UNIVERSIDADE DE SÃO PAULO
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Bounds on the critical parameter for the frog model on trees

Caio Augusto de Carvalho Pena

Tese de Doutorado do Programa Interinstitucional de Pós-Graduação em Estatística (PIPGEs)

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Bounds on the critical parameter for the frog model on trees

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Limitantes para parâmetro crítico para o modelo do sapo em árvores

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This work is dedicated to
my three little ones and my wife.

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“Start where you are.
Use what you have.
Do what you can.”
(Arthur Ashe)

RESUMO

PENA, C. A. **Limitantes para parâmetro crítico para o modelo do sapo em árvores**. 2024. 62 p. Tese (Doutorado em Estatística – Programa Interinstitucional de Pós-Graduação em Estatística) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

Consideramos o modelo do sapo com tempo de vida, em árvores infinitas. Neste modelo, sapos (partículas), quando acordados, realizam um passeio aleatório simétrico na árvore, acordando todos os sapos adormecidos dos sítios visitados, até morrerem. Consideramos variações do modelo, mudando a estrutura da árvore (orientada ou não, aleatória ou não) e a distribuição de sobrevida. Nestes modelos, a probabilidade de sobrevida de uma sapo é controlada por um parâmetro $p \in [0,1]$, e existe um valor crítico p_c tal que, se $p < p_c$, então apenas finitos sapos são acordados com probabilidade 1, enquanto se $p > p_c$, infinitos sapos são acordados com probabilidade positiva. A tese está dedicada à obtenção de limitantes inferiores e/ou superiores para este parâmetro crítico em função de constantes estruturantes dos modelos considerados.

Palavras-chave: Modelos do sapo, Teoria da renovação, Processo de ramificação multi-tipos, Parâmetro crítico.

ABSTRACT

PENA, C. A. **Bounds on the critical parameter for the frog model on trees**. 2024. 62 p. Tese (Doutorado em Estatística – Programa Interinstitucional de Pós-Graduação em Estatística) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

We consider the frog model with a lifetime on infinite trees. In this model, frogs (particles), when awake, perform a symmetric random walk on the tree, waking up all dormant frogs at the sites visited until they die. We consider variations of the model by changing the structure of the tree (oriented or not, random or not) and the survival distribution. In these models, the survival probability of a frog is controlled by a parameter $p \in [0,1]$, and there is a critical value p_c such that if $p < p_c$, then only finitely many frogs are awakened with probability 1, while if $p > p_c$, infinitely many frogs are awakened with positive probability. The thesis is dedicated to obtaining lower and/or upper bounds for this critical parameter as a function of structural constants of the considered models.

Keywords: Frog models, Renewal theory, Multi-types branching processes, Critical parameter.

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CHAPTER 1

INTRODUCTION

Frog models are simple models for the propagation of information (or rumor, or disease) through a graph: active (informed/ill) particles perform independent random walks on the graph, activating (informing/infecting) frogs of the visited vertices. The idea is that an active particle has an information and, when it comes into contact with an inactive/sleepy particle, it shares that information. At some point, the informed particles die, thus leaving the system. This class of models seems to have been first introduced by Telcs and Wormald (1999) under the name of *egg model*: particles making independent simple symmetric random walks on $\mathbb{Z}^d$ transform eggs of the visited vertices into active particles. In our terminology, particles are frogs and eggs (inactive particles) are inactive frogs.

The literature on frog models has grown very on over the last two decades, with variations depending on the lifetime of the frogs, the underlying graph structure on which the frogs wander around, the type of random walk the frogs perform, the initial number of frogs per vertex, *etc*...

The questions that are asked depend on the variation of the model under consideration. For instance, when considering the frog with an infinite lifetime, the main interest lies in recurrence (with probability 1, infinitely many particles visit the root) or transience (with probability 1, only finitely many particles visit the root). See for instance, Hoffman, Johnson and Junge (2017) who show that in homogeneous $d+1$-regular trees, the model is recurrent when $d=2$ and transient when $d>4$. This issue of transience and recurrence is also studied in $\mathbb{Z}^d$ (see Döbler and Pfeifroth (2014)). When the frog's lifetime is almost surely finite, the question arises of its survival. By survival, we mean that at any given time, there will always be an active frog. In their paper, Alves, Machado and Popov (2002a) demonstrated that on certain graphs, if the frog's lifetime follows a geometric distribution with parameter $1-p$, then a phase transition occurs. For values of p below a critical value $p_c \in (0,1)$, the model dies with probability 1, while for values of p above p_c, the model has a positive probability of survival. For finite graphs, see Hermon (2018), Benjamini *et al.* (2020), Carvalho and Machado (2023).

In this thesis we will consider the frog model on infinite trees. For each model we give bounds on the critical parameter of the lifetime as functions of structural constants of the model.

- For the case of homogeneous trees $\mathbb{T}_d$ Alves, Machado and Popov (2002a) proved that

  $$\frac{d+1}{2d+1} \leq p_c \leq \frac{d+1}{2d-2}. \tag{1.1}$$

  This paper was followed by a sequence of works improving the upper bound (Lebensztayn, Machado and Popov (2005), Gallo and Rodriguez (2018), Lebensztayn and Utria (2019)). Our main result concerning this model (see Theorem 2) gives new lower and upper bounds which improves on those of the literature (Gallo and Pena (2023)).

- We also consider a modification of the lifetime distribution. At birth, the frog has a probability p of making k steps ($k \in \mathbb{N}$, fixed), and a probability $1-p$ of dying instantaneously. For this model, in directed trees, we derived a polynomial equation that determines the exact critical parameter (see Theorem 5), from which we obtained lower and upper bounds (see Corollary 6). In the case of undirected trees, we obtained a lower bound for the critical parameter (see Theorem 7). As far as we know, nobody has considered this case before.

- For the frog model on directed and regular trees, Gallo and Rodriguez (2018) present an equation that allows the exact determination of the critical parameter (see Theorem 10), which they use to obtain explicit lower and upper bounds. We generalized these results to the case where random sub-trees of $\vec{\mathscr{T}}_d \subset \vec{\mathbb{T}}_d$ are considered (see Theorem 8) and obtained, from this, lower and upper bounds for the critical parameter (see Corollary 9).

This thesis is organized as follows: Chapter 2 contains the basic definitions and notation, descriptions of the models, and statements of the results. Chapter 3 includes all the proofs, and finally, Chapter 4 provides a discussion concerning the results and the proofs.

CHAPTER 2

MODELS AND RESULTS

This chapter is organized as follows. In Section 2.1.1, we provide a concise overview of graph theory, and in Section 2.1.2, a general definition of the frog model with lifetime, establishing the notation that will be used throughout the text. Section 2.2 contains our main results and is split into four subsections according to four variants of the model, varying the lifetime distribution and the underlying structure (oriented/random trees).

2.1 Preliminaries

In this section we define the notation that is used throughout the text with regard to graphs, in particular trees. In addition, we define the frog model with lifetimes on trees.

2.1.1 Graphs

In this work, we will study frogs (particles) performing random walks on graphs. We therefore need to introduce the notation and basic definitions for these structures. The nomenclature and notation used are mainly based on Lyons and Peres (2021).

A **graph** is formally defined as a pair $G = (V, E)$, where V represents the set of *vertices* and E is a subset of $V \times V$, called the *edges* set. A graph is said to be **undirected** when $(x,y) \in E$ if and only if $(y,x) \in E$; in this case, we say that the edge (x,y) is **unoriented**. An edge is said to be **directed** and $(x,y) \in E$ then $(y,x) \notin E$. To differentiate an unoriented edge, we use the notation $[x,y]$, whereas for an oriented edge, the notation is $\langle x,y \rangle$. If $(x,y) \in E$ we say that x and y are **neighbors** or **adjacents**. The **degree** of a vertex is the number of neighbors of that vertex. A graph is said to be **locally finite** if all the vertices have finite degree. When the degree of all vertices is equal to d, we say that the graph is **homogeneous** or more specifically **d-regular**. A **subgraph** of a graph $G = (V, E)$ is a graph in which the set of vertices is a subset of V and the set of edges is a subset of E.

A **path** of size n is a sequence of vertices $v_0, v_1, \dots, v_n$ such that $(v_i, v_{i+1}) \in E$ for $0 \leq i \leq n-1$. When a path does not pass through any vertex (resp., edge) more than once, we will call this vertex **simple** (resp., edge simple). A **cycle** in a graph is a closed sequence of edges and vertices where the first and last vertices are the same, and all intermediate vertices and edges are distinct. More formally, a cycle is a sequence of vertices $v_1, v_2, \dots, v_n$ such that $v_1 = v_n$ and for each i from 1 to $n-1$, there is an edge connecting v_i to v_{i+1}. Additionally, no vertex (except the first and last) and no edge appears more than once in the sequence. A graph is said to be **connected** if there is a path between any pair of vertices. In other words, for any two distinct vertices x and y in the graph, there exists a sequence of edges that connects x to y. The **distance** between two vertices x and y is the number of edges of the shortest path connecting x to y, denoted by $d(x,y)$ or $dist(x,y)$. By "shortest," we mean the smallest number of edges. We are interested in a particular type of graph, namely trees. **Trees** are connected graphs that have no cycles. It is well known that, given two distinct vertices x and y in a tree, there is exactly one unique path between them, denoted by $x \rightsquigarrow y$.

The trees we use throughout the text are **rooted**, meaning that one of the vertices is designated as the root, denoted by $\boldsymbol{o}$. We say $v \leq v'$ if v is one of the vertices on the path between the root $\boldsymbol{o}$ and v'; naturally, we say $v < v'$ if $v \leq v'$ and $v \neq v'$. The **level** of a vertex v is the distance between that vertex and the origin.

To visualize a graph, we represent the vertices as points. Regarding the edges, we have two situations: the edge $[x,y]$ is represented by a line connecting x to y, and the edge $\langle x,y \rangle$ is represented by an arrow pointing from x to y. In Figure 1, we show a part of a tree rooted with infinite vertices that is {d+1}-regular and its edges are unoriented, denoted as $\mathbb{T}_d$. In Figure 2, we depict part of a $\{d+1\}$-regular rooted tree with infinitely many vertices, where the edges are directed away from the root, denoted as $\overrightarrow{\mathbb{T}}_d$.

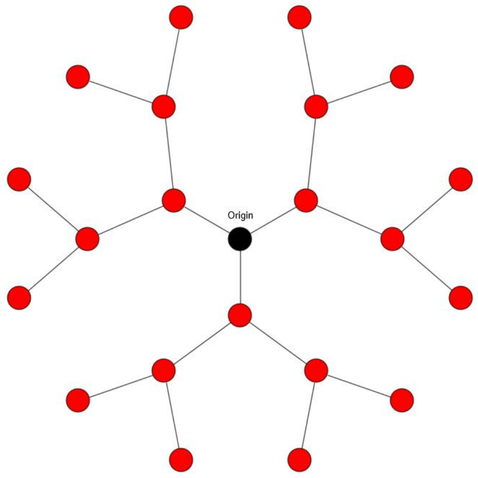


Figure 1 – Part of a 3-regular rooted tree, $\mathbb{T}_2$. The root of the tree is highlighted in black.

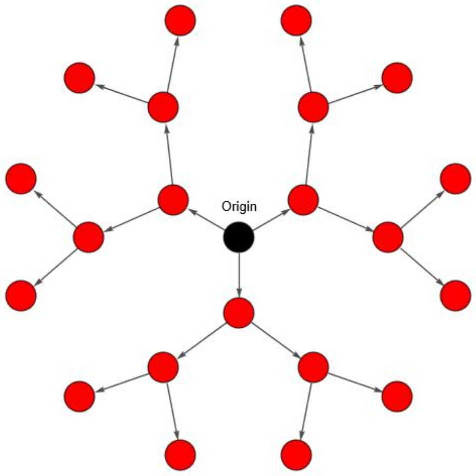


Figure 2 – Part of a 3-regular rooted tree, $\overrightarrow{\mathbb{T}}_2$. The root of the tree is highlighted in black.

2.1.2 Frog Model with Lifetime on Trees

In this section we define the system of interacting random walk called **frog model**. The version of this model we are interested in can be described as follows. We put one particle per vertex in a tree with infinitely many vertices, $\mathbb{T}$, and initially, they are all inactive (asleep), except for the one at the root, which is active (awake). While active, a particle x performs a symmetric random walk $\{S_n^x, n \geq 1\}$ on $\mathbb{T}$, activating all the inactive particles it encounters along its path, until it dies out and is removed from the system. We denote the lifetime of particle x by $\tau_x(p)$, where p is the parameter determining whether a particle survives and performs a jump, or dies with probability $1-p$. For each active particle x, the sets $\{S_n^x, n \geq 1\}$ and $\tau_x(p)$ are independent. We will denote it by FM$(\mathbb{T}, \tau(p))$ if the tree is not oriented, and by FM$(\overrightarrow{\mathbb{T}}, \tau(p))$ if the tree is oriented.

We say that the frog model **survives** if there are infinitely many frogs activated at the end of the process. Equivalently, this means that there is at least one active frog at each time t in the process. Note that the probability of survival is a non-decreasing function of p (see Lemma 1).

Lemma 1. The probability of survival, $\mathbb{P}_p$(FM survival), is non-decreasing in p.

Proof. To demonstrate this, consider a fixed tree $\mathbb{T}$. Let $(X_t^p)_{t\geq 0}$ be the random variable representing the number of active particles at time t in the frog model with survival parameter p on $\mathbb{T}$. Similarly, let $(X_t^q)_{t\geq 0}$ represent the number of active particles in the frog model with survival parameter q on the same tree, where $q < p$.

We aim to show that $X_t^p \geq X_t^q$ for all t using the coupling technique.

Consider a sequence $\mathbf{U} = (U_t)_{t\geq 1}$ of independent random variables uniformly distributed in $[0,1]$. Fix a realization of $\mathbf{U}$.

- At time $t = 0$, by definition, we have $X_0^p = X_0^q = 1$.

- For $t = 1$, we analyze the value of U_1 and the following cases:

1. If $0 < U_1 \leq q < p$, then particles in both models survive, so $X_1^p = X_1^q$.
2. If $0 < q < U_1 \leq p$, then only the particle in the model with parameter p survives. Thus, $X_1^p \geq X_1^q$.
3. If $0 < q < p < U_1$, then particles in both models die. Thus, $X_1^p \geq X_1^q$ in this case as well.

Therefore, $X_1^p \geq X_1^q$. Assuming the hypothesis holds for time t, we use induction to show it holds for time $t+1$. Each active particle at time t in the model with parameter q can be associated with at least one active particle at time t in the model with parameter p. By repeating the argument for time $t = 1$ for each particle, we conclude that $X_{t+1}^p \geq X_{t+1}^q$. By induction, $X_t^p \geq X_t^q$ for all t. Hence, the probability of survival is a non-decreasing function of p. □

Thus, we can define p_c as

$$p_c := \inf\{p > 0;\ \text{FM}(\mathbb{T}, p)\ \textit{survives}\}.$$

As we said before, our problem then is to obtain explicit bounds on p_c as a function of parameters of the model.

2.2 Main Results

In this section, we present five variations of the frog model. For each variation of the frog model, we provide bounds for the critical parameter, and, when possible, we discuss the history of the model and compare to results of the literature.

2.2.1 *Original Frog Model on Regular Trees*

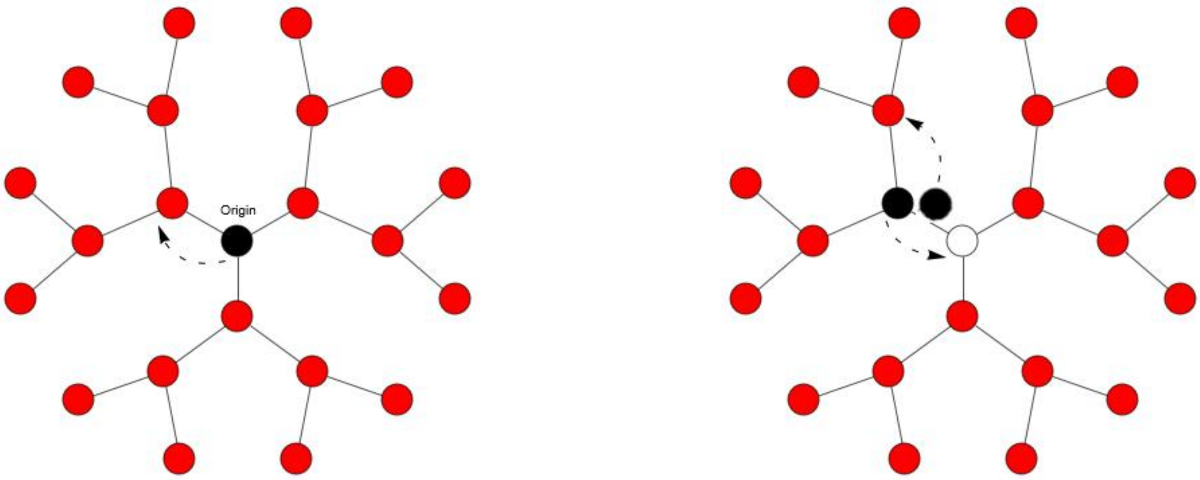
The frog model, denoted by $\text{FM}(\mathbb{T}_d, p)$, is defined on the tree $\mathbb{T}_d$, with the lifetime of the frog following a geometric distribution with parameter $1-p$. This particular instance is referred to as the **original frog model**. In Figure 3, we present the first three steps of a realization of the frog model on $\mathbb{T}_2$.

This model was studied by Alves, Machado and Popov (2002a), who provided the following bounds for the critical parameter on $\mathbb{T}_d$,

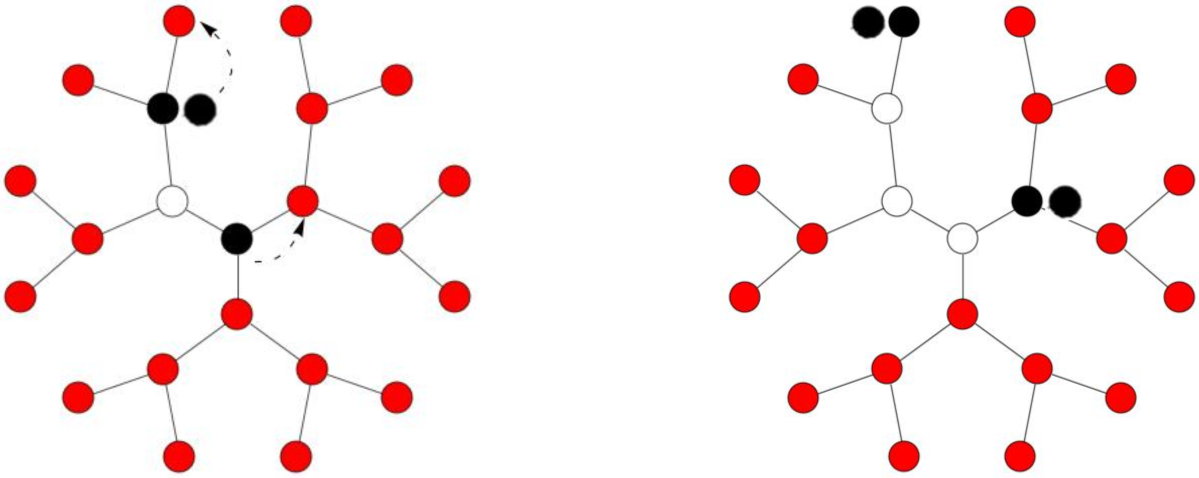
$$\frac{d+1}{2d+1} \leq p_c \leq \frac{d+1}{2d-2}. \tag{2.1}$$

Three years later, Lebensztayn, Machado and Popov (2005) obtained the following improvement of the upper bound

$$p_c \leq \frac{d+1}{2d}.$$



(a) Time $t = 0$ (b) Time $t = 1$



(c) Time $t = 2$ (d) Time $t = 3$

Figure 3 – The initial steps of a realization of the frog model on $\mathbb{T}_2$. The black ball represents an active particle, the red ball an inactive particle, and the white ball denotes a visited vertex without any particles.

Gallo and Rodriguez (2018), obtained as a consequence of the study of a percolation model on *oriented* trees:

$$p_c \leq \frac{(d+1)\left[(7d-1)-\sqrt{(7d-1)^2-14}\right]}{d(7d-1)^2-7d+2-d(7d-1)\sqrt{(7d-1)^2-14}}$$

and one year later, Lebensztayn and Utria (2019) devoted their work to a further improvement of the upper bound, but the expression they obtained is very complicated (see Definition 2.1 and Theorem 2.2 therein).

Notice that the above improvements all refer to the upper bound of p_c. The lower bound in display (2.1), obtained by Alves, Machado and Popov (2002a), has not been improved so far. In (GALLO; PENA, 2023), we improve the *both* bounds, as shown in Theorem 2.

Theorem 2. For any $d \geq 2$, consider the tree $\mathbb{T}_d$. Then,

$$\frac{2(d+1)}{\sqrt{4d^2+4d-3}+2d+1} \leq p_c(d) \leq \frac{(d+1)\left(4d+\frac{1}{14d^2}-2\right)(8d-5)}{d(8d-5)^2+\left(4d+\frac{1}{14d^2}-2\right)^2}. \tag{2.2}$$

Remark 1. From (2.1), it follows that $p_c(d)$ converges to $1/2$ when d diverges, and for d large enough it gives

$$\frac{1}{2}+\frac{1}{4d}-\frac{1}{8d^2}+O\left(\frac{1}{d^3}\right) \leq p_c(d) \leq \frac{1}{2}+\frac{1}{d}+\frac{1}{d^2}+O\left(\frac{1}{d^3}\right).$$

Theorem 2 gives, after some algebra,

$$\frac{1}{2}+\frac{1}{4d}+O\left(\frac{1}{d^4}\right) \leq p_c(d) \leq \frac{1}{2}+\frac{7}{16d}-\frac{5}{128d^2}+O\left(\frac{1}{d^3}\right), \tag{2.3}$$

we can conclude that, as d approaches infinity, both the lower bound and the upper bound converge to 1/2. Where $O(\cdot)$ is the Landau notation.

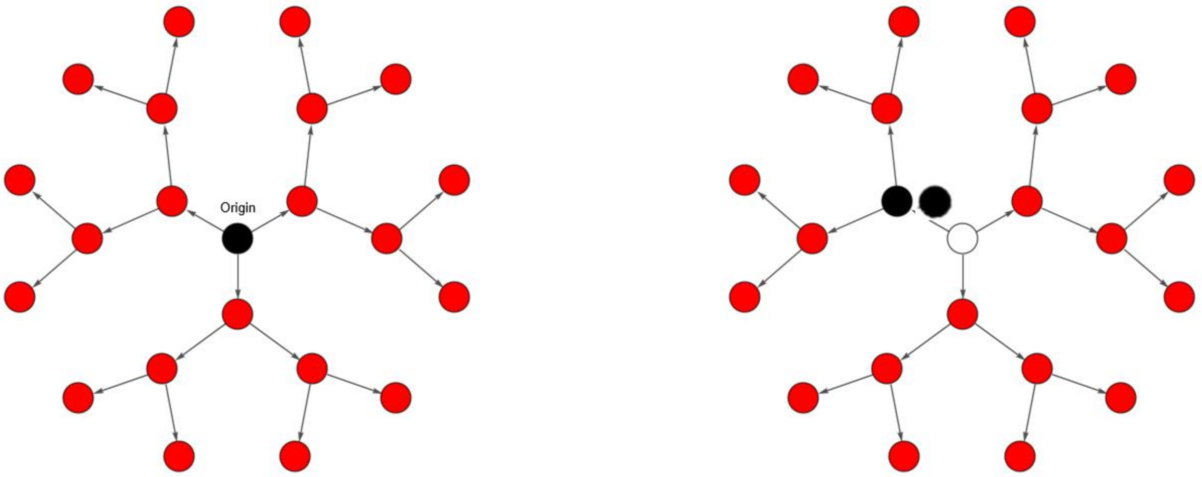
For illustrative purposes, in Table 7, we have calculated numerically the lower and upper bounds for the critical parameter of the original frog model for several values of d.

Table 1 – Lower bound (LB) and upper bound (UB) of p_c from Theorem 2 and lower bound on $\hat{p}_c$ from Lemma 12.

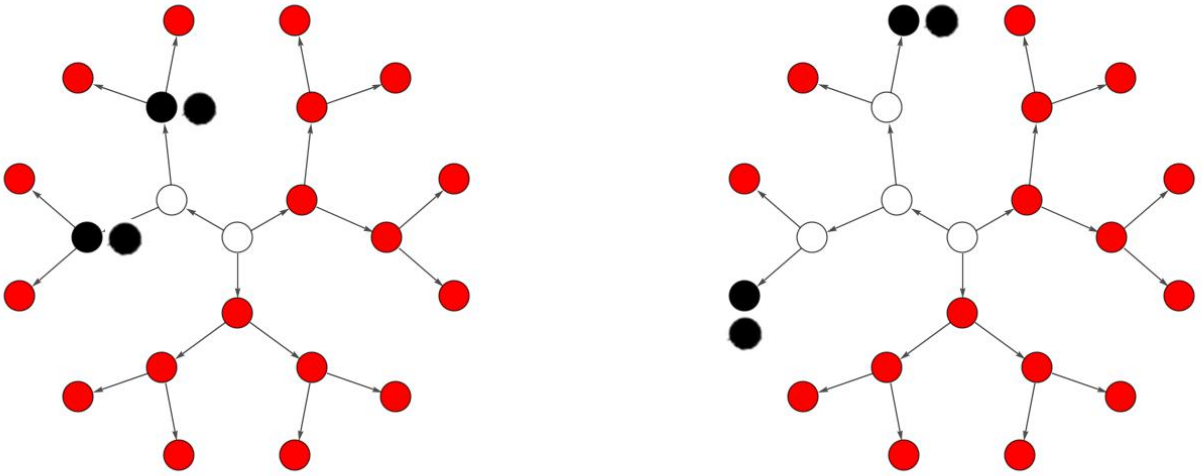
d	LB on p_c	LB on $\hat{p}_c$	UB on $\hat{p}_c$ and p_c
2	0.6261364	0.7103674	0.7137989
3	0.5835921	0.6419859	0.6428580
4	0.5625890	0.6071563	0.6074957
5	0.5500385	0.5860557	0.5862210
6	0.5416859	0.5719015	0.5719940
7	0.5357250	0.5617475	0.5618043
8	0.5312564	0.5541074	0.5541448
9	0.5277818	0.5481503	0.5481761
10	0.5250027	0.5433751	0.5433937
20	0.5125001	0.5217793	0.5217815
50	0.5050000	0.5087345	0.5087346
100	0.5025000	0.5043711	0.5043711

2.2.2 Frog Model on the Oriented Regular Trees

In the frog model on directed (oriented) homogeneous trees active particles can only move to adjacent vertices that are furthest from the root. We denote it by FM($\overrightarrow{\mathbb{T}}_d, p$). In Figure 4, we present the first three steps of a realization of the frog model on $\mathbb{T}_2$.



(a) Time $t = 0$ (b) Time $t = 1$



(c) Time $t = 2$ (d) Time $t = 3$

Figure 4 – The first steps of a realization of the oriented frog model on $\overrightarrow{\mathbb{T}}_2$. The black ball represents an active particle, the red ball represents an inactive particle, and the white ball represents a vertex that has already been visited and is without any particles.

In their paper, Gallo and Rodriguez (2018) present an elegant result that provides an exact method for determining the critical parameter for the frog model in directed trees.

Theorem 3 (Theorem 1 of (GALLO; RODRIGUEZ, 2018))**.** For all $d \geq 2$, the critical parameter for the oriented frog model $\mathrm{FM}(\overrightarrow{\mathbb{T}}_d, p)$ is given by

$$\sum_{k\geq 1} p_c^k \prod_{i=1}^{k-1} \left[1 - \left(\frac{p_c}{d}\right)^i\right] = 1. \tag{2.4}$$

We found no direct method to solve this equation as a function of p, so we manipulated the equation to derive lower and upper bounds for p_c. The result we obtained is presented in Corollary 4.

Corollary 4. For $d \geq 2$

$$\frac{5-8d-16d^2+64d^3}{12-20d-48d^2+128d^3} \leq p_c \leq \frac{56d^3-28d^2+1}{14d^2(8d-5)}.$$

Table 2 presents values of the lower and upper bounds for the critical parameter in the deterministic case, varying the value of d. These values are derived from Corollary 4. In the column labeled Theorem 3 (with 30 terms), we used software MATHEMATICA to expand the sum in Equation 2.4 up to its first 30 terms. Subsequently, we equated the expression to 1 and recorded the numerical values obtained.

Table 2 – Numerical values for the lower and upper bounds of the critical parameter for the directed frog model, along with the numerical solution of the equation considering only its first 30 terms.

d	Lower bound	Theorem 3 (with 30 parcels)	Upper bound
2	0.543532	0.545747	0.547078
3	0.525874	0.526334	0.526734
4	0.518352	0.518518	0.518684
5	0.514206	0.514283	0.514367
6	0.511584	0.511626	0.511674
7	0.509777	0.509802	0.509833
8	0.508457	0.508473	0.508493
9	0.50745	0.507462	0.507476
10	0.506658	0.506666	0.506676
20	0.503225	0.503226	0.503227
30	0.502127	0.502128	0.502128
40	0.501587	0.501587	0.501587
50	0.501266	0.501266	0.501266
100	0.500629	0.500629	0.500629

2.2.3 Frog Model with Lifetime 0 or k

In the previous sections, we explored variations in the frog model concerning whether a particle could move towards the root of the tree. In this section, we propose a modification to the lifetime of the particles. In this modification of the lifetime, an active particle jumps exactly k steps and dies with probability p, or dies (without jumping) with probability $1-p$.

We investigate this variation of the frog model in directed and undirected homogeneous trees. The notations used are $\text{FM}(\overrightarrow{\mathbb{T}}_d, p, k)$ and $\text{FM}(\mathbb{T}_d, p, k)$, respectively.

Note first that in case $k=1$, that is, if the particle survives, it only takes one step and dies; the process dies out with probability 1. In fact,

$$\begin{aligned}
\mathbb{P}(\text{FM}(\overrightarrow{\mathbb{T}}_d, p, k=1) \textit{ survives}) &= \lim_n \mathbb{P}_{p,k=1}(A_n) \\
&= \lim_n p^n \qquad (2.5) \\
&= 0,
\end{aligned}$$

where in the first equality A_n is the event that some particle at distance n from the origin is activated. This also can be seen observing that, in this case, the model can be paralleled with a Galton-Watson random tree with offspring in $\{0,1\}$, which therefore cannot survive.

Similar to the previous models, our objective is to determine the critical parameter p_c, which, in this case, depends on both d and k. We have derived a finite-degree equation that precisely provides this parameter.

Theorem 5. Consider FM$(\overrightarrow{\mathbb{T}}_d, p, k)$. Given $d, k \geq 2$ we have that

$$\sum_{i=1}^{k} p_c \prod_{j=1}^{i-1} \left(1 - \frac{p_c}{d^j}\right) = 1. \tag{2.6}$$

For $k = 2$ we have an exact equation for the critical parameter,

$$p_c(d,2) = d - \sqrt{d^2 - d}. \tag{2.7}$$

For $k \geq 3$, Equation 2.6 becomes somewhat intricate to solve. For instance, when $k = 3$, we obtain the polynomial $p\left(3d^3 - 2d^2 p - dp + p^2\right) = d^3$. The solution to this polynomial, within the interval $(0,1)$, is already too complicated to be written here.

For large k, we tried to obtain approximations. The next result provides explicit bounds for p_c as a function of d and k for the critical parameter.

Corollary 6. Consider FM$(\overrightarrow{\mathbb{T}}_d, p, k)$. For $k \geq 3$ and $d \geq 2$ we have

$$\frac{2\sqrt{d}}{\sqrt{d}k + \sqrt{k(dk-4)+4}} \leq p_c(d,k) \leq \frac{2}{\sqrt{\frac{-4d^{1-k} + (d-1)^2 k^2 - 4dk + 4d + 4k}{(d-1)^2}} + k} \tag{2.8}$$

When $k = 2$, the limits coincide, giving the formula presented in Theorem 5.

Remark 2. Let us analyze the asymptotic behavior of the bounds obtained in Equation 2.8. Note that the term d^{1-k} can be bounded by $\frac{1}{2}$. Substituting this value into the upper bound and expanding the bounds in a series for sufficiently large d, we obtain the following expressions:

$$\frac{1}{k} + \frac{(k-1)}{dk^3} + O\left(\frac{1}{d^{3/2}}\right) \leq p_c \leq \frac{1}{k} + \frac{(k-1)}{dk^3} + O\left(\frac{1}{d^2}\right). \tag{2.9}$$

We can conclude that, as d approaches infinity, both the lower bound and the upper bound converge to $\frac{1}{k}$.

When we keep d fixed and k large, we obtain the following expressions:

$$\frac{1}{k} + \frac{1}{dk^2} + O\left(\frac{1}{k^3}\right) \leq p_c \leq \frac{1}{k} + \frac{1}{(d-1)k^2} + O\left(\frac{1}{k^3}\right). \tag{2.10}$$

Thus, as k approaches infinity, the critical parameter converges to 0.

Table 3 – We computed numerical values, utilizing software MATHEMATICA, for the lower and upper limits of the critical parameter for the model with a fixed lifetime, considering different values of d while maintaining $k = 5$.

d	Lower Bound - Corollary 6	Corollary 5	Upper Bound - Corollary 6
2	0.219224	0.231255	0.233353
3	0.211983	0.216068	0.216422
4	0.208712	0.210753	0.210874
5	0.206846	0.208067	0.208122
6	0.205638	0.206451	0.206481
7	0.204793	0.205373	0.205390
8	0.204168	0.204602	0.204614
9	0.203688	0.204025	0.204033
10	0.203307	0.203576	0.203582
20	0.201626	0.201690	0.201690
30	0.201078	0.201106	0.201106
40	0.200806	0.200822	0.200822
50	0.200644	0.200654	0.200654
100	0.200321	0.200323	0.200323

In Table 3, we present values of the critical parameter when we vary the value of d while keeping $k = 5$.

Table 4 shows some numerical values for the critical parameter limits, keeping $d = 8$ fixed and varying the values of k. As expected, as the value of k increases, i.e. the lifetime of the particles increases, the critical parameter decreases.

Table 4 – Some numerical values for the critical parameter limits, keeping $d = 8$ fixed and varying the values of k.

k	Lower Bound - Corollary 6	Corollary 5	Upper Bound - Corollary 6
2	0.516685	0.516685	0.516685
3	0.343146	0.343771	0.343799
4	0.256151	0.256707	0.256726
5	0.204168	0.204602	0.204614
6	0.169665	0.170002	0.170009
7	0.145113	0.145379	0.145384
8	0.126757	0.126971	0.126974
9	0.112518	0.112692	0.112695
10	0.101151	0.101296	0.101298
20	0.0503005	0.0503411	0.0503413
30	0.0334687	0.0334874	0.0334874
40	0.0250766	0.0250873	0.0250873
50	0.0200492	0.0200561	0.0200562
100	0.0100124	0.0100142	0.0100142

For the frog model on undirected trees, $\mathrm{FM}(\mathbb{T}_d, p, k)$, we obtained a lower bound for the critical parameter, which is presented in Theorem 7.

Theorem 7. Consider $\mathrm{FM}(\mathbb{T}_d, p, k)$. Given $d \geq 2$ and $k \geq 2$ we have

$$\frac{d+1}{dk} \leq p_c(d,k). \tag{2.11}$$

We did not find any existing literature on the frog model with a fixed lifetime.

Remark 3. From Equation 2.11 we have that

$$\frac{1}{k} + \frac{1}{dk} \leq p_c(d,k). \tag{2.12}$$

We can conclude that, as d approaches infinity, the critical parameter is bounded below by $1/k$.

We believe it is possible to use other distributions for the lifetime in the frog model, and the techniques used to obtain the results presented in this thesis would still be applicable.

2.2.4 Frog Model on the Oriented Random Tree

In Sections 2.2.1 and 2.2.2, we examined the frog model in homogeneous trees with degree $d+1$, in undirected and directed trees, respectively. Now, we will explore the frog model on random directed subtrees of $\overrightarrow{\mathbb{T}}_d$.

The random subtree, denoted by $\overrightarrow{\mathscr{T}}_d$, is constructed as follows: we start with one vertex $\boldsymbol{o}$, which serves as the root of the tree. The distribution of the number of offspring (neighbors) of the root is represented by a discrete random variable $G^{\boldsymbol{o}}$, where $P(G^{\boldsymbol{o}} = k) = q_k$ with $k = 0, 1, 2, ..., d+1$, and $\sum_{k=0}^{d+1} q_k = 1$, thus, if $G^o = k$, we randomly associate, to each of these k children, a neighboring vertex to the origin in T_d. To each of these v_i offspring (neighbors) (if any), we associate a random variable G_{v_i}, where $G \sim G_{v_i}$, $i = 1, 2, ..., d$, follows the distribution of the number of offspring (neighbors) given by $P(G = i) = p_i$, for $i = 0, 1, 2, \ldots, d$, and $\sum_{i=0}^{d} p_i = 1$. If $G_{v_i} = n$, we randomly select n neighbors of v_i and connect them to v_i. We then repeat this process recursively. These trees are known as Galton-Watson trees, see Figure 5 for an example. As we are interested in studying the behavior of the frog on infinite trees, we establish the condition that $\mathbb{E}[G] > 1$. As it is well known, this implies that with positive probability, the Galton-Watson tree is infinite. The notation we use for this model is $\mathrm{FM}(\overrightarrow{\mathscr{T}}_d, p)$.

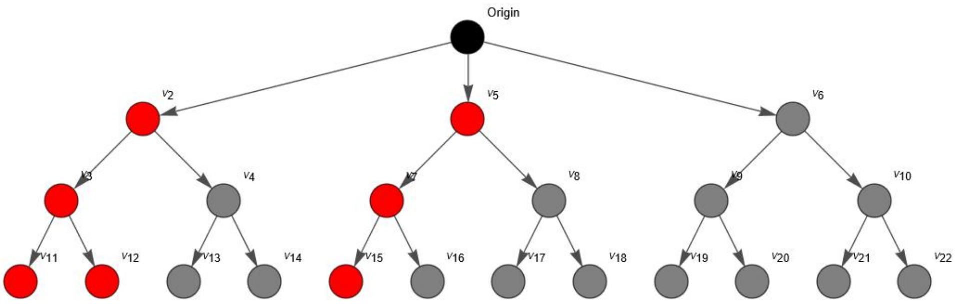


Figure 5 – The random subtree $\overrightarrow{\mathscr{T}}_3$. In this example $G^{\boldsymbol{o}} = 2, G_5 = 1, \ldots$.

Concerning the frog model with directed random tree lifetimes, as far as we know, no literature has been found regarding limits for the critical parameter. However, in this context, Proposition 1.2 of (ALVES; MACHADO; POPOV, 2002b) provides a lower bound for the (undirected) frog model on trees with maximum degree of k. They showed that $p_c \geq \frac{k}{1+2(k-1)}$.

We obtained an expression that precisely gives the critical parameter for the directed random tree frog model, as presented in Theorem 8.

Theorem 8. The critical parameter for frog model on random directed tree is given by

$$\sum_{k\geq 1} \psi^k \mathbb{E}\left[\frac{p_c^k}{\prod_{i=1}^{k} G_i^*} \prod_{j=1}^{k-1}\left(1-\frac{p_c^j}{\prod_{i=1}^{j} G_i^*}\right)\right] = 1. \tag{2.13}$$

where $\psi := d\,\mathbb{P}(\{v_1 \rightsquigarrow v_2 \in \overrightarrow{\mathscr{T}}_d\})$ and $\mathbb{P}(G_i^* = a) = \mathbb{P}(G_i = a | \boldsymbol{o} \rightsquigarrow v_{i+1} \in \overrightarrow{\mathscr{T}}_d)$.

Note that Theorem 8 generalizes Theorem 3 by setting $G \equiv d$. This implies $\overrightarrow{\mathscr{T}}_d = \overrightarrow{\mathbb{T}}_d$ and $\psi = d$.

We aim to derive bounds for p_c from this equation. These bounds should depend exclusively on measures related to G and the conditions for the existence of a path between the origin and a vertex v located at a distance n from the origin. We obtained Corollary 9.

Corollary 9. Given $d \geq 2$ we have that

$$\frac{2-\sqrt{2}\alpha\sqrt{\dfrac{2\psi-1}{\alpha^2\psi}}}{\alpha} < p_c(d), \tag{2.14}$$

and

$$p_c(d) < \; -\frac{\alpha}{4\beta} - \frac{1}{2}\sqrt{\frac{\alpha^2}{4\beta^2} + \frac{\Delta}{3\sqrt[3]{2}\beta^2\psi^2} + \frac{2\sqrt[3]{2}\left(\alpha^2\psi+2\beta\right)}{\beta\Delta}} - \frac{1}{2}\sqrt{\frac{\alpha^2}{2\beta^2} - \frac{\Delta}{3\sqrt[3]{2}\beta^2\psi^2} - \frac{2\sqrt[3]{2}\left(\alpha^2\psi+2\beta\right)}{\beta\Delta} - \frac{16\alpha\beta-\alpha^3\psi}{4\beta^3\psi\sqrt{\frac{\alpha^2}{4\beta^2}+\frac{\Delta}{3\sqrt[3]{2}\beta^2\psi^2}+\frac{2\sqrt[3]{2}(\alpha^2\psi+2\beta)}{\beta\Delta}}}}, \tag{2.15}$$

where

$$\begin{aligned}
\alpha &= \mathbb{E}\left(\frac{1}{G_i^*}\right),\\
\beta &= \mathbb{E}\left(\frac{1}{G_i^{*2}}\right),\\
\Delta &= \sqrt[3]{135\alpha^2\beta^2\psi^4 + \sqrt{18225\alpha^4\beta^4\psi^8 - 4\left(6\alpha^2\beta\psi^3 + 12\beta^2\psi^2\right)^3}},\\
\psi &= d\,\mathbb{P}(\{v_1 \rightsquigarrow v_2 \in \overrightarrow{\mathscr{T}}_d\}).
\end{aligned}$$

Example 1. In this example, our $\overrightarrow{\mathscr{T}}_d$ subtree will be constructed as follows. For each edge $e \in \overrightarrow{\mathbb{T}}_d$, we associate a random variable X_e with a Bernoulli distribution with parameter q. Thus,

$e \in \mathscr{T}_d$ if and only if $X_e = 1$. In this case, the random tree $\mathscr{T}_{(d,q)}$ is a Galton-Watson tree with a binomial offspring distribution.

In the tree $\overrightarrow{\mathscr{T}}_{(d,q)}$, we consider the frog model as described in Section 2.2.2, and denote this model by $\mathrm{FM}(\overrightarrow{\mathscr{T}}_{(d,q)}, p)$.

We must emphasize that our interest lies in determining the conditions for the survival of the frog model within the tree $\overrightarrow{\mathscr{T}}_{(d,q)}$. This necessitates the existence of at least one infinite connected component in $\overrightarrow{\mathscr{T}}_{(d,q)}$. To establish this, we use the well-known result of percolation, to state that, in a tree $\overrightarrow{\mathscr{T}}_{(d,q)}$, there exists an infinite connected component with positive probability if and only if $q > \frac{1}{d}$.

This follows directly from the fact that $\mathbb{E}(G) > 1$ if and only if $q > 1/d$ since G is binomial.

Another important observation is that, for any two vertices v and w in $\overrightarrow{\mathbb{T}}_d$ both at a distance n from the origin, the probability of having a path in $\overrightarrow{\mathscr{T}}_{(d,q)}$ from the origin to v is identical to the probability of having a path from the origin to w, ensuring path symmetry.

Let us calculate the values of ψ, α, and β separately for this example.

$$\begin{aligned} \psi &= \lim_n \sqrt[n]{\mathbb{P}(\{o \rightsquigarrow v\} \in \mathbb{T}_{q,d}) d^n} \\ &= \lim_n \sqrt[n]{\mathbb{P}(\langle 0, v_1\rangle, \langle v_1, v_2\rangle, \ldots, \langle v_{n-1}, v_n\rangle \in \mathbb{T}_{q,d}) d^n} \\ &\overset{ind}{=} \lim_n \sqrt[n]{\mathbb{P}(\langle 0, v_1\rangle \in \mathbb{T}_{q,d})^n d^n} \\ &= dq. \end{aligned} \tag{2.16}$$

Note that G^* is equivalent to the distribution of $G'+1$, where $G' \sim \text{Binomial}[d-1, q]$. Thus, that $\alpha = \mathbb{E}\left(\frac{1}{G'+1}\right)$ and $\beta = \mathbb{E}\left(\frac{1}{(G'+1)^2}\right)$ so, using the software MATHEMATICA, we calculate both

$$\alpha = -\frac{(1-q)^d - 1}{dq}, \text{ and } \beta = -\frac{(1-q)^d \, {}_3F_2\left(1,1,1-d;2,2;\frac{q}{q-1}\right)}{q-1},$$

where ${}_pF_q(a;b;z)$ is the generalized hypergeometric function.

Since we have α, β and ψ, and since the chosen distribution is symmetrical, we can apply Corollary 9. We conclude that

$$-\frac{2dq}{(1-q)^d - 1} - \sqrt{2}\sqrt{\frac{dq(2dq-1)}{\left((1-q)^d - 1\right)^2}} < p_c(d,q). \tag{2.17}$$

Substituting α, β, and ψ in Expression 2.15 results in an overly complex expression; therefore, we omit it.

Table 5 presents the lower and upper bounds for the directed frog model on the random tree when we keep the expectation constant. Note that, while keeping the average degree fixed at 2, even if we increase the value of d, the decrease in the value of q has a greater impact on the limit. This results in higher limits, indicating that the model exhibits greater difficulty in propagation.

Table 5 – Some numerical values for the lower and upper limits for the directed frog model when the expectation of vertex number is constant equal to 2.

q	d	Corollary 9 - Lower Bound	Corollary 9 - Upper Bound
1	2	0.535898	0.554413
2/3	3	0.55651	0.59611
1/2	4	0.571625	0.619889
2/5	5	0.581083	0.634585
1/3	6	0.587474	0.644534
2/7	7	0.592064	0.651712
1/4	8	0.595517	0.657134
2/9	9	0.598208	0.661374
1/5	10	0.600362	0.664781
1/10	20	0.610068	0.68028
1/15	30	0.613305	0.68551
1/20	40	0.614924	0.688138
1/25	50	0.615894	0.689718
1/50	100	0.617835	0.692889

Table 6 presents some numerical values for the lower and upper bounds for the directed frog model in a random tree when $q = 1/2$.

Table 6 – Some numerical values for the lower and upper bounds for the directed frog model when q is constantly set to $1/2$.

q	d	Corollary 9 - Lower Bound	Corollary 9 - Upper Bound
1/2	2	0.781049	1
1/2	3	0.629155	1
1/2	4	0.571625	0.619889
1/2	5	0.544892	0.570772
1/2	6	0.531072	0.54747
1/2	7	0.523348	0.534591
1/2	8	0.518711	0.526808
1/2	9	0.515726	0.521772
1/2	10	0.513669	0.518317
1/2	20	0.506412	0.507291
1/2	30	0.504238	0.504597
1/2	40	0.503165	0.50336
1/2	50	0.502525	0.502647
1/2	100	0.501256	0.501286

CHAPTER 3

PROOFS

This chapter is dedicated to the proofs of the main results.

3.1 Proof of Theorem 2 - Lower Bound

The idea of the proof is to slightly modify the dynamics of the frog model so that it can be easily coupled with a two-type branching process (TTBP) in such a way that the latter dominates the former. This TTBP is defined in Subsection 3.1.1. In Subsection 3.1.2, we give the alternative construction of the frog model, and although it is not a branching process, we call it the *frog model branching process* (FMBP) by abuse of terminology. We prove the lower bound of Theorem 2 in Subsection 3.1.3, by constructing the coupling between the TTBP and the FMBP.

3.1.1 A Two-Type Branching Process

Consider a two-type branching process in which, at each time step, exactly one particle is chosen, gives birth to a random number of individuals of each type, independently of everything else and dies. For $a = 1,2$, we denote by $p_a(i,j)$ the probability that a particle of type a generates i particles of type 1 and j particles of type 2:

$$p_1(0,0) = 1-p,\ p_1(1,0) = 0,\ p_1(2,0) = \frac{pd}{d+1},\ p_1(0,1) = \frac{p}{d+1}. \tag{3.1}$$

$$p_2(0,0) = 1-p,\ p_2(1,0) = \frac{p}{d+1},\ p_2(2,0) = \frac{p(d-1)}{d+1},\ p_2(0,1) = \frac{p}{d+1}. \tag{3.2}$$

The proposed distributions for each type a were motivated by the possible configurations formed by active particles observed individually in the frog model. For the type 1 particle, the imagined configuration is when the particle is positioned in such a way that, *in front of it*, these are the neighbors of the vertex where the particle is located, which are the farthest from the root, no vertex has been visited until that moment and, *behind it*, these are the neighbors of the vertex where the particle is located that are closest to the root, there is a vertex that has already been visited previously, as shown in Figure 6."

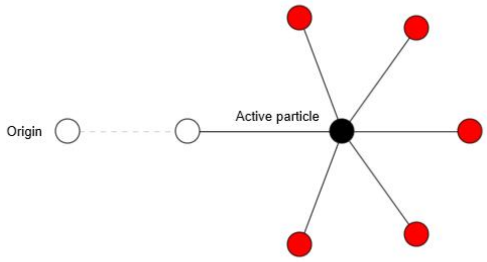


Figure 6 – Configuration that motivated the probability distribution of type 1 particles on $\mathbb{T}_3$. The red circles represent vertices that have not yet been visited, the black circle represents an active particle while the white circle represents a vertex that has already been visited.

For the type 2 particle, the imagined configuration is when the particle is positioned in such a way that, *in front of it*, a vertex has already been visited and, *behind it*, there is a vertex that has already been visited previously, as shown in Figure 7.

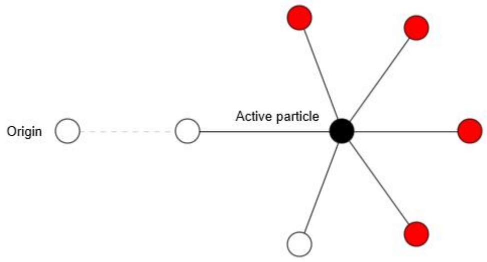


Figure 7 – Configuration that motivated the probability distribution of type 2 particles on $\mathbb{T}_3$. The red circles represent vertices that have not yet been visited, the black circle represents an active particle while the white circle represents a vertex that has already been visited.

It is well-known that a multi-types Galton-Watson process has probability zero to survive if, and only if, the largest eigenvalue of the first moment matrix is smaller or equal to 1 (see Athreya and Ney (2004) for instance). Simple calculations show that this matrix is

$$M = \begin{pmatrix} \frac{2dp}{d+1} & \frac{p}{d+1} \\ \frac{(2d-1)p}{d+1} & \frac{p}{d+1} \end{pmatrix},$$

and its largest eigenvalue is

$$\rho_M = \frac{\left(\sqrt{4d^2+4d-3}+2d+1\right)p}{2(d+1)}.$$

In other words, if

$$p \leq \frac{2(d+1)}{\sqrt{4d^2+4d-3}+2d+1},$$

then the TTBP defined above will generate finitely many individuals with probability 1.

3.1.2 Modifying the Dynamics of the Frog Model

We define the following modification in the dynamics of the frog model, which does not alter its survival probability but simplifies the construction of a coupling with the TTBP.

1. We consider the frog model in a way that frogs move one at a time, and at each time, the frog that is chosen to make the step is arbitrary. That is, the frog of the origin makes a move (with probability p), activating the sleeping frog of the visited site. Then, we choose any frog among the activated ones, to make a move (a move that is made with probability p too) while the others remain frozen (not moving) and active, and can be chosen to make a move at a future time step. This procedure slows down the process in the sense that it propagates more slowly on the tree, but since the random walks of each activated frog are independent, this does not change anything in terms of survival.

2. For ease of comparison with a two-type branching process, we will interpret "moves" differently. At each step, instead of saying that the chosen frog moves to a neighboring site, we will think that with probability p, gives birth to frogs at one neighboring site, one frog if the chosen neighboring site has been already visited, and two frogs otherwise and then dies.

3. For any $t \geq 0$, we denote by $\mathscr{T}_t \subset \mathbb{T}_d$ the set of visited sites at time t. Notice that $\mathscr{T}_t \subseteq \mathscr{T}_{t+1}, t \geq 0$.

4. We consider the model starting at the random time $K = \inf\{t \geq 0 : |\mathscr{T}_t| = d+3\}$ where $|\cdot|$ denotes the cardinality of the set. This ensures that at least one frog within a distance of 2 from the root has been activated (see the next item). Notice that $K = \infty$ has a positive probability to occur.

5. At each time step t, frogs that are at the tip of $\mathscr{T}_t$ (meaning they have d unvisited neighboring sites) are classified as Type 1, and the other frogs are classified as Type 2. Note that once we consider times $t \geq K$, at least one neighboring site of a Type 2 frog is surrounded by at least two already visited sites. This means that any activated frog in the system can be classified as either type 1 or type 2.

Note that, if a frog, when born, is of type 1, depending on the position of its vertex v in the evolving set $\mathscr{T}_t, t \geq 1$, it may transform into type 2 (notice the difference in the preceding sentence between "transform" and "give birth to"). However, the reverse cannot occur, a frog that is born type 2 cannot, over time, transform into a type 1 frog.

Fix $t \geq K$ and $\mathscr{T}_t = T$. We denote by $p_v(i, j|T)$ the offspring distribution of the frog located at v inside T which has been chosen to make a move, where i and j are respectively the numbers of offspring of types 1 and 2. For any v, the distribution p_v lives on

$\{(0,0),(1,0),(0,1),(2,0)\}$. The location of v inside $\mathscr{T}_t = T$ specifies the type 1 or 2 of the frog:

- Suppose it has type 1. Then, independently of v and T

$$p_v(0,0|T) = 1-p,\; p_v(1,0|T) = 0,\; p_v(2,0|T) = \frac{pd}{d+1},\; p_v(0,1|T) = \frac{p}{d+1}. \tag{3.3}$$

- Suppose it has type 2. Then, there exist two integers (which depend on the location of v in T) a,b, with $a \geq 1$ and $a+b \geq 2$, such that

$$p_v(0,0|T) = 1-p,\; p_v(1,0|T) = \frac{pb}{d+1},\; p_v(2,0|T) = p-p\frac{a+b}{d+1},\; p_v(0,1|T) = \frac{pa}{d+1}. \tag{3.4}$$

A geometric interpretation of the coefficient b is that it represents the number of neighbors in front of v that have been visited up to time t.

3.1.3 Proof of the Lower Bound

We are now ready to prove our lower bound.

Proof of the lower bound of Theorem 2. Let $M_t^i, i=1,2$ (*resp.* $N_t^i, i=1,2$) count the number of active particles of type i in the system at time t in the FMBP (*resp.* in the TTBP).

Since we are only interested in bounding the critical parameter of the frog model, instead of starting from $\mathscr{T}_0 = \{o\}$ we can start the frog model from any of the configurations satisfying $|\mathscr{T}_0| = d+3$. So let us start from any vector (M_0^1, M_0^2) having positive probability to be produced in the FM with $|\mathscr{T}_0| = d+3$, and put $(N_0^1, N_0^2) = (M_0^1, M_0^2)$. If we can couple these processes in such a way that, for any $t \geq 0$, $N_t^1 \geq M_t^1$ and $N_t^1 + N_t^2 \geq M_t^1 + M_t^2$, then, in particular, the total number of particles in the TTBP is at least as large as the total number of activated frogs in the FMBP, at each time step. Together with what we said in Subsection 3.1.1, if

$$p \leq \frac{2(d+1)}{\sqrt{4d^2+4d-3}+2d+1},$$

then the FMBP would not survive with probability 1, which would conclude the proof of the theorem.

So it only remains to prove that we can couple these processes in such a way that, for any $t \geq 0$, $N_t^1 \geq M_t^1$ and $N_t^1 + N_t^2 \geq M_t^1 + M_t^2$. The inequalities are satisfied at $t=0$ by definition. We assume that the inequalities are satisfied at time t, and we now want to prove that they are still satisfied at time $t+1$. To couple the processes at $t+1$, we need to couple the probability distributions p_1, p_2 and $p_v(\cdot|T)$. We do this using a random variable U_{t+1} uniformly distributed in $[0,1]$ (independent of everything) and several partitions of $[0,1]$. To construct p_1, p_2 we use the following partitions

$$\mathscr{P}_1 = \{I_1^{0,1}, I_1^{2,0}, I_1^{0,0}\}$$
$$\mathscr{P}_2 = \{I_2^{0,1}, I_2^{1,0}, I_2^{2,0}, I_2^{0,0}\}$$

where

$$I_1^{0,1} = \left[0, \frac{p}{d+1}\right[\ , \ I_1^{2,0} = \left[\frac{p}{d+1}, p\right[\ , \ I_1^{0,0} = [p,1]$$

and

$$I_2^{0,1} = \left[0, \frac{p}{d+1}\right[\ , \ I_2^{1,0} = \left[\frac{p}{d+1}, \frac{2p}{d+1}\right[\ , \ I_2^{2,0} = \left[\frac{2p}{d+1}, p\right[\ , \ I_2^{0,0} = [p,1] \, .$$

We refer to Figure 8 for a pictorial representation of these partitions.

For the coupling, if a type $k = 1,2$ particle is chosen at time $t+1$ from the TTBP, its offspring is chosen with U_{t+1}, *using* partition $\mathscr{P}_k$ in the following way: we put (i,j) if and only if $U_{t+1} \in I_k^{i,j}$. This gives the correct offspring distribution since

$$P(U_{t+1} \in I_1^{i,j}) = p_1(i,j) \ \text{ and } \ P(U_{t+1} \in I_2^{i,j}) = p_2(i,j).$$

For moving frogs of type 1 in the FMBP, no matter what is the pair (v,T), we can use partition $\mathscr{P}_1$ similarly (and with the same uniform U_{t+1}) since for such frogs $p_v(\cdot|T) = p_1$.

According to (3.4), for moving frogs of type 2 we define the partition (recall that $a \geq 1$ and $a+b \geq 2$)

$$\mathscr{P}_{(a,b)} = \{I_{(a,b)}^{0,1}, I_{(a,b)}^{1,0}, I_{(a,b)}^{2,0}, I_{(a,b)}^{0,0}\}$$

where

$$I_{(a,b)}^{0,1} = \left[0, \frac{ap}{d+1}\right[\ , \ I_{(a,b)}^{1,0} = \left[\frac{ap}{d+1}, \frac{(a+b)p}{d+1}\right[\ , \ I_{(a,b)}^{2,0} = \left[\frac{(a+b)p}{d+1}, p\right[\ , \ I_{(a,b)}^{0,0} = [p,1] \, .$$

Observe that, for any a,b and i,j, we have

$$P(U_{t+1} \in I_{(a,b)}^{i,j}) = p_v(i,j|T) \, .$$

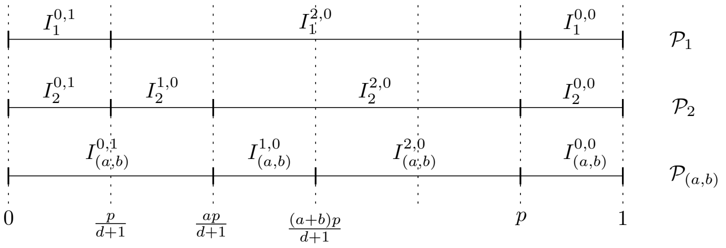


Figure 8 – Pictorial representation of the partitions $\mathscr{P}_1, \mathscr{P}_2$ and $\mathscr{P}_{(a,b)}$, when $d = 4, a = 2$ and $b = 1$.

The coupling is thus performed by updating both processes at time $t+1$ using the same uniform random variable U_{t+1}. We can now establish the recursion from t to $t+1$, recalling that the recursion hypothesis is $N_t^1 \geq M_t^1$ and $N_t^1 + N_t^2 \geq M_t^1 + M_t^2$:

1. If the chosen frog for the next FMBP move is of type 1, then a TTBP type 1 particle is chosen to give birth to its offspring. Since $N_t^1 \geq M_t^1$, this choice is always possible. In this case, according to our coupling, the two chosen particles (one in each process) give birth to the same offspring, thus maintaining the inequalities.

2. Suppose now that the chosen frog is of type 2, then, either a type 2 particle of the branching process is chosen to give birth to its offspring, or, if there is none, a type 1 particle is chosen.

   According to the location v in T of the chosen frog of type 2, a pair (a,b) of integers is associated, as explained before. For any fixed pair (a,b), we have the following possibilities:

   a) The particle of the TTBP is also a type 2 particle: then

      - if $U_{t+1} \leq \frac{p}{d+1}$, then one particle of type 2 is created in each process;
      - if $U_{t+1} \geq p$, then the chosen particle dies in each process;
      - if $U_{t+1} \in \left[\frac{p}{d+1}, \frac{ap}{d+1}\right]$ (and therefore $a \geq 2$ since otherwise this interval half open to the right is empty) the branching process produces two type 1 particles while the frog model produces one type 2 particle;
      - if $U_{t+1} \in I_{(a,b)}^{1,0}$ then either one (if $a = 1$) or two (if $a \geq 2$) type 1 particles are created in the branching process, while the frog model creates one type 1 particle;
      - if $U_{t+1} \in I_{(a,b)}^{2,0}$ then two type 1 particles are created in both processes.

      Therefore, for this case, no matter where the uniform random variable U_{t+1} falls, the inequalities of the recursion hypothesis are maintained at time $t+1$.

   b) There is no more type 2 particles in the TTBP, and we have to choose a type 1 particle. In this case, the reasoning is the same as above, observing moreover that, necessarily, $N_t^1 \geq M_t^1 + 1$.

   Thus, in any case, we have $N_{t+1}^1 \geq M_{t+1}^1$ and $N_{t+1}^1 + N_{t+1}^2 \geq M_{t+1}^1 + M_{t+1}^2$, establishing the recursion.

The proof of the lower bound of Theorem 2 is concluded. ☐

Remark 4. We could assume that the number of particles per vertex is random and the technique to find the lower bound for this model with more particles would be the same.

3.2 Proof of Theorem 2 - Upper Bound

We will proceed in three steps. In the first step of the proof, we explain the comparison of the frog model with its oriented version. This part of the proof is common to (LEBENSZTAYN; MACHADO; POPOV, 2005), (LEBENSZTAYN; UTRIA, 2019) and (GALLO; RODRIGUEZ, 2018). In the second step, we state Theorem 10, a very nice result that identifies the critical parameter of the oriented version with the root of a power series. Notice that, although this result was already proved in (GALLO; RODRIGUEZ, 2018) (see Theorem 1 therein), we will partially prove it to give support to our discussion in Chapter 4. Based on this theorem, the two last steps are dedicated to obtaining bounds for the critical parameter of the oriented version, and these steps only rely on calculus. This is done by first finding finite degree polynomials (of degree 6 and degree 5) which approximate the power series from below and from above, and next, finding root approximations for these polynomials.

3.2.1 Step One: Oriented Version of the Frog Model

Consider the following modification of the frog model, that we call *oriented frog model*: when a frog, initially at vertex $v \in \mathbb{T}_d$ is activated and makes its random walk, it only activates the frogs of vertices v' such that $v' > v$.

It is obvious that, if this oriented model survives, then the original model survives as well, so if we find an upper bound for the critical parameter $\hat{p}_c$ of the oriented model, it will also be an upper bound for the critical parameter of the original model.

Our objective in the remaining of the proof will be to find tight upper and lower bounds for $\hat{p}_c$. Notice that even though the lower bound on $\hat{p}_c$ is not necessary to get our upper bound for p_c, we will partially provide it to show that the upper bound of $\hat{p}_c$ is already very accurate as an estimate of $\hat{p}_c$ (see the discussion in Chapter 4).

3.2.2 Step Two: $\hat{p}_c$ as the Root of a Power Series

For any vertices v', v, we denote by $\{v \to v'\}$ the event that the frog at v, if it were active, would visit v' during his random walk. Lemma 2.1 of Lebensztayn, Machado and Popov (2005) states that for any v, v' such that $d(v, v') = n \geq 1$,

$$\mathbb{P}_p(v \to v') = r^n \tag{3.5}$$

where

$$r = r(p,d) := \frac{d+1-\sqrt{(d+1)^2-4dp^2}}{2dp}, \tag{3.6}$$

a fact that will have importance later on. Notice in particular that, writing $r_c = r(\hat{p}_c, d)$, we can now focus on r_c directly since r is a continuous bijection on $[0,1]$.

The objective of the present step is to prove the following very nice theorem which gives the critical parameter as the root of a power series. It is proved, indirectly, in (GALLO; RODRIGUEZ, 2018), so we include its proof here for the sake of completeness.

Theorem 10.

$$\sum_{k\geq 1} d^k r(\hat{p}_c,d)^k \prod_{i=1}^{k-1}(1-r(\hat{p}_c,d)^i) = 1.$$

Proof. We denote by $\{v \overset{c}{\to} v'\}$ the event that either $\{v \to v'\}$, or there exist $k \geq 1$ and a sequence of vertices $v_1, \dots, v_k$ such that $v =: v_0 < v_1 < \dots < v_k < v_{k+1} := v'$ such that $\bigcap_{i=0}^{k} \{v_i \to v_{i+1}\}$. In words, $\{v \overset{c}{\to} v'\}$ means that v has started a chain of activation of frogs which in particular activates v'. By symmetry, we can use $u_n = u_n(r,d) := \mathbb{P}_p(o \overset{c}{\to} v)$ for any v such that $d(o,v) = n$.

On the one hand, we have

$$\begin{aligned} P_p(\text{survival of the oriented model}) &= \mathbb{P}_p\left(\bigcap_{n\geq 1} \bigcup_{v:d(o,v)=n} \{o \overset{c}{\to} v\}\right) \\ &= \lim_{n\to\infty} \mathbb{P}_p\left(\bigcup_{v:d(o,v)=n} \{o \overset{c}{\to} v\}\right) \end{aligned}$$

Thus

$$\mathbb{P}_p\left(\bigcup_{v:d(o,v)=n} \{o \overset{c}{\to} v\}\right) \leq d^n \mathbb{P}_p(o \overset{c}{\to} v) =: d^n u_n.$$

In other words,

$$d^n u_n \to 0 \Rightarrow \mathbb{P}_p(\text{survival of the oriented model}) = 0. \tag{3.7}$$

On the other hand, the expected number of vertices at distance n of the root which have been visited by frogs is, also by symmetry

$$\sum_{v:d(o,v)=n} \mathbb{P}_p(o \overset{c}{\to} v) = d^n u_n.$$

Fix some $N \geq 1$ and consider the following process, obtained from the oriented model *via* the following recursive procedure:

- At each vertex of level N which has been visited (at time N, since the process is oriented), we keep only one activated frog.
- Each frog activated at level $iN, i \geq 1$ is started, and at each vertex of level $(i+1)N$ which has been visited, we keep only one activated frog.

Observe first that this modified process is dominated by the oriented process since at each step $i \geq 1$ of the recursive procedure we only keep 1 activated frog at the visited vertices at distance iN of the root. Moreover, the frog we keep at iN, given it has reached this level, can be substituted by a new active frog, because of the loss of memory property of the geometric distribution (3.5). It follows that the number of visited vertices at level $iN, i \geq 1$ of this modified process has the same distribution as the number of individuals at the i^{th} generation in a Galton-Watson tree with expected offspring $d^N u_N$. If $d^N u_N > 1$, the Galton-Watson has a positive probability to survive, and thus so does the oriented model. In other words

$$\text{there exist } N \geq 1 : d^N u_N > 1 \Rightarrow \mathbb{P}_p(\text{survival of the oriented model}) > 0. \tag{3.8}$$

To continue, we need to study $u_n = \mathbb{P}_p(o \overset{c}{\to} v)$ and its limiting properties, and it is precisely in that study that (GALLO; RODRIGUEZ, 2018) differs from (LEBENSZTAYN;

MACHADO; POPOV, 2005) and (LEBENSZTAYN; UTRIA, 2019) (see Chapter 4 for a discussion). Indeed, Gallo and Rodriguez (2018) used a result of Gallo *et al.* (2014) implying that u_n is the probability that an undelayed renewal sequence, with inter-renewal distribution $f_k = r^k \prod_{i=1}^{k-1}(1-r^i), k \geq 1$, has a renewal at time n. This allowed them to conclude, using renewal theory (see Section 4 in (GALLO; RODRIGUEZ, 2018)) that $u_\infty(r,d) := \lim u_n^{1/n}$ exists, is continuous in r, and it satisfies the equality

$$\sum_{k\geq 1} r^k u_\infty^{-k} \prod_{i=1}^{k-1}(1-r^i) = 1. \tag{3.9}$$

Together with (3.7) and (3.8), the existence of $u_\infty(r,d)$ gives us

$$u_\infty > 1/d \Rightarrow \mathbb{P}_p(\text{survival of the oriented model}) > 0$$
$$u_\infty < 1/d \Rightarrow \mathbb{P}_p(\text{survival of the oriented model}) = 0.$$

The continuity of $u_\infty(r,d)$ allows us conclude that r_c satisfies

$$u_\infty(r_c,d) = \frac{1}{d}. \tag{3.10}$$

Putting (3.10) together with (3.9) concludes the proof of the theorem. □

3.2.3 Third Step: Bounds for $\hat{p}_c$ as Zeros of Polynomials

Using (3.6) and and Theorem 10, the way to proceed now is to find $\underline{r} = \underline{r}(d)$ and $\overline{r} = \overline{r}(d)$ such that

$$\underline{r} \leq r_c \leq \overline{r} \tag{3.11}$$

which yields

$$\frac{(d+1)\underline{r}}{1+d\underline{r}^2} \leq \hat{p}_c \leq \frac{(d+1)\overline{r}}{1+d\overline{r}^2}. \tag{3.12}$$

The first characterization that we will give of $\underline{r}$ and $\overline{r}$ is as zeros of polynomials.

Proposition 11. Let $\underline{r}$ (*resp.* $\overline{r}$), denote the unique root of the polynomial $L(r) := -d^3r^6 + d^3r^5 + d^2r^3 - 2dr + 1$ (*resp.* of the polynomial $U(r) := d^3r^5 + d^2r^3 - 2dr + 1$) in $r \in (0, 1/d)$. Then

$$\underline{r} \leq r_c \leq \overline{r}.$$

Proof. We have the inequalities for any $k \geq 3$

$$1 - r - r^2 < (1 - r - r^2 + r^k \leq) \prod_{i=1}^{k-1}\left(1-r^i\right) \leq (1-r)(1-r^2) \tag{3.13}$$

where the lower bound between parenthesis follows by recursion and the two others are trivial. Inequalities (3.13) imply that

$$f_{\inf}(r) \leq \sum_{k\geq 1} d^k r^k \prod_{i=1}^{k-1}(1-r^i) \leq f_{\sup}(r)$$

with

$$f_{\inf}(r) = dr + d^2(1-r)r^2 + (1-r-r^2)\left(-\frac{d^3r^3}{dr-1}\right)$$

and

$$f_{\sup}(r) = dr + d^2(1-r)r^2 + (1-r)(1-r^2)\left(-\frac{d^3r^3}{dr-1}\right).$$

With Theorem 10 we conclude that

$$f_{\inf}(r_c) \leq 1 \leq f_{\sup}(r_c).$$

With some further algebra, we notice that

$$\begin{aligned} f_{\inf}(r_c) \leq 1 &\Leftrightarrow d^3r_c^5 + d^2r_c^3 - 2dr_c + 1 \geq 0 \\ f_{\sup}(r_c) \geq 1 &\Leftrightarrow -d^3r_c^6 + d^3r_c^5 + d^2r_c^3 - 2dr + 1 \leq 0. \end{aligned}$$

But since both polynomials are decreasing on $(0,1/d)$, then we conclude that the root $\underline{r}$ of $L(r) = -d^3r^6 + d^3r^5 + d^2r^3 - 2dr + 1$ and the root $\overline{r}$ of $U(r) = d^3r^5 + d^2r^3 - 2dr + 1$ will satisfy

$$\underline{r} \leq r_c \leq \overline{r}$$

as stated by the lemma. □

3.2.4 *Fourth (Last) Step: Explicit Bounds for $\hat{p}_c$ as Functions of d*

In principle, we could simply seek for the exact expression of the zeros stated in Proposition 11. However, this leads to very complicated expressions. So we will do one further step to get approximations of $\overline{r}$ and $\underline{r}$.

Lemma 12. For $d \geq 2$

$$\underline{r}(d) \geq r_L := \frac{5 - 8d - 16d^2 + 64d^3}{12d - 20d^2 - 48d^3 + 128d^4}$$

and

$$\overline{r}(d) \leq r_U := \frac{2 - \frac{1}{14d^2} - 4d}{5d - 8d^2}.$$

Putting Display (3.12), Proposition 11 and Lemma 12 together, we get explicit lower and upper bounds for $\hat{p}_c$ as a function of d. In conjunction with Step 1, this in particular proves the upper bound given in Theorem 2.

3.3 Proof of Theorem 8

Proof of Theorem 8. Throughout the demonstration, let us consider the following sets:

$$\begin{aligned} A_n &:= \{\text{some frog at distance } n \text{ from the origin is activated}\}, \\ A_\infty &:= \{\text{an infinite number of frogs are activated}\}, \\ A_n^v &:= \{\text{the frog at vertex } v \text{ at distance } n \text{ from the root is activated}\}. \end{aligned}$$

First, Let us establish a sufficient condition to derive a lower bound for the critical parameter. Note that:

$$\begin{array}{rcl} \mathbb{P}(\text{FM}(\overrightarrow{\mathscr{T}}_d,q) \text{ survives}) & = & \mathbb{P}(\bigcap_n A_n) \\ & = & \lim_n \mathbb{P}(A_n) \\ & = & \lim_n \mathbb{P}(\bigcup_{v:d(\boldsymbol{o},v)=n} A_n^v) \\ & \leq & \lim_n \sum_{v:d(\boldsymbol{o},v)=n} \mathbb{P}(A_n^v) \\ & = & \lim_n d^n \mathbb{P}(A_n^v) \\ & = & \lim_n d^n \mathbb{P}(A_n^v \cap \{\boldsymbol{o} \rightsquigarrow v \in \overrightarrow{\mathscr{T}}_d \cup \boldsymbol{o} \rightsquigarrow v \notin \overrightarrow{\mathscr{T}}_d\}) \\ & = & \lim_n d^n \mathbb{P}(A^v | \boldsymbol{o} \rightsquigarrow v \in \overrightarrow{\mathscr{T}}_d) \mathbb{P}(\boldsymbol{o} \rightsquigarrow v \in \overrightarrow{\mathscr{T}}_d). \end{array} \tag{3.14}$$

The sixth equality is a consequence of the symmetry of the model. Let us denote $\psi_n := \mathbb{P}_q(\boldsymbol{o} \rightsquigarrow v \in \overrightarrow{\mathscr{T}}_d)d^n$ and $p_{q,n} := \mathbb{P}_q(A_n^v | \boldsymbol{o} \rightsquigarrow v \in \overrightarrow{\mathscr{T}}_d)$.

From the inequality provided by (3.14), we can conclude that:

$$\text{if } q \text{ is such that } \psi_n p_{q,n} \to 0, \text{ then } p_c(\text{FM}(\overrightarrow{\mathscr{T}}_d)) > q. \tag{3.15}$$

As a second step, we will present a sufficient condition to obtain an upper bound. To derive a condition providing an upper bound for the critical parameter, let us compare the frog model with a branching process (BP_N) in the following way.

The individuals in the branching process will be associated with the vertices of the tree as follows. The origin of the tree is associated with the patriarch of the branching process. The children of the patriarch is the set of vertices v such that $d(\boldsymbol{o},v) = N$ which was visited by an active frog at some point in the process. In other words, the children are vertices v such that A_N^v occurred. Recursively, the children of a vertex v at distance kN from the origin are vertices w such that $A_{(k+1)N}^w$ occurred. It is important to note that there may be more than one active particle at a given vertex v at distance N from the origin. Even when this occurs, we will count that vertex as a single child.

The constructed branching process is dominated by the frog model, i.e., if the branching process survives, then the frog model also survives.

It is well known that the branching process has a positive probability of surviving if and only if the expectation of the offspring is greater than 1. For BP_N, we have,

$$\begin{array}{rcl} \mathbb{E}_q\left(\sum_{v:d(\boldsymbol{o},v)=N} \mathbb{I}_{(A_N^v)}\right) & = & \sum_{v:d(\boldsymbol{o},v)=N} \mathbb{E}_q(\mathbb{I}_{(A_N^v)}) \\ & = & d^N \mathbb{P}_q(\boldsymbol{o} \rightsquigarrow v \in \overrightarrow{\mathscr{T}}_d) p_{q,N}, \end{array}$$

the second equality is due to the symmetry of the model.

Thus, a sufficient condition to obtain an upper bound for the frog model is given by:

$$\text{there exists } N \text{ and } q \text{ such that } \psi_N p_{q,N} > 1, \text{ then } q \geq p_c(\text{FM}(\overrightarrow{\mathscr{T}}_d)). \tag{3.16}$$

The condition represented by 3.16 can be interpreted as follows: If, for a given q and a specific value of N, the inequality $\psi_N p_{q,N} > 1$ holds, then the branching process has a positive probability of survival. Consequently, as the branching process survives, we can deduce that $p_c(BP_N) < q$. Conversely, considering that $p_c(\mathrm{FM}(\overrightarrow{\mathscr{T}}_d)) < p_c(BP_N)$, it follows that $p_c(\mathrm{FM}(\overrightarrow{\mathscr{T}}_d)) < q$.

The conditions outlined in (3.15) and (3.16) for lower and upper bounds, respectively, depend on ψ_n and $p_{q,n}$. Let us start analyzing of $p_{q,n}$.

In words, $p_{q,n}$ denotes the probability that an active frog visits a vertex situated at a distance n from the origin, under the condition that the path from the origin to v is part of the tree $\overrightarrow{\mathscr{T}}_d$.

Consider a path $v_0 \rightsquigarrow v_n$, which, by construction of the proof, is contained in $\overrightarrow{\mathscr{T}}_d$ and the event $\{v_0 \xrightarrow{c} v_n\}$, which signifies that an active frog starts at vertex v_0 and triggers a chain of activations that, in particular, activates the frog positioned at v_n (this definition was used in the proof of Theorem 10). Let $v_0, v_1, ..., v_n$ be the vertices in $v_0 \rightsquigarrow v_n$. From these vertices, we define another process given by $Y^{(n)} = (Y_i^{(n)})_{0\le i\le n}$, where $Y_0^{(n)} = 1$ and $Y_i^{(n)} = \mathbf{1}_{\{v_{n-i}\xrightarrow{c} v_n\}}$ for $i = 1, 2, \dots, n$. The process $Y^{(n)}$ is the auxiliary of the process of propagation of frogs along the path $v_0 \rightsquigarrow v_n$. We now define auxiliary random variables S_k, $k \ge 0$, where S_k represents the k-th natural number for which $Y_{S_k}^{(n)} = 1$, with $S_0 = 0$. Using the variable S_k, we define the random variable $T_k = S_k - S_{k-1}$. Note that T_k measures the distance between the $(k-1)$-th and the k-th 1 in the auxiliary process. We have that

$$Y_i^{(n)} = 1 \text{ if and only if } \sum_{k=1}^{l} T_k = i, \text{ for some } l.$$

We now argue that the process Y is a renewal process, which amounts to say that the $(T_k)_{k\ge1}$ are i.i.d. To show this the sequence, we associate to each vertex v_i the following random variables: $G_i\mathbb{I}_{G_i\ge1}$, which represents the number of neighbors of vertex v_i, $B_{(i,j)}$, which is a Bernoulli variable with parameter $1-p$, telling whether the j-th active frog at vertex i survives or dies, and $U_{(i,j)}$, which follows a uniform distribution on $[0,1]$. In words, the realization $u_{(i,j)}$ of $U_{(i,j)}$ determines to which vertex the j-th surviving active frog will jump. Supposing that $G_i\mathbb{I}_{G_i\ge1} = k$, we divide $[0,1]$ equally into k parts. This partition is in the same way as we did in Section 3.1.3. The event $T_i = k$ depends on the variables $(U_{(l,j)}, B_{(l,j)}, G_{(l,j)})$ for $l \in \{S_{i-1}+1, \dots, S_i\}$ in exactly the same way that $T_1 = k$ depends on the variables $(U_{(l,j)}, B_{(l,j)}, G_{(l,j)})$ for $l \in \{S_0 + 1 = 1, \dots, S_1\}$. Since these sets of random variables are i.i.d., it follows that the sequence $T_i, i \ge 1$ is i.i.d.

The distribution of T_1 is given by

$$\begin{aligned}\mathbb{P}(T_1 = k) &= \mathbb{P}(S_1 - S_0 = k)\\ &= \mathbb{P}(Y_k^{(n)} = 1, Y_{k-1}^{(n)} = 0, ..., Y_1^{(n)} = 0 \mid Y_0^{(n)} = 1)\\ &= \mathbb{E}\left(\frac{p^k}{\prod_{i=1}^{k} G_i\mathbb{I}_{G_i\ge1}}\prod_{j=1}^{k-1}\left(1-\frac{p^j}{\prod_{i=1}^{j} G_i\mathbb{I}_{G_i\ge1}}\right)\right).\end{aligned} \tag{3.17}$$

In what follows we will parallel the proof given in the work Gallo and Rodriguez (2018). Given that $Y^{(n)}$ is a renewal process, we know from Gallo and Rodriguez (2018), that there exists a parameter γ such that

$$\log\gamma = -\lim_{n}\frac{1}{n}\log\mathbb{P}(Y_n = 1).$$

Now, Let us examine a comparison between γ and ψ.

Case 1: Suppose $\frac{\psi}{\gamma} > 1$, then there is $\varepsilon > 0$ such that $\frac{\psi}{\gamma}e^{-\varepsilon} > 1$. On the other hand, by the definition of a limit, we know that, given $\varepsilon > 0$, there is N such that, for all $n \geq N$ we have

$$\begin{array}{rcccl} -\varepsilon & < & \log\gamma - (-\frac{1}{n}\log\mathbb{P}(Y_n = 1)) & < & \varepsilon \\ -\varepsilon & < & \log\gamma + \frac{1}{n}\log\mathbb{P}(Y_n = 1) & < & \varepsilon \\ -\log\gamma - \varepsilon & < & \frac{1}{n}\log\mathbb{P}(Y_n = 1) & < & \varepsilon - \log\gamma \\ n(-\log\gamma - \varepsilon) & < & \log\mathbb{P}(Y_n = 1) & < & n(\varepsilon - \log\gamma) \\ e^{-n(\log\gamma + \varepsilon)} & < & \mathbb{P}(Y_n = 1) & < & e^{-n(-\varepsilon + \log\gamma)} \\ \left(\frac{1}{\gamma}e^{-\varepsilon}\right)^n & < & \mathbb{P}(Y_n = 1) & < & \left(\frac{1}{\gamma}e^{\varepsilon}\right)^n. \end{array} \tag{3.18}$$

Multiplying the last inequality by ψ^n, we get

$$\left(\frac{\psi}{\gamma}e^{-\varepsilon}\right)^n < \psi^n\mathbb{P}(Y_n = 1) < \left(\frac{\psi}{\gamma}e^{\varepsilon}\right)^n. \tag{3.19}$$

We therefore have the following sequence of implications

$$\begin{aligned} \frac{\psi}{\gamma} > 1 \quad &\Longrightarrow \quad \text{there exists } \varepsilon : \frac{\psi}{\gamma}e^{-\varepsilon} > 1 \\ &\Longrightarrow \quad \text{there exists } N : \psi^n\mathbb{P}(Y_n = 1) > 1 \end{aligned} \tag{3.20}$$

Case 2: In case $\dfrac{\psi}{\gamma} < 1$.

$$\begin{aligned} \frac{\psi}{\gamma} < 1 \quad &\Longrightarrow \quad \text{there exists } \varepsilon : \frac{\psi}{\gamma}e^{\varepsilon} < 1 \\ &\Longrightarrow \quad \psi^n\mathbb{P}(Y_n = 1) \leq \left(\frac{\psi}{\gamma}e^{\varepsilon}\right)^n \to 0. \end{aligned} \tag{3.21}$$

Note that,

$$\gamma > \psi \text{ (Case2)} \implies p < p_c \text{ the implication follows from 3.15,}$$

and

$$\gamma < \psi \text{ (Case1)} \implies p > p_c \text{ the implication follows from 3.16.}$$

In the article by Gallo and Rodriguez (2018), Lemma 1 and Lemma 2, it was demonstrated that

$$\sum_{k\geq 1}\gamma(p)^k\mathbb{P}(T_1 = k) = 1, \tag{3.22}$$

additionally, it has been established that $\gamma(p)$ is continuous. Due to the continuity of $\gamma(p)$, 3.15 and 3.16, it follows that $\gamma(p_c)=\psi$.

We have determined that the critical parameter for the directed frog model is given by

$$\sum_{k\geq 1}\psi^k\mathbb{E}\left(\frac{p^k}{\prod_{i=1}^{k}G_i\mathbb{I}_{G_i\geq 1}}\prod_{j=1}^{k-1}\left(1-\frac{p^j}{\prod_{i=1}^{j}G_i\mathbb{I}_{G_i\geq 1}}\right)\right)=1. \tag{3.23}$$

□

3.4 Proof of Corollary 9

Proof of Corollary 9. Our goal is to establish bounds for Equation (2.13). Note that, for $k\geq 2$, we have:

$$1-\frac{p}{G_1}-\frac{p^2}{G_1G_2} \;\leq\; \prod_{j=1}^{k-1}\left(1-\frac{p^j}{\prod_{i=1}^{j}G_i}\right) \;\leq\; 1-\frac{p}{G_1}. \tag{3.24}$$

On one hand, we obtain (providing us with a lower bound):

$$\begin{aligned}
1 &= \sum_{k\geq 1}\psi^k\mathbb{E}\left(\frac{p^k}{\prod_{i=1}^{k}G_i}\prod_{j=1}^{k-1}\left(1-\frac{p^j}{\prod_{i=1}^{j}G_i}\right)\right)\\
&= \psi p\alpha+\sum_{k\geq 2}\psi^k\mathbb{E}\left(\frac{p^k}{\prod_{i=1}^{k}G_i}\prod_{j=1}^{k-1}\left(1-\frac{p^j}{\prod_{i=1}^{j}G_i}\right)\right)\\
&\leq \psi p\alpha+\sum_{k\geq 2}\psi^k\mathbb{E}\left(\frac{p^k}{\prod_{i=1}^{k}G_i}\left(1-\frac{p}{G_1}\right)\right)\\
&= \psi p\alpha+\sum_{k\geq 2}\psi^k\mathbb{E}\left(\frac{p^k}{\prod_{i=1}^{k}G_i}-\frac{p^{k+1}}{G_1^2\prod_{i=1}^{k-1}G_i}\right)\\
&= \psi p\alpha+\sum_{k\geq 2}\psi^k\mathbb{E}\left(\frac{p^k}{\prod_{i=1}^{k}G_i}-\frac{p^{k+1}}{G_1^2\prod_{i=1}^{k-1}G_i}\right)\\
&= \psi p\alpha+\left(1-\frac{p\beta}{\alpha}\right)\sum_{k\geq 2}\psi^k p^k\alpha^k\\
&= \psi p\alpha+\left(1-\frac{p\beta}{\alpha}\right)\left(-\frac{\alpha^2p^2\psi^2}{\alpha p\psi-1}\right),
\end{aligned} \tag{3.25}$$

where $\alpha=\mathbb{E}\left(\frac{1}{G\mathbb{I}_{G\geq 1}}\right)$ and $\beta=\mathbb{E}\left(\frac{1}{(G\mathbb{I}_{G\geq 1})^2}\right)$.

Let us further manipulate the inequality to derive an explicit expression as a function of

p.

$$\begin{aligned}
1 &\leq \psi p\alpha + \left(1-\frac{p\beta}{\alpha}\right)\left(-\frac{\alpha^2p^2\psi^2}{\alpha p\psi-1}\right)\\
1-\psi p\alpha &\leq \left(1-\frac{p\beta}{\alpha}\right)\left(-\frac{\alpha^2p^2\psi^2}{\alpha p\psi-1}\right)\\
(1-\psi p\alpha)^2 &\leq \left(1-\frac{p\beta}{\alpha}\right)\alpha^2p^2\psi^2\\
1-\psi p\alpha &\leq \left(1-\frac{p\beta}{\alpha}\right)^{\frac{1}{2}}\alpha p\psi\\
1-\psi p\alpha &\overset{(1)}{\leq} (1-p\alpha)^{\frac{1}{2}}\alpha p\psi\\
1-\psi p\alpha &\overset{(2)}{\leq} \left(1-\frac{p\alpha}{2}\right)\alpha p\psi
\end{aligned} \tag{3.26}$$

After some algebraic manipulations (using Mathematica software), by isolating the variable p, we obtain a lower bound for the critical parameter.

$$\frac{2-\sqrt{2}\alpha\sqrt{\dfrac{2\psi-1}{\alpha^2\psi}}}{\alpha} < p_c. \tag{3.27}$$

(1) $\alpha^2 \leq \beta$ Jensen.

(2) It follows from the Maclaurin series expansion of

$$(1-x)^{\frac{1}{2}} = 1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\cdots$$

. On the other hand we have that

$$\begin{aligned}
1 &= \sum_{k\geq 1}\psi^k\mathbb{E}\left(\frac{p_c^k}{\prod_{i=1}^k G_i}\prod_{j=1}^{k-1}\left(1-\frac{p_c^j}{\prod_{i=1}^j G_i}\right)\right)\\
&\geq \psi p\alpha + \sum_{k\geq 2}\psi^k\mathbb{E}\left(\frac{p_c^k}{\prod_{i=1}^k G_i}\left(1-\frac{p}{G_1}-\frac{p^2}{G_1G_2}\right)\right)\\
&= \psi p\alpha + \sum_{k\geq 2}\psi^k\mathbb{E}\left(\frac{p_c^k}{\prod_{i=1}^k G_i}-\frac{p_c^{k+1}}{G_1^2\prod_{i=1}^{k-1}G_i}-\frac{p_c^{k+2}}{G_1^2G_2^2\prod_{i=1}^{k-2}G_i}\right)\\
&= \psi p\alpha + \left(1-\frac{p\beta}{\alpha}-\frac{p^2\beta^2}{\alpha^2}\right)\sum_{k\geq 1}\psi^k p^k\alpha^k\\
&= \psi p\alpha + \left(1-\frac{p\beta}{\alpha}-\frac{p^2\beta^2}{\alpha^2}\right)\left(-\frac{\alpha^2p^2\psi^2}{\alpha p\psi-1}\right)
\end{aligned} \tag{3.28}$$

Let us manipulate the inequality above in order to obtain an explicit expression for the upper limit.

$$\begin{aligned}
1 &\geq \psi p\alpha + \left(1-\frac{p\beta}{\alpha}-\frac{p^2\beta^2}{\alpha^2}\right)\left(-\frac{\alpha^2p^2\psi^2}{\alpha p\psi-1}\right)\\
(1-\alpha p\psi)^2 &\geq \left(1-\frac{p\beta}{\alpha}-\frac{p^2\beta^2}{\alpha^2}\right)\alpha^2p^2\psi^2\\
1-\alpha p\psi &\geq \left(1-\frac{p\beta}{\alpha}-\frac{p^2\beta^2}{\alpha^2}\right)^{\frac{1}{2}}\alpha p\psi\\
1-\alpha p\psi &\geq \left(1-\tfrac{\beta^4p^4}{7\alpha^4}-\tfrac{2\beta^3p^3}{7\alpha^3}-\tfrac{9\beta^2p^2}{14\alpha^2}-\tfrac{\beta p}{2\alpha}\right)\alpha p\psi.\\
\beta^2p^4\psi^2+\alpha\beta p^3\psi^2-2\alpha p\psi+1 &\geq 0
\end{aligned} \tag{3.29}$$

Thus, the unique root between 0 and 1 of the polynomial $\beta^2p^4\psi^2+\alpha\beta p^3\psi^2-2\alpha p\psi+1$ is an upper bound. We obtained the exact root for this polynomial using Mathematica

$$\begin{aligned}
p_c(d) < \quad & -\tfrac{\alpha}{4\beta}-\tfrac{1}{2}\sqrt{\tfrac{\alpha^2}{4\beta^2}+\tfrac{\Delta}{3\sqrt[3]{2}\beta^2\psi^2}+\tfrac{2\sqrt[3]{2}(\alpha^2\psi+2\beta)}{\beta\Delta}}-\\
& \tfrac{1}{2}\sqrt{\tfrac{\alpha^2}{2\beta^2}-\tfrac{\Delta}{3\sqrt[3]{2}\beta^2\psi^2}-\tfrac{2\sqrt[3]{2}(\alpha^2\psi+2\beta)}{\beta\Delta}-\tfrac{16\alpha\beta-\alpha^3\psi}{4\beta^3\psi\sqrt{\tfrac{\alpha^2}{4\beta^2}+\tfrac{\Delta}{3\sqrt[3]{2}\beta^2\psi^2}+\tfrac{2\sqrt[3]{2}(\alpha^2\psi+2\beta)}{\beta\Delta}}}},
\end{aligned}$$

where $\Delta = \sqrt[3]{135\alpha^2\beta^2\psi^4+\sqrt{18225\alpha^4\beta^4\psi^8-4\left(6\alpha^2\beta\psi^3+12\beta^2\psi^2\right)^3}}$.

□

3.5 Proof of the Corollary 5

Proof of the Theorem Corollary 5. Analogously to the approach taken in the proof of Theorem 8 in Section 3.3, we can assert that the process along a single branch is a renewal process, with the distinction lying in the distribution of T_1.

Let us construct a distribution for $T_1 = i$. Fix $k \in \mathbb{N}$ and consider $i \in \mathbb{N}$. Note that if $k < i$, then $\mathbb{P}(T_1 = i) = 0$. For $k \geq i$, we have:

$$\begin{aligned}
\mathbb{P}(T_1 = i) &= \mathbb{P}(S_1 - S_0 = i)\\
&= \mathbb{P}(Y_{S_1} = 1, Y_{S_1-1} = 0, \ldots, Y_1 = 0 \mid Y_{S_0} = 1)\\
&= \frac{p}{d^i}\prod_{j=1}^{i-1}\left(1-\frac{p}{d^j}\right).
\end{aligned} \tag{3.30}$$

In what follows, we will follow a similar path as in the proof provided in (GALLO; RODRIGUEZ, 2018).

By substituting the result from 3.30 into Equation 3.22, and considering that if $k < i$ then $\mathbb{P}(T_1 = i) = 0$, we obtain:

$$\sum_{i=1}^{k}\gamma(p,k)^i\frac{p}{d^i}\prod_{j=1}^{i-1}\left(1-\frac{p}{d^j}\right) = 1. \tag{3.31}$$

From Lemma 2, we know that $\gamma(p,k)$ is continuous, hence we can conclude that $\gamma(p_c,k) = d$. Substituting this result into Equation 3.31, we conclude that:

$$\sum_{i=1}^{k} p_c \prod_{j=1}^{i-1}\left(1-\frac{p_c}{d^j}\right) = 1. \tag{3.32}$$

□

3.6 Proof of the Corollary 6

Proof of the Corollary 6. Let us take a look at the product that appears in

$$\sum_{i=1}^{k} p \prod_{j=1}^{i-1}\left(1-\frac{p}{d^j}\right) = 1.$$

Let us begin by establishing the upper limit. It is noteworthy that for $i=1$, we have $\prod_{j=1}^{0}\left(1-\frac{p}{d^j}\right) = 1$. For $i=2$, the expression becomes $\prod_{j=1}^{1}\left(1-\frac{p}{d^j}\right) = 1-\frac{p}{d}$, and for $i \geq 3$, we have:

$$\begin{array}{rcl} \prod_{j=1}^{i-1}\left(1-\frac{p}{d^j}\right) & = & 1-\frac{p}{d}-\frac{p}{d^2}-\ldots--\frac{p}{d^{i-1}}+r(i) \\ & \geq & 1-\sum_{i=1}^{j-1}\frac{p}{d^i} \\ & = & 1-\frac{pd^{-j}\left(d^j-d\right)}{d-1}. \end{array} \tag{3.33}$$

The inequality stems from the fact that $r(i-1) \geq 0$ (demonstrated through an inductive argument) for all $i \geq 3$, where r is a polynomial function of p with a degree of $i-1$. Then

$$\begin{array}{rcl} 1 & = & \sum_{i=1}^{k} p \prod_{j=1}^{i-1}\left(1-\frac{p}{d^j}\right) \\ & = & p\left[1+(1-p)+\sum_{i=3}^{k}\prod_{j=1}^{i-1}\left(1-\frac{p}{d^j}\right)\right] \\ & \geq & p\left[1+(1-p)+\sum_{i=3}^{k}\left(1-\frac{pd^{-j}\left(d^j-d\right)}{d-1}\right)\right] \\ & = & p\left(\frac{d^{-k-1}\left((k-2)(p+1)d^{k+1}-(k-2)(p+2)d^{k+2}+(k-2)d^{k+3}+pd^k-d^2p\right)}{(d-1)^2}+\left(1-\frac{p}{d}\right)+1\right). \end{array} \tag{3.34}$$

Solving the equation

$$1 = p\left(\frac{d^{-k-1}\left((k-2)(p+1)d^{k+1}-(k-2)(p+2)d^{k+2}+(k-2)d^{k+3}+pd^k-d^2p\right)}{(d-1)^2}+\left(1-\frac{p}{d}\right)+1\right),$$

the solution is

$$p = -\frac{(d-1)^2 d^k\left(\sqrt{\frac{-4d^{1-k}+d^2k^2-2d\left(k^2+2k-2\right)+k(k+4)}{(d-1)^2}}-k\right)}{2\left((k-1)d^{k+1}-kd^k+d\right)} \geq p_c(d,k).$$

To derive the lower limit, we employ a similar rationale, with the limit now being generated by

$$\prod_{j=2}^{i-1}\left(1-\frac{p}{d^j}\right) \leq 1-\frac{p}{d}. \tag{3.35}$$

And we conclude that

$$p_c(d,k) \geq \frac{dk - \sqrt{d}\sqrt{dk^2 - 4k + 4}}{2(k-1)}. \tag{3.36}$$

□

3.7 Proof of the Theorem 7

Proof of the Theorem 7. The goal is to compare the model $\mathrm{FM}(\mathbb{T}_d, p, k)$ with a branching process $\left(BP_{(k)}\right)$ as follows. We know that the particle survives with probability p and dies with probability $1-p$. Once it survives, it always takes k steps, which, in this model, can be away from the root or near the root.

Let us denote $X_{(k)}$ as the random variable representing the number of offspring in $BP_{(k)}$. The distribution of $X_{(k)}$ is given by:

$$P(X_{(k)} = i) = \begin{cases} p\binom{k}{i}\left(\dfrac{d}{d+1}\right)^i\left(1 - \dfrac{d}{d+1}\right)^{k-i} & if\ i \neq 0, \\ p\left(\dfrac{1}{d+1}\right)^k + 1 - p & if\ i = 0. \end{cases} \tag{3.37}$$

The motivation for this distribution given by Equation 3.37 comes from the dynamics of $\mathrm{FM}(\mathbb{T}_d, p, k)$. An active particle first "decides" whether it will survive or not with probability p and $1-p$ respectively. If it dies, it does not generate children, but if it survives, it takes k steps in $\mathbb{T}_d$. With each step she takes, we associate a random variable with a Bernoulli distribution with parameter $\frac{d}{d+1}$, which means that, with each jump away from the root, it generates a child and, if it jumps towards the root, it does not generate a child.

Clearly, $BP_{(k)}$ propagates more than $\mathrm{FM}(\mathbb{T}_d, p, k)$, so the critical parameter for $BP_{(k)}$ will be a lower bound for $\mathrm{FM}(\mathbb{T}_d, p, k)$.

The expectation for the offspring is given by

$$\begin{aligned} \mathbb{E}[X_{(k)}] &= \textstyle\sum_{i=0}^{k} i \cdot P(X_k = i) \\ &= \textstyle\sum_{i=0}^{k} i \cdot p\binom{k}{i}\left(\frac{d}{d+1}\right)^i\left(1 - \frac{d}{d+1}\right)^{k-i} \\ &= \frac{dkp}{d+1}. \end{aligned} \tag{3.38}$$

Using the fact that a branching process propagates if and only if its expectation is greater than 1.

We can conclude that a lower limit for the critical parameter for $\mathrm{FM}(\mathbb{T}_d, p, k)$ would be

$$\frac{d+1}{dk} \leq p_c(d,k). \tag{3.39}$$

□

CHAPTER 4

CONCLUSION

4.1 Further literature on related models

Under the name “frog model”, a vast literature has been developed in recent years. Here we highlight two other papers focussing on trees. The first one, (HERMON, 2018), considers the speed of the spread and the final proportion of activated frogs, on finite trees and with frogs having a.s. finite lifetime (not necessarily starting with one sleeping frog *per* site). Although the situation and problem are slightly different than ours here, the paper also gives a nice account of the recent literature for general frog models. The second one, much more related to ours, is (HOFFMAN; JOHNSON; JUNGE, 2017), which proves the transience and recurrence of the frog model on infinite trees when the frogs have an infinite lifetime (and starting with one sleeping frog *per* site). In that paper, the authors use a similar argument as the one used here to get the upper bound, designing a multi-type branching process that dominates the frog model of interest. We will come back to this in Section 4.3 when we will discuss lower bounds.

4.2 The upper bounds and their proofs

Our objective here is to compare the literature’s upper bounds (and their proofs). Specifically, we have to compare (LEBENSZTAYN; MACHADO; POPOV, 2005), (GALLO; RODRIGUEZ, 2018), (LEBENSZTAYN; UTRIA, 2019), and the present work.

As already mentioned, to obtain their upper bounds, all the above-cited works used Step 1 in their proofs: they consider the oriented version of the frog model. The main difference is in Step 2. Lebensztayn, Machado and Popov (2005) used the fact that, finding a solution, in p, for $d^k u_k(p,d) = 1$ inside the interval $(0,1/d)$, yields an upper bound for p_c. This is the content of Theorem 3.1 therein, and it is a fact that can also be concluded from (3.8) above. The problem is that this yields a bound that depends on k, and as they observe, it is not obvious whether this sequence of upper bounds is decreasing, so they cannot make the limiting procedure at this step. Instead, they consider a sequence $v_k \leq u_k$ and work on the asymptotics of the sequence of solutions of $d^k v_k$ as a function of r, or, equivalently, of p (see Lemmas 4.1, 4.2 and 4.3 therein). It is interesting how this approach differs from ours. Formally, what Lebensztayn, Machado and Popov (2005) obtained is that (see Display (4.2) therein)

$$u_k = r^k \prod_{i=1}^{k-1}(1-r^i) + \sum_{j=1}^{k-1} r^{k-j} u_j \prod_{l=1}^{k-j-1}(1-r^l). \tag{4.1}$$

At this point what they decided to do, instead of studying the asymptotic behavior in k, is to take

$$v_k = r^k \prod_{i=1}^{k-1}(1-r) + \sum_{i=1}^{k-1} r^{k-j} u_j \prod_{l=1}^{k-j-1}(1-r) = r^k(1-r)^{k-1} + \sum_{i=1}^{k-1} r^{k-j} u_j (1-r)^{k-j-1}$$

which clearly satisfies $v_k \leq u_k$. Observe that this amounts simply to substitute r^i and r^l by r into the products of (4.1). They can then work asymptotically with the solutions of $d^k v_k = 1$ and obtain their bound $r_c \leq 1 - \sqrt{(d-1)/d}$, which is actually the solution of $dr(2-r) = 1$ for $r \in (0, 1/d)$. It was also remarked by Lebensztayn, Machado and Popov (2005) that substituting the r^i and r^l by r^2 for any $k \geq 2$ would naturally yield tighter, yet more complicated bounds for r_c. Indeed, they state that r_c would be the root $\bar{r}_U$ of $\bar{U}(r) := dr^4 - d(d+1)r^3 + 2dr - 1$ in $(0, 1/d)$. It is precisely what Lebensztayn and Utria (2019) used, yielding yet another refinement of the bound, although very complicated (see Definition 2.1 and Theorem 2.2 therein).

On the contrary, what we do (and what was done by Gallo and Rodriguez (2018)) in Step 2 is that we directly work asymptotically on (4.1) using renewal theory, and this yields Theorem 10. In particular, notice that the bounds in (LEBENSZTAYN; MACHADO; POPOV, 2005) and (LEBENSZTAYN; UTRIA, 2019) are direct consequences of Theorem 10 as well: indeed, substituting, in the products, r^i and r^l by r, yields the polynomial $dr(2-r) = 1$ used in (LEBENSZTAYN; MACHADO; POPOV, 2005), and substituting by r^i and r^l by r^2 (for $i, l \geq 2$) yields the polynomial $\bar{U}(r) := dr^4 - d(d+1)r^3 + 2dr - 1$ used in (LEBENSZTAYN; UTRIA, 2019).

Our upper bound (2.2) is simpler than the one of Lebensztayn and Utria (2019). To see that it is also tighter, it is enough to notice (we did this using Mathematica) that $\bar{U}(r_U) < 0$ and that $\bar{U}$ is monotonically increasing on $(0, 1/d)$ (see Lemma 12 for the definition of r_U).

4.3 The lower bounds and their proofs

The original lower bound of Alves, Machado and Popov (2002a) was based on a simple coupling with a one-type branching process in which each particle could have 0 offspring with probability $1-p$, 1 offspring with probability $p/(d+1)$ (for frogs coming back) and 2 offsprings with probability $pd/(d+1)$. In other words, their coupling took into account the fact that except for the frog initially at the root, an active frog necessarily has at least one adjacent vertex which has already been visited. What we noticed is that after a certain number of steps (almost-surely finite), any activated frog which is not at the tip of the visited cluster has at least two visited neighboring sites. To take this into account, we needed to consider a two-type branching process.

The frog model can be seen as an "infinite types" branching process, and it is naturally possible to improve further our reasoning. The idea would be to compare the frog model with branching processes having more and more types. For instance, a simple modification of the multi-type branching process used by Hoffman, Johnson and Junge (2017) in the proof of their Proposition 19 could be used as well in our setting. This amounts essentially to adding types corresponding to keeping track of two simultaneously activated particles. Doing so, we are able, for instance, to prove that $p_c(2) < p_c(3)$, but it is impossible to get explicit expression holding for any d based on so many types. More generally, we tried such a method up to a certain level and obtained slightly tighter lower bounds with very involved expressions, and we preferred to keep it simple at the cost of precision.

4.4 Concluding remark

Tighter bounds on $\hat{p}_c$ can be obtained from Theorem 10, this is a matter of root approximation for the power series of Theorem 10. However, as pointed out by Table 7, such improvements would be almost insignificant compared to the distance to the lower bounds of p_c since we are limited by the lower bound on $\hat{p}_c$, already very close to the upper bound. This is also made clear by Lemma 12 which yield, for large d's,

$$r_c(d) = \frac{1}{2d} + \frac{1}{16d^2} + \frac{5}{128d^3} + O\left(\frac{1}{d^4}\right)$$

and therefore, together with Display (3.12),

$$\hat{p}_c(d) = \frac{1}{2} + \frac{7}{16d} - \frac{5}{128d^2} + O\left(\frac{1}{d^3}\right).$$

For this reason, the upper bound cannot be further improved based on the comparison with the oriented model. Future works should either improve the lower bound or find another approach to get upper bounds for p_c.

There is yet another motivation behind the search for tighter bounds, which we now explain. Fontes, Machado and Sarkar (2004) proved that the critical parameter of the frog model, in general graphs, is not always a monotonic function of the graph. For homogeneous trees, the question of whether or not $p_c(d)$ is monotonically decreasing in d remains open. Denote by $l(d)$ (*resp.* $r(d)$) the lower (*resp.* upper) bound of (2.2). What we can show (using the software Mathematica) is that $l(d) > r(ad)$ for any $a \geq 1.75$. This implies that $p_c(d) > p_c(ad)$ for those values of a, meaning for instance that $p_c(2) > p_c(4) > p_c(7) > p_c(13)$.... A proof of monotonicity using this argument needs the bounds to be strengthened, and we believe the present paper also provides new insights in this direction.

Table 7 – Lower bound (LB) and upper bound (UB) of p_c from Theorem 2 and lower bound on $\hat{p}_c$ from Lemma 12.

d	**LB on** p_c	**LB on** $\hat{p}_c$	**UB on** $\hat{p}_c$ **and** p_c
2	0.6261364	0.7103674	0.7137989
3	0.5835921	0.6419859	0.6428580
4	0.5625890	0.6071563	0.6074957
5	0.5500385	0.5860557	0.5862210
6	0.5416859	0.5719015	0.5719940
7	0.5357250	0.5617475	0.5618043
8	0.5312564	0.5541074	0.5541448
9	0.5277818	0.5481503	0.5481761
10	0.5250027	0.5433751	0.5433937
20	0.5125001	0.5217793	0.5217815
50	0.5050000	0.5087345	0.5087346
100	0.5025000	0.5043711	0.5043711

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APPENDIX A

PROOF OF THE LEMMA 12

Proof of the Lemma 12. For the lower bound, we use the Newton-Raphson method (see for instance (HILDEBRAND, 1987, Section 10.11)) for root approximation of $L(r) = -r^3 r^6 + d^3 r^5 + d^2 r^3 - 2dr + 1$. The second derivative of $L(r)$ is positive for all $r \in (0, 1/d)$, so $L(r)$ is convex on this interval. Thus, the approximation calculated from the Newton-Raphson method will be smaller than the root $\underline{r}$. Recall that the Newton-Raphson iterative method is started from some value t_0 and for any $n \geq 1$

$$t_n \quad = \quad t_{n-1} - \frac{L(t_{n-1})}{L'(t_{n-1})}.$$

Starting with $t_0 = 0$ and iterating two times, we get after some algebraic manipulations

$$t_2 = \frac{5 - 8d - 16d^2 + 64d^3}{12d - 20d^2 - 48d^3 + 128d^4}.$$

This concludes the proof of the lower bound.

For the upper bound, the proof of the Newton-Raphson method is much longer so we will proceed simply proving that $r_U \geq \bar{r}$, which is faster. We proceed in two steps: we first show that the polynomial U is monotonically decreasing in r around the root of interest, and secondly, that $U(r_U) < 0$ for any $d \geq 2$.

For the first step, notice that

$$U'(r) < 0 \Leftrightarrow 5d^2 r^4 + 3dr^2 - 2 < 0,$$

This is equivalent to $r < \sqrt{1/(5d)}$. Thus it only remains to show that $r_U < \sqrt{1/(5d)}$. To see this, we come back to the proof of Proposition 11 and notice that

$$(1-r)^{k-1} \leq \prod_{i=1}^{k-1} (1 - r^i)$$

leads to an intermediary polynomial $f_{\text{inf}}(r) \leq dr(2-r) \leq \sum_{k \geq 1} d^k r^k \prod_{i=1}^{k-1} (1 - r^i)$. Thus $dr_U(2 - r_U) \leq 1$, meaning that indeed $r_U \leq 1 - \sqrt{(d-1)/d}$ which is strictly smaller than $\sqrt{1/(5d)}$ for any $d \geq 2$.

To prove the second step, we used the software Mathematica to write $U(r_U(d)) < 0$ as

$$-\frac{1}{537824\,d^{12}(8d-5)^5}\bar{U}(d) < 0$$

where

$$\begin{aligned}\bar{U}(d) = {} & 211441664\,d^{14} - 801511424\,d^{13} + 988904672\,d^{12} - 496642048\,d^{11} \\ & + 72342816\,d^{10} + 14993216\,d^{9} - 2579360\,d^{8} - 918064\,d^{7} \\ & + 203840\,d^{6} + 26460\,d^{5} - 7840\,d^{4} - 280\,d^{3} + 140\,d^{2} - 1.\end{aligned}$$

We are done if we prove that $\bar{U}(d) > 0$ for any $d \geq 2$. Using Cauchy bounds, we know that the largest root of this polynomial is smaller or equal to (for $i = 0, \ldots, 14$ we write a_i for the coefficient of order i)

$$m := 1 + \max\left\{\left|\frac{a_{13}}{a_{14}}\right|, \ldots, \left|\frac{a_0}{a_{14}}\right|\right\} = 1 + \frac{988904672}{211441664} < 6.$$

The proof is concluded noticing that for $d = 2, \ldots, 6$, $\bar{U}(d) > 0$.

□


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