UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Lebesgue solvability of equations associated to elliptic and canceling linear differential operators with measure data

Victor Sandrin Biliatto

São Carlos-SP
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"Le savant n'étudie pas la nature parce que cela est utile; il l'étudie parce qu'il y prend plaisir et il y prend plaisir parce qu'elle est belle. Si la nature n'était pas belle, elle ne vaudrait pas la peine d'être connue, la vie ne vaudrait pas la peine d'être vécue. Je ne parle pas ici, bien entendu, de cette beauté qui frappe les sens, de la beauté des qualités et des apparences; non que j'en fasse fi, loin de là, mais elle n'a rien à faire avec la science; je veux parler de cette beauté plus intime qui vient de l'ordre harmonieux des parties, et qu'une intelligence pure peut saisir."

Henri Poincaré, in *Science et Méthode*

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Eu nunca teria imaginado o quanto meu doutorado seria diferente do mestrado. Não do ponto de vista da matemática, da pesquisa, pois desse eu já esperava que fosse mais complicado, mais trabalhoso, até mais frustrante em certos momentos em que as ideias não funcionassem. O inesperado veio do lado social, humano. O período do meu doutorado, entre o segundo semestre de 2019 e o primeiro semestre de 2024, concorreu em seu começo com a pandemia da COVID-19 e, por isso, após um mísero semestre de disciplinas cursadas, foi preciso nos isolarmos em nossas casas por um longo tempo e realizarmos todas as atividades de maneira remota.

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***

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Resumo

Nesta tese, apresentamos novos resultados sobre a resolubilidade da equação $A^*(x,D)f = \mu$ para $f \in L^p$, dada uma medida complexa μ, associada a um operador diferencial linear elíptico $A(x,D)$ de ordem m com coeficientes complexos suaves. Nosso método se baseia no controle da energia$-(m,p)$ de μ oferecendo condições suficientes para a existência de soluções quando $1 \leqslant p < \infty$. Um estudo particular sobre resolubilidade global em espaços de Lebesgue da equação para equação $A^*(D)f = \mu$, no qual $A(D)$ é um operador diferencial homogêneo com coeficientes constantes também é apresentado. Obtemos também condições suficientes no caso limite $p = \infty$ usando novas estimativas L^1 (globais e locais) em medidas para operadores elípticos e cancelantes, que são de particular interesse.

Palavras-chave: Campos vetoriais de medida-divergência, resolubilidade em espaços de Lebesgue, estimativas L^1, equações elípticas, operadores cancelantes.

Abstract

In this thesis, we present new results on the solvability of the equation $A^*(x,D)f = \mu$ for $f \in L^p$, with complex measure data μ, associated to an elliptic linear differential operator $A(x,D)$ of order m with variable complex coefficients. Our method is based on $(m,p)-$energy control of μ giving sufficient conditions for solutions when $1 \leqslant p < \infty$. A particular study is presented in the global setting of Lebesgue solvability for the equation $A^*(D)f = \mu$, where $A(D)$ is a homogeneous differential operator with constant coefficients. We also obtain sufficient conditions in the limiting case $p = \infty$ using new L^1 (global and local) estimates on measures for elliptic and canceling operators, which are interesting on their own.

Keywords: Divergence-measure vector fields, Lebesgue solvability, L^1 estimates, elliptic equations, canceling operators.

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Introduction

This thesis is comprised by the results obtained by the author and his collaborators during his PhD. These results are collected in three articles:

- BILIATTO, V.; PICON, T. **A note on Lebesgue Solvability of Elliptic Homogeneous Linear Equations with Measure Data**. *J. Geom. Anal.*, v. 34, n. 1, 22, 2024;
- BILIATTO, V.; MOONENS, L.; PICON, T. **Hausdorff dimension of removable sets for elliptic and canceling homogeneous differential operators in the class of bounded functions**. *Submitted*, https://doi.org/10.48550/arXiv.2312.02560;
- BILIATTO, V.; PICON, T. **Sufficient Conditions for Local Lebesgue Solvability of Canceling and Elliptic Linear Differential Equations with Measure Data**. *Submitted*, https://dx.doi.org/10.2139/ssrn.4710804.

The contents of these papers are connected by a common thread: the study of sufficient conditions on a (vector-valued) complex Borel measure μ in order to obtain a function $f \in L^p$, for $1 \leqslant p \leqslant \infty$, which solves the equation

$$A^*(x,D)f = \mu$$

in distributional sense. Here, $A^*(\cdot,D)$ is the formal adjoint operator associated to a linear differential operator of order m on Ω, $N \geqslant 2$ and $1 \leqslant m < N$, from a finite dimensional complex vector space E to a finite dimensional complex vector space F, given by

$$A(x,D) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\partial^\alpha,$$

where $a_\alpha \in C^\infty(\Omega, \mathscr{L}(E,F))$ are smooth complex coefficients.

This research was motivated by the article *Characterizations of the Existence and Removable Singularities of Divergence-measure Vector Fields* [38], due to N. Phuc and M. Torres, previously studied by the author in his masters dissertation. In that work, they obtained characterizations for the existence of L^p solutions to the equation

$$\operatorname{div} \vec{f} = \nu.$$

A natural question that arose from this study was wondering if it would be possible to expand their results to a more general class of differential operators that includes the gradient $A(D) = \nabla$, whose formal adjoint is the divergence $A^*(D) = div$.

The results in [6] deal with a first, simpler case, where $A(\cdot,D)$ is homogeneous and has constant coefficients, i.e.

$$A(D) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha,$$

as is the case of the gradient. The main theorems are stated below. First, the case for $1 \leqslant p < \infty$:

Theorem A. *Let $A(D)$ be a homogeneous linear differential operator of order $1 \leqslant m < N$ on $\mathbb{R}^N$, $N \geqslant 2$, from E to F, and $\mu \in \mathscr{M}(\mathbb{R}^N, E^*)$ a complex-valued Borel measure.*

(i) If $1 \leqslant p \leqslant N/(N-m)$, $\mu \in \mathscr{M}_+(\mathbb{R}^N, E^*)$ and $f \in L^p(\mathbb{R}^N, F^*)$ is a solution for

$$A^*(D)f = \mu, \tag{1}$$

then $\mu \equiv 0$.

(ii) If $N/(N-m) < p < \infty$ and $f \in L^p(\mathbb{R}^N, F^*)$ is a solution for (1), then μ has finite $(m,p)-$energy. Conversely, if $|\mu|$ has finite $(m,p)-$energy and $A(D)$ is elliptic, then there exists a function $f \in L^p(\mathbb{R}^N, F^*)$ solving (1).

The endpoint case $p = \infty$ is treated separately, assuming an extra condition related to canceling operators satisfying a special L^1 type estimate.

Theorem B. *Let $A(D)$ be a homogeneous linear differential operator of order $1 \leqslant m < N$ on $\mathbb{R}^N$ from E to F and $\mu \in \mathscr{M}(\mathbb{R}^N, E^*)$. If $A(D)$ is elliptic and canceling, and μ satisfies*

$$\|\mu\|_{0,N-m} \doteq \sup_{r>0} \frac{|\mu|(B_r)}{r^{N-m}} < \infty,$$

and the potential control

$$\int_0^{|y|/2} \frac{|\mu|(B(y,r))}{r^{N-m+1}}\, dr \lesssim 1, \quad \textit{uniformly on } y,$$

then, there exists $f \in L^\infty(\mathbb{R}^N, F^*)$ solving (1).

In [38], the solvability results are used to characterize removable singularities for the divergence equation. In the same spirit, in [5] we use Theorem B to prove the following necessary condition:

Theorem C. *Assume that $A(D)$ is an elliptic and canceling homogeneous differential operator on $\mathbb{R}^N$ of order $1 \leqslant m < N$, from E to F. If the closed set $S \subseteq \mathbb{R}^N$ is removable for the equation $A^*(D)f = 0$ in $L^\infty(\mathbb{R}^N, F^*)$, then S has Hausdorff dimension less than or equal to $N - m$.*

Finally, in [7] we step into an even more general case, considering $A(\cdot,D)$ defined on an open subset $\Omega \subset \mathbb{R}^N$ with variable smooth complex coefficients a_α. We obtain the following sufficient conditions for local solvability:

Theorem D. *Let $A(\cdot,D)$ be an elliptic linear differential operator of order $1 \leqslant m < N$ on Ω from E to F, $1 < p < \infty$ and $\mu \in \mathscr{M}(\Omega, E^*)$. If, for each $x_0 \in \Omega$, there exists an open neighborhood $U \ni x_0$ of Ω such that $|\mu|$ has finite strong $(m,p)-$energy on U, then the equation*

$$A^*(x,D)f = \mu \tag{2}$$

is L^p locally solvable in Ω.

Analogously to the homogeneous operators with constant coefficients, we state a version for the case $p = \infty$.

Theorem E. *Let $A(\cdot,D)$ be a linear differential operator of order $1 \leqslant m < N$ on Ω from E to F and $\mu \in \mathscr{M}(\Omega, E^*)$. Suppose that $A(\cdot,D)$ is elliptic and canceling in Ω and μ satisfies*

$$\|\mu\|_{\Omega,N-m} \doteq \sup_{B(x,r)\subset\Omega} \frac{|\mu|(B(x,r))}{r^{N-m}} < \infty.$$

Then, for each fixed $x_0 \in \Omega$, there exists an open neighborhood $U \ni x_0$ in Ω such that, if the potential condition

$$\int_0^{a|y-x_0|} \frac{|\mu|(B(y,r))}{r^{N-m+1}}\, dr \lesssim 1,$$

where a is some constant between 0 and 1, is satisfied uniformly for almost every $y \in U$, then there exists a function $f \in L^\infty(U,F^*)$ solving (2).

The goal of this thesis is to prove the Theorems A-E introducing a new machinery in the setting of higher order operators. The text is organized as follows:

In Chapter 1, definitions and results which are necessary for the main proofs are presented. It contains sections about the results in [38], vector-valued measures, measures with finite energy, the definition of elliptic, canceling and cocanceling operators, Stein-Weiss inequalities, Riesz transforms and pseudo-differential operators.

Chapter 2 is devoted to the results from [6]. Theorem A is shown first, then a Stein-Weiss type inequality is proved in order to obtain Theorem B. The chapter ends with some applications and comments, including a reciprocal to Theorem B for first order operators.

In Chapter 3, the results from [5] are exhibited. The definition of removable singularity is introduced, together with some previously known results. Then, the proof of a version of Frostman's lemma with a decay condition is followed by Theorem C.

Chapter 4 focus on the results from [7]. First, some topics from Chapter 1 are revisited, namely, a stronger definition for measures with finite energy and local definitions of ellipticity, cancelation and cocancelation are given. Then, Theorems D and E are proved, followed by comments and applications.

An appendix at the end of the text outlines the proof of an important estimate from [38]. Although this estimate inspired the reasoning behind the proofs of Theorems B and E, its proof highlights why

Phuc and Torres's argument for ∇ does not work in the general case for $A(\cdot,D)$.

Notation: throughout this work, Ω always denotes an open subset of $\mathbb{R}^N$. The symbol $f \lesssim g$ means that there exists a constant $C > 0$, depending neither on f nor on g, such that $f \leqslant Cg$. Given a set $A \subset \mathbb{R}^N$ we denote by $|A|$ its Lebesgue measure. We write $B = B(x,R)$ for the open ball with center x and radius $R > 0$. By B_R we mean the ball $B(0,R)$. We fix $\fint_Q f(x)dx := \frac{1}{|Q|}\int_Q f(x)dx$, where Q is a measurable set. We write $K \subset\subset \Omega$ to say that K is a compact subset of Ω.

CHAPTER 1

Preliminaries

In this chapter we present some definitions and results that will be necessary throughout this work.

1.1 Characterizations of the existence of divergence-measure vector fields

N. Phuc and M. Torres in [38] characterized the existence of solutions in Lebesgue spaces for the divergence equation

$$\operatorname{div} \vec{f} = \nu, \tag{1.1}$$

where $\nu \in \mathscr{M}_+(\mathbb{R}^N)$, the set of scalar positive Borel measures on $\mathbb{R}^N$, and $f \in L^p(\mathbb{R}^N, \mathbb{R}^N)$. The method is based on controlling the $(1,p)-$ energy of ν defined by $\|I_1\nu\|_{L^p}$, where I_1 is the Riesz potential operator. In fact, $\|I_1\nu\|_{L^p}$ finite is a necessary condition for solvability in L^p, since from (1.1) we have

$$I_1\nu = c_N \sum_{j=1}^{N} R_j f_j \tag{1.2}$$

and the control in norm follows as a direct consequence of the continuity of Riesz transform operators R_j in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$. The following result was proved in [38, Theorems 3.1 and 3.2]:

Theorem 1.1. *If $f \in L^p(\mathbb{R}^N, \mathbb{R}^N)$ satisfies* (1.1) *for some $\nu \in \mathscr{M}_+(\mathbb{R}^N)$, then*

(i) $\nu = 0$, assuming $1 \leqslant p \leqslant N/(N-1)$;

(ii) ν has finite $(1,p)$-energy, assuming $N/(N-1) < p < \infty$. Conversely, if $\nu \in \mathscr{M}_+(\mathbb{R}^N)$ has finite $(1,p)-$energy, then there is a vector field $f \in L^p(\mathbb{R}^N, \mathbb{R}^N)$ satisfying (1.1).

The previous result does not cover the case $p = \infty$, since the proof breaks down once the Riesz transform is not bounded in $L^\infty(\mathbb{R}^N)$. However from Gauss-Green theorem, if $f \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ is a solution of (1.1) then for any ball $B(x,r)$ there exists $C = C(N) > 0$ such that

$$\nu(B(x,r)) = \int_{\partial B(x,r)} f \cdot n \, d\mathscr{H}^{N-1} \leqslant C\|f\|_{L^\infty} r^{N-1}.$$

It is easy to check that $\|I_1\nu\|_{L^\infty} < \infty$ implies the following control of the measure ν on balls

$$\nu(B(x,r)) \leqslant Cr^{N-1}, \tag{1.3}$$

where the constant is independent of $x \in \mathbb{R}^N$ and $r > 0$. Indeed,

$$I_1\nu(x) \geqslant C\int_{B(x,r)} \frac{1}{|x-y|^{N-1}}\,d\nu(y) \geqslant C\int_{B(x,r)} \frac{1}{r^{N-1}}\,d\nu(y) = \frac{C\,\nu(B(x,r))}{r^{N-1}},$$

hence we have (1.3). A non-trivial argument (see Appendix A) is sufficient to show that (1.3) implies

$$\left|\int_{\mathbb{R}^n} u(x)d\nu\right| \leqslant C\|\nabla u\|_{L^1}, \quad \forall u \in C_c^\infty(\mathbb{R}^n) \tag{1.4}$$

and from a standard duality argument a solution f for (1.1) in $L^\infty(\mathbb{R}^N,\mathbb{R}^N)$ is obtained. Hence, they proved the following result in this case [38, Theorem 3.3]:

Theorem 1.2. *If $f \in L^\infty(\mathbb{R}^N,\mathbb{R}^N)$ satisfies* (1.1) *for some $\nu \in \mathscr{M}_+(\mathbb{R}^N)$, then ν satisfies* (1.3) *for every $x \in \mathbb{R}^N$, $r > 0$ and some constant C independent of x and r. Conversely, if $\nu \in \mathscr{M}_+(\mathbb{R}^N)$ has the property* (1.3)*, then there is a vector field $f \in L^\infty(\mathbb{R}^N,\mathbb{R}^N)$ satisfying* (1.1).

In Phuc and Torres's proof of (1.4), the fact that one is dealing with the divergence operator plays a very specific role through the co-area formula (see Appendix A), suggesting that their argument does not adapt easily into obtaining a solvability result for other operators than the divergence operator.

1.2 Vector-valued measures and total variation

Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We denote by $\mathscr{M}(\Omega)$ the set of signed (i.e., real-valued) Borel measures on Ω. We add the subscript $\mathscr{M}_+(\Omega)$ to denote the set of positive Borel measures on Ω. We write $\mathscr{M}(\Omega,\mathbb{C})$ for the set of complex-valued Borel measures on Ω given by $\mu = \mu^{\mathrm{Re}} + i\mu^{\mathrm{Im}}$, where $\mu^{\mathrm{Re}}, \mu^{\mathrm{Im}} \in \mathscr{M}(\Omega)$. By $\mathscr{M}_+(\Omega,\mathbb{C})$ we mean the set of measures $\mu \in \mathscr{M}(\Omega,\mathbb{C})$ such that $\mu^{\mathrm{Re}}, \mu^{\mathrm{Im}} \in \mathscr{M}_+(\Omega)$. Let X be a complex vector space with $\dim_{\mathbb{C}} X = d < \infty$. We denote by $\mathscr{M}(\Omega,X)$ the set of all X-valued complex measures on Ω, $\mu = (\mu_1,\ldots,\mu_d)$, where $\mu_\ell = \mu_\ell^{\mathrm{Re}} + i\mu_\ell^{\mathrm{Im}} \in \mathscr{M}(\Omega,\mathbb{C})$ for all $\ell = 1,\ldots,d$. Similarly, $\mathscr{M}_+(\Omega,X)$ is the set of measures $\mu \in \mathscr{M}(\Omega,X)$ such that $\mu_\ell \in \mathscr{M}_+(\Omega,\mathbb{C})$ for all $\ell = 1,\ldots,d$. Here we are implicitly interchanging X and $\mathbb{C}^d$.

The theory of vector-valued measures has substantial differences in comparison to that of scalar-valued ones. Within the scope of this text, however, the classic properties and results we know for scalar-values measures remain valid for countably additive vector-valued measures (see [3]). Let $\mu \in \mathscr{M}(\Omega,X)$. If f is a scalar-valued function defined on Ω, then

$$\int f\,d\mu = \left(\int f\,d\mu_1, \cdots, \int f\,d\mu_d\right).$$

If $g = (g_1, \ldots, g_d)$ is an X-valued function defined on Ω, then

$$\int g\, d\mu = \left(\int g_1\, d\mu_1, \cdots, \int g_d\, d\mu_d \right).$$

If $\nu \in \mathscr{M}(\Omega, \mathbb{C})$, the *total variation of* ν is the positive measure defined, for each ν-measurable set A, by $|\nu|(A) = \sup \sum_k |\nu(A_k)|$, where the supremum is taken over all partitions $\{A_k\}$ of A into measurable sets. The total variation of ν is, by construction, the smallest positive measure λ such that $|\nu(A)| \leqslant \lambda(A)$ for every ν-measurable set A. It is known that, for any $\nu \in \mathscr{M}(\Omega, \mathbb{C})$, one has $|\nu|(\Omega) < \infty$ and, therefore, any complex measure is bounded. For X-valued measures, the definition is similar. If $\mu \in \mathscr{M}(\Omega, X)$, the *total variation of* μ is the positive measure defined, for each μ-measurable set A, by $|\mu|(A) = \sup \sum_k |\mu(A_k)| = \sup \sum_k \sqrt{\sum_{\ell=1}^d |\mu_\ell(A_k)|^2}$, where the supremum is now taken over all partitions $\{A_k\}$ of A into *a finite number* of measurable sets. One interesting property is that $|\mu|$ is comparable with $\sum_{\ell=1}^d |\mu_\ell|$. More specifically, $|\mu| \leqslant \sum_{\ell=1}^d |\mu_\ell| \lesssim |\mu|$. Indeed, $|\mu_\ell| \leqslant |\mu|$ for all $\ell = 1, \ldots, d$, since $|\mu_\ell(A_k)| \leqslant |\mu(A_k)|$, thus $\sum_{\ell=1}^d |\mu_\ell| \lesssim |\mu|$. For the converse,

$$\sum_k |\mu(A_k)| \leqslant \sum_k \sum_{\ell=1}^d |\mu_\ell(A_k)| = \sum_{\ell=1}^d \sum_k |\mu_\ell(A_k)|.$$

The summations can be swapped as the sum in k converges absolutely by the definition of complex measures. Hence

$$|\mu|(A) \leqslant \sup \sum_{\ell=1}^d \sum_k |\mu_\ell(A_k)| \leqslant \sum_{\ell=1}^d \sup \sum_k |\mu_\ell(A_k)| = \sum_{\ell=1}^d |\mu_\ell|(A).$$

Definition 1.3. We say that a measure $\mu \in \mathscr{M}(\Omega)$ is λ*-Ahlfors regular*, for $1 \leqslant \lambda < \infty$, if it satisfies the Morrey control given by

$$\|\mu\|_\lambda \doteq \sup_B \frac{|\mu|(B(x,r))}{r^\lambda} < \infty,$$

where the supremum is taken over all open balls $B = B(x,r)$ in Ω.

In other words, μ is λ-Ahlfors regular if $|\mu|(B(x,r)) \leqslant Cr^\lambda$ for every $x \in \Omega$ and $r > 0$, where C is independent of x and r. For $\Omega = \mathbb{R}^N$, we introduce the notation

$$\|\mu\|_{0,\lambda} \doteq \sup_{r>0} \frac{|\mu|(B_r)}{r^\lambda}$$

for the case when the supremum is taken only over balls centered at the origin.

1.3 Energy and potentials of measures

For any $0 < m < N$ and any function f in the Schwartz space $\mathscr{S}(\mathbb{R}^N)$, consider the fractional integrals called *Riesz potential* operators given by the action of the multiplier $\widehat{I_m f}(\xi) = |\xi|^{-m} \widehat{f}(\xi)$. Thus, $I_m f$ is defined by

$$I_m f(x) = \frac{1}{\gamma(m)} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-m}} dy,$$

with $\gamma(m) := \pi^{N/2}\, 2^m\, \Gamma(m/2)/\Gamma((N-m)/2)$, where Γ is the standard Gamma function.

It is important to point out that the constant in the previous formula depends on the definition of Fourier transform that is being used. Here, for $f \in \mathscr{S}$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} f(x)\, dx, \tag{1.5}$$

and the inversion formula becomes

$$f(x) = \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi}\, \widehat{f}(\xi)\, d\xi. \tag{1.6}$$

We extend this definition for measures. Let $\eta \in \mathscr{M}(\Omega, X)$. Then, we define the Riesz potential of η by

$$I_m\eta(x) := \frac{1}{\gamma(m)} \int_\Omega \frac{1}{|x-y|^{N-m}} d\eta(y)$$

if $X = \mathbb{C}$, and $I_m\eta := (I_m\eta_1, \ldots, I_m\eta_d)$ for a general vector space X.

Definition 1.4. Let $1 \leqslant p < \infty$ and $0 < m < N$. We say that $\mu \in \mathscr{M}(\Omega, X)$ has *finite* $(m,p)-$*energy* if

$$\|I_m\mu\|_{L^p} := \left(\int_{\mathbb{R}^N} |I_m\mu(x)|^p\, dx \right)^{1/p} < \infty,$$

and μ has *finite* $(m,1)^*-$ *energy* if

$$\|I_m\mu\|_{L^{1,\infty}} \doteq \sup_{\lambda>0} \lambda\; |\{x : |I_m\mu(x)| > \lambda\}| < \infty.$$

From the previous definitions follows $\|I_m\mu_\ell^{\mathrm{Re}}\|_{L^p} + \|I_m\mu_\ell^{\mathrm{Im}}\|_{L^p} \lesssim \|I_m\mu\|_{L^p}$ for $\ell = 1, \ldots, d$. The same control holds replacing L^p by $L^{1,\infty}$.

Proposition 1.5. *If $\mu \in \mathscr{M}_+(\Omega, X)$ has finite $(m,p)-$energy for some $1 < p \leqslant N/(N-m)$ or $(m,1)^*-$ energy, then $\mu \equiv 0$ on Ω.*

Proof. Let $R > 0$ and, by simplicity, we assume $\mu_\ell \in \mathscr{M}_+(\Omega)$ for each $\ell \in \{1, \ldots, d\}$. We have

$$\begin{aligned} I_m\mu_\ell(x) &\gtrsim \int_{B_R\cap\Omega} \frac{1}{|x-y|^{N-m}}\, d\mu_\ell(y) \\ &\geqslant \int_{B_R\cap\Omega} \frac{1}{(|x|+R)^{N-m}}\, d\mu_\ell(y) \\ &= \frac{\mu_\ell(B_R\cap\Omega)}{(|x|+R)^{N-m}}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^N} |I_m\mu(x)|^p\, dx \gtrsim \int_{\mathbb{R}^N} [I_m\mu_\ell(x)]^p\, dx &\gtrsim \int_{\mathbb{R}^N} \left[\frac{\mu_\ell(B_R\cap\Omega)}{(|x|+R)^{N-m}} \right]^p dx \\ &= [\mu_\ell(B_R\cap\Omega)]^p \int_{\mathbb{R}^N} \frac{1}{(|x|+R)^{(N-m)p}}\, dx. \end{aligned}$$

Observe that

$$\int_{\mathbb{R}^N} \frac{1}{(|x|+R)^{(N-m)p}}\,dx = c(N)\int_0^\infty \frac{r^{N-1}}{(r+R)^{(N-m)p}}\,dr$$
$$= c(N)\int_R^\infty \frac{(r-R)^{N-1}}{r^{(N-m)p}}\,dr$$

and for $1 < p \leqslant N/(N-m)$ the last integral blows up to infinity, as $N-1-(N-m)p \geqslant -1$. Hence we must have $\mu_\ell(B_R \cap \Omega) = 0$, since $\|I_m\mu\|_{L^p} < \infty$. For the case $p = 1$, we have

$$\sup_{\lambda>0} \lambda \left| \left\{ x \in \mathbb{R}^N : \frac{\mu_\ell(B_R \cap \Omega)}{(|x|+R)^{N-m}} > \lambda \right\} \right| \lesssim \|I_m\mu\|_{L^{1,\infty}} < \infty.$$

However,

$$\lambda \left| \left\{ x : \frac{\mu_\ell(B_R \cap \Omega)}{(|x|+R)^{N-m}} > \lambda \right\} \right| = \lambda \left| B\left(0, \left(\frac{\mu_\ell(B_R\cap\Omega)}{\lambda}\right)^{\frac{1}{N-m}} - R\right) \right|$$
$$= \lambda^{-\frac{m}{N-m}} \left| B\left(0, \mu_\ell(B_R\cap\Omega)^{\frac{1}{N-m}} - \lambda^{\frac{1}{N-m}} R\right) \right|,$$

which blows-up to infinity when $\lambda > 0$ is small and $\mu_\ell(B_R \cap \Omega) \neq 0$. Given that $R > 0$ was arbitrarily chosen, and that $\Omega = \bigcup_{k\in\mathbb{N}} [B_k \cap \Omega]$, we conclude that $\mu_\ell \equiv 0$ on Ω for every $\ell \in \{1, \ldots, d\}$. Therefore, $\mu \equiv 0$. □

L. Hedberg and T. Wolff introduced in [21] a notion of potential within the framework of nonlinear potential operators. For a positive Borel measure ν on $\mathbb{R}^N$, $1 < p < \infty$ and $\alpha > 0$, the *Wolff potential* $W_{\alpha,p}$ *of* ν is defined as

$$W_{\alpha,p}\nu(x) = \int_0^\infty \left[\frac{\nu(B(x,r))}{r^{N-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dr}{r}, \qquad \text{for } x \in \mathbb{R}^N.$$

There is also a truncated version of this potential that works fine on bounded domains $\Omega \subset \mathbb{R}^N$, where the integration is done in a bounded interval $(0,t)$ for some fixed $t > 0$:

$$W^t_{\alpha,p}\nu(x) = \int_0^t \left[\frac{\nu(B(x,r))}{r^{N-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dr}{r}, \qquad \text{for } x \in \Omega.$$

One of the hypothesis we introduced in our solvability results can be understood as an uniform control of the truncated Wolff potential. Applications of the Wolff potential can be found, for instance, in [1, 40].

1.4 Elliptic, canceling and cocanceling operators

Let $A(D)$ be a homogeneous linear differential operator of order m on $\mathbb{R}^N$, $N \geqslant 2$, from a finite dimensional complex vector space E to a finite dimensional complex vector space F, given by

$$A(D) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha : C^\infty(\mathbb{R}^N, E) \to C^\infty(\mathbb{R}^N, F).$$

The nomenclature *homogeneous* emphasizes that all the partial derivatives in $A(D)$ have the same order m. The coefficients a_α belong to the set $\mathscr{L}(E,F)$ of linear transformations from E to F. They are constant, in the sense that they do not depend on $x \in \mathbb{R}^N$. An important function associated to $A(D)$ is its *symbol*: a linear transformation $A(\xi): E \to F$ defined, for each $\xi \in \mathbb{R}^N$, by

$$A(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha,$$

which is, in essence, the Fourier transform of $A(D)$, but avoiding the multiplicative constants.

Definition 1.6. A homogeneous linear differential operator $A(D)$ on $\mathbb{R}^N$ from E to F is said to be *elliptic* if, for every $\xi \in \mathbb{R}^N \setminus \{0\}$, its symbol $A(\xi)$ is injective.

We present some examples of elliptic homogeneous operators.

Example 1.7. The gradient operator $\nabla : C^\infty(\mathbb{R}^N,\mathbb{R}) \to C^\infty(\mathbb{R}^N,\mathbb{R}^N)$ is elliptic.

Observe that

$$\nabla u = \sum_{j=1}^{N} e_j \partial_{x_j} u,$$

therefore the symbol $A(\xi): \mathbb{R} \to \mathbb{R}^N$ is given by $A(\xi)(t) = t\,\xi$, which is obviously injective for every $\xi \in \mathbb{R}^N \setminus \{0\}$.

Example 1.8. The Laplace operator $\Delta : C^\infty(\mathbb{R}^N,\mathbb{R}) \to C^\infty(\mathbb{R}^N,\mathbb{R})$ is elliptic.

Since

$$\Delta u = \sum_{j=1}^{N} \partial_{x_j}^2 u,$$

we have $A(\xi): \mathbb{R} \to \mathbb{R}$ given by $A(\xi)(t) = |\xi|^2 t$, which is injective for every $\xi \in \mathbb{R}^N \setminus \{0\}$.

Example 1.9. The vector Laplace operator $\Delta : C^\infty(\mathbb{R}^N,\mathbb{R}^M) \to C^\infty(\mathbb{R}^N,\mathbb{R}^M)$, given by

$$\Delta f = (\Delta f_1, \ldots, \Delta f_M),$$

is elliptic as $A(\xi): \mathbb{R}^M \to \mathbb{R}^M$ is given by $A(\xi)(v) = |\xi|^2 v$, which is injective for every $\xi \in \mathbb{R}^N \setminus \{0\}$.

Example 1.10. Let $C^\infty(\Omega,\Lambda^k\mathbb{R}^N)$, for $k \in \{0,\ldots,N\}$, be the space of k-forms on $\mathbb{R}^N$ with smooth coefficients defined on an open subset $\Omega \subseteq \mathbb{R}^N$. A k-form $f \in C^\infty(\Omega,\Lambda^k\mathbb{R}^N)$ can be written as

$$f = \sum_{|I|=k} f_I \, dx_I,$$

where $I = \{i_1,\ldots,i_k\}$ is an ordered set of strictly increasing indices $i_\ell \in \{1,\ldots,N\}$, $f_I \in C^\infty(\Omega)$ and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is the wedge product. The exterior derivative operators $d_k : C^\infty(\Omega,\Lambda^k\mathbb{R}^N) \to C^\infty(\Omega,\Lambda^{k+1}\mathbb{R}^N)$ are defined by $d_0 f \doteq \sum_{j=1}^N \partial_{x_j} f \, dx_j$ for $f \in C^\infty(\Omega) = C^\infty(\Omega,\Lambda^0\mathbb{R}^N)$, and

$$d_k f \doteq \sum_{|I|=k} (d_0 f_I)\, dx_I = \sum_{|I|=k} \sum_{j=1}^{N} \partial_{x_j} f_I \, dx_j \wedge dx_I$$

for $f \in C^\infty(\Omega, \Lambda^k \mathbb{R}^N)$, $1 \leqslant k \leqslant N-1$. Consider also, for $0 \leqslant k \leqslant N-1$, the co-exterior derivative operators $d_k^* : C^\infty(\Omega, \Lambda^{k+1}\mathbb{R}^N) \to C^\infty(\Omega, \Lambda^k \mathbb{R}^N)$ defined by

$$\int d_k u \cdot v \, dx = \int u \cdot d_k^* v \, dx, \quad u \in C_c^\infty(\mathbb{R}^N, \Lambda^k \mathbb{R}^N) \text{ and } v \in C_c^\infty(\mathbb{R}^N, \Lambda^{k+1}\mathbb{R}^N)$$

where the dot indicates the standard pairing on forms of the same degree. For each $f \in C^\infty(\Omega, \Lambda^{k+1}\mathbb{R}^N)$ given by $\sum_{|I|=k+1} f_I \, dx_I$, we may write

$$d_k^* f \doteq \sum_{|I|=k+1} \sum_{j \in I} -\partial_{x_j} f_I \, dx_j \vee dx_I.$$

Above, for each $j_\ell \in I = \{j_1, \dots, j_{k+1}\}$,

$$dx_{j_\ell} \vee dx_I \doteq (-1)^{\ell+1} dx_{j_1} \wedge \cdots \wedge dx_{j_{\ell-1}} \wedge dx_{j_{\ell+1}} \wedge \cdots \wedge dx_{j_{k+1}}.$$

The chain $\{d_k\}_k$ defines a complex of differential operators, in the sense that $d_{k+1} \circ d_k = 0$, called *de Rham complex*.

Consider, for $0 \leqslant k \leqslant N$, the operator

$$A(D) = (d_k, d_{k-1}^*) : C_c^\infty(\mathbb{R}^N, \Lambda^k \mathbb{R}^N) \to C_c^\infty(\mathbb{R}^N, \Lambda^{k+1}\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N, \Lambda^{k-1}\mathbb{R}^N).$$

When $k = 0$ and $k = N$, the operators d_{-1}^* and d_N, respectively, must be understood as zero. We claim that the operator $A(D)$ is elliptic. In fact, the symbol $A(\xi) : \Lambda^k(\mathbb{R}^N) \to \Lambda^{k+1}(\mathbb{R}^N) \times \Lambda^{k-1}(\mathbb{R}^N)$ is given by $A(\xi)(v) = (\xi \wedge v, \star(\xi \wedge \star v))$, where $\star$ denotes the Hodge star operator (see [28, Section 1.7]). The ellipticity follows from the Lagrange identity

$$|\xi|^2 |v|^2 = |\xi \wedge v|^2 + |\star(\xi \wedge \star v)|^2.$$

Example 1.11. The Laplace-Beltrami operator $(d_k^* \, d_k, d_{k-1} \, d_{k-1}^*) : C_c^\infty(\mathbb{R}^N, \Lambda^k \mathbb{R}^N) \to C_c^\infty(\mathbb{R}^N, \Lambda^k \mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N, \Lambda^k \mathbb{R}^N)$ is elliptic for $k \in \{1, \dots, N-1\}$. In fact, it is consequence of the identity

$$\Delta_k = d_{k-1} \, d_{k-1}^* + d_k^* \, d_k,$$

where $(\Delta_k f)_I = \Delta f_I$ for each $|I| = k$ (see [25, Lemma 3.1]).

Example 1.12. The Korn-Sobolev-Strauss operator $D_s : C_c^\infty(\mathbb{R}^N, \mathbb{R}^N) \to C_c^\infty(\mathbb{R}^N, \mathbb{R}^{N(N+1)/2})$ given by $D_s u(x) \doteq f(x)$ with

$$f_{j,k}(x) \doteq \frac{\partial_{x_j} u_k(x) + \partial_{x_k} u_j(x)}{2}, \quad 1 \leqslant j \leqslant k \leqslant N,$$

is elliptic. Its symbol $D_s(\xi) : \mathbb{R}^N \to \mathbb{R}^{N(N+1)/2}$ is given by

$$D_s(\xi)_{j,k}(v) = \frac{\xi_j v_k + \xi_k v_j}{2}, \quad 1 \leqslant j \leqslant k \leqslant N.$$

Let $\xi \neq 0$ and $v \in \mathbb{R}^N$ such that $D_s(\xi)(v) = 0$. Then, in particular, $D_s(\xi)_{j,j}(v) = \xi_j v_j = 0$ for every $1 \leqslant j \leqslant N$. Without loss of generality, suppose $\xi_1 \neq 0$. Then $v_1 = 0$, which means that $D_s(\xi)_{1,k}(v) = \frac{1}{2}\xi_1 v_k = 0$ for every $1 \leqslant k \leqslant N$. Therefore, $v = 0$.

Example 1.13. Consider, for $m \in \mathbb{N}$ and $0 \leqslant k \leqslant N$, the Lanzani-Raich operator

$$A(D) = (d_k(d_k^* d_k)^m, (d_{k-1}^* d_{k-1})^m d_{k-1}^*) : C_c^\infty(\mathbb{R}^N, \Lambda^k \mathbb{R}^N) \to C_c^\infty(\mathbb{R}^N, \Lambda^{k+1}\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N, \Lambda^{k-1}\mathbb{R}^N)$$

as a higher order div-curl operator (see [30]). Analogous to the case $m = 0$, the operator is elliptic.

Definition 1.14. A homogeneous linear differential operator $A(D)$ on $\mathbb{R}^N$ from E to F is said to be *canceling* if

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)[E] = \{0\}.$$

The theory of canceling operators was introduced by J. Van Schaftingen, motivated by studies of some L^1 *a priori* estimates for vector fields with divergence free and chain complexes. J. Bourgain and H. Brezis proved in [8, Theorem 5] the following solvability result:

Theorem 1.15. *If $N \geqslant 2$ and $1 \leqslant k \leqslant N-1$ we have*

$$d_k[\dot{W}^{1,N}(\Lambda^k \mathbb{R}^N)] = d_k[(\dot{W}^{1,N} \cap L^\infty)(\Lambda^k \mathbb{R}^N)].$$

More precisely, given $X \in \dot{W}^{1,N}(\Lambda^k \mathbb{R}^N)$, there exist some $Y \in (\dot{W}^{1,N} \cap L^\infty)(\Lambda^k \mathbb{R}^N)$ and a constant $C > 0$ such that one has $d_k X = d_k Y$ as well as:

$$\|\nabla Y\|_{L^N} + \|Y\|_{L^\infty} \leqslant C \|d_k X\|_{L^N}. \tag{1.7}$$

Here, $\dot{W}^{k,p}$ denotes the homogeneous Sobolev space $W^{k,p}$. Clearly the result fails for $k = 0$, i.e. for $d_0 = \nabla$ (see [8]), and the same statement holds for the operator d_k^* when $2 \leqslant k \leqslant N$. The proof of Theorem 1.15 is a particular case of the following result from [8, Theorem 10]:

Theorem 1.16. *Let $S : \bigoplus_{s=1}^r \dot{W}^{1,N}(\mathbb{R}^N) \to Y$ to be a bounded operator into a Banach space Y with closed range. Assume further that for each $s = 1,..,r$ there is an index $i_s \in \{1,...,N\}$ such that:*

$$\|Sf\| \leqslant C \max_{1 \leqslant s \leqslant r} \max_{i \neq i_s} \|\partial_i f_s\|_{L^N}.$$

Then for all $\vec{f} \in \bigoplus_{s=1}^r \dot{W}^{1,N}(\mathbb{R}^N)$ there exist $\vec{g} \in \bigoplus_{s=1}^r (\dot{W}^{1,N} \cap L^\infty)(\mathbb{R}^N)$ and constants $C, C' > 0$ satisfying $S\vec{f} = S\vec{g}$ and:

$$\|\nabla g\|_{L^N} + \|g\|_{L^\infty} \leqslant C\|S\vec{g}\| \leqslant C'\|\nabla f\|_{L^N}. \tag{1.8}$$

As a consequence of the previous theorem and by the Hahn-Banach theorem, for the first-order div-curl operator $A(D) = (d_k, d_{k-1}^*)$ for $N \geqslant 4$ and $2 \leqslant k \leqslant N-2$ the [8, Corollary 24] asserts that the estimate

$$\|u\|_{L^{N/N-1}} \leqslant C \left(\|d_k u\|_{L^1 + W^{-1,N/N-1}} + \|d_{k-1}^* u\|_{L^1 + W^{-1,N/N-1}} \right), \quad \forall u \in C_c^\infty(\mathbb{R}^N, \Lambda^k \mathbb{R}^N), \tag{1.9}$$

holds where $\|h\|_{L^1 + W^{-1,d/d-1}} := \inf \left\{ \|f\|_{L^1} + \|g\|_{W^{-1,d/d-1}} \text{ such that } h = f + g \right\}$. In particular,

$$\|u\|_{L^{N/N-1}} \leqslant C \left(\|d_k u\|_{L^1} + \|d_{k-1}^* u\|_{L^1} \right), \quad \forall u \in C_c^\infty(\mathbb{R}^N, \Lambda^k \mathbb{R}^N), \tag{1.10}$$

extending the classical Sobolev-Gagliardo-Nirenberg estimates taking $k = 0$, i.e $d_0 = \nabla$ and $d^*_{-1} = 0$. Independently, using a simple approach, Lanzani and Stein in [31] proved the inequality (1.10) for $N \geqslant 3$ when k is neither 1 nor $N-1$, moreover

$$\|u\|_{L^{N/N-1}} \leqslant C\left(\|d_1 u\|_{L^1} + \|d_0^* u\|_{H^1}\right), \quad \forall u \in C_c^\infty(\mathbb{R}^N, \Lambda^1\mathbb{R}^N), \tag{1.11}$$

and

$$\|u\|_{L^{N/N-1}} \leqslant C\left(\|d_{N-1} u\|_{H^1} + \|d_{N-2}^* u\|_{L^1}\right), \quad \forall u \in C_c^\infty(\mathbb{R}^N, \Lambda^{N-1}\mathbb{R}^N), \tag{1.12}$$

where H^1 is the Hardy space when $p = 1$. Several other L^1 inequalities of the type

$$\|u\|_{L^{N/(N-1)}} \leqslant C\|A(D)u\|_{L^1}, \quad \forall u \in C_c^\infty(\mathbb{R}^N, E),$$

for first order operators were obtained with some additional compatibility assumption (see [8, Corollary 26] for the Korn's inequalities) as in the case of div-curl operator $A(D) = (d_k, d^*_{k-1})$. Estimates of the type

$$\|D^{m-1}u\|_{L^{p^*}} \leqslant C_p\|A(D)u\|_{L^p}, \quad u \in C_c^\infty(\mathbb{R}^N, E),$$

for homogeneous differential operators $A(D) : C_c^\infty(\mathbb{R}^N, E) \to C_c^\infty(\mathbb{R}^N, F)$ with order $1 \leqslant m < N$ and $1 \leqslant p < \frac{N}{m}$, where $\dfrac{1}{p^*} := \dfrac{1}{p} - \dfrac{m}{N}$, are characterized by ellipticity as a consequence of the standard Sobolev embedding and the classical result due Calderón and Zygmund in [10]:

Theorem 1.17. *Let $1 < p < N$. Then the estimate*

$$\|D^m u\|_{L^p} \leqslant C_p\|A(D)u\|_{L^p}, \quad \forall u \in C_c^\infty(\mathbb{R}^N, E),$$

holds if and only if $A(D)$ is elliptic.

The latter estimate fails in general for $p = 1$ as presented by Ornstein in [37]. However, Van Schaftingen in [45, Theorem 1.3] characterized the classical Sobolev-Gagliardo-Nirenberg inequality

$$\|D^{m-1}\, u\|_{L^{N/(N-1)}} \leqslant C\|A(D)u\|_{L^1}, \quad \forall u \in C_c^\infty(\mathbb{R}^N, E)$$

if and only if the operator $A(D)$ is elliptic and canceling.

Some examples of canceling operators are the following:

Example 1.18. The gradient operator $\nabla : C^\infty(\mathbb{R}^N, \mathbb{R}) \to C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ is canceling if and only if $N \geqslant 2$.

We have seen that, in this case, $A(\xi) = \xi$. Thus, for every $\xi \in \mathbb{R}^N \setminus \{0\}$, $A(\xi)[\mathbb{R}] = \mathbb{R}\xi$ is the straight line through the origin determined by the vector ξ. Therefore,

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)[\mathbb{R}] = \begin{cases} \{0\}, & \text{if } N \geqslant 2, \\ \mathbb{R}, & \text{if } N = 1. \end{cases}$$

Example 1.19. The operator $A(D) = (d_k, d^*_{k-1})$ from Example 1.10 is canceling if and only if $k \in \{2,\dots,N-2\}$. Remember that $A(\xi)(v) = (\xi \wedge v, \star(\xi \wedge \star v))$. If

$$(f,g) \in \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)\left[\Lambda^k(\mathbb{R}^N)\right],$$

then $\xi \wedge f = 0$ and $\xi \wedge \star g = 0$ for every $\xi \in \mathbb{R}^N$. Since $2 \leqslant k \leqslant N-2$, we conclude that $f = 0$ and $g = 0$, as $d_{k+1} \circ d_k = d^*_k \circ d^*_{k+1} = 0$. Notice that if $k = 1$ we cannot prove that $g = 0$, so the operator is not canceling. Analogously, for the case $k = N-1$ we cannot prove $f = 0$.

Example 1.20. The Laplace-Beltrami operator $(d^*_k\, d_k, d_{k-1}\, d^*_{k-1})$ from Example 1.11 is canceling for $k \in \{1,\dots,N-1\}$.

Example 1.21. The Korn-Sobolev-Strauss operator D_s from Example 1.12 is canceling. Recall that

$$D_s(\xi)_{j,k}(v) = \frac{\xi_j v_k + \xi_k v_j}{2}, \quad 1 \leqslant j \leqslant k \leqslant N.$$

Let

$$w \in \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} D_s(\xi)\left[\mathbb{R}^N\right].$$

Denote by e_ℓ, $\ell = 1,\dots,N$, the unit vector $e_\ell = (0,\dots,0,1,0,\dots,0)$ with 1 in the ℓ^{th} entry. Observe that

$$D_s(e_1)_{j,k}(v) = \begin{cases} v_1, & j = k = 1 \\ v_k/2, & 1 = j < k \leqslant N \\ 0, & 1 < j \leqslant k \leqslant N. \end{cases}$$

Since, in particular, $w \in D_s(e_1)\left[\mathbb{R}^N\right]$, we get $w_{j,k} = 0$ for $1 < j \leqslant k \leqslant N$. Similarly,

$$D_s(e_N)_{j,k}(v) = \begin{cases} 0, & 1 \leqslant j \leqslant k < N \\ v_j/2, & 1 \leqslant j < k = N \\ v_N, & j = k = N, \end{cases}$$

and we get $w_{1,k} = 0$ for $1 \leqslant k < N$, as $w \in D_s(e_N)\left[\mathbb{R}^N\right]$. Finally, since $D_s(e_2)_{1,N}(v) = 0$ and $w \in D_s(e_2)\left[\mathbb{R}^N\right]$, we obtain $w_{1,N} = 0$, concluding that $w = 0$.

Example 1.22. The scalar Laplacian $\Delta = \sum_{j=1}^N \partial^2 x_j$ in $\mathbb{R}^N$ is elliptic but it is not canceling, as

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)[\mathbb{R}] = \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} |\xi|^2 \mathbb{R} = \mathbb{R}.$$

Example 1.23. The vector Laplace operator from Example 1.9 is not canceling as well, since

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)\left[\mathbb{R}^M\right] = \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} |\xi|^2 \mathbb{R}^M = \mathbb{R}^M.$$

Other examples of canceling operators can be found in [45].

One fundamental property of elliptic and canceling operators $A(D)$ is the existence of a homogeneous linear differential operator $L(D) : C^\infty(\mathbb{R}^N, F) \to C^\infty(\mathbb{R}^N, V)$, for some finite dimensional complex vector space V, such that

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} ker L(\xi) = \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)[E] = \{0\}. \tag{1.13}$$

More precisely:

Proposition 1.24. *Let $A(D)$ be a homogeneous differential operator on $\mathbb{R}^N$ from E to F. If $A(D)$ is elliptic, then there exists a finite-dimensional vector space V and a homogeneous differential operator $L(D)$ on $\mathbb{R}^N$ from F to V such that*

$$ker L(\xi) = A(\xi)[E] \tag{1.14}$$

for every $\xi \in \mathbb{R}^N \setminus \{0\}$.

The proof for the above result can be found in [45, Proposition 4.2] as well as the next definition, also introduced by Van Schaftingen:

Definition 1.25. A homogeneous linear differential operator $L(D)$ on $\mathbb{R}^N$ from F to V is said to be *cocanceling* if

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} ker L(\xi) = \{0\}.$$

As a consequence, if $A(D)$ is elliptic and canceling then Proposition 1.24 implies (1.13). This means that there exists a cocanceling operator $L(D)$ such that, for every $u \in C_c^\infty(\mathbb{R}^N, E)$,

$$L(D)(A(D)u) = 0. \tag{1.15}$$

In [45, Remark 4.1], Van Schaftingen gives a particular expression for the symbol of an operator satisfying (1.14):

$$L(\xi) = \det(A(\xi)^* \circ A(\xi)) \, \mathrm{Id} - A(\xi) \circ \mathrm{adj}(A(\xi)^* \circ A(\xi)) \circ A(\xi)^*, \tag{1.16}$$

where $\mathrm{adj}(A(\xi)^* \circ A(\xi)) = \det(A(\xi)^* \circ A(\xi))(A(\xi)^* \circ A(\xi))^{-1}$ denotes the adjugate operator of $A(\xi)^* \circ A(\xi)$.

The operator $L(D)$ and the associated space V obtained from his construction are not the only pair with the desired property and might not even be the most practical one to work with, but it is interesting in that it is built exclusively from the ellipticity of $A(D)$.

For instance, if $A(D) = \nabla$, Van Schaftingen's recipe gives $L(\xi) : \mathbb{R}^N \to \mathbb{R}^N$, where $[L(\xi)v]_j = |\xi|^2 v_j - \xi_j \sum_{k=1}^N \xi_k v_k$, for each $j = 1, \dots, N$. Therefore, $L(D) : C_c^\infty(\mathbb{R}^N, \mathbb{R}^N) \to C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$ is given by

$$[L(D)f]_j = \Delta f_j - \sum_{k=1}^N \partial_{x_j} \partial_{x_k} f_k$$

for each $j=1,\dots,N$. One can verify that if $\xi \neq 0$ and $v \in ker L(\xi)$, then $v \in \mathbb{R}\xi$. Indeed, if $L(\xi)v=0$, then, for each $j=1,\dots,N$,

$$|\xi|^2 v_j - \xi_j \sum_{k=1}^{N} \xi_k v_k = 0,$$

hence $v_j = \left(\frac{\xi \cdot v}{|\xi|^2}\right)\xi_j$. Thus, $v = \left(\frac{\xi \cdot v}{|\xi|^2}\right)\xi \in \mathbb{R}\xi$. Conversely, $t\xi \in ker L(\xi)$ for every $t \in \mathbb{R}$. Therefore, $ker L(\xi) = \mathbb{R}\xi = A(\xi)[\mathbb{R}]$. Note also that $L(D)(\nabla u) = 0$.

Examples of cocanceling operators are:

Example 1.26. The divergence operator $L(D) = \operatorname{div} : C^\infty(\mathbb{R}^N, \mathbb{R}^N) \to C^\infty(\mathbb{R}^N, \mathbb{R})$ is cocanceling.

Indeed, observe that

$$\operatorname{div} f = \sum_{j=1}^{N} \partial_{x_j} f_j$$

and therefore $L(\xi) : \mathbb{R}^N \to \mathbb{R}$ is given by $L(\xi)(v) = \xi \cdot v$. Then clearly

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} ker L(\xi) = \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} \xi^\perp = \{0\}.$$

Example 1.27. Let $k \in \{0,\dots,N-1\}$. The operator $L(D) = d_k$, the exterior derivative defined in Example 1.10, is cocanceling. We have $L(\xi)(v) = \xi \wedge v$. If $v \in \Lambda^k(\mathbb{R}^N)$ for $k \leqslant N-1$ and $\xi \wedge v = 0$ for every $\xi \in \mathbb{R}^N$, then $v = 0$.

Example 1.28. The higher order divergence operator $L(D) : C^\infty(\mathbb{R}^N, \mathbb{R}^M) \to C^\infty(\mathbb{R}^N, \mathbb{R})$, where $M = \binom{N+k-1}{k}$ is the number of multi-indices α of length N and $|\alpha| = k$, given by

$$L(D)f = \sum_{|\alpha|=k} \partial^\alpha f_\alpha$$

is cocanceling. Its symbol $L(\xi) : \mathbb{R}^M \to \mathbb{R}$ is given by

$$L(\xi)(v) = \sum_{|\alpha|=k} \xi^\alpha v_\alpha.$$

Let

$$v \in \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} ker L(\xi).$$

Then

$$\sum_{|\alpha|=k} \xi^\alpha v_\alpha = 0$$

for every $\xi \in \mathbb{R}^N$. The properties of multivariate polynomials imply that $v = 0$.

More examples are available in [45].

1.5 Stein-Weiss inequality in L^1 norm

The study of two-weight inequalities for the Riesz potential operator started with Stein and Weiss in [43], where they proved the following inequality involving power weights:

Theorem 1.29 ([43, Theorem B*]). *Let $N \geqslant 1$, $0 < \ell < N$ and $1 < p \leqslant q < \infty$. Assume α, β satisfying the conditions*

(i) $\alpha < \frac{N}{p'}$ *and* $\beta < \frac{N}{q}$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$*;*

(ii) $\alpha + \beta \geqslant 0$*;*

(iii) $\frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta - \ell}{N}$.

Then, there exists $C > 0$, depending only on the parameters p, q, α, β, such that

$$\||x|^{-\beta} I_\ell f\|_{L^q(\mathbb{R}^N)} \leqslant C \||x|^{\alpha} f\|_{L^p(\mathbb{R}^N)}, \qquad \forall f \in \mathscr{S}(\mathbb{R}^N), \tag{1.17}$$

Naturally, one asks if (1.17) holds for $p = 1$. The answer is *no*, in general. For $p = 1$ and $\alpha = 0$, for instance, let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be a positive smooth function supported on the unit ball B_1 such that $\int_{\mathbb{R}^N} \varphi(x)\, dx = 1$ and, for each $\varepsilon > 0$, consider $\varphi_\varepsilon(x) := \varepsilon^{-N}\varphi(\varepsilon^{-1}x)$. Then, applying (1.17) for φ_ε and using the scaling invariance, we have

$$\||x|^{-\beta} I_\ell \varphi_\varepsilon\|_{L^q(\mathbb{R}^N)} \leqslant C\|\varphi_\varepsilon\|_{L^1(\mathbb{R}^N)} = C$$

uniformly. Taking $\varepsilon \searrow 0$, we know that $I_\ell \varphi_\varepsilon(x) \to |x|^{-N+\ell}$ almost everywhere, which implies

$$\||x|^{-(\beta+N-\ell)}\|_{L^q(\mathbb{R}^N)} = \||x|^{-N/q}\|_{L^q(\mathbb{R}^N)} \lesssim 1,$$

a contradiction.

Inequality (1.17) can be rewritten as

$$\left(\int_{\mathbb{R}^N} \left|\int_{\mathbb{R}^N} K(x,y) f(y)\, dy\right|^q |x|^{-\beta q}\, dx\right)^{1/q} \leqslant C \left(\int_{\mathbb{R}^N} |f(x)|^p |x|^{\alpha p}\, dx\right)^{1/p},$$

where $K(x,y) := \gamma(\ell)^{-1}|x-y|^{-N+\ell}$ and $0 < \ell < N$. De Nápoli and Picon in [11] studied the Stein-Weiss inequality for the Riesz potential in the case $p = 1$, and characterized this inequality for a class of vector fields associated to cocanceling operators. Their main result was the following:

Theorem 1.30 ([11, Theorem 1.2]). *Let $N \geqslant 2$, $0 < \ell < N$, $0 \leqslant \alpha < 1$, $\beta < N/q$, $\alpha + \beta > 0$ and $\frac{1}{q} = 1 + \frac{\alpha + \beta - \ell}{N}$. Then if $L(D)$ is cocanceling, there exists $C > 0$ such that*

$$\left(\int_{\mathbb{R}^N} |I_\ell f(x)|^q |x|^{-\beta q}\, dx\right)^{1/q} \leqslant C \int_{\mathbb{R}^N} |f(x)|\, |x|^{\alpha}\, dx, \tag{1.18}$$

for all $|x|^\alpha f \in L^1(\mathbb{R}^N, F)$ satisfying $L(D)f = 0$ in the sense of distributions. Conversely, if for every non zero $|x|^\alpha f \in L^1(\mathbb{R}^N, F)$ satisfying $L(D)f = 0$ the inequality (1.18) holds for $\alpha = 0$, then $L(D)$ is cocanceling.

The converse in the case $0<\alpha<1$ is an open question. However, the theorem fails for $\alpha=1$. Let $\varphi\in C_c^\infty(B_1)$ a non-negative function with $\int_{\mathbb{R}^N}\varphi(x)\,dx=1$ and $\varphi_\varepsilon(x)=\varepsilon^{-N}\varphi(x/\varepsilon)$ for $\varepsilon>0$. The vector field $\vec{f}_\varepsilon$ on $\mathbb{R}^N$ with components $f_{1,\varepsilon}(x)=\partial_{x_2}(\varphi_\varepsilon(x))$, $f_{2,\varepsilon}(x)=-\partial_{x_1}(\varphi_\varepsilon(x))$ and $f_{j,\varepsilon}(x)=0$ for $j=3,\dots,N$ satisfies $\operatorname{div}\vec{f}_\varepsilon=0$ for all $\varepsilon>0$. We showed in Example 1.26 that the divergence operator is cocanceling. Then, assuming Theorem 1.30 holds for $\alpha=1$, we have

$$\left(\int_{\mathbb{R}^N}|I_\ell f_\varepsilon(x)|^q\,|x|^{-\beta q}\,dx\right)^{1/q}\lesssim\sum_{j=1,2}\int_{\mathbb{R}^N}|f_{j,\varepsilon}(x)|\,|x|\,dx,\quad\forall\varepsilon>0. \tag{1.19}$$

But $f_{1,\varepsilon}(x)=\varepsilon^{-1}(\partial_{x_2}\varphi)_\varepsilon(x)$ and $f_{2,\varepsilon}(x)$ has a similar expression, hence

$$\sum_{j=1,2}\int_{\mathbb{R}^N}|f_{j,\varepsilon}(x)|\,|x|\,dx\lesssim\int_{\mathbb{R}^N}|\nabla\varphi(x)|\,|x|\,dx<\infty, \tag{1.20}$$

independently of ε. However, writing $I_\ell f_{j,\varepsilon}(x)=C_{N,\ell,j}(K_j*\varphi_\varepsilon)(x)$, with $K_j(x)=x_j/|x|^{N-\ell+2}$, and taking $\varepsilon\to0$ we obtain, for the left-hand side of (1.19),

$$\left(\int_{\mathbb{R}^N}|x_j|^q\,|x|^{(-N+\ell-2-\beta)q}\,dx\right)^{1/q}=\left(C_N\int_0^\infty r^{(-N+\ell-1-\beta)q}\,r^{N-1}\,dr\right)^{1/q}=\left(C_N\int_0^\infty r^{-1}\,dr\right)^{1/q},$$

as $(-N+\ell-1-\beta)q+N=0$. This integral diverges, contradicting the (1.20). The previous theorem follows directly from the next Fundamental Lemma:

Lemma 1.31 ([11, Lemma 3.2]). *Assume $N\geqslant2$, $0<\ell<N$ and $K(x,y)\in L^1_{loc}(\mathbb{R}^N\times\mathbb{R}^N,\mathscr{L}(F,V))$ satisfying*

$$|K(x,y)|\leqslant C\,|x-y|^{\ell-N},\quad x\neq y \tag{1.21}$$

and

$$|K(x,y)-K(x,0)|\leqslant C\,\frac{|y|}{|x|^{N-\ell+1}},\quad 2|y|\leqslant|x|. \tag{1.22}$$

Suppose $0\leqslant\alpha<1$, $\beta<N/q$, $\alpha+\beta>0$ and $\frac{1}{q}=1+\frac{\alpha+\beta-\ell}{N}$. If $L(D)$ is cocanceling, then there exists $C>0$ such that

$$\left(\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}K(x,y)f(y)\,dy\right|^q|x|^{-\beta q}\,dx\right)^{1/q}\leqslant C\int_{\mathbb{R}^N}|f(x)|\,|x|^\alpha\,dx, \tag{1.23}$$

for all $|x|^\alpha f\in L^1(\mathbb{R}^N,F)$ satisfying $L(D)f=0$ in the sense of distributions.

The inequality (1.18) follows from Lemma 1.31 observing that the Riesz potential kernel $K(x,y)=\gamma(\ell)^{-1}|x-y|^{-N+\ell}$ satisfies (1.21) and (1.22).

The following results are examples of applications for Lemma 1.31 and can be seen in [11]:

Theorem 1.32 (Hardy-Littlewood-Sobolev inequality in L^1 norm for canceling operators)**.** *Let $A(D)$ be an elliptic homogeneous linear differential operator of order ν on $\mathbb{R}^N$, $N\geqslant2$, from E to F and assume that $0\leqslant\alpha<1$, $0<\ell<N$ and $\ell\leqslant\nu$. If $A(D)$ is canceling, then the estimate*

$$\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{(\nu-\ell)/2}u(x)\right|^q|x|^{-N+(N-\ell-\alpha)q}\,dx\right)^{1/q}\leqslant C\int_{\mathbb{R}^N}|x|^\alpha|A(D)u(x)|\,dx,$$

holds for every $u\in C_c^\infty(\mathbb{R}^N,E)$, some $C>0$ and $1\leqslant q<\frac{N}{N+\alpha-\ell}$.

In the statement above, $g = (-\Delta)^{a/2} f$ is the positive fractional power of the Laplacian defined from the multiplier $\widehat{g}(\xi) = |\xi|^a \widehat{f}(\xi)$ for $f \in \mathscr{S}'(\mathbb{R}^N)$, the space of tempered distributions, and $a \geqslant 0$.

The next is a previously known result from [46, Theorem 1.2] regarding convolution kernels, but Lemma 1.31 allows a much easier proof.

Theorem 1.33 (Fractional integral operator for L^1 vector fields)**.** *Let $N \geqslant 2$, $0 < \ell < N$, $0 \leqslant \alpha < 1$, $\beta < N/q$, $\alpha + \beta > 0$ and $\dfrac{1}{q} = 1 + \dfrac{\alpha + \beta - \ell}{N}$. Suppose that $K(x)$ satisfies*

(a) $|K(x)| \leqslant C\,|x|^{\ell - N}, \quad x \neq 0$*;*

(b) $|K(x-y) - K(x)| \leqslant C\,\frac{|y|}{|x|^{N-\ell+1}}, \quad 2|y| \leqslant |x|$.

If $T_\ell f(x) = \int_{\mathbb{R}^N} K(x-y) f(y)\,dy$, then there exists $C > 0$ such that

$$\left\| |x|^{-\beta} T_\ell f \right\|_{L^q} \leqslant C \, \| |x|^\alpha f \|_{L^1} + \left\| |x|^\alpha \nabla (-\Delta)^{-1} \operatorname{div} f \right\|_{L^1},$$

for all $f \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$.

To prove this, we first use the Helmholtz decomposition $f = g + h$ where $h = \nabla(-\Delta)^{-1} \operatorname{div} f$. Then $g = f - \nabla(-\Delta)^{-1} \operatorname{div} f$ satisfies $\operatorname{div} g = 0$. Since div is cocanceling, Lemma 1.31 gives

$$\left\| |x|^{-\beta} T_\ell g \right\|_{L^q} \leqslant C \, \| |x|^\alpha f \|_{L^1} + \left\| |x|^\alpha \nabla (-\Delta)^{-1} \operatorname{div} f \right\|_{L^1}.$$

For h, as ∇ is elliptic and canceling, there exists a cocanceling operator $L(D)$ such that $L(D)h = 0$ and (1.18) implies

$$\left\| |x|^{-\beta} T_\ell h \right\|_{L^q} \leqslant C \, \| |x|^\alpha h \|_{L^1}.$$

1.6 Some classes of operators

In this section we present two important classes that will be fundamental is this thesis.

1.6.1 Riesz transforms

One important class of singular integral operators that will be useful for us are the *Riesz transforms* R_j, for $j = 1, \ldots, N$, given by

$$R_j f(x) = \lim_{\varepsilon \to 0^+} c_N \int_{|x-y| > \varepsilon} f(y) \frac{x_j - y_j}{|x-y|^{N+1}}\, dy$$

for $f \in \mathscr{S}(\mathbb{R}^N)$ and $c_N = \Gamma\left(\frac{N+1}{2}\right) / \pi^{(N+1)/2}$. Its Fourier transform is given by

$$\widehat{(R_j f)}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi).$$

A property of the Riesz transforms that will be key for some proofs in this text is that they are bounded from L^p to itself for $1 < p < \infty$ (see [12, Corollary 4.8] or [18, Corollary 5.2.8]), that is, the operator can be extended to $L^p(\mathbb{R}^N)$, for all $1 < p < \infty$, so that $\|R_j f\|_{L^p} \leqslant C(p,N)\|f\|_{L^p}$ for all $f \in L^p(\mathbb{R}^N)$. The Riesz transforms are also of type weak$(1,1)$ (see [18, Section 5.3]), which means that for every $f \in L^1(\mathbb{R}^N)$ and $j = 1,\dots,N$,

$$\|R_j f\|_{L^{1,\infty}} \doteq \sup_{\lambda>0} \lambda \left|\{x : |R_j f(x)| > \lambda\}\right| \lesssim \|f\|_{L^1}.$$

This is a simple application of Calderón-Zygmund Theorem [12, Theorem 5.1].

With the above properties, one is allowed to consider compositions of Riesz transforms. Let $\alpha = (\alpha_1,\dots,\alpha_N)$ be a multi-index. Then, the *Riesz transform of order* α is the operator

$$R^\alpha f \doteq \left(R_1^{\alpha_1} \circ R_2^{\alpha_2} \circ \cdots \circ R_N^{\alpha_N}\right) f,$$

for $f \in L^p(\mathbb{R}^N)$, where $R_j^{\alpha_j}$ is the composition $R_j \circ R_j \circ \cdots \circ R_j$ for α_j times. The boundedness properties of R_j naturally extend to R^α. It is straightforward to conclude that

$$\widehat{(R^\alpha f)}(\xi) = (-i)^{|\alpha|} \frac{\xi^\alpha}{|\xi|^{|\alpha|}} \hat{f}(\xi),$$

where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_N^{\alpha_N}$ and $|\alpha| = \sum_{j=1}^N \alpha_j$.

1.6.2 Pseudo-differential operators

In this subsection, we present some basic tools of the theory of pseudo-differential operators in the Hörmander classes, with examples and properties that will be necessary in some proofs of this text. We refer to [23, 24, 44] for a deeper study on the topic.

Definition 1.34. Let $\Omega \subset \mathbb{R}^N$ be open and let $m, \rho, \delta \in \mathbb{R}$ with $0 < \rho \leqslant 1$ and $0 \leqslant \delta < 1$. The set of *symbols of order m and type* ρ, δ, denoted by $S^m_{\rho,\delta}(\Omega)$, called the *Hörmander classes*, is the set of all $a \in C^\infty(\Omega \times \mathbb{R}^N)$ such that, for every $K \subset\subset \Omega$ and all multi-indices α, β, the estimate

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leqslant C_{\alpha,\beta,K} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}, \quad \text{for } x \in K, \xi \in \mathbb{R}^N, \tag{1.24}$$

holds for some constant $C_{\alpha,\beta,K} > 0$. Here, $\langle\xi\rangle \doteq (1+|\xi|^2)^{1/2}$ denotes the so-called *japanese bracket*.

In particular, we write $S^m = S^m_{1,0}$ and simply say that S^m is the set of *symbols of order m*. We also define $S^\infty_{\rho,\delta} \doteq \bigcup_m S^m_{\rho,\delta}$ and $S^{-\infty} \doteq \bigcap_m S^m_{\rho,\delta}$. Notice that the definition of $S^{-\infty}$ does not depend on ρ and δ.

Example 1.35. Let $a(x,\xi) = \sum_{|\gamma|\leqslant m} a_\gamma(x)\xi^\gamma$, with $a_\gamma \in C^\infty(\Omega)$. Then, fixed $K \subset\subset \Omega$ and taking $x \in K$, we have

$$\partial_x^\beta \partial_\xi^\alpha a(x,\xi) = C_\alpha \sum_{\substack{|\gamma|\leqslant m \\ \alpha\leqslant\gamma}} \partial^\beta a_\gamma(x)\, \xi^{\gamma-\alpha},$$

where the notation $\alpha \leqslant \gamma$ means that $\alpha_j \leqslant \gamma_j$ for every $j = 1, \dots, N$. By the compactness of K,

$$\begin{aligned} \left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| &\leqslant C_{\alpha,\beta,K} \sum_{\substack{|\gamma|\leqslant m \\ \alpha\leqslant\gamma}} |\xi|^{|\gamma|-|\alpha|} \\ &\leqslant C_{\alpha,\beta,K} \sum_{\substack{|\gamma|\leqslant m \\ \alpha\leqslant\gamma}} (1+|\xi|^2)^{\frac{1}{2}(|\gamma|-|\alpha|)} \\ &\leqslant \tilde{C}_{\alpha,\beta,K} \langle\xi\rangle^{m-|\alpha|}. \end{aligned}$$

Hence, $a \in S^m(\Omega)$.

Example 1.36. If $a \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ is positively homogeneous of degree m for $|\xi| \geqslant 1$, i.e.,

$$a(x,t\xi) = t^m a(x,\xi)$$

for $|\xi| \geqslant 1$ and $t \geqslant 1$, then, fixed $K \subset\subset \mathbb{R}^N$ and taking $x \in K$, we have:

- if $|\xi| \leqslant 1$, then $\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leqslant C_{1,\alpha,\beta,K}$, since $K \times \overline{B_1}$ is compact;
- if $\xi = t\eta$, with $|\eta| = 1$ and $t = |\xi| > 1$, then

$$t^{|\alpha|}\partial_\xi^\alpha a(x,\xi) = \partial_\eta^\alpha a(x,t\eta) = \partial_\eta^\alpha [t^m a(x,\eta)] = t^m \partial_\eta^\alpha a(x,\eta) = |\xi|^m \partial_\eta^\alpha a(x,\eta),$$

  hence, $\partial_x^\beta \partial_\xi^\alpha a(x,\xi) = |\xi|^{m-|\alpha|}\partial_x^\beta \partial_\eta^\alpha a(x,\eta)$, but since $K \times \mathbb{S}^{N-1}$ is compact, we have $\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leqslant C_{2,\alpha,\beta,K}|\xi|^{m-|\alpha|}$.

Combining both cases, we have

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leqslant C_{\alpha,\beta,K}\langle\xi\rangle^{m-|\alpha|}, \quad \text{for } x \in K, \xi \in \mathbb{R}^N,$$

that is, $a \in S^m(\mathbb{R}^N)$.

From the previous example, the symbol $a(x,\xi) = |\xi|^{2k}$, $k \in \mathbb{N}$, belongs to S^{2k}.

Example 1.37. Let $A > 0$ and $a(x,\xi) = (1 + A|\xi|^2)^{m/2}$ for $m \in \mathbb{R}$. If $m = 0$, it is straightforward that $a \in S^0$. If $m \neq 0$, since a is independent of x, we only need to analyze $\partial_\xi^\alpha a(x,\xi)$. Using Faà di Bruno's formula,

$$\partial_\xi^\alpha a(x,\xi) = \sum_{1\leqslant\ell\leqslant|\alpha|}\left[C_\ell(1+A|\xi|^2)^{\frac{m}{2}-\ell} \sum_{\substack{\gamma^1+\cdots+\gamma^\ell=\alpha \\ |\gamma^j|\geqslant 1; j=1,\dots,\ell}} \left[\left(\partial_\xi^{\gamma^1}|\xi|^2\right)\cdots\left(\partial_\xi^{\gamma^\ell}|\xi|^2\right)\right]\right].$$

It follows from the previous example that $|\xi|^2 \in S^2$, thus $\left|\partial_\xi^{\gamma^j}|\xi|^2\right| \leqslant C\langle\xi\rangle^{2-|\gamma^j|}$. Also,

$$(1+A|\xi|^2)^{\frac{m}{2}-\ell} \leqslant C_\ell(1+|\xi|^2)^{\frac{m}{2}-\ell} = C_\ell\langle\xi\rangle^{m-2\ell}$$

Hence,

$$\begin{aligned}\left|\partial_\xi^\alpha a(x,\xi)\right| &\leqslant \sum_{1\leqslant \ell\leqslant |\alpha|}\left[C_\ell\langle\xi\rangle^{m-2\ell}\sum_{\substack{\gamma^1+\cdots+\gamma^\ell=\alpha\\ |\gamma^j|\geqslant 1;\, j=1,\ldots,\ell}} C_\alpha\left[\langle\xi\rangle^{2-|\gamma^1|}\cdots\langle\xi\rangle^{2-|\gamma^\ell|}\right]\right]\\ &= \sum_{1\leqslant \ell\leqslant |\alpha|}\left[C_{\alpha,\ell}\langle\xi\rangle^{m-2\ell}\langle\xi\rangle^{2\ell-|\alpha|}\right]\\ &= C_\alpha\langle\xi\rangle^{m-|\alpha|}.\end{aligned}$$

Therefore, $a\in S^m$.

We can now define the main subject of this section.

Definition 1.38. To a symbol $a\in S^m_{\rho,\delta}(\Omega)$ we associate the operator

$$P(a)f(x)=\int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi}\,a(x,\xi)\widehat{f}(\xi)\,d\xi,\quad f\in\mathscr{S}(\Omega). \tag{1.25}$$

$P(a):\mathscr{S}(\Omega)\to\mathscr{S}(\Omega)$ is called a *pseudo-differential operator of order m and type* ρ,δ. The set of all such operators is denoted by $Op\,S^m_{\rho,\delta}(\Omega)$.

Example 1.39. Let $a(x,\xi)=\sum_{|\alpha|=m} a_\alpha\xi^\alpha$ be the symbol in Example 1.35 with all a_α constant and $a_\alpha=0$ for $|\alpha|<m$. Then

$$P(a)f(x)=\int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi}\left(\sum_{|\alpha|=m} a_\alpha\xi^\alpha\widehat{f}(\xi)\right)d\xi.$$

From Fourier inversion formula (1.6),

$$P(a)f(x)=\int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi}\widehat{\left(C\sum_{|\alpha|=m} a_\alpha\partial^\alpha f\right)}(\xi)\,d\xi=C\sum_{|\alpha|=m} a_\alpha\partial^\alpha f(x).$$

Hence, the homogeneous linear differential operator of order m given by $A(D)=\sum_{|\alpha|=m} a_\alpha\partial^\alpha$ is a pseudo-differential operator in $Op\,S^m$.

More examples will appear in future chapters.

Pseudo-differential operators are bounded from $\mathscr{S}(\Omega)$ to $\mathscr{S}(\Omega)$ (see [23, Theorem 18.1.6]). If we replace $\widehat{f}$ by its definition in (1.25), we get

$$\begin{aligned}P(a)f(x) &= \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi}\,a(x,\xi)\left(\int_{\mathbb{R}^N} e^{-2\pi i y\cdot\xi} f(y)\,dy\right)d\xi\\ &= \int_{\mathbb{R}^N} K(x,y)f(y)\,dy,\end{aligned}$$

where $K(x,y)=\int_{\mathbb{R}^N} e^{2\pi i(x-y)\cdot\xi}\,a(x,\xi)\,d\xi\in\mathscr{S}'(\Omega\times\mathbb{R}^N)$ is called the *distribution kernel* of $P(a)$. It enjoys the following properties:

Theorem 1.40 ([2, Theorem 1.1]). *Let $T \in OpS^m_{\rho,\delta}(\Omega)$, $0 < \rho \leqslant 1$, $0 \leqslant \rho < 1$, be a pseudo-differential operator with symbol $a(x,\xi)$ and let $K(x,y)$ be its distribution kernel.*

(i) (Pseudo-local property) K is smooth outside the diagonal. Moreover, given $\alpha,\beta \in \mathbb{Z}^N_+$, there is $n_0 \in \mathbb{Z}_+$ such that for each $n \geqslant n_0$,

$$\sup_{x\neq y} |x-y|^n |\partial^\alpha_x \partial^\beta_y K(x,y)| < \infty.$$

(ii) Suppose a has compact support in ξ uniformly with respect to x. Then K is smooth, and given $\alpha,\beta \in \mathbb{Z}^N_+$, $n \in \mathbb{Z}_+$, there is $C > 0$ such that

$$|\partial^\alpha_x \partial^\beta_y K(x,y)| \leqslant C(1+|x-y|)^{-n}.$$

(iii) Suppose that $m+M+N < 0$ for some $M \in \mathbb{Z}_+$. Then K is a bounded continuous function with bounded continuous derivatives of order up to M.

(iv) Suppose that $m+M+N = 0$ for some $M \in \mathbb{Z}_+$. Then, there is $C > 0$ such that

$$\sup_{|\alpha+\beta|=M} |\partial^\alpha_x \partial^\beta_y K(x,y)| \leqslant C\big|\ln|x-y|\big|, \quad x \neq y.$$

(v) Suppose that $m+M+N > 0$ for some $M \in \mathbb{Z}_+$. Then, there is $C > 0$ such that

$$\sup_{|\alpha+\beta|=M} |\partial^\alpha_x \partial^\beta_y K(x,y)| \leqslant C|x-y|^{-(m+M+N)/\rho}, \quad x \neq y.$$

Next we state a comprehensive result on the L^p boundedness of pseudo-differential operators, due to Álvarez and Hounie, which will be very useful in future proofs.

Theorem 1.41 ([2, Theorems 3.2 and 3.5]). *Let $T \in OpS^m_{\rho,\delta}(\Omega)$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$ and set $\lambda = \max\{0,(\delta-\rho)/2\}$. Then*

(i) T is of type weak$(1,1)$ if $m \leqslant -N\left[\dfrac{1-\rho}{2} + \lambda\right]$,

and it continuously maps $L^p(\Omega)$ to $L^q(\Omega)$, for $1 < p \leqslant q < \infty$, in the following cases:

(ii) if $p \leqslant 2 \leqslant q$ and $m \leqslant -N\left(\dfrac{1}{p} - \dfrac{1}{q} + \lambda\right)$;

(iii) if $2 \leqslant p \leqslant q$ and $m \leqslant -N\left[\dfrac{1}{p} - \dfrac{1}{q} + (1-\rho)\left(\dfrac{1}{2} - \dfrac{1}{p}\right) + \lambda\right]$;

(iv) if $p \leqslant q \leqslant 2$ and $m \leqslant -N\left[\dfrac{1}{p} - \dfrac{1}{q} + (1-\rho)\left(\dfrac{1}{q} - \dfrac{1}{2}\right) + \lambda\right]$.

Particular cases of Theorem 1.41 that are noteworthy are:

- $T \in OpS^m_{\rho,\delta}(\Omega)$ is continuous from $L^p(\Omega)$ to itself, $1 < p < \infty$, if

$$m \leqslant -N\left[(1-\rho)\left|\frac{1}{p}-\frac{1}{2}\right|+\lambda\right].$$

- $T \in OpS^m(\Omega)$ is continuous from $L^p(\Omega)$ to itself, $1 < p < \infty$, and is of type weak$(1,1)$, if $m \leqslant 0$.

Regarding L^1 continuity, we also have the following:

Theorem 1.42 ([35, Theorem 6.1]). *Let $T \in OpS^m(\mathbb{R}^N)$. If $m < 0$, then T maps continuously $L^1(\mathbb{R}^N)$ to itself.*

To finish this subsection, we connect the elliptic operators defined beforehand with the theory of pseudo-differential operators in order to obtain an important property.

Definition 1.43. Let $T \in OpS^m_{\rho,\delta}(\Omega)$ and $a \in S^m_{\rho,\delta}(\Omega)$. We say a is a *principal symbol* of T if

$$T - P(a) \in OpS^{m+\delta-\rho}_{\rho,\delta}(\Omega).$$

Example 1.44. $A(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha \in S^m$ is a principal symbol of $A(D) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha \in OpS^m$.

Definition 1.45. Let $\rho > \delta$ and $a \in S^m_{\rho,\delta}(\Omega)$. We say a is an *elliptic symbol of order m* if, for every $K \subset\subset \Omega$, there are positive constants C and r such that

$$|a(x,\xi)| > C\langle\xi\rangle^m, \quad \text{for } x \in K, |\xi| > r.$$

If $T \in OpS^m_{\rho,\delta}(\Omega)$, we say T is *elliptic of order m* if it has an elliptic principal symbol of order m.

Example 1.46. $A(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ is an elliptic symbol of order m if and only if $A(\xi) \neq 0$ for $\xi \in \mathbb{R}^N \setminus \{0\}$. This is in line with the definition of $A(D)$ being elliptic given in Definition 1.6.

Theorem 1.47 ([23, Theorem 18.1.8]). *Let $\rho \geqslant \delta$ and $a_j \in S^{m_j}_{\rho,\delta}(\Omega)$, $j = 1,2$. Then*

$$P(a_1)P(a_2) \in OpS^{m_1+m_2}_{\rho,\delta}(\Omega).$$

The next theorem shows that elliptic operators are invertible, modulo an operator in $OpS^{-\infty}$.

Theorem 1.48 ([23, Theorem 18.1.9]). *Let $\rho > \delta$, $a \in S^m_{\rho,\delta}(\Omega)$ and $b \in S^{-m}_{\rho,\delta}(\Omega)$. Then the conditions below are equivalent:*

(i) $P(a)P(b) - I \in OpS^{-\infty}(\Omega)$;

(ii) $P(b)P(a) - I \in OpS^{-\infty}(\Omega)$,

and a determines $b \mod S^{-\infty}(\Omega)$. Here, I is the identity operator $P(1)$. Both (i) and (ii) imply

(iii) $a(x,\xi)b(x,\xi) - 1 \in S^{-1}_{\rho,\delta}(\Omega)$,

which then imply that a is an elliptic symbol of order m. Conversely, if $a \in S^m_{\rho,\delta}(\Omega)$ is elliptic of order m, then one can find $b \in S^{-m}_{\rho,\delta}(\Omega)$ satisfying (i), (ii) and (iii).

CHAPTER 2

Global solvability for homogeneous linear operators with constant coefficients

Throughout this chapter, $A(D)$ denotes an elliptic homogeneous linear differential operator of order m on $\mathbb{R}^N$, $N \geqslant 2$ and $1 \leqslant m < N$, with constant coefficients, from a finite dimensional complex vector space E to a finite dimensional complex vector space F. Since the vector spaces have finite dimension we will use, for simplicity, X in the place of X^*.

Inspired by Theorems 1.1 and 1.2, we will study the Lebesgue solvability for the equation

$$A^*(D)f = \mu, \tag{2.1}$$

where $A^*(D)$ is the formal adjoint operator associated to the homogeneous linear differential operator $A(D)$:

Definition 2.1. The *formal adjoint* of a differential operator $L : C_c^\infty(\Omega, E) \to C_c^\infty(\Omega, F)$ is the differential operator $L^* : C_c^\infty(\Omega, F) \to C_c^\infty(\Omega, E)$ determined by

$$\int_\Omega L\varphi \cdot \overline{\psi} = \int_\Omega \varphi \cdot \overline{L^*\psi}$$

for every $\varphi \in C_c^\infty(\Omega, E)$ and $\psi \in C_c^\infty(\Omega, F)$. In other words, $L^* = \overline{L^t}$, where L^t is the formal transpose of L and $\overline{L^t}$ denotes the operator obtained by conjugating the coefficients of L^t.

Example 2.2. If $L = \nabla : C_c^\infty(\mathbb{R}^N, \mathbb{R}) \to C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$, then

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla\varphi \cdot \overline{\psi} &= \sum_{j=1}^N \int_{\mathbb{R}^N} (\partial_j \varphi)\, \psi_j = -\sum_{j=1}^N \int_{\mathbb{R}^N} \varphi\, (\partial_j \psi_j) \\ &= -\int_{\mathbb{R}^N} \varphi \operatorname{div} \psi = \int_{\mathbb{R}^N} \varphi\, \overline{(-\operatorname{div} \psi)},\end{aligned}$$

and, therefore, $L^* = -\operatorname{div}$.

The first result of this chapter concerns the Lebesgue solvability for the equation (2.1) when $1 \leqslant p < \infty$ and is the focus of Section 2.1.

Theorem 2.3. *Let $A(D)$ be a homogeneous linear differential operator of order $1 \leqslant m < N$ on $\mathbb{R}^N$, $N \geqslant 2$, from E to F and $\mu \in \mathscr{M}(\mathbb{R}^N, E)$.*

(i) If $1 \leqslant p \leqslant N/(N-m)$, $f \in L^p(\mathbb{R}^N, F)$ is a solution for (2.1) *and $\mu \in \mathscr{M}_+(\mathbb{R}^N, E)$, then $\mu \equiv 0$.*

(ii) If $N/(N-m) < p < \infty$ and $f \in L^p(\mathbb{R}^N, F)$ is a solution for (2.1)*, then μ has finite $(m,p)-$energy. Conversely, if $|\mu|$ has finite $(m,p)-$energy and $A(D)$ is elliptic, then there exists a function $f \in L^p(\mathbb{R}^N, F)$ solving* (2.1).

In particular, Theorem 2.3 recovers Theorem 1.1 taking $A(D) = -\nabla$, where $E = \mathbb{R}$, $F = \mathbb{R}^N$ and $A^*(D) = \text{div}$.

The second and main result of this chapter deals with the case $p = \infty$ and is proved in Section 2.2.

Theorem 2.4. *Let $A(D)$ be a homogeneous linear differential operator of order $1 \leqslant m < N$ on $\mathbb{R}^N$ from E to F and $\mu \in \mathscr{M}(\mathbb{R}^N, E)$. If $A(D)$ is elliptic and canceling, and μ satisfies*

$$\|\mu\|_{0,N-m} \doteq \sup_{r>0} \frac{|\mu|(B_r)}{r^{N-m}} < \infty, \tag{2.2}$$

and the potential control

$$\int_0^{|y|/2} \frac{|\mu|(B(y,r))}{r^{N-m+1}}\, dr \lesssim 1, \quad \textit{uniformly on } y, \tag{2.3}$$

then, there exists $f \in L^\infty(\mathbb{R}^N, F)$ solving (2.1).

We point out that the assumption (2.2) is weaker in comparison to $(N-m)$-Ahlfors regularity, $\|\mu\|_{N-m} < \infty$, since here it is only necessary to take the supremum over balls centered at the origin. The condition (2.3) can be understood as an uniform control of the truncated Wolff potential associated to $|\mu|$.

2.1 The $1 \leqslant p < \infty$ case

The section is dedicated to the proof of Theorem 2.3. For the next proposition, the following lemma will be necessary.

Lemma 2.5 ([42, p. 73])**.** *Let $P_k(x)$ be a homogeneous harmonic polynomial of degree k, $k \geqslant 1$. Then*

$$\left(\frac{P_k(\cdot)}{|\cdot|^{k+N-\alpha}} \right)^{\widehat{}}(\xi) = \gamma_{k,\alpha} \frac{P_k(\xi)}{|\xi|^{k+\alpha}},$$

where $\gamma_{k,\alpha} = i^k \pi^{N/2-\alpha} \dfrac{\Gamma(k/2+\alpha/2)}{\Gamma(k/2+N/2-\alpha/2)}$.

Proposition 2.6. *Let $1 \leqslant p \leqslant N/(N-m)$. If $\mu \in \mathscr{M}_+(\mathbb{R}^N, E)$ and $f \in L^p(\mathbb{R}^N, F)$ is a solution for $A^*(D)f = \mu$, then $\mu \equiv 0$.*

Proof. From the identity $(N-m)\displaystyle\int_{|x-y|}^{\infty} \frac{1}{r^{N-m+1}}\,dr = \frac{1}{|x-y|^{N-m}}$ and the Fubini's theorem we may write

$$\begin{aligned} I_m\mu(x) &= \frac{1}{\gamma(m)} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-m}}\,d\mu(y) = c_{N,m} \int_{\mathbb{R}^N} \left(\int_{|x-y|}^{\infty} \frac{1}{r^{N-m+1}}\,dr \right) d\mu(y) \\ &= c_{N,m} \int_{\mathbb{R}^N} \left(\int_0^{\infty} \frac{\chi_{\{r>|x-y|\}}(r)}{r^{N-m+1}}\,dr \right) d\mu(y) = c_{N,m} \int_0^{\infty} \left(\int_{\mathbb{R}^N} \frac{\chi_{\{r>|x-y|\}}(r)}{r^{N-m+1}}\,d\mu(y) \right) dr \\ &= c_{N,m} \int_0^{\infty} \left(\int_{B(x,r)} \frac{1}{r^{N-m+1}}\,d\mu(y) \right) dr = c_{N,m} \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\infty} \frac{\mu(B(x,r))}{r^{N-m+1}}\,dr. \end{aligned}$$

Now, using the Gauss-Green theorem, we have

$$\begin{aligned} \mu(B(x,r)) &= \int_{B(x,r)} A^*(D)f(y)\,dy = \sum_{|\alpha|=m} a^*_\alpha \int_{B(x,r)} \partial^\alpha f(y)\,dy \\ &= \sum_{|\alpha|=m} a^*_\alpha \int_{\partial B(x,r)} \partial^{\alpha - e_{j_\alpha}} f(y)\, \frac{y_{j_\alpha} - x_{j_\alpha}}{|y-x|}\,d\omega(y), \end{aligned}$$

where we choose, for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$, a number $j_\alpha \in \{1, \ldots, N\}$ such that $\alpha_{j_\alpha} \neq 0$ in a way that $\partial^\alpha f = \partial_{x_{j_\alpha}}(\partial^{\alpha - e_{j_\alpha}} f)$. Summarizing

$$\begin{aligned} I_m\mu(x) &= c_{N,m} \sum_{|\alpha|=m} a^*_\alpha \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\infty} \left(\int_{|x-y|=r} \partial^{\alpha - e_{j_\alpha}} f(y)\, \frac{y_{j_\alpha} - x_{j_\alpha}}{|y-x|^{N-m+2}}\,d\omega(y) \right) dr \\ &= c_{N,m} \sum_{|\alpha|=m} a^*_\alpha \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} \partial^{\alpha - e_{j_\alpha}} f(y)\, \frac{x_{j_\alpha} - y_{j_\alpha}}{|x-y|^{N-m+2}}\,dy \\ &= c_{N,m} \sum_{|\alpha|=m} a^*_\alpha \left(K_{j_\alpha} * \partial^{\alpha - e_{j_\alpha}} f \right)(x), \end{aligned}$$

where $K_{j_\alpha}(x) := x_{j_\alpha}/|x|^{N-m+2}$. Thus from Lemma 2.5 we have $\widehat{K_{j_\alpha}}(\xi) = c_{N,m}\, \xi_{j_\alpha}/|\xi|^m$ and hence, renaming the constant $c_{N,m}$, we have

$$(K_{j_\alpha} * \partial^{\alpha - e_{j_\alpha}} f)^{\widehat{}}(\xi) = c_{N,m} \frac{\xi_{j_\alpha}}{|\xi|^m} \xi^{\alpha - e_{j_\alpha}} \widehat{f}(\xi) = c_{N,m} \frac{\xi^\alpha}{|\xi|^m} \widehat{f}(\xi) = \widehat{(R^\alpha f)}(\xi).$$

In this way,

$$I_m\mu = c_{N,m} \sum_{|\alpha|=m} a^*_\alpha R^\alpha f. \tag{2.4}$$

In particular for $m = 1$,

$$I_1\mu(x) = c_N \sum_{j=1}^{N} a^*_j \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} f(y)\, \frac{x_j - y_j}{|x-y|^{N+1}}\,dy = c_N \sum_{j=1}^{N} a^*_j R_j f(x)$$

for almost every $x \in \mathbb{R}^N$.

Since each R_j is bounded from L^p to itself for $1 < p < \infty$ and of type weak$(1,1)$, we conclude that $\|I_m\mu\|_{L^p} \lesssim \|f\|_{L^p} < \infty$, that is, μ has finite $(m,p)-$energy for $1 < p \leqslant N/(N-m)$ and $\|I_m\mu\|_{L^{1,\infty}} \lesssim \|f\|_{L^1} < \infty$ for $p = 1$. Notice that up until this point we only needed $\mu \in \mathscr{M}(\mathbb{R}^N, E)$. If $\mu \in \mathscr{M}_+(\mathbb{R}^N, E)$, it follows from Proposition 1.5 that $\mu \equiv 0$ in $\mathbb{R}^N$. □

Next we prove the second part of the Theorem 2.3. The following proposition will be necessary. Its proof can be found in [12, p. 71].

Proposition 2.7. *If T is a tempered distribution homogeneous of degree α, then its Fourier transform is homogeneous of degree $-N-\alpha$.*

Proposition 2.8. *Let $N/(N-m) < p < \infty$ and $\mu \in \mathscr{M}(\mathbb{R}^N, E)$. If $f \in L^p(\mathbb{R}^N, F)$ is a solution for $A^*(D)f = \mu$, then μ has finite $(m,p)-$energy. Conversely, if $|\mu|$ has finite $(m,p)-$energy, then there exists a function $f \in L^p(\mathbb{R}^N, F)$ solving $A^*(D)f = \mu$.*

Proof. The first part follows from identity (2.4) and the boundedness of order α Riesz transform operators. For the converse, consider the function $\xi \mapsto H(\xi) \in \mathscr{L}(F,E)$ defined by

$$H(\xi) = (A^* \circ A)^{-1}(\xi) A^*(\xi)$$

that is smooth in $\mathbb{R}^N \backslash \{0\}$ and homogeneous of degree $-m$. Here $A^*(\xi)$ is the symbol of the adjoint operator $A^*(D)$. Since we are assuming that $1 \leqslant m < N$, then H is a locally integrable tempered distribution and its inverse Fourier transform $K(x)$ is a locally integrable tempered distribution homogeneous of degree $-N+m$ (Proposition 2.7) that satisfies

$$u(x) = \int_{\mathbb{R}^N} K(x-y)[A(D)u(y)]\, dy, \quad u \in C_c^\infty(\mathbb{R}^N, E). \tag{2.5}$$

and clearly $|u(x)| \leqslant I_m|A(D)u|(x)$.

Let $w_A^{m,p'}(\mathbb{R}^N, E)$ be the closure of $C_c^\infty(\mathbb{R}^N, E)$ with respect to the norm $\|u\|_{m,p'} \doteq \|A(D)u\|_{L^{p'}}$. Thus,

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u(x)\, d\mu(x) \right| \lesssim \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} \frac{|A(D)u(y)|}{|x-y|^{N-m}}\, dy \right] d|\mu|(x) &\lesssim \int_{\mathbb{R}^N} |A(D)u(y)|\, I_m|\mu|(y)\, dy \\ &\leqslant \|u\|_{m,p'} \|I_m|\mu|\|_{L^p} \lesssim \|u\|_{m,p'}, \end{aligned}$$

since $|\mu|$ has finite $(m,p)-$energy, following that $\mu \in [w_A^{m,p'}(\mathbb{R}^N, E)]^*$. Since $A(D) : w_A^{m,p'}(\mathbb{R}^N, E) \to L^{p'}(\mathbb{R}^N, F)$ is a linear isometry, its adjoint $A^*(D) : L^p(\mathbb{R}^N, F) \to [w_A^{m,p'}(\mathbb{R}^n, E)]^*$ is surjective. Therefore, there exists $f \in L^p(\mathbb{R}^N, F)$ such that $A^*(D)f = \mu$. □

In the end of the previous proof, the following lemma was used. It is a direct consequence of Hahn-Banach Theorem:

Lemma 2.9. *Let X and Y be normed vector spaces and $T : X \to Y$ a linear isometry. Then its adjoint application $T^* : Y^* \to X^*$ is surjective.*

Observe that Proposition 2.8, hence Theorem 2.3(ii), becomes a characterization for the existence of an L^p solution ($N/(N-m) < p < \infty$) for (2.1) if $|\mu| = \mu$. This is the case when μ is a positive scalar measure:

Corollary 2.10. *Let $A(D)$ be an elliptic homogeneous linear differential operator of order $1 \leqslant m < N$ on $\mathbb{R}^N$, $N \geqslant 2$, from E to F, where E and F are finite dimensional real vector spaces, with $\dim_{\mathbb{R}} E = 1$. Let $\mu \in \mathscr{M}_+(\mathbb{R}^N, E^*)$ and $N/(N-m) < p < \infty$. Then μ has finite $(m,p)-$energy if and only if there exists a function $f \in L^p(\mathbb{R}^N, F^*)$ solving $A^*(D)f = \mu$.*

2.2 The $p = \infty$ case

In this section we will prove Theorem 2.4. The main ingredient of the proof is to investigate sufficient conditions on μ in order to obtain

$$\left| \int_{\mathbb{R}^N} u(x)\, d\mu(x) \right| \lesssim \|A(D)u\|_{L^1}, \qquad \forall u \in C_c^\infty(\mathbb{R}^N, E). \tag{2.6}$$

The strategy used by Phuc and Torres to prove (1.4) is of no use here, as the co-area formula used by them is not applicable for a general operator $A(D)$. However, (2.6) is pretty similar to the Stein-Weiss inequality (1.18) studied by de Nápoli and Picon in [11] for $q = 1$. The twist here is that we have $d\mu$ instead of $|x|^{-\beta}dx$, i.e. in their case the (scalar) positive measure is given by a special weighted power for some $\beta > 0$. In order to prove Theorem 2.4 it is enough to show that (2.6) holds. In fact, assuming the validity of that inequality, we conclude that $\mu \in [w_A^{m,1}(\mathbb{R}^N, E)]^*$ and, following the argument used in the proof of Proposition 2.8, there exists $f \in L^\infty(\mathbb{R}^N, F)$ such that $A^*(D)f = \mu$. From the identity (2.5), since $A(D)$ is elliptic, the inequality (2.6) is equivalent to

$$\left| \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} K(x-y)g(y)\, dy \right] d\mu(x) \right| \lesssim \|g\|_{L^1} \tag{2.7}$$

where $g := A(D)u$, for all $u \in C_c^\infty(\mathbb{R}^N, E)$ and moreover

$$|K(x-y)| \leqslant C\, |x-y|^{m-N}, \quad x \neq y \tag{2.8}$$

and

$$|\partial_y K(x-y)| \leqslant C\, |x-y|^{m-N-1}, \quad 2|y| \leqslant |x|. \tag{2.9}$$

To see why (2.9) holds, notice that $\widehat{(\partial_y K)} = c\xi \widehat{K}$ is homogeneous of degree $1-m$. From Proposition 2.7 we conclude that $\partial_y K$ is homogeneous of degree $m-N-1$.

The proof reduces to obtaining inequality (2.7) invoking a special class of vector fields in L^1 norm associated to an elliptic and canceling operator $A(D)$ and μ satisfying (2.2) and (2.3).

2.2.1 A Hardy-type inequality

The first step in the proof of Theorem 2.4 is an extension of a Hardy-type inequality [11, Lemma 2.1] on two measures.

Lemma 2.11. *Let $1 \leqslant q < \infty$ and ν be a σ-finite real positive measure. Suppose $\tilde{u}$ and $\tilde{v}$ are measurable and non-negative almost everywhere. Then*

$$\left[\int_{\mathbb{R}^N}\left(\int_{B_{|x|/2}}\tilde{g}(y)\,dy\right)^q\tilde{u}(x)\,d\nu(x)\right]^{1/q} \lesssim \int_{\mathbb{R}^N}\tilde{g}(x)\tilde{v}(x)\,dx \tag{2.10}$$

holds for all $\tilde{g} \geqslant 0$ if and only if

$$C := \sup_{R>0}\left(\int_{(B_R)^c}\tilde{u}(x)\,d\nu(x)\right)^{1/q}\left(\sup_{x\in B_R}[\tilde{v}(x)]^{-1}\right) < \infty. \tag{2.11}$$

Analogously

$$\left[\int_{\mathbb{R}^N}\left(\int_{(B_{|x|/2})^c}\tilde{g}(y)\,dy\right)^q\tilde{u}(x)\,d\nu(x)\right]^{1/q} \lesssim \int_{\mathbb{R}^N}\tilde{g}(x)\tilde{v}(x)dx \tag{2.12}$$

holds for all $\tilde{g} \geqslant 0$ if and only if

$$\tilde{A} := \sup_{R>0}\left(\int_{B_R}\tilde{u}(x)\,d\nu(x)\right)^{1/q}\left(\sup_{x\in(B_R)^c}[\tilde{v}(x)]^{-1}\right) < \infty. \tag{2.13}$$

Proof. First we prove (2.11) implies (2.10). By Minkowski inequality we have

$$\begin{aligned}\left[\int_{\mathbb{R}^N}\left(\int_{B_{|x|/2}}\tilde{g}(y)\,dy\right)^q\tilde{u}(x)\,d\nu(x)\right]^{1/q} &= \left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\tilde{g}(y)\,\chi_{\{2|y|<|x|\}}(x,y)\,dy\right)^q\tilde{u}(x)\,d\nu(x)\right]^{1/q}\\ &\leqslant \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}[\tilde{g}(y)]^q\,\chi_{\{2|y|<|x|\}}(x,y)\,\tilde{u}(x)\,d\nu(x)\right)^{1/q}dy\\ &= \int_{\mathbb{R}^N}\tilde{g}(y)\left(\int_{(B_{2|y|})^c}\tilde{u}(x)\,d\nu(x)\right)^{1/q}dy\\ &\leqslant C\int_{\mathbb{R}^N}\tilde{g}(y)\,\tilde{v}(y)\,dy,\end{aligned}$$

since

$$\left(\int_{(B_{2|y|})^c}\tilde{u}(x)\,d\nu(x)\right)^{1/q}[\tilde{v}(y)]^{-1} \leqslant \left(\int_{(B_{2|y|})^c}\tilde{u}(x)\,d\nu(x)\right)^{1/q}\left(\sup_{x\in B_{2|y|}}[\tilde{v}(x)]^{-1}\right) \leqslant C.$$

Conversely, for $R > 0$ consider $S(R) := \operatorname*{ess\,sup}_{z\in B_R}[\tilde{v}(z)]^{-1}$. For each $n \in \mathbb{N}$, we define the set

$$\widetilde{M_n} := \left\{z \in B_R : [\tilde{v}(z)]^{-1} > S(R) - \frac{1}{n}\right\}.$$

From the definition follows $|\widetilde{M_n}| > 0$, hence there exist $M_n \subseteq \widetilde{M_n}$ with $0 < |M_n| < \infty$. Choosing $\widetilde{g}(y) = \chi_{M_n}(y)$ and using (2.10), we have

$$\begin{aligned}\left(\int_{(B_{2R})^c} \tilde{u}(x)\, d\nu(x)\right)^{1/q} &= \frac{1}{|M_n|}\left[\int_{(B_{2R})^c}\left(\int_{B_R} \chi_{M_n}(y)\, dy\right)^q \tilde{u}(x)\, d\nu(x)\right]^{1/q} \\ &\leqslant \frac{1}{|M_n|}\left[\int_{\mathbb{R}^N}\left(\int_{B_{|x|/2}} \chi_{M_n}(y)\, dy\right)^q \tilde{u}(x)\, d\nu(x)\right]^{1/q} \\ &\lesssim \fint_{M_n} \tilde{v}(x)\, dx \lesssim \left(S(R) - \frac{1}{n}\right)^{-1}.\end{aligned}$$

Taking $n \to \infty$ we get $\left(\int_{(B_{2R})^c} \tilde{u}(x)\, d\nu(x)\right)^{1/q} S(R) \lesssim 1$ and the result follows since the control is uniform on $R > 0$. The proof is analogous for (2.12) $\iff$ (2.13). □

Observe that, to prove (2.10) above, it would suffice to ask for the weaker condition

$$\left(\int_{(B_{2|y|})^c} \tilde{u}(x)\, d\nu(x)\right)^{1/q} \lesssim \tilde{v}(y)$$

for almost every $y \in \mathbb{R}^N$.

2.2.2 A Stein-Weiss-type inequality

The following peculiar estimate for vector fields belonging to the kernel of some cocanceling operator was presented at [11, Lemma 3.1].

Lemma 2.12. *Let $L(D)$ be a cocanceling homogeneous linear differential operator of order m on $\mathbb{R}^N$ from F to V. Then there exists $C > 0$ such that, for every $\varphi \in C_c^m(\mathbb{R}^N, F)$, we have*

$$\left|\int_{\mathbb{R}^N} \varphi(y) \cdot f(y)\, dy\right| \leqslant C \sum_{j=1}^{m} \int_{\mathbb{R}^N} |f(y)|\, |y|^j\, |D^j \varphi(y)|\, dy \tag{2.14}$$

for all functions $f \in L^1(\mathbb{R}^N, F)$ satisfying $L(D)f = 0$ in the sense of distributions.

The second step to obtain (2.7) is an improvement of Lemma 1.31 ([11, Lemma 3.2], [27, Lemma 2.1]) in the setting of positive Borel measures.

Lemma 2.13. *Assume $N \geqslant 2$, $0 < \ell < N$ and $K(x,y) \in L^1_{loc}(\mathbb{R}^N \times \mathbb{R}^N, \mathscr{L}(F,V))$ satisfying*

$$|K(x,y)| \leqslant C\, |x-y|^{\ell - N}, \quad x \neq y \tag{2.15}$$

and

$$|K(x,y) - K(x,0)| \leqslant C\, \frac{|y|}{|x|^{N-\ell+1}}, \quad 2|y| \leqslant |x|. \tag{2.16}$$

Suppose $1 \leqslant q < \infty$ *and let* $\nu \in \mathscr{M}_+(\mathbb{R}^N)$ *satisfying*

$$\|\nu\|_{0,(N-\ell)q} < \infty, \tag{2.17}$$

and the following uniform potential condition

$$[[\nu]]_{(N-\ell)q} := \sup_{y\in\mathbb{R}^N} \int_0^{|y|/2} \frac{\nu(B(y,r))}{r^{(N-\ell)q+1}} dr < \infty. \tag{2.18}$$

If $L(D)$ *is cocanceling, then there exists* $\widetilde{C} > 0$ *such that*

$$\left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} K(x,y) g(y)\, dy \right|^q d\nu(x) \right)^{1/q} \leqslant \widetilde{C} \int_{\mathbb{R}^N} |g(x)|\, dx, \tag{2.19}$$

for all $g \in L^1(\mathbb{R}^N, F)$ *satisfying* $L(D)g = 0$ *in the sense of distributions.*

Remark 2.14. A stronger condition satisfying (2.18) is given by

$$\nu(B(y,R)) \leqslant C_2\, |y|^{(N-\ell)q-N} R^N \tag{2.20}$$

when $R < |y|/2$. The integration boundary $|y|/2$ in (2.18) can be swapped to $a|y|$, where a is a fixed constant $0 < a < 1$. In this case, (2.20) must hold for $R < a|x|$ to imply (2.18).

Let us present an example of positive measures satisfying (2.17) and (2.18). Suppose $N \geqslant 2$, $0 < \ell < N$, $1 \leqslant q \leqslant N/(N-\ell)$ and define $d\nu = |x|^{(N-\ell)q-N} dx$. The control (2.17) is obvious for the case when $q = N/(N-\ell)$, since ν is simply the Lebesgue measure and $(N-\ell)q = N$. Otherwise,

$$\nu(B_R) = \int_{B_R} |x|^{(N-\ell)q-N}\, dx \lesssim \int_0^R r^{(N-\ell)q-1}\, dr \lesssim R^{(N-\ell)q}.$$

For (2.18) we note that, if $|y| < R < |x|/2$, then $|x|/2 < |x+y| < 3|x|/2$. Thus,

$$\nu(B(x,R)) = \int_{B(x,R)} |y|^{(N-\ell)q-N}\, dy = \int_{B_R} |x+y|^{(N-\ell)q-N}\, dy \lesssim |x|^{(N-\ell)q-N} R^N.$$

In order to prove the inequality (2.7), and consequently the Theorem 2.4, we estimate

$$\left| \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} K(x-y) g(y)\, dy \right] d\mu(x) \right| \leqslant \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} K(x-y) g(y)\, dy \right| d|\mu|(x)$$

and we apply the Lemma 2.13 for $q = 1$ and $\nu = |\mu|$, taking $K(x,y) = K(x-y)$ given by identity (2.5) that, for $\ell = m$, satisfies (2.8), which obviously implies (2.15), and (2.9), which by the Mean Value Inequality and the fact that $|x-\eta| \geqslant |x|/2$ for $|\eta| \leqslant |y| \leqslant |x|/2$, implies that

$$|K(x,y) - K(x,0)| \lesssim |y| \sup_{\eta\in[0,y]} |\partial_y K(x,\eta)| \lesssim |y|/|x|^{N-\ell+1},$$

that is (2.16). Note that (2.17) and (2.18) come naturally from (2.2) and (2.3). The conclusion follows taking $g := A(D)u$ that belongs to the kernel of some cocanceling operator $L(D)$ from (1.15).

Now we present the proof of Lemma 2.13.

Proof of Lemma 2.13. Let $\psi \in C_c^\infty(B_{1/2},\mathbb{R})$ be a cut-off function such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ on $B_{1/4}$, and write $K(x,y) = K_1(x,y) + K_2(x,y)$ with $K_1(x,y) = \psi(y/|x|)K(x,0)$. We claim that

$$J_j \doteq \left(\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N} K_j(x,y)g(y)\,dy\right|^q d\nu(x)\right)^{1/q} \lesssim \int_{\mathbb{R}^N}|g(x)|\,dx \tag{2.21}$$

for $j=1,2$ and $g \in L^1(\mathbb{R}^N,F)$ satisfying $L(D)g=0$ in the sense of distributions.

Using the control (2.15) we estimate

$$\begin{aligned}
J_1 &= \left(\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\psi\left(\frac{y}{|x|}\right)g(y)\,dy\right|^q |K(x,0)|^q\,d\nu(x)\right)^{1/q}\\
&\lesssim \left(\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\psi\left(\frac{y}{|x|}\right)g(y)\,dy\right|^q |x|^{(\ell-N)q}\,d\nu(x)\right)^{1/q}\\
&\lesssim \left(\int_{\mathbb{R}^N}\left[\int_{B_{|x|/2}}\frac{|y|}{|x|}|g(y)|\,dy\right]^q |x|^{(\ell-N)q}\,d\nu(x)\right)^{1/q}\\
&= \left(\int_{\mathbb{R}^N}\left[\int_{B_{|x|/2}}|y||g(y)|\,dy\right]^q |x|^{(\ell-N-1)q}\,d\nu(x)\right)^{1/q},
\end{aligned} \tag{2.22}$$

where the second inequality follows from (2.14) for $\varphi(y) = \psi(y/|x|)\eta$, where for a fixed x, η is a unit vector in F chosen so that

$$\left|\int_{\mathbb{R}^N}\psi\left(\frac{y}{|x|}\right)\eta\cdot g(y)\,dy\right| = \left|\int_{\mathbb{R}^N}\psi\left(\frac{y}{|x|}\right)g(y)\,dy\right|.$$

Since for any multi-index α we have $\partial^\alpha\varphi(y) = |x|^{-|\alpha|}\partial_y^\alpha\psi(y/|x|)$ and $\psi \in C_c^\infty(B_{1/2},\mathbb{R})$, (2.14) gives us

$$\begin{aligned}
\left|\int_{\mathbb{R}^N}\psi\left(\frac{y}{|x|}\right)g(y)\,dy\right| &\lesssim \sum_{j=1}^m\int_{B_{|x|/2}}|g(y)|\frac{|y|^j}{|x|^j}\left|D_y^j\psi\left(\frac{y}{|x|}\right)\right|dy\\
&\lesssim \int_{B_{|x|/2}}|g(y)|\frac{|y|}{|x|}\,dy.
\end{aligned}$$

In order to control (2.22) we use the first part of Lemma 2.11, taking $\tilde{u}(x) = |x|^{(\ell-N-1)q}$, $\tilde{g}(x) = |x||g(x)|$ and $\tilde{v}(x) = |x|^{-1}$. So checking (2.11) we have

$$\begin{aligned}
\left(\int_{(B_R)^c}\tilde{u}(x)\,d\nu(x)\right)^{1/q} &= \left(\sum_{k=1}^\infty\int_{2^{k-1}R\leqslant|x|<2^kR}|x|^{(\ell-N-1)q}\,d\nu(x)\right)^{1/q}\\
&\leqslant \left(\sum_{k=1}^\infty(2^{k-1}R)^{(\ell-N-1)q}\nu(B_{2^kR})\right)^{1/q}\\
&\leqslant \|\nu\|_{0,(N-\ell)q}^{1/q}\left(\sum_{k=1}^\infty(2^{k-1}R)^{(\ell-N-1)q}(2^kR)^{(N-\ell)q}\right)^{1/q}\\
&\lesssim \|\nu\|_{0,(N-\ell)q}^{1/q}\left\{\sup_{x\in B_R}[\tilde{v}(x)]^{-1}\right\}^{-1},
\end{aligned}$$

where the last step follows from $\sup_{x\in B_R}[\tilde{v}(x)]^{-1}=R$. Hence,

$$J_1 \lesssim \left(\int_{\mathbb{R}^N}\left[\int_{B_{|x|/2}}|y||g(y)|\,dy\right]^q |x|^{(\ell-N-1)q}\,d\nu(x)\right)^{1/q} \lesssim \|\nu\|_{0,(N-\ell)q}^{1/q}\int_{\mathbb{R}^N}|g(x)|\,dx.$$

Now for J_2, using Minkowski's Inequality we get

$$J_2 \leqslant \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}|K_2(x,y)|^q\,d\nu(x)\right)^{1/q}|g(y)|\,dy.$$

It remains to be shown that

$$\int_{\mathbb{R}^N}|K_2(x,y)|^q\,d\nu(x)\leqslant C \tag{2.23}$$

for some constant $C>0$ uniformly on y. Recall that $K_2(x,y)=K(x,y)-\psi(y/|x|)K(x,0)$. If $2|y|>|x|$, then $\psi(y/|x|)=0$, thus $|K_2(x,y)|=|K(x,y)|$. If $|x|\geqslant 4|y|$, then $\psi(y/|x|)=1$, thus $|K_2(x,y)|=|K(x,y)-K(x,0)|$. In the region $2|y|\leqslant|x|<4|y|$ we have

$$K_2(x,y)=\left[1-\psi\left(\frac{y}{|x|}\right)\right]K(x,y)+\psi\left(\frac{y}{|x|}\right)[K(x,y)-K(x,0)].$$

For each $y\in\mathbb{R}^N$ we get the following upper estimate for the previous integration

$$\begin{aligned}\int_{\mathbb{R}^N}|K_2(x,y)|^q\,d\nu(x) &\leqslant \int_{|x|<2|y|}|K(x,y)|^q\,d\nu(x)\\ &\quad+\int_{2|y|\leqslant|x|<4|y|}(|K(x,y)|^q+|K(x,y)-K(x,0)|^q)\,d\nu(x)\\ &\qquad+\int_{|x|\geqslant 4|y|}|K(x,y)-K(x,0)|^q\,d\nu(x)\\ &=\int_{|x|<4|y|}|K(x,y)|^q\,d\nu(x)+\int_{|x|\geqslant 2|y|}|K(x,y)-K(x,0)|^q\,d\nu(x)\\ &:=(\mathrm{I})+(\mathrm{II}).\end{aligned}$$

From conditions (2.16) and (2.17) we have

$$\begin{aligned}(\mathrm{II}) &\lesssim |y|^q\int_{(B_{2|y|})^c}|x|^{(\ell-N-1)q}\,d\nu(x)\\ &=|y|^q\sum_{k=1}^{\infty}\int_{2^k|y|\leqslant|x|<2^{k+1}|y|}|x|^{(\ell-N-1)q}\,d\nu(x)\\ &\leqslant |y|^q\sum_{k=1}^{\infty}(2^k|y|)^{(\ell-N-1)q}\nu(B_{2^{k+1}|y|})\\ &\lesssim \|\nu\|_{0,(N-\ell)q}|y|^{(\ell-N)q}\sum_{k=1}^{\infty}2^{k(\ell-N-1)q}(2^{k+1}|y|)^{(N-\ell)q}\\ &=\|\nu\|_{0,(N-\ell)q}2^{(N-\ell)q}\sum_{k=1}^{\infty}2^{-kq}\\ &\lesssim \|\nu\|_{0,(N-\ell)q}\end{aligned}$$

while from condition (2.15)

$$
\begin{aligned}
(\mathrm{I}) &\lesssim \int_{B_{4|y|}} |x-y|^{(\ell-N)q}\, d\nu(x) \\
&= \underbrace{\int_{B(y,|y|/2)} |x-y|^{(\ell-N)q}\, d\nu(x)}_{(I_a)} + \underbrace{\int_{B_{4|y|}\setminus B(y,|y|/2)} |x-y|^{(\ell-N)q}\, d\nu(x)}_{(I_b)}.
\end{aligned}
$$

The second part is straightforward:

$$
(I_b) \leqslant \frac{1}{(|y|/2)^{(N-\ell)q}} \int_{B_{4|y|}} d\nu(x) = \frac{\nu(B_{4|y|})}{(|y|/2)^{(N-\ell)q}} \lesssim \|\nu\|_{0,(N-\ell)q}.
$$

Finally, writing $A_x := \{r \in \mathbb{R} : r > |x-y|\}$ and pointing out that $B(y,|y|/2) \subset B_{2|y|}$, we obtain from (2.17) and (2.18)

$$
\begin{aligned}
(I_a) &= \int_{B(y,|y|/2)} (N-\ell)q \left(\int_{|x-y|}^{\infty} r^{(\ell-N)q-1}\, dr \right) d\nu(x) \\
&= (N-\ell)q \int_{\mathbb{R}^N} \chi_{B(y,|y|/2)}(x) \left(\int_0^{\infty} \frac{\chi_{A_x}(r)}{r^{(N-\ell)q+1}}\, dr \right) d\nu(x) \\
&= (N-\ell)q \int_0^{\infty} \left(\int_{B(y,|y|/2)\cap B(y,r)} \frac{1}{r^{(N-\ell)q+1}}\, d\nu(x) \right) dr \\
&= (N-\ell)q \left(\int_0^{|y|/2} \frac{\nu(B(y,r))}{r^{(N-\ell)q}} \frac{dr}{r} + \nu(B(y,|y|/2)) \int_{|y|/2}^{\infty} \frac{1}{r^{(N-\ell)q+1}}\, dr \right) \\
&\lesssim (N-\ell)q \left[[[\nu]]_{(N-\ell)q} + \nu(B_{2|y|}) \frac{1}{(N-\ell)q} (|y|/2)^{(\ell-N)q} \right] \\
&\lesssim [[\nu]]_{(N-\ell)q} + \|\nu\|_{0,(N-\ell)q},
\end{aligned}
$$

concluding (2.23) and thus $J_2 \lesssim \left([[\nu]]_{(N-\ell)q} + \|\nu\|_{0,(N-\ell)q} \right)^{1/q} \int_{\mathbb{R}^N} |g(y)|\, dy.$ □

2.3 Applications and general comments

2.3.1 Avoiding the Wolff potential condition

We can get a similar result to Lemma 2.13 without the potential condition (2.18). In this case, however, we must extend the Ahlfors regularity hypothesis (2.17) to every ball and the conclusion (2.19) has an extra power weight on the right-hand side of the inequality.

Lemma 2.15. *Assume $N \geqslant 2$, $0 < \ell < N$ and $K(x,y) \in L^1_{loc}(\mathbb{R}^N \times \mathbb{R}^N, \mathscr{L}(F,V))$ satisfying*

$$
|K(x,y)| \leqslant C\, |x-y|^{\ell-N}, \quad x \neq y \tag{2.24}
$$

and

$$|K(x,y)-K(x,0)| \leqslant C\,\frac{|y|}{|x|^{N-\ell+1}}, \quad 2|y| \leqslant |x|. \tag{2.25}$$

Suppose $0<\alpha<1$, $1\leqslant q<\infty$, *and let* $\nu \in \mathscr{M}_+(\mathbb{R}^N)$ *satisfying*

$$\|\nu\|_{(N-\ell+\alpha)q} < \infty. \tag{2.26}$$

If $L(D)$ *is cocanceling then there exists* $\widetilde{C}>0$ *such that*

$$\left(\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N} K(x,y)g(y)\,dy\right|^q d\nu(x)\right)^{1/q} \leqslant \widetilde{C}\int_{\mathbb{R}^N}|g(x)|\,|x|^{\alpha}\,dx, \tag{2.27}$$

for all $g\,|x|^{\alpha}\in L^1(\mathbb{R}^N,F)$ *satisfying* $L(D)g=0$ *in the sense of distributions.*

Proof. With the necessary adaptations, the proof follows the same steps of Lemma 2.13, except when estimating (I_a).

$$\begin{aligned}
(I_a) &= \int_{B(y,|y|/2)} |x-y|^{(\ell-N)q}\,d\nu(x)\\
&= \sum_{k=1}^{\infty}\int_{B(y,2^{-k}|y|)\setminus B(y,2^{-(k+1)}|y|)} |x-y|^{(\ell-N)q}\,d\nu(x)\\
&\leqslant \sum_{k=1}^{\infty}\int_{B(y,2^{-k}|y|)} (2^{-(k+1)}|y|)^{(\ell-N)q}\,d\nu(x)\\
&= |y|^{(\ell-N)q}\sum_{k=1}^{\infty}(2^{-(k+1)})^{(\ell-N)q}\,\nu(B(y,2^{-k}|y|))\\
&\leqslant \|\nu\|_{(N-\ell+\alpha)q}\,|y|^{(\ell-N)q}\sum_{k=1}^{\infty}(2^{-(k+1)})^{(\ell-N)q}\,(2^{-k}|y|)^{(N-\ell+\alpha)q}\\
&= 2^{(N-\ell)q}\,\|\nu\|_{(N-\ell+\alpha)q}\,|y|^{\alpha q}\sum_{k=1}^{\infty}(2^{-\alpha q})^k\\
&= 2^{(N-\ell)q}\,\|\nu\|_{(N-\ell+\alpha)q}\,(2^{\alpha q}-1)^{-1}|y|^{\alpha q}.
\end{aligned}$$

□

2.3.2 First order operators

It remains as an open question whether (2.2) or (2.3) are necessary conditions to obtain a L^∞ solution to (2.1) for homogeneous differential operator $A(D)$ with order $m>1$. For $m=1$, however, we show that certain (expected) decay regularity on μ is necessary:

Theorem 2.16. *Let* $A(D)$ *be a first order homogeneous linear differential operator on* $\mathbb{R}^N$ *from* E *to* F *and* $\mu\in\mathscr{M}(\mathbb{R}^N,E)$. *If there exists* $f\in L^\infty(\mathbb{R}^N,F)$ *solving* (2.1), *then there is a constant* $C>0$ *such that*

$$|\mu(B(x,r))| \leqslant Cr^{N-1}$$

for every $x\in\mathbb{R}^N$ *and* $r>0$.

Proof. Denoting $A(D) = \sum_{j=1}^{N} a_j \partial_j$ we have, for every $x \in \mathbb{R}^N$ and almost every $r > 0$,

$$\mu(B(x,r)) = \int_{B(x,r)} A^*(D) f(y)\, dy = -\sum_{j=1}^{N} \int_{B(x,r)} a_j^* \partial_j f(y)\, dy$$
$$= -\sum_{j=1}^{N} \int_{\partial B(x,r)} a_j^* f(y) \frac{y_j - x_j}{|y-x|}\, dS(y),$$

hence $|\mu(B(x,r))| \leqslant C_N \|f\|_{L^\infty} r^{N-1}$.

To extend this estimate for every $r > 0$, let $M \subset \mathbb{R}_+$ be the zero-measure set of values $r > 0$ for which the previous estimate does not hold. Given $x \in \mathbb{R}^N$ and $r > 0$ we can write $B[x,r] = \cap_j B(x,r_j)$, where $(r_j)_j \subset \mathbb{R}_+ \setminus M$ is a decreasing sequence converging to r (note that $\mathbb{R}_+ \setminus M$ is dense in $\mathbb{R}_+$). Thus, simplifying the notation assuming $\mu_\ell \in \mathscr{M}(\mathbb{R}^N)$ for each $j = 1, ..., d$ we have

$$\mu_\ell(B(x,r)) \leqslant \lim_{j\to\infty} |\mu(B(x,r_j))| \leqslant C_N \|f\|_{L^\infty} \lim_{j\to\infty} r_j^{N-1} = C_N \|f\|_{L^\infty} r^{N-1}.$$

Summarizing

$$|\mu(B(x,r))| \leqslant (2d)^{1/2} C_N \|f\|_{L^\infty} r^{N-1}.$$

□

2.3.3 De Rham complex

Recall the operator

$$A(D) = (d_k, d_{k-1}^*) : C_c^\infty(\mathbb{R}^N, \Lambda^k \mathbb{R}^N) \to C_c^\infty(\mathbb{R}^N, \Lambda^{k+1}\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N, \Lambda^{k-1}\mathbb{R}^N)$$

from Examples 1.10 and 1.19. It was shown that, for $k \in \{2, \dots, N-2\}$, $A(D)$ is elliptic and canceling. Its adjoint

$$A^*(D) : C_c^\infty(\mathbb{R}^N, \Lambda^{k+1}\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N, \Lambda^{k-1}\mathbb{R}^N) \to C_c^\infty(\mathbb{R}^N, \Lambda^k \mathbb{R}^N)$$

is given by

$$A^*(D)(f,g) = d_k^* f + d_{k-1} g.$$

Hence, we have the following corollary of Theorems 2.4 and 2.16:

Corollary 2.17. *Let $d_k : C^\infty(\mathbb{R}^N, \Lambda^k\mathbb{R}^N) \to C^\infty(\mathbb{R}^N, \Lambda^{k+1}\mathbb{R}^N)$ and $d_k^* : C^\infty(\mathbb{R}^N, \Lambda^{k+1}\mathbb{R}^N) \to C^\infty(\mathbb{R}^N, \Lambda^k\mathbb{R}^N)$ be the exterior and co-exterior derivatives defined in Example 1.10 and $\mu \in \mathscr{M}(\mathbb{R}^N, \Lambda^k\mathbb{R}^N)$. If $k \in \{2, \dots, N-2\}$, and μ satisfies*

$$\|\mu\|_{0,N-1} \doteq \sup_{r>0} \frac{|\mu|(B_r)}{r^{N-1}} < \infty,$$

and the potential control

$$\int_0^{|y|/2} \frac{|\mu|(B(y,r))}{r^N}\, dr \lesssim 1, \quad \textit{uniformly on } y,$$

then, there exists $(f,g) \in L^\infty(\mathbb{R}^N, \Lambda^{k+1}\mathbb{R}^N) \times L^\infty(\mathbb{R}^N, \Lambda^{k-1}\mathbb{R}^N)$ *solving*

$$d_k^* f + d_{k-1} g = \mu. \tag{2.28}$$

Conversely, if there exists $(f,g) \in L^\infty(\mathbb{R}^N, \Lambda^{k+1}\mathbb{R}^N) \times L^\infty(\mathbb{R}^N, \Lambda^{k-1}\mathbb{R}^N)$ *solving* (2.28)*, then there is a constant* $C > 0$ *such that*

$$|\mu(B(x,r))| \leqslant C r^{N-1}$$

for every $x \in \mathbb{R}^N$ *and* $r > 0$.

2.3.4 Limiting case of trace inequalities for vector fields

F. Gmeineder, B. Raiţă and J. Van Schaftingen (see [17, Theorem 1.1]) characterized an inequality similar to (2.6) involving positive Borel scalar measures. Precisely: if $q = \frac{N-s}{N-1}$ and $0 \leqslant s < 1$ then the estimate

$$\left(\int_{\mathbb{R}^N} \left| D^{m-1} u(x) \right|^q d\nu(x) \right)^{1/q} \lesssim \|\nu\|_{q(N-1)}^{1/q} \|A(D)u\|_{L^1}, \tag{2.29}$$

for all $u \in C_c^\infty(\mathbb{R}^N, E)$ and all $q(N-1)-$Ahlfors regular measure ν, holds if and only if $A(D)$ is elliptic and canceling. Besides the authors claim that it seems to be not simple to obtain a generalization for $s = 1$, i.e $q = 1$, in particular the inequality holds for the total derivative operator $A(D) = D^m$ that is elliptic and canceling (see Remark 2.19).

Next we present the validity of the inequality (2.29) for $q = 1$ (see [17, Theorem 1.1]) under $(N-1)-$Ahlfors regularity and an additional uniform potential condition on ν.

Theorem 2.18. *Let* $A(D)$ *be a homogeneous linear differential operator of order* m *on* $\mathbb{R}^N$, $N \geq 2$, *from* E *to* F. *Then for all* $\nu \in \mathscr{M}_+(\mathbb{R}^N)$ *satisfying* (2.17) *and* (2.18)*, with* $\ell = q = 1$*, there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}^N} \left| D^{m-1} u(x) \right| d\nu \leqslant C \|A(D)u\|_{L^1}, \quad \forall u \in C_c^\infty(\mathbb{R}^N, E). \tag{2.30}$$

Proof. The inequality follows by the combination of the identity $D^{m-1}u(x) = \int_{\mathbb{R}^N} K(x-y)[A(D)u(y)]\,dy$ where $\widehat{K}(\xi) := \sum_{|\alpha|=m-1} \xi^\alpha (A^* \circ A)^{-1}(\xi) A^*(\xi)$ that satisfies (2.15) and (2.16) for $\ell = 1$ and then the estimate (2.30) follows by Lemma 2.13 for $q = 1$, as showed in the proof of inequality (2.7). □

As a consequence of the previous proof we can estimate the constant at inequality (2.30) by

$$C \lesssim \|\nu\|_{0,N-1} + [[\nu]]_{N-1}.$$

Remark 2.19. Let $D^m := (D^\alpha)_{|\alpha|=m}$ the total derivative operator that is an elliptic and canceling homogeneous linear differential operator. Using (1.4) it follows directly that

$$\int_{\mathbb{R}^N} \left| D^{m-1} u(x) \right| d\nu \lesssim \|\nu\|_{N-1} \|D^m u\|_{L^1}, \tag{2.31}$$

for all $u \in C_c^\infty(\mathbb{R}^N)$ and $\nu \in \mathscr{M}_+(\mathbb{R}^N)$. Although the assumption that ν is $(N-1)-$Ahlfors regular contrasts with $\|\nu\|_{0,N-1} < \infty$ at Theorem 2.18, the uniform potential condition (2.18) is not necessary to the validity of (2.31).

In the same spirit of [27, Theorem A] the inequality (2.30) can be extended for the following:

Theorem 2.20. *Let $A(D)$ be a homogeneous linear differential operator of order m on $\mathbb{R}^N$, $N \geq 2$, from E to F, and assume that $1 \leqslant q < \infty$, $0 < \ell < N$ and $\ell \leqslant m$. Then for all $\nu \in \mathscr{M}_+(\mathbb{R}^N)$ satisfying* (2.17) *and* (2.18) *there exists $C > 0$ such that*

$$\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{(m-\ell)/2} u(x)\right|^q d\nu\right)^{1/q} \leqslant C\|A(D)u\|_{L^1}, \quad \forall u \in C_c^\infty(\mathbb{R}^N, E). \tag{2.32}$$

The proof follows the same steps when proving Theorem 2.18 and will be omitted. In particular, the inequality (2.32) recovers the inequality (1.5) in *Local Hardy-Littlewood-Sobolev inequalities for canceling elliptic differential operators* [27] taking $d\mu = |x|^{-N+(N-\ell)q} dx$ for $1 \leqslant q < N/(N-\ell)$ (see Remark 4.1).

CHAPTER 3

Removable singularities

Given a linear differential operator $A(x,D) = \sum_{|\alpha|\leqslant m} a_\alpha(x)\partial^\alpha$ with smooth coefficients in $\mathbb{R}^N$, $N \geqslant 2$, one calls a closed set $S \subseteq \mathbb{R}^N$ *removable* for the equation $A(x,D)f = 0$ with respect to a space $\mathscr{F}$ of locally integrable functions (scalar or vector-valued), provided that for any $f \in \mathscr{F}$ satisfying (in the sense of distributions) the equation $A(x,D)f = 0$ outside S, one has $A(x,D)f = 0$ in $\mathbb{R}^N$ (in the sense of distributions).

The following result dates back to Harvey and Polking [20, Theorem 4.1(b)], where $\mathscr{H}^s$ will stand for the s-dimensional Hausdorff (outer) measure in $\mathbb{R}^N$.

Theorem 3.1. *If $A(x,D)$ is a linear differential operator of order $m < N$ with smooth coefficients and if the closed set $S \subseteq \mathbb{R}^N$ satisfies $\mathscr{H}^{N-m}(S) = 0$, then S is removable for the equation $A(x,D)f = 0$ with respect to the space $L^\infty_{loc}(\mathbb{R}^N)$ of locally (essentially) bounded functions.*

Removable sets for several linear equations have been studied, and sometimes characterized completely, in the literature.

3.1 The divergence case

It was first proven by Moonens in [34] that a compact set $S \subseteq \mathbb{R}^N$ is removable for the equation $\operatorname{div} f = 0$ with respect to $L^\infty(\mathbb{R}^N, \mathbb{R}^N)$, if and only if one has $\mathscr{H}^{N-1}(S) = 0$. The proof, however, heavily relies on the fact that one deals with the divergence operator, and cannot be carried out to other differential operators (even of order one).

Shortly after, Phuc and Torres [38] obtained as an application of Theorem 1.2, among other results, a new proof for the above characterization of compact removable sets for the divergence equation with respect to bounded vector fields, this time relying on a new strategy to prove that a compact set $S \subseteq \mathbb{R}^N$ with $\mathscr{H}^{N-1}(S) > 0$ cannot be removable for the divergence equation. Their argument uses the famous Frostman's lemma [32, Theorem 8.8]:

Lemma 3.2 (Frostman's lemma)**.** *Let B be a Borel set in $\mathbb{R}^N$. Then $\mathscr{H}^s(B) > 0$ if and only if there exists a non-trivial Radon measure $\nu \in \mathscr{M}_+(\mathbb{R}^N)$ (i.e., $0 < \nu(\mathbb{R}^N) < \infty$) such that ν is compactly supported on B and $\nu(B(x,r)) \leqslant r^s$ for $x \in \mathbb{R}^N$ and $r > 0$.*

If a compact set $S \subseteq \mathbb{R}^n$ has $\mathscr{H}^{N-1}(S) > 0$, then Frostman's lemma gives a non-trivial Radon measure ν supported on S with the property that $\nu(B(x,r)) \leqslant r^{N-1}$ for all $x \in \mathbb{R}^N$ and $r > 0$. Then, by Theorem 1.2, the equation $\operatorname{div} f = \nu$ has a solution in $L^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Since $\operatorname{div} f = 0$ outside S (as ν is supported in S), if S were removable for the divergence equation with respect to L^∞, this would imply $\operatorname{div} f = 0$ in $\mathbb{R}^N$, a contradiction with the fact that $\operatorname{div} f = \nu \neq 0$ in $\mathbb{R}^N$. Once again, the presence of the divergence operator prevents this proof being extended to other operators.

3.2 The $A^*(D)$ case

As an application of Theorem 2.4, we can prove a necessary condition for a set $S \subseteq \mathbb{R}^N$ to be removable for the equation $A^*(D)f = 0$ associated to an elliptic and canceling homogeneous differential operator $A(D)$:

Theorem 3.3. *Assume that $A(D)$ is an elliptic and canceling homogeneous differential operator on $\mathbb{R}^N$ of order $1 \leqslant m < N$, from a finite-dimensional vector space E to a finite-dimensional vector space F. If the closed set $S \subseteq \mathbb{R}^N$ is removable for the equation $A^*(D)f = 0$ in $L^\infty(\mathbb{R}^N, F)$, then S has Hausdorff dimension less than or equal to $N - m$.*

Recall that the Hausdorff dimension of a set S is defined as the infimum of all $s \geqslant 0$ such that $\mathscr{H}^s(S) = 0$.

It follows from Theorem 3.1 that if $\mathscr{H}^{N-m}(S) = 0$, then S is removable for the equation $A^*(D)f = 0$ with respect to L^∞. Such a set has Hausdorff dimension less than or equal to $N - m$. It hence only remains open whether or not some sets with Hausdorff dimension $N - m$, yet positive $(N - m)$-dimensional Hausdorff measure, may be removable in this context.

3.2.1 A version of Frostman's lemma with decay

If one were to use Frostman's lemma to obtain a measure satisfying the hypothesis from Theorem 2.4, condition (2.3) would be missing. Remember (see Remark 2.14) that a sufficient condition for (2.3) to be fulfilled in this case is given by

$$\nu(B(x,r)) \lesssim |x|^{-m} r^N \tag{3.1}$$

when $r < |x|/2$. Because of that, we provide a result ensuring at least that, given integers $1 \leqslant m < N$ and a closed set $S \subseteq \mathbb{R}^N$ satisfying $\mathscr{H}^{N-m+\alpha}(S) > 0$ for some $\alpha > 0$, there exists a non-trivial Radon measure supported on S and satisfying conditions (2.2) and (2.3). This will result from observing that one can impose, in the statement of Frostman's lemma, a decay condition.

Lemma 3.4 (Frostman's lemma with power weight decay)**.** *Assume that $0 < \alpha < s < N$ are fixed and that $B \subseteq \mathbb{R}^N$ is a Borel set satisfying $\mathscr{H}^s(B) > 0$. Then there exists a non-trivial Radon measure $\mu \in \mathscr{M}_+(\mathbb{R}^N)$ supported on B satisfying:*

$$\|\mu\|_{0,s-\alpha} = \sup_{r>0} \frac{\mu(B_r)}{r^{s-\alpha}} < \infty, \tag{3.2}$$

and such that, for any $x \in \mathbb{R}^N$ and any $0 < r < |x|/2$, one has

$$\mu(B(x,r)) \lesssim |x|^{-\alpha} r^s. \tag{3.3}$$

Proof. Using Lemma 3.2 we find a non-trivial Radon measure ν supported on B satisfying $\nu(B(x,r)) \leqslant r^s$ for all $x \in \mathbb{R}^N$ and all $r > 0$. Now define $A_k := \{x \in \mathbb{R}^N : k \leqslant |x| < k+1\}$ for $k \in \mathbb{N} \cup \{0\}$ and introduce the Radon measure μ defined by:

$$\mu := \sum_{k=0}^{\infty} 2^{-k\alpha} \nu \llcorner A_k.$$

Observe first that, for $0 < r < 1$, $B_r \subset A_0$. Hence, one has

$$\frac{\mu(B_r)}{r^{s-\alpha}} = \frac{\nu(B_r)}{r^{s-\alpha}} \leqslant \frac{r^s}{r^{s-\alpha}} = r^\alpha \leqslant 1,$$

while if one has $j \leqslant r < j+1$, for some $j \in \mathbb{N}$, there holds

$$\begin{aligned} \mu(B_r) &= \sum_{k=0}^{j-1} 2^{-k\alpha} \nu(A_k) + 2^{-j\alpha} \nu(B_r \cap A_j) \\ &\leqslant \sum_{k=0}^{j-1} 2^{-k\alpha} \nu(B_{k+1}) + 2^{-j\alpha} \nu(B_r) \\ &\leqslant \sum_{k=0}^{j-1} 2^{-k\alpha} (k+1)^s + 2^{-j\alpha} r^s, \end{aligned}$$

thus,

$$\begin{aligned} \frac{\mu(B_r)}{r^{s-\alpha}} &\leqslant \frac{1}{r^{s-\alpha}} \left[\sum_{k=0}^{j-1} 2^{-k\alpha} (k+1)^s + 2^{-j\alpha} r^s \right] \\ &= \frac{1}{r^{s-\alpha}} \sum_{k=0}^{j-1} 2^{-k\alpha} (k+1)^s + 2^{-j\alpha} r^\alpha \\ &\leqslant \sum_{k=0}^{j-1} 2^{-k\alpha} (k+1)^s + [2^{-j}(j+1)]^\alpha \\ &\leqslant C_{\alpha,s} < \infty, \end{aligned}$$

with, for instance, $C_{\alpha,s} := 1 + \sum_{k=0}^{\infty} 2^{-k\alpha}(k+1)^s$, since one has $2^{-j}(j+1) \leqslant 1$ for all $j \in \mathbb{N}$ and $\sum_{k=0}^{\infty} 2^{-k\alpha}(k+1)^s$ converges (by the ratio test). Therefore, (3.2) holds.

To prove (3.3), fix $x \in \mathbb{R}^N$ and $0 < r \leqslant |x|/2$. Choosing $j \in \mathbb{N} \cup \{0\}$ such that one has $j \leqslant |x| < j+1$, one finds $r < \frac{j+1}{2}$ and hence also, for $y \in B(x,r)$,

$$|y| \geqslant |x| - |x-y| > j - \frac{j+1}{2} = \frac{j-1}{2} \quad \text{and} \quad |y| \leqslant |x| + |y-x| < j+1+\frac{j+1}{2} = \frac{3}{2}(j+1),$$

so that there holds $B(x,r) \cap A_k = \varnothing$ for $k < m_j := \left\lfloor \frac{j-1}{2} \right\rfloor$ and $k > n_j := \left\lceil \frac{3}{2}(j+1) \right\rceil$, where $\lfloor a \rfloor$ denotes the greatest integer less than or equal to a and $\lceil a \rceil$ is the smallest integer greater than or equal to a.

We can hence compute

$$\mu(B(x,r)) \leqslant \sum_{k=m_j}^{n_j} 2^{-k\alpha} \nu(B(x,r)) \leqslant r^s \sum_{k=m_j}^{n_j} 2^{-k\alpha}. \tag{3.4}$$

Yet one has

$$\begin{aligned}\sum_{k=m_j}^{n_j} 2^{-k\alpha} &= 2^{-m_j\alpha} \sum_{k=0}^{n_j-m_j} 2^{-k\alpha} = 2^{-m_j\alpha} \frac{1-2^{-[1+(n_j-m_j)]\alpha}}{1-2^{-\alpha}} \\ &= \frac{2^{-m_j\alpha} - 2^{-(n_j+1)\alpha}}{1-2^{-\alpha}} \leqslant \frac{1}{1-2^{-\alpha}} 2^{-m_j\alpha} \leqslant \frac{1}{1-2^{-\alpha}} 2^{-\left(\frac{j-1}{2}-1\right)\alpha} \\ &= \frac{2^{\frac{3}{2}\alpha}}{1-2^{-\alpha}} 2^{-\frac{j}{2}\alpha}. \end{aligned} \tag{3.5}$$

Writing then

$$2^{-\frac{j}{2}\alpha} = |x|^{-\alpha} \left(\frac{|x|}{2^{\frac{j}{2}}}\right)^{\alpha} \leqslant |x|^{-\alpha} \left(\frac{j+1}{2^{\frac{j}{2}}}\right)^{\alpha} \leqslant \left(\frac{3}{2}\right)^{\alpha} |x|^{-\alpha}, \tag{3.6}$$

since one has $\frac{k+1}{2^{\frac{k}{2}}} \leqslant \frac{3}{2}$ for any $k \in \mathbb{N} \cup \{0\}$, we finally get, combining (3.4), (3.5) and (3.6),

$$\begin{aligned}\mu(B(x,r)) &\leqslant r^s \frac{2^{\frac{3}{2}\alpha}}{1-2^{-\alpha}} 2^{-\frac{j}{2}\alpha} \\ &\leqslant \frac{2^{\frac{\alpha}{2}} \cdot 3^{\alpha}}{1-2^{-\alpha}} \cdot |x|^{-\alpha} r^s,\end{aligned}$$

which establishes (3.3). □

3.2.2 Hausdorff dimension of removable sets for $A^*(D)$

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. If the Hausdorff dimension of S were larger than $N-m$, then there would exist $\alpha > 0$ such that $\mathscr{H}^{N-m+\alpha}(S) > 0$. The above Frostman's lemma with power weight decay - Lemma 3.4 - applied to $B = S$ and $s = N-m+\alpha$ ensures the existence of a non-trivial real-valued Radon measure ν supported on S satisfying

$$\sup_{r>0} \frac{\nu(B_r)}{r^{N-m}} < \infty,$$

and such that, for any $x \in \mathbb{R}^N$ and any $0 < r < |x|/2$, one has

$$\nu(B(x,r)) \lesssim |x|^{-\alpha} r^{N-m+\alpha}.$$

Yet then, if $e \in E$ is fixed, and if one defines the vector-valued measure $\mu(B) := \nu(B)e$ for any $B \subseteq \mathbb{R}^N$, there holds

$$\sup_{r>0} \frac{|\mu|(B_r)}{r^{N-m}} = \|e\|_E \sup_{r>0} \frac{\nu(B_r)}{r^{N-m}} < \infty,$$

meaning that (2.2) is fulfilled.

We also get, for any $x \in \mathbb{R}^N$, $x \neq 0$,

$$\int_0^{\frac{|x|}{2}} \frac{|\mu|(B(x,r))}{r^{N-m+1}}\, dr = \int_0^{\frac{|x|}{2}} \frac{\nu(B(x,r))}{r^{N-m+1}}\, dr \lesssim |x|^{-\alpha} \int_0^{\frac{|x|}{2}} r^{\alpha-1}\, dr = \frac{1}{2^\alpha \alpha},$$

so that (2.3) is also satisfied uniformly in $x \in \mathbb{R}^N$, $x \neq 0$.

Hence it follows from Theorem 2.4 that there exists $f \in L^\infty(\mathbb{R}^N, F)$ solving $A^*(D)f = \mu$, which implies that S is *not* removable for the equation $A^*(D)f = 0$, since, as argued in the divergence case, one has $A^*(D)f = 0$ outside S (in the sense of distributions) but $A^*(D)f = \mu \neq 0$ in $\mathbb{R}^N$ (in the sense of distributions). □

CHAPTER 4

Local solvability for non-homogeneous linear operators with variable coefficients

In Chapter 2, we dealt with homogeneous operators with constant coefficients defined in the whole euclidean space. We now want to take a step further towards a more general case. If one decides to consider operators with variable coefficients, it will be naïve to expect solvability results that hold in the whole $\mathbb{R}^N$. That is why the results addressed in this chapter are all local, meaning that they only hold in open subsets of $\mathbb{R}^N$. From now on, Ω always denotes an open subset of $\mathbb{R}^N$.

Let $A(\cdot, D)$ be a linear differential operator of order m on Ω, $N \geqslant 2$ and $1 \leqslant m < \infty$, from a finite dimensional complex vector space E to a finite dimensional complex vector space F, given by

$$A(x,D) = \sum_{|\alpha| \leqslant m} a_\alpha(x) \partial^\alpha : C_c^\infty(\Omega, E) \to C_c^\infty(\Omega, F), \tag{4.1}$$

where the coefficients are now smooth functions $a_\alpha \in C^\infty(\Omega, \mathscr{L}(E,F))$. We will study the local Lebesgue solvability for the equation

$$A^*(x,D) f = \mu, \tag{4.2}$$

where we denote $A^*(x,D) = \sum_{|\alpha| \leqslant m} a_\alpha^*(x) \partial^\alpha$, with $a_\alpha^* : \Omega \to \mathscr{L}(F,E)$. We shall say that equation (4.2) is L^p *locally solvable* in Ω if, for each $x_0 \in \Omega$, there exist an open neighborhood $U \subseteq \Omega$ and $f \in L^p(U,F)$ such that

$$\int_U \varphi \, d\mu = \int_U f \cdot A(x,D) \, \varphi \, dx, \qquad \varphi \in C_c^\infty(U,E). \tag{4.3}$$

Continuous solvability for (4.2), in distributional sense, was recently characterized by Moonens and Picon for elliptic and canceling operators:

Theorem 4.1 ([36, Theorem 1.3])**.** *Let $A(\cdot, D)$ be an elliptic and canceling linear differential operator with smooth coefficients. Then, every point $x_0 \in \Omega$ admits an open neighborhood $U \subset \Omega$ such that, for any distribution $\mu \in \mathscr{D}'(U,E)$, the equation* (4.2) *is continuously solvable in U if and only if for* *every $\varepsilon > 0$ and every compact set $K \subset\subset U$, there exists $\theta = \theta(K,\varepsilon) > 0$ such that one has*

$$|\mu(\varphi)| \leqslant \theta \|\varphi\|_{W^{m-1,1}} + \varepsilon \|A(\cdot,D)\varphi\|_{L^1}, \tag{4.4}$$

for any $\varphi \in C_K^\infty(U,E)$, i.e. the space of smooth functions φ in U vanishing outside K. Here, $W^{m-1,1}(U)$ is the homogeneous Sobolev space defined by $L^1(U)$ functions whose weak derivatives of order $m-1$ belong to $L^1(U)$.

One must be wary of the definition of ellipticity, cancelation and cocancelation for non-homogeneous operators with variable coefficients. This will be addressed in Section 4.2.

The first result in this chapter presents sufficient conditions on μ to guarantee the local Lebesgue solvability for the equation (4.2) when $1 < p < \infty$. Proving it is the goal of Section 4.3.

Theorem 4.2. *Let $A(\cdot,D)$ be an elliptic linear differential operator of order $1 \leqslant m < N$ on Ω from E to F as in* (4.1)*, $1 < p < \infty$ and $\mu \in \mathscr{M}(\Omega,E)$. If, for each $x_0 \in \Omega$, there exists an open neighborhood $U \ni x_0$ of Ω such that $|\mu|$ has finite strong $(m,p)-$energy on U, then the equation* (4.2) *is L^p locally solvable in Ω.*

The strong $(m,p)-$energy is slightly different from the $(m,p)-$energy defined in Section 1.2, replacing the Riesz potentials by Bessel potentials in order to attain a better behavior at infinity. Section 4.1 will focus on that matter. As in the setting of operators with constant coefficients, the proof resumes to obtaining the *a priori* local estimate

$$\left| \int_U u(x)\, d\mu(x) \right| \leqslant C \|A(\cdot,D)u\|_{L^{p'}}, \quad \forall u \in C_c^\infty(U,E) \tag{4.5}$$

for some $C = C(U,p,N) > 0$, and the solvability follows from a duality argument. Once again, the case when $p = \infty$ is treated separately and is the main result of this chapter, being proved in Section 4.4.

Theorem 4.3. *Let $A(\cdot,D)$ be a linear differential operator of order $1 \leqslant m < N$ on Ω from E to F as in* (4.1) *and $\mu \in \mathscr{M}(\Omega,E)$. Suppose that $A(\cdot,D)$ is elliptic and canceling in Ω and μ satisfies*

$$\|\mu\|_{\Omega,N-m} \doteq \sup_{B(x,r)\subset\Omega} \frac{|\mu|(B(x,r))}{r^{N-m}} < \infty. \tag{4.6}$$

Then, for each fixed $x_0 \in \Omega$, there exists an open neighborhood $U \ni x_0$ in Ω such that, if the potential condition

$$\int_0^{a|y-x_0|} \frac{|\mu|(B(y,r))}{r^{N-m+1}}\, dr \lesssim 1, \tag{4.7}$$

where a is some constant between 0 and 1, is satisfied uniformly for almost every $y \in U$, then there exists a function $f \in L^\infty(U,F)$ solving (4.2).

In contrast to (2.2) in Theorem 2.4, the hypothesis (4.6) is the $(N-m)$-Ahlfors regularity of μ on Ω, taking the supremum over every ball (not only those centered at the origin).

Although the proof follows similar steps in comparison to the global case, the work is far from trivial since the machinery in the setting of operators with variable coefficients involves new arguments. An example of measure satisfying (4.6) and (4.7) is given in Example 4.23.

4.1 Strong $(m,p)-$energy

For any $m \geqslant 0$ and any function f in the Schwartz space $\mathscr{S}(\mathbb{R}^N)$, the *Bessel potential* operators J_m are given by the action of the multiplier $\widehat{J_m f}(\xi) = [\xi]^{-m}\widehat{f}(\xi)$, where $[\xi] = \left(1+4\pi^2|\xi|^2\right)^{1/2}$. We write $J_m f = G_m * f$, if $m > 0$, and $J_0 f = f$, where

$$G_m(x) = c_m \int_0^\infty e^{-\pi|x|^2/\delta}\, h_m(\delta)\, d\delta, \tag{4.8}$$

with $c_m = (4\pi)^{-m/2}\Gamma(m/2)^{-1}$ and $h_m(\delta) = e^{-\delta/4\pi}\,\delta^{(-N+m-2)/2}$. The kernel $G_m \in L^1(\mathbb{R}^N)$ and is clearly positive and radially symmetric.

Remark 4.4. From Example 1.37, we know that $J_m f$ is a pseudo-differential operator in OpS^{-m}. From Theorem 1.41, it follows that J_m is of weak type$(1,1)$ and is bounded from L^p to itself, for $1 < p < \infty$.

To point out an important property of Bessel potentials, we introduce a notation to describe the asymptotic behavior of a given function.

Definition 4.5. Let $f : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ and $\phi : \Omega \to (0,\infty)$ and let a be an accumulation point of Ω or $a = \pm\infty$ (when Ω is unbounded). We say that:

(i) $f(x) = O(\phi(x))$ as $x \to a$ (read "f is big-oh of ϕ") if there exist a neighborhood $a \in U \subset \Omega$ and a positive constant A such that $|f(x)| \leqslant A\,\phi(x)$ for every $x \in U$;

(ii) $f(x) = o(\phi(x))$ as $x \to a$ (read "f is little-oh of ϕ") if, for each $\varepsilon > 0$, there exists a neighborhood $a \in U \subset \Omega$ such that $|f(x)| \leqslant \varepsilon\,\phi(x)$ for every $x \in U$.

It is often useful to interpret $O(\cdot)$ and $o(\cdot)$ as limits: $f(x) = O(\phi(x))$ as $x \to a$ is equivalent to

$$\limsup_{x\to a} \frac{|f(x)|}{\phi(x)} < \infty,$$

whereas $f(x) = o(\phi(x))$ as $x \to a$ is equivalent to

$$\lim_{x\to a} \frac{|f(x)|}{\phi(x)} = 0.$$

The first important property of the Bessel potential kernel is that, for $0 < m < N$, its behavior near the origin is similar to the Riesz potential kernel. More precisely, if $0 < m < N$,

$$G_m(x) = \frac{|x|^{-N+m}}{\gamma(m)} + o(|x|^{-N+m}) \text{ as } |x| \to 0, \tag{4.9}$$

where $\gamma(m) := \pi^{N/2} 2^m \Gamma(m/2)/\Gamma((N-m)/2)$. The second important property regards its decay at infinity:

$$G_m(x) = O(e^{-|x|/2}) \text{ as } |x| \to \infty, \tag{4.10}$$

meaning that it has exponential decay, a useful property that Riesz potentials lack. More on Bessel potential operators can be seen in [42].

In a similar fashion to the Riesz potential, if $\eta \in \mathscr{M}(\Omega, \mathbb{C})$, we define

$$J_m\eta(x) := \int_\Omega G_m(x-y)\, d\eta(y)$$

and, for $\mu \in \mathscr{M}(\Omega, X)$, $J_m\mu := (J_m\mu_1, \ldots, J_m\mu_d)$.

Definition 4.6. Let $1 \leqslant p < \infty$, $0 < m < N$ and $\Omega \subseteq \mathbb{R}^N$. We say that $\mu \in \mathscr{M}(\Omega, X)$ has *finite strong $(m,p)-$energy* on a subset $U \subset \mathbb{R}^N$ if

$$\|J_m\mu\|_{L^p(U)} := \left(\int_U |J_m\mu(x)|^p\, dx\right)^{1/p} < \infty,$$

and μ has *finite strong $(m,1)^*-$energy* on U if

$$\|J_m\mu\|_{L^{1,\infty}(U)} \doteq \sup_{\lambda>0} \lambda\, |\{x \in U : |J_m\mu(x)| > \lambda\}| < \infty.$$

We say a measure $\mu \in \mathscr{M}_+(\Omega, X)$ is translation-invariant if, for every $x, y \in \Omega$ and $r > 0$ such that $B(x,r), B(y,r) \subset \Omega$, we have $\mu(B(x,r)) = \mu(B(y,r))$. Clearly the Lebesgue measure restricted to Ω is translation-invariant. The next proposition shows that, in $\mathbb{R}^N$, a certain property is still valid if we replace the Riesz potential for the Bessel potential.

Proposition 4.7. *Let $0 < m < N$. If $\mu \in \mathscr{M}_+(\mathbb{R}^N, X)$ is translation-invariant and has finite strong $(m,p)-$energy for some $1 < p \leqslant N/(N-m)$ or strong $(m,1)^*-$energy on $\mathbb{R}^N$, then $\mu \equiv 0$.*

Proof. Fix a point $x_0 \in \mathbb{R}^N$ and suppose by simplicity that the components of μ are real measures, i.e. $\mu_\ell \in \mathscr{M}_+(\mathbb{R}^N)$ (otherwise take μ_ℓ^{Re} or μ_ℓ^{Im}). By (4.9), given $\varepsilon > 0$ there exists a small $R > 0$ such that

$$\frac{|x|^{-N+m}}{\gamma(m)} - G_m(x) \leqslant \left| G_m(x) - \frac{|x|^{-N+m}}{\gamma(m)} \right| \leqslant \varepsilon |x|^{-N+m}$$

whenever $|x| < R$. Thus, choosing ε small enough, we have

$$G_m(x) \geqslant \left(\frac{1}{\gamma(m)} - \varepsilon\right) |x|^{-N+m} \geqslant 0 \quad \text{when } |x| < R. \tag{4.11}$$

By (4.11) and the translation-invariance of the measure, we have

$$\begin{aligned} J_m\mu_\ell(x) &\geqslant \int_{B(x,R)} G_m(x-y)\, d\mu_\ell(y) \geqslant \int_{B(x,R)} \frac{c_{N,m}}{|x-y|^{N-m}}\, d\mu_\ell(y) \\ &\geqslant \int_{B(x,R)} \frac{c_{N,m}}{(|x|+R)^{N-m}}\, d\mu(y) = c_{N,m} \frac{\mu_\ell(B(x,R))}{(|x|+R)^{N-m}} = c_{N,m} \frac{\mu_\ell(B(x_0,R))}{(|x|+R)^{N-m}}. \end{aligned}$$

We then get

$$\begin{aligned}\int_{\mathbb{R}^N} |J_m\mu(x)|^p\,dx &\gtrsim \int_{\mathbb{R}^N} [J_m\mu_\ell(x)]^p\,dx \geqslant \int_{\mathbb{R}^N}\left[\frac{c_{N,m}\,\mu_\ell(B(x_0,R))}{(|x|+R)^{N-m}}\right]^p dx\\ &= [c_{N,m}\mu_\ell(B(x_0,R))]^p \int_{\mathbb{R}^N} \frac{1}{(|x|+R)^{(N-m)p}}\,dx\\ &= [c_{N,m}\mu_\ell(B(x_0,R))]^p \int_0^\infty \frac{r^{N-1}}{(r+R)^{(N-m)p}}\,dr\\ &= [c_{N,m}\mu_\ell(B(x_0,R))]^p \int_R^\infty \frac{(r-R)^{N-1}}{r^{(N-m)p}}\,dr\end{aligned}$$

and for $1 < p \leqslant N/(N-m)$ the last integral blows up to infinity, as $N-1-(N-m)p \geqslant -1$ (independent of $R > 0$). Thus, as $\mu \in \mathscr{M}_+(\mathbb{R}^N, X)$ has finite strong $(m,p)-$energy, we must have $\mu_\ell(B(x_0,R)) = 0$. For the case $p = 1$, we have

$$\sup_{\lambda>0} \lambda \left|\left\{x \in \mathbb{R}^N : \frac{\mu_\ell(B(x_0,R))}{(|x|+R)^{N-m}} > \lambda\right\}\right| \lesssim \|J_m\mu\|_{L^{1,\infty}} < \infty.$$

However,

$$\begin{aligned}\lambda\left|\left\{x : \frac{\mu_\ell(B(x_0,R))}{(|x|+R)^{N-m}} > \lambda\right\}\right| &= \lambda\left|B\left(0, \left(\frac{\mu_\ell(B(x_0,R))}{\lambda}\right)^{\frac{1}{N-m}} - R\right)\right|\\ &= \lambda^{-\frac{m}{N-m}}\left|B\left(0, \mu_\ell(B(x_0,R))^{\frac{1}{N-m}} - \lambda^{\frac{1}{N-m}}R\right)\right|,\end{aligned}$$

which blows-up to infinity when $\lambda > 0$ is small and $\mu_\ell(B(x_0,R)) \neq 0$. Since $\mathbb{R}^N$ is separable, then $\mu_\ell(\mathbb{R}^N) = 0$ for each $\ell = 1,\dots,d$. Therefore, $\mu \equiv 0$ on $\mathbb{R}^N$. □

4.2 The pointwise notion of ellipticity, cancelation and cocancelation

As alluded previously, the definitions for elliptic, canceling and cocanceling operators are slightly different in this framework. Actually, these are pointwise definitions that specialize to the simpler versions seen when the operator is homogeneous and has constant coefficients.

Let $A(\cdot,D)$ be a linear differential operator of order m on Ω as in (4.1). Its *principal part* is the homogeneous operator of order m on Ω

$$A_m(x,D) = \sum_{|\alpha|=m} a_\alpha(x)\partial^\alpha.$$

The *symbol* of $A(\cdot,D)$ is the linear transformation $A(x,\xi) : E \to F$ defined, for each $(x,\xi) \in \Omega\times\mathbb{R}^N$, by

$$A(x,\xi) := \sum_{|\alpha|\leqslant m} a_\alpha(x)\,\xi^\alpha,$$

and its *principal symbol* is

$$A_m(x,\xi) := \sum_{|\alpha|=m} a_\alpha(x)\,\xi^\alpha,$$

Remark 4.8. From Example 1.35, $A(\cdot,D)$ is a pseudo-differential operator in OpS^m. Notice that the definition of its principal symbol is in accordance with the one given in Definition 1.43.

Definition 4.9. A linear differential operator $A(\cdot,D)$ on Ω from E to F is said to be *elliptic at* $x_0 \in \Omega$ if, for every $\xi \in \mathbb{R}^N \setminus \{0\}$, its principal symbol at x_0, $A_m(x_0,\xi) : E \to F$, is injective. We say that $A(\cdot,D)$ is *elliptic in* Ω if it is elliptic at every $x_0 \in \Omega$.

Remark 4.10. Once again, as in the homogeneous constant coefficient case, this definition of ellipticity is equivalent to that of $A(\cdot,D)$ being a elliptic pseudo-differential operator. Recall from Example 1.46 that $A_m(x_0,\xi)$ is an elliptic symbol of order m if and only if $A_m(x_0,\xi) \neq 0$ for $\xi \in \mathbb{R}^N \setminus \{0\}$. By continuity, $A_m(x,\xi) \neq 0$ for $\xi \in \mathbb{R}^N \setminus \{0\}$ and x in a neighborhood of x_0. Thus, Theorem 1.48 is applicable when $A(\cdot,D)$ is elliptic in the sense of Definition 4.9.

Definition 4.11. A linear differential operator $A(\cdot,D)$ on Ω from E to F is said to be *canceling at* $x_0 \in \Omega$ if its principal part evaluated at x_0, $A_m(x_0,D)$, is canceling in the sense of homogeneous constant coefficient operators, i.e. if

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A_m(x_0,\xi)[E] = \{0\}.$$

We say that $A(\cdot,D)$ is *canceling in* Ω if it is canceling at every $x_0 \in \Omega$.

Let $L(\cdot,D) : C^\infty(\Omega,F) \to C^\infty(\Omega,V)$, where V is another finite dimensional complex vector space, be a linear differential operator of order κ given by

$$L(x,D) = \sum_{|\alpha| \leqslant \kappa} b_\alpha(x)\partial^\alpha, \tag{4.12}$$

where $b_\alpha \in C^\infty(\Omega,\mathscr{L}(F,V))$.

Definition 4.12. We say that $L(\cdot,D)$ is *cocanceling at* $x_0 \in \Omega$ if, for every $f \in F \setminus \{0\}$, the polynomial $p : \mathbb{R}^N \setminus \{0\} \to V$, with coefficients in V, defined by

$$p(\xi) \doteq L_\kappa(x_0,\xi)f$$

does not vanish identically. We say that $L(\cdot,D)$ is *cocanceling in* Ω if it is cocanceling at every $x_0 \in \Omega$.

A polynomial vanishes identically if and only if all its coefficients vanish. Since the number of multi-indices $\alpha \in \mathbb{N}^N$ with $|\alpha| = \kappa$ is $M \doteq \binom{N+\kappa-1}{\kappa}$, then $L(\cdot,D)$ is cocanceling at x_0 if and only if $\widetilde{L}(x_0) : F \to V^M$ given by $\widetilde{L}(x_0)f = (b_\alpha(x_0)f)_{|\alpha|=\kappa}$ is injective.

Lemma 4.13 ([26, Lemma 2.4])**.** *Let $L(\cdot,D)$ be an operator as in* (4.12) *with smooth coefficients $b_\alpha \in C^\infty(\Omega,\mathscr{L}(F,V))$. If $L(\cdot,D)$ is cocanceling at $x_0 \in \Omega$, then there exist a ball $B = B(x_0,r) \subset \Omega$ and functions $k_\alpha \in C^\infty(B,\mathscr{L}(V,F))$ such that*

$$\sum_{|\alpha|=\kappa} k_\alpha(x)b_\alpha(x) = I_F, \quad x \in B, \tag{4.13}$$

where I_F denotes the identity in F.

Proof. From the hypothesis, we know that $\widetilde{L}(x_0) : F \to V^M$ defined above is injective. Then the same is true for x sufficiently close to x_0. We can see $\widetilde{L}(x)$ as a rectangular matrix $S(x)$ with coefficients that depend smoothly on x. Hence, there exists $r > 0$ sufficiently small such that, for $x \in B = B(x_0, r)$, there is a matrix $K(x)$ with coefficients depending smoothly on x representing a left inverse $K(x) : V^M \to F$ of $\widetilde{L}(x)$, that is, $K(x)S(x) = I_F$ for $x \in B$. Writing $K(x)$ as $(k_\alpha(x))_{|\alpha|=\kappa}$, we get (4.13). □

An important property of cocanceling operators is that, if $\mathscr{X} := ker L(\cdot, D) \cap C_c^\infty(\Omega, F)$, then for every $K \subset\subset \Omega$ there exists a constant $C = C(K) > 0$ such that

$$\left| \int_\Omega f(x) \cdot \varphi(x) dx \right| \leqslant C \|f\|_{L^1} \|\nabla \varphi\|_{L^N}, \quad f \in \mathscr{X}, \ \varphi \in C_c^\infty(K, F). \tag{4.14}$$

A proof for this property is given in [26, Theorem 2.3].

Similar to homogeneous constant coefficient operators, there is a crucial relation between elliptic and canceling operators and cocanceling operators in Ω. That relation arises in the proof of the following theorem, asserting that the ellipticity and cancelation of $A(\cdot, D)$ in Ω is characterized by the local Sobolev-Gagliardo-Nirenberg inequality:

Theorem 4.14 ([26, Theorem 4.2])**.** *Let $A(\cdot, D)$ be a linear differential operator of order m in Ω. Then $A(\cdot, D)$ is elliptic and canceling if and only if every point $x_0 \in \Omega$ is contained in a ball $B = B(x_0, r) \subset \Omega$ such that the a priori estimate*

$$\|\varphi\|_{W^{m-1, N/(N-1)}} \leqslant C \|A(\cdot, D)\varphi\|_{L^1}, \quad \varphi \in C_c^\infty(B, E), \tag{4.15}$$

holds for some $C = C(B) > 0$.

In the statement above, $W^{m,p}(\Omega)$ denotes the homogeneous Sobolev space of functions $f \in L^p(\Omega)$ such that all weak derivatives of f up to order m are $L^p(\Omega)$ functions, equipped with the norm

$$\|f\|_{W^{m,p}} \doteq \sum_{|\beta|=m} \|\partial^\beta f\|_{L^p}. \tag{4.16}$$

During the proof of the last theorem, from the ellipticity of $A(\cdot, D)$, Hounie and Picon show that there exist a finite dimensional complex vector space V and a linear differential operator $L(\cdot, D)$: $C^\infty(\Omega, F) \to C^\infty(\Omega, V)$ of order $\kappa = 2mN$ such that

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A_m(x_0, \xi)[E] = \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} ker L_\kappa(x_0, \xi), \quad \text{for each } x_0 \in \Omega, \tag{4.17}$$

and since $A(\cdot, D)$ is canceling in Ω, i.e. the intersection in the left-hand side is $\{0\}$, the operator $L(\cdot, D)$ is cocanceling in Ω. Their construction generalizes Van Schaftingen's operator (1.16) for operators with variable coefficients.

Using the previous motivation, in order to study the validity of the inequality (4.5) for $p' = 1$, we prove a local version of Lemma 2.12 in the setting of cocanceling vector fields.

Lemma 4.15. *Let $L(\cdot,D)$ be a cocanceling linear differential operator of order κ on Ω from F to V as in* (4.12). *Then, for each $x_0 \in \Omega$, there exists an open ball $B = B(x_0,r) \subset \Omega$ such that*

$$\left|\int_B \varphi(x)\cdot f(x)\,dx\right| \lesssim \sum_{j=1}^{\kappa} \int_B |f(x)||x-x_0|^j |D^j\varphi(x)|\,dx + \int_B |f(x)||x-x_0||\varphi(x)|\,dx,$$

for all $\varphi \in C_c^\kappa(B,F)$ and $f \in \mathscr{X} := \ker L_\kappa(\cdot,D) \cap C_c^\kappa(B,F)$.

Proof. For a fixed $x_0 \in \Omega$, Lemma 4.13 guarantees there is a ball $B = B(x_0,r) \subset \Omega$ and functions $k_\alpha \in C^\infty(B,\mathscr{L}(V,F))$ such that (4.13) holds. Without loss of generality, we can consider $r<1$. Let $P : B \to \mathscr{L}(F,V)$ be given by

$$P(x) = \sum_{|\beta|=\kappa} \frac{(x-x_0)^\beta}{\beta!} k_\beta^*(x),$$

where $k_\beta^*(x) \in \mathscr{L}(F,V)$ denotes the adjoint of $k_\beta(x)$ with respect to the inner product. For $|\alpha| = \kappa$, we have

$$\begin{aligned}
\partial^\alpha P(x) &= \sum_{|\beta|=\kappa} \frac{1}{\beta!}\left[\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}[\partial^\gamma (x-x_0)^\beta]\partial^{\alpha-\gamma}k_\beta^*(x) + [\partial^\alpha (x-x_0)^\beta]k_\beta^*(x)\right] \\
&= \left[\sum_{|\beta|=\kappa}\sum_{\substack{\gamma<\alpha\\ \gamma<\beta}} c_{\alpha,\beta,\gamma}(x-x_0)^{\beta-\gamma}\partial^{\alpha-\gamma}k_\beta^*(x)\right] + k_\alpha^*(x) \\
&\doteq Q_\alpha(x) + k_\alpha^*(x),
\end{aligned}$$

where the second equality is attained observing that, if $\gamma < \alpha$, then γ never equals β, $\partial^\gamma (x-x_0)^\beta = c_{\beta,\gamma}(x-x_0)^{\beta-\gamma}$ when $\gamma < \beta$ and is zero otherwise, and, if $|\beta| = |\alpha|$, then $\partial^\alpha (x-x_0)^\beta = 0$ unless $\beta = \alpha$, when it equals $\alpha!$. It is noticeable that Q_α is of order at most κ and has no term without a power of x. Hence, we have

$$\begin{aligned}
L_\kappa^*(x,D)(P(x)) &= \sum_{|\alpha|=\kappa} b_\alpha^*(x)\partial^\alpha(P(x)) \\
&= \sum_{|\alpha|=\kappa} b_\alpha^*(x)Q_\alpha(x) + \sum_{|\alpha|=\kappa} b_\alpha^*(x)k_\alpha^*(x) \\
&= \sum_{|\alpha|=\kappa} b_\alpha^*(x)Q_\alpha(x) + \left[\sum_{|\alpha|=\kappa} k_\alpha(x)b_\alpha(x)\right]^*.
\end{aligned}$$

Writing $R(x) \doteq \sum_{|\alpha|=\kappa} b_\alpha^*(x)Q_\alpha(x)$ and observing (4.13), we conclude that

$$L_\kappa^*(x,D)(P(x)) = I_F + R(x), \quad x \in B. \tag{4.18}$$

Let $\varphi \in C^\kappa(B\setminus\{x_0\},F)$ and $f \in C_c^\kappa(B,F)$ such that $L_\kappa(x,D)f = 0$. Then,

$$\begin{aligned}
\int_B \varphi(x)\cdot f(x)\,dx &= \int_B f(x)\cdot[L_\kappa^*(x,D)(P(x))\varphi(x) - R(x)\varphi(x)]\,dx \\
&= \underbrace{\int_B f(x)\cdot[L_\kappa^*(x,D)(P(x))\varphi(x)]\,dx}_{\text{(I)}} - \underbrace{\int_B f(x)\cdot[R(x)\varphi(x)]\,dx}_{\text{(II)}}.
\end{aligned}$$

We first observe that

$$\begin{aligned}
\text{(I)} &= \int_B f(x)\cdot [L_\kappa^*(x,D)(P(x))\varphi(x) - L_\kappa^*(x,D)(P(x)\varphi(x))]\,dx + \int_B f(x)\cdot L_\kappa^*(x,D)(P(x)\varphi(x))\,dx \\
&= \int_B f(x)\cdot [L_\kappa^*(x,D)(P(x))\varphi(x) - L_\kappa^*(x,D)(P(x)\varphi(x))]\,dx + \int_B L_\kappa(x,D)(f(x))\cdot P(x)\varphi(x)\,dx \\
&= \int_B f(x)\cdot [L_\kappa^*(x,D)(P(x))\varphi(x) - L_\kappa^*(x,D)(P(x)\varphi(x))]\,dx,
\end{aligned}$$

since $f \in ker L_\kappa(\cdot,D)$. We now proceed to calculate $L_\kappa^*(x,D)(P(x)\varphi(x))$. First,

$$\partial^\alpha(P(x)\varphi(x)) = \sum_{0<\gamma\leqslant\alpha} \partial^{\alpha-\gamma}P(x)\,\partial^\gamma\varphi(x) + \partial^\alpha P(x)\,\varphi(x),$$

where

$$\partial^{\alpha-\gamma}P(x) = \sum_{|\beta|=\kappa}\sum_{\substack{\eta\leqslant\alpha-\gamma\\ \eta\leqslant\beta}} \frac{1}{(\beta-\eta)!}(x-x_0)^{\beta-\eta}\,\partial^{\alpha-\gamma-\eta}k_\beta^*(x).$$

Hence,

$$\begin{aligned}
L_\kappa^*(x,D)(P(x)\varphi(x)) &= \sum_{|\alpha|=\kappa} b_\alpha^*(x)\partial^\alpha(P(x)\varphi(x)) \\
&= \sum_{|\alpha|=\kappa}\sum_{0<\gamma\leqslant\alpha}\sum_{|\beta|=\kappa}\sum_{\substack{\eta\leqslant\alpha-\gamma\\ \eta\leqslant\beta}} \frac{1}{(\beta-\eta)!}\,b_\alpha^*(x)\,(x-x_0)^{\beta-\eta}\,\partial^{\alpha-\gamma-\eta}k_\beta^*(x)\,\partial^\gamma\varphi(x) \\
&\qquad + \sum_{|\alpha|=\kappa} b_\alpha^*(x)\,\partial^\alpha P(x)\,\varphi(x).
\end{aligned}$$

Since the second part is just $L_\kappa^*(x,D)(P(x))\varphi(x)$ and, on B, k_β^* and b_α^* are C^∞ functions, we have

$$|\text{(I)}| \lesssim \int_B |f(x)| \sum_{|\alpha|=\kappa}\sum_{0<\gamma\leqslant\alpha}\sum_{|\beta|=\kappa}\sum_{\substack{\eta\leqslant\alpha-\gamma\\ \eta\leqslant\beta}} |x-x_0|^{|\beta-\eta|}\,|\partial^\gamma\varphi(x)|\,dx.$$

However, as $|\eta| \leqslant |\alpha-\gamma| = |\alpha|-|\gamma| = \kappa-|\gamma|$, we get $|\beta-\eta| = |\beta|-|\eta| = \kappa-|\eta| \geqslant |\gamma|$. As we took B with radius smaller than 1, we have

$$|\text{(I)}| \lesssim \int_B |f(x)| \sum_{|\alpha|=\kappa}\sum_{0<\gamma\leqslant\alpha} |x-x_0|^{|\gamma|}\,|\partial^\gamma\varphi(x)|\,dx \lesssim \sum_{j=1}^{\kappa}\int_B |f(x)||x-x_0|^j\,|D^j\varphi(x)|\,dx.$$

Meanwhile, $|\text{(II)}| \leqslant \int_B |f(x)|\,|R(x)|\,|\varphi(x)|\,dx$. Observe that, for $x \in S \doteq \operatorname{supp}(f)$,

$$|R(x)| \lesssim \sum_{|\alpha|=\kappa}\sum_{|\beta|=\kappa}\sum_{\substack{\gamma<\alpha\\ \gamma<\beta}} \|b_\alpha^*\|_{L^\infty(S)}\,|x-x_0|^{|\beta-\gamma|}\,\|\partial^{\alpha-\gamma}k_\beta^*\|_{L^\infty(S)} \lesssim |x-x_0|,$$

since $|\beta-\gamma| = |\beta|-|\gamma| \geqslant \kappa-(\kappa-1) = 1$ and $|x-x_0| < r < 1$. Therefore,

$$\left|\int_B \varphi(x)\cdot f(x)\,dx\right| \lesssim \sum_{j=1}^{\kappa}\int_B |f(x)||x-x_0|^j\,|D^j\varphi(x)|\,dx + \int_B |f(x)||x-x_0||\varphi(x)|\,dx.$$

□

Remark 4.16. When $L(x,D) = L(D)$, i.e. L has constant coefficients, one has that the k_α from Lemma 4.13 are constants as well. In this case, $Q_\alpha \equiv 0$ and, consequently, $R \equiv 0$ in the previous proof. So, letting $x_0 = 0$ and $B = \mathbb{R}^N$,

$$\left| \int_{\mathbb{R}^N} \varphi(x) \cdot f(x)\, dx \right| \lesssim \sum_{j=1}^{\kappa} \int_{\mathbb{R}^N} |f(x)||x|^j \, |D^j \varphi(x)| \, dx,$$

recovering Lemma 2.12.

4.3 The $1 < p < \infty$ case

The focus of this section is to prove Theorem 4.2. We will make use of the famous Sobolev-Gagliardo-Nirenberg inequality:

Theorem 4.17 ([9, Theorem 9.9])**.** *Let* $1 \leqslant p < \infty$. *Then,*

$$W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N), \quad \text{where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N},$$

and there exists a constant $C = C(N,p)$ *such that*

$$\|f\|_{L^{p^*}} \leqslant C \|\nabla f\|_{L^p}, \quad \forall\, f \in W^{1,p}(\mathbb{R}^N). \tag{4.19}$$

Suppose $A(\cdot,D)$ is an elliptic operator of order m on Ω as in (4.1) and consider the order $2m$ differential operator $\Delta_A \doteq A_m^*(\cdot,D)\, A_m(\cdot,D)$, which may be regarded as an elliptic pseudo-differential operator with symbol in $S^{2m}(\Omega)$. So, from Theorem 1.48, there exist pseudo-differential operators $q(\cdot,D) \in OpS^{-2m}(\Omega)$ and $r(\cdot,D) \in OpS^{-\infty}(\Omega)$ such that

$$u(x) = q(x,D)\Delta_A u(x) + r(x,D)u(x), \quad \forall\, u \in C^\infty(\Omega,E). \tag{4.20}$$

For the next proposition, recall from (4.16) that

$$\|f\|_{W^{m,p}} \doteq \sum_{|\beta|=m} \|\partial^\beta f\|_{L^p}.$$

Proposition 4.18. *Let* $A_m(\cdot,D)$ *as before and* $1 < p < \infty$. *Then, for every point* $x_0 \in \Omega$, *there exist an open ball* $B = B(x_0,\ell) \subset \Omega$ *and a constant* $C = C(B) > 0$ *such that*

$$\|\varphi\|_{W^{m,p}} \leqslant C \|A_m(\cdot,D)\varphi\|_{L^p}, \quad \forall\, \varphi \in C_c^\infty(B,E). \tag{4.21}$$

Proof. Fixed $x_0 \in \Omega$, let $\ell > 0$ such that $B = B(x_0,\ell) \subset \Omega$. It follows from Hölder inequality and Sobolev-Gagliardo-Nirenberg inequality (4.19) that

$$\int_B |\varphi(x)|^p dx \leqslant |B|^{1-p/p^*} \left(\int_B |\varphi(x)|^{p^*} dx \right)^{p/p^*} \leqslant C|B|^{p/N} \|\nabla \varphi\|_{L^p}^p,$$

for all $\varphi \in C_c^\infty(B,E)$, where $C > 0$ is an universal constant depending on N and p. Hence,

$$\|\varphi\|_{L^p} \leqslant C|B|^{1/N}\|\nabla\varphi\|_{L^p}.$$

Bootstrapping the previous argument, we get

$$\|\varphi\|_{L^p} \leqslant C|B|^{m/N}\|\varphi\|_{W^{m,p}}, \tag{4.22}$$

and if we start with $\partial^\alpha\varphi$ instead of φ, we obtain

$$\|\partial^\alpha\varphi\|_{L^p} \leqslant C|B|^{(m-|\alpha|)/N}\|\varphi\|_{W^{m,p}}, \quad \forall\,\varphi \in C_c^\infty(B,E) \tag{4.23}$$

for all $|\alpha| \leqslant m$. Now, from identity (4.20), we may write, for $|\beta| \leqslant m$,

$$\partial^\beta\varphi(x) = \tilde{q}(x,D)[A_m(x,D)\varphi(x)] + \tilde{r}(x,D)\varphi(x)$$

where $\tilde{q}(\cdot,D) := \partial^\beta q(\cdot,D)A_m^*(\cdot,D) \in OpS^{-(m-|\beta|)}(\Omega)$ and $\tilde{r}(\cdot,D) := \partial^\beta r(\cdot,D) \in OpS^{-\infty}(\Omega)$. Thus,

$$\begin{aligned}\|\partial^\beta\varphi\|_{L^p} &\leqslant \|\tilde{q}(\cdot,D)[A_m(\cdot,D)\varphi]\|_{L^p} + \|\tilde{r}(\cdot,D)\varphi\|_{L^p}\\ &\lesssim \|A_m(\cdot,D)\varphi\|_{L^p} + \|\varphi\|_{L^p},\end{aligned}$$

since $\tilde{q}(\cdot,D)$ and $\tilde{r}(\cdot,D)$ are bounded from L^p to itself for $1 < p < \infty$ (Theorem 1.41), including the case $p = 1$ if $|\beta| < m$ (Theorem 1.42). Hence, we get

$$\|\varphi\|_{W^{m,p}} \lesssim \|A_m(\cdot,D)\varphi\|_{L^p} + \|\varphi\|_{L^p},$$

and using (4.22),

$$\|\varphi\|_{W^{m,p}} \lesssim \|A_m(\cdot,D)\varphi\|_{L^p} + |B|^{m/N}\|\varphi\|_{W^{m,p}}.$$

Shrinking the radius of B to absorb the second term on the right-hand side, we conclude the desired estimate. □

Let $A(\cdot,D)$ be as in (4.1) and suppose that all coefficients a_α are bounded at some neighborhood of $x_0 \in \Omega$, namely $B = B(x_0,\ell) \subseteq \Omega$, and consider $C := \sum_{|\alpha|<m}\|a_\alpha\|_{L^\infty(B)}$. For $1 < p < \infty$ and $u \in C_c^\infty(B,E)$, we have

$$\begin{aligned}\|A_m(\cdot,D)u\|_{L^p} &= \Big\|A(\cdot,D)u - \sum_{|\alpha|<m} a_\alpha\partial^\alpha u\Big\|_{L^p}\\ &\leqslant \|A(\cdot,D)u\|_{L^p} + C\sum_{|\alpha|<m}\|\partial^\alpha u\|_{L^p}\end{aligned}$$

Then, decreasing the radius ℓ if necessary, (4.23) and (4.21) give

$$\begin{aligned}\|A_m(\cdot,D)u\|_{L^p} &\lesssim \|A(\cdot,D)u\|_{L^p} + \sum_{|\alpha|<m}|B|^{(m-|\alpha|)/N}\|u\|_{W^{m,p}}.\\ &\lesssim \|A(\cdot,D)u\|_{L^p} + |B|^{1/N}\|u\|_{W^{m,p}}\\ &\lesssim \|A(\cdot,D)u\|_{L^p} + |B|^{1/N}\|A_m(\cdot,D)u\|_{L^p}\end{aligned}$$

Shrinking ℓ conveniently to absorb the second term on the right-hand side, we have

$$\|A_m(\cdot,D)u\|_{L^p} \leqslant C\|A(\cdot,D)u\|_{L^p}, \quad \forall u \in C_c^\infty(B,E), \tag{4.24}$$

for all $1 < p < \infty$ and $C = C(B) > 0$. The case $p = 1$ also holds for (4.24) if, besides being elliptic $A(\cdot,D)$ is also canceling. This is consequence of Theorem 4.14. Thus, using the previous argument, it is sufficient to remark that, if $|\alpha| < m$, then, from (4.23), Hölder inequality and (4.15),

$$\begin{aligned} \|\partial^\alpha u\|_{L^1} &\lesssim |B|^{(m-1-|\alpha|)/N}\|u\|_{W^{m-1,1}} \\ &\leqslant |B|^{(m-|\alpha|)/N}\|u\|_{W^{m-1,N/(N-1)}} \\ &\leqslant C|B|^{(m-|\alpha|)/N}\|A(\cdot,D)u\|_{L^1}, \end{aligned} \tag{4.25}$$

for all $u \in C_c^\infty(B,E)$, and the proof follows analogously as before.

Now we are ready to demonstrate the main result of this section.

Proof of Theorem 4.2. Let $w_A^{m,p'}(B,E)$ be the closure of $C_c^\infty(B,E)$ with respect to the norm $\|u\|_{m,p'} \doteq \|A(\cdot,D)u\|_{L^{p'}}$ for some open ball B in Ω. We claim that, for each $x_0 \in \Omega$, there exists an open ball $B = B(x_0,\ell) \subseteq \Omega$ such that

$$\left|\int_B u(x)\,d\mu(x)\right| \lesssim \|A(\cdot,D)u\|_{L^{p'}}\|J_m|\mu|\|_{L^p(B)}, \quad \forall u \in C_c^\infty(B,E). \tag{4.26}$$

Since $|\mu|$ has finite strong $(m,p)-$energy on B, it follows that $\mu \in [w_A^{m,p'}(B,E)]^*$. It is clear that $A(\cdot,D): w_A^{m,p'}(B,E) \to L^{p'}(B,F)$ is a linear isometry, hence its adjoint $A^*(\cdot,D): L^p(B,F) \to [w_A^{m,p'}(B,E)]^*$ is surjective. Therefore, there exists $f \in L^p(B,F)$ such that $A^*(x,D)f = \mu$ in B.

Thus, all that is left to do is to prove (4.26). Fixed $B(x_0,\ell) \subseteq \Omega$, for all $u \in C_c^\infty(B,E)$ we may write, using (4.20), the identity

$$\begin{aligned} u(x) &= q(x,D)\Delta_A(x,D)u(x) + r(x,D)u(x) \\ &= J_m(D)[q_0(x,D)A_m(x,D)u(x) + r_0(x,D)u(x)] \\ &= J_m(D)\tilde{u}(x), \end{aligned}$$

where

$$q_0(x,\xi) \doteq [\xi]^m q(x,\xi)A_m^*(x,\xi) \in S^0(\Omega),$$
$$r_0(x,\xi) \doteq [\xi]^m r(x,\xi) \in S^{-\infty}(\Omega) \text{ and}$$
$$\tilde{u}(x) \doteq q_0(x,D)A_m(x,D)u(x) + r_0(x,D)u(x).$$

Thus,

$$\begin{aligned}\left|\int_B u(x)\,d\mu(x)\right| &= \left|\int_B J_m(D)\tilde{u}(x)\,d\mu(x)\right| \\ &\lesssim \int_B \left[\int_{\mathbb{R}^N} G_m(x-y)|\tilde{u}(y)|\,dy\right] d|\mu|(x) \\ &= \int_{\mathbb{R}^N} |\tilde{u}(y)| \left[\int_B G_m(x-y)\,d|\mu|(x)\right] dy \\ &\lesssim \int_B |\tilde{u}(y)|\,J_m|\mu|(y)\,dy \\ &\lesssim \left[\|q_0(\cdot,D)A_m(\cdot,D)u\|_{L^{p'}} + \|r_0(\cdot,D)u\|_{L^{p'}}\right] \|J_m|\mu|\|_{L^p(B)} \\ &\lesssim \left[\|A_m(\cdot,D)u\|_{L^{p'}} + \|u\|_{L^{p'}}\right] \|J_m|\mu|\|_{L^p(B)},\end{aligned}$$

where in the last inequality we used that $q_0(\cdot,D)$ and $r_0(\cdot,D)$ are bounded operators from $L^{p'}$ to itself, since $1<p'<\infty$. It follows from (4.22), (4.21) and (4.24), shrinking the radius ℓ if necessary,

$$\begin{aligned}\left|\int_B u(x)\,d\mu(x)\right| &\lesssim \left[\|A_m(\cdot,D)u\|_{L^{p'}} + \|u\|_{W^{m,p'}}\right] \|J_m|\mu|\|_{L^p(B)} \\ &\lesssim \|A_m(\cdot,D)u\|_{L^{p'}} \|J_m|\mu|\|_{L^p(B)} \\ &\lesssim \|A(\cdot,D)u\|_{L^{p'}} \|J_m|\mu|\|_{L^p(B)}, \quad \forall u \in C_c^\infty(B,E),\end{aligned}$$

completing the proof. □

4.4 The $p=\infty$ case

Now we turn to the proof of Theorem 4.3. Suppose $A(\cdot,D)$ is an elliptic and canceling operator of order m on Ω as in (4.1). In order to accomplish the proof, it is sufficient to show that, for each $x_0\in\Omega$ satisfying (4.7), there exists an open ball $B=B(x_0,\ell)\subseteq\Omega$ such that

$$\left|\int_B u(x)\,d\mu(x)\right| \lesssim \|A_m(\cdot,D)u\|_{L^1}, \quad \forall u \in C_c^\infty(B,E), \tag{4.27}$$

then from (4.24), decreasing ℓ if necessary,

$$\left|\int_B u(x)\,d\mu(x)\right| \lesssim \|A(\cdot,D)u\|_{L^1}, \quad \forall u \in C_c^\infty(B,E) \tag{4.28}$$

meaning that $\mu\in[w_A^{m,1}(B,E)]^*$ and, following the argument used in the proof of Theorem 4.2, there exists $f\in L^\infty(B,F)$ such that $A^*(x,D)f=\mu$ in B.

From Theorem 1.48, there exist pseudo-differential operators $Q_1(\cdot,D)\in OpS^{-m}(\Omega)$ and $Q_2(\cdot,D)\in OpS^{-\infty}(\Omega)$ such that

$$u(x) = Q_1(x,D)A_m(x,D)\,u(x) + Q_2(x,D)\,u(x), \quad \forall u \in C^\infty(\Omega,E). \tag{4.29}$$

In view of the previous identity, in order to obtain (4.27) it is enough to prove that, for some $B = B(x_0,\ell)$ and $C = C(B) > 0$, the estimates

$$\int_B |Q_2(x,D)\,u(x)|\,d|\mu|(x) \leqslant C\|A_m(\cdot,D)u\|_{L^1} \tag{4.30}$$

and

$$\int_B |Q_1(x,D)\,A_m(x,D)\,u(x)|\,d|\mu|(x) \leqslant C\|A_m(\cdot,D)u\|_{L^1} \tag{4.31}$$

hold for all $u \in C_c^\infty(B,E)$.

First we prove (4.30). Since $Q_2(\cdot,D) \in OpS^{-\infty}(\Omega)$, Theorem 1.41 says it is bounded from $L^{N/(N-1)}$ to itself. Also, there exists $B = B(x_0,\ell)$ such that (4.15) holds. We can take $\ell < 1$. Thus, using Hölder inequality, (4.6), (4.22) and (4.15), we get

$$\begin{aligned}
\int_B |Q_2(x,D)u(x)|\,d|\mu|(x) &\leqslant \|Q_2(\cdot,D)u\|_{L^{N/(N-1)}(B)}\,|\mu|(B)^{1/N}\\
&\lesssim \|u\|_{L^{N/(N-1)}}\,\ell^{\frac{N-m}{N}}\\
&\lesssim \ell^{\frac{N-m}{N}}\,|B|^{\frac{m-1}{N}}\,\|u\|_{W^{m-1,N/(N-1)}}\\
&\lesssim \ell^{\frac{N-m}{N}+m-1}\,\|u\|_{W^{m-1,N/(N-1)}}\\
&\lesssim \|A_m(\cdot,D)u\|_{L^1},
\end{aligned}$$

for all $u \in C_c^\infty(B,E)$.

Now, to prove (4.31), let us recall that $Q_1(\cdot,D) \in S^{-m}(\Omega)$ can be written in terms of its distribution kernel $K(x,y)$:

$$Q_1(x,D)\,A_m(x,D)\,u(x) = \int_\Omega K(x,y)\,A_m(y,D)\,u(y)\,dy.$$

From Theorem 1.40 (i), K is smooth outside the diagonal $\{(x,x) \in \Omega\times\Omega\}$ and, from Theorem 1.40 (v), it satisfies the estimates

$$|K(x,y)| \leqslant C_1|x-y|^{m-N}, \quad x \neq y, \tag{4.32}$$

and

$$|\partial_y K(x,y)| \leqslant C_2|x-y|^{m-N-1}, \quad x \neq y, \tag{4.33}$$

for some $C_1, C_2 > 0$. Thus, the proof of (4.31) reduces to obtaining

$$\int_B \left| \int_\Omega K(x,y)\,g(y)\,dy \right| d|\mu|(x) \leqslant C\|g\|_{L^1}, \tag{4.34}$$

where $g := A_m(\cdot,D)u$, for all $u \in C_c^\infty(B,E)$.

4.4.1 A local Hardy-type inequality

First, we show a local version of the Hardy-type inequality from Lemma 2.11.

Lemma 4.19. *Let $1 \leqslant q < \infty$, ν be a σ-finite real positive measure on an open set $U \subset \mathbb{R}^N$ and $B(x_0,r) \subseteq U$ a fixed ball. Let $\tilde{u}$ and $\tilde{v}$ be measurable and non-negative almost everywhere on U. Then, for $0<\delta \leqslant 1$, there exists $A = A(\delta) > 0$ such that the inequality*

$$\left[\int_{B(x_0,r)} \left(\int_{B(x_0,\delta|x-x_0|)} \tilde{g}(y)\,dy\right)^q \tilde{u}(x)\,d\nu\right]^{1/q} \leqslant A \int_{B(x_0,r)} \tilde{g}(x)\tilde{v}(x)\,dx \tag{4.35}$$

holds for all $\tilde{g} \geqslant 0$ if

$$\left(\int_{B^c(x_0,\delta^{-1}|y-x_0|)\cap U} \tilde{u}(x)\,d\nu\right)^{1/q} \leqslant A\tilde{v}(y), \quad a.e.\ y \in B(x_0,\delta r). \tag{4.36}$$

Proof. By Minkowski inequality, we have

$$\begin{aligned}
&\left[\int_{B(x_0,r)} \left(\int_{B(x_0,\delta|x-x_0|)} \tilde{g}(y)\,dy\right)^q \tilde{u}(x)\,d\nu\right]^{1/q} \\
&\qquad = \left[\int_{B(x_0,r)} \left(\int_U \tilde{g}(y)\,\chi_{\{|y-x_0|<\delta|x-x_0|\}}(x,y)\,dy\right)^q \tilde{u}(x)\,d\nu\right]^{1/q} \\
&\qquad \leqslant \int_{B(x_0,\delta r)} \left(\int_U [\tilde{g}(y)]^q \chi_{\{|y-x_0|<\delta|x-x_0|\}}(x,y)\,\tilde{u}(x)\,d\nu\right)^{1/q} dy \\
&\qquad = \int_{B(x_0,\delta r)} \tilde{g}(y) \left(\int_{B^c(x_0,\delta^{-1}|y-x_0|)\cap U} \tilde{u}(x)\,d\nu\right)^{1/q} \tilde{v}(y)[\tilde{v}(y)]^{-1}\,dy \\
&\qquad \leqslant A \int_{B(x_0,r)} \tilde{g}(y)\,\tilde{v}(y)\,dy,
\end{aligned}$$

where in the last inequality we used (4.36) and $\delta \leqslant 1$. □

Example 4.20. Let $U = B(x_0,r) \subset \mathbb{R}^N$ with $r<1$, $\delta = r/2$, and consider ν a positive measure such that $\nu(B(x_0,R)) \leqslant CR^t$ for all $0<R<r$ and some $t>0$. We claim that (4.36) is satisfied for $\tilde{u}(x) := |x-x_0|^{-t-q}$ and $\tilde{v}(y) := |y-x_0|^{-1}$. Indeed, denote by $\tilde{\nu}$ the natural extension of the measure ν in $\mathbb{R}^N$ given by $\tilde{\nu}(S) := \nu(S\cap U)$. Clearly, for any $R>0$, we have $\tilde{\nu}(B(x_0,R)) \leqslant CR^t$. Thus, since $B^c(x_0,\delta^{-1}|y-x_0|)\cap B(x_0,r) \neq \varnothing$ for $y \in B(x_0,\delta r)$, denoting

$$A^k := \{x : \delta^{-1}|y-x_0|2^k \leqslant |x-x_0| \leqslant \delta^{-1}|y-x_0|2^{k+1}\},$$

we have

$$
\begin{aligned}
\left(\int_{B^c(x_0,\delta^{-1}|y-x_0|)\cap U}\tilde{u}(x)\,d\nu\right)^{1/q} &= \left(\int_{B^c(x_0,\delta^{-1}|y-x_0|)}\frac{1}{|x-x_0|^{t+q}}\,d\tilde{\nu}\right)^{1/q}\\
&= \left(\sum_{k=0}^{\infty}\int_{A^k}|x-x_0|^{-t-q}\,d\tilde{\nu}\right)^{1/q}\\
&\leqslant \left(\sum_{k=0}^{\infty}\left(\delta^{-1}|y-x_0|2^k\right)^{-t-q}\tilde{\nu}\left(B(x_0,\delta^{-1}|y-x_0|2^{k+1})\right)\right)^{1/q}\\
&\leqslant C\left(\sum_{k=0}^{\infty}\left(\delta^{-1}|y-x_0|2^k\right)^{-t-q}\left(\delta^{-1}|y-x_0|2^{k+1}\right)^{t}\right)^{1/q}\\
&= C\delta 2^{t/q}|y-x_0|^{-1}\left(\sum_{k=0}^{\infty}2^{-kq}\right)^{1/q}\\
&= C\delta 2^{(t/q)+1}(2^q-1)^{-1/q}\tilde{v}(y)\\
&\leqslant A\tilde{v}(y),
\end{aligned}
$$

with A independent of δ.

4.4.2 A local Stein-Weiss-type inequality

Finally, in order to obtain (4.34), we prove a local version of Lemma 2.13.

Lemma 4.21. *Assume $N \geqslant 2$, $0<\ell<N$ and $K(x,y)\in L^1_{loc}(\mathbb{R}^N\times\Omega,\mathscr{L}(F,V))$ satisfying*

$$|K(x,y)|\leqslant C_1\,|x-y|^{\ell-N},\quad x\neq y \tag{4.37}$$

and

$$|K(x,y)-K(x,z)|\leqslant C_2\,\frac{|y-z|}{|x-z|^{N-\ell+1}},\quad 2|y-z|\leqslant|x-z|. \tag{4.38}$$

Suppose $1\leqslant q<N/(N-\ell)$ and let $\nu\in\mathscr{M}_+(\Omega)$ satisfying

$$\|\nu\|_{\Omega,(N-\ell)q}\doteq\sup_{B(x,R)\subset\Omega}\frac{\nu(B(x,R))}{R^{(N-\ell)q}}<\infty. \tag{4.39}$$

If $L(\cdot,D)$ is cocanceling then, for each $x_0\in\Omega$, there exist an open neighborhood $x_0\in U\subset\Omega$ and $C=C(U)>0$ such that, if

$$\int_0^{a|y-x_0|}\frac{\nu(B(y,s))}{s^{(N-\ell)q+1}}\,ds\leqslant C_3, \tag{4.40}$$

where a is some constant between 0 and 1, is a uniform control for almost every $y\in U$, then

$$\left(\int_U\left|\int_U K(x,y)g(y)\,dy\right|^q d\nu(x)\right)^{1/q}\leqslant C\int_U|g(x)|\,dx, \tag{4.41}$$

holds for all $g\in C^\infty_c(U,F)$ satisfying $L_\kappa(\cdot,D)g=0$ in the sense of distributions.

Remark 4.22. A stronger condition satisfying (4.40) is given by

$$\nu(B(y,R)) \leqslant C\,|y-x_0|^{(N-\ell)q-N}R^N \tag{4.42}$$

when $R < a|y-x_0|$. In fact,

$$\int_0^{a|y-x_0|} \frac{\nu(B(y,s))}{s^{(N-\ell)q+1}}\,ds \leqslant C|y-x_0|^{(N-\ell)q-N}\int_0^{a|y-x_0|} s^{-(N-\ell)q-1+N}\,ds = C\frac{a^{N-(N-\ell)q}}{N-(N-\ell)q},$$

since $N-(N-\ell)q > 0$.

Example 4.23. Let $U = B(x_0,r)$. The positive scalar measure given by the weighted power $d\nu := |x-x_0|^{t-N}dx$, for $0 < t < N$, satisfies

$$\nu(B(x_0,R)) \leqslant CR^t, \quad \text{for all } 0 < R < r \tag{4.43}$$

and

$$\nu(B(y,R)) \leqslant C|y-x_0|^{t-N}R^N, \quad \text{for all } R < a|y-x_0| \tag{4.44}$$

and $0 < a < 1$.

Notice that

$$\nu(B(x_0,R)) = \int_{B(x_0,R)} |x-x_0|^{t-N}\,dx = \left|\mathbb{S}^{N-1}\right|\int_0^R r^{t-1}\,dr = t^{-1}\left|\mathbb{S}^{N-1}\right|R^t$$

for any $R > 0$. For (4.44), we note that, if $|x| < R < a|y-x_0|$, then $(1-a)|y-x_0| < |x+y-x_0| < (1+a)|y-x_0|$. Thus,

$$\nu(B(y,R)) = \int_{B(y,R)} |x-x_0|^{t-N}\,dx = \int_{B_R} |x+y-x_0|^{t-N}\,dx \lesssim |y-x_0|^{t-N}|B_R| = C|y-x_0|^{t-N}R^N.$$

Assume the validity of Lemma 4.21. Let $A(\cdot,D)$ and μ be as in the statement of Theorem 4.3. Take K as the distribution kernel of Q_1 (obtained in (4.29)). Hence (4.32) and (4.33) imply that K satisfies (4.37) and (4.38) with $\ell = m$, (4.6) and (4.7) imply that $\nu = |\mu|$ satisfies (4.39) and (4.39) with $\ell = m$ and $q = 1$, and, since $A(\cdot,D)$ is elliptic and canceling, (4.17) implies that there exists a cocanceling operator $L(\cdot,D)$ of order $\kappa = 2mN$ such that $g = A_m(\cdot,D)u \in ker\,L_\kappa(\cdot,D)$ for all $u \in C_c^\infty(U,E)$. Putting it all together, we obtain (4.41) in the particular form

$$\int_U \left|\int_U K(x,y)A_m(y,D)u(y)\,dy\right| d|\mu|(x) \leqslant C\|A_m(\cdot,D)u\|_{L^1},$$

which is exactly (4.34). As we have already discussed, this implies (4.31) which, together with (4.30), shows (4.27), and this proves Theorem 4.3.

Proof of Lemma 4.21. For each $x_0 \in \Omega$, let $U \doteq B(x_0,r)$ be the neighborhood from Lemma 4.15. We can always choose $r < 1$ small enough such that we have $B(x_0,(1+a)r) \subset \Omega$. Consider $\psi \in$

$C_c^\infty(B(x_0,r/2))$ be a cut-off function such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ on $B(x_0,r/4)$, and write $K(x,y) = K_1(x,y)+K_2(x,y)$ with $K_1(x,y) = \psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)K(x,x_0)$. To prove (4.41), it is enough to show that

$$J_j \doteq \left(\int_U \left|\int_U K_j(x,y)g(y)\,dy\right|^q d\nu(x)\right)^{1/q} \lesssim \int_U |g(x)|\,dx, \quad j=1,2,$$

for all $g \in C_c^\infty(U,F)$ such that $L_\kappa(\cdot,D)g = 0$. First, we point out that

$$\begin{aligned} J_1 &= \left(\int_U \left|\int_U \psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)g(y)\,dy\right|^q |K(x,x_0)|^q\,d\nu(x)\right)^{1/q} \\ &\leqslant C_1\left(\int_U \left|\int_U \psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)g(y)\,dy\right|^q \frac{1}{|x-x_0|^{(N-\ell)q}}\,d\nu(x)\right)^{1/q}, \end{aligned}$$

in which the inequality follows from (4.37). Now we use Lemma 4.15 for $\varphi(y) = \psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)\eta$ where, for a fixed $x \in U$, η is a unit vector in F chosen so that

$$\left|\int_U \psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)\eta\cdot g(y)\,dy\right| = \left|\int_U \psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)g(y)\,dy\right|.$$

Thus,

$$\begin{aligned} \left|\int_U \psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)g(y)\,dy\right| &\lesssim \sum_{j=1}^{\kappa}\int_U |g(y)||y-x_0|^j\,|D^j\varphi(y)|\,dy + \int_U |g(y)||y-x_0||\varphi(y)|\,dy \\ &\lesssim \sum_{j=1}^{\kappa}\int_U |g(y)|\frac{|y-x_0|^j}{|x-x_0|^j}\left|D^j\psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)\right|dy \\ &\quad + \int_U |g(y)||y-x_0|\left|\psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)\right|dy \\ &\lesssim \int_{B\left(x_0,\frac{r|x-x_0|}{2}\right)} |g(y)|\frac{|y-x_0|}{|x-x_0|}\,dy + \int_{B\left(x_0,\frac{r|x-x_0|}{2}\right)} |g(y)||y-x_0|\,dy \\ &\lesssim \int_{B\left(x_0,\frac{r|x-x_0|}{2}\right)} |g(y)|\frac{|y-x_0|}{|x-x_0|}\,dy, \end{aligned}$$

where in the second to last inequality we use that $\psi \in C_c^\infty(B(x_0,r/2))$ and $r<1$ to get $\frac{|y-x_0|}{|x-x_0|} < 1$ and, in the last one, that $x \in U$ implies $|y-x_0| < \frac{|y-x_0|}{|x-x_0|}$. Replacing it in the previous inequality, we have

$$J_1 \lesssim \left[\int_U \left(\int_{B\left(x_0,\frac{r|x-x_0|}{2}\right)} |g(y)||y-x_0|\,dy\right)^q \frac{1}{|x-x_0|^{(N-\ell+1)q}}\,d\nu(x)\right]^{1/q}.$$

Now we use Lemma 4.19 for $\tilde{g}(y) = |g(y)||y-x_0|$, $\tilde{u}(x) = |x-x_0|^{-(N-\ell+1)q}$ and $\tilde{v}(y) = |y-x_0|^{-1}$, which is exactly Example 4.20 for $t = (N-\ell)q$. Therefore,

$$J_1 \lesssim \int_U |g(x)|\,dx, \quad \text{for all } g \in C_c^\infty(U,F) \text{ satisfying } L_\kappa(\cdot,D)g = 0.$$

For J_2 we are going to analyze

$$K_2(x,y) = K(x,y) - K_1(x,y) = K(x,y) - \psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)K(x,x_0).$$

If $x \in B\left(x_0, \frac{2}{r}|y-x_0|\right)$, then $\psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)=0$ and we have $|K_2(x,y)|=|K(x,y)|$. Otherwise, if $x \in B^c\left(x_0, \frac{4}{r}|y-x_0|\right)$, then $\psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)=1$ and $|K_2(x,y)|=|K(x,y)-K(x,x_0)|$. In the intermediate region $B^c\left(x_0, \frac{2}{r}|y-x_0|\right) \cap B\left(x_0, \frac{4}{r}|y-x_0|\right)$, the following identity holds:

$$K_2(x,y)=\left[1-\psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)\right]K(x,y)+\psi\left(\frac{y-x_0}{|x-x_0|}+x_0\right)[K(x,y)-K(x,x_0)].$$

Hence, using Minkowski's inequality,

$$J_2=\left(\int_U\left|\int_U K_2(x,y)g(y)\,dy\right|^q d\nu(x)\right)^{1/q} \leqslant \int_U\left(\int_U |K_2(x,y)|^q\,d\nu(x)\right)^{1/q}|g(y)|\,dy.$$

If we prove that $\int_U |K_2(x,y)|^q\,d\nu(x) \leqslant C$ for almost every $y \in U$ with a uniform constant $C>0$, we will obtain

$$J_2 \lesssim \int_U |g(y)|\,dy, \quad \text{for all } g \in C_c^\infty(U,F),$$

concluding the proof.

Estimating $K_2(x,y)$ in the three regions featured above and using (4.37) and (4.38), we can write

$$\begin{aligned}
\int_U |K_2(x,y)|^q\,d\nu(x) &= \int_{U\cap B\left(x_0,\frac{2}{r}|y-x_0|\right)} |K_2(x,y)|^q\,d\nu(x) \\
&\quad + \int_{U\cap B^c\left(x_0,\frac{2}{r}|y-x_0|\right)\cap B\left(x_0,\frac{4}{r}|y-x_0|\right)} |K_2(x,y)|^q\,d\nu(x) \\
&\qquad + \int_{U\cap B^c\left(x_0,\frac{4}{r}|y-x_0|\right)} |K_2(x,y)|^q\,d\nu(x) \\
&\lesssim \int_{U\cap B\left(x_0,\frac{2}{r}|y-x_0|\right)} |K(x,y)|^q\,d\nu(x) \\
&\quad + \int_{U\cap B^c\left(x_0,\frac{2}{r}|y-x_0|\right)\cap B\left(x_0,\frac{4}{r}|y-x_0|\right)} (|K(x,y)|^q+|K(x,y)-K(x,x_0)|^q)\,d\nu(x) \\
&\qquad + \int_{U\cap B^c\left(x_0,\frac{4}{r}|y-x_0|\right)} |K(x,y)-K(x,x_0)|^q\,d\nu(x) \\
&= \int_{U\cap B\left(x_0,\frac{4}{r}|y-x_0|\right)} |K(x,y)|^q\,d\nu(x) \\
&\quad + \int_{U\cap B^c\left(x_0,\frac{2}{r}|y-x_0|\right)} |K(x,y)-K(x,x_0)|^q\,d\nu(x) \\
&\lesssim \underbrace{\int_{U\cap B\left(x_0,\frac{4}{r}|y-x_0|\right)} \frac{1}{|x-y|^{(N-\ell)q}}\,d\nu(x)}_{\text{(I)}} + \underbrace{\int_{U\cap B^c\left(x_0,\frac{2}{r}|y-x_0|\right)} \frac{|y-x_0|^q}{|x-x_0|^{(N-\ell+1)q}}\,d\nu(x)}_{\text{(II)}}.
\end{aligned}$$

Notice that

$$U\cap B\left(x_0,\frac{4}{r}|y-x_0|\right)=\begin{cases} B\left(x_0,\frac{4}{r}|y-x_0|\right), & \text{if } |y-x_0|<r^2/4 \\ U, & \text{otherwise}\end{cases}$$

and

$$U\cap B^c\left(x_0,\frac{2}{r}|y-x_0|\right)=\begin{cases} A\left(x_0,\frac{2}{r}|y-x_0|,r\right), & \text{if } |y-x_0|<r^2/2 \\ \varnothing, & \text{otherwise,}\end{cases}$$

where $A(p,r_1,r_2)$ denotes the annulus $\{x : r_1 \leqslant |x-p| < r_2\}$.

To estimate (II), we only need to consider the case $|y-x_0| < r^2/2$. In this situation, proceeding just like in the calculations from Example 4.20, we get (II) $\leqslant C r^q 2^{(N-\ell)q}(2^q-1)^{-1}$.

For (I), consider first $|y-x_0| < r^2/4$. In this case, $B(y,a|y-x_0|) \subset B\left(x_0, \frac{4}{r}|y-x_0|\right) \subset U$ and we can isolate the singularity to obtain

$$\text{(I)} = \underbrace{\int_{B(y,a|y-x_0|)} \frac{1}{|x-y|^{(N-\ell)q}}\, d\nu(x)}_{(*)} + \underbrace{\int_{B\left(x_0,\frac{4}{r}|y-x_0|\right)\cap B^c(y,a|y-x_0|)} \frac{1}{|x-y|^{(N-\ell)q}}\, d\nu(x)}_{(**)}.$$

In $(**)$, we have $|x-y| \geqslant a|y-x_0|$, hence

$$\begin{aligned}(**) \leqslant (a|y-x_0|)^{-(N-\ell)q} \int_{B\left(x_0,\frac{4}{r}|y-x_0|\right)} d\nu(x) &= (a|y-x_0|)^{-(N-\ell)q}\, \nu\left(B\left(x_0, \frac{4}{r}|y-x_0|\right)\right) \\ &\leqslant C a^{-(N-\ell)q}\left(\frac{4}{r}\right)^{(N-\ell)q} \\ &\leqslant C\left(\frac{4}{ar}\right)^{(N-\ell)q}.\end{aligned}$$

For $(*)$, let $A_x = \{s \in \mathbb{R} : s > |x-y|\}$. Then we may write

$$\begin{aligned}(*) &= \int_{B(y,a|y-x_0|)} (N-\ell)q \left(\int_{|x-y|}^{\infty} s^{(\ell-N)q-1}\, ds\right) d\nu(x) \\ &= (N-\ell)q \int_{\Omega} \chi_{B(y,a|y-x_0|)}(x) \left(\int_0^{\infty} \frac{\chi_{A_x}(s)}{s^{(N-\ell)q+1}}\, ds\right) d\nu(x) \\ &= (N-\ell)q \int_0^{\infty} \left(\int_{\Omega} \frac{\chi_{B(y,a|y-x_0|)}(x)\, \chi_{A_x}(s)}{s^{(N-\ell)q+1}}\, d\nu(x)\right) ds \\ &= (N-\ell)q \int_0^{\infty} \left(\int_{B(y,a|y-x_0|)\cap B(y,s)} \frac{1}{s^{(N-\ell)q+1}}\, d\nu(x)\right) ds \\ &= (N-\ell)q \left[\int_0^{a|y-x_0|} \frac{\nu(B(y,s))}{s^{(N-\ell)q+1}}\, ds + \nu(B(y,a|y-x_0|)) \int_{a|y-x_0|}^{\infty} \frac{1}{s^{(N-\ell)q+1}}\, ds\right] \\ &\leqslant (N-\ell)q \left[C_3 + \|\nu\|_{\Omega,(N-\ell)q}\, (a|y-x_0|)^{(N-\ell)q}\, (a|y-x_0|)^{(\ell-N)q}\, \frac{1}{(N-\ell)q}\right] \\ &= C_3\,(N-\ell)q + \|\nu\|_{\Omega,(N-\ell)q}.\end{aligned}$$

For the case $|y-x_0| \geqslant r^2/4$, if y is close enough to the boundary of U, the ball $B(y,a|y-x_0|)$ will not be entirely contained in U. But, from our choice of r, $B(y,a|y-x_0|) \subset B(x_0,(1+a)r) \subset \Omega$, so that it makes sense to estimate

$$\text{(I)} \leqslant \underbrace{\int_{B(y,a|y-x_0|)} \frac{1}{|x-y|^{(N-\ell)q}}\, d\nu(x)}_{(*')} + \underbrace{\int_{U\cap B^c(y,a|y-x_0|)} \frac{1}{|x-y|^{(N-\ell)q}}\, d\nu(x)}_{(**')}.$$

The calculations for $(*')$ are exactly the same as before, whereas for $(**')$,

$$\begin{aligned}(**') &\leqslant (a|y-x_0|)^{-(N-\ell)q}\int_U d\nu(x)\\ &\leqslant C\left(\frac{r}{a|y-x_0|}\right)^{(N-\ell)q}\\ &\leqslant C\left(\frac{4r}{ar^2}\right)^{(N-\ell)q}\\ &= C\left(\frac{4}{ar}\right)^{(N-\ell)q},\end{aligned}$$

completing the proof. □

4.5 Applications and general comments

4.5.1 A necessary condition

A natural question arises about necessary conditions on μ in order to obtain local Lebesgue solvability for the equation (4.2). Suppose that, for each $x_0 \in \Omega$, there exists an open ball $B = B(x_0,R) \subseteq \Omega$ such that $f \in L^p(B,F^*)$ is a local solution for $A^*(\cdot,D)f = \mu$ in the sense of (4.3). If the identity

$$\int_B J_m\varphi\, d\mu = \int_B f(x)\cdot A(x,D)J_m\varphi(x)\,dx, \quad \forall\,\varphi \in C_c^\infty(B;E) \tag{4.45}$$

were valid formally, then we should conclude that $J_m(\mu \llcorner B) = J_m A^*(\cdot,D)f$ and, since $J_m \circ A^*(\cdot,D)$ is a pseudo-differential operator of order zero (which is bounded from L^p to itself for $1 < p < \infty$), it would imply $\|J_m(\mu \llcorner B)\|_{L^p(B)} \lesssim \|f\|_{L^p(B)}$, then $\mu \llcorner B$ would have finite strong $(m,p)-$energy on B. However, the identity (4.45) is not a consequence of (4.3), since the smooth function $J_m\varphi$ does not have compact support on B for $\varphi \in C_c^\infty(B,E)$. In order to avoid this technical problem, we state the following result with a notion of local strong $(m,p)-$energy of μ.

Theorem 4.24. *Assume that $A(\cdot,D)$ from E to F is a linear differential operator of order $m < N$ as in* (4.1) *and $\mu \in \mathscr{M}(\Omega,E)$. If, for each $x_0 \in \Omega$, there exists an open ball $B \subset \Omega$ centered at x_0 such that $f \in L^p(B,F)$ is a solution for $A^*(\cdot,D)f = \mu$ for $1 \leqslant p < \infty$, then $\|J_m(\mu \llcorner \widetilde{B})\|_{L^p(B)} < \infty$ for any $\widetilde{B} \subset B$.*

Proof. For each $\widetilde{B} \subset B$, let $\varepsilon > 0$ such that $\widetilde{B}_\varepsilon := \widetilde{B} + B(0,\varepsilon) \subset B$ and $\psi_\varepsilon := \chi_{\widetilde{B}} * \phi_\varepsilon \in C_c^\infty(\widetilde{B}_\varepsilon)$, where $\phi \in C_c^\infty(B(0,1))$, is radial, positive with $\|\phi\|_{L^1} = 1$ and $\phi_\varepsilon(x) = \varepsilon^{-N}\phi(x/\varepsilon)$. Clearly, $\psi_\varepsilon(x) \equiv 1$ on $\widetilde{B}$ and, for $d\mu_\varepsilon := \psi_\varepsilon(x)\,d\mu$ we have $\mu_\varepsilon(A) = (\mu \llcorner \widetilde{B})(A)$ for any $A \subseteq \widetilde{B}$. We know that

$$\|J_m(\mu \llcorner \widetilde{B})\|_{L^p(B)} = \sup\left|\int_B J_m(\mu \llcorner \widetilde{B})(x)\,\varphi(x)\,dx\right|,$$

where the supremum is taken over all $\varphi \in C_c^\infty(B,E)$ with $\|\varphi\|_{L^{p'}(B)} \leqslant 1$. From Fubini's Theorem, and the fact that G_m is radially symmetric, follows

$$\begin{aligned}\int_B J_m(\mu \llcorner \widetilde{B})(x)\,\varphi(x)\,dx &= \int_B \left[\int_{\widetilde{B}} G_m(x-y)d\mu(y)\right]\,\varphi(x)\,dx\\ &= \int_{\widetilde{B}} \left[\int_B G_m(y-x)\,\varphi(x)\,dx\right]\,d\mu(y)\\ &= \int_{\widetilde{B}} J_m\varphi(y)\,d\mu(y)\\ &= \lim_{\varepsilon\to 0}\int_B (\psi_\varepsilon\cdot J_m\varphi)\,(y)\,d\mu(y).\end{aligned}$$

From the fact that $\psi_\varepsilon \cdot J_m\varphi \in C_c^\infty(\widetilde{B}_\varepsilon)$ and $\widetilde{B}_\varepsilon \subset B$ then, from (4.3), we have

$$\begin{aligned}\left|\int_B (\psi_\varepsilon\cdot J_m\varphi)\,(y)\,d\mu(y)\right| &= \left|\int_B f(x)\cdot A(x,D)\,(\psi_\varepsilon\cdot J_m\varphi)\,(x)\,dx\right|\\ &\leqslant \|f\|_{L^p(B)}\,\|A(\cdot,D)\,(\psi_\varepsilon\cdot J_m\varphi)\,\|_{L^{p'}(B)}\\ &\leqslant C(\varepsilon)\|f\|_{L^p(B)}\,\|\varphi\|_{L^{p'}(B)}\end{aligned}$$

since $A(\cdot,D)\,(\psi_\varepsilon\cdot J_m)$ is a pseudo-differential operator with order zero, thus bounded in L^p for all $1<p<\infty$ (Theorem 1.41). Notice that the constant $C(\varepsilon)$ may blow-up as $\varepsilon\to 0$. Hence, we have $\|J_m(\mu\llcorner\widetilde{B})\|_{L^p(B)} \lesssim \|f\|_{L^p(B)} < \infty$ for any $\widetilde{B}\subset B$. □

4.5.2 Fractional estimate with measures

The following L^1 estimate for pseudo-differential operators was obtained by Hounie and Picon:

Theorem 4.25 ([27, Theorem C]). *Let $A(\cdot,D)$ be a differential operator of order m as in* (4.1) *and assume that* $0<\ell<N$ *and* $\ell\leqslant m$. *If $A(\cdot,D)$ is elliptic and canceling in Ω, then for every $x_0\in\Omega$, $1\leqslant q<N/(N-\ell)$, and any properly supported pseudo-differential operator $P_{m-\ell}(x,D)\in \mathrm{Op}\,S^{m-\ell}_{1,\delta}(\Omega)$,* $0\leqslant\delta<1$, *there exists a neighborhood $U\ni x_0$ and $C>0$ such that*

$$\left(\int_{\mathbb{R}^N} |P_{m-\ell}\,u(x)|^q|x|^{(N-\ell)q-N}\,dx\right)^{1/q} \leqslant C\int_{\mathbb{R}^N}|A(x,D)u(x)|\,dx \tag{4.46}$$

holds for every $u\in C_c^\infty(U,E)$.

The proof of inequality (4.46) is a direct consequence of the method used in the proof of Theorem 4.3 and we describe the main steps. Using the ellipticity and Theorem 1.48, we may exhibit properly supported pseudo-differential operators $Q_1(\cdot,D)\in OpS^{-\ell}_{1,\delta}(U)$ and $Q_2(\cdot,D)\in OpS^{-\infty}(U)$ such that

$$P_{m-\ell}u = Q_1(\cdot,D)[A_m(\cdot,D)u] + Q_2(\cdot,D)u, \quad u\in C^\infty(U,E).$$

Thus, in order to obtain the estimate (4.46) it is sufficient to prove the controls (4.31) and (4.30), where $d\nu := |x-x_0|^{(N-\ell)q-N}dx$. Thanks to the calculus presented in Example 4.23, the measure $\nu\in\mathscr{M}_+(\Omega)$ satisfies (4.39) and (4.40), so the controls follow as before. Using this argument, we obtain the following L^1 Sobolev estimate for pseudo-differential operators with measures:

Theorem 4.26. *Let $A(\cdot,D)$ be a differential operator of order m as in* (4.1)*, assume that $0<\ell<N$, $\ell\leqslant m$ and $\nu\in\mathscr{M}_+(\Omega)$ satisfying* (4.39) *and* (4.40)*. If $A(\cdot,D)$ is elliptic and canceling in Ω, then for every $x_0\in\Omega$, $1\leqslant q<N/(N-\ell)$, and any properly supported pseudo-differential operator $P_{m-\ell}(\cdot,D)\in \mathrm{Op}\,S^{m-\ell}_{1,\delta}(\Omega)$, $0\leq\delta<1$, there exists a neighborhood $U\ni x_0$ and $C>0$ such that*

$$\left(\int_{\mathbb{R}^N}|P_{m-\ell}u(x)|^q d\nu\right)^{1/q}\leqslant C\int_{\mathbb{R}^N}|A(x,D)u(x)|\,dx$$

holds for every $u\in C_c^\infty(U,E)$.

4.5.3 Divergence-type equations associated to systems of complex vector fields

Consider n complex vector fields $L_1,\dots,L_n$, $n\geqslant 2$, with smooth coefficients defined on $\Omega\subseteq\mathbb{R}^N$ with $N\geqslant 2$. We will assume that the system of vector fields $\mathscr{L}\doteq\{L_1,\dots,L_n\}$ is linearly independent. Consider the gradient $\nabla_{\mathscr{L}}:C^\infty(\Omega)\to C^\infty(\Omega,\mathbb{C}^n)$ given by $\nabla_{\mathscr{L}}u\doteq(L_1u,..,L_nu), u\in C^\infty(\Omega)$ and its formal complex adjoint operator, defined for $v\in C^\infty(\Omega,\mathbb{C}^n)$ by

$$\mathrm{div}_{\mathscr{L}^*}v\doteq L_1^*v_1+\ldots+L_n^*v_n.$$

Moonens and Picon obtained a characterization for the local continuous solvability result of the equation

$$\mathrm{div}_{\mathscr{L}^*}v=f. \tag{4.47}$$

It is a particular case of Theorem 4.1.

Theorem 4.27 ([35, Theorem 1.2])**.** *Assume that $\mathscr{L}$ is an elliptic system of vector fields. Then every point $x_0\in\Omega$ is contained in an open neighborhood $U\subset\Omega$ such that, for any $f\in\mathscr{D}'(U)$, the equation* (4.47) *is continuously solvable in U if and only if, for every $\varepsilon>0$ and every compact set $K\subset\subset U$, there exists $\theta=\theta(K,\varepsilon)>0$ such that one has, for every $\varphi\in C_K^\infty(U)$:*

$$|f(\varphi)|\leqslant\theta\|\varphi\|_{L^1}+\varepsilon\|\nabla_{\mathscr{L}}\varphi\|_{L^1}. \tag{4.48}$$

The ellipticity means that, for any *real* 1-form ω such that $\langle\omega,L_j\rangle=0$, one has $\omega=0$. Consequently, the number n of vector fields must satisfy $\frac{N}{2}\leqslant n\leqslant N$. The ellipticity of the system is equivalent to the second order operator $\Delta_{\mathscr{L}}\doteq L_1^*L_1+\cdots+L_n^*L_n$ being elliptic in the classical sense. Since the system $\mathscr{L}$ is linearly independent, the following lemma shows that $\nabla_{\mathscr{L}}$ is a canceling operator.

Lemma 4.28 ([26, Lemma 5.1])**.** *Let $x_0\in\Omega$ and let $\ell(x,\xi)$ denote the principal symbol of $\nabla_{\mathscr{L}}$. The following properties are equivalent:*

(i) $\nabla_{\mathscr{L}}$ is canceling at x_0;

(ii) the range of the map $\xi\mapsto\ell(x_0,\xi)\in\mathbb{C}^n$ has dimension $\geqslant 2$;

(iii) there exist two vector fields $L_{j_1}, L_{j_2} \in \mathscr{L}$ *that are linearly independent at* x_0.

Reducing the neighborhood U (for instance, take $K = \overline{B(x_0,r)} \subset U$ and redefine $U = B(x_0,r)$), and from the fact that $\|\varphi\|_{L^1} \leqslant C(U)\|\nabla_{\mathscr{L}}\varphi\|_{L^1}$ locally (see (4.25)), then, if $\nu \in \mathscr{M}_+(U,\mathbb{C})$ satisfies (4.48), we have

$$\left|\int_U \varphi(x)\, d\nu\right| = |\nu(\varphi)| \leqslant (\theta C(U) + \varepsilon)\|\nabla_{\mathscr{L}}\varphi\|_{L^1}, \quad \varphi \in C_c^\infty(U),$$

thus, as argued for (4.28), there exists $u \in L^\infty(U)$ solution of $\mathrm{div}_{\mathscr{L}^*} u = \nu$. Clearly, local continuous solutions are bounded, however the converse is not true. A similar argument shows that (4.4) implies (4.28) locally for elliptic and canceling operators.

APPENDIX A

Proof of estimate (1.4)

As mentioned earlier, showing that (1.3) implies (1.4) is far from trivial. This appendix is devoted to present an outline for this proof. Some results will have their demonstrations omitted as they heavily rely on techniques from Geometric Measure Theory, which would require a lot of background and is not the scope of this text, but all of them will be given a reference prior to their statement.

Lemma A.1 ([33, Theorem 1.2.1/2])**.** *Let $G \subset \mathbb{R}^N$ be a bounded open subset with smooth boundary. There exists a covering of G by a sequence of balls with radii r_j, $j = 1,2,\ldots$, such that*

$$\sum_j r_j^{N-1} \leqslant c\mathscr{H}^{N-1}(\partial G),$$

where $c = c(N)$ and $\mathscr{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure.

The next theorem is known as the *co-area formula for functions of bounded variation*.

Theorem A.2 ([47, Theorem 5.4.4])**.** *Let $\Omega \subset \mathbb{R}^N$ be open and $f \in \mathscr{BV}(\Omega)$. Then*

$$\|Df\|(\Omega) = \int_{\mathbb{R}} \|\partial(\Omega \cap \mathscr{L}_t)\|(\Omega)\,dt,$$

where $\mathscr{L}_t = \{x \in \Omega : f(x) > t\}$.

Here, $\mathscr{BV}(\Omega)$ denotes the *set of functions of bounded variation* in Ω, $\|Df\|$ is the *variation measure* of f and $\|\partial\mathscr{L}_t\|(\Omega)$ is the *perimeter measure* of $\mathscr{L}_t$. Their precise definitions and properties will be omitted, but can be found in [13] or [47]. What is important in our case is that, if $f \in C_c^\infty(\Omega)$, then $\|Df\|(\Omega) = \|\nabla f\|_{L^1(\Omega)}$ and $\|\partial(\Omega \cap \mathscr{L}_t)\|(\Omega) = \mathscr{H}^{N-1}(\partial\mathscr{L}_t)$.

Lemma A.3 ([33, Corollary 1.2.2])**.** *Let $f \in C_c^\infty(\Omega)$. Then, for almost all $t \in \mathbb{R}$, the sets $\partial\mathscr{L}_t$ are C^∞-compact manifolds.*

Theorem A.4. *Let ν be a positive Borel measure in $\mathbb{R}^N$, $q \geqslant 1$, $\Omega \subset \mathbb{R}^N$ an open subset and $\mathscr{G}$ the collection of subsets G of Ω such that $\bar{G} \subset \Omega$ are compact and each G is bounded by a C^∞ manifold. If*

$$\sup_{G\in\mathscr{G}} \frac{\nu(G)^{1/q}}{\mathscr{H}^{N-1}(\partial G)} < \infty,$$

then, for all $u \in C_c^\infty(\Omega)$,

$$\|u\|_{L^q(\Omega,\nu)} \leqslant C\|\nabla u\|_{L^1(\Omega)},$$

where

$$C \leqslant \sup_{G\in\mathscr{G}} \frac{\nu(G)^{1/q}}{\mathscr{H}^{N-1}(\partial G)}.$$

Proof. Let

$$\mathscr{L}_t = \{x \in \Omega : |u(x)| > t\}.$$

Then

$$\begin{aligned}
\|u\|_{L^q(\Omega,\nu)} &= \left(\int_\Omega |u(x)|^q \, d\nu(x)\right)^{\frac{1}{q}} = \left(\int_\Omega \left(\int_0^\infty \chi_{(0,|u(x)|^q)}(\tau)\, d\tau\right) d\nu(x)\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \left(\int_\Omega \chi_{(0,|u(x)|^q)}(\tau)\, d\nu(x)\right) d\tau\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \nu(\{x\in\Omega : |u(x)|^q > \tau\})\, d\tau\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \nu(\{x\in\Omega : |u(x)| > t\})\, d(t^q)\right)^{\frac{1}{q}} = \left(\int_0^\infty \nu(\mathscr{L}_t)\, d(t^q)\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\nu(\mathscr{L}_t) \leqslant \nu(\mathscr{L}_s)$ when $s \leqslant t$, we have

$$\begin{aligned}
\int_0^\infty \nu(\mathscr{L}_t)\, d(t^q) &= \int_0^\infty [\nu(\mathscr{L}_t)^{1/q}]^q \, d(t^q) \\
&= \int_0^\infty q[t\nu(\mathscr{L}_t)^{1/q}]^{q-1} \nu(\mathscr{L}_t)^{1/q}\, dt \\
&= \int_0^\infty q\left[\int_0^t \nu(\mathscr{L}_t)^{1/q}\, ds\right]^{q-1} \nu(\mathscr{L}_t)^{1/q}\, dt \\
&\leqslant \int_0^\infty q\left[\int_0^t \nu(\mathscr{L}_s)^{1/q}\, ds\right]^{q-1} \nu(\mathscr{L}_t)^{1/q}\, dt \\
&= \int_0^\infty \frac{d}{dt}\left(\left[\int_0^t \nu(\mathscr{L}_s)^{1/q}\, ds\right]^q\right) dt \\
&= \left(\int_0^\infty \nu(\mathscr{L}_t)^{1/q}\, dt\right)^q.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\|u\|_{L^q(\Omega,\nu)} &\leqslant \int_0^\infty \nu(\mathscr{L}_t)^{1/q}\,dt \\
&= \int_0^\infty \frac{\nu(\mathscr{L}_t)^{1/q}}{\mathscr{H}^{N-1}(\partial\mathscr{L}_t)} \mathscr{H}^{N-1}(\partial\mathscr{L}_t)\,dt \\
&\leqslant \sup_{G\in\mathscr{G}} \frac{\nu(G)^{1/q}}{\mathscr{H}^{N-1}(\partial G)} \int_0^\infty \mathscr{H}^{N-1}(\partial\mathscr{L}_t)\,dt,
\end{aligned}
$$

as, by Lemma A.3, almost all the sets $\mathscr{L}_t$ belong to the collection $\mathscr{G}$. Therefore, by the coarea formula (Theorem A.2), we have

$$
\begin{aligned}
\|u\|_{L^q(\Omega,\nu)} &\leqslant C\int_0^\infty \mathscr{H}^{N-1}(\partial\mathscr{L}_t)\,dt \\
&= C\|\nabla u\|_{L^1(\Omega)}.
\end{aligned}
$$

□

In the case when $\Omega = \mathbb{R}^N$, the hypothesis on the previous theorem can be weakened, only taking the supremum over balls.

Theorem A.5. *Let ν be a positive Borel measure in $\mathbb{R}^N$ and $q \geqslant 1$. If*

$$
\sup_{x\in\mathbb{R}^N; r>0} \frac{\nu(B(x,r))}{r^{(N-1)q}} < \infty,
$$

then for all $u \in C_c^\infty(\mathbb{R}^N)$

$$
\|u\|_{L^q(\nu)} \leqslant C\|\nabla u\|_{L^1},
$$

where

$$
C^q \leqslant c^q \sup_{x\in\mathbb{R}^N;\, r>0} \frac{\nu(B(x,r))}{r^{(N-1)q}},
$$

with $c = c(N) > 0$.

Proof. Given $G \in \mathscr{G}$ as in the previous theorem, let $\{B(x_j, r_j)\}$ be the covering of G given by Lemma A.1. Since

$$
\left(\sum_j a_j\right)^{1/q} \leqslant \sum_j a_j^{1/q}
$$

for $a_j \geqslant 0$, we obtain

$$
\begin{aligned}
\nu(G) &\leqslant \sum_j \nu(B(x_j,r_j)) \leqslant \left(\sum_j \nu(B(x_j,r_j))^{1/q}\right)^q \\
&= \left(\sum_j \left[r_j^{(1-N)q}\nu(B(x_j,r_j))\right]^{1/q} r_j^{N-1}\right)^q \\
&\leqslant \sup_{x\in\mathbb{R}^N;\, r>0} \frac{\nu(B(x,r))}{r^{(N-1)q}} \left(\sum_j r_j^{N-1}\right)^q.
\end{aligned}
$$

Hence, from Lemma A.1,

$$\nu(G) \leqslant c^q \sup_{x\in\mathbb{R}^N;\, r>0} \frac{\nu(B(x,r))}{r^{(N-1)q}} \left(\mathscr{H}^{N-1}(\partial G)\right)^q .$$

Thus,

$$\sup_{G\in\mathscr{G}} \frac{\nu(G)^{1/q}}{\mathscr{H}^{N-1}(\partial G)} \leqslant c \left(\sup_{x\in\mathbb{R}^N;\, r>0} \frac{\nu(B(x,r))}{r^{(N-1)q}} \right)^{1/q} < \infty.$$

The proof is completed applying Theorem A.4. □

Finally, (1.3) ⇒ (1.4) follows from Theorem A.5 for $q = 1$.

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