UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Unicidade para Equações dos Tipos: Burgers, Kuramoto-Sivashinsky e Schrödinger

Miguel Angel Cuayla Zapata

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Sumário

Resumo

Com base nas estimativas de Carleman e sob certas condições de decaimento exponencial linear provamos, unicidade para equações do tipo: Burgers, Kuramoto-Sivashinsky e Schrödinger.

Palavra-chave: Soluções, Unicidade, Equações dissipativas, Desigualdade de Carleman.

Abstract

Based on Carleman's estimates and under certain conditions of linear exponential decay we prove uniqueness for equations of type: Burgers, Kuramoto-Sivashinsky and Schrödinger.

Key words: Solutions, Uniqueness, Dissipatives equations, Carleman's estimates.

Lista de Símbolos

$\Omega =$ subconjunto não vazio aberto de um espaço euclidiano $\mathbb{R}^n$;

$C^k(\Omega) = \{u : \Omega \to \mathbb{R}; u$ é continuamente k vezes diferenciável$\}$;

$C^\infty(\Omega) = \{u : \Omega \to \mathbb{R}; u$ é infinitamente diferenciável$\}$;

$C_b^k(\Omega) = \left\{u \in C^k(\Omega); \partial^\alpha u \text{ é limitada}, \forall |\alpha| \leq k\right\}$;

$C_b^\infty(\Omega) = \left\{u \in C^\infty(\Omega); \partial^\alpha u \text{ é limitada}, \forall \alpha \in \mathbb{N}^d\right\}$;

$C_0^k(\Omega) = \left\{u \in C^k(\Omega); \partial^\alpha u \text{ se anula no infinito}, \forall |\alpha| \leq k\right\}$;

$C_0^\infty(\Omega) = \left\{u \in C^\infty(\Omega); \partial^\alpha u \text{ se anula no infinito}, \forall \alpha \in \mathbb{N}^d\right\}$;

$C_c^k(\Omega) = \left\{u \in C^k(\Omega); \text{supp}(u) \text{ é compacto}\right\}$;

$C_c^\infty(\Omega) = \{u \in C^\infty(\Omega); \text{supp}(u) \text{ é compacto}\}$;

$L^p(\Omega) = \left\{u : \Omega \to \mathbb{R}; u \text{ é mensurável e } \|u\|_{L^p(\Omega)} = \left[\int_\Omega u(x)^p dx\right]^{\frac{1}{p}} < \infty\right\}$;

$L_{loc}^p(\Omega) = \left\{u : \Omega \to \mathbb{R}; u \text{ é mensurável e } \int_K u(x)^p dx < \infty, \forall K \subset \Omega \text{ compacto}\right\}$;

$W^{s,p}(\Omega)$ representa o espaço de Sobolev de índices s e p;

$H^s(\Omega) = W^{s,2}(\Omega)$;

$C^1([0,\infty), L^2(\mathbb{R})) =$ O fecho do espaço $C_c^1([0,\infty) \times \mathbb{R})$ na norma $\|u\| = \sup_t \|u(t,\cdot)\|_{L^2} + \sup_t \|\partial_t u(t,\cdot)\|_{L^2}$;

$C^1([0,\infty), H^1(\mathbb{R})) =$ O fecho do espaço $C_c^1([0,\infty) \times \mathbb{R})$ na norma $\|u\| = \sup_t \|u(t,\cdot)\|_{H^1} + \sup_t \|\partial_t u(t,\cdot)\|_{H^1}$;

$\widehat{u}$ representa a transformada de Fourier de u com relação a variável espacial x;

$[A, B] = AB - BA$;

$\mathtt{P}(x_1, \ldots, x_n)$ representa um polinômio nas variáveis $(x_1, \ldots, x_n)$;

$A \gtrsim B$ denota que existe uma constante $C > 0$ tal que $A \geq CB$;

$\mathcal{L}(X, Y)$ espaço de funções lineares $f : X \to Y$.

Introdução

Em 1939, Torsten Carleman introduziu em [5] algumas estimativas de energia com peso exponencial para provar resultados de unicidade para algumas equações diferenciais elípticas (PDE) com coeficientes suficientemente regulares em dimensão 2. Este tipo de estimativa, agora chamada como estimativa de Carleman, foi generalizada e sistematizada por Lars Hörmander e outros autores para uma classe maior de operadores diferenciais de dimensão arbitrária ver [15, 16, 46]. A técnica de estimativas de Carleman foi além das equações elípticas e tem sido aplicadas para obter resultados como continuidade única, e no caso de equações do tipo parabólicas torna-se essencial. Também é utilizado na teoria de controle de EDP, para equações de evolução, com soluções num espaço determinado para as condições iniciais.

Nesta tese, primeiramente, estudamos a seguinte equação, chamada aqui de equação do tipo Burgers,

$$\partial_t u + \partial_x^2 u = a(u)\partial_x u, \tag{1}$$

com a uma função real de classe $C^\infty(\mathbb{R})$, tal que $|a(s)| \leq M_1 \left(|s| + |s|^l\right)$, para algum $l \in \mathbb{Z}^+$, $l \geq 2$.

No caso de $a(s) = s$, a equação (1) é chamada equação de Burgers. De fato, fazendo a seguinte mudança de variável $t = -\tau$ obtemos

$$\partial_\tau u + u\partial_x u = \partial_x^2 u; \tag{2}$$

Inicialmente, consideremos

$$f(t,x) = e^{\lambda\varphi(x-bt)}u(t,x), \tag{3}$$

com u solução da equação (1), $\varphi \in C^4(\mathbb{R})$ e b um parâmetro em função de $\lambda > 0$, que será escolhido apropriadamente. A seguir geramos uma nova equação utilizando (3) em (1)

$$\partial_t f - (S_\lambda + A_\lambda) f = a(u)\left(-\lambda\, \varphi' \cdot f + \partial_x f\right), \tag{4}$$

com S_λ e A_λ termos simétricos e anti-simétricos, respectivamente.

Assim, para cada t, sob hipóteses de decaimento em x, obtemos

$$\int_1^\infty \int_{\mathbb{R}} |u(t,x)|^2 \, \mathtt{dxdt} < \infty. \tag{5}$$

Logo, utilizando as propriedades da equação (1), resulta que

$$u \equiv 0, \tag{6}$$

ver Capítulo 2.

No Capítulo 3, consideramos uma equação do tipo Kuramoto-Sivashinsky, a saber

$$\partial_t u + \partial_x^2 u - \partial_x^4 u = a(u)\partial_x u, \tag{7}$$

com $|a(s)| \leq M_1(|s| + |s|^l)$, $a \in C^\infty(\mathbb{R}, \mathbb{R})$, para algum $l \in \mathbb{Z}^+,\ l \geq 2$.
Observamos que a equação (7) é uma modificação da equação introduzida inicialmente por Kuramoto, i.e.,

$$\partial_t v + \nabla_x^2 v + 4\nabla_x^4 v + v\nabla_x v = 0, \tag{8}$$

considerada em [28, 29, 30], para o estudo da turbulência de fase na reação de Belousov-Zhabotinsky. Uma extensão da equação (8), para duas ou mais dimensões espaciais, foi desenvolvida, por Sivashinsky em [10, 34, 37] no estudo da teoria de propagação de chamas no fluxo turbulento de uma mistura de combustível gasoso. A equação de Kuramoto-Sivashinsky tem algumas aplicações em áreas que incluem a representação de uma classe de formação de padrões, ver [40, 42], o modelo de bifurcação e caos, ver [1, 32], na descrição de flutuações da posição da frente de chamas, ver [10, 11], no movimento de fluidos numa parede vertical e reações químicas com oscilações espacialmente uniformes sobre um meio homogêneo.

Uma classe mais geral do modelo da equação (8) foi introduzida e discutida em [14]

$$\partial_t u + \beta u \partial_x u + \alpha \partial_x^2 u + \gamma \partial_x^4 u = 0, \tag{9}$$

com $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$, para $n = 1$.

Observamos que a equação (7) é um caso particular de (9), quando $a(u) = u$.

Os primeiros resultados de existência global de soluções clássicas de (8) podem ser vistos em [33] e [38]. Além disso a equação (8) foi estudada para existência de soluções periódicas em [4] e [27].

Assim, de maneira análoga ao Capítulo 2, estudamos também condições de decaimento na solução para a equação (7), suficientes para que sejam satisfeitas apenas pela solução nula. Demonstraremos que impondo um decaimento exponencial linear relativo a uma função peso de Carleman podemos concluir a unicidade.

Para isso tomamos

$$f(t,x) := e^{\lambda \varphi(r)} u(t,x), \tag{10}$$

com u solução de (7), $r = x - bt$ e b um parâmetro em função de $\lambda > 0$, que será escolhido apropriadamente. Em seguida, fazemos uma descomposição do operador em termos simétricos e anti-simétricos denotados por S_λ e A_λ, respectivamente, para gerar desigualdades para f com uma função peso para f (como foi utilizado em [46]), ou seja

$$\partial_t f - (S_\lambda + A_\lambda) f = a(u) \left(-\lambda \, \partial_x \theta \cdot f + \partial_x f \right);$$

deste modo, podemos garantir a integrabilidade com respeito a t da norma $L^2(\mathbb{R})$ de u, isto é

$$\int_1^\infty \int_{\mathbb{R}} |u(t,x)|^2 \, \mathtt{dxdt} < \infty.$$

Logo, usamos as propriedades da equação (7) e sob hipóteses de decaimento em x garantimos que u é a solução nula.

Observamos que o b acima citado não pode ser arbitrário. De fato, no caso da equação de Kawahara

$$\partial_t u + \delta\partial_x^3 u - \xi\partial_x^5 u = a(u)\partial_x u, \tag{11}$$

com $\delta \in \mathbb{R}\diagdown\{0\}$ e $\xi > 0$, em [8] obteve-se que sob a hipótese $b \geq C\lambda^5$, com C uma constante positiva e com condições de decaimento da solução, a única solução é a nula, e tal hipótese não pode ser retirada.

Note que, a equação KdV é um caso particular da equação de Kawahara quando $\delta = 1$ e $\xi = 0$, ou seja,

$$\partial_t u + \partial_x^3 u = a(u)\partial_x u;$$

esse caso foi estudado em [23]; assim temos que para todo $b \in \mathbb{R}$ e sob condições de decaimento segue que a única solução é a nula.

No Capítulo 4, será aplicada uma técnica baseada na proposta de Escauriaza, Kenig, Ponce e Vega em [6], para demonstrar condições de decaimento para equação linear de Schrödinger de quarta ordem com potencial real

$$\left(\partial_t - \mathtt{i}\left(\partial_x^2 + \partial_x^4\right)\right) u = \mathtt{i} V(t,x)u, \quad x \in \mathbb{R},\ t \in [0,\infty), \tag{12}$$

com $u : [0,\infty) \times \mathbb{R} \to \mathbb{C}$, e demonstrar que a única solução é nula.

A equação (12) foi estudada inicialmente por Karpman em [22] e Karpman-Shagalov em [21],

$$\mathtt{i}\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \lambda\frac{\partial^4\psi}{\partial x^4} + |\psi|^2 = 0, \tag{13}$$

considerada de maneira unidimensional, para o estudo da influência de dispersão de ordem maior em ondas solitárias em fenômenos de instabilidade e colapso .

Além disso, um estudo do problema de valor inicial para equação de Schrödinger de quarta ordem com diferentes termos não lineares foi feito em [24], [18], [13], [19], [44], [43] e [45].

A presente tese está organizada da seguinte forma: no Capítulo 1, revisamos alguns conceitos básicos e definições, como por exemplo, definição e propriedades dos espaços L^p

e Sobolev, além de alguns resultados importantes para o desenvolvimento do trabalho. No Capítulo 2, apresentamos propriedades e um teorema de unicidade para a equação do tipo Burgers. No Capítulo 3, apresentamos propriedades da solução da equação do tipo Kuramoto-Sivashinsky, assim como desigualdades com uma função peso adequada para f e finalmente apresentamos um resultado de unicidade para ela. No Capítulo 4, apresentamos um teorema de unicidade para a equação linear de Schrödinger de quarta ordem com potencial real. Finalizamos, apresentando um apêndice, com as demonstrações de alguns resultados utilizados e que foram omitidas ao longo do texto.

Capítulo 1

Preliminares

Neste capítulo, apresentamos alguns conceitos e resultados que serão úteis nas demonstrações ao longo do trabalho.

1.1 Espaço $L^p(\Omega)$

O espaço L^p forma uma classe de espaços de Banach de funções f com a norma definida em termos da integral de f.

Considere o espaço mensurável $(X, \mathcal{M}, \mu)$[1]. Se f é uma função mensurável em X e $1 \leq p \leq \infty$, definimos

$$\|f\|_{L^p(X)} = \left(\int_X |f|^p \, \mathtt{d}\mu \right)^{1/p}, \quad 1 \leq p < \infty,$$

e

$$\|f\|_{L^\infty(X)} = \mathtt{supess}_{x \in X} \{|f(x)|\}, \quad \text{quando } p = \infty.$$

(permitimos a possibilidade que $\|f\|_{L^p(X)} = \infty$), e definimos

$$L^p(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{C} : f \text{ é mensurável e } \|f\|_{L^p(X)} < \infty\}, \tag{1.1}$$

abreviamos $L^p(X, \mathcal{M}, \mu)$ por $L^p(X)$, ou simplesmente L^p quando isto não cause confusão.

Proposição 1 (Desigualdade de Hölder). *Suponha que $1 \leq p \leq \infty$ e $\frac{1}{p} + \frac{1}{p'} = 1$(isto é, $p' = p/(p-1)$). Se f e g são funções mensuráveis em X, então*

$$\|fg\|_{L^1(X)} \leq \|f\|_{L^p(X)} \|g\|_{L^{p'}(X)}.$$

[1] μ denota uma medida

Ver [9, pág. 182].

Proposição 2 (Desigualdade de Minkowsky). *Se* $1 \leq p \leq \infty$ *e* $f,\ g \in L^p(X)$, *então*

$$\|f + g\|_{L^p(X)} \leq \|f\|_{L^p(X)} + \|g\|_{L^p(X)}\,. \tag{1.2}$$

Ver [9, pág. 183].

Proposição 3. $C_c^\infty(\mathbb{R}^d)$ *é denso em* $L^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$.

Ver [9, pág. 245].

Definição 1 (Derivada fraca). *Seja* $\Omega \subset \mathbb{R}^d$, α *multi-índice, dizemos que a distribuição* v *é a derivada fraca de ordem* α *de* u *se* $v \in L^1_{loc}(\Omega)$ *e*

$$\int_\Omega u \partial^\alpha \varphi \ \mathtt{dx} = (-1)^{|\alpha|} \int_\Omega v\varphi \ \mathtt{dx},$$

para todo $\varphi \in C_c^\infty(\Omega)$.

1.2 Espaços de Sobolev

Nesta seção, apresentamos alguns resultados dos espaços de Sobolev que facilitarão a apresentação dos resultados desta tese. As demonstrações dos resultados aqui enunciados podem ser encontrados nas referências [2, 3, 9, 17, 31, 39].

Introduzimos o espaço de Sobolev de ordem inteira e algumas das suas mais importantes propriedades. Estes espaços podem ser definidos em qualquer $\Omega \subset \mathbb{R}^d$ e são subespaços de $L^p(\Omega)$.

1.2.1 Definição e propriedades básicas

Definição 2 (Norma de Sobolev). *Sejam* $\|\cdot\|_{m,p}$ *com* m *um inteiro positivo e* $1 \leq p \leq \infty$, *definido por*

$$\|v\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha v\|_p^p \right)^{1/p}, \quad se \ \ 1 \leq p < \infty, \tag{1.3}$$

$$\|v\|_{m,\infty} = \max_{0 \leq |\alpha| \leq m} \|\partial^\alpha v\|_\infty\,, \quad se \ \ p = \infty. \tag{1.4}$$

para qualquer função v onde o lado direito faça sentido, $\|\cdot\|_p$ denota a norma $L^p(\Omega)$.

Definição 3 (Espaço de Sobolev). *Para qualquer inteiro $m \geq 0$ e $1 \leq p \leq \infty$ consideremos três espaços vetoriais com norma $\|\cdot\|_{m,p}$.*

(a) $H^{m,p}(\Omega) \equiv$ o completamento de $\{v \in C^m(\Omega) \ : \ \|v\|_{m,p} < \infty\}$ com respeito à norma $\|\cdot\|_{m,p}$,

(b) $W^{m,p}(\Omega) \equiv \{v \in L^p(\Omega) \ : \ \partial^\alpha v \in L^p(\Omega)$ para $0 \leq |\alpha| \leq m\}$, onde $\partial^\alpha v$ é a derivada parcial fraca na Definição 1 , e

(c) $W_0^{m,p}(\Omega) \equiv$ o fecho de $C_c^\infty(\Omega)$ no espaço $W^{m,p}(\Omega)$.

Assim, cada um destes espaços junto a norma de Sobolev, são chamados de espaços de Sobolev sobre Ω. Claramente $W_0^{0,p}(\Omega) = L^p(\Omega)$, pela densidade de $C_c^\infty(\Omega)$ em $L^p(\Omega)$, se $1 \leq p < \infty$.

Para qualquer m inteiro não negativo, temos o seguinte mergulho

$$W_0^{m,p} \hookrightarrow W^{m,p}(\Omega) \hookrightarrow L^p(\Omega).$$

Seguirá do Teorema 2, ver seção 1.2.2, que $H^{m,p}(\Omega) = W^{m,p}(\Omega)$ para cada Ω.

1.2.2 O espaço de Sobolev $W^{-m,p'}(\Omega)$

Nesta seção, consideramos m inteiro e $1 < p < \infty$ fixos, p' tal que $\frac{1}{p'} + \frac{1}{p} = 1$ e

$$\langle u, v \rangle = \int_\Omega u(x)v(x)\mathtt{dx},$$

para quaisquer funções u e v, para o qual o lado direito faça sentido.

Dado o inteiro $n \geq 1$ e $m \geq 0$, seja $N = N(n,m)$ o número de multi-índice $\alpha = (\alpha_1, \ldots, \alpha_n)$ com $|\alpha| \leq m$. Para cada multi-índice α seja $\Omega_\alpha = \Omega$, em uma cópia diferente de $\mathbb{R}^n$, assim os Ω_α serão mutuamente disjuntos, $\Omega^{(m)} = \cup_{|\alpha| \leq m} \Omega_\alpha$.

Definição 4 (Dual de $L^p(\Omega^{(m)})$). *Para cada $L \in (L^p(\Omega))'$, com $1 \leq p < \infty$, existe um único $v \in L^{p'}(\Omega^{(m)})$ tal que para cada $u \in L^p(\Omega^{(m)})$,*

$$L(u) = \int_{\Omega^{(m)}} u(x)v(x)\mathtt{dx} = \sum_{|\alpha| \leq m} \int_{\Omega_\alpha} u_\alpha(x)v_\alpha(x)\mathtt{dx} = \sum_{|\alpha| \leq m} \langle u_\alpha, v_\alpha \rangle,$$

com u_α , v_α as restrições de u e v, respectivamente, para cada Ω_α. Mais ainda

$$\|L\|_{\left(L^p(\Omega^{(m)})\right)'} = \|v\|_{L^{p'}(\Omega^{(m)})}.$$

Assim,

$$\left(L^p(\Omega^{(m)})\right)' \equiv L^{p'}(\Omega^{(m)}).$$

Teorema 1. *Seja $1 \leq p < \infty$. Para cada $L \in (W^{m,p}(\Omega))'$, existe um elemento $v \in L^{p'}(\Omega^{(m)})$, tal que a restrição de v em $\Omega^{(m)}$ é v_α, temos para todo $u \in W^{m,p}(\Omega)$ que*

$$L(u) = \sum_{0 \leq |\alpha| \leq m} \langle \partial^\alpha u, v_\alpha \rangle ,$$

Além disso,

$$\|L\|_{(W^{m,p}(\Omega))'} = \inf \|v\|_{L^{p'}(\Omega^{(m)})} = \min \|v\|_{L^{p'}(\Omega^{(m)})}.$$

Ver [2, pág. 63].

Teorema 2. *Se $1 \leq p < \infty$, então*

$$H^{m,p}(\Omega) = W^{m,p}(\Omega). \tag{1.5}$$

Ver [2, pág. 67].

Corolário 1. $W_0^{m,p}(\mathbb{R}^d) = W^{m,p}(\mathbb{R}^d)$.

Ver [2, pág. 70].
Observamos que $C_c^\infty(\mathbb{R}^d)$ é denso em $W^{m,p}(\mathbb{R}^d)$. E no caso $p = 2$, denotamos $H^{m,p}(\Omega)$ e $W^{m,p}(\Omega)$ por $H^m(\Omega)$ e $W^m(\Omega)$, respectivamente.

Definição 5. *Seja s um número real. O espaço Sobolev $H^s(\mathbb{R}^d)$ consiste das distribuições temperadas u tal que $\widehat{u} \in L^2_{loc}(\mathbb{R}^d)$ e*

$$\|u\|^2_{H^s} := \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s |\widehat{u}(\xi)|^2 \, \mathtt{d}\xi < \infty. \tag{1.6}$$

Ver [3, pág. 38].

Proposição 4. *O espaço $H^s(\mathbb{R}^d)$ com o produto escalar*

$$\langle u, v \rangle_{H^s} := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} \mathtt{d}\xi,$$

é um espaço Hilbert.

Ver [3, pág. 38].

A família dos espaços H^s é decrescente com respeito a s. Além disso, temos a seguinte proposição.

Proposição 5. *Se $s_0 \leq s \leq s_1$, então temos que*

$$\|u\|_{H^s} \leq \|u\|_{H^{s_0}}^{1-\theta} \|u\|_{H^{s_1}}^{\theta} \quad \textit{com} \quad s = (1-\theta)s_0 + \theta s_1.$$

Ver [3, pág. 38].

Quando s é um inteiro não negativo, usando a fórmula de Fourier-Plancherel, segue que o espaço H^s coincide com o espaço das funções, $u \in L^2$ e $\partial^\alpha u \in L^2$ para cada $\alpha \in \mathbb{N}^d$ com $|\alpha| \leq s$, ou seja, como descreve-se na seguinte proposição.

Proposição 6. *Seja s um inteiro não negativo, então*

$$H^s(\mathbb{R}^d) := \left\{u \in L^2(\mathbb{R}^d) : \ \partial^\alpha u \in L^2(\mathbb{R}^d), \ \forall\, |\alpha| \leq s\right\}, \tag{1.7}$$

com a seguinte norma

$$\|u\|_{H^s(\mathbb{R}^d)} := \left(\sum_{|\alpha|\leq s} \|\partial^\alpha u\|_{L^2(\mathbb{R}^d)}^2\right)^{1/2}, \tag{1.8}$$

ver [39, pág. 2]. Devemos notar que a norma (1.8) é equivalente a (1.6).

No caso em que s é um inteiro negativo, o espaço H^s é descrito pela seguinte proposição.

Proposição 7. *Seja s um inteiro positivo. O espaço $H^{-s}(\mathbb{R}^d)$ consiste das distribuições que são as somas de funções $L^2(\mathbb{R}^d)$ e de derivadas de ordem menor ou igual a s de funções $L^2(\mathbb{R}^d)$, isto é*

$$H^{-s}(\mathbb{R}^d) := \left\{u \in \mathcal{D}' : u = \sum_{|\alpha|\leq s} \partial^\alpha g_\alpha, \ g_\alpha \in L^2(\mathbb{R}^d), \ \forall\, |\alpha| \leq s\right\},$$

com a seguinte norma, que é equivalente com (1.6),

$$\|u\|_{H^{-s}(\mathbb{R}^d)} := \inf\left\{\left(\sum_{|\alpha|\leq s} \|g_\alpha\|_{L^2(\mathbb{R}^d)}\right)^{1/2} : u = \sum_{|\alpha|\leq s} \partial^\alpha g_\alpha, \ g_\alpha \in L^2(\mathbb{R}^d), \quad \forall\, |\alpha| \leq s\right\}. \tag{1.9}$$

Proposição 8. *Se $s > d/2$, então para cada $u \in H^s(\mathbb{R}^d)$, temos que*

$$\|u\|_{L^\infty} < \infty.$$

Demonstração: Basta demonstrar que $\widehat{u} \in L^1(\mathbb{R})$. De fato

$$\begin{aligned}
\|\widehat{u}\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}^d} |\widehat{u}(\xi)| \, \mathtt{d}\xi \\
&= \int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^{s/2} |\widehat{u}(\xi)| \left(1+|\xi|^2\right)^{-s/2} \mathtt{d}\xi \\
&\le \left(\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^{s} |\widehat{u}(\xi)|^2 \, \mathtt{d}\xi\right)^{1/2} \left(\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^{-s} \mathtt{d}\xi\right)^{1/2} \\
&\le C \|u\|_{H^s(\mathbb{R}^d)} \, .
\end{aligned} \tag{1.10}$$

Logo, aplicando transformada de Fourier inversa e (1.10), temos

$$\begin{aligned}
|u(x)| &= (2\pi)^{-1} \left| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \, \widehat{u}(\xi) \mathtt{d}\xi \right| \\
&\le C \|u\|_{H^s(\mathbb{R}^d)} \, , \quad \forall x \in \mathbb{R}^d.
\end{aligned}$$

■

Teorema 3 (Mergulho do espaço Sobolev)**.** *Suponhamos que $s > k + \frac{d}{2}$.*

a. Se $u \in H^s(\mathbb{R}^d)$ então $\widehat{(\partial^\alpha u)} \in L^1$ e $\left\|\widehat{\partial^\alpha u}\right\|_{L^1(\mathbb{R}^d)} \le C \|u\|_{H^s(\mathbb{R}^d)}$ para $|\alpha| \le k$, onde C depende somente de $k - s$.

b. $H^s(\mathbb{R}^d) \subset C_0^k(\mathbb{R}^d)$ e além disso, para todo $u \in H^s(\mathbb{R}^d)$ existe uma constante $C > 0$ tal que

$$\sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^2(\mathbb{R}^d)} \le C \|u\|_{H^s(\mathbb{R}^d)} \, .$$

Ver [9, pág. 302].

Proposição 9. *Sejam $F \in C^\infty(\mathbb{R}^d)$ e $F(0) = 0$ e $s > d/2$. Então, para $u \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, tem-se*

$$\|F(u)\|_{H^s(\mathbb{R}^d)} \le C(s) \left(\|u\|_{L^\infty(\mathbb{R}^d)}\right) \left(1 + \|u\|_{H^s(\mathbb{R}^d)}\right).$$

Ver [39, pág. 12].

Teorema 4. *Sejam $1 \leq p < d$ e $\frac{1}{p} + \frac{1}{p'} = 1$. Então, existe $C > 0$ tal que*

$$\|u\|_{L^{p'}(\mathbb{R}^d)} \leq C \|Du\|_{L^p(\mathbb{R}^d)}, \forall u \in C_c^1(\mathbb{R}^d),$$

sendo C uma constante de depende de p e d.

Ver [7, pág. 265].

Proposição 10. *Sejam $g \in C^1(\mathbb{R})$ e $\varepsilon > 0$. Então, existe uma constante $C > 0$, tal que*

$$\int_{|x|\leq\varepsilon} |g(x)|^2 \,\mathtt{dx} \leq \varepsilon C \left(\int_{\varepsilon\leq|x|\leq 2\varepsilon} |g(x)|^2 \,\mathtt{dx} + \int_{|x|\leq 2\varepsilon} |g'(x)|^2 \,\mathtt{dx} \right). \tag{1.11}$$

Demonstração: Seja $\beta : \mathbb{R} \to [0, \infty)$ e $\beta \in C^1(\mathbb{R})$

$$\beta(x) = \begin{cases} 1, & |x| \leq \varepsilon \\ 0, & 2\varepsilon \leq |x| \end{cases};$$

segue da definição de β que $\mathtt{supp}(\beta) \subset \{x \in \mathbb{R} : |x| \leq 2\varepsilon\}$ e do Teorema 4 para $\beta g \in C_c^1(\mathbb{R})$; assim

$$\begin{aligned}
\int_{|x|\leq\varepsilon} |g(x)|^2 \,\mathtt{dx} \leq & \int_{|x|\leq 2\varepsilon} |\beta(x)g(x)|^2 \,\mathtt{dx} \\
\leq & 4\varepsilon \int_{|x|\leq 2\varepsilon} \left|(\beta(x)g(x))'\right|^2 \,\mathtt{dx} \\
\leq & 2^3\varepsilon \left(\int_{|x|\leq 2\varepsilon} |\beta'(x)g(x)|^2 \,\mathtt{dx} + \int_{|x|\leq 2\varepsilon} |\beta(x)g'(x)|^2 \,\mathtt{dx} \right) \\
= & 2^3\varepsilon \left(\int_{\varepsilon\leq|x|\leq 2\varepsilon} |\beta'(x)g(x)|^2 \,\mathtt{dx} + \int_{|x|\leq 2\varepsilon} |\beta(x)g'(x)|^2 \,\mathtt{dx} \right) \\
\leq & 2^3\varepsilon \left(\max_{\varepsilon\leq|x|\leq 2\varepsilon} \{|\beta'(x)|^2\} \int_{\varepsilon\leq|x|\leq 2\varepsilon} |g(x)|^2 \,\mathtt{dx} + \int_{|x|\leq 2\varepsilon} |\beta(x)g'(x)|^2 \,\mathtt{dx} \right) \\
\leq & \varepsilon\, C_{\beta,\varepsilon} \left(\int_{\varepsilon\leq|x|\leq 2\varepsilon} |g(x)|^2 \,\mathtt{dx} + \int_{|x|\leq 2\varepsilon} |g'(x)|^2 \,\mathtt{dx} \right),
\end{aligned}$$

com $C_{\beta,\varepsilon} = \max\left\{ 2^3 \max_{\varepsilon\leq|x|\leq 2\varepsilon} \{|\beta'(x)|^2\}, 2^3 \right\}$. ■

Capítulo 2

Uma equação do tipo Burgers

Nesta capítulo, trabalhamos com a equação do tipo Burgers

$$\partial_t u + \partial_x^2 u = a(u)\partial_x u, \quad x \in \mathbb{R},\ t \in [0,\infty),\ u \in C^1([0,\infty), H^1(\mathbb{R})), \tag{2.1}$$

em que a é uma função real de classe $C^\infty(\mathbb{R})$, $a(0) = 0$ e

$$|a(s)| \leq M_1 \left(|s| + |s|^l\right),\ \text{para algum } l \geq 2,\ l \in \mathbb{Z}^+, \tag{2.2}$$

com M_1 uma constante positiva. Se u for solução fraca da equação (2.1) impomos condições de decaimento exponencial linear tanto para u como para as derivadas de u.

A seguir, enunciamos o teorema principal deste capítulo.

Teorema 5. *Seja $u \in C^1([0,\infty), H^1(\mathbb{R}))$ uma solução de (2.1), com $a(s)$ satisfazendo (2.2). Se existem $\lambda_0 \geq 1$, $c_0 > 0$, $M_2 > 0$ e $M_3 > 0$, tais que*

$$\sup_{t\geq 0} \|u(t,\cdot)\|^2_{H^1(\mathbb{R})} \leq M_2, \tag{2.3}$$

$$\sup_{t\geq 0} \int_{\mathbb{R}} e^{|x-bt|} u^2(t,x)\mathtt{dx} \leq M_3, \tag{2.4}$$

$$\sup_{t\geq 0} \int_{\mathbb{R}} e^{2\lambda|x-bt|} (\partial_x^j u)^2(t,x)\mathtt{dx} < \infty,\ \textit{com } j \in \{0,1,2\}, \tag{2.5}$$

para

$$\lambda \geq \max\left\{\lambda_0, c_0 M_1^2 \left(1 + M_2^{2(l-1)}\right)\left(M_3 + (M_2 M_3)^{1/2}\right)\right\}\ \textit{e}\ \ b \geq 3\lambda, \tag{2.6}$$

então,

$$u \equiv 0. \tag{2.7}$$

Agora, para o caso $b = 0$, necessitamos acrescentar uma hipótese adicional.

Teorema 6. *Seja $u \in C^1([0,\infty), H^1(\mathbb{R}))$ solução de (2.1), com $a(u)$ uma função real, satisfazendo (2.2).Se existem $\lambda_0 \geq 1$, $c_0 > 0$, $M_2 > 0$, $M_3 > 0$ e $\delta > 0$, tais que*

$$u(t,x) = 0, \quad \forall\, t \in [0,\infty) \text{ e } |x| \leq \delta, \tag{2.8}$$

$$\sup_{t\geq 0} \|u(t,\cdot)\|^2_{H^1(\mathbb{R})} \leq M_2, \tag{2.9}$$

$$\sup_{t\geq 0} \int_{\mathbb{R}} e^{|x|} u^2(t,x)\mathtt{dx} \leq M_3, \tag{2.10}$$

$$\sup_{t\geq 0} \int_{\mathbb{R}} e^{2\lambda|x|} (\partial_x^j u)^2(t,x)\mathtt{dx} < \infty, \quad \text{com } j \in \{0,1,2\}, \tag{2.11}$$

para

$$\lambda \geq \max\left\{\lambda_0, c_0 M_1^2 \left(1 + M_2^{2(l-1)}\right)\left(M_3 + (M_2 M_3)^{1/2}\right)\right\}, \tag{2.12}$$

então,

$$u \equiv 0.$$

Observação 1. *Na hipótese do Teorema 5 (respectivamente o Teorema 6), se M_3 for suficientemente pequeno, podemos substituir (2.6) (respectivamente (2.12)) por $\lambda > \max\{\lambda_0, c_0 M_1^2\}$ e obter o mesmo resultado.*

Note que o Teorema 6 tem uma hipótese adicional (2.8), em comparação do Teorema 5. Antes de iniciarmos a demonstração desses teoremas, apresentamos alguns resultados necessários que serão utilizados ao longo deste capítulo. Mais precisamente, usando as hipóteses (2.3)-(2.5) e definindo uma nova função $f := f(t,x) = e^{\lambda\theta(t,x)}u(t,x)$ desenvolveremos desigualdades satisfatórias para f.

2.1 Termos simétricos e anti-simétricos

Esta seção será dedicada à construção de uma nova equação semilinear para uma função na forma $f = e^{\lambda\theta}u$, com u solução da equação (2.1). Definimos S_λ e A_λ como sendo os termos simétricos e anti-simétricos, respectivamente, para a nova equação semilinear e assim apresentamos uma função φ_0 com certas propriedades específicas que permitirá definir θ.

Seja u solução da equação (2.1), consideramos a seguinte função

$$f(t,x) := e^{\lambda\theta(t,x)}u(t,x), \tag{2.13}$$

com $\theta(t,x)$ uma função suave a ser construída posteriormente.

A seguir, substituindo (2.13) em (2.1), temos

$$\begin{aligned}
\partial_t f =& e^{\lambda\theta}\lambda\partial_t\theta \cdot u + e^{\lambda\theta}\partial_t u \\
=& e^{\lambda\theta}\lambda\partial_t\theta \cdot (e^{-\lambda\theta}f) + e^{\lambda\theta}(a(u)\partial_x u - \partial_x^2 u) \\
=& \lambda\partial_t\theta \cdot f - e^{\lambda\theta}\partial_x^2\left(e^{-\lambda\theta}f\right) + e^{\lambda\theta}a(e^{-\lambda\theta}f)\partial_x\left(e^{-\lambda\theta}f\right) \\
=& \lambda\partial_t\theta \cdot f - e^{\lambda\theta}\partial_x^2\left(e^{-\lambda\theta}f\right) + a(e^{-\lambda\theta}f)\left(-\lambda\partial_x\theta \cdot f + \partial_x f\right);
\end{aligned}$$

disto, temos

$$\partial_t f - \lambda\partial_t\theta \cdot f + e^{\lambda\theta}\partial_x^2\left(e^{-\lambda\theta}f\right) = F, \tag{2.14}$$

com $F := a(e^{-\lambda\theta}f)\left(-\lambda\partial_x\theta \cdot f + \partial_x f\right)$.

Vamos, associar lado esquerdo de (2.14) os termos simétrico e anti-simétrico com a finalidade de gerar desigualdades com uma função peso para f (como foi feito em [46]). Para isto, definimos os seguintes operadores que estão bem definidos pela escolha de θ :

$$\begin{array}{rcl}
R_1 : [0,\infty) & \longrightarrow & \mathcal{L}(H^2(\mathbb{R}), L^2(\mathbb{R})) \\
t & \longmapsto & \left(g(t,\cdot) \mapsto e^{\lambda\theta(t,\cdot)}\partial_x^2(e^{-\lambda\theta(t,\cdot)}g(t,\cdot))\right)
\end{array}, \tag{2.15}$$

e

$$\begin{array}{rcl}
R_1^* : [0,\infty) & \longrightarrow & \mathcal{L}(H^2(\mathbb{R}), L^2(\mathbb{R})) \\
t & \longmapsto & \left(g(t,\cdot) \mapsto e^{-\lambda\theta(t,\cdot)}\partial_x^2(e^{\lambda\theta(t,\cdot)}g(t,\cdot))\right)
\end{array}. \tag{2.16}$$

Usando (2.15) e (2.16), temos o seguinte Lema.

Lema 2.1. *Sejam $g,\ h \in H^2(\mathbb{R})$, segue que*

$$\int_{\mathbb{R}} (R_1(t)g)(x)h(x)\mathtt{dx} = \int_{\mathbb{R}} g(x)(R_1^*(t)h)(x)\mathtt{dx}. \tag{2.17}$$

Demonstração: Utilizando $H^2(\mathbb{R}) \subset C_0^1(\mathbb{R})$ e integração por partes, resulta a demonstração do Lema. ∎

Logo, para definir as partes simétricas e anti-simétricas, tomamos os seguintes termos:

$$
\begin{aligned}
S_\lambda =& -\lambda\partial_t\theta + \frac{R_1 + R_1^*}{2} \\
=& -\lambda\partial_t\theta + \frac{\left(\lambda^2\left(\partial_x\theta\right)^2 - \lambda\partial_x^2\theta - 2\lambda\partial_x\theta\partial_x + \partial_x^2\right) + \left(\lambda^2\left(\partial_x\theta\right)^2 + \lambda\partial_x^2\theta + 2\lambda\partial_x\theta\partial_x + \partial_x^2\right)}{2} \\
=& -\lambda\partial_t\theta + \lambda^2\left(\partial_x\theta\right)^2 + \partial_x^2,
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
\widetilde{A}_\lambda =& \frac{R_1 - R_1^*}{2} \\
=& \frac{\left(\lambda^2\left(\partial_x\theta\right)^2 - \lambda\partial_x^2\theta - 2\lambda\partial_x\theta\partial_x + \partial_x^2\right) - \left(\lambda^2\left(\partial_x\theta\right)^2 + \lambda\partial_x^2\theta + 2\lambda\partial_x\theta\partial_x + \partial_x^2\right)}{2} \\
=& -\lambda\left(\partial_x^2\theta\right) - 2\,\lambda\left(\partial_x\theta\right)\partial_x,
\end{aligned}
\tag{2.19}
$$

$$
A_\lambda = \partial_t + \widetilde{A}_\lambda. \tag{2.20}
$$

Então, utilizando (2.18)-(2.20), tomamos S_λ como a parte simétrica e A_λ como a parte anti-simétrica.

Logo, segue do Lema 2.1 que:

$$
\int_{\mathbb{R}} (S_\lambda(t)g)(x)h(x)\mathtt{dx} = \int_{\mathbb{R}} g(x)(S_\lambda(t)h)(x)\mathtt{dx}, \quad \forall g, h \in H^2(\mathbb{R}), \tag{2.21}
$$

e

$$
\int_{\mathbb{R}} (\widetilde{A}_\lambda(t)g)(x)h(x)\mathtt{dx} = -\int_{\mathbb{R}} g(x)(\widetilde{A}_\lambda(t)h)(x)\mathtt{dx}, \quad \forall g, h \in H^2(\mathbb{R}). \tag{2.22}
$$

Em seguida, precisamos de uma função adequada para gerar desigualdades satisfatórias para f, com propriedades muito específicas, para isto apresentamos o seguinte lema.

Lema 2.2. *Existe uma função $\varphi_0 \in C^4(\mathbb{R})$ par e não negativa com as seguintes propriedades:*

(i) $\inf\limits_{|r|\geq 1} |\varphi_0'(r)| \geq 1$,

(ii) $|\varphi_0'(r)| \leq 3, \qquad \forall\, r \in \mathbb{R}$,

(iii) $0 < \varphi_0^{(2)}(r) \leq 1, \qquad \forall\, r \in \mathbb{R},$

(iv) $\varphi_0^{(2)}(r) = 1, \qquad \forall |r| \leq \frac{3}{2},$

(v) $\varphi_0^{(2)}$ *é decrescente para* $r > \frac{3}{2}$,

(vi) $\exists\, C_1 > 0$ *constante, tal que*

$$e^{\varphi_0(r)} \leq C_1 e^{3|r|}, \qquad \forall\, r \in \mathbb{R}, \tag{2.23}$$

$$e^{\frac{-\varphi_0(r)}{6}} \leq C_1 \varphi_0^{(2)}(r), \qquad \forall\, r \in \mathbb{R}, \tag{2.24}$$

$$\left|\frac{d^k \varphi_0(r)}{dr^k}\right| \leq C_1 \varphi_0^{(2)}(r), \qquad \forall\, r \in \mathbb{R}, \qquad k = 3, 4. \tag{2.25}$$

A construção de φ_0 pode ser vista no Apêndice A.1.

A partir das propriedades de φ_0 , apresentadas no Lema 2.2, consideremos as seguintes escolhas $\varphi = \frac{\varphi_0}{3}$ e

$$\theta(t, x) = \varphi(x - bt), \tag{2.26}$$

com $b > 0$ que será escolhido na Proposição 12.

Logo, utilizando as propriedades da função φ seguem as propriedades de grande utilidade para a demonstração do Teorema 5.

2.2 Propriedades da solução

Nesta seção, apresentamos as propriedades da solução. Inicialmente consideramos $u(t, \cdot) \in H^1(\mathbb{R})$, utilizando as hipóteses de decaimento (do Teorema 5) obtemos maior regularidade de u, isto é, $u(t, \cdot) \in H^2(\mathbb{R}),\ \forall\, t \in [0, \infty)$. A seguir faremos isto com mais detalhe.

Proposição 11. *Seja u solução de* (2.1) *tal que* $u \in C^1([0, \infty), H^1(\mathbb{R}))$ *e* $\varphi \in C^4(\mathbb{R})$ *definido no Lema 2.2.*

(i) Se u satisfaz (2.3), *então* $u \in L^\infty([0, \infty) \times \mathbb{R})$.

(ii) Se u satisfaz (2.5), *então* $u(t, \cdot)$, $e^{\lambda\varphi(\cdot - bt)} u(t, \cdot) \in H^2(\mathbb{R})$, $\forall\, t \in [0, \infty)$.

Demonstração: Vamos demonstrar (i). Pelo Teorema 3 segue que $\widehat{u} \in L^1$. Logo, aplicando a transformada de Fourier inversa, temos que existe uma constante $C > 0$ para todo $t \in [0, \infty)$,

$$\begin{aligned} |u(t,x)| &= (2\pi)^{-1} \left| \int_{\mathbb{R}} e^{ix\xi} \widehat{u}(t,\xi) d\xi \right| \\ &\leq (2\pi)^{-1} \left\| \widehat{u}(t,\cdot) \right\|_{L^1(\mathbb{R})} \\ &\leq C \left\| u(t,\cdot) \right\|_{H^1(\mathbb{R})}, \qquad \forall \ x \in \mathbb{R}. \end{aligned}$$

Logo, usando (2.3) segue

$$\|u\|_{L^\infty([0,\infty)\times\mathbb{R})} \leq C \sup_{t\in[0,\infty)} \|u(t,\cdot)\|_{H^1(\mathbb{R})} \leq C M_2^{\frac{1}{2}}.$$

Agora vamos demonstrar (ii). De (2.5) e (2.23) temos

$$\begin{aligned} \int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} (\partial_x^k u)^2(t,x) \mathtt{dx} &\leq C_1^{2\lambda} \int_{\mathbb{R}} e^{2\lambda|x-bt|} (\partial_x^k u)^2(t,x) \mathtt{dx} \\ &< \infty, \quad k = 0, 1, 2. \end{aligned} \tag{2.27}$$

Agora, como $\lambda > 0$, φ uma função positiva e vale (2.27), segue que

$$\int_{\mathbb{R}} \left(\partial_x^k u(t,x)\right)^2 \mathtt{dx} \leq \int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} (\partial_x^k u(t,x))^2 \mathtt{dx} < \infty, \quad k = 0, 1, 2.$$

Daí, $u(t,\cdot) \in H^2(\mathbb{R}),\ \forall\, t \in [0,\infty)$.

Por outro lado, como

$$\begin{aligned} \partial_x^k \left(e^{\lambda\varphi(x-bt)} u\right) &= \sum_{j=0}^{k} \binom{k}{j} \partial_x^j \left(e^{\lambda\varphi(x-bt)}\right) \partial_x^{k-j} u \\ &= e^{\lambda\varphi(x-bt)} \partial_x^k u + \sum_{j=1}^{k} \binom{k}{j} \mathtt{P}(\lambda, \varphi', \varphi^{(2)}, \ldots, \varphi^{(j)}) e^{\lambda\varphi(x-bt)} \partial_x^{k-j} u, \end{aligned}$$

utilizando as propriedades de φ no Lema 2.2, obtemos uma constante $\widetilde{C} > 0$ tal que

$$\int_{\mathbb{R}} \left|\partial_x^k \left(e^{\lambda\varphi(x-bt)} u(t,x)\right)\right|^2 \mathtt{dx} \leq \widetilde{C} \sum_{j=0}^{k} \int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} (\partial_x^j u)^2(t,x) \mathtt{dx}, \quad k = 0, 1, 2.$$

Portanto, por (2.27), concluímos que $e^{\lambda\varphi(\cdot-bt)} u(t,\cdot) \in H^2(\mathbb{R})$, $\forall\, t \in [0,\infty)$.

■

2.3 O comutador $[S_\lambda, A_\lambda]$

Nesta seção, estudaremos o comutador $[S_\lambda, A_\lambda]$. Para demonstrar a positividade de $\langle [S_\lambda, A_\lambda]f, f\rangle$ será suficiente considerar $b \geq 3\lambda$.

Primeiro, faremos algumas manipulações algébricas no comutador:

$$\begin{aligned}
[S_\lambda, A_\lambda] &= S_\lambda A_\lambda - A_\lambda S_\lambda \\
&= 4\lambda^3(\partial_x\theta)^2\partial_x^2\theta - 4\lambda^2\partial_x\theta\partial_{x,t}^2\theta - \lambda\partial_x^4\theta + \lambda\partial_t^2\theta - 4\lambda(\partial_x^3\theta)\partial_x - 4\lambda(\partial_x^2\theta)\partial_x^2 \\
&= 4\,\lambda^3\,(\partial_x\theta)^2\left(\partial_x^2\theta\right) - 4\,\lambda^2\partial_x\theta\partial_{x,t}^2\theta - \lambda\partial_x^4\theta + \lambda\partial_t^2\theta + \partial_x\left(-4\partial_x^2\theta\,\partial_x\right). \qquad (2.28)
\end{aligned}$$

Daí, com as derivadas de θ limitadas, verificamos que o comutador pode ser expressado na forma

$$[S_\lambda, A_\lambda] = b_0(t,x) + \partial_x(b_1(t,x)\partial_x), \tag{2.29}$$

com $b_j \in C_b(\mathbb{R})$, $j = 0, 1$ e

$$\begin{aligned}
b_0 &= 4\lambda^3(\partial_x\theta)^2\partial_x^2\theta - 4\lambda^2\partial_x\theta\partial_{x,t}^2\theta - \lambda\partial_x^4\theta + \lambda\partial_t^2\theta \quad \text{e} \\
b_1 &= -4\partial_x^2\theta.
\end{aligned}$$

A equação (2.29) será utilizada para demonstrar a positividade de $\langle [S_\lambda, A_\lambda]f, f\rangle$, ponto chave para demonstração do teorema principal.

Em seguida, descrevemos o comutador no espaço $H^2(\mathbb{R})$.

Lema 2.3. *Para cada $t \in [0,\infty)$ fixo, o comutador*

$$\begin{aligned}
[S_\lambda(t), A_\lambda(t)] : \quad H^2(\mathbb{R}) &\longrightarrow H^{-1}(\mathbb{R}) \\
f &\longmapsto [S_\lambda(t), A_\lambda(t)]f
\end{aligned}$$

é uma função linear e contínua.

Demonstração:

Iniciamos a demonstração notando que para cada $t \in [0,\infty)$, existem funções $\widetilde{b}_j \in C_b(\mathbb{R})$ de modo que podemos escrever $[S_\lambda, A_\lambda]$ da seguinte forma

$$[S_\lambda, A_\lambda] = \sum_{j=0}^{2} \widetilde{b}_j(t,x)\partial_x^j,$$

com

$$\begin{aligned}
\widetilde{b}_0 &= b_0, \\
\widetilde{b}_1 &= \partial_x b_1, \\
\widetilde{b}_2 &= b_1,
\end{aligned}$$

disto segue que $[S_\lambda, A_\lambda]$ é linear.

Agora, provaremos a continuidade. Seja $f \in H^2(\mathbb{R})$, então podemos escrever

$$\widetilde{b}_2 \partial_x^2 f = \partial_x(\widetilde{b}_2 \partial_x f) - \partial_x \widetilde{b}_2 \partial_x f.$$

Daí,

$$\begin{aligned}
[S_\lambda, A_\lambda] f &= \widetilde{b}_0 f + \widetilde{b}_1 \partial_x f + \widetilde{b}_2 \partial_x^2 f \\
&= \widetilde{b}_0 f + \widetilde{b}_1 \partial_x f + \partial_x(\widetilde{b}_2 \partial_x f) - \partial_x \widetilde{b}_2 \partial_x f \\
&= \widetilde{b}_0 f + \left(\widetilde{b}_1 - \partial_x \widetilde{b}_2\right) \partial_x f + \partial_x(\widetilde{b}_2 \partial_x f).
\end{aligned}$$

Assim, geramos as seguintes funções $f_j \in L^2(\mathbb{R})$ e uma constante $C > 0$ tais que

$$[S_\lambda, A_\lambda] f = \sum_{j=0}^{1} \partial_x^j f_j,$$

e

$$\|f_j\|_{L^2(\mathbb{R})} \leq C \|f\|_{H^2(\mathbb{R})}, \qquad \forall\, j = 0, 1,$$

sendo

$$\begin{aligned}
f_0 &= \widetilde{b}_0 f + \left(\widetilde{b}_1 - \partial_x \widetilde{b}_2\right) \partial_x f \quad \text{e} \\
f_1 &= \widetilde{b}_2 \partial_x f.
\end{aligned}$$

Assim, utilizando a Proposição 7 resulta que

$$[S_\lambda, A_\lambda] f \in H^{-1}(\mathbb{R}),$$

e existe uma constante $C > 0$ tal que

$$\|[S_\lambda, A_\lambda] f\|_{H^{-1}(\mathbb{R})} \leq C \|f\|_{H^2(\mathbb{R})}.$$

∎

Agora, definimos $\langle\ ,\ \rangle$ de modo que tenha um comportamento sesquilinear contínuo para $H^{-1}(\mathbb{R})\times H^{2}(\mathbb{R})$. Para isto, definimos a forma sesquilinear

$$\begin{array}{rccl}\langle\ ,\ \rangle: & H^{-s}(\mathbb{R})\times H^{s}(\mathbb{R}) & \longrightarrow & \mathbb{C} \\ & (g,h) & \longmapsto & \displaystyle\int_{\mathbb{R}} g(x)\overline{h(x)}\mathtt{dx}\end{array} \tag{2.30}$$

de (2.30) e da Definição 5, segue que $\langle\ ,\ \rangle$ é contínuo.

Agora para o caso $s=2$, como $H^{-1}(\mathbb{R})\subset H^{-2}(\mathbb{R})$ temos que $\langle\ ,\ \rangle$ é uma forma sesquilinear contínua em $H^{-1}(\mathbb{R})\times H^{2}(\mathbb{R})$.

A seguir utilizando (2.30) para $[S_\lambda, A_\lambda]f\in H^{-1}(\mathbb{R})$ e $f\in H^{2}(\mathbb{R})$ com imagem real, apresentamos o seguinte lema.

Lema 2.4. *Para cada $t\in[0,\infty)$ e $f\in H^2(\mathbb{R})$, tem-se*

$$\langle [S_\lambda, A_\lambda]f, f\rangle = \sum_{j=0}^{1}(-1)^j\int_{\mathbb{R}} b_j(t,x)(\partial_x^j f)^2\mathtt{dx}.$$

Demonstração: Sejam $t\in[0,\infty)$ e $f\in H^2(\mathbb{R})$. Utilizando a continuidade de $\langle\ ,\ \rangle$ temos

$$\begin{aligned}\langle [S(t)_\lambda, A_\lambda(t)]f, f\rangle &= \left\langle \sum_{j=0}^{1}\partial_x^j(b_j(t,x)\partial_x^j f), f\right\rangle \\ &= \sum_{j=0}^{1}\left\langle \partial_x^j(b_j(t,x)\partial_x^j f), f\right\rangle \\ &= \sum_{j=0}^{1}(-1)^j\left\langle (b_j(t,x)\partial_x^j f), \partial_x^j f\right\rangle \\ &= \sum_{j=0}^{1}(-1)^j\int_{\mathbb{R}} b_j(t,x)\left(\partial_x^j f\right)^2\mathtt{dx}.\end{aligned}$$

■

2.4 Desigualdades com peso para f.

Nesta seção, apresentamos proposições fundamentais para as demonstrações do Teorema 5 e do Teorema 6. A seguir, a seguinte proposição demonstrará a positividade de $\langle [S_\lambda, A_\lambda]f, f\rangle$ utilizando $b\gtrsim\lambda$ e (2.26). A demonstração é de muita importância para concluir o Teorema 5. Por outro lado, para facilitar a notação, a partir daqui denotamos por S_λ e A_λ ao invés de $S_\lambda(t)$ e $A_\lambda(t)$, respectivamente.

Proposição 12. *Sejam $f \in H^2(\mathbb{R})$ e φ como no Lema 2.2. Então, existem constantes $\lambda_0 \geq 1$ e $A_0 > 0$ tais que para todo $\lambda > \lambda_0$ e $b \geq 3\lambda$, tem-se:*

$$\langle [S_\lambda, A_\lambda]\, f, f\rangle \geq A_0 \int_{\mathbb{R}} \varphi^{(2)}(x-bt)\left[\lambda^3 f^2 + \lambda(\partial_x f)^2\right] \mathtt{dx},$$

com

$$S_\lambda = \lambda^2(\varphi')^2 + b\lambda\varphi' + \partial_x^2, \tag{2.31}$$

$$A_\lambda = \partial_t - 2\lambda\, \varphi'\partial_x - \lambda\varphi^{(2)}, \tag{2.32}$$

$$r = x - bt.$$

Demonstração: Utilizando (2.28) e φ, temos

$$\begin{aligned}\langle [S_\lambda, A_\lambda]\, f, f\rangle &= \left\langle \left[S_\lambda, \partial_t + \widetilde{A}_\lambda\right] f, f\right\rangle \\ &= \underbrace{4\lambda^3 \int_{\mathbb{R}} \left(\varphi'(r)\right)^2 \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(1)} + \underbrace{4b\lambda^2 \int_{\mathbb{R}} \left(\varphi'(r)\right) \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(2)} \\ &\quad + \underbrace{b^2\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(3)} \underbrace{-\lambda \int_{\mathbb{R}} \varphi^{(4)}(r) f^2 \mathtt{dx}}_{(4)} \\ &\quad + \underbrace{4\lambda \int_{\mathbb{R}} \varphi^{(2)}(r)(\partial_x f)^2 \mathtt{dx}}_{(5)}.\end{aligned} \tag{2.33}$$

com $r = x - bt$.

Por outro lado, como $b \geq 3\lambda$ e $|\varphi'(r)| \leq 1$, segue que

$$\begin{aligned}\left(b + 2\lambda\varphi'(r)\right)^2 &\geq \left(b - 2\lambda\,|\varphi'(r)|\right)^2 \\ &\geq (3\lambda - 2\lambda)^2 \\ &= \lambda^2.\end{aligned} \tag{2.34}$$

Logo, de (2.25) e (2.34), existe uma constante $C > 0$ tal que

$$\begin{aligned}(1) + (2) + (3) &= \int_{\mathbb{R}} \left(b + 2\lambda\varphi'(r)\right)^2 \lambda\, \varphi^{(2)}(r) f^2 \mathtt{dx} \\ &\geq \lambda^3 \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx},\end{aligned} \tag{2.35}$$

e

$$(4) \leq C\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx}. \tag{2.36}$$

Daí, de (2.35)-(2.36), temos que

$$\begin{aligned}(1)+(2)+(3)+(4) &\geq (1)+(2)+(3)-|(4)| \\ &\geq \int_{\mathbb{R}} (b+2\lambda\varphi'(r))^2 \, \lambda \, \varphi^{(2)}(r) f^2 \mathtt{dx} - C\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx} \\ &\geq (\lambda^3 - C\lambda) \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx}. \end{aligned} \tag{2.37}$$

Daí, existe $\lambda_0 > 0$ tal que $\forall \, \lambda > \lambda_0$, obtemos

$$\lambda^3 - C\lambda > \lambda^3/2.$$

Assim,

$$(1)+(2)+(3)+(4) \geq \frac{\lambda^3}{2} \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx}. \tag{2.38}$$

Além disso, de (2.33) e (2.38) temos que existem $\lambda_0 > 1$ e $A_0 > 0$, tais que para todo $\lambda > \lambda_0$, resulta que

$$(1)+(2)+(3)+(4)+(5) \geq A_0 \int_{\mathbb{R}} \varphi^{(2)}(r) \left[\lambda^3 f^2 + \lambda(\partial_x f)^2\right] \mathtt{dx}. \tag{2.39}$$

com $A_0 = \min\{\frac{1}{2}, 4\}$, disso segue da demonstração da Proposição 12. ∎

Proposição 13. *Sejam $f \in H^2(\mathbb{R})$, φ como no Lema 2.2 e $b = 0$. Então, existem constantes $\lambda_0 > 1$ e $A_0 > 0$ tais que para todo $\lambda > \lambda_0$, resulta:*

$$\langle [S_\lambda, A_\lambda] f, f \rangle \geq A_0 \int_{\mathbb{R}} \varphi^{(2)}(x) \left[\lambda^3 (\varphi')^2 f^2 + \lambda(\partial_x f)^2\right] \mathtt{dx}, \tag{2.40}$$

com

$$S_\lambda = \lambda^2 (\varphi')^2 + \partial_x^2, \tag{2.41}$$

$$A_\lambda = \partial_t - 2\lambda \, \varphi' \partial_x - \lambda \varphi^{(2)}, \tag{2.42}$$

$$r = x.$$

Demonstração: Para o caso $b = 0$ em (2.33) vale:

$$
\begin{aligned}
\langle [S_\lambda, A_\lambda]\, f, f\rangle &= \left\langle \left[S_\lambda, \partial_t + \widetilde{A}_\lambda\right] f, f \right\rangle \\
&= \underbrace{4\lambda^3 \int_{\mathbb{R}} \left(\varphi'(r)\right)^2 \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(1)} \underbrace{-\lambda \int_{\mathbb{R}} \varphi^{(4)}(r) f^2 \mathtt{dx}}_{(2)} \\
&\quad \underbrace{+ 4\lambda \int_{\mathbb{R}} \varphi^{(2)}(r)(\partial_x f)^2 \mathtt{dx}}_{(3)}. \qquad (2.43)
\end{aligned}
$$

Observe que $(1), (3) \geq 0$.

Por outro lado, resulta de (2.25) que existe uma constante $C > 0$ tal que

$$
(2) \leq C\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx}; \qquad (2.44)
$$

note que pelo Lema 2.2 para $|r| \leq 1$, tem-se $\varphi^{(4)}(r) = 0$.

Logo, para demonstrar a Proposição 13, dividimos $\mathbb{R}$ nos seguintes intervalos:

Para $|r| \leq 1$, segue de (2.43) e do Lema 2.2 que

$$
\begin{aligned}
\langle [S_\lambda, A_\lambda]\, f, f\rangle\Big|_{|r|\leq 1} &= \left\langle \left[S_\lambda, \partial_t + \widetilde{A}_\lambda\right] f, f \right\rangle\Big|_{|r|\leq 1} \\
&= 4\lambda^3 \int_{|r|\leq 1} \left(\varphi'(r)\right)^2 \varphi^{(2)}(r) f^2 \mathtt{dx} - \lambda \int_{|r|\leq 1} \varphi^{(4)}(r) f^2 \mathtt{dx} \\
&\quad + 4\lambda \int_{|r|\leq 1} \varphi^{(2)}(r)(\partial_x f)^2 \mathtt{dx} \\
&= 4\lambda^3 \int_{|r|\leq 1} \left(\varphi'(r)\right)^2 \varphi^{(2)}(r) f^2 \mathtt{dx} + 4\lambda \int_{|r|\leq 1} \varphi^{(2)}(r)(\partial_x f)^2 \mathtt{dx}. \qquad (2.45)
\end{aligned}
$$

Para $|r| \geq 1$, segue de (2.43) , (2.25), $(i), (iv)$ no Lema 2.2 que

$$
\begin{aligned}
\left|\lambda \int_{|r|\geq 1} \varphi^{(4)}(r) f^2 \mathtt{dx}\right| &\leq \lambda \int_{|r|\geq 1} \left|\varphi^{(4)}(r)\right| f^2 \mathtt{dx} \\
&= 9\lambda \int_{|r|\geq 1} \frac{1}{9} \left|\varphi^{(4)}(r)\right| f^2 \mathtt{dx} \qquad (2.46) \\
&\leq 9\lambda \int_{|r|\geq 1} (\varphi'(r))^2 \left|\varphi^{(4)}(r)\right| f^2 \mathtt{dx} \\
&\leq 9C\lambda \int_{|r|\geq 1} (\varphi'(r))^2 \varphi^{(2)}(r) f^2 \mathtt{dx}. \qquad (2.47)
\end{aligned}
$$

Daí, resulta que

$$
\begin{aligned}
\left.\langle [S_\lambda, A_\lambda]\, f, f\rangle\right|_{|r|\geq 1} &= \left.\left\langle \left[S_\lambda, \partial_t + \widetilde{A}_\lambda\right] f, f\right\rangle\right|_{|r|\geq 1} \\
&= 4\lambda^3 \int_{|r|\geq 1} \left(\varphi'(r)\right)^2 \varphi^{(2)}(r) f^2 \mathtt{dx} - \lambda \int_{|r|\geq 1} \varphi^{(4)}(r) f^2 \mathtt{dx} \\
&\quad + 4\lambda \int_{|r|\geq 1} \varphi^{(2)}(r)(\partial_x f)^2 \mathtt{dx} \\
&\geq 4\lambda^3 \int_{|r|\geq 1} \left(\varphi'(r)\right)^2 \varphi^{(2)}(r) f^2 \mathtt{dx} - \left|\lambda \int_{|r|\geq 1} \varphi^{(4)}(r) f^2 \mathtt{dx}\right| \\
&\quad + 4\lambda \int_{|r|\geq 1} \varphi^{(2)}(r)(\partial_x f)^2 \mathtt{dx} \\
&\geq 4\lambda^3 \int_{|r|\geq 1} \left(\varphi'(r)\right)^2 \varphi^{(2)}(r) f^2 \mathtt{dx} - 9C\lambda \int_{|r|\geq 1} (\varphi'(r))^2 \varphi^{(2)}(r) f^2 \mathtt{dx} \\
&\quad + 4\lambda \int_{|r|\geq 1} \varphi^{(2)}(r)(\partial_x f)^2 \mathtt{dx} \\
&= \int_{|r|\geq 1} \varphi^{(2)}(r) \left[\left(4\lambda^3 - 9C\lambda\right)(\varphi'(r))^2 f^2 + 4\lambda \left(\partial_x f\right)^2\right] \mathtt{dx}. \qquad (2.48)
\end{aligned}
$$

Logo, em (2.48) segue que existe $\lambda_0 > 1$ tal que $\forall\ \lambda > \lambda_0$

$$
4\lambda^3 - 9C\lambda > 2\lambda^3. \qquad (2.49)
$$

Então, de (2.49) em (2.48) obtemos

$$
\left.\langle [S_\lambda, A_\lambda]\, f, f\rangle\right|_{|r|\geq 1} \geq \int_{|r|\geq 1} \varphi^{(2)}(r) \left[2\lambda^3 (\varphi'(r))^2 f^2 + 4\lambda \left(\partial_x f\right)^2\right] \mathtt{dx}. \qquad (2.50)
$$

Assim, de (2.45) e (2.50)

$$
\begin{aligned}
\langle [S_\lambda, A_\lambda]\, f, f\rangle &= \left.\langle [S_\lambda, A_\lambda]\, f, f\rangle\right|_{|r|\leq 1} + \left.\langle [S_\lambda, A_\lambda]\, f, f\rangle\right|_{|r|\geq 1} \\
&\geq \int_{|r|\leq 1} \varphi^{(2)}(r) \left[4\lambda^3 \left(\varphi'(r)\right)^2 f^2 \mathtt{dx} + 4\lambda \left(\partial_x f\right)^2\right] \mathtt{dx} \\
&\quad + \int_{|r|\geq 1} \varphi^{(2)}(r) \left[2\lambda^3 (\varphi'(r))^2 f^2 + 4\lambda \left(\partial_x f\right)^2\right] \mathtt{dx} \\
&\geq \int_{\mathbb{R}} \varphi^{(2)}(r) \left[2\lambda^3 \left(\varphi'\right)^2 f^2 + 4\lambda \left(\partial_x f\right)^2\right] \mathtt{dx};
\end{aligned}
$$

disto segue o resultado da Proposição 13 com $A_0 = 2$. ∎

A seguir, apresentamos uma proposição que permitirá estimar $\langle F, f\rangle$, com F dado em (2.14) a parte não linear da equação (2.1).

Proposição 14. *Existe uma constante $C > 0$, tal que*

$$|\langle a(u)\left(-\lambda(\partial_r\varphi)f + \partial_x f\right), f\rangle| < C\lambda, \tag{2.51}$$

onde C depende das constantes M_1, M_2 e M_3 dadas em (2.2)-(2.4).

Demonstração:

De fato, como $f = e^{\lambda\varphi}u$, definimos a seguinte função

$$\alpha(s) := \begin{cases} \dfrac{1}{s^2}\displaystyle\int_0^s a(\tau)\tau \mathtt{d}\tau & , \quad s \neq 0 \\ 0 & , \quad s = 0 \end{cases}; \tag{2.52}$$

temos

$$e^{\lambda\varphi}\partial_x u = \partial_x f - \lambda\varphi' f, \tag{2.53}$$

$$\partial_x(u^2\alpha(u)) = a(u)u\partial_x u. \tag{2.54}$$

Além disso, usando a Proposição 11 sabemos que $e^{\lambda\varphi(\cdot - bt)}u(t,\cdot) \in H^2(\mathbb{R}) \cap C_0^1(\mathbb{R})$ e como $\alpha(u)$ é limitado, resulta que

$$\lim_{d\to\pm\infty} e^{\lambda\varphi(d-bt)}u^2(t,d)\alpha(u(t,d)) = 0. \tag{2.55}$$

Logo, usando integração por partes e por (2.53)-(2.55), temos que

$$\begin{aligned}
\int_{\mathbb{R}} a(u)(\partial_x f - \lambda\varphi' f)f\mathtt{dx} &= \int_{\mathbb{R}} a(u)e^{\lambda\varphi}\partial_x u\; e^{\lambda\varphi}u\mathtt{dx}, \\
&= \int_{\mathbb{R}} e^{2\lambda\varphi}a(u)u\partial_x u\mathtt{dx}, \\
&= \lim_{d\to\infty} (e^{2\lambda\varphi}u^2)\alpha(u)\Big|_{-d}^{d} - \int_{\mathbb{R}} 2\lambda\varphi'(r)e^{2\lambda\varphi}\; u^2\alpha(u)\mathtt{dx} \\
&= -2\lambda \int_{\mathbb{R}} \varphi'(r)e^{2\lambda\varphi}\; u^2\alpha(u)\mathtt{dx}.
\end{aligned} \tag{2.56}$$

Assim, por (2.56), $\alpha(u)$ limitado, $e^{2\lambda\varphi}u \in H^2(\mathbb{R})$ e $|\varphi'| \leq 1$, temos que existe uma constante $C > 0$ tal que

$$\begin{aligned}
\left|\int_{\mathbb{R}} a(u)(\partial_x f - \lambda\varphi' f)f\mathtt{dx}\right| &\leq 2\lambda\int_{\mathbb{R}} \left|\varphi'(r)e^{2\lambda\varphi}\; u^2\alpha(u)\right| \mathtt{dx} \\
&< \lambda\, C.
\end{aligned} \tag{2.57}$$

■

A seguir, enunciamos a Proposição 15 que permitirá estimar $\langle F, F\rangle$ superiormente por $C\int_{\mathbb{R}}\varphi^{(2)}(r)\left[\lambda^3 f^2+\lambda\left(\partial_x f\right)^2\right]\mathtt{dx}$ (com F dado em (2.14) e C é uma constante positiva), termo que aparece como limitante inferior de $\langle[S_\lambda, A_\lambda]f, f\rangle$ na Proposição 12. Isto junto à Proposição 17 permitirá obter

$$\langle[S_\lambda, A_\lambda]f, f\rangle-\langle F, F\rangle \geq A_0\int_{\mathbb{R}}\varphi^{(2)}(r)\left[\lambda^3 f^2+\lambda\left(\partial_x f\right)^2\right]\mathtt{dx},$$

com A_0 definido na demonstração da Proposição 12, resultado útil para a demonstração do Teorema 5.

Proposição 15. *Existe uma constante $C>0$ tal que*

$$\|a(u)(\partial_x f-\lambda\varphi'(r)\cdot f)\|_{L^2}^2 \leq C\int_{\mathbb{R}}\varphi^{(2)}(r)\left[\lambda^3 f^2+\lambda\left(\partial_x f\right)^2\right]\mathtt{dx}, \tag{2.58}$$

sendo $C=C_0\frac{M_1^2}{\lambda}\left(1+M_2^{2(j-1)}\right)\left(M_3+(M_2M_3)^{1/2}\right)$ e C_0 uma constante positiva.

Demonstração:

Para facilitar a demonstração, procedemos por etapas:

Etapa 1. Provaremos que, existe uma constante $C_1>0$, tal que

$$e^{\varphi(x-bt)/2}u^2(t,x) \leq 2C_1^{1/6}\left((M_2M_3)^{1/2}+M_3\right). \tag{2.59}$$

De fato, usando Proposição 11 temos que $e^{\lambda\varphi(\cdot-bt)}u(\cdot,t)\in H^2(\mathbb{R})\cap C_0^1(\mathbb{R})$ e pelo Teorema Fundamental do Cálculo, resulta que

$$\begin{aligned}
e^{\frac{\varphi(x-bt)}{2}}u^2(t,x) &= e^{\frac{\varphi(d-bt)}{2}}u^2(t,d)+\int_d^x \frac{\mathtt{d}}{\mathtt{dx}}\left(e^{\frac{\varphi(x-bt)}{2}}u^2\right)\mathtt{dx}\\
&\leq e^{\frac{\varphi(d-bt)}{2}}u^2(t,d)+\int_d^x \left|\frac{\mathtt{d}}{\mathtt{dx}}\left(e^{\frac{\varphi(x-bt)}{2}}u^2\right)\right|\mathtt{dx}\\
&\leq e^{\frac{\varphi(d-bt)}{2}}u^2(t,d)+\int_{-\infty}^{\infty} \left|\frac{\mathtt{d}}{\mathtt{dx}}\left(e^{\frac{\varphi(x-bt)}{2}}u^2\right)\right|\mathtt{dx}.
\end{aligned} \tag{2.60}$$

Segue das hipóteses (2.3)-(2.4) e (2.23)

$$\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \leq M_2^{1/2}, \tag{2.61}$$

$$\left\|e^{\frac{\varphi(\cdot-bt)}{2}}u(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C_1^{1/6}M_3^{1/2}, \tag{2.62}$$

$$\left\|e^{\frac{\varphi(\cdot-bt)}{4}}u(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C_1^{1/12}M_3^{1/2}. \tag{2.63}$$

Daí, por (2.61)-(2.63), Proposição 1 e fazendo $d \to -\infty$ em (2.60), obtemos que

$$
\begin{aligned}
e^{\frac{\varphi(x-bt)}{2}}u^2(t,x) &\leq \int_{-\infty}^{\infty}\left|\frac{\mathtt{d}}{\mathtt{dx}}\left(e^{\frac{\varphi(x-bt)}{2}}u^2(t,x)\right)\right|\mathtt{dx},\\
&= \int_{-\infty}^{\infty}\left|2\partial_x u\left(e^{\varphi(x-bt)/2}u\right)+\frac{1}{2}\varphi' e^{\varphi(x-bt)/2}u^2\right|\mathtt{dx},\\
&\leq \int_{-\infty}^{\infty}\left[\left|2\partial_x u\left(e^{\varphi(x-bt)/2}u\right)\right|+\frac{1}{2}\left|\varphi'\right| e^{\varphi(x-bt)/2}u^2\right]\mathtt{dx},\\
&\leq 2\left\|\partial_x u\right\|_{L^2(\mathbb{R})}\left\|e^{\frac{\varphi(x-bt)}{2}}u\right\|_{L^2(\mathbb{R})}+\left\|e^{\frac{\varphi(x-bt)}{4}}u\right\|^2_{L^2(\mathbb{R})},\\
&\leq 2C_1^{1/6}\left((M_2M_3)^{1/2}+M_3\right).
\end{aligned}
$$

Etapa 2. Existe uma constante $C_0 > 0$, tal que

$$
e^{-\varphi(r)/2} \leq C_0\varphi^{(2)}(r). \tag{2.64}
$$

Inicialmente, consideramos o caso $0 \leq r \leq 2$. Sendo $\varphi \in C^4(\mathbb{R})$ segue que $\varphi^{(2)}$ é limitada no compacto e não nula, então existe uma constante C_0 tal que

$$
1 \leq C_0\varphi^{(2)}(r).
$$

Logo, como $e^{-\frac{1}{2}r} \leq 1$, segue o resultado.

Em seguida, para $2 < r$ o resultado segue do fato que

$$
\lim_{r\to+\infty} e^{r/2}\varphi^{(2)}(r) = +\infty.
$$

Então, para cada $r > 2$, existe uma constante $1/C_0 > 0$ tal que

$$
\frac{1}{C_0} \leq e^{r/2}\varphi^{(2)}(r), \tag{2.65}
$$

daí, segue o resultado.

Etapa 3. Existe uma constante $C_0 > 0$ tal que

$$
|a(u)|^2 \leq C_0M_1^2\left(1+M_2^{2(l-1)}\right)\left((M_2M_3)^{1/2}+M_3\right)\varphi^{(2)}. \tag{2.66}
$$

Logo, usando (i) da Proposição 11, (2.59) e (2.64) (resultados das Etapas 1 e 2 respectivamente), temos que

$$\begin{aligned}
|a(u)|^2 &\leq M_1^2 \left(|u| + |u|^l\right)^2,\\
&\leq 2M_1^2 \left(|u|^2 + |u|^{2l}\right),\\
&= 2M_1^2 \left(e^{\varphi(x-bt)/2} |u|^2 e^{-\varphi(x-bt)/2} + |u|^{2(l-1)} e^{\varphi(x-bt)/2} |u|^2 e^{-\varphi(x-bt)/2}\right),\\
&= 2M_1^2 e^{\varphi(x-bt)/2} |u|^2 \left[e^{-\varphi(x-bt)/2} + e^{-\varphi(x-bt)/2} |u|^{2(l-1)}\right],\\
&\leq 2M_1^2 e^{\varphi(x-bt)/2} |u|^2 \left[C_0\varphi^{(2)} + M_2^{2(l-1)} C_0 \varphi^{(2)}\right],\\
&\leq 2M_1^2 C_0 \varphi^{(2)} \left(1 + M_2^{2(l-1)}\right) e^{\varphi(x-bt)/2} |u|^2,\\
&= C_0 M_1^2 \left(1 + M_2^{2(l-1)}\right) \left(2(M_2M_3)^{1/2} + \frac{1}{2}M_3\right) \varphi^{(2)},\\
&\leq C_0 M_1^2 \left(1 + M_2^{2(l-1)}\right) \left(M_3 + (M_2M_3)^{1/2}\right) \varphi^{(2)}.
\end{aligned}$$

Etapa 4. Existe uma constante $C_0 > 0$ tal que

$$|f_x - \lambda\varphi' f|^2 \leq C_0 \left(|f_x|^2 + \lambda^2 |f|^2\right). \tag{2.67}$$

Segue da limitação $|\varphi'| < 1$ no Lema 2.2

$$\begin{aligned}
|f_x - \lambda\varphi' f|^2 &\leq 2 \left(|f_x|^2 + \lambda^2 (\varphi')^2 |f|^2\right)\\
&\leq 2 \left(|f_x|^2 + \lambda^2 |f|^2\right).
\end{aligned}$$

Voltamos para a demonstração da Proposição 15. Segue da Proposição 14, Etapa 1, Etapa 2, Etapa 3, Etapa 4 e as propriedades de φ que

$$\begin{aligned}
\|a(u)(f_x - \lambda\varphi' f)\|_{L^2}^2 &= \int_{\mathbb{R}} |a(u)(f_x - \lambda\varphi' f)|^2 \,\mathtt{dx},\\
&= \int_{\mathbb{R}} |a(u)|^2 |(f_x - \lambda\varphi' f)|^2 \,\mathtt{dx},\\
&\leq \int_{\mathbb{R}} C_0 \varphi^{(2)} M_1^2 \left(1 + M_2^{2(l-1)}\right) \left(2(M_2M_3)^{1/2} + \frac{1}{2}M_3\right) \left(|f_x|^2 + \lambda^2 |f|^2\right) \mathtt{dx},\\
&\leq C_0 \frac{M_1^2}{\lambda} \left(1 + M_2^{2(l-1)}\right) \left(2(M_2M_3)^{1/2} + \frac{1}{2}M_3\right),\\
&\quad \times \int_{\mathbb{R}} \varphi^{(2)} \left(\lambda |f_x|^2 + \lambda^3 |f|^2\right) \mathtt{dx}.
\end{aligned}$$

Daí, segue o resultado. ■

Agora, apresentamos uma proposição similar ao resultado na Proposição 17, porém consideramos $b = 0$, isto é, $r = x$. Isto permitirá obter resultados para o caso $b = 0$ no Teorema 6.

Proposição 16. *Existe uma constante $C > 0$ tal que*

$$\|a(u)(\partial_x f - \lambda\varphi'(r)\cdot f)\|_{L^2}^2 \leq C\int \varphi^{(2)}(x)\left[\lambda^3\left(\varphi'(r)\right)^2 f^2 + \lambda\left(\partial_x f\right)^2\right]\mathtt{dx}, \tag{2.68}$$

sendo $C = C_0\frac{M_1^2}{\lambda}\left(1 + M_2^{2(l-1)}\right)\left(M_3 + (M_2M_3)^{1/2}\right)$ *e* C_0 *uma constante positiva.*

Demonstração:

Usando a Proposição 14, o resultado da Etapa 4 na Proposição 15 e as propriedades de φ, resulta que

$$\begin{aligned}
\|a(u)(f_x - \lambda\varphi' f)\|_{L^2}^2 &= \int_{\mathbb{R}} |a(u)(f_x - \lambda\varphi' f)|^2\,\mathtt{dx},\\
&= \int_{\mathbb{R}} |a(u)|^2\,|(f_x - \lambda\varphi' f)|^2\,\mathtt{dx},\\
&\leq \int_{\mathbb{R}} C_0\varphi^{(2)}M_1^2\left(1 + M_2^{2(l-1)}\right)\left(2(M_2M_3)^{1/2} + \frac{1}{2}M_3\right)\\
&\quad\times\left(|f_x|^2 + \lambda^2\left(\varphi'\right)^2|f|^2\right)\mathtt{dx},\\
&\leq C_0\frac{M_1^2}{\lambda}\left(1 + M_2^{2(l-1)}\right)\left(2(M_2M_3)^{1/2} + \frac{1}{2}M_3\right)\\
&\quad\times\int_{\mathbb{R}}\varphi^{(2)}\left(\lambda|f_x|^2 + \lambda^3\left(\varphi'\right)^2|f|^2\right)\mathtt{dx},
\end{aligned}$$

disto, segue o resultado. ■

Em seguida enunciamos a Proposição 17 que nos permitirá ter um controle da constante C, na Proposição 15, podendo fazer ela suficientemente pequena para todo $\lambda > \lambda_0^1$.

Proposição 17. *Existe $\lambda_0^1 > 0$ tal que $\forall\,\lambda > \lambda_0^1$, resulta que*

$$\|a(u)\left(\partial_x f - \lambda\varphi'\cdot f\right)\|_{L^2(\mathbb{R})}^2 \leq \frac{A_0}{4}\int_{\mathbb{R}}\varphi^{(2)}(x - bt)\left[\lambda^3 f^2 + \lambda(\partial_x f)^2\right]\mathtt{dx}, \tag{2.69}$$

com A_0 definido na Proposição 12, $\lambda_0^1 = c_0M_1^2\left(1 + M_2^{2(l-1)}\right)\left((M_2M_3)^{1/2} + M_3\right)$ *e c_0 constante positiva, que depende de A_0 e l, dada em (2.2).*

Demonstração: Utilizando a constante A_0 dada na Proposição 12, temos que existe $\lambda_0^1 > 0$, que depende de l e A_0, tal que $\forall\,\lambda > \lambda_0^1$

$$C_0\frac{M_1^2}{\lambda}\left(1 + M_2^{2(l-1)}\right)\left((M_2M_3)^{1/2} + M_3\right) \leq \frac{A_0}{4}. \tag{2.70}$$

com $\lambda_0^1 = c_0 M_1^2 \left(1 + M_2^{2(l-1)}\right)\left((M_2 M_3)^{1/2} + M_3\right)$ e $c_0 = 4C_0/A_0$ constante positiva que depende de A_0.

Logo, segue da Proposição 15 e (2.70) que

$$\|a(u)\,(\partial_x f - \lambda\varphi' \cdot f)\|^2_{L^2(\mathbb{R})} \leq \frac{A_0}{4}\int_{\mathbb{R}} \varphi^{(2)}(x - bt)\left[\lambda^3 f^2 + \lambda(\partial_x f)^2\right].$$

Daí segue o resultado. ■

De maneira similar, utilizando a Proposição 16 e $b = 0$, obtemos a seguinte proposição.

Proposição 18. *Existe $\lambda_0^1 > 0$ tal que $\forall\ \lambda > \lambda_0^1$, resulta que*

$$\|a(u)\,(\partial_x f - \lambda\varphi' \cdot f)\|^2_{L^2(\mathbb{R})} \leq \frac{A_0}{4}\int_{\mathbb{R}} \varphi^{(2)}(x)\left[\lambda^3\left(\varphi'\right)^2 f^2 + \lambda(\partial_x f)^2\right]\mathtt{dx}. \tag{2.71}$$

com A_0 definido na Proposição 12, $\lambda_0^1 = c_0 M_1^2 \left(1 + M_2^{2(l-1)}\right)\left((M_2 M_3)^{1/2} + M_3\right)$ e c_0 constante positiva, que dependem de A_0 e l, dada em (2.2)*.*

Demonstração: Segue de maneira similar que a demonstração da Proposição 17. ■

A seguinte proposição demonstrará propriedades específicas a respeito de $\|u(t,\cdot)\|_{L^2(\mathbb{R})}$, fundamental para finalizar a demonstração que a solução de (2.1) é nula.

Proposição 19. *Seja $u \in C^1([0,\infty), H^1(\mathbb{R}))$ uma solução da equação* (2.1) *com u satisfazendo* (2.11)*. Então*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})} \geq \|u(0,\cdot)\|_{L^2(\mathbb{R})}, \quad \forall\ t \in [0,\infty). \tag{2.72}$$

Demonstração: Inicialmente, multiplicamos por u na equação (2.1) e obtemos

$$\partial_t\left(u^2/2\right) + \partial_x^2\left(u^2/2\right) - \left(\partial_x u\right)^2 = a(u)u\partial_x u. \tag{2.73}$$

Logo, definimos a seguinte função

$$\begin{array}{rcl} G\ :\mathbb{R} & \longrightarrow & \mathbb{R} \\ z & \longmapsto & \displaystyle\int_0^z a(s)s\mathtt{ds}, \end{array}$$

com a dado em (2.2). Disto segue que $G \in C^\infty(\mathbb{R})$ e $G(0) = 0$.

Como $u(t,\cdot) \in H^2(\mathbb{R}) \cap C_0^1(\mathbb{R})$ para cada $t \in [0,\infty)$ segue da Proposição 9, que

$$G(u(t,\cdot)) \in H^2(\mathbb{R}) \cap C_0^1(\mathbb{R}), \quad \forall\, t \in [0,\infty). \tag{2.74}$$

Daí, segue que

$$\lim_{x\to\pm\infty} G(u(t,x)) = 0. \tag{2.75}$$

Utilizando (2.75), $u(t,\cdot) \in H^2(\mathbb{R}) \cap C_0^1(\mathbb{R})$ e integrando com respeito a x em (2.73), obtemos que

$$\begin{gathered}
\frac{1}{2}\frac{\mathtt{d}}{\mathtt{dt}}\left(\int_{\mathbb{R}} (u(t,x))^2\,\mathtt{dx}\right) + \int_{\mathbb{R}} \partial_x^2\left(\frac{u^2(t,x)}{2}\right)\mathtt{dx} - \int_{\mathbb{R}} (\partial_x u)^2\,\mathtt{dx} = \int_{\mathbb{R}} \partial_x(G(u(t,x)))\mathtt{dx}\\
\frac{1}{2}\frac{\mathtt{d}}{\mathtt{dt}}\left(\int_{\mathbb{R}} (u(t,x))^2\,\mathtt{dx}\right) + \partial_x\left(\frac{u^2(t,x)}{2}\right)\Bigg|_{-\infty}^{+\infty} - \int_{\mathbb{R}} (\partial_x u)^2\,\mathtt{dx} = G(u(t,x))|_{-\infty}^{+\infty}\\
\frac{1}{2}\frac{\mathtt{d}}{\mathtt{dt}}\left(\int_{\mathbb{R}} (u(t,x))^2\,\mathtt{dx}\right) - \int_{\mathbb{R}} (\partial_x u)^2\,\mathtt{dx} = 0.
\end{gathered}$$

Então,

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})} \geq \|u(0,\cdot)\|_{L^2(\mathbb{R})}.$$

■

2.5 Demonstração do Teorema 5

Para a demonstração, consideramos $u \in C^1([0,\infty), H^1(\mathbb{R}))$ uma solução de (2.1), $\lambda \geq \max\left\{\lambda_0,\, c_0 M_1^2\big(1 + M_2^{2(l-1)}\big)\left((M_2M_3)^{1/2} + M_3\right)\right\}$ e $f(t,\cdot) \in H^2(\mathbb{R}) \cap C_0^1(\mathbb{R})$, sendo f tal que

$$\partial_x f = -\left(S_\lambda + \widetilde{A}_\lambda\right) f + F,$$

com $F = a(u)\left(\partial_x f - \lambda\varphi'(x-bt)f\right)$, $S_\lambda^* = S_\lambda$ e $\widetilde{A}_\lambda^* = -\widetilde{A}_\lambda$.

Daí temos identidades necessárias para a demonstração do teorema:

$$\frac{\mathtt{d}}{\mathtt{dt}}\langle f, f\rangle = -\,2\langle S_\lambda f, f\rangle + 2\langle F, f\rangle, \tag{2.76}$$

$$\frac{\mathtt{d}}{\mathtt{dt}}\langle S_\lambda f, f\rangle = -\,\langle [S_\lambda, A_\lambda]\, f, f\rangle - 2\langle S_\lambda f, S_\lambda f\rangle + 2\langle F, S_\lambda f\rangle, \tag{2.77}$$

com $[S_\lambda, A_\lambda] = \left[S_\lambda, \widetilde{A}_\lambda\right] - [\partial_t, S]$.

Em seguida, construímos uma sequência $T_n \uparrow \infty$ tal que

$$\|S_\lambda f(T_n)\|^2_{L^2(\mathbb{R})} \leq \widetilde{C}\lambda, \quad \forall\, n \in \mathbb{N}; \tag{2.78}$$

com $\widetilde{C}$, independente de n.

Para isto, usamos (2.76), (2.77), uma função suave $\eta : [a,b] \to \mathbb{R}$ e por integração por partes temos que

$$\int_a^b \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} = -\left.\frac{1}{2}\left(\eta'\langle f,f\rangle\right)\right|_a^b + \int_a^b \eta'(t)\langle F, f\rangle \mathtt{dt} + \frac{1}{2}\int_a^b \eta''(t)\langle f,f\rangle \mathtt{dt}, \tag{2.79}$$

$$\begin{aligned}\int_a^b \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} = {}& \left(\eta\langle S_\lambda f, f\rangle\right)\big|_a^b + \int_a^b \eta(t)\langle [S_\lambda, A_\lambda]\, f, f\rangle \mathtt{dt} \\ & + 2\int_a^b \eta(t)\langle S_\lambda f, S_\lambda f\rangle \mathtt{dt} - 2\int_a^b \eta(t)\langle F, S_\lambda f\rangle \mathtt{dt}.\end{aligned} \tag{2.80}$$

Antes de proceder com a demonstração, devemos observar que (2.79) e (2.80) nos permitirá, respectivamente, limitar superiormente e inferiormente o termo $\int_a^b \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt}$. Por outro lado, segue das Proposições 12 e 17 a seguinte desigualdade

$$\begin{aligned}\langle [S_\lambda, A_\lambda]\, f, f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle \geq {}& \langle S_\lambda f, S_\lambda f\rangle \\ + \frac{A_0}{2}\int_{\mathbb{R}} \varphi^{(2)}(x-bt)\left[\lambda^3 f^2 + \lambda\left(\partial_x f\right)^2\right];\end{aligned} \tag{2.81}$$

a demonstração de (2.81) pode ser vista no Apêndice A.1.

No passo seguinte, vamos procurar por uma estimativa superior para integral de $\langle S_\lambda f, S_\lambda f\rangle$ em cada intervalo $[n-1/2, n+1/2]$; para isto definimos a seguinte função:

$$\begin{array}{rccl}\eta_n : & \left[n-\frac{1}{2}, n+\frac{1}{2}\right] & \longrightarrow & \mathbb{R}, \\ & t & \longmapsto & \sin\left(\pi\left(t-n+\frac{1}{2}\right)\right).\end{array} \tag{2.82}$$

com a seguinte propriedade $\eta_n\left(n \pm \frac{1}{2}\right) = 0$ e utilizando (2.79) e (2.80), obtemos

$$\begin{aligned}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n'(t)\langle S_\lambda f, f\rangle \mathtt{dt} = {}& \left(\eta_n\langle S_\lambda f, f\rangle\right)\big|_{n-\frac{1}{2}}^{n+\frac{1}{2}} + \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\langle [S_\lambda, A_\lambda]\, f, f\rangle \mathtt{dt} \\ & + 2\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\langle S_\lambda f, S_\lambda f\rangle \mathtt{dt} - 2\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\langle F, S_\lambda f\rangle \mathtt{dt},\end{aligned}$$

$$= \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t) \left\{ \langle [S_\lambda, A_\lambda]\, f, f \rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle \right\} \mathtt{dt}$$

$$\geq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t) \left\{ \langle S_\lambda f, S_\lambda f\rangle + \frac{A_0}{2} \int_{\mathbb{R}} \varphi^{(2)} \left[\lambda^3 f^2 + \lambda\,(\partial_x f)^2\right] \mathtt{dx} \right\} \mathtt{dt}$$

$$\geq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\, \|S_\lambda f(t)\|^2_{L^2_x(\mathbb{R})}\, \mathtt{dt}. \tag{2.83}$$

Para limitar superiormente $\int_a^b \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt}$, usamos as limitações de η_n, $\eta_n'(t)$ e $\eta_n''(t)$ uniforme por uma constante que não depende de n e (2.79); segue que existe uma constante $C_2 > 0$, independente de n, tal que

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n'(t)\langle S_\lambda f, f\rangle \mathtt{dt} \leq\; C_2\lambda, \qquad \forall\, n \in \mathbb{N}; \tag{2.84}$$

portanto, segue de (2.83) e (2.84)

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\, \|S_\lambda f(t)\|^2_{L^2_x(\mathbb{R})}\, \mathtt{dt} \leq C_2\lambda, \qquad \forall\, n \in \mathbb{N}.$$

Pelo Teorema do Valor Médio, $\exists\, T_n \in \left[n, n+\frac{1}{4}\right]$ tais que $T_n \geq 1,\ T_n \uparrow \infty$ e

$$\begin{aligned} \frac{\eta_n(T_n)}{4}\, \|S_\lambda f(T_n)\|^2_{L^2_x(\mathbb{R})} &= \int_n^{n+\frac{1}{4}} \eta_n(t)\, \|S_\lambda f(t)\|^2_{L^2_x(\mathbb{R})}\, \mathtt{dt} \\ &\leq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\, \|S_\lambda f(t)\|^2_{L^2_x(\mathbb{R})}\, \mathtt{dt} \\ &\leq C_2\lambda. \end{aligned}$$

Assim, desde que $T_n \in \left[n, n+\frac{1}{4}\right]$, temos

$$\eta_n(T_n) \geq \frac{\sqrt{2}}{2}, \qquad \forall\, n \in \mathbb{N},$$

e consequentemente,

$$\|S_\lambda f(T_n)\|^2_{L^2_x(\mathbb{R})} \leq \frac{8\lambda C_2}{\sqrt{2}} = \lambda \widetilde{C}, \qquad \forall\, n \in \mathbb{N}, \tag{2.85}$$

para alguma constante $\widetilde{C} > 0$, independente de n.

Para concluir, será necessário uma estimativa no espaço tempo para f. Para isto, consideramos uma função $\eta : [0, \infty] \to \mathbb{R}$ suave e positiva tal que $\eta(0) = 0$ e $\eta(t) = 1$, se $t \geq 1$; logo, utilizando (2.79) temos

$$\int_0^{T_n} \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} = -\,\frac{1}{2}\left(\eta'\langle f, f\rangle\right)\Big|_0^{T_n} + \int_0^{T_n} \eta'(t)\langle F, f\rangle \mathtt{dt} + \frac{1}{2}\int_0^{T_n} \eta''(t)\langle f, f\rangle \mathtt{dt}. \tag{2.86}$$

Logo, de (2.5), (2.85) e Proposição 14 em (2.86), existe uma constante $C_3 > 0$, independente de n, tal que

$$\int_0^{T_n} \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} \leq C_3, \qquad \forall\, n \in \mathbb{N}. \tag{2.87}$$

Por outro lado, utilizando (2.80) e (2.81),

$$\begin{aligned}\int_0^{T_n} \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} = & \left(\eta(t)\langle S_\lambda f, f\rangle\right)\big|_0^{T_n} + \int_0^{T_n} \eta(t)\left\{\langle [S_\lambda, A_\lambda]\, f, f\rangle\right.\\ & \left. + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle\right\} \mathtt{dt}\\ \geq & \langle\, S_\lambda f(T_n,\cdot), f(T_n,\cdot)\rangle + \frac{A_0}{2}\int_0^{T_n} \eta(t)\int_{\mathbb{R}} \varphi^{(2)}(x-bt)\left[\lambda^3 f^2\right.\\ & \left. + \lambda\left(\partial_x f\right)^2\right] \mathtt{dxdt}.\end{aligned} \tag{2.88}$$

Então, de (2.87) em (2.88) temos que

$$C_3 - \langle S_\lambda f(T_n,\cdot), f(T_n,\cdot)\rangle \geq \frac{A_0}{2}\int_0^{T_n} \eta(t)\int_{\mathbb{R}} \varphi^{(2)}(x-bt)\left[\lambda^3 f^2 + \lambda\left(\partial_x f\right)^2\right] \mathtt{dx},$$

e pela desigualdade de Cauchy-Schwarz, (2.85) e (2.88) existe uma constante $C_4 > 0$, independente de n, tal que

$$|\langle S_\lambda f(T_n,\cdot), f(T_n,\cdot)\rangle| \leq \|S_\lambda f(T_n,\cdot)\|\,\|f(T_n,\cdot)\| \leq \lambda C_4, \qquad \forall\, n \in \mathbb{N}.$$

Assim, obtemos que

$$\int_0^{T_n} \eta(t)\int_{\mathbb{R}} \varphi^{(2)}(x-bt)\left[\lambda^3 f^2 + \lambda\left(\partial_x f\right)^2\right] \mathtt{dxdt} \leq C_5,$$

para alguma constante $C_5 > 0$, independente de n.

Tomando $T_n \to \infty$ e observando que $\eta(t) = 1$, $\forall\, t \geq 1$, obtemos

$$\int_1^{\infty}\int_{\mathbb{R}} \varphi^{(2)}(x-bt)\left[\lambda^3 f^2 + \lambda\left(\partial_x f\right)^2\right] \mathtt{dxdt} \leq C_5,$$

isto é,

$$\int_1^{\infty}\int_{\mathbb{R}} \varphi^{(2)}(x-bt)\left[\lambda^3 f^2 + \lambda\left(\partial_x f\right)^2\right] \mathtt{dxdt} < \infty.$$

Como $f = e^{\lambda\varphi(x-bt)}u$, resulta que

$$\int_1^{\infty}\int_{\mathbb{R}} \varphi^{(2)}(x-bt)e^{2\lambda\varphi(x-bt)}u^2(t,x)\mathtt{dxdt} < \infty. \tag{2.89}$$

Assim, usando (2.24) em (2.89), resulta que

$$\int_1^\infty \int_{\mathbb{R}} u^2(t,x)\mathtt{dxdt} < \infty. \tag{2.90}$$

Agora, finalizaremos a demonstração do anulamento da solução u. Utilizando a Proposição 19 temos a desigualdade

$$\|u(t,\cdot)\|_{L^2}^2 \geq \|u(0,\cdot)\|_{L^2}^2, \quad \forall\, t \geq 0. \tag{2.91}$$

Assim, de (2.90) e (2.91), temos que

$$u(0,\cdot) = 0. \tag{2.92}$$

Logo, dado $t_0 > 0$ e considerando $u_{t_0}(t,x) = u(t+t_0, x+bt_0)$ para $(t,x) \in [0,\infty) \times \mathbb{R}$, temos que u_{t_0} satisfaz a mesma equação e as hipóteses (2.3), (2.4) e (2.5).

Então, de maneira análoga com em (2.92), segue que para cada $t_0 \in [0,\infty)$

$$u_{t_0}(0,x) = u(t_0, x+bt_0) = 0, \quad \forall\, x \in \mathbb{R},$$

e por conseguinte,

$$u(t,x) \equiv 0, \quad \forall\, (t,x) \in [0,\infty) \times \mathbb{R}.$$

∎

Agora, para outra escolha de b no Teorema 5, apresentamos a demonstração do Teorema 5 que estuda o caso particular de $b = 0$.

2.6 Demonstração do Teorema 6

De maneira similar como na demonstração do Teorema 5, consideramos $u \in C^1([0,\infty), H^1(\mathbb{R}))$ uma solução de (2.1), $\lambda \geq \max\left\{\lambda_0, c_0 M_1^2\big(1 + M_2^{2(l-1)}\big)\left((M_2M_3)^{1/2} + M_3\right)\right\}$ e com $f(t,\cdot) \in H^2(\mathbb{R}) \cap C_0^1(\mathbb{R})$, satisfazendo

$$\partial_x f = -\left(S_\lambda + \widetilde{A}_\lambda\right) f + F,$$

com $F = a(u)\left(\partial_x f - \lambda\varphi'(x-bt)f\right)$, $S_\lambda^* = S_\lambda$ e $\widetilde{A}_\lambda^* = -\widetilde{A}_\lambda$.

Daí, temos (identidades suficientes para a demonstração do teorema)

$$\frac{d}{\mathtt{dt}}\langle f,f\rangle = -\,2\langle S_\lambda f,f\rangle + 2\langle F,f\rangle, \tag{2.93}$$

$$\frac{d}{\mathtt{dt}}\langle S_\lambda f,f\rangle = -\,\langle [S_\lambda,A_\lambda]\,f,f\rangle - 2\langle S_\lambda f,S_\lambda f\rangle + 2\langle F,S_\lambda f\rangle, \tag{2.94}$$

com $[S_\lambda,A_\lambda]=\left[S_\lambda,\widetilde{A}_\lambda\right]-[\partial_t,S]$.

Inicialmente, construímos uma sequência $T_n\uparrow\infty$ tal que

$$\|S_\lambda f(T_n)\|^2_{L^2(\mathbb{R})}\le \widetilde{C}\lambda,\quad \forall\, n\in\mathbb{N}; \tag{2.95}$$

com $\widetilde{C}$, independente de n.

Usando (2.93) e (2.94) para uma função suave $\eta:[a,b]\to\mathbb{R}$ e da integração por partes temos que

$$\int_a^b \eta'(t)\langle S_\lambda f,f\rangle\mathtt{dt} = -\,\frac{1}{2}\left(\eta'\langle f,f\rangle\right)\Big|_a^b + \int_a^b \eta'(t)\langle F,f\rangle\mathtt{dt} + \frac{1}{2}\int_a^b \eta''(t)\langle f,f\rangle\mathtt{dt}, \tag{2.96}$$

$$\begin{aligned}\int_a^b \eta'(t)\langle S_\lambda f,f\rangle\mathtt{dt} &= \left(\eta\langle S_\lambda f,f\rangle\right)\big|_a^b + \int_a^b \eta(t)\langle [S_\lambda,A_\lambda]\,f,f\rangle\mathtt{dt}\\ &\quad + 2\int_a^b \eta(t)\langle S_\lambda f,S_\lambda f\rangle\mathtt{dt} - 2\int_a^b \eta(t)\langle F,S_\lambda f\rangle\mathtt{dt}.\end{aligned} \tag{2.97}$$

Observamos que (2.96) e (2.97) nos permitirá, respectivamente, estimar superiormente e inferiormente o termo $\int_a^b \eta'(t)\langle S_\lambda f,f\rangle\mathtt{dt}$.

Por outro lado, segue da Proposição 13 e Proposição 18 a seguinte desigualdade

$$\begin{aligned}\langle [S_\lambda,A_\lambda]\,f,f\rangle + 2\langle S_\lambda f,S_\lambda f\rangle - 2\langle F,S_\lambda f\rangle &\ge \langle S_\lambda f,S_\lambda f\rangle\\ + \frac{A_0}{2}\int_{\mathbb{R}}\varphi^{(2)}(x)\left[\lambda^3\left(\varphi'\right)^2 f^2 + \lambda\left(\partial_x f\right)^2\right];\end{aligned} \tag{2.98}$$

para detalhes da demonstração de (2.98), ver o Apêndice A.1.

No passo seguinte, encontramos uma estimativa superior da integral com respeito a t de $\langle S_\lambda f,S_\lambda f\rangle$ no intervalo $[n-1/2,n+1/2]$; para isto definimos a seguinte função suave:

$$\begin{aligned}\eta_n:\quad \left[n-\tfrac{1}{2},n+\tfrac{1}{2}\right] &\longrightarrow \mathbb{R},\\ t &\longmapsto \sin\left(\pi\left(t-n+\tfrac{1}{2}\right)\right),\end{aligned} \tag{2.99}$$

com a seguinte propriedade $\eta_n\left(n\pm\frac{1}{2}\right)=0$ e utilizando (2.96) e (2.97), obtemos

$$\begin{aligned}
&\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n'(t)\langle S_\lambda f,f\rangle\mathtt{dt}=\left(\eta_n\langle S_\lambda f,f\rangle\right)\Big|_{n-\frac{1}{2}}^{n+\frac{1}{2}}+\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n(t)\langle[S_\lambda,A_\lambda]\,f,f\rangle\mathtt{dt}\\
&+2\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n(t)\langle S_\lambda f,S_\lambda f\rangle\mathtt{dt}-2\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n(t)\langle F,S_\lambda f\rangle\mathtt{dt}\\
&=\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n(t)\left\{\langle[S_\lambda,A_\lambda]\,f,f\rangle+2\langle S_\lambda f,S_\lambda f\rangle-2\langle F,S_\lambda f\rangle\right\}\mathtt{dt}\\
&\geq\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n(t)\left\{\langle S_\lambda f,S_\lambda f\rangle+\frac{A_0}{2}\int_{\mathbb{R}}\varphi^{(2)}(x)\left[\lambda^3\left(\varphi'\right)^2f^2+\lambda\left(\partial_x f\right)^2\right]\mathtt{dx}\right\}\mathtt{dt}\\
&\geq\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n(t)\left\|S_\lambda f(t)\right\|_{L_x^2(\mathbb{R})}^2\mathtt{dt}.
\end{aligned}\tag{2.100}$$

Logo, usando as limitações de η_n, $\eta_n'(t)$ e $\eta_n''(t)$ por uma constante C positiva que não depende de n, (2.5) e Proposição 14 em (2.96), segue que existe uma constante $C_2>0$, independente de n, tal que

$$\begin{aligned}
&\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n'(t)\langle S_\lambda f,f\rangle\mathtt{dt}\\
&=-\frac{1}{2}\left(\eta_n'\langle f,f\rangle\right)\Big|_{n-1/2}^{n+1/2}+\int_{n-1/2}^{n+1/2}\eta_n'(t)\langle F,f\rangle\mathtt{dt}+\frac{1}{2}\int_{n-1/2}^{n+1/2}\eta_n''(t)\langle f,f\rangle\mathtt{dt}\\
&\leq C\sup_{t\geq0}\int_{\mathbb{R}}e^{2\lambda\varphi}u^2\mathtt{dx}+C\lambda+\frac{C}{2}\sup_{t\geq0}\int_{\mathbb{R}}e^{2\lambda\varphi}u^2\mathtt{dx}\\
&\leq\ C_2\lambda,\qquad\forall\,n\in\mathbb{N}.
\end{aligned}\tag{2.101}$$

Portanto, segue de (2.100) e (2.101)

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n(t)\left\|S_\lambda f(t)\right\|_{L_x^2(\mathbb{R})}^2\mathtt{dt}\leq C_2\lambda,\qquad\forall\,n\in\mathbb{N}.$$

Pelo Teorema do Valor Médio, $\exists\,T_n\in\left[n,n+\frac{1}{4}\right]$ tal que $T_n\geq1,\ T_n\uparrow\infty$ e

$$\begin{aligned}
\frac{\eta_n(T_n)}{4}\left\|S_\lambda f(T_n)\right\|_{L_x^2(\mathbb{R})}^2&=\int_n^{n+\frac{1}{4}}\eta_n(t)\left\|S_\lambda f(t)\right\|_{L_x^2(\mathbb{R})}^2\mathtt{dt}\\
&\leq\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n(t)\left\|S_\lambda f(t)\right\|_{L_x^2(\mathbb{R})}^2\mathtt{dt}\\
&\leq C_2\lambda.
\end{aligned}$$

Assim, desde que $T_n\in\left[n,n+\frac{1}{4}\right]$, temos que

$$\eta_n(T_n)\geq\frac{\sqrt{2}}{2},\qquad\forall\,n\in\mathbb{N},$$

portanto,

$$\|S_\lambda f(T_n)\|^2_{L^2_x(\mathbb{R})} \leq \frac{8\lambda C_2}{\sqrt{2}} = \lambda\widetilde{C}, \qquad \forall\, n \in \mathbb{N}, \tag{2.102}$$

para alguma constante $\widetilde{C} > 0$, independente de n.

Para concluir, será necessário uma estimativa no espaço tempo para f. Para isto, consideramos uma função $\eta : [0, \infty] \to \mathbb{R}$ suave e positiva tal que $\eta(0) = 0$ e $\eta(t) = 1$, se $t \geq 1$, logo utilizando (2.96) temos

$$\int_0^{T_n} \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} = -\frac{1}{2}\left(\eta'\langle f, f\rangle\right)\Big|_0^{T_n} + \int_0^{T_n} \eta'(t)\langle F, f\rangle \mathtt{dt} + \frac{1}{2}\int_0^{T_n} \eta''(t)\langle f, f\rangle \mathtt{dt}. \tag{2.103}$$

Logo, de (2.11), (2.102), Proposição 14 e propriedade de η em (2.103), existe uma constante $C_3 > 0$, independente de n, tal que

$$\int_0^{T_n} \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} \leq C_3\lambda, \qquad \forall\, n \in \mathbb{N}. \tag{2.104}$$

Por outro lado, utilizando (2.97) e (2.98), temos uma desigualdade inferior

$$\begin{aligned}\int_0^{T_n} \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} &= \left(\eta(t)\langle S_\lambda f, f\rangle\right)\Big|_0^{T_n} + \int_0^{T_n} \eta_n(t)\left\{\langle [S_\lambda, A_\lambda]\, f, f\rangle\right.\\ &\quad \left.+\, 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle\right\} \mathtt{dt}\\ &\geq \langle S_\lambda f(T_n,\cdot), f(T_n,\cdot)\rangle + \frac{A_0}{2}\int_0^{T_n} \eta(t)\int_{\mathbb{R}} \varphi^{(2)}(x)\left[\lambda^3\left(\varphi'\right)^2 f^2\right.\\ &\quad \left.+\lambda\left(\partial_x f\right)^2\right] \mathtt{dxdt}.\end{aligned} \tag{2.105}$$

Então, de (2.104) e (2.105) resulta

$$C_3\lambda - \langle S_\lambda f(T_n,\cdot), f(T_n,\cdot)\rangle \geq \frac{A_0}{2}\int_0^{T_n} \eta(t)\int_{\mathbb{R}} \varphi^{(2)}(x)\left[\lambda^3\left(\varphi'\right)^2 f^2 + \lambda\left(\partial_x f\right)^2\right] \mathtt{dx},$$

e pela desigualdade de Cauchy-Schwarz, (2.102) e (2.105) existe uma constante $C_4 > 0$, independente de n, tal que

$$|\langle S_\lambda f(T_n,\cdot), f(T_n,\cdot)\rangle| \leq \|S_\lambda f(T_n,\cdot)\|_{L^2(\mathbb{R})}\,\|f(T_n,\cdot)\|_{L^2(\mathbb{R})} \leq \lambda C_4, \qquad \forall\, n \in \mathbb{N}.$$

Assim, obtemos que

$$\int_0^{T_n} \eta(t)\int_{\mathbb{R}} \varphi^{(2)}(r)\left[\lambda^3\left(\varphi'\right)^2 f^2 + \lambda\left(\partial_x f\right)^2\right] \mathtt{dxdt} \leq C_5,$$

para alguma constante $C_5 > 0$, independente de n.

Daí, fazendo $T_n \to \infty$ e observando que $\eta(t) = 1, \ \forall \ t \geq 1$, obtemos

$$\int_1^\infty \int_{\mathbb{R}} \varphi^{(2)}(x) \left[\lambda^3 (\varphi')^2 f^2 + \lambda (\partial_x f)^2 \right] \mathtt{dxdt} \leq C_5.$$

Logo,

$$\int_1^\infty \int_{\mathbb{R}} \varphi^{(2)}(x) \lambda^3 (\varphi'(x))^2 (f(t,x))^2 \, \mathtt{dxdt} < \infty.$$

Como $f = e^{\lambda\varphi(x)} u$, tem-se

$$\int_1^\infty \int_{\mathbb{R}} \varphi^{(2)}(x) e^{2\lambda\varphi(x)} (\varphi'(x))^2 u^2(t,x) \mathtt{dxdt} < \infty.$$

Então, usando (2.8), (2.24) e $0 < (\varphi'(\delta))^2 \leq (\varphi'(x))^2, \ \ \forall \ |x| \geq \delta$, resulta que

$$\begin{aligned} \infty &> \int_1^\infty \int_{\mathbb{R}} (\varphi'(x))^2 u^2(t,x) \mathtt{dxdt} \\ &= \int_1^\infty \int_{|x|\geq\delta} (\varphi'(x))^2 u^2(t,x) \mathtt{dxdt} \\ &\geq \min_{|x|\geq\delta} \left\{ (\varphi'(x))^2 \right\} \int_1^\infty \int_{|x|\geq\delta} u^2(t,x) \mathtt{dxdt}. \end{aligned} \tag{2.106}$$

Assim, de (2.8) e (2.106) obtemos que

$$\int_1^\infty \int_{\mathbb{R}} u^2(t,x) \mathtt{dxdt} < \infty. \tag{2.107}$$

Agora, na maneira análoga à demonstração do Teorema 5, finalizamos a demonstração do anulamento da solução u. Utilizando o resultado da Proposição 19 para u solução de (2.1).

Usando a Proposição 19 para u solução da (2.1), segue a seguinte desigualdade

$$\|u(t,\cdot)\|_{L^2}^2 \geq \|u(0,\cdot)\|_{L^2}^2, \quad \forall \ t \geq 0. \tag{2.108}$$

Assim, de (2.107)-(2.108), temos que

$$u(0,\cdot) = 0. \tag{2.109}$$

Logo, dado $t_0 > 0$ e considerando $u_{t_0}(t,x) = u(t+t_0,x)$ para $(t,x) \in [0,\infty) \times \mathbb{R}$, temos que u_{t_0} satisfaz a mesma equação e as hipóteses (2.8), (2.9), (2.10) e (2.11).

Então, de maneira análoga como em (2.109), temos que para cada $t_0 \in [0, \infty)$

$$u_{t_0}(0, x) = u(t_0, x) = 0, \quad \forall\, x \in \mathbb{R}.$$

Consequentemente,

$$u(t, x) \equiv 0, \quad \forall\, (t, x) \in [0, \infty) \times \mathbb{R}.$$

Capítulo 3

Equação do tipo Kuramoto-Sivashinsky

Neste capítulo, o nosso objetivo será procurar por condições de decaimento para soluções da equação do tipo Kuramoto-Sivashinsky(K-S) para que ela seja nula. Após impormos condições de decaimento exponencial linear obtemos os resultados desejados.

Inicialmente, consideramos a seguinte equação (K-S)

$$\partial_t u + \partial_x^2 u - \partial_x^4 u = a(u)\partial_x u, \tag{3.1}$$

com a é uma função real da classe C^∞, tal que

$$|a(u)| \leq M_1 \left(|u| + |u|^l\right), \quad \text{para algum } \; l \geq 2 \; \text{ e } l \in \mathbb{Z}^+, \tag{3.2}$$

onde M_1 é uma constante positiva.

Enunciamos o seguinte resultado principal deste Capítulo.

Teorema 7. *Seja $u \in C^1([0,\infty), H^1(\mathbb{R}))$ uma solução de* (3.1) *onde $a(u)$ é uma função real satisfazendo* (3.2). *Se existem $\lambda_0 \geq 1$, $c_0 > 0$, $M_2 > 0$ e $M_3 > 0$ tais que*

$$\sup_{t\geq 0} \|u(t,\cdot)\|_{H^1}^2 \leq M_2, \tag{3.3}$$

$$\sup_{t\geq 0} \int e^{|x-bt|} u^2(t,x)\mathtt{dx} \leq M_3, \tag{3.4}$$

$$\sup_{t\geq 0} \int e^{2\lambda|x-bt|} (\partial_x^j u)^2(t,x)\mathtt{dx} < \infty \; \textit{com } j \in \{0,1,2,3,4\}, \tag{3.5}$$

para

$$\lambda \geq \left\{\lambda_0, c_0 M_1^2 \left(1 + M_2^{2(l-1)}\right)\left(M_3 + (M_2 M_3)^{1/2}\right)\right\} \quad e \quad b \gtrsim \lambda^4. \tag{3.6}$$

Então,

$$u \equiv 0.$$

Observação 2. *Na hipótese do Teorema 7, se M_3 for suficientemente pequeno, podemos substituir* (3.6) *por* $\lambda > \max\{\lambda_0, c_0 M_1^2\}$ *e obter o mesmo resultado.*

Antes de demonstrar o Teorema 7, apresentamos alguns resultados prévios nas seguintes seções, que auxiliarão na demonstração do Teorema 7.

3.1 Termos simétricos e anti-simétricos

Nesta seção, procuramos gerar uma caracterização da equação (3.1), de modo que obtemos uma desigualdade conveniente para demonstrar o Teorema 7.

Seja u solução da equação (3.1), definimos a seguinte função,

$$f(t,x) := e^{\lambda\theta(t,x)}u(t,x), \tag{3.7}$$

com $\theta(x,t)$ uma função diferenciável com derivadas limitadas a ser construída posteriormente.

Observamos que

$$\begin{aligned}
\partial_t f &= e^{\lambda\theta}\lambda\partial_t\theta \cdot u + e^{\lambda\theta}\partial_t u \\
&= e^{\lambda\theta}\lambda\partial_t\theta \cdot (e^{-\lambda\theta}f) + e^{\lambda\theta}\left(a(e^{-\lambda\theta}f)\partial_x u - \partial_x^2 u + \partial_x^4 u\right) \\
&= \lambda\partial_t\theta \cdot f - e^{\lambda\theta}\left(\partial_x^2 - \partial_x^4\right)\left(e^{-\lambda\theta}f\right) + e^{\lambda\theta}a(e^{-\lambda\theta}f)\partial_x(e^{-\lambda\theta}f) \\
&= \lambda\partial_t\theta \cdot f - e^{\lambda\theta}\left(\partial_x^2 - \partial_x^4\right)\left(e^{-\lambda\theta}f\right) + e^{\lambda\theta}a(e^{-\lambda\theta}f)\left(e^{-\lambda\theta}\left(-\lambda\partial_x\theta \cdot f + \partial_x f\right)\right) \\
&= \lambda\partial_t\theta \cdot f - e^{\lambda\theta}\left(\partial_x^2 - \partial_x^4\right)\left(e^{-\lambda\theta}f\right) + a(e^{-\lambda\theta}f)\left(-\lambda\partial_x\theta \cdot f + \partial_x f\right) \\
&= \lambda\partial_t\theta \cdot f - e^{\lambda\theta}\partial_x^2(e^{-\lambda\theta}f) + e^{\lambda\theta}\partial_x^4(e^{-\lambda\theta}f) + a(e^{-\lambda\theta}f)\left(-\lambda\partial_x\theta \cdot f + \partial_x f\right).
\end{aligned}$$

Daí,

$$\partial_t f - \lambda\partial_t\theta \cdot f + e^{\lambda\theta}\partial_x^2\left(e^{-\lambda\theta}f\right) - e^{\lambda\theta}\partial_x^4\left(e^{-\lambda\theta}f\right) = a(e^{-\lambda\theta}f)\left(-\lambda\partial_x\theta \cdot f + \partial_x f\right).$$

A seguir, realizamos uma separação nos termos simétricos e anti-simétricos com a finalidade de obter as desigualdades com uma função peso para f (como foi feito em [46]).

Para $j \in \{1, 2\}$ consideramos as seguintes funções

$$\begin{array}{rccl} R_j: & [0,\infty) & \to & \mathcal{L}(H^4(\mathbb{R}), L^2(\mathbb{R})) \\ & t & \mapsto & \big(g(t,\cdot) \mapsto e^{\lambda\theta(t,\cdot)}\partial_x^{2j}(e^{-\lambda\theta(t,\cdot)}g(t,\cdot))\big)\,, \end{array}$$

e

$$\begin{array}{rccl} R_j^*: & [0,\infty) & \to & \mathcal{L}(H^4(\mathbb{R}), L^2(\mathbb{R})) \\ & t & \mapsto & \big(g(t,\cdot) \mapsto e^{-\lambda\theta(t,\cdot)}\partial_x^{2j}(e^{\lambda\theta(t,\cdot)}g(t,\cdot))\big)\,. \end{array}$$

Lema 3.1. *Para $j \in \{1, 2\}$ e $g, h \in H^4(\mathbb{R})$, temos que*

$$\int_{\mathbb{R}} (R_j(t)g)(x)h(x)\mathtt{dx} = \int_{\mathbb{R}} g(x)(R_j^*(t)h)(x)\mathtt{dx}.$$

Demonstração: Como $H^4(\mathbb{R}) \subset C_0^3(\mathbb{R})$ e aplicando integração por partes, obtemos a demonstração do Lema 3.1. ■

Logo, para definir as partes simétricas e anti-simétricas, definimos os seguintes termos:

$$\begin{aligned} S_1 =& \frac{R_1 + R_1^*}{2} \\ =& \lambda^2 (\partial_x\theta)^2 + \partial_x^2, \end{aligned} \tag{3.8}$$

$$\begin{aligned} A_1 =& \frac{R_1 - R_1^*}{2} \\ =& -\lambda\left(\partial_x^2\theta\right) - 2\lambda\,(\partial_x\theta)\,\partial_x, \end{aligned} \tag{3.9}$$

$$\begin{aligned} S_2 =& -\lambda\partial_t\theta + \frac{R_2 + R_2^*}{2}, \\ =& \lambda^4 (\partial_x\theta)^4 + 4\lambda^2\left(\partial_x^3\theta\right)(\partial_x\theta) + 3\lambda^2\left(\partial_x^2\theta\right)^2, \\ & + 12\lambda^2\left(\partial_x^2\theta\right)(\partial_x\theta)\,\partial_x + 6\lambda^2(\partial_x\theta)^2\,\partial_x^2 + \partial_x^4, \end{aligned} \tag{3.10}$$

$$\begin{aligned}
\widetilde{A}_2 =& \frac{R_2 - R_2^*}{2}, \\
=& -\lambda \left(6\, (\partial_x \theta)^2 \left(\partial_x^2 \theta\right) \lambda^2 + 4\, \lambda^2 \left(\partial_x \theta\right)^3 (\partial_x)\right., \\
& + \partial_x^4 \theta + 4\, (\partial_x \theta)\, \partial_x^3 + 6\, \left(\partial_x^2\right) \partial_x^2 \theta, \\
& \left.+4\, \left(\partial_x^3 \theta\right) \partial_x\right).
\end{aligned} \tag{3.11}$$

Daí, definimos

$$A_2 = \partial_t + \widetilde{A}_2, \quad \widetilde{A} = A_1 + \widetilde{A}_2, \quad S_\lambda = S_1 + S_2, \quad A_\lambda = A_1 + A_2, \quad . \tag{3.12}$$

Logo, segue do Lema 3.1 que

$$\int_{\mathbb{R}} (S_\lambda(t) g)(x) h(x) \mathtt{dx} = \int_{\mathbb{R}} g(x) (S_\lambda(t) h)(x) \mathtt{dx}, \quad \forall\, g, h \in H^4(\mathbb{R}), \tag{3.13}$$

e

$$\int_{\mathbb{R}} (\widetilde{A}(t) g)(x) h(x) \mathtt{dx} = -\int_{\mathbb{R}} g(x) (\widetilde{A}(t) h)(x) \mathtt{dx}, \quad \forall\, g, h \in H^4(\mathbb{R}). \tag{3.14}$$

De aqui na frente, definimos uma função adequada para obter desigualdades satisfatórias para f, com propriedades muito específicas.

Lema 3.2. *Existe uma função $\varphi_0 \in C^8(\mathbb{R})$, par e não negativa com as seguintes propriedades:*

(i) $\inf_{|r| \geq 1} |\varphi_0'(r)| \geq 1$,

(ii) $|\varphi_0'(r)| \leq 3, \qquad \forall\, r \in \mathbb{R}$,

(iii) $0 < \varphi_0^{(2)}(r) \leq 1, \qquad \forall\, r \in \mathbb{R}$,

(iv) $\varphi_0^{(2)}(r) = 1, \qquad \forall\, |r| \leq \frac{3}{2}$,

(v) $\varphi_0^{(2)}(r)$ *é decrescente para* $r > \frac{3}{2}$,

(vi) $\exists\, C_1 > 0$ *constante, tal que*

$$\begin{aligned}
e^{\varphi_0(r)} \leq& C_1 e^{3|r|}, \qquad \forall r \in \mathbb{R}, && (3.15)\\
e^{\frac{-\varphi_0(r)}{6}} \leq& C_1 \varphi_0^{(2)}(r), \qquad \forall r \in \mathbb{R}, && (3.16)\\
\left|\frac{d^k \varphi_0(r)}{dr^k}\right| \leq& C_1 \varphi_0^{(2)}(r), \qquad \forall r \in \mathbb{R}, \qquad k = 3, 4, 5, 6, 7, 8. && (3.17)
\end{aligned}$$

A construção de φ_0 pode ser vista no Apêndice A.2.

Utilizando as propriedades de φ_0 , definimos $\varphi := \varphi_0/3$ e fazemos a seguinte escolha

$$\theta(t,x) = \varphi(x - bt), \tag{3.18}$$

com $b > 0$ a ser escolhido.

3.2 Propriedades da solução

Nesta seção, apresentamos propriedades da solução da equação do tipo K-S. Inicialmente, consideramos que $u(t,\cdot) \in H^1(\mathbb{R})$, daí segue das condições de decaimento (hipóteses do Teorema 7) obtemos que $u(t,\cdot) \in H^4(\mathbb{R})$, $\forall\ t \in [0,+\infty)$. A seguir, descrevemos os detalhes para cada um dos resultados.

Proposição 20. *Sejam u uma solução de* (3.1) *e φ como no Lema 3.2 tal que $u \in C^1([0,\infty), H^1(\mathbb{R}))$.*

(i) Se u satisfaz (3.3)*, então $u \in L^\infty([0,\infty) \times \mathbb{R})$.*

(ii) Se u satisfaz (3.5)*, então $u(t,\cdot)$, $e^{\lambda\varphi(\cdot - bt)}u(t,\cdot) \in H^4(\mathbb{R})$, $\forall\ t \in [0,\infty)$.*

Demonstração: Para (i), pela Teorema 3, temos que $\widehat{u} \in L^1$. Logo, aplicando transformada de Fourier inversa, temos que

$$\begin{aligned}
|u(t,x)| &= (2\pi)^{-1}\left|\int_{\mathbb{R}} e^{ix\xi}\widehat{u}(t,\xi)d\xi\right| \\
&\le (2\pi)^{-1}\left\|\widehat{u}(t,\cdot)\right\|_{L^1(\mathbb{R})} \\
&\le C\left\|u(t,\cdot)\right\|_{H^1(\mathbb{R})}, \qquad \forall\ x \in \mathbb{R}\ \text{ e }\ \forall\ t \in [0,\infty),
\end{aligned}$$

para alguma constante $C > 0$.

Daí, usando (3.3), segue-se

$$\|u\|_{L^\infty([0,\infty)\times\mathbb{R})} \le C \sup_{t\in[0,\infty)} \|u(t,\cdot)\|_{H^1(\mathbb{R})} \le CM_2^{\frac{1}{2}}.$$

Para demonstrar (ii), utilizamos (3.5) e (3.15) para obter

$$\begin{aligned}
\int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)}(\partial_x^k u)^2(t,x)\mathtt{dx} &\le C_1^{2\lambda/3}\int_{\mathbb{R}} e^{2\lambda|x-bt|}(\partial_x^k u)^2(t,x)\mathtt{dx} \\
&< \infty, \quad k = 0,1,\ldots,4.
\end{aligned} \tag{3.19}$$

Como $\lambda > 0$, φ uma função positiva e (3.19), segue que

$$\int_{\mathbb{R}} \left(\partial_x^k u(t,x)\right)^2 \mathtt{dx} \leq \int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} (\partial_x^k u(t,x))^2 \mathtt{dx} < \infty.$$

com $k \in \{0,1,\ldots,4\}$. Daí, temos $u(t,\cdot) \in H^4(\mathbb{R}), \ \forall\, t \in [0,\infty)$.

Por outro lado, como

$$\begin{aligned}
\partial_x^k \left(e^{\lambda\varphi(x-bt)} u\right) &= \sum_{j=0}^{k} \binom{k}{j} \partial_x^j \left(e^{\lambda\varphi(x-bt)}\right) \partial_x^{k-j} u \\
&= e^{\lambda\varphi(x-bt)} \partial_x^k u + \sum_{j=1}^{k} \binom{k}{j} \mathtt{P}(\lambda, \varphi', \varphi^{(2)}, \ldots, \varphi^{(j)}) e^{\lambda\varphi(x-bt)} \partial_x^{k-j} u,
\end{aligned}$$

utilizando as propriedades de φ no Lema 3.2, obtemos uma constante $\widetilde{C} > 0$ tal que

$$\int_{\mathbb{R}} \left|\partial_x^k \left(e^{\lambda\varphi(x-bt)} u(t,x)\right)\right|^2 \mathtt{dx} \leq \widetilde{C} \sum_{j=0}^{k} \int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} (\partial_x^j u)^2(t,x) \mathtt{dx}, \quad k = 0,1,...,4.$$

Portanto, por (3.19), concluímos que $e^{\lambda\varphi(\cdot-bt)} u(t,\cdot) \in H^4(\mathbb{R})$, $\forall\, t \in [0,\infty)$.

■

3.3 O comutador $[S_\lambda, A_\lambda]$

Nesta seção, estudaremos o comutador $[S_\lambda(t), A_\lambda(t)]$, de aqui em diante, por simplicidade de notação, usamos S_λ e A_λ ao invés de $S_\lambda(t)$ e $A_\lambda(t)$ respectivamente. Verificaremos que uma condição suficiente para demonstrar a positividade do produto interno $\langle [S_\lambda, A_\lambda] f, f\rangle$, é considerar $b \gtrsim \lambda^4$, isto é, $\exists\, C > 0$ tal que $b \geq C\lambda^4$.

Primeiro fazemos os cálculos usando (3.8)-(3.12)

$$\begin{aligned}
[S_\lambda, A_\lambda] &= S_\lambda A_\lambda - A_\lambda S_\lambda \\
&= (S_1 + S_2) \circ (A_1 + A_2) - (A_1 + A_2) \circ (S_1 + S_2) \\
&= S_1 A_1 - A_1 S_1 + S_1 A_2 - A_2 S_1 + S_2 A_1 - A_1 S_2 + S_2 A_2 - A_2 S_2 \\
&= [S_1, A_1] + [S_1, A_2] + [S_2, A_1] + [S_2, A_2],
\end{aligned}$$

com

$$
\begin{aligned}
[S_1, A_1] =& \lambda \left(4\,\lambda^2 \left(\partial_x\theta\right)^2 \left(\partial_x^2\theta\right) - 4\, \left(\partial_x^2\theta\right) \partial_x^2 - \left(\partial_x^4\theta\right) - 4\, \left(\partial_x^3\theta\right) \partial_x\right) \\
=& \, 4\,\lambda^3 \left(\partial_x\theta\right)^2 \left(\partial_x^2\theta\right) - \lambda\partial_x^4\theta + \partial_x \left(-4\lambda\partial_x^2\theta\partial_x\right), \qquad (3.20)
\end{aligned}
$$

$$
\begin{aligned}
[S_1, A_2] =& \lambda \left(8\,\lambda^4 \left(\partial_x\theta\right)^4 \left(\partial_x^2\theta\right) + 2\,\lambda^2 \left(\partial_x\theta\right)^2 \left(\partial_x^4\theta\right) + 8\,\lambda^2 \left(\partial_x\theta\right) \left(\partial_x^2\theta\right) \partial_x^3\theta \right.\\
& -2\,\lambda\, \left(\partial_x\theta\right) \partial_{x,t}^2\theta - 8\, \left(\partial_x^2\theta\right) \partial_x^4 - \left(\partial_x^6\theta\right) - 6\, \left(\partial_x^5\theta\right) \partial_x - 14\, \left(\partial_x^4\theta\right) \partial_x^2 \\
& \left. -16\, \left(\partial_x^3\theta\right) \partial_x^3\right) \\
=& \, 8\,\lambda^5 \left(\partial_x\theta\right)^4 \left(\partial_x^2\theta\right) + 2\,\lambda^3 \left(\partial_x\theta\right)^2 \left(\partial_x^4\theta\right) + 8\,\lambda^3 \left(\partial_x\theta\right) \left(\partial_x^2\theta\right) \partial_x^3\theta \\
& + \partial_x \left(-6\lambda\partial_x^4\theta\partial_x\right) + \partial_x^2 \left(-8\lambda\partial_x^2\theta\partial_x^2\right), \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
[S_2, A_1] =& \, 8\,\lambda^5 \left(\partial_x\theta\right)^4 \left(\partial_x^2\theta\right) + 2\,\lambda^3 \left(\partial_x\theta\right)^2 \left(\partial_x^4\theta\right) + 8\,\lambda^3 \left(\partial_x\theta\right) \left(\partial_x^2\theta\right) \partial_x^3\theta \\
& + \partial_x \left(-6\lambda\partial_x^4\theta\partial_x\right) + \partial_x^2 \left(-8\lambda\partial_x^2\theta\partial_x^2\right), \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
[S_2, A_2] =& \lambda \left(16\, \left(\partial_x\theta\right)^6 \left(\partial_x^2\theta\right) \lambda^6 - 4\, \left(\partial_x\theta\right)^4 \left(\partial_x^4\theta\right) \lambda^4 - 64\, \left(\partial_x\theta\right)^3 \left(\partial_x^2\theta\right) \left(\partial_x^3\theta\right) \lambda^4 \right.\\
& -48\, \left(\partial_x\theta\right)^2 \left(\partial_x^2\theta\right)^3 \lambda^4 - 8\, \left(\partial_x\theta\right)^3 \left(\partial_{x,t}^2\theta\right) \lambda^3 + 4\, \left(\partial_x\theta\right)^2 \left(\partial_x^6\theta\right) \lambda^2 \\
& +24\, \left(\partial_x\theta\right) \left(\partial_x^2\theta\right) \left(\partial_x^5\theta\right) \lambda^2 + 32\, \left(\partial_x\theta\right) \left(\partial_x^4\theta\right) \left(\partial_x^3\theta\right) \lambda^2 \\
& -36\, \left(\partial_x^2\theta\right)^2 \left(\partial_x^4\theta\right) \lambda^2 - 80\, \left(\partial_x^2\theta\right) \left(\partial_x^3\theta\right)^2 \lambda^2 - 8\, \left(\partial_x\theta\right) \left(\partial_{x^3,t}^2\theta\right) \lambda \\
& \left. -12\, \left(\partial_x^2\theta\right) \left(\partial_{x^2,t}^3\theta\right) \lambda - 8\, \left(\partial_x^3\theta\right) \left(\partial_{x,t}^2\theta\right) \lambda - \partial_x^8\theta + \partial_t^2\theta\right) \\
& + \partial_x \left(-48\lambda^5(\partial_x\theta)^4\partial_x^2\theta\partial_x\right) + \partial_x \left(24\lambda^3(\partial_x\theta)^2\partial_x^4\theta\partial_x\right) \\
& + \partial_x \left(96\lambda^3(\partial_x\theta)(\partial_x^2\theta)(\partial_x^3\theta)\right) + \partial_x \left(-48\lambda^3(\partial_x^2\theta)^3\partial_x\right) \\
& + \partial_x \left(-24\lambda^2\partial_x\theta\partial_{x,t}^2\theta\partial_x\right) + \partial_x \left(-8\partial_x^6\theta\partial_x\right) \\
& + \partial_x^2 \left(-68\lambda\partial_x^4\theta\partial_x^2\right) + \partial_x^2 \left(48\lambda^3(\partial_x\theta)^2(\partial_x^2\theta)\partial_x^2\right) + \partial_x^2 \left(48\lambda(\partial_x^4\theta)\partial_x^2\right) \\
& + \partial_x^3 \left(-16\lambda\partial_x^2\theta\partial_x^3\right). \qquad (3.23)
\end{aligned}
$$

Daí, verificamos que o comutador pode ser expressado por

$$
[S_\lambda, A_\lambda] = \sum_{j=0}^{3} \partial_x^j (b_j(t,x)\partial_x^j), \qquad (3.24)
$$

com $b_j \in C_b(\mathbb{R})$ e $j = 0, 1, 2, 3$.

A equação (3.24) será utilizada para demonstrar que o produto interno $\langle [S_\lambda, A_\lambda]f, f\rangle$ é positivo que é a chave para demonstração do teorema principal.

Em seguida, descrevemos o comutador no espaço $H^4(\mathbb{R})$.

Lema 3.3. *Para cada $t \in \mathbb{R}$ fixo, o comutador*

$$\begin{aligned}[S_\lambda(t), A_\lambda(t)] : \quad H^4(\mathbb{R}) &\longrightarrow H^{-2}(\mathbb{R})\\ f &\longmapsto [S_\lambda(t), A_\lambda(t)]f,\end{aligned}$$

é uma função linear e contínua.

Demonstração: Para cada $t \in [0, \infty)$ fixo, existem funções $\widetilde{b}_j \in C_b(\mathbb{R})$ de modo que podemos escrever $[S_\lambda(t), A_\lambda(t)]$ da seguinte forma

$$[S_\lambda(t), A_\lambda(t)] = \sum_{j=0}^{6} \widetilde{b}_j(t,x)\partial_x^j.$$

Como $f \in H^4(\mathbb{R})$, escrevemos

$$\begin{aligned}\widetilde{b}_5\partial_x^5 f =& \partial_x(\widetilde{b}_5\partial_x^4 f) - \partial_x\widetilde{b}_5\partial_x^4 f,\\ \widetilde{b}_6\partial_x^6 f =& \partial_x^2(\widetilde{b}_6\partial_x^4 f) - \partial_x(2\partial_x\widetilde{b}_6\partial_x^4 f) + \partial_x^2\widetilde{b}_6\partial_x^4 f.\end{aligned}$$

Daí, temos que

$$\begin{aligned}[S_\lambda(t), A_\lambda(t)]f =& \widetilde{b}_0 f + \widetilde{b}_1\partial_x f + \widetilde{b}_2\partial_x^2 f + \widetilde{b}_3\partial_x^3 f + (\widetilde{b}_4 - \partial_x\widetilde{b}_4 + \partial_x^2\widetilde{b}_4)\partial_x^4 f\\ &+ \partial_x(\widetilde{b}_4\partial_x^4 f - 2\partial_x\widetilde{b}_4\partial_x^4 f) + \partial_x^2(\widetilde{b}_4\partial_x^4 f).\end{aligned}$$

Assim, obtemos as seguintes funções $f_j \in L^2(\mathbb{R})$ e uma constante $C > 0$ tais que

$$[S_\lambda(t), A_\lambda(t)]f = \sum_{j=0}^{2} \partial_x^j f_j,$$

e

$$\|f_j\|_{L^2(\mathbb{R})} \leq C\,\|f\|_{H^4(\mathbb{R})}, \qquad \forall\, j = 0, 1, 2,$$

onde

$$\begin{aligned}f_0 =& b_0 f + b_1\partial_x f + b_2\partial_x^2 f + b_3\partial_x^3 f + (b_4 - \partial_x b_4 + \partial_x^2 b_4)\partial_x^4 f\\ f_1 =& b_4\partial_x^4 f - 2\partial_x b_4\partial_x^4 f\\ f_2 =& b_4\partial_x^4 f.\end{aligned}$$

Segue da Proposição 7 que

$$[S_\lambda(t), A_\lambda(t)]f \in H^{-2}(\mathbb{R}),$$

e existe uma constante $C > 0$ tal que

$$\|[S_\lambda(t), A_\lambda(t)]f\|_{H^{-2}(\mathbb{R})} \leq C \|f\|_{H^4(\mathbb{R})}.$$

∎

Agora, definimos $\langle\ ,\ \rangle$ de modo que tenha um comportamento sesquilinear contínuo para $H^{-2}(\mathbb{R}) \times H^4(\mathbb{R})$. Para isto, definimos a forma sesquilinear

$$\begin{array}{rccl} \langle\ ,\ \rangle: & H^{-s}(\mathbb{R}) \times H^s(\mathbb{R}) & \longrightarrow & \mathbb{C} \\ & (g,h) & \longmapsto & \displaystyle\int_{\mathbb{R}} g(x)\overline{h(x)}\mathtt{dx} \end{array} \tag{3.25}$$

de (3.25) e da Definição 5, segue que $\langle\ ,\ \rangle$ é contínuo.

Agora para o caso $s = 4$, como $H^{-2}(\mathbb{R}) \subset H^{-4}(\mathbb{R})$ temos que $\langle\ ,\ \rangle$ é uma forma sesquilinear contínua em $H^{-2}(\mathbb{R}) \times H^4(\mathbb{R})$.

A seguir utilizando (3.25) para $[S_\lambda, A_\lambda]f \in H^{-2}(\mathbb{R})$ e $f \in H^4(\mathbb{R})$ cuja imagem é real, apresentamos o seguinte lema.

Lema 3.4. *Da definição de* (3.25) *e* (3.24)*. Para cada* $t \in [0,\infty)$ *e* $f \in H^4(\mathbb{R})$*, resulta que*

$$\langle [S_\lambda, A_\lambda]f, f\rangle = \sum_{j=0}^{3}(-1)^j \int_{\mathbb{R}} b_j(t,x)(\partial_x^j f)^2 \mathtt{dx}.$$

Demonstração: Sejam $t \in [0,\infty)$, $f \in H^4(\mathbb{R})$ e utilizando a continuidade de $\langle\ ,\ \rangle$ temos que

$$\begin{aligned} \langle [S(t)_\lambda, A_\lambda(t)]f, f\rangle &= \left\langle \sum_{j=0}^{3} \partial_x^j(b_j(t,x)\partial_x^j f), f \right\rangle \\ &= \sum_{j=0}^{3} \left\langle \partial_x^j(b_j(t,x)\partial_x^j f), f \right\rangle \\ &= \sum_{j=0}^{3} (-1)^j \left\langle (b_j(t,x)\partial_x^j f), \partial_x^j f \right\rangle \\ &= \sum_{j=0}^{3} (-1)^j \int_{\mathbb{R}} b_j(t,x) \left(\partial_x^j f\right)^2 \mathtt{dx}. \end{aligned}$$

■

A seguir, enunciamos o Lema 3.5 que irá auxiliar a demonstração da Proposição 21.

Lema 3.5. *Seja $h \in C_0^2(\mathbb{R})$ não negativa tal que*

$$\left|h^{(k)}(x)\right| \leq C_k h(x)\ ,\quad C_k > 0 \text{ e } k \in \{1,2\}.$$

Então, $\exists$ C>0 tal que

$$\int_{\mathbb{R}} h(x)(g'(x))^2 \mathtt{dx} \leq C\left\{\int_{\mathbb{R}} h(x)(g^{(2)}(x))^2 \mathtt{dx} + \int_{\mathbb{R}} h(x)(g(x))^2 \mathtt{dx}\right\},\quad \forall\, g \in H^4(\mathbb{R}) \cap C_0^3(\mathbb{R}). \tag{3.26}$$

Demonstração:

Como $\left|h^{(k)}(x)\right| \leq C_k h(x)$, temos que $h' \in C_0^1(\mathbb{R})$.

Logo, segue da integração por partes, de $g \in C_0^3(\mathbb{R})$, $g' \in C_0^2(\mathbb{R})$ e $h \in C_0^2(\mathbb{R})$,

$$\begin{aligned}
\int_{\mathbb{R}} h(x)(g'(x))^2 \mathtt{dx} &= \int_{\mathbb{R}} h(x)g'(x)g'(x)\mathtt{dx} = \underbrace{hg'g\big|_{-\infty}^{\infty}}_{=0} - \int_{\mathbb{R}} (h'g'g + hg^{(2)}g)\mathtt{dx} \\
&= -\int_{\mathbb{R}} h'g'g\mathtt{dx} - \int_{\mathbb{R}} hg^{(2)}g\mathtt{dx} \\
&= -\int_{\mathbb{R}} h'\left(\frac{g^2}{2}\right)' \mathtt{dx} - \int_{\mathbb{R}} hg^{(2)}g\mathtt{dx} \\
&= -\underbrace{h'\frac{g^2}{2}\Big|_{-\infty}^{\infty}}_{=0} + \int_{\mathbb{R}} \underbrace{h^{(2)}\left(\frac{g^2}{2}\right)}_{\left|h^{(2)}\right| \leq C_2 h} \mathtt{dx} - \int_{\mathbb{R}} hg^{(2)}g\mathtt{dx}.
\end{aligned}$$

Assim, como $2ab \leq a^2 + b^2$ para $a,\ b \in \mathbb{R}$ temos que

$$\begin{aligned}
\left|\int_{\mathbb{R}} h(x)(g'(x))^2 \mathtt{dx}\right| &\leq \frac{1}{2}\int_{\mathbb{R}} C_2 hg^2 \mathtt{dx} + \int_{\mathbb{R}} h^{1/2}\left|g^{(2)}\right| h^{1/2}|g|\,\mathtt{dx} \\
&\leq \frac{1}{2}\int_{\mathbb{R}} C_2 hg^2 \mathtt{dx} + \frac{1}{2}\int_{\mathbb{R}} h(g^{(2)})^2 \mathtt{dx} + \frac{1}{2}\int_{\mathbb{R}} hg^2 \mathtt{dx} \\
&= \frac{1}{2}\int_{\mathbb{R}} h(x)(g^{(2)}(x))^2 \mathtt{dx} + \frac{(C_2+1)}{2}\int_{\mathbb{R}} h(x)g^2(x)\mathtt{dx} \\
&\leq C\left(\int_{\mathbb{R}} h(x)(g^{(2)}(x))^2 \mathtt{dx} + \int_{\mathbb{R}} h(x)(g(x))^2 \mathtt{dx}\right),
\end{aligned}$$

com $C = \max\{\frac{C_2+1}{2}, \frac{1}{2}\}$. ■

Observação 3. *Para cada $\varepsilon > 0$, pelo Lema 3.5, resulta que*

$$\int_{\mathbb{R}} h(x)(g'(x))^2 \mathtt{dx} \leq C\left\{\varepsilon^2 \int_{\mathbb{R}} h(x)(g^{(2)}(x))^2 \mathtt{dx} + \frac{1}{\varepsilon^2}\int_{\mathbb{R}} h(x)(g(x))^2 \mathtt{dx}\right\}.$$

Para demonstrar a observação 3 denotamos $y = \varepsilon x$, $g_0(x) = g(\varepsilon x)$ e $h_0(x) = h(\varepsilon x)$ e aplicando o Lema 3.26 para h_0 e g_0 , obtemos que

$$\int_{\mathbb{R}} h_0(x)(g_0'(x))^2 \mathtt{dx} = C\left(\int_{\mathbb{R}} h_0(x)(g_0^{(2)}(x))^2 \mathtt{dx} + \int_{\mathbb{R}} h_0(x)\left(g_0(x)\right)^2 \mathtt{dx}\right) \tag{3.27}$$

$$= C\left(\varepsilon^3 \int_{\mathbb{R}} h(y)(g^{(2)}(y))^2 \mathtt{dy} + \frac{1}{\varepsilon}\int_{\mathbb{R}} h(y)\left(g(y)\right)^2 \mathtt{dy}\right). \tag{3.28}$$

e como

$$\int_{\mathbb{R}} h_0(x)(g_0'(x))^2 \mathtt{dx} = \int_{\mathbb{R}} \varepsilon h(y)(g'(y))^2 \mathtt{dy}. \tag{3.29}$$

Assim, de (3.27) e (3.29) obtemos o resultado.

3.4 Desigualdades com peso para f.

Em seguida, apresentaremos as proposições fundamentais para a demonstração do Teorema 7. Na seguinte proposição, utilizando (3.18), demonstramos a positividade de $\langle [S_\lambda, A_\lambda]\, f, f\rangle$ para $b \gtrsim \lambda^4$.

Proposição 21. *Seja $f \in H^4(\mathbb{R})$. Então, existem constantes $\lambda_0 \geq 1$ e $A_0 > 0$ tais que para todo $\lambda > \lambda_0$ e $b \gtrsim \lambda^4$, resulta que:*

$$\langle [S_\lambda, A_\lambda]\, f, f\rangle \geq A_0 \int_{\mathbb{R}} \varphi^{(2)}(r)\left[\lambda^9 f^2 + \lambda(\partial_x f)^2 + \lambda(\partial_x^2 f)^2 + \lambda(\partial_x^3 f)^2\right] \mathtt{dx},$$

onde

$$\begin{aligned} S_\lambda =& \Big((\partial_x\varphi)^2 - 4\,(\partial_x\varphi)\,\partial_x^3\varphi - 3\,\left(\partial_x^2\varphi\right)^2 - 6\,(\partial_x\varphi)^2\,\partial_x^2 - 12\,(\partial_x\varphi)\left(\partial_x^2\varphi\right)\partial_x\Big)\lambda^2 \\ &- (\partial_x\varphi)^4\,\lambda^4 + \partial_x^2 - \partial_x^4 - (\partial_t\varphi)\,\lambda, \end{aligned} \tag{3.30}$$

$$\begin{aligned} A_\lambda =& \lambda\left(-\partial_x^2\varphi + \partial_x^4\varphi - 2\,(\partial_x\varphi)\,\partial_x + 4\,(\partial_x\varphi)\,\partial_x^3 + 6\,\left(\partial_x^2\varphi\right)\partial_x^2 + 4\,(\partial_x)\,\partial_x^3\varphi\right) \\ &+ \lambda^3\left(6\,(\partial_x\varphi)^2\,\partial_x^2\varphi + 4\,(\partial_x\varphi)^3\,\partial_x\right) + \partial_t. \end{aligned} \tag{3.31}$$

Demonstração:

Primeiramente, consideremos o produto interno $\langle [S_\lambda, A_\lambda] f, f\rangle$. Temos que

$$
\begin{aligned}
\langle [S_\lambda, A_\lambda] f, f\rangle &= \left\langle \left[S_\lambda, \partial_t + \widetilde{A}_\lambda\right] f, f\right\rangle \\
&= \langle [S_1; A_1]f, f\rangle + 2\langle [S_2; A_1]f, f\rangle + \langle [S_2; A_2]f, f\rangle \\
&= \underbrace{16\lambda^7 \int_{\mathbb{R}} (\varphi'(r))^6 \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(1)} \underbrace{-64\lambda^5 \int_{\mathbb{R}} (\varphi'(r))^3 \varphi^{(2)}(r)\varphi^{(3)}(r) f^2 \mathtt{dx}}_{(2)} \\
&\quad \underbrace{-16\lambda^5 \int_{\mathbb{R}} (\varphi'(r))^4 \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(3)} \underbrace{-4\lambda^5 \int_{\mathbb{R}} (\varphi'(r))^4 \varphi^{(4)}(r) f^2 \mathtt{dx}}_{(4)} \\
&\quad \underbrace{-48\lambda^5 \int_{\mathbb{R}} (\varphi'(r))^2 \left(\varphi^{(2)}(r)\right)^3 f^2 \mathtt{dx}}_{(5)} \underbrace{-8b\,\lambda^4 \int_{\mathbb{R}} (\varphi'(r))^3 \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(6)} \\
&\quad \underbrace{-80\lambda^3 \int_{\mathbb{R}} \left(\varphi^{(3)}(r)\right)^2 \left(\varphi^{(2)}(r)\right) f^2 \mathtt{dx}}_{(7)} \underbrace{-16\lambda^3 \int_{\mathbb{R}} \left(\varphi^{(3)}(r)\right) (\varphi'(r)) \left(\varphi^{(2)}(r)\right) f^2 \mathtt{dx}}_{(8)} \\
&\quad + \underbrace{32\lambda^3 \int_{\mathbb{R}} \left(\varphi^{(3)}(r)\right) \varphi'(r) \left(\varphi^{(4)}(r)\right) f^2 \mathtt{dx}}_{(9)} + \underbrace{4\lambda^3 \int_{\mathbb{R}} (\varphi'(r))^2 \left(\varphi^{(2)}(r)\right) f^2 \mathtt{dx}}_{(10)} \\
&\quad \underbrace{-4\lambda^3 \int_{\mathbb{R}} (\varphi'(r))^2 \left(\varphi^{(4)}(r)\right) f^2 \mathtt{dx}}_{(11)} \underbrace{+4\lambda^3 \int_{\mathbb{R}} (\varphi'(r))^2 \left(\varphi^{(6)}(r)\right) f^2 \mathtt{dx}}_{(12)} \\
&\quad + \underbrace{24\lambda^3 \int_{\mathbb{R}} (\varphi'(r)) \left(\varphi^{(2)}(r)\right) \left(\varphi^{(5)}(r)\right) f^2 \mathtt{dx}}_{(13)} \underbrace{-36\lambda^3 \int_{\mathbb{R}} \left(\varphi^{(2)}(r)\right)^2 \left(\varphi^{(4)}(r)\right) f^2 \mathtt{dx}}_{(14)} \\
&\quad \underbrace{-8b\lambda^2 \int_{\mathbb{R}} \left(\varphi^{(3)}(r)\right) \left(\varphi^{(2)}(r)\right) f^2 \mathtt{dx}}_{(15)} + \underbrace{4b\lambda^2 \int_{\mathbb{R}} (\varphi'(r)) \left(\varphi^{(2)}(r)\right) f^2 \mathtt{dx}}_{(16)} \\
&\quad \underbrace{-8b\lambda^2 \int_{\mathbb{R}} (\varphi'(r)) \left(\varphi^{(4)}(r)\right) f^2 \mathtt{dx}}_{(17)} \underbrace{-12b\lambda^2 \int_{\mathbb{R}} \left(\varphi^{(2)}(r)\right) \left(\varphi^{(3)}(r)\right) f^2 \mathtt{dx}}_{(18)} \\
&\quad \underbrace{-\lambda \int_{\mathbb{R}} \varphi^{(8)}(r) f^2 \mathtt{dx}}_{(19)} \underbrace{-\lambda \int_{\mathbb{R}} \varphi^{(4)}(r) f^2 \mathtt{dx}}_{(20)} + \underbrace{2\lambda \int_{\mathbb{R}} \varphi^{(6)}(r) f^2 \mathtt{dx}}_{(21)} \\
&\quad + \underbrace{b^2\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(22)} + \underbrace{48\lambda^5 \int_{\mathbb{R}} (\varphi'(r))^4 \left(\varphi^{(2)}(r)\right) f_x^2 \mathtt{dx}}_{(23)} \\
&\quad \underbrace{-96\lambda^3 \int_{\mathbb{R}} \left(\varphi^{(3)}(r)\right) (\varphi'(r)) \left(\varphi^{(2)}(r)\right) f_x^2 \mathtt{dx}}_{(24)}
\end{aligned}
$$

$$\underbrace{-24\lambda^3 \int_{\mathbb{R}} (\varphi'(r))^2 \left(\varphi^{(4)}(r)\right) f_x^2 \mathtt{dx}}_{(25)} + \underbrace{48\lambda^3 \int_{\mathbb{R}} \left(\varphi^{(2)}(r)\right)^3 f_x^2 \mathtt{dx}}_{(26)}$$
$$+ \underbrace{24b\lambda^2 \int_{\mathbb{R}} (\varphi'(r)) \left(\varphi^{(2)}(r)\right) f_x^2 \mathtt{dx}}_{(27)} + \underbrace{4\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) f_x^2 \mathtt{dx}}_{(28)}$$
$$\underbrace{-12\lambda \int_{\mathbb{R}} \varphi^{(4)}(r) f_x^2 \mathtt{dx}}_{(29)} + \underbrace{8\lambda \int_{\mathbb{R}} \varphi^{(6)}(r) f_x^2 \mathtt{dx}}_{(30)}$$
$$+ \underbrace{48\lambda^3 \int_{\mathbb{R}} (\varphi'(r))^2 \varphi^{(2)}(r) f_{xx}^2 \mathtt{dx}}_{(31)} + \underbrace{16\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) f_{xx}^2 \mathtt{dx}}_{(32)}$$
$$\underbrace{-20\lambda \int_{\mathbb{R}} \varphi^{(4)}(r) f_{xx}^2 \mathtt{dx}}_{(33)} + \underbrace{16\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) f_{xxx}^2 \mathtt{dx}}_{(34)}. \tag{3.32}$$

com $r = x - bt$, $f_x^2 = (\partial_x f)^2$, $f_{xx}^2 = (\partial_x^2 f)^2$ e $f_{xxx}^2 = (\partial_x^3 f)^2$. Note que em (3.32) os termos (10), (22), (23), (26) (28), (31), (32), (34) são positivos e (27) é um termo que precisa ser estimado de uma forma adequada para facilitar a demonstração.

Para isto, segue de (3.15)-(3.17) (propriedades de φ) e $(\partial_x f)^2 = \partial_x^2 \left(\frac{f^2}{2}\right) - f\partial_x^2 f$, a seguinte equação

$$\begin{aligned}
(27) &= 24b\lambda^2 \int_{\mathbb{R}} \varphi'(r)\varphi^{(2)}(r)(\partial_x f)^2 \mathtt{dx} \\
&= 24b\lambda^2 \int_{\mathbb{R}} \varphi'(r)\varphi^{(2)}(r)\partial_x^2 \left(\frac{f^2}{2}\right) \mathtt{dx} - 24b\lambda^2 \int_{\mathbb{R}} \varphi'(r)\varphi^{(2)}(r) f\partial_x^2 f \mathtt{dx} \\
&= 12b\lambda^2 \int_{\mathbb{R}} \partial_x^2(\varphi'(r)\varphi^{(2)}(r)) f^2 \mathtt{dx} - 24b\lambda^2 \int_{\mathbb{R}} \varphi'(r)\varphi^{(2)}(r) f\partial_x^2 f \mathtt{dx} \\
&= 12b\lambda^2 \int_{\mathbb{R}} \left(\varphi^{(3)}(r)\varphi^{(2)}(r) + 2\varphi^{(2)}(r)\varphi^{(3)}(r) + \varphi'(r)\varphi^{(4)}(r)\right) f^2 \mathtt{dx} \\
&\quad - 24b\lambda^2 \int_{\mathbb{R}} \varphi'(r)\varphi^{(2)}(r) f\partial_x^2 f \mathtt{dx} \\
&= 36b\lambda^2 \int_{\mathbb{R}} \varphi^{(3)}(r)\varphi^{(2)}(r) f^2 \mathtt{dx} + 12b\lambda^2 \int_{\mathbb{R}} \varphi'(r)\varphi^{(4)}(r) f^2 \mathtt{dx} \\
&\quad - 24b\lambda^2 \int_{\mathbb{R}} \varphi'(r)\varphi^{(2)}(r) f\partial_x^2 f \mathtt{dx}.
\end{aligned} \tag{3.33}$$

Logo, usando que $2ab \leq a^2 + b^2$ em (3.33), temos

$$\begin{aligned}\left|24b\lambda^2 \int_{\mathbb{R}} \varphi'(r)\varphi^{(2)}(r) f \partial_x^2 f \mathtt{dx}\right| &\leq \int b\lambda^{1/2}(\varphi^{(2)}(r))^{1/2} f \cdot 24\lambda^{3/2} \left|\varphi'(r)\right| (\varphi^{(2)}(r))^{1/2}(\partial_x^2 f)\mathtt{dx} \\ &\leq \frac{24}{2\cdot 13}\lambda b^2 \int \varphi^{(2)}(r) f^2 \mathtt{dx} \\ &\quad + \frac{24\cdot 13}{2}\lambda^3 \int (\varphi'(r))^2 \varphi^{(2)}(r)(\partial_x^2 f)^2 \mathtt{dx}. \qquad (3.34)\end{aligned}$$

Daí,

$$\begin{aligned}(27) \geq 36b\lambda^2 \int_{\mathbb{R}} \varphi^{(3)}(r)\varphi^{(2)}(r) f^2 \mathtt{dx} + 12b\lambda^2 \int_{\mathbb{R}} \varphi'(r)\varphi^{(4)}(r) f^2 \mathtt{dx} \\ - \frac{12}{13}\lambda b^2 \int \varphi^{(2)}(r) f^2 \mathtt{dx} - 156\lambda^3 \int (\varphi'(r))^2 \varphi^{(2)}(r)(\partial_x^2 f)^2 \mathtt{dx}. \qquad (3.35)\end{aligned}$$

Assim, usando (3.35) em (3.32), obtemos

$$\begin{aligned}\langle [S_\lambda, A_\lambda] f, f\rangle \geq\ & \underbrace{16\lambda^7 \int_{\mathbb{R}} (\varphi'(r))^6 \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(1)} \underbrace{-64\lambda^5 \int_{\mathbb{R}} (\varphi'(r))^3 \varphi^{(2)}(r)\varphi^{(3)}(r) f^2 \mathtt{dx}}_{(2)} \\ & \underbrace{-16\lambda^5 \int_{\mathbb{R}} (\varphi'(r))^4 \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(3)} \underbrace{-4\lambda^5 \int_{\mathbb{R}} (\varphi'(r))^4 \varphi^{(4)}(r) f^2 \mathtt{dx}}_{(4)} \\ & \underbrace{-48\lambda^5 \int_{\mathbb{R}} (\varphi'(r))^2 \left(\varphi^{(2)}(r)\right)^3 f^2 \mathtt{dx}}_{(5)} \underbrace{-8b\,\lambda^4 \int_{\mathbb{R}} (\varphi'(r))^3 \varphi^{(2)}(r) f^2 \mathtt{dx}}_{(6)} \\ & \underbrace{-80\lambda^3 \int_{\mathbb{R}} \left(\varphi^{(3)}(r)\right)^2 \left(\varphi^{(2)}(r)\right) f^2 \mathtt{dx}}_{(7)} \underbrace{-16\lambda^3 \int_{\mathbb{R}} \left(\varphi^{(3)}(r)\right)(\varphi'(r))\left(\varphi^{(2)}(r)\right) f^2 \mathtt{dx}}_{(8)} \\ & + \underbrace{32\lambda^3 \int_{\mathbb{R}} \left(\varphi^{(3)}(r)\right)\varphi'(r)\left(\varphi^{(4)}(r)\right) f^2 \mathtt{dx}}_{(9)} + \underbrace{4\lambda^3 \int_{\mathbb{R}} (\varphi'(r))^2 \left(\varphi^{(2)}(r)\right) f^2 \mathtt{dx}}_{(10)} \\ & \underbrace{-4\lambda^3 \int_{\mathbb{R}} (\varphi'(r))^2 \left(\varphi^{(4)}(r)\right) f^2 \mathtt{dx}}_{(11)} \underbrace{+4\lambda^3 \int_{\mathbb{R}} (\varphi'(r))^2 \left(\varphi^{(6)}(r)\right) f^2 \mathtt{dx}}_{(12)} \\ & + \underbrace{24\lambda^3 \int_{\mathbb{R}} (\varphi'(r))\left(\varphi^{(2)}(r)\right)\left(\varphi^{(5)}(r)\right) f^2 \mathtt{dx}}_{(13)} \underbrace{-36\lambda^3 \int_{\mathbb{R}} \left(\varphi^{(2)}(r)\right)^2 \left(\varphi^{(4)}(r)\right) f^2 \mathtt{dx}}_{(14)} \\ & \underbrace{+28b\lambda^2 \int_{\mathbb{R}} \left(\varphi^{(3)}(r)\right)\left(\varphi^{(2)}(r)\right) f^2 \mathtt{dx}}_{(15)} + \underbrace{4b\lambda^2 \int_{\mathbb{R}} (\varphi'(r))\left(\varphi^{(2)}(r)\right) f^2 \mathtt{dx}}_{(16)}\end{aligned}$$

$$
\begin{aligned}
&\underbrace{+4b\lambda^2\int_{\mathbb{R}}(\varphi'(r))\left(\varphi^{(4)}(r)\right)f^2\mathtt{dx}}_{(17)}\underbrace{-12b\lambda^2\int_{\mathbb{R}}\left(\varphi^{(2)}(r)\right)\left(\varphi^{(3)}(r)\right)f^2\mathtt{dx}}_{(18)}\\
&\underbrace{-\lambda\int_{\mathbb{R}}\varphi^{(8)}(r)f^2\mathtt{dx}}_{(19)}\underbrace{-\lambda\int_{\mathbb{R}}\varphi^{(4)}(r)f^2\mathtt{dx}}_{(20)}+\underbrace{2\lambda\int_{\mathbb{R}}\varphi^{(6)}(r)f^2\mathtt{dx}}_{(21)}\\
&+\underbrace{\frac{1}{13}b^2\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)f^2\mathtt{dx}}_{(22)}+\underbrace{48\lambda^5\int_{\mathbb{R}}(\varphi'(r))^4\left(\varphi^{(2)}(r)\right)f_x^2\mathtt{dx}}_{(23)}\\
&\underbrace{-96\lambda^3\int_{\mathbb{R}}\left(\varphi^{(3)}(r)\right)(\varphi'(r))\left(\varphi^{(2)}(r)\right)f_x^2\mathtt{dx}}_{(24)}\\
&\underbrace{-24\lambda^3\int_{\mathbb{R}}(\varphi'(r))^2\left(\varphi^{(4)}(r)\right)f_x^2\mathtt{dx}}_{(25)}+\underbrace{48\lambda^3\int_{\mathbb{R}}\left(\varphi^{(2)}(r)\right)^3 f_x^2\mathtt{dx}}_{(26)}\\
&+\underbrace{4\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)f_x^2\mathtt{dx}}_{(27)}\\
&\underbrace{-12\lambda\int_{\mathbb{R}}\varphi^{(4)}(r)f_x^2\mathtt{dx}}_{(28)}+\underbrace{8\lambda\int_{\mathbb{R}}\varphi^{(6)}(r)f_x^2\mathtt{dx}}_{(29)}\\
&-\underbrace{108\lambda^3\int_{\mathbb{R}}(\varphi'(r))^2\varphi^{(2)}(r)f_{xx}^2\mathtt{dx}}_{(30)}+\underbrace{16\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)f_{xx}^2\mathtt{dx}}_{(31)}\\
&\underbrace{-20\lambda\int_{\mathbb{R}}\varphi^{(4)}(r)f_{xx}^2\mathtt{dx}}_{(32)}+\underbrace{16\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)f_{xxx}^2\mathtt{dx}}_{(33)}.
\end{aligned}
\tag{3.36}
$$

Em seguida, observamos que (1), (10), (22), (23), (26), (27), (31) e (33) são positivos.

Agora, estimamos inferiormente alguns termos de (3.36), usando as propriedades de $\left|\varphi^{(4)}(r)\right| \leq C_1\varphi^{(2)}(r)$ e $3/2 \leq |\varphi'(r)|$ para $|r| \geq 3/2$. Assim, temos

$$
\begin{aligned}
|(32)| =&\leq \left|-20\lambda\int_{\mathbb{R}}\varphi^{(4)}(r)(\partial_x^2 f)^2\mathtt{dx}\right|\\
&\leq\frac{40C_1}{3}\int_{\mathbb{R}}(\varphi'(r))^2\varphi^{(2)}(r)(\partial_x^2 f)^2\mathtt{dx}.
\end{aligned}
$$

Por outro lado, fazemos uma escolha de b tal que $b \geq C_4\lambda^4$ com $C_4 > 0$.

Logo, desde que $C_6 < \frac{C_4^2}{13}$, $\exists\, \lambda_0^1 > 0$, tal que $\forall\, \lambda > \lambda_0^1$

$$\begin{aligned}
&(22) - |(2) + \cdots + (9) + (11) + (13) + \cdots + (21)| \\
&\geq \Bigg[\frac{1}{13}\lambda b^2 - \big(64 \cdot 8C_1\lambda^5 + 256\lambda^5 + 4 \cdot 16C_1\lambda^5 + 48 \cdot 4\lambda^5 + 8 \cdot 8b\lambda^4 + 80C_1^2\lambda^3 \\
&\quad +16 \cdot 2C_1\lambda^3 + 32 \cdot 2C_1^2\lambda^3 + 4 \cdot 4C_1\lambda^3 + 24 \cdot 2C_1\lambda^3 + 36C_1\lambda^3 + 28C_1 b\lambda^2 \\
&\quad +8b\lambda^2 + 8C_1 b\lambda^2 + 12C_1 b\lambda^2 + C_1\lambda + C_1\lambda + 2C_1\lambda\Big] \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx} \\
&\geq \Bigg[\frac{1}{13}b^2\lambda - \big((512C_1 + 256 + 64C_1 + 192)\lambda^5 + 64b\lambda^4 + \big(80C_1^2 32C_1 + 64C_1^2 \\
&\quad +16C_1 + 48C_1 + 36C_1\big)\lambda^3 + (28C_1 b + 8b + 8C_1 b + 12C_1 b)\,\lambda^2 \\
&\quad +4C_1\lambda\Big] \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx} \\
&\geq C_6\lambda^9. \qquad (3.37)
\end{aligned}$$

Além disso, existe $\lambda_0^2 > 0$ tal que $\forall \lambda > \lambda_0^2$

$$\begin{aligned}
(23) - |(24) + (25) + (28) + (29)| &\geq \left(48\lambda^5 - \frac{256}{9}C_1\lambda^3 - \frac{32}{3}C_1\lambda^3 - \frac{64}{27}C_1\lambda \right. \\
&\quad \left. - \frac{128}{81}C_1\lambda\right) \int_{\mathbb{R}} (\varphi'(r))^4 \varphi^{(2)}(r) (\partial_x^2 f)^2 \mathtt{dx} \\
&> 0, \qquad (3.38)
\end{aligned}$$

e existe $\lambda_0^3 > 0$ tal que para todo $\lambda > \lambda_0^3$

$$\begin{aligned}
(30) + (32) &\geq (30) - |(32)| \\
&\geq -\left(108\lambda^3 + \frac{40C_1}{3}\lambda\right) \int_{\mathbb{R}} (\varphi'(r))^2 \varphi^{(2)}(r) \left(\partial_x^2 f\right)^2 \mathtt{dx} \\
&= -C_5\lambda^3 \int_{\mathbb{R}} (\varphi'(r))^2 \varphi^{(2)}(r) \left(\partial_x^2 f\right)^2 \mathtt{dx}. \qquad (3.39)
\end{aligned}$$

Utilizando (3.37)-(3.39) em (3.35), temos que

$$\begin{aligned}
\langle [S_\lambda, A_\lambda] f, f \rangle &\geq (1) + (2) + (33) \\
&\geq \underbrace{(1) + (10)}_{\geq 0} + \underbrace{(22) - |(2) + \cdots + (9) + (11) + \cdots + (21)|}_{> C_6\lambda^9 \int_{\mathbb{R}} \varphi^{(2)} f^2 \mathtt{dx}} \\
&\quad + \underbrace{(23) - |(24) + (25) + (28) + (29)|}_{> 0} + \underbrace{(26) + (27)}_{\geq 0} \\
&\quad + \underbrace{(30) - |(32)|}_{\leq 0} + \underbrace{(31) + (33)}_{\geq 0}
\end{aligned}$$

$$
\begin{aligned}
&\geq C_6\lambda^9\int_{\mathbb{R}}\varphi^{(2)}(r)f^2\mathtt{dx}+4\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)(\partial_x f)^2\mathtt{dx}+16\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)(\partial_x^2 f)^2\mathtt{dx}\\
&\quad+16\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)(\partial_x^3 f)^2\mathtt{dx}\\
&\quad-\left(108\lambda^3+\frac{40C_1}{3}\lambda\right)\int_{\mathbb{R}}\left(\varphi'(r)\right)^2\varphi^{(2)}(r)\left(\partial_x^2 f\right)^2\mathtt{dx}.
\end{aligned}
$$

Então,

$$
\begin{aligned}
&\langle[S_\lambda,A_\lambda]f,f\rangle+\left(108\lambda^3+\frac{40C_1}{3}\lambda\right)\int_{\mathbb{R}}\left(\varphi'(r)\right)^2\varphi^{(2)}(r)\left(\partial_x^2 f\right)^2\mathtt{dx}\\
&\geq\ C_6\lambda^9\int_{\mathbb{R}}\varphi^{(2)}(r)f^2\mathtt{dx}+4\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)(\partial_x f)^2+16\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)(\partial_x^2 f)^2\mathtt{dx}\\
&\quad+16\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)(\partial_x^3 f)^2\mathtt{dx}. \qquad (3.40)
\end{aligned}
$$

Usando o Lema 3.5 para $g=\partial_x f$, $\varepsilon_1=\frac{1}{C_2\lambda}$, $f(t,\cdot)\in H^4(\mathbb{R})\cap C_0^3(\mathbb{R})$ e $h(x)=\varphi^{(2)}(x-bt)$, resulta que

$$
\begin{aligned}
&\lambda^3\int_{\mathbb{R}}(\varphi'(r))^2\varphi^{(2)}(r)\left(\partial_x^2 f\right)^2\mathtt{dx}\\
&\leq\ C_7\left\{\lambda^3\varepsilon_1^2\int_{\mathbb{R}}\varphi^{(2)}(r)\left(\partial_x^3 f(x)\right)^2\mathtt{dx}+\lambda^3\left(\frac{1}{\varepsilon_1}\right)^2\int_{\mathbb{R}}\varphi^{(2)}(r)\left(\partial_x f(x)\right)^2\mathtt{dx}\right\}\\
&=\ C_7\left\{\lambda^3\left(\frac{1}{C_2\lambda}\right)^2\int_{\mathbb{R}}\varphi^{(2)}(r)\left(\partial_x^3 f(x)\right)^2\mathtt{dx}+\lambda^3\left(\frac{1}{\frac{1}{C_2\lambda}}\right)^2\int_{\mathbb{R}}\varphi^{(2)}(r)\left(\partial_x f(x)\right)^2\mathtt{dx}\right\}\\
&=\ \frac{C_7}{C_2^2}\,\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)\left(\partial_x^3 f(x)\right)^2\mathtt{dx}+(C_7\cdot C_2^2)\lambda^5\int\varphi^{(2)}(r)\left(\partial_x f(x)\right)^2\mathtt{dx}. \qquad (3.41)
\end{aligned}
$$

Daí, aplicando o Lema 3.5 com $g=f$, $\varepsilon_2=\frac{1}{C_3\lambda^2}$ e $h(x)=\varphi^{(2)}(x-bt)$, obtemos que

$$
\begin{aligned}
&\lambda^5\int\varphi^{(2)}(r)\left(\partial_x f\right)^2\mathtt{dx}\\
&\leq\ C_7\left\{\lambda^5\varepsilon_2^2\int_{\mathbb{R}}\varphi^{(2)}(r)\left(\partial_x^2 f(x)\right)^2\mathtt{dx}+\lambda^5\left(\frac{1}{\varepsilon_2}\right)^2\int_{\mathbb{R}}\varphi^{(2)}(r)\left(f(x)\right)^2\mathtt{dx}\right\}\\
&=\ C_7\left\{\lambda^5\left(\frac{1}{C_3\lambda^2}\right)^2\int_{\mathbb{R}}\varphi^{(2)}(r)\left(\partial_x^2 f(x)\right)^2\mathtt{dx}+\lambda^5\left(\frac{1}{\frac{1}{C_3\lambda^2}}\right)^2\int_{\mathbb{R}}\varphi^{(2)}(r)\left(f(x)\right)^2\mathtt{dx}\right\}\\
&=\ \frac{C_7}{C_3^2}\lambda\int_{\mathbb{R}}\varphi^{(2)}(r)\left(\partial_x^2 f(x)\right)^2\mathtt{dx}+(C_7\,C_3^2)\cdot\lambda^9\int_{\mathbb{R}}\varphi^{(2)}(r)\left(f(x)\right)^2\mathtt{dx}. \qquad (3.42)
\end{aligned}
$$

Então, segue de (3.41) e (3.42) que

$$
\begin{aligned}
&108\,\lambda^3 \int (\varphi'(r))^2 \varphi^{(2)}(r) \left(\partial_x^2 f\right)^2 \mathtt{dx} \\
&\leq\ 432\,\lambda^3 \int \varphi^{(2)}(r) \left(\partial_x^2 f\right)^2 \mathtt{dx} \\
&\leq\ 432 \left\{ \frac{C_7}{C_2^2}\,\lambda \int \varphi^{(2)}(r) \left(\partial_x^3 f(x)\right)^2 \mathtt{dx} + (C_7 \cdot C_2^2)\lambda^5 \int \varphi^{(2)}(r) \left(\partial_x f(x)\right)^2 \mathtt{dx} \right\} \\
&\leq\ 432\, \frac{C_7}{C_2^2}\,\lambda \int \varphi^{(2)}(r) \left(\partial_x^3 f(x)\right)^2 \mathtt{dx} + 432\,(C_7 \cdot C_2^2) \left\{ \frac{C_7}{C_3^2}\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) \left(\partial_x^2 f(x)\right)^2 \mathtt{dx} \right. \\
&\left. +(C_7\, C_3^2) \cdot \lambda^9 \int \varphi^{(2)}(r) \left(f(x)\right)^2 \mathtt{dx} \right\} \\
&\leq\ 432\, \frac{C_7}{C_2^2}\,\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) \left(\partial_x^3 f(x)\right)^2 \mathtt{dx} + 432\, \frac{C_7^2 \cdot C_2^2}{C_3^2}\lambda \int \varphi^{(2)}(r) \left(\partial_x^2 f(x)\right)^2 \mathtt{dx} \\
&+ 432 \cdot (C_7^2\, C_2^2\, C_3^2) \cdot \lambda^9 \int \varphi^{(2)}(r) \left(f(x)\right)^2 \mathtt{dx}. \qquad (3.43)
\end{aligned}
$$

Assim, utilizando (3.43) em (3.40) com $432\, C_1^2\, C_2^2\, C_3^2 < C_6/2$, $432\, \frac{C_7^2 \cdot C_2^2}{C_3^2} < 1$ e $432\, \frac{C_1}{C_2^2} < 1$, resulta que

$$
\begin{aligned}
\langle [S_\lambda, A_\lambda]\, f, f\rangle \geq\ & C_6 \lambda^9 \int_{\mathbb{R}} \varphi^{(2)}(r) f^2 \mathtt{dx} + 4\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) (\partial_x f)^2 \mathtt{dx} \\
& + 16\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) (\partial_x^2 f)^2 \mathtt{dx} + 16\lambda \int_{\mathbb{R}} \varphi^{(2)}(r) (\partial_x^3 f)^2 \mathtt{dx} \\
& - 108\,\lambda^3 \int_{\mathbb{R}} \left(\varphi'(r)\right)^2 \varphi^{(2)}(r) \left(\partial_x^2 f\right)^2 \mathtt{dx} \\
\geq & \int \varphi^{(2)}(r) \left[\left(C_6 \lambda^9 - 432 \cdot (C_7^2\, C_2^2\, C_3^2)\, \lambda^9\right) f^2 + 4\lambda (\partial_x f)^2 \right. \\
& \left. + \left(16\lambda - 432\, \frac{C_7^2 \cdot C_2^2}{C_3^2}\lambda\right) (\partial_x^2 f)^2 + \left(16\lambda - 432\, \frac{C_7}{C_2^2}\,\lambda\right) (\partial_x^3 f)^2 \right] \mathtt{dx} \\
\geq & \int \varphi^{(2)}(r) \left[\frac{C_6}{2} \lambda^9 f^2 + 4\lambda (\partial_x f)^2 + 15\lambda (\partial_x^2 f)^2 + 15\lambda (\partial_x^3 f)^2 \right] \mathtt{dx} \\
\geq & A_0 \int \varphi^{(2)}(r) \left[\lambda^9 f^2 + \lambda f_x^2 + \lambda (\partial_x^2 f)^2 + \lambda (\partial_x^3 f)^2 \right] \mathtt{dx},
\end{aligned}
$$

com $A_0 = \min\{\frac{C_6}{2}, 15, 4\}$ e $\lambda_0 = \max\{\lambda_0^1, \lambda_0^2, \lambda_3^2\}$ segue o resultado.

■

Proposição 22. *Existe uma constante $C > 0$, tal que*

$$
|\langle a(u)\, (-\lambda(\partial_r \varphi) f + \partial_x f)\,, f\rangle| \leq C\lambda, \qquad (3.44)
$$

onde C depende das constantes M_1, M_2 e M_3 dadas em (3.2)-(3.5).

Demonstração:

Como $f = e^{\lambda\varphi}u$, definimos a seguinte função

$$\begin{array}{rccc} \alpha: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & s & \longmapsto & \alpha(s), \end{array}$$

com

$$\alpha(s) := \begin{cases} \dfrac{1}{s^2}\displaystyle\int_0^s a(\tau)\tau \mathtt{d}\tau & , \quad s \neq 0 \\ 0 & , \quad s = 0 \end{cases}; \tag{3.45}$$

de (3.45), segue que $\alpha(u)$ é limitado e

$$e^{\lambda\varphi}\partial_x u = \partial_x f - \lambda\varphi' f, \tag{3.46}$$

$$\partial_x(u^2\alpha(u)) = a(u)u\partial_x u. \tag{3.47}$$

Além disso, usando a Proposição 20, e sabendo que $e^{\lambda\varphi(\cdot - bt)}u(t,\cdot) \in H^4(\mathbb{R}) \cap C_0^3(\mathbb{R})$ e como $\alpha(u)$ é limitado, resulta que

$$\lim_{d\to\pm\infty} e^{\lambda\varphi(d-bt)}u^2(t,d)\alpha(u(t,d)) = 0. \tag{3.48}$$

Logo, usando integração por partes e por (3.46)-(3.48), obtemos que

$$\begin{aligned} \int_{\mathbb{R}} a(u)(\partial_x f - \lambda\varphi' f) f \mathtt{dx} &= \int_{\mathbb{R}} a(u) e^{\lambda\varphi}\partial_x u \; e^{\lambda\varphi} u \mathtt{dx}, \\ &= \int_{\mathbb{R}} e^{2\lambda\varphi} a(u) u \partial_x u \mathtt{dx}, \\ &= \lim_{d\to\infty} e^{2\lambda\varphi}(u^2\alpha(u))\big|_{-d}^{d} - \int_{\mathbb{R}} 2\lambda\varphi'(r) e^{2\lambda\varphi} \; u^2\alpha(u)\mathtt{dx} \\ &= -2\lambda \int_{\mathbb{R}} \varphi'(r) e^{2\lambda\varphi} \; u^2\alpha(u)\mathtt{dx}. \end{aligned} \tag{3.49}$$

Assim, por (3.49), $|\varphi'| \leq 1$, $\alpha(u)$ é limitado e (3.5), temos que existe uma constante $C > 0$ tal que

$$\begin{aligned} \left| \int_{\mathbb{R}} a(u)(\partial_x f - \lambda\varphi' f) f \mathtt{dx} \right| &\leq 2\lambda \int_{\mathbb{R}} \left| \varphi'(r) e^{2\lambda\varphi} \; u^2\alpha(u) \right| \mathtt{dx} \\ &= \lambda\, C. \end{aligned} \tag{3.50}$$

■

Proposição 23. *Existe uma constante $C > 0$ tal que*

$$\|a(u)(\partial_x f - \lambda\varphi'(r)\cdot f)\|_{L^2}^2 \leq C \int \varphi^{(2)}(r) \left[\lambda^3 f^2 + \lambda\,(\partial_x f)^2\right] \mathtt{dx}, \tag{3.51}$$

com $C = C_1 \frac{M_1}{\lambda}(1 + M_2^{2(l-1)})\left((M_2M_3)^{1/2} + M_3\right)$ e $l \geq 2$, $l \in \mathbb{Z}^+$, onde C_1 uma constante positiva.

Demonstração:

Para demonstrar esta proposição, dividimos por etapas.

Etapa 1. Provaremos que

$$e^{\varphi(x-bt)/2}u^2(t,x) \le 2C_1^{1/6}\left((M_2M_3)^{1/2} + M_3\right). \tag{3.52}$$

De fato, usando Proposição 20 temos que $e^{\lambda\varphi(\cdot-bt)}u(t,\cdot) \in H^4(\mathbb{R}) \cap C_0^3(\mathbb{R})$ e pelo Teorema Fundamental do Cálculo resulta que

$$\begin{aligned} e^{\frac{\varphi(x-bt)}{2}}u^2(t,x) &= e^{\frac{\varphi(d-bt)}{2}}u^2(t,d) + \int_d^x \frac{\mathtt{d}}{\mathtt{dx}}\left(e^{\frac{\varphi(x-bt)}{2}}u^2\right)\mathtt{dx} \\ &\le e^{\frac{\varphi(d-bt)}{2}}u^2(t,d) + \int_d^x \left|\frac{\mathtt{d}}{\mathtt{dx}}\left(e^{\frac{\varphi(x-bt)}{2}}u^2\right)\right|\mathtt{dx} \\ &\le e^{\frac{\varphi(d-bt)}{2}}u^2(t,d) + \int_{-\infty}^{\infty} \left|\frac{\mathtt{d}}{\mathtt{dx}}\left(e^{\frac{\varphi(x-bt)}{2}}u^2\right)\right|\mathtt{dx}. \end{aligned} \tag{3.53}$$

Segue das hipóteses (3.3)-(3.5) que

$$\|\partial_x u\|_{L^2(\mathbb{R})} \le M_2^{1/2}, \tag{3.54}$$

$$\left\|e^{\frac{\varphi(\cdot-bt)}{2}}u(t,\cdot)\right\|_{L^2(\mathbb{R})} \le C_1^{1/6}M_3^{1/2}, \tag{3.55}$$

$$\left\|e^{\frac{\varphi(\cdot-bt)}{4}}u(t,\cdot)\right\|_{L^2(\mathbb{R})} \le C_1^{1/12}M_3^{1/2}. \tag{3.56}$$

Daí, por (3.54)-(3.56) e fazendo $d \to -\infty$ em (3.53), resulta que

$$\begin{aligned} e^{\frac{\varphi(x-bt)}{2}}u^2 &\le \int_{-\infty}^{\infty} \left|\frac{\mathtt{d}}{\mathtt{dx}}\left(e^{\frac{\varphi(x-bt)}{2}}u^2\right)\right|\mathtt{dx}, \\ &\le 2\,\|\partial_x u\|_{L^2(\mathbb{R})}\left\|e^{\frac{\varphi(x-bt)}{2}}u\right\|_{L^2(\mathbb{R})} + \left\|e^{\frac{\varphi(x-bt)}{2}}u\right\|_{L^2(\mathbb{R})}^2, \\ &\le 2C_1^{1/6}\left((M_2M_3)^{1/2} + M_3\right). \end{aligned}$$

Etapa 2. Existe uma constante C_0 tal que

$$e^{-\varphi(r)/2} \le C_0\varphi^{(2)}(r). \tag{3.57}$$

De fato, inicialmente trabalhamos para o caso $0 \le r \le 2$ sendo $\varphi \in C^8(\mathbb{R})$, segue que $\varphi^{(2)}$ é limitada num compacto e não nula, então existe uma constante C_0 tal que

$$1 \le C_0\varphi^{(2)}(r).$$

Logo, como $e^{-\frac{1}{2}r} \le 1$, segue-se o resultado.

Em seguida, para $2 < r$ o resultado segue do fato que

$$\lim_{r\to+\infty} e^{r/2}\varphi^{(2)}(r) = +\infty.$$

Assim, para todo $r > 2$, existe uma constante $1/C_0 > 0$ tal que

$$\frac{1}{C_0} \leq e^{r/2}\varphi^{(2)}(r), \tag{3.58}$$

logo segue-se o resultado.

Etapa 3. Existe uma constante $C_0 > 0$ tal que

$$|a(u)|^2 \leq C_0 M_1^2(1 + M_2^{2(l-1)})\varphi^{(2)}\left((M_2M_3)^{1/2} + M_3\right), \quad \text{para algum } l \geq 2,\ l \in \mathbb{Z}^+. \tag{3.59}$$

De fato, usando os resultados: (i) da Proposição 20, (3.52) e (3.57) dadas nas Etapas 1 e 2, respectivamente, temos que

$$\begin{aligned}
|a(u)|^2 &\leq M_1^2\left(|u| + |u|^l\right)^2, \\
&\leq 2M_1^2\left(|u|^2 + |u|^{2l}\right), \\
&= 2M_1^2\left(e^{\varphi(x-bt)/2}|u|^2 e^{-\varphi(x-bt)/2} + |u|^{2(l-1)} e^{\varphi(x-bt)/2}|u|^2 e^{-\varphi(x-bt)/2}\right), \\
&= 2M_1^2 e^{\varphi(x-bt)/2}|u|^2\left[e^{-\varphi(x-bt)/2} + e^{-\varphi(x-bt)/2}|u|^{2(l-1)}\right], \\
&\leq 2M_1^2 e^{\varphi(x-bt)/2}|u|^2\left[C_0\varphi^{(2)} + M_2^{2(l-1)}C_0\varphi^{(2)}\right], \\
&\leq 2M_1^2 C_0\varphi^{(2)}\left(1 + M_2^{2(l-1)}\right)e^{\varphi(x-bt)/2}|u|^2, \\
&= C_0M_1^2\left(1 + M_2^{2(j-1)}\right)\varphi^{(2)}\left(2(M_2M_3)^{1/2} + \frac{1}{2}M_3\right), \\
&\leq C_0M_1^2(1 + M_2^{2(l-1)})\left(M_3 + (M_2M_3)^{1/2}\right)\varphi^{(2)}.
\end{aligned}$$

Etapa 4. Existe uma constante $C_0 > 0$ tal que

$$|f_x - \lambda\varphi' f|^2 \leq C_0\left(|f_x|^2 + \lambda^2|f|^2\right). \tag{3.60}$$

Segue da limitação de φ' no Lema 3.2

$$\begin{aligned}
|f_x - \lambda\varphi' f|^2 &\leq 2\left(|f_x|^2 + \lambda^2(\varphi')^2|f|^2\right) \\
&\leq 2\left(|f_x|^2 + \lambda^2|f|^2\right).
\end{aligned}$$

Logo, voltando para demonstração da Proposição 23, segue dos resultados: Etapa 1, Etapa 2, Etapa 3, Etapa 4 e a Proposição 22 que

$$\begin{aligned}
\|a(u)(f_x - \lambda\varphi' f)\|_{L^2}^2 &= \int_{\mathbb{R}} |a(u)(f_x - \lambda\varphi' f)|^2 \,\mathtt{dx}, \\
&= \int_{\mathbb{R}} |a(u)|^2 \, |(f_x - \lambda\varphi' f)|^2 \,\mathtt{dx}, \\
&\leq \int_{\mathbb{R}} C_0 \varphi^{(2)} M_1^2 \left(1 + M_2^{2(j-1)}\right) \left(2(M_2 M_3)^{1/2} + \frac{1}{2} M_3\right) \\
&\quad \times \left(|f_x|^2 + \lambda^2 |f|^2\right) \mathtt{dx}, \\
&\leq C_0 \frac{M_1^2}{\lambda} \left(1 + M_2^{2(j-1)}\right) \left(2(M_2 M_3)^{1/2} + \frac{1}{2} M_3\right) \\
&\quad \times \int_{\mathbb{R}} \varphi^{(2)} \left(\lambda |f_x|^2 + \lambda^3 |f|^2\right) \mathtt{dx}, \\
&\leq C_0 \frac{M_1^2}{\lambda} \left(1 + M_2^{2(l-1)}\right) \left(M_3 + (M_2 M_3)^{1/2}\right) \\
&\quad \times \int_{\mathbb{R}} \varphi^{(2)}(r) \left(\lambda |f_x|^2 + \lambda^3 |f|^2\right) \mathtt{dx};
\end{aligned}$$

daí segue o resultado. ■

A seguinte proposição mostrará uma limitação superior da norma $L^2(\mathbb{R})$ da parte não linear $F = a(u)(\partial_x f - \lambda\varphi' f)$ da equação e com a Proposição 21 obtemos uma limitação inferior do termo

$$\langle [S_\lambda, A_\lambda] f, f\rangle - \langle F, F\rangle \geq \frac{3A_0}{4} \int_{\mathbb{R}} \varphi^{(2)}(r) \left[\lambda^9 f^2 + \lambda(\partial_x f)^2 + \lambda(\partial_x^2 f)^2 + \lambda(\partial_x^3 f)^2\right] \mathtt{dx},$$

notamos que, o lado esquerdo será útil na demonstração do Teorema 7.

Proposição 24. *Existe $\lambda_0^1 > 0$ tal que $\forall\ \lambda > \lambda_0^1$ resulta que*

$$\|a(u)\,(\partial_x f - \lambda\varphi' \cdot f)\|_{L^2(\mathbb{R})} \leq \frac{A_0}{4} \int_{\mathbb{R}} \varphi^{(2)}(r) \left[\lambda^9 f^2 + \lambda(\partial_x f)^2 + \lambda(\partial_x^2 f)^2 + \lambda(\partial_x^3 f)^2\right] \mathtt{dx}, \tag{3.61}$$

com A_0 definido na Proposição 21.

Demonstração: Utilizando a constante A_0 dado na Proposição 21, temos que existe $\lambda_0^1 > 0$, que depende de l, tal que $\forall\ \lambda > \lambda_0^1$

$$C_0 \frac{M_1^2}{\lambda} \left(1 + M_2^{2(l-1)}\right) \left((M_2 M_3)^{1/2} + M_3\right) \leq \frac{A_0}{4},$$

com $\lambda_0^1 = c_0 M_1^2 \left(1 + M_2^{2(l-1)}\right)\left((M_2 M_3)^{1/2} + M_3\right)$ e $c_0 = 4C_0/A_0$. ■

Na sequência, a Proposição 25 dará uma desigualdade importante para $\|u(t,\cdot)\|_{L^2(\mathbb{R})}$, que será fundamental para finalizarmos a demonstração.

Proposição 25. *Seja $u \in C^1([0,\infty), H^1(\mathbb{R}))$ uma solução da equação* (3.1) *com u satisfazendo* (3.5)*. Então*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})} \geq \|u(0,\cdot)\|_{L^2(\mathbb{R})}, \quad \forall\, t \in [0,\infty). \tag{3.62}$$

Demonstração: De fato, multiplicando por u na equação (3.1), resulta que

$$\partial_t\left(u^2/2\right) + \partial_x^2\left(u^2/2\right) - (\partial_x u)^2 + \partial_x^3(u^2/2) - \partial_x(\partial_x u)^2 - \left(\partial_x^2 u\right)^2 = a(u)u\partial_x u. \tag{3.63}$$

Logo, definindo a seguinte função

$$\begin{array}{rcl} G \;:\mathbb{R} & \longrightarrow & \mathbb{R} \\ z & \longmapsto & \displaystyle\int_0^z a(s)s\mathtt{ds}, \end{array}$$

observamos que $G \in C^\infty(\mathbb{R})$ e $G(0) = 0$.

Como $u(t,\cdot) \in H^2(\mathbb{R}) \cap C_0^1(\mathbb{R})$ para cada $t \in [0,\infty)$, segue da Proposição 9

$$G(u(t,\cdot)) \in H^2(\mathbb{R}) \cap C_0^1(\mathbb{R}), \quad \forall\, t \in [0,\infty). \tag{3.64}$$

Assim,

$$\lim_{x\to\pm\infty} G(u(t,x)) = 0. \tag{3.65}$$

Utilizando (3.65), $u(t,\cdot) \in H^4(\mathbb{R}) \cap C_0^3(\mathbb{R})$ e integrando com respeito a x, o lado esquerdo da equação (3.63) é equivalente a:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathtt{d}}{\mathtt{dt}}\left(\int_{\mathbb{R}} (u(t,x))^2\,\mathtt{dx}\right) + \int_{\mathbb{R}} \partial_x^2\left(\frac{u^2(t,x)}{2}\right)\mathtt{dx} - \int_{\mathbb{R}} (\partial_x u)^2\,\mathtt{dx} + \int_{\mathbb{R}} \partial_x^3(u^2/2)\mathtt{dx} \\
&- \int_{\mathbb{R}} \partial_x(\partial_x u)^2\,\mathtt{dx} - \int_{\mathbb{R}} \left(\partial_x^2 u\right)^2\mathtt{dx} \\
&= \frac{1}{2}\frac{\mathtt{d}}{\mathtt{dt}}\left(\int_{\mathbb{R}} (u(t,x))^2\,\mathtt{dx}\right) + \partial_x\left(\frac{u^2(t,x)}{2}\right)\Bigg|_{-\infty}^{+\infty} - \int_{\mathbb{R}} (\partial_x u)^2\,\mathtt{dx} + \partial_x^2\left(\frac{u^2}{2}\right)\Bigg|_{-\infty}^{\infty} \\
&- (\partial_x u)^2\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} \left(\partial_x^2 u\right)^2\mathtt{dx} \\
&= \frac{1}{2}\frac{\mathtt{d}}{\mathtt{dt}}\left(\int_{\mathbb{R}} (u(t,x))^2\,\mathtt{dx}\right) - \int_{\mathbb{R}} (\partial_x u)^2\,\mathtt{dx} - \int_{\mathbb{R}} \left(\partial_x^2 u\right)^2\mathtt{dx}.
\end{aligned} \tag{3.66}$$

e utilizando (3.65) no lado direito de (3.63), tem-se que

$$\begin{aligned}\int_{\mathbb{R}} \partial_x(G(u(t,x)))\mathtt{dx} &= G(u(t,x))|_{-\infty}^{+\infty} \\ &= 0. \end{aligned} \tag{3.67}$$

Então, segue de (3.66)-(3.67) que $\forall\, t \in [0,\infty)$

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})} \geq \|u(0,\cdot)\|_{L^2(\mathbb{R})}.$$

■

3.5 Demonstração do Teorema 7

Para a demonstração, consideramos $u \in C^1([0,\infty), H^1(\mathbb{R}))$ uma solução de (3.1), com $f(t,\cdot) \in H^4(\mathbb{R}) \cap C_0^3(\mathbb{R})$ e $\lambda \geq \max\left\{\lambda_0, c_0 M_1^2 \left(1 + M_2^{2(l-1)}\right)\left((M_2M_3)^{1/2} + M_3\right)\right\}$, lembremos que a função f satisfaz

$$\partial_x f = -\left(S_\lambda + \widetilde{A}_\lambda\right) f + F,$$

com $F = a(u)\left(\partial_x f - \lambda\varphi'(x-bt)f\right)$, $S_\lambda^* = S_\lambda$ e $\widetilde{A}_\lambda = -\widetilde{A}_\lambda$.

Daí temos, identidades necessárias para a demonstração do teorema como:

$$\frac{\mathtt{d}}{\mathtt{dt}}\langle f, f\rangle = -\,2\langle S_\lambda f, f\rangle + 2\langle F, f\rangle, \tag{3.68}$$

$$\frac{\mathtt{d}}{\mathtt{dt}}\langle S_\lambda f, f\rangle = -\,\langle [S_\lambda, A_\lambda]\, f, f\rangle - 2\langle S_\lambda f, S_\lambda f\rangle + 2\langle F, S_\lambda f\rangle, \tag{3.69}$$

com $[S_\lambda, A_\lambda] = \left[S_\lambda, \widetilde{A}_\lambda\right] - [\partial_t, S]$.

Inicialmente, construímos uma sequência $T_n \uparrow \infty$ tal que

$$\|S_\lambda(T_n)\|_{L^2(\mathbb{R})} \leq C_2\lambda, \quad n \in \mathbb{N},$$

com C_2 é uma constante positiva, independente de n.

Para isto, usamos (3.68), (3.69) e uma função diferenciável $\eta : [a,b] \to \mathbb{R}$ e após de aplicar integração por partes, segue que

$$\int_a^b \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} = -\frac{1}{2}\left(\eta'\langle f, f\rangle\right)\Big|_a^b + \int_a^b \eta'(t)\langle F, f\rangle \mathtt{dt} + \frac{1}{2}\int_a^b \eta''(t)\langle f, f\rangle \mathtt{dt}, \tag{3.70}$$

$$\begin{aligned}\int_a^b \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} &= \left(\eta\langle S_\lambda f, f\rangle\right)\big|_a^b + \int_a^b \eta(t)\langle [S_\lambda, A_\lambda] f, f\rangle \mathtt{dt} \\ &\quad + 2\int_a^b \eta(t)\langle S_\lambda f, S_\lambda f\rangle \mathtt{dt} - 2\int_a^b \eta(t)\langle F, S_\lambda f\rangle \mathtt{dt},\end{aligned} \tag{3.71}$$

isto permitirá estimar superiormente e inferiormente $\int_a^b \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt}$.

Por outro lado, segue das Proposições 21 e 24 a seguinte desigualdade

$$\begin{aligned}&\langle [S_\lambda, A_\lambda] f, f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle \geq \langle S_\lambda f, S_\lambda f\rangle \\ &+ \frac{A_0}{2}\int_{\mathbb{R}} \varphi^{(2)}(x-bt)\left[\lambda^9 f^2 + \lambda\left(\partial_x f\right)^2 + \lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right].\end{aligned} \tag{3.72}$$

a demonstração de (3.72) pode ser vista no Apêndice A.1.

No seguinte passo, encontraremos uma estimativa uniforme de $\langle S_\lambda f, S_\lambda f\rangle$, para isto definimos a seguinte função suave:

$$\begin{array}{llll}\eta_n: & \left[n-\frac{1}{2}, n+\frac{1}{2}\right] & \longrightarrow & \mathbb{R}, \\ & t & \longmapsto & \sin\left(\pi\left(t-n+\frac{1}{2}\right)\right),\end{array} \tag{3.73}$$

com a seguinte propriedade $\eta_n\left(n \pm \frac{1}{2}\right) = 0$ e utilizando (3.70) e (3.71), obtemos

$$\begin{aligned}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n'(t)\langle S_\lambda f, f\rangle \mathtt{dt} &= \left(\eta_n\langle S_\lambda f, f\rangle\right)\big|_{n-\frac{1}{2}}^{n+\frac{1}{2}} + \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\langle [S_\lambda, A_\lambda] f, f\rangle \mathtt{dt} \\ &\quad + 2\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\langle S_\lambda f, S_\lambda f\rangle \mathtt{dt} - 2\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\langle F, S_\lambda f\rangle \mathtt{dt}, \\ &= \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\left\{\langle [S_\lambda, A_\lambda] f, f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle\right\} \mathtt{dt} \\ &\geq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\Bigg\{\langle S_\lambda f, S_\lambda f\rangle + \frac{A_0}{2}\int_{\mathbb{R}} \varphi^{(2)}(r)\left[\lambda^9 f^2 + \lambda\left(\partial_x f\right)^2\right. \\ &\quad \left. + \lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right]\Bigg\} \mathtt{dt} \\ &\geq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t)\left\|S_\lambda f(t)\right\|_{L_x^2(\mathbb{R})}^2 \mathtt{dt}.\end{aligned} \tag{3.74}$$

Logo, usando as limitações de η_n, $\eta_n'(t)$ e $\eta_n''(t)$ uniforme por uma constante que não depende de n, Proposição 22 e (3.5) em (3.70), segue que existe $C_2 > 0$, independente de

n, tal que

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n'(t)\langle S_\lambda f, f\rangle \mathtt{dt} \leq C_2\lambda, \qquad \forall\, n \in \mathbb{N}. \tag{3.75}$$

Então, utilizando (3.74) e (3.75) resulta que

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t) \left\| S_\lambda f(t) \right\|_{L_x^2(\mathbb{R})}^2 \mathtt{dt} \leq C_2\lambda, \qquad \forall\, n \in \mathbb{N}.$$

Pelo teorema do valor médio, $\exists\, T_n \in \left[n, n+\frac{1}{4}\right]$ tal que $T_n \geq 1,\ T_n \to \infty$ e

$$\begin{aligned}
\frac{\eta_n(T_n)}{4} \left\| S_\lambda f(T_n) \right\|_{L_x^2(\mathbb{R})}^2 &= \int_{n}^{n+\frac{1}{4}} \eta_n(t) \left\| S_\lambda f(t) \right\|_{L_x^2(\mathbb{R})}^2 \mathtt{dt} \\
&\leq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t) \left\| S_\lambda f(t) \right\|_{L_x^2(\mathbb{R})}^2 \mathtt{dt} \\
&\leq C_2\lambda.
\end{aligned}$$

Assim, desde que $T_n \in \left[n, n+\frac{1}{4}\right]$, temos

$$\eta_n(T_n) \geq \frac{\sqrt{2}}{2}, \qquad \forall\, n \in \mathbb{N},$$

portanto,

$$\left\| S_\lambda f(T_n) \right\|_{L_x^2(\mathbb{R})}^2 \leq \frac{8\lambda C_2}{\sqrt{2}} = \lambda \widetilde{C}, \qquad \forall\, n \in \mathbb{N}, \tag{3.76}$$

para alguma constante $\widetilde{C} > 0$, independente de n.

Continuando com a demonstração do teorema, é necessário uma estimativa espaço tempo para f. Para isto consideramos uma função $\eta : [0, \infty] \to \mathbb{R}$ suave e positiva tal que $\eta(0) = 0$ e $\eta(t) = 1$, se $t \geq 1$. Logo, utilizando (3.70) temos

$$\int_0^{T_n} \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} = -\left.\frac{1}{2}\left(\eta'\langle f, f\rangle\right)\right|_0^{T_n} + \int_0^{T_n} \eta'(t)\langle F, f\rangle \mathtt{dt} + \frac{1}{2}\int_0^{T_n} \eta''(t)\langle f, f\rangle \mathtt{dt}. \tag{3.77}$$

De (3.77), (3.76) e Proposição 22, existe uma constante $C_3 > 0$, independente de n, tal que

$$\int_0^{T_n} \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} \leq C_3, \qquad \forall\, n \in \mathbb{N}. \tag{3.78}$$

Por outro lado, utilizando (3.71) e (3.72),

$$\begin{aligned}\int_0^{T_n} \eta'(t)\langle S_\lambda f, f\rangle \mathtt{dt} =& \left(\eta(t)\langle S_\lambda f, f\rangle\right)\Big|_0^{T_n} + \int_0^{T_n} \eta(t)\left\{\langle [S_\lambda, A_\lambda] f, f\rangle \right.\\ &\left. + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle\right\} \mathtt{dt}\\ \geq& \langle S_\lambda f(T_n,\cdot), f(T_n,\cdot)\rangle + \frac{A_0}{2}\int_0^{T_n}\eta(t)\int_{\mathbb{R}}\varphi^{(2)}(r)\left[\lambda^9 f^2\right.\\ &\left. +\lambda\left(\partial_x f\right)^2 + \lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right] \mathtt{dxdt}. \end{aligned} \tag{3.79}$$

Então, segue de (3.78) e (3.79)

$$\begin{aligned} C_3 - \langle S_\lambda f(T_n,\cdot), f(T_n,\cdot)\rangle \geq& \frac{A_0}{2}\int_0^{T_n}\eta(t)\int_{\mathbb{R}}\varphi^{(2)}(x-bt)\left[\lambda^9 f^2 + \lambda\left(\partial_x f\right)^2\right.\\ &\left. +\lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right] \mathtt{dxdt}. \end{aligned} \tag{3.80}$$

Logo, pela desigualdade de Cauchy-Schwarz, (3.76) e (3.5) em (3.80), existe uma constante $C_4 > 0$, independente de n, tal que

$$|\langle S_\lambda f(T_n,\cdot), f(T_n,\cdot)\rangle| \leq \|S_\lambda f(T_n,\cdot)\|_{L^2(\mathbb{R})}\|f(T_n,\cdot)\|_{L^2(\mathbb{R})} \leq \lambda C_4, \qquad \forall\, n \in \mathbb{N}.$$

Assim, resulta que

$$\int_0^{T_n}\eta(t)\int_{\mathbb{R}}\varphi^{(2)}(x-bt)\left[\lambda^9 f^2 + \lambda\left(\partial_x f\right)^2 + \lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right]\mathtt{dxdt} \leq C_5,$$

para alguma constante $C_5 > 0$, independente de n.

Daí, fazendo $T_n \to \infty$ e observando que $\eta(t) = 1$, $\forall\, t \geq 1$, obtemos

$$\int_1^{\infty}\int_{\mathbb{R}}\varphi^{(2)}(x-bt)\left[\lambda^9 f^2 + \lambda\left(\partial_x f\right)^2 + \lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right]\mathtt{dxdt} \leq C_5,$$

como $f = e^{\lambda\varphi(x-bt)}u$, então

$$\int_1^{\infty}\int_{\mathbb{R}}\varphi^{(2)}(x-bt)\left[\lambda^9 f^2 + \lambda\left(\partial_x f\right)^2 + \lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right]\mathtt{dxdt} < \infty.$$

o que implica

$$\int_1^{\infty}\int_{\mathbb{R}}\varphi^{(2)}(x-bt)e^{2\lambda\varphi(x-bt)}u^2(t,x)\mathtt{dxdt} < \infty. \tag{3.81}$$

Assim, usando (3.16) em (3.81), resulta que

$$\int_1^{\infty}\int_{\mathbb{R}} u^2(t,x)\mathtt{dxdt} < \infty. \tag{3.82}$$

Da Proposição 25 temos

$$\|u(t,\cdot)\|_{L^2}^2 \geq \|u(0,\cdot)\|_{L^2}^2 , \quad \forall t > 0. \tag{3.83}$$

Assim, de (3.82) e (3.83) temos que

$$u(0,x) = 0, \quad \forall\, x \in \mathbb{R}.$$

Logo, dado $t_0 > 0$ e considerando $u_{t_0}(t,x) = u(t+t_0, x+bt_0)$ para $(t,x) \in [0,\infty) \times \mathbb{R}$, temos que u_{t_0} satisfaz a mesma equação e as hipóteses (3.3), (3.4) e (3.5).

Então, segue o resultado

$$u_{t_0}(0,x) = u(t_0, x+bt_0) = 0, \quad \forall\, x \in \mathbb{R},$$

e consequentemente,

$$u(t,x) \equiv 0, \quad \forall\, (t,x) \in [0,\infty) \times \mathbb{R}.$$

Capítulo 4

Equação de Schrödinger de Quarta Ordem

Neste capítulo, o objetivo é obter condições de decaimento para a solução da equação linear de Schrödinger de quarta ordem com potencial real, com a finalidade de que a única solução seja nula.

Considere a seguinte equação linear de Schrödinger de quarta ordem

$$\partial_t u - i\partial_x^2 u - i\partial_x^4 u = \mathtt{i}V(t,x)u, \quad x \in \mathbb{R},\ t \in [0,\infty), \tag{4.1}$$

com $V(t,x)$ uma função potencial real e $u : [0,\infty) \times \mathbb{R} \to \mathbb{C}$.

A seguir, apresentamos o resultado principal deste Capítulo que é similar com [6], onde é obtido um resultado de unicidade para a equação de Schrödinger com potencial V real com $p \in (1, 4/3]$.

Teorema 8. *Seja $u \in C^1([0,\infty), L^2(\mathbb{R}))$ solução de (4.1) com potencial real $V \in L^\infty([0,\infty)\times \mathbb{R})$ tal que*

$$|V(t,x)| \leq \frac{C_1}{\langle x \rangle^{4-3p}} = \frac{C_1}{(1+|x|^2)^{\frac{4-3p}{2}}}, \tag{4.2}$$

com $p \in (1, 8/7]$.

Se existe uma constante $\lambda_0 > 0$, que depende de $\|V\|_{L^\infty([0,\infty)\times\mathbb{R})}$, $C_1 > 0$ e p com

$$\sup_{t\geq 0} \int_{\mathbb{R}} e^{\lambda_0 |x|^p} \left|\partial_x^j u(t,x)\right|^2 \mathtt{dx} < +\infty, \tag{4.3}$$

para $j \in \{0,1,2,3,4\}$, *então*

$$u \equiv 0.$$

Antes de demonstrar o Teorema 8, apresentamos alguns resultados prévios nas seguintes seções, que auxiliarão na demonstração do mesmo.

4.1 Termos simétricos e anti-simétricos

Nesta seção, obtemos uma nova equação linear a partir de (4.1). Para isto definimos a função

$$f(t,x) := e^{\lambda\theta(x)}u(t,x), \tag{4.4}$$

com u solução de (4.1), λ um parâmetro positivo e θ uma função diferenciável a ser escolhida posteriormente. Serão geradas desigualdades que auxiliarão na demonstração do Teorema 8.

Observamos de (4.4) que

$$\begin{aligned}\partial_t f &= e^{\lambda\theta}\partial_t u\\ &= e^{\lambda\theta}(\mathtt{i}V(e^{-\lambda\theta}f) + \mathtt{i}\partial_x^2(e^{-\lambda\theta}f) + \mathtt{i}\partial_x^4(e^{-\lambda\theta}f))\\ &= \mathtt{i}V\cdot f + \mathtt{i}e^{\lambda\theta}\left(\partial_x^2+\partial_x^4\right)(e^{-\lambda\theta}f)\\ &= \mathtt{i}V\cdot f + \mathtt{i}e^{\lambda\theta}\partial_x^2(e^{-\lambda\theta}f) + \mathtt{i}e^{\lambda\theta}\partial_x^4(e^{-\lambda\theta}f).\end{aligned}$$

Daí obtemos uma nova equação linear

$$\partial_t f = \mathtt{i}e^{\lambda\theta}\partial_x^4(e^{-\lambda\theta}f) + \mathtt{i}e^{\lambda\theta}\partial_x^4(e^{-\lambda\theta}f) + iV(t,x)f. \tag{4.5}$$

Em seguida, utilizando a expressão do lado direito de (4.5), obtemos uma separação em fatores simétricos e anti-simétricos, como foi obtido em [46], com a finalidade de gerar desigualdades para f.

Considere as seguintes funções

$$\begin{aligned}&R_j : [0,\infty) \longrightarrow \mathcal{L}(H^4(\mathbb{R}), L^2(\mathbb{R}))\\ &R_j(t)g(x) = \mathtt{i}e^{\lambda\theta(x)}\partial_x^{2j}(e^{-\lambda\theta(x)}g(x)),\end{aligned} \tag{4.6}$$

e

$$\begin{aligned} &R_j^*: [0,\infty) \longrightarrow \mathcal{L}(H^4(\mathbb{R}), L^2(\mathbb{R})) \\ &R_j^*(t)g(x) = -\mathtt{i}e^{-\lambda\theta(x)}\partial_x^{2j}(e^{\lambda\theta(x)}g(x)), \end{aligned} \tag{4.7}$$

para $j \in \{1,2\}$.
A seguir, enunciamos o Lema 4.1 que descreve a relação das expressões (4.6) e (4.7).

Lema 4.1. *Para $j \in \{1,2\}$ e $g, h \in H^4(\mathbb{R};\mathbb{C})$, temos que*

$$\int_{\mathbb{R}} (R_j(t)g)(x)\overline{h(x)}\mathtt{dx} = \int_{\mathbb{R}} g(x)\overline{(R_j^*(t)h)(x)}\mathtt{dx}. \tag{4.8}$$

Em seguida, definimos as partes simétricas e anti-simétricas de (4.5), que são definidos por:

$$\begin{aligned} S_1 &= \frac{R_1 + R_1^*}{2} \\ &= \mathtt{i}\lambda\left(-\partial_x\theta - 2\partial_x\theta\partial_x\right), \end{aligned} \tag{4.9}$$

$$\begin{aligned} A_1 &= \frac{R_1 - R_1^*}{2} \\ &= \mathtt{i}\left(\lambda^2(\partial_x\theta)^2 + \partial_x^2\right), \end{aligned} \tag{4.10}$$

$$\begin{aligned} S_2 &= \frac{R_2 + R_2^*}{2} \\ &= \mathtt{i}\left[-6\lambda^3(\partial_x\theta)^2\partial_x^2\theta + \left(-4\lambda^3(\partial_x\theta)^3 - 4\lambda\partial_x^3\theta\right)\partial_x - 6\lambda\partial_x^2\theta\partial_x^2 - 4\lambda\partial_x\theta\partial_x^3\right], \end{aligned} \tag{4.11}$$

$$\begin{aligned} A_2 &= \frac{R_2 + R_2^*}{2} \\ &= \mathtt{i}\left[\left(\lambda^4(\partial_x\theta)^4 + 4\lambda^2\partial_x\theta\partial_x^3\theta + 3\lambda^2(\partial_x^2\theta)^2\right) + 12\lambda^2\partial_x\theta\partial_x^2\theta\partial_x + 6\lambda^2(\partial_x\theta)^2\partial_x^2 + \partial_x^4\right]. \end{aligned} \tag{4.12}$$

A partir das expressões (4.9)-(4.12), definimos

$$S_\lambda = S_1 + S_2 \quad \text{e} \quad A_\lambda = A_1 + A_2.$$

Daí, temos

$$\begin{aligned} S_\lambda &= \frac{R_1 + R_1^* + R_2 + R_2^*}{2} \\ &= \mathtt{i}\lambda\left(-\partial_x^2\theta - \partial_x^4\theta - 2\partial_x\theta\partial_x - 4(\partial_x\theta)^3\partial_x - 6\partial_x^2\theta\partial_x^2 - 4\partial_x\theta\partial_x^3\right) \\ &\quad + \mathtt{i}\lambda^3\left(-6(\partial_x\theta)^2\partial_x^2\theta - 4(\partial_x\theta)^3\partial_x\right), \end{aligned} \tag{4.13}$$

$$\begin{aligned} A_\lambda &= \frac{R_1 - R_1^* + R_2 - R_2^*}{2} \\ &= \mathtt{i}\lambda^2\left((\partial_x\theta)^2 + 4\partial_x\theta\partial_x^3\theta + 3(\partial_x^2\theta)^2 + 12\partial_x\theta\partial_x^2\theta\partial_x + 6(\partial_x\theta)^2\partial_x^2\right) \\ &\quad + \mathtt{i}(\partial_x^2 + \partial_x^4) + \mathtt{i}\lambda^2(\partial_x\theta)^4. \end{aligned} \tag{4.14}$$

Logo, segue do Lema 4.1 que

$$\int_{\mathbb{R}} (S_\lambda(t)g)(x)\overline{h(x)}\mathtt{dx} = \int_{\mathbb{R}} g(x)\overline{(S_\lambda^*(t)h)(x)}\mathtt{dx}, \quad \forall\, g, h \in H^4(\mathbb{R}),$$

e

$$\int_{\mathbb{R}} (A_\lambda(t)g)(x)\overline{h(x)}\mathtt{dx} = -\int_{\mathbb{R}} g(x)\overline{(A_\lambda^*(t)h)(x)}\mathtt{dx}, \quad \forall\, g, h \in H^4(\mathbb{R}).$$

Na seção seguinte, utilizando S_λ e A_λ definidos na seção anterior, vamos demonstrar algumas propriedades do comutador $[S_\lambda, A_\lambda]$, para f definido em (4.4).

4.2 O Comutador $[S_\lambda, A_\lambda]f$

Estudamos as propriedades do comutador $[S_\lambda, A_\lambda]$ com $1 < p \leq 8/7$, obtendo resultados satisfatórios.

$$\begin{aligned} [S_\lambda, A_\lambda] &= S_\lambda A_\lambda - A_\lambda S_\lambda \\ &= [S_1, A_1] + [S_1, A_2] + [S_2, A_1] + [S_2, A_2] \\ &= [S_1, A_1] + 2[S_1, A_2] + [S_2, A_2] \end{aligned}$$

em que

$$\begin{aligned}
[S_1, A_1] &= 4\lambda^3\partial_x^2\theta(\partial_x\theta)^2 - \lambda\partial_x^4\theta - 4\lambda\partial_x^3\theta\partial_x - 4\lambda\partial_x^2\theta\partial_x^2 \\
&= 4\lambda^3\partial_x^2\theta(\partial_x\theta)^2 - \lambda\partial_x^4\theta - \partial_x\left(4\lambda\partial_x^2\theta\partial_x\right), \qquad (4.15)
\end{aligned}$$

$$\begin{aligned}
[S_1, A_2] &= 8\lambda^5(\partial_x\theta)^4\partial_x^2\theta + 2\lambda^3(\partial_x\theta)^2\partial_x^4\theta + 8\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^3\theta - \lambda\partial_x^6\theta - 6\lambda\partial_x^5\theta\partial_x \\
&\quad - 14\lambda\partial_x^4\theta\partial_x^2 - 16\partial_x^3\theta\partial_x^3 - 8\lambda\partial_x^2\theta\partial_x^4 \\
&= 8\lambda^5(\partial_x\theta)^4\partial_x^2\theta + 2\lambda^3(\partial_x\theta)^2\partial_x^4\theta + 8\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^3\theta - \lambda\partial_x^6\theta \\
&\quad + \partial_x(-6\lambda\partial_x^4\theta\partial_x) + \partial_x^2(-8\lambda\partial_x^2\theta\partial_x^2), \qquad (4.16)
\end{aligned}$$

$$\begin{aligned}
[S_2, A_1] &= 8\lambda^5(\partial_x\theta)^4\partial_x^2\theta + 2\lambda^3(\partial_x\theta)^2\partial_x^4\theta + 8\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^3\theta - \lambda\partial_x^6\theta \\
&\quad + \partial_x(-6\lambda\partial_x^4\theta\partial_x) + \partial_x^2(-8\lambda\partial_x^2\theta\partial_x^2), \qquad (4.17)
\end{aligned}$$

$$\begin{aligned}
[S_2, A_2] &= 16\lambda^7(\partial_x\theta)^6\partial_x^2\theta - 48\lambda^5(\partial_x\theta)^2(\partial_x^2\theta)^3 - 64\lambda^5(\partial_x\theta)^3\partial_x^2\theta\partial_x^3\theta \\
&\quad - 4\lambda^5(\partial_x\theta)^4\partial_x^4\theta - 36\lambda^3(\partial_x^2\theta)^2\partial_x^4\theta + 24\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^5\theta - 80\lambda^3\partial_x^2\theta(\partial_x^3\theta)^2 \\
&\quad + 4\lambda^3(\partial_x\theta)^2\partial_x^6\theta + 36\lambda^3\partial_x\theta\partial_x^3\theta\partial_x^4\theta - \lambda\partial_x^8\theta \\
&\quad + \partial_x\Big[\big(-48\lambda^5(\partial_x\theta)^4\partial_x^2\theta - 48\lambda^3(\partial_x^2\theta)^3 + 96\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^3\theta + 24\lambda^3(\partial_x\theta)^2\partial_x^4\theta \\
&\quad -8\lambda\partial_x^6\theta\big)\partial_x\Big] \\
&\quad + \partial_x^2\big[\big(48\lambda^3(\partial_x\theta)^2\partial_x^2\theta - 20\lambda\partial_x^4\theta\big)\partial_x^2\big] + \partial_x^3\big[\big(48\lambda^3(\partial_x\theta)^2\partial_x^2\theta - 68\lambda\partial_x^4\theta\big)\partial_x^3\big]. \qquad (4.18)
\end{aligned}$$

Observemos que $[S_1, A_2] = [S_2, A_1]$.

Em seguida, verificamos que o comutador pode ser expresso por

$$[S_\lambda, A_\lambda] = \sum_{j=0}^{3} \partial_x^j\left(b_j(x)\partial_x^j\right), \qquad (4.19)$$

com $b_j \in C_b(\mathbb{R})$ e

$$\begin{aligned}
b_0 &= 4\lambda^3\partial_x^2\theta(\partial_x\theta)^2 - \lambda\partial_x^4\theta + 16\lambda^5(\partial_x\theta)^4\partial_x^2\theta + 4\lambda^3(\partial_x\theta)^2\partial_x^4\theta + 16\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^3\theta - 2\lambda\partial_x^6\theta \\
&\quad + 16\lambda^7(\partial_x\theta)^6\partial_x^2\theta - 48\lambda^5(\partial_x\theta)^2(\partial_x^2\theta)^3 - 64\lambda^5(\partial_x\theta)^3\partial_x^2\theta\partial_x^3\theta \\
&\quad - 4\lambda^5(\partial_x\theta)^4\partial_x^4\theta - 36\lambda^3(\partial_x^2\theta)^2\partial_x^4\theta + 24\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^5\theta - 80\lambda^3\partial_x^2\theta(\partial_x^3\theta)^2 \\
&\quad + 4\lambda^3(\partial_x\theta)^2\partial_x^6\theta + 36\lambda^3\partial_x\theta\partial_x^3\theta\partial_x^4\theta - \lambda\partial_x^8\theta,
\end{aligned}$$

$$
\begin{aligned}
b_1 &= 4\lambda\partial_x^2\theta - 12\lambda\partial_x^4\theta + \big(- 48\lambda^5(\partial_x\theta)^4\partial_x^2\theta - 48\lambda^3(\partial_x^2\theta)^3 + 96\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^3\theta \\
&\quad + 24\lambda^3(\partial_x\theta)^2\partial_x^4\theta - 8\lambda\partial_x^6\theta\big), \\
b_2 &= - 16\lambda\partial_x^2\theta + 48\lambda^3(\partial_x\theta)^2\partial_x^2\theta - 20\lambda\partial_x^4\theta, \\
b_3 &= 48\lambda^3(\partial_x\theta)^2\partial_x^2\theta - 68\lambda\partial_x^4\theta.
\end{aligned}
$$

Na sequência, descrevemos o comutador no espaço $H^4(\mathbb{R})$, cuja demonstração é análoga a do Lema 3.3.

Lema 4.2. *Para cada $t \in [0, \infty)$ temos que o comutador está bem definido como*

$$
\begin{aligned}
[S_\lambda(t), A_\lambda(t)] : \quad H^4(\mathbb{R}) &\longrightarrow H^{-2}(\mathbb{R}) \\
f(t,\cdot) &\longmapsto [S_\lambda(t), A_\lambda(t)]f(t,\cdot),
\end{aligned}
$$

e é um operador contínuo.

A demonstração segue de maneira análoga ao Lema 2.3.

A seguinte identificação segue da equação (4.19) e a demonstração é análoga ao Lema 2.4.

Lema 4.3. *O produto interno $\langle [S_\lambda(t), A_\lambda(t)]f, f\rangle$ pode ser escrito*

$$
\langle [S_\lambda(t), A_\lambda(t)]f, f\rangle = \sum_{j=0}^{3} \int_{\mathbb{R}} a_j(x) \left|\partial_x^j f(t,x)\right|^2 \mathtt{dx}, \tag{4.20}
$$

com $a_j \in C_b(\mathbb{R})$.

A seguir, definimos uma função adequada para gerar as desigualdades satisfatórias para f.

Lema 4.4. *Existe uma função $\varphi_0 \in C^8(\mathbb{R})$, par e não negativa com as seguintes propriedades.*

(i) $\varphi_0(r) = r^p + \beta,$ *para* $r \geq 1,$ $p \in (1, 8/7]$ *e* $\beta = a_1 + \cdots + a_8 - 1,$

(ii) $\varphi_0(r) = a_0 + a_1 r^2 + \cdot + a_8 r^{16}, \quad r \in [0,1],$

(iii) $\varphi_0(0) = 0,$

(iv) $\varphi_0(r) > 0,$ *para* $r > 0,$

(v) $\varphi_0^{(2)}(r) > 0, \quad \forall\, r \in (0,1),$

(vi) $\varphi_0(r) \leq Mr^p, \quad \forall\, r \in [0,\infty)$ *e* $M \geq 3/2$.

Para maiores detalhes da demonstração do Lema 4.4, veja o Apêndice B.4.

A partir das propriedades de φ_0 no Lema 4.4, fazemos a seguinte escolha $\varphi = \varphi_0/(2M)$, e por conveniência fazemos a seguinte definição

$$\theta(x) := \varphi(r), \tag{4.21}$$

com $r = |x|$ e M definido na Afirmação 6.

Na seção seguinte, enunciaremos algumas propriedades da solução da equação (4.1).

4.3 Propriedade da solução (4.1)

Sob a hipótese de decaimento em (4.3), enunciamos a seguinte proposição.

Proposição 26. *Seja u solução de* (4.1) *tal que* $u \in C^1([0,\infty), L^2(\mathbb{R}))$*:*

(i) Se u satisfaz (4.3)*, então* $u(t,\cdot), e^{\lambda\varphi(\cdot)}u(t,\cdot) \in H^4(\mathbb{R}) \cap C_0^3(\mathbb{R}), \quad \forall\, t \in [0,\infty).$

(ii) Se u satisfaz (4.3)*, então* $u(t,\cdot) \in L^\infty(\mathbb{R}), \quad \forall\, t \in [0,\infty).$

Demonstração: De maneira análoga a demonstração da Proposição 11. ■

4.4 Desigualdades para f

Nesta seção, desenvolvemos desigualdades necessárias para a demonstração do Teorema 8. Inicialmente, o comutador é expresso em 27 termos como segue:

$$\begin{aligned}\big\langle [S_\lambda, A_\lambda] f, f\big\rangle &= (1) + (2) + \cdots + (27)\\ &= \underbrace{\int_{\mathbb{R}} 16\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\,|f|^2\,\mathtt{dx}}_{(1)} \underbrace{- \int_{\mathbb{R}} 4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r)\,|f|^2\,\mathtt{dx}}_{(2)}\\ &\quad \underbrace{- \int_{\mathbb{R}} 64\lambda^5\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\,|f|^2\,\mathtt{dx}}_{(3)} + \underbrace{\int_{\mathbb{R}} 16\lambda^5(\partial_x\varphi(r))^4\partial_x^2\varphi(r)\,|f|^2\,\mathtt{dx}}_{(4)}\end{aligned}$$

$$
\begin{aligned}
&\underbrace{-\int_{\mathbb{R}} 48\lambda^5(\varphi'(r))^2\varphi^{(2)}(r))^3\,|f|^2\,\mathtt{dx}}_{(5)}+\underbrace{\int_{\mathbb{R}} 32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r)\,|f|^2\,\mathtt{dx}}_{(6)}\\
&+\underbrace{\int_{\mathbb{R}} 4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)\,|f|^2\,\mathtt{dx}}_{(7)}\underbrace{-\int_{\mathbb{R}} 36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r)\,|f|^2\,\mathtt{dx}}_{(8)}\\
&\underbrace{-\int_{\mathbb{R}} 80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2\,|f|^2\,\mathtt{dx}}_{(9)}+\underbrace{\int_{\mathbb{R}} 16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\,|f|^2\,\mathtt{dx}}_{(10)}\\
&+\underbrace{\int_{\mathbb{R}} 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\,|f|^2\,\mathtt{dx}}_{(11)}+\underbrace{\int_{\mathbb{R}} 4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r)\,|f|^2\,\mathtt{dx}}_{(12)}\\
&+\underbrace{\int_{\mathbb{R}} 24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r)\,|f|^2\,\mathtt{dx}}_{(13)}\underbrace{-\int_{\mathbb{R}} \lambda\varphi^{(4)}(r)\,|f|^2\,\mathtt{dx}}_{(14)}\\
&\underbrace{-\int_{\mathbb{R}} 2\lambda\varphi^{(6)}(r)\,|f|^2\,\mathtt{dx}}_{(15)}\underbrace{-\int_{\mathbb{R}} \lambda\varphi^{(8)}(r)|f|^2\mathtt{dx}}_{(16)}\\
&+\underbrace{\int_{\mathbb{R}} 48\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\,|\partial_x f|^2\,\mathtt{dx}}_{(17)}\underbrace{-24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)\,|\partial_x f|^2\,\mathtt{dx}}_{(18)}\\
&\underbrace{-\int_{\mathbb{R}} 96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\,|\partial_x f|^2\,\mathtt{dx}}_{(19)}+\underbrace{\int_{\mathbb{R}} 48\lambda^3(\varphi^{(2)}(r))^3\,|\partial_x f|^2\,\mathtt{dx}}_{(20)}\\
&+\underbrace{\int_{\mathbb{R}} 12\lambda\varphi^{(4)}(r)\,|\partial_x f|^2\,\mathtt{dx}}_{(21)}+\underbrace{\int_{\mathbb{R}} 4\lambda\varphi^{(2)}(r)\,|\partial_x f|^2\,\mathtt{dx}}_{(22)}\\
&+\underbrace{\int_{\mathbb{R}} 8\lambda\varphi^{(6)}(r)|\partial_x f|^2\mathtt{dx}}_{(23)}\\
&+\underbrace{\int_{\mathbb{R}} \left[48\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\,\left|\partial_x^2 f\right|^2\,\mathtt{dx}\right.}_{(24)}\underbrace{-\int_{\mathbb{R}} 16\lambda\varphi^{(2)}(r)\,\left|\partial_x^2 f\right|^2\,\mathtt{dx}}_{(25)}\\
&\underbrace{-\int_{\mathbb{R}} \left.20\lambda\varphi^{(4)}(r)\right]|\partial_x^2 f|^2\mathtt{dx}}_{(26)}\\
&+\underbrace{\int_{\mathbb{R}} \left[16\lambda\varphi^{(2)}(r)\right]|\partial_x^3 f|^2\mathtt{dx}}_{(27)}
\end{aligned}
\tag{4.22}
$$

para abreviar as notações deixamos subentendido que as derivadas de φ são com relação a r.

Assim, como $1 < p \leq 8/7$, obtemos que

$$
\begin{aligned}
\big\langle \big[S_\lambda, A_\lambda\big] f, f \big\rangle = & \int_{\mathbb{R}} \big[16\lambda^7(\varphi')^6\varphi^{(2)} - 4\lambda^5(\varphi')^4\varphi^{(4)} - 64\lambda^5\varphi'\varphi^{(2)}\varphi^{(3)} \\
& + 16\lambda^5(\varphi')^4\varphi^{(2)} - 48\lambda^5(\varphi')^2(\varphi^{(2)})^3 + 32\lambda^3\varphi'\varphi^{(2)}\varphi^{(4)} \\
& + 4\lambda^3(\varphi')^2\varphi^{(4)} - 36\lambda^3(\varphi^{(2)})^2\varphi^{(4)} - 80\lambda^3(\varphi)^{(2)}(\varphi^{(3)})^2 \\
& + 16\lambda^3\varphi'\varphi^{(2)}\varphi^{(3)} + 4\lambda^3(\varphi')^2\varphi^{(2)} + 4\lambda^3(\varphi')^2\varphi^{(6)} \\
& + 24\lambda^3\varphi'\varphi^{(2)}\varphi^{(5)} - \lambda\varphi^{(4)} - 2\lambda\varphi^{(6)} - \lambda\varphi^{(8)}\big]|f|^2\mathtt{dx} \\
& + \int_{\mathbb{R}} \big[48\lambda^5(\varphi')^4\varphi^{(2)} - 24\lambda^3(\varphi')^2\varphi^{(4)} - 96\lambda^3\varphi'\varphi^{(2)}\varphi^{(3)} \\
& + 48\lambda^3(\varphi^{(2)})^3 + 12\lambda\varphi^{(4)} + 4\lambda\varphi^{(2)} + 8\lambda\varphi^{(6)}\big]|\partial_x f|^2\mathtt{dx} \\
& + \int_{\mathbb{R}} \big[48\lambda^3(\varphi')^2\varphi^{(2)} - 16\lambda\varphi^{(2)} - 20\lambda\varphi^{(4)}\big]|\partial_x^2 f|^2\mathtt{dx} \\
& + \int_{\mathbb{R}} \big[16\lambda\varphi^{(2)}\big]|\partial_x^3 f|^2\mathtt{dx} \\
\leq\ & C\big(\lambda^7 \left\|(\varphi')^6\varphi^{(2)}\right\|_\infty + \lambda^5 \left\|(\varphi')^4\varphi^{(4)}\right\|_\infty + \lambda^5 \left\|\varphi'\varphi^{(2)}\varphi^{(3)}\right\|_\infty \\
& + \lambda^5 \left\|(\varphi')^4\varphi^{(2)}\right\|_\infty + \lambda^5 \left\|(\varphi')^4\varphi^{(2)}\right\|_\infty + \lambda^5 \left\|(\varphi')^2(\varphi^{(2)})^3\right\|_\infty \\
& + \lambda^3 \left\|\varphi'\varphi^{(2)}\varphi^{(4)}\right\|_\infty + \lambda^3 \left\|(\varphi')^2\varphi^{(4)}\right\|_\infty + \lambda^3 \left\|(\varphi^{(2)})^2\varphi^{(4)}\right\|_\infty + \lambda^3 \left\|(\varphi)^{(2)}(\varphi^{(3)})^2\right\|_\infty \\
& + \lambda^3 \left\|\varphi'\varphi^{(2)}\varphi^{(3)}\right\|_\infty + \lambda^3 \left\|(\varphi')^2\varphi^{(2)}\right\|_\infty + \lambda^3 \left\|(\varphi')^2\varphi^{(6)}\right\|_\infty \\
& + \lambda^3 \left\|\varphi'\varphi^{(2)}\varphi^{(5)}\right\|_\infty + \lambda \left\|\varphi^{(4)}\right\|_\infty + \lambda \left\|\varphi^{(6)}\right\|_\infty + \lambda \left\|\varphi^{(8)}\right\|_\infty\big) \int_{\mathbb{R}} |f|^2\mathtt{dx} \\
& + C\left(\lambda^5 \left\|(\varphi')^4\varphi^{(2)}\right\|_\infty + \lambda^3 \left\|(\varphi')^2\varphi^{(4)}\right\|_\infty + \lambda^3 \left\|\varphi'\varphi^{(2)}\varphi^{(3)}\right\|_\infty \right.\\
& \left. +\lambda^3 \left\|(\varphi^{(2)})^3\right\|_\infty + \lambda \left\|\varphi^{(4)}\right\|_\infty + \lambda \left\|\varphi^{(2)}\right\|_\infty + \lambda \left\|\varphi^{(6)}\right\|_\infty\right) \int_{\mathbb{R}} |\partial_x f|^2\, \mathtt{dx} \\
& + C\big(\lambda^3 \left\|(\varphi')^2\varphi^{(2)}\right\|_\infty + \lambda \left\|\varphi^{(2)}\right\|_\infty + \lambda \left\|\varphi^{(4)}\right\|_\infty\big) \int_{\mathbb{R}} |\partial_x^2 f|^2\mathtt{dx} \\
& + C\lambda \left\|\varphi^{(2)}\right\|_\infty \int_{\mathbb{R}} |\partial_x^3 f|^2\mathtt{dx},
\end{aligned}
$$

com $C > 0$ uma constante.

A seguir, apresentamos os seguintes resultados para estimar $\int_{T_1}^\infty \int_{\mathbb{R}} |f|^2 / \langle x\rangle^{4-3p}\mathtt{dxdt}$.

Proposição 27. *$\forall\, T_0, T_1 \in \mathbb{R}$ com $T_0 < T_1$ e $f = f(t,x)$ temos;*

$$
\begin{aligned}
& \int_{T_0}^{T_1} \int_{\mathbb{R}} [S_\lambda, A_\lambda] f\overline{f}\mathtt{dxdt} + \int_{T_0}^{T_1} \int_{\mathbb{R}} |S_\lambda f|^2\, \mathtt{dxdt} \leq \int_{T_0}^{T_1} \int_{\mathbb{R}} |\partial_t f - (S_\lambda + A_\lambda) f|^2\, \mathtt{dxdt} \\
& + \left|\int_{\mathbb{R}} S_\lambda f(T_1)\overline{f(T_1)}\mathtt{dx}\right| + \left|\int_{\mathbb{R}} S_\lambda f(T_0)\overline{f(T_0)}\mathtt{dx}\right|. \qquad (4.23)
\end{aligned}
$$

Demonstração: Segue das propriedades de S_λ e f solução da equação (4.1), que

$$\begin{aligned}
\frac{\mathtt{d}}{\mathtt{dt}}\langle S_\lambda f, f\rangle &= \langle \partial_t S_\lambda f, f\rangle + \langle S_\lambda f, \partial_t f\rangle\\
&= \langle S_\lambda \partial_t f, f\rangle + \langle S_\lambda f, \partial_t f\rangle\\
&= \langle \partial_t f, S_\lambda f\rangle + \langle S_\lambda f, \partial_t f\rangle\\
&= 2\Re\langle \partial_t f, S_\lambda f\rangle\\
&= 2\Re\langle \partial_t f - (S_\lambda + A_\lambda) f, S_\lambda f\rangle + 2\Re\langle (S_\lambda + A_\lambda) f, S_\lambda f\rangle\\
&= 2\Re\langle \partial_t f - (S_\lambda + A_\lambda) f, S_\lambda f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle + \langle S_\lambda A_\lambda f, f\rangle\\
&\quad - \langle A_\lambda S_\lambda f, f\rangle\\
&= 2\Re\langle \partial_t f - (S_\lambda + A_\lambda) f, S_\lambda f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle + \langle [S_\lambda, A_\lambda] f, f\rangle.
\end{aligned}$$

Integrando em $[T_0, T_1]$ e como $2\,|ab| \le |a|^2 + |b|^2$, segue que

$$\begin{aligned}
&\int_{T_0}^{T_1} \langle [S_\lambda, A_\lambda] f, f\rangle \mathtt{dt} + 2\int_{T_0}^{T_1} \langle S_\lambda f, S_\lambda f\rangle \mathtt{dt}\\
&= -2\Re\int_{T_0}^{T_1} \langle \partial_t f - (S_\lambda + A_\lambda) f, S_\lambda f\rangle \mathtt{dt} + \langle S_\lambda f, f\rangle\Big|_{T_0}^{T_1}\\
&\le \int_{T_0}^{T_1}\int_{\mathbb{R}} |\partial_t f - (S_\lambda + A_\lambda) f|^2\,\mathtt{dxdt} + \int_{T_0}^{T_1}\int_{\mathbb{R}} |S_\lambda f|^2\,\mathtt{dxdt} + \left|\int_{\mathbb{R}} S_\lambda f(T_1)\overline{f(T_1)}\right|\\
&\quad + \left|\int_{\mathbb{R}} S_\lambda f(T_0)\overline{f(T_0)}\right|;
\end{aligned}$$

logo, tem-se o resultado.

■

Para facilitar a demonstração do Teorema 8, definimos uma função $\eta(t)$ tal que para todo $T > 1/2$,

$$\begin{aligned}
\eta:\ [T-1/2, T+1/2] &\longrightarrow \mathbb{R}\\
t &\longmapsto (t - (T-1/2))((T+1/2) - t),
\end{aligned}$$

satisfaz as seguintes propriedades

1) $\eta(T-1/2) = \eta(T+1/2) = 0$, para todo $t \in [T-1/2, T+1/2]$,

2) $0 \le \eta(t) \le 1/4$,

3) $\eta'(t) \le 1$,

4) $\eta''(t) = -2$.

A seguir, apresentaremos a seguinte proposição que nos permitirá gerar uma sequência $\{T_n : n \in \mathbb{N}\}$ tal que $T_n \uparrow \infty$, de modo que facilitará a demonstração do Teorema 8.

Proposição 28. $\forall\, T > 1/2$

$$\begin{aligned}
&\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \eta(t)(|S_\lambda f|^2 + [S_\lambda, A_\lambda]f\overline{f})\mathtt{dxdt} + \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |f|^2\,\mathtt{dxdt} \\
&\leq\ 2\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |\partial_t f - (S_\lambda + A_\lambda)f|^2\,\mathtt{dxdt} + 2\left|\left(\int_{\mathbb{R}} |f|^2\Big|_{T-1/2}^{T+1/2}\mathtt{dx}\right)\right|.
\end{aligned} \tag{4.24}$$

Demonstração:

Inicialmente, derivando $\langle f, f\rangle$ com relação a t e usando as propriedades da equação (4.1) obtemos:

$$\begin{aligned}
\frac{\mathtt{d}}{\mathtt{dt}}\langle f, f\rangle &= \langle \partial_t f, f\rangle + \langle f, \partial_t f\rangle \\
&= \langle \partial_t f - (S_\lambda + A_\lambda)f, f\rangle + \langle f, \partial_t f - (S_\lambda + A_\lambda)f\rangle + \langle (S_\lambda + A_\lambda)f, f\rangle \\
&\quad + \langle f, (S_\lambda + A_\lambda)f\rangle \\
&= 2\Re\langle \partial_t f - (S_\lambda + A_\lambda)f, f\rangle + 2\langle S_\lambda f, f\rangle + \langle A_\lambda f, f\rangle + \langle f, A_\lambda f\rangle \\
&= 2\Re\langle \partial_t f - (S_\lambda + A_\lambda)f, f\rangle + 2\langle S_\lambda f, f\rangle.
\end{aligned} \tag{4.25}$$

Daí, multiplicando por $\eta'(t)$ e integrando em $[T-1/2, T+1/2]$, segue que

$$\begin{aligned}
\int_{T-1/2}^{T+1/2} -2\langle S_\lambda f, f\rangle\eta'(t)\mathtt{dt} &= \int_{T-1/2}^{T+1/2} 2\Re\langle \partial_t f - (S_\lambda + A_\lambda)f, f\rangle\eta'(t)\mathtt{dt} \\
&\quad - \int_{T-1/2}^{T+1/2} \eta'(t)\partial_t\langle f, f\rangle\mathtt{dt};
\end{aligned} \tag{4.26}$$

usando integração por partes e as propriedades de η, temos

$$-\int_{T-1/2}^{T+1/2} \eta'(t)\partial_t\langle f, f\rangle\mathtt{dt} = -\langle f, f\rangle\eta'(t)\Big|_{T-1/2}^{T+1/2} + \int_{T-1/2}^{T+1/2}\langle f, f\rangle\eta''(t)\mathtt{dt} \tag{4.27}$$

e

$$\begin{aligned}
&-\int_{T-1/2}^{T+1/2} 2\langle S_\lambda f, f\rangle\eta'(t)\mathtt{dt} = -2\langle S_\lambda f, f\rangle\eta(t)\Big|_{T-1/2}^{T+1/2} + 2\int_{T-1/2}^{T+1/2}\eta\partial_t\langle S_\lambda f, f\rangle\mathtt{dt} \\
&= 2\int_{T-1/2}^{T+1/2}\eta(t)\left(2\Re\langle \partial_t f - (S_\lambda + A_\lambda)f, S_\lambda f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle + \langle [S_\lambda, A_\lambda]f, f\rangle\right)\mathtt{dt}.
\end{aligned} \tag{4.28}$$

Logo, de (4.27)-(4.28) em (4.26), segue que

$$\begin{aligned}
&\int_{T-1/2}^{T+1/2} \left[\left(4\langle S_\lambda f, S_\lambda f\rangle + 2\langle [S_\lambda, A_\lambda] f, f\rangle \right) \eta(t) + 4\Re\langle \partial_t f - (S_\lambda + A_\lambda) f, S_\lambda f\rangle \eta(t) \right] \mathtt{dt} \\
&= \int_{T-1/2}^{T+1/2} 2\Re\langle \partial_t f - (S_\lambda + A_\lambda) f, f\rangle \eta'(t) \mathtt{dt} - \langle f, f\rangle \eta'(t) \Big|_{T-1/2}^{T+1/2} \\
&\quad + \int_{T-1/2}^{T+1/2} \langle f, f\rangle \eta''(t) \mathtt{dt}. \qquad (4.29)
\end{aligned}$$

Assim, de (4.29) resulta que

$$\begin{aligned}
&\int_{T-1/2}^{T+1/2} \eta(t) \left[\left(4\langle S_\lambda f, S_\lambda f\rangle + 2\langle [S_\lambda, A_\lambda] f, f\rangle \right) \right] \mathtt{dt} \\
&= -4\Re \int_{T-1/2}^{T+1/2} \eta(t) \langle \partial_t f - (S_\lambda + A_\lambda) f, S_\lambda f\rangle \mathtt{dt} \\
&\quad + \int_{T-1/2}^{T+1/2} 2\Re\langle \partial_t f - (S_\lambda + A_\lambda) f, f\rangle \eta'(t) \mathtt{dt} - \langle f, f\rangle \eta'(t) \Big|_{T-1/2}^{T+1/2} \\
&\quad + \int_{T-1/2}^{T+1/2} \langle f, f\rangle \eta''(t) \mathtt{dt}; \qquad (4.30)
\end{aligned}$$

e como

$$\begin{aligned}
&-4\Re \int_{T-1/2}^{T+1/2} \eta(t) \langle \partial_t f - (S_\lambda + A_\lambda) f, S_\lambda f\rangle \mathtt{dt} \\
&= -2 \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} \eta(t) 2\Re \left[(\partial_t f - (S_\lambda + A_\lambda) f) \cdot \overline{S_\lambda f} \right] \mathtt{dxdt} \\
&\le 2 \int_{T-1/2}^{T+1/2} \eta(t) \left(\int_{\mathbb{R}} |\partial_t f - (S_\lambda + A_\lambda) f|^2 \, \mathtt{dx} + \int_{\mathbb{R}} |S_\lambda f|^2 \, \mathtt{dx} \right) \mathtt{dt}. \qquad (4.31)
\end{aligned}$$

Então, usando as propriedades de η e (4.30)-(4.31) em (4.29), temos

$$\begin{aligned}
&2 \int_{T-1/2}^{T+1/2} \eta(t) \left(\langle S_\lambda f, S_\lambda f\rangle + \langle [S_\lambda, A_\lambda] f, f\rangle \right) \mathtt{dt} + 2 \int_{T-1/2}^{T+1/2} \langle f, f\rangle \mathtt{dt} \\
&= -4\Re \int_{T-1/2}^{T+1/2} \eta(t) \langle \partial_t f - (S_\lambda + A_\lambda) f, S_\lambda f\rangle \mathtt{dt} + 2\Re \int_{T-1/2}^{T+1/2} \langle \partial_t f - (S_\lambda + A_\lambda) f, f\rangle \eta'(t) \mathtt{dt} \\
&\quad - \langle f, f\rangle \eta'(t) \Big|_{T-1/2}^{T+1/2} - 2 \int_{T-1/2}^{T+1/2} \eta(t) \langle S_\lambda f, S_\lambda f\rangle \mathtt{dt} \\
&\le 2 \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} \eta(t) \, |\partial_t f - (S_\lambda + A_\lambda) f|^2 \, \mathtt{dxdt} + 2 \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} \eta(t) \, |S_\lambda f|^2 \, \mathtt{dxdt} \\
&\quad + \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} |\partial_t f - (S_\lambda + A_\lambda) f|^2 \, \mathtt{dxdt} + \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} |f|^2 \, \mathtt{dxdt} - \langle f, f\rangle \eta'(t) \Big|_{T-1/2}^{T+1/2} \\
&\quad - 2 \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} \eta(t) \, |S_\lambda f|^2 \, \mathtt{dxdt}
\end{aligned}$$

$$\leq \frac{5}{2}\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |\partial_t f - (S_\lambda + A_\lambda)f|^2 \,\mathtt{dxdt} - \langle f, f\rangle \eta'(t)\Big|_{T-1/2}^{T+1/2} + \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |f|^2 \,\mathtt{dx}. \tag{4.32}$$

Assim,

$$\begin{aligned}
&\int_{T-1/2}^{T+1/2} \eta(t)\left(\langle S_\lambda f, S_\lambda f\rangle + \langle [S_\lambda, A_\lambda]f, f\rangle\right)\mathtt{dt} + \frac{1}{2}\int_{T-1/2}^{T+1/2}\langle f, f\rangle \mathtt{dt} \\
&\leq \frac{5}{4}\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |\partial_t f - (S_\lambda + A_\lambda)f|^2 \,\mathtt{dxdt} + \frac{1}{2}\left|\int_{\mathbb{R}} |f|^2\Big|_{T-1/2}^{T+1/2} \mathtt{dx}\right|.
\end{aligned}$$

■

A seguir, demonstraremos o Teorema 8, resultado principal deste Capítulo.

4.5 Demonstração do Teorema 8

Para facilitar a demonstração do Teorema 8, dividimos por etapas:

Na primeira etapa, fazemos uso da Proposição 28 e construímos uma sequência de $T_j \uparrow \infty$ de modo que se verifique (4.33).

Etapa 1: Se

$$\sup_{t\geq 0}\int_{\mathbb{R}} e^{\lambda |x|^p}\left|\partial_x^j u(t,x)\right|^2 \mathtt{dx} \leq C_\lambda, \quad j \in \{0,1,2,3,4\},$$

então, $\exists\ \{T_j; j\in\mathbb{N}\}$ com $T_j \uparrow \infty$ quando $j\to\infty$ tal que

$$\sup_{j\in\mathbb{N}}\int_{\mathbb{R}} |S_\lambda f(T_j, x)|^2 \,\mathtt{dx} \leq \widetilde{C}_\lambda, \tag{4.33}$$

com $f(t,x) = e^{\lambda\varphi(x)}u(t,x)$ e $\widetilde{C}_\lambda$ uma constante que depende de $C_\lambda, \lambda, \|V\|_\infty$ e p.

Demonstração:

Segue da Proposição 28,

$$\begin{aligned}
&\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \eta(t)\left(|S_\lambda f|^2 + [S_\lambda, A_\lambda]f\overline{f}\right)\mathtt{dxdt} + \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |f|^2 \,\mathtt{dxdt} \\
&\leq 2\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |\partial_t f - (S_\lambda + A_\lambda)f|^2 \,\mathtt{dxdt} + 2\left|\int_{\mathbb{R}} |f|^2\Big|_{T-1/2}^{T+1/2}\mathtt{dx}\right| \\
&= 2\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |V|^2\,|f|^2 \,\mathtt{dxdt} + 2\left|\int_{\mathbb{R}} |f|^2\Big|_{T-1/2}^{T+1/2}\mathtt{dx}\right|.
\end{aligned} \tag{4.34}$$

Logo, segue da inequação (4.34), das hipóteses (4.3) e as propriedades de $[S_\lambda, A_\lambda]f$ que

$$\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \eta(t)\,|S_\lambda f|^2\,\mathtt{dxdt} \leq \widetilde{C}_\lambda, \tag{4.35}$$

onde $\widetilde{C}_\lambda$ depende de $\|V\|_{L^\infty}$, λ, p e C_λ. Para maior detalhe da demonstração ver o Apêndice B.1.

Disto, utilizando o teorema do valor médio, (4.35) e $\eta(t) \geq 3/16$ no intervalo $[T-1/4, T+1/4]$, resulta que existe um $T^* \in [T-1/4; T+1/4]$

$$\begin{aligned}\widetilde{C}_\lambda &\geq \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \eta(t)\,|S_\lambda f(t,x)|^2\,\mathtt{dxdt} \geq \int_{T-1/4}^{T+1/4}\int_{\mathbb{R}} \eta(t)\,|S_\lambda f(t,x)|^2\,\mathtt{dxdt}\\ &\geq \frac{3}{16}\int_{\mathbb{R}} |S_\lambda f(T^*,x)|^2\,\mathtt{dx}.\end{aligned} \tag{4.36}$$

Então, podemos construir uma sequência de $T_j \uparrow +\infty$, tal que

$$\int_{\mathbb{R}} |S_\lambda f(T_j,x)|^2\,\mathtt{dx} \leq \widetilde{C}_\lambda. \tag{4.37}$$

■

Na seguinte etapa, demonstraremos uma estimativa para a integral $\int_{T_1}^{T_j}\int_{\mathbb{R}} |f|^2/\langle x\rangle^{4-3p}\mathtt{dx}$, para todo $j \in \mathbb{Z}^+$.

Etapa 2: $\exists\, \lambda_0 > 0$, tal que se $\lambda \geq \lambda_0$ então para qualquer $j \in \mathbb{Z}^+$

$$\int_{T_1}^{T_j}\int_{\mathbb{R}} \frac{e^{2\lambda\varphi}\,|u(t,x)|^2}{\langle x\rangle^{4-3p}}\mathtt{dxdt} \leq C_\lambda,$$

com C_λ, independente de $j \in \mathbb{Z}^+$.

Afirmação 1.

$$\int_{T_1}^{T_j}\int_{\mathbb{R}} [S_\lambda, A_\lambda] f\overline{f}\,\mathtt{dxdt} \leq \int_{T_1}^{T_j}\int_{\mathbb{R}} \left|e^{\lambda\varphi}Vu\right|^2\mathtt{dxdt} + C_\lambda^1, \tag{4.38}$$

com C_λ^1, independente de $j \in \mathbb{Z}^+$.

De fato, segue da Proposição 27, de $\partial_t f - (S_\lambda + A_\lambda)f = \mathtt{i}Vf$ e de $f = e^{\lambda\varphi}u$ que

$$\begin{aligned}\int_{T_1}^{T_j}\int_{\mathbb{R}} [S_\lambda, A_\lambda] f\overline{f}\,\mathtt{dxdt} &\leq \int_{T_1}^{T_j}\int_{\mathbb{R}} |\partial_t f - (S_\lambda + A_\lambda)f|\,\mathtt{dxdt} + \left|\int_{\mathbb{R}} S_\lambda f(T_j)\overline{f}(T_j)\mathtt{dx}\right|\\ &\quad + \left|\int_{\mathbb{R}} S_\lambda f(T_1)\overline{f}(T_1)\mathtt{dx}\right| - \int_{T_1}^{T_j}\int_{\mathbb{R}} |S_\lambda f|^2\,\mathtt{dxdt}\\ &\leq \int_{T_1}^{T_j}\int_{\mathbb{R}} |\mathtt{i}Vf|^2\,\mathtt{dxdt} + \int_{\mathbb{R}} |S_\lambda f(T_j)|^2\,\mathtt{dx} + \int_{\mathbb{R}} |f(T_j)|^2\,\mathtt{dx}\\ &\quad + \int_{\mathbb{R}} |S_\lambda f(T_1)|^2\,\mathtt{dx} + \int_{\mathbb{R}} |f(T_1)|^2\,\mathtt{dx}.\end{aligned} \tag{4.39}$$

Utilizando $\sup_{j\in\mathbb{Z}^+}\langle S_\lambda f(T_j), S_\lambda f(T_j)\rangle \leq \widetilde{C}_\lambda$ e $\sup_{t\geq 0}\int_{\mathbb{R}}|f|^2\,\mathtt{dx} \leq C$ em (4.39), resulta que

$$\int_{T_1}^{T_j}\int_{\mathbb{R}}[S_\lambda, A_\lambda]f\overline{f}\,\mathtt{dxdt} \leq \int_{T_1}^{T_j}\int_{\mathbb{R}}\left|e^{\lambda\varphi}Vu\right|^2\mathtt{dx} + C_\lambda^1; \tag{4.40}$$

onde C_λ^1, independe de $j \in \mathbb{Z}^+$.

De agora em diante chamaremos de termos positivos de $\langle[S_\lambda, A_\lambda]f, f\rangle$ (ver (4.22)), os seguintes termos:

$$\begin{aligned}
&\int_{\mathbb{R}}\left[8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right]|f|^2\,\mathtt{dx}\\
&+\int_{\mathbb{R}}\left[24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right]|\partial_x f|^2\,\mathtt{dx}\\
&+\int_{\mathbb{R}}24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\left|\partial_x^2 f\right|^2\mathtt{dx} + \int_{\mathbb{R}}16\lambda\varphi^{(2)}(r)\left|\partial_x^3 f\right|^2\mathtt{dx},
\end{aligned} \tag{4.41}$$

podendo mudar os coeficientes em cada uma das integrais.

A seguir, estimamos os termos positivos de $\langle[S_\lambda, A_\lambda]f, f\rangle$ por termos cujo domínio de integração é $|x| \leq \varepsilon$.

Afirmação 2. *Se $1 < p \leq 8/7$, existe $\lambda_0 > 0$ tal que $\forall\ \lambda > \lambda_0$ temos que:*

$$\begin{aligned}
&\int_{\mathbb{R}}\left[8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right]|f|^2\,\mathtt{dx}\\
&+\int_{\mathbb{R}}\left[24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right]|\partial_x f|^2\,\mathtt{dx}\\
&+\int_{\mathbb{R}}24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\left|\partial_x^2 f\right|^2\mathtt{dx} + \int_{\mathbb{R}}16\lambda\varphi^{(2)}(r)\left|\partial_x^3 f\right|^2\mathtt{dx}\\
\leq\ \ & C_\lambda + C'\left(\lambda^5\left\|(\varphi')^4\varphi^{(4)}\right\|_\infty + \lambda^5\left\|(\varphi')^3\varphi^{(2)}\varphi^{(3)}\right\|_\infty + \lambda^5\left\|(\varphi')^2(\varphi^{(2)})^3\right\|_\infty\right.\\
&+\lambda^3\left\|\varphi'\varphi^{(3)}\varphi^{(4)}\right\|_\infty + \lambda^3\left\|(\varphi')^2\varphi^{(4)}\right\|_\infty + \lambda^3\left\|(\varphi^{(2)})^2\varphi^{(4)}\right\| + \lambda^3\left\|\varphi^{(2)}(\varphi^{(3)})^2\right\|_\infty\\
&+\lambda^3\left\|\varphi'\varphi^{(2)}\varphi^{(3)}\right\|_\infty + \lambda\left\|(\varphi')^2\varphi^{(6)}\right\|_\infty + \lambda^3\left\|\varphi'\varphi^{(2)}\varphi^{(5)}\right\|_\infty + \lambda\left\|\varphi^{(4)}\right\| + \lambda\left\|\varphi^{(6)}\right\|_\infty\\
&\left.+\lambda\left\|\varphi^{(8)}\right\|_\infty\right)\int_{|x|\leq\varepsilon}|f|^2\,\mathtt{dx}\\
&+C''\left(\lambda^3\left\|(\varphi')^2\varphi^{(4)}\right\|_\infty + \lambda^3\left\|\varphi'\varphi^{(2)}\varphi^{(3)}\right\| + \lambda\left\|\varphi^{(4)}\right\|_\infty + \lambda\left\|\varphi^{(6)}\right\|_\infty\right)\int_{|x|\leq\varepsilon}|\partial_x f|^2\,\mathtt{dx}\\
&+C'''\left(\lambda\left\|\varphi^{(2)}\right\|_\infty + \lambda\left\|\varphi^{(4)}\right\|_\infty\right)\int_{|x|\leq\varepsilon}\left|\partial_x^2 f\right|^2\mathtt{dx},
\end{aligned} \tag{4.42}$$

para detalhar a demonstração da desigualdade (4.42) veja o Apêndice B.2.

Utilizando os termos positivos de $\int_{T_1}^{T_j} \langle [S_\lambda, A_\lambda] f, f \rangle \mathtt{dt}$, a desigualdade (4.42) e adicionando os termos $\int_{T_1}^{T_j} \int_{|x|\leq 1} |f|^2 \mathtt{dxdt}$, $\int_{T_1}^{T_j} \int_{|x|\leq 1} |\partial_x f|^2 \mathtt{dxdt}$ e $\int_{T_1}^{T_j} \int_{|x|\leq 1} |\partial_x^2 f|^2 \mathtt{dxdt}$, obtemos a seguinte estimativa uniforme para todo $j \in \mathbb{Z}^+$:

$$\int_{T_1}^{T_j} \int_{\mathbb{R}} |f|^2 / \langle x \rangle^{4-3p} \mathtt{dx} \leq C_\lambda.$$

Antes de continuar, observamos a seguinte igualdade

$$\begin{aligned}
&\int_{T_1}^{T_j} \int_{\mathbb{R}} \left[4\lambda^7 (\varphi'(r))^6 \varphi^{(2)}(r) + 16\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) + 4\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r)\right] |f|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} \left[12\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) + 48\lambda^3 (\varphi^{(2)}(r))^3 + 4\lambda \varphi^{(2)}(r)\right] |\partial_x f|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\mathbb{R}} 16\lambda \varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |\partial_x f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&= \int_{T_1}^{T_j} \int_{\mathbb{R}} \left[8\lambda^7 (\varphi'(r))^6 \varphi^{(2)}(r) + 16\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) + 4\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r)\right] |f|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} \left[24\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) + 48\lambda^3 (\varphi^{(2)}(r))^3 + 4\lambda \varphi^{(2)}(r)\right] |\partial_x f|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} 24\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\mathbb{R}} 16\lambda \varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |\partial_x f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&- \int_{T_1}^{T_j} \int_{\mathbb{R}} 4\lambda^7 (\varphi'(r))^6 \varphi^{(2)}(r) |f|^2 \mathtt{dxdt} - \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) |\partial_x f|^2 \mathtt{dxdt} \\
&- \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt}. \qquad (4.43)
\end{aligned}$$

Logo, aplicando o resultado da equação (4.42) em (4.43), segue que

$$\begin{aligned}
&\int_{T_1}^{T_j} \int_{\mathbb{R}} \left[4\lambda^7 (\varphi'(r))^6 \varphi^{(2)}(r) + 16\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) + 4\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r)\right] |f|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} \left[12\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) + 48\lambda^3 (\varphi^{(2)}(r))^3 + 4\lambda \varphi^{(2)}(r)\right] |\partial_x f|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\mathbb{R}} 16\lambda \varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |\partial_x f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt}
\end{aligned}$$

$$\leq C_\lambda + \int_{T_1}^{T_j}\int_{|x|\leq\varepsilon} \big[\,|V|^2 + 4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r) + 64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)$$

$$+ 48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3 - 32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r) - 4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)$$

$$+ 36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r) + 80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2 - 16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$

$$- 4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r) - 24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r) + \lambda\varphi^{(4)}(r) + 2\lambda\varphi^{(6)}(r)$$

$$+ \lambda\varphi^{(8)}(r)\big]|f|^2\mathtt{dxdt}$$

$$+ \int_{T_1}^{T_j}\int_{|x|\leq\varepsilon} \big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r) + 96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$

$$- 12\lambda\varphi^{(4)}(r) - 4\lambda\varphi^{(2)}(r) - 8\lambda\varphi^{(6)}(r)\big]|\partial_x f|^2\mathtt{dxdt}$$

$$+ \int_{T_1}^{T_j}\int_{|x|\leq\varepsilon} \big[16\lambda\varphi^{(2)}(r) + 20\lambda\varphi^{(4)}(r)\big]|\partial_x^2 f|^2\mathtt{dxdt}$$

$$+ \int_{T_1}^{T_j}\int_{|x|\leq 1} \lambda^2\,|f|^2\,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{|x|\leq 1} \lambda^2\,|\partial_x f|^2\,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{|x|\leq 1} \lambda^2\,\left|\partial_x^2 f\right|^2\mathtt{dxdt}$$

$$- \int_{T_1}^{T_j}\int_{\mathbb{R}} 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\,|f|^2\,\mathtt{dxdt} - \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\,|\partial_x f|^2\,\mathtt{dxdt}$$

$$- \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\,\left|\partial_x^2 f\right|^2\mathtt{dxdt}$$

$$\leq C_\lambda + \int_{T_1}^{T_j}\int_{|x|\leq\varepsilon} \big[\,|V|^2 + 4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r) + 64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)$$

$$+ 48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3 - 32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r) - 4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)$$

$$+ 36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r) + 80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2 - 16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$

$$- 4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r) - 24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r) + \lambda\varphi^{(4)}(r) + 2\lambda\varphi^{(6)}(r)$$

$$+ \lambda\varphi^{(8)}(r) + \lambda^2\big]|f|^2\mathtt{dxdt}$$

$$+ \int_{T_1}^{T_j}\int_{|x|\leq\varepsilon} \big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r) + 96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$

$$- 12\lambda\varphi^{(4)}(r) - 4\lambda\varphi^{(2)}(r) - 8\lambda\varphi^{(6)}(r) + \lambda^2\big]|\partial_x f|^2\mathtt{dxdt}$$

$$+ \int_{T_1}^{T_j}\int_{|x|\leq\varepsilon} \big[16\lambda\varphi^{(2)}(r) + 20\lambda\varphi^{(4)}(r) + \lambda^2\big]|\partial_x^2 f|^2\mathtt{dxdt}$$

$$+ \int_{T_1}^{T_j}\int_{\varepsilon\leq|x|\leq 1} \lambda^2\,|f|^2\,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{\varepsilon\leq|x|\leq 1} \lambda^2\,|\partial_x f|^2\,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{\varepsilon\leq|x|\leq 1} \lambda^2\,\left|\partial_x^2 f\right|^2\mathtt{dxdt}$$

$$- \int_{T_1}^{T_j}\int_{\mathbb{R}} 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\,|f|^2\,\mathtt{dxdt} - \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\,|\partial_x f|^2\,\mathtt{dxdt}$$

$$- \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\,\left|\partial_x^2 f\right|^2$$

$$
\begin{aligned}
&\leq\; C_\lambda + \left[\|P_1\|_\infty + \|V\|_\infty^2 + \lambda^2\right] \int_{T_1}^{T_j} \int_{|x|\leq\varepsilon} |f|^2 \,\mathtt{dxdt} + \left[\|P_2\|_\infty + \lambda^2\right] \int_{T_1}^{T_j} \int_{|x|\leq\varepsilon} |\partial_x f|^2 \,\mathtt{dxdt} \\
&\left[\|P_3\|_\infty + \lambda^2\right] \int_{T_1}^{T_j} \int_{|x|\leq\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\varepsilon\leq|x|\leq 1} \lambda^2 |f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\varepsilon\leq|x|\leq 1} \lambda^2 |\partial_x f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\varepsilon\leq|x|\leq 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&- \int_{T_1}^{T_j} \int_{\mathbb{R}} 4\lambda^7 (\varphi'(r))^6 \varphi^{(2)}(r) |f|^2 \,\mathtt{dxdt} - \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) |\partial_x f|^2 \,\mathtt{dxdt} \\
&- \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r) \left|\partial_x^2 f\right|^2,
\end{aligned} \tag{4.44}
$$

em que

$$
\begin{aligned}
P_1 &= 4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r) + 64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r) \\
&\quad + 48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3 - 32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r) - 4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r) \\
&\quad + 36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r) + 80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2 - 16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r) \\
&\quad - 4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r) - 24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r) + \lambda\varphi^{(4)}(r) + 2\lambda\varphi^{(6)}(r) \\
&\quad + \lambda\varphi^{(8)}(r), && (4.45) \\
P_2 &= 24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r) + 96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r) \\
&\quad - 12\lambda\varphi^{(4)}(r) - 4\lambda\varphi^{(2)}(r) - 8\lambda\varphi^{(6)}(r), && (4.46) \\
P_3 &= 16\lambda\varphi^{(2)}(r) + 20\lambda\varphi^{(4)}(r). && (4.47)
\end{aligned}
$$

Logo das propriedades de φ existem as constantes positivas $C_1,\ C_2$ e C_3 tais que $\|P_1\|_\infty \leq C_1\lambda^5$, $\|P_2\|_\infty \leq C_2\lambda^3$ e $\|P_3\|_\infty \leq C_3\lambda$. Daí, resulta a seguinte desigualdade

$$
\begin{aligned}
&\int_{T_1}^{T_j} \int_{\mathbb{R}} \left[\, 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2 \,\mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} \left[\, 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2 \,\mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\mathbb{R}} 16\lambda\varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |\partial_x f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&\leq C_\lambda + \left(C_1\lambda^5 + \lambda^2 + \|V\|_\infty\right) \int_{T_1}^{T_j} \int_{|x|\leq\varepsilon} |f|^2 \,\mathtt{dxdt} + \left(C_2\lambda^3 + \lambda^2\right) \int_{T_1}^{T_j} \int_{|x|\leq\varepsilon} |\partial_x f|^2 \,\mathtt{dxdt}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(C_3\lambda + \lambda^2\right) \int_{T_1}^{T_j} \int_{|x|\le\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\varepsilon\le|x|\le1} \lambda^2 |f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\varepsilon\le|x|\le1} \lambda^2 |\partial_x f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\varepsilon\le|x|\le1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&- \int_{T_1}^{T_j} \int_{\mathbb{R}} 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) |f|^2 \mathtt{dxdt} - \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) |\partial_x f|^2 \mathtt{dxdt} \\
&- \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt}. \qquad (4.48)
\end{aligned}
$$

Depois, utilizamos as três últimas integrais do lado direito de (4.48), para estimar em $[\varepsilon, 1]$.

Por outro lado, utilizando a Proposição 10 para f, $\partial_x f$ e $\partial_x^2 f$, temos as seguintes desigualdades:

$$
\int_{|x|\le\varepsilon} |f|^2 \mathtt{dx} \le \varepsilon C_\varphi \int_{|x|\le2\varepsilon} |\partial_x f|^2 \mathtt{dx} + C_\varphi \int_{\varepsilon\le|x|\le2\varepsilon} |f|^2 \mathtt{dx}, \qquad (4.49)
$$

$$
\int_{|x|\le\varepsilon} |\partial_x f|^2 \mathtt{dx} \le \varepsilon C_\varphi \int_{|x|\le2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dx} + C_\varphi \int_{\varepsilon\le|x|\le2\varepsilon} |\partial_x f|^2 \mathtt{dx}, \qquad (4.50)
$$

$$
\int_{|x|\le\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dx} \le \varepsilon C_\varphi \int_{|x|\le2\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dx} + C_\varphi \int_{\varepsilon\le|x|\le2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dx}. \qquad (4.51)
$$

A seguir, utilizando as desigualdades (4.49)-(4.51) em (4.48), resulta que

$$
\begin{aligned}
&\int_{T_1}^{T_j} \int_{\mathbb{R}} \left[\, 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} \left[\, 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\mathbb{R}} 16\lambda\varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{|x|\le1} \lambda^2 |f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\le1} \lambda^2 |\partial_x f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\le1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&\le\; C_\lambda + \left(C_1\lambda^5 + \lambda^2\right) \left(\varepsilon C_\varphi \int_{T_1}^{T_j} \int_{|x|\le2\varepsilon} |\partial_x f|^2 \mathtt{dxdt} + \varepsilon C_\varphi \int_{T_1}^{T_j} \int_{\varepsilon\le|x|\le2\varepsilon} |f|^2 \mathtt{dxdt}\right) \\
&+ \left(C_2\lambda^3 + \lambda^2\right) \left(\varepsilon C_\varphi \int_{T_1}^{T_j} \int_{|x|\le2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \varepsilon C_\varphi \int_{T_1}^{T_j} \int_{\varepsilon\le|x|\le2\varepsilon} |\partial_x f|^2 \mathtt{dxdt}\right) \\
&+ \left(C_3\lambda + \lambda^2\right) \left(\varepsilon C_\varphi \int_{T_1}^{T_j} \int_{|x|\le2\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dxdt} + \varepsilon C_\varphi \int_{T_1}^{T_j} \int_{\varepsilon\le|x|\le2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt}\right) \\
&+ \int_{T_1}^{T_j} \int_{\varepsilon\le|x|\le1} \lambda^2 |f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\varepsilon\le|x|\le1} \lambda^2 |\partial_x f|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\varepsilon\le|x|\le1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt}
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{T_1}^{T_j} \int_{\mathbb{R}} 4\lambda^7 (\varphi'(r))^6 \varphi^{(2)}(r) \, |f|^2 \, \mathtt{dxdt} - \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) \, |\partial_x f|^2 \, \mathtt{dxdt} \\
&- \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt}. \qquad (4.52)
\end{aligned}
$$

Para finalizar a demonstração, precisamos da seguinte afirmação.

Afirmação 3. *Da desigualdade* (4.52) *e* $\varepsilon = 1/\lambda^5$, *existe* $\lambda_0 > 0$ *tal que para todo* $\lambda > \lambda_0$, *vale:*

$$
\begin{aligned}
&\int_{T_1}^{T_j} \int_{\mathbb{R}} \left[4\lambda^7 (\varphi'(r))^6 \varphi^{(2)}(r) + 16\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) + 4\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r) \right] |f|^2 \, \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} \left[12\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) + 48\lambda^3 (\varphi^{(2)}(r))^3 + 4\lambda \varphi^{(2)}(r) \right] |\partial_x f|^2 \, \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\mathbb{R}} 16\lambda \varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |f|^2 \, \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |\partial_x f|^2 \, \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&\leq \; C_\lambda, \qquad (4.53)
\end{aligned}
$$

com C_λ, independente de T_j para todo $j \in \mathbb{N}$. Para ver mais detalhes da demonstração veja o Apêndice B.3.

Então, segue de (4.53)

$$
\begin{aligned}
C_\lambda &\geq \int_{T_1}^{T_j} \int_{\mathbb{R}} \left[4\lambda^7 (\varphi'(r))^6 \varphi^{(2)}(r) \right] |f|^2 \, \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |f|^2 \, \mathtt{dxdt} \\
&\geq \int_{T_1}^{T_j} \int_{1\leq|x|} 4\lambda^7 \frac{C(p)}{\langle x\rangle^{4-3p}} |f|^2 \, \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |f|^2 \, \mathtt{dxdt} \\
&\geq C(p)\lambda^2 \int_{T_1}^{T_j} \int_{\mathbb{R}} \frac{|f|^2}{\langle x\rangle^{4-3p}} \mathtt{dxdt},
\end{aligned}
$$

com $C(p)$ uma constante positiva, que depende de p.

Na próxima etapa demonstraremos a conservação de $\|u(t,\cdot)\|_{L^2}$ da equação (4.1), que junto com o resultado da Etapa 2, demonstraremos que a solução é nula .

Etapa 3: Conservação da norma.

Seja u uma solução da equação (4.1) e V real. Então

$$\|u(t,\cdot)\|_{L^2} = \|u(0,\cdot)\|_{L^2}, \quad \forall\, t \in [0,\infty). \tag{4.54}$$

Demonstração da Etapa 3

$$\begin{aligned}
\frac{\mathtt{d}}{\mathtt{dt}} \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} =& \frac{\mathtt{d}}{\mathtt{dt}} \langle u(t,\cdot), u(t,\cdot)\rangle \\
=& \langle \partial_t u, u\rangle + \langle u, \partial_t u\rangle \\
=& \left\langle \mathtt{i}\left(\partial_x^2 u + \partial_x^4 u + Vu\right), u\right\rangle + \left\langle u, \mathtt{i}\left(\partial_x^2 u + \partial_x^4 u + Vu\right)\right\rangle \\
=& \left\langle \mathtt{i}\partial_x^2 u, u\right\rangle + \left\langle u, \mathtt{i}\partial_x^2 u\right\rangle + \left\langle \mathtt{i}\partial_x^4 u, u\right\rangle + \left\langle u, \mathtt{i}\partial_x^4 u\right\rangle + \langle \mathtt{i}Vu, u\rangle + \langle u, \mathtt{i}Vu\rangle \\
=& -\langle \mathtt{i}\partial_x u, \partial_x u\rangle - \langle \partial_x u, \mathtt{i}\partial_x u\rangle - \left\langle \mathtt{i}\partial_x^2 u, \partial_x^2 u\right\rangle - \left\langle \partial_x^2 u, \mathtt{i}\partial_x^2 u\right\rangle + \langle \mathtt{i}Vu, u\rangle + \langle u, \mathtt{i}Vu\rangle \\
=& -\int_{\mathbb{R}} \mathtt{i}\,(\partial_x u)^2\,\mathtt{dx} + \int_{\mathbb{R}} \mathtt{i}\,(\partial_x u)^2\,\mathtt{dx} - \int_{\mathbb{R}} \mathtt{i}\left(\partial_x^2 u\right)^2\mathtt{dx} + \int_{\mathbb{R}} \mathtt{i}\left(\partial_x^2 u\right)^2\mathtt{dx} \\
&+ \int_{\mathbb{R}} \mathtt{i}Vu^2\mathtt{dx} - \int_{\mathbb{R}} \mathtt{i}Vu^2\mathtt{dx} \\
=& 0;
\end{aligned}$$

daí, segue que $\|u(t,\cdot)\|_{L^2(\mathbb{R})} = cte$.

Portanto, $\forall\, t \in [0,+\infty)$ resulta que

$$\|u(t,\cdot)\|_{L^2} = \|u(0,\cdot)\|_{L^2}.$$

Na próxima etapa, demonstramos que a solução da equação (4.1) é nula.

Etapa 4:

$$u(t,x) \equiv 0, \quad \forall\, (t,x) \in ([0,\infty) \times \mathbb{R}).$$

Como $V = V(t,x) \in \mathbb{R}$ e pelas Etapas 2-3, segue que

$$\begin{aligned}
(T_j - T_1)\|u(0,\cdot)\|^2_{L^2} &= \int_{T_1}^{T_j} \|u(0,\cdot)\|^2_{L^2}\,\mathtt{dt} \\
&= \int_{T_1}^{T_j} \|u(t,\cdot)\|^2_{L^2}\,\mathtt{dt} \\
&= \int_{T_1}^{T_j}\int_{\mathbb{R}} |u(t,x)|^2\,\mathtt{dxdt} \\
&= \int_{T_1}^{T_j}\int_{\mathbb{R}} |u(t,x)|^2\,\frac{e^{2\lambda\varphi}}{\langle x\rangle^{4-3p}}\,\langle x\rangle^{4-3p}\,e^{-2\lambda\varphi}\mathtt{dxdt}
\end{aligned}$$

$$\leq \sup_{x\in\mathbb{R}} \left(\langle x\rangle^{4-3p} e^{-2\lambda\varphi}\right) \int_{T_1}^{T_j} \int_{\mathbb{R}} |u(t,x)|^2 \frac{e^{2\lambda\varphi}}{\langle x\rangle^{4-3p}} \mathtt{dxdt} \leq C_\lambda.$$

Logo, fazendo $T_j \uparrow \infty$, temos que

$$u \equiv 0.$$

■

Apêndice A

A.1 Demonstração da equação (2.81) e (3.72)

Inicialmente, demonstramos a desigualdade (3.72)

$$
\begin{aligned}
&\langle [S_\lambda, A_\lambda]\, f, f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle \\
&\geq \langle S_\lambda f, S_\lambda f\rangle + \frac{A_0}{2}\int_{\mathbb{R}} \varphi^{(2)}(x-bt)\left[\lambda^9 f^2 + \lambda\left(\partial_x f\right)^2 + \lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right]\mathtt{dx}.
\end{aligned}
$$

Utilizando (3.69), segue que

$$
\begin{aligned}
&\langle [S_\lambda, A_\lambda]\, f, f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle \\
&= -\frac{\partial}{\partial t}\langle S_\lambda f, f\rangle \\
&= -\langle \partial_t S_\lambda f, f\rangle - \langle S_\lambda f, \partial_t f\rangle \\
&= -\langle \partial_t S_\lambda f, f\rangle + \langle S_\lambda \partial_t f, f\rangle - 2\langle S_\lambda f, \partial_t f\rangle \\
&= \langle -\left(\partial_t S_\lambda - S_\lambda \partial_t\right) f, f\rangle - 2\langle S_\lambda f, \partial_t f\rangle \\
&= -\langle [\partial_t, S_\lambda]\, f, f\rangle - 2\langle S_\lambda f, \partial_t f\rangle + \langle \widetilde{A}_\lambda f, S_\lambda f\rangle - \langle \widetilde{A}_\lambda f, S_\lambda f\rangle \\
&= -\langle [\partial_t, S_\lambda]\, f, f\rangle + \langle [S_\lambda, \widetilde{A}_\lambda] f, f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle S_\lambda f, F\rangle \\
&= \langle [S_\lambda, A_\lambda]\, f, f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle S_\lambda f, F\rangle \\
&= \langle [S_\lambda, A_\lambda]\, f, f\rangle + \langle S_\lambda f, S_\lambda f\rangle - \langle S_\lambda f, A_\lambda f\rangle - \langle S_\lambda f, F\rangle \\
&= \langle [S_\lambda, A_\lambda]\, f, f\rangle + \langle S_\lambda f, S_\lambda f\rangle + \langle A_\lambda f, A_\lambda f\rangle - \langle F, F\rangle,
\end{aligned}
\tag{A.1}
$$

onde $\langle S_\lambda f, A_\lambda f\rangle + \langle S_\lambda f, F\rangle = -\langle A_\lambda f, A_\lambda f\rangle + \langle F, F\rangle$.

Assim, segue das Proposições 21 e 24 em (A.1), que

$$\begin{aligned}
&\langle [S_\lambda, A_\lambda] f, f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle \\
&\geq \langle [S_\lambda, A_\lambda] f, f\rangle + \langle S_\lambda f, S_\lambda f\rangle + \langle A_\lambda f, A_\lambda f\rangle - \langle F, F\rangle \\
&= \langle [S_\lambda, A_\lambda] f, f\rangle - \langle F, F\rangle + \langle S_\lambda f, S_\lambda f\rangle + \langle A_\lambda f, A_\lambda f\rangle \\
&\geq A_0 \int_{\mathbb{R}} \varphi^{(2)}(r) \left[\lambda^9 f^2 + \lambda\left(\partial_x f\right)^2 + \lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right] \mathtt{dx} \\
&\quad - \frac{A_0}{4} \int_{\mathbb{R}} \varphi^{(2)}(r) \left[\lambda^9 f^2 + \lambda\left(\partial_x f\right)^2 + \lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right] \mathtt{dx} + \langle S_\lambda f, S_\lambda f\rangle \\
&\geq \langle S_\lambda f, S_\lambda f\rangle + \frac{A_0}{2} \int_{\mathbb{R}} \varphi^{(2)}(r) \left[\lambda^9 f^2 + \lambda\left(\partial_x f\right)^2 + \lambda\left(\partial_x^2 f\right)^2 + \lambda\left(\partial_x^3 f\right)^2\right] \mathtt{dx},
\end{aligned}$$

disto segue a desigualdade (3.72).

Em seguida, demonstramos a desigualdade (2.81). De maneira análoga utilizando as Proposições 12 e 17 em (A.1) resulta que

$$\begin{aligned}
&\langle [S_\lambda, A_\lambda] f, f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle \\
&\geq \langle [S_\lambda, A_\lambda] f, f\rangle + \langle S_\lambda f, S_\lambda f\rangle + \langle A_\lambda f, A_\lambda f\rangle - \langle F, F\rangle \\
&= \langle [S_\lambda, A_\lambda] f, f\rangle - \langle F, F\rangle + \langle S_\lambda f, S_\lambda f\rangle + \langle A_\lambda f, A_\lambda f\rangle \\
&\geq A_0 \int_{\mathbb{R}} \varphi^{(2)}(x) \left[\lambda^3 f^2 + \lambda\left(\partial_x f\right)^2\right] \mathtt{dx} \\
&\quad - \frac{A_0}{4} \int_{\mathbb{R}} \varphi^{(2)}(x) \left[\lambda^3 f^2 + \lambda\left(\partial_x f\right)^2\right] \mathtt{dx} + \langle S_\lambda f, S_\lambda f\rangle \\
&\geq \langle S_\lambda f, S_\lambda f\rangle + \frac{A_0}{2} \int_{\mathbb{R}} \varphi^{(2)}(x) \left[\lambda^3 f^2 + \lambda\left(\partial_x f\right)^2\right] \mathtt{dx},
\end{aligned}$$

disto, segue a desigualdade (2.81).

Logo, finalizamos demonstrando a desigualdade (2.98). Utilizando as Proposições 13 e 18 segue que

$$\begin{aligned}
&\langle [S_\lambda, A_\lambda] f, f\rangle + 2\langle S_\lambda f, S_\lambda f\rangle - 2\langle F, S_\lambda f\rangle \\
&\geq \langle [S_\lambda, A_\lambda] f, f\rangle + \langle S_\lambda f, S_\lambda f\rangle + \langle A_\lambda f, A_\lambda f\rangle - \langle F, F\rangle \\
&= \langle [S_\lambda, A_\lambda] f, f\rangle - \langle F, F\rangle + \langle S_\lambda f, S_\lambda f\rangle + \langle A_\lambda f, A_\lambda f\rangle \\
&\geq A_0 \int_{\mathbb{R}} \varphi^{(2)}(r) \left[\lambda^3 (\varphi')^2 f^2 + \lambda\left(\partial_x f\right)^2\right] \mathtt{dx}
\end{aligned}$$

$$- \frac{A_0}{4} \int_{\mathbb{R}} \varphi^{(2)}(r) \left[\lambda^3 (\varphi')^2 f^2 + \lambda (\partial_x f)^2\right] \mathtt{dx} + \langle S_\lambda f, S_\lambda f \rangle$$
$$\geq \langle S_\lambda f, S_\lambda f \rangle + \frac{A_0}{2} \int_{\mathbb{R}} \varphi^{(2)}(r) \left[\lambda^3 (\varphi')^2 f^2 + \lambda (\partial_x f)^2\right] \mathtt{dx},$$

disto, segue a desigualdade (2.98)

A.2 Existência da função peso de Carleman dos lemas 2.2 e 3.2

Faremos a demonstração do Lema 3.2. A demonstração do Lema 2.2 segue de maneira análoga. **Demonstração:** Seja $0 < \epsilon$ pequeno e consideramos a seguinte função contínua

$$\begin{aligned} \psi_0 \ : [0,\infty) &\longrightarrow \mathbb{R} \\ r \quad &\longmapsto \psi_0(r), \end{aligned}$$

definida por

$$\psi_0(r) = \begin{cases} 1, & r \in [0, \frac{3}{2} + 2\epsilon], \\ ar + b, & r \in [\frac{3}{2} + 2\epsilon, 2 - 2\epsilon], \\ \frac{\ln 2}{4r(\ln r)^2}, & r \in [2 - 2\epsilon, \infty), \end{cases} \tag{A.2}$$

com $a, b \in \mathbb{R}$, tais que que satisfazem a seguinte equação

$$\begin{cases} a\left(\frac{3}{2} + 2\epsilon\right) + b = 1, \\ a(2 - 2\epsilon) + b = \frac{\ln 2}{4(2-2\epsilon)[\ln(2-2\epsilon)]^2}, \end{cases} \tag{A.3}$$

disto, temos que $\psi_0 \in C([0,\infty))$.

Agora, definimos a função $\psi : [0,\infty) \to \mathbb{R}$ por

$$\psi(r) := \int_0^r \int_0^s \psi_0(t) \mathtt{dtds}.$$

Calculando, obtemos

$$\psi(r) = \begin{cases} \dfrac{r^2}{2}, & r \in [0, \frac{3}{2} + 2\epsilon], \\ \dfrac{ar^3}{6} + \dfrac{br^2}{2} + c_1 r + c_2, & r \in [\frac{3}{2} + 2\epsilon, 2 - 2\epsilon], \\ -\displaystyle\int_{2-2\epsilon}^r \frac{\ln 2}{4 \ln s} \mathtt{ds} + c_3 r + c_4, & r \in [2 - 2\epsilon, \infty), \end{cases} \tag{A.4}$$

com as constantes c_1, c_2, c_3 e c_4 são dadas por

$$\begin{aligned}
c_1 =& -\frac{a}{2}\left(\frac{3}{2}+2\epsilon\right)^2 - b\left(\frac{3}{2}+2\epsilon\right) + \frac{3}{2}+2\epsilon,\\
c_2 =& -\frac{a}{6}\left(\frac{3}{2}+2\epsilon\right)^3 - \frac{b}{2}\left(\frac{3}{2}+2\epsilon\right)^2 + \frac{1}{2}\left(\frac{3}{2}+2\epsilon\right)^2 - c_1\left(\frac{3}{2}+2\epsilon\right),\\
c_3 =& \frac{a}{2}\left(2-2\epsilon\right)^2 + b\left(2-2\epsilon\right) - \frac{a}{2}\left(\frac{3}{2}+2\epsilon\right)^2 - b\left(\frac{3}{2}+2\epsilon\right) + \frac{3}{2}+2\epsilon + \frac{\ln 2}{4\ln\left(2-2\epsilon\right)},\\
c_4 =& \frac{a}{6}(2-2\epsilon)^3 + \frac{b}{2}\left(2-2\epsilon\right)^2 + c_1\left(2-2\epsilon\right) + c_2 - c_3(2-2\epsilon).
\end{aligned}$$

Para podermos estimar as constantes c_1, c_2 e c_3, tomamos $\epsilon = 0$, disto temos

$$c_3 = 2 + \frac{1}{32\ln 2} \qquad \text{e} \qquad |\psi'(r)| \leq 2 + \frac{1}{32\ln 2}, \quad \forall r \geq 0.$$

Portanto, utilizando a continuidade de ψ, podemos tomar $\epsilon > 0$ suficientemente pequeno e obter

$$2 < c_3 < 3 \qquad \text{e} \qquad |\psi'(r)| < 3, \quad \forall r \geq 0. \tag{A.5}$$

Então, definindo $\psi(r) = \psi(-r)$, para todo $r < 0$, temos ψ, par e não negativa.

Por outro lado, tomando $\psi_\epsilon \in C_c^\infty(\mathbb{R})$ tal que

$$\psi_\epsilon \geq 0, \quad \psi_\epsilon(x) = \psi_\epsilon(-x), \quad \operatorname{supp}(\psi_\epsilon) \subset [-\epsilon, \epsilon] \quad \text{e} \quad \int_{\mathbb{R}} \psi_\epsilon(x)dx = 1$$

definimos

$$\begin{array}{rccl}
\varphi_0 : & \mathbb{R} & \longrightarrow & [0,\infty) \\
& r & \longmapsto & \psi * \psi_\epsilon(r).
\end{array} \tag{A.6}$$

Então, temos que: $\varphi_0 \in C^\infty(\mathbb{R})$, par e não negativa.

A seguir, provaremos que φ_0 é a função procurada.

De fato, por (A.2), observamos que $\psi'' > 0$, consequentemente, ψ' é uma função crescente. Assim, para $r \geq 1$ e $y \in [-\epsilon, \epsilon]$, temos $r - y \geq 1 - \epsilon$ e utilizando as propriedades de ψ_ε, resulta que

$$\varphi_0'(r) = \psi' * \psi_\epsilon(r) = \int_{-\epsilon}^{\epsilon} \psi'(r-y)\psi_\epsilon(y)dy \geq \psi'(1-\epsilon) = 1 - \epsilon, \qquad \forall r \geq 1. \tag{A.7}$$

Por outro lado, para $r \leq -1$ e $y \in [-\epsilon, \epsilon]$, segue que $r - y \leq -1 + \epsilon$. Então

$$\varphi_0'(r) = \psi' * \psi_\epsilon(r) = \int_{-\epsilon}^{\epsilon} \psi'(r-y)\psi_\epsilon(y)dy \leq \psi'(-1+\epsilon) = -1+\epsilon, \qquad \forall r \leq -1, \quad \text{(A.8)}$$

logo, segue de (A.7) e (A.8)

$$|\varphi_0'(r)| \geq 1 - \varepsilon, \quad \forall \varepsilon > 0 \tag{A.9}$$

onde concluímos a validade de (i), isto é

$$\inf_{|r|\geq 1} |\varphi_0'(r)| \geq 1.$$

Por outro lado, de (A.5) obtemos (ii) e observando que $\psi''(r) = 1$, para todo $|r| \leq \frac{3}{2} + 2\epsilon$, segue (iii). Para obter (iv) é suficiente observar que $0 < \psi''(r) \leq 1$, para todo $r \in \mathbb{R}$. O item (v) segue do fato da função ψ_0 ser decrescente para $r \geq 0$. Para concluir a demonstração, resta-nos obter (vi). Primeiro, provaremos que existe uma constante $\widetilde{C}_1 > 0$ tal que

$$e^{\varphi_0(r)} \leq \widetilde{C}_1 e^{3r}, \qquad \forall r \geq 0.$$

Como este resultado é imediato no intervalo compacto $[0, 2]$, consideremos $r \geq 2$. Então, para $y \in [-\epsilon, \epsilon]$, segue que $r - y > 2 - 2\epsilon$.

Assim, por (A.4) e (A.5), obtemos

$$\begin{aligned}
\varphi_0(r) = \psi * \psi_\epsilon(r) &= \int_{-\epsilon}^{\epsilon} \psi(r-y)\psi_\epsilon(y)dy \\
&= \int_{-\epsilon}^{\epsilon} \left[-\int_{2-2\epsilon}^{r-y} \frac{\ln 2}{4\ln s} ds + c_3(r-y) + c_4 \right] \psi_\epsilon(y)dy \\
&\leq \int_{-\epsilon}^{\epsilon} (3r + c_4)\psi_\epsilon(y)dy \\
&= 3r + c_4,
\end{aligned}$$

e portanto, $e^{\varphi_0(r)} \leq e^{c_4} e^{3r}$, para todo $r \geq 2$.

Agora, provaremos que existe uma constante $\widetilde{C}_2 > 0$ tal que

$$\widetilde{C}_2 \varphi_0^{(2)}(r) e^{\frac{\varphi_0(r)}{6}} \geq 1, \qquad \forall r \geq 0.$$

Para obter este resultado, é suficiente que

$$\lim_{r\to\infty} \varphi_0^{(2)}(r)e^{\frac{\varphi_0(r)}{6}} = \infty, \tag{A.10}$$

então, a seguir verificamos este limite. Dados $r \geq 2$ e $y \in [-\epsilon, \epsilon]$, temos $r - y > 2 - 2\epsilon$. Assim,

$$\begin{aligned} \varphi_0^{(2)}(r) = \psi^{(2)} * \psi_\epsilon(r) &= \int_{-\epsilon}^{\epsilon} \psi^{(2)}(r-y)\psi_\epsilon(y)dy \\ &= \int_{-\epsilon}^{\epsilon} \frac{\ln 2}{4(r-y)[\ln(r-y)]^2}\psi_\epsilon(y)dy \\ &\geq \frac{\ln 2}{4(r+\epsilon)[\ln(r+\epsilon)]^2}. \end{aligned} \tag{A.11}$$

Além disso, lembrando que $c_3 > 2$, por (A.5), obtemos

$$\begin{aligned} \varphi_0(r) &= \int_{-\epsilon}^{\epsilon} \psi(r-y)\psi_\epsilon(y)dy \\ &\geq \int_{-\epsilon}^{\epsilon} \left[-\int_{2-2\epsilon}^{r-y} \frac{\ln 2}{4\ln s}ds + 2(r-y) + c_4 \right] \psi_\epsilon(y)dy. \end{aligned} \tag{A.12}$$

Por outro lado, observamos que

$$2r - \int_{2-2\epsilon}^{r} \frac{\ln 2}{4\ln s}ds \geq r, \qquad \forall r \geq 2-2\epsilon,$$

dai, utilizando (A.12) temos que

$$\varphi_0(r) \geq r + c_4, \qquad \forall r \geq 2. \tag{A.13}$$

Logo, por (A.11) e (A.13), segue que

$$\varphi_0^{(2)}(r)e^{\frac{\varphi_0(r)}{6}} \geq \frac{e^{(r+c_4)/6}\ln 2}{4(r+\epsilon)[\ln{(r+\epsilon)}]^2}, \qquad \forall r \geq 2,$$

e, portanto,

$$\lim_{r\to\infty} \varphi_0^{(2)}(r)e^{\frac{\varphi_0(r)}{6}} = \infty.$$

Para concluirmos a demonstração do lema, agora, provaremos que existe uma constante $\widetilde{C}_3 > 0$ tal que

$$\left|\frac{d^k\varphi_0(r)}{dr^k}\right| \leq \widetilde{C}_3\varphi_0^{(2)}(r), \qquad \forall r \geq 0, \qquad k = 3,4,5,6,7,8. \tag{A.14}$$

De fato, se $r \in [0, \frac{3}{2}+\epsilon)$ e $y \in [-\epsilon, \epsilon]$, então $-\epsilon \leq r - y < \frac{3}{2} + 2\epsilon$ e, disto,

$$\varphi_0^{(3)}(r) = \int_{-\epsilon}^{\epsilon} \psi^{(3)}(r-y)\psi_\varepsilon(y)dy = 0.$$

Assim, concluímos que

$$\varphi_0^{(k)}(r) = 0, \quad \forall r \in \left[0, \frac{3}{2} + \epsilon\right), \qquad \forall k \geq 3.$$

Sendo $\varphi_0^{(2)}(r) > 0$, para todo $r \geq 0$, obtemos (A.14) para o intervalo $\left[0, \frac{3}{2} + \epsilon\right)$. O caso $r \in \left[\frac{3}{2} + \epsilon, 2 + \epsilon\right]$ é imediato, uma vez que as derivadas $\varphi_0^{(k)}$ são limitadas neste intervalo e $\varphi_0^{(2)} > 0$. Portanto, suponhamos $r \in (2 + \epsilon, \infty)$. Neste caso, temos $r - y > 2$, para todo $y \in [-\epsilon, \epsilon]$, e assim

$$\begin{aligned}
\varphi_0^{(k)}(r) &= \int_{-\epsilon}^{\epsilon} \psi^{(k)}(r - y)\psi_\epsilon(y)dy \\
&= \int_{-\epsilon}^{\epsilon} \psi_0^{(k-2)}(r - y)\psi_\epsilon(y)dy, \qquad \forall\, k \in \{3, 4, \ldots, 8\}.
\end{aligned}$$

Por outro lado, derivando, observamos que existe uma constante $\widetilde{C}_3 > 0$ tal que

$$|\psi_0^{(k-2)}(r)| \leq \widetilde{C}_3 \psi_0(r), \qquad \forall\, r > 2, \quad k = 3, \ldots, 8,$$

e portanto,

$$\begin{aligned}
|\varphi_0^{(k)}(r)| &\leq \widetilde{C}_3 \int_{-\epsilon}^{\epsilon} \psi_0(r - y)\psi_\varepsilon(y)dy \\
&= \widetilde{C}_3 \varphi_0^{(2)}(r), \qquad \forall\, r > 2, \quad k = 3, \ldots, 8.
\end{aligned}$$

Tomando $C_1 = \max\left\{\widetilde{C}_1, \quad \widetilde{C}_2, \quad \widetilde{C}_3\right\}$, concluímos a demonstração do Lema 3.2. ■

Apêndice B

B.1 Demonstração da equação (4.35)

$$\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}}\eta(t)\,|S_\lambda f(x,t)|^2\,\mathtt{dxdt}\le \widetilde{C}_\lambda.$$

Demonstração: Sabemos que

$$\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}}\eta(t)\,|S_\lambda f|^2\,\mathtt{dxdt}\le 2\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}}|V|^2\,|f|^2\,\mathtt{dxdt}+2\left|\int_{\mathbb{R}}|f|^2\Big|_{T-1/2}^{T+1/2}\mathtt{dx}\right|$$
$$-\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}}[S_\lambda,A_\lambda]f\overline{f}\mathtt{dxdt}. \tag{B.1}$$

Em seguida, estimaremos $-\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}}[S_\lambda,A_\lambda]f\overline{f}\mathtt{dxdt}$, para isto primeiro estimamos os termos negativos ou com sinal não definido em $-\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}}[S_\lambda,A_\lambda]f\overline{f}\mathtt{dxdt}$. Para isto utilizamos $1<p\le 8/7$, (4.22) e (4.3)

$$\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}}\big[4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r)+64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)$$
$$+48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3-32\lambda^3\varphi'(r)\varphi^{(3)}(r)\varphi^{(4)}(r)-4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)$$
$$+36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r)+80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2-16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$
$$-4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r)-24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r)+\lambda\varphi^{(4)}(r)$$
$$+2\lambda\varphi^{(6)}(r)+\lambda\varphi^{(8)}(r)\big]|f|^2\mathtt{dxdt}$$
$$+\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}}\big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)+96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$
$$-12\lambda\varphi^{(4)}(r)-4\lambda\varphi^{(2)}(r)-8\lambda\varphi^{(6)}(r)\big]|\partial_x f|^2\mathtt{dxdt}$$
$$+\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}}\big[16\lambda\varphi^{(2)}(r)+20\lambda\varphi^{(4)}(r)\big]|\partial_x^2 f|^2\mathtt{dxdt}$$

$$
\begin{aligned}
&\leq \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \big[4\lambda^5 \left\|(\varphi'(r))^4\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 64\lambda^5 \left\|(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&+ 48\lambda^5 \left\|(\varphi'(r))^2(\varphi^{(2)}(r))^3\right\|_{L^\infty(\mathbb{R})} + 32\lambda^3 \left\|\varphi'(r)\varphi^{(3)}(r)\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&+ 4\lambda^3 \left\|(\varphi'(r))^2\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 36\lambda^3 \left\|(\varphi^{(2)}(r))^2\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&+ 80\lambda^3 \left\|(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2\right\|_{L^\infty(\mathbb{R})} + 16\lambda^3 \left\|\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&- 4\lambda^3 \left\|(\varphi'(r))^2\varphi^{(6)}(r)\right\|_{L^\infty(\mathbb{R})} + 24\lambda^3 \left\|\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r)\right\|_{L^\infty(\mathbb{R})} + \lambda \left\|\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&+ 2\lambda \left\|\varphi^{(6)}(r)\right\|_{L^\infty(\mathbb{R})} + \lambda \left\|\varphi^{(8)}(r)\right\|_{L^\infty(\mathbb{R})}\big]|f|^2\mathtt{dxdt} \\
&+ \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \big[24\lambda^3 \left\|(\varphi'(r))^2\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 96\lambda^3 \left\|\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&- 12\lambda \left\|\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 4\lambda \left\|\varphi^{(2)}(r)\right\|_{L^\infty(\mathbb{R})} + 8\lambda \left\|\varphi^{(6)}(r)\right\|_{L^\infty(\mathbb{R})}\big]|\partial_x f|^2\mathtt{dxdt} \\
&+ \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \big[16\lambda \left\|\varphi^{(2)}(r)\right\|_{L^\infty(\mathbb{R})} + 20\lambda \left\|\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})}\big]|\partial_x^2 f|^2\mathtt{dxdt} \\
&\leq \big(4\lambda^5 \left\|(\varphi'(r))^4\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 64\lambda^5 \left\|(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&+ 48\lambda^5 \left\|(\varphi'(r))^2(\varphi^{(2)}(r))^3\right\|_{L^\infty(\mathbb{R})} + 32\lambda^3 \left\|\varphi'(r)\varphi^{(3)}(r)\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&+ 4\lambda^3 \left\|(\varphi'(r))^2\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 36\lambda^3 \left\|(\varphi^{(2)}(r))^2\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&+ 80\lambda^3 \left\|(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2\right\|_{L^\infty(\mathbb{R})} + 16\lambda^3 \left\|\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&- 4\lambda^3 \left\|(\varphi'(r))^2\varphi^{(6)}(r)\right\|_{L^\infty(\mathbb{R})} + 24\lambda^3 \left\|\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r)\right\|_{L^\infty(\mathbb{R})} + \lambda \left\|\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&+ 2\lambda \left\|\varphi^{(6)}(r)\right\|_{L^\infty(\mathbb{R})} + \lambda \left\|\varphi^{(8)}(r)\right\|_{L^\infty(\mathbb{R})}\big) \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |f|^2\mathtt{dxdt} \\
&+ \big(24\lambda^3 \left\|(\varphi'(r))^2\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 96\lambda^3 \left\|\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&- 12\lambda \left\|\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 4\lambda \left\|\varphi^{(2)}(r)\right\|_{L^\infty(\mathbb{R})} + 8\lambda \left\|\varphi^{(6)}(r)\right\|_{L^\infty(\mathbb{R})}\big) \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |\partial_x f|^2\mathtt{dxdt} \\
&+ \big(24\lambda^3 \left\|(\varphi'(r))^2\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 96\lambda^3 \left\|\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\right\|_{L^\infty(\mathbb{R})} \\
&- 12\lambda \left\|\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 4\lambda \left\|\varphi^{(2)}(r)\right\|_{L^\infty(\mathbb{R})} + 8\lambda \left\|\varphi^{(6)}(r)\right\|_{L^\infty(\mathbb{R})}\big) \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |\partial_x f|^2\mathtt{dxdt} \\
&+ \big(16\lambda \left\|\varphi^{(2)}(r)\right\|_{L^\infty(\mathbb{R})} + 20\lambda \left\|\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})}\big) \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |\partial_x^2 f|^2\mathtt{dxdt}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \big(4\lambda^5 \left\|(\varphi'(r))^4\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 64\lambda^5 \left\|(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\right\|_{L^\infty(\mathbb{R})}\\
&+ 48\lambda^5 \left\|(\varphi'(r))^2(\varphi^{(2)}(r))^3\right\|_{L^\infty(\mathbb{R})} + 32\lambda^3 \left\|\varphi'(r)\varphi^{(3)}(r)\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})}\\
&+ 4\lambda^3 \left\|(\varphi'(r))^2\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 36\lambda^3 \left\|(\varphi^{(2)}(r))^2\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})}\\
&+ 80\lambda^3 \left\|(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2\right\|_{L^\infty(\mathbb{R})} + 16\lambda^3 \left\|\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\right\|_{L^\infty(\mathbb{R})}\\
&+ 4\lambda^3 \left\|(\varphi'(r))^2\varphi^{(6)}(r)\right\|_{L^\infty(\mathbb{R})} + 24\lambda^3 \left\|\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r)\right\|_{L^\infty(\mathbb{R})} + \lambda \left\|\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})}\\
&+ 2\lambda \left\|\varphi^{(6)}(r)\right\|_{L^\infty(\mathbb{R})} + \lambda \left\|\varphi^{(8)}(r)\right\|_{L^\infty(\mathbb{R})}\big) \sup_{t\geq 0} \int_{\mathbb{R}} |f|^2 \mathtt{dx}\\
&+ \big(24\lambda^3 \left\|(\varphi'(r))^2\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 96\lambda^3 \left\|\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\right\|_{L^\infty(\mathbb{R})}\\
&+ 12\lambda \left\|\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})} + 4\lambda \left\|\varphi^{(2)}(r)\right\|_{L^\infty(\mathbb{R})} + 8\lambda \left\|\varphi^{(6)}(r)\right\|_{L^\infty(\mathbb{R})}\big) \sup_{t\geq 0} \int_{\mathbb{R}} |\partial_x f|^2 \mathtt{dx}\\
&+ \big(16\lambda \left\|\varphi^{(2)}(r)\right\|_{L^\infty(\mathbb{R})} + 20\lambda \left\|\varphi^{(4)}(r)\right\|_{L^\infty(\mathbb{R})}\big) \sup_{t\geq 0} \int_{\mathbb{R}} |\partial_x^2 f|^2 \mathtt{dx}\\
&\leq C_{1,\lambda} \sup_{t\geq 0} \int_{\mathbb{R}} |f|^2 \mathtt{dx} + C_{2,\lambda} \sup_{t\geq 0} \int_{\mathbb{R}} |\partial_x f|^2 \mathtt{dx} + C_{3,\lambda} \sup_{t\geq 0} \int_{\mathbb{R}} |\partial_x^2 f|^2 \mathtt{dx}\\
&\leq \widetilde{C}_\lambda < \infty,
\end{aligned} \tag{B.2}
$$

com $C_{1,\lambda}$, $C_{2,\lambda}$ e $C_{3,\lambda}$ constantes positivas que dependem de λ, p.

Logo, utilizando a equação (B.2), temos

$$
\begin{aligned}
&- \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} [S_\lambda; A_\lambda] f \overline{f} \mathtt{dxdt}\\
&= - \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} \left[16\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2 \mathtt{dxdt}\\
&- \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} \left[48\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2 \mathtt{dxdt}\\
&- \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} \left[48\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |\partial_x^2 f|^2 \mathtt{dxdt} - \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} \left[16\lambda\varphi^{(2)}(r)\right] |\partial_x^3 f|^2 \mathtt{dxdt}\\
&+ \int_{T-1/2}^{T+1/2} \int_{\mathbb{R}} \big[4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r) + 64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&+ 48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3 - 32\lambda^3\varphi'(r)\varphi^{(3)}(r)\varphi^{(4)}(r) - 4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)\\
&+ 36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r) + 80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2 - 16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&- 4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r) - 24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r) + \lambda\varphi^{(4)}(r)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\lambda\varphi^{(6)}(r) + \lambda\varphi^{(8)}(r)\big]|f|^2\mathtt{dxdt}\\
&+ \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r) + 96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&- 12\lambda\varphi^{(4)}(r) - 4\lambda\varphi^{(2)}(r) - 8\lambda\varphi^{(6)}(r)\big]|\partial_x f|^2\mathtt{dxdt}\\
&+ \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \big[16\lambda\varphi^{(2)}(r) + 20\lambda\varphi^{(4)}(r)\big]|\partial_x^2 f|^2\mathtt{dxdt}\\
&\leq \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \big[4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r) + 64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&+ 48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3 - 32\lambda^3\varphi'(r)\varphi^{(3)}(r)\varphi^{(4)}(r) - 4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)\\
&+ 36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r) + 80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2 - 16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&- 4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r) - 24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r) + \lambda\varphi^{(4)}(r)\\
&+ 2\lambda\varphi^{(6)}(r) + \lambda\varphi^{(8)}(r)\big]|f|^2\mathtt{dxdt}\\
&+ \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r) + 96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&- 12\lambda\varphi^{(4)}(r) - 4\lambda\varphi^{(2)}(r) - 8\lambda\varphi^{(6)}(r)\big]|\partial_x f|^2\mathtt{dxdt}\\
&+ \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \big[16\lambda\varphi^{(2)}(r) + 20\lambda\varphi^{(4)}(r)\big]|\partial_x^2 f|^2\mathtt{dxdt}\\
&\leq \widetilde{C}_\lambda;
\end{aligned}
\tag{B.3}
$$

daí, segue pela equação (B.3) e (4.3) em (B.1) que

$$
\begin{aligned}
\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} \eta(t)\,|S_\lambda f|^2\,\mathtt{dxdt} &\leq 2\int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} |V|^2\,|f|^2\,\mathtt{dxdt} + 2\left|\int_{\mathbb{R}} |f|^2\,\mathtt{dx}\Big|_{T-1/2}^{T+1/2}\right|\\
&\quad - \int_{T-1/2}^{T+1/2}\int_{\mathbb{R}} [S_\lambda, A_\lambda]f\overline{f}\mathtt{dxdt}\\
&\leq\ (2\,\|V\|_{L^\infty} + 2)\sup_{t\geq 0}\int_{\mathbb{R}} |f|^2\,\mathtt{dxdt} + C_\lambda\\
&\leq \widetilde{C}_\lambda.
\end{aligned}
$$

onde, $\widetilde{C}_\lambda$ depende de $C_\lambda,\ \lambda,\ p$ e $\|V\|_\infty$. ■

B.2 Prova da desigualdade (4.42)

Nesta seção, observamos que os termos positivos de $\langle [S_\lambda, A_\lambda] f, f\rangle$ pode ser estimado por integrais em $[0,\varepsilon]$, isto é, existe $\lambda_0 > 0$ tal que $\forall\ \lambda > \lambda_0$, segue a seguinte desigualdade

$$\begin{aligned}
&\int \left[8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2 \,\mathtt{dx} \\
&+ \int \left[24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2 \,\mathtt{dx} \\
&+ \int 24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dx} + \int 16\lambda\varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dx} \\
\leq\ \ & C_\lambda + C' \left(\lambda^5 \left\|(\varphi')^4\varphi^{(4)}\right\|_\infty + \lambda^5 \left\|(\varphi')^3\varphi^{(2)}\varphi^{(3)}\right\|_\infty + \lambda^5 \left\|(\varphi')^2(\varphi^{(2)})^3\right\|_\infty\right. \\
&\ + \lambda^3 \left\|\varphi'\varphi^{(3)}\varphi^{(4)}\right\|_\infty + \lambda^3 \left\|(\varphi')^2\varphi^{(4)}\right\|_\infty + \lambda^3 \left\|(\varphi^{(2)})^2\varphi^{(4)}\right\| + \lambda^3 \left\|\varphi^{(2)}(\varphi^{(3)})^2\right\|_\infty \\
&\ + \lambda^3 \left\|\varphi'\varphi^{(2)}\varphi^{(3)}\right\|_\infty + \lambda \left\|(\varphi')^2\varphi^{(6)}\right\|_\infty + \lambda^3 \left\|\varphi'\varphi^{(2)}\varphi^{(5)}\right\|_\infty + \lambda \left\|\varphi^{(4)}\right\|_\infty + \lambda \left\|\varphi^{(6)}\right\|_\infty \\
&\ \left. +\lambda \left\|\varphi^{(8)}\right\|_\infty\right) \int_{|x|\leq\varepsilon} |f|^2 \,\mathtt{dx} \\
&\ + C'' \left(\lambda^3 \left\|(\varphi')^2\varphi^{(4)}\right\|_\infty + \lambda^3 \left\|\varphi'\varphi^{(2)}\varphi^{(3)}\right\|_\infty + \lambda \left\|\varphi^{(4)}\right\|_\infty + \lambda \left\|\varphi^{(6)}\right\|_\infty\right) \int_{|x|\leq\varepsilon} |\partial_x f|^2 \,\mathtt{dx} \\
&\ + C''' \left(\lambda \left\|\varphi^{(2)}\right\|_\infty + \lambda \left\|\varphi^{(4)}\right\|_\infty\right) \int_{|x|\leq\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dx}.
\end{aligned} \tag{B.4}$$

Para facilitar a demonstração faremos por etapas:

Etapa 1:

Nesta Etapa, demonstraremos a seguinte desigualdade:

$$\begin{aligned}
&\int_{\mathbb{R}} \left[8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2 \,\mathtt{dx} \\
&+ \int_{\mathbb{R}} \left[24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2 \,\mathtt{dx} \\
&+ \int_{\mathbb{R}} 24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dx} + \int_{\mathbb{R}} 16\lambda\varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dx} \\
&\leq C_\lambda + \int_{\mathbb{R}} |Vf|^2 \,\mathtt{dx} + \int_{\mathbb{R}} \left[4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r) + 64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\right. \\
&+ 48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3 - 32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r) - 4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r) \\
&+ 36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r) + 80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2 - 16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r) \\
&- 4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r) - 24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r) + \lambda\varphi^{(4)}(r) + 2\lambda\varphi^{(6)}(r) \\
&\left. + \lambda\varphi^{(8)}(r)\right] |f|^2 \mathtt{dx}
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}}\big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)+96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&-12\lambda\varphi^{(4)}(r)-4\lambda\varphi^{(2)}(r)-8\lambda\varphi^{(6)}(r)\big]|\partial_x f|^2\mathtt{dx}\\
&+\int_{\mathbb{R}}\big[16\lambda\varphi^{(2)}(r)+20\lambda\varphi^{(4)}(r)\big]|\partial_x^2 f|^2\mathtt{dxdt}\\
&-\int_{\mathbb{R}}8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\,|f|^2\,\mathtt{dxdt}-\int_{\mathbb{R}}24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\,|\partial_x f|^2\,\mathtt{dxdt}\\
&-\int_{\mathbb{R}}24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\,\left|\partial_x^2 f\right|^2\mathtt{dx}.
\end{aligned}
\tag{B.5}
$$

De fato, segue da definição de $\langle[S_\lambda,A_\lambda]f,f\rangle$ em (4.22), que o lado esquerdo de (B.5) é:

$$
\begin{aligned}
&\int_{\mathbb{R}}\big[8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)+16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)+4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\big]\,|f|^2\,\mathtt{dx}\\
&+\int_{\mathbb{R}}\big[24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)+48\lambda^3(\varphi^{(2)}(r))^3+4\lambda\varphi^{(2)}(r)\big]\,|\partial_x f|^2\,\mathtt{dx}\\
&+\int_{\mathbb{R}}24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\,\left|\partial_x^2 f\right|^2\mathtt{dx}+\int_{\mathbb{R}}16\lambda\varphi^{(2)}(r)\,\left|\partial_x^3 f\right|^2\mathtt{dx}\\
&=\int_{\mathbb{R}}[S_\lambda,A_\lambda]f\overline{f}\mathtt{dx}+\int_{\mathbb{R}}\big[4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r)\\
&+64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)+48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3-32\lambda^3\varphi'(r)\varphi^{(3)}(r)\varphi^{(4)}(r)\\
&-4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)+36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r)+80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2\\
&-16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)-4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r)-24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r)+\lambda\varphi^{(4)}(r)\\
&+2\lambda\varphi^{(6)}(r)+\lambda\varphi^{(8)}(r)\big]|f|^2\mathtt{dx}\\
&+\int_{\mathbb{R}}\big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)+96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&-12\lambda\varphi^{(4)}(r)-4\lambda\varphi^{(2)}(r)-8\lambda\varphi^{(6)}(r)\big]|\partial_x f|^2\mathtt{dx}\\
&+\int_{\mathbb{R}}\big[16\lambda\varphi^{(2)}(r)+20\lambda\varphi^{(4)}(r)\big]|\partial_x^2 f|^2\mathtt{dx}\\
&-\int_{\mathbb{R}}8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\,|f|^2\,\mathtt{dxdt}-\int_{T_1}^{T_j}\int_{\mathbb{R}}24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\,|\partial_x f|^2\,\mathtt{dxdt}\\
&-\int_{\mathbb{R}}24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\,\left|\partial_x^2 f\right|^2\mathtt{dx}.
\end{aligned}
\tag{B.6}
$$

Daí, utilizando a (4.38) em (B.6), resulta que

$$
\begin{aligned}
&\int_{\mathbb{R}} \left[8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2 \,\mathtt{dx} \\
&+ \int_{\mathbb{R}} \left[24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2 \,\mathtt{dx} \\
&+ \int_{\mathbb{R}} 24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dx} + \int_{\mathbb{R}} 16\lambda\varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dx} \\
&\leq C_\lambda + \int_{\mathbb{R}} |Vf|^2 \,\mathtt{dx} + \int_{\mathbb{R}} \left[4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r) + 64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\right. \\
&+ 48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3 - 32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r) - 4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r) \\
&+ 36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r) + 80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2 - 16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r) \\
&- 4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r) - 24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r) + \lambda\varphi^{(4)}(r) + 2\lambda\varphi^{(6)}(r) \\
&\left.+ \lambda\varphi^{(8)}(r)\right]|f|^2\mathtt{dx} \\
&+ \int_{\mathbb{R}} \left[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r) + 96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\right. \\
&\left.- 12\lambda\varphi^{(4)}(r) - 4\lambda\varphi^{(2)}(r) - 8\lambda\varphi^{(6)}(r)\right]|\partial_x f|^2\mathtt{dx} \\
&+ \int_{\mathbb{R}} \left[16\lambda\varphi^{(2)}(r) + 20\lambda\varphi^{(4)}(r)\right]|\partial_x^2 f|^2\mathtt{dxdt} \\
&- \int_{\mathbb{R}} 8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) \,|f|^2 \,\mathtt{dxdt} - \int_{\mathbb{R}} 24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) \,|\partial_x f|^2 \,\mathtt{dxdt} \\
&- \int_{\mathbb{R}} 24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dx};
\end{aligned}
\tag{B.7}
$$

com isto finalizamos a demonstração da Etapa 1.

A seguir, no lado direito de (B.7) dividimos a integral respeito a x, nos intervalos $[0, \varepsilon]$, $[\varepsilon, 1]$ e $[1, +\infty)$, isto é

$$
\begin{aligned}
&\int_{\mathbb{R}} \left[8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2 \,\mathtt{dx} \\
&+ \int_{\mathbb{R}} \left[24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2 \,\mathtt{dx} \\
&+ \int_{\mathbb{R}} 24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dx} + \int 16\lambda\varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dx} \\
&\leq\; C_\lambda + \int_{T_1}^{T_j} \int_{|x|\leq\varepsilon} \left[\,|V|^2 + 4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r) + 64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\right. \\
&+ 48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3 - 32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r) - 4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)
\end{aligned}
$$

$$
\begin{aligned}
&+36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r)+80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2-16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&-4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r)-24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r)+\lambda\varphi^{(4)}(r)+2\lambda\varphi^{(6)}(r)\\
&+\lambda\varphi^{(8)}(r)\big]|f|^2\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{|x|\le\varepsilon}\big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)+96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&-12\lambda\varphi^{(4)}(r)-4\lambda\varphi^{(2)}(r)-8\lambda\varphi^{(6)}(r)\big]|\partial_x f|^2\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{|x|\le\varepsilon}\big[16\lambda\varphi^{(2)}(r)+20\lambda\varphi^{(4)}(r)\big]|\partial_x^2 f|^2\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1}\big[\,|V|^2+4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r)+64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&+48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3-32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r)-4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)\\
&+36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r)+80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2-16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&-4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r)-24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r)+\lambda\varphi^{(4)}(r)+2\lambda\varphi^{(6)}(r)\\
&+\lambda\varphi^{(8)}(r)-8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\big]|f|^2\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1}\big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)+96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&-12\lambda\varphi^{(4)}(r)-4\lambda\varphi^{(2)}(r)-8\lambda\varphi^{(6)}(r)-24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\big]|\partial_x f|^2\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1}\big[16\lambda\varphi^{(2)}(r)+20\lambda\varphi^{(4)}(r)-24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\big]|\partial_x^2 f|^2\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{1\le|x|}\big[\,|V|^2+4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r)+64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&+48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3-32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r)-4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)\\
&+36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r)+80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2-16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&-4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r)-24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r)+\lambda\varphi^{(4)}(r)+2\lambda\varphi^{(6)}(r)\\
&+\lambda\varphi^{(8)}(r)\big]|f|^2\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{1\le|x|}\big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)+96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\
&-12\lambda\varphi^{(4)}(r)-4\lambda\varphi^{(2)}(r)-8\lambda\varphi^{(6)}(r)-24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\big]|\partial_x f|^2\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{1\le|x|}\big[16\lambda\varphi^{(2)}(r)+20\lambda\varphi^{(4)}(r)-24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\big]|\partial_x^2 f|^2\mathtt{dxdt}\\
&-\int_{\mathbb{R}}8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\,|f|^2\,\mathtt{dxdt}-\int_{\mathbb{R}}24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\,|\partial_x f|^2\,\mathtt{dxdt}\\
&-\int_{\mathbb{R}}24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\,\left|\partial_x^2 f\right|^2\,\mathtt{dxdt}. \qquad \text{(B.8)}
\end{aligned}
$$

Depois, para estimar o lado direito de (B.8), usamos as seguintes desigualdades para λ suficientemente grande e $\varepsilon > 0$ suficientemente pequeno.

Etapa 2:

No conjunto $\{x \in \mathbb{R} : \varepsilon \leq |x| \leq 1\}$ temos que existem $\lambda_0^1,\ \lambda_0^2$ e λ_0^3 positivos tais que para todo $\lambda > \max\{\lambda_0^1, \lambda_0^2, \lambda_0^3\}$, segue as seguintes desigualdades:

$$\begin{aligned}\lambda \leq\ & 8\lambda^6(\varphi'(r))^6\varphi^{(2)}(r) - 4\lambda^4(\varphi'(r))^4\varphi^{(4)}(r) - 64\lambda^4(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\\ & - 48\lambda^4(\varphi'(r))^2(\varphi^{(2)}(r))^3 + 32\lambda^2\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r) + 4\lambda^2(\varphi'(r))^2\varphi^{(4)}(r)\\ & - 36\lambda^2(\varphi^{(2)}(r))^2\varphi^{(4)}(r) - 80\lambda^2(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2 + 16\lambda^2\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\ & + 4\lambda^2(\varphi'(r))^2\varphi^{(6)}(r) + 24\lambda^2\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r) - \varphi^{(4)}(r) - 2\varphi^{(6)}(r)\\ & - \varphi^{(8)}(r) - |V|^2,\end{aligned} \tag{B.9}$$

$$\begin{aligned}\lambda \leq\ & 24\lambda^4(\varphi'(r))^4\varphi^{(2)}(r) - 24\lambda^2(\varphi'(r))^2\varphi^{(4)}(r) - 96\lambda^2\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\ & + 12\varphi^{(4)}(r) + 4\varphi^{(2)}(r) + 8\varphi^{(6)}(r),\end{aligned} \tag{B.10}$$

$$\lambda \leq 24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) - 16\varphi^{(2)}(r) - 20\varphi^{(4)}(r); \tag{B.11}$$

isto segue do fato que: $8\lambda^6(\varphi'(r))^6\varphi^{(2)}(r)$, $24\lambda^4(\varphi'(r))^4\varphi^{(2)}(r)$, $24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)$ são estritamente positivos em $[\varepsilon, 1]$.

Por outro lado, para conjunto $\{x \in \mathbb{R} : 1 \leq |x|\}$ temos que existem $\lambda_0^4,\ \lambda_0^5$ e λ_0^6 positivos, tais que para todo $\lambda > \max\{\lambda_0^4, \lambda_0^5, \lambda_0^6\}$, segue as seguintes desigualdades:

$$\begin{aligned}\frac{\lambda}{\langle x\rangle^{8-7p}} \leq\ & 8\lambda^6(\varphi'(r))^6\varphi^{(2)}(r) - 4\lambda^4(\varphi'(r))^4\varphi^{(4)}(r) - 64\lambda^4(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)\\ & - 48\lambda^4(\varphi'(r))^2(\varphi^{(2)}(r))^3 + 32\lambda^2\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r) + 4\lambda^2(\varphi'(r))^2\varphi^{(4)}(r)\\ & - 36\lambda^2(\varphi^{(2)}(r))^2\varphi^{(4)}(r) - 80\lambda^2(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2 + 16\lambda^2\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)\\ & + 4\lambda^2(\varphi'(r))^2\varphi^{(6)}(r) + 24\lambda^2\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r) - \varphi^{(4)}(r) - 2\varphi^{(6)}(r)\\ & - \varphi^{(8)}(r) - |V|^2,\end{aligned} \tag{B.12}$$

$$\frac{\lambda}{\langle x\rangle^{6-5p}} \leq 24\lambda^4(\varphi'(r))^4\varphi^{(2)}(r) - 24\lambda^2(\varphi'(r))^2\varphi^{(4)}(r) - 96\lambda^2\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$
$$+ 12\varphi^{(4)}(r) + 4\varphi^{(2)}(r) + 8\varphi^{(6)}(r), \tag{B.13}$$

$$\frac{\lambda}{\langle x\rangle^{4-3p}} \leq 24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) - 16\varphi^{(2)}(r) - 20\varphi^{(4)}(r); \tag{B.14}$$

isto segue das propriedades de φ em $[1,\infty)$.

Logo, utilizando as desigualdades (B.9)-(B.11) e (B.12)-(B.14) da Etapa 2 em (B.8), resulta que

$$\int_{\mathbb{R}} \left[8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2\,\mathtt{dx}$$
$$+ \int_{\mathbb{R}} \left[24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2\,\mathtt{dx}$$
$$+ \int_{\mathbb{R}} 24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\left|\partial_x^2 f\right|^2\mathtt{dx} + \int_{\mathbb{R}} 16\lambda\varphi^{(2)}(r)\left|\partial_x^3 f\right|^2\mathtt{dx}$$
$$\leq\ C_\lambda + \int_{T_1}^{T_j}\int_{|x|\leq\varepsilon} \big[\,|V|^2 + 4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r) + 64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)$$
$$+ 48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3 - 32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r) - 4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)$$
$$+ 36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r) + 80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2 - 16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$
$$- 4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r) - 24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r) + \lambda\varphi^{(4)}(r) + 2\lambda\varphi^{(6)}(r)$$
$$+ \lambda\varphi^{(8)}(r)\big]|f|^2\mathtt{dxdt}$$
$$+ \int_{T_1}^{T_j}\int_{|x|\leq\varepsilon} \big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r) + 96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$
$$- 12\lambda\varphi^{(4)}(r) - 4\lambda\varphi^{(2)}(r) - 8\lambda\varphi^{(6)}(r)\big]|\partial_x f|^2\mathtt{dxdt}$$
$$+ \int_{T_1}^{T_j}\int_{|x|\leq\varepsilon} \big[16\lambda\varphi^{(2)}(r) + 20\lambda\varphi^{(4)}(r)\big]|\partial_x^2 f|^2\mathtt{dxdt}$$
$$+ \int_{T_1}^{T_j}\int_{\varepsilon\leq|x|\leq 1} \big[-\lambda^2 + (1-\lambda)\,|V|^2\,\big]|f|^2\mathtt{dxdt}$$
$$+ \int_{T_1}^{T_j}\int_{\varepsilon\leq|x|\leq 1} \big(-\lambda^2\big)|\partial_x f|^2\mathtt{dxdt}$$
$$+ \int_{T_1}^{T_j}\int_{\varepsilon\leq|x|\leq 1} \big(-\lambda^2\big)|\partial_x^2 f|^2\mathtt{dxdt}$$

$$+\int_{T_1}^{T_j}\int_{1\leq|x|}\left[(1-\lambda)\,|V|^2-\frac{\lambda^2}{\langle x\rangle^{8-7p}}\right]|f|^2\mathtt{dxdt}+\int_{T_1}^{T_j}\int_{1\leq|x|}\frac{-\lambda^2}{\langle x\rangle^{6-5p}}|\partial_x f|^2\mathtt{dxdt}$$
$$+\int_{T_1}^{T_j}\int_{1\leq|x|}\frac{-\lambda^2}{\langle x\rangle^{4-3p}}|\partial_x^2 f|^2\mathtt{dxdt}. \tag{B.15}$$

Assim, segue que $\forall\ \lambda>\max\{\lambda_0^1,\ldots,\lambda_0^6,1\}$

$$\int_{\mathbb{R}}\left[8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)+16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)+4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right]|f|^2\,\mathtt{dx}$$
$$+\int_{\mathbb{R}}\left[24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)+48\lambda^3(\varphi^{(2)}(r))^3+4\lambda\varphi^{(2)}(r)\right]|\partial_x f|^2\,\mathtt{dx}$$
$$+\int_{\mathbb{R}}24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\left|\partial_x^2 f\right|^2\mathtt{dx}+\int_{\mathbb{R}}16\lambda\varphi^{(2)}(r)\left|\partial_x^3 f\right|^2\mathtt{dx}$$
$$\leq\ C_\lambda+\int_{T_1}^{T_j}\int_{|x|\leq\varepsilon}\big[\,|V|^2+4\lambda^5(\varphi'(r))^4\varphi^{(4)}(r)+64\lambda^5(\varphi'(r))^3\varphi^{(2)}(r)\varphi^{(3)}(r)$$
$$+48\lambda^5(\varphi'(r))^2(\varphi^{(2)}(r))^3-32\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(4)}(r)-4\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)$$
$$+36\lambda^3(\varphi^{(2)}(r))^2\varphi^{(4)}(r)+80\lambda^3(\varphi^{(2)}(r))(\varphi^{(3)}(r))^2-16\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$
$$-4\lambda^3(\varphi'(r))^2\varphi^{(6)}(r)-24\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(5)}(r)+\lambda\varphi^{(4)}(r)+2\lambda\varphi^{(6)}(r)$$
$$+\lambda\varphi^{(8)}(r)\big]|f|^2\mathtt{dxdt}$$
$$+\int_{T_1}^{T_j}\int_{|x|\leq\varepsilon}\big[24\lambda^3(\varphi'(r))^2\varphi^{(4)}(r)+96\lambda^3\varphi'(r)\varphi^{(2)}(r)\varphi^{(3)}(r)$$
$$-12\lambda\varphi^{(4)}(r)-4\lambda\varphi^{(2)}(r)-8\lambda\varphi^{(6)}(r)\big]|\partial_x f|^2\mathtt{dxdt}$$
$$+\int_{T_1}^{T_j}\int_{|x|\leq\varepsilon}\big[16\lambda\varphi^{(2)}(r)+20\lambda\varphi^{(4)}(r)\big]|\partial_x^2 f|^2\mathtt{dxdt}.$$

Portanto, como $p\in(1,8/7]$ obtemos

$$\int\left[8\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)+16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)+4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right]|f|^2\,\mathtt{dx}$$
$$+\int\left[24\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)+48\lambda^3(\varphi^{(2)}(r))^3+4\lambda\varphi^{(2)}(r)\right]|\partial_x f|^2\,\mathtt{dx}$$
$$+\int 24\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\left|\partial_x^2 f\right|^2\mathtt{dx}+\int 16\lambda\varphi^{(2)}(r)\left|\partial_x^3 f\right|^2\mathtt{dx}$$
$$\leq\quad C_\lambda+C'\left(\lambda^5\left\|(\varphi')^4\varphi^{(4)}\right\|_\infty+\lambda^5\left\|(\varphi')^3\varphi^{(2)}\varphi^{(3)}\right\|_\infty+\lambda^5\left\|(\varphi')^2(\varphi^{(2)})^3\right\|_\infty\right.$$
$$+\lambda^3\left\|\varphi'\varphi^{(3)}\varphi^{(4)}\right\|_\infty+\lambda^3\left\|(\varphi')^2\varphi^{(4)}\right\|_\infty+\lambda^3\left\|(\varphi^{(2)})^2\varphi^{(4)}\right\|+\lambda^3\left\|\varphi^{(2)}(\varphi^{(3)})^2\right\|_\infty$$
$$+\lambda^3\left\|\varphi'\varphi^{(2)}\varphi^{(3)}\right\|_\infty+\lambda\left\|(\varphi')^2\varphi^{(6)}\right\|_\infty+\lambda^3\left\|\varphi'\varphi^{(2)}\varphi^{(5)}\right\|_\infty+\lambda\left\|\varphi^{(4)}\right\|_\infty+\lambda\left\|\varphi^{(6)}\right\|_\infty$$
$$\left.+\lambda\left\|\varphi^{(8)}\right\|_\infty\right)\int_{|x|\leq\varepsilon}|f|^2\,\mathtt{dx}$$
$$+C''\left(\lambda^3\left\|(\varphi')^2\varphi^{(4)}\right\|_\infty+\lambda^3\left\|\varphi'\varphi^{(2)}\varphi^{(3)}\right\|_\infty+\lambda\left\|\varphi^{(4)}\right\|_\infty+\lambda\left\|\varphi^{(6)}\right\|_\infty\right)\int_{|x|\leq\varepsilon}|\partial_x f|^2\,\mathtt{dx}$$
$$+C'''\left(\lambda\left\|\varphi^{(2)}\right\|_\infty+\lambda\left\|\varphi^{(4)}\right\|_\infty\right)\int_{|x|\leq\varepsilon}\left|\partial_x^2 f\right|^2\mathtt{dx}, \tag{B.16}$$

sendo C' uma constante positiva que depende de $\|V\|_{L^\infty}$, C'' e C''' são constantes positivas e $\lambda_0 = \max\{\lambda_0^1, \ldots, \lambda_0^6, 1\}$.

B.3 Demonstração da desigualdade (4.53).

A desigualdade (4.53) estima uniformemente para cada $j \in \mathbb{Z}^+$ os termos positivos de $\int_{T_1}^{T_j} \langle [S_\lambda, A_\lambda] f, f \rangle \mathtt{dt}$, isto é, existe $\lambda_0 > 0$ tal que para todo $\lambda > \lambda_0$

$$\begin{aligned}
&\int_{T_1}^{T_j}\int_{\mathbb{R}} \left[\, 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2 \,\mathtt{dxdt} \\
&+ \int_{T_1}^{T_j}\int_{\mathbb{R}} \left[\, 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2 \,\mathtt{dxdt} \\
&+ \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \int_{T_1}^{T_j}\int_{\mathbb{R}} 16\lambda\varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j}\int_{|x|\le 1} \lambda^2 |f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{|x|\le 1} \lambda^2 |\partial_x f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{|x|\le 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&\le\; C_\lambda.
\end{aligned} \tag{B.17}$$

Utilizando a desigualdade (4.52), temos que

$$\begin{aligned}
&\int_{T_1}^{T_j}\int_{\mathbb{R}} \left[\, 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2 \,\mathtt{dxdt} \\
&+ \int_{T_1}^{T_j}\int_{\mathbb{R}} \left[\, 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2 \,\mathtt{dxdt} \\
&+ \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \int_{T_1}^{T_j}\int_{\mathbb{R}} 16\lambda\varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j}\int_{|x|\le 1} \lambda^2 |f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{|x|\le 1} \lambda^2 |\partial_x f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{|x|\le 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&\le\; C_\lambda + \left(C_1\lambda^5 + \lambda^2\right)\left(\varepsilon C_\varphi \int_{T_1}^{T_j}\int_{|x|\le 2\varepsilon} |\partial_x f|^2 \,\mathtt{dxdt} + \varepsilon C_\varphi \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 2\varepsilon} |f|^2 \,\mathtt{dxdt}\right) \\
&+ \left(C_2\lambda^3 + \lambda^2\right)\left(\varepsilon C_\varphi \int_{T_1}^{T_j}\int_{|x|\le 2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \varepsilon C_\varphi \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 2\varepsilon} |\partial_x f|^2 \,\mathtt{dxdt}\right) \\
&+ \left(C_3\lambda + \lambda^2\right)\left(\varepsilon C_\varphi \int_{T_1}^{T_j}\int_{|x|\le 2\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dxdt} + \varepsilon C_\varphi \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt}\right) \\
&+ \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1} \lambda^2 |f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1} \lambda^2 |\partial_x f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&- \int_{T_1}^{T_j}\int_{\mathbb{R}} 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\, |f|^2 \,\mathtt{dxdt} - \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\, |\partial_x f|^2 \,\mathtt{dxdt} \\
&- \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2.
\end{aligned} \tag{B.18}$$

Por outro lado, utilizando a seguinte desigualdade

$$\int_{|x|\le 2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dx} \le \left(\varepsilon C_\varphi \int_{|x|\le 4\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dx} + \varepsilon C_\varphi \int_{2\varepsilon\le|x|\le 4\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dx}\right), \tag{B.19}$$

em (B.18), temos que

$$\begin{aligned}
&\int_{T_1}^{T_j}\int_{\mathbb{R}} \left[\, 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r) + 16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right] |f|^2 \,\mathtt{dxdt} \\
&+ \int_{T_1}^{T_j}\int_{\mathbb{R}} \left[\, 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r) + 48\lambda^3(\varphi^{(2)}(r))^3 + 4\lambda\varphi^{(2)}(r)\right] |\partial_x f|^2 \,\mathtt{dxdt} \\
&+ \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \int_{T_1}^{T_j}\int_{\mathbb{R}} 16\lambda\varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j}\int_{|x|\le 1} \lambda^2 |f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{|x|\le 1} \lambda^2 |\partial_x f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{|x|\le 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&\le\; C_\lambda + \left(C_1\lambda^5 + \lambda^2\right)\left(\varepsilon C_\varphi \int_{T_1}^{T_j}\int_{|x|\le 2\varepsilon} |\partial_x f|^2 \,\mathtt{dxdt} + \varepsilon C_\varphi \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 2\varepsilon} |f|^2 \,\mathtt{dxdt}\right) \\
&+ \left(C_2\lambda^3 + \lambda^2\right)\left[\varepsilon C_\varphi \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 2\varepsilon} |\partial_x f|^2 \,\mathtt{dxdt} + \varepsilon C_\varphi \left(\varepsilon C_\varphi \int_{T_1}^{T_j}\int_{|x|\le 4\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dxdt}\right.\right. \\
&\left.\left. + \varepsilon C_\varphi \int_{T_1}^{T_j}\int_{2\varepsilon\le|x|\le 4\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt}\right)\right] \\
&+ \left(C_3\lambda + \lambda^2\right)\left(\varepsilon C_\varphi \int_{T_1}^{T_j}\int_{|x|\le 2\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dxdt} + \varepsilon C_\varphi \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt}\right) \\
&+ \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1} \lambda^2 |f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1} \lambda^2 |\partial_x f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&- \int_{T_1}^{T_j}\int_{\mathbb{R}} 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\, |f|^2 \,\mathtt{dxdt} - \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\, |\partial_x f|^2 \,\mathtt{dxdt} \\
&- \int_{T_1}^{T_j}\int_{\mathbb{R}} 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 .
\end{aligned} \tag{B.20}$$

Observamos que no lado direito de (B.20), para os termos que contém como domínio de integração $\{x \in \mathbb{R} : \varepsilon \le |x| \le 2\varepsilon\}$. Seguirá das propriedades de φ e ε pequeno, que existem as constantes C_ε e c_ε positivas, tais que $c_\varepsilon \le (\varphi'(r))^2\varphi^{(2)}(r) \le C_\varepsilon$ e para λ suficientemente grande temos que:

$$\left[-4c_\varepsilon^3\lambda^7 + \varepsilon C_\varphi(C_1\lambda^5 + \lambda^2)\right] \int_{\varepsilon\le|x|\le 2\varepsilon} |f|^2 \,\mathtt{dx} < 0,$$

$$\left[-12c_\varepsilon^2\lambda^5 + \varepsilon C_\varphi(C_1\lambda^5 + \lambda^2) + \varepsilon C_\varphi(C_3\lambda + \lambda^2)\right] \int_{\varepsilon\le|x|\le 2\varepsilon} |\partial_x f|^2 \,\mathtt{dx} < 0.$$

Analogamente, para $\{x \in \mathbb{R} : 2\varepsilon \le |x| \le 4\varepsilon\}$ temos que

$$\left[-12c_\varepsilon\lambda^3 + \varepsilon C_\varphi\right] \int_{2\varepsilon\le|x|\le 4\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dx} < 0.$$

Agora, para os termos que contém o domínio de integração $\{x \in \mathbb{R} : |x| \leq 2\varepsilon\}$ e $\{x \in \mathbb{R} : |x| \leq 4\varepsilon\}$. Utilizando (4.42), temos que $\forall\ \lambda > \lambda_0 = \max\{\lambda_0^1, \lambda_0^2, \dots, \lambda_0^6, 1\}$, segue que

$$\begin{aligned}
&C_\varphi \int_{|x|\leq 2\varepsilon} |\partial_x f|^2 \,\mathtt{dx} \\
&\leq -\frac{(8\lambda - C_2\lambda^3\varepsilon^2 C_\varphi)}{48\lambda^3 C_\varphi^2 - C_1\lambda^5\varepsilon} C_\varphi \int_{|x|\leq 4\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dx} - \frac{8\lambda - C_3\lambda\varepsilon}{48\lambda^3 C_\varphi^2 - C_1\lambda^5\varepsilon} C_\varphi \int_{|x|\leq 2\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dx} \\
&\quad + \frac{1}{48\lambda^3 C_\varphi^2 - C_1\lambda^5\varepsilon} \left[C_\lambda + C_1\lambda^5\varepsilon C_\varphi \int_{\varepsilon\leq|x|\leq 2\varepsilon} |f|^2 \,\mathtt{dx} + C_2\lambda^3\varepsilon^2 C_\varphi^2 \int_{2\varepsilon\leq|x|\leq 4\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dx} \right. \\
&\quad \left. + C_2\lambda^3\varepsilon^2 C_\varphi^2 \int_{\varepsilon\leq|x|\leq 2\varepsilon} |\partial_x f|^2 \,\mathtt{dx} + C_3\lambda\varepsilon C_\varphi \int_{\varepsilon\leq|x|\leq 2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dx} \right]. \qquad \text{(B.21)}
\end{aligned}$$

Então, usando (B.21) em (B.20), temos que

$$\begin{aligned}
&\int_{T_1}^{T_j} \int_{\mathbb{R}} \left[4\lambda^7 (\varphi'(r))^6 \varphi^{(2)}(r) + 16\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) + 4\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r) \right] |f|^2 \,\mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} \left[12\lambda^5 (\varphi'(r))^4 \varphi^{(2)}(r) + 48\lambda^3 (\varphi^{(2)}(r))^3 + 4\lambda \varphi^{(2)}(r) \right] |\partial_x f|^2 \,\mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{\mathbb{R}} 12\lambda^3 (\varphi'(r))^2 \varphi^{(2)}(r) \left|\partial_x^2 f\right|^2 \mathtt{dxdt} + \int_{T_1}^{T_j} \int_{\mathbb{R}} 16\lambda \varphi^{(2)}(r) \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 |\partial_x f|^2 \,\mathtt{dxdt} + \int_{T_1}^{T_j} \int_{|x|\leq 1} \lambda^2 \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&\leq\ C_\lambda + \left(C_1\lambda^5 + \lambda^2\right) \left\{ \varepsilon \left[-\frac{(8\lambda - C_2\lambda^3\varepsilon^2 C_\varphi)}{48\lambda^3 C_\varphi^2 - C_1\lambda^5\varepsilon} C_\varphi \int_{T_1}^{T_j} \int_{|x|\leq 4\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \right.\right. \\
&- \frac{8\lambda - C_3\lambda\varepsilon}{48\lambda^3 C_\varphi^2 - C_1\lambda^5\varepsilon} C_\varphi \int_{T_1}^{T_j} \int_{|x|\leq 2\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \\
&+ \frac{1}{48\lambda^3 C_\varphi^2 - C_1\lambda^5\varepsilon} \left(C_\lambda + C_1\lambda^5\varepsilon C_\varphi \int_{T_1}^{T_j} \int_{\varepsilon\leq|x|\leq 2\varepsilon} |f|^2 \,\mathtt{dxdt} \right. \\
&+ C_2\lambda^3\varepsilon^2 C_\varphi^2 \int_{T_1}^{T_j} \int_{2\varepsilon\leq|x|\leq 4\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \\
&\left.\left. + C_2\lambda^3\varepsilon^2 C_\varphi^2 \int_{T_1}^{T_j} \int_{\varepsilon\leq|x|\leq 2\varepsilon} |\partial_x f|^2 \,\mathtt{dxdt} + C_3\lambda\varepsilon C_\varphi \int_{\varepsilon\leq|x|\leq 2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \right) \right] \\
&\left. + \varepsilon C_\varphi \int_{T_1}^{T_j} \int_{\varepsilon\leq|x|\leq 2\varepsilon} |f|^2 \,\mathtt{dxdt} \right\} \\
&+ \left(C_2\lambda^3 + \lambda^2\right) \left[\varepsilon C_\varphi \int_{T_1}^{T_j} \int_{\varepsilon\leq|x|\leq 2\varepsilon} |\partial_x f|^2 \,\mathtt{dxdt} + \varepsilon C_\varphi \left(\varepsilon C_\varphi \int_{T_1}^{T_j} \int_{|x|\leq 4\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dxdt} \right.\right. \\
&\left.\left. + \varepsilon C_\varphi \int_{T_1}^{T_j} \int_{2\varepsilon\leq|x|\leq 4\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \right) \right] \\
&+ \left(C_3\lambda + \lambda^2\right) \left(\varepsilon C_\varphi \int_{T_1}^{T_j} \int_{|x|\leq 2\varepsilon} \left|\partial_x^3 f\right|^2 \mathtt{dxdt} + \varepsilon C_\varphi \int_{T_1}^{T_j} \int_{\varepsilon\leq|x|\leq 2\varepsilon} \left|\partial_x^2 f\right|^2 \mathtt{dxdt} \right)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1}\lambda^2|f|^2\,\mathtt{dxdt}+\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1}\lambda^2|\partial_x f|^2\,\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1}\lambda^2\left|\partial_x^2 f\right|^2\mathtt{dxdt}\\
&-\int_{T_1}^{T_j}\int_{\mathbb{R}}4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\,|f|^2\,\mathtt{dxdt}-\int_{T_1}^{T_j}\int_{\mathbb{R}}12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\,|\partial_x f|^2\,\mathtt{dxdt}\\
&-\int_{T_1}^{T_j}\int_{\mathbb{R}}12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\left|\partial_x^2 f\right|^2\mathtt{dxdt}\\
&\le C_\lambda\left[1+\frac{1}{48\lambda^3C_\varphi^2-C_1\lambda^5\varepsilon}\right]+\frac{(C_1\lambda^5+\lambda^2)}{\left(48\lambda^3C_\varphi^2-C_1\lambda^5\varepsilon\right)}\varepsilon^2C_1\lambda^5C_\varphi\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 2\varepsilon}|f|^2\,\mathtt{dxdt}\\
&+\left(C_1\lambda^5+\lambda^2\right)\varepsilon C_\varphi\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 2\varepsilon}|f|^2\,\mathtt{dxdt}+\left[\frac{(C_1\lambda^5+\lambda^2)}{(48\lambda^3C_\varphi^2-C_1\lambda^5\varepsilon)}\left(C_2\lambda^3\varepsilon^3C_\varphi^2\right)\right.\\
&\left.+(C_2\lambda^3+\lambda^2)\varepsilon C_\varphi\right]\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 2\varepsilon}|\partial_x f|^2\,\mathtt{dxdt}\\
&+\left[\frac{(C_1\lambda^5+\lambda^2)}{\left(48\lambda^3C_\varphi^2-C_1\lambda^5\varepsilon\right)}\left(C_2\lambda^3\varepsilon^3C_\varphi^2\right)+\left(C_2\lambda^3+\lambda^2\right)\varepsilon^2C_\varphi^2\right]\int_{T_1}^{T_j}\int_{2\varepsilon\le|x|\le 4\varepsilon}\left|\partial_x^2 f\right|^2\mathtt{dxdt}\\
&+\left[\frac{(C_1\lambda^5+\lambda^2)}{\left(48\lambda^3C_\varphi^2-C_1\lambda^5\varepsilon\right)}\left(C_3\lambda\varepsilon^2C_\varphi\right)+\left(C_3\lambda+\lambda^2\right)\varepsilon C_\varphi\right]\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 2\varepsilon}\left|\partial_x^2 f\right|^2\mathtt{dxdt}\\
&+\left[\frac{-\left(C_1\lambda^5+\lambda^2\right)}{\left(48\lambda^3C_\varphi^2-C_1\lambda^5\varepsilon\right)}\varepsilon\left(8\lambda-C_3\lambda\varepsilon\right)C_\varphi+\left(C_3\lambda+\lambda^2\right)\varepsilon C_\varphi\right]\int_{T_1}^{T_j}\int_{|x|\le 2\varepsilon}\left|\partial_x^3 f\right|^2\mathtt{dxdt}\\
&+\left[\frac{-\left(C_1\lambda^5+\lambda^2\right)}{\left(48\lambda^3C_\varphi^2-C_1\lambda^5\varepsilon\right)}\varepsilon\left(8\lambda-C_3\lambda\varepsilon\right)C_\varphi+\left(C_2\lambda^3+\lambda^2\right)\varepsilon^2C_\varphi^2\right]\int_{T_1}^{T_j}\int_{|x|\le 4\varepsilon}\left|\partial_x^3 f\right|^2\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{\varepsilon\le|x|\le 1}\left(\lambda^2|f|^2+\lambda^2|\partial_x f|^2+\lambda^2\left|\partial_x^2 f\right|^2\right)\mathtt{dxdt}\\
&-\int_{T_1}^{T_j}\int 4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)\,|f|^2\,\mathtt{dxdt}-\int_{T_1}^{T_j}\int 12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)\,|\partial_x f|^2\,\mathtt{dxdt}\\
&-\int_{T_1}^{T_j}\int 12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\left|\partial_x^2 f\right|^2\mathtt{dxdt}
\end{aligned}
\tag{B.22}
$$

Assim, para $\varepsilon = 1/\lambda^5$ em (B.22) conseguimos que

$$
\begin{aligned}
&\int_{T_1}^{T_j}\int_{\mathbb{R}}\left[\,4\lambda^7(\varphi'(r))^6\varphi^{(2)}(r)+16\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)+4\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\right]|f|^2\,\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{\mathbb{R}}\left[\,12\lambda^5(\varphi'(r))^4\varphi^{(2)}(r)+48\lambda^3(\varphi^{(2)}(r))^3+4\lambda\varphi^{(2)}(r)\right]|\partial_x f|^2\,\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{\mathbb{R}}12\lambda^3(\varphi'(r))^2\varphi^{(2)}(r)\left|\partial_x^2 f\right|^2\mathtt{dxdt}+\int_{T_1}^{T_j}\int_{\mathbb{R}}16\lambda\varphi^{(2)}(r)\left|\partial_x^3 f\right|^2\mathtt{dxdt}\\
&+\int_{T_1}^{T_j}\int_{|x|\le 1}\lambda^2|f|^2\,\mathtt{dxdt}+\int_{T_1}^{T_j}\int_{|x|\le 1}\lambda^2|\partial_x f|^2\,\mathtt{dxdt}+\int_{T_1}^{T_j}\int_{|x|\le 1}\lambda^2\left|\partial_x^2 f\right|^2\mathtt{dxdt}\\
&\le C_\lambda.
\end{aligned}
$$

B.4 Demonstração do Lema 4.4

Nesta seção, construímos uma função $\varphi \in C^8([0,\infty))$ estritamente convexa em $[0,1]$ tal que $\varphi(0) = 0$. Para isto, definimos o seguinte polinômio

$$\varphi(r) = a_0 + a_1 r^2 + a_2 r^4 + \cdots + a_8 r^{16}, \quad r \in [0,1], \tag{B.23}$$

com as seguintes restrições em $r = 1$

$$b = \begin{bmatrix} \varphi(1) \\ \varphi'(1) \\ \varphi^{(2)}(1) \\ \varphi^{(3)}(1) \\ \varphi^{(4)}(1) \\ \varphi^{(5)}(1) \\ \varphi^{(6)}(1) \\ \varphi^{(7)}(1) \\ \varphi^{(8)}(1) \end{bmatrix}_{9\times 1} = \begin{bmatrix} 1+\beta \\ p \\ p(p-1) \\ p(p-1)(p-2) \\ p(p-1)(p-2)(p-3) \\ p(p-1)(p-2)(p-3)(p-4) \\ p(p-1)(p-2)(p-3)(p-4)(p-5) \\ p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6) \\ p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7) \end{bmatrix}_{9\times 1}$$

Daí, temos o seguinte sistema de equação $A(1)\cdot x = b$, com:

$$A(r) = \begin{bmatrix} 1 & r^2 & r^4 & r^6 & r^8 & r^{10} & r^{12} & r^{14} & r^{16} \\ 0 & 2\,r & 4\,r^3 & 6\,r^5 & 8\,r^7 & 10\,r^9 & 12\,r^{11} & 14\,r^{13} & 16\,r^{15} \\ 0 & 2 & 12\,r^2 & 30\,r^4 & 56\,r^6 & 90\,r^8 & 132\,r^{10} & 182\,r^{12} & 240\,r^{14} \\ 0 & 0 & 24\,r & 120\,r^3 & 336\,r^5 & 720\,r^7 & 1320\,r^9 & 2184\,r^{11} & 3360\,r^{13} \\ 0 & 0 & 24 & 360\,r^2 & 1680\,r^4 & 5040\,r^6 & 11880\,r^8 & 24024\,r^{10} & 43680\,r^{12} \\ 0 & 0 & 0 & 720\,r & 6720\,r^3 & 30240\,r^5 & 95040\,r^7 & 240240\,r^9 & 524160\,r^{11} \\ 0 & 0 & 0 & 720 & 20160\,r^2 & 151200\,r^4 & 665280\,r^6 & 2162160\,r^8 & 5765760\,r^{10} \\ 0 & 0 & 0 & 0 & 40320\,r & 604800\,r^3 & 3991680\,r^5 & 17297280\,r^7 & 57657600\,r^9 \\ 0 & 0 & 0 & 0 & 40320 & 1814400\,r^2 & 19958400\,r^4 & 121080960\,r^6 & 518918400\,r^8 \end{bmatrix} \tag{B.24}$$

e

$$x = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_8 \end{bmatrix}. \tag{B.25}$$

Assim, a solução é

$$x = \begin{bmatrix} a_0 \\ \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} \frac{p^8}{10321920} - \frac{p^7}{143360} + \frac{13\,p^6}{61440} - \frac{9\,p^5}{2560} + \frac{1069\,p^4}{30720} - \frac{267\,p^3}{1280} + \frac{29531\,p^2}{40320} \\ -\frac{761\,p}{560} + 1 + \beta \\ -\frac{p(p-12)(p-14)(p-4)(p-16)(p-6)(p-8)(p-10)}{1290240} \\ \frac{p(p-12)(p-2)(p-14)(p-16)(p-6)(p-8)(p-10)}{368640} \\ -\frac{p(p-12)(p-2)(p-14)(p-4)(p-16)(p-8)(p-10)}{184320} \\ \frac{p(p-12)(p-2)(p-14)(p-4)(p-16)(p-6)(p-10)}{147456} \\ -\frac{p(p-12)(p-2)(p-14)(p-4)(p-16)(p-6)(p-8)}{184320} \\ \frac{p(p-2)(p-14)(p-4)(p-16)(p-6)(p-8)(p-10)}{368640} \\ -\frac{p(p-12)(p-2)(p-4)(p-16)(p-6)(p-8)(p-10)}{1290240} \\ \frac{p(p-12)(p-2)(p-14)(p-4)(p-6)(p-8)(p-10)}{10321920} \end{bmatrix} \tag{B.26}$$

A seguir demonstramos que $\varphi^{(2)}(r)$ é decrescente em $(0,1)$.

Afirmação:

$$\varphi^{(3)}(r) < 0, \quad \forall\, r \in (0,1). \tag{B.27}$$

De fato, segue da solução em (B.26)

$$\varphi^{(3)}(r) = \frac{1}{15360}\, p(p-2)\, r \left(\widetilde{a}_2 + \widetilde{a}_3 r^2 + \widetilde{a}_4 r^4 + \widetilde{a}_5 r^6 + \widetilde{a}_6 r^8 + \widetilde{a}_7 r^{10} + \widetilde{a}_8 r^{12}\right), \tag{B.28}$$

em que

$$
\begin{aligned}
\widetilde{a}_2 &= (p-6)(p-8)(p-10)(p-12)(p-14)(p-16)\\
\widetilde{a}_3 &= -10(p-4)(p-8)(p-10)(p-12)(p-14)(p-16)\\
\widetilde{a}_4 &= 35(p-4)(p-6)(p-10)(p-12)(p-14)(p-16)\\
\widetilde{a}_5 &= -60(p-4)(p-6)(p-8)(p-12)(p-14)(p-16)\\
\widetilde{a}_6 &= 55(p-4)(p-6)(p-8)(p-10)(p-14)(p-16)\\
\widetilde{a}_7 &= -26(p-4)(p-6)(p-8)(p-10)(p-12)(p-16)\\
\widetilde{a}_7 &= 5(p-4)(p-6)(p-8)(p-10)(p-12)(p-14)
\end{aligned}
$$

como $p \in (1, 8/7]$.

Para facilitar a demonstração de $\varphi^{(3)}(r) < 0$ em $r \in (0,1)$, dividimos a demonstração por etapas:

Na primeira etapa, demonstramos que

$$\widetilde{a}_2 + \widetilde{a}_3 r^2 + \widetilde{a}_4 r^4 > 0, \quad \forall\, r \in (0,1). \tag{B.29}$$

De fato, realizando manipulações algébricas do discriminante do polinômio quadrático em r^2, segue que

$$D = -40(p-4)(p-8)(p^2 - 12p + 46) < 0, \tag{B.30}$$

pois $p \in (1, 8/7]$.
O que implica que o polinômio (B.29) não tem pontos críticos no intervalo $(0,1)$ e como $\widetilde{a}_2 > 0$, segue o resultado.

Na segunda etapa, demonstraremos que

$$\widetilde{a}_5 r^6 + \widetilde{a}_6 r^8 > 0. \tag{B.31}$$

De fato,

$$\widetilde{a}_5 r^6 + \widetilde{a}_6 r^8 = (p-4)(p-6)(p-8)(p-14)(p-16) r^6 \left[-60(p-12) + 55(p-10) r^2\right]$$

e como $-60(p-12)+55(p-10)r^2$ não tem raízes reais no intervalo $r \in (0,1)$ e é estritamente negativo, segue então o resultado.

Finalmente, na terceira etapa demonstramos que

$$\widetilde{a}_7 r^{10} + \widetilde{a}_8 r^{12} > 0. \tag{B.32}$$

De fato, como

$$\widetilde{a}_7 r^{10} + \widetilde{a}_8 r^{12} = (p-4)(p-6)(p-8)(p-10)(p-12)r^{10}\left[-26(p-16)+5(p-14)r^2\right]$$

e como $-26(p-16)+5(p-14)r^2$ não tem raiz real no intervalo $(0,1)$ e é estritamente negativo, segue então o resultado.

Utilizando os resultados de (B.29),(B.31) e (B.32), resulta que $\forall\, r \in (0,1)$

$$\varphi^{(3)}(r) = \frac{1}{15360}\, p(p-2)\, r\left(\widetilde{a}_2 + \widetilde{a}_3 r^2 + \widetilde{a}_4 r^4 + \widetilde{a}_5 r^6 + \widetilde{a}_6 r^8 + \widetilde{a}_7 r^{10} + \widetilde{a}_8 r^{12}\right) < 0$$

o que implica que $\varphi^{(2)}(r)$ é decrescente em $(0,1)$.

Logo, verificamos

$$\varphi^{(2)}(0) = 2a_1 > p, \quad p > \varphi^{(2)}(1) = p(p-1), \quad \varphi^{(2)}(r) > 0 \quad \forall\, r \in (0,1)$$

$\forall\, p \in (1, 8/7]$, e como $\varphi^{(3)}(r) < 0$ para todo $r \in (0,1)$, o que implica que $\varphi^{(2)}(r)$ é decrescente e não nula.

Assim, verifica-se que φ é estritamente convexa em $r \in [0,1]$ com $p \in (1, 8/7]$.

Afirmação 4.

$$\varphi'(r) > 0, \quad \forall\, r \in (0,1). \tag{B.33}$$

De fato, como $\varphi^{(2)}(r) > 0$, $\forall\, r \in (0,1)$ e como $\varphi'(0) = 0$, segue a afirmação.

Afirmação 5.

$$\varphi(0) = 0$$

De fato, como $\varphi(1) = 1+\beta$ e escolhendo $\beta = a_1+\cdots+a_8-1$, segue que $a_0 = 0$ e $\varphi(0) = 0$, disto segue a afirmação.

Afirmação 6.

$$\varphi(r) \leq Mr^p, \quad \forall r \in [0,\infty). \tag{B.34}$$

De fato, para $r \in [1,\infty)$. Existe uma constante positiva C, tal que $\beta \leq C$, daí segue que

$$\begin{aligned} \varphi(r) &= r^p + \beta \\ &\leq (1+C)r^p. \end{aligned}$$

Logo, para $0 < r < 1$, seja $M = \max\{M_0, \beta, 3/2\}$ com $M_0 = \sup_{r\in[0,1]} \left|\varphi^{(2)}(r)\right|$ e $\beta = a_1 + \cdots + a_8 - 1$. Disto, definimos a seguinte função

$$\begin{array}{rccl} F: & [0,1] & \longrightarrow & \mathbb{R} \\ & r & \longmapsto & Mr^p - \varphi(r), \end{array} \tag{B.35}$$

logo, segue que $\forall\, r \in (0,1)$

$$\begin{aligned} F^{(2)}(r) &= Mp(p-1)r^{p-2} - \varphi^{(2)}(r) \\ &= Mp(p-1)r^{p-2} - \left|\varphi^{(2)}(r)\right| \\ &\geq \left(Mr^{p-2} - M_0\right) p(p-1) \\ &\geq \left(M - M_0\right) p(p-1) \\ &> 0, \end{aligned}$$

e $F'(0) = F(0) = 0$, o que implica que $F'(r)$ é crescente em $(0,1)$ e $F(r) > 0$, para todo $r \in (0,1)$.

Em seguida, pela definição em (B.35), resulta que

$$\varphi(r) \leq Mr^p, \quad \forall\, r \in [0,1).$$

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