UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Pseudo-parallel immersions of Lorentzian manifolds in pseudo-Riemannian space forms

Mynor Ademar Melara Estrada

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Mynor Ademar Melara Estrada

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To my family and friends.
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Resumo

Nesta tese de Doutorado, estudamos subvariedades pseudo-paralelas em formas espaciais pseudo-Riemannianas. Damos uma caracterização das superfícies Lorentzianas pseudo-paralelas com fibrado normal não flat em formas espaciais pseudo-Riemannianas como superfícies λ-isotrópicas, estendendo um resultado análogo de Asperti-Lobos-Mercuri no caso Riemanniano. Consequentemente, para este tipo de superfícies Lorentzianas damos uma caracterização usando o conceito de hipérbola de curvatura e obtemos um resultado de não existência quando o espaço ambiente é uma forma espacial Lorentziana. Em particular, quando o espaço ambiente é uma forma espacial pseudo-Riemanniana de dimensão 4, obtemos que qualquer superfície Lorentziana pseudo-paralela com fibrado normal não flat é superextremal, ou seja, uma superfície λ-isotrópica com campo vetorial de curvatura média identicamente nulo, e o espaço ambiente deve ter métrica de índice 2. No caso em que a função de pseudo-paralelismo é constante, descrevemos explicitamente essas superfícies com codimensão dois, obtendo que são superfícies paralelas e existem em formas espaciais não flat, e para o caso em que a função de pseudo-paralelismo não é constante, damos exemplos explícitos dessas superfícies no espaço pseudo-euclidiano de dimensão 4 com métrica de índice 2. Um exemplo de uma superfície Lorentziana pseudo-paralela extremal e flat com fibrado normal não flat que não é semi-paralela é dada em codimensão três. Continuamos o estudo das hipersuperfícies Lorentzianas pseudo-paralelas em formas espaciais Lorentzianas iniciado por Lobos, completando a caracterização do operador de Weingarten inclusive quando este não é diagonalizável. Então, consideramos o caso em que a função de pseudo-paralelismo é constante e distinta da curvatura do espaço ambiente e damos a classificação local dessas hipersuperfícies sob a hipótese de serem boas no sentido de Ryan. Também damos uma classificação das hipersuperfícies Lorentzianas semiparalelas completas e conexas do espaço de Minkowski e uma classificação local das hipersuperfícies Lorentzianas pseudo-paralelas com função de pseudo-paralelismo constante e curvatura média constante nas formas espaciais Lorentzianas.

Palavras-chave: Espaço pseudo-Riemanniano, superfície pseudo-paralela, hipersuperfície pseudo-paralela, subvariedade Lorentziana, superfície λ-isotrópica, hipérbola de curvatura normal, imersão extremal, superfície rotacional geral, hipersuperfície isoparamétrica.

Abstract

In this Ph.D. thesis, we study pseudo-parallel submanifolds in pseudo-Riemannian space forms. We give a characterization of pseudo-parallel Lorentzian surfaces with non-flat normal bundle in pseudo-Riemannian space forms as λ-isotropic surfaces, extending an analogous result by Asperti-Lobos-Mercuri in the Riemannian case. Consequently, for this kind of Lorentzian surfaces we give a characterization using the concept of hyperbola of curvature and get a non-existence result when the ambient space is a Lorentzian space form. In particular, when the ambient space is a 4-dimensional pseudo-Riemannian space form, we obtain that any pseudo-parallel Lorentzian surface with non-flat normal bundle is super-extremal, i.e., a λ-isotropic surface with everywhere vanishing mean curvature vector field, and the ambient space must have metric of index 2. In the case where the pseudo-parallelism function is constant, we explicitly describe these surfaces with codimension two, obtaining that they are parallel surfaces and exist in non-flat space forms, and for the case where the pseudo-parallelism function is non-constant we give explicit examples of these surfaces in the 4-dimensional pseudo-Euclidean space with metric of index 2. An example of an extremal and flat pseudo-parallel Lorentzian surface with non-flat normal bundle which is not semi-parallel is given in codimension three. We continue the study of pseudo-parallel Lorentzian hypersurfaces in Lorentzian space forms started by Lobos, by completing the characterization of the Weingarten operator even when it is non-diagonalizable. Then, we consider the case where the pseudo-parallelism function is constant and different from the curvature of the ambient space and give the local classification of these hypersurfaces under the hypothesis of being good in the sense of Ryan. We also give a classification of the connected complete semi-parallel Lorentzian hypersurfaces of the Minkowski space and a local classification of the pseudo-parallel Lorentzian hypersurfaces with constant pseudo-parallelism function and constant mean curvature in Lorentzian space forms.

Keywords: Pseudo-Riemannian space, pseudo-parallel surface, pseudo-parallel hypersurface, Lorentzian submanifold, λ-isotropic surface, hyperbola of normal curvature, extremal immersion, general rotational surface, isoparametric hypersurface.

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$\mathbb{Q}^m(c)$: m-dimensional Riemannian space form of constant curvature c;

$\mathbb{E}^m$: m-dimensional Euclidean space;

$\mathbb{E}^m_s$: m-dimensional pseudo-Euclidean space with metric of index s;

$\mathbb{Q}^m_s(c)$: m-dimensional pseudo-Riemannian space form of constant curvature c and metric of index s;

M^n_t : n-dimensional pseudo-Riemannian manifold with metric of index t;

T_xM : Tangent space to manifold M at point x;

$N_fM(x)$: Normal space to immersion f from manifold M at point x;

TM : Tangent bundle of manifold M;

N_fM : Normal bundle to immersion f of manifold M;

$N^1(x)$: First normal space of a immersion at point x;

α : Second fundamental form;

α_{ij} : Second fundamental form $\alpha(e_i, e_j)$;

$\mathcal{H}$: Mean curvature vector field;

H: Mean curvature function;

$I_fM^2_1(x)$: Invariant subspace of the normal space to surface M^2_1 expanded by α_{12} and $\alpha_{11}+\alpha_{22}$;

$\mathscr{H}_x$: Hyperbola of normal curvature at point x;

λ : Isotropy function;

ψ : pseudo-parallelism function;

I_n : Identity operator in a n-dimensional real vector space;

∇ : Levi-Civita connection;

$\nabla^\perp$: Normal connection;

$\overline{\nabla}$: Van der Waerden-Bortolotti connection.

Introduction

Among the Riemannian manifolds, those that present some type of symmetry or homogeneity tend to be very interesting in themselves and are also important for applications due to their simplicity (see [54]). For example, n-dimensional Euclidean spaces, spheres and hyperbolic spaces, which are connected complete simply-connected manifolds with constant sectional curvature, are the most fundamental Riemannian manifolds and much of the research in geometry is related to them.

It usually happens in mathematics that the variation of an object is represented by some type of derivation and then the simplest objects are those for which that derivation vanishes. In this way, a tensor field in a Riemannian manifold is called parallel if its covariant derivative vanishes. Thus, a natural generalization for the perfect symmetry of manifolds with constant sectional curvature is given by the class of Riemannian manifolds whose curvature tensor R is parallel, i.e., $\nabla R = 0$. A manifold in this class is called a locally symetric space. When all the Riemannian manifold is reflectionally symmetric around any point, then it is called a symmetric space. Locally symmetric spaces were extensively studied by É. Cartan around 1925-1930 and he also classified symmetric spaces.

Works in (locally) symmetric spaces led to new research in two directions: intrinsically, they were generalized to semi-symmetric spaces, introduced by É. Cartan in [16] and classified by Z.I. Szabó (see [69] and [70]). E. Cartan and H. Takagi presented examples of semi-symmetric manifolds that are not locally-symmetric (see [73]). Secondly, in Submanifold Theory, (locally) parallel immersions were introduced by D. Ferus (see [32] and [33]) as an extrinsic analogue of locally symmetric space, that is, as immersions with parallel second fundamental form α (i.e., with $\overline{\nabla}\alpha = 0$, where $\overline{\nabla}$ is the Van der Waerden-Bortolotti connection of the immersion). The same author obtained a local classification of such immersions in Euclidean spaces and spheres of constant sectional curvature. In the hyperbolic spaces two classifications were obtained independently by Backes-Reckziegel (see [10]) and M. Takeuchi (see [74]). Parallel immersions are related to symmetric spaces in the sense that any parallel submanifold of a Riemannian space form is intrinsically a locally symmetric space.

Next, semi-parallel immersions were defined by J. Deprez in [23] satisfying an analogous condition to that for semi-symmetry. Many results on semi-parallel immersions can be found, for example, in [7, 23, 24, 30, 51, 52, 54]. Even when a full classification is not available yet, we can find a complete classification of semi-parallel hypersurfaces in [24] for the Euclidean space and in [30] for Riemannian space forms. Also, Riemannian cylinders $\mathbb{H}^n(c)\times\mathbb{R}$ and $\mathbb{S}^n(c)\times\mathbb{R}$ are examples of symmetric spaces and a classification of parallel and semi-parallel immersions in these cylinders is given in [15] and [76].

Again in the case of intrinsic geometry, investigation of several properties of semi-symmetric spaces gave rise to a more general class of manifolds, that is, the class of pseudo-symmetric spaces. For example, these spaces appear naturally from study of totally umbilical submanifolds of a semi-symmetric space with parallel mean curvature vector (see [2]). The class of pseudo-symmetric manifolds is very large, and many examples of pseudo-symmetric manifolds which are not semi-symmetric have been constructed (see e.g. [25], [26] and references therein). Many particular results are known, see, for example, [22, 25, 26, 27, 28, 29], but a full classification is not available yet. Finally, pseudo-parallel immersions were introduced by Asperti-Lobos-Mercuri in [8] as a generalization of semi-parallel immersions and as an extrinsic analogue of pseudo-symmetric spaces.

On the other hand, with the publication of his Especial Relativity Theory in 1905, Einstein gave a solution to the difficulties that the Classical Newtonian Physics have to do around the property of invariance of the speed of light, introducing an innovative way to change space and time coordinates, which led to conceiving models of the space-time with three spatial dimensions and one time dimension, having a metric tensor which is negative definite in the time direction (see [62]). Thus, from Einstein's work, the positiveness of the inner product induced from Riemannian metrics was weakened and the research involving pseudo-Riemannian manifolds, i.e., smooth manifolds furnished with a non-degenerate metric tensor, had a growing interest that has reached our time. Particularly, when the largest integer that is the dimension of a subspace of the tangent space at any point of the variety in which the metric is negative defined, called the *index* of the metric, is 1, the manifold is called a *Lorentzian* manifold. All the classes of symmetric spaces and parallel immersions, as well as their successive generalizations including pseudo-symmetric spaces and pseudo-parallel immersions, can be defined in this more general context of the pseudo-Riemannian geometry.

An isometric immersion $f: M_t^n \to \widetilde{M}_s^m$ between pseudo-Riemannian manifolds with dimensions n and m and metric tensors of index t and s, respectively, is said to be *pseudo-parallel* if its second fundamental form α satisfies the following condition:

$$\overline{R}(X,Y)\cdot\alpha = \psi(X\wedge Y)\cdot\alpha, \tag{1}$$

for some smooth real-valued function ψ on M_t^n and for all tangent vector fields X,Y of M_t^n, where $\overline{R}$ is the curvature tensor corresponding to the Van der Waerden-Bortolotti connection $\overline{\nabla}$ of the immersion,

$X \wedge Y$ denotes the endomorphism defined by

$$(X \wedge Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y$$

and $\overline{R}(X,Y)$, $X \wedge Y$ are considered in (1) as fields of linear operators acting as derivations on α. Sometimes we will say that the immersion is ψ-pseudo-parallel to specify the parallelism function.

Asperti-Lobos-Mercuri in [9] proved that pseudo-parallel surfaces in Riemannian space forms are surfaces with flat normal bundle (i.e., with vanishing normal curvature tensor) or λ-isotropic surfaces in the sense of O'Neill in [61] (i.e., for each point x of the surface, $||\alpha(X,X)||$ does not depends on the choice of the unit tangent vector X of the surface at x, that is, the ellipse of normal curvature at x is a circle centered at the mean curvature vector $\mathcal{H}(x)$ and orthogonal to $\mathcal{H}(x)$). In particular, they proved that pseudo-parallel surfaces of Riemannian space forms with non-flat normal bundle in codimension two are superminimal in the sense of Bryant in [12] (i.e., minimal and λ-isotropic). As a consequence, they gave a characterization of the Veronese surface in codimension two. Also, they classified pseudo-parallel surfaces in codimension three with constant ψ. Next, Lobos-Tassi-Yucra Hancco in [50], extended the study of pseudo-parallel surfaces to the case where the ambient space is a cylinder $\mathbb{H}^n(c) \times \mathbb{R}$ or $\mathbb{S}^n(c) \times \mathbb{R}$.

Pseudo-parallel hypersurfaces of a Riemannian space form were characterized by Asperti-Lobos-Mercuri in [9] as quasi-umbilical hypersurfaces or cyclides of Dupin. Also, pseudo-parallel real hypersurfaces in complex space forms were classified by Lobos-Ortega in [48] and the study of pseudo-parallel hypersurfaces in cylinders $\mathbb{S}^m(c) \times \mathbb{R}$ and $\mathbb{H}^m(c) \times \mathbb{R}$ was started by F. Lin and B. Yang in [43], including a classification and the geometric description of this kind of hypersurfaces under the condition of having at most two distinct principal curvatures. Then, the complete description was given by Lobos-Tassi in [49] and as an application, they obtained a classification of pseudo-parallel hypersurfaces in $\mathbb{S}^m(c) \times \mathbb{R}$ and $\mathbb{H}^m(c) \times \mathbb{R}$ with constant mean curvature function.

Chacón-Lobos in [17] studied pseudo-parallel Lagrangian submanifolds in a complex space form. For the case of dimension 2, they showed that the minimal Lagrangian surfaces are pseudo-parallel. In particular, they proved that the semi-parallel Lagrangian surfaces are totally geodesic or flat and gave examples of pseudo-parallel Lagrangian surfaces which are not semi-parallel. Also, in this work they conjectured that every Lagrangian pseudo-parallel submanifold of dimension at least 3 of a complex space form is semi-parallel, which was later proven by Dillen-Van der Veken-Vrancken in [31].

In [44], Lobos started the study of pseudo-parallel hypersurfaces in pseudo-Riemannian space forms where the situation is richer and new possibilities arise from the fact that the Weingarten operator is not diagonalizable in general for pseudo-Riemannian hypersurfaces. Indeed, Lobos showed a class of pseudo-parallel Lorentzian hypersurfaces into Lorentzian space forms which are not semi-parallel and have non-diagonalizable Weingarten operator. These examples introduced by Alías-Ferrández-Lucas in [5], are closely related with certain classes of generalized umbilical hypersurfaces in n-dimensional Lorentz-Minkowski space, introduced by M. Magid in [55].

Also in [44], for a pseudo-parallel hypersurface $f : M_t^n \to \mathbb{Q}_s^{n+1}(c)$, with $s \in \{t, t+1\}$, Lobos showed that if at each point $x \in M_t^n$ the Weingarten operator A in the η-direction, with η a unit normal vector field to f, satisfies an identity of the form

$$A^2 = \lambda A + \mu I_n, \tag{2}$$

where $\lambda, \mu \in \mathbb{R}$, $\varepsilon = \langle \eta, \eta \rangle = \pm 1$ and I_n is the identity operator in T_xM, then the hypersurface is pseudo-parallel. It is interesting that hypersurfaces whose Weingarten operator satisfies particular cases of (2) have been also studied: when $\lambda = \mu = 1$, the so-called *golden-shaped* hypersurfaces were classified by Yang-Fu in [81] for Lorentzian space forms and when λ and μ are positive integers the so-called *metallic shaped* hypersurfaces were classified by Özgür-Özgür in [64] for Lorentzian space forms (see also [19, 63], for the Riemannian case). It is worth observing that for metallic shaped hypersurfaces in Lorentzian space forms, which generalize golden-shaped hypersurfaces, the Weingarten operator is always diagonalizable.

Other works on surfaces or hypersurfaces that are worth mentioning in the pseudo-Riemannian context are the following: E. Safiulina in [72] studied and gave a classification of parallel and semi-parallel spacelike surfaces in pseudo-Euclidean spaces. Ü. Lumiste in [53] obtained a classification of semi-parallel Lorentzian (timelike) surfaces in Lorentzian space forms. The λ-isotropy condition was studied for the pseudo-Riemannian case by Y. Kim in [40] and by Cabrerizo-Fernández-Gómez in [13] and [14]. Also, K. Hasegawa in [38] showed a characterization of the Lorentzian surfaces of the Veronese type in four-dimensional manifolds of neutral signature, as extremal (i.e. with mean curvature vector field zero) and isotropic with negative spin immersions of constant Gaussian curvature. The isotropy with negative spin condition was also studied by G.R. Jensen and M. Rigoli in [39]. On the other hand, Al-shehri and Guediri in [6], studied semi-symmetric Lorentzian hypersurfaces in Lorentzian space with constant curvature and obtained some classification results, especially when the ambient space has non-zero curvature, using analogous techniques of those used by Ryan in [66]. The semi-symmetric case when the ambient space is a Minkowski space was studied by Van de Woestijne and Verstraelen in [75], under the condition that the rank of the Weingarten operator is greater than two.

The aim of this work is study pseudo-parallel immersions between pseudo-Riemannian manifolds, being particularly interested in those pseudo-parallel immersions of Lorentzian manifolds into the m-dimensional pseudo-Riemannian space form $\mathbb{S}^m_s(c)$, $\mathbb{H}^m_s(c)$ or $\mathbb{E}^m_s$, with constant curvature c and index s, denoted for short by $\mathbb{Q}^m_s(c)$. Specifically, we study pseudo-parallel Lorentzian surfaces (see [46]) and pseudo-parallel Lorenzian hypersurfaces in the ambient spaces $\mathbb{Q}^m_s(c)$, continuing the work in [44].

It is natural to ask the following questions:

(i) Any Lorentzian surface with non-flat normal bundle in a pseudo-Riemannian space form $\mathbb{Q}^m_s(c)$ is pseudo-parallel if and only if it is λ-isotropic?

(ii) Which are all the pseudo-parallel Lorentzian surfaces with non-flat normal bundle in a 4 dimensional or 5-dimensional pseudo-Riemannian space form with constant pseudo-parallelism function ψ?

(iii) Are there pseudo-parallel Lorentzian surfaces with non-flat normal bundle in a 4 dimensional pseudo-Riemannian space form with non constant pseudo-parallelism function ψ?

(iv) Which are all the pseudo-parallel hypersurfaces of a pseudo-Riemannian space form?

In this work, we answer affirmatively to the first question and give partial answers to questions (ii) and (iii): the case of codimension two in question (ii) and the case $c = 0$ in question (iii). Question (iv) is still an open problem in general, but here we almost completely solve the particular situation corresponding to the following question:

(v) Which are all the pseudo-parallel Lorentzian hypersurfaces of a Lorentzian space form $\mathbb{Q}^{n+1}_1(c)$ with constant pseudo-parallelism function ψ?

Let M^2_1 be a Lorentzian surface and let $\mathbb{Q}^m_s(c)$ an m-dimensional pseudo-Riemannian space form of constant sectional curvature c and index s, with $1 \le s \le m-1$. We begin by observing that any isometric immersion $f : M^2_1 \to \mathbb{Q}^m_s(c)$ with flat normal bundle is pseudo-parallel. Then, we obtain analogous results of those given for Riemannian pseudo-parallel surfaces with non-flat normal bundle in [9] and [50]. We recall that f is called λ-*isotropic*, in this pseudo-Riemannian context, if for any point x of the surface we have that $\langle \alpha(X,X), \alpha(X,X) \rangle = \lambda(x)$, for all unit tangent vector X of M^2_1 at x and for some smooth real-valued function λ on M^2_1 (see [40]). The following is the main result that we obtain for the case of surfaces:

Theorem 0.1. *An isometric immersion $f : M^2_1 \to \mathbb{Q}^m_s(c)$ which has non-flat normal bundle on any open subset of M^2_1 is ψ-pseudo-parallel if and only if it is λ-isotropic. Moreover, for such an immersion we have that f is pseudo-umbilical and*

(a) if $\psi \neq K$, then $2 \le s \le m-2$ and

$$\lambda = -3\psi - c + 4K, \tag{3}$$

$$\langle \mathcal{H}, \mathcal{H} \rangle = -2\psi - c + 3K; \tag{4}$$

(b) if $\psi = K$, then $3 \leq s \leq m-3$ and $\lambda = \langle \mathcal{H}, \mathcal{H} \rangle = K - c$,

where K is the Gaussian curvature of M_1^2, $\mathcal{H}$ is the mean curvature vector field of f and λ is a smooth real-valued function on M_1^2.

We remark that there are no pseudo-parallel Lorentzian surfaces with non-flat normal bundle in Lorentzian space forms. In the Riemannian case, condition $\psi = K$ implies that the pseudo-parallel surface has flat normal bundle (see [9]). In Example 2.5, we show a pseudo-parallel Lorentzian surface with non-flat normal bundle and $\psi = K = 0$ in a 6-dimensional pseudo-Euclidean space.

As a consequence of Theorem 0.1, we obtain the following geometric characterization for pseudo-parallel Lorentzian surfaces with non-flat normal bundle in terms of the hyperbola of normal curvature.

Corollary 0.2. *Let $f : M_1^2 \to \mathbb{Q}_s^m(c)$ be an isometric immersion with Gaussian curvature K. f is ψ-pseudo-parallel with non-flat normal bundle on any open subset of M_1^2 if and only if, for each $x \in M_1^2$, the set*

$$\mathscr{H}_x = \{\langle X, X \rangle \alpha(X, X) : X \in T_xM \text{ with } \langle X, X \rangle = \pm 1\}$$

is a non-degenerate hyperbola with center at the mean curvature vector $\mathcal{H}(x)$, which lies in a 2-dimensional affine subspace $\mathcal{V}$ of $N_fM(x)$ orthogonal to $\mathcal{H}(x)$, such that

(a) either $\mathcal{V} - \mathcal{H}(x)$ is Lorentzian and $\mathscr{H}_x$ is an equilateral hyperbola satisfying that $\langle W - \mathcal{H}(x), W - \mathcal{H}(x) \rangle = r(x) \neq 0$ does not depend on $W \in \mathscr{H}_x$. In this case, $2 \leq s \leq m-2$, $r(x) = K - \psi$ and if $m = 4$, then $s = 2$ and f is extremal;

(b) or all non-zero vectors of $\mathcal{V} - \mathcal{H}(x)$ are lightlike. In this case, $3 \leq s \leq m-3$, $\psi = K$ and if $m = 6$, then $s = 3$ and $\langle \mathcal{H}(x), \mathcal{H}(x) \rangle = 0$.

In particular, for $m = 4$, we obtain that any pseudo-parallel Lorentzian surface with non-flat normal bundle in $\mathbb{Q}_s^4(c)$ is super-extremal, i.e., extremal and λ-isotropic, and $s = 2$. In this case and under the hypothesis that the pseudo-parallelism function is constant, using a classification result by Hasegawa in [38] for extremal and isotropic with negative spin immersion, we show the next result:

Corollary 0.3. *Let $f : M_1^2 \to \mathbb{Q}_s^4(c)$ be an isometric immersion with $R^\perp \neq 0$. f is ψ-pseudo-parallel if and only if $s = 2$ and f is an extremal and isotropic with negative spin immersion. Moreover, if ψ is constant, then $K = \dfrac{c}{3} \neq 0$ and locally $f(M_1^2)$ is congruent to an open set of the Veronese type surface given in Example 3.1.*

Considering Corollary 0.3, it is natural to ask if there are ψ-pseudo-parallel Lorentzian surfaces with non-flat normal bundle and non-constant ψ in a pseudo-Riemannian space form $\mathbb{Q}_2^4(c)$, especially for $c = 0$. We answer affirmatively to this question, showing the first explicit examples of this kind of surfaces in $\mathbb{E}^4_2$. For this, we study the class of extremal general rotational surfaces in $\mathbb{E}^4_2$, such that the meridian m lies in a two-dimensional plane, i.e., $m : x(u) = (f(u), 0, g(u), 0)$, $u \in J \subset \mathbb{R}$. General rotational surfaces as a source of examples of surfaces in the 4-dimensional Euclidean space were introduced by Moore and analogous Lorentzian surfaces in $\mathbb{E}^4_2$ were defined by Aleksieva-Milousheva-Turgay in [4], where some classification results were given. Using classifications in [4] and looking at the λ-isotropy condition, we obtain the following results:

Theorem 0.4. *Let $\mathcal{M}_1$ be a general rotational Lorentzian surface of elliptic type in $\mathbb{E}^4_2$, defined by (1.23). Then $\mathcal{M}_1$ is pseudo-parallel with $R^\perp \neq 0$ if and only if the meridian m is determined by $(f+g)^2 = a(f-g)^2 + b$, with $fg' \neq gf'$ everywhere, $a \neq 0$ constant, b constant, $\theta = \beta$. In this case, $\psi = \frac{3}{2}K = \dfrac{3(fg'-gf')^2}{(f'^2-g'^2)(g^2-f^2)^2}$.*

Theorem 0.5. *Let $\mathcal{M}_2$ be a general rotational surface of hyperbolic type in $\mathbb{E}^4_2$, defined by (1.28). Then $\mathcal{M}_2$ is pseudo-parallel with $R^\perp \neq 0$ if and only if the meridian m is determined by*

(i) $f = cg^k$, *$c \neq 0$ constant,* $k = \pm\frac{\theta}{\beta} \neq \pm 1$, *or*

(ii) $\arctan\left(\frac{f'}{g'}\right) = -\arctan\left(\frac{f}{g}\right) + b$, *with $fg' \neq gf'$ everywhere, b constant, $\theta = \beta$.*

For any of these cases, we have $\psi = \frac{3}{2}K = \dfrac{-3\theta^2\beta^2(fg'-gf')^2}{(f'^2+g'^2)(\beta^2 g^2+\theta^2 f^2)^2}$.

Note that all the surfaces in Theorem 0.4 and Theorem 0.5 are not semi-parallel and the Veronese type surface mentioned in Corollary 0.3 is a parallel immersion. To get an example of a ψ-pseudo-parallel surface with non flat normal bundle and constant ψ which is not semi-parallel, we must look at codimension three. We give an example of a such surface in $\mathbb{S}^5_2(c)$. The case with constant ψ in codimension three for ψ-pseudo-parallel Lorentzian surfaces with non flat normal bundle is more complicated than in the Riemannian case and we do not give a complete classification; indeed, to our knowledge there is no classification of λ-isotropic Lorentzian surfaces with constant λ in $\mathbb{Q}^5_s(c)$, with $s = 2, 3$. We want to remark that Simons' formula was used by Sakamoto in [68] for the classification of λ-isotropic surfaces with constant λ in $\mathbb{Q}^5(c)$, but the question remains whether a Simons type formula can be obtained for Lorentzian surfaces. It is worth to mention that as part of our study of Simons' formula, a generalization of a result by Asperti-Lobos-Mercuri in Theorem 1.1 of [8] was obtained in a joint work with M.R. Santos for spacelike pseudo-parallel immersions of any codimension in pseudo-Riemannian warped product spaces. Such a result can be found in Theorem 2 of [47] and and it gives conditions for the mean curvature vector and the pseudo-parallelism function ψ to guarantee that a point of a pseudo-parallel immersion is a geodesic point.

For a pseudo-parallel Lorentzian hypersurface $f : M^n_1 \to \mathbb{Q}^{n+1}_s(c)$, with $s \in \{0, 1\}$, we complete in Proposition 4.6 a result partially given by Lobos in [44], which provides a specific description of the pseudo-parallelism condition in terms of the Weingarten operator at each point of the hypersurface, covering both the diagonalizable and non-diagonalizable cases. Then, we shall focus on pseudo-parallel Lorentzian hypersurfaces with constant ψ in Lorentzian space forms $\mathbb{Q}_1^{n+1}(c)$.

For the case $\psi = c = 0$, we give the following classification of connected complete semi-parallel Lorentzian hypersurfaces in the Minkowski space, using essentially the technique in [75, 59], for the case where the rank of the Weingarten operator is greater than 1, and the classification of the complete Lorentzian hypersurfaces with constant curvature zero in $\mathbb{E}_1^{n+1}$, for the case where the rank of the Weingarten operator is at most 1.

Theorem 0.6. *Let $f : M_1^n \to \mathbb{E}_1^{n+1}$ be a connected and complete semi-parallel Lorentzian hypersurface in $\mathbb{E}_1^{n+1}$, with $n \geq 3$. Then, $f(M_1^n)$ is congruent to one of the following Lorentzian submanifolds:*

(i) $\mathbb{E}_1^n = \{x \in \mathbb{E}_1^{n+1} : x_{n+1} = 0\}$;

(ii) $\mathbb{S}_1^n(a^2) = \left\{ x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{a^2} \right\}$ *with* $a \neq 0$;

(iii) $\mathbb{S}^k(a^2) \times \mathbb{E}_1^{n-k} = \left\{ x \in \mathbb{E}_1^{n+1} : \sum_{i=2}^{k+2} x_i^2 = \frac{1}{a^2} \right\}$, *with* $a \neq 0$ *and* $2 \leq k \leq n-1$;

(iv) $\mathbb{S}_1^k(a^2) \times \mathbb{E}^{n-k} = \left\{ x \in \mathbb{E}_1^{n+1} : -x_1^2 + \sum_{i=2}^{k+1} x_i^2 = \frac{1}{a^2} \right\}$, *with* $a \neq 0$ *and* $2 \leq k \leq n-1$;

(v) $\mathbb{E}_1^{n-2} \times h(\mathbb{E}^2)$, *where $h(\mathbb{E}^2)$ is a Euclidean cylinder in a subspace $\mathbb{E}^3$ of $\mathbb{E}_1^{n+1}$ orthogonal to $\mathbb{E}_1^{n-2}$; or $\mathbb{E}^{n-2} \times h(\mathbb{E}_1^2)$, where $h(\mathbb{E}_1^2)$ is a Lorentzian cylinder or a B-scroll in a subspace $\mathbb{E}_1^3$ of $\mathbb{E}_1^{n+1}$ orthogonal to $\mathbb{E}^{n-2}$.*

Next, using an approach analogous to that by Ryan, Al-shehri and Guediri in [66, 6], we obtain the following local classification results for the case where $\psi \neq c$ and the hypersurface is good in the sense of Ryan.

Theorem 0.7. *Let M_1^n be a ψ-pseudo-parallel Lorentzian hypersurface in $\mathbb{Q}_1^{n+1}(c)$, with $n \geq 3$ and constant $\psi < c$. Then M_1^n is either good and locally congruent to one of the following Lorentzian manifolds*

(i) $\mathbb{S}_1^n(a^2+c) = \left\{ x \in \mathbb{S}_1^{n+1}(c) \subset \mathbb{E}_1^{n+2} : x_{n+2} = \sqrt{\frac{1}{c} - \frac{1}{a^2+c}} \right\}$ *with* $a \in \mathbb{R}$, *if* $c > 0$;

(ii) $\mathbb{E}_1^n = \{x \in \mathbb{E}_1^{n+1} : x_{n+1} = 0\}$ *or* $\mathbb{S}_1^n(a^2) = \left\{ x \in \mathbb{E}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{a^2} \right\}$ *with* $a \neq 0$, *if* $c = 0$;

(iii) $\mathbb{H}_1^n(a^2+c) = \left\{ x \in \mathbb{H}_1^{n+1}(c) \subset \mathbb{E}_2^{n+2} : x_{n+2} = \sqrt{\frac{1}{c} - \frac{1}{a^2+c}} \right\}$, *with* $|a| < \sqrt{-c}$, *or* $\mathbb{S}_1^n(a^2+c) = \left\{ x \in \mathbb{H}_1^{n+1}(c) \subset \mathbb{E}_2^{n+2} : x_1 = \sqrt{\frac{1}{a^2+c} - \frac{1}{c}} \right\}$, *with* $a^2 > -c$. *In this case* $c < 0$;

(iv) *a totally umbilical hypersurface of the form* $\{y \in H_1^{n+1}(c) : \langle y, X\rangle = a\}$, *where* $a = \pm\sqrt{-c}$ *and* X *is a parallel vector field in* $\mathbb{E}_2^{n+2}$, *which satisfies* $\langle X, X\rangle = 0$. *In this case* $c < 0$;

(v) $\mathbb{S}^k(a^2+c)\times\mathbb{S}_1^{n-k}(b^2+c) = \left\{x \in \mathbb{S}_1^{n+1}(c) \subset \mathbb{E}_1^{n+2} : \sum_{i=2}^{k+2} x_i^2 = \frac{1}{a^2+c}, -x_1^2 + \sum_{i=k+3}^{n+2} x_i^2 = \frac{1}{b^2+c}\right\}$, *where* $c > 0$, $\psi = ab + c = 0$ *and* $1 < k < n-1$;

or else, M_1^n *is a bad hypersurface foliated either by* $(n-1)$*-dimensional Riemannian hyperspheres or by* $(n-1)$*-dimensional de Sitter spaces.*

Theorem 0.8. *Let* M_1^n *be a good* ψ*-pseudo-parallel Lorentzian hypersurface in* $\mathbb{Q}_1^{n+1}(c)$, *with* $n \geq 3$ *and constant* $\psi > c$. *Then,* M_1^n *is locally congruent to one of the following Lorentzian manifolds*

(i) *a totally geodesic or totally umbilical hypersurface as described in parts (i), (ii), (iii) and (iv) of Theorem 0.7;*

(ii) $\mathbb{H}_1^k(a^2+c)\times\mathbb{S}^{n-k}\left(\frac{c^2}{a^2}+c\right) = \left\{x \in \mathbb{H}_1^{n+1}(c) \subset \mathbb{E}_2^{n+2} : -\sum_{i=1}^{2} x_i^2 + \sum_{i=3}^{k+1} x_i^2 = \frac{1}{a^2+c}, \sum_{i=k+2}^{n+2} x_i^2 = \frac{a^2}{c^2+ca^2}\right\}$, *where* $c < 0$, $|a| < \sqrt{-c}$, $\psi = 0$ *and* $1 < k < n-1$;

(iii) $\mathbb{S}_1^k(a^2+c)\times\mathbb{H}^{n-k}\left(\frac{c^2}{a^2}+c\right) = \left\{x \in \mathbb{H}_1^{n+1}(c) \subset \mathbb{E}_2^{n+2} : -x_1^2 + \sum_{i=3}^{k+2} x_i^2 = \frac{1}{a^2+c}, -x_2^2 + \sum_{i=k+3}^{n+2} x_i^2 = \frac{a^2}{c^2+ca^2}\right\}$, *where* $c < 0$, $|a| > \sqrt{-c}$, $\psi = 0$ *and* $1 < k < n-1$;

(iv) *a generalized umbilical hypersurface of degree* 2 *as in* (1.36), (1.37) *or* (1.38), *where* $\psi = c + a^2$, $a = -\tau \neq 0$, *at each connected component of the open subset of non-umbilical points.*

To obtain the classifications in Theorem 0.7 and Theorem 0.8, the study of isoparametric Lorentzian hypersurfaces carried out in [37, 55, 80, 42, 5, 1] will be useful. In fact, for the case $\psi \neq c$, the constancy of the pseudo-parallelism function will imply that the principal curvatures of the hypersurface will be constant as well, provided that the hypersurface is good.

In Example 4.24, we give a Lorentzian hypersurface $f : U \to \mathbb{E}_1^4$ with U a neighborhood of 0 in $\mathbb{R}^3$, parameterized by

$$f(s,u,z) = \gamma(s) + uB(s) + z\widetilde{e}_4 + C(s) - \sqrt{1-z^2}C(s). \tag{5}$$

where

$$\begin{aligned}\gamma(s) = \frac{1}{6}s&\left\{2\,{}_0F_1\left(;\frac{1}{3},-\frac{s^3}{18}\right)^2 + 4\,{}_0F_1\left(;\frac{1}{3},-\frac{s^3}{18}\right){}_0F_1\left(;\frac{4}{3},-\frac{s^3}{18}\right) + s^3\,{}_0F_1\left(;\frac{4}{3},-\frac{s^3}{18}\right)^2\right\}\widetilde{e}_1\\ &+\left\{8\,{}_0F_1\left(;\frac{2}{3},-\frac{s^3}{18}\right)^2 - 8\,{}_0F_1\left(;\frac{2}{3},-\frac{s^3}{18}\right){}_0F_1\left(;\frac{5}{3},-\frac{s^3}{18}\right) + s^3\,{}_0F_1\left(;\frac{5}{3},-\frac{s^3}{18}\right)^2\right\}\widetilde{e}_2\\ &+\frac{1}{6}\left\{-2\,{}_0F_1\left(;\frac{1}{3},-\frac{s^3}{18}\right)\left[2\,{}_0F_1\left(;\frac{1}{3},-\frac{s^3}{18}\right) + s^3\,{}_0F_1\left(;\frac{4}{3},-\frac{s^3}{18}\right)\right]\right.\\ &\left.+s^3\left[{}_0F_1\left(;\frac{1}{3},-\frac{s^3}{18}\right) + {}_0F_1\left(;\frac{4}{3},-\frac{s^3}{18}\right)\right]{}_0F_1\left(;\frac{5}{3},-\frac{s^3}{18}\right)\right\}\widetilde{e}_3,\end{aligned} \tag{6}$$

$\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ is a pseudo-orthonormal frame in $\mathbb{E}^4_1$ and $\{T(s), B(s), C(s), \widetilde{e}_4\}$ is a pseudo-orthonormal frame associated to γ, satisfying $T(s) = \frac{d}{ds}\gamma(s)$ and $\frac{d}{ds}C(s) = -T(s) + sB(s)$. The curve $\gamma(s)$ and the vector fields $T(s), B(s), C(s)$ were obtained as solutions of a initial value problem using the software Mathematica. This hypersurface f is a generalized umbilical hypersurface of degree 2 out of the non empty set of umbilical points, in fact, the Weingarten operator A is non diagonalizable almost everywhere, except when the parameter s vanishes and then A degenerates to a multiple of the identity. Thus, f is isoparametric in the sense of Hahn (see [37]), but the minimal polynomial of the Weingarten operator is not constant.

Finally, we study the particular case of pseudo-parallel Lorentzian hypersurfaces in $\mathbb{Q}^{n+1}_1(c)$ with constant ψ and constant mean curvature function H. Here, the classification of all immersions of $\mathbb{E}^n_1$ into $\mathbb{E}^{n+1}_1$ given in [34] will also be useful, in addition to the study of isoparametric hypersurfaces mentioned above. We obtain the following results.

Theorem 0.9. *Let $f : M^n_1 \to \mathbb{Q}^{n+1}_1(c)$ be a ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi = c$. If f has nonzero constant mean curvature, then $f(M^n_1)$ is locally congruent to one of the following Lorentzian hypersurfaces:*

(i) A totally umbilical $\mathbb{S}^n_1(a^2) = \left\{x \in \mathbb{E}^{n+1}_1 : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{a^2}\right\}$, $a = H \neq 0$, if $c = 0$.

(ii) A totally umbilical $\mathbb{S}^n_1(a^2+c) = \left\{x \in \mathbb{S}^{n+1}_1(c) \subset \mathbb{E}^{n+2}_1 : x_{n+2} = \sqrt{\frac{1}{c} - \frac{1}{a^2+c}}\right\}$, with $a = H \neq 0$, if $c > 0$.

(iii) A totally umbilical $\mathbb{H}^n_1(a^2+c) = \left\{x \in \mathbb{H}^{n+1}_1(c) \subset \mathbb{E}^{n+2}_2 : x_{n+2} = \sqrt{\frac{1}{c} - \frac{1}{a^2+c}}\right\}$, with $0 < |a| < \sqrt{-c}$, or $\mathbb{S}^n_1(a^2+c) = \left\{x \in \mathbb{H}^{n+1}_1(c) \subset \mathbb{E}^{n+2}_2 : x_1 = \sqrt{\frac{1}{a^2+c} - \frac{1}{c}}\right\}$, with $a^2 > -c$. In this case $a = H \neq 0$ and $c < 0$.

(iv) A totally umbilical hypersurface of the form $\{y \in H^{n+1}_1(c) : \langle y, X\rangle = a\}$, where $a = \pm\sqrt{-c} = H$ and X is a parallel vector field in $\mathbb{E}^{n+2}_2$, which satisfies $\langle X, X\rangle = 0$. In this case $c < 0$.

(v) A cylinder $\mathbb{S}^k_\tau(a^2) \times \mathbb{E}^{n-k}_{1-\tau} = \left\{x \in \mathbb{E}^{n+1}_1 : -\sum_{i=2-\tau}^{1} x_i^2 + \sum_{i=2}^{k+2-\tau} x_i^2 = \frac{1}{a^2}\right\}$, $a = \frac{nH}{k} \neq 0$ and $1 \leq k \leq n-1$. In this case $c = 0$.

Theorem 0.10. *Let $f : M^n_1 \to \mathbb{Q}^{n+1}_1(c)$ be a ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi = c$. If f has mean curvature $H = 0$, then $f(M^n_1)$ is either totally geodesic or*

(i) $c = 0$ and $f(M^n_1)$ is a generalized cylinder given by $\mathbb{E}^{n-2} \times h(\mathbb{E}^2_1)$, where $h(\mathbb{E}^2_1)$ is a B-scroll in a subspace $\mathbb{E}^3_1$ of $\mathbb{E}^{n+1}_1$ orthogonal to $\mathbb{E}^{n-2}$, that is, locally, the hypersurface $f : U \to \mathbb{E}^{n+1}_1$, U

a neighborhood of 0 *in* $\mathbb{R}^n$*, parameterized by*

$$f(s,y,z_3,\ldots,z_n) = \gamma(s) + yB(s) + \sum_i z_i Z_i(s)$$

where γ *is a null curve in* $\mathbb{E}_1^{n+1}$ *with an associated pseudo-orthonormal frame* $\{T(s),B(s),Z_3(s),\ \ldots,Z_n(s),C(s)\}$ *of tangent vectors of* $\mathbb{E}_1^{n+1}$ *along* γ*, such that* $T(S)$ *and* $B(s)$ *are lightlike vectors with* $\langle T(S),B(S)\rangle = -1$*,* $T(s) = \frac{d}{ds}\gamma(s)$ *and* $\frac{d}{ds}C(s) = -\kappa(s)B(s)$*.*

(ii) $c < 0$*, and at the open subset of non-geodesic points, locally* $f(M_1^n)$ *is an open piece of a Lorentzian hypersurface* $f : U \to \mathbb{H}_1^{n+1}(c) \subset \mathbb{E}_2^{n+2}$*, with* U *an open neighborhood of* 0 *in* $\mathbb{R}^n$*, parameterized by*

$$f(s,u,z) = \sqrt{1 - c\sum_{i=3}^{n} z_i^2}\,\gamma(s) + uB(s) + \sum_{i=3}^{n} z_i Z_i(s), \tag{7}$$

where $z = (z_3,\ldots,z_n)$*,* $\gamma(s)$ *is a null curve in* $\mathbb{H}_1^{n+1}(c)$ *with an associated pseudo-orthonormal frame* $\{T(s),\ B(s),Z_3(s),\ldots,Z_n(s),C(s)\}$ *of tangent vector fields of* $\mathbb{H}_1^{n+1}(c)$ *along* γ*, such that* $\langle T(s),T(s)\rangle = \langle B(s),B(s)\rangle = 0$*,* $\langle T(s),B(s)\rangle = -1$*,* $\langle Z_i(s),Z_i(s)\rangle = \langle C(s),C(s)\rangle = 1$*, all other inner products are zero along* $\gamma(s)$*,* $\frac{d}{ds}\gamma(s) = T(s)$ *and* $\frac{d}{ds}C(s) = \kappa(s)B(s)$*, where* $\frac{d}{ds}$ *denote the ordinary derivation in* $\mathbb{E}_2^{n+1}$ *and* $\kappa(s) \neq 0$*.*

(iii) $c > 0$*, and at the open subset of non-geodesic points, locally* $f(M_1^n)$ *is an open piece of a Lorentzian hypersurface* $f : \Omega = (a,b)\times\mathbb{R}\times\mathbb{S}_+^{n-2} \to \mathbb{S}_1^{n+1}(c) \subset \mathbb{E}_1^{n+2}$*, with* $\mathbb{S}_+^{n-2}(c) = \{y = (y_3,\ldots,y_{n+1}) \in \mathbb{S}^{n-2}(c) : y_3 > 0\}$*, parameterized by*

$$\begin{aligned} f(t,u,y) &= y_3E_3(t) + uE_2(t) + \sum_{i=4}^{n+1} y_iE_i(t) \\ &= \sqrt{\frac{1}{c} - \sum_{i=4}^{n+1} y_i^2}\,E_3(t) + uE_2(t) + \sum_{i=4}^{n+1} y_iE_i(t), \end{aligned} \tag{8}$$

where $\rho(t)$*, with* $t \in (a,b)$*, is a parameterized curve satisfying* $\langle\rho(t),\rho(t)\rangle = 0$*, for all* t*, and* $\frac{d}{dt}\rho(t)$ *is a spacelike curve, with an associate pseudo-orthonormal frame* $\{E_1(t),E_2(t) = \rho(t),E_3(t),\ldots,E_{n+2}(t)\}$ *of tangent vector fields to* $\mathbb{E}_1^{n+2}$ *along* ρ*, such that* $\langle E_1,E_1\rangle = \langle E_2,E_2\rangle = 0$*,* $\langle E_1,E_2\rangle = -1$*,* $\langle E_j,E_j\rangle = 1$*, for* $4 \le j \le n+2$*, all other inner products are zero along* $\rho(s)$ *and*

$$\begin{aligned} &E_1' = C_1E_3 + C_2E_4 + \cdots + C_nE_{n+2}, \\ &E_2' = E_3, \quad E_3' = E_1 + C_1E_2, \quad E_j' = C_{j-2}E_2, \text{ for } 4 \le j \le n+2, \end{aligned} \tag{9}$$

where $C_1,\ldots,C_n$ *are functions in the variable* t *and* $E_i' = \frac{d}{dt}E_i$*.*

Theorem 0.11. *Let* $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$ *be a* ψ*-pseudo-parallel Lorentzian hypersurface, with* $n \ge 3$ *and constant* $\psi < c$*. If* f *has constant mean curvature, then* $f(M_1^n)$ *is locally congruent to one of the following Lorentzian hypersurfaces:*

(i) A totally geodesic or totally umbilical hypersurface as described in parts (i), (ii), (iii) and (iv) of Theorem 0.7, with $a = H$.

(ii) $\mathbb{S}^k(a^2+c)\times\mathbb{S}_1^{n-k}\left(\frac{c^2}{a^2}+c\right) = \left\{x\in\mathbb{S}_1^{n+1}(c)\subset\mathbb{E}_1^{n+2} : \sum_{i=2}^{k+2} x_i^2 = \frac{1}{a^2+c}, -x_1^2+\sum_{i=k+3}^{n+2} x_i^2 = \frac{a^2}{c^2+ca^2}\right\}$, *where* $c>0$, $\psi=0$, $a=\frac{nH\pm\sqrt{n^2H^2+4k(n-k)c}}{2k}$ *and* $1\le k\le n-1$.

Theorem 0.12. *Let $f: M_1^n \to \mathbb{Q}_1^{n+1}(c)$ be a ψ-pseudo-parallel Lorentzian hypersurface, with $n\ge 3$ and constant $\psi>c$. If f has constant mean curvature, then $f(M_1^n)$ is locally congruent to one of the following Lorentzian hypersurfaces:*

(i) A totally geodesic or totally umbilical hypersurface as described in parts (i), (ii), (iii) and (iv) of Theorem 0.7, with $a = H$.

(ii) $\mathbb{H}_1^k(a^2+c)\times\mathbb{S}^{n-k}\left(\frac{c^2}{a^2}+c\right) = \left\{x\in\mathbb{H}_1^{n+1}(c)\subset\mathbb{E}_2^{n+2} : -\sum_{i=1}^{2} x_i^2+\sum_{i=3}^{k+1} x_i^2 = \frac{1}{a^2+c}, \sum_{i=k+2}^{n+2} x_i^2 = \frac{a^2}{c^2+ca^2}\right\}$, *where* $c<0$, $|a|<\sqrt{-c}$, $\psi=0$, $a=\frac{nH\pm\sqrt{n^2H^2+4k(n-k)c}}{2k}$ *and* $1\le k\le n-1$.

(iii) $\mathbb{S}_1^k(a^2+c)\times\mathbb{H}^{n-k}\left(\frac{c^2}{a^2}+c\right) = \left\{x\in\mathbb{H}_1^{n+1}(c)\subset\mathbb{E}_2^{n+2} : -x_1^2+\sum_{i=3}^{k+2} x_i^2 = \frac{1}{a^2+c}, -x_2^2+\sum_{i=k+3}^{n+2} x_i^2 = \frac{a^2}{c^2+ca^2}\right\}$, *where* $c<0$, $|a|>\sqrt{-c}$, $\psi=0$, $a=\frac{nH\pm\sqrt{n^2H^2+4k(n-k)c}}{2k}$ *and* $1\le k\le n-1$.

(iv) A generalized umbilical hypersurface of degree 2 *as in* (1.36), (1.37) *or* (1.38), *where $\psi = c+a^2$, $\tau = a = H \neq 0$, in the open subset of non-umbilical points.*

We remark that the classification of ψ-pseudo-parallel Lorentzian hypersurfaces in $\mathbb{Q}_1^{n+1}(c)$, with constant $\psi = c \neq 0$ is still an open problem. We state the following conjecture:

Conjecture 0.13. *Any connected ψ-pseudo-parallel Lorentzian hypersurface $f: M_1^n\to\mathbb{Q}_1^{n+1}(c)$, with $n\ge 3$ and $\psi = c\neq 0$, is congruent to a totally umbilical hypersurface of $\mathbb{Q}_1^{n+1}(c)$ or $k(x)\le 1$ everywhere on M_1^n.*

The thesis is organized in five chapters. In Chapter one, we introduce the notations we use along the whole work and recall some concepts of the Submanifold Theory. We state the definitions of pseudo-parallel immersions and other related extrinsic notions. We also recall the pseudo-Riemannian space forms as well the Fundamental Equations for surfaces and hypersurfaces in these ambient spaces, which will be useful in the next chapters. In the following sections, we will make a summary about some particular surfaces and hypersurfaces of pseudo-Riemannian space forms which will be a source of examples for our work, namely, the general rotational surfaces with plane meridians in 4-dimensional pseudo-Euclidean spaces as well the B-scrolls, generalized cylinders and generalized umbilical hypersurfaces of degree 2 in Lorentzian space forms, also we recall classification results for hypersurfaces of constant curvature in Lorentzian space forms and for isoparametric Lorentzian hypersurfaces in Lorentzian space forms, in particular those such that the Weingarten operator is diagonalizable and has at most two different principal curvatures or the minimal polynomial is $(t-a)^2$, with $a \in \mathbb{R}$ constant.

In Chapter two, first we present in Section 2.1 some basic results about Lorentizan surfaces in pseudo-Riemannian space forms. Mainly, we characterized the condition of the normal bundle being non-flat in terms of the linear independence of two normal vector fields given by the second fundamental form and we reduce the pseudo-parallelism condition to equations involving the normal curvature tensor, the Gaussian curvature, the second fundamental form α and the pseudo-parallelism function ψ. We show in Example 2.5 a pseudo-parallel Lorentzian surface with non-flat normal bundle and $\psi = K$. In Section 2.2, we characterize λ-isotropic Lorentzian surfaces by providing several equivalent conditions to λ-isotropy. In particular, we study the hyperbola of normal curvature of λ-isotropic Lorentzian surfaces. Finally, in Section 2.3, we prove Theorem 0.1 and obtain as corollary a characterization of pseudo-parallel Lorentzian surfaces with non-flat normal bundle in terms of the hyperbola of normal curvature.

In Chapter three, we will present examples of pseudo-parallel Lorentzian surfaces with non-flat normal bundle in a 4-dimensional or 5-dimensional pseudo-Riemannian space form. First, in Section 3.1, we recall the definition of isotropy with negative (positive) spin. Next, in Example 3.1, we prove that Lorentzian surfaces of the Veronese type in codimension two, which are extremal and isotropic with negative spin immersion, are also parallel and λ-isotropic. Then, we study the case of pseudo-parallel Lorentzian surface with constant ψ and prove Corollary 0.3. In Section 3.1, we study pseudo-parallel general rotational surfaces with plane meridians in $\mathbb{E}^4_2$ and prove Theorem 0.4 and Theorem 0.5. In section 3.3, we explore the case of codimension three and give, in Example 3.10, a flat extremal pseudo-parallel Lorentzian surface with non-flat normal bundle in $\mathbb{S}^5_2(c)$ which is not a semi-parallel surface.

In Chapter four, we first recall in Section 4.1 some basic notions about pseudo-parallel hypersurfaces in pseudo-Riemannian space forms. Since reference [44] is quite difficult to find, for the sake of completeness we recall some results therein. In Section 4.2, we obtain in Proposition 4.6 a complete characterization of the Weingarten operator of a pseudo-parallel Lorentzian hypersurface in pseudo-Riemannian space forms. In Section 4.3, we begin describing in detail the Weingarten operator of a Lorentzian hypersurface in $\mathbb{Q}^{n+1}_1(c)$, with $n \geq 3$, in Lemma 4.7. Then, we obtain in Theorem 4.10 the classification of all connected complete semi-parallel Lorentzian hypersurfaces of $\mathbb{E}^{n+1}_1$. Next, in Section 4.4, we study the case where the pseudo-parallelism function ψ is constant and does not coincide with the curvature of the ambient space, proving that the set of bad points of the pseudo-parallel

Lorentzian hypersurface is open if $\psi \neq c$ and also closed if $\psi < c$. Then we show in Proposition 4.17 for $\psi \neq c$, that the rank of the Weingarten operator is either zero everywhere or n everywhere on the hypersurface. Using this, we prove Theorem 0.7 and Theorem 0.8. Finally, in Section 4.5, we study ψ-pseudo-parallel Lorentzian hypersurfaces in $\mathbb{Q}_1^{n+1}(c)$ with constant ψ and constant mean curvature function H and prove Theorem 0.9 and Theorem 0.10, for the case $\psi = c$, and Theorem 0.11 and Theorem 0.12, for the cases $\psi < c$ and $\psi > c$, respectively.

CHAPTER 1

Preliminaries and basic notations

In this chapter, we recall basic notions of Submanifold Theory in pseudo-Riemannian manifolds and fix the notation we use along this work. For the reader interested in a detailed introduction to Theory of Riemannian or pseudo-Riemannian submanifolds we recommend [20] and [62].

1.1 Basics of theory of pseudo-Riemannian submanifolds

A *scalar product* B in a finite dimensional real vector space V is a non-degenerate symmetric bilinear form. The dimension of the largest subspace $W \subset V$ on which $B|_W$ is negative definite, i.e., $B(v,v) < 0$ for all nonzero vector $v \in W$, is called the *index* of B. A vector $v \in V$ is said to be *timelike* if satisfies $B(v,v) < 0$ or *lightlike* (null) if $v \neq 0$ and $B(v,v) = 0$, in other case v is called a *spacelike* vector. Non-degeneracy of B means that $v \in V$ with $B(u,v) = 0$ for all $u \in V$ implies $v = 0$.

A *non-degenerate metric tensor* $\widetilde{g}$ in a m-dimensional smooth manifold $\widetilde{M}$, is a symmetric non-degenerate $(0,2)$ tensor field on $\widetilde{M}$ of constant index, i.e., $\widetilde{g}$ assigns to each point $x \in \widetilde{M}$ a scalar product $\widetilde{g}_x$ on $T_x\widetilde{M}$, and the index of $\widetilde{g}_x$ is the same for all $x \in \widetilde{M}$. A *pseudo-Riemannian manifold* is a smooth manifold furnished with a non-degenerate metric tensor $\widetilde{g}$.

Let $\widetilde{M}^m_s$ be a m-dimensional pseudo-Riemannian manifold with metric $\widetilde{g}$ of index s and let M^n_t an n-dimensional pseudo-Riemannian manifold with metric g of index t, with $n < m$, $0 \leq t \leq n$ and $t \leq s \leq m$. We say that a smooth map $f : M^n_t \to \widetilde{M}^m_s$ is an *immersion* if its differential $f_* : T_xM \to T_{f(x)}\widetilde{M}$ is injective, for all $x \in M^n_t$.

Moreover, an immersion $f : M^n_t \to \widetilde{M}^m_s$ is said to be an *isometric immersion* provided that

$$\widetilde{g}(f_*X, f_*Y) = g(X,Y),$$

for all $X,Y \in T_xM$, for all $x \in M^n_t$.

From now until the end of this section, we consider $f : M_t^n \to \widetilde{M}_s^m$ an isometric immersion. We denote the tangent bundles of M_t^n and $\widetilde{M}_s^m$ by TM and $T\widetilde{M}$, respectively, and denote by $f^*T\widetilde{M}$ the induced bundle over M_t^n, whose fiber at $x \in M_t^n$ is $T_{f(x)}\widetilde{M}$.

For each point $x \in M_t^n$, we denote by $N_fM(x)$ the orthogonal complement of f_*T_xM in $T_{f(x)}\widetilde{M}$ which is called the normal space of f an x. The normal bundle of f, denoted by N_fM, is the vector subbundle of $f^*T\widetilde{M}$ whose fiber at each point $x \in M_t^n$ is $N_fM(x)$. Smooth sections of TM are called tangent vector fields and smooth sections of N_fM are called normal vector fields.

The Levi–Civita connection of $\widetilde{M}_s^m$ induce a connection $\widetilde{\nabla}$ on $f^*T\widetilde{M}$. Given tangent vector fields X,Y of M_t^n, we can make the decomposition

$$\widetilde{\nabla}_{f_*X} f_*Y = (\widetilde{\nabla}_{f_*X} f_*Y)^T + (\widetilde{\nabla}_{f_*X} f_*Y)^\perp,$$

respect to the decomposition as a orthogonal direct sum

$$f^*T\widetilde{M} = f_*TM \oplus N_fM.$$

The tangent part $\nabla_X Y = (f_*)^{-1}(\widetilde{\nabla}_{f_*X} f_*Y)^T$, coincides with the Levi-Civita connection of M_t^n. The symmetric 2-tensor field defined by the normal part

$$\alpha(X,Y) = (\widetilde{\nabla}_{f_*X} f_*Y)^\perp,$$

is called the **second fundamental form** of f. Thus, we have the **Gauss formula**

$$\widetilde{\nabla}_{f_*X} f_*Y = f_*\nabla_X Y + \alpha(X,Y), \tag{1.1}$$

for all tangent vector fields X,Y of M_t^n.

At each point $x \in M_t^n$, α defines a symmetric bilinear map $\alpha : T_xM \times T_xM \to N_fM(x)$, which we also call the second fundamental form of f at x.

For any $\xi \in N_fM(x)$, the corresponding Weingarten operator of f at x in the ξ-direction, denoted by A_ξ, is defined by

$$\widetilde{g}(\alpha(X,Y),\xi) = g(A_\xi X,Y), \tag{1.2}$$

for all $X,Y \in T_xM$, where $A_\xi X \in T_xM$.

It follows that $-f_*A_\xi X$ is the tangent part of $\widetilde{\nabla}_{f_*X}\xi$. On the other hand, the normal component

$$\nabla_X^\perp \xi = (\widetilde{\nabla}_{f_*X}\xi)^\perp,$$

where $X \in TM$ and ξ is a normal vector field of M_t^n, defines a torsion-free connection $\nabla^\perp$ in $N_f(M)$ compatible with $\widetilde{g}$, called the **normal connection** of f. Thus, we have the **Weingarten formula**:

$$\widetilde{\nabla}_{f_*X}\xi = -f_*A_\xi X + \nabla^\perp_X \xi. \tag{1.3}$$

Let $\{X_1,\dots,X_n\}$ a local frame of M_t^n and denote by $g_{ij} = g(X_i,X_j)$. The mean curvature vector field of f is the normal vector field $\mathcal{H}$, defined by

$$\mathcal{H} = \frac{1}{n}\mathrm{trace}(\alpha) = \frac{1}{n}\sum_{i,j=1}^{n} g^{ij}\alpha(X_i,X_j),$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

We say that f is *extremal* (minimal or maximal) if $\mathcal{H} = 0$.

We denote by R the curvature tensor at $x \in M_t^n$ corresponding to TM and adopt the sign convention

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for all $X,Y,Z \in T_xM$.

The sectional curvature $K(X,Y)$ of M_t^n with respect to spam$\{X,Y\} \subset T_xM$ is defined by

$$K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

We denote by $R^\perp$ the curvature tensor of the normal bundle N_fM, and it is given by

$$R^\perp(X,Y)\xi = \nabla^\perp_X\nabla^\perp_Y\xi - \nabla^\perp_Y\nabla^\perp_X\xi - \nabla^\perp_{[X,Y]}\xi,$$

for all $X,Y \in T_xM$ and any normal vector field ξ of f.

We say that f has **flat normal bundle**, or **vanishing normal curvature** if $R^\perp = 0$ on M_t^n.

Let $\widetilde{R}$ the curvature tensor of $T\widetilde{M}$, from Gauss and Weingarten formulas, we can deduce the following three important equations, called compatibility equations of the isometric immersion f:

GAUSS EQUATION:

$$(\widetilde{R}(f_*X,f_*Y)f_*Z)^T = f_*R(X,Y)Z + A_{\alpha(X,Z)}Y - A_{\alpha(Y,Z)}X; \tag{1.4}$$

CODAZZI-MAINARDI EQUATION:

$$(\widetilde{R}(f_*X,f_*Y)f_*Z)^\perp = (\overline{\nabla}_X\alpha)(Y,Z) - (\overline{\nabla}_Y\alpha)(X,Z), \tag{1.5}$$

where

$$(\overline{\nabla}_X\alpha)(Y,Z) = \nabla^{\perp}_X\alpha(Y,Z) - \alpha(\nabla_X Y,Z) - \alpha(Y,\nabla_X Z);$$

RICCI EQUATION:

$$(\widetilde{R}(f_*X,f_*Y)\xi)^{\perp} = R^{\perp}(X,Y)\xi + \alpha(A_\xi X,Y) - \alpha(X,A_\xi Y), \tag{1.6}$$

for $X,Y,Z \in T_xM$ and $\xi \in N_fM(x)$.

1.2 Pseudo-parallel submanifolds

Let $f : M^n_t \to \widetilde{M}^m_s$ be an isometric immersion. We consider the decomposition of induced bundle $f^*T\widetilde{M}$ over M^n_t as a Whitney sum $f^*T\widetilde{M} = f_*TM \oplus N_fM$ and we denote by $\overline{R} = R \oplus R^{\perp}$ the curvature tensor corresponding to the Van der Waerden-Bortoletti connection $\overline{\nabla} = \nabla \oplus \nabla^{\perp}$ of f.

f is said to be:

1. *Totally geodesic* if
$$\alpha(X,Y) = 0; \tag{1.7}$$

2. *Totally umbilical* if
$$\alpha(X,Y) = \langle X,Y\rangle \mathcal{H}; \tag{1.8}$$

3. *Pseudo-umbilical* if
$$\langle \alpha(X,Y),\mathcal{H}\rangle = \langle \mathcal{H},\mathcal{H}\rangle\langle X,Y\rangle; \tag{1.9}$$

4. λ*-isotropic* if
$$\langle \alpha(X,X),\alpha(X,X)\rangle = \lambda(x) \tag{1.10}$$
for some smooth real-valued function λ on M^n_t and for any $X \in T_xM$ with $\|X\| = \sqrt{|\langle X,X\rangle|} = 1$, for all $x \in M^n_t$.

5. *Locally parallel* if
$$(\overline{\nabla}_X\alpha)(Y,Z) = 0, \tag{1.11}$$
with $(\overline{\nabla}_X\alpha)(Y,Z)$ defined as in (1.5);

6. *Semi-parallel* if
$$(\overline{R}(X,Y)\cdot\alpha)(Z,W) = 0; \tag{1.12}$$

7. *Pseudo-parallel* if
$$(\overline{R}(X,Y)\cdot\alpha)(Z,W) = \psi((X\wedge Y)\cdot\alpha)(Z,W), \tag{1.13}$$
for some smooth real-valued function ψ on M^n_t and for any $X,Y,Z,W \in T_xM$, for all $x \in M^n_t$.

Here the notation means

$$(\overline{R}(X,Y)\cdot\alpha)(Z,W) = R^{\perp}(X,Y)\alpha(Z,W) - \alpha(R(X,Y)Z,W) - \alpha(Z,R(X,Y)W),$$
$$((X\wedge Y)\cdot\alpha)(Z,W) = -\alpha((X\wedge Y)Z,W) - \alpha(Z,(X\wedge Y)W)$$
$$(X\wedge Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y.$$

1.3 Pseudo-Riemannian space forms

Let $\mathbb{E}^N_s$ be the N-dimensional pseudo-Euclidean space with the semi-Riemannian metric of index s given by

$$\langle x,y\rangle = -\sum_{i=1}^{s} x_i y_i + \sum_{i=s+1}^{N} x_i y_i, \tag{1.14}$$

where $x=(x_1,\dots,x_N), y=(y_1,\dots,y_N)\in\mathbb{E}^N_s$.

We will consider a standard pseudo-Riemannian space form $\mathbb{Q}^m_s(c)$ as a complete m-dimensional pseudo-Riemannian manifold with constant sectional curvature c and index s, such that

$$\mathbb{Q}^m_s(c) = \begin{cases} \mathbb{H}^m_s(c)\subset\mathbb{E}^{m+1}_{s+1}, & \text{if } c<0,\\ \mathbb{E}^m_s, & \text{if } c=0,\\ \mathbb{S}^m_s(c)\subset\mathbb{E}^{m+1}_s, & \text{if } c>0,\end{cases}$$

where the m-dimensional pseudo-sphere $\mathbb{S}^m_s(c)$, $c>0$, is a connected component of $\{x\in\mathbb{E}^{m+1}_s : \langle x,x\rangle = \frac{1}{c}\}$ with the induced metric of index s and the m-dimensional pseudo-hyperbolic space $\mathbb{H}^m_s(c)$, $c<0$, is a connected component of $\left\{x\in\mathbb{E}^{m+1}_{s+1} : \langle x,x\rangle = \frac{1}{c}\right\}$ with the induced metric of index s. We remark that $\mathbb{S}^m_{m-1}(c)$, $c>0$, and $\mathbb{H}^m_1(c)$, $c<0$, are not simply connected.

For $c=0$, we denote by $\widetilde{\nabla}$ the usual directional derivative in $\mathbb{E}^m_s$. For $c\neq 0$, the outward pointing unit normal vector of $\mathbb{Q}^m_s(c)$ in $\mathbb{E}^{m+1}_{s+\sigma}$, where $\sigma=0$ if $c>0$ and $\sigma=1$ if $c<0$, at any point $x=(x_1,\dots,x_n)$, is given by normalization of the position vector $u=\sqrt{|c|}i(x)=\sqrt{|c|}(x_1,\dots,x_n)$. With respect to this unit normal vector, we have that the inclusion $i:\mathbb{Q}^m_s(c)\to\mathbb{E}^{m+1}_{s+\sigma}$ is umbilical with Weingarten operator $-\sqrt{|c|}I_m$, where I_m is the identity in T_xM, and thus, its second fundamental form at x is given by

$$\alpha^i(X,Y) = \langle u,u\rangle\langle\alpha^i(X,Y),u\rangle u = -c\langle X,Y\rangle i(x),$$

for all $X,Y\in T_x\mathbb{Q}^m_s(c)$.

Now, denoting by $\hat{\nabla}$ the usual directional derivative in $\mathbb{E}^{m+1}_{s+\sigma}$, we can recover the Levi-Civita connection $\widetilde{\nabla}$ of $\mathbb{Q}^m_s(c)$, $c\neq 0$, using Gauss formula for immersion i

$$i_*\widetilde{\nabla}_X Y = \hat{\nabla}_{i_*X} i_*Y - \alpha^i(X,Y) = \hat{\nabla}_{i_*X} i_*Y + c\langle X,Y\rangle i(x).$$

On the other hand, the curvature tensor $\widetilde{R}$ of a pseudo-Riemannian space form $\mathbb{Q}_s^m(c)$ at a point x, is given by

$$R(X,Y)Z = c(X \wedge Y)Z,$$

for all $X,Y,Z \in T_x\mathbb{Q}_s^m(c)$.

We will study pseudo-parallel immersions in $\mathbb{Q}_s^m(c)$ in two special cases. When immersion f is a surface (i.e., f from a 2-dimensional pseudo-Riemannian submanifold) and when f is a hypersurface (i.e., f from a n-dimensional pseudo-Riemannian manifold with $m = n+1$).

For the first case, let $f : M_t^2 \to \mathbb{Q}_s^m(c)$ be an isometric immersion. Let $\{e_1, e_2\}$ be an orthonormal local frame for M_t^2 and denote $\alpha_{ij} = \alpha(e_i, e_j)$. The compatibility equations of f can be expressed as following (see for instance [58] and [62]):

GAUSS:

$$R(e_1,e_2)e_k = c(e_1 \wedge e_2)e_k + A_{\alpha_{2k}}e_1 - A_{\alpha_{1k}}e_2. \tag{1.15}$$

CODAZZI-MAINARDI:

$$(\overline{\nabla}_{e_1}\alpha)(e_2,e_k) = (\overline{\nabla}_{e_2}\alpha)(e_1,e_k). \tag{1.16}$$

RICCI:

$$R^{\perp}(e_1,e_2)\xi = \alpha(e_1, A_\xi e_2) - \alpha(A_\xi e_1, e_2), \tag{1.17}$$

for all $\xi \in N_fM$.

Now, for the case when f is a hypersurface, consider an insometric immersion $f : M_t^n \to \mathbb{Q}_s^{n+1}(c)$. In this case, a smooth unit normal vector field $\eta \in N_fM$ is locally unique, up to sign. Let A the Weingarten operator corresponding to the η-direction and let $\varepsilon = \langle \eta, \eta \rangle$, we write the Gauss and the Weingarten formulas as

$$\widetilde{\nabla}_{f_*X} f_*Y = f_*\nabla_X Y + \varepsilon\langle AX, Y\rangle\eta,$$

and

$$\widetilde{\nabla}_{f_*X}\eta = -f_*AX,$$

respectively, where X,Y are tangent vector fields of M_t^n, and the mean curvature vector can be write (locally) as

$$\mathcal{H}(x) = \varepsilon H(x)\eta(x),$$

where $x \in M_t^n$ and $H(x)$ is called the **mean curvature** of f at x, with respect to η.

The compatibility equations for the hypersurface f are given by

GAUSS:

$$R(X,Y)Z = c(X \wedge Y)Z + \varepsilon(AX \wedge AY)Z. \tag{1.18}$$

CODAZZI-MAINARDI:

$$(\nabla_X \cdot A)Y = (\nabla_Y \cdot A)X, \tag{1.19}$$

where $(\nabla_X \cdot A)Y = \nabla_X(AY) - A(\nabla_X Y)$. We observed that Ricci Equation is trivially satisfied for the case of hypersurfaces.

The eigenvalues of the Weingarten operator A are called the **principal curvatures** of the Lorentzian hypersurface M^n_t. For each $x \in M^n_t$, the subspace $T_0(x) = \{X \in T_xM : A_xX = 0\}$ is called the relative nullity space at x. The dimension of the subspace $T_0(x)$ is called the **index of relative nullity** at x, while the rank of the shape operator A_x is called the **type number** at x and it is denoted by $k(x)$.

1.4 General rotational surfaces with plane meridians in $\mathbb{E}^4_2$

General rotational surfaces of Moore type in the pseudo- Euclidean 4-space $\mathbb{E}^4_2$, are defined in [4] as follows. Let $Oe_1e_2e_3e_4$ be an orthonormal frame of $\mathbb{E}^4_2$, such that $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1$ and $\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = -1$. Let $m : x(u) = (x_1(u), x_2(u), x_3(u), x_4(u))$, $u \in J \subset \mathbb{R}$, be a smooth spacelike or timelike curve in $\mathbb{E}^4_2$, and θ, β constants. A general rotational surface of elliptic type can be defined by:

$$X(u,v) = (X_1(u,v), X_2(u,v), X_3(u,v), X_4(u,v)), \tag{1.20}$$

where

$$\begin{aligned} X_1(u,v) &= x_1(u)\cos\theta v - x_2(u)\sin\theta v; \\ X_2(u,v) &= x_1(u)\sin\theta v + x_2(u)\cos\theta v; \\ X_3(u,v) &= x_3(u)\cos\beta v - x_4(u)\sin\beta v; \\ X_4(u,v) &= x_3(u)\sin\beta v + x_4(u)\cos\beta v. \end{aligned} \tag{1.21}$$

In the present section we shall consider Lorentzian general rotational surfaces of elliptic type for which $\theta > 0, \beta > 0, x_2(u) = x_4(u) = 0$. In this case the meridian m lies in a two-dimensional plane.

Similarly to the general rotational surfaces of elliptic type we define general rotational surfaces of hyperbolic type in $\mathbb{E}^4_2$ as follows:

$$\begin{aligned}
X_1(u,v) &= x_1(u)\cosh\theta v - x_3(u)\sinh\theta v;\\
X_2(u,v) &= x_2(u)\cosh\beta v + x_4(u)\sinh\beta v;\\
X_3(u,v) &= x_1(u)\sinh\theta v + x_3(u)\cosh\theta v;\\
X_4(u,v) &= x_2(u)\sinh\beta v + x_4(u)\cosh\beta v.
\end{aligned} \tag{1.22}$$

We shall consider Lorentzian general rotational surfaces of hyperbolic type with plane meridians m for which $\theta > 0$, $\beta > 0$, $x_3(u) = x_4(u) = 0$.

In this section, we denote by $\widetilde{\nabla}$ the usual directional derivative in $\mathbb{E}^4_2$.

1.4.1 General rotational surfaces of elliptic type with plane meridians

Now we shall consider general rotational surfaces of elliptic type with plane meridians. Let $\mathcal{M}_1$ be the surface in $\mathbb{E}^4_2$ defined by

$$\mathcal{M}_1 : z(u,v) = (f(u)\cos\theta v, f(u)\sin\theta v, g(u)\cos\beta v, g(u)\sin\beta v), \tag{1.23}$$

where $u \in J \subset \mathbb{R}$, $v \in [0,2\pi)$, θ and β are positive constants and $f(u)$, $g(u)$ are smooth non-vanishing functions satisfying

$$\theta^2 f^2(u) - \beta^2 g^2(u) < 0 \text{ and } f'^2(u) - g'^2(u) > 0. \tag{1.24}$$

A tangent frame field in $T\mathcal{M}_1$ is determined by the vector fields

$$\begin{aligned}
z_u &= (f'(u)\cos\theta v, f'(u)\sin\theta v, g'(u)\cos\beta v, g'(u)\sin\beta v)\\
z_v &= (-\theta f(u)\sin\theta v, \theta f(u)\cos\theta v, -\beta g(u)\sin\beta v, \beta g(u)\cos\beta v)
\end{aligned}$$

The coefficients of the first fundamental form of $\mathcal{M}_1$ are expressed by

$$\begin{aligned}
E &= \langle z_u, z_u\rangle = f'^2(u) - g'^2(u) > 0;\\
F &= \langle z_u, z_v\rangle = 0;\\
G &= \langle z_v, z_v\rangle = \theta^2 f^2(u) - \beta^2 g^2(u) < 0.
\end{aligned}$$

So, $\mathcal{M}_1$ is a Lorentzian surface in $\mathbb{E}^4_2$.

We consider the following tangent frame fields

$$X = \frac{z_u}{\sqrt{E}}; \quad Y = \frac{z_v}{\sqrt{-G}},$$

which satisfy $\langle X,X\rangle = 1$, $\langle Y,Y\rangle = -1$ and $\langle X,Y\rangle = 0$. Let η_1 and η_2 be the normal vector fields defined by

$$\eta_1 = \frac{1}{\sqrt{-G}}(\beta g(u)\sin\theta v, -\beta g(u)\cos\theta v, \theta f(u)\sin\beta v, -\theta f(u)\cos\beta v);$$
$$\eta_2 = \frac{1}{\sqrt{E}}(g'(u)\cos\theta v, g'(u)\sin\theta v, f'(u)\cos\beta v, f'(u)\sin\beta v).$$

Note that $\langle\eta_1,\eta_1\rangle = 1$, $\langle\eta_2,\eta_2\rangle = -1$ and $\langle\eta_1,\eta_2\rangle = 0$.

From [4], we have the equations:

$$\begin{array}{ll}
\widetilde{\nabla}_X X = -\nu_1\eta_2; & \widetilde{\nabla}_X \eta_1 = \mu Y;\\
\widetilde{\nabla}_X Y = \mu\eta_1; & \widetilde{\nabla}_X \eta_2 = -\nu_1 X;\\
\widetilde{\nabla}_Y X = -\gamma_2 Y + \mu\eta_1; & \widetilde{\nabla}_Y \eta_1 = -\mu X + \beta_2\eta_2;\\
\widetilde{\nabla}_Y Y = -\gamma_2 X - \nu_2\eta_2. & \widetilde{\nabla}_Y \eta_2 = \nu_2 Y + \beta_2\eta_1.
\end{array} \tag{1.25}$$

Thus,

$$\alpha(X,X) = -\nu_1\eta_2; \quad \alpha(X,Y) = \mu\eta_1; \quad \alpha(Y,Y) = -\nu_2\eta_2. \tag{1.26}$$

The Gaussian curvature K, the curvature of the normal connection $K^\perp$, and the mean curvature vector field $\mathcal{H}$ of the general rotational surface $\mathcal{M}_1$ are expressed in terms of the geometric functions ν_1, ν_2 and μ as follows (see also Proposition 3.1 of [3]):

$$K = \nu_1\nu_2 + \mu^2; \quad K^\perp = -\mu(\nu_1+\nu_2); \quad \mathcal{H} = \frac{\nu_2-\nu_1}{2}\eta_2.$$

We use the following notations:

$$\begin{array}{ll}
\nu_1 = \dfrac{g'f''-f'g''}{(f'^2-g'^2)^{\frac{3}{2}}}; & \nu_2 = \dfrac{-(\theta^2 fg'-\beta^2 gf')}{\sqrt{f'^2-g'^2}(\beta^2g^2-\theta^2f^2)};\\
\mu = \dfrac{\theta\beta(fg'-gf')}{\sqrt{f'^2-g'^2}(\beta^2g^2-\theta^2f^2)}; & \gamma_2 = \dfrac{\theta^2 ff'-\beta^2 gg'}{\sqrt{f'^2-g'^2}(\beta^2g^2-\theta^2f^2)};\\
\beta_2 = \dfrac{\theta\beta(ff'-gg')}{\sqrt{f'^2-g'^2}(\beta^2g^2-\theta^2f^2)}. &
\end{array} \tag{1.27}$$

1.4.2 General rotational surfaces of hyperbolic type with plane meridians

Now we shall consider general rotational surfaces of hyperbolic type with plane meridians. Let $\mathcal{M}_2$ be the surface in $\mathbb{E}^4_2$ defined by

$$\mathcal{M}_2 : z(u,v) = (f(u)\cosh\theta v, g(u)\cosh\beta v, f(u)\sinh\theta v, g(u)\sinh\beta v), \tag{1.28}$$

where $u\in J\subset\mathbb{R}$, $v\in[0,2\pi)$, θ and β are positive constants and $f(u)$, $g(u)$ are smooth non-vanishing functions satisfying

$$\theta^2 f^2(u) + \beta^2 g^2(u) > 0 \text{ and } f'^2(u) + g'^2(u) > 0. \tag{1.29}$$

The tangent frame field $T_p\mathcal{M}_2$ is determined by the vector fields

$$z_u = (f'(u)\cosh\theta v, g'(u)\cosh\beta v, f'(u)\sinh\theta v, g'(u)\sinh\beta v)$$
$$z_v = (\theta f(u)\sinh\theta v, \beta g(u)\sinh\beta v, \theta f(u)\cosh\theta v, \beta g(u)\cosh\beta v)$$

The coefficients of the first fundamental form of $\mathcal{M}_1$ are expressed by

$$\begin{aligned}
E &= \langle z_u, z_u\rangle = f'^2(u) + g'^2(u) > 0;\\
F &= \langle z_u, z_v\rangle = 0;\\
G &= \langle z_v, z_v\rangle = -(\theta^2 f^2(u) + \beta^2 g^2(u)) < 0.
\end{aligned}$$

So, $\mathcal{M}_2$ is a Lorentzian surface in $\mathbb{E}^4_2$.

We consider the following tangent frame fields

$$X = \frac{z_u}{\sqrt{E}}; \quad Y = \frac{z_v}{\sqrt{-G}},$$

which satisfy $\langle X, X\rangle = 1$, $\langle Y, Y\rangle = -1$ and $\langle X, Y\rangle = 0$. Let η_1 and η_2 be the normal vector fields defined by

$$\begin{aligned}
\eta_1 &= \frac{1}{\sqrt{E}}(g'(u)\cosh\theta v, -f'(u)\cosh\beta v, g'(u)\sinh\theta v, -f'(u)\sinh\beta v);\\
\eta_2 &= \frac{1}{\sqrt{-G}}(\beta g(u)\sinh\theta v, -\theta f(u)\sinh\beta v, \beta g(u)\cosh\theta v, -\theta f(u)\cosh\beta v).
\end{aligned}$$

Note that $\langle\eta_1, \eta_1\rangle = 1$, $\langle\eta_2, \eta_2\rangle = -1$ and $\langle\eta_1, \eta_2\rangle = 0$.

From [4], we have the equations:

$$\begin{aligned}
\widetilde{\nabla}_X X &= \nu_1\eta_1; & \widetilde{\nabla}_X \eta_1 &= -\nu_1 X;\\
\widetilde{\nabla}_X Y &= -\mu\eta_2; & \widetilde{\nabla}_X \eta_2 &= \mu Y;\\
\widetilde{\nabla}_Y X &= -\gamma_2 Y - \mu\eta_2; & \widetilde{\nabla}_Y \eta_1 &= \nu_2 Y - \beta_2\eta_2;\\
\widetilde{\nabla}_Y Y &= -\gamma_2 X + \nu_2\eta_1; & \widetilde{\nabla}_Y \eta_2 &= -\mu X - \beta_2\eta_1.
\end{aligned} \tag{1.30}$$

Thus,

$$\alpha(X, X) = \nu_1\eta_1; \quad \alpha(X, Y) = -\mu\eta_2; \quad \alpha(Y, Y) = \nu_2\eta_1. \tag{1.31}$$

The Gaussian curvature K, the curvature of the normal connection $K^\perp$, and the mean curvature vector field $\mathcal{H}$ of the general rotational surface $\mathcal{M}_2$ are expressed in terms of the geometric functions ν_1, ν_2 and μ as follows:

$$K = -(\nu_1\nu_2 + \mu^2); \quad K^\perp = \mu(\nu_1 + \nu_2); \quad \mathcal{H} = \frac{\nu_1 - \nu_2}{2}\eta_1.$$

We use the following notations:

$$
\begin{aligned}
\nu_1 &= \frac{g'f''-f'g''}{(f'^2+g'^2)^{\frac{3}{2}}}; & \nu_2 &= \frac{\theta^2 fg'-\beta^2 gf'}{\sqrt{f'^2+g'^2}(\theta^2 f^2+\beta^2 g^2)};\\
\mu &= \frac{\theta\beta(fg'-gf')}{\sqrt{f'^2+g'^2}(\theta^2 f^2+\beta^2 g^2)}; & \gamma_2 &= -\frac{\theta^2 ff'+\beta^2 gg'}{\sqrt{f'^2+g'^2}(\theta^2 f^2+\beta^2 g^2)};\\
\beta_2 &= -\frac{\theta\beta(ff'+gg')}{\sqrt{f'^2+g'^2}(\theta^2 f^2+\beta^2 g^2)}.
\end{aligned}
\tag{1.32}
$$

1.5 Lorentzian hypersurfaces in Lorentzian space forms with type II Weingarten operator

Let V be a vector space over $\mathbb{R}$ endowed with a nondegenerate inner product $\langle\ ,\ \rangle$. We recall that an endomorphism A of $(V,\langle\ ,\ \rangle)$ is said to be self-adjoint if it satisfies $\langle AX,Y\rangle = \langle X,AY\rangle$, for all $X,Y\in V$.

It is well known that a self-adjoint endomorphism in a indefinite vector space $(V,\langle\ ,\ \rangle)$ does not need to be diagonalizable. In particular, we recall the following well known classification result for self-adjoint endomorphisms in Lorentzian vector spaces (see for instance [62] or [65]):

Lemma 1.1. *Let V be a n-dimensional real vector space with a Lorentzian inner product $\langle\ ,\ \rangle$. A self-adjoint endomorphism A on V can take only one of the following forms:*

I.

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix},$$

II.

$$\begin{pmatrix} a & 0 & & & \\ 1 & a & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{pmatrix},$$

III.

$$\begin{pmatrix} a & 0 & 0 & & & \\ 0 & a & 1 & & & \\ -1 & 0 & a & & & \\ & & & a_4 & & \\ & & & & \ddots & \\ & & & & & a_n \end{pmatrix},$$

IV.

$$\begin{pmatrix} a & b & & & \\ -b & a & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{pmatrix}, b\neq 0.$$

Forms I and IV correspond to an orthonormal basis $\{E_1,\dots,E_n\}$, where $\langle E_1,E_1\rangle=-1$, $\langle E_i,E_j\rangle=\delta_{ij}$ for $i,j\geq 2$ and $\langle E_1,E_i\rangle=0$ for $i\geq 2$. Forms II and III correspond to a pseudo-orthonormal basis $\{X,Y,E_3,\dots,E_n\}$, where $\langle X,X\rangle=\langle Y,Y\rangle=0$, $\langle X,Y\rangle=-1$, $\langle E_i,E_j\rangle=\delta_{ij}$ for $i,j\geq 3$ and $\langle X,E_i\rangle=\langle Y,E_i\rangle=0$ for $i\geq 3$. In cases I, II and III, all the eigenvalues are real, while in case IV there are two complex eigenvalues $a+bi$ and $a-bi$.

It follows that the Weingarten operator of a Lorentzian hypersurface in a pseudo-Riemannian space form $\mathbb{Q}_s^{n+1}(c)$, $s \in \{1,2\}$, must be of one of the forms in Lemma 1.1, at each point of the hypersurface. In particular, form II will be important in our study of pseudo-parallel Lorentzian hypersurfaces, especially when the minimal polynomial of the Weingarten operator is t^2 or $(t-a)^2$, $a \neq 0$, so we give in this section the fundamental examples of such hypersurfaces for the case where the ambient space is also a Lorentzian space, i.e., when $s = 1$. In order to this, we need to present first some concepts about null curves with an associated pseudo-orthonormal frame in $\mathbb{Q}_1^{n+1}(c)$.

A *null* (lightlike) curve in a Lorentzian space form is a curve $s \mapsto \gamma(s)$ all of whose tangent vectors are lightlike.

1.5.1 B-scrolls and generalized cylinders

Let $\gamma(s)$ a null curve in the Minkowski space $\mathbb{E}_1^3$ such that $T(s) = \dfrac{d\gamma(s)}{ds}$ and $T'(s) = \dfrac{dT(s)}{ds}$ are never colinear. Since $\langle T, T\rangle = 0$, it follows that $\langle T, T'\rangle = 0$ and thus, T' is everywhere spacelike. Let $C(s)$ be the unit spacelike vector field along γ satisfying $T'(s) = \kappa(s)C(s)$, with $\kappa(s) = ||T(s)||$. Finally, for each s, consider the 2-dimensional subspace $C(s)^\perp$ of $T_{\gamma(s)}\mathbb{E}_1^3$ orthogonal to $C(s)$. Since $C(s)^\perp$ is a Lorentzian plane, it contains a lightlike vector $B(s)$ such that $\langle T(s), B(s)\rangle = -1$. Let $\tau(s) = \langle B'(s), C(s)\rangle$, we obtain that $B'(s) = \tau(s)C(s)$ and $C'(s) = \tau(s)T(s) + \kappa(s)B(s)$. Therefore, $\{T(s), B(s), C(s)\}$ is a Cartan frame for $\gamma(s)$. If in addition $\tau(s) = 0$, for all s, that is, $B(s)$ is parallel, then the null curve $\gamma(s)$ with the Cartan frame $\{T(s), B(s), C(s)\}$ is called a *generalized cubic* in this particular case when the ambient space is $\mathbb{E}_1^3$. With these notations, we have the following immersion:

Definition 1.2. The immersion given by the parametrization $h(s,u) = \gamma(s) + uB(s)$ is called *B-scroll* associated to γ. Since $h_*\left(\frac{\partial}{\partial s}\right) = T(s) + u\tau(s)C(s)$ and $h_*\left(\frac{\partial}{\partial u}\right) = B(s)$, the metric of the B-scroll h is Lorentzian.

It was proved in Theorem 3.10 of [34] that a B-scroll h is flat if and only if $\tau(s) = 0$ for all s. In this case, h is an isometric immersion of $\mathbb{E}_1^2$ onto $\mathbb{E}_1^3$ and we also have that $C(s)$ is a unit normal vector field to h. Since $h_*\left(\frac{\partial}{\partial_s}\right) = T(s)$ and $h_*\left(\frac{\partial}{\partial_u}\right) = B(s)$, the Weingarten operator $A = A_{C(s)}$ is given in the frame $\left\{h_*\left(\frac{\partial}{\partial_s}\right), h_*\left(\frac{\partial}{\partial_u}\right)\right\}$ by

$$A = \begin{pmatrix} 0 & 0 \\ -\kappa(s) & 0 \end{pmatrix}.$$

It was proved in Theorem 9.7 of [34] that the B-scroll immersions with $\tau(s) = 0$ for all s are the only isometric immersions of $\mathbb{E}_1^2$ onto $\mathbb{E}_1^3$ such that the Weingarten operator satisfies identity $t^2 = 0$. Moreover, when $\kappa(s) \neq 0$, the minimal polynomial of the Weingarten operator is t^2, that is, A takes the form *II* in Lemma 1.1 with only 0 being eigenvalue.

It is known that the only parallel surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ are open sets of planes, spheres or right circular cylinders. On the other hand, the next result provides a classification of parallel Lorentzian surfaces in $\mathbb{E}^3_1$ given by Chen-Van der Veken in [18], where the essential examples, other than totally umbilical surfaces and cylinders, are the B-scroll immersions.

Proposition 1.3 (Chen-Van der Veken)**.** *A non-degenerate parallel Lorentzian surface in $\mathbb{E}^3_1$ is congruent to an open part of one of the following five types of surfaces:*

(i) a Lorentzian plane $\mathbb{E}^2_1$ in $\mathbb{E}^3_1$ given by $L=(u,v,0)$;

(ii) a totally umbilical de Sitter space $\mathbb{S}^2_1$ in $\mathbb{E}^3_1$ given by

$$L=b(\sinh u,\cosh u\cos v,\cosh u\sin v),\ \textit{with } b>0;$$

(iii) a flat cylinder $\mathbb{E}^1_1\times\mathbb{S}^1$ in $\mathbb{E}^3_1$ given by $L=(u,a\cos v,a\sin v)$ with $a>0$;

(iv) a flat cylinder $\mathbb{S}^1_1\times\mathbb{E}^1$ given by $L=(a\sinh u,a\cosh u,v)$ with $a>0$;

(v) a flat minimal Lorentzian surface in $\mathbb{E}^3_1$ given by

$$L=\left(\frac{1}{6}(u-v)^3+u,\frac{1}{6}(u-v)^3+v,\frac{1}{2}(u-v)^2\right).$$

Surfaces in *(v)* of Proposition 1.3 are precisely B-scroll immersions. After reparametrizing, we can write surface L in *(v)* as the B-scroll $h(s,u)=\gamma(s)+uB(s)$ with $B=\frac{1}{\sqrt{2}}(1,1,0)$ and $\gamma(s)=\frac{1}{\sqrt{2}}\left(\frac{2}{3}s^3+s,\frac{2}{3}s^3-s,\sqrt{2}s^2\right)$, in this case we have $\kappa(s)=2$ (see Figure 1.1).

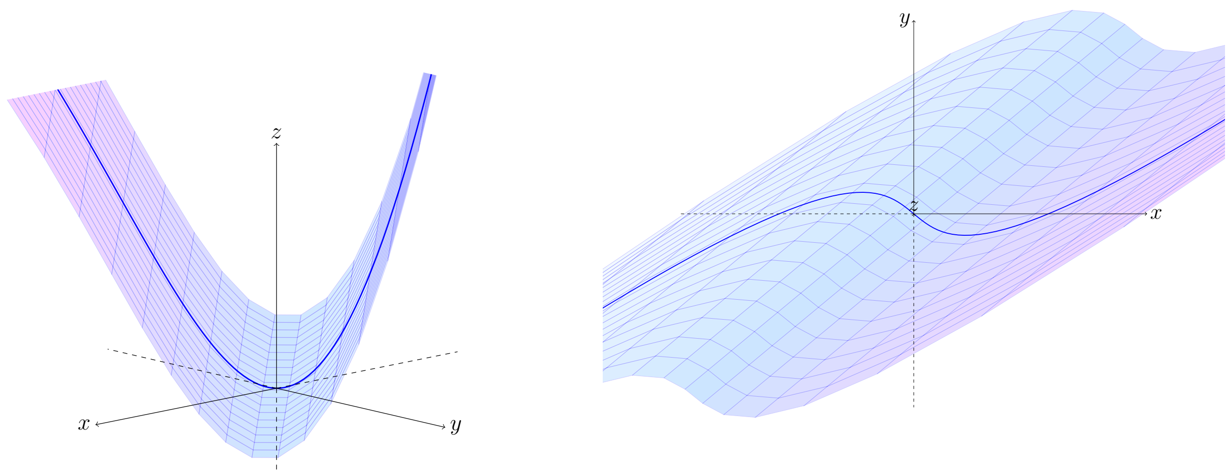

Figure 1.1: Two different views of the B-scroll immersion in *(v)* of Proposition 1.3

Now, to get higher dimensional examples of Lorentzian hypersurfaces whose Weingarten operator has minimal polynomial t^2, we simply consider the isometric immersion

$$f=h\times I_{n-2}:\mathbb{E}^2_1\times\mathbb{E}^{n-2}\to\mathbb{E}^3_1\times\mathbb{E}^{n-2},$$

where $h : \mathbb{E}_1^2 \to \mathbb{E}_1^3$ is a B-scroll immersion (see [34]). Such an isometric immersion is defined by

$$f(s,u,z) = \gamma(s) + uB(s) + \sum_{i=3}^{n} z_i Z(s),$$

where $z = (z_3, \ldots, z_n)$ and $\{T(s), B(s), Z_3(s), \ldots, Z_n(s), C(s)\}$ is a pseudo-orthonormal frame of vector fields of $\mathbb{E}_1^{n+1}$ along a null curve γ, such that $T(s), B(s)$ are lightlike vector fields with $\langle T(s), B(s)\rangle = -1$, $T(s) = \frac{d}{ds}\gamma(s)$ and $\frac{d}{ds}C(s) = \kappa(s)B(s)$. f is called a *generalized cylinder* (see [55]) and if $\kappa(s) \neq 0$, then the minimal polynomial of f is t^2. In fact, we have that $f_*\left(\frac{\partial}{\partial_s}\right) = T(s) + u\frac{d}{ds}B(s) + \sum_{i=3}^n z_i \frac{d}{ds} Z_i(s)$, $f_*\left(\frac{\partial}{\partial_u}\right) = B(s)$ and $f_*\left(\frac{\partial}{\partial_{z_i}}\right) = Z_i(s)$. We note that $C(s)$ is a unit spacelike normal vector field to f, since using that $\frac{d}{ds}C(s) = \kappa(s)B(s)$ we obtain $\langle \frac{d}{ds}B(s), C(s)\rangle = \langle \frac{d}{ds}Z_i(s), C(s)\rangle = 0$. Then, for the Weingarten operator $A = A_{C(s)}$ of f, using the Weingarten formula, we have that $Af_*\left(\frac{\partial}{\partial_s}\right) = -\frac{\partial}{\partial_s}C(s) = -k(s)B(s)$, $Af_*\left(\frac{\partial}{\partial_u}\right) = -\frac{\partial}{\partial_u}C(s) = 0$ and $Af_*\left(\frac{\partial}{\partial_{z_i}}\right) = -\frac{\partial}{\partial_{z_i}}C(s) = 0$, for $3 \leq i \leq n$. Therefore, A takes the form

$$A = \begin{pmatrix} 0 & 0 \\ -\kappa(s) & 0 \end{pmatrix} \oplus 0_{n-2},$$

expressed in the frame $\left\{ f_*\left(\frac{\partial}{\partial_s}\right), f_*\left(\frac{\partial}{\partial_u}\right), f_*\left(\frac{\partial}{\partial_{z_3}}\right), \ldots, f_*\left(\frac{\partial}{\partial_{z_n}}\right) \right\}$.

Now, we will show analogous examples to the generalized cylinders for $c \neq 0$.

Example 1.4. For $c < 0$, let $\gamma(s)$ be a null curve in $\mathbb{H}_1^{n+1}(c) \subset \mathbb{E}_2^{n+2}$ with a pseudo-orthonormal frame $\{T(s), B(s), Z_3(s), \ldots, Z_n(s), C(s)\}$ of tangent vector fields of $\mathbb{H}_1^{n+1}(c)$ along γ, such that $T(s), B(s)$ are lightlike vector fields with $\langle T(s), B(s)\rangle = -1$, $\frac{d}{ds}\gamma(s) = T(s)$ and $\frac{d}{ds}C(s) = \kappa(s)B(s)$, where $\frac{d}{ds}$ denote the ordinary derivation in $\mathbb{E}_2^{n+1}$. As in [80], we consider the Lorentzian hypersurface $f : U \to \mathbb{H}_1^{n+1}(c) \subset \mathbb{E}_2^{n+2}$, with U an open neighborhood of 0 in $\mathbb{R}^n$, parameterized by

$$f(s,u,z) = \sqrt{1 - c\sum_{i=3}^{n} z_i^2}\,\gamma(s) + uB(s) + \sum_{i=3}^{n} z_i Z_i(s), \tag{1.33}$$

where $z = (z_3, \ldots, z_n)$. If $\kappa(s) \neq 0$, it follows from [80] that the minimal polynomial of f is t^2. Indeed, since $\frac{d}{ds}C(s) = \kappa(s)B(s)$, it follows that the vector field $f_*\left(\frac{\partial}{\partial_s}\right) = \sqrt{1 - c\sum_{i=3}^n z_i^2}\,T(s) + u\frac{d}{ds}B(s) + \sum_{i=3}^n z_i \frac{d}{ds} Z_i(s)$ does not contain terms with $C(s)$, as $f_*\left(\frac{\partial}{\partial_u}\right) = B(s)$ and $f_*\left(\frac{\partial}{\partial_{z_i}}\right) = \frac{-cz_i}{\sqrt{1 - c\sum_i z_i^2}}\gamma(s) + Z_i(s)$ as well. This means that $C(s)$ is a unit spacelike normal vector field to f in $\mathbb{H}_1^{n+1}(c)$. Then, for the Weingarten operator $A = A_{C(s)}$ of f, we have that $Af_*\left(\frac{\partial}{\partial_s}\right) = -\frac{\partial}{\partial_s}C(s) = -k(s)B(s)$, $Af_*\left(\frac{\partial}{\partial_u}\right) = -\frac{\partial}{\partial_u}C(s) = 0$ and $Af_*\left(\frac{\partial}{\partial_{z_i}}\right) = -\frac{\partial}{\partial_{z_i}}C(s) = 0$, for $3 \leq i \leq n$. Therefore, A takes the form

$$A = \begin{pmatrix} 0 & 0 \\ -\kappa(s) & 0 \end{pmatrix} \oplus 0_{n-2},$$

expressed in the frame $\left\{ f_*\left(\frac{\partial}{\partial_s}\right), f_*\left(\frac{\partial}{\partial_u}\right), f_*\left(\frac{\partial}{\partial_{z_3}}\right), \ldots, f_*\left(\frac{\partial}{\partial_{z_n}}\right) \right\}$.

Example 1.5. For $c > 0$, accordingly with [42], we consider this time a light cone curve in $\mathbb{E}_1^{n+2}$, that is, a parameterized curve $\rho(t)$, with $t \in (a,b)$, satisfying $\langle \rho(t), \rho(t) \rangle = 0$, for all t. If $\rho(t)$ is never colinear with $\frac{d}{dt}\rho(t)$, then $\rho(t)$ is a spacelike curve, since $\langle \frac{d}{dt}\rho(t), \frac{d}{dt}\rho(t) \rangle > 0$. We can assume that t is the arc lenght parameter of $\rho(t)$. As in Lemma 6.1 of [42], we can associate to ρ a pseudo-orthonormal frame $\{E_1(t), E_2(t) = \rho(t), E_3(t), \dots, E_{n+2}(t)\}$ of vector fields of $\mathbb{E}_1^{n+2}$ along ρ, such that $E_1(t), E_2(t)$ are lightlike vector fields with $\langle E_1(t), E_2(t) \rangle = -1$, and

$$\begin{aligned} &E_1' = C_1 E_3 + C_2 E_4 + \cdots + C_n E_{n+2}, \\ &E_2' = E_3, \quad E_3' = E_1 + C_1 E_2, \quad E_j' = C_{j-2} E_2, \text{ for } 4 \leq j \leq n+2, \end{aligned} \tag{1.34}$$

where $C_1, \dots, C_n$ are functions in the variable t and $E_i' = \frac{d}{dt}E_i$.

Let $\mathbb{S}^{n-2}(c) \subset \mathbb{R}^{n-1}$ the $(n-2)$-dimensional Euclidean sphere of constant curvature c and denote by $\mathbb{S}_+^{n-2}(c) = \{y = (y_3, \dots, y_{n+1}) \in \mathbb{S}^{n-2}(c) : y_3 > 0\}$. We consider the Lorentzian hypersurface $f : \Omega = (a,b) \times \mathbb{R} \times \mathbb{S}_+^{n-2} \to \mathbb{S}_1^{n+1}(c) \subset \mathbb{E}_1^{n+2}$, parameterized by

$$\begin{aligned} f(t,u,y) &= y_3 E_3(t) + u E_2(t) + \sum_{i=4}^{n+1} y_i E_i(t) \\ &= \sqrt{\frac{1}{c} - \sum_{i=4}^{n+1} y_i^2}\, E_3(t) + u E_2(t) + \sum_{i=4}^{n+1} y_i E_i(t). \end{aligned} \tag{1.35}$$

From Theorem 6.4 of [42] and its proof, we have that $M_1^n = f(\Omega)$ is a Lorentzian hypersurface of $\mathbb{S}_1^{n+1}(c)$, $E_{n+2}(t)$ is a unit spacelike normal vector field to f and the minimal polynomial of the Weingarten operator A of f is t^2, that is, A takes the form

$$A = \begin{pmatrix} 0 & 0 \\ -C_n(t) & 0 \end{pmatrix} \oplus 0_{n-2}.$$

Note that if $C_1 = 0$, then $\frac{1}{\sqrt{c}} E_3$ is a null curve in $\mathbb{S}_1^{n+1}(c)$ and (1.35) takes a form as (1.33).

1.5.2 Generalized umbilical hypersurfaces of degree 2

The notion of generalized umbilical hypersurfaces in Lorentz-Minkowski space $\mathbb{E}_1^{n+1}$ has been introduced by Magid in [55], for some kind of hypersurfaces satisfying that all principal curvatures are equal. Then, analogous hypersurfaces were given by Alías-Ferrández-Lucas in [5] for non-flat Lorentzian space forms (see also [80], for the case $c < 0$).

For $c \neq 0$, let $\gamma : J \subset \mathbb{R} \to \mathbb{Q}_1^{n+1}(c) \subset \mathbb{E}_\sigma^{n+2}$, $\sigma = 1 + \frac{c-|c|}{2c}$, be a null curve with a pseudo-orthonormal frame $\{T(s), B(s), Z_3(s), \dots, Z_n(s), C(s)\}$ of tangent vector fields of $\mathbb{Q}_1^{n+1}(c)$ along γ, such that $T(s), B(s)$ are lightlike vector fields with $\langle T(s), B(s) \rangle = -1$, $\frac{d}{ds}\gamma(s) = T(s)$ and $\frac{d}{ds}C(s) =$

$\tau T(s)+\kappa(s)B(s)$, where $\frac{d}{ds}$ denote the ordinary derivation in $\mathbb{E}_\sigma^{n+2}$, with $\kappa(s)\neq 0$ and τ is a nonzero constant. If $c+\tau^2\neq 0$, then the map $f: J\times\mathbb{R}\times\mathbb{R}^{n-2}\to\mathbb{Q}_1^{n+1}(c)$ defined by

$$f(s,u,z)=\left(\frac{\tau^2+cg(z)}{c+\tau^2}\right)\gamma(s)+uB(s)+\sum_{i=3}^{n} z_i Z_i(s)-\frac{\tau(1-g(z))}{c+\tau^2}C(s), \tag{1.36}$$

where $z=(z_2,\dots,z_n)$ and $g(z)=\sqrt{1-(c+\tau^2)\sum_{i=3}^{n} z_i^2}$, parameterizes, in a neighborhood of the origin, a Lorentzian hypersurface M_1^n of $\mathbb{Q}_1^{n+1}(c)$.

A unit normal vector field to M_1^n in $\mathbb{Q}_1^{n+1}(c)$ is given by

$$\eta(s,u,z)=-\frac{c\tau(1-g(z))}{c+\tau^2}\gamma(s)+u\tau B(s)+\tau\sum_{i=3} z_i Z_i(s)+\frac{c+\tau^2 g(z)}{c+\tau^2}C(s).$$

If $\tau=\pm\sqrt{-c}$, with $c<0$, then we define the map $f: J\times\mathbb{R}\times\mathbb{R}^{n-2}\to\mathbb{H}_1^{n+1}(c)$ by

$$f(s,u,z)=\left(1-\frac{c}{2}\sum_{i=3}^{n} z_i^2\right)\gamma(s)+uB(s)+\sum_{i=3}^{n} z_i Z_i(s)-\frac{\tau}{2}\sum_{i=3}^{n} z_i^2 C(s), \tag{1.37}$$

which parameterizes, in a neighborhood of the origin, a Lorentzian hypersurface M_1^n in $\mathbb{H}_1^{n+1}(c)$. In this case, a unit normal vector field to M_1^n in $\mathbb{H}_1^{n+1}(c)$ is given by

$$\eta(s,u,z)=-\frac{\tau c}{2}\left(\sum_{i=3}^{n} z_i^2\right)\gamma(s)+u\tau B(s)+\tau\sum_{i=3}^{n} z_i Z_i(s)+\left(1+\frac{c}{2}\sum_{i=3}^{n} z_i^2\right)C(s).$$

We have from [5] that the Weingarten operator $A=A_\eta$ of M_1^n, given by (1.36) or (1.37), satisfies the equation $A^2=-2\tau A-\tau^2 I_n$. Indeed, the minimal polynomial of M_1^n is $(t+\tau)^2$ and

$$\begin{aligned}
Af_*\left(\frac{\partial}{\partial_s}\right)&=-\frac{\partial}{\partial_s}\eta=-\tau f_*\left(\frac{\partial}{\partial_s}\right)-k(s)B(s),\\
Af_*\left(\frac{\partial}{\partial_u}\right)&=-\frac{\partial}{\partial_u}\eta=-\tau f_*\left(\frac{\partial}{\partial_u}\right),\\
Af_*\left(\frac{\partial}{\partial_{z_i}}\right)&=-\frac{\partial}{\partial_{z_i}}\eta=-\tau f_*\left(\frac{\partial}{\partial_{z_i}}\right),
\end{aligned}$$

for $3\leq i\leq n$. From Proposition 4.2, we conclude that the hypersurface M_1^n is ψ-pseudo-parallel with $\psi=c+\tau^2$. M_1^n is called a *generalized umbilical hypersurface of degree 2*.

For $c=0$, we consider M_1^n as the Lorentzian hypersurface $f: J\times\mathbb{R}\times\mathbb{R}^{n-2}\to\mathbb{E}_1^{n+1}$, parameterized by

$$f(s,u,z)=\gamma(s)+uB(s)+\sum_{i=3}^{n} z_i Z_i(s)-\frac{1}{\tau}\left(1-\sqrt{1-\tau^2\sum_{i=3}^{n} z_i^2}\right)C(s), \tag{1.38}$$

where γ is a null curve in $\mathbb{E}_1^{n+1}$ with a pseudo-orthonormal frame $\{T(s),B(s),Z_3(s),\dots,Z_n(s),C(s)\}$ of vector fields of $\mathbb{E}_1^{n+1}$ along γ, such that $T(s),B(s)$ are lightlike vector fields with $\langle T(s),B(s)\rangle=-1$,

$\frac{d}{ds}\gamma(s) = T(s)$ and $\frac{d}{ds}C(s) = \tau T(s) + \kappa(s)B(s)$, where $\frac{d}{ds}$ denote the ordinary derivation in $\mathbb{E}_1^{n+1}$, $\kappa(s) \neq 0$ and τ is a nonzero constant.

A unit normal vector field to M_1^n in $\mathbb{E}_1^{n+1}$ is given by

$$\eta(s,u,z) = u\tau B(s) + \sqrt{1-\tau^2\sum_{i=3}^{n} z_i^2}C(s) + \tau\sum_{i=3}^{n} z_i Z_i(s).$$

We have from [55], that the minimal polynomial of the Weingarten operator of the hypersurface M_1^n given by (1.38), is $(t+\tau)^2$ and M_1^n is a generalized umbilical hypersurface of $\mathbb{E}_1^{n+1}$. Again from Proposition 4.2, we conclude that M_1^n is ψ-pseudo-parallel with $\psi = c+\tau^2$.

Observation 1.6. Note that if X,Y are lightlike vectors with $\langle X,Y\rangle = -1$, then $rX, \frac{1}{r}Y$ satisfy the same conditions for any real number $r \neq 0$. From here, we can see that a pseudo-orthonomal frame associated to a null curve γ is not uniquely determined (after reparametrizing γ), as well as the function $\kappa(s)$ if $\kappa(s) \neq 0$. Also, function $C_n(t)$ in Example 1.4 is not uniquely determined. Thus, if the Weingaten operator A of a Lorentzian hypersurface takes the form II in Lemma 4.6, as it may happen for the examples above of B-scrolls, generalized cylinders and generalized umbilical hypersurfaces of degree 2, then the component under the diagonal of A can be changed to be any nonzero real number.

1.5.3 Lorentzian hypersurfaces with constant curvature in $\mathbb{Q}_1^{n+1}(c)$

Complete Lorentzian hypersurfaces of constant curvature in Lorentz-Minkowski space $\mathbb{E}_1^{n+1}$ and in Lorentzian spheres $\mathbb{S}_1^{n+1}(c)$ are described in [6], as following

Proposition 1.7 (Al-shehri-Guediri)**.** *In $\mathbb{S}_1^{n+1}(c)$, $n \geq 3$, a connected complete Lorentzian hypersurface of constant curvature is a small or a great hypersphere.*

Proposition 1.8 (Al-shehri-Guediri)**.** *Let M_1^n, $n \geq 3$, be a n-dimensional connected complete Lorentzian hypersurface of constant curvature $\hat{c}$ in $\mathbb{E}_1^{n+1}$. Then, necessarily $\hat{c} \geq 0$ and we have:*

(a) If $\hat{c} = 0$, M_1^n is isometric to $\mathbb{R}_1^n$ or to one of the following products:

(i) $\mathbb{E}_1^{n-2} \times g(\mathbb{E}^2)$, where $g(\mathbb{E}^2)$ is a Euclidean cylinder in a subspace $\mathbb{E}^3$ of $\mathbb{E}_1^{n+1}$ orthogonal to $\mathbb{E}_1^{n-2}$.

(ii) $\mathbb{E}^{n-2} \times g(\mathbb{E}_1^2)$, where $g(\mathbb{E}_1^2)$ is a Lorentzian cylinder or a B-scroll in a subspace $\mathbb{E}_1^3$ of $\mathbb{E}_1^{n+1}$ orthogonal to $\mathbb{E}_1^{n-2}$.

1. If $\hat{c} > 0$, M_1^n is isometric to $\mathbb{S}_1^n(\hat{c})$.

For $c < 0$, situation is more complicate due to the absence of any precise result which classifies isometric immersions of $\mathbb{H}_1^n(c)$ into $\mathbb{H}_1^{n+1}(c)$ (see Remark 3.4 in [6]), so in this case we just can say that

Proposition 1.9 (Al-shehri-Guediri)**.** *Let M_1^n, $n \geq 3$, be a n-dimensional connected complete Lorentzian hypersurface of constant curvature $\hat{c}$ in $\mathbb{H}_1^{n+1}(c)$. Then, either*

(a) $\hat{c} = c$ and the number type $k(x) \leq 1$, for all $x \in M_1^n$, or

(b) $\hat{c} > c$ and M_1^n is totally umbilical, with Weingarten operator $A_x = \sqrt{\hat{c} - c} I_n$, for all $x \in M_1^n$.

For a classification of totally umbilical immersions in $\mathbb{H}_1^{n+1}(c)$ we refer to [1] (see also [67]).

1.6 Some classification results for isoparametric hypersurfaces in $\mathbb{Q}_1^{n+1}(c)$

Let M_t^n be a pseudo-Riemannian hypersurface in $\mathbb{Q}_s^{n+1}(c)$ with a (local) unit normal vector field η and consider the following possible conditions on M_t^n:

(A) All the principal curvatures with their algebraic multiplicities are constant on M_t^n.

(B) All the parallel hypersurfaces M_r defined, at least locally and for sufficiently small $r \in \mathbb{R}$, by $\Psi_r : M_t^n \to \mathbb{Q}_1^{n+1}(c)$, $p \to \exp_p(r\eta_p)$, have constant mean curvature.

(C) The minimal polynomial of the Weingarten operator $A = A_\eta$ is constant on M_t^n.

Hypersurfaces in Riemannian space forms satisfying any of these conditions were firstly studied by Cartan and Münzner. Later, Nomizu in 1981 extended the study of these conditions to spacelike hypersurfaces in Lorentzian space forms (see [60]). Hahn in 1984, with his work in [37], proved that conditions (A) and (B) above are equivalents in the case of pseudo-Riemannian hypersurfaces of $\mathbb{Q}_s^{n+1}(c)$, so the next definition makes sense.

Definition 1.10. A pseudo-Riemannian hypersurface in $\mathbb{Q}_s^{n+1}(c)$ satisfying condition (A) or (B) is called *isoparametric*.

Moreover, Hahn proved that conditions (A) and (B) are more general than condition (C), by showing the existence of isoparametric pseudo-Riemannian hypersurfaces in pseudo-Riemannian space forms which do not satisfy (C). Classification of isoparametric hypersurfaces in pseudo-Riemannian space forms is an open problem in general. For the case of isoparametric Lorentzian hypersurfaces in Lorentzian space forms, there are significant advances made by Magid, Abe-Koike, Xiao and Li in [55], [1], [80] and [42], respectively, but the classification is not complete.

In 1985, Magid in [55] classified isoparametric Lorentzian hypersurfaces in Lorentz-Minkowski space satisfying condition (C) (in fact, Magid used (C) as definition and so did Xiao and Li). Magid proved that these particular isoparametric Lorentzian hypersurfaces in $\mathbb{E}_1^{n+1}$ are cylinders or umbilical hypersurfaces when the Weingarten operator is diagonalizable or they are some kind of hypersurfaces with properties close to cylinders and umbilical hypersurfaces when the Weingarten operator is non-diagonalizable and has constant minimal polynomial. In order to obtain this classification, Magid proved that the Weingarten operator of any Lorentzian hypersurface with constant minimal polynomial in $\mathbb{E}_1^{n+1}$ cannot have complex eigenvalues (Theorem 4.10 of [55]) and at most one real eigenvalue is nonzero (Corollary 2.7 of [55]).

In particular, in Theorem 4.5 of [55], it was proved the next result which will be useful in our study of pseudo-parallel Lorentzian hypersurfaces

Proposition 1.11 (Magid)**.** *If the Weingarten operator of a Lorentzian hypersurface M_1^n of $\mathbb{E}_1^{n+1}$ has $(t-a)^2$, with $a \neq 0$ constant, as its minimal polynomial, then in a neighborhood of any point, M_1^n is a generalized umbilical hypersurface of degree 2 as in (1.38), with $a = -\tau$.*

For $c \neq 0$, an analogous result was proved by Alías-Ferrández-Lucas in Theorem 5.5 of [5], which we state below

Proposition 1.12 (Alías-Ferrández-Lucas)**.** *Let M_1^n be a Lorentzian hypersurface of $\mathbb{Q}_1^{n+1}(c)$ and let $(t-a)^2$, with $a \neq 0$ constant, be the minimal polynomial of its Weingarten operator. Then, in a neighborhood of any point, M_1^n is a generalized umbilical hypersurface of degree 2 as in (1.36) and (1.37), with $a = -\tau$.*

Result in Theorem 1.12 was also showed by Xiao in Theorem 4.1 and Theorem 4.2 of [80], for the case $c < 0$.

For an isometric immersion $f : \mathbb{E}_1^n \to \mathbb{E}_1^{n+1}$, Graves in [34] proved that the set W of non-geodesic points of f is open and it is the union of a family of parallel $(n-1)$-hyperplanes, where each hyperplane is a leaf of the relative nullity foliation. Thus, a metric is degenerate in all these hyperplanes or it is non-degenerate in all of them. Since the points $x \in W$ with degenerate relative nullity are exactly the points where the minimal polynomial is t^2, Graves gave in Theorem 9.8 the next classification result:

Proposition 1.13 (Graves)**.** *Up to a proper motion of $\mathbb{E}_1^{n+1}$, the isometric immersions $f : \mathbb{E}_1^n \to \mathbb{E}_1^{n+1}$ with degenerate relative nullity have the form*

$$h \times I_{n-2} : \mathbb{E}_1^2 \times \mathbb{E}^{n-2} \to \mathbb{E}_1^3 \times \mathbb{E}^{n-2},$$

where the factors in each product are orthogonal and $h : \mathbb{E}_1^2 \to \mathbb{E}_1^3$ is a B-scroll immersion as in Definition 1.2.

Thus, the only isometric immersions from $\mathbb{E}_1^n$ into $\mathbb{E}_1^{n+1}$ whose minimal polynomial of the Weingarten operator is t^2 are the generalized cylinders over a B-scroll immersion, which are also defined by (1.33), associated to a generalized cubic γ with $\kappa(s) \neq 0$, for all s. We note, from the proof of the degenerate case in [34], that if the generalized cubic γ has $\kappa(s) = 0$ for some s, then all points of the hyperplane passing through that point of γ and contained in the generalized cylinder $h \times Id$ are geodesic points.

Observation 1.14. Analogously, it is known that the only Lorentzian hypersurfaces in $\mathbb{Q}_1^{n+1}(c)$, $c \neq 0$, with t^2 as the minimal polynomial of the Weingarten operator are the parameterized hypersurfaces given by 1.33 and 1.35. A proof of this can be found in Theorem 4.2 and case (2) in Theorem 4.1 of [80], for $c < 0$, and in Theorem 6.8 of [42] for the case $c > 0$.

For a classification of isoparametric hypersurfaces in $\mathbb{Q}_1^{n+1}(c)$ with diagonalizable Weingarten operator having only one or two distinct principal curvatures, we refer to [1]. There, Abe-Koike-Yamaguchi in Theorem 5.1 proved that, locally, any such hypersurface is either a totally umbilical hypersurface or it is isometric to the product of two real space forms of constant curvature, according to the models presented there. In the same reference, a global classification result was obtained in Theorem 5.2, but it is a partial result because some models are not simply connected.

CHAPTER 2

Pseudo-parallel Lorentzian surfaces in pseudo-Riemannian space forms

In this chapter, we begin our study of pseudo-parallel Lorentzian surfaces in pseudo-Riemannian space forms $\mathbb{Q}^m_s(c)$. In the first section we present some basic results about Lorentizan surfaces involving their second fundamental form, their Gaussian curvature and their normal curvature tensor, as well as the pseudo-parallelism condition. In the second section we characterize λ-isotropic Lorentzian surfaces by providing several equivalent conditions to λ-isotropy. In particular, we study the hyperbola of normal curvature of λ-isotropic Lorentzian surfaces. In the third section, we prove our principal theorem of characterization of pseudo-parallel Lorentzian surfaces with non flat normal bundle in $\mathbb{Q}^m_s(c)$ and in particular the non-existence of pseudo-parallel Lorentzian surfaces with non flat normal bundle in Lorentzian space forms.

2.1 Lorentzian surfaces in $\mathbb{Q}^m_s(c)$ and the pseudo-parallelism condition

In this section we prove some auxiliary results concerning pseudo-parallel Lorentzian surfaces in $\mathbb{Q}^m_s(c)$, with $1 \leq s \leq m-1$, that will be useful later. Specifically, we determine whether the normal bundle of a surface is non-flat depending on whether a particular subspace of the normal space at each point on the surface, which is generated by two normal vectors related to the second fundamental form α, is two-dimensional or not. Also, working with an orthonormal frame of the tangent space to the Lorenzian surface, we reduce the pseudo-parallelism condition to few equations involving the normal curvature tensor, the Gaussian curvature, the second fundamental form α and the pseudo-parallelism function ψ. We explore some differences that these equations imply with respect to the Riemannian case, due to the possible existence of lightlike vectors in the normal space at each point of the surface when the metric of the ambient space is just non-degenerate. Finally, we study how the pseudo-parallelism condition behaves with respect to the composition of immersions, which can be used to obtain examples of pseudo-parallel immersions.

Let $f : M_1^2 \to \mathbb{Q}_s^m(c)$ be an isometric immersion. Let $\{e_1, e_2\}$ be an orthonormal local frame for M_1^2, where $\varepsilon_1 = \langle e_1, e_1\rangle = 1$, $\varepsilon_2 = \langle e_2, e_2\rangle = -1$ and $\langle e_1, e_2\rangle = 0$.

It follows from the Ricci equation (1.17) that

$$R^{\perp}(e_1, e_2)\xi \in \operatorname{span}\{\alpha(X,Y) : X, Y \in TM\}, \text{ for all } \xi \in N_f M_1^2.$$

For the mean curvature vector field, we have

$$\mathcal{H} = \frac{1}{2}\operatorname{trace}(\alpha) = \frac{1}{2}(\alpha_{11} - \alpha_{22}).$$

Lemma 2.1. *Let $f : M_1^2 \to \mathbb{Q}_s^m(c)$ be an isometric immersion. We have the following equations:*

$$K = c - \langle \alpha_{11}, \alpha_{22}\rangle + \langle \alpha_{12}, \alpha_{12}\rangle, \tag{2.1}$$

$$R^{\perp}(e_1, e_2)\xi = ((\alpha_{11} + \alpha_{22}) \wedge \alpha_{12})\xi, \tag{2.2}$$

for all $\xi \in N_f M$. Moreover, $\alpha_{11} + \alpha_{22}$ and α_{12} are linearly dependent if and only if f has vanishing normal curvature.

Proof. For the Gaussian curvature K of M_1^2, we have

$$K = \frac{\langle R(e_1, e_2)e_2, e_1\rangle}{\langle e_1, e_1\rangle\langle e_2, e_2\rangle - \langle e_1, e_2\rangle^2} = -\langle R(e_1, e_2)e_2, e_1\rangle.$$

It follows from Gauss equation (1.15) that

$$\begin{aligned} K &= -c\langle (e_1 \wedge e_2)e_2, e_1\rangle - \langle \alpha_{11}, \alpha_{22}\rangle + \langle \alpha_{12}, \alpha_{12}\rangle \\ &= -c\langle e_2, e_2\rangle\langle e_1, e_1\rangle + c\langle e_1, e_2\rangle\langle e_2, e_1\rangle - \langle \alpha_{11}, \alpha_{22}\rangle + \langle \alpha_{12}, \alpha_{12}\rangle \\ &= c - \langle \alpha_{11}, \alpha_{22}\rangle + \langle \alpha_{12}, \alpha_{12}\rangle. \end{aligned}$$

Next, using Ricci equation (1.17) we get

$$\begin{aligned} R^{\perp}(e_1, e_2)\xi &= \alpha(e_1, A_\xi e_2) - \alpha(A_\xi e_1, e_2) \\ &= \sum_k \alpha(e_1, \varepsilon_k\langle A_\xi e_2, e_k\rangle e_k) - \sum_k \alpha(\varepsilon_k\langle A_\xi e_1, e_k\rangle e_k, e_2) \\ &= \sum_k \varepsilon_k\langle \alpha(e_2, e_k), \xi\rangle \alpha(e_1, e_k) - \sum_k \varepsilon_k\langle \alpha(e_1, e_k), \xi\rangle \alpha(e_k, e_2) \\ &= \langle \alpha_{12}, \xi\rangle(\alpha_{11} + \alpha_{22}) - (\langle \alpha_{11}, \xi\rangle + \langle \alpha_{22}, \xi\rangle)\alpha_{12} \\ &= \langle \alpha_{12}, \xi\rangle(\alpha_{11} + \alpha_{22}) - \langle \alpha_{11} + \alpha_{22}, \xi\rangle\alpha_{12} \\ &= ((\alpha_{11} + \alpha_{22}) \wedge \alpha_{12})\xi, \end{aligned}$$

for all $\xi \in N_f M$.

Now, if $\alpha_{11}+\alpha_{22}$ and α_{12} are linearly dependent, then either $\alpha_{12}=0$ or for some $\mu\in\mathbb{R}$ we have

$$R^{\perp}(e_1,e_2)\xi=((\alpha_{11}+\alpha_{22})\wedge\alpha_{12})\xi=(\mu\alpha_{12}\wedge\alpha_{12})\xi=0.$$

Conversely, if $R^{\perp}(e_1,e_2)\xi=0$ for all $\xi\in N_fM$, it follows from (2.2) that

$$\langle\alpha_{12},\xi\rangle(\alpha_{11}+\alpha_{22})-\langle\alpha_{11}+\alpha_{22},\xi\rangle\alpha_{12}=0.$$

If there exists $\xi\in N_fM$ such that $\langle\alpha_{12},\xi\rangle\neq 0$ or $\langle\alpha_{11}+\alpha_{22},\xi\rangle\neq 0$, then $\alpha_{11}+\alpha_{22}$ and α_{12} are linearly dependent. If $\langle\alpha_{12},\xi\rangle=\langle\alpha_{11}+\alpha_{22},\xi\rangle=0$, for all $\xi\in N_fM$, from non-degeneracy we have that $\alpha_{12}=\alpha_{11}+\alpha_{22}=0$. Therefore, $\alpha_{11}+\alpha_{22}$ and α_{12} are linearly dependent. □

Lemma 2.2. *Let $f:M_1^2\to\mathbb{Q}_s^m(c)$ be an isometric immersion. f is ψ-pseudo-parallel if and only if the following equations are satisfied:*

$$R^{\perp}(e_1,e_2)\alpha_{11}=R^{\perp}(e_1,e_2)\alpha_{22}=2(\psi-K)\alpha_{12},\tag{2.3}$$

$$R^{\perp}(e_1,e_2)\alpha_{12}=(\psi-K)(\alpha_{11}+\alpha_{22}).\tag{2.4}$$

Proof. In fact, equation (1.13) is equivalent to

$$\begin{aligned}R^{\perp}(X,Y)\alpha(Z,W)=\alpha(R(X,Y)Z,W)+\alpha(Z,R(X,Y)W)\\ -\psi\alpha((X\wedge Y)Z,W)-\psi\alpha(Z,(X\wedge Y)W).\end{aligned}$$

Therefore, for the orthonormal frame $\{e_1,e_2\}$, we have

$$\begin{aligned}R^{\perp}(e_1,e_2)\alpha_{12}&=\alpha(R(e_1,e_2)e_1,e_2)+\alpha(e_1,R(e_1,e_2)e_2)\\ &\quad-\psi\alpha((e_1\wedge e_2)e_1,e_2)-\psi\alpha(e_1,(e_1\wedge e_2)e_2)\\ &=\varepsilon_2\langle R(e_1,e_2)e_1,e_2\rangle\alpha(e_2,e_2)+\varepsilon_1\langle R(e_1,e_2)e_2,e_1\rangle\alpha(e_1,e_1)\\ &\quad-\psi\langle e_2,e_1\rangle\alpha(e_1,e_2)+\psi\langle e_1,e_1\rangle\alpha(e_2,e_2)\\ &\quad-\psi\langle e_2,e_2\rangle\alpha(e_1,e_1)+\psi\langle e_1,e_2\rangle\alpha(e_1,e_2)\\ &=-K(\alpha(e_2,e_2)+\alpha(e_1,e_1))+\psi(\alpha(e_2,e_2)+\alpha(e_1,e_1))\\ &=(\psi-K)(\alpha_{22}+\alpha_{11}).\end{aligned}$$

And for $i=1,2$, we have

$$\begin{aligned}R^{\perp}(e_1,e_2)\alpha_{ii}&=\alpha(R(e_1,e_2)e_i,e_i)+\alpha(e_i,R(e_1,e_2)e_i)\\ &\quad-\psi\alpha((e_1\wedge e_2)e_i,e_i)-\psi\alpha(e_i,(e_1\wedge e_2)e_i)\\ &=\langle R(e_1,e_2)e_2,e_1\rangle\alpha(e_2,e_1)+\langle R(e_1,e_2)e_2,e_1\rangle\alpha(e_1,e_2)\\ &\quad-\psi\langle e_2,e_i\rangle\alpha(e_1,e_i)+\psi\langle e_1,e_i\rangle\alpha(e_2,e_i)\\ &\quad-\psi\langle e_2,e_i\rangle\alpha(e_i,e_1)+\psi\langle e_1,e_i\rangle\alpha(e_i,e_2)\\ &=-2K\alpha(e_1,e_2)+2\psi\alpha(e_1,e_2)\\ &=2(-K+\psi)\alpha_{12}.\end{aligned}$$

□

As a consequence of Lemma 2.2, we get:

Proposition 2.3. *Let $f : M_1^2 \to \mathbb{Q}_s^m(c)$ be an isometric immersion. If f has flat normal bundle, then f is a ψ-pseudo-parallel immersion with $\psi = K$. Moreover, if f is totally umbilical, then f is a ψ-pseudo parallel immersion for any ψ.*

Proof. If f has $R^\perp = 0$, taking $\psi = K$, we conclude from equations (2.3) and (2.4) that f is ψ-pseudo-parallel. □

Observation 2.4. The converse of Proposition 2.3 is not true in general, because in the pseudo-Riemannian case it is possible to have $R^\perp(e_1,e_2)\xi = 0$ for all $\xi \in \operatorname{span}\{\alpha(X,Y) : X,Y \in TM\}$ and $R^\perp \neq 0$, as we show in Example 2.5.

We consider the first normal spaces $N^1(x) := \{N^0(x)\}^\perp \subset N_f M(x)$, $x \in M_1^2$, where $N^0(x) = \{\eta \in N_f M(x) : A_\eta = 0\}$. Thus, $N^1(x) = \operatorname{span}\{\alpha(X,Y) : X,Y \in T_x M\}$.

Example 2.5. Consider the isometric immersion $f : \mathbb{E}_1^2 \to \mathbb{E}_3^6$, defined by

$$f(x_1,x_2) = (e^{x_1+x_2}, x_1^2, x_1, x_2, x_1^2, e^{x_1+x_2}).$$

We have that f is semi-parallel with $K = 0$, $\langle \mathcal{H},\mathcal{H}\rangle = 0$ and $R^\perp \neq 0$. In fact, we have

$$\begin{aligned} e_1 &= df(\partial_{x_2}) = (e^{x_1+x_2}, 0, 0, 1, 0, e^{x_1+x_2}), \\ e_2 &= df(\partial_{x_1}) = (e^{x_1+x_2}, 2x_1, 1, 0, 2x_1, e^{x_1+x_2}), \end{aligned}$$

with $\langle e_1,e_1\rangle = 1$, $\langle e_2,e_2\rangle = -1$ and $\langle e_1,e_2\rangle = 0$.

Also,

$$\begin{aligned} \alpha_{11} &= (e^{x_1+x_2}, 0, 0, 0, 0, e^{x_1+x_2}), \\ \alpha_{22} &= (e^{x_1+x_2}, 2, 0, 0, 2, e^{x_1+x_2}), \\ \alpha_{12} &= (e^{x_1+x_2}, 0, 0, 0, 0, e^{x_1+x_2}). \end{aligned}$$

Thus, $\alpha_{11}+\alpha_{22}$ and α_{12} are linearly independent, which means that $R^\perp \neq 0$, $\mathcal{H} = -(0,1,0,0,1,0)$ is lightlike and $\langle \alpha_{12},\alpha_{12}\rangle = \langle \alpha_{11}+\alpha_{22},\alpha_{11}+\alpha_{22}\rangle = \langle \alpha_{11}+\alpha_{22},\alpha_{12}\rangle = 0$. Then, for all $x = (x_1,x_2) \in \mathbb{E}_1^2$, we have

$$N^1(x) = \operatorname{span}\{2(e^{x_1+x_2}, 1, 0, 0, 1, e^{x_1+x_2}), (e^{x_1+x_2}, 0, 0, 0, 0, e^{x_1+x_2})\},$$

$$N^0(x) = \{N^1(x)\}^\perp = N^1(x),$$

and

$$N_f \mathbb{E}_1^2(x) = N^1(x) \oplus \operatorname{span}\{\eta_1, \eta_2\},$$

where

$$\eta_1 = (x_1 e^{x_1+x_2}, x_1^2 - \frac{1}{4}, x_1, 0, x_1^2 + \frac{1}{4}, x_1 e^{x_1+x_2}),$$
$$\eta_2 = -2\eta_1 + \left(-\frac{1}{2e^{x_1+x_2}}, 0, 1, -1, 0, \frac{1}{2e^{x_1+x_2}}\right),$$

with $\langle \eta_1, \eta_1 \rangle = \langle \eta_1, \eta_2 \rangle = \langle \eta_2, \eta_2 \rangle = \langle \alpha_{11} + \alpha_{22}, \eta_2 \rangle = \langle \alpha_{12}, \eta_1 \rangle = 0$ and $\langle \alpha_{12}, \eta_2 \rangle = \langle \alpha_{11} + \alpha_{22}, \eta_1 \rangle = 1$.

Thus, for all $\xi \in \operatorname{span}\{\alpha(X,Y) : X, Y \in TM\}$, using Ricci equation (2.2) we obtain

$$R^{\perp}(e_1, e_2)\xi = \langle \alpha_{12}, \xi \rangle (\alpha_{11} + \alpha_{22}) - \langle \alpha_{11} + \alpha_{22}, \xi \rangle \alpha_{12} = 0.$$

Moreover, from Gauss equation (2.1) we get $K = 0$, then f satisfies equations (2.3) and (2.4) with $\psi = 0$, thus f is a semi-parallel immersion.

In particular, we have

$$R^{\perp}(e_1, e_2)\eta_1 = \langle \alpha_{12}, \eta_1 \rangle (\alpha_{11} + \alpha_{22}) - \langle \alpha_{11} + \alpha_{22}, \eta_1 \rangle \alpha_{12} = -\alpha_{12} \neq 0.$$

Observation 2.6. Isometric immersion f in Example 2.5 was obtain from Remark 3.4 in [14], where Cabrerizo-Fernández-Gómez gave examples of λ-isotropic immersions in pseudo-Euclidean spaces with isotropy function $\lambda = \langle \mathcal{H}, \mathcal{H} \rangle = 0$ which are not totally umbilical (see also Example 4.3 in [13]). We will study λ-isotropic immersions in Section 2.2.

Observation 2.7. Taking $\psi = 0$ in equations (2.3) and (2.4), we get that any non-flat semi-parallel Lorentzian surface M_1^2 in $\mathbb{Q}_s^m(c)$ with $R^{\perp} = 0$ is umbilical. Then, any non-flat and non-umbilical surface with flat normal bundle is an example of a pseudo-parallel immersion which is not semi-parallel.

Since any hypersurface has $R^{\perp} = 0$, it follows from Proposition 2.3 that

Corollary 2.8. *Any isometric immersion $f : M_1^2 \to \mathbb{Q}_s^3(c)$ is pseudo-parallel.*

As another consequence of Lemma 2.2, we get

Lemma 2.9. *Let $f : M_1^2 \to \mathbb{Q}_s^m(c)$ be a pseudo-parallel isometric immersion. The mean curvature vector field $\mathcal{H}$ satisfies $R^{\perp}(X,Y)\mathcal{H} = 0$, for all $X, Y \in TM$.*

Proof. Since $\mathcal{H} = \frac{1}{2}(\alpha_{11} - \alpha_{22})$, it follows from (2.3) and (2.4) that

$$R^{\perp}(e_1, e_2)\mathcal{H} = \frac{1}{2}(R^{\perp}(e_1, e_2)\alpha_{11} - R^{\perp}(e_1, e_2)\alpha_{22}) = 0.$$

□

The following proposition is useful to construct examples of pseudo-parallel submanifolds.

Proposition 2.10. *Let $f: M_t^n \to \mathbb{Q}_s^N(c)$ be a isometric immersion and $h: \mathbb{Q}_s^N(c) \to \mathbb{Q}_{\hat{s}}^{\hat{N}}(\hat{c})$ be a totally umbilical immersion. If f is ψ-pseudo-parallel, then $h \circ f: M_t^n \to \mathbb{Q}_{\hat{s}}^{\hat{N}}(\hat{c})$ is ψ-pseudo-parallel.*

Proof. We denote by α^f, α^h and $\alpha^{h\circ f}$ the second fundamental forms of f, h and $h \circ f$, respectively. In the same way, we denote by $R_f^\perp$ and $R_{h\circ f}^\perp$ the normal curvature tensors of f and $h \circ f$, respectively. Let $\overline{R}_f$ and $\overline{R}_{h\circ f}$ be the curvature tensors of the Var der Waerden-Bortolotti connections $\overline{\nabla}^f$ and $\overline{\nabla}^{h\circ f}$ of the respective bundles $TM \oplus N_f M$ and $TM \oplus N_{h\circ f} M$. Since h is a totally umbilical immersion, we have the following relations:

$$\begin{aligned}
\alpha^{h\circ f}(Z,W) &= \langle Z,W\rangle \mathcal{H}^h + h_*\alpha^f(Z,W),\\
R_{h\circ f}^\perp(X,Y)h_*\alpha^f(Z,W) &= h_*R_f^\perp(X,Y)\alpha^f(Z,W),\\
R_{h\circ f}^\perp(X,Y)\mathcal{H}^h &= 0.
\end{aligned}$$

Then, for $X,Y,Z,W \in TM$, we get that the immersion $h \circ f$ satisfies pseudo-parallelism condition (1.13), since

$$\begin{aligned}
\left[\overline{R}_{h\circ f}(X,Y)\cdot\alpha^{h\circ f}\right](Z,W) &= R_{h\circ f}^\perp(X,Y)\alpha^{h\circ f}(Z,W) - \alpha^{h\circ f}(R(X,Y)Z,W)\\
&\quad - \alpha^{h\circ f}(Z,R(X,Y)W)\\
&= h_*R_f^\perp(X,Y)\alpha^f(Z,W) + \langle Z,W\rangle R_{h\circ f}^\perp(X,Y)H^h\\
&\quad - h_*\alpha^f(R(X,Y)Z,W) - \langle R(X,Y)Z,W\rangle H^h\\
&\quad - h_*\alpha^f(Z,R(X,Y)W) - \langle Z,R(X,Y)W\rangle H^h\\
&= h_*\psi[X\wedge Y\cdot\alpha^f](Z,W)\\
&= \psi\Big\{-h_*\alpha^f((X\wedge Y)Z,W - \alpha^f(Z,(X\wedge Y)W))\\
&\qquad -\langle (X\wedge Y)Z,W\rangle\mathcal{H}^h - \langle Z,(X\wedge Y)W\rangle\mathcal{H}^h\Big\}\\
&= \psi\Big\{\alpha^{h\circ f}((X\wedge Y)Z,W) - \alpha^{h\circ f}(Z,(X\wedge Y)W)\Big\}\\
&= \psi[X\wedge Y\cdot\alpha^{h\circ f}](Z,W).
\end{aligned}$$

In fact, let ∇, $\widetilde{\nabla}$ and $\hat{\nabla}$ be the Levi-Civita connections of M_t^n, $\mathbb{Q}_s^N(c)$ and $\mathbb{Q}_{\hat{s}}^{\hat{N}}(\hat{c})$, respectively. We have

$$\begin{aligned}
\alpha^{h\circ f}(Z,W) &= \hat{\nabla}_Z h_*(f_*W) - h_*(f_*\nabla_Z W)\\
&= \hat{\nabla}_Z h_*(f_*W) - h_*\widetilde{\nabla}_Z f_*W + h_*\alpha^f(Z,W)\\
&= \alpha^h(f_*Z,f_*W) + h_*\alpha^f(Z,W).
\end{aligned}$$

Now, let $\xi \in N_f M$ and $\eta \in N_h\mathbb{Q}_s^N(c)$ be two normal vector fields of f and h, respectively. Note that $h_*\xi \in N_{h\circ f}M$ and $\langle \eta, h_*\xi\rangle = 0$.

The Weingarten operators of f, h and $h\circ f$ are related in the following way:

$$\begin{aligned}\langle A^{h\circ f}_{h_*\xi}X,Y\rangle &= \langle \alpha_{h\circ f}(X,Y),h_*\xi\rangle\\ &= \langle \alpha^h(f_*X,f_*Y),h_*\xi\rangle + \langle h_*\alpha^f(X,Y),h_*\xi\rangle\\ &= \langle \alpha^f(X,Y),\xi\rangle\\ &= \langle A^f_\xi X,Y\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle A^{h\circ f}_{\eta}X,Y\rangle &= \langle \alpha_{h\circ f}(X,Y),\eta\rangle\\ &= \langle \alpha^h(f_*X,f_*Y),\eta\rangle + \langle h_*\alpha^f(X,Y),\eta\rangle\\ &= \langle \alpha^h(f_*X,f_*Y),\eta\rangle\\ &= \langle A^h_\eta f_*X,f_*Y\rangle.\end{aligned}$$

Hence,

$$A^{h\circ f}_{h_*\xi}X = A^f_\xi X \text{ and } f_*A^{h\circ f}_\eta X = A^h_\eta f_*X, \text{ for all } X\in TM.$$

On the other hand, the normal connections of f, h and $h\circ f$ satisfy the following relations:

$$\begin{aligned}\nabla^{\perp h\circ f}_X h_*\xi &= \hat{\nabla}_X h_*\xi + h_*f_*A^{h\circ f}_{h_*\xi}X\\ &= h_*\widetilde{\nabla}_X\xi + \alpha_h(f_*X,\xi) + h_*f_*A^{h\circ f}_{h_*\xi}X\\ &= -h_*f_*A^f_\xi X + h_*\nabla^{\perp f}_X\xi + \alpha_h(f_*X,\xi) + h_*f_*A^{h\circ f}_{h_*\xi}X\\ &= -h_*f_*A^f_\xi X + h_*\nabla^{\perp f}_X\xi + \alpha_h(f_*X,\xi) + h_*f_*A^f_\xi X\\ &= h_*\nabla^{\perp f}_X\xi + \alpha_h(f_*X,\xi),\end{aligned}$$

and

$$\begin{aligned}\nabla^{\perp h\circ f}_X \eta &= h_*f_*A^{h\circ f}_\eta X + \hat{\nabla}_X\eta\\ &= h_*f_*A^{h\circ f}_\eta X - h_*A^h_\eta f_*X + \nabla^{\perp h}_{f_*X}\eta\\ &= h_*A^h_\eta f_*X - h_*A^h_\eta f_*X + \nabla^{\perp h}_{f_*X}\eta\\ &= \nabla^{\perp h}_{f_*X}\eta.\end{aligned}$$

In particular, if h is an umbilical immersion and $\mathcal{H}^h$ denotes the mean curvature vector of h, then we have

$$\alpha^{h\circ f}(Z,W) = \langle Z,W\rangle\mathcal{H}^h + h_*\alpha^f(Z,W),$$

$$\nabla^{\perp h\circ f}_X h_*\xi = h_*\nabla^{\perp f}_X\xi.$$

Putting all this information together, including the hypothesis of h being umbilical, we conclude that:

$$R^{\perp}_{h\circ f}(X,Y)h_*\alpha^f(Z,W) = h_*R^{\perp}_f(X,Y)\alpha^f(Z,W),$$

and

$$R^{\perp}_{h\circ f}(X,Y)\mathcal{H}^h = R^{\perp}_h(f_*X,f_*Y)\mathcal{H}^h = 0.$$

□

2.2 The λ-isotropy condition and the hyperbola of normal curvature

In this section, we study λ-isotropic Lorentzian surfaces in pseudo-Riemannian space forms and their hyperbolas of curvature, obtaining conditions equivalent to λ-isotropy.

First, we state the following lemma, which provide a characterization of λ-isotropic Lorentzian surfaces and is analogous to a result by O'Neill in [61] for the Riemannian case and extended by Cabrerizo-Fernández-Gómez in [13] for pseudo-Euclidean spaces. We prove this lemma with a technique analogous to the one used by Lobos-Tassi-Yucra Hancco in [50].

Lemma 2.11. *Let $f: M^2_1 \to \mathbb{Q}^m_s(c)$ be an isometric immersion and let $\{e_1,e_2\}$ be an orthonormal frame of M^2_1, with $\langle e_1,e_1\rangle = 1$, $\langle e_2,e_2\rangle = -1$ and $\langle e_1,e_2\rangle = 0$. Then, the following conditions are equivalent:*

(a) f *is* λ*-isotropic.*

(b) $\langle \alpha_{ii},\alpha_{12}\rangle = 0$ *and* $\langle \alpha_{ii},\alpha_{ii}\rangle = -2\langle \alpha_{12},\alpha_{12}\rangle - \langle \alpha_{11},\alpha_{22}\rangle$*, where* $i = 1,2$.

(c) $\langle \alpha_{11}+\alpha_{22},\alpha_{11}+\alpha_{22}\rangle = -4\langle \alpha_{12},\alpha_{12}\rangle$ *and* $\{\mathcal{H},\alpha_{11}+\alpha_{22},\alpha_{12}\}$ *is an orthogonal set.*

(d) $\langle \alpha(X,X),\alpha(X,X)\rangle = \langle X,X\rangle^2\lambda(x)$*, for all* $X \in T_xM$ *and for all* $x \in M^2_1$.

Moreover, if f is λ-isotropic, then $\lambda = \langle \mathcal{H},\mathcal{H}\rangle - \langle \alpha_{12},\alpha_{12}\rangle = \dfrac{1}{2}(c - K + 3\langle \mathcal{H},\mathcal{H}\rangle)$, where K is the Gaussian curvature of M^2_1 and $\mathcal{H}$ is the mean curvature vector field of f.

Proof. *(a)* $\Rightarrow$ *(b)*. Let us suppose that $f: M^2_1 \to \mathbb{Q}^m_s(c)$ is λ-isotropic and fix an arbitrary point $x \in M^2_1$. For $X \in T_xM$, such that $X = \cosh t e_1 + \sinh t e_2$, we have that $\|X\| = 1$ for all $t \in \mathbb{R}$. Also,

$$\alpha(X,X) = \cosh^2 t\,\alpha_{11} + 2\cosh t \sinh t\,\alpha_{12} + \sinh^2 t\,\alpha_{22},$$

and we have

$$\begin{aligned}\lambda &= \langle \alpha(X,X), \alpha(X,X)\rangle \\ &= \left(\cosh^4 t + \sinh^4 t\right)\lambda + 4\cosh^2 t \sinh^2 t \langle \alpha_{12}, \alpha_{12}\rangle \\ &\quad + 4\cosh^3 t \sinh t \langle \alpha_{11}, \alpha_{12}\rangle + 4\cosh t \sinh^3 t \langle \alpha_{12}, \alpha_{22}\rangle \\ &\quad + 2\cosh^2 t \sinh^2 t \langle \alpha_{11}, \alpha_{22}\rangle.\end{aligned}$$

Since λ does not depend on t, from the λ-isotropy condition (1.10), taking the derivative with respect to t, we get

$$\begin{aligned}0 &= \frac{d}{dt}\langle \alpha(X,X), \alpha(X,X)\rangle \\ &= 4(\cosh^3 t \sinh t + \cosh t \sinh^3 t)\lambda \\ &\quad + 2(\cosh^3 t \sinh t + \cosh t \sinh^3 t)(4\langle \alpha_{12}, \alpha_{12}\rangle \\ &\quad + 2\langle \alpha_{11}, \alpha_{22}\rangle) + 4(\cosh^4 t + 3\cosh^2 t \sinh^2 t)\langle \alpha_{11}, \alpha_{12}\rangle \\ &\quad + 4(3\cosh^2 + t \sinh^2 t + \sinh^4 t)\langle \alpha_{12}, \alpha_{22}\rangle \\ &= \sinh(4t)(\lambda + 2\langle \alpha_{12}, \alpha_{12}\rangle + \langle \alpha_{11}, \alpha_{22}\rangle) \\ &\quad + 2(\cosh(2t)+1)(2\cosh(2t)-1)\langle \alpha_{11}, \alpha_{12}\rangle \\ &\quad + 2(\cosh(2t)-1)(2\cosh(2t)+1)\langle \alpha_{12}, \alpha_{22}\rangle.\end{aligned}$$

Then,

$$0 = \left.\frac{d\lambda}{dt}\right|_{t=0} = \left.\frac{d}{dt}\langle \alpha(X,X), \alpha(X,X)\rangle\right|_{t=0} = 4\langle \alpha_{11}, \alpha_{12}\rangle. \tag{2.5}$$

Now, for $\hat{t} \neq 0$, we have

$$\left.\frac{d}{dt}\langle \alpha(X,X), \alpha(X,X)\rangle\right|_{t=-\hat{t}} = 0 = \left.\frac{d}{dt}\langle \alpha(X,X), \alpha(X,X)\rangle\right|_{t=\hat{t}}. \tag{2.6}$$

Using (2.5) and (2.6), we get

$$\begin{aligned}(\lambda + 2\langle \alpha_{12}, \alpha_{12}\rangle + \langle \alpha_{11}, \alpha_{22}\rangle)\sinh 4\hat{t} &= 2\langle \alpha_{12}, \alpha_{22}\rangle(1-\cosh 2\hat{t})(1+2\cosh 2\hat{t}) \\ &= -(\lambda + 2\langle \alpha_{12}, \alpha_{12}\rangle + \langle \alpha_{11}, \alpha_{22}\rangle)\sinh 4\hat{t}.\end{aligned}$$

Since $\hat{t} \neq 0$, we deduce that

$$\langle \alpha_{ii}, \alpha_{ii}\rangle = \lambda = -2\langle \alpha_{12}, \alpha_{12}\rangle - \langle \alpha_{11}, \alpha_{22}\rangle, \tag{2.7}$$

and

$$\langle \alpha_{12}, \alpha_{22}\rangle = 0. \tag{2.8}$$

So, we have *(b)*. Also, it follows from (2.7) that

$$\langle \mathcal{H}, \mathcal{H}\rangle = \frac{1}{2}(\lambda - \langle \alpha_{11}, \alpha_{22}\rangle) = \lambda + \langle \alpha_{12}, \alpha_{12}\rangle. \tag{2.9}$$

Moreover, from (2.9) and Gauss equation (2.1), we get

$$3\langle \mathcal{H},\mathcal{H}\rangle = 2\lambda - \langle \alpha_{11},\alpha_{22}\rangle + \langle \alpha_{12},\alpha_{12}\rangle = 2\lambda + K - c.$$

(b) ⇒ (c). Let us assume that the isometric immersion $f : M_1^2 \to \mathbb{Q}_s^m(c)$ satisfies $\langle \alpha_{ii},\alpha_{12}\rangle = 0$ and $\langle \alpha_{ii},\alpha_{ii}\rangle = -2\langle \alpha_{12},\alpha_{12}\rangle - \langle \alpha_{11},\alpha_{22}\rangle$, for $i = 1,2$. We have that

$$2\langle \mathcal{H},\alpha_{11}+\alpha_{22}\rangle = \langle \alpha_{11}-\alpha_{22},\alpha_{11}+\alpha_{22}\rangle = \langle \alpha_{11},\alpha_{11}\rangle - \langle \alpha_{22},\alpha_{22}\rangle = 0,$$

$$2\langle \mathcal{H},\alpha_{12}\rangle = \langle \alpha_{11}-\alpha_{22},\alpha_{12}\rangle = 0,$$

and

$$\langle \alpha_{11}+\alpha_{22},\alpha_{12}\rangle = 0.$$

Thus, $\{\mathcal{H},\alpha_{11}+\alpha_{22},\alpha_{12}\}$ is an orthogonal set. Also, we have that

$$\begin{aligned}\langle \alpha_{11}+\alpha_{22},\alpha_{11}+\alpha_{22}\rangle &= \langle \alpha_{11},\alpha_{11}\rangle + 2\langle \alpha_{11},\alpha_{22}\rangle + \langle \alpha_{22},\alpha_{22}\rangle\\ &= -4\langle \alpha_{12},\alpha_{12}\rangle,\end{aligned}$$

and we have *(c)*.

(c) ⇒ (a). Let $x \in M_1^2$ be fixed. For all $X \in T_xM$ with $\langle X,X\rangle = 1$, there exists $t \in \mathbb{R}$ and $\epsilon_1 \in \{-1,1\}$ such that $X = \epsilon_1 \cosh t e_1 + \sinh t e_2$. Thus

$$\begin{aligned}\alpha(X,X) &= \alpha(\epsilon_1 \cosh t e_1 + \sinh t e_2, \epsilon_1 \cosh t e_1 + \sinh t e_2)\\ &= \cosh^2 t\,\alpha_{11} + \epsilon_1 2\cosh t \sinh t\,\alpha_{12} + \sinh^2 t\,\alpha_{22}\\ &= \left(\frac{\cosh(2t)+1}{2}\right)\alpha_{11} + \epsilon_1 2\cosh t \sinh t\,\alpha_{12} + \left(\frac{\cosh(2t)-1}{2}\right)\alpha_{22}\\ &= \frac{1}{2}(\alpha_{11}-\alpha_{22}) + \frac{1}{2}\cosh(2t)(\alpha_{11}+\alpha_{22}) + \epsilon_1 \sinh(2t)\alpha_{12}\\ &= \mathcal{H}(x) + \frac{1}{2}\cosh(2t)(\alpha_{11}+\alpha_{22}) + \epsilon_1 \sinh(2t)\alpha_{12}.\end{aligned} \tag{2.10}$$

If $X \in T_xM$ with $\langle X,X\rangle = -1$, there exists $t \in \mathbb{R}$ and $\epsilon_1 \in \{-1,1\}$ such that $X = \sinh t e_1 + \epsilon_1 \cosh t e_2$. Thus

$$\begin{aligned}\alpha(X,X) &= \alpha(\sinh t e_1 + \epsilon_1 \cosh t e_2, \sinh t e_1 + \epsilon_1 \cosh t e_2)\\ &= \sinh^2 t\,\alpha_{11} + \epsilon_1 2\cosh t \sinh t\,\alpha_{12} + \cosh^2 t\,\alpha_{22}\\ &= \left(\frac{\cosh(2t)-1}{2}\right)\alpha_{11} + \epsilon_1 2\cosh t \sinh t\,\alpha_{12} + \left(\frac{\cosh(2t)+1}{2}\right)\alpha_{22}\\ &= \frac{1}{2}(-\alpha_{11}+\alpha_{22}) + \frac{1}{2}\cosh(2t)(\alpha_{11}+\alpha_{22})\epsilon_1 \sinh(2t)\alpha_{12}\\ &= -\mathcal{H}(x) + \frac{1}{2}\cosh(2t)(\alpha_{11}+\alpha_{22}) + \epsilon_1 \sinh(2t)\alpha_{12}.\end{aligned} \tag{2.11}$$

Hence, in both cases we have

$$\alpha(X,X) = \langle X,X\rangle \mathcal{H}(x) + \frac{1}{2}\cosh(2t)(\alpha_{11}+\alpha_{22}) + \epsilon_1 \sinh(2t)\alpha_{12}. \tag{2.12}$$

Now, if $\{\mathcal{H}, \alpha_{11}+\alpha_{22}, \alpha_{12}\}$ is an orthogonal set and $\langle \alpha_{11}+\alpha_{22}, \alpha_{11}+\alpha_{22}\rangle = -4\langle \alpha_{12}, \alpha_{12}\rangle$, from equation (2.12) follows that $\langle \alpha(X,X), \alpha(X,X)\rangle = \langle \mathcal{H},\mathcal{H}\rangle - \langle \alpha_{12},\alpha_{12}\rangle$ does not depend on t and f is a λ-isotropic surface with $\lambda = \langle \mathcal{H},\mathcal{H}\rangle - \langle \alpha_{12},\alpha_{12}\rangle$, so we get *(a)*.

(a) $\Leftrightarrow$ *(d)*. One of the implications is immediate. Next, if f is λ-isotropic and $\|X\| \neq 0$, we get *(d)* by using $\frac{X}{\|X\|}$. It only remains to prove that $\langle \alpha(X,X), \alpha(X,X)\rangle = 0$ when f is λ-isotropic and $X \in T_xM$ is lightlike, but this case follows by continuity, like in [13]. □

In the Riemannian case, for an isometric immersion $f : M^2 \to \widetilde{M}^m$ of a surface M^2 in a Riemannian manifold $\widetilde{M}^m$, the λ-isotropy condition has a geometric interpretation in terms of the indicatrix of normal curvature at each point $x \in M^2$, which is the set $\{\alpha(X,X) : X \in T_xM \text{ with } ||X|| = 1\}$ contained in the normal space to f at x. In this case, the indicatrix is an ellipse, maybe degenerate, with center at the mean curvature vector $\mathcal{H}$, so more precisely it is called *ellipse of normal curvature*. If f is λ-isotropic then, at each point $x \in M^2$, the ellipse of normal curvature is a circle orthogonal to the mean curvature vector. In particular, if for $x \in M^2$ this circle has radius 0, that is, when the ellipse of normal curvature degenerates to a point, then x is a umbilical point, thus, λ-isotropic immersions generalize totally umbilical immersions. When the ellipse of normal curvature is a circle with positive radius, then the λ-isotropic immersion f has non flat normal bundle. Observe that at $x \in M^2$, $\alpha(X,X)$ is the normal curvature vector of the surface along a curve in M^2 whose speed vector at x is X. Thus, in a way, the geometry of λ-isotropic surfaces is the same regardless of direction. For instance, the only λ-isotropic surfaces in the Euclidean space $\mathbb{E}^3$ are the totally umbilical ones (see Figure 2.1).

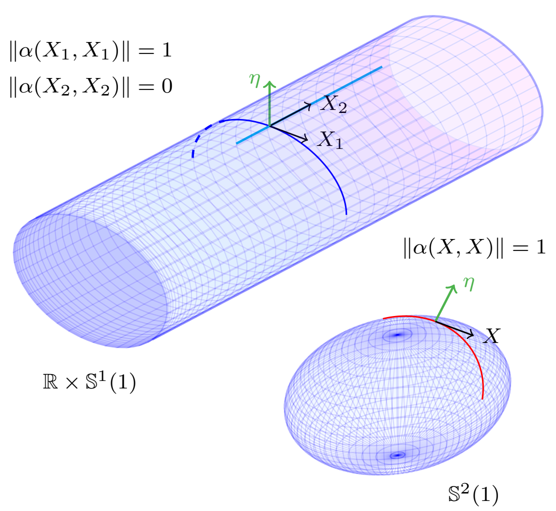

Figure 2.1: Cylinder $\mathbb{R}\times\mathbb{S}^1(1)$ is not a λ-isotropic surface in $\mathbb{E}^3$, but sphere $\mathbb{S}^2(1)$ is totally umbilical and λ-isotropic in $\mathbb{E}^3$ with $\lambda = 1$.

Now, for a Lorentzian surface $f : M_1^2 \to \mathbb{Q}_s^m(c)$, we observe from the right side of equation (2.12), that the image of the map $X \to \alpha(X,X)$ from the set of tangent unit vectors of M_1^2 at x into $N_fM(x)$ is composed by branches of hyperbolas with center at $\mathcal{H}(x)$ if $\langle X,X\rangle = 1$ and at $-\mathcal{H}(x)$ if $\langle X,X\rangle = -1$ (see Figure 2.2). Thus, for each $x \in M_1^2$, we consider the indicatrix of normal curvature of f as the set

$$\mathscr{H}_x = \{\langle X,X\rangle\alpha(X,X) : X \in T_xM \text{ with } \langle X,X\rangle = \pm 1\}, \tag{2.13}$$

which is an hyperbola in $N_fM(x)$ centered at $\mathcal{H}(x)$, so we refer to $\mathscr{H}_x$ as the *hyperbola of normal curvature* of f at x.

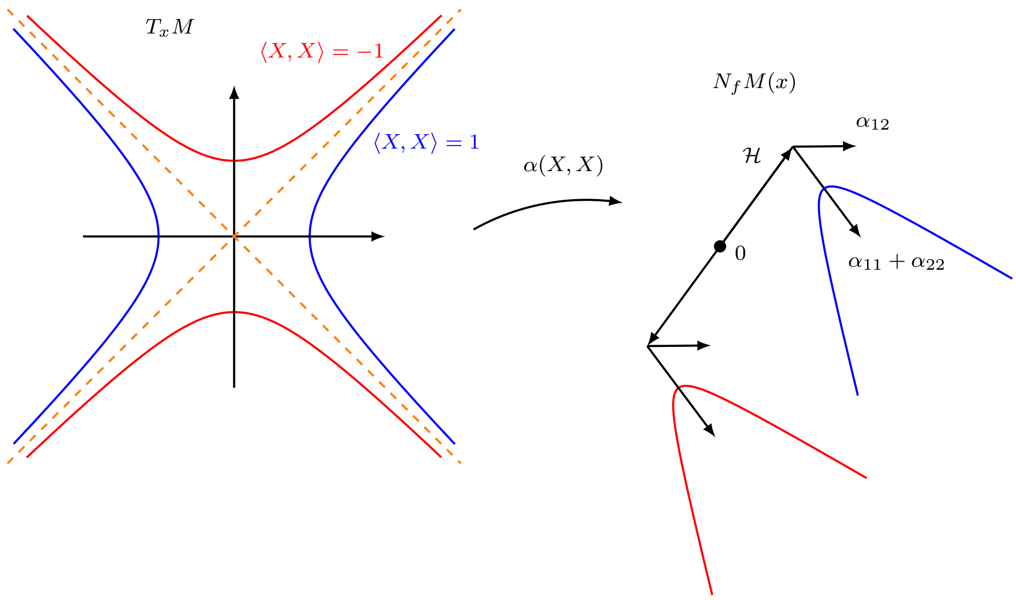

Figure 2.2: Branches of hyperbolas as image of the map $X \mapsto \alpha(X,X)$ from the set of unit tangent vectors of M_1^2 at x into $N_fM(x)$.

Notice that in this definition, the hyperbola of normal curvature can also be degenerate, in the sense that it can be contained in a right line or be a point. From here to the end of this section, we study the geometric relation between the λ-isotropy condition and the concept of hyperbola of normal curvature for Lorentzian surfaces.

The frame $\{e_1, e_2\}$ in Lemma 2.11 can be transformed according to

$$\begin{aligned} e_1' &= \epsilon_1 \cosh t e_1 + \sinh t e_2, \\ e_2' &= \epsilon_3 \sinh t e_1 + \epsilon_1\epsilon_3 \cosh t e_2, \end{aligned}$$

where $\epsilon_1, \epsilon_3 \in \{1,-1\}$. Let's denote $\alpha_{ij}' = \alpha(e_i', e_j')$, it follows from (2.10), (2.11) and analogous calculations that

$$\begin{aligned} \frac{\alpha_{11}'+\alpha_{22}'}{2} &= \cosh(2t)\left(\frac{\alpha_{11}+\alpha_{22}}{2}\right) + \epsilon_1 \sinh(2t)\alpha_{12}, \\ \alpha_{12}' &= \epsilon_1\epsilon_3 \sinh(2t)\left(\frac{\alpha_{11}+\alpha_{22}}{2}\right) + \epsilon_3 \cosh(2t)\alpha_{12}. \end{aligned}$$

Thus, $I_f M_1^2(x) := \mathrm{span}\{\alpha_{11}+\alpha_{22}, \alpha_{12}\}$ is an invariant vector subspace of $N_f M(x)$ which contains the hyperbola of normal curvature $\mathscr{H}_x$.

Proposition 2.12. *Let $f : M_1^2 \to \mathbb{Q}_s^m(c)$ be an isometric immersion with Gaussian curvature K. f is λ-isotropic with flat normal bundle if and only if, for each $x \in M_1^2$, either f is umbilical at x or $I_f M_1^2(x)$ is 1-dimensional, lightlike and orthogonal to $\mathcal{H}(x)$. Moreover, we have that $\lambda = \langle \mathcal{H}, \mathcal{H} \rangle = K - c$.*

Proof. Fix an arbitrary point $x \in M_1^2$. In fact, if $R^\perp = 0$ then $\{\alpha_{11}+\alpha_{22}, \alpha_{12}\}$ is linearly dependent. If f is λ-isotropic, for any orthonormal frame $\{e_1, e_2\}$ of M_1^2 nearly of x, it follows from Lemma 2.11 that $\langle \alpha_{11}+\alpha_{22}, \alpha_{11}+\alpha_{22} \rangle = -4\langle \alpha_{12}, \alpha_{12} \rangle$ and $\{\alpha_{11}+\alpha_{22}, \alpha_{12}\}$ is orthogonal to $\mathcal{H}$. If $\alpha_{11}+\alpha_{22} = 0$, we have that $-4\langle \alpha_{12}, \alpha_{12} \rangle = \langle \alpha_{11}+\alpha_{22}, \alpha_{11}+\alpha_{22} \rangle = 0$, thus, $\alpha_{12} = 0$ or α_{12} is lightlike. Analogously, if $\alpha_{12} = 0$, then $\alpha_{11}+\alpha_{22} = 0$ or $\alpha_{11}+\alpha_{22}$ is lightlike. If $\alpha_{11}+\alpha_{22} = \mu\alpha_{12}$ with $\mu \neq 0$, we have that $\mu^2\langle \alpha_{12}, \alpha_{12} \rangle = \langle \alpha_{11}+\alpha_{22}, \alpha_{11}+\alpha_{22} \rangle = \langle \alpha_{11}+\alpha_{22}, \alpha_{12} \rangle = 0$, then α_{12} and $\alpha_{11}+\alpha_{22}$ are zero or lightlike. Now, if $\alpha_{11}+\alpha_{22} = \alpha_{12} = 0$ then f is umbilical. If $\alpha_{11}+\alpha_{22}$ is lightlike or α_{12} is lightlike, hence $I_f M_1^2(x) = \mathrm{span}\{\alpha_{11}+\alpha_{22}, \alpha_{12}\}$ is 1-dimensional and lightlike.

The converse is true from Ricci equation (2.2) and Lemma 2.11, since $\{\mathcal{H}, \alpha_{11}+\alpha_{22}, \alpha_{12}\}$ is an orthogonal set and $\langle \alpha_{11}+\alpha_{22}, \alpha_{11}+\alpha_{22} \rangle = 0 = -4\langle \alpha_{12}, \alpha_{12} \rangle$. Also, from Lemma 2.11 we get $\lambda = \langle \mathcal{H}, \mathcal{H} \rangle = K - c$. □

Proposition 2.13. *Let $f : M_1^2 \to \mathbb{Q}_s^m(c)$ be an isometric immersion. f is λ-isotropic with non-flat normal bundle on any open subset of M_1^2 if and only if, for each $x \in M_1^2$, the set $\mathscr{H}_x$, given by* (2.13)*, is a non-degenerate hyperbola with center at the mean curvature vector $\mathcal{H}(x)$, which lies in a 2-dimensional affine subspace $\mathcal{V}$ of $N_f M(x)$ orthogonal to $\mathcal{H}(x)$, such that*

(a) either $\mathcal{V} - \mathcal{H}(x)$ is Lorentzian and $\mathscr{H}_x$ is an equilateral hyperbola satisfying that $\langle W - \mathcal{H}(x), W - \mathcal{H}(x) \rangle = r(x) \neq 0$ does not depend on $W \in \mathscr{H}_x$. In this case, $2 \leq s \leq m-2$, $r(x) = \lambda(x) - \langle \mathcal{H}(x), \mathcal{H}(x) \rangle$ and if $m = 4$, then $s = 2$ and f is extremal;

(b) or all non-zero vectors of $\mathcal{V} - \mathcal{H}(x)$ are lightlike. In this case, $3 \leq s \leq m-3$, $\lambda(x) = \langle \mathcal{H}(x), \mathcal{H}(x) \rangle$ and if $m = 6$, then $s = 3$ and $\lambda(x) = 0$.

Proof. Let us suppose that $f : M_1^2 \to \mathbb{Q}_s^m(c)$ is λ-isotropic with $R^\perp \neq 0$ and fix an arbitrary point $x \in M_1^2$. Let $\{e_1, e_2\}$ be an orthonormal frame of M_1^2, with $\langle e_1, e_1 \rangle = 1$, $\langle e_2, e_2 \rangle = -1$ and $\langle e_1, e_2 \rangle = 0$. It follows from equation (2.12) that $\mathscr{H}_x = \{\mathcal{H}(x) + \epsilon_2 y(t) : t \in \mathbb{R}, \epsilon_2 = \pm 1\}$, where

$$y(t) := \frac{1}{2}\cosh(2t)(\alpha_{11}+\alpha_{22}) + \sinh(2t)\alpha_{12}.$$

Since $R^\perp \neq 0$, using Ricci equation (2.2), we obtain that $\{\alpha_{11}+\alpha_{22}, \alpha_{12}\}$ is linearly independent, hence $I_f M_1^2(x) = \mathrm{span}\{\alpha_{11}+\alpha_{22}, \alpha_{12}\}$ is 2-dimensional and $\{y(t), t \in \mathbb{R}\}$ is a branch of a non-degenerate hyperbola, this means that it is not contained in a 1-dimensional space. Thus, $\mathscr{H}_x$ is a non-degenerate hyperbola with center $\mathcal{H}(x)$, which lies in the affine subspace $\mathcal{V}$ of $N_fM(x)$ generated by $\{\alpha_{11}+\alpha_{22},\alpha_{12}\}$ and containing $\mathcal{H}(x)$.

From Lemma 2.11, we have that $\{\mathcal{H}(x),\alpha_{11}+\alpha_{22},\alpha_{12}\}$ is an orthogonal set and

$$\langle\alpha_{11}+\alpha_{22},\alpha_{11}+\alpha_{22}\rangle=-4\langle\alpha_{12},\alpha_{12}\rangle=4(\lambda(x)-\langle\mathcal{H}(x),\mathcal{H}(x)\rangle).$$

Thus, we obtain that $\mathcal{V}$ is orthogonal to $\mathcal{H}(x)$ and for $W=\mathcal{H}(x)+\epsilon_2 y(t)$, we have that $\langle W-\mathcal{H}(x),W-\mathcal{H}(x)\rangle=\langle y(t),y(t)\rangle=\lambda(x)-\langle\mathcal{H}(x),\mathcal{H}(x)\rangle=r(x)$ does not depend on t.

If $r(x)\neq 0$, then $\langle\alpha_{11}+\alpha_{22},\alpha_{11}+\alpha_{22}\rangle\langle\alpha_{12},\alpha_{12}\rangle<0$ and $I_fM_1^2(x)=\mathcal{V}-\mathcal{H}(x)$ is Lorentzian, thus $2\leq s\leq m-2$ and $\mathscr{H}_x$ is a equilateral hyperbola. In particular, if $m=4$, we have that $N_fM(x)=I_fM_1^2(x)$ and using that $\mathcal{H}(x)$ is orthogonal to $\alpha_{11}+\alpha_{22}$ and α_{12}, we can conclude that $\mathcal{H}(x)=0$.

If $r(x)=0$, then any non-zero vector in $I_fM_1^2(x)=\mathcal{V}-\mathcal{H}(x)$ is lightlike and $I_fM_1^2(x)\subset\{I_fM_1^2(x)\}^{\perp}$. Since $\dim I_fM_1^2(x)+\dim\left\{I_fM_1^2(x)\right\}^{\perp}=\dim N_fM(x)$, it follows that $\dim N_fM(x)\geq 4$ and $3\leq s\leq m-3$. If $m=6$, then $\left\{I_fM_1^2(x)\right\}^{\perp}=I_fM_1^2(x)$ and using that $\mathcal{H}(x)$ is orthogonal to $\alpha_{11}+\alpha_{22}$ and α_{12}, we can conclude that $\mathcal{H}(x)\in I_fM_1^2(x)$.

Conversely, if for $x\in M_1^2$, the set $\mathscr{H}_x$ is a non-degenerate hyperbola centered at $\mathcal{H}(x)$, which lies in a 2-dimensional affine subspace $\mathcal{V}$ of $N_fM(x)$ orthogonal to $\mathcal{H}(x)$, from equation (2.12) we have that $\mathscr{H}_x-\mathcal{H}(x)=\{\epsilon_2 y(t):t\in\mathbb{R},\epsilon_2=\pm1\}$ is orthogonal to $\mathcal{H}(x)$, i.e., $\langle\mathcal{H}(x),y(t)\rangle=0$, for all $t\in\mathbb{R}$, where $y(t):=\frac{1}{2}\cosh(2t)(\alpha_{11}+\alpha_{22})+\sinh(2t)\alpha_{12}$.

Also, for all $X\in T_xM$ with $\langle X,X\rangle=\epsilon_2=\pm1$, there exists $t\in\mathbb{R}$ such that, using equation (2.12), we can write $\alpha(X,X)=\epsilon_2\mathcal{H}(x)+y(t)$. Then

$$\langle\alpha(X,X),\alpha(X,X)\rangle=\langle\mathcal{H}(x),\mathcal{H}(x)\rangle+\langle y(t),y(t)\rangle. \tag{2.14}$$

Since $\mathscr{H}_x$ is a non-degenerate hyperbola, we have that $\{\alpha_{11}+\alpha_{22},\alpha_{12}\}$ is linearly independent and it follows from Ricci equation (1.17) that $R^{\perp}\neq 0$.

If all non-zero vectors of $\mathcal{V}-\mathcal{H}(x)$ are lightlike, then $\langle y(t),y(t)\rangle=0$ and f is λ-isotropic with $\lambda=\langle\mathcal{H}(x),\mathcal{H}(x)\rangle$. On the other hand, if $\mathcal{V}-\mathcal{H}(x)$ is Lorentzian and $\mathscr{H}_x$ is an equilateral hyperbola in $\mathcal{V}$ such that $\langle W-\mathcal{H}(x),W-\mathcal{H}(x)\rangle=r(x)$ does not depend on $W=\mathcal{H}(x)+\epsilon_2 y(t)\in\mathscr{H}_x$, we have that $\langle y(t),y(t)\rangle$ does not depend on t. Therefore, we can conclude that f is λ-isotropic with $\lambda=\langle\mathcal{H}(x),\mathcal{H}(x)\rangle+\langle y(t),y(t)\rangle$. □

2.3 Characterization of pseudo-parallel surfaces in $\mathbb{Q}_s^m(c)$

In this section we present our main result concerning pseudo-parallel Lorentzian surfaces in $\mathbb{Q}_s^m(c)$ with nowhere vanishing normal curvature, which is stated below. As a consequence, we obtain a characterization of this kind of Lorentzian surfaces in terms of the hyperbola of normal curvature.

Theorem 2.14. *An isometric immersion $f: M_1^2 \to \mathbb{Q}_s^m(c)$ which has non-flat normal bundle on any open subset of M_1^2 is ψ-pseudo-parallel if and only if it is λ-isotropic. Moreover, for such an immersion we have that f is pseudo-umbilical and*

(a) if $\psi \neq K$, then $2 \leq s \leq m-2$ and

$$\lambda = -3\psi - c + 4K, \tag{2.15}$$

$$\langle \mathcal{H}, \mathcal{H} \rangle = -2\psi - c + 3K; \tag{2.16}$$

(b) if $\psi = K$, then $3 \leq s \leq m-3$ and $\lambda = \langle \mathcal{H}, \mathcal{H} \rangle = K - c$,

where K is the Gaussian curvature of M_1^2, $\mathcal{H}$ is the mean curvature vector field of f and λ is a smooth real-valued function on M_1^2.

Observation 2.15. In particular, Theorem 2.14 states that there are no pseudo-parallel Lorentzian surfaces with non-flat normal bundle in Lorentzian space forms.

Observation 2.16. Case *(b)* in Theorem 2.14 represents a significant difference with respect to the Riemannian case. Example 2.5 shows that pseudo-parallel Lorentzian surfaces as in *(b)* actually exist.

The proof of the Theorem 2.14 will be carried out using techniques analogous to those used by Asperti-Lobos-Mercuri in [9] and Lobos-Tassi-Yucra Hancco in [50]:

Proof of Theorem 2.14. Let us suppose that $f: M_1^2 \to \mathbb{Q}_s^m(c)$ is pseudo-parallel with non-flat normal bundle. For $x \in M_1^2$ and an orthonormal frame $\{e_1, e_2\}$ of M_1^2, with $\langle e_1, e_1 \rangle = 1$, $\langle e_2, e_2 \rangle = -1$ and $\langle e_1, e_2 \rangle = 0$, combining equations (2.2), (2.3) and (2.4), we get for $i = 1, 2$

$$\begin{aligned} R^{\perp}(e_1, e_2)\alpha_{ii} - 2(\psi - K)\alpha_{12} &= 0 \\ \langle \alpha_{12}, \alpha_{ii} \rangle(\alpha_{11} + \alpha_{22}) - \langle \alpha_{11} + \alpha_{22}, \alpha_{ii} \rangle \alpha_{12} - 2(\psi - K)\alpha_{12} &= 0 \\ \langle \alpha_{12}, \alpha_{ii} \rangle(\alpha_{11} + \alpha_{22}) - \left(\langle \alpha_{11} + \alpha_{22}, \alpha_{ii} \rangle + 2(\psi - K)\right)\alpha_{12} &= 0 \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} R^{\perp}(e_1, e_2)\alpha_{12} - (\psi - K)(\alpha_{11} + \alpha_{22}) &= 0 \\ \langle \alpha_{12}, \alpha_{12} \rangle(\alpha_{11} + \alpha_{22}) - \langle \alpha_{11} + \alpha_{22}, \alpha_{12} \rangle \alpha_{12} - (\psi - K)(\alpha_{11} + \alpha_{22}) &= 0 \\ \left(\langle \alpha_{12}, \alpha_{12} \rangle - (\psi - K)\right)(\alpha_{11} + \alpha_{22}) - \langle \alpha_{11} + \alpha_{22}, \alpha_{12} \rangle \alpha_{12} &= 0. \end{aligned} \tag{2.18}$$

Since f has non-flat normal bundle, from Ricci equation (2.2), we can conclude that $\alpha_{11} + \alpha_{22}$ and α_{12} are linearly independent, in particular $\alpha_{11} + \alpha_{22} \neq 0$ and $\alpha_{12} \neq 0$. Using this and equations (2.17) and (2.18), we get

$$\langle \alpha_{12}, \alpha_{11} \rangle = \langle \alpha_{12}, \alpha_{22} \rangle = 0, \tag{2.19}$$

$$\langle \alpha_{11} + \alpha_{22}, \alpha_{ii} \rangle = -2(\psi - K), \tag{2.20}$$

$$\langle \alpha_{12}, \alpha_{12} \rangle = \psi - K. \tag{2.21}$$

From Gauss equation (2.1), we have

$$\begin{aligned} \langle \alpha_{11}, \alpha_{22} \rangle &= -K + c + \langle \alpha_{12}, \alpha_{12} \rangle \\ \langle \alpha_{11}, \alpha_{22} \rangle &= -2K + c + \psi, \end{aligned} \tag{2.22}$$

$$\begin{aligned} \langle \alpha_{11} + \alpha_{22}, \alpha_{11} \rangle &= -2(\psi - K) \\ \langle \alpha_{11}, \alpha_{11} \rangle &= -2(\psi - K) - \langle \alpha_{22}, \alpha_{11} \rangle \\ \langle \alpha_{11}, \alpha_{11} \rangle &= -2(\psi - K) - (-2K + c + \psi) \\ \langle \alpha_{11}, \alpha_{11} \rangle &= -3\psi + 4K - c, \end{aligned} \tag{2.23}$$

$$\begin{aligned} \langle \alpha_{11} + \alpha_{22}, \alpha_{22} \rangle &= -2(\psi - K) \\ \langle \alpha_{22}, \alpha_{22} \rangle &= -2(\psi - K) - \langle \alpha_{11}, \alpha_{22} \rangle \\ \langle \alpha_{22}, \alpha_{22} \rangle &= -2(\psi - K) - (-2K + c + \psi) \\ \langle \alpha_{22}, \alpha_{22} \rangle &= -3\psi + 4K - c, \end{aligned} \tag{2.24}$$

$$\langle \alpha_{11} + \alpha_{22}, \alpha_{11} + \alpha_{22} \rangle = -4(\psi - K), \tag{2.25}$$

$$\begin{aligned} \langle \mathcal{H}, \mathcal{H} \rangle &= \frac{1}{4} \langle \alpha_{11} - \alpha_{22}, \alpha_{11} - \alpha_{22} \rangle \\ &= \frac{1}{4} (\langle \alpha_{11}, \alpha_{11} \rangle - 2\langle \alpha_{11}, \alpha_{22} \rangle + \langle \alpha_{22}, \alpha_{22} \rangle) \\ &= \frac{1}{4} (-3\psi + 4K - c - 2(-2K + c + \psi) - 3\psi + 4K - c) \\ &= \frac{1}{4} (-8\psi - 4c + 12K) \\ &= -2\psi - c + 3K. \end{aligned} \tag{2.26}$$

In particular, from equations (2.19), (2.20) and (2.21), we have that $\alpha_{11} + \alpha_{22}$ and α_{12} are orthogonal, $\langle \alpha_{11} + \alpha_{22}, \alpha_{11} + \alpha_{22} \rangle = -4(\psi - K) = -4\langle \alpha_{12}, \alpha_{12} \rangle$ and from equations (2.19) and (2.20) we obtain that $\mathcal{H}$ is orthogonal to $\{\alpha_{11} + \alpha_{22}, \alpha_{12}\}$. Also, for all $X \in T_xM$ with $\|X\| = 1$, it follows from equation (2.12) that

$$\begin{aligned} \lambda(x) &= \langle \alpha(X,X), \alpha(X,X) \rangle \\ &= \langle \mathcal{H}, \mathcal{H} \rangle + \frac{1}{4} \cosh^2(2t) \langle \alpha_{11} + \alpha_{22}, \alpha_{11} + \alpha_{22} \rangle + \sinh^2(2t) \langle \alpha_{12}, \alpha_{12} \rangle \\ &= \langle \mathcal{H}, \mathcal{H} \rangle - \langle \alpha_{12}, \alpha_{12} \rangle \\ &= -2\psi - c + 3K - (\psi - K) \\ &= -3\psi - c + 4K. \end{aligned}$$

Therefore, f is λ-isotropic with $\lambda = -3\psi - c + 4K$ and by Corollary 4.6 of [14], f is pseudo-umbilical.

Also, we have that $\lambda - \langle \mathcal{H}, \mathcal{H} \rangle = K - \psi$. It follows from Proposition 2.13 that if $\psi - K \neq 0$, then $2 \leq s \leq m-2$, and if $\psi - K = 0$, then $3 \leq s \leq m-3$.

Conversely, let us suppose that f is λ-isotropic. From Lemma 2.11, we have that

$$\langle \alpha_{ii}, \alpha_{12} \rangle = 0 \tag{2.27}$$

and

$$\lambda = -2\langle \alpha_{12}, \alpha_{12} \rangle - \langle \alpha_{11}, \alpha_{22} \rangle. \tag{2.28}$$

Then, using Gauss equation (2.1), we get

$$\langle \alpha_{12}, \alpha_{12} \rangle = -\frac{1}{3}(c + \lambda - K).$$

From Ricci equation (2.2), for $i = 1, 2$, we obtain

$$\begin{aligned} R^{\perp}(e_1, e_2)\alpha_{ii} &= ((\alpha_{11} + \alpha_{22}) \wedge \alpha_{12})\alpha_{ii} \\ &= -\langle \alpha_{11} + \alpha_{22}, \alpha_{ii} \rangle \alpha_{12} \\ &= -(\lambda + \langle \alpha_{11}, \alpha_{22} \rangle)\alpha_{12} \\ &= -(\lambda - \lambda - 2\langle \alpha_{12}, \alpha_{12} \rangle)\alpha_{12} \\ &= 2\langle \alpha_{12}, \alpha_{12} \rangle \alpha_{12} \\ &= -\frac{2}{3}(c + \lambda - K)\alpha_{12}, \end{aligned}$$

and

$$\begin{aligned} R^{\perp}(e_1, e_2)\alpha_{12} &= ((\alpha_{11} + \alpha_{22}) \wedge \alpha_{12})\alpha_{12} \\ &= \langle \alpha_{12}, \alpha_{12} \rangle (\alpha_{11} + \alpha_{22}) \\ &= -\frac{1}{3}(c + \lambda - K)(\alpha_{11} + \alpha_{22}). \end{aligned}$$

Therefore, taking $\psi = -\frac{1}{3}(c + \lambda - 4K)$, we conclude that f is ψ-pseudo-parallel according to Lemma 2.2. □

Considering $\psi = 0$ in Theorem 2.14 and equations (2.3), (2.4), we get the next corollary which is a generalization of a result given by J. Deprez in [23]. Also, E. Safiulina in [72] obtained an analogous result for semi-parallel spacelike surfaces in pseudo-Euclidean spaces.

Corollary 2.17. *Let $f : M_1^2 \to \mathbb{Q}_s^m(c)$ be an isometric immersion. f is semi-parallel if and only if there exists an open and dense subset U of M_1^2, such that the connected components of U are of the following types:*

(a) Totally umbilical Lorentzian surfaces.

(b) Lorentzian surfaces with $K = 0$ and $R^{\perp}(X,Y)(N^1(x)) = 0$, for all $X,Y \in T_xM$ and for all $x \in M_1^2$. In this case, if $R^{\perp} \neq 0$, then $3 \leq s \leq m-3$ and f is λ-isotropic with $\lambda = -c = \langle \mathcal{H}, \mathcal{H} \rangle$ and pseudo-umbilical.

(c) λ-isotropic Lorentzian surfaces with non-flat normal bundle and $\lambda = -c + 4K = \langle \mathcal{H}, \mathcal{H} \rangle + K$. In this case, $2 \leq s \leq m-2$ and f is pseudo-umbilical.

Here, K is the Gaussian curvature of M_1^2, $\mathcal{H}$ is the mean curvature vector field of f, $N^1(x)$ is the first normal space of f at x and λ is a smooth real-valued function on M_1^2.

Proof. First, let us assume that f is semi-parallel and fix an arbitrary point $x \in M_1^2$. Let $\{e_1, e_2\}$ be an orthonormal frame of M_1^2 at x. If $R^{\perp} = 0$ and $K \neq 0$, it follows from equations (2.3) and (2.4) that f is umbilical. If $R^{\perp} \neq 0$, it follows from Theorem 2.14 by considering $\psi = 0$ that f is a λ-isotropic and pseudo-umbilical Lorentzian surface with $\lambda = -c + 4K$, $\langle \mathcal{H}, \mathcal{H} \rangle = -c + 3K$ and if $K \neq 0$, then $2 \leq s \leq m-2$, if $K = 0$, then $3 \leq s \leq m-3$.

Now, consider the set $U_1 = \{x \in M_1^2 : K(x) \neq 0, \langle \mathcal{H}, \mathcal{H} \rangle \neq -c + 3K\}$. U_1 is an open subset of M_1^2 and we have that f is umbilical in all $x \in U_1$. Then, K and $\langle \mathcal{H}, \mathcal{H} \rangle$ are constant in the connected components of U_1. In fact, if f is umbilical, from Codazzi-Mainardi equation (1.16) we have $\nabla^{\perp}_{e_i} \mathcal{H} = \langle e_j, e_j \rangle \nabla^{\perp}_{e_i} \mathcal{H} = \langle e_i, e_j \rangle \nabla^{\perp}_{e_j} \mathcal{H} = 0$, with $\{i, j\} = \{1, 2\}$, then $\langle \mathcal{H}, \mathcal{H} \rangle$ is constant and from Gauss equation (2.1), we have that $K = c + \langle \mathcal{H}, \mathcal{H} \rangle$ is constant. Thus, U_1 is also closed in M_1^2. We can assume that M_1^2 is connected. Then, $U_1 = M_1^2$ or $U_1 = \emptyset$. In the first case, we get *a)*. In the second case, let $U_2 = \{x \in M_1^2 : K(x) \neq 0\}$ and $U_3 = M_1^2 \setminus \overline{U_2}$, where $\overline{U_2}$ is the closure of U_2, then U_2 and U_3 are open subsets of M_1^2 and $U = U_2 \cup U_3$ is an open and dense subset of M_1^2. For all connected component U_γ of U, we have that $U_\gamma \subset U_2$ and we have *(c)* or $U_\gamma \subset U_3$ and from equations (2.3) and (2.4) we have *(b)*. This proves one of the implications. The converse follows from Theorem 2.14 and equations (2.3) and (2.4). □

Observation 2.18. The semi-parallel Lorentzian surface in Example 2.5 shows that case (b) in Corollary 2.17 with the non-flat normal bundle condition is not empty at least for $c = 0$.

As another consequence of Theorem 2.14, we give below a geometric characterization of pseudo-parallel Lorentzian surfaces with $R^{\perp} \neq 0$ in pseudo-Riemannian space forms in terms of their hyperbolas of normal curvature.

Corollary 2.19. *Let $f : M_1^2 \to \mathbb{Q}_s^m(c)$ be an isometric immersion with Gaussian curvature K. f is ψ-pseudo-parallel with non-flat normal bundle on any open subset of M_1^2 if and only if, for each $x \in M_1^2$, the set*

$$\mathscr{H}_x = \{\langle X, X \rangle \alpha(X,X) : X \in T_xM \text{ with } \langle X, X \rangle = \pm 1\}$$

is a non-degenerate hyperbola with center at the mean curvature vector $\mathcal{H}(x)$, which lies in a 2-dimensional affine subspace $\mathcal{V}$ of $N_fM(x)$ orthogonal to $\mathcal{H}(x)$, such that

(a) either $\mathcal{V}-\mathcal{H}(x)$ is Lorentzian and $\mathscr{H}_x$ is an equilateral hyperbola satisfying that $\langle W-\mathcal{H}(x), W-\mathcal{H}(x)\rangle = r(x) \neq 0$ does not depend on $W \in \mathscr{H}_x$. In this case, $2 \leq s \leq m-2$, $r(x) = K-\psi$ and if $m=4$, then $s=2$ and f is extremal;

(b) or all non-zero vectors of $\mathcal{V}-\mathcal{H}(x)$ are lightlike. In this case, $3 \leq s \leq m-3$, $\psi = K$ and if $m=6$, then $s=3$ and $\langle \mathcal{H}(x), \mathcal{H}(x)\rangle = 0$.

Proof. From Theorem 2.14 we have that f is ψ-pseudo-parallel with $R^\perp \neq 0$ if and only if f is a λ-isotropic immersion with $R^\perp \neq 0$. Moreover, we have that $K-\psi = \lambda - \langle \mathcal{H}, \mathcal{H}\rangle$. The result follows from Proposition 2.13. $\square$

CHAPTER 3

Pseudo-parallel Lorentzian surfaces with non flat normal bundle in low codimension

In this chapter, we present examples of pseudo-parallel Lorentzian surfaces with non-flat normal bundle in a 4-dimensional or 5-dimensional pseudo-Riemannian space form. In the first section, we study pseudo-parallel Lorentzian surfaces with $R^{\perp} \neq 0$ in $\mathbb{Q}_s^4(c)$ and we prove that these surfaces, which are extremal and only exists for $s = 2$, also satisfy the condition of being isotropic with negative spin immersions. As a consequence, we obtain that any pseudo-parallel Lorentzian surface with $R^{\perp} \neq 0$ in a 4-dimensional pseudo-Riemannian space form with constant pseudo-parallelism function ψ is locally congruent to a surface of Veronese type, which is only defined for $c \neq 0$. In the second section, we obtain examples of pseudo-parallel Lorentzian surfaces with non-flat normal bundle and non-constant $\psi \neq 0$, by studying general rotational surfaces with plane meridians in $\mathbb{E}_2^4$. Finally, in the third section, we give an example of a flat and extremal pseudo-parallel Lorentzian surface with $R^{\perp} \neq 0$ and constant ψ in $\mathbb{S}_2^5(c)$ which is not semi-parallel.

3.1 Pseudo-parallel Lorentzian surfaces in $\mathbb{Q}_2^4(c)$

3.1.1 Lorentzian surface of Veronese type in $\mathbb{Q}_2^4(c)$, $c \neq 0$

Let $f : M_1^2 \to \mathbb{Q}_2^4(c)$ be an isometric immersion such that M_1^2 is oriented. Let $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ be a (local) frame adapted to the orientation of $\mathbb{Q}_2^4(c)$, such that $\{\widetilde{e}_1, \widetilde{e}_3\}$ is a pseudo-orthonormal frame that defines the orientation of M_1^2 and $\{\widetilde{e}_2, \widetilde{e}_4\}$ is a pseudo-orthonormal frame for $N_f M_1^2$. Setting for $i, j \in \{1, 3\}$,

$$\widetilde{\alpha}_{ij} := \alpha(\widetilde{e}_i, \widetilde{e}_j) = \sum_{k=2,4} \widetilde{\alpha}_{ij}^k \widetilde{e}_k.$$

We say that f is *isotropic with negative spin* at $x \in M_1^2$ if

$$\widetilde{\alpha}_{11}^2(x) = 0 = \widetilde{\alpha}_{33}^4(x),$$

and we say that f is *isotropic with positive spin* at $x \in M_1^2$ if it is isotropic with negative spin at x with respect to the opposite orientation of $\mathbb{Q}_2^4(c)$ (see [39]).

We say that f is *isotropic with negative (positive) spin* if it is isotropic with negative (positive) spin at every point of M_1^2.

Example 3.1. K. Miura in [58] defined an extremal isometric immersion $f : \mathbb{S}_1^2(1) \to \mathbb{S}_2^4(3)$ by

$$f(x,y,z) = (xy, xz, yz, \frac{\sqrt{3}}{6}(2x^2+y^2+z^2), \frac{1}{2}(y^2-z^2)),$$

which corresponds to the Veronese immersion in Riemannian geometry (see also [11] and [57]). K. Hasegawa in [38] proved that f is isotropic with negative spin. Also, composing with homotethies and anti-homotethies of $\mathbb{S}_1^2(1)$ and $\mathbb{S}_2^4(3)$, Hasegawa obtained extremal and isotropic with negative spin immersions of the Veronese type from $\mathbb{Q}_1^2(\frac{c}{3})$ to $\mathbb{Q}_2^4(c)$, $c \neq 0$.

Moreover, we have that f is a parallel immersion. Thus, f is ψ-pseudo-parallel with $\psi = 0$ and λ-isotropic with $\lambda = 1$.

In fact, let $\gamma_x(s)$ and $\gamma_\theta(s)$ be curves parameterized by arc length in $\mathbb{S}_1^2(1)$, defined by

$$\gamma_x(s) = \left(x, \sqrt{x^2+1}\cos\left(\frac{s}{\sqrt{x^2+1}}\right), \sqrt{x^2+1}\sin\left(\frac{s}{\sqrt{x^2+1}}\right)\right), \; x \text{ constant}.$$
$$\gamma_\theta(s) = (\sinh(s), \cosh(s)\cos\theta, \cosh(s)\sin\theta), \; \theta \text{ constant}.$$

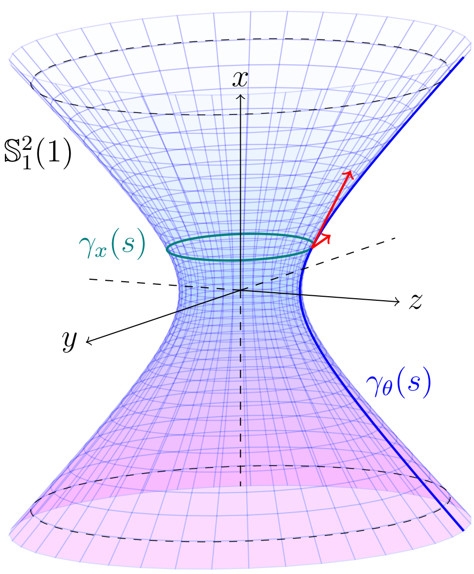

Figure 3.1: Parameterized curves $\gamma_x(s)$ and $\gamma_\theta(s)$ whose velocity vectors provide an orthonormal frame of the tangent space to $\mathbb{S}_1^2(1)$.

Taking the derivative respect to s, we get the velocity vectors

$$\gamma_x'(s) = \left(0, -\sin\left(\frac{s}{\sqrt{x^2+1}}\right), \cos\left(\frac{s}{\sqrt{x^2+1}}\right)\right) = \frac{1}{\sqrt{x^2+1}}(0,-z,y),$$
$$\gamma_\theta'(s) = (\cosh(s), \sinh(s)\cos\theta, \sinh(s)\sin\theta) = \frac{1}{\sqrt{x^2+1}}\left(x^2+1, xy, xz\right),$$

since $1 \leq \cosh(s) = \sqrt{y^2+z^2} = \sqrt{x^2+1}$ and $\sinh(s) = x$, $\cosh(s)\cos\theta = y$, $\cosh(s)\sin\theta = z$ (see Figure 3.1). Thus, we have that $\{T_1, T_2\}$ is an orthonormal frame of $\mathbb{S}_1^2(1)$, where

$$T_1(x,y,z) = \frac{1}{\sqrt{x^2+1}}(0,-z,y),$$
$$T_2(x,y,z) = \frac{1}{\sqrt{x^2+1}}\left(x^2+1, xy, xz\right),$$

with $\langle T_1, T_1\rangle = 1$, $\langle T_2, T_2\rangle = -1$ and $\langle T_1, T_2\rangle = 0$

Next, we apply the function f to obtain the following parameterized curves in $\mathbb{S}_2^4(3)$:

$$\begin{aligned} f(\gamma_x(s)) = \Bigg(& x\sqrt{x^2+1}\cos\left(\frac{s}{\sqrt{x^2+1}}\right), x\sqrt{x^2+1}\sin\left(\frac{s}{\sqrt{x^2+1}}\right), \\ & \frac{(x^2+1)}{2}\sin\left(\frac{2s}{\sqrt{x^2+1}}\right), \frac{\sqrt{3}}{6}(1+3x^2), \\ & \frac{(x^2+1)}{2}\cos\left(\frac{2s}{\sqrt{x^2+1}}\right)\Bigg), \\ f(\gamma_\theta(s)) = \Bigg(& \frac{1}{2}\sinh(2s)\cos\theta, \frac{1}{2}\sinh(2s)\sin\theta, \frac{1}{2}\cosh^2(s)\sin(2\theta), \\ & \frac{\sqrt{3}}{6}(1+3\sinh^2(s)), \frac{1}{2}\cosh^2(s)\cos(2\theta)\Bigg). \end{aligned}$$

Taking the derivative respect to s, we get

$$\begin{aligned} (f\circ\gamma_x)'(s) &= \Bigg(-x\sin\left(\frac{s}{\sqrt{x^2+1}}\right), x\cos\left(\frac{s}{\sqrt{x^2+1}}\right), \\ &\qquad \sqrt{x^2+1}\cos\left(\frac{2s}{\sqrt{x^2+1}}\right), 0, -\sqrt{x^2+1}\sin\left(\frac{2s}{\sqrt{x^2+1}}\right)\Bigg) \\ &= \frac{1}{\sqrt{x^2+1}}(-xz, xy, y^2-z^2, 0, -2yz), \end{aligned}$$

and

$$\begin{aligned} (f\circ\gamma_\theta)'(s) &= \Bigg(\cosh(2s)\cos\theta, \cosh(2s)\sin\theta, \frac{1}{2}\sinh(2s)\sin(2\theta), \\ &\qquad \frac{\sqrt{3}}{2}\sinh(2s), \frac{1}{2}\sinh(2s)\cos(2\theta)\Bigg) \\ &= \frac{1}{\sqrt{x^2+1}}((2x^2+1)y, (2x^2+1)z, 2xyz, \sqrt{3}x(x^2+1), x(y^2-z^2)). \end{aligned}$$

Thus, we get the following tangent vector fields of $f(\mathbb{S}^2_1(1))$

$$\begin{aligned} e_1 = df(T_1) &= \frac{1}{\sqrt{x^2+1}}(-xz, xy, y^2-z^2, 0, -2yz), \\ e_2 = df(T_2) &= \frac{1}{\sqrt{x^2+1}}((2x^2+1)y, (2x^2+1)z, 2xyz, \sqrt{3}x(x^2+1), x(y^2-z^2)). \end{aligned}$$

We observe that $\{e_1, e_2\}$ define an orthonormal frame of $f(\mathbb{S}^2_1(1))$ with $\langle e_1, e_1\rangle = 1$, $\langle e_2, e_2\rangle = -1$ and $\langle e_1, e_2\rangle = 0$. Thus, f is an isometric immersion.

Now, denoting by $\check{\nabla}$ the usual directional derivative in $\mathbb{E}^3_1$, we obtain

$$\begin{aligned} \check{\nabla}_{T_1}T_1 &= -\frac{z}{\sqrt{x^2+1}}\check{\nabla}_{\partial_y}\left(0, -\frac{z}{\sqrt{x^2+1}}, \frac{y}{\sqrt{x^2+1}}\right) + \frac{y}{\sqrt{x^2+1}}\check{\nabla}_{\partial_z}\left(0, -\frac{z}{\sqrt{x^2+1}}, \frac{y}{\sqrt{x^2+1}}\right) \\ &= -\frac{1}{x^2+1}(0, y, z), \\ \check{\nabla}_{T_2}T_1 &= \sqrt{x^2+1}\check{\nabla}_{\partial_x}\left(0, -\frac{z}{\sqrt{x^2+1}}, \frac{y}{\sqrt{x^2+1}}\right) + \frac{xy}{\sqrt{x^2+1}}\check{\nabla}_{\partial_y}\left(0, -\frac{z}{\sqrt{x^2+1}}, \frac{y}{\sqrt{x^2+1}}\right) \\ &\quad + \frac{xz}{\sqrt{x^2+1}}\check{\nabla}_{\partial_z}\left(0, -\frac{z}{\sqrt{x^2+1}}, \frac{y}{\sqrt{x^2+1}}\right) \\ &= (0,0,0). \end{aligned}$$

A unit normal vector field to $\mathbb{S}^2_1(1)$ in $\mathbb{E}^3_1$ is given by the position vector $p = (x, y, z)$ and we have that

$$\langle \check{\nabla}_{T_1}T_1, p\rangle = -\frac{1}{x^2+1}\langle (0, y, z), (x, y, z)\rangle = -\frac{1}{x^2+1}(y^2+z^2) = -1.$$

Thus, for the Levi-Civita conection ∇ of $\mathbb{S}^2_1(1)$, we have that

$$\begin{aligned} \nabla_{T_1}T_1 &= -\frac{1}{x^2+1}(0, y, z) + (x, y, z) = \left(x, \frac{x^2y}{x^2+1}, \frac{x^2z}{x^2+1}\right) = \frac{x}{\sqrt{x^2+1}}T_2, \\ \nabla_{T_2}T_1 &= 0, \end{aligned}$$

and considering immersion f, we write

$$\begin{aligned} \nabla_{e_1}e_1 &= df\left(\nabla_{T_1}T_1\right) = \frac{x}{\sqrt{x^2+1}}df(T_2) \\ &= \frac{1}{x^2+1}((2x^2+1)xy, (2x^2+1)xz, 2x^2yz, \sqrt{3}x^2(x^2+1), x^2(y^2-z^2)), \\ \nabla_{e_2}e_1 &= df\left(\nabla_{T_2}T_1\right) = 0. \end{aligned}$$

Let $x_1 = xy$, $x_2 = xz$, $x_3 = yz$, $x_4 = \frac{\sqrt{3}}{6}(3x^2+1)$ and $x_5 = \frac{y^2-z^2}{2}$, then

$$\begin{aligned} e_1 = df(T_1) &= \sqrt{\frac{2x_5}{x_1^2-x_2^2+2x_5}}(-x_2, x_1, 2x_5, 0, -2x_3), \\ \nabla_{e_1}e_1 &= \frac{2}{x_1^2-x_2^2+2x_5}\Big((x_1^2-x_2^2+x_5)x_1, (x_1^2-x_2^2+x_5)x_2, (x_1^2-x_2^2)x_3, \\ &\qquad \frac{\sqrt{3}}{2}(x_1^2-x_2^2)\left(\frac{x_1^2-x_2^2}{2x_5}+1\right), (x_1^2-x_2^2)x_5\Big). \end{aligned}$$

Next, let $\hat{\nabla}$ be the usual directional derivative in $\mathbb{E}_2^5$, we have that

$$\begin{aligned}
\hat{\nabla}_{e_1}e_1 &= \frac{1}{\sqrt{\frac{x_1^2-x_2^2}{2x_5}+1}}\left(-x_2\hat{\nabla}_{\partial_{x_1}}e_1 + x_1\hat{\nabla}_{\partial_{x_2}}e_1 + 2x_5\hat{\nabla}_{\partial_{x_3}}e_1 - 2x_3\hat{\nabla}_{\partial_{x_5}}e_1\right)\\
&= \frac{1}{\sqrt{\frac{x_1^2-x_2^2}{2x_5}+1}}\Bigg\{-x_2\Bigg(\frac{x_1x_2}{2x_5\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}}, \frac{1}{\sqrt{\frac{x_1^2-x_2^2}{2x_5}+1}} - \frac{x_1^2}{2x_5\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}}, \frac{-x_1}{\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}},\\
&\quad 0, \frac{x_1x_3}{x_5\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}}\Bigg) + x_1\Bigg(\frac{-1}{\sqrt{\frac{x_1^2-x_2^2}{2x_5}+1}} - \frac{x_2^2}{2x_5\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}}, \frac{x_1x_2}{2x_5\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}},\\
&\quad \frac{x_2}{\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}}, 0, \frac{-x_2x_3}{x_5\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}}\Bigg) + 2x_5\left(0,0,0,0,-\frac{2}{\sqrt{\frac{x_1^2-x_2^2}{2x_5}+1}}\right)\\
&\quad -2x_3\Bigg(-\frac{x_2(x_1^2-x_2^2)}{4x_5^2\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}}, \frac{x_1(x_1^2-x_2^2)}{4x_5^2\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}}, \frac{2}{\sqrt{\frac{x_1^2-x_2^2}{2x_5}+1}} + \frac{(x_1^2-x_2^2)}{2x_5\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}}, 0,\\
&\quad -\frac{x_3(x_1^2-x_2^2)}{2x_5^2\left(\frac{x_1^2-x_2^2}{2x_5}+1\right)^{3/2}}\Bigg)\Bigg\}\\
&= \left(\frac{-2x_1x_2^2x_5}{(x_1^2-x_2^2+2x_5)^2}, \frac{2x_1^2x_2x_5}{(x_1^2-x_2^2+2x_5)^2} - \frac{2x_2x_5}{x_1^2-x_2^2+2x_5}, \frac{4x_1x_2x_5^2}{(x_1^2-x_2^2+2x_5)^2}, 0, \frac{-4x_1x_2x_3x_5}{(x_1^2-x_2^2+2x_5)^2}\right)\\
&\quad + \left(\frac{-2x_1x_5}{x_1^2-x_2^2+2x_5} - \frac{2x_1x_2^2x_5}{(x_1^2-x_2^2+2x_5)^2}, \frac{2x_1^2x_2x_5}{(x_1^2-x_2^2+2x_5)^2}, \frac{4x_1x_2x_5^2}{(x_1^2-x_2^2+2x_5)^2}, 0, \frac{-4x_1x_2x_3x_5}{(x_1^2-x_2^2+2x_5)^2}\right)\\
&\quad - \left(0,0,0,0,\frac{8x_5^2}{x_1^2-x_2^2+2x_5}\right) + \left(\frac{2x_2x_3(x_1^2-x_2^2)}{(x_1^2-x_2^2+2x_5)^2}, \frac{-2x_1x_3(x_1^2-x_2^2)}{(x_1^2-x_2^2+2x_5)^2}, \frac{-8x_3x_5}{x_1^2-x_2^2+2x_5}\right.\\
&\quad \left. - \frac{4x_3x_5(x_1^2-x_2^2)}{(x_1^2-x_2^2+2x_5)^2}, 0, \frac{4x_3^2(x_1^2-x_2^2)}{(x_1^2-x_2^2+2x_5)^2}\right),\\
&= -\frac{2x_5}{x_1^2-x_2^2+2x_5}(x_1,x_2,4x_3,0,4x_5).
\end{aligned}$$

A normal vector field to $\mathbb{S}_2^4(3)$ in $\mathbb{E}_2^5$ is given by $f(p) = (x_1,x_2,x_3,x_4,x_5)$, with $\langle f(p), f(p)\rangle = \dfrac{1}{3}$. Then, we have that

$$\langle\hat{\nabla}_{e_1}e_1, f(p)\rangle = \frac{-2x_5}{x_1^2-x_2^2+2x_5}\langle(x_1,x_2,4x_3,0,4x_5),(x_1,x_2,x_3,x_4,x_5)\rangle = -1,$$

and we obtain

$$\begin{aligned}
\alpha_{11} &= \hat{\nabla}_{e_1}e_1 - 3\langle\hat{\nabla}_{e_1}e_1, f(p)\rangle f(p) - \nabla_{e_1}e_1\\
&= \left(xy, xz, yz\left(\frac{x^2-1}{x^2+1}\right), \frac{\sqrt{3}}{2}(x^2+1), \frac{y^2-z^2}{2}\left(\frac{x^2-1}{x^2+1}\right)\right).
\end{aligned}$$

Analogously, we obtain

$$\alpha_{12} = \left(-z, y, \frac{x(y^2-z^2)}{x^2+1}, 0, -\frac{2xyz}{x^2+1}\right).$$

Since f is an extremal immersion, i.e., $\mathcal{H} = 0$, we get that $\alpha_{11} = \alpha_{22}$.

Also, we have that $\{\alpha_{11}, \alpha_{12}\}$ is an orthonormal frame of $N_f\mathbb{S}^2_1(1)$, with

$$\langle \alpha_{11}, \alpha_{11}\rangle = 1, \langle \alpha_{12}, \alpha_{12}\rangle = -1, \text{ and } \langle \alpha_{11}, \alpha_{12}\rangle = 0.$$

Then, there exists a 1-form Φ, such that

$$\nabla^\perp_X \alpha_{11} = \Phi(X)\alpha_{12} \quad \text{and} \quad \nabla^\perp_X \alpha_{12} = \Phi(X)\alpha_{11}.$$

Also, it follows from Ricci equation (2.2) that $R^\perp \neq 0$.

Now, we have

$$(\overline{\nabla}_{e_1}\cdot\alpha)(e_1,e_1) = \nabla^\perp_{e_1}\alpha_{11} - 2\alpha(\nabla_{e_1}e_1,e_1) = \left(\Phi(e_1) - \frac{2x}{\sqrt{x^2+1}}\right)\alpha_{12}, \tag{3.1}$$

$$\begin{aligned}(\overline{\nabla}_{e_1}\cdot\alpha)(e_1,e_2) &= \nabla^\perp_{e_1}\alpha_{12} - \alpha(\nabla_{e_1}e_1,e_2) - \alpha(e_1,\nabla_{e_1}e_2)\\ &= \left(\Phi(e_1) - \frac{2x}{\sqrt{x^2+1}}\right)\alpha_{11},\end{aligned} \tag{3.2}$$

$$(\overline{\nabla}_{e_2}\cdot\alpha)(e_1,e_1) = \nabla^\perp_{e_2}\alpha_{11} - 2\alpha(\nabla_{e_2}e_1,e_1) = \Phi(e_2)\alpha_{12}, \tag{3.3}$$

$$(\overline{\nabla}_{e_2}\cdot\alpha)(e_1,e_2) = \nabla^\perp_{e_2}\alpha_{12} - \alpha(\nabla_{e_2}e_1,e_2) - \alpha(e_1,\nabla_{e_2}e_2) = \Phi(e_2)\alpha_{11}. \tag{3.4}$$

Also,

$$A_{\alpha_{11}} = \begin{pmatrix} \langle A_{\alpha_{11}}e_1, e_1\rangle & \langle A_{\alpha_{11}}e_2, e_1\rangle \\ -\langle A_{\alpha_{11}}e_1, e_2\rangle & -\langle A_{\alpha_{11}}e_2, e_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$A_{\alpha_{12}} = \begin{pmatrix} \langle A_{\alpha_{12}}e_1, e_1\rangle & \langle A_{\alpha_{12}}e_2, e_1\rangle \\ -\langle A_{\alpha_{12}}e_1, e_2\rangle & -\langle A_{\alpha_{12}}e_2, e_2\rangle \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore,

$$A_{\alpha_{11}}e_1 = e_1, \quad A_{\alpha_{11}}e_2 = -e_2, \quad A_{\alpha_{12}}e_1 = e_2 \quad \text{and} \quad A_{\alpha_{12}}e_2 = -e_1.$$

On the other hand, using that

$$\begin{aligned}\alpha_{11} = \Bigg(& x_1, x_2, x_3\left(\frac{x_1^2-x_2^2-2x_5}{x_1^2-x_2^2+2x_5}\right), \frac{\sqrt{3}}{2}\left(\frac{x_1^2-x_2^2}{2x_5}+1\right),\\ & x_5\left(\frac{x_1^2-x_2^2-2x_5}{x_1^2-x_2^2+2x_5}\right)\Bigg),\end{aligned}$$

$$e_1 = \sqrt{\frac{2x_5}{x_1^2-x_2^2+2x_5}}(-x_2, x_1, 2x_5, 0, -2x_3),$$

we obtain that

$$\hat{\nabla}_{e_1}\alpha_{11} = \frac{1}{\sqrt{x^2+1}}\left(-xz, xy, \frac{(y^2-z^2)(x^2-1)}{x^2+1}, 0, -\frac{2yz(x^2-1)}{x^2+1}\right).$$

Next, for the Levi-Civita connection $\widetilde{\nabla}$ of $\mathbb{S}^4_2(3)$, we have

$$\nabla^{\perp}_{e_1}\alpha_{11} = \widetilde{\nabla}_{e_1}\alpha_{11} + A_{\alpha_{11}}e_1 = \hat{\nabla}_{e_1}\alpha_{11} + e_1 = \frac{2x}{\sqrt{x^2+1}}\alpha_{12},$$

since $\langle \hat{\nabla}_{e_1}\alpha_{11}, f(x,y,z)\rangle = 0$. Thus, $\Phi(e_1) = \dfrac{2x}{\sqrt{x^2+1}}$. It follows from equations (3.1) and (3.2) that

$$(\overline{\nabla}_{e_1}\cdot\alpha)(e_1,e_1) = (\overline{\nabla}_{e_1}\cdot\alpha)(e_1,e_2) = 0.$$

Analogously,

$$\nabla^{\perp}_{e_2}\alpha_{11} = \widetilde{\nabla}_{e_2}\alpha_{11} + A_{\alpha_{11}}e_2 = e_2 - e_2 = 0.$$

Thus, $\Phi(e_2) = 0$ and, from equations (3.3) and (3.4), we obtain

$$(\overline{\nabla}_{e_2}\cdot\alpha)(e_1,e_1) = (\overline{\nabla}_{e_2}\cdot\alpha)(e_1,e_2) = 0.$$

Then, we conclude that f is a parallel immersion.

Also, we have that $\langle \alpha_{11}+\alpha_{22}, \alpha_{11}+\alpha_{22}\rangle = 4\langle \alpha_{11},\alpha_{11}\rangle = 4 = -4\langle \alpha_{12},\alpha_{12}\rangle$, $\langle \alpha_{11}+\alpha_{22},\alpha_{12}\rangle = 0$ and $\mathcal{H} = 0$. Therefore, from Lemma 2.11, f is a λ-isotropic immersion with $\lambda = -\langle \alpha_{12},\alpha_{12}\rangle = 1$.

3.1.2 Pseudo-parallel Lorentzian surfaces with constant pseudo-parallelism function ψ in $\mathbb{Q}^4_s(c)$

Hasegawa in [38], showed that an extremal and isotropic with negative spin Lorentzian surface with constant Gaussian curvature in $\mathbb{Q}^4_s(c)$ is congruent to a piece of a surface of the Veronese type from Example 3.1. Also, isotropy with negative spin has an interpretation using the concept of hyperbola of curvature, but the plane of the hyperbola is not necessarily orthogonal to $\mathcal{H}$ (see [39]). We use Hasegawa's characterization to get the next result:

Corollary 3.2. *Let $f : M^2_1 \to \mathbb{Q}^4_s(c)$ be an isometric immersion with $R^{\perp} \neq 0$. f is ψ-pseudo-parallel if and only if $s = 2$ and f is an extremal and isotropic with negative spin immersion. Moreover, if ψ is constant, then $K = \dfrac{c}{3} \neq 0$ and locally $f(M^2_1)$ is congruent to an open set of the Veronese type surface given in Example 3.1.*

Proof. If $f : M^2_1 \to \mathbb{Q}^4_s(c)$ is ψ-pseudo-parallel with $R^{\perp} \neq 0$, from Theorem 2.14 we have that f is λ-isotropic, $s = 2$ and from Corollary 2.19 f is extremal. For $x \in M^2_1$, let $\{\widetilde{e}_1, \widetilde{e}_3\}$ be a local pseudo-orthonormal frame that defines the orientation of U, with $x \in U$ and U open in M^2_1, such that

$\langle\widetilde{e}_1,\widetilde{e}_1\rangle = \langle\widetilde{e}_3,\widetilde{e}_3\rangle = 0$ and $\langle\widetilde{e}_1,\widetilde{e}_3\rangle = 1$. Denote $\widetilde{\alpha}_{ij} = \widetilde{\alpha}(\widetilde{e}_i,\widetilde{e}_j)$ for $i,j \in \{1,3\}$, it follows from Ricci equation that

$$\begin{aligned} R^{\perp}(\widetilde{e}_1,\widetilde{e}_3)\xi &= \alpha(\widetilde{e}_1,A_\xi\widetilde{e}_3) - \alpha(A_\xi\widetilde{e}_1,\widetilde{e}_3) \\ &= \langle A_\xi\widetilde{e}_3,\widetilde{e}_3\rangle\widetilde{\alpha}_{11} + \langle A_\xi\widetilde{e}_3,\widetilde{e}_1\rangle\widetilde{\alpha}_{13} - \langle A_\xi\widetilde{e}_1,\widetilde{e}_3\rangle\widetilde{\alpha}_{13} - \langle A_\xi\widetilde{e}_1,\widetilde{e}_1\rangle\widetilde{\alpha}_{33} \\ &= \langle\widetilde{\alpha}_{33},\xi\rangle\widetilde{\alpha}_{11} + \langle\widetilde{\alpha}_{13},\xi\rangle\widetilde{\alpha}_{13} - \langle\widetilde{\alpha}_{13},\xi\rangle\widetilde{\alpha}_{13} - \langle\widetilde{\alpha}_{11},\xi\rangle\widetilde{\alpha}_{33} \\ &= \langle\widetilde{\alpha}_{33},\xi\rangle\widetilde{\alpha}_{11} - \langle\widetilde{\alpha}_{11},\xi\rangle\widetilde{\alpha}_{33} \\ &= (\widetilde{\alpha}_{11}\wedge\widetilde{\alpha}_{33})\xi. \end{aligned}$$

Then, we have that $\widetilde{\alpha}_{11}$ and $\widetilde{\alpha}_{33}$ are linearly independent since $R^{\perp} \neq 0$. Moreover, from isotropy condition and Lemma 2.11 we have that $\widetilde{\alpha}_{11}$ and $\widetilde{\alpha}_{33}$ are lightlike vectors. Thus, choosing conveniently the orientation in $\mathbb{Q}_2^4(c)$, we conclude that f is isotropic with negative spin at x (see [39]).

Conversely, let $f : M_1^2 \to \mathbb{Q}_2^4(c)$ be an extremal and isotropic with negative spin isometric immersion and $R^{\perp} \neq 0$. For $x \in M_1^2$, let $\{\widetilde{e}_1,\widetilde{e}_3\}$ be a local pseudo-orthonormal frame that defines the orientation of U, with $x \in U$ and U open in M_1^2, such that $\langle\widetilde{e}_1,\widetilde{e}_1\rangle = \langle\widetilde{e}_3,\widetilde{e}_3\rangle = 0$ and $\langle\widetilde{e}_1,\widetilde{e}_3\rangle = 1$. Then,

$$\mathcal{H} = \frac{1}{2}\text{trace}(\alpha) = \widetilde{\alpha}_{13}. \tag{3.5}$$

For any unit tangent vector $X \in T_xM$, there exists $t \in \mathbb{R}$ and $\epsilon_1,\epsilon_3 \in \{-1,1\}$ such that $X = \frac{1}{\sqrt{2}}(\epsilon_1 t\widetilde{e}_1 + \frac{\epsilon_3}{t}\widetilde{e}_3)$. Since f is extremal, we have

$$\alpha(X,X) = \epsilon_1\epsilon_3\mathcal{H} + \frac{t^2}{2}\widetilde{\alpha}_{11} + \frac{1}{2t^2}\widetilde{\alpha}_{33} = \frac{t^2}{2}\widetilde{\alpha}_{11} + \frac{1}{2t^2}\widetilde{\alpha}_{33}. \tag{3.6}$$

Since f is isotropic with negative spin we have that $\langle\widetilde{\alpha}_{11},\widetilde{\alpha}_{11}\rangle = \langle\widetilde{\alpha}_{33},\widetilde{\alpha}_{33}\rangle = 0$ and we get

$$\langle\alpha(X,X),\alpha(X,X)\rangle = \frac{1}{2}\langle\widetilde{\alpha}_{11},\widetilde{\alpha}_{33}\rangle. \tag{3.7}$$

Therefore, f is λ-isotropic with $\lambda = \frac{1}{2}\langle\widetilde{\alpha}_{11},\widetilde{\alpha}_{33}\rangle$ and by Theorem 2.14 f is ψ-pseudo-parallel, since $R^{\perp} \neq 0$.

If in addition ψ is constant, from Theorem 2.14, we have that $K = \frac{2\psi + c}{3}$ is constant in M_1^2 and $K \neq c$ since $K \neq \psi$. Then, the claim follows from Corollary 4 in [38]. □

3.2 Examples of pseudo-parallel Lorentzian surfaces in $\mathbb{E}_2^4$ with non-constant ψ

For general rotational surfaces of elliptic type with plane meridians in $\mathbb{E}_2^4$, defined by 1.23, we are looking for those satisfying condition of being extremal and λ-isotropic.

Cabrerizo-Fernández-Gómez in Corollary 6.3, Corollary 6.6 and Proposition 6.8 of [13], proved that

- non-totally umbilical λ-isotropic Lorentzian surfaces in $\mathbb{E}^4_2$ with constant Gaussian curvature or constant λ are 0-isotropic surfaces with $K=0$. Moreover, at each non-geodesic point, all non-zero vectors in the first normal space $Im(\alpha)$ are lightlike vectors, thus, $Im(\alpha)$ is one-dimensional in this case;

- any non-totally umbilical λ-isotropic Lorentzian surface in $\mathbb{E}^4_2$ with $\lambda \neq 0$ (or $K \neq 0$) everywhere, has $\mathcal{H}=0$ and $Im(\alpha)$ is all of the normal space to the surface. In this case, $K=-2\lambda$.

Note that the mean curvature vector field of $\mathcal{M}_1$, given by $\mathcal{H}=\dfrac{\nu_2-\nu_1}{2}\eta_2$, is never lightlike. Analogously, $\alpha(X,Y)$ and $\alpha(X,X)+\alpha(Y,Y)$ are not lightlike. So, we get

Proposition 3.3. *Let $\mathcal{M}_1$ be a general rotational surface of elliptic type with plane meridians in $\mathbb{E}^4_2$. If $\mathcal{M}_1$ is a non-totally umbilical λ-isotropic surface, then $\mathcal{H}=0$, K is non-constant with $K=-2\lambda$ and in non-geodesic points the second fundamental form α is a surjective mapping.*

If $\mathcal{H}=0$ we have $\nu_1=\nu_2=\nu$ and from [4] we have the next result for extremal general rotational surfaces.

Theorem 3.4 (Aleksieva-Milousheva-Turgay [4]). *Let $\mathcal{M}_1$ be a general rotational surface of elliptic type in $\mathbb{E}^4_2$, defined by (1.23). Then $\mathcal{M}_1$ has $\mathcal{H}=0$ if and only if the meridian m is determined by one of the following:*

(i) $f=cg^k$, *$c\neq 0$ constant,* $k=\pm\frac{\theta}{\beta}\neq\pm 1$*;*

(ii) $\arcsin\left(\frac{\theta f}{\sqrt{A}}\right)=\pm\frac{\theta}{\beta}\arcsin\left(\frac{\beta g}{\sqrt{A}}\right)+C$*; C constant, $A>0$ constant, $\theta\neq\beta$;*

(iii) $(f+g)^2=a(f-g)^2+b$*, where $a\neq 0$ constant, b constant, $\theta=\beta$.*

From [9], we have that a Lorentzian surface with $R^{\perp}\neq 0$ in $\mathbb{E}^4_2$ is ψ-pseudo-parallel if and only if it is λ-isotropic with $\mathcal{H}=0$. In this case, we have that $\psi=-3\lambda=\frac{3}{2}K$. Moreover, condition $R^{\perp}\neq 0$ for $\mathcal{M}_1$ is equivalent to the fact that $\mathrm{span}\{\alpha(X,X)+\alpha(Y,Y),\alpha(X,Y)\}$ is all of the normal space to the surface in $\mathbb{E}^4_2$.

On the other hand, for each unit tangent vector Z of $\mathcal{M}_1$, there exists $t\in\mathbb{R}$ and $\epsilon_1\in\{-1,1\}$, such that

$$\langle Z,Z\rangle\alpha(Z,Z)=\mathcal{H}+\langle Z,Z\rangle[\cosh(2t)\frac{\alpha(X,X)+\alpha(Y,Y)}{2}+\epsilon_1\sinh(2t)\alpha(X,Y)].$$

Thus, if $\mathcal{M}_1$ is extremal, then it is also λ-isotropic if and only if $\langle\alpha(X,X)+\alpha(Y,Y),\alpha(X,Y)\rangle=0$ and $\langle\alpha(X,X)+\alpha(Y,Y),\alpha(X,X)+\alpha(Y,Y)\rangle=-4\langle\alpha(X,Y),\alpha(X,Y)\rangle$ (see Lemma 8 in [9]).

Since $\{\eta_1, \eta_2\}$ is an orthonormal frame, using equations (1.26), we have that $\mathcal{M}_1$ is extremal with $R^\perp \neq 0$ if and only if $\nu = \nu_1 = \nu_2 \neq 0$ and $\mu \neq 0$. In this case, the λ-isotropic condition for $\mathcal{M}_1$ is equivalent to the following equation being satisfied:

$$-4\nu^2 = -(\nu_1+\nu_2)^2 = \langle \alpha(X,X)+\alpha(Y,Y), \alpha(X,X)+\alpha(Y,Y)\rangle = -4\langle \alpha(X,Y), \alpha(X,Y)\rangle = -4\mu^2.$$

This means that $\mathcal{M}_1$ is λ-isotropic with $\mathcal{H} = 0$ and $R^\perp \neq 0$, and so it is pseudo-parallel, if and only if $\nu^2 = \mu^2 \neq 0$, with $\nu = \nu_1 = \nu_2$.

Now, we note from the proof of Theorem 3.4 in [4] that $\nu^2 \neq \mu^2$ in case *(ii)*, thus, $\mathcal{M}_1$ is not pseudo-parallel with $R^\perp \neq 0$ in this case. On the other hand, $\theta = \beta$ in case *(iii)* and it follows from (1.27) that $\mu = -\nu_2 = -\nu$. Finally, $\mu^2 - \nu^2 = 0$ in case *(i)*, but this case is empty. In fact, in *(i)*, we have that $f' = \pm c\frac{\theta}{\beta} g^{\pm\frac{\theta}{\beta}-1} g'$. Next, from conditions for f, g, f' and g' given in (1.24) we obtain

$$\begin{aligned}
G = \theta^2 f^2(u) - \beta^2 g^2(u) &= \theta^2 c^2 g^{\pm 2\frac{\theta}{\beta}}(u) - \beta^2 g^2(u) \\
&= g^2(u)\left(\theta^2 c^2 g^{2\left(\pm\frac{\theta}{\beta}-1\right)}(u) - \beta^2\right) < 0, \\
E = f'^2(u) - g'^2(u) &= c^2 \frac{\theta^2}{\beta^2} g^{2\left(\pm\frac{\theta}{\beta}-1\right)}(u) g'^2(u) - g'^2(u) \\
&= \frac{g'^2(u)}{\beta^2}\left(\theta^2 c^2 g^{2\left(\pm\frac{\theta}{\beta}-1\right)}(u) - \beta^2\right) > 0.
\end{aligned}$$

Then, we have a contradiction and *(i)* in Theorem 3.4 cannot happen.

Thus, for pseudo-parallel surfaces with $R^\perp \neq 0$, we obtain:

Theorem 3.5. *Let $\mathcal{M}_1$ be a general rotational Lorentzian surface of elliptic type in $\mathbb{E}^4_2$, defined by (1.23). Then $\mathcal{M}_1$ is pseudo-parallel with $R^\perp \neq 0$ if and only if the meridian m is determined by $(f+g)^2 = a(f-g)^2 + b$, with $fg' \neq gf'$ everywhere, $a \neq 0$ constant, b constant and $\theta = \beta$. In this case, $\psi = \frac{3}{2}K = 3\mu^2$.*

Note that, for example, functions $g(u) = u$ and $f(u) = \sqrt{1-u^2}$, $u \in J = (\frac{1}{\sqrt{2}}, 1)$, satisfy conditions in Theorem 3.5 for $a = -1$, $b = 2$, $\theta = \beta = 1$ and satisfy constraints in (1.24).

Observation 3.6. Since $fg' \neq gf'$, we have that $\mu \neq 0$ everywhere and all the surfaces in Theorem 3.5 are not semi-parallel (see also Corollary 13 in [9]). From Proposition 3.3, we have that K is non-constant.

For extremal general rotational surfaces of hyperbolic type, we have from [4] the next result

Theorem 3.7 (Aleksieva-Milousheva-Turgay, [4]). *Let $\mathcal{M}_2$ be a general rotational surface of hyperbolic type in $\mathbb{E}^4_2$, defined by (1.28). Then $\mathcal{M}_2$ has $\mathcal{H} = 0$ if and only if the meridian m is determined by one of the following:*

(i) $f = cg^k$, $c \neq 0$ *constant,* $k = \pm\frac{\theta}{\beta} \neq \pm 1$;

(ii) $\theta f + \sqrt{\theta^2 f^2 - A} = C\left(\beta g + \sqrt{\beta^2 g^2 + A}\right)^{\pm\frac{\theta}{\beta}}$, *C constant, A constant* $AC \neq 0$, $\theta \neq \beta$;

(iii) $\arctan\left(\frac{f'}{g'}\right) = -\arctan\left(\frac{f}{g}\right) + b$, *where b constant,* $\theta = \beta$.

We observe that, in Theorem 3.7, case *(i)* is not empty, since the conditions in (1.29) only require that at least one of f or g is non-zero and that at least one of f' or g' is non-zero, thus, $g(u) = u$ and $f(u) = u^2$ satisfy equation in case *(i)*, with $\theta = 2\beta$, $c = 1$ and $u > 0$. Also, $g(u) = u$ and $f(u) = \frac{1}{u}$ satisfy equation in case *(iii)* with $b = 0$ and $u > 0$. An analysis analogous to that for Theorem 3.5 leads to the next result:

Theorem 3.8. *Let $\mathcal{M}_2$ be a general rotational surface of hyperbolic type in $\mathbb{E}^4_2$, defined by (1.28). Then $\mathcal{M}_2$ is pseudo-parallel with $R^\perp \neq 0$ if and only if the meridian m is determined by*

(i) $f = cg^k$, $c \neq 0$ *constant,* $k = \pm\frac{\theta}{\beta} \neq \pm 1$;

(ii) $\arctan\left(\frac{f'}{g'}\right) = -\arctan\left(\frac{f}{g}\right) + b$, $fg' \neq gf'$ *everywhere, b constant,* $\theta = \beta$.

For any of these cases, we have $\psi = \frac{3}{2}K = -3\mu^2$.

Observation 3.9. Note that an analogous result to Proposition 3.3 can be obtained for general rotational surfaces of hyperbolic type with plane meridians. Thus, we have that for a surface $\mathcal{M}_2$ as in Theorem 3.8, K is non-constant. Moreover, $\mathcal{M}_2$ is not semi-parallel since $\mu \neq 0$ everywhere.

3.3 A extremal and flat pseudo-parallel Lorentzian surface in $\mathbb{S}^5_2(c)$ which is not semi-parallel

In [9], Asperti-Lobos-Mercuri gave an example of a pseudo-parallel surface in a 5-dimensional Riemannian space form, which is not semi-parallel, namely the standard flat tori. Now, we give an analogous example in the pseudo-Riemannian case which was used by L. Vrancken in [77], for the study of Lagrangian submanifolds in indefinite complex space forms.

Example 3.10. Consider the immersion $T : \mathbb{E}^2_1 \to \mathbb{S}^5_2(c) \subset (\mathbb{E}^6_2, (-,-,+,+,+,+))$, defined by

$$T(x,y) = \frac{2}{\sqrt{6c}}(\cos u \sinh v, \sin u \sinh v, \cos u \cosh v, \sin u \cosh v, \frac{\sqrt{2}}{2}\cos 2u, -\frac{\sqrt{2}}{2}\sin 2u),$$

where $u = \sqrt{\frac{c}{2}}x, v = \frac{\sqrt{6c}}{2}y$. From [77], we have that T is a flat and extremal Lorentzian surface. A direct calculation shows that T is ψ-pseudo-parallel and λ-isotropic with $\psi = -\lambda = -\dfrac{c}{2} \neq 0$ and $R^\perp \neq 0$. Thus, T is not a semi-parallel immersion. Composing T with anti-homotethies of $\mathbb{S}^5_2(c)$, we obtain pseudo-parallel immersions with non-flat normal bundle and constant ψ in $\mathbb{H}^5_3(-c)$.

In fact,

$$\begin{aligned}\langle T(x,y),T(x,y)\rangle &= \frac{2}{3c}(-\sinh^2 v(\cos^2 u+\sin^2 u)+\cosh^2 v(\cos^2 u+\sin^2 u)\\ &\quad +\frac{1}{2}(\cos^2 2u+\sin^2 2u)) = \frac{1}{c},\\ e_1 = dT(\partial_x) &= \frac{1}{\sqrt{3}}(-\sin u\sinh v,\cos u\sinh v,-\sin u\cosh v,\cos u\cosh v,\\ &\quad -\sqrt{2}\sin 2u,-\sqrt{2}\cos 2u),\\ e_2 = dT(\partial_y) &= (\cos u\cosh v,\sin u\cosh v,\cos u\sinh v,\sin u\sinh v,0,0)\end{aligned}$$

and $\langle e_1,e_1\rangle = 1$, $\langle e_2,e_2\rangle = -1$, $\langle e_1,e_2\rangle = 0$. Thus, T is a Lorentzian surface in $\mathbb{S}^5_2(c)$ and $\{e_1,e_2\}$ is an orthonormal frame of T.

Now, setting

$$\begin{aligned}x_1 &= \frac{2}{\sqrt{6c}}\cos u\sinh v,\ x_2 = \frac{2}{\sqrt{6c}}\sin u\sinh v,\ x_3 = \frac{2}{\sqrt{6c}}\cos u\cosh v,\\ x_4 &= \frac{2}{\sqrt{6c}}\sin u\cosh v,\ x_5 = \frac{1}{\sqrt{3c}}\cos 2u \quad\text{and}\quad x_6 = -\frac{1}{\sqrt{3c}}\sin 2u,\end{aligned}$$

we have

$$\begin{aligned}e_1 &= \sqrt{\frac{c}{2}}(-x_2,x_1,-x_4,x_3,2x_6,-2x_5),\\ e_2 &= \frac{\sqrt{6c}}{2}(x_3,x_4,x_1,x_2,0,0),\end{aligned}$$

and

$$\begin{aligned}-x_1^2-x_2^2+x_3^2+x_4^2+x_5^2+x_6^2 &= \frac{1}{c},\\ 3x_5^2+3x_6^2 &= \frac{1}{c}.\end{aligned}$$

Let $\hat{\nabla}$ be the usual directional derivative in $\mathbb{E}^6_2$. We have

$$\hat{\nabla}_{e_1}e_1 = -\frac{c}{2}(x_1,x_2,x_3,x_4,4x_5,4x_6).$$

and

$$\langle\hat{\nabla}_{e_1}e_1,e_1\rangle = \langle\hat{\nabla}_{e_1}e_1,e_2\rangle = 0.$$

Thus,

$$\nabla_{e_1}e_1 = 0$$

and

$$\begin{aligned}
\alpha_{11} &= \hat{\nabla}_{e_1}e_1 - c\langle\hat{\nabla}_{e_1}e_1, T(x,y)\rangle T(x,y)\\
&= -\frac{c}{2}(x_1,x_2,x_3,x_4,4x_5,4x_6) + c(x_1,x_2,x_3,x_4,x_5,x_6)\\
&= \frac{c}{2}(x_1,x_2,x_3,x_4,-2x_5,-2x_6)\\
&= \sqrt{\frac{c}{6}}(\cos u\sinh v, \sin u\sinh v,\\
&\qquad \cos u\cosh v, \sin u\cosh v, -\sqrt{2}\cos 2u, \sqrt{2}\sin 2u).
\end{aligned}$$

Analogously,

$$\begin{aligned}
\hat{\nabla}_{e_2}e_2 &= \frac{3c}{2}(x_1,x_2,x_3,x_4,0,0),\\
\hat{\nabla}_{e_2}e_1 = \hat{\nabla}_{e_1}e_2 &= \frac{\sqrt{3}}{2}c(-x_4,x_3,-x_2,x_1,0,0).
\end{aligned}$$

Thus,

$$\nabla_{e_2}e_2 = 0, \nabla_{e_2}e_1 = \nabla_{e_1}e_2 = 0$$

and

$$\begin{aligned}
\alpha_{22} &= \frac{c}{2}(x_1,x_2,x_3,x_4,-2x_5,-2x_6) = \alpha_{11},\\
\alpha_{12} &= \frac{\sqrt{3}}{2}c(-x_4,x_3,-x_2,x_1,0,0)\\
&= \sqrt{\frac{c}{2}}(-\sin u\cosh v, \cos u\cosh v, -\sin u\sinh v, \cos u\sinh v, 0, 0),
\end{aligned}$$

with $\langle\alpha_{11},\alpha_{11}\rangle = \frac{c}{2}, \langle\alpha_{11},\alpha_{12}\rangle = 0, \langle\alpha_{12},\alpha_{12}\rangle = -\frac{c}{2}$.

Then, we have that $R^{\perp} \neq 0$ since $\alpha_{11}+\alpha_{22} = 2\alpha_{11}$ and α_{12} are linearly independent. Also, $\nabla_{e_1}e_1 = \nabla_{e_2}e_2 = \nabla_{e_2}e_1 = \nabla_{e_1}e_2 = 0$ means that T is flat, i.e., $K = 0$.

Moreover, $\mathcal{H} = \frac{1}{2}(\alpha_{11}-\alpha_{22}) = \frac{1}{2}(\alpha_{11}-\alpha_{11}) = 0$, i.e., T is extremal, $\{H, \alpha_{11}+\alpha_{22}, \alpha_{12}\}$ is an orthogonal set and

$$\langle\alpha_{11}+\alpha_{22}, \alpha_{11}+\alpha_{22}\rangle = 4\langle\alpha_{11},\alpha_{11}\rangle = 2c = -4\langle\alpha_{12},\alpha_{12}\rangle.$$

Thus, from Lemma 2.11 we have that T is λ-isotropic with $\lambda = -\langle\alpha_{12},\alpha_{12}\rangle = \frac{c}{2}$. Finally, using Ricci equation (2.2) we get

$$R^{\perp}(e_1,e_2)\alpha_{11} = R^{\perp}(e_1,e_2)\alpha_{22} = -c\alpha_{12}, \tag{3.8}$$

$$R^{\perp}(e_1,e_2)\alpha_{12} = -\frac{c}{2}(\alpha_{11}+\alpha_{22}). \tag{3.9}$$

It follows from Lemma 2.2 that T is ψ-pseudo-parallel with $\psi = -\frac{c}{2}$, that is, T is not a semi-parallel surface.

Example 3.11. Let $f : \mathbb{Q}_1^2(c) \to \mathbb{Q}_2^4(3c)$, $c \neq 0$ be an immersion of the Veronese type of Example 3.1, if $i : \mathbb{Q}_2^4(3c) \to \mathbb{E}_{2+\sigma}^5$, $\sigma = \frac{c-|c|}{2c}$, and $j : \mathbb{Q}_2^4(3c) \to \mathbb{Q}_2^{4+m}(3c)$, $m \geq 1$, are the inclusions, we have from Proposition 2.10 that $i \circ f$ is a pseudo-parallel Lorentzian surface of $\mathbb{E}_{2+\sigma}^5$ and $j \circ f$ is a pseudo-parallel Lorentzian surface of $\mathbb{Q}_2^{4+m}(3c)$.

Observation 3.12. Classification of ψ-pseudo-parallel Lorentzian surfaces with non-flat normal bundle in $\mathbb{Q}_s^5(c)$, with $s = 2, 3$, is still an open problem. Even in the case of constant ψ, the question still remains as to whether there are other surfaces of this type apart from those presented in Example 3.10 and $j \circ f$ with $m = 1$ in Example 3.11.

CHAPTER 4

Pseudo-parallel Lorentzian hypersurfaces in pseudo-Riemannian space forms

In this chapter, we study pseudo-parallel hypersurfaces in pseudo-Riemannian space forms, especially when the hypersurface and the ambient space both have metric of index 1. Then, we consider the case where the pseudo-parallelism function is constant and different from the curvature of the ambient space and give the classification of such hypersurfaces under the hypothesis of being good in the sense of Ryan. We also give a classification of the complete semi-parallel Lorentzian hypersurfaces in the Minkowski space and of the pseudo-parallel Lorentzian hypersurfaces with constant pseudo-parallelism function and constant mean curvature in Lorentzian space forms.

4.1 Pseudo-parallel hypersurfaces in pseudo-Riemannian space forms

For the case of hypersurfaces, the pseudo-parallel condition (1.13) can be stated in terms of the Weingarten operator A in the locally unique, up a sign, normal η-direction. We say that a hypersurface $f : M_t^n \to \mathbb{Q}_s^{n+1}(c)$ is ψ-pseudo-parallel if it satisfies the condition:

$$R(U,V)\cdot A = \psi(U \wedge V)\cdot A, \tag{4.1}$$

for all $U,V \in TM$, where

$$(R(U,V)\cdot A)Z = R(U,V)AZ - AR(U,V)Z,$$
$$((U \wedge V)\cdot A)Z = (U \wedge V)AZ - A(U \wedge V)Z,$$

for all $U,V,Z \in TM$. Next, using Gauss Equation (1.18), we observe that:

$$\begin{aligned}
(R(U,V)\cdot A)Z-\psi((U\wedge V)\cdot A)Z&\\
&= R(U,V)AZ - AR(U,V)Z - \psi((U\wedge V)AZ - A(U\wedge V)Z)\\
&= c(\langle V,AZ\rangle U - \langle U,AZ\rangle V) + \varepsilon(\langle AV,AZ\rangle AU - \langle AU,AZ\rangle AV)\\
&\quad - A\left(c(\langle V,Z\rangle U - \langle U,Z\rangle V) + \varepsilon(\langle AV,Z\rangle AU - \langle AU,Z\rangle AV)\right)\\
&\quad - \psi(\langle V,AZ\rangle U - \langle U,AZ\rangle V - A(\langle V,Z\rangle U - \langle U,Z\rangle V))\\
&= -(c-\psi)\langle U,AZ\rangle V + (c-\psi)\langle V,AZ\rangle U - (\varepsilon\langle AU,AZ\rangle - (c-\psi)\langle U,Z\rangle)AV\\
&\quad + (\varepsilon\langle AV,AZ\rangle - (c-\psi)\langle V,Z\rangle)AU + \varepsilon\langle AU,Z\rangle A^2V - \varepsilon\langle AV,Z\rangle A^2U.
\end{aligned} \tag{4.2}$$

Thus, from (4.1) and (4.2) we conclude that the hypersurface f is pseudo-parallel if and only if it satisfies

$$\begin{aligned}
0 &= -(c-\psi)\langle U,AZ\rangle V + (c-\psi)\langle V,AZ\rangle U - (\varepsilon\langle AU,AZ\rangle - (c-\psi)\langle U,Z\rangle)AV\\
&\quad + (\varepsilon\langle AV,AZ\rangle - (c-\psi)\langle V,Z\rangle)AU + \varepsilon\langle AU,Z\rangle A^2V - \varepsilon\langle AV,Z\rangle A^2U,
\end{aligned} \tag{4.3}$$

for all $U,V \in TM$.

As pointed out in the Introduction, we recall some results given in [44].

Proposition 4.1 (Lobos, [44])**.** *Let $f : M_t^n \to \mathbb{Q}_s^{n+1}(c)$, $n \geq 2$, be a pseudo-parallel hypersurface. Then, the Weingarten operator A satisfies the polynomial equation:*

$$\varepsilon\ \mathrm{trace}(A)A^2 + \left(n(c-\psi) - \varepsilon\ \mathrm{trace}(A^2)\right)A - (c-\psi)\ \mathrm{trace}(A)I_n = 0, \tag{4.4}$$

where I_n is the identity operator in TM.

Proof. The Weingarten operator A can be expressed in an (local) orthonormal frame $\{E_1,\ldots,E_n\}$ on M_t^n. Let us $\varepsilon_i = \langle E_i,E_i\rangle$. Using (4.2), for all $W \in TM$ we have

$$\begin{aligned}
&\sum_i(\varepsilon_i(R(W,E_i)\cdot A)E_i - \varepsilon_i\psi((W\wedge E_i)\cdot A)E_i) = \sum_i \varepsilon_i\,(-(c-\psi)\langle W,AE_i\rangle E_i + (c-\psi)\langle E_i,AE_i\rangle W\\
&\quad - (\varepsilon\langle AW,AE_i\rangle - (c-\psi)\langle W,E_i\rangle)AE_i + (\varepsilon\langle AE_i,AE_i\rangle - (c-\psi)\langle E_i,E_i\rangle)AW\\
&\quad +\varepsilon\langle AW,E_i\rangle A^2E_i - \varepsilon\langle AE_i,E_i\rangle A^2W\big)\\
&= \sum_i \varepsilon_i\,(-(c-\psi)\langle AW,E_i\rangle E_i + (c-\psi)\langle AE_i,E_i\rangle W - \varepsilon A\left(\langle A^2W,E_i\rangle E_i\right) + (c-\psi)A\left(\langle W,E_i\rangle E_i\right)\\
&\quad + \varepsilon\langle A^2E_i,E_i\rangle AW - (c-\psi)A(\langle E_i,E_i\rangle W) + \varepsilon A^2(\langle AW,E_i\rangle E_i) - \varepsilon\langle AE_i,E_i\rangle A^2W\big)\\
&= -(c-\psi)AW + (c-\psi)\ \mathrm{trace}(A)W - \varepsilon A^3W + (c-\psi)AW + \varepsilon\ \mathrm{trace}(A^2)AW - n(c-\psi)AW\\
&\quad + \varepsilon A^3w - \varepsilon\ \mathrm{trace}(A)A^2W\\
&= -\left(\varepsilon\ \mathrm{trace}(A)A^2 + (n(c-\psi) - \varepsilon\ \mathrm{trace}(A^2))A - (c-\psi)\ \mathrm{trace}(A)I\right)W,
\end{aligned}$$

and from pseudo-parallelism condition (4.1) we obtain the polynomial equation (4.4). $\square$

Proposition 4.2 (Lobos, [44])**.** *Let $f : M_t^n \to \mathbb{Q}_s^{n+1}(c)$, $n \geq 2$, be a hypersurface. If for a point $x \in M_t^n$ there exist real numbers λ, μ such that $A^2 = \lambda A + \mu I_n$ at x, where I_n denotes the identity operator, then f satisfies the pseudo-parallelism condition at x with $\psi(x) = c - \varepsilon\mu$.*

Proof. Using Gauss equation, for $U, V, Z \in T_xM$ we have that

$$\begin{aligned}
(R(U,V)\cdot A)Z =& R(U,V)(AZ) - A(R(U,V)Z)\\
=& c(U\wedge V)AZ + \varepsilon(AU\wedge AV)AZ - A(c(U\wedge V)Z + \varepsilon(AU\wedge AV)Z)\\
=& c(U\wedge V)AZ + \varepsilon(\langle AV, AZ\rangle AU - \langle AU, AZ\rangle AV)\\
&- cA((U\wedge V)Z) - \varepsilon(\langle AV, Z\rangle A^2U - \langle AU, Z\rangle A^2V)\\
=& c(U\wedge V)AZ + \varepsilon(\langle V, \lambda AZ + \mu Z\rangle AU - \langle U, \lambda AZ + \mu Z\rangle AV)\\
&- cA((U\wedge V)Z) - \varepsilon(\langle AV, Z\rangle(\lambda AU + \mu U) - \langle AU, Z\rangle(\lambda AV + \mu V))\\
=& c(U\wedge V)AZ + \varepsilon\mu\langle V, Z\rangle AU - \varepsilon\mu\langle U, Z\rangle AV\\
&- cA((U\wedge V)Z) - \varepsilon\mu\langle V, AZ\rangle U) + \varepsilon\mu(\langle U, AZ\rangle V)\\
=& (c-\varepsilon\mu)(U\wedge V)AZ - A(U\wedge V)Z))\\
=& (c-\varepsilon\mu)((U\wedge V)\cdot A)Z.
\end{aligned}$$

Therefore, f satisfies the pseudo-parallelism condition (4.1) at x with $\psi(x) = c - \varepsilon\mu$. □

Observation 4.3. If $f(M_t^n)$ is totally umbilical, it can be seen from (4.1) that f is pseudo-parallel with any smooth function ψ.

We recall that a manifold M_t^n is pseudo-symmetric if there exist a real valued smooth function ψ on M_t^n, such that

$$R(U,V)\cdot R = \psi(U\wedge V)\cdot R, \tag{4.5}$$

for all $U, V \in TM$, where

$$\begin{aligned}
(R(U,V)\cdot R)(X,Y,Z) &= R(U,V)(R(X,Y)Z) - R(R(U,V)X,Y)Z - R(X,R(U,V)Y)Z\\
&\quad - R(X,Y)R(U,V)Z,\\
((U\wedge V)\cdot R)(X,Y,Z) &= (U\wedge V)R(X,Y)Z - R((U\wedge V)X,Y)Z - R(X,(U\wedge V)Y)Z\\
&\quad - R(X,Y)(U\wedge V)Z.
\end{aligned}$$

As in the Riemannian case (see [8]), pseudo-parallel Lorentzian hypersurfaces are intrinsically characterized by the next result:

Proposition 4.4 (Lobos, [44])**.** *Let $f : M_t^n \to \mathbb{Q}_s^{n+1}(c)$, $n \geq 2$, be a ψ-pseudo-parallel hypersurface, then M_t^n is a ψ-pseudo-symmetric manifold.*

Proof. Using Gauss Equation (1.18) in the definition of $(R(U,V)\cdot R)(X,Y,Z)$, for all $X,Y,Z \in TM^n_t$, we have that

$$
\begin{aligned}
&(R(U,V)\cdot R)(X,Y,Z)\\
=&R(U,V)(R(X,Y)Z) - R(R(U,V)X,Y)Z - R(X,R(U,V)Y)Z - R(X,Y)R(U,V)Z\\
=&R(U,V)(c(X\wedge Y)Z + \varepsilon(AX\wedge AY)Z) - c(R(U,V)X\wedge Y)Z - \varepsilon(AR(U,V)X\wedge AY)Z\\
&-c(X\wedge R(U,V)Y)Z - \varepsilon(AX\wedge AR(U,V)Y)Z - c(X\wedge Y)R(U,V)Z - \varepsilon(AX\wedge AY)R(U,V)Z\\
=&c(R(U,V)(X\wedge Y)Z - ((R(U,V)X)\wedge Y)Z - (X\wedge R(U,V)Y)Z - (X\wedge Y)R(U,V)Z)\\
&+\varepsilon R(U,V)(AX\wedge AY)Z - \varepsilon(AR(U,V)X\wedge AY)Z - \varepsilon(AX\wedge AR(U,V)Y)Z - \varepsilon(AX\wedge AY)R(U,V)Z\\
=&\varepsilon\langle AY,Z\rangle R(U,V)AX - \varepsilon\langle AX,Z\rangle R(U,V)AY\\
&-\varepsilon\langle AY,Z\rangle AR(U,V)X + \varepsilon\langle AR(U,V)X,Z\rangle AY\\
&-\varepsilon\langle AR(U,V)Y,Z\rangle AX + \varepsilon\langle AX,Z\rangle AR(U,V)Y\\
&-\varepsilon\langle AY,R(U,V)Z\rangle AX + \varepsilon\langle AX,R(U,V)Z\rangle AY\\
=&\varepsilon\langle AY,Z\rangle(R(U,V)AX - AR(U,V)X) - \varepsilon\langle AX,Z\rangle(R(U,V)AY - AR(U,V)Y)\\
&-\varepsilon\langle R(U,V)AX - AR(U,V)X,Z\rangle AY + \varepsilon\langle R(U,V)AY - AR(U,V)Y,Z\rangle AX,
\end{aligned}
$$

since $R(U,V)(X\wedge Y)Z - ((R(U,V)X)\wedge Y)Z - (X\wedge R(U,V)Y)Z - (X\wedge Y)R(U,V)Z = 0$. Next, using pseudo-parallelism condition (4.1) and Gauss equation once more, we obtain

$$
\begin{aligned}
&(R(U,V)\cdot R)(X,Y,Z)\\
=&\varepsilon\psi\langle AY,Z\rangle(U\wedge V(AX) - A(U\wedge V(X))) - \varepsilon\psi\langle AX,Z\rangle(U\wedge V(AY) - A(U\wedge V(Y)))\\
&-\varepsilon\psi\langle U\wedge V(AX) - A(U\wedge V(X)),Z\rangle AY + \varepsilon\psi\langle U\wedge V(AY) - A(U\wedge V(Y)),Z\rangle AX\\
=&\varepsilon\psi\langle AY,Z\rangle U\wedge V(AX) - \varepsilon\psi\langle AX,Z\rangle U\wedge V(AY)\\
&-\varepsilon\psi\langle AY,Z\rangle A(U\wedge V(X)) + \varepsilon\psi\langle A(U\wedge V(X)),Z\rangle AY\\
&-\varepsilon\psi\langle A(U\wedge V(Y)),Z\rangle AX + \varepsilon\psi\langle AX,Z\rangle A(U\wedge V(Y))\\
&-\varepsilon\psi\langle(U\wedge V)Z,AY\rangle AX + \varepsilon\psi\langle(U\wedge V)Z,AX\rangle AY\\
=&\psi U\wedge V(\varepsilon(AX\wedge AY)Z) - \varepsilon\psi(A(U\wedge V(X))\wedge AY)Z - \varepsilon\psi(AX\wedge A(U\wedge V(Y)))Z\\
&-\varepsilon\psi(AX\wedge AY)((U\wedge V)Z)\\
=&\psi((U\wedge V)R(X,Y)Z - R((U\wedge V)X,Y)Z - R(X,(U\wedge V)Y)Z - R(X,Y)(U\wedge V)Z)\\
&-c\psi((U\wedge V)((X\wedge Y)Z) - ((U\wedge V)X\wedge Y)Z - (X\wedge(U\wedge V)Y)Z - (X\wedge Y)((U\wedge V)Z))\\
=&\psi((U\wedge V)\cdot R)(X,Y,Z),
\end{aligned}
$$

since $(U\wedge V)((X\wedge Y)Z)) - ((U\wedge V)X\wedge Y)Z - (X\wedge(U\wedge V)Y)Z - (X\wedge Y)((U\wedge V)Z) = 0$. Thus, M^n_t is ψ-pseudo-symmetric. □

4.2 Reduction of the pseudo-parallelism condition for Lorentzian hypersurfaces in $\mathbb{Q}_s^{n+1}(c)$

Considering the classification of Weingarten operator types for Lorentzian manifolds, provided by Lemma 1.1, we study the pseudo-parallelism condition for Lorentzian hypersurfaces in $\mathbb{Q}_s^{n+1}(c)$ according to each case.

Observation 4.5. From Corollary 2.8, any Lorentzian surface in $\mathbb{Q}_s^3(c)$, $s \in \{1,2\}$, is pseudo-parallel.

For n-dimensional Lorentzian hypersurfaces in pseudo-Riemannian space forms, with $n \geq 3$, we complete a partial result from [44].

Proposition 4.6. *Let $f : M_1^n \to \mathbb{Q}_s^{n+1}(c)$ be a pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and $s \in \{1,2\}$. Then, at each point $x \in M_1^n$,*

(i) either the Weingarten operator A is diagonalizable with principal curvature functions $a_1, \dots, a_n$, and in this case:

$$(c - \psi + \varepsilon a_i a_j)(a_j - a_i) = 0, \textit{ for all } i \neq j. \tag{4.6}$$

Consequently f has at most two distinct principal curvatures at x and, if it has exactly two, their product is $\varepsilon(\psi - c)$;

(ii) or the Weingarten operator A has the form:

$$A = \begin{pmatrix} a & 0 & & & \\ 1 & a & & & \\ & & a & & \\ & & & \ddots & \\ & & & & a \end{pmatrix},$$

with $a^2 = \varepsilon(\psi - c)$.

Proof. Since f is pseudo-parallel in $\mathbb{Q}_s^{n+1}(c)$, it satisfies

$$(R(U,V) \cdot A)Z - \psi(U \wedge V \cdot A)Z = 0, \text{ for all } U,V,Z \in TM.$$

Let x be a fixed point of M_1^n. The proof will be separated in four cases, corresponding to the four possible forms I, II, III and IV for the Weingarten operator A at x, given in Lemma 1.1.

Case 1: In the case where A is diagonalizable, we have $AE_i = a_i E_i$, for all $i = 1, 2, \dots, n$. Then, for $U = Z = E_i$ and $V = E_j$, $i \neq j$, from pseudo-parallelism condition (4.3), it follows that:

$$\begin{aligned} 0 &= -(c-\psi)a_i\langle E_i, E_i\rangle E_j + (c-\psi)a_i\langle E_j, E_i\rangle E_i \\ &\quad - (\varepsilon a_i^2\langle E_i, E_i\rangle - (c-\psi)\langle E_i, E_i\rangle)a_j E_j + (\varepsilon a_j a_i\langle E_j, E_i\rangle - (c-\psi)\langle E_j, E_i\rangle)a_i E_i \\ &\quad + \varepsilon a_i\langle E_i, E_i\rangle a_j^2 E_j - \varepsilon a_j\langle E_j, E_i\rangle a_i^2 E_i \\ &= (-(c-\psi)a_i - \varepsilon a_i^2 a_j + (c-\psi)a_j + \varepsilon a_i a_j^2)\langle E_i, E_i\rangle E_j. \end{aligned}$$

Since $\langle E_i, E_i \rangle \neq 0$, we obtain identity (4.6), that is

$$(c - \psi + \varepsilon a_i a_j)(a_j - a_i) = 0, i \neq j.$$

If there exists distinct a_i, a_j, a_k, we can assume $a_i \neq 0$. Then, cyclic permuting the indices in (4.6) and adding the results, we have $a_i(a_j - a_k) = 0$, which is a contradiction, so at most two of the a_i's are distinct.

If there exists exactly two unequal eigenvalues a_i, a_j of A, that is, M_1^n has two unequal principal curvatures, then $\psi = c + \varepsilon a_i a_j$. If all eigenvalues of A are equal, then M_1^n is umbilical and ψ is arbitrary.

Case 2: If A takes the form II, in a pseudo-orthonormal basis $\{X, Y, E_3, \dots, E_n\}$ at x, for $i \geq 3$ we have

$$AE_i = a_i E_i,\ AX = aX + Y,\ AY = aY,$$
$$A^2 E_i = a_i^2 E_i,\ A^2 X = a^2 X + 2aY,\ A^2 Y = a^2 Y.$$

if $U = Z = X$, $V = E_i$ (since $n \geq 3$), from pseudo-parallelism condition (4.3), it follows that:

$$\begin{aligned}
0 &= -(c-\psi)\langle X, aX+Y\rangle Y + (c-\psi)\langle Y, aX+Y\rangle X \\
&\quad - (\varepsilon\langle aX+Y, aX+Y\rangle - (c-\psi)\langle X, X\rangle)aY + (\varepsilon\langle aY, aX+Y\rangle - (c-\psi)\langle Y, X\rangle)(aX+Y) \\
&\quad + \varepsilon\langle aX+Y, X\rangle a^2 Y - \varepsilon\langle aY, X\rangle(a^2X + 2aY) \\
&= (c-\psi)Y - (c-\psi)aX + 2\varepsilon a^2 Y - (\varepsilon a^2 - (c-\psi))(aX+Y) - \varepsilon a^2 Y + \varepsilon a(a^2X + 2aY) \\
&= 2(c - \psi + \varepsilon a^2)Y.
\end{aligned}$$

Thus, we have

$$c - \psi = -\varepsilon a^2 \tag{4.7}$$

Now, if $U = X$, $V = Z = E_i$, $i \geq 3$, from pseudo-parallelism condition (4.3), it follows that:

$$\begin{aligned}
0 &= -(c-\psi)\langle X, a_i E_i\rangle E_i + (c-\psi)a_i\langle E_i, E_i\rangle X \\
&\quad - (\varepsilon\langle aX+Y, a_i E_i\rangle - (c-\psi)\langle X, E_i\rangle)a_i E_i + (\varepsilon a_i^2\langle E_i, E_i\rangle - (c-\psi)\langle E_i, E_i\rangle)(aX+Y) \\
&\quad + \varepsilon\langle aX+Y, E_i\rangle a_i^2 E_i - \varepsilon a_i\langle E_i, E_i\rangle(a^2X + 2aY) \\
&= ((c-\psi)a_i + \varepsilon a_i^2 a - (c-\psi)a - \varepsilon a_i a^2)X + (\varepsilon a_i^2 - (c-\psi) - 2\varepsilon a_i a)Y \\
&= (c - \psi + \varepsilon a_i a)(a_i - a)X + (\varepsilon a_i^2 - (c-\psi) - 2\varepsilon a_i a))Y
\end{aligned}$$

Thus,

$$c - \psi = \varepsilon a_i^2 - 2\varepsilon a_i a. \tag{4.8}$$

Combining (4.7) with (4.8), we obtain

$$\varepsilon a_i^2 - 2\varepsilon a_i a = -\varepsilon a^2,$$

which is equivalent to

$$(a-a_i)^2 = 0. \tag{4.9}$$

Thus, we have $a = a_i$, for all $i \geq 3$, and A has the form

$$A = \begin{pmatrix} a & 0 & & & \\ 1 & a & & & \\ & & a & & \\ & & & \ddots & \\ & & & & a \end{pmatrix},$$

with $\psi = c + \varepsilon a^2$.

Case 3: If A takes the form III, in a pseudo-ortonormal basis $\{X, Y, E_3, \ldots, E_n\}$ at x, for $i \geq 4$ we have

$$AE_i = a_i E_i,\ AX = aX - E_3,\ AY = aY,\ AE_3 = Y + aE_3,$$

$$A^2 E_i = a_i^2 E_i,\ A^2 X = a^2 X - Y - 2aE_3,\ A^2 Y = a^2 Y,\ A^2 E_3 = 2aY + a^2 E_3.$$

Now, choosing $U = Z = X$ and $V = E_3$, from pseudo-parallelism condition (4.3) it follows that:

$$\begin{aligned}
0 &= -(c-\psi)\langle X, aX - E_3\rangle E_3 + (c-\psi)\langle E_3, aX - E_3\rangle X \\
&\quad - (\varepsilon\langle aX - E_3, aX - E_3\rangle - (c-\psi)\langle X, X\rangle)(Y + aE_3) + (\varepsilon\langle Y + aE_3, aX - E_3\rangle - (c-\psi)\langle E_3, X\rangle)(aX - E_3) \\
&\quad + \varepsilon\langle aX - E_3, X\rangle(2aY + a^2 E_3) - \varepsilon\langle Y + aE_3, X\rangle(a^2 X - Y - 2aE_3) \\
&= -(c-\psi)X - \varepsilon(Y + aE_3) - 2\varepsilon a(aX - E_3) + \varepsilon(a^2 X - Y - 2aE_3) \\
&= (-\varepsilon a^2 - c + \psi)X - 2\varepsilon Y - \varepsilon a E_3,
\end{aligned}$$

which is a contradiction since $\varepsilon \neq 0$. Thus, A can not take the form III.

Case 4: If A takes the form IV, with two complex conjugate eigenvalues, for $i \geq 3$ we have

$$AE_i = a_i E_i,\ AE_1 = aE_1 - bE_2,\ AE_2 = bE_1 + aE_2,$$

$$A^2 E_i = a_i^2 E_i,\ A^2 E_1 = (a^2 - b^2)E_1 - 2abE_2,\ A^2 E_2 = 2abE_1 + (a^2 - b^2)E_2.$$

If $U = Z = E_1$, $V = E_2$, from pseudo-parallelism condition (4.3) it follows that:

$$\begin{aligned}
0 &= -(c-\psi)\langle E_1, aE_1 - bE_2\rangle E_2 + (c-\psi)\langle E_2, aE_1 - bE_2\rangle E_1 \\
&\quad - (\varepsilon\langle aE_1 - bE_2, aE_1 - bE_2\rangle - (c-\psi)\langle E_1, E_1\rangle)(bE_1 + aE_2) \\
&\quad + (\varepsilon\langle bE_1 + aE_2, aE_1 - bE_2\rangle - (c-\psi)\langle E_2, E_1\rangle)(aE_1 - bE_2) \\
&\quad + \varepsilon\langle aE_1 - bE_2, E_1\rangle(2abE_1 + (a^2 - b^2)E_2) - \varepsilon\langle bE_1 + aE_2, E_1\rangle((a^2 - b^2)E_1 - 2abE_2) \\
&= (-(c-\psi)b + \varepsilon b(a^2 - b^2) - b(c-\psi) - 2\varepsilon a^2 b - 2\varepsilon a^2 b + \varepsilon b(a^2 - b^2))E_1 \\
&\quad + ((c-\psi)a + \varepsilon a(a^2 - b^2) - (c-\psi)a + 2\varepsilon ab^2 - \varepsilon a(a^2 - b^2) - 2\varepsilon ab^2)E_2 \\
&= -2b(c - \psi + \varepsilon(a^2 + b^2))E_1.
\end{aligned}$$

Thus, $b \neq 0$ implies

$$c-\psi=-\varepsilon(a^2+b^2). \tag{4.10}$$

On the other hand, since $n \geq 3$, if $U=E_1$, $V=Z=E_i$, $i \geq 3$, from pseudo-parallelism condition (4.3), it follows that:

$$\begin{aligned}
0 &= -(c-\psi)a_i\langle E_1,E_i\rangle E_i+(c-\psi)a_i\langle E_i,E_i\rangle E_1 \\
&\quad -(\varepsilon\langle aE_1-bE_2,a_iE_i\rangle-(c-\psi)\langle E_1,E_i\rangle)a_iE_i+(\varepsilon a_i^2\langle E_i,E_i\rangle-(c-\psi)\langle E_i,E_i\rangle)(aE_1-bE_2) \\
&\quad +\varepsilon\langle aE_1-bE_2,E_i\rangle a_i^2E_i-\varepsilon a_i\langle E_i,E_i\rangle((a^2-b^2)E_1-2abE_2) \\
&= ((c-\psi)a_i+\varepsilon aa_i^2-(c-\psi)a-\varepsilon a_i(a^2-b^2))E_1+(-\varepsilon ba_i^2+(c-\psi)b+2\varepsilon a_iab)E_2 \\
&= ((c-\psi+\varepsilon aa_i)(a_i-a)+\varepsilon a_ib^2)E_1+b(-\varepsilon a_i^2+(c-\psi)+2\varepsilon a_ia)E_2
\end{aligned}$$

Using that $b \neq 0$, we obtain

$$c-\psi=-\varepsilon(2a_ia-a_i^2) \tag{4.11}$$

Next, from (4.10) and (4.11), we obtain that

$$a^2+b^2=2a_ia-a_i^2 \iff a^2-2aa_i+a_i^2+b^2=0 \iff (a-a_i)^2+b^2=0,$$

which is not possible, since $b \neq 0$. Thus, A can not have a complex eigenvalue. $\square$

4.3 Pseudo-parallel Lorentzian hypersurfaces in $\mathbb{Q}_1^{n+1}(c)$ with constant ψ

Now, we focus on the case where the ambient space form is Lorentzian, i.e., $s=1$. The following results are consequences of Proposition 4.6.

Lemma 4.7. *Let $f: M_1^n \to \mathbb{Q}_1^{n+1}(c)$, $n \geq 3$, be a pseudo-parallel Lorentzian hypersurface. Then, at each point $x \in M_1^n$, either*

(i) the Weingarten operator A_p is diagonalizable and the principal curvatures a_i of M_1^n, $1 \leq i \leq n$, satisfy the identity:

$$(c-\psi+a_ia_j)(a_i-a_j)=0, \text{ for all } i \neq j. \tag{4.12}$$

Consequently, in an orthonormal basis at x, either

- *$A_p=aI_n$, $a \in \mathbb{R}$, or*
- *$A_p=aI_k \oplus bI_{n-k}$, where $1 \leq k \leq n-1$, $ab=\psi-c$ and $a \neq b$.*

(ii) or $\psi = c$ and for a pseudo-orthonormal basis at x

$$A_p = \begin{pmatrix} 0 & 0 & & & \\ 1 & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix},$$

(iii) or $\psi > c$ and for a pseudo-orthonormal basis at x

$$A_p = \begin{pmatrix} a & 0 & & & \\ 1 & a & & & \\ & & a & & \\ & & & \ddots & \\ & & & & a \end{pmatrix}, \text{ where } a^2 = \psi - c > 0.$$

Corollary 4.8. *Let $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$, $n \geq 3$, be a ψ-pseudo-parallel Lorentzian hypersurface. Then, for any $x \in M_1^n$ such that $\psi(x) \neq c$, we have that $k(x) = 0$ or $k(x) = n$. If $k(x) = n$, then A_x has at most two distinct eigenvalues.*

Proof. If A_x is diagonalizable, we assume to the contrary that $1 \leq k(x) \leq n-1$, Let a_i be a nonzero eigenvalue of A_x, from Lemma 4.7, with $a_j = 0$, it follows that $a_i(c - \psi(x)) = 0$, which is a contradiction. Thus, $k(x) = 0$ or $k(x) = n$. If $k(x) = n$, we have that $(c - \psi(x) + a_i a_j)(a_i - a_j) = 0$, for any $i \neq j$, then $a_i = a_j$ or $a_i = \frac{\psi(x)-c}{a_j} \neq 0$. It follows that at most two eigenvalues of A_x are distinct.

If A_x is non-diagonalizable, then the proposition follows from *(iii)* in Lemma 4.7. □

Corollary 4.9. *Let $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$, $n \geq 3$, be a Lorentzian hypersurface and let $x \in M_1^n$.*

(i) If $k(x) \leq 1$, we have that $R(X,Y) \cdot A = cX \wedge Y \cdot A$, for all $X, Y \in T_x M_1^n$.

(ii) If $k(x) \geq 2$ and $R(X,Y) \cdot A = cX \wedge Y \cdot A$, for all $X, Y \in T_x M_1^n$, then A_x is diagonalizable and the nonzero principal curvatures are equal.

Proof. First, suppose that A_x is diagonalizable. If $k(x) \leq 1$, that is, if at most one of a_i's is nonzero, say a, then we have that $A_x^2 - aA_x = 0$ and it follows from Proposition 4.2 that the pseudo-parallelism condition is satisfied at x with $\psi(x) = c$. If $k(x) \geq 2$, then for any $i \neq j$, such that $a_i a_j \neq 0$, it follows from the identity (4.12) that $a_i - a_j = 0$, if the pseudo-parallelism condition is satisfied in x with $\psi(x) = c$.

Now, suppose that A_x is non-diagonalizable. It follows from Lemma 4.7 that if the pseudo-parallelism condition is satisfied at x with $\psi = c$, then $k(x) = 1$. Conversely, if $k(x) = 1$, since A_x only can take the form II in Lemma 1.1, with $a = a_3 = \cdots = a_n = 0$, we have that $A_x^2 = 0$ and it follows from Proposition 4.2 that the pseudo-parallelism condition is satisfied at x with $\psi = c$. □

For the case $\psi = c = 0$, we obtain the next result.

Theorem 4.10. *Let $f : M_1^n \to \mathbb{E}_1^{n+1}$ be a connected and complete semi-parallel Lorentzian hypersurface in $\mathbb{E}_1^{n+1}$, with $n \geq 3$. Then, $f(M_1^n)$ is congruent to one of the following Lorentzian submanifolds:*

(i) $\mathbb{E}_1^n = \{x \in \mathbb{E}_1^{n+1} : x_{n+1} = 0\}$;

(ii) $\mathbb{S}_1^n(a^2) = \left\{ x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{a^2} \right\}$ *with* $a \neq 0$;

(iii) $\mathbb{S}^k(a^2) \times \mathbb{E}_1^{n-k} = \left\{ x \in \mathbb{E}_1^{n+1} : \sum_{i=2}^{k+2} x_i^2 = \frac{1}{a^2} \right\}$, *with* $a \neq 0$ *and* $2 \leq k \leq n-1$;

(iv) $\mathbb{S}_1^k(a^2) \times \mathbb{E}^{n-k} = \left\{ x \in \mathbb{E}_1^{n+1} : -x_1^2 + \sum_{i=2}^{k+1} x_i^2 = \frac{1}{a^2} \right\}$, *with* $a \neq 0$ *and* $2 \leq k \leq n-1$;

(v) $\mathbb{E}_1^{n-2} \times h(\mathbb{E}^2)$, *where $h(\mathbb{E}^2)$ is a Euclidean cylinder in a subspace $\mathbb{E}^3$ of $\mathbb{E}_1^{n+1}$ orthogonal to $\mathbb{E}_1^{n-2}$; or $\mathbb{E}^{n-2} \times h(\mathbb{E}_1^2)$, where $h(\mathbb{E}_1^2)$ is a Lorentzian cylinder or a B-scroll in a subspace $\mathbb{E}_1^3$ of $\mathbb{E}_1^{n+1}$ orthogonal to $\mathbb{E}^{n-2}$.*

Proof. First, consider the case where $k(x_0) \geq 2$ for some point $x_0 \in M_1^n$. Note that the proof of Theorem 2 of [75] can be carried out without major changes for the weaker condition $k(x) \geq 2$, provided that the Weingarten operator at any point $x \in M_1^n$ with $k(x) \geq 2$ takes the form $A_x = a(x) I_{k(x)} \oplus 0_{n-k(x)}$, with $a(x) \neq 0$, which is the case, from Lemma 4.7.

Then, if $k(x_0) \geq 2$ at some point $x_0 \in M_1^n$, let $\pi : \widetilde{M}_1^n \to M_1^n$ be the universal covering of M_1^n and consider the immersion $\widetilde{f} = f \circ \pi$. Proceeding for $\widetilde{f}$ as in [75, 59], as we show in Proposition A.1 in the Appendix A, we obtain that $k(x)$ and the nonzero eigenvalue $a(x)$ are constant, say $k(x) = k$, $a(x) = a$, and $\widetilde{f} : \widetilde{M}_1^n \to \mathbb{E}_1^{n+1}$ is a surjective map onto either a totally umbilical $\mathbb{S}_1^n(a^2)$, with $a \neq 0$, or a product as in *(iii)* and *(iv)*. Note that if $\widetilde{f}(\widetilde{M}_1^n)$ is also simply connected, we can deduce that π is one-to-one, thus, f is an isometry and M_1^n is simply connected as well. If $\widetilde{f}(\widetilde{M}_1^n) = \mathbb{S}_1^2(a^2) \times \mathbb{E}^{n-2}$, then $\widetilde{f} = f \circ \pi$ is just a surjective map and π is not necessarily one-to-one, so M_1^n is not necessarily simply connected in this case, but still we have $f(M_1^n) = \mathbb{S}_1^2(a^2) \times \mathbb{E}^{n-2}$.

On the other hand, if $k(x) \leq 1$ for all $x \in M_1^n$, we have that M_1^n has constant curvature 0. From Corollary 4.9, we know that any such hypersurface in $\mathbb{E}_1^{n+1}$ is always semi-parallel. It follows from the classification of connected complete Lorentzian hypersuperfaces with constant curvature of $\mathbb{E}_1^{n+1}$ in Theorem 3.5 of [6] that $f(M_1^n)$ is a totally geodesic $\mathbb{E}_1^n$ or has one of the forms in *(v)* (in this case, the subset of geodesic points of $f(M_1^n)$ is not necessarily empty). This completes the proof of the theorem. □

4.4 Good pseudo-parallel Lorentzian hypersurfaces with constant $\psi \neq c$

From here, we will study ψ-pseudo-parallel Lorentzian hypersurfaces in $\mathbb{Q}_1^{n+1}(c)$ with constant ψ using the techniques in [6, 66] to obtain analogous classification results when $\psi \neq c$.

Let $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$ be a Lorentzian hypersurface, with $n \geq 3$. In the sense of Ryan [66], we say that a point $x \in M_1^n$ is *bad* if A_x is non-singular and has at least one eigenvalue of multiplicity one. Otherwise, x is said to be *good*. If all points of M_1^2 are good, then we say that f is a *good* hypersurface.

Observation 4.11. Let $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$ be a ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi \neq c$. From Lemma 4.7, we have shown that at each point $x \in M_1^n$ the Weingarten operator A_x can be one of the following.

1. $A_x = aI_n \neq 0$,
2. $A_x = 0$,
3. A_x is diagonalizable and has two unequal nonzero eigenvalues a, b, each of multiplicity greater than 1. In this case, we have $ab = \psi - c$.
4. A_x is diagonalizable and has two unequal nonzero eigenvalues a, b, of multiplicity 1 and $n-1$. In this case, we have $ab = \psi - c$.
5. A_x is non-diagonalizable and has only one eigenvalue a which is nonzero. In this case, we have that $\psi - c = a^2 > 0$ and the minimal polynomial of A_x is $(t-a)^2$.

Definition 4.12. We say that $x \in M_1^n$ is a point of type $1, 2, \ldots, 5$ whenever A_x has, respectively, the form $1, 2, \ldots, 5$ in the list in Remark 4.11. The set of points of type i will be denoted by C_i.

Observation 4.13. If $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$ is a ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and $\psi \neq c$, the set of bad points of M_1^n is precisely C_4.

Proposition 4.14. *Let $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$ be a ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and $\psi \neq c$. Then, C_4 is open.*

Proof. Assume without loss of generality that M_1^n is orientable. Since f is ψ-pseudo-parallel with $\psi \neq c$, from Lemma 4.7, all eigenvalues of A are real numbers. Therefore, we may define n principal curvature functions $\{a_1, \ldots, a_n\}$ as done in Lemma 2.1 of [66] and using the same arguments there, we conclude that the a_i's are continuous functions.

Note that, also from Lemma 4.7, only two eigenvalues of A can be distinct. Let x_0 be a bad point such that (without loss of generality) $a(x) = a_1(x_0) > a_2(x_0) = a_3(x_0) = \cdots = a_n(x_0) = b(x)$. By continuity, there is an open neighborhood U of x_0, in which we have $a = a_1 > a_j$, for all $j \in \{2,3,\ldots,n\}$. We conclude that $a_2 = a_3 = \cdots = a_n = b$ in U, thus, the multiplicities of eigenvalues a, b are constant in U. Moreover, U can be chosen so that $a_i \neq 0$ in U since $a_i(x_0) \neq 0$ for all $1 \leq i \leq n$. Then, by checking the list in Remark 4.11, we conclude that U consists entirely of type 4 points. □

Observation 4.15. The same argument above shows that the set of type 3 points is open and the multiplicities remain constant in a sufficiently small neighborhood of a point of that type.

Proposition 4.16. *Let $f : M^n_1 \to \mathbb{Q}^{n+1}_1(c)$ be a ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi < c$. Then, the set of bad points is closed.*

Proof. Let $\{x_i\}$ be a sequence of bad points converging to some point $x \in M^n_1$. For any $i \geq 1$, A_{x_i} is diagonalizable and has two unequal eigenvalues $a(x_i)$, $b(x_i)$, such that $a(x_i)b(x_i) = \psi - c$. By continuity, we have $a(x)b(x) = \psi - c$. Since ψ is constant and $\psi - c < 0 \leq a^2(x)$, we have $a(x) \neq b(x)$ and $a(x)b(x) \neq 0$. It follows that A_x is diagonalizable with two unequal nonzero eigenvalues. Thus, x is type 3 or type 4. Since the set of type 3 points is open, it follows that x must be of type 4, i.e., x is a bad point. □

Proposition 4.17. *Let $f : M^n_1 \to \mathbb{Q}^{n+1}_1(c)$ be a connected good ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi \neq c$. Then, either $k(x) = 0$ for all $x \in M^n_1$, or $k(x) = n$ for all $x \in M^n_1$.*

Proof. A) If $\psi < c$, the argument is analogous to that in Proposition 4.7 of [66]. In fact, from the list in Remark 4.11, we can define $W = \{x \in M^n_1 : k(x) = 0\} = \{x \in M^n_1 : \det(A_x) = 0\}$. We have that W is closed. Since M^n_1 is connected, it will be sufficient to show that W is open. First consider a sequence of points $\{y_i\}$ of type 3 converging to some point y_0. Since the principal curvatures are continuous and satisfy $a(y_i)b(y_i) = \psi - c < 0$, for all i, it follows that $a(y_0)b(y_0) \neq 0$. Thus, y_0 can not lie in W. Then, for a given $x_0 \in W$, we can choose a connected neighborhood U of x_0 which contains no points of type 3. We will now show that U has no points of type 1. Suppose there is a point y of type 1 in U. Let $V = \{x \in U : \det(A_x) = \det(A_y)\}$. We have that V is closed in U. Choose an arbitrary $z \in V$. Since A_z is non-singular, z has a (connected) neighborhood $U' \subset U$ where A is non-singular. U' consists entirely of umbilical poits. By Proposition D.4 of [41], the principal curvature function a of A is constant in U' and is equal to $a(z)$, thus $U' \subset U$. This shows that V is open and hence $V = U$. This can not happen since x_0 belongs to W. We conclude that there are no type 1 points in U. Thus, from the list in Remark 4.11 and since M^n_1 is good, we have that $U \subset W$ and W is open.

B) If $\psi > c$, the argument is analogous to that in Proposition 5.7 in [6]. As above, the subset $W = \{x \in M^n_1 : k(x) = 0\} = \{x \in M^n_1 : \det(A_x) = 0\}$ is closed. It remains to show that W is also open.

Analogously to the case $\psi < c$, we have that $W \cap \overline{C}_3 = \emptyset$, where $\overline{C}_3$ denotes the closure of C_3. Then, for each $x_0 \in W$, we can choose a connected neighborhood U of x_0 which contains no points of type 3. We claim that $U \cap C_1 = U \cap C_5 = \emptyset$. Indeed, let $x_1 \in U \cap (C_1 \cup C_5)$, and set $V_{x_1} = \{x \in U : \det(A_x) = \det(A_{x_1}) = (a(x_1))^n\}$, which is a closed subset of U. Choose an arbitrary $z \in V_{x_1}$. Since A_{x_1} is non-singular, z has a (connected) neighborhood $U' \subset U$ where A is non-singular. If $\{y_i\}$ is a sequence in C_5 converging to z, since the principal curvature a satisfies $a(y_i)^2 = \psi - c$ for all i, then we get by continuity that $a(z) = \sqrt{\psi - c}$. Since z is arbitrary, we may assume that either $U' \subset C_1$ or else $a = \sqrt{\psi - c}$ in U'. If $U' \subset C_1$, then U' is totally umbilical. By Proposition D.4 of [41], the principal curvature function a of A is constant in U' and is equal to $a(z)$, thus $U' \subset U$. This shows that V_{x_1} is open and hence $V_{x_1} = U$. This is a contradiction since $x_0 \in W$. Hence, we conclude in this case that $U \cap (C_1 \cup C_5) = \emptyset$ as claimed. If $a = \sqrt{\psi - c}$ in U', then $a(w) = a(z) = \sqrt{\psi - c}$, for all $w \in U'$, and we have that $\det(A_w) = \det(A_z) = \det(A_{x_1})$, for all $w \in U'$, consequently, $U' \subset V_{x_1}$. This also shows that V_{x_1} is open and hence $V_{x_1} = U$. This is a contradiction since $x_0 \in W$. We therefore conclude in this case that $U \cap (C_1 \cup C_5) = \emptyset$. Hence, $U \subset W$ and W is open. The proof of the proposition is complete. □

Proposition 4.18. *Let $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$ be a connected ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi < c$. If M_1^n has at least one good point, then either $f(M_1^n)$ is totally geodesic, totally umbilical or M_1^n consists entirely of points of type 3.*

Proof. Since M_1^n has at least one good point, it follows from Proposition 4.14, Proposition 4.16 and the connectedness of M_1^n that f is a good hypersurface. If $k(x) = 0$ for all $x \in M_1^n$, then $f(M_1^n)$ is totally geodesic. On the other hand, suppose that $k(x_0) \geq 1$, for some $x_0 \in M_1^n$, it follows from Proposition 4.17 that $k(x) = n$ for all $x \in M_1^n$. Note that M_1^n does not contain type 5 points, since $\psi - c < 0$. Thus, in this case, M_1^n only can contain type 3 points or totally umbilical points. Since C_3 is closed and open, the proposition follows again by the connectedness of M_1^n. □

Theorem 4.19. *Let $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$ be a ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi < c$. Then $f(M_1^n)$ is locally congruent to either a good hypersurface which is one of the following*

(i) $\mathbb{S}_1^n(a^2+c) = \left\{ x \in \mathbb{S}_1^{n+1}(c) \subset \mathbb{E}_1^{n+2} : x_{n+2} = \sqrt{\dfrac{1}{c} - \dfrac{1}{a^2+c}} \right\}$ *with $a \in \mathbb{R}$, if $c > 0$;*

(ii) $\mathbb{E}_1^n = \{x \in \mathbb{E}_1^{n+1} : x_{n+1} = 0\}$ *or* $\mathbb{S}_1^n(a^2) = \left\{ x \in \mathbb{E}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \dfrac{1}{a^2} \right\}$ *with $a \neq 0$, if $c = 0$;*

(iii) $\mathbb{H}_1^n(a^2+c) = \left\{ x \in \mathbb{H}_1^{n+1}(c) \subset \mathbb{E}_2^{n+2} : x_{n+2} = \sqrt{\dfrac{1}{c} - \dfrac{1}{a^2+c}} \right\}$*, with $|a| < \sqrt{-c}$, or* $\mathbb{S}_1^n(a^2+c) = \left\{ x \in \mathbb{H}_1^{n+1}(c) \subset \mathbb{E}_2^{n+2} : x_1 = \sqrt{\dfrac{1}{a^2+c} - \dfrac{1}{c}} \right\}$*, with $a^2 > -c$. In this case $c < 0$;*

(iv) *a flat totally umbilical hypersurface of the form* $\{y \in \mathbb{H}^{n+1}_1(c) : \langle y, X\rangle = a\}$, *where* $a = \pm\sqrt{-c}$ *and* X *is a parallel vector field in* $\mathbb{E}^{n+2}_2$, *which satisfies* $\langle X, X\rangle = 0$. *In this case* $c < 0$;

(v) $\mathbb{S}^k_1(a^2+c) \times \mathbb{S}^{n-k}(b^2+c) = \left\{ x \in \mathbb{S}^{n+1}_1(c) \subset \mathbb{E}^{n+2}_1 : -x_1^2 + \sum_{i=2}^{k+1} x_i^2 = \frac{1}{a^2+c}, \sum_{i=k+2}^{n+2} x_i^2 = \frac{1}{b^2+c} \right\}$, *where* $c > 0$, $\psi = ab + c = 0$ *and* $1 < k < n-1$;

or else, $f(M^n_1)$ *is locally congruent to a bad hypersurface foliated either by* $(n-1)$*-dimensional Riemannian spaces of constant curvature greater than c or by* $(n-1)$*-dimensional Lorentzian spaces of constant curvature greater than c.*

Proof. We can assume that M^n_1 is connected. First, suppose that M^n_1 contains at least one good point. It follows from Proposition 4.18 that M^n_1 is either totally geodesic, totally umbilical or else M^n_1 consists entirely of points of type 3 and the two unequal nonzero eigenvalues a, b of the Weingarten operator A satisfy $ab = \psi - c < 0$. If M^n_1 consists entirely of points of type 3, then the eigenvalues a, b have constant multiplicities. We deduce from Proposition D.4 of [41], that the eigenspace distributions T_a and T_b, given by $T_a = \{X \in TM : AX = aX\}$ and $T_b = \{X \in TM : AX = bX\}$, are differentiable and integrable with a and b constant on each leaf of the corresponding eigenspace distribution. Since $ab = \psi - c$ is constant and a, b are nonzero, it follows that a is constant if and only if b is constant, thus a, b are constant everywhere in M^n_1. Therefore, f is isoparametric with diagonalizable Weingarten operator and *(i)*, *(ii)*, *(iii)*, *(iv)* and *(v)* follow from Theorem 5.1 of [1] combined with Theorem 3.1 of [80]. In particular, from Theorem 3.5 of [1] we have that $\psi = ab + c = 0$ if all point of M^n_1 are type 3 points.

On the other hand, if all the points of M^n_1 are bad points, as in Theorem 5.10 of [36], let a and b the two distinct nonzero eigenvalues of A, with constant multiplicities $n-1$ and 1, respectively. Then, by Proposition D.4 of [41], we have that the distribution T_a is differentiable and integrable with a constant on each leaf of T_a. Moreover, each integral manifold M_a of T_a is a non-degenerate totally geodesic hypersurface of M^n_1, thus the curvature tensor R_a of M_a coincides with the restriction of R to M_a. This means that for any $X, Y \in T_a$, we have that $R_a(X,Y) = R(X,Y) = (a^2+c)X \wedge Y$. It follows that M_a is a $(n-1)$-dimensional space of constant curvature $a^2 + c$. This completes the proof of the theorem, since the metric of M_a can be Riemannian or Lorentzian and all the leaves must have the same metric. □

Observation 4.20. The immersion $f : \mathbb{E}^n_1 \to \mathbb{H}^n_1(c)$, $x \mapsto (\sqrt{-c}\langle x,x\rangle + \frac{5}{4\sqrt{-c}}, x, \sqrt{-c}\langle x,x\rangle + \frac{3}{4\sqrt{-c}})$ is totally umbilical, with Weingarten operator $A = \pm\sqrt{-c}I_n$, and the vector field $X = (\pm 2c, 0, \ldots, 0, \pm 2c) \in \mathbb{E}^n_2$ satisfies conditions in *(iv)* of Theorem 4.19.

Corollary 4.21. *Let* $f : M^n_1 \to \mathbb{Q}^{n+1}_1(c)$ *be a connected complete* ψ*-pseudo-parallel Lorentzian hypersurface, with* $n \geq 3$, $c \geq 0$ *and constant* $\psi < c$, *containing at least one good point. Then,* $f(M^n_1)$ *is congruent to one of the hypersurfaces described in (i), (ii), (iii), (iv) and (v) of Theorem 4.19.*

Proof. Since M_1^n has at least one good point, we have that f is isoparametric with diagonalizable Weingarten operator and $f(M_1^n)$ is locally congruent to one of the models in Theorem 4.19. If M_1^n is not locally congruent with $\mathbb{H}_1^n(a^2+c)$ or $\mathbb{S}_1^2(a^2+c)\times\mathbb{S}^{n-2}(b^2+c)$, the result follows from Theorem 5.2 of [1].

If M_1^n is locally congruent to $\mathbb{S}_1^2(a^2+c)\times\mathbb{S}^{n-2}(b^2+c)$, with a,b constant, we have that $ab+c=0$ from Theorem 4.19. Then, let $\pi:\widetilde{M}_1^n\to M_1^n$ be its universal covering, we have the immersion $\widetilde{f}=f\circ\pi:\widetilde{M}_1^n\to\mathbb{S}_1^{n+1}(c)$. In this case, the unit normal vector field η to $\widetilde{f}$ and the distribution T_a and T_b are globally defined with dimensions greater than 1. It follows from Wu's extension of the de Rham decomposition theorem to pseudo-Riemannian manifolds in [78] that $\widetilde{M}_1^n$ is congruent to the simply connected product $\widetilde{\mathbb{S}}_1^2(a^2+c)\times\mathbb{S}^{n-2}(b^2+c)$, where $\widetilde{\pi}:\widetilde{\mathbb{S}}_1^2(a^2+c)\to\mathbb{S}_1^2(a^2+c)$ is the universal covering. Then, consider $\pi_2:\widetilde{\mathbb{S}}_1^2(a^2+c)\times\mathbb{S}_1^{n-2}(b^2+c)\to\mathbb{S}_1^2(a^2+c)\times\mathbb{S}^{n-2}(b^2+c)$, given by $\pi_2=(\widetilde{\pi},\mathrm{id})$. Note that the inclusion i of $\mathbb{S}_1^2(a^2+c)\times\mathbb{S}^{n-2}(b^2+c)=\left\{(y_1,y_2,y_3,z_4,\dots,z_{n+2})\in\mathbb{E}_1^{n+2}:-y_1^2+y_2^2+y_3^2=\dfrac{1}{a^2+c},\sum_{i=4}^{n+2}z_i^2=\dfrac{1}{b^2+c}\right\}$ into $\mathbb{S}_1^{n+1}(c)$ does in fact exist, since $\dfrac{1}{a^2+c}+\dfrac{1}{b^2+c}=-\dfrac{1}{ab}=\dfrac{1}{c}$. Note that $\mathbb{S}_1^2(a^2+c)\times\mathbb{S}^{n-2}(b^2+c)$ is connected and complete. Consider then $\hat{f}=i\circ\pi_2$. Since $k(x)=n$ everywhere (because all points are type 3 points), we can apply Theorem 1.3 of [66] to deduce that $\widetilde{f}=\hat{f}$. Thus, $f(M_1^n)=\widetilde{f}(\widetilde{M}_1^n)$ is congruent to $\mathbb{S}^2(a^2+c)\times\mathbb{S}_1^{n-2}(b^2+c)$ and we obtain the missing case $k=2$, in *(v)*.

Now, suppose that $c<0$. If $f(M_1^n)$ is a totally geodesic hypersurface of $\mathbb{H}_1^{n+1}(c)$, then it is congruent to the standard imbedding of $\mathbb{H}_1^n(c)$ into $\mathbb{H}_1^{n+1}(c)$, given in *(iii)* of Theorem 4.19 with $a=0$ (in fact, $f(M_1^2)$ is totally umbilical in $\mathbb{E}_1^{n+2}$ with codimension two). If M_1^n is locally congruent to $\mathbb{H}_1^n(a^2+c)$, with $a\neq0$, let $\pi:\widetilde{M}_1^n\to M_1^n$ be its universal covering. Then, $\widetilde{M}_1^n$ is a connected complete simply connected space of constant curvature a^2+c, and hence congruent to $\widetilde{\mathbb{H}}_1^n(a^2+c)$, where $\widetilde{\pi}:\widetilde{\mathbb{H}}_1^n(a^2+c)\to\mathbb{H}_1^n(a^2+c)$ is the universal covering. Note that $k(x)=n$ everywhere. Since we have the inclusion $i:\mathbb{H}_1^n(a^2+c)\to\mathbb{H}_1^{n+1}(c)\subset\mathbb{E}_2^{n+2}$, given by $x\mapsto\left(x,\sqrt{\frac{1}{c}-\frac{1}{a^2+c}}\right)$, it follows from Theorem 1.3 of [66] that $f(M_1^n)=f\circ\pi(\widetilde{M}_1^n)$ is congruent to $i\circ\widetilde{\pi}(\widetilde{M}_1^n)=i(\mathbb{H}_1^n(a^2+c))$. This completes the proof of the corollary. □

Now, for $\psi>c$ we have the next result.

Theorem 4.22. *Let $f:M_1^n\to\mathbb{Q}_1^{n+1}(c)$ be a good ψ-pseudo-parallel Lorentzian hypersurface, with $n\geq3$ and constant $\psi>c$. Then, for each connected component C of M_1^n containing at least some point x such that $k(x)\geq1$, we have that f is isoparametric and either*

(i) totally umbilical,

(ii) locally congruent to a product of two spaces each of constant curvature greater than c. In this case, the ambient space is $\mathbb{H}_1^{n+1}(c)$.

(iii) or the open subset of non umbilical points of C consists entirely of points of type 5.

Proof. Let C be a connected component of M_1^n, which contains at least one point x such that $k(x) \geq 1$. It follows from Proposition 4.17 that $k(x) = n$ everywhere in C. Since f is pseudo-parallel and $\psi > c$, according to Lemma 4.7, we conclude that A has at most two unequal eigenvalues, which are real nonzero, and if at some point $y \in M_1^n$ exactly two are distinct then A_y is diagonalizable.

Suppose first that there is a type 3 point x_0 in C. Define two continuous functions $a = a_1$ and $b = a_n$ (the largest and smallest eigenvalues respectively), and note that $a \geq b$. Let

$$W_3 = \{x \in C : a(x) = a(x_0) \text{ and } b(x) = b(x_0)\}.$$

We have that W_3 is a closed subset in C. By continuity, x_0 has a neighborhood U consisting of points of type 3 (C_3 is open). Indeed, by applying an analogous argument of that in Proposition 2.2 of [66], we have that a and b are differentiable and have constant multiplicities, say k and $n-k$, in U, where $1 < k < n-1$ since all points are good. Thus, U consists entirely of points of type 3, as desired.

Let $a_1 = a_2 = \cdots = a_k = a$ and $a_{k+1} = a_{k+2} = \cdots = a_n = b$. From proposition D.4 of [41] and since a, b satisfy $b = \frac{\psi - c}{a}$ in U, it follows that $a = a(x_0)$ and $b = b(x_0)$ near x_0. Observe that x_0 could have been any point in W_3, thus, W_3 is also open and hence is all of C. Thus, f is then isoparametric in C with A diagonalizable and the unequal principal curvatures satisfy $ab > 0$. Assertion *(ii)* of the theorem follows from Theorem 5.1 of [1], including the fact that $\psi = c + ab = 0$. Thus, $c < 0$ in this case.

Suppose now that C does not contains points of type 3. Since M_1^n is a good hypersurface, we see that C consists entirely of umbilical points and type 5 points. Let $a \neq 0$ be the unique eigenvalue of A in C. By Proposition D.4 of [41], we have that a is constant in C, say $a(x) = a_0$, thus we obtain that f is isoparametric in C. Since a point x of type 5 must satisfy the equation $a^2(x) = \psi - c > 0$, we get that C consists entirely of umbilical points if $a_0 \neq \pm\sqrt{\psi - c}$. On the other hand, if $a_0 = \sqrt{\psi - c}$, consider the subsets

$$U = \{x \in C : A_x = \sqrt{\psi - c} I_n\},$$
$$V = \{x \in C : A_x = \begin{pmatrix} \sqrt{\psi - c} & 0 \\ 1 & \sqrt{\psi - c} \end{pmatrix} \oplus \sqrt{\psi - c} I_{n-2}\}.$$

We have that U and V are complementary with U closed. Thus, the open set V of type 5 points of C, which can be empty, is open. This proves assertions *(i)* and *(iii)* of the theorem. □

Theorem 4.23. *Let $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$ be a good ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi > c$. Then, $f(M_1^n)$ is locally congruent to one of the following Lorentzian hypersurfaces:*

(i) A totally geodesic or totally umbilical hypersurface as described in parts (i), (ii), (iii) and (iv) of Theorem 4.19.

(ii) $\mathbb{H}_1^k(a^2+c)\times\mathbb{S}^{n-k}\left(\frac{c^2}{a^2}+c\right)=\left\{x\in\mathbb{H}_1^{n+1}(c)\subset\mathbb{E}_2^{n+2}: -\sum_{i=1}^{2}x_i^2+\sum_{i=3}^{k+1}x_i^2=\frac{1}{a^2+c},\sum_{i=k+2}^{n+2}x_i^2=\frac{a^2}{c^2+ca^2}\right\}$, *where* $c<0$, $|a|<\sqrt{-c}$, $\psi=0$ *and* $1<k<n-1$.

(iii) $\mathbb{S}_1^k(a^2+c)\times\mathbb{H}^{n-k}\left(\frac{c^2}{a^2}+c\right)=\left\{x\in\mathbb{H}_1^{n+1}(c)\subset\mathbb{E}_2^{n+2}: -x_1^2+\sum_{i=3}^{k+2}x_i^2=\frac{1}{a^2+c},-x_2^2+\sum_{i=k+3}^{n+2}x_i^2=\frac{a^2}{c^2+ca^2}\right\}$, *where* $c<0$, $|a|>\sqrt{-c}$, $\psi=0$ *and* $1<k<n-1$.

(iv) A generalized umbilical hypersurface of degree 2 *as in* (1.36), (1.37) *or* (1.38), *in the open subset of non-umbilical points.*

Proof. We can assume that M_1^n is connected. We observe first that if there exists a point $x\in M_1^n$ with $k(x)\geq 1$, then f is necessarily isoparametric, according to each possibility in Theorem 4.22, that is, either the Weingarten operator is diagonalizable with two unequal nonzero eigenvalues everywhere, f is totally umbilical or all the non umbilical points are type 5 points. In other case, $f(M_1^n)$ is a totally geodesic hypersurface. Thus, the result follows from Theorem 5.1 of [1], combined with Theorem 4.5 of [55], Theorem 5.5 of [5] and Theorem 3.1 and Theorem 4.2 of [80]. □

In the last part of Corollary 4.23, we were forced to omit the presence of umbilical points, due to the possible existence of pseudo-parallel Lorentzian hypersurfaces containing type 5 points which converge to umbilical points, as shown by the following example, at least for the case $c=0$.

Example 4.24. Consider the matrix

$$M(s)=\begin{pmatrix}0&0&-1\\0&0&s\\s&-1&0\end{pmatrix}.$$

We denote

$$X(s)=\begin{pmatrix}x_1&x_2&x_3\\y_1&y_2&y_3\\z_1&z_2&z_3\end{pmatrix},$$

and let us solve the initial value problem:

$$X'(s)=X(s)M(s),\qquad X(0)=I_3. \tag{4.13}$$

We have that

$$\begin{aligned}
x_2'&=-x_3, & x_3'&=-x_1+sx_2, & x_1'&=sx_3,\\
y_2'&=-y_3, & y_3'&=-y_1+sy_2, & y_1'&=sy_3,\\
z_2'&=-z_3, & z_3'&=-z_1+sz_2, & z_1'&=sz_3.
\end{aligned}$$

We affirm that there is a matrix $X(s)$ which is a solution of the initial value problem (4.13) in a neighborhood of $s=0$. Let $T(s),B(s),C(s)$ be the vector fields given by the columns of $X(s)$ in a pseudo-orthonormal frame $\{\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3,\widetilde{e}_4\}$ of $\mathbb{E}^4_1$, where $\widetilde{e}_1$ and $\widetilde{e}_2$ are lightlike vector fields, that is, $T(s)=x_1\widetilde{e}_1+y_1\widetilde{e}_2+z_1\widetilde{e}_3$, $B(s)=x_2\widetilde{e}_1+y_2\widetilde{e}_2+z_2\widetilde{e}_3$ and $C(s)=x_3\widetilde{e}_1+y_3\widetilde{e}_2+z_3\widetilde{e}_3$. We have that these vector fields satisfy $T'(s)=sC(s)$, $B'(s)=-C(s)$ and $C'(s)=-T(s)+sB(s)$, with $T(s)$ and $B(s)$ lightlike, so the curve $\gamma(s)=\int_0^s T(t)dt$ is a null curve and $\{T(s),B(s),C(s),\widetilde{e}_4\}$ is a pseudo-orthonormal frame associated to γ, with $\tau=-1$ and $\kappa(s)=s$. In fact, using Mathematica [79] to solve (4.13), we obtain explicit expressions for $T(s)$, $B(s)$ and $C(s)$ and we also obtain that the null curve γ is given in terms of the hypergeometric ${}_0F_1$ function as follows.

$$\begin{aligned}\gamma(s)=\frac{1}{6}s&\left\{2\,{}_0F_1\left(;\frac{1}{3},-\frac{s^3}{18}\right)^2+4\,{}_0F_1\left(;\frac{1}{3},-\frac{s^3}{18}\right){}_0F_1\left(;\frac{4}{3},-\frac{s^3}{18}\right)+s^3\,{}_0F_1\left(;\frac{4}{3},-\frac{s^3}{18}\right)^2\right\}\widetilde{e}_1\\&+\left\{8\,{}_0F_1\left(;\frac{2}{3},-\frac{s^3}{18}\right)^2-8\,{}_0F_1\left(;\frac{2}{3},-\frac{s^3}{18}\right){}_0F_1\left(;\frac{5}{3},-\frac{s^3}{18}\right)+s^3\,{}_0F_1\left(;\frac{5}{3},-\frac{s^3}{18}\right)^2\right\}\widetilde{e}_2\\&+\frac{1}{6}\left\{-2\,{}_0F_1\left(;\frac{1}{3},-\frac{s^3}{18}\right)\left[2\,{}_0F_1\left(;\frac{1}{3},-\frac{s^3}{18}\right)+s^3\,{}_0F_1\left(;\frac{4}{3},-\frac{s^3}{18}\right)\right]\right.\\&\left.+s^3\left[{}_0F_1\left(;\frac{1}{3},-\frac{s^3}{18}\right)+{}_0F_1\left(;\frac{4}{3},-\frac{s^3}{18}\right)\right]{}_0F_1\left(;\frac{5}{3},-\frac{s^3}{18}\right)\right\}\widetilde{e}_3,\end{aligned}\tag{4.14}$$

where we can choose $\widetilde{e}_1=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)$, $\widetilde{e}_2=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0\right)$, $\widetilde{e}_3=(0,0,1,0)$ and $\widetilde{e}_4=(0,0,0,1)$, given in the standard coordinates of $\mathbb{E}^4_1$.

Then, we consider $f:U\to\mathbb{E}^4_1$ with U a neighborhood of 0 in $\mathbb{R}^3$, given by

$$f(s,u,z)=\gamma(s)+uB(s)+z\widetilde{e}_4+C(s)-\sqrt{1-z^2}C(s).\tag{4.15}$$

A unit normal vector field η to f is given by

$$\eta(s,u,z)=-uB(s)+\sqrt{1-z^2}C(s)-z\widetilde{e}_4.$$

Considering the frame $f_*\left(\frac{\partial}{\partial s}\right)$, $f_*\left(\frac{\partial}{\partial u}\right)$, $f_*\left(\frac{\partial}{\partial z}\right)$, and using again Mathematica, we obtain that the first fundamental form of f is

$$I(s,u,z)=\begin{pmatrix}u^2+2s(1-z^2-\sqrt{1-z^2}) & -\sqrt{1-z^2} & -\dfrac{uz}{\sqrt{1-z^2}}\\ -\sqrt{1-z^2} & 0 & 0\\ -\dfrac{uz}{\sqrt{1-z^2}} & 0 & \dfrac{1}{1-z^2}\end{pmatrix},$$

and the Weingarten operator $A=A_\eta$ takes the form

$$A(s,u,z)=\begin{pmatrix}1&0&0\\-s&1&0\\0&0&1\end{pmatrix}.$$

Observe that the determinant of $I(s,u,z)$ is -1 and $f_*\left(\frac{\partial}{\partial z}\right)$ is spacelike for small z, thus, $f(U)$ is a Lorentzian hypersurface in $\mathbb{E}^4_1$, even for $s=0$. We note that $f(U)$ is a generalized umbilical hypersurface of degree 2 for $s\neq0$, such that the Weingarten operator has minimal polynomial $(t-1)^2$, but all the points of $f(U)$ at the slice with $s=0$ are umbilical points and the minimal polynomial is $t-1$. Using Mathematica [79], we draw the slice $z=0$ of the immersion f that we show in Figure 4.1.

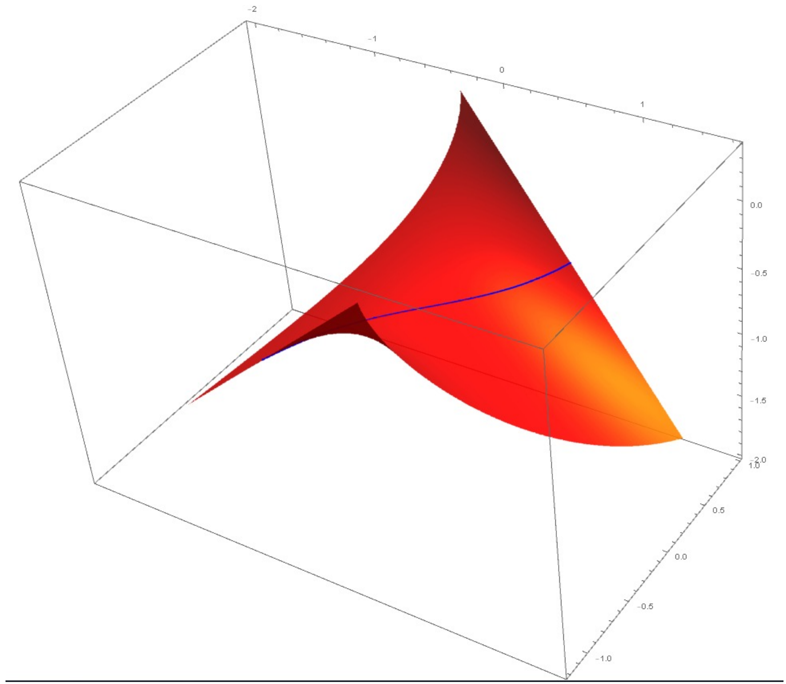

Figure 4.1: Slice $z = 0$ of the hypersurface f of $\mathbb{E}^4_1$ given by (4.15) from Example 4.24.

4.5 Classification of ψ-pseudo-parallel Lorentzian hypersurfaces with constant ψ and constant mean curvature in Lorentzian space forms

Let $f : M^n_1 \to \mathbb{Q}^{n+1}_1(c)$ be a Lorentzian hypersurface. We consider the mean curvature function H of M^n_1, defined by

$$H = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i \langle \alpha(E_i, E_i), \eta\rangle = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i \langle AE_i, E_i\rangle,$$

where $\{E_1, \dots, E_n\}$ is an orthonormal frame in TM with $\varepsilon_i = \langle E_i, E_i\rangle$ and η is a unit (spacelike) normal vector field. For a pseudo-orthonormal frame $\{X, Y, E_3, \dots, E_n\}$, where $\langle X,Y\rangle = -1$, $\langle X,X\rangle = \langle Y,Y\rangle = \langle X,E_i\rangle = \langle Y,E_i\rangle = 0$, and $\langle E_i,E_j\rangle = \delta_{ij}$, $3 \le i, j \le n$, we have that

$$H = \frac{1}{n}\left(-2\langle AX,Y\rangle + \sum_{i=3}^{n} \langle AE_i, E_i\rangle\right),$$

Theorem 4.25. *Let $f : M^n_1 \to \mathbb{Q}^{n+1}_1(c)$ be a ψ-pseudo-parallel Lorentzian hypersurface, with $n \ge 3$ and constant $\psi = c$. If f has nonzero constant mean curvature, then $f(M^n_1)$ is locally congruent to one of the following Lorentzian hypersurfaces:*

(i) A totally umbilical $\mathbb{S}^n_1(a^2) = \left\{ x \in \mathbb{E}^{n+1}_1 : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{a^2} \right\}$, *$a = H \neq 0$, if $c = 0$.*

(ii) A totally umbilical $\mathbb{S}^n_1(a^2 + c) = \left\{ x \in \mathbb{S}^{n+1}_1(c) \subset \mathbb{E}^{n+2}_1 : x_{n+2} = \sqrt{\frac{1}{c} - \frac{1}{a^2+c}} \right\}$, *with $a = H \neq 0$, if $c > 0$.*

(iii) *A totally umbilical* $\mathbb{H}^n_1(a^2+c) = \left\{x \in \mathbb{H}^{n+1}_1(c) \subset \mathbb{E}^{n+2}_2 : x_{n+2} = \sqrt{\frac{1}{c} - \frac{1}{a^2+c}}\right\}$, *with* $0 < |a| < \sqrt{-c}$, *or* $\mathbb{S}^n_1(a^2+c) = \left\{x \in \mathbb{H}^{n+1}_1(c) \subset \mathbb{E}^{n+2}_2 : x_1 = \sqrt{\frac{1}{a^2+c} - \frac{1}{c}}\right\}$, *with* $a^2 > -c$. *In this case* $a = H \neq 0$ *and* $c < 0$.

(iv) *A totally umbilical hypersurface of the form* $\{y \in H^{n+1}_1(c) : \langle y, X\rangle = a\}$, *where* $a = \pm\sqrt{-c} = H$ *and* X *is a parallel vector field in* $\mathbb{E}^{n+2}_2$, *which satisfies* $\langle X, X\rangle = 0$. *In this case* $c < 0$.

(v) *A cylinder* $\mathbb{S}^k_\tau(a^2) \times \mathbb{E}^{n-k}_{1-\tau} = \left\{x \in \mathbb{E}^{n+1}_1 : -\sum_{i=2-\tau}^{1} x_i^2 + \sum_{i=2}^{k+2-\tau} x_i^2 = \frac{1}{a^2}\right\}$, $a = \frac{nH}{k} \neq 0$ *and* $1 \leq k \leq n-1$. *In this case* $c = 0$.

Proof. We can assume that M^n_1 is connected. Since f is pseudo-parallel, it follows from Lemma 4.7 that A_x has at most one nonzero eigenvalue $a(x)$, for $x \in M^n_1$. Thus, we have that $k(x)a(x) = nH \neq 0$ and so $a(x) \neq 0$ with $1 \leq k(x) \leq n$. Then, again from Lemma 4.7, we have that A_x is diagonalizable and takes the form $A_x = a(x)I_{k(x)} \oplus 0_{n-k(x)}$. As in [59] (or [66]), we can show that $k(x)$ is a locally constant function and, since M^n_1 is connected, we have that $k(x)$ is constant in all of M^n_1, say $k(x) = k$. Again, using that $k(x)a(x) = nH$ is constant, we have that $a(x)$ is constant in all of M^n_1, say $a(x) = a$. This means that f is isoparametric. The theorem follows from the classification of isoparametric hypersurfaces with diagonalizable Weingarten operator in Theorem 5.1 of [1]. □

In the case of $\psi = c$ and degenerate relative nullity, we have the following result:

Theorem 4.26. *Let* $f : M^n_1 \to \mathbb{Q}^{n+1}_1(c)$ *be a* ψ*-pseudo-parallel Lorentzian hypersurface, with* $n \geq 3$ *and constant* $\psi = c$. *If* f *has mean curvature* $H = 0$, *then* $f(M^n_1)$ *is either totally geodesic or*

(i) $c = 0$ *and* $f(M^n_1)$ *is a generalized cylinder given by* $\mathbb{E}^{n-2} \times h(\mathbb{E}^2_1)$, *where* $h(\mathbb{E}^2_1)$ *is a B-scroll in a subspace* $\mathbb{E}^3_1$ *of* $\mathbb{E}^{n+1}_1$ *orthogonal to* $\mathbb{E}^{n-2}$, *that is, locally, the hypersurface* $f : U \to \mathbb{E}^{n+1}_1$, U *a neighborhood of* 0 *in* $\mathbb{R}^n$, *parameterized by*

$$f(s, y, z_3, \dots, z_n) = \gamma(s) + yB(s) + \sum_i z_i Z_i(s)$$

where γ *is a null curve in* $\mathbb{E}^{n+1}_1$ *with an associated pseudo-orthonormal frame* $\{T(s), B(s), Z_3(s), \dots, Z_n(s), C(s)\}$ *of vector fields of* $\mathbb{E}^{n+1}_1$ *along* γ, *such that* $T(S)$ *and* $B(s)$ *are lightlike vector fields with* $\langle T(S), B(S)\rangle = -1$, $T(s) = \frac{d}{ds}\gamma(s)$ *and* $\frac{d}{ds}C(s) = \kappa(s)B(s)$.

(ii) $c \neq 0$ *and at the open subset of non-geodesic points, locally* $f(M^n_1)$ *is an open piece of a hypersurface as described in Example 1.4 or Example 1.5.*

Proof. Under the assumptions of the theorem, it follows from Lemma 4.7 that the Weingarten operator A only have one eigenvalue 0 and f is isoparametric with minimal polynomial t or t^2. In fact, as in the proof of Theorem 4.25, if A_x has one nonzero eigenvalue $a(x)$, for $x \in M^n_1$, we have that

$k(x)a(x) = nH = 0$, which is a contradiction and $A_x = 0_n$ or A_x takes the form in *(ii)* of Lemma 4.7. Thus, for $c = 0$, the results follows from Theorem 9.8 of [34] (see also the observation after the proof of Theorem 4.4 of [55]). For $c \neq 0$, the result about the open subset of non-geodesic points follows from the classification of isoparametric Lorentzian hypersurfaces in Theorem 4.2 and case (2) in Theorem 4.1 of [80], for $c < 0$, and in Theorem 6.8 of [42] for the case $c > 0$. □

Note that in *(i)* of Theorem 4.26, $\kappa(s)$ can be zero. The next example shows that B-scrolls containing non-geodesic points which converge to geodesic points indeed exists, at least for the case $c = 0$.

Example 4.27. Consider the matrix

$$M(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ s & 0 & 0 \end{pmatrix}.$$

We denote

$$X(s) = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix},$$

and let us solve the initial value problem:

$$X'(s) = X(s)M(s), \quad X(0) = I_3.$$

We have that

$$\begin{aligned} x_2' &= 0, & x_3' &= sx_2, & x_1' &= sx_3, \\ y_2' &= 0, & y_3' &= sy_2, & y_1' &= sy_3, \\ z_2' &= 0, & z_3' &= sz_2, & z_1' &= sz_3. \end{aligned}$$

We obtain: $x_2 = c_1$ constant, then $x_3' = c_1 s$ and we have $x_3 = \frac{1}{2}c_1 s^2 + a$. It follows that $x_1' = \frac{1}{2}c_1 s^3 + as$ and we obtain $x_1 = \frac{1}{8}c_1 s^4 + \frac{1}{2}as^2 + K$. Using that $M(0) = I_3$, it follows that $c_1 = 0$, $a = 0$ and $K = 1$. Thus, $x_1 = 1$, $x_2 = 0$ and $x_3 = 0$.

Analogously, we obtain $y_1 = \frac{1}{8}s^4$, $y_2 = 1$, $y_3 = \frac{1}{2}s^2$ and $z_1 = \frac{1}{2}s^2$, $z_2 = 0$ and $z_3 = 1$.

Now, let $A(s) = x_1\widetilde{e}_1 + y_1\widetilde{e}_2 + z_1\widetilde{e}_3 = \widetilde{e}_1 + \frac{1}{8}s^4\widetilde{e}_2 + \frac{1}{2}s^2\widetilde{e}_3$ in a pseudo-ortonormal frame $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$ of $\mathbb{E}^3_1$, where $\langle\widetilde{e}_1, \widetilde{e}_1\rangle = \langle\widetilde{e}_2, \widetilde{e}_2\rangle = 0$, $\langle\widetilde{e}_1, \widetilde{e}_2\rangle = -1$, $\langle\widetilde{e}_1, \widetilde{e}_3\rangle = \langle\widetilde{e}_2, \widetilde{e}_3\rangle = 0$ and $\langle\widetilde{e}_3, \widetilde{e}_3\rangle = 1$. We can see that $A(s)$ satisfies $\langle A(s), A(s)\rangle = -2x_1y_1 + z_1^2 = 0$. So the curve

$$\gamma(s) = \int_0^s A(t)dt = (s, \frac{1}{40}s^5, \frac{1}{6}s^3) = s\widetilde{e}_1 + \frac{1}{40}s^5\widetilde{e}_2 + \frac{1}{6}s^3\widetilde{e}_3,$$

is a null curve with $\kappa(s) = s$ (in fact γ and the columns of $X(s)$ define a generalized null cubic) and $f(s,u) = \gamma(s) + uB(s)$ is a B-scroll, where $B(s) = x_2\widetilde{e}_1 + y_2\widetilde{e}_2 + z_2\widetilde{e}_3 = \widetilde{e}_2$ is a parallel lightlike vector field in $\mathbb{E}^3_1$ and $C(s,u) = C(s) = x_3\widetilde{e}_1 + y_3\widetilde{e}_2 + z_3\widetilde{e}_3 = \frac{1}{2}s^2\widetilde{e}_2 + \widetilde{e}_3$ is a unit spacelike normal vector field to f. Note that $f_*\left(\frac{\partial}{\partial_s}\right) = A(s)$, $f_*\left(\frac{\partial}{\partial_u}\right) = B(s)$ and $\langle A(s), B(s)\rangle = \langle \widetilde{e}_1, \widetilde{e}_2\rangle = -1$, for all $s \in \mathbb{R}$. Thus, the metric of f is Lorentzian even for $s = 0$, that is, s can be zero, and note that the points at the image of f with $s = 0$ are geodesic points, since the Weingarten operator A takes the form

$$A(s,u) = \begin{pmatrix} 0 & 0 \\ -s & 0 \end{pmatrix}.$$

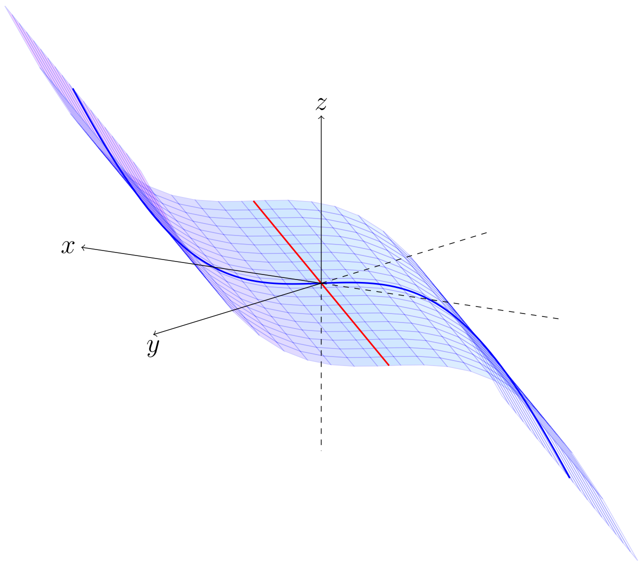

Figure 4.2: B-scroll with geodesic points from Example 4.27

If $\psi < c$ and H constant, we have the following classification result:

Theorem 4.28. *Let $f : M^n_1 \to \mathbb{Q}^{n+1}_1(c)$ be a ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi < c$. If f has constant mean curvature, then $f(M^n_1)$ is locally congruent to one of the following Lorentzian hypersurfaces:*

(i) A totally geodesic or totally umbilical hypersurface as described in parts (i), (ii), (iii) and (iv) of Theorem 4.19, with $a = H$.

(ii) $$\mathbb{S}^k(a^2+c) \times \mathbb{S}^{n-k}_1\left(\frac{c^2}{a^2}+c\right) = \left\{x \in \mathbb{S}^{n+1}_1(c) \subset \mathbb{E}^{n+2}_1 : \sum_{i=2}^{k+2} x_i^2 = \frac{1}{a^2+c}, -x_1^2 + \sum_{i=k+3}^{n+2} x_i^2 = \frac{a^2}{c^2+ca^2}\right\},$$

where $c > 0$, $\psi = 0$, $a = \dfrac{nH \pm \sqrt{n^2H^2 + 4k(n-k)c}}{2k}$ and $1 \leq k \leq n-1$.

Proof. We can assume that M^n_1 is connected. First, suppose that there exists at least one good point $x \in M^n_1$. It follows from Proposition 4.14 and Proposition 4.16 that the set of bad points is open and closed, thus, f is a good hypersurface because of the connectedness of M^n_1. Then, it follows from Theorem 4.19 that $f(M^n_1)$ is locally congruent either to a totally geodesic or totally umbilical hypersurface as described in parts *(i)*, *(ii)*, *(iii)* and *(iv)* of that same theorem, with $a = H$ in this case, or else $f(M_1^n)$ is locally a product of two spaces each of constant curvature as in *(v)* of the same theorem, for $1 < k < n-1$, and the Weingarten operator has two nonzero eigenvalues a and $-\frac{c}{a}$, which satisfy $ka - (n-k)\frac{c}{a} = nH$. Thus, $a = \dfrac{nH \pm \sqrt{n^2H^2 + 4k(n-k)c}}{2k}$.

On the other hand, if all the points of M_1^n are bad, without loss of generality, let a and b be the two distinct nonzero eigenvalues of the Weingarten operator with multiplicities 1 and $n-1$, which satisfy $b = \dfrac{\psi - c}{a}$ from Lemma 4.7. Since we have that $a + (n-1)b = nH$, we obtain $a + (n-1)\dfrac{\psi - c}{a} = nH$. Thus, $a = \dfrac{nH \pm \sqrt{n^2H^2 - 4(n-1)(\psi - c)}}{2}$ is constant in M_1^n and so also b is constant, because H and ψ are constant and M_1^n is connected. From the classification of isoparametric hypersurfaces with diagonalizable Weingarten operator and at most two real principal curvatures in Theorem 5.1 of [1], we have that $f(M_1^n)$ is locally congruent to a hypersurface as in *(ii)* of this theorem for $k = 1$ or $k = n-1$. It follows from this classification that $\psi = 0$ in this case. □

Now, if $\psi < c$, to obtain a classification of pseudo-parallel Lorentzian hypersurfaces with constant ψ and constant H, we prove first the next result:

Proposition 4.29. *Let $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$ be a connected ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi > c$. If f has constant mean curvature and contains at least one bad point, then all points of M_1^n are bad points.*

Proof. Consider the set C_4 of bad points of M_1^n, which is open from Proposition 4.14. Let C be a connected component of C_4, we have that C is open because M_1^n is locally connected. Since M_1^n is connected, we just need to prove that C is also closed and the proposition follows. In fact, let $\{x_i\}$ be a sequence of bad points in C converging to some point $x \in M_1^n$. From Lemma 4.7, A is diagonalizable and has two unequal eigenvalues a, b, such that $ab = \psi - c$ on C. Using an argument as in Proposition 2.2 of [66], we can show that the multiplicities of a and b are locally constant functions near non-umbilical points and, thus, we have that a and b have constant multiplicities in C. We can assume that 1 and $n-1$ are the multiplicities of a and b, respectively. Since ψ is constant, H is constant and $a + (n-1)b = nH$ and $b = \dfrac{\psi - c}{a}$ in C, we have that $a = \dfrac{nH \pm \sqrt{n^2H^2 - 4(n-1)(\psi - c)}}{2}$ is constant in C and so also b is constant in C, because C is connected. By continuity, we have that $a(x) = a$, $b(x) = b$ and also the multiplicities of a and b at x are 1 and $n-1$, respectively. Therefore, x is also a bad point and C is also closed. □

Theorem 4.30. *Let $f : M_1^n \to \mathbb{Q}_1^{n+1}(c)$ be a ψ-pseudo-parallel Lorentzian hypersurface, with $n \geq 3$ and constant $\psi > c$. If f has constant mean curvature, then $f(M_1^n)$ is locally congruent to one of the following Lorentzian hypersurfaces:*

(i) A totally geodesic or totally umbilical hypersurface as described in parts (i), (ii), (iii) and (iv) of Theorem 4.19, with $a = H$.

(ii) $\mathbb{H}^k_1(a^2+c)\times\mathbb{S}^{n-k}\left(\dfrac{c^2}{a^2}+c\right)=\left\{x\in\mathbb{H}^{n+1}_1(c)\subset\mathbb{E}^{n+2}_2 : -\sum_{i=1}^{2}x_i^2+\sum_{i=3}^{k+1}x_i^2=\dfrac{1}{a^2+c},\sum_{i=k+2}^{n+2}x_i^2=\dfrac{a^2}{c^2+ca^2}\right\}$, *where* $c<0$, $|a|<\sqrt{-c}$, $\psi=0$, $a=\dfrac{nH\pm\sqrt{n^2H^2+4k(n-k)c}}{2k}$ *and* $1\leq k\leq n-1$.

(iii) $\mathbb{S}^k_1(a^2+c)\times\mathbb{H}^{n-k}\left(\dfrac{c^2}{a^2}+c\right)=\left\{x\in\mathbb{H}^{n+1}_1(c)\subset\mathbb{E}^{n+2}_2 : -x_1^2+\sum_{i=3}^{k+2}x_i^2=\dfrac{1}{a^2+c},-x_2^2+\sum_{i=k+3}^{n+2}x_i^2=\dfrac{a^2}{c^2+ca^2}\right\}$, *where* $c<0$, $|a|>\sqrt{-c}$, $\psi=0$, $a=\dfrac{nH\pm\sqrt{n^2H^2+4k(n-k)c}}{2k}$ *and* $1\leq k\leq n-1$.

(iv) A generalized umbilical hypersurface of degree 2 *as in* (1.36), (1.37) *or* (1.38), *where* $\psi=c+a^2$, $\tau=a=H\neq 0$, *in the open subset of non-umbilical points.*

Proof. We can assume that M^n_1 is connected. From Proposition 4.29, we have that either M^n_1 is a good hypersurface or else all the points of M^n_1 are bad points. Thus, using an analogous argument as in the proof of Theorem 4.28, we can deduce the result from Theorem 4.23 and Theorem 5.1 of [1]. □

Observation 4.31. The classification of ψ-pseudo-parallel Lorentzian hypersurfaces in $\mathbb{Q}^{n+1}_1(c)$, with constant $\psi=c\neq 0$ is still an open problem.

From the remark after the models of isoparametric hypersurfaces with diagonalizable Weingarten operator with at most two unequal eigenvalues in [1] and Theorem 5.2 of the same reference, we have that any such Lorentzian hypersurface M^n_1 either is totally umbilical or the two distinct eigenvalues a,b satisfy $ab=-c$. If in addition, M^n_1 is a ψ-pseudo-parallel hypersurface, from Lemma 4.7, we have necessarily that $ab=\psi-c$ and so $\psi=0$. Then, we cannot have two distinct constant eigenvalues if $\psi=c\neq 0$. Therefore, it is reasonable to state the following conjecture:

Conjecture 4.32. *Any connected ψ-pseudo-parallel Lorentzian hypersurface $f:M^n_1\to\mathbb{Q}^{n+1}_1(c)$, with $n\geq 3$ and $\psi=c\neq 0$, is congruent to a totally umbilical hypersurface of $\mathbb{Q}^{n+1}_1(c)$ or $k(x)\leq 1$ everywhere on M^n_1.*

APPENDIX A

Semi-parallel hypersurfaces in $\mathbb{E}_1^{n+1}$ with rank of the Weingarten operator ≥ 2

We will prove here the following proposition, which is part of the classification of connected and complete semi-parallel Lorentzian hypersurfaces of the Minkowski space given in Theorem 4.10.

Proposition A.1. *Let M_1^n, $n \geq 3$, be a connected complete Lorentzian manifold and let $f : M_1^n \to \mathbb{E}_1^{n+1}$ be an isometric immersion. Suppose that the type number $k(x) \geq 2$ at least at one point $x \in M_1^n$. Then f is semi-parallel if and only if f is either an isometry and M_1^n is congruent to*

(i) $\mathbb{S}^k(a^2) \times \mathbb{E}_1^{n-k}$, $2 \leq k \leq n-1$,

(ii) $\mathbb{S}_1^k(a^2) \times \mathbb{E}^{n-k}$, $3 \leq k \leq n$,

for some a, k constants, $a \neq 0$, or else $f(M_1^n)$ is congruent to $\mathbb{S}_1^2(a^2) \times \mathbb{E}^{n-2}$.

The proof of Proposition A.1 can be carried out practically without changes from that by Van de Woestijne, Verstraelen and Nomizu in [75, 59]. Without loss of generality, we may suppose that M_1^n is simply connected and thus a unit normal vector field η to f can be globally defined, since in case that M_1^n is not simply connected we can always work with the universal covering instead. Note that f is not necessarily an injective map.

First, we will prove Proposition A.1 under the assumption that $k(x) \geq 2$ for all $x \in M_1^n$. Suppose that f is semi-parallel. It follows from Corollary 4.7, that the Weingarten operator takes the form $A_x = a(x) I_{k(x)} \oplus 0_{n-k(x)}$, for all $x \in M_1^n$. Since all eigenvalues of A are real numbers, we may define n principal curvature functions $\{a_1, \ldots, a_n\}$ as done in Lemma 2.1 of [66] and using the same arguments there, we conclude that the a_i's are continuous functions. We have that $k(x)$ is locally constant in M_1^n. In fact, if $k(y) = n$ for some $y \in M_1^n$, since $a(y) \neq 0$, we have that $a_i(x) = a(x) \neq 0$, for all $1 \leq i \leq n$, and $k(x) = n$ on some neighborhood of y. If $2 \leq k(y) \leq n-1$, we have that $a(y) = a_1(y) = \cdots = a_{k(y)}(y) > a_{k(y)+1}(y) = \cdots = a_n(y) = 0$. By continuity, there is an open neighborhood U of y, on which

$k(y)$ principal curvatures have absolute value greater than $\frac{1}{2}|a(y)|$ and $n-k(y)$ principal curvatures have absolute value smaller than $\frac{1}{2}|a(y)|$. It follows that $k(x) \geq k(y) \geq 2$ for all $x \in U$, and since all the nonzero principal curvatures are equal, we have that the $n-k(y)$ principal curvatures with absolute value smaller than $\frac{1}{2}|a(y)|$ are 0. Thus, $k(x) = k(y)$ constant on U. By the connectedness of M_1^n, it follows that $k(x)$ is constant in M_1^n, say, $k(x) = k$, and the only nonzero eigenvalue $a(x)$ of A_x, defines a differentiable function $a(x) = \frac{1}{k}\operatorname{trace}(A_x)$ on M_1^n. Now, we consider the distributions T_0 and T_1 which are defined by

$$T_0(x) = \{X \in T_x M_1^n : AX = 0\},$$
$$T_1(x) = \{X \in T_x M_1^n : AX = aX\}.$$

If $2 \leq k \leq n-1$, as in Proposition 2.3 of [66], we can prove that these distributions on M_1^n are differentiable and involutive (i.e., integrable). This is also true if $k = n$, since in this case $T_1(x) = T_x M_1^n$. From Proposition D.4 of [41], since dimension of $T_1(x)$ is greater than 1, it follows that $X(a) = 0$ for all $X \in T_1$, i.e., a is constant on each maximal integral manifold of T_1. Also, we have that $T_x M_1^n = T_0(x) \oplus T_1(x)$, for all $x \in M_1^n$. For any $Z \in TM_1^n$, $(Z)_0$ and $(Z)_1$ denote the component of Z in $T_0(x)$ and $T_1(x)$, respectively).

We will prove that a is also constant on each maximal integral manifold of T_0 and so it is constant on M_1^n. As in [59], it follows that:

Lemma A.2. *(i) If $X \in T_1, Y \in T_0$, then $A(\nabla_X Y) = -Y(a)X$.*

(ii) If $Y \in T_0$, then $\nabla_Y(T_1) \subset T_1$.

(iii) If $Y \in T_0$, then $\nabla_Y(T_0) \subset T_0$.

(iv) If $Y \in T_0$, $X \in T_1$ and $[X,Y] = 0$, then $\nabla_X Y \in T_1$.

Proof. Let $X \in T_1$, $Y \in T_0$ and compute both sides of the Codazzi equation:

$$\begin{aligned}(\nabla_X A)Y &= -A(\nabla_X Y) = -a(\nabla_X Y)_1,\\ (\nabla_Y A)X &= \nabla_Y(aX) - A(\nabla_Y X) = Y(a)X + a(\nabla_Y X) - a(\nabla_Y X)_1\\ &= Y(a)X + a(\nabla_Y X)_0.\end{aligned}$$

Thus, we obtain

$$(\nabla_Y X)_0 = 0, \text{ that is, } \nabla_Y X \in T_1$$

and also

$$Y(a)X = -a(\nabla_X Y)_1 = -A(\nabla_X Y).$$

Then, we have *(i)* and *(ii)*.

Now, *(iii)* follows from *(ii)*, since $(T_0)^\perp = T_1$. In fact, since $\langle Z, X\rangle = \frac{1}{a}\langle Z, AX\rangle = \frac{1}{a}\langle AZ, X\rangle = 0$ for $Z \in T_0$, we have that $\langle \nabla_Y Z, X\rangle = -\langle Z, \nabla_Y X\rangle = 0$, for all $X \in T_1$.

Finally, *(iv)* follows from $\nabla_X Y = \nabla_Y X + [X,Y] = \nabla_Y X \in T_1$. □

Also, as in [59], we have the following result.

Lemma A.3. *If $Y(a) = 0$ for all $Y \in T_0$, then $\nabla_X T_0 \subset T_0$ and $\nabla_X T_1 \subset T_1$ for all vector $X \in TM$.*

Proof. Under the assumption, *(i)* of Lemma A.2 implies that $A(\nabla_X Y) = 0$, that is $\nabla_X Y \in T_0$ for $X \in T_1$ and $Y \in T_0$. Thus, $\nabla_X(T_0) \subset T_0$ for $X \in T_1$. Since $(T_0)^\perp = T_1$, analogously to *(iii)* of Lemma A.2, we also have that $\nabla_X(T_1) \subset T_1$. □

The next lemma is essential and its proof use strongly that the ambient space has zero curvature.

Lemma A.4. *Let Y, Z vector fields in T_0, such that $\nabla_Y Z = \nabla_Z Y = 0$. If there is a non-vanishing vector field X belonging to T_1, such that $[X,Y] = [X,Z] = 0$, then $(YZ)\left(\frac{1}{a}\right) = 0$.*

Proof. Since $AY = 0$, we have that $R(X,Y) = AX \wedge AY = 0$. On the other hand, using that $\nabla_Y Z = 0$ and $[X,Y] = 0$, we have that $0 = R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z = -\nabla_Y(\nabla_X Z)$. From *(i)* of Lemma A.2, we have that $-Z(a)X = A(\nabla_X Z)$. From *(iv)* of Lemma A.2, we have $A(\nabla_X Z) = a(\nabla_X Z)$. Thus, we obtain that $\nabla_X Z = -\frac{Z(a)}{a}X$.
Hence, we have $\nabla_Y\left(\frac{Z(a)}{a}X\right) = 0$, which implies

$$\frac{aYZ(a) - Y(a)Z(a)}{a^2}X + \frac{Z(a)}{a}\nabla_Y X = 0.$$

Since $[X,Y] = 0$, we have that $\nabla_Y X = \nabla_X Y = -\frac{Y(a)}{a}X$ (in the same way as for $\nabla_X Z = -\frac{Z(a)}{a}X$.). Then, the equation above reduces to

$$(aYZ(a) - 2Y(a)Z(a))X = 0.$$

Since X is non-vanishing, we obtain that

$$aYZ(a) - 2Y(a)Z(a) = 0.$$

Therefore,

$$YZ\left(\frac{1}{a}\right) = -\frac{aYZ(a) - 2Y(a)Z(a)}{a^3} = 0. \tag{A.1}$$

□

We will denote by $M_0(x)$ and $M_1(x)$ the maximal integral submanifolds of M_1^n corresponding respectively to T_0 and T_1, passing through the point x. Then, as in [75, 59], we have the following result.

Proposition A.5. *(i) $M_0(x)$ is complete and totally geodesic in M_1^n.*

(ii) The restriction to $M_0(x)$ of the isometrical immersion f of M_1^n in $\mathbb{E}_1^{n+1}$, is an isometry of $M_0(x)$ to $\mathbb{E}^{n-k}(x)$ or to $\mathbb{E}_1^{n-k}(x)$.

Proof. (i) From *(iii)* of Lemma A.2, we have that $\nabla_Y(T_0) \subset T_0$ for all $Y \in T_0$. This means that $M_0(x)$ is totally geodesic (that is, $\alpha_{M_0(x)}(Y,Z) = (\nabla_Y Z)^\perp = 0$ for $Z \in T_0$, since $TM_0(x) = T_0$). $M_0(x)$ is complete as a maximal integral submanifold which is totally geodesic. Indeed, let $y(s)$ be a geodesic in $M_0(x)$. As a geodesic in M_1^n, $y(s)$ it is infinitely extendible. Denote $s_0 = \sup\{s_1 : y(t) \in M_0(x) \text{ for } s < s_1\}$. Choosing local coordinates $\{x^1,\ldots,x^k,x^{k+1},\ldots,x^n\}$ with origin $y(s_0)$, such that $\{\frac{\partial}{\partial x^1}\ldots,\frac{\partial}{\partial x^k}\}$ e $\{\frac{\partial}{\partial x^{k+1}}\ldots,\frac{\partial}{\partial x^n}\}$ are local frames for T_1 and T_0. Since $y(s)$, $s < s_0$ is a geodesic lying in the T_0-direction, we have $y^i(s) = c^i$ constant, $1 \leq i \leq k$, for $s_0 - \delta < s < s_0$, with $\delta > 0$. Since in $y(s_0)$ all coordinates are 0, it follows that $y^i(s) \to 0$ as $s \to s_0$, and since all c^i's are constants in this interval, we obtain that $c^1 = \cdots = c^k = 0$, that is, $y(s) = (0,\ldots,0,y^{k+1}(s),\ldots,y^n(s))$, for $s_0 - \delta < s \leq s_0$, and the tangent vector to $y(s)$ in s_0 is still in the T_0-direction. Therefore, the geodesic continues to lie in $M_0(x)$.

(ii) Note that $f(M_0(x))$, which is $(n-k)$-dimensional, is also a totally geodesic submanifold of the Minkowski space $\mathbb{E}_1^{n+1}$, since $\alpha(X,Y) = \langle AX,Y\rangle\eta = 0$, for all $X,Y \in T_0(X) = TM_0(x)$. Consequently, every geodesic of $M_0(x)$ is mapped under the immersion f to a straight line in $\mathbb{E}_1^{n+1}$. The restriction of the metric on $M_0(x)$ is Euclidean or Lorentzian. Accordingly, by the completeness of $M_0(x)$, we have that $f(M_0(x)) = \mathbb{E}^{n-k}(x)$ or $f(M_0(x)) = \mathbb{E}_1^{n-k}(x)$. It follows that f is a covering map (see Corollary 29 in p. 202 of [62]) and so it is an isometry of $M_0(x)$ to $\mathbb{E}^{n-k}(x)$ or $\mathbb{E}_1^{n-k}(x)$.

□

We now come to the crucial step of the proof.

Proposition A.6. *For any $Y \in T_0$, we have $Y(a) = 0$.*

Proof. For a point $x \in M_1^n$, Let $\{y^1,\ldots,y^k,y^{k+1},\ldots,y^n\}$ be a coordinate system with origin x in a neighborhood U of x, such that $\{\frac{\partial}{\partial y^1}\ldots,\frac{\partial}{\partial y^k}\}$ and $\{\frac{\partial}{\partial y^{k+1}}\ldots,\frac{\partial}{\partial y^n}\}$ are local bases for T_1 and T_0. Following Proposition A.5, we have to consider two cases:

A) If $M_0(x)$ is isometric to $\mathbb{E}^{n-k}(x)$, we may assume that the restriction of $\{y^k,\ldots,y^{k+1}\}$ to $M_0(x) \cap U$ is rectangular, that is

$$\left\langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \right\rangle = \delta_{\alpha\beta}, \text{ for } k+1 \leq \alpha,\beta \leq n.$$

We will show that the restriction of $\{y^k,\ldots,y^{k+1}\}$ a $M_0(y) \cap U$, for any $y \in M_1(x) \cap U$, is also rectangular. Denote functions $g_{\alpha\beta} = \left\langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \right\rangle$, $k+1 \leq \alpha,\beta \leq n$, we have

$$\frac{\partial g_{\alpha\beta}}{\partial y_i} = \left\langle \nabla_{\frac{\partial}{\partial y^i}} \left(\frac{\partial}{\partial y^\alpha}\right), \frac{\partial}{\partial y^\beta} \right\rangle + \left\langle \frac{\partial}{\partial y^\alpha}, \nabla_{\frac{\partial}{\partial y^i}} \left(\frac{\partial}{\partial y^\beta}\right) \right\rangle.$$

Now, Lemma A.2, *(iv)*, implies that $\nabla_{\frac{\partial}{\partial y^i}}\left(\frac{\partial}{\partial y^\alpha}\right) \in T_1$, para $1 \leq i \leq k$. Thus,

$$\left\langle \nabla_{\frac{\partial}{\partial y^i}}\left(\frac{\partial}{\partial y^\alpha}\right), \frac{\partial}{\partial y^\beta}\right\rangle = 0$$

and, similarly, $\left\langle \frac{\partial}{\partial y^\alpha}, \nabla_{\frac{\partial}{\partial y^i}}\left(\frac{\partial}{\partial y^\beta}\right)\right\rangle = 0$. Thus, we have that $\dfrac{\partial g_{\alpha\beta}}{\partial y_i} = 0$, $k+1 \leq \alpha, \beta \leq n$, that is, functions $g_{\alpha\beta}$ are constant respect to variables $y^1, \ldots, y^k$, thus

$$g_{\alpha\beta}(y^1, \ldots, y^k, y^{k+1}, \ldots, y^n) = g_{\alpha\beta}(0, \ldots, 0, y^{k+1}, \ldots, y^n) = \delta_{\alpha\beta}.$$

Now, let $Y = \frac{\partial}{\partial y^\alpha}$, where $k+1 \leq \alpha \leq n$, and $X = \frac{\partial}{\partial y^i}$, where $1 \leq i \leq k$. Since $\{y^{k+1}, \ldots, y^n\}$ is rectangular in each $M_0(y) \cap U$ (thus, for each y, $\hat{\nabla}_Y Y = 0$, where $\hat{\nabla}$ is the connection on $M_0(y) \cap U$), which is totally geodesic in M_1^n, we have that $\nabla_Y Y = 0$ (these vector fields are parallel in $M_0(y)$, since $M_0(y)$ is Euclidean, so they are also parallel in M_1^n). Applying Lemma A.4 to X, Y and $Z = Y$, ($[X,Y] = 0$ for the rectangular coordinates), we obtain that $Y^2\left(\frac{1}{a}\right) = 0$.

If L is a straight line in $M_0(x)$ (i.e., a geodesic), let Y be the parallel vector field in the direction of L on the Euclidean space $M_0(x)$. For any point of L, we may choose suitable local coordinates $\{y^1, \ldots, y^n\}$ and show by the argument above that $Y^2\left(\frac{1}{a}\right) = 0$. This means that if s is the length parameter of L, which can take any real value by the completeness of $M_0(x)$, then $\frac{d^2}{ds^2}\left(\frac{1}{a}\right) = 0$. Thus, $\frac{1}{a} = us + v$, for all $s \in \mathbb{R}$, where u, v are constant. If $u \neq 0$, then $\frac{1}{a}$ will be 0 for $s = -\frac{v}{u}$, which is a contradiction. We have thus shown that a is constant in L. Since L is an arbitrary straight line in $M_0(x)$ starting from x, we can conclude that a is constant in $M_0(x)$. Therefore, $Y(a) = 0$, for any $Y \in T_0$.

B) If $M_0(x)$ is isometric to the Minkowski space $\mathbb{E}_1^{n-k}(x)$, we may assume that the restriction of $\{y^k, \ldots, y^{k+1}\}$ to $M_0(x) \cap U$ is rectangular in the Lorentzian sense, i.e.,

$$\left\langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right\rangle = \varepsilon_\alpha \delta_{\alpha\beta}, \text{ for } k+1 \leq \alpha, \beta \leq n,$$

where $\varepsilon_\gamma = 1$, for $\gamma \in \{k+2, \ldots, n\}$ and $\varepsilon_1 = -1$. As in case A), we can obtain that $Y^2\left(\frac{1}{a}\right) = 0$, for any $Y = \frac{\partial}{\partial y^\alpha}$.

Consequently, a is constant on all straight lines in $M_0(x)$, which pass through x but do not lye on the null cone through x. Considering this and the fact that a is continuous on M_1^n (even differenciable), it follows that a is a constant function on the whole of $M_0(x)$.

□

Observation A.7. Since $X(a) = 0$ for all $X \in T_1$, it follows that $Z(a) = 0$ for any tangent vector Z of M_1^n. Thus, a is constant in M_1^n, which means that M_1^n is isoparametric with diagonalizable Weingarten operator.

Proposition A.8. *Let $M_0(x)$ and $M_1(x)$ be the maximal integral submanifolds of T_0 and T_1, respectively, through $x \in M_1^n$.*

(i) $M_1(x)$ is complete and totally geodesic in M_1^n for any $x \in M_1^n$.

(ii) For any point $w \in M_1^n$, let $M_0 = M_0(w)$ and $M_1 = M_1(w)$. Then M_1^n is isometric to $M_0 \times M_1$.

(iii) For all $x \in M_1$, the spaces $f(M_0(x)) = \mathbb{E}^{n-k}(x)$, respectively, $f(M_0(x)) = \mathbb{E}_1^{n-k}(x)$, given in Proposition A.5, are all parallel.

(iv) In case M_0 is isometric to $\mathbb{E}^{n-k}$, the restriction f_1 of f to M_1, is a covering map of M_1 onto $\mathbb{S}_1^k \subset \mathbb{E}_1^{k+1}$, which is an isometry if $k \geq 3$, where $\mathbb{E}_1^{k+1}$ is orthogonal to $\mathbb{E}^{n-k}$ in $\mathbb{E}_1^{n+1}$.

(v) In case M_0 is isometric to $\mathbb{E}_1^{n-k}$, the restriction f_1 of f to M_1, is a covering map of M_1 onto $\mathbb{S}^k \subset \mathbb{E}^{k+1}$, which is an isometry, where $\mathbb{E}^{k+1}$ is orthogonal to $\mathbb{E}_1^{n-k}$ in $\mathbb{E}_1^{n+1}$.

(vi) If f_0 is the restriction of f to M_0, then $f = f_0 \times f_1$, that is, $f(y,x) = (f_0(y), f_1(x))$, for every point $(y,x) \in M_0 \times M_1 = M_1^n$.

Proof. (i) By Proposition A.6 and Lemma A.3, we know that $\nabla_X(T_1) \subset T_1$, for any vector field X belonging to T_1. This means that $M_1(x)$ is totally geodesic. The completeness can be proved in the same way as for $M_0(x)$.

(ii) Lemma A.2 and Lemma A.3 together imply that T_0 and T_1 are parallel. Since M_1^n is simply connected and complete and the restrictions of the metric on T_wM to $T_0(w)$ and $T_1(w)$ are non-degenerate, by Wu's extension of the de Rham decomposition theorem to pseudo-Riemannian manifolds in [78], we can conclude that M_1^n is isometric to $M_0 \times M_1$.

(iii) Let $Y \in T_0(w)$ and let Y_t be the family of tangent vectors parallel to Y along a curve $x(t)$ in M_1. By (ii), we have that $Y_t \in T_0(x(t))$. Let $\hat{\nabla}$ the ordinary derivation in $\mathbb{E}_1^{n+1}$ and considering f locally, we get (denoting by $\overrightarrow{x_t}$ the tangent vector of the curve $x(t)$)

$$\hat{\nabla}_{f_*(\overrightarrow{x_t})} f_*(Y_t) = f_*(\nabla_{\overrightarrow{x_t}} Y_t) + \langle AY_t, \overrightarrow{x_t} \rangle \eta = 0,$$

since $\nabla_{\overrightarrow{x_t}} Y_t = 0$ and $\langle AY_t, \overrightarrow{x_t} \rangle = 0$ (in fact $AY_t = 0$). Thus, $f_*(Y_t)$ is parallel in $\mathbb{E}_1^{n+1}$. Since the flat subspaces $\mathbb{E}_\sigma^{n-k}(x) = f(M_0(x))$, $\sigma \in \{0,1\}$, have $f(T_0(x))$ as the tangent space at $f(x)$, we conclude that all $\mathbb{E}_\sigma^{n+1}(x)$, $x \in M_1$, are parallel.

(iv) Consider the $\mathbb{E}_1^{n+1}$-valued vector function $x \mapsto \eta_x + af(x)$ on M_1. For any tangent vector X to M_1, we have

$$\hat{\nabla}_{f_*(X)}(\eta + af) = f_*(-AX + aX) = 0,$$

which shows that $\eta + af$ is a constant vector v in $\mathbb{E}_1^{n+1}$ and $f(M_1)$ lies in the hypersphere $\mathbb{S}_1^n(a^2)$ with center $\frac{1}{a}v$ and radius $|\frac{1}{a}|$. Since $f(M_1)$ is orthogonal to $f(M_0(x)) = \mathbb{E}^{n-k}(x)$, $x \in M_1$, at each point of $f(M_1)$, and $\mathbb{E}^{n-k}(x)$ are all parallel to $\mathbb{E}^{n-k}$. It follows that $f(M_1)$ is also contained in the linear subspace $\mathbb{E}_1^{k+1}$ of $\mathbb{E}_1^{n+1}$ which passes through $f(w)$ and is orthogonal to $\mathbb{E}^{n-k}$. Hence, $f(M_1)$ lies in the sphere $\mathbb{S}_1^k(a^2) = \mathbb{S}_1^n(a^2) \cap \mathbb{E}_1^{k+1}$. Since M_1 is complete, it follows that $f_1 : M_1 \to \mathbb{S}_1^k(a^2)$ is a covering map of M_1 onto $\mathbb{S}_1^k(a^2)$ (see p. 202 of [62]). Particularly, if $k \geq 3$, then $\mathbb{S}_1^k(a^2)$ is simply connected and f_1 is one-to-one, thus, f_1 is an isometry in this case.

(v) This proof is similar to the one in *(iv)*.

(vi) Let $(y,x) \in M_0 \times M_1$. Let $y = \exp_w sY_0$, where Y_0 is a unit vector in $T_0(w)$. Then, the point (y,x) is equal to $\exp_x sY$, where Y is the unit vector in $T_0(x)$, which is parallel to Y_0. By *(iii)*, we know that f_*Y_0 and f_*Y are parallel in $\mathbb{E}_1^{n+1}$. Since f maps every geodesic in $M_0(x)$ onto a straight line in $\mathbb{E}_\sigma^{n-k}(x)$, we see that $f(y,x) = \exp_{f_1(x)} sf_*Y$ and this is equal to $(f_0(y), f_1(x))$, since $f_0(y) = \exp_{f(w)} sf_*Y_0$. We have thus shown that $f(y,x) = (f_0(y), f_1(x))$. □

So far, we proved Proposition A.1 under the assumption that the type number $k(x)$ of the Weingarten operator is greater than 1 at every point of the Lorentzian hypersurface M_1^n.

The proof under the weaker assumption that there exists a point $x \in M_1^n$ where $k(x) \geq 2$, can be adapted from [59] using arguments similar to those given above, as follows. Let $W = \{x : k(x) \geq 2\}$, which is an open set (multiplicities are locally constant for $k(x) \geq 1$). Let x_0 be a point with $k(x_0) \geq 2$ and let W_0 be the connected component of x_0 in W. We have that $k(x)$ is constant in W_0, $a(x)$ is a differentiable function, and the distributions T_0 and T_1 defined in W_0 are differentiable and involutive. Lemmas above are valid. Then, we can prove the following result:

Proposition A.9. *Let M_0 and M_1 be the maximal integral submanifolds of T_0 and T_1, respectively, through x_0.*

(i) *M_0 is totally geodesic in M_1^n and locally flat.*

(ii) *On a geodesic $L(s)$ in M_0 with arc length parameter s, we have $a(s) = \frac{1}{us+v}$.*

(iii) *M_0 is complete and a is constant in M_0.*

(iv) *$k(x) \geq 2$ for all $x \in M_1^n$.*

Proof. *(i)* M_0 is totally geodesic by *(iii)* in Lemma A.2. Hence the curvature tensor of M_0 is the restriction of the curvature tensor R of M_1^n to M_0. Since we have $R(X,Y) = AX \wedge AY = 0$ for any X

and Y tangents to M_0, it follows that M_0 is locally flat.

(ii) For any geodesic $L(s)$ in M_0 with arc length parameter s, we may show that $\frac{d}{ds}\left(\frac{1}{a}\right) = 0$ by using the essentially same argument as for Proposition A.6.

(iii) Let $L(s)$ be a geodesic in M_0 starting from x_0. As a geodesic in M_1^n, it is infinitely extendible. If this entire geodesic does not lie in W_0, let s_0 be such that $L(s) \in W_0$ (and from the argument in the proof of *(i)* of Proposition A.5, $L(s) \in M_0$) for $s < s_0$, but $L(s_0) \notin W_0$. The characteristic polynomial of A in $L(s)$, $s < s_0$, is $(t - a(s))^k t^{n-k}$. Setting $s \to s_0$, we obtain that the characteristic polynomial of A in s_0 is $(t - a(s_0))^k t^{n-k}$. On the other hand, $a(s_0) = \lim_{s \to s_0} a(s) = \lim_{s \to s_0} \dfrac{1}{us + v}$ can not be 0. This shows that $k(L(s_0)) \geq 2$. It follows that $L(s_0) \in W_0$ and, again as in the proof that $M_0(x)$ is complete when $k \geq 2$ everywhere in *(i)* of Proposition A.5, we have that $L(s) \in M_0$. Thus, M_0 is complete. Also, as in the proof of Proposition A.6, we can prove that constant u has to be 0, that is, a is constant in M_0.

(iv) Since a is constant on any maximal integral manifold of T_0 (defined on W_0), we have that $Y(a) = 0$ for all $Y \in T_0$. Since we also have that $X(a) = 0$ for all $X \in T_1$, hence a is a constant function on W_0. We now show that W_0 is actually equal to M_1^n. Suppose $W_0 \neq M_1^n$ and let $\{y_i\}$ be a sequence of points in W_0 converging to some point $y \in M_1^n$. By the continuity argument for the characteristic polynomial of A and using that $a(y_i) = a \neq 0$ constant, we can show that $k(y) \geq 2$. Thus, W_0 is open and closed so that $W_0 = M_1^n$, completing the proof of the proposition. $\square$

Proposition A.9 shows that the assumption that $k(x) \geq 2$ for some point x in M_1^n actually implies that $k(x) \geq 2$ everywhere in M_1^n, with $k(x) = k$ and $a \neq 0$ constants in M_1^n. Thus, Proposition A.1 has been proved.

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