Newton Polyhedra and Invariants of Determinantal Singularities

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I dedicate this work to my mother, my grandmother and my siblings.

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Abstract

In this work, we study invariants of determinantal singularities, by analysing the Newton polyhedra which arise from the entries of a given matrix. The main contribution of this work is providing sufficient conditions, which guarantee the Whitney equisingularity of a family of isolated determinantal singularities (IDS) in terms of Newton polyhedra. We also introduce a formula to compute the local Euler obstruction of IDS in terms of Newton polyhedra and we simplify this formula for some classes of singularities, which must satisfy a condition on its Newton polyhedra. Lastly, we present an implementation on the software OSCAR in order to compute relative mixed volumes of pairs of polyhedra, to verify the non-degeneracy of a 2×3 matrix and to compute the local Euler obstruction of an IDS defined by a 2×3 matrix.

Keywords: Determinantal Singularities, Local Euler Obstruction, Newton polyhedra, OSCAR, Whitney equisingularity.

Resumo

Neste trabalho, estudamos invariantes de singularidades determinantais, analisando os poliedros de Newton que surgem a partir das entradas de uma dada matriz. A principal contribuição deste trabalho é fornecer condições suficientes que garantem a Whitney equisingularidade de uma família de singularidades determinantais isoladas (IDS) em termos de poliedros de Newton. Também introduzimos uma fórmula para calcular a obstrução de Euler local de IDS em termos de poliedros de Newton e simplificamos esta fórmula para algumas classes de singularidades, que devem satisfazer uma condição em seus poliedros de Newton. Por fim, apresentamos uma implementação no software OSCAR para calcular volumes mistos relativos de pares de poliedros, verificar a não degeneracidade de uma matriz de ordem 2×3 e calcular a obstrução de Euler local de uma IDS definida por uma matriz de ordem 2×3.

Palavras-chave: Singularidades Determinantais, Obstrução de Euler Local, Poliedros de Newton, OSCAR, Whitney equisingularidade.

Zusammenfassung

In dieser Arbeit untersuchen wir Invarianten von Determinantiellen Singularitäten, indem wir die Newton-Polyeder analysieren, die aus den Einträgen einer gegebenen Matrix entstehen. Der Hauptbeitrag dieser Arbeit besteht darin, ausreichende Bedingungen zu liefern, die die Whitney-Äquisingularität einer Familie isolierter Determinantiellen Singularitäten (IDS) in Bezug auf die Newton-Polyeder garantieren. Wir stellen auch eine Formel vor, um die lokale Euler-Obstruktion von IDS in Bezug auf die Newton-Polyeder zu berechnen und vereinfachen diese Formel für einige Klassen von Singularitäten, die eine Bedingung an ihre Newton-Polyeder erfüllen müssen. Schließlich präsentieren wir eine Implementierung in der Software OSCAR, um relative gemischte Volumina von Paaren von Polyedern zu berechnen, die Nichtdegeneriertheit einer 2×3-Matrix zu überprüfen und die lokale Euler-Obstruktion einer durch eine 2×3-Matrix definierten IDS zu berechnen.

Schlüsselwörter: Determinantiellen Singularitäten, Lokale Euler-Obstruktion, Newton-Polyeder, OSCAR, Whitney-Äquisingularität.

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Introduction

Singularity Theory is am important branch of mathematics. The original foundations of Singularity Theory can be traced back to various areas of mathematics, such as Commutative Algebra, Algebraic Geometry, Topology, Complex Analysis and Differential Equations.

While other fields focuses in general situations, Singularity Theory focus on studying the singular points of analytic spaces of functions. A space with a singular point locally does not look like $\mathbb{C}^m$, therefore it is not as simple as possible.

While sufficiently small neighbourhoods of smooth points of the same dimension are all isomorphic, singular points fall in to a great number of different isomorphism classes. A full classification of all infinitely many family of cases is unthinkable and invariants help to put some order in those cases.

Although they pose an obstruction to general techniques, the invariants hold a wealth of information about the singular case, much richer than in a smooth point. There is a variety of invariants used to characterize singularities. Some of them are coarse, such as multiplicity and local Euler obstruction, and other are finer, such as Milnor number and intersection number.

Along the years, many authors have pursued a formula to compute some of those invariants with algebraic tools. The Milnor number is, for instance, an example of a topological invariant which can be computed using algebraic methods. In this direction, Newton polyhedra of Newton non-degenerate functions play an important role, since it provides an algebraic as well as a visual and computational approach to understanding the behaviour of a singularity.

In this work, we study the so called determinantal singularities, which are singularities defined by equations given by minors of a matrix. Determinantal singularities are a natural extension of complete intersection singularities. Therefore, searching for an understanding on which properties of complete intersection singularities also hold for determinantal singularities is a natural way to follow. Our task is to study the Whitney equisingularity of a family of determinantal singularities as well as the invariants such as multiplicity and local Euler obstruction of determinantal singularities, by analysing the Newton polyhedron arising from the entries of the matrix which defines the singularity.

Polar multiplicities are important tools in the study of Whitney equisingularity, for many classes of spaces. For instance, in [24, 25], Gaffney showed that a family of d-dimensional isolated complete intersection singularities (ICIS) $\{(X_t,0)\}_{t\in D}$ is Whitney equisingular if, and only if, the polar multiplicities (see Definition 1.54) $m_i(X_t,0)$, $i=0,\dots,d$ are constant on this family, where D is an open disc around the origin in $\mathbb{C}$. Later, Nuño-Ballesteros, Oréfice-Okamoto and Tomazella [46] extended Gaffney's result to isolated determinantal singularities (IDS), proving that a good family of d-dimensional IDS $\{(X^s_{A_t},0)\}_{t\in D}$ is Whitney equisingular if and only if the polar multiplicities $m_i(X^s_{A_t},0)$, $i=0,\dots,d$, do not depend on t.

Whitney equisingularity of families of varieties is also strongly related to Newton polyhedra. As noted by Eyral and Oka [21], "in unpublished notes Briançon showed that a family of Newton non-degenerate isolated hypersurface singularities (see Definition 1.34) with constant Newton polyhedron is Whitney equisingular". Eyral and Oka [21] extended Briançon's result to families of possibly non-isolated non-degenerate hypersurface singularities.

The main contribution of this work is in Chapter 2, where we extend the ideas of Eyral and Oka in order to prove that a family of Newton non-degenerate determinantal singularities with constant Newton polyhedra is smooth over the admissible coordinate spaces (see definition 2.4). We combine this fact with the results presented by Nuño-Ballesteros, Oréfice-Okamoto and Tomazella [46] to prove the following theorem.

Theorem 2.30. *Let $\left\{(X^s_{A_t},0)\right\}_{t\in D}$, be a family of determinantal singularities, defined by the germ of matrices $A_t=((a_{i,j})_t):(\mathbb{C}^m,0)\to(M_{n,k},0)$ with holomorphic entries. Suppose that $X^s_{A_0}$ has an isolated singularity at 0 and, for all $t\in D$, the matrix A_t satisfies the following conditions:*

(i) *the Newton polyhedra Δ^t_j of $(a_{i,j})_t$ are convenient and independent of t;*

(ii) *the matrix A_t is Newton non-degenerate (see Definition 2.2).*

Then the family $\left\{(X^s_{A_t},0)\right\}_{t\in D}$ is Whitney equisingular.

Whitney equisingularity is a geometric feature, for this reason it can not be checked directly by a computer. On the other hand, the non-degeneracy assumption provides an algebraic tool to work with computational methods.

The second contribution of this work is providing a method to compute the local Euler obstruction of an isolated determinantal singularity in terms of Newton polyhedra. Many authors have presented formulas to compute some invariants of singularities using newton polyhedra. In 1976, Kouchnirenko [39] introduced a formula to compute the Milnor number of a Newton non-degenerate hypersurface singularity in terms of its Newton polyhedron. Later, Oka [48, 49] extended Kouchnirenko's result to isolated complete intersection singularities. In 2007, Esterov [18, 19] followed the same approach to give formulas to compute the multiplicity of an IDS as well as the Euler characteristic of the Milnor fiber of a function restricted to an isolated determinantal singularity.

One can compute the local Euler obstruction of a variety in terms of its polar multiplicities using the formula introduced by Lê and Teissier [40]. Furthermore, Brasselet, Lê and Seade presented a formula to compute the local Euler obstruction, by computing the Euler characteristic of the Milnor fiber on each stratum of a Whitney stratification [6]. In Chapter 3, we combine the formula presented by Brasselet, Lê and Seade with Esterov's formula in order to obtain the following result.

Corollary 3.21. *Let $(X_A^n,0)$ be an isolated determinantal singularity defined by the matrix germ $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{n,k},0)$, where A has holomorphic entries. Suppose that the Newton polyhedra of $a_{i,j}$ do not depend on i and the functions $a_{i,j}$ are convenient, $i=1,\dots,n$, $j=1,\dots,k$. If the matrix A is Newton non-degenerate, then*

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{\{j_1,\dots,j_q\}\subset\{1,\dots,k\}}\sum_{\substack{I\subset\{1,\dots,m\}\\|I|\geq q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}$$
$$\times\sum_{\substack{a_{j_1},\dots,a_{j_q}\in\mathbb{N}\\a_{j_1}+\cdots+a_{j_q}=|I|-a}}|I|!\cdot(L^I)^a(\widetilde{\Delta}^I_{j_1})^{a_{j_1}}\cdots(\widetilde{\Delta}^I_{j_q})^{a_{j_q}}.$$

In addition, we use $\mathscr{G}$-equivalence (see Definition 1.25) to present a method, which is based on the method introduced by Kouchnirenko [39], to compute this Euler obstruction, in the case where the entries of a matrix are not necessarily convenient. Furthermore, we use the same method to compute the vanishing Euler characteristic of an isolated determinantal singularity.

In Chapter 4, we provide methods in order to simplify the computations involved in the above theorem. We use GL-equivalence of matrices (see Definition 4.1) in order to define the Newton polyhedron of a matrix and compute the local Euler obstruction of an IDS in terms of this polyhedron. Moreover, we use the ideas of Chen [12], in order to unmix the relative mixed volume computations and also simplify the computations involved in the above formula. As a result we obtain the following theorem.

Theorem 4.27. *Let $(X_A^n,0)$ be the IDS defined by the germ of a Newton non-degenerate matrix $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{n,k},0)$. Let $\Delta_{i,j}$ be the Newton polyhedron of $a_{i,j}$, Δ_j be the convex hull conv$(\Delta_{1,j},\dots,\Delta_{n,j})$ and Δ_A be the convex hull conv$(\Delta_{1,1},\dots,\Delta_{n,k})$.*

(i) *Suppose that the polyhedron Δ_j is convenient, $j=1,\dots,k$. If, for each $j=1,\dots,k$, the polyhedra $\Delta_{1,j},\dots,\Delta_{n,j}$ are interlaced (see Definition 4.23), then*

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{\{j_1,\dots,j_q\}\subset\{1,\dots,k\}}\sum_{\substack{I\subset\{1,\dots,m\}\\|I|\geq q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}$$
$$\times\sum_{\substack{a_{j_1},\dots,a_{j_q}\in\mathbb{N}\\a_{j_1}+\cdots+a_{j_q}=|I|-a}}|I|!\cdot(L^I)^a(\widetilde{\Delta}^I_{j_1})^{a_{j_1}}\cdots(\widetilde{\Delta}^I_{j_q})^{a_{j_q}}.$$

(ii) *Suppose that the polyhedron Δ_A is convenient. If the polyhedra $\Delta_{i,j}$, $i=1,\dots,n$ and $j=1,\dots,k$, are interlaced, then*

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{q=k-n+1}^{k}\sum_{\substack{I\subset\{1,\dots,m\}\\|I|\geq q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}$$
$$\times\binom{|I|-a-1}{q-1}\binom{k}{q}|I|!(L^I)^a(\widetilde{\Delta}^I_A)^{|I|-a}.$$

Lastly, there is an implementation on the mathematical software OSCAR [51] to verify the non-degeneracy condition, to compute mixed volumes of pairs of polyhedra and to compute the local Euler obstruction of a determinantal singularity using the above theorems.

CHAPTER 1

Background

We use this chapter to revisit key concepts to the development of this work. In Section 1, in order to introduce the concept of determinantal singularities, which is our object of study, we present definitions and results concerning algebraic sets and analytic spaces. In Section 2, we present the Newton polyhedron of a function and the mixed volume of polyhedra. These objects will appear in later chapters, when we compute some invariants of determinantal singularities. In Section 3, we present the polar varieties, which will be important in Chapter 2, where we study of the Whitney equisingularity of a family of isolated determinantal singularities. In Section 4, we introduce Whitney stratification and Whitney equisingularity. Whitney stratification is an important object to compute the local Euler obstruction of an analytic space. In section 5, we present concepts and results regarding local Euler obstruction, which is the main invariant that we compute, in Chapter 3 and Chapter 4, in terms of Newton polyhedra. Lastly, in Section 6, we present the vanishing Euler characteristic of an isolated determinantal singularity, which plays an important role in Chapter 2 to guarantee that under some conditions the polar multiplicities are constant along a family of determinantal singularities.

1 Complex analytic space germs

Throughout this work, we study the analytic structure of determinantal varieties, which are geometric objects given by the zero locus of the set of minors of a given matrix with holomorphic entries. In this section, we will present definitions and important results regarding algebraic varieties and analytic spaces as well as space germs. Lastly, we will present the class of determinantal singularities.

In comparison to analytic sets, algebraic sets a are easier to understand. Therefore, for didactic reasons, we present firstly concepts of algebraic geometry. For this part, we will follow [36, Chapter 1].

Let $\mathbb{C}[x_1,\dots,x_m]$ be the polynomial ring in m-variables over $\mathbb{C}$. We can see the elements of $\mathbb{C}[x_1,\cdots,x_m]$ as functions from the space $\mathbb{C}^m$ on $\mathbb{C}$. Therefore, given $f\in\mathbb{C}[x_1,\dots,x_m]$, the zero locus of f, which we denote by $V(f)=\{p\in\mathbb{C}^m; f(p)=0\}$, makes sense. In general, if $S\subset\mathbb{C}[x_1,\cdots,x_m]$, we can define the zero locus of S as

$$V(S) = \{p \in \mathbb{C}^m : f(p) = 0, \forall f \in S\}.$$

It is easy to show that, if I is the ideal of $\mathbb{C}[x_1,\dots,x_m]$ generated by S, then $V(S) = V(I)$. Moreover, since $\mathbb{C}[x_1,\dots,x_m]$ is a noetherian ring, any ideal I is finitely generated by some $f_1,\dots,f_r$, thus $V(S)$ can be written as the zero locus of a finite set of polynomials.

Definition 1.1. *A subset Y of $\mathbb{C}^m$ is said to be an* **algebraic set** *if there exists a subset $S \subset \mathbb{C}[x_1,\dots,x_m]$ such that $Y = V(S)$.*

Example 1.2. *Let $I =< y - x^2 >$ be an ideal of $\mathbb{C}[x,y]$. The set $Y = V(I) \subset \mathbb{C}^2$ is an algebraic set. If an ideal I of $\mathbb{C}[x_1,\dots,x_m]$ is generated by only one element, then the set $V(I) \subset \mathbb{C}^m$ is called* **hypersurface**

We will use the following more complicated example as running example throughout this work.

Example 1.3. *Let $I =< xz - y^2, yw - z^2, xw - yz >$ be an ideal of $\mathbb{C}[x,y,z,w]$. Then $Y = V(I)$ is an algebraic set.*

The following proposition (see [36, Proposition 1.1] for a proof) shows that algebraic sets satisfy the axioms of a topology as closed sets. This topology is called the **Zariski topology**.

Proposition 1.4. *Consider the polynomial ring $\mathbb{C}[x_1,\dots,x_m]$.*

(i) *Let $S_1, S_2 \subset \mathbb{C}[x_1,\dots,x_m]$, then $V(S_1) \cup V(S_2) = V(S_1 \cdot S_2)$, where $S_1 \cdot S_2 = \{f_1 \cdot f_2 : f_1 \in S_1$ and $f_2 \in S_2\}$. Therefore, the union of two algebraic sets is an algebraic set.*

(ii) *Let $\{S_i\}_{i \in I}$ be an arbitrary collection of subsets of $\mathbb{C}[x_1,\dots,x_m]$, then $V(\cup_{i \in I} S_i) = \cap_{i \in i} V(S_i)$. Thus, the intersection of an arbitrary family of algebraic sets also is an algebraic set.*

(iii) *$V(0) = \mathbb{C}^m$ and $V(1) = \emptyset$. This means that both the empty set and the whole space are algebraic sets.*

Definition 1.5. *We say that a non-empty set Y of a topological space X is* **irreducible** *if it can not be written as an union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.*

As one can see in [36, Corollary 1.4], $V(I)$ is irreducible if and only if I is a prime ideal.

Example 1.6. *The set $V(< I >) \subset \mathbb{C}^4$, where $I =< xz - y^2, yw - z^2, xw - yz >$, is irreducible, because I is a prime ideal. On the other hand, the set $V(< xy, yz, xz >) \subset \mathbb{C}^3$ is the union of the three coordinate axes, therefore, it is reducible.*

Definition 1.7. *An* **affine algebraic variety** *is an irreducible subset of $\mathbb{C}^m$ which is given by zeros of polynomial functions.*

Example 1.8. *The first set in Example 1.6 is an affine algebraic variety, while the second set is not.*

Similar to polynomial functions, we can also study the zero locus of one or more analytic functions. The analytic ring inherits some nice properties from the polynomial ring, such as the noetherian property, power series representations, Zariski topology, among others.

For this subject, we follow [34]. Before we continue, it is important to observe that, by Osgood's Lemma, a continuous function $f : U \subset \mathbb{C}^m \to \mathbb{C}$ is analytic if and only if it is holomorphic, where $U \subset \mathbb{C}^m$ is an open set. In this work, analytic and holomorphic have the same meaning.

Definition 1.9. *Consider the set of pairs (V_α, U_α), where U_α and V_α are analytic subsets containing the origin in $\mathbb{C}^m$, U_α is an open neighbourhood of the origin in $\mathbb{C}^m$ and $V_\alpha \subset U_\alpha$. Two such pairs (V_1, U_1) e (V_2, U_2) are equivalent if there exists a neighbourhood $W \subset U_1 \cap U_2$ of the origin such that $V_1 \cap W = V_2 \cap W$. An equivalence class of these pairs is said to be a* **germ at the origin** *in $\mathbb{C}^m$.*

We can also define germs of functions at the origin $0 \in \mathbb{C}^m$, as equivalence classes in the set of differentiable functions from $\mathbb{C}^m$ to $\mathbb{C}$.

Definition 1.10. *Let $f, g : \mathbb{C}^m \to \mathbb{C}$ be differentiable functions, we say that f is* **equivalent** *to g if there exists a neighbourhood U of 0 in $\mathbb{C}^m$, where f and g coincide. We denote by f the germ which its representative is the function $f : \mathbb{C}^m \to \mathbb{C}$.*

We denote the local ring of germs at 0 of analytic functions in $\mathbb{C}^m$ by $\mathscr{O}_m$. If $f_1, \ldots, f_r \in \mathscr{O}_m$, the equivalence class of the set $\{x \in \mathbb{C}^m : f_1(x) = \cdots = f_r(x) = 0\}$, where $f_1, \ldots, f_r : \mathbb{C}^m \to \mathbb{C}$ are representatives of the germs $f_1, \ldots, f_r$, respectively, is denoted by $\mathscr{V}(f_1, \ldots, f_r)$. If f_i and g_i are representative of the same germ, where $i = 1, \ldots, r$, then the sets $\mathscr{V}(f_1, \ldots, f_r)$ and $\mathscr{V}(g_1, \ldots, g_r)$ coincide.

Definition 1.11. *A* **germ of an analytic space** $(V, 0)$ **around the origin** *is the germ of the subset*

$$V = \mathscr{V}(f_1, \ldots, f_r),$$

for $f_1, \ldots, f_r \in \mathscr{O}_m$.

Our purpose is to study the nature of such analytic spaces in the neighbourhood of some fixed point in $\mathbb{C}^m$, which without loss of generality we can consider to be the origin.

We say that a germ of an analytic space V is irreducible when for any germs V_1 and V_2 such that $V = V_1 \cup V_2$ then $V = V_1$ or $V = V_2$, in this case we say that V is an **analytic variety**.

The next proposition states that a germ of an analytic space can always be decomposed in irreducible components. One can find a proof for it in [34, page 89].

Proposition 1.12. *Let V be a germ of an analytic space, then there exists a positive integer s and irreducible varieties $V_1, \dots, V_s$, with V_i not contained in V_j, for all $i \neq j$, such that $V = V_1 \cup \cdots \cup V_s$. Such varieties are uniquely determined, up to their order, and they are called* **irreducible components** *of V.*

A germ of an analytic space at x is a germ of a set V at x such that, for some neighbourhood U of x, the space $V \cap U$ can be described by $\mathscr{V}(f_1, \dots, f_r)$, for some $f_1, \dots, f_r \in \mathscr{O}_m$.

Definition 1.13. *A germ $V = \mathscr{V}(f_1, \dots, f_r)$ is called* **reduced** *if the $\mathbb{C}$-quotient algebra $\mathscr{O}_m / \langle f_1, \dots, f_r \rangle$ does not contain nilpotent elements.*

A function $f : X \to \mathbb{C}$ defined on an analytic variety X is **holomorphic** if for all $x \in X$ there exists a neighbourhood V of x in $\mathbb{C}^m$ such that $f|_{X \cap V}$ is the restriction of a holomorphic function in V.

Definition 1.14. *A map $F = (f_1 \dots, f_k) : X \to \mathbb{C}^k$ is called* **holomorphic** *if $f_i : X \to \mathbb{C}$ for $i = 1, \dots, k$ are holomorphic functions. A holomorphic map $F : X \to \mathbb{C}^m$ is said to be* **biholomorphic** *if it is bijective and its inverse is also holomorphic.*

We recall that the Jacobian matrix of a holomorphic mapping $F = (f_1, \dots, f_k) : U \subset \mathbb{C}^m \to \mathbb{C}^k$ at a point z is

$$J(F)(z) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(z) & \cdots & \frac{\partial f_1}{\partial x_m}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1}(z) & \cdots & \frac{\partial f_k}{\partial x_m}(z) \end{bmatrix},$$

where $\frac{\partial f}{\partial x_i}(z)$ is the partial derivative of the function f with respect to x_i at the point z.

Definition 1.15. *A point $z \in U$ is said to be a* **regular** *point of the map F if the rank of $J(F)(z)$ is maximal. Otherwise, z is a* **singular** *point of F. We denote the singular set of F by $S(F)$.*

We say that a point z of a germ of an analytic space V is a **regular** or **smooth point** if for some neighbourhood U of z, the germ $U \cap V$ can be described as the zero locus of a finite number of germs of analytic functions which have z as a regular point. A non-regular point of V is called **singular point of** V.

Definition 1.16. *The* **dimension** *of an analytic variety V is the dimension of the tangent space to V at a regular point of V.*

Example 1.17. *Let $X = \mathscr{V}(f_1, \dots, f_r)$ be a germ of an analytic variety, where $f_i : \mathbb{C}^m \to \mathbb{C}$, $i = 1, \dots, r$. We say that X is a* **complete intersection** *if $\dim(X) = m - r$.*

Definition 1.18. *We say that a germ of an analytic space V is* **equidimensional** *when all of its irreducible components have the same dimension.*

Example 1.19. *Let $(X,0)$ be a germ around the origin defined by $X = V(xy,yz,xz) \subset \mathbb{C}^3$. The germ $(X,0)$ contains three irreducible components, which are all 1-dimensional. Therefore, the germ $(X,0)$ is equidimensional.*

An important class of analytic spaces is the class of determinantal varieties. Determinantal varieties have been widely studied by researchers in Commutative Algebra and Algebraic Geometry (see [9, 10]). In Singularity Theory there are countless articles with the purpose of studying those varieties, we can quote, for instance, Ebeling and Gusein-Zade [15], Frühbis-Krüger and Neumer [22], Nuño-Ballesteros, Oréfice-Okamoto and Tomazella [45], Pereira and Ruas [53] and Zach [58].

Let

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,k} \end{bmatrix}$$

denote the matrix of $n \times k$ indeterminates over $\mathbb{C}$. The set of all such matrices is denoted by $M_{n,k}$.

Definition 1.20. *The subset $M^s_{n,k} = \{A \in M_{n,k} : \mathrm{rank}(A) < s\}$ is called* **generic determinantal variety**.

The set $M^s_{n,k}$ is an irreducible subvariety of $M_{n,k}$ with codimension $(n-s+1)(k-s+1)$ (see [10]). Moreover, the singular set of $M^s_{n,k}$ is exactly $M^{s-1}_{n,k}$.

Definition 1.21. *Let $A = (a_{i,j}(x))$ be a $n \times k$ matrix, whose entries are complex analytic functions in $U \subset \mathbb{C}^m$, $0 \in U$ and F be a map defined by the s size minors of A. We say that $F^{-1}(0)$ is a* **determinantal variety** *if it has codimension $(n-s+1)(k-s+1)$.*

Some of the invariants studied along this work, such as the top polar multiplicity and the vanishing Euler characteristic, depend on determinantal structure of the variety $F^{-1}(0)$, therefore, we denote the determinantal variety defined by the s size minor of a matrix A by X^s_A.

We can see the matrix $A = (a_{i,j}(x))$ as a map $A : \mathbb{C}^m \to M_{n,k}$, with $A(0) = 0$. Therefore, the determinantal variety in $\mathbb{C}^m$ is the set $X^s_A = A^{-1}(M^s_{n,k})$.

Remark 1.22. Let $X = f^{-1}(0)$, where $f = (f_1,\ldots,f_k) : \mathbb{C}^m \to \mathbb{C}^k$, be a complete intersection variety. Then X is a determinantal variety defined by the matrix $A = [\; f_1 \;\; \cdots \;\; f_k \;]$.

Definition 1.23. *Let $(X^s_A,0) \subset (\mathbb{C}^m,0)$ be a determinantal variety satisfying the condition*

$$s = 1 \text{ or } m < (n-s+2)(k-s+2).$$

The variety $(X^s_A,0)$ is said to be an **isolated determinantal singularity** *(IDS) if X^s_A is smooth at x and $\mathrm{rank}\, A(x) = s-1$ for all $x \neq 0$ in a neighbourhood of the origin.*

Example 1.24. *Consider the algebraic variety $X = V(I) \subset \mathbb{C}^4$, where I is the ideal generated by*

$$f_1 = xw - yz, \;\; f_2 = xz - y^2 \;\text{ e }\; f_3 = yw - z^2.$$

Note that, f_1, f_2 e f_3 are exactly the size 2 minors of the matrix

$$A = \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}.$$

We can show that X has an isolated singularity at the origin $0 \in \mathbb{C}^4$ and that the tangent space on its regular points have dimension 2, which means that X is a surface in $\mathbb{C}^4$. Furthermore, we observe that $(2-2+1)(3-2+1) = 2$. Therefore, X is a determinantal variety in $\mathbb{C}^4$.

In the following we present the essential notions of equivalence for matrices and the associated concepts of finite determinacy. The main references for this subject are [23] and [52].

We denote by $GL_p(\mathscr{O}_m)$ the group of $p \times p$ invertible matrices with entries in $\mathscr{O}_m$. Let $\mathscr{R}$ the group of change of coordinates in $\mathbb{C}^m$, *i.e.,* $\mathscr{R}$ is the group composed by the germs of diffeomorphisms, which are analytic.

In addition, consider $\mathscr{H} = GL_k(\mathscr{O}_m) \times GL_n(\mathscr{O}_m)$ and denote by $M_{n,k}(\mathscr{O}_m)$ the set of all $n \times k$ matrices with entries in $\mathscr{O}_m$.

Definition 1.25. *Let $\mathscr{G}_{(n,k)} = \mathscr{R} \times GL_k(\mathscr{O}_m) \times GL_n(\mathscr{O}_m)$. Two germs of singularities $A, B \in M_{n,k}(\mathscr{O}_m)$ are* $\mathscr{G}_{(n,k)}$**-equivalent** *if and only if there exists $(\varphi, P, Q) \in \mathscr{G}_{(n,k)}$ such that $A = Q^{-1}(\varphi^* B)P$.*

Given $(\phi, R, S), (\varphi, P, Q) \in \mathscr{G}_{(n,k)}$ we can define the composition

$$(\phi, R, S) \circ (\varphi, P, Q) = (\varphi \circ \phi, (\phi^* P)R, (\phi^* Q)S).$$

With this operation, the set $\mathscr{G}_{(n,k)}$ has a group structure. Therefore, the mapping

$$\begin{array}{ccc} \mathscr{G}_{(n,k)} \times M_{n,k}(\mathscr{O}_m) & \to & M_{n,k}(\mathscr{O}_m) \\ (\varphi, P, Q, M) & \mapsto & Q^{-1}(\varphi^* M)P \end{array}$$

is an action from the group $\mathscr{G}_{(n,k)}$ on the space of $n \times k$ matrices with entries in $\mathscr{O}_m$. Hence, two germs of matrices are $\mathscr{G}_{(n,k)}$-equivalent if and only if they belong to the same orbit under this action. When there is no risk of misunderstandings, $\mathscr{G}_{(n,k)}$ is denoted simply by $\mathscr{G}$.

We observe that in the above definition the varieties X_A^s and X_B^s defined by matrices A and B are not necessarily determinantal. If they are indeed determinantal singularities, then, by the following lemma (see [23, Lemma 2.1.3] for a proof), the varieties X_A^s and X_B^s are isomorphic as germs.

Lemma 1.26. *Let $A \in M_{n,k}$ be a matrix. For any pair of invertible matrices $P \in GL_n(\mathbb{C})$ and $Q \in GL_k(\mathbb{C})$ and every number s one has*

$$< (P \cdot A \cdot Q^{-1})^{\wedge s} > = < A^{\wedge s} >,$$

where $A^{\wedge s}$ denotes the exterior power $\wedge s : \bigwedge^s(\mathbb{C}^k) \to \bigwedge^s(\mathbb{C}^n)$.

Definition 1.27. *A germ of a matrix $A \in M_{n,k}(\mathscr{O}_m)$ is l-**determined**, or $l-\mathscr{G}$-**determined** if for every matrix B such that*

$$j^l A(0) = j^l B(0),$$

*B is $\mathscr{G}$-equivalent to A. If M is l-determined for some l, we say that A is $\mathscr{G}$-**finitely determined**. The smallest l such that the germ A is l-determined is called **determinacy bound** of A.*

Theorem 1.28. *Isolated determinantal singularities are $\mathscr{G}$-finitely determined and, therefore, they have a polynomial representative.*

A detailed proof can be found in [52, Corollary 2.4.1].

2 Newton polyhedra and mixed volumes of polyhedra

In this section we present concepts, definitions and results regarding convex geometry, such as Newton polyhedra, mixed volumes of polyhedra and relative mixed volumes of pairs of polyhedra.

Newton polyhedra of polynomial functions are important objects which can be very useful to compute some mathematical objects, such as the Milnor number ([39]), the principal zeta function of monodromy ([48]), multiplicities ([18], [4]), among others. For this section, we take as reference [49].

The monomial $x_1^{a_1}\cdots x_m^{a_m}$ is denoted by x^a, where $a=(a_1,\ldots,a_m)\in\mathbb{Z}^m$. We denote by $\mathbb{R}^m_+$, the positive orthant of $\mathbb{R}^m$. A subset $\Delta\subset\mathbb{R}^m_+$ is called a **Newton polyhedron** when there exists some $P\subset\mathbb{Z}^m_+$ such that Δ is the convex hull of the set $\{p+v : p\in P \text{ and } v\in\mathbb{R}^m_+\}$. In this case, Δ is said to be the **Newton polyhedron determined by** P.

Definition 1.29. *If $f\in\mathscr{O}_m$ is a germ of a polynomial function $f(x)=\sum_{p\in\mathbb{Z}^m_+} c_p x^p$, then the **support** of f is $\mathrm{supp(f)} := \{\mathrm{p}\in\mathbb{Z}^{\mathrm{m}}_+ \,|\, \mathrm{c_p}\neq 0\}$. The **Newton polyhedron** of f, Δ_f, is the Newton polyhedron determined by $\mathrm{supp(f)}$.*

Definition 1.30. *A polyhedron is said to be **convenient** if it touches all coordinate axes. A function $f:\mathbb{C}^m\to\mathbb{C}$ is **convenient** if its Newton polyhedron is convenient.*

Example 1.31. *Consider the function $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ defined by $f(x,y)=x^3-y^2$. We construct step by step the Newton polyhedron of f. Firstly, we note that $f(x,y)=x^3y^0-x^0y^2$, then $supp(f)=\{(3,0,(0,2)\}$. The first step is to indicate those points in the positive orthant $\mathbb{R}^2_+$, as shown Figure 1.1. Secondly, we draw the orthant $\mathbb{R}^2_+$ at each of those two points. As we can see in the Figure 1.2.*

Lastly, we take the convex hull of the union of $(3,0)+\mathbb{R}^2_+$ and $(0,2)+\mathbb{R}^2_+$ and we obtain the Newton polyhedron of f.

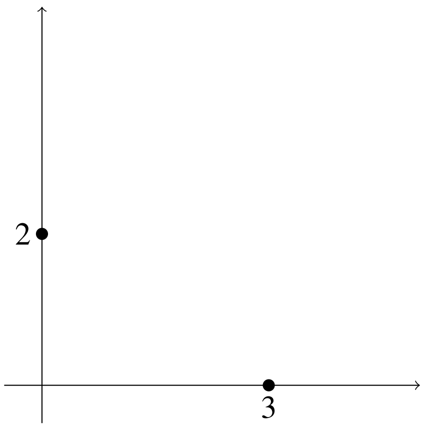


Figure 1.1: Support of f.

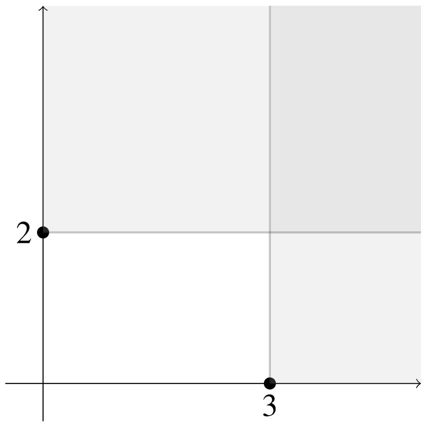


Figure 1.2: Sum with with $\mathbb{R}^2_+$.

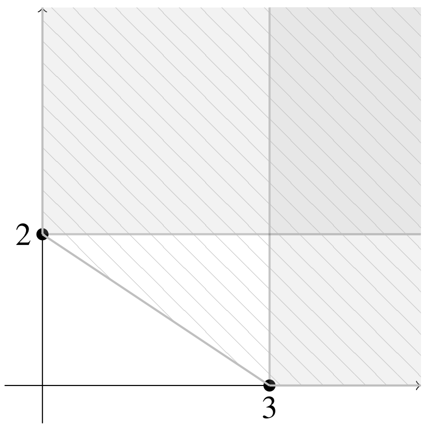


Figure 1.3: Newton polyhedron of f.

Example 1.32. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be defined by $f(x,y,z) = ax + by + cz$, where a, b and c are non-zero. Then the following picture illustrates the Newton polyhedron of f.*

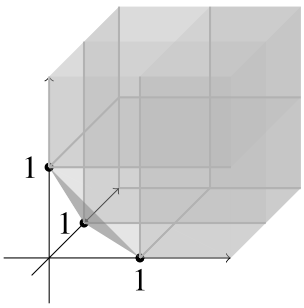


Figure 1.4: Newton polyhedron of f.

Let $P = (p_1, \ldots, p_m)$ be a weight vector. For each $x \in \mathbb{R}^m$, we define $P(x) = \sum_{i=1}^{m} p_i \cdot x_i$. For a positive weight ($p_i \geq 0$, for $i = 1, \ldots, m$), we define $d(P; f)$ as the minimal value of the restriction $P|_{\Delta_f}$, *i.e.*, $d(P; f) = \min\{P(x) : x \in \Delta_f\}$. Let

$$\Gamma(P; f) = \{x \in \Delta_f : P(x) = d(P; f)\}$$

be the face of Δ_f, where P takes the minimal value $d(P;f)$. For a strictly positive weight P ($p_i > 0$, for $i = 1,\dots,m$), $\Gamma(P;f)$ is a bounded face of Δ_f. When there is no risk of misunderstandings, we denote $\Gamma(P;f)$ simply by Γ. We define

$$f|_\Gamma(x) = \sum_{a\in\Gamma} c_a x^a$$

and we call $f|_\Gamma$ the **face function** of f with respect to Γ.

Example 1.33. *Let $f : (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be the function defined by $f(x,y) = x^3 - y^2$. Consider the weight $P = (p_1, p_2)$, with $p_1 > 0$ and $p_2 > 0$ and the function $P(x,y) = p_1 \cdot x + p_2 \cdot y$. We have three cases.*

(i) *If $p_1 > p_2$, then $d(P;f) = 2\cdot p_2$, $\Gamma_1 = \Gamma(P;f) = \{(0,2)\}$ and $f|_{\Gamma_1} = -y^2$.*

(ii) *If $p_1 < p_2$, then $d(P;f) = 3\cdot p_1$, $\Gamma_2 = \Gamma(P;f) = \{(3,0)\}$ and $f|_{\Gamma_2} = x^3$.*

(iii) *If $p_1 = p_2$, then $d(P;f) = 3p_1 + 2p_2$, $\Gamma_3 = \Gamma(P;f) = \overline{(3,0)(0,2)}$ and $f|_{\Gamma_3} = x^3 - y^2$, where $\overline{AB}$ denotes the line segment between the points A and B.*

If $p_1 = 0$, for example, $d(P;f) = 0$ and the face $\Gamma(P;f) = (3,0) + (\mathbb{R}_+,0)$ is not a bounded face of the Newton polyhedron Δ_f.

Definition 1.34. *Let $f : (\mathbb{C}^m,0) \to (\mathbb{C},0)$ be the germ of a holomorphic function, with Newton polyhedron Δ_f. The function f is said to be* **Newton non-degenerate** *if for every bounded face Γ of Δ_f the restriction of $f|_\Gamma$ to the set $\{x \in (\mathbb{C}\setminus\{0\})^m : f|_\Gamma(x) = 0\}$ has no critical points.*

Example 1.35. *Consider the function $f : (\mathbb{C}^2,0) \to (\mathbb{C},0)$ defined by $f(x,y) = x^3 - y^2$. As seen in Example 1.33, the Newton polyhedron Δ_f contains three bounded faces Γ_1, Γ_2 and Γ_3. Firstly, we have*

$$\{(x,y) \in (\mathbb{C}\setminus\{0\})^2 : f|_{\Gamma_1}(x,y) = -y^2 = 0\} = \{(x,y) \in (\mathbb{C}\setminus\{0\})^2 : f|_{\Gamma_2}(x,y) = x^3 = 0\} = \emptyset.$$

Moreover, since $f|_{\Gamma_3}(x,y) = x^3 - y^2$, we have

$$df|_{\Gamma_3} = \begin{bmatrix} 3x^2 & -2y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \Leftrightarrow (x,y) = (0,0).$$

As $(0,0) \notin (\mathbb{C}\setminus\{0\})^2$, the restriction of $df|_{\Gamma_3}$ to the set $\{(x,y) \in (\mathbb{C}\setminus\{0\})^2 : f|_{\Gamma_3}(x) = 0\}$ has no zeros. Then $f|_{\Gamma_3}$ has no critical points in $\{(x,y) \in (\mathbb{C}\setminus\{0\})^2 : f|_{\Gamma_3}(x) = 0\}$. Therefore, the function f is Newton non-degenerate.

Remark 1.36. The Newton non-degeneracy of a function depends on the choice of the coordinates. For instance, in Example 1.35, $f(x,y) = x^3 - y^2$ is Newton non-degenerate. However, if we make the change of coordinates

$$\begin{array}{rccc} \phi : & \mathbb{C}^2 & \to & \mathbb{C}^2 \\ & (x,y) & \mapsto & (x,x+y) \end{array},$$

we obtain the function $g = (f\circ\phi(x,y)) = x^3 - (x+y)^2$, which is not Newton non-degenerate.

Mixed volumes of polyhedra are a concept from convex geometry which have several important applications in mathematics and related fields. They are a generalization of the notion of volume to higher dimensions, and they provide a way to measure the interaction between multiple convex bodies. In algebraic geometry, mixed volumes play an important role in questions related to intersection theory and intersection multiplicities.

The set of all convex bounded polyhedra in $\mathbb{R}^m$ is denoted by $\mathscr{C}$. There are many natural ways to define, on the set $\mathscr{C}$, interesting operations of addition and scalar products. For $A,B \in \mathbb{C}$ and $\lambda \in \mathbb{R}$, the **Minkowski addition** and the ordinary scalar product are defined in the following:

$$\begin{aligned} A+B &= \{x+y : x \in A, y \in B\}, \\ \lambda \cdot A &= \{\lambda \cdot x : x \in A\}. \end{aligned}$$

Proposition 1.37 (see [32, Proposition 6.1]for a proof) and Theorem 1.38 (a proof can be found in [32, Theorem 6.1]) show that the set $\mathscr{C}$ with the Minkowski addition is a semigroup.

Proposition 1.37. *Let $A,B \in \mathscr{C}$ and $\lambda \in \mathbb{R}$. Then $A+B, \lambda \cdot A \in \mathscr{C}$.*

Theorem 1.38. *$\mathscr{C}$, endowed with Minkowski addition, is a commutative semigroup with cancellation law.*

Minkowski's theorem on mixed volumes (one can find a proof in [32, Theorem 6.5]) states that the volume of a linear combination of convex bodies is a polynomial in the coefficients of the linear combination.

Theorem 1.39 (Minkowski Theorem)**.** *Let $P_1 \dots, P_m \in \mathscr{C}$. Then there are coefficients, $MV(P_{i_1}, \dots, P_{i_d})$, $1 \le i_1, \dots, i_d \le m$, which are symmetric in the indices and such that*

$$Vol(\lambda_1 \cdot P_1 + \cdots + \lambda_m \cdot P_m) = \sum_{i_1,\dots,i_d=1}^{m} MV(P_{i_1}, \dots, P_{i_d}) \cdot \lambda_{i_1} \cdots \lambda_{i_d}, \tag{2.1}$$

for $\lambda_1, \dots, \lambda_m \ge 0$.

Definition 1.40. *The coefficient $MV(P_{i_1}, \dots, P_{i_d})$ in the above theorem is called* **mixed volume**.

The mixed volume can be more explicitly expressed in terms of volumes of Minkowski sums, through the following polarization formula.

Lemma 1.41.

$$MV(P_1, \dots, P_m) = \frac{1}{m!} \sum_{r=1}^{m} (-1)^{m-r} \sum_{1 \le i_1 \le \cdots \le i_r \le m} \mathrm{Vol}(\mathrm{P}_{\mathrm{i}_1} + \cdots + \mathrm{P}_{\mathrm{i}_\mathrm{r}}).$$

One can find a proof in [54, Lemma 5.1.4]. For more properties of the mixed volume, we suggest [27], [32] and [54].

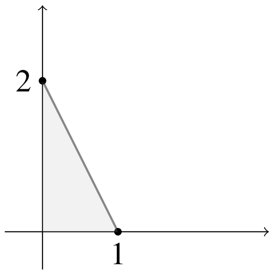


Figure 1.5: Polyhedron P_1.

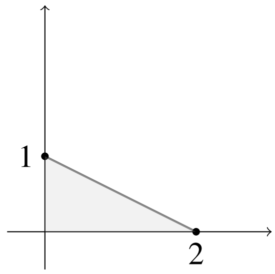


Figure 1.6: Polyhedron P_2.

Example 1.42. *Consider the sets* $S_1 = \{(0,0),(1,0),(0,2)\}$ *and* $S_2 = \{(0,0),(2,0),(1,2)\}$. *Let* P_1 *and* P_2 *be polyhedra given by* $P_1 = conv(S_1)$ *and* $P_2 = conv(S_2)$. *Figure 1.5 and Figure 1.6 illustrate polyhedra* P_1 *and* P_2*, respectively.*

By Lemma 1.41, the mixed volume of P_1 *and* P_2 *is given by*

$$MV(P_1,P_2) = \frac{1}{2!}(Vol(P_1+P_2) - Vol(P_1) - Vol(P_2)).$$

As illustrated in Figure 1.7, the Minkowski sum

$$P_1 + P_2 = conv(S_1+S_2) = conv(\{(0,0),(1,0),(0,1),(2,0),(0,2),(3,0),(0,3),(1,1),(2,2)\}).$$

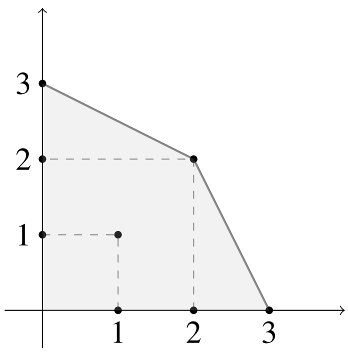


Figure 1.7: Minkowski sum $P_1 + P_2$.

Therefore,

$$MV(P_1,P_2) = \frac{1}{2}(6-1-1) = 2.$$

Computing mixed volumes can be a very complicated task in high dimension spaces. Therefore, establishing certain sufficient conditions to simplify those computations are very helpful. In his article, Chen [12] introduced conditions under which $MV(P_1,\ldots,P_m)$ is exactly equal $Vol(conv(P_1,\ldots,P_m))$, where $conv(P_1,\ldots,P_m)$ denotes the convex hull of the union $\cup_{i=1}^m P_i$.

Theorem 1.43. *Given non-empty finite sets* $S_1,\ldots,S_m \subset \mathbb{Q}^m$*, let* $\widetilde{S} = S_1 \cup \cdots \cup S_m$. *If every positive dimensional face* F *of* $conv(\widetilde{S})$ *satisfies one of the following conditions:*

(i) $F \cap S_i \neq \emptyset$, *for all* $i \in \{1,\dots,m\}$*;*

(ii) $F \cap S_i$ *is a singleton for some* $i \in \{1,\dots,m\}$*;*

(iii) *For each* $i \in I := \{i : F \cap S_i \neq \emptyset\}$, $F \cap S_i$ *is contained in a common coordinate subspace of dimension* $|I|$*, and the projection of* F *to this subspace is of dimension less than* $|I|$*;*

then

$$MV(conv(S_1),\dots,conv(S_m)) = Vol(conv(\widetilde{S})).$$

One can find a detailed proof in [12, Theorem 1.3]. With few changes, it is possible to generalize the above result to the case where the union is taken over a subset of the polytopes, therefore transforming the mixed volume into semi-mixed volume.

Corollary 1.44. *Given non-empty finite sets* $S_{i,j} \subset \mathbb{Q}^m$ *for* $i = 1,\dots,r$ *and* $j = 1,\dots,k_i$ *with* $k_i \in \mathbb{Z}_+$ *and* $k_1 + \cdots + k_r = m$*, let* $Q_{i,j} = conv(S_{i,j})$*,* $\widetilde{S_i} = \cup_{j=1}^{k_i} S_{i,j}$*, and* $\widetilde{Q_i} = conv(\widetilde{S_i})$*. If for each* i*, every positive dimensional face of* $\widetilde{Q_i}$ *that intersects* $S_{i,j}$ *for some* j *on at least two points must intersect all* $S_{i,1},\dots,S_{i,k_i}$*, then*

$$MV(Q_{1,1},\dots,Q_{m,k_m}) = MV(\underbrace{\widetilde{Q}_1,\dots,\widetilde{Q}_1}_{k_1},\dots,\underbrace{\widetilde{Q}_m,\dots,\widetilde{Q}_m}_{k_m}).$$

The proof can be found in [12, Corollary 5.1].

Example 1.45. *Consider* $P_1 = conv(S_1)$ *and* $P_2 = conv(S_2)$ *be as in Example 1.42. Let* $P = conv(S_1,S_2)$ *(Figure 1.8), we have*

$$MV(P_1,P_2) = Vol(P) = 2.$$

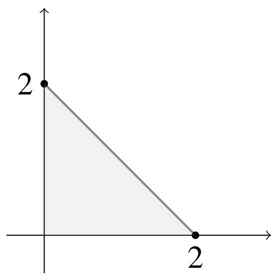


Figure 1.8: Polyhedron $P = conv(S_1\, S_2)$.

In [18], [17], Esterov introduces a relative version of the mixed volume, where the polyhedra involved are not necessarily bounded, with the purpose of proving a relative version of Bernstein' formula [3]. In the following we present essential definitions and results necessaries to understand this concept.

Definition 1.46. *A pair of polyhedra* (A,B) *in* $\mathbb{R}^m$ *is called* **bounded** *if both* $A \setminus B$ *and* $B \setminus A$ *are bounded.*

Let $N \subset \mathbb{R}^m$ be a convex polyhedron. Its **support function** $N(\cdot)$ is defined as

$$N(P) = \inf_{x \in N} P(x),$$

for every weight vector P in the dual space $(\mathbb{R}^m)^*$. The set

$$N^P = \{x \in N : P(x) = N(P)\}$$

is called the **support face** of the polyhedron N with respect to the weight vector $P \in (\mathbb{R}^m)^*$. The set $\{P : N(P) > -\infty\} \subset (\mathbb{R}^m)^*$ is called the **support cone** of N.

Consider the set $\mathscr{C}_\Gamma$ of all ordered pairs of polyhedra (A,B) with a given support cone $\Gamma \in (\mathbb{R}^m)^*$, such that the differences $A \setminus B$ and $B \setminus A$ are bounded.

Definition 1.47. (i) *The* **Minkowski sum** $(A_1,B_1)+(A_2,B_2)$ *of two pairs* (A_1,B_1) *and* (A_2,B_2) *in* $\mathscr{C}_\Gamma$ *is the pair* (A_1+A_2,B_1+B_2) *in* $\mathscr{C}_\Gamma$.

(ii) *The* **volume** $Vol(A,B)$ *of a bounded pair* (A,B) *is the difference* $Vol(A \setminus B) - Vol(B \setminus A)$.

$\mathscr{C}_\Gamma$ is a semigroup with respect to Minkowski addition of two pairs of polyhedra.

Definition 1.48. *The* **mixed volume** *of pairs of polyhedra with the support cone* $\Gamma \subset (\mathbb{R}^m)^*$ *is the symmetric multilinear function*

$$MV : \underbrace{\mathscr{C}_\Gamma \times \cdots \times \mathscr{C}_\Gamma}_{m} \rightarrow \frac{\mathbb{Z}}{m!} ,$$

such that $MV((A,B),\dots,(A,B)) = Vol(A,B)$ *for every pair* $(A,B) \in \mathscr{C}_\Gamma$.

As for the mixed volume, we can explicitly compute the relative mixed volume of polyhedra through a polarized formula (see [17, Lemma 3] for a proof).

Lemma 1.49. *The mixed volume of pairs is the following polarization of the volume of a pair (as a function on the semigroup of polyhedron pairs).*

$$MV((A_1,B_1),\dots,(A_m,B_m)) = \frac{1}{m!}\sum_{r=1}^{m}(-1)^{m-r}\sum_{1\le i_1\le\cdots\le i_r\le m}\mathrm{Vol}((\mathrm{A}_{\mathrm{i}_1},\mathrm{B}_{\mathrm{i}_1})+\cdots+(\mathrm{A}_{\mathrm{i}_r},\mathrm{B}_{\mathrm{i}_r})).$$

Example 1.50. *Consider* $f_1, f_2 : \mathbb{C}^2 \to \mathbb{C}$ *defined by* $f_1(x,y) = x+y^2$ *and* $f_2(x,y) = x^2+y$. *Let* Δ_{f_1} *and* Δ_{f_2} *be their Newton polyhedra.*

The mixed volume of the pairs of polyhedra $(\mathbb{R}^2_+,\Delta_{f_1})$ *and* $(\mathbb{R}^2_+,\Delta_{f_2})$ *is given by*

$$\begin{aligned} MV((\mathbb{R}^2_+,\Delta_{f_1}),(\mathbb{R}^2_+,\Delta_{f_2})) &= \frac{1}{2!}(Vol(\mathbb{R}^2_+,\Delta_{f_1}+\Delta_{f_2}) - Vol(\mathbb{R}^2_+,\Delta_{f_1}) - Vol(\mathbb{R}^2_+,\Delta_{f_2})) \\ &= \frac{1}{2}(3-1-1) = \frac{1}{2}. \end{aligned}$$

For brevity, we denote the mixed volume

$$\mathrm{MV}(\underbrace{\Gamma_1,\dots,\Gamma_1}_{a_1},\dots,\underbrace{\Gamma_r,\dots,\Gamma_r}_{a_r}) := \Gamma_1^{a_1}\cdots\Gamma_r^{a_r}.$$

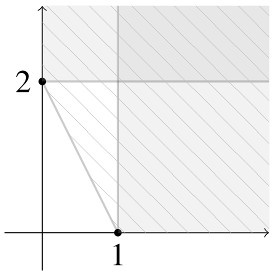


Figure 1.9: Newton polyhedron of f_1.

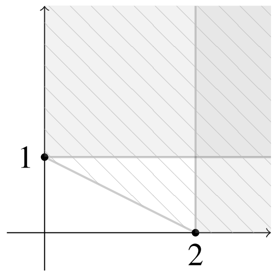


Figure 1.10: Newton polyhedron of f_2.

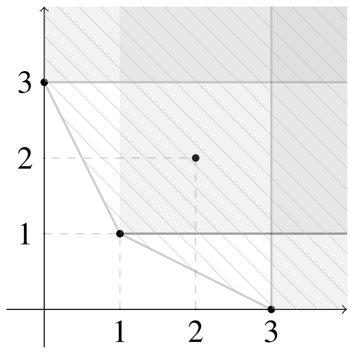


Figure 1.11: $\Delta_{f_1} + \Delta_{f_2}$.

3 Polar varieties

The notion of polar variety was used by Lê and Teissier in the 70's, ([40, 56]) with the purpose of studying singularities of analytic varieties. In this section, we present the definition of polar varieties introduced by Teissier in [56].

Let $f : X \to S$ be a morphism of reduced complex analytic spaces, such that the fibers of f are smooth with dimension $d = \dim X - \dim S$, outside a closed subset, not dense $F \subset X$. In general, we can embed $X \subset S \times \mathbb{C}^r$ as the diagram

$$\begin{array}{ccc} X & \longrightarrow & S \times \mathbb{C}^r \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle p_1} & \\ S & & \end{array}$$

Let D_{d-k+1} be a vector subspace of $\mathbb{C}^r$ of codimension $d-k+1$, with $0 \leq k \leq d$, and $p : \mathbb{C}^r \to \mathbb{C}^{d-k+1}$ be a generic linear projection (in the sense of [56, Proposition 1.3.2]), whose kernel is D_{d-k+1}. For $x \in X \setminus F$ the fiber $X_x = f^{-1}(f(x))$ of f in X is non-singular and contained in $\{f(x)\} \times \mathbb{C}^r$. We denote by $p_x : X_x \to \mathbb{C}^{d-k+1}$ the restriction of the projection p to X_x . We define

$$P_k(f,p,0)^0 = \{y \in X \setminus F;\ \ y \in \Sigma(p_x)\},$$

where $\Sigma(p_x)$ denotes the set of critical points of p_x, and by $P_k(f,p,0)$ its closure in X.

Proposition 1.51. *Let $f : (X,0) \to (S,0)$ be a smooth morphism, i.e., plane with smooth fibers, in* *every point of $X \setminus F$. Then $P_k(f,p,0)$ is a closed analytic subset, empty or of pure codimension k in X.*

The proof can be found in [56, Corollary 1.3.2]. Teissier [56] introduced the definition of relative polar variety.

Definition 1.52. *Let $f:(X,0)\to(S,0)$ be a morphism as above, a S-embedding $(X,0)\subset(S,0)\times\mathbb{C}^r$ is a generic linear subspace $D_{d-k+1}\subset\mathbb{C}^r$ of codimension $d-k+1$, where D_{d-k+1} is the kernel of a generic linear projection $p:\mathbb{C}^r\to\mathbb{C}^{d-k+1}$. The closed analytic subset $P_k(f,p,0)$ of X is called a* **relative polar variety** *of X with codimension k associated to f and D_{d-k+1}. When S is a point, we call $P_k(f,p,0)$ an* **absolute polar variety**.

Example 1.53. *This example was extracted from [33, Example 1.42]. Consider the surface $V=\{(x,y,t)\in\mathbb{C}^3 : f(x,y,t)=y^2-x^3-t^2x^2=0\}$ and the projection parallel to the y-axis. The singular set of V is the t-axis. Let $(x_0,y_0,t_0)\in V_{reg}$, the tangent plane to V at this point is given by*

$$x\cdot\frac{\partial f}{\partial x}(x_0,y_0,t_0)+y\cdot\frac{\partial f}{\partial y}(x_0,y_0,t_0)+t\cdot\frac{\partial f}{\partial z}(x_0,y_0,t_0)=0.$$

In order to the projection π restrict to the tangent plane $T_{p_0}V_{reg}$, where $p_0=x\cdot\dfrac{\partial f}{\partial x}(x_0,y_0,t_0)$, not to be surjective, the tangent plane must contain the y-axis, i.e., points of the form $(0,y,0)$ must satisfy the plane equation, then we want $y\cdot\frac{\partial f}{\partial y}(x_0,y_0,t_0)=0$. Therefore, the polar variety $P_1(V,P)$ associated to the projection π is the closure of the solution of the following system of equations,

$$\begin{cases} y\cdot\frac{\partial f}{\partial y}(x,y,t)=0 \\ f(x,y,t)=0 \end{cases},\ \text{where } (x,y,t)\in V_{reg}.$$

Thus,

$$\begin{cases} y=0 \\ y^2-x^3-t^2x^2=0)=0 \end{cases},\ \text{where } (x,y,t)\in V_{reg}.$$

Therefore, $P_1(V,P)=\overline{\{(x,y,t)\in V_{reg} : x^2(x-t^2)=0\}}$.

Definition 1.54. *Let X be a subvariety with dimension k contained on a regular variety M. Let D be a small polydisc (cartesian product of discs) centered at p. Consider a projection π on a k-dimensional generic plane. Let q be a point in $\pi(D)\setminus\pi(p)$ and δ a small polydisc centered at q. The* **multiplicity** *of X at p, denoted by $m(X,p)$, is the number of fibers of $\pi^{-1}(\delta)\cap D\cap X$.*

The key invariant of the polar variety $P_k(f,p,0)$ is its multiplicity at 0, $m_0(P_k(f,p,0))$, which is called k^{th} **relative polar multiplicity** of X and denoted by $m_k(X,f,0)$. If f is a constant map, we denote the multiplicity by $m_k(X,0)$. For a generic linear projection, the multiplicity is independent of D_{d-k+1} and, indeed, it is an analytic invariant of X.

Example 1.55. *We shall compute the polar multiplicities of the surface $V=\{(x,y,t)\in\mathbb{C}^3 : f(x,y,t)=y^2-x^3-t^2x^2=0\}$. We have $P_0(V,D_2)=V$ and $P_1(C,D_1)=\{(x,y,t)\in\mathbb{C}^3 : y=0 \text{ and } x-t^2=0\}$, where D_2 is the zero vectorial space and D_1 is the line defined by the y-axis. Therefore, $m_0(V,0)=2$ and $m_1(V,0)=1$.*

4 Whitney stratification and Whitney equisingularity

In this section, we introduce the notion of Whitney stratification, which was presented by Whitney in [57] and widely used since then. Another reference to this subject is [28]. In addition we also study the Whitney equisingularity of a family of singularities.

Definition 1.56. *Let M be a smooth variety and $V \subset M$. A* **locally finite stratification** *of V is a partition of V in subvarieties of M (called* **strata***) such that, for each point of V there exists a neighbourhood in M which meets only finitely many strata.*

Example 1.57. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function given by $f(x,y,z) = x^2 - y^2 z$, then the variety V defined by the zero locus of f is $V = \left\{(x,y,z) \in \mathbb{R}^3;\ x^2 = y^2 z\right\}$. Considering subvarieties $V_\alpha, V_\beta \subset \mathbb{R}^3$ defined by $V_\beta = \left\{(0,0,z) \in \mathbb{R}^3\right\} \subset V$ and $V_\alpha = V \setminus V_\beta$, we have that V_α and V_β form a finite and, therefore, locally finite stratification of V.*

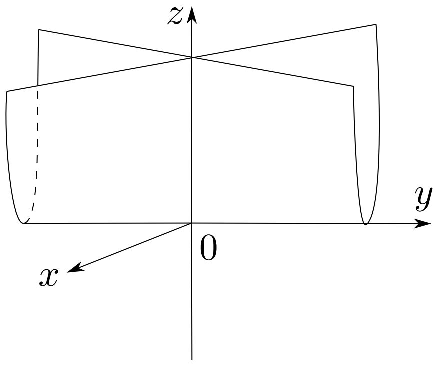


Figure 1.12: Variety $V \subset \mathbb{R}^3$.

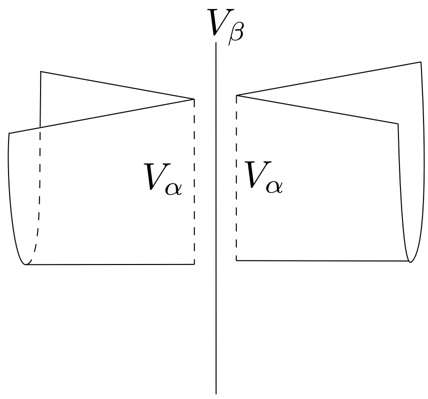


Figure 1.13: Stratification of V.

We are interested in stratifications, where points in the same stratum have homeomorphic neighbourhoods. This is granted by the boundary condition.

Definition 1.58. *We say that a stratification $\{V_\alpha\}$ of V* **satisfies the boundary condition***, if for any two strata V_α and V_β, such that $V_\alpha \cap \overline{V}_\beta \neq \emptyset$ then $V_\alpha \subset \overline{V}_\beta$.*

Once these strata are disjoint, either $V_\alpha = V_\beta$ or $V_\alpha \subset \overline{V_\beta} \setminus V_\beta$.

Definition 1.59. *A stratification $\{V_\alpha\}$* **satisfies the Whitney conditions** *if for every pair of strata (V_α, V_β), such that $V_\beta \subset \overline{V_\alpha}$ and for every point $y \in V_\beta$ we have:*

a) For every sequence of points $\{x_i\}$ of V_α converging to y, such that the limit

$$\lim_{i \to \infty} T_{x_i}(V_\alpha) = T$$

exists in the corresponding Grassmannian, T contains $T_y(V_\beta)$.

b) If we also have a sequence $\{y_i\}$ of points in V_β with limit y and such that the limit of directions

$$\lim_{i\to\infty} \overline{x_i y_i} = \lambda$$

exists in the projective space, then T contains λ.

These conditions are called **conditions** (a) **and** (b) **of Whitney**.

A stratification which satisfies the boundary condition and the conditions of Whitney is called a **Whitney stratification**.

Whitney showed, in his work [57], that every complex analytic variety admits a stratification which satisfies conditions $a)$ and $b)$ of Whitney. Moreover, if X is a locally finite collection of analytic subsets of U, then we can choose a stratification $\mathscr{V}$ such that each element of X is a union of strata of $\mathscr{V}$ (see [31]). Such stratification is called **stratification adapted to** X.

Next, we present some examples of stratification.

Example 1.60. *Consider the cone C with vertex at the origin, and the stratification $\{V_1, V_2\}$, where V_1 is a generatrix of the cone and $V_2 = C \setminus V_1$. In this case, the conditions (a) and (b) of Whitney are not satisfied. It is enough to consider a sequence $\{x_i\}$ of points in C, which are over a generatrix L of the cone which is not V_1, whose limit is the origin, such that the segment $\overline{x_i y_i}$ has always the same direction λ.The condition (a) is not satisfied, once the limit of tangent spaces $T_{x_i}V_2$ does not contain the space $T_0(V_1)$, the condition (b) is also not satisfied, because λ is not contained in the limit of tangent spaces $T_{x_i}V_2$.*

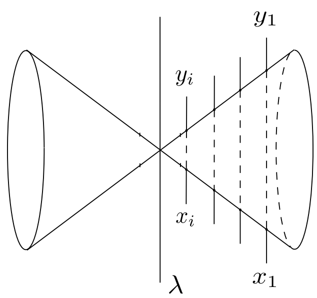


Figure 1.14: Singular variety.

Example 1.61. *Consider the variety V in $\mathbb{C}^3$ defined by the equation $y^2 - x^3 - t^2x^2 = 0$.*

If we take the t-axis as a stratum V_1 and the regular part of V, V_{reg}, as the other stratum, the stratification $\{V_1, V_2\}$ (Figure 1.16) satisfies the condition (a), but does not satisfy the condition (b). However, if we add a zero dimensional stratum, the origin of $\mathbb{C}^3$, we have both conditions satisfied (Figure 1.17).

Whitney equisingularity is a concept which deals with the behaviour of singularities in families of germs of varieties as parameters vary. It is a condition that ensures that the singularities change in a

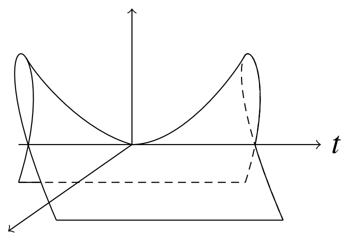


Figure 1.15: Singular variety V.

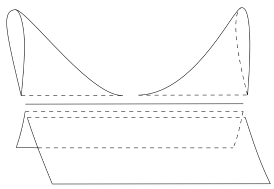


Figure 1.16: Stratification of V.

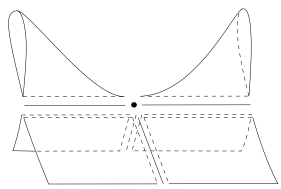


Figure 1.17: Whitney stratification of V.

controlled and compatible way, making it a valuable tool for understanding the behaviour of singular points. In the following, we present the Whitney equisingularity of a family of functions.

Consider a function $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ and let $U \subset \mathbb{C}^m$ be an open neighbourhood of $0 \in \mathbb{C}^m$. Let $D \subset \mathbb{C}$ be an open disc centered at $0 \in \mathbb{C}$ and consider the map germ $F : (U \times D, 0) \to (\mathbb{C}, 0)$ such that $F(z, 0) = f(z)$ for all $z \in \mathbb{C}^m$. Then we set $f_t(z) = F(z,t)$.

Definition 1.62. *We say that the family* $\{(V(f_t), 0)\}_{t\in D}$ *is* **Whitney equisingular**, *if the stratification* $\mathscr{V} = \{F^{-1}(0,0) \setminus T, T\}$ *satisfies the Whitney conditions, where* $T = D \times \{0\} \subset \mathbb{C} \times \mathbb{C}^m$.

Example 1.63. (i) *Consider the family* $\{(V(f_t), 0)\}_{t\in D}$ *defined by* $f_t(x,y) = y^2 - x^3$. *This family is clearly Whitney equisingular.*

(ii) *Consider the family* $\{(V(f_t), 0)\}_{t\in D}$ *defined by* $f_t(x,y) = y^2 - x^3 - t^2x^2$. *As showed in Example 1.61, the stratification* $\mathscr{V} = \{F^{-1}(0,0) \setminus T, T\}$ *does not satisfy condition* $b)$ *of Whitney. Therefore, this family is not Whitney equisingular.*

5 Local Euler obstruction

An important invariant studied in singularity theory is the local Euler obstruction, which was defined by MacPherson in [42] as one of the main ingredients of its proof for the Deligne and Grothendieck's conjecture. This conjecture concerns the existence and uniqueness of the Chern classes in the singular case. Some basic references about the development of the local Euler obstruction are [5], [13] and [33].

In [7] Brasselet and Schwartz, using vector fields, introduced an equivalent definition for this invariant, which we will consider in this work. This point of view allows us to study the local Euler obstruction in the class of index of vector fields.

Before presenting the definition of the local Euler obstruction introduced in [7] we discuss the concepts of Nash modification and Nash bundle. For results on Nash modification and resolution of singularities, we suggest [55].

We denote by $G(d,n)$ the Grassnannian of d-planes of $\mathbb{C}^m$ (see [41] for more details on Grassmannian varieties). Let X be a representative of a germ of a complex analytic space $(X,0)$, equidimensional of complex dimension d with $X \subset U$, where U is an open subset of $\mathbb{C}^m$.

Over the regular part X_{reg} of X, we can define the Gauss map $\phi : X_{reg} \to U \times G(d,n)$ by:

$$\phi(x) = (x, T_x X_{reg}).$$

$$\begin{array}{ccc} & & U \times G(d,n) \\ & \overset{\phi}{\nearrow} & \downarrow \pi_1 \\ X_{reg} & \xrightarrow[i]{} & U \end{array}$$

Definition 1.64. *The* **Nash modification** $\widetilde{X}$ *is defined as the closure of the image of* ϕ *in* $U \times G(d,n)$.

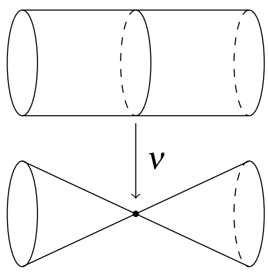
Remark 1.65. As observed in [8, pg. 130], $\widetilde{X}$ is a complex analytic space with a natural analytic projection $\nu : \widetilde{X} \to X$, whose restriction $\nu|_{\nu^{-1}(X_{reg})}$ is holomorphic, bijective and its inverse is also holomorphic.

Denote by $U(d,n)$ the tautological bundle over $G(d,n)$, *i.e.,* the bundle whose fiber in $P \in G(d,n)$ is the set of all vectors of P, and by T the bundle corresponding to the trivial extension of $U(d,n)$ over $U \times G(d,n)$, *i.e.,* T contains the elements (x,P,v), where $(x,P) \in U \times G(d,n)$ and $v \in P$.

Definition 1.66. *The* **Nash bundle** $\widetilde{T}$ *with basis* $\widetilde{X}$ *is the restriction of* T *over* $\widetilde{X}$*, then we have the diagram:*

$$\begin{array}{ccc} \widetilde{T} & \hookrightarrow & T \\ \downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & U \times G(d,n) \\ \nu \downarrow & & \downarrow \nu \\ X & \hookrightarrow & U \end{array}$$

Example 1.67. *The Nash modification of the cone is a cylinder.*



It is important to remark the fact that the Nash modification of a singular set is not always regular, *i.e.,* not all Nash modifications solve a singularity of an algebraic or analytic variety.

Example 1.68. *Consider the plane curve formed by the cusp C and by a line l, such that l is the limit*

$$\lim_{p\to 0} l_p,$$

where l_p is the tangent line to the cusp at p, this curve is illustrated in the following figure.

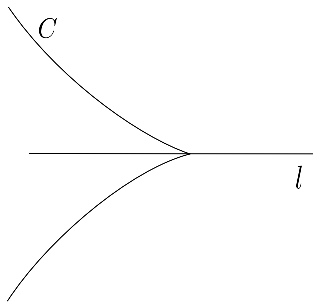


Figure 1.18: Plane curve formed by the cusp C and the line l.

The Nash modification of this curve, is the space curve represented in the Figure 1.18. Note that $(0,l)$ is still a singular point of the Nash modification, since it is still a crossing point between two curves.

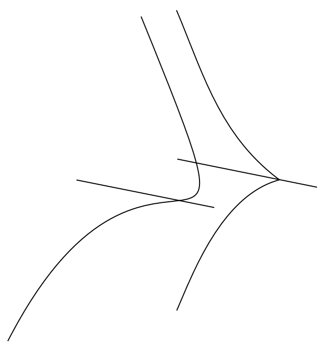


Figure 1.19: Nash modification of the curve formed by the cusp C and by the line l.

Let $(X,0) \subset (\mathbb{C}^m,0)$ be a germ of a complex analytic variety, equidimensional and reduced of codimension d in an open subset $U \subset \mathbb{C}^m$. We consider a Whitney stratification $\mathscr{V} = \{V_i\}$ of U adapted to X and assume that $\{0\}$ is a stratum. We choose a small representative of $(X,0)$ such that 0 belongs to the closure of all strata. We denote such representative by X and write $X = \cup_{i=0}^{q} V_i$ where $V_0 = \{0\}$ and $V_q = X_{\text{reg}}$ is the set of regular points of X. We assume that the strata $V_0,\dots,V_{q-1}$ are connected and the analytic sets $\overline{V_0},\dots,\overline{V_{q-1}}$ are reduced.

We denote by $TU|_X$ the restriction to X of the tangent bundle of U. We know that a stratified vector field v in X is a continues section of $TU|_X$ such that if $x \in V_\alpha \cap X$, then $v(x) \in T_x(V_\alpha)$. From the Whitney conditions we have the following lemma (one can find a proof in [7].):

Lemma 1.69. *Every stratified vector field v without singularities in a subset $A \subset X$ has a canonical lifting to a section $\widetilde{v}$, of the Nash bundle $\widetilde{T}$, without singularities in $\nu^{-1}(A) \subset \widetilde{X}$.*

Now consider the radial stratified vector field $v(x)$ in a neighbourhood of $\{0\}$ in X, *i.e.,* there exists ε_0 such that for all $0 < \varepsilon \leq \varepsilon_0$, $v(x)$ points to outside the ball $\mathbb{B}_\varepsilon$ over the boundary $\mathbb{S}_\varepsilon := \partial\mathbb{B}_\varepsilon$.

In [7] Brasselet and Schwartz defined the local Euler obstruction in the following way:

Definition 1.70. *Let v be a radial vector field in $X \cap \mathbb{S}_\varepsilon$ and $\widetilde{v}$ be the lifting of v in $\nu^{-1}(X \cap \mathbb{S}_\varepsilon)$ to a section of the Nash bundle. The* **local Euler obstruction** $\mathrm{Eu}_X(0)$ *is defined as the obstruction to extend $\widetilde{v}$ as a non zero section of $\widetilde{T}$ over $\nu^{-1}(X \cap \mathbb{B}_\varepsilon)$.*

More precisely, let $\mathscr{O}(\widetilde{v}) \in H^{2d}(\nu^{-1}(X \cap \mathbb{B}_\varepsilon), \nu^{-1}(X \cap \mathbb{S}_\varepsilon))$ be a cocycle of obstruction to extend $\widetilde{v}$ as a non zero section of $\widetilde{T}$ to the interior of $\nu^{-1}(X \cap \mathbb{B}_\varepsilon)$. The local Euler obstruction $\mathrm{Eu}_X(0)$ is defined as the cocycle of evaluation $\mathscr{O}(\widetilde{v})$ in the fundamental class of the pair $(\nu^{-1}(X \cap \mathbb{B}_\varepsilon), \nu^{-1}(X \cap \mathbb{S}_\varepsilon))$. The local Euler obstruction is an integer number.

Remark 1.71. Some important properties of the local Euler obstruction:

- The local Euler obstruction at a regular point is equal to 1.
- The local Euler obstruction at a point of a curve is exactly the multiplicity of this point over this curve [30].
- The local Euler obstruction is constant along each stratum of a Whitney stratification [7].
- $\mathrm{Eu}_{X\times Y}(a,b) = \mathrm{Eu}_X(a) \cdot \mathrm{Eu}_Y(b), \forall a \in X, \forall b \in Y.$

The local Euler obstruction is not easily computed using its definition, which motivated many authors to find formulas to facilitate its computation. In [6], Brasselet, Lê and Seade provided a Lefschetz type formula, therefore a topological formula, for the local Euler obstruction. Before presenting that formula we present the definition of transversality and a lemma concerning generic linear forms.

Definition 1.72. *Let Y_1, Y_2 be smooth subvarieties of a smooth variety M. The variety Y_1 is* **transversal** *to Y_2 in M, if for each point $p \in Y_1 \cap Y_2$, we have*

$$T_pX_1 + T_pY_2 = T_pM.$$

Lemma 1.73. *There exists a non-empty Zariski open set W in the space of complex linear functions on $\mathbb{C}^m$ such that for every $l \in W$, there exists a representative X of $(X,0)$ such that:*

(i) *for each $x \in X$, the hyperplane $l^{-1}(0)$ is transversal in $\mathbb{C}^m$ to every limit of tangent spaces in $T(X_{reg})$ of points in X_{reg} converging to x;*

(ii) *for each y in the closure $\overline{V_i}$ in X of each strata V_i, $i = 1, \ldots, q$, the hyperplane $l^{-1}(0)$ is transversal in $\mathbb{C}^m$ to every limit of tangent spaces in TV_i of points converging to y, where $\{V_i\}$ is a Whitney stratification of X.*

In particular, for each $l \in W$ one has

$$\widetilde{X} \subset \mathbb{C}^m \times (G(d,m) - H^*)$$

where $H^* := \{T \in G(d,N) : l(T) = 0\}$.

One can find a proof in [6, Lemma 1.3].

Theorem 1.74. *Let $(X,0)$ be a germ of a complex analytic variety and $\{V_i\}$ a Whitney stratification of X. Let $\ell : U \to \mathbb{C}$ be a generic linear function (in the sense of Lemma 1.73), where, U is an open neighbourhood of 0 in $\mathbb{C}^m$. Then*

$$\mathrm{Eu}_X(0) = \sum_i \chi(V_i \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_i),$$

where $\mathbb{B}_\varepsilon$ is a small closed ball around the origin in $\mathbb{C}^m$, $t_0 \in \mathbb{C}\backslash\{0\}$ such that $\|t_0\| << 1$, $\mathrm{Eu}_X(V_i)$ is the local Euler obstruction of X at any point of the stratum V_i and the above sum extends to all strata such that $0 \in \overline{V}_i$.

The proof can be found in [6, Theorem 3.1].

Example 1.75. *This example was extracted from [33, Example 2.16]. Let $f : \mathbb{C}^3 \to \mathbb{C}$ be defined by*

$$f(x,y,t) = y^2 - x^3 - t^2x^2$$

and consider $X = f^{-1}(0)$. Consider also the Whitney stratification of X given by

$$\{V_0 = \{0\}, V_1 = \{t-\text{axis}\} \setminus \{0\}, V_2 = V_{reg}\}.$$

Let $\ell : \mathbb{C}^3 \to \mathbb{C}$ be the generic linear function defined by $\ell(x,y,t) = t$,then, by Theorem 1.74, we have

$$\begin{aligned} \mathrm{Eu}_X(0) &= \chi(V_0 \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_0) \\ &+ \chi(V_1 \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_1) \\ &+ \chi(V_2 \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_2). \end{aligned}$$

Since $V_0 \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0) = \emptyset$, we have $\chi(V_0 \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0)) = 0$. In addition, $V_1 \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0) = \{(0,0,t_0)\}$, then, $\chi(V_1 \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0)) = 1$. On the other hand, $V_2 \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0) = \{(x,y,t)/y^2 - x^3 - t_0^2x^2 = 0\}\backslash\{(0,0,t_0)\}$. With help from [35, Theorem 2], which presents a formula to compute the Euler characteristic of a plane curve, we can compute $\chi(V_2 \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0)) = -1$. By the first item of Remark 1.71 we have $\mathrm{Eu}_X(V_2) = \mathrm{Eu}_X(V_{reg}) = 1$. To compute $\mathrm{Eu}_X(V_1)$, we take $t_0 \in V_1$, note that in a small neighbourhood of t_0 we have that V_1 is diffeomorphic to the product of a small disc containing t_0 and contained in t-axis with the curve $X \cap \ell^{-1}(t_0)$, using the second and the last item of Remark 1.71, we obtain $\mathrm{Eu}_X(V_1) = 2$. Therefore,

$$\mathrm{Eu}_X(0) = 0 \cdot \mathrm{Eu}_X(0) + 1 \cdot 2 + (-1) \cdot 1 = 1.$$

Hence, for any non zero section $\widetilde{v}$ of $\widetilde{T}$ over $\nu^{-1}(X \cap \partial\mathbb{B}_\varepsilon)$, the lifting of a radial vector field v at $0 \in X$ over the set $X \cap \partial\mathbb{B}_\varepsilon$, can not be extended over $\nu^{-1}(X \cap \mathbb{B}_\varepsilon)$ without singularities.

Lê and Teissier [40, Corollary 5.1.2] provided the following formula to compute the local Euler obstruction of an analytic space in terms of its polar multiplicities.

$$\mathrm{Eu}_X(0) = \sum_{k=0}^{d-1} (-1)^k m_k(X,0). \tag{5.1}$$

Example 1.76. *Let* $V = \{(x,y,t) \in \mathbb{C}^3 : f(x,y,t) = y^2 - x^3 - t^2x^2 = 0\}$. *As we have seen in Example 1.55,* $m_0(V,0) = 2$ *and* $m_1(V,0) = 1$, *therefore*

$$\mathrm{Eu}_V(0) = 2 - 1 = 1.$$

6 Vanishing Euler characteristic and top polar multiplicity

In this section, we present concepts, definitions and results regarding vanishing Euler characteristic and its relation to the top polar multiplicity. In addition, we study the relation between top polar multiplicity and the other polar multiplicities as well as their relation with Whitney equisingularity of families of IDS.

The vanishing Euler characteristic of an isolated determinantal singularity was introduced by Nuño-Ballesteros, Oréfice-Okamoto and Tomazella [45]. In the same article, the authors showed that the vanishing Euler characteristic satisfies properties which hold for the Milnor number of an ICIS, such as the relations with polar multiplicities and a Lê–Greuel type formula.

Let $(X_A^s, 0)$ be an IDS defined by the germ of a matrix $A : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$. Choose a small enough representative $A : B_\varepsilon \to (M_{n,k}, 0)$, where B_ε is a small enough open ball centered at the origin in $\mathbb{C}^m$. For a matrix $B \in M_{n,k}$, denote

$$\begin{array}{rccl} A+B: & B_\varepsilon & \to & M_{n,k} \\ & x & \mapsto & (A+B)(x) = A(x) + B \end{array}$$

Lemma 1.77. *There exists a non-empty Zariski open subset* $W \subset M_{n,k}$ *such that:*

(i) X_{A+B}^s *is smooth and* $\mathrm{rank}(A(x)+B) = s-1$, *for all* $x \in X_{A+B}^s$ *and* $A \in W$*;*

(ii) *the Euler characteristic of* X_{A+B}^s, $\chi(X_{A+B}^s)$, *does not depend on* $A \in W$.

The proof can be found in [45, Lemma 3.1]. When $s = 1$, the IDS $(X_A^s, 0)$ is an ICIS and we can see X_{A+B}^s as the Milnor fiber of $(X_A^s, 0)$. In this case, X_{A+B}^s has the homotopy type of a bouquet of μ spheres, where μ is the Milnor number of X_A^s (see for instance [45]). Therefore, the Milnor number coincides with the so-called vanishing Euler characteristic, *i.e.,*

$$\mu(X_A^s, 0) = (-1)^d (\chi(X_{A+B}^s - 1)),$$

where $d = \dim(X_A^s, 0)$. Based on this fact and Lemma 1.77, Nuño-Ballesteros, Oréfice-Okamoto and Tomazella defined the vanishing Euler characteristic of an IDS as follows.

Definition 1.78. *The* **vanishing Euler characteristic** *of the d-dimensional IDS* $(X_A^s, 0)$ *is defined by*

$$\nu(X_A^s, 0) = (-1)^d(\chi(X_{A+B}^s) - 1),$$

where $A \in W$ *and* W *is given in Lemma 1.77.*

For codimension 2 IDS in $\mathbb{C}^4$ and $\mathbb{C}^5$, the vanishing Euler characteristic coincides with the Milnor number introduced by Pereira and Ruas [53].

Consider a map germ $\mathscr{A} : (\mathbb{C}^m \times \mathbb{C}, 0) \to (M_{n,k}, 0)$ such that $\mathscr{A}(x, 0) = A(x)$ for all $x \in \mathbb{C}^m$. When X_A^s is a determinantal variety, the projection

$$\begin{array}{cccc} \pi : & (X_{\mathscr{A}}^s, 0) & \to & (\mathbb{C}, 0) \\ & (x,t) & \mapsto & t \end{array}$$

is called a **determinantal deformation of** X_A^s . If we fix a small enough representative $A : B_\varepsilon \to (M_{n,k}, 0)$, where B_ε is the open ball centered at the origin with radius $\varepsilon > 0$, then we set $A_t(x) := \mathscr{A}(x,t)$ and $X_{A_t}^s = A_t^{-1}(M_{n,k}^s)$.

Definition 1.79. *Let* $\mathscr{A}$ *be as in the above construction. We say that* $\mathscr{A}$ *defines a* **determinantal smoothing** *of* $(X_A^s, 0)$ *if* X_t *is smooth and* $\operatorname{rank}(A_t(x)) = s - 1$ *for all* $x \in X_t$ *and all* $t \neq 0$ *small enough.*

The above formula to compute the vanishing Euler characteristic is also valid when we consider a general determinantal smoothing X_t instead of the special smoothing X_{A+B}^s. A proof for the following theorem can be found in [45, Theorem 3.4].

Theorem 1.80. *Let* $\mathscr{A} : (\mathbb{C}^m \times \mathbb{C}, 0) \to (M_{n,k}, 0)$ *be a determinantal smoothing of* $(X_A^s, 0)$*. Then, for all* $t \neq 0$ *small enough,*

$$\nu(X_A^s, 0) = (-1)^d(\chi(X_{A_t}^s) - 1).$$

Let $(X_A^s, 0) \subset (\mathbb{C}^m, 0)$ be an IDS and let $p : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be a generic linear function. Then $X_A^s \cap p^{-1}(0)$ is an IDS of type $(n,k;s)$ inside the hyperplane $p^{-1}(0)$. Therefore, it makes sense to consider the vanishing Euler characteristic $\nu(X_A^s \cap p^{-1}(0))$. The proof of the following lemma can be found in [45, Lemma 4.2].

Lemma 1.81. *Let* $p : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ *be a generic linear function,* $B \in M_{n,k}$ *a generic matrix and* $c \in \mathbb{C} \setminus \{0\}$ *sufficiently small. Then*

$$\begin{aligned} \nu(X_A^s \cap p^{-1}(0)) &= (-1)^{d-1}(\chi(X_A^s \cap p^{-1}(c))) \\ &= (-1)^{d-1}(\chi(X_{A+B}^s \cap p^{-1}(0))) \\ &= (-1)^{d-1}(\chi(X_{A+B}^s \cap p^{-1}(c))), \end{aligned}$$

where $d = \dim(X_A^s)$.

Theorem 1.82 (see [46, Theorem 4.2] for a proof) and Corollary 1.83 (its proof is in [46, Corollary 4.3]) show under which circumstances, the vanishing Euler characteristic is constant on a family of isolated determinantal singularities.

Theorem 1.82. *Let $X^s_{A_0}$ be a d-dimensional IDS and let $X^s_{A_t}$ be any determinantal deformation. Then,*

$$\nu(X^s_{A_0},0)-\sum_{x\in S(X^s_{A_t})}\nu(X^s_{A_t},0)=(-1)^d(\chi(X^s_{A_t})-1),$$

where $S(\cdot)$ denotes the singular set.

Corollary 1.83. *Let $X^s_{A_t}$ be a determinantal deformation of the IDS $X^s_{A_0}$. Then, the sum*

$$\sum_{x\in S(X^s_{A_t})}\nu(X^s_{A_t},x)$$

is constant on t if and only if $\chi(X^s_{A_t})=1$.

In the following, we present the definition of top polar multiplicity or d^{th} polar multiplicity of an IDS $(X^s_A,0)$, as the number of critical points of a morsification of generic linear map defined on the smoothing X^s_{A+B}. For more information on Morse functions and morsification see [45, Appendix A].

Firstly, it is important to point out that for a complex analytic variety $X\subset\mathbb{C}^m$ of dimension d, the top polar variety of X of codimension d consists of a finite number of points or it is empty. In both cases, its multiplicity is not defined. However, in [24] Gaffney introduced the d^{th} polar multiplicity tin the following way.

Let $\mathfrak{X}\subset\mathbb{C}^m\times\mathbb{C}^s$ be a complex analytic variety of dimension $d+s$ and $f:\mathfrak{X}\to\mathbb{C}^s$ an analytic function such that $f^{-1}(0)=X$. Then, we have the following definition:

Definition 1.84. *The* **top polar multiplicity** *of X, which is denoted by $m_d(X,p,0)$, is defined as the multiplicity $m_0(P_d(f,p,0))$, where $P_d(f,p,0)$ is the polar variety of $\mathfrak{X}$ related to (f,p). If p is generic, we denote $m_d(X,p,0)$ by $m_d(X,0)$.*

Nuño-Ballesteros, Oréfice-Okamoto and Tomazella defined the top polar multiplicity of an IDS trough the following construction.

Lemma 1.85. *There exists a linear function $p:\mathbb{C}^m\to\mathbb{C}$ and a non-empty Zariski open subset $W\subset M_{n,k}$ such that X_{A+B} is smooth and $p|_{X_{A+B}}$ is a Morse function for all $B\in W$.*

The proof can be found in [45, Lemma 4.1].

Definition 1.86. *The* **top polar multiplicity of** $(X^s_A,0)$ *is defined by*

$$m_d(X^s_A,0):=\sharp\Sigma(p|_{X_{A+B}}),$$

where $p:\mathbb{C}\to\mathbb{C}$ is a generic linear map, $B\in M_{n,k}$ is a generic matrix, where $\sharp\Sigma(p|_{X_{A+B}})$ denotes the number of critical points of $p|_{X_{A+B}}$ and d is the dimension of X^s_A.

The top polar multiplicity is related to the vanishing Euler characteristic and the local Euler obstruction of an IDS. These relations are stated in the following theorems.

Theorem 1.87. *Let $p:\mathbb{C}\to\mathbb{C}$ be a generic linear function. Then*

$$m_d(X_A^s,0)=\nu(X_A^s,0)+\nu(X_A^s\cap p^{-1}(0),0).$$

The proof of this theorem can be found in [45, Theorem 4.3].

Theorem 1.88. *Let $(X_A^s,0)$ be an IDS of dimension d. Then,*

$$\mathrm{Eu}(X_A^s,0)+(-1)^d m_d(X_A^s,0)=1+(-1)^d\nu(X_A^s,0).$$

One can find a proof in [45, Theorem 4.5]. The last two theorems show that m_d does not depend neither on choice of the smoothing or the generic linear function.

Example 1.89. *This example was computed in [45, Example 4.6]. Consider $X_A^2\subset\mathbb{C}^4$ the determinantal variety defined by the matrix*

$$A(x,y,z,w)=\begin{bmatrix} x & y & z\\ y & z & w\end{bmatrix}.$$

Take

$$B=\frac{1}{100}\begin{bmatrix} 6 & -8 & 5\\ 1 & 8 & 7\end{bmatrix}$$

and $p(x,y,z,w)=2x-3y+4z-w$. Making computations with Mathematica, we see that

$$m_2(X_A^s,0)=\sharp\Sigma(p|_{X^2_{A+B}})=3.$$

Since $X\cap p^{-1}(0)$ is a curve, following [47] one can compute its Milnor number in terms of the Milnor number $\mu(g|_{X_A^2\cap p^{-1}(0)})$ and the local degree $deg(g|_{X_A^2\cap p^{-1}(0)})$ of a function $g:(\mathbb{C}^4,0)\to(\mathbb{C},0)$ on the curve $X_A^2\cap p^{-1}(0)$: $\mu(g|_{X_A^2\cap p^{-1}(0)})=\nu(X_A^2\cap p^{-1}(0),0)+deg(g|_{X_A^2\cap p^{-1}(0)})-1$. Taking $g(x,y,z,w)=x-2y+5z+5w$, we have that

$$\mu(g|_{X_A^2\cap p^{-1}(0)})=\sharp\Sigma(g|_{X^2_{A+B}\cap p^{-1}(0)})=4,\ deg(g|_{X\cap p^{-1}(0)})=\sharp\{X^2_{A+B}\cap p^{-1}(0)\cap g^{-1}(0)\}=3.$$

Therefore, $\nu(X_A^2\cap p^{-1}(0),0)=\mu(X_A^2\cap p^{-1}(0),0)=4-3+1=2$. Hence

$$\mathrm{Eu}_{X_A^2}(0)=1-\nu(X_A^2\cap p^{-1}(0),0)=-1$$

and

$$\nu(X_A^2,0)=m_2(X_A^2)-\nu(X_A^2\cap p^{-1}(0),0)=1$$

For an isolated determinantal singularity, there exists a relation between its top polar multiplicity of generic hyperplane sections and its polar multiplicities. The following lemma was introduced by Gaffney, Grulha Jr. and Ruas [26, Lemma 3.6].

Lemma 1.90. *Suppose X_A^s is a d-dimensional IDS, $\pi_2:(X_A^s,0)\to(\mathbb{C}^2,0)$ defines the polar curve $P_1(X_A^s)$. If H is a generic hyperplane then*

$$m_{d-1}(X_A^s\cap H,0)=m_0(P_1(X_A^s),0).$$

The above lemma is actually valid for a larger class of determinantal singularities called essentially isolated determinantal singularities.

The polar multiplicities $m_i(X^s_A,0)$, $i=0,\dots,d$, play an important role in the study of equisingularity of families of d-dimensional determinantal singularities (see [46]).

Let $X^s_{A_t}$ be a determinantal deformation of $(X^s_{A_0},0)$ as above, we say that:

(i) $X^s_{A_t}$ is **origin preserving** if $0\in S(X^s_{A_t})$, for all t in D, where $S(X^s_{A_t})$ denotes the singular set of $X^s_{A_t}$ and $D\subset\mathbb{C}$ is a disc around the origin. Then $\left\{(X^s_{A_t},0)\right\}_{t\in D}$ is called a 1-*parameter family* of IDS;

(ii) $\left\{(X^s_{A_t},0)\right\}_{t\in D}$ is a **good family** if there exists $\varepsilon>0$ with $S(X^s_{A_t})=\{0\}$ on B_ε, for all t in D;

(iii) $\left\{(X^s_{A_t},0)\right\}_{t\in D}$ is **Whitney equisingular** if it is a good family and $\left\{X^s_{\mathscr{A}}\setminus T,T\right\}$ satisfies the Whitney conditions, where $T=\{0\}\times D$.

Theorem 1.91. *A good family of d-dimensional IDS $\{(X^s_{A_t},0)\}_{t\in D}$ is Whitney equisingular if and only if all the polar multiplicities $m_i(X^s_{A_t},0)$, $i=0,\dots,d$ are constant on $t\in D$.*

The proof to this Theorem can be found in [46, Theorem 5.3].

CHAPTER 2

Newton polyhedra and Whitney equisingularity

Our purpose in this chapter is to prove that a family of IDS $\{(X^s_{A_t},0)\}_{t\in D}$, where $D\subset\mathbb{C}$ is a neighbourhood of the origin, defined by Newton non-degenerate matrices with convenient entries is Whitney equisingular (Theorem 2.30).

1 Families of determinantal singularities

In order to prove the Whitney equisingularity of a family of IDS $\{(X^s_{A_t},0)\}_{t\in D}$, we would like to show that the vanishing Euler characteristic is constant on this family, by applying Theorem 1.91 to the family $\{(X^s_{A_t},0)\}_{t\in D}$. By Corollary 1.83, we need to prove that $\chi(X^s_{A_t},0)=1$ for all $t\in D$.

We start with the following example, which shows a necessary condition in order to the vanishing Euler characteristic be constant on a family of singularities.

Example 2.1. *Consider the cusp defined by the function*

$$\begin{array}{rccc} f: & (\mathbb{C}^2,0) & \to & (\mathbb{C},0) \\ & (x,y) & \mapsto & y^2-x^3 \end{array}$$

and the cusp deformation defined by

$$\begin{array}{rccc} F: & (\mathbb{C}^2\times\mathbb{C},0) & \to & (\mathbb{C},0) \\ & (x,y,t) & \mapsto & f_t(x,y)=y^2-x^3-t^2x^2 \end{array}.$$

As we have seen in Example 1.63, the family $\{(V(f_t),0)\}_{t\in D}$ is not Whitney equisingular. Since m_0 is constant on this family, then the top polar multiplicity m_1 must be non-constant on this family. Therefore, the vanishing Euler characteristic $\nu(V(f_t)),0)$ depends on $t\in D$.

We note that, for each $t\in D$, $V(f_t)$ has an isolated singularity at the origin. Thus, there exists, for each t, $\varepsilon_t>0$ such that B_{ε_t} is a Milnor ball, i.e., $V(f_t)\cap B_{\varepsilon_t}$ is smooth outside the origin and intersects transversally S_r, for all $r<B_{\varepsilon_t}$. Therefore,

$$\chi(V(f_t)\cap B_{\varepsilon_t})=1,$$

for all $t \in D$.

Example 2.1 raises a natural question: why is the vanishing Euler characteristic not constant in this family? The answer to this question is in the fact that, in this example, there is not an uniform $\varepsilon > 0$, such that $\chi(V(f_t) \cap B_\varepsilon) = 1$, for all $t \in D$. Therefore, finding conditions which guarantee this uniformity can be very helpful to study the Whitney equisingularity of families of varieties.

We proceed by extending the Newton non-degeneracy condition to determinantal singularities. In [18, Definition 1.16], Esterov introduces a non-degeneracy condition which comprehends determinantal singularities given by the maximal minors of a matrix $A(x) = (a_{i,j}(x))$, under the assumption that, all the entries of a column j has the same Newton polyhedron Δ_j. Here we still consider that, for each $j = 1, \dots, k$, the Newton polyhedron of $a_{i,j}(x)$ is Δ_j, for all $i = 1, \dots, n$. However, we do not consider determinantal varieties given only by the maximal minors of a matrix.

Definition 2.2. *Let $A = (a_{i,j}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a germ of a holomorphic matrix, which defines a determinantal singularity X_A^s of type $(n,k;s)$. We denote by $\Delta_j \subset \mathbb{R}_+^m$ the Newton polyhedron of $a_{i,j}$, for all $i = 1, \dots, n$. The matrix A is said to be* **Newton non-degenerate** *if, for each collection of faces $\Gamma_j \subset \Delta_j$ such that the sum $\sum_{j=1}^{k} \Gamma_j$ is a bounded face of the polyhedron $\sum_{j=1}^{k} \Delta_j$, the polynomial matrix $(a_{i,j}|_{\Gamma_j})$ defines a non-singular determinantal variety of type $(n,k;s)$ in $(\mathbb{C} \setminus 0)^m$. In this case, we say that X_A^s is a* **Newton non-degenerate determinantal singularity**.

Example 2.3. *Let X_A^2 be the determinantal variety of type $(2,3;2)$ defined by the matrix*

$$A = \begin{bmatrix} x+y+z+w & x+2y-z+3w & 3x+2y+z-w \\ 2x+3y+z+2w & x+y+3z-w & x-y+2z-2w \end{bmatrix}.$$

We shall verify if A is Newton non-degenerate.

Firstly, we observe that, since $\Delta_1 = \Delta_2 = \Delta_3$, then $\Delta_1 + \Delta_2 + \Delta_3 = 3 \cdot \Delta_1$. Moreover, let Γ_1, Γ_2 and Γ_3 be bounded faces of Δ_1, Δ_2 and Δ_3, respectively, since these polyhedra are the same, then $\Gamma_1 + \Gamma_2 + \Gamma_3$ is a bounded face of $3 \cdot \Delta_1$ only if $\Gamma_1 = \Gamma_2 = \Gamma_3$. Take for instance the line segment connecting the points $(3,0,0,0)$ and $(0,3,0,0)$, which we denote by σ. This line segment is a bounded face of $3 \cdot \Delta_1$ and its the set of points satisfying $(1-s) \cdot (3,0,0,0) + s \cdot (0,3,0,0)$, for $s \in [0,1]$. We can rewrite this expression as $3 \cdot [(1-s) \cdot (1,0,0,0) + s \cdot (0,1,0,0)]$, for $s \in [0,1]$. Therefore, $\sigma = \sigma_1 + \sigma_2 + \sigma_3$, where $\sigma_1 = \sigma_2 = \sigma_3$ is the line segment connecting the points $(1,0,0,0)$ and $(0,1,0,0)$. The following figures illustrate this example in the xy-plane.

Therefore, we must verify that the polynomial matrix $(a_{i,j}|_\Gamma)$ defines a non-singular determinantal variety of type $(2,3;2)$ in $(\mathbb{C} \setminus 0)^4$, where Γ is a bounded face of $\Delta_1 = \Delta_2 = \Delta_3$. We write $A = \{(1,0,0,0)\}$, $B = \{(0,1,0,0)\}$, $C = \{(0,0,1,0)\}$ and $D = \{(0,0,0,1)\}$, those are the 0-dimensional bounded faces of Δ_1. The 1-dimensional bounded faces of Δ_1 are the line segments $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, $\overline{CD}$. The 2-dimensional bounded faces of Δ_1 are the triangles $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, $\triangle BCD$. Lastly, the 3-dimensional bounded face of Δ_1 is the tetrahedron connecting the vertices A, B, C and

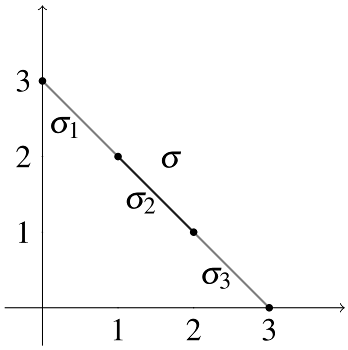


Figure 2.1: Face σ.

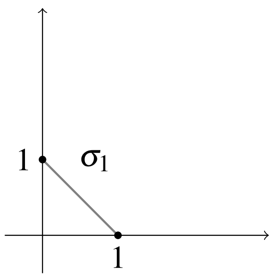


Figure 2.2: Face σ_1.

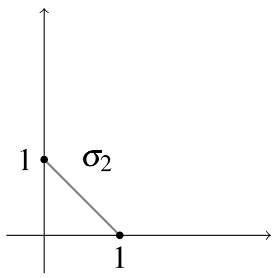


Figure 2.3: Face σ_2.

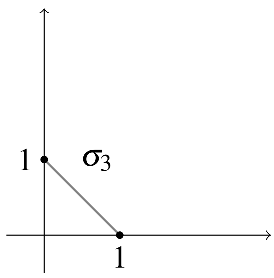


Figure 2.4: Face σ_3.

D, which we denote simply by Γ. We have

$$(a_{i,j}|_A) = \begin{bmatrix} x & x & 3x \\ 2x & x & x \end{bmatrix}$$

and $X^2_{(a_{i,j}|_A)} = \{(x,y,z,w) \in \mathbb{C}^4 : x^2 = 0\}$. Therefore, $X^2_{(a_{i,j}|_A)} \cap (\mathbb{C} \setminus 0)^4 = \emptyset$. The remaining 0-dimensional faces are analogous, then

$$X^2_{(a_{i,j}|_B)} \cap (\mathbb{C} \setminus 0)^4 = X^2_{(a_{i,j}|_C)} \cap (\mathbb{C} \setminus 0)^4 = X^2_{(a_{i,j}|_D)} \cap (\mathbb{C} \setminus 0)^4 = \emptyset.$$

Moreover,

$$(a_{i,j}|_{\overline{AB}}) = \begin{bmatrix} x+y & x+2y & 3x+2y \\ 2x+3y & x+y & x-y \end{bmatrix}$$

and $X^2_{(a_{i,j}|_{\overline{AB}})} = \{(x,y,z,w) \in \mathbb{C}^4 : -x^2-5xy-5y^2 = -5x^2-13xy-7y^2 = -2x^2-4xy-4y^2 = 0\}$. Therefore, $X^2_{(a_{i,j}|_{\overline{AB}})} \cap (\mathbb{C} \setminus 0)^4 = \emptyset$. Analogously,

$$X^2_{(a_{i,j}|_{\overline{AC}})} \cap (\mathbb{C}\setminus 0)^4 = X^2_{(a_{i,j}|_{\overline{AD}})} \cap (\mathbb{C}\setminus 0)^4 = X^2_{(a_{i,j}|_{\overline{BC}})} \cap (\mathbb{C}\setminus 0)^4 = X^2_{(a_{i,j}|_{\overline{BD}})} \cap (\mathbb{C}\setminus 0)^4 = X^2_{(a_{i,j}|_{\overline{CD}})} \cap (\mathbb{C}\setminus 0)^4 = \emptyset.$$

In addition,

$$(a_{i,j}|_{\triangle ABC}) = \begin{bmatrix} x+y+z & x+2y-z & 3x+2y+z \\ 2x+3y+z & x+y+3z & x-y+2z \end{bmatrix}$$

and $X^2_{(a_{i,j}|_{\triangle ABC})} = \{(x,y,z,w) \in \mathbb{C}^4 : -x^2-5xy+5xz-5y^2+5yz+4z^2 = -5x^2-13xy-2xz-7y^2-4yz+z^2 = -2x^2-4xy-9xz-4y^2-2yz-5z^2 = 0\}$. With the help of OSCAR [51], we obtain that $X^2_{(a_{i,j}|_{\triangle ABC})} \cap (\mathbb{C} \setminus 0)^4$ is a 2-dimensional non-singular variety. Analogously $X^2_{(a_{i,j}|_{\triangle ABD})} \cap (\mathbb{C} \setminus 0)^4$,

$X^2_{(a_{i,j}|_{\triangle ACD})} \cap (\mathbb{C}\setminus 0)^4$ and $X^2_{(a_{i,j}|_{\triangle BCD})} \cap (\mathbb{C}\setminus 0)^4$ are also 2-dimensional non-singular varieties. Lastly, $(a_{i,j}|_\Gamma) = A$ and we use OSCAR [51] to show that $X_A \cap (\mathbb{C}\setminus 0)^4$ is a 2-dimensional non-singular variety. Therefore, the matrix A is Newton non-degenerate.

The smoothness along coordinate spaces plays an important role in the study of Whitney equisingularity. This concept were introduced by Eyral and Oka [21, 49] for hypersurfaces and complete intersection singularities and we present it in the next paragraphs.

For any subset $I \subset \{1,\dots,m\}$, we define

$$C^I = \{(x_1,\dots,m) \in \mathbb{C}^m : x_i = 0,\ i \notin I\},$$
$$C^{*I} = \{(x_1,\dots,m) \in \mathbb{C}^m : x_i = 0 \text{ if and only if } i \notin I\}.$$

In particular, $\mathbb{C}^{\emptyset} = \mathbb{C}^{*\emptyset} = \{0\}$ and $\mathbb{C}^{*\{1,\dots,m\}} = \mathbb{C}^{*m}$, where $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$.

Definition 2.4. *Let $f : (\mathbb{C}^m,0) \to (\mathbb{C},0)$ be a germ of an analytic function and $I \subset \{1,\dots,m\}$. We say that $\mathbb{C}^I$ is an* **admissible coordinate subspace for** *f if $f^I := f|_{\mathbb{C}^I\cap U}$ is not constantly zero, where U is a neighbourhood of the origin in $\mathbb{C}^m$.*

Example 2.5. *Let $f : (\mathbb{C}^3,0) \to (\mathbb{C},0)$ be defined by $f(x,y,t) = y^2 - x^3 - x^2t^2$. $\mathbb{C}^{\{3\}}$ is not admissible while $\mathbb{C}^{\{1,2\}}$ is admissible for f.*

If the function $f : (\mathbb{C}^m,0) \to (\mathbb{C},0)$ is convenient, then $\mathbb{C}^I$ is admissible for f, for all subsets $I \subset \{1,\dots,m\}$.

Oka, in [49, Chapter III, Lemma 2.2] and [50, Theorem 19], proved that a hypersurface defined by a Newton non-degenerate function is smooth along its admissible coordinate spaces.

Theorem 2.6. *Let $f : (\mathbb{C}^m,0) \to (\mathbb{C},0)$ be a germ of a Newton non-degenerate function. Then there exists a positive number R such that for every admissible coordinate space $\mathbb{C}^I$, $V(f^I) \cap B_R$ is smooth and the sphere S_r with $0 < r \le R$ intersects $V(f^I) \cap B_R$ transversally.*

In addition, Eyral and Oka [21, Proposition 3.1], introduced an uniform version of the above theorem.

Proposition 2.7. *Suppose that for all t sufficiently small, the following two conditions are satisfied:*

(i) *the Newton polyhedron Δ_t of f_t at 0 is independent of t;*

(ii) *the polynomial function f_t is Newton non-degenerate.*

Then there exists a positive number $R > 0$ such that for any admissible coordinate space of f_0 and any t sufficiently small, the set $V(f_t) \cap \mathbb{C}^{*I} \cap B_R$ is non-singular and intersects transversely with S_r for any $r < R$, where B_R (respectively, S_r) is the open ball (respectively, the sphere) with center at the origin $0 \in \mathbb{C}^m$ and radius R (respectively, r).

Our next goal is to extend the previous concepts and results to determinantal singularities.

Definition 2.8. *Let $A = (a_{i,j}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a germ of a holomorphic map, we say that $\mathbb{C}^I$ is an* **admissible coordinate space for** *A if $\mathbb{C}^I$ is admissible for each $a_{i,j}$, $i = 1, \ldots, n$ and $j = 1, \ldots, k$.*

Along this work, we will use the following notation:

(i) $A^I = (a^I_{i,j})$, where $a^I_{i,j} = a_{i,j}|_{\mathbb{C}^I \cap U}$;

(ii) $A^{*I} = (a^{*I}_{i,j})$, where $a^{*I}_{i,j} = a_{i,j}|_{\mathbb{C}^{*I} \cap U}$.

In the following Lemma we show that a Newton non-degenerate matrix is also Newton non-degenerate on the admissible coordinate spaces $\mathbb{C}^I \subset \mathbb{C}$.

Lemma 2.9. *Let $A : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a germ of a holomorphic matrix and $f_1, \ldots, f_l : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be germs of functions. Let $\mathbb{C}^I$ be an admissible coordinate subspace for A. If the matrix A is Newton non-degenerate, then A^I is Newton non-degenerate.*

Proof. For each $j = 1, \ldots, k$ let Γ_j be the faces of $\Delta_{a^I_{i,j}}$ such that $\sum_{j=1}^k \Gamma_j$ is a bounded face of $\sum_{j=1}^k \Delta_{a^I_{i,j}}$. Since $\Delta_{a^I_{i,j}} = \Delta_{a_{i,j}} \cap \mathbb{R}^I$, Γ_j is also a face of $\Delta_{a_{i,j}}$ such that $\sum_{j=1}^k \Gamma_j$ is a bounded face of $\sum \Delta_{a_{i,j}}$ for each $j = 1, \ldots, k$. Since the matrix A is Newton non-degenerate, then $(a^I_{i,j}|_{\Gamma_j})$ defines a non-singular determinantal variety in $(\mathbb{C} \setminus 0)^m$. Hence, A^I is Newton non-degenerate. □

We finally have all the tools necessaries to state the version of Proposition 2.7 to determinantal singularities.

Proposition 2.10. *Suppose that for all t sufficiently small, the following two conditions are satisfied:*

(i) *the matrix $A_t = ((a_{i,j})_t)$ is Newton non-degenerate;*

(ii) *the Newton polyhedron Δ^t_j of $(a_{i,j})_t$ is independent of t, for all $j = 1, \ldots, k$.*

Then there exists a positive number $R > 0$ such that for any admissible coordinate subspace $\mathbb{C}^I$ of A_0 and for any t sufficiently small, the set $X^s_{A_t} \cap \mathbb{C}^{*I} \cap B_R$ is non-singular and intersects transversely with S_r for any $r < R$, where B_R (respectively, S_r) is the open ball (respectively, the sphere) with center at the origin $0 \in \mathbb{C}^m$ and radius R (respectively, r).

Proof. The proof is given in Section 2. □

Corollary 2.11. *In addition to the conditions of Proposition 2.10, if the functions $(a_{i,j})_t$ are convenient (see Definition 1.30), then there exists a positive number $R > 0$ such that for any t sufficiently small, the set $X^s_{A_t} \cap B_R$ is smooth outside the origin and intersects transversely with S_r for any $r < R$, where B_R (respectively, S_r) is the open ball (respectively, the sphere) with center at the origin $0 \in \mathbb{C}^m$ and radius R (respectively, r).*

Proof. Since the functions $(a_{i,j})_t$ are convenient, then, for each $t \in D$, the coordinate subspace $\mathbb{C}^I$ is admissible for $A_t = (a_{i,j})_t$ for all subsets $I \subset \{1,\dots,k\}$, $i = 1,\dots,n,\ j = 1,\dots,k$ and the result follows. □

Corollary 2.12. *If for all t sufficiently small, the following two conditions are satisfied:*

(i) *the matrix* $A_t = ((a_{i,j})_t)$*, is Newton non-degenerate;*

(ii) *the Newton polyhedron* Δ_j^t *of* $(a_{i,j})_t$ *is convenient and independent of t, for all* $j = 1,\dots,k$.

Then the vanishing Euler characteristic of $X^s_{A_t}$ *is independent of t, i.e.,*

$$\nu(X^s_{A_t},0) = \nu(X^s_{A_0},0).$$

Proof. It follows directly from Corollary 1.83 and Corrolary 2.11. □

2 Proof of Proposition 2.10

In order to prove Proposition 2.10, we follow the steps of the proof of Proposition 2.7, making the necessary adjustments to determinantal singularities.

As Δ_j^t is independent of t, the set of admissible systems is also independent of t and, since there are only finitely many subsets $I \subset \{1,\dots,m\}$, it is sufficient to show the result for a fixed $I = \{1,\dots,r\}$, $r \leq m$.

We divide this proof in four main steps. In the first two steps we show that $X^s_{A_t} \cap \mathbb{C}^{*I} \cap B_R$ is smooth and in the last two steps we prove that this variety is transversal to S_r for $r < R$. Both steps are proved by contradiction. After each step we present examples, where we make the computations with a 2×3 matrix.

Firstly, consider the determinantal deformation $\mathscr{A} : (\mathbb{C}^m \times \mathbb{C},0) \to (M_{n,k},0)$ of A. Suppose that for all $R > 0$ the intersection $X^s_{A_t} \cap \mathbb{C}^{*I} \cap B_R$ has a singular point. Consider the sequence $\{(t_R, z_R)\}$ of points in $X^s_{\mathscr{A}} \cap (D \times \mathbb{C}^{*I})$ converging to $(0,0)$, where z_R is a singularity of $X^s_{A_t} \cap \mathbb{C}^{*I} \cap B_R$. Then $(0,0)$ is in the closure of the set

$$W = \{(t,z) \in D \times \mathbb{C}^{*I} : z \in X^s_{A_t} \text{ and } z \text{ is a singular point of } X^s_{A_t}\}.$$

Then, by the Curve Selection Lemma [44], there exists an analytic curve

$$\begin{array}{rccc} p: & [0,\varepsilon] & \to & W \\ & q & \mapsto & (t(q), z(q)) \end{array}$$

such that $p(q) = (t(q), z_1(q), \dots, z_r(q), 0, \dots, 0)$, for all $q \neq 0$, and $p(0) = (0,0)$. For $1 \leq i \leq r$, consider the Taylor expansions

$$t(q) = t_0 q^{\omega} + \cdots, \qquad z_i(q) = a_i q^{w_i} + \cdots,$$

where $t_0, a_i \neq 0$ and $\omega, w_i > 0$, for $i = 1, \dots, r$. Here, the three centered dots stand for the higher order terms. Choose $a = (a_1, \dots, a_r, 0, \dots, 0) \in \mathbb{C}^{*I}$ and $w = (w_1, \dots, w_r, 0, \dots, 0) \in \mathbb{N}^m \setminus \{0\}$. Consider the face Γ_j of $(\Delta_j^{t(q)})^I = (\Delta_j^0)^I$ defined as the set where the map

$$\begin{array}{rccc} W_j^I : & (\Delta_j^0)^I & \to & \mathbb{R}_+ \\ & x := (x_1, \dots, x_r, 0, \dots, 0) & \mapsto & \sum_{i=1}^{r} x_i w_i \end{array}$$

takes its minimal value d_j and such that $\sum_{j=1}^{k} \Gamma_j$ is a bounded face of $\sum_{j=1}^{k} \Delta_j$.

Lemma 2.13. *The point $a \in \mathbb{C}^{*I}$ belongs to the variety $X^s_{A_0^{*I}|_\Gamma} = ((a_{i,j}^{*I})_0|_{\Gamma_j})$.*

Proof. Firstly, consider the $n \times k$ matrix

$$(A_{t(q)}^{*I})|_\Gamma(z(q)) = ((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(z(q)).$$

Now, note that

$$(a_{i,j})_t(z) = ((a_{i,j})_t)|_{\Gamma_j}(z) + \sum_{\alpha \notin \Gamma_j} c_\alpha z^\alpha. \tag{2.1}$$

Consider a monomial component

$$(c_\alpha z^\alpha)|_{\Gamma_j} = c_\alpha z_1^{\alpha_1} \dots z_r^{\alpha_r} \tag{2.2}$$

of the face function $a_{i,j}|_{\Gamma_j}$. Then over the curve p we have

$$\begin{aligned} (c_\alpha z(q)^\alpha)|_{\Gamma_j} &= c_\alpha (a_1 q^{w_1} + \cdots)^{\alpha_1} \dots (a_r q^{w_r} + \cdots)^{\alpha_r} \\ &= c_\alpha a_1^{\alpha_1} \dots a_r^{\alpha_r} q^{d_j} + \cdots . \end{aligned}$$

Therefore,

$$(a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(z(q)) = (a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a) q^{d_j} + \cdots . \tag{2.3}$$

Consider the set

$$\mathscr{C} = \{(\mathscr{I}, \mathscr{J}) : \mathscr{I} \subset \{1, \dots, n\},\ \mathscr{J} \subset \{1, \dots, k\} \text{ and } |\mathscr{I}| = |\mathscr{J}| = s\}.$$

Since the Newton polyhedron of $a_{i,j}^{*I}$ is Δ_j^I for all $i = 1, \dots, n$ and $z(q)$ belongs to the determinantal variety $X^s_{A_{t(q)}^{*I}|_\Gamma}$, which is defined by the s size minors of $A_{t(q)}^{*I}|_\Gamma$, for each $(\mathscr{I}, \mathscr{J}) \in \mathscr{C}$, we have the zero polynomial

$$\det(((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a) q^{d_j} + \cdots)_{i \in \mathscr{I},\, j \in \mathscr{J}}) = \det(((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a))_{i \in \mathscr{I},\, j \in \mathscr{J}}) q^{\sum_{j \in \mathscr{J}} d_j} + \cdots = 0.$$

This implies that all the coefficients of this polynomial are equal to zero, in particular

$$\det(((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a))_{i \in \mathscr{I},\, j \in \mathscr{J}}) = 0.$$

Taking $q \to 0$, we have

$$\det(((a_{i,j}^{*I})_0|_{\Gamma_j}(a))_{i \in \mathscr{I},\, j \in \mathscr{J}}) = 0,$$

for all $(\mathscr{I}, \mathscr{J}) \in \mathscr{C}$. Therefore, the point a belongs to the determinantal variety $X^s_{A_0^{*I}|_\Gamma}$. □

We exemplify the previous computations with a determinantal singularity of type $(2,3;2)$.

Example 2.14. *Consider the determinantal singularity X_A^2, defined by the matrix germ $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{2,3},0)$. Then over the curve p, we have*

$$A^{*I}_{t(q)}|_{\Gamma}(z(q))=\begin{bmatrix}(a^{*I}_{11})_{t(q)}|_{\Gamma_1}(z(q)) & (a^{*I}_{12})_{t(q)}|_{\Gamma_2}(z(q)) & (a^{*I}_{13})_{t(q)}|_{\Gamma_3}(z(q))\\ (a^{*I}_{21})_{t(q)}|_{\Gamma_1}(z(q)) & (a^{*I}_{22})_{t(q)}|_{\Gamma_2}(z(q)) & (a^{*I}_{23})_{t(q)}|_{\Gamma_3}(z(q))\end{bmatrix}.$$

By Eq. (2.3), *we have*

$$A^{*I}_{t(q)}|_{\Gamma}(z(q))=\begin{bmatrix}(a^{*I}_{11})_{t(q)}|_{\Gamma_1}(a)q^{d_1}+\cdots & (a^{*I}_{12})_{t(q)}|_{\Gamma_2}(a)q^{d_2}+\cdots & (a^{*I}_{13})_{t(q)}|_{\Gamma_3}(a)q^{d_3}+\cdots\\ (a^{*I}_{21})_{t(q)}|_{\Gamma_1}(a)q^{d_1}+\cdots & (a^{*I}_{22})_{t(q)}|_{\Gamma_2}(a)q^{d_2}+\cdots & (a^{*I}_{23})_{t(q)}|_{\Gamma_3}(a)q^{d_3}+\cdots\end{bmatrix}.$$

Now, we take the 2 *size minors given by the two first columns of $A^{*I}_{t(q)}|_{\Gamma}(z(q))$. Thus, we have*

$$\begin{aligned}
&\det(((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a)q^{d_j}+\cdots)_{i\in\{1,2\},\, j\in\{1,2\}})\\
&= ((a^{*I}_{11})_{t(q)}|_{\Gamma_1}(a)q^{d_1}+\cdots)\cdot((a^{*I}_{22})_{t(q)}|_{\Gamma_2}(a)q^{d_2}+\cdots)\\
&\qquad -((a^{*I}_{12})_{t(q)}|_{\Gamma_2}(a)q^{d_2}+\cdots)\cdot((a^{*I}_{21})_{t(q)}|_{\Gamma_1}(a)q^{d_1}+\cdots)\\
&= [((a^{*I}_{11})_{t(q)}|_{\Gamma_1}(a))\cdot((a^{*I}_{22})_{t(q)}|_{\Gamma_2}(a))-((a^{*I}_{12})_{t(q)}|_{\Gamma_2}(a))\cdot((a^{*I}_{21})_{t(q)}|_{\Gamma_1}(a)]q^{d_1+d_2}+\cdots\\
&=\det(((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})q^{d_1+d_2}+\cdots=0.
\end{aligned}$$

Therefore,

$$\det(((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})=0.$$

Taking $q\to 0$, we obtain

$$\det(((a^{*I}_{i,j})_0|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})=0.$$

Repeating the process to the other 2 *size minors of $A^{*I}_{t(q)}|_{\Gamma}(z(q))$*

$$\begin{cases}\det(((a^{*I}_{i,j})_0|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})=0\\ \det(((a^{*I}_{i,j})_0|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}})=0.\\ \det(((a^{*I}_{i,j})_0|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{2,3\}})=0\end{cases}$$

Therefore,

$$a\in X^2_{A_0^{*I}|_{\Gamma}}.$$

Lemma 2.15. *The point a is a singularity of $X^s_{A_0^{*I}|_{\Gamma}}$.*

Proof. We start by taking the partial derivative of Eq. (2.2), which gives us the following equation

$$\frac{\partial}{\partial z_l}(c_\alpha z^\alpha)=\alpha_l c_\alpha z_1^{\alpha_1}\ldots z_l^{\alpha_l-1}\ldots z_r^{\alpha_r}.$$

Then, over the curve p

$$\begin{aligned}
\frac{\partial}{\partial z_l}(c_\alpha z(q)^\alpha)|_{\Gamma_j}&=\alpha_l c_\alpha(a_1q^{w_1}+\cdots)^{\alpha_1}\cdots(a_lq^{w_l}+\cdots)^{\alpha_l-1}\cdots(a_rq^{w_r}+\cdots)^{\alpha_r}\\
&=\alpha_l a_1^{\alpha_1}\cdots a_l^{\alpha_l-1}\cdots a_r^{\alpha_r}q^{d_j-w_l}+\cdots\\
&=\frac{\partial}{\partial z_l}(c_\alpha z(q)^\alpha)|_{\Gamma_j}(a)\cdot q^{d_j-w_l}+\cdots.
\end{aligned}$$

Thus, the following equation holds

$$\frac{\partial}{\partial z_l}((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(z(q)) = \frac{\partial}{\partial z_l}((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a)) \cdot q^{d_j - w_l} + \cdots . \tag{2.4}$$

It follows from Eq. (2.1) that

$$((a_{i,j}^{*I})_{t(q)}(z(q)) = ((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(z(q)) + \sum_{\alpha \notin \Gamma_j} c_\alpha z(q)^\alpha$$

and

$$\frac{\partial}{\partial z_l}((a_{i,j}^{*I})_{t(q)}(z(q)) = \frac{\partial}{\partial z_l}((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(z(q)) + \frac{\partial}{\partial z_l}(\sum_{\alpha \notin \Gamma_j} c_\alpha z(q)^\alpha).$$

Furthermore, by Eq. (2.3), Eq. (2.4) and the derivative product law, we have

$$\frac{\partial}{\partial z_l}(\det((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(z(q))_{i\in\mathscr{I},\, j\in\mathscr{J}})) = \frac{\partial}{\partial z_l}(\det((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a))_{i\in\mathscr{I},\, j\in\mathscr{J}}) \cdot q^{(\Sigma_{j\in\mathscr{J}} d_j) - w_l} + \cdots .$$

Then the jacobian matrix of the map given by the s size minors of the matrix

$$A_{t(q)}^{*I}|_\Gamma(z(q)) = ((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a)q^{d_j} + \cdots)_{i\in\mathscr{I},\, j\in\mathscr{J}}$$

is the $\binom{k}{s}\binom{n}{s} \times m$ matrix

$$\left(\frac{\partial}{\partial z_l}(\det((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a))_{i\in\mathscr{I},\, j\in\mathscr{J}}) \cdot q^{(\Sigma_{j\in\mathscr{J}} d_j) - w_l} + \cdots\right)_{(\mathscr{I},\mathscr{J})\in\mathscr{C}, l\in\{1,\ldots,m\}}.$$

Since $z(q)$ is a singularity of the determinantal variety $X^s_{A^{*I}_{t(q)}|_\Gamma}$, this matrix has rank less than $(n-s+1)(k-s+1)$. This means that all the $(n-s+1)(k-s+1)$ size minors of this jacobian matrix are equal to zero.

Note that, for each $\theta = 1, \ldots, (n-s+1)(k-s+1)$, the zero polynomial

$$\begin{aligned}\det\Big(\frac{\partial}{\partial z_{l_\theta}}\big(\det(((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a))_{i\in\mathscr{I}_\theta,\, j\in\mathscr{J}_\theta}) \cdot q^{(\Sigma_{j\in\mathscr{J}_\theta} d_j) - w_{l_\theta}} + \cdots)_{(\mathscr{I}_\theta,\mathscr{J}_\theta)\in\mathscr{C}}\big)_\theta\Big) \\ = \det\left(\frac{\partial}{\partial z_{l_\theta}}\left(\det(((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a))_{i\in\mathscr{I}_\theta,\, j\in\mathscr{J}_\theta})_{\mathscr{I}_\theta,\mathscr{J}_\theta\in\mathscr{C}}\right)_\theta\right) \\ \times q^{(\Sigma_{\theta=1}^{(n-s+1)(k-s+1)} \Sigma_{j\in\mathscr{J}_\theta} d_j) - \Sigma_{l_\theta=1}^{(n-s+1)(k-s+1)} w_{l_\theta}} + \cdots = 0,\end{aligned}$$

implies that everyone of its coefficients are equal to zero, in particular

$$\det\left(\frac{\partial}{\partial z_{l_\theta}}(\det(((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a))_{i\in\mathscr{I}_\theta,\, j\in\mathscr{J}_\theta})_{\mathscr{I}_\theta,\mathscr{J}_\theta\in\mathscr{C}})_\theta\right) = 0.$$

Therefore, the matrix

$$\left(\frac{\partial}{\partial z_l}(\det((a_{i,j}^{*I})_{t(q)}|_{\Gamma_j}(a))_{i\in\mathscr{I},\, j\in\mathscr{J}})\right)_{(\mathscr{I},\mathscr{J})\in\mathscr{C}, l\in\{1,\ldots,m\}}$$

has also rank less than $(n-s+1)(k-s+1)$. Taking again $q \to 0$, the following matrix has also rank less than $(n-s+1)(k-s+1)$

$$\left(\frac{\partial}{\partial z_l} (\det((a_{i,j}^{*I})_0 |\Gamma_j(a))_{i\in\mathscr{I},\, j\in\mathscr{J}}) \right)_{(\mathscr{I},\mathscr{J})\in\mathscr{C}, l\in\{1,\dots,m\}}$$

Since this matrix is the jacobian matrix of $A_0^{*I}|_\Gamma(a)$, the point $a \in \mathbb{C}^{*I}$ is a singularity of the determinantal variety $X^s_{A_0^{*I}|_\Gamma}$. □

By Lemma 2.13 and Lemma 2.15, the matrix A_0^{*I} is not Newton non-degenerate, which contradicts Lemma 2.9.

Now we continue with the matrix from Example 2.14.

Example 2.16. *Consider the 2 size minor given by the two first columns of $A_{t(q)}^{*I}|_\Gamma(z(q))$. We have the following partial derivative*

$$\frac{\partial}{\partial z_l}(\det((a_{i,j}^{*I})_{t(q)}|\Gamma_j(z(q))_{i\in\{1,2\},\, j\in\{1,2\}})) = \frac{\partial}{\partial z_l}(\det((a_{i,j}^{*I})_{t(q)}|\Gamma_j(a))_{i\in\{1,2\},\, j\in\{1,2\}}) \cdot q^{d_1+d_2-w_l} + \cdots .$$

Of course, we can do the same for $j \in \{1,3\}$ and $j \in \{2,3\}$. Therefore, the jacobian matrix of $A_{t(q)}^{*I}|_\Gamma(z(q))$ is the following $3\times m$ matrix

$$\begin{bmatrix} \frac{\partial}{\partial z_1}(\det((a_{i,j}^{*I})_{t(q)}|\Gamma_j(a))_{i\in\{1,2\},\, j\in\{1,2\}}) \cdot q^{d_1+d_2-w_1} + \cdots & \cdots & \frac{\partial}{\partial z_m}(\det((a_{i,j}^{*I})_{t(q)}|\Gamma_j(a))_{i\in\{1,2\},\, j\in\{1,2\}}) \cdot q^{d_1+d_2-w_m} + \cdots \\ \frac{\partial}{\partial z_1}(\det((a_{i,j}^{*I})_{t(q)}|\Gamma_j(a))_{i\in\{1,2\},\, j\in\{1,3\}}) \cdot q^{d_1+d_3-w_1} + \cdots & \cdots & \frac{\partial}{\partial z_m}(\det((a_{i,j}^{*I})_{t(q)}|\Gamma_j(a))_{i\in\{1,2\},\, j\in\{1,3\}}) \cdot q^{d_1+d_3-w_m} + \cdots \\ \frac{\partial}{\partial z_1}(\det((a_{i,j}^{*I})_{t(q)}|\Gamma_j(a))_{i\in\{1,2\},\, j\in\{2,3\}}) \cdot q^{d_2+d_3-w_1} + \cdots & \cdots & \frac{\partial}{\partial z_m}(\det((a_{i,j}^{*I})_{t(q)}|\Gamma_j(a))_{i\in\{1,2\},\, j\in\{2,3\}}) \cdot q^{d_2+d_3-w_m} + \cdots \end{bmatrix}.$$

Since, the variety $X^2_{A_{t(q)}^{*I}|_\Gamma}$ has codimension 2, this jacobian matrix has rank equal to 2. This means that its size 2 minors vanish. We compute now, the 2 size minor defined by the two first lines and two first *columns of* $A^{*I}_{t(q)}|_{\Gamma}(z(q))$. *Therefore, we have*

$$\left(\frac{\partial}{\partial z_1}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})\cdot q^{d_1+d_2-w_1}+\cdots\right)$$
$$\times\left(\frac{\partial}{\partial z_2}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}})\cdot q^{d_1+d_3-w_2}+\cdots\right)$$
$$-\left(\frac{\partial}{\partial z_1}(\det(a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}})\cdot q^{d_1+d_3-w_1}+\cdots\right)$$
$$\times\left(\frac{\partial}{\partial z_2}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})\cdot q^{d_1+d_2-w_2}+\cdots\right)$$
$$=\left(\frac{\partial}{\partial z_1}(\det(((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})\right.$$
$$\left.\times\frac{\partial}{\partial z_2}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}})\right)\cdot q^{(d_1+d_2)+(d_1+d_3)-w_1-w_2}+\cdots$$
$$-\left(\frac{\partial}{\partial z_1}(\det(a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}})\right.$$
$$\left.\times\frac{\partial}{\partial z_2}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})\right)\cdot q^{(d_1+d_2)+(d_1+d_3)-w_1-w_2}+\cdots$$
$$=\left(\frac{\partial}{\partial z_1}(\det(((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})\right.$$
$$\times\frac{\partial}{\partial z_2}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}})$$
$$-(\frac{\partial}{\partial z_1}(\det(a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}})$$
$$\left.\times\frac{\partial}{\partial z_2}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})\right)\cdot q^{(d_1+d_2)+(d_1+d_3)-w_1-w_2}+\cdots$$
$$=0$$

Therefore,

$$(\frac{\partial}{\partial z_1}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})\cdot\frac{\partial}{\partial z_2}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}})$$
$$-(\frac{\partial}{\partial z_1}(\det(a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}})\cdot\frac{\partial}{\partial z_2}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})$$
$$=0.$$

Repeating the same process for all 2 *size minors of this jacobian matrix, we obtain that the following matrix has also rank less than* 2.

$$\begin{bmatrix}\frac{\partial}{\partial z_1}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}}) & \cdots & \frac{\partial}{\partial z_m}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,2\}})\\ \frac{\partial}{\partial z_1}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}}) & \cdots & \frac{\partial}{\partial z_m}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{1,3\}})\\ \frac{\partial}{\partial z_1}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{2,3\}}) & \cdots & \frac{\partial}{\partial z_m}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\{1,2\},\, j\in\{2,3\}})\end{bmatrix}$$

Since the above matrix is the jacobian matrix of the variety $X^2_{A^{*I}_{t(q)}|_{\Gamma}}$ *at* a*, the point* a *is a singularity of* $X^2_{A^{*I}_{t(q)}|_{\Gamma}}$.

The objective of the following steps is to prove the transversality of $X^s_{A_t} \cap \mathbb{C}^{*I} \cap B_R$ with the sphere S_r, for $r < R$. Suppose that there exists a sequence $\{(t_R, z_R)\}$ of points in $X^s_{\mathscr{A}} \cap (D \times \mathbb{C}^{*I})$ converging to $(0,0)$ and such that $X^s_{A_{t_R}} \cap \mathbb{C}^{*I}$ does not intersect the sphere $S_{||z_R||}$ transversally at z_R. Thus

$$(T_{z_R} S_{||z_R||})^{\perp} \subseteq (T_{z_R}(X^s_{A_{t_R}} \cap \mathbb{C}^{*I}))^{\perp}.$$

The orthogonal space $(T_{z_R}(X^s_{A_{t_R}} \cap \mathbb{C}^{*I}))^{\perp}$ is generated by the set

$$G = \{grad(\det((a^{*I}_{i,j})_{t_R}(z_R))_{i\in\mathscr{I}, j\in\mathscr{J}}) : (\mathscr{I}, \mathscr{J}) \in \mathscr{C}\}.$$

where the gradient of a function f is $grad(f(z)) = \left(\overline{\frac{\partial f}{\partial z_1}(z)}, \ldots, \overline{\frac{\partial f}{\partial z_m}(z)}\right)$ and $\overline{\frac{\partial f}{\partial z_i}(z)}$ denotes the complex conjugation of $\frac{\partial f}{\partial z_i}(z)$, $i = 1, \ldots, m$. For simplification purposes, we use the notation $\overline{\frac{\partial f}{\partial z_i}(z)} = \frac{\overline{\partial} f}{\partial z_i}(z)$, $i = 1, \ldots, m$.

For determinantal singularities in general, the set G is not linearly independent, then G is not a basis for $(T_{z_R}(X^s_{A_{t_R}} \cap \mathbb{C}^{*I}))^{\perp}$. Our first task is finding, if necessary, a subsequence $(t_{R_\gamma}, z_{R_\gamma})$ of (t_R, z_R), such that

$$G_\gamma = \{grad(\det((a^{*I}_{i,j})_{t_{R_\gamma}}(z_{R_\gamma}))_{i\in\mathscr{I}, j\in\mathscr{J}}) : (\mathscr{I}, \mathscr{J}) \in \mathscr{C}_\gamma\}$$

is a basis for $(T_{z_{R_\gamma}}(X^s_{A_{t_{R_\gamma}}} \cap \mathbb{C}^{*I}))^{\perp}$.

Firstly, we divide $\mathbb{N}$ into subsets $\mathscr{P}_\gamma$ such that

$$G_\gamma = \{grad(\det((a^{*I}_{i,j})_{t_{R_\gamma}}(z_{R_\gamma}))_{i\in\mathscr{I}, j\in\mathscr{J}}) : (\mathscr{I}, \mathscr{J}) \in \mathscr{C}_\gamma\}$$

is a basis for $(T_{z_{R_\gamma}}(X^s_{A_{t_{R_\gamma}}} \cap \mathbb{C}^{*I}))^{\perp}$, where $\mathscr{C}_\gamma \subset \mathscr{C}$ and $1/R_\gamma \in \mathscr{P}_\gamma$. Since there are only finitely many subsets $\mathscr{P}_\gamma$, there exists γ_0 such that $\mathscr{P}_{\gamma_0}$ is not finite. Therefore, $\{(t_{R_{\gamma_0}}, z_{R_{\gamma_0}})\}_{\frac{1}{R_{\gamma_0}} \in \mathscr{P}_{\gamma_0}}$ is the desired subsequence. To simplify, we denote this subsequence by $(t_{R_\gamma}, z_{R_\gamma})$. Therefore, for each γ, we can write the vector $z_{R_\gamma} \in (T_{z_{R_\gamma}} S_{||z_{R_\gamma}||})^{\perp} \subseteq (T_{z_R}(X^s_{A_{t_R}} \cap \mathbb{C}^{*I}))^{\perp}$ uniquely as a linear combination of vectors from G_γ, *i.e.,* there exist $\lambda_{(\mathscr{I},\mathscr{J})}$ satisfying

$$z_{R_\gamma} = \sum_{(\mathscr{I},\mathscr{J})\in\mathscr{C}_\gamma} \lambda_{(\mathscr{I},\mathscr{J})} \cdot grad(\det((a^{*I}_{i,j})_{t_{R_\gamma}}(z_{R_\gamma}))_{i\in\mathscr{I}, j\in\mathscr{J}}).$$

We observe that some of the coefficients $\lambda_{(\mathscr{I},\mathscr{J})}$ in the above linear combination may be zero. However, by the same arguments, if necessary, we can consider a subsequence of $(t_{R_{\gamma\alpha}}, z_{R_{\gamma\alpha}})$ of $(t_{R_\gamma}, z_{R_\gamma})$ such that

$$z_{R_{\gamma\alpha}} = \sum_{(\mathscr{I},\mathscr{J})\in\mathscr{C}_{\gamma\alpha}} \lambda_{(\mathscr{I},\mathscr{J})} \cdot grad(\det((a^{*I}_{i,j})_{t_{R_{\gamma\alpha}}}(z_{R_{\gamma\alpha}}))_{i\in\mathscr{I}, j\in\mathscr{J}}),$$

with $\lambda_{\mathscr{I},\mathscr{J}} \neq 0$, for all $(\mathscr{I}, \mathscr{J}) \in \mathscr{C}_{\gamma\alpha}$ and $1/R_{\gamma\alpha} \in \mathscr{P}_{\gamma\alpha}$. We will denote this subsequence simply by $\{(t_{R_\alpha}, z_{R_\alpha})\}$ and we denote by $\mathscr{C}_\alpha$ the set of $(\mathscr{I}, \mathscr{J}) \in \mathscr{C}$ such that

$$\{grad(\det((a^{*I}_{i,j})_{t_{R_\alpha}}(z_{R_\alpha}))_{i\in\mathscr{I}, j\in\mathscr{J}})\}$$

is linearly independent and $\lambda_{(\mathscr{I},\mathscr{J})} \neq 0$, for all $(\mathscr{I},\mathscr{J}) \in \mathscr{C}_\alpha$, and $1/\alpha \in \mathscr{P}_\alpha$.

Since the subsequence $(t_{R_\alpha}, z_{R_\alpha})$ also converges to $(0,0)$, the point $(0,0)$ belongs to the closure of the set consisting of points $(t,z) \in D \times \mathbb{C}^{*I}$ such that

$$z \in X^s_{A_t} \text{ and } z = \sum_{(\mathscr{I},\mathscr{J}) \in \mathscr{C}_\alpha} \lambda_{(\mathscr{I},\mathscr{J})} \cdot grad(\det((a^{*I}_{i,j})_t(z))_{i \in \mathscr{I}, j \in \mathscr{J}}).$$

By the Curve Selection Lemma [44], there exists a real analytic curve

$$(t(q), z(q)) = (t(q), z_1(q), \ldots, z_r(q), 0, \ldots, 0)$$

and Laurent series $\lambda_{(\mathscr{I},\mathscr{J})}$, $(\mathscr{I},\mathscr{J}) \in \mathscr{C}_\alpha$, such that

a) $(t(0), z(0)) = (0,0)$;

b) $(t(q), z(q)) \in D \times \mathbb{C}^{*I}$, for $q \neq 0$;

c) $z(q) \in X^s_{A_{t(q)}}$;

d) $z(q) = \sum_{(\mathscr{I},\mathscr{J}) \in \mathscr{C}_\alpha} \lambda_{(\mathscr{I},\mathscr{J})} \cdot grad(\det(((a^{*I}_{i,j})_{t(q)}(z(q)))_{i \in \mathscr{I}, j \in \mathscr{J}})).$

Consider the Taylor expansions

$$t(q) = t_0 q^{\omega} + \cdots, \qquad z_i(q) = a_i q^{w_i} + \cdots,$$

where $t_0, a_i \neq 0$ and $\nu, w_i > 0$, for $i = 1, \ldots, r$. Consider also the Laurent expansions

$$\lambda_{(\mathscr{I},\mathscr{J})}(q) = \beta_{(\mathscr{I},\mathscr{J})} \cdot q^{u(\mathscr{I},\mathscr{J})} + \cdots,$$

where $\beta_{(\mathscr{I},\mathscr{J})} \neq 0$.

We define $a = (a_1, \ldots, a_r, 0, \ldots, 0) \in \mathbb{C}^{*I}$, $w = (w_1, \ldots, w_r, 0, \ldots, 0) \in \mathbb{N}^m$ and d_j the minimum value of the function $l^j_w : (\Delta^{t(q)}_j)^I \to \mathbb{R}_+$, with Γ_j being the face of $(\Delta^{t(q)}_j)^I = (\Delta^0_j)^I$ where this function takes this minimum value and such that $\sum_{j=1}^{k} \Gamma_j$ is a bounded face of $\sum_{j=1}^{k} \Delta_k$.

As usual, we exemplify the above construction with a determinantal singularity defined by a 2×3 matrix.

Example 2.17. *We proceed with the determinantal variety X^2_A of type $(2,3;2)$ from Example 2.14. Since this variety has codimension 2, the orthogonal space $(T_{z_R}(X^s_{A_{t_R}} \cap \mathbb{C}^{*I}))^{\perp}$ has dimension 2. Therefore, every basis for it has 2 vectors. Since, the matrix $A^{*I}_{t_R}$ has three size 2 minors, for each z_R, one of the gradient vectors is a linear combination of the other 2. In general, the gradient vectors which form a basis for $(T_{z_R}(X^s_{A_{t_R}} \cap \mathbb{C}^{*I}))^{\perp}$ vary depending on t_R and z_R. We want a subsequence of (t_R, z_R) such that a the gradient vectors of the same minors form a basis for the orthogonal space. Since the variety $X^2_{A^{*I}_{t_R}}$ is defined by the three size 2 minors of $A^{*I}_{t_R}$, we can take a partition of the set $\mathbb{N}$ with three* *subsets. Since this partition has a finite number of subsets, one of them must be infinite and we can take our subsequence using this infinite set.*

Suppose, without loss of generality, that the subsequence $(t_{R_\gamma}, z_{R_\gamma})$ *is such that the gradient of the minors defined by the subset* $\mathscr{C}_\gamma = \{(\mathscr{I}, \mathscr{J}_1), (\mathscr{I}, \mathscr{J}_2)\} = \{(\{1,2\},\{1,2\}), (\{1,2\},\{1,3\})\}$ *forms a basis for* $(T_{z_{R_\gamma}}(X^s_{A_{t_{R_\gamma}}} \cap \mathbb{C}^{*I}))^\perp$*, i.e., the set*

$$\{grad(\det((a^{*I}_{i,j})_{t_{R_\gamma}}(z_{R_\gamma}))_{i\in\mathscr{I}, j\in\mathscr{J}_1}), grad(\det((a^{*I}_{i,j})_{t_{R_\gamma}}(z_{R_\gamma}))_{i\in\mathscr{I}, j\in\mathscr{J}_2})\}$$

is linearly independent. Therefore, we can write the vector $z_{R_\gamma} \in (T_{z_{R_\gamma}} S_{||z_{R_\gamma}||})^\perp$ *uniquely as a linear combination of vectors from this set, i.e., there exist* λ_1 *and* λ_2 *satisfying*

$$z_{R_\gamma} = \sum_{k=1}^{2} \lambda_k \cdot grad(\det((a^{*I}_{i,j})_{t_{R_\gamma}}(z_{R_\gamma}))_{i\in\mathscr{I}, j\in\mathscr{J}_k}).$$

We remark that λ_1 *or* λ_2 *might be zero, (in the case, where* z_{R_γ} *is a multiple of the vector* $grad(\det((a^{*I}_{i,j})_{t_{R_\gamma}}(z_{R_\gamma}))_{i\in\mathscr{I}, j\in\mathscr{J}_k})$*, for* $k = 1$ *or* $k = 2$*). If that is the case, we take another subsequence* $(t_{R_{\gamma_\alpha}}, z_{R_{\gamma_\alpha}}))$ *such that for each member of this sequence, one of the following items is true.*

1. $\lambda_1 \neq 0$ *and* $\lambda_2 = 0$*;*

2. $\lambda_1 = 0$ *and* $\lambda_2 \neq 0$*.*

We take the case, where $\lambda_1 \neq 0$ *and* $\lambda_2 \neq 0$ *(for items* 1. *and* 2. *the procedure is the same as in the proof of Proposition 2.7. Since* $(t_{R_\gamma}, z_{R_\gamma})$ *converges to* $(0,0)$*, the point* $(0,0)$ *belongs to the closure of the set consisting of points* $(t,z) \in D \times \mathbb{C}^{*I}$ *such that*

$$z \in X^s_{A_t} \text{ and } z = \sum_{k=1}^{2} \lambda_k \cdot grad(\det((a^{*I}_{i,j})_t(z))_{i\in\mathscr{I}, j\in\mathscr{J}_k}).$$

By the Curve Selection Lemma [44], there exists a real analytic curve

$$(t(q), z(q)) = (t(q), z_1(q), \ldots, z_r(q), 0, \ldots, 0)$$

and Laurent series $\lambda_1(q)$ *and* $\lambda_2(q)$*, such that*

(i) $(t(0), z(0)) = (0,0)$*;*

(ii) $(t(q), z(q)) \in D \times \mathbb{C}^{*I}$*, for* $q \neq 0$*;*

(iii) $z(q) \in X^2_{A_{t(q)}}$*;*

(iv) $z(q) = \sum_{k=1}^{2} \lambda_k(q) \cdot grad(\det(((a^{*I}_{i,j})_{t(q)}(z(q)))_{i\in\mathscr{I}, j\in\mathscr{J}_k}))$.

Therefore, we can consider the Taylor expansions

$$t(q) = t_0 q^{\omega} + \cdots, \qquad z_i(q) = a_i q^{w_i} + \cdots,$$

where $t_0, a_i \neq 0$ and $\nu, w_i > 0$, for $i = 1, \ldots, r$. We also consider the Laurent expansions

$$\lambda_1(q) = \beta_1 \cdot q^{u_1} + \cdots, \lambda_2(q) = \beta_2 \cdot q^{u_2} + \cdots,$$

where $\beta_k \neq 0$, for $k = 1, 2$.

After the last example, we can return to the proof of Proposition 2.10.

Lemma 2.18. *There exists $\widetilde{\mathscr{C}} \subseteq \mathscr{C}_\alpha$ such that*

$$\sum_{(\mathscr{I},\mathscr{J})\in\widetilde{\mathscr{C}}} \beta_{(\mathscr{I},\mathscr{J})} \sum_{l=1}^{r} w_l \overline{a_l} \frac{\overline{\partial}}{\partial z_l} (\det((a^{*I}_{i,j})_0|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) \neq 0.$$

Proof. Indeed, since

$$grad(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(z(q)))_{i\in\mathscr{I},j\in\mathscr{J}}) =$$
$$(\frac{\overline{\partial}}{\partial z_1}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) \cdot q^{(\Sigma_{j\in\mathscr{J}} d_j)-w_1} + \cdots, \cdots,$$
$$\frac{\overline{\partial}}{\partial z_r}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) \cdot q^{(\Sigma_{j\in\mathscr{J}} d_j)-w_r} + \cdots, 0, \cdots, 0)),$$

by d), we have

$$a_l q^{w_l} + \cdots = z_l(q) = \sum_{(\mathscr{I},\mathscr{J})\in\mathscr{C}_\alpha} \beta_{(\mathscr{I},\mathscr{J})} \frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) \cdot q^{\Sigma_{j\in\mathscr{J}} d_j + u_{(\mathscr{I},\mathscr{J})} - w_l} + \cdots,$$

for $l = 1, \ldots, m$.

We choose the set $\widetilde{\mathscr{C}} \subset \mathscr{C}_\alpha$, such that, for each $(\mathscr{I},\mathscr{J}), (\widetilde{\mathscr{I}}, \widetilde{\mathscr{J}})$ in $\widetilde{\mathscr{C}}$, we have

$$(\sum_{j\in\mathscr{J}} d_j) + u_{(\mathscr{I},\mathscr{J})} = (\sum_{j\in\widetilde{\mathscr{J}}} d_j) + u_{\widetilde{\mathscr{I}},\widetilde{\mathscr{J}}} = \min\{(\sum_{j\in\mathscr{J}} d_j) + u_{\mathscr{I},\mathscr{J}} : (\mathscr{I},\mathscr{J}) \in \mathscr{C}_\alpha\}.$$

Then, $w_l = (\sum_{j\in\mathscr{J}} d_j) + u_{(\mathscr{I},\mathscr{J})} - w_l$, for all $(\mathscr{I},\mathscr{J}) \in \widetilde{\mathscr{C}}$.

We may reorder $w_1, \ldots, w_r$, if necessary, such that $w_1 = \cdots = w_b < w_c$ $(b < c \leq r)$. Therefore,

$$\sum_{(\mathscr{I},\mathscr{J})\in\widetilde{\mathscr{C}}} \beta_{(\mathscr{I},\mathscr{J})} \frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) = \begin{cases} a_l, & 1 \leq l \leq b, \\ 0, & b < l \leq r \end{cases}.$$

Multiplying both sides of the last equality by $w_l \overline{a_l}$, which is non-zero, we have

$$\sum_{(\mathscr{I},\mathscr{J})\in\widetilde{\mathscr{C}}} \beta_{\mathscr{I},\mathscr{J}} w_l \overline{a_l} \frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) = \begin{cases} w_l|a_l|^2, & 1 \leq l \leq b, \\ 0, & b < l \leq r \end{cases}.$$

Taking a sum over $1 \leq l \leq r$ and taking $q \to 0$ we have

$$\sum_{(\mathscr{I},\mathscr{J}) \in \widetilde{\mathscr{C}}} \beta_{\mathscr{I},\mathscr{J}} \sum_{l=1}^{r} w_l \overline{a_l} \frac{\overline{\partial}}{\partial z_l} (\det((a^{*I}_{i,j})_0 |\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) = \sum_{l=1}^{r} w_l |a_l|^2 \neq 0.$$

$\square$

Next we continue with our computations with the family of determinantal singularities from Example 2.14.

Example 2.19. *For $k = 1,2$, we have*

$$grad(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(z(q)))_{i\in\mathscr{I},j\in\mathscr{J}_k}) =$$
$$(\frac{\overline{\partial}}{\partial z_1}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}_k}) \cdot q^{(\Sigma_{j\in\mathscr{J}_k} d_j)-w_1} + \cdots, \cdots,$$
$$\frac{\overline{\partial}}{\partial z_r}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}_k}) \cdot q^{(\Sigma_{j\in\mathscr{J}_k} d_j)-w_r} + \cdots, 0, \cdots, 0).$$

Then, by (iv),

$$a_l q^{w_l} + \cdots = z_l(q) = \sum_{k=1}^{2} \beta_k \frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}_k}) \cdot q^{\Sigma_{j\in\mathscr{J}_k} d_j + u_k - w_l} + \cdots, \qquad (2.5)$$

for $l = 1,\dots,m$. We have again three possible cases, which are the following.

1. $d_1 + d_2 + u_1 = d_1 + d_3 + u_2$;
2. $d_1 + d_2 + u_1 < d_1 + d_3 + u_2$;
3. $d_1 + d_2 + u_1 > d_1 + d_3 + u_2$.

If $d_1 + d_2 + u_1 < d_1 + d_3 + u_2$, then $w_l = d_1 + d_2 - w_l$, because w_l is the smallest degree of a monomial of $a_l q^{w_l} + \cdots$. Therefore, for each $l = 1,\dots,m$, the coefficients a_l and β_1 must be equal. Thus, we can proceed exactly the same as [21, Proposition 3.1]. Of course, the procedure to the case, where $d_1 + d_2 + u_1 > d_1 + d_3 + u_2$ is analogous.

We take special care of the case, where $d_1 + d_2 + u_1 = d_1 + d_3 + u_2$. In this case, Eq. (2.5) *implies that, for each $l = 1,\dots,m$*

$$a_l = \sum_{k=1}^{2} \beta_k \frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}_k}).$$

Now, we may reorder $w_1,\dots,w_r$, if necessary, such that $w_1 = \cdots = w_b < w_c$ $(b < c \leq r)$. Therefore,

$$\sum_{k=1}^{2} \beta_k \frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}_k}) = \begin{cases} a_l, & 1 \leq l \leq b, \\ 0, & b < l \leq r \end{cases}.$$

Multiplying both sides of the last equality by $w_l\overline{a_l}$, we have

$$\sum_{k=1}^{2}\beta_k w_l\overline{a_l}\frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}_k}) = \begin{cases} w_l|a_l|^2, & 1\le l\le b,\\ 0, & b<l\le r\end{cases}.$$

Taking a sum over $1\le l\le r$ and taking $q\to 0$ we have

$$\sum_{k=1}^{2}\beta_k\sum_{l=1}^{r}w_l\overline{a_l}\frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_0|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}_k}) = \sum_{l=1}^{r}w_l|a_l|^2\neq 0.$$

The last step of the proof is to prove the following equality.

Lemma 2.20. *For each $(\mathscr{I},\mathscr{J})\in\widetilde{\mathscr{C}}$, the following equality holds*

$$\sum_{l=1}^{r}w_la_l\frac{\partial}{\partial z_l}(\det((a^{*I}_{i,j})_0|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) = 0.$$

Proof. The polynomial $\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}$ is weighted homogeneous with respect to the weight w and it has weighted degree $\sum_{j\in\mathscr{J}} d_j$, therefore, it follows from the Euler identity that

$$(\sum_{j\in\mathscr{J}}d_j)\cdot\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}} = \sum_{l=1}^{r}w_la_l\frac{\partial}{\partial z_l}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}).$$

Taking $q\to 0$, by the same arguments of Lemma 1, $a\in X^s_{A^{*I}_0}$. Then

$$\det((a^{*I}_{i,j})_0|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}} = 0.$$

Hence

$$\sum_{l=1}^{r}w_la_l\frac{\partial}{\partial z_l}(\det((a^{*I}_{i,j})_0|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) = 0.$$

□

Combining Lemma 2.18 and Lemma 2.20, we get the contradiction

$$0 = \sum_{(\mathscr{I},\mathscr{J})\in\widetilde{\mathscr{C}}}\beta_{(\mathscr{I},\mathscr{J})}\sum_{l=1}^{r}w_l\overline{a_l}\frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_0|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}})\neq 0.$$

Hence, there exists a positive number $R>0$ such that for any admissible coordinate subspace $\mathbb{C}^I$ of A_0 and for any t sufficiently small, the set $X^s_{A_t}\cap\mathbb{C}^{*I}\cap B_R$ is non-singular and intersects transversely with S_r for any $r<R$.

3 Families of fibers on determinantal singularities

In this section, we will extend the concepts from Section 1 to a family of fibers of functions on determinantal singularities.

Now, given a determinantal deformation of X_A^s, $\mathscr{A}$, consider the functions $f_{k+1},\dots,f_p : (\mathbb{C}^m,0)\to(\mathbb{C},0)$ and function germs $F_{k+1},\dots,F_p : (\mathbb{C}^m\times\mathbb{C},0)\to(\mathbb{C},0)$ such that $F_i(x,0)=f_i(x)$ for all $x\in\mathbb{C}^m$ and $i=k+1,\dots,p$. For each $i=k+1,\dots,p$, we fix small enough representatives $f_i : B_\varepsilon\to(\mathbb{C},0)$, where B_ε is the open ball centered at the origin with radius $\varepsilon>0$, and we set $f_{(i,t)}(x) := F_i(x,t)$, $i=k+1,\dots,p$. Therefore, we can consider the map germ $\widetilde{\mathscr{A}} := (\mathscr{A},F_{k+1},\dots,F_p) : (\mathbb{C}^m\times\mathbb{C},0)\to(M_{n,k}\times\mathbb{C}^p,0)$ and we have a family of fibers

$$\{(X_{A_t}^s\cap f_{(k+1,t)}^{-1}(0)\cap\cdots\cap f_{(p,t)}^{-1}(0),0)\}_{t\in D}.$$

The properties of fibers of functions defined on determinantal singularities were also studied, for instance, by Ament, Nuño-Ballesteros, Oréfice-Okamoto and Tomazella [2], Carvalho, Nuño-Ballesteros, Oréfice-Okamoto and Tomazella [11] and Menegon and Pereira [43].

Definition 2.21. *Let $A : (\mathbb{C}^m,0)\to(M_{n,k},0)$ be a germ of a matrix with holomorphic entries and let Δ_j be the Newton polyhedron of $a_{i,j}$. Let $f_{k+1},\dots,f_p : (\mathbb{C}^m,0)\to(\mathbb{C},0)$ be germs of holomorphic functions with Newton polyhedra $\Delta_{k+1},\dots,\Delta_p$, respectively. The variety $X_A^s\cap f_{k+1}^{-1}(0)\cap\cdots\cap f_p^{-1}(0)$ is said to be* **Newton non-degenerate** *if, A is a Newton non-degenerate matrix and for each collection of faces $\Gamma_j\subset\Delta_j$, $j=1,\dots,k,k+1,\dots,p$ such that the sum $\sum_{j=1}^{p}\Gamma_j$ is a bounded face of the polyhedron $\sum_{j=1}^{p}\Delta_j$, the variety*

$$X^s_{(a_{i,j}|_{\Gamma_j})}\cap(f_{k+1}|_{\Gamma_{k+1}})^{-1}(0)\cap\cdots\cap(f_p|_{\Gamma_p})^{-1}(0)$$

is a non-singular variety in $(\mathbb{C}\setminus 0)^m$ with the same dimension as $X_A^s\cap f_{k+1}^{-1}(0)\cap\cdots\cap f_p^{-1}(0)$.

When $l=1$, the definition introduced by Esterov ([18, Definition 1.16]) of f being non-degenerate with respect to a matrix A implies that the variety $X_A^s\cap f^{-1}(0)$ is a Newton non-degenerate variety.

Remark 2.22. Given a Newton non-degenerate matrix A and a generic linear form with respect to X_A^s $h : (\mathbb{C}^m,0)\to(\mathbb{C},0)$, the restriction of h to X_A^s may be degenerate if we eliminate one variable using $h=0$ (see [49, Example (I-2)]). However the variety $X_A^s\cap h^{-1}(0)\subset\mathbb{C}^m$ is Newton non-degenerate.

From now on, we will denote by $\Delta_1^t,\dots,\Delta_k^t,\Delta_{k+1}^t,\dots,\Delta_p^t$ the Newton polyhedra of the columns of the matrix $A_t=((a_{i,j})_t)$ and the Newton polyhedra of the functions $f_{(k+1,t)},\dots,f_{(p,t)}$, respectively. With this notation, we introduce the following result.

Lemma 2.23. *Let $A : (\mathbb{C}^m,0)\to(M_{n,k},0)$ be a germ of a holomorphic matrix and $f_1,\dots,f_l : (\mathbb{C}^m,0)\to(\mathbb{C},0)$ be germs of functions. Let $\mathbb{C}^I$ be an admissible coordinate subspace for $A,f_1,\dots,f_l$. If the variety $X_A^s\cap f_1^{-1}(0)\cap\cdots\cap f_l^{-1}(0)$ is Newton non-degenerate, then $X_{A^I}^s\cap(f_1^I)^{-1}(0)\cap\cdots\cap(f_l^I)^{-1}(0)$ is a Newton non-degenerate variety.*

Proof. The proof follows by the same arguments Lemma 2.9. □

Proposition 2.24. *Suppose that for all t sufficiently small, the following two conditions are satisfied:*

(i) *the variety* $X^s_{A_t} \cap f^{-1}_{(k+1,t)}(0) \cap \cdots \cap f^{-1}_{(p,t)}(0)$ *is Newton non-degenerate, where* $A_t = ((a_{i,j})_t)$*;*

(ii) *the Newton polyhedra* $\Delta^t_1, \ldots, \Delta^t_k, \Delta^t_{k+1}, \ldots, \Delta^t_p$ *are independent of* t*.*

Then there exists a positive number $R > 0$ *such that for any admissible coordinate subspace* $\mathbb{C}^I$ *of* A_0*,* $f_{(k+1,0)}, \ldots, f_{(p,0)}$ *and any* t *sufficiently small, such that the variety*

$$X^s_{A_t} \cap f^{-1}_{(k+1,t)}(0) \cap \cdots \cap f^{-1}_{(p,t)}(0) \cap \mathbb{C}^{*I} \cap B_R$$

is non-singular and intersects transversely with S_r *for any* $r < R$*, where* B_R *(respectively,* S_r*) is the open ball (respectively, the sphere) with center at the origin* $0 \in \mathbb{C}^m$ *and radius* R *(respectively,* r*).*

Proof. The proof is presented in Section 2.24. □

Corollary 2.25. *In addition to the conditions of Proposition 2.24, if the Newton polyhedra*

$$\Delta^t_1, \ldots, \Delta^t_k, \Delta^t_{k+1}, \ldots, \Delta^t_p$$

are convenient, then there exists a positive number $R > 0$ *such that for any* t *sufficiently small, the variety* $X^s_{A_t} \cap f^{-1}_{k+1}(0) \cap \cdots \cap f^{-1}_p(0) \cap \mathbb{C}^{*I} \cap B_R$ *is smooth outside the origin and intersects transversely with* S_r *for any* $r < R$*, where* B_R *(respectively,* S_r*) is the open ball (respectively, the sphere) with center at the origin* $0 \in \mathbb{C}^m$ *and radius* R *(respectively,* r*).*

Proof. Since the polyhedra

$$\Delta^t_1, \ldots, \Delta^t_k, \Delta^t_{k+1}, \ldots, \Delta^t_p$$

are convenient, then every coordinate space is admissible for $A, f_{k+1}, \ldots, f_p$. □

Let $h : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be a generic linear form with respect to X^s_A and for each $c \in \mathbb{C}$ denote $A^c(x_1, \ldots, x_{m-1}) = A(x_1, \ldots, x_{m-1}, c)$. Then the variety $X^s_A \cap h^{-1}(0)$ is biholomorphic to the IDS $X^s_{A^0} \subset \mathbb{C}^{m-1}$.

Since, h is a generic linear form, the variety $X^s_A \cap h^{-1}(0)$ is Newton non-degenerate and we can assume that the Newton polyhedron Δ_h of h is convenient. Moreover, if $X^s_{A_t}$ is a determinantal deformation of X^s_A, we can consider a deformation $X^s_{A_t} \cap h_t^{-1}(0)$ of $X^s_A \cap h^{-1}(0)$ where for each t $h_t : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ is a generic linear form with respect to $X^s_{A_t}$ such that Δ_{h_t} is convenient for all t. Therefore, we have the following corollary.

Corollary 2.26. *Suppose that for all* t *sufficiently small, the following conditions are satisfied:*

(i) *the variety* $X^s_{A_t}$ *is Newton non-degenerate;*

(ii) *the Newton polyhedra* Δ^t_j *of* $(a_{i,j})_t$ *are convenient and independent of* t*, for all* $j = 1, \ldots, k$*.*

Then the vanishing Euler characteristic of $X^s_{A_t} \cap h_t^{-1}(0)$ *and consequently the top polar multiplicity of* $X^s_{A_t}$ *are independent of* t, *i.e.,*

$$\nu(X^s_{A_t} \cap h_t^{-1}(0), 0) = \nu(X^s_{A_0} \cap h_0^{-1}(0), 0)$$

and

$$m_d(X^s_{A_t}, 0) = m_d(X^s_{A_0}, 0).$$

Proof. The first equality follows directly from Corollary 1.83 and Corollary 2.25. The second equation follows directly from the first equality, Theorem 1.87 and Corollary 2.12. □

4 Proof of Proposition 2.24

The proof is a combination of the proof of Proposition 2.10 and [21, Proposition 3.1]. Again it is sufficient showing the result for a fixed $I = \{1, \dots, r\}$, $r \leq m$, since there are only finitely many subsets $I \subset \{1, \dots, m\}$.

Let $\mathscr{A} : (\mathbb{C}^m \times \mathbb{C}, 0) \to (M_{n,k}, 0)$ a determinantal deformation of X^s_A and $F_l : (\mathbb{C}^m \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a deformation of $V(f_l)$ for $l = k+1, \dots, p$.

Suppose that there exists a sequence $\{(t_R, z_R)\}$ of points in $X^s_{\mathscr{A}} \cap F^{-1}_{(k+1,t)}(0) \cap \cdots \cap F^{-1}_{(p,t)}(0) \cap (D \times \mathbb{C}^{*I})$ converging to $(0,0)$, where z_R is a singularity of $X^s_{A_t} \cap f^{-1}_{(k+1,t)}(0) \cap \cdots \cap f^{-1}_{(p,t)}(0) \cap \mathbb{C}^{*I} \cap B_R$. Then $(0,0)$ is in the closure of the set

$$\begin{aligned} W = \{(t,z) \in D \times \mathbb{C}^{*I} :\ & z \in X^s_{A_t} \cap f^{-1}_{(k+1,t)}(0) \cap \cdots \cap f^{-1}_{(p,t)}(0) \\ & \text{and } z \text{ is a singular point of } X^s_{A_t} \cap f^{-1}_{(k+1,t)}(0) \cap \cdots \cap f^{-1}_{(p,t)}(0)\}. \end{aligned}$$

Then, by the Curve Selection Lemma [44], there exists an analytic curve

$$(t(q), z(q)) = (t(q), z_1(q), \dots, z_r(q), 0, \dots, 0),$$

for all $q \neq 0$, and $(t(0), z(0)) = (0,0)$. For $1 \leq i \leq r$, consider the Taylor expansions

$$t(q) = t_0 q^{\omega} + \cdots, \qquad z_i(q) = a_i q^{w_i} + \cdots,$$

where $t_0, a_i \neq 0$ and $\omega, w_i > 0$. Choose $a = (a_1, \dots, a_r, 0, \dots, 0) \in \mathbb{C}^{*I}$ and $w = (w_1, \dots, w_r, 0, \dots, 0) \in \mathbb{N}^m \setminus \{0\}$ and consider the face Γ_j of $(\Delta_j^{t(q)})^I = (\Delta_j^0)^I$ defined as the set where the map

$$\begin{array}{cccc} l_w^j : & (\Delta_j^{t(q)})^I & \to & \mathbb{R}_+ \\ & x := (x_1, \dots, x_r, 0, \dots, 0) & \mapsto & \sum_{i=1}^r x_i w_i \end{array}$$

takes its minimal value d_j, $j = 1, \dots, k, k+1, \dots, p$, and such that $\sum_{j=1}^p \Gamma_j$ is a bounded face of $\sum_{j=1}^p \Delta_j$.

Lemma 2.27. *The point a belongs to the fiber* $X^s_{A_0^{*I}|_\Gamma} \cap (f^{*I}_{(k+1,0)}|_{\Gamma_{k+1}})^{-1}(0) \cap \cdots \cap (f^{*I}_{(p,0)}|_{\Gamma_p})^{-1}(0)$.

Proof. Since $z(q) \in X^s_{A^{*I}_{t(q)}|_\Gamma}$, by Lemma 2.13, $a \in X^s_{A^{*I}_0|_\Gamma}$. Moreover, $z(q) \in (f^{*I}_{(i,t(q))}|_{\Gamma_i})^{-1}(0)$, for all $i = k+1,\ldots,p$. Since, every hypersurface singularity is also a determinantal singularity, by Lemma 2.13, $a \in (f^{*I}_{(i,0)}|_{\Gamma_i})^{-1}(0)$, for all $i = k+1,\ldots,p$. Therefore,

$$a \in X^s_{A^{*I}_0|_\Gamma} \cap (f^{*I}_{(k+1,0)}|_{\Gamma_{k+1}})^{-1}(0) \cap \cdots \cap (f^{*I}_{(p,0)}|_{\Gamma_p})^{-1}(0).$$

□

Lemma 2.28. *The point a is a singularity of $X^s_{A^{*I}_0|_\Gamma} \cap (f^{*I}_{(k+1,0)}|_{\Gamma_{k+1}})^{-1}(0) \cap \cdots \cap (f^{*I}_{(p,0)}|_{\Gamma_p})^{-1}(0)$.*

Proof. Consider the jacobian matrix of the map given by the s size minors of $A^{*I}_{t(q)}|_\Gamma$ at $z(q)$,

$$J_{A^{*I}_{t(q)}|_\Gamma}(z(q)) = (M_{(\mathscr{I},\mathscr{J}),l}(z(q))),$$

where

$$M_{(\mathscr{I},\mathscr{J}),l}(z(q)) = \frac{\partial}{\partial z_l}(\det((a^{*I}_{i,j})_{t(q)}|_{\Gamma_j}(a))_{i\in\mathscr{I},\, j\in\mathscr{J}}) \cdot q^{(\Sigma_{j\in\mathscr{J}} d_j)-w_l} + \cdots,$$

$(\mathscr{I},\mathscr{J}) \in \mathscr{C}$ and $l = 1,\ldots m$. In addition, consider the jacobian matrix of the map

$$f^{*I}_{t(q)} = (f^{*I}_{(k+1,t(q))}|_{\Gamma_{k+1}},\ldots,f^{*I}_{(p,t(q))}|_{\Gamma_p})$$

at $z(q)$,

$$J_{f^{*I}_{t(q)}}(z(q)) = (\frac{\partial}{\partial z_l}(f^{*I}_{(\alpha,t(q))}|_{\Gamma_j}(a) \cdot q^{d_\alpha - w_l} + \cdots),$$

where, $\alpha = k+1,\ldots,p$ and $l = 1,\ldots,m$. Therefore, the jacobian matrix of the map given by the s size minors of $A^{*I}_{t(q)}|_\Gamma$ together with the map $f^{*I}_{t(q)}$ at $z(q)$ is

$$J_{A^{*I}_{t(q)}|_\Gamma, f^{*I}_{t(q)}}(z(q)) = \left[\begin{array}{c} J_{A^{*I}_{t(q)}|_\Gamma}(z(q)) \\ \hline J_{f^{*I}_{t(q)}}(z(q)) \end{array}\right].$$

Since rank of $J_{A^{*I}_{t(q)}|_\Gamma, f^{*I}_{t(q)}}(a)$ is the same as rank of $J_{A^{*I}_{t(q)}|_\Gamma, f^{*I}_{t(q)}}(z(q))$ and $z(q)$ is a singularity of $X^s_{A^{*I}_{t(q)}|_\Gamma} \cap (f^{*I}_{(k+1,t(q))}|_{\Gamma_{k+1}})^{-1}(0) \cap \cdots \cap (f^{*I}_{(p,t(q))}|_{\Gamma_p})^{-1}(0)$, the point a is also a singularity of

$$X^s_{A^{*I}_{t(q)}|_\Gamma} \cap (f^{*I}_{(k+1,t(q))}|_{\Gamma_{k+1}})^{-1}(0) \cap \cdots \cap (f^{*I}_{(p,t(q))}|_{\Gamma_p})^{-1}(0).$$

Therefore, taking $q \to 0$, a is a singularity of $X^s_{A^{*I}_0|_\Gamma} \cap (f^{*I}_{(k+1,0)}|_{\Gamma_{k+1}})^{-1}(0) \cap \cdots \cap (f^{*I}_{(p,0)}|_{\Gamma_p})^{-1}(0)$. □

By Lemma 2.27 and Lemma 2.28, $a \in \mathbb{C}^{*I}$ is a singularity of $X^s_{A^{*I}_0|_\Gamma} \cap (f^{*I}_{(k+1,0)}|_{\Gamma_{k+1}})^{-1}(0) \cap \cdots \cap (f^{*I}_{(p,0)}|_{\Gamma_p})^{-1}(0)$. Therefore, this variety is not Newton non-degenerate which contradicts of Lemma 2.23.

To prove the "transversality", we suppose that there exists a sequence $\{(t_R, z_R)\}$ of points in $X^s_{\mathscr{A}} \cap F^{-1}_{k+1}(0) \cap \cdots \cap F^{-1}_p(0) \cap (D \times \mathbb{C}^{*I})$ converging to $(0,0)$ and such that $X^s_{A_{t_R}} \cap f^{-1}_{(k+1,t_R)}(0) \cap \cdots \cap f^{-1}_{(p,t_R)}(0) \cap \mathbb{C}^{*I}$ does not intersect the sphere $S_{||z_R||}$ transversally at z_R. Thus

$$(T_{z_R} S_{||z_R||})^\perp \subseteq (T_{z_R}(X^s_{A_{t_R}} \cap f^{-1}_{(k+1,t_R)}(0) \cap \cdots \cap f^{-1}_{(p,t_R)}(0) \cap \mathbb{C}^{*I}))^\perp.$$

Moreover, $(T_{z_R}(X^s_{A_{t_R}}\cap f^{-1}_{(k+1,t_R)}(0)\cap\cdots\cap f^{-1}_{(p,t_R)}(0)\cap\mathbb{C}^{*I}))^{\perp}$ is generated by the set $G=G^1\cup G^2$, where $G^1=\{grad(\det((a^{*I}_{i,j})_{t_R}(z_R))_{i\in\mathscr{I},j\in\mathscr{J}}):(\mathscr{I}\,\mathscr{J})\in\mathscr{C}\}$ and $G^2=\{grad(f^{*I}_{(\rho,t_R)}(z_R)):k+1\leq\rho\leq p\}$.

Once again, we are looking for subsets $\mathscr{C}_\gamma\subset\mathscr{C}$ and $\tau_\gamma\subset\{k+1,\dots,p\}$ such that $G_\gamma=G^1_\gamma\cup G^2_\gamma$ is a basis for

$$(T_{z_{R_\gamma}}(X^s_{A_{t_{R_\gamma}}}\cap f^{-1}_{(k+1,t_{R_\gamma})}(0)\cap\cdots\cap f^{-1}_{(p,t_{R_\gamma})}(0)\cap\mathbb{C}^{*I}))^{\perp},$$

where $G^1_\gamma=\{grad(\det((a^{*I}_{i,j})_{t_{R_\gamma}}(z_{R_\gamma}))_{i\in\mathscr{I},j\in\mathscr{J}}):(\mathscr{I},\mathscr{J})\in\mathscr{C}_\gamma\}$ and $G^2_\gamma=\{grad(f^{*I}_{(\tau,t_{R_\gamma})}(z_{R_\gamma})):\tau\subset\tau_\gamma\}$.

As early, there exists a subsequence $\{(t_{R_\gamma},z_{R_\gamma})\}$ of $\{(t_R,z_R)\}$ such that G_γ is a basis for this orthogonal space. Therefore, we can write $z_R\in(T_{z_{R_\gamma}}S_{||z_{R_\gamma}||})^{\perp}$ uniquely as linear combination of those gradient vectors, *i.e.,* there exist $\lambda_{(\mathscr{I},\mathscr{J})}$ and μ_ρ satisfying

$$z_{R_\gamma}=\sum_{(\mathscr{I},\mathscr{J})\subset\mathscr{C}_\gamma}\lambda_{(\mathscr{I},\mathscr{J})}\cdot grad(\det((a^{*I}_{i,j})_{t_{R_\gamma}}(z_{R_\gamma}))_{i\in\mathscr{I},j\in\mathscr{J}})+\sum_{\rho\in\tau_\gamma}\mu_\rho\cdot grad(f^{*I}_{(\rho,t_{R_\gamma})}|_{\Gamma_\rho}(z_{R_\gamma})).$$

We observe that some of the coefficients $\lambda_{(\mathscr{I},\mathscr{J})},\mu_\rho$ may be zero in the above linear combination. If $\mu_\rho=0$ for all $\rho=\tau_1,\dots,\tau_\gamma$ we proceed in the same way as Section 2. If not, we can take a subsequence $\{(t_{R_\alpha},z_{R_\alpha})\}$ of $\{(t_{R_\gamma},z_{R_\gamma})\}$ such that

$$z_{R_\alpha}=\sum_{(\mathscr{I},\mathscr{J})\in\mathscr{C}_\alpha}\lambda_{(\mathscr{I},\mathscr{J})}\cdot grad(\det((a^{*I}_{i,j})_{t_{R_\alpha}}(z_{R_\alpha}))_{i\in\mathscr{I},j\in\mathscr{J}})+\sum_{\rho=\tau_\alpha}\mu_\rho\cdot grad(f^{*I}_{(\rho,t_{R_\alpha})}|_{\Gamma_\rho}(z_{R_\alpha})),$$

with $\lambda_{(\mathscr{I},\mathscr{J})}\neq0$ and $\mu_\rho\neq0$ for all $(\mathscr{I},\mathscr{J})\in\mathscr{C}_\alpha\subset\mathscr{C}_\gamma$ and $\rho\in\tau_\alpha\subset\tau_\gamma\subset\{k+1,\dots,p\}$.

Since $(t_{R_\alpha},z_{R_\alpha})\to(0,0)$, $(0,0)$ belongs to the closure of the set consisting of points $(t,z)\in D\times\mathbb{C}^{*I}$ such that

$$z\in X^s_{A_t}\cap f^{-1}_{(k+1,t)}(0)\cap\cdots\cap f^{-1}_{(p,t)}(0)\text{ and}$$

$$z=\sum_{(\mathscr{I},\mathscr{J})\in\mathscr{C}_\alpha}\lambda_{(\mathscr{I},\mathscr{J})}\cdot grad(\det((a^{*I}_{i,j})_t(z))_{i\in\mathscr{I},j\in\mathscr{J}})+\sum_{\rho=\tau_1}^{\tau_\alpha}\mu_\rho\cdot grad(f^{*I}_{(\rho,t)}|_{\Gamma_\rho}(z)).$$

By the Curve Selection Lemma [44], there exists a real analytic curve

$$(t(q),z(q))=(t(q),z_1(q),\dots,z_r(q),0,\dots,0)$$

and Laurent series $\lambda_{(\mathscr{I},\mathscr{J})}$, $(\mathscr{I},\mathscr{J})\in\mathscr{C}_\alpha$, and μ_ρ, $\rho=\tau_1,\dots,\tau_\alpha$, such that

(i) $(t(0),z(0))=(0,0)$;

(ii) $(t(q),z(q))\in D\times\mathbb{C}^{*I}$, for $q\neq0$;

(iii) $z(q)\in X^s_{A_{t(q)}}\cap(f^{*I}_{(k+1,t(q))})^{-1}(0)\cap\cdots\cap(f^{*I}_{(p,t(q))})^{-1}(0)$;

(iv) $z(q)=\sum_{(\mathscr{I},\mathscr{J})\in\mathscr{C}_\alpha}\lambda_{(\mathscr{I},\mathscr{J})}(q)\cdot grad(\det((a^{*I}_{i,j})_{t(q)}(z(q)))_{i\in\mathscr{I},j\in\mathscr{J}})$
$+\sum_{\rho=\tau_1}^{\tau_\alpha}\mu_\rho(q)\cdot grad(f^{*I}_{(\rho,t(q))}|_{\Gamma_\rho}(z(q)))$.

Consider the Taylor expansions

$$t(q) = t_0 q^{\omega} + \cdots, \qquad z_i(q) = a_i q^{w_i} + \cdots,$$

where $t_0, a_i \neq 0$ and $\omega, w_i > 0$, for $i = 1, \dots, r$. Consider also the Laurent expansions

$$\lambda_{(\mathscr{I},\mathscr{J})}(q) = \beta_{(\mathscr{I},\mathscr{J})} \cdot q^{u(\mathscr{I},\mathscr{J})} + \cdots, \ \mu_\rho(q) = \zeta_\rho q^{v_\rho} + \cdots$$

where $\beta_{(\mathscr{I},\mathscr{J})} \neq 0$ and $\zeta_\rho \neq 0$. Choose $a = (a_1, \dots, a_r, 0, \dots, 0) \in \mathbb{C}^{*I}$ and $w = (w_1, \dots, w_r, 0, \dots, 0) \in \mathbb{N}^m \setminus \{0\}$ and consider the face Γ_j of $(\Delta_j^{t(q)})^I = (\Delta_j^0)^I$ defined as the set where the map

$$\begin{array}{rccc} l_w^j : & (\Delta_j^{t(q)})^I & \to & \mathbb{R}_+ \\ & x := (x_1, \dots, x_r, 0, \dots, 0) & \mapsto & \sum_{i=1}^{r} x_i w_i \end{array}$$

takes its minimal value d_j, $j = 1, \dots, k, k+1, \dots, p$, and such that $\sum_{j=1}^{p} \Gamma_j$ is a bounded face of $\sum_{j=1}^{p} \Delta_j$.

Lemma 2.29. *There exist subsets $\widetilde{\mathscr{C}} \subset \mathscr{C}_\alpha$ and $\widetilde{\tau} \subset \tau_\alpha$ such that*

$$\sum_{(\mathscr{I},\mathscr{J}) \subset \widetilde{\mathscr{C}}} \beta_{(\mathscr{I},\mathscr{J})} \sum_{l=1}^{r} w_l \overline{a_l} \frac{\overline{\partial}}{\partial z_l} (\det((a_{i,j}^{*I})_0 |_{\Gamma_j}(a))_{i \in \mathscr{I}, j \in \mathscr{J}}) + \sum_{\rho \in \widetilde{\tau}} \zeta_\rho \sum_{l=1}^{r} w_l \overline{a_l} \frac{\overline{\partial}}{\partial z_l} (f_{(\rho,0)}^{*I} |_{\Gamma_\rho}(a)) \neq 0.$$

Proof. Since

$$grad(\det((a_{i,j}^{*I})_{t(q)} |_{\Gamma_j}(z(q)))_{i \in \mathscr{I}, j \in \mathscr{J}}) =$$
$$\Big(\frac{\overline{\partial}}{\partial z_1} (\det((a_{i,j}^{*I})_{t(q)} |_{\Gamma_j}(a))_{i \in \mathscr{I}, j \in \mathscr{J}}) \cdot q^{(\Sigma_{j \in \mathscr{J}} d_j) - w_1} + \cdots, \cdots,$$
$$\frac{\overline{\partial}}{\partial z_r} (\det((a_{i,j}^{*I})_{t(q)} |_{\Gamma_j}(a))_{i \in \mathscr{I}, j \in \mathscr{J}}) \cdot q^{(\Sigma_{j \in \mathscr{J}} d_j) - w_r} + \cdots, 0, \cdots, 0\Big)$$

and

$$grad(f_{(\rho,t(q))}^{*I} |_{\Gamma_\rho}(z(q))) =$$
$$\Big(\frac{\overline{\partial}}{\partial z_1} (f_{(\rho,t(q))}^{*I} |_{\Gamma_\rho}(a)) \cdot q^{d_\rho - w_1} + \cdots, \cdots, \frac{\overline{\partial}}{\partial z_r} (f_{(\rho,t(q))}^{*I} |_{\Gamma_\rho}(a)) \cdot q^{d_\rho - w_r} + \cdots, 0, \cdots, 0\Big)$$

by (iv), we have

$$a_l q^{w_l} + \cdots = z_l(q) = \sum_{(\mathscr{I},\mathscr{J}) \in \mathscr{C}_\alpha} \beta_{\mathscr{I},\mathscr{J}} \frac{\overline{\partial}}{\partial z_l} (\det((a_{i,j}^{*I})_{t(q)} |_{\Gamma_j}(a))_{i \in \mathscr{I}, j \in \mathscr{J}}) \cdot q^{\Sigma_{j \in \mathscr{J}} d_j + u_{\mathscr{I},\mathscr{J}} - w_l} + \cdots$$
$$+ \sum_{\rho \in \tau_\alpha} \zeta_\rho \frac{\overline{\partial}}{\partial z_l} (f_{(\rho,t(q))}^{*I} |_{\Gamma_\rho}(a)) \cdot q^{d_\rho + v_\rho - w_r} + \cdots,$$

for $l = 1, \dots, m$.

We choose the sets $\widetilde{\mathscr{C}} \subset \mathscr{C}_\alpha$ and $\widetilde{\tau} \subset \tau_\alpha$, such that, for each $(\mathscr{I},\mathscr{J}), (\widetilde{\mathscr{I}},\widetilde{\mathscr{J}}) \in \widetilde{\mathscr{C}}$ and $\rho, \widetilde{\rho} \in \widetilde{\tau}$, we have

$$(\sum_{j\in\mathscr{J}} d_j)+u_{(\mathscr{I},\mathscr{J})} = (\sum_{j\in\widetilde{\mathscr{J}}} d_j)+u_{(\widetilde{\mathscr{I}},\widetilde{\mathscr{J}})} = \min\{(\sum_{j\in\mathscr{J}} d_j)+u_{(\mathscr{I},\mathscr{J})} : (\mathscr{I},\mathscr{J})\in\mathscr{C}_\alpha\}$$
$$= \min\{d_j+\nu_j : j\in\tau_\alpha\} = d_\rho+\nu_\rho = d_{\widetilde{\rho}}+\nu_{\widetilde{\rho}}.$$

Then, $w_l = (\sum_{j\in\mathscr{J}} d_j)+u_{(\mathscr{I},\mathscr{J})} - w_l = d_\rho+\nu_\rho$, for all $(\mathscr{I},\mathscr{J})\in\widetilde{\mathscr{C}}$ and $\rho\in\widetilde{\tau}$.

We can reorder $w_1,\dots,w_r$, if necessary, such that $w_1=\cdots=w_b<w_c (b<c\le r)$. Therefore,

$$\sum_{(\mathscr{I},\mathscr{J})\in\widetilde{\mathscr{C}}_\alpha} \beta_{\mathscr{I},\mathscr{J}} \frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_{t(q)}|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) + \sum_{\rho\in\widetilde{\tau}} \zeta_\rho \frac{\overline{\partial}}{\partial z_l}(f^{*I}_{(\rho,t(q))}|\Gamma_\rho(a)) = \begin{cases} a_l, & 1\le l\le b,\\ 0, & b<l\le r\end{cases}$$

Repeating the same process as in Lemma 2.18, we multiply this equality by $w_l\overline{a_l}$, take the sum over $1\le l\le r$ and take $q\to 0$ to obtain

$$\sum_{(\mathscr{I},\mathscr{J})\subset\widetilde{\mathscr{C}}} \beta_{(\mathscr{I},\mathscr{J})} \sum_{l=1}^{r} w_l\overline{a_l}\frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_0|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}})$$
$$+\sum_{\rho\in\widetilde{\tau}}\zeta_\rho\sum_{l=1}^{r} w_l\overline{a_l}\frac{\overline{\partial}}{\partial z_l}(f^{*I}_{(\rho,0)}|\Gamma_\rho(a)) = \sum_{l=1}^{r} w_l|a_l|^2\neq 0.$$

□

By Lemma 2.20, we have, for each $(\mathscr{I},\mathscr{J})\in\widetilde{\mathscr{C}}$,

$$\sum_{l=1}^{r} w_l a_l \frac{\partial}{\partial z_l}(\det((a^{*I}_{i,j})_0|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) = 0$$

and, for each $\rho\in\widetilde{\tau}$,

$$\sum_{l=1}^{r} w_l\overline{a_l}\frac{\overline{\partial}}{\partial z_l}(f^{*I}_{(\rho,0)}|\Gamma_\rho(a)) = 0,$$

since a hypersurface is also a determinantal variety.

Hence, we conclude the proof of Lemma 2.23 with the contraction

$$0 = \sum_{(\mathscr{I},\mathscr{J})\subset\widetilde{\mathscr{C}}} \beta_{(\mathscr{I},\mathscr{J})} \sum_{l=1}^{r} w_l\overline{a_l}\frac{\overline{\partial}}{\partial z_l}(\det((a^{*I}_{i,j})_0|\Gamma_j(a))_{i\in\mathscr{I},j\in\mathscr{J}}) + \sum_{\rho\in\widetilde{\tau}}\zeta_\rho\sum_{l=1}^{r} w_l\overline{a_l}\frac{\overline{\partial}}{\partial z_l}(f^{*I}_{(\rho,0)}|\Gamma_\rho(a)) \neq 0.$$

5 Whitney equisingularity

Finally, we have all the necessary tools to prove the main theorem of this chapter.

Theorem 2.30. *Let $\left\{(X^s_{A_t},0)\right\}_{t\in D}$, be a d-dimensional family of determinantal singularities, defined by the germ of matrices $A_t = ((a_{i,j})_t) : (\mathbb{C}^m,0)\to(M_{n,k},0)$ with holomorphic entries. Suppose that $X^s_{A_0}$ has an isolated singularity at 0 and, for all $t\in D$, the matrix A_t satisfies the following conditions:*

(i) *the Newton polyhedra Δ^t_j of $(a_{i,j})_t$ are convenient and independent of t;*

(ii) *the matrix A_t is Newton non-degenerate.*

Then the family $\left\{(X^s_{A_t},0)\right\}_{t\in D}$ *is Whitney equisingular.*

Proof. Firstly, since the Newton polyhedra Δ^t_j are convenient and independent of t for each $j = 1,\ldots,k$ and the matrix A_t is Newton non-degenerate, then by Corollary 2.11, there exists a positive number R such that for any t sufficiently small, the set $X^s_{A_t} \cap B_R$ is smooth outside the origin, where B_R is the open ball with center at the origin and radius R. Therefore, this family is good. Moreover, by Corollary 2.26,

$$m_d(X^s_{A_t},0) = m_d(X^s_{A_0},0). \tag{5.1}$$

Now, applying successively Corollary 2.26 we have the following relation,

$$\nu(X^s_{A_t} \cap h^{-1}_{(t,1)}(0) \cap \cdots \cap h^{-1}_{(t,l)}(0),0) = \nu(X^s_{A_0} \cap h^{-1}_{(0,1)}(0) \cap \cdots \cap h^{-1}_{(0,l)}(0),0),$$

where $h_{(t,i)}$ are generic linear forms with respect to

$$X^s_{A_t} \cap h^{-1}_{(t,1)}(0) \cap \cdots \cap h^{-1}_{(t,i-1)}(0)$$

for $i = 1,\ldots,l$. Hence,

$$m_{d-l}(X^s_{A_t} \cap h^{-1}_{(t,1)}(0) \cap \cdots \cap h^{-1}_{(t,l)}(0),0) = m_{d-l}((X^s_{A_0} \cap h^{-1}_{(0,1)}(0) \cap \cdots \cap h^{-1}_{(0,l)}(0),0).$$

In addition, applying successively Lemma 1.90, we have

$$m_{d-l}(X^s_{A_t} \cap h^{-1}_{(t,1)}(0) \cap \cdots \cap h^{-1}_{(t,l)}(0),0) = m_{d-l}(X^s_{A_t},0),$$

for $l = 1,\ldots,d$. Combining both equations above, we have

$$\begin{aligned} m_{d-l}(X^s_{A_t},0) &= m_{d-l}(X^s_{A_t} \cap h^{-1}_{(t,1)}(0) \cap \cdots \cap h^{-1}_{(t,l)}(0),0) \\ &= m_{d-l}((X^s_{A_0} \cap h^{-1}_{(0,1)}(0) \cap \cdots \cap h^{-1}_{(0,l)}(0),0) \\ &= m_{d-l}(X^s_{A_0},0), \end{aligned}$$

for all $l = 1,\ldots,d$.

Therefore, $m_j(X^s_{A_t},0) = m_j(X^s_{A_0},0)$, for all $0 \le j \le d$. Hence, the family $\left\{(X^s_{A_t},0)\right\}_{t\in D}$ is Whitney equisingular, by Theorem 1.91. □

Example 2.31. *Consider the classical Briançon-Speder example, i.e., the family of functions* $f_t : (\mathbb{C}^3,0) \to (\mathbb{C},0)$ *defined by* $f_t(x,y,z) = x^5 + t\cdot xy^6 + y^7z + z^{15}$*.The family* $\{(V(f_t),0)\}_{t\in D}$ *is topologically trivial, but it is not Whitney equisingular.*

We observe that the function f_t is not convenient and the Newton polyhedron of f_t depends on t (see Figure 2.5 and Figure 2.6).

This example illustrates how essential is Condition (i) of Theorem 2.30.

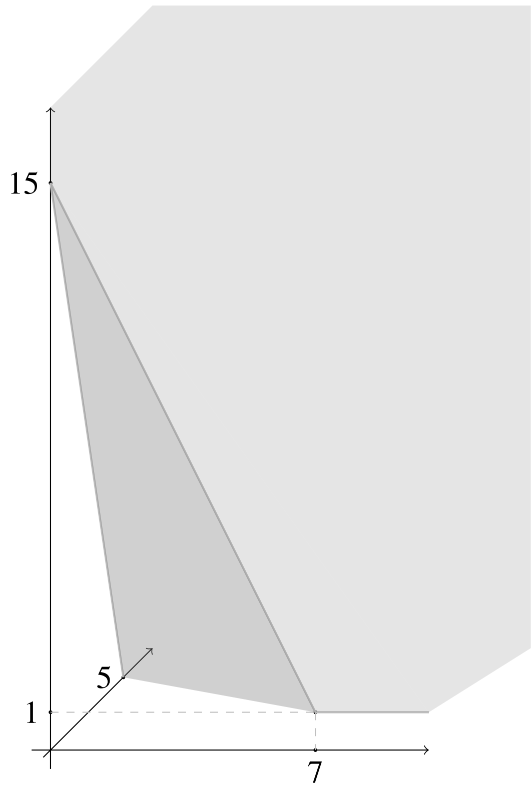


Figure 2.5: Newton polyhedron of f_0.

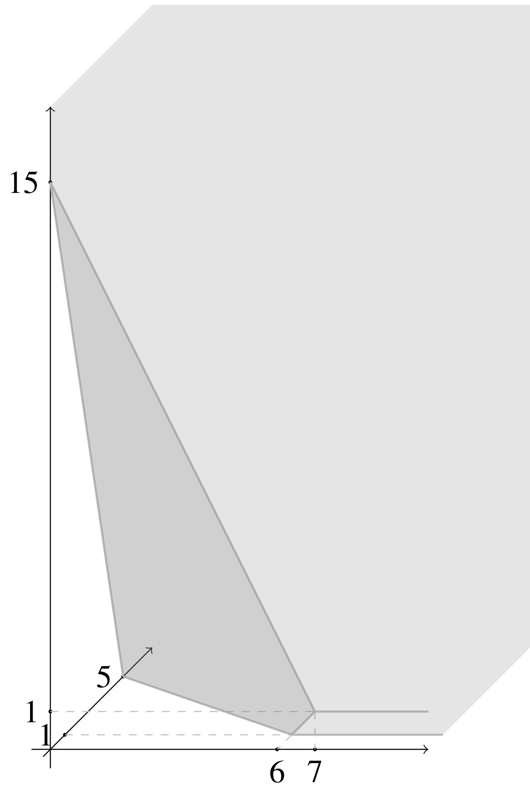


Figure 2.6: Newton polyhedron of f_t, for $t \neq 0$.

Example 2.32. *Let $\left\{(X^2_{A_t},0)\right\}_{t\in D}$ be the family of 2-dimensional IDS defined by the matrices $A_t : (\mathbb{C}^4,0) \to (M_{2,3},0)$, where*

$$A_t = \begin{bmatrix} x+y+z+w+ty^4 & 2x+y+z+3w+tx^3 & 5x+7y+z+w+tw^2 \\ 2x+3y+7z+8w+tz^2 & 9x+5y+7z+11w+ty^2 & 13x+15y+9z+11w+tx^4 \end{bmatrix}.$$

The matrix A_t is Newton non-degenerate and the polyhedron Δ^t_j, $j=1,2,3$, is convenient and constant on t, for all $t\in D$. Then, by Theorem 2.30, $\left\{(X^2_{A_t},0)\right\}_{t\in D}$ is Whitney equisingular.

Example 2.33. *Let $\left\{(X^2_{A_t},0)\right\}_{t\in D}$ be the family of 2-dimensional determinantal singularities defined by the germ $A_t : (\mathbb{C}^4,0) \to (M_{2,3},0)$, given by the matrix*

$$\begin{bmatrix} 2x+2y^3+ty^4+z^2-3w^4 & 2x+3y^3+2ty^4+2z^2-5w^4 & 3x+2y^3+ty^4+2z^2-3w^4 \\ 3x+3y^3+ty^4+2z^2-4w^4 & 3x+4y^3+2ty^4+4z^2-7w^4 & 5x+3y^3+ty^4+3z^2-3w^4 \end{bmatrix}.$$

For all $t\in D$, the matrix A_t is Newton non-degenerate (see Example A.3), Δ^t_j is convenient and independent of t, for all $j=1,2,3$. Hence, this family is Whitney equisingular.

CHAPTER 3

Newton polyhedra and determinantal singularities

Newton polyhedra of polynomial functions have been a powerful tool to compute invariants of singularities. Along the years, many authors have provided formulas to compute, for instance, the number of solutions of a system of polynomial equations [3], the genus of a complete intersection [37], the Milnor number of isolated hypersurfaces singularities [39] and of isolated complete intersection singularities [49], the Euler characteristic of the Milnor fiber of a function restricted to an isolated determinantal singularity [18], among others.

1 Newton polyhedra and determinantal singularities

This section is devoted to present the formulas introduced by Esterov (see [18] and the extended version [19]). The results introduced by Esterov to compute invariants of determinantal singularities are derived from his results on a resultantal singularity. Therefore, we start this section with its definition.

Let $B \subset \mathbb{Z}^n$ be a finite set. Denote the set of all Laurent polynomials of the form $\sum\limits_{b\in B} c_b t^b$ by $\mathbb{C}[B]$. Let $B_1,\ldots,B_k \subset \mathbb{Z}^n$ be finite sets. Let $\Sigma(B_1,\ldots,B_n)$ be the closure of the set of all collections $(p_1,\ldots,p_k) \in \mathbb{C}[B_1]\oplus\cdots\oplus\mathbb{C}[B_k]$ such that the set $\{t \in \mathbb{C}^{*n} : p_1(t) = \cdots = p_k(t) = 0\}$ is not empty.

Definition 3.1. *Let $B_1,\ldots B_k \subset \mathbb{Z}^n$ be finite sets. The germ of an analytic set $M \subset \mathbb{C}^m$ in a neighbourhood of the origin is called $(B_1,\ldots,B_k)$-**resultantal** if, for some analytic germ $f : (\mathbb{C}^m,0) \to \mathbb{C}[B_1]\oplus\cdots\oplus\mathbb{C}[B_k]$ the set M is equal $f^{-1}(\Sigma(B_1,\ldots,B_k))$ and its codimension in $\mathbb{C}^m$ is the same as the codimension of $\Sigma(B_1,\ldots,B_k)$ in the space $\mathbb{C}[B_1]\oplus\cdots\oplus\mathbb{C}[B_k]$.*

Determinantal varieties of type $(n,k;n)$ are identified with resultantal singularities trough the following construction.

Remark 3.2. Consider $B_1 = \cdots = B_k = \{e_0, e_1, \ldots, e_{n-1}\} \subset \mathbb{Z}^{n-1}$, where $e_j = (\underbrace{0,\ldots,0}_{j-1},1,0,\ldots,0)$ and $n < k$. Then we can identify the space of collections of linear functions

$$(a_{1,1} + \sum_{i=2}^{n} a_{i,1}t_{i-1}, \ldots, a_{0,k} + \sum_{i=2}^{n} a_{i,k}t_{i-1}) \in \mathbb{C}[B_1] \oplus \cdots \oplus \mathbb{C}[B_k]$$

with the space of $n \times k$ matrices $(a_{i,j})$, the set $\Sigma(B_1, \ldots, B_k)$ with the set of all degenerate matrices, and $(B_1, \ldots, B_k)$-resultantal singularities with determinantal singularities of type $(n,k;n)$.

Example 3.3. *Consider the sets $B_1 = B_2 = B_3 = \{0,1\} \subset \mathbb{Z}$ and the germ $f : \mathbb{C}^4 \to \mathbb{C}[B_1] \oplus \mathbb{C}[B_2] \oplus \mathbb{C}[B_3]$ defined by $f(x,y,z,w) = (x + y\cdot t, y + z\cdot t, z + w\cdot t)$. The variety $M = f^{-1}(\Sigma(B_1,B_2,B_3))$ is a (B_1,B_2,B_3)-resultantal singularity. Moreover, M is the set of points in $\mathbb{C}^4$ such that the system of equations*

$$\begin{cases} x + y\cdot t = 0 \\ y + z\cdot t = 0 \\ z + w\cdot t = 0 \end{cases}$$

has no trivial solutions. Therefore, M is the set of points of $\mathbb{C}^4$ such that the matrix

$$A = \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}$$

has rank less than 2. Hence, $M = \{(x,y,z,w) \in \mathbb{C}^4 : \operatorname{rank}(A(x,y,z,w)) < 2\} = X_A^2$ is a determinantal variety of type $(2,3;2)$.

In the following, we present the definition of a essential collection of sets in $\mathbb{Z}^n$. In [18], Esterov proves that "each resultantal set is $(A_1,\ldots,A_k)$-resultantal for some non-degenerate collection of finite sets $A_1,\ldots,A_k \subset \mathbb{Z}^n$".

Definition 3.4. (i) *A sublattice $L \subset \mathbb{Z}^n$ is* **generated** *by a set $B \in \mathbb{Z}^n$ if it is generated by all vectors of the form $a - b$, where $a \in B$ and $b \in B$.*

(ii) *The* **dimension** *of a set $B \in \mathbb{Z}^n$ is the dimension of the sublattice generated by A.*

(iii) *The* **sum** *of sets $B_i \in \mathbb{Z}^n$ is the set of all sums of the form $\sum_i b_i$, where $b_i \in B_i$.*

(iv) *The* **codimension** *of a collection of finite sets $B_1,\ldots,B_k \subset \mathbb{Z}^n$ is the difference $k - \dim(\sum_i B_i)$, if $k \neq 0$. The codimension of the empty collection is 0.*

Definition 3.5. *A collection of finite sets $B_1,\ldots,B_k \subset \mathbb{Z}^n$ is said to be* **essential** *if its codimension is greater than the codimension of every subcollection $B_{i_1},\ldots,B_{i_r}$, $\{i_1,\ldots,i_r\} \subsetneq \{1,\ldots,k\}$.*

Definition 3.6. *A (m,n)-***Newton pile** *$\mathscr{P} = (B.,\Delta.)$ is a collection of finite sets $B_i \subset \mathbb{Z}^n$ and polyhedra $\Delta_a \subset \mathbb{R}^m_+$, where $i = 1,\ldots,k$ and a runs over all pairs (b,i) such that $i = 1,\ldots,k$ and $b \in B_i$. A Newton pile is called* **essential** *if the collection $\{B_1,\ldots,B_k\}$ is essential and the sum $\sum_i B_i$ contains the origin and generates $\mathbb{Z}^n$. A Newton pile is called* **convenient** *if the difference $\mathbb{R}^m_+ \setminus \Delta_{(b,i)}$ is bounded for all pairs (b,i).*

For a point $p=(p_1,\dots,p_k)\in\mathbb{C}[B_1]\oplus\cdots\oplus\mathbb{C}[B_k]$, denote the coefficient of the monomial t^b in the polynomial p_i by $c_{(b,i)}(p)$. For a (m,n)-newton pile $\mathscr{P}=(B.,\Delta.)$, let $\mathbb{C}\{\mathscr{P}\}$ be the set of germs of all analytic maps $f:\mathbb{C}^m\to\mathbb{C}[B_1]\oplus\cdots\oplus\mathbb{C}[B_k]$ such that the components $c_{(b,i)}\circ f$ are contained in the space $\mathbb{C}\{\Delta_{(b,i)}\}$, where $\mathbb{C}\{\Delta\}$ is the the space of all complex analytic functions of the form $\sum\limits_{b\in\Delta\cap\mathbb{Z}^m} c_b x^b$.

Definition 3.7. *"Almost all collections of germs $f_1\in\mathbb{C}\{\Delta_1\},\dots,f_k\in\mathbb{C}\{\Delta_k\}$" means "all collections of germs $(f_1,\dots,f_k)$, $f_i=\sum_{b\in\Delta_i}c_{(i,b)}x^b$, such that $P(c_{a_1},\dots,c_{a_N})\neq 0$, where P is some non-zero polynomial in N variables". "Almost all maps in $\mathbb{C}\{B\}$" means "almost all collections of components $c(b,i)\circ f$ in $\oplus_{i=1,\dots,k,\in B_i}\mathbb{C}\{\Delta_{(b,i)}\}$".*

For a (m,n)-newton pile $\mathscr{P}$ denote the convex hull of the union $\cup_{b\in B_i}\{b\}\times\Delta_{(b,i)}\subset\mathbb{R}^n\oplus\mathbb{R}^m$ by $\mathscr{P}(i)$. Denote the (m,n)-Newton pile $(B_1,\dots,B_k,\mathbb{R}^m_+,\dots,\mathbb{R}^m_+)$ by $\mathscr{P}_0$.

Theorem 3.8. *Let $\mathscr{P}$ be a convenient essential (m,n)-Newton pile such that $k=m+n$. Then for almost all maps $f\in\mathbb{C}\{\mathscr{P}\}$, the intersection number of the germ $f(\mathbb{C}^m)$ and the set $\Sigma(B_1,\dots,B_k)$ in the space $\oplus_i\mathbb{C}[B_i]$ is well defined and equal to the mixed volume of pairs*

$$k!(\mathscr{P}_0(1),\mathscr{P}(1))^1\cdots(\mathscr{P}_0(k),\mathscr{P}(k))^1.$$

One can find the proof in [18, Theorem 3.5]. Since determinantal singularities of type $(n,k;n)$ can be identified with resultantal singularities, Theorem 3.8 can be rewritten for a determinantal singularity, by setting $B_1=\cdots=B_k=\{e_0,\dots,e_{n-1}\}$.

For a collection of positive weights $w=(w_1,\dots,w_m)$ assigned to the variables $x_1,\dots,x_m$, the lowest order non-zero w-quasihomogeneous component of the function $f=\sum\limits_{a\in\mathbb{Z}^m}c_a x^a$ is denoted by f^w.

Definition 3.9. *The system of polynomial functions $f_1,\dots,f_r:\mathbb{C}^m\to\mathbb{C}$ is* **Newton non-degenerate** *if, for every collection of positive weights $w=(w_1,\dots,w_m)$, the polynomial equations $f_1^w=\cdots=f_r^w=0$ have no common zeros in $\mathbb{C}^{*m}$.*

Let $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{n,k},0)$ be a germ of a matrix and $\Delta_{i,j}$ be the Newton polyhedron of $a_{i,j}$. For each $j=1,\dots,k$, denote by $\Delta_{1,j}*\cdots*\Delta_{n,j}$ the convex hull of the union $\cup_{i=1}^n\{e_{n-1}\}\times\Delta_{i,j}\subset\mathbb{R}^{n-1}\times\mathbb{R}^m_+$. This polyhedron is called **Cayley polyhedron**.

Corollary 3.10. *Let $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{n,k},0)$ be a germ of a matrix such that $m=k-n+1$ and let $\Delta_{i,j}$ be the Newton polyhedron of $a_{i,j}$. If the system of equations $a_{i,j}$, $i=1,\dots,n$ and $j=1,\dots,k$ is Newton non-degenerate and the pairs of polyhedra $(\mathbb{R}^m_+,\Delta_{i,j})$ are bounded, then the intersection number of the germ $A(\mathbb{C}^m)$ and $M^n_{n,k}$ is equal*

$$k!(S_{n-1}\times\mathbb{R}^m_+,\Delta_{1,1}*\cdots*\Delta_{n,1})^1\cdots(S_{n-1}\times\mathbb{R}^m_+,\Delta_{1,k}*\cdots*\Delta_{n,k})^1,$$

where S_m is the standard m-dimensional simplex.

One can find a proof in [17] and, with more details, in the extended version [19, Corollary 3.10]. Moreover, if the Newton polyhedron of $a_{i,j}$ does not depend on i, Esterov [18] introduces a formula to compute the multiplicity of a determinantal singularity.

Definition 3.11. *The matrix $A = (a_{i,j})$ is said to be* **weakly Newton non-degenerate**, *if, for each collection w of positive weights and every subset $\mathscr{I} \subset \{1,\dots,n\}$, the set of all points $x \in \mathbb{C}^{*m}$, such that the matrix is $(a^w_{i,j}(x))_{\substack{i\in\mathscr{I} \\ j\in\{1,\dots,k\}}}$ is degenerate, has the maximal possible codimension $k - |\mathscr{I}| + 1$.*

Remark 3.12. Under the conditions of the above definition, the matrix A is weakly Newton non-degenerate for almost all collections $a_{i,j} \in \mathbb{C}\{\Delta_j\}$ ([18, Theorem 1.17.1]).

For a polyhedron $\Delta \subset \mathbb{R}^m_+$, denote the pair $(\mathbb{R}^m_+, \Delta)$ by $\widetilde{\Delta}$. Denote the pair $(\mathbb{R}^m_+, \overline{\mathbb{R}^m_+ \setminus S_m})$ by L. The proof of the following corollary can be found in [18, Theorem 1.9].

Corollary 3.13. *Let $A : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a germ of a matrix with holomorphic entries. Suppose that the Newton polyhedron of $a_{i,j}$ does not depend on i and let Δ_j be the Newton polyhedra of $a_{i,j}$, for all $i = 1,\dots,n$, $j = 1,\dots,k$. If $\Delta_j \subset \mathbb{R}^m_+$, $j = 1,\dots,k$, touches all the coordinate axes and A is weakly Newton non-degenerate, then X^n_A is a determinantal singularity with multiplicity*

$$\sum_{1\le j_0\le\cdots\le j_{k-n}\le k} m!\cdot(\widetilde{\Delta}_{j_0})^1\cdots(\widetilde{\Delta}_{j_{k-n}})^1(L)^{m-k+n-1}$$

Using the same ideas, Esterov [18] also presents a formula to compute the Euler characteristic of a function restricted to a resultantal singularity in terms of its Newton pile.

Definition 3.14. *Let $A \subset \mathbb{Z}^n$ be a finite set and let B be a subset of A. The* **dual cone** *Γ of the set B is the set of weight vectors $\gamma \in (\mathbb{Z}^n)^*$ such that $\{a \in A : \gamma(a) = min\ \gamma(A)\} = B$. If a cone Γ' is contained in Γ, then B is called* **support set** *of Γ' and it is denoted by $A^{\Gamma'}$.*

Suppose that the integer polyhedra $\Delta_0,\dots,\Delta_k \subset \mathbb{R}^m$ are parallel to each other, the dimension of $\sum \Delta_i$ equals $p+1$, and Δ_0 is contained in a pointed cone C such that $C \setminus \Delta_0$ is bounded. Denote the sum of mixed volume of pairs

$$(-1)^{k-p-1} \sum_{\substack{a_0+\cdots+a_k=p+1 \\ a_0,\dots,a_k\in\mathbb{N}}} (p+1)!(C,\Delta_0)^{a_0}(\Delta_1,\Delta_1)^{a_1}\cdots(\Delta_k,\Delta_k)^{a_k}$$

by $\chi(\Delta_0,\Delta_1,\dots,\Delta_k)$.

For a set $I \subset \{1,\dots,m\}$ and a polyhedron $\Delta \subset \mathbb{R}^m_+$, denote the polyhedron $\Delta \cap \mathbb{R}^I$ by Δ^I. For a (m,n)-newton pile $\mathscr{P} = (B_1,\dots,B_k,\Delta.)$ and a cone $\Gamma \subset (\mathbb{Z}^n)^*$ denote the $(|I|,n)$-Newton pile $(B^\Gamma_1,\dots,B^\Gamma_k,\Delta^I.)$ by $\mathscr{P}^{\Gamma,I}$.

Theorem 3.15. *Let $(B_1,\dots,B_k,\Delta.)$ be a convenient essential (m,n)-Newton pile. Let $\Delta \subset \mathbb{R}^m_+$ be a polyhedron which touches all the coordinate axes. Denote the polyhedron $\{0\}\times\Delta \subset \mathbb{R}^n \oplus \mathbb{R}^m$ by $\widetilde{\Delta}$.*

Let Φ be the dual fan of the convex hull conv$(\sum_j B_j)$. Then the Euler characteristic of the Milnor fiber of $g|_{f^{-1}(\Sigma(B_1,\dots,B_k))}$ is

$$\sum_{\substack{\Gamma\in\Phi\\ I\subset\{1,\dots,m\}}} |I|!\cdot\chi(\widetilde{\Delta},\mathscr{P}^{\Gamma,I}(1),\cdots,\mathscr{P}^{\Gamma,I}(k))$$

for almost all pairs $(g,f)\in\mathbb{C}\{\Delta\}\times\mathbb{C}\{\mathscr{P}\}$.

One can find a proof in [18, Theorem 3.19]. Let $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{n,k},0)$ be a germ of a matrix with holomorphic entries and let $\Delta_{i,j}$ be the Newton polyhedron of the entry $a_{i,j}$. If, for each $j=1,\dots,k$, the polyhedron $\Delta_{i,j}$ depends on i, we can compute the Euler characteristic of the Milnor fiber of $f|_{X_A^n}$ applying Theorem 3.15, for the Newton pile $(B_1,\dots,B_k,\Delta.)$, where $B_1=\cdots=B_k=\{e_0,\dots,e_{n-1}\}$.

Definition 3.16. *The matrix $A=(a_{i,j})$ is said to be* **Newton non-degenerate**, *if, for every collection of positive weights w, the polynomial matrix $(a^w_{i,j})$ defines a non-singular determinantal set in $\mathbb{C}^{*m}$.*

In this case, a function $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ is called **Newton non-degenerate with respect to** *A, if, for every collection w of positive weights, the restriction of f^w to the determinantal set, defined by the matrix $(a^w_{i,j})$ in $\mathbb{C}^{*m}$, has no critical points.*

If $m\le 2(k-n+2)$, then the function f is Newton non-degenerate with respect to A for almost all collections $f\in\mathbb{C}\{\Delta_0\}$, $a_{i,j}\in\mathbb{C}\{\Delta_j\}$ ([18, Theorem 1.17.2]).

Corollary 3.17. *Denote the Newton polyhedron of the entry $a_{i,j}$ of a matrix $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{n,k},0)$ with holomorphic entries, $n\le k$ by $\Delta_{i,j}$. Suppose that $a_{i,j}$ is convenient, for $i=1,\dots,n$ and $j=1,\dots,k$, and $m\le 2(k-n+2)$.*

(i) *If the A is Newton non-degenerate, then $(X^n_A,0)$ is smooth outside of the origin.*

(ii) *If the germ of a convenient function $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ is Newton non-degenerate with respect to A, the Euler characteristic of a Milnor fiber of $f|_{X^n_A}$ is*

$$\sum_{\substack{\Gamma\in\Phi\\ I\subset\{1,\dots,m\}}} \chi(\widetilde{\Delta}^I,\Delta^{\Gamma,I}_{1,1}*\cdots*\Delta^{\Gamma,I}_{n,1},\cdots,\Delta^{\Gamma,I}_{1,k}*\cdots*\Delta^{\Gamma,I}_{n,k}),$$

where Δ is the Newton polyhedron of f and $\Delta^{\Gamma,I}_{1,j}*\cdots*\Delta^{\Gamma,j}_{n,1}$ denotes the Newton pile $(B^\Gamma_1,\dots,B^\Gamma_k,\Delta^I.)$ for $B_1=\cdots B_k=\{e_0,\mathrm{e}_1,\dots,e_{n-1}\}$.

The proof follows the same construction as Corollary 3.10. Moreover, if the Newton polyhedron of $a_{i,j}$ does not depend on i, we have the simpler formula.

Corollary 3.18. *Denote the Newton polyhedron of the entry $a_{i,j}$ of a matrix $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{n,k},0)$ with holomorphic entries, $n\le k$ by Δ_j. Suppose that $a_{i,j}$ is convenient, for $i=1,\dots,n$ and $j=1,\dots,k$, and $m\le 2(k-n+2)$.*

(i) *If the A is Newton non-degenerate, then* $(X^n_A,0)$ *is smooth outside of the origin.*

(ii) *If the germ of a convenient function* $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ *is Newton non-degenerate with respect to A, then the Euler characteristic of a Milnor fiber of* $f|_{X^n_A}$ *is*

$$\sum_{\{j_1,\ldots,j_q\}\subset\{1,\ldots,k\}}\ \sum_{\substack{I\subset\{1,\ldots,m\}\\ |I|\geq q+1}}\ \sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}$$
$$\times\sum_{\substack{a_{j_1},\ldots,a_{j_q}\in\mathbb{N}\\ a_{j_1}+\cdots+a_{j_q}=|I|-a}}|I|!\cdot(\widetilde{\Delta}^I_0)^a(\widetilde{\Delta}^I_{j_1})^{a_{j_1}}\cdots(\widetilde{\Delta}^I_{j_q})^{a_{j_q}}.$$

The proof can be found in [18, Theorem 1.12].

2 Local Euler obstruction

Theorem 1.74, relates the local Euler obstruction of an analytic space X with the Euler characteristic of the Milnor fiber restricted to each stratum of a Whitney stratification of X. In addition, Theorem 3.15 computes the Euler characteristic of the Milnor fiber restricted to a resultantal singularity in terms of its Newton pile. The main purpose of this section is combining these theorems in order to present a formula to compute the local Euler obstruction of an IDS $(X^n_A,0)$ in terms of the Newton polyhedra of the entries of the matrix A. We start stating a theorem which computes the local Euler obstruction of resultantal varieties with an isolated singularity in terms of its Newton pile.

Theorem 3.19. *Let* $\mathscr{P}=(B_1,\ldots,B_k,\Delta.)$ *be a convenient essential* (m,n)*-Newton pile and consider the germ of an isolated resultantal singularity* $X=f^{-1}(\Sigma(B_1,\ldots,B_k))$*. Let* Φ *be the dual fan of the convex hull conv*$(\sum_j B_j)$*. Then the local Euler obstruction of X is*

$$\mathrm{Eu}_X(0)=\sum_{\substack{\Gamma\in\Phi\\ I\subset\{1,\ldots,m\}}}\chi(\widetilde{\mathscr{L}^I},\mathscr{P}^{\Gamma,I}(1),\cdots,\mathscr{P}^{\Gamma,I}(k)),$$

for almost all $f\in\mathbb{C}\{\mathscr{P}\}$*, where* $\mathscr{L}=\mathbb{R}^m_+\setminus S_m$ *and* $\widetilde{\mathscr{L}}=\{e_0\}\times\mathscr{L}$*.*

Proof. Since X has an isolated singularity at the origin, the partition

$$\mathscr{V}=\{\{0\},X\setminus\{0\}\}$$

is a Whitney stratification of X. Thus, if $l:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ is a generic linear form, by Theorem 1.74

$$\mathrm{Eu}_X(0)=\chi(\{0\}\cap B_\varepsilon\cap l^{-1}(t_0))\cdot\mathrm{Eu}_X(\{0\})+\chi((X^n_A\setminus\{0\})\cap B_\varepsilon\cap l^{-1}(t_0))\cdot\mathrm{Eu}_{X^n}(X\setminus\{0\}).$$

On the other hand, as $t_0\neq 0$, then $\{0\}\cap B_\varepsilon\cap l^{-1}(t_0)=\emptyset$. Therefore,

$$\chi(\{0\}\cap B_\varepsilon\cap l^{-1}(t_0))=0.$$

Moreover, the stratum $X\setminus\{0\}$ is the smooth part of X, then $\mathrm{Eu}_X(X\setminus\{0\})=1$. Consequently,

$$\mathrm{Eu}_X(0)=\chi(X\setminus\{0\}\cap B_\varepsilon\cap l^{-1}(t_0)).$$

Therefore in order to compute the Euler obstruction of an isolated singularity X, we can compute the Euler characteristic of the Milnor fiber of a generic linear form on X.

Moreover, by Lemma 1.73, we can choose an appropriate linear form $l:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ which is convenient (its Newton polyhedron is $\mathscr{L}=\mathbb{R}^m_+\setminus S_m$). Therefore, the result follows from Theorem 3.15. □

Applying the same process as in the previous section, we can rewrite the above theorem for isolated determinantal singularities.

Corollary 3.20. *Denote the Newton polyhedron of the entry $a_{i,j}$ of a matrix $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{n,k},0)$ with holomorphic entries, $n\le k$, by $\Delta_{i,j}$. If the matrix A is Newton non-degenerate, then the local Euler obstruction of $(X^n_A,0)$ is*

$$\mathrm{Eu}_{X^n_A}(0)=\sum_{\substack{\Gamma\in\Phi\\ I\subset\{1,\dots,m\}}}\chi(\widetilde{\mathscr{L}^I},\Delta^{\Gamma,I}_{1,1}*\cdots*\Delta^{\Gamma,I}_{n,1},\cdots,\Delta^{\Gamma,I}_{1,k}*\cdots*\Delta^{\Gamma,I}_{n,k}).$$

In addition, if the germ of a matrix A is such that the Newton polyhedron of $a_{i,j}$ is independent of i, for all $j=1,\dots,k$, we have a much simpler formula.

Corollary 3.21. *Let X^n_A be an isolated determinantal singularity defined by the matrix germ $A:(\mathbb{C}^m,0)\to(M_{n,k},0)$, where A has holomorphic entries. Suppose that the Newton polyhedra $a_{i,j}$ does not depend on i and the function $a_{i,j}$ is convenient, $i=1,\dots,n$, $j=1,\dots,k$. Denote by Δ_j the Newton polyhedron of $a_{i,j}$, $j=1,\dots,k$. If the matrix A is Newton non-degenerate, then*

$$\mathrm{Eu}_{X^n_A}(0)=\sum_{\{j_1,\dots,j_q\}\subset\{1,\dots,k\}}\sum_{\substack{I\subset\{1,\dots,m\}\\ |I|\ge q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}$$
$$\times\sum_{\substack{a_{j_1},\dots,a_{j_q}\in\mathbb{N}\\ a_{j_1}+\dots+a_{j_q}=|I|-a}}|I|!\cdot(L^I)^a(\widetilde{\Delta}^I_{j_1})^{a_{j_1}}\cdots(\widetilde{\Delta}^I_{j_q})^{a_{j_q}}.$$

Example 3.22. *Let $(X^2_A,0)$ be the IDS defined by the matrix germ $A:(\mathbb{C}^4,0)\to(M_{2,3},0)$, where*

$$A=\begin{bmatrix} x+y+z+w & 2x+y+z+3w & 5x+7y+z+w\\ 2x+3y+7z+8w & 9x+5y+7z+11w & 13x+15y+9z+11w\end{bmatrix}.$$

The matrix A is Newton non-degenerate and its entries are convenient. Therefore, by Corollary 3.21,

$$\begin{aligned}
\mathrm{Eu}_{X_A^2}(0) = 3![&(L^{\{1,2,3\}})^1(\widetilde{\Delta}_1^{\{1,2,3\}})^1(\widetilde{\Delta}_2^{\{1,2,3\}})^1 + (L^{\{1,2,3\}})^1(\widetilde{\Delta}_1^{\{1,2,3\}})^1(\widetilde{\Delta}_3^{\{1,2,3\}})^1\\
&+(L^{\{1,2,3\}})^1(\widetilde{\Delta}_1^{\{1,2,3\}})^2(\widetilde{\Delta}_3^{\{1,2,3\}})^1 + (L^{\{1,2,4\}})^1(\widetilde{\Delta}_1^{\{1,2,4\}})^1(\widetilde{\Delta}_2^{\{1,2,4\}})^1\\
&+(L^{\{1,2,4\}})^1(\widetilde{\Delta}_1^{\{1,2,4\}})^1(\widetilde{\Delta}_3^{\{1,2,4\}})^1 + (L^{\{1,2,4\}})^1(\widetilde{\Delta}_1^{\{1,2,4\}})^2(\widetilde{\Delta}_3^{\{1,2,4\}})^1\\
&+(L^{\{1,3,4\}})^1(\widetilde{\Delta}_1^{\{1,3,4\}})^1(\widetilde{\Delta}_2^{\{1,3,4\}})^1 + (L^{\{1,3,4\}})^1(\widetilde{\Delta}_1^{\{1,3,4\}})^1(\widetilde{\Delta}_3^{\{1,3,4\}})^1\\
&+(L^{\{1,3,4\}})^1(\widetilde{\Delta}_1^{\{1,3,4\}})^2(\widetilde{\Delta}_3^{\{1,3,4\}})^1 + (L^{\{2,3,4\}})^1(\widetilde{\Delta}_1^{\{2,3,4\}})^1(\widetilde{\Delta}_2^{\{2,3,4\}})^1\\
&+(L^{\{2,3,4\}})^1(\widetilde{\Delta}_1^{\{2,3,4\}})^1(\widetilde{\Delta}_3^{\{2,3,4\}})^1 + (L^{\{2,3,4\}})^1(\widetilde{\Delta}_1^{\{2,3,4\}})^2(\widetilde{\Delta}_3^{\{2,3,4\}})^1]\\
&-4![(L)^2(\widetilde{\Delta}_1)^1(\widetilde{\Delta}_2)^1 + (L)^1(\widetilde{\Delta}_1)^1(\widetilde{\Delta}_2)^1 + (L)^1(\widetilde{\Delta}_1)^1(\widetilde{\Delta}_2)^2 + (L)^2(\widetilde{\Delta}_1)^1(\widetilde{\Delta}_3)^1\\
&+(L)^1(\widetilde{\Delta}_1)^1(\widetilde{\Delta}_3)^1 + (L)^1(\widetilde{\Delta}_1)^1(\widetilde{\Delta}_3)^2 + (L)^2(\widetilde{\Delta}_2)^1(\widetilde{\Delta}_3)^1 + (L)^1(\widetilde{\Delta}2)^1(\widetilde{\Delta}_3)^1 + (L)^1(\widetilde{\Delta}_2)^1(\widetilde{\Delta}_3)^2]\\
&-4\cdot 4!(L)^1(\widetilde{\Delta}_1)^1(\widetilde{\Delta}_2)^1(\widetilde{\Delta}_3)^1\\
&= 12-9-4=-1.
\end{aligned}$$

3 $\mathscr{G}$-equivalence and matrices with non-convenient entries

In 1976, Kouchnirenko [39] introduced a formula to compute the Milnor number of a germ of Newton non-degenerate function $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ in terms of its Newton polyhedron. If $f(x_1,\dots,x_m)$ is non-convenient, the author dealt with this function in the following way: he would add a function $\widetilde{f}(x_1,\dots,x_m) = \alpha_1 x_1^M + \cdots + \alpha_m x_m^M$ to f. In this way the function $g = f + \widetilde{f}$ is convenient. Moreover, Kouchnirenko proved that the Milnor number of f and g are equal for M big enough and, for sufficiently general coefficients $\alpha_1,\dots,\alpha_m$, the function g is Newton non-degenerate.

In this section, we use the ideas introduced by Kouchnirenko [39] and $\mathscr{G}$-equivalence of matrices to deal with matrices, which do not have convenient entries.

We start this section with a motivational example. Consider the IDS X_A^2 defined by the matrix

$$A = \begin{bmatrix} x+y & y+z & z+w \\ 2x+3y & 5y+7z & 9z+11w \end{bmatrix}.$$

The matrix A is Newton non-degenerate (see A.2), but its entries are not convenient. Consider now the matrix

$$\widetilde{A} = \begin{bmatrix} x^2+y^2+z^2+w^2 & x^2+y^2+z^2+w^2 & x^2+y^2+z^2+w^2w \\ 2x^2+y^2+2z^2+3w^2 & 5x^2+y^2+z^2+7w^2 & 9x^2+11y^2+z^2+w^2 \end{bmatrix}.$$

The matrix ${}_2A = A + \widetilde{A}$ is Newton non-degenerate and its entries are convenient. Moreover, A is $\mathscr{G}$-finitely determined and its determinacy bound is 1 (see Definition 1.27). Therefore, the germs A and ${}_2A$ are $\mathscr{G}$-equivalents and, for this reason, the determinantal singularities X_A^2 and $X_{{}_2A}^2$ have the same Euler obstruction. Therefore, we can apply Corollary 3.21 to the matrix ${}_2A$.

We can extend the above construction to any isolated determinantal singularity. Let ${}_MA = ({}_Ma_{i,j})$ denote the matrix defined by ${}_Ma_{i,j} = a_{i,j} + \sum_{l=1}^{m} \alpha_{i,j}^l x_l^M$, we denote by ${}_M\Delta_{i,j}$ the Newton polyhedron of ${}_MA$.

Theorem 3.23. *Let X_A^n be the IDS defined by the matrix germ $A:(\mathbb{C}^m,0)\to(M_{n,k},0)$ with non-convenient entries. If the matrix A is Newton non-degenerate, then the local Euler obstruction of $(X_A^n,0)$ is*

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{\substack{\Gamma\in\Phi\\ I\subset\{1,\dots,m\}}}\chi(\widetilde{\mathscr{L}^I},{}_M\Delta_{1,1}^{\Gamma,I}*\cdots*{}_M\Delta_{n,1}^{\Gamma,I},\dots,{}_M\Delta_{1,k}^{\Gamma,I}*\cdots*{}_M\Delta_{n,k}^{\Gamma,I}),$$

where M is greater than the determinacy bound of A.

Proof. Let $(X_A^n,0)$ be an isolated determinantal singularity, defined by the matrix $A=(a_{i,j})$. By Theorem 1.28, X_A^n is finitely determined. Consider the matrix $\widetilde{A}=(\widetilde{a}_{i,j})$, where $\widetilde{a}_{i,j}=\sum_{l=1}^{m}\alpha_{i,j}^l x_i^M$, $i=1,\dots,n$ and $j=1,\dots,k$. Suppose that M is greater than the determinacy bound of A, then the matrices A and ${}_MA=A+\widetilde{A}$ are $\mathscr{G}$-equivalents. Therefore, X_A^n is isomorph to $X_{{}_MA}^n$ and they have the same Euler obstruction.

Since A is Newton non-degenerate, then ${}_MA$ is Newton non-degenerate for almost all coefficients $\alpha_{i,j}^l$ (see [39, Theorem 3.7]), therefore, we can choose appropriate $\alpha_{i,j}^l$ such that each entry ${}_Ma_{i,j}$ is convenient, $i=1,\dots,n$ and $j=1,\dots,k$, and the matrix ${}_MA$ is Newton non-degenerate. Therefore, by Corollary 3.20, the local Euler obstruction of X_A^n is

$$\mathrm{Eu}_{X_A^n}(0)=\mathrm{Eu}_{X_{{}_MA}^n}(0)=\sum_{\substack{\Gamma\in\Phi\\ I\subset\{1,\dots,m\}}}\chi(\widetilde{\mathscr{L}^I},{}_M\Delta_{1,1}^{\Gamma,I}*\cdots*{}_M\Delta_{n,1}^{\Gamma,I},\dots,{}_M\Delta_{1,k}^{\Gamma,I}*\cdots*{}_M\Delta_{n,k}^{\Gamma,I}).$$

□

As usual, if the Newton polyhedron of $a_{i,j}$ does not depend on i, we have the following simpler formula.

Corollary 3.24. *Let $(X_A^n,0)$ be the IDS defined by the matrix germ $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{n,k},0)$ with non-convenient entries. Denote by Δ_j the Newton polyhedron of $a_{i,j}$. If the matrix A is Newton non-degenerate, then the local Euler obstruction of X_A^n is*

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{\{j_1,\dots,j_q\}\subset\{1,\dots,k\}}\ \sum_{\substack{I\subset\{1,\dots,m\}\\ |I|\geq q+1}}\ \sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}$$
$$\times\sum_{\substack{a_{j_1},\dots,a_{j_q}\in\mathbb{N}\\ a_{j_1}+\cdots+a_{j_q}=|I|-a}}|I|!\cdot(L^I)^a({}_M\widetilde{\Delta}_{j_1}^I)^{a_{j_1}}\cdots({}_M\widetilde{\Delta}_{j_q}^I)^{a_{j_q}}.$$

Example 3.25. *Consider the matrix germ A from the above construction, i.e.,*

$$A=\begin{bmatrix} x+y & y+z & z+w\\ 2x+3y & 5y+7z & 9z+11w\end{bmatrix}.$$

Then, by Corollary 3.24,

$$\mathrm{Eu}_{X_A^2}(0) = \sum_{\{j_1,\dots,j_q\}\subset\{1,2,3\}} \sum_{\substack{I\subset\{1,2,3,4\}\\ |I|\geq q+1}} \sum_{a=1}^{|I|-q} (-1)^{|I|+1} \binom{|I|+q-a-2}{q-2} \times \sum_{\substack{a_{j_1},\dots,a_{j_q}\in\mathbb{N}\\ a_{j_1}+\cdots+a_{j_q}=|I|-a}} |I|!\cdot (L^I)^a ({}_2\widetilde{\Delta}^I_{j_1})^{a_{j_1}} \cdots ({}_2\widetilde{\Delta}^I_{j_q})^{a_{j_q}} = -1.$$

We used OSCAR [51] to compute this local Euler obstruction (see Example A.6).

4 Vanishing Euler characteristic and Newton polyhedra

The purpose of this section is to use the same ideas as in Section 3 and apply $\mathscr{G}$-equivalence of matrices also to compute the vanishing Euler characteristic of an IDS X^s_A.

Let $\mathscr{A}:(\mathbb{C}^m\times\mathbb{C},0)\to(M_{n,k},0)$ be a determinantal smoothing of X^n_A. We take a small enough representative $\mathscr{A}:B_\varepsilon\times D\to\mathbb{C}$, where B_ε is the open ball with radius ε centered at $0\in\mathbb{C}^m$ and D is small enough open ball centered at the origin in $\mathbb{C}$, such that $X^n_{A_t}$ is smooth and $\mathrm{rank}(A_t(x))=n-1$ for all $x\in X^n_{A_t}$ and all $t\in D\setminus\{0\}$. By Theorem 1.80

$$\nu(X^n_A,0)=(-1)^d(\chi(X_t)-1),$$

for all $t\in D\setminus\{0\}$, where d is the dimension of X^s_A.

Since $\mathscr{A}$ is a determinantal smoothing, then $(X^n_{\mathscr{A}},0)\subset\mathbb{C}^{m+1}$ is an isolated determinantal singularity. Moreover, the determinantal smoothing $\mathscr{A}$ can be chosen such that the matrix $\mathscr{A}$ is Newton non-degenerate and the variable t appears in every entry of $\mathscr{A}$ (see [45, Proof of Theorem 3.4]). We consider the projection onto the second factor

$$\begin{array}{rccc}\pi: & (\mathbb{C}^m\times\mathbb{C},0) & \to & (\mathbb{C},0)\\ & (x,t) & \mapsto & t\end{array},$$

which is a finite map germ whose generic fiber is $\pi|^{-1}_{X^n_{\mathscr{A}}}(t)=X^n_{A_t}$. Therefore, we can compute the vanishing Euler characteristic, by computing the Euler characteristic of the Milnor fiber of π restricted to the total space $X^n_{\mathscr{A}}$, *i.e.*,

$$\nu(X^n_A,0)=(-1)^d(\chi(X^n_{\mathscr{A}}\cap\pi^{-1}(\delta)\cap B_\varepsilon)-1),$$

where $\delta\in\mathbb{C}\setminus\{0\}$ is such that $|\delta|<<1$.

We observe that the set of critical points of $\pi|X^n_{\mathscr{A}}$ is $X^n_A\times\{0\}\subset\mathbb{C}^m\times\mathbb{C}$. Therefore, if the matrix A is Newton non-degenerate, then the projection π is Newton non-degenerate with respect to $\mathscr{A}$.

Our first challenge to compute this Euler characteristic using Corollary 3.17 comes from the fact that π is not convenient. However, we apply $\mathscr{G}$-equivalence of matrices to overcome this challenge.

Firstly, consider the function ${}_M\pi : (\mathbb{C}^m \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ defined by ${}_M\pi(x_1, \dots, x_m, t) = t + \beta_1 x_1^M + \cdots + \beta_m x_m^M$. Consider ${}_M\mathscr{A} = \mathscr{A} + \widetilde{A}$, where $\widetilde{A} = (\widetilde{a}_{i,j})$ and $\widetilde{a}_{i,j} = \sum_{l=1}^{m} \alpha_{i,j}^l x_l^M$ (we do not need to add a monomial αt^M, because we can choose a determinantal smoothing with t in every entry). Suppose that M is greater than the determinacy bound of $\mathscr{A}$, therefore, $X^n_{\mathscr{A}}$ and $X^n_{{}_MA}$ are isomorph. Moreover, since M is greater than the determinacy bound of $\mathscr{A}$, then ${}_M\pi^{-1}(t) \cap X^n_{\mathscr{A}}$ is isomorph to $X^n_{A_t}$. Therefore, we can replace the Euler characteristic of $X^n_{A_t}$ by the Euler characteristic of ${}_M\pi^{-1}(t) \cap X^n_{\mathscr{A}}$ to compute the vanishing Euler characteristic of X^n_A. Hence,

$$\nu(X^n_A, 0) = (-1)^d (\chi(X^n_{\mathscr{A}} \cap {}_M\pi^{-1}(\delta) \cap B_\varepsilon) - 1).$$

In addition, if A is Newton non-degenerate, then the projection π is Newton non-degenerate with respect to $\mathscr{A}$ and we can choose the coefficients β_l and $\alpha^l_{i,j}$, $l = 1, \dots, m$ such that ${}_M\pi$ is Newton non-degenerate with respect to ${}_M\mathscr{A}$ (see [39, Theorem 3.7]). Hence, we have the following result.

Theorem 3.26. *Let $A : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a germ of a Newton non-degenerate matrix. Suppose that $(X^n_A, 0)$ is an isolated determinantal singularity and let $\mathscr{A} = (\mathscr{A}_{i,j}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be its determinantal smoothing. Denote by $\Delta_{i,j}$ the Newton polyhedron of $\mathscr{A}_{i,j}$. Then we have the following relation.*

$$1 + (-1)^{\dim X^n_A} \nu(X^n_A, 0) = \sum_{\substack{\Gamma \in \Phi \\ I \subset \{1, \dots, m\}}} \chi({}_M\widetilde{\Delta}^I, {}_M\Delta^{\Gamma,I}_{1,1} * \cdots * {}_M\Delta^{\Gamma,I}_{n,1}, \dots, {}_M\Delta^{\Gamma,I}_{1,k} * \cdots * {}_M\Delta^{\Gamma,I}_{n,k}),$$

where ${}_M\Delta$ is the Newton polyhedron of ${}_M\pi$.

In the case, where the Newton polyhedron of $\mathscr{A}_{i,j}$ does not depend on i, for all $j = 1, \dots, k$, we have the following formula.

Corollary 3.27. *Let $A : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a germ of a Newton non-degenerate matrix. Suppose that $(X^n_A, 0)$ is an isolated determinantal singularity and $\mathscr{A} = (\mathscr{A}_{i,j}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ is its determinantal smoothing. Let Δ_j be the Newton polyhedron of $\mathscr{A}_{i,j}$. Then*

$$\begin{aligned} 1 + (-1)^{\dim X^n_A} \nu(X^n_A, 0) = \sum_{\{j_1, \dots, j_q\} \subset \{1, \dots, k\}} \sum_{\substack{I \subset \{1, \dots, m\} \\ |I| \geq q+1}} \sum_{a=1}^{|I|-q} (-1)^{|I|+k-n} \binom{|I| + q - a - 2}{n + q - k - 1} \\ \times \sum_{\substack{a_{j_1}, \dots, a_{j_q} \in \mathbb{N} \\ a_{j_1} + \cdots + a_{j_q} = |I| - a}} |I|! \cdot ({}_M\widetilde{\Delta}^I_0)^a ({}_M\widetilde{\Delta}^I_{j_1})^{a_{j_1}} \cdots ({}_M\widetilde{\Delta}^I_{j_q})^{a_{j_q}}. \end{aligned}$$

Example 3.28. *Consider the matrix germ*

$$A = \begin{bmatrix} x+y & y+z & z+w \\ 2x+3y & 5y+7z & 9z+11w \end{bmatrix}.$$

$X^2_A \subset \mathbb{C}^4$ is an isolated determinantal singularity. Let $\mathscr{A} = A + B$ denote its smoothing, where

$$B = \frac{1}{100} \begin{bmatrix} 6t & -8t & 5t \\ t & 8t & 7t \end{bmatrix}.$$

As in the previous section, we consider the matrix

$$\widetilde{A} = \begin{bmatrix} x^2+y^2+z^2+w^2 & x^2+y^2+z^2+w^2 & x^2+y^2+z^2+w^2w \\ 2x^2+y^2+2z^2+3w^2 & 5x^2+y^2+z^2+7w^2 & 9x^2+11y^2+z^2+w^2 \end{bmatrix}.$$

The matrix ${}_2\mathscr{A} = \mathscr{A} + \widetilde{A}$ *is Newton non-degenerate (see Example A.4) and it is* $\mathscr{G}$*-equivalent to* $\mathscr{A}$*.*

Consider the projection $\pi : \mathbb{C}^4 \times \mathbb{C} \to \mathbb{C}$ *defined by* $\pi(x,y,z,w,t) = t$ *and the function* ${}_2\pi : \mathbb{C}^4 \times \mathbb{C} \to \mathbb{C}$*, where* ${}_2\pi(x,y,z,w,t) = x^2+y^2+z^2+w^2+t$*. Consider* $0 < |\delta| << 1$ *and set* $t = \delta - x^2 - y^2 - z^2 - w^2$*. Therefore,*

$${}_2\pi^{-1}(\delta) \cap X^2_{{}_2\mathscr{A}} \cong X^2_{{}_2A_\delta},$$

where ${}_2A_\delta = A + \widetilde{A} + B_\delta$ *and*

$$B_\delta = \frac{1}{100}\begin{bmatrix} 6(\delta-x^2-y^2-z^2-w^2) & -8(\delta-x^2-y^2-z^2-w^2) & 5(\delta-x^2-y^2-z^2-w^2) \\ (\delta-x^2-y^2-z^2-w^2) & 8(\delta-x^2-y^2-z^2-w^2) & 7(\delta-x^2-y^2-z^2-w^2) \end{bmatrix}.$$

We have

$$X^2_{A_t} \cong \pi^{-1}(t) \cap X^2_{\mathscr{A}} \cong {}_2\pi^{-1}(t) \cap X^2_{{}_2\mathscr{A}}.$$

Moreover, since 2 is greater than the determinacy bound of A_δ*, the matrix* ${}_2A_\delta$ *is* $\mathscr{G}$*-equivalent to* A_δ*. Thus* $X^2_{A_\delta} \cong X^2_{{}_2A_\delta}$*. In addition, setting* $\pi(x,y,z,w,t) = t = \delta$*, we obtain* $X^2_{A_t} \cap \pi^{-1}(\delta) \cong X^2_{A_\delta}$*.*

Therefore,

$$\nu(X^2_A, 0) = (-1)^2(\chi(X^2_{A_t} \cap \pi^{-1}(\delta) \cap B_\varepsilon) - 1) = (-1)^2(\chi(X^2_{{}_2\mathscr{A}} \cap {}_2\pi^{-1}(\delta) \cap B_\varepsilon) - 1).$$

Thus, computing the Euler characteristic of the Milnor fiber of ${}_2\pi$ *restricted to* $X^2_{{}_2\mathscr{A}}$*, we obtain*

$$1 + \nu(X^2_A, 0) = \sum_{\{j_1,\dots,j_q\}\subset\{1,2,3\}} \sum_{\substack{I\subset\{1,\dots,5\} \\ |I|\geq q+1}} \sum_{a=1}^{|I|-q} (-1)^{|I|+1} \binom{|I|+q-a-2}{q-2} \times \sum_{\substack{a_{j_1},\dots,a_{j_q}\in\mathbb{N} \\ a_{j_1}+\cdots+a_{j_q}=|I|-a}} |I|! \cdot ({}_2\widetilde{\Delta}^I_0)^a ({}_2\widetilde{\Delta}^I_{j_1})^{a_{j_1}} \cdots ({}_2\widetilde{\Delta}^I_{j_q})^{a_{j_q}} = 2.$$

Hence, $\nu(X^2_A, 0) = 2 - 1 = 1$*.*

The above Euler characteristic of the Milnor fiber was computed with OSCAR [51] (see Example A.7).

CHAPTER 4

GL-equivalence and Newton polyhedra

The purpose of this chapter is to present methods which allow us to compute more concrete examples, where we can apply the results from Chapter 3. We apply row and column operations to the germ of a matrix $A = (a_{i,j})$ in order to obtain a new matrix $\widetilde{A}$ such that up to constants all the monomial components of each $a_{i,j}$ appear on each entry of $\widetilde{A}$ and, therefore, the Newton polyhedron of each entry of $\widetilde{A}$ will be the same. After this process, the condition that the Newton polyhedron of each column of the matrix $\widetilde{A}$ is convenient will be more satisfiable.

1 Equivalent matrices and Newton polyhedra

We start this section defining an equivalence of matrices, which is slightly different from the definition of $\mathscr{G}$-equivalence (Definition 1.25).

Definition 4.1. *The germs of matrices* $A, \widetilde{A} : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ *are said to be* **GL-equivalent** *if they belong to the same equivalence class of the following relation:*

$$A \sim \widetilde{A} \Leftrightarrow \exists P \in GL_n(\mathbb{C}), \exists Q \in GL_k(\mathbb{C}) : \widetilde{A} = P \cdot A \cdot Q,$$

where $GL_l(\mathbb{C})$ *is the group of order* l *invertible matrices with entries in* $\mathbb{C}$.

In contrast to Definition 1.25, the above definition does not involve a change of coordinates on the source. Therefore, if the germs $A, \widetilde{A} : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ are GL-equivalent, then $X^s_A = X^s_{\widetilde{A}}$.

Furthermore, the action $GL_n(\mathbb{C}) \times GL_k(\mathbb{C})$ on the space of matrices $M_{n,k}(\mathscr{O}_m)$ is a subgroup of the $\mathscr{G}$-action (see [14, 52]). Since the invariants such as polar multiplicities, Euler obstruction, vanishing Euler characteristic only depend on the $\mathscr{G}$-equivalence class, the GL-equivalence of matrices does not alter them.

Definition 4.2. *If $A = (a_{i,j}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ is a germ of a matrix with polynomial entries, we denote by*

$$\mathrm{supp(A)} := \bigcup_{\substack{i \in \{1,\dots,n\} \\ j \in \{1,\dots,k\}}} \mathrm{supp(a_{i,j})}.$$

The **Newton polyhedron** *of A, which we denote by Δ_A, is the Newton polyhedron determined by* supp(A).

Given a germ of a matrix A, there is always a germ $\widetilde{A}$ which is GL-equivalent to A such that the Newton polyhedron of each entry of $\widetilde{A}$ is equal to Δ_A. Since both the matrices A and $\widetilde{A}$ define the same singularity, therefore, if A defines a determinantal singularity, then whenever we need the Newton polyhedron of each entry of a matrix A and $\widetilde{A}$ is Newton non-degenerate, we can replace them by Δ_A.

Definition 4.3. *Let A be a germ of a matrix with polynomial entries. We say that $\widetilde{A}$ is* **GL-equivalent to** *A* **with respect to** *Δ_A, if $\widetilde{A}$ is GL-equivalent to A and the Newton polyhedron of each entry of $\widetilde{A}$ is equal to Δ_A.*

Example 4.4. *Consider the determinantal singularity given by the germ $A : (\mathbb{C}^4, 0) \to (M_{2,3}, 0)$, where*

$$A = \begin{bmatrix} x-z & y-w & z-w \\ y-w & z-w & w+x \end{bmatrix}.$$

None of the entries of A has convenient Newton polyhedron $\Delta_{i,j}$. Now, consider the matrices

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}. \tag{1.1}$$

We obtain the germ $\widetilde{A} = P \cdot A \cdot Q$, given by

$$\widetilde{A} = \begin{bmatrix} 2x+2y+z-3w & 2x+3y+2z-5w & 3x+2y+2z-3w \\ 3x+3y+2z-4w & 3x+4y+4z-7w & 5x+3y+3z-3w \end{bmatrix}.$$

In this case, every function $\widetilde{a}_{i,j}$ is convenient and has the same Newton polyhedron, $i = 1,\dots,n$, $j = 1,\dots,k$.

We observe that the GL-equivalence may affect the Newton non-degeneracy conditions. In the following we exemplify this fact in the next example.

Example 4.5. (i) *In Example 4.4, both matrices A and $\widetilde{A}$ are Newton non-degenerate. Then we can apply the results from Chapter 3, directly to $\widetilde{A}$.*

(ii) *Now, consider the determinantal singularity given by the germ of the matrix*

$$B = \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}.$$

For every collection of positive weights $w = (w_1, w_2, w_3, w_4)$*, the matrix* $B^w = (b^w_{i,j}) = B$*. Since* $X^2_B \cap (\mathbb{C}^*)^4$ *is a smooth determinantal variety, the matrix* B *is Newton non-degenerate.*

Using the matrices P *and* Q *from* Eq. (1.1)*, we obtain the germ* $\widetilde{B} = P \cdot B \cdot Q$ *given by the matrix*

$$\begin{bmatrix} x+2y+2z+w & x+3y+3z+w & x+2y+3z+2w \\ x+3y+3z+2w & x+4y+5z+2w & x+3y+4z+4w \end{bmatrix}.$$

Consider the weight $w = (1,2,2,2)$*, then*

$$\widetilde{B}^w = \begin{bmatrix} x & x & x \\ x & x & x \end{bmatrix}.$$

Therefore $X^2_{\widetilde{B}^w} \cap \mathbb{C}^{*4} = \mathbb{C}^4 \cap \mathbb{C}^{*4}$ *is not a determinantal variety. Hence,* $\widetilde{B}$ *is not Newton non-degenerate.*

Motivated by this example, we present the following definition.

Definition 4.6. *Let* $A = (a_{i,j}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ *be a germ of a matrix with polynomial entries.*

(i) *The matrix* A *is* **weakly Newton non-degenerate with respect to** Δ_A *if for every bounded face* Γ *of* Δ_A *the matrix* $A|_\Gamma$ *is Newton non-degenerate;*

(ii) *The matrix* A *is* **Newton non-degenerate with respect to** Δ_A *if for every bounded face* Γ *of* Δ_A *the matrix* $A|_\Gamma$ *is strongly Newton non-degenerate;*

(iii) *The germ of a function* f *is* **Newton non-degenerate with respect to** A *and* Δ_A *if for every bounded face* Γ *of* Δ_A *the function* $f|_\Gamma$ *is Newton non-degenerate with respect to the matrix* $A|_\Gamma$*.*

We choose to define Newton non-degeneracy trough the bounded faces of Δ_A, in order to be able to verify this condition directly to the matrix A. The definitions trough bounded faces and strictly positive weights are equivalent (see [18] and [19]).

Example 4.7. *Let* A *be the matrix from Example 4.4 is Newton non-degenerate with respect to its Newton polyhedron* Δ_A*.*

Using this ideas we show, in the next sections, how we can compute invariants of determinantal singularities using the polyhedron Δ_A.

2 Multiplicity

Corollary 3.13 computes the multiplicity of a determinantal variety X^n_A when A is weakly Newton non-degenerate, each entry $a_{i,j}$ is convenient and the Newton polyhedron of $a_{i,j}$ does not depend on i, $i = \{1, \ldots, n\}$ $j = 1, \ldots, k$. Using this result and equivalence of matrices we present the next result, which depends only on the Newton polyhedron of A.

Proposition 4.8. *Let $A = (a_{i,j}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a germ of a matrix with polynomial entries such that its Newton polyhedron Δ_A touches all coordinate axes. If A is weakly Newton non-degenerate with respect to Δ_A, then A defines the germ of a determinantal singularity X_A^n, whose multiplicity is*

$$m(X_A^n, 0) = \binom{k}{k-n+1} \cdot m! \cdot \widetilde{\Delta}_A^{k-n+1} L^{m-k+n-1}.$$

Proof. Let $\widetilde{A}$ be a germ of a matrix equivalent to A with respect to Δ_A. Then $\widetilde{A}$ is weakly Newton non-degenerate and the entries $\widetilde{a}_{i,j}$ are convenient. By Corollary 3.13, the variety $X_{\widetilde{A}}^n$ is a determinantal singularity and it has multiplicity

$$m(X_{\widetilde{A}}^n, 0) = \sum_{0<j_0<\cdots<j_{k-n}\leq k} m! \cdot \widetilde{\Delta}_{j_0}^1 \cdots \widetilde{\Delta}_{j_{k-n}}^1 L^{m-k+n-1}, \tag{2.1}$$

where Δ_j is the Newton polyhedron of the function $\widetilde{a}_{i,j} : \mathbb{C}^m \to \mathbb{C}$. Since $X_A^n = X_{\widetilde{A}}^n$ and $\Delta_j = \Delta_A$, for all $j = 1, \ldots, k$, by Eq. (2.1), the multiplicity of $X_A^n = X_{\widetilde{A}}^n$ is

$$m(X_A^n, 0) = m(X_{\widetilde{A}}^n, 0) = \sum_{0<j_0<\cdots<j_{k-n}\leq k} m! \cdot \widetilde{\Delta}_A^{k-n+1} L^{m-k+n-1} = \binom{k}{k-n+1} \cdot m! \cdot \widetilde{\Delta}_A^{k-n+1} L^{m-k+n-1}.$$

□

Example 4.9. *Let A be the germ given by the matrix*

$$A = \begin{bmatrix} x - z^2 & y^3 - w^4 & z^2 - w^4 \\ y^3 - w^4 & z^2 - w^4 & w^4 + x \end{bmatrix}$$

and P and Q from Eq. (1.1), *then we obtain the germ $\widetilde{A} = P \cdot A \cdot Q$, given by the matrix*

$$\begin{bmatrix} 2x + 2y^3 + z^2 - 3w^4 & 2x + 3y^3 + 2z^2 - 5w^4 & 3x + 2y^3 + 2z^2 - 3w^4 \\ 3x + 3y^3 + 2z^2 - 4w^4 & 3x + 4y^3 + 4z^2 - 7w^4 & 5x + 3y^3 + 3z^2 - 3w^4 \end{bmatrix}.$$

The Newton polyhedron of each entry of $\widetilde{A}$, Δ_A, is convenient and the matrix $\widetilde{A}$ is Newton non-degenerate (see A.3), then the matrix A is Newton non-degenerate with respect to Δ_A. By Proposition 4.8, X_A^2 is a determinantal singularity and its multiplicity is

$$m(X_A^2, 0) = \binom{3}{2} \cdot 4! \cdot \widetilde{\Delta}_A^2 L^2 = \binom{3}{2} = 3 \cdot 4! \cdot \frac{2}{4!} = 6.$$

Here, we use OSCAR [51] to compute the mixed volume $\widetilde{\Delta}_A^2 L^2$ (see Example A.5).

We observe that, we could not apply Corollary 3.13 to the matrix A. However, we can apply to a germ $\widetilde{A}$, which is equivalent to A with respect to Δ_A. This is the same as applying Proposition 4.8 directly to the matrix A.

Now, we compute the multiplicity of a determinantal singularity defined by a matrix A with linear initial part. This class of determinantal singularities includes the ones defined by matrices with linear entries, which is very important, for instance, in [1], the authors find a class of essentially isolated determinantal singularities defined by a matrix with homogeneous entries using a matrix with linear entries.

Corollary 4.10. *Under the conditions of the Proposition 4.8, if the matrix germ* $A : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ *has linear initial part, then the multiplicity of* X_A^n *is given by*

$$m(X_A^n, 0) = \binom{k}{k-n+1}.$$

Proof. It follows directly from the fact that, in this case, $\widetilde{\Delta}_A = L$. □

Example 4.11. *Consider the germ* A*, given in Example 4.4. The Newton polyhedron* Δ_A *is convenient,* A *is Newton non-degenerate with respect to* Δ_A *and* $\widetilde{\Delta}_A = L$*. By the last corollary,* X_A^2 *is a determinantal singularity and its multiplicity is*

$$m(X_A^2, 0) = \binom{3}{2} = 3.$$

In the following, we present an example where the matrix A is not Newton non-degenerate with respect to Δ_A, but we use the equivalence of matrices to find an equivalent matrix which is Newton non-degenerate, using one Newton polyhedra for the first two columns of A and another Newton polyhedron for the last column.

Example 4.12. *Consider the germ* A *given by the matrix*

$$A = \begin{bmatrix} z+w-y & y+x+z & x-y-w \\ x-y-w & w-z-y & y^2+z^k \end{bmatrix}$$

and the matrix

$$Q = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

Then, we have the equivalent matrix $\widetilde{A} = Q \cdot A$ *given by*

$$\begin{bmatrix} 2x-3y+z+w & x-y-z+w & x-y+y^2+2z^k-w \\ 3x-4y+z+2w & x-2y-2z+2w & x-y+2y^2+3z^k-w \end{bmatrix}.$$

The matrix $\widetilde{A}$ *is Newton non-degenerate, in this case* $\widetilde{\Delta}_1 = \widetilde{\Delta}_2 = L$ *and* Δ_3 *is the Newton polyhedron with vertices* $\{(1,0,0,0),(0,1,0,0),(0,0,k,0),(0,0,0,1),(0,0,0,0)\}$*. Therefore, we can apply Corollary 3.13 to* $\widetilde{A}$ *and obtain*

$$m(X_A^n, 0) = 4! \cdot L^4 + 4! \cdot \widetilde{\Delta}_3^1 L^3 + 4! \cdot \widetilde{\Delta}_3^1 L^3 = 3.$$

3 Local Euler obstruction

We can apply the same process as in Proposition 4.8, in order to compute the Euler characteristic of the Milnor of a function restricted to a determinantal singularity and, consequently, compute the local Euler obstruction of an isolated determinantal singularity in terms of the Newton polyhedra of the matrix.

Proposition 4.13. *Let X_A^n be a determinantal singularity given by the matrix $A=(a_{i,j}):(\mathbb{C}^m,0)\to(M_{n,k},0)$, where A has holomorphic entries and Newton polyhedron Δ_A. Suppose that Δ_A touches all the coordinate axes and $m\leq 2(k-n+2)$.*

i) If A is Newton non-degenerate with respect to Δ_A, then X_A^n is smooth outside the origin.

ii) If the map germ $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ is convenient and f is Newton non-degenerate with respect to A and Δ_A, then the Euler characteristic of the Milnor fiber of $f|_{X_A^n}$ is given by

$$\chi(F_0)=\sum_{q=k-n+1}^{k}\sum_{\substack{I\subset\{1,\dots,m\}\\|I|\geq q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}\times\binom{|I|-a-1}{q-1}\binom{k}{q}\cdot|I|!\cdot(\widetilde{\Delta}_f^I)^a(\widetilde{\Delta}_A^I)^{|I|-a}.$$

Proof. Let $\widetilde{A}$ be a germ equivalent to A with respect to Δ_A. By Corollary 3.18, the determinantal singularity $X_A^n=X_{\widetilde{A}}^n$ is smooth outside the origin and

$$\chi(F_0)=\sum_{\substack{a\in\mathbb{N},\,I\subset\{1,\dots,m\}\\\{j_1,\dots,j_q\}\subset\{1,\dots,k\}}}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}\times\sum_{\substack{a_{j_1},\dots,a_{j_q}\in\mathbb{N}\\a_{j_1}+\cdots+a_{j_q}=|I|-a}}|I|!\cdot(\widetilde{\Delta}_f^I)^a(\widetilde{\Delta}_{j_1}^I)^{a_{j_1}}\cdots(\widetilde{\Delta}_{j_q}^I)^{a_{j_q}},$$

where Δ_j is the Newton polyhedron of $\widetilde{a}_{i,j}$. Since $\Delta_A=\Delta_j$, for all $j\in\{1,\dots,k\}$, we have

$$\chi(F_0)=\sum_{\substack{a\in\mathbb{N},\,I\subset\{1,\dots,m\}\\\{j_1,\dots,j_q\}\subset\{1,\dots,k\}}}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}\times\sum_{\substack{a_{j_1},\dots,a_{j_q}\in\mathbb{N}\\a_{j_1}+\cdots+a_{j_q}=|I|-a}}|I|!\cdot(\widetilde{\Delta}_f^I)^a(\widetilde{\Delta}_A^I)^{a_{j_1}}\cdots(\widetilde{\Delta}_A^I)^{a_{j_q}}.$$

Furthermore, the number of combinations for the sum $a_{j_1}+\cdots+a_{j_q}=|I|-a$ is $\binom{|I|-a-1}{q-1}$, then we have

$$\chi(F_0)=\sum_{\substack{a\in\mathbb{N},\,I\subset\{1,\dots,m\}\\\{j_1,\dots,j_q\}\subset\{1,\dots,k\}}}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}\binom{|I|-a-1}{q-1}\binom{k}{q}\cdot|I|!\cdot(\widetilde{\Delta}_f^I)^a(\widetilde{\Delta}_A^I)^{|I|-a}.$$

We assume $\binom{r}{s}=0$ for $r\notin\{0,\dots,s\}$, then, all terms in this sum are equal to zero, except the terms with $|I|-a\geq q>k-n$. Hence,

$$\chi(F_0)=\sum_{q=k-n+1}^{k}\sum_{\substack{I\subset\{1,\dots,m\}\\|I|\geq q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}\binom{|I|-a-1}{q-1}\binom{k}{q}\cdot|I|!\cdot(\widetilde{\Delta}_f^I)^a(\widetilde{\Delta}_A^I)^{|I|-a}.$$

□

As a consequence of Proposition 4.13, we can also compute the local Euler obstruction of a determinantal variety with isolated singularity using the Newton polyhedron of A.

Corollary 4.14. *Let X_A^n be an isolated determinantal singularity defined by the germ $A:(\mathbb{C}^m,0)\to(M_{n,k},0)$, where A has holomorphic entries. Suppose that Δ_A touches all coordinate axes. If the matrix A is Newton non-degenerate with respect to Δ_A, then*

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{q=k-n+1}^{k}\sum_{\substack{I\subset\{1,\dots,m\}\\ |I|\geq q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}\times\binom{|I|-a-1}{q-1}\binom{k}{q}\cdot|I|!\cdot(L^I)^a(\widetilde{\Delta}_A^I)^{|I|-a}.$$

Example 4.15. *Let $(X,0)\subset(\mathbb{C}^m,0)$ be an ICIS defined by the polynomial functions $f_1,\dots,f_k$, where $f_i:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ for $i=1,\dots,k$. Since $(X,0)$ is an ICIS, $(X,0)$ is also a determinantal singularity given by $X=X_A^1$, where $A=\begin{bmatrix} f_1 & \cdots & f_k\end{bmatrix}$. If Δ_A touches all the coordinate axes and A is Newton non-degenerate with respect to Δ_A, then the local Euler obstruction of X is*

$$\mathrm{Eu}_X(0)=\sum_{\substack{I\subset\{1,\dots,m\}\\ |I|\geq k+1}}\sum_{a=1}^{|I|-k}(-1)^{|I|+k-1}\cdot\binom{|I|-a-1}{k-1}\cdot|I|!\cdot(L^I)^a(\widetilde{\Delta}_A^I)^{|I|-a}.$$

For instance, consider the ICIS X_A^1 where $A:(\mathbb{C}^3,0)\to(M_{1,2},0)$ is defined by

$$A(x,y,z)=\begin{bmatrix} x^2+y^2 & xy+z^k\end{bmatrix}.$$

The matrix A is Newton non-degenerate with respect to Δ_A, therefore

$$\mathrm{Eu}_{X_A^1}(0)=\binom{1}{1}\cdot 3!\cdot(L)^1(\widetilde{\Delta}_A)^2=3!\cdot\frac{4}{3!}=4.$$

Another way to make this computation is using Eq. (5.1) *together with Proposition 4.8. Since, X_A^1 is a curve, then*

$$\mathrm{Eu}_{X_A^1}(0)=m_0(X_A^1,0)=\binom{2}{2}3!(\widetilde{\Delta}_A)^2(L)^1=4.$$

Remark 4.16. In the last example, we can see that the local Euler obstruction can be independent of k in the A_k, D_k and S_k series of singularities. There are many cases where the local Euler obstruction depends on characteristics which are not related to every exponent of every monomial. For instance, the Euler obstruction of affine toric surfaces depends only in the minimum dimension of the embedding (see [29]) and the local Euler obstruction of images of stable maps with corank 1 is always 1 (see [38]).

In the following we present a class of IDS for which the local Euler obstruction is given just as a sum of binomial coefficients.

Corollary 4.17. *When a germ A satisfies the conditions of Corollary 4.14 and A has linear initial part, we have the following formula for the local Euler obstruction of X_A^n:*

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{q=k-n+1}^{k}\sum_{|I|=q+1}^{m}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}\binom{|I|-a-1}{q-1}\binom{k}{q}\binom{m}{|I|}.$$

Example 4.18. *Consider A, the germ given in Example 4.4. Since the matrix A have linear entries and the matrix A is Newton non-degenerate with respect to Δ_A, by Corollary 4.17,*

$$\mathrm{Eu}_{X_A^2}(0)=\sum_{q=2}^{3}\sum_{|I|=q+1}^{4}\sum_{a=1}^{|I|-q}(-1)^{|I|+1}\binom{|I|+q-a-2}{q-2}\binom{|I|-a-1}{q-1}\binom{3}{q}\binom{4}{|I|}=-1.$$

4 Whitney equisingularity

Lastly, we can also present a version of Theorem 2.30 using the Newton polyhedron of a matrix.

Corollary 4.19. *Let $\left\{(X_{A_t}^s,0)\right\}_{t\in D}$, be a d-dimensional family of determinantal singularities, defined by the germ of matrices $A_t=(a_{i,j}^t):(\mathbb{C}^m,0)\to(M_{n,k},0)$ with holomorphic entries. Suppose that $X_{A_0}^s$ has an isolated singularity at 0 and, for all $t\in D$, the matrix A_t satisfies the following conditions:*

(i) *the Newton polyhedron Δ_{A_t} of A_t is convenient and independent of t;*

(ii) *the matrix A_t is strongly Newton non-degenerate with respect to Δ_{A_t}.*

Then the family $\left\{(X_{A_t}^s,0)\right\}_{t\in D}$ is Whitney equisingular.

Using Corollary 4.19, Example 4.4 and the elements from the previous sections, we present an example of a Whitney equisingular family.

Example 4.20. *Let $\left\{(X_{A_t}^2,0)\right\}_{t\in D}$ be the family of 2-dimensional determinantal singularities defined by the germ $A_t:(\mathbb{C}^4,0)\to(M_{2,3},0)$, with*

$$A_t=\begin{bmatrix} x-z & y+ty^2-w & z-w\\ y-w & z-w & w+x\end{bmatrix}.$$

For all $t\in D$, the matrix A_t is strongly Newton non-degenerate with respect to Δ_{A_t}, Δ_{A_t} is convenient and independent of t. Then, by Corollary 4.19, $\left\{(X_{A_t}^2,0)\right\}_{t\in D}$ is Whitney equisingular.

5 Unmixing the relative mixed volume computations

The purpose of this section is to present a condition in terms of Newton polyhedra which allows us to replace the mixed volume involving polyhedra $\Delta_{1,1},\dots,\Delta_{n,1},\dots,\Delta_{1,k},\dots,\Delta_{n,k}$ by the mixed volume involving the convex hulls $conv(\Delta_{1,1},\dots,\Delta_{n,1}),\dots,conv(\Delta_{1,k},\dots,\Delta_{n,k})$. As we have seen in Chapter 1, Section 2, Chen [12] introduced a condition on the polyhedra $\Delta_1,\dots,\Delta_m\subset\mathbb{R}^m$ in order to unmix the classic mixed volume computation. Using the definition of interlaced polyhedra, Esterov [16] introduced a method to unmix the relative mixed volume computations in the case where we have a complete intersection singularity. In the next paragraphs we extend their results in order to unmix the relative mixed volume computations for determinantal singularities.

We start by choosing an appropriate resultantal singularity to our set up. Let $B_0 = \{e_0\}$ and $B = \{e_0, e_1, \ldots, e_{n-1}\}$. We denote the space $\underbrace{\mathbb{C}[B] \oplus \cdots \oplus \mathbb{C}[B]}_{r}$ by $\mathbb{C}[B]^r$. A point in $\mathbb{C}[B_0]^{a_0} \oplus \mathbb{C}[B]^k$ is a collection of Laurent polynomials of the form $(s_1, \ldots, s_{a_0}, a_{1,1} + \sum_{i=2}^{n} a_{i,1}t_{i-1}, \ldots, a_{1,k} + \sum_{i=2}^{n} a_{i,k}t_{i-1})$, where $s_l, a_{i,j}$, $l = 1, \ldots, a_0$, $i = 1, \ldots, n$, $j = 1, \ldots, k$, is the natural system of coordinates in $\mathbb{C}[B_0]^{a_0} \oplus \mathbb{C}[B]^k$. A germ of a holomorphic map $f : (\mathbb{C}^m, 0) \to (\mathbb{C}[B_0]^{a_0} \oplus \mathbb{C}[B]^k, 0)$ is given by its components $s_l = f_l$ and $a_{i,j} = f_{i,j}$, in this coordinate system.

By Theorem 3.8, the intersection number $f(\mathbb{C}^m) \cap \Sigma(\underbrace{B_0, \ldots, B_0}_{a_0}, \underbrace{B, \ldots, B}_{k})$ is equal

$$(k+a_0)!(\{e_0\} \times \mathbb{R}^m_+, \Delta_0)^{a_0}(S_{n-1} \times \mathbb{R}^m_+, \Delta_{1,1} * \cdots * \Delta_{n,1})^1 \cdots (S_{n-1} \times \mathbb{R}^m_+, \Delta_{1,k} * \cdots * \Delta_{n,k})^1, \quad (5.1)$$

if the system of equations $s_l, f_{i,j}$ is Newton non-degenerate. Note that this intersection number only makes sense for $m = k - n + a_0 + 1$, otherwise the intersection number is not defined.

Consider the cone $\{e_0\} \times \mathbb{R}^m_+$. We observe that, for each $j = 1, \ldots, k$, the polyhedra $\Delta_{1,j} * \cdots * \Delta_{n,j}$ is the convex hull of the union $\cup_{a: c_a \neq 0}\, a + \{e_0\} \times \mathbb{R}^m_+$, where $f_j = \sum_{i=1}^{n} \sum_{a \in \mathbb{Z}^m} c_a x^a t_i$. The next lemma, which was introduced by Esterov [20, Lemma 2.6], shows that the cone $\{e_0\} \times \mathbb{R}^m_+$ plays the role of the unit in the semigroup of pairs of convex bounded polyhedra.

Lemma 4.21. $(\{e_0\} \times \mathbb{R}^m_+, Q_1)^1 (P_2, Q_2)^1 \cdots (P_m, Q_m)^1 = (\{e_0\} \times \mathbb{R}^m_+, Q_1)^1 (Q_2, Q_2)^1 \cdots, (Q_m, Q_m)^1$, *i.e., the left-hand side does not depend on the choice of* $P_2, \ldots, P_m$ *.*

Therefore, the intersection number $f(\mathbb{C}^m) \cap \Sigma(\underbrace{B_0, \ldots, B_0}_{a_0}, \underbrace{B, \ldots, B}_{k})$ is

$$(k+a_0)! \cdot (\{e_0\} \times \mathbb{R}^m_+, \Delta_0)^{a_0}(\Delta_{1,1} * \cdots * \Delta_{n,1}, \Delta_{1,1} * \cdots * \Delta_{n,1})^1 \cdots (\Delta_{1,k} * \cdots * \Delta_{n,k}, \Delta_{1,k} * \cdots * \Delta_{n,k})^1, \quad (5.2)$$

provided that the system of equations $s_l, f_{i,j}$ is Newton non-degenerate.

Lemma 4.22. *Let* $A : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ *be a matrix germ,* $P \in GL_n(\mathbb{C})$ *and* $Q \in GL_k(\mathbb{C})$*. Consider the map germs*

$$\begin{array}{rccc} f: & (\mathbb{C}^m,0) & \to & (\mathbb{C}[B_0]^{a_0} \oplus \mathbb{C}[B]^k, 0) \\ & x & \mapsto & (s_0(x), a_{1,1}(x) + \sum_{i=2}^{n} a_{i,1}(x)t_{i-1}, \ldots, a_{1,k}(x) + \sum_{i=2}^{n} a_{i,k}(x)t_{i-1}), \end{array}$$

and

$$\begin{array}{rccc} \widetilde{f} & (\mathbb{C}^m,0) & \to & (\mathbb{C}[B_0]^{a_0} \oplus \mathbb{C}[B]^k, 0) \\ & x & \mapsto & (s_0(x), \widetilde{a}_{1,1}(x) + \sum_{i=2}^{n} \widetilde{a}_{i,1}(x)t_{i-1}, \ldots, \widetilde{a}_{1,k}(x) + \sum_{i=2}^{n} \widetilde{a}_{i,k}(x)t_{i-1}), \end{array}$$

where $\widetilde{A} = (\widetilde{a}_{i,j}) = P \cdot A \cdot Q$. *Then we have the following isomorphism*

$$\widetilde{f}(\mathbb{C}^m) \cap \Sigma(\underbrace{B_0,\dots,B_0}_{a_0},\underbrace{B,\dots,B}_{k}) \cong f(\mathbb{C}^m) \cap \Sigma(\underbrace{B_0,\dots,B_0}_{a_0},\underbrace{B,\dots,B}_{k}).$$

Proof. Consider the homomorphism

$$\begin{array}{rccc} \phi: & f(\mathbb{C}^m) & \to & \widetilde{f}(\mathbb{C}^m) \\ & f(x) & \mapsto & \widetilde{f}(x) \end{array}.$$

The homomorphism ϕ is an isomorphism with inverse

$$\begin{array}{lccc} \phi^{-1}: & \widetilde{f}(\mathbb{C}^m) & \to & f(\mathbb{C}^m) \\ & (s_0(x), \widetilde{a}_{1,1}(x) + \sum_{i=2}^{n} \widetilde{a}_{i,1}(x)t_{i-1}, \dots, \widetilde{a}_{1,k}(x) + \sum_{i=2}^{n} \widetilde{a}_{i,k}(x)t_{i-1}) & \mapsto & (s_0(x), m_{1,1}(x) + \sum_{i=2}^{n} m_{i,1}(x)t_{i-1}, \dots, m_{1,k}(x) + \sum_{i=2}^{n} m_{i,k}(x)t_{i-1}) \end{array},$$

where $M = (m_{i,j}) = P^{-1} \cdot \widetilde{A} \cdot Q^{-1}$.

Since $\widetilde{f}(\mathbb{C}^m)$ and $f(\mathbb{C}^m)$ are isomorphic, their intersection with $\Sigma(\underbrace{B_0,\dots,B_0}_{a_0},\underbrace{B,\dots,B}_{k})$ are also isomorphic. □

The following definition was introduced by Chen [12]. Esterov [16] also introduced a notion of interlaced polyhedra, which contains the cases covered by Definition 4.23.

Definition 4.23. *For a collection of polyhedra* $\Delta_1,\dots,\Delta_k \subset \mathbb{R}^m_+$. *Denote by* S_j *the set of vertices of the polyhedron* Δ_j, $j = 1,\dots,k$. *A collection of polyhedra* $\Delta_1,\dots,\Delta_k \subset \mathbb{C}^m$ *is called* **interlaced** *if every l-dimensional face of* $conv(\Delta_1,\dots,\Delta_k)$, $l > 0$, *which intersects* S_j *for some* j *in at least two points must intersect all* $S_1,\dots,S_k$.

Example 4.24. *Let* $\Delta_1, \Delta_2 \subset \mathbb{R}^2_+$ *be the Newton polyhedra of the functions* $f_1 = x + y^2$ *and* $f_2 = x^2 + y$, *respectively (see Figure 4.1), and let* Δ *be the convex hull* $conv(\Delta_1, \Delta_2)$ *(see Figure 4.2).*

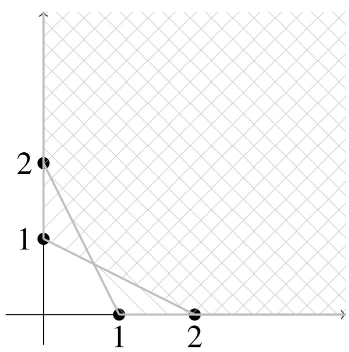


Figure 4.1: Polyhedra Δ_1 and Δ_2.

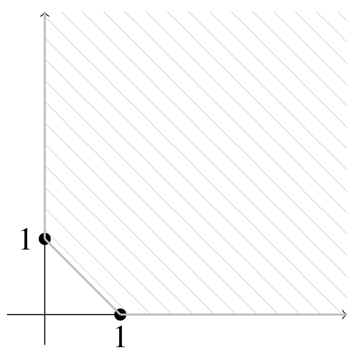


Figure 4.2: Polyhedron Δ.

Note that, the polyhedron Δ *has two 0-dimensional faces* $(1,0)$ *and* $(0,1)$ *and one 1-dimensional* $\overline{AB}$ *proper face* $\overline{(1,0)(0,1)}$. *In addition, the face* $\overline{(1,0)(0,1)}$ *intersects both* S_1 *and* S_2. *Therefore, polyhedra* Δ_1 *and* Δ_2 *are interlaced.*

For a polyhedron $P \subset \mathbb{R}^m_+$ and a strictly positive weight $\gamma = (\gamma_1, \ldots, \gamma_m)$, define $d(\gamma, P) := min\{< p, \gamma >: p \in P\}$ and $(P)_\gamma := \{p \in P :< p, \gamma >= d(\gamma, p)\}$. Since γ is strictly positive, then $(P)_\gamma$ is a bounded face of P.

Lemma 4.25. *Let $S_1, \ldots, S_k \subset \mathbb{Q}^m$ be the sets of vertices of polyhedra $\Delta_1, \ldots, \Delta_k$, respectively. Let Γ be a proper face of $conv(\Delta_1, \ldots, \Delta_k)$ and let γ be a strictly positive weight vector. For each j such that $\Gamma \cap S_j \neq \emptyset$, we have $\Gamma \cap \Delta = (\Delta_j)_\gamma$.*

The proof can be found in [12, Lemma 4.1]. Now we are ready to unmix the relative mixed volume computations. We follow the steps of Chen [12, Corollary 5.1], adapting to relative mixed volumes of polyhedra.

Theorem 4.26. *Let $\Delta_0, \Delta_{i,j} \subset \mathbb{R}^m_+$, $i = 1, \ldots, n$ and $j = 1, \ldots, k$, be convenient polyhedra. Denote the polyhedron $\{e_0\} \times \Delta_0 \subset \mathbb{R}^{n-1} \oplus \mathbb{R}^m_+$ by $\widetilde{\Delta}_0$ and by $\Delta_{1,j} * \cdots * \Delta_{n,j}$ the convex hull $conv(\cup_{i=1}^n \{e_{i-1}\} \times \Delta_{i,j})$.*

(i) *For each $j = 1, \ldots, k$, denote the convex hull $conv(\Delta_{1,j}, \ldots, \Delta_{n,j})$ by Δ_j. If, for each $j = 1, \ldots, k$, the polyhedra $\Delta_{i,j} \subset \mathbb{R}^m_+$, $i = 1, \ldots, n$, are interlaced, then*

$$(\{0\} \times \mathbb{R}^m_+, \widetilde{\Delta}_0)^{a_0}(\Delta_{1,1} * \cdots * \Delta_{1,n}, \Delta_{1,1} * \cdots * \Delta_{n,1})^{a_1} \cdots (\Delta_{1,k} * \cdots * \Delta_{n,k}, \Delta_{1,k} * \cdots * \Delta_{n,k})^{a_k} = (\{0\} \times \mathbb{R}^m_+, \widetilde{\Delta})^{a_0}(S_{n-1} \times \Delta_1, S_{n-1} \times \Delta_1)^{a_1} \cdots (S_{n-1} \times \Delta_k, S_{n-1} \times \Delta_k)^{a_k},$$

where S_{n-1} is the standard $(n-1)$-dimensional simplex and $a_0 + a_1 + \cdots + a_k = m$.

(ii) *If the polyhedra $\Delta_{i,j} \subset \mathbb{R}^m_+$, $i = 1, \ldots, n$ and $j = 1, \ldots, k$, are interlaced, then*

$$(\{0\} \times \mathbb{R}^m_+, \widetilde{\Delta}_0)^{a_0}(\Delta_{1,1} * \cdots * \Delta_{1,n}, \Delta_{1,1} * \cdots * \Delta_{n,1})^{a_1} \cdots (\Delta_{1,k} * \cdots * \Delta_{n,k}, \Delta_{1,k} * \cdots * \Delta_{n,k})^{a_k} = (\{0\} \times \mathbb{R}^m_+, \widetilde{\Delta}_0)^{a_0}(S_{n-1} \times \Delta, S_{n-1} \times \Delta)^{m-a_0},$$

where Δ is the convex hull $conv(\Delta_{1,1}, \ldots, \Delta_{n,k})$.

Proof. Let S_0 and $S_{i,j}$ be the sets of vertices of the polyhedra Δ_0 and $\Delta_{i,j}$, respectively. Let $P = (p_0^l, p_{i,j}^l)$, $i = 1, \ldots, n$, $j = 1, \ldots, k$ and $l = 1, \ldots, a_j$, be a Newton non-degenerate system of Laurent polynomials such that, for each i, j and l, $p_0^l = \sum\limits_{a \in S_1} c_{0,a}^l x^a$ and $p_{i,j}^l = \sum\limits_{a \in S_{i,j}} c_{i,j,a}^l x^a$, $a_1 + \cdots + a_k = m - a_0$. Define $P_j^l = (p_{1,j}^l, \ldots, p_{n,j}^l)$, for each $j = 1, \ldots, k$ and $l = 1, \ldots, a_j$.

Let $f : (\mathbb{C}^m, 0) \to (\mathbb{C}[B_0]^{a_0} \oplus \mathbb{C}[B]^{m-a_0}, 0)$ be a function germ defined by

$$f(x) = \Big(s_0^1(x), \ldots, s_0^{a_0}(x), a_{1,1}^1(x) + \sum_{i=2}^n a_{i,1}^1(x) t_{i-1}, \ldots, a_{1,1}^{a_1}(x) + \sum_{i=2}^n a_{i,1}^{a_1}(x) t_{i-1}, \ldots, a_{1,k}^1(x) + \sum_{i=2}^n a_{i,k}^1(x) t_{i-1}, \ldots, a_{1,k}^{a_k}(x) + \sum_{i=2}^n a_{i,k}^{a_k}(x) t_{i-1}\Big).$$

Then the Newton polyhedron of s_0^l is Δ_0 and the Newton polyhedron of $a_{i,j}^l$ is $\Delta_{i,j}$, $i = 1,\dots,n$, $j = 1,\dots,k$, for all $l = 1,\dots,a_j$ (this means that those polyhedra do not depend on l). Therefore, it follows from Eq. (5.2) that the intersection number $f(\mathbb{C}^m) \cap \Sigma(\underbrace{B_0,\dots,B_0}_{a_0},\underbrace{B,\dots,B}_{k-a_0})$ is

$$m!(\{e_0\} \times \mathbb{R}_+^m, \widetilde{\Delta}_0)^{a_0}(\Delta_{1,1} * \cdots * \Delta_{n,1}, \Delta_{1,1} * \cdots * \Delta_{n,1})^{a_1} \cdots (\Delta_{1,1} * \cdots * \Delta_{n,1}, \Delta_{1,k} * \cdots * \Delta_{n,k})^{a_k}. \quad (5.3)$$

(i) Consider a matrix $Q \in GL_n(\mathbb{C})$ such that the Newton polyhedron of $\widetilde{a}_{i,j}^l$ is equal Δ_j, $j = 1,\dots,k$, for all $i = 1,\dots,n$ and $l = 1,\dots,a_j$, where $A = (a_{i,j}^l)$ and $\widetilde{A} = (\widetilde{a}_{i,j}^l) = Q \cdot A$ (we can choose the matrix Q such that there is no cancellations of terms of terms in $Q \cdot P_j^l$).

Consider the system of polynomial equations $M \cdot P$ induced by the non-singular $m \cdot n \times m \cdot n$ block matrix

$$M = \begin{bmatrix} I_{a_0} & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix}.$$

Our first goal is proving that the system of polynomial equations $M \cdot P$ is Newton non-degenerate. For a strictly positive weight vector γ, consider the face $\Gamma_j = (\Delta_j)_\gamma$, $j = 1,\dots,k$.

We divide in two possible cases. Firstly, suppose that Γ_j is a vertex for some $j = 1,\dots,k$, then each Laurent polynomial in $init_\gamma(Q \cdot P_j^l)$ has only one term, and $V^*(init_\gamma(Q \cdot P_j^l)) = \emptyset$, where $V^*(g_1,\dots,g_r) = \{x \in \mathbb{C}^{*m} : g_1(x) = \cdots = g_r(x) = 0\}$ and the initial part $init_\gamma(g)$ is the restriction of the function g to the face $(\Delta)_\gamma$. Since

$$init_\gamma(M \cdot P) = \begin{bmatrix} init_\gamma(p_0^1) \\ \vdots \\ init_\gamma(p_0^{a_0}) \\ \vdots \\ init_\gamma(Q \cdot P_k^1) \\ \vdots \\ init_\gamma(Q \cdot P_k^{a_k}) \end{bmatrix},$$

$V^*(init_\gamma(M \cdot P)) \subset V^*(init_\gamma(Q \cdot P_j^l)) = \emptyset$. Therefore, $V^*(init_\gamma(M \cdot P)) = \emptyset$.

Secondly, suppose that $\Gamma_1,\dots,\Gamma_k$ are positive dimensional. For each $j \in \{1,\dots,k\}$, let $I_j = \{i \in \{1,\dots,n\} : \Gamma_j \cap S_{i,j} \neq \emptyset\}$, for $j = 1,\dots,k$. Thus, by Lemma 4.25,

$$init_\gamma(Q \cdot P_j^l) = \begin{bmatrix} \sum_{r=1}^n q_{1,r} \sum_{a \in \Gamma_j \cap S_{i,j}} c_{i,r,a}^l x^a \\ \vdots \\ \sum_{r=1}^n q_{n,r} \sum_{a \in \Gamma_j \cap S_{i,j}} c_{i,r,a}^l x^a \end{bmatrix} = Q_{I_j} \cdot \begin{bmatrix} \sum_{a \in \Gamma_j \cap S_{i,j}} c_{i,r,a}^l x^a \end{bmatrix}_{i \in I_j},$$

where Q_{I_j} is the matrix containing columns of $Q = (q_{i,j})$ indexed by I_j. Since Q is non-singular, then $\operatorname{rank}(Q_{I_j})$ is $|I_j| \leq n$. Therefore, $init_\gamma(Q \cdot P_j^l) = 0$ if and only if $\sum_{a \in \Gamma_j \cap S_{i,j}} c_{i,r,a}^l x^a = 0$, for each $i \in I_j$.

If $|\Gamma_j \cap S_{i,j}| = 1$ for all $i \in I_j$, then each one of the Laurent polynomials from above has again only one term and, therefore, $V^*(init_\gamma(Q \cdot P_j^l)) = \emptyset$. Hence $V(init_\gamma(M \cdot P)) = \emptyset$.

Now, suppose that, for each $j = 1, \ldots, k$, $|\Gamma_j \cap S_{i,j}| > 1$ for at least one $i \in I_j$, then since $\Delta_{1,j}, \ldots, \Delta_{n,j}$ are interlaced, then each Γ_j must intersect each of $S_{i,j}$, for all $i = 1, \ldots, n$. Thus $I_j = \{1, \ldots, n\}$, for each $j = 1, \ldots, k$. Since M is non-singular, then $init_\gamma(M \cdot P) = 0$ if and only if $init_\gamma(P) = 0$. Therefore $V^*(init_\gamma(B \cdot P)) = V^*(init_\gamma(P)) = \emptyset$. Hence, the system $M \cdot P$ is Newton non-degenerate.

Let $\widetilde{f}(\mathbb{C}^m, 0) \to (\mathbb{C}[B_0]^{a_0} \oplus \mathbb{C}[B]^{m-a_0}, 0)$ be a function germ defined by

$$\widetilde{f}(x) = (s_0^1(x), \ldots, s_0^{a_0}(x), \widetilde{a}_{1,1}^1(x) + \sum_{i=2}^n \widetilde{a}_{i,1}^1(x) t_{i-1}, \ldots, \widetilde{a}_{1,1}^{a_1}(x) + \sum_{i=2}^n \widetilde{a}_{i,1}^{a_1}(x) t_{i-1}, \ldots,$$
$$\widetilde{a}_{1,k}^1(x) + \sum_{i=2}^n \widetilde{a}_{i,k}^1(x) t_{i-1}, \ldots, \widetilde{a}_{1,k}^{a_k}(x) + \sum_{i=2}^n \widetilde{a}_{i,k}^{a_k}(x) t_{i-1}).$$

Then the Newton polyhedron of s_0^l is Δ_0 and the Newton polyhedron of $\widetilde{a}_{i,j}^l$ is Δ_j, $j = 1, \ldots, k$, for all $i = 1, \ldots, n$ and $l = 1, \ldots, a_j$. Therefore, by Eq. (5.2), the intersection number $\widetilde{f}(\mathbb{C}^m) \cap \Sigma(\underbrace{B_0, \ldots, B_0}_{a_0}, \underbrace{B, \ldots, B}_{k-a_0})$ is equal

$$m!(\{0\} \times \mathbb{R}_+^m, \widetilde{\Delta})^{a_0} (S_{n-1} \times \Delta_1, S_{n-1} \times \Delta_1)^{a_1} \cdots (S_{n-1} \times \Delta_k, S_{n-1} \times \Delta_k)^{a_k}. \tag{5.4}$$

In addition Q is non-singular, then, by Lemma 4.22, we have the isomorphism

$$\widetilde{f}(\mathbb{C}^m) \cap \Sigma(\underbrace{B_0, \ldots, B_0}_{a_0}, \underbrace{B, \ldots, B}_{k-a_0}) \cong f(\mathbb{C}^m) \cap \Sigma(\underbrace{B_0, \ldots, B_0}_{a_0}, \underbrace{B, \ldots, B}_{k-a_0}).$$

Therefore, combining Eq. (5.3) and Eq. (5.4) we obtain

$$(\{0\} \times \mathbb{R}_+^m, \widetilde{\Delta}_0)^{a_0} (\Delta_{1,1} * \cdots * \Delta_{1,n}, \Delta_{1,1} * \cdots * \Delta_{n,1})^{a_1} \cdots (\Delta_{1,k} * \cdots * \Delta_{n,k}, \Delta_{1,k} * \cdots * \Delta_{n,k})^{a_k} =$$
$$(\{0\} \times \mathbb{R}_+^m, \widetilde{\Delta})^{a_0} (S_{n-1} \times \Delta_1, S_{n-1} \times \Delta_1)^{a_1} \cdots (S_{n-1} \times \Delta_k, S_{n-1} \times \Delta_k)^{a_k}.$$

(ii) the proof is analogous to item i).

□

Theorem 4.27. *Let $(X_A^n, 0)$ be the IDS defined by the germ of a Newton non-degenerate matrix $A = (a_{i,j}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$. Let $\Delta_{i,j}$ be the Newton polyhedron of $a_{i,j}$, Δ_j be the convex hull $conv(\Delta_{1,j}, \ldots, \Delta_{n,j})$, for each $j = 1, \ldots, k$, and Δ_A be the convex hull $conv(\Delta_{1,1}, \ldots, \Delta_{n,k})$.*

(i) *Suppose that the polyhedron* Δ_j *is convenient,* $j=1,\ldots,k$. *If, for each* $j=1,\ldots,k$*, the polyhedra* $\Delta_{1,j},\ldots,\Delta_{n,j}$ *are interlaced, then*

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{\{j_1,\ldots,j_q\}\subset\{1,\ldots,k\}}\ \sum_{\substack{I\subset\{1,\ldots,m\}\\|I|\geq q+1}}\ \sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}$$
$$\times\sum_{\substack{a_{j_1},\ldots,a_{j_q}\in\mathbb{N}\\a_{j_1}+\cdots+a_{j_q}=|I|-a}}|I|!\cdot(L^I)^a(\widetilde{\Delta}^I_{j_1})^{a_{j_1}}\cdots(\widetilde{\Delta}^I_{j_q})^{a_{j_q}}.$$

(ii) *Suppose that the polyhedron* Δ_A *is convenient. If the polyhedra* $\Delta_{i,j}$*,* $i=1,\ldots,n$ *and* $j=1,\ldots,k$*, are interlaced, then*

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{q=k-n+1}^{k}\ \sum_{\substack{I\subset\{1,\ldots,m\}\\|I|\geq q+1}}\ \sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}$$
$$\times\binom{|I|-a-1}{q-1}\binom{k}{q}|I|!(L^I)^a(\widetilde{\Delta}^I_A)^{|I|-a}.$$

Proof. Firstly, let M be an integer number, which is greater than the determinacy bound of A. Consider the matrix ${}_MA=({}_Ma_{i,j})$, where ${}_Ma_{i,j}=a_{i,j}+\sum_{l=1}^{m}\alpha^l_{i,j}x_l^M$. Let ${}_M\Delta_{i,j}$ be the Newton polyhedron of the entry ${}_Ma_{i,j}$. By Theorem 3.23, the local Euler obstruction of X_A^n is

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{\substack{\Gamma\in\phi\\I\subset\{1,\ldots,m\}}}\chi(\widetilde{\mathscr{L}^I},{}_M\Delta^{\Gamma,I}_{1,1}*\cdots*{}_M\Delta^{\Gamma,I}_{n,1},\ldots,{}_M\Delta^{\Gamma,I}_{1,1}*\cdots*{}_M\Delta^{\Gamma,I}_{n,1}).\tag{5.5}$$

(i) Moreover, the polyhedra $\Delta_{1,j},\ldots,\Delta_{n,j}$ are interlaced, for each $j=1,\ldots,k$, then the polyhedra ${}_M\Delta_{1,j},\ldots,{}_M\Delta_{n,j}$ are also interlaced. In addition, Δ_j is convenient, then $\Delta_j={}_M\Delta_j=conv({}_M\Delta_{1,j},\ldots,{}_M\Delta_{n,j})$. Therefore, by Theorem 4.26

$$(\{e_0\}\times\mathbb{R}^I_+,\{e_0\}\times\mathscr{L}^I)^{a_0}({}_M\Delta^{\Gamma,I}_{1,1}*\cdots*{}_M\Delta^{\Gamma,I}_{1,n},{}_M\Delta^{\Gamma,I}_{1,1}*\cdots*{}_M\Delta^{\Gamma,I}_{n,1})^{a_1}\cdots$$
$$({}_M\Delta^{\Gamma,I}_{1,k}*\cdots*{}_M\Delta^{\Gamma,I}_{n,k},{}_M\Delta^{\Gamma,I}_{1,k}*\cdots*{}_M\Delta^{\Gamma,I}_{n,k})^{a_k}=$$
$$(\{e_0\}\times\mathbb{R}^I_+,\{e_0\}\times\mathscr{L}^I)^{a_0}(S_{n-1}\times\Delta^{\Gamma,I}_1,S_{n-1}\times\Delta^{\Gamma,I}_1)^{a_1}\cdots(S_{n-1}\times\Delta^{\Gamma,I}_k,S_{n-1}\times\Delta^{\Gamma,I}_k)^{a_k}.$$

Thus, we can compute the Euler obstruction taking Δ_j to be the Newton polyhedron of the column j of A, for each $j=1,\ldots,k$. Therefore, we can reduce Eq. (5.5) to

$$\mathrm{Eu}_{X_A^n}(0)=\sum_{\{j_1,\ldots,j_q\}\subset\{1,\ldots,k\}}\ \sum_{\substack{I\subset\{1,\ldots,m\}\\|I|\geq q+1}}\ \sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}$$
$$\times\sum_{\substack{a_{j_1},\ldots,a_{j_q}\in\mathbb{N}\\a_{j_1}+\cdots+a_{j_q}=|I|-a}}|I|!\cdot(L^I)^a(\widetilde{\Delta}^I_{j_1})^{a_{j_1}}\cdots(\widetilde{\Delta}^I_{j_q})^{a_{j_q}}.$$

(ii) Now, suppose that Δ_A is convenient. By assumption, the polyhedra $\Delta_{1,1},\ldots,\Delta_{n,k}$ are interlaced. Since Δ_A is convenient, then $\Delta_A = {}_M\Delta_A = conv({}_M\Delta_{1,1},\ldots,{}_M\Delta_{n,k})$. Thus, by Theorem 4.26

$$(\{e_0\}\times\mathbb{R}^I_+,\{e_0\}\times\mathscr{L}^I)^{a_0}({}_M\Delta^{\Gamma,I}_{1,1}*\cdots*{}_M\Delta^{\Gamma,I}_{1,n},{}_M\Delta^{\Gamma,I}_{1,1}*\cdots*{}_M\Delta^{\Gamma,I}_{n,1})^{a_1}\cdots$$
$$({}_M\Delta^{\Gamma,I}_{1,k}*\cdots*{}_M\Delta^{\Gamma,I}_{n,k},{}_M\Delta^{\Gamma,I}_{1,k}*\cdots*{}_M\Delta^{\Gamma,I}_{n,k})^{a_k}=$$
$$(\{e_0\}\times\mathbb{R}^I_+,\{e_0\}\times\mathscr{L}^I)^{a_0}(S_{n-1}\times\Delta^{\Gamma,I}_A,S_{n-1}\times\Delta^{\Gamma,I}_A)^{|I|-a_0}.$$

Once again, we can reduce Eq. (5.5) to obtain

$$\mathrm{Eu}_{X^n_A}(0)=\sum_{q=k-n+1}^{k}\sum_{\substack{I\subset\{1,\ldots,m\}\\|I|\geq q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}\times\binom{|I|-a-1}{q-1}\binom{k}{q}|I|!(L^I)^a(\widetilde{\Delta}^I_A)^{|I|-a}.$$

□

With the help of Theorem 4.27, in the following, we compute more concrete examples.

Example 4.28. *Consider the matrix germ* $A=(a_{i,j}):(\mathbb{C}^4,0)\to(M_{2,3},0)$, *defined by*

$$A=\begin{bmatrix} w & y & x\\ z & w & y\end{bmatrix}.$$

Denote by $\Delta_{i,j}$ *the Newton polyhedron of* $a_{i,j}$. *We observe that, the polyhedra* $\Delta_{1,1},\Delta_{1,2},\Delta_{1,3},\Delta_{2,1},\Delta_{2,2},\Delta_{2,3}$ *are interlaced and* $conv(\Delta^1_{1,1},\ldots,\Delta^1_{2,3})$ *is convenient. In addition, the matrix A is Newton non-degenerate. Therefore,*

$$\mathrm{Eu}_{X^2_A}(0)=\sum_{q=k-n+1}^{k}\sum_{\substack{I\subset\{1,\ldots,m\}\\|I|\geq q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}\times\binom{|I|-a-1}{q-1}\binom{k}{q}|I|!(L^I)^a(\widetilde{\Delta}^I_A)^{|I|-a}=-1.$$

We used OSCAR [51] to compute the above formula (see Example A.8).

Example 4.29. *Let* $X^n_A\subset\mathbb{C}^{n\times k}$ *be a generic determinantal variety, i.e.,* X^n_A *is defined by the n size minors of the matrix*

$$A(x_{1,1},\ldots,x_{n,k})=\begin{bmatrix} x_{1,1} & \cdots & x_{1,k}\\ \vdots & \ddots & \vdots\\ x_{n,1} & \cdots & x_{n,k}\end{bmatrix}.$$

Denote by $\Delta_{i,j}$ *the Newton polyhedra of* $a_{i,j}$. *The polyhedra* $\Delta_{i,j}$, $i=1,\ldots,n$ *and* $j=1,\ldots,k$, *are interlaced and* Δ_A *is such that* $\mathbb{R}^{n\times k}_+\setminus\Delta_A=S_{n\times k}$. *Therefore, by Theorem 4.27,*

$$\mathrm{Eu}_{X^n_A}(0)=\sum_{q=k-n+1}^{k}\sum_{|I|=q+1}^{n\times k}\sum_{a=1}^{|I|-q}(-1)^{|I|+k-n}\binom{|I|+q-a-2}{n+q-k-1}\binom{|I|-a-1}{q-1}\binom{k}{q}\binom{n\times k}{|I|}.$$

We computed the above formula in OSCAR [51], for some values of n and k (see Example A.10).

Example 4.30. *Let* $A : (\mathbb{C}^m, 0) \to (M_{2,k}, 0)$ *be the matrix germ defined by*

$$A(x_1,\dots,x_m) = \begin{bmatrix} x_1 & x_2 & \cdots & x_{m-1} \\ x_2 & x_3 & \cdots & x_m \end{bmatrix}.$$

The matrix A *is Newton non-degenerate, the Newton polyhedra of its entries are interlaced and* $\Delta_A = S_m$*. Therefore,*

$$\mathrm{Eu}_{X_A^2}(0) = \sum_{q=m-2}^{m-1} \sum_{|I|=q+1}^{m} \sum_{a=1}^{|I|-q} (-1)^{|I|+m-3} \binom{|I|+q-a-2}{q-m+2} \binom{|I|-a-1}{q-1} \binom{m-1}{q} \binom{m}{|I|} = 3-m.$$

We observe that X_A^2 *is a toric surface in* $\mathbb{C}^m$*, therefore, its Euler obstruction is indeed* $3-m$ *(see [29]). We also computed the above formula, for various values of* m*, with OSCAR [51] (see A.11).*

In addition, we can also use the previous construction to compute the vanishing Euler characteristic of an IDS.

Theorem 4.31. *Let* X_A^n *be the IDS defined by the germ of a Newton non-degenerate matrix* $A = (a_{i,j}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ *and* $\mathscr{A} : (\mathbb{C}^{m+1}, 0) \to (M_{n,k}, 0)$ *its determinantal smoothing. Let* $\Delta_{i,j}$ *be the Newton polyhedron of* $\mathscr{A}_{i,j}$*,* Δ_j *be the convex hull conv*$(\Delta_{1,j},\dots,\Delta_{n,j})$ *and* $\Delta_{\mathscr{A}}$ *be the convex hull conv*$(\Delta_{1,1},\dots,\Delta_{n,k})$*.*

(i) *Suppose that the polyhedron* Δ_j *is convenient,* $j = 1,\dots,k$*. If, for each* $j = 1,\dots,k$*, the polyhedra* $\Delta_{1,j},\dots,\Delta_{n,j}$ *are interlaced, then*

$$1 + (-1)^{\dim X_A^n} \nu(X_A^n, 0) = \sum_{\{j_1,\dots,j_q\} \subset \{1,\dots,k\}} \sum_{\substack{I \subset \{1,\dots,m\} \\ |I| \geq q+1}} \sum_{a=1}^{|I|-q} (-1)^{|I|+k-n} \binom{|I|+q-a-2}{n+q-k-1} \times \sum_{\substack{a_{j_1},\dots,a_{j_q} \in \mathbb{N} \\ a_{j_1}+\dots+a_{j_q} = |I|-a}} |I|! \cdot ({}_M\widetilde{\Delta}_0^I)^a (\widetilde{\Delta}_{j_1}^I)^{a_{j_1}} \cdots (\widetilde{\Delta}_{j_q}^I)^{a_{j_q}}.$$

(ii) *Suppose that the polyhedron* Δ_A *is convenient. If the polyhedra* $\Delta_{i,j}$*,* $i = 1,\dots,n$ *and* $j = 1,\dots,k$*, are interlaced, then*

$$1 + (-1)^{\dim X_A^n} \nu(X_A^n, 0) = \sum_{q=k-n+1}^{k} \sum_{\substack{I \subset \{1,\dots,m\} \\ |I| \geq q+1}} \sum_{a=1}^{|I|-q} (-1)^{|I|+k-n} \binom{|I|+q-a-2}{n+q-k-1} \times \binom{|I|-a-1}{q-1} \binom{k}{q} |I|! ({}_M\widetilde{\Delta}_0^I)^a (\widetilde{\Delta}_{\mathscr{A}}^I)^{|I|-a}.$$

Example 4.32. *Let* X_A^2 *be the IDS defined by the matrix germ* $A : (\mathbb{C}^4, 0) \to (M_{2,3}, 0)$*, where*

$$A = \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}.$$

The matrix A is Newton non-degenerate and the polyhedra $\Delta_{1,1},\dots,\Delta_{2,3}$ are interlaced. Denote by Δ_A the convex hull $conv(\Delta_{1,1},\dots,\Delta_{2,3})$.

Consider the determinantal smoothing of X_A^2 $\mathscr{A} : (\mathbb{C}^4 \times \mathbb{C},0) \to (M_{2,3},0)$ defined by

$$\mathscr{A}(x,y,z,w,t) = \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix} + \frac{1}{100}\begin{bmatrix} 6t & -8t & 5t \\ t & 8t & 7t \end{bmatrix}.$$

Denote by ${}_2\Delta_0$ the Newton polyhedra of $t+\beta_1 x^2+\beta_2 y^2+\beta_3 z^2+\beta_4 w^2$, by ${}_2\Delta_{i,j}$ the Newton polyhedra of $\mathscr{A}_{i,j}+\alpha_{i,j}^1 x^2+\alpha_{i,j}^2 y^2+\alpha_{i,j}^3 z^2+\alpha_{i,j}^4 w^2$. Moreover, the matrix $\mathscr{A}$ is Newton non-degenerate and the polyhedra ${}_2\Delta_{1,1},\dots,{}_2\Delta_{2,3}$ are interlaced. Therefore,

$$1+\nu(X_A^2,0) = \sum_{q=k-n+1}^{k} \sum_{\substack{I\subset\{1,\dots,m\}\\ |I|\geq q+1}} \sum_{a=1}^{|I|-q} (-1)^{|I|+k-n} \binom{|I|+q-a-2}{n+q-k-1} \times \binom{|I|-a-1}{q-1}\binom{k}{q} |I|!({}_2\widetilde{\Delta}_0^I)^a(\widetilde{\Delta}_{\mathscr{A}}^I)^{|I|-a} = 2$$

The above expression was computed with OSCAR [51] (see Example A.9). Hence, $\nu(X_A^2,0)=1$. With the local Euler obstruction and the vanishing Euler characteristic of X_A^2, we can compute the top polar multiplicity of X_A^2.

$$m_2(X_A^2,0) = 1+\nu(X_A^2,0)-\mathrm{Eu}_{X_A^2}(0) = 1+1-(-1) = 3.$$

APPENDIX A

Implementation on OSCAR

This appendix is dedicated to presenting functions implemented in OSCAR [51]. One can find the code files in `https://github.com/MaicomVarella/implementation_on_OSCAR`. Firstly, we generate the Newton polyhedron of a function and verify if a function is Newton non-degenerate. As an extension, we also verify if a 2×3 matrix is Newton non-degenerate. Moreover, we introduce a function to compute the relative mixed volume of polyhedra. Lastly, we compute the local Euler obstruction of isolated determinantal singularities defined by 2×3 matrices.

1 Non-degeneracy of a function

We start by generating the Newton polyhedron, $\Delta(f)$, of a polynomial function, according with Definition 1.29. For a polynomial function $f = \sum_{a\in\mathbb{N}^n} c_a x^a$, the command

```
Newton_polytope(f::MPolyElemLoc)
```

returns the polyhedron $conv(\bigcup_{a\in supp(f)} a+\mathbb{R}^n_+)$, where $supp(f) = \{a\in\mathbb{N}^n : c_a \neq 0\}$.

Then we compute its Newton diagram, $\Gamma(f)$, which is the set of all bounded faces of the polyhedron. For a polynomial function f, the function

```
Newton_diagram(f::MPolyElemLoc)
```

returns the set of bounded faces of $\Delta(f)$.

In the next step, we verify if a function is convenient, *i.e.,* its Newton polyhedra touches all the coordinate axis. For a polynomial function f, the command

```
is_convenient(f::MPolyElemLoc)
```

returns “true” if its Newton polyhedron meets all the coordinate axis and it returns “false” otherwise.

Lastly, we verify if a polynomial function is Newton non-degenerate. Before that, we compute the face function with respect to a polyhedron. For a polynomial function $f = \sum_{a\in\mathbb{N}^n} c_a x^a$ and a polyhedron P, the command

```
face_function(f::MPolyElemLoc,P::Polyhedron)
```

returns the function $f_P = \sum_{a \in P} c_a x^a$. For a polynomial function f, the command

```
is_Newton_non_degenerate(f::MPolyElemLoc)
```

returns“true” if, for all faces $P \in \Gamma(f)$, the germ $(V(f_P),0)$ is non-singular outside the coordinate hyperplanes $\mathbb{C}^{*n}$.

Example A.1. *Let $f : (\mathbb{C}^3,0) \to (\mathbb{C},0)$ be the function germ defined by $f(x,y,z) = x^4 + y^4 + z^4 + xyz + y^2z^2 + x^5$.*

```
julia> R, (x,y,z) = QQ["x", "y", "z"]
(Multivariate Polynomial Ring in x, y, z over Rational Field,
fmpq_mpoly[x, y, z])

julia> f = x^4+y^4+z^4+x*y*z+y^2*z^2+x^5
x^5 + x^4 + x*y*z + y^4 + y^2*z^2 + z^4

julia> Newton_polytope(f)
A polyhedron in ambient dimension 3

julia> Newton_diagram(f)
13-element Vector{Polyhedron{fmpq}}:
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3
 A polyhedron in ambient dimension 3

julia> is_convenient(f)
true

julia> is_Newton_non_degenerate(f)
true
```

2 Non-degeneracy of a 2×3 matrix

Analogously, we can verify the non-degeneracy of a matrix (see Definition 2.2). We make the following construction for a 2×3 matrix, but it can also be made to any size matrix. We verify the

Newton non-degeneracy of matrices $(a_{i,j})$ such that the Newton polyhedron of $a_{i,j}$ does not depend on i.

For a 2×3 matrix, a, with polynomial entries, the function

```
all_faces(a)
```

returns the collection of all vectors, where the entry j, $j = 1,2,3$, is a face of the Newton diagram $\Gamma(a_{1,j})$.

For a 2×3 matrix a with polynomial entries, the command

```
faces_to_keep(a)
```

returns a boolean vector, where an entry is "true" if the vector of all_faces (a) contains only faces σ_j of $\Gamma(a_{1,j})$ such that $\sum_{j=1,2,3} \sigma_j$ belongs to the Newton diagram $\Gamma(a_{1,1} * a_{1,2} * a_{1,3})$ and "false" otherwise.

For a 2×3 matrix a with polynomial entries, the function

```
bounded_faces(a)
```

returns a subcollection of all_faces (a) containing only the faces σ_j of $\Gamma(a_{1,j})$ such that $\sum_{j=1,2,3} \sigma_j$ belongs to the Newton diagram $\Gamma(a_{1,1} * a_{1,2} * a_{1,3})$.

We can also define a face matrix, where each entry of a matrix is a face function for some polyhedron. For a 2×3 matrix a with polynomial entries and polyhedra P, Q and R, the command

```
matrix_function(a,P::Polyhedron,Q::Polyhedron,S::Polyhedron)
```

returns a matrix, $a_{P,Q,R}$, with entries $(a_{1,1})_P$, $(a_{2,1})_P$, $(a_{1,2})_Q$, $(a_{1,2})_Q$, $(a_{1,3})_R$ and $(a_{2,3})_R$.

Finally, we can verify if a 2×3 matrix is Newton non-degenerate. For a 2×3 matrix a with polynomial entries, the function

```
is_non_degenerate(a)
```

returns "true" if for all vectors [P Q R] of bounded_faces(a) the determinantal variety defined by $a_{P,Q,R}$ is non-singular outside the coordinate hyperplanes $\mathbb{C}^{*n}$.

Example A.2. *Consider the matrix germ* $A : (\mathbb{C}^4, 0) \to (M_{2,3}, 0)$ *defined by the matrix*

$$A = \begin{bmatrix} x+y & y+z & z+w \\ 2x+3y & 5y+7z & 9z+11w \end{bmatrix}.$$

```
julia> A = R[x+y y+z z+w; 2x+3y 5y+7z 9z+11w]
[      x + y          y + z           z + w]
[2*x + 3*y    5*y + 7*z    9*z + 11*w]

julia> is_non_degenerate(A)
true
```

Example A.3.
```
julia> A = R[2x+2y^3+z^2-3w^4 2x+3y^3+2z^2-5w^4
3x+2y^3+2z^2-3w^4; 3x+3y^3+2z^2-4w^4 3x+4y^3+4z^2-7w^4
5x+3y^3+3z^2-3w^4]
[  2*x + 2*y^3 + z^2 - 3*w^4   2*x + 3*y^3 + 2*z^2 - 5*w^4
3*x + 2*y^3 + 2*z^2 - 3*w^4]
[3*x + 3*y^3 + 2*z^2 - 4*w^4   3*x + 4*y^3 + 4*z^2 - 7*w^4
5*x + 3*y^3 + 3*z^2 - 3*w^4]

julia> is_non_degenerate(A)
true
```

Example A.4.
```
julia> A = R[x+y y+z z+w; 2x+3y 5y+7z 9z+11w]
[      x + y          y + z          z + w]
[2*x + 3*y    5*y + 7*z    9*z + 11*w]

julia> B = R[x^2+y^2+z^2+w^2 x^2+y^2+z^2+w^2 x^2+y^2+z^2+w^2;
x^2+y^2+2z^2+3w^2 5x^2+y^2+z^2+7w^2 9x^2+11y^2+z^2+w^2]
[x^2 + y^2 + z^2 + w^2 x^2 + y^2 + z^2 + w^2
x^2 + y^2 + z^2 + w^2]
[x^2 + y^2 + 2*z^2 + 3*w^2 5*x^2 + y^2 + z^2 + 7*w^2
9*x^2 + 11*y^2 + z^2 + w^2]

julia> C = A+B
[x^2 + x + y^2 + y + z^2 + w^2 x^2 + y^2 + y + z^2 + z + w^2
x^2 + y^2 + z^2 + z + w^2 + w]
[x^2 + 2*x + y^2 + 3*y + 2*z^2 + 3*w^2   5*x^2 + y^2 + 5*y + z^2
 + 7*z + 7*w^2  9*x^2 + 11*y^2 + z^2 + 9*z + w^2 + 11*w]

julia> is_non_degenerate(C)
true
```

3 Relative mixed volume of pairs of polyhedra

In this section, we present the implementation on the formula 1.49, in order to compute the normalized relative mixed volume of pairs of polyhedra $\widetilde{\Delta}^J = (\mathbb{R}^J_+, \Delta^J)$, where $\Delta^J = \mathbb{R}^J_+ \cap \Delta$ and $J \subset \{1,\ldots,n\}$.

For a polyhedron P and a vector J, the function

```
matrix(P::Polyhedron, J)
```

returns the matrix with the vertices of the intersection between P and the coordinate hyperplane $\mathbb{C}^J$.

For a polyhedron P and a vector J, the function

```
NP(P::Polyhedron, J)
```

returns the polyhedron generated by the lines of `matrix(P,J)`.

For a polyhedron P and a vector J, the function

```
NP0(P::Polyhedron,J)
```

returns the polyhedron generated by the lines of matrix(P,J) and the origin.

For a polyhedron P and a vector J, the command

```
Vol(P::Polyhedron,J)
```

returns the difference of volumes volume(NP0(P,J))−volume(NP(P,J)).

For a set of polyhedra $p_1,\dots,p_{|J|}$ and a vector J, the function

```
MV(J,p...)
```

returns the sum

$$\sum_{r=1}^{m}(-1)^{|J|-r}\sum_{1\le i_1\le\cdots\le i_r\le|J|}\mathrm{Vol}(\mathrm{J},\mathrm{p}_1+\cdots+\mathrm{p}_r).$$

Example A.5.
```
julia> p = Newton_polytope(x+y+z+w)
A polyhedron in ambient dimension 4

julia> q = Newton_polytope(x+y^3+z^2-5w^4)
A polyhedron in ambient dimension 4

julia> MV([1,2,3,4],p,p,q,q)
2
```

4 Euler characteristic of the Milnor fiber

Finally, we compute the Euler characteristic of the Milnor fiber of a function f restricted to an isolated determinantal singularity X_A^2, defined by a 2×3 matrix, by applying Corollary 3.18.

For a polynomial function f and a 2×3 matrix, the command

```
Euler_characteristic_Milnor_fiber(f,a)
```

returns the sum

$$\sum_{\{j_1,\dots,j_q\}\subset\{1,2,3\}}\sum_{\substack{I\subset\{1,\dots,m\}\\|I|\ge q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+1}\binom{|I|+q-a-2}{q-2}$$
$$\times\sum_{\substack{a_{j_1},\dots,a_{j_q}\in\mathbb{N}\\a_{j_1}+\cdots+a_{j_q}=|I|-a}}MV(I,\underbrace{p_0,\dots,p_0}_{a},\underbrace{p_{j_1},\dots,p_{j_1}}_{a_{j_1}}\cdots\underbrace{p_{j_q},\dots,p_{j_q}}_{a_{j_q}}),$$

where p_0 is the Newton polyhedron of f and p_j is the Newton polyhedron of $a_{1,j}$, for $j=1,2,3$.

Example A.6.
```
julia> A = R[x+y y+z z+w; 2x+3y 5y+7z 9z+11w]
[     x + y        y + z         z + w]
[2*x + 3*y    5*y + 7*z    9*z + 11*w]

julia> B = R[x^2+y^2+z^2+w^2 x^2+y^2+z^2+w^2 x^2+y^2+z^2+w^2;
```

```
x^2+y^2+2z^2+3w^2  5x^2+y^2+z^2+7w^2  9x^2+11y^2+z^2+w^2]
[      x^2 + y^2 + z^2 + w^2          x^2 + y^2 + z^2 + w^2
x^2 + y^2 + z^2 + w^2]
[x^2 + y^2 + 2*z^2 + 3*w^2    5*x^2 + y^2 + z^2 + 7*w^2
9*x^2 + 11*y^2 + z^2 + w^2]

julia> C = A+B
[x^2 + x + y^2 + y + z^2 + w^2  x^2 + y^2 + y + z^2 + z + w^2
 x^2 + y^2 + z^2 + z + w^2 + w]
[x^2 + 2*x + y^2 + 3*y + 2*z^2 + 3*w^2  5*x^2 + y^2 + 5*y + z^2
+ 7*z + 7*w^2    9*x^2 + 11*y^2 + z^2 + 9*z + w^2 + 11*w]

julia> f = x+y+z+w
x + y + z + w

julia> Euler_characteristic_Milnor_fiber(f,C)
-1
```

Example A.7.

```
julia> A = R[x+y  y+z  z+w;  2x+3y  5y+7z  9z+11w]
[      x + y          y + z          z + w]
[2*x + 3*y    5*y + 7*z    9*z + 11*w]

julia> B = 1//100*(R[6t  -8t  5t;  t  8t  7t])
[ 3//50*t     -2//25*t      1//20*t]
[1//100*t      2//25*t     7//100*t]

julia> C = R[x^2+y^2+z^2+w^2  x^2+y^2+z^2+w^2  x^2+y^2+z^2+w^2;
x^2+y^2+2z^2+3w^2  5x^2+y^2+z^2+7w^2  9x^2+11y^2+z^2+w^2]
[      x^2 + y^2 + z^2 + w^2          x^2 + y^2 + z^2 + w^2
x^2 + y^2 + z^2 + w^2]
[x^2 + y^2 + 2*z^2 + 3*w^2    5*x^2 + y^2 + z^2 + 7*w^2
9*x^2 + 11*y^2 + z^2 + w^2]

D = A+B+C
[x^2 + x + y^2 + y + z^2 + w^2 + 3//50*t  x^2 + y^2 + y + z^2 + z
+ w^2 - 2//25*t  x^2 + y^2 + z^2 + z + w^2 + w + 1//20*t]
[x^2 + 2*x + y^2 + 3*y + 2*z^2 + 3*w^2 + 1//100*t  5*x^2 + y^2
+ 5*y + z^2 + 7*z + 7*w^2 + 2//25*t    9*x^2 + 11*y^2 + z^2 + 9*z
+ w^2 + 11*w + 7//100*t]

julia> f = x^2+y^2+z^2+w^2+t
x^2 + y^2 + z^2 + w^2 + t

julia> Euler_characteristic_Milnor_fiber(f,D)
2
```

Let $A = (a_{i,j}) : (\mathbb{C}^m, 0) \to (M_{n,k}, 0)$ be a matrix germ. If the Newton polyhedra of the entries $a_{i,j}$ are interlaced, for $i = 1, \ldots, n$ and for $j = 1, \ldots, k$ or A is Newton non-degenerate with respect to its Newton polyhedron Δ_A and Δ_A is convenient, we can compute the Euler characteristic of the Milnor fiber of a function restricted to the determinantal singularity $(X_A^n, 0)$ in terms of the Newton polyhedron of f and Δ_A (see Theorem 4.27).

For a polynomial function f and a 2×3 matrix a with polynomial entries, the command

```
Euler_characteristic_Milnor_fiber_matrix(f,a)
```

returns

$$\sum_{q=2}^{3}\sum_{\substack{I\subset\{1,\dots,m\}\\|I|\geq q+1}}\sum_{a=1}^{|I|-q}(-1)^{|I|+1}\binom{|I|+q-a-2}{q-2}\binom{|I|-a-1}{q-1}\binom{3}{q}MV(I,\underbrace{p_0,\dots,p_0}_{a},\underbrace{p,\dots,p}_{|I|-a}),$$

where p_0 is the Newton polyhedron of f and p is the Newton polyhedron of a.

Example A.8.

```
julia> A = R[x y z; y z w]
[x   y   z]
[y   z   w]

julia> f = x+y+z+w
x + y + z + w

julia> Euler_characteristic_Milnor_fiber_matrix(f,A)
-1
```

Example A.9.

```
julia> A = R[x y z; y z w]
[x   y   z]
[y   z   w]

julia> B = 1//100*(R[6t -8t 5t; t 8t 7t])
[ 3//50*t    -2//25*t     1//20*t]
[1//100*t     2//25*t    7//100*t]

julia> C = A+B
[ x + 3//50*t    y - 2//25*t     z + 1//20*t]
[y + 1//100*t    z + 2//25*t    w + 7//100*t]

julia> f = x^2+y^2+z^2+w^2
x^2 + y^2 + z^2 + w^2

julia> f = x^2+y^2+z^2+w^2+t
x^2 + y^2 + z^2 + w^2 + t

julia> Euler_characteristic_Milnor_fiber_matrix(f,C)
2
```

Example A.10. *Using Example 4.29, we compute the Euler obstruction of a generic determinantal variety of type $(n,k;n)$ for certain n and k such that the binomial computations do not overflow.*

```
julia> generic_determinantal(n,k) = sum(sum(sum((-1)^
(p+k-n)*binomial(p+q-a-2,n+q-k-1)*binomial(p-a-1,q-1)binomial(k,q)
*binomial(n*k,p) for a in 1:p-q) for p in q+1:n*k) for q in k-n+1
:k)
generic_determinantal (generic function with 1 method)

julia> [generic_determinantal(2,k) for k in 2:33] ==
[2 for k in 2:33]
true

julia> [generic_determinantal(3,k) for k in 3:22] ==
[3 for k in 3:22]
true

julia> [generic_determinantal(4,k) for k in 4:16] ==
[4 for k in 4:16]
true

julia> [generic_determinantal(5,k) for k in 5:13] ==
[5 for k in 5:13]
true

julia> [generic_determinantal(6,k) for k in 6:11] ==
[6 for k in 6:11]
true

julia> [generic_determinantal(7,k) for k in 7:9] ==
[7 for k in 7:9]
true

julia> [generic_determinantal(8,k) for k in 8:8] ==
[8 for k in 8:8]
true
```

Example A.11. *Applying Example 4.30, we compute the Euler obstruction of the toric variety defined by the matrix*

$$A(x_1,\dots,x_m) = \begin{bmatrix} x_1 & x_2 & \cdots & x_{m-1} \\ x_2 & x_3 & \cdots & x_m \end{bmatrix}.$$

```
julia> toric_determinantal(m) = sum(sum(sum((-1)^(p+m-3)*binomial
(p+q-a-2,q-m+2)*binomial(p-a-1,q-1)*binomial(m-1,q)*binomial(m,p)
for a in 1:p-q) for p in q+1:m) for q in m-2:m-1)
toric_determinantal (generic function with 1 method)

julia> [toric_determinantal(m) for m in 4:100000000] ==
[3-m for m in 4:100000000]
true
```

Bibliography

[1] I. Ahmed and M. A. S. Ruas. Determinacy of determinantal varieties. *Manuscripta Math.*, 159(1-2):269–278, 2019.

[2] D. A. H. Ament, J. J. Nuño-Ballesteros, B. Oréfice-Okamoto, and J. N. Tomazella. The Euler obstruction of a function on a determinantal variety and on a curve. *Bull. Braz. Math. Soc. (N.S.)*, 47(3):955–970, 2016.

[3] D. N. Bernstein. The number of roots of a system of equations. *Funkcional. Anal. i Priložen.*, 9(3):1–4, 1975.

[4] C. Bivià-Ausina. The integral closure of modules, Buchsbaum-Rim multiplicities and Newton polyhedra. *J. London Math. Soc. (2)*, 69(2):407–427, 2004.

[5] J. P. Brasselet. Local Euler obstruction, old and new. In *XI Brazilian Topology Meeting (Rio Claro, 1998)*, pages 140–147. World Sci. Publ., River Edge, NJ, 2000.

[6] J.-P. Brasselet, D. T. Lê, and J. Seade. Euler obstruction and indices of vector fields. *Topology*, 39(6):1193–1208, 2000.

[7] J.-P. Brasselet and M.-H. Schwartz. Sur les classes de Chern d'un ensemble analytique complexe. In *The Euler-Poincaré characteristic (French)*, volume 82 of *Astérisque*, pages 93–147. Soc. Math. France, Paris, 1981.

[8] J.-P. Brasselet, J. Seade, and T. Suwa. *Vector fields on singular varieties*, volume 1987 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[9] W. Bruns and J. Herzog. Cohen-Macaulay rings. *Revised edition, Cambridge University Press, New York*, 1998.

[10] W. Bruns and U. Vetter. Determinantal rings. *Springer-Verlang, New York*, 1998.

[11] R. S. Carvalho, J. J. Nuño-Ballesteros, B. Oréfice-Okamoto, and J. N. Tomazella. Equisingularity of families of functions on isolated determinantal singularities. *Bull. Braz. Math. Soc. (N.S.)*, 53(1):1–20, 2022.

[12] T. Chen. Unmixing the mixed volume computation. *Discrete Comput. Geom.*, 62(1):55–86, 2019.

[13] T. M. Dalbelo. *Superfícies nultitóricas, obstrução de Euler e aplicações*. PhD thesis, 2014.

[14] J. Damon. The unfolding and determinacy theorems for subgroups of $\mathscr{A}$ and $\mathscr{K}$. *Mem. Amer. Math. Soc.*, 50(306):x+88, 1984.

[15] W. Ebeling and S. M. Gusein-Zade. On indices of 1-forms on determinantal singularities. *Tr. Mat. Inst. Steklova*, 267:119–131, 2009.

[16] A. Esterov. Indices of 1-forms and Newton polyhedra. *Rev. Mat. Complut.*, 18(1):233–242, 2005.

[17] A. Esterov. Indices of 1-forms, intersection indices, and Newton polyhedra. *Mat. Sb.*, 197(7):137–160, 2006.

[18] A. Esterov. Determinantal singularities and Newton polyhedra. *Tr. Mat. Inst. Steklova*, 259(Anal. i Osob. Ch. 2):20–38, 2007.

[19] A. Esterov. Determinantal singularities and Newton polyhedra. *arXiv:0906.5097v2*, 2010.

[20] A. Esterov. Newton polyhedra of discriminants of projections. *Discrete Comput. Geom.*, 44(1):96–148, 2010.

[21] C. Eyral and M. Oka. Non-compact Newton boundary and Whitney equisingularity for non-isolated singularities. *Adv. Math.*, 316:94–113, 2017.

[22] A. Frühbis-Krüger and A. Neumer. Simple Cohen-Macaulay codimension 2 singularities. *Comm. Algebra*, 38(2):454–495, 2010.

[23] A. Frühbis-Krüger and M. Zach. Determinantal singularities. In *Handbook of Geometry and Topology of Singularities IV*, pages 45–159. Springer International Publishing, 2023.

[24] T. Gaffney. Polar multiplicities and equisingularity of map germs. *Topology*, 32(1):185–223, 1993.

[25] T. Gaffney. Multiplicities and equisingularity of ICIS germs. *Invent. Math.*, 123(2):209–220, 1996.

[26] T. Gaffney, N. G. Grulha Jr., and M. A. S. Ruas. The local Euler obstruction and topology of the stabilization of associated determinantal varieties. *Math. Z.*, 291(3-4):905–930, 2019.

[27] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. Reprint of the 1994 edition.

[28] C. G. Gibson, K. Wirthmüller, A. A. du Plessis, and E. J. N. Looijenga. *Topological stability of smooth mappings*. Lecture Notes in Mathematics, Vol. 552. Springer-Verlag, Berlin-New York, 1976.

[29] G. González-Sprinberg. Calcul de l'invariant local d'Euler pour les singularités quotient de surfaces. *C. R. Acad. Sci. Paris Sér. A-B*, 288(21):A989–A992, 1979.

[30] G. González-Sprinberg. L'obstruction locale d'Euler et le théorème de MacPherson. In *The Euler-Poincaré characteristic (French)*, volume 82 of *Astérisque*, pages 7–32. Soc. Math. France, Paris, 1981.

[31] M. Goresky and R. MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.

[32] P. M. Gruber. *Convex and discrete geometry*, volume 336 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Berlin, 2007.

[33] N. G. Grulha Jr. *Obstrução de Euler de aplicações analíticas*. PhD thesis, 2007.

[34] R. C. Gunning and H. Rossi. *Analytic functions of several complex variables*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.

[35] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández. On the topology of germs of meromorphic functions and its applications. *Algebra i Analiz*, 11(5):92–99, 1999.

[36] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[37] A. G. Hovanskiĭ. Newton polyhedra, and the genus of complete intersections. *Funktsional. Anal. i Prilozhen.*, 12(1):51–61, 1978.

[38] V. H. Jorge-Pérez and M. J. Saia. Euler obstruction, polar multiplicities and equisingularity of map germs in $\mathscr{O}(n,p)$, $n<p$. *Internat. J. Math.*, 17(8):887–903, 2006.

[39] A. G. Kouchnirenko. Polyèdres de Newton et nombres de Milnor. *Invent. Math.*, 32(1):1–31, 1976.

[40] D. T. Lê and B. Teissier. Variétés polaires locales et classes de Chern des variétés singulières. *Ann. of Math. (2)*, 114(3):457–491, 1981.

[41] E. L. Lima. *Variedades diferenciáveis*, volume 15 of *Monografías de Matemática [Mathematical Monographs]*. Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1977.

[42] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.

[43] A. Menegon and M. S. Pereira. On deformations of isolated singularity function. *São Paulo J. Math. Sci.*, 2023.

[44] J. Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.

[45] J. J. Nuño-Ballesteros, B. Oréfice-Okamoto, and J. N. Tomazella. The vanishing Euler characteristic of an isolated determinantal singularity. *Israel J. Math.*, 197(1):475–495, 2013.

[46] J. J. Nuño-Ballesteros, B. Oréfice-Okamoto, and J. N. Tomazella. Equisingularity of families of isolated determinantal singularities. *Math. Z.*, 289(3-4):1409–1425, 2018.

[47] J. J. Nuño-Ballesteros and J. N. Tomazella. The Milnor number of a function on a space curve germ. *Bull. Lond. Math. Soc.*, 40(1):129–138, 2008.

[48] M. Oka. Principal zeta-function of nondegenerate complete intersection singularity. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 37(1):11–32, 1990.

[49] M. Oka. *Non-degenerate complete intersection singularity*. Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1997.

[50] M. Oka. Non-degenerate mixed functions. *Kodai Math. J.*, 33(1):1–62, 2010.

[51] Oscar – open source computer algebra research system, version 0.14.0-dev, 2023.

[52] M. S. Pereira. *Variedades determinantais e singularidades de matrizes*. PhD thesis, 2010.

[53] M. S. Pereira and M. A. S. Ruas. Codimension two determinantal varieties with isolated singularities. *Math. Scand.*, 115(2):161–172, 2014.

[54] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[55] M. Spivakovsky. Sandwiched singularities and desingularization of surfaces by normalized Nash transformations. *Ann. of Math. (2)*, 131(3):411–491, 1990.

[56] B. Teissier. Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney. In *Algebraic geometry (La Rábida, 1981)*, volume 961 of *Lecture Notes in Math.*, pages 314–491. Springer, Berlin, 1982.

[57] H. Whitney. Tangents to an analytic variety. *Ann. of Math. (2)*, 81:496–549, 1965.

[58] M. Zach. Bouquet decomposition for determinantal Milnor fibers. *Journal of Singularities*, 22:190–204, 2020.