A}_k = \{0, 1, \cdots k-1\}$.

Notation 1. *The symbol $|$ in Definition 13 reads "such that".*

The Markovian property says that the conditional probability of any future "event", given "any" past events and the present state $\mathscr{X}_m = i$, is independent of past events and depends only on the current state. A sequence of random variables $(\mathscr{X}_m)_{m\in\mathbb{N}}$ is a **Markov chain** if it has the Markovian property. A Markov chain is defined by its transition matrix $W = (w(i,j))_{k\times k}$, where

$$w(i,j) = Prob\,(\mathscr{X}_{m+1} = j \mid \mathscr{X}_m = i) \quad \forall i,j \in \mathscr{A}_k \quad \text{and} \quad m \in \mathbb{N}.$$

In order to compute the n$-$th transition matrix, we can simply use matrix multiplication to get the desired result and $w_{ij}^{(n)}$ will denote the element of row i and column j of W^n.

Definition 14. *Let $(\mathscr{X}_n)_{n\in\mathbb{N}}$ be a Markov chain with state space $\mathscr{A}_k$. State j is said to be **accessible** from state i if*

$$w_{ij}^{(m)} = Prob(\mathscr{X}_m = i \mid \mathscr{X}_0 = j) > 0 \quad \textit{for some} \quad m \geq 0.$$

*We say that the states i,j **communicate** if*

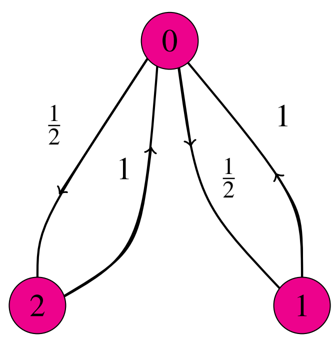
$$w_{ij}^{(m)} = Prob(\mathscr{X}_m = i \mid \mathscr{X}_0 = j) > 0 \quad \textit{and} \quad w_{ij}^{(\widetilde{m})} = Prob(\mathscr{X}_{\widetilde{m}} = j \mid \mathscr{X}_0 = i) > 0,$$

*for some $m, \widetilde{m} \geq 0$. A Markov chain is said to be **irreducible** if all its states communicate.*

Now, we will introduce an important property. Suppose $\mathscr{A}_k = \{0,1,2\}$ such that we can only return to the state 0 in, say, even time. It's not too hard to believe that this type of chain exists. In this case, the state 0 presents a very peculiar behavior, the visit time to the state 0 displays a periodicity. Below we will define this feature rigorously and then we will present examples to improve understanding.

Definition 15. *Consider the set $\mathfrak{W}(i) := \{n \geq 1 \mid w^{(n)}(i,i) > 0\}$ the set of times in the chain in which it is possible to return to the initial position i. The Period of state i is defined as the greatest common divisor of the set $\mathfrak{W}(i)$. For an irreducible chain, the period of the chain is defined to be the period that is common to all states. The chain will be called **aperiodic** if all states have period 1. If a chain is not aperiodic we will call it **periodic**.*

Example 1. *Consider the Markov chain represented by the following graph :*



The transition matrix is

$$W = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

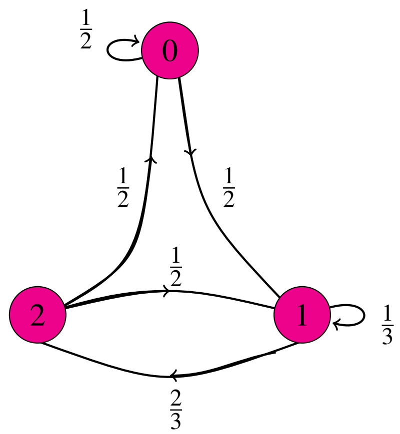
It is not difficult to see that this chain is periodic with period 2 *and*

$$W^{2n} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad W^{2n-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proposition 3. *Any irreducible Markov chain that has at least one "self-loop" (i.e., one state i for which $Prob(\mathscr{X}_n = i | \mathscr{X}_{n-1} = i) > 0$), is aperiodic.*

Proof. Suppose state i has a self-loop. From any state j, the chain can eventually get to i (by irreducibility), and use the self-loop any number of times, and then return to j (by irreducibility), rendering the greatest common divisor of timesteps at which we could have returned to state j to be 1. □

Example 2. *Consider the Markov chain represented by the following graph :*



The transition matrix is

$$W = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Note that

$$W^2 = \begin{pmatrix} \frac{1}{4} & \frac{5}{12} & \frac{1}{3} \\ \frac{1}{3} & \frac{4}{9} & \frac{2}{9} \\ \frac{1}{4} & \frac{5}{2} & \frac{1}{3} \end{pmatrix}.$$

Since each entry of the matrix W^2 is positive, it follows that W is an irreducible matrix, and therefore the same happens for the Markov chain. The chain is aperiodic since there is a self-loop, e.g., $w_{11} > 0$.

Definition 16. *The probability of first visit at state j after m steps, starting from state i,is:*

$$r_{ij}^{(m)} = Prob(\mathscr{X}_n = j, \mathscr{X}_1 \neq j, \mathscr{X}_2 \neq j, \cdots, \mathscr{X}_{m-1} \neq j \mid \mathscr{X}_0 = i).$$

The expected number of steps to arrive for the first time at state j starting from i is:

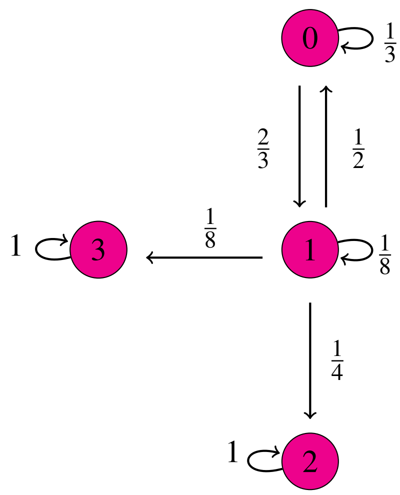
$$h_{ij} = \sum_{m>0} m \cdot r_{ij}^{(m)}.$$

The probability of a visit (not necessarily for the first time) at state j, starting from state i, is:

$$f_{ii} = \sum_{m>0} r_{ij}^{(m)}.$$

If $f_{ij} < 1$ then there is a positive probability that the Markov chain never arrives at state j, so in this case $h_{ij} = \infty$. A state i for which $f_{ii} < 1$ (i.e. the chain has positive probability of never visiting state i again) is a **transient state** . If $f_{ii} = 1$ then the state is called **recurrent**. More so, if state i is recurrent, but $h_{ii} = \infty$ is **recurring null**. If is recurrent and $h_{ii} \neq \infty$ is **positive recurrent**. Every state is either recurrent or transient.

Example 3. *Consider a Markov chain represented by the following graph:*



The transition matrix is

$$W = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The probability of starting from 0*, moving to* 1*, staying there for one time step and then moving back to* 0 *is:*

$$Prob\left(\mathscr{X}_3 = 0, \mathscr{X}_2 = 1, \mathscr{X}_1 = 1 \mid \mathscr{X}_0 = 0\right) = w_{12} \cdot w_{22} \cdot w_{21} = \frac{2}{3} \cdot \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{24}.$$

The probability of moving from 0 *to* 0 *in two steps is:*

$$w_{11}^{(2)} = w_{11} \cdot w_{11} + w_{12} \cdot w_{21} = \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} = \frac{4}{9}.$$

The first visit probability from 0 *to* 1 *in two steps is:*

$$r_{01}^{(2)} = w_{11}w_{12} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9},$$

while

$$r_{01}^{(7)} = (w_{11})^6(w_{12}) = \left(\frac{1}{3}\right)^6 \cdot \frac{2}{3} = \frac{2}{3^7},$$

and

$$r_{10}^{(m)} = (w_{22})^{m-1}(w_{21}) = \left(\frac{1}{8}\right)^{m-1} \cdot \frac{1}{2} = \frac{1}{3^{3m-2}},$$

for $m \geq 1$ *(since* $w_{21}^{(0)} = 0$*).*

Therefore, the probability of (eventually) visiting state 0 *from* 1 *is:*

$$f_{01} = \sum_{m>0} \frac{1}{3^{3m-2}} = \frac{4}{7}$$

and the expected number of steps to move from 0 *to* 1 *is:*

$$h_{01} = \sum_{m>0} m r_{01}^{(m)} = \sum_{m>0} m(w_{11})^{m-1} w_{12} = \frac{3}{2}.$$

Note that in this example, 0 *can only be reached from* 1, *(the directed graph is not strongly connected) which makes the Markov chain not irreducible.*

When the state space of a Markov chain is finite, there is an important result that we present below

Theorem 1. *Every irreducible Markov chain with a finite state space is positive recurrent.*

Proof. See [45]. □

Given a discrete distribution $\pi = (\pi(i))_{i \in \mathscr{A}_k}$, and define the tail of the π by $\mathfrak{c}_m \equiv \sum_{i \geq m} \pi(i)$. A fundamental result proved in [33] is that there exists a number $0 < \theta < \infty$ such that the limit

$$\lim_{m \to \infty} \frac{1}{m} \mathfrak{c}_m$$

exist and is equal to $-\theta$ independent of the states. If the Markov chain in $\mathscr{A}_k$ is irreducible, then it is positive recurrent, with the unique stationary distribution π, exist some constant η_i such that

$$\kappa(m) \leq \eta_i e^{-m \cdot \theta^*} \tag{2.5}$$

for each $i \in \mathscr{A}_k$, where $0 < \theta^* < \infty$ is the largest constant for which the inequality (2.5) is satisfied.

Now we will present a simple way to calculate the unique stationary distribution for a Markov chain on $\mathscr{A}_k$. Let $(\mathscr{X}_m)_{m \in \mathbb{N}}$ be a finite irreducible aperiodic Markov chain with state space $\mathscr{A}_k$ and transition matrix W, and let π_m be the distribution of $\mathscr{X}_m$:

$$\pi_m(i) = Prob(\mathscr{X}_m = i) \quad \text{for all } i \in \mathscr{A}_k.$$

By conditioning on the possible predecessors of the $(m+1)$-st state, we see that

$$\pi_{m+1}(j) = \sum_{i \in \mathscr{A}_k} Prob(\mathscr{X}_m = i) w(i,j) = \sum_{i \in \mathscr{A}_k} \pi_m(i) w(i,j) \quad \text{for all} \ \ j \in \mathscr{A}_k. \tag{2.6}$$

The vectorial form of (2.6) is

$$\pi_{m+1} = \pi_m W, \quad \text{for} \quad m \geq 0. \tag{2.7}$$

and hence

$$\pi_m = \pi_0 W^m \quad \text{for} \quad m \geq 0.$$

The Convergence Theorem (Theorem 4.9 of [37]) implies that for a sufficiently large time, a finite irreducible aperiodic Markov chain, with distribution π, satisfies $\lim\limits_{m\to\infty} w_{ij}^{(m)} = \pi(j)$, where $j \in \mathscr{A}_k$. So,

$$\lim_{m\to\infty} \pi_m(j) = \sum_{i \in \mathscr{A}_k} \pi_0(i) \lim_{m\to\infty} w_{ij}^{(m)} = \pi(j),$$

since $\sum\limits_{i \in \mathscr{A}_k} \pi_0(i) = 1$. Therefore, from (2.7) and the uniqueness of the limit, we obtain that π is stationary if and only if $\pi = \pi W$.

Definition 17. *The* ***total variation distance*** *between two probability distributions π and $\widetilde{\pi}$ on $\mathscr{A}_k$ is defined as*

$$\|\pi - \widetilde{\pi}\|_{TV} := \max_{\mathscr{E} \subseteq \mathscr{A}_k} \mid \pi(\mathscr{E}) - \widetilde{\pi}(\mathscr{E}) \mid. \tag{2.8}$$

This definition is explicitly probabilistic: the distance between π and $\widetilde{\pi}$ is the maximum difference between the probabilities assigned to a single event by the two distributions.

It is useful to introduce a parameter that measures the time required by a Markov chain so that the distance to stationarity is small. Thus we present the following definition.

Definition 18. *Let W be an irreducible, aperiodic transition matrix on $\mathscr{A}_k$, and π a stationary distribution. Define the* ***distance function*** *for all $m \in \mathbb{N}$ by:*

$$\kappa(m) := \max_{i \in \mathscr{A}_k} \|w^m(i,\cdot) - \pi\|_{TV}.$$

The ***mixing time*** *(parameterized by ε) of a Markov chain with transition matrix W is defined as*

$$t_{mix}(\varepsilon) := \min\{m : \kappa(m) \leq \varepsilon\}. \tag{2.9}$$

In the Definition 18 weusually take $\varepsilon = \frac{1}{4}$ and in this case we write $t_{mix} := t_{mix}\left(\frac{1}{4}\right)$. The choice of $\frac{1}{4}$ is rather arbitrary any number strictly smaller than $\frac{1}{2}$ would serve. For more details see [37].

Given a Markov chain and a random walk over it, it is interesting to study the cases in which we can do the "reverse chain". Before formally defining it, consider the following situation.

Consider an irreducible Markov chain $(\mathscr{X}_m)_{m\in\mathbb{N}}$ in finite state space $\mathscr{A}_k$ with transition probability matrix W. Fix a positive integer N and define reversed chain $\mathscr{Y}_m := \mathscr{X}_{N-m}$ for $0,1,\cdots N$. Then, $(\mathscr{Y}_0,\cdots,\mathscr{Y}_N) = (\mathscr{X}_N,\cdots,\mathscr{X}_0)$, so $(\mathscr{Y}_m)_{m=0}^N$ is the sequence of states we observe if, starting at time N, we run the original Markov chain "backwards". To justify the name "reversed chain", we present:

Theorem 2. *If the irreducible Markov chain $(\mathscr{X}_n)_{n\in\mathbb{N}}$ starts from the stationary distribution π, then the reverse chain $(\mathscr{Y}_m)_{m=0}^N$ is an irreducible Markov chain with transition probabilities*

$$\widehat{w}(i,j) = \frac{\pi(j)w(j,i)}{\pi(i)} \quad \textit{for } i,j\in\mathscr{A}_k.$$

The stationary distribution for the reverse chain is also π.

Proof. See [37]. □

The Theorem 2 provides conditions for a Markov chain to "look the same" regardless of whether we look into the past or into the future. But for the transition probabilities to be the same in both chains, we need $\widehat{w}(i,j) = w(i,j)$ for all $i,j\in\mathscr{A}_k$, or equivalently,

$$\pi(i)w(i,j) = \pi(j)w(j,i) \quad \text{for all } i,j\in\mathscr{A}_k. \tag{2.10}$$

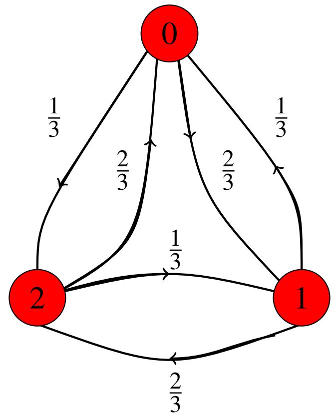
Definition 19. *A Markov chain whose stationary distribution π and transition probability matrix W satisfy (2.10) is called **reversible**.*

Remark 4. *Equation (2.10) is often called the **detailed balance**.*

Proposition 4. *If W and π satisfy (2.10), then π is invariant to W.*

Proof. See [39]. □

Example 4. *Consider the Markov chain represented by the following graph :*



The transition matrix is

$$W = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Note that

$$W^2 = \begin{pmatrix} \frac{4}{9} & \frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{pmatrix}.$$

Since each entry of the matrix W^2 is positive, it follows that W is an irreducible matrix, and therefore the same happens for the Markov chain. Note that $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is invariant, so by Theorem 2, $\widehat{W} = W^T$. However, W is not symmetric, and it follows that $W \neq \widehat{W}$, i. e., (2.10) is not satisfied and therefore this chain is not reversible.

Example 5. *Consider the Markov chain with diagram where $0 < p = 1 - q < 1$. Note that W for this*

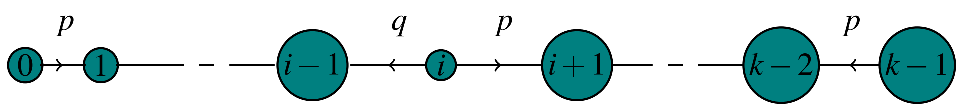


diagram is irreducible and that the non-zero detailed balance equation reads

$$\pi(i)w(i,i+1) = \pi(i+1)w(i+1,i) \quad \textit{for } i = 0, 1, \cdots k-2.$$

So a solution is given by

$$\pi = \left(\left(\frac{p}{q} \right)^i, i = 0, 1, \cdots, k-1 \right)$$

and this may be normalised to give a distribution in the detailed balance with W. Hence this chain is reversible.

Definition 20. ***(Big-O)*** *$g(n) = O(h(n))$ means there exists some constant δ such that $g(n) \leq \delta h(n)$, for large enough n (that is, as $n \to \infty$). We say "g of n is Big-O of h of n".*

Definition 21. ***(Big-Θ)*** *$g(n) = \Theta(h(n))$ means there exists some constants δ_1 and δ_2 such that $\delta_2 h(n) \leq g(n) \leq \delta_1 h(n)$. We say "g of n is Big-Theta of h of n".*

2.1.3 Some words about symbolic dynamics

Consider a set $\mathscr{A}_k = \{0, 1, \cdots, k-1\}$ with k elements and with the discrete topology. Let $\pi = (\pi_1, \cdots, \pi_n)$ be the probability distribution on $\mathscr{A}_k$. Now, consider $\mathscr{A}_k^{\mathbb{N}}$, i.e., all the sequences $x = (x_j)_{j \in \mathbb{N}}$, with $x_j \in \mathscr{A}_k$, for all $j \in \mathbb{N}$ and the product topology of all discrete topologies.

The σ-algebra $\mathscr{B}$ defined on it would be generated by finite unions of cylinder sets where a cylinder set is a subset of $\mathscr{A}_k^{\mathbb{N}}$ determined by a finite number of values, such that:

$$C = \{(x_n)_{n \in \mathbb{N}} : x_i = c_i, -m \leq i \leq n\},$$

where c_i is any fixed symbol in the alphabet. Therefore, there exist a unique measure $\mu_\pi^{\mathbb{N}}$, called the Bernoulli measure, such that if C is a cylinder set, then $\mu_\pi^{\mathbb{N}}(C) = \prod_{i=-m}^{n} \pi_{c_i}$.

Definition 22. *Let $x = (x_j)_{j \in \mathbb{N}}$ and $y = (y_j)_{j \in \mathbb{N}}$ two elements of $\mathscr{A}_k^{\mathbb{N}}$. Define $d : \mathscr{A}_k^{\mathbb{N}} \times \mathscr{A}_k^{\mathbb{N}} \to \mathbb{R}$ by:*

$$d(x,y) = \sum_{i \in \mathbb{N}} \frac{|\, x_i - y_i \,|}{2^i}.$$

Definition 23. *Define $\sigma_k^+ : \mathscr{A}_k^{\mathbb{N}} \to \mathscr{A}_k^{\mathbb{N}}$ given by $\sigma_k^+((a_j)) = (b_j)$, where $b_j = a_{j+1}$. The map σ_k^+ is called one-sided shift and the discrete flow $(\mathscr{A}_k^{\mathbb{N}}, \sigma_k^+)$ is called one-sided shift system.*

The measure space $\left(\mathscr{A}_k^{\mathbb{N}}, \mathscr{B}, \mu_\pi^{\mathbb{N}}, \sigma_k^+\right)$ will be called the **Bernoulli shift** space with distribution π.

Remark 5. *We will use the notation $(M, \mathscr{M}, \mu, g)$, to refer to a system formed by a measure space M (which can be a probability space), a σ -algebra $\mathscr{M}$ on this set, a measure μ (which can be a probability measure) and a continous map $g : M \to M$. Whenever the measure of the system $(M.\mathscr{M}, med, g)$ is the Lebesgue measure med, the σ-algebra $\mathscr{M}$ considered will be the σ-Lebesgue algebra.*

Remark 6. *The same construction as above can be done for $\mathbb{Z}$, i.e., all the sequences $x = (x_j)_{j \in \mathbb{Z}}$, with $x_j \in \mathscr{A}_k$, for all $j \in \mathbb{Z}$ and then map $\sigma_k : \mathscr{A}_k^{\mathbb{Z}} \to \mathscr{A}_k^{\mathbb{Z}}$ is called a two-sided shift.*

In the same way as we did above for a finite type set, we can have shifts on an infinite type set.

Let $\mathscr{A}_\infty \subset \mathbb{R}$ (usually an interval). Consider $\mathscr{A}_\infty^{\mathbb{N}}$ the space of all infinite sequences of real numbers and let $\sigma_\infty^+ : \mathscr{A}_\infty^{\mathbb{N}} \to \mathscr{A}_\infty^{\mathbb{N}}$ be the one-side shift, that is, $\sigma_\infty^+((x_j)_{j \in \mathbb{N}}) = (y_j)_{j \in \mathbb{N}}$, where $y_j = x_{j+1}$.

It is known that σ_∞^+ is measurable with respect to the σ-Borel algebra $\mathscr{B}_\infty$, defined as the smallest σ-algebra containing all events $\{x : x_n \in B\}$, where B is a one-dimensional Borel set. If ξ is a Borel probability measure in $\mathbb{R}$, then the product measure ξ^∞ in $\mathscr{B}_\infty$ is the only probability measure such that

$$\xi^\infty(B_0 \times B_1 \times \cdots \times B_n \times \mathscr{A}_\infty \times \mathscr{A}_\infty \times \cdots) = \prod_{i=0}^{m} \xi(B_i),$$

for all one-dimensional Borel sets $B_0, B_1, \cdots$. Furthermore the shift σ_∞^+ preserves the product measure ξ^∞. The existence and uniqueness of such a measure follows from the Caratheodory Extension Theorem.

Definition 24. *The sequence $(\mathscr{X}_n)_{n\in\mathbb{N}}$ on $\left(\mathscr{A}_\infty^{\mathbb{N}}, \mathscr{B}_\infty, \xi^\infty, \sigma_\infty^+\right)$ is called stationary , if σ_∞^+ is a measure-preserving in $\left(\mathscr{A}_\infty^{\mathbb{N}}, \mathscr{B}_\infty, \xi^\infty, \sigma_\infty^+\right)$.*

A collection of random variables is **independent and identically distributed (i.i.d.)** if each random variable has the same probability distribution as the others and all are mutually independent, i.e., the outcome of one event does not affect the outcome of another.

On the space $\mathscr{A}_\infty^{\mathbb{N}}$ we consider the same metric from those spaces before, that is, given $(x_j)_{j\in\mathbb{N}}$, $(y_j)_{j\in\mathbb{N}} \in \mathscr{A}_\infty^{\mathbb{N}}$ the distance between them is the real number $d(x,y) = \sum\limits_{j\in\mathbb{N}} \frac{|x_j - y_j|}{2^j}$.

Definition 25. *Given $g : M \to M$ and any measure μ in M, we denote by $g_*\mu$ and call iterated (or image) of μ by g, the measure defined by $g_*\mu(B) = \mu(g^{-1}(B))$ for each measurable set $B \subset M$. Note that μ is invariant to g if and only if $g_*\mu = \mu$.*

Definition 26. *Let μ and ν be probability measures invariant under measurable maps $g : M \to M$ and $h : N \to N$, respectively. We say that the systems $(M, \mathfrak{M}, \mu, g)$ and $(N, \mathfrak{N}, \nu, h)$ are **ergodically equivalent** if*

(i) One can find measurable sets $A \subset M$ and $B \subset N$ that are invariant by g and h respectively, i.e., $g(A) \subset A$ and $h(B) \subset B$, with $\mu(A) = 1$ and $\nu(B) = 1$;

(ii) There exists a bijection $L : M \to N$ such that L and L^{-1} are measurable, in such a way that, $h_*\mu = \nu$ and $L \circ g = h \circ L$.

We say that a measurable map $g : M \to M$ is a **Bernoulli map** if $(M, \mathfrak{M}, \mu, g)$ it ergodically equivalent to $\left(\mathscr{A}_k^{\mathbb{N}}, \mathscr{B}, \mu_\pi^{\mathbb{N}}, \sigma_k^+\right)$.

2.1.4 Perron-Frobenius Theory

In probability theory, we call **covariance** of two random variables, $\mathscr{X}$ and $\mathscr{Y}$, the number

$$C(\mathscr{X}, \mathscr{Y}) = \mathbb{E}(\mathscr{X} - \mathbb{E}[\mathscr{X}])(\mathscr{Y} - \mathbb{E}[\mathscr{Y}]) = \mathbb{E}[\mathscr{X}\mathscr{Y}] - \mathbb{E}[\mathscr{X}]\mathbb{E}[\mathscr{Y}].$$

Covariance is a statistical measure where you can compare two random variables, allowing you to understand how they relate to each other. Therefore, given an invariant probability μ of a dynamical system $g : M \to M$ and given measurable functions $\mathfrak{j}, \mathfrak{h} : M \to \mathbb{R}$, we will analyze the evolution of the covariance

$$C_n(\mathfrak{j}, \mathfrak{h}) = C(\mathfrak{j} \circ g^n, \mathfrak{h})$$

when time n goes to infinity. If $\mathfrak{j} = \chi_A$ and $\mathfrak{h} = \chi_B$ are characteristic functions, then $\mathfrak{j}(x)$ gives information about the position of the initial point x, while $\mathfrak{h}(g^n(x))$ tells us the position of the n-th iterate $g^n(x)$.

Definition 27. *Let $g : M \to M$ be a measurable map and let μ be an invariant probability. The sequence of covariances of two measurable functions $\mathfrak{j}, \mathfrak{h} : M \to \mathbb{R}$ is defined by*

$$C(\mathfrak{j}, \mathfrak{h}) = \int (\mathfrak{j} \circ g^n)\, \mathfrak{h}\, d\mu - \int \mathfrak{j}\, d\mu \int \mathfrak{h}\, d\mu \quad ; \quad n \in \mathbb{N}.$$

We say that the system (g, μ) is

(*i*) *weakly-mixing if* $\lim\limits_{n\to\infty} C_n(\chi_A, \chi_B) = \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=0}^{n-1} \mid \mu\left(g^{-i}(A) \cap B\right) - \mu(A)\mu(B) \mid = 0;$

(*ii*) *strong-mixing if* $\lim\limits_{n\to\infty} C_n(\chi_A, \chi_B) = \lim\limits_{n\to\infty} \mu\left(g^{-n}(A) \cap B\right) - \mu(A)\mu(B) = 0.$

for any measurable sets $A, B \subset M$.

Proposition 5. *Let σ_∞^+ be the one-sided shift in $\left(\mathscr{A}_\infty^{\mathbb{N}}, \mathscr{B}_\infty, \xi^\infty\right)$, where the probability measure ξ^∞ is the measure of the product such that under ξ the coordinate variables are i.i.d. with distribution ξ. Then coordinate variables are stationary. Furthermore, σ_∞^+ is a strong-mixing, a weakly-mixing, and therefore ergodic.*

Proof. See [42]. □

We present a topological version of the notion of mixing system.

Definition 28. *Assume that M is a topological space. A map $g : M \to M$ is **topologically mixing**, if given any $\emptyset \neq U, V \subset M$, there exists $n_0 \in \mathbb{N}$ such that $g^{-n}(U) \cap V \neq \emptyset$, for all $n \geq n_0$.*

Definition 29. *Let $(M, \mathfrak{d})$ be a compact metric space and $g : M \to M$ be a continuous map. Let $\psi : M \to \mathbb{R}$ be a continous function, which we call **potential** in M. For the given g and ψ, we can define a linear operator $\mathscr{L} = \mathscr{L}_{g,\psi} : \mathscr{C}^0(M) \to \mathscr{C}^0(M)$ defined in the $\mathscr{C}^0(M)$ space of complex continuous functions by*

$$\mathscr{L}\phi(x) = \sum_{x \in g^{-1}(y)} \psi(x)\phi(x)$$

*for ϕ in a suitable function space on M. The operator we just defined is called a **transfer operator**. If ψ is positive, i.e., $\psi(x) > 0$ for all x in M, then the operator is a positive operator, this means that it maps a positive function to a positive function. A positive transfer operator is also called a **Ruelle-Perron-Frobenius (RPF)** operator.*

Definition 30. *A continuous map $g : M \to M$ in a compact metric space $(M, \mathfrak{d})$ is an expanding map if there exist constants $\delta > 1$ and $\eta > 0$ such that for every $p \in M$ the image of the ball $B(p,\eta)$ contains a neighborhood of the closure of $B(g(p),\eta)$ and*

$$\mathfrak{d}(g(x), g(y)) \geq \delta \mathfrak{d}(x,y) \quad \textit{for every } \ x,y \in B(p,\eta).$$

When $g : M \to M$ a topological mixing and expanding map, a special case of the RPF operator is given by taking $psi(x) = e^{\Psi(x)}$. Thus, we can redefine the RPF operator as follows:

$$\mathscr{L}\phi(y) = \sum_{x \in g^{-1}(y)} e^{\Psi(x)}\phi(x).$$

Finally by the Riesz-Markov Theorem (Theorem 0.3.12 of [53]), the dual of the Banach space $\mathscr{C}^0(M)$ is identified with the vector space $\mathscr{M}(M)$ of measurements complex borelians. So, the dual of the RPF operator is the operator linear $\mathscr{L}^* : \mathscr{M}(M) \to \mathscr{M}(M)$ defined by

$$\int_M hd\,(\mathscr{L}^*(\theta)) = \int_M \mathscr{L}(h)\,d(\theta) \text{ for all } h \in \mathscr{C}^0(M) \text{ and } \theta \in \mathscr{M}(M).$$

The concept of the generalized RPF operator is analogue of the transfer matrix method of classical statistical mechanics. In this work we will use a even more particular case of RPF operator, taking $\Psi = -\beta \log \mid \det J_\mu g \mid$, where $J_\mu g$ is the Jacobian of g with respect to the reference measure μ. So, $\Psi = -\beta \log \mid \det J_\mu g \mid$, is called **geometric potential**. We will also adopt the Lebesgue measure as a reference measure associated with the RPF operator.

Theorem A 1. ***(Perron-Frobenius Theorem)*** *Let $W = [w_{ij}]$ be a irreducible square matrix of order k.*

(i) *There is a nonnegative eigenvalue λ such that no eigenvalue of W has absolute value greater than λ;*

(ii) *We have* $\min_i \left(\sum_{j=1}^k w_{ij}\right) \leq \lambda \leq \max_i \left(\sum_{j=1}^k w_{ij}\right)$;

(iii) *Corresponding to the eigenvalue λ there is a nonnegative left (row) eigenvalue $u = (u_1, \cdots, u_k)$ and a irreducible right (column) eigenvector $v^T = (v_1 \cdots v_k)$;*

(iv) *If W is irreducible then λ is a simple eigenvalue and the corresponding eingenvector are strictly positive (i.e., $u_i > 0, v_i > 0$ for all i);*

(v) *If W is irreducible then λ is the eigenvalue of W corresponding to a irreducible eigenvector.*

Proof. See [54]. □

Lemma 1. *Let W be the transition matrix of a finite Markov chain.*

(i) *If λ is an eigenvalue of W, then $\mid \lambda \mid \leq 1$;*

(ii) *If W is irreducible, the vector space of eigenfunctions corresponding to the eigenvalue* 1 *is the one-dimensional space generated by the column vector* $1^T := (1 \cdots 1)$;

(iii) *If W is irreducible and aperiodic, then* -1 *is not an eigenvalue of* W.

Proof. See [37]. □

By Lemma 1 that the eigenvalues of a transition matrix are in the range $[-1,1]$, so we can name them as follows $1 = \lambda_0 \geq \lambda_1, \cdots \lambda_{k-1} \geq -1$.

Definition 31. *We call* ***absolute spectral gap*** *the difference* $\vartheta_* := 1 - \rho_{ess}(W)$, *where* $\rho_{ess}(W) := \max\{|\ \lambda\ |:$ *is an eigenvalue of* $W, \lambda \neq \pm 1\}$. *And the* ***spectral gap*** *of a reversible chain is defined by* $\vartheta := 1 - \lambda_1$. *The* ***relaxation time*** t_{rel} *of a reversible Markov chain with absolute spectral gap* ϑ_* *is defined as* $t_{rel} := \frac{1}{\vartheta_*}$.

2.1.5 Basic Notation on Topological Pressure

Definition 32. *Let* $\mathfrak{A}$ *and* $\mathfrak{V}$ *be open covers of a compact set* M. *We define its* ***join***, *as the collection of all sets of the form* $U \cup V$, *where* $U \in \mathfrak{A}$ *and* $V \in \mathfrak{V}$, *and denote it by* $\mathfrak{A} \vee \mathfrak{V}$. *Note that this join is a refinement of both coverages. This allows us to construct refinements of a single open cover* $\mathfrak{A}$. *For each* $n \in \mathbb{N}$, *we define*

$$\mathfrak{U}^n := \bigvee_{i=0}^{n-1} g^{-i}(\mathfrak{U}),$$

where $g^{-i}(\mathfrak{A}) := \{g^{-i}(U) : U \in \mathfrak{A}\}$ *and* $g : M \to M$ *be a continuous map.*

Let $g : M \to M$ be a continuous map in a compact metric space with metric $\mathfrak{d}$ and $\phi : M \to \mathbb{R}$ be a potential. For each $n \in \mathbb{N}$, we define $\phi_n : M \to \mathbb{R}$ by $\phi_n(x) = \sum_{i=0}^{n-1} \phi(g^i(x))$ to be the n-th Birkoff sum evaluated at a point $x \in M$ for the potential ϕ. Furthermore, given any nonempty set $\mathfrak{O} \subset M$, we denote

$$\phi_n(\mathfrak{O}) = \sup\{\phi_n(x) : x \in \mathfrak{O}\}.$$

Definition 33. *Let* g *be a continuous transformation on a compact metric space* $(M, \mathfrak{d})$. *Let* ϕ *a potential,* $n \in \mathbb{N}$ *and* $\mathfrak{A}$ *be an open cover of* M. *We denote*

$$P_n(g, \phi, \mathfrak{A}) := \inf \left\{ \sum_{U \in \mathfrak{V}} e^{\phi_n(x)} : \mathfrak{V} \text{ a finite subcover of } \mathfrak{U}^n \right\}. \tag{2.11}$$

Since ϕ is bounded in M by compactness, $P_n(g, \phi, \mathfrak{A})$ is the infimum over a subset of bounded real numbers. Thus $P_n(g, \phi, \mathfrak{A}) < \infty$. We define **the pressure of the potential ϕ with respect to g and the open cover $\mathfrak{A}$** by

$$P_{top}(g,\phi,\mathfrak{A}) := \limsup_{n\to\infty}\frac{1}{n}\log P_n(g,\phi,\mathfrak{A}). \tag{2.12}$$

We want to calculate the pressure as we set the diameter of the cover $\mathfrak{A}$ to zero. The following lemma ensures that this limit exists and does not depend on the choice of covers (see [54]).

Definition 34. *Let $(M,\mathfrak{d})$ be a metric space, we call the diameter of an open cover the sum of the diameters of its elements.*

Lemma 2. *Let $\{\mathfrak{A}_n\}_{n\in\mathbb{N}}$ be any sequence of open covers of M such that*

$$diam(\mathfrak{A}_n) \to 0, \text{ when } n \to \infty.$$

Then the limit $\lim_{n\to\infty} P_{top}(g,\phi,\mathfrak{A}_n)$ exists in $\mathbb{R}\cup\{\infty\}$ and does not depend on the choice of the sequence.

Definition 35. *Let g be a continuous map on a compact metric space $(M,\mathfrak{d})$. Let ϕ be a potential and $\{\mathfrak{A}_k\}_{k\in\mathbb{N}}$ be a sequence of open covers of M such that $diam(\mathfrak{A}_n) \to 0$, when $n \to \infty$. We define **the topological pressure of the potential ϕ with respect to** g as*

$$P_{top}(g,\phi) = \lim_{n\to\infty} P_{top}(g,\phi,\mathfrak{A}_n). \tag{2.13}$$

We now introduce the concept of pressure through sets separated by (n,ε) and spanning sets which will be very important throughout this work.

Definition 36. *Let $g : M \to M$ be a continuous map on the metric space $(M,\mathfrak{d})$. A set $E \subset M$ is called $(n,\varepsilon)-$separated for f for n a positive integer and $\varepsilon > 0$ provided that for every pair of distinct points $x,y \in E$, $x \neq y$, there is at least one m with $0 \le m < n$ such that $\mathfrak{d}(g^m(x),g^m(y)) > \varepsilon$.*

Another way of expressing this concept is to introduce the distance

$$\mathfrak{d}_{n,g}(x,y) = \sup_{0\le j<n} \mathfrak{d}(g^j(x),g^j(y)).$$

Using this distance, a set $S \subset M$ is $(n,\varepsilon)-$separated for g provided $\mathfrak{d}_{n,g}(x,y) > \varepsilon$ for every pair of distinct points $x,y \in E$, $x \neq y$.

Definition 37. *Let $g : M \to M$ be a continuous map on the space M with metric $\mathfrak{d}$. Let $K \subset M$ be a subset. For a positive integer r, let*

$$\mathfrak{d}_{r,g}(w,z) = \sup_{0\le j<r} \mathfrak{d}(g^j(w),g^j(z))$$

as we defined above. A set $E \subset K$ is said to $(n,\varepsilon)-$spanning K for n a positive integer and $\varepsilon > 0$ provided for each $x \in K$ there exists a $y \in E$ such that $d_{n,g}(x,y) \le \varepsilon$.

Thus, given $n \geq 1$ and $\varepsilon > 0$, define

$$G_n(g,\phi,\varepsilon) = \inf\left\{\sum_{x\in E} e^{\phi_n(x)} : \ E \text{ is subset } (n,\varepsilon) - \text{spanning of } \ M\right\} \quad \text{and} \tag{2.14}$$

$$S_n(g,\phi,\varepsilon) = \inf\left\{\sum_{x\in E} e^{\phi_n(x)} : \ E \text{ is subset } (n,\varepsilon) - \text{separated of } \ M\right\}. \tag{2.15}$$

Then we can define

$$G(g,\phi,\varepsilon) = \lim_{n\to\infty} \sup \frac{1}{n} \log G_n(g,\phi,\varepsilon) \qquad \text{and} \tag{2.16}$$

$$S(g,\phi,\varepsilon) = \lim_{n\to\infty} \sup \frac{1}{n} \log S_n(g,\phi,\varepsilon). \tag{2.17}$$

Also,

$$G(g,\phi) = \lim_{\varepsilon\to 0} G(g,\phi,\varepsilon) \qquad \text{and} \tag{2.18}$$

$$S(g,\phi) = \lim_{\varepsilon\to 0} S(g,\phi,\varepsilon). \tag{2.19}$$

Proposition 6. *$P_{top}(g,\phi) = G(g,\phi) = S(g,\phi)$ for all potential ϕ in M.*

Proof. See [53]. □

Finally, we present some of the properties that the topological pressure satisfies and that will be necessary for the main results (see [54]).

Proposition 7. *Let g be a continuous map on a compact metric space $(M,\mathfrak{d})$ and let ϕ be a potential. Then:*

(i) *$P_{top}(g,0) = h_{top}(g)$, where h_{top} is topological entropy;*

(i) *Let $(M_1,\mathfrak{d}_1)$ and $(M_2,\mathfrak{d}_2)$ compact metric spaces with continuous maps $g_i : M_i \to M_i$, for $i = 1,2$. If $h : M_1 \to M_2$ is a surjective continuous map with $h \circ g_1 = g_2 \circ h$. Then for every potential ϕ of M_2 we have $P_{top}(g_1,\phi\circ h) \geq P_{top}(g_2,\phi)$. The equality holds if h is a homeomorphism.*

Note that, if we consider $Y = (X^+,X^+)$ then Y is a smooth vector field.

Definition 38. *Let $Y = (X^+,X^+)$ be a vector field, and ϕ_Y is the flow defined by this field, we define the time-one map of this field by $F^1(x) = \phi_Y(1,x)$. And the topological pressure of the flow is defined by $P_{top}(Y) := P_{top}(F^1, -\beta \log|\det J_{med} F^1|)$.*

2.1.6 Carathéodory Construction, Hausdorff Measure and Minkowski Dimension

The Carathéodory construction of outer-measures is a general framework with which one can construct many of the standard geometric outer-measures including the Hausdorff measures.

Definition 39. *Let $(M, \mathfrak{d})$ be a metric space, $\mathfrak{F} \subseteq \mathscr{P}(M)$ and $\zeta : \mathfrak{F} \to [0,\infty)$ (potentially Hausdorff) such that:*

(*i*) *For all $\delta > 0$ there exist $\{\mathfrak{A}_i\} \subset \mathfrak{F}$ such that $M \subset \cup_i \mathfrak{A}_i$ and $\mathfrak{d}(\mathfrak{A}_i) \leq \delta$;*

(*ii*) *For all $\delta > 0$ there exist $\mathfrak{A} \in \mathfrak{F}$ such that $\zeta(U) \leq \delta$.*

For $\delta > 0$ we define

$$\psi_\delta : \mathscr{P}(M) \to [0,\infty], \textit{ with } \psi_\delta(A) = inf\left\{\Sigma_i \zeta(\mathfrak{A}_i) : A \subset \cup_i \mathfrak{A}_i, \mathfrak{d}(\mathfrak{A}_i) < \delta \, \{\mathfrak{A}_i\} \subset \mathfrak{F}\right\}.$$

By a δ-cover in the context of the Carathéodory construction we mean a countable collection of sets $\{\mathfrak{A}_i\} \subset \mathfrak{F}$ such that $\zeta(\mathfrak{A}_i) \leq \delta$ and $\mathfrak{d}(\mathfrak{A}_i) \leq \delta$. This definition is dependent on $\mathfrak{F}$, if this is ambiguous we will refer to such covers as $(\mathfrak{F}, \delta)$-covers. For brevity we write $\psi_\delta(A) = \inf \sum_i \zeta(\mathfrak{A}_i)$ where $\{\mathfrak{A}_i\}$ is understood to be a $(\mathfrak{F}, \delta)$-cover of A. In cases where this notation is ambiguous we will use an appropriately descriptive unambiguous version of the definition above.

Definition 40. *Let $(M, \mathfrak{d})$ be a metric space, $\mathfrak{F} \subseteq \mathscr{P}(M)$ and $\zeta_c(\cdot) = \mathfrak{d}(\cdot)^c$, then for each $c \in (0,\infty)$ we construct the c-dimensional size δ approximating measures $\mathscr{H}^c_\delta$ and the c-dimensional Hausdorff Measure, $\mathscr{H}^c$ via the Carathéodory construction.*

One should note immediately that rather than constructing one outer-measure we are actually constructing a family of outer-measures parameterized by $c \in [0,\infty)$. This family has the interesting property, which will be shown in [56], that each outer-measure provides useful information about a different family of subsets of M.

Definition 41. *The Hausdorff Dimension (or Hausdorff-Besicovitch Dimension) of a set A is the unique $c \in [0,\infty)$ such that*

$$\mathscr{H}^t(A) = \begin{cases} \infty & for\ all\ \ 0 \leq t < c \\ 0 & for\ all\ \ t > c \end{cases}.$$

We denote the Hausdorff dimension of a set A by $dim_{\mathscr{H}}(A)$.

Next we will talk about Minkowski dimension. In our discussion of Minkowski dimension, also known as box dimension, we only consider compact subsets A of some Euclidean space $\mathbb{R}^n$. The definitions also make sense in a metric space. Since a manifold M can be embedded in some Euclidean space $\mathbb{R}^n$, our definitions apply to compact manifolds.

Definition 42. *For $\varepsilon > 0$, consider the subdivision of $\mathbb{R}^n$ into boxes or cubes of sides of length ε: for $(j_1,\cdots,j_n) \in \mathbb{Z}^n$, let*

$$R_{j_1,\cdots,j_n} = \{(x_1,\cdots,x_n) : j_i\varepsilon \le x_i < (j_i+1)\,\varepsilon\,;\, 1 \le i \le n\}.$$

A box of this kind is said to be a box from the ε-grid. Let $N(\varepsilon,A)$ be the number of boxes R_j among all the choices of $j \in \mathbb{Z}^n$ such that $A \cap R_j \neq \emptyset$.

Definition 43. *For a general compact subset $A \subset \mathbb{R}^n$, we define the* ***upper Minkowski dimension*** *of A, as*

$$\overline{dim}_M(A) = \limsup_{\varepsilon\to 0} \frac{\log(N(\varepsilon,A))}{\log(\varepsilon^{-1})}$$

and the ***lower Minkowski dimension***

$$\underline{dim}_M(A) = \liminf_{\varepsilon\to 0} \frac{\log(N(\varepsilon,A))}{\log(\varepsilon^{-1})}.$$

If the two values agree, the common value is simply called the Minkowski dimension of A and denoted by $dim_M(A)$.

Now, consider the metric $\widetilde{d}$, which is compatible with the product topology on $\mathscr{A}_k^{\mathbb{N}}$, as follows: for every $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$

$$\widetilde{d}(x,y) = \begin{cases} 2^{-m} & , \quad m = min\,\{i;\, x_i \neq y_i\} \\ 0 & if\; x_i = y_i \quad \forall\, i \in \mathbb{N} \end{cases}.$$

Let $\mathscr{K} \subset \mathscr{A}_k^{\mathbb{N}}$ be a closed σ_k-invariant subset. Fustenberg ([29] Proposition $III-1$) proved the following relationship among entropy, Hausdorff and Minkowski dimensions of $\mathscr{K}$ with respect to $\widetilde{d}$:

$$dim_H(\mathscr{K},\widetilde{d}) = dim_M(\mathscr{K},\widetilde{d}) = h_{top}(\sigma_k^+|_{\mathscr{K}},\sigma_k), \tag{2.20}$$

where $h_{top}(\sigma_k^+|_{\mathscr{K}},\sigma_k)$ is the topological entropy of $(\mathscr{K},\sigma_k)$.

How we know, $d(x,y) = \sum\limits_{i\in\mathbb{N}} \frac{|x_i-y_i|}{2^i}$ is also a metric of $\mathscr{A}_k^{\mathbb{N}}$. Besides, $1 \le d(x,y) \le 2(k-1)$.

Lemma 3. *The previous metrics d and $\widetilde{d}$ are equivalent.*

Proof. If $x = y$ there is nothing to do. So, consider any $x \neq y$. Take $s = 1$ and $0 < t \le \frac{1}{(k-1)2^{m+1}}$, $m = min\{i;\, x_i \neq y_i\}$.

Note that

$$\widetilde{d}(x,y) = 2^{-m} \le 1 \le 1.d(x,y).$$

Also,

$$t \le \frac{1}{(k-1)2^{m+1}} \Rightarrow 2(k-1)t \le 2^{-m} = \widetilde{d}(x,y) \Rightarrow td(x,y) \le 2t(k-1) \le 2^{-m} = \widetilde{d}(x,y).$$

So,

$$td(x,y) \le \widetilde{d}(x,y) \le 1 \cdot d(x,y),$$

and it follows that d and $\widetilde{d}$ are equivalent. □

Therefore, the relationship (2.20) is also valid for $\mathscr{K}$ with respect to metric d.

CHAPTER 3

Topological pressure, Hausdorff dimension, relation time and mixing time for PSVFs

Before presenting the main results of this work, consider the following construction and its consequences.

Remark 7 (Construction of the "petals"). *Consider the PSVF:*

$$Z(x,y) = \begin{cases} X^+(x,y) &= (1,1-x) \text{ for } y \geq 0 \\ X^-(x,y) &= (-1,1-x) \text{ for } y \leq 0 \end{cases}, \tag{3.1}$$

where $\Sigma = \{y=0\}$. Note that X^+ and X^- they are symmetric and the point $(1,0)$ is an invisible-invisible two-fold, and there is a closed trajectory that goes through the origin $(0,0)$. Consider six rays from the origin (one at each multiple of $\frac{\pi}{3}$). In that way, the plane is divided into six different regions. Number each region from 1 to 6, counter clockwise. We define X^+ in region 1, and X^- in region 6. Now, define a vector field in each one of these regions, such that in regions 3 and 5 we have a phase portrait that are rotations (of angles $\pi/3$ and $2\pi/3$, respectively) of X^+, and in regions 2 and 4 the phase portrait considered are rotations (of angles $\pi/3$ and $2\pi/3$, respectively) of X^- (see Figure 3.1). Now there are three closed arcs that goes through the origin. It defines a PSVF $\widetilde{Z}_3$ with six different regions such that, apart from three invisible two-folds, and the origin, every other point is either sewing or crossing, and every trajectory that does not go through the origin is closed.

For a better understanding of the reader, we will reproduce here some details of the previously mentioned construction made in detail by [3]. Firstly, note that the 6 semi-straight lines (rays) can be parameterized by

$$r_j = \left\{ \left(t\cdot\cos\left(\frac{j\pi}{3},\right), t\cdot\sin\left(\frac{j\pi}{3},\right)\right); t\geq 0 \right\} \quad j = 0,1,2,3,4,5.$$

These rays divide the plane into 6 open regions $R_0, R_1, R_2, R_3, R_4, R_5$ such that each R_j is bounded by r_j and r_{j+1} for $j = 0,1,2,3,4$ and R_5 is bounded by r_5 and r_0, (see 3.1) whose rotation matrix of the angle $\frac{2\pi}{3}$ is:

$$R_{\frac{2\pi}{3}} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}. \tag{3.2}$$

Thus, in the region R_{2k} we consider the vector field $\widetilde{X}_k^+ = R_{2\pi/3}^k.X^+$, with $k = 0, 1, \cdots, (n-1)$ and $n-1 = 2$ in this case. So,

$$\widetilde{X}_0^+ = X^+ = (1, 1-x),$$

$$\widetilde{X}_1^+ = \left(\frac{1}{2}(\sqrt{3}(x-1)-1), \frac{1}{2}(x+\sqrt{3}-1)\right)$$

and

$$\widetilde{X}_2^+ = \left(\frac{1}{2}(\sqrt{3}(-x)+\sqrt{3}-1), \frac{1}{2}(x-\sqrt{3}-1)\right).$$

Analogously, the region R_{2k+1} we consider the vector field $\widetilde{X}_k^- = R_{2\pi/3}^k.X^-$, with $k = 0, 1, 2$. So,

$$\widetilde{X}_0^- = X^- = (1, 1-x),$$

$$\widetilde{X}_1^- = \left(\frac{1}{2}(\sqrt{3}(x-1)+1), \frac{1}{2}(x-\sqrt{3}-1)\right)$$

and

$$\widetilde{X}_2^- = \left(\frac{1}{2}(\sqrt{3}(-x)+\sqrt{3}+1), \frac{1}{2}(x+\sqrt{3}-1)\right).$$

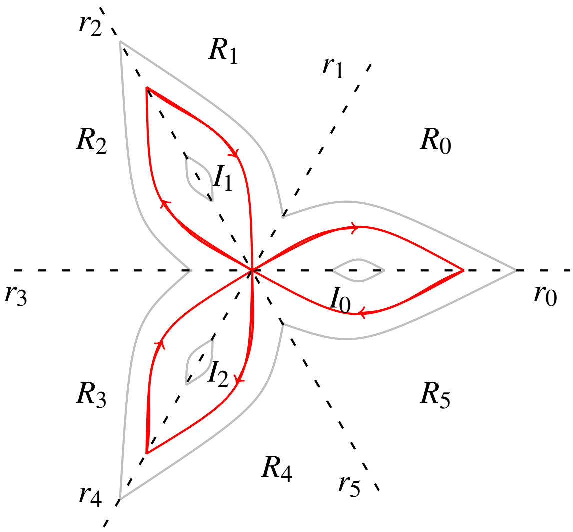


Figure 3.1: Consider 6 semi-straight lines (rays) highlighted in red is invariant for $\widetilde{Z}_3$.

Note that, $\widetilde{Z}_3$ expends a total time $\widetilde{t} = 4$ to get out of an initial condition, travel using one of the curves I_0, I_1, I_2 and return to this point. A simple reparametrization on time can produce a new PSVF with an analogous time $\widetilde{t} = 1$. For sake of simplicity on the calculation we keep $\widetilde{Z}_3 = (\widetilde{X}_0^+, \widetilde{X}_1^-, \widetilde{X}_1^+, \widetilde{X}_2^-, \widetilde{X}_2^+, \widetilde{X}_0^-)$. The union of the curves I_0, I_1, I_2 is invariant for the PSVF $\widetilde{Z}_3$ composed by $\widetilde{X}_0^+, \widetilde{X}_1^-, \widetilde{X}_1^+, \widetilde{X}_2^-, \widetilde{X}_2^+, \widetilde{X}_0^-$. Note that a trajectory of $\widetilde{Z}_3$ is the amalgamation of these three different arcs in every possible combination.

In general, the same construction above can be done for any $k \in \mathbb{N}, k \geq 2$. This way take $\mathscr{V} = \bigcup_{n=0}^{k-1} I_n$, where each arc I_n is a "petal" (local closed orbit). Our main goal is to prove that there is a conjugacy between the time-one map of the fields $\widetilde{Z}_k$ restricted to $\mathscr{V}$ and a one-sided shift space. But, since a PSVF does not give uniqueness of trajectory through a point, the a time-one map of $\widetilde{Z}_k$, $\widetilde{Z}_k^1 : \mathscr{V} \to \mathscr{V}$ such that $\widetilde{Z}_k^1(x) = \phi_{\widetilde{Z}_k}(1,x)$, (here $\phi_{\widetilde{Z}_k}(0,x) = x$ and $\phi_{\widetilde{Z}_k}$ is the flow of $\widetilde{Z}_k$), is not well-defined, because it may have more than one image (depending on the flow chosen). One way of avoiding this is to work with a subset of the space of all possible trajectories in such a way that T_1 will be well-defined when restricted to this subset. So, consider

$$\Upsilon_k = \left\{ \gamma \mid \gamma \text{ is a positive global trajectory of } \widetilde{Z}_k \text{ with } \gamma(0) \in \mathscr{V} \right\},$$

and then we can define the time-one map in Υ_k, similarly to what was done before, that is, $T_1 : \Upsilon_k \to \Upsilon_k$, $T_1(\gamma)(\cdot) = \gamma(\cdot + 1)$.

Remark 8. *Note that given $\gamma \in \Upsilon_k$, by the construction of the PSVF $\widetilde{Z}_k$, for every $t \in \mathbb{R}^+$, there is a unique $t^* \in [t, t+1)$ such that $\gamma(t^*) = 0$.*

Definition 44. *Let $\mathfrak{s} : \Upsilon_k \to \mathscr{A}_k^{\mathbb{N}}$ be given by $\mathfrak{s}(\gamma) = (\mathfrak{s}_j(\gamma))_{j \in \mathbb{N}}$, where :*

$$\mathfrak{s}_j(\gamma) = \begin{cases} n \text{ if } \gamma(j) \in I_n \\ m \text{ if } \gamma(j) = 0 \text{ and } \gamma\left(j + \frac{1}{2}\right) \in I_m \end{cases}.$$

The sequence $\mathfrak{s}(\gamma)$ is called the itinerary of γ.

Since $k \in \mathbb{N}, k \geq 2$, clearly $\mathfrak{s}$ is well defined. On the other hand, given $\gamma \in \Upsilon_k$, there are infinitely many trajectories with the same itinerary as γ, just by changing its initial condition, without modifying the compartment where it is located. To avoid such a situation, we consider the equivalence relation:

Definition 45. *Let $\gamma_1, \gamma_2 \in \Upsilon_k$. We say $\gamma_1 \sim \gamma_2$ if and only if $\mathfrak{s}(\gamma_1) = \mathfrak{s}(\gamma_2)$. Denote $\overline{\Upsilon}_k = \Upsilon_k / \sim$.*

In fact, for $\gamma_1 \in \Upsilon_k$, we have $\gamma_1 \sim \gamma_1$ because $\mathfrak{s}(\gamma_1) = \mathfrak{s}(\gamma_1)$. Also for all $\gamma_1, \gamma_2 \in \Upsilon_k$, if $\gamma_1 \sim \gamma_2$, we have $\mathfrak{s}(\gamma_1) = \mathfrak{s}(\gamma_2)$, and in this way, $\gamma_2 \sim \gamma_1$. And finally for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon_k$, if $\gamma_1 \sim \gamma_2$, we have $\mathfrak{s}(\gamma_1) = \mathfrak{s}(\gamma_2)$ and if $\gamma_2 \sim \gamma_3$, we have $\mathfrak{s}(\gamma_2) = \mathfrak{s}(\gamma_3)$, and in this way, $\mathfrak{s}(\gamma_1) = \mathfrak{s}(\gamma_2) = \mathfrak{s}(\gamma_3)$, and therefore $\gamma_1 \sim \gamma_3$.

Remark 9. *Observe that, given $\gamma \in \Upsilon_k$, there exists a representative $\gamma^* \in \overline{\gamma}$ such that $\gamma^*(0) = 0$, because if $\gamma(0) = 0$, simply take $\gamma^* = \gamma$, if not, by Remark 8 there are unique t^* and $\varepsilon > 0$ such that $0 \leq t^* < \varepsilon$ such that $\gamma(t^*) = 0$. Moreover, if $\mathfrak{s}(\gamma) = (\mathfrak{s}_j)_{j \in \mathbb{N}}$, then $\gamma((t^* + j, t^* + j + 1)) = I_{\mathfrak{s}_j}$. Which implies that $\gamma((j, j+1)) = I_{\mathfrak{s}_j}$ and, consequently, $\gamma^*(j + \frac{1}{2}) \in I_{\mathfrak{s}_j}$. Then $\mathfrak{s}(\gamma^*) = \mathfrak{s}(\gamma)$.*

Definition 46. *The Hausdorff distance between the sets A and B is given by*

$$d_H(A,B) = max \left\{ \sup_{x \in A} \inf_{y \in B} d(x,y), \sup_{y \in B} \inf_{x \in A} d(x,y) \right\} = max \left\{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A) \right\}.$$

Definition 47. *Now, define $\rho_k : \overline{\Upsilon}_k \times \overline{\Upsilon}_k \to \mathbb{R}$, by*

$$\rho_k(\overline{\gamma_1}, \overline{\gamma_2}) = \sum_{i\in\mathbb{N}} \frac{d_i(\overline{\gamma_1}, \overline{\gamma_2})}{2^i},$$

where $d_i(\overline{\gamma_1}, \overline{\gamma_2}) = d_H\left(\gamma_1^*([i,i+1]), \gamma_2^*([i,i+1])\right)$, d_H is the Hausdorff distance and γ_1^*, γ_2^* are those representatives given in Remark 9.

To simplify the notation, from now on we refer only to $\gamma \in \overline{\Upsilon}_k$, as the equivalence class $\overline{\gamma}$ with the representative γ^* .

Proposition 8. $\left(\overline{\Upsilon}_k, \rho_k\right)$ *is a metric space.*

Proof. Let us show that ρ_k is well-defined, i.e., the series in the previous definition converges. Since $k \in \mathbb{N}, k \geq 2$, there exists $\mathfrak{M} > 0$ such that $d_i(\gamma_1, \gamma_2) \leq \mathfrak{M}$, for any $i \in \mathbb{N}$, since $\gamma_1([i,i+1]), \gamma_2([i,i+1])$ are closed subsets of $\mathscr{V}$ which is compact. So $\rho_k(\gamma_1, \gamma_2) \leq \mathfrak{M}\left(\sum\limits_{i=0}^{\infty} \frac{1}{2^i}\right) = 2\mathfrak{M}$, i.e., it converges for any $\gamma_1, \gamma_2 \in \overline{\Upsilon}_k$.

Now, let us show it is a metric. Note that every summand in the definition of ρ_k is non-negative, then $\rho_k(\gamma_1, \gamma_2) = 0$ if and only if $d_i(\gamma_1, \gamma_2) = 0$ for every $i \in \mathbb{N}$. Hence $\gamma_1([i,i+1]) = \gamma_2([i,i+1])$, for all $i \in \mathbb{N}$, then $\gamma_1 = \gamma_2$. From $d_i(\gamma_1, \gamma_2) = d_i(\gamma_2, \gamma_1)$, we obtain $\rho_k(\gamma_1, \gamma_2) = \rho_k(\gamma_2, \gamma_1)$. Now, for every $i \in \mathbb{N}$, we get:

$$d_i(\gamma_1, \gamma_2) \leq d_i(\gamma_1, \gamma_3) + d_i(\gamma_3, \gamma_2) \Rightarrow$$

$$\sum_{i=0}^{N} \frac{d_i(\gamma_1, \gamma_2)}{2^i} \leq \sum_{i=0}^{N} \frac{d_i(\gamma_1, \gamma_3)}{2^i} + \sum_{i=0}^{N} \frac{d_i(\gamma_3, \gamma_2)}{2^i} \; \forall N \Rightarrow \rho_k(\gamma_1, \gamma_2) \leq \rho_k(\gamma_1, \gamma_3) + \rho_k(\gamma_3, \gamma_2).$$

□

Let $\overline{T}_1 : \overline{\Upsilon}_k \to \overline{\Upsilon}_k$ be the function induced by T_1, that is, $\overline{T}_1(\overline{\gamma}) = \overline{T_1(\gamma)}$. Note that the induced function does not depends on the representative. In fact if $\mathfrak{s}(\gamma_1) = \mathfrak{s}(\gamma_2) = (\mathfrak{s}_j)_{j\in\mathbb{N}}$, then, for all $j \in \mathbb{N}$ it happens

$$\begin{aligned} &\gamma_1(j), \gamma_2(j) \in I_{\mathfrak{s}_j} \Rightarrow \gamma_1(j+1), \gamma_2(j+1) \in I_{\mathfrak{s}_{j+1}} \Rightarrow \\ &T_1(\gamma_1)(j), T_1(\gamma_2)(j) \in I_{\mathfrak{s}_{j+1}} \Rightarrow \mathfrak{s}(T_1(\gamma_1)) = \mathfrak{s}(T_1(\gamma_2)). \end{aligned}$$

Proposition 9. *The function $\overline{T}_1$ given above is continuous.*

Proof. First note that

$$d_i\left(\overline{T}_1(\gamma_1), \overline{T}_1(\gamma_2)\right) = d_H\left(T_1(\gamma_1)([i,i+1]), T_1(\gamma_2)([i,i+1])\right) =$$

$$d_H\left(\gamma_1([i+1,i+2]), \gamma_2([i+1,i+2])\right) = d_{i+1}(\gamma_1, \gamma_2).$$

Now,

$$\rho_k\left(\overline{T}_1(\gamma_1), \overline{T}_1(\gamma_2)\right) = \sum_{i=0}^{\infty} \frac{d_i(T_1(\gamma_1), T_1(\gamma_2))}{2^i} = \sum_{i=0}^{\infty} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^i} =$$

$$\lim_{n\to\infty}\left(\sum_{i=0}^{n}\frac{d_{i+1}(\gamma_1,\gamma_2)}{2^i}\right)=\lim_{n\to\infty}\left(2\sum_{i=0}^{n}\frac{d_{i+1}(\gamma_1,\gamma_2)}{2^{i+1}}\right)=\lim_{n\to\infty}\left(2\sum_{j=1}^{n+1}\frac{d_j(\gamma_1,\gamma_2)}{2^j}\right)\leq$$

$$\lim_{n\to\infty}\left(2\sum_{j=1}^{n+1}\frac{d_j(\gamma_1,\gamma_2)}{2^j}\right)+d_0(\gamma_1,\gamma_2)=2\lim_{n\to\infty}\left(\sum_{j=1}^{n+1}\frac{d_j(\gamma_1,\gamma_2)}{2^j}+\frac{d_0(\gamma_1,\gamma_2)}{2}\right)=2\rho_k(\gamma_1,\gamma_2).$$

Hence $\overline{T}_1$ is continuous. □

Now let $\bar{\mathfrak{s}}:\overline{\Upsilon}_k\to\mathscr{A}_k^{\mathbb{N}}$ be the function induced by $\mathfrak{s}$, that is, $\bar{\mathfrak{s}}(\bar{\gamma})=\overline{\mathfrak{s}(\gamma)}$. Note that the induced function does not depend on the representative, because of the equivalence relation and because it is one-to-one.

Proposition 10. *The map $\bar{\mathfrak{s}}$ is a homeomorphism onto its image.*

Proof. Put $\varpi=\min\left\{d_H(I_l,I_j)\ l\neq l\,\text{and}\,l,j=0,1\cdots,k-1\right\}$ and $\gamma_1,\gamma_2\in\overline{\Upsilon}_k$. Suppose $\rho_k(\gamma_1,\gamma_2)$ $<\frac{\varpi}{2^N}$. Then, for any $0\leq i\leq N: d_i(\gamma_1,\gamma_2)<\varpi$, because, on the contrary, we have $\rho_k(\gamma_1,\gamma_2)=\sum\frac{d_i(\gamma_1,\gamma_2)}{2^i}\geq\frac{\varpi}{2^N}$. Now, $d_i(\gamma_1,\gamma_2)<\varpi\Rightarrow d_i(\gamma_1,\gamma_2)=0$ and therefore, $\gamma_1((i,i+1))=\gamma_2((i,i+1))$ implying that $\mathfrak{s}_i(\gamma_1)=\mathfrak{s}_i(\gamma_2)$, for all $0\leq i\leq N$. So,

$$d(\mathfrak{s}(\gamma_1),\mathfrak{s}(\gamma_2))=\sum_{i\in\mathbb{N}}\frac{\mid\mathfrak{s}_i(\gamma_1)-\mathfrak{s}_i(\gamma_2)\mid}{2^i}=\sum_{\substack{i\in\mathbb{N}\\ i>N}}\frac{\mid\mathfrak{s}_i(\gamma_1)-\mathfrak{s}_i(\gamma_2)\mid}{2^i}\leq\frac{k-1}{2^{N-1}}.$$

This proves that $\bar{\mathfrak{s}}$ is continuous. The same argument reverses itself in order to show $\bar{\mathfrak{s}}$ is open; let $\gamma_1,\gamma_2\in\overline{\Upsilon}_k$, and $N\in\mathbb{N}$, $d(\bar{\mathfrak{s}}(\gamma_1),\bar{\mathfrak{s}}(\gamma_2))\leq\frac{1}{2^N}\Rightarrow\mathfrak{s}_i(\gamma_1)=\mathfrak{s}(\gamma_2)\quad\forall\,0\leq i\leq N$. Then $\gamma_1([i,i+1])=\gamma_2([i,i+1])$, for all $0\leq i\leq N$. Hence

$$\rho_k(\gamma_1,\gamma_2)=\sum_{\substack{i\in\mathbb{N}\\ i>N}}\frac{d_i(\gamma_1,\gamma_2)}{2^i}\leq\frac{\mathfrak{N}}{2^{N-1}}.$$

Where $\mathfrak{N}>0$, such that $diam(\mathscr{V})<\mathfrak{N}$. Therefore $\bar{\mathfrak{s}}$ in a homeomorphism over its image. □

Finally, we are now in a position to prove the following proposition:

Proposition 11. *The function $\bar{\mathfrak{s}}:\overline{\Upsilon}_k\to\bar{\mathfrak{s}}\left(\overline{\Upsilon}_k\right)$ is a conjugacy between $\overline{T}_1$ and σ_k^+, i.e., $\bar{\mathfrak{s}}\circ\overline{T}_1=\sigma_k^+\circ\bar{\mathfrak{s}}$.*

Proof. Let $\gamma\in\overline{\Upsilon}_k,(a_j)_{j\in\mathbb{N}}=\bar{\mathfrak{s}}(\gamma)$ and $(b_j)_{j\in\mathbb{N}}=\bar{\mathfrak{s}}(T_1(\gamma))$, then :

$$\begin{array}{ccc}\overline{\Upsilon}_k & \xrightarrow{\overline{T}_1} & \overline{\Upsilon}_k\\ \Big\downarrow{\scriptstyle\bar{\mathfrak{s}}} & & \Big\downarrow{\scriptstyle\bar{\mathfrak{s}}}\\ \bar{\mathfrak{s}}(\overline{\Upsilon}_k) & \xrightarrow{\sigma_k^+} & \bar{\mathfrak{s}}(\overline{\Upsilon}_k)\end{array}$$

$$b_j=r\,\text{if}\,\overline{T}_1(\gamma)(j)\in I_r\Rightarrow b_j=r\,\text{if}\,\gamma(j+1)\in I_r\Rightarrow b_j=a_{j+1}\Rightarrow(b_j)_{j\in\mathbb{N}}=\sigma_k^+\left((a_j)_{j\in\mathbb{N}}\right).$$

It remains to show that $\bar{\mathfrak{s}}\left(\overline{\Upsilon}_k\right)$ is a subshift. First, $\bar{\mathfrak{s}}$ is continuous and $\overline{\Upsilon}_k$ is compact. Then $\bar{\mathfrak{s}}\left(\overline{\Upsilon}_k\right)$ is closed in $\mathscr{A}_k^{\mathbb{N}}$. And, sice $\bar{s}$ is a homeomorphism onto its image, there exists $\bar{\mathfrak{s}}^{-1}:\bar{\mathfrak{s}}\left(\overline{\Upsilon}_k\right)\to\overline{\Upsilon}_k$ and it is continuous. By the first part of this proof, we have $\sigma_k^+=\bar{\mathfrak{s}}\circ\overline{T}_1\circ\bar{\mathfrak{s}}^{-1}$, which proves the invariant part. Hence $\bar{\mathfrak{s}}$ is a conjugacy between both systems. □

Now, for each $k \in \mathbb{N}$, $k \geq 2$ consider PSVF

$$Z_k(x,y) = \begin{cases} X_k^+(x,y) &= \left(1, P_k'(x)\right) \text{ for } y \geq 0 \\ X_-^k(x,y) &= \left(-1, P_k'(x)\right) \text{ for } y \leq 0 \end{cases}, \tag{3.3}$$

where

$$P_k(x) = -\left(x + \frac{k-1}{2}\right)\left(x - \frac{k-1}{2}\right)\prod_{i=1}^{k-1}\left(x - \left(i - \frac{k}{2}\right)\right)^2, \; k \in \mathbb{N}, k \geq 2.$$

Note that, P_k has $2k$ roots, being 2 simple roots at $r_0 = \frac{1-k}{2}$ and $r_1 = \frac{k-1}{2}$ and $k-1$ roots of multiplicity two at $p_j = j - \frac{k}{2}$ for $j = 1, \cdots, k-1$. Moreover, $P_k'(r_0) > 0$, $P_k'(r_1) < 0$, $P_k'(p_j) = 0$, and $P_k''(p_j) > 0$, for every $j = 1, \cdots, k-1$. In addition, by Lemma 1 of [2] $(r_0, 0)$ and $(r_1, 0)$ are crossing points of Z_k, the points $(p_j, 0)$ are visible-visible two folds of Z_k, $j = 1, 2, \cdots k-1$. For each $k \in \mathbb{N}, k \geq 2$, consider $\gamma_k^{X^+} = \{(x, P_k(x)) \mid x \in [r_0, r_1]\}$ and $\gamma_k^{X^-}\{(x, -P_k(x)) \mid x \in [r_0, r_1]\}$. Define $\Lambda_k = \gamma_k^{X^+} \cup \gamma_k^{X^-}$, and note that Λ_k is an invariant set for Z_k.

Our main goal is to prove the conjugacy between the time-one map of the fields Z_k restricted to Λ_k and a one-sided shift map. Take the set

$$\Omega_k = \{\gamma \mid \gamma \text{ is a positive global trajectory of } Z_k \text{ with } \gamma(0) \in \Lambda_k\}.$$

Proposition 12. *For any $k \in \mathbb{N}, k \geq 2$ let $\gamma \in \Omega_k$, then for all $t \in \mathbb{R}^+$, there exists an unique $t^* \in [t, t+1)$ such that $\gamma(t^*) \in \{(p_j, 0) \mid j = 1, \cdots, k-1\}$.*

Proof. Follows immediately from the expression of Z_k and Lemmas 1 and 2 of the reference [2]. □

The region Λ_k can be partitioned into arcs that goes from p_j to the adjacent ones (p_{j+1} and p_{j-1}) or to itself. So, consider k to be fixed and let $I_0 = \{(x, P_k(x)),\ x \in [r_0, p_1)\} \cup \{(x, -P_k(x)),\ x \in [r_0, p_1)\}$, the arc from p_1 to itself passing through r_0. For any $j = 1, \cdots, k-2$, let $I_{2j-1} = \{(x, P_k(x)),\ x \in (p_j, p_{j+1})\}$ and $I_{2j} = \{(x, -P_k(x)),\ x \in (p_j, p_{j+1})\}$, the arcs from p_j to p_{j+1} and from p_{j+1} to p_j, respectively. And, $I_{2k-3} = \{(x, P_k(x)),\ x \in (p_{k-1}, r_1]\} \cup \{(x, -P_k(x)),\ x \in (p_{k-1}, r_1]\}$. In short, we enumerate these arcs top to bottom, left to right. See Figure 3.2.

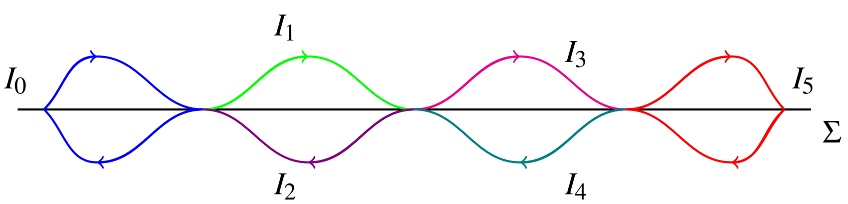


Figure 3.2: Case $k = 4$.

Definition 48. *Let $s : \Omega_k \to \mathscr{A}_{2(k-1)}^{\mathbb{N}}$ be given by $s(\gamma) = (s_j(\gamma))_{j \in \mathbb{N}}$, where :*

$$s_j(\gamma) = \begin{cases} n \text{ if } \gamma(j) \in I_n \\ m \text{ if } \gamma(j) \in \{(p_l, 0) \,|\, l = 1, \cdots, k-1\} \text{ and } \gamma\left(j + \frac{1}{2}\right) \in I_m \end{cases}.$$

The sequence $s(\gamma)$ is called the itinerary of γ.

According to Definition 48, given $\gamma \in \Omega_k$, there exist infinitely many distinct trajectories with the same itinerary of γ, simply because the initial conditions belong to the same arc I_n. In order to avoid this problem we will consider the following definition:

Definition 49. *Let $\gamma_1, \gamma_2 \in \Omega_k$. We say that $\gamma_1 \sim \gamma_2$ if and only if $s(\gamma_1) = s(\gamma_2)$. Denote $\overline{\Omega}_k = \Omega_k / \sim$.*

The relation in Definition 49 is an equivalence relation. The proof is similar to the one made in the Definition 45.

Proposition 13. *Given $\overline{\gamma} \in \overline{\Omega}_k$, there exists a representative γ^* such that $\gamma^*(0) \in \{(p_j, 0), j = 1, 2, \cdots, k-1\}$.*

Proof. Since, $\gamma \in \Omega_k$, there exists a representative $\gamma^* \in \overline{\gamma}$ such that $\gamma^*(0) \in \{(p_j, 0), j = 1, 2, \cdots, k-1\}$, because if $\gamma(0) \in \{(p_j, 0), j = 1, 2, \cdots, k-1\}$, simply take $\gamma^* = \gamma$, otherwise, by Proposition 12 there are unique t^* and $\varepsilon > 0$ such that $0 \leq t^* < \varepsilon$ such that $\gamma(t^*) \in \{(p_j, 0), j = 1, 2, \cdots, k-1\}$. Moreover, if $s(\gamma) = (s_j)_{j \in \mathbb{N}}$, then $\gamma((t^* + j, t^* + j + 1)) = I_{s_j}$. Which implies that $\gamma((j, j+1)) = I_{s_j}$ and, consequently, $\gamma^*(j + \frac{1}{2}) \in I_{s_j}$. Then $s(\gamma^*) = s(\gamma)$. □

Definition 50. *Now, define $\rho_k : \overline{\Omega}_k \times \overline{\Omega}_k \to \mathbb{R}$, by*

$$\rho_k(\overline{\gamma_1}, \overline{\gamma_2}) = \sum_{i \in \mathbb{N}} \frac{d_i(\overline{\gamma_1}, \overline{\gamma_2})}{2^i},$$

where $d_i(\overline{\gamma_1}, \overline{\gamma_2}) = d_H\left(\gamma_1^*([i, i+1]), \gamma_2^*([i, i+1])\right)$, d_H is the Hausdorff distance and γ_1^*, γ_2^* are those representatives given in Proposition 13.

For simplicity of notation, again we will refer only to $\gamma \in \overline{\Omega}_k$, meaning the equivalence class $\overline{\gamma}$ with the representative γ^* given in Proposition 13.

In an entirely analogous way to Proposition 8, $(\overline{\Omega}_k, \rho_k)$ is a metric space. Let $\overline{T_1} : \overline{\Omega}_k \to \overline{\Omega}_k$ be the function induced by T_1, that is, $\overline{T_1}(\overline{\gamma}) = \overline{T_1(\gamma)}$. Note that the induced function does not depends on the representative. In fact if $s(\gamma_1) = s(\gamma_2) = (s_j)_{j \in \mathbb{N}}$, then, for all $j \in \mathbb{N}$ it happens

$$\begin{array}{l} \gamma_1(j), \gamma_2(j) \in I_{s_j} \Rightarrow \gamma_1(j+1), \gamma_2(j+1) \in I_{s_{j+1}} \Rightarrow \\ T_1(\gamma_1)(j), T_2(\gamma_2)(j) \in I_{s_{j+1}} \Rightarrow s(T_1(\gamma_1)) = s(T_1(\gamma_2)). \end{array}$$

The map $\overline{T}_1$ given above is continuous, the proof is similar to the proof of Proposition 9.

Now let $\overline{s} : \overline{\Omega}_k \to \{0, 1, \cdots, 2k-3\}^{\mathbb{N}}$ be the function induced by s, that is, $\overline{s}(\overline{\gamma}) = \overline{s(\gamma)}$. Note that the induced function does not depend on the representative, because of the equivalence relation and the fact that it is one-to-one.

From now on we will separate the case $k = 2$ from the case $k \geq 3$. This will be done because as we will see in the following two propositions, $\overline{s}$ is a conjugacy between $\overline{\Omega}_2$ and the shift of two symbols $\{0, 1\}^{\mathbb{N}}$. Now for $k \geq 3$ $\overline{s}$ is a conjugacy between $\overline{\Omega}_k$ with a subshift $\overline{s}(\overline{\Omega}_k) \subseteq \mathscr{A}_k^{\mathbb{N}}$.

Proposition 14. *The function $\overline{s} : \overline{\Omega}_2 \to \{0, 1\}^{\mathbb{N}}$ is a conjugacy between $\overline{T}_1$ and σ_2^+, i.e., $\overline{s} \circ \overline{T}_1 = \sigma_2^+ \circ \overline{s}$.*

Proof. Let us show that $\bar{s}(\overline{\Omega}_2) = \{0,1\}^{\mathbb{N}}$. Given $(s_j) \in \{0,1\}^{\mathbb{N}}$, construct γ by concatenating the arcs I_0 and I_1 according to (s_j), so γ is a positive trajectory with $\gamma(0) = p_1$, and $\gamma((0,1)) = I_{s_0}$. Then $\gamma(1) = p_1$ and $\gamma((1,2)) = I_{s_1}$. In general, for all $j \in \mathbb{N}, \gamma(j) = p_1$ and $\gamma((j, j+1)) = I_{s_j}$. By Definition 4, γ is a positive global trajectory of Z_2 and therefore $\gamma \in \overline{\Omega}_2$. Moreover, $s(\gamma) = \bar{s}(\gamma) = (s_j)_{j\in\mathbb{N}}$.

$$\begin{tikzcd}
\overline{\Omega}_2 \arrow[r, "\overline{T}_1"] \arrow[d, "\bar{s}"'] & \overline{\Omega}_2 \arrow[d, "\bar{s}"] \\
\{0,1\}^{\mathbb{N}} \arrow[r, "\sigma_2^+"] & \{0,1\}^{\mathbb{N}}
\end{tikzcd}$$

Now, for the commutative par $\gamma \in \overline{\Omega}_2, (a_j)_{j\in\mathbb{N}} = \bar{s}(\gamma)$ and $(b_j)_{j\in\mathbb{N}} = \bar{s}(T_1(\gamma))$, then:

$$b_j = \begin{cases} 0 \text{ if } \overline{T}_1(\gamma)(j) \in I_0 \\ 1 \text{ if } \overline{T}_1(\gamma)(j) \in I_1 \end{cases} = \begin{cases} 0 \text{ if } \gamma(j+1) \in I_0 \\ 1 \text{ if } \gamma(j+1) \in I_1 \end{cases} = a_{j+1} \Rightarrow$$

$(b_j)_{j\in\mathbb{N}} = \sigma_2^+((a_j)_{j\in\mathbb{N}})$, i.e., $\bar{s} \circ \overline{T}_1 = \sigma_2^+ \circ \bar{s}$. □

Proposition 15. *The function $\bar{s} : \overline{\Omega}_k \to \bar{s}\left(\overline{\Omega}_k\right)$ is a conjugacy between $\overline{T}_1$ and σ_r^+, ($r \leq 2k-3$) i.e., $\bar{s} \circ \overline{T}_1 = \sigma_r^+ \circ \bar{s}$.*

Proof. Let $\gamma \in \overline{\Omega}_k, (a_j)_{j\in\mathbb{N}} = \bar{s}(\gamma)$ and $(b_j)_{j\in\mathbb{N}} = \bar{s}(T_1(\gamma))$, then :

$$\begin{tikzcd}
\overline{\Omega}_k \arrow[r, "\overline{T}_1"] \arrow[d, "\bar{s}"'] & \overline{\Omega}_k \arrow[d, "\bar{s}"] \\
\bar{s}(\overline{\Omega}_k) \arrow[r, "\sigma_r^+"] & \bar{s}(\overline{\Omega}_k)
\end{tikzcd}$$

$$b_j = x \text{ if } \overline{T}_1(\gamma)(j) \in I_x \Rightarrow b_j = x \text{ if } \gamma(j+1) \in I_x \Rightarrow b_j = a_{j+1} \Rightarrow (b_j)_{j\in\mathbb{N}} = \sigma_r^+\left((a_j)_{j\in\mathbb{N}}\right).$$

It remains to show that $\bar{s}\left(\overline{\Omega}_k\right)$ is a subshift. First, $\bar{s}$ is continuous and $\overline{\Omega}_k$ is compact. Then $\bar{s}\left(\overline{\Omega}_k\right)$ is closed in $\mathscr{A}_k^{\mathbb{N}}$. And, sice $\bar{s}$ is a homeomorphism onto its image, there exists $\bar{s}^{-1} : \bar{s}\left(\overline{\Omega}_k\right) \to \overline{\Omega}_k$ and it is continuous. By the first part of this proof, we have $\sigma_r^+ = \bar{s} \circ \overline{T}_1 \circ \bar{s}^{-1}$, which proves the invariant part. Hence $\bar{s}$ is a conjugacy between both systems. □

Remark 10. *In [2], the authors relate time-one maps of the PSVFs (3.3) and two-sided (sub)shifts. In the results above, we were inspired by the techniques developed in [2] to relate time-one maps of the PSVFs (3.3) and unilateral (sub)shifts. In addition we did the same for the PSVFs $\widetilde{Z}_k$ as in the construction done in Remark 7. This is important in order to conclude that the RPF operator is well defined, as we will show later.*

Definition 51. *Let $Z = (X^+, X^-)$ be a PSVF defined over a compact 2-dimensional surface M and $\widetilde{\Omega} \subseteq \Omega = \{\gamma :$ positive global trajectory of $Z\}$. We define the topological pressure P_{top} of Z on M, as the topological pressure of the map $\overline{T}_1$ in $\widetilde{\Omega} \subseteq \Omega$, that is,*

$$P_{top}(Z) := P_{top}\left(\overline{T}_1\big|_{\widetilde{\Omega}}, -\beta \log|\det J_{med}\overline{T}_1|\right).$$

Remark 11. *The pressure is a weighted version of topological entropy, where the "weights" are determined by the potential, Definition 51 agrees with Definition* 4.2 *of entropy for PSVF given in [3].*

Before presenting the main results of this thesis, we will make the following very important lemma.

Lemma 4. *Let* $Z = (X^+, X^-)$ *be a PSVF defined over a compact* 2*-dimensional surface* M *and* $\widetilde{\Omega} \subseteq \Omega = \{\gamma : \text{positive global trajectory of } Z\}$. *Then* $\overline{T}_1|_{\widetilde{\Omega}}$ *is an expanding map and topologically mixing.*

Proof. It suffices to note that the properties are topological invariant. □

3.0.1 Main results

In the following theorem we will use the construction made in Remark 7 and the topological conjugacy $\bar{\mathfrak{s}} : \overline{\Upsilon}_k \to \bar{\mathfrak{s}}(\overline{\Upsilon}_k)$, to show that the systems $(\overline{\Upsilon}_k, \mathscr{D}, med, \overline{T}_1)$ and $(\mathscr{A}_k^{\mathbb{N}}, \mathscr{B}, \mu_\pi^{\mathbb{N}}, \sigma_k^+)$ are ergodically equivalent.

Theorem 3. *Given* $k \in \mathbb{N}$, $k \geq 2$ *there exists a PSVF* $\widetilde{Z}_k$, *as in the construction done in Remark 7, with* k *petals, such that the system* $(\overline{\Upsilon}_k, \mathscr{D}, med, \overline{T}_1)$ *is ergodically equivalent to the system* $(\mathscr{A}_k^{\mathbb{N}}, \mathscr{B}, \mu_\pi^{\mathbb{N}}, \sigma_k^+)$, *i.e.,* $\overline{T}_1 : \overline{\Upsilon}_k \to \overline{\Upsilon}_k$ *is a Bernoulli map. Furthermore,* $\overline{T}_1$ *is strong-mixing and therefore weakly-mixing and ergodic.*

Now, since $\sigma_k^+ : \mathscr{A}_k^{\mathbb{N}} \to \mathscr{A}_k^{\mathbb{N}}$ is a topologically mixing and expanding map, we can use this fact and the Theorem 3, to obtain the following corollary.

Corollary 1. *With the hypotheses of the previous theorem, we get that* $\overline{T}_1 : \overline{\Upsilon}_k \to \overline{\Upsilon}_k$ *is topologically mixing.*

Under these conditions we show that $\overline{T}_1 : \overline{\Upsilon}_k \to \overline{\Upsilon}_k$ is a topologically mixing and expanding. So, we can define the RPF operator for this map. Therefore we can calculate the topological pressure and consequently the topological entropy. Furthermore, we can obtain a Markov chain, whose states are the trajectory arcs of the field $\widetilde{Z}_k$ and use the properties of this chain to define and estimate the mixer and the relaxation time for these PSVFs.

Corollary 2. *Given* $k \in \mathbb{Z}$, $k \geq 2$ *there exists a PSVF* $\widetilde{Z}_k$, *as in the construction done in Remark 7, with* k *petals. Then exist* $\widetilde{\Omega} \subseteq \overline{\Upsilon}_k$ *such that:*

(*i*) $P_{top}(\overline{T}_1|_{\widetilde{\Omega}}, -\beta \log |\det J_{med}\overline{T}_1|) = \log(\rho(A))$, *where* A *is an irreducible matrix associated with an oriented graph given by the trajectories of* $\widetilde{\Omega}$.

(*ii*) *If* A *is a matrix of a Markov chain formed by the arcs of trajectories in* $\widetilde{\Omega}$, *irreducible, aperiodic and reversible, then* $e^{-t_{rel}} = \rho_{ess}(A)$.

(iii) *There is a chain of random transposition in arcs composing a global trajectory such that,*

$$t_{mix} \leq (2+O(1))k\log k.$$

Also for $0<\varepsilon<1$ *it happens*

$$t_{mix}(\varepsilon) \geq \frac{k+1}{2}\log\left(\frac{1-\varepsilon}{6}k\right),$$

for sufficiently large k, where $O(1)$ *means that there exists some constant* δ *such that* $O(1)\leq\delta$.

(iv) *Suppose that* $\lambda\neq 1$ *is an eigenvalue for the transition matrix A of an irreducible, reversible and aperiodic Markov chain. Then*

(a) $(t_{rel}-1)\log(2)\leq t_{mix}\leq t_{rel}\log\left(\frac{4}{\pi_{min}}\right)$, *where* $\pi_{min}=\min_{i\in\mathscr{A}_k}\{\pi(i)\}$;

(b) $t_{mix}=\Theta(k\log k)$ *and* $t_{rel}=O(k\log k)$.

The next two theorems and their corollaries are analogous to Theorem 3 and its Corollaries 1, 2. But although the results are similar, we have to do them separately, since the nature of piecewise smooth vector fields are different.

Theorem 4. *Consider the PSVF (3.3), with* $k=2$. *Then the system* $(\overline{\Omega}_2,\mathscr{D},med,\overline{T}_1)$ *is ergodically equivalent to the system* $(\{0,1\}^{\mathbb{N}},\mathscr{B},\mu_\pi^{\mathbb{N}},\sigma_k^+)$, *i.e.,* $\overline{T}_1:\overline{\Omega}_2\to\overline{\Omega}_2$ *is a Bernoulli map. Furthermore,* $\overline{T}_1$ *is strong-mixing and therefore weakly-mixing and ergodic.*

Corollary 3. *With the hypotheses of the previous theorem we get that* $\overline{T}_1:\overline{\Omega}_2\to\overline{\Omega}_2$ *is topologically mixing.*

Corollary 4. *There exists* $\widetilde{\Omega}\subseteq\overline{\Omega}_2$ *such that analogous versions of items* (i) *and* $(iii)-$ *subitem* (a) *of Corollary 2 are valid.*

Theorem 5. *Consider the PSVF (3.3). Then, for* $k\in\mathbb{N},k\geq 3$, *the system* $(\overline{\Omega}_k,\mathscr{D},med,\overline{T}_1)$ *is ergodically equivalent to the system* $(\mathscr{A}_k^{\mathbb{N}},\mathscr{B},\mu_\pi^{\mathbb{N}},\sigma_k^+)$, *i.e.,* $\overline{T}_1:\overline{\Omega}_k\to\overline{\Omega}_k$ *is a Bernoulli map. Furthermore,* $\overline{T}_1$ *is strong-mixing and therefore weakly-mixing and ergodic.*

Corollary 5. *With the hypotheses of the previous theorem, we get that* $\overline{T}_1:\overline{\Omega}_k\to\overline{\Omega}_k$ *is topologically mixing.*

Corollary 6. *There exists* $\widetilde{\Omega}\subseteq\overline{\Omega}_k$ *such that analogous versions of items* $(i)-(iii)$ *of Corollary 2 are valid.*

In Theorem 6 below, we cannot use the RPF operator to calculate entropy, since the matrix of order 2 that represents the RPF operator has a finite number of possibilities. However, as $\bar{s} : \overline{\Omega}_2 \to \{0,1\}^{\mathbb{N}}$ is a topological conjugacy between $\overline{T}_1$ and σ_2^+, we can use the entropy properties of σ_2^+ in $\{0,1\}^{\mathbb{N}}$, to obtain the same properties as $\overline{T}_1$ in $\overline{\Omega}_2$.

Theorem 6. *Given $1 < \alpha \leq 2$, there are trajectories of Z_2 ($k = 2$ in (3.3)) such that $h_{top}(Z_2) = \log \alpha$. Furthermore for each $c \in (0, \log 2]$ there exists a set A_c such that $dim_{\mathscr{H}}(A_c) = \log \alpha = h_{top}(Z_2) = dim_M(A_c)$.*

3.1 Proof of the main results

3.1.1 Proof of Theorem 3

Proof. Let $C \subset \mathscr{A}_k^{\mathbb{N}}$ be measurable and invariant in the σ-algebra $\mathscr{B}$. As every Bernoulli shift is ergodic, we can assume that $\mu_\pi^{\mathbb{N}}(C) = 1$. Note that from Proposition 11 the map $\bar{\mathfrak{s}}^{-1} : \bar{\mathfrak{s}}(\overline{\Upsilon}_k) \to \overline{\Upsilon}_k$ is is a topological conjugacy, i.e., $\overline{T}_1 \circ \bar{\mathfrak{s}}^{-1} = \bar{\mathfrak{s}}^{-1} \circ \sigma_k^+$.

Taking $B = \bar{\mathfrak{s}}^{-1}(C) \subset \overline{\Upsilon}_k$, then we have $\overline{T}_1(B) = (\bar{\mathfrak{s}}^{-1} \circ \sigma_k^+ \circ \bar{\mathfrak{s}})(B) = \bar{\mathfrak{s}}^{-1}(\sigma_k^+(\bar{\mathfrak{s}}(B))) \subset \bar{\mathfrak{s}}^{-1}(C) = B$. As every σ-algebra is also an algebra, it follows from Theorem 1.1 [40] that there are $\widehat{\Upsilon}_1, \widehat{\Upsilon}_2, \dots \in \mathscr{D}$, with $\widehat{\Upsilon}_i \subseteq \widehat{\Upsilon}_{i+1}$ such that $\mathscr{D} = \bigcup_{i=1}^{\infty} \widehat{\Upsilon}_i$ and $med(\widehat{\Upsilon}_i) < \infty$, for all i. Put $\widehat{\mu} = \bar{\mathfrak{s}}_*^{-1} \mu_\pi^{\mathbb{N}}$ which is a measure in $\overline{\Upsilon}_k$. Since $\bar{\mathfrak{s}}^{-1} : \bar{\mathfrak{s}}(\overline{\Upsilon}_k) \to \overline{\Upsilon}_k$ is a bijection, follows that $\widehat{\mu}(\widehat{\Upsilon}_i) < \infty$, for all i. Therefore, by Theorem 2.4 [40], we get $med = \widehat{\mu}$. In this way it follows that $med(B) = \bar{\mathfrak{s}}_*^{-1} \mu_\pi^{\mathbb{N}}(B) = \mu_\pi^{\mathbb{N}}((\bar{\mathfrak{s}}^{-1})^{-1}(B)) = \mu_\pi^{\mathbb{N}}(\bar{\mathfrak{s}}(B)) = \mu_\pi^{\mathbb{N}}(C) = 1$. Therefore, $\bar{\mathfrak{s}}^{-1}$ is a bijection, restricted to a subset of total measure, and both it and its inverse are measurable.

In this way, $\overline{T}_1$ is a Bernoulli map, that is, the systems $(\overline{\Upsilon}_k, \mathscr{D}, med, \overline{T}_1)$ and $(\mathscr{A}_k^{\mathbb{N}}, \mathscr{B}, \mu_\pi^{\mathbb{N}}, \sigma_k^+)$ are equivalent.

In order to conclude the result, it is enough to show that $\overline{T}_1$ is strong-mixing, becuase a strong-mixing map is also weakly-mixing and all weakly-mixing is ergodic. But, we get that $\overline{T}_1$ is strong-mixing, since σ_k^+ is strong-mixing. Indeed, given any measurable sets $\Omega_A, \Omega_B \in \mathscr{D}$ we get

$$med\left((\overline{T}_1)^{-n}(\Omega_A) \cap \Omega_B\right) = \mu_\pi^{\mathbb{N}}\left((\bar{\mathfrak{s}}^{-1})^{-1}\left((\overline{T}_1)^{-n}(\Omega_A) \cap \Omega_B\right)\right) =$$

$$\mu_\pi^{\mathbb{N}}\left((\sigma_k^+)^{-n}(\bar{\mathfrak{s}}(\Omega_A)) \cap \bar{\mathfrak{s}}(\Omega_B)\right) \to \mu_\pi^{\mathbb{N}}\left(\bar{\mathfrak{s}}(\Omega_A)\right) \mu_\pi^{\mathbb{N}}\left(\bar{\mathfrak{s}}(\Omega_B)\right) = med(\Omega_A) med(\Omega_B),$$

when $n \to \infty$.

□

3.1.2 Proof of Corollary 1

Proof. The proof follows directly from the previous Theorem 1 and Proposition 7.1.6 of [53]. □

3.1.3 Proof of Corollay 2

Proof. (i) : By Lemma 4 the RPF operator is well defined for $\overline{T}_1\big|_{\widetilde{\Omega}}$. Now, consider a Markov partition $\mathscr{P} = \{P_0, \cdots, P_{k-1}\}$ in the domain of $\overline{T}_1$ such that $|\mathscr{P}| = k$, $med\left(\mathscr{P}_j\right) > 0\, \forall j \in \{0, 1, \cdots, k-1\}$ and

$$\phi_{\widetilde{Z}_k}(\mathscr{P}_j) = \bigcup_{B \in \mathscr{C} \subset \mathscr{P}} B \quad \text{for all} \quad \mathscr{P}_j \in \mathscr{P}.$$

So we will have the matrix that represents RPF operator for the geometric potential given by :

$$\left[\mathscr{L}^*_{\overline{T}_1|_{\widetilde{\Omega}}, -\beta \log|\det J_{med}\overline{T}_1|}\right]_{k\times k} := a_{ij} = \begin{cases} \left(\frac{med\left(\mathscr{P}_j \cap \phi_Z^{-1}(\mathscr{P}_i)\right)}{med\left(\mathscr{P}_j\right)}\right)^{\beta}, & med\left(\mathscr{P}_j \cap \phi^{-1}_{\widetilde{Z}_k}(\mathscr{P}_i)\right) \neq 0 \\ 0, & med\left(\mathscr{P}_j \cap \phi^{-1}_{\widetilde{Z}_k}(\mathscr{P}_i)\right) = 0 \end{cases}.$$

In this way we get the following matrix:

$$A = \left[\mathscr{L}^*_{\overline{T}_1|_{\widetilde{\Omega}}, -\beta \log|\det J_{med}\overline{T}_1|}\right] = \begin{pmatrix} p_{11}^{\beta} & p_{12}^{\beta} & \cdots & p_{1k}^{\beta} \\ p_{21}^{\beta} & p_{22}^{\beta} & \cdots & p_{2k}^{\beta} \\ \vdots & \vdots & \ddots & \vdots \\ p_{k1}^{\beta} & p_{k2}^{\beta} & \cdots & p_{kk}^{\beta} \end{pmatrix}, \tag{3.4}$$

such that $\sum_{i=1}^{k} p_{ij} = 1$, p_{ij} can be saw as the probability of $\phi_{\widetilde{Z}_k}(x) \in I_j$ if $x \in I_i$, for all $1 \leq i, j \leq k$, where k is the number of petals. (for a reference, the red curve on Figure 3.1 has 3 petals). Note that each subset of trajectories $\widetilde{\Omega} \subset \overline{\Omega}$ is associated with a graph and further by Lemma 5.5.1 of [19] (see the example 9), each graph is associated with a subshift of finite type. Therefore, by the Perron-Frobenius Theorem A 1 and by the Corollary 2.3 of [18], we get $P_{top}(\overline{T}_1\big|_{\widetilde{\Omega}}, -\beta \log|\det J_{med}\overline{T}_1|) = \log(\rho(A))$. Furthermore if $\beta = 0$, the transfer matrix is an adjacency's matrix and for $\beta = 1$ this matrix is the stochastic matrix. As this operator admits a finite representation, the topological pressure is given by

$$P_{top}(\beta) = \log\rho\left(\mathscr{L}^*_{-\beta \log|\det J_{med}T_1|}\right) = \log\rho\left(\mathscr{L}_{-\beta \log|\det J_{med}T_1|}\right) = \log(\rho(A)).$$

Moreover, if $\beta = 1$, by Perron-Frobenius Theorem A 1, we get $\rho(A) = 1$ and the topological pressure is zero.

(ii) : As we saw in the previous item, when $\beta = 1$ A is a stochastic matrix, in addition, since A is an irreducible, aperiodic and reversible matrix, $\vartheta = \vartheta_* > 0$ (see Definition 31). Finally, as the Markov chain is positive recurrent, then $e^{-t_{rel}} = \rho_{ess}(A)$, that is, $\rho_{ess}(A)$ characterizes a rate of exponential decay.

(iii) : Let $I_0, I_1, \cdots, I_{k-1}$ be the k trajectory arcs of the PSVF $\widetilde{Z}_k$. Let $\gamma \in \overline{\Upsilon}_k$, whose concatenation of its arcs is given as follows: $\gamma(0) \in I_i$ and there is no repetition in the concatenation of the following $k-1$ trajectory arcs. In this way, we establish a certain "ordination" for the set formed by the trajectory arcs. Once this "order" is established, for the next k concatenations, take the trajectory arc I_i and place it uniformly randomly in some other position (see Figure 3.3), i.e. the set of trajectory arcs is the same, however the "order" for the concatenation will change. Thus, from position k to position $2k-1$ the concatenations will follow this new "order". By repeating this process, we end up "shuffling", the set formed by $I_0, I_1, \cdots, I_{k-1}$ and each "shuffling" step establishes how the next k concatenations should be done. Note that the successive arrangements of the trajectory arcs that form γ generate a sequence of random variables $(\mathscr{X}_m)_{m\in\mathbb{N}}$ in the S_k group of $k!$ possible permutations.

We can then estimate how long we should "shuffle" using this method until the formation of γ is completely random. To do this, let $\tau_{top} \in \mathbb{N}$ be the time of a move after the first occasion on which the last arc of the original trajectory was "moved" to the beginning. From Proposition 6.1 by [37], the array formed by the arcs of γ in time τ_{top}, represented by the random variable $\mathscr{X}_{top}$ has uniform stationary distribution on the set S_k of all permutations of $\{1, 2\cdots, k\}$ and, moreover, this time τ_{top} is independent of $\mathscr{X}_{top}$.

It is a well know fact that the number of fixed points in a random permutation on S_k is 1, independent of how many elements are being permuted. Thus, let $\mathfrak{F}(\zeta)$ be the number of fixed points of the permutation ζ. If ζ is obtained from the identity by applying t random transpositions, then $\mathfrak{F}(\zeta)$ is at least as large as the arc number of trajectories that were not touched by any of the transpositions, i.e, none of these arcs of trajectories have been moved or some that have been moved may have returned to their original positions. Our "shuffle" chain defines the transpositions by choosing completely independent and uniformly random pairs of trajectories. In this way, the result is entirely analogous to the proofs of Proposition 8.4 and Corollary 8.10 of [37].

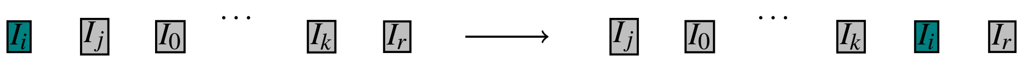


Figure 3.3: Scheme for the shuffling

(iv) The first part follows from Corollary 8.10 and Proposition 8.4 of [37]. The second part follows directly from item (ii)applied to chain of random transpositions in k trajectory arcs. □

3.1.4 Proof of Theorem 4

Proof. Analogous to the proof of Theorem 3. □

3.1.5 Proof of Corollary 3

Proof. Analogous to the proof of Corollary 1. □

3.1.6 Proof of Corollary 4

Proof. (i) : By Proposition 15, $\overline{T}_1$ and σ_2^+ are topologically conjugated by $\bar{s}$, exist a subset $\widetilde{\Omega} \subset \overline{\Omega}_2$, such that $\overline{T}_1\big|_{\widetilde{\Omega}}$ the RPF operator is well defined for $\overline{T}_1\big|_{\widetilde{\Omega}}$. See that such the shift is associated to the transition matrix:

$$A=\left[\mathscr{L}^{*}_{\overline{T}_1|_{\widetilde{\Omega}},-\beta\log|\det J_{med}\overline{T}_1|}\right]=\begin{pmatrix}(1-p_{11})^{\beta} & p_{12}^{\beta}\\ p_{21}^{\beta} & (1-p_{21})^{\beta}\end{pmatrix},$$

where p_{11} can be saw as the probability if $x\in I_0$ and p_{21} can be saw as the probability if $x\in I_1$. Again note that A also represents a matrix of an oriented graph, where each entry represents a_{ij} the

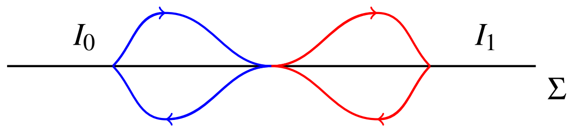


Figure 3.4: PSVF from Theorem 6.

probability that there is a connection (graph edge) between the arc I_0 and the arc I_1 (vertices of the graph). Furthermore each oriented graph is conjugated to the finite subshift of two symbols (see [19], Lemma 5.5.1). So, from the Perron-Frobenius Theorem A 1 and from the Corollary 2.3 of [18], it follows that $P_{top}(\overline{T}_1|_{\widetilde{\Omega}},-\beta\log|\det J_{med}\overline{T}_1|)=\log(\rho(A))$.

The items (ii) and (iv) – subitem (a) are entirely analogous to the proof of Corollary 2. $\square$

3.1.7 Proof of Theorem 5

Proof. Analogous to the proof of Theorem 3. $\square$

3.1.8 Proof of Corollary 5

Proof. Analogous to the proof of Corollary 1. $\square$

3.1.9 Proof of Corollary 6

Proof. $(i):$ By Proposition 15, $\overline{T}_1$ and σ_r^{+} are topologically conjugated by $\bar{s}$, exist a subset $\widetilde{\Omega}\subset\overline{\Omega}_k$, such that $\overline{T}_1|_{\widetilde{\Omega}}$ the RPF operator is well defined for $\overline{T}_1|_{\widetilde{\Omega}}$. See that such subshift is associated to the transition matrix:

$$A=\left[\mathscr{L}^{*}_{\overline{T}_1|_{\widetilde{\Omega}},-\beta\log|\det J_{med}\overline{T}_1|}\right]=[a_{ij}]_{m\times m}$$

$$=\begin{pmatrix}p_{11}^{\beta} & (1-p_{11})^{\beta} & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & p_{23}^{\beta} & (1-p_{23})^{\beta} & \cdots & 0 & 0\\ p_{31}^{\beta} & (1-p_{31})^{\beta} & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & p_{m(m-1)}^{\beta} & (1-p_{m(m-1)})^{\beta}\end{pmatrix},$$

such that $\sum_{i=1}^{m}p_{ij}=1$, p_{ij} can be saw as the probability of $\phi_{Z_k}(x)\in I_j$ if $x\in I_i$, for all $1\le i,j\le m$, where m is the number of trajectory arcs.

Note that A also represents a matrix of an oriented graph, where each entry represents a_{ij} the probability that there is a connection (graph edge) between the arc I_i and the arc I_j (vertices of the

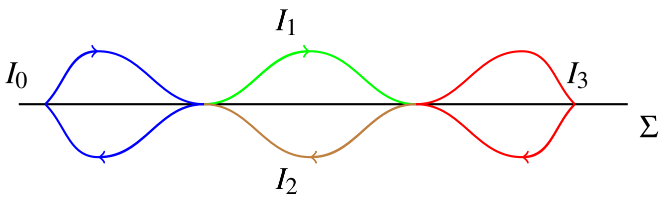


Figure 3.5: Case $k = 3$.

graph). Furthermore each oriented graph is conjugated to a finite subshift (see [19], Lemma 5.5.1) which in turn is conjugated to the space of all trajectories of the vector field Z_k (see [3]). So, from the Perron-Frobenius Theorem A 1 and from the Corollary 2.3 of [18], it follows that $P_{top}(\overline{T}_1 |_{\widetilde{\Omega}}, -\beta \log |\det J_{med}\overline{T}_1 |) = \log(\rho(A))$.

Items (ii), (iii) and (iv) are entirely analogous to the proof of Corollary 2. □

3.1.10 Proof of Theorem 6

Before proving Theorem 6 we will prove the following lemma.

Lemma 5. *Consider the family of maps*

$$\mathfrak{H}_\alpha(x) := \begin{cases} \alpha x \ \textit{for} \ 0 \le x \le \frac{1}{2} \\ \alpha(1-x) \ \textit{for} \ \frac{1}{2} \le x \le 1 \end{cases},$$

on $[0,1]$. *So for every* $\alpha \in (1,2]$, *the system* $([0,1], \mathscr{D}, med, \mathfrak{H}_\alpha)$ *is ergodically equivalent to the system* $(\{0,1\}^{\mathbb{N}}, \mathscr{B}, \mu_\pi^{\mathbb{N}}, \sigma_2^+)$, *for the probability vector* $\pi = (1/\alpha, 1-1/\alpha)$. *This family of maps is called generalized tent maps.*

Proof. We define $\Phi = (\Phi_n)_{n\in\mathbb{N}} : [0,1] \to \{0,1\}^{\mathbb{N}}$ by

$$\Phi_n(x) := \begin{cases} \alpha x, \ \text{if} \ 0 \le \mathfrak{H}_\alpha^n(x) \le \frac{1}{2} \\ \alpha(1-x), \ \text{if} \ \frac{1}{2} \le \mathfrak{H}_\alpha^n(x) \le 1 \end{cases}.$$

The conclusion of the proof follows from analogous to Example 12.5 by [21]. □

Proof of Theorem 6. Using Lemma 5 and Theorem 8.2.7 of [19], we conclude that $h_{top}(\mathfrak{H}_\alpha) = h_{top}(\sigma_2^+)$. Therefore, from Section 8.3.4 of [19], it follows that if $1 < \alpha \le 2$ then $h_{top}(\mathfrak{H}_\alpha) = \log \alpha$. In this way there is a connection between the "world" of one-sided shift spaces and the "world" of the generalized tent map. Then, since $\bar{s} : \overline{\Omega}_2 \to \{0,1\}^{\mathbb{N}}$ is a topological conjugacy, for every $1 < \alpha \le 2$ we have subsets $\widetilde{\Omega} \subseteq \overline{\Omega}_2$ whose entropy $h_{top}(Z_2) = h_{top}(T_1 |_{\widetilde{\Omega}}) = \log \alpha$.

Finally, from Corollary 5.1.18 of [56], for every $c \in (0, \log 2] \subset [0,1]$ there exists a set $A_c \subset (0, \log 2]$ such that $dim_{\mathscr{H}}(A_c) = c$. Furthermore, from what we saw in the previous paragraph and by (2.20), for every $c \in (0, \log 2]$, there is $1 < \alpha \le 2$ such that $c = \log \alpha$. So it occurs that

$$dim_{\mathscr{H}}(A_c) = \log \alpha = h_{top}(\mathfrak{H}_\alpha) = h_{top}(T_1 |_{\widetilde{\Omega}}) = dim_M(A_c).$$

□

Example 6. *In Corollary 2, take $\alpha = 4$. Let $\widetilde{\Omega} \subset \overline{\Upsilon}_4$ be such that $\widetilde{\Omega} = \{$orbits that: from I_0 go to I_1, I_2, I_3 and from I_j only goes to I_0, $j = 1,2,3\}$. Note that this subset of trajectories is associated with an ori-*

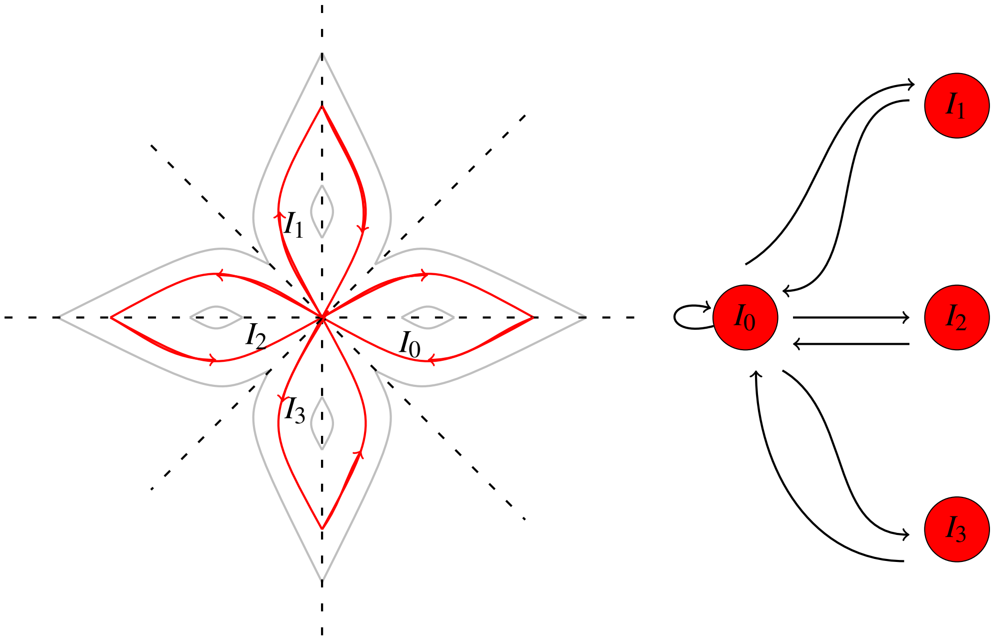


Figure 3.6: Trajectory arcs for $\widetilde{Z}_4$ and the related graph.

ented graph (see Figure 3.6), whose transition matrix is as follows:

$$A = \left[\mathscr{L}^*_{T_1|_{\widetilde{\Omega}}}\right] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

For A^2 we have $a_{ij} > 0$. So, it follows that A is an irreducible matrix and in addition the eigenvalues of A are $\lambda_1 = 0$ with multiplicity 2, $\lambda_2 = \frac{1+\sqrt{13}}{2}$ and $\lambda_3 = \frac{1-\sqrt{13}}{2}$. Therefore $h_{top}(T_1|_{\widetilde{\Omega}}) = \log(\rho(A)) = \log\left(\frac{1+\sqrt{13}}{2}\right)$.

Remark 12. *When A is defined by 1 in the entire first row and in the entire first column, plus all other entries are null, the shift associated with A is called* ***golden mean shift****.*

Example 7. *In Corollary 6, consider the case $k = 3$. Then $P_3(x) = -x^6 + \frac{3x^4}{2} - \frac{9x^2}{15} + \frac{1}{16}$ with roots ± 1 and $\pm\frac{1}{2}$, where the first three have multiplicity 2 and the last ones are simple. Moreover the PSVF Z_3 is:*

$$Z_3(x,y) = \begin{cases} X_3^+(x,y) & = \left(1, -6x^5 + 6x^3 - \frac{9x}{8} + \frac{9x^2}{2}\right) \text{ for } y \geq 0 \\ X_3^-(x,y) & = \left(-1, -6x^5 + 6x^3 - \frac{9x}{8} + \frac{9x^2}{2}\right) \text{ for } y \leq 0 \end{cases}.$$

The points $p_1 = (-\frac{1}{2}, 0)$, $p_2 = (\frac{1}{2}, 0)$ are visible-visible fold-folds and the other ones are crossing points of Z_3. The invariant region is the set:

$$\Lambda_3 = \{(x, P_3(x)) | -1 \leq x \leq 1\} \cup \{(x, -P_3(x)) | -1 \leq x \leq 1\},$$

that is partitioned into the arcs

$$I_0 = \left\{(x, P_3(x)) | -1 \le x < -\frac{1}{2}\right\} \cup \left\{(x, -P_3(x)) | -1 \le x < -\frac{1}{2}\right\},$$

$$I_1 = \left\{(x, P_3(x)) | -\frac{1}{2} \le x < \frac{1}{2}\right\},\ I_2 = \left\{(x, -P_3(x)) | -\frac{1}{2} \le x < \frac{1}{2}\right\} \text{ and}$$

$$I_3 = \left\{(x, P_3(x)) | \frac{1}{2} \le x < 1\right\} \cup \left\{(x, -P_3(x)) | \frac{1}{2} \le x < 1\right\} \text{ as shows Figure 3.7.}$$

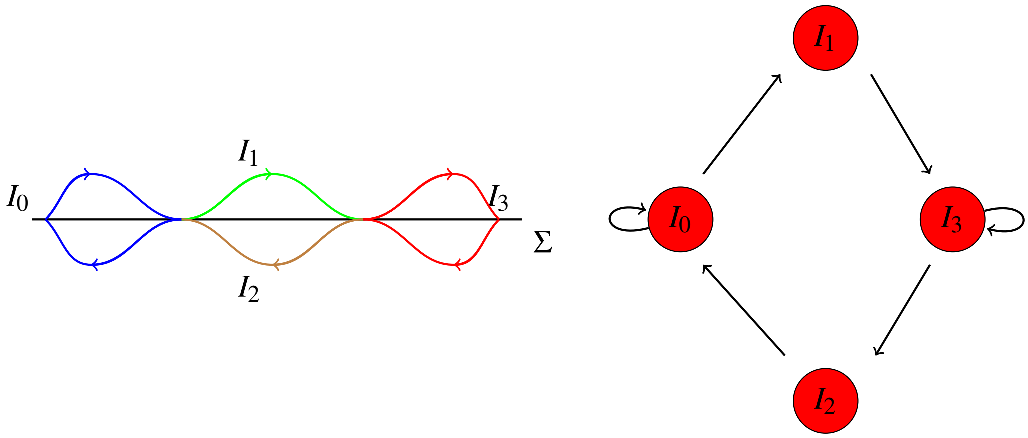


Figure 3.7: Trajectory arcs for Z_3 and the related graph.

Now, Ω_3 is the set of all trajectories contained in Λ_3 and $s : \Omega_3 \to \{0,1,2,3\}^{\mathbb{N}}$. If we take $\overline{\Omega}_3 = \Omega_3/\sim$ and the functions $\bar{s}$ and $\overline{T}_1$ as before, we have that $\bar{s}(\overline{\Omega}_3)$ is a subshift of $\{0,1,2,3\}^{\mathbb{N}}$ and $\bar{s}$ is a conjugacy between $\overline{T}_1$ and the shift σ_r^+ $(r \le 3)$. In fact, it is easy to see that such subshift is associated to the transition matrix:

$$A = \left[\mathscr{L}^*_{\overline{T}_1|_{\widetilde{\Omega}}, -\beta \log |J_{med}\overline{T}_1|}\right] = \begin{pmatrix} p_{11}^{\beta} & (1-p_{11})^{\beta} & 0 & 0 \\ 0 & 0 & p_{23}^{\beta} & (1-p_{23})^{\beta} \\ (1-p_{32})^{\beta} & p_{32}^{\beta} & 0 & 0 \\ 0 & 0 & (1-p_{44})^{\beta} & p_{44}^{\beta} \end{pmatrix}.$$

So, when $p_1 = p_{11} = p_{44}$ and $p_2 = p_{23} = p_{32}$, it follows that

$$P_{top}\left(\overline{T}_1|_{\widetilde{\Omega}}, -\beta \log |J_{med}\overline{T}_1|\right) = \log\left(\frac{(p_1^{\beta} + p_2^{\beta}) + \sqrt{(p_1^{\beta} - p_2^{\beta})^2 + 4(1-p_1)^{\beta}(1-p_2)^{\beta}}}{2}\right).$$

When $\beta = 0$, the matrix A is given by :

$$A = \left[\mathscr{L}^*_{\overline{T}_1|_{\widetilde{\Omega}}, 0}\right] = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Since $a_{ij} > 0$ for A^2, it follows that A is an irreducible matrix and in addition the eigenvalues of A are $\lambda_1 = 0$ with multiplicity 3, $\lambda_2 = 2$. Therefore $h_{top}(T_1|_{\widetilde{\Omega}}) = \log(\rho(A)) = \log 2$.

Example 8. *Consider in Corollary 2 that the matrix A has the following form $A = \frac{1}{k}B$, where B is the unitary matrix, i.e., all matrix entries are* 1*. So,*

$$p_A(x) = \begin{vmatrix} \left(\frac{1}{k}-x\right) & \frac{1}{k} & \cdots & \frac{1}{k} \\ \frac{1}{k} & \left(\frac{1}{k}-x\right) & \cdots & \frac{1}{k} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{k} & \frac{1}{k} & \cdots & \left(\frac{1}{k}-x\right) \end{vmatrix}.$$

Using elementary operations adding to one row a non-zero multiple of another, the determinate of the resulting matrix will not change. So we can use this tool to find $p_A(x)$. The determinant of the matrix is equal to the element found in the last row after the matrix has been reduced to echelon form using the formula

$$a_{ij} = \frac{a_{rc} \cdot a_{ij} - a_{ic} \cdot a_{rj}}{\mathfrak{p}},$$

where r and c are the row and column numbers of the supporting element, and $\mathfrak{p}$ is the pivot of i in i-th step of the process. So, we get

$$p_A(x) = (-1)^k x^{k-1}(x-1).$$

Therefore, all eigenvalues of A are $\lambda_0 = 1$ and $\lambda_1 = 0$ with multiplicity $k-1$. As all entries of A are equal to $\frac{1}{k}$, A is symmetric, so A is irreducible, aperiodic and reversible, whose probability distribution is $\pi(i) = 1/k$ for all i, moreover, by definition $t_{mix} = 0$, since $\kappa(m) = 0$. Note also that although all states in this chain are positive recurrent, we do not have exponential decay, since its second largest eigenvalue is zero. But we can still calculate t_{rel} using the Definition 31, i.e., $t_{rel} = \frac{1}{1-\rho_{ess}(A)} = 1$. When the probability transition matrix A is symmetric the distribution $\pi(i) = 1/k$ for all i this is called uniform distribution, that is, the transition matrix A satisfies $\sum_i p_{ij} = 1$ and $\sum_j p_{ij} = 1$.

Remark 13. *Something similar happens if the probabilities of the matrix A of Corollary 4 are equal to $\frac{1}{2}$.*

Example 9. *Let $\widetilde{\Omega} \subset \overline{\Omega}_2$ be such that $\widetilde{\Omega} = \{$orbits that: $I_0 \leftrightarrow I_0, I_0 \leftrightarrow I_1\}$.*

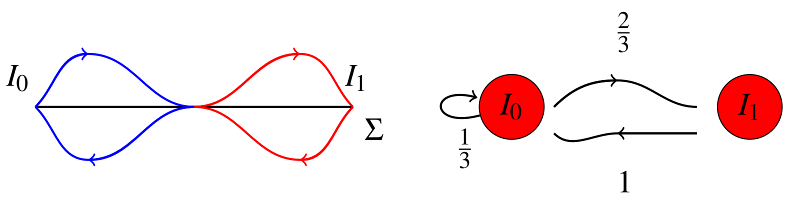


Figure 3.8: Markov chain for $\widetilde{\Omega}$ from Corollary 6.

Note that this subset of trajectories is associated with an oriented graph (see Figure 3.8), whose Markov chain transition matrix is as follows:

$$A = \left[\mathcal{L}^*_{T_1|_{\widetilde{\Omega}}, -\log|\det J_{med}\overline{T}_1|}\right] = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{pmatrix}.$$

Note that for A^2 *we have* $a_{ij} > 0$*, so* A *is irreducible and aperiodic. Furthermore,* A *is reversible, as every irreducible two-state Markov chain is reversible and all its states are positive recurrent. Calculating the eigenvalues of* A *we have* $\lambda_0 = 1$ *and* $\lambda_1 = -\frac{2}{3}$.

Since the stationary distribution exists and is unique, we can find it by solving the equation $\pi = \pi A$, $\pi = (\pi_0\,\pi_1)$*, together with* $\pi_0 + \pi_1 = 1$*. So, we obtain* $\pi_0 = \frac{3}{5}$ *and* $\pi_1 = \frac{2}{5}$*. Therefore, by Corollary 6,* $\rho_{ess}(A) = e^{-t_{rel}} \Rightarrow \frac{2}{3} = e^{-t_{rel}} \Leftrightarrow t_{rel} = \log\left(\frac{3}{2}\right)$ *and then again by Corollary 6,*

$$\left(\log\left(\frac{3}{2}\right) - 1\right)\log(2) \leq t_{mix} \leq \log\left(\frac{3}{2}\right)\log\left(\frac{5}{3}\right).$$

CHAPTER 4

Piecewise smooth vector fields with sliding motion preserving measures

As we already said, PSVFs with a sliding region can transform certain sets of $\mathbb{R}^n$ with positive measure on a line segment with zero measure. This fact adds extra difficulty when studying invariant measures. However, the escaping region has the power to transform that same straight line into a set of positive measures. Therefore, in this chapter we will show that there are PSVFs with $\Sigma^e \cup \Sigma^s \neq \emptyset$ such that the measure is preserved, as long as the regions Σ^e and Σ^s are "connected" in some way. The next definition will be the **keypoint** in the sequence.

Definition 52. *Consider an n-dimensional PSVF $Z \in \mathscr{Z}^r$ presenting a sliding region Σ^s and an escaping region Σ^e.*

(i) *Z presents an* ***sliding-escaping connection*** *between Σ_1^s and Σ_1^e if there exist open connected components $\Sigma_1^s \subset \Sigma^s$ and $\Sigma_1^e \subset \Sigma^e$, such that*

$$\forall p \in \Sigma_1^s, \exists q \in \Sigma_1^e \text{ such that } \phi_Z(t,p) = q \text{ for some Z-trajectory/flow } \phi_Z \text{ and some } t > 0$$

and

$$\forall \widetilde{p} \in \Sigma_1^e, \exists \widetilde{q} \in \Sigma_1^s \text{ such that } \widetilde{\phi_Z}(\widetilde{t},\widetilde{p}) = \widetilde{q} \text{ for some Z-trajectory/flow } \widetilde{\phi_Z} \text{ and some } \widetilde{t} > 0.$$

(ii) *When there is an sliding-escaping connection for Z between Σ_1^s and Σ_1^e, we call $\mathscr{A} = Sat\left(\Sigma_1^s\right)$ a* ***connecting domain****.*

Example 10. *Consider the PSVF*

$$Z(x,y) = \begin{cases} X^+(x,y) = (1,-2x) \text{ for } y \geq 0 \\ X^-(x,y) = \left(-2,-4x^3+2x\right) \text{ for } y \leq 0 \end{cases}, \tag{4.1}$$

presented in [3].

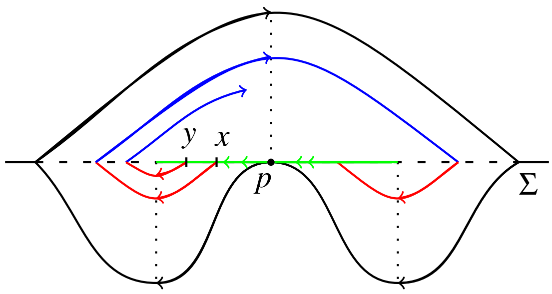


Figure 4.1: Arc of trajectory γ, such that, $\gamma(0) = x$, and $\gamma(\eta_\gamma) = y$. Both x and y are escape points from Σ^e to Σ^- to γ.

Let $\Delta = \{(x,y) \in \mathbb{R}^2; -1 \leq x \leq 1$ *and* $x^4/2 - x^2/2 \leq y \leq 1 - x^2\}$. *From Proposition* 1 *of [12] we know that* Δ *is a minimal set of* (4.1).

Now consider the following construction:

(i) γ_1 *an arc of trajectory of* X^- *connecting the point* $x = \gamma_1(0) \in \Sigma^e$ *with a point* $\widetilde{y} \in \Sigma$;

(ii) γ_2 *an arc of trajectory of* X^+ *connecting the point* $\widetilde{y}$ *with a point* $\widetilde{u} \in \Sigma$;

(iii) γ_3 *an arc of trajectory of* X^- *connecting the point* $\widetilde{u}$ *with a point* $v \in \Sigma^s$;

(iv) γ_4 *an arc of trajectory of* X^- *connecting the point* $y = \gamma_1(\eta_{\gamma_1}) \in \Sigma^e$ *with a point* $\widetilde{x} \in \Sigma$;

(v) γ_5 *an arc of trajectory of* X^+ *connecting the point* $\widetilde{x}$ *with a point* $\widetilde{v} \in \Sigma$;

(vi) γ_6 *an arc of trajectory of* X^- *connecting the point* $\widetilde{v}$ *with a point* $u \in \Sigma^s$.

The gray region is the set bounded by the union of trajectory arcs above, together with the trajectories of the sliding vector field Z^s *and the escaping vector field* Z^e*, (see Figure 4.2).*

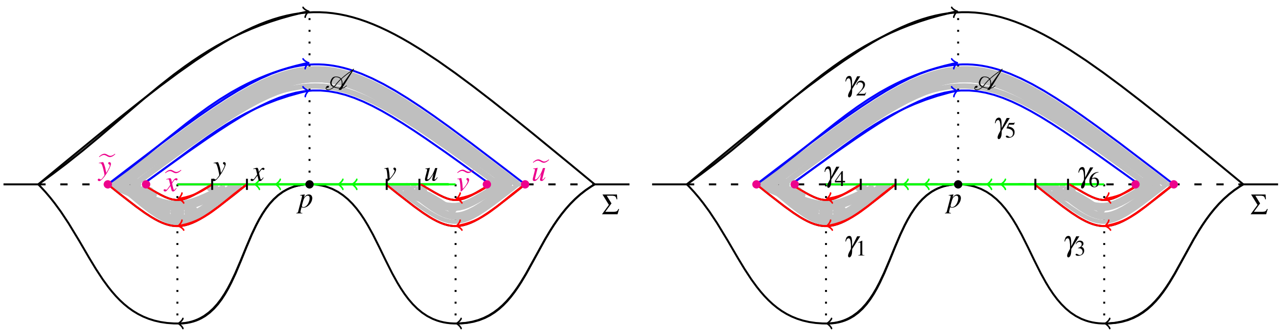


Figure 4.2: Connection domain

Note that (v,u) *and* (y,x) *are open connected components of* Σ^s *and* Σ^e *respectively. Furthermore, by construction, for every* $a \in (v,u)$*, there exists* $b \in (y,x)$ *such that* $\widetilde{\phi_Z}(t,a) = b$ *for some Z-trajectory* ϕ_Z *and some* $t > 0$*. Analogously, for every* $\widetilde{a} \in (y,x)$*, there is* $\widetilde{b} \in (v,u)$ *such that* $\phi_Z(\widetilde{t},\widetilde{a}) = \widetilde{b}$ *for some Z-trajectory* $\widetilde{\phi_Z}$ *and some* $\widetilde{t} > 0$*. This way, the connection domain is* $\mathscr{A} = Sat((v,u))$*.*

Let $T \in \mathbb{R}$ fixed and $\widetilde{\Theta} \subset \Lambda$ a set of global trajectories of the PSVF Z. We denote

$$Sat(A,T,\widetilde{\Theta}) = \bigcup_{\phi_Z \in \widetilde{\Theta}} \bigcup_{p \in A} \phi_Z(T,p).$$

Note that $Sat(A,T,\widetilde{\Theta}) \subset Sat(A)$.

From now on, we will always work with the Lebesgue measure, which will be denoted by *med*. The next result is the main result of this chapter and it will proved in Section 4.1.

Theorem 7. *Given an n-dimensional PSVF* $Z = (X^+,X^-) \in \mathscr{Z}^r$*, let* $\mathscr{A}$ *be a connecting domain between* Σ_1^s *and* Σ_1^e *such that the sets* Σ_1^s *and* Σ_1^e *have the same dimension and the divergent* $div(X^{\pm}) = 0$*. Then for all* $\alpha \in [0,\beta]$*, with* $\beta > 1$*, and a subset* $\widetilde{\Lambda} \subset \Lambda$*, where* Λ *is the set of all trajectories of Z, we have*

$$med(Sat(A,T,\widetilde{\Lambda})) = \alpha.med(A)$$

for a fixed time $T \in \mathbb{R}$ *such that* $A \subset \mathscr{A} \setminus \Sigma$ *is a compact set,* $med(A) > 0$ *and* $Sat(A,T,\widetilde{\Lambda}) \cap \Sigma \neq \emptyset$*.*

There are a lot of important results that can be obtained as consequences of the previous theorem. First of all, since α is a positive real number (including zero), it is possible to obtain:

- a suitable set $\widetilde{\Lambda}$ such that $Sat(A,T,\widetilde{\Lambda})$ has null measure (i.e $\alpha = 0$ in Theorem 7).
- $med(Sat(A,T_1,\widetilde{\Lambda})) = K_1$ and, since $Sat(A,T_1,\widetilde{\Lambda})$ can be saw like a new set $B \subset \mathscr{A} \setminus \Sigma$, we get $med(Sat(A,T_2,\widetilde{\Lambda})) = K_2$ with $K_1 \neq K_2$. When $K_1 > 1$ and $K_2 < 1$, the set $Sat(A,T,\widetilde{\Lambda})$ "*expands*" for all T in a suitable interval T_1 and $Sat(A,T,\widetilde{\Lambda})$ "*contracts*" for all T in a suitable interval T_2. This generates a "*horse-shoes like*" dynamics.

Let ϕ_X^t be the flow of a smooth vector field X defined over a Riemannian manifold M and μ a measure in M. We say that a flow ϕ_X^t preserves a measure μ if: for any Borel set $A \subset M$, $\mu(\phi_X^t(A)) = \mu(A)$, $\forall t \in \mathbb{R}$. However, when we consider Filippov systems, we saw that for a given initial condition $p_0 \in M$, it is possible that there is not an unique solution passing through p_0. Consequently, the previous definition of flow preservation and measurement fails. Therefore, to get around this difficulty, considering the analogous definition of a preserving measure for flow, we say that Filippov systems (2.1) preserves the measure *med* if for $T \in \mathbb{R}$ fixed and $\widetilde{\Lambda} \subset \Lambda$ an specific choice of trajectories of the PSVF Z if,

$$med(Sat(A,T,\widetilde{\Lambda})) = med(A). \tag{4.2}$$

Furthermore we denote

$$\phi_Z^T(p) = \phi_Z(T,p) = Sat(p,T,\widetilde{\Lambda}) = \bigcup_{\phi_Z \in \widetilde{\Lambda}} \phi_Z(T,p), \ \forall p \ \in A. \tag{4.3}$$

Therefore, from (4.2) and (4.3), we say that a PSVF preserves a measure *med* if $med\left(\phi_Z^T(A)\right) = med(A)$, for any Borel subset $A \subset M$.

Next result provides conditions in order to obtain a set of trajectories of a PSVF preserving measure.

Corollary 7. *With the hypothesis of Theorem 7, there exists a subset $\widetilde{\Lambda}$ of the set of all trajectories Λ of Z, such that $med\left(Sat\left(A,T,\widetilde{\Lambda}\right)\right) = med(A)$.*

Proof. It is enough to take $\alpha = 1$ in the proof of Theorem 7. □

Let $Z = (X^+, X^-)$ be defined over a compact n dimensional surface $M \subseteq \mathbb{R}^n$ and, as before, $\Lambda = \{\gamma \mid \gamma \text{ is a global trajectory of } Z\}$. Consider the following definition:

Definition 53. *Define $\rho : \Lambda \times \Lambda \to \mathbb{R}$ by:*

$$\rho(\gamma_1(t), \gamma_2(t)) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \int_i^{i+1} \| \gamma_1(t) - \gamma_2(t) \| \, dt,$$

where $\| \gamma_1(t) - \gamma_2(t) \|$ denotes the usual distance between the points $\gamma_1(t)$ and $\gamma_2(t)$ for all fixed t.

Proposition 16. *The space (Λ, ρ) is a metric space.*

Proof. Let $\gamma_1, \gamma_2 \in \Lambda$. Observe that M being compact, implies $\| \gamma_1(t) - \gamma_2(t) \|$ is bounded for all t. Thus, the series above converges for any γ_1, γ_2. If $\rho(\gamma_1, \gamma_2) = 0$ then $\int_i^{i+1} \| \gamma_1(t) - \gamma_2(t) \| \, dt = 0$ for all $i \in \mathbb{Z}$, which implies $\gamma_1(t) = \gamma_2(t)$ for all t and therefore $\gamma_1 = \gamma_2$. The proof of $\rho(\gamma_1, \gamma_2) = \rho(\gamma_2, \gamma_1)$ follows directly from $\| \gamma_1(t) - \gamma_2(t) \| = \| \gamma_2(t) - \gamma_1(t) \|$.

Finally, using the triangular inequality $\| \gamma_1(t) - \gamma_2(t) \| \leq \| \gamma_1(t) - \gamma_3(t) \| + \| \gamma_3(t) - \gamma_2(t) \|$ for all t, we get the inequality $\rho(\gamma_1, \gamma_2) \leq \rho(\gamma_1, \gamma_3) + \rho(\gamma_3, \gamma_2)$. □

In this space of all global trajectories we can define the action of the group $(\mathbb{R},+)$ on Λ by $\mathscr{T}:(\mathbb{R},+)\times\Lambda\to\Lambda,\ \mathscr{T}(t,\gamma)(\cdot)=\gamma(\cdot+t)$.

Remark 14. *Let $\mathscr{O}=\{$positive global trajectories of $Z\}$. Note that $(\mathscr{O},\rho)$, is also a metric space, where in this case, $\rho(\gamma_1(t),\gamma_2(t))=\sum\limits_{i\in\mathbb{N}}\frac{1}{2^i}\int_i^{i+1}\parallel\gamma_1(t)-\gamma_2(t)\parallel dt$. Additionally, notice that $\mathscr{T}:\mathbb{R}^+\times\mathscr{O}\to\mathscr{O},\ \mathscr{T}(t,\gamma)(\cdot)=\gamma(\cdot+t)$ is well defined, where $\mathbb{R}^+=\{t\in\mathbb{R}\,|\,t\geq 0\}$.*

Definition 54. *We say that a property is valid at med-almost every point whether it is valid on all $\mathfrak{A}$ except possibly on a set of zero measure.*

Definition 55. *We say that a trajectory γ escapes $\overline{\Sigma^e}$ at a time s_0 if $\gamma(s_0)\in\overline{\Sigma^e}$ and there is $s_1>s_0$, such that $\gamma(s)\notin\overline{\Sigma^e}$, for all $s_0<s<s_1$. We may also add that γ escapes to Σ^+ or Σ^-, depending on which of these regions the points $\gamma(s)$ are located in. Alternatively, we may say that $\gamma(s_0)$ is a escape point of γ.*

Let A be a compact invariant set. Consider the set

$$\Omega=\left\{\gamma\,\text{positive global trajectory of }\ Z\,|_{Sat\left(A,T,\widetilde{\Lambda}\right)}\right\}.$$

Remark 15. *We get that Ω is not compact, because it is not closed. In fact, consider $(\gamma_{p_n})_{n\in\mathbb{N}}\subset\Omega$ a sequence of closed simple trajectories through p, such that the period of each n is half of the previous period. So, $\int_i^{i+1}\parallel(\gamma_{p_n})-p\parallel dt\to 0$. Then $(\gamma_{p_n})_{n\in\mathbb{N}}$ converges to a fixed trajectory $\widetilde{\gamma}(t)\equiv p$, but $\widetilde{\gamma}(t)\notin\Omega$.*

For each $p\in\overline{\Sigma^e}$ there is a $(n-1)$-dimensional block B^p containing p. Without loss of generality, we can take such a block as a unitary block $B_{[0,1]}$, because if it is not, there is a family of homeomorphisms h_i that takes each edge $[a_i,b_i]$ of B^p on each edge $[0,1]$ of $B^p_{[0,1]}$. Furthermore, since Z presents a sliding-escaping connection, given a trajectory $\gamma\in\Omega$, there exists $t>0$ such that γ reaches p.

Put $\mathfrak{G}=\bigcup\limits_{i=0}^{\infty}B^{p_i}_{[0,1]}$. Since every local trajectory intersects it and every point $p\in\mathfrak{G}$ reaches $\widetilde{p}\in A$ in a finite positive time, we get that every trajectory visits it infinitely many times. Thus, the set $\mathfrak{G}$ will play the role of $\{p_j\}$ obtained in Theorem A of [2]. However, we cannot adjust the expression of a PSVF Z to make the time between visits equal to some constant, as was done in the Theorem A of [2].

In order to deal with this, define the function $\eta:\Omega\to\mathbb{R}^+$ such that $\eta(\gamma)=\eta_\gamma=\min\{t>0\mid\gamma(t)$ escapes $\overline{\Sigma^e}$ at $p\in\mathfrak{G}\}$, in other words, the time η_γ is the "next escape time" for γ. Now, define the map $\mathscr{T}:\Omega\to\Omega$, as $\mathscr{T}(\gamma)(\cdot)=\gamma(\cdot+\eta_\gamma)$.

Remark 16. *Given $\gamma\in\Omega$, we have $\mathscr{T}^2(\gamma)(t)=\mathscr{T}\circ\mathscr{T}(\gamma)(t)=\mathscr{T}(\gamma)(t+\eta_{\mathscr{T}(\gamma)})=\gamma(t+\eta_{\mathscr{T}(\gamma)}+\eta_\gamma)$. So, $\mathscr{T}^n(\gamma)(t)=\mathscr{T}\circ\mathscr{T}^{n-1}(\gamma)(t)=\gamma(t+\eta_{\mathscr{T}^{n-1}(\gamma)}+\cdots+\eta_{\mathscr{T}(\gamma)}+\eta_\gamma)$.*

The following Propositions and Lemmas are adaptations and generalizations of results found in [2].

Lemma 6. *Given a positive global trajectory $\gamma \in \Omega$, there exists a infinite increasing sequence $(t_j^{\gamma})_{j\in\mathbb{N}}$ and $q \in \Sigma_1^s$, such that $\gamma_q(t_j^{\gamma_q})$ escapes $\overline{\Sigma^e}$ at $p = \mathscr{T}^j(\gamma_q)(0) \in \mathfrak{G}$, where γ_q is a global trajectory passing through q.*

Proof. Since Z presents a sliding-escaping connection, given a trajectory $\gamma \in \Omega$, there exists $t > 0$ such that γ_q reaches p. Therefore, to conclude the proof, we just need to show that $\gamma_q(t_j^{\gamma_q}) = p$. Thus, for all $j > 0$, let $t_j^{\gamma_q} = \sum_{n=0}^{j} \eta_{\mathscr{T}^n(\gamma_q)}$. We get

$$\gamma_q(t_j^{\gamma_q}) = \gamma_q\left(\eta_{\gamma_q} + \eta_{\mathscr{T}(\gamma_q)} + \cdots + \eta_{\mathscr{T}^{j-1}(\gamma_q)}\right) = \mathscr{T}(\gamma_q)\left(\eta_{\mathscr{T}(\gamma_q)} + \cdots + \eta_{\mathscr{T}^{j-1}(\gamma_q)}\right) =$$

$$\mathscr{T}^2(\gamma_q)\left(\eta_{\mathscr{T}^2(\gamma_q)} + \cdots + \eta_{\mathscr{T}^{j-1}(\gamma_q)}\right) = \cdots = \mathscr{T}^{j-1}\left(\eta_{\mathscr{T}^{j-1}(\gamma_q)}\right) = \mathscr{T}^j(\gamma_q)(0) = p \in \mathfrak{G}.$$

□

Lemma 7. *Given a positive global trajectory $\gamma \in \Omega$ and $(t_j^{\gamma})_{j\in\mathbb{N}}$ given above, consider the sequence $(t_j^{\mathscr{T}(\gamma)})$. It holds $t_j^{\mathscr{T}(\gamma)} = t_{j+1}^{\gamma} - \eta_\gamma$.*

Proof. In fact, $t_0^{\gamma} = 0 = \eta_\gamma - \eta_\gamma = t_1^{\gamma} - \eta_\gamma$.

For $j > 0$,

$$t_j^{\mathscr{T}(\gamma)} = \sum_{n=0}^{j-1} \eta_{\mathscr{T}^{n+1}(\gamma)} = \sum_{n=1}^{j} \eta_{\mathscr{T}^n(\gamma)} - \eta_\gamma = t_{j+1}^{\gamma} - \eta_\gamma.$$

□

Remark 17. *Sometimes, when we find it necessary, we will use the simplified notation $\gamma(t_j^{\gamma})$ to refer to the trajectories that pass through $q \in \Sigma^s$ and escape through $p \in \mathfrak{G}$.*

Definition 56. *Consider $\theta : \overline{\Sigma^e} \cap \mathfrak{G} \to [0,1]$ the projection on θ-coordinate that is, $\theta(y_1, y_2, \cdots, y_\theta, \cdots, y_{n-1}) = y_\theta$ and define the itinerary map $\mathfrak{b} : \Omega \to [0,1]^{\mathbb{N}}$, as $\mathfrak{b}(\gamma) = (\mathfrak{b}_j(\gamma))_{j\in\mathbb{N}}$, where $\mathfrak{b}_j(\gamma) = \theta(\gamma(t_j^{\gamma}))$. In this way, every single trajectory can be encoded by the θ-coordinates of its beats on $\mathfrak{G}$.*

Note that $\mathfrak{b}$ is well-defined and, by construction, it is onto because given a sequence $(x_j)_{j\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ it is possible to construct a trajectory $\gamma \in \Omega$ by concatenating the correct arcs in order to get $\mathfrak{b}(\gamma) = (x_j)_{j\in\mathbb{N}}$. Furthermore, there are infinitely many trajectories that describes the same curve, simply by changing the initial condition (a shift in time), and the function $\mathfrak{b}$ takes all of these to the same sequence. So, we consider the quotient set $\overline{\Omega} = \Omega/\mathfrak{b}$, where two trajectories γ_1, γ_2 are in the same equivalence class if $\mathfrak{b}(\gamma_1) = \mathfrak{b}(\gamma_2)$, i.e., two trajectories are in the same equivalence class if and only if they have the same itinerary map associated, i.e., they are homeomorphic to the same closed interval.

In fact, it is an equivalence relation. For $\gamma_1 \in \Omega$, we have $\gamma_1 \sim \gamma_1$ because $\mathfrak{b}(\gamma_1) = \mathfrak{b}(\gamma_1)$. Also for all $\gamma_1, \gamma_2 \in \Omega$, if $\gamma_1 \sim \gamma_2$, we have $\mathfrak{b}(\gamma_1) = \mathfrak{b}(\gamma_2)$, and in this way, $\gamma_2 \sim \gamma_1$. Finally, for all $\gamma_1, \gamma_2, \gamma_3 \in \Omega$, if $\gamma_1 \sim \gamma_2$, we have $\mathfrak{b}(\gamma_1) = \mathfrak{b}(\gamma_2)$ and if $\gamma_2 \sim \gamma_3$, we have $\mathfrak{b}(\gamma_2) = \mathfrak{b}(\gamma_3)$. In this way, $\mathfrak{b}(\gamma_1) = \mathfrak{b}(\gamma_2) = \mathfrak{b}(\gamma_3)$, and therefore $\gamma_1 \sim \gamma_3$.

Definition 57. *Let $\overline{\gamma_1}, \overline{\gamma_2} \in \overline{\Omega}$, define $\rho : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$, by*

$$\rho(\overline{\gamma_1}, \overline{\gamma_2}) = \sum_{i \in \mathbb{N}} \frac{d_i(\gamma_1^*, \gamma_2^*)}{2^i},$$

where $d_i(\gamma_1^*, \gamma_2^*) = d_H\left(\gamma_1^*([t_i^{\gamma_1^*}, t_{i+1}^{\gamma_1^*}]), \gamma_2^*([t_i^{\gamma_2^*}, t_{i+1}^{\gamma_2^*}])\right)$. That is, at each step i, we take the Hausdorff distance between the i-th loops of both trajectories.

Proposition 17. $(\overline{\Omega}, \rho)$ *is a metric space.*

Proof. The function ρ defined above is well-defined, since A is a compact set, which implies there exists $\mathfrak{C} > 0$, such that $d_i(\gamma_1^*, \gamma_2^*) < \mathfrak{C}$, for all $i \in \mathbb{N}$. The proof of those properties for it to be a metric is analogous to the one given in Proposition 7 of [2]. $\square$

Let $\overline{\mathscr{T}} : \overline{\Omega} \to \overline{\Omega}$ the function induced by $\mathscr{T}$, that is, $\overline{\mathscr{T}}(\overline{\gamma}) = \overline{\mathscr{T}(\gamma)}$. It does not depend on the representative, for if $\mathfrak{b}(\gamma_1) = \mathfrak{b}(\gamma_2) = (\mathfrak{b}_j)_{j \in \mathbb{N}}$, then, for all $j \in \mathbb{N}$, $\theta\left(\gamma_1((t_j^{\gamma_1}))\right) = \theta\left(\gamma_2((t_j^{\gamma_2}))\right) = \mathfrak{b}_j$. In fact,

$$\theta\left(\mathscr{T}(\gamma_1)(t_j^{\mathscr{T}(\gamma_1)})\right) = \theta\left(\gamma_1(t_j^{\mathscr{T}(\gamma_1)} + \eta_{\gamma_1})\right) = \theta\left(\gamma_1(t_{j+1}^{\mathscr{T}(\gamma_1)} - \eta_{\gamma_1} + \eta_{\gamma_1})\right)$$

$$= \theta\left(\gamma_1(t_{j+1}^{\gamma_1})\right) = \theta\left(\gamma_2(t_{j+1}^{\gamma_2})\right) = \theta\left(\mathscr{T}(\gamma_2)(t_j^{\mathscr{T}(\gamma_2)})\right) \Rightarrow \mathfrak{b}(\mathscr{T}(\gamma_1)) = \mathfrak{b}(\mathscr{T}(\gamma_2)).$$

Proposition 18. *The function $\overline{\mathscr{T}} : \overline{\Omega} \to \overline{\Omega}$ is continous.*

Proof. Note that $d_i(\mathscr{T}(\gamma_1), \mathscr{T}(\gamma_2)) = d_H\left(\mathscr{T}(\gamma_1)([t_i^{\mathscr{T}(\gamma_1)}, t_{i+1}^{\mathscr{T}(\gamma_1)}]), \mathscr{T}(\gamma_2)([t_i^{\mathscr{T}(\gamma_2)}, t_{i+1}^{\mathscr{T}(\gamma_2)}])\right)$

$$= d_H\left(\mathscr{T}(\gamma_1)([t_{i+1}^{\mathscr{T}(\gamma_1)}, t_{i+2}^{\mathscr{T}(\gamma_1)}]), \mathscr{T}(\gamma_2)([t_{i+1}^{\mathscr{T}(\gamma_2)}, t_{i+2}^{\mathscr{T}(\gamma_2)}])\right) = d_{i+1}(\gamma_1, \gamma_2).$$

So,

$$\rho\left(\overline{\mathscr{T}}(\gamma_1), \overline{\mathscr{T}}(\gamma_2)\right) = \sum_{i \in \mathbb{N}} \frac{d_i(\mathscr{T}(\gamma_1), \mathscr{T}(\gamma_2))}{2^i} = \sum_{i \in \mathbb{N}} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^i}.$$

$$\lim_{n \to \infty}\left(\sum_{i=0}^{n} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^i}\right) = \lim_{n \to \infty}\left(2\sum_{i=0}^{n} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{i+1}}\right) =$$

$$\lim_{n \to \infty}\left(2\sum_{j=1}^{n+1} \frac{d_j(\gamma_1, \gamma_2)}{2^j}\right) \leq \lim_{n \to \infty}\left(2\sum_{j=1}^{n+1} \frac{d_j(\gamma_1, \gamma_2)}{2^j}\right) + d_0(\gamma_1, \gamma_2) =$$

$$2\lim_{n \to \infty}\left(\sum_{j=1}^{n+1} \frac{d_j(\gamma_1, \gamma_2)}{2^j} + \frac{d_0(\gamma_1, \gamma_2)}{2}\right) = 2\rho(\gamma_1, \gamma_2).$$

Hence $\mathscr{T}$ is continuous. $\square$

Proposition 19. $\overline{\mathfrak{b}} : \overline{\Omega} \to [0,1]^{\mathbb{N}}$ *is a homeomorphism.*

Proof. The function $\mathfrak{b}$ is onto, which implies that $\overline{\mathfrak{b}}$ is onto and it is injective because it is defined on the quotient $\Omega/\mathfrak{b}$. So, we just have to show the continuity.

Note that $\| \mathfrak{b}_j(\gamma_1) - \mathfrak{b}_j(\gamma_2) \| = \| \gamma_1(t_j^{\gamma_1}) - \gamma_2(t_j^{\gamma_2}) \| \leq d_j(\gamma_1,\gamma_2)$ for all $j \in \mathbb{N}$. Then:

$$d(\mathfrak{b}(\gamma_1),\mathfrak{b}(\gamma_2)) = \sum_{j\in\mathbb{N}} \frac{\| \mathfrak{b}_j(\gamma_1) - \mathfrak{b}_j(\gamma_2) \|}{2^i} \leq \sum_{j\in\mathbb{N}} \frac{d_j(\gamma_1,\gamma_2)}{2^j} = \rho(\gamma_1,\gamma_2).$$

Thus $\overline{\mathfrak{b}}$ is continuous. Now let us show that $\overline{\mathfrak{b}}^{-1}$ is continuous. In order to do it, let us show that $\overline{\mathfrak{b}}$ is open: let $\gamma_1,\gamma_2 \in \overline{\Omega}$, and $\varepsilon > 0$ such that $d(\overline{\mathfrak{b}}(\gamma_1),\overline{\mathfrak{b}}(\gamma_2)) < \varepsilon$. Then $d_i(\gamma_1,\gamma_2) < \mathfrak{I}\varepsilon$, where $\mathfrak{I} > 0$ is such that $diam(A) < \mathfrak{I}$. Hence

$$\rho(\gamma_1,\gamma_2) = \sum_{i\in\mathbb{N}} \frac{d_j(\gamma_1,\gamma_2)}{2^i} < 3\mathfrak{I}\varepsilon.$$

□

Proposition 20. *Let $\sigma_\infty^+ : [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ be the shift map. Then $\overline{\mathfrak{b}} \circ \overline{\mathscr{T}} = \sigma_\infty^+ \circ \overline{\mathfrak{b}}$.*

Proof. Let $\gamma \in \overline{\Omega}$, $(x_j)_{j\in\mathbb{N}} = \overline{\mathfrak{b}}(\gamma)$ and $(y_j)_{j\in\mathbb{N}} = \overline{\mathfrak{b}}(\overline{\mathscr{T}}(\gamma))$, then :

$$\begin{tikzcd}
\overline{\Omega} \arrow[r, "\overline{T}_1"] \arrow[d, "\overline{\mathfrak{b}}"'] & \overline{\Omega} \arrow[d, "\overline{\mathfrak{b}}"] \\
{[0,1]^{\mathbb{N}}} \arrow[r, "\sigma_\infty^+"] & {[0,1]^{\mathbb{N}}}
\end{tikzcd}$$

$$y_j = \mathfrak{b}_j(\overline{\mathscr{T}}(\gamma)) = \theta\left(\overline{\mathscr{T}}(\gamma)(t_j^{\overline{\mathscr{T}}(\gamma)})\right) = \theta\left(\overline{\mathscr{T}}(\gamma)(t_j^{\overline{\mathscr{T}}(\gamma)} + \eta_\gamma)\right) =$$
$$\theta\left(\overline{\mathscr{T}}(\gamma)(t_j^{\overline{\mathscr{T}}(\gamma)} - \eta_\gamma + \eta_\gamma)\right) = \mathfrak{b}_{j+1}(\gamma) = x_{j+1}.$$

□

We say that *med* is a probability measure if $med(A) = 1$. As we will only deal with finite measures, that is, such that $med(A) < \infty$. In this case we can always convert *med* into a probability measure ν_{med}. For that, just set

$$\nu_{med}(\mathfrak{A}) = \frac{med(\mathfrak{A})}{med(A)}, \quad \text{for each measurable set } \mathfrak{A} \subset A. \tag{4.4}$$

We say that *med* is a **measure invariant** by $\overline{\mathscr{T}}$ if there exists $\widetilde{\Omega} \subseteq \Omega$ and $\mathfrak{A}$ such that $med(Sat(\mathfrak{A}, T,\widetilde{\Omega})) = med(\mathfrak{A})$ and $med(\widetilde{\Omega}) = med(\overline{\mathscr{T}}^{-1}(\widetilde{\Omega}))$. Furthermore if every $\mathfrak{A} \subset A$ satisfies (4.4), then ν_{med} is said to be an **invariant probability measure**.

We will introduce the notion of recurrence, which is very clear in the context of smooth and discrete dynamical systems, but in the non-smooth scenario it must be clarified to avoid misunderstandings.

Definition 58. *Given a PSVF $Z \in \mathscr{Z}^r$, we say that a point $p \in V$ is recurrent by Z if exists $\widetilde{\Omega} \subset \overline{\Omega}$ and a sequence $(t_j^{\gamma})_{j\in\mathbb{N}} \in I_{max}$ such that $(t_j^{\gamma})_{j\in\mathbb{N}} \to \tau^+(p)$ in $\mathbb{R}^+$ and $\overline{\mathscr{T}}(\gamma_p)(t_j^{\gamma_p}) \to p$ when $j \to \infty$, where $\gamma_p \in \widetilde{\Omega}$, is a global trajectory passing through p. A set $A \subset V$ is recurrent if every $p \in A$ is recurrent.*

So, we can state the following result:

Theorem 8 (Like-Poincaré's Recurrence Theorem for PSVFs). *Let $Z = (X^+, X^-)$ be defined over a compact n dimensional surface $M \subseteq \mathbb{R}^n$ that satisfies the Corollary 7, where med is an invariant measure by $\overline{\mathscr{T}}$. Then, med-almost every point $p \in A \subset \mathscr{A} \setminus \Sigma$ is recurrent by Z.*

Theorem 8 will be proved in Section 4.1. Now, we can state the following result:

Theorem 9. *Let Z be a PSVF satisfying the hypothesis of Corollary 7. If $\mathscr{D}$ is a σ-algebra associated to $\overline{\Omega}$ then the system $(\overline{\Omega}, \mathscr{D}, \nu_{med}, \overline{\mathscr{T}})$ is ergodically equivalent to the system $([0,1]^{\mathbb{N}}, \mathscr{B}_\infty, \xi^\infty, \sigma_\infty^+)$. Furthermore, ν_{med} a probability measure invariant by $\overline{\mathscr{T}}$ and $\overline{\mathscr{T}}$ is strong-mixing, therefore weakly-mixing and ergodic.*

Theorem 9 will be proved in Section 4.1.

We call *average visit time* from p to $\mathfrak{A} \subset A \subseteq \mathscr{A} \setminus \Sigma$, with the hypothesis of the Corollary 7, the value

$$\mathfrak{T}(\mathfrak{A}, p) = \lim_{n\to\infty} \frac{1}{n} card \left\{0 \le j \le n; \mathscr{T}^j(\gamma_p)(t) \in \mathfrak{A}\right\}, \tag{4.5}$$

where *card*, denotes the cardinality of the set.

Note that it is necessary to check if the limit (4.5) exists (we will show the existence later). Call $\chi_{\mathfrak{Y}}$ the characteristic function defined in $\mathfrak{A}$, we can write the expression on the right hand side of (4.5) as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{\mathfrak{Y}}\left(\mathscr{T}^j(\gamma_p)(t)\right). \tag{4.6}$$

We will show in the Corollary 8 that there is convergence at ν_{med}-almost every point, for the map $\overline{\mathscr{T}}$. In particular, the limit on (4.6) exists and it is well defined for *med*-almost every point $p \in \mathfrak{A}$.

The next result also provides a version of a classical result to the context of PSVFs.

Corollary 8. ***(Like-Birkhoff's Theorem for PSVFs)*** *Under the hypothesis of the previous theorem, let $(\overline{\Omega}, \mathscr{D}, \nu_{med}, \overline{\mathscr{T}})$ be a probability space. Since $\overline{\mathscr{T}} : \overline{\Omega} \to \overline{\Omega}$ a measure-preserving map. The following are equivalent:*

(i) *$\overline{\mathscr{T}}$ is ergodic;*

(ii) *For any $G \in \mathscr{L}^1(\overline{\Omega}, \mathscr{D}, \nu_{med}, \overline{\mathscr{T}})$ and ν_{med}-almost every $\gamma_p \in \overline{\Omega}$,*

$$\widetilde{G}(\gamma_p) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} G\left(\mathscr{T}^j(\gamma_p)(t)\right) = \int G \, d\nu_{med}.$$

Corollary 8 will be proved in the next section. Note that this guarantees that the limit 4.6 exists, just take $G = \chi_{\mathfrak{Y}}$. Thus, the item (ii) above generalizes Equation (4.6) to the case where G is any integrable function. The limit $\widetilde{G}$ is called *temporal mean*, or *orbital mean*, of G. The previous result shows that the average times are constant over trajectories (orbits), at *med*-almost every point $p \in \mathfrak{A}$.

4.1 Proof of the main results

Lemma 8. *A trajectory of an n-dimensional PSVF $Z = (X^+, X^-) \in \mathscr{Z}^r$ enters an escaping region Σ^e in a tangency point placed at the boundary of $\overline{\Sigma^e}$.*

Proof. It is straightforward since all trajectories of (the open set) Σ^e are departing from it. □

Lemma 9. *If $\mathscr{A}$ is a connecting domain, $\mathscr{A} \cap \Sigma^s$ has no pseudo equilibrium points.*

Proof. Let $p \in \mathscr{A} \cap \Sigma$ be an equilibrium point of Z^s, since $\mathscr{A} = Sat(\Sigma_1^s)$, it follows that $p \in \Sigma_1^s$ and in this case, there should be $q \in \Sigma^e$ such that $\phi_Z(t, p) = q$, which does not happen, because p is a pseudo equilibrium and so $\phi_Z(t, p) = p$ for all $t \geq 0$.

□

Proof of Theorem 7. Given $A \subseteq \mathscr{A} \setminus \Sigma$ we will separate the proof when either $\alpha = 0$ or $\alpha > 0$.

• Consider $\alpha > 0$. Let $Im(A) \subset \Sigma^s$ be the set where A first intersects Σ.

Since A is compact, there exists a first time $t_1 > 0$ such that $\phi_Z^{-t_1}(Im(A)) \cap A \neq \emptyset$ and $\phi_Z^{-t_1-h}(Im(A)) \cap A = \emptyset$ for all $h > 0$.

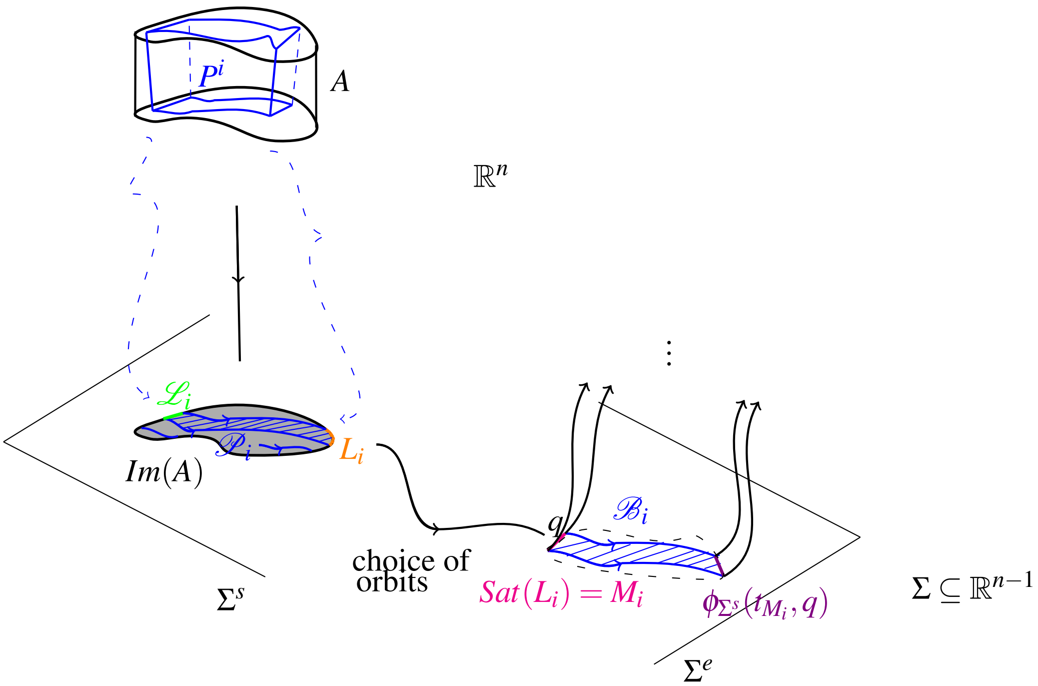


Figure 4.3: Constuction of the Theorem 7.

From Lemma 9, there exists a partition $\mathscr{P} = \{\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m\}$ of $Im(A)$ such that the boundary $\partial\mathscr{P}_i$ of the component $\mathscr{P}_i$ is a set of trajectories of the sliding vector field and pieces of the boundary $\partial(Im(A))$ of $Im(A)$. Of course, each $\mathscr{P}_i$ is a flow-box for the sliding vector field.

Consider $j \in \{1, 2, \ldots, m\}$ fixed. We define

$$P^j = \{\phi_Z^{-t}\left(\partial\mathscr{P}_j\right) \mid \phi_Z^{-t}\left(\partial\mathscr{P}_j\right) \cap A \neq \emptyset \text{ for } 0 < t \leq t_1\}.$$

With such a construction we obtain, see Figure 4.3, that

$$A = \bigcup_{j=1}^{m} P^j.$$

Note that

$$med(A) = med\left(\bigcup_{j=1}^{m} P^j\right) = \sum_{j=1}^{m} med(P^j),$$

since the interior of the subsets P^j are mutually disjoint.

Each $\mathscr{P}_i \subset \Sigma^s$ is a flow-box using the flow of the sliding vector field. Let us call $\mathscr{L}_i$ the transversal section where the sliding vector field is entering $\mathscr{P}_i$ and L_i the transversal section where the sliding vector field is departing from $\mathscr{P}_i$. By construction, $\mathscr{L}_i$ and L_i belongs to $\partial(Im(A))$.

The hypothesis that $A \subset \mathscr{A}$ ensures that L_i has an image $M_i \subset \Sigma^e$. In order to obtain M_i we have to consider an specific choice of trajectories. This choice will determine the set $\widetilde{\Lambda}$. Consider t_{M_i} the maximal time such that $\phi_{Z^s}(t, p)$ is contained in Σ^e for all $\phi_{Z^s}(0, p) = p \in M_i$ and $t \in [0, t_{M_i}]$.

From the definition of connecting domain, given a fixed point $q \in M_i$, the Z^s-flow-box $\mathscr{B}_i = M_i \times \phi_{Z^s}(t_{M_i}, q)$ has positive measure in Σ, given by m_i.

Now, let $q_i \in \mathscr{B}_i$ a fixed point and consider $Y_{q_i} > 0$ the maximal time such that $\phi_{X^+}(Y_{q_i}, p) \in \Sigma^+$ for all $p \in \mathscr{B}_i$. Take a reparametrization on time h_i such that $h_i(T) = Y_{q_i}$ (this reparametrization can be reached using, for example, arc length parameterization). Consider the Z-flow-box $\mathscr{B}_i \times \phi_{X^+}(h_i(T), q_i)$ and call $\beta_i.med(P^i)$ its measure. The number β_i has a limitation according to the maximum time taken for the points of $\mathscr{B}_i$ to reach Σ^s again (this time is finite according to the definition of connecting domain).

Consider $\mathscr{B} = \bigcup\limits_{j=1}^{m} \mathscr{B}_j$. From the definition of connecting domain, we get $A \subset \mathscr{B}$. So,

$$A \subset \mathscr{B} = Sat\left(A, T, \widetilde{\Lambda}\right) \Rightarrow 0 < med(A) < med\left(Sat\left(A, T, \widetilde{\Lambda}\right)\right) \Rightarrow$$

$$med(\mathscr{B}) = med\left(Sat\left(A, T, \widetilde{\Lambda}\right)\right) = \beta med(A) \text{ for some } \beta > 1.$$

Considering $H_i(t) = \frac{\alpha}{\beta} h_i(t)$, with $\alpha \in (0, \beta]$, and the new box $\widetilde{\mathscr{B}}_i \times \phi_{X^+}(H_i(T), q_i)$ we have $\widetilde{\mathscr{B}} = \bigcup\limits_{j=1}^{m} \widetilde{\mathscr{B}}_j$ and $med(\widetilde{\mathscr{B}}) = \frac{\alpha}{\beta} med(\mathscr{B})$ for a new choice of trajectories contained in $\widetilde{\widetilde{\Lambda}}$. So,

$$med\left(Sat\left(A, T, \widetilde{\widetilde{\Lambda}}\right)\right) = med(\widetilde{\mathscr{B}}) = \frac{\alpha}{\beta} med(\mathscr{B}) = \frac{\alpha}{\beta}.\beta med(A) = \alpha.med(A).$$

• If $\alpha = 0$. The hypothesis that $A \subset \mathscr{A}$ ensures that there exists $\mathfrak{M}_i \subset \Sigma^e$ being the image of L_i. However, in order to obtain $\mathfrak{M}_i$ we have to consider an specific choice of trajectories, that collapses every flow-box $\mathscr{P}_i \subset \Sigma^s$ and that escape from Σ^e for every $\mathfrak{M}_i$. This choice will determine the set $\widetilde{\widetilde{\Lambda}}$. But since Σ^e is a manifold with codimension one, it has measure zero and consequently, the same happens for each $\mathfrak{M}_i$. In such a case, call $\mathfrak{B} = \bigcup_{i=1}^{m} \mathfrak{M}_i$ and

$$med\left(Sat\left(A,T,\widetilde{\widetilde{\Lambda}}\right)\right) = med\left(\mathfrak{B}\right) = \sum_{i=1}^{m} med\left(\mathfrak{M}_i\right) = 0.$$

□

Example 11. *Note that it is extremely important that the sliding and escaping regions have the same dimension, otherwise (e.g. $dim(\Sigma^s) < dim(\Sigma^e)$ see Figure 4.4) since Z present a sliding-escaping connection, the Z^s-flow-box $\mathscr{B}_i = M_i \times \phi_{Z^s}(t_{M_i}, q)$ will have measure zero in Σ, for any choice of orbits.*

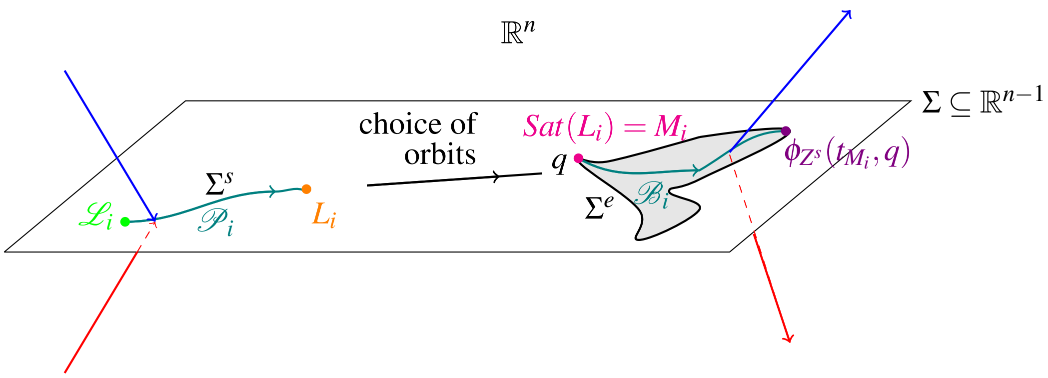


Figure 4.4: Z presents a sliding-escaping and $dim(\Sigma^s) < dim(\Sigma^e)$.

Proof of Theorem 8 . Consider $(U_j)_{j\geq 1}$ an enumerable open basis of A. Since med is an invariant measure by $\overline{\mathscr{T}}$, there are subsets $\widetilde{\Omega}_j \subset \Omega$, such that $med\left(Sat\left(U_j, T, \widetilde{\Omega}_j\right)\right) = med(U_j)$ and $med(\widetilde{\Omega}_j) = med(\overline{\mathscr{T}}^{-1}(\widetilde{\Omega}_j))$ for every $j \geq 1$. Consider the set

$$\widetilde{U}_j = \{p \in U_j : \exists \widetilde{t}_j > 0 \text{ such that } \{\overline{\mathscr{T}}(\gamma_p)(t) : |\, t\,| > \widetilde{t}_j \text{ and } \gamma \in \widetilde{\Omega}_j\} \cap U_j = \emptyset\}.$$

Since Z presents a sliding-escaping connection, there are subsets of trajectories Ω_j^* passing by points of $\widetilde{U}_j$ such that $med\left(Sat\left(\widetilde{U}_j, T, \Omega_j^*\right)\right) = med(\widetilde{U}_j) = 0$, and then $med\left(\bigcup_{j=1}^{\infty} \widetilde{U}_j\right) = 0$. To conclude the proof, we just need to prove that every point $p \in A \setminus \bigcup_{j=1}^{\infty} \widetilde{U}_j$ is recurrent for Z .

For that, take $p \in A \setminus \bigcup_{j=1}^{\infty} \widetilde{U}_j$. So $p \notin \widetilde{U}_j$ for all $j \geq 1$. We note that $p \notin \widetilde{U}_j$ means that for every $\widetilde{t}_j > 0$ there exists $t_j^{\gamma_p} > \widetilde{t}_j$ such that $\mathscr{T}(\gamma_p)(t_j^{\gamma_p}) \in U_j$. But given a neighborhood U of p there exists an open set U_j from the basis $(U_j)_{j\geq 1}$ such that $p \in U_j \subset U$. Thus, $p \in U_j \setminus \widetilde{U}_j$, that is, p is recurrent by Z. □

Proof of Theorem 9. From Corollary 7, there is a subset $\widetilde{\Omega} \subset \Omega$ such that $med(Sat(A,T,$ $\widetilde{\Omega}) = med(A) < \infty$.

Take $\mathscr{Y} = (\mathscr{Y}_0, \mathscr{Y}_1, \cdots,)$ a stationary sequence of random variables i.i.d. defined on $[0,1]^{\mathbb{N}}$ as in Proposition 5. From Proposition 20, each term $\mathscr{Y}_k$ of the sequence $\mathscr{Y}$ is taken by $\overline{\mathfrak{b}}$ in $\mathscr{X}_n = \mathscr{Y}_n \circ \overline{\mathfrak{b}}$, so we get a sequence $\mathscr{X} = (\mathscr{X}_0, \mathscr{X}_1, \cdots)$ defined on $\overline{\Omega}$.

$$\begin{tikzcd} \overline{\Omega} \arrow[r, "\overline{\mathfrak{b}}"] \arrow[dr, "\mathscr{X}_n"'] & {[0,1]^{\mathbb{N}}} \arrow[d, "\mathscr{Y}_n"] \\ & \mathbb{R} \end{tikzcd}$$

Furthermore, since the sequence $(\mathscr{Y}_0, \mathscr{Y}_1, \cdots)$ is stationary, it follows that $\mathscr{X}_n$ has the same distribution with respect to the Borel σ-algebra $\mathscr{B}_\infty$, and the unique measure of induced probability

$$\nu_{med} = \overline{\mathfrak{b}}_*^{-1} \xi^\infty.$$

Now, given $x \in \mathbb{R}$, for each term $\mathscr{Y}_n$ of the sequence, $C = \mathscr{Y}_n^{-1}(-\infty, x] \in \mathscr{B}_\infty$. Since σ_∞^+ is measure-preserving for ξ^∞, we get

$$\xi^\infty((\sigma_\infty^+)^{-1}(C)) = \xi^\infty(C).$$

Furthermore for every $B \in \overline{\Omega}$, we have

$$\nu_{med}(\overline{\mathscr{T}}^{-1}(B)) = \overline{\mathfrak{b}}_*^{-1} \xi^\infty(\overline{\mathscr{T}}^{-1}(B)) = \xi^\infty((\overline{\mathfrak{b}}^{-1})^{-1}(\overline{\mathscr{T}}^{-1}(B))) =$$

$$\xi^\infty(\overline{\mathfrak{b}}(\overline{\mathscr{T}}^{-1}(B))) = \xi^\infty(\overline{\mathfrak{b}}(\overline{\mathfrak{b}}^{-1}(\sigma_\infty^+)^{-1}(\overline{\mathfrak{b}}(B)))) = \xi^\infty((\sigma_\infty^+)^{-1}(\overline{\mathfrak{b}}(B)) =$$

$$\xi^\infty(\overline{\mathfrak{b}}(B)) = \overline{\mathfrak{b}}_*^{-1} \xi^\infty(B) = \nu_{med}(B).$$

Therefore, ν_{med} is invariant probability measure by $\overline{\mathscr{T}}$.

Since σ_∞^+ is measure-preserving for ξ^∞, there is $\widetilde{\mathscr{Y}}$ which is stationary. If loss of generality, take $\widetilde{C} = \widetilde{\mathscr{Y}}_n^{-1} \subset [0,1]^{\mathbb{N}}$ measurable and invariant on σ-algebra $\mathscr{B}_\infty$, such that $\xi^\infty(\widetilde{C}) = 1$. Now put $\overline{\mathfrak{b}}(B) = \widetilde{C}$, so

$$\nu_{med}(B) = \overline{\mathfrak{b}}_*^{-1} \xi^\infty(B) = \xi^\infty(\overline{\mathfrak{b}}(B)) = \xi^\infty(\widetilde{C}) = 1.$$

In addition,

$$\overline{\mathscr{T}}(B) = (\overline{\mathfrak{b}}^{-1} \circ \sigma_\infty^+ \circ \overline{\mathfrak{b}})(B) = \overline{\mathfrak{b}}^{-1}(\sigma_\infty^+(\overline{\mathfrak{b}}(B))) = \overline{\mathfrak{b}}^{-1}(\sigma_\infty^+(\widetilde{C})) \subset \overline{\mathfrak{b}}^{-1}(\widetilde{C}) = B.$$

Therefore, using Proposition 20, $\overline{\mathfrak{b}}^{-1}$ is a bijection, which restricted to a subset of total measure, both it and its inverse are measurable. In this way, the systems $(\overline{\Omega}, \mathscr{D}, \nu_{med}, \overline{\mathscr{T}})$ and $([0,1]^{\mathbb{N}}, \mathscr{B}_\infty,$ $\xi^\infty, \sigma_\infty^+)$ are equivalent.

In order to conclude the proof, it is enough to show that $\overline{\mathscr{T}}$ is strong-mixing, since a strong-mixing map is also weakly-mixing and all weakly-mixing is ergodic. But, we get that $\overline{\mathscr{T}}$ is strong-mixing, since σ_∞^+ is strong-mixing. Indeed, given any measurable sets $\Omega_A, \Omega_B \in \mathscr{D}$ we get

$$\nu_{med}\left((\overline{\mathscr{T}})^{-n}(\Omega_A) \cap \Omega_B\right) = \xi^\infty\left((\overline{\mathfrak{b}}^{-1})^{-1}\left((\overline{\mathscr{T}})^{-n}(\Omega_A) \cap \Omega_B\right)\right) =$$

$$\xi^{\infty}\left((\sigma_{\infty}^{+})^{-n}(\overline{\mathfrak{b}}(\Omega_A))\cap\overline{\mathfrak{b}}(\Omega_B)\right)\to\xi^{\infty}\left(\overline{\mathfrak{b}}(\Omega_A)\right)\xi^{\infty}\left(\overline{\mathfrak{b}}(\Omega_B)\right)=\nu_{med}(\Omega_A)\nu_{med}(\Omega_B),$$

when $n\to\infty$.

$\square$

Proof of Corollary 8 . Applying Birkhoff's Theorem (Theorem 1.6 of [42]) and since σ_{∞}^{+} is ergodic, the system $([0,1]^{\mathbb{N}},\mathscr{B}_{\infty},\xi^{\infty},\sigma_{\infty}^{+})$ is ergodic if and only if for any $H\in\mathscr{L}^1([0,1]^{\mathbb{N}},\mathscr{B}_{\infty},\xi^{\infty},\sigma_{\infty}^{+})$ and ξ^{∞}-almost every $(x_i)_{i\in\mathbb{N}}\in[0,1]^{\mathbb{N}}$,

$$\widetilde{H}((x_i)_{i\in\mathbb{N}})=\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}G\left((\sigma_{\infty}^{+})^{j}((x_i)_{i\in\mathbb{N}})\right)=\int H\,d\xi^{\infty}.$$

Therefore, from Theorem 9 it follows that (i) and (ii) are equivalent. $\square$

4.2 Applications

Example 12. *Consider the PSVF of Example 10.*

Let $\Omega=\{\gamma\mid\gamma$ is a positive global trajectory of $Z\mid_{Sat(A,T,\widetilde{\Lambda})}\}$, where $A\subset\mathscr{A}\setminus\Sigma\subset\Delta$. And let

$$\widetilde{A}=\{\gamma\in\Omega\mid\forall t>0,\ \text{we get}\ \gamma(t)\in\mathscr{A}\ \text{and}\ \gamma\ \text{scape from}\ \Sigma^e\ \text{to}\ \ \Sigma^-\ \text{at}\ t=0\}.$$

Using the proof of Proposition 6.3 of [3], there is a topological conjugacy between the itinerary functions of elements of $\widetilde{A}$ and $[0,1]^{\mathbb{N}}$.

Since $div(X^{\pm})=0$ and Z has an escape connection, it follows that Z satisfies the assumptions of Theorem 7. Thus, Corollary 7 ensures that there is $\widetilde{\Omega}\subset\Omega$ such that $med\left(Sat\left(A,T,\widetilde{\Omega}\right)\right)=med(A)$. So, using (4.4), for each measurable set $\mathfrak{A}\subset A$ we get $\nu_{med}(\mathfrak{A})\leq 1$. Thus, Z satisfies Theorems 8 and 9 and, consequently, the Corollary 8. As consequence, the Like-Poincaré's Recurrence Theorem and the Like-Birkhoff's Theorem for PSVFs are valid in this context.

Example 13. *Let us consider the linear vector fields*

$$X^{+}(x,y,z)=(z,0,-x)\quad\text{and}\quad X^{-}(x,y,z)=\left(-\frac{1}{2}(\sqrt{3}y+z),\frac{\sqrt{3}x}{2},\frac{x}{2}\right)$$

with $(x,y,z)\in\mathbb{S}^2$ which is presented in [22]. Let Σ^1 and Σ^2 be the curves on $\mathbb{S}^2$ given by the intersection of $\mathbb{S}^2$ with the planes $z=1/2$ and $z=-1/2$, respectively.

Now, put $Z=(X^+,X^-,X^+)$ be a PSVF with three zones on $\mathbb{S}^2$ being X^+ defined on $R_1=\{(x,y,z)\in\mathbb{S}^2;z\geq 1/2\}$ and $R_3=\{(x,y,z)\in\mathbb{S}^2;z\leq -1/2\}$ and X^- defined on $R_2=\{(x,y,z)\in\mathbb{S}^2;\mid z\mid\leq 1/2\}$.

By Theorem A of [22] it follows that Z is topologically transitive on $\mathbb{S}^2$. So, from Theorem B of [22], Z has a sliding-escaping connection. Also, notice that $div(X^{\pm})=0$ and Z satisfies the hypotheses of Theorem 7. In fact the Theorem B of [22] states that

"If Z is a transitive PSVF on $\mathbb{S}^2$ having a finite number of tangency points on Σ, then the following statements hold:

(*i*) *The sliding and escaping regions are non-empty sets;*

(*ii*) *Every sliding and escaping regions are connected by some trajectory of Z. Moreover there are an uncountable number of trajectories of Z connecting sliding and escaping regions".*

So every PSVF Z that satisfies Theorem B of [22] has a sliding-escaping connection in $\mathbb{S}^2$ *and if in addition,* $div(X^{\pm}) = 0$*, satisfies the hypotheses of Theorem 7 .*

Now, Corollary 7 ensures that there is $\widetilde{\Omega} \subset \Omega$ *such that* $med\left(Sat\left(A, T, \widetilde{\Omega}\right)\right) = med(A)$*. So, the same analysis done in Example 4.1 can be repeated here and the PSVF* (4.1) *satisfies Theorems 8 and 9 and, consequently, the Corollary 8. As consequence, the Like-Poincaré's Recurrence Theorem and the Like-Birkhoff's Theorem for PSVFs are valid in this context.*

CHAPTER 5

Concluding Remarks and Future Work

In the course of this work, despite the results obtained and compiled in the preprints [25, 26, 27, 28], we had questions that remain open. Below we present some of them.

In Chapter 3 using the time-one map induced on the quotient space of all positive global trajectories, we construct conditions for a subset of trajectories of a PSVF to be associated with a subshift of finite type through a topological conjugacy. Once this topological conjugacy is established, we actually show that, under certain conditions, there is an ergodic equivalence between the space of displacements with the Bernoulli measure and the space of the global trajectories of a PSVF with the Lebesgue measure. This allowed us to obtain entropy and topological pressure for PSVFs. Furthermore, we present examples of PSVFs whose entropy is r, for a positive **real** number r in the quotient space. In view of this, the first question arises: what happens to the calculations of entropy and pressure in the space of all trajectories outside the quotient space?

An interesting future work is to verify the validity of the variational principle and spectral theory for the PSVFs discussed in Chapter 3. Also in this chapter, what is the behavior of a symmetry group acting in the space of all trajectories of the piecewise smooth vector field $\widetilde{Z}_k$ given by Remark 7 ?

We use the relationship between topological entropy, Hausdorff dimension and Minkowski dimensions (box dimension) for displacements of the finite type given by Simpson [48] to calculate such quantities for PSVFs. However, a question remains: can the concept of Hausdorff dimension and Minkowski dimension be adapted to the space of all trajectories of PSVFs?

In Chapter 4 we study PSVFs Z that present sliding-escaping connections whose Lebesgue measure are preserved. In [38] the authors conjecture that *if a PSVF admits an invariant measure, then the set formed by all possible trajectories that contain nonuniqueness solution points has zero measurement*. Furthermore, they proved that the conjecture is valid for Lipschitzian differential inclusion. This fact motivate us to try to prove it when Z present a sliding-escaping connection (see Theorem B [38]).

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