UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

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Euclidean Hypersurfaces with Genuine Conformal Deformations in Codimension Two.

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Tese apresentada ao Programa de Pós-Graduação em Matemática da Universidade Federal de São Carlos como parte dos requisitos para a obtenção do título de Doutor em Matemática.

Advisor: Prof. Dr. Ruy Tojeiro de Figueiredo Júnior

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Folha de Aprovação

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Resumo

Classificamos as hipersuperfícies $f : M^n \to \mathbb{R}^{n+1}$ que possuem uma curvatura principal de multiplicidade $n-2$ que admitem uma deformação conforme genuína $\tilde{f} : M^n \to \mathbb{R}^{n+2}$. Uma deformação conforme $\tilde{f} : M^n \to \mathbb{R}^{n+2}$ de f é genuína se em nenhum aberto $U \subset M^n$ a restrição $\tilde{f}|_U$ é uma composição $\tilde{f}|_U = h \circ f|_U$ de $f|_U$ com uma immersão conforme $h\colon V \to \mathbb{R}^{n+2}$ de um aberto $V \subset \mathbb{R}^{n+1}$ que contém $f(U)$.

Abstract

We classify hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ with a principal curvature of multiplicity $n-2$ that admit a genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$. That a conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$ of f is genuine means that there does not exist any open subset $U \subset M^n$ such that $\tilde{f}|_U$ is a composition $\tilde{f}|_U = h \circ f|_U$ of $f|_U$ with a conformal immersion $h\colon V \to \mathbb{R}^{n+2}$ of an open subset $V \subset \mathbb{R}^{n+1}$ containing $f(U)$.

Contents

Introduction

Hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ that admit an isometric (respectively, conformal) deformation $g : M^n \to \mathbb{R}^{n+1}$ that is not isometrically congruent (respectively, conformally congruent) to f on any open subset of M^n are called *Sbrana-Cartan Hypersurfaces* (respectively, *Cartan Hypersurfaces*). These two types of hypersurfaces have been classified in the beginning of the twentieth century: in the isometric case by Sbrana in [1] and Cartan [2] for $n \geq 3$, and in the conformal one by Cartan in [3] for $n \geq 5$. The most interesting classes of Sbrana-Cartan (respectively, Cartan) hypersurfaces are envelopes of certain two-parameter congruences of affine hyperplanes (respectively, hyperspheres), which may admit either a one-parameter family of isometric (respectively, conformal) deformations, or a single one. Partial results on Cartan hypersurfaces of dimensions four and three were also obtained by Cartan in [4] and [5], respectively.

The classification of Sbrana-Cartan hypersurfaces was extended to the case of nonflat ambient space forms by Dajczer-Florit-Tojeiro in [6]. Moreover, among other things, in that paper the problem of determining whether Sbrana-Cartan hypersurfaces that allow a single deformation do exist, which was not addressed by Sbrana or Cartan, was given an affirmative answer.

A nonparametric description of Cartan hypersurfaces of dimension $n \geq 5$ of $\mathbb{R}^{n+1}$ was given in [7], where it was shown that any such hypersurface arises by intersecting the light-cone $\mathbb{V}^{n+2}$ in Lorentzian space $\mathbb{L}^{n+3}$ with a flat space-like submanifold of codimension two of $\mathbb{L}^{n+3}$.

We also refer to [6] and [7], as well as to [8], for modern accounts of the classifications of Sbrana-Cartan and Cartan hypersurfaces. Our presentation in this thesis is close in spirit to that in [8].

When studying isometric or conformal deformations of a Euclidean submanifold with codimension greater than one, one has to take into account that any submanifold of a deformable submanifold already possesses the isometric deformations induced by the latter. Therefore, it is necessary to restrict the study to those deformations that are "genuine",

$$
\begin{tikzcd}
 & & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "f"] \arrow[r, hook, "j"] \arrow[drr, "\tilde{f}"'] & N^{n+r} \arrow[ur, "F"'] \arrow[dr, "\tilde{F}"] & \\
 & & \mathbb{R}^{n+q}.
\end{tikzcd}
$$

Figure 1: f and $\tilde{f}$ not genuinely conformally congruent

that is, those which are not induced by deformations of an "extended" submanifold. It is also of interest to consider deformations of a submanifold that take place in a possibly different codimension.

These ideas have been made precise in [9] in the isometric case, and extended to the conformal realm in [10] as follows. Let $f : M^n \to \mathbb{R}^{n+p}$ be a conformal immersion of an n-dimensional Riemannian manifold M^n into Euclidean space. A conformal immersion $\tilde{f} : M^n \to \mathbb{R}^{n+q}$ is said to be a *genuine conformal deformation* of f if f and $\tilde{f}$ are nowhere (i.e., on no open subset of M^n) compositions, $f = F \circ j$ and $\tilde{f} = \tilde{F} \circ j$, of a conformal embedding $j : M^n \to N^{n+r}$ into a Riemannian manifold N^{n+r} with $r > 0$ and conformal immersions $F : N^{n+r} \to \mathbb{R}^{n+p}$ and $\tilde{F} : N^{n+r} \to \mathbb{R}^{n+q}$ (see figure 1).

In this work we are interested in the case where $p = 1$ and $q = 2$. In the isometric realm, from the assumption that $f : M^n \to \mathbb{R}^{n+1}$ admits a genuine isometric deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$, it follows from Theorem 1 in [11] that $\operatorname{rank} f \leq 3$. The situation in which $\operatorname{rank} f = 2$ was solved some years ago in [12].

In the conformal instance, from Theorem 1 of [13] it follows that $f : M^n \to \mathbb{R}^{n+1}$ must have a principal curvature λ with multiplicity greater than or equal to $n-3$ if it admits a genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$. We will study the particular case in which the multiplicity is $n-2$. For the case $n-3$, it seems better to start by attempting to solve the analogous problem in the isometric realm, which is also still open.

Hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ having a principal curvature λ of multiplicity $n-2$ are envelopes of two-parameter congruences of hyperspheres (see Chapter 1). These are given by a focal function $h : L^2 \to \mathbb{R}^{n+1}$ and a radius function $r \in C^\infty(L)$, where $L^2 = M^n/\Delta^{n-2}$ is the quotient space of leaves of the umbilical eigendistribution distribution Δ associated to λ. In terms of the model of Euclidean space $\mathbb{R}^{n+1}$ as a hypersurface of the light-cone $\mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$, the congruence of hyperspheres (h, r) can be represented by a surface

$s : L^2 \to \mathbb{Q}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ in the de Sitter space. With the aid of the conformal Gauss parametrization, the hypersurface f can be recovered back from the surface s. Our approach is to determine which surfaces $s : L^2 \to \mathbb{Q}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ give rise to hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ that admit genuine conformal deformations $\tilde{f} : M^n \to \mathbb{R}^{n+2}$.

In the proof, we follow similar steps to those of the isometric case. We show in Chapter 3 that the fact that a hypersurface $f : M^n \to \mathbb{R}^{n+1}$ having a principal curvature λ of multiplicity $n-2$ admits a genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$ can be encoded by a triple (D_1, D_2, ψ) satisfying several conditions, where $D_i \in \Gamma(\mathrm{End}(\Delta^\perp))$, $1 \leq i \leq 2$, and ψ is a one-form on M^n. This requires the preliminary algebraic step of determining the structure of the second fundamental form of the isometric light-cone representative of a genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$ of f, which is carried out in Chapter 2.

The next step is to prove that the triple (D_1, D_2, ψ) can be projected down to a triple $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ on the quotient space L^2, and to express the conditions on (D_1, D_2, ψ) in terms of simpler ones on $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ (see Chapter 4). The last step is then to characterize the surfaces $s : L^2 \to \mathbb{Q}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ that carry a triple $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ satisfying those conditions. This is done in Chapter 5. For the statement and proof of our classification of Euclidean hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ that admit genuine conformal deformations $\tilde{f} : M^n \to \mathbb{R}^{n+1}$ in Chapter 6, all that was needed was to put together the steps accomplished in the previous chapters.

The main theorem of this thesis is, as far as we know, the first classification result for a class of submanifolds admitting genuine conformal deformations, apart from the classical one by Cartan of the Euclidean hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ that admit genuine conformal deformations $\tilde{f} : M^n \to \mathbb{R}^{n+1}$. In the isometric realm, besides the isometric version of our result in [12], isometric immersions $f : M^n \to \mathbb{R}^{n+2}$ of rank two that admit genuine isometric deformations $\tilde{f} : M^n \to \mathbb{R}^{n+2}$ have been classified in [14], [15] and [16].

We hope that, after reading this thesis, the reader will have learned about some of the tools used in the conformal theory of submanifolds. For that reason we have included most of the necessary background material in Chapter 1, trying to make the presentation as self-contained as possible. Without more chit chat, lets start!

Chapter 1

Prerequisites

Studying the topic at hand requires a lot of background knowledge. Including in this chapter all the material needed to understand the present work would be an impossible endeavour, due to the space and time that it would require to accomplish that enterprise. Therefore, we had to draw a starting line about what we assume the reader knows. In choosing that starting point we took into consideration our wish that the present work would be understandable to any mathematician in the area of differential geometry. Hence, we will require the reader to know about smooth manifolds and Riemmanian geometry. With that setting in mind, we are confident enough that we will be able to supplement other knowledge needs in order for the reader to understand this thesis.

As discussed in the introduction, we will work with hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ carrying a principal curvature of multiplicity $n-2$. Some properties of their shape operators and of the associated eigendistributions will be needed in later chapters. In this chapter, we will start by deriving those properties for the more general setting of isometric immersions $f : M^n \to \mathbb{R}^{n+p}$ that carry a Dupin principal normal vector field.

As the title suggests, we will work with conformal immersions $\tilde{f} : M^n \to \mathbb{R}^{n+2}$. However, in practice, we will replace those conformal maps by their isometric light-cone representatives $\tilde{F} : M^n \to \mathbb{V}^{n+3} \subset \mathbb{R}^{n+4}$. The reason is simple enough, in this way we can use all the isometric theory behind them. Of course, the price paid in exchange is to work with Lorentzian ambient spaces. For that motive, we will introduce to the reader the model of Euclidean space $\mathbb{R}^m$ as a hypersurface of the Lorentz light-cone $\mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$, and then use this model to define the isometric light-cone representative of a conformal immersion $f : M^n \to \mathbb{R}^{n+1}$.

As the reader will see from a proposition in the section on Principal normals, a hypersurface $f : M^n \to \mathbb{R}^{n+1}$ carrying a principal curvature of multiplicity $n-2$ is the envelope of a two-parameter congruence of hyperspheres. Another reason to introduce the Euclidean Model $\Psi : \mathbb{R}^m \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ is because hyperspheres have a neat description in terms of vectors in the Sitter space $\mathbb{Q}^{n+2}_{1,1} \subset \mathbb{L}^{n+3}$, and so a two-parameter congruence of hyperspheres will have a simple representation as a surface $s : L^2 \to \mathbb{Q}^{n+2}_{1,1}$.

Finally, we will discuss how to recover the hypersurface $f : M^n \to \mathbb{R}^{n+1}$ from the congruence of hyperspheres with the help of the Conformal Gauss Parametrization. This is analogous to the Gauss Parametrization of hypersurfaces having constant index of relative nullity, which can be parametrized by their Gauss map and support function. Hopefully, this will close all the gaps of knowledge the reader will need to understand this work.

One last word: all the material that was mentioned before can be found in [8], which was a keystone in my study. Without that book, it would have been impossible for me to accomplish the present work. Of course, in that book the prerequisites listed before are explained in more detail and treated with greater generality, but since the work at hand will be available before the book is published, we decided to include this prerequisite chapter.

1.1 Principal Normals

Given an isometric immersion $f : M^n \to N^m$, a vector η in the normal space $N_f M(x)$ of f at x is called a *principal normal* if the subspace

$$E_\eta(x) = \{X \in T_x M : \alpha(X,Y) = \langle X, Y\rangle \eta \quad \text{for all } Y \in T_x M\} \tag{1.1}$$

is non-trivial. A section η of $N_f M$ is called a principal normal vector field of f with multiplicity $q > 0$ if the subspace $E_\eta(x)$ is q-dimensional at each $x \in M^n$.

In terms of the shape operators of f, the subspace $E_\eta(x)$ can be expressed as

$$E_\eta(x) = \bigcap_{\gamma \in N_f M(x)} \ker\left(A_\gamma - \langle \gamma, \eta\rangle I\right), \tag{1.2}$$

as one can easily deduce: If $X \in \cap_{\gamma \in N_f M(x)} \ker\left(A_\gamma - \langle \gamma, \eta\rangle I\right)$, then for all $\gamma \in N_f M(x)$ and $Y \in T_x M$ we have

$$\langle \alpha(X,Y), \gamma\rangle = \langle A_\gamma X, Y\rangle = \langle X, Y\rangle \langle \eta, \gamma\rangle .$$

This leads us to conclude that $\alpha(X,Y) = \langle X, Y\rangle \eta$ for any $Y \in T_x M$, or, in another words, that $X \in E_\eta(x)$. For the other inclusion, let $X \in E_\eta(x)$, then from the definition of the subspace $E_\eta(x)$, we have

$$\alpha(X,Y) = \langle X,Y \rangle \eta,$$

for any $Y \in T_xM$. Taking inner product with an arbitrary $\gamma \in N_fM(x)$ we conclude

$$\langle A_\gamma X,Y \rangle = \langle X,Y \rangle \langle \eta,\gamma \rangle$$

and the other inclusion follows.

In the case where f is an oriented hypersurface with a unit normal vector field N, then a principal normal $\eta \in \Gamma(N_fM)$ can be expressed as $\eta = \lambda N$. From equation (1.2), we have that $X \in E_\eta(x)$ if and only if $AX = \lambda X$, that is, $X \in E_\lambda(x)$, where

$$E_\lambda(x) = \{X \in T_xM : AX = \lambda X\}.$$

Therefore, $\eta = \lambda N$ is a principal normal of f at $x \in M^n$ if and only if λ is a principal curvature of f at $x \in M^n$. We can think of principal normals as the generalization to higher codimensions of principal curvatures.

A principal normal vector field $\eta \in \Gamma(N_fM)$ is said to be *Dupin* if η is parallel along E_η in the normal connection. In the specific case where f is a hypersurface, if $\eta = \lambda N$ is a Dupin principal normal vector field, then λ is a principal curvature with constant multiplicity and for any $T \in E_\eta$, we have

$$0 = \nabla^\perp_T \eta = T(\lambda)N.$$

Therefore, λ is constant along E_η. The other way around is also valid, if λ is a principal curvature with constant multiplicity and constant along E_λ, then $\eta = \lambda N$ is a Dupin principal normal vector field.

A smooth distribution E on a Riemannian manifold M^n is called *umbilical* if there exists a section δ of $E^\perp$, named the *mean curvature vector field* of E, such that

$$\langle \nabla_S T, X \rangle = \langle S,T \rangle \langle X,\delta \rangle$$

for all S, $T \in \Gamma(E)$ and $X \in \Gamma(E^\perp)$. The distribution E is integrable, since

$$\langle [S,T], X \rangle = \langle \nabla_S T, X \rangle - \langle \nabla_T S, X \rangle = \langle S,T \rangle \langle X,\delta \rangle - \langle T,S \rangle \langle X,\delta \rangle = 0,$$

for all S, $T \in \Gamma(E)$, and so $[S,T] \in \Gamma(E)$. Moreover, if σ^k is a leaf of E and $j : \sigma^k \to M^n$ is the inclusion map, then for all S, $T \in \Gamma(E)$ we have

$$j^*\nabla_S j_* T = j_* \nabla_S T + \alpha^j(S,T).$$

Taking inner product with $X \in \Gamma(E^\perp)$, we get

$$\langle S,T\rangle \langle X,\delta\rangle = \left\langle \alpha^j(S,T), X\right\rangle,$$

which means that σ^k is an umbilical submanifold of M^n with mean curvature vector field δ.

An umbilical immersion j is called an *extrinsic sphere* if its mean curvature vector field δ is parallel in the normal connection. If an umbilical distribution E also satisfies $(\nabla_X \delta)_{E^\perp} = 0$, where δ is the mean curvature vector field, then

$$j^*\nabla_X \delta = -j_* A_\delta X +^j \nabla_X^\perp \delta = -j_* A_\delta X.$$

Hence ${}^j\nabla_X^\perp \delta = 0$, and σ^k is an extrinsic sphere. With this in mind, we call an umbilical distribution E *spherical* if its mean curvature vector field δ satisfies

$$(\nabla_X \delta)_{E^\perp} = 0.$$

We finish this section by proving a proposition that can be found in [8]. We will not just cite the result, because we will need some facts that appear during the proof of item (ii).

Proposition 1.1 (Proposition 1.22 in [8]). *Let $f : M^n \to \mathbb{Q}_c^m$ be an isometric immersion with a principal normal vector field η of multiplicity q. Then the following assertions hold:*

(i) The distribution $x \mapsto E_\eta(x)$ is smooth.

(ii) The principal normal vector field η is Dupin if and only if E_η is a spherical distribution and f maps each leaf of E_η into an extrinsic sphere of $\mathbb{Q}_c^m$.

(iii) If $q \geq 2$ then η is a Dupin principal normal vector field.

(iv) If η is a Dupin principal normal vector field and $c = 0$, then the map $h : M^n \to \mathbb{R}^m$ defined as

$$h = f + \frac{1}{||\eta||^2}\eta$$

is constant along E_η.

Proof. First, using equation (1.2), we will show that

$$(E_\eta(x))^\perp = \operatorname{span}\{A_\gamma X - \langle \gamma, \eta\rangle X : X \in T_xM \,\&\, \gamma \in N_fM(x)\}. \tag{1.3}$$

Let

$$Y \in E_\eta(x) = \cap_{\gamma \in N_fM(x)} \ker(A_\gamma - \langle \gamma, \eta\rangle I).$$

Then, for arbitrary $X \in T_xM$ and $\gamma \in N_fM(x)$ we have

$$\langle Y, A_\gamma X - \langle \gamma, \eta\rangle X\rangle = \langle A_\gamma Y - \langle \gamma, \eta\rangle Y, X\rangle = 0,$$

so

$$A_\gamma X - \langle \gamma, \eta\rangle X \in \left(\cap_{\gamma \in N_fM(x)} \ker(A_\gamma - \langle \gamma, \eta\rangle I)\right)^\perp = (E_\eta(x))^\perp.$$

For the other inclusion, let

$$Y \in (\operatorname{span}\{A_\gamma X - \langle \gamma, \eta\rangle X : X \in T_xM \,\&\, \gamma \in N_fM\})^\perp,$$

then, for any $\gamma \in N_fM(x)$ and $X \in T_xM$ we have

$$\langle A_\gamma Y - \langle \gamma, \eta\rangle Y, X\rangle = \langle Y, A_\gamma X - \langle \gamma, \eta\rangle X\rangle = 0.$$

Hence, $A_\gamma Y - \langle \gamma, \eta\rangle Y = 0$ for any $\gamma \in N_fM(x)$ and our affirmation follows.

Lets start by proving item (i). It is enough to prove that the distribution

$$x \mapsto (E_\eta(x))^\perp$$

is smooth. From equation (1.3), just choose pairs (X_i, γ_i) for $i = 1, \cdots, k$ such that $X_i \in T_xM$, $\gamma_i \in N_fM(x)$ and $\{A_{\gamma_i} X_i - \langle \gamma_i, \eta\rangle X_i : i = 1, \cdots, k\}$ is a basis for $(E_\eta(x))^\perp$. Then, extend (X_i, γ_i) for $i = 1, \cdots, k$ smoothly in a neighborhood of $x \in M^n$. Maybe in a smaller neighborhood, $\{A_{\gamma_i} X_i - \langle \gamma_i, \eta\rangle X_i : i = 1, \cdots, k\}$ will be a frame for $E_\eta^\perp$, which shows that the distribution is smooth.

Lets now prove item (ii). Suppose that η is a Dupin principal vector field, and let $\eta = \lambda\zeta$ where $\zeta \in \Gamma(N_fM)$ is of unit length. Then, for any $T \in \Gamma(E_\eta)$ we have

$$0 = \nabla_T^\perp \eta = T(\lambda)\zeta + \lambda \nabla_T^\perp \zeta.$$

Hence,

$$0 = T(\lambda) = \langle \operatorname{grad}\lambda, T\rangle ,$$

or, in another words, $\operatorname{grad}\lambda \in \Gamma(E_\eta^\perp)$.

If S, $T \in \Gamma(E_\eta)$ and $X \in \mathfrak{X}(M)$, using the Codazzi equation for A_ζ and equation (1.2) we get

$$\begin{aligned}
0 &= \left\langle \nabla_X A_\zeta T, S\right\rangle - \left\langle A_\zeta \nabla_X T, S\right\rangle - \left\langle A_{\nabla_X^\perp \zeta} T, S\right\rangle - \left\langle \nabla_T A_\zeta X, S\right\rangle + \left\langle A_\zeta \nabla_T X, S\right\rangle \\
&\quad + \left\langle A_{\nabla_T^\perp \zeta} X, S\right\rangle \\
&= \langle \nabla_X \lambda T, S\rangle - \lambda \langle \nabla_X T, S\rangle - \langle \nabla_T A_\zeta X, S\rangle + \lambda \langle \nabla_T X, S\rangle \\
&= \langle T, S\rangle \langle \operatorname{grad}\lambda, X\rangle - T\langle A_\zeta X, S\rangle + \langle A_\zeta X, \nabla_T S\rangle + \lambda T\langle X, S\rangle - \lambda \langle X, \nabla_T S\rangle \\
&= \langle T, S\rangle \langle \operatorname{grad}\lambda, X\rangle + \langle (A_\zeta - \lambda I)\nabla_T S, X\rangle .
\end{aligned}$$

Therefore,

$$(A_\zeta - \lambda I)\nabla_T S = -\langle T, S\rangle \operatorname{grad}\lambda \tag{1.4}$$

for all T, $S \in \Gamma(E_\eta)$. Similarly, the Codazzi equation for A_ξ, where $\xi \in \Gamma(N_f M)$ is a section orthogonal to η, applied to $T \in \Gamma(E_\eta)$, $X \in \mathfrak{X}(M)$, and taking inner product with $S \in \Gamma(E_\eta)$, yields

$$\begin{aligned}
0 &= \left\langle \nabla_X A_\xi T, S\right\rangle - \left\langle A_\xi \nabla_X T, S\right\rangle - \left\langle A_{\nabla_X^\perp \xi} T, S\right\rangle - \left\langle \nabla_T A_\xi X, S\right\rangle + \left\langle A_\xi \nabla_T X, S\right\rangle \\
&\quad + \left\langle A_{\nabla_T^\perp \xi} X, S\right\rangle \\
&= -\left\langle \nabla_X^\perp \xi, \eta\right\rangle \langle T, S\rangle + \langle X, A_\xi \nabla_T S\rangle .
\end{aligned}$$

Consequently,

$$\langle A_\xi \nabla_T S, X\rangle = \lambda \langle T, S\rangle \left\langle \nabla_X^\perp \xi, \zeta\right\rangle , \tag{1.5}$$

for any T, $S \in \Gamma(E_\eta)$ and $X \in \mathfrak{X}(M)$.

Taking into account our alternate definition of the subspace E_η given by equation (1.2), from equations (1.4) and (1.5) we conclude that $\nabla_T S \in \Gamma(E_\eta)$ for any pair of orthogonal T, $S \in \Gamma(E_\eta)$. Defining $\beta : \Gamma(E_\eta) \times \Gamma(E_\eta) \to \Gamma(E_\eta^\perp)$ by

$$\beta(T, S) = (\nabla_T S)_{E_\eta^\perp},$$

we have $\beta(T, S) = 0$ for any orthogonal pair T, S. Moreover, since this bilinear form is $C^\infty(M)$-linear, it is, in fact, a tensor. Consider $\{T_1, \cdots, T_k\}$ to be an orthonormal frame for E_η, then

$$0 = \beta(T_i + T_j, T_i - T_j) = \beta(T_i, T_i) - \beta(T_j, T_j).$$

Hence,

$$\beta(T_i, T_i) = \beta(T_j, T_j)$$

for any choice of $i \neq j$. So, without ambiguity we can define $\delta = \beta(T_i, T_i) = (\nabla_{T_i} T_i)_{E_\eta^\perp}$. Then, we have

$$\begin{aligned}\beta(T, S) &= \sum_{i,j} a_i b_j \beta(T_i, T_j) \\ &= \sum_i a_i b_i \beta(T_i, T_i) \\ &= \langle T, S \rangle \, \delta.\end{aligned}$$

The above equation means that the distribution E_η is umbilical, because for all T, $S \in \Gamma(E_\eta)$ and $X \in \Gamma(E_\eta^\perp)$ we have

$$\langle \nabla_T S, X \rangle = \langle \beta(T, S), X \rangle = \langle T, S \rangle \langle X, \delta \rangle ,$$

where $\delta = (\nabla_T T)_{E_\eta^\perp}$ for any $T \in \Gamma(E_\eta)$ of unit length.

From equations (1.4) and (1.5), and using the alternative definition of the subspace E_η given in equation (1.2), we arrive to

$$(A_\zeta - \lambda I)\delta = (A_\zeta - \lambda I)(\nabla_T T)_{E_\eta^\perp} = (A_\zeta - \lambda I)(\nabla_T T) = -\operatorname{grad} \lambda, \tag{1.6}$$

and

$$\big\langle A_\xi \delta, X \big\rangle = \big\langle A_\xi (\nabla_T T)_{E_\eta^\perp}, X \big\rangle = \big\langle A_\xi \nabla_T T, X \big\rangle = \lambda \big\langle \nabla_X^\perp \xi, \zeta \big\rangle, \tag{1.7}$$

for $T \in \Gamma(E_\eta)$ of unit length. In particular, the equation (1.6) will be fundamental on a later chapter, so the reader should keep it in mind.

We must now prove that E_η is a spherical distribution, that is, that the mean curvature vector field δ satisfies

$$(\nabla_X \delta)_{E^\perp} = 0.$$

Utilizing the Codazzi equation for A_ζ applied to $T \in \Gamma(E_\eta)$, $X \in \mathfrak{X}(M)$, then taking inner product with δ, using equation (1.7) and that η is a Dupin principal vector field we obtain

$$\begin{aligned}
0 &= \big\langle \nabla_X A_\zeta T, \delta \big\rangle - \big\langle A_\zeta \nabla_X T, \delta \big\rangle - \big\langle A_{\nabla_X^\perp \zeta} T, \delta \big\rangle - \big\langle \nabla_T A_\zeta X, \delta \big\rangle + \big\langle A_\zeta \nabla_T X, \delta \big\rangle \\
&\quad + \big\langle A_{\nabla_T^\perp \zeta} X, \delta \big\rangle \\
&= \lambda \big\langle \nabla_X T, \delta \big\rangle - \big\langle A_\zeta \nabla_X T, \delta \big\rangle - \big\langle \nabla_T A_\zeta X, \delta \big\rangle + \big\langle A_\zeta \nabla_T X, \delta \big\rangle + \big\langle A_{\nabla_T^\perp \zeta} X, \delta \big\rangle \\
&= - \langle \nabla_T (A_\zeta - \lambda I) X, \delta \rangle - \lambda \langle \nabla_T X, \delta \rangle + \lambda \langle \nabla_X T, \delta \rangle + \langle [T, X], A_\zeta \delta \rangle \\
&\quad + \lambda \big\langle \nabla_X^\perp \nabla_T^\perp \zeta, \zeta \big\rangle \\
&= - \langle \nabla_T (A_\zeta - \lambda I) X, \delta \rangle - \lambda \langle [T, X], \delta \rangle + \langle [T, X], A_\zeta \delta \rangle - \lambda \big\langle \nabla_T^\perp \zeta, \nabla_X^\perp \zeta \big\rangle \\
&= - \langle \nabla_T (A_\zeta - \lambda I) X, \delta \rangle + \langle (A_\zeta - \lambda I) \delta, [T, X] \rangle,
\end{aligned}$$

for any $T \in \Gamma(E_\eta)$ and $X \in \mathfrak{X}(M)$. Hence, using the above equality, equation (1.6) and that λ is constant along E_η we get

$$\begin{aligned}
\langle \nabla_T \delta, (A_\zeta - \lambda I) X \rangle &= T \langle (A_\zeta - \lambda I) \delta, X \rangle - \langle \delta, \nabla_T (A_\zeta - \lambda I) X \rangle \\
&= T \langle (A_\zeta - \lambda I) \delta, X \rangle - \langle (A_\zeta - \lambda I) \delta, [T, X] \rangle \\
&= -T \langle \operatorname{grad} \lambda, X \rangle + \langle \operatorname{grad} \lambda, [T, X] \rangle \\
&= -T X(\lambda) + [T, X](\lambda) \\
&= 0.
\end{aligned} \tag{1.8}$$

Using the Codazzi equation for A_ξ, where ξ is a section orthogonal to η, applied to $T \in \Gamma(E_\eta)$, $X \in \mathfrak{X}(M)$, taking inner product with the mean curvature vector field δ and using equation (1.7) we obtain

$$\begin{aligned}
0 &= \big\langle \nabla_X A_\xi T, \delta \big\rangle - \big\langle A_\xi \nabla_X T, \delta \big\rangle - \big\langle A_{\nabla_X^\perp \xi} T, \delta \big\rangle - \big\langle \nabla_T A_\xi X, \delta \big\rangle + \big\langle A_\xi \nabla_T X, \delta \big\rangle \\
&\quad + \big\langle A_{\nabla_T^\perp \xi} X, \delta \big\rangle \\
&= \big\langle A_\xi \delta, [T, X] \big\rangle - \big\langle \delta, \nabla_T A_\xi X \big\rangle + \big\langle A_{\nabla_T^\perp \xi} X, \delta \big\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \nabla_T \delta, A_\xi X \rangle &= T \langle A_\xi \delta, X \rangle - \langle \delta, \nabla_T A_\xi X \rangle \\
&= T \big\langle A_\xi \delta, X \big\rangle - \big\langle A_{\nabla_T^\perp \xi} \delta, X \big\rangle - \big\langle A_\xi \delta, [T, X] \big\rangle,
\end{aligned}$$

for any $T \in \Gamma(E_\eta)$ and $X \in \mathfrak{X}(M)$. Using equation (1.7) and the Ricci equation for ξ and

ζ, we get

$$\begin{aligned}\left\langle \nabla_T \delta, A_\xi X \right\rangle &= T\left\langle A_\xi \delta, X \right\rangle - \left\langle A_{\nabla^{\perp}_T \xi} \delta, X \right\rangle - \left\langle A_\xi \delta, [T, X] \right\rangle \\ &= \lambda T \left\langle \nabla^{\perp}_X \xi, \zeta \right\rangle - \lambda \left\langle \nabla^{\perp}_X \nabla^{\perp}_T \xi, \zeta \right\rangle - \lambda \left\langle \nabla^{\perp}_{[T,X]} \xi, \zeta \right\rangle \\ &= \lambda \left\langle R^{\perp}(T, X)\xi, \zeta \right\rangle \\ &= \lambda \left\langle [A_\xi, A_\zeta]T, X \right\rangle \\ &= 0,\end{aligned} \tag{1.9}$$

for all $T \in \Gamma(E_\eta)$ and $X \in \mathfrak{X}(M)$. Since

$$(E_\eta(x))^{\perp} = \text{span}\left\{A_\gamma X - \langle \gamma, \eta \rangle X : X \in T_x M \ \& \ \gamma \in N_f M(x)\right\},$$

using equations (1.8) and (1.9) we conclude that $\nabla_T \delta \in \Gamma(E_\eta)$ for any $T \in \Gamma(E_\eta)$, so E_η is an spherical distribution.

We must prove now that the restriction of f to each leaf σ^k generated by the distribution E_η is an extrinsic sphere. Let $j : \sigma^k \to M^n$ be the inclusion map and define $\tilde{f} = f \circ j : \sigma^k \to \mathbb{Q}^m_c$. Since,

$$\alpha^{\tilde{f}}(T, S) = f_* \alpha^j(T, S) + \alpha^f(j_* T, j_* S) = \langle T, S \rangle f_* \delta + \langle T, S \rangle \eta,$$

we get that $\tilde{f}$ is umbilical with mean curvature vector field $\gamma = f_* \delta + \eta$. Now, because $j_* T \in \Gamma(E_\eta)$ and η is a Dupin principal, we have

$$\begin{aligned}\tilde{f}^* \tilde{\nabla}_T \left(f_* \delta + \eta\right) &= f^* \tilde{\nabla}_{j_* T} f_* \delta + f^* \tilde{\nabla}_{j_* T} \eta \\ &= f_* \nabla_{j_* T} \delta + \alpha^f(j_* T, \delta) - f_* A_\eta j_* T +^f \nabla^{\perp}_{j_* T} \eta \\ &= -||\delta||^2 \tilde{f}_* T - ||\eta||^2 \tilde{f}_* T,\end{aligned}$$

where we used that $\nabla_{j_* T} \delta \in \Gamma(E_\eta)$. We conclude that ${}^{\tilde{f}}\nabla^{\perp}_T \left(f_* \delta + \eta\right) = 0$ and so $\tilde{f}$ is an extrinsic sphere.

For the converse of item (ii), let η be a principal normal vector field of multiplicity q with associated spherical distribution E_η such that f maps each leaf σ^k generated by the distribution E_η to an extrinsic sphere of $\mathbb{Q}^m_c$. We must prove that ${}^f\nabla^{\perp}_T \eta = 0$, for any $T \in \Gamma(E_\eta)$. Since E_η is in particular an umbilical distribution, we have

$$\alpha^{\tilde{f}}(S, T) = f_* \alpha^j(S, T) + \alpha^f(j_* S, j_* T) = \langle S, T \rangle \left(f_* \delta + \eta\right),$$

for any $S,\, T \in \Gamma(E_\eta)$ where δ is the mean curvature vector field of the distribution E_η. Therefore, $\tilde{f}$ is umbilical. Since the image of the leaf σ^k is an extrinsic sphere, we must have ${}^{\tilde{f}}\nabla^{\perp}_{T}\left(f_*\delta + \eta\right) = 0$. Now, because η is a principal normal vector field

$$\tilde{f}^*\tilde{\nabla}_T\left(f_*\delta + \eta\right) = f^*\tilde{\nabla}_{j_*T} f_*\delta + f^*\tilde{\nabla}_{j_*T}\eta = f_*\nabla_{j_*T}\delta - f_*A_\eta j_*T +^f \nabla^{\perp}_{j_*T}\eta,$$

so from ${}^{\tilde{f}}\nabla^{\perp}_{T}\left(f_*\delta + \eta\right) = 0$ we conclude ${}^{f}\nabla^{\perp}_{j_*T}\eta = 0$.

We now prove item (iii). The Codazzi equation for A_ζ and $S,\, T \in \Gamma(E_\eta)$ gives us

$$\begin{aligned}
0 &= \left\langle \nabla_S A_\zeta T, S\right\rangle - \left\langle A_\zeta \nabla_S T, S\right\rangle - \left\langle A_{\nabla^{\perp}_{S}\zeta} T, S\right\rangle - \left\langle \nabla_T A_\zeta S, S\right\rangle + \left\langle A_\zeta \nabla_T S, S\right\rangle \\
&\quad + \left\langle A_{\nabla^{\perp}_{T}\zeta} S, S\right\rangle \\
&= S(\lambda)\left\langle T, S\right\rangle + \lambda\left\langle \nabla_S T, S\right\rangle - \lambda\left\langle \nabla_S T, S\right\rangle - T(\lambda)\left\langle S, S\right\rangle - \lambda\left\langle \nabla_T S, S\right\rangle \\
&\quad + \lambda\left\langle \nabla_T S, S\right\rangle \\
&= S(\lambda)\left\langle T, S\right\rangle - T(\lambda)\left\langle S, S\right\rangle.
\end{aligned}$$

Taking a pair of orthogonal unit vectors $S,\, T \in \Gamma(E_\eta)$ we conclude that $T(\lambda) = 0$. The Codazzi equation for $S,\, T \in \Gamma(E_\eta)$ and A_ξ gives us

$$\begin{aligned}
0 &= \left\langle \nabla_S A_\xi T, S\right\rangle - \left\langle A_\xi \nabla_S T, S\right\rangle - \left\langle A_{\nabla^{\perp}_{S}\xi} T, S\right\rangle - \left\langle \nabla_T A_\xi S, S\right\rangle + \left\langle A_\xi \nabla_T S, S\right\rangle \\
&\quad + \left\langle A_{\nabla^{\perp}_{T}\xi} S, S\right\rangle \\
&= -\left\langle \nabla^{\perp}_{S}\xi, \eta\right\rangle\left\langle T, S\right\rangle + \left\langle \nabla^{\perp}_{T}\xi, \eta\right\rangle\left\langle S, S\right\rangle.
\end{aligned}$$

Again, for a pair of orthogonal unit vectors $S,\, T \in \Gamma(E_\eta)$ we get $\left\langle \nabla^{\perp}_{T}\xi, \zeta\right\rangle = 0$ for any $\xi \in \Gamma(N_f M)$ orthogonal to ζ. Hence, we get $\nabla^{\perp}_{T}\zeta = 0$, so

$$\nabla^{\perp}_{T}\eta = T(\lambda)\zeta + \lambda\nabla^{\perp}_{T}\zeta = 0.$$

Lasty, lets prove item (iv). Since η is a Dupin principal, we have $T\left\langle \eta, \eta\right\rangle = 2\left\langle \nabla^{\perp}_{T}\eta, \eta\right\rangle = 0$, for any $T \in \Gamma(E_\eta)$. From differentiating the function $h : M^n \to \mathbb{R}^m$, we get

$$\begin{aligned}
h_*T &= f_*T - \frac{1}{||\eta||^2} f_* A_\eta T \\
&= 0,
\end{aligned}$$

for any $T \in \Gamma(E_\eta)$. Therefore, the function h is constant along E_η. □

Just as an advance of what will come in the future, we will mention that the last item of the Proposition 1.1 show us that if $h : M^n \to \mathbb{R}^m$ is an isometric immersion with a Dupin Principal vector field η, then each leaf σ^k of the spherical distribution E_η "lies" on an hypersphere of center $h(p)$ and radius $1/||\eta(p)||$ where $p \in \sigma^k$. Precise definitions will be given in the Hyperspheres Representation section.

1.2 The Euclidean Model in the Light-Cone

As mentioned before, we will characterize the Euclidean space as a hypersurface of the light-cone in the Lorentz space. Let $\mathbb{L}^{m+2}$ be the $(m+2)$-dimensional Lorentz space with metric signature $(m+1, 1)$ and denote the light-cone associated to it by

$$\mathbb{V}^{m+1} = \{p \in \mathbb{L}^{m+2} : \langle p, p \rangle = 0 \text{ and } p \neq 0\}.$$

Given $w \in \mathbb{V}^{m+1}$, define a hypersurface $\mathbb{E}^m_w$ by

$$\mathbb{E}^m_w = \mathbb{V}^{m+1} \cap \left\{p \in \mathbb{L}^{m+2} : \langle p, w \rangle = 1\right\}.$$

Therefore, $\mathbb{E}^m_w$ is the intersection of the light cone and a plane having w as a normal. Fix $v_0 \in \mathbb{E}^m_w$, since by definition we have $\langle v_0, w \rangle = 1$, v_0 and w are linearly independent light-like vectors. Hence, they generate a Lorentzian plane.

Let $\Psi = \Psi_{v_0,w,C} : \mathbb{R}^m \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ be defined by

$$\Psi(x) = v_0 + Cx - \frac{||x||^2}{2} w, \tag{1.10}$$

where $C : \mathbb{R}^m \to \{v_0, w\}^\perp$ is any linear isometry. We affirm that Ψ is an isometric embedding with $\Psi(\mathbb{R}^m) = \mathbb{E}^m_w$. First, observe that

$$\Psi_* X = CX - \langle X, x \rangle w. \tag{1.11}$$

Since $C(\mathbb{R}^m) = \{v_0, w\}^\perp$ and w is a light-like vector, immediately we conclude that Ψ is an isometric immersion. From,

$$\begin{aligned}
\langle \Psi(x), \Psi(x) \rangle &= \left\langle v_0 + Cx - \frac{||x||^2}{2} w, v_0 + Cx - \frac{||x||^2}{2} w \right\rangle \\
&= -\frac{||x||^2}{2} + \langle Cx, Cx \rangle - \frac{||x||^2}{2} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}\langle \Psi(x), w\rangle &= \left\langle v_0 + Cx - \frac{||x||^2}{2} w, w \right\rangle \\ &= \langle v_0, w\rangle \\ &= 1\end{aligned}$$

we have that $\Psi(\mathbb{R}^m) \subset \mathbb{E}^m_w$. For the other inclusion, let $v \in \mathbb{E}^m_w$, then v is a light-like vector and $\langle v, w\rangle = 1$. Define $z \in \mathbb{L}^{m+2}$ by $z = v - v_0 + kw$, where $k = -\langle v, v_0\rangle$. The choice of that k was not randomly done, in fact, it was chosen to make $z \in \{v_0, w\}^\perp$. Therefore, let $x \in \mathbb{R}^m$ such that $Cx = z$. We have

$$\begin{aligned}\langle Cx, Cx\rangle &= \langle v - v_0 + kw, v - v_0 + kw\rangle \\ &= -\langle v, v_0\rangle + k - \langle v_0, v\rangle - k + k - k \\ &= -2\langle v, v_0\rangle ,\end{aligned}$$

so $||x||^2 = 2k$ and then

$$Cx = v - v_0 + \frac{||x||^2}{2} w.$$

Hence, $v = \Psi(x)$.

To finish our introduction to the Euclidean model in the light-cone section, let us see what properties the isometric immersion Ψ has.

Proposition 1.2 (Proposition 9.1 in [8])**.** *Let $f : M^n \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ be an isometric immersion of a Riemannian manifold. Then the position vector field f is a light-like parallel normal vector field satisfying*

$$\left\langle \alpha^f(X, Y), f \right\rangle = -\langle X, Y\rangle$$

for all X, $Y \in \mathfrak{X}(M)$.

Proof. Since f is a map into the light-cone, we have that the position vector field f is light-like. Differentiating $\langle f, f\rangle = 0$ we obtain that $f \in \Gamma(N_f M)$. Differentiating once more $\langle f_* X, f\rangle = 0$, we obtain

$$\left\langle \alpha^f(X, Y), f \right\rangle = -\langle X, Y\rangle$$

and so

$$f_* X = f^* \tilde{\nabla}_X f = -f_* A_f X + \nabla^\perp_X f = f_* X + \nabla^\perp_X f,$$

which shows that f is parallel. $\square$

Therefore, applying the above proposition, we get that Ψ is a light-like, parallel, normal vector field such that $A_\Psi = -I$. From $\langle \Psi(x), w\rangle = 1$, differentiating once we obtain that $w \in \Gamma(N_\Psi \mathbb{R}^m)$. Differentiating twice, we get $\left\langle \alpha^\Psi(X,Y), w\right\rangle = 0$. Therefore, the second fundamental form of Ψ is given by

$$\alpha^\Psi(X,Y) = -\langle X, Y\rangle\, w. \tag{1.12}$$

1.3 Hyperspheres Representation

In this section we will see that hyperspheres in $\mathbb{R}^m$ are in one-to-one correspondence with vectors in the de Sitter space $\mathbb{Q}^{m+1}_{1,1}$.

Let $\mathbb{S} \subset \mathbb{R}^m$ be an hypersphere with unit normal vector field N and mean curvature h (or hyperplane if $h = 0$) and $j : \mathbb{S} \to \mathbb{R}^m$ the inclusion map. Define $S : \mathbb{S} \to \mathbb{L}^{m+2}$ by

$$S(x) = \Psi_*(j(x))N(x) + h(\Psi \circ j)(x). \tag{1.13}$$

Then, for $X \in \mathfrak{X}(\mathbb{S})$, using equation (1.12) and considering the orthogonality of j_*X and N, differentiating the above equation we obtain

$$\begin{aligned} S_*X &= (\Psi \circ j)^*\tilde{\nabla}_X S \\ &= (\Psi \circ j)^*\tilde{\nabla}_X \Psi_* N + h\Psi_* j_* X \\ &= \Psi^*\tilde{\nabla}_{j_*X}\Psi_* N + h\Psi_* j_* X \\ &= \Psi_*\bar{\nabla}_{j_*X} N + \alpha^\Psi(j_*X, N) + h\Psi_* j_* X \\ &= -h\Psi_* j_* X + h\Psi_* j_* X = 0. \end{aligned}$$

Hence, the map S is constant, that is, $S(\mathbb{S}) = \{v\}$.

Now, observe that

$$\langle S, S\rangle = \langle \Psi_* N + h(\Psi \circ j), \Psi_* N + h(\Psi \circ j)\rangle = \langle N, N\rangle = 1$$

and

$$\langle \Psi \circ j, S\rangle = \langle \Psi \circ j, \Psi_* N + h(\Psi \circ j)\rangle = 0,$$

so v belongs to the de Sitter space and $\Psi(\mathbb{S}) \subset \mathbb{E}^m_w \cap \{v\}^\perp$. From the definition of S we have that $\mathbb{S}$ is an hyperplane if and only if $\langle v, w\rangle = 0$.

A remark is in order: the map S takes into account the orientation N given to the hypersphere. If we change the orientation to $-N$, the mean curvature h also changes sign. Hence, $S(\mathbb{S}, N) = -S(\mathbb{S}, -N)$. Generally we will assign the unit normal pointing inwards to the sphere.

Let $\mathbb{S}$ be the non-degenerate hypersphere with x_0 as center and radius $r > 0$. Then, the unit normal vector pointing inwards at $x \in \mathbb{S}$ is given by

$$N(x) = \frac{x_0 - x}{r}$$

with mean curvature vector $h = 1/r$. From equations (1.10) and (1.11) we get

$$\begin{aligned} v &= \Psi_*(j(x))N(x) + h(\Psi \circ j)(x) \\ &= C\left(\frac{x_0 - x}{r}\right) - \left\langle \frac{x_0 - x}{r}, x \right\rangle w + \frac{1}{r}\left(v_0 + Cx - \frac{||x||^2}{2}w\right) \\ &= \frac{1}{r}\Psi(x_0) + \frac{||x_0||^2}{2r}w - \frac{\langle x_0, x\rangle}{r}w + \frac{||x||^2}{2r}w \\ &= \frac{1}{r}\Psi(x_0) + \frac{r}{2}w. \end{aligned} \tag{1.14}$$

We obtain a formula relating v with the center and radius of the hypersphere. In the case where the hypersphere is degenerate, that is, it is an hyperplane with unit normal vector N, then

$$v = CN - \langle N, x\rangle\, w = CN - cw, \tag{1.15}$$

where $c = \langle N, x\rangle$ is constant as the reader can see by differentiating along a vector on the plane. Hence, we obtain a formula relating v, the unit normal vector N and the distance towards the hyperplane at the origin defined by N.

We are now ready to prove that, given any v in the de Sitter space, there exist an oriented hypersphere $\mathbb{S}$ such that $S(\mathbb{S}) = \{v\}$. Since we have proved the other inclusion, we will conclude

$$\Psi(\mathbb{S}) = \mathbb{E}^m_w \cap \{v\}^\perp.$$

We will do it by considering three cases.

Suppose first that $\langle v, w\rangle > 0$, then define $r^{-1} = \langle v, w\rangle$ and

$$z = r\left(v - \frac{r}{2}w\right).$$

From the definition of r and because $\langle v, v\rangle = 1$, we have $z \in \mathbb{E}^m_w$, so there exist $x_0 \in \mathbb{R}^m$ such that $\Psi(x_0) = z$. Let the oriented hypersphere $\mathbb{S}$ be the one with center in x_0 and radius r and unit normal vector pointing inwards. Then, from equation (1.14) we have $S(\mathbb{S}) = v$.

If $\langle v, w\rangle < 0$, then work with $-v$. We will conclude that there exist an oriented hypersphere $\mathbb{S}$ such that $S(\mathbb{S}) = -v$. Then, just change the orientation to obtain v.

If $\langle v, w\rangle = 0$, then we are in the case of an hyperplane. Define $z = v + cw$, where $c = -\langle v, v_0\rangle$ was chosen in order to have $z \in \{v_0, w\}^{\perp}$. Since z is of unit length, there exist N of unit length such that $CN = z$. Define

$$\mathbb{P} = \{x \in \mathbb{R}^m : \langle N, x\rangle = -\langle v, v_0\rangle\}$$

and we have $S(\mathbb{P}) = CN - \langle N, x\rangle w = z + \langle v, v_0\rangle w = v$.

From the equations (1.14) and (1.15) we see that the correspondence assigning hyperspheres to vectors in the de Sitter space is one-onto-one.

1.4 Envelopes of Congruences of Hyperspheres

In this section we define what we understand by a congruence of hyperspheres and show that the hypersurfaces we will be working along are the envelopes of two-parameter congruences of hyperspheres.

Let $h : M^n \to \mathbb{R}^m$ be a smooth function, where M^n is a Riemannian manifold and $r \in C^\infty(M)$ a positive function. Then, the assignment

$$x \mapsto S\left(h(x), r(x)\right)$$

where $x \in M^n$ and $S\left(h(x), r(x)\right)$ is the hypersphere centered at $h(x)$ and radius $r(x)$ is called a *congruence of hyperspheres*. From the discussion at the start of the section, the reader must has already guessed that we will use our model of hyperspheres in the de Sitter space. Therefore, the congruence of hyperspheres $S(h(x), r(x))$ can be identified with a map $S : M^n \to \mathbb{Q}^{m+1}_{1,1}$ defined by

$$S(x) = \frac{1}{r(x)}\Psi(h(x)) + \frac{r(x)}{2}w$$

where from now and onwards let the de Sitter space be denoted by $\mathbb{Q}^{m+1}_{1,1}$. The congruence of hyperspheres $S\left(h(x), r(x)\right)$ will be called a *k-parameter congruence of hyperspheres* if

S has rank k everywhere. Because

$$S_*X = -\frac{X(r)}{r^2(x)}\Psi(h(x)) + \frac{1}{r(x)}\Psi_*h_*X + \frac{X(r)}{2}w,$$

we have $\ker S_* = \ker r_* \cap \ker h_*$. Similarly, we can define a k-parameter congruence of hyperplanes. Since we will not use that definition, we leave the job for the avid reader.

An isometric immersion $f : M^n \to \mathbb{R}^m$ is said to *envelop* a congruence of hyperspheres determined by a focal function $h : M^n \to \mathbb{R}^m$ and radius $r \in C^\infty(M)$ if

$$f(x) \in S\left(h(x), r(x)\right) \quad \text{and} \quad f_*T_xM \leq T_{f(x)}S\left(h(x), r(x)\right), \tag{1.16}$$

or in equations,

$$||f(x) - h(x)||^2 = r^2(x) \quad \text{and} \quad \langle f_*X, f(x) - h(x)\rangle = 0, \tag{1.17}$$

for all $x \in M^n$ and $X \in T_xM$. Differentiating the first equation above and using the second equation, we get

$$-\langle h_*X, f(x) - h(x)\rangle = rX(r),$$

so, $\ker h_* \leq \ker r_*$ and $\ker S_* = \ker h_*$ if f envelops S.

The hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ that will be matter of study has a Dupin principal of multiplicity $n-2$. With the following proposition, we can conclude that f is an envelope of a two-parameter congruence of hyperspheres:

Proposition 1.3 (Proposition 9.4 in [8])**.** *If a hypersurface $f : M^n \to \mathbb{R}^{n+1}$ envelops a k-parameter congruence of (non-degenerate) hyperspheres $S : M^n \to \mathbb{Q}^{n+2}_{1,1}$, $1 \leq k \leq n-1$, then f has a principal curvature λ such that $\ker S_*(x) \leq E_\lambda(x)$ for all $x \in M^n$, with $\ker S_*(x) = E_\lambda(x)$ for all x in an open dense subset of M^n, on which λ is constant along E_λ.*

Conversely, any hypersurface $f : M^n \to \mathbb{R}^{n+1}$ that carries a non-null Dupin principal curvature of multiplicity $n-k$ envelopes a k-parameter congruence of hyperspheres.

Proof. Let us start proving the converse which is the important case to us. Then, let λ be the principal curvature of multiplicity $n-k$ of the hypersurface f. Define $h : M^n \to \mathbb{R}^{m+1}$ by

$$h = f + \frac{1}{\lambda}N$$

and $r = \lambda^{-1} \in C^\infty(M)$. Since $h - f = \lambda^{-1}N$, it is straightforward that f envelops the congruence of hyperspheres determined by the focal map h and radius r.

We have to prove that this congruence of hyperspheres is in fact a k-parameter congruence of hyperspheres. In order to do so, we will show that $E_\lambda = \ker h_*$. By Proposition 1.1, we have $E_\lambda \leq \ker h_*$. If $X \in \ker h_*$, then

$$0 = f_* X - \lambda^{-2} X(\lambda) N - \lambda^{-1} f_* AX.$$

Hence, $AX = \lambda X$ and $X \in E_\lambda$, thus proving that $E_\lambda = \ker h_*$.

For the other way around, let $h : M^n \to \mathbb{R}^{n+1}$ be the focal map and $r \in C^\infty(M)$ the radius function of the congruence of hyperspheres enveloped by f. Then, the equations in (1.17) are valid. Therefore,

$$N = \frac{1}{r}(h - f)$$

is a unit normal vector field of f. Differentiating the equation above for $X \in \ker S_* = \ker h_* \cap \ker r_*$ we obtain,

$$-f_* AX = -\frac{X(r)}{r^2}(h - f) + \frac{1}{r}(h_* X - f_* X) = -\frac{1}{r} f_* X.$$

We conclude that $\ker S_* \leq E_\lambda$ with $\lambda = r^{-1}$. Now, suppose λ is constant along E_λ and let $X \in E_\lambda$, then

$$h_* X = f_* X - \lambda^{-2} X(\lambda) N - \lambda^{-1} f_* AX = 0,$$

so $X \in \ker h_* = \ker S_*$ which concludes the proof. □

1.5 The Light-Cone Representative

We will be working with conformal immersions $f : M^n \to \mathbb{R}^m$. However, in practice, we will replace those conformal immersions with isometric immersions $F : M^n \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$. In this section we will see how this is done.

Let M^n be a Riemannian manifold and $f : M^n \to \mathbb{R}^m$ a conformal immersion with conformal factor $\varphi \in C^\infty(M)$, that is, φ is a positive function such that

$$\langle f_* X, f_* Y \rangle = \varphi^2 \langle X, Y \rangle_M ,$$

for any X, $Y \in \mathfrak{X}(M)$. The map $F : M^n \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ defined by

$$F = \frac{1}{\varphi}(\Psi \circ f)$$

is called the *isometric light-cone representative* of f. It is an isometric immersion because

$$\begin{aligned}\langle F_*X, F_*Y\rangle &= \left\langle X(\varphi^{-1})(\Psi\circ f)+\varphi^{-1}\Psi_*f_*X, Y(\varphi^{-1})(\Psi\circ f)+\varphi^{-1}\Psi_*f_*Y\right\rangle\\ &= \varphi^{-2}\langle f_*X, f_*Y\rangle\\ &= \langle X, Y\rangle .\end{aligned}$$

Therefore, from a conformal immersion $f : M^n \to \mathbb{R}^m$ we get an isometric immersion $F : M^n \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$.

Transforming isometric immersions into the light-cone to conformal immersions into Euclidean space also work, but some care is needed. First, fix an Euclidean model $\Psi = \Psi_{v_0,w,C} : \mathbb{R}^m \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ such that $\mathbb{E}^m_w \subset \mathbb{V}^{m+1}_+$ and define the projection $\Pi : \mathbb{V}^{m+1} - \mathbb{R}w \to \mathbb{E}^m_w$ of the light-cone (with the exception of a line) onto $\mathbb{E}^m_w$ by

$$\Pi(u) = \frac{u}{\langle u, w\rangle}.$$

Given an isometric immersion $F : M^n \to \mathbb{V}^{m+1} - \mathbb{R}w$ we can define a map $f : M^n \to \mathbb{R}^{m+1}$ by the identity

$$\Psi\circ f = \Pi\circ F.$$

Then, using the properties for F given in proposition 1.2, we get

$$\begin{aligned}\langle f_*X, f_*Y\rangle &= \langle \Psi_*f_*X, \Psi_*f_*Y\rangle\\ &= \langle \Pi_*F_*X, \Pi_*F_*Y\rangle\\ &= \left\langle -\frac{\langle F_*X, w\rangle}{\langle F(x), w\rangle^2}F(x)+\frac{1}{\langle F(x), w\rangle}F_*X, -\frac{\langle F_*Y, w\rangle}{\langle F(x), w\rangle^2}F(x)+\frac{1}{\langle F(x), w\rangle}F_*Y\right\rangle\\ &= \frac{1}{\langle F(x), w\rangle^2}\langle X, Y\rangle\end{aligned}$$

so, $f : M^n \to \mathbb{R}^m$ is a conformal immersion with conformal factor $\langle F, w\rangle^{-1}$.

After all the previous discussion, we are now ready to state a result.

Proposition 1.4 (Proposition 9.9 in [8]). *Let M^n be a Riemannian manifold. Then, the following holds:*

(i) Any conformal immersion $f : M^n \to \mathbb{R}^m$ with conformal factor $\varphi \in C^\infty(M)$ gives rise to an isometric immersion $\mathcal{I}(f) : M^n \to \mathbb{V}^{m+1}_+$ given by

$$\mathcal{I}(f) = \frac{1}{\varphi}\Psi\circ f.$$

(ii) Any isometric immersion $F : M^n \to \mathbb{V}^{m+1}_+ - \mathbb{R}w$ *give rise to a conformal immersion* $\mathcal{C}(F) : M^n \to \mathbb{R}^m$ *with conformal factor* $1/\langle F, w\rangle$ *given by*

$$\Psi \circ \mathcal{C}(F) = \Pi \circ F.$$

(iii) For any conformal immersion $f : M^n \to \mathbb{R}^m$ *and for any isometric immersion* $F : M^n \to \mathbb{V}^{m+1}_+ - \mathbb{R}w$ *one has*

$$\mathcal{C}(\mathcal{I}(f)) = f \quad \textit{and} \quad \mathcal{I}(\mathcal{C}(F)) = F.$$

Proof. The only thing left to prove is item (iii). For the first one, notice that

$$\begin{aligned} \Pi \circ \mathcal{I}(f) &= \Pi\left(\frac{1}{\varphi}\Psi \circ f\right) \\ &= \Psi \circ f, \end{aligned}$$

so by item (ii) we get $\mathcal{C}(\mathcal{I}(f)) = f$. For the second identity, from item (i) and (ii) we get

$$\begin{aligned} \mathcal{I}(\mathcal{C}(F)) &= \langle F, w\rangle\, \Psi \circ \mathcal{C}(F) \\ &= \langle F, w\rangle\, \Pi \circ F \\ &= F, \end{aligned}$$

which concludes the proof. □

To finish this section, we will just enunciate a proposition from [8] that gives an equivalent condition about when two conformal immersions $f,\, g : M^n \to \mathbb{R}^m$ are *conformally congruent*, that is, there exist conformal immersion $\tau : \mathbb{R}^m \to \mathbb{R}^m$ such that $f = \tau \circ g$.

Proposition 1.5 (Proposition 9.18 in [8])**.** *Let* $f,\, g : M^n \to \mathbb{R}^m$ *be conformal immersions. Then the immersions* f *and* g *are conformally congruent if and only if their isometric light-cone representatives* $\mathcal{I}(f), \mathcal{I}(g) : M^n \to \mathbb{V}^{m+1}_+ \subset \mathbb{L}^{m+2}$ *are isometrically congruent.*

1.6 Conformal Gauss Parametrization

As mentioned before, we will work with hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$, $n \geq 6$, that have a principal curvature λ of multiplicity $n-2$. From item (iii) of proposition 1.1, the principal normal vector field $\eta = \lambda N$ will be a Dupin principal normal vector field, hence we can apply the converse of proposition 1.3 to conclude that the hypersurface f is the envelope of a two-parameter congruence of hyperspheres $S(h(x), r(x))$, where we will call from now $h : M^n \to \mathbb{R}^{n+1}$ the focal map and $r \in C^\infty(M)$ the radius function of the hypersphere.

If the reader is acquainted with the Gauss parametrization, he knows that a hypersurface with constant relative nullity can be parametrized in terms of the Gauss map and support function. Conversely, the Gauss map and support function are enough to recover the hypersurface. We want to show a similar result, this time for hypersurfaces having a Dupin principal curvature, and use it in later chapters in the context given before.

Let $f : M^n \to \mathbb{R}^{n+1}$ be an oriented hypersurface carrying a nowhere vanishing Dupin principal curvature λ with constant multiplicity $n-k$. From proposition 1.3, the hypersurface f is the envelope of a k-parameter congruence of hyperspheres determined by the focal map $h : M^n \to \mathbb{R}^{n+1}$ defined by

$$h(x) = f(x) + \frac{1}{\lambda(x)} N(x) \tag{1.18}$$

and the radius function $s \in C^\infty(M)$ given by $s(x) = \lambda^{-1}(x)$.

Let $L^k = M^n/E_\lambda$ be the quotient space of leaves of the distribution E_λ and $\pi : M^n \to L^k$ the corresponding projection. From item (iv) of proposition 1.1, and since λ is constant along E_λ, we can define $g : L^k \to \mathbb{R}^{n+1}$ and $r \in C^\infty(L)$ by

$$g \circ \pi = h \quad \text{and} \quad r \circ \pi = \lambda^{-1}.$$

Therefore, from equation (1.18) for $\bar{x} = \pi(x)$, we get

$$f(x) = g \circ \pi(x) - (r \circ \pi)(x) N(x) = g(\bar{x}) - r(\bar{x}) N(x). \tag{1.19}$$

Also notice that g is an immersion, because if $0 = g_* \bar{X} = g_* \pi_* X = h_* X$, then $X \in \ker h_* \leq \ker s_*$. So, differentiating equation (1.18), we get $X \in E_\lambda$ and $\bar{X} = 0$. We will give L^k the metric induced by g.

Differentiating equation (1.18) and taking inner product with the unit normal vector field of f given from its orientation, we arrive to

$$\begin{aligned}
0 &= \langle f_* Y, N \rangle \\
&= \left\langle h_* Y - Y(\lambda^{-1}) N + \lambda^{-1} N_* Y, N \right\rangle \\
&= \langle g_* \pi_* Y, N \rangle - \pi_* Y(r) \\
&= \langle g_* \pi_* Y, N - g_* \mathrm{grad}\, r \rangle,
\end{aligned}$$

for all $Y \in T_xM$. Thus, if we express $N_fM(x) = g_*T_xM \oplus N_gM(x)$, then the projection of $N(x)$ into g_*T_xM is given by

$$N^T(x) = g_*\mathrm{grad}\, r(\bar{x}). \tag{1.20}$$

In particular, we have $||\mathrm{grad}\, r|| < 1$.

Define by $N^\perp(x)$ the projection of $N(x)$ into $N_gM(x)$. From $||N^T(x)|| = ||g_*\mathrm{grad}\, r(\bar{x})||$ we have

$$||N^\perp(x)|| = \sqrt{1 - ||g_*\mathrm{grad}\, r(\bar{x})||^2}.$$

Therefore, we can define a map $\Phi : M^n \to N_g^1 L$ into the unit normal bundle of g by $\Phi(x) = (\bar{x}, u)$, where

$$u = \frac{1}{\sqrt{1 - ||g_*\mathrm{grad}\, r(\bar{x})||^2}} N^\perp(x).$$

If $\Phi(x) = \Phi(z)$, then x and z belong to the same leaf. From this observation, the definition of Φ and the equality in the second variable of $\Phi(x) = \Phi(z)$, we get $N^\perp(x) = N^\perp(z)$. Since $N^T(x) = N^T(z)$, we have $N(x) = N(z)$ and so, using equation (1.19), we conclude that $f(x) = f(z)$. Since the restriction of f to any leaf is an extrinsic sphere (proposition 1.1), we have $x = z$ and hence, the map Φ is injective.

Now, we will prove that Φ is an immersion. Let $X \in E_\lambda$, then from the expression of $N^T(x)$, we have $N^T(x)_*X = 0$. Therefore,

$$\Phi_*X = (0, N(x)_*X) = -\lambda(0, f_*X)$$

and Φ has at least rank $n - k$. Since $\pi : M^n \to L^k$ is a submersion, we conclude our affirmation. Together with the fact that Φ is injective, we get that Φ is a diffeomorphism onto an open set U of $N_g^1 L$.

Let $\theta : U \to M^n$ be the inverse of Φ. Then

$$\begin{aligned}
(\bar{x}, u) &= \Phi \circ \theta(\bar{x}, u) \\
&= \left(\overline{\theta(\bar{x}, u)}, \frac{1}{\sqrt{1 - ||g_*\mathrm{grad}\, r\left(\overline{\theta(\bar{x}, u)}\right)||^2}} N^\perp\left(\theta(\bar{x}, u)\right) \right) \\
&= \left(\bar{x}, \frac{1}{\sqrt{1 - ||g_*\mathrm{grad}\, r\left(\bar{x}\right)||^2}} N^\perp\left(\theta(\bar{x}, u)\right) \right).
\end{aligned}$$

Then, from equations (1.19), (1.20) we have

$$f \circ \theta(\bar{x}, u) = g(\bar{x}) - r(\bar{x}) g_* \operatorname{grad} r(\bar{x}) - r(\bar{x}) \sqrt{1 - ||g_* \operatorname{grad} r(\bar{x})||^2}\, u, \tag{1.21}$$

for all $(\bar{x}, u) \in U$. This map is called the *Conformal Gauss Parametrization* of f.

We have proved the converse part of the following theorem:

Theorem 1.6 (Theorem 9.6 in [8]). *Let $g : V^k \to \mathbb{R}^{n+1}$ be an isometric immersion and let $r \in C^\infty(V)$ be a positive function such that $||grad\, r|| < 1$. Then the map $\phi : N_g^1 V \to \mathbb{R}^{n+1}$ defined by*

$$\phi(y, u) = g(y) - r(y) g_* grad\, r(y) - r(y) \sqrt{1 - ||grad\ r(y)||^2}\, u \tag{1.22}$$

parametrizes, on the open subset of regular points, a hypersurface that carries a Dupin principal curvature of multiplicity $n - k$.

Conversely, if $f : M^n \to \mathbb{R}^{n+1}$ is an orientable hypersurface with a Dupin principal curvature of multiplicity $n - k$ then there exist an isometric immersion $g : V^k \to \mathbb{R}^{n+1}$, a positive function $r \in C^\infty(V)$ with $||grad\, r|| < 1$ and a diffeomorphism $\theta : U \to M^n$ of an open subset $U \subset N_g^1 V$ such that $f \circ \theta$ is given by (1.22).

Proof. Motivated by the demonstration of the converse of this theorem, define a vector field $N \in \Gamma(\phi^* T\mathbb{R}^{n+1})$ by

$$N(y, u) = g_* \operatorname{grad} r(y) + \sqrt{1 - ||\operatorname{grad} r||^2}\, u.$$

It is of unit length, because

$$\begin{aligned} \langle N, N \rangle &= ||g_* \operatorname{grad} r||^2 + 1 - ||\operatorname{grad} r||^2 \\ &= 1. \end{aligned}$$

From the definition of the vector field N, we have

$$\phi(y, u) = g(y) - r(y) N(y, u).$$

We want to show that $N \in N_\phi(N_g^1 V)$. We will do it in two steps: Let $X \in T_{(y,u)} N_g^1 V$ vertical vector, that is $\pi_* X = 0$ where $\pi : N_g^1 V \to V$, we get

$$\phi_* X = -r N_* X. \tag{1.23}$$

Hence, $\langle \phi_* X, N \rangle = 0$. On the other hand, any non-vertical vector can be written as $\zeta_* Y$

where $\zeta : V^k \to N^1_g V$ is a section with $\zeta(y) = u$. Therefore,

$$\phi_* \zeta_* Y = g_* Y - Y(r)N - rN_* \zeta_* Y \tag{1.24}$$

and from the definition of N we conclude

$$\langle \phi_* \zeta_* Y, N \rangle = \langle g_* Y, N \rangle - Y(r) = \langle Y, \operatorname{grad} r \rangle - Y(r) = 0,$$

showing that N is a normal vector field.

From equation (1.23) we get

$$\phi_* X = r\phi_* A^{\phi}_N X,$$

that is, all vertical vectors belong to E_λ, where $\lambda = r^{-1}$. On the other hand, for any $Y \in \mathfrak{X}(V)$ we have

$$r\phi_*(A - \lambda I)\zeta_* Y = Y(r)N - g_* Y = Y(r)g_* \operatorname{grad} r + Y(r)\sqrt{1 - ||\operatorname{grad} r||^2}u - g_* Y.$$

If $\zeta_* Y \in E_\lambda$, then the equation above must be zero. From the hypothesis that $||\operatorname{grad} r|| < 1$ we get $Y(r) = 0$, and so $g_* Y = 0$. Since g is an immersion, we have $Y = 0$ and that means that $\zeta_* Y$ is a vertical vector. This shows that, on the open set where ϕ_* is injective, ϕ is a hypersurface that has a Dupin principal curvature of multiplicity $n - k$. $\square$

As said earlier the hypersurfaces we will work with are envelopes of a two-parameter congruence of hyperspheres. With the information from the focal map and radius function we can recover back the hypersurface via this theorem.

Chapter 2

Light-cone representatives of conformal deformations

In this chapter we show how nongenuine conformal deformations $\tilde{f} : M^n \to \mathbb{R}^{n+p}$ of a conformal immersion $f : M^n \to \mathbb{R}^{n+1}$ can be characterized in terms of their isometric light-cone representatives $F : M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ and $\tilde{F} : M^n \to \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$, and study the structure of the second fundamental form of the isometric light-cone representative of a genuine conformal deformation.

2.1 Characterizing nongenuine conformal deformations

Let $f : M^n \to \mathbb{R}^{n+1}$ and $\tilde{f} : M^n \to \mathbb{R}^{n+p}$ be conformal immersions. The following result of [13] characterizes, in terms of their isometric light-cone representatives $F : M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ and $\tilde{F} : M^n \to \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$, when $\tilde{f}$ is the composition $\tilde{f} = h \circ f$ of f with a conformal immersion $h : V \to \mathbb{R}^{n+p}$ of an open subset $V \supset f(M^n)$ of $\mathbb{R}^{n+1}$.

Proposition 2.1 (Proposition 2 in [13]). *Let $f\colon M^n \to \mathbb{R}^{n+1}$ and $\tilde{f}\colon M^n \to \mathbb{R}^{n+p}$ be conformal immersions. Endow M^n with the metric induced by f. Consider $F\colon M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ and $\tilde{F}\colon M^n \to \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$ the light-cone representatives of f and $\tilde{f}$, respectively. Given an open set $U \subset M^n$, there exists a conformal immersion $h : V \to \mathbb{R}^{n+p}$ of an open subset $V \supset f(U)$ of $\mathbb{R}^{n+1}$ such that $\tilde{f}|_U = h \circ f|_U$ if and only if there exists an isometric immersion $H : W \to \mathbb{V}^{n+p+1}$ of an open subset $W \supset F(U)$ of $\mathbb{V}^{n+2}$ such that $\tilde{F}|_U = H \circ F|_U$.*

Proof. We will first prove the sufficiency part. If $H : W \to \mathbb{V}^{n+p+1}$ is an isometric immersion of an open subset $W \supset F(U)$ of $\mathbb{V}^{n+2}$ such that $\tilde{F}|_U = H \circ F|_U$, define $V = \Psi^{-1}(W)$ and consider $H \circ \Psi : V \to \mathbb{V}^{n+p+1}$. Then $h = \mathcal{C}(H \circ \Psi) : V \to \mathbb{R}^{n+p}$ is a conformal immersion and

$$\tilde{f}|_U = \mathcal{C}(\tilde{F}|_U) = \mathcal{C}(H \circ F|_U) = \mathcal{C}(H \circ \Psi) \circ f|_U = h \circ f|_U.$$

To prove the converse, let $h : V \to \mathbb{R}^{n+p}$ be a conformal immersion of an open subset $V \supset f(U)$ of $\mathbb{R}^{n+1}$ such that $\tilde{f}|_U = h \circ f|_U$. Let $H : \Psi(V) \to \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$ be defined by

$$\mathcal{I}(h) = H \circ \Psi.$$

Then

$$\mathcal{C}(H \circ F|_U) = \mathcal{C}(H \circ \Psi) \circ f|_U = h \circ f|_U = \tilde{f}|_U.$$

Therefore, from Proposition 1.4, we get $\tilde{F}|_U = H \circ F|_U$. Now, extend H to an isometric immersion $H\colon W \subset \mathbb{V}^{n+2} \to \mathbb{V}^{n+p+1}$ by setting $H(t\Psi(x)) = tH(\Psi(x))$ for any $x \in V$. $\square$

In order to apply Proposition 2.1, one must have sufficient conditions on a pair of isometric immersions $F\colon M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ and $\tilde{F}\colon M^n \to \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$ which imply the existence of an isometric immersion $H : W \to \mathbb{V}^{n+p+1}$ of an open subset $W \supset F(M^n)$ of $\mathbb{V}^{n+2}$ such that $\tilde{F} = H \circ F$. This is the content of the following lemma in the case of interest for us in this work, namely, the case $p = 2$.

Lemma 2.2. *Let $F : M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ and $\tilde{F} : M^n \to \mathbb{V}^{n+3} \subset \mathbb{L}^{n+4}$ be isometric immersions, and suppose that F is an embedding. Assume that there exist $\xi \in \Gamma(N_{\tilde{F}}M)$ of unit length and a vector bundle isometry $T : N_F M \to L = \{\xi\}^\perp$, which is parallel in the induced connection on L and satisfies $TF = \tilde{F}$, such that*

(i) $\langle \xi, \tilde{F} \rangle = 0$,

(ii) $\operatorname{rank} A^{\tilde{F}}_\xi = 1$,

(iii) ${}^{\tilde{F}}\nabla^\perp_Z \xi = 0$ *for all* $Z \in \ker A^{\tilde{F}}_\xi$,

(iv) $\alpha_{\tilde{F}} = T \circ \alpha_F + \langle A^{\tilde{F}}_\xi , \rangle \xi$.

Then, there exists an isometric immersion $H : W \to \mathbb{V}^{n+3}$ of an open subset $W \subset \mathbb{V}^{n+2}$ containing $F(M^n)$ such that $\tilde{F} = H \circ F$.

Proof. Let $Y \in (\ker A_\xi)^\perp$ be an eigenvector of A_ξ having β as the unique non-zero eigenvalue. Define the subspace

$$W = \operatorname{span}\left\{ \left(\tilde{F}^*\tilde{\nabla}_X\xi\right)_{\tilde{F}_*TM\oplus L} : X \in \mathfrak{X}(M) \right\}.$$

If $X \in \ker A_\xi$, then from item (iii) we get

$$\tilde{F}^*\tilde{\nabla}_X\xi = -\tilde{F}_*A_\xi X + \nabla^\perp_X\xi = 0.$$

Taking into account that $\nabla^\perp_Z\xi \in \Gamma(L)$ for any $Z \in \mathfrak{X}(M)$, we arrive at the conclusion that W is a one-dimensional subbundle of $\mathbb{R}(\tilde{F}_*Y) \oplus L$ and it is spanned by the vector field $-\beta\tilde{F}_*Y + \nabla^\perp_Y\xi$.

Let Γ be the orthogonal complement of W in $\mathbb{R}(\tilde{F}_*Y) \oplus L$, so it is a 3-rank vector subbundle. From the expression of W we have $\Gamma \cap \tilde{F}_*TM = \{0\}$. Notice that for any section δ of Γ, we have $\tilde{F}^*\tilde{\nabla}_X\delta \in \tilde{F}_*TM \oplus L$, because

$$\left\langle \tilde{F}^*\tilde{\nabla}_X\delta, \xi \right\rangle = \left\langle \delta, \tilde{F}^*\tilde{\nabla}_X\xi \right\rangle = 0.$$

Since the position vector field $\tilde{F}$ is parallel in the normal connection and is everywhere orthogonal to ξ by condition (i), it is a section of Γ.

Define

$$\mathcal{T} : F_*TM \oplus N_FM \to \tilde{F}_*TM \oplus L$$

by

$$\mathcal{T} = \tilde{F}_*F_*^{-1} \oplus T.$$

Observe that $\mathcal{T}$ is a vector bundle isometry, because F, $\tilde{F}$ are isometric immersions and T is an isometry. Set $\Omega = \mathcal{T}^{-1}(\Gamma)$. Since $\Gamma \cap \tilde{F}_*TM = \{0\}$, we see that Ω is transversal to F_*TM. Also, from our assumption on T, we conclude that the position vector field F is a section of Ω. Because F is an embedding, the map $G : \Omega \to \mathbb{L}^{n+3}$ defined by

$$G(\beta(x)) = F(x) + \beta(x)$$

parametrizes a tubular neighborhood of $F(M^n)$ if restricted to a neighborhood U of the 0-section of Ω. Give U the Lorentzian metric induced by G. For a vertical vector $Z \in T_{\beta(x)}\Omega$ we have

$$G_*(\beta(x))Z = Z,$$

while for a non-vertical vector $Z \in T_{\beta(x)}\Omega$ we get

$$\begin{aligned} G_*(\beta(x))Z &= F_*\pi_*Z + F^*\tilde{\nabla}_{\pi_*Z}(F_*X + \eta) \\ &= F_*\left(\pi_*Z + \nabla_{\pi_*Z}X - A^F_\eta \pi_*Z\right) + \alpha^F(\pi_*Z, X) +^F \nabla^\perp_{\pi_*Z}\eta, \end{aligned}$$

for $F_*X + \eta \in \Gamma(\Omega)$.

We claim that the map $\tilde{G} : \Omega \to \mathbb{L}^{n+4}$ defined by

$$\tilde{G}(\beta(x)) = \tilde{F}(x) + \mathcal{T}(\beta(x))$$

is an isometric immersion on U. This fact follows from

$$\tilde{G}_*(\beta(x))Z = TZ$$

for any vertical $Z \in T_{\beta(x)}\Omega$, while for a non-vertical $Z \in T_{\beta(x)}\Omega$, taking into account that $\tilde{F}^*\tilde{\nabla}_X\delta \in \tilde{F}_*TM \oplus L$ for any $\delta \in \Gamma$, that T is parallel in the induced connection of L and condition (iv), we get

$$\begin{aligned} \tilde{G}_*(\beta(x))Z &= \tilde{F}_*\pi_*Z + \tilde{F}^*\tilde{\nabla}_{\pi_*Z}(\tilde{F}_*X + T\eta) \\ &= \tilde{F}_*\left(\pi_*Z + \nabla_{\pi_*Z}X - A^{\tilde{F}}_{T\eta}\pi_*Z\right) + \alpha^{\tilde{F}}_L(\pi_*Z, X) + ({}^{\tilde{F}}\nabla^\perp_{\pi_*Z}T\eta)_L \\ &= \tilde{F}_*F_*^{-1}F_*\left(\pi_*Z + \nabla_{\pi_*Z}X - A^F_\eta\pi_*Z\right) + T\left(\alpha^F(\pi_*Z, X) +^F \nabla^\perp_{\pi_*Z}\eta\right). \end{aligned}$$

Therefore, $||\tilde{G}_*(\beta(x))Z|| = ||G_*(\beta(x))Z||$, and our claim follows.

Now define $H\colon G(U) \subset \mathbb{L}^{n+3} \to \mathbb{L}^{n+4}$ by

$$H = \tilde{G}|_U \circ (G|_U)^{-1}.$$

The map H is an isometric immersion and $\tilde{F} = H \circ F$.

Define an open set in $\mathbb{V}^{n+2}$ by $W = G(U) \cap \mathbb{V}^{n+2}$. Because $F(M^n) \subset G(U)$ and $F : M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$, it is clear that $F(M^n) \subset W$. The only thing left to prove is that $H(W) \subset \mathbb{V}^{n+3}$. To see this, choose local sections δ_1, δ_2 of Γ such that $\{\tilde{F}, \delta_1, \delta_2\}$ is a frame for Γ. Then $\{F, \bar{\delta}_1, \bar{\delta}_2\}$, where $\mathcal{T}(\bar{\delta}_i) = \delta_i$, is a frame for Ω. From the definition of G and because $G(U)$ is a tubular neighborhood of $F(M^n)$, we may write $G : U \times I^3 \to \mathbb{L}^{n+3}$ as

$$G(x, t, s_1, s_2) = (1 + t)F(x) + s_1\bar{\delta}_1 + s_2\bar{\delta}_2$$

and $\tilde{G} : U \times I^3 \to \mathbb{L}^{n+4}$ as

$$\tilde{G}(x,t,s_1,s_2) = (1+t)\tilde{F}(x) + s_1\delta_1 + s_2\delta_2,$$

where I is an interval containing zero. Since $\tilde{G} = H \circ G$, we have

$$\langle \delta_1, \delta_2 \rangle = \langle \tilde{G}_* \partial_{s_1}, \tilde{G}_* \partial_{s_2} \rangle = \langle G_* \partial_{s_1}, G_* \partial_{s_2} \rangle = \langle \bar{\delta}_1, \bar{\delta}_2 \rangle$$

and

$$\langle \tilde{F}, \delta_i \rangle = \langle \tilde{G}_* \partial_t, \tilde{G}_* \partial_{s_i} \rangle = \langle G_* \partial_t, G_* \partial_{s_i} \rangle = \langle F, \delta_i \rangle.$$

Hence, $\langle H(G), H(G) \rangle = \langle \tilde{G}, \tilde{G} \rangle = \langle G, G \rangle$, which implies that $H(W) \subset \mathbb{V}^{n+3}$ as we wanted. $\square$

We will also need the following slightly more general version of Lemma 2.2.

Lemma 2.3. *Let $\tilde{F} : M^n \to \mathbb{V}^{n+3} \subset \mathbb{L}^{n+4}$ be an isometric immersion. Assume there exist $\xi \in \Gamma(N_{\tilde{F}}M)$ of unit length such that*

(i) $\langle \xi, \tilde{F} \rangle = 0$,

(ii) $\operatorname{rank} A^{\tilde{F}}_{\xi} = 1$,

(iii) ${}^{\tilde{F}}\nabla^{\perp}_{Z}\xi = 0$ *for all* $Z \in \ker A^{\tilde{F}}_{\xi}$.

Suppose further that the vector subbundle $L = \{\xi\}^{\perp}$, the connection on L induced by the normal connection of $\tilde{F}$, and the L-valued symmetric bilinear form $\alpha_L = \pi_L \circ \alpha^{\tilde{F}}$, satisfy the Gauss, Codazzi and Ricci equations for an isometric immersion of M^n into $\mathbb{L}^{n+3}$. Then, there exist an open set $V \subset M^n$ and locally isometric immersions $F : V \subset M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ and $H : W \subset \mathbb{V}^{n+2} \to \mathbb{V}^{n+3}$ with $F(V) \subset W$, such that $\tilde{F} = H \circ F|_V$.

Proof. Let $U \subset M^n$ be a simply connected open set. By the Fundamental Theorem of Submanifolds, there exists an isometric immersion $F : U \to \mathbb{L}^{n+3}$ and a vector bundle isometry $\phi : L \to N_F U$ such that

$$\alpha^F = \phi \circ \alpha_L \quad \text{and} \quad {}^F\nabla^{\perp}\phi = \phi({}^{\tilde{F}}\nabla^{\perp})_L. \tag{2.1}$$

From the definition of the vector bundle L and item (i), we have that the position vector field $\tilde{F}$ is a section of L. Taking that information into account, we get

$$F^*\tilde{\nabla}_X \phi(\tilde{F}) = -F_* A_{\phi(\tilde{F})} X +^F \nabla^{\perp}_X \phi(\tilde{F}) = F_* X.$$

Therefore, the section $F - \phi(\tilde{F})$ is constant and equal to $P_0 \in \mathbb{L}^{n+3}$. Since ϕ is a vector bundle isometry and $\tilde{F}$ is a light-like section, it follows that $F - P_0 \in \mathbb{V}^{n+2}$. Without loss of generality we may assume that $P_0 = 0$ and so $\phi(\tilde{F}) = F$.

Define $T : N_F M \to L$ by $T \circ \phi = I$. Since $N_F U$ and L have the same dimension and $T : N_F U \to L$, $\phi : L \to N_F U$ are vector bundle isometries with $T \circ \phi = I$, we have $\phi \circ T = I$. Then

$$\phi({}^{\tilde{F}}\nabla^{\perp} T)_L =^F \nabla^{\perp}(\phi \circ T) =^F \nabla^{\perp}$$

and $TF = \tilde{F}$. Moreover, applying T to both sides of the last equation, we get

$$({}^{\tilde{F}}\nabla^{\perp} T)_L = T({}^F\nabla^{\perp}),$$

which means that T is parallel in the induced connection. From equation (2.1) we get

$$\alpha^{\tilde{F}}(X, Y) = \pi_L \circ \alpha^{\tilde{F}}(X, Y) + \langle A_\xi X, Y \rangle \xi = T \circ \alpha^F(X, Y) + \langle A_\xi X, Y \rangle \xi.$$

We finish by applying the last lemma to $F|_V$, where $V \subset U$ is an open set where $F|_V$ is an embedding. $\square$

Remark 2.4. An observation is in order when we apply this lemma later. Vector fields $\xi \in \Gamma(N_F M)$ and $\zeta \in \Gamma(L)$ will be called correspondent if $\phi(\zeta) = \xi$. From equation (2.1) we have that their shape operators are the same, that is, $A^F_\xi = A^{\tilde{F}}_\zeta$ and $\phi({}^{\tilde{F}}\nabla^{\perp}\zeta)_L =^F \nabla^{\perp}\xi$. Since ϕ is a vector bundle isometry, correspondent orthonormal (respectively, pseudo-orthonormal) frames will be orthonormal (respectively, pseudo-orthonormal) frames.

2.2 Structure of the second fundamental form

Let $f : M^n \to \mathbb{R}^{n+1}$ be a hypersurface with a principal curvature λ of multiplicity $n-2$. Assume that f is neither a Sbrana-Cartan nor a Cartan hypersurface and admits a genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$. Our aim in this section is to describe the structure of the second fundamental form of the isometric light-cone representative $\tilde{F} = \mathcal{I}(\tilde{f}) : M^n \to \mathbb{V}^{n+3} \subset \mathbb{L}^{n+4}$ of $\tilde{f}$.

We will make use of the following equivalent forms of a basic result on flat bilinear forms known as the Main Lemma.

Lemma 2.5 (Main Lemma in [8]). *Let $\beta : V^n \times V^n \to W^{p,q}$ be a symmetric flat bilinear* *form such that* $\mathcal{S}(\beta) = W^{p,q}$. *If* $p \leq 5$ *and* $p+q<n$, *then*

$$\dim \mathcal{N}(\beta) \geq \dim V - \dim W = n-p-q.$$

Lemma 2.6 (Main Lemma bis in [8])**.** *Let* $\beta : V^n \times V^n \to W^{p,q}$, $1 \leq p \leq 5$ *and* $p+q<n$, *be a symmetric flat bilinear form. If* $\dim \mathcal{N}(\beta) \leq n-p-q-1$, *then there is an orthogonal descomposition*

$$W^{p,q} = W_1^{l,l} \oplus W_2^{p-l,q-l}, \; 1 \leq l \leq p,$$

such that the W_j*-components* β_j *of* β *satisfy:*

1. β_1 *is non-zero and null .*
2. β_2 *is flat and* $\dim \mathcal{N}(\beta_2) \geq \dim V - \dim W_2$.

The remaining of this section is devoted to prove the following result.

Proposition 2.7. *Let* $f : M^n \to \mathbb{R}^{n+1}$, $n \geq 6$, *be an oriented hypersurface having a principal curvature* $\lambda \in \mathbb{R}$ *of constant multiplicity* $n-2$ *with respect to a unit normal vector field* N. *Assume that* f *is neither a Cartan nor a Sbrana-Cartan hypersurface on any open subset of* M^n *and that there exists a genuine conformal deformation* $\tilde{f} : M^n \to \mathbb{R}^{n+2}$ *of* f. *Then the following assertions hold for its isometric light-cone representative* $\tilde{F} = \mathcal{I}(\tilde{f}) : M^n \to \mathbb{V}^{n+3} \subset \mathbb{L}^{n+4}$*:*

(i) With possibly the exception of a set with empty interior or in the boundary points, at each point y *of a closed subset* $\mathcal{V} \subset M^n$, *there exist a pseudo-orthonormal basis* ζ_0,ζ_1,ζ_2,$\tilde{F}$ *of* $N_{\tilde{F}}M(y)$, *with*

$$\langle \zeta_2, \zeta_2 \rangle = 0, \; \langle \zeta_2, \tilde{F} \rangle = 1$$

such that the component of $\alpha^{\tilde{F}}$ *with respect to* $L = span\{\zeta_2, \tilde{F}\}$ *satisfies*

$$\alpha_L^{\tilde{F}}(X,Y) = -\langle X,Y \rangle \zeta_2 \tag{2.2}$$

for all X,$Y \in T_yM$, *and* $\ker A \cap \ker A_{\zeta_0} \cap \ker A_{\zeta_1}$ *has dimension* $n-2$, *where* A *is the shape operator of* f *with respect to* N.

(ii) For each $x \in \mathcal{U} = M^n - \mathcal{V}$ *there exist a space-like vector* $\mu \in N_{\tilde{F}}M(x)$ *of unit length and a flat bilinear form* $\gamma : T_xM \times T_xM \to span\{\mu\}^\perp$ *such that*

$$\alpha^{\tilde{F}}(X,Y) = \langle AX,Y \rangle \mu + \gamma(X,Y) \tag{2.3}$$

for all $X,Y \in T_xM$. Moreover, $\lambda = -\langle \mu, \tilde{F} \rangle^{-1}$ is a non-zero principal curvature of f and $\Delta = \mathcal{N}(\gamma)$ is the $(n-2)$-dimensional eigenspace E_λ of λ.

Proof. Differentiating $\tilde{F} = \varphi^{-1}(\Psi \circ \tilde{f})$ we get

$$\tilde{F}_* X = X(\varphi^{-1})(\Psi \circ \tilde{f}) + \varphi^{-1}\Psi_* \tilde{f}_* X.$$

Thus, the normal bundle $N_{\tilde{F}}M$ of $\tilde{F}$ splits orthogonally as

$$N_{\tilde{F}}M = \Psi_* N_{\tilde{f}}M \oplus \mathbb{L}^2$$

where $\mathbb{L}^2$ is a Lorentzian plane bundle having the position vector field $\tilde{F}$ as a section. Thus, there exist unique sections ξ and η of $\mathbb{L}^2$ such that

$$\langle \xi, \xi \rangle = -1, \quad \langle \xi, \eta \rangle = 0 \quad \text{and} \quad \langle \eta, \eta \rangle = 1$$

and $\tilde{F}$ is a multiple of $\xi + \eta$.

At any $x \in M^n$, endow $W(x) = N_fM(x) \oplus N_{\tilde{F}}M(x)$ with the indefinite metric of type $(2,3)$ given by

$$\langle\langle\, , \rangle\rangle_{W(x)} = \langle\, , \rangle_{N_fM(x)} - \langle\, , \rangle_{N_{\tilde{F}}M(x)}\,.$$

Define a symmetric bilinear form by

$$\beta = \alpha^f \oplus \alpha^{\tilde{F}} : T_xM \times T_xM \to W(x).$$

From

$$\left\langle \alpha^{\tilde{F}}(X,Y), \tilde{F} \right\rangle = -\langle X, Y \rangle$$

we deduce that $\mathcal{N}(\alpha^{\tilde{F}}) = \{0\}$, and from $\mathcal{N}(\beta) \le \mathcal{N}(\alpha^{\tilde{F}})$ we conclude that β has a trivial kernel. Moreover, using the Gauss equations for f and $\tilde{F}$, we have

$$\begin{aligned}
\langle \beta(X,Y), \beta(Z,W) \rangle - \langle \beta(X,W), \beta(Z,Y) \rangle& \\
&= \left\langle \alpha^f(X,Y), \alpha^f(Z,W) \right\rangle - \left\langle \alpha^{\tilde{F}}(X,Y), \alpha^{\tilde{F}}(Z,W) \right\rangle \\
&\quad - \left\langle \alpha^f(X,W), \alpha^f(Z,Y) \right\rangle + \left\langle \alpha^{\tilde{F}}(X,W), \alpha^{\tilde{F}}(Z,Y) \right\rangle \\
&= \langle R(X,Z)W, Y \rangle - \langle R(X,Z)W, Y \rangle \\
&= 0.
\end{aligned}$$

Therefore β is flat symmetric bilinear form.

From Lemma 2.6 for the case $(p,q)=(2,3)$, and since $n\geq 6$, it follows that $\mathcal{S}(\beta)$ is degenerate, that is, the isotropic vector subspace

$$\Omega=\mathcal{S}(\beta)\cap\mathcal{S}(\beta)^{\perp}$$

is non-trivial. Since the metric $\langle\langle\,,\rangle\rangle$ is positive definite on $W_1=\operatorname{span}\{N,\xi\}$ and negative definite on $W_2=\operatorname{span}\{\Psi_*N_1^{\tilde f},\Psi_*N_2^{\tilde f},\eta\}$, where $N_1^{\tilde f}$,$N_2^{\tilde f}$ is an orthonormal frame of $N_{\tilde f}M$, the orthogonal projections $P_1:W\to W_1$ and $P_2:W\to W_2$ map Ω isomorphically onto $P_1(\Omega)$ and $P_2(\Omega)$, respectively.

From the fact that (Lemma 22 in [17])

$$\dim\mathcal{S}(\beta)+\dim\mathcal{S}(\beta)^{\perp}=5,$$

it follows that $\dim\Omega=1$ or $\dim\Omega=2$. Our first step is to show that our assumption that $\tilde f$ is a genuine conformal deformation of f implies that the second possibility can not occur at any point of M^n.

Assume first that $\dim\Omega=2$ and that β is null on some open subset $U\subset M^n$. Then $P_1|_\Omega$ is an isomorphism onto W_1 along U, due to dimensional reasons. Let $\zeta\in\Omega$ be such that $P_1(\zeta)=\xi$. Since

$$\big\langle\zeta,\zeta\big\rangle=0=\big\langle\beta(X,Y),\zeta\big\rangle=-\big\langle\alpha^{\tilde F}(X,Y),\zeta\big\rangle,$$

then ζ is a light-like vector in $\mathcal{S}(\alpha^{\tilde F})^{\perp}$. Moreover, from

$$\big\langle\alpha^{\tilde F}(X,Y),\tilde F\big\rangle=-\big\langle X,Y\big\rangle,$$

we conclude that $\tilde F$ and $\zeta_2=\big\langle\zeta,\tilde F\big\rangle^{-1}\zeta$ are linearly independent light-like vectors with $\big\langle\zeta_2,\tilde F\big\rangle=1$.

Let $\nu\in\Omega$ be such that $P_1(\nu)=N$. Then, $\nu=N+\tilde\mu$ where $\tilde\mu\in N_{\tilde F}U$ is a space-like vector of unit length. Since

$$0=\big\langle\beta(X,Y),N+\tilde\mu\big\rangle=\big\langle\alpha^f(X,Y),N\big\rangle-\big\langle\alpha^{\tilde F}(X,Y),\tilde\mu\big\rangle,$$

we conclude that $A_N=A^{\tilde F}_{\tilde\mu}$. Because ν, $\zeta\in\Omega$ we have

$$0=\langle\nu,\zeta\rangle=\langle\tilde\mu,\zeta\rangle=\langle\tilde\mu,\zeta_2\rangle\,.$$

Define $\mu=\tilde\mu-\big\langle\tilde\mu,\tilde F\big\rangle\zeta_2$ and choose a space-like vector $\zeta_1\in\{\mu,\zeta_2,\tilde F\}^{\perp}$ of unit length.

Then $\{\mu, \zeta_1, \zeta_2, \tilde{F}\}$ is a pseudo-orthonormal frame with respect to which the second fundamental of $\tilde{F}$ is given by

$$\alpha^{\tilde{F}}(X,Y) = \langle A_N X, Y\rangle\, \mu + \langle A_{\zeta_1} X, Y\rangle\, \zeta_1 - \langle X, Y\rangle\, \zeta_2.$$

Since β is a null bilinear symmetric form, then

$$\langle \beta(X,Y), \beta(X,Y)\rangle = 0.$$

Using this fact and the expression of the second fundamental form of $\tilde{F}$ we conclude that $A_{\zeta_1} = 0$, and therefore

$$\alpha^{\tilde{F}}(X,Y) = \langle A_N X, Y\rangle\, \mu - \langle X, Y\rangle\, \zeta_2.$$

Because $A_{\zeta_1} = 0$, from the Codazzi equation of f and $\tilde{F}$ for $A_N = A_\mu$ we get

$$\left\langle \nabla^{\perp}_X \mu, \zeta_2\right\rangle Y = \left\langle \nabla^{\perp}_Y \mu, \zeta_2\right\rangle X.$$

Hence, $\left\langle \nabla^{\perp}_X \mu, \zeta_2\right\rangle = 0$.

From the Codazzi equation for $A_{\zeta_1} = 0$, we arrive to

$$\left\langle \nabla^{\perp}_X \zeta_1, \mu\right\rangle A_N Y - \left\langle \nabla^{\perp}_X \zeta_1, \zeta_2\right\rangle Y = \left\langle \nabla^{\perp}_Y \zeta_1, \mu\right\rangle A_N X - \left\langle \nabla^{\perp}_Y \zeta_1, \zeta_2\right\rangle X.$$

Picking an orthonormal frame of eigenvectors $X_1, \cdots, X_n$ of A with eigenvalues $\lambda_1, \cdots, \lambda_n$ respectively, we conclude for $i \neq j$

$$\lambda_j \left\langle \nabla^{\perp}_{X_i} \zeta_1, \mu\right\rangle = \left\langle \nabla^{\perp}_{X_i} \zeta_1, \zeta_2\right\rangle.$$

If $\left\langle \nabla^{\perp}_{X_i} \zeta_1, \zeta_2\right\rangle \neq 0$ for some $i = 1, \cdots, n$, then we would have a principal curvature of multiplicity at least $n-1$, a contradiction, so $\left\langle \nabla^{\perp}_{X_i} \zeta_1, \zeta_2\right\rangle = 0$ for all $i = 1, \cdots, n$. Since there are at least two non-zero principal curvatures, we also have $\left\langle \nabla^{\perp}_{X_i} \zeta_1, \mu\right\rangle = 0$ for $i = 1, \cdots, n$.

From the information we deduced in the last two paragraphs, we conclude that μ,ζ_1,ζ_2 and $\tilde{F}$ are parallel normal sections. Let $\bar{f}\colon U \to \mathbb{R}^{n+2}$ be the composition of $f|_U$ with a totally geodesic inclusion $i\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$. Then the second fundamental form of its isometric light-cone representative $\bar{F}\colon U \to \mathbb{V}^{n+3} \subset \mathbb{L}^{n+4}$ is given by

$$\alpha^{\bar{F}}(X,Y) = \left\langle A^f_N X, Y\right\rangle \Psi_* i_* N - \left\langle X, Y\right\rangle w.$$

Let $\bar{N}$ be a unit normal vector field to i along $f|_U$. Then, the vector bundle isometry $\tau\colon N_{\bar{F}}U \to N_{\tilde{F}}U$ given by

$$\tau\Psi_* i_* N = \mu, \;\; \tau\Psi_*\bar{N} = \zeta_1, \;\; \tau\bar{F} = \tilde{F} \;\text{ and }\; \tau w = \zeta_2$$

is parallel and satisfies $\tau\alpha^{\bar{F}} = \alpha^{\tilde{F}|_U}$. It follows from the Fundamental Theorem of Submanifolds (Theorem 1.10 in [8]) that $\tilde{F}|_U$ and $\bar{F}$ are congruent, and hence $\tilde{f}|_U$ is conformally congruent to $\bar{f} = i\circ f|_U$ by Proposition 9.18 in [8], which contradicts the assumption that $\tilde{f}$ is a genuine conformal deformation of f.

Now assume that $\dim\Omega = 2$ and β is not null on some open subset $U \subset M^n$. As in the previous case, there exists a pseudo-orthonormal frame $\{\mu, \zeta_1, \zeta_2, \tilde{F}\}$ with respect to which the second fundamental form of $\tilde{F}$ is given by

$$\alpha^{\tilde{F}}(X,Y) = \langle AX, Y\rangle\,\mu + \langle A_{\zeta_1}X, Y\rangle\,\zeta_1 - \langle X, Y\rangle\,\zeta_2.$$

By Lemma 2.6, we have $\dim\ker A_{\zeta_1} \geq n-1$. Since we are now assuming that β is not null, we must have $\dim\ker A_{\zeta_1} = n-1$.

From the Codazzi equation for $A = A_\mu$ we get

$$A_{\nabla^\perp_X\mu}Y = A_{\nabla^\perp_Y\mu}X.$$

Since $\tilde{F}$ is parallel, we obtain

$$\left\langle \nabla^\perp_X\mu, \zeta_1\right\rangle A_{\zeta_1}Y - \left\langle \nabla^\perp_X\mu, \zeta_2\right\rangle Y = \left\langle \nabla^\perp_Y\mu, \zeta_1\right\rangle A_{\zeta_1}X - \left\langle \nabla^\perp_Y\mu, \zeta_2\right\rangle X.$$

For X, $Y \in \ker A_{\zeta_1}$ we conclude that $\ker A_{\zeta_1} \leq \ker\omega_2$, where ω_i are the one-forms defined by $\omega_i(Y) = \left\langle \nabla^\perp_Y\mu, \zeta_i\right\rangle$ for $i = 1,2$. With this new information, for $X \in \ker A_{\zeta_1}$ and Y a unit eigenvector of A_{ζ_1} having the unique non-zero eigenvalue, we get

$$\left\langle \nabla^\perp_X\mu, \zeta_1\right\rangle A_{\zeta_1}Y = -\left\langle \nabla^\perp_Y\mu, \zeta_2\right\rangle X.$$

Therefore, $\omega_2 = 0$ and $\ker A_{\zeta_1} \leq \ker\omega_1$.

Let $F\colon M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ be the isometric light-cone representative of $f\colon M^n \to \mathbb{R}^{n+1}$, whose second fundamental form is given by

$$\alpha^F(X,Y) = \langle AX, Y\rangle\,\Psi_* N - \langle X, Y\rangle\,w$$

for all $X,Y\in\mathfrak{X}(M)$. Define a vector bundle isometry $T: N_FM\to L=\{\zeta_1\}^\perp$ by setting

$$T(F)=\tilde F,\quad T(\Psi_*N)=\mu\quad\text{and}\quad T(w)=\zeta_2.$$

Then the second fundamental forms of F and $\tilde F$ are related by

$$\alpha^{\tilde F}=T\circ\alpha^F+\langle A_{\zeta_1}\cdot,\cdot\rangle\,\zeta_1.$$

Moreover, using that $\omega_2=0$ one can easily check that T is parallel with respect to the induced connection on L. Since $\ker A_{\zeta_1}\le\ker\omega_1$, it follows from Lemma 2.2 that, restricted to any open subset $U\subset M^n$ where F is an embedding, $\tilde F|_U$ is a composition $\tilde F|_U = H\circ F|_U$ of $F|_U$ with an isometric immersion $H: W\subset\mathbb{V}^{n+2}\to\mathbb{V}^{n+3}$ with $F(U)\subset W$. By Proposition 2.1, there exists a conformal immersion $h: V\to\mathbb{R}^{n+p}$ of an open subset $V\supset f(U)$ of $\mathbb{R}^{n+1}$ such that $\tilde f=h\circ f$.

In summary, we have shown that the subspace Ω must be one-dimensional at any point of M^n. The next step is to show that β can not be not null at any point of M^n.

Assume otherwise that β is null at $x\in M^n$, and suppose first that $P_1(\Omega)=\mathrm{span}\{\xi\}$. Then, $\mathcal{S}(\beta)=\Omega$ projects onto $\mathrm{span}\{\xi\}$ under P_1. Therefore $A=0$, a contradiction with the fact that f has a principal curvature with multiplicity $n-2$.

Suppose now that $P_1(\Omega)\neq\mathrm{span}\{\xi\}$. This is equivalent to requiring that the orthogonal projection $\Pi_1: W\to N_fM$ maps Ω isomorphically onto N_fM, say, $N=\Pi_1(\nu)$ for some $\nu\in\Omega$. Set $\mu=\Pi_2(\nu)$, where $\Pi_2: W\to N_{\tilde F}M$ is the orthogonal projection onto $N_{\tilde F}M$. Then $A=A^{\tilde F}_\mu$ for $N+\mu=\nu\in\Omega=\mathcal{S}(\beta)\subset\mathcal{S}(\beta)^\perp$, and hence

$$\beta(X,Y)=\big(\alpha^f(X,Y),\alpha^{\tilde F}(X,Y)\big)=\big(\langle AX,Y\rangle N,\langle AX,Y\rangle\mu\big).$$

Therefore,

$$-\big\langle X,Y\big\rangle=\big\langle\alpha^{\tilde F}(X,Y),\tilde F\big\rangle=\big\langle AX,Y\big\rangle\big\langle\mu,\tilde F\big\rangle,$$

again a contradiction with the assumption on the multiplicity of one of the principal curvatures of f.

We have thus proved so far that $\dim\Omega=1$ and that β is not null at any point of M^n. Let $\mathcal{V}\subset M^n$ be the closed subset where $P_1(\Omega)=\mathrm{span}\{\xi\}$. We will show that item (i) in the statement holds at any $x\in\mathcal{V}$. Since $P_1(\Omega)=\mathrm{span}\{\xi\}$ at x, there exists a light-like $\zeta\in\Omega$ such that $\zeta\in\mathcal{S}(\alpha^{\tilde F})^\perp$, and from

$$\big\langle\alpha^{\tilde F}(X,Y),\tilde F\big\rangle=-\big\langle X,Y\big\rangle$$

it follows that $\tilde{F} \notin \Omega$. Defining
$$\zeta_2 = \left\langle \zeta, \tilde{F} \right\rangle^{-1} \zeta$$
we have a pseudo-orthonormal frame $\{\zeta_2, \tilde{F}\}$ for a Lorentzian plane L. The projection of the second fundamental form onto L is given by

$$\alpha_L^{\tilde{F}}(X, Y) = -\langle X, Y \rangle \zeta_2,$$

hence
$$\alpha^{\tilde{F}}(X, Y) = \langle A_{\zeta_0} X, Y \rangle \zeta_0 + \langle A_{\zeta_1} X, Y \rangle \zeta_1 - \langle X, Y \rangle \zeta_2,$$

where $\{\zeta_0, \zeta_1, \zeta_2, \tilde{F}\}$ is a pseudo-orthonormal basis of $N_{\tilde{F}} M(x)$. Since $\dim \Omega = 1$, the bilinear form
$$\hat{\beta} : T_x M \times T_x M \to \operatorname{span}\{N, \zeta_0, \zeta_1\}$$
defined by
$$\hat{\beta} = \alpha^f \oplus \left\langle \alpha^{\tilde{F}}, \zeta_0 \right\rangle \zeta_0 \oplus \left\langle \alpha^{\tilde{F}}, \zeta_1 \right\rangle \zeta_1$$
is flat an non-degenerate, hence $\dim \mathcal{N}(\hat{\beta}) \geq n - 3$. From

$$\mathcal{N}(\hat{\beta}) = \ker A \cap \ker A_{\zeta_0} \cap \ker A_{\zeta_1}$$

it follows that the principal curvature of f with multiplicity $n-2$ must be zero, and that $\mathcal{N}(\hat{\beta})$ is contained in the corresponding eigenspace. It remains to prove that $\dim \mathcal{N}(\hat{\beta}) = n - 2$ at x. In other words, it suffices to show that the case $\dim \mathcal{N}(\hat{\beta}) = n - 3$ cannot occur on any open subset.

Let us assume, by contradiction, that $\dim \mathcal{N}(\hat{\beta}) = n-3$ on some open subset $U \subset M^n$. Before advancing any further, since this part will require some work, let us start by giving an idea of what we are planning to do. We will prove with the aid of Lemma 2.3 that $\tilde{f}|_U = h \circ g$, where $g : U \to \mathbb{R}^{n+1}$ is a genuine isometric deformation of $f|_U$ and $h : V \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ is a conformal immersion of an open subset $V \supset g(U)$. But this implies that f is a Sbrana-Cartan hypersurface, which is ruled out by our hypotheses.

Denote by Δ the $(n-3)$-dimensional vector subspace $\mathcal{N}(\hat{\beta})$. Since E_λ properly contains Δ, we can pick a non-trivial $T \in E_\lambda \cap \Delta^\perp$. Hence,

$$\{0\} \neq \hat{\beta}_T(T_x M) \leq \operatorname{span}\{\zeta_0, \zeta_1\}.$$

Suppose that we have $\hat{\beta}_T(T_x M) = \operatorname{span}\{\zeta_0, \zeta_1\}$. From the well known rank-nullity theorem of linear algebra, there exists a non-trivial $X \in \ker \hat{\beta}_T \cap \Delta^\perp$. From the flatness of $\hat{\beta}$ we conclude

$$0 = \langle \hat{\beta}(X,T), \hat{\beta}(W,Z) \rangle = \langle \hat{\beta}(X,W), \hat{\beta}(T,Z) \rangle,$$

for arbitrary W, $Z \in \mathfrak{X}(U)$. Combined with the fact that $\dim \mathcal{N}(\hat{\beta}) = n-3$ we get

$$\hat{\beta}_X(T_xM) = \operatorname{span}\{N\}.$$

Again, from the rank-nullity theorem of linear algebra, pick linearly independent Y, $Z \in \ker \hat{\beta}_X \cap \Delta^\perp$. Using the flatness of $\hat{\beta}$ and the assumption regarding the dimension of $\mathcal{N}(\hat{\beta})$ we obtain

$$\{0\} \neq \hat{\beta}_Y(T_xM), \hat{\beta}_Z(T_xM) \leq \operatorname{span}\{\zeta_0, \zeta_1\}.$$

We conclude that $\dim E_\lambda \geq n-1$ since Y, Z and $T \in E_\lambda$, a contradiction. Therefore, $\hat{\beta}_T(T_xM)$ must be a one-dimensional subspace.

Redefining the original pseudo-orthonormal frame $\{\zeta_0, \zeta_1, \zeta, \tilde{F}\}$ of $N_{\tilde{F}}M$, we can suppose that

$$\hat{\beta}_T(T_xM) = \operatorname{span}\{\zeta_0\}.$$

From the rank-nullity theorem, we can choose two linearly independent vectors X, $Y \in \ker \hat{\beta}_T \cap \Delta^\perp$. From the flatness of the symmetric bilinear form $\hat{\beta}$ we get

$$\{0\} \neq \hat{\beta}_X(T_xM), \hat{\beta}_Y(T_xM) \leq \operatorname{span}\{N, \zeta_1\}.$$

Therefore, X, $Y \in \ker A_{\zeta_0}$ and from the fact that $\mathcal{N}(\hat{\beta}) \leq \ker A_{\zeta_0}$ we conclude that $\dim \ker A_{\zeta_0} = n-1$, or equivalently, $\operatorname{rank} A_{\zeta_0} = 1$. From this fact, $\dim E_\lambda = n-2$ and the non-degeneracy of $\hat{\beta}$ we have

$$A \neq \pm A_{\zeta_1}, \tag{2.4}$$

which will be a key element in proving that f is Sbrana-Cartan.

Define the symmetric bilinear form

$$\gamma = \hat{\beta} - \left\langle \alpha^{\tilde{F}}, \zeta_0 \right\rangle \zeta_0 = \alpha^f \oplus \left\langle \alpha^{\tilde{F}}, \zeta_1 \right\rangle \zeta_1 : T_xM \times T_xM \to \operatorname{span}\{N, \zeta_1\}.$$

We will show first that the bilinear form γ is non-degenerate, that is, that for all W,$Z \in \mathfrak{X}(M)$ such that $\gamma(W,Z) \neq 0$ we must find R, $S \in \mathfrak{X}(M)$ satisfying

$$0 \neq \langle \gamma(W,Z), \gamma(R,S) \rangle .$$

For all $W \in \mathfrak{X}(M)$ and $R \in \mathcal{N}(\hat{\beta}) \oplus \mathbb{R}\{T\}$ we have

$$\gamma(R, W) = 0$$

so we do not have to worry about this case. For $R, S \in \operatorname{span}\{X, Y\}$, where X and $Y \in \ker A_{\zeta_0}$ are linearly independent, we have

$$\gamma(R, S) = \big(\hat{\beta} - \big\langle \alpha^{\tilde{F}}, \zeta_0 \big\rangle \zeta_0\big)(R, S) = \hat{\beta}(R, S).$$

Suppose that $\gamma(R, S) = \hat{\beta}(R, S) \neq 0$. Because of the non-degeneracy of $\hat{\beta}$, there exist Z, $W \in \mathfrak{X}(M)$ such that

$$\big\langle \hat{\beta}(R, S), \hat{\beta}(Z, W) \big\rangle \neq 0.$$

Then,

$$\big\langle \big(\hat{\beta} - \big\langle \alpha^{\tilde{F}}, \zeta_0 \big\rangle \zeta_0\big)(R, S), \big(\hat{\beta} - \big\langle \alpha^{\tilde{F}}, \zeta_0 \big\rangle \zeta_0\big)(Z, W) \big\rangle = \big\langle \hat{\beta}(R, S), \hat{\beta}(Z, W) \big\rangle \neq 0$$

and, as consequence, we obtain the non-degeneracy of γ.

The bilinear form γ is also flat. If $R \in \mathcal{N}(\hat{\beta}) \oplus \mathbb{R}T$ and S, Z, $W \in \mathfrak{X}(M)$, then

$$\langle \gamma(R, S), \gamma(Z, W) \rangle = 0 = \langle \gamma(R, W), \gamma(Z, S) \rangle .$$

So, suppose R, S, Z, $W \in \operatorname{span}\{X, Y\}$ where X and $Y \in \ker A_{\zeta_0}$ are linearly independent. Then, from the flatness of $\hat{\beta}$ we have

$$\begin{aligned} &\langle \gamma(R, S), \gamma(Z, W) \rangle - \langle \gamma(R, W), \gamma(Z, S) \rangle \\ &\qquad = \big\langle \hat{\beta}(R, S), \hat{\beta}(Z, W) \big\rangle - \big\langle \hat{\beta}(R, W), \hat{\beta}(Z, S) \big\rangle = 0. \end{aligned}$$

Using the Main Lemma 2.5 we have

$$\dim \mathcal{N}(\gamma) \geq n - 2.$$

Since $\mathcal{N}(\gamma) = \ker A \cap \ker A_{\zeta_1}$, we conclude that $\operatorname{rank} A_{\zeta_1} \leq 2$. If $\operatorname{rank} A_{\zeta_1} \leq 1$, then using the same argument as the one used above, we conclude that α^f is flat, which means, again from the Main Lemma 2.5, that f has an $(n-1)$-dimensional relative nullity, a contradiction. Therefore, we must have $\operatorname{rank} A_{\zeta_1} = 2$. Observe that $\ker A_{\zeta_1}$ is not contained in $\ker A_{\zeta_0}$, because the existence of the vector T denies this fact.

Pick two linearly independent vectors X, $Y \in \ker A_{\zeta_0} \cap \Delta^\perp$. If $A_{\zeta_1}X$, $A_{\zeta_1}Y$ are linearly dependent, then a linear combination of X, Y would be in the kernel of A_{ζ_1}. Since $\dim\ker A_{\zeta_0} = n-1$, $\dim\ker A_{\zeta_1} = n-2$ and $\Delta \leq \ker A_{\zeta_0} \cap \ker A_{\zeta_1}$ we get $\ker A_{\zeta_1} \leq \ker A_{\zeta_0}$. This is a contradiction to the observation done in the last paragraph. Therefore, given two linearly independent vectors X, $Y \in \ker A_{\zeta_0} \cap \Delta^\perp$ we have that $A_{\zeta_1}X$, $A_{\zeta_1}Y$ spans the image of A_{ζ_1}.

Suppose $\operatorname{Img} A_{\zeta_0} \leq \operatorname{Img} A_{\zeta_1}$. Let X, Y be unit length orthogonal eigenvectors having α and β as non-zero eigenvalues for the shape operator A_{ζ_1}. Then, any vector $Z \in \Delta^\perp$ orthogonal to the plane spanned by X and Y would belong in the subspace $\ker A_{\zeta_0} \cap \ker A_{\zeta_1} \cap \Delta^\perp$, a contradiction because $\ker A_{\zeta_1}$ is not contained in $\ker A_{\zeta_0}$. Therefore, $\operatorname{Img} A_{\zeta_0} \cap \operatorname{Img} A_{\zeta_1} = \{0\}$.

Now, let us use the Codazzi equations to gain information about the normal connection. From the Codazzi equation for $A_{\zeta_2} = 0$ we have

$$A_{\nabla^\perp_Z \zeta_2} W = A_{\nabla^\perp_W \zeta_2} Z$$

for all Z, $W \in \mathfrak{X}(M)$, or equivalently, by expanding in terms of our pseudo-orthonormal base $\{\zeta_0, \zeta_1, \zeta_2, \tilde{F}\}$,

$$\left\langle \nabla^\perp_Z \zeta_2, \zeta_0 \right\rangle A_{\zeta_0} W + \left\langle \nabla^\perp_Z \zeta_2, \zeta_1 \right\rangle A_{\zeta_1} W = \left\langle \nabla^\perp_W \zeta_2, \zeta_0 \right\rangle A_{\zeta_0} Z + \left\langle \nabla^\perp_W \zeta_2, \zeta_1 \right\rangle A_{\zeta_1} Z. \tag{2.5}$$

For $Z \in \Delta$ and $W \in \ker A_{\zeta_0}$ we have

$$\left\langle \nabla^\perp_Z \zeta_2, \zeta_1 \right\rangle = 0 \quad \text{for } Z \in \Delta.$$

For two linearly independent Z, $W \in \ker A_{\zeta_0} \cap \Delta^\perp$,

$$\left\langle \nabla^\perp_Z \zeta_2, \zeta_1 \right\rangle A_{\zeta_1} W = \left\langle \nabla^\perp_W \zeta_2, \zeta_1 \right\rangle A_{\zeta_1} Z.$$

Since $\{A_{\zeta_1} Z, A_{\zeta_1} W\}$ is a basis for $\operatorname{Img} A_{\zeta_1}$, we get

$$\left\langle \nabla^\perp_Z \zeta_2, \zeta_1 \right\rangle = 0 \quad \text{for } Z \in \ker A_{\zeta_0}.$$

Now, for $Z \in \ker A_{\zeta_0} \cap \Delta^\perp$ and $W \in \mathfrak{X}(M)$, we arrive to

$$\left\langle \nabla^\perp_Z \zeta_2, \zeta_0 \right\rangle A_{\zeta_0} W = \left\langle \nabla^\perp_W \zeta_2, \zeta_1 \right\rangle A_{\zeta_1} Z.$$

Since $\operatorname{Img} A_{\zeta_0} \cap \operatorname{Img} A_{\zeta_1} = \{0\}$, we conclude that

$$\left\langle \nabla^{\perp}_{W}\zeta_2, \zeta_1 \right\rangle = 0 \quad \text{for } W \in \mathfrak{X}(M). \tag{2.6}$$

Applying the information found in the last equation to the original Codazzi equation (2.5), we arrive to

$$\left\langle \nabla^{\perp}_{Z}\zeta_2, \zeta_0 \right\rangle A_{\zeta_0} W = \left\langle \nabla^{\perp}_{W}\zeta_2, \zeta_0 \right\rangle A_{\zeta_0} Z$$

for Z, $W \in \mathfrak{X}(M)$. Therefore,

$$\left\langle \nabla^{\perp}_{W}\zeta_2, \zeta_0 \right\rangle = 0 \quad \text{for } W \in \ker A_{\zeta_0}. \tag{2.7}$$

The reader might suspect that we are trying to prove all the hypothesis of lemma 2.3.

Now, let us work with the Codazzi equation for A_{ζ_0}:

$$\nabla_Z A_{\zeta_0} W - A_{\zeta_0} \nabla_Z W - A_{\nabla^{\perp}_{Z}\zeta_0} W = \nabla_W A_{\zeta_0} Z - A_{\zeta_0} \nabla_W Z - A_{\nabla^{\perp}_{W}\zeta_0} Z,$$

or equivalently,

$$\begin{aligned} \nabla_Z A_{\zeta_0} W - A_{\zeta_0} \nabla_Z W - \left\langle \nabla^{\perp}_{Z}\zeta_0, \zeta_1 \right\rangle A_{\zeta_1} W + \left\langle \nabla^{\perp}_{Z}\zeta_0, \zeta_2 \right\rangle W \\ = \nabla_W A_{\zeta_0} Z - A_{\zeta_0} \nabla_W Z - \left\langle \nabla^{\perp}_{W}\zeta_0, \zeta_1 \right\rangle A_{\zeta_1} Z + \left\langle \nabla^{\perp}_{W}\zeta_0, \zeta_2 \right\rangle Z. \end{aligned}$$

As we have done in the first Codazzi equation let $Z \in \Delta$ and $W \in \ker A_{\zeta_0} \cap \Delta^{\perp}$, using equation (2.7) we get

$$-A_{\zeta_0} \nabla_Z W - \left\langle \nabla^{\perp}_{Z}\zeta_0, \zeta_1 \right\rangle A_{\zeta_1} W = -A_{\zeta_0} \nabla_W Z.$$

Since $\operatorname{Img} A_{\zeta_0} \cap \operatorname{Img} A_{\zeta_1} = \{0\}$, we obtain that

$$\left\langle \nabla^{\perp}_{Z}\zeta_0, \zeta_1 \right\rangle = 0 \quad \text{for } Z \in \Delta.$$

Now, if Z, $W \in \ker A_{\zeta_0} \cap \Delta^{\perp}$ are linearly independet, we have

$$-A_{\zeta_0} \nabla_Z W - \left\langle \nabla^{\perp}_{Z}\zeta_0, \zeta_1 \right\rangle A_{\zeta_1} W = -A_{\zeta_0} \nabla_W Z - \left\langle \nabla^{\perp}_{W}\zeta_0, \zeta_1 \right\rangle A_{\zeta_1} Z.$$

Again, since $\operatorname{Img} A_{\zeta_0} \cap \operatorname{Img} A_{\zeta_1} = \{0\}$, we arrive to

$$\left\langle \nabla^{\perp}_{Z}\zeta_0, \zeta_1 \right\rangle = 0 \quad \text{for } Z \in \ker A_{\zeta_0}. \tag{2.8}$$

That is all the information we need regarding the normal connection.

We claim that that the vector subbundle $L = \mathrm{span}\{\zeta_0\}^\perp$, the connection on L induced by the normal connection of $\tilde{F}$, and the L-valued symmetric bilinear form $\alpha_L = \pi_L \circ \alpha^{\tilde{F}}$, satisfy the Gauss, Codazzi and Ricci equations for an isometric immersion of M^n into $\mathbb{L}^{n+3}$.

First, let us prove the Gauss equation. Using that $\mathrm{rank}\, A_{\zeta_0} = 1$ and the isometric immersion $\tilde{F}$ satisfies that equation, we get

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= \left\langle \alpha^{\tilde{F}}(X,W), \alpha^{\tilde{F}}(Y,Z)\right\rangle - \left\langle \alpha^{\tilde{F}}(X,Z), \alpha^{\tilde{F}}(Y,W)\right\rangle \\ &= \left\langle \alpha_L^{\tilde{F}}(X,W), \alpha_L^{\tilde{F}}(Y,Z)\right\rangle - \left\langle \alpha_L^{\tilde{F}}(X,Z), \alpha_L^{\tilde{F}}(Y,W)\right\rangle.\end{aligned}$$

Therefore, the Gauss equation is trivially satisfied for bilinear symmetric form $\alpha_L = \pi_L \circ \alpha^{\tilde{F}}$.

Let us move on to the Codazzi equations. Since $\tilde{F}$ is parallel in the normal connection, it is trivial that $A_{\tilde{F}}$ satisfies the Codazzi equation for the connection $(\nabla^\perp)_L$, since $A_{\tilde{F}}$ already satisfies the Codazzi equation for the connection $\nabla^\perp$. Now, for the Codazzi equation for A_{ζ_i}, for $i = 1$, 2 we have to show

$$\nabla_X A_{\zeta_i} Y - A_{\zeta_i}\nabla_X Y - A_{(\nabla_X^\perp \zeta_i)_L} Y = \nabla_Y A_{\zeta_i} X - A_{\zeta_i}\nabla_Y X - A_{(\nabla_Y^\perp \zeta_i)_L} X.$$

Since we already have

$$\nabla_X A_{\zeta_i} Y - A_{\zeta_i}\nabla_X Y - A_{\nabla_X^\perp \zeta_i} Y = \nabla_Y A_{\zeta_i} X - A_{\zeta_i}\nabla_Y X - A_{\nabla_Y^\perp \zeta_i} X,$$

the validity of the Codazzi equations for A_{ζ_i} for $i = 1$, 2 in the connection $(\nabla^\perp)_L$ is equivalent to showing

$$\left\langle \nabla_X^\perp \zeta_i, \zeta_0\right\rangle A_{\zeta_0} Y = \left\langle \nabla_Y^\perp \zeta_i, \zeta_0\right\rangle A_{\zeta_0} X.$$

From equations (2.7) and (2.8) and because $\mathrm{rank}\, A_{\zeta_0} = 1$ the last equation is true, hence we have finished showing the validity of all the Codazzi equations.

Lastly, lets prove the Ricci equations. Since $\tilde{F}$ is parallel in the normal connection, we must only prove the Ricci equation involving ζ_1 and ζ_2. From the Ricci equation for the isometric immersion $\tilde{F}$ we have

$$\begin{aligned}
\langle [A_{\zeta_1}, A_{\zeta_2}]X, Y\rangle &= \left\langle R^{\perp}(X,Y)\zeta_1, \zeta_2\right\rangle \\
&= \left\langle \nabla^{\perp}_X\nabla^{\perp}_Y\zeta_1, \zeta_2\right\rangle - \left\langle \nabla^{\perp}_Y\nabla^{\perp}_X\zeta_1, \zeta_2\right\rangle - \left\langle \nabla^{\perp}_{[X,Y]}\zeta_1, \zeta_2\right\rangle \\
&= \left\langle \nabla^{\perp}_X\big(\left\langle \nabla^{\perp}_Y\zeta_1, \zeta_0\right\rangle\zeta_0 + \left\langle \nabla^{\perp}_Y\zeta_1, \zeta_2\right\rangle\tilde{F}\big), \zeta_2\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_Y\big(\left\langle \nabla^{\perp}_X\zeta_1, \zeta_0\right\rangle\zeta_0 + \left\langle \nabla^{\perp}_X\zeta_1, \zeta_2\right\rangle\tilde{F}\big), \zeta_2\right\rangle - \left\langle \nabla^{\perp}_{[X,Y]}\zeta_1, \zeta_2\right\rangle \\
&= \left\langle \nabla^{\perp}_Y\zeta_1, \zeta_0\right\rangle \left\langle \nabla^{\perp}_X\zeta_0, \zeta_2\right\rangle - \left\langle \nabla^{\perp}_X\zeta_1, \zeta_0\right\rangle \left\langle \nabla^{\perp}_Y\zeta_0, \zeta_2\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_{[X,Y]}\zeta_1, \zeta_2\right\rangle.
\end{aligned}$$

From equations (2.7) and (2.8) and rank $A_{\zeta_0} = 1$ we get

$$\langle [A_{\zeta_1}, A_{\zeta_2}]X, Y\rangle = -\left\langle \nabla^{\perp}_{[X,Y]}\zeta_1, \zeta_2\right\rangle,$$

which is precisely the Ricci equation for ζ_1 and ζ_2 for the connection $(\nabla^{\perp})_L$.

It follows from Lemma 2.3 that there exist an open set $V \subset M^n$ and isometric immersions $G : V \subset M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$, $H : W \subset \mathbb{V}^{n+2} \to \mathbb{V}^{n+3}$ with $G(V) \subset W$, such that $\tilde{F} = H \circ G$. From the remark 2.4, the correspondent normal vector field to $\tilde{F}$ is G. The correspondent normal vector field to ζ_2 is a light-like normal vector field that has a vanishing shape operator. It is also parallel in the normal connection, because from equation (2.6) we have $(\nabla^{\perp}_X\zeta_2)_L = 0$. Hence, it is a constant light-like vector $w \in \mathbb{V}^{n+2}$, with $\langle G, w\rangle = 1$. Therefore, we have $G(M^n) \subset \mathbb{E}^{n+1}_w$ and from Proposition 1.4, $G = \Psi \circ g$ for some isometric immersion $g\colon M^n \to \mathbb{R}^{n+1}$.

It remains to prove that g is a genuine deformation of f and hence, f is a Sbrana-Cartan hypersurface. If we denote by ξ correspondent vector fields to ζ_1 then, again from the Remark 2.4, $\{\xi, w, G\}$ is a pseudo-orthonormal frame of N_GM, where ξ is of unit length, $A_\xi = A_{\zeta_1}$, $A_w = A_{\zeta_2} = 0$, $A_G = A_{\tilde{F}} = -I$. Hence,

$$\alpha^G(X,Y) = \langle A_{\zeta_1}X, Y\rangle\, \xi - \langle X, Y\rangle\, w.$$

If f and g are isometrically congruent, then in particular they are conformally congruent. From Proposition 1.5, F and G are isometrically congruent, that is there exist an isometry $T : \mathbb{L}^{n+3} \to \mathbb{L}^{n+3}$ such that $T \circ F = G$. Therefore, $T \circ \alpha^F = \alpha^G$ and because $F = \Psi \circ f$, we have

$$\alpha^F(X,Y) = \langle AX, Y\rangle\, \Psi_* N - \langle X, Y\rangle\, w.$$

If we express

$$\begin{cases} T\Psi_* N = a_1\xi + a_2 w + a_3 G \\ Tw = b_1\xi + b_2 w + b_3 G \\ TF = G \end{cases}$$

then because $\{\Psi_* N, w, F\}$ and $\{\xi, w, G\}$ are pseudo-orthonormal basis and T is an isometry we must have $a_2 = 0$, $b_2 = 1$, $a_1^2 = 1$, $a_1 b_1 + a_3 = 0$ and $b_1^2 + 2b_3 = 0$. From the condition $T \circ \alpha^F = \alpha^G$, we get

$$\begin{cases} a_1 A - b_1 I = A_{\zeta_1} \\ a_3 A - b_3 I = 0. \end{cases}$$

Then, $a_1 = \pm 1$, $b_1 = \mp a_3$, $b_1^2 + 2b_3 = 0$ and

$$\begin{cases} \pm A - b_1 I = A_{\zeta_1} \\ \mp b_1 A - b_3 I = 0. \end{cases}$$

From the last system of equations, by multiplying the first one by b_3 and the last one by $-b_1$ we obtain $\pm(b_3 + b_1^2)A = b_3 A_{\zeta_1}$. So, using $b_1^2 + 2b_3 = 0$, we get $A = \pm A_{\zeta_1}$ or $b_3 = 0$. If $b_3 = 0$, then $b_1 = 0$ and we also get $A = \pm A_{\zeta_1}$, which is a contradiction from equation (2.4). Therefore, g is a genuine isometric deformation and f is a Sbrana-Cartan hypersurface.

Finally, we will show that the conditions in item (ii) hold at any point x of the open subset $\mathcal{U} = M^n - \mathcal{V}$ where $P_1(\Omega) \neq \operatorname{span}\{\xi\}$. The latter condition is equivalent to requiring that the orthogonal projection $\Pi_1 : W \to N_f M$ maps Ω isomorphically onto $N_f M$, say $N = \Pi_1(\nu)$ for some $\nu \in \Omega$. Set $\mu = \Pi_2(\nu)$, where $\Pi_2 : W \to N_{\tilde{F}} M$ is the orthogonal projection into $N_{\tilde{F}} M$. Then, $A^f = A^{\tilde{F}}_\mu$ where $N + \mu = \nu \in \Omega$ and

$$\alpha^{\tilde{F}}(X,Y) = \langle AX, Y \rangle \mu + \gamma(X,Y)$$

for

$$\gamma : T_x M \times T_x M \to \{\mu\}^\perp$$

flat, non-degenerate bilinear form. Hence, from Lemma 2.6, we get $\mathcal{N}(\gamma) \geq n-3$. Pick $T \in \mathcal{N}(\gamma)$, then

$$-\big\langle T, Y \big\rangle = \big\langle \alpha^{\tilde{F}}(T,Y), \tilde{F} \big\rangle = \big\langle AT, Y \big\rangle \big\langle \mu, \tilde{F} \big\rangle.$$

Therefore, $\langle \mu, \tilde{F} \rangle$ is non-zero and $\lambda = -\langle \mu, \tilde{F} \rangle^{-1}$ is a principal curvature of f. From the hypothesis regarding the multiplicity of the principal curvatures of f and item (iii) of Proposition 1.1, we conclude that λ is a Dupin principal curvature of multiplicity $n-2$ and $\mathcal{N}(\gamma) \leq E_\lambda$. This fact and the assumption on the principal curvatures multiplicity of f leave us with two possibilities: $\dim \mathcal{N}(\gamma) = n-3$ or $\dim \mathcal{N}(\gamma) = n-2$. If we show that $\dim \mathcal{N}(\gamma) = n-2$, then we will have completed the proof that item (ii) holds on $\mathcal{U}$.

So assume, by contradiction, that $\dim \mathcal{N}(\gamma) = n-3$ on some open subset, and denote $\mathcal{N}(\gamma)$ by Δ. Since Δ is contained properly in E_λ, there exist $T \in E_\lambda \cap \Delta^\perp$. Then,

$$-\langle T, Y \rangle = \langle \alpha^{\tilde{F}}(T, Y), \tilde{F} \rangle = \lambda \langle T, Y \rangle \langle \mu, \tilde{F} \rangle + \langle \gamma(T, Y), \tilde{F} \rangle$$

and we conclude that

$$\langle \gamma(T, Y), \tilde{F} \rangle = 0.$$

Therefore, $\gamma_T(T_x M)$ is orthogonal to $\tilde{F}$. If we define $\zeta = \lambda \tilde{F} + \mu$, we have

$$\langle \zeta, \zeta \rangle = -1, \quad \langle \zeta, \mu \rangle = 0, \quad \langle \gamma(T, Y), \zeta \rangle = 0$$

and $A_\zeta = A - \lambda I$. Consider an orthonormal frame $\{\mu, \zeta_1, \zeta_2, \zeta\}$ of $N_{\tilde{F}} M$.

We will prove that $\tilde{f} = h \circ g$ where $g : M^n \to \mathbb{R}^{n+1}$ is a genuine conformal deformation of f, that is, f is a Cartan Hypersurface, and $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ is a conformal immersion. We will use Lemma 2.3 to conclude those facts. The steps we will follow are similar to those where $\dim \mathcal{N}(\hat{\beta}) = n-3$.

From the facts above, since $\langle \gamma(T, Y), \zeta \rangle = 0$ and $T \in \Delta^\perp$ we obtain

$$0 \neq \gamma_T(T_x M) \leq \operatorname{span}\{\zeta_1, \zeta_2\}.$$

Suppose that we have the equality. From the rank-nullity theorem of linear algebra, there exist $X \in \ker \gamma_T \cap \Delta^\perp$. From the flatness of the bilinear form γ we get

$$0 = \langle \gamma(T, X), \gamma(Z, W) \rangle = \langle \gamma(T, W), \gamma(X, Z) \rangle$$

for Z, $W \in \mathfrak{X}(M)$. It follows that

$$\gamma_X(T_x M) \leq \operatorname{span}\{\zeta\}.$$

In fact, we have the equality, because if $\gamma_X(T_xM) = \{0\}$, then

$$-\big\langle X, Y \big\rangle = \big\langle \alpha^{\tilde{F}}(X,Y), \tilde{F} \big\rangle = \big\langle AX, Y \big\rangle \langle \mu, \tilde{F} \rangle$$

and $X \in E_\lambda$, a contradiction since E_λ would be $(n-1)$-dimensional. Now, from the rank-nullity theorem consider Y, $Z \in \ker \gamma_X \cap \Delta^\perp$ linearly independent. From the flatness of γ we obtain

$$\gamma_Y(T_xM), \gamma_Z(T_xM) \leq \operatorname{span}\{\zeta_1, \zeta_2\},$$

that is

$$\big\langle \gamma(Y,W), \tilde{F} \big\rangle = 0 = \big\langle \gamma(Z,W), \tilde{F} \big\rangle \quad \text{for} \quad W \in \mathfrak{X}(M).$$

Arguing as before, this means that Y, $Z \in E_\lambda$, a contradiction regarding the dimension of this subspace. We deduce that the subspace $\gamma_T(T_xM) \leq \operatorname{span}\{\zeta_1, \zeta_2\}$ must be one-dimensional. By suitably redefining the orthonormal frame $\{\mu, \zeta_1, \zeta_2, \zeta\}$, we can assume that $\gamma_T(T_xM) = \operatorname{span}\{\zeta_1\}$.

From the rank-nullity theorem we can pick X, $Y \in \ker \gamma_T \cap \Delta^\perp$ linearly independent. From the flatness of γ we have

$$\gamma_X(T_xM), \gamma_Y(T_xM) \leq \operatorname{span}\{\zeta_2, \zeta\},$$

that is $X, Y \in \ker A_{\zeta_1}$. Notice that $\mathcal{N}(\gamma) \leq \ker A_{\zeta_1}$ and $A_{\zeta_1}T \neq 0$, therefore $\dim\ker A_{\zeta_1} = n-1$, or equivalently, $\operatorname{rank} A_{\zeta_1} = 1$. Since $\operatorname{rank} A_{\zeta_1} = 1$ and $\operatorname{rank} A_\zeta = \operatorname{rank}(A - \lambda I) = 2$, we cannot have

$$A_{\zeta_2} \neq \pm A_\zeta, \tag{2.9}$$

otherwise γ would be degenerate.

Define the symmetric bilinear form

$$\beta = \gamma - \langle \gamma, \zeta_1 \rangle\, \zeta_1 = \langle \gamma, \zeta_2 \rangle\, \zeta_2 - \langle \gamma, \zeta \rangle\, \zeta : T_xM \times T_xM \to \operatorname{span}\{\zeta_1, \zeta\}.$$

We will show that this bilinear form is non-degenerate and flat. From the fact that $\gamma_T(T_xM) = \operatorname{span}\{\zeta_1\}$ we have

$$\beta(R, W) = 0 \quad \text{for } R \in \Delta \oplus \mathbb{R}T.$$

So, suppose $\beta(Z,W) \neq 0$ for Z, $W \in \operatorname{span}\{X, Y\}$ where X, $Y \in \ker A_{\zeta_1} \cap \Delta^\perp$ are linearly independent. Then, $\beta(Z,W) = \gamma(Z,W)$, so from the non-degeneracy of γ there exist R,

$S \in \mathfrak{X}(M)$ such that

$$\langle \beta(Z,W), \beta(R,S)\rangle = \langle \gamma(Z,W), \beta(R,S)\rangle = \langle \gamma(Z,W), \gamma(R,S)\rangle \neq 0.$$

We conclude that β is non-degenerate.

For flatness, we have to prove

$$\langle \beta(Z,W), \beta(R,S)\rangle - \langle \beta(Z,S), \beta(R,W)\rangle .$$

If $R \in \Delta \oplus \mathbb{R}T$, then $\beta_T(T_xM) = \{0\}$, so the above equation is trivially satisfied. Therefore, we can suppose R, S, Z, $W \in \operatorname{span}\{X, Y\}$ where X, $Y \in \ker A_{\zeta_1} \cap \Delta^{\perp}$ are linearly independent. Then, from the flatness of γ

$$\begin{aligned}\langle \beta(Z,W), \beta(R,S)\rangle &- \langle \beta(Z,S), \beta(R,W)\rangle \\ &= \langle \gamma(Z,W), \gamma(R,S)\rangle - \langle \gamma(Z,S), \gamma(R,W)\rangle = 0\end{aligned}$$

and we have the flatness of β.

Using the Main Lemma 2.5 we conclude that $\dim \mathcal{N}(\beta) \geq n-2$ and since $\mathcal{N}(\beta) \leq \ker A_{\zeta_2}$, we get $\operatorname{rank} A_{\zeta_2} \leq 2$. If $\operatorname{rank} A_{\zeta_2} \leq 1$, using the same argument, we conclude that $\beta - \langle \gamma, \zeta_2\rangle \zeta_2 = -\langle \gamma, \zeta\rangle \zeta$ is flat. Also, it is non-degenerate, because the metric is negative definite, thus from the Main Lemma 2.5 $\dim \mathcal{N}(A_\zeta) \geq n-1$. This is impossible, because $A_\zeta = A - \lambda I$ has rank two. Therefore, $\operatorname{rank} A_{\zeta_2} = 2$. Also $\mathcal{N}(\beta) = \ker A_{\zeta_2} \cap \ker A_\zeta$. Since $\dim \ker A_{\zeta_2} = \dim \ker A_\zeta = n-2$ we have that $\ker A_{\zeta_2} = \ker A_\zeta$. Observe that $\ker A_{\zeta_2}$ is not contained in $\ker A_{\zeta_1}$ because the vector field T denies this fact.

Pick two linearly independent vectors $X, Y \in \ker A_{\zeta_1} \cap \Delta^{\perp}$. If $A_{\zeta_2}X$, $A_{\zeta_2}Y$ are linearly dependent, then a linear combination of X, Y belongs to the $\ker A_{\zeta_2}$. This would mean that $\ker A_{\zeta_2} \leq \ker A_{\zeta_1}$, a contradiction from what we have said in the last paragraph. Therefore, given two linearly independent vectors $X, Y \in \ker A_{\zeta_1} \cap \Delta^{\perp}$, then $A_{\zeta_2}X$, $A_{\zeta_2}Y$ are linearly independent.

Suppose $\operatorname{Img} A_{\zeta_1} \leq \operatorname{Img} A_{\zeta_2}$. Let X, $Y \in \Delta^{\perp}$ orthogonal unit eigenvectors of A_{ζ_2} having α and β as non-zero eigenvalues, respectively. Let $Z \in \Delta^{\perp}$ orthogonal to X and Y. Then, $Z \in \ker A_{\zeta_1} \cap \ker A_{\zeta_2}$, which is a contradiction. So, we must conclude that $\operatorname{Img} A_{\zeta_1} \cap \operatorname{Img} A_{\zeta_2} = \{0\}$.

Now let us use the Codazzi equations to gain information about the normal covariant connection. From the Codazzi equation $A_\mu = A$ we have

$$A_{\nabla^\perp_X \mu} Y = A_{\nabla^\perp_Y \mu} X,$$

or taking into consideration that $\nabla^\perp_X \zeta = X(\lambda)\tilde{F} + \nabla^\perp_X \mu$, we get

$$\begin{aligned}
&\left\langle \nabla^\perp_X \mu, \zeta_1 \right\rangle A_{\zeta_1} Y + \left\langle \nabla^\perp_X \mu, \zeta_2 \right\rangle A_{\zeta_2} Y - \lambda^{-1} X(\lambda) A_\zeta Y \\
&\qquad = \left\langle \nabla^\perp_Y \mu, \zeta_1 \right\rangle A_{\zeta_1} X + \left\langle \nabla^\perp_Y \mu, \zeta_2 \right\rangle A_{\zeta_2} X - \lambda^{-1} Y(\lambda) A_\zeta X.
\end{aligned}$$

For $Y = R \in \Delta$ and $X \in \ker A_{\zeta_1}$, taking into account that $\Delta \leq E_\lambda$ and λ is a Dupin principal curvature, we obtain

$$0 = \left\langle \nabla^\perp_R \mu, \zeta_2 \right\rangle A_{\zeta_2} X.$$

For a suitable choice of X we conclude

$$\left\langle \nabla^\perp_R \mu, \zeta_2 \right\rangle = 0 \quad \text{for } R \in \Delta. \tag{2.10}$$

For $Y = R \in \Delta$, $X \in \mathfrak{X}(M)$ and using equation (2.10) we have

$$0 = \left\langle \nabla^\perp_R \mu, \zeta_1 \right\rangle A_{\zeta_1} X,$$

so

$$\left\langle \nabla^\perp_R \mu, \zeta_1 \right\rangle = 0, \quad \text{for } R \in \Delta. \tag{2.11}$$

The Codazzi equation for A_{ζ_1} gives

$$\nabla_X A_{\zeta_1} Y - A_{\zeta_1} \nabla_X Y - A_{\nabla^\perp_X \zeta_1} Y = \nabla_Y A_{\zeta_1} X - A_{\zeta_1} \nabla_Y X - A_{\nabla^\perp_Y \zeta_1} X.$$

Because $\left\langle \nabla^\perp_X \zeta_1, \mu \right\rangle = \left\langle \nabla^\perp_X \zeta_1, \zeta \right\rangle$ and $A_\zeta = A - \lambda I$, working a bit more on the Codazzi equation A_{ζ_1}, we get

$$\begin{aligned}
&\nabla_X A_{\zeta_1} Y - A_{\zeta_1} \nabla_X Y - \lambda \left\langle \nabla^\perp_X \zeta_1, \mu \right\rangle Y - \left\langle \nabla^\perp_X \zeta_1, \zeta_2 \right\rangle A_{\zeta_2} Y \\
&\qquad = \nabla_Y A_{\zeta_1} X - A_{\zeta_1} \nabla_Y X - \lambda \left\langle \nabla^\perp_Y \zeta_1, \mu \right\rangle X - \left\langle \nabla^\perp_Y \zeta_1, \zeta_2 \right\rangle A_{\zeta_2} X.
\end{aligned}$$

For $Y = R \in \Delta$ and $X \in \ker A_{\zeta_1}$ and using equation (2.11) we get

$$-A_{\zeta_1} \nabla_X R - \lambda \left\langle \nabla^\perp_X \zeta_1, \mu \right\rangle R = -A_{\zeta_1} \nabla_R X - \left\langle \nabla^\perp_R \zeta_1, \zeta_2 \right\rangle A_{\zeta_2} X,$$

so we conclude

$$\left\langle \nabla^{\perp}_{X}\zeta_1, \mu\right\rangle = 0 \quad \text{for } X \in \ker A_{\zeta_1} \tag{2.12}$$

and

$$\left\langle \nabla^{\perp}_{R}\zeta_1, \zeta_2\right\rangle = 0 \quad \text{for } R \in \Delta. \tag{2.13}$$

Now for X, $Y \in \ker A_{\zeta_1}$ and using equation (2.12), we have

$$-A_{\zeta_1}\nabla_X Y - \left\langle \nabla^{\perp}_{X}\zeta_1, \zeta_2\right\rangle A_{\zeta_2}Y = -A_{\zeta_1}\nabla_Y X - \left\langle \nabla^{\perp}_{Y}\zeta_1, \zeta_2\right\rangle A_{\zeta_2}X,$$

so

$$\left\langle \nabla^{\perp}_{X}\zeta_1, \zeta_2\right\rangle = 0 \quad \text{for } X \in \ker A_{\zeta_1}. \tag{2.14}$$

From equations (2.12), (2.14) and $\left\langle \nabla^{\perp}_{X}\zeta_1, \mu\right\rangle = \left\langle \nabla^{\perp}_{X}\zeta_1, \zeta\right\rangle$ we conclude that ζ_1 is parallel along $\ker A_{\zeta_1}$.

Define the rank-3 subbundle L by $L = \{\zeta_1\}^{\perp}$. We already have conditions (i)-(iii) of Lemma 2.3 satisfied. The only remaining thing to prove is that $(\alpha_L, (\nabla^{\perp})_L)$ satisfies the Gauss, Codazzi and Ricci equations.

First, let us prove the Gauss equation. Since

$$\alpha^{\tilde{F}}(X,Y) = \langle AX, Y\rangle \mu + \langle A_{\zeta_1}X, Y\rangle \zeta_1 + \langle A_{\zeta_2}X, Y\rangle \zeta_2 - \langle A_{\zeta}X, Y\rangle \zeta$$

satisfies the Gauss equation and $\operatorname{rank} A_{\zeta_1} = 1$, then the symmetric, bilinear section

$$\alpha_L(X,Y) = \langle AX, Y\rangle \mu + \langle A_{\zeta_2}X, Y\rangle \zeta_2 - \langle A_{\zeta}X, Y\rangle \zeta$$

also satisfies the Gauss equation.

Let us move on to the Codazzi equations. First, let us prove the Codazzi equation for $A_\mu = A$. We must show that

$$A_{(\nabla^{\perp}_{X}\mu)_L}Y = A_{(\nabla^{\perp}_{Y}\mu)_L}X,$$

or equivalently,

$$\left\langle \nabla^{\perp}_{X}\mu, \zeta_2\right\rangle A_{\zeta_2}Y - \left\langle \nabla^{\perp}_{X}\mu, \zeta\right\rangle A_{\zeta}Y = \left\langle \nabla^{\perp}_{Y}\mu, \zeta_2\right\rangle A_{\zeta_2}X - \left\langle \nabla^{\perp}_{Y}\mu, \zeta\right\rangle A_{\zeta}X.$$

Because,

$$A_{\nabla^{\perp}_{X}\mu}Y = A_{\nabla^{\perp}_{Y}\mu}X,$$

or expressed in term of the orthonormal frame $\{\mu, \zeta_1, \zeta_2, \zeta\}$ of $N_{\tilde{F}}M$,

$$\begin{aligned}&\left\langle \nabla^{\perp}_X \mu, \zeta_1 \right\rangle A_{\zeta_1} Y + \left\langle \nabla^{\perp}_X \mu, \zeta_2 \right\rangle A_{\zeta_2} Y - \left\langle \nabla^{\perp}_X \mu, \zeta \right\rangle A_{\zeta} Y \\ &\qquad = \left\langle \nabla^{\perp}_Y \mu, \zeta_1 \right\rangle A_{\zeta_1} X + \left\langle \nabla^{\perp}_Y \mu, \zeta_2 \right\rangle A_{\zeta_2} X - \left\langle \nabla^{\perp}_Y \mu, \zeta \right\rangle A_{\zeta} X\end{aligned}$$

we must only prove

$$\left\langle \nabla^{\perp}_X \mu, \zeta_1 \right\rangle A_{\zeta_1} Y = \left\langle \nabla^{\perp}_Y \mu, \zeta_1 \right\rangle A_{\zeta_1} X.$$

Because $\dim \ker A_{\zeta_1} = n - 1$ and ζ_1 is parallel along $\ker A_{\zeta_1}$, the above equation is valid.

The other Codazzi equations are proved in a similar way. For the sake of clarity, we will prove them. For the Codazzi equation of A_{ζ_2} we have to show

$$\nabla_X A_{\zeta_2} Y - A_{\zeta_2} \nabla_X Y - A_{(\nabla^{\perp}_X \zeta_2)_L} Y = \nabla_Y A_{\zeta_2} X - A_{\zeta_2} \nabla_Y X - A_{(\nabla^{\perp}_Y \zeta_2)_L} X.$$

Since we know that $\alpha^{\tilde{F}}$ satisfies the Codazzi equations, we have

$$\nabla_X A_{\zeta_2} Y - A_{\zeta_2} \nabla_X Y - A_{\nabla^{\perp}_X \zeta_2} Y = \nabla_Y A_{\zeta_2} X - A_{\zeta_2} \nabla_Y X - A_{\nabla^{\perp}_Y \zeta_2} X.$$

So, as before, it is enough to demonstrate

$$\langle \nabla_X \zeta_2, \zeta_1 \rangle A_{\zeta_1} Y = \langle \nabla_Y \zeta_2, \zeta_1 \rangle A_{\zeta_1} X.$$

Again, since ζ_1 is parallel along $\ker A_{\zeta_1}$ and $\dim \ker A_{\zeta_1} = n - 1$, the above equation is valid.

To show that A_ζ satisfies the Codazzi equation we must prove

$$\nabla_X A_{\zeta} Y - A_{\zeta} \nabla_X Y - A_{(\nabla^{\perp}_X \zeta)_L} Y = \nabla_Y A_{\zeta} X - A_{\zeta} \nabla_Y X - A_{(\nabla^{\perp}_Y \zeta)_L} X.$$

By the same arguments, we only have to show

$$\left\langle \nabla^{\perp}_X \zeta, \zeta_1 \right\rangle A_{\zeta_1} Y = \left\langle \nabla^{\perp}_Y \zeta, \zeta_1 \right\rangle A_{\zeta_1} X.$$

The same argument as in the last two cases shows the legitimacy of the above equation This concludes the verification of the Codazzi equations.

Let us move on to the Ricci equations. Let us start with the one involving μ and ζ_2.

Notice that

$$\begin{aligned}
\left\langle R^{\perp}(X,Y)\mu,\zeta_2\right\rangle &= \left\langle \nabla^{\perp}_X\nabla^{\perp}_Y\mu,\zeta_2\right\rangle - \left\langle \nabla^{\perp}_Y\nabla^{\perp}_X\mu,\zeta_2\right\rangle - \left\langle \nabla^{\perp}_{[X,Y]}\mu,\zeta_2\right\rangle \\
&= \left\langle \nabla^{\perp}_X\left(\left\langle \nabla^{\perp}_Y\mu,\zeta_1\right\rangle\zeta_1 + \left\langle \nabla^{\perp}_Y\mu,\zeta_2\right\rangle\zeta_2 - \left\langle \nabla^{\perp}_Y\mu,\zeta\right\rangle\zeta\right),\zeta_2\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_Y\left(\left\langle \nabla^{\perp}_X\mu,\zeta_1\right\rangle\zeta_1 + \left\langle \nabla^{\perp}_X\mu,\zeta_2\right\rangle\zeta_2 - \left\langle \nabla^{\perp}_X\mu,\zeta\right\rangle\zeta\right),\zeta_2\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_{[X,Y]}\mu,\zeta_2\right\rangle \\
&= \left\langle \nabla^{\perp}_X\zeta_1,\zeta_2\right\rangle\left\langle \nabla^{\perp}_Y\mu,\zeta_1\right\rangle + X\left\langle \nabla^{\perp}_Y\mu,\zeta_2\right\rangle - \left\langle \nabla^{\perp}_X\zeta,\zeta_2\right\rangle\left\langle \nabla^{\perp}_Y\mu,\zeta\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_X\mu,\zeta_1\right\rangle\left\langle \nabla^{\perp}_Y\zeta_1,\zeta_2\right\rangle - Y\left\langle \nabla^{\perp}_X\mu,\zeta_2\right\rangle - \left\langle \nabla^{\perp}_X\mu,\zeta\right\rangle\left\langle \nabla^{\perp}_Y\zeta,\zeta_2\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_{[X,Y]}\mu,\zeta_2\right\rangle.
\end{aligned}$$

and

$$\begin{aligned}
\left\langle R_L(X,Y)\mu,\zeta_2\right\rangle &= \left\langle (\nabla^{\perp}_X)_L(\nabla^{\perp}_Y)_L\mu,\zeta_2\right\rangle - \left\langle (\nabla^{\perp}_Y)_L(\nabla^{\perp}_X)_L\mu,\zeta_2\right\rangle - \left\langle (\nabla^{\perp}_{[X,Y]})_L\mu,\zeta_2\right\rangle \\
&= \left\langle (\nabla^{\perp}_X)_L\left(\left\langle \nabla^{\perp}_Y\mu,\zeta_2\right\rangle\zeta_2 - \left\langle \nabla^{\perp}_Y\mu,\zeta\right\rangle\zeta\right),\zeta_2\right\rangle \\
&\quad - \left\langle (\nabla^{\perp}_Y)_L\left(\left\langle \nabla^{\perp}_X\mu,\zeta_2\right\rangle\zeta_2 - \left\langle \nabla^{\perp}_X\mu,\zeta\right\rangle\zeta\right),\zeta_2\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_{[X,Y]}\mu,\zeta_2\right\rangle \\
&= X\left\langle \nabla^{\perp}_Y\mu,\zeta_2\right\rangle - \left\langle \nabla^{\perp}_X\zeta,\zeta_2\right\rangle\left\langle \nabla^{\perp}_Y\mu,\zeta\right\rangle - Y\left\langle \nabla^{\perp}_X\mu,\zeta_2\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_X\mu,\zeta\right\rangle\left\langle \nabla^{\perp}_Y\zeta,\zeta_2\right\rangle - \left\langle \nabla^{\perp}_{[X,Y]}\mu,\zeta_2\right\rangle.
\end{aligned}$$

Because $\left\langle R^{\perp}(X,Y)\mu,\zeta_2\right\rangle = \langle [A_\mu, A_{\zeta_2}]X, Y\rangle$, we must prove

$$\left\langle \nabla^{\perp}_X\zeta_1,\zeta_2\right\rangle\left\langle \nabla^{\perp}_Y\mu,\zeta_1\right\rangle - \left\langle \nabla^{\perp}_X\mu,\zeta_1\right\rangle\left\langle \nabla^{\perp}_Y\zeta_1,\zeta_2\right\rangle = 0,$$

which is true because $\dim\ker A_{\zeta_1} = n-1$ and ζ_1 is parallel along $\ker A_{\zeta_1}$.

The second equation to be proved is the Ricci equation for μ and ζ. Notice that

$$\begin{aligned}
\left\langle R^{\perp}(X,Y)\mu,\zeta\right\rangle &= \left\langle \nabla^{\perp}_X\nabla^{\perp}_Y\mu,\zeta\right\rangle - \left\langle \nabla^{\perp}_Y\nabla^{\perp}_X\mu,\zeta\right\rangle - \left\langle \nabla^{\perp}_{[X,Y]}\mu,\zeta\right\rangle \\
&= \left\langle \nabla^{\perp}_X\left(\left\langle \nabla^{\perp}_Y\mu,\zeta_1\right\rangle\zeta_1 + \left\langle \nabla^{\perp}_Y\mu,\zeta_2\right\rangle\zeta_2 - \left\langle \nabla^{\perp}_Y\mu,\zeta\right\rangle\zeta\right),\zeta\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_Y\left(\left\langle \nabla^{\perp}_X\mu,\zeta_1\right\rangle\zeta_1 + \left\langle \nabla^{\perp}_X\mu,\zeta_2\right\rangle\zeta_2 - \left\langle \nabla^{\perp}_X\mu,\zeta\right\rangle\zeta\right),\zeta\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_{[X,Y]}\mu,\zeta\right\rangle \\
&= \left\langle \nabla^{\perp}_X\zeta_1,\zeta\right\rangle\left\langle \nabla^{\perp}_Y\mu,\zeta_1\right\rangle + \left\langle \nabla^{\perp}_X\zeta_2,\zeta\right\rangle\left\langle \nabla^{\perp}_Y\mu,\zeta_2\right\rangle + X\left\langle \nabla^{\perp}_Y\mu,\zeta\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_X\mu,\zeta_1\right\rangle\left\langle \nabla^{\perp}_Y\zeta_1,\zeta\right\rangle - \left\langle \nabla^{\perp}_X\mu,\zeta_2\right\rangle\left\langle \nabla^{\perp}_Y\zeta_2,\zeta\right\rangle - Y\left\langle \nabla^{\perp}_X\mu,\zeta\right\rangle \\
&\quad - \left\langle \nabla^{\perp}_{[X,Y]}\mu,\zeta\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle R_L(X,Y)\mu,\zeta\rangle &= \left\langle(\nabla^\perp_X)_L(\nabla^\perp_Y)_L\mu,\zeta\right\rangle - \left\langle(\nabla^\perp_Y)_L(\nabla^\perp_X)_L\mu,\zeta\right\rangle - \left\langle(\nabla^\perp_{[X,Y]})_L\mu,\zeta\right\rangle\\
&= \left\langle(\nabla^\perp_X)_L\left(\left\langle\nabla^\perp_Y\mu,\zeta_2\right\rangle\zeta_2 - \left\langle\nabla^\perp_Y\mu,\zeta\right\rangle\zeta\right),\zeta\right\rangle\\
&\quad - \left\langle(\nabla^\perp_Y)_L\left(\left\langle\nabla^\perp_X\mu,\zeta_2\right\rangle\zeta_2 - \left\langle\nabla^\perp_X\mu,\zeta\right\rangle\zeta\right),\zeta\right\rangle - \left\langle\nabla^\perp_{[X,Y]}\mu,\zeta\right\rangle\\
&= \left\langle\nabla^\perp_X\zeta_2,\zeta\right\rangle\left\langle\nabla^\perp_Y\mu,\zeta_2\right\rangle + X\left\langle\nabla^\perp_Y\mu,\zeta\right\rangle - \left\langle\nabla^\perp_X\mu,\zeta_2\right\rangle\left\langle\nabla^\perp_Y\zeta_2,\zeta\right\rangle\\
&\quad - Y\left\langle\nabla^\perp_X\mu,\zeta\right\rangle - \left\langle\nabla^\perp_{[X,Y]}\mu,\zeta\right\rangle.
\end{aligned}$$

As before, it is enough to prove

$$\left\langle\nabla^\perp_X\zeta_1,\zeta\right\rangle\left\langle\nabla^\perp_Y\mu,\zeta_1\right\rangle - \left\langle\nabla^\perp_X\mu,\zeta_1\right\rangle\left\langle\nabla^\perp_Y\zeta_1,\zeta\right\rangle = 0,$$

which again is valid because of the fact that $\dim\ker A_{\zeta_1} = n-1$ and ζ_1 is parallel along $\ker A_{\zeta_1}$.

The last Ricci equation to be proved is the one involving ζ_2 and ζ. Observe that

$$\begin{aligned}
\left\langle R^\perp(X,Y)\zeta_2,\zeta\right\rangle &= \left\langle\nabla^\perp_X\nabla^\perp_Y\zeta_2,\zeta\right\rangle - \left\langle\nabla^\perp_Y\nabla^\perp_X\zeta_2,\zeta\right\rangle - \left\langle\nabla^\perp_{[X,Y]}\zeta_2,\zeta\right\rangle\\
&= \left\langle\nabla^\perp_X\left(\left\langle\nabla^\perp_Y\zeta_2,\mu\right\rangle\mu + \left\langle\nabla^\perp_Y\zeta_2,\zeta_1\right\rangle\zeta_1 - \left\langle\nabla^\perp_Y\zeta_2,\zeta\right\rangle\zeta\right),\zeta\right\rangle\\
&\quad - \left\langle\nabla^\perp_Y\left(\left\langle\nabla^\perp_X\zeta_2,\mu\right\rangle\mu + \left\langle\nabla^\perp_X\zeta_2,\zeta_1\right\rangle\zeta_1 - \left\langle\nabla^\perp_X\zeta_2,\zeta\right\rangle\zeta\right),\zeta\right\rangle\\
&\quad - \left\langle\nabla^\perp_{[X,Y]}\zeta_2,\zeta\right\rangle\\
&= \left\langle\nabla^\perp_X\mu,\zeta\right\rangle\left\langle\nabla^\perp_Y\zeta_2,\mu\right\rangle + \left\langle\nabla^\perp_X\zeta_1,\zeta\right\rangle\left\langle\nabla^\perp_Y\zeta_2,\zeta_1\right\rangle + X\left\langle\nabla^\perp_Y\zeta_2,\zeta\right\rangle\\
&\quad - \left\langle\nabla^\perp_X\zeta_2,\mu\right\rangle\left\langle\nabla^\perp_Y\mu,\zeta\right\rangle - \left\langle\nabla^\perp_X\zeta_2,\zeta_1\right\rangle\left\langle\nabla^\perp_Y\zeta_1,\zeta\right\rangle - Y\left\langle\nabla^\perp_X\zeta_2,\zeta\right\rangle\\
&\quad - \left\langle\nabla^\perp_{[X,Y]}\zeta_2,\zeta\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle R_L(X,Y)\zeta_2,\zeta\rangle &= \left\langle(\nabla^\perp_X)_L(\nabla^\perp_Y)_L\zeta_2,\zeta\right\rangle - \left\langle(\nabla^\perp_Y)_L(\nabla^\perp_X)_L\zeta_2,\zeta\right\rangle - \left\langle(\nabla^\perp_{[X,Y]})_L\zeta_2,\zeta\right\rangle\\
&= \left\langle(\nabla^\perp_X)_L\left(\left\langle\nabla^\perp_Y\zeta_2,\mu\right\rangle\mu - \left\langle\nabla^\perp_Y\zeta_2,\zeta\right\rangle\zeta\right),\zeta\right\rangle\\
&\quad - \left\langle(\nabla^\perp_Y)_L\left(\left\langle\nabla^\perp_X\zeta_2,\mu\right\rangle\mu - \left\langle\nabla^\perp_X\zeta_2,\zeta\right\rangle\zeta\right),\zeta\right\rangle - \left\langle\nabla^\perp_{[X,Y]}\zeta_2,\zeta\right\rangle\\
&= \left\langle\nabla^\perp_X\mu,\zeta\right\rangle\left\langle\nabla^\perp_Y\zeta_2,\mu\right\rangle + X\left\langle\nabla^\perp_Y\zeta_2,\zeta\right\rangle - \left\langle\nabla^\perp_X\zeta_2,\mu\right\rangle\left\langle\nabla^\perp_Y\mu,\zeta\right\rangle\\
&\quad - Y\left\langle\nabla^\perp_X\zeta_2,\zeta\right\rangle - \left\langle\nabla^\perp_{[X,Y]}\zeta_2,\zeta\right\rangle.
\end{aligned}$$

Because,

$$\left\langle R^\perp(X,Y)\zeta_2,\zeta\right\rangle = \langle[A_{\zeta_2},A_\zeta],X,Y\rangle$$

we must prove that

$$\left\langle \nabla^{\perp}_{X}\zeta_1, \zeta \right\rangle \left\langle \nabla^{\perp}_{Y}\zeta_2, \zeta_1 \right\rangle - \left\langle \nabla^{\perp}_{X}\zeta_2, \zeta_1 \right\rangle \left\langle \nabla^{\perp}_{Y}\zeta_1, \zeta \right\rangle = 0.$$

Again by the same argument, the above equation is true since $\dim\ker A_{\zeta_1} = n-1$ and ζ_1 is parallel along $\ker A_{\zeta_1}$.

It follows from Lemma 2.3 that there exist an open set $U \subset M^n$ and locally isometric immersions $G : U \subset M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ and $H : W \subset \mathbb{V}^{n+2} \to \mathbb{V}^{n+3}$ with $G(U) \subset W$, such that $\tilde{F} = H \circ G|_U$. Writing $g = \mathcal{C}(G)$, we have $G = \mathcal{I}(g)$, and from Lemma 2.1 we conclude that there exists a conformal immersion $h\colon V \to \mathbb{R}^{n+2}$ of an open subset $V \subset \mathbb{R}^{n+1}$ containing $g(U)$ such that $\tilde{f} = h \circ g|_U$.

Now, we will prove that g is a genuine conformal deformation of f. Suppose, on the contrary, that f and g are conformally congruent. Then, from Proposition 2.1, their isometric light-cone representatives F and G are isometrically congruent, that is, there exist an isometry $T : \mathbb{L}^{n+3} \to \mathbb{L}^{n+3}$ such that $G = T \circ F$. Considering the Remark 2.4, suppose that the frame correspondent to the orthonormal frame $\{\mu, \zeta_2, \zeta\}$ is $\{\xi, \eta, \theta\}$, in that order. Therefore, we have

$$\alpha^G(X,Y) = \langle AX, Y\rangle\, \xi + \langle A_{\zeta_2}X, Y\rangle\, \eta - \langle A_{\zeta}X, Y\rangle\, \theta.$$

Since the vector field correspondent to $\tilde{F}$ is G and $\zeta = \lambda\tilde{F} + \mu$, we get $\theta = \lambda G + \xi$. Expressing $T\Psi_*N$, Tw and TF in terms of the orthonormal frame $\{\xi, \eta, \theta\}$ of N_GM

$$\begin{cases} T\Psi_*N = a_1\xi + a_2\eta + a_3\theta \\ Tw = b_1\xi + b_2\eta + b_3\theta \\ TF = -\frac{1}{\lambda}\xi + \frac{1}{\lambda}\theta \end{cases}$$

and taking into account that $\{T\Psi_*N, Tw, TF\}$ is a pseudo-orthonormal frame, we obtain $a_1 = -a_3$, $-\lambda = b_1 + b_3$, $a_1^2 + a_2^2 - a_3^2 = 1$, $a_1b_1 + a_2b_2 - a_3b_3 = 0$ and $b_1^2 + b_2^2 - b_3^2 = 0$. Replacing a_3 and b_3 in those equations we get $a_2 = \pm 1$, $\pm b_2 = a_1\lambda$ and $b_2^2 = \lambda^2 + 2\lambda b_1$. Using those two equations we get $b_1 = \lambda(a_1^2 - 1)/2$, so we have all variables expressed in terms of a_1. Lets use now the condition $\alpha^G = T \circ \alpha^F$ where

$$\alpha^F(X,Y) = \langle AX, Y\rangle\, \Psi_*N - \langle X, Y\rangle\, w.$$

We obtain

$$\begin{cases} (a_1-1)A = b_1 I \\ a_2 A - b_2 I = A_{\zeta_2} \\ a_3 A - b_3 I = -A_\zeta. \end{cases}$$

The last equation is equivalent to the first one, so lets use the first two equations. Using the expression of b_1 in terms of a_1 we get

$$(a_1-1)A = \frac{\lambda(a_1^2-1)}{2} I.$$

If $a_1 = 1$, then in the second equation taking into account that $a_2 = \pm 1$ and $\pm b_2 = a_1\lambda$, we get $\pm A_\zeta = A_{\zeta_2}$. On the contrary, if $a_1 \neq 1$, we get

$$A = \frac{\lambda(a_1+1)}{2} I$$

and

$$A_\zeta = \frac{\lambda(a_1-1)}{2} I.$$

Replacing in the second equation $a_2 = \pm 1$,$b_2 = \pm a_1\lambda$ and the expression of A in terms of the identity we get

$$\mp A_\zeta = \pm \frac{\lambda(-a_1+1)}{2} I = A_{\zeta_2}.$$

So, in both cases $A_{\zeta_2} = \pm A_\zeta$, a contradiction with equation (2.9) that proves that g must be a genuine conformal deformation of f. □

Some final comments to round up the chapter are in order. Regarding item (i) of proposition 2.7, we are discarding sets of empty interior or points that are in the boundary of the closed set $\mathcal{V}$, however this does not pose any problem. During the demonstration of this item we show that in those sets f has a $(n-2)$-dimensional relative nullity. We can just compose f with an inversion (conformal map) in order to have a non-null principal curvature λ in those points and be in item (ii).

Second, if we fall into item (i), we will show that $\tilde{f} = h \circ g$, with $g : M^n \to \mathbb{R}^{n+2}$ and isometric immersion and $h : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ a conformal map. Thus, we are in fact in the isometric case treated in [12]. With the same notation of item (i) and writing the Codazzi equation in the direction of ζ_i, for i, $j = 0, 1$, $i \neq j$, $T \in \Delta = \ker A \cap \ker A_{\zeta_0} \cap \ker A_{\zeta_1}$ and $X \in \Delta^\perp$, we have

$$\nabla_T A_{\zeta_i} X - A_{\zeta_i} \nabla_T X - \left\langle \nabla_T^\perp \zeta_i, \zeta_j \right\rangle A_{\xi_j} X + \left\langle \nabla_T^\perp \zeta_i, \zeta_2 \right\rangle X = -A_{\zeta_i} \nabla_X T + \left\langle \nabla_X^\perp \zeta_i, \zeta_2 \right\rangle T$$

Taking inner product with T and since Δ is a totally geodesic distribution, we conclude

$$\left\langle \nabla^{\perp}_{X}\zeta_i, \zeta_2 \right\rangle = 0, \text{ for } X \in \Delta^{\perp},\ i = 1, 2.$$

The Codazzi equation for A_{ζ_2} for $T \in \Delta$ and $X \in \Delta^{\perp}$ gives us

$$0 = \left\langle \nabla^{\perp}_{T}\zeta_2, \zeta_0 \right\rangle A_{\zeta_0}X + \left\langle \nabla^{\perp}_{T}\zeta_2, \zeta_1 \right\rangle A_{\zeta_1}X.$$

We can then affirm that one of two cases happens: $\tilde{\rho} \in \text{span}\{\zeta_0, \zeta_1\}$ such that $A_\rho = 0$ or $\nabla^{\perp}_{X}\zeta_2 = 0$ for any X.

If we suppose that the first case is valid, then decompose $\tilde{\rho} = \Psi_*\rho + \rho_1$, with $\rho \in \Gamma(N_{\tilde{f}}M)$ and $\rho_1 \in \Gamma(\mathbb{L}^2)$, according to the orthogonal decomposition of $N_{\tilde{F}}M$ as

$$N_{\tilde{F}}M = \Psi_* N_{\tilde{f}}M \oplus \mathbb{L}^2,$$

where $\mathbb{L}^2$ is a Lorentzian plane bundle having the position vector field $\tilde{F}$ as a section. Since $\tilde{\rho}$ and $\tilde{F}$ are orthogonal, we have $\rho_1 = \left\langle \rho_1, \tilde{\zeta} \right\rangle \tilde{F}$, where $\{\tilde{\zeta}, \tilde{F}\}$ is a pseudo-orthonormal frame of $\mathbb{L}^2$. Because the $\Psi_* N_{\tilde{f}}M$-component of $\alpha^{\tilde{F}}$ is $\varphi^{-1}\Psi_*\alpha^f$, from $A_{\tilde{\rho}} = 0$ we get

$$0 = \varphi^{-1}\langle A_\rho X, Y\rangle - \langle X, Y\rangle\langle \tilde{\zeta}, \rho_1\rangle,$$

for all X, $Y \in \mathfrak{X}(M)$. In particular, since $\tilde{\rho}$ is not trivial, the normal vector field ρ can not be trivial either. We conclude that $A_\rho = \beta I$, with $\beta = \varphi\langle \tilde{\zeta}, \rho_1\rangle$.

If β vanishes, from the Codazzi equation for $\tilde{f}$ we get

$$\left\langle \nabla^{\perp}_{X}\rho, \theta \right\rangle A_\theta Y = \left\langle \nabla^{\perp}_{Y}\rho, \theta \right\rangle A_\theta X,$$

where $\{\rho, \theta\}$ is an orthonormal frame of $N_{\tilde{f}}M$. If the rank of A_θ is at least two, then $\left\langle \nabla^{\perp}_{X}\rho, \theta \right\rangle = 0$ for any $X \in T_xM$, and using Corollary 2.2 in [8] we can reduce the codimension of $\tilde{f}$, a contradiction because f is not a Cartan hypersurface. Otherwise, the relative nullity distribution of $\tilde{f}$ is greater than $n-2$. This is also a contradiction, because M^n would be conformally flat and as consequence f would have a principal curvature with multiplicity at least $n-1$ by Theorem 16.5 in [8].

If $\beta \neq 0$, then ρ cannot be parallel in the normal connection, otherwise $\tilde{f}(M)$ would be contained in an hypersphere and, as a consequence, f would be a Cartan Hypersurface (just consider $g = \tau \circ \tilde{f}$ where τ is the stereographic projection of the sphere $\mathbb{S}^{n+1}$ onto Euclidean space $\mathbb{R}^{n+1}$). On the other hand, if ρ is not parallel, then by Theorem 14 in [11]

M^n would be conformally flat, a contradiction because the multiplicity of the principal curvature λ of f is $n-2$ (Theorem 16.5 in [8]).

In the second case, if $\nabla^{\perp}_X\zeta_2 = 0$, then from $A_{\zeta_2} = 0$, we conclude that ζ_2 is a constant light-like normal vector field. Just consider the Euclidean model in the light-cone $\hat{\Psi}_{\hat{v}_0,\zeta_2,\hat{C}}$ and let $\hat{f} = \hat{\mathcal{C}}(\tilde{F}) : M^n \to \mathbb{R}^{n+2}$ be the conformal map that is defined using that model. Since $\langle \tilde{F}, \zeta_2 \rangle = 1$, in fact this is an isometric immersion. Now,

$$\tilde{f} = \mathcal{C}(\tilde{F}) = \mathcal{C}(\hat{\Psi} \circ \hat{f}) = \mathcal{C}(T \circ \Psi) \circ \hat{f},$$

so $\tilde{f}$ and $\hat{f}$ are conformal.

Lastly, regarding the hypothesis that $f : M^n \to \mathbb{R}^{n+1}$ is not a Sbrana-Cartan nor a Cartan hypersurface. Whenever we used those assumptions, in fact we proved that the genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$ was a composition $\tilde{f} = h \circ g$, with $g : M^n \to \mathbb{R}^{n+1}$ a genuine isometric or conformal deformation of f. Passing the definitions given in the paper [16], we can define an *honest conformal deformation* of f to be a genuine conformal deformation that is nowhere a composition. Using this new definition, we can remove the hypothesis of $f : M^n \to \mathbb{R}^{n+1}$ not being a Sbrana-Cartan or a Cartan hypersurface and ask $\tilde{f} : M^n \to \mathbb{R}^{n+2}$ to be a honest conformal deformation of f.

Chapter 3

The triple (D_1, D_2, ψ)

In the classification of hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ that admit genuine isometric deformations $\tilde{f} \colon M^n \to \mathbb{R}^{n+2}$, accomplished in [12], the first step was to show that, excluding some trivial cases, the existence of such a deformation is equivalent to f being a hyperbolic or elliptic hypersurface and to the existence of a pair of tensors D_1, D_2 and a one-form ψ on M^n satisfying certain equations. The aim of this chapter is to prove an analogous result in the conformal realm. As we shall see, the proof becomes significantly more involved due to the fact that working with isometric light-cone representatives requires adding two extra dimensions on the normal bundles. However this difficulty will be overcome since the second fundamental form in those directions behave nicely.

Before stating the main result of this chapter (Proposition 3.2 below), we need some definitions.

Let $f : M^n \to \mathbb{R}^{n+1}$ be a hypersurface that carries a principal curvature of multiplicity $n-2$, let Δ denote the corresponding eigenbundle and

$$C : \Gamma(\Delta) \to \Gamma(\mathrm{End}(\Delta^\perp))$$

be the splitting tensor, defined by

$$C_T X = -\nabla^h_X T,$$

with $T \in \Gamma(\Delta)$, $X \in \Gamma(\Delta^\perp)$ and $\nabla^h_X T = (\nabla_X T)^h$, where h is the projection onto $\Delta^\perp$. The hypersurface f is said to be *hyperbolic* (respectively, *parabolic* or *elliptic*) if there exists $J \in \Gamma(\mathrm{End}(\Delta^\perp))$ satisfying the following conditions:

(i) $J^2 = I$ (respectively, $J^2 = 0$, with $J \neq 0$, and $J^2 = -I$).

(ii) $\nabla^h_T J = 0$ for all $T \in \Gamma(\Delta)$.

(iii) $C_T \in \mathrm{span}\{I, J\}$ for all $T \in \Gamma(\Delta)$.

A hypersurface $f : M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, is said to be *conformally ruled* if it carries an umbilical distribution L of rank $n-1$ such that the restriction of f to each leaf of L is also umbilical.

A hypersurface $f : M^n \to \mathbb{R}^{n+1}$ is *conformally surface-like* if $f(M)$ is the image by a Möbius transformation of $\mathbb{R}^{n+1}$ of an open subset of one of the following:

1. a cylinder $M^2 \times \mathbb{R}^{n-2}$ over a surface $M^2 \subset \mathbb{R}^3$;

2. a cylinder $CM^2 \times \mathbb{R}^{n-3}$, where $CM^2 \subset \mathbb{R}^4$ denotes the cone over a surface $M^2 \subset \mathbb{S}^3$;

3. a rotation hypersurface over a surface $M^2 \subset \mathbb{R}^3_+$.

We will need the following characterization of conformally surface-like hypersurfaces, which is a consequence of a more general result of [8] (see Corollary 9.27).

Proposition 3.1 (Corollary 9.32 in [8]). *A hypersurface $f : M^n \to \mathbb{R}^{n+1}$ is* conformally surface-like *if and only if it has a principal curvature λ of multiplicity $n-2$ whose eigendistribution $\Delta = ker(A - \lambda I)$ has the property that the distribution $\Delta^\perp$ is umbilical.*

In the remaining of this chapter we prove the following result.

Proposition 3.2. *Let $f : M^n \to \mathbb{R}^{n+1}$ be an oriented hypersurface with a nowhere vanishing principal curvature λ of constant multiplicity $n-2$. Assume that f is not a Cartan hypersurface on any open subset of M^n. If f admits a genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$, then, on each connected component of an open dense subset, it is either hyperbolic or elliptic with respect to a tensor $J \in \Gamma(End(\Delta^\perp))$, where $\Delta = \ker(A - \lambda I)$, and there exist a unique (up to signs and permutation) pair (D_1, D_2) of tensors in $\Gamma(End(\Delta^\perp))$ contained in span$\{I, J\}$ and a unique one-form ψ on M^n satisfying the following conditions:*

(i) $\Delta \leq \ker \psi$,

(ii) $\det D_i = \frac{1}{2}$,

(iii) $\nabla^h_T D_i = 0 = [D_i, C_T]$ *for all* $T \in \Delta$,

(iv) $(\nabla_X (A - \lambda I) D_i) Y - (\nabla_Y (A - \lambda I) D_i) X$
$= (X \wedge Y) D_i^t grad\, \lambda + (-1)^j (A - \lambda I) \left(\psi(X) D_j Y - \psi(Y) D_j X\right),$

(v) $\langle (\nabla_Y D_i)X - (\nabla_X D_i)Y, grad\,\lambda\rangle + Hess\,\lambda(D_iX, Y) - Hess\,\lambda(X, D_iY)$
$+ (-1)^j\psi(X)\,\langle D_jY, grad\,\lambda\rangle - (-1)^j\psi(Y)\,\langle D_jX, grad\,\lambda\rangle$
$= \lambda\,(\langle AX, (A-\lambda I)D_iY\rangle - \langle (A-\lambda I)D_iX, AY\rangle),$

(vi) $d\psi(Z,T) = 0$ *for all* $Z \in \mathfrak{X}(M)$ *and* $T \in \Delta$,

(vii) $d\psi(X,Y) = \langle [(A-\lambda I)D_1, (A-\lambda I)D_2]X, Y\rangle$.

(viii) $D_2^2 \neq \pm D_1^2$.

(ix) $rank\,(D_1^2 + D_2^2 - I) = 2$.

Conversely, let $f : M^n \to \mathbb{R}^{n+1}$ be a simply connected hypersurface that is not conformally surface-like and carries a nowhere vanishing principal curvature of constant multiplicity $n-2$. If f is hyperbolic or elliptic with respect to $J \in End(\Delta^\perp)$, where $\Delta = ker(A-\lambda I)$, and there exist a triple (D_1, D_2, ψ), with $D_i \in span\{I, J\}$, satisfying items (i)-(ix), then f admits a genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$. Moreover, distinct triples (up to sign and permutation) yield non conformally congruent conformal deformations.

Remark 3.3. From the observation given in the last page of the last chapter, we do not require the principal curvature λ to be nowhere vanishing. However, we include it as part of the proposition because item (i) of proposition 2.7 gives us useful information about the behavior of the genuine deformation when λ is null. Also, as mentioned before, in the direct statement, we can remove the hypothesis about the hypersurface f not being Sbrana-Cartan or Cartan and add the hypothesis that $\tilde{f}$ is an honest deformation.

Proof. Let $\tilde{F} : M^n \to \mathbb{V}^{n+3} \subset \mathbb{L}^{n+4}$ be the isometric light-cone representative of $\tilde{f}$. Since the principal curvature λ of f with multiplicity $n-2$ is nowhere vanishing, it follows from Proposition 2.7 that for each $x \in M^n$ there exist a space-like $\mu \in N_{\tilde{F}}M(x)$ of unit length and a flat bilinear form $\gamma : T_xM \times T_xM \to \mathrm{span}\{\mu\}^\perp$ such that

$$\alpha^{\tilde{F}}(X,Y) = \langle AX, Y\rangle\,\mu + \gamma(X,Y)$$

for all X,$Y \in T_xM$. Moreover, $\lambda = -\langle \mu, \tilde{F}\rangle^{-1}$ and $\Delta = \mathcal{N}(\gamma)$ is the $(n-2)$-dimensional eigenspace E_λ of λ.

Denote $\zeta = \lambda\tilde{F} + \mu$. Then it is straightforward to see that

$$\langle \zeta, \zeta\rangle = -1 \quad \text{and} \quad \langle \zeta, \mu\rangle = 0.$$

Also, notice that $\tilde{F}$ belongs to the Lorentzian plane spanned by ζ and μ, and

$$A_\zeta = \lambda A_{\tilde{F}} + A_\mu = A - \lambda I.$$

Consider the Riemannian plane-bundle $\mathbb{P} = \{\zeta, \mu\}^\perp$. For each $\xi \in \mathbb{P}$, define

$$D_\xi = (A - \lambda I)^{-1} A_\xi = A_\zeta^{-1} A_\xi \in \Gamma(\mathrm{End}(\Delta^\perp))$$

and let

$$W = \mathrm{span}\{D_\xi : \xi \in \mathbb{P}\}.$$

Lemma 3.4. *The subspace W has dimension two on an open dense subset of M^n.*

Proof. Since D_{ζ_1}, D_{ζ_2} spans the subspace W for any frame $\{\zeta_1, \zeta_2\}$ of $\mathbb{P}$, the dimension of W is at most two. Suppose W is one-dimensional. Then, there exists $D_\eta \in W$ that spans W, hence

$$\begin{aligned}
\alpha^{\tilde{F}}(X,Y) &= \langle AX, Y\rangle \mu + \langle A_{\zeta_1} X, Y\rangle \zeta_1 + \langle A_{\zeta_2} X, Y\rangle \zeta_2 - \langle A_\zeta X, Y\rangle \zeta \\
&= \langle AX, Y\rangle \mu + \langle (A - \lambda I) D_{\zeta_1} X, Y\rangle \zeta_1 + \langle (A - \lambda I) D_{\zeta_2} X, Y\rangle \zeta_2 - \langle A_\zeta X, Y\rangle \zeta \\
&= \langle AX, Y\rangle \mu + a(\zeta_1) \langle (A - \lambda I) D_\eta X, Y\rangle \zeta_1 + a(\zeta_2) \langle (A - \lambda I) D_\eta X, Y\rangle \zeta_2 \\
&\quad - \langle A_\zeta X, Y\rangle \zeta \\
&= \langle AX, Y\rangle \mu + \langle (A - \lambda I) D_\eta X, Y\rangle (a(\zeta_1)\zeta_1 + a(\zeta_2)\zeta_2) - \langle A_\zeta X, Y\rangle \zeta
\end{aligned}$$

where $\{\zeta_1, \zeta_2\}$ is an orthonormal frame of the plane-bundle $\mathbb{P}$. Thus, there exists a non-trivial $\tilde{\rho} \in \Gamma(\mathbb{P})$ such that $A_{\tilde{\rho}} = 0$. If W is trivial, then any non-trivial $\tilde{\rho} \in \Gamma(\mathbb{P})$ has this property.

Decompose $\tilde{\rho} = \Psi_* \rho + \rho_1$, with $\rho \in \Gamma(N_{\tilde{f}} M)$ and $\rho_1 \in \Gamma(\mathbb{L}^2)$, according to the orthogonal decomposition of $N_{\tilde{F}} M$ as

$$N_{\tilde{F}} M = \Psi_* N_{\tilde{f}} M \oplus \mathbb{L}^2,$$

where $\mathbb{L}^2$ is a Lorentzian plane bundle having the position vector field $\tilde{F}$ as a section. Since $\tilde{\rho}$ and $\tilde{F}$ are orthogonal, we have $\rho_1 = \big\langle \rho_1, \tilde{\zeta} \big\rangle \tilde{F}$, where $\{\tilde{\zeta}, \tilde{F}\}$ is a pseudo-orthonormal frame of $\mathbb{L}^2$. Because the $\Psi_* N_{\tilde{f}} M$-component of $\alpha^{\tilde{F}}$ is $\varphi^{-1} \Psi_* \alpha^f$, from $A_{\tilde{\rho}} = 0$ we get

$$0 = \varphi^{-1} \big\langle A_\rho X, Y \big\rangle - \big\langle X, Y \big\rangle \big\langle \tilde{\zeta}, \rho_1 \big\rangle,$$

for all X, $Y \in \mathfrak{X}(M)$. In particular, since $\tilde{\rho}$ is not trivial, the normal vector field ρ can not be trivial either. We conclude that $A_\rho = \beta I$, with $\beta = \varphi\langle\tilde{\zeta}, \rho_1\rangle$.

If β vanishes, from the Codazzi equation for $\tilde{f}$ we get

$$\left\langle \nabla_X^\perp \rho, \theta \right\rangle A_\theta Y = \left\langle \nabla_Y^\perp \rho, \theta \right\rangle A_\theta X,$$

where $\{\rho, \theta\}$ is an orthonormal frame of $N_{\tilde{f}}M$. If the rank of A_θ is at least two, then $\left\langle \nabla_X^\perp \rho, \theta \right\rangle = 0$ for any $X \in T_xM$, and using Corollary 2.2 in [8] we can reduce the codimension of $\tilde{f}$, a contradiction because f is not a Cartan hypersurface. Otherwise, the relative nullity distribution of $\tilde{f}$ is greater than $n-2$. This is also a contradiction, because M^n would be conformally flat and as consequence f would have a principal curvature with multiplicity at least $n-1$ by Theorem 16.5 in [8].

If $\beta \neq 0$, then ρ cannot be parallel in the normal connection, otherwise $\tilde{f}(M)$ would be contained in an hypersphere and, as a consequence, f would be a Cartan Hypersurface (just consider $g = \tau \circ \tilde{f}$ where τ is the stereographic projection of the sphere $\mathbb{S}^{n+1}$ onto Euclidean space $\mathbb{R}^{n+1}$). On the other hand, if ρ is not parallel, then by Theorem 14 in [11] M^n would be conformally flat, a contradiction because the multiplicity of the principal curvature λ of f is $n-2$ (Theorem 16.5 in [8]). □

In the next lemma, we derive some properties of the tensors D_ξ that will be useful in the sequel.

Lemma 3.5. *The following holds:*

(i) $[D_\xi, C_T] = 0$ *for all* $T \in \Gamma(\Delta)$.

(ii) $\nabla_T^h D_\xi = 0$ *for all* $T \in \Gamma(\Delta)$ *and* $\xi \in \Gamma(N_{\tilde{F}}M)$ *parallel along* Δ.

Proof. First, let us get an expression for the projection onto $\Delta^\perp$ of the covariant derivatives of the tensors A and A_ξ, for $\xi \in \mathbb{P}$:

$$\begin{aligned}(\nabla_T^h A)X &= (\nabla_X^h A)T \\ &= \nabla_X^h AT - A\nabla_X^h T \\ &= \nabla_X^h \lambda T + AC_TX \\ &= (A - \lambda I)C_TX\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}(\nabla_T^h A)(X, \xi) &= (\nabla_X^h A)(T, \xi) \\ &= A_\xi C_T X,\end{aligned} \tag{3.2}$$

for all $X \in \Gamma(\Delta^\perp)$. In particular, $(A - \lambda I)C_T$ and $A_\xi C_T$ are symmetric, because

$$\begin{aligned}
\left\langle (\nabla^h_T A)X, Y \right\rangle &= \langle \nabla_T AX, Y \rangle - \langle A\nabla_T X, Y \rangle \\
&= T\,\langle AX, Y \rangle - \langle AX, \nabla_T Y \rangle - T\,\langle X, AY \rangle + \langle X, \nabla_T AY \rangle \\
&= \left\langle X, (\nabla^h_T A)Y \right\rangle
\end{aligned}$$

and

$$\begin{aligned}
\left\langle (\nabla^h_T A)(X, \xi), Y \right\rangle &= \langle \nabla_T A_\xi X, Y \rangle - \langle A_\xi \nabla_T X, Y \rangle - \left\langle A_{\nabla^\perp_T \xi} X, Y \right\rangle \\
&= T\,\langle A_\xi X, Y \rangle - \langle A_\xi X, \nabla_T Y \rangle - T\,\langle X, A_\xi Y \rangle + \langle X, \nabla_T A_\xi Y \rangle \\
&\quad - \left\langle X, A_{\nabla^\perp_T \xi} Y \right\rangle \\
&= \left\langle X, (\nabla^h_T A)(Y, \xi) \right\rangle.
\end{aligned}$$

Therefore, using the symmetries that we just found and the definition of D_ξ, we arrive to the following expression

$$\begin{aligned}
(A - \lambda I)D_\xi C_T &= A_\xi C_T \\
&= C^t_T A_\xi \\
&= C^t_T (A - \lambda I)D_\xi \\
&= (A - \lambda I)C_T D_\xi
\end{aligned}$$

which proves the first item, because, from the multiplicity of λ, the endomorphism $A - \lambda I$ is an isomorphism when restricted to $\Delta^\perp$.

For a section $\xi \in N_{\tilde{F}} M$ parallel along Δ, we have

$$\begin{aligned}
(A - \lambda I)D_\xi C_T &= A_\xi C_T \\
&= \nabla^h_T A_\xi \\
&= \nabla^h_T (A - \lambda I)D_\xi \\
&= \nabla^h_T A D_\xi - \lambda \nabla^h_T D_\xi
\end{aligned}$$

where in the second equality we have used the assumption that ξ is parallel along Δ, and in the fourth equality we have used that λ is a Dupin principal curvature. From equation (3.1), we also have

$$(A - \lambda I)C_T D_\xi = (\nabla^h_T A)D_\xi.$$

Subtracting both identities we get

$$\begin{aligned}(A-\lambda I)[D_\xi, C_T] &= A\nabla^h_T D_\xi - \lambda\nabla^h_T D_\xi\\ &= (A-\lambda I)\nabla^h_T D_\xi,\end{aligned}$$

which proves the second item for any section $\xi \in \Gamma(N_{\tilde{F}}M)$ parallel along the distribution Δ. □

Lemma 3.6. *There exists an endomorphism J on $\Delta^\perp$ such that $J^2 = \epsilon I$, with $\epsilon \in \{1, 0, -1\}$, and*

$$\mathrm{span}\{\mathrm{I}\} < \mathrm{C}(\Delta) \leq \mathrm{span}\{\mathrm{I}, \mathrm{J}\} = \mathrm{W}.$$

Proof. Since the hypersurface is not conformally surface-like on any open subset of M^n, otherwise f would be a Cartan hypersurface, by Corollary 3.1 the distribution $\Delta^\perp$ is not umbilical, and hence $C(\Delta) \neq \mathrm{span}\{I\}$.

Let

$$S = \{A \in \mathrm{End}(\Delta^\perp) : AB = BA \text{ for } B \in W\}$$

be the commutator of the subspace W. By part (i) of Lemma 3.5, $C(\Delta) \leq S$. From Lemma 3.4, we know that $\dim W = 2$. Using this information we claim that, if $I \notin W$, then $S = \mathrm{span}\{I\}$, a contradiction since $C(\Delta) \leq S$ and $C(\Delta) \neq \mathrm{span}\{I\}$. We will prove the claim by contradiction, so suppose that $\{A, B\}$ is a basis of W and let $T \in S$ with $T \neq rI$. By the definition of the subspace S, the endomorphism T commutes with A and B. Put A in Jordan canonical form, so we have three cases, depending on whether A has two different real eigenvalues, one real eigenvalue of multiplicity two or two complex conjugate eigenvalues:

$$\mathbf{A} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \lambda_1 \neq \lambda_2, \text{ or } \mathbf{A} = \begin{pmatrix}\lambda & 1\\ 0 & \lambda,\end{pmatrix} \text{ or } \mathbf{A} = \begin{pmatrix} r & s\\ -s & r,\end{pmatrix} s \neq 0.$$

If

$$\mathbf{T} = \begin{pmatrix} a & b\\ c & d\end{pmatrix},$$

then from $AT = TA$ for the first case, we get

$$\begin{pmatrix}\lambda_1 a & \lambda_1 b\\ \lambda_2 c & \lambda_2 d\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = \begin{pmatrix}\lambda_1 a & \lambda_2 b\\ \lambda_1 c & \lambda_2 d\end{pmatrix}.$$

Since $\lambda_1 \neq \lambda_2$, we conclude that $b = c = 0$, so

$$\mathbf{T} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

with $a \neq d$ from the supposition that $T \neq rI$. If

$$\mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

from $TB = BT$ and using the same argument as before and since B cannot be a multiple of the identity, we get that $f = g = 0$, so

$$\mathbf{B} = \begin{pmatrix} e & 0 \\ 0 & h \end{pmatrix},$$

with $e \neq h$. Without losing generality, suppose $\lambda_1 \neq 0$, otherwise $\lambda_2 \neq 0$ and we can rename variables. Let us find constants a and b such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + b \begin{pmatrix} e & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} a\lambda_1 + be & 0 \\ 0 & a\lambda_2 + bh \end{pmatrix},$$

so,

$$1 - \frac{1 - be}{\lambda_1}\lambda_2 = bh.$$

Therefore,

$$\frac{\lambda_1 - \lambda_2}{\lambda_1} = b\left(\frac{h\lambda_1 - e\lambda_2}{\lambda_1}\right).$$

If $h\lambda_1 - e\lambda_2 = 0$, then A and B would be linearly dependent. So, $h\lambda_1 - e\lambda_2 \neq 0$ and we can solve the equations for the unknowns a and b. This means that the identity belongs to the subspace W, a contradiction.

Lets now move to the second case, that is, to the case where A has one real eigenvalue of multiplicity two. From $AT = TA$, we get

$$\begin{pmatrix} \lambda a + c & \lambda b + d \\ \lambda c & \lambda d \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda a & a + \lambda b \\ \lambda c & c + \lambda d \end{pmatrix}.$$

Therefore, $c = 0$ and $a = d$, so

$$\mathbf{T} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

with $b \neq 0$. From $TB = BT$ we have

$$\begin{pmatrix} ae+bg & af+bh \\ ag & ah \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} ae & be+af \\ ag & bg+ah \end{pmatrix}.$$

We conclude that $bg = 0$ and $bh = be$. So, $g = 0$, $h = e$, $f \neq 0$ and

$$\mathbf{B} = \begin{pmatrix} e & f \\ 0 & e \end{pmatrix}.$$

In fact, dividing by f we can assume that $f = 1$. Then, $\mathbf{A} - \mathbf{B} = (\lambda - e)\mathbf{I} \neq 0$, a contradiction.

For the third case, from $AT = TA$ we have

$$\begin{pmatrix} ra+sc & rb+sd \\ -sa+rc & -sb+rd \end{pmatrix} = \begin{pmatrix} r & s \\ -s & r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r & s \\ -s & r \end{pmatrix} = \begin{pmatrix} ra-sb & sa+rb \\ rc-sd & sc+rd \end{pmatrix}.$$

Therefore, $c = -b$, $a = d$, $b \neq 0$ and

$$\mathbf{T} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

With the same argument, we can conclude that

$$\mathbf{B} = \begin{pmatrix} e & f \\ -f & e \end{pmatrix},$$

then $f\mathbf{A} - s\mathbf{B} = (rf - se)\mathbf{I} \neq 0$, a contradiction. Since we have proved our claim for the three cases, we have $I \in W$.

For the three cases of the Jordan decomposition of A,

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \lambda_1 \neq \lambda_2, \text{ or } \mathbf{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda, \end{pmatrix} \text{ or } \mathbf{A} = \begin{pmatrix} r & s \\ -s & r, \end{pmatrix} s \neq 0,$$

we got that $\{\mathbf{I}, \mathbf{A}\}$ is a basis for W. For those three cases, we will find $\mathbf{J}$ such that $\mathbf{J}^2 = \epsilon I$ and that $\{\mathbf{I}, \mathbf{J}\}$ is a basis for W. In the first case, notice that

$$W = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

For this case, $\mathbf{J}^2 = \epsilon I$, with $\mathbf{J} \neq \pm I$ and $\mathbf{J} \neq 0$ can only happen with $\epsilon = 1$ and, up to sign,

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can generate this matrix with the following linear combination of $\mathbf{I}$ and $\mathbf{A}$:

$$\frac{2}{\lambda_1 - \lambda_2}\mathbf{A} - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\mathbf{I} = \frac{2}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For the second case, we have that

$$W = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \text{ for } a, b \in \mathbb{R} \right\}.$$

Since

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^2 = \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix},$$

we can only have $\mathbf{J}^2 = 0$ with $\mathbf{J} \neq 0$ when

$$\mathbf{J} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$

or $\mathbf{J}^2 = \mathbf{I}$ when $\mathbf{J} = \pm\mathbf{I}$. The latter case would lead to a contradiction, because we are looking for a basis of W. In the former case, we will choose $b = 1$. We can generate $\mathbf{J} = \mathbf{A} - \lambda\mathbf{I}$.

For the last case, we have

$$W = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ for } a, b \in \mathbb{R} \right\}.$$

Since

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^2 = \begin{pmatrix} a^2 - b^2 & 2ab \\ -2ab & a^2 - b^2 \end{pmatrix},$$

we can only have $\mathbf{J}^2 = \mathbf{I}$ with $\mathbf{J} = \pm\mathbf{I}$ or, up to sign, $\mathbf{J}^2 = -\mathbf{I}$ with

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which can be generated by $\mathbf{J} = b^{-1}(\mathbf{A} - a\mathbf{I})$.

Let $a\mathbf{I} + b\mathbf{J} \in W$, then for any $c\mathbf{I} + d\mathbf{J} \in W$ we have

$$(a\mathbf{I} + b\mathbf{J})(c\mathbf{I} + d\mathbf{J}) = (c\mathbf{I} + d\mathbf{J})(a\mathbf{I} + b\mathbf{J})$$

so $W \leq S$. Working through each case for W, we find that $\dim S = 2$, so in fact we have equality. We conclude that

$$C(\Delta) \leq S = W = \operatorname{span}\{\mathbf{I}, \mathbf{J}\}$$

which ends the demonstration of this lemma. □

Lemma 3.7. *For any orthonormal frame $\{\xi_1, \xi_2\}$ of $\mathbb{P}$ we have*

$$1 = \det D_{\xi_1} + \det D_{\xi_2}. \tag{3.3}$$

Proof. Since γ is a flat bilinear form and

$$\det A_\zeta = \det A_{\xi_1} + \det A_{\xi_2},$$

the conclusion follows. □

Now, consider any orthonormal frame $\{\xi_1, \xi_2\}$ of $\mathbb{P}$ and define the one-forms

$$\tilde{\psi}(X) = \left\langle \nabla^\perp_X \xi_1, \xi_2 \right\rangle, \quad \tilde{\omega_1}(X) = \left\langle \nabla^\perp_X \xi_1, \mu \right\rangle \quad \text{and} \quad \tilde{\omega_2}(X) = \left\langle \nabla^\perp_X \xi_2, \mu \right\rangle.$$

Since afterwards we will fix a convenient frame for $\mathbb{P}$, we will reserve the symbols ω_1, ω_2 and ψ for that specific orthonormal frame, and use $\tilde{\omega_1}$, $\tilde{\omega_2}$ and $\tilde{\psi}$ for an arbitrary orthonormal frame.

From the definition of the section $\zeta = \lambda \tilde{F} + \mu$, we have

$$\nabla^\perp_X(\zeta - \mu) = X(\lambda)\tilde{F} = \lambda^{-1} X(\lambda)(\zeta - \mu).$$

So, we obtain

$$\begin{aligned} \nabla^\perp_X \xi_1 &= \left\langle \nabla^\perp_X \xi_1, \mu \right\rangle \mu + \left\langle \nabla^\perp_X \xi_1, \xi_2 \right\rangle \xi_2 - \left\langle \nabla^\perp_X \xi_1, \zeta \right\rangle \zeta \\ &= \tilde{\omega_1}(X)(\mu - \zeta) + \tilde{\psi}(X)\xi_2, \end{aligned} \tag{3.4}$$

$$\begin{aligned}\nabla^\perp_X \xi_2 &= \left\langle \nabla^\perp_X \xi_2, \mu\right\rangle \mu + \left\langle \nabla^\perp_X \xi_2, \xi_1\right\rangle \xi_1 - \left\langle \nabla^\perp_X \xi_2, \zeta\right\rangle \zeta \\ &= \tilde{\omega}_2(X)(\mu - \zeta) - \tilde{\psi}(X)\xi_1,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\nabla^\perp_X \mu &= \left\langle \nabla^\perp_X \mu, \xi_1\right\rangle \xi_1 + \left\langle \nabla^\perp_X \mu, \xi_2\right\rangle \xi_2 - \left\langle \nabla^\perp_X \mu, \zeta\right\rangle \zeta \\ &= -\tilde{\omega}_1(X)\xi_1 - \tilde{\omega}_2(X)\xi_2 - \lambda^{-1}X(\lambda)\zeta\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\nabla^\perp_X \zeta &= \left\langle \nabla^\perp_X \zeta, \mu\right\rangle \mu + \left\langle \nabla^\perp_X \zeta, \xi_1\right\rangle \xi_1 + \left\langle \nabla^\perp_X \zeta, \xi_2\right\rangle \xi_2 \\ &= -\lambda^{-1}X(\lambda)\mu - \tilde{\omega}_1(X)\xi_1 - \tilde{\omega}_2(X)\xi_2,\end{aligned} \tag{3.7}$$

for all X, $Y \in \mathfrak{X}(M)$.

Lets write down the Codazzi equation of $A = A_\mu$ with respect to the frame $\{\mu, \xi_1, \xi_2, \zeta\}$. Since A already satisfies the Codazzi equation and with the aid of the identity (3.6), we get

$$\begin{aligned}0 &= A_{\nabla^\perp_X \mu} Y - A_{\nabla^\perp_Y \mu} X \\ &= -\tilde{\omega}_1(X)A_{\xi_1}Y - \tilde{\omega}_2(X)A_{\xi_2}Y - \lambda^{-1}X(\lambda)A_\zeta Y \\ &\quad + \tilde{\omega}_1(Y)A_{\xi_1}X + \tilde{\omega}_2(Y)A_{\xi_2}X + \lambda^{-1}Y(\lambda)A_\zeta X,\end{aligned} \tag{3.8}$$

for arbitrary X, $Y \in \mathfrak{X}(M)$. By performing $\lambda(A - \lambda I)^{-1}$ on both sides of the equation and conveniently rearranging we obtain

$$(X \wedge Y)\operatorname{grad}\lambda = D_{\xi_1}\left(\lambda\tilde{\omega}_1(X)Y - \lambda\tilde{\omega}_1(Y)X\right) + D_{\xi_2}\left(\lambda\tilde{\omega}_2(X)Y - \lambda\tilde{\omega}_2(Y)X\right), \tag{3.9}$$

for X, $Y \in \Gamma(\Delta^\perp)$.

Let us move on to the Codazzi equation for $A_\zeta = A - \lambda I$. Using the Codazzi equation for A and equation (3.7), we obtain

$$\begin{aligned}0 &= (\nabla_X A_\zeta)Y - (\nabla_Y A_\zeta)X - A_{\nabla^\perp_X \zeta}Y + A_{\nabla^\perp_Y \zeta}X \\ &= (\nabla_X(A - \lambda I))Y - (\nabla_Y(A - \lambda I))X - A_{\nabla^\perp_X \zeta}Y + A_{\nabla^\perp_Y \zeta}X \\ &= -(\nabla_X \lambda I)Y + (\nabla_Y \lambda I)X - A_{\nabla^\perp_X \zeta}Y + A_{\nabla^\perp_Y \zeta}X \\ &= -X(\lambda)Y + Y(\lambda)X + \lambda^{-1}X(\lambda)AY + \tilde{\omega}_1(X)A_{\xi_1}Y + \tilde{\omega}_2(X)A_{\xi_2}Y - \lambda^{-1}Y(\lambda)AX \\ &\quad - \tilde{\omega}_1(Y)A_{\xi_1}X - \tilde{\omega}_2(Y)A_{\xi_2}X,\end{aligned} \tag{3.10}$$

or equivalently,

$$\begin{aligned}(X \wedge Y)\operatorname{grad}\lambda = \lambda^{-1}Y(\lambda)AX + \tilde{\omega}_1(Y)A_{\xi_1}X + \tilde{\omega}_2(Y)A_{\xi_2}X \\ - \lambda^{-1}X(\lambda)AY - \tilde{\omega}_1(X)A_{\xi_1}Y - \tilde{\omega}_2(X)A_{\xi_2}Y\end{aligned} \tag{3.11}$$

for arbitrary X, $Y \in \mathfrak{X}(M)$.

Finally, using equations (3.4), (3.5) and that $A_\zeta = A - \lambda I$, the Codazzi equation for ξ_i, $i = 1, 2$, is

$$\begin{aligned}(\nabla_X A_{\xi_i})Y - (\nabla_Y A_{\xi_i})X &= A_{\nabla^\perp_X \xi_i}Y - A_{\nabla^\perp_Y \xi_i}X \\ &= \tilde{\omega}_i(X)AY - \tilde{\omega}_i(X)A_\zeta Y + (-1)^j\tilde{\psi}(X)A_{\xi_j}Y \\ &\quad - \tilde{\omega}_i(Y)AX + \tilde{\omega}_i(Y)A_\zeta X - (-1)^j\tilde{\psi}(Y)A_{\xi_j}X \\ &= \lambda\left(\tilde{\omega}_i(X)Y - \tilde{\omega}_i(Y)X\right) + (-1)^j\left(\tilde{\psi}(X)A_{\xi_j}Y - \tilde{\psi}(Y)A_{\xi_j}X\right)\end{aligned} \tag{3.12}$$

for arbitrary X, $Y \in \mathfrak{X}(M)$ and $j = 1, 2$ with $j \neq i$.

Let us use the newly gathered information to prove an important property about the kernel of the tensors $\tilde{\omega}_i$.

Lemma 3.8. *For the one-forms $\tilde{\omega}_1(X) = \langle \nabla^\perp_X \xi_1, \mu\rangle$ and $\tilde{\omega}_2(X) = \langle \nabla^\perp_X \xi_2, \mu\rangle$, where $\{\xi_1, \xi_2\}$ is an arbitrary orthonormal frame for $\mathbb{P}$, we have*

$$\Delta \leq \ker\tilde{\omega}_1 \cap \ker\tilde{\omega}_2. \tag{3.13}$$

Proof. For arbitrary $T \in \Delta$ and any $X \in \Delta^\perp$, from the Codazzi equation (3.8) we get

$$\tilde{\omega}_1(T)A_{\xi_1}X + \tilde{\omega}_2(T)A_{\xi_2}X = 0.$$

Performing $(A - \lambda I)^{-1}$ on both sides of the equation above, we get

$$\tilde{\omega}_1(T)D_{\xi_1} + \tilde{\omega}_2(T)D_{\xi_2} = 0.$$

Since the subspace W spanned by the endomorphisms D_ξ, $\xi \in \mathbb{P}$, has dimension two by Lemma 3.4, and hence D_{ξ_1} and D_{ξ_2} is a basis for W for any frame $\{\xi_1, \xi_2\}$ of $\mathbb{P}$, we conclude that $\tilde{\omega}_i(T) = 0$. □

It is now time to compute the Ricci equations for μ and ξ for the isometric immersion

$\tilde{F}$. Using equations (3.4), (3.5), (3.6) and (3.7) we get:

$$\begin{aligned}
\left\langle R^{\perp}(X,Y)\mu, \xi_i \right\rangle &= \left\langle \nabla^{\perp}_X \nabla^{\perp}_Y \mu, \xi_i \right\rangle - \left\langle \nabla^{\perp}_Y \nabla^{\perp}_X \mu, \xi_i \right\rangle - \left\langle \nabla^{\perp}_{[X,Y]} \mu, \xi_i \right\rangle \\
&= \left\langle \nabla^{\perp}_X \left(-\tilde{\omega}_1(Y)\xi_1 - \tilde{\omega}_2(Y)\xi_2 - \lambda^{-1} Y(\lambda)\zeta \right), \xi_i \right\rangle \\
&\quad - \left\langle \nabla^{\perp}_Y \left(-\tilde{\omega}_1(X)\xi_1 - \tilde{\omega}_2(X)\xi_2 - \lambda^{-1} X(\lambda)\zeta \right), \xi_i \right\rangle \\
&\quad - \left\langle -\tilde{\omega}_1([X,Y])\xi_1 - \tilde{\omega}_2([X,Y])\xi_2 - \lambda^{-1}[X,Y](\lambda)\zeta, \xi_i \right\rangle \\
&= -X\tilde{\omega}_i(Y) - \tilde{\omega}_j(Y) \left\langle \nabla^{\perp}_X \xi_j, \xi_i \right\rangle - \lambda^{-1} Y(\lambda) \left\langle \nabla^{\perp}_X \zeta, \xi_i \right\rangle + Y\tilde{\omega}_i(X) \\
&\quad + \tilde{\omega}_j(X) \left\langle \nabla^{\perp}_Y \xi_j, \xi_i \right\rangle + \lambda^{-1} X(\lambda) \left\langle \nabla^{\perp}_Y \zeta, \xi_i \right\rangle + \tilde{\omega}_i([X,Y]) \\
&= -\mathrm{d}\tilde{\omega}_i(X,Y) - (-1)^i \tilde{\omega}_j(Y)\tilde{\psi}(X) + (-1)^i \tilde{\omega}_j(X)\tilde{\psi}(Y) + \lambda^{-1} Y(\lambda)\tilde{\omega}_i(X) \\
&\quad - \lambda^{-1} X(\lambda)\tilde{\omega}_i(Y).
\end{aligned}$$

Because $1 \leq i \neq j \leq 2$, we can change $-(-1)^i$ by $(-1)^j$ we arrive to the following conclusion:

$$\begin{aligned}
\langle [A_\mu, A_{\xi_i}]X, Y \rangle = &-d\tilde{\omega}_i(X,Y) + (-1)^j \tilde{\omega}_j(Y)\tilde{\psi}(X) - (-1)^j \tilde{\omega}_j(X)\tilde{\psi}(Y) \\
&+ \lambda^{-1} Y(\lambda)\tilde{\omega}_i(X) - \lambda^{-1} X(\lambda)\tilde{\omega}_i(Y),
\end{aligned} \tag{3.14}$$

for all X, $Y \in \mathfrak{X}(M)$.

Since $\zeta = \lambda\tilde{F} + \mu$ and $\tilde{F}$ is parallel in the normal connection, it follows that

$$\left\langle R^{\perp}(X,Y)\zeta, \mu \right\rangle = 0.$$

On the other hand, $[A_\mu, A_\zeta] = 0$, because $A_\zeta = A - \lambda I$ and $A_\mu = A$. Therefore the Ricci equation for μ and ζ brings no new information.

The next equation to deduce is the one for ξ_1 and ξ_2. Using equations (3.4), (3.6) and (3.7), we obtain

$$\begin{aligned}
\left\langle R^{\perp}(X,Y)\xi_1, \xi_2 \right\rangle &= \left\langle \nabla^{\perp}_X \nabla^{\perp}_Y \xi_1, \xi_2 \right\rangle - \left\langle \nabla^{\perp}_Y \nabla^{\perp}_X \xi_1, \xi_2 \right\rangle - \left\langle \nabla^{\perp}_{[X,Y]} \xi_1, \xi_2 \right\rangle \\
&= \tilde{\omega}_1(Y) \left\langle \nabla^{\perp}_X (\mu - \zeta), \xi_2 \right\rangle + X\tilde{\psi}(Y) \\
&\quad - \tilde{\omega}_1(X) \left\langle \nabla^{\perp}_Y (\mu - \zeta), \xi_2 \right\rangle - Y\tilde{\psi}(X) - \tilde{\psi}([X,Y]) \\
&= \mathrm{d}\tilde{\psi}(X,Y),
\end{aligned}$$

so we conclude

$$\langle [A_{\xi_1}, A_{\xi_2}]X, Y \rangle = \mathrm{d}\tilde{\psi}(X,Y), \tag{3.15}$$

for X, $Y \in \mathfrak{X}(M)$.

The last equation is the one relating ξ_i and ζ. Using equations (3.4) and (3.5), we get

$$\begin{aligned}\left\langle R^{\perp}(X,Y)\xi_i,\zeta\right\rangle &= \left\langle \nabla^{\perp}_X\nabla^{\perp}_Y\xi_i,\zeta\right\rangle - \left\langle \nabla^{\perp}_Y\nabla^{\perp}_X\xi_i,\zeta\right\rangle - \left\langle \nabla^{\perp}_{[X,Y]}\xi_i,\zeta\right\rangle \\ &= X\tilde{\omega}_i(Y) + \tilde{\omega}_i(Y)\lambda^{-1}X(\lambda) + (-1)^j\tilde{\psi}(Y)\left\langle \nabla^{\perp}_X\xi_j,\zeta\right\rangle \\ &\quad - Y\tilde{\omega}_i(X) - \tilde{\omega}_i(X)\lambda^{-1}Y(\lambda) - (-1)^j\tilde{\psi}(X)\left\langle \nabla^{\perp}_Y\xi_j,\zeta\right\rangle - \tilde{\omega}_i([X,Y]) \\ &= \mathrm{d}\tilde{\omega}_i(X,Y) + \tilde{\omega}_i(Y)\lambda^{-1}X(\lambda) + (-1)^j\tilde{\psi}(Y)\tilde{\omega}_j(X) \\ &\quad - \tilde{\omega}_i(X)\lambda^{-1}Y(\lambda) - (-1)^j\tilde{\psi}(X)\tilde{\omega}_j(Y),\end{aligned}$$

for $i \neq j$. Therefore,

$$\begin{aligned}\langle [A_{\xi_i}, A_\zeta]X, Y\rangle = d\tilde{\omega}_i(X,Y) &+ \tilde{\omega}_i(Y)\lambda^{-1}X(\lambda) + (-1)^j\tilde{\psi}(Y)\tilde{\omega}_j(X) \\ &- \tilde{\omega}_i(X)\lambda^{-1}Y(\lambda) - (-1)^j\tilde{\psi}(X)\tilde{\omega}_j(Y),\end{aligned} \tag{3.16}$$

for X, $Y \in \mathfrak{X}(M)$.

We now pick a suitable orthonormal frame $\{\xi_1, \xi_2\}$ of the Riemannian plane bundle $\mathbb{P}$ that has a nice behavior with the normal connection: those sections will be parallel along Δ.

Lemma 3.9. *There exists a unique (up to sign and permutation) orthonormal frame ξ_1,ξ_2 of $\mathbb{P}$ such that $D_i = D_{\xi_i}$, for $i = 1, 2$, satisfy*

$$\det D_1 = \frac{1}{2} = \det D_2.$$

Moreover, $D_2^2 \neq -D_1^2$ and ξ_i, for $i = 1, 2$, is parallel along Δ.

Proof. Since W is two-dimensional, we have $D_1 \neq \pm D_2$. We will show that $D_2^2 \neq -D_1^2$ by contradiction, so suppose $D_2^2 = -D_1^2$. Since $W = \operatorname{span}\{I, J\}$, let $D_1 = aI + bJ$ and $D_2 = cI + dJ$. From our hypothesis we get

$$(c^2 + \epsilon d^2)I + 2cdJ = -(a^2 + \epsilon b^2)I - 2abJ.$$

If $\epsilon = 1$, we end up with $a = b = c = d = 0$, a contradiction because D_1 and D_2 would be the trivial endomorphisms, which means that W is trivial. If $\epsilon = 0$, then $a = c = 0$ and W would be spanned by J, also a contradiction. So we are left with the case where J has two complex conjugate eigenvalues.

Denote by $\hat{D}_i$ the complex linear extension of D_i. Then,

$$\hat{D}_1 = \begin{pmatrix} \theta & 0 \\ 0 & \bar{\theta} \end{pmatrix} \quad \text{and} \quad \hat{D}_2 = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}.$$

From $D_2^2 = -D_1^2$, we get $\alpha^2 = -\theta^2$, or by suitably changing sign of ξ_2 if necessary, $\alpha = i\theta$. By Lemma 3.7, the sum of their determinants is 1, then $1 = 2|\theta|^2$, so we can suppose

$$\sqrt{2}\hat{D}_1 = \begin{pmatrix} \theta & 0 \\ 0 & \bar{\theta} \end{pmatrix} \quad \text{and} \quad \sqrt{2}\hat{D}_2 = \begin{pmatrix} i\theta & 0 \\ 0 & -i\bar{\theta} \end{pmatrix}$$

for $\theta \in \mathbb{S}^1$. Writing $\theta = e^{i\beta}$ we have

$$\sqrt{2}D_1 = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \quad \text{and} \quad \sqrt{2}D_2 = \begin{pmatrix} -\sin\beta & \cos\beta \\ -\cos\beta & -\sin\beta \end{pmatrix}.$$

Then,

$$\cos\beta\sqrt{2}D_1 - \sin\beta\sqrt{2}D_2 = I \quad \text{and} \quad \sin\beta\sqrt{2}D_1 + \cos\beta\sqrt{2}D_2 = J.$$

Hence, the orthonormal frame $\{\xi, \eta\}$ of $\mathbb{P}$ defined by

$$\xi = \cos\beta\xi_1 - \sin\beta\xi_2 \quad \text{and} \quad \eta = \sin\beta\xi_1 + \cos\beta\xi_2$$

satisfies

$$\sqrt{2}D_\xi = I \quad \text{and} \quad \sqrt{2}D_\eta = J,$$

or equivalently,

$$\sqrt{2}A_\xi = (A - \lambda I) \quad \text{and} \quad \sqrt{2}A_\eta = (A - \lambda I)J.$$

From the above identities, we are hinted to see what we get from the Codazzi equation for $\sqrt{2}A_\xi = A_\zeta$. Using equation (3.4) with $\xi_1 = \xi$ and $\xi_2 = \eta$ and taking into account that A is a Codazzi tensor, we obtain

$$\begin{aligned} 0 &= (\nabla_X\sqrt{2}A_\xi)Y - (\nabla_Y\sqrt{2}A_\xi)X - \sqrt{2}A_{\nabla^\perp_X\xi}Y + \sqrt{2}A_{\nabla^\perp_Y\xi}X \\ &= (\nabla_X(A - \lambda I))\,Y - (\nabla_Y(A - \lambda I))\,X - \sqrt{2}A_{\nabla^\perp_X\xi}Y + \sqrt{2}A_{\nabla^\perp_Y\xi}X \\ &= (\nabla_X A)Y - (\nabla_X\lambda I)Y - (\nabla_Y A)X + (\nabla_Y\lambda I)X - \sqrt{2}A_{\nabla^\perp_X\xi}Y + \sqrt{2}A_{\nabla^\perp_Y\xi}X \\ &= -X(\lambda)Y + Y(\lambda)X - \sqrt{2}\tilde{\omega}_1(X)AY + \sqrt{2}\tilde{\omega}_1(X)A_\zeta Y - \sqrt{2}\tilde{\psi}(X)A_\eta Y \\ &\quad + \sqrt{2}\tilde{\omega}_1(Y)AX - \sqrt{2}\tilde{\omega}_1(Y)A_\zeta X + \sqrt{2}\tilde{\psi}(Y)A_\eta X, \end{aligned}$$

for arbitrary X, $Y \in \mathfrak{X}(M)$. Therefore, rearranging the above equation we get

$$\begin{aligned}(X \wedge Y)\operatorname{grad}\lambda = {} & \sqrt{2}\tilde{\omega}_1(X)AY - \sqrt{2}\tilde{\omega}_1(X)A_\zeta Y + \sqrt{2}\tilde{\psi}(X)A_\eta Y \\ & - \sqrt{2}\tilde{\omega}_1(Y)AX + \sqrt{2}\tilde{\omega}_1(Y)A_\zeta X - \sqrt{2}\tilde{\psi}(Y)A_\eta X,\end{aligned}$$

Using $A_\zeta = A - \lambda I$, we have

$$(X \wedge Y)\operatorname{grad}\lambda = \sqrt{2}\lambda\tilde{\omega}_1(X)Y + \sqrt{2}\tilde{\psi}(X)A_\eta Y - \sqrt{2}\lambda\tilde{\omega}_1(Y)X - \sqrt{2}\tilde{\psi}(Y)A_\eta X$$

or, expressed in another way,

$$\big[\big(Y(\lambda)+\sqrt{2}\lambda\tilde{\omega}_1(Y)\big)I+\sqrt{2}\tilde{\psi}(Y)A_\eta\big]X = \big[\big(X(\lambda)+\sqrt{2}\lambda\tilde{\omega}_1(X)\big)I+\sqrt{2}\tilde{\psi}(X)A_\eta\big]Y. \quad (3.17)$$

Using Lemma 3.8 for $Y = T \in \Delta$ and $X \in \Delta^\perp$ in the above equation, we get

$$\sqrt{2}\tilde{\psi}(T)A_\eta X = \big(X(\lambda) + \sqrt{2}\lambda\tilde{\omega}_1(X)\big)T.$$

Since $\sqrt{2}A_\eta = (A - \lambda I)J$ is an isomorphism on $\Delta^\perp$,

$$\Delta \leq \ker\tilde{\psi} \quad \text{and} \quad X(\lambda) + \sqrt{2}\lambda\tilde{\omega}_1(X) = 0, \quad \text{for } X \in \Delta^\perp.$$

Replacing the last identity in equation (3.17) for X and $Y \in \Delta^\perp$, we obtain

$$\tilde{\psi}(Y)A_\eta X = \tilde{\psi}(X)A_\eta Y.$$

Because $\sqrt{2}A_\eta = (A - \lambda I)J$ is an isomorphism on $\Delta^\perp$, it follows that $\ker\tilde{\psi} = \mathfrak{X}(M)$. From the equation (3.15) we conclude

$$\big\langle R^\perp(X,Y)\xi,\eta\big\rangle = \langle[A_\xi, A_\eta]X, Y\rangle = \mathrm{d}\tilde{\psi}(X,Y) = 0.$$

From $[A_\xi, A_\eta] = 0$, then $[(A-\lambda I),(A-\lambda I)J] = 0$. This means that A and J commute. Then, the endomorphism $A - \lambda I$ in $\Delta^\perp$ would be a multiple of the identity. It cannot be the null operator, because f has a principal curvature λ with multiplicity $n-2$. Therefore, $A - \lambda I = \beta I$ in $\Delta^\perp$, with $\beta \neq 0$. Using the identity $(A - \lambda I)C_T = \nabla^h_T A$, we get

$$\beta C_T = \nabla^h_T(\beta + \lambda)I = T(\beta + \lambda)I.$$

We have a contradiction, because f is not conformally surface-like. Therefore, $D_2^2 \neq -D_2^2$.

We will prove the existence and uniqueness of an orthonormal frame $\{\xi_1, \xi_2\} \subset \mathbb{P}$ such that $\det D_{\xi_i} = 1/2$. Let us start with the existence. Pick an arbitrary orthonormal frame $\{\xi, \eta\}$ for $\mathbb{P}$. Since

$$1 = \det D_\xi + \det D_\eta,$$

and from Lemma 3.7 we do not have to do any work if D_ξ or D_η has determinant 1/2. So, suppose without loosing generality that $\det D_\xi < 1/2$ and $\det D_\eta > 1/2$. Then, define an orthonormal frame on $\mathbb{P}$ by rotating ξ and η by an arbitrary angle $0 < \theta < \pi/2$, that is, $\xi_1(\theta) = \cos\theta\,\xi + \sin\theta\,\eta$ and $\xi_2(\theta) = -\sin\theta\,\xi + \cos\theta\,\eta$. We have

$$\det D_\xi = \det D_{\xi_1(0)} < \det D_{\xi_1(\frac{\pi}{2})} = \det D_\eta,$$

so by continuity we get existence.

We are left with uniqueness. We have to do it with a case by case analysis, depending on whether the tensor J has two distinct eigenvectors, one eigenvector of multiplicity two or two complex eigenvectors ($J^2 = I$, $J^2 = 0$, or $J^2 = -I$, respectively).

Suppose first $J^2 = I$ and, by the existence part, let $\{\xi_1, \xi_2\}$ be the orthonormal frame on $\mathbb{P}$ such that D_1 and D_2 have determinant 1/2. Then,

$$D_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

with $ab = cd = 1/2$. Let us rotate the orthonormal frame $\{\xi_1, \xi_2\}$ and see if the endomorphisms they induce also have determinant 1/2. So, define $\xi = \cos\theta\,\xi_1 + \sin\theta\,\xi_2$ and $\eta = -\sin\theta\,\xi_1 + \cos\theta\,\xi_2$ and therefore

$$D_\xi = \begin{pmatrix} a\cos\theta + c\sin\theta & 0 \\ 0 & b\cos\theta + d\sin\theta \end{pmatrix}$$

with $\det D_\xi = ab\cos^2\theta + cd\sin^2\theta + \cos\theta\sin\theta(ad + bc) = 1/2 + \cos\theta\sin\theta(ad + bc)$. We conclude that $ad = -bc$. Multiply both sides of this equation by bd, so we end up with $abd^2 = -b^2cd$. Since $ab = cd = 1/2$, we conclude that $d^2 = -b^2$, or $b = 0 = d$, a contradiction because the determinants of D_1 and D_2 are not zero.

For $J^2 = 0$, we have

$$D_1 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}$$

with $a^2 = c^2 = 1/2$. Then,

$$D_\xi = \begin{pmatrix} a\cos\theta + c\sin\theta & b\cos\theta + d\sin\theta \\ 0 & a\cos\theta + c\sin\theta \end{pmatrix}$$

with $\det D_\xi = a^2\cos^2\theta + c^2\sin^2\theta + 2ac\cos\theta\sin\theta = 1/2 + 2ac\cos\theta\sin\theta$. If $a = 0$ or $c = 0$, then D_1 or D_2 would have a null determinant, contradicting our assumption.

For the last case $J^2 = -I$,

$$D_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

with $a^2 + b^2 = c^2 + d^2 = 1/2$. Then,

$$D_\xi = \begin{pmatrix} a\cos\theta + c\sin\theta & b\cos\theta + d\sin\theta \\ -b\cos\theta - d\sin\theta & a\cos\theta + c\sin\theta \end{pmatrix}$$

with $\det D_\xi = (a^2+b^2)\cos^2\theta + (c^2+d^2)\sin^2\theta + 2\cos\theta\sin\theta(ac+bd) = 1/2 + 2\cos\theta\sin\theta(ac+bd)$. If $ac = -bd$, then

$$cD_1 = c(aI + bJ) = -b(dI - cJ) = bJD_2.$$

Since both endomorphisms have the same determinant, we get $c = \pm b$. If $c = b = 0$, then $a = \pm d$, $D_1 = aI$ and $D_2 = dJ$. So we have $D_2^2 = -d^2 I = -a^2 I = -D_1^2$, a contradiction. If $c = \pm b \neq 0$, then $a = \mp d$ and $D_1^2 = -D_2^2$, same contradiction.

We are left to prove that ξ_1 and ξ_2 are parallel along Δ. For this, we will use the uniqueness of the orthonormal frame $\{\xi_1, \xi_2\}$. Given $x \in M^n$, $T \in \Delta$ and an integral curve γ of T starting at x, let $\hat{\xi}_i(t)$ denote the parallel transport of $\xi_i(x)$ along $\gamma(t)$. By Lemma 3.5, we have that $\nabla^h_{\gamma'(t)} D_{\hat{\xi}_i(t)} = 0$, or equivalently, $\nabla^h_{\gamma'(t)} D_{\hat{\xi}_i(t)} X = D_{\hat{\xi}_i(t)} \nabla^h_{\gamma'(t)} X$. Given an orthonormal frame $\{X, Y\}$ of $\mathfrak{X}(M)$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \det D_{\hat{\xi}_i(t)} &= \frac{\mathrm{d}}{\mathrm{d}t}\big[\big\langle D_{\hat{\xi}_i(t)} X, X\big\rangle \big\langle D_{\hat{\xi}_i(t)} Y, Y\big\rangle - \big\langle D_{\hat{\xi}_i(t)} X, Y\big\rangle \big\langle D_{\hat{\xi}_i(t)} Y, X\big\rangle\big] \\
&= \big[\big\langle D_{\hat{\xi}_i(t)} \nabla^h_{\gamma'(t)} X, X\big\rangle + \big\langle D_{\hat{\xi}_i(t)} X, \nabla^h_{\gamma'(t)} X\big\rangle\big]\big\langle D_{\hat{\xi}_i(t)} Y, Y\big\rangle \\
&\quad + \big\langle D_{\hat{\xi}_i(t)} X, X\big\rangle\big[\big\langle D_{\hat{\xi}_i(t)} \nabla^h_{\gamma'(t)} Y, Y\big\rangle + \big\langle D_{\hat{\xi}_i(t)} Y, \nabla^h_{\gamma'(t)} Y\big\rangle\big] \\
&\quad - \big[\big\langle D_{\hat{\xi}_i(t)} \nabla^h_{\gamma'(t)} X, Y\big\rangle + \big\langle D_{\hat{\xi}_i(t)} X, \nabla^h_{\gamma'(t)} Y\big\rangle\big]\big\langle D_{\hat{\xi}_i(t)} Y, X\big\rangle \\
&\quad - \big\langle D_{\hat{\xi}_i(t)} X, Y\big\rangle\big[\big\langle D_{\hat{\xi}_i(t)} \nabla^h_{\gamma'(t)} Y, X\big\rangle + \big\langle D_{\hat{\xi}_i(t)} Y, \nabla^h_{\gamma'(t)} X\big\rangle\big].
\end{aligned}$$

Using now that $\{X, Y\}$ is an orthonormal frame and decomposing the vector fields $\nabla^h_{\gamma'(t)} X$ and $\nabla^h_{\gamma'(t)} Y$ in this frame, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \det D_{\hat{\xi}_i(t)} = 0.$$

Therefore, the determinant is constant and equal to $1/2$ along the parallel transport. Since ξ_1 and ξ_2 are unique (up to signs and permutation) with this property, by continuity we must have $\hat{\xi}_i(t) = \xi_i(\gamma(t))$ for any t. It follows that $\nabla^\perp_T \xi_i = 0$ for any $T \in \Delta$, $i = 1, 2$. $\square$

From now on, we fix the privileged orthonormal frame $\{\xi_1, \xi_2\}$ of $\mathbb{P}$ given by the above lemma and omit the tilde notation in ω_1, ω_2 and ψ when using this frame. Also, the notation D_1 and D_2 refers to the privileged frame $\{\xi_1, \xi_2\}$, so that $D_i = D_{\xi_i}$ for $i = 1, 2$.

We will show that the pair (D_1, D_2) and the one-form ψ satisfy conditions (i)-$(viii)$ in the statement, and leave item (ix) for later. From Lemma 3.8 and because ξ_i are parallel along Δ, we get

$$\Delta \leq \ker \psi \cap \ker \omega_1 \cap \ker \omega_2. \tag{3.18}$$

In particular, condition (i) is satisfied. From Lemma 3.9 we have item (ii) and from Lemma 3.5 we have item (iii).

Now that item (i) has been proved, we will show that the tensors D_i carry the information of the one-forms ω_i. Therefore, the tensors D_i and the one-form ψ furnish the information about the normal covariant derivative. Using Codazzi Equation (3.12) for $Y = T \in \Gamma(\Delta)$, a unit length section, and $X \in \Gamma(\Delta^\perp)$, we get

$$\begin{aligned} 0 &= \lambda\omega_i(X)T - (\nabla_X A_{\xi_i})T + (\nabla_T A_{\xi_i})X \\ &= \lambda\omega_i(X)T + A_{\xi_i}\nabla_X T + \nabla_T A_{\xi_i} X - A_{\xi_i}\nabla_T X. \end{aligned}$$

Taking the inner product with T of both sides of the above equation and using equation (1.6) of Proposition 1.1, we obtain

$$\begin{aligned} 0 &= \lambda\omega_i(X) + \langle \nabla_T A_{\xi_i} X, T\rangle \\ &= \lambda\omega_i(X) - \langle A_{\xi_i} X, \nabla_T T\rangle \\ &= \lambda\omega_i(X) - \langle (A - \lambda I) D_i X, \delta\rangle \\ &= \lambda\omega_i(X) + \langle D_i X, \operatorname{grad}\lambda\rangle, \end{aligned}$$

or equivalently,

$$\omega_i(X) = -\frac{1}{\lambda}\langle D_i X, \operatorname{grad}\lambda\rangle, \tag{3.19}$$

for $X \in \Gamma(\Delta^\perp)$.

Let us prove the rest of the items of the statement. From the Codazzi equation (3.12) we have

$$\begin{aligned}(\nabla_X(A-\lambda I)D_i)Y &- (\nabla_Y(A-\lambda I)D_i)X \\ &= \lambda\left(\omega_i(X)Y - \omega_i(Y)X\right) + (-1)^j\left(\psi(X)A_{\xi_j}Y - \psi(Y)A_{\xi_j}X\right),\end{aligned}$$

for X, $Y \in \Gamma(\Delta^\perp)$. From equation (3.19) we get

$$\begin{aligned}\lambda\left(\omega_i(X)Y - \omega_i(Y)X\right) &= \langle D_iY, \operatorname{grad}\lambda\rangle X - \langle D_iX, \operatorname{grad}\lambda\rangle Y \\ &= (X \wedge Y)D_i^t \operatorname{grad}\lambda.\end{aligned}$$

Because $A_{\xi_j} = (A-\lambda I)D_j$, combining the last two equations we conclude that

$$\begin{aligned}(\nabla_X(A-\lambda I)D_i)Y &- (\nabla_Y(A-\lambda I)D_i)X \\ &= (X \wedge Y)D_i^t \operatorname{grad}\lambda + (-1)^j(A-\lambda I)\left(\psi(X)D_jY - \psi(Y)D_jX\right)\end{aligned}$$

for all X, $Y \in \Gamma(\Delta^\perp)$, which is item (iv).

The Ricci equation (3.14) gives

$$\begin{aligned}\langle [A_{\xi_i}, A_\mu]X, Y\rangle &= \mathrm{d}\omega_i(X,Y) - \lambda^{-1}Y(\lambda)\omega_i(X) + \lambda^{-1}X(\lambda)\omega_i(Y) \\ &\quad + (-1)^j\psi(Y)\omega_j(X) - (-1)^j\psi(X)\omega_j(Y),\end{aligned}$$

for all X, $Y \in \mathfrak{X}(M)$. From equation (3.19) we get

$$\begin{aligned}Y\omega_i(X) &= -Y(\lambda^{-1}\langle D_iX, \operatorname{grad}\lambda\rangle) \\ &= \lambda^{-2}Y(\lambda)\langle D_iX, \operatorname{grad}\lambda\rangle - \lambda^{-1}\langle \nabla_Y D_iX, \operatorname{grad}\lambda\rangle - \lambda^{-1}\operatorname{Hess}\lambda(D_iX, Y) \\ &= -\lambda^{-1}Y(\lambda)\omega_i(X) - \lambda^{-1}\langle \nabla_Y D_iX, \operatorname{grad}\lambda\rangle - \lambda^{-1}\operatorname{Hess}\lambda(D_iX, Y),\end{aligned}$$

for all X, $Y \in \Gamma(\Delta^\perp)$. Therefore,

$$\begin{aligned}\mathrm{d}\omega_i&(X,Y) - \lambda^{-1}Y(\lambda)\omega_i(X) + \lambda^{-1}X(\lambda)\omega_i(Y) \\ &= \mathrm{d}\omega_i(X,Y) + Y\omega_i(X) + \lambda^{-1}\langle \nabla_Y D_iX, \operatorname{grad}\lambda\rangle + \lambda^{-1}\operatorname{Hess}\lambda(D_iX, Y) \\ &\quad - X\omega_i(Y) - \lambda^{-1}\langle \nabla_X D_iY, \operatorname{grad}\lambda\rangle - \lambda^{-1}\operatorname{Hess}\lambda(D_iY, X) \\ &= \frac{1}{\lambda}\left(\langle (\nabla_Y D_i)X - (\nabla_X D_i)Y, \operatorname{grad}\lambda\rangle + \operatorname{Hess}\lambda(D_iX, Y) - \operatorname{Hess}\lambda(X, D_iY)\right),\end{aligned}$$

for all X, $Y \in \Gamma(\Delta^\perp)$. We also have

$$(-1)^j\psi(Y)\omega_j(X) - (-1)^j\psi(X)\omega_j(Y) \\ = (-1)^j\psi(X)\lambda^{-1}\langle D_jY, \operatorname{grad}\lambda\rangle - (-1)^j\psi(Y)\lambda^{-1}\langle D_jX, \operatorname{grad}\lambda\rangle\,,$$

for all X, $Y \in \Gamma(\Delta^\perp)$. Combining these results, we conclude (v):

$$\lambda\left(\langle AX, (A-\lambda I)D_iY\rangle - \langle (A-\lambda I)D_iX, AY\rangle\right) \\ = \langle (\nabla_Y D_i)X - (\nabla_X D_i)Y, \operatorname{grad}\lambda\rangle + \operatorname{Hess}\lambda(D_iX, Y) - \operatorname{Hess}\lambda(X, D_iY) \\ + (-1)^j\psi(X)\langle D_jY, \operatorname{grad}\lambda\rangle - (-1)^j\psi(Y)\langle D_jX, \operatorname{grad}\lambda\rangle\,,$$

for all X, $Y \in \Gamma(\Delta^\perp)$.

The Ricci equation (3.15) gives

$$\langle [A_{\xi_1}, A_{\xi_2}]X, Y\rangle = \mathrm{d}\psi(X, Y),$$

for X, $Y \in \mathfrak{X}(M)$. If $Y = T \in \Gamma(\Delta)$, then the left side is zero, and item (vi) is proven. Also, item (vii) follows from the same equation, because $A_{\xi_i} = (A - \lambda I)D_i$.

We have from Lemma 3.9 that $D_2^2 \neq -D_1^2$. For $D_2^2 \neq D_1^2$, let $D_1 = aI + bJ$ and $D_2 = cI + dJ$. Suppose we have equality. Then

$$(a^2 + b^2\epsilon)I + 2abJ = (c^2 + d^2\epsilon)I + 2cdJ.$$

In the case where $\epsilon = 0$, $ab = cd$ and $a^2 = c^2$. It follows that $a = \pm c$. If $a = c = 0$, then $W = \operatorname{span}\{J\}$, a contradiction. If not, then $b = \pm d$, and again W would be one-dimensional. If $\epsilon = 1$, then $a^2 + b^2 = c^2 + d^2$ and $ab = cd$. From the determinant value of D_i, we get $a^2 - b^2 = c^2 - d^2$. Then $a^2 = c^2$, and we can get the same contradiction of the preceding case. The same for $\epsilon = -1$. So, $(viii)$ is proven.

In Lemma 3.6, we have found a candidate $J \in \Gamma(\operatorname{End}(\Delta^\perp))$ that will make our hypersurface $f : M^n \to \mathbb{R}^{n+1}$ hyperbolic, parabolic or elliptic, depending on whether $J^2 = I$, $J^2 = 0$ or $J^2 = -I$, respectively. We will now prove the condition that is missing: $\nabla^h_T J = 0$.

Lemma 3.10. *The tensor J satisfies $\nabla^h_T J = 0$.*

Proof. We will have to treat each case separately. Let us start with the case $J^2 = I$. Without loss of generality, suppose $D_1 = aI + bJ$ with $b \neq 0$. We can do this because

$\dim W = 2$. Take $X \in \Gamma(\Delta^\perp)$ of unit length such that $D_1X = \theta X$. Then,

$$\begin{aligned}0 &= (\nabla^h_T D_1)X \\ &= \nabla^h_T \theta X - D_1 \nabla^h_T X \\ &= (T\theta)X - (D_1 - \theta I)\nabla^h_T X.\end{aligned}$$

If $\nabla^h_T X \neq 0$, since $(D_1 - \theta I)X = 0$ and $\nabla^h_T X$ is orthogonal to X, we would have

$$(D_1 - \theta I)^2 = 0,$$

and because $\det D_1 = 1/2$,

$$0 = \left(\frac{1}{2\theta} - \theta\right)^2 = \theta^2 - 1 + \frac{1}{4\theta^2}.$$

Solving this equation leads to $\theta^2 = 1/2$ or $\theta = \pm 1/\sqrt{2}$. Taking into account that $\det D_1 = 1/2$, we conclude that $D_1 = \pm\sqrt{2}^{-1} I$, a contradiction with our initial assumption. Therefore, $\nabla^h_T X = 0$ and $T\theta = 0$. We have $D_1 = aI + bJ$ with

$$a = \frac{\theta + \frac{1}{2\theta}}{2} \quad \text{and} \quad b = \frac{\theta - \frac{1}{2\theta}}{2},$$

so $0 = \nabla^h_T D_1 = (Ta)I + (Tb)J + b\nabla^h_T J = b\nabla^h_T J$. We conclude that $\nabla^h_T J = 0$.

Now, let us prove the case $J^2 = 0$. Without loss of generality, assume that there exists an orthonormal frame $\{X, Y\}$ for $\Gamma(\Delta^\perp)$ such that

$$JX = Y \quad \text{and} \quad JY = 0.$$

We can do this because we can pick $X \in (\ker J)^\perp$ and $Y \in \ker J$ such that $JX = \beta Y$ and $JY = 0$ with β non-null function. Then, we redefine $J = \beta^{-1}J$ and still have $\operatorname{span}\{I, J\} = W$ and $J^2 = 0$.

Suppose $D_1 = \sqrt{2}^{-1} I + bJ$ with $b \neq 0$. From $0 = \nabla^h_T D_1$ we get

$$\begin{aligned}0 &= (\nabla^h_T D_1)X \\ &= \nabla^h_T(\sqrt{2}^{-1} X + bY) - D_1\left(\left\langle \nabla^h_T X, Y\right\rangle Y\right) \\ &= \sqrt{2}^{-1}\left\langle \nabla^h_T X, Y\right\rangle Y + (Tb)Y + b\left\langle \nabla^h_T Y, X\right\rangle X - \sqrt{2}^{-1}\left\langle \nabla^h_T X, Y\right\rangle Y.\end{aligned}$$

Therefore, $\left\langle \nabla^h_T Y, X\right\rangle = 0$, and consequently $\nabla^h_T Y = 0$.

Now it is an easy task to show that $\nabla^h_T J = 0$. We have,

$$\begin{aligned}(\nabla^h_T J)X &= \nabla^h_T Y - J\left\langle \nabla^h_T X, Y\right\rangle Y\\ &= 0\end{aligned}$$

and

$$\begin{aligned}(\nabla^h_T J)Y &= -J\left\langle \nabla^h_T Y, X\right\rangle X\\ &= 0,\end{aligned}$$

which proves our statement.

Now, let us move on to the last case, $J^2 = -I$. From the fact that $\dim W = 2$, and because $\det D_i = 1/2$, we can suppose

$$D_1 = \frac{\cos\theta}{\sqrt{2}} I + \frac{\sin\theta}{\sqrt{2}} J \tag{3.20}$$

with $\sin\theta \neq 0$. Let X,$Y \in \Gamma(\Delta^\perp)$ be such that

$$JX = -Y \quad \text{and} \quad JY = X.$$

Then,

$$D_1 X = \frac{\cos\theta X - \sin\theta Y}{\sqrt{2}} \quad \text{and} \quad D_1 Y = \frac{\cos\theta Y + \sin\theta X}{\sqrt{2}}. \tag{3.21}$$

From the condition $\nabla^h_T D_1 = 0$, and using the formulas (3.20), (3.21) we get two equations:

$$\begin{aligned}0 &= (\nabla^h_T D_1)X\\ &= \nabla^h_T D_1 X - D_1 \nabla^h_T X\\ &= \frac{1}{\sqrt{2}}\left((T\cos\theta)X - (T\sin\theta)Y - \sin\theta\nabla^h_T Y - \sin\theta J\nabla^h_T X\right)\end{aligned}$$

and

$$\begin{aligned}0 &= (\nabla^h_T D_1)Y\\ &= \nabla^h_T D_1 Y - D_1 \nabla^h_T Y\\ &= \frac{1}{\sqrt{2}}\left((T\cos\theta)Y + (T\sin\theta)X + \sin\theta\nabla^h_T X - \sin\theta J\nabla^h_T Y\right).\end{aligned}$$

Multiplying on both sides of the first equation by $\sqrt{2}J$, we get

$$-(T\cos\theta)Y - (T\sin\theta)X - \sin\theta J\nabla^h_T Y + \sin\theta\nabla^h_T X = 0.$$

We conclude that $T(\cos\theta) = 0 = T(\sin\theta)$, and therefore

$$0 = (\nabla^h_T D_1) = \frac{\sin\theta}{\sqrt{2}}\nabla^h_T J.$$

$\square$

We have thus found a tensor $J \in \Gamma(\mathrm{End}(\Delta^\perp))$ that makes our hypersurface hyperbolic, parabolic or elliptic, depending on whether $J^2 = I$, $J^2 = 0$ or $J^2 = -I$, respectively. We now prove that the second possibility occurs precisely when f is conformally ruled, which can not happen by the assumption that f is not a Cartan hypersurface.

Lemma 3.11. *Let $f : M^n \to \mathbb{R}^{n+1}$ be an oriented hypersurface with a nowhere vanishing principal curvature of constant multiplicity $n-2$. Assume that f is not a Cartan hypersurface on any open subset of M^n and that it admits a genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$. If f is parabolic with respect to a tensor $J \in \Gamma(End(\Delta^\perp))$, then it is conformally ruled. Moreover, all genuine conformal deformations $\tilde{f} : M^n \to \mathbb{R}^{n+2}$ of f are conformally ruled with the same rulings.*

Proof. If f is parabolic, then there exists a tensor $J \in \Gamma(\mathrm{End}(\Delta^\perp))$ satisfying the following conditions:

(i) $J^2 = 0$.

(ii) $\nabla^h_T J = 0$ for all $T \in \Gamma(\Delta)$.

(iii) $C_T \in \mathrm{span}\{I, J\}$ for all $T \in \Gamma(\Delta)$.

The tensor J given might not be the same found in Lemma 3.6, but since f is not conformally surface-like on any open subset of M^n, by the assumption that it is not a Cartan hypersurface, the subspace generated by I and J must be W, the subspace generated by our endomorphisms D_ξ, where $\xi \in \mathbb{P}$. In fact, the tensor J must be a function multiple of the tensor found in Lemma 3.6.

Pick $\{X, Y\}$ orthonormal frame of $\Gamma(\Delta^\perp)$ such that $JY = 0$ and $JX = \delta Y$ with $\delta \neq 0$. We will prove that the distribution

$$L(x) = \Delta(x) \oplus Y(x)$$

is umbilical, that is there exist $Z \in \Gamma(L^\perp)$, and therefore Z is a multiple of X, such that

$$\langle \nabla_U V, X\rangle = \langle U, V\rangle \langle X, Z\rangle$$

for U,$V \in \Gamma(L)$.

Since $C_T \in \operatorname{span}\{I, J\}$ and $JY = 0$, it follows that $\langle C_T Y, X\rangle = 0$ for all $T \in \Gamma(\Delta)$. Hence

$$\langle \nabla_Y T, X\rangle = -\langle C_T Y, X\rangle = 0 \quad \text{for } T \in \Delta. \tag{3.22}$$

Because $D_i \in \operatorname{span}\{I, J\}$ with $\det D_i = 1/2$, we can suppose by changing signs on our privileged frame if necessary that

$$\sqrt{2} D_i = I + b_i J, \tag{3.23}$$

so $\sqrt{2} D_i Y = Y$. Since $\dim W = 2$, we can suppose that D_1 is not a multiple of the identity, or equivalently $b_1 \neq 0$. From $\nabla^h_T D_1 = 0$ we get

$$0 = \nabla^h_T Y - \sqrt{2} D_1 \nabla^h_T Y.$$

Since $\nabla^h_T Y$ is orthogonal to Y and $\sqrt{2} D_1$ is not the identity endomorphism, we conclude that $\nabla^h_T Y = 0$, or the equivalent equation

$$\langle \nabla_T Y, X\rangle = 0. \tag{3.24}$$

This is to be expected, because $\langle T, Y\rangle = 0$ and we want to prove that the distribution L is umbilical.

From $(A - \lambda I)C_T = \nabla^h_T A$ (equation (3.1)), we have that $(A - \lambda I)C_T$ is symmetric and from Lemma 3.6

$$\operatorname{span}\{I\} < C(\Delta) \leq \operatorname{span}\{I, J\},$$

we conclude that $(A - \lambda I)J$ is symmetric. Therefore,

$$\langle (A - \lambda I)Y, Y\rangle = \delta^{-1} \langle (A - \lambda I)JX, Y\rangle = \delta^{-1} \langle X, (A - \lambda I)JY\rangle = 0. \tag{3.25}$$

It follows that in the orthonormal frame $\{X, Y\}$ of $\Delta^\perp$ we have

$$A - \lambda I = \begin{pmatrix} \beta & \mu \\ \mu & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \beta + \lambda & \mu \\ \mu & \lambda \end{pmatrix} \tag{3.26}$$

with $\mu \neq 0$, because $A - \lambda I$ restricted to $\Delta^{\perp}$ is an isomorphism. Now, from the form of D_i given in (3.23) and the form of $A - \lambda I$ given in (3.26) we get

$$\sqrt{2} A_{\xi_i} Y = (A - \lambda I)\sqrt{2} D_i Y = (A - \lambda I)Y = \mu X$$

and

$$\sqrt{2} A_{\xi_i} X = (A - \lambda I)\sqrt{2} D_i X = (A - \lambda I)(X + b_i \delta Y) = (\beta + b_i \delta \mu)X + \mu Y.$$

Define $\theta = b_1 \delta \mu \neq 0$ and $\tilde{\theta} = b_2 \delta \mu$, so in the orthonormal frame $\{X, Y\}$ we have

$$\sqrt{2} A_{\xi_1} = \begin{pmatrix} \beta + \theta & \mu \\ \mu & 0 \end{pmatrix} \quad \text{and} \quad \sqrt{2} A_{\xi_2} = \begin{pmatrix} \beta + \tilde{\theta} & \mu \\ \mu & 0 \end{pmatrix}. \tag{3.27}$$

Let us use the Codazzi equation of A applied to $T \in \Gamma(\Delta)$ of unit length and $Y \in \Gamma(\Delta^{\perp})$ and then take the inner product with T. Doing that and using equation (3.26) we get

$$\begin{aligned} 0 &= \langle \nabla_T AY, T \rangle - \langle A \nabla_T Y, T \rangle - \langle \nabla_Y AT, T \rangle + \langle A \nabla_Y T, T \rangle \\ &= \langle \nabla_T (\mu X + \lambda Y), T \rangle + \lambda \langle \nabla_T T, Y \rangle - Y(\lambda) \\ &= -\mu \langle \nabla_T T, X \rangle - \lambda \langle \nabla_T T, Y \rangle + \lambda \langle \nabla_T T, Y \rangle - Y(\lambda), \end{aligned}$$

so we conclude

$$\mu \langle \nabla_T T, X \rangle = -Y\lambda. \tag{3.28}$$

This equation tell us that the candidate for mean curvature of L would be

$$Z = -\frac{Y\lambda}{\mu} X.$$

Now, the Codazzi equation for A applied to X, $Y \in \Gamma(\Delta^{\perp})$ and taking inner product with Y, together with equation (3.26) yields

$$\begin{aligned} 0 &= \langle \nabla_X AY, Y \rangle - \langle A \nabla_X Y, Y \rangle - \langle \nabla_Y AX, Y \rangle + \langle A \nabla_Y X, Y \rangle \\ &= \langle \nabla_X (\mu X + \lambda Y), Y \rangle - \langle \nabla_X Y, \mu X + \lambda Y \rangle - \langle \nabla_Y ((\beta + \lambda)X + \mu Y), Y \rangle \\ &\quad + \langle \nabla_Y X, \mu X + \lambda Y \rangle \\ &= \mu \langle \nabla_X X, Y \rangle + X(\lambda) + \mu \langle \nabla_X X, Y \rangle + (\beta + \lambda) \langle \nabla_Y Y, X \rangle - Y(\mu) - \lambda \langle \nabla_Y Y, X \rangle. \end{aligned}$$

So, we arrive to

$$0 = 2\mu \langle \nabla_X X, Y\rangle + X(\lambda) + \beta \langle \nabla_Y Y, X\rangle - Y(\mu). \tag{3.29}$$

Next, we need the information given by the Codazzi equation for A_{ξ_i} applied to X, $Y \in \Gamma(\Delta^\perp)$. Lets start first with $i = 1$. Using equations (3.12) and (3.27), we obtain

$$\begin{aligned}
0 &= \big\langle \big(\nabla_X \sqrt{2} A_{\xi_1}\big) Y, Y\big\rangle - \big\langle \big(\nabla_Y \sqrt{2} A_{\xi_1}\big) X, Y\big\rangle - \sqrt{2}\lambda\omega_1(X) - \sqrt{2}\psi(X)\big\langle A_{\xi_2} Y, Y\big\rangle \\
&\quad + \sqrt{2}\psi(Y)\langle A_{\xi_2} X, Y\rangle \\
&= \langle \nabla_X \mu X, Y\rangle - \langle \nabla_X Y, \mu X\rangle - \langle \nabla_Y ((\beta+\theta)X + \mu Y), Y\rangle + \langle \nabla_Y X, \mu X\rangle - \sqrt{2}\lambda\omega_1(X) \\
&\quad + \mu\psi(Y) \\
&= 2\mu \langle \nabla_X X, Y\rangle + (\beta+\theta)\langle \nabla_Y Y, X\rangle - Y(\mu) - \sqrt{2}\lambda\omega_1(X) + \mu\psi(Y).
\end{aligned}$$

For $i = 2$, using equations (3.12) and (3.27) we get

$$\begin{aligned}
0 &= \big\langle (\nabla_X \sqrt{2} A_{\xi_2}) Y, Y\big\rangle - \big\langle (\nabla_Y \sqrt{2} A_{\xi_2}) X, Y\big\rangle - \sqrt{2}\lambda\omega_2(Y) + \sqrt{2}\psi(X)\big\langle A_{\xi_1} Y, Y\big\rangle \\
&\quad - \sqrt{2}\psi(Y)\big\langle A_{\xi_1} X, Y\big\rangle \\
&= \big\langle \nabla_X \sqrt{2} A_{\xi_2} Y, Y\big\rangle - \big\langle \sqrt{2} A_{\xi_2} \nabla_X Y, Y\big\rangle - \big\langle \nabla_Y \sqrt{2} A_{\xi_2} X, Y\big\rangle + \big\langle \sqrt{2} A_{\xi_2} \nabla_Y X, Y\big\rangle \\
&\quad - \sqrt{2}\lambda\omega_2(Y) - \mu\psi(Y) \\
&= \big\langle \nabla_X \mu X, Y\big\rangle - \big\langle \nabla_X Y, \mu X\big\rangle - \big\langle \nabla_Y \big((\beta+\tilde{\theta})X + \mu Y\big), Y\big\rangle + \big\langle \nabla_Y X, \mu X\big\rangle \\
&\quad - \sqrt{2}\lambda\omega_2(Y) - \mu\psi(Y),
\end{aligned}$$

Hence,

$$0 = 2\mu \langle \nabla_X X, Y\rangle + (\beta+\theta)\langle \nabla_Y Y, X\rangle - Y(\mu) - \sqrt{2}\lambda\omega_1(X) + \mu\psi(Y). \tag{3.30}$$

and

$$0 = 2\mu \langle \nabla_X X, Y\rangle + (\beta+\tilde{\theta})\langle \nabla_Y Y, X\rangle - Y(\mu) - \sqrt{2}\lambda\omega_2(X) - \mu\psi(Y). \tag{3.31}$$

Replacing equation (3.29) into equations (3.30) and (3.31) we obtain

$$\theta \langle \nabla_Y Y, X\rangle - X(\lambda) - \sqrt{2}\lambda\omega_1(X) + \mu\psi(Y) = 0$$

and

$$\tilde{\theta} \langle \nabla_Y Y, X\rangle - X(\lambda) - \sqrt{2}\lambda\omega_2(X) - \mu\psi(Y) = 0.$$

Adding both equations,

$$(\theta + \tilde{\theta}) \langle \nabla_Y Y, X \rangle - 2X(\lambda) - \sqrt{2}\lambda \left(\omega_1(X) + \omega_2(X)\right) = 0.$$

Now using equation (3.19), the form of D_i given in (3.23) and that $(\theta + \tilde{\theta}) = (b_1 + b_2)\delta\mu$ we get

$$\begin{aligned}
(\theta + \tilde{\theta}) \langle \nabla_Y Y, X \rangle &= 2X(\lambda) + \sqrt{2}\lambda \left(\omega_1(X) + \omega_2(X)\right) \\
&= 2X(\lambda) + \lambda\big(- \lambda^{-1}\big\langle \sqrt{2}D_1 X, \operatorname{grad}\lambda\big\rangle - \lambda^{-1}\big\langle \sqrt{2}D_2 X, \operatorname{grad}\lambda\big\rangle\big) \\
&= 2X(\lambda) - \langle X + b_1\delta Y, \operatorname{grad}\lambda\rangle - \langle X + b_2\delta Y, \operatorname{grad}\lambda\rangle \\
&= 2X(\lambda) - X(\lambda) - b_1\delta Y(\lambda) - X(\lambda) - b_2\delta Y(\lambda) \\
&= -(b_1 + b_2)\delta Y(\lambda) \\
&= -\frac{\theta + \tilde{\theta}}{\mu} Y(\lambda),
\end{aligned}$$

so

$$(\theta + \tilde{\theta}) \left(\mu \langle \nabla_Y Y, X \rangle + Y\lambda\right) = 0. \tag{3.32}$$

If $(\theta + \tilde{\theta}) \neq 0$, then from equations (3.22), (3.24), (3.28) and (3.32) we get that L is an umbilical distribution with mean curvature vector $Z = -(Y\lambda/\mu)X$.

We will show now that $\theta \neq -\tilde{\theta}$, so suppose by contradiction that $\theta = -\tilde{\theta}$. From equation (3.27), we have

$$\sqrt{2}A_{\xi_1} = \begin{pmatrix} \beta + \theta & \mu \\ \mu & 0 \end{pmatrix} \quad \text{and} \quad \sqrt{2}A_{\xi_2} = \begin{pmatrix} \beta - \theta & \mu \\ \mu & 0 \end{pmatrix}.$$

Define an orthonormal frame $\{\xi, \eta\}$ for $\mathbb{P}$ by

$$\xi = \frac{1}{\sqrt{2}} \left(\xi_1 + \xi_2\right) \quad \text{and} \quad \eta = \frac{1}{\sqrt{2}} \left(\xi_1 - \xi_2\right).$$

Then, from equation (3.26) we have

$$A_\xi = \begin{pmatrix} \beta & \mu \\ \mu & 0 \end{pmatrix} = (A - \lambda I) \quad \text{and} \quad A_\eta = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.33}$$

Time put into work this new information in the Codazzi equations for the isometric immersion $\tilde{F}$. For $A = A_\mu$, using equation (3.9) and that $D_\xi = (A - \lambda I)^{-1} A_\xi = I$ we get

$$(X \wedge Y)\operatorname{grad}\lambda = \lambda\tilde{\omega}_1(X)Y - \lambda\tilde{\omega}_1(Y)X + D_\eta\left(\lambda\tilde{\omega}_2(X)Y - \lambda\tilde{\omega}_2(Y)X\right) \tag{3.34}$$

From equation (3.33), we have

$$(A - \lambda I)^{-1} = -\frac{1}{\mu^2}\begin{pmatrix} 0 & -\mu \\ -\mu & \beta \end{pmatrix}.$$

From the definition of D_η and equation (3.33), we get

$$D_\eta X = (A - \lambda I)^{-1} A_\eta X = \theta (A - \lambda I)^{-1} X = \frac{\theta}{\mu} Y$$

and

$$D_\eta Y = (A - \lambda I)^{-1} A_\eta Y = 0.$$

Therefore, from equation (3.34)

$$Y(\lambda)X - X(\lambda)Y = \lambda\tilde{\omega}_1(X)Y - \lambda\tilde{\omega}_1(Y)X - \frac{\lambda\theta}{\mu}\tilde{\omega}_2(Y)Y$$

Hence,

$$\tilde{\omega}_1(Y) + \lambda^{-1}Y(\lambda) = 0 \quad \text{and} \quad \tilde{\omega}_1(X) + \lambda^{-1}X(\lambda) - \frac{\theta}{\mu}\tilde{\omega}_2(Y) = 0. \tag{3.35}$$

Now, from the Codazzi equation of $A_\xi = A - \lambda I$ we have

$$(\nabla_Z A_\xi)W - (\nabla_W A_\xi)Z = A_{\nabla^\perp_Z \xi} W - A_{\nabla^\perp_W \xi} Z$$

where Z, $W \in \mathfrak{X}(M)$. For the left part of the equation,

$$(\nabla_Z A_\xi)W - (\nabla_W A_\xi)Z = (\nabla_Z (A - \lambda I))W - (\nabla_W (A - \lambda I))Z = (Z \wedge W)\operatorname{grad}\lambda.$$

On the other hand, from equation (3.12) we get

$$A_{\nabla^\perp_Z \xi} W - A_{\nabla^\perp_W \xi} Z = \lambda\tilde{\omega}_1(Z)W + \tilde{\psi}(Z)A_\eta W - \lambda\tilde{\omega}_1(W)Z - \tilde{\psi}(W)A_\eta Z.$$

Combining both parts we arrive to

$$(Z \wedge W)\operatorname{grad}\lambda = \lambda\tilde{\omega}_1(Z)W + \tilde{\psi}(Z)A_\eta W - \lambda\tilde{\omega}_1(W)Z - \tilde{\psi}(W)A_\eta Z.$$

For vectors $Z = T \in \Delta$ and $W = X$, using equation (3.33) and Lemma 3.8 we conclude that

$$X(\lambda)T = \tilde{\psi}(T)\theta X - \lambda\tilde{\omega}_1(X)T,$$

or equivalently,

$$X(\lambda) = -\lambda\tilde{\omega}_1(X) \quad \text{and} \quad \Delta \leq \ker\tilde{\psi}. \tag{3.36}$$

Replacing now $Z = X$ and $W = Y$ and using equation (3.33) we get

$$(X \wedge Y)\operatorname{grad}\lambda = \lambda\tilde{\omega}_1(X)Y - \lambda\tilde{\omega}_1(Y)X - \theta\tilde{\psi}(Y)X.$$

From this equation, we arrive to

$$Y(\lambda) = -\theta\tilde{\psi}(Y) - \lambda\tilde{\omega}_1(Y) \quad \text{and} \quad -X(\lambda) = \lambda\tilde{\omega}_1(X). \tag{3.37}$$

Therefore using equations (3.35), (3.36) and (3.37) we conclude

$$\Delta \oplus \operatorname{span}\{Y\} \leq \ker\tilde{\psi} \cap \ker\tilde{\omega}_2 \quad \text{and} \quad \lambda^{-1}Z(\lambda) + \tilde{\omega}_1(Z) = 0, \text{ for } Z \in \mathfrak{X}(M), \tag{3.38}$$

for $Z \in \mathfrak{X}(M)$.

Now, the second fundamental form of $\tilde{F}$ is given by

$$\begin{aligned}\alpha^{\tilde{F}}(X,Y) &= \langle AX, Y\rangle\,\mu + \langle (A-\lambda I)X, Y\rangle\,\xi + \langle A_\eta X, Y\rangle\,\eta - \langle (A-\lambda I)X, Y\rangle\,\zeta \\ &= \langle AX, Y\rangle\,(\mu+\xi-\zeta) - \lambda\,\langle X, Y\rangle\,(\xi-\zeta) + \langle A_\eta X, Y\rangle\,\eta.\end{aligned}$$

From equations (3.4), (3.6), (3.7) and (3.38) we get

$$\nabla^\perp_X(\mu+\xi-\zeta) = \lambda^{-1}X(\lambda)(\mu-\zeta) + \tilde{\omega}_1(X)(\mu-\zeta) + \tilde{\psi}(X)\eta = \tilde{\psi}(X)\eta, \tag{3.39}$$

for $X \in \mathfrak{X}(M)$. While using equations (3.4), (3.7) and (3.38) we get

$$\begin{aligned}\nabla^\perp_X\lambda(\xi-\zeta) &= X(\lambda)(\xi-\zeta) + \lambda\nabla^\perp_X(\xi-\zeta) \\ &= X(\lambda)(\xi-\zeta) - X(\lambda)(\mu-\zeta) + \lambda\tilde{\psi}(X)\eta + X(\lambda)\mu - X(\lambda)\xi + \lambda\tilde{\omega}_2(X)\eta \\ &= \lambda\big(\tilde{\psi}(X) + \tilde{\omega}_2(X)\big)\eta,\end{aligned} \tag{3.40}$$

for $X \in \mathfrak{X}(M)$.

The second fundamental form of the isometric light-cone representative $F\colon M^n \to$

$\mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ is given by

$$\alpha^F(X,Y) = \langle AX, Y\rangle\, \Psi_* N - \langle X, Y\rangle\, w.$$

Define a vector-bundle isometry $\tau\colon N_F M \to L = \{\eta\}^\perp$ by setting

$$\tau\Psi_* N = \mu + \xi - \zeta, \quad \tau w = \lambda(\xi - \zeta) \quad \text{and} \quad \tau F = \tilde{F}.$$

From equations (3.39) and (3.40) the vector bundle isometry is parallel in the induced connection on L. By Lemma 2.2, there exists an isometric immersion $H : W \subset \mathbb{V}^{n+2} \to \mathbb{V}^{n+3}$ with $F(M^n) \subset W$, such that $\tilde{F} = H \circ F$. It follows from Proposition 2.1 that there exists a conformal immersion $h : V \to \mathbb{R}^{n+p}$ of an open subset $V \supset f(M^n)$ of $\mathbb{R}^{n+1}$ such that $\tilde{f} = h \circ f$, contradicting the assumption that $\tilde{f}$ is a genuine conformal deformation of f. Hence, $\theta \neq -\tilde{\theta}$ and L is an umbilical distribution.

We will prove now that the restriction of f to each leaf of L is also umbilical. Define $g = f \circ i : \sigma^{n-1} \to \mathbb{R}^{n+1}$ where σ is a leaf generated by the umbilical distribution L. From equation (3.25) we get

$$\alpha^g(Y,Y) = f_*\alpha^i(Y,Y) + \alpha^f(Y,Y) = f_*Z + \lambda N$$

and

$$\alpha^g(T,S) = f_*\alpha^i(T,S) + \alpha^f(T,S) = \langle T,S\rangle\, f_*Z + \lambda\,\langle T,S\rangle\, N,$$

for T, $S \in \Delta$. So g is umbilical with $f_*Z + \lambda N$ as mean curvature. We conclude that f is conformally ruled.

Let us now prove that $\tilde{f}$ is conformally ruled with the same rulings as f. From equation (3.25) and (3.27) we have

$$\alpha^{\tilde{F}}(S,T) = \lambda\,\langle S,T\rangle\,\mu, \quad \text{and} \quad \alpha^{\tilde{F}}(Y,Y) = \lambda\mu$$

for S, $T \in \Delta$. Therefore,

$$\alpha^{\tilde{F}}(Z,W) = \lambda\,\langle Z,W\rangle\,\mu,$$

for Z, $W \in L$. It follows that $\langle A_{\xi_i} Z, W\rangle = 0$, for Z, $W \in L$. As in the demonstration of Lemma 3.4, consider $\xi_i = \Psi_*\rho_i + \eta_i$, for $i = 1,2$, orthogonal decomposition, where $\rho_i \in N_{\tilde{f}} M$ and $\eta_i \in \operatorname{span}\{\tilde{F}, \tilde{\zeta}\}$. Then,

$$0 = \varphi^{-1}\big\langle A^{\tilde{f}}_{\rho_i} Z, W\big\rangle - \big\langle Z, W\big\rangle\big\langle \tilde{\zeta}, \eta_i\big\rangle.$$

From the above equation ρ_i cannot be trivial, otherwise ξ_i would be trivial. We conclude that

$$A^{\tilde{f}}_{\rho_i}\big|_L = \beta_i I.$$

Also from the fact that $\langle \xi_1, \xi_2 \rangle = 0$, $\left\langle \xi_i, \tilde{F} \right\rangle = 0$ and $\eta_i \in \operatorname{span}\{\tilde{F}, \tilde{\zeta}\}$, we get that η_i is a multiple of $\tilde{F}$ and hence ρ_1 and ρ_2 has unit length and are orthogonal. As before, let $g = \tilde{f} \circ i : \sigma^{n-1} \to \mathbb{R}^{n+2}$ where σ^{n-1} the leaf generated by the umbilical distribution L and $i : \sigma^{n-1} \to M^n$ is the inclusion. Then,

$$\alpha^g(Z, W) = \tilde{f}_* \alpha^i(Z, W) + \alpha^{\tilde{f}}(Z, W) = \langle Z, W \rangle \tilde{f}_* Z + \beta_1 \langle Z, W \rangle \rho_1 + \beta_2 \langle Z, W \rangle \rho_2,$$

where Z, $W \in L$. So, g is umbilical with $\tilde{f}_* Z + \beta_1 \rho_1 + \beta_2 \rho_2$ as mean curvature vector, and therefore $\tilde{f}$ is also conformally ruled with the same rulings as f. $\square$

It remains to prove condition (ix).

Lemma 3.12. *The tensors D_1 and D_2 satisfy*

$$\mathit{rank}\,(D_1^2 + D_2^2 - I) = 2.$$

Proof. We will divide the proof in two cases, depending on whether f is elliptic or hyperbolic.

Elliptic Case

This case is almost trivial. Let $D_1 = aI + bJ$ and $D_2 = cI + dJ$. Since $\det D_i = 1/2$ we must have $a^2 + b^2 = c^2 + d^2 = 1/2$, then

$$D_1^2 + D_2^2 - I = \begin{pmatrix} a^2 - b^2 + c^2 - d^2 - 1 & 2(ab + cd) \\ -2(ab + cd) & a^2 - b^2 + c^2 - d^2 - 1 \end{pmatrix}.$$

We cannot have $\operatorname{rank}(D_1^2 + D_2^2 - I) = 1$, because neither column is a multiple of the other. If the endomorphism $D_1^2 + D_2^2 - I$ has rank zero, then from

$$a^2 + b^2 + c^2 + d^2 = 1$$

and

$$a^2 - b^2 + c^2 - d^2 = 1$$

we conclude that $b = d = 0$. This is a contradiction, because we cannot have $W = \text{span}\{I\}$.

Hyperbolic Case

Suppose that $\text{rank}\,(D_1^2 + D_2^2 - I) < 2$ and let

$$\sqrt{2}D_1 = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1^{-1} \end{pmatrix} \quad \text{and} \quad \sqrt{2}D_2 = \begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_2^{-1} \end{pmatrix}. \tag{3.41}$$

Then,

$$2D_1^2 + 2D_2^2 - 2I = \begin{pmatrix} \theta_1^2 + \theta_2^2 - 2 & 0 \\ 0 & \theta_1^{-2} + \theta_2^{-2} - 2 \end{pmatrix}, \tag{3.42}$$

so without loss of generality (if not interchange θ_i with θ_i^{-1}) suppose $\theta_1^2 + \theta_2^2 = 2$.

Notice that if we define

$$\sqrt{2}\xi = \theta_1\xi_1 + \theta_2\xi_2 \quad \text{and} \quad \sqrt{2}\eta = -\theta_2\xi_1 + \theta_1\xi_2$$

we have that $\{\xi, \eta\}$ is an orthonormal frame of $\mathbb{P}$ with $D_\xi = I$ and D_η has rank equal to one (D_η cannot be null otherwise $\dim W = 1$). Therefore, $A_\xi = A - \lambda I$ and A_η has rank equal to one. As a consequence, there exist orthogonal eigenvectors X, $Y \in \Gamma(\Delta^\perp)$ with $A_\eta X = 0$ and $A_\eta Y \neq 0$.

Time to input this information into the Codazzi equations for the isometric immersion $\tilde{F}$ and see what we can find. For $A = A_\mu$ using equation (3.8), $A_\xi = A - \lambda I$ and that $A_\eta X = 0$ we get

$$\begin{aligned} 0 &= -\tilde{\omega}_1(X)A_\xi Y - \tilde{\omega}_2(X)A_\eta Y - \lambda^{-1}X(\lambda)A_\varsigma Y \\ &\quad + \tilde{\omega}_1(Y)A_\xi X + \tilde{\omega}_2(Y)A_\eta X + \lambda^{-1}Y(\lambda)A_\varsigma X \\ &= -\left[\tilde{\omega}_1(X) + \lambda^{-1}X(\lambda)\right](A - \lambda I)Y - \tilde{\omega}_2(X)A_\eta Y + \left[\tilde{\omega}_1(Y) + \lambda^{-1}Y(\lambda)\right](A - \lambda I)X. \end{aligned}$$

Operating both sides of the equation by $(A - \lambda I)^{-1}$ and rearranging appropriately we obtain

$$\left[\tilde{\omega}_1(X) + \lambda^{-1}X(\lambda)\right]Y + \tilde{\omega}_2(X)D_\eta Y = \left[\tilde{\omega}_1(Y) + \lambda^{-1}Y(\lambda)\right]X. \tag{3.43}$$

Now, let us work on the Codazzi equation of $A_\xi = A - \lambda I$. From equation (3.12) we have

$$(\nabla_Z A_\xi)W - (\nabla_W A_\xi)X = \lambda\tilde{\omega}_1(Z)W + \tilde{\psi}(Z)A_\eta W - \lambda\tilde{\omega}_1(W)Z - \tilde{\psi}(W)A_\eta Z,$$

for Z, $W \in \mathfrak{X}(M)$. Since $A_\xi = A - \lambda I$, the left part of the above equation is

$$\begin{aligned}(\nabla_Z A_\xi)W - (\nabla_W A_\xi)Z &= (\nabla_Z(A - \lambda I))W - (\nabla_W(A - \lambda I))Z \\ &= W(\lambda)Z - Z(\lambda)W \\ &= (Z \wedge W)\operatorname{grad}\lambda.\end{aligned}$$

Combining both parts we get

$$(Z \wedge W)\operatorname{grad}\lambda = \lambda\tilde{\omega}_1(Z)W + \tilde{\psi}(Z)A_\eta W - \lambda\tilde{\omega}_1(W)Z - \tilde{\psi}(W)A_\eta Z. \tag{3.44}$$

For $Z = X$ and $W = T \in \Delta$, using Lemma 3.8, $A_\eta X = 0$ and that $T(\lambda) = 0$ we obtain the equality

$$-X(\lambda)T = \lambda\tilde{\omega}_1(X)T$$

and so $-X(\lambda) = \lambda\tilde{\omega}_1(X)$. Replacing in equation (3.43), we get

$$\tilde{\omega}_2(X)D_\eta Y = \left[\tilde{\omega}_1(Y) + \lambda^{-1}Y(\lambda)\right] X. \tag{3.45}$$

For $Z = T$ and $W = Y$ in equation (3.44) we get

$$Y(\lambda)T = \tilde{\psi}(T)A_\eta Y - \lambda\tilde{\omega}_1(Y)T,$$

so $\Delta \leq \ker\tilde{\psi}$ and $-Y(\lambda) = \lambda\tilde{\omega}_1(Y)$. Therefore, taking into account $A_\eta Y \neq 0$, replacing in equation (3.45) we obtain $\tilde{\omega}_2(X) = 0$. Lastly, for $Z = X$ and $W = Y$,

$$(X \wedge Y)\operatorname{grad}\lambda = \lambda\tilde{\omega}_1(X)Y + \tilde{\psi}(X)A_\eta Y - \lambda\tilde{\omega}_1(Y)X$$

or $\tilde{\psi}(X) = 0$. In summary, we have

$$\Delta \oplus \operatorname{span}\{X\} \leq \ker\tilde{\psi} \cap \ker\tilde{\omega}_2 \tag{3.46}$$

and

$$\lambda^{-1}Z(\lambda) + \tilde{\omega}_1(Z) = 0, \tag{3.47}$$

for $Z \in \mathfrak{X}(M)$. Using equations (3.4), (3.6), (3.7) and (3.47) we observe that

$$\begin{aligned}\nabla^\perp_Z(\mu + \xi - \zeta) &= \lambda^{-1}Z(\lambda)(\mu - \zeta) + \tilde{\omega}_1(Z)(\mu - \zeta) + \tilde{\psi}(Z)\eta \\ &= \tilde{\psi}(Z)\eta,\end{aligned} \tag{3.48}$$

for $Z \in \mathfrak{X}(M)$. Similarly, using equations (3.4), (3.7) and (3.47) we get

$$\begin{aligned}\nabla^{\perp}_{Z}\lambda(\xi-\zeta) &= Z(\lambda)(\xi-\zeta)+\lambda\tilde{\omega}_1(Z)(\mu-\zeta)+\lambda\tilde{\psi}(Z)\eta+Z(\lambda)\mu+\lambda\tilde{\omega}_1(Z)\xi+\lambda\tilde{\omega}_2(Z)\eta\\ &= \lambda\left(\tilde{\psi}(Z)+\tilde{\omega}_2(Z)\right)\eta, \end{aligned}\tag{3.49}$$

for $Z \in \mathfrak{X}(M)$.

Let us rearrange the expression of the second fundamental form of $\tilde{F}$ to

$$\begin{aligned}\alpha^{\tilde{F}}(X,Y) &= \langle AX,Y\rangle\,\mu+\langle (A-\lambda I)X,Y\rangle\,\xi-\langle A_\eta X,Y\rangle\,\eta-\langle (A-\lambda I)X,Y\rangle\,\zeta\\ &= \langle AX,Y\rangle\,(\mu+\xi-\zeta)+\langle A_\eta X,Y\rangle\,\eta-\lambda\,\langle X,Y\rangle\,(\xi-\zeta).\end{aligned}$$

Let $L=\operatorname{span}\{\eta\}^{\perp}$ and let F be the isometric light-cone representative of f. Define a vector bundle isometry $\tau\colon N_FM\to L$ by setting

$$\tau\Psi_*N=\mu+\xi-\zeta,\quad \tau w=\lambda(\xi-\zeta)\quad\text{and}\quad \tau F=\tilde{F}.$$

From equations (3.48) and (3.49) the vector bundle isometry τ is parallel in the induced connection on L. We have all the conditions of Lemma 2.2, where item (iii) follows from (3.46). Therefore, there exists an isometric immersion $H : W \subset \mathbb{V}^{n+2} \to \mathbb{V}^{n+3}$ with $F(M^n) \subset W$, such that $\tilde{F} = H \circ F$. By Lemma 2.1, there exist a conformal immersion $h : V \to \mathbb{R}^{n+p}$ of an open subset $V \supset f(M^n)$ of $\mathbb{R}^{n+1}$ such that $\tilde{f} = h \circ f$, contradicting the assumption that $\tilde{f}$ is a genuine conformal deformation of f. □

We have finished demonstrating the direct implication of the proposition 3.2. We will now prove the converse of Proposition 3.2. As the reader might already suspect, the idea is to use the tensors D_i, for $i = 1, 2$, and the one form ψ to define a compatible connection $\hat{\nabla}$ and a symmetric form $\hat{\alpha}$ with the same formulas we got in the direct implication. With the aid of items (i) to $(viii)$, we will show that they satisfy the Gauss, Codazzi and Ricci equations and this will provide us with an application $\tilde{F} : M^n \to \mathbb{L}^{n+4}$. With a bit more of work, we will ensure that its image is in the light-cone $\mathbb{V}^{n+3}$. We will use item (ix) to ensure that the conformal immersion

$$\tilde{f}=\mathcal{C}(\tilde{F}) : M^n \to \mathbb{R}^{n+2}$$

is a genuine conformal deformation of f.

Choose an orthonormal frame μ, ξ_1, ξ_2 and ζ of the trivial bundle $E = M^n \times \mathbb{L}^4$ where ζ is a time-like vector. Extend the definition of the tensors D_i to Δ by requiring

$\Delta \leq \ker D_i$. Motivated by equation (3.19) define a one-form

$$\omega_i(X) = -\frac{1}{\lambda} \langle D_i X, \operatorname{grad} \lambda \rangle ,$$

where λ is the non-null principal curvature of constant multiplicity $n-2$ of the hypersurface $f : M^n \to \mathbb{R}^{n+1}$. Also drived by equations (3.4), (3.5), (3.6) and (3.7) define a compatible connection $\hat{\nabla}$ on E by declaring

$$\begin{aligned}
\hat{\nabla}_X \mu &= -\omega_1(X)\xi_1 - \omega_2(X)\xi_2 - \lambda^{-1} X(\lambda)\zeta, \\
\hat{\nabla}_X \xi_1 &= \omega_1(X)(\mu - \zeta) + \psi(X)\xi_2, \\
\hat{\nabla}_X \xi_2 &= \omega_2(X)(\mu - \zeta) - \psi(X)\xi_1, \\
\hat{\nabla}_X \zeta &= -\lambda^{-1} X(\lambda)\mu - \omega_1(X)\xi_1 - \omega_2(X)\xi_2,
\end{aligned}$$

for $X \in \mathfrak{X}(M)$, and the extending the definition. Since $n \geq 6$, we have that λ is a Dupin principal curvature (Proposition 1.1). Taking into account this fact, the definition of the connection, the definition of the one-forms ω_i, together with item (i), we conclude that μ, ξ_1, ξ_2 and ζ are parallel sections along Δ on the connection $\hat{\nabla}$.

Let

$$\hat{\alpha} : TM \times TM \to E$$

be the bilinear form defined by

$$\begin{aligned}
\hat{\alpha}(X, Y) = {} & \langle AX, Y \rangle \mu + \langle (A - \lambda I) D_1 X, Y \rangle \xi_1 + \langle (A - \lambda I) D_2 X, Y \rangle \xi_2 \\
& - \langle (A - \lambda I) X, Y \rangle \zeta.
\end{aligned}$$

From the symmetry of $(A - \lambda I)C_T$ (see equation (3.1)), and because f is hyperbolic or elliptic and not conformally surface-like, we obtain that $(A - \lambda I)J$ is also symmetric. Since $D_i \in \operatorname{span}\{I, J\}$, we get that $(A - \lambda I)D_i$ is also symmetric. Then, we get the symmetry of $\hat{\alpha}$.

From Proposition 11 and Proposition 12 of Chapter 4 in [18], in order to prove that $\hat{\alpha}$ satisfies the Gauss equation it is enough to show that $K(X, Y) = \hat{K}(X, Y)$ for all orthonormal vectors X,$Y \in TM$, where K is the sectional curvature of M^n (the knowledge of all sectional curvatures determines the curvature tensor). That equality is clear if X or Y belongs to Δ, because

$$\Delta = \ker(A - \lambda I)D_1 \cap \ker(A - \lambda I)D_2 \cap \ker(A - \lambda I). \tag{3.50}$$

So, for orthonormal X, $Y \in \Delta^{\perp}$, and using item (ii), we have

$$\begin{aligned}\hat{K}(X,Y) &= \langle \hat{\alpha}(X,X), \hat{\alpha}(Y,Y)\rangle - \langle \hat{\alpha}(X,Y), \hat{\alpha}(X,Y)\rangle \\ &= K(X,Y) + \det(A-\lambda I)D_1 + \det(A-\lambda I)D_2 - \det(A-\lambda I) \\ &= K(X,Y),\end{aligned}$$

which proves our claim. Therefore $\hat{\alpha}$ satisfies the Gauss equation.

Let us move on to the Codazzi equations. First notice from the definition of $\hat{\alpha}$ we must prove that

$$A_{\mu} = A, \quad A_{\xi_1} = (A-\lambda I)D_1, \quad A_{\xi_2} = (A-\lambda I)D_2 \quad \text{and} \quad A_{\zeta} = A - \lambda I.$$

satisfy the Codazzi equations. In order to prove that $A_{\mu} = A$ satisfies the Codazzi equation, we must show that

$$A_{\hat{\nabla}_Z \mu} W = A_{\hat{\nabla}_W \mu} Z,$$

for all Z, $W \in \mathfrak{X}(M)$. Using the definition of the connection, equation (3.50), and the fact that λ is a Dupin principal curvature, we get

$$\begin{aligned}A_{\hat{\nabla}_Z \mu} T - A_{\hat{\nabla}_T \mu} Z &= -\omega_1(Z)A_{\xi_1}T - \omega_2(Z)A_{\xi_2}T - \lambda^{-1}Z(\lambda)A_{\zeta}T \\ &\quad + \omega_1(T)A_{\xi_1}Z + \omega_2(T)A_{\xi_2}Z + \lambda^{-1}T(\lambda)A_{\zeta}Z \\ &= 0,\end{aligned}$$

for all $T \in \Gamma(\Delta)$ and $Z \in \mathfrak{X}(M)$. On the other hand, using the definition of ω_i, item (ii) and the property

$$D_i X \wedge D_i Y = \det D_i (X \wedge Y),$$

for $i = 1, 2$, we get

$$\begin{aligned}A_{\hat{\nabla}_X \mu} Y - A_{\hat{\nabla}_Y \mu} X &= -\omega_1(X)A_{\xi_1}Y - \omega_2(X)A_{\xi_2}Y - \lambda^{-1}X(\lambda)A_{\zeta}Y \\ &\quad + \omega_1(Y)A_{\xi_1}X + \omega_2(Y)A_{\xi_2}X + \lambda^{-1}Y(\lambda)A_{\zeta}X \\ &= \lambda^{-1}(A-\lambda I)\left(D_1X(\lambda)D_1Y + D_2X(\lambda)D_2Y - X(\lambda)Y\right) \\ &\quad + \lambda^{-1}(A-\lambda I)\left(-D_1Y(\lambda)D_1X - D_2Y(\lambda)D_2X + Y(\lambda)X\right) \\ &= \lambda^{-1}(A-\lambda I)\left(-(D_1X \wedge D_1Y)\operatorname{grad}\lambda - (D_2X \wedge D_2Y)\operatorname{grad}\lambda\right) \\ &\quad + \lambda^{-1}(A-\lambda I)(X \wedge Y)\operatorname{grad}\lambda \\ &= 0,\end{aligned}$$

for all X, $Y \in \Gamma(\Delta^\perp)$. This proves the Codazzi equation for μ.

Let us prove the Codazzi equation of $A_\zeta = A - \lambda I$. Using the Codazzi equation for A, taking into account that ζ is parallel along Δ, that λ is a Dupin principal and the definition of the connection $\hat{\nabla}$, we obtain

$$\begin{aligned}
(\nabla_Z A_\zeta)T &- (\nabla_T A_\zeta)Z - A_{\hat{\nabla}_Z\zeta}T + A_{\hat{\nabla}_T\zeta}Z \\
&= (\nabla_Z(A-\lambda I))T - (\nabla_T(A-\lambda I))Z - A_{\hat{\nabla}_Z\zeta}T \\
&= -Z(\lambda)T + T(\lambda)Z + \lambda^{-1}Z(\lambda)AT + \omega_1(Z)A_{\xi_1}T + \omega_2(Z)A_{\xi_2}T \\
&= 0,
\end{aligned}$$

for all $T \in \Gamma(\Delta)$ and $Z \in \mathfrak{X}(M)$. We are left the case when X, $Y \in \Gamma(\Delta^\perp)$. Using the Codazzi equation for A, the definition of the connection $\hat{\nabla}$, the definition of the one-forms ω_i and item (ii) we get

$$\begin{aligned}
(\nabla_X A_\zeta)Y &- (\nabla_Y A_\zeta)X - A_{\hat{\nabla}_X\zeta}Y + A_{\hat{\nabla}_Y\zeta}X \\
&= (\nabla_X(A-\lambda I))Y - (\nabla_Y(A-\lambda I))X - A_{\hat{\nabla}_X\zeta}Y + A_{\hat{\nabla}_Y\zeta}X \\
&= -X(\lambda)Y + Y(\lambda)X + \lambda^{-1}X(\lambda)AY + \omega_1(X)A_{\xi_1}Y + \omega_2(X)A_{\xi_2}Y \\
&\quad - \lambda^{-1}Y(\lambda)AX - \omega_1(Y)A_{\xi_1}X - \omega_2(Y)A_{\xi_2}X \\
&= \lambda^{-1}(A-\lambda I)\left(X(\lambda)Y - Y(\lambda)X - D_1X(\lambda)D_1Y - D_2X(\lambda)D_2Y\right) \\
&\quad + \lambda^{-1}(A-\lambda I)\left(D_1Y(\lambda)D_1X + D_2Y(\lambda)D_2X\right) \\
&= \lambda^{-1}(A-\lambda I)\left(-(X\wedge Y)\operatorname{grad}\lambda\right) \\
&\quad + \lambda^{-1}(A-\lambda I)\left((D_1X\wedge D_1Y)\operatorname{grad}\lambda + (D_2X\wedge D_2Y)\operatorname{grad}\lambda\right) \\
&= 0.
\end{aligned}$$

This concludes the proof for the Codazzi equation of A_ζ.

Now, it is time to prove the Codazzi equation for A_{ξ_i}. We must show that

$$(\nabla_Z A_{\xi_i})W - (\nabla_W A_{\xi_i})Z = A_{\hat{\nabla}_Z\xi_i}W - A_{\hat{\nabla}_W\xi_i}Z.$$

First, let us suppose $Z = T$, $W = S \in \Gamma(\Delta)$. Then, because ξ_i is parallel along Δ, the right hand side of the equation is zero. Since $\Delta \leq \ker A_{\xi_i}$, we must show that

$$A_{\xi_i}\nabla_S T - A_{\xi_i}\nabla_T S = 0.$$

From the symmetry of A_{ξ_i}, $\operatorname{Img} A_{\xi_i} \leq \Delta^\perp$ and because the distribution Δ is umbilical, for $X \in \Gamma(\Delta^\perp)$ we have

$$\langle A_{\xi_i}\nabla_S T, X\rangle - \langle A_{\xi_i}\nabla_T S, X\rangle = \langle S, T\rangle \langle \delta, A_{\xi_i}X\rangle - \langle T, S\rangle \langle \delta, A_{\xi_i}X\rangle = 0,$$

where δ is the mean curvature vector field of Δ. For $R \in \Gamma(\Delta)$, since $A_{\xi_i}R = 0$, we have

$$\langle A_{\xi_i}\nabla_S T, R\rangle - \langle A_{\xi_i}\nabla_T S, R\rangle = 0.$$

This shows that, at least for S, $T \in \Gamma(\Delta)$, the Codazzi equation for A_{ξ_i} is valid.

Now, suppose $Z = X \in \Gamma(\Delta^\perp)$ and $W = T \in \Gamma(\Delta)$. Then, from the definition of the connection $\hat{\nabla}$, the fact that ξ_i is parallel along Δ and the definition A_{ξ_i}, we get

$$\begin{aligned}&(\nabla_X A_{\xi_i})T - (\nabla_T A_{\xi_i})X - A_{\hat{\nabla}_X \xi_i}T + A_{\hat{\nabla}_T \xi_i}X\\ &\quad = -(A - \lambda I)D_i\nabla_X T - \nabla_T(A - \lambda I)D_i X + (A - \lambda I)D_i\nabla_T X - \lambda\omega_i(X)T.\end{aligned}$$

Taking the inner product with $S \in \Gamma(\Delta)$, the definition of the one-form ω_i, and using equation (1.6), we get

$$\begin{aligned}\langle (A - \lambda I)D_i X, \nabla_T S\rangle &- \lambda\omega_i(X)\langle T, S\rangle\\ &= \langle T, S\rangle \langle (A - \lambda I)D_i X, \delta\rangle - \lambda\omega_i(X)\langle T, S\rangle\\ &= -\langle T, S\rangle \langle D_i X, \operatorname{grad}\lambda\rangle - \lambda\omega_i(X)\langle T, S\rangle\\ &= 0.\end{aligned}$$

For the horizontal component, we must prove

$$(A - \lambda I)D_i C_T X - \big(\nabla^h_T(A - \lambda I)\, D_i\big)X = 0.$$

Now,

$$\begin{aligned}\nabla^h_T(A - \lambda I)D_i &= \nabla^h_T A D_i - \lambda\nabla^h_T D_i\\ &= (\nabla^h_T A)D_i\\ &= (A - \lambda I)C_T D_i\\ &= (A - \lambda I)D_i C_T\end{aligned}$$

where we have used that λ is Dupin, equation (3.1) and item (iii). Therefore, the horizontal and vertical components are zero, which proves the equation for $X \in \mathfrak{X}(M)$ and

$T \in \Gamma(\Delta)$.

The last case is when X, $Y \in \Gamma(\Delta^{\perp})$. On one hand, working on one side of the Codazzi equation we get

$$\begin{aligned}
A_{\hat{\nabla}_X \xi_i} Y - A_{\hat{\nabla}_Y \xi_i} X &= \omega_i(X)AY - \omega_i(X)A_\zeta Y + (-1)^j \psi(X) A_{\xi_j} Y \\
&\quad - \omega_i(Y)AX + \omega_i(Y)A_\zeta X - (-1)^j \psi(Y) A_{\xi_j} X \\
&= \lambda\omega_i(X)Y - \lambda\omega_i(Y)X + (-1)^j (A - \lambda I)\left(\psi(X)D_jY - \psi(Y)D_jX\right) \\
&= -D_iX(\lambda)Y + D_iY(\lambda)X + (-1)^j (A - \lambda I)\left(\psi(X)D_jY - \psi(Y)D_jX\right) \\
&= (X \wedge Y)D_i^t \operatorname{grad}\lambda + (-1)^j (A - \lambda I)\left(\psi(X)D_jY - \psi(Y)D_jX\right).
\end{aligned}$$

On the other hand, from item (iv), we already have the left part of the equation:

$$(\nabla_X A_{\xi_i})Y - (\nabla_Y A_{\xi_i})X = (X \wedge Y)D_i^t \operatorname{grad}\lambda + (-1)^j (A - \lambda I)(\psi(X)D_jY - \psi(Y)D_jX).$$

Comparing both sides of the equation, we conclude that the Codazzi equation for A_{ξ_i} has been proved.

Now, let us move on to the Ricci equations. Let us start with the Ricci equation for μ and ζ. Since $A_\mu = A$ and $A_\zeta = (A - \lambda I)$ commute, we have $\langle [A_\mu, A_\zeta]Z, W\rangle = 0$. On the other hand, from the definition of the connection $\hat{\nabla}$ we have

$$\begin{aligned}
\left\langle \hat{R}(Z,W)\mu, \zeta \right\rangle &= \left\langle \hat{\nabla}_Z \hat{\nabla}_W \mu, \zeta \right\rangle - \left\langle \hat{\nabla}_W \hat{\nabla}_Z \mu, \zeta \right\rangle - \left\langle \hat{\nabla}_{[Z,W]} \mu, \zeta \right\rangle \\
&= \left\langle \hat{\nabla}_Z \big(-\omega_1(W)\xi_1 - \omega_2(W)\xi_2 - \lambda^{-1} W(\lambda)\zeta \big), \zeta \right\rangle \\
&\quad - \left\langle \hat{\nabla}_W \big(-\omega_1(Z)\xi_1 - \omega_2(Z)\xi_2 - \lambda^{-1} Z(\lambda)\zeta \big), \zeta \right\rangle - \lambda^{-1}[Z,W](\lambda) \\
&= -\omega_1(W)\omega_1(Z) - \omega_2(W)\omega_2(Z) - \lambda^{-2} Z(\lambda)W(\lambda) + \lambda^{-1} ZW(\lambda) \\
&\quad + \omega_1(Z)\omega_1(W) + \omega_2(Z)\omega_2(W) + \lambda^{-2} W(\lambda)Z(\lambda) - \lambda^{-1} WZ(\lambda) \\
&\quad - \lambda^{-1}[Z,W](\lambda) \\
&= 0,
\end{aligned}$$

for all Z, $W \in \mathfrak{X}(M)$. Combining both equations, we have shown that the Ricci equation for μ and ζ is valid.

Let us prove the Ricci equation for A_μ and A_{ξ_i}. First, let us prove for X, $Y \in \Gamma(\Delta^{\perp})$. Using the symmetry of A and $(A - \lambda I)D_i$, on one hand we have

$$\langle [A_{\xi_i}, A_\mu]X, Y\rangle = \langle AX, (A - \lambda I)D_iY\rangle - \langle (A - \lambda I)D_iX, AY\rangle.$$

This expression is given by item (v). On the other hand, from the definition of the connection $\hat{\nabla}$ and the definition of the one forms ω_i we get

$$\begin{aligned}\big\langle \hat{R}(X,Y)\xi_i, \mu\big\rangle &= \big\langle \hat{\nabla}_X\hat{\nabla}_Y\xi_i, \mu\big\rangle - \big\langle \hat{\nabla}_Y\hat{\nabla}_X\xi_i, \mu\big\rangle - \big\langle \hat{\nabla}_{[X,Y]}\xi_i, \mu\big\rangle \\ &= X\omega_i(Y) + \omega_i(Y)\big\langle \hat{\nabla}_X(\mu-\zeta), \mu\big\rangle + (-1)^j\psi(Y)\big\langle \hat{\nabla}_X\xi_j, \mu\big\rangle \\ &\quad - Y\omega_i(X) - \omega_i(X)\big\langle \hat{\nabla}_Y(\mu-\zeta), \mu\big\rangle - (-1)^j\psi(X)\big\langle \hat{\nabla}_Y\xi_j, \mu\big\rangle - \omega_i([X,Y]) \\ &= X\omega_i(Y) + \omega_i(Y)\lambda^{-1}X(\lambda) + (-1)^j\psi(Y)\omega_j(X) \\ &\quad - Y\omega_i(X) - \omega_i(X)\lambda^{-1}Y(\lambda) - (-1)^j\psi(X)\omega_j(Y) - \omega_i([X,Y])\end{aligned} \tag{3.51}$$

Simplifying further the above equation, this time using the definition of the one-forms ω_i, we get

$$\begin{aligned}\big\langle \hat{R}(X,Y)\xi_i, \mu\big\rangle &= -X(\lambda^{-1}\langle D_iY, \operatorname{grad}\lambda\rangle) - \lambda^{-2}\langle D_iY, \operatorname{grad}\lambda\rangle X(\lambda) \\ &\quad - (-1)^j\lambda^{-1}\psi(Y)\langle D_jX, \operatorname{grad}\lambda\rangle + Y(\lambda^{-1}\langle D_iX, \operatorname{grad}\lambda\rangle) \\ &\quad + \lambda^{-2}\langle D_iX, \operatorname{grad}\lambda\rangle Y(\lambda) + (-1)^j\lambda^{-1}\psi(X)\langle D_jY, \operatorname{grad}\lambda\rangle \\ &\quad - \omega_i([X,Y]) \\ &= -\lambda^{-1}\langle \nabla_X D_iY, \operatorname{grad}\lambda\rangle - \lambda^{-1}\operatorname{Hess}\lambda(X, D_iY) \\ &\quad + \lambda^{-1}\langle \nabla_Y D_iX, \operatorname{grad}\lambda\rangle + \lambda^{-1}\operatorname{Hess}\lambda(Y, D_iX) - \omega_i([X,Y]) \\ &\quad + (-1)^j\lambda^{-1}\psi(X)\langle D_jY, \operatorname{grad}\lambda\rangle - (-1)^j\lambda^{-1}\psi(Y)\langle D_jX, \operatorname{grad}\lambda\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\big\langle \hat{R}(X,Y)\xi_i, \mu\big\rangle &= \lambda^{-1}(\langle(\nabla_Y D_i)X - (\nabla_X D_i)Y, \operatorname{grad}\lambda\rangle) \\ &\quad + \lambda^{-1}(\operatorname{Hess}\lambda(Y, D_iX) - \operatorname{Hess}\lambda(X, D_iY)) \\ &\quad + \lambda^{-1}\big((-1)^j\psi(X)\langle D_jY, \operatorname{grad}\lambda\rangle - (-1)^j\psi(Y)\langle D_jX, \operatorname{grad}\lambda\rangle\big).\end{aligned}$$

Comparing the expression given in item (v) and the equation above we conclude that the Ricci equation is valid for ξ_i and μ for X, $Y \in \Gamma(\Delta^\perp)$.

Now for $X \in \Gamma(\Delta^\perp)$ and $T \in \Gamma(\Delta)$, we have, on one hand,

$$\langle [A_{\xi_i}, A_\mu]X, T\rangle = 0$$

while, on the other hand, from the fourth equality in equation (3.51),

$$\big\langle \hat{R}(X,T)\xi_i, \mu\big\rangle = -T\omega_i(X) - \omega_i([X,T]).$$

From the definition of the one-form ω_i, $\nabla^h_T D_i X = D_i \nabla^h_T X$ and the fact that C_T and D_i commute, we get

$$\begin{aligned}
-T\omega_i(X) - \omega_i([X,T]) &= T\left(\frac{1}{\lambda}\left\langle D_i X, \operatorname{grad}\lambda\right\rangle\right) + \frac{1}{\lambda}\left\langle D_i[X,T], \operatorname{grad}\lambda\right\rangle \\
&= \frac{1}{\lambda}\left(\left\langle \nabla_T D_i X, \operatorname{grad}\lambda\right\rangle + \operatorname{Hess}\lambda(T, D_i X) + \left\langle D_i[X,T], \operatorname{grad}\lambda\right\rangle\right) \\
&= \frac{1}{\lambda}\left(D_i X T(\lambda) - \left\langle \nabla_{D_i X} T, \operatorname{grad}\lambda\right\rangle + \left\langle D_i \nabla_X T, \operatorname{grad}\lambda\right\rangle\right) \\
&= \frac{1}{\lambda}\left(\left\langle C_T D_i X, \operatorname{grad}\lambda\right\rangle - \left\langle D_i C_T X, \operatorname{grad}\lambda\right\rangle\right) = 0.
\end{aligned}$$

Lastly, for T and $S \in \Gamma(\Delta)$, on one hand, $\langle [A_{\xi_i}, A_\mu]T, S\rangle = 0$ because $\ker A_{\xi_i} = \Delta$. On the other hand, $\left\langle \hat{R}(T,S)\xi_i, \mu\right\rangle = 0$ because ξ_i is parallel along Δ and $[T,S] \in \Gamma(\Delta)$.

Now, let us move forward to the Ricci equation between A_{ξ_1} and A_{ξ_2}. We have

$$\begin{aligned}
\left\langle \hat{R}(Z,W)\xi_1, \xi_2\right\rangle &= \left\langle \hat{\nabla}_Z \hat{\nabla}_W \xi_1, \xi_2\right\rangle - \left\langle \hat{\nabla}_W \hat{\nabla}_Z \xi_1, \xi_2\right\rangle - \left\langle \hat{\nabla}_{[Z,W]} \xi_1, \xi_2\right\rangle \\
&= \omega_1(W)\left\langle \hat{\nabla}_Z(\mu - \zeta), \xi_2\right\rangle + Z\psi(W) - \omega_1(Z)\left\langle \hat{\nabla}_W(\mu - \zeta), \xi_2\right\rangle \\
&\quad - W\psi(Z) - \psi([Z,W]) \\
&= \mathrm{d}\psi([Z,W]).
\end{aligned}$$

If Z or W belongs to $\Gamma(\Delta)$, then item (vi) proves the Ricci equation since $\langle [A_{\xi_1}, A_{\xi_2}]Z, W\rangle = 0$. If both $Z = X$ and $W = Y$ belong to $\Gamma(\Delta^\perp)$, then item (vii) proves the Ricci equation of A_{ξ_1} and A_{ξ_2},

Lastly, we will show that the Ricci equation for A_ζ and A_{ξ_i} is equivalent to the Ricci equation for A_μ and A_{ξ_i}. On one hand, we have

$$\begin{aligned}
\left\langle \hat{R}(X,Y)\xi_i, \zeta\right\rangle &= \left\langle \hat{\nabla}_X \hat{\nabla}_Y \xi_i, \zeta\right\rangle - \left\langle \hat{\nabla}_Y \hat{\nabla}_X \xi_i, \zeta\right\rangle - \left\langle \hat{\nabla}_{[X,Y]} \xi_i, \zeta\right\rangle \\
&= \left\langle \hat{\nabla}_X\big(\omega_i(Y)(\mu - \zeta) + (-1)^j \psi(Y)\xi_j\big), \zeta\right\rangle \\
&\quad - \left\langle \hat{\nabla}_Y\big(\omega_i(X)(\mu - \zeta) + (-1)^j \psi(X)\xi_j\big), \zeta\right\rangle - \omega_i([X,Y]) \\
&= X\omega_i(Y) + \omega_i(Y)\left\langle \hat{\nabla}_X(\mu - \zeta), \zeta\right\rangle + (-1)^j\psi(Y)\left\langle \hat{\nabla}_X \xi_j, \zeta\right\rangle \\
&\quad - Y\omega_i(X) - \omega_i(X)\left\langle \hat{\nabla}_Y(\mu - \zeta), \zeta\right\rangle - (-1)^j\psi(X)\left\langle \hat{\nabla}_Y \xi_j, \zeta\right\rangle \\
&\quad - \omega_i([X,Y]) \\
&= X\omega_i(Y) + \omega_i(Y)\lambda^{-1}X(\lambda) + (-1)^j\psi(Y)\omega_j(X) \\
&\quad - Y\omega_i(X) - \omega_i(X)\lambda^{-1}Y(\lambda) - (-1)^j\psi(X)\omega_j(Y) \\
&\quad - \omega_i([X,Y]).
\end{aligned}$$

If we compare this expression with the equation (3.51) we conclude that

$$\left\langle \hat{R}(X,Y)\xi_i, \mu \right\rangle = \left\langle \hat{R}(X,Y)\xi_i, \zeta \right\rangle.$$

On the other hand,

$$\langle [A_{\xi_i}, A_\zeta]X, Y\rangle = \langle [A_{\xi_i}, A - \lambda I]X, Y\rangle = \langle [A_{\xi_i}, A_\mu]X, Y\rangle .$$

So, we have finished proving all the Ricci equations.

Using the Fundamental Theorem of Submanifolds (Theorem 1.25 in [8]), there exist an isometric immersion $\tilde{F} : M^n \to \mathbb{L}^{n+4}$ and a vector bundle isometry $\Phi : E \to N_{\tilde{F}}M$ such that

$$\Phi \circ \hat{\alpha} = \alpha^{\tilde{F}} \quad \text{and} \quad \Phi\hat{\nabla} = \nabla^{\perp}\Phi.$$

Moreover, the vector field $\rho = \lambda^{-1}\Phi(\zeta - \mu)$ satisfies

$$\begin{aligned}
\lambda(\tilde{F}^*\tilde{\nabla})_X\rho &= \lambda X(\lambda^{-1})\Phi(\zeta-\mu) + (\tilde{F}^*\tilde{\nabla})_X\Phi(\zeta-\mu)\\
&= -\lambda^{-1}X(\lambda)\Phi(\zeta-\mu) - \tilde{F}_*A_{\Phi(\zeta-\mu)}X + \nabla^{\perp}_X\Phi(\zeta-\mu)\\
&= -X(\lambda)\rho - \tilde{F}_*A_{\zeta-\mu}X + \Phi\hat{\nabla}_X(\zeta-\mu)\\
&= -X(\lambda)\rho + \lambda\tilde{F}_*X + \lambda^{-1}X(\lambda)\Phi(\zeta-\mu)\\
&= \lambda F_*X
\end{aligned}$$

for all $X \in \mathfrak{X}(M)$. Therefore,

$$(\tilde{F}^*\tilde{\nabla})_X(\tilde{F} - \rho) = 0$$

for all $X \in \mathfrak{X}(M)$, and hence $\tilde{F} - \rho$ is a constant vector $P_0 \in \mathbb{L}^{n+4}$. It follows that

$$\left\langle \tilde{F} - P_0, \tilde{F} - P_0 \right\rangle = \left\langle \rho, \rho \right\rangle = \lambda^{-2}\left\langle \zeta - \mu, \zeta - \mu \right\rangle = 0,$$

that is, $\tilde{F}$ takes values in $P_0 + \mathbb{V}^{n+3}$. Without loss of generality, suppose $P_0 = 0$, otherwise redefine $\tilde{F}$ by $\tilde{F} - P_0$. Thus, $\tilde{F}$ gives rise to a conformal immersion $\tilde{f} = \mathcal{C}(\tilde{F}) : M^n \to \mathbb{R}^{n+2}$ by Proposition 1.4. From now and until we finish the proof, without loss of generality we identify the vectors in E with those in $N_{\tilde{F}}M$.

Now it is time to prove the last statement of Proposition 3.2. First, suppose that distinct triples (D_1, D_2, ψ) and $(\hat{D}_1, \hat{D}_2, \hat{\psi})$ give rise to congruent conformal immersions $\tilde{f}$ and $\tilde{g}$. Then, by Proposition 2.1, their isometric light-cone representatives $\tilde{F}$ and $\tilde{G}$ are congruent isometric immersions, that is, there exist $T \in O_1^+(m+4)$ such that

$\tilde{G} = T \circ \tilde{F}$. Hence, $\alpha^{\tilde{G}} = T \circ \alpha^{\tilde{F}}$ and from Exercise 1.6 in [8], $\hat{\nabla}^{\perp} T = T\nabla^{\perp}$. From the equality regarding second fundamental forms applied to $(T,T) \in \Delta \times \Delta$ we conclude that $T(\mu) = \hat{\mu}$. Taking into account the last fact, from the equality $\tilde{G} = T \circ \tilde{F}$ we get $T(\zeta) = \hat{\zeta}$. Now, from

$$\left\langle A^{\tilde{G}}_{T(\xi_i)} X, Y \right\rangle = \left\langle \alpha^{\tilde{G}}(X,Y), T(\xi_i) \right\rangle = \left\langle \alpha^{\tilde{F}}(X,Y), \xi_i \right\rangle = \left\langle A^{\tilde{F}}_{\xi_i} X, Y \right\rangle$$

and the uniqueness of the sections $\hat{\xi}_i$ such that $\det D_{\hat{\xi}_i} = 1/2$, we conclude that $T(\xi_i) = \hat{\xi}_i$ and $D_i = \hat{D}_i$. From $\hat{\nabla}^{\perp} T = T\nabla^{\perp}$ we conclude that ψ and $\hat{\psi}$ must be also equal, a contradiction.

For the converse, suppose non-congruent conformal immersions $\tilde{f}$ and $\tilde{g}$ have the same triples. From the uniqueness of the frame ξ_1 and ξ_2, define $T : N_{\tilde{F}}M \to N_{\tilde{G}}M$ such that $T(\mu) = \hat{\mu}$, $T(\xi_i) = \hat{\xi}_i$ and $T(\zeta) = \hat{\zeta}$. Since we have same triples, we have $\hat{\nabla}^{\perp} T = T\nabla^{\perp}$ and $\alpha^{\tilde{G}} = T \circ \alpha^{\tilde{F}}$. Therefore, the isometric light-cone representatives are congruent, a contradiction.

The only thing left to prove is that the conformal immersion $\tilde{f} = \mathcal{C}(\tilde{F})$ is a genuine deformation of f. For that we must use item (ix). Before doing so, we need to prove a lemma.

Lemma 3.13. *Let $\tilde{F} : M^n \to \mathbb{V}^{n+3} \subset \mathbb{L}^{n+4}$ the isometric immersion that comes from the triple (D_1, D_2, ψ). If $\tilde{f} = \mathcal{C}(\tilde{F})$ is not a genuine deformation of f, then there exist an orthonormal frame $\{\xi_1, \xi_2\}$ of $\mathbb{P} = \{\mu, \zeta\}^{\perp}$ such that $D_{\xi_1} = I$ and $\operatorname{rank} D_{\xi_2} \leq 1$.*

Proof. Assume that $\tilde{f}$ is not a genuine conformal deformation of f. Then, by Proposition 2.1, there exist an open set $U \subset M^n$ and an isometric immersion $H : W \to \mathbb{V}^{n+3}$, with $W \supset F(U)$ open in $\mathbb{V}^{n+2}$, such that $\tilde{F}|_U = H \circ F|_U$. Without loss of generality, we will suppose $U = M^n$. Because M^n has been endowed with the metric induced by f, by Proposition 1.4 the isometric light-cone representative of f is given by $F = \Psi \circ f$. We conclude that there exists an isometric immersion $T = H \circ \Psi : V \subset \mathbb{R}^{n+1} \to \mathbb{V}^{n+3} \subset \mathbb{L}^{n+4}$ such that $\tilde{F} = T \circ f$.

Since T is an isometric immersion into the light-cone, the position vector field T is a section of its normal bundle $N_T\mathbb{R}^{n+1}$. Complete it to a pseudo-orthonormal frame $\{\rho, T, \tilde{\zeta}\}$ of $\Gamma(N_T\mathbb{R}^{n+1})$, where $\tilde{\zeta}$ is a light-like vector field such that $\left\langle \tilde{\zeta}, T \right\rangle = 1$. We can associate to this frame an orthonormal frame given by

$$\left\{ \rho, \frac{T + \tilde{\zeta}}{\sqrt{2}}, \frac{T - \tilde{\zeta}}{\sqrt{2}} \right\}.$$

From the Gauss equation of the isometric immersion T, we get that α^T is flat. Since $\left\langle \alpha^T(Z,W), T\right\rangle = -\langle Z, W\rangle$, for all Z, $W \in \mathfrak{X}(\mathbb{R}^{n+1})$, then $\mathcal{N}(\alpha^T) = \{0\}$. Using the Main Lemma bis 2.6 we conclude that

$$\dim\Omega = \dim\left(\mathcal{S}(\alpha^T) \cap \mathcal{S}(\alpha^T)^{\perp}\right) = 1.$$

Let

$$W_1 = \operatorname{span}\left\{\rho, \frac{T+\tilde{\zeta}}{\sqrt{2}}\right\} \quad \text{and} \quad W_2 = \operatorname{span}\left\{\frac{T-\tilde{\zeta}}{\sqrt{2}}\right\}$$

be subspaces of $N_T\mathbb{R}^{n+1}$ and observe that the projections $P_i : N_T\mathbb{R}^{n+1} \to W_i$, for $i = 1, 2$ restricted to Ω are isomorphisms onto their images. Then, by dimensional reasons, $P_2|_{\Omega}$ is an isomorphism.

Let $\beta \in \Omega$ such that $P_2(\beta) = (T - \tilde{\zeta})/\sqrt{2}$. From the definition, we have that β is a light-like vector field with $A^T_\beta = 0$. In terms of the orthonormal frame, we have

$$\beta = \cos\theta\rho + \sin\theta\frac{T+\tilde{\zeta}}{\sqrt{2}} + \frac{T-\tilde{\zeta}}{\sqrt{2}},$$

where $\theta \in [0, 2\pi)$. Let us rearrange our original orthonormal frame to another orthonormal frame $\{\gamma, \delta, \tilde{\gamma}\}$, where

$$\gamma = \cos\theta\rho + \sin\theta\frac{T+\tilde{\zeta}}{\sqrt{2}}, \quad \delta = -\sin\theta\rho + \cos\theta\frac{T+\tilde{\zeta}}{\sqrt{2}} \quad \text{and} \quad \tilde{\gamma} = \frac{T-\tilde{\zeta}}{\sqrt{2}}.$$

The shape operators of the immersion T with respect to the elements of this frame have some interesting properties. First, since $\beta = \gamma + \tilde{\gamma}$ and $A^T_\beta = 0$, then $A^T_\gamma = -A^T_{\tilde{\gamma}}$. Second, because

$$\begin{aligned}\alpha^T(Z,W) &= \left\langle A^T_\delta Z, W\right\rangle \delta + \left\langle A^T_\gamma Z, W\right\rangle \gamma - \left\langle A^T_{\tilde{\gamma}} Z, W\right\rangle \tilde{\gamma} \\ &= \left\langle A^T_\delta Z, W\right\rangle \delta + \left\langle A^T_\gamma Z, W\right\rangle \beta,\end{aligned}$$

where Z, $W \in \mathfrak{X}(\mathbb{R}^{n+1})$, using the Main Lemma bis 2.6 we conclude that $\dim\ker A^T_\delta = n$. Also, from Proposition 1.2 and the definition of δ and β we have

$$\begin{aligned}-\langle Z, W\rangle &= \left\langle A^T_\delta Z, W\right\rangle \langle\delta, T\rangle + \left\langle A^T_\gamma Z, W\right\rangle \langle\beta, T\rangle \\ &= \frac{\cos\theta}{\sqrt{2}}\left\langle A^T_\delta Z, W\right\rangle + \frac{\sin\theta - 1}{\sqrt{2}}\left\langle A^T_\gamma Z, W\right\rangle\end{aligned} \tag{3.52}$$

If we replace Z, W by f_*X, f_*Y for X, $Y \in \mathfrak{X}(M)$, respectively, in the above equation we get

$$-\langle f_*X, f_*Y\rangle = \frac{\cos\theta}{\sqrt{2}}\left\langle A^T_\delta f_*X, f_*Y\right\rangle + \frac{\sin\theta - 1}{\sqrt{2}}\left\langle A^T_\gamma f_*X, f_*Y\right\rangle.$$

Hence,

$$-f_* = \frac{\cos\theta}{\sqrt{2}}(A^T_\delta f_*)^T + \frac{\sin\theta - 1}{\sqrt{2}}(A^T_\gamma f_*)^T, \tag{3.53}$$

where we are decomposing $T_{f(x)}\mathbb{R}^{n+1} = f_*T_xM^n \oplus \mathbb{R}N(x)$ and Z^T denotes the projection onto the first component of the above decomposition for $Z \in T_{f(x)}\mathbb{R}^{n+1}$.

We have a natural decomposition of $N_{\tilde{F}}M$ given by

$$N_{\tilde{F}}M(x) = T_*N_fM(x) \oplus N_T\mathbb{R}^{n+1}(f(x)).$$

So, if $\eta \in N_T\mathbb{R}^{n+1}$ is a normal section of T, then $\eta \circ f \in N_{\tilde{F}}M$ is a normal section of $\tilde{F}$. Therefore, $\{T_*N, (\gamma\circ f), (\delta\circ f), (\tilde{\gamma}\circ f)\}$ is an orthonormal frame for $N_{\tilde{F}}M$, and in this frame we have

$$\begin{aligned}\alpha^{\tilde{F}}(X,Y) &= T_*\alpha^f(X,Y) + \alpha^T(f_*X, f_*Y) \\ &= \langle AX, Y\rangle T_*N + \left\langle A^T_\delta f_*Xf_*Y\right\rangle(\delta\circ f) + \left\langle A^T_\gamma f_*X, f_*Y\right\rangle((\gamma\circ f) + (\tilde{\gamma}\circ f)).\end{aligned} \tag{3.54}$$

Since f is an isometric immersion, we get

$$\begin{aligned}(A^T_\gamma f_*X)^T &= f_*A^{\tilde{F}}_{\gamma\circ f}X,\\ (A^T_\gamma f_*X)^T &= -f_*A^{\tilde{F}}_{\tilde{\gamma}\circ f}X,\\ (A^T_\delta f_*X)^T &= f_*A^{\tilde{F}}_{\delta\circ f}X,\end{aligned} \tag{3.55}$$

From the above identities, we conclude

$$A^{\tilde{F}}_{\gamma\circ f} = -A^{\tilde{F}}_{\tilde{\gamma}\circ f}X \quad\text{and}\quad \operatorname{rank} A^{\tilde{F}}_{\delta\circ f} \leq 1. \tag{3.56}$$

The normal space of $\tilde{F}$ also has another orthonormal frame, namely the one we used to define the isometric immersion $\tilde{F}$, that is, $\{\mu, \xi_1, \xi_2, \zeta\}$. Our aim now is to find an expression of $\tilde{F}$ and μ in terms of the other orthonormal frame of $N_{\tilde{F}}M$. The first one is straight, since

$$T = \frac{\sqrt{2}}{2}(\cos\theta\delta + \sin\theta\gamma + \tilde{\gamma})$$

and $\tilde{F} = T \circ f$, we have

$$\tilde{F} = \frac{\sqrt{2}}{2}\left(\cos\theta(\delta \circ f) + \sin\theta(\gamma \circ f) + (\tilde{\gamma} \circ f)\right). \tag{3.57}$$

For the second one, taking into account that f is an isometric immersion, $\dim \Delta = n - 2$ and $\dim \ker A_\delta^T = n$, there exist $T \in \Delta$ of unit length such that $f_* T \in \ker A_\delta^T$. One one hand, we have

$$\begin{aligned}\alpha^{\tilde{F}}(T,T) &= \langle AT, T\rangle \mu + \langle (A - \lambda I)D_1 T, T\rangle \xi_1 + \langle (A - \lambda I)D_2 T, T\rangle \xi_2 - \langle (A - \lambda I)T, T\rangle \zeta \\ &= \lambda\mu.\end{aligned}$$

On the other hand, using equations (3.52) and (3.54) we get

$$\begin{aligned}\alpha^{\tilde{F}}(T,T) &= \lambda T_* N + \left\langle A_\gamma^T f_* T, f_* T\right\rangle ((\gamma \circ f) + (\tilde{\gamma} \circ f)) \\ &= \lambda T_* N - \frac{\sqrt{2}}{\sin\theta - 1}\left((\gamma \circ f) + (\tilde{\gamma} \circ f)\right).\end{aligned}$$

Hence, combining the last two identities, we conclude

$$\mu = T_* N - \frac{\sqrt{2}}{\lambda(\sin\theta - 1)}\left((\gamma \circ f) + (\tilde{\gamma} \circ f)\right). \tag{3.58}$$

Now that we have expressions for μ and $\tilde{F}$ given in equations (3.57) and (3.58), we can find out where the Riemannian plane $\mathbb{P} = \{\mu, \zeta\}^\perp$ is located, since this plane is orthogonal to μ and $\tilde{F}$. It is straightforward to verify that

$$\begin{aligned}\xi_1 = T_* N &+ \left(\frac{\lambda\cos^2\theta}{\sqrt{2}(1 - \sin\theta)} - \frac{\lambda}{\sqrt{2}}\right)(\gamma \circ f) + \left(\frac{\lambda\cos^2\theta}{\sqrt{2}(1 - \sin\theta)} - \frac{\lambda\sin\theta}{\sqrt{2}}\right)(\tilde{\gamma} \circ f) \\ &+ \frac{\lambda\cos\theta}{\sqrt{2}}(\delta \circ f)\end{aligned}$$

and

$$\xi_2 = \frac{\cos\theta}{1 - \sin\theta}\left((\gamma \circ f) + (\tilde{\gamma} \circ f)\right) + (\delta \circ f).$$

is an orthonormal frame for $\mathbb{P}$. From the properties given in equation (3.56), and the fact that the shape operator of $\tilde{F}$ in the direction $T_* N$ is given by A, we have

$$\begin{aligned}A_{\xi_1}^{\tilde{F}} &= A + \frac{\lambda}{\sqrt{2}}\left((\sin\theta - 1)A_{\gamma\circ f}^{\tilde{F}} + \cos\theta A_{\delta\circ f}^{\tilde{F}}\right) \\ A_{\xi_2}^{\tilde{F}} &= A_{\delta\circ f}^{\tilde{F}}.\end{aligned}$$

From equation (3.56) we have that the rank of $D_{\xi_2} = (A - \lambda I)A_{\xi_2}$ is less than or equal to one. For the other shape operator, from equations (3.53) and (3.55) we obtain

$$\begin{aligned} f_* A_{\xi_1} &= f_* A + \frac{\lambda}{\sqrt{2}} \left((\sin\theta - 1) f_* A^{\tilde F}_{\gamma\circ f} + \cos\theta f_* A^{\tilde F}_{\delta\circ f} \right) \\ &= f_* A + \frac{\lambda}{\sqrt{2}} \left((\sin\theta - 1)(A^T_\gamma f_*)^T + \cos\theta (A^T_\delta f_*)^T \right) \\ &= f_* A + \frac{\lambda}{\sqrt{2}} (-\sqrt{2} f_*) \\ &= f_* \left(A - \lambda I \right). \end{aligned}$$

Hence $A_{\xi_1} = A - \lambda I$ and $D_{\xi_1} = I$. □

Suppose now that rank $D_1^2 + D_2^2 - I = 2$, and assume by contradiction that $\tilde f$ is not a genuine deformation of f. By Lemma 3.13, there exists an orthonormal frame $\{\xi_1, \xi_2\}$ of $\mathbb{P}$ such that $D_{\xi_1} = I$ and $\operatorname{rank} D_{\xi_2} \leq 1$. Let $\theta \in [0, \pi/2]$ be such that

$$D_1 = \cos\theta D_{\xi_1} + \sin\theta D_{\xi_2}$$

and

$$D_2 = -\sin\theta D_{\xi_1} + \cos\theta D_{\xi_2},$$

where D_1 and D_2 are our tensors with determinant 1/2. Then,

$$\begin{aligned} D_1^2 + D_2^2 - I = {} & \cos^2\theta D_{\xi_1}^2 + \sin^2\theta D_{\xi_2}^2 + \cos\theta\sin\theta D_{\xi_1} D_{\xi_2} + \cos\theta\sin\theta D_{\xi_2} D_{\xi_1} \\ & + \sin^2\theta D_{\xi_1}^2 + \cos^2\theta D_{\xi_2}^2 - \cos\theta\sin\theta D_{\xi_1} D_{\xi_2} - \cos\theta\sin\theta D_{\xi_2} D_{\xi_1} - I \\ = {} & D_{\xi_2}^2 \end{aligned}$$

and this means that rank $D_1^2 + D_2^2 - I < 2$, a contradiction. This completes the proof of Proposition 3.2. □

Chapter 4

The Reduction

Let $f : M^n \to \mathbb{R}^{n+1}$ be a hypersurface that is not conformally surface-like and envelops a two-parameter congruence of hyperspheres $s : L^2 \to S_1^{n+2} \subset \mathbb{L}^{n+3}$. In this chapter, the problem of finding a pair of tensors (D_1, D_2) and a one-form ψ on M^n satisfying all the conditions in Proposition 3.2 is reduced to a similar but easier one on the surface L^2. Lets begin with some definitions.

Let $\pi : M \to L$ be a submersion. A vector field $X \in \mathfrak{X}(M)$ is *projectable* if it is π-related to a vector field $\bar{X} \in \mathfrak{X}(L)$. A tensor D on M is *projectable* if there exist a tensor $\bar{D}$ on L such that $\bar{D} \circ \pi_* = \pi_* \circ D$. Similarly, a one-form ω on M is *projectable* if there exist a one-form $\bar{\omega}$ on L such that $\bar{\omega} \circ \pi_* = \omega$.

We will need the following results, which give conditions for tensors and one-forms to be projectable.

Proposition 4.1 (Corollary 11.6 in [8])**.** *Let Δ be an integrable distribution on a Riemannian manifold M and let $L = M/\Delta$ be the (local) quotient space of leaves of Δ. A tensor $D \in \Gamma(End(\Delta^\perp))$ is projectable if and only if*

$$\nabla^h_T D = [D, C_T]$$

for all $T \in \Gamma(\Delta)$.

Proposition 4.2 (Corollary 12 in [12])**.** *Let Δ be an integrable distribution on a differentiable manifold M, let $L = M/\Delta$ be the (local) quotient space of leaves of Δ and let $\pi : M \to L$ be the quotient map. Then a one-form ω on M is projectable if and only if $\omega(T) = 0$ and $d\omega(T, X) = 0$ for any $T \in \Delta$ and $X \in \Delta^\perp$.*

The reduction lemma is as follows.

Lemma 4.3. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a hypersurface that is not conformally surface-like and envelops a two-parameter congruence of hyperspheres $s : L^2 \to \mathbb{Q}^{n+2}_{1,1} \subset \mathbb{L}^{n+3}$. Let Δ be the eigenbundle of f correspondent to its principal curvature λ of multiplicity $n-2$. If f is hyperbolic (respectively, elliptic) with respect to $J \in \Gamma(End(\Delta^\perp))$ and there exists a triple (D_1, D_2, ψ) with D_1,$D_2 \in \Gamma(End(\Delta^\perp))$, D_1, $D_2 \in span\{I, J\}$, and ψ a one-form on M^n satisfying (i)-(ix) in Proposition 3.2, then J, D_1 and D_2 are the horizontal lifts of tensors $\bar{J}$, $\bar{D}_1$, $\bar{D}_2 \in span\{\bar{I}, \bar{J}\}$ on L^2, with $\bar{J}^2 = I$ (respectively, $\bar{J}^2 = -I$) and ψ is the horizontal lift of a one-form $\bar{\psi}$ on L^2 such that s is hyperbolic (respectively, elliptic) with respect to $\bar{J}$ and the triple $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ satisfies:*

(a) $\det \bar{D}_i = 1/2$,

(b) $(\nabla'_X \bar{D}_i)Y - (\nabla'_Y \bar{D}_i)X = (-1)^j \left((\bar{\psi}(X)\bar{D}_j Y - \bar{\psi}(Y)\bar{D}_j(X)\right)$,

(c) $d\bar{\psi}(X, Y) = \left\langle \bar{D}_2 X, \bar{D}_1 Y\right\rangle' - \left\langle \bar{D}_1 X, \bar{D}_2 Y\right\rangle'$,

(d) $\bar{D}_2^2 \neq \pm \bar{D}_1^2$,

(e) $rank(\bar{D}_1^2 + \bar{D}_2^2 - \bar{I}) = 2$.

Conversely, if $s : L^2 \to \mathbb{Q}^{n+2}_{1,1} \subset \mathbb{L}^{n+3}$ is hyperbolic (respectively, elliptic) with respect to a tensor $\bar{J}$ on L^2 satisfying $\bar{J}^2 = \bar{I}$ (respectively, $\bar{J}^2 = -\bar{I}$), then the hypersurface f is hyperbolic (respectively, elliptic) with respect to the horizontal lift J of $\bar{J}$, and the horizontal lifts D_1 and D_2 of tensors $\bar{D}_1$, $\bar{D}_2 \in span\{\bar{I}, \bar{J}\}$ and the one-form $\psi = \bar{\psi} \circ \pi_*$ satisfying items (a) to (e) have all the properties (i) to (ix) in Proposition 3.2.

Proof. Conditions (i) and (vi) of Proposition 3.2, together with Proposition 4.2, assure us that the one-form ψ is projectable with respect to the canonical projection $\pi : M \to L^2$ onto the (local) quotient of leaves of the distribution Δ, that is, there exists a one-form $\bar{\psi}$ on L^2 such that

$$\bar{\psi} \circ \pi_* = \psi.$$

The tensors D_1 and D_2 are also projectable, because of item (iii) of Proposition 3.2 and Proposition 4.1, that is, there exist tensors $\bar{D}_1$ and $\bar{D}_2$ on L^2 such that

$$\bar{D}_1 \circ \pi_* = \pi_* \circ D_1 \quad \text{and} \quad \bar{D}_2 \circ \pi_* = \pi_* \circ D_2. \tag{4.1}$$

From item (iii) we have that the tensors D_i commute with the tensors C_T. Since the tensors D_i are generated by the endomorphisms I and J, and taking into account item (viii), at least one D_i is of the form $D_i = a_i I + b_i J$ with b_i not null. It follows that C_T

and J commute, or equivalently, $[C_T, J] = 0$. The fact that $f : M^n \to \mathbb{R}^{n+1}$ is hyperbolic or elliptic gives us that $\nabla^h_T J = 0$. Therefore J is projectable onto $\bar{J}$, with

$$\pi_* \circ J = \bar{J} \circ \pi_*.$$

Since $D_i \in \operatorname{span}\{I, J\}$ and D_i, I and J are projectable onto $\bar{D}_i$, $\bar{I}$ and $\bar{J}$, respectively, we get that $\bar{D}_i \in \operatorname{span}\{\bar{I}, \bar{J}\}$. Because $f : M^n \to \mathbb{R}^{n+1}$ is hyperbolic or elliptic, we have that $J^2 = \epsilon I$, where $\epsilon = 1$, if f is hyperbolic, and $\epsilon = -1$, if f is elliptic. Then,

$$\bar{J}^2\bar{X} = \pi_* J^2 X = \epsilon \pi_* X = \epsilon \bar{X}.$$

Hence $\bar{J}^2 = \epsilon \bar{I}$, with $\epsilon = 1$, if f is hyperbolic, and $\epsilon = -1$, if f is elliptic.

The linear operator $\pi_*|_{\Delta^\perp}$ is an isomorphism, so items (a), (d) and (e) follow without trouble from (4.1). Essentially, D_i is indistinguishable from $\bar{D}_i$.

Let $S\colon M^n \to \mathbb{Q}^{n+2}_{1,1} \subset \mathbb{L}^{n+3}$ be the two-parameter congruence of hyperspheres enveloped by f, so that $S = s \circ \pi$. Then, the proof of Proposition 1.3 gives us that

$$S(x) = \Psi_*(f(x))N(x) + \lambda(x)\Psi(f(x)). \tag{4.2}$$

Differentiating the equation (4.2) with respect to $Y \in \Gamma(\Delta^\perp)$ we obtain

$$\begin{aligned} S_*Y &= (\Psi \circ f)^* \tilde{\nabla}_Y S \qquad (4.3)\\ &= (\Psi \circ f)^* \tilde{\nabla}_Y (\Psi_* \circ f)N + Y(\lambda)\Psi \circ f + \lambda \Psi_* f_* Y\\ &= \Psi^* \tilde{\nabla}_{f_*Y} \Psi_* N + Y(\lambda)\Psi \circ f + \lambda \Psi_* f_* Y\\ &= \Psi_* f^* \bar{\nabla}_Y N + \alpha^\Psi(f_*Y, N) + Y(\lambda)\Psi \circ f + \lambda \Psi_* f_* Y\\ &= -\Psi_* f_* AY + Y(\lambda)\Psi \circ f + \lambda \Psi_* f_* Y\\ &= -\Psi_* f_*(A - \lambda I)Y + Y(\lambda)\Psi \circ f. \end{aligned}$$

Replacing Y by D_iY in (4.3) we get

$$\Psi_* f_*(A - \lambda I)D_iY = \langle D_iY, \operatorname{grad} \lambda \rangle\, \Psi \circ f - S_* D_i Y. \tag{4.4}$$

This equation and the one before it will give us two ways of differentiating vector fields of the form $\Psi_* f_*(A - \lambda I)X$, for $X \in \mathfrak{X}(M)$. Comparing both expressions will give us the results we seek. So, differentiating one more time the equation (4.4) with respect to

$X \in \Gamma(\Delta^\perp)$ yields

$$\begin{aligned}(\Psi\circ f)^*\tilde\nabla_X\Psi_*f_*(A-\lambda I)D_iY &= \langle\nabla_X D_iY, \operatorname{grad}\lambda\rangle\,\Psi\circ f \\ &+ \operatorname{Hess}\lambda(X, D_iY)\Psi\circ f + \langle D_iY, \operatorname{grad}\lambda\rangle\,\Psi_*f_*X - (\Psi\circ f)^*\tilde\nabla_X S_*D_iY.\end{aligned}\tag{4.5}$$

Let $i:\mathbb{Q}^{n+2}_{1,1}\to\mathbb{L}^{n+3}$ be the inclusion. If we denote the vector field S_*D_iY and $s_*\bar D_i\pi_*Y$ by $g: M^n\to\mathbb{L}^{n+3}$ and $h: L^2\to\mathbb{L}^{n+3}$, respectively, then $g = h\circ\pi$. If $\langle\cdot,\cdot\rangle'$ is the metric on L^2 induced by s and ∇' its Levi-Civita connection, we then get

$$\begin{aligned}(\Psi\circ f)^*\tilde\nabla_X S_*D_iY &= X(g)\\ &= \pi_*X(h)\\ &= (i\circ s)^*\tilde\nabla_{\pi_*X}s_*\bar D_i\pi_*Y\\ &= s^*\tilde\nabla_{\pi_*X}s_*\bar D_i\pi_*Y - \left\langle\pi_*X, \bar D_i\pi_*Y\right\rangle' s\circ\pi\\ &= s_*\nabla'_{\pi_*X}\bar D_i\pi_*Y + \alpha'(\pi_*X, \bar D_i\pi_*Y) - \left\langle\pi_*X, \bar D_i\pi_*Y\right\rangle' s\circ\pi,\end{aligned}\tag{4.6}$$

where X, $Y\in\Gamma(\Delta^\perp)$ are projectable vector fields. Because we have endowed L^2 with the metric induced by s, Ψ and f are isometric immersions, and equation (4.3) we obtain

$$\begin{aligned}\left\langle\pi_*X, \bar D_i\pi_*Y\right\rangle' &= \left\langle s_*\pi_*X, s_*\bar D_i\pi_*Y\right\rangle\\ &= \langle X(\lambda)\Psi\circ f - \Psi_*f_*(A-\lambda I)X, D_iY(\lambda)\Psi\circ f - \Psi_*f_*(A-\lambda I)D_iY\rangle\\ &= \langle(A-\lambda I)X, (A-\lambda I)D_iY\rangle\,.\end{aligned}\tag{4.7}$$

Therefore, replacing into equation (4.5) the equations (4.6) and (4.7),

$$\begin{aligned}&(\Psi\circ f)^*\tilde\nabla_X\Psi_*f_*(A-\lambda I)D_iY\\ &= \langle\nabla_X D_iY, \operatorname{grad}\lambda\rangle\,\Psi\circ f + \operatorname{Hess}\lambda(X, D_iY)\Psi\circ f + \langle D_iY, \operatorname{grad}\lambda\rangle\,\Psi_*f_*X\\ &\quad - s_*\nabla'_{\pi_*X}\bar D_i\pi_*Y - \alpha'(\pi_*X, \bar D_i\pi_*Y) + \langle(A-\lambda I)X, (A-\lambda I)D_iY\rangle\,(\Psi_*N + \lambda(\Psi\circ f)).\end{aligned}\tag{4.8}$$

Hence,

$$\begin{aligned}&(\Psi\circ f)^*\tilde\nabla_X\Psi_*f_*(A-\lambda I)D_iY - (\Psi\circ f)^*\tilde\nabla_Y\Psi_*f_*(A-\lambda I)D_iX\\ &= \langle\nabla_X D_iY, \operatorname{grad}\lambda\rangle\,\Psi\circ f + \operatorname{Hess}\lambda(X, D_iY)\Psi\circ f + \langle D_iY, \operatorname{grad}\lambda\rangle\,\Psi_*f_*X\\ &\quad - s_*\nabla'_{\pi_*X}\bar D_i\pi_*Y - \alpha'(\pi_*X, \bar D_i\pi_*Y) + \langle(A-\lambda I)X, (A-\lambda I)D_iY\rangle\,(\Psi_*N + \lambda(\Psi\circ f)).\\ &\quad - \langle\nabla_Y D_iX, \operatorname{grad}\lambda\rangle\,\Psi\circ f - \operatorname{Hess}\lambda(Y, D_iX)\Psi\circ f - \langle D_iX, \operatorname{grad}\lambda\rangle\,\Psi_*f_*Y\\ &\quad + s_*\nabla'_{\pi_*Y}\bar D_i\pi_*X + \alpha'(\pi_*Y, \bar D_i\pi_*X) - \langle(A-\lambda I)Y, (A-\lambda I)D_iX\rangle\,(\Psi_*N + \lambda(\Psi\circ f)).\end{aligned}\tag{4.9}$$

On the other hand, using the formula for the second fundamental form of Ψ given in equation (1.12), together with equation (4.4), it follows that

$$\begin{aligned}
&(\Psi\circ f)^*\tilde\nabla_X\Psi_*f_*(A-\lambda I)D_iY \qquad (4.10)\\
&\quad=\Psi_*f^*\bar\nabla_Xf_*(A-\lambda I)D_iY+\alpha^\Psi(f_*X,f_*(A-\lambda I)D_iY)\\
&\quad=\Psi_*f_*\nabla_X(A-\lambda I)D_iY+\langle AX,(A-\lambda I)D_iY\rangle\,\Psi_*N-\langle X,(A-\lambda I)D_iY\rangle\,w\\
&\quad=\Psi_*f_*(\nabla_X(A-\lambda I)D_i)Y+\Psi_*f_*(A-\lambda I)D_i\nabla_XY+\langle AX,(A-\lambda I)D_iY\rangle\,\Psi_*N\\
&\qquad-\langle X,(A-\lambda I)D_iY\rangle\,w\\
&\quad=\Psi_*f_*(\nabla_X(A-\lambda I)D_i)Y+\langle D_i\nabla_XY,\operatorname{grad}\lambda\rangle\,\Psi\circ f-S_*D_i\nabla_XY\\
&\qquad+\langle AX,(A-\lambda I)D_iY\rangle\,\Psi_*N-\langle X,(A-\lambda I)D_iY\rangle\,w\\
&\quad=\Psi_*f_*(\nabla_X(A-\lambda I)D_i)Y+\langle D_i\nabla_XY,\operatorname{grad}\lambda\rangle\,\Psi\circ f-s_*\bar D_i\pi_*\nabla_XY\\
&\qquad+\langle AX,(A-\lambda I)D_iY\rangle\,\Psi_*N-\langle X,(A-\lambda I)D_iY\rangle\,w.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\Psi\circ f)^*\tilde\nabla_X\Psi_*f_*(A-\lambda I)D_iY-(\Psi\circ f)^*\tilde\nabla_Y\Psi_*f_*(A-\lambda I)D_iX \qquad (4.11)\\
&\quad=\Psi_*f_*\left((\nabla_X(A-\lambda I)D_i)Y-(\nabla_Y(A-\lambda I)D_i)X\right)+\langle D_i[X,Y],\operatorname{grad}\lambda\rangle\,\Psi\circ f\\
&\qquad-s_*\bar D_i\pi_*[X,Y]+\left(\langle AX,(A-\lambda I)D_iY\rangle-\langle AY,(A-\lambda I)D_iX\rangle\right)\Psi_*N\\
&\qquad-\left(\langle X,(A-\lambda I)D_iY\rangle-\langle Y,(A-\lambda I)D_iX\rangle\right)w.
\end{aligned}$$

As mentioned before, comparing the expressions just obtained in equations (4.9) and (4.11), we get

$$\begin{aligned}
&\Psi_*f_*B(X,Y)+\theta(X,Y)\Psi_*N+\varphi(X,Y)\Psi\circ f-\lambda^{-1}\theta(X,Y)w \qquad (4.12)\\
&\quad=s_*((\nabla'_{\pi_*Y}\bar D_i)\pi_*X-(\nabla'_{\pi_*X}\bar D_i)\pi_*Y)+\alpha'(\pi_*Y,\bar D_i\pi_*X)-\alpha'(\pi_*X,\bar D_i\pi_*Y)
\end{aligned}$$

where

$$B(X,Y)=(\nabla_X(A-\lambda I)D_i)Y-(\nabla_Y(A-\lambda I)D_i)X-X\wedge Y(D_i^t\operatorname{grad}\lambda),$$

$$\theta(X,Y)=\lambda(\langle X,(A-\lambda I)D_iY\rangle-\langle Y,(A-\lambda I)D_iX\rangle),$$

$$\begin{aligned}
\varphi(X,Y)&=\langle(\nabla_YD_i)X-(\nabla_XD_i)Y,\operatorname{grad}\lambda\rangle+\operatorname{Hess}\lambda(D_iX,Y)-\operatorname{Hess}\lambda(X,D_iY)\\
&\qquad-\lambda(\langle(A-\lambda I)X,(A-\lambda I)D_iY\rangle-\langle(A-\lambda I)D_iX,(A-\lambda I)Y\rangle),
\end{aligned}$$

for all $X, Y \in \Gamma(\Delta^\perp)$ that are projectable.

Notice that, in proving the identity in equation (4.12), we have only used that D_i are projectable onto $\bar{D}_i$ and the formula of the congruence of hyperspheres given in equation (4.2). Let us now use the properties that f and the triple (D_1, D_2, ψ) satisfy.

From equation (3.1), we get the symmetry of $(A - \lambda I)C_T$. Because f is hyperbolic or elliptic and not surface-like we have that there exist $T \in \Delta$, such that $C_T = aI + bJ$ with b not null, hence the operator $(A - \lambda I)J$ is symmetric. Since $D_i \in \operatorname{span}\{I, J\}$, we get that $(A - \lambda I)D_i$ are symmetric. From the last fact, items (iv) and (v) of Proposition 3.2 and equation (4.12) we obtain

$$\begin{aligned}(-1)^j \Psi_* f_* (A - \lambda I)(\psi(X) D_j Y - \psi(Y) D_j X)\\ + ((-1)^j \psi(Y) \langle D_j X, \operatorname{grad} \lambda \rangle - (-1)^j \psi(X) \langle D_j Y, \operatorname{grad} \lambda \rangle) \Psi \circ f\\ = s_*((\nabla'_{\pi_* Y} \bar{D}_i)\pi_* X - (\nabla'_{\pi_* X} \bar{D}_i)\pi_* Y) + \alpha'(\pi_* Y, \bar{D}_i \pi_* X) - \alpha'(\pi_* X, \bar{D}_i \pi_* Y).\end{aligned} \tag{4.13}$$

From equation (4.3) we get

$$\begin{aligned}(-1)^j \psi(X)(\langle D_j Y, \operatorname{grad} \lambda \rangle \Psi \circ f - S_* D_j Y) - (-1)^j \psi(Y)(\langle D_j X, \operatorname{grad} \lambda \rangle \Psi \circ f - S_* D_j X)\\ + ((-1)^j \psi(Y) \langle D_j X, \operatorname{grad} \lambda \rangle - (-1)^j \psi(X) \langle D_j Y, \operatorname{grad} \lambda \rangle) \Psi \circ f\\ = s_*((\nabla'_{\pi_* Y} \bar{D}_i)\pi_* X - (\nabla'_{\pi_* X} \bar{D}_i)\pi_* Y) + \alpha'(\pi_* Y, \bar{D}_i \pi_* X) - \alpha'(\pi_* X, \bar{D}_i \pi_* Y),\end{aligned}$$

so, simplifying, we end up with

$$\begin{aligned}(-1)^j \bar{\psi}(\pi_* Y) s_* \bar{D}_j \pi_* X - (-1)^j \bar{\psi}(\pi_* X) s_* \bar{D}_j \pi_* Y\\ = s_*((\nabla'_{\pi_* Y} \bar{D}_i)\pi_* X - (\nabla'_{\pi_* X} \bar{D}_i)\pi_* Y) + \alpha'(\pi_* Y, \bar{D}_i \pi_* X) - \alpha'(\pi_* X, \bar{D}_i \pi_* Y).\end{aligned}$$

Comparing the tangent and normal components we get the identities

$$(\nabla'_{\pi_* Y} \bar{D}_i)\pi_* X - (\nabla'_{\pi_* X} \bar{D}_i)\pi_* Y = (-1)^j \bar{\psi}(\pi_* Y) \bar{D}_j \pi_* X - (-1)^j \bar{\psi}(\pi_* X) \bar{D}_j \pi_* Y$$

and

$$\alpha'(\pi_* Y, \bar{D}_i \pi_* X) = \alpha'(\pi_* X, \bar{D}_i \pi_* Y).$$

The first equation above gives us (b), while the second one means that s is hyperbolic or elliptic with respect to $\bar{J}$, because $\bar{D}_i \in \operatorname{span}\{\bar{I}, \bar{J}\}$.

The only thing left to prove is condition (c). Using that ψ is projectable onto $\bar{\psi}$, item

(vii) of Proposition 3.2 and equation (4.4) we obtain

$$\begin{aligned}
\mathrm{d}\bar{\psi}(\bar{X},\bar{Y}) &= \mathrm{d}\psi(X,Y) \qquad (4.14)\\
&= \langle[(A-\lambda I)D_1,(A-\lambda I)D_2]X,Y\rangle\\
&= \langle(A-\lambda I)D_2X,(A-\lambda I)D_1Y\rangle - \langle(A-\lambda I)D_1X,(A-\lambda I)D_2Y\rangle\\
&= \langle\langle D_2X,\operatorname{grad}\lambda\rangle\,\Psi\circ f - S_*D_2X, \langle D_1Y,\operatorname{grad}\lambda\rangle\,\Psi\circ f - S_*D_1Y\rangle\\
&\quad - \langle\langle D_1X,\operatorname{grad}\lambda\rangle\,\Psi\circ f - S_*D_1X, \langle D_2Y,\operatorname{grad}\lambda\rangle\,\Psi\circ f - S_*D_2Y\rangle\\
&= \langle S_*D_2X,S_*D_1Y\rangle - \langle S_*D_1X,S_*D_2Y\rangle\\
&= \left\langle \bar{D}_2\bar{X},\bar{D}_1\bar{Y}\right\rangle' - \left\langle \bar{D}_1\bar{X},\bar{D}_2\bar{Y}\right\rangle'.
\end{aligned}$$

This completes the proof of the direct statement.

Let us now prove the converse. As was mentioned after we showed equation (4.12), we can use it in the proof of the converse statement. Using it, and taking into account condition (b) and the fact that s is hyperbolic or elliptic, we have

$$\begin{aligned}
&\Psi_*f_*B(X,Y)+\theta(X,Y)\Psi_*N+\varphi(X,Y)\Psi\circ f-\lambda^{-1}\theta(X,Y)w \qquad (4.15)\\
&= s_*((\nabla'_{\pi_*Y}\bar{D}_i)\pi_*X-(\nabla'_{\pi_*X}\bar{D}_i)\pi_*Y)+\alpha'(\pi_*Y,\bar{D}_i\pi_*X)-\alpha'(\pi_*X,\bar{D}_i\pi_*Y)\\
&= (-1)^j s_*\left(\bar{\psi}(\pi_*Y)\bar{D}_j(\pi_*X)-\bar{\psi}(\pi_*X)\bar{D}_j\pi_*Y\right)\\
&= (-1)^j\left(\psi(Y)S_*D_jX-\psi(X)S_*D_jY\right).
\end{aligned}$$

From equation (4.4) we have

$$\begin{aligned}
&(-1)^j\left(\psi(Y)S_*D_jX-\psi(X)S_*D_jY\right)\\
&\quad= (-1)^j\psi(Y)\left(\langle D_jX,\operatorname{grad}\lambda\rangle\,\Psi\circ f-\Psi_*f_*(A-\lambda I)D_jX\right)\\
&\qquad - (-1)^j\psi(X)\left(\langle D_jY,\operatorname{grad}\lambda\rangle\,\Psi\circ f-\Psi_*f_*(A-\lambda I)D_jY\right).
\end{aligned}$$

Therefore, if we arrange equation (4.15) with this new information, we end up with

$$\Psi_*f_*\tilde{B}(X,Y)+\theta(X,Y)\Psi_*N+\tilde{\varphi}(X,Y)\Psi\circ f-\lambda^{-1}\theta(X,Y)w=0$$

where $\tilde{B}$ and $\tilde{\varphi}$ are proper modifications of B and φ. Because the above equation is expressed as an orthogonal decomposition, we conclude that

$$0=\theta(X,Y)=\lambda(\langle X,(A-\lambda I)D_iY\rangle-\langle Y,(A-\lambda I)D_iX\rangle),$$

for all projectable vector fields X,$Y \in \Gamma(\Delta^\perp)$. Since θ is a tensor, we conclude that

$(A-\lambda I)D_i$ is symmetric.

Let $J \in \Gamma(\mathrm{End}(\Delta^\perp))$ (respectively, $D_i \in \Gamma(\mathrm{End}(\Delta^\perp))$ be the horizontal lift of $\bar{J}$ (respectively, $\bar{D}_i$) and ψ the horizontal lift of $\bar{\psi}$. Since $\bar{D}_1$, $\bar{D}_2 \in \mathrm{span}\,\{\bar{I}, \bar{J}\}$ and $\pi_*|_{\Delta^\perp}$ is an isomorphism, we have that D_1, $D_2 \in \mathrm{span}\,\{I, J\}$ and $J^2 = \epsilon I$, depending on whether s is hyperbolic or elliptic. Let us prove that D_i and ψ satisfy (i) to (ix), and that f is hyperbolic (respectively, elliptic) with respect to J. Items (i) and (vii) are clear because ψ projects to $\bar{\psi}$. From item (a) we get item (ii), item (e) gives item (ix) and from item (d) we get item (viii).

To prove item (iii), since D_i are projectable tensors, we have

$$\nabla^h_T D_i = [D_i, C_T]$$

for all $T \in \Gamma(\Delta)$. On the other hand, because of $\nabla^h_T A = \nabla^h_T (A-\lambda I)$ and equation (3.1), we get

$$\begin{aligned}
\nabla^h_T (A-\lambda I)D_i &- (A-\lambda I)D_i C_T \\
&= (\nabla^h_T (A-\lambda I) - (A-\lambda I)C_T)D_i + (A-\lambda I)(\nabla^h_T D_i - [D_i, C_T]) \\
&= 0.
\end{aligned}$$

Hence,

$$\nabla^h_T (A-\lambda I)D_i = (A-\lambda I)D_i C_T.$$

In particular, this implies that $(A-\lambda I)D_i C_T$ is symmetric. Therefore,

$$\begin{aligned}
(A-\lambda I)D_i C_T &= C_T^t D_i^t (A-\lambda I) \\
&= C_T^t (A-\lambda I)D_i \\
&= (A-\lambda I)C_T D_i.
\end{aligned}$$

Since $(A-\lambda I)$ is an isomorphism when restricted to $\Delta^\perp$, we obtain item (iii).

Observe that there exists $i = 1, 2$ such that $D_i = a_i I + b_i J$ with b_i not null, otherwise, $\bar{D}_1$ and $\bar{D}_2$ would be multiples of the identity endomorphism, and from item (a) we would end up with $\bar{D}_1 = \pm\bar{D}_2$, a contradiction with item (d). Since $[D_i, C_T] = 0$, for all $T \in \Gamma(\Delta)$, it follows that $\nabla^h_T J = [J, C_T] = 0$. Because f is not conformally surface-like, $\{I, J\}$ must be linearly independent. If we put J into Jordan canonical form with the condition $J^2 = \pm I$, and see what kind of matrices commute with J, we get $C(\Gamma(\Delta)) \leq \mathrm{span}\{I, J\}$. Thus f is hyperbolic (respectively, elliptic) with respect to J.

Since s is hyperbolic (respectively, elliptic) with respect to $\bar{J}$ and $\bar{D}_i \in \operatorname{span}\{\bar{I}, \bar{J}\}$, then

$$\begin{aligned}\alpha'(\bar{D}_i\pi_*X, \pi_*Y) - \alpha'(\pi_*X, \bar{D}_i\pi_*Y) &= \alpha'((a_i\bar{I} + b_i\bar{J})\pi_*X, \pi_*Y) - \alpha'(\pi_*X, (a_i\bar{I} + b_i\bar{J})\pi_*Y)\\ &= b_i\left(\alpha'(\bar{D}_i\pi_*X, \pi_*Y) - \alpha'(\pi_*X, \bar{D}_i\pi_*Y)\right),\end{aligned}$$

for all X,$Y \in \Delta^\perp$, where we choose D_i such that $D_i = a_iI + b_iJ$ with b_i not null. Hence,

$$\alpha'(\bar{D}_i\pi_*X, \pi_*Y) = \alpha'(\pi_*X, \bar{D}_i\pi_*Y).$$

From equation (4.12), and taking into account that θ is null because of the symmetry of $(A - \lambda I)D_i$, we get

$$\begin{aligned}&\Psi_*f_*\left((\nabla_X(A - \lambda I)D_i)\,Y - (\nabla_Y(A - \lambda I)D_i)\,X - X \wedge Y(D_i^t\operatorname{grad}\lambda)\right)\\ &\quad + \left(\langle(\nabla_YD_i)X - (\nabla_XD_i)Y, \operatorname{grad}\lambda\rangle + \operatorname{Hess}\lambda(D_iX, Y) - \operatorname{Hess}\lambda(X, D_iY)\right)\Psi\circ f\\ &\quad - \lambda\left(\langle(A - \lambda I)X, (A - \lambda I)D_iY\rangle - \langle(A - \lambda I)D_iX, (A - \lambda I)Y\rangle\right)\Psi\circ f\\ &= s_*\left(\left(\nabla'_{\pi_*Y}\bar{D}_i\right)\pi_*X - \left(\nabla'_{\pi_*X}\bar{D}_i\right)\pi_*Y\right).\end{aligned} \tag{4.16}$$

Using item (b), equation (4.4), and the fact that D_1, D_2 and ψ project to $\bar{D}_1$, $\bar{D}_2$ and $\bar{\psi}$, respectively, we obtain

$$\begin{aligned}&s_*\left(\left(\nabla'_{\pi_*Y}\bar{D}_i\right)\pi_*X - \left(\nabla'_{\pi_*X}\bar{D}_i\right)\pi_*Y\right)\\ &\quad = (-1)^js_*\left(\bar{\psi}(\pi_*Y)\bar{D}_j\pi_*X - \bar{\psi}(\pi_*X)\bar{D}_j\pi_*Y\right)\\ &\quad = (-1)^j\psi(Y)S_*D_jX - (-1)^j\psi(X)S_*D_jY\\ &\quad = (-1)^j\psi(Y)\left(\langle D_jX, \operatorname{grad}\lambda\rangle\,\Psi\circ f - \Psi_*f_*(A - \lambda I)D_jX\right)\\ &\qquad - (-1)^j\psi(X)\left(\langle D_jY, \operatorname{grad}\lambda\rangle\,\Psi\circ f - \Psi_*f_*(A - \lambda I)D_jY\right).\end{aligned} \tag{4.17}$$

Combining equations (4.16) and (4.17), we get

$$\begin{aligned}0 &= \Psi_*f_*\left((\nabla_X(A - \lambda I)D_i)\,Y - (\nabla_Y(A - \lambda I)D_i)\,X - X \wedge Y(D_i^t\operatorname{grad}\lambda)\right)\\ &\quad + (-1)^j\Psi_*f_*(A - \lambda I)\left(\psi(Y)D_jX - \psi(X)D_jY\right)\\ &\quad + (-1)^j\left(\psi(X)\langle D_jY, \operatorname{grad}\lambda\rangle - \psi(Y)\langle D_jX, \operatorname{grad}\lambda\rangle\right)\Psi\circ f\\ &\quad + \left(\langle(\nabla_YD_i)X - (\nabla_XD_i)Y, \operatorname{grad}\lambda\rangle + \operatorname{Hess}\lambda(D_iX, Y) - \operatorname{Hess}\lambda(X, D_iY)\right)\Psi\circ f\\ &\quad - \lambda\left(\langle(A - \lambda I)X, (A - \lambda I)D_iY\rangle - \langle(A - \lambda I)D_iX, (A - \lambda I)Y\rangle\right)\Psi\circ f.\end{aligned}$$

Taking into account the symmetry of the operator $(A-\lambda I)D_i$, and that the above identity is expressed in an orthogonal basis, we get items (iv) and (v) of Proposition 3.2. Going the other way around in equation (4.14) gives us (vii). This concludes the proof. $\square$

Chapter 5

The Subset $\mathcal{C}_S$

The aim of this chapter is to characterize hyperbolic or elliptic surfaces $s : L^2 \to \mathbb{Q}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ that admit a triple $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ satisfying items (a) to (e) of Lemma 4.3. We follow closely the proof of Proposition 9 in [12].

Let us start with the case in which $s : L^2 \to \mathbb{Q}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ is an hyperbolic surface with respect to the tensor $\bar{J}$. Let (u, v) be coordinates in a neighborhood W of $(0, 0)$ whose coordinate vector fields $\{\partial_u, \partial_v\}$ are eigenvectors of $\bar{J}$ with eigenvalues 1 and -1, respectively. Since the surface s is hyperbolic,

$$\alpha'(\partial_u, \partial_v) = \alpha'(J\partial_u, \partial_v) = \alpha'(\partial_u, J\partial_v) = -\alpha'(\partial_u, \partial_v),$$

and hence

$$\alpha'(\partial_u, \partial_v) = 0.$$

The coordinates (u, v) are called *real-conjugate coordinates*. Write

$$\nabla_{\partial_u}\partial_v = \Gamma^1\partial_u + \Gamma^2\partial_v, \tag{5.1}$$

where Γ^i are the Christoffel symbols in terms of the frame $\{\partial_u, \partial_v\}$. As usual, we denote $F = \langle\partial_u, \partial_v\rangle$, and please do not confuse with the isometric light-cone representative of f. Define the differential operator

$$Q(\theta) = \operatorname{Hess}\theta(\partial_u, \partial_v) + F\theta = \theta_{uv} - \Gamma^1\theta_u - \Gamma^2\theta_v + F\theta. \tag{5.2}$$

For each pair of smooth functions $U = U(u)$ and $V = V(v)$, define

$$\varphi^U(u, v) = U(u)e^{2\int_0^v \Gamma^1(u,s)\mathrm{d}s} \quad \text{and} \quad \phi^V(u, v) = V(v)e^{2\int_0^u \Gamma^2(s,v)\mathrm{d}s}. \tag{5.3}$$

These functions satisfy

$$\varphi_v^U = 2\Gamma^1 \varphi^U \quad \text{and} \quad \phi_u^V = 2\Gamma^2 \psi^V \tag{5.4}$$

with initial conditions $\varphi^U(u,0) = U(u)$ and $\phi^V(0,v) = V(v)$. In particular, U and V can be recovered from φ^U and ϕ^V. Assume, in addition, that one of the following conditions holds:

$$U, V > 0 \quad \text{or} \quad 0 < 2\varphi^U < -(2\phi^V + 1) \quad \text{or} \quad 0 < 2\phi^V < -(2\varphi^U + 1). \tag{5.5}$$

Under one of these conditions, one can define

$$\rho^{UV} = \sqrt{|2(\varphi^U + \phi^V) + 1|} \tag{5.6}$$

and

$$\mathcal{C}_s = \left\{ (U,V) : (5.5)\,\text{holds and}\, Q\left(\rho^{UV}\right) = 0 \right\}.$$

Now, let us suppose that $s : L^2 \to \mathbb{Q}_{1,1}^{n+2}$ is an elliptic surface with respect to a tensor J. Let (u,v) be coordinates around $(0,0)$ whose coordinate vector fields satisfy $J\partial_u = \partial_v$ and $J\partial_v = -\partial_u$. Extend the definition of J, ∇ and α^s to the complex field, that is,

$$J(X + iY) = JX + iJY,$$

$$\nabla_{X+iY}(Z + iW) = \nabla_X Z - \nabla_Y W + i\nabla_Y Z + i\nabla_X W$$

and

$$\alpha^s(X + iY, Z + iW) = \alpha^s(X,Z) - \alpha^s(Y,W) + i\alpha^s(Y,Z) + i\alpha^s(X,W).$$

If we define $\partial_z = (\partial_u - i\partial_v)/2$ and $\partial_{\bar{z}} = (\partial_u + i\partial_v)/2$, then we have

$$J\partial_z = \frac{J\partial_u - iJ\partial_v}{2} = \frac{\partial_v + i\partial_u}{2} = \frac{i(\partial_u - i\partial_v)}{2} = i\partial_z$$

and

$$J\partial_{\bar{z}} = \frac{J\partial_u + iJ\partial_v}{2} = \frac{\partial_v - i\partial_u}{2} = \frac{-i(\partial_u + i\partial_v)}{2} = -i\partial_{\bar{z}}.$$

Hence, ∂_z and $\partial_{\bar{z}}$ are eigenvectors of the complexified tensor J with eigenvalues i and $-i$, respectively, and from the fact that s is elliptic we have

$$i\alpha^s(\partial_z, \partial_{\bar{z}}) = \alpha^s(J\partial_z, \partial_{\bar{z}}) = \alpha^s(\partial_z, J\partial_{\bar{z}}) = -i\alpha^s(\partial_z, \partial_{\bar{z}}),$$

so, $\alpha^s(\partial_z, \partial_{\bar{z}}) = 0$. As in the hyperbolic case, the coordinates (u,v) also receive a special name: they are called *complex-conjugate coordinates*.

Let Γ_{ij}^{j} be the Christoffel symbols symbols of the connection ∇ with respect to the basis ∂_u and ∂_v. Then,

$$\nabla_{\partial_z}\partial_{\bar z} = \frac{1}{4}\nabla_{\partial_u - i\partial_v}(\partial_u + i\partial_v),$$

thus

$$\begin{aligned}
\nabla_{\partial_z}\partial_{\bar z} &= \frac{1}{4}\left(\nabla_{\partial_u}\partial_u + \nabla_{\partial_v}\partial_v - i\nabla_{\partial_v}\partial_u + i\nabla_{\partial_u}\partial_v\right)\\
&= \frac{1}{4}\left(\Gamma_{11}^1\partial_u + \Gamma_{11}^2\partial_v + \Gamma_{22}^1\partial_u + \Gamma_{22}^2\partial_v - i\Gamma_{12}^1\partial_u - i\Gamma_{12}^2\partial_v + i\Gamma_{12}^1\partial_u + i\Gamma_{12}^2\partial_v\right)\\
&= \frac{1}{4}\left(\left(\Gamma_{11}^1 + \Gamma_{22}^1\right)\partial_u + \left(\Gamma_{11}^2 + \Gamma_{22}^2\right)\partial_v\right)\\
&= \left(\frac{\Gamma_{11}^2 + \Gamma_{22}^1}{4} - i\frac{\Gamma_{11}^2 + \Gamma_{22}^2}{4}\right)\partial_{\bar z} + \left(\frac{\Gamma_{11}^2 + \Gamma_{22}^1}{4} + i\frac{\Gamma_{11}^2 + \Gamma_{22}^2}{4}\right)\partial_z.
\end{aligned}$$

Therefore, we can define a complex-valued Christoffel symbol $\Gamma : W \subset L^2 \to \mathbb{C}$ such that

$$\nabla_{\partial_z}\partial_{\bar z} = \Gamma\partial_z + \bar\Gamma\partial_{\bar z}.$$

Set $F = \langle \partial_z, \partial_{\bar z}\rangle$, where $\langle\, ,\rangle$ is the complexified extension of the metric induced by s, and define the differential operator

$$Q(\theta) = \operatorname{Hess}\theta(\partial_z, \partial_{\bar z}) + F\theta = \theta_{z\bar z} - \Gamma\theta_z - \bar\Gamma\theta_{\bar z} + F\theta,$$

where $\theta : W \subset L^2 \to \mathbb{C}$ is a smooth function. For each holomorphic function ζ, let $\varphi^\zeta(z, \bar z)$ be the unique complex valued function by

$$\varphi^\zeta_{\bar z} = 2\Gamma\varphi^\zeta \quad \text{and} \quad \varphi^\zeta(z, 0) = \zeta(z).$$

Assume further that

$$\varphi^\zeta \neq -\frac{1}{2} \quad \text{and} \quad 4\mathrm{Re}\,(\varphi^\zeta) + 1 < 0. \tag{5.7}$$

In this case, define

$$\rho^\zeta = \sqrt{-(4\mathrm{Re}\,(\varphi^\zeta) + 1)}$$

and

$$\mathcal{C}_s = \left\{\zeta \text{ holomorphic} : \text{ equation (5.7) holds and } Q(\rho^\zeta) = 0\right\}.$$

We are now ready to state and prove the main result of the chapter.

Proposition 5.1 (Modification of Proposition 9 in [10])**.** *Let $s : L^2 \to \mathbb{Q}^{n+2}_{1,1} \subset \mathbb{L}^{n+3}$ be an elliptic or hyperbolic surface. Then there exists a triple $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ satisfying all conditions in Lemma 4.3 if and only if $\mathcal{C}_s$ is nonempty. Distinct triples (up to signs and permutation) give rise to distinct elements of $\mathcal{C}_s$, and conversely.*

Proof. We will divide the proof into cases, depending on whether s is hyperbolic or elliptic.

Hyperbolic case

Assume that s is hyperbolic with respect to $\bar{J}$, and let $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ satisfy all conditions in Lemma 4.3. Let (u, v) be real-conjugate coordinates whose coordinate vector fields are eigenvectors of $\bar{J}$. Since $\bar{D}_1$, $\bar{D}_2 \in \operatorname{span}\{\bar{I}, \bar{J}\}$, they are also eigenvectors of $\bar{D}_i$, $1 \le i \le 2$. From condition (a), we can suppose that the endomorphisms $\bar{D}_i$ are represented in this basis by

$$\sqrt{2}\bar{D}_1 = \begin{pmatrix} \theta_1 & 0 \\ 0 & 1/\theta_1 \end{pmatrix} \quad \text{and} \quad \sqrt{2}\bar{D}_2 = \begin{pmatrix} \theta_2 & 0 \\ 0 & 1/\theta_2 \end{pmatrix}. \tag{5.8}$$

From item (e), that is, the assumption that rank $\bar{D}_1^2 + \bar{D}_2^2 - \bar{I} = 2$, and

$$(\sqrt{2}\bar{D}_1)^2 + (\sqrt{2}\bar{D}_2)^2 - 2\bar{I} = \begin{pmatrix} \theta_1^2 + \theta_2^2 - 2 & 0 \\ 0 & 1/\theta_1^2 + 1/\theta_2^2 - 2 \end{pmatrix}$$

we infer that $\theta_1^2 + \theta_2^2 \neq 2$ and $1/\theta_1^2 + 1/\theta_2^2 \neq 2$. Also, from item (d), we get $\theta_1 \neq \pm\theta_2$.

Taking into account that the Lie bracket of coordinate vector fields is zero, the equation of item (b) can be written as

$$\nabla'_{\partial_u} \bar{D}_i \partial_v - \nabla'_{\partial_v} \bar{D}_i \partial_u = (-1)^j \left(\bar{\psi}^u \bar{D}_j \partial_v - \bar{\psi}^v \bar{D}_j \partial_u \right), \; i \neq j,$$

where $\bar{\psi}^u = \bar{\psi}(\partial_u)$ and $\bar{\psi}^v = \bar{\psi}(\partial_v)$. Multiplying both sides of the above equation by $\sqrt{2}$, and using the information about how the endomorphisms D_i act on $\{\partial_u, \partial_v\}$, we get

$$\nabla'_{\partial_u} \theta_i^{-1} \partial_v - \nabla'_{\partial_v} \theta_i \partial_u = (-1)^j \left(\bar{\psi}^u \theta_j^{-1} \partial_v - \bar{\psi}^v \theta_j \partial_u \right), \; i \neq j.$$

Working further on the above equation, we obtain

$$\begin{aligned} -\frac{(\theta_i)_u}{\theta_i^2}\partial_v + \theta_i^{-1}(\Gamma^1\partial_u + \Gamma^2\partial_v) - (\theta_i)_v\partial_u - \theta_i(\Gamma^1\partial_u + \Gamma^2\partial_v) \\ = (-1)^j \left(\bar{\psi}^u \theta_j^{-1} \partial_v - \bar{\psi}^v \theta_j \partial_u \right), \; i \neq j. \end{aligned}$$

From the equality of the components of both sides of the preceding equation with respect to the coordinate vector fields, we get that item (b) is equivalent to the system of partial differential equations

$$\frac{(\theta_i)_u}{\theta_i^2} + \left(\theta_i - \frac{1}{\theta_i}\right)\Gamma^2 = -(-1)^j \frac{\bar{\psi}^u}{\theta_j}, \tag{5.9}$$

$$(\theta_i)_v + \left(\theta_i - \frac{1}{\theta_i}\right)\Gamma^1 = (-1)^j \bar{\psi}^v \theta_j, \tag{5.10}$$

with $i \neq j$. Defining $\tau_i = \theta_i^2$, and multiplying the first equation by $-2/\theta_i$ and the second equation by $2\theta_i$, the preceding system becomes

$$\left(\frac{1}{\tau_i}\right)_u + 2\left(\frac{1}{\tau_i} - 1\right)\Gamma^2 = 2(-1)^j \frac{\bar{\psi}^u}{\theta_1\theta_2}, \tag{5.11}$$

$$(\tau_i)_v + 2(\tau_i - 1)\Gamma^1 = 2(-1)^j \bar{\psi}^v \theta_1\theta_2, \quad 1 \leq i \neq j \leq 2. \tag{5.12}$$

Considering the equation (5.11) for the cases $i = 1$ and $i = 2$ and summing them up we obtain

$$\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)_u + 2\left(\frac{1}{\tau_1} + \frac{1}{\tau_2} - 2\right)\Gamma^2 = 0. \tag{5.13}$$

With the same procedure, but using instead equation (5.12), we get

$$(\tau_1 + \tau_2)_v + 2(\tau_1 + \tau_2 - 2)\Gamma^1 = 0. \tag{5.14}$$

Defining $\alpha = \tau_1 + \tau_2$ and $\beta = 1/\tau_1 + 1/\tau_2$, the preceding equations can be written as

$$\beta_u + 2(\beta - 2)\Gamma^2 = 0 \quad \text{and} \quad \alpha_v + 2(\alpha - 2)\Gamma^1 = 0. \tag{5.15}$$

From the definition of τ_i we have that α, $\beta > 0$. Moreover, since $\theta_1^2 \neq \theta_2^2$, we have that τ_1 and τ_2 are distinct real roots of

$$\tau^2 - (\tau_1 + \tau_2)\tau + \tau_1\tau_2 = 0,$$

or, by expressing in terms of α and β,

$$\tau^2 - \alpha\tau + (\alpha/\beta) = 0.$$

From the discriminant condition, we get $\alpha\beta > 4$, and τ_1 and τ_2 can be recovered from α

and β by solving the second degree polynomial, that is,

$$2\tau_i = \alpha - (-1)^i \sqrt{\frac{\alpha}{\beta}\,(\alpha\beta - 4)}, \quad 1 \leq i \leq 2. \tag{5.16}$$

Since $\theta_1^2 + \theta_2^2 \neq 2$ and $1/\theta_1^2 + 1/\theta_2^2 \neq 2$, we have that $\alpha \neq 2$ and $\beta \neq 2$. Then, we can define

$$\varphi = \frac{1}{\alpha - 2} \quad \text{and} \quad \phi = \frac{1}{\beta - 2}. \tag{5.17}$$

From $\alpha > 0$, $\beta > 0$, $\alpha\beta - 4 > 0$,

$$\alpha = 2 + \frac{1}{\varphi} \quad \text{and} \quad \beta = 2 + \frac{1}{\phi},$$

and noticing that φ and ϕ cannot be both negative, we get

$$0 < \frac{2}{\varphi} + \frac{2}{\phi} + \frac{1}{\varphi\phi} = \frac{1}{\varphi\phi}\left(2\phi + 2\varphi + 1\right),$$

and hence (φ, ϕ) satisfies (5.5). Moreover,

$$\frac{\varphi_v}{\varphi} = -\frac{\alpha_v}{\alpha - 2} \quad \text{and} \quad \frac{\phi_u}{\phi} = -\frac{\beta_u}{\beta - 2},$$

so, from equation (5.15), we get

$$\frac{\varphi_v}{\varphi} = 2\Gamma^1 \quad \text{and} \quad \frac{\phi_u}{\phi} = 2\Gamma^2.$$

We still have not used condition (c) in Lemma 4.3. Let us work first on the left side of item (c). Since the expression of $\bar{\psi}$ on the basis $\{\partial_u, \partial_v\}$ is given by

$$\bar{\psi} = \bar{\psi}^u \mathrm{d}u + \bar{\psi}^v \mathrm{d}v,$$

differentiating the one-form $\bar{\psi}$ we get

$$2\mathrm{d}\bar{\psi}(\partial_u, \partial_v) = 2(\bar{\psi}^v_u - \bar{\psi}^u_v)\mathrm{d}u \wedge \mathrm{d}v(\partial_u, \partial_v) = 2(\bar{\psi}^v_u - \bar{\psi}^u_v).$$

On the other hand,

$$\left\langle \sqrt{2}\bar{D}_2\partial_u, \sqrt{2}\bar{D}_1\partial_v \right\rangle - \left\langle \sqrt{2}\bar{D}_1\partial_u, \sqrt{2}\bar{D}_2\partial_v \right\rangle = \left(\frac{\theta_2}{\theta_1} - \frac{\theta_1}{\theta_2}\right) F = \frac{\tau_2 - \tau_1}{\theta_1\theta_2} F.$$

Therefore, we conclude that item (c) is equivalent to

$$2(\bar{\psi}^v_u - \bar{\psi}^u_v) = \frac{\tau_2 - \tau_1}{\theta_1\theta_2}F. \tag{5.18}$$

Set

$$\rho = \sqrt{|2(\varphi + \phi) + 1|} = \sqrt{\left|\frac{2}{\alpha - 2} + \frac{2}{\beta - 2} + 1\right|} = \frac{\sqrt{\alpha\beta - 4}}{\sqrt{|(\alpha - 2)(\beta - 2)|}}. \tag{5.19}$$

We want to show now that

$$Q(\rho) = \rho_{uv} - \Gamma^1\rho_u - \Gamma^2\rho_v + F\rho = 0.$$

In order to do so, we will express the functions ρ, Γ^1 and Γ^2 in terms of θ_i, and then replace into the differential equation. First, by definition we have $\tau_i = \theta_i^2$. Also by definition, $\alpha = \tau_1 + \tau_2 = \theta_1^2 + \theta_1^2$ and $\beta = 1/\theta_1^2 + 1/\theta_2^2$. Using equations (5.11) and (5.12) we get

$$\Gamma^1 = -\frac{\theta_1(\theta_1)_v + \theta_2(\theta_2)_v}{\theta_1^2 + \theta_2^2 - 2}, \tag{5.20}$$

$$\Gamma^2 = -\frac{\theta_1^3(\theta_2)_u + \theta_2^3(\theta_1)_u}{\theta_1\theta_2(2\theta_2^2\theta_1^2 - \theta_2^2 - \theta_1^2)}, \tag{5.21}$$

$$\bar{\psi}^u = \frac{(\theta_2)_u\theta_1^3 - (\theta_1)_u\theta_2^3 - (\theta_2)_u\theta_1 + (\theta_1)_u\theta_2}{2\theta_2^2\theta_1^2 - \theta_2^2 - \theta_1^2} \tag{5.22}$$

and

$$\bar{\psi}^v = -\frac{(\theta_2)_v\theta_2\theta_1^2 - (\theta_1)_v\theta_2^2\theta_1 - \theta_2(\theta_2)_v + \theta_1(\theta_1)_v}{\theta_1\theta_2(\theta_1^2 + \theta_2^2 - 2)}. \tag{5.23}$$

Using equation (5.18), we can express F in terms of functions that are defined in terms of the θ_i:

$$F = \frac{2\theta_1\theta_2(\bar{\psi}^v_u - \bar{\psi}^u_v)}{\theta_2^2 - \theta_1^2}. \tag{5.24}$$

Lastly, using equation (5.19) we have

$$\rho = \sqrt{\frac{(\theta_1^2 + \theta_2^2)^2/\theta_1^2\theta_2^2 - 4}{\left|(\theta_1^2 + \theta_2^2 - 2)(\frac{1}{\theta_1^2} + \frac{1}{\theta_2^2} - 2)\right|}}. \tag{5.25}$$

Replacing those identities into

$$Q(\rho) = \rho_{uv} - \Gamma^1\rho_u - \Gamma^2\rho_v + F\rho, \tag{5.26}$$

and with the help of the Maple program, we obtain that $Q(\rho) = 0$. Thus, the set $\mathcal{C}_s$ is non-empty.

Now, let us move on to the converse of the statement. Since $s : L^2 \to \mathbb{Q}^{n+2}_{1,1} \subset \mathbb{L}^{n+3}$ is hyperbolic, there exist real conjugate coordinates (u, v). If $(U, V) \in \mathcal{C}_s$, then

$$\varphi^U(u, v) = U(u)e^{2\int_0^v \Gamma^1(u,s)\mathrm{d}s} \quad \text{and} \quad \phi^V(u, v) = V(v)e^{2\int_0^u \Gamma^2(s,v)\mathrm{d}s}$$

must satisfy equation (5.4) and, together with the functions U and V, also satisfy equation (5.5). From the definition of the set $\mathcal{C}_s$, we must have $Q(\rho) = 0$, where $\rho = \sqrt{|2(\varphi^U + \phi^V) + 1|}$. Set

$$\alpha = 2 + \frac{1}{\varphi^U} \quad \text{and} \quad \beta = 2 + \frac{1}{\phi^V}$$

which are well defined because U, V, φ^U and ϕ^V satisfy one of the equations in (5.5), and therefore, φ^U and ϕ^V cannot vanish at any point.

Since (φ^U, ϕ^V) satisfies equation (5.5), we affirm that we must have $\alpha > 0$, $\beta > 0$ and $\alpha\beta - 4 > 0$. In the first possiblity, namely, if U, $V > 0$, then $\varphi^U > 0$ and $\phi^V > 0$, and using the definition of α and β we conclude the validity of our affirmation. If

$$0 < 2\varphi^U < -(2\phi^V + 1),$$

then we immediately see that $\alpha > 0$. We also have $\psi^V < -1/2$, so $\beta > 0$. Lastly,

$$\alpha\beta - 4 = \frac{2}{\varphi^U} + \frac{2}{\phi^V} + \frac{1}{\varphi^U\phi^V} = \frac{1}{\varphi^U\phi^V}\left(2\varphi^U + 2\phi^V + 1\right).$$

Since, $\varphi^U > 0$, $\phi^V < 0$ and $2\varphi^U + 2\phi^V + 1 < 0$, we conclude that $\alpha\beta - 4 > 0$. Because the other case is symmetric, we have finished the proof of the affirmation.

With the information $\alpha > 0$, $\beta > 0$ and $\alpha\beta - 4 > 0$, we can define the functions τ_i by equation (5.16), that is, τ_i are roots of the second degree polynomial $\tau^2 - \alpha\tau + \alpha/\beta = 0$, for $i = 1$ and $i = 2$. We then conclude that $\tau_1 + \tau_2 = \alpha$ and $\tau_1\tau_2 = \alpha/\beta$.

As before, write $\tau_i = (\theta_i)^2$ and let $\bar{\psi}^u$ and $\bar{\psi}^v$ be given by equations (5.11) and (5.12), respectively. Replacing τ_i by θ_i^2 in those equations, we arrive at the same equations as in the direct statement, so we can express Γ^1, Γ^2, $\bar{\psi}^u$ and $\bar{\psi}^v$ in terms of the θ_i by the identities (5.20), (5.21), (5.22) and (5.23). From the fact that $\tau_1 + \tau_2 = \alpha$ and $\tau_1\tau_2 = \alpha/\beta$, we get

$$\alpha = \theta_1^2 + \theta_2^2 \quad \text{and} \quad \beta = 1/\theta_1^2 + 1/\theta_2^2.$$

From the definition of ρ, we have that equation (5.19) is valid, and so, replacing α and β is terms of the θ_i, we also obtain equation (5.25). Since ρ cannot vanish at any point, and from $Q(\rho) = 0$, we obtain

$$F = -\frac{\rho_{uv} - \Gamma^1 \rho_u - \Gamma^2 \rho_v}{\rho} \tag{5.27}$$

which can also be expressed in terms of the θ_i using equations (5.25), (5.20) and (5.21).

Replacing those identities in

$$2(\bar{\psi}^v_u - \bar{\psi}^u_v) - \frac{\tau_2 - \tau_1}{\theta_1 \theta_2} F, \tag{5.28}$$

and using Maple, we get that the above equation is identically zero, so equation (5.18) is satisfied. Let $\bar{D}_1$ and $\bar{D}_2$ be defined by equation (5.8) with respect to the frame $\{\partial_u, \partial_v\}$ of coordinate vector fields, and set $\bar{\psi} = \bar{\psi}^u \mathrm{d}u + \bar{\psi}^v \mathrm{d}v$. Then condition (a) is clear from the definition of $\bar{D}_i$, whereas condition (b) follows from the validity of equations (5.11) and (5.12). Condition (c) is a consequence of equation (5.18). Since $\alpha > 0$, we have $\tau_1 \neq -\tau_2$, so $\bar{D}^1_1 \neq -\bar{D}^2_2$. Because the discriminant is $\alpha\beta - 4 > 0$, τ_1 and τ_2 are not equal, so $\bar{D}^1_1 \neq \bar{D}^2_2$, and item (d) is proved. From the definition of α and β we cannot have $\alpha = 2$ or $\beta = 2$, so item (e) follows. Distinct pairs (φ, ϕ) give rise to distinct 4-tuples $(\tau^1, \tau^2, \bar{\psi}^u, \bar{\psi}^v)$, and hence to distinct triples $(\bar{D}_1, \bar{D}_2, \bar{\psi})$. This completes the proof for the hyperbolic case.

Elliptic case

Nearly all the ideas used in the hyperbolic case will be applied to this case. In fact, we will arrive at the same equations, as we will soon see. The difference is that we will need to work with the complex extensions of the tensors in order to have complex eigenvectors.

Suppose $s : L^2 \to \mathbb{Q}^{n+2}_{1,1} \subset \mathbb{L}^{n+3}$ is an elliptic surface, and that there exists a triple $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ satisfying all conditions in Lemma 4.3. Since we will use complex conjugate operation, let us omit the bar notation on the triple just for now. We can assume that there exist complex-conjugate coordinates (u, v) around $(0, 0)$ on L^2 such that

$$\partial_z = \frac{\partial_u - i\partial_v}{2} \quad \text{and} \quad \partial_{\bar{z}} = \frac{\partial_u + i\partial_v}{2}$$

are eigenvectors of the complex linear extension of the tensor J with eigenvectors i and

$-i$, respectively. From item (a) of Lemma 4.3 we can assume that

$$\sqrt{2}D_i = a_i I + b_i J,$$

where $a_i^2 + b_i^2 = 1$. Considering the complex extension of the tensor D_i, which we will denote by the same symbol, we have

$$\sqrt{2}D_i\partial_z = (a_i + ib_i)\partial_z \quad \text{and} \quad \sqrt{2}D_i\partial_{\bar{z}} = (a_i - ib_i)\partial_{\bar{z}}.$$

Therefore, we can write

$$\sqrt{2}D_1 = \begin{pmatrix} \theta_1 & 0 \\ 0 & \bar{\theta}_1 \end{pmatrix} \quad \text{and} \quad \sqrt{2}D_2 = \begin{pmatrix} \theta_2 & 0 \\ 0 & \bar{\theta}_2 \end{pmatrix}, \tag{5.29}$$

with respect to the frame $\{\partial_z, \partial_{\bar{z}}\}$, where $\theta_i : L^2 \to \mathbb{S}^1$. Moreover, from item (d) of Lemma 4.3, we must have $\theta_1 \neq \pm\theta_2$.

We will now prove some properties of the functions θ_i and of the one-form ψ. Set $\psi^z = \psi(\partial_z)$ and $\psi^{\bar{z}} = \psi(\partial_{\bar{z}})$. If $\psi = \psi^u \mathrm{d}u + \psi^v \mathrm{d}v$, then

$$\psi^z = 1/2(\psi^u - i\psi^v) \quad \text{and} \quad \psi^{\bar{z}} = 1/2(\psi^u + i\psi^v),$$

so

$$\overline{\psi^z} = \psi^{\bar{z}}. \tag{5.30}$$

Similarly, if $\theta_j = x^j + iy^j$, where x^j, $y^j : L^2 \to \mathbb{R}$, then

$$(\theta_j)_z = 1/2\left(x^j_u - ix^j_v + iy^j_u + y^j_v\right) \quad \text{and} \quad (\bar{\theta}_j)_{\bar{z}} = 1/2(x^j_u + ix^j_v - iy^j_u + y^j_v).$$

Also,

$$(\theta_j)_{\bar{z}} = 1/2\left(x^j_u + ix^j_v + iy^j_u - y^j_v\right) \quad \text{and} \quad (\bar{\theta}_j)_z = 1/2\left(x^j_u - ix^j_v - iy^j_u - y^j_v\right).$$

Therefore,

$$\overline{(\theta_j)_z} = (\bar{\theta}_j)_{\bar{z}} \quad \text{and} \quad \overline{(\theta_j)_{\bar{z}}} = (\bar{\theta}_j)_z. \tag{5.31}$$

As mentioned before, we can define a complex valued Christoffel symbol Γ by

$$\nabla_{\partial_z}\partial_{\bar{z}} = \Gamma\partial_z + \bar{\Gamma}\partial_{\bar{z}}.$$

As in the hyperbolic case, define $\tau^i = \theta_i^2$, $1 \leq i \leq 2$. Then, from item (b) of Lemma 4.3, and taking into account how the endomorphisms $\sqrt{2}D_i$ act on the frame $\{\partial_z, \partial_{\bar{z}}\}$, we get

$$\nabla_{\partial_z}\bar{\theta}_i\partial_{\bar{z}} - \nabla_{\partial_{\bar{z}}}\theta_i\partial_z = (-1)^j \left(\psi^z\bar{\theta}_j\partial_{\bar{z}} - \psi^{\bar{z}}\theta_j\partial_z\right),$$

which is equivalent to

$$(\bar{\theta}_i)_z\partial_{\bar{z}} + \bar{\theta}_i\left(\Gamma\partial_z + \bar{\Gamma}\partial_{\bar{z}}\right) - (\theta_i)_{\bar{z}}\partial_z - \theta_i\left(\Gamma\partial_z + \bar{\Gamma}\partial_{\bar{z}}\right) = (-1)^j \left(\psi^z\bar{\theta}_j\partial_{\bar{z}} - \psi^{\bar{z}}\theta_j\partial_z\right).$$

Therefore, we get a system of partial differential equations

$$(\theta_i)_{\bar{z}} - \bar{\theta}_i\Gamma + \theta_i\Gamma = (-1)^j\psi^{\bar{z}}\theta_j \quad \text{and} \quad (\bar{\theta}_i)_z + \bar{\theta}_i\bar{\Gamma} - \theta_i\bar{\Gamma} = (-1)^j\psi^z\bar{\theta}_j. \tag{5.32}$$

Those equations are equivalent, as the reader can see by performing the conjugate operation on any of those identities and using equations (5.30) and (5.31). Like in the hyperbolic case, multiplying both sides of the equation (5.32) by $2\theta_i$ and taking into account that $\theta_i \in S^1$, we get

$$(\tau_i)_{\bar{z}} + 2\left(\tau_i - 1\right)\Gamma = 2(-1)^j\psi^{\bar{z}}\theta_1\theta_2. \tag{5.33}$$

Now, we will use item (c) of Lemma 4.3. On one hand, since

$$\mathrm{d}\psi = (\psi_u^v - \psi_v^u)\mathrm{d}u \wedge \mathrm{d}v,$$

we obtain

$$2\mathrm{d}\psi(\partial_z, \partial_{\bar{z}}) = \frac{1}{2}\mathrm{d}\psi(\partial_u - i\partial_v, \partial_u + i\partial_v) = i\mathrm{d}\psi(\partial_u, \partial_v) = i(\psi_u^v - \psi_v^u).$$

Because $\psi^z = 1/2(\psi^u - i\psi^v)$, then $\psi_{\bar{z}}^z = 1/4(\psi_u^u + i\psi_v^u - i\psi_u^v + \psi_v^v)$. Therefore, we have $2\mathrm{d}\psi(\partial_z, \partial_{\bar{z}}) = -4i\mathrm{Im}\,\psi_{\bar{z}}^z$. On the other hand,

$$\left\langle\sqrt{2}D_2\partial_z, \sqrt{2}D_1\partial_{\bar{z}}\right\rangle - \left\langle\sqrt{2}D_1\partial_z, \sqrt{2}D_2\partial_{\bar{z}}\right\rangle = \left(\bar{\theta}_1\theta_2 - \theta_1\bar{\theta}_2\right)F = \frac{\tau_2 - \tau_1}{\theta_1\theta_2}F.$$

Using item (c) of Lemma (4.3) and multiplying both sides by i, we conclude

$$4\mathrm{Im}\,\psi_{\bar{z}}^z = i\frac{\tau_2 - \tau_1}{\theta_1\theta_2}F. \tag{5.34}$$

As in the hyperbolic case, define $\alpha = \tau_1 + \tau_2$. Then, summing up the cases $i = 1$ and

$i = 2$ in equation (5.33), we obtain

$$\alpha_{\bar{z}} + 2(\alpha - 2)\Gamma = 0. \tag{5.35}$$

Because $\theta_i \in S^1$, also $\tau_i \in S^1$. From condition (d) in Lemma 4.3, we have $\tau_i \neq \pm\tau_2$. Hence, $0 < |\alpha| = |\tau_1 + \tau_2| < 2$. Thus,

$$\varphi = \frac{1}{\alpha - 2}$$

is well defined and satisfies

$$\frac{\varphi_{\bar{z}}}{\varphi} = -\frac{\alpha_{\bar{z}}}{\alpha - 2} = 2\Gamma.$$

Since

$$4\mathrm{Re}\,\varphi + 1 = 2\frac{\alpha + \bar{\alpha} - 4}{|\alpha - 2|^2} + 1 = \frac{|\alpha|^2 - 4}{|\alpha - 2|^2}$$

and $|\alpha| < 2$, we conclude that $4\mathrm{Re}\,\varphi + 1 < 0$. Since $\alpha \neq 0$, we have $\varphi \neq -1/2$, thus we obtain the conditions in equation (5.7). From the equation $x^2 - (\tau_1 + \tau_2)x + \tau_1\tau_2 = 0$, we can recover τ_1 and τ_2. Because $\tau_1 + \tau_2 = \alpha$, $\tau_i \in \mathbb{S}^1$ and

$$\tau_1\tau_2 = \frac{\tau_1 + \tau_2}{1/\tau_1 + 1/\tau_2} = \frac{\tau_1 + \tau_2}{\bar{\tau}_1 + \bar{\tau}_2} = \frac{\alpha}{\bar{\alpha}},$$

we can recover τ_1 and τ_2 from the equation

$$x^2 - \alpha x + \frac{\alpha}{\bar{\alpha}} = 0,$$

that is solving the second degree polynomial, we obtain

$$\tau_j = \frac{\alpha}{2}\left(1 - (-1)^j i\frac{\sqrt{4 - |\alpha|^2}}{|\alpha|}\right). \tag{5.36}$$

Now set

$$\rho = \sqrt{-\left(4\mathrm{Re}\,\varphi + 1\right)} = \frac{\sqrt{4 - |\alpha|^2}}{|\alpha - 2|}, \tag{5.37}$$

we must prove that $Q(\rho) = 0$, in order to show that $\mathcal{C}_s$ is non-empty. In order to do so, as in the hyperbolic case let us express Γ, $\psi^{\bar{z}}$, F and ρ in terms of the functions θ_i. First, notice that $\alpha = \theta_1^2 + \theta_2^2$ and $\bar{\alpha} = 1/\theta_1^2 + 1/\theta_2^2$. From equation (5.35), and replacing α in terms of θ_i, we get

$$\Gamma = -\frac{(\theta_1^2+\theta_2^2)_{\bar z}}{2\,(\theta_1^2+\theta_2^2-2)} = -\frac{\theta_1(\theta_1)_{\bar z}+\theta_2(\theta_2)_{\bar z}}{\theta_1^2+\theta_2^2-2}. \tag{5.38}$$

If we take the complex conjugate of Γ, use that $\bar\theta_i = \theta_i^{-1}$ and the equations (5.31) we get

$$\bar\Gamma = -\frac{\bar\theta_1(\bar\theta_1)_z+\bar\theta_2(\bar\theta_2)_z}{1/\theta_1^2+1/\theta_2^2-2} = -\frac{\theta_2^3(\theta_1)_z+\theta_1^3(\theta_2)_z}{\theta_1\theta_2\,(2\theta_1^2\theta_2^2-\theta_1^2-\theta_2^2)}. \tag{5.39}$$

Using equation (5.33) with $i=1$ and the Γ expression in terms of the θ_i, we obtain

$$\psi^{\bar z} = \frac{\theta_1\theta_2^2(\theta_1)_{\bar z}-\theta_1^2\theta_2(\theta_2)_{\bar z}-\theta_1(\theta_1)_{\bar z}+\theta_2(\theta_2)_{\bar z}}{\theta_1\theta_2\,(\theta_1^2+\theta_2^2-2)}. \tag{5.40}$$

Taking the complex conjugate of the above equation and using equation (5.30) we get

$$\psi^{z} = \frac{\theta_2(\theta_1)_z-\theta_1(\theta_2)_z-\theta_2^3(\theta_1)_z+\theta_1^3(\theta_2)_z}{2\theta_1^2\theta_2^2-\theta_1^2-\theta_2^2}. \tag{5.41}$$

Observe that

$$(\psi^{\bar z})_z-(\psi^z)_{\bar z} = \overline{(\psi^z)_{\bar z}}-(\psi^z)_{\bar z} = -2i\mathrm{Im}\,(\psi^z)_{\bar z}.$$

Therefore, using equation (5.34) we get

$$2\left((\psi^{\bar z})_z-(\psi^z)_{\bar z}\right) = -4i\mathrm{Im}\,(\psi^z)_{\bar z} = \frac{\theta_2^2-\theta_1^2}{\theta_1\theta_2.}F$$

Solving for F we conclude

$$F = \frac{2\theta_1\theta_2\left((\psi^{\bar z})_z-(\psi^z)_{\bar z}\right)}{\theta_2^2-\theta_1^2}. \tag{5.42}$$

From equation (5.37) and the expression of α and $\bar\alpha$ in terms of θ_i we have

$$\rho = \sqrt{\frac{4-(\theta_1^2+\theta_2^2)/\theta_1^2\theta_2^2}{(\theta_1^2+\theta_2^2-2)\,(1/\theta_1^2+1/\theta_2^2-2)}} = i\sqrt{\frac{(\theta_1^2+\theta_2^2)/\theta_1^2\theta_2^2-4}{|\,(\theta_1^2+\theta_2^2-2)\,(1/\theta_1^2+1/\theta_2^2-2)\,|}}. \tag{5.43}$$

If we compare the expressions we got for Γ, $\bar\Gamma$, $\psi^{\bar z}$, ψ^z, F and ρ, except for constant multiple i in the ρ, they are the same equations as (5.20), (5.21), (5.22), (5.23), (5.24) and (5.25) we have found in the hyperbolic case, when we replace $(z,\bar z)$, $(\Gamma,\bar\Gamma)$, $(\psi^z,\psi^{\bar z})$ for (u,v), (Γ^1,Γ^2) and (ψ^u,ψ^v), respectively. Therefore,

$$Q(\rho) = \rho_{z\bar z}-\Gamma\rho_z-\bar\Gamma\rho_{\bar z}+F\rho = 0,$$

as one can confirm using Maple. This shows that $\mathcal{C}_s$ is non-empty.

Let us move on to the converse of the theorem. Consider complex-conjugate coordinates (u, v) for the surface $s : L^2 \to \mathbb{Q}^{n+2}_{1,1}$. If $\zeta \in \mathcal{C}_s$ is an holomorphic function, then the complex valued function $\varphi^\zeta(z, \bar{z})$ defined by

$$\varphi^{z}_{\bar{z}} = 2\Gamma\varphi^\zeta \quad \text{and} \quad \varphi^\zeta(z, 0) = \zeta$$

satisfies equation (5.7), and $\rho^\zeta = \sqrt{-(4\mathrm{Re}\,\varphi^\zeta + 1)}$ is a function such that $Q(\rho^\zeta) = 0$.

Define

$$\alpha = 2 + \frac{1}{\varphi^\zeta},$$

then from the first condition of equation (5.7) we have that α is not null. Since

$$|\alpha|^2 = \alpha\bar{\alpha} = \left(2 + \frac{\overline{\varphi^\zeta}}{|\varphi^\zeta|^2}\right)\left(2 + \frac{\varphi^\zeta}{|\varphi^\zeta|^2}\right) = 4 + \frac{4\mathrm{Re}\,\varphi^\zeta + 1}{|\varphi^\zeta|^2},$$

from the second condition of equation (5.7), we get $|\alpha| < 2$.

Set τ_j for $j = 1, 2$ by equation (5.36), that is, τ_j are solutions of the second degree polynomial

$$x^2 - \alpha x + \frac{\alpha}{\bar{\alpha}} = 0.$$

Therefore, $\alpha = \tau_1 + \tau_2$. From the definition of τ_j, we have

$$|\tau_j| = \frac{|\alpha|}{2}\sqrt{\left(1 + \frac{4 - |\alpha|^2}{|\alpha|^2}\right)} = 1,$$

for $j = 1, 2$. Also from the definition of τ_j and because $|\alpha| < 2$, we have $\tau_1 \neq \pm\tau_2$.

Write $\tau_j = \theta_j^2$, and define $\psi^{\bar{z}}$ by equation (5.33). In order to define the one-form ψ, remember that in the direct statement we had $\psi^{\bar{z}} = 1/2(\psi^u + i\psi^v)$. Therefore, thinking backwards, define $\psi^u = 2\mathrm{Re}\,\psi^{\bar{z}}$ and $\psi^v = 2\mathrm{Im}\,\psi^{\bar{z}}$. Define the complex extensions $\sqrt{2}D_j$ by equation (5.29). To recover the original $\sqrt{2}D_j$ just remember that $\sqrt{2}D_j = a_jI + b_jJ$ for $\theta_j = a_j + ib_j$. So, we get a triple (D_1, D_2, ψ). We have to show that this triple satisfies conditions (a) to (e) of Lemma (4.3).

Since $|\tau_j| = 1$, then $|\theta_j| = 1$, so $\det\sqrt{2}D_j = 1$ and we obtain (a). Because equation (5.33) is satisfied, item (b) follows. From the fact that $\tau_1 \neq \pm\tau_2$ and how τ_j is defined we get item (d). For item (e), if

$$\sqrt{2}D_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$$

with $\theta_j = a_j + ib_j$ and rank $(\sqrt{2}D_1)^2 + (\sqrt{2}D_2)^2 - 2I < 2$, then

$$(\sqrt{2}D_1)^2 + (\sqrt{2}D_2)^2 - 2I = \begin{pmatrix} a_1^2 - b_1^2 + a_2^2 - b_2^2 - 2 & 2a_1b_1 + 2a_2b_2 \\ -2a_1b_1 - 2a_2b_2 & a_1^2 - b_1^2 + a_2^2 - b_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $|\theta_j| = 1$ we have

$$a_1^2 - b_1^2 + a_2^2 - b_2^2 = 2 \quad \text{and} \quad a_1^2 + b_1^2 + a_2^2 + b_2^2 = 2,$$

and hence, $b_1 = 0 = b_2$ and $a_j = \pm 1$. Therefore, $\theta_1 = \pm\theta_2$, a contradiction because $\tau_1 \neq \tau_2$, which proves (e).

Let us prove item (c). Since $\varphi^\zeta = 1/(\alpha - 2)$, and from the definition of ρ^ζ, we get equation (5.37). Therefore, because $\alpha = \theta_1^2 + \theta_2^2$, we get equation (5.43). Since $\psi^{\bar{z}}$ and Γ satisfy equation (5.33), we have the validity of equations (5.38), (5.39), (5.40) and (5.41). From the condition $Q(\rho) = 0$, we get

$$F = -\frac{-\rho_{z\bar{z}} - \Gamma\rho_z - \bar{\Gamma}\rho_{\bar{z}}}{\rho}, \tag{5.44}$$

so we can express F in terms of θ_i using equations (5.43), (5.38) and (5.39). Notice that the ρ used in the hyperbolic case differs from this ρ by a multiple of i. We arrive at the same equations as in proof of the converse statement of the hyperbolic case, with $(z, \bar{z})$, $(\Gamma, \bar{\Gamma})$, $(\psi^z, \psi^{\bar{z}})$ instead of (u, v), (Γ^1, Γ^2) and (ψ^u, ψ^v), respectively. Thus, equation (5.34) is valid, and so is item (c).

Distinct $\zeta's$ give rise to distinct $\varphi^{\zeta}{}'s$, and so distinct $\alpha's$. Since the τ_i are defined by $x^2 - \alpha x + \frac{\alpha}{\bar{\alpha}} = 0$, we get distinct $\tau_i's$. So, we get distinct $\theta_i's$, and so a distinct triple (D_1, D_2, ψ). This concludes the proof. □

Before finishing the current chapter, we give an explicit example of an hyperbolic surface $s : L^2 \to \mathbb{Q}^m_{1,1}$ whose associated subset $\mathcal{C}_s$ is noempty. In the next chapter, this example and the classification theorem 6.1 will provide us with an example of a hypersurface f that admits a genuine conformal deformation in codimension two. This means that the objects we are studying do exist.

Let us start by orthogonally decomposing

$$\mathbb{L}^{m+1} = \mathbb{R}^{m_1} \times \mathbb{L}^{m_2}$$

and considering a curve $\alpha : I_1 \to \mathbb{S}^{m_1-1} \subset \mathbb{R}^{m_1}$ parametrized by arc length. Denote

$\tilde{\alpha} = i \circ \alpha$, where $i : \mathbb{R}^{m_1} \to \mathbb{L}^{m+1}$ is the inclusion, and consider the flat parallel vector subbundle $\mathcal{L} \subset N_{\tilde{\alpha}} I$ of rank $k = m_2 + 1$ whose fiber at $v \in I_1$ is

$$\mathcal{L}(v) = \mathbb{R}\tilde{\alpha}(v) \oplus \mathbb{L}^{m_2}. \tag{5.45}$$

If $\{\xi_1, \cdots, \xi_k\}$ is an orthonormal frame of parallel sections of $\mathcal{L}$, with $\xi_1(v) = \tilde{\alpha}(v)$, then we can define a parallel vector bundle isometry $\phi : I_1 \times \mathbb{L}^k \to \mathcal{L}$ by

$$\phi(v, Y) = \phi_v(Y) = \sum_{i=1}^{k} Y^i \xi_i(v).$$

Let $e \in \mathbb{L}^k$ be such that $\phi_v(e) = \tilde{\alpha}(v) = \xi_1(v)$ for all $v \in I_1$, and denote

$$\Omega^0(\tilde{\alpha}) = \{Y \in \mathbb{L}^k : \langle Y, e\rangle > 0\}.$$

Consider $\beta : I_0 \to \mathbb{Q}^{k-1}_{1,1} \cap \Omega^0(\tilde{\alpha}) \subset \mathbb{L}^k$, another curve parametrized by arc length. Define $s : I_0 \times I_1 \to \mathbb{Q}^m_{1,1} \subset \mathbb{L}^{m+1}$ by

$$s(u, v) = \phi_v(\beta(u)).$$

Then we have

$$s_*\partial_u = \phi_v(\beta'(u)) \quad \text{and} \quad s_*\partial_v = \langle \beta(u), e\rangle\, \tilde{\alpha}'(v),$$

hence s is an immersion with induced metric

$$ds^2 = du^2 + \rho^2(u) dv^2,$$

where $\rho(u) = \langle \beta(u), e\rangle$. Moreover, differentiating, say, the first of the preceding equations with respect to v gives that

$$\alpha^s(\partial_u, \partial_v) = 0.$$

By a suitable change of coordinates $\tilde{u} = \gamma(u)$, we can pass to isothermal coordinates with respect to which the metric is written as

$$ds^2 = e^{2\lambda(\tilde{u})}(\mathrm{d}\tilde{u}^2 + \mathrm{d}v^2)$$

for some smooth function $\lambda = \lambda(\tilde{u})$, and we still have $\alpha^s(\partial_{\tilde{u}}, \partial_v) = 0$. Thus, the surface s is an hyperbolic surface and $(\tilde{u}, v)$ are real-conjugate coordinates. For simplicity, we rewrite $\tilde{u}$ by u.

Let us show that, for the above surface $s : I_0 \times I_1 \to \mathbb{Q}^m_{1,1} \subset \mathbb{L}^{m+1}$, the subset $\mathcal{C}_s$ is non-empty. If we define by

$$E = \langle \partial_u, \partial_u \rangle = e^{2\lambda(u)}, \quad F = \langle \partial_u, \partial_v \rangle = 0 \quad \text{and} \quad G = \langle \partial_v, \partial_v \rangle = e^{2\lambda(u)},$$

then the Christoffel symbols Γ^1 and Γ^2 defined by (5.1) satisfy

$$0 = E_v = 2\Gamma^1 E \quad \text{and} \quad 2\lambda' e^{2\lambda} = G_u = 2\Gamma^2 G.$$

Hence,

$$\Gamma^1 = 0 \quad \text{and} \quad \Gamma^2 = \lambda'.$$

Given a pair of smooth functions $\tilde{U} = \tilde{U}(u)$ and $V = V(v)$, from the definition of $\varphi^{\tilde{U}}$ and φ^V for the hyperbolic case from equation (5.3), we get

$$\varphi^{\tilde{U}} = \tilde{U} \qquad \text{and} \quad \varphi^V = Ve^{2\lambda}.$$

By suitably modifying $\tilde{U}$ we have

$$\varphi^{\tilde{U}} = e^{2\lambda}U \qquad \text{and} \quad \varphi^V = e^{2\lambda}V,$$

so, taking into account the definition of ρ (equation (5.6)), we obtain

$$\rho = \rho^{\tilde{U}V} = \sqrt{2e^{2\lambda}(U+V)+1}.$$

From the expression of Γ^1 and Γ^2 and the definition of Q given in equation (5.2), we have

$$Q(\theta) = \theta_{uv} - \Gamma^1\theta_u - \Gamma^2\theta_v + F\theta = \theta_{uv} - \lambda'\theta_v.$$

Now,

$$\rho_v = \frac{e^{2\lambda}V_v}{\sqrt{2e^{2\lambda}(U+V)+1}}$$

and so

$$\rho_{uv} = \frac{2\lambda' e^{2\lambda}V_v\left(2e^{2\lambda}(U+V)+1\right) - V_v e^{2\lambda}\left(2\lambda' e^{2\lambda}(U+V) + e^{2\lambda}U_u\right)}{(2e^{2\lambda}(U+V)+1)^{3/2}},$$

which implies that the equation $0 = Q(\rho) = \rho_{uv} - \lambda'\rho_v$ reduces to

$$V_v(2\lambda' - U_u e^{2\lambda}) = 0,$$

which is satisfied for $V = k$, where k is a constant, or for $U = c - e^{-2\lambda}$. With this we have shown that $\mathcal{C}_s$ is nonempty.

We point out that other examples of warped products of curves $s : L^2 \to \mathbb{Q}^m_{1,1}$ as above can be obtained by considering other types of orthogonal decompositions in equation (5.45).

Chapter 6

The Classification

We are now in a position to state and prove the classification of hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ that carry a nowhere vanishing principal curvature of multiplicity $n-2$ and admit a genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$.

Theorem 6.1. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a hypersurface with a nowhere vanishing principal curvature of multiplicity $n-2$. Assume that f is not a Cartan hypersurface on any open subset of M^n, and that it admits a genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$. Then, on each connected component of an open dense subset of M^n, it envelops a two-parameter congruence of hyperspheres $s : L^2 \to \mathbb{Q}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ which is either an elliptic or hyperbolic surface and whose associated set $\mathcal{C}_s$ is non-empty.*

Conversely, any simply connected hypersurface f that envelops a two parameter congruence of hyperspheres $s : L^2 \to \mathbb{Q}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ that is either an elliptic or hyperbolic surface and is such that the set $\mathcal{C}_s$ is non-empty admits genuine conformal deformations in $\mathbb{R}^{n+2}$ which are parametrized by $\mathcal{C}_s$.

Remark 6.2. See remark 3.3.

Proof. It follows from Propositions 2.7 and 3.2 that, on an open dense subset of M^n, the hypersurface is either elliptic or hyperbolic and there exists a unique (up to sign and permutation) triple (D_1, D_2, ψ) satisfying all conditions in Proposition 3.2. By Lemma 4.3, the two-paramenter congruence of hyperspheres $s : L^2 \to \mathbb{Q}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$ that is enveloped by f is either an elliptic or hyperbolic surface, respectively, and the triple (D_1, D_2, ψ) projects to a (unique) triple $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ on L^2 satisfying all conditions in Lemma 4.3. We conclude from Proposition 5.1 that $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ gives rise to a unique element of $\mathcal{C}_s$.

Conversely, assume that $f : M^n \to \mathbb{R}^{n+1}$ is a simply connected hypersurface that envelops a two-parameter congruence of hyperspheres $s : L^2 \to \mathbb{Q}_{1,1}^{n+2} \subset \mathbb{L}^{n+3}$, which is either an elliptic or hyperbolic surface, and is such that the set $\mathcal{C}_s$ is non-empty. By Proposition 5.1, each element of $\mathcal{C}_s$ gives rise to a unique triple $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ on L^2 satisfying all conditions in Lemma 4.3. Then, it follows from Proposition 4.3 that f is either elliptic or hyperbolic, respectively, and that $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ can be lifted to a unique triple (D_1, D_2, ψ) on M^n satisfying all conditions in Proposition 3.2. By Proposition 3.2, such triple yields a unique (up to a conformal transformation of $\mathbb{R}^{n+2}$) genuine conformal deformation $\tilde{f} : M^n \to \mathbb{R}^{n+2}$ of f.

Finally, we also have that (congruence classes of) genuine conformal deformations of f are in one-to-one correspondence with triples $(\bar{D}_1, \bar{D}_2, \bar{\psi})$ on L^2 satisfying all conditions in Lemma 4.3 (up to signs and permutation), and these are in turn in one-to-one correspondence with elements of $\mathcal{C}_s$. $\square$

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