UNIVERSIDADE FEDERAL DE SÃO CARLOS
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PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Higher order div-curl type estimates for elliptic linear differential operators on localizable Hardy spaces

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Catarina Barbosa Machado

Advisor: Tiago Henrique Picon

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Prof. Dr. Tiago Henrique Picon (USP)

Profa. Dra. Galia Dafni (CONCORDIA)

Prof. Dr. Emmanuel Russ (IMM)

Prof. Dr. Gabriel Cueva Candido Soares de Araújo (USP)

Prof. Dr. Gustavo Hoepfner (UFSCar)

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Resumo

Nesta tese de doutorado investigamos estimativas do tipo divergente-rotacional (div-curl) associadas a operadores diferenciais parciais lineares nos espaços de Hardy-Sobolev. O resultado principal apresenta uma extensão local, nos espaços localizáveis de Hardy-Sobolev, das estimativas div-curl para operadores diferenciais parciais lineares homogêneos elípticos com coeficientes complexos suaves inspiradas em um resultado clássico de Coifman, Lions, Meyer & Semmes. Essa versão recupera resultados estabelecidos na literatura relativos a operadores de primeira ordem associados a um sistema elíptico de campos vetoriais suaves complexos. Como ferramenta foram desenvolvidas uma nova decomposição de Calderón-Zygmund e desigualdades do tipo Poincaré para espaços de Hardy-Sobolev. Um segundo resultado principal apresenta uma versão global não homogênea para estimativas do tipo div-curl nos espaços localizáveis de Hardy quando $p = 1$ associada a campos vetoriais complexos com coeficientes constantes. Como aplicação, uma nova caracterização do espaço local *bmo* no espaço euclidiano associado aos operadores div-curl é estabelecida.

Palavras-chave: estimativas divergente-rotacional; espaços de Hardy-Sobolev; decomposição atômica; espaço *bmo*; desigualdade de Poincaré; operadores elípticos; campos vetoriais complexos.

Abstract

In this PhD thesis, we investigate div-curl type estimates associated with linear partial differential operators on Hardy-Sobolev spaces. The main result establishes a local extension of div-curl estimates for elliptic homogeneous linear partial differential operators with smooth complex coefficients in localizable Hardy-Sobolev spaces, inspired by the classical result due to Coifman, Lions, Meyer and Semmes. This version recovers known results in the literature concerning first-order operators associated with elliptic systems of complex vector fields. As key tools, we develop a new Calderón-Zygmund decomposition and a Poincaré-type inequality in localizable Hardy-Sobolev spaces. A second main result presents a global nonhomogeneous version of div-curl type estimates in the localizable Hardy space for $p = 1$, associated with complex vector fields with constant coefficients. As an application, we provide a new characterization of the local *bmo* space in Euclidean space associated with div-curl terms.

Keywords: div-curl estimates; Hardy-Sobolev spaces; atomic decomposition; *bmo* space; Poincaré inequality; elliptic operators; complex vector fields.

Contents

Introduction

In the classical work due to Coifman, Lions, Meyer & Semmes in [13], some nonlinear inequalities were studied, in particular the called div-curl estimates on Hardy space in $\mathbb{R}^N$, precisely: (Theorem II.3) if

$$\frac{N}{N+1} < p < \infty, \quad 1 < q \leq \infty \quad \text{and} \quad \frac{1}{r} := \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N} \tag{1}$$

then there exists $C > 0$ such that

$$\|u \cdot v\|_{H^r} \leq C \|u\|_{H^p} \|v\|_{H^q}, \tag{2}$$

where $u \in H^p(\mathbb{R}^N, \mathbb{R}^N)$ and $v \in H^q(\mathbb{R}^N, \mathbb{R}^N)$ satisfy curl $u = 0$ and div $v = 0$. We remark that, if curl $u = 0$ then we may always write $u = \nabla \phi$ and since $\operatorname{div} v = 0$ we have $u \cdot v = \nabla \phi \cdot v = \operatorname{div}(\phi\, v)$ and the estimate (2) may be written equivalently as

$$\|\nabla \phi \cdot v\|_{H^r} \leq C \|\nabla \phi\|_{H^p} \|v\|_{H^q}, \tag{3}$$

under assumption $\operatorname{div} v = 0$. Note that this restriction condition can be understood as v belonging to the kernel of the formal adjoint of the gradient operator. We point out that in the particular case $r = 1$, the previous inequalities substantially improve the information from Hölder's inequality. We recall that the Hardy space $H^1(\mathbb{R}^N)$ is a strict subspace of $L^1(\mathbb{R}^N)$. Estimates of this type have been the object of extensions as nonhomogeneous version for local Hardy spaces and BMO ([11], [12], [14]), paraproducts ([8], [9], [50]), Lipschitz domains in $\mathbb{R}^N$ ([3], [33], [34]), weighted Hardy spaces and the endpoint case (when $q = \infty$) in ([7]), and applications in the Navier-Stokes equation ([36]).

In [24] the authors obtained a local version of inequality (3) in the setting of systems and complexes of vector fields with complex variable coefficients. Suppose that $\mathscr{L} := \{L_1, \dots, L_n\}$ is a system of linearly independent vector fields with smooth complex coefficients defined on an open set $\Omega \subset \mathbb{R}^N$ and consider the operators

$$\nabla_{\mathscr{L}} u := (L_1 u, \dots, L_n u), \quad \text{for} \quad u \in C^\infty(\Omega)$$

and

$$\operatorname{div}_{\mathscr{L}^*} v := \sum_{j=1}^{n} L_j^* v_j, \quad \text{for} \quad v \in C^\infty(\Omega, \mathbb{R}^n),$$

which are precisely the operators ∇ and div when $n = N$ and $L_j = \partial_{x_j}$ for $j = 1, \dots, n$. Here $L_j^* = \overline{L_j}$ denotes the vector field obtained from L_j by conjugating its coefficients. Since the classical Hardy spaces $H^p(\mathbb{R}^N)$ are not closed by the multiplication by test functions and the related operators have variable coefficient, it is appropriate to consider the nonhomogeneous version of Hardy spaces denoted by $h^p(\mathbb{R}^N)$, also called localizable Hardy spaces, introduced by Goldberg in [22]. [24, Theorem A] asserts that if $\mathscr{L}$ is an elliptic system of complex vector fields defined on $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and r, p, q satisfying (1), then for every point $x_0 \in \Omega$ there exist an open neighborhood $x_0 \in U \subset \Omega$ and $C(U) > 0$ such that

$$\|\nabla_{\mathscr{L}}\phi \cdot v\|_{h^r} \leq C\|\nabla_{\mathscr{L}}\phi\|_{h^p}\|v\|_{h^q} \tag{4}$$

holds for any $\phi \in C_c^\infty(U)$ and $v \in C_c^\infty(U, \mathbb{C}^n)$ satisfying $\operatorname{div}_{\mathscr{L}^*} v = 0$. The ellipticity of the system $\{L_1, \ldots, L_n\}$ means that for any *real* 1-form ω, satisfying $\langle \omega, L_j \rangle = 0$ for all $j = 1, \ldots, n$ implies $\omega = 0$, which is equivalent to saying that the second order operator

$$\Delta_{\mathscr{L}} := L_1^* L_1 + \cdots + L_n^* L_n$$

is elliptic in the classical sense, or even that the operator $\nabla_{\mathscr{L}}$ is elliptic in the classical sense. Extensions of the inequality (4) for complexes associated to the system $\mathscr{L}$, inspired in the global setting for de Rham complex (see [13]), were also presented in [24].

A natural question arises on how to consider analogous inequalities when the gradient is replaced by higher order linear differential operators with variable coefficients. Partial results were obtained by Taylor [46, Chapter 8, Section 8] in the setting of pseudo-differential operators and by Kozono & Yanagisawa in [32] on Lebesgue spaces, including first order linear differential operators with smooth variable coefficients.

In the first half of this thesis, our main goal is to answer the question by presenting a type of div-curl estimates for high order linear differential operators with variable coefficients in Hardy spaces in the same spirit as (3). In order to obtain estimates analogous to (4), for operators with variable coefficients one must deal with local estimates rather than global ones, thus the localizable version on Hardy spaces is appropriate.

Let $\Omega \subseteq \mathbb{R}^N$ an open set and $A(\cdot, D)$ a homogeneous linear differential operator of order m with smooth complex coefficients in Ω denoted by:

$$A(x,D) = \sum_{|\alpha|=m} a_\alpha(x)\partial^\alpha : C^\infty(\Omega, E) \mapsto C^\infty(\Omega, F),$$

where E is a complex vector space of finite dimension n_E and F is a complex vector space of finite dimension $n_F \geq n_E$. The first result of this thesis is the following:

Theorem A. *Let $A(\cdot,D)$ an elliptic homogeneous linear differential operator as before on $\Omega\subset\mathbb{R}^N$, with order $m\geq 1$ and let r,p,q satisfying*

$$\frac{N}{N+m}<p\leq\frac{N}{m},\quad 1<q\leq\infty\ \textit{and}\ \frac{1}{r}:=\frac{1}{p}+\frac{1}{q}<1+\frac{m}{N}.$$

Then for each $x_0\in\Omega$ there exists an open neighborhood $x_0\in U\subset\Omega$ and $C>0$ such that

$$\|A(\cdot,D)\phi\cdot_F v\|_{h^r}\leq C\,\|A(\cdot,D)\phi\|_{h^p}\,\|v\|_{W^{m-1,q}} \tag{5}$$

for any $\phi\in C_c^\infty(U,E)$ and $v\in C_c^\infty(U,F)$, satisfying $A^*(\cdot,D)v=0$.

We say that $A(\cdot,D)$ is elliptic in Ω if for each $x_0\in\Omega$ its principal symbol $a(x_0,\cdot)$ is injective, that is,

$$a(x_0,\xi):=\sum_{|\alpha|=m}a_\alpha(x_0)\xi^\alpha\neq 0,\ \text{for all}\ \xi\in\mathbb{R}^N\neq 0.$$

Here $\|v\|_{W^{k,q}}:=\sum_{|\gamma|\leq k}\|D^\gamma v\|_{L^q}$ is the nonhomogeneous Sobolev norm in L^q with order $k\in\mathbb{N}$. We use the notation $A^*:=\overline{A}^t$ where $\overline{A}$ denotes the operator obtained from A by conjugating its coefficients and A^t is its formal transpose, meaning that for all smooth functions φ and ψ having compact support in Ω and taking values in E and F respectively, we have:

$$\int_\Omega A(x,D)\varphi(x)\cdot_F\psi(x)\,dx=\int_\Omega\varphi(x)\cdot_E A^*(x,D)\psi(x)\,dx.$$

A nonhomogeneous version of the inequality (5) is stated at Theorem 3.6, where the condition $A^*(\cdot,D)v=0$ is avoided. The proof of this result and some applications for first order operators associated with the system of complex vector fields will be presented in Section 3.3. In Section 3.3.2 we will present in more detail the notations, the spaces of differential k-forms and how these estimates can be obtained from the first result of this thesis. Note that the estimate (5) is exactly (4) when $A(\cdot,D)=\nabla_{\mathscr{L}}$ that implies $m=1$ and $A^*(\cdot,D)=\operatorname{div}_{\mathscr{L}^*}$, recovering Theorem **A**.

We point out that the critical point $N/(N+m)$ in Theorem **A** approaches 0 as the order of the operator m gets higher. It is well known that in Hardy spaces some cancellation conditions are required when $p\searrow 0$ (see [15, 16] for more details), which means the necessity of new treatment in comparison to the case $m=1$. Moreover, it should be mentioned here that by allowing in (5) a much larger class of higher order linear differential operators, the mechanism of proof has to be substantially refined.

The heart of the proof of Theorem **A** is given by a Poincaré type inequality for the localizable Hardy-Sobolev spaces, namely: for each $x\in\mathbb{R}^N$, $0<t<1$ and $f\in C_c^\infty(\mathbb{R}^N)$ there exist a special polynomial $P_{x,t,f}$ with degree less than $m-1$ and C a positive (universal) constant such that

$$\left(\int_{\mathbb{R}^N}\left[\sup_{0<t<1}\fint_{B_x^t}\left\{\frac{1}{t^m}|f(y)-P_{x,t,f}(y)|\right\}^\alpha dy\right]^{p/\alpha}dx\right)^{1/p}\leq C\|D^m f\|_{h^p}, \tag{6}$$

where $\alpha\in[1,p^*)$ with $p^*=Np/(N-mp)$ and m,p as in the statement of Theorem **A**. Here $D^m f:=(\partial^\alpha f)_{|\alpha|=m}$ is the total derivative operator with order m. Lemma 3.1 is devoted to state the inequality

(6) and its proof follows from reduction of uniform control for bounded atoms and a special smooth atomic decomposition on localizable Hardy spaces.

To present the second part of this thesis, let us go back to the classical div-curl inequality (2) in the particular case when $r = 1$, precisely: if $V \in L^p(\mathbb{R}^N, \mathbb{R}^N)$ and $W \in L^{p'}(\mathbb{R}^N, \mathbb{R}^N)$ are vector fields satisfying $\operatorname{div} V = 0$ and $\operatorname{curl} W = 0$, in the sense of distributions, for some $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, then $V \cdot W$ belongs to the Hardy space $H^1(\mathbb{R}^N)$ and moreover there exists a constant $C > 0$ such that

$$\|V \cdot W\|_{H^1} \leq C \|V\|_{L^p} \|W\|_{L^{p'}}. \tag{7}$$

Note that $\operatorname{div} V = 0$ and $\operatorname{curl} W = 0$ imply that $\int V \cdot W = 0$, which is a necessary condition for $V \cdot W$ to belong to $H^1(\mathbb{R}^N)$. A natural question arises about a version of the inequality (7) when $\operatorname{div} V$ and $\operatorname{curl} W$ are not null. A first answer was presented in [13], namely: if V and W satisfy

$$V \in L^p_{loc}(\mathbb{R}^N, \mathbb{R}^N), \quad W \in L^{p'}_{loc}(\mathbb{R}^N, \mathbb{R}^N), \quad \operatorname{div} V \in W^{-1,r}_{loc} \quad \text{and} \quad \operatorname{curl} W \in W^{-1,s}_{loc},$$

for $r > p$ and $s > p'$, then $f := V \cdot W$ belongs to $\mathscr{H}^1_{loc}(\mathbb{R}^N)$, *i.e.*, $f \log f \in L^1_{loc}(\mathbb{R}^N)$. Afterwards, a nonhomogeneous estimate was presented by Dafni in [14, Theorem 5] and we state as follows:

Theorem 0.1. *Suppose V and W are vector fields on $\mathbb{R}^N$ satisfying*

$$V \in L^p(\mathbb{R}^N, \mathbb{R}^N),\ W \in L^{p'}(\mathbb{R}^N, \mathbb{R}^N),\ 1 < p < \infty \text{ and } \frac{1}{p} + \frac{1}{p'} = 1.$$

If there exists a function $f \in L^p(\mathbb{R}^N)$ and a matrix-valued function A with components in $L^{p'}(\mathbb{R}^N)$ such that, in the sense of distributions,

$$\operatorname{div} V = f, \quad \operatorname{curl} W = A,$$

then $V \cdot W$ belongs to the local Hardy space $h^1(\mathbb{R}^N)$, with

$$\|V \cdot W\|_{h^1} \leq C \left(\|V\|_{L^p} \|W\|_{L^{p'}} + \|f\|_{L^p} \|W\|_{L^{p'}} + \|V\|_{L^p} \|A\|_{L^{p'}} \right) \tag{8}$$

Note that the estimate (7) implies (8) in the case $\operatorname{div} V = 0$ and $\operatorname{curl} W = 0$, since $H^1(\mathbb{R}^N)$ is continuously embedded in $h^1(\mathbb{R}^N)$. The novelty here lies in considering the appropriate space $h^1(\mathbb{R}^N)$ instead of $H^1(\mathbb{R}^N)$, since in the nonhomogeneous setting it is not guaranteed that $\int_{\mathbb{R}^N} V \cdot W = 0$.

Consider again $\mathscr{L} := \{L_1, \dots, L_n\}$ be a system of linearly independent vector fields with complex coefficients defined on $\mathbb{R}^N$ and the operators $\nabla_{\mathscr{L}}$ and $\operatorname{div}_{\mathscr{L}^*}$ as before. Naturally, we may also define the curl operator associated with $\mathscr{L}$ given by the matrix

$$\operatorname{curl}_{\mathscr{L}} v := \left(L_i v_j - L_j v_i \right)_{ij}, \quad \text{for } v \in C^\infty(\mathbb{R}^N, \mathbb{C}^n).$$

Note that when $n = N$ and $L_j = \partial_{x_j}$ for $j = 1, \dots, n$, we get $\operatorname{curl}_{\mathscr{L}} = \operatorname{curl}$.

The next goal of this thesis will be to address the following question: for which systems of vector fields $\mathscr{L}$ the global estimate

$$\|V\cdot W\|_{h^1}\leq C\left(\|V\|_{L^p}\|W\|_{L^{p'}}+\|\operatorname{div}_{\mathscr{L}^*}V\|_{L^p}\|W\|_{L^{p'}}+\|V\|_{L^p}\|\operatorname{curl}_{\mathscr{L}}W\|_{L^{p'}}\right) \tag{9}$$

holds? The second result of this thesis is the following:

Theorem B. *Let $\{L_1,\ldots,L_n\}$ be an elliptic system of complex vector fields on $\mathbb{R}^N$ with constant complex coefficients with $n\geq 2$. If $V\in L^p(\mathbb{R}^N,\mathbb{C}^n)$ and $W\in L^{p'}(\mathbb{R}^N,\mathbb{C}^n)$ with $1<p<\infty$ satisfy*

$$\operatorname{div}_{\mathscr{L}^*}V\in L^p(\mathbb{R}^N)\ \textit{and}\ \operatorname{curl}_{\mathscr{L}}W\in L^{p'}(\mathbb{R}^N,\mathbb{C}^{n\times n}),$$

then $V\cdot W$ belongs to $h^1(\mathbb{R}^N)$. Moreover, there exists a constant $C>0$ such that (9) *holds.*

Observe that the local version of inequality (9) is a direct consequence of Theorem **A** in its nonhomogeneous version, that it will be stated at Theorem 3.6 in Section 3.3. In particular, note that the estimate (4), proved in [24, Theorem A], corresponds to the local version when $W=\nabla_{\mathscr{L}}\phi$ and $\operatorname{div}_{\mathscr{L}^*}v=0$, where $\mathscr{L}$ is an elliptic system of complex vector fields with smooth variable coefficients. We remark that in this setting the operator $\operatorname{curl}_{\mathscr{L}}W$ is not necessary null. In fact,

$$\operatorname{curl}_{\mathscr{L}}(\nabla_{\mathscr{L}}\phi)=\left([L_i,L_j]\phi\right)_{i,j},$$

where $[L_i,L_j]:=L_iL_j-L_jL_i$ is the commutator of the vector fields. Clearly, if the vector fields $\{L_1,\ldots,L_n\}$ have constant coefficients then $\operatorname{curl}_{\mathscr{L}}W=\operatorname{curl}_{\mathscr{L}}(\nabla_{\mathscr{L}}\phi)=0$ and then (4) recover (9) locally, assuming $\operatorname{div}_{\mathscr{L}^*}v=0$.

In the same spirit of [14, Theorem 5], the proof of Theorem B is simplified by reducing it to two specific cases of the inequality (9), namely Theorems 4.2 and 4.3 stated in Chapter 4 . The first case is a global nonhomogeneous version of the inequality (4), in which $W=\nabla_{\mathscr{L}}\phi$. The second simplification is a reduction of the inequality (9) for general $W\in L^{p'}(\mathbb{R}^N,\mathbb{C}^n)$ and $\operatorname{div}_{\mathscr{L}^*}V=0$. The conclusion of (9) will follow by a Hodge type decomposition $V=V_1+V_2$ given by Lemma 4.1 for each $V\in L^p(\mathbb{R}^N,\mathbb{C}^n)$, in which $\operatorname{div}_{\mathscr{L}^*}V_1=0$ and $V_2=\nabla_{\mathscr{L}}\phi$.

In the third part of this thesis, we are interested in applying the Theorem **B** in order to generate a new decomposition of the *bmo* space, since it is the dual space of the local Hardy space $h^1(\mathbb{R}^N)$. In [13], the authors proved a type of converse of inequality (7), called div-curl lemma, that asserts each $f\in H^1(\mathbb{R}^N)$ can be written as

$$f=\sum_{j=1}^{\infty}\lambda_k f_k$$

in the sense of distribution, where the sequence $\{\lambda_k\}_k\in\ell^1(\mathbb{R})$ and $f_k:=V_k\cdot W_k$ with $W_k,V_k\in L^2(\mathbb{R}^N,\mathbb{R}^N)$ satisfying $\operatorname{div}V_k=0$ and $\operatorname{curl}W_k=0$. This result is a direct consequence from the duality $BMO(\mathbb{R}^N)=(H^1(\mathbb{R}^N))^*$ and a characterization of the BMO norm given by

$$\|g\|_{BMO}\approx\sup_{V,W}\int_{\mathbb{R}^N}g(x)(V\cdot W)(x)dx,$$

where the supremum is taken over all vector fields $W, V \in L^2(\mathbb{R}^N, \mathbb{R}^N)$ satisfying $\operatorname{div} \mathrm{V} = 0$, $\operatorname{curl} \mathrm{W} = 0$ and $\|V\|_{L^2}, \|W\|_{L^2} \leq 1$.

Now, let $\mathscr{L} = \{L_1, \dots, L_n\}$ as in the statement of Theorem **B** and denote by $(\mathscr{D}C_{\mathscr{L}})^p_{0,1}$ the family of all functions which can be written in the form $V \cdot W$, where $V \in L^p(\mathbb{R}^N, \mathbb{C}^n)$ and $W \in L^{p'}(\mathbb{R}^N, \mathbb{C}^n)$ are vector fields satisfying $\|V\|_{L^p}, \|W\|_{L^{p'}} \leq 1$ with $\operatorname{div}_{\mathscr{L}^*} V = 0$ and $\|\operatorname{curl}_{\mathscr{L}} W\|_{L^{p'}} \leq 1$. Analogously, we define $(\mathscr{D}C_{\mathscr{L}})^p_{1,0}$ the family of all functions $V \cdot W$, where $V \in L^p(\mathbb{R}^N, \mathbb{C}^n)$ and $W \in L^{p'}(\mathbb{R}^N, \mathbb{C}^n)$ are vector fields satisfying $\|V\|_{L^p}, \|W\|_{L^{p'}} \leq 1$ with $\|\operatorname{div}_{\mathscr{L}^*} V\|_{L^p} \leq 1$ and $W := \nabla_{\mathscr{L}} \phi$. The third result of this thesis is the following:

Theorem C. *Let $\mathscr{L} = \{L_1, ..., L_n\}$ be an elliptic system of complex vector fields on $\mathbb{R}^N$ with complex constant coefficients with $n \geq 2$. If $g \in bmo(\mathbb{R}^N)$, then*

$$\|g\|_{bmo} \simeq \sup_{f \in (\mathscr{D}C_{\mathscr{L}})^p_{1,0}} \left| \int_{\mathbb{R}^N} g(x) f(x) dx \right| \simeq \sup_{f \in (\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left| \int_{\mathbb{R}^N} g(x) f(x) dx \right|,$$

for any $1 < p < \infty$.

We recall the dual of $h^1(\mathbb{R}^N)$ can be identified with the space $bmo(\mathbb{R}^N)$ given by the set of locally integrable functions f that satisfy

$$\|g\|_{bmo} := \sup_{|B| \leq 1} ⨍_B |g(x) - g_B| dx + \sup_{|B| > 1} ⨍_B |g(x)| dx < \infty, \tag{10}$$

where $g_B := \dfrac{1}{|B|} \displaystyle\int_B g(x) dx$.

Finally, as a direct consequence of the previous characterization and duality, we announce the following div-curl lemma associated with an elliptic system of complex vector fields.

Corollary D. *Let $\mathscr{L} = \{L_1, ..., L_n\}$ be an elliptic system of complex vector fields on $\mathbb{R}^N$ with complex constant coefficients with $n \geq 2$ and $1 < p < \infty$. For each $f \in h^1(\mathbb{R}^N)$ there exist a sequence $\{\lambda_k\}_k \in \ell^1(\mathbb{C})$ and a sequence $\{f_k\}_k \in (\mathscr{D}C_{\mathscr{L}})^p_{1,0}$ such that*

$$f = \sum_{k=1}^{\infty} \lambda_k f_k, \tag{11}$$

in the sense of distributions. The same decomposition holds replacing $(\mathscr{D}C_{\mathscr{L}})^p_{1,0}$ by $(\mathscr{D}C_{\mathscr{L}})^p_{0,1}$.

The organization of the thesis is as follows. In Chapter 1 we recall some basic definitions on Hardy spaces, in addition to presenting some definitions and results about elliptic linear differential operators, pseudo-differential operators theory and Sobolev type inequalities on Lebesgue and localizable Hardy spaces. The Sections 1.6 and 2.2 are devoted to a generalization of the Poincaré inequality following the works due to Miyachi in [37, 38]. In Chapter 2 we introduce the Hardy-Sobolev spaces, besides some elliptic estimates in Hardy norms. We also present a new smooth atomic decomposition on Hardy-Sobolev spaces that it will be useful to prove the inequality (6). The proof of first main result of this thesis, namely the Theorem **A**, is presented at the Chapter 3. Still in the same chapter, the Section 3.3 is destined for a nonhomogeneous version of Theorem **A** and two applications: local estimates for first order differential operators associated with a system of complex vector fields and for pseudo-complex operators also associated with a system of complex vector fields. From Chapter 4 onwards, the content covers the second part of the thesis. Initially we present a Hodge decomposition associated with system of complex vector fields, followed by the proof of Theorem **B** and expected future results. In the last chapter, we prove the Theorem **C** and consequently we present the proof of Corollary **D**.

CHAPTER 1

Preliminaries

1.1 Notations

Consider $\mathbb{R}^N = \{(x_1, x_2, \ldots, x_N) : x_j \in \mathbb{R}, j = 1, 2, \ldots N\}$ with the usual norm $|x| := \left(\sum_{j=1}^N x_j^2\right)^{1/2}$. Throughout the work, we will use the notation $\Omega \subseteq \mathbb{R}^N$ for an open set and B_x^t for an open ball $B(x,t)$ centered at x and radius $t > 0$. For a generic ball we will use B, with $|B|$ being its Lebesgue measure. In a similar way, $Q(x,r)$ denotes a cube centered at x and side $\ell(Q) = r$. For each $x \in \mathbb{R}^N$, we denote $<\!x\!> := \left(1 + |x|^2\right)^{1/2}$. If $s \in \mathbb{R}$, we denote by $\lfloor s \rfloor$ the largest integer less than s. The complex inner product is denoted by $z \cdot w = \sum_{j=1}^N z_j \overline{w_j}$, with $z, w \in \mathbb{C}^N$, and the inner product on functions spaces is given by $\langle f, g \rangle = \int f \overline{g}$.

By using the multi-index notation, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_N}^{\alpha_N}$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$, where $x \in \mathbb{R}^N$ and $\alpha \in \mathbb{Z}_+^N$, with the norm $|\alpha| := \sum_{j=1}^N \alpha_j$. Furthermore, we denote $D^\alpha = (-i)^{|\alpha|} \partial^\alpha$. Let $\alpha, \beta \in \mathbb{Z}_+^N$, we say $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for all $j = 1, 2, \ldots, N$. In this case, we can use the following multi-index operations:

$$\alpha - \beta := (\alpha_1 - \beta_1, \ldots, \alpha_N - \beta_N) \text{ and } \binom{\alpha}{\beta} := \binom{\alpha_N}{\beta_N} \cdots \binom{\alpha_N}{\beta_N}.$$

Still with respect to multi-index notation, we point out the Leibniz formula, widely used throughout this work to apply the product rule of higher-order derivatives:

$$\partial^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha - \beta} f \, \partial^\beta g.$$

With respect to some function spaces and distribution spaces, we set $C_c^\infty(\mathbb{R}^N)$ the space of all smooth functions with compact support, also called *test functions space*, and $\mathscr{D}'(\mathbb{R}^N)$ the distributions space, *i.e.*, the set of all continuous linear functionals $u : C_c^\infty(\mathbb{R}^N) \mapsto \mathbb{C}$. We set $S(\mathbb{R}^N)$ the Schwartz space and $S'(\mathbb{R}^N)$ the tempered distributions set, while $\mathscr{E}'(\mathbb{R}^N)$ denotes the set of distributions with compact support. For each $1 \leq p < \infty$, a measurable functions f on Ω belongs to $L^p(\Omega)$ if $\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p\right)^{\frac{1}{p}} < \infty$ and f belongs to $L_{loc}^p(\Omega)$ if $\|f\|_{L^p(K)} < \infty$ for all $K \subset \Omega$ compact. Moreover, f belongs to $L^\infty(\Omega)$ if

$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf\{c \geq 0 : |\{x \in \Omega : |f(x)| > c\}| = 0\ \} < \infty.$$

When $\Omega = \mathbb{R}^N$ the domain will often be omitted from the notation. For each $0 < p < \infty$, we denote by p' its conjugate exponent, that is, $1/p + 1/p' = 1$. In particular, if $p = 1$ we consider $p' = \infty$. A basic tool we shall need is the Hölder inequality, which says that if $f \in L^p$ and $g \in L^{p'}$, for $1 \leq p \leq \infty$, then $f.g \in L^1$ and

$$\|f.g\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

In order to simplify notation, if A is a matrix-valued function with components A_{ij} in L^p, we denote $\|A\|_{L^p} := \sum_{i,j} \|A_{ij}\|_{L^p}$.

We set $\mathscr{P}_k$ the space of polynomials of degree at most k from $\mathbb{R}^N$ to complex values and $W^{k,p}(\mathbb{R}^N)$ denotes the classical (nonhomogeneous) Sobolev space, with $k \in \mathbb{N}$, $p \geq 1$ and $\|f\|_{W^{k,p}} = \sum_{|\alpha|\leq k} \|\partial^\alpha f\,\|_{L^p}$. The notation $h^p_c(B)$ and $h^{k,p}_c(B)$ will stand for $h^p(\mathbb{R}^N) \cap \mathscr{E}'(B)$ and $h^{k,p}(\mathbb{R}^N) \cap \mathscr{E}'(B)$, respectively, and the definition of the spaces $h^p(\mathbb{R}^N)$ and $h^{k,p}(\mathbb{R}^N)$ will be presented on Sections 1.2 and 2.1. The closure of $C^\infty_c(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by $W^{k,p}_0(\Omega)$ and its dual space, *i.e.* the space of distributions in which all derivatives with order less or equal than k belongs $L^{p'}$, is denoted by $W^{-k,p'}(\Omega)$.

Let f, g measurable functions on $\mathbb{R}^N$, we denote by $f * g$ its convolution, given by

$$f * g(x) := \int_{\mathbb{R}^N} f(x-y)g(y)\, dy,$$

when the integral is well-defined. In particular, if $f \in L^p_{loc}$ for some $p > 0$ and g has compact support the operator is well-defined. The definition of convolution can also be extended to distributions, precisely: if $u \in \mathscr{D}'$ and $\varphi \in C^\infty_c$ or if $u \in \mathscr{E}'$ and $\varphi \in C^\infty$, then

$$u * \varphi(x) := \langle u, \varphi^x_- \rangle,$$

where $\varphi^x_-(y) = \varphi(x-y)$. We will also frequently use the notation $\varphi_t(x) = t^{-N}\varphi(x/t)$.

The Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx$$

for $f \in L^1(\mathbb{R}^N)$ and, in particular when $f \in S(\mathbb{R}^N)$ its inverse is well-defined and is given by $\mathscr{F}^{-1}u(x) = (2\pi)^{-N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}u(\xi)d\xi$.

Let E a subset of $\mathbb{R}^N$, we define $\mathbb{X}_E$ the characteristic function by

$$\mathbb{X}_E := \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{otherwise.} \end{cases}$$

If f is a measurable function, we say that the mean of f on a measurable set E with $0 < |E| < \infty$, f_E, is $⨍_E f := \frac{1}{|E|}\int_E f$. In particular when $E = B(x,t)$ we can also denote f_E as f^t_x.

A basic tool we shall need is the Hardy-Littlewood maximal operator M. For a function $f \in L^1_{loc}(\mathbb{R}^N)$ we define

$$Mf(x) := \sup_{r>0} \fint_{B(x,r)} |f(y)|dy, \quad \text{a.e. } x \in \mathbb{R}^N. \tag{1}$$

It is well known that $M : f \mapsto Mf$ is a bounded operator in L^p, for $1 < p \leq \infty$, and when $f \in L^\infty$ we have the trivial estimate $Mf(x) \leq \|f\|_\infty$, a.e. $x \in \mathbb{R}^N$.

In an analogous way, for each $q > 0$ we define the maximal operator M_q on Ω by

$$M_q g(x) := \sup_{t>0} \left(\frac{1}{t^N} \int_{B(x,t)\cap\Omega} |g(y)|^q dy \right)^{\frac{1}{q}}, \quad \text{a.e. } x \in \Omega, \tag{2}$$

which is well-defined for all $g \in L^q_{loc}(\Omega)$ and is bounded in $L^p(\Omega)$, for all $q < p \leq \infty$.

We denote by $\mathscr{M}$ the Grand Maximal Function associated to a collection $\mathfrak{S} \subset S(\mathbb{R}^N)$ of rapidly decreasing functions, that is

$$\mathscr{M}f(x) = \sup_{\phi\in\mathfrak{S}} M_\phi f(x) := \sup_{\phi\in\mathfrak{S}} \left[\sup_{t>0} |(u * \phi_t)(x)| \right],$$

where $\mathfrak{S} = \{\phi \in S(\mathbb{R}^N) : \left\| x^\alpha \partial^\beta \phi \right\|_{L^\infty} \leq 1$, for all $|\alpha|, |\beta| \leq C\}$. It is well known that for each fixed $N/(N-1) < p \leq 1$ there is a appropriated constant $C = C_{N,p}$ (bigger than N/p) such that $\|f\|_{h^p} \simeq \|\mathscr{M}f\|_{L^p}$ (see [22, Theorem 1] for the details).

Let $f : \Omega \mapsto \mathbb{C}$ and $g : \Omega \mapsto \mathbb{C}^N$ differentiable functions, we denote the gradient function of f by $\nabla f(x) = (\partial_{x_1} f(x), \ldots, \partial_{x_N} f(x))$, and we denote the divergence and curl functions of g by, respectively,

$$\operatorname{div} g(x) = \sum_{j=1}^{N} \partial_{x_j} g_j(x) \quad \text{and} \quad \operatorname{curl} g(x) = \left[(\partial_{x_i} g_j(x) - \partial_{x_j} g_i(x))_{ij} \right]_{N\times N},$$

for $x \in \Omega$. The Laplace operator is given by $\Delta f(x) = \sum_{j=1}^N \partial^2_{x_j} f(x)$.

A fundamental solution of Δ, *i. e.*, $\Delta G = \delta_0$, is the function $G \in L^1_{loc}(\mathbb{R}^N)$ given by

$$G(x) = \begin{cases} \dfrac{\ln|x|}{2\pi} & \text{if } N = 2; \\[2ex] \dfrac{|x|^{-N+2}}{(2-N)|\mathbb{S}^{N-1}|} & \text{if } N \geq 3. \end{cases} \tag{3}$$

Here $|\mathbb{S}^{N-1}|$ denotes the surface measure (of the unit sphere in $\mathbb{R}^N$) and δ_0 denotes the Dirac delta distribution, given by $\langle \delta_0, \phi \rangle := \phi(0)$, for all $\phi \in C_c^\infty(\mathbb{R}^N)$. For each fixed $1 < p < \infty$ it is well known that $G * f$ is well defined for all $f \in L^p(\mathbb{R}^N)$ and also belongs to $L^p(\mathbb{R}^N)$. Furthermore, D^2G is a bounded operator in L^p, in the sense of $\left\| \partial_{x_i} \partial_{x_j} G * f \right\|_{L^p} \lesssim \|f\|_{L^p}$, for all $f \in L^p(\mathbb{R}^N)$ and for all $i, j \in \{1, \ldots, N\}$.

We will denote by $V = (V_1, \ldots, V_N)$ a vector field on $\Omega \subset \mathbb{R}^N$ with locally integrable components, that is, $V_j \in L^1_{loc}(\Omega)$, for all $j = 1, \ldots, N$. Its divergence is defined, in the sense of distributions on Ω, by

$$\langle \operatorname{div} V, \phi \rangle := -\int_\Omega V \cdot \nabla\phi,$$

and its curl is defined as the matrix distribution whose components are given by

$$\langle (\operatorname{curl} V)_{ij}, \phi \rangle := \int_{\Omega} V_j \partial_{x_i} \phi - V_i \partial_{x_j} \phi,$$

for $\phi \in C_c^\infty(\Omega)$ and $i, j \in \{1, 2, \ldots, N\}$.

Regarding vector function and distributions, let E be a complex vector space with finite dimension n_E. We say that $f = (f_1, ..., f_{n_E}) : \mathbb{R}^N \mapsto E$ belongs to vector spaces $C_c^\infty(\mathbb{R}^N; E)$, $S(\mathbb{R}^N; E)$ and $L^p(\mathbb{R}^N; E)$ if each coordinate function f_i belongs to $C_c^\infty(\mathbb{R}^N)$, $S(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$, respectively. In a similar way we denote the E-valued distributions spaces $\mathscr{D}'(\mathbb{R}^N; E), S'(\mathbb{R}^N; E)$ and $\mathscr{E}'(\mathbb{R}^N; E)$. We set $\mathscr{P}_k^E$ the space of polynomials $P : \mathbb{R}^N \mapsto E$ of degree at most k from $\mathbb{R}^N$ to values of the complex vector space E, in particular when $E = \mathbb{C}$ we return to the notation $\mathscr{P}_k$.

1.2 Hardy spaces and localizable Hardy spaces

In this section, we will define Hardy spaces $H^p(\mathbb{R}^N)$ through maximal functions and present some of their main properties. Our reference at this subject will be [44, Chapter 3]. Concurrently, we will present the localizable Hardy spaces $h^p(\mathbb{R}^N)$, introduced by Goldberg in [22].

Fix, once for all, a radial non-negative function $\varphi \in C_c^\infty(\mathbb{R}^N)$ supported in the unit ball $B(0,1)$ with $\int \varphi = 1$, and denote $\varphi_t(x) = t^{-N} \varphi(x/t)$. For a tempered distribution $u \in S'(\mathbb{R}^N)$, we will start defining the *maximal function* $M_\varphi u$ and the *small maximal function* $m_\varphi u$ given by, respectively,

$$M_\varphi u(x) := \sup_{t>0} |(u * \varphi_t)(x)| \quad \text{and} \quad m_\varphi u(x) := \sup_{0<t<1} |(u * \varphi_t)(x)|.$$

Definition 1.1 (Hardy spaces and localizable Hardy spaces)**.** *Let $0 < p < \infty$. We say that a tempered distribution $u \in S'(\mathbb{R}^N)$ belongs to the Hardy space $H^p(\mathbb{R}^N)$ when $M_\varphi u \in L^p(\mathbb{R}^N)$, i.e. $\|u\|_{H^p} := \|M_\varphi u\|_{L^p} < \infty$. In the particular case $p = \infty$, we set $H^\infty(\mathbb{R}^N) := L^\infty(\mathbb{R}^N)$. Analogously, we say that a tempered distribution $u \in S'(\mathbb{R}^N)$ belongs to the localizable Hardy space $h^p(\mathbb{R}^N)$ when $m_\varphi u \in L^p(\mathbb{R}^N)$, i.e. $\|u\|_{h^p} := \|m_\varphi u\|_{L^p} < \infty$. In the particular case $p = \infty$, we set $h^\infty(\mathbb{R}^N) := L^\infty(\mathbb{R}^N)$.*

In a more general way, the spaces $H^p(\mathbb{R}^N)$ and $h^p(\mathbb{R}^N)$ can be defined using any $\varphi \in S(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \varphi \neq 0$ and the spaces are independent of the choice of φ. When $0 < p \leq 1$, the spaces $H^p(\mathbb{R}^N)$ and $h^p(\mathbb{R}^N)$ are complete metric spaces with the distances $d(u_1, u_2) = \|u_1 - u_2\|_{H^p}^p$ and $d'(v_1, v_2) = \|v_1 - v_2\|_{h^p}^p$, for $u_1, u_2 \in H^p(\mathbb{R}^N)$ and $v_1, v_2 \in h^p(\mathbb{R}^N)$ and when $1 < p \leq \infty$, the functionals $\|\cdot\|_{H^p}$ and $\|\cdot\|_{h^p}$ are norms and these spaces are complete normed spaces. Furthermore, $H^p(\mathbb{R}^N)$ is continuously embedded in $h^p(\mathbb{R}^N)$, for all $0 < p \leq \infty$ and $H^p(\mathbb{R}^N) = h^p(\mathbb{R}^N) = L^p(\mathbb{R}^N)$, for all $1 < p \leq \infty$, with comparable norms. In particular, when $p = 1$, the space $h^1(\mathbb{R}^N)$ is densely contained in $L^1(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N) \subset h^1(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$ strictly. Although $h^p(\mathbb{R}^N)$ is not locally convex for $0 < p < 1$ and $\|\cdot\|_{h^p}$ is truly a quasi-norm (see [49]), we will still refer to $\|\cdot\|_{h^p}$ as a norm by simplicity. We point out that $\|\cdot\|_{h^p}$ is translation invariant.

In contrast to Hardy space, the localizable version is closed under multiplication by test functions. Precisely, if $\phi \in C_c^\infty(\mathbb{R}^N)$, $f \in H^p(\mathbb{R}^N)$ and $g \in h^p(\mathbb{R}^N)$ then in general $\phi f \notin H^p(\mathbb{R}^N)$ but $\phi g \in h^p(\mathbb{R}^N)$.

Next, we will describe the dual spaces of $H^p(\mathbb{R}^N)$ and $h^p(\mathbb{R}^N)$, more details can be found in [20] and [22].

Definition 1.2 (Hölder spaces)**.** *Let* $0 < r < 1$*. A continuous function* f *belongs to the homogeneous Hölder (or Lipschitz) space* $\dot{\Lambda}^r(\mathbb{R}^N)$ *if there exists* $C > 0$ *such that*

$$|f(x+h) - f(x)| \leq C|h|^r, \quad \textit{for every } x, h \in \mathbb{R}^N.$$

For $r = 1$*,* $f \in \dot{\Lambda}^1(\mathbb{R}^N)$ *(also called Zygmund space) if there exists* $C > 0$ *such that for every* $x, h \in \mathbb{R}^N$

$$|f(x+h) + f(x-h) - 2f(x)| \leq C|h|.$$

If $r = k + s$ *for* $k \in \mathbb{N}$ *and* $0 < s \leq 1$*, we say that* $f \in \dot{\Lambda}^r(\mathbb{R}^N)$ *if all derivatives* $\partial^\alpha f \in \dot{\Lambda}^s(\mathbb{R}^N)$ *for* $|\alpha| = k$*.*

The homogeneous Hölder space $\tilde{\dot{\Lambda}}^r(\mathbb{R}^N)$ is a locally convex topological vector space with the semi-norms

$$[f]_{k+s} := \sum_{|\alpha|=k} \sup_{\substack{x,h \in \mathbb{R}^N \\ h \neq 0}} \frac{|\partial^\alpha f(x+h) - \partial^\alpha f(x)|}{|h|^s}, \qquad 0 < s < 1,$$

or

$$[f]_{k+s} := \sum_{|\alpha|=k} \sup_{\substack{x,h \in \mathbb{R}^N \\ h \neq 0}} \frac{|\partial^\alpha f(x+h) + \partial^\alpha f(x-h) - 2\partial^\alpha f(x)|}{|h|}, \qquad s = 1,$$

modulo the subspace of the functions satisfying $[f]_r = 0$, which are all polynomials of degree less than or equal to k.

When $0 < p < 1$, the dual space of $H^p(\mathbb{R}^N)$ may be identified with the normed homogeneous Hölder space $\tilde{\dot{\Lambda}}^r(\mathbb{R}^N) := \dot{\Lambda}^r(\mathbb{R}^N)/\mathscr{P}_k$, with $r = \gamma_p := N\left(\frac{1}{p} - 1\right)$ (see [20, Chapter III, Section 5] for more details). When $0 < p < 1$, the dual space of $h^p(\mathbb{R}^N)$ may be identified with the nonhomogeneous Hölder space $\Lambda^r(\mathbb{R}^N) := \tilde{\dot{\Lambda}}^r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, with $r = \gamma_p$, equipped with the norm $\|f\|_r = [f]_r + \|f\|_{L^\infty}$, (see [22, Theorem 5] for more details).

In the particular case when $p = 1$, the dual of $H^1(\mathbb{R}^N)$ can be identified with the $BMO(\mathbb{R}^N)$ space, that is, the space of functions of bounded mean oscillation, defined as the space of locally integrable functions f which satisfy

$$\|f\|_{BMO} := \sup_{x \in \mathbb{R}^N} \left[\sup_{Q \ni x} \fint_Q |f - f_Q| \right] < \infty,$$

where the inner supremum is taken over all cubes with sides parallel to the axes that contain x. The dual of $h^1(\mathbb{R}^N)$ can be identified with the space $bmo(\mathbb{R}^N)$, or local $BMO(\mathbb{R}^N)$, defined as the space of locally integrable functions f which satisfy

$$\|f\|_{bmo} := \sup_{|Q|<1} \frac{1}{|Q|}\int_Q |f - f_Q| + \sup_{|Q|\geq 1} \frac{1}{|Q|}\int_Q |f| < \infty.$$

The proof of the duality of the previous spaces uses an important characterization of Hardy spaces, their decomposition into well-behaved functions, called atoms, which will be the topic of the next subsection.

1.2.1 Atomic decomposition

Since we have the equivalence of the spaces $H^p(\mathbb{R}^N)$, $h^p(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ for $p > 1$, in this subsection we will always consider $0 < p \leq 1$.

Definition 1.3. *An H^p-atom is a measurable function $a(x)$ supported in some ball $B = B(x_0, r)$ satisfying the following properties:*

(i) $\|a\|_{L^\infty} \leq |B|^{-1/p}$ *;*

(ii) $\int x^\beta a(x)\,dx = 0$*, for all* $\beta \in \mathbb{Z}_+^N$ *satisfying* $|\beta| \leq N_p := \lfloor N(p^{-1} - 1)\rfloor$.

The propriety (i) concerns a size condition, while the propriety (ii) is a moment condition.

Definition 1.4. *A bounded $h^p(\mathbb{R}^N)$-atom, in the sense of Goldberg (see [22], [44]), is a measurable function $a(x)$ supported at some ball $B = B(x_0, r)$ satisfying the following properties:*

(i) $\|a\|_{L^\infty} \leq |B|^{-1/p}$*;*

(ii) if $r < 1$ *then* $\int x^\beta a(x)\,dx = 0$*, for all* $\beta \in \mathbb{Z}_+^N$ *satisfying* $|\beta| \leq N_p := \lfloor N(p^{-1} - 1)\rfloor$.

This type of bounded atom is usually called standard h^p-atom or simply h^p-atom. In contrast to $H^p(\mathbb{R}^N)$-atoms, the moment conditions are only required for atoms supported on balls with radius $r < 1$. Atoms supported in balls with $r > 1$ without any moment condition as (ii) will be called rough atoms and atoms with radius $r < 1$ will be called standard atoms (see [1] for the terminology). For a deep discussion on moment conditions of atoms in $h^p(\mathbb{R}^N)$ we refer [15] and [16]. Note that an atom a (both H^p-atom and h^p-atom) belongs to $L^q(\mathbb{R}^N)$, for all $q > 0$, since $\|a\|_{L^q} \leq \|a\|_{L^\infty} |B|^{1/q} \leq |B|^{1/q-1/p}$ and, in particular, $\|a\|_{L^p} \leq 1$. As expected, an H^p-atom belongs to $H^p(\mathbb{R}^N)$ space. Moreover, H^p-atoms are uniformly bounded, *i.e.* there is a positive constant $C = C(N,p)$ such that $\|a\|_{H^p} \leq C$, for all a H^p-atom. The same happens with standard h^p-atoms in localizable Hardy space $h^p(\mathbb{R}^N)$.

The next result is known as standard atomic decomposition theorem and its proof can be verified in [44, p. 107].

Theorem 1.5. *Let $0 < p \leq 1$, then every $f \in H^p(\mathbb{R}^N)$ can be written as a sum of H^p-atoms $\{a_k\}_k$, i.e., there is a sequence of complex numbers $\{\lambda_k\}_k \in \ell^p$ such that*

$$f = \sum_k \lambda_k a_k$$

in H^p norm. Moreover, there is a positive constant $c = c(N,p)$ such that $\|\{\lambda_k\}_k\|_{\ell^p} = \left(\sum_k |\lambda_k|^p\right)^{1/p} \leq c\,\|f\|_{H^p}$.

Note that the conclusion does not guarantee the uniqueness of the atomic decomposition of f. In fact, it is true that

$$\|f\|_{H^p}^p \simeq \inf \sum_k |\lambda_k|^p,$$

in which the infimum is taken over all possible atomic decompositions.

In the same way, we have an atomic decomposition result for localizable Hardy spaces $h^p(\mathbb{R}^N)$ (see [22, Lemma 5]) which means: any $f \in h^p$ can be written as an infinite linear combination of h^p-atoms, more precisely, there exist scalars λ_k and h^p-atoms a_k such that $\sum_k |\lambda_k|^p < \infty$ and the series $\sum_k \lambda_k a_k$ converges to f both in $h^p(\mathbb{R}^N)$ and consequently in $S'(\mathbb{R}^N)$. Furthermore, $\|f\|_{h^p}^p \simeq \inf \sum_k |\lambda_k|^p$. Another useful fact is that the previous atoms may be assumed to be smooth functions, in particular the inclusions $C_c^\infty(\mathbb{R}^N) \subset S(\mathbb{R}^N) \subset h^p(\mathbb{R}^N)$ are dense (see the proofs in [27, p. 79] and [22, p. 35], respectively).

The referred atomic decomposition of $h^p(\mathbb{R}^N)$ is very similar to the atomic decomposition of $H^p(\mathbb{R}^N)$ in terms of H^p-atoms, the change being that the notion of h^p-atom is less restrictive than that of H^p-atom, since an H^p-atom $a(x)$ must satisfy (i) and a stronger form of (ii): condition moments are required, regardless of the size of the radius. The next result is a technical tool due to Goldberg in [22] useful for establishing a connection between the $H^p(\mathbb{R}^N)$ and $h^p(\mathbb{R}^N)$ spaces.

Lemma 1.6 (Lemma 4,[22])**.** *Let $0 < p \leq 1$ and $\phi \in S(\mathbb{R}^N)$ satisfying $\int \phi = 1$ and $\int x^\alpha \phi(x)dx = 0$ for all $\alpha \neq 0$, then $\|f - \phi * f\|_{H^p} \leq C\,\|f\|_{h^p}$. So, in particular if $f \in h^p(\mathbb{R}^N)$ then $f - \phi * f$ belongs to $H^p(\mathbb{R}^N)$.*

As a consequence of the lemma, we can present such a connection as follows: let $f \in h^p(\mathbb{R}^N)$, for $0 < p \leq 1$, then there exist $u \in H^p(\mathbb{R}^N)$ and $v \in (C^\infty \cap h^p)(\mathbb{R}^N)$ satisfying $f = u + v$ and there is a constant $C > 0$ such that

$$\|u\|_{H^p} + \|v\|_{h^p} \leq C\,\|f\|_{h^p}. \tag{1.2.1}$$

It is a direct consequence of previous lemma taking $v = \phi * f$ and $u = f - v$.

Remark 1.7. *We say that the linear operator $T : C_c^\infty(\mathbb{R}^N) \mapsto \mathscr{D}'(\mathbb{R}^N)$ is weakly continuous if $\langle \phi_j, \psi \rangle \longrightarrow \langle \phi, \psi \rangle$, for all $\psi \in C_c^\infty(\mathbb{R}^N)$, implies that $\langle T\phi_j, \psi \rangle \longrightarrow \langle T\phi, \psi \rangle$, for all $\psi \in C_c^\infty(\mathbb{R}^N)$.*

Finally, we observe that the decomposition of localizable Hardy spaces into atoms is not sufficient to guarantee the extension of bounded linear operators on atoms into bounded operators on the whole space (see [10, Theorem 2] for a famous counterexample). However, we will present a result that gives us a condition of when such an extension is true.

Theorem 1.8. *Let* $T : C_c^\infty(\mathbb{R}^N) \mapsto \mathscr{D}'(\mathbb{R}^N)$ *be a weakly continuous linear operator. Given* $0 < p_1 \leq 1$ *and* $p_1 \leq p_2 \leq \infty$*, if there exists a positive constant* $C > 0$ *such that* $\|Ta\|_{h^{p_2}} \leq C$*, for all smooth* h^{p_1}*-atom* a*, then* T *can be extended as a bounded operator* $T : h^{p_1}(\mathbb{R}^N) \mapsto h^{p_2}(\mathbb{R}^N)$.

The most general proof can be found at [10, p. 3540] and in this case precisely [24, p. 779].

1.3 Elliptic linear differential operators

Consider $A(\cdot, D)$ a linear differential operator of order $m \geq 1$ with smooth complex coefficients given by

$$A(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha : C^\infty(\Omega, E) \mapsto C^\infty(\Omega, F),$$

where $a_\alpha(x) \in \mathscr{L}(E,F)$ is a linear map for each $x \in \Omega$ and E and F are complex vector spaces of finite dimensions (n_E and n_F respectively). We will denote by

$$A_\nu(x,D) = \sum_{|\alpha|=m} a_\alpha(x)\partial^\alpha$$

the principal part of $A(\cdot,D)$ and when $A(\cdot,D) = A_\nu(\cdot,D)$ we say that A is a homogeneous operator. The adjoint operator of A is denoted by

$$A^*(x,D) = \sum_{|\alpha|\leq m} a_\alpha^*(x)\partial^\alpha : C^\infty(\Omega, F^*) \mapsto C^\infty(\Omega, E^*),$$

with $a_\alpha^*(x) \in \mathscr{L}(F^*, E^*)$ for each $x \in \Omega$. Since the complex vector spaces E and F have finite dimension, to simplify the notation we write $E^* = E$ and $F^* = F$ as convenient. Since the complex vector spaces have finite dimension, by simplicity. we may consider $E \cong \mathbb{C}^{n_E}$ and $F \cong \mathbb{C}^{n_F}$, and naturally the inner products may be regarded as

$$u \cdot_E v = \sum_{j=1}^{n_E} u_j\overline{v_j} \quad \text{and} \quad w \cdot_F z = \sum_{j=1}^{n_F} w_j\overline{z_j},$$

for all $u = (u_1, ..., u_{n_E}), v = (v_1, ..., v_{n_E}) \in E$ and $w = (w_1, ..., w_{n_F}), z = (z_1, ..., z_{n_F}) \in F$, as well as the complex conjugation.

We also use the notation $A^* = \overline{A^t}$, where $\overline{A}$ denotes the operator obtained from A by conjugating its coefficients and A^t is its formal transpose. It means that

$$\int_\Omega A(x,D)f(x) \cdot_F g(x)\, dx = \int_\Omega f(x) \cdot_E A^*(x,D)g(x)\, dx, \tag{1.3.1}$$

for all $f \in C_c^\infty(\Omega, E)$ and $g \in C_c^\infty(\Omega, F)$.

With respect to estimates for the product of functions, note that in the case where $A(\cdot,D)$ is a homogeneous linear differential operator of order $m \geq 1$ with smooth complex coefficients, this holds by the Leibniz rule:

$$A^*(x,D) = (-1)^m \sum_{|\alpha|=m} \left[a^*_\alpha(x)\partial^\alpha + \sum_{\beta<\alpha} c_\beta(x)\partial^\beta \right],$$

where $a^*_\alpha := \overline{a^t_\alpha}$ and $c_\beta \in C^\infty(\Omega, \mathscr{L}(F,E))$ are given by $c_\beta(x) \doteq \binom{\alpha}{\beta} \partial^{\alpha-\beta} a^*_\alpha(x)$. Moreover, let $g \in C^\infty_c(\Omega,\mathbb{C})$ and $h \in C^\infty(\Omega,F)$, so by the Leibniz rule and uniform boundedness in compact sets of a^*_α, g and all theirs partial derivatives, there exists a positive constant $C = C(A,g,m,n_E,n_F)$ such that

$$|A^*(x,D)(gh)(x)| \leq C \left(|A^*(x,D)h(x)| + \sum_{0\leq|\beta|<m} |\partial^\beta h(x)| \right),$$

for all $x \in \operatorname{supp}(g)$.

Definition 1.9 (Elliptic operator)**.** *We say that $A(\cdot,D)$ is an elliptic linear differential operator in $x_0 \in \Omega$ if its principal symbol*

$$a(x_0,\xi) := \sum_{|\alpha|=m} a_\alpha(x_0)\xi^\alpha$$

is an injective matrix-valued function, that is, $a(x_0,\xi) \neq \mathbf{0}_{n_F\times n_E}$, for all $\xi \in \mathbb{R}^N \neq 0$. We say that $A(\cdot,D)$ is an elliptic operator in Ω if is elliptic for all $x_0 \in \Omega$.

Below we present some examples of elliptic linear differential operators.

Example 1.10. *The gradient operator $\nabla : C^\infty(\mathbb{R}^N,\mathbb{C}) \mapsto C^\infty(\mathbb{R}^N,\mathbb{C}^N)$ given by $\nabla f(x) = (\partial_{x_1} f(x),\ldots, \partial_{x_N} f(x))$ is an elliptic linear differential operator on $\mathbb{R}^N$, since for any $x_0 \in \mathbb{R}^N$,*

$$a(x_0,\xi) = \xi \neq 0, \textit{ for all } \xi \in \mathbb{R}^N \neq 0.$$

Example 1.11. *The Cauchy-Riemann operator $A(D) : C^\infty(\mathbb{R}^2) \mapsto C^\infty(\mathbb{R}^2)$ given by $A(D) = \frac{1}{2}(\partial_x + i\partial_y)$ is an elliptic linear differential operator on $\mathbb{R}^2$, since for each $(x_0,y_0) \in \mathbb{R}^2$,*

$$a((x_0,y_0),(\xi_1,\xi_2)) = \frac{1}{2}(\xi_1 + i\xi_2) \neq 0, \textit{ for all } (\xi_1,\xi_2) \neq (0,0).$$

Example 1.12. *The Laplace operator $\Delta : C^\infty(\mathbb{R}^N) \mapsto C^\infty(\mathbb{R}^N)$ given by $\Delta f(x) = \sum_{j=1}^N \partial^2_{x_j} f(x)$ is an elliptic linear differential operator on $\mathbb{R}^N$, since for each $x_0 \in \mathbb{R}^N$,*

$$a(x_0,\xi) = \sum_{|\alpha|=2} a_\alpha(x_0)\xi^\alpha = \sum_{j=1}^N \xi_j^2 \neq 0, \textit{ for all } \xi \neq 0.$$

Example 1.13. *The directional type operator* $A_m : C^\infty(\mathbb{R}^N) \mapsto C^\infty(\mathbb{R}^N)$ *of order* m *given by* $A_m f(x) = \sum_{k=1}^K (y_k \cdot \nabla)^m f(x)$, *in which* y_k *are fixed vectors in* $\mathbb{R}^N$ *called directions, is an elliptic linear differential operator on* $\mathbb{R}^N$ *if* m *is even and* $\{y_1,...,y_K\}$ *spans* $\mathbb{R}^N$. *Indeed, for each* x_0 *and* $\xi \in \mathbb{R}^N$, *we denote its symbol*

$$a(x_0,\xi) = \sum_{k=1}^{K} (y_k \cdot \xi)^m.$$

If $\{y_1,...,y_K\}$ *does not span* $\mathbb{R}^N$, *there exists a point* $\xi_0 \neq 0$ *orthogonal to all* y_k, *then* $y_k \cdot \xi_0 = 0$ *for all* $k = 1,...,K$ *and* $a(x_0,\xi_0) = 0$. *Consider now* $\{y_1,...,y_K\}$ *a generator of* $\mathbb{R}^N$ *and* m *even. Note that* $(y_k \cdot \xi)^m \geq 0$ *for all* ξ *and* $k \in \{1,...,K\}$. *So,* $a(x_0,\xi) = 0$ *implies that each* $(y_k \cdot \xi)^m$ *is null. Since* $\{y_1,...,y_K\}$ *spans* $\mathbb{R}^N$ *then* $\xi = 0$, *concluding the ellipticity of the operator* A_m.

Example 1.14. *Let* $A(\cdot,D) : C_c^\infty(\mathbb{R}^2,\mathbb{C}^2) \mapsto C_c^\infty(\mathbb{R}^2,\mathbb{C}^3)$ *the nonhomogeneous linear differential operator of order* 4 *in* $\mathbb{R}^2$ *given by*

$$A(x,D)u(x) = \left(\partial_{x_1}^3 u_1(x), \frac{x_2\partial_{x_2}u_1(x) + x_1\partial_{x_1}u_2(x)}{2}, \partial_{x_2}^4 u_2(x)\right),$$

such that $u(x) = (u_1(x),u_2(x))$. *Note that* $A(x,D)u(x)$ *can be also write as*

$$\begin{pmatrix} 0 & 0 \\ 0 & x_1/2 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_{x_1}u_1 \\ \partial_{x_1}u_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ x_2/2 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_{x_2}u_1 \\ \partial_{x_2}u_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_{x_1}^3 u_1 \\ \partial_{x_1}^3 u_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \partial_{x_2}^4 u_1 \\ \partial_{x_2}^4 u_2 \end{pmatrix}$$

and its symbol is given by

$$a(x,\xi) = \begin{pmatrix} \xi_1^3 & 0 \\ x_2\xi_2/2 & x_1\xi_1/2 \\ 0 & \xi_2^4 \end{pmatrix}.$$

Thus, for each $x_0 \in \mathbb{R}^2$, *we have* $a(x_0,\xi) \neq \boldsymbol{0}_{3\times 2}$ *for all* $\xi \in \mathbb{R}^2 - \{0\}$ *and therefore* $A(\cdot,D)$ *is an elliptic operator.*

Let us now look at the concept of ellipticity in the k-forms setting.

Example 1.15. *Consider the de Rham complex given by the sequence of differential operators:*

$$0 \longrightarrow C^\infty(\mathbb{R}^N) \xrightarrow{d} C^\infty(\mathbb{R}^N,\Lambda^1(\mathbb{R}^N)) \xrightarrow{d} \cdots \xrightarrow{d} C^\infty(\mathbb{R}^N,\Lambda^N(\mathbb{R}^N)) \xrightarrow{d} 0,$$

where $C^\infty(\mathbb{R}^N,\Lambda^k(\mathbb{R}^N))$ *is the space of smooth differential forms of degree* k *in* $\mathbb{R}^N$ *and* d *is the exterior operator given by*

$$dw = \sum_{i_1<\cdots<i_k} \sum_{j=1}^{N} \partial_{x_j} w_{i_1\ldots i_k} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and satisfying $d^2 = 0$. *Thus, the de Rham complex is elliptic, in the following sense. For any* $x_0 \in \mathbb{R}^N$, *we define the symbol* $a(x_0,\xi) : \Lambda^k(\mathbb{R}^N) \mapsto \Lambda^{k+1}(\mathbb{R}^N)$ *as the exterior product:*

$$a(x_0,\xi)(w) = i\xi \wedge w.$$

We also denote by $i_\xi : \Lambda^{k+1}(\mathbb{R}^N) \mapsto \Lambda^k(\mathbb{R}^N)$ *the inner contraction given by*

$$i_\xi(dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}}) = \sum_r (-1)^{r-1} \xi_{i_r} dx_{i_1} \wedge \cdots \vee dx_{i_r} \vee \cdots \wedge dx_{i_{k+1}}.$$

Suppose that $\xi \wedge w = 0$*, for some* $\xi \neq 0$*, then by identity*

$$i_\xi(\xi \wedge w) + \xi \wedge i_\xi(w) = |\xi|^2 w$$

we get $|\xi|^2 w = \xi \wedge i_\xi(w) = 0$*, which implies that* $w = 0$*, since* $\xi \neq 0$*. Therefore,* $a(x_0, \xi)$ *is injective, concluding the ellipticity of the de Rham complex.*

Example 1.16. *For each* $k \in \{1, \ldots, N-1\}$ *we define the second order operator*

$$A(D) = (dd^*, d^*d) : C^\infty(\mathbb{R}^N, \Lambda^k(\mathbb{R}^N)) \to C^\infty(\mathbb{R}^N, \Lambda^k(\mathbb{R}^N)) \times C^\infty(\mathbb{R}^N, \Lambda^k(\mathbb{R}^N)),$$

in which d *and* d^* *are the exterior and co-exterior derivatives, respectively. Note that* $A^*(D) = dd^* + d^*d = \Delta$ *is an elliptic operator, then* $A(D)$ *is also an elliptic operator.*

Next, we will present an important class of elliptic homogeneous differential operators of order 1 that will be explored with special attention throughout this thesis.

1.3.1 Elliptic system of complex vector fields

Consider n complex vector fields $L_1, \ldots, L_n$, $n \geq 2$, with smooth coefficients defined on $\Omega \subseteq \mathbb{R}^N$, in which $N \geq 2$, *i.e.* for each $j = 1, 2, \ldots n$,

$$L_j := \sum_{k=1}^{N} a_{jk}(x) \partial_{x_k},$$

for each $x \in \Omega$, with $a_{jk} \in C^\infty(\Omega)$. We will always assume in this thesis that the system of vector fields $\mathscr{L} := \{L_1, \ldots, L_n\}$ is linearly independent.

Remark 1.17. *Let* $\mathscr{L} = \{L_1, ..., L_n\}$ *be a system of vector fields with complex smooth coefficients defined on* Ω*. We remark that for each* $x_0 \in \Omega$ *there is a neighborhood* $U_0 \subset \Omega$ *such that, choosing appropriate generators and reordering the coordinates* $\{x_1, x_2, ..., x_N\}$*, we have*

$$\mathrm{span}_{C^\infty(U_0)}\{L_1, ..., L_n\} = \mathrm{span}_{C^\infty(U_0)}\{\widetilde{L_1}, ..., \widetilde{L_n}\},$$

where

$$\widetilde{L_j} := \frac{\partial}{\partial x_j} + \sum_{k=1}^{m} a_{jk} \frac{\partial}{\partial x_{n+k}}, \tag{1.3.2}$$

for $j = 1, ..., n$ *and* $m := N - n$*. As reference we cite the construction presented in the proof of [5, Proposition 4.4]. In particular, when the coefficients are constant, this property holds globally. We will be interested in systems of vector fields in which* L_j *has representation as* (1.3.2) *in the second main result, in Chapter 4.*

Definition 1.18. *Let $\mathscr{L} = \{L_1, \dots, L_n\}$ be a system of vector fields with complex smooth coefficients defined on Ω. We define $\nabla_{\mathscr{L}} : C^\infty(\mathbb{R}^N, \mathbb{C}) \mapsto C^\infty(\mathbb{R}^N, \mathbb{C}^n)$ the gradient operator associated with $\mathscr{L}$ given by $\nabla_{\mathscr{L}} := (L_1, \dots, L_n)$.*

For each x_0 and $\xi \in \mathbb{R}^N$ and $j \in \{1, \dots, n\}$ we denote by $\ell_j(x_0, \xi) := \sum_{k=1}^N a_{jk}(x_0)\xi_k$ the symbol associated with L_j. Thus, the symbol associated with the operator $\nabla_{\mathscr{L}}$ can be write as

$$\ell(x_0, \xi) := (\ell_1(x_0, \xi), \dots, \ell_n(x_0, \xi)).$$

Definition 1.19. *We say the system $\mathscr{L} = \{L_1, \dots, L_n\}$ is elliptic in Ω if the operator $\nabla_{\mathscr{L}}$ is elliptic for each $x_0 \in \Omega$, that is, if its symbol $\ell(x_0, \xi) = (\ell_1(x_0, \xi), \dots, \ell_n(x_0, \xi))$ is not null, for all $\xi \in \mathbb{R}^N - \{0\}$.*

Thus, the system being elliptic is equivalent to saying that the symbols of all vector fields do not have a common zero. If the system $\mathscr{L}$ is elliptic, then the number n of vector fields must satisfy $N/2 \leq n \leq N$ and the justification can be consulted in [39, p. 1076].

Example 1.20. *The Cauchy-Riemann system in $\mathbb{C}^r \cong \mathbb{R}^{2r}$ is given by*

$$L_j = \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial x_{r+j}}, \text{ for } j = 1, ..., r$$

where $N = 2r$, and defines an elliptic system.

Example 1.21. *Consider the system $\mathscr{L}$ given by*

$$L_j = \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial x_{r+j}}, \text{ for } j = 1, ..., r \quad \text{and} \quad L_{2r+j} = \frac{\partial}{\partial x_{2r+j}}, \text{ for } j = 1, ..., s$$

where $N = 2r + s$. Note that $\mathscr{L}$ defines an elliptic system and, in particular, when $s = 0$ we obtain the Cauchy-Riemann system above.

Example 1.22. *Let $\mathscr{L} = \{L_1, L_2\}$ be a system of vector fields with complex smooth coefficients defined on $\mathbb{R}^3$ given by*

$$L_1 = \partial_{x_1} + i\,\partial_{x_3} \text{ and } L_2 = \partial_{x_2} + i\,x_2\,\partial_{x_3}.$$

Then $\mathscr{L}$ is an elliptic system. Indeed, for each $x \in \mathbb{R}^3$, the symbol

$$a(x, \xi) = (\xi_1 + i\,\xi_3\ ,\ \xi_2 + i\,x_2\,\xi_3)$$

is null if and only if $\xi_1 = \xi_3 = 0$ and $\xi_2 = x_2\xi_3 = 0$, since each ξ_j is a real number. So, $\xi = 0$, regardless of the choice of x.

Example 1.23. *Let $\mathscr{L} = \{L_1, L_2\}$ be a system of vector fields with complex smooth coefficients defined on $\mathbb{R}^3$ given by*

$$L_1 = \partial_{x_1} + i\,x_2\,\partial_{x_3} \text{ and } L_2 = \partial_{x_2}.$$

Then $\mathscr{L}$ is not an elliptic system. Indeed, note that, for each $x \in \mathbb{R}^3$, the symbol

$$a(x,\xi) = (\xi_1 + i\, x_2\, \xi_3\ ,\ \xi_2)$$

is null if and only if $\xi_1 = x_2\xi_3 = 0$ and $\xi_2 = 0$, since each ξ_j is a real number. In particular, $a(x,\xi) = 0$, for all $\xi \in \mathbb{R}^N - \{0\}$ and for all x belonging to the hyperplane $M = \{(x_1, 0, x_3)\}$.

Definition 1.24. *We say that the system $\mathscr{L}$ is involutive if each commutator operator*

$$[L_j, L_\ell] := L_j \circ L_\ell - L_\ell \circ L_j$$

is a linear combination of $L_1, \ldots, L_n$, for all $j, \ell \in \{1, \ldots, n\}$.

Clearly, if $\mathscr{L}$ is a system of vector fields with constant complex coefficients then $[L_j, L_\ell] = 0$, for all $j, \ell \in \{1, \ldots, n\}$. As a generalization of Example 1.22, we can also present the following example:

Example 1.25. *Let $\mathscr{L} = \{L_1, ..., L_{N-1}\}$ be a system of vector fields with complex smooth coefficients defined on $\mathbb{R}^N$ given by*

$$L_j = \partial_{x_j} + i\,(x_j)^{j-1}\,\partial_{x_N},\ for\ j = 1, \ldots N-1.$$

Then $\mathscr{L}$ is an elliptic and involutive system. Indeed, for each $x \in \mathbb{R}^N$, the symbol

$$a(x,\xi) = \big(\xi_1 + i\,\xi_N\ ,\ \xi_2 + i\,x_2\,\xi_N\ ,\ \ldots\ ,\ \xi_{N-1} + i\,(x_{N-1})^{N-2}\,\xi_N\big)$$

is null if and only if $\xi_1 = \xi_N = 0$, $\xi_2 = x_2\xi_N = 0$, $\ldots$, $\xi_{N-1} = (x_{N-1})^{N-2}\xi_N = 0$, since each ξ_j is a real number. This way, $\xi = 0$, regardless of the choice of x. Furthermore, it is easy to see that $[L_j, L_\ell] = 0$, for all $j, \ell \in \{1, \ldots, N-1\}$.

Note that the ellipticity of the system $\mathscr{L}$ does not imply that each component L_j is elliptic. Indeed, in the previous example, anyone L_j is not elliptic at the origin, but the system $\mathscr{L}$ is elliptic at the origin.

Remark 1.26. *Let $\mathscr{L} = \{L_1, ..., L_n\}$ be an elliptic and involutive system of vector fields with smooth complex coefficients defined on Ω. A combination between Frobenius Theorem and Newlander-Nirenberg Theorem shows that there exists a local coordinate*

$$\{x_1, \ldots, x_r, y_1, \ldots, y_r, t_1, \ldots, t_s\}$$

such that the system $\mathscr{L}$ can be generated by the system defined on Example 1.21. The details can be verified in [29, Section 4.1].

The previous remark shows that in some situations studying local estimates for the operator $\nabla_{\mathscr{L}}$, in the case of an elliptic and involutive system, is equivalent to considering a system of fields with constant coefficients (see [29]). The study in this particular case of constant vector fields will be presented in Chapters 4 and 5, precisely in Theorems **B** and **C**.

Let $\mathscr{L} := \{L_1, \dots, L_n\}$ be a system of linearly independent vector fields with complex smooth coefficients defined on Ω. The operator $\nabla_{\mathscr{L}}$ can be written as $A(\cdot, D)$ before for $m = 1$, $E = \mathbb{C}$, $F = \mathbb{C}^n$ and each $a_\alpha : \Omega \mapsto \mathscr{L}(\mathbb{C}, \mathbb{C}^n)$ can be written in the form $a_{e_k}(x) = (a_{1k}(x), \dots, a_{nk}(x))$, for $k = 1, 2, \dots, N$. Furthermore, note that its adjoint operator is the divergence operator associated with $\mathscr{L}$ given by

$$\operatorname{div}_{\mathscr{L}^*} g(x) := \sum_{j=1}^{n} L_j^* g_j(x), \text{ for } g \in C^\infty(\Omega, \mathbb{C}^n)$$

and $x \in \Omega$. Here $L_j^* := \overline{L_j^t}$ denotes the adjoint operator of L_j, in the sense of $\int L_j f\, \overline{g} = \int f\, \overline{L_j^* g}$. In particular, if the coefficients of L_j are constant then $L_j^* := -\overline{L_j}$. This case where

$$A(\cdot, D) = \nabla_{\mathscr{L}} \text{ and } A^*(\cdot, D) = \operatorname{div}_{\mathscr{L}^*} \tag{1.3.3}$$

will be cited several times as applications of the results of this thesis.

We define the curl operator associated with $\mathscr{L}$ as the matrix given by

$$\operatorname{curl}_{\mathscr{L}} g(x) := \Big[\big(L_i g_j(x) - L_j g_i(x) \big)_{ij} \Big]_{n \times n},$$

for $g \in C^\infty(\Omega, \mathbb{C}^n)$ and $x \in \Omega$. We also define the operator

$$\Delta_{\mathscr{L}} := L_1^* L_1 + \cdots + L_n^* L_n.$$

The ellipticity of the system $\mathscr{L}$ (defined in 1.19) is equivalent to saying that the second order operator $\Delta_{\mathscr{L}}$ is elliptic (defined in 1.9). In particular, when the coefficients are constant, it follows from Remark 1.17 that the ellipticity of $\Delta_{\mathscr{L}}$ means that there exists $C > 0$ such that

$$\sum_{j=1}^{n} \left| \xi_j + \sum_{k=1}^{m} a_{jk} \xi_{n+k} \right|^2 \geq C |\xi|^2, \quad \forall \xi \in \mathbb{R}^N.$$

Note that, in the particular case of the system defined in Example 1.21, $\Delta_{\mathscr{L}}$ is a multiple of Laplacian operator Δ (see [5] for more details on this subject). Of course, if $n = N$ and $L_j = \partial_{x_j}$ for all $j = 1, \dots, N$, then $\nabla_{\mathscr{L}} = \nabla$, $\operatorname{div}_{\mathscr{L}^*} = \operatorname{div}$, $\operatorname{curl}_{\mathscr{L}} = \operatorname{curl}$ and $\Delta_{\mathscr{L}} = \Delta$.

The next subsection will be devoted only to local and global estimates of the operator $\nabla_{\mathscr{L}}$ with constant coefficients in L^p and $W^{-1,p}$ spaces. Such controls will be fundamental in the proof of Theorem **B**, in Chapter 4.

1.3.2 Elliptic estimates in Sobolev spaces with constant coefficients

Consider an elliptic system of linearly independent complex vector fields $\mathscr{L} := \{L_1, ..., L_n\}$ with constant complex coefficients in $\mathbb{R}^N$ for $N \geq 2$, where $L_j := \sum_{k=1}^{N} a_{jk} \partial_{x_k}$ and $n \leq N$. Note that the operator $\Delta_{\mathscr{L}} = L_1^* L_1 + \cdots + L_n^* L_n$ is a slight variation of Laplacian operator and it has a fundamental solution $E(x)$, *i.e.* $\Delta_{\mathscr{L}} E = \delta_0$, which is locally integrable tempered distribution homogeneous of degree $-N+2$ for $N \geq 3$ and $\log|x|$ type for $N = 2$. In particular, $\partial^2 E$ is a bounded operator from $L^p(\mathbb{R}^N)$ to itself for $1 < p < \infty$.

Indeed, consider the symbol of the operator $\Delta_{\mathscr{L}}$ given by

$$\delta(\xi) = -\sum_{j=1}^{n} \sum_{\ell,i=1}^{N} \overline{a_{j\ell}} a_{ji} \xi_\ell \xi_i.$$

We recall that, from [Definition 4.2, [17]], a distribution (or tempered distribution) T is homogeneous of degree k if for every test function ϕ it satisfies $\langle T, \phi_\lambda \rangle = \lambda^k \langle T, \phi \rangle$, in which $\phi_\lambda(x) := \lambda^{-N} \phi\left(\lambda^{-1} x\right)$. Thus, $1/\delta(\xi)$ is a smooth function in $\mathbb{R}^N - \{0\}$ (since $\Delta_{\mathscr{L}}$ is elliptic) and homogeneous of degree -2. Furthermore, it follows from [17, Proposition 4.3, p. 71] that its inverse Fourier transform, denoted by E, is a locally integrable tempered distribution homogeneous of degree $-N+2$ and satisfies

$$(\Delta_{\mathscr{L}} E)\widehat{\ } = (\mathscr{F}^{-1}\delta * E)\widehat{\ } = (\mathscr{F}^{-1}\delta)\widehat{\ }\, \widehat{E} = \delta\, \widehat{E} = 1 = \widehat{\delta_0},$$

in the sense of

$$u(x) = \int_{\mathbb{R}^N} E(x-y) \Delta_{\mathscr{L}} u(y) dy, \text{ for all } u \in C_c^\infty(\mathbb{R}^N).$$

In other words, if $N \geq 3$, then $E(x) := C|x|^{2-N}$ is a fundamental solution of $\Delta_{\mathscr{L}}$, where C is a constant depending only of N and the coefficients of $\mathscr{L}$. In the case when $N = 2$ and $n = 2$, we can write $\delta(\xi) = |v|^2$, where $v = M^{-1}\xi$ and M is a change-of-variable matrix. By the linear independence of the system $\mathscr{L}$, M is invertible and hence a linear isomorphism. Consequently, $E(x) := log|M^{-1}x|$. We observe that $E * f$ belongs to $L^p(\mathbb{R}^N)$, for all $f \in L^p(\mathbb{R}^N)$, and that $D^2 E$ is a bounded operator in L^p, in the sense of

$$\left\| \partial_{x_i} \partial_{x_j} E * f \right\|_{L^p} \lesssim \|f\|_{L^p}, \tag{1.3.4}$$

for all $f \in L^p(\mathbb{R}^N)$ and $i, j \in \{1, \ldots, N\}$. The proof uses a detailed treatment of singular integral operators with homogeneous Calderón–Zygmund kernels (including logarithmic terms), and can be found, for instance, in [17], [44] and [45], in addition to [41, Chapter 3], where all calculations are given explicitly. Moreover, if $N = 2$ we define $g(w) := f(Mw)$, then

$$\|\partial_i \partial_j E * f\|_{L^p} = |\det M|^{1/p} \|\partial_i \partial_j G * g\|_{L^p} \lesssim |\det M|^{1/p} \|g\|_{L^p} = \|f\|_{L^p},$$

when G is the fundamental solution of Δ defined in (3).

Next we will proof important *a priori* estimates of the operator $\nabla_{\mathscr{L}}$ on L^p and $W^{-1,p}$ spaces that will be useful.

Proposition 1.27. *Let $\mathscr{L} = \{L_1, L_2, ..., L_n\}$ be an elliptic system of complex vector fields on $\mathbb{R}^N$ with constant complex coefficients and $1 < p < \infty$. Then there exists $C > 0$ such that*

$$\|\nabla \phi\|_{L^p} \leq C \|\nabla_{\mathscr{L}} \phi\|_{L^p}, \textit{ for all } \nabla \phi \in L^p(\mathbb{R}^N).$$

Proof. Consider first $\nabla\phi \in S(\mathbb{R}^N)$. Using the fundamental solution of $\Delta_{\mathscr{L}}$ and that the vector fields have constant coefficients, we may write

$$\nabla\phi = \nabla\mathrm{div}_{\mathscr{L}^*}\,(E * \nabla_{\mathscr{L}}\phi) = (\nabla\mathrm{div}_{\mathscr{L}^*}\;E) * \nabla_{\mathscr{L}}\phi$$

and since D^2E is a bounded operators on $L^p(\mathbb{R}^N)$ for $1 < p < \infty$, in the sense of (1.3.4), the estimate follows for all $\nabla\phi \in S(\mathbb{R}^N)$. Now, by density of the space, consider $\nabla\phi \in L^p(\mathbb{R}^N)$ and $\{\phi_j\}_j \in S(\mathbb{R}^N)$ such that $\|\partial_k(\phi - \phi_j)\|_{L^p} \to 0$, for all $k \in \{1,\dots,N\}$. Since each L_j is a combination of standard derivatives (L_j has constant coefficients) then $\|\nabla_{\mathscr{L}}\phi_j\|_{L^p} \to \|\nabla_{\mathscr{L}}\phi\|_{L^p}$ and therefore

$$\|\nabla\phi\|_{L^p} \leq \|\nabla(\phi - \phi_j)\|_{L^p} + \|\nabla_{\mathscr{L}}\phi_j\|_{L^p} \leq \|\nabla_{\mathscr{L}}\phi\|_{L^p}.$$

□

Proposition 1.28. *Consider $\mathscr{L} = \{L_1,...,L_n\}$ be a system of complex vector fields on $\mathbb{R}^N$ with complex constant coefficients and $1 < p < \infty$. Then for each ball $B \subset \mathbb{R}^N$, there is a constant $C = C(B,\mathscr{L}) > 0$ such that*

$$\|\nabla g\|_{W^{-1,p}(B)} \leq C \sum_{j=1}^{n} \left\|L_j^* g\right\|_{W^{-1,p}(B)}.$$

We recall that $\|\nabla g\|_{W^{-1,p}(B)} := \sum_{j=1}^{N} \left\|\partial_{x_j} g\right\|_{W^{-1,p}(B)}$, where

$$\|\partial_{x_i} g\|_{W^{-1,p}(B)} = \sup_{\substack{\|u\|_{W^{1,p'}(B)} \leq 1 \\ u \in C_c^\infty(B)}} |\langle g, \partial_{x_i} u\rangle| = \sup_{\substack{\|u\|_{W^{1,p'}(B)} \leq 1 \\ u \in C_c^\infty(B)}} \left|\int g(x)\,\overline{\partial_{x_i} u(x)}dx\right|. \tag{1.3.5}$$

Proof. Using again the fundamental solution of $\Delta_{\mathscr{L}}$ and since the vector fields have constant coefficients, we may write, for all $u \in C_c^\infty(\mathbb{R}^N)$,

$$\partial_{x_i} u = \sum_{j=1}^{n} L_j h_{ij}, \text{ where } h_{ij} := \partial_{x_i} L_j^* E * u.$$

Thus,

$$\begin{aligned}\left|\int_B g(x)\,\overline{\partial_{x_i} u(x)}dx\right| \leq \sum_{j=1}^{n} \left|\int_B g\,\overline{L_j h_{ij}(x)}dx\right| &= \sum_{j=1}^{n} \left|\langle L_j^* g, \mathbb{X}_B h_{ij}\rangle\right| \\ &\leq \sum_{j=1}^{n} \left\|L_j^* g\right\|_{W^{-1,r}(B)} \left\|h_{ij}\right\|_{W^{1,r'}(B)}.\end{aligned}$$

As $h_{ij} = \sum_{k=1}^{N} -\overline{a_{kj}}\left(\partial^2_{x_i x_k} E * u\right)$ and noting that $\|\partial^2_{x_i x_k} E * u\|_{L^{r'}} \leq C_{ik}\|u\|_{L^{r'}}$, we have

$$\left|\int_B g(x)\,\overline{\partial_{x_i} u(x)}dx\right| \lesssim \sum_{j=1}^{n} \left\|L_j^* g\right\|_{W^{-1,r}(B)} \|u\|_{W^{1,r'}(B)}$$

for all $u \in C_c^\infty(B)$. Substituting in (1.3.5) we conclude the result. □

1.4 Pseudo-differential operators

In this section, we recall the special class of Hörmander symbols. After, we will present the definition of operators associated with this class, called Pseudo-Differential Operators. Finally, we will write such operators in terms of their kernel and present some results about the decay control of these kernels. These estimates will be a very important tool in proving our Theorem **A**. Our main references at this subject will be [2], [22] and [26].

Definition 1.29 (Hörmander's class)**.** *Let $\Omega \subset \mathbb{R}^N$ an open set, $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$. We define the Hörmander's class $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ the set of functions $\phi \in C^\infty(\Omega \times \mathbb{R}^N)$ such that*

$$\sup_{\substack{x\in K\\ \xi\in\mathbb{R}^N}} (1+|\xi|)^{-m+\rho.|\beta|-\delta.|\alpha|}|\partial_x^\alpha \partial_\xi^\beta \phi(x,\xi)| = C(K,\alpha,\beta) < \infty, \tag{1.4.1}$$

for all $K \subset \Omega$ compact and $\alpha, \beta \in \mathbb{Z}^N_+$ multi-indexes. The functions $\phi \in S^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ are called symbols of order m and type ρ, δ.

When the domain space is evident, it can be omitted, writing just $S^m_{\rho,\delta}$. Symbols of type $(1,0)$ will be denoted by S^m. We also denote the sets

$$S^\infty_{\rho,\delta}(\Omega\times\mathbb{R}^N) = \bigcup_m S^m_{\rho,\delta}(\Omega\times\mathbb{R}^N) \quad \text{and} \quad S^{-\infty}_{\rho,\delta}(\Omega\times\mathbb{R}^N) = \bigcap_m S^m_{\rho,\delta}(\Omega\times\mathbb{R}^N).$$

Example 1.30. *For each $m \in \mathbb{Z}$, define $\phi_m(x,\xi) = <\xi>^m = (1+|\xi|^2)^{m/2}$, thus $\phi_m \in S^m$.*

A trivial but useful property is that if $\phi \in S^m_{\rho,\delta}$ then $\partial_x^\alpha \partial_\xi^\beta \phi \in S^{m-\rho.|\beta|+\delta.|\alpha|}_{\rho,\delta}$ for all $\alpha, \beta \in \mathbb{Z}^N_+$ multi-indexes. Moreover, if $m < m'$ then $S^m_{\rho,\delta} \subset S^{m'}_{\rho,\delta}$.

Definition 1.31. *We define a pseudo-differential operator $P(x,D)$ associated to symbol $a \in S^m_{\rho,\delta}$ as*

$$P(x,D)u(x) = \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi, \tag{1.4.2}$$

with $u \in S(\mathbb{R}^N)$. We denote the class of operators as before associated with symbols in $S^m_{\rho,\delta}$ by $OpS^m_{\rho,\delta}$.

Example 1.32. *Let $A(D)$ a linear differential operator of order m with constant coefficients $A(D) = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha$, then $A(D)$ is a pseudo-differential operator, associated to the symbol*

$$a(\xi) = \frac{1}{(2\pi)^N} \sum_{|\alpha|\le m} a_\alpha i^\alpha \xi^\alpha \in S^m.$$

Indeed, by the inverse Fourier transform we may write

$$\begin{aligned} A(D)u(x) &= \frac{1}{(2\pi)^N}\int e^{ix\cdot\xi}\widehat{A(D)u}(\xi)d\xi \\ &= \frac{1}{(2\pi)^N}\int e^{ix\cdot\xi}\sum_{|\alpha|\le m} a_\alpha \widehat{\partial^\alpha u}(\xi)d\xi \\ &= \int e^{ix\cdot\xi}\left(\frac{1}{(2\pi)^N}\sum_{|\alpha|\le m} a_\alpha i^\alpha \xi^\alpha\right)\widehat{u}(\xi)d\xi. \end{aligned}$$

Example 1.33. *For each $m \in \mathbb{N}$, the operator associated to the symbol $a_m(\xi) = <\xi>^{-m} := (1+|\xi|^2)^{-m/2} \in S^{-m}$ is called Bessel Potential Operator of order m, and denoted by J_m.*

In general, by the Schwartz Kernel Theorem (see [48, Theorem 51.7] for the proof), every pseudo-differential operator in $OpS^m_{\rho,\delta}$, with $0 < \rho \leq 1, 0 \leq \delta < 1$, can be written as

$$P(x,D)u(x) = \int K(x,y)u(y)dy,$$

where K is its kernel, given by $K(x,y) = \int e^{i(x-y)\xi} a(x,\xi)\, d\xi$. In the particular case, when $a(x,\xi) = a(\xi)$, the kernel is obtained through the inverse Fourier transform of the symbol, *i.e.* $P(D)u = K_0 * u$, where $K_0 = (2\pi)^N \mathscr{F}^{-1} a$ and $K(x,y) = K_0(x-y)$. The following result gives us important estimates of the kernel of pseudo-differential operators.

Theorem 1.34. *[2, Theorem 1.1] Let $P(x,D) \in OpS^m_{\rho,\delta}$, with $0 < \rho \leq 1, 0 \leq \delta < 1$ and $K(x,\xi)$ its respective kernel. Then, the following properties hold.*

a) The distribution K is smooth outside the diagonal. Moreover, given $\alpha, \beta \in \mathbb{Z}^N_+$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x \neq y} |x-y|^n \, |\partial_x^\alpha \partial_x^\beta K(x,y)| < \infty,$$

for all $n \geq n_0$.

b) If $\operatorname{supp} a(x,\cdot)$ is uniformly compact with respect to x, then K is smooth and, given $\alpha, \beta \in \mathbb{Z}^N_+$ and $n \in \mathbb{N}$ such that

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \leq C(1+|x-y|)^{-n}$$

c) Let $j \in \mathbb{N}$ such that $m+j+N < 0$, then K is a bounded continuous function with bounded continuous derivatives of order at most j.

d) Let $j \in \mathbb{N}$ such that $m+j+N = 0$, then there is a constant $C > 0$ such that

$$\sup_{|\alpha+\beta|=j} |\partial_x^\alpha \partial_y^\beta K(x,y)| \leq C \left|\log|x-y|\right|,$$

for all $x \neq y$.

e) Let $j \in \mathbb{N}$ such that $m+j+N > 0$, then there is a constant $C > 0$ such that

$$\sup_{|\alpha+\beta|=j} |\partial_x^\alpha \partial_y^\beta K(x,y)| \leq C|x-y|^{-(m+j+N)/\rho},$$

for all $x \neq y$.

Next, we will see the boundedness of pseudo-differential operators on Lebesgue spaces and localizable Hardy spaces, depending on their type and order.

Theorem 1.35. *[2, Theorem* 3.4*] Given* $1 < p < \infty$ *and* $P(x,D) \in OpS^m_{\rho,\delta}$*, with* $0 < \rho \leq 1, 0 \leq \delta < 1$*, if*

$$m \leq -N\left[(1-\rho)\left|\frac{1}{p}-\frac{1}{2}\right| + \lambda\right],$$

in which $\lambda = \max\left\{0, \frac{(\delta-\rho)}{2}\right\}$*, then* $P(x,D)$ *is continuous from* L^p *to itself.*

Theorem 1.36. *[22, Theorem* 4*] If* $P(x,D) \in OpS^0$ *then* $\|Tf\|_{h^p} \leq C\|f\|_{h^p}$*, for all* $f \in h^p(\mathbb{R}^N)$*.*

Remark 1.37. *The operators in* $OpS^{-m}_{1,0}$ *are bounded in* h^p*-norm, for* $0 < p < \infty$ *and* $m \geq 0$*.*

Indeed, for $p > 1$ this is a consequence of Theorem 1.35, for $\rho = 1$ and $\delta = 0$. Note that $\lambda = 0$ and $-N\left[(1-\rho)\left|\frac{1}{p}-\frac{1}{2}\right| + \lambda\right] = 0$. Consider now $0 < p \leq 1$. The case $m = 0$ is the Theorem 1.36. For any $m > 0$, it is enough to see that $OpS^{-m}_{1,0} \subset OpS^0_{1,0}$.

For a deeper discussion see [28], where the result is extended to $0 \leq \delta < 1$.

Remember that the Bessel Potential operator J_m is associated to the symbol $a_m(\xi) = <\xi>^{-m}$, with $m \in \mathbb{N}$. By an abuse of notation, we define

$$J_{-m}f(x) = \int_{\mathbb{R}^N} e^{ix\cdot\xi} <\xi>^m \hat{f}(\xi)d\xi,$$

that defines an operator in the class OpS^m.

Proposition 1.38. *Let* $0 < p < \infty$*, then* $\|J_{-m}f\|_{h^p} \simeq \sum_{j=0}^m \left\|D^j f\right\|_{h^p}$*.*

Proof. Firstly, note that

$$<\xi>^{2m} = \left(1+|\xi|^2\right)^m = 1 + \sum_{j=1}^m \binom{m}{j}|\xi|^{2j}.$$

Thus,

$$<\xi>^m = \frac{<\xi>^{2m}}{<\xi>^m} = \frac{1}{<\xi>^m} + \sum_{j=1}^m \binom{m}{j}\frac{|\xi|^{2j}}{<\xi>^m}$$

and

$$<\xi>^m\hat{f}(\xi) = \frac{1}{<\xi>^m}\hat{f}(\xi) + \sum_{j=1}^m \binom{m}{j}\frac{|\xi|^{2j}\hat{f}(\xi)}{<\xi>^m}.$$

For each $j = 1,...,m$, let is write

$$|\xi|^{2j} = \sum_{i_j=1}^N \cdots \sum_{i_2=1}^N \sum_{i_1=1}^N \xi_{i_1}^2.\xi_{i_2}^2 \ldots \xi_{i_j}^2 \tag{1.4.3}$$

and define the pseudo-differential operator $P_j \in OpS^{j-m}_{1,0}$ associated to the symbol

$$a_j(\xi) = \binom{m}{j}\frac{1}{<\xi>^m}\sum_{i_j=1}^N \cdots \sum_{i_2=1}^N \sum_{i_1=1}^N \xi_{i_1}.\xi_{i_2} \ldots \xi_{i_j}.$$

Finally, as

$$<\xi>^m \hat{f}(\xi) = \frac{1}{<\xi>^m}\hat{f}(\xi) + \sum_{j=1}^{m} \binom{m}{j} \frac{\sum_{i_j=1}^{N} \cdots \sum_{i_2=1}^{N} \sum_{i_1=1}^{N} \xi_{i_1}.\xi_{i_2} \ldots \xi_{i_j} \widehat{(\partial^j_{i_1,i_2,\ldots,i_j} f)}(\xi)}{<\xi>^m}$$

then, by definition of pseudo-differential operator,

$$J_{-m}f = J_m f + \sum_{j=1}^{m} P_j(\partial^j_{i_1,i_2,\ldots,i_j} f).$$

Now, it is enough we apply Remark 1.37 to conclude that

$$\|J_{-m}f\|_{h^p} \lesssim \|f\|_{h^p} + \sum_{j=1}^{m} \left\|\partial^j_{i_1,i_2,\ldots,i_j} f\right\|_{h^p} = \sum_{j=0}^{m} \left\|D^j f\right\|_{h^p}.$$

On the other hand, for each $j = 0, 1, \ldots, m$, we have

$$D^j f = (D^j \circ J_m) \circ J_{-m}(f),$$

with $D^j \circ J_m \in OpS^{j-m}_{1,0}$. Again, by Remark 1.37, $\left\|D^j f\right\|_{h^p} \lesssim \|J_{-m}f\|_{h^p}$ and therefore,

$$\|J_{-m}f\|_{h^p} \simeq \sum_{j=0}^{m} \left\|D^j f\right\|_{h^p} = \|f\|_{h^{m,p}_{<}}.$$

Lastly, we will justify the identity (1.4.3). For $j = 1$, it is clear that $|\xi|^2 = \sum_{i_1=1}^{N} \xi_{i_1}^2$. For $j = 2$ we have

$$|\xi|^4 = |\xi|^2|\xi|^2 = \left[\sum_{i_1=1}^{N} \xi_{i_1}^2\right]\left[\sum_{i_2=1}^{N} \xi_{i_2}^2\right] = \sum_{i_2=1}^{N}\left[\sum_{i_1=1}^{N} \xi_{i_1}^2\right]\xi_{i_2}^2 = \sum_{i_2=1}^{N}\sum_{i_1=1}^{N} \xi_{i_1}^2 \xi_{i_2}^2.$$

Since the indices of the sums are independent of each other, the identity extends to any j. □

From here on, we will need to present some definitions and properties of special pseudo-differential operators, called parametrices. The main reference used was [26], in addition to [25, Chapter 3].

Definition 1.39. *Let $\Omega \subset \mathbb{R}^N$ an open subset, $m \in \mathbb{Z}$, $\delta < \rho$ and $a \in S^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$. We say that $a(x,\xi)$ is an elliptic symbol of order m if, for all $K \subset \Omega$ compact, there exist positive constants C e ε such that*

$$|a(x,\xi)| > C(1+|\xi|)^m, \textit{ for all } x \in K, |\xi| > \varepsilon.$$

Furthermore, we say that $P(x,D) \in OpS^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ is elliptic of order m if the principal part of its associated symbol is elliptic of order m.

Example 1.40. *Let $A(x,D) = \sum_{|\alpha| \le m} a_\alpha(x)\partial^\alpha$, with $a_\alpha \in C^\infty(\Omega)$, then the principal part of its associated symbol is*

$$a_\nu(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x) i^\alpha \xi^\alpha$$

and $A(x,D)$ is an elliptic pseudo-differential operator if for each $x \in \Omega$, $a_\nu(x,\xi) \neq 0$, for all $\xi \neq 0$. In other words, every elliptic linear differential operator is an elliptic pseudo-differential operator.

Definition 1.41. *Let $\Omega \subset \mathbb{R}^N$ an open subset, $m \in \mathbb{Z}$, $\delta < \rho$ and $P \in OpS^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$. We say that the operator $Q \in OpS^{m'}_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ is a parametrix of P if $P \circ Q \equiv Q \circ P \equiv I \mod OpS^{-\infty}(\Omega \times \mathbb{R}^N)$.*

Example 1.42. *Let $A(D) = \partial_x^4 - \partial_y^2 + 1 \in OpS^4(\mathbb{R}^2, \mathbb{R}^2)$ and consider B the pseudo-differential operator associated with the symbol $b(\xi) = (\xi_1^4 + \xi_2^2 + 1)^{-1}$, then*

$$\begin{aligned}
B \circ Af(x) &= \int e^{ix\cdot\xi} \frac{1}{\xi_1^4 + \xi_2^2 + 1} \left(\partial_x^4 f - \partial_y^2 f + f\right)\widehat{\ }(\xi) d\xi \\
&= \int e^{ix\cdot\xi} \frac{1}{\xi_1^4 + \xi_2^2 + 1} (i^4 \xi_1^4 - i^2 \xi_2^2 + 1)\widehat{f}(\xi) d\xi \\
&= \int e^{ix\cdot\xi} \widehat{f}(\xi) d\xi = f(x),
\end{aligned}$$

for all $f \in S(\mathbb{R}^2)$. Thus, B is a parametrix of A.

Proposition 1.43 (Proposition 12.1, [26])**.** *Let $\Omega \subset \mathbb{R}^N$ an open subset, $m \in \mathbb{Z}$, $\delta < \rho$ and $P(x,D) \in OpS^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$. The following conditions are equivalent:*

(i) $P(x,D)$ is elliptic;

(ii) $P(x,D)$ has a parametrix $Q(x,D) \in OpS^{-m}_{\rho,\delta}(\Omega \times \mathbb{R}^N)$;

(iii) There are $Q(x,D) \in OpS^{-m}_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ and $\varepsilon > 0$ such that $P \circ Q \in OpS^{-\varepsilon}_{\rho,\delta}(\Omega \times \mathbb{R}^N)$.

Paraphrasing for the notation that will be used in the following chapters, for each $P(\cdot,D) \in OpS^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ elliptic, with $0 < \rho \leq 1, 0 \leq \delta < 1$ and $m > 0$, there exist pseudo-differential operators $Q(\cdot,D) \in OpS^{-m}_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ called parametrix and $R(\cdot,D) \in OpS^{-\infty}_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ called regularizing such that

$$u(x) = Q(x,D) \circ P(x,D)u(x) + R(x,D)u(x), \tag{1.4.4}$$

for all $u \in C^\infty(\Omega)$.

All the previous definitions and properties of pseudo-differential operators were presented in the scalar setting. In what follows, we extend these frameworks to the vector-value case. Our reference at this subject will be [47, Chapter 1 - Section 7 and Chapter 2 - Section 2].

Once for all, fix E_1 and E_2 complex vector spaces of finite dimension, n_1 and n_2, respectively. We denote by $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^N; E_1, E_2)$ the set of functions $a : \Omega \times \mathbb{R}^N \mapsto \mathscr{L}(E_1, E_2)$ of C^∞-class satisfying

$$\sup_{\substack{x \in K \\ \xi \in \mathbb{R}^N}} (1 + |\xi|)^{-m+\rho.|\beta|-\delta.|\alpha|} \|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\|_{\mathscr{L}(E_1,E_2)} = C(K,\alpha,\beta) < \infty.$$

Of course, since each $a \in S^m_{\rho,\delta}(\Omega \times \mathbb{R}^N; E_1, E_2)$ can be represented by a $n_1 \times n_2$ matrix whose entries a_{ij} are continuous linear mappings, then

$$\|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\|_{\mathscr{L}(E_1,E_2)} = \left(\sum_{j=1}^{n_2} \sum_{i=1}^{n_1} \left(\partial_x^\alpha \partial_\xi^\beta a_{ij}(x,\xi) \right)^2 \right)^{1/2},$$

which is equivalent to the sum norm, a less than a constant depending only on the dimensions n_1 and n_2. This way, we denote by $OpS^m_{\rho,\delta}(\Omega;E_1,E_2)$ the class of all pseudo-differential operator $P(x,D)$ associated with a symbol $a(x,\xi)$ in $S^m_{\rho,\delta}(\Omega\times\mathbb{R}^N;E_1,E_2)$, defined as in (1.4.2), with $u\in S(\mathbb{R}^N;E_1)$. Note that now $P(\cdot,D)u:\Omega\mapsto E_2$. In particular when $E_1=E_2$, we denote only by $S^m_{\rho,\delta}(\Omega\times\mathbb{R}^N;E_1)$ and $OpS^m_{\rho,\delta}(\Omega;E_1)$.

The theory developed for scalar pseudo-differential operators extends almost unchanged to operators in $OpS^m_{\rho,\delta}(\Omega;E_1,E_2)$, the only care required concerns matrix non-commutativity.

Definition 1.44. *If E_1 and E_2 have the same dimension then we say that $P\in OpS^m_{\rho,\delta}(\Omega;E_1,E_2)$ is elliptic if the principal part of its associated symbol $a(x,\xi)$ is a bijection of E_1 onto E_2.*

Theorem 1.45 (Theorem 2.4, p. 101, [47])**.** *Let $P\in OpS^m_{\rho,\delta}(\Omega;E_1,E_2)$ an elliptic pseudo-differential operator associated with the symbol $a(x,\xi)$. There is an elliptic operator $Q\in OpS^{-m}_{\rho,\delta}(\Omega;E_2,E_1)$ (called parametrix) associated with the symbol $a(x,\xi)^{-1}$ (the inverse $a(x,\xi)$ in $\mathscr{L}(E_1,E_2)$) such that*

$$Q\circ P\equiv I_{E_1} \mod OpS^{-\infty}_{\rho,\delta}(\Omega;E_1) \quad \textit{and} \quad P\circ Q\equiv I_{E_2} \mod OpS^{-\infty}_{\rho,\delta}(\Omega;E_2).$$

By the previous theorem, we can now conclude a vector-value version of (1.4.4): let E be a complex vector spaces of finite dimension n_E, for each $P(\cdot,D)\in OpS^m_{\rho,\delta}(\Omega;E)$ elliptic, with $0<\rho\leq 1, 0\leq\delta<1$ and $m>0$, there exist pseudo-differential operators $Q(\cdot,D)\in OpS^{-m}_{\rho,\delta}(\Omega;E)$ called parametrix and $R(\cdot,D)\in OpS^{-\infty}_{\rho,\delta}(\Omega;E)$ called regularizing such that

$$u(x)=Q(x,D)\circ P(x,D)u(x)+R(x,D)u(x),$$

for all $x\in\Omega$ and $u\in C^\infty(\mathbb{R}^N;E)$.

1.4.1 Riesz potential operator

In this section we saw the Bessel Potential operator and its boundedness on the h^p norm. Now we will see another important potential operator, called the Riesz Potential, as well as a pointwise control involving the Hardy-Littlewood Maximal Operator defined in (1).

Definition 1.46. *Let $0<\gamma<N$, the Riesz Potential I_γ operator is defined by*

$$I_\gamma f(y)=\frac{1}{c_\gamma}\int_{\mathbb{R}^N}\frac{f(\eta)}{|y-\eta|^{N-\gamma}}d\eta,$$

where $f\in S(\mathbb{R}^N)$ and $c_\gamma=\pi^{\frac{N}{2}}2^{\gamma-N}\dfrac{\Gamma(\gamma/2)}{\Gamma((N-\gamma)/2)}$.

Note that $I_\gamma f(y)$ can also be denoted as $f*|\cdot|^{-(N-\gamma)}(y)$. Let us write the Riesz Potential operator through its multiplier $m(x,\xi)$. For this we will use a well-known property of Fourier transform: $(|\cdot|^\alpha)\widehat{}(\xi)=C|\xi|^{\alpha-N}$ (for more details about the proof and the constant C see [23, Example 2.4.9]).

As already discussed above Theorem 1.34, $I_\gamma f = K_0 * f$, where $K_0 = (2\pi)^N \mathscr{F}^{-1} m$ and $K_0(x-y) = \dfrac{c_\gamma^{-1}}{|x-y|^{N-\gamma}}$. Thus, by the Fourier transform

$$m(\xi) = (2\pi)^{-N} \widehat{\left(\frac{c_\gamma^{-1}}{|\cdot|^{N-\gamma}} \right)}(\xi) = \frac{C}{c_\gamma (2\pi)^N |\xi|^\gamma} = |\xi|^{-\gamma}.$$

Note that the definition of c_γ was chosen precisely to cancel out with the constant C. This way, unless a constant that depends only on the dimension N, the symbol of the Riesz Potential operator is $a(\xi) = |\xi|^{-\gamma}$.

Proposition 1.47. *Let $0 < \gamma < N$ and $1 < p,q < \infty$ such that $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{N}$, then*

$$|I_\gamma f|(x) \lesssim [Mf(x)]^{\frac{p}{q}} \, \|f\|_{L^p}^{1-\frac{p}{q}}.$$

The proof can be found at [44, p. 354] and the inequality to be proved is the estimate (33). In an even more general way, the Riesz operator is bounded in $L^q - L^p$-norm, *i.e.* for each $0 < \gamma < N$ and $1 < p < q < \infty$ satisfying $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{N}$, there is a constant $C = C(p,q,N)$ such that

$$\left\| I_\gamma f \right\|_{L^q} \leq C \left\| f \right\|_{L^p},$$

for all $f \in L^p(\mathbb{R}^N)$. The proof can be found at [44, p. 354] and now the inequality to be proved is the estimate (31).

1.5 Sobolev inequalities on Lebesgue and Hardy spaces

In this section we present Sobolev type inequalities on Lebesgue and localizable Hardy spaces, with bounded domain or in all $\mathbb{R}^N$. The next five results consider $p \geq 1$ and their proofs can be consulted in [18], at Sections 5.6 and 5.8, while the last result consider $0 < p \leq 1$, our reference for this case will be [24].

Theorem 1.48 (Sobolev-Gagliardo-Nirenberg inequality)**.** *Let $1 \leq p < N$. Then there exists a positive constant $C = C(p,N)$ such that*

$$\|f\|_{L^{p^*}(\mathbb{R}^N)} \leq C \, \|Df\|_{L^p(\mathbb{R}^N)},$$

for all $f \in C_c^1(\mathbb{R}^N)$, where $p^* = \dfrac{Np}{N-p}$.

The proof can be found in [18, p. 263].

Theorem 1.49 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^N$ bounded, connected and open subset and $f \in W^{1,p}(\Omega)$, for some $1 \leq p \leq \infty$. Then, there exists a positive constant $C = C(p,N,\Omega)$ such that*

$$\|f - f_\Omega\|_{L^p(\Omega)} \leq C \, \|Df\|_{L^p(\Omega)}.$$

The proof can be found in [18, p. 275].

Theorem 1.50 (Poincaré inequality for a ball)**.** *Let* $1 \le p \le \infty$ *and* $f \in W^{1,p}(B(x_0,r))$*, then there exists a positive constant* $C = C(N,p)$ *such that*

$$\|f - f_B\|_{L^p(B(x_0,r))} \le Cr\,\|Df\|_{L^p(B(x_0,r))}\,.$$

The three previous theorems are well-known and their proof can be found in [18, pp. 263, 275 and 276], respectively.

Theorem 1.51. *Let* $\Omega \subset \mathbb{R}^N$ *bounded and open subset such that* $\partial\Omega \in C^1$ *and* $f \in W^{1,p}(\Omega)$*, for some* $1 \le p < N$*. Then, there exists a positive constant* $C = C(p,N,\Omega)$ *such that*

$$\|f\|_{L^{p^*}(\Omega)} \le C\,\|f\|_{W^{1,p}(\Omega)}\,. \tag{1.5.1}$$

The proof can be found in [18, p. 265]. Furthermore, it follows from (1.5.1) that, for each $1 \le q < p^*$, there exists a positive constant $C = C(p,q,N,\Omega)$ such that

$$\|f\|_{L^q(\Omega)} \le C\,\|Df\|_{L^p(\Omega)}\,.$$

Indeed, as $q < p^*$ then we can apply the Hölder's Inequality for $\gamma = \frac{p^*}{q}$, getting

$$\int_\Omega |f(x)|^q dx \;\le\; \left(\int_\Omega |f(x)|^{q\cdot\gamma} dx\right)^{\frac{1}{\gamma}} |\Omega|^{\frac{1}{\gamma'}} = \left(\int_\Omega |f(x)|^{p^*} dx\right)^{\frac{q}{p^*}} |\Omega|^{1-\frac{q}{p^*}} = \|f\|^q_{L^{p^*}(\Omega)}\,|\Omega|^{1-\frac{q}{p^*}}.$$

Thus,

$$\|f\|_{L^q(\Omega)} \le \|f\|_{L^{p^*}(\Omega)}\,|\Omega|^{\frac{1}{q}-\frac{1}{p^*}}.$$

By the Theorem 1.51, exists a constant $c = c(p,N,\Omega)$ such that $\|f\|_{L^{p^*}(\Omega)} \le c\,\|Df\|_{L^p(\Omega)}$, so

$$\|f\|_{L^q(\Omega)} \le C\,\|Df\|_{L^p(\Omega)}\,,$$

being $C = C(p,q,N,\Omega) = c|\Omega|^{\frac{1}{q}-\frac{1}{p^*}} < \infty$.

Proposition 1.52 (Poincaré inequality)**.** *If* $1 \le p < N$ *then for each* $u \in W^{1,p}(B_x^t)$ *and* $1 \le q \le p^*$ *there exists a positive constant* $C = C(N,p,q)$ *such that*

$$\left(⨍_{B(x,t)} \left|\frac{1}{t}\left(u - u_x^t\right)\right|^q\right)^{\frac{1}{q}} \le C\left(⨍_{B(x,t)} |\nabla u|^p\right)^{\frac{1}{p}} \tag{1.5.2}$$

Proof. Indeed, by the Theorem 1.51, if U is a bounded, open subset of $\mathbb{R}^N$ and $v \in W^{1,p}(U)$ for some $1 \le p < N$, then $\|v\|_{L^q(U)} \le C\,\|\nabla v\|_{L^p(U)}$ for each $1 \le q \le p^*$ and the constant C depends only on

p,q,N and U. In particular, if $U = B_0^1$ and $v(y) = u(x+ty)$, then $\left\|v - v_0^1\right\|_{L^q(B_0^1)} \leq C\left\|\nabla v\right\|_{L^p(B_0^1)}$ and $C = C(p,q,N)$. However,

$$\begin{aligned}\left\|v - v_0^1\right\|_{L^q(B_0^1)} &= \left(\int_{B(0,1)} \left|v(y) - v_0^1\right|^q \, dy\right)^{\frac{1}{q}} = \left(\int_{B(0,1)} \left|u(x+ty) - u_x^t\right|^q \, dy\right)^{\frac{1}{q}} \\ &= \left(\int_{B(x,t)} \frac{1}{t^N}\left|u(z) - u_x^t\right|^q \, dz\right)^{\frac{1}{q}} = \left(\fint_{B(x,t)} \left|u(z) - u_x^t\right|^q \, dz\right)^{\frac{1}{q}}\end{aligned}$$

and

$$\begin{aligned}\left\|\nabla v\right\|_{L^p(B_0^1)} &= \left(\int_{B(0,1)} \left|\nabla_y[u(x+ty)]\right|^p \, dy\right)^{\frac{1}{p}} = \left(\int_{B(0,1)} \left|t\,\nabla_y u(x+ty)\right|^p \, dy\right)^{\frac{1}{p}} \\ &= t\left(\int_{B(x,t)} \frac{1}{t^N}\left|\nabla_z u(z)\right|^p \, dz\right)^{\frac{1}{p}} = t\left(\fint_{B(x,t)} \left|\nabla u(z)\right|^p \, dz\right)^{\frac{1}{p}}.\end{aligned}$$

Thus,

$$\left(\fint_{B(x,t)} \left|u - u_x^t\right|^q\right)^{\frac{1}{q}} \leq Ct\left(\fint_{B(x,t)} \left|\nabla u\right|^p\right)^{\frac{1}{p}},$$

for each $1 \leq q \leq p^*$. □

Theorem 1.53. *[24, Theorem 3.1] Let $N \geq 2, 0 < p \leq 1$ and $B = B(x,r) \subset \mathbb{R}^N$ an open ball, then there is a continuous embedding*

$$h_c^{1,p}(B) \subset h_c^{p^*}(B).$$

The proof can be found in [24, p. 780]. In particular, this result guarantees us that there exists a positive constant $C = C(p,N,B)$ such that $\|f\|_{h^{p^*}} \leq C\|\nabla f\|_{h^p}$, for all $f \in h_c^{1,p}(B)$ and

$$\|f\|_{h^p} \leq C'\|\nabla f\|_{h^p}, \text{ for all } f \in C_c^\infty(B). \tag{1.5.3}$$

Of course such estimates are also valid for E-valued vector functions f, computing the derivatives of each component. As an example, let us show the vector version of the estimate (1.5.3): let $g = (g_1, ..., g_{n_E}) \in C_c^\infty(B,E)$, then

$$\|g\|_{h^p} \lesssim \sum_{j=1}^{n_E} \left\|g_j\right\|_{h^p} \leq C' \sum_{j=1}^{n_E} \left\|\nabla g_j\right\|_{h^p} \lesssim \|Dg\|_{h^p}.$$

For the above inequalities, we use that the h^p-norm of a vector function is bounded by the sum of the h^p-norm of each coordinate function. The omitted constants depend only on p and N.

1.6 Hardy spaces on open sets and Poincaré inequality

The classical Poincaré inequality for balls on Sobolev spaces, stated in Theorem 1.50, establishes this type of control

$$\int_B \left| \frac{1}{r}(f(x) - f_B) \right|^p dx \leq C_{N,p} \int_B |Df(x)|^p \, dx,$$

for $f \in W^{1,p}(B)$ and $1 \leq p \leq \infty$. Our interest is to study estimates of this type for higher order derivatives, that is, we will subtract polynomials that depend on f instead of its mean f_B, f belongs to a Hardy-Sobolev space (to be defined in the next chapter) and the limiting term will depend on the m-th order derivatives of f. The details of this result will be presented only in Section 2.2.

Before that, in this section we will do a study of Hardy spaces on an open domain Ω. We will present a series of constructions and well-know results due to Miyachi in [37]. The observations proved in this section will be used in Section 2.2 to conclude an extension of Poincaré's inequality, stated in the Theorem 2.2.

Definition 1.54 (Maximal functions on Ω). *Let Ω an open set in $\mathbb{R}^N$ and fix $\phi \in C_c^\infty(B(0,1))$ with $\int \phi = 1$. For each $0 < a \leq \infty$ and $f \in \mathscr{D}'(\Omega)$ we define the maximal functions*

(i) $f^+_{\phi,\Omega}(x) = \sup\limits_{0<t<d(x,\partial\Omega)} |\langle f, \phi_t(x - \cdot)\rangle|$, *for* $x \in \Omega$;

(ii) $f^{+,a}_{\phi,\Omega}(x) = \sup\limits_{0<t<\min\{a,d(x,\partial\Omega)\}} |\langle f, \phi_t(x - \cdot)\rangle|$, *for* $x \in \Omega$.

Furthermore, for each $x \in \mathbb{R}^N$, $t > 0$ and $L \in \mathbb{N}$, we define

$$\mathscr{F}_{L,\Omega}(x,t) = \{\phi \in C_c^\infty(B(x,t) \cap \Omega), \|\partial^\alpha \phi\|_\infty \leq t^{-N-|\alpha|}, |\alpha| \leq L\},$$

and the following function

(iii) $f^*_{L,\Omega,a}(x) = \sup \left\{ |\langle f, \phi \rangle| \; ; \; \phi \in \bigcup\limits_{0<t<a} \mathscr{F}_{L,\Omega}(x,t) \right\}.$

When $\Omega = \mathbb{R}^N$ we simplify the notations $f^+_{\phi,\Omega}, f^{+,a}_{\phi,\Omega}$ and $f^*_{L,\Omega,a}$ by respectively $f^+_\phi, f^{+,a}_\phi$ and $f^*_{L,a}$. The following definition was presented by Miyachi in [37, Section 2].

Definition 1.55. *Let $0 < p < \infty$ and $f \in \mathscr{D}'(\Omega)$. We say that $f \in H^p(\Omega)$ (Hardy space) or $f \in h^p(\Omega)$ (nonhomogeneous Hardy space or localizable Hardy space) if $f^+_{\phi,\Omega} \in L^p(\Omega)$ or if $f^{+,1}_{\phi,\Omega} \in L^p(\Omega)$, respectively. Furthermore, we say that $f \in H^p_{loc}(\Omega)$ if, for each $x \in \Omega$, exists an open neighborhood $U_x \subset \Omega$ such that $f\big|_{U_x} \in H^p(U_x)$.*

When $\Omega = \mathbb{R}^N$ then $f^+_{\phi,\mathbb{R}^N} = M_\phi f$ and $f^{+,1}_{\phi,\mathbb{R}^N} = m_\phi f$, thus we recover the classical Hardy spaces $H^p(\mathbb{R}^N)$ and $h^p(\mathbb{R}^N)$, respectively.

Proposition 1.56. *With respect to maximal functions, the following properties hold* *(i)* *If* $a \le b$, *then* $f^{*,a}_{M,\Omega}(x) \le f^{*,b}_{M,\Omega}(x)$, *for each* $x \in \Omega$;

(ii) $f^{+,a}_{\phi,\Omega}(x) \le C_{\phi,L} f^{*}_{L,\Omega,a}(x)$, *for each* $x \in \Omega$;

(iii) *If* $f \in L^\infty(\Omega)$, *then* $f^{*}_{L,\Omega,a}(x) \le \|f\|_{L^\infty(\Omega)}$;

(iv) *For each* $0 < \gamma < \infty$, $L > \frac{N}{\gamma} - N$ *and* $a > 0$, *we have*

$$\|f^{*}_{L,\Omega,a}\|_{L^\gamma(\Omega)} \le C_{\phi,L,a} \|f^{+,1}_{\phi,\Omega}\|_{L^\gamma(\Omega)}. \tag{1.6.1}$$

Proof. The proof of items $(i), (ii)$ and (iii) are trivial, since they come directly from the definition. The property (iv) comes from a non trivial argument. By [38, Corollary 2], we have the pointwise control

$$f^{*}_{L,\Omega,a}(x) \le C_{\phi,L} M_{\frac{N}{N+L}}\left[f^{+,a}_{\phi,\Omega}\right](x),$$

for any $x \in \Omega$ and $a, L > 0$, where $M_{\frac{N}{N+L}}$ is the fractional Maximal Operator defined in (2) with $q = \frac{N}{N+L}$. Since M_q is bounded in $L^p(\Omega)$ for all $p > q$, then the inequality (1.6.1) holds for all L and γ satisfying $\gamma > \frac{N}{N+L}$.

$\square$

In particular, the properties (ii) and (iv) show that $H^p(\Omega)$ or $h^p(\Omega)$ does not depend on the choice of ϕ, proving the good definition.

Remark 1.57. *Note that* M_q *is bounded operator in* $L^p(\Omega)$ *for all* $p > q$, *even if both are less than* 1. *Indeed, we can denote* $h = |g|^q$ *and* $M_q g(x) = [Mh(x)]^{\frac{1}{q}}$, *in which* M *is the Hardy-Littlewood maximal operator. Since* M *is bounded in* L^p *for all* $1 < p \le \infty$ *then*

$$\|M_q g\|_{L^p} = \left\|[Mh]^{\frac{1}{q}}\right\|_{L^p} = \left(\int |Mh|^{\frac{p}{q}}\right)^{\frac{q}{p}\frac{1}{q}} = \left[\|Mh\|_{L^{\frac{p}{q}}}\right]^{\frac{1}{q}} \lesssim \left[\|h\|_{L^{\frac{p}{q}}}\right]^{\frac{1}{q}} = \|g\|_{L^p}.$$

Given $f \in \mathscr{D}'(\Omega)$ and $x \in \Omega$, if the limit $\lim_{t \searrow 0} f * \phi_t(x)$ exists and does not depend of the choice of ϕ then we define

$$[f](x) := \lim_{t \searrow 0} f * \phi_t(x).$$

We point out that if $f \in L^1_{loc}(\mathbb{R}^N)$ then $[f]$ exists and $[f] = f$ almost everywhere. The following construction is also due to Miyachi.

Lemma 1.58. *[37, Lemma 4] Let* $m \in \mathbb{N}$. *Then for each cube* $Q \subset \mathbb{R}^N$ *there exists a function* ϕ_Q *on* $\mathbb{R}^N \times \mathbb{R}^N$ *which has the following properties:*

(i) $\phi_Q \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ *and supp* $\phi_Q(x, \cdot) \subset \mathring{Q}$, *for each* $x \in \mathbb{R}^N$. *Here* $\mathring{Q}$ *denotes the interior of* Q.

(ii) *For each* $f \in \mathscr{D}'(\mathring{Q})$, *the function* $P_{Q,f} : \mathbb{R}^N \mapsto \mathbb{C}$ *given by* $P_{Q,f}(x) = \langle f, \phi_Q(x, \cdot)\rangle$ *belongs to* $\mathscr{P}_{m-1}$, *i.e. it is a polynomial of degree at most* $m - 1$.

(iii) $P(x) = \int P(y)\phi_Q(x,y)dy$, *for all* $P \in \mathscr{P}_{m-1}$.

(iv) Let $a > 0$ *and* $x \in aQ$, *then*

$$|\partial_x^\alpha \partial_y^\beta \phi_Q(x,y)| \leq C_{m,a,\alpha,\beta}\ell(Q)^{-N-|\alpha|-|\beta|},$$

for all α, β *multi-indexes.*

Proof. Let Q_0 the unit cube centered at the origin in $\mathbb{R}^N$ and fix $\eta \in C_c^\infty(Q_0)$ such that $\eta \geq 0$ and $\int \eta = 1$. Consider $\mathscr{P}_{m-1}$ the Hilbert space of the polynomials of degree at most $m-1$ with inner product

$$\langle p, q\rangle_M = \int p(x)\bar{q}(x)\eta(x)dx$$

and $\{\pi_i\}_i$ an orthonormal basis of $(\mathscr{P}_{m-1}, \langle\cdot,\cdot\rangle_M)$. For each cube $Q \subset \mathbb{R}^N$, we define

$$\phi_Q(x,y) = \ell(Q)^{-N}\left[\sum_{i=1}^m \pi_i\left(\frac{x-x_Q}{\ell(Q)}\right)\overline{\pi_i\left(\frac{y-x_Q}{\ell(Q)}\right)}\right]\eta\left(\frac{y-x_Q}{\ell(Q)}\right),$$

that satisfies all the properties desired. ☐

In particular, if we take $\eta = \zeta$, being ζ defined in [44, p. 102] as follows: Fix a positive smooth function ζ that equals 1 in the cube of side length 1 centered at the origin and vanishes outside the concentric cube of side length $\tilde{a}$. We set $\zeta_k(x) = \zeta\left(\frac{x-x_k}{\ell_k}\right)$, where x_k is the center of the cube Q_k and ℓ_k is its side length. Write $\eta_k = \dfrac{\zeta_k}{\sum_j \zeta_j}$ [...] and $\tilde{\eta_k} = \dfrac{\eta_k}{(\int \eta_k)}$.

Then the orthonormal basis $\{e_i\}$ utilized in [44, p. 104] given by $\langle e_i, e_j\tilde{\eta_k}\rangle = \delta_{ij}$ can be rewritten as

$$e_i(x) = \left(\frac{\int \eta_k}{\ell_k^N}\right)^{\frac{1}{2}}\pi_i\left(\frac{x-x_k}{\ell_k}\right),$$

in which $\{\pi_i\}$ is the orthonormal basis defined in [37]. So, the polynomial created in [44] to prove the generalized of Calderón-Zygmund decomposition (and consequently to prove a version of atomic decomposition, with atoms having higher moment condition) is a particular case of the polynomial $P_{Q,f}$ defined in Lemma 1.58. This relation will be fundamental in the proof of Theorem 2.6, when we will present a higher order version of the Calderón-Zygmund decomposition.

An important consequence of this lemma is the following result:

Proposition 1.59. *Let* $0 < p \leq 1$ *and* $m \in \mathbb{N}$. *If* $f \in \mathscr{D}'(\Omega)$ *such that* $\partial^\alpha f \in H^p_{loc}(\Omega)$ *for all* $|\alpha| = m$, *then for each* $Q \subset \Omega$ *and* $L \in \mathbb{N}$, *there exist a polynomial* $P_{Q,f} \in \mathscr{P}_{m-1}$ *and a positive constant* $C = C(m,p,L)$ *satisfying*

$$|[f](x) - P_{Q,f}(x)| \leq C\ell(Q)^m \sum_{|\alpha|=m}(\partial^\alpha f)^*_{L,\Omega,a}(x), \quad a := \sqrt{N}\ell(Q), \tag{1.6.2}$$

for $x \in Q$ *almost everywhere.*

The proof can be found at [37, p. 86 inequality (5.15)]. We are interested to use this result when $\Omega = \mathbb{R}^N$ and for functions in $(L^1_{loc} \cap h^p)(\mathbb{R}^N)$. In order to do it, we present the following definition presented in [37].

Definition 1.60. *Let $\Omega \subset \mathbb{R}^N$ an open subset, we say that Ω satisfies the condition $(\star)$ if for each $x \in \Omega$ such that $d(x, \partial\Omega) < A^{-1}$, there is a $x' \in \Omega^c$ such that*

$$d(x,x') < A\, d(x,\partial\Omega) \quad \textit{and} \quad d(x',\Omega) > A^{-1}\, d(x,\partial\Omega),$$

where A is a positive constant bigger than 1.

Example 1.61. *Any open ball on $\mathbb{R}^N$ satisfies the condition $(\star)$.*

Indeed, let $B(x_0,r) \in \mathbb{R}^N$ an open ball. We need to find a positive constant $A > 1$ such that, for each $x \in B(x_0,r)$ with $d(x,\partial B) < A^{-1}$, there exists $x' \in B^c$ that satisfies

$$d(x,x') < A\, d(x,\partial B) \quad \text{and} \quad d(x',B) > A^{-1}\, d(x,\partial B).$$

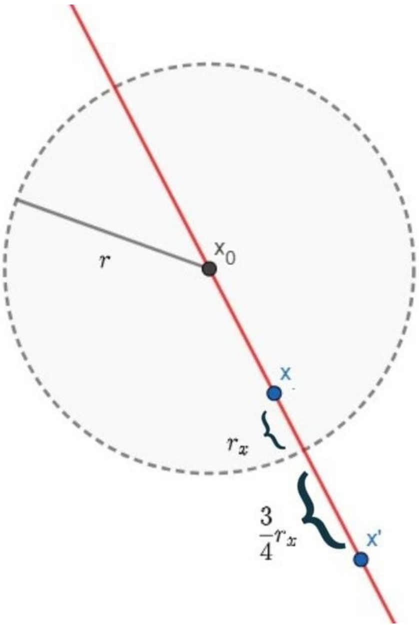
For this, we denote $r_x = d(x,\partial B)$ and define $x' = c(x-x_0)+x_0$, where

$$c = \frac{7}{4}\frac{r}{d(x,x_0)} - \frac{3}{4},$$

i.e. x' belongs to the line that passes through x and x_0. In particular, we can choose $A = 2$. Thus,

$$d(x,x') = \frac{7}{4}r_x < 2r_x = Ar_x \quad \text{and} \quad d(x',B) = \frac{3}{4}r_x > \frac{1}{2}r_x = A^{-1}r_x.$$

The picture below shows the idea of the choosing x' for $N = 2$.



Proposition 1.62. *With respect to relation between the spaces $H^p(\Omega)$, $h^p(\Omega)$ and $H^p_{loc}(\Omega)$, the following properties hold:*

(i) $h^p(\mathbb{R}^N)\big|_{\Omega} = h^p(\Omega)$ *if* Ω *satisfies the condition* $(\star)$*;*

(ii) $h^p(\Omega) = H^p(\Omega)$*, if* Ω *is a bounded set.*

The details can be verified in Remarks (b) and (c), from [37, p. 77]. A complete characterization of extension domains for H^p spaces can be also found in [43].

Proposition 1.63. $H^p(\mathbb{R}^N) \subset h^p(\mathbb{R}^N) \subset H^p_{loc}(\mathbb{R}^N)$.

Proof. It is clear that $H^p(\mathbb{R}^N) \subset h^p(\mathbb{R}^N)$. Let $f \in h^p(\mathbb{R}^N)$ and for each x in $\mathbb{R}^N$ denote $U_x := B(x,1)$. Thus, by the previous proposition,

$$f\big|_{U_x} \in h^p(\mathbb{R}^N)\big|_{U_x} = h^p(U_x) = H^p(U_x)$$

and, by definition, $f \in H^p_{loc}(\mathbb{R}^N)$.

□

CHAPTER 2

Hardy-Sobolev spaces: definitions, extensions and refinements

In this chapter we are interested in studying properties of tempered distributions whose higher order derivatives belong to localizable Hardy spaces. We will start by defining the Hardy-Sobolev spaces and then we will present an extension of the Poincaré inequality, using notations and results presented in Section 1.6. After that, we will see local estimates of elliptic linear differential operators with smooth coefficients on Hardy-Sobolev spaces, to be used in the proof of Theorem **A**. Next, we will present a new atomic decomposition. For that, a particular version of the Calderón-Zygmund Decomposition will be proved, with more rigorous controls for higher-order derivatives, and finally, it will show that Hardy-Sobolev spaces can be approximated by a more refined sequence of atoms.

2.1 Hardy-Sobolev spaces

Let $k \in \mathbb{N}$, $0 < p < \infty$ and consider on $C_c^\infty(\mathbb{R}^N)$ the functional

$$\|f\|_{h^{k,p}(\mathbb{R}^N)} := \sum_{|\alpha|=k} \|\partial^\alpha f\|_{h^p(\mathbb{R}^N)}.$$

The completion of $C_c^\infty(\mathbb{R}^N)$ for the functional $\|f\|_{h^{k,p}(\mathbb{R}^N)}$ may be identified with a subspace of $S'(\mathbb{R}^N)$ denoted by $h^{k,p}(\mathbb{R}^N)$ and called by us as the (homogeneous) Hardy-Sobolev space of order k. The construction presented in this section is a generalization of the space identified as the homogeneous localizable Hardy-Sobolev space $h^{1,p}(\mathbb{R}^N)$ in [24]. We claim if $f \in h^{k,p}(\mathbb{R}^N)$ then $D^k f \in S'(\mathbb{R}^N)$ and

$$\|f\|_{h^{k,p}(\mathbb{R}^N)} = \|D^k f\|_{h^p}.$$

for $k < N$. Indeed, suppose (f_j) is a sequence of test functions which is a Cauchy sequence for the functional $\|f_j\|_{h^{k,p}} = \|D^k f_j\|_{h^p}$ and fix $\psi \in S(\mathbb{R}^N)$. The next result stated was proved in [45] and, inspired by [6], we will use it to generate the following decomposition of ψ:

$$\psi = \sum_{|\alpha|=k} K_\alpha * D^\alpha \psi, \tag{2.1.1}$$

where $K_\alpha(x) = C_{\alpha,N}\, x^\alpha/|x|^N$.

Theorem 2.1 (Theorem 4.1, [45]). *Suppose $w \in \mathbb{C}$ is such that $0 < Re(w) < N$ and $P(x)$ is a homogeneous harmonic polynomial of degree k on E_N (a N-dimensional real Euclidean space). If $K(x) = \dfrac{P(x)}{|x|^{N+k-w}}$ then $\widehat{K}(\xi) = c_{k,\alpha,N}\dfrac{P(\xi)}{|\xi|^{k+w}}$, in which $c_{k,\alpha,N} = i^{-k}\pi^{N/2-w}\,\Gamma\left(\frac{k+w}{2}\right)/\Gamma\left(\frac{N+k-w}{2}\right)$.*

In particular, for each α such that $|\alpha| = k$, we consider $P_\alpha(x) = {c_{k,\alpha,N}}^{-1}\, x^\alpha$ and $w = k$. Thus,

$$\begin{aligned}\left(\sum_{|\alpha|=k} K_\alpha * D^\alpha\psi\right)^{\widehat{}}(x) &= \sum_{|\alpha|=k} (K_\alpha * D^\alpha\psi)^{\widehat{}}(x) = \sum_{|\alpha|=k} \widehat{K_\alpha}(x)\widehat{D^\alpha\psi}(x)\\ &= \sum_{|\alpha|=k} \frac{x^\alpha}{|x|^{2k}}\, x^\alpha\,\widehat{\psi}(x) = \frac{\sum_{|\alpha|=k} x^{2\alpha}}{|x|^{2k}}\,\widehat{\psi}(x) = \widehat{\psi}(x),\end{aligned}$$

obtaining the decomposition 2.1.1. Thus,

$$\langle f_j, \psi\rangle = (-1)^k \sum_{|\alpha|=k} \langle D^\alpha f_j, K_\alpha * \psi\rangle$$

and $K_\alpha * \psi$ is a smooth bounded function with bounded derivatives of all orders. This comes from the fact that $|K_\alpha(x)| \lesssim |x|^{k-N}$ for any $|\alpha| = k$ and $\psi \in S(\mathbb{R}^N)$, so

$$\begin{aligned}|D^\beta(K_\alpha * \psi)(x)| &= |K_\alpha * D^\beta\psi(x)| = \left|\int_{\mathbb{R}^N} K_\alpha(x-y)D^\beta\psi(y)dy\right|\\ &\le \int_{\mathbb{R}^N} \frac{|x-y|^k}{|x-y|^N}|D^\beta\psi(y)|dy \le \int_{\mathbb{R}^N} \frac{|x-y|^{k+1}}{|x-y|^{N+1}}|D^\beta\psi(y)|dy\\ &\le c_{k,\beta,\psi}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+1}}dy < \infty.\end{aligned}$$

That way, $K_\alpha * \psi \in \Lambda^r(\mathbb{R}^N)$ for any $r > 0$. If $0 < p < 1$ and $\gamma_p := N(p^{-1} - 1)$, hence by duality

$$|\langle f_j - f_k, \psi\rangle| \le \sum_{|\alpha|=k} |\langle D^\alpha(f_j - f_k), K_\alpha * \psi\rangle| \le \|f_j - f_k\|_{h^{k,p}} \sum_{|\alpha|=k} \|K_\alpha * \psi\|_{\Lambda^{\gamma_p}}$$

showing that $\{f_j\}_j$ is a Cauchy sequence in $S'(\mathbb{R}^N)$.

Summarizing, $f \in S'(\mathbb{R}^N)$ belongs to $h^{k,p}(\mathbb{R}^N)$ if there exists a sequence $f_j \in C_c^\infty(\mathbb{R}^N)$ such that $\{f_j\}_j$ is a Cauchy sequence for the norm $g \mapsto \|D^k g\|_{h^p}$ and $f_j \to f$ in $S'(\mathbb{R}^N)$, exactly as shown. It follows that if $f \in h^{k,p}(\mathbb{R}^N)$ then $D^k f \in h^p(\mathbb{R}^N)$ and we set $\|f\|_{h^{k,p}} := \|D^k f\|_{h^p}$. When $p = 1$ the proof is similar by considering $bmo(\mathbb{R}^N)$ instead of $\Lambda^r(\mathbb{R}^N)$.

We point out this is a Hausdorff topology, that means it is sufficient to show if a polynomial $\theta \in \mathscr{P}_{k-1}$ (the class of polynomials with order less than or equal to $k-1$) belongs to $h^{k,p}(\mathbb{R}^N)$ then $\theta = 0$. In fact, let $(f_j)_j \in C_c^\infty(\mathbb{R}^N)$ be a Cauchy sequence with norm $g \mapsto \|D^k g\|_{h^p}$ that converges to θ in $S'(\mathbb{R}^N)$. Thus, $(D^k f_j)_j$ is a Cauchy sequence in $h^p(\mathbb{R}^N)$ and, in particular, $\|D^k f_j\|_{h^p} \to \|D^k\theta\|_{h^p} = 0$. Now consider the alternate sequence $(f_1, 0, f_2, 0, \dots)$, that is Cauchy sequence of test functions for the norm $g \mapsto \|D^k g\|_{h^p}$ that converges in $S'(\mathbb{R}^N)$ to 0 and has a subsequence converging to θ, thus showing that $\theta = 0$.

Alternatively, we may present another natural definition of Hardy-Sobolev spaces (see [31] for the case $k = 1$). Consider the space of tempered distributions $f \in S'(\mathbb{R}^N)$ such that $D^k f \in h^p(\mathbb{R}^N)$ equipped with the seminorm $\|D^k f\|_{h^p}$ and denote it by $\mathrm{h}^{k,p}(\mathbb{R}^N)$. In order to turn it into a Hausdorff space, we must take the quotient $\mathrm{h}^{k,p}(\mathbb{R}^N)/\{\mathscr{P}_{k-1}\}$. Moreover, we have shown that there is exactly one element of $h^{k,p}(\mathbb{R}^N)$ in each equivalence class in $\mathrm{h}^{k,p}(\mathbb{R}^N)/\{\mathscr{P}_{k-1}\}$, so we get the inclusion $h^{k,p}(\mathbb{R}^N) \subset \mathrm{h}^{k,p}(\mathbb{R}^N)/\{\mathscr{P}_{k-1}\}$. The second inclusion follows using the same arguments from [31, Remark 8]. In summary, any $f \in \mathrm{h}^{k,p}(\mathbb{R}^N)$ can be it can be approximated by choose smooth convolution approximations $\{f_j\}_j$ such that $\|D^k f_j - D^k f\|_{h^p(\mathbb{R}^N)} \leq 2^{-j}$ for all $j \geq 1$. So, we conclude the isometry $h^{k,p}(\mathbb{R}^N) \simeq \mathrm{h}^{k,p}(\mathbb{R}^N)/\{\mathscr{P}_{k-1}\}$.

The Hardy-Sobolev spaces $h^{k,p}(\mathbb{R}^N)$ defined here are the *homogeneous* localizable (also called local or nonhomogeneous) Hardy-Sobolev spaces which in the literature are often denoted by $\dot{h}^{k,p}(\mathbb{R}^N)$. Throughout this thesis we will denote $\|\cdot\|_{h^{k,p}_{<}} = \sum_{j=0}^{k} \|\cdot\|_{h^{j,p}}$.

2.2 Extension of Poincaré inequality on Hardy-Sobolev spaces for $\Omega = \mathbb{R}^N$

The goal of this section will be to apply the Proposition 1.59 to functions in $L^1_{loc}(\mathbb{R}^N) \cap h^{m,p}(\mathbb{R}^N)$, with $0 < p \leq 1$, in order to obtain a pointwise control to get our extension of the Poincaré inequality, stated in Theorem 2.2.

As seen in Proposition 1.63, $h^p(\mathbb{R}^N) \subset H^p_{loc}(\mathbb{R}^N)$, so if $f \in L^1_{loc}(\mathbb{R}^N) \cap h^{m,p}(\mathbb{R}^N)$ then in particular $\partial^\alpha f \in H^p_{loc}$ for all $|\alpha| = m$ and in fact we can apply the Proposition 1.59 in our case of interest.

Let $0 < p \leq 1$, $m \in \mathbb{N}$ and $f \in L^1_{loc}(\mathbb{R}^N)$ such that $\partial^\beta f \in h^p(\mathbb{R}^N)$ for all $|\beta| = m$. For each $B(x,t) \subset \mathbb{R}^N$ with $0 < t < T$ for some fixed $T > 0$, there exists a polynomial $P_{x,t,f} \in \mathscr{P}_{m-1}$ and a constant $C_{m,N,T,p} > 0$ satisfying

$$|f(y) - P_{x,t,f}(y)| \leq C_{m,N,T,p}\, t^m \sum_{|\alpha|=m} (\partial^\beta f)^*_{N,a}(y), \quad a := 2\sqrt{N}t, \tag{2.2.1}$$

for $y \in B(x,t)$ almost everywhere.

In fact, for each open ball $B(x,t) \subset \mathbb{R}^N$ and $f \in L^1_{loc}(\mathbb{R}^N)$ we can consider the open cube $Q := Q(x,2t)$ centered at x and side $\ell(Q) = 2t$ (since $B(x,t) \subset Q(x,2t)$) and denote $\tilde{f} = f\, \mathbb{X}_{B(x,t)}$. Since $\tilde{f} \in \mathscr{D}'(Q)$, Lemma 1.58 ensures that there exits a test function ϕ_Q such that $P_{Q,\tilde{f}}(y) = \langle \tilde{f}, \phi_Q(y,\cdot) \rangle$ is a polynomial of degree at most $m-1$. In particular,

$$P_{Q,\tilde{f}}(y) = \int_Q \tilde{f}(z)\phi_Q(y,z)dz = \int_{B_x^t} f(z)\phi_Q(y,z)dz.$$

By simplicity, we denote $P_{x,t,f} := P_{Q(x,2t),\tilde{f}}$ and we can apply Proposition 1.59 for $\tilde{f} \in \mathscr{D}'(\mathbb{R}^N)$, noting that $[\tilde{f}] = f$ in $B(x,t)$ almost everywhere, choosing $L = N$ and getting the inequality (2.2.1). Our second step will be to integrate inequality (2.2.1) into the ball $B(x,t)$ for any $\alpha \geq 1$, getting

$$\left(\fint_{B(x,t)} \left| f(y) - P_{x,t,f}(y) \right|^{\alpha} dy \right)^{\frac{1}{\alpha}} \leq \left(\fint_{B(x,t)} \left[C_{m,N,T,p}\, t^m \sum_{|\beta|=m} (\partial^{\beta} f)^{*}_{N,a}(y) \right]^{\alpha} dy \right)^{\frac{1}{\alpha}}.$$

Thus,

$$\begin{aligned} \left(\fint_{B(x,t)} \left| \frac{1}{t^m} \left[f(y) - P_{x,t,f}(y) \right] \right|^{\alpha} dy \right)^{\frac{1}{\alpha}} &\leq \frac{C}{|B_x^t|^{\frac{1}{\alpha}}} \left\| \sum_{|\beta|=m} (\partial^{\beta} \tilde{f})^{*}_{N,2\sqrt{N}t} \right\|_{L^{\alpha}(B_x^t)} \\ &\leq \frac{C}{|B_x^t|^{\frac{1}{\alpha}}} \left\| \sum_{|\beta|=m} (\partial^{\alpha} \tilde{f})^{*}_{N,2\sqrt{N}T} \right\|_{L^{\alpha}(B_x^t)} \\ &\leq \frac{C}{|B_x^t|^{\frac{1}{\gamma}}} \left\| \sum_{|\beta|=m} (\partial^{\beta} \tilde{f})^{*}_{N,2\sqrt{N}T} \right\|_{L^{\gamma}(B_x^t)}, \end{aligned}$$

for any γ satisfying $\gamma = \alpha$ if $\alpha > 1$ and $\gamma > \alpha$ if $\alpha = 1$. By the proof of inequality (1.6.1),

$$\begin{aligned} \left\| \sum_{|\beta|=m} (\partial^{\beta} \tilde{f})^{*}_{N,2\sqrt{N}T} \right\|_{L^{\gamma}(B_x^t)} &\leq C_{\phi,T,N} \sum_{|\beta|=m} \left\| M_{\frac{1}{2}} \left[(\partial^{\beta} \tilde{f})^{+,1}_{\phi,\mathbb{R}^N} \right] \right\|_{L^{\gamma}(B_x^t)} \\ &\leq C_{\phi,T,N} \sum_{|\beta|=m} \left\| M_{\frac{1}{2}} \left[(\partial^{\beta} \tilde{f})^{+,1}_{\phi,\mathbb{R}^N} \right] \right\|_{L^{\gamma}(\mathbb{R}^N)} \\ &\lesssim \sum_{|\beta|=m} \left\| (\partial^{\beta} \tilde{f})^{+,1}_{\phi,\mathbb{R}^N} \right\|_{L^{\gamma}(\mathbb{R}^N)}, \text{ since } 1/2 < \gamma \\ &\lesssim \sum_{|\beta|=m} \left\| \partial^{\beta} \tilde{f} \right\|_{L^{\gamma}(\mathbb{R}^N)}, \text{ since } 1 < \gamma \\ &\lesssim \|D^m f\|_{L^{\gamma}(B_x^t)}, \end{aligned}$$

Summarizing, the result can be stated as follows

Theorem 2.2 (Extension of Poincaré inequality). *Let $f \in L^1_{loc}(\mathbb{R}^N)$ such that $\partial^{\beta} f \in h^p(\mathbb{R}^N)$ for all $|\beta| = m$ and for some $0 < p \leq 1$. For each $B(x,t) \subset \mathbb{R}^N$ with $0 < t < T$, there exist a polynomial $P_{x,t,f} \in \mathscr{P}_{m-1}$ and a constant $C = C(N,p,m,T,\alpha) > 0$ such that*

$$\left(\fint_{B_x^t} \left| \frac{1}{t^m} [f(y) - P_{x,t,f}(y)] \right|^{\alpha} dy \right)^{\frac{1}{\alpha}} \leq C \left(\fint_{B_x^t} |D^m f(y)|^{\gamma} dy \right)^{\frac{1}{\gamma}}, \tag{2.2.2}$$

where $\gamma = \alpha > 1$ or $\gamma > \alpha = 1$.

2.3 Local estimates of elliptic linear differential operators in Hardy-Sobolev spaces

The goal of this section is to show that if $A(\cdot,D)$ is an elliptic linear differential operator then for each $x_0 \in \Omega$ there exist an open neighborhood $U \subset \Omega$ and a positive constant $C = C(p,N,m)$ such that

$$\|u\|_{h^{m,p}_{<}} := \sum_{j=0}^{m} \|u\|_{h^{j,p}} = \sum_{|\beta|\leq m} \|\partial^{\beta}u\|_{h^p} \leq C\|A(\cdot,D)u\|_{h^p}, \quad \forall u \in C_c^{\infty}(U,E). \tag{2.3.1}$$

For that, suppose $A(\cdot,D)$ an elliptic operator, A_ν its principal part and consider the order $2m$ differential operator

$$\Delta_A := A_\nu^*(\cdot,D) \circ A_\nu(\cdot,D),$$

which may be regarded as an elliptic pseudo-differential operator with symbol in the Hörmander class $S^{2m}_{1,0}(\Omega \times \mathbb{R}^N; E, E^*)$ (see Section 1.4 for more details). Thus, it follows by Theorem 1.45 that there exist pseudo-differential operators $q(\cdot,D) \in OpS^{-2m}_{1,0}(\Omega; E^*, E)$ (parametrix) and $r(\cdot,D) \in OpS^{-\infty}(\Omega; E)$ (regularizing) such that

$$u(x) = q(x,D)\Delta_A u(x) + r(x,D)u(x), \quad \forall\, u \in C^{\infty}(\Omega,E). \tag{2.3.2}$$

Proposition 2.3. *Assume $N \geq 2$, $m \geq 1$ and $A_\nu(\cdot,D)$ as before. Then, for every point $x_0 \in \Omega$, there exist an open ball $B = B(x_0,\ell) \subset \Omega$ and a constant $C = C(B,p,N) > 0$ such that*

$$\|u\|_{h^{m,p}_{<}} \leq C\|A_\nu(\cdot,D)u\|_{h^p}, \quad \forall u \in C_c^{\infty}(B,E). \tag{2.3.3}$$

Proof. Fixed $x_0 \in \Omega$, let $\ell > 0$ such that $B = B(x_0,\ell) \subset \Omega$. Firstly, we will prove that, for each fixed $j = \{0,1,\ldots,m\}$, there exists a constant $C = C(p,N,j,\ell)$ such that

$$\|u\|_{h^p} \leq C\left\|D^j u\right\|_{h^p}, \text{ for all } u \in C_c^{\infty}(B,E). \tag{2.3.4}$$

For each $u \in C_c^{\infty}(B(x_0,\ell))$, it follows from Theorem 1.50 and the proof of Theorem 1.51 that $\|u\|_{L^p} \leq C_0\ell\,\|Du\|_{L^p}$, for all $p > 1$ and it follows from (1.5.3) that

$$\|u\|_{h^p} \leq C_0\ell\,\|Du\|_{h^p}, \text{ for all } 0 < p \leq 1.$$

Bootstrapping this inequalities for all $j \in \mathbb{N}$ we get

$$\|u\|_{h^p} \leq C_0\ell^j\left\|D^j u\right\|_{h^p}, \text{ for all } 0 < p < \infty,$$

and we just denote $C = C_0\ell^j$.

Now, by identity (2.3.2) we may write, for each β with $|\beta| = j$,

$$\partial^{\beta}u(x) = \tilde{q}(x,D)[A_\nu(x,D)u(x)] + \tilde{r}(x,D)u(x)$$

where $\tilde{q}(x,D) := \partial^\beta \circ q(x,D) \circ A_\nu^*(x,D) \in OpS_{1,0}^{-m+j}(B)$ and $\tilde{r}(x,D) := \partial^\beta \circ r(x,D) \in OpS^{-\infty}(B)$. Thus

$$\|\partial^\beta u\|_{h^p} \leq \|\tilde{q}(\cdot,D)A_\nu(\cdot,D)u\|_{h^p} + \|\tilde{r}(\cdot,D)u\|_{h^p} \lesssim \|A_\nu(\cdot,D)u\|_{h^p} + \|u\|_{h^p},$$

since $\tilde{q}(\cdot,D), \tilde{r}(\cdot,D)$ are bounded operators from h^p to itself for $0 < p < \infty$, by Remark 1.37.

Using (2.3.4) and shrinking ℓ to absorb the second term on the right-hand side, we conclude the desired estimate. □

Remark 2.4. *Let $A(\cdot,D)$ as before and suppose that all coefficients a_α are bounded at some neighborhood of $x_0 \in \Omega$, namely $B = B(x_0,\ell) \subseteq \Omega$, and consider $C := \sum_{|\alpha|<m} \|a_\alpha\|_{L^\infty(B)}$. For $0 < p < \infty$ we have, from* (2.3.4),

$$\begin{aligned}
\|A_\nu(\cdot,D)u\|_{h^p} &\leq \|A(\cdot,D)u\|_{h^p} + C\sum_{|\alpha|<m} \|\partial^\alpha u\|_{h^p} \\
&\lesssim \|A(\cdot,D)u\|_{h^p} + \sum_{|\alpha|<m} |B|^{(m-|\alpha|)/N} \|u\|_{h^{m,p}}. \\
&\lesssim \|A(\cdot,D)u\|_{h^p} + |B|^{1/N} \|u\|_{h^{m,p}}.
\end{aligned}$$

Using (2.3.3) *and absorbing the constant, decreasing the radius if necessary, we have*

$$\|A_\nu(\cdot,D)u\|_{h^p} \leq C\|A(\cdot,D)u\|_{h^p}, \quad \forall u \in C_c^\infty(B,E), \tag{2.3.5}$$

for $0 < p < \infty$ and $C = C(B,p) > 0$.

By the inequality (2.3.5) and the Proposition 2.3 we conclude the control (2.3.1).

2.4 New smooth atomic decomposition on Hardy-Sobolev spaces

2.4.1 Relation between the atomic decompositions of H^p and h^p

Let us first recall the standard atomic decomposition on $h^p(\mathbb{R}^N)$ due to Goldberg in ([22, Lemmas 4 and 5]). By inequality (1.2.1) any $f \in h^p(\mathbb{R}^N)$ can be decompose as $f = f_1 + f_2$ so that

$$\|f_1\|_{H^p} + \|f_2\|_{h^p} \leq C\|f\|_{h^p},$$

with $C > 0$ independent of f. The choice of f_1 and f_2 is relatively simple: we can fix $\varphi \in C_c^\infty(\mathbb{R}^N)$ satisfying $\int \varphi(x)\,dx = 1$ and $\int x^\alpha \varphi(x)\,dx = 0$ for all $0 < |\alpha| \leq L(p)$ sufficiently large (bigger than $N_p = \lfloor N(p^{-1}-1)\rfloor$) and write $f_1 := f - \varphi * f \in H^p(\mathbb{R}^N)$ and $f_2 := \varphi * f \in h^p(\mathbb{R}^N)$. Thus, the atomic decomposition of f is obtained by choosing an H^p-atomic decomposition of f_1 and an h^p-atomic decomposition of f_2, which will be discussed next.

Choose $0 \le \psi(x) \le 1 \in C_c^\infty(\mathbb{R}^N)$ supported in the cube $Q_0^2 := Q(0,2)$ centered at the origin with side length 2 such that $\sum_{k\in\mathbb{Z}^N} \psi(x-k) = 1$ for all $x \in \mathbb{R}^N$. Writing $\psi_k(x) := \psi(x-k) \in C_c^\infty(Q_k^2)$, we may define

$$\mu_k := |Q_k^2|^{1/p} \|\psi_k f_2\|_{L^\infty(Q_k^2)}$$

and

$$a_k := \frac{\psi_k f_2}{\mu_k}\text{ , for } \psi_k f_2 \neq 0,$$

obtaining the following decomposition

$$f_2 = \sum_k \psi_k f_2 = \sum_k \mu_k a_k.$$

Clearly $\psi_k f_2 \in C_c^\infty(Q_k^2)$ and a_k is a rough atom, since $\|a_k\|_{L^\infty} \le |Q_k^2|^{-1/p}$. We claim that

$$\sum_k |\mu_k|^p \le C\|f\|_{h^p}^p. \tag{2.4.1}$$

To show this control, we recall that given $R > 0$ there exists $C > 0$ such that $|\varphi * f(x)| \le C\mathscr{M}f(y)$, for $|x-y| \le R$, where $\mathscr{M}$ denotes the Grand Maximal Function and then

$$\sup_{Q_k^2} |\varphi * f(x)|^p \le C\frac{1}{|Q_k^2|}\int_{Q_k^2} [\mathscr{M}f(y)]^p\, dy,$$

that implies $\sum_k |Q_k^2| \sup_{Q_k^2} |\varphi * f(x)|^p \le C\|\mathscr{M}f\|_{L^p}^p \simeq \|f\|_{h^p}^p$, because the family of cubes $\{Q_k^2\}$ has the bounded intersection property. This concludes (2.4.1).

In the previous proof, we should simplify taking $\psi := \mathbb{X}_{B(0,1)}$. The advantage of using a cut-off function is the following: for $f \in C_c^\infty(\mathbb{R}^N)$ we may conclude $a_k \in C_c^\infty(Q_k^2)$ and the series $\sum_k \mu_k a_k$ reduces to a finite sum. In this particular setting, we may obtain also a finite atomic decomposition for bounded atoms for f_1 in H^p (see [35, Theorem 3.1 and Remarks 3.2 and 3.3] for more details). Invoking the atomic decomposition on H^p we may write

$$f = \sum_k \mu_{1,k} a_{1,k} + \sum_k \mu_{2,k} a_{2,k}, \tag{2.4.2}$$

where $a_{1,k}$ are H^p-atoms, $a_{2,k}$ are rough atoms and

$$\sum_k \left(|\mu_{1,k}|^p + |\mu_{2,k}|^p\right) \le C\|f\|_{h^p}^p.$$

Additionally, we may take an atomic decomposition where the sum (2.4.2) is finite assuming, for instance, $f \in (h^p \cap C_c^\infty)(\mathbb{R}^N)$.

Remark 2.5. *If $f \in S(\mathbb{R}^N)$ then clearly $f_2 := \varphi * f \in S(\mathbb{R}^N)$ and the partial sums $\sum_{|k|\le M} \mu_k a_k$ converge to f_2 in $S(\mathbb{R}^N)$ and a fortiori in $\Lambda^r(\mathbb{R}^N)$ for any $r > 0$. In fact, first we claim $\sum_{|k|\le j} \psi_k f_2$ converges to*

f_2 when $j \to \infty$ in $S(\mathbb{R}^N)$, where ψ_k was given previously, i.e. for each $\alpha, \beta \in \mathbb{Z}_+^N$, the sequence

$$x^\alpha \partial^\beta \left(\sum_{|k| \le j} \psi_k(x) f_2(x) \right)$$

converges uniformly to $x^\alpha \partial^\beta f_2(x)$ when $j \to \infty$. We point out that, by the compact support of ψ, for each $x \in \mathbb{R}^N$ there are at most 2^N indexes $k \in \mathbb{Z}^N$ such that $\psi_k(x) := \psi(x-k) \neq 0$. Thus, the sum $\sum_{k \in \mathbb{Z}^N} \psi(x-k) = 1$ is always finite and for each $\gamma \in \mathbb{Z}^N - \{0\}$ we have

$$\sum_{k \in \mathbb{Z}^N} \partial^\gamma \psi_k(x) = \sum_{k \in \mathbb{Z}^N} \partial^\gamma \psi(x-k) = \partial^\gamma \left(\sum_{k \in \mathbb{Z}^N} \psi(x-k) \right) = 0.$$

The conclusion of the uniform convergence follows by the identity

$$\partial^\beta \left(\sum_{|k| \le j} \psi_k(x) f_2(x) \right) = \left(\sum_{|k| \le j} \psi_k(x) \right) \partial^\beta f_2(x) + \sum_{0 < \gamma \le \beta} \binom{\beta}{\gamma} \left(\sum_{|k| \le j} \partial^\gamma \psi_k(x) \right) \partial^{\beta - \gamma} f_2(x).$$

The second claim follows by canonical embedding, for instance if $0 < r < 1$ then for any $\delta > 0$ we have

$$\begin{aligned}
[f]_r &= \sup_{\substack{x,h \in \mathbb{R}^N \\ h \neq 0}} \frac{|f(x+h) - f(x)|}{|h|^r} \\
&\le \sup_{\substack{x \in \mathbb{R}^N \\ h \in B(0,\delta) - \{0\}}} \frac{|f(x+h) - f(x)|}{|h|^r} + \sup_{\substack{x \in \mathbb{R}^N \\ h \in B(0,\delta)^c}} \frac{|f(x+h) - f(x)|}{|h|^r} \\
&\le \|Df\|_{L^\infty} \sup_{\substack{x \in \mathbb{R}^N \\ h \in B(0,\delta) - \{0\}}} |h|^{1-r} + \delta^{-r} \sup_{\substack{x \in \mathbb{R}^N \\ h \in B(0,\delta)^c}} |f(x+h)| + |f(x)| \\
&\le \delta^{1-r} \|Df\|_{L^\infty} + 2\delta^{-r} \|f\|_{L^\infty}.
\end{aligned} \tag{2.4.3}$$

In particular, if we take $\delta = 1$ then

$$[f]_r \le \|Df\|_{L^\infty} + 2\|f\|_{L^\infty} \le 4 \left(\|Df\|_{L^\infty} + \|f\|_{L^\infty} \right).$$

In a similar way,

$$\begin{aligned}
[f]_1 &= \sup_{\substack{x,h \in \mathbb{R}^N \\ h \neq 0}} \frac{|f(x+h) + f(x-h) - 2f(x)|}{|h|} \le 2 \sup_{\substack{x,h \in \mathbb{R}^N \\ h \neq 0}} \frac{|f(x+h) - f(x)|}{|h|} \\
&\le 2 \left(\sup_{\substack{x \in \mathbb{R}^N \\ h \in B(0,1) - \{0\}}} \frac{|f(x+h) - f(x)|}{|h|} + \sup_{\substack{x \in \mathbb{R}^N \\ h \in B(0,1)^c}} \frac{|f(x+h) - f(x)|}{|h|} \right) \\
&\le 2 \left(\|Df\|_{L^\infty} + \sup_{\substack{x \in \mathbb{R}^N \\ h \in B(0,1)^c}} |f(x+h)| + |f(x)| \right) \\
&\le 4 \left(\|Df\|_{L^\infty} + \|f\|_{L^\infty} \right).
\end{aligned}$$

Furthermore, if $r > 1$, then we may write $r = k + s$, for some $k \in \mathbb{N}$ and $0 < s \leq 1$ and

$$[f]_r = \sum_{|\alpha|=k} [\partial^\alpha f]_s \leq 4 \left(\left\| D^{k+1} f \right\|_{L^\infty} + \left\| D^k f \right\|_{L^\infty} \right).$$

Summarizing, if f and $Df \in L^\infty(\mathbb{R}^N)$ then $f \in \Lambda^r(\mathbb{R}^N)$ for all $0 < r \leq 1$, and if $f, Df, ..., D^k f \in L^\infty(\mathbb{R}^N)$ then $f \in \Lambda^r(\mathbb{R}^N)$ for all $r > 1$, in which

$$k = \begin{cases} \lfloor r \rfloor, & \text{if } r > \lfloor r \rfloor, \\ r - 1, & \text{if } r = \lfloor r \rfloor. \end{cases}$$

Returning to the proof, denote $R_j(x) := \sum_{|k| \leq j} \psi_k(x) f_2(x) - f_2(x)$. Thus, by the uniform convergence of $(x^\alpha \partial^\beta R_j)_j$ to zero for all $\alpha, \beta \in \mathbb{Z}_+^N$, we have that

$$\|R_j\|_{\Lambda^r} = [R_j]_{\Lambda^r} + \|R_j\|_{L^\infty} \leq 4 \left(\left\| D^{k+1} f \right\|_{L^\infty} + \left\| D^k f \right\|_{L^\infty} \right) + \|R_j\|_{L^\infty}$$

converge to zero when $j \to \infty$, for all $r > 0$.

2.4.2 A generalized Calderón-Zygmund decomposition

Consider $m \in \mathbb{N}$ and $N/(N+m) < p < 1$, we denote

$$\gamma_p := N \left(\frac{1}{p} - 1 \right), \ N_p := \lfloor \gamma_p \rfloor \text{ and } r_p \ := \ \gamma_p - N_p.$$

Note that $0 < \gamma_p < m$, $N_p \leq m - 1$ and $0 \leq r_p < 1$. We recall that if $N_p < \gamma_p$ then $f \in \Lambda^{\gamma_p}(\mathbb{R}^N)$ means $\partial^\alpha f \in \Lambda^{r_p}(\mathbb{R}^N)$ for all $|\alpha| = N_p$, and if $N_p = \gamma_p$ then $\partial^\alpha f \in \Lambda^1(\mathbb{R}^N)$ for all $|\alpha| = \gamma_p - 1$. Furthermore, when $p = 1$ we consider $bmo(\mathbb{R}^N)$ instead of $\Lambda^{\gamma_p}(\mathbb{R}^N)$, (for more details, see Section 1.2 of Chapter 1).

Next, we will announce the goal of this section: a particular version of the Calderón-Zygmund decomposition, which establishes controls for higher-order derivatives of the so-called "bad part" of the decomposition. But first, let us fix some notations in the spirit of Stein's book [44, Chapter III, Section 2], that is: $\mathscr{M}_0 = M_\varphi$ is the maximal function associated to a single function $\varphi \in S(\mathbb{R}^N)$ with $\int \varphi \neq 0$, *i.e.* $M_\varphi f(x) = \sup_{t>0} |f * \varphi_t(x)|$, while $\mathscr{M}$ denotes the Grand Maximal Function associated to collection $\mathfrak{S} \subset S(\mathbb{R}^N)$ of rapidly decreasing functions:

$$\mathscr{M} f(x) = \sup_{\phi \in \mathfrak{S}} M_\phi f(x)$$

and $\mathfrak{S} = \{\phi \in S(\mathbb{R}^N) : \left\| x^\alpha \partial^\beta \phi \right\|_{L^\infty} \leq 1$, for all $|\alpha|, |\beta| \leq C_{N,p}\}$, for some $C_{N,p}$ sufficiently large (bigger than N/p). We may assume without loss of generality that $\mathfrak{S}$ is a bounded subset of $L^1(\mathbb{R}^N)$, in particular, if $u \in L^\infty(\mathbb{R}^N)$ then $\mathscr{M} u \in L^\infty(\mathbb{R}^N)$.

Theorem 2.6 (Higher order Calderón-Zygmund decomposition)**.** *Let $m \in \mathbb{N}$ and $N/(N+m) < p < 1$. For each $f \in (C_c^m \cap H^p)(\mathbb{R}^N)$ and $\alpha > 0$ there exist a decomposition $f = g + b$, $b := \sum_{k=1}^\infty \beta_k$ and a collection of cubes $\{Q_k^*\}_k$ with sides of length ℓ_k parallel to the axes such that* *(i)* $|g(x)| \leq c\,\alpha$ *almost everywhere;*

(ii) Each $\beta_k \in C_c^m(Q_k^*)$ *satisfies the following properties:*

$$\|D^i\beta_k\|_{L^\infty} \leq c\sum_{j=0}^{i} \ell_k^{m+j-i}\,\left\|D^{m+j}f\right\|_{L^\infty}, \quad \forall i \in \{0,1,...,m-1\}, \tag{2.4.4}$$

$$\|D^i\beta_k\|_{L^\infty} \leq c\left\|D^i f\right\|_{L^\infty}, \quad \forall i \in \{1,2,...,m\}, \tag{2.4.5}$$

$$\int_{\mathbb{R}^N} \mathscr{M}_0(\beta_k)^p\,dx \leq c\int_{Q_k^*} \mathscr{M}(f)^p\,dx, \tag{2.4.6}$$

$$\int x^\gamma \beta_k(x)dx = 0, \quad \forall |\gamma| \leq m-1. \tag{2.4.7}$$

(iii) the $\{Q_k^*\}$ *have the bounded intersection property and*

$$\mathscr{O} := \{x :\ \mathscr{M}f(x) > \alpha\} = \bigcup_k Q_k^*.$$

Proof. We define $b = \sum_k \beta_k$ and $g = f - b$ analogously as in the generalized Calderón-Zygmund decomposition [44, p. 101]. Thus, fix $0 \leq \eta_k(x) \leq 1$ a partition of unity for $\mathscr{O}$ subordinated for the covering $\{Q_k^*\}$ satisfying $\operatorname{supp}\eta_k \subset Q_k^*$ and $|\partial^\beta \eta_k| \leq C_\beta \ell_k^{-|\beta|}$ and define

$$\beta_k(x) := (f(x) - P_{f,k}(x))\eta_k(x), \tag{2.4.8}$$

where $P_{f,k}(x) = \displaystyle\int f(y)\phi_{Q_k}(x,y)dy$ is the polynomial with degree at most $m-1$ given by Lemma 1.58 and

$$\phi_{Q_k}(x,y) = \ell_k^{-N}\left[\sum_{i=1}^m \pi_i\left(\frac{x-x_k}{\ell_k}\right)\overline{\pi_i\left(\frac{y-x_k}{\ell_k}\right)}\right]\eta_k(y) = \left[\sum_{i=1}^m e_i(x)\overline{e_i(y)}\right]\tilde{\eta}_k(y).$$

Here $\tilde{\eta}_k := \eta_k\left(\int \eta_k dx\right)^{-1}$ and $e_i(x) := \left(\int \eta_k dx\right)\ell_k^{-N/2}\pi_i\left(\frac{x-x_k}{\ell_k}\right)$ and then $\{e_1,...,e_m\}$ defines an orthonormal basis of $(\mathscr{P}_{m-1}, \langle\cdot,\cdot\rangle_S)$, where

$$\langle p,q\rangle_S := \int p\bar{q}\tilde{\eta}_k.$$

The conclusion of the claims (i), (iii) and of the inequality (2.4.6) follows *is bis idem* as in the [44] and will be omitted. The property (2.4.7) follows directly from (2.4.8), precisely

$$\begin{aligned}
\int P_{f,k}(x)\eta_k(x)\overline{e_{j_0}}(x)\,dx &= \int\left(\int f(y)\left[\sum_{i=1}^m e_i(x)\overline{e_i}(y)\right]\tilde{\eta}_k(y)\,dy\right)\eta_k(x)\overline{e_{j_0}}(x)\,dx \\
&= \int f(y)\tilde{\eta}_k(y)\left[\sum_{i=1}^m \overline{e_i}(y)\left(\int e_i(x)\,\eta_k(x)\overline{e_{j_0}}(x)\,dx\right)\right]dy \\
&= \int f(y)\eta_k(y)\left[\sum_{i=1}^m \overline{e_i}(y)\left(\int e_i(x)\,\tilde{\eta}_k(x)\overline{e_{j_0}}(x)\,dx\right)\right]dy \\
&= \int f(y)\eta_k(y)\overline{e_{j_0}}(y)\,dy,
\end{aligned}$$

for each $j_0 \in \{1,...,m\}$, thus $\int \beta_k(x)q(x)dx = 0$ for all $q \in \mathscr{P}_{m-1}$. Since $N_p \leq m-1$ we get the desired property.

It follows from inequality (1.6.2) and the uniform control $f^*_{L,a}(x) \leq C(N)\|f\|_{L^\infty}$ for all $f \in L^\infty(\mathbb{R}^N)$ that

$$|\beta_k(x)| \leq |f - P_{f,k}|(x) \leq C_{m,p,L}\, \ell_k^m \sum_{|\alpha|=m} (\partial^\alpha f)^*_{L,\sqrt{N}\ell_k}(x) \lesssim \ell_k^m \|D^m f\|_{L^\infty},$$

proving the inequality (2.4.4) for $i = 0$.

For each fixed $k \in \mathbb{N}$ and $j \in \{1,2,...,m\}$, the Taylor polynomial of f centered at x_k with degree $j-1$ will denote by $P^{j-1}_{x_k,f}$. Remember that

$$\left|f(x) - P^{j-1}_{x_k,f}(x)\right| \lesssim |x - x_k|^j \sup_{|\gamma|=j} |\partial^\gamma f(\tilde{x_k})|,$$

for some $\tilde{x_k} \in [x, x_k]$. In particular, $\left|f(x) - P^{j-1}_{x_k,f}(x)\right| \lesssim \ell_k^j \|D^j f\|_{L^\infty}$ for each $x \in Q_k^*$. It follows from the Leibniz rule that, for each $\alpha \in \mathbb{Z}^N_+$ with $|\alpha| = j$,

$$\begin{aligned} |\partial^\alpha \beta_k(x)| &\lesssim \sum_{0\leq\gamma<\alpha} |\partial^\gamma (f - P_{f,k})(x) \partial^{\alpha-\gamma}\eta_k(x)| + \sum_{|\gamma|=j} |\partial^\gamma f(x)\eta_k(x)| \\ &\lesssim \sum_{0\leq\gamma<\alpha} \ell_k^{|\gamma|-j} |\partial^\gamma (f - P_{f,k})(x)| + \|D^j f\|_{L^\infty} \\ &\lesssim \sum_{0\leq\gamma<\alpha} \ell_k^{|\gamma|-j} \left[\underbrace{|\partial^\gamma f(x) - \partial^\gamma P^{j-1}_{x_k,f}(x)|}_{(A)} + \underbrace{|\partial^\gamma (P^{j-1}_{x_k,f} - P_{f,k})(x)|}_{(B)} \right] + \|D^j f\|_{L^\infty}. \end{aligned} \tag{2.4.9}$$

Before we estimate (A), remember that $\partial^\gamma P^{j-1}_{x_k,f} = P^{j-|\gamma|-1}_{x_k,\partial^\gamma f}$, for all $|\gamma| < j$. Indeed,

$$\begin{aligned} \partial^\gamma P^{j-1}_{x_k,f}(x) &= \partial^\gamma \left[\sum_{0\leq|\alpha|\leq j-1} \frac{\partial^\alpha f(x_k)}{\alpha!}(x - x_k)^\alpha \right] \\ &= \sum_{0\leq|\alpha|\leq j-1} \frac{\partial^\alpha f(x_k)}{\alpha!} \partial^\gamma [(x - x_k)^\alpha] \\ &= \sum_{\substack{0\leq|\alpha|\leq j-1 \\ \gamma<\alpha}} \frac{\partial^\alpha f(x_k)}{\alpha!} \binom{\alpha}{\gamma} \gamma! (x - x_k)^{\alpha-\gamma} \\ &= \sum_{\substack{0\leq|\alpha|\leq j-1 \\ \gamma<\alpha}} \frac{\partial^\alpha f(x_k)}{(\alpha-\gamma)!}(x - x_k)^{\alpha-\gamma} \\ &= \sum_{0\leq|\theta|\leq j-1-\gamma} \frac{\partial^{\theta+\gamma} f(x_k)}{\theta!}(x - x_k)^\theta = P^{j-1-|\gamma|}_{x_k,\partial^\gamma f}(x). \end{aligned}$$

Thus,

$$(A) = |\partial^\gamma f(x) - P^{j-|\gamma|-1}_{x_k,\partial^\gamma f}(x)| \lesssim \ell_k^{j-|\gamma|} \|D^{j-|\gamma|}(\partial^\gamma f)\|_{L^\infty} \simeq \ell_k^{j-|\gamma|} \|D^j f\|_{L^\infty}.$$

On the other hand, it follows from (iii) of Lemma 1.58 that

$$P^{j-1}_{x_k,f}(x) = \int_{Q_k} P^{j-1}_{x_k,f}(y)\phi_{Q_k}(x,y)\,dy, \text{ for all } x \in Q_k.$$

This way,

$$P^{j-1}_{x_k,f}(x) - P_{f,k}(x) = \int_{Q_k}\left[P^{j-1}_{x_k,f}(y) - f(y)\right]\phi_{Q_k}(x,y)\,dy$$

and

$$\begin{aligned}(B) &= |\partial^\gamma(P^{j-1}_{x_k,f} - P_{f,k})(x)| = \left|\int_{Q_k}\left(P^{j-1}_{x_k,f}(y) - f(y)\right)\partial^\gamma_x\phi_{Q_k}(x,y)\,dy\right| \\ &\lesssim \ell^j_k\|D^j f\|_{L^\infty}\ell_k^{-N-|\gamma|}\ell_k^N = \ell_k^{j-|\gamma|}\|D^j f\|_{L^\infty}\end{aligned}$$

Replacing in (2.4.9), we obtain

$$|\partial^\alpha\beta_k(x)| \lesssim \sum_{0\le\gamma<\alpha}\ell_k^{|\gamma|-j}\left[\ell_k^{j-|\gamma|}\|D^j f\|_{L^\infty} + \ell_k^{j-|\gamma|}\|D^j f\|_{L^\infty}\right] + \|D^j f\|_{L^\infty},$$

getting (2.4.5).

Let us now prove the inequality (2.4.4). For each $\alpha \in \mathbb{Z}^N_+$ with $|\alpha| = i \in \{1,2,...,m-1\}$, we have by the Leibniz rule that

$$|\partial^\alpha\beta_k(x)| \lesssim \sum_{0\le\gamma\le\alpha}\left|\partial^\gamma(f - P_{f,k})(x)\partial^{\alpha-\gamma}\eta_k(x)\right| \lesssim \sum_{0\le\gamma\le\alpha}\ell_k^{|\gamma|-i}|\partial^\gamma f(x) - \partial^\gamma P_{f,k}(x)|. \quad (2.4.10)$$

Since $\partial^\gamma P_{f,k}(x) = P_{\partial^\gamma f,k}(x) + (II)$, in which

$$|(II)| \le C_{m,L,\gamma}\,\ell_k^{m-|\gamma|}\sum_{|\beta|=m}(\partial^\beta f)^*_{L,\sqrt{N}\ell_k}(x) \lesssim \ell_k^{m-|\gamma|}\|D^m f\|_{L^\infty},$$

(see [37, p. 87] for more details of this (II)'s pointwise control), we get

$$|\partial^\gamma f(x) - \partial^\gamma P_{f,k}(x)| \le |\partial^\gamma f(x) - P_{\partial^\gamma f,k}(x)| + |(II)| \lesssim \ell_k^m\sum_{|\theta|=m}\left\|\partial^{\theta+\gamma}f\right\|_{L^\infty} + \ell_k^{m-|\gamma|}\|D^m f\|_{L^\infty}.$$

Plugging it at inequality (2.4.10) we have

$$\begin{aligned}|\partial^\alpha\beta_k(x)| &\lesssim \sum_{0\le\gamma\le\alpha}\ell_k^{|\gamma|-i}\left[\ell_k^m\sum_{|\theta|=m}\left\|\partial^{\theta+\gamma}f\right\|_{L^\infty} + \ell_k^{m-|\gamma|}\|D^m f\|_{L^\infty}\right] \\ &\le \sum_{\substack{0\le\gamma\le\alpha \\ |\gamma|=j}}\ell_k^{m+j-i}\ell_k^m\left\|D^{m+j}f\right\|_{L^\infty} + \ell_k^{m-i}\|D^m f\|_{L^\infty}\end{aligned}$$

and therefore

$$\|D^i\beta_k\|_{L^\infty} \lesssim \sum_{j=0}^{i}\ell_k^{m+j-i}\left\|D^{m+j}f\right\|_{L^\infty},$$

concluding the proof with the inequality (2.4.4). □

Corollary 2.7. *Given* $\alpha > 0$ *and* $\varepsilon > 0$*, there is a decomposition* $f = g + b^{\#} + \rho$ *from Theorem 2.6 where* $b^{\#} = \sum_{k \in F} \beta_k$ *for some finite set* $F \subset \mathbb{N}$ *so that* $\|\rho\|_{\Lambda^{\gamma_p}} \leq \varepsilon$*, when* $N/(N+m) < p < 1$*, and* $\|\rho\|_{bmo} \leq \varepsilon$*, when* $p = 1$.

Proof. Let F be the set of indices $k \in \mathbb{Z}$ such that $\ell_k \geq \delta$ where $0 < \delta < 1$ is a number to be determined. Note that F is finite because $\mathscr{O} = \{x : \mathscr{M} f(x) > \alpha\}$ has compact closure. Set $\rho := b - b^{\#}$ and $c_f := \max\limits_{0 \leq i \leq m} \left\|D^i f\right\|_{L^\infty} < \infty$, since $f \in C_c^m(\mathbb{R}^N)$. As the cubes $\{Q_k^*\}$ that contain $\operatorname{supp} \beta_k$ have the bounded intersection property then the number of cubes with side $\ell_k < \delta$ is finite (and the number does not depend on α).

Since $N_p + 1 \leq m$, it follows from the inequality (2.4.5) that

$$\|D^{N_p+1}\rho\|_{L^\infty} = \left\| \sum_{\substack{k \in \mathbb{Z}/ \\ \ell_k < \delta}} D^{N_p+1}\beta_k \right\|_{L^\infty} \lesssim c_f$$

and it from (2.4.4) that $\|D^i \rho\|_{L^\infty} \lesssim c_f \delta^{m-i}$, for all $i \in \{0, 1, ..., m-1\}$. In particular,

$$\|D^{N_p}\rho\|_{L^\infty} \lesssim \delta^{m-N_p} \lesssim \delta \quad \text{and} \quad \|\rho\|_{L^\infty} \lesssim \delta^m \lesssim \delta^{1-r_p},$$

because

$$1 - r_p \leq (1 - r_p) + \gamma_p = 1 + N_p \leq m.$$

Note that, by inequality (2.4.3) from the Remark 2.5 and the previous estimate, we have

$$[\rho]_{\Lambda^{\gamma_p}} \;=\; [D^{N_p}\rho]_{\Lambda^{r_p}} \leq \|D^{N_p+1}\rho\|_{L^\infty}\delta^{1-r_p} + \|D^{N_p}\rho\|_{L^\infty}\delta^{-r_p} \lesssim \delta^{1-r_p}.$$

Thus, $\|\rho\|_{\Lambda^{\gamma_p}} = [\rho]_{\Lambda^{\gamma_p}} + \|\rho\|_{L^\infty} \lesssim \delta^{1-r_p} = \varepsilon$, chosen an appropriated δ. For the case $p = 1$, it is enough to note that $\|\rho\|_{bmo} \leq 3\|\rho\|_{L^\infty} \lesssim \varepsilon$. □

2.4.3 Special approximation method: atom refinement

Our interest in this section is to prove the following result:

Proposition 2.8. *Let* m, p *and* γ_p *as before. Given* $f \in (C_c^m \cap h^p)(\mathbb{R}^N)$ *there exists a sequence of functions* $f_n \in C_c^m(\mathbb{R}^N)$ *having the following properties*

(i) there exists $C > 0$ *such that* $\|f_n\|_{h^p} \leq C\|f\|_{h^p}$*;*

(ii) $\lim\limits_{n \to \infty} \|f - f_n\|_{\Lambda^{\gamma_p}} = 0$*, when* $N/(N+m) < p < 1$ *and* $\lim\limits_{n \to \infty} \|f - f_n\|_{bmo} = 0$*, when* $p = 1$*;*

The proof of this proposition will heavily utilize the results obtained in the previous subsection. It is from these results that we will obtain a more refined sequence of atoms, which we will use in the proof of Theorem **A**, specifically the Lemma 3.3.

Proof. The proof is reduced for the case that $f \in (H^p \cap C_c^m)(\mathbb{R}^N)$. In fact, using the decomposition of inequality (1.2.1) we may split $f = f_1 + f_2$ with $f_1 \in (H^p \cap C_c^m)(\mathbb{R}^N)$ and for $f_2 \in C_c^\infty(\mathbb{R}^N)$ the conclusion follows by Remark 2.5. To simplify the notation, we write f instead of f_1.

Let $f \in (H^p \cap C_c^m)(\mathbb{R}^N)$ and fix a small number $\varepsilon > 0$. For each $i \in \mathbb{N}$, we pick a decomposition $f = g_i + b_i = g_i + b_i^\# + \rho_i$ with the properties granted by the Theorem 2.6 and Corollary 2.7, for $\alpha_i = 2^i$ and $\varepsilon_i = \varepsilon 2^i$, *i.e.*

- $\mathscr{O}^i \subseteq \mathscr{O}^{i+1}$;
- $|g_i(x)| \leq c2^i$;
- $\|\rho_i\|_{\Lambda^{\gamma_p}} \leq \varepsilon 2^i$ (or $\|\rho_i\|_{bmo} \leq \varepsilon 2^i$, when $p = 1$);
- $b_i^\# = \sum_{\substack{k\in\mathbb{Z}\\ \ell_k \geq \delta_i}} \beta_{ik}$, in which $\delta_i = \left(\varepsilon 2^i\right)^{\frac{1}{1-r_p}}$.

Given two consecutive values of i and the corresponding decompositions $f = g_{i+1} + b_{i+1}^\# + \rho_{i+1} = g_i + b_i^\# + \rho_i$ and look at the difference $b_i - b_{i+1} = g_{i+1} - g_i$ we see that

$$\|b_i - b_{i+1}\|_{L^\infty} \leq \|g_{i+1}\|_{L^\infty} + \|g_i\|_{L^\infty} \lesssim 2^i.$$

On the other hand, it follows from (2.4.5) and the bounded intersection property of $\{Q_{i,k}^*\}$ that $\|D^j b_i\|_{L^\infty} \leq C$, with C independent of i, for all $j \in \{1,2,...,m\}$, which implies that

$$\|D^{N_p+1}(b_i - b_{i+1})\|_{L^\infty} \leq C.$$

To estimate $D^{N_p}(b_i - b_{i+1})$, note that $\delta_i < \delta_{i+1}$, this way

$$b_i^\# = \sum_{\substack{k\in\mathbb{Z}\\ \ell_k\geq\delta_i}} \beta_{ik} = \sum_{\substack{k\in\mathbb{Z}\\ \ell_k\geq\delta_{i+1}}} \beta_{ik} + \sum_{\substack{k\in\mathbb{Z}\\ \delta_i\leq\ell_k<\delta_{i+1}}} \beta_{ik} \; = b_{i+1}^\# + \sum_{\substack{k\in\mathbb{Z}\\ \delta_i\leq\ell_k<\delta_{i+1}}} \beta_{ik}.$$

Thus, it follows from (2.4.4) that

$$\begin{aligned}
\left\|D^{N_p}(b_i - b_{i+1})\right\|_{L^\infty} &\leq \left\|D^{N_p}(\rho_i - \rho_{i+1})\right\|_{L^\infty} + \left\|D^{N_p}(b_i^\# - b_{i+1}^\#)\right\|_{L^\infty} \\
&\lesssim \delta_i^{m-N_p} + \left\|D^{N_p}\left(\sum_{\substack{k\in\mathbb{Z}\\ \delta_i\leq\ell_k<\delta_{i+1}}} \beta_{ik}\right)\right\|_{L^\infty} \\
&\lesssim \delta_i^{m-N_p} + \delta_{i+1}^{m-N_p} \lesssim c_{r_p}\, \delta_i,
\end{aligned}$$

for all $i \leq 0$. Again, by inequality (2.4.3) from the Remark 2.5, we have

$$\|b_i - b_{i+1}\|_{\Lambda^{\gamma_p}} \leq \|b_i - b_{i+1}\|_{L^\infty} + \|D^{N_p+1}(b_i - b_{i+1})\|_{L^\infty} M_i^{1-r_p} + \|D^{N_p}(b_i - b_{i+1})\|_{L^\infty} M_i^{-r_p},$$

for any $M_i > 0$. In particular, we can choose $M_i = \delta_i$. By the previous inequalities

$$\|b_i - b_{i+1}\|_{\Lambda^{\gamma_p}} \lesssim 2^i + \delta_i^{1-r_p} + \delta_i \delta_i^{-r_p} \simeq 2^i + \varepsilon 2^i,$$

for all $i \leq 0$. Thus, we see that

$$\|g_{i+1} - g_i\|_{\Lambda^{\gamma_p}} = \|b_i - b_{i+1}\|_{\Lambda^{\gamma_p}} \leq c\,2^i,$$

implying that the sum $\sum_{-\infty}^{L}(g_{i+1} - g_i)$ converges to 0 in $\Lambda^{\gamma_p}(\mathbb{R}^N)$ as $L \to -\infty$.

For the case $p = 1$,

$$\|g_{i+1} - g_i\|_{bmo} \lesssim \|g_{i+1} - g_i\|_{L^\infty} \lesssim 2^i$$

and the sum $\sum_{-\infty}^{L}(g_{i+1} - g_i)$ converges to 0 also in $bmo(\mathbb{R}^N)$ as $L \to -\infty$.

Since f is bounded, $\mathscr{M} f$ is also bounded and there is $N_0 \in \mathbb{Z}$ such that $g_{i+1} \equiv g_i$ for all $i \geq N_0$. Therefore, we may write

$$\begin{aligned}
f &= \sum_{-\infty}^{L}(g_{i+1} - g_i) + \sum_{L+1}^{N_0}(g_{i+1} - g_i) \\
&= \sum_{i=L+1}^{N_0}(b_i^{\#} - b_{i+1}^{\#}) + \sum_{i=L+1}^{N_0}(\rho_i - \rho_{i+1}) + \sum_{-\infty}^{L}(g_{i+1} - g_i) \\
&= \sum_{i=L+1}^{N_0}\left(\sum_{k\in F_i}\beta_{i,k} - \sum_{l\in F_{i+1}}\beta_{i+1,l}\right) + \sum_{-\infty}^{L}(g_{i+1} - g_i) \\
&:= f^\flat + R,
\end{aligned}$$

where $\|R\|_{\Lambda^{\gamma_p}}$ (or $\|R\|_{bmo}$, for $p = 1$) can be made arbitrary small by taking ε small and L close enough to $-\infty$. The next step is to write $f^\flat$ as a finite sum

$$f^\flat := \sum_{i=L+1}^{N_0}\left(\sum_{k\in F_i}\beta_{i,k} - \sum_{l\in F_{i+1}}\beta_{i+1,l}\right) = \sum_{i=L+1}^{N_0}\sum_{k}\lambda_{i,k}\,a_{i,k}$$

where $\left\{a_{i,k}\right\}_{i,k}$ are H^p-atoms and $\left\{\lambda_{i,k}\right\}_{i,k}$ satisfy $\sum\limits_{i,k}|\lambda_{i,k}|^p \leq C\|f\|_{H^p}^p$. It is convenient to redefine the functions $\beta_{i,k}$ by setting $\beta_{i,k} \equiv 0$ if $k \notin F_i$ for each fixed i, so the approximation $f^\flat$ of f may be written now as

$$f^\flat = \sum_{i=L+1}^{N_0}\left(\sum_{k}(f - P_{f,i,k})\eta_{i,k} - \sum_{\ell}(f - P_{f,i+1,\ell})\eta_{i+1,\ell}\right) = \sum_{i=L+1}^{N_0}\sum_{k}A_{i,k} \qquad (2.4.11)$$

where

$$A_{i,k} = (f - P_{f,i,k})\eta_{i,k} \;-\; \sum_{l}(f - P_{f,i+1,\ell})\eta_{i+1,\ell}\eta_{i,k} \;+\; \sum_{\ell}Q_{f,k,\ell}\,\eta_{i+1,\ell},$$

with

$$Q_{f,k,\ell}(x) = P_{(h\,\eta_{i,k}),i+1,\ell} \quad \text{and} \quad h := f - P_{f,i+1,\ell},$$

and it is understood that $P_{f,i,k} = 0$ and $Q_{f,k,\ell} = 0$ if $k \notin F_i$ or $\ell \notin F_{i+1}$. However, in order to prove that (2.4.11) holds one needs to assume that

$$\sum_{k \in F_i} \eta_{i,k} \equiv 1 \text{ on the support of } \eta_{\ell,k+1} \text{ whenever } \ell \in F_{i+1}. \tag{2.4.12}$$

This can be achieved inductively, by adding to F_{N_0} a finite number of extra indexes to obtain (2.4.12) when $i = N_0$, then finitely many additional indexes to F_{N_0-1} so (2.4.12) holds for $i = N_0 - 1$, etc. Note that the additional terms $\beta_{i,k}$ added to $f^\flat$ in this way do not increase the norm $\|R\|_{\Lambda^{\gamma_p}}$, which remains at least as small as in its original definition.

Let $B_{i,k}$ be the smallest ball containing $Q^*_{i,k}$. Exactly the same arguments on pages 108 and 109 of [44] used to deal with an infinite decomposition show now, in the finite decomposition case, that setting

$$a_{i,k} = c^{-1} 2^{-i} |B_{i,k}|^{-1/p} A_{i,k} \text{ and } \lambda_{i,k} = c 2^i |B_{i,k}|^{1/p}.$$

Thus, we have $f^\flat = \sum_{i=L+1}^{N_0} \sum_k \lambda_{i,k} a_{i,k}$. Furthermore,

$$\sum_{i,k} |\lambda_{i,k}|^p = c \sum_{i,k} 2^{ip} |Q_{i,k}| \leq c' \int (\mathscr{M} f)^p(x)\, dx \leq C \|f\|^p_{H^p}.$$

Finally, for each $n \in \mathbb{N}$ we may take f_n as the function $f^\flat$ in the previous construction where R satisfies the estimative $\|R\|_{\Lambda^{\gamma_p}} < 1/n$. Then, the sequence $(f_n)_n$ satisfies the properties (i) and (ii) required in the Proposition 2.8. □

CHAPTER 3

The proof of Theorem A and consequences

At the beginning of this chapter we will proof Theorem **A**, stated in the introduction (Theorem 1). The first section corresponds to the primary estimates of the term $A(\cdot,D)\phi\cdot v$ and in the statement of a fundamental lemma (Lemma 3.1) and the second section continues the proof of the lemma, reducing it to the uniform control of a sublinear operator for atoms. The third section states and demonstrates a nonhomogeneous version of Theorem **A**, where the hypothesis $A^*(\cdot,D)v=0$ is removed. Next, two applications of the result are presented and lastly, an expected global version of the theorem is stated in the case where the operator $A(\cdot,D)$ has constant coefficients.

3.1 Proof of Theorem A

Let $\psi\in C_c^\infty(B(0,1))$ with $\int\psi=1$ and $\psi_t(x) := t^{-N}\psi(x/t)$, $0<t<1$. By simplicity we consider $x_0=0$, $U=B(0,\rho)$ with radius $\rho<1/2$ to be chosen later. Indeed, suppose that

$$\|A(\cdot,D)\phi\cdot_F v\|_{h^r}\le C\,\|A(\cdot,D)\phi\|_{h^p}\,\|v\|_{W^{m-1,q}}$$

for any $\phi\in C_c^\infty(B(0,\rho),E)$ and $v\in C_c^\infty(B(0,\rho),F)$, satisfying $A^*(\cdot,D)v=0$. Let now $\varphi\in C_c^\infty(B(x_0,\rho),E)$ and $w\in C_c^\infty(B(x_0,\rho),F)$, satisfying $A^*(\cdot,D)w=0$. We will denote $\varphi_{x_0}(x):=\varphi(x-x_0)$, $w_{x_0}(x):=w(x-x_0)$, $A_{x_0}(x,D):=A(x-x_0,D)$ and $A^*_{x_0}(x,D):=A^*(x-x_0,D)$. Note that $A^*(\cdot,D)w=0$ is equivalent to $A^*_{x_0}(\cdot,D)w_{x_0}=0$. Due to translation invariance in h^p and Sobolev norm, we obtain

$$\begin{aligned}\|A(\cdot,D)\varphi\cdot_F w\|_{h^r} &= \|A_{x_0}(\cdot,D)\varphi_{x_0}\cdot_F w_{x_0}\|_{h^r}\\ &\le C\,\|A_{x_0}(\cdot,D)\varphi_{x_0}\|_{h^p}\,\|w_{x_0}\|_{W^{m-1,q}}\\ &= C\,\|A(\cdot,D)\varphi\|_{h^p}\,\|w\|_{W^{m-1,q}}.\end{aligned}$$

Let $P:\Omega\mapsto E$ a polynomial of degree at most $m-1$ such that $A(\cdot,D)P=0$ on U, also to be adjusted later. In fact, by the homogeneity hypothesis, $A(\cdot,D)P=0$ for all $P\in\mathscr{P}^E_{m-1}$. Then, we may write

$$\overline{\psi_t*[A(x,D)\phi\cdot_F v]\,(x)}=\int_{B(x,t)}A^*(y,D)\,[\psi_t\,(x-y)\,v(y)]\,\cdot_E\,(\phi-P)(y)dy.$$

Indeed, by definition of convolution,

$$\begin{aligned}
\overline{\psi_t * [A(x,D)\phi \cdot_F v]\,(x)} &= \psi_t * [v \cdot_F A(x,D)\phi]\,(x) \\
&= \int [\psi_t(x-y)v(y)] \cdot_F A(y,D)\phi(y)dy \\
&= \int [\psi_t(x-y)v(y)] \cdot_F A(y,D)(\phi-P)(y)dy \\
&= \frac{1}{t^N}\int_{B(x,t)} A^*(y,D)\left[\psi\left(\frac{x-y}{t}\right)v(y)\right] \cdot_E (\phi-P)(y)dy.
\end{aligned}$$

Applying the product rule under assumption $A^*(\cdot,D)v=0$ and using the uniform local boundedness of the functions $\partial^\theta a^*_\alpha$ and $\partial^\theta \psi$ for $0\le|\theta|\le m$, and taking $0<t\le 1$ we may estimate

$$|\psi_t * [A(\cdot,D)\phi \cdot_F v]|\,(x) \lesssim \frac{C(\psi)}{t^{N+m}}\int_{B^t_x}\left(\sum_{0\le|\gamma|<m}|\partial^\gamma v(y)|\right)|\phi(y)-P(y)|\,dy,$$

where

$$C(\psi) := \max_{\substack{|\alpha|\le m\\ 0\le\gamma,\theta\le\alpha}} \sup_{z\in\mathbb{R}^N}\left\{(1+\|\partial^\gamma a^*_\alpha\|_{L^\infty})\,|\partial^\theta\psi|(z)\right\}.$$

Now, fixed p,q and r satisfying the hypotheses, take $1<\beta<q$ and $1<\beta'<p^*$ satisfying $\beta'<\beta$ and $1/\beta+1/\beta'=1$, where $p^*:=Np/(N-mp)$. Thus, by Hölder inequality

$$|\varphi_t * [A(x,D)\phi \cdot_F v]\,(x)| \lesssim \frac{1}{t^{N+m}}\left[\int_{B(x,t)}\left(\sum_{0\le|\gamma|<m}|\partial^\gamma v(y)|\right)^\beta dy\right]^{\frac{1}{\beta}}\left[\int_{B(x,t)}|\phi(y)-P(y)|^{\beta'}dy\right]^{\frac{1}{\beta'}}.$$

Since $t^N=|B^t_x|^{1/\beta}|B^t_x|^{1/\beta'}|B^1_0|^{-1}$ we can write

$$|\psi_t * [A(\cdot,D)\phi \cdot_F v]|\,(x) \lesssim \left[\fint_{B^t_x}\left(\sum_{0\le|\gamma|<m}|\partial^\gamma v(y)|\right)^\beta dy\right]^{\frac{1}{\beta}}\left[\fint_{B^t_x}\left|\frac{1}{t^m}(\phi(y)-P(y))\right|^{\beta'}dy\right]^{\frac{1}{\beta'}}$$

which implies that $\sup\limits_{0<t<1}|\psi_t * A(\cdot,D)\phi\cdot v|^r\,(x)$ is bounded by

$$C(\psi)\left[M\left(\left(\sum_{0\le|\gamma|<m}|\partial^\gamma v|\right)^\beta\right)(x)\right]^{\frac{r}{\beta}}\left[\sup_{0<t<1}\fint_{B^t_x}\left|\frac{1}{t^m}\,[\phi(y)-P(y)]\right|^{\beta'}dy\right]^{\frac{r}{\beta'}},$$

where M is the Hardy-Littlewood maximal operator defined in (1). Integrating on $\mathbb{R}^N$ and using Hölder inequality for qr^{-1} and pr^{-1} (since $qr^{-1},pr^{-1}>1$ and $1/r=1/p+1/q$), we have the control

$$\|A(\cdot,D)\phi \cdot_F v\|^r_{h^r} \lesssim \|M(D^m_< v)\|^{\frac{r}{\beta}}_{L^{\frac{q}{\beta}}}\left(\int_{\mathbb{R}^N}\left[\sup_{0<t<1}\fint_{B^t_x}\left|\frac{1}{t^m}\,[\phi(y)-P(y)]\right|^{\beta'}dy\right]^{\frac{p}{\beta'}}dx\right)^{\frac{r}{p}}, \tag{3.1.1}$$

where $D^m_< v := \left(\sum_{0 \le |\gamma| < m} |\partial^\gamma v| \right)^\beta$. Indeed,

$$
\begin{aligned}
\|A(\cdot,D)\phi \cdot_F v\|^r_{h^r} &\lesssim \left[\int_{\mathbb{R}^N} [M(D^m_< v)(x)]^{\frac{r}{\beta}\cdot\frac{q}{r}} \, dx \right]^{\frac{r}{q}} \\
&\cdot \left[\int_{\mathbb{R}^N} \left(\sup_{0<t<1} \fint_{B^t_x} \left| \frac{1}{t^m} [\phi(y) - P(y)] \right|^{\beta'} dy \right)^{\frac{r}{\beta'}\cdot\frac{p}{r}} dx \right]^{\frac{r}{p}} \\
&= \left[\int_{\mathbb{R}^N} [M(D^m_< v)(x)]^{\frac{q}{\beta}} \, dx \right]^{\frac{r}{q}} \\
&\cdot \left(\int_{\mathbb{R}^N} \left[\sup_{0<t<1} \fint_{B^t_x} \left| \frac{1}{t^m} [\phi(y) - P(y)] \right|^{\beta'} dy \right]^{\frac{p}{\beta'}} dx \right)^{\frac{r}{p}}.
\end{aligned}
$$

Now, we will use the boundedness of M in $L^{\frac{q}{\beta}}$, since $1 < \beta < q$. The first term on the right-hand side of inequality (3.1.1) without the exponent r can be estimated by

$$
\|M(D^m_< v)\|^{\frac{1}{\beta}}_{L^{\frac{q}{\beta}}} \le \left\| \left(\sum_{0 \le |\gamma| < m} |\partial^\gamma v| \right)^\beta \right\|^{\frac{1}{\beta}}_{L^{\frac{q}{\beta}}} = \left\| \sum_{0 \le |\gamma| < m} |\partial^\gamma v| \right\|_{L^q} \lesssim \|v\|_{W^{m-1,q}} \tag{3.1.2}
$$

In order to conclude the proof of Theorem A, we need to estimate the second term on the right-hand side of (3.1.1). This follows from a fundamental inequality that holds for scalar local Hardy-Sobolev spaces and is an ersatz for [13, Lemma II.2]:

Lemma 3.1 (Fundamental Lemma)**.** *Let $m \in \mathbb{N}$, $\frac{N}{N+m} < p \le \frac{N}{m}$ and $\alpha \in [1, p^*)$. Then, there exist a constant $C > 0$ and a polynomial $P_{x,t,f} \in \mathscr{P}_{m-1}$ such that the estimate*

$$
\int_{\mathbb{R}^N} \left[\sup_{0<t<1} \fint_{B^t_x} \left\{ \frac{1}{t^m} |f(y) - P_{x,t,f}(y)| \right\}^\alpha dy \right]^{\frac{p}{\alpha}} dx \le C \|f\|^p_{h^{m,p}} \tag{3.1.3}
$$

holds for every $f \in C^\infty_c(\mathbb{R}^N)$.

For now, let us assume this lemma and conclude the proof of Theorem **A**. Using Remark 3.2 stated next (for $a = \beta'$ and $a = p/\beta'$) and the fact that the supremum of the sum loses to the sum of the supremum, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} \left[\sup_{0<t<1} \fint_{B^t_x} \left| \frac{1}{t^m} [\phi(y) - P(y)] \right|^{\beta'} dy \right]^{\frac{p}{\beta'}} dx &\lesssim \int_{\mathbb{R}^N} \left[\sup_{0<t<1} \fint_{B^t_x} \sum_{j=1}^{n_E} \left| \frac{1}{t^m} [\phi_j(y) - P_j(y)] \right|^{\beta'} dy \right]^{\frac{p}{\beta'}} dx \\
&\le \int_{\mathbb{R}^N} \left[\sum_{j=1}^{n_E} \sup_{0<t<1} \fint_{B^t_x} \left| \frac{1}{t^m} [\phi_j(y) - P_j(y)] \right|^{\beta'} dy \right]^{\frac{p}{\beta'}} dx \\
&\lesssim \sum_{j=1}^{n_E} \int_{\mathbb{R}^N} \left[\sup_{0<t<1} \fint_{B^t_x} \left| \frac{1}{t^m} [\phi_j(y) - P_j(y)] \right|^{\beta'} dy \right]^{\frac{p}{\beta'}} dx.
\end{aligned}
$$

Now we can apply Lemma 3.1 for each component in the previous sum, choosing $P_j = P_{x,t,\phi_j}$. Thus,

$$\sum_{j=1}^{n_E} \int_{\mathbb{R}^N} \left[\sup_{0<t<1} \fint_{B_x^t} \left| \frac{1}{t^m} \left[\phi_j(y) - P_j(y) \right] \right|^{\beta'} dy \right]^{\frac{p}{\beta'}} dx \le C \sum_{j=1}^{n_E} \|\phi_j\|_{h^{m,p}}^p \le \|\phi\|_{h^{m,p}}^p.$$

By the inequality (2.3.1) proved in the previous chapter, there exist a ball $B_0^\rho = B(0,\rho)$ for ρ sufficiently small and a constant $C = C(B,p,N,m) > 0$ such that

$$\left(\int_{\mathbb{R}^N} \left[\sup_{0<t<1} \fint_{B_x^t} \left\{ \frac{1}{t^m} \left| \phi(y) - P_{x,t,\phi}(y) \right| \right\}^{\alpha} dy \right]^{\frac{p}{\alpha}} dx \right)^{\frac{1}{p}} \le C \|A(\cdot,D)\phi\|_{h^p}, \tag{3.1.4}$$

for all $\phi \in C_c^\infty(B_0^\rho, E)$. Plugging the inequalities (3.1.4) and (3.1.2) at (3.1.1) we conclude the proof of Theorem **A**.

□

Remark 3.2. *Let $h \in C_c^\infty(\Omega, E)$ and $a > 0$, then there exists a positive constant C_{a,n_E}, independent of h, such that*

$$|h(x)|^a \le C_{a,n_E} \sum_{j=1}^{n_E} |h_j(x)|^a.$$

Note that this is a simple but technical remark. In fact, it follows by Hölder inequality that

$$\left| \sum_{j=1}^{n_E} w_j \right|^b \le c_{b,n_E} \sum_{j=1}^{n_E} |w_j|^b,$$

for all $w = (w_1, \ldots, w_{n_E}) \in E$ and $b > 0$, in which $c_{b,n_E} = 1$ if $0 < b \le 1$ and $c_{b,n_E} = (n_E)^{b-1}$ if $b \ge 1$. For each $a > 0$ we denote $b = a/2$, then

$$|h(x)|^a = \left| \sum_{j=1}^{n_E} |h_j(x)|^2 \right|^{a/2} \le c_{b,n_E} \sum_{j=1}^{n_E} |h_j(x)|^{2b} = c_{a,n_E} \sum_{j=1}^{n_E} |h_j(x)|^a.$$

The remainder of this chapter is devoted to proving Lemma 3.1.

Proof. For each $f \in C_c^\infty(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and $0 < t < 1$ consider $P_{x,t,f}(y) := \langle f, \phi_{Q(x,2t)}(y,\cdot) \rangle$ the polynomial of degree at most $m-1$ given by the Lemma 1.58 and define the sublinear operator

$$Tf(x) = \sup_{0<t<1} \left\{ \fint_{B(x,t)} \left| \frac{1}{t^m} \left[f(y) - P_{x,t,f}(y) \right] \right|^{\alpha} dy \right\}^{\frac{1}{\alpha}},$$

for fixed $1 \le \alpha < p^*$. Rewriting the inequality (3.1.3), it is sufficient to prove that

$$\int_{\mathbb{R}^N} |Tf(x)|^p \, dx \le C \|f\|_{h^{m,p}}^p, \quad \forall f \in C_c^\infty(\mathbb{R}^N). \tag{3.1.5}$$

Firstly, let us consider $m < N$, $1 < p \leq \frac{N}{m}$ and denote $\tilde{p} = p - \lfloor p \rfloor$. Since $f \in L^1_{loc}(\mathbb{R}^N)$ and $\partial^\beta f \in h^{\tilde{p}}$, for all $|\beta| = m$, it follows from the Theorem 2.2 with $T = 1$ that

$$\left(\fint_{B(x,t)} \left| \frac{1}{t^m} \left[f(y) - P_{x,t,f}(y) \right] \right|^\alpha dy \right)^{\frac{1}{\alpha}} \leq C_{N,\tilde{p},m,\alpha} \left(\fint_{B(x,t)} |D^m f|^\gamma dy \right)^{\frac{1}{\gamma}},$$

for any $1 < \gamma < p$. Applying the supremum for all $0 < t < 1$ we get the pointwise control

$$Tf(x) \lesssim \left[M\left(|D^m f|^\gamma \right)(x) \right]^{\frac{1}{\gamma}}.$$

Thus, using the boundedness of M in $L^{\frac{p}{\gamma}}(\mathbb{R}^N)$,

$$\begin{aligned} \|T\|^p_{L^p(\mathbb{R}^N)} &\lesssim \left\| \left[M\left(|D^m f|^\gamma \right) \right]^{\frac{1}{\gamma}} \right\|^p_{L^p(\mathbb{R}^N)} = \left\| M\left(|D^m f|^\gamma \right) \right\|^{p/\gamma}_{L^{p/\gamma}(\mathbb{R}^N)} \\ &\lesssim \left\| |D^m f|^\gamma \right\|^{p/\gamma}_{L^{p/\gamma}(\mathbb{R}^N)} = \|D^m f\|^p_{L^p(\mathbb{R}^N)}, \end{aligned}$$

obtaining (3.1.5).

For the second and more extensive part of the proof, we need to prove for the case $\frac{N}{N+m} < p \leq \min\left\{1, \frac{N}{m}\right\}$. Note that this case includes both $N \leq m$ and $m < N$. Consider J_m the Bessel potential operator defined in Example 1.33 and denote the sublinear operator $T' := T \circ J_m$. Writing the identities $f = J_m g$ and $D^m f = (D^m \circ J_m) g$, by Proposition 1.38 we have

$$\|f\|_{h^{m,p}_<} := \sum_{j=0}^{m} \left\| D^j f \right\|_{h^p} \simeq \|J_{-m} f\|_{h^p} = \|(J_{-m} \circ J_m) g\|_{h^p} \lesssim \|g\|_{h^p},$$

since pseudo-differential operators of order zero are bounded from $h^p(\mathbb{R}^N)$ to itself for $0 < p \leq 1$. Thus, to obtain (3.1.5) it is enough to prove the existence of (uniform) $C > 0$ such that for every compact subset $K \subset \mathbb{R}^N$ the control

$$\int_K \left| T' g(x) \right|^p dx \leq C \|g\|^p_{h^p}, \quad \forall g \in C^\infty_c(\mathbb{R}^N) \tag{3.1.6}$$

holds. Indeed, suppose that (3.1.6) holds for all $K \subset\subset \mathbb{R}^N$, denote $h(x) := |T' g(x)|^p$ and $h_n(x) := h(x) \mathscr{X}_{B[0,n]}(x)$ for each $n \in \mathbb{N}$. Thus, by Fatou Lemma, we have

$$\int_{\mathbb{R}^N} \left| T' g(x) \right|^p dx = \int_{\mathbb{R}^N} h(x) dx \leq \limsup_n \int_{\mathbb{R}^N} h_n(x) dx \leq C \|g\|^p_{h^p}$$

and then

$$\int_{\mathbb{R}^N} |Tf(x)|^p dx = \int_{\mathbb{R}^N} \left| T' g(x) \right|^p dx \leq C \|g\|^p_{h^p} \lesssim \|f\|^p_{h^{m,p}}$$

proving (3.1.5).

The proof of (3.1.6) follows by two fundamental steps. The first is the reduction of the estimate to the case in which g is a bounded h^p-atom, *i.e.* there exists a uniform constant $C > 0$ such that

$$\int_K \left| T' a(x) \right|^p dx \leq C, \tag{3.1.7}$$

for all *a* h^p-atom (see Sections 1.2.1 and 2.4.3).

This point is crucial, since there is an example of a linear operator uniformly bounded on standard atoms but that cannot be extended for Hardy spaces (see [10]). In the same spirit presented at [24, Appendix A1] we state the following result:

Lemma 3.3. *Given $f \in C_c^\infty(\mathbb{R}^N)$ there exist a sequence of functions $f_n \in (C_c^m \cap h^p)(\mathbb{R}^N)$ and a uniform constant $C > 0$ such that*

(i) $\|f_n\|_{h^p} \leq C\|f\|_{h^p}$*;*

(ii) $\lim\limits_{n\to\infty} \left\|T'(f_n - f)\right\|_{L^p(K)} = 0$*, for all* $K \subset\subset \mathbb{R}^N$*;*

(iii) $\|T'f_n\|_{L^p(\mathbb{R}^N)} \leq C\,\|f\|_{h^p}$.

The estimate (3.1.6) follows directly from (ii) and (iii). Indeed, assuming the previous lemma, given $g \in C_c^\infty$ there is a sequence $\{g_n\}_n$ in $C_c^m \cap h^p$ such that

$$\begin{aligned}
\int_K \left|T'g(x)\right|^p dx &\leq \int_K \left|T'(g_n - g)(x)\right|^p dx + \int_K \left|T'(g)(x)\right|^p dx \\
&\leq \int_K \left|T'(g_n - g)(x)\right|^p dx + \int_{\mathbb{R}^N} \left|T'(g)(x)\right|^p dx \\
&\leq \left\|T'(g_n - g)\right\|_{L^p(K)}^p + C\,\|g\|_{h^p}^p .
\end{aligned}$$

Applying the limit $n \to \infty$, we obtain $\int_K |T'g(x)|^p\, dx \leq C\|g\|_{h^p}^p$.

Let us prove Lemma 3.3.

Proof. It follows from Proposition 2.8 that given $f \in C_c^\infty(\mathbb{R}^N)$ there exists a sequence of functions $f_n \in (C_c^m \cap h^p)(\mathbb{R}^N)$ such that (i) holds and $\lim\limits_{n\to\infty} \|f - f_n\|_{\Lambda^{\gamma_p}} = 0$. We also claim that

$$\lim_{n\to\infty} T'(f_n - f)(x) = 0 \tag{3.1.8}$$

uniformly over compact subsets $K \subset \mathbb{R}^N$ that implies directly (ii). In order to prove (3.1.8), it is enough to show that the sublinear operator

$$T'f(x) = \sup_{0<t<1} \left(\fint_{B_x^t} \left\{ \frac{1}{t^m} \left| J_m f(y) - P_{x,t,J_m f}(y) \right| \right\}^\alpha dy \right)^{1/\alpha}$$

satisfies $\sup\limits_K T'f(x) \leq C\|f\|_{\Lambda^{\gamma_p}}$. By the inequality (2.2.2) applied for $J_m f$ and the boundedness of the pseudo-differential operators $D^m \circ J_m$ of order zero on Hölder spaces (see [4]), we get

$$T'f(x) \leq \|D^m \circ J_m f\|_{L^\infty} \leq \|D^m \circ J_m f\|_{\Lambda^{\gamma_p}} \leq C\|f\|_{\Lambda^{\gamma_p}}$$

for all $x \in K \subset\subset \mathbb{R}^N$. For a moment, we assume the validity of (3.1.7) in order to prove (iii). Since $f_n \in C_c^m(\mathbb{R}^N)$, we can decompose f_n as a *finite* sum of smooth bounded h^p-atoms, namely $f_n =$

$\sum_{k=1}^{k_n}\lambda_{k,n}a_{k,n}$ satisfying $\sum_k|\lambda_{k,n}|^p \leq C\|f\|_{h^p}$ with $C>0$ independent of k and n (see the discussion in Section 2). Thus,

$$\|T'f_n\|_{L^p}^p \leq \sum_{k=1}^{k_n}|\lambda_{k,n}|^p\|T'a_{k,n}\|_{L^p}^p \leq C\,\|f\|_{h^p}^p.$$

□

Hence, the last step to complete the proof of Lemma 3.1 is to show the inequality (3.1.7). This uniform control, however, will be proved in the next section.

3.2 Uniform control of T' on atoms

Let a an h^p-atom compactly supported in a ball $B=B(x_0,r)$ defined in Section 2.4.3, which means that a follows the standard Definition 1.4, but it has more moment conditions, that is, if $r<1$ then

$$\int_{\mathbb{R}^N} x^\beta a(x)\,dx = 0,$$

for all $\beta\in\mathbb{Z}_+^N$ satisfying $|\beta|\leq m-1$, instead of $|\beta|\leq N_p=\lfloor N(p^{-1}-1)\rfloor$, since $N_p\leq m-1$.

Without loss of generality, we may assume $x_0=0$. Indeed, suppose that

$$\int_{\mathbb{R}^N}|T'a(x)|^p\,dx \leq C, \tag{3.2.1}$$

for all a h^p-atom supported in ball $B(0,r)$. Given b an h^p-atom supported in ball $B(x_0,r)$, we denote $\tilde{a}(x):=b(x-x_0)$. Clearly $\tilde{a}$ is also an h^p-atom satisfying (3.2.1) (due to translation invariance) and, therefore, the uniform boundedness holds for all h^p-atoms supported in ball $B(x_0,r)$.

Let us prove the inequality (3.2.1). Consider first $|x|\leq 10r$ and set $b:=J_m a$. It follows from the inequality (2.2.2) that

$$\left(\fint_{B_x^t}\left|\frac{1}{t^m}\left[b(y)-P_{x,t,b}(y)\right]\right|^\alpha dy\right)^{\frac{1}{\alpha}} \lesssim \left(\fint_{B_x^t}|D^m b(y)|^\alpha dy\right)^{\frac{1}{\alpha}},$$

for all $1<\alpha<p^*$. Taking the supremum for all $0<t<1$, we get

$$T'a(x)\lesssim\left[M(|D^m b|^\alpha)(x)\right]^{\frac{1}{\alpha}}.$$

Hence, if $\alpha<\beta<\infty$ then we have

$$\begin{aligned}
\|\left[M(|D^m b|^\alpha)\right]^{1/\alpha}\|_{L^\beta} &= \|M(|D^m b|^\alpha)\|_{L^{\beta/\alpha}}^{1/\alpha} \lesssim \||D^m b|^\alpha\|_{L^{\beta/\alpha}}^{1/\alpha}\\
&= \|D^m J_m a\|_{L^\beta} \lesssim \|a\|_{L^\beta}\\
&\lesssim \|a\|_{L^\infty}|B_0^r|^{\frac{1}{\beta}} \lesssim r^{N\left(\frac{1}{\beta}-\frac{1}{p}\right)},
\end{aligned}$$

where the second inequality follows from the fact that $D^m \circ J_m$ is a pseudo-differential operator of order zero and, by Remark 1.37, it is bounded in L^β for any $1 < \beta < \infty$. Thus,

$$\left(\int_{|x|\leq 10r} |T'a(x)|^p dx\right)^{1/p} \lesssim r^{N\left(\frac{1}{p}-\frac{1}{\beta}\right)} \left(\int_{|x|\leq 10r} |T'a(x)|^\beta dx\right)^{1/\beta}$$
$$\lesssim r^{N\left(\frac{1}{p}-\frac{1}{\beta}\right)} \|M(|D^m b|^\alpha)^{1/\alpha}\|_{L^\beta} \lesssim 1,$$

uniformly for any an h^p-atom. We point out that only the size condition of the atom is required in the previous control.

Moving on for the set $|x| \geq 10r$, we will distinguish atoms according to the support $B(0,r)$, since atoms with $r \leq 1$ are H^p-atoms, thus satisfy moment conditions, and atoms with $r > 1$ do not.

Recall that the kernel of Bessel potential $K_m(y,z) \in S'(\mathbb{R}^N)$ is smooth off the diagonal and, by Theorem 1.34, satisfies estimates

$$\sup_{|\alpha|+|\beta|=M} \left|D_y^\alpha D_z^\beta K_m(y,z)\right| \leq C_{\alpha,\beta} |y-z|^{-(M+N-m)}, \quad y \neq z \tag{3.2.2}$$

for all $M > m - N$ and

$$\sup_{|y-z|\geq 1/2} |y-z|^L \left|D_x^\alpha D_y^\beta K_m(y,z)\right| \leq C_{\alpha,\beta,L}, \tag{3.2.3}$$

for any $\alpha, \beta \in \mathbb{Z}_+^N$ and $L \geq N_{\alpha,\beta} \in \mathbb{N}$. If an h^p-atom $a(z)$ is supported in $B(0,r)$ with $r \geq 1$, a consequence from (3.2.3) is that there exists $N_m \in \mathbb{N}$ such that, for any $L \geq N_m$, there exists a constant $C_{N,m,L} > 0$ such that

$$\sup_{w \in B_x^t} |D^m b(w)| \leq C_{N,m,L}\, r^{-\frac{N}{p}+N} |x|^{-L}, \quad \forall\, 0 < t < 1 \text{ and } |x| > 10r. \tag{3.2.4}$$

Indeed, let us prove that affirmation: for $y \in B(x,t)$ and $0 < t < 1$ we have

$$|y-x| < t < 1 \leq r < \frac{1}{10}|x| \Longrightarrow |y| > |x| - |x-y| > \frac{9}{10}|x|$$

and, for $|w| < r$, it follows

$$|y-w| \geq |y| - |w| > \frac{9}{10}|x| - r > \frac{4}{5}|x|.$$

Thus, we can apply the control (3.2.3) and

$$\begin{aligned}
|D^m b(y)| &= \left|\int_{B(0,r)} D^m K_m(y,w)a(w)dw\right| \\
&\leq \int_{B(0,r)} |D^m K_m(y,w)a(w)|\,dw \\
&\leq \|a\|_{L^\infty}\int_{B(0,r)} |D^m K_m(y,w)|\,dw \\
&\lesssim |B(0,r)|^{-\frac{1}{p}}\int_{B(0,r)} |D^m K_m(y,w)|\,dw \\
&\lesssim C_{L,m}|B(0,r)|^{-\frac{1}{p}}\int_{B(0,r)} \frac{1}{|y-w|^L}dw \\
&\leq \frac{5C_{L,m}}{4}\,|B(0,r)|^{-\frac{1}{p}}\int_{B(0,r)} \frac{1}{|x|^L}dw \\
&\lesssim C_{L,m}|B(0,r)|^{1-\frac{1}{p}}\frac{1}{|x|^L} = C_{N,m,L}\,\frac{r^{N\cdot\left(1-\frac{1}{p}\right)}}{|x|^L},
\end{aligned}$$

for all $y \in B_x^t$, proving the inequality (3.2.4).

Applying again the inequality (2.2.2), by the previous affirmation we have

$$\begin{aligned}
T'a(x) &= \left[\sup_{0<t<1}\left\{⨍_{B(x,t)}\left|\frac{1}{t^m}\left[b(y)-P_{x,t,b}(y)\right]\right|^\alpha dy\right\}\right]^{\frac{1}{\alpha}} \\
&\lesssim \left[\sup_{0<t<1}\left\{⨍_{B(x,t)}|D^m b(y)|^\alpha dy\right\}\right]^{\frac{1}{\alpha}} \lesssim \frac{r^{-\frac{N}{p}+N}}{|x|^L},
\end{aligned}$$

that implies

$$\int_{|x|\geq 10r}|T'a(x)|^p dx \;\lesssim\; r^{(N-L)p}\int_{|x|\geq 10}\frac{1}{|x|^{Lp}}dx$$

where the estimate is uniformly bounded for $r \geq 1$ choosing $L > \max\{N+m, N_m\}$. Indeed, since $p > \frac{N}{N+m}$, if we take $L > N+m$ we have $L > N/p$. So $N - Lp < 0$ and $p(N-L) < 0$, because

$$L > N+m \Rightarrow L > N \Rightarrow N-L < 0 \Rightarrow p(N-L) < 0.$$

Then,

$$\begin{aligned}
\int_{|x|\geq 10r}|T'(a)(x)|^p dx &\lesssim r^{N(p-1)}\int_{|x|\geq 10r}\frac{1}{|x|^{Lp}}\,dx \\
&\lesssim r^{N(p-1)}\,|S^{N-1}|\int_{10r}^{\infty}\rho^{N-1-Lp}d\rho \\
&= r^{N(p-1)}\,|S^{N-1}|\left[-\frac{(10r)^{N-Lp}}{N-Lp}\right] \\
&\lesssim r^{N(p-1)+N-Lp} \\
&= r^{p(N-L)} \leq 1.
\end{aligned}$$

Continuing, assume now that $a(z)$ is an h^p-atom supported in $B(0,r)$ with $r<1$ that satisfies the moment condition $\int_{\mathbb{R}^N} a(z)z^\beta dz=0$, for all $|\beta|\le m-1$. Fix w_0 and denote P_{w_0,K_m} the Taylor polynomial of degree $m-1$ of K_m centered at w_0, thus

$$|K_m(w)-P_{w_0,K_m}(w)|\lesssim |w-w_0|^m \sup_{|\gamma|=m}|\partial^\gamma K_m(\widetilde{w_0})|,$$

for some $\widetilde{w_0}\in[w,w_0]$. For each $y\in B(0,10r)^c$, we consider $w_0:=y$ and $w:=y-z$ with $z\in B(0,r)$ then by (3.2.2) taking $M=m$ and reproducing (3.2.4) we have

$$|b(y)|\lesssim \int_{B(0,r)}|a(z)|z|^m \sup_{w\in[y-z,y]}|D_y^m K_m(w)|dz\lesssim \frac{r^{-\frac{N}{p}+N+m}}{|y|^N}, \tag{3.2.5}$$

for all $|y|>10r$. More than that, next we will prove that $|b(y)|\lesssim r^{-\frac{N}{p}+N+m}|y|^{-N}$, for all $|y|\ge 5r$.

Let us prove the inequality (3.2.5), for all $|y|\ge 5r$. First, note that if $y\in B(0,5r)^c$ and $z\in supp(a)\subset B(0,r)$ then

$$|y|\le \frac{5}{4}|w|, \quad \text{for all } w\in[y-z,y]. \tag{3.2.6}$$

Indeed, if $w\in[y-z,y]$ then there is a $s\in[0,1]$ such that $w=sy+(1-s)(y-z)=y-(1-s)z$ and, by the triangular inequality,

$$|w|\ \ge\ |y|-|(1-s)z|\ \ge\ |y|-(1-s)r\ \ge\ |y|-(1-s)\frac{|y|}{5}\ \ge\ \frac{4+s}{5}|y|\ \ge\ \frac{4}{5}|y|.$$

By the moment condition of a,

$$\begin{aligned}\int_{B(0,r)}P_{y,K_m}(y-z)a(z)dz &= \int_{B(0,r)}\left(\sum_{|\gamma|\le m-1}\frac{1}{\gamma!}\partial^\gamma K_m(y)(-z)^\gamma\right)a(z)dz\\ &= \sum_{|\gamma|\le m-1}\frac{-1}{\gamma!}\partial^\gamma K_m(y)\left(\int_{B(0,r)}(z)^\gamma a(z)dz\right)\\ &= \sum_{|\gamma|\le m-1}\frac{-1}{\gamma!}\partial^\gamma K_m(y)0=0.\end{aligned}$$

This way,

$$\begin{aligned}
|b(y)| &= \left|\int_{B(0,r)} K_m(y-z)a(z)dz\right| \\
&= \left|\int_{B(0,r)} K_m(y-z)a(z)dz - \int_{B(0,r)} P_{y,K_m}(y-z)a(z)dz\right| \\
&= \left|\int_{B(0,r)} a(z)\left(K(y-z) - P_{y,K_m}(y-z)\right)dz\right| \\
&\leq \int_{B(0,r)} |a(z)|\,|K(y-z) - P_{y,K_m}(y-z)|dz \\
&= \int_{B(0,r)} |a(z)|\left|\sum_{|\gamma|=m} \frac{1}{\gamma!} R^{\gamma}_{y,K_m}(y-z)(-z)^{\gamma}\right| dz \\
&\leq \int_{B(0,r)} |a(z)|\, c_m |z|^m \sup_{w\in[y-z,y]} \{|D^m K_m(w)|\} dz \\
&\leq \int_{B(0,r)} |a(z)|\, \tilde{c_m} |z|^m \sup_{w\in[y-z,y]} \left\{\frac{1}{|w|^{m+N-m}}\right\} dz, \text{ by the control (3.2.2)} \\
&\simeq \int_{B(0,r)} |a(z)||z|^m \sup_{w\in[y-z,y]} \left\{\frac{1}{|w|^{N}}\right\} dz.
\end{aligned}$$

And, by the estimate (3.2.6),

$$\begin{aligned}
|b(y)| &\lesssim \int_{B(0,r)} |a(z)||z|^m \frac{1}{|y|^N} dz \leq \frac{r^m}{|y|^N}\, \|a\|_{L^\infty}\, |B(0,r)| \\
&\leq \frac{r^m}{|y|^N}\, |B(0,r)|^{1-\frac{1}{p}} \lesssim \frac{r^{m-\frac{N}{p}+N}}{|y|^N},
\end{aligned}$$

concluding the inequality (3.2.5), for all $|y| \geq 5r$.

Similarly, we will prove that for each $j \in \mathbb{N}$

$$|D^j b(y)| \leq C\, \frac{r^{-\frac{N}{p}+N+m}}{|y|^{N+j}}, \qquad \text{for all } |y| \geq 10r. \tag{3.2.7}$$

Indeed,

$$\begin{aligned}
|D^j b(y)| &= \left|\int_{B(0,r)} D^j K_m(y-\eta)a(\eta)\,d\eta\right| \\
&= \left|\int_{B(0,r)} D^j K_m(y-\eta)a(\eta)d\eta - \int_{B(0,r)} P_{y,D^jK_m}(y-\eta)a(\eta)\,d\eta\right| \\
&= \left|\int_{B(0,r)} a(\eta)\big(D^jK(y-\eta)-P_{y,D^jK_m}(y-\eta)\big)\,d\eta\right| \\
&\leq \int_{B(0,r)} |a(\eta)|.|D^jK(y-\eta)-P_{y,D^jK_m}(y-\eta)|\,d\eta \\
&\leq \int_{B(0,r)} |a(\eta)|.|\eta|^m.c_m.\sup_{w\in[y-\eta,y]}\{|D^{m+j}K_m(w)|\}\,d\eta \\
&\leq \int_{B(0,r)} |a(\eta)|.|\eta|^m.\widetilde{c_m}.\sup_{w\in[y-\eta,y]}\left\{\frac{1}{|w|^{m+j+N-m}}\right\}d\eta, \text{ by Theorem 1.34} \\
&\simeq \int_{B(0,r)} |a(\eta)|.|\eta|^m.\sup_{w\in[y-\eta,y]}\left\{\frac{1}{|w|^{N+j}}\right\}d\eta \\
&\lesssim \int_{B(0,r)} |a(\eta)|.|\eta|^m\frac{1}{|y|^{N+j}}\,d\eta \\
&\leq \frac{r^m}{|y|^{N+j}}\,\|a\|_{L^\infty}\,|B(0,r)| \\
&\leq \frac{r^m}{|y|^{N+j}}\,|B(0,r)|^{1-\frac{1}{p}} \\
&\lesssim \frac{r^{m-\frac{N}{p}+N}}{|y|^{N+j}}.
\end{aligned}$$

Next, our goal will be to prove the following global estimate

$$|b(y)| = |J_m a(y)| \lesssim r^{-\frac{N}{p}+m}, \qquad \text{for all } y\in\mathbb{R}^N. \tag{3.2.8}$$

In a first case, we will consider $m<N$. The control (3.2.2) for $M=0$ implies that $|J_m a(y)| \lesssim I_m|a|(y)$, for all $y\in\mathbb{R}^N$, where $I_\gamma f = f * |y|^{-(N-\gamma)}$ is the Riesz Potential (see Section 1.4.1). We apply Proposition 1.47 for $\beta>1$ and $\beta^* = \dfrac{\beta N}{N-m\beta}$, then

$$\begin{aligned}
|J_m a(y)| &\lesssim |I_m(|a|)(y)| \lesssim [M|a|(y)]^{\frac{\beta}{\beta^*}}\,\|a\|_{L^\beta}^{1-\frac{\beta}{\beta^*}} \\
&\leq \|a\|_{L^\infty}^{\frac{\beta}{\beta^*}}\,\|a\|_{L^\infty}^{1-\frac{\beta}{\beta^*}}\,r^{\frac{N}{\beta}(1-\frac{\beta}{\beta^*})} \\
&= \|a\|_{L^\infty}\,r^\gamma \lesssim r^{-\frac{N}{p}+m}, \qquad \text{for all } y\in\mathbb{R}^N.
\end{aligned}$$

Now, we consider the case $N\leq m$. Applying the Theorem 1.50 j times for $p=1$, where $j=m-N+$

$1 \geq 1$, and applying the item (e) of the Theorem 1.34 we get

$$\begin{aligned}
|b(y)| &\leq \int_{B(0,r)} |K_m(y,z)a(z)|dz \\
&\lesssim r^{-\frac{N}{p}} \int_{B(0,r)} |K_m(y,z)|dz \\
&\lesssim r^{j-\frac{N}{p}} \int_{B(0,r)} |D^j K_m(y,z)|dz \\
&\lesssim r^{j-\frac{N}{p}} \int_{B(0,r)} \frac{1}{|y-z|}dz \\
&= r^{j-\frac{N}{p}} I_{N-1}(\mathbb{X}_{B(0,r)})(y).
\end{aligned}$$

Again, by the Proposition 1.47, we have $|I_{N-1}(\mathbb{X}_{B(0,r)})(y)| \lesssim \|\mathbb{X}_{B(0,r)}\|_{L^\infty} r^{N-1}$. Thus,

$$|b(y)| \lesssim r^{j-\frac{N}{p}+N-1} = r^{-\frac{N}{p}+m}, \quad \text{for all } y \in \mathbb{R}^N,$$

concluding the inequality (3.2.8).

Next, we can see that the estimate (3.2.8) may be combined with (3.2.5) to yield a new global estimate, getting

$$|b(y)| \leq C\frac{r^{-\frac{N}{p}+N+m}}{(r+|y|)^N}, \quad \text{for all } y \in \mathbb{R}^N. \tag{3.2.9}$$

Indeed, for $y \in B(0,10r)^c$, we have

$$\begin{aligned}
|y| \geq 10r \;&\Rightarrow\; r \leq \frac{1}{10}|y| \\
&\Rightarrow\; r+|y| \leq \frac{11}{10}|y| \\
&\Rightarrow\; \frac{1}{|y|} \leq \frac{11}{10}\frac{1}{r+|y|} \\
&\Rightarrow\; \frac{1}{|y|^N} \leq \left(\frac{11}{10}\right)^N \frac{1}{(r+|y|)^N}
\end{aligned}$$

Thus, by the estimate (3.2.5),

$$|b(y)| \lesssim \left(\frac{11}{10}\right)^N \frac{r^{m-\frac{N}{p}+N}}{(r+|y|)^N},\; y \in B(0,10r)^c.$$

Now, to the other case when $y \in B(0,10r)$, we have

$$\begin{aligned}
|y| \leq 10r \;&\Rightarrow\; r+|y| \leq 11r \\
&\Rightarrow\; \frac{1}{11} \leq \frac{r}{r+|y|} \\
&\Rightarrow\; \frac{1}{11^N} \leq \frac{r^N}{(r+|y|)^N}.
\end{aligned}$$

By the inequality (3.2.8),

$$\begin{aligned} |b(y)| &\lesssim r^{m-\frac{N}{p}} \\ &= r^{m-\frac{N}{p}} \cdot \frac{11^N}{11^N} \\ &\leq r^{m-\frac{N}{p}}\, 11^N \,\frac{r^N}{(r+|y|)^N} \\ &= 11^N \frac{r^{m-\frac{N}{p}+N}}{(r+|y|)^N}, \quad \text{if } y \in B(0,10r) \end{aligned}$$

and we conclude that

$$|b(y)| \leq c\frac{r^{m-\frac{N}{p}+N}}{(r+|y|)^N}, \qquad \text{for all } y \in \mathbb{R}^N,$$

proving the inequality (3.2.9). Finally, we can also conclude that

$$|b(ry)| \leq C\frac{r^{-\frac{N}{p}+m}}{(1+|y|)^N}, \quad y \in \mathbb{R}^N. \tag{3.2.10}$$

After all these estimates, let us go back to the proof that $\int_{B(0,10r)^c} |T'(a)|^p \lesssim 1$. For this, consider now $|x| \geq 10$, thus

$$\begin{aligned} T'(a)(rx) &= \sup_{0<t<1} \left\{ \left[\fint_{B(rx,t)} \left| \frac{1}{t^m}[b(\tilde{y}) - P_{rx,t,b}(\tilde{y})] \right|^{\alpha} d\tilde{y} \right]^{\frac{1}{\alpha}} \right\} \\ &= \sup_{0<t<1} \left\{ \left[\fint_{B\left(x,\frac{t}{r}\right)} \left| \frac{1}{t^m}[b(ry) - P_{rx,t,b}(ry)] \right|^{\alpha} dy \right]^{\frac{1}{\alpha}} \right\} \\ &\leq \underbrace{\sup_{0<t<\frac{r|x|}{2}} \left\{ \left[\fint_{B\left(x,\frac{t}{r}\right)} \left| \frac{1}{t^m}[b(ry) - P_{rx,t,b}(ry)] \right|^{\alpha} dy \right]^{\frac{1}{\alpha}} \right\}}_{(I)} \\ &+ \underbrace{\sup_{\frac{r|x|}{2}<t<1} \left\{ \left[\fint_{B\left(x,\frac{t}{r}\right)} \left| \frac{1}{t^m}[b(ry) - P_{rx,t,b}(ry)] \right|^{\alpha} dy \right]^{\frac{1}{\alpha}} \right\}}_{(II)}. \end{aligned}$$

Analogously to the previous arguments, we get

$$(I) \lesssim \frac{r^{-\frac{N}{p}}}{|x|^{N+m}}, \tag{3.2.11}$$

reinforcing that $y \in B(x,t/r)$ implies that $ry \in B(0,5r)^c$, so we need to apply the estimate (3.2.5) for all $B(0,5r)^c$ to conclude (3.2.11). Indeed, we are in the case where $|x| \geq 10$ and $t \leq \frac{r|x|}{2}$, so

$$|y| \;=\; |x-(x-y)| \;\geq\; |x|-|x-y| \;\geq\; |x|-\frac{t}{r} \;\geq\; |x|-\frac{|x|}{2} \;=\; \frac{|x|}{2} \;\geq\; 5. \tag{3.2.12}$$

Thus, we can apply the inequality (3.2.7) for $j=m$, getting

$$\begin{aligned}
\left[\fint_{B\left(x,\frac{t}{r}\right)}\left|\frac{1}{t^m}(b(ry)-P_{rx,t,b}(ry))\right|^{\alpha}dy\right]^{\frac{1}{\alpha}} &\leq C_m\left[\fint_{B\left(x,\frac{t}{r}\right)}|D^m b(ry)|^{\alpha}dy\right]^{\frac{1}{\alpha}}\\
&\leq C_m\left[\fint_{B\left(x,\frac{t}{r}\right)}\left|\frac{r^{m-\frac{N}{p}+N}}{|ry|^{N+m}}\right|^{\alpha}dy\right]^{\frac{1}{\alpha}}\\
&\leq C_m\left[\fint_{B\left(x,\frac{t}{r}\right)}\left|\frac{r^{-\frac{N}{p}}}{|y|^{N+m}}\right|^{\alpha}dy\right]^{\frac{1}{\alpha}}\\
&\leq C_m r^{-\frac{N}{p}}.\left[\fint_{B\left(x,\frac{t}{r}\right)}\left|\frac{1}{|y|^{N+m}}\right|^{\alpha}dy\right]^{\frac{1}{\alpha}}.
\end{aligned}$$

Note that, by inequality (3.2.12), $|y|\geq |x|/2$, so

$$\begin{aligned}
(I) &= \sup_{0<t<\frac{r|x|}{2}}\left\{\left[\fint_{B\left(x,\frac{t}{r}\right)}\left|\frac{1}{t^m}(b(ry)-P_{rx,t,b}(ry))\right|^{\alpha}dy\right]^{\frac{1}{\alpha}}\right\}\\
&\lesssim r^{-\frac{N}{p}}\sup_{0<t<\frac{r|x|}{2}}\left\{\left[\fint_{B\left(x,\frac{t}{r}\right)}\left(\frac{1}{|x|^{N+m}}\right)^{\alpha}dy\right]^{\frac{1}{\alpha}}\right\} = \frac{r^{-\frac{N}{p}}}{|x|^{N+m}}.
\end{aligned}$$

concluding (3.2.11).

To estimate (II), note that

$$\begin{aligned}
(II) &= r^{-m}\sup_{\frac{|x|}{2}<t<\frac{1}{r}}\left\{\left[\fint_{B(x,t)}\left|\frac{1}{t^m}[b(ry)-P_{rx,rt,b}(ry)]\right|^{\alpha}dy\right]^{\frac{1}{\alpha}}\right\}\\
&\lesssim r^{-m}\sup_{\frac{|x|}{2}<t<\frac{1}{r}}\left\{\frac{1}{t^{m+\frac{N}{\alpha}}}\left(\|b(r\cdot)\|_{L^{\alpha}(B(x,t))}+\left\|P_{rx,rt,b}(r\cdot)\right\|_{L^{\alpha}(B(x,t))}\right)\right\}.
\end{aligned}$$

We will prove that

(i) $\|(b(r\cdot)\|_{L^{\alpha}(B(x,t))}\lesssim r^{m-\frac{N}{p}}$;

(ii) $\left\|P_{rx,rt,b}(r\cdot)\right\|_{L^{\alpha}(B(x,t))}\lesssim \dfrac{r^{m-\frac{N}{p}}}{t^{N(1-\frac{1}{\alpha})}}[1+\log t]$.

Indeed, it follows from the inequality (3.2.10) that

$$
\begin{aligned}
\|(b(r\cdot)\|_{L^\alpha(B(x,t))} &= \left(\int_{B(x,t)} |b(ry)|^\alpha dy\right)^{\frac{1}{\alpha}} \\
&\lesssim \left(\int_{B(x,t)} \left(\frac{r^{m-\frac{N}{p}+N}}{(r+|ry|)^N}\right)^\alpha dy\right)^{\frac{1}{\alpha}} \\
&\leq r^{m-\frac{N}{p}} \left(\int_{\mathbb{R}^N} \left(\frac{1}{(1+|y|)^N}\right)^\alpha dy\right)^{\frac{1}{\alpha}}.
\end{aligned}
$$

Since $\alpha > 1$ then

$$
\begin{aligned}
\int_{\mathbb{R}^N} \left(\frac{1}{(1+|y|)^N}\right)^\alpha dy &= \int_{B(0,r_0)} \frac{1}{(1+|y|)^{N\alpha}} dy + \int_{B(0,r_0)^c} \frac{1}{(1+|y|)^{N\alpha}} dy \\
&\leq |B(0,r_0)| + \int_{B(0,r_0)^c} \frac{1}{(1+|y|)^{N\alpha}} dy \\
&\leq |B(0,r_0)| + c_m \cdot \int_{r_0}^\infty \frac{\rho^{N-1}}{(1+\rho)^{N\alpha}} d\rho < \infty,
\end{aligned}
$$

proving (i).

Since $|P_{z,s,f}(y)| \lesssim \fint_{Q(z,2s)} |f(w)| dw$, for all $y \in B(z,s)$, and $\int_{B(z,s)} \frac{1}{(1+|w|)^N} dw \lesssim \log s$ then

$$\begin{aligned}
\left\|P_{rx,rt,b}(r\cdot)\right\|_{L^\alpha(B(x,t))} &= \left(\int_{B(x,t)} |P_{rx,rt,b}(ry)|^\alpha dy\right)^{\frac{1}{\alpha}} \\
&= \left(\frac{1}{r^N}\int_{B(rx,rt)} |P_{rx,rt,b}(\tilde{y})|^\alpha d\tilde{y}\right)^{\frac{1}{\alpha}} \\
&\lesssim \left(\frac{1}{r^N}\int_{B(rx,rt)} \left(\fint_{Q(rx,2rt)} |b(\tilde{z})|d\tilde{z}\right)^\alpha d\tilde{y}\right)^{\frac{1}{\alpha}} \\
&= \frac{|B(rx,rt)|^{\frac{1}{\alpha}}}{r^{\frac{N}{\alpha}}} \fint_{Q(rx,2rt)} |b(\tilde{z})|d\tilde{z} \\
&= \frac{|B(x,t)|^{\frac{1}{\alpha}}}{|Q(rx,2rt)|} \int_{Q(rx,2rt)} |b(\tilde{z})|d\tilde{z} \\
&= \frac{|B(x,t)|^{\frac{1}{\alpha}}}{|Q(rx,2rt)|}\, r^N \int_{Q(x,2t)} |b(rz)|dz \\
&\simeq \frac{1}{t^{N(1-\frac{1}{\alpha})}} \int_{Q(x,2t)} |b(rz)|dz \\
&\lesssim \frac{1}{t^{N(1-\frac{1}{\alpha})}} \int_{Q(x,2t)} \frac{r^{m-\frac{N}{p}+N}}{(r+|rz|)^N}dz \\
&= \frac{r^{m-\frac{N}{p}}}{t^{N(1-\frac{1}{\alpha})}} \int_{Q(x,2t)} \frac{1}{(1+|z|)^N}dz \\
&\leq \frac{r^{m-\frac{N}{p}}}{t^{N(1-\frac{1}{\alpha})}} \int_{B(x,(N+1)t)} \frac{1}{(1+|z|)^N}dz \\
&\lesssim \frac{r^{m-\frac{N}{p}}}{t^{N(1-\frac{1}{\alpha})}} \log[(N+1)t] \\
&= \frac{r^{m-\frac{N}{p}}}{t^{N(1-\frac{1}{\alpha})}} [1+\log t],
\end{aligned}$$

proving (ii).

The estimates (i) and (ii) imply

$$\begin{aligned}
(II) &\lesssim r^{-m} \sup_{\frac{|x|}{2}<t<\frac{1}{r}} \left\{\frac{1}{t^{m+\frac{N}{\alpha}}}\left(r^{m-\frac{N}{p}} + \frac{r^{m-\frac{N}{p}}}{t^{N(1-\frac{1}{\alpha})}}[1+\log t]\right)\right\} \\
&= r^{-\frac{N}{p}} \sup_{\frac{|x|}{2}<t<\frac{1}{r}} \left\{\frac{1}{t^{m+\frac{N}{\alpha}}} + \frac{1+\log t}{t^{N+m}}\right\}
\end{aligned}$$

and, therefore

$$(II) \lesssim r^{-\frac{N}{p}} \left[\frac{1}{|x|^{m+\frac{N}{\alpha}}} + \frac{1+\log|x|}{|x|^{m+N}}\right]. \tag{3.2.13}$$

This way, by the inequalities (3.2.11) and (3.2.13), we have

$$\begin{aligned}\int_{|x|\geq 10r} |T'(a)(x)|^p dx &= r^N \int_{|x|\geq 10} |T'(a)(rx)|^p dx \\ &\lesssim \int_{|x|\geq 10} \left(\frac{1+\log|x|}{|x|^{N+m}} + \frac{1}{|x|^{m+\frac{N}{\alpha}}} \right)^p dx < \infty,\end{aligned}$$

since the hypotheses $N/(N+m) < p$ and $\alpha < p^*$ imply and $(N+m)p > N$ and $\left(m+\dfrac{N}{\alpha}\right)p > N$.

Lastly, for $\alpha = 1$ we may choose $\gamma > 1$ and then by the inequality (2.2.2) we have

$$T'(a)(x) \lesssim [M(|D^m b|^{\gamma})(x)]^{\frac{1}{\gamma}}.$$

and the proof follows *is bis idem* except now we will have

(iii) $\|(b(r\cdot)\|_{L^1(B(x,t))} \lesssim r^{m-\frac{N}{p}} \ \log t$;

(iv) $\left\|P_{rx,rt,b}(r\cdot)\right\|_{L^1(B(x,t))} \lesssim r^{m-\frac{N}{p}} \ [1+\log t]$.

Indeed, it again follows from the inequality (3.2.10) that

$$\|(b(r\cdot)\|_{L^1(B(x,t))} = \int_{B(x,t)} |b(ry)|dy \lesssim \int_{B(x,t)} \frac{r^{m-\frac{N}{p}+N}}{(r+|ry|)^N} dy \leq r^{m-\frac{N}{p}} \int_{B(x,t)} \frac{1}{(1+|y|)^N} dy.$$

Since $\int_{B(x,s)} \frac{1}{(1+|z|)^N} dz \lesssim \log s$, then $\|(b(r\cdot)\|_{L^1(B(x,t))} \lesssim r^{m-\frac{N}{p}} \ \log t$, proving (iii).

As before, since $|P_{z,s,f}(y)| \lesssim ⨍_{Q(z,2s)} |f(w)|dw$, for all $y \in B(z,s)$ then

$$
\begin{aligned}
\left\|P_{rx,rt,b}(r\cdot)\right\|_{L^1(B(x,t))} &= \int_{B(x,t)} |P_{rx,rt,b}(ry)|^{\alpha}\, dy \\
&= \frac{1}{r^N}\int_{B(rx,rt)} |P_{rx,rt,b}(\tilde{y})|\, d\tilde{y} \\
&\lesssim \frac{1}{r^N}\int_{B(rx,rt)} \left(\fint_{Q(rx,2rt)} |b(\tilde{z})|\, d\tilde{z} \right) d\tilde{y} \\
&= \frac{|B(rx,rt)|}{r^N} \fint_{Q(rx,2rt)} |b(\tilde{z})| d\tilde{z} \\
&= \frac{|B(x,t)|}{|Q(rx,2rt)|} \int_{Q(rx,2rt)} |b(\tilde{z})| d\tilde{z} \\
&\simeq \frac{1}{r^N}\, r^N \int_{Q(x,2t)} |b(rz)| dz \\
&\simeq \int_{Q(x,2t)} |b(rz)| dz \\
&\lesssim \int_{Q(x,2t)} \frac{r^{m-\frac{N}{p}+N}}{(r+|rz|)^N} dz \\
&= r^{m-\frac{N}{p}} \int_{Q(x,2t)} \frac{1}{(1+|z|)^N} dz \\
&\leq r^{m-\frac{N}{p}} \int_{B(x,(N+1)t)} \frac{1}{(1+|z|)^N} dz \\
&\lesssim r^{m-\frac{N}{p}} \log[(N+1)t] \\
&= r^{m-\frac{N}{p}} [1+\log t],
\end{aligned}
$$

proving (iv). Now we have

$$
\begin{aligned}
(II) &\lesssim r^{-m} \sup_{\frac{|x|}{2}<t<\frac{1}{r}} \left\{ \frac{1}{t^{m+N}} \left(r^{m-\frac{N}{p}} \log t + r^{m-\frac{N}{p}} [1+\log t] \right) \right\} \\
&= r^{-\frac{N}{p}} \sup_{\frac{|x|}{2}<t<\frac{1}{r}} \left\{ \frac{1+2\log t}{t^{N+m}} \right\} \\
&= r^{-\frac{N}{p}} \left(\frac{1+2\log|x|}{|x|^{m+N}} \right)
\end{aligned}
$$

and, therefore

$$
\begin{aligned}
\int_{|x|\geq 10r} |T'(a)(x)|^p dx &= r^N \int_{|x|\geq 10} |T'(a)(rx)|^p dx \\
&\lesssim \int_{|x|\geq 10} \left(\frac{1+\log|x|}{|x|^{N+m}} \right)^p dx < \infty,
\end{aligned}
$$

by hypothesis $N/(N+m) < p$.

□

Remark 3.4. *An observation to be made is that the hypotheses required about β and β' throughout the proof still assure us that the exponents p and q satisfy the hypotheses of the Theorem A.*

In fact, throughout the proofs we assume that $\beta, \beta' > 1$, $\beta < q$, $1 \leq \beta' < p^* = \frac{Np}{N-mp}$, $\beta' < \beta$ and $\frac{1}{\beta} + \frac{1}{\beta'} = 1$. We need that $\frac{1}{p} + \frac{1}{q} < 1 + \frac{m}{N}$, $\frac{N}{N+m} < p \leq \frac{N}{m}$ and $q \geq 1$.

(i) Since $\beta' < p^*$ then $\frac{1}{\beta'} - \frac{1}{p^*} > 0$ and

$$\frac{1}{q} < \frac{1}{\beta} < \frac{1}{\beta} + \left(\frac{1}{\beta'} - \frac{1}{p^*}\right) = 1 - \frac{1}{p^*} = 1 - \left(\frac{1}{p} - \frac{m}{N}\right) = 1 + \frac{m}{N} - \frac{1}{p}.$$

Then, $\frac{1}{p} + \frac{1}{q} < 1 + \frac{m}{N}$.

(ii) Since $\frac{1}{q} > 0$ then, by the previous item,

$$\frac{1}{p} < 1 + \frac{m}{N} - \frac{1}{q} < 1 + \frac{m}{N} = \frac{N+m}{N}.$$

This way, $p > \frac{N}{N+m}$.

(iii) Now suppose by absurdity that $q < 1$, then $\beta < q < 1$ implies that $\frac{1}{\beta} > 1$. Furthermore, $\beta' \geq 1 > 0 \Rightarrow \frac{1}{\beta'} > 0$. So,

$$\frac{1}{\beta} + \frac{1}{\beta'} > 1 + 0 = 1,$$

and it is an absurd. Therefore, $q \geq 1$.

Remark 3.5. *After finishing the proof, it is worth noting that in the case where $1 < q < N$, it follows from Poincaré inequality and Theorem **A** that the estimate*

$$\|A(\cdot,D)\phi \cdot v\|_{h^r} \leq C \|A(\cdot,D)\phi\|_{h^p} \|D^{m-1}v\|_{L^q}$$

holds for any $\phi \in C_c^\infty(U,E)$ and $v \in C_c^\infty(U,F)$, satisfying $A^*(\cdot,D)v = 0$, in the sense of distributions.

3.3 Nonhomogeneous estimate of Theorem A and applications

We start presenting a nonhomogeneous version of Theorem **A** where the assumption $A^*(\cdot,D)v = 0$ (in the sense of distributions) is removed.

Theorem 3.6. *Let $A(\cdot,D)$ an elliptic homogeneous linear differential operator as before on $\Omega \subset \mathbb{R}^N$, with order $m > 0$ and let r, p, q satisfying*

$$\frac{N}{N+m} < p \leq \frac{N}{m}, \quad 1 < q \leq \infty \text{ and } \frac{1}{r} := \frac{1}{p} + \frac{1}{q} < 1 + \frac{m}{N}.$$

Then for each $x_0 \in \Omega$ there exist an open neighborhood $x_0 \in U \subset \Omega$ and $C > 0$ such that

$$\|A(\cdot,D)\phi \cdot_F v\|_{h^r} \leq C \|A(\cdot,D)\phi\|_{h^p} \left(\|v\|_{W^{m-1,q}} + \|A^*(\cdot,D)v\|_{L^q}\right)$$

for any $\phi \in C_c^\infty(U,E)$ and $v \in C_c^\infty(U,F)$.

Proof. As in the previous case, let $\psi \in C_c^\infty(B(0,1))$ with $\int \psi = 1$ and $\psi_t(x) := t^{-N}\psi(x/t)$, $0<t<1$. Following the notation of Theorem **A**, we may write

$$\psi_t * [A(\cdot,D)\phi \cdot_F v](x) = \int_{B(x,t)} A^*(y,D)\left[\psi_t(x-y)\overline{v(y)}\right] \cdot_E (\phi - P)(y)dy,$$

where P is a polynomial of degree at most $m-1$. By the uniform local boundedness of the functions $\partial^\theta a_\alpha^*$ and $\partial^\theta \psi$ for $0 \le |\theta| \le m$, and taking $0<t\le 1$ we have

$$|\psi_t * A(\cdot,D)\phi \cdot_F v|(x) \lesssim \frac{C(\psi)}{t^{N+m}} \int_{B_x^t} \left(\sum_{0\le|\gamma|<m} |\partial^\gamma v(y)| + |A^*(y,D)v(y)| \right) |\phi(y) - P(y)|\, dy,$$

and the conclusion follows *is bis idem* as before. □

In the next two subsections, we will see that the Theorems **A** and 3.6 extend two particular cases:

(i) When $m=1$ and $A(\cdot,D) = \nabla_{\mathscr{L}}$, as in the Example 1.3.3, extending [24, Theorem A];

(ii) When $A(\cdot,D) = (d_{\mathscr{L},k}, d^*_{\mathscr{L},k-1})$, where $\{d_{\mathscr{L},k}\}_k$ is the pseudo-complex associated with $\mathscr{L}$, extending [24, Theorem B and C].

3.3.1 Application 1: elliptic systems of complex vector fields

Let $\mathscr{L} := \{L_1, ..., L_n\}$ be an elliptic system of linearly independent complex vector fields with complex smooth coefficients defined on Ω for $N \ge 2$. As already seen in Example 1.3.3, the gradient operator associated with $\mathscr{L}$ given by $\nabla_{\mathscr{L}} = (L_1, \dots, L_n)$ is an example of an elliptic linear differential operator $A(\cdot,D)$, for $m=1$, $E=\mathbb{C}$ and $F=\mathbb{C}^n$. Furthermore, its formal complex adjoint operator, defined for $v \in C^\infty(\Omega, \mathbb{C}^n)$, is given by:

$$\mathrm{div}_{\mathscr{L}^*} v := L_1^* v_1 + L_2^* v_2 + ... + L_n^* v_n.$$

As a direct consequence of the previous nonhomogeneous version we extend the [24, Theorem A], stated as follows:

Corollary 3.7. *Assume that the system of complex vector fields $\{L_1, \dots, L_n\}$ is elliptic on $\Omega \subset \mathbb{R}^N$ and let p, q satisfy*

$$\frac{N}{N+1} < p < \infty, \quad 1 < q \le \infty, \quad \frac{1}{r} := \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N}.$$

Then for every point $x_0 \in \Omega$ there exist an open neighborhood $x_0 \in U \subset \Omega$ and a positive constant $C>0$ such that

$$\|\nabla_{\mathscr{L}}\phi \cdot v\|_{h^r} \le C \|\nabla_{\mathscr{L}}\phi\|_{h^p} \left(\|v\|_{h^q} + \|\mathrm{div}_{\mathscr{L}^*} v\|_{h^q}\right), \tag{3.3.1}$$

holds for any $\phi \in C_c^\infty(U)$ and $v \in C_c^\infty(U, \mathbb{C}^n)$.

3.3.2 Application 2: pseudo-complexes associated with vector fields

First, let us present the basic notation used in the subject of pseudo-complexes associated to the system $\mathscr{L}$. Denote by $C^\infty(\Omega,\Lambda^k\mathbb{R}^n)$ the space of k-forms on $\mathbb{R}^n$ with smooth complex coefficients defined on Ω and $0\le k\le n$. Each $f\in C^\infty(\Omega,\Lambda^k\mathbb{R}^n)$ may be written as

$$f=\sum_{|I|=k} f_I dx_I, \qquad \text{with } dx_I := dx_{i_1}\wedge\cdots\wedge dx_{i_k},$$

where $f_I\in C^\infty(\Omega)$ and $I=\{i_1,...,i_k\}$ is a set of strictly increasing indices with $i_\ell\in\{1,...,n\}$, for $\ell=1,...,k$.

Fixed a system of complex vector fields $\mathscr{L}=\{L_1,\dots,L_n\}$, consider the differential operators

$$d_{\mathscr{L},k}: C^\infty(\Omega,\Lambda^k\mathbb{R}^n)\mapsto C^\infty(\Omega,\Lambda^{k+1}\mathbb{R}^n)$$

defined by

$$d_{\mathscr{L},0}f := \sum_{j=1}^n (L_j f)dx_j,$$

for $f\in C^\infty(\Omega)$ and for each $1\le k\le n-1$

$$d_{\mathscr{L},k}f := \sum_{|I|=k}(d_{\mathscr{L},0}f_I)dx_I = \sum_{|I|=k}\sum_{j=1}^n (L_j f_I)\, dx_j\wedge dx_I,$$

for $f=\sum_{|I|=k} f_I dx_I\in C^\infty(\Omega,\Lambda^k\mathbb{R}^n)$.

We also define the dual pseudo-complex $d^*_{\mathscr{L},k}: C^\infty(\Omega,\Lambda^{k+1}\mathbb{R}^n)\mapsto C^\infty(\Omega,\Lambda^k\mathbb{R}^n)$, for $0\le k\le n-1$, determined by the following relation for any $u\in C^\infty_c(\Omega,\Lambda^k\mathbb{R}^n)$ and $v\in C^\infty_c(\Omega,\Lambda^{k+1}\mathbb{R}^n)$:

$$\int d_{\mathscr{L},k}u\cdot\bar{v} = \int u\cdot\overline{d^*_{\mathscr{L},k}v},$$

where the dot indicates the standard pairing on forms of the same degree. This is to say that given $f=\sum_{|J|=k} f_J dx_J$, one has:

$$d^*_{\mathscr{L},k}f = \sum_{|J|=k}\sum_{j\in J} L^*_j f_J dx_j\vee dx_J,$$

where, for each $j_\ell\in J=\{j_1,...,j_k\}$ and $\ell\in\{1,...,k\}$, $dx_{j_\ell}\vee dx_J$ is defined by:

$$dx_{j_\ell}\vee dx_J := (-1)^{\ell+1}dx_1\wedge...\wedge dx_{j_{\ell-1}}\wedge dx_{j_{\ell+1}}\wedge...\wedge dx_{j_k}.$$

Now, suppose that the system $\mathscr{L}$ is involutive, then the chain $\{d_{\mathscr{L},k}\}_k$ defines a complex of differential operators associated to the system $\mathscr{L}$, which is precisely the *de Rham complex* when $n=N$ and $L_j=\partial_{x_j}$ (see [5] for more details). In the non-involutive situation, we do not get a complex in general, and the fundamental complex property $d_{\mathscr{L},k+1}\circ d_{\mathscr{L},k}=0$ might not hold. On the other hand, this chain still satisfies a “pseudo-complex” property in the sense that $d_{\mathscr{L},k+1} \circ d_{\mathscr{L},k}$ is a differential of operator of order one rather than two, as it is generically expected. We will refer to $(d_{\mathscr{L},k}, C^\infty(\Omega, \Lambda^k\mathbb{R}^n))$ as the pseudo-complex $\{d_{\mathscr{L}}\}$ associated with $\mathscr{L}$ on Ω.

Furthermore, we can also define the operator $\Delta_k : C^\infty(\mathbb{R}^N, \Lambda^k\mathbb{R}^n) \mapsto C^\infty(\mathbb{R}^N, \Lambda^k\mathbb{R}^n)$ given by

$$\Delta_k = d_{\mathscr{L},k-1} d^*_{\mathscr{L},k-1} + d^*_{\mathscr{L},k} d_{\mathscr{L},k},$$

with $0 \leq k \leq n$ and considering $d^*_{\mathscr{L},-1} = d_{\mathscr{L},n} = 0$ for the case $k = 0$ and $k = n$. Note that if $f \in C^\infty(\mathbb{R}^N, \Lambda^k\mathbb{R}^n)$, then $\Delta_k f$ also belongs to $C^\infty(\mathbb{R}^N, \Lambda^k\mathbb{R}^n)$ and can be written as

$$\Delta_k f = \sum_{|I|=k} (\Delta_k f)_I dx_I.$$

Lemma 3.8 (Lemma 3.1, [30]). *Let* $f = \sum_{|I|=k} f_I dx_I \in C^\infty(\mathbb{R}^N, \Lambda^k\mathbb{R}^n)$, *then*

$$(\Delta_k f)_I = \Delta_{\mathscr{L}}(f_I) + \sum_{j\in I} \sum_{\ell \in I} [L_j, L^*_\ell] f_{\{\ell\}\cup I - \{j\}}.$$

In particular, if $\mathscr{L}$ is a system with complex constant coefficients then $[L_j, L^*_\ell] = 0$, for all $j, \ell \in I$ and $(\Delta_k f)_I = \Delta_{\mathscr{L}}(f_I)$, for all $|I| = k$. Thus, by denoting $\Delta_{\mathscr{L}} f = \sum_{|I|=k} \Delta_{\mathscr{L}}(f_I) dx_I$, where $f \in C^\infty(\mathbb{R}^N, \Lambda^k\mathbb{R}^n)$, we conclude $\Delta_{\mathscr{L}} f = \Delta_k f$.

Proposition 3.9 (Hodge decomposition). *Let $\mathscr{L}$ be a system with complex constant coefficients and $f \in C^\infty(\mathbb{R}^N, \Lambda^k\mathbb{R}^n)$, then there are g, h k-forms satisfying $d_{\mathscr{L},k} g = 0$ and $d^*_{\mathscr{L},k-1} h = 0$ such that $f = g + h$.*

Proof. Consider the operator $\Delta_k = d_{\mathscr{L},k-1} d^*_{\mathscr{L},k-1} + d^*_{\mathscr{L},k} d_{\mathscr{L},k}$ as before. By the previous Lemma, since the coefficients are constant we get $\Delta_k f = \Delta_{\mathscr{L}} f$, for all f k-form. Thus,

$$\begin{aligned} f &= \delta_0 * f = \Delta_{\mathscr{L}} E * f = \Delta_{\mathscr{L}}(E * f) = \Delta_k(E * f) \\ &= d_{\mathscr{L},k-1} d^*_{\mathscr{L},k-1}(E * f) + d^*_{\mathscr{L},k} d_{\mathscr{L},k}(E * f) \\ &:= g + h, \end{aligned}$$

where E is the fundamental solution of $\Delta_{\mathscr{L}}$. Furthermore, note that

$$d_{\mathscr{L},k}\, g = d_{\mathscr{L},k} d_{\mathscr{L},k-1}[d^*_{\mathscr{L},k-1}(E * f)] = 0$$

and

$$d^*_{\mathscr{L},k-1}\, h = d^*_{\mathscr{L},k-1} d^*_{\mathscr{L},k}[d_{\mathscr{L},k}(E * f)] = 0.$$

□

In particular, passing through an isomorphism, we can see $d_{\mathscr{L},0}$ as $\nabla_{\mathscr{L}}$ and $d^*_{\mathscr{L},0}$ as $\text{div}_{\mathscr{L}^*}$. Thus, all f 1-form can be written as $f = g + h$, in which

$$g = d_{\mathscr{L},0}[d^*_{\mathscr{L},0}(E * f)] = \nabla_{\mathscr{L}}[d^*_{\mathscr{L},0}(E * f)]$$

and

$$div_{\mathscr{L}^*} h = d^*_{\mathscr{L},0} h = d^*_{\mathscr{L},0} d^*_{\mathscr{L},1}[d_{\mathscr{L},1}(E * f)] = 0.$$

Therefore, each f vector fields can be written as $f = h + \nabla_{\mathscr{L}}\phi$, in which $div_{\mathscr{L}^*} h = 0$ and $\phi \in C^\infty(\mathbb{R}^N)$. We will return to this discussion in Section 4.1.

Now, let $\mathscr{L} := \{L_1, ..., L_n\}$ be an elliptic system of linearly independent complex vector fields with complex smooth coefficients defined on Ω. Consider the operator

$$A(\,\cdot\,, D) = (d_{\mathscr{L},k}, d^*_{\mathscr{L},k-1}) : C_c^\infty(\Omega, \Lambda^k\mathbb{R}^n) \to C_c^\infty(\Omega, \Lambda^{k+1}\mathbb{R}^n) \times C_c^\infty(\Omega, \Lambda^{k-1}\mathbb{R}^n),$$

for $0 \leq k \leq n$. Here the operator $d_{\mathscr{L},-1} = d^*_{\mathscr{L},-1}$ is understood to be zero. From [30, Lemma 3.1] the operator $A(\cdot, D)$ is elliptic. It is clear that for $(u, v) \in C_c^\infty(\Omega, \Lambda^{k+1}\mathbb{R}^n) \times C_c^\infty(\Omega, \Lambda^{k-1}\mathbb{R}^n)$ we have

$$A^*(\cdot, D)(u, v) = d^*_{\mathscr{L},k} u + d_{\mathscr{L},k-1} v.$$

As a direct consequence of the previous nonhomogeneous version, we have:

Corollary 3.10. *Assume that the system of vector fields $\mathscr{L} = \{L_1, \ldots, L_n\}$ is elliptic and $0 \leq k \leq n$, and p, q, r are as in Theorem A. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that the estimate*

$$\|d_{\mathscr{L},k}\phi \cdot v_1 + d_{\mathscr{L},k}\phi \cdot v_2\|_{h^r} \leq C\left(\|d_{\mathscr{L},k}\phi\|_{h^p} + \|d^*_{\mathscr{L},k-1}\phi\|_{h^p}\right)\left(\|v\|_{h^q} + \|d^*_{\mathscr{L},k} v_1\|_{h^q} + \|d_{\mathscr{L},k-1} v_2\|_{h^q}\right).$$

holds for some $C > 0$ and for every $\phi \in C_c^\infty(U, \Lambda^k\mathbb{R}^n)$, $v_1 \in C_c^\infty(U, \Lambda^{k+1}\mathbb{R}^n)$ and $v_2 \in C_c^\infty(U, \Lambda^{k-1}\mathbb{R}^n)$.

In particular taking $v_2 = 0$ and $v = v_1$ we have

$$\|d_{\mathscr{L},k}\phi \cdot v\|_{h^r} \leq C\left(\|d_{\mathscr{L},k}\phi\|_{h^p} + \|d^*_{\mathscr{L},k-1}\phi\|_{h^p}\right)\left(\|v\|_{h^q} + \|d^*_{\mathscr{L},k} v\|_{h^q}\right)$$

for all $\phi \in C_c^\infty(U, \Lambda^k\mathbb{R}^n)$ and $v \in C_c^\infty(U, \Lambda^{k+1}\mathbb{R}^n)$, recovering the Theorems B and C in [24], where in the second we assume that the system $\mathscr{L}$ is involutive.

3.4 Additional remarks

In a natural way, we may expect a global version of the Theorem **A** in Hardy space in the case where the homogeneous linear differential operator has constant coefficients. We expect the validity of the following result: let $A(D)$ an elliptic homogeneous linear differential operator with constant complex coefficients in $\mathbb{R}^N$ with order $m < N$ and let r, p, q satisfying

$$\frac{N}{N+m} < p \leq 1, \quad 1 < q \leq \infty \text{ and } \frac{1}{r} := \frac{1}{p} + \frac{1}{q} < 1 + \frac{m}{N}.$$

Then for any $\phi \in C_c^\infty(\mathbb{R}^N, E)$ and $v \in C_c^\infty(\mathbb{R}^N, F)$, satisfying $A^*(\cdot, D)v = 0$, in the sense of distributions, there exists $C > 0$ such that holds

$$\|A(D)\phi \cdot_F v\|_{H^r} \leq C \, \|A(D)\phi\|_{H^p} \, \|v\|_{W^{m-1,q}}.$$

Besides the formal formulation is expected, it is not clear if the proof is a consequence of the tools presented in the local case. Global controls requires a more sophisticated machinery, with a new version of Fundamental Lemma in $H^p(\mathbb{R}^N)$.

CHAPTER 4

The proof of Theorem B

In this chapter we will proof Theorem **B**, the second main result of the thesis stated in the introduction (Theorem 2). The first section briefly highlights how vector fields in L^p can be decomposed into terms that relate to the operators $\mathrm{div}_{\mathscr{L}^*}$ and $\nabla_{\mathscr{L}}$, defined in Example 1.3.3. Next, Theorem **B** is proved, assuming two particular cases stated as Theorems 4.2 and 4.3 and the next two sections are dedicated to proving such cases, the first when $W = \nabla_{\mathscr{L}}\phi$ and the second when $\mathrm{div}_{\mathscr{L}^*} V = 0$.

4.1 Hodge decomposition on div-curl terms

Here we will present a Hodge decomposition for vector fields in the div-curl setting. The more general case has already been introduced in the context of pseudo-complexes associated with systems of vector fields, in Proposition 3.9.

Lemma 4.1. *Let $\mathscr{L} = \{L_1, L_2, ..., L_n\}$ be an elliptic system of complex vector fields on $\mathbb{R}^N$ with constant complex coefficients. Each $V \in L^p(\mathbb{R}^N, \mathbb{C}^n)$ for $1 < p < \infty$ can be decomposed as*

$$V = V_1 + V_2,$$

with V_1, V_2 satisfying $div_{\mathscr{L}^*} V_1 = 0$ and $V_2 = \nabla_{\mathscr{L}}\varphi_2$. Moreover

$$\|V_i\|_{L^p} \lesssim \|V\|_{L^p}, \quad for\ i = 1, 2. \tag{4.1.1}$$

Proof. Using the fundamental solution of $\Delta_{\mathscr{L}}$, we may define $V_2 := \nabla_{\mathscr{L}}\varphi_2$ with $\varphi_2 = E * div_{\mathscr{L}^*} V$ and $V_1 := V - V_2$. Clearly

$$div_{\mathscr{L}^*} V_2 = div_{\mathscr{L}^*} \nabla_{\mathscr{L}}\varphi_2 = \Delta_{\mathscr{L}}\varphi_2 = div_{\mathscr{L}^*} V,$$

thus $div_{\mathscr{L}^*} V_1 = 0$. The estimate (4.1.1) follows directly from the boundedness of $\partial^2 E$ from $L^p(\mathbb{R}^N)$ to itself for $1 < p < \infty$. $\square$

4.2 Proof of Theorem B

In the same spirit of [14, Theorem 5], the proof of Theorem B is simplified by reducing it to two specific cases of the inequality (9). The first case is a global nonhomogeneous version of the inequality (4), namely:

Theorem 4.2 (Case $W = \nabla_{\mathscr{L}}\phi$)**.** *Let* $\mathscr{L} = \{L_1, ..., L_n\}$ *be an elliptic system of complex vector fields on* $\mathbb{R}^N$ *with complex constant coefficients with* $n \geq 2$*. If* $V \in L^p(\mathbb{R}^N, \mathbb{C}^n)$ *and* $div_{\mathscr{L}^*}\, V \in L^p(\mathbb{R}^N)$ *with* $1 < p < \infty$*, then the inequality*

$$\|V \cdot \nabla_{\mathscr{L}}\phi\|_{h^1} \leq C\left(\|V\|_{L^p} + \|div_{\mathscr{L}^*}\, V\|_{L^p}\right)\|\nabla_{\mathscr{L}}\phi\|_{L^{p'}}$$

holds for all function ϕ *such that* $\nabla_{\mathscr{L}}\phi \in L^{p'}(\mathbb{R}^N, \mathbb{C}^n)$.

The second simplification is a reduction of the inequality (9) for general $W \in L^{p'}(\mathbb{R}^N, \mathbb{C}^n)$ and $div_{\mathscr{L}^*}V = 0$, being stated as:

Theorem 4.3 (Case $\mathrm{div}_{\mathscr{L}^*}\, V = 0$)**.** *Let* $\mathscr{L} = \{L_1, ..., L_n\}$ *be an elliptic system of complex vector fields on* $\mathbb{R}^N$ *with complex constant coefficients with* $n \geq 2$*. If* $W \in L^{p'}(\mathbb{R}^N, \mathbb{C}^n)$ *and* $curl_{\mathscr{L}}\, W \in L^{p'}(\mathbb{R}^N)$ *with* $1 < p < \infty$*, then the inequality*

$$\|V \cdot W\|_{h^1} \leq C\|V\|_{L^p}\left(\|W\|_{L^{p'}} + \|curl_{\mathscr{L}}W\|_{L^{p'}}\right)$$

holds for all $V \in L^p(\mathbb{R}^N, \mathbb{C}^n)$ *which satisfies* $div_{\mathscr{L}^*}V = 0$.

In order to obtain the proof of Theorem **B**, we assume the validity of Theorems 4.2 and 4.3. Using the Hodge decomposition from Lemma 4.1, we may write $V = V_1 + V_2$ and $W = W_1 + W_2$ with

$$div_{\mathscr{L}^*}V_1 = div_{\mathscr{L}^*}W_2 = 0 \text{ and } V_2 = \nabla_{\mathscr{L}}\phi_2, W_1 = \nabla_{\mathscr{L}}\phi_1,$$

in the sense of distributions, for some $\phi_1 \in L^{p'}(\mathbb{R}^N)$ and $\phi_2 \in L^p(\mathbb{R}^N)$. Then,

$$V \cdot W = V_1 \cdot W + V_2 \cdot W_1 + V_2 \cdot W_2,$$

and from Theorem 4.3 we have

$$\begin{aligned}\|V_1 \cdot W\|_{h^1} &\lesssim \|V_1\|_{L^p}\left(\|W\|_{L^{p'}} + \|curl_{\mathscr{L}}W\|_{L^{p'}}\right)\\ &\lesssim \|V\|_{L^p}\left(\|W\|_{L^{p'}} + \|curl_{\mathscr{L}}W\|_{L^{p'}}\right)\end{aligned}$$

since $div_{\mathscr{L}^*}V_1 = 0$. By Theorem 4.2 we have

$$\begin{aligned}\|V_2 \cdot W_1\|_{h^1} &\lesssim \|W_1\|_{L^{p'}}\left(\|V_2\|_{L^p} + \|div_{\mathscr{L}}V_2\|_{L^p}\right)\\ &= \|W_1\|_{L^{p'}}\left(\|V_2\|_{L^p} + \|div_{\mathscr{L}}V\|_{L^p}\right)\\ &\lesssim \|W\|_{L^{p'}}\left(\|V\|_{L^p} + \|div_{\mathscr{L}}V\|_{L^p}\right)\end{aligned}$$

since $W_1 = \nabla_{\mathscr{L}}\phi_1$ and

$$\begin{aligned}
\|V_2 \cdot W_2\|_{h^1} &\lesssim \|V_2\|_{L^p}\left(\|W_2\|_{L^{p'}} + \|div_{\mathscr{L}^*}W_2\|_{L^p}\right)\\
&\lesssim \|V_2\|_{L^p}\|W_2\|_{L^{p'}}\\
&\lesssim \|V\|_{L^p}\|W\|_{L^{p'}}
\end{aligned}$$

since $V_2 = \nabla_{\mathscr{L}}\phi_2$ and $div_{\mathscr{L}^*}W_2 = 0$. Combining the previous estimates, we obtain

$$\begin{aligned}
\|V \cdot W\|_{h^1} &\leq \|V_1 \cdot W\|_{h^1} + \|V_2 \cdot W_1\|_{h^1} + \|V_2 \cdot W_2\|_{h^1}\\
&\lesssim \|V\|_{L^p}\left(\|W\|_{L^{p'}} + \|curl_{\mathscr{L}}W\|_{L^{p'}}\right) + \|div_{\mathscr{L}^*}\,V\|_{L^p}\|W\|_{L^{p'}}\\
&= \|V\|_{L^p}\|W\|_{L^{p'}} + \|V\|_{L^p}\|curl_{\mathscr{L}}W\|_{L^{p'}} + \|div_{\mathscr{L}^*}\,V\|_{L^p}\|W\|_{L^{p'}},
\end{aligned}$$

getting the desired estimate.

□

Next we will define a local maximal function on $W^{-1,s}_{loc}(\mathbb{R}^N)$ that we will use in the proof of the Theorems 4.2 and 4.3. Fixed $\varphi \in C_c^\infty(B(0,1))$ with $\varphi \geq 0$ and $\int \varphi = 1$, denote for each $x \in \mathbb{R}^N$ and $t > 0$ the function $\varphi_t^x(y) := \dfrac{1}{t^N}\varphi\left(\frac{x-y}{t}\right)$. Given $1 < s \leq \infty$ and $f \in W^{-1,s}_{loc}(\mathbb{R}^N)$, we define by $M^{loc}_{W^{-1,s}}f(x)$ a local maximal operator as the smaller constant $C > 0$ which satisfies

$$|\langle f, \varphi_t^x(\phi - \phi_{B_t^x})\rangle| \leq C\left(\fint_{B(x,t)} |\bigtriangledown\phi|^{s'}\right)^{\frac{1}{s'}},$$

for all $0 < t < 1$ and $\phi \in W^{1,s'}_{loc}(\mathbb{R}^N)$. The boundedness of $M^{loc}_{W^{-1,s}}$ on $L^p(\mathbb{R}^N)$ was proved by Dafni in [14], precisely:

Lemma 4.4. *If $1 < s < p^*$ for $1 < p < N$ or $1 < s < \infty$ for $p \geq N$ then there exists $C = C(p,s,N) > 0$ such that $\left\|M^{loc}_{W^{-1,s}}f\right\|_{L^p} \leq C\|f\|_{L^p}$, for all $f \in L^p(\mathbb{R}^N)$.*

We recall that for each $u \in W^{1,p}(\mathbb{R}^N)$ with $1 \leq p < N$ there exists a constant $C = C(N,p) > 0$ such that

$$\left(\fint_B \left|\frac{1}{r_B}(u - u_B)\right|^{p^*}\right)^{\frac{1}{p^*}} \leq C\left(\fint_B |\nabla u|^p\right)^{\frac{1}{p}}$$

for any ball B where r_B is its radius. This inequality is known as the Sobolev-Poincaré inequality (see Proposition 1.52 for a quick proof).

4.3 Proof of Theorem 4.2 (case $W = \nabla_{\mathscr{L}}\phi$)

Let ϕ be a function such that $\nabla_{\mathscr{L}}\phi \in L^{p'}(\mathbb{R}^N)$ which is equivalent to $\nabla\phi \in L^{p'}(\mathbb{R}^N)$ by Proposition 1.27. For each $x \in \mathbb{R}^N$ and $0 < t < 1$ we define

$$\Phi_t^x(y) := \varphi_t^x(y)(\phi(y) - \phi_{B_x^t})$$

that is supported on $B(x,t)$. From the definition of $div_{\mathscr{L}^*}\,V$, we have

$$\langle div_{\mathscr{L}^*}\,V,\Phi_t^x\rangle=\sum_{j=1}^{n}\langle L_j^*V_j\;\overline{\Phi_t^x}\rangle=\sum_{j=1}^{n}\int_{B(x,t)}V_j\overline{L_j(\Phi_t^x)}=\int_{B(x,t)}V\cdot\overline{\nabla_{\mathscr{L}}\Phi_t^x}$$

and taking the product

$$\begin{aligned}\nabla_{\mathscr{L}}\Phi_t^x(y)&=\nabla_{\mathscr{L}}\left[\varphi_t^x(y)(\phi(y)-\phi_{B_x^t})\right]\\&=\left[-\frac{1}{t^{N+1}}\nabla_{\mathscr{L}}\varphi\left(\frac{x-y}{t}\right).(\phi(y)-\phi_{B_x^t})+\varphi_t^x(y)\,\nabla_{\mathscr{L}}\phi(y)\right]\end{aligned}$$

that implies

$$\begin{aligned}\langle div_{\mathscr{L}^*}\,V,\Phi_t^x\rangle&=-\frac{1}{t^{N+1}}\int_{B(x,t)}V\cdot\left[\nabla_{\mathscr{L}}\varphi\left(\frac{x-y}{t}\right)(\phi(y)-\phi_{B_x^t})\right]dy\\&+\int_{B(x,t)}\varphi_t^x(y)\,(V\cdot\nabla_{\mathscr{L}}\phi)\,(y)\,dy\\&=-\frac{1}{t^{N+1}}\int_{B(x,t)}V\cdot\overline{\left[\nabla_{\mathscr{L}}\varphi\left(\frac{x-y}{t}\right)(\phi(y)-\phi_{B_x^t})\right]}dy+\varphi_t*\overline{(V\cdot\nabla_{\mathscr{L}}\phi)(x)}.\end{aligned}$$

then

$$\begin{aligned}\varphi_t*(V\cdot\nabla_{\mathscr{L}}\phi)(x)&=\overline{\langle div_{\mathscr{L}^*}\,V,\Phi_t^x\rangle}\\&+\frac{1}{t^{N+1}}\int_{B(x,t)}\left[\nabla_{\mathscr{L}}\varphi\left(\frac{x-y}{t}\right)\right]\cdot\left[\overline{V(y)}(\phi(y)-\phi_{B_x^t})\right]dy.\end{aligned}\tag{4.3.1}$$

Let $1<\alpha<p$, $1<\beta<p'$ satisfying $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=1+\dfrac{1}{N}$. Note that $\beta^*=\alpha'$ and $\beta<N$. We point out that $\phi\in L^{\alpha'}_{loc}(\mathbb{R}^N)$. In fact, if $1<p'<N$ and $\nabla\phi\in L^{p'}(\mathbb{R}^N)$ then by Sobolev-Gagliardo-Nirenberg inequality $\phi\in L^{p'_*}(\mathbb{R}^N)$ with

$$\frac{1}{p'_*}:=\frac{1}{p'}-\frac{1}{N}<\frac{1}{\beta}-\frac{1}{N}=\frac{1}{\alpha'}$$

that implies $\alpha'<p^{'}_{*}$ and consequently $\phi\in L^{\alpha'}_{loc}(\mathbb{R}^N)$. Otherwise, if $p'\geq N$ then $\phi\in L^q_{loc}(\mathbb{R}^N)$ for any $1\leq q<\infty$. Applying the Hölder inequality and the Sobolev-Poincaré inequality, the second term in (4.3.1) can be controlled by

$$\begin{aligned}\frac{\|\nabla_{\mathscr{L}}\varphi\|_{L^\infty}}{t^{N+1}}\int_{B(x,t)}\left|\overline{V(y)}(\phi(y)-\phi_{B_x^t})\right|dy&\lesssim\left(⨍_{B(x,t)}|V(y)|^{\alpha}dy\right)^{\frac{1}{\alpha}}\left(⨍_{B(x,t)}\left|\frac{1}{t}(\phi(y)-\phi_{B_x^t})\right|^{\alpha'}dy\right)^{\frac{1}{\alpha'}}\\&=\left(⨍_{B(x,t)}|V(y)|^{\alpha}dy\right)^{\frac{1}{\alpha}}\left(⨍_{B(x,t)}\left|\frac{1}{t}(\phi(y)-\phi_{B_x^t})\right|^{\beta^*}dy\right)^{\frac{1}{\beta^*}}\\&\lesssim\left(⨍_{B(x,t)}|V(y)|^{\alpha}dy\right)^{\frac{1}{\alpha}}\left(⨍_{B(x,t)}|\nabla\phi(y)|^{\beta}dy\right)^{\frac{1}{\beta}}\\&\lesssim\left[M(|V|^{\alpha})(x)\right]^{\frac{1}{\alpha}}\left[M(|\nabla\phi|^{\beta})(x)\right]^{\frac{1}{\beta}},\end{aligned}$$

where M denotes the Hardy-Littlewood maximal function. From the definition of $M^{loc}_{W^{-1,s}}(div_{\mathscr{L}^*} V)$, the first term (4.3.1) is controlled by

$$\begin{aligned}|\langle div_{\mathscr{L}^*} V, \Phi_t^x\rangle| &\le M^{loc}_{W^{-1,s}}(div_{\mathscr{L}^*} V)(x) \left(\fint_{B(x,t)} |\nabla\phi(y)|^{s'} dy \right)^{\frac{1}{s'}} \\ &\lesssim M^{loc}_{W^{-1,s}}(div_{\mathscr{L}^*} V)(x) \left[M\left(|\nabla\phi|^{s'}\right)(x)\right]^{\frac{1}{s'}},\end{aligned}$$

for some $1 < s < \infty$ to be chosen later. Taking the supremum for $0 < t < 1$ we have

$$m_\varphi(V \cdot \nabla_{\mathscr{L}}\phi)(x) \lesssim M^{loc}_{W^{-1,s}}(div_{\mathscr{L}^*} V)(x) \left[M\left(|\nabla\phi|^{s'}\right)(x)\right]^{\frac{1}{s'}} + \left[M(|V|^\alpha)(x)\right]^{\frac{1}{\alpha}} \left[M(|\nabla\phi|^\beta)(x)\right]^{\frac{1}{\beta}}, \tag{4.3.2}$$

and to compute the norm $\|V \cdot \nabla_{\mathscr{L}}\phi\|_{h^1}$ it is sufficient estimate each term in the right side hand in L^1 norm. Using the Hölder's inequality for the first term, we have

$$\begin{aligned}\|M^{loc}_{W^{-1,s}}(div_{\mathscr{L}^*} V) \left[M\left(|\nabla\phi|^{s'}\right)\right]^{\frac{1}{s'}} \|_{L^1} &\le \|M^{loc}_{W^{-1,s}}(div_{\mathscr{L}^*} V)\|_{L^p} \| \left[M\left(|\nabla\phi|^{s'}\right)\right]^{\frac{1}{s'}} \|_{L^{p'}} \\ &= \|M^{loc}_{W^{-1,s}}(div_{\mathscr{L}^*} V)\|_{L^p} \|M\left(|\nabla\phi|^{s'}\right)\|^{1/s'}_{L^{p'/s'}} \\ &\le \|M^{loc}_{W^{-1,s}}(div_{\mathscr{L}^*} V)\|_{L^p} \|\nabla\phi\|_{L^{p'}},\end{aligned}$$

where in the last inequality we used the boundedness of Hardy-Littlewood maximal function since $p' > s'$ that is equivalent to $p < s$. Note that if $1 < p < N$ then we may choose some $p < s < p^*$, otherwise if $p \ge N$ we choose any $1 < s < p$. Thus from Lemma 4.4 we have

$$\begin{aligned}\|M^{loc}_{W^{-1,s}}(div_{\mathscr{L}^*} V) \left[M\left(|\nabla\phi|^{s'}\right)\right]^{\frac{1}{s'}} \|_{L^1} &\lesssim \|M^{loc}_{W^{-1,s}}(div_{\mathscr{L}^*} V)\|_{L^p} \|\nabla\phi\|_{L^{p'}} \\ &\lesssim \|div_{\mathscr{L}^*} V\|_{L^p} \|\nabla\phi\|_{L^{p'}}.\end{aligned}$$

For the second term, we use the Hölder's inequality and the boundedness of maximal operator M again to conclude that

$$\| \left[M(|V|^\alpha)\right]^{\frac{1}{\alpha}} \left[M(|\nabla\phi|^\beta)\right]^{\frac{1}{\beta}} \|_{L^1} \lesssim \|M(|V|^\alpha)\|^{1/\alpha}_{L^{p/\alpha}} \|M(|\nabla\phi|^\beta)\|^{1/\beta}_{L^{p'/\beta}} \lesssim \|V\|_{L^p} \|\nabla\phi\|_{L^{p'}}.$$

Combining the previous control in norm L^1 and using the Proposition 1.27 we have

$$\|V \cdot \nabla_{\mathscr{L}}\phi\|_{h^1} \lesssim \|div_{\mathscr{L}^*} V\|_{L^p} \|\nabla\phi\|_{L^{p'}} + \|V\|_{L^p} \|\nabla\phi\|_{L^{p'}} \lesssim \left(\|V\|_{L^p} + \|div_{\mathscr{L}^*} V\|_{L^p}\|\right) \|\nabla_{\mathscr{L}}\phi\|_{L^{p'}},$$

as desired. □

4.4 Proof of Theorem 4.3 (case $\mathrm{div}_{\mathscr{L}^*} V = 0$)

Let $V := (V_1, V_2, ..., V_n)$ and $U_i := -E * V_i$, where E is the fundamental solution of $\Delta_{\mathscr{L}}$. Clearly $-\Delta_{\mathscr{L}} U_i = V_i$ and $\left\|\partial^2 U_i\right\|_{L^p} \le C \|V_i\|_{L^p}$, for any $1 < p < \infty$ and for each $i = 1, ..., n$. Note that $U :=$

$(U_1, U_2, ..., U_n)$ satisfies $div_{\mathscr{L}^*}\, U = div_{\mathscr{L}^*}\, V = 0$. Consider now $B := curl_{\mathscr{L}}\, U = (B_{ij})_{1\le i,j\le n}$ with $B_{ij} := L_jU_i - L_iU_j$ and denote $B_j := (B_{1j}\ B_{2j}\ ...\ B_{nj})$ the j-th column of the (symmetric) matrix B. Thus

$$div_{\mathscr{L}^*}\, B_j = \sum_{i=1}^{n} L_i^* B_{ij} = L_j(div_{\mathscr{L}^*}\, U) - \Delta_{\mathscr{L}}\, U_j = V_j.$$

This way,

$$\begin{aligned}
\overline{V\cdot W} = \sum_{j=1}^{n} (\overline{div_{\mathscr{L}^*}\, B_j})W_j &= -\sum_{j=1}^{n} (div_{\mathscr{L}}\, \overline{B_j})W_j \\
&= -\sum_{j=1}^{n} div_{\mathscr{L}}\, (\overline{B_j}W_j) + \sum_{i,j=1}^{n} \overline{B_{ij}}L_i(W_j) \\
&= -\sum_{j=1}^{n} div_{\mathscr{L}}\, (\overline{B_j}W_j) + \sum_{i<j} \overline{B_{ij}}\left(L_iW_j - L_jW_i\right) \\
&= -\sum_{j=1}^{n} div_{\mathscr{L}}\, (\overline{B_j}W_j) + \sum_{i<j} \overline{B_{ij}}(\mathrm{curl}_{\mathscr{L}}\, W)_{ij}.
\end{aligned}$$

Now, let $\widetilde{B}$ the symmetric matrix given by $\widetilde{B}_{ij} := B_{ij} - (B_{ij})_{B_x^t}$ that satisfies $div_{\mathscr{L}^*}\, \tilde{B}_j = div_{\mathscr{L}^*}\, B_j = V_j$. It is clear that

$$\overline{V\cdot W} = -\sum_{j=1}^{n} div_{\mathscr{L}}\, (\overline{\widetilde{B}_j}W_j) + \sum_{i<j} \overline{\widetilde{B}_{ij}}(\mathrm{curl}_{\mathscr{L}}\, W)_{ij}.$$

Let $\varphi \in C_c^\infty(B(0,1))$ with $\varphi \ge 0$ and $\int \varphi = 1$, then we may write

$$\begin{aligned}
\sum_{i<j} \langle (\mathrm{curl}_{\mathscr{L}}\, W)_{ij}, \varphi_t^x \widetilde{B}_{ij}\rangle &= \sum_{i<j} \int_{B(x,t)} (\mathrm{curl}_{\mathscr{L}}\, W)_{ij}(y)\ \overline{\varphi_t^x(y)\widetilde{B}_{ij}(y)}\, dy \\
&= \int_{B(x,t)} \varphi_t^x(y) \sum_{i<j} \overline{\widetilde{B}_{ij}}(y)(\mathrm{curl}_{\mathscr{L}}\, W)_{ij}(y)\, dy \\
&= \int_{B(x,t)} \varphi_t^x(y) \left(\sum_{j=1}^{n} div_{\mathscr{L}}\, (\overline{\widetilde{B}_j}W_j)(y) + \overline{V\cdot W}(y) \right) dy \\
&= \sum_{j=1}^{n} \int_{B(x,t)} \varphi_t^x(y)\, div_{\mathscr{L}}\, (\overline{\widetilde{B}_j}W_j)(y)\, dy + \overline{\varphi_t * V\cdot W}(x) \\
&= -\sum_{j=1}^{n} \int_{B(x,t)} \nabla_{\mathscr{L}}\varphi_t^x(y)\cdot(\widetilde{B}_j\overline{W_j})(y)\, dy + \overline{\varphi_t * V\cdot W}(x).
\end{aligned}$$

that implies

$$\begin{aligned}
\left|\left(\varphi_t * V\cdot W\right)(x)\right| &\le \sum_{i<j} \left| \langle (\mathrm{curl}_{\mathscr{L}}\, W)_{ij}, \varphi_t^x \widetilde{B}_{ij}\rangle \right| + \frac{\|\nabla_{\mathscr{L}}\varphi\|_{L^\infty}}{t^{N+1}} \int_{B(x,t)} \left|(\tilde{B}_j\overline{W_j})(y)\right| dy \\
&\lesssim \sum_{i<j} M^{loc}_{W^{-1,s}}((\mathrm{curl}_{\mathscr{L}}\, W)_{ij})(x) \left(\fint_{B(x,t)} |\nabla\widetilde{B}_{ij}(y)|^{s'} dy \right)^{\frac{1}{s'}} \\
&+ \frac{1}{t} \fint_{B(x,t)} \left|(\tilde{B}_j\overline{W_j})(y)\right| dy,
\end{aligned}$$

where in the first inequality we used the definition of the operator $M^{loc}_{W^{-1,s}}$ to some s to be chosen later.

Consider $1<\alpha<p$ and $1<\beta<p'$, analogous in the proof of Theorem 4.2. Applying the Hölder's inequality and the Sobolev-Poincaré inequality where $\beta'=\alpha^*$ we have

$$\begin{aligned}
\frac{1}{t}\fint_{B(x,t)}\left|(\tilde{B}_j\overline{W_j})(y)\right|dy &\lesssim \left(\fint_{B(x,t)}\left|W_j(y)\right|^{\beta}dy\right)^{\frac{1}{\beta}}\left(\fint_{B(x,t)}\left|\frac{1}{t}(B_{ij}(y)-(B_{ij})_{B^t_x})\right|^{\beta'}dy\right)^{\frac{1}{\beta'}}\\
&= \left(\fint_{B(x,t)}\left|W_j(y)\right|^{\beta}dy\right)^{\frac{1}{\beta}}\left(\fint_{B(x,t)}\left|\frac{1}{t}(B_{ij}(y)-(B_{ij})_{B^t_x})\right|^{\alpha^*}dy\right)^{\frac{1}{\alpha^*}}\\
&\lesssim \left(\fint_{B(x,t)}\left|W_j(y)\right|^{\beta}dy\right)^{\frac{1}{\beta}}\left(\fint_{B(x,t)}\left|\nabla B_{ij}(y)\right|^{\alpha}dy\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

Plugging this inequality in the previous control and taking the supremum for $0<t<1$ we have

$$\begin{aligned}
m_{\varphi}(V\cdot W)(x) \lesssim \sum_{i<j}M^{loc}_{W^{-1,s}}((\operatorname{curl}_{\mathscr{L}}W)_{ij})(x)\left(M\left(|\nabla\tilde{B}_{ij}|^{s'}\right)(x)\right)^{\frac{1}{s'}}\\
+\sum_{i,j=1}^{n}\left(M(\left|W_j\right|^{\beta})(x)\right)^{\frac{1}{\beta}}\left(M(\left|\nabla B_{ij}\right|^{\alpha})(x)\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

Taking the same choice of s in the previous theorem (in fact, just replace p by p' in the mentioned calculations) and using the Hölder's inequality, we may conclude

$$\begin{aligned}
\|V\cdot W\|_{h^1} &\lesssim \sum_{i,j=1}^{n}\left(\left\|M^{loc}_{W^{-1,s}}((\operatorname{curl}_{\mathscr{L}}W)_{ij})\right\|_{L^{p'}}+\left\|W_j\right\|_{L^{p'}}\right)\left\|\nabla B_{ij}\right\|_{L^p}\\
&\lesssim \sum_{i,j=1}^{n}\left(\|\operatorname{curl}_{\mathscr{L}}W)_{ij}\|_{L^{p'}}+\left\|W_j\right\|_{L^{p'}}\right)\left\|\nabla B_{ij}\right\|_{L^p}.
\end{aligned}$$

From the definition of B_{ij} we have $\left\|\nabla B_{ij}\right\|_{L^p}\lesssim\|U\|_{W^{2,p}}\lesssim\|V\|_{L^p}$, thus

$$\|V\cdot W\|_{h^1}\lesssim\left(\|W\|_{L^{p'}}+\sum_{i,j}\left\|(\operatorname{curl}_{\mathscr{L}}W)_{ij}\right\|_{L^{p'}}\right)\|V\|_{L^p},$$

as desired. □

4.5 Additional remarks

It is natural and in our interest to prove a local version of Theorem **B** when $\mathscr{L}=\{L_1,...,L_n\}$ is an elliptic system of complex vector fields on Ω with smooth complex coefficients. Note firstly that the local version of Theorem 4.2 can be stated as

Theorem 4.5. *Let $\mathscr{L}=\{L_1,...,L_n\}$ be a system of complex vector fields on $\Omega\subset\mathbb{R}^N$ and suppose V a vector field on $\Omega\subset\mathbb{R}^N$ satisfying*

$$V \text{ and } div_{\mathscr{L}^*}V\in L^p(\Omega,\mathbb{C}^n),$$

for some $1 < p < \infty$. *For each point* $x_0 \in \Omega$ *there exists an open neighborhood* $U_0 \subset \Omega$ *of* x_0 *such that,* $V \cdot \nabla_{\mathscr{L}} \phi$ *belongs to the local Hardy space* h^1, *with*

$$\|V \cdot \nabla_{\mathscr{L}} \phi\|_{h^1} \leq C\left(\|V\|_{L^p} + \|div_{\mathscr{L}^*} V\|_{L^p}\right) \|\nabla_{\mathscr{L}} \phi\|_{L^{p'}},$$

for all $\phi \in C_c^\infty(U_0)$ *satisfying* $\nabla_{\mathscr{L}} \phi \in L^{p'}(\Omega, \mathbb{C}^n)$, *in which* $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

The proof follows as usual. Instead of inequality (4.3.1) we have

$$\begin{aligned} \varphi_t * (V \cdot W)(x) &= \langle f, \Phi_t^x \rangle \\ &+ \frac{1}{t^{N+1}} \int_{B(x,t)} \left[\nabla_{\mathscr{L}^{x,t}} \varphi\left(\frac{x-y}{t}\right)\right] \cdot \left[V(y)(\overline{\phi}(y) - \overline{\phi_{B_x^t}})\right] dy, \end{aligned} \tag{4.5.1}$$

where $\mathscr{L}^{x,t}$ denotes the translated system given by $L_j^{x,t}(y,D) := \sum_{k=1}^{N} a_{jk}(x-ty)\partial_{x_k}$ for each fixed $x \in \mathbb{R}^N$ and $0 < t < 1$. Thus,

$$\frac{1}{t^{N+1}} \int_{B(x,t)} \left| \left[\nabla_{\mathscr{L}^{x,t}} \varphi\left(\frac{x-y}{t}\right)\right] \cdot \left[V(y)(\overline{\phi}(y) - \overline{\phi_{B_x^t}})\right] \right| dy \lesssim \left[M(|V|^\alpha)(x)\right]^{\frac{1}{\alpha}} \left[M(|\nabla \phi|^\beta)(x)\right]^{\frac{1}{\beta}}$$

and the proof follows the same.

As for the local case of Theorem 4.3, the proof remains open. In this case, since the operators L_i^* and L_j do not commute, we lose the identity $div_{\mathscr{L}^*} B_j = V_j$.

CHAPTER 5

The proof of Theorem C and consequences

In this chapter we show a proof of Theorem **C**, which decomposes the bmo space by a family of functions written in $\operatorname{div}_{\mathscr{L}^*} - \operatorname{curl}_{\mathscr{L}}$ -terms. The second section proves, as a consequence of the previous result, a div-curl lemma, which characterizes the $h^1(\mathbb{R}^N)$ space, through its duality. Finally, in the last section we comment on future results, which extend Theorem **C** to elliptic complexes.

5.1 Proof of Theorem C

Let $g \in bmo(\mathbb{R}^N)$ and assume $f \in (\mathscr{D}C_{\mathscr{L}})^p_{1,0} \cup (\mathscr{D}C_{\mathscr{L}})^p_{0,1} \subset h^1(\mathbb{R}^N)$ from Theorem B. By the duality $bmo(\mathbb{R}^N) = (h^1(\mathbb{R}^N))^*$ it follows

$$\left|\int_{\mathbb{R}^N} g(x)\overline{f(x)}dx\right| \leq C\|g\|_{bmo}, \quad \forall f \in (\mathscr{D}C_{\mathscr{L}})^p_{1,0} \cup (\mathscr{D}C_{\mathscr{L}})^p_{0,1}.$$

So now, it is sufficient to prove that

$$\|g\|_{bmo} \leq C \sup_{f\in X}\left|\int_{\mathbb{R}^N} g(x)\overline{f(x)}dx\right|$$

for $X = (\mathscr{D}C_{\mathscr{L}})^p_{1,0}$ or $X = (\mathscr{D}C_{\mathscr{L}})^p_{0,1}$. In order to estimate $\|g\|_{bmo}$, from the definition in (10), we split in two cases : balls $B := B(x_0, R)$ with $R \leq 1$ and $R > 1$.

Let $B^* := B(x_0, 2R)$. The Theorem III.2 in [13] asserts that

$$\left(⨍_B |g(x) - g_B|^2 dx\right)^{\frac{1}{2}} \leq C \sup_{V,W}\left|\int g(x)(V\cdot W)(x)dx\right|,$$

where the supremum is taken over all real vector fields V, W in $C_c^\infty(B^*)$, with $\|V\|_{L^2}, \|W\|_{L^2} \leq 1$, satisfying $\operatorname{div} V = 0$ and $\operatorname{curl} W = 0$. We will adapt this argument in our setting. It follows from [21, Corollary 2.1, p. 20] and Proposition 1.28 that

$$\|g - g_B\|_{L^2(B)} \lesssim \|\nabla g\|_{W^{-1,2}(B)} \lesssim \sum_{i=1}^{n} \|L_i^* g\|_{W^{-1,2}(B)} = \sup_{\substack{\|\nabla u\|_{L^2(B)}\leq 1\\ u\in C_c^\infty(B)}} \left|\int g(x)\,\overline{L_i u}(x)dx\right|. \tag{5.1.1}$$

We claim that for each $u \in C_c^\infty(B)$ with $\|\nabla u\|_{L^2(B)} \leq 1$ and $1 < p < \infty$ there exist vector fields V, W satisfying $div_{\mathscr{L}^*}\, V = 0$ with $\|V\|_{L^p} \leq 1$ and $curl_{\mathscr{L}}\, W = 0$ with $\|W\|_{L^{p'}} \leq 1$ such that

$$V \cdot W = C|B|^{-\frac{1}{2}}\overline{L_i u}, \tag{5.1.2}$$

for some constant $C > 0$. Plugging into (5.1.1) we have

$$\left(\fint_B |g(x) - g_B|^2 dx\right)^{\frac{1}{2}} \lesssim \sum_{i=1}^{n} \|L_i^* g\|_{W^{-1,2}(B)} \lesssim \sup_{f \in (\mathscr{D}C_{\mathscr{L}})^p_{1,0} \cap (\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left|\int g(x)\overline{f(x)}dx\right|. \tag{5.1.3}$$

Consider a function $u \in C_c^\infty(B)$ with $\|\nabla u\|_{L^2(B)} \leq 1$ and $\eta \in C_c^\infty(B(0,2))$ such that $\eta \equiv 1$ in $B(0,1)$ and $\|\eta\|_{L^\infty(B(0,2))} \leq 1$. Denote $\eta_B(w) := \eta\left(\dfrac{w - w_0}{R}\right)$ and define the vector fields

$$V := \frac{|B|^{\frac{1}{2}-\frac{1}{p}}}{2C}\left(\overline{L_i u}\, e_j - \overline{L_j u}\, e_i\right) \text{ and } W := \gamma|B|^{-\frac{1}{p'}}\nabla_{\mathscr{L}}\left(\left(x_j - x_j^0\right)\eta_B(x)\right), \tag{5.1.4}$$

for $i, j \in \{1, ..., n\}$ with $i \neq j$, where $\{e_1, \ldots, e_n\}$ denotes de canonical basis of $\mathbb{R}^n$ and C, γ are appropriate positive constants to be chosen later. We claim that

$$V \cdot W = \frac{\gamma}{2C}|B|^{-\frac{1}{2}}\overline{L_i u},$$

where $div_{\mathscr{L}^*}\, V = 0$ with $\|V\|_{L^p} \leq 1$ and $curl_{\mathscr{L}}\, W = 0$ with $\|W\|_{L^{p'}} \leq 1$, for $1 < p \leq 2$. Clearly

$$div_{\mathscr{L}^*}\, V = L_j^* V_j + L_i^* V_i = \frac{|B|^{\frac{1}{2}-\frac{1}{p}}}{2C}\left[L_i^*, L_j^*\right]\overline{u} = 0$$

and $|V| \leq |B|^{\frac{1}{2}-\frac{1}{p}}|\nabla u|$ choosing $C := \max\limits_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} \left\{1, |a_{jk}|\right\}$. Since $\operatorname{supp}(V) \subseteq B$ follows from Hölder's inequality that $\|V\|_{L^p(B)} \leq |B|^{\frac{1}{p}-\frac{1}{2}}\|V\|_{L^2(B)} \leq |B|^{\frac{1}{p}-\frac{1}{2}}|B|^{\frac{1}{2}-\frac{1}{p}}\|\nabla u\|_{L^2(B)} \leq 1$. It is easy to see that $curl_{\mathscr{L}}\, W = \gamma|B|^{-\frac{1}{p'}}curl_{\mathscr{L}}(\nabla_{\mathscr{L}}\varphi) = \gamma|B|^{-\frac{1}{p'}}\left([L_i, L_j]\varphi\right)_{ij} = 0$. Note that $\operatorname{supp}(W) \subseteq \operatorname{supp}(\eta_B) \subseteq B^*$ and $L_\ell\left(x_j - x_j^0\right) = \delta_{\ell j}$. Furthermore, for each $x \in B^*$ we have

$$\begin{aligned} \sum_{k=1}^{n}\left|L_k\left(\left(x_j - x_j^0\right)\eta_B(x)\right)\right| &= \sum_{k=1}^{n}\left|\delta_{kj}\eta_B(x) + \left(x_j - x_j^0\right)L_k\eta_B(x)\right| \\ &\leq nC|\eta_B(x)| + 2R\sum_{k=1}^{n}\frac{1}{R}\left|L_k\eta\left(\frac{x - x_0}{R}\right)\right| \\ &= nC + 2\sum_{k=1}^{n}\left|L_k\eta\left(\frac{x - x_0}{R}\right)\right| \end{aligned}$$

and choosing $\gamma := 2^{-\frac{N}{p'}}(2\|\nabla_{\mathscr{L}}\eta\|_{L^\infty} + nC)^{-1}$, follows $|W| \leq 2^{-\frac{N}{p'}}|B|^{-\frac{1}{p'}}$ that implies

$$\|W\|_{L^{p'}} \leq |B^*|^{\frac{1}{p'}}\|W\|_{L^\infty(B^*)} = 2^{\frac{N}{p'}}|B|^{\frac{1}{p'}}\|W\|_{L^\infty(B^*)} \leq 2^{\frac{N}{p'}}|B|^{\frac{1}{p'}}\, 2^{-\frac{N}{p'}}|B|^{-\frac{1}{p'}} = 1.$$

Lastly, we point out that $V \cdot W = V_i \overline{W_i} + V_j \overline{W_j}$ and $V_i = V_j = 0$ on $\mathbb{R}^N \backslash B$. Furthermore, as $\eta_B \equiv 1$ in B then for each $x \in B$ we have

$$W_k(x) = \gamma |B|^{-\frac{1}{p'}} \left(\delta_{kj} \eta_B(x) + (x_j - x_j^0) L_k \eta_B(x) \right) = \gamma |B|^{-\frac{1}{p'}} \delta_{kj}$$

for $k = i, j$. In particular, as we are assuming $\mathscr{L}$ as in (1.3.2) thus $W_i = 0$ and $W_j = \gamma |B|^{-\frac{1}{p'}}$ on B. Therefore,

$$V \cdot W \;\; = \;\; V_j \, W_j = \frac{|B|^{\frac{1}{2} - \frac{1}{p}}}{2C} \overline{L_i u} \, \gamma |B|^{-\frac{1}{p'}} = \frac{\gamma}{2C} |B|^{-\frac{1}{2}} \overline{L_i u}.$$

Now we adapt the previous construction to attend $p > 2$. Consider the vector fields

$$V = \gamma' |B|^{-\frac{1}{p}} \left[L_i^* \left(\eta_B(x) \left(x_j - x_j^0 \right) \right) e_j - L_j^* \left(\eta_B(x) \left(x_j - x_j^0 \right) \right) e_i \right] \text{ and } W = \frac{|B|^{\frac{1}{2} - \frac{1}{p'}}}{C} \nabla_{\mathscr{L}} u, \qquad (5.1.5)$$

with γ', C are appropriate constants to be chosen. Analogously as proved before, we have $div_{\mathscr{L}^*} \, V = curl_{\mathscr{L}} \, W = 0$. Clearly, $\operatorname{supp}(V) \subset B^*$ and since

$$\begin{aligned} L_\ell^* \left(\eta_B(x) \left(x_j - x_j^0 \right) \right) &= \left(x_j - x_j^0 \right) L_\ell^* \eta_B(x) + \eta_B(x) L_\ell^* \left(x_j - x_j^0 \right) \\ &= \frac{\left(x_j - x_j^0 \right)}{R} \left(L_\ell^* \eta_B \right) \left(\frac{x - x_0}{R} \right) - \eta_B(x) \delta_{\ell j} \end{aligned}$$

that implies $|V| \leq \gamma' |B|^{-\frac{1}{p}} \left(\, 1 + 4 \|\nabla_{\mathscr{L}^*} \eta\|_{L^\infty} \right) = 2^{-\frac{N}{p}} |B|^{-\frac{1}{p}} = |B^*|^{-\frac{1}{p}}$ and then $\|V\|_{L^p} \leq 1$. Since $\operatorname{supp}(W) \subseteq B$ and $1 < p' < 2$ follows from Hölder's inequality that $\|W\|_{L^{p'}} \leq C^{-1} |B|^{\frac{1}{p} - \frac{1}{2}} \|W\|_{L^2} \leq C^{-1} |B|^{\frac{1}{p} - \frac{1}{2}} |B|^{\frac{1}{2} - \frac{1}{p}} \|\nabla_{\mathscr{L}} u\|_{L^2} \leq 1$, where $C > 0$ is the constant from the control $\|\nabla_{\mathscr{L}} u\|_{L^2} \leq C \|\nabla u\|_{L^2}$ given by $C := N\sqrt{n} \max\limits_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} \left\{ 1, |a_{jk}| \right\}$. In the same way,

$$V \cdot W = \gamma' C^{-1} |B|^{-\frac{1}{2}} \overline{L_i u}.$$

Indeed, $V \cdot W = V_i \overline{W_i} + V_j \overline{W_j}$ and now $W_i = W_j = 0$ on $\mathbb{R}^N \backslash B$. As $\eta_B \equiv 1$ in B then for each $x \in B$ we have

$$V_k(x) = \gamma' |B|^{-\frac{1}{p}} \left(\delta_{ki} \eta_B(x) \delta_{jj} - \delta_{kj} \eta_B(x) \delta_{ij} \right) = \gamma' |B|^{-\frac{1}{p}} \delta_{ki}$$

and $W_k = C^{-1} |B|^{\frac{1}{2} - \frac{1}{p'}} L_k u$, for $k = i, j$. Therefore,

$$V \cdot W \;\; = \;\; V_i \overline{W_i} = \gamma' |B|^{-\frac{1}{p}} C^{-1} |B|^{\frac{1}{2} - \frac{1}{p'}} \overline{L_i u} = \frac{\gamma'}{C} |B|^{-\frac{1}{2}} \overline{L_i u}.$$

We conclude the identity (5.1.2) taking $C := \max\{\gamma, \gamma'\}$. We remark that (5.1.3) holds for any ball

B. Denote $C_{N,p,\eta} := C|B|^{\frac{1}{2}}\ \max\{2\gamma^{-1}, \gamma'^{-1}\}$. Thus, for any $1 < p < \infty$, we get

$$\begin{aligned}
\|L_i^* g\|_{W^{-1,2}(B)} &= \sup_{\|u\|_{W^{1,2}(B)}\leq 1} |\langle L_i^* g, u\rangle| = \sup_{\|\nabla u\|_{L^2(B)}\leq 1} |\langle g, L_i u\rangle| \\
&= \sup_{\substack{\|\nabla u\|_{L^2(B)}\leq 1\\ u\in C_c^\infty(B)}} \left|\int g\,\overline{L_i u}\right| \\
&\leq C_{N,p,\eta} \sup_{\substack{\|\nabla u\|_{L^2(B)}\leq 1\\ u\in C_c^\infty(B)}} \left|\int g\, V\cdot W\right| \qquad (5.1.6)\\
&\leq C_{N,p,\eta} \sup_{V\cdot W\in(\mathscr{D}C_{\mathscr{L}})^p_{1,0}\cap(\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left|\int g\, V\cdot W\right|.
\end{aligned}$$

In equation (5.1.6), we use V and W to denote the vector fields defined on (5.1.4) and (5.1.5), which depend of p.

Now let us move on assuming $R \geq 1$. We claim that

$$\left(\fint_{B(x_0,R)} |g(w)|^p dw\right)^{\frac{1}{p}} \leq C \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left|\int g(x)\overline{f(x)}dx\right|. \qquad (5.1.7)$$

Firstly, we will prove the control (5.1.7) when $B = B(0,1)$, denoted by B_1. It follows from [40, Theorem 1, p. 108] that the inequality

$$\|g\|_{L^r(B_1)} \leq C\left[\|g\|_{W^{-1,r}(B_1)} + \sum_{i=1}^n \|L_i^* g\|_{W^{-1,r}(B_1)}\right], \qquad (5.1.8)$$

holds for any $1 < r < \infty$. The estimates for $\|L_i^* g\|_{W^{-1,p}(B_1)}$ are analogous to those presented in (5.1.3) replacing $W^{-1,2}(B_1)$ by $W^{-1,p}(B_1)$. In fact, we claim that for each $u \in C_c^\infty(B_1)$ with $\|\nabla u\|_{L^{p'}(B_1)} \leq 1$ and $1 < p' < \infty$ there exist vector fields V, W satisfying $div_{\mathscr{L}^*}\, V = 0$ with $\|V\|_{L^p} \leq 1$ and $curl_{\mathscr{L}}\, W = 0$ with $\|W\|_{L^{p'}} \leq 1$ such that

$$V\cdot W = \widetilde{C}|B_1|^{-\frac{1}{p}}\overline{L_i u}, \qquad (5.1.9)$$

for some constant $\widetilde{C} > 0$ and then

$$\sum_{i=1}^n \|L_i^* g\|_{W^{-1,p}(B_1)} \lesssim \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{1,0}\cap(\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left|\int g(x)\overline{f(x)}dx\right|. \qquad (5.1.10)$$

As before, consider a function $u \in C_c^\infty(B_1)$ with $\|\nabla u\|_{L^{p'}(B_1)} \leq 1$ and $\eta \in C_c^\infty(B_1^*)$ such that $\eta \equiv 1$ in B_1 and $\|\eta\|_{L^\infty(B_1^*)} \leq 1$. Define the vector fields

$$V = \gamma'|B_1|^{-\frac{1}{p}}\left[L_i^*\left(\eta(x)x_j\right)e_j - L_j^*\left(\eta(x)x_j\right)e_i\right] \text{ and } W = C^{-1}\nabla_{\mathscr{L}} u, \qquad (5.1.11)$$

for $i, j \in \{1, ..., n\}$ with $i \neq j$, and C, γ' are appropriate positive constants to be chosen later. Analogously as proved before, we have $div_{\mathscr{L}^*}\, V = curl_{\mathscr{L}}\, W = 0$. Clearly, $\operatorname{supp}(V) \subset B_1^*$ and since

$$L_\ell^*\left(\eta(x)x_j\right) = x_j L_\ell^*\eta(x) + \eta(x)L_\ell^* x_j = x_j L_\ell^*\eta(x) - \eta(x)\delta_{\ell j}$$

we have $|V| \leq \gamma'|B_1|^{-\frac{1}{p}}\left(1+4\|\nabla_{\mathscr{L}^*}\eta\|_{L^\infty}\right) = |B_1^*|^{-\frac{1}{p}}$, choosing $\gamma' = 2^{-\frac{N}{p}}(1+4\|\nabla_{\mathscr{L}}\eta\|_{L^\infty})^{-1}$, that implies $\|V\|_{L^p} \leq 1$. Taking the constant from the control $\|\nabla_{\mathscr{L}}u\|_{L^{p'}} \leq C\|\nabla u\|_{L^{p'}}$ given by

$$C := N\sqrt{n} \max_{\substack{1\leq k\leq m\\ 1\leq j\leq n}} \left\{1, |a_{jk}|\right\},$$

then $\|W\|_{L^{p'}} \leq C^{-1}\,\|\nabla_{\mathscr{L}}u\|_{L^{p'}} \leq 1$.

To prove $V\cdot W = \gamma' C^{-1}|B_1|^{-\frac{1}{p}}\overline{L_i u}$, note that $V\cdot W = V_i\overline{W_i} + V_j\overline{W_j}$ and $W_i = W_j = 0$ on $\mathbb{R}^N\backslash B_1$. As $\eta \equiv 1$ in B_1 then for each $x\in B_1$ we have

$$V_k(x) = \gamma'|B_1|^{-\frac{1}{p}}\left(\delta_{ki}\eta(x)\delta_{jj} - \delta_{kj}\eta(x)\delta_{ij}\right) = \gamma'|B_1|^{-\frac{1}{p}}\delta_{ki}$$

and $W_k = C^{-1}L_k u$, for $k = i, j$. Therefore,

$$V\cdot W \;=\; V_i\overline{W_i} = \gamma'|B|^{-\frac{1}{p}}C^{-1}\overline{L_i u} = \frac{\gamma'}{C}|B|^{-\frac{1}{p}}\overline{L_i u}.$$

We conclude the identity (5.1.9) taking $\widetilde{C} := \dfrac{\gamma'}{C}|B|^{-\frac{1}{p}}$.

Lemma 5.1. *If* $\phi\in C_c^\infty(B(0,1))$ *then we can write* $\phi = V_1\cdot W_1$*, where* V_1, W_1 *are smooth vector fields satisfying the following properties:*

(i) *supp* $V_1 \subset B(0,1)$ *and supp* $W_1 \subset B(0,2)$*;*

(ii) *curl*$_{\mathscr{L}}$ $W_1 = 0$ *and* $\|W_1\|_{L^{p'}} \leq C_1$*, for some* $C_1 > 0$ *independent of* ϕ*;*

(iii) $\|V_1\|_{L^p} = \|\phi\|_{L^p}$ *with* $\|$*div*$_{\mathscr{L}^*}$ $V_1\|_{L^p} \leq \|\nabla_{\mathscr{L}^*}\phi\|_{L^p}$.

Analogously, we may write $\phi = V_2\cdot W_2$*, where* V_2, W_2 *are smooth vector fields satisfying:*

(iv) *supp* $W_2 \subset B(0,1)$ *and supp* $V_2 \subset B(0,2)$*;*

(v) *div*$_{\mathscr{L}^*}$ $V_2 = 0$ *and* $\|V_2\|_{L^p} \leq C_2$*, for some* $C_2 > 0$ *independent of* ϕ*;*

(vi) $\|W_2\|_{L^{p'}} = \|\phi\|_{L^{p'}}$ *with* $\|$*curl*$_{\mathscr{L}}$ $W_2\|_{L^{p'}} \leq 2\,\|\nabla_{\mathscr{L}}\phi\|_{L^{p'}}$*;*

A direct consequence of the previous lemma shows that for each $\phi\in C_c^\infty(B(0,1))$ with $\|\phi\|_{W^{1,p'}} \leq 1$, there exists a constant $C_2 > 0$ independent of ϕ such that $C_2\phi = V_2\cdot W_2 \in (\mathscr{D}C_{\mathscr{L}})^p_{0,1}$ for $1 < p < \infty$. Then for $B_1 := B(0,1)$ we have

$$\begin{aligned}
\|g\|_{W^{-1,p}(B_1)} = \sup_{\substack{\|\phi\|_{W^{1,p'}(B_1)}\leq 1\\ \phi\in C_c^\infty(B_1)}} \left|\int g(x)\,\overline{\phi(x)}dx\right| &= (C_2)^{-1} \sup_{\substack{\|\phi\|_{W^{1,p'}(B_1)}\leq 1\\ \phi\in C_c^\infty(B_1)}} \left|\int g(x)\,\overline{(V_2\cdot W_2)}(x)dx\right| \\
&\leq (C_2)^{-1} \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left|\int g(x)\,\overline{f(x)}dx\right|.
\end{aligned} \tag{5.1.12}$$

Using the first part of the lemma, the previous control follows the same replacing $(\mathscr{D}C_{\mathscr{L}})^p_{0,1}$ by $(\mathscr{D}C_{\mathscr{L}})^p_{1,0}$, that is,

$$\|g\|_{W^{-1,p'}(B_1)} \leq (C_1)^{-1} \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{1,0}} \left|\int g(x)\,\overline{f(x)}dx\right|. \tag{5.1.13}$$

Proof. Fix $\phi \in C_c^\infty(B_1)$ and $\eta \in C_c^\infty(B_1^*)$ such that $\eta \equiv 1$ in B_1 and $\|\eta\|_{L^\infty(B_1^*)} \leq 1$. We define $V_1(x) := \phi(x)e_1$ and $W_1(x) := \nabla_{\mathscr{L}}(x_1\eta(x))$. Clearly $curl_{\mathscr{L}}\, W_1 = 0$, $\|V_1\|_{L^p} = \|\phi\|_{L^p}$ and $\|div_{\mathscr{L}^*}\, V_1\|_{L^p} = \|L_1^*\phi\|_{L^p} \leq \|\nabla_{\mathscr{L}^*}\phi\|_{L^p}$. Note that for $x \in B_1$ we have

$$L_1[x_1\eta(x)] = \eta(x) + x_1[L_1\eta](x) = 1,$$

since $\operatorname{supp} V_1 \subset B_1$ we have $(V_1 \cdot W_1)(x) = \phi(x)L_1[x_1\eta(x)] = \phi(x)$. Moreover

$$|W_1(x)| = \sum_{j=1}^n |L_j(x_1\eta(x))| \leq |L_jx_1||\eta| + |x_1| \sum_{j=1}^n |L_j\eta(x)| \lesssim |\eta(x)| + |x_1||\nabla_{\mathscr{L}}\eta(x)|$$

and as $\operatorname{supp} W_1 \subset B_1^*$, we have

$$\|W_1\|_{L^{p'}} \leq |B_1^*|^{\frac{1}{p'}} \|W_1\|_{L^\infty} \leq |B_1^*|^{\frac{1}{p'}} (1 + 2\|\nabla_{\mathscr{L}}\eta\|_{L^\infty})$$

For the second part we define $V_2(x) = L_1^*(x_1\eta(x))e_2 - L_2^*(x_1\eta(x))e_1$, $W_2(x) = \phi(x)e_2$ that satisfies (by definition) $\|W_2\|_{L^{p'}} = \|\phi\|_{L^{p'}}$, $\|curl_{\mathscr{L}}\, W_2\|_{L^{p'}} \leq 2\|\nabla_{\mathscr{L}}\phi\|_{L^{p'}}$ and $div_{\mathscr{L}^*}\, V_2 = 0$. Since

$$L_\ell^*[x_1\eta(x)] = \delta_{\ell 1}(x)\eta(x) + x_1[L_\ell^*\eta](x)$$

we have $|V_2(x)| \leq |x_1|(|L_1^*\eta(x)| + |L_2^*\eta(x)|) + |\eta(x)|$ and $\operatorname{supp} V_2 \subset B_1^*$ that implies

$$\|V_2\|_{L^p} \leq (2\|\nabla_{\mathscr{L}^*}\eta\|_{L^\infty} + 1)|B_1^*|^{\frac{1}{p}}.$$

Note that $\operatorname{supp} W_2 \subset B_1$ and since $L_\ell^*[x_1\eta(x)] = 1$ for $x \in B_1$ we have $\phi = V_2 \cdot W_2$.

□

Now, we move on for a ball $B(x_0, R)$, with $R \geq 1$. For each $\phi \in C_c^\infty(B(x_0,R))$ we may define $\widetilde{\phi} \in C_c^\infty(B(0,1))$ given by $\widetilde{\phi}(y) := \phi(x_0 + yR)$ and applying Lemma 5.1 there exist vector fields $\widetilde{V}_i, \widetilde{W}_i$ for $i = 1,2$ satisfying (i)-(vi) above such that $\widetilde{\phi} = \widetilde{V}_i \cdot \widetilde{W}_i$. Defining $V_i(x) := R^{-\frac{N}{p}}\widetilde{V}_i\left(\frac{x-x_0}{R}\right)$ and $W_i(x) := R^{-\frac{N}{p'}}\widetilde{W}_i\left(\frac{x-x_0}{R}\right)$ we have that there exist constants $C_i > 0$ independent of ϕ such that $C_1\phi = V_1 \cdot W_1 \in (\mathscr{D}C_{\mathscr{L}})^p_{1,0}$ and $C_2\phi = V_2 \cdot W_2 \in (\mathscr{D}C_{\mathscr{L}})^p_{0,1}$.

For each $g \in L^1_{loc}(\mathbb{R}^N)$, we define $\widetilde{g}(y) = g(x_0 + Ry)$ and then

$$\int_{B(x_0,R)} g(x)\overline{(V_i \cdot W_i)(x)}\, dx \;=\; \int_{B(0,1)} \widetilde{g}(y)\overline{(\widetilde{V}_i \cdot \widetilde{W}_i)(y)}\, dy. \tag{5.1.14}$$

Furthermore, using change of variables and the inequality (5.1.8) for $B_1 := B(0,1)$ we have

$$\begin{aligned}\left(\fint_{B(x_0,R)} |g(x)|^p dx\right)^{\frac{1}{p}} = \left(\fint_{B_1} |\widetilde{g}(y)|^p dy\right)^{\frac{1}{p}} &= C_N\|\widetilde{g}\|_{L^p(B_1)} \\ &\leq C\left[\|\widetilde{g}\|_{W^{-1,p}(B_1)} + \sum_{i=1}^n \|L_i^*\widetilde{g}\|_{W^{-1,p}(B_1)}\right]\end{aligned}$$

From (5.1.12) and the identity (5.1.14) we have

$$\|\widetilde{g}\|_{W^{-1,p}(B_1)} \lesssim \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left| \int g(x)\, \overline{f(x)} dx \right| \tag{5.1.15}$$

and by the inequality (5.1.10) we have

$$\sum_{i=1}^{n} \|L_i^* \widetilde{g}\|_{W^{-1,p}(B_1)} \lesssim \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{1,0}\cap(\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left| \int g(x)\overline{f(x)} dx \right|. \tag{5.1.16}$$

Combining the previous estimate, we may conclude

$$\begin{aligned}
\|g\|_{bmo} &\leq \sup_{|B(x_0,R)|\leq 1} \left(\fint_{B(x_0,R)} |g(x)-g_B|^2 dx \right)^{\frac{1}{2}} + \sup_{|B(x_0,R)|>1} \left(\fint_{B(x_0,R)} |g(x)|^p dx \right)^{\frac{1}{p}} \\
&\lesssim \left(\sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{0,1}\cap(\mathscr{D}C_{\mathscr{L}})^p_{1,0}} \left| \int g(x)\overline{f(x)} dx \right| + \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left| \int g(x)\overline{f(x)} dx \right| \right) \\
&\lesssim \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left| \int_{\mathbb{R}^N} g(x)\overline{f(x)} dx \right|.
\end{aligned}$$

The same arguments hold replacing $(\mathscr{D}C_{\mathscr{L}})^p_{0,1}$ by $(\mathscr{D}C_{\mathscr{L}})^p_{1,0}$, by taking p' instead of p in (5.1.8). In more detail, when we take the vector fields

$$V := (2C)^{-1} \left(\overline{L_i u}\, e_j - \overline{L_j u}\, e_i \right) \text{ and } W := \gamma \nabla_{\mathscr{L}} \left((x_j - x_j^0)\, \eta_B(x) \right),$$

with $\gamma := |B^*|^{-\frac{1}{p'}} (2\,\|\nabla_{\mathscr{L}}\eta\|_{L^\infty} + nC)^{-1}$ and C as before, we prove that

$$\sum_{i=1}^{n} \|L_i^* g\|_{W^{-1,p'}(B)} \lesssim \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{1,0}\cap(\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left| \int g(x)\overline{f(x)} dx \right|.$$

Replacing the above inequality and the inequality (5.1.13) into the inequality (5.1.8) for $r = p'$, we obtain

$$\|g\|_{bmo} \lesssim \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{1,0}} \left| \int_{\mathbb{R}^N} g(x)\overline{f(x)} dx \right|. \tag{5.1.17}$$

Indeed, we claim that for each $u \in C_c^\infty(B_1)$ with $\|\nabla u\|_{L^p(B_1)} \leq 1$ and $1 < p < \infty$ there exist vector fields V, W satisfying $div_{\mathscr{L}^*}\, V = 0$ with $\|V\|_{L^p} \leq 1$ and $curl_{\mathscr{L}}\, W = 0$ with $\|W\|_{L^{p'}} \leq 1$ such that

$$V \cdot W = C|B_1|^{-\frac{1}{p'}} \overline{L_i u}, \tag{5.1.18}$$

for some constant $C > 0$. Obtaining

$$\frac{1}{|B_1|^{\frac{1}{p'}}} \sum_{i=1}^{n} \|L_i^* g\|_{W^{-1,p'}(B_1)} \lesssim \sup_{f\in(\mathscr{D}C_{\mathscr{L}})^p_{1,0}\cap(\mathscr{D}C_{\mathscr{L}})^p_{0,1}} \left| \int g(x)\overline{f(x)} dx \right|. \tag{5.1.19}$$

As before, consider a function $u \in C_c^\infty(B_1)$ with $\|\nabla u\|_{L^p(B_1)} \leq 1$ and $\eta \in C_c^\infty(B(0,2))$ such that $\eta \equiv 1$ in $B(0,1)$ and $\|\eta\|_{L^\infty(B(0,2))} \leq 1$. Define the vector fields

$$V := (2C)^{-1} \left(\overline{L_i u}\, e_j - \overline{L_j u}\, e_i \right) \text{ and } W := \gamma |B_1|^{-\frac{1}{p'}} \nabla_{\mathscr{L}} \left(x_j \eta(x) \right),$$

for $i, j \in \{1,...,n\}$ with $i \neq j$, where $\{e_1,\dots,e_n\}$ denotes de canonical basis of $\mathbb{R}^n$ and C, γ are appropriate positive constants to be chosen later. Clearly $div_{\mathscr{L}^*}\, V = L_j^* V_j + L_i^* V_i = 0$ and $|V| \leq |\nabla u|$ choosing as always $C := N \max\limits_{\substack{1\leq k\leq m\\ 1\leq j\leq n}} \{1, |a_{jk}|\}$. Since $\mathrm{supp}\,(V) \subseteq B$, it follows that $\|V\|_{L^p(B)} \leq \|\nabla u\|_{L^p(B)} \leq 1$. It is easy to see that $curl_{\mathscr{L}}\, W = \gamma |B|^{-\frac{1}{p'}} curl_{\mathscr{L}}(\nabla_{\mathscr{L}}\varphi) = \gamma |B|^{-\frac{1}{p'}} \left([L_i, L_j]\varphi\right)_{ij} = 0$. Note that $\mathrm{supp}\,(W) \subseteq \mathrm{supp}\,(\eta) \subseteq B^*$ and $L_\ell\left(x_j\right) = \delta_{\ell j}$. Furthermore, for each $x \in B^*$ we have

$$\sum_{k=1}^{n} \left|L_k\left(x_j \eta(x)\right)\right| = \sum_{k=1}^{n} \left|\delta_{kj}\eta(x) + x_j L_k \eta(x)\right| \leq n + 2\sum_{k=1}^{n} |L_k \eta\,(x)|$$

and choosing $\gamma := 2^{-\frac{N}{p'}}(2\,\|\nabla_{\mathscr{L}}\eta\|_{L^\infty} + n)^{-1}$, follows $|W| \leq 2^{-\frac{N}{p'}}|B|^{-\frac{1}{p'}}$ that implies

$$\|W\|_{L^{p'}} \leq |B^*|^{\frac{1}{p'}}\, \|W\|_{L^\infty(B^*)} \leq 1.$$

Lastly, we point out that $V \cdot W = V_i \overline{W_i} + V_j \overline{W_j}$ and $V_i = V_j = 0$ on $\mathbb{R}^N \backslash B$. Furthermore, as $\eta \equiv 1$ in B then for each $x \in B$ we have

$$W_k(x) = \gamma |B|^{-\frac{1}{p'}} \left(\delta_{kj}\eta(x) + x_j L_k \eta(x)\right) = \gamma |B|^{-\frac{1}{p'}} \delta_{kj}$$

for $k = i, j$. In particular, as we are assuming $\mathscr{L}$ as in (1.3.2) thus $W_i = 0$ and $W_j = \gamma |B|^{-\frac{1}{p'}}$ on B. Therefore,

$$V \cdot W \;=\; V_j\, W_j = \frac{\gamma}{2C}|B|^{-\frac{1}{p'}}\overline{L_i u}.$$

We conclude the identity (5.1.18) taking $C := \max\{\gamma, \gamma'\}$. Thus, replacing the inequality (5.1.19) and the inequality (5.1.13) into the inequality (5.1.8) for $r = p'$, we conclude (5.1.17). □

5.2 Proof of Corollary D ($\mathrm{div}_{\mathscr{L}^*} - \mathrm{curl}_{\mathscr{L}}$ lemma)

Let us prove Corollary 4. To simplify the notation, consider $V := (\mathscr{D}C_{\mathscr{L}})^p_{1,0}$ and $F := h^1(\mathbb{R}^N)$. A direct consequence of Theorem A implies that V is a bounded symmetric (*i.e.* $h \in V$ then $-h \in V$) subset of F. If we prove that the closure of V in the norm F, denoted by $\overline{V}$, contains the unit ball of F, it follows from Lemma III.1 in [13], each $\|f\|_{h^1} \leq 1$ can be decomposed by

$$f = \sum_{k=1}^{\infty} 2^{-k} f_k, \quad f_k \in V, \tag{5.2.1}$$

with convergence in F. Now, from Lemma III.2 in [13], the closed convex hull $\widetilde{V}$ contains the unit ball of F if and only if $\|g\|_{(h^1)^*}$ is equivalent to the functional

$$\sup_{f\in V}\left|\int_{\mathbb{R}^N} g(x) f(x) dx\right|,$$

that is exactly the conclusion of Theorem B, since $(h^1(\mathbb{R}^N))^* = bmo(\mathbb{R}^N)$. The decomposition (11) follows taking $\lambda_k := 2^{-k}\|f\|_{h^1} \in \ell^1(\mathbb{C})$, for every $f \in h^1(\mathbb{R}^N)$. Clearly $\|\lambda\|_{\ell^1} \leq \|f\|_{h^1}$ and the convergence in (11) holds also in the sense of tempered distributions. The same conclusion holds for $V = (\mathscr{D}C_{\mathscr{L}})^p_{0,1}$.

5.3 Additional remarks

Similarly to Application 2 (Section 3.3.2), it is natural to try to apply the result to differential forms instead of vector fields. Again, consider the operator

$$A(\cdot,D)=(d_{\mathscr{L},k},d^*_{\mathscr{L},k-1}):C_c^\infty(\Omega,\Lambda^k\mathbb{R}^n)\to C_c^\infty(\Omega,\Lambda^{k+1}\mathbb{R}^n)\times C_c^\infty(\Omega,\Lambda^{k-1}\mathbb{R}^n),$$

for $0\leq k\leq n$. In particular, since $A(\cdot,D)$ is elliptic and canceling operator for $k\notin\{1,n-1\}$, it follows from [30, Theorem B] that for each $x_0\in\Omega$ there exist an neighborhood $U\subset\Omega$ of x_0 and $C>0$ such that the inequality

$$\|u\|_{L^{N/N-1}}\leq C(\|d_{\mathscr{L},k}u\|_{L^1}+\|d^*_{\mathscr{L},k-1}u\|_{L^1}),$$

holds for any $u\in C_c^\infty(U,\Lambda^k\mathbb{R}^n)$. As an extension of such inequality, it is expected to obtain a generalization of inequality (9) to k-forms, stated below

Corollary 5.2. *Let $0\leq k\leq n$ and $\mathscr{L}=\{L_1,\dots,L_n\}$ an elliptic system of complex vector fields on Ω with smooth complex coefficients. Then every point $x_0\in\Omega$ is contained in an open neighborhood $U\subset\Omega$ such that the estimate*

$$\|d_{\mathscr{L},k}\phi\cdot v_1+d_{\mathscr{L},k}\phi\cdot v_2\|_{h^r}\leq C\left(\|d_{\mathscr{L},k}\phi\|_{h^p}+\|d^*_{\mathscr{L},k-1}\phi\|_{h^p}\right)\left(\|v\|_{h^q}+\|d^*_{\mathscr{L},k}v_1\|_{h^q}+\|d_{\mathscr{L},k-1}v_2\|_{h^q}\right).$$

holds for some $C>0$ and for every $\phi\in C_c^\infty(U,\Lambda^k\mathbb{R}^n)$, $v_1\in C_c^\infty(U,\Lambda^{k+1}\mathbb{R}^n)$ and $v_2\in C_c^\infty(U,\Lambda^{k-1}\mathbb{R}^n)$.

In particular taking $v_2=0$ and $v=v_1$ we have

$$\|d_{\mathscr{L},k}\phi\cdot v\|_{h^r}\leq C\left(\|d_{\mathscr{L},k}\phi\|_{h^p}+\|d^*_{\mathscr{L},k-1}\phi\|_{h^p}\right)\left(\|v\|_{h^q}+\|d^*_{\mathscr{L},k}v\|_{h^q}\right)$$

for all $\phi\in C_c^\infty(U,\Lambda^k\mathbb{R}^n)$ and $v\in C_c^\infty(U,\Lambda^{k+1}\mathbb{R}^n)$, recovering the Theorems B and C in [24], where in the second we assume that the system $\mathscr{L}$ is involutive.

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