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UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

The study of periodic orbits in piecewise continuous differential systems

Gabriela Lye Watanabe

São Carlos-SP
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Gabriela Lye Watanabe

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I dedicate this work
to my family.

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Resumo

Nos últimos anos, o estudo de sistemas dinâmicos suaves por partes avançou rapidamente. Um dos problemas investigados nesta área é o cálculo de ciclos limites que bifurcam a partir da perturbação de um foco fraco ou de uma órbita periódica. Nesta dissertação, apresentaremos dois métodos utilizados para calcular o número de ciclos limites que bifurcam a partir da perturbação de um sistema, nomeadamente o método das constantes de Lyapunov e o método da função de Melnikov.

Palavras-chave: Sistemas contínuos por partes; bifurcação; centro; constantes de Lyapunov; funções de Melnikov.

Abstract

In recent years, the study of piecewise dynamical systems has advanced rapidly. One of the problems investigated in this field is the calculation of limit cycles that bifurcate from the perturbation of a weak focus or of a periodic orbit. In this dissertation, we will present two methods used to compute the number of limit cycles that bifurcate from the perturbation of a system, namely Lyapunov constants method and the Melnikov function method.

Keywords: Piecewise systems; bifurcation; center; Lyapunov constants; Melnikov functions.

Contents

List of Symbols

$\Lambda^k(V)$: Set of alternating k-tensors on the finite-dimensional vector space V;

C^∞: Set of infinitely smooth functions;

C^ω: Set of analytic functions;

$\text{int}X$: Interior of the set X;

$\overline{X}$: Closure of the set X;

∂X: Boundary of the set X;

$B(p,r)$: Open ball centered at p with radius r;

Df or f': Derivative of the function f;

$\frac{\partial}{\partial x_i} f$ or f_{x_i}: Partial derivative of a function $f(x_1,\ldots,x_n)$ respect to x_i;

∇f: The gradient of a function f;

Π_X: Poincaré or return map associated with a vector field X;

$\langle \cdot,\cdot \rangle$: Usual inner product in $\mathbb{R}^n$;

$\widehat{AB}$: Arc from A to B;

$\overrightarrow{AB}$: Oriented segment from A to B;

$O(\cdot)$: Big O function.

Introduction

In 1988, the Russian mathematician Aleksei F. Filippov published the book "*Differential Equations with Discontinuous Right-Hand Sides*" [9], which laid the groundwork for the theory of piecewise systems. In his work, Filippov defines the vector field of such systems using switching manifold defined in an open set, which divides the phase space into distinct regions. Due to the importance of this contribution, there exists a class of piecewise dynamical systems called Filippov systems, which study Filippov vector fields.

Consider the following equation

$$\dot{x} = F(x),$$

where F is a vector field on $\mathbb{R}^2$.

In the linear case, that is, when $F(x) = Ax$, with $A \in M_2(\mathbb{R})$, the classification of phase portraits, as saddles, nodes and centers, depends essentially on the trace and the determinant of the matrix A. For nonlinear systems, if $p \in \mathbb{R}^2$ is an isolated singularity of F, the Hartman-Grobman Theorem guarantees that, when the Jacobian matrix $B = DF(p)$ is hyperbolic (that is, all eigenvalues with real part nonzero), then the phase portrait of F is locally conjugate to the linear system $\dot{x} = Bx$ in a neighborhood of p. However, if B is not hyperbolic, it is not possible to distinguish whether the singularity corresponds to a center or a focus. This is precisely the center-focus problem.

This problem is related to Hilbert's 16th problem, which concerns determining the number of limit cycles. A limit cycle is an isolated periodic orbit within the set of all closed orbits of the system. In the problem proposed by Hilbert, the question is: "what is the maximum number of limit cycles for a planar smooth system with polynomials of degree n?". This remains a famous open problem in the qualitative theory of ordinary differential equations.

As in the theory of continuous dynamical systems, a central problem of piecewise systems is the calculation of limit cycles and the classification of singularities as centers or focus. Currently, we can use some methods to investigate the calculation of limit cycles such as: the averaging theory, the Abelian integrals, the Melnikov function, or the Lyapunov constants. In this work, we will specifically address the Lyapunov constants method and the Melnikov function method.

In the Lyapunov constants method, we study the article "*Center-focus problem for discontinuous planar differential equations*" [13]. In this paper, Armengol Gasull and Joan Torregrosa study the number of limit cycles which bifurcate from a weak focus at the origin in planar piecewise polynomial systems. Their computations of Lyapunov constants, which coincide with some coefficients in the series expansion of the Poincaré map, is based on a suitable decomposition of certain one-forms associated with the expression of the system in polar coordinates. However, since the Lyapunov constants provide only the conditions for the systems have a center at the origin, it is necessary to verify whether the origin is a center for systems satisfying the conditions established by these constants. One way to show this is proving that the system is reversible, either by a change of variables or by finding the first integral associated with the system.

In addition, in the Melnikov function method, we study the article "*Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems*" [18] by Xia Liu and Maoan Han. In this paper, the authors consider a general perturbation of piecewise Hamiltonian systems on the plane. When the unperturbed system has a family of periodic orbits, similar to the perturbations of smooth system, an expression of the first order Melnikov function is obtained, which can be used to study the number of limit cycles bifurcated from the periodic orbits.

From this, the dissertation is structured into four chapters. In Chapter 1, we will present the fundamental concepts and main results of Filippov's Theory. In Chapter 2, we will explain the Lyapunov constants method and illustrate its application with planar examples. In Chapter 3, dedicated to the Melnikov function method, we will define this function for piecewise Hamiltonian systems and apply the theory to planar examples. Finally, in Chapter 4, we conclude the principal ideas of this dissertation. It is important to observe that, in both methods, we replicate the applications of the papers. And these applications were the subject of oral presentations given at scientific events.

CHAPTER 1

Filippov Systems

In this chapter, we will present the main definitions related to Filippov systems. We begin by defining what a Filippov system is. Then, we study the notions of trajectory and periodic orbit. Next, we examine the critical points and tangency points of a Filippov system. Finally, we discuss separatrices and the concept of topological equivalence for these systems.

1.1 Local trajectory

In this section, we will define the Filippov system and the local trajectory of an autonomous Filippov vector field.

Consider the following example.

Example 1.1. *Let $\dot{x} = 1 - 2\,\mathrm{sgn}\,x$. Then,*

$$\dot{x} = \begin{cases} -1 & \text{if } x > 0 \\ 3 & \text{if } x < 0 \end{cases} \implies x(t) = \begin{cases} 3t + c_1 & \text{if } x > 0, \\ -t + c_2 & \text{if } x < 0. \end{cases}$$

Thus, the differential equation $\dot{x} = f(t,x)$ is discontinuous in x, as we can see in Figure 1.1.

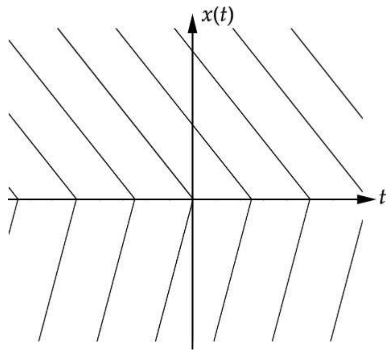


Figure 1.1: Phase portrait of equation $\dot{x} = 1 - 2\,\mathrm{sgn}(x)$.

From Example 1.1 we observe that there exist differential equations whose solutions may lose certain properties – such as differentiability and or even uniqueness – at points of discontinuity in the vector field. This motivates the study of piecewise systems.

Since we work with planar systems and the dynamics will occur around the origin, we consider an open, bounded and connected set $U \subset \mathbb{R}^2$ which contains the origin. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^r, where $r \geq 1$, which has 0 as a regular value, and $\Sigma = f^{-1}(0) \cap U$ is a submanifold of U of codimension 1. Then, the curve Σ splits U into two open subsets:

$$\Sigma^+ = \{(x,y) \in U \mid f(x,y) > 0\} \qquad \text{and} \qquad \Sigma^- = \{(x,y) \in U \mid f(x,y) < 0\}.$$

Now, consider the piecewise vector field:

$$Z(x,y) = \begin{cases} X(x,y), \text{ if } (x,y) \in \Sigma^+, \\ Y(x,y), \text{ if } (x,y) \in \Sigma^-, \end{cases} \tag{1.1}$$

where X and Y are C^r, for $r > 1$, in $\overline{\Sigma^+}$ and $\overline{\Sigma^-}$, respectively, and Σ is called the **switching manifold** of (1.1). We will define the Filippov convention to define the Filippov vector field on $\mathbb{R}^2$ based on [9, page 50].

Let $u \in \mathbb{R}^2$, $I \subset \mathbb{R}$ be an open interval and $G \subset \mathbb{R}^2$ be an open subset. The solution $\varphi : I \to G$ of the differential system $\dot{u} = Z(u)$ corresponds to the solution of the differential inclusion

$$\dot{u} \in \mathscr{F}_Z(u), \qquad u \in G, \tag{1.2}$$

where $\mathscr{F}_Z : G \to \mathbb{R}^2$ is a set-valued function given by

$$\mathscr{F}_Z : G \to \mathbb{R}^2 := \begin{cases} \{X(u)\}, & \text{if } f(u) > 0, \\ \{\alpha X(u) + (1-\alpha)Y(u) \mid \alpha \in [0,1]\}, & \text{if } f(u) = 0, \\ \{Y(u)\}, & \text{if } f(u) < 0. \end{cases}$$

Furthermore, the function $\varphi : I \to G$ is the solution of the differential inclusion (1.2) if it is an absolutely continuous set-valued function, which satisfies $\dot{\varphi}(t) \in \mathscr{F}_Z(\varphi(t))$, for every $t \in I$.

Definition 1.2. *A **Filippov vector field** is a piecewise vector field of the form* (1.1) *whose dynamics is governed by the Filippov convention. That is, its trajectories are solutions of the differential inclusion* (1.2)*. Analogously, we refer to* (1.1) *as a **Filippov system**.*

Sometimes it is convenient to denote a Filippov vector field by $Z = (X,Y)$ in order to clarify its components, or by $Z = (X,Y,f)$ to clearly state the dependence of Σ on f. The space of all piecewise smooth systems of the form (1.1) is denoted by $\mathfrak{Z}^r = \mathfrak{X}^r \times \mathfrak{X}^r$ with the product topology, where $\mathfrak{X}^r$ is a space of C^r vector fields.

Now, we want to understand the geometric interpretation of the trajectory of a Filippov vector field. To do this, we will define the following regions.

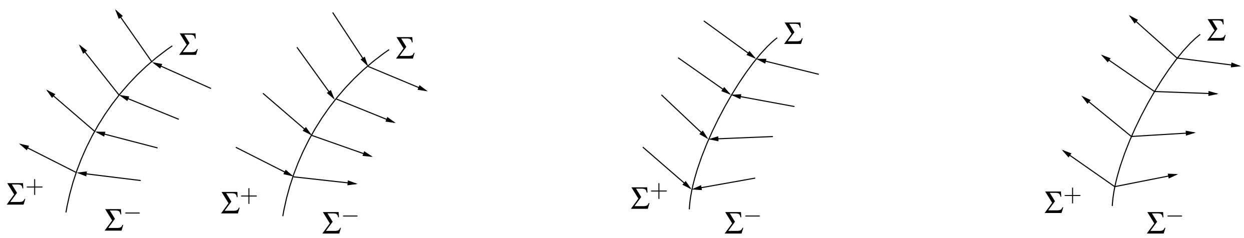
Definition 1.3. *(Guardia et al. [14], page 1970). The **Lie derivative** of f with respect to the vector field X is given by $Xf(p) = \langle X(p), \nabla f(p)\rangle$ and $X^i f(p) = \langle \nabla X^{i-1} f(p), X(p)\rangle$, for all integer $i \geq 2$.*

Definition 1.4. *(Guardia et al. [14], page 1970). Let $Z = (X,Y,f)$ be a Filippov vector field and Σ a switching manifold. We split Σ into three parts depending on whether or not the vector field points towards it:*

(i) ***Crossing or Sewing region**: $\Sigma^c = \{p \in \Sigma \mid Xf(p)\cdot Yf(p) > 0\}$ (see Figure 1.2a);*

(ii) ***Sliding region**: $\Sigma^s = \{p \in \Sigma \mid Xf(p) < 0$ and $Yf(p) > 0\}$ (see Figure 1.2b);*

(iii) ***Escaping region**: $\Sigma^e = \{p \in \Sigma \mid Xf(p) > 0$ and $Yf(p) < 0\}$ (see Figure 1.2c).*



(a) Crossing or Sewing region (b) Sliding region (c) Escaping region

Figure 1.2: Regions of Σ.

Observe that if $p \in \Sigma$ such that $Xf(p) > 0$ or $Xf(p) < 0$, then by the continuity of $Xf : \Sigma \to \mathbb{R}$, there exists an open set $U \subset \Sigma$ containing p such that $Xf(q) > 0$ or $Xf(q) < 0$, for all $q \in U$, respectively. So, Σ^c, Σ^s and Σ^e are relatively open subsets of Σ, and each may have several connected components.

We now define the sliding vector field Z^s, which will be necessary for the definition of the local trajectory. According to Filippov [9, page 51], the vectors of Z^s are given by the convex combination of the vectors of X and Y. In other words, given a Filippov vector field $Z = (X,Y,f)$ and a point $p \in \Sigma$, there exists $\alpha \in [0,1]$ such that $Z^s(p) = \alpha X(p) + (1-\alpha)Y(p)$. We will present here the method to find α based on Buzzi et al. [2, page 3].

Let $Z = (X,Y,f)$ be a Filippov vector field, and let $p \in \Sigma^s$. We can associate to the point p the vectors $X(p)$ and $Y(p)$. The sliding vector field associated with the Filippov vector field Z at p is the vector $Z^s(p)$, which is tangent to Σ^s at p. Assume that the gradient of f is directed toward the domain Σ^+. Since the gradient vector $\nabla f(p)$ is perpendicular to the level curve $f(x,y) = 0$ at a point p, it follows that $\nabla f(p)$ is orthogonal to $Z^s(p)$, that is, $\langle \nabla f(p), Z^s(p)\rangle = 0$.

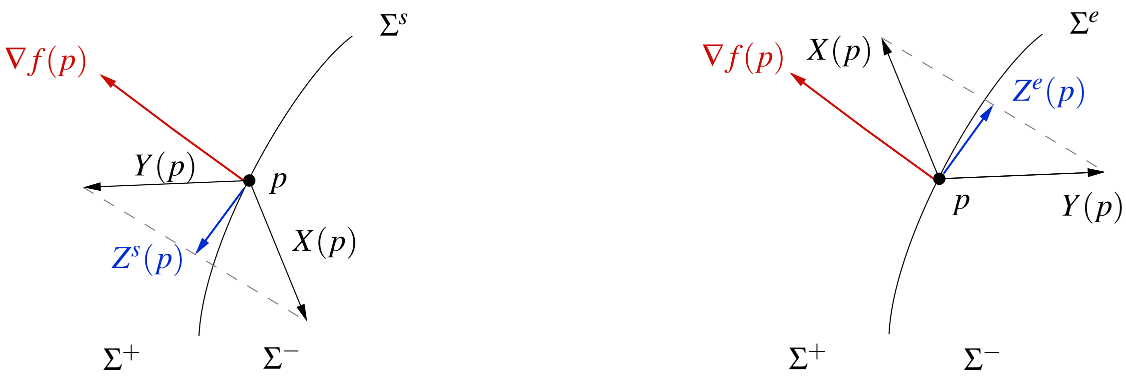
Then,

$$\begin{aligned}
\langle \nabla f(p), Z^s(p)\rangle = 0 &\Longrightarrow \langle \nabla f(p), \alpha X(p) + (1-\alpha)Y(p)\rangle = 0\\
&\Longrightarrow \alpha\langle \nabla f(p), X(p)\rangle + (1-\alpha)\langle \nabla f(p), Y(p)\rangle = 0\\
&\Longrightarrow \alpha\langle \nabla f(p), X(p)\rangle + \langle \nabla f(p), Y(p)\rangle - \alpha\langle \nabla f(p), Y(p)\rangle = 0\\
&\Longrightarrow \alpha\langle \nabla f(p), X(p) - Y(p)\rangle + \langle \nabla f(p), Y(p)\rangle = 0\\
&\Longrightarrow \left\{\alpha = \frac{\langle \nabla f(p), Y(p)\rangle}{\langle \nabla f(p), Y(p) - X(p)\rangle} = \frac{Yf(p)}{Yf(p) - Xf(p)}\right\} \qquad \text{and}\\
&\qquad \left\{1-\alpha = -\frac{\langle \nabla f(p), X(p)\rangle}{\langle \nabla f(p), Y(p) - X(p)\rangle} = -\frac{Xf(p)}{Yf(p) - Xf(p)}\right\}.
\end{aligned}$$

Consequently,

$$Z^s(p) = \frac{1}{Yf(p) - Xf(p)} \cdot [Yf(p)X(p) - Xf(p)Y(p)]. \tag{1.3}$$

Furthermore, if $p \in \Sigma^s$, then $p \in \Sigma^e$ for $-Z$, where $-Z$ is the vector field whose vectors are oriented in the opposite direction of those of Z. Thus, we can define the escaping vector field Z^e on Σ^e associated with Z by $Z^e = -(-Z^s)$. We will denote by Z^Σ both the sliding vector field Z^s and the escaping vector field Z^e (see Figure 1.3).



(a) Vector field Z^s.

(b) Vector field Z^e.

Figure 1.3: Vector field Z^Σ.

Now that we know what the structure of the vector field is like along our space with the switching manifold, we will define the concept of trajectory and orbit for a specific type of systems: the autonomous systems.

Let $p \in \mathbb{R}^2$, X, Y be smooth autonomous vector fields on open subsets $U, V \subset \mathbb{R}^2$, respectively. We denote by $\varphi_X(t,p)$ and $\varphi_Y(t,p)$ as the flows associated with X and Y, respectively, such that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_X(t,p) = X(\varphi_X(t,p)),\\ \varphi_X(0,p) = p, \end{cases} \qquad \text{and} \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_Y(t,p) = Y(\varphi_Y(t,p)),\\ \varphi_Y(0,p) = p, \end{cases} \tag{1.4}$$

where it is defined in time for $t \in I \subset \mathbb{R}$ and $I = I(p,X)$ is a real interval which depends on the point $p \in U$ and the vector fields X.

The local trajectory or orbital solution of a Filippov vector field of the form (1.1) through a point p satisfying (1.4) is the continuous concatenation curve described as follows:

(i) For $p \in \Sigma^+$ and $p \in \Sigma^-$ such that $X(p) \neq (0,0)$ and $Y(p) \neq (0,0)$, respectively, the trajectory is given by $\varphi_Z(t,p) = \varphi_X(t,p)$ and $\varphi_Z(t,p) = \varphi_Y(t,p)$, respectively, for $t \in I \subset \mathbb{R}$;

(ii) For $p \in \Sigma^c$ such that $Xf(p), Yf(p) > 0$ and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t,p) = \varphi_Y(t,p)$, for $t \in I \cap \{t \leq 0\}$, and $\varphi_Z(t,p) = \varphi_X(t,p)$, for $t \in I \cap \{t \geq 0\}$. For the case $Xf(p), Yf(p) < 0$, the definition is the same reversing time;

(iii) For $p \in \Sigma^e$, the trajectory is defined as $\varphi_Z(t,p) = \varphi_{Z^\Sigma}(t,p)$, for $t \leq 0$, and $\varphi_Z(t,p)$ is either $\varphi_X(t,p)$ or $\varphi_Y(t,p)$ or $\varphi_{Z^\Sigma}(t,p)$, for $t \geq 0$. For $p \in \Sigma^s$, the definition is the same reversing time;

(iv) For $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$, the trajectory is defined as $\varphi_Z(t,p) = \varphi_1(t,p)$, for $t \leq 0$, and $\varphi_Z(t,p) = \varphi_2(t,p)$, for $t \geq 0$, where each φ_1, φ_2 is either φ_X or φ_Y or φ_{Z^Σ};

(v) For any other point, $\varphi_Z(t,p) = p$ for all $t \in I$.

Note that by this description the local trajectory is uniquely determined by a suitable concatenation of the local trajectories of X and Y at p.

Definition 1.5. *(Guardia et al. [14], page 1971). The* ***local orbit*** *of a point $p \in U$ is the set*

$$\gamma(p) = \{\varphi_Z(t,p) \mid t \in I\}.$$

Definition 1.6. *(Guardia et al. [14], page 1972). Given a trajectory $\varphi_Z(t,q) \in \Sigma^+ \cup \Sigma^-$ and a point $p \in \Sigma$, we say that p is a* ***departing point*** *of $\varphi(t,q)$ if there exists $t_0 < 0$ such that $\lim\limits_{t \to t_0^+} \varphi_Z(t,q) = p$, and that it is an* ***arrival point*** *of $\varphi_Z(t,q)$ if there exists $t_0 > 0$ such that $\lim\limits_{t \to t_0^-} \varphi_Z(t,q) = p$.*

Furthermore, if $p \in \Sigma^c$, p is a departing point of $\varphi_Z(t,q)$ for any q belonging to the forward orbit

$$\gamma^+(p) = \{\varphi_Z(t,p) \mid t \in I \cap \{t \geq 0\}\},$$

and is an arrival point of $\varphi_Z(t,q)$ for any q belonging to the backward orbit

$$\gamma^-(p) = \{\varphi_Z(t,p) \mid t \in I \cap \{t \leq 0\}\}.$$

1.2 Critical points and tangency points

In this section, we will study the definition of critical points and tangency points of Filippov systems. Critical points are points $p \in \Sigma^s \cup \Sigma^e$ such that $Z^\Sigma(p) = (0,0)$. They can be classified as stable pseudo-nodes, unstable pseudo-nodes, and saddle pseudo-nodes, which exhibit behavior similar to that of stable nodes, unstable nodes, and saddles in smooth systems.

In addition, if $p \in \Sigma$ and either $Xf(p) = 0$ or $Yf(p) = 0$, then p is called a tangency point, which can be classified as either a singular tangency or a regular tangency. An interesting property of such points is that the trajectory leaves one region (sliding, escaping or crossing) of Σ and enters another region at the tangency point. This occurs because the sign of Xf or Yf must change, and since these functions are continuous, the change in trajectory occurs exactly at the tangency point, as we will see later.

Definition 1.7. *(Guardia et al. [14], page 1971). The points $p \in \Sigma^s \cup \Sigma^e$ which satisfy $Z^\Sigma(p) = (0,0)$, that is, the critical points of the Z^Σ, will be called* ***pseudo-equilibria*** *of Z.*

Observe that if p is a pseudo-equilibria of Z, then $Z^\Sigma(p) = (0,0)$, $Xf(p) \neq 0$ and $Yf(p) \neq 0$, because $p \in \Sigma^s \cup \Sigma^e$. Hence, $Yf(p)X(p) - Xf(p)Y(p) = (0,0)$, which implies that

$$X(p) = \frac{Xf(p)}{Yf(p)}\, Y(p).$$

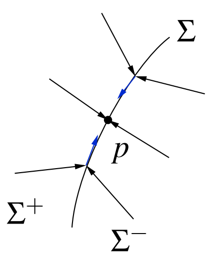
Thus, at this point the vector fields X and Y must be collinear.

Definition 1.8. *(Guardia et al. [14], page 1971). Let $p \in \Sigma^s \cup \Sigma^e$.*

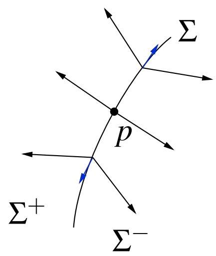
(i) The point $p \in \Sigma^s$ is a ***stable pseudo-node*** *if p is stable and $Z^s(p) = (0,0)$ (see Figure 1.4a);*

(ii) The point $p \in \Sigma^e$ is an ***unstable pseudo-node*** *if p is unstable and $Z^e(p) = (0,0)$ (see Figure 1.4b);*

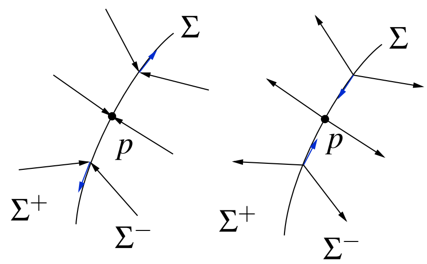
(iii) If either $p \in \Sigma^s$, $Z^s(p) = (0,0)$ and p is unstable, or $p \in \Sigma^e$, $Z^e(p) = (0,0)$ and p is stable, then p is a ***saddle pseudo-node*** *(see Figure 1.4c).*



(a) Stable pseudo-node



(b) Unstable pseudo-node



(c) Saddle pseudo-node

Figure 1.4: Pseudo-nodes.

Definition 1.9. *(Guardia et al. [14], page 1970). The point $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$ which satisfies $Xf(p) = 0$ or $Yf(p) = 0$ is called a* ***tangency point****.*

Definition 1.10. *(Perez [19], page 25). Let $p \in \Sigma$ be a tangency point. If there exists an orbit γ of the vector field X (respectively, Y) that passes through the point p after a finite time t_0, such that γ remains in Σ^- (respectively, Σ^+), for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, then p is called a* ***visible tangency point****. Otherwise, if p is not visible, that is, $\gamma \subset \Sigma^+$ (respectively, Σ^-), for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, it is an* ***invisible tangency point****.*

Definition 1.11. *(Guardia et al. [14], page 1970). Let X be a smooth vector field and let Σ be a switching manifold. A point $p \in \Sigma$ such that $Xf(p) = 0$ and $X^2 f(p) \neq 0$ is a* ***fold or quadratic tangency*** *of X. If $Xf(p) = X^2 f(p) = 0$ and $X^3 f(p) \neq 0$, then X has a* ***cusp or cubic tangency****.*

Definition 1.12. *(Buzzi et al. [2], page 4). Let $p \in \Sigma$ be a tangency point. If p is an invisible tangency point, then p is* ***singular****. On the other hand, the tangency point p is* ***regular*** *if its is not singular.*

According to Guardia et al. [14, page 1970], in more degenerate systems (of infinite codimension), there may exist a continuum of tangency points. In contrast, for low-codimensional bifurcations in planar Filippov systems, the tangency points are isolated in Σ. Thus, we will adopt this assumption throughout the text.

Now, we look at some examples of regular tangent points. In all of them, we take $p = (0,0)$ and $\Sigma = \{(x,y) \in U \mid y = 0\}$. Since the switching manifold corresponds to the subset of axis x and let f be the function associated with Σ, we can suppose without loss generality that $\nabla f = (0,1)$.

Example 1.13. *Consider*

$$Z_1 = \begin{cases} X_1 = \begin{pmatrix} 1 \\ x^2 \end{pmatrix}, & \text{if } y > 0, \\ Y_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \text{if } y < 0. \end{cases}$$

Then, for $y > 0$, we have that

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = x^2 \end{cases} \implies \begin{cases} \int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t & = \int \mathrm{d}t \\ \dot{y} & = x^2 \end{cases} \implies \begin{cases} x = t + C_1 \\ y = \int (t+C_1)^2 \mathrm{d}t \end{cases} \implies \begin{cases} x = t + C_1, \\ y = \frac{t^3}{3} + C_1 t^2 + C_1 t + C_2. \end{cases}$$

And, for $y < 0$

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 1 \end{cases} \implies \begin{cases} \int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t & = \int \mathrm{d}t \\ \int \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t & = \int \mathrm{d}t \end{cases} \implies \begin{cases} x = t + C_1, \\ y = t + C_2. \end{cases}$$

Furthermore,

$$\begin{array}{rclclcl} X_1 f(x,y) & = & \langle X_1(x,y), \nabla f(x,y) \rangle & = & \langle (1,x^2),(0,1) \rangle & = & x^2, \\ X_1^2 f(x,y) & = & \langle X_1(x,y), \nabla X_1 f(x,y) \rangle & = & \langle (1,x^2),(2x,0) \rangle & = & 2x, \\ X_1^3 f(x,y) & = & \langle X_1(x,y), \nabla X_1^2 f(x,y) \rangle & = & \langle (1,x^2),(2,0) \rangle & = & 2, \\ Y_1 f(x,y) & = & \langle Y_1(x,y), \nabla f(x,y) \rangle & = & \langle (1,1),(0,1) \rangle & = & 1. \end{array}$$

Hence, $X_1 f(x,y) \cdot Y_1 f(x,y) > 0$, for all $(x,y) \in \Sigma \setminus \{(0,0)\}$. In other words, $\Sigma \setminus \{(0,0)\} = \Sigma^c$, and $(0,0)$ is a cubic tangency of X_1 (see Figure 1.5).

Example 1.14. *Consider*

$$Z_2 = \begin{cases} X_2 = \begin{pmatrix} 1 \\ 2x \end{pmatrix}, & \text{if } y > 0, \\ Y_2 = \begin{pmatrix} 2 \\ 7x \end{pmatrix}, & \text{if } y < 0. \end{cases}$$

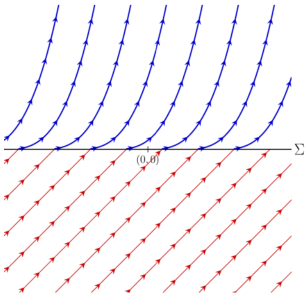


Figure 1.5: Phase portraits of the Filippov vector field Z_1.

Then, for $y > 0$, we have that

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 2x \end{cases} \implies \begin{cases} \int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t & = \int \mathrm{d}t \\ \dot{y} & = 2x \end{cases} \implies \begin{cases} x = t + C_1 \\ y = \int 2(t + C_1)\mathrm{d}t \end{cases} \implies \begin{cases} x = t + C_1, \\ y = t^2 + 2C_1 t + C_2. \end{cases}$$

And, for $y < 0$

$$\begin{cases} \dot{x} = 2 \\ \dot{y} = 7x \end{cases} \implies \begin{cases} \int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t & = \int 2\mathrm{d}t \\ \dot{y} & = 7x \end{cases} \implies \begin{cases} x = 2t + 2C_1 \\ y = \int 7(2t + 2C_1)\mathrm{d}t \end{cases} \implies \begin{cases} x = 2t + 2C_1, \\ y = 7t^2 + 14C_1 t + C_2. \end{cases}$$

Furthermore,

$$\begin{array}{rclclcl} X_2 f(x,y) & = & \langle X_2(x,y), \nabla f(x,y) \rangle & = & \langle (1,2x), (0,1) \rangle & = & 2x, \\ X_2^2 f(x,y) & = & \langle X_2(x,y), \nabla X_2 f(x,y) \rangle & = & \langle (1,2x), (2,0) \rangle & = & 2, \\ Y_2 f(x,y) & = & \langle Y_2(x,y), \nabla f(x,y) \rangle & = & \langle (2,7x), (0,1) \rangle & = & 7x, \\ Y_2^2 f(x,y) & = & \langle Y_2(x,y), \nabla Y_2 f(x,y) \rangle & = & \langle (2,7x), (7,0) \rangle & = & 14. \end{array}$$

Hence, $X_2 f(x,y) \cdot Y_2 f(x,y) = 14x^2 > 0$, for all $(x,y) \in \Sigma \setminus \{(0,0)\}$. In other words, $\Sigma \setminus \{(0,0)\} = \Sigma^c$, and $(0,0)$ is a visible fold tangency of X_2 and an invisible fold tangency of Y_2 (see Figure 1.6).

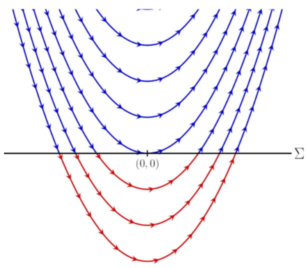


Figure 1.6: Phase portraits of the Filippov vector field Z_2.

Example 1.15. *Consider*

$$Z_3 = \begin{cases} X_3 = \begin{pmatrix} 1 \\ -x^2 \end{pmatrix}, & \text{if } y > 0, \\ Y_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \text{if } y < 0. \end{cases}$$

Then, for $y > 0$, we have that

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = -x^2 \end{cases} \Longrightarrow \begin{cases} \int \frac{dx}{dt} dt & = \int dt \\ \dot{y} & = -x^2 \end{cases} \Longrightarrow \begin{cases} x = t + C_1 \\ y = -\int (t + C_1)^2 dt \end{cases} \Longrightarrow \begin{cases} x = t + C_1, \\ y = -\frac{t^3}{3} - C_1 t^2 - C_1 t + C_2. \end{cases}$$

And, for $y < 0$

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 1 \end{cases} \Longrightarrow \begin{cases} \int \frac{dx}{dt} dt & = \int dt \\ \int \frac{dy}{dt} dt & = \int dt \end{cases} \Longrightarrow \begin{cases} x = t + C_1, \\ y = t + C_2. \end{cases}$$

Furthermore,

$$\begin{array}{rclclcl} X_3 f(x,y) & = & \langle X_3(x,y), \nabla f(x,y) \rangle & = & \langle (1,-x^2),(0,1) \rangle & = & -x^2, \\ X_3^2 f(x,y) & = & \langle X_3(x,y), \nabla X_3 f(x,y) \rangle & = & \langle (1,-x^2),(-2x,0) \rangle & = & -2x, \\ X_3^3 f(x,y) & = & \langle X_3(x,y), \nabla X_3^2 f(x,y) \rangle & = & \langle (1,-x^2),(-2,0) \rangle & = & -2, \\ Y_3 f(x,y) & = & \langle Y_3(x,y), \nabla f(x,y) \rangle & = & \langle (1,1),(0,1) \rangle & = & 1. \end{array}$$

Hence, $X_3 f(x,y) \cdot Y_3 f(x,y) = -x^2 < 0$, for all $(x,y) \in \Sigma \setminus \{(0,0)\}$. Thus, $\Sigma \setminus \{(0,0)\} = \Sigma^s$, and $(0,0)$ is a cubic tangency of X_3.

Moreover,

$$Z^\Sigma(x,0) = \frac{1}{1-(-x^2)} \cdot (1(1,-x^2) - (-x^2)(1,1)) = \left(\frac{1+x^2}{1+x^2}, \frac{0}{1+x^2} \right) = (1,0).$$

Therefore, the vector field Z^Σ is made up of vectors parallel to $(1,0)$, which implies that $\varphi_Z(t,p) = (t,0)^T$ (see Figure 1.7).

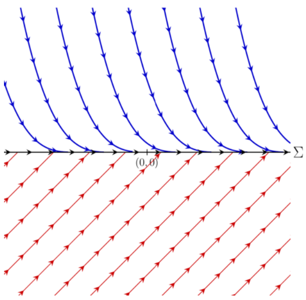


Figure 1.7: Phase portraits of the Filippov vector field Z_3.

Example 1.16. *Consider*

$$Z_4 = \begin{cases} X_4 = \begin{pmatrix} 1 \\ 2x \end{pmatrix}, & \text{if } y > 0, \\ Y_4 = \begin{pmatrix} -2 \\ -7x \end{pmatrix}, & \text{if } y < 0. \end{cases}$$

Then, for $y > 0$, we have that

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 2x \end{cases} \Longrightarrow \begin{cases} \int \frac{dx}{dt} dt & = \int dt \\ \dot{y} & = 2x \end{cases} \Longrightarrow \begin{cases} x = t + C_1 \\ y = \int 2(t + C_1) dt \end{cases} \Longrightarrow \begin{cases} x = t + C_1, \\ y = t^2 + 2C_1 t + C_2. \end{cases}$$

And, for $y<0$

$$\begin{cases}\dot{x}=-2\\ \dot{y}=-7x\end{cases} \implies \begin{cases}x=-2t-2C_1,\\ y=-7t^2-14C_1t+C_2.\end{cases}$$

Furthermore,

$$\begin{array}{rclcrcr} X_4f(x,y) &=& \langle X_4(x,y),\nabla f(x,y)\rangle &=& \langle(1,2x),(0,1)\rangle &=& 2x,\\ X_4^2f(x,y) &=& \langle X_4(x,y),\nabla X_4f(x,y)\rangle &=& \langle(1,2x),(2,0)\rangle &=& 2,\\ Y_4f(x,y) &=& \langle Y_4(x,y),\nabla f(x,y)\rangle &=& \langle(-2,-7x),(0,1)\rangle &=& -7x,\\ Y_4^2f(x,y) &=& \langle Y_4(x,y),\nabla Y_4f(x,y)\rangle &=& \langle(-2,-7x),-7,0\rangle &=& 14.\end{array}$$

Hence, $X_4f(x,y)\cdot Y_4f(x,y)=-14x^2<0$, for all $(x,y)\in\Sigma\setminus\{(0,0)\}$. Thus, for $x>0$, $(x,0)\in\Sigma^e$, and for $x<0$, $(x,0)\in\Sigma^s$. Since the lie derivatives Xf and Yf are continuous functions, their signs change exactly at the tangency point, which is $(0,0)$. Moreover, $(0,0)$ is a visible fold tangency of X_4 and an invisible fold tangency of Y_4.

And,

$$Z^{\Sigma}(x,0)=\frac{1}{-7x-2x}\cdot((-7x)(1,2x)-(2x)(-2,-7x))=\left(\frac{-3x}{-9x},\frac{0}{-9x}\right)=\left(\frac{1}{3},0\right).$$

Therefore, the vector field Z^{Σ} is made up of vectors parallel to $(1/3,0)$, which implies that $\varphi_Z(t,p)=(t/3,0)^T$ (see Figure 1.8).

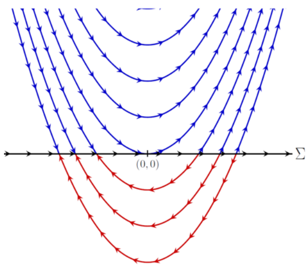


Figure 1.8: Phase portraits of the Filippov vector field Z_4.

Now, let's look at some examples of singular tangent points. As well as in previous examples, we will take $p=(0,0)$, $\Sigma=\{(x,y)\in U\mid y=0\}$, and $\nabla f=(0,1)$.

Example 1.17. *Consider*

$$Z_5=\begin{cases}X_5=\begin{pmatrix}1\\-2x\end{pmatrix}, & \text{if } y>0,\\ Y_5=\begin{pmatrix}-1\\-x+x^2\end{pmatrix}, & \text{if } y<0.\end{cases}$$

Then, for $y>0$, we have that

$$\begin{cases}\dot{x}=1\\ \dot{y}=-2x\end{cases} \implies \begin{cases}\int\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t &=\int\mathrm{d}t\\ \dot{y} &=-2x\end{cases} \implies \begin{cases}x=t+C_1\\ y=\int-2(t+C_1)\mathrm{d}t\end{cases} \implies \begin{cases}x &=t+C_1,\\ y &=-t^2-2C_1t+C_2.\end{cases}$$

And, for $y<0$

$$\begin{cases}\dot{x}=-1\\ \dot{y}=-x+x^2\end{cases} \implies \begin{cases}\int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t &= -\int \mathrm{d}t\\ \dot{y} &= -x+x^2\end{cases} \implies \begin{cases}x=-t+C_1\\ y=\int -(-t+C_1)+(-t+C_1)^2\mathrm{d}t\end{cases}$$

$$\implies \begin{cases}x=-t+C_1\\ y=\int t^2+(-2C_1+1)t+(C_1^2-C_1)\mathrm{d}t\end{cases}$$

$$\implies \begin{cases}x=-t+C_1,\\ y=\frac{t^3}{3}+(-2C_1+1)\frac{t^2}{2}+(C_1^2-C_1)t+C_2.\end{cases}$$

Furthermore,

$$\begin{aligned} X_5f(x,y) &= \langle X_5(x,y),\nabla f(x,y)\rangle = \langle (1,-2x),(0,1)\rangle = -2x,\\ Y_5f(x,y) &= \langle Y_5(x,y),\nabla f(x,y)\rangle = \langle (-1,-x+x^2),(0,1)\rangle = -x+x^2.\end{aligned}$$

Hence, $X_5f(x,y)\cdot Y_5f(x,y)=-2x^3+2x^2$, *which is positive for* $x\in(-\infty,0)\cup(0,1)$ *and negative for* $x\in(1,+\infty)$, *where* $X_5f(x,y)<0$ *and* $Y_5f(x,y)>0$. *Thus,* $(x,0)\in\Sigma^c$ *for* $x\in(-\infty,0)\cup(0,1)$, *and* $(x,0)\in\Sigma^s$ *for* $x\in(1,+\infty)$. *Since the lie derivatives* Xf *and* Yf *are continuous functions, their signs change exactly at the tangency point* $(1,0)$. *And, the trajectories spiral around* $p=(0,0)$ *as it happens around a focus for smooth systems (see Figure 1.9).*

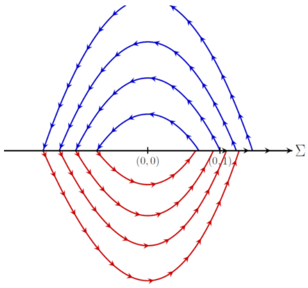


Figure 1.9: Phase portraits of the Filippov vector field Z_5.

Example 1.18. *Consider*

$$Z_6^{\pm}=\begin{cases}X_6^{\pm}=\begin{pmatrix}\pm1\\ x\end{pmatrix}, & \text{if } y>0,\\ Y_6=\begin{pmatrix}0\\ 1\end{pmatrix}, & \text{if } y<0.\end{cases}$$

Then, for $y>0$, *we have that*

$$\begin{cases}\dot{x}=\pm1\\ \dot{y}=x\end{cases} \implies \begin{cases}\int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t &= \pm\int \mathrm{d}t\\ \dot{y} &= x\end{cases} \implies \begin{cases}x=\pm t+C_1\\ y=\int \pm t+C_1\mathrm{d}t\end{cases} \implies \begin{cases}x &= \pm t+C_1,\\ y &= \pm\frac{t^2}{2}+C_1t+C_2.\end{cases}$$

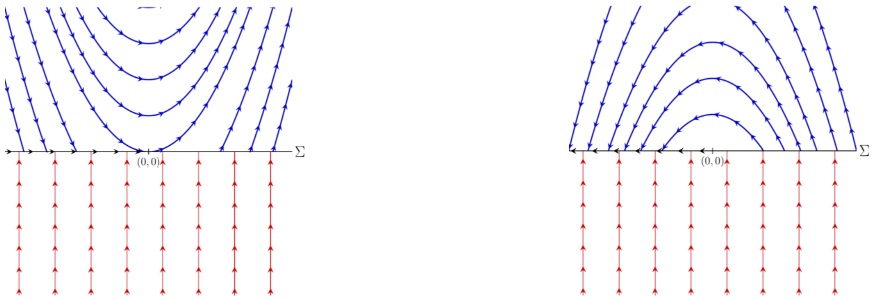
And, for $y<0$

$$\begin{cases}\dot{x}=0\\ \dot{y}=1\end{cases} \implies \begin{cases}\int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t &= \int 0\mathrm{d}t\\ \int \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t &= \int 1\mathrm{d}t\end{cases} \implies \begin{cases}x=0\\ y=t+C_1\end{cases}$$

Furthermore,

$$\begin{array}{rcccccl} X_6^{\pm} f(x,y) & = & \langle X_6^{\pm}(x,y), \nabla f(x,y)\rangle & = & \langle(\pm 1, x),(0,1)\rangle & = & x, \\ Y_6 f(x,y) & = & \langle Y_6(x,y), \nabla f(x,y)\rangle & = & \langle(0,1),(0,1)\rangle & = & 1. \end{array}$$

Hence, $X_6 f(x,y) \cdot Y_6 f(x,y) = x$, which is negative for $x \in (-\infty, 0)$ and positive for $x \in (0, +\infty)$, where $X_6 f(x,y) < 0$ and $Y_6 f(x,y) > 0$ for $x \in (-\infty, 0)$. Thus, $(x,0) \in \Sigma^c$ for $x \in (0, +\infty)$, and $(x,0) \in \Sigma^s$ for $x \in (-\infty, 0)$. Since the lie derivatives Xf and Yf are continuous functions, their signs change exactly at the tangency point, which is $(0,0)$ (see Figure 1.10).



(a) Phase portrait of Z_6^+. (b) Phase portrait of Z_6^-.

Figure 1.10: Phase portraits of the Filippov vector field $Z_6^{\pm}$.

Example 1.19. *Consider*

$$Z_7 = \begin{cases} X_7 = \begin{pmatrix} 1 \\ x \end{pmatrix}, & \text{if } y > 0, \\ Y_7 = \begin{pmatrix} -1 \\ x \end{pmatrix}, & \text{if } y < 0. \end{cases}$$

Then, for $y > 0$, we have that

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = x \end{cases} \Longrightarrow \begin{cases} \int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t & = \int \mathrm{d}t \\ \dot{y} & = x \end{cases} \Longrightarrow \begin{cases} x = t + C_1 \\ y = \int t + C_1 \mathrm{d}t \end{cases} \Longrightarrow \begin{cases} x & = t + C_1, \\ y & = \frac{t^2}{2} + C_1 t + C_2. \end{cases}$$

And, for $y < 0$

$$\begin{cases} \dot{x} = -1 \\ \dot{y} = x \end{cases} \Longrightarrow \begin{cases} \int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t & = -\int \mathrm{d}t \\ \dot{y} & = x \end{cases} \Longrightarrow \begin{cases} x = -t + C_1 \\ y = \int -t + C_1 \mathrm{d}t \end{cases} \Longrightarrow \begin{cases} x & = -t + C_1, \\ y & = -\frac{t^2}{2} + C_1 t + C_2. \end{cases}$$

Furthermore,

$$\begin{array}{rcccccl} X_7 f(x,y) & = & \langle X_7(x,y), \nabla f(x,y)\rangle & = & \langle(1, x),(0,1)\rangle & = & x, \\ Y_7 f(x,y) & = & \langle Y_7(x,y), \nabla f(x,y)\rangle & = & \langle(-1,x),(0,1)\rangle & = & x. \end{array}$$

Hence, $X_7 f(x,y) \cdot Y_7 f(x,y) = x^2$, which is positive for all $x \in \mathbb{R}$. Thus, $(x,0) \in \Sigma^c$, for all $x \in \mathbb{R}$ (see Figure 1.11).

Now, we will give a generalization of the definition of singularity. Roughly speaking, a singularity can be characterized by being the zero of a suitable function.

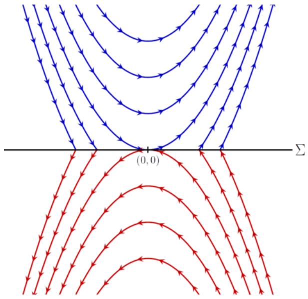


Figure 1.11: Phase portraits of the Filippov vector field Z_7.

Definition 1.20. *(Guardia et al. [14], page 1972). The* ***singularities*** *of a Filippov vector field* (1.1) *are:*

(i) $p \in \Sigma^{\pm}$ *such that* p *is an equilibrium of* X *or* Y*, that is* $X(p) = (0,0)$ *or* $Y(p) = (0,0)$*, respectively;*

(ii) $p \in \Sigma^s \cup \Sigma^e$ *such that* p *is a pseudoequilibrium, that is,* $Z^{\Sigma}(p) = (0,0)$*;*

(iii) $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$*, that is,* p *is a (regular and singular) tangency point (i.e.,* $Xf(p) = 0$ *or* $Yf(p) = 0$*).*

Any other point will be called ***regular point****.*

As noted by Guardia et al. [14, page 1972], in smooth dynamical systems, the singularities correspond to the zeros of the vector field. This implies that the trajectory and, as consequence, the orbit through these points is just the point itself. Nevertheless, in Filippov systems there exist singularities, such as regular tangency points, which have as orbit such that $\gamma(p) \neq \{p\}$. Thus, we will classify the singularities as distinguished and non-distinguished.

Definition 1.21. *(Guardia et al. [14], page 1972). Let* p *be a singularity of a Filippov vector field* (1.1)*. If* $\gamma(p) = \{p\}$*, then* p *is a* ***distinguished singularity****. If* p *is not a distinguished, then* p *is called* ***non-distinguished singularity****.*

In particular, we can observe that the non-distinguished singularity are the regular tangency points, which although they are not regular points, their local orbit is homeomorphic to $\mathbb{R}$.

Definition 1.22. *(Guardia et al. [14], page 1972). We can classify a distinguished singularity* p *as:*

(i) $p \in \Sigma^{\pm}$ *such that* p *is an equilibrium of* X *or* Y*, that is,* $X(p) = (0,0)$ *or* $Y(p) = (0,0)$*, respectively;*

(ii) $p \in \Sigma^s \cup \Sigma^e$ *such that* p *is a pseudoequilibrium, that is,* $Z^{\Sigma}(p) = (0,0)$*;*

(iii) $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$ *such that it is a singular tangency point.*

Definition 1.23. *(Guardia et al. [14], page 1975). A* ***(maximal) regular orbit*** *of Z is a piecewise smooth curve γ such that:*

(i) $\gamma \cap \Sigma^+$ *and* $\gamma \cap \Sigma^-$ *are a union of orbits of the smooth vector field X and Y, respectively;*

(ii) The intersection $\gamma \cap \Sigma$ *consists only of crossing points and regular tangency points in* $\partial \Sigma^c$*;*

(iii) γ *is maximal with respect to these conditions.*

Note that a regular orbit never hits Σ^s nor Σ^e.

Definition 1.24. *(Guardia et al. [14], page 1975). A* ***(maximal) sliding orbit*** *or* ***singular orbit*** *of Z is a smooth curve* $\gamma \subset \overline{\Sigma^s} \cup \overline{\Sigma^e}$ *such that it is a maximal orbit of the smooth vector field* Z^Σ.

Furthermore, in the next chapter, we work with a specific type of singularity known as Σ-*monodromic singularity*. Its definition requires the concept of a Σ-characteristic orbit.

Definition 1.25. *(Buzzi et al. [3], page 3). Let p be an isolated singularity point in* Σ *of a piecewise analytic vector field* $Z = (X,Y)$*. We say that p has a* ***Σ-characteristic orbit*** *if one of the following conditions holds:*

(i) There exists a regular trajectory γ *of X (resp. Y), with* $p \in \gamma(t_0)$*, for* $t_0 \in \mathbb{R}$*, and* $p \in \overline{\gamma \cap \Sigma^+}$ *(resp.* $p \in \overline{\gamma \cap \Sigma^-}$*);*

(ii) There exists a regular trajectory γ *of X (resp. Y) with* $\lim_{t \to \pm\infty} \gamma(t) = p$ *and there exists a neighborhood V of p such that* $\gamma \cap V \subset \cap \Sigma^+$ *(resp.* $\gamma \cap V \subset \cap \Sigma^-$*);*

(iii) For all neighborhood V of p, there exists $q \in V \cap \Sigma$ *such that* $Xf(p) \cdot Yf(p) < 0$.

Definition 1.26. *(Buzzi et al. [3], page 3). A singularity point* $p \in \Sigma$ *is a* ***Σ-monodromic singularity point*** *of Z if Z does not have* Σ*-characteristic orbits associated with p.*

Definition 1.27. *(Buzzi et al. [3], page 4). A* Σ*-monodromic singularity point p of Z is a*

(i) ***center*** *if there exists a neighborhood V of p such that* $V \setminus \{p\}$ *is fulfilled of regular orbits of Z;*

(ii) ***focus*** *if there exists a neighborhood V of p such that for all orbits* γ *of Z by points in V spirals toward or backward p;*

(iii) ***center-focus*** *if there exists a sequence of regular orbits* γ_n *of Z, with* γ_{n+1} *in the interior of* γ_n *such that* $\gamma_n \to p$ *as* $n \to \infty$ *and such that every trajectory between* γ_n *and* γ_{n+1} *spirals toward* γ_n *or* γ_{n+1} *as t increase or decrease.*

By Definitions 1.25 and 1.27, a singularity point $p \in \Sigma$ is a Σ-monodromic singularity point if and only if it is a center, or focus, or center-focus.

Proposition 1.28. *(Buzzi et al. [3], page 9). Let $Z=(X,Y)$ be a piecewise analytic vector filed and $p_0 \in \Sigma$ be a singularity point Without loss of generality we assume that p_0 is the origin and the orbits of Z turn around the origin in counterclockwise sense. If the following conditions hold:*

(i) there exists a continuous change of coordinate such that the switching manifold Σ in these new coordinates, is a subset of $\{y=0\}$; and

(ii) there are integers p,q,v and w such that in the wight polar coordinates $(x,y)=(r^p\cos\theta, r^v\sin\theta)$ in $\Sigma\cup\Sigma^+$ and $(x,y)=(r^q\cos\theta, r^w\sin\theta)$ in $\Sigma\cup\Sigma^-$ the systems associated with the vector fields X and Y are equivalent to differential equations

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{F^{\pm}(r,\theta)}{G^{\pm}(r,\theta)}, \tag{1.5}$$

where $F^{\pm}$ and $G^{\pm}$ are analytic functions with $F^{\pm}(0,\theta)=0$, for all $\theta\in\mathbb{R}$, $G^{+}(0,\rho)\neq 0$, for all $\theta\in[0,\pi]$, and $G^{-}(0,\theta)\neq 0$, for all $\theta\in[-\pi,0]$;

then Z has an isolated Σ-monodromic singularity at p_0.

See Buzzi et al. [3, page 9] for a proof of Proposition 1.28.

1.3 Separatrices, periodic orbits and cycles

In this section, we will define the concepts of separatrix and periodic orbit for planar Filippov systems.

Definition 1.29. *(Guardia et al. [14], page 1975). An **unstable separatrix** is either:*

(i) A regular orbit Γ which is the unstable invariant manifold of a regular saddle point $p\in\overline{\Sigma^+}$ of X or $p\in\overline{\Sigma^-}$ of Y, that is,

$$\Gamma=\left\{q\in U \text{ such that } \varphi_Z(t,q) \text{ is defined for } t\in(-\infty,0) \text{ and } \lim_{t\to-\infty}\varphi_Z(t,q)=p\right\}.$$

We denote it by $W^u(p)$; or

(ii) A regular orbit which has a distinguished singularity $p\in\Sigma$ as a departing point. We denote it by $W^u_{\pm}(p)$, where the subscript $\pm$ means that it leaves p from $\Sigma^{\pm}$.

*A **stable separatrix** is defined analogously. If a separatrix is simultaneously stable and unstable it is a **separatrix connection**. If unstable separatrices arrive at the same point p they are related.*

As observed by Guardia et al. [14, page 1975], the trajectory lying in the separatrix reaches p in infinite time when the α-limit set is a regular saddle point (Case (i)) whereas in the second case, since distinguished singularity is a point p such that $\gamma(p)=\{p\}$, then the trajectory lying in the separatrix may reach the singularity in finite time.

From now on, we will explore how we can generalize the concept of periodic orbits in Filippov systems, which are divided into two cases. The first one is the regular periodic orbit.

Definition 1.30. *(Guardia et al. [14], page 1976). A* ***regular periodic orbit*** *is a regular orbit* $\gamma = \{\varphi_Z(t,p) \mid t \in \mathbb{R}\}$*, which therefore belongs to* $\Sigma^+ \cup \Sigma^- \cup \overline{\Sigma}^c$ *and satisfies* $\varphi_Z(t+T,p) = \varphi_Z(t,p)$ *for some* $T > 0$.

Furthermore, Guardia et al. [14, page 1976] adds that the second case is the sliding periodic orbit when Σ is homeomorphic to $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and $\Sigma = \Sigma^s$ or $\Sigma = \Sigma^e$ in such a way that the sliding vector field does not have critical points. Thus, the whole Σ is a periodic orbit.

Definition 1.31. *(Guardia et al. [14], page 1976). A* ***periodic cycle*** *is the closure of a finite set of pieces of orbits* $\gamma_1, \ldots, \gamma_n$ *such that* γ_{2k} *is a piece of sliding orbit,* γ_{2k+1} *is a maximal regular orbit and the departing and arrival points of* γ_{2k+1} *belong to* $\overline{\gamma_{2k}}$ *and* $\overline{\gamma_{2k+2}}$*, respectively (see Figure 1.12).*

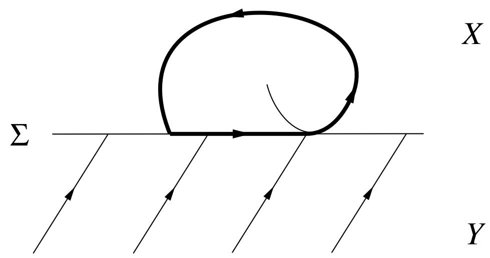


Figure 1.12: Example of a cycle.

Besides cycles and periodic orbits, there exists another distinguished geometric object which is important when one studies topological equivalence and bifurcations in Filippov Systems.

Definition 1.32. *(Guardia et al. [14], page 1976). A* ***pseudocycle*** *is the closure of a set of regular orbits* $\gamma_1, \ldots, \gamma_n$ *such that their edges, that is the arrival and departing points, of any* γ_i*, coincide with one of the edges of* γ_{i-1} *and one of the edges of* γ_{i+1} *(and also between* γ_1 *and* γ_n*) forming a curve homeomorphic to* $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$*, in such a way that in some point coincide two departing or two arrival points (see Figure 1.13).*

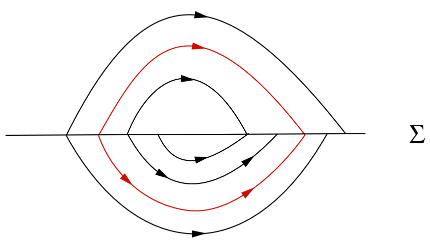


Figure 1.13: Example of a pseudocycle.

1.4 Topological equivalence of Filippov Systems

In this section, we will define topological equivalence and Σ-equivalence, under which objects such as periodic orbits, cycles and pseudo-cycles are preserved.

Definition 1.33. *(Guardia et al. [14], page 1977). Two Filippov vector fields Z and $\widetilde{Z}$ of $\mathfrak{Z}^r$ defined in open sets U and $\widetilde{U}$ and with switching manifolds $\Sigma \subset U$ and $\widetilde{\Sigma} \subset \widetilde{U}$ respectively are **Σ-equivalent** if there exists an orientation preserving homeomorphism $h : U \to \widetilde{U}$ which sends Σ to $\widetilde{\Sigma}$ and sends orbits of Z to orbits of $\widetilde{Z}$.*

It follows from the definition of Σ-equivalence that regular orbits are taken to regular orbits and distinguished singularities to distinguished singularities. Moreover, since it sends arrival and departing points to arrival and departing points, the sets $\overline{\Sigma^c}, \overline{\Sigma^s}$ and $\overline{\Sigma^e}$ are preserved. Consequently, it also sends sliding orbits to sliding orbits and preserves separatrices, separatrix connections, periodic orbits, cycles and pseudocycles.

The definition of Σ-equivalence is strict, as it requires the preservation of the switching manifold. Nevertheless, topologically, for Z and $\widetilde{Z}$ to display similar qualitative behavior, it is not necessary to preserve the crossing region $\overline{\Sigma^c}$. The flow near a point in the crossing region is the same as the flow near a regular point in Σ^+ or Σ^-, where the vector field is smooth. Thus, we will consider also the classical concept of topological equivalence.

Definition 1.34. *(Guardia et al. [14], page 1977). Two Filippov vector fields Z and $\widetilde{Z}$ of $\mathfrak{Z}^r$ defined in open sets U and $\widetilde{U}$ and with switching manifolds $\Sigma \subset U$ and $\widetilde{\Sigma} \subset \widetilde{U}$ respectively are **topologically equivalent** if there exists an orientation preserving homeomorphism $h : U \to \widetilde{U}$ which sends orbits of Z to orbits of $\widetilde{Z}$.*

From these definitions, we conclude that if two vector fields are Σ-equivalent, then they are also topologically equivalent but the reciprocal is not true. For an example of the non-validity of the reciprocal we refer to Section 9 of [14], on page 1998. And, analogously to Σ-equivalences, topological equivalences preserve $\overline{\Sigma^s}$ and $\overline{\Sigma^e}$, which implies that they preserve $\Sigma^+ \cup \Sigma^- \cup \overline{\Sigma^c}$, regular orbits, sliding orbits, distinguished singularities, separatrices, separatrix connections, periodic orbits, cycles and pseudocycles.

Definition 1.35. *(Guardia et al. [14], page 1977). Let X and $\widetilde{X}$ be smooth vector fields, and $\varphi_X(t,x)$ and $\varphi_{\widetilde{X}}(t,x)$ their corresponding flows. We say that they are C^r-conjugated if there exists a C^r homeomorphism h such that $h(\varphi_X(t,x)) = \varphi_{\widetilde{X}}(t,h(x))$.*

According to Perez [19, page 34], since h is a smooth homeomorphism, then it follows that

$$\begin{array}{rcll} h(\varphi_X(t,x)) & = & \varphi_{\widetilde{X}}(t,h(x)) & \Longrightarrow \\ \frac{\mathrm{d}}{\mathrm{d}t}(h(\varphi_X(t,x))) & = & \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_{\widetilde{X}}(t,h(x))) & \Longrightarrow \\ Dh(\varphi_X(t,x))\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_X(t,x)) & = & \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_{\widetilde{X}}(t,h(x))) & \Longrightarrow \\ Dh(\varphi_X(t,x))X(\varphi_X(t,x)) & = & \widetilde{X}(\varphi_{\widetilde{X}}(t,h(x))) & \overset{t=0}{\Longrightarrow} \\ Dh(p)X(p) & = & \widetilde{X}(h(p)), & \end{array}$$

where Dh denotes the differential of h. From the bijectivity of h, there exists a point $\widetilde{p}$ in the domain of h such that

$$Dh(h^{-1}(\widetilde{p}))X(h^{-1}(\widetilde{p})) = \widetilde{X}(\widetilde{p}).$$

Thus, $h_*X = \widetilde{X}$, where $h_*X(\widetilde{p}) = Dh(h^{-1}(\widetilde{p}))X(h^{-1}(\widetilde{p}))$ (see Figure 1.14).

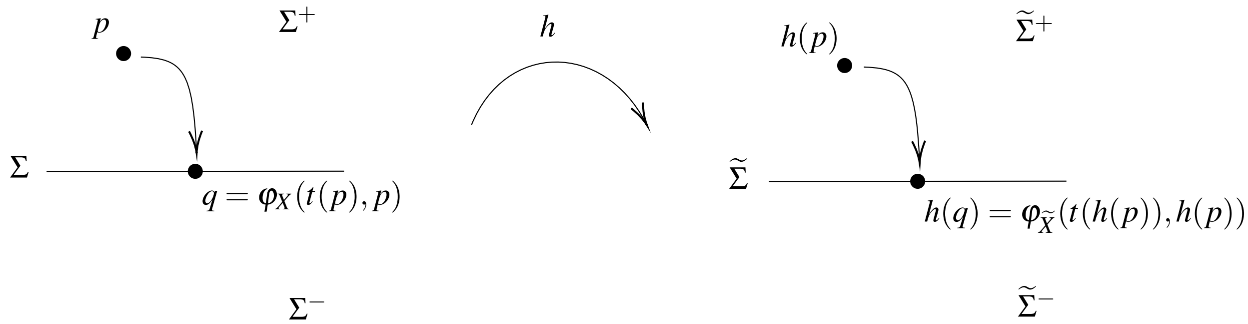


Figure 1.14: C^r-conjugation h.

Proposition 1.36. *(Guardia et al. [14], page 1978). We consider any diffeomorphism $h : U \to \widetilde{U}$ which conjugates on one hand, X in $\Sigma^+ \subset U$ and $\widetilde{X}$ in $\widetilde{\Sigma} \subset \widetilde{U}$ and, in the other hand, Y in $\Sigma^- \subset U$ and $\widetilde{Y}$ in $\widetilde{\Sigma} \subset \widetilde{U}$. Then, it also conjugates the sliding vector fields Z^Σ and $\widetilde{Z}^\Sigma$, and therefore h gives a topological equivalence between $Z = (X, Y)$ and $\widetilde{Z} = (\widetilde{X}, \widetilde{Y})$.*

See Perez [19, page 34] for a proof of Proposition 1.36.

Now that we have studied some of the main concepts of piecewise systems, we will study Lyapunov constants and Melnikov functions for the computation of limit cycles. We will then apply these methods to examples of planar systems. In all cases we consider, the switching manifold will correspond either to the subset of x-axis or to the subset of y-axis. Consequently, we will adjust the notation for a Filippov systems to $Z = (X^+, X^-)$ or $Z = (Y^+, Y^-)$, respectively.

CHAPTER 2

Lyapunov coefficients for piecewise systems

In this chapter, we will study a perturbation problem involving perturbations of a weak focus. In particular, consider the following class of discontinuous planar systems of ordinary differential equations

$$Z = \begin{cases} X^+ = \begin{pmatrix} -y + P^+(x,y) \\ x + Q^+(x,y) \end{pmatrix}, & \text{if } y \geq 0, \\ X^- = \begin{pmatrix} -y + P^-(x,y) \\ x + Q^-(x,y) \end{pmatrix}, & \text{if } y \leq 0. \end{cases} \tag{2.1}$$

where P^+, Q^+, P^-, Q^- are analytic functions starting at least with second order terms. The above system has the origin as a critical point. We are interested in the following two problems:

($\mathbf{Q_1}$) The center-focus problem, that is, to determine if the origin of system (2.1) is either a center, or an attractor, or a repeller.

($\mathbf{Q_2}$) The cyclicity problem, that is, fix a class of systems of type (2.1) and determine the maximum number of limit cycles which bifurcate from the origin under the variation of the parameters inside this class of systems.

Furthermore, if any linear term is added to system (2.1) in X^+ and X^-, then the linear parts of the vector fields X^+ and X^- at the singular point $(0,0)$, given by $DX^+((0,0))$ and $DX^-((0,0))$ respectively, can have zero trace and positive determinant. In this case, the origin can be a Σ-monodromic singularity of the system (2.1), that is, the origin can be a center, a focus, or a center-focus. In particular, if origin is a center, then one more limit cycle can appear.

2.1 Definitions and main results

In this section we will see the method to compute the Lyapunov constants. Consider the piecewise system (2.1).

Note that the origin is a critical point. By applying a polar coordinate transformation to (2.1), with $x = r\cos\theta$ and $y = r\sin\theta$, we have that:

$$\begin{cases} x^2+y^2=r^2 \\ \dfrac{y}{x}=\tan\theta \end{cases} \implies \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(x^2+y^2\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{y}{x}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\tan\theta\right) \end{cases} \implies \begin{cases} 2x\dot{x}+2y\dot{y}=2r\dot{r} \\ \dfrac{\dot{y}x-y\dot{x}}{x^2}=\sec^2\theta\,\dot{\theta} \end{cases}$$

$$\implies \begin{cases} \dot{r}=\dfrac{x\dot{x}+y\dot{y}}{r} \\ \dot{\theta}=\dfrac{\dot{y}x-y\dot{x}}{r^2\cos^2\theta\cdot\dfrac{1}{\cos^2\theta}} \end{cases} \implies \begin{cases} \dot{r}=\dfrac{x\dot{x}+y\dot{y}}{r}, \\ \dot{\theta}=\dfrac{\dot{y}x-y\dot{x}}{r^2}. \end{cases}$$

For $y \geq 0$, we have that $\theta \in [0,\pi]$ and

$$\begin{cases} \dot{r}=\dfrac{[r\cos\theta(-r\sin\theta+P^+(r\cos\theta,r\sin\theta))]+[r\sin\theta(r\cos\theta+Q^+(r\cos(\theta,r\sin\theta)))]}{r} \\ \dot{\theta}=\dfrac{[r\cos\theta+Q^+(r\cos\theta,r\sin\theta)r\cos\theta]-[r\sin\theta(-r\sin\theta+P^+(r\cos\theta,r\sin\theta))]}{r^2} \end{cases} \implies$$

$$\begin{cases} \dot{r}=\dfrac{r[\cos\theta P^+(r\cos\theta,r\sin\theta)+\sin\theta Q^+(r\cos\theta,r\sin\theta))]}{r} \\ \dot{\theta}=\dfrac{r^2[\sin^2\theta+\cos^2\theta]}{r^2}+\dfrac{r\cos\theta Q^+(r\cos\theta,r\sin\theta)-r\sin\theta P^+(r\cos\theta,r\sin\theta)]}{r^2} \end{cases} \implies$$

$$\begin{cases} \dot{r}=\cos\theta P^+(r\cos\theta,r\sin\theta)+\sin\theta Q^+(r\cos\theta,r\sin\theta)) \\ \dot{\theta}=1+\dfrac{1}{r}[\cos\theta Q^+(r\cos\theta,r\sin\theta)-\sin\theta P^+(r\cos\theta,r\sin\theta)] \end{cases} \implies$$

$$\begin{cases} \dot{r}=R^+(r,\theta), \\ \dot{\theta}=1+\Theta^+(r,\theta), \end{cases}$$

where

$$\begin{aligned} R^+(r,\theta) &= \cos\theta P^+(r\cos\theta,r\sin\theta)+\sin\theta Q^+(r\cos\theta,r\sin\theta)), \\ \Theta^+(r,\theta) &= \frac{1}{r}[\cos\theta Q^+(r\cos\theta,r\sin\theta)-\sin\theta P^+(r\cos\theta,r\sin\theta)]. \end{aligned}$$

Applying the same idea to the case $y \leq 0$, we have that $\theta \in [\pi,2\pi]$ and

$$\begin{cases} \dot{r}=R^-(r,\theta), \\ \dot{\theta}=1+\Theta^-(r,\theta). \end{cases}$$

So, the expression of (2.1) in polar coordinates is given by

$$\widetilde{Z}=\begin{cases} \widetilde{X^+}=\begin{pmatrix} R^+(r,\theta) \\ 1+\Theta^+(r,\theta) \end{pmatrix}, & \text{if } \theta\in[0,\pi], \\ \widetilde{X^-}=\begin{pmatrix} R^-(r,\theta) \\ 1+\Theta^-(r,\theta) \end{pmatrix}, & \text{if } \theta\in[\pi,2\pi], \end{cases} \tag{2.2}$$

where R^+, R^-, Θ^+ and Θ^- are analytic functions in r, $\sin\theta$ and $\cos\theta$.

As Chicone [5, page 508] observes, Θ^+ and Θ^- have a removable singularity at $r = 0$. By the change to polar coordinates, the singularity at the origin in the plane has been blow-up to the circle $\{0\}\times\mathbb{S}^1$ on the phase cylinder $\mathbb{R}\times\mathbb{S}^1$. In this case, the singularity at the origin corresponds to the family of the periodic orbits on the cylinder given by the solutions $r(t) \equiv 0$ and $\theta(t) = t + \theta_0$. Taking this into account, unless stated otherwise, from now on we will assume that $\theta_0 = 0$. In addition, the Poincaré section $\theta = 0$ corresponds to the Poincaré section $\{y = 0\}$ of the family (2.1). Thus, there is a correspondence between the returns maps and the displacement function associated with this respective Poincaré section.

Since $\dot{\theta} = 1 + \Theta^+(r,\theta)$, for $\theta \in [0,\pi]$, and $\dot{\theta} = 1 + \Theta^-(r,\theta)$, for $\theta \in [\pi,2\pi]$, are not vanishing for $t = 0$, then θ is local invertible on some bounded time interval, by the Inverse Function Theorem (see Theorem A.4). So, we can remove the dependence on time t from the equation (2.2), thus obtaining

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \begin{cases} \dfrac{R^+(r,\theta)}{1+\Theta^+(r,\theta)}, & \text{if } \theta \in [0,\pi], \\ \dfrac{R^-(r,\theta)}{1+\Theta^-(r,\theta)}, & \text{if } \theta \in [\pi,2\pi]. \end{cases} \tag{2.3}$$

By Proposition 1.28, we have that the origin is Σ-monodromic singularity of Z.

Now, consider the following initial value problems

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \begin{cases} \dfrac{R^+(r,\theta)}{1+\Theta^+(r,\theta)}, & \text{if } \theta \in [0,\pi], \\ r^+(\rho,0) = \rho, \end{cases} \qquad \frac{\mathrm{d}r}{\mathrm{d}\theta} = \begin{cases} \dfrac{R^-(r,\theta)}{1+\Theta^-(r,\theta)}, & \text{if } \theta \in [\pi,2\pi]. \\ r^+(\rho,0) = \rho, \end{cases} \tag{2.4}$$

We will find the systems of differential one-forms with the initial condition equivalent to these systems. By Cartan [4, page 88], the equation (2.3) is often written in the form

$$\mathrm{d}r = \begin{cases} \dfrac{R^+(r,\theta)}{1+\Theta^+(r,\theta)}\mathrm{d}\theta, & \text{if } \theta \in [0,\pi], \\ \dfrac{R^-(r,\theta)}{1+\Theta^-(r,\theta)}\mathrm{d}\theta, & \text{if } \theta \in [\pi,2\pi]. \end{cases}$$

The solution is the function $r(\theta)$ which annihilate the differential forms

$$\mathrm{d}r - \frac{R^+(r,\theta)}{1+\Theta^+(r,\theta)}\mathrm{d}\theta, \qquad \mathrm{d}r - \frac{R^-(r,\theta)}{1+\Theta^-(r,\theta)}\mathrm{d}\theta,$$

which is equivalent to annihilate the differentials forms

$$r\mathrm{d}r - \left(r \cdot \frac{R^+(r,\theta)}{1+\Theta^+(r,\theta)}\right)\mathrm{d}\theta, \qquad r\mathrm{d},r - \left(r \cdot \frac{R^-(r,\theta)}{1+\Theta^-(r,\theta)}\right)\mathrm{d}\theta. \tag{2.5}$$

Let $H(r) = r^2/2$, then $\mathrm{d}H = r\mathrm{d}r$, and we have that $[r \cdot R^\pm(r,\theta)]/[1+\Theta^\pm(r,\theta)]\mathrm{d}\theta$ are analytic one-forms, 2π-periodic in θ and polynomial in r. Thus, the solution curves of system (2.2) can be obtained as solutions of the next system of differential one-forms:

$$\begin{cases} \mathrm{d}H + \sum_{i\geq 1} \omega_i^+ = 0, & \text{if } \theta \in [0,\pi], \\ r^+(\rho,0) = \rho, \end{cases} \qquad \begin{cases} \mathrm{d}H + \sum_{i\geq 1} \omega_i^- = 0, & \text{if } \theta \in [\pi,2\pi], \\ r^-(\rho,\pi) = \rho. \end{cases} \tag{2.6}$$

Let $r^+(\rho,\theta)$ (respectively, $r^-(\rho,\theta)$) be the solution of the initial value problems. Since $\mathrm{d}r/\mathrm{d}\theta$ is a quotient of the analytic functions $R^+, R^-, \Theta^+, \Theta^-$, then $r^\pm(\rho,\theta)$ admit power series expansions for $\rho > 0$ sufficiently small, given by

$$\begin{cases} r^+(\rho,\theta) = \rho + \sum_{i=2}^{\infty} p_i^+(\theta)\rho^i, & \text{if } \theta \in [0,\pi], \\ r^-(\rho,\theta) = \rho + \sum_{i=2}^{\infty} p_i^-(\theta)\rho^i, & \text{if } \theta \in [\pi,2\pi], \end{cases}$$

where $p_i^+(0) = 0$ and $p_i^-(\pi) = 0$, for all $i \geq 1$.

Definition 2.1. *(Gasull and Torregrosa [13], page 1756). The* ***positive half-return (or half-Poincaré) map*** *and the* ***negative half-return (or half-Poincaré) map*** *are functions such that*

$$\begin{cases} \Pi^+(\rho) = r^+(\rho,\pi) = \rho + \sum_{i\geq 2} p_i^+(\pi)\rho^i, \\ \Pi^-(\rho) = r^-(\rho,2\pi) = \rho + \sum_{i\geq 2} p_i^-(2\pi)\rho^i. \end{cases}$$

Definition 2.2. *(Gasull and Torregrosa [13], page 1756). The* ***complete return map or complete Poincaré map*** *associated with system* (2.1) *— or equivalently to system* (2.6) *— is given by the composition of these two maps:*

$$\Pi(\rho) = \Pi^-\left(\Pi^+(\rho)\right) := \rho + \sum_{i\geq 2} p_i(\theta)\rho^i, \tag{2.7}$$

as depicted in Figure 2.1.

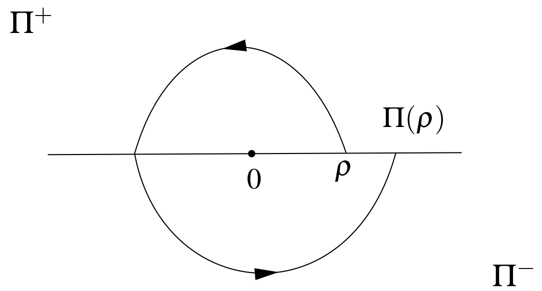


Figure 2.1: The return map of system (2.1)

Since Π^+, Π^- and Π depend on r^+ or r^-, then they are analytic maps. If a return of a half-return map $\Pi^\pm$ is associated with a vector field X, then we write Π_X or $\Pi_X^\pm$.

Definition 2.3. *(Gasull and Torregrosa [13], page 1756). The first nonzero $p_k(\theta)$ is called the* ***kth-Lyapunov constant*** *of system* (2.1)*, and is denoted by V_k.*

This definition implies that $V_1 = \cdots = V_{k-1} = 0$. Furthermore, in these cases, if there exists $k \geq 2$ such that $V_k \neq 0$, then the origin of system is a weak focus of order k. Otherwise, the origin is a center. In addition, the weak focus is an attractor, if $V_k < 0$, and it is a repeller, if $V_k > 0$.

According to Gasull and Torregrosa [13, page 1756], a main difference between the smooth case and the general system (2.1) is that while in the first case the first nonzero V_k occurs always for odd k, for the second case k can be any natural number greater than 1.

In view of this, we need a method to compute Π^+ and Π^-, and then compose them for to obtain these constants. The next Lemma shows a way to simplify both problems, but before we state and prove it, we need the following result.

Lemma 2.4. *(Coll et al. [6], page 1756). Let* $f(x) = x + \sum_{i\geq 2} f_i x^i$ *and* $g(x) = x + \sum_{i\geq 2} g_i x^i$ *be the power series expansion of* f *and* g*, respectively, defined on a real domain containing* $x = 0$*. Then, the following statements are equivalent*

(i) There exist $a \in \mathbb{R}$, $a \neq 0$ *and* $j \in \mathbb{N}$ *such that* $(g \circ f)(x) = x + ax^j + O(x^{j+1})$*;*

(ii) There exist $a \in \mathbb{R}$, $a \neq 0$ *and* $j \in \mathbb{N}$ *such that* $f(x) - g^{-1}(x) = ax^j + O(x^{j+1})$*.*

See Coll et al. [6, page 1756] for a proof of the Lemma 2.4.

Lemma 2.5. *(Gasull and Torregrosa [13], page 1757). The first nonzero term of the map* $\Pi_Z(\rho) - \rho$ *defined in* (2.7) *(see Figure 2.2) coincides with the first nonzero term of the map*

$$\Pi^+_{X^+}(\rho) - \left(\Pi^-_{X^-}\right)^{-1}(\rho) = \Pi^+_{X^+}(\rho) - \Pi^+_{-X^-(x,-y)}(\rho).$$

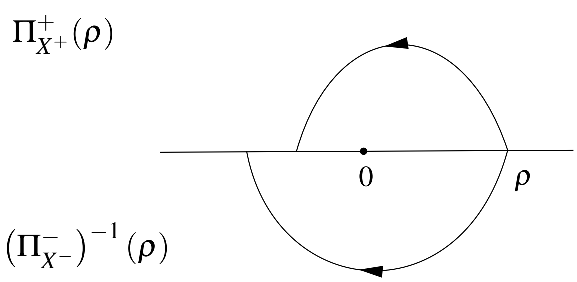


Figure 2.2: Half-return maps Π^+ and $(\Pi^-)^{-1}$ of system (2.1)

Proof. Consider the change variables $(\bar{x},\bar{y},\bar{t}) : (x,y,t) \mapsto (x,-y,-t)$ in the differential equation associated with X^-, then

$$\dot{\bar{x}} = \frac{\mathrm{d}}{\mathrm{d}t}x(-t) = -\dot{x}(-t),$$

$$\dot{\bar{y}} = \frac{\mathrm{d}}{\mathrm{d}t} - y(-t) = \dot{y}(-t).$$

Thus,

$$\begin{cases}\dot{x}(t)=-y(t)+P^-(x(t),y(t))\\ \dot{y}(t)=x(t)+Q^-(x(t),y(t))\end{cases} \xrightarrow{(\bar{x},\bar{y},\bar{t})} \begin{cases}\dot{\bar{x}}(\bar{t})=-[-\bar{y}(\bar{t})+P^-(\bar{x}(\bar{t}),\bar{y}(\bar{t}))]\\ \dot{\bar{y}}(\bar{t})=\bar{x}(\bar{t})+Q^-(\bar{x}(\bar{t}),\bar{y}(\bar{t}))\end{cases}$$
$$\Longrightarrow \begin{cases}\dot{x}(-t)=-y(-t)-P^-(x(-t),-y(-t))\\ \dot{y}(-t)=x(-t)+Q^-(x(-t),-y(-t))\end{cases}$$
$$\xrightarrow{-P^-=S^-} \begin{cases}\dot{x}(-t)=-y(-t)+S^-(x(-t),-y(-t)),\\ \dot{y}(-t)=x(-t)+Q^-(x(-t),-y(-t)).\end{cases}$$

Now, consider

$$x(-t)=r(-t)\cos\theta(-t) \quad \text{and} \quad y(-t)=r(-t)\sin\theta(-t).$$

Since $\theta(t)\in[\pi,2\pi]$ in X^-, it follows that $\sin\theta(t)<0$. Consequently, $-y(-t)=-\sin\theta(-t)>0$ and $\theta(-t)\in[0,\pi]$.

Furthermore,

$$\begin{cases}x(-t)=r(-t)\cos\theta(-t)\\ y(-t)=r(-t)\sin\theta(-t)\end{cases} \Longrightarrow \begin{cases}x^2(-t)+y^2(-t)=(r(-t))^2\\ \dfrac{y(-t)}{x(-t)}=\tan\theta(-t)\end{cases}$$
$$\Longrightarrow \begin{cases}\dot{r}(-t)=\dfrac{\dot{x}(-t)x(-t)+\dot{y}(-t)y(-t)}{r(-t)},\\ \dot{\theta}(-t)=\dfrac{\dot{y}(-t)x(-t)-y(-t)\dot{x}(-t)}{(r(-t))^2}.\end{cases}$$

Let $\widetilde{r}=r(-t)$ and $\widetilde{\theta}=\theta(-t)$. We have that

$$\begin{cases}\dot{\widetilde{r}}=\cos\widetilde{\theta}\,S^-(\widetilde{r}\cos\widetilde{\theta},\widetilde{r}\sin\widetilde{\theta})-\sin\widetilde{\theta}\,Q^-(\widetilde{r}\cos\widetilde{\theta},\widetilde{r}\sin\widetilde{\theta}))\\ \dot{\widetilde{\theta}}(-t)=1+\dfrac{1}{\widetilde{r}}\left(Q^-(\widetilde{r}\cos\widetilde{\theta},\widetilde{r}\sin\widetilde{\theta})+S^-(\widetilde{r}\cos\widetilde{\theta},\widetilde{r}\sin\widetilde{\theta})\right)\end{cases} \Longrightarrow \left(\Pi^-_{X^-(x,y)}\right)^{-1}=\Pi^+_{-X^-(x,-y)},$$

that is, the inverse of the map $\Pi^-_{X^-(x,y)}$ comes from the fact that r and θ are functions depending on $-t$.

Now, consider $f(\rho)=\Pi^+_{X^+}(\rho)$ and $g(\rho)=\Pi^-_{X^-}(\rho)$. Then, f and g are analytic functions vanishing at zero and such that $f'(0)=g'(0)=1$. Furthermore,

$$\Pi_X(\rho)-\rho=\Pi^-_{X^-}\left(\Pi^+_{X^+}(\rho)\right)-\rho=g(f(\rho))-\rho.$$

If there are no nonzero terms in $\Pi_X(\rho)-\rho$, then the origin is a singularity of center type. Otherwise, if there exists a nonzero term in the map $\Pi_X(\rho)-\rho$ defined in (2.7), then by Lemma 2.4 the first nonzero term of the map $\Pi_X(\rho)-\rho$ coincides with the first nonzero term of the map

$$f(\rho)-g^{-1}(\rho)=\Pi^+_{X^+}(\rho)-\left(\Pi^-_{X^-}\right)^{-1}(\rho)=\Pi^+_{X^+}(\rho)-\Pi^+_{-X^-(x,-y)}(\rho).$$

□

By this lemma, when $V_k \neq 0$, the weak focus is an attractor, if $V_k > 0$, and a repeller, if $V_k < 0$. Regarding the study of limit cycles in this context, we define the displacement function.

Definition 2.6. *The* ***displacement function*** *is the analytic function defined by*

$$\Delta(\rho) = \Pi^+_{X^+}(\rho) - \left(\Pi^-_{X^-}\right)^{-1}(\rho).$$

In particular, as noted by Gasull and Torregrosa [13, page 1757], this lemma reduces the computation of $\Pi^+_{X^+}$ (resp. $(\Pi^-_{X^-})^{-1}$) to the study of the half-positive return map of a smooth planar differential equation of the form:

$$\mathrm{d}H + \sum_{i\geq 1} \omega_i = 0,$$

associated with $X^+(x,y)$ (respectively, $-X^-(x,-y)$).

The following result reduces this problem to the study of a perturbation of the Hamiltonian system $\mathrm{d}H = 0$, where $H(r) = r^2/2$.

Lemma 2.7. *(Gasull and Torregrosa [13], page 1757). Let*

$$\Pi^+(\rho) = \rho + \sum_{i\geq 2} p_i^+(\pi)\rho^i$$

be the positive half-return map associated with the polar expression of a smooth system of type (2.1)*, written as*

$$\mathrm{d}H + \sum_{i\geq 1} \omega_i = 0. \tag{2.8}$$

Let $r(\theta,\varepsilon,\rho) = \sum_{i\geq 0} r_i(\theta,\rho)\varepsilon^i$ be the solution of the initial value problem

$$\begin{cases} \mathrm{d}H + \sum_{i\geq 1} \varepsilon^i \omega_i^+ = 0, \\ r(0,\varepsilon,\rho) = \rho. \end{cases} \tag{2.9}$$

Then, the half-return map is given by $\Pi^+(\varepsilon\rho) = \varepsilon r(\pi,\varepsilon,\rho)$. Consequently, for $i \geq 2$, the coefficients are determined by $p_i^+(\pi) = r_{i-1}(\pi,\rho)/\rho^i$.

Proof. For each fixed ε and ρ, let $r_{\mathrm{sol}}(\theta)$ be the trajectory such that $r_{\mathrm{sol}}(0) = \varepsilon\rho$ and $r_{\mathrm{sol}}(\pi) = \Pi^+(\varepsilon\rho)$. Let $\bar{r}(\theta) = r_{\mathrm{sol}}(\theta)/\varepsilon$. It follows that:

$$\bar{r}(0) = \frac{r_{\mathrm{sol}}(0)}{\varepsilon} = \frac{\varepsilon\rho}{\varepsilon} = \rho.$$

Next, we derive the ordinary differential equation satisfied by $\bar{r}(\theta)$. Substituting r_{sol} into (2.8) and noting that $H(r) = r^2/2$ implies $\mathrm{d}H = r\,\mathrm{d}r$, we have:

$$r \to r_{\mathrm{sol}} = \varepsilon\bar{r} \implies \mathrm{d}r \to \mathrm{d}(r_{\mathrm{sol}}) = \mathrm{d}(\varepsilon\bar{r}) = \varepsilon\ \mathrm{d}\bar{r}.$$

Consequently,

$$\mathrm{d}H(r_{\mathrm{sol}}) = r_{\mathrm{sol}}\ \mathrm{d}r_{\mathrm{sol}} = (\varepsilon\bar{r})(\varepsilon\ \mathrm{d}\bar{r}) = \varepsilon^2(\bar{r}\ \mathrm{d}\bar{r}) = \varepsilon^2\ \mathrm{d}H(\bar{r}).$$

Furthermore, the analytic one-forms ω_i are derived from the polynomial terms of system (2.1). Since $P^{\pm}(x,y)$ and $Q^{\pm}(x,y)$ start with terms of degree 2 or higher, each ω_i is a homogeneous polynomial in r of degree $i+1$. Thus, we obtain:

$$\omega_i(r_{\text{sol}},\theta) = \omega_i(\varepsilon\bar{r},\theta) = \varepsilon^{i+1}\omega_i(\bar{r},\theta).$$

Moreover, since $R^{\pm}$ and $\Theta^{\pm}$ start at least with terms of second and first order in r, respectively, the term $[r\cdot R^{\pm}(r,\theta)]/[1+\Theta^{\pm}(r,\theta)]$ in equation (2.5) starts with second order in r. Thus, the ω_i are polynomial functions in r of order $i+1$. Consequently, $\omega_i(\varepsilon\bar{r},\theta)$ are polynomial functions where ε has order $i+1$, which implies:

$$\omega_i(r_{\text{sol}},\theta) = \omega_i(\varepsilon\bar{r},\theta) = \varepsilon^{i+1}\omega_i(\bar{r},\theta).$$

Therefore, r_{sol} satisfies:

$$\begin{aligned} dH(r_{\text{sol}}) + \sum_{i\geq 1}\omega_i^+(r_{\text{sol}},\theta) = 0 &\implies \varepsilon^2 dH(\bar{r}) + \sum_{i\geq 1}\varepsilon^{i+2}\omega_i^+(\bar{r},\theta) = 0 \\ &\implies dH(\bar{r}) + \sum_{i\geq 1}\varepsilon^{i}\omega_i^+(\bar{r},\theta) = 0. \end{aligned}$$

Hence, $\bar{r} = r_{\text{sol}}/\varepsilon$ is the solution of (2.9). It follows that $\Pi^+(\varepsilon\rho) = r_{\text{sol}}(\pi) = \varepsilon\bar{r}(\pi)$. As a consequence, for $i\geq 2$ and setting $p_1^+(\pi) = 1$, we have:

$$\begin{aligned} \Pi^+(\varepsilon\rho) = \varepsilon\rho + \sum_{i\geq 2} p_i^+(\pi)\varepsilon^i\rho^i &\implies \varepsilon r(\pi,\varepsilon,\rho) = \varepsilon\left(\rho + \sum_{i\geq 2} p_i^+(\pi)\varepsilon^{i-1}\rho^i\right) \\ &\implies r(\pi,\varepsilon,\rho) = \rho + \sum_{i\geq 2} p_i^+(\pi)\varepsilon^{i-1}\rho^i \\ &\implies \sum_{i\geq 0} r_i(\pi,\rho)\varepsilon^i = \sum_{i\geq 1} p_i^+(\pi)\varepsilon^{i-1}\rho^i \\ &\implies \sum_{i\geq 1} r_{i-1}(\pi,\rho)\varepsilon^{i-1} = \sum_{i\geq 1} p_i^+(\pi)\varepsilon^{i-1}\rho^i \\ &\implies r_{i-1}(\pi,\rho) = p_i^+(\pi)\rho^i \\ &\implies p_i^+(\pi) = \frac{r_{i-1}(\pi,\rho)}{\rho^i}. \end{aligned}$$

Finally, we conclude that $\Pi^+(\varepsilon\rho) = \varepsilon r(\pi,\varepsilon,\rho)$ and, for $i\geq 2$, $p_i^+(\pi) = r_{i-1}(\pi,\rho)/\rho^i$. □

The proof of our main result also relies on the following technical lemma regarding the decomposition of arbitrary one-forms. Such a decomposition is reminiscent of the methods employed by Françoise [10, 11, 12].

Lemma 2.8. *Let $\Omega = \alpha(r,\theta)\,\mathrm{d}r + \beta(r,\theta)\,\mathrm{d}\theta$ be an arbitrary analytic one-form, 2π-periodic in θ, and $H(r) = r^2/2$. Then there exist functions $h(r,\theta)$, $S(r,\theta)$, and $F(r)$, also 2π-periodic in θ, defined by*

$$F(r) = \frac{1}{2\pi}\int_0^{2\pi}\beta(r,\psi)\,\mathrm{d}\psi, \quad S(r,\theta) = \int_0^{\theta}\beta(r,\psi)\,\mathrm{d}\psi - F(r)\theta, \quad h(r,\theta) = \frac{\alpha(r,\theta) - \partial S(r,\theta)/\partial r}{H'(r)},$$

such that

$$\Omega = \Omega^0 + \Omega^1,$$

where $\Omega^0 = h\,\mathrm{d}H + \mathrm{d}S$ *and* $\Omega^1 = F(r)\,\mathrm{d}\theta$*, satisfying*

$$\int_{H=\rho} \Omega^0 = 0, \quad \int_{H=\rho} \Omega^1 = \int_{H=\rho} \Omega.$$

Proof. In fact, we can write:

$$\begin{aligned}
\Omega =&\alpha(r,\theta)\ \mathrm{d}r + \beta(r,\theta)\ \mathrm{d}\theta \\
=&\alpha(r,\theta)\ \mathrm{d}r + \beta(r,\theta)\ \mathrm{d}\theta + \left[\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi - \frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right]\ \mathrm{d}\theta \\
=&\alpha(r,\theta)\ \mathrm{d}r + \left[\beta(r,\theta) - \frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right]\ \mathrm{d}\theta + \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\ \mathrm{d}\theta \\
=&\alpha(r,\theta)\ \mathrm{d}r + \left[\frac{\partial}{\partial\theta}\left(\int_0^{\theta} \beta(r,\psi)\ \mathrm{d}\psi - \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\theta\right)\right]\ \mathrm{d}\theta \\
&+ \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\ \mathrm{d}\theta \\
=&\alpha(r,\theta)\ \mathrm{d}r + \left[\frac{\partial}{\partial\theta}\left(\int_0^{\theta} \beta(r,\psi)\ \mathrm{d}\psi - \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\theta\right)\right]\ \mathrm{d}\theta \\
&+ \left\{\left[\frac{\partial}{\partial r}\left(\int_0^{\theta} \beta(r,\psi)\ \mathrm{d}\psi - \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\theta\right)\right]\ \mathrm{d}r\right. \\
&- \left.\left[\frac{\partial}{\partial r}\left(\int_0^{\theta} \beta(r,\psi)\ \mathrm{d}\psi - \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\theta\right)\right]\ \mathrm{d}r\right\} \\
&+ \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\ \mathrm{d}\theta \\
=&\alpha(r,\theta)\ \mathrm{d}r + \mathrm{d}\left[\int_0^{\theta} \beta(r,\psi)\ \mathrm{d}\psi - \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\theta\right] \\
&- \left[\frac{\partial}{\partial r}\left(\int_0^{\theta} \beta(r,\psi)\ \mathrm{d}\psi - \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\theta\right)\right]\ \mathrm{d}r \\
&+ \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\ \mathrm{d}\theta \\
=&\frac{1}{H'(r)}\left[\alpha(r,\theta) - \frac{\partial}{\partial r}\left(\int_0^{\theta} \beta(r,\psi)\ \mathrm{d}\psi - \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\theta\right)\right]\ r\,\mathrm{d}r \\
&+ \mathrm{d}\ \left[\int_0^{\theta} \beta(r,\psi)\ \mathrm{d}\psi - \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\theta\right] \\
&+ \left(\frac{1}{2\pi}\int_0^{2\pi} \beta(r,\psi)\ \mathrm{d}\psi\right)\ \mathrm{d}\theta.
\end{aligned}$$

By setting $F(r)$, $S(r,\theta)$, and $h(r,\theta)$ as in the statement, we obtain:

$$\Omega = h\ \mathrm{d}H + \mathrm{d}S + F(r)\ \mathrm{d}\theta = \Omega^0 + \Omega^1.$$

Furthermore, let $D = B((0,0),\rho)$. By Stokes's Theorem (see Theorem A.11), we have:

$$\begin{aligned}
\int_{H=\rho} \Omega^0 &= \int_D \mathrm{d}\Omega^0 \\
&= \int_D \mathrm{d}(h\mathrm{d}H) + \mathrm{d}\,(\mathrm{d}S) \\
&= \int_D [(\mathrm{d}h \wedge \mathrm{d}H) + (h\,\mathrm{d}\,(\mathrm{d}H))] + 0 \\
&= \int_D [(\mathrm{d}h \wedge \mathrm{d}H) + 0] \\
&= \int_D \left(\left(\frac{\partial}{\partial r} h(r,\theta)\,\mathrm{d}r + \frac{\partial}{\partial \theta} h(r,\theta)\,\mathrm{d}\theta \right) \wedge (r\,\mathrm{d}r) \right) \\
&= \int_D r \cdot \frac{\partial}{\partial \theta} h(r,\theta)\,\mathrm{d}\theta \wedge \mathrm{d}r \\
&= \int_0^{\sqrt{\rho}} \int_0^{2\pi} r \cdot \frac{\partial}{\partial \theta} h(r,\theta)\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_0^{\sqrt{\rho}} r \cdot [h(r,2\pi) - h(r,0)]\,\mathrm{d}r \\
&= \int_0^{\sqrt{\rho}} r \cdot 0\,\mathrm{d}r \\
&= 0.
\end{aligned}$$

Consequently,

$$\int_{H=\rho} \Omega = \int_{H=\rho} \Omega^1.$$

□

Theorem 2.9. *Let $r(\theta,\varepsilon,\rho)$ be the solution of the initial value problem*

$$\begin{cases} \mathrm{d}H + \sum_{i\geq 1} \varepsilon^i \omega_i = 0, \\ r(0,\varepsilon,\rho) = \rho, \end{cases} \tag{2.10}$$

where $H(r) = r^2/2$ and the one-forms $\omega_i = \omega_i(r,\theta)$ are 2π-periodic in θ. Then, for any $n \in \mathbb{N}$, the solution $r(\theta,\varepsilon,\rho)$ satisfies the implicit equation

$$\frac{r^2(\theta,\varepsilon,\rho) - \rho^2}{2} + O(\varepsilon^{n+1}) = \sum_{i=1}^{n} \varepsilon^i \left[\int_0^\theta F_i(r(\psi,\varepsilon,\rho))\mathrm{d}\psi + S_i(r(\psi,\varepsilon,\rho),\psi)\Big|_{\psi=0}^{\psi=\theta} \right],$$

where the one-forms Ω_i and the functions $F_i(r)$, $h_i(r,\theta)$, and $S_i(r,\theta)$ are defined inductively as follows: let $h_0 = 1$ and set

$$-\Omega_i := -\sum_{j=1}^{i} \omega_j h_{i-j} = h_i\,\mathrm{d}H + \mathrm{d}S_i + F_i\,\mathrm{d}\theta,$$

for $i = 1,2,\ldots,n$, where each $-\Omega_i$ is decomposed according to Lemma 2.8.

Proof. Denote by $\gamma_\varepsilon = \gamma_\varepsilon(\theta,\rho)$ the curve $\{r(\psi,\varepsilon,\rho), \psi \in [0,\theta]\}$, the solution of (2.10). Consider the one-forms $-\Omega_i$ for $i = 1,\ldots,n$ and their decompositions given in Lemma 2.8. Since $\mathrm{d}H + \sum_{i\geq 1} \varepsilon^i \omega_i =$

0, we have:

$$
\begin{aligned}
0 &= \int_{\gamma_\varepsilon} \left[\left(1+\sum_{i=1}^{n}\varepsilon^i h_i\right)\left(\mathrm{d}H+\sum_{i\geq 1}\varepsilon^i\omega_i\right)\right]\\
&= \int_{\gamma_\varepsilon}\left(\mathrm{d}H+\sum_{i\geq 1}\varepsilon^i\omega_i+\sum_{i=1}^{n}\varepsilon^i h_i\,\mathrm{d}H+\sum_{i=1}^{n}\varepsilon^i h_i\sum_{i\geq 1}\varepsilon^i\omega_i\right)\\
&= \int_{\gamma_\varepsilon}\left[\mathrm{d}H+\left(\sum_{i=1}^{n}\varepsilon^i\omega_i+O\left(\varepsilon^{n+1}\right)\right)+\sum_{i=1}^{n}\varepsilon^i h_i\,\mathrm{d}H+\left(\sum_{i=1}^{n}\sum_{j\geq 1}\varepsilon^{i+j}h_i\omega_j\right)\right]\\
&= \int_{\gamma_\varepsilon}\left\{\mathrm{d}H+\sum_{i=1}^{n}\varepsilon^i\left(\omega_i+h_i\,\mathrm{d}H\right)+\left[\left(\sum_{k=2}^{n}\varepsilon^k\sum_{j=1}^{k-1}h_{k-j}\omega_j\right)+O(\varepsilon^{n+1})\right]+O\left(\varepsilon^{n+1}\right)\right\}\\
&= \int_{\gamma_\varepsilon}\left\{\mathrm{d}H+\sum_{i=1}^{n}\varepsilon^i\left(\omega_i+h_i\,\mathrm{d}H\right)+\left[\left(\sum_{k=2}^{n}\varepsilon^k\left(\sum_{j=1}^{k}h_{k-j}\omega_j-\omega_k h_0\right)\right)+O(\varepsilon^{n+1})\right]+O\left(\varepsilon^{n+1}\right)\right\}\\
&= \int_{\gamma_\varepsilon}\left\{\mathrm{d}H+\sum_{i=1}^{n}\varepsilon^i\left(\omega_i+h_i\,\mathrm{d}H\right)+\left[\left(\sum_{k=2}^{n}\varepsilon^k\left(\Omega_k-\omega_k\cdot 1\right)\right)+O(\varepsilon^{n+1})\right]+O\left(\varepsilon^{n+1}\right)\right\}\\
&= \int_{\gamma_\varepsilon}\left[\mathrm{d}H+\sum_{i=1}^{n}\varepsilon^i\left(\omega_i+h_i\,\mathrm{d}H\right)+\sum_{k=2}^{n}\varepsilon^k\left(\Omega_k-\omega_k\right)+O\left(\varepsilon^{n+1}\right)\right]\\
&= \int_{\gamma_\varepsilon}\left[\mathrm{d}H+\varepsilon\left(\omega_1+h_1\,\mathrm{d}H\right)+\sum_{l=2}^{n}\varepsilon^l\left(\omega_l-\omega_l+\Omega_l+h_l\,\mathrm{d}H\right)+O\left(\varepsilon^{n+1}\right)\right]\\
&= \int_{\gamma_\varepsilon}\left[\mathrm{d}H+\varepsilon\left(\Omega_1+h_1\,\mathrm{d}H\right)+\sum_{l=2}^{n}\varepsilon^l\left(\Omega_l+h_l\,\mathrm{d}H\right)+O\left(\varepsilon^{n+1}\right)\right]\\
&= \int_{\gamma_\varepsilon}\left(\mathrm{d}H+\sum_{l=1}^{n}\varepsilon^l\left(\Omega_l+h_l\,\mathrm{d}H\right)+O\left(\varepsilon^{n+1}\right)\right)\\
&= \int_{\gamma_\varepsilon}\left(\mathrm{d}H-\sum_{l=1}^{n}\varepsilon^l\left(F_l(r)\,\mathrm{d}\theta+\mathrm{d}S_l(r,\theta)\right)+O\left(\varepsilon^{n+1}\right)\right)\\
&= \int_{\gamma_\varepsilon}\mathrm{d}H-\left[\sum_{l=1}^{n}\varepsilon^l\left(\int_{\gamma_\varepsilon}F_l(r)\,\mathrm{d}\theta+\int_{\gamma_\varepsilon}\mathrm{d}S_l(r,\theta)\right)\right]+O\left(\varepsilon^{n+1}\right)\\
&= \left[H(r(\theta,\varepsilon,\rho))-H(\rho)\right]-\left[\sum_{l=1}^{n}\varepsilon^l\left(\int_0^\theta F_l(r(\psi,\varepsilon,\rho))\,\mathrm{d}\psi+\left(S_l(r(\theta,\varepsilon,\rho),\theta)-S_l(\rho,0)\right)\right)\right]+O\left(\varepsilon^{n+1}\right)\\
&= \left[\frac{r^2(\theta,\varepsilon,\rho)-\rho^2}{2}\right]-\left[\sum_{l=1}^{n}\varepsilon^l\left(\int_0^\theta F_l(r(\psi,\varepsilon,\rho))\mathrm{d}\psi+S_l(r(\psi,\varepsilon,\rho),\psi)\Big|_{\psi=0}^{\psi=\theta}\right)\right]+O\left(\varepsilon^{n+1}\right),
\end{aligned}
$$

which implies that

$$
\frac{r^2(\theta,\varepsilon,\rho)-\rho^2}{2}+O(\varepsilon^{n+1})=\sum_{i=1}^{n}\varepsilon^i\left[\int_0^\theta F_i(r(\psi,\varepsilon,\rho))\mathrm{d}\psi+S_i(r(\psi,\varepsilon,\rho),\psi)\Big|_{\psi=0}^{\psi=\theta}\right].
$$

□

Corollary 2.10. *Let $r(\theta,\varepsilon,\rho)=\sum_{i\geq 0}r_i(\theta,\rho)\varepsilon^i$ be the solution of the initial value problem* (2.10)*. Assume that the functions $r_0(\theta,\rho)=\rho, r_1(\theta,\rho),\ldots,r_{n-1}(\theta,\rho)$ are known. Then, $r_n(\theta,\rho)$ can be obtained by equating the ε^n-terms in the implicit expression of $r(\theta,\varepsilon,\rho)$ given in Theorem 2.9. Specifi-*

cally, the equation takes the form

$$\rho r_n(\theta,\rho) = \mathscr{F}_n(\theta,\rho,r_1,\dots,r_{n-1}),$$

where $\mathscr{F}_n$ *depends on the one-forms* $\omega_1,\omega_2,\dots,\omega_n$ *through the corresponding functions* F_i, S_i*, and* $r_i = r_i(\theta,\rho)$ *for* $i = 1,2,\dots,n$*. In particular, we have* $\mathscr{F}_1 = F_1(\rho)\theta + S_1(\rho,\theta) - S_1(\rho,0)$ *and*

$$\mathscr{F}_2 = F_2(\rho)\theta + \left[S_2(\rho,\psi) + \frac{\partial S_1}{\partial r}(\rho,\psi) r_1(\psi,\rho)\right]\Big|_{\psi=0}^{\psi=\theta} - \frac{1}{2} r_1(\theta,\rho)^2 + F_1'(\rho)\int_0^\theta r_1(\psi,\rho)\mathrm{d}\psi.$$

Proof. Let $r(\theta,\varepsilon,\rho) = \sum_{i\geq 0} r_i(\theta,\rho)\varepsilon^i$. Then, we have:

$$\begin{aligned} r^2(\theta,\varepsilon,\rho) &= \left(\sum_{i\geq 0} r_i(\theta,\rho)\varepsilon^i\right)^2 = \left(\rho + \sum_{i\geq 1} r_i(\theta,\rho)\varepsilon^i\right)^2 \\ &= \rho^2 + 2\rho\sum_{i\geq 1} r_i(\theta,\rho)\varepsilon^i + \sum_{k\geq 2}\left(\sum_{i+j\geq k} r_i r_j(\theta,\rho)\right)\varepsilon^k, \end{aligned}$$

which implies

$$\begin{aligned} \frac{r^2(\theta,\varepsilon,\rho)-\rho^2}{2} &= \rho\sum_{i\geq 1} r_i(\theta,\rho)\varepsilon^i + \frac{1}{2}\sum_{k\geq 2}\left(\sum_{i+j\geq k} r_i r_j(\theta,\rho)\right)\varepsilon^k \\ &= \rho r_1(\theta,\rho)\varepsilon + \sum_{k\geq 2}\left(\rho r_k + \frac{1}{2}\sum_{i+j\geq k} r_i r_j(\theta,\rho)\right)\varepsilon^k. \end{aligned}$$

Since F_i and S_i are analytic, they admit power series representations. Using $r(\theta,\varepsilon,\rho) - \rho = \sum_{k\geq 1} r_k(\theta,\rho)\varepsilon^k$ and fixing ψ, we expand F_i around ρ:

$$\begin{aligned} F_i(r(\psi,\varepsilon,\rho)) &= F_i(\rho) + F_i'(\rho)(r-\rho) + \frac{1}{2}F_i''(\rho)(r-\rho)^2 + \dots \\ &= F_i(\rho) + F_i'(\rho)(r_1\varepsilon + r_2\varepsilon^2 + \dots) + \frac{1}{2}F_i''(\rho)(r_1\varepsilon + \dots)^2 + \dots \\ &= F_i(\rho) + [F_i'(\rho)r_1]\varepsilon + [F_i'(\rho)r_2 + \frac{1}{2}F_i''(\rho)r_1^2]\varepsilon^2 + O(\varepsilon^3). \end{aligned}$$

Similarly, fixing ψ, the Taylor expansion for S_i is:

$$\begin{aligned} S_i(r(\psi,\varepsilon,\rho),\psi) &= S_i(\rho,\psi) + \frac{\partial S_i}{\partial r}(\rho,\psi)(r-\rho) + \frac{1}{2}\frac{\partial^2 S_i}{\partial r^2}(\rho,\psi)(r-\rho)^2 + \dots \\ &= S_i(\rho,\psi) + \left[\frac{\partial S_i}{\partial r}(\rho,\psi)r_1(\psi,\rho)\right]\varepsilon \\ &\quad + \left[\frac{\partial S_i}{\partial r}(\rho,\psi)r_2(\psi,\rho) + \frac{1}{2}\frac{\partial^2 S_i}{\partial r^2}(\rho,\psi)r_1^2(\psi,\rho)\right]\varepsilon^2 + O(\varepsilon^3). \end{aligned}$$

By Theorem 2.9, substituting these expansions into the implicit equation:

$$\rho r_1\varepsilon + \sum_{k=2}^{n}\left(\rho r_k + \frac{1}{2}\sum_{i+j=k} r_i r_j\right)\varepsilon^k = \sum_{i=1}^{n}\varepsilon^i\left[\int_0^\theta F_i(r)\,\mathrm{d}\psi + S_i(r,\psi)\Big|_{\psi=0}^{\psi=\theta}\right].$$

Equating the coefficients of ε^n, we obtain the recursive relation:

$$\rho r_n(\theta,\rho) = \mathscr{F}_n(\theta,\rho,r_1,\ldots,r_{n-1}),$$

where $\mathscr{F}_n$ depends on the one-forms $\omega_1,\omega_2,\ldots,\omega_n$, by the corresponding F_i, S_i, and $r_i = r_i(\theta,\rho)$, for $i = 1,2,\ldots,n$.

Specifically, for ε^1:

$$\mathscr{F}_1 = \rho r_1(\theta,\rho) = \int_0^\theta F_1(\rho)\,\mathrm{d}\psi + S_1(\rho,\psi)\Big|_{\psi=0}^{\psi=\theta} = F_1(\rho)\theta + S_1(\rho,\theta) - S_1(\rho,0).$$

For ε^2, collecting terms from the expansion of F_1, S_1 and F_2, S_2:

$$\mathscr{F}_2 = F_2(\rho)\theta + \left[S_2(\rho,\psi) + \frac{\partial S_1}{\partial r}(\rho,\psi) r_1\right]\Bigg|_{\psi=0}^{\psi=\theta} - \frac{1}{2}r_1^2 + F_1'(\rho)\int_0^\theta r_1(\psi,\rho)\,\mathrm{d}\psi.$$

□

2.1.1 Method for computation Lyapunov constants

Based on Andronov et al. [1, page 250] and on Gasull and Torregrosa [13, page 1758], for the computation of the Lyapunov constants, we proceed as follows:

1. Write system (2.1) in the polar form (2.3), which can be expressed as:

$$\begin{cases} \mathrm{d}H + \sum_{i\geq 1}\omega_i^+ = 0, & \text{if } \theta\in[0,\pi], \\ \mathrm{d}H + \sum_{i\geq 1}\omega_i^- = 0, & \text{if } \theta\in[\pi,2\pi]. \end{cases} \tag{2.11}$$

2. Consider the expression of (2.11) in the upper half-plane and denote by X^+ its associated vector field.

3. Associate with this vector field the initial value problem (2.10) and calculate $r(\theta,\varepsilon,\rho)$ by using Corollary 2.10.

4. Use Lemma 2.8 to obtain $\Pi_X^+(\rho)$ from $r(\pi,\varepsilon,\rho)$.

5. Consider the expression (2.11) in the lower half-plane and denote by X^+ its associated vector field. Take the transformation $-X^-(x,-y)$ and repeat steps 3 and 4 to obtain $\Pi^+_{-X^-(x,-y)}(\rho)$.

6. Compute the difference $\Pi^+_{X^+}(\rho) - \Pi^+_{-X^-(x,-y)}(\rho)$. According to Lemma 2.5, its series expansion yields the Lyapunov constants for system (2.1).

In practice, let $r^+(\rho,\theta)$ and $r^-(\rho,\theta)$ be the solutions of the following initial value problems, respectively:

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}\theta} = R^+(\rho,\theta), \\ r^+(\rho,0) = \rho, \end{cases} \quad \text{if } \theta\in[0,\pi], \qquad \begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}\theta} = R^-(\rho,\theta), \\ r^-(\rho,\pi) = \rho, \end{cases} \quad \text{if } \theta\in[\pi,2\pi], \tag{2.12}$$

where

$$\begin{aligned} R^+(\rho,\theta) &= R_1^+(\theta)\rho + R_2^+(\theta)\rho^2 + \cdots, \quad \text{if } \theta \in [0,\pi], \\ R^-(\rho,\theta) &= R_1^-(\theta)\rho + R_2^-(\theta)\rho^2 + \cdots, \quad \text{if } \theta \in [\pi,2\pi], \end{aligned} \tag{2.13}$$

Recall that the solutions can be expressed as power series in ρ:

$$\begin{aligned} r^+(\theta,\rho) &= p_1^+(\theta)\rho + p_2^+(\theta)\rho^2 + \cdots, \quad \text{if } \theta \in [0,\pi], \\ r^-(\theta,\rho) &= p_1^-(\theta)\rho + p_2^-(\theta)\rho^2 + \cdots, \quad \text{if } \theta \in [\pi,2\pi], \end{aligned} \tag{2.14}$$

where, according to the initial conditions in (2.12), the functions $p_i^{\pm}(\theta)$ must satisfy:

$$\begin{aligned} p_1^+(0) = 1, &\quad p_2^+(0) = p_3^+(0) = \cdots = 0, \\ p_1^-(\pi) = 1, &\quad p_2^-(\pi) = p_3^-(\pi) = \cdots = 0. \end{aligned} \tag{2.15}$$

Substituting the expressions (2.13) and (2.14) into (2.12), and equating the coefficients of corresponding powers of ρ, we obtain the following recursive differential equations for the functions $p_i^{\pm}(\theta)$, $i = 1,2,3,\ldots$:

$$\begin{aligned} &\begin{cases} (p_1^+)'(\theta) = R_1^+(\theta)p_1^+(\theta), \\ (p_2^+)'(\theta) = R_1^+(\theta)p_2^+(\theta) + R_2^+(\theta)(p_1^+(\theta))^2, \\ (p_3^+)'(\theta) = R_1^+(\theta)p_3^+(\theta) + 2R_2^+(\theta)p_1^+(\theta)p_2^+(\theta) + R_3(\theta)(p_1^+(\theta))^2, \\ \cdots \end{cases} \quad \text{if } \theta \in [0,\pi], \\ &\begin{cases} (p_1^-)'(\theta) = R_1^-(\theta)p_1^-(\theta), \\ (p_2^-)'(\theta) = R_1^-(\theta)p_2^-(\theta) + R_2^-(\theta)(p_1^-(\theta))^2, \\ (p_3^-)'(\theta) = R_1^-(\theta)p_3^-(\theta) + 2R_2^-(\theta)p_1^-(\theta)p_2^-(\theta) + R_3(\theta)(p_1^-(\theta))^2, \\ \cdots \end{cases} \quad \text{if } \theta \in [\pi,2\pi]. \end{aligned} \tag{2.16}$$

The condition (2.15) may be considered as the initial conditions for the functions $p_i^{\pm}(\theta)$ satisfying differential equations (2.16). Using these initial conditions and successively integrating equations (2.15) as linear differential equations for the corresponding functions, we obtain

$$\begin{aligned} &\begin{cases} p_1^+(\theta) = \exp\left[\int_0^\theta R_1^+(s)\,\mathrm{d}s\right], \\ p_2^+(\theta) = p_1^+(\theta)\int_0^\theta R_2^+(s)p_1^+(s)\,\mathrm{d}s, \\ p_3^+(\theta) = p_1^+(\theta)\int_0^\theta \left[2R_2(s)p_2^+(s) + R_3(s)(p_1^+(s))^2\right]\,\mathrm{d}s, \\ \cdots \end{cases} \quad \text{if } \theta \in [0,\pi], \\ &\begin{cases} p_1^-(\theta) = \exp\left[\int_\pi^\theta R_1^-(s)\,\mathrm{d}s\right], \\ p_2^-(\theta) = p_1^-(\theta)\int_\pi^\theta R_2^-(s)p_1^-(s)\,\mathrm{d}s, \\ p_3^-(\theta) = p_1^-(\theta)\int_\pi^\theta \left[2R_2(s)p_2^-(s) + R_3(s)(p_1^-(s))^2\right]\,\mathrm{d}s, \\ \cdots \end{cases} \quad \text{if } \theta \in [\pi,2\pi]. \end{aligned} \tag{2.17}$$

Hence, the half-return maps $\Pi_{X^+}^+(\rho)$ and $\left(\Pi_{X^-}^-\right)^{-1}(\rho)$ are given by:

$$\begin{aligned} \Pi_{X^+}^+(\rho) &= r^+(\pi,\rho) = p_1^+(\pi)\rho + p_2^+(\pi)\rho^2 + \cdots, \\ \left(\Pi_{X^-}^-(\rho)\right)^{-1} &= \Pi^+_{-X^-(x,-y)} = \tilde{p}_1^-(\pi)\rho + \tilde{p}_2^-(\pi)\rho^2 + \cdots, \end{aligned} \tag{2.18}$$

where the coefficients $\tilde{p}_i^-(\pi)$ are determined by the system:

$$\begin{cases} \tilde{p}_1^-(\pi) = \exp\left[\int_0^\pi R_1^-(s)\,\mathrm{d}s\right], \\ \tilde{p}_2^-(\pi) = \tilde{p}_1^-(\pi)\int_0^\pi R_2^-(s)p_1^-(s)\,\mathrm{d}s, \\ \tilde{p}_3^-(\pi) = \tilde{p}_1^-(\pi)\int_0^\pi \left[2R_2(s)p_2^-(s) + R_3(s)(p_1^-(s))^2\right]\,\mathrm{d}s, \\ \cdots \end{cases}$$

Finally, the k-th Lyapunov constant V_k is derived from the expansion of the full displacement map as:

$$V_k = p_k^+(\pi) - \tilde{p}_k^-(\pi).$$

For a detailed implementation of this algorithm in Maple, see Appendix B.

2.2 Applications

In this section, we apply the method described above to solve the center-focus problem and to obtain small-amplitude limit cycles for two cases of discontinuous differential systems: quadratic systems and Liénard equations.

2.2.1 Quadratic systems

In this example, we classify the centers of a family of discontinuous quadratic system. For this family, we also find an example with five limit cycles.

Proposition 2.11. *Consider the system*

$$Z = \begin{cases} X^+ = \begin{pmatrix} -y + p_{20}x^2 + p_{11}xy + p_{02}y^2 \\ x + q_{20}x^2 + q_{11}xy + q_{02}y^2 \end{pmatrix}, & y \geq 0, \\ X^- = \begin{pmatrix} -y \\ x \end{pmatrix}, & y \leq 0. \end{cases}$$

Then, the origin is a center for this system if one of the following conditions holds:

(i) $p_{11} = q_{20} = q_{02} = 0$;

(ii) $p_{20} = p_{11} + q_{20} = p_{02} + q_{11} = q_{02} = 0$;

(iii) $2p_{20} + q_{11} = p_{11} + 2q_{02} = q_{20} = 0$;

(iv) $p_{20} = -p_{11} + q_{20} = q_{02} + q_{20} = p_{02} = 0$;

(v) $2p_{11}q_{20} + 3p_{20}^2 - 2q_{20}^2 = 2q_{11} + 5p_{20} = 8p_{02}q_{20}^2 - 3p_{20}^3 + 8p_{20}q_{20}^2 = 4q_{02}q_{20} - 3p_{20}^2 + 4q_{20}^2 = 0.$

Proof. To show that there are no additional centers within this family, we compute several Lyapunov constants by using the Appendix B. Since the system has six parameters, Maple may not simplify the expressions for the Lyapunov constants. If this happen, the reader should employ to use the Gröbner basis techniques to manually simplify the constants. To perform this simplification, replace the sections "Calculation of simplified Lp[j] and Lm[j] coefficients" and "Calculation of Lyapunov coefficients", in the code with the following:

```
        #Calculation of simplified Lp[j] and Lm[j] coefficients

        vars:=[p11, q11, p02, q02, p20, q20]:
        ord := tdeg(op(vars)): #we are defining an order for the Groebner basis

        #Positive case
        V1p := 0:
        V2p := factor(NormalForm(Lp[2], Basis([V1p], ord), ord)):
        V3p := factor(NormalForm(Lp[3], Basis([V1p, V2p], ord), ord)):
        V4p := factor(NormalForm(Lp[4], Basis([V1p, V2p, V3p], ord), ord)):
        V5p := factor(NormalForm(Lp[5], Basis([V1p, V2p, V3p, V4p], ord), ord)):
        V6p := factor(NormalForm(Lp[6], Basis([V1p, V2p, V3p, V4p, V5p], ord), ord)):
        V7p := factor(NormalForm(Lp[7], Basis([V1p, V2p, V3p, V4p, V5p, V6p], ord), ord)):

        #Negative case
        V1m := 0:
        V2m := factor(NormalForm(Lm[2], Basis([V1m], ord), ord)):
        V3m := factor(NormalForm(Lm[3], Basis([V1m, V2m], ord), ord)):
        V4m := factor(NormalForm(Lm[4], Basis([V1m, V2m, V3m], ord), ord)):
        V5m := factor(NormalForm(Lm[5], Basis([V1m, V2m, V3m, V4m], ord), ord)):
        V6m := factor(NormalForm(Lm[6], Basis([V1m, V2m, V3m, V4m, V5m], ord), ord)):
        V7m := factor(NormalForm(Lm[7], Basis([V1m, V2m, V3m, V4m, V5m, V6m], ord), ord)):

        #Calculation of Lyapunov coefficients
        #If N>7, then add V8:=V8p+V8m, and so on...
        V1:=V1p+V1m;
        V2:=V2p+V2m;
        V3:=V3p+V3m;
        V4:=V4p+V4m;
        V5:=V5p+V5m;
        V6:=V6p+V6m;
        V7:=V7p+V7m;
```

Using this procedure, we obtain the following Lyapunov constants:

$$\begin{aligned}
V_2 =& \frac{2}{3}\left(p_{11}+q_{20}+2q_{02}\right),\\
V_3 =& -\frac{\pi}{8}\left(2p_{20}q_{02}+q_{02}q_{11}+3p_{20}q_{20}+q_{11}q_{20}+p_{02}q_{20}\right),\\
V_4 =& \frac{1}{15}\left(2q_{20}\left(6p_{20}p_{02}+3p_{20}^2-4q_{02}^2-4q_{02}q_{20}\right)\right),\\
V_5 =& \frac{1}{64}q_{20}p_{20}\pi\left(p_{20}^2-2p_{20}q_{11}-8q_{02}q_{20}-8q_{20}^2\right),\\
V_6 =& \frac{1}{175}16p_{20}q_{20}^2\left(2p_{02}q_{20}-p_{20}q_{02}+q_{20}p_{20}\right),\\
V_7 =& 0.
\end{aligned}$$

Solving the nonlinear system $\{V_2 = V_3 = V_4 = V_5 = V_6 = V_7 = 0\}$:

```
            solve({V2 = 0, V3 = 0, V4 = 0, V5 = 0, V6 = 0, V7 = 0});

```

yields exactly the families stated in the theorem, which completes the proof. □

During the development of this research, item (v) of Proposition 2.11 was corrected. However, in order to prove the converse of the result, it would be necessary to find a first integral associated with system under the conditions given in item (v). Since we are still working on obtaining this first integral, we decided to present only the direct implication of the proposition.

2.2.2 Liénard equations

In this section, we study the center-focus problem for Liénard discontinuous systems of the form

$$Z = \begin{cases} X^+ = \begin{pmatrix} -y+\sum_{i=2}^n a_i x^i \\ x \end{pmatrix}, & \text{if } y \geq 0, \\ X^- = \begin{pmatrix} -y+\sum_{i=2}^n b_i x^i \\ x \end{pmatrix}, & \text{if } y \leq 0. \end{cases} \tag{2.19}$$

According to Coll et al. [6, page 1752], the following families of systems of this type have a center at the origin for

(i) $a_{2k+1} = b_{2k+1} = 0$; or

(ii) $a_{2k} + b_{2k} = 0$;

for all $k \in \mathbb{N}$. Also, the authors try to prove that the above families are the only centers inside (2.19). In particular, they show that for the following particular systems

$$Z = \begin{cases} X^+ = \begin{pmatrix} -y+x^{2j+1}+x^{2(k-j)} \\ x \end{pmatrix}, & \text{if } y \geq 0, \\ X^- = \begin{pmatrix} -y-x^{2j+1} \\ x \end{pmatrix}, & \text{if } y \leq 0, \end{cases}$$

for $1 \leq j < k$, the Lyapunov constant $V_{2k} = C_{k,j}$ is not zero, and then indeed the above families are the only centers for system (2.19). With this aim they compute some $C_{k,j}$ for $k \leq 6$, obtaining in all cases a number different from zero. The methods of Gasull and Torregrosa [13] allow to compute these values for larger values of k. In particular in Table 2.1, the values of $C_{k,j}$ for $1 \leq j < k \leq 10$ are given. To calculate these constants, in the section "Non-linear part of system" and "Variables" of Appendix B, replace them with:

```

        #Non-linear part of system
        k := ; #Enter a number greater than or equal to 2 here.
        j := ; #Enter a number here that is greater than or equal to 1 and
    less than k.
        N := 2*k;

        Ap := x^(2*j + 1) + x^(2*(k - j));
        Bp := 0;
        Am := -x^(2*j + 1);
        Bm := 0;

        #Variables
        xp := -y + Ap;
        yp := x + Bp;
        xm := -y + Am;
        ym := x + Bm;

```

Table 2.1: The values of $C_{k,j}$ for $1 \leq j < k \leq 10$.

$k\backslash j$	1	2	3	4	5	6	7	8	9
2	$\frac{14}{15}$								
3	$\frac{58}{525}$	$\frac{26}{21}$							
4	$-\frac{702}{1225}$	$\frac{578}{945}$	$\frac{38}{27}$						
5	$-\frac{84806}{72765}$	$\frac{1774}{24255}$	$\frac{446}{495}$	$\frac{50}{33}$					
6	$-\frac{2516806}{1486485}$	$-\frac{381454}{945945}$	$\frac{20506}{45045}$	$\frac{2338}{2145}$	$\frac{62}{39}$				
7	$-\frac{10882038}{5010005}$	$-\frac{5201926}{6243237}$	$\frac{22138}{405405}$	$\frac{17746}{25025}$	$\frac{3578}{2925}$	$\frac{74}{45}$			
8	$-\frac{390159442}{149324175}$	$-\frac{80616454}{65702637}$	$-\frac{1571630}{5054049}$	$\frac{1388402}{3828825}$	$\frac{20666}{23205}$	$\frac{562}{425}$	$\frac{86}{51}$		
9	$-\frac{123363871018}{40811445675}$	$-\frac{3311635214}{2080583505}$	$-\frac{809489458}{1248350103}$	$\frac{6970394}{160044885}$	$\frac{3868322}{6613425}$	$\frac{174142}{169575}$	$\frac{6778}{4845}$	$\frac{98}{57}$	
10	$-\frac{36344996758}{10667118605}$	$-\frac{788914022}{4134146445}$	$-\frac{2575685746}{2675035935}$	$\frac{58491854}{231175945}$	$\frac{13153354}{43648605}$	$\frac{1609546}{2136645}$	$\frac{89678}{79135}$	$\frac{8738}{5985}$	$\frac{110}{63}$

Proposition 2.12. *For $n = 7$ the only centers of system* (2.19) *are the ones satisfying*

(i) either $a_3 = b_3 = a_5 = b_5 = a_7 = b_7 = 0$*; or*

(ii) $a_i + b_i = 0$, $i = 2, \dots, 7$.

Proof. By the results of Coll et al. [6, page 1752], we know that both families have a center at the origin. We have to prove that there are no additional centers. So, we need to calculate the Lyapunov constants. Thus, in the section "Non-linear part of system" and "Variables" of Appendix B, replace them with:

```
#Non-linear part of system
  M := 7;
  N := 12;
  Ap := sum(a[i]*x^i, i = 2 .. M);
  Bp := 0;
  Am := sum(b[i]*x^i, i = 2 .. M);
  Bm := 0;

#Variables
  xp := -y + Ap;
  yp := x;
  xm := -y + Am;
  ym := x;

```

Thus, we obtain

$$\begin{aligned}
V_2 &= 0,\\
V_3 &= \frac{3}{8}\pi\left(a_3 + b_3\right),\\
V_4 &= \frac{14}{15}\left(a_2 a_3 + b_2 b_3\right),\\
V_5 &= \frac{5}{16}\pi\left(a_5 + b_5\right),\\
V_6 &= \frac{58}{525}\left(a_3 a_4 + b_3 b_4\right) + \frac{26}{21}\left(a_2 a_5 + b_2 b_5\right),\\
V_7 &= \frac{35}{128}\pi(a_7 + b_7),\\
V_8 &= \frac{38}{27}\left(a_2 a_7 + b_2 b_7\right) - \frac{702}{1225}\left(a_3 a_6 + b_3 b_6\right) + \frac{578}{945}\left(a_4 a_5 + b_4 b_5\right),\\
V_9 &= 0,\\
V_{10} &= \frac{1774}{24255}\left(a_5 a_6 + b_5 b_6\right) + \frac{446}{495}\left(a_4 a_7 + b_4 b_7\right),\\
V_{11} &= 0,\\
V_{12} &= \frac{20506}{45045}\left(a_6 a_7 + b_6 b_7\right).
\end{aligned}$$

Now, we need to solve the system $\{V_2 = V_3 = \cdots = V_{11} = V_{12} = 0\}$. For that, we will use the code:

```
solve({V[3] = 0, V[4] = 0, V[5] = 0, V[6] = 0, V[7] = 0, V[8] = 0, V[9] = 0, V[10] = 0, V[11] = 0, V[12] = 0});

```

whose solution is

$\{a_2 = a_2, a_3 = 0, a_4 = a_4, a_5 = 0, a_6 = a_6, a_7 = 0, b_2 = b_2, b_3 = 0, b_4 = b_4, b_5 = 0, b_6 = b_6, b_7 = 0\}$,
$\{a2 = b_2, a_3 = -b_3, a_4 = b_4, a_5 = -b_5, a_6 = b_6, a_7 = -b_7, b_2 = b_2, b_3 = b_3, b_4 = b_4, b_5 = b_5,$
$b_6 = b_6, b_7 = b_7\}.$

This concludes the proof of the proposition. Note that the rational numbers involved in the even Lyapunov constants are the ones given in Table 2.1. □

CHAPTER 3

Melnikov Function for piecewise systems

In this chapter, we study a perturbation of a periodic annulus. In such systems, limit cycles can bifurcate either from the boundary or from the interior of the annulus. According to Han and Yu [15, page 261], in both cases, the limit cycles under consideration correspond to fixed points of the Poincaré return map, which is an analytic function near any periodic orbit in the period annulus.

Thus, the perturbation problem can be reduced to the analysis of an analytic near-Hamiltonian system H. For this class of systems, the first order Melnikov function can be employed to determine the number of bifurcated limit cycles. Consider the general form of the near-Hamiltonian system on the plane

$$\begin{cases} \dot{x} = H_y(x,y) + \varepsilon f(x,y), \\ \dot{y} = -H_x(x,y) + \varepsilon g(x,y), \end{cases} \tag{3.1}$$

where H, f, and g are C^∞ functions, and $\varepsilon \geq 0$ is a small real parameter. Suppose that for $\varepsilon = 0$, there exists a family of periodic orbits L_h surrounding the origin, defined by $H(x,y) = h$, for $h > 0$. As is well known, the function

$$\widetilde{M}(h) = \oint\limits_{L_h} g \, \mathrm{d}x - f \, \mathrm{d}y$$

is called *the first order Melnikov function* of system (3.1). This function plays an important role in the study of the bifurcations of limit cycles. For instance, if $M(h)$ has an isolated zero h_0 with odd multiplicity, then system (3.1) has a limit cycle near L_{h_0}.

Therefore, in this chapter, we seek to answer the following questions:

($\mathbf{Q_1}$) Given a piecewise perturbed Hamiltonian system, how many limit cycles can we obtain from it?

($\mathbf{Q_2}$) Can we determine the exact, maximum, or minimum number of limit cycles?

3.1 Definitions and main results

In this section, we derive the general expression of a Melnikov function $M(h)$ applied to perturbed piecewise Hamiltonian systems and show how it can be applied to compute limit cycles.

Consider the piecewise Hamiltonian system:

$$Z = \begin{cases} Y^+ = \begin{pmatrix} H_y^+(x,y) \\ -H_x^+(x,y) \end{pmatrix}, & \text{if } x > 0, \\ Y^- = \begin{pmatrix} H_y^-(x,y) \\ -H_x^-(x,y) \end{pmatrix}, & \text{if } x \leq 0, \end{cases} \tag{3.2}$$

where $H^\pm \in C^\infty$ and with the piecewise Hamiltonian function:

$$H(x,y) = \begin{cases} H^+(x,y), & \text{if } x > 0, \\ H^-(x,y), & \text{if } x \leq 0, \end{cases}$$

with $H^\pm(0,0) = 0$.

Now, consider the perturbed piecewise Hamiltonian system:

$$Z_\varepsilon = \begin{cases} Y_\varepsilon^+ = \begin{pmatrix} H_y^+(x,y) + \varepsilon f^+(x,y) \\ -H_x^+(x,y) + \varepsilon g^+(x,y) \end{pmatrix}, & \text{if } x > 0, \\ Y_\varepsilon^- = \begin{pmatrix} H_y^-(x,y) + \varepsilon f^-(x,y) \\ -H_x^-(x,y) + \varepsilon g^-(x,y) \end{pmatrix}, & \text{if } x \leq 0, \end{cases} \tag{3.3}$$

where $f^\pm, g^\pm \in C^\infty$.

We will suppose that (3.2) has a family of periodic orbits near the origin, and we will study the bifurcation problem of limit cycles of (3.3). For system (3.2), we make the following assumptions:

(I) There exist an interval $J = (\alpha, \beta)$, and two points $A(h) = (0, a(h))$ and $A_1(h) = (0, a_1(h))$ such that for $h \in J$,

$$H^+(A(h)) = H^+(A_1(h)) = h,$$
$$H^-(A(h)) = H^-(A_1(h)),$$

where $a(h) \neq a_1(h)$.

(II) If $x > 0$, then the system Y^+ has an orbital arc L_h^+ starting from $A(h)$ and ending at $A_1(h)$ defined by $L_h^+ : H^+(x,y) = h$. If $x \leq 0$, then the system Y^- has an orbital arc L_h^- starting from $A_1(h)$ and ending at $A(h)$ defined by $L_h^- : H^-(x,y) = H^-(A_1(h))$.

Similarly to the continuous case, for $x > 0$, L_h^+ approaches an elementary center point, denoted by L_α, as $h \to \alpha$, and an invariant curve, denoted by L_β^+, as $h \to \beta$. Analogously, for $x \leq 0$, L_h^- approaches an elementary center point L_α, as $H^-(A_1(h)) \to \alpha$, and an invariant curve, denoted by L_β^-, as $H^-(A_1(h)) \to \beta$ (see Figure 3.1).

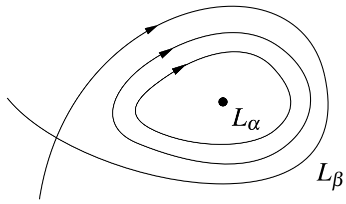


Figure 3.1: Example of a phase portrait.

Under the above assumptions (I) and (II), system (3.2) has a family of periodic orbits $L_h = L_h^+ \cup L_h^-$, for $h \in J$. Since $H^\pm \in C^\infty$, then $H^\pm$ are continuous, which implies that $L_h^\pm = (H^\pm)^{-1}(\{0\})$ are closed. Consequently, L_h is a closed curve. In general, L_h is piecewise smooth. Further, without loss of generality, suppose that L_h has a clockwise orientation (see Figure 3.2).

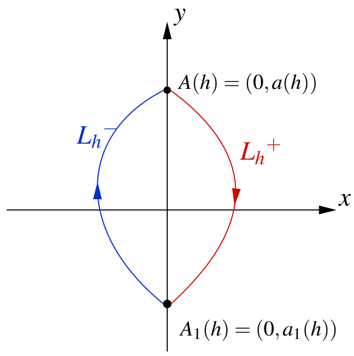


Figure 3.2: The closed orbits of (3.2).

Furthermore, consider an open set G such that $G = \bigcup_{\alpha<h<\beta} L_h$. We want to study the number of limit cycles of (3.3) in a neighborhood of the closure $\overline{G}$ of G, for $\varepsilon > 0$ small. According to Han and Yu [15, page 263], if (3.3) has a limit cycle $L(\varepsilon)$ for $\varepsilon > 0$ sufficiently small, then as $\varepsilon \to 0$ the limit cycle tends either to the center L_α, or to a periodic orbit L_h with $h \in (\alpha, \beta)$, or to the boundary $L_\beta = L_\beta^+ \cup L_\beta^-$, that is

$$\lim_{\varepsilon \to 0} L(\varepsilon) = L_h, \qquad h \in [\alpha, \beta].$$

In this case, it is said that the limit cycle $L(\varepsilon)$ is generated from L_h. Thus, in order to study the number of limit cycles, we first need to study the number of limit cycles generated from each L_h.

Now, we define the bifurcation function $F(h, \varepsilon)$ of system (3.3) as follows (see Figure 3.3). Consider the orbit of system Y_ε^+, starting from $A(h)$, and denote by A_ε its first intersection point with the negative y-axis. Next, let B_ε denote the first intersection point of the orbit starting from A_ε of system Y_ε^-, with the positive y-axis. Writing $A_\varepsilon = (0, a_\varepsilon(h))$ and $B_\varepsilon = (0, b_\varepsilon(h))$, we define:

$$H^+(B_\varepsilon) - H^+(A) = \varepsilon F(h, \varepsilon). \tag{3.4}$$

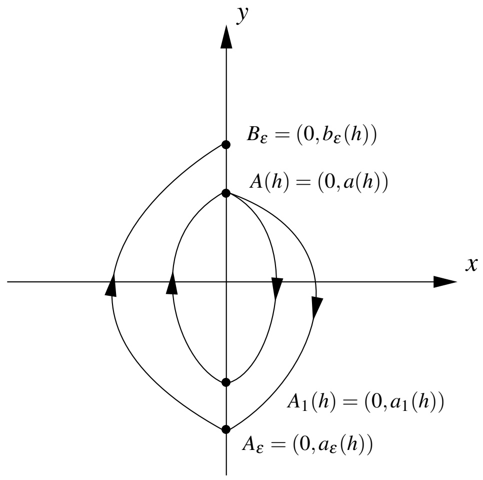


Figure 3.3: The Poincaré map related to the section $x = 0$.

Note that both A_ε and B_ε are smooth in (ε, h) which implies that the function F in (3.4) is also smooth. Moreover, since $\Sigma = \{x \in U \mid x = 0\}$, the Poincaré function of the system (3.3) is given by

$$\Pi_{Z_\varepsilon} = \Pi^-_{Y^-_\varepsilon} \circ \Pi^+_{Y^+_\varepsilon} : \Sigma \to \Sigma,$$

where $\Pi_{Z_\varepsilon}(A) = B_\varepsilon$.

Hence, system (3.3) has a periodic orbit near L_{h_0}, for $h_0 \in J$, if and only if $B_\varepsilon = A$, for (h, ε) near $(h_0, 0)$. The stability of a limit cycle in this case can be defined in the same way as for the smooth case. Moreover, from (3.4), under assumptions (I) and (II), one can see that a zero (an isolated zero, respectively) of F corresponds to a periodic orbit (a limit cycle, respectively) of (3.3). Therefore, we obtain the following lemma.

Lemma 3.1. *(Liu and Han [18], page 1381). For $|\varepsilon|$ small fixed and $h \in J$, then $F \in C^\infty$. The system* (3.3) *has a periodic orbit (a limit cycle, respectively) near L_{h_0} for $h_0 \in J$ if and only if the equation $F(h, \varepsilon) = 0$ has a root in h near h_0.*

Proof. Since $F(h, \varepsilon) = [H^+(B_\varepsilon) - H^+(A)]/\varepsilon$ and $H^+ \in C^\infty$, then $F \in C^\infty$, for $|\varepsilon|$ small.

In addition, let $A_0 \in L_{h_0}$, where $h_0 \in (\alpha, \beta)$. Consider $\Sigma = \{x \in U \mid x = 0\}$ the Poincaré section that passes through A_0, and define

$$n_1 = (H^+_y(A_0), -H^+_x(A_0)), \qquad n_0 = (H^-_x(A_0), H^-_y(A_0)).$$

Since $(\dot{x}, \dot{y}) = (H^+_y(x, y), -H^+_x(x, y))$, then the vector n_1 is tangent to L_{h_0}, and n_0 is the gradient vector normal to L_{h_0} at A_0. Let n_Σ denote a unit vector parallel to Σ, as shown in Figure 3.4. Hence, $n_0 \cdot n_\Sigma \neq 0$, because y-axis is a Poincaré section.

And we can write

$$\Sigma = \{A_0 + un_\Sigma, u \in \mathbb{R}, \text{ for } |u| \text{ small}\}.$$

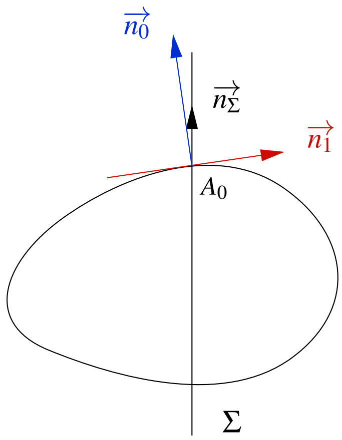


Figure 3.4: The Poincaré section.

Consider the function $G(h,u) = H^+(A_0 + un_\Sigma) - h$. Since $A_0 \in L_{h_0}$, then $H^+(A_0) = h_0$. Hence,

$$G(h_0, 0) = H^+(A_0) - h_0 = h_0 - h_0 = 0,$$
$$\frac{\partial G}{\partial u}(h_0, 0) = \frac{\partial H^+}{\partial u}(A_0) - \frac{\partial h_0}{\partial u} = (DH^+(A_0) \cdot n_\Sigma) - 0 = n_0 \cdot n_\Sigma \neq 0.$$

By the Implicit Function Theorem (see Theorem A.5), the equation $G(h,u) = 0$ defines a unique function $u = a(h)$, with $a(h) \in C^\infty$ and $a(h_0) = 0$. Let

$$A(h) = A_0 + a(h)n_\Sigma, \text{ for } |h - h_0| \text{ small}.$$

Then, $A(h) = L_h \cap \Sigma \in C^\infty$, with $A(h_0) = A_0$.

For $|\varepsilon|$ small, let $\varphi(t, A, \varepsilon)$ denote the solution of (3.3) satisfying $\varphi(0, A, \varepsilon) = A$. This solution is C^∞ in its variables. Let $T = T(h)$ denote the period of L_h. It follows that

$$\varphi(0, A, 0) = \varphi(T, A, 0) = A.$$

Now, consider the function $G_1(t, h, \varepsilon) = (\varphi(t, A(h), \varepsilon) - A(h)) \cdot n_\Sigma^\perp$, where $n_\Sigma^\perp \neq 0$ is a vector normal to n_Σ. Observe that

$$G_1(T(h_0), h_0, 0) = (\varphi(0, A_0, 0) - A(h_0)) \cdot n_\Sigma^\perp = (A(h_0) - A(h_0)) \cdot n_\Sigma^\perp = (0,0) \cdot n_\Sigma^\perp = 0,$$
$$\frac{\partial G_1}{\partial t}(T(h_0), h_0, 0) = \left(\frac{\partial \varphi}{\partial t}(T(h_0), A_0, 0) - \frac{\partial A}{\partial t}(h_0)\right) \cdot n_\Sigma^\perp = \big((H_y^+(A_0), -H_x^+(A_0)) - (0,0)\big) \cdot n_\Sigma^\perp$$
$$= n_1 \cdot n_\Sigma^\perp \neq 0.$$

Using the Implicit Function Theorem (see Theorem A.5) again, we know that there exists $t = \tau(h, \varepsilon) = T(h_0) + O(|\varepsilon| + |h - h_0|) \in C^\infty$, such that

$$G_1(\tau, h, \varepsilon) = 0 \text{ or } (\varphi(\tau, A, \varepsilon) - A) \cdot n_\Sigma^\perp = 0.$$

This shows that the vector $\varphi(\tau, A, \varepsilon) - A$ is parallel to n_Σ. Since $A \in \Sigma$, it follows that $\varphi(\tau, A, \varepsilon) \in \Sigma$. Moreover, $T(h) = \tau(h, 0)$ is a C^∞ function. Let B_ε denote the first intersection point of the orbit starting from A of system Y_ε^- with Σ, that is, $B_\varepsilon = \varphi(\tau, A, \varepsilon)$. We can write $B_\varepsilon = A + b(h)n_\Sigma$. Then, by the Mean Value Theorem (see Theorem A.2), we obtain

$$DH(C) = \frac{H^+(B_\varepsilon) - H^+(A)}{B_\varepsilon - A},$$

where $C = A + O(B_\varepsilon - A)$ and $B_\varepsilon - A = (0, O(\varepsilon))$.

Since $H^\pm \in C^\infty$ and $A = A_0 + a(h)n_\Sigma$, for $|h - h_0|$ small, we can expand DH around A_0, that is

$$\begin{aligned}
H^+(B_\varepsilon) - H^+(A) =& DH^+(A + O(B_\varepsilon - A)) \cdot (B_\varepsilon - A) \\
=& DH^+(A + (0, O(\varepsilon))) \cdot (B_\varepsilon - A) \\
=& \Bigg(DH^+(A_0) + (DH^+)^2(A_0) \cdot \frac{\|A + (0, O(\varepsilon)) - A_0\|^1}{1} \\
& + O\left(\|A + (0, O(\varepsilon)) - A_0\|^2\right) \Bigg) \cdot (B_\varepsilon - A) \\
=& \left(DH^+(A_0) + O(\|A_0 + a(h)n_\Sigma + (0, O(\varepsilon)) - A_0\|)\right) \cdot (B_\varepsilon - A) \\
=& \left(DH^+(A_0) + O(\|a(h)n_\Sigma + (0, O(\varepsilon))\|)\right) \cdot (B_\varepsilon - A).
\end{aligned}$$

Since $(0, O(\varepsilon)) \in \Sigma$, then $(0, O(\varepsilon))$ and $a(h)n_\Sigma$ have the same direction. As a consequence,

$$\begin{aligned}
H^+(B_\varepsilon) - H^+(A) &= \left(DH^+(A_0) + O(\|a(h)n_\Sigma + (0, O(\varepsilon))\|)\right) \cdot (B_\varepsilon - A) \\
&= \left(DH^+(A_0) + O(|a(h)|\|n_\Sigma\| + \|O(0, O(\varepsilon))\|)\right) \cdot (B_\varepsilon - A) \\
&= \left(DH^+(A_0) + O(|h - h_0| \cdot 1 + O(|\varepsilon|))\right) \cdot (B_\varepsilon - A) \\
&= \left(DH^+(A_0) + O(|h - h_0| + |\varepsilon|)\right) \cdot (B_\varepsilon - A) \\
&= (n_0 + O(|h - h_0| + |\varepsilon|)) \cdot (A_0 + b(h)n_\Sigma - (A_0 - a(h)n_\Sigma)) \\
&= (n_0 + O(|h - h_0| + |\varepsilon|)) \cdot (b(h) - a(h))n_\Sigma \\
&= [n_0 \cdot n_\Sigma + n_\Sigma O(|h - h_0| + |\varepsilon|)] \cdot (b(h) - a(h)) \\
&= [n_0 \cdot n_\Sigma + O(|h - h_0| + |\varepsilon|)] \cdot (b(h) - a(h)).
\end{aligned}$$

Recall that $O(|h - h_0| + |\varepsilon|)$ is a function such that the coefficients are $(DH^+)^2(A_0)$, $(DH^+)^3(A_0)$, $\ldots$, which are vectors. Then, $(DH^+)^2(A_0) \cdot n_\Sigma$, $(DH^+)^3(A_0) \cdot n_\Sigma$, $\ldots$ are numbers. In this sense, we committed an abuse of language in the last equality. Finally, since $n_0 \cdot n_\Sigma \neq 0$, we obtain that

$$A = B_\varepsilon \iff a(h) = b(h) \iff H^+(B_\varepsilon) = H^+(A),$$

which occur if and only if (3.3) has a periodic orbit near L_{h_0}, for $h_0 \in J$. □

The previous lemma also applies to $F, f^\pm, g^\pm \in C^\omega$.

As in the smooth case, let $M(h)$ be the first order Melnikov function. We define $M(h) := F(h, 0)$.

We now present some theorems concerning the Melnikov function. Theorem 3.2 gives a general expression of $M(h)$, Theorem 3.3 gives a formal development of $M(h)$ at $h = 0$, and Theorem 3.4 provides a condition for limit cycle bifurcation of a class of perturbed piecewise Hamiltonian systems.

Note that the expansion given by Theorem 3.3 is expressed in terms of powers of $\sqrt{h}$. This is due to the study of limit cycles for sufficiently small h, and because the result will be applied to systems that require such an expansion.

Theorem 3.2. *(Liu and Han [18], page 1381). Under the conditions* (I) *and* (II)*, for the first order Melnikov function of system* (3.3)*, we have*

$$M(h)=\frac{H_y^+(A)}{H_y^-(A)}\left[\frac{H_y^-(A_1)}{H_y^+(A_1)}\int_{\widehat{AA_1}} g^+\,\mathrm{d}x-f^+\,\mathrm{d}y+\int_{\widehat{A_1A}} g^-\mathrm{d}x-f^-\,\mathrm{d}y\right]. \tag{3.5}$$

Further, if $M(h_0)=0$ and $M'(h_0)\neq 0$ for some $h_0\in J$, then for $|\varepsilon|$ small (3.3) *has a unique limit cycle near L_{h_0}. If h_0 is a zero of $M(h)$ having odd multiplicity, then for $|\varepsilon|$ small* (3.3) *has at least one limit cycle near L_{h_0}.*

Proof. Adding zeros, we have that

$$\begin{aligned}H^+(B_\varepsilon)-H^+(A)=&[H^+(B_\varepsilon)-H^-(B_\varepsilon)]+[H^-(B_\varepsilon)-H^-(A_\varepsilon)]+[H^-(A_\varepsilon)-H^+(A_\varepsilon)]\\&+[H^+(A_\varepsilon)-H^+(A)].\end{aligned}$$

Now, let $l_1=H^+(B_\varepsilon)-H^-(B_\varepsilon)$, $l_2=H^-(B_\varepsilon)-H^-(A_\varepsilon)$, $l_3=H^-(A_\varepsilon)-H^+(A_\varepsilon)$, and $l_4=H^+(A_\varepsilon)-H^+(A)$.

Observe that, by the Fundamental Theorem of Calculus (see Theorem A.3) and since $H^\pm,f^\pm,g^\pm:\mathbb{R}^2\to\mathbb{R}$, we have

$$\begin{aligned}l_4=H^+(A_\varepsilon)-H^+(A)&=\int_{\widehat{AA_\varepsilon}}\mathrm{d}H^+=\int_{\widehat{AA_\varepsilon}}H_x^+\mathrm{d}x+H_y^+\mathrm{d}y\\&\overset{(3.3)}{=}\int_{\widehat{AA_\varepsilon}}\left[H_x^+\left(H_y^++\varepsilon f^+\right)+H_y^+\left(-H_x^++\varepsilon g^+\right)\right]\,\mathrm{d}t\\&=\int_{\widehat{AA_\varepsilon}}\left[H_x^+H_y^++\varepsilon H_x^+f^+-H_y^+H_x^++\varepsilon H_y^+g^+\right]\,\mathrm{d}t\\&=\varepsilon\int_{\widehat{AA_\varepsilon}}\left[H_x^+f^++H_y^+g^+\right]\mathrm{d}t\\&\overset{(3.3)}{=}\int_{\widehat{AA_\varepsilon}}\left[\left(-\frac{\mathrm{d}y}{\mathrm{d}t}+\varepsilon g^+\right)f^++\left(\frac{\mathrm{d}x}{\mathrm{d}t}-\varepsilon f^+\right)g^+\right]\,\mathrm{d}t\\&=\int_{\widehat{AA_\varepsilon}}\left[-f^+\mathrm{d}y+g^+\mathrm{d}x\right]\\&=\int_{\widehat{AA_\varepsilon}}g^+\mathrm{d}x-f^+\mathrm{d}y.\end{aligned}$$

Moreover, since $A_\varepsilon=(0,a_\varepsilon)=A_1+(0,O(\varepsilon))$, we have

$$\begin{aligned}l_4=H^+(A_\varepsilon)-H^+(A)&=\int_{\widehat{AA_\varepsilon}}g^+\mathrm{d}x-f^+\mathrm{d}y\\&=\varepsilon\left[\int_{\widehat{AA_1}}g^+\mathrm{d}x-f^+\mathrm{d}y+O(\varepsilon)\right],\end{aligned}$$

which implies

$$\left.\frac{\partial l_4}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y. \tag{3.6}$$

On the other hand, using $l_4 = H^+(A_\varepsilon) - H^+(A)$ and $A_\varepsilon = (0, a_\varepsilon(h)) = (0, a_1(h) + O(\varepsilon))$, we get

$$\begin{aligned}\left.\frac{\partial l_4}{\partial \varepsilon}\right|_{\varepsilon=0} &= \left.\frac{\partial H^+(A_\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} - \left.\frac{\partial H^+(A)}{\partial \varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial H^+(0, a_1(h) + O(\varepsilon))}{\partial \varepsilon}\right|_{\varepsilon=0} - \left.\frac{\partial H^+(0, a_1(h))}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &= H_y^+(0, a_1(h)) \left.\frac{\partial a_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} - 0 = H_y^+(A_1) \left.\frac{\partial a_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0},\end{aligned}$$

and, hence,

$$\left.\frac{\partial a_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{\int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y}{H_y^+(A_1)}. \tag{3.7}$$

Similarly, for $B_\varepsilon = (0, b_\varepsilon) = A + (0, O(\varepsilon))$, we have

$$l_2 = \left[H^-(B_\varepsilon) - H^-(A_\varepsilon)\right] = \int_{\widehat{A_\varepsilon B_\varepsilon}} \mathrm{d}H^- = \varepsilon \left[\int_{A_1 A} g^- \mathrm{d}x - f^- \mathrm{d}y + O(\varepsilon)\right],$$

and

$$\left.\frac{\partial l_2}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_{\widehat{A_1 A}} g^- \mathrm{d}x - f^- \mathrm{d}y. \tag{3.8}$$

Proceeding in a similar way for $l_3 = H^-(A_\varepsilon) - H^+(A_\varepsilon)$, $A_\varepsilon = (0, a_\varepsilon)$ and (3.7), we get

$$\left.\frac{\partial l_3}{\partial \varepsilon}\right|_{\varepsilon=0} = \left[H_y^-(A_1) - H_y^+(A_1)\right] \left.\frac{\partial a_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = \left(\frac{H_y^-(A_1)}{H_y^+(A_1)} - 1\right) \int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y. \tag{3.9}$$

Similarly, from $l_2 = H^-(B_\varepsilon) - H^-(A_\varepsilon)$, $A_\varepsilon = (0, a_\varepsilon)$ and $B_\varepsilon = (0, b_\varepsilon)$, we obtain

$$\left.\frac{\partial l_2}{\partial \varepsilon}\right|_{\varepsilon=0} = H_y^-(A) \left.\frac{\partial b_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} - H_y^-(A_1) \left.\frac{\partial a_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}\left.\frac{\partial b_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} &= \frac{1}{H_y^-(A)} \left.\frac{\partial l_2}{\partial \varepsilon}\right|_{\varepsilon=0} + \frac{H_y^-(A_1)}{H_y^-(A)} \left.\frac{\partial a_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &= \frac{1}{H_y^-(A)} \left.\frac{\partial l_2}{\partial \varepsilon}\right|_{\varepsilon=0} + \frac{H_y^-(A_1)}{H_y^-(A)} \left(\frac{\int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y}{H_y^+(A_1)}\right) \\ &= \frac{1}{H_y^-(A)} \int_{\widehat{A_1 A}} g^- \mathrm{d}x - f^- \mathrm{d}y + \frac{H_y^-(A_1)}{H_y^-(A) H_y^+(A_1)} \int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y.\end{aligned} \tag{3.10}$$

Then, using $l_1 = H^+(B_\varepsilon) - H^-(B_\varepsilon)$ and (3.10), we have

$$\begin{aligned}\left.\frac{\partial l_1}{\partial \varepsilon}\right|_{\varepsilon=0} &= \left[H_y^+(A) - H_y^-(A)\right] \left.\frac{\partial b_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &= \left[H_y^+(A) - H_y^-(A)\right] \left[\frac{1}{H_y^-(A)} \int_{\widehat{A_1 A}} g^- \mathrm{d}x - f^- \mathrm{d}y + \frac{H_y^-(A_1)}{H_y^-(A) H_y^+(A_1)} \int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y\right] \\ &= \left(\frac{H_y^+(A)}{H_y^-(A)} - 1\right) \int_{\widehat{A_1 A}} g^- \mathrm{d}x - f^- \mathrm{d}y + \left(\frac{H_y^+(A) H_y^-(A_1)}{H_y^-(A) H_y^+(A_1)} - \frac{H_y^-(A_1)}{H_y^+(A_1)}\right) \int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y.\end{aligned} \tag{3.11}$$

Hence, from (3.4), we obtain

$$H^+(B_\varepsilon) - H^+(A) = \varepsilon F(h,\varepsilon) = \varepsilon(F(h,0)+O(\varepsilon)) = \varepsilon(M(h)+O(\varepsilon)),$$

where

$$\begin{aligned}
M(h) &= \sum_{j=1}^{4} \frac{\partial l_j}{\partial \varepsilon}\bigg|_{\varepsilon=0} \\
&= \left[\left(\frac{H_y^+(A)}{H_y^-(A)} - 1\right)\int_{\widehat{A_1A}} g^- \mathrm{d}x - f^- \mathrm{d}y + \left(\frac{H_y^+(A)H_y^-(A_1)}{H_y^-(A)H_y^+(A_1)} - \frac{H_y^-(A_1)}{H_y^+(A_1)}\right)\int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y\right] \\
&\quad + \left[\int_{\widehat{A_1A}} g^- \mathrm{d}x - f^- \mathrm{d}y\right] + \left[\left(\frac{H_y^-(A_1)}{H_y^+(A_1)} - 1\right)\int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y\right] + \left[\int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y\right] \\
&= \left[\left(\frac{H_y^+(A)H_y^-(A_1)}{H_y^-(A)H_y^+(A_1)}\right)\int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y\right] + \left[\left(\frac{H_y^+(A)}{H_y^-(A)}\right)\int_{\widehat{A_1A}} g^- \mathrm{d}x - f^- \mathrm{d}y\right] \\
&= \frac{H_y^+(A)}{H_y^-(A)}\left[\frac{H_y^-(A_1)}{H_y^+(A_1)}\int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y + \int_{\widehat{A_1A}} g^- \mathrm{d}x - f^- \mathrm{d}y\right].
\end{aligned}$$

In addition, by Lemma 3.1, there is no limit cycle near L_{h_0}, for $|\varepsilon'|$ small, if $M(h_0) \neq 0$. Then, by the contrapositive, if $M(h_0) = 0$ for $|\varepsilon_1|$ small, then there exist a limit cycle near L_{h_0}. Moreover, since $M'(h_0) \neq 0$, the Inverse Function Theorem (see Theorem A.4) implies that M is a local diffeomorphism. Hence, there is ε_2 such that there exists a unique h_0 satisfying $M(h_0) = 0$, for $|\varepsilon_2|$ small. Setting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we obtain the uniqueness.

Finally, let h_0 be a multiple zero of $M(h)$ with odd multiplicity. Then, for $\varepsilon_0 > 0$ small, we have

$$M(h_0 - \varepsilon_0)\cdot M(h_0 + \varepsilon_0) < 0,$$

since $M(h) = F(h,0)$ and $F(h,\varepsilon) = F(h,0) + O(\varepsilon)$, then

$$F(h_0 - \varepsilon_0, \varepsilon)\cdot F(h_0 + \varepsilon_0, \varepsilon) < 0,$$

for $0 < \varepsilon < \varepsilon_0$, with ε_0 is sufficiently small.

Thus, the function $F(h,\varepsilon) = 0$ has a root $h^* \in (h_0 - \varepsilon_0, h_0 + \varepsilon)$. □

By the Green formula (see Theorem A.12), we also have

$$\int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y + \int_{\overrightarrow{A_1A}} g^+ \mathrm{d}x - f^+ \mathrm{d}y = \iint_{\mathrm{int}(\widehat{AA_1}\cup\overrightarrow{A_1A})} (f_x^+ + g_y^+)\,\mathrm{d}x\,\mathrm{d}y,$$

which implies that

$$\begin{aligned}
\int_{\widehat{AA_1}} g^+ \mathrm{d}x - f^+ \mathrm{d}y &= \iint_{\mathrm{int}(\widehat{AA_1}\cup\overrightarrow{A_1A})} (f_x^+ + g_y^+)\mathrm{d}x\,\mathrm{d}y - \int_{\overrightarrow{A_1A}} g^+ \mathrm{d}x - f^+ \mathrm{d}y \\
&= \iint_{\mathrm{int}(\widehat{AA_1}\cup\overrightarrow{A_1A})} (f_x^+ + g_y^+)\mathrm{d}x\,\mathrm{d}y - \left[\int_{\overrightarrow{A_1A}} g^+ \mathrm{d}x - \int_{\overrightarrow{A_1A}} f^+ \mathrm{d}y\right] \\
&= \iint_{\mathrm{int}(\widehat{AA_1}\cup\overrightarrow{A_1A})} (f_x^+ + g_y^+)\mathrm{d}x\,\mathrm{d}y - \left[0 - \int_{\overrightarrow{A_1A}} f^+ \mathrm{d}y\right] \\
&= \iint_{\mathrm{int}(\widehat{AA_1}\cup\overrightarrow{A_1A})} (f_x^+ + g_y^+)\mathrm{d}x\,\mathrm{d}y + \int_{\overrightarrow{A_1A}} f^+(0,y)\mathrm{d}y.
\end{aligned}$$

Hence,

$$\int_{\widehat{AA_1}} g^+ \,\mathrm{d}x - f^+ \,\mathrm{d}y = \iint_{\mathrm{int}(\widehat{AA_1}\cup\overrightarrow{A_1A})} (f_x^+ + g_y^+)\,\mathrm{d}x\,\mathrm{d}y + \int_{\overrightarrow{A_1A}} f^+(0,y)\,\mathrm{d}y \equiv M^+(h). \tag{3.12}$$

Similarly,

$$\int_{\widehat{A_1A}} g^- \,\mathrm{d}x - f^- \,\mathrm{d}y = \iint_{\mathrm{int}(\widehat{A_1A}\cup\overrightarrow{AA_1})} (f_x^- + g_y^-)\,\mathrm{d}x\,\mathrm{d}y + \int_{\overrightarrow{A_1A}} f^-(0,y)\,\mathrm{d}y \equiv M^-(h). \tag{3.13}$$

Thus, the expression of $M(h)$ can be rewritten as

$$M(h) = \frac{H_y^+(A)}{H_y^-(A)}\left[\frac{H_y^-(A_1)}{H_y^+(A_1)}M^+(h) + M^-(h)\right]. \tag{3.14}$$

For the next theorems, we suppose that the piecewise Hamiltonian system (3.2) has an elementary center at the origin. Since $(0,0)$ is a singularity point, then $H_x^\pm(0,0) = H_y^\pm(0,0) = 0$. Furthermore, system (3.2) is given by $A = \left(H_y^+, -H_x^+\right)$ for $x>0$, or $A = \left(H_y^-, -H_x^-\right)$ for $x \le 0$. Because $H^\pm \in C^\infty$, it follows that there exists

$$DA(0,0) = \begin{cases} \begin{pmatrix} \frac{\partial H_y^+}{\partial x}(0,0) & \frac{\partial H_y^+}{\partial y}(0,0) \\ -\frac{\partial H_x^+}{\partial x}(0,0) & -\frac{\partial H_x^+}{\partial y}(0,0) \end{pmatrix}, & \text{if } x>0, \\ \begin{pmatrix} \frac{\partial H_y^-}{\partial x}(0,0) & \frac{\partial H_y^-}{\partial y}(0,0) \\ -\frac{\partial H_x^-}{\partial x}(0,0) & -\frac{\partial H_x^-}{\partial y}(0,0) \end{pmatrix}, & \text{if } x\le 0. \end{cases}$$

Since $(0,0)$ is an elementary center, the matrix $DA(0,0)$ is either similar to the matrix

$$B^+ = \begin{pmatrix} 0 & \beta^+ \\ -\beta^+ & 0 \end{pmatrix},$$

for $x>0$, or similar to the matrix

$$B^- = \begin{pmatrix} 0 & \beta^- \\ -\beta^- & 0 \end{pmatrix},$$

for $x \le 0$. In any case, the eigenvalues of $DA(0,0)$ are purely imaginary and non-vanishing (see Dumortier et al. [8, page 15]). Then, $\det(DA(0,0)) = \det(B) > 0$. In other words, $H^\pm(x,y)$ satisfy

$$\begin{aligned} H_x^\pm(0,0) = H_y^\pm(0,0) = 0, \\ \det(DA(0,0)) > 0. \end{aligned} \tag{3.15}$$

We now state the second theorem.

Theorem 3.3. *(Liu and Han [18], page 1381). Let $f^\pm(0,0) = g^\pm(0,0) = 0$. Under the conditions* (I)*,* (II)*, and* (3.15) *with $J = (0,\beta)$, $\beta > 0$, there exists a C^∞ function $N(r) = rO(r)$ for $0 < r \ll 1$ such that $M(h) = N(\sqrt{h})$. Therefore, we have formally $M(h) = \sqrt{h}\sum_{i\ge 1} b_i h^{\frac{i}{2}}$, where b_i are coefficients independent of h.*

Proof. Let $A(h) = (0, a(h)) = (0, y_0)$, where $y_0 > 0$. From (3.15) and $J = (0, \beta)$, we can suppose, without loss of generality, that $H^+_{yy}(0,0) > 0$, $H^+_{xx}(0,0) > 0$, and $H^+_{xy}(0,0) = 0$. Since H^+_{yy} is continuous, then there exists $\varepsilon > 0$ such that, for every $h \in (h_0 - \varepsilon, h_0 + \varepsilon)$, $H^+_{yy}(x,y) > 0$. So, we can expand H^+ at the origin and evaluate it at $A(h)$, that is

$$\begin{aligned} H^+(0,y_0) =& H^+(0,0) + \frac{1}{1!}\left[H^+_x(0,0)(0-0) + H^+_y(0,0)(y_0-0)\right] + \\ &+ \frac{1}{2!}\left[H^+_{xx}(0,0)(0-0)^2 + 2H^+_{xy}(0,0)(0-0)(y_0-0) + H^+_{yy}(0,0)(y_0-0)^2\right] + \cdots \\ =& 0 + [0+0] + \frac{0+0+H^+_{yy}(0,0)(y-y_0)^2}{2!} + \frac{0+0+0+H_{yyy}(0,0)(y_0-0)^3}{3!} + \cdots \\ =& \frac{H^+_{yy}(0,0)}{2!}y_0^2 + \frac{H^+_{yyy}(0,0)}{3!}y_0^3 + \cdots \end{aligned}$$

Setting $b = H^+_{yy}(0,0)/2!$, we have that $b > 0$.

From (3.15) and $H^+(A(h)) = h$, we have that

$$H^+(0,y_0) = by_0^2 + \sum_{j\geq 3} h^+_{0j} y_0^j = h,$$

which implies

$$\begin{aligned} y_0^2\left(1 + \frac{1}{b}\sum_{j\geq 3} h^+_{0j} y_0^{j-2}\right) = \frac{h}{b} \implies |y_0|\left(1 + \frac{1}{b}\sum_{j\geq 3} h^+_{0j} y_0^{j-2}\right)^{\frac{1}{2}} = \sqrt{\frac{h}{b}} \\ \xrightarrow{y_0>0} y_0\left(1 + \frac{1}{b}\sum_{j\geq 3} h^+_{0j} y_0^{j-2}\right)^{\frac{1}{2}} = \sqrt{\frac{h}{b}}. \end{aligned}$$

Consider $G(y,v) = y\left(1 + (1/b)\sum_{j\geq 3} h^+_{0j} y_0^{j-2}\right)^{1/2} - \sqrt{v/b}$. Then,

$$\begin{aligned} G(y_0,h) &= y_0\left(1 + \frac{1}{b}\sum_{j\geq 3} h^+_{0j} y_0^{j-2}\right)^{\frac{1}{2}} - \sqrt{\frac{h}{b}} = 0, \\ \frac{\partial G}{\partial y}(y_0,h) &= \left(1 + \frac{1}{b}\sum_{j\geq 3} h^+_{0j} y_0^{j-2}\right)^{\frac{1}{2}} = \frac{1}{y_0}\sqrt{\frac{h}{b}} \neq 0. \end{aligned}$$

Hence, by the Implicit Function Theorem (see Theorem A.5), there exists a C^∞ function $\phi_0(v) = (v/\sqrt{b}) + \sum_{i\geq 2} e_i v^i$ such that

$$y_0 = \phi_0(\sqrt{h}) = \sqrt{\frac{h}{b}} + \sum_{i\geq 2} e_i h^{\frac{i}{2}}, \tag{3.16}$$

where the coefficients e_i depend on h^+_{0j} and b.

By the definition of the points $A_\varepsilon(h)$ and $B_\varepsilon(h)$, under conditions in (3.15), there exist C^∞ functions $\phi(y_0,\varepsilon)$ and $\psi(y_0,\varepsilon)$ such that, if (3.16) holds, then $a_\varepsilon(h) = \phi(y_0,\varepsilon)$ and $b_\varepsilon(h) = \psi(y_0,\varepsilon)$.

Furthermore, note that $\psi(y_0,0)=y_0$, since $a_\varepsilon(h)|_{\varepsilon=0}=a(h)+O(\varepsilon)=a(h)$. Therefore, for system (3.3), we introduce a displacement function as follows

$$d(y_0,\varepsilon)=\psi(y_0,\varepsilon)-\psi(y_0,0)=\psi(y_0,\varepsilon)-y_0=\varepsilon\overline{d}(y_0,\varepsilon),$$

where $\overline{d}$ is C^∞ in (y_0,ε), for $|\varepsilon|+|y_0|$ small.

Now, consider the auxiliary function $g(t)=H^+(0,t)$. Then,

$$g'(t)=H_y^+(0,t)\cdot 1=H_y^+(0,t).$$

By the Fundamental Theorem of Calculus (see Theorem A.3),

$$H^+(0,\psi(y_0,\varepsilon))-H^+(0,y_0)=\int_{y_0}^{\psi(y_0,\varepsilon)} H_y^+(0,t)\,\mathrm{d}t.$$

Now, consider the change of coordinates $t=y_0+s(\psi(y_0,\varepsilon)-y_0)$, where $s\in[0,1]$. Notice that when $s=0$, then $t=y_0$, and when $s=1$, then $t=\psi(y_0,\varepsilon)$. Furthermore, $\mathrm{d}t=(\psi(y_0,\varepsilon)-y_0)\,\mathrm{d}s=\varepsilon\overline{d}(y_0,\varepsilon)\,\mathrm{d}s$. Hence,

$$H^+(0,\psi(y_0,\varepsilon))-H^+(0,y_0)=\int_0^1 \left(H_y^+(0,y_0+s\varepsilon\overline{d}(y_0,\varepsilon))\right)\cdot\left(\varepsilon\overline{d}(y_0,\varepsilon)\right)\,\mathrm{d}s.$$

By (3.4), under (3.16) we obtain

$$\begin{aligned}\varepsilon F(h,\varepsilon)&=\int_0^1 \left(H_y^+(0,y_0+s\varepsilon\overline{d}(y_0,\varepsilon))\right)\cdot\left(\varepsilon\overline{d}(y_0,\varepsilon)\right)\,\mathrm{d}s \implies\\ F(h,\varepsilon)&=\int_0^1 \left(H_y^+(0,y_0+s\varepsilon\overline{d}(y_0,\varepsilon))\right)\cdot\left(\overline{d}(y_0,\varepsilon)\right)\,\mathrm{d}s\\ &\equiv\tilde{F}(y_0,\varepsilon).\end{aligned}\tag{3.17}$$

Observe that $\tilde{F}$ is C^∞ in (y_0,ε) near $(y_0,\varepsilon)=(0,0)$. In addition, it follows from (3.4) and (3.17) that

$$M(h)=F(h,0)=\tilde{F}(y_0,0)=H_y^+(y_0,0)\overline{d}(y_0,0)\equiv\overline{\phi}(y_0).\tag{3.18}$$

So, $\overline{\phi}\in C^\infty$ with $\overline{\phi}(0)=0$.

Now, consider $N(v)=\overline{\phi}(\phi_0(v))$. Then, $N\in C^\infty$ and we can expand N at the origin

$$N(v)=N(0)+N'(0)v+O(v^2).$$

Observe that

$$\begin{aligned}N(0)&=\overline{\phi}(\phi_0(0))\overset{\phi_0(0)=0}{=}\overline{\phi}(0)\overset{\overline{\phi}(0)=0}{=}0,\\ N'(0)&=\overline{\phi}'(\phi_0(0))\cdot\phi_0'(0)=\overline{\phi}'(0)\cdot\phi_0'(0)\overset{(3.16)}{=}\frac{1}{\sqrt{b}}\cdot\overline{\phi}_0'(0)\\ &\overset{(3.18)}{=}\frac{1}{\sqrt{b}}\left[H_{yy}^+(0,0)\cdot\overline{d}(0,0)+H_y^+(0,0)\cdot\overline{d}_{y_0}(0,0)\right]\\ &\overset{H_y^+(0,0)=0}{=}\frac{1}{\sqrt{b}}\left[H_{yy}^+(0,0)\cdot\left(\frac{\psi(0,0)-0}{\varepsilon}\right)+0\cdot\overline{d}_{y_0}(0,0)\right]\\ &\overset{\psi(0,0)=0}{=}\frac{1}{\sqrt{b}}\left[H_{yy}^+(0,0)\cdot 0+0\cdot\overline{d}_{y_0}(0,0)\right]\\ &=0.\end{aligned}$$

Since $N(0) = N'(0) = 0$, the expansion of N start with v^2, that is, $N(v) = O(v^2)$. So,

$$N(\sqrt{h}) = O(\sqrt{h}^2) = O(h) = \sqrt{h}O(\sqrt{h}).$$

For h is small, it is sufficient to take $r = \sqrt{h}$ such that $M(h) = N(\sqrt{h})$. Therefore, $M(h) = \sqrt{h}\sum_{i\geq 1} b_i h^{i/2}$, where b_i are coefficients independent of h. □

Now, we consider the nonsmooth system with multiple parameters

$$Z_\delta = \begin{cases} Y_\delta^+ = \begin{pmatrix} H_y^+(x,y) + \varepsilon f^+(x,y,\delta) \\ -H_x^+(x,y) + \varepsilon g^+(x,y,\delta) \end{pmatrix}, & \text{if } x > 0, \\ Y_\delta^- = \begin{pmatrix} H_y^-(x,y) + \varepsilon f^-(x,y,\delta) \\ -H_x^-(x,y) + \varepsilon g^-(x,y,\delta) \end{pmatrix}, & \text{if } x \leq 0, \end{cases} \tag{3.19}$$

where $f^\pm$ and $g^\pm$ are C^∞ functions, $\varepsilon > 0$ is small and $\delta \in \mathbb{R}^m$, with $m \geq 1$. By (3.5), the first order Melnikov function also depends on δ, and we denote it by $M(h,\delta)$. Further, by Theorem 3.2 we have

$$M(h,\delta) = \sqrt{h}\left(\sum_{k=1}^{N} B_{k-1}(\delta)h^{\frac{k}{2}} + O\left(h^{\frac{N+1}{2}}\right)\right), \tag{3.20}$$

for any integer $N > 1$. We have the following theorem.

Theorem 3.4. *(Liu and Han [18], page 1381). Let $f^\pm(0,0,\delta) = g^\pm(0,0,\delta) = 0$. Under the conditions* (I), (II)*, and* (3.15) *with $J = (0,\beta)$, if there exist an integer $k \geq 1$ and $\delta_0 \in \mathbb{R}^m$ such that*

$$B_j(\delta_0) = 0, \qquad j = 0,\dots,k-1, \qquad B_k(\delta_0) \neq 0,$$

and

$$\operatorname{rank}\left(\frac{\partial(B_0,\dots,B_{k-1})}{\partial(\delta_1,\dots,\delta_m)}(\delta_0)\right) = k, \tag{3.21}$$

where $\delta = (\delta_1,\dots,\delta_m)$, $m \geq k$, then system (3.19) *has at most k limit cycles in a neighborhood of the origin for all (ε,δ) near $(0,\delta_0)$, and k limit cycles appear for some (ε,δ) near $(0,\delta_0)$.*

Proof. For system (3.19), the function F in (3.4) depends on δ, and we denote it by $F(h,\varepsilon,\delta)$.

Then, by (3.17) we have

$$F(h,\varepsilon,\delta) = \tilde{F}(y_0,\varepsilon,\delta),$$

where $\tilde{F}$ is C^∞ function of (y_0,ε,δ). Substituting (3.16) into $\tilde{F}$ yields

$$\tilde{F}(y_0,\varepsilon,\delta) = \sqrt{h}\left(\sum_{k\geq 1} a_{k-1}(\varepsilon,\delta)h^{\frac{k}{2}}\right), \tag{3.22}$$

where by $M(h,\delta) = F(h,0,\delta)$ and by (3.20), we have $a_k(\varepsilon,\delta) = B_k(\delta) + O(\varepsilon)$, for $k \geq 0$.

Since (3.21) holds and $m \geq k$, the Jacobian matrix $\partial(B_0,\dots,B_{k-1})/\partial(\delta_1,\dots,\delta_k)(\delta_0)$ has k linearly independent columns. Consequently, its leading $k \times k$ submatrix invertible. As a consequence,

$$\det\left(\frac{\partial(B_0,\dots,B_{k-1})}{\partial(\delta_1,\dots,\delta_k)}(\delta_0)\right) \neq 0.$$

By the Inverse Function Theorem (see Theorem A.4), the change of parameters

$$b_j = a_j(\varepsilon, \delta), \qquad j = 0, \ldots, k-1,$$

has the inverse $\delta_j = \delta_j(b_0, \ldots, b_{k-1}, \delta_{k+1}, \ldots, \delta_m)$, where $j = 1, 2, \ldots, k$. Then, setting $r = h$, and by (3.17) and (3.22), we have

$$\begin{aligned}\tilde{F}(y_0, \varepsilon, \delta) &= r\left(b_0 + \cdots + b_{k-1} r^{k-1} + \tilde{b}_k r^k + O\left(r^{k+1}\right)\right) \\ &= r\overline{F}(r, \varepsilon, b_0, \ldots, b_{k-1}, \delta_{k+1}, \ldots, \delta_m),\end{aligned}$$

where $\tilde{b}_k = B_k(\delta_0) \neq 0$ as $\varepsilon = b_0 = \cdots = b_{k-1} = 0$ and $\delta_j = \delta_{j0}$ for $j = k+1, \ldots, m$.

Since $r = \sqrt{h} > 0$, the zeros of $\tilde{F}$ coincide with the zeros of $b_0 + \cdots + b_{k-1} r^{k-1} + \tilde{b}_k r^k + O\left(r^{k+1}\right)$. Let $G(t) = b_0 + \cdots + b_{k-1} t^{k-1} + \tilde{b}_k t^k + O(t^{k+1})$. Assuming G is a polynomial function of at least degree k, then G possesses at least k zeros. If G were to have more than k zeros, then by Rolle's Theorem (see Theorem A.1), it follows that G' has at least k zeros, $G^{(2)}$ has at least $k-1$ zeros, and, proceeding by induction, $G^{(k)}$ must have at least one zero. However,

$$\frac{\mathrm{d}^k G}{\mathrm{d}t^k}(0) = k!\, \tilde{b}_k \neq 0,$$

which implies that $G^{(k)}(t) \neq 0$ for t sufficiently close to 0. This contradicts the previous assertion. Thus, G has at most k zeros for t near $t = 0$. Consequently, $\tilde{F}$ has at most k zeros in the variable r near $r = 0$, which implies that $\tilde{F}$ has at most k zeros in h near $h = 0$. □

3.2 Applications

In this section, we apply the Melnikov function to compute the number of limit cycles.

3.2.1 Polynomial systems

Proposition 3.5. *(Liu and Han [18], page 1384) Consider a piecewise polynomial system of the form*

$$Z = \begin{cases} Y^+ = \begin{pmatrix} b^+ y + \varepsilon \sum\limits_{i+j=0}^{n} a_{ij}^+ x^i y^j \\ -b^+ x + \varepsilon \sum\limits_{i+j=0}^{n} a_{ij}^+ x^i y^j \end{pmatrix}, & \text{if } x > 0, \\ Y^- = \begin{pmatrix} b^- y + \varepsilon \sum\limits_{i+j=0}^{n} a_{ij}^- x^i y^j \\ -b^- x + \varepsilon \sum\limits_{i+j=0}^{n} a_{ij}^- x^i y^j \end{pmatrix}, & \text{if } x \leq 0, \end{cases} \tag{3.23}$$

where $b^{\pm} > 0$. Then, the maximal number of isolated zeros of the first order Melnikov function $M(h)$ is n.

Proof. For $\varepsilon = 0$, note that the Hamiltonian function associated with system (3.23) is given by

$$H(x,y) = \begin{cases} H^+(x,y) = \dfrac{1}{2}b^+(x^2+y^2), & \text{for } x > 0, \\ H^-(x,y) = \dfrac{1}{2}b^-(x^2+y^2), & \text{for } x \le 0. \end{cases}$$

Under assumption (II), let $A(h) = (0, a(h))$ and $A_1(h) = (0, a_1(h))$ be the points where the orbit with energy h intersects the switching line. It follows that

$$H^+(A(h)) = h \implies H^+(0, a(h)) = h \implies \frac{b^+(0^2 + (a(h))^2)}{2} = h \xrightarrow{a(h)>0} a(h) = \sqrt{\frac{2h}{b^+}},$$

$$H^-(A(h)) = H^-(A_1(h)) \implies \frac{b^-\left(0^2 + \left(\sqrt{\frac{2h}{b^+}}\right)^2\right)}{2} = \frac{b^-(0^2 + (a_1(h))^2)}{2}$$

$$\xrightarrow{a_1(h)<0} a_1(h) = -\sqrt{\frac{2h}{b^+}}.$$

Hence, $A(h) = (0, \sqrt{2h/b^+})$ and $A_1(h) = (0, -\sqrt{2h/b^+})$.

By Theorem (3.2), we have that:

$$\begin{aligned} M(h) =& \frac{b^+\left(\sqrt{\frac{2h}{b^+}}\right)}{b^-\left(\sqrt{\frac{2h}{b^+}}\right)} \frac{b^-\left(-\sqrt{\frac{2h}{b^+}}\right)}{b^+\left(-\sqrt{\frac{2h}{b^+}}\right)} \int_{\widehat{AA_1}} \left(\sum_{i+j=0}^{n} b^+_{ij} x^i y^j \mathrm{d}x - \sum_{i+j=0}^{n} a^+_{ij} x^i y^j \mathrm{d}y\right) \\ &+ \frac{b^+\left(\sqrt{\frac{2h}{b^+}}\right)}{b^-\left(\sqrt{\frac{2h}{b^+}}\right)} \int_{\widehat{A_1A}} \left(\sum_{i+j=0}^{n} b^-_{ij} x^i y^j \mathrm{d}x - \sum_{i+j=0}^{n} a^-_{ij} x^i y^j \mathrm{d}y\right) \\ =& \int_{\widehat{AA_1}} \left(\sum_{i+j=0}^{n} b^+_{ij} x^i y^j \mathrm{d}x - \sum_{i+j=0}^{n} a^+_{ij} x^i y^j \mathrm{d}y\right) \\ &+ \left[\left(\frac{b^+}{b^-}\right) \int_{\widehat{A_1A}} \left(\sum_{i+j=0}^{n} b^-_{ij} x^i y^j \mathrm{d}x - \sum_{i+j=0}^{n} a^-_{ij} x^i y^j \mathrm{d}y\right)\right]. \end{aligned}$$

Setting $x = \sqrt{2h/b^+}\cos\theta$ and $y = \sqrt{2h/b^+}\sin\theta$, then

$$\mathrm{d}x = -\sqrt{2h/b^+}\sin\theta \ \mathrm{d}\theta, \quad \mathrm{d}y = \sqrt{2h/b^+}\cos\theta \ \mathrm{d}\theta.$$

Furthermore,

$$\begin{cases} \sqrt{\frac{2h}{b^+}}\cos\theta = 0 \\ \sqrt{\frac{2h}{b^+}}\sin\theta = \sqrt{\frac{2h}{b^+}} \end{cases} \implies \begin{cases} \cos\theta = 0 \\ \sin\theta = 1 \end{cases} \implies \theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z},$$

$$\begin{cases} \sqrt{\frac{2h}{b^+}}\cos\theta = 0 \\ \sqrt{\frac{2h}{b^+}}\sin\theta = -\sqrt{\frac{2h}{b^+}} \end{cases} \implies \begin{cases} \cos\theta = 0 \\ \sin\theta = -1 \end{cases} \implies \theta = -\frac{\pi}{2} + 2l\pi, \quad l \in \mathbb{Z}.$$

Hence,

$$
\begin{aligned}
M(h) =& \int_{\widehat{AA_1}} \left(\sum_{i+j=0}^{n} b_{ij}^{+} x^i y^j \mathrm{d}x - \sum_{i+j=0}^{n} a_{ij}^{+} x^i y^j \mathrm{d}y \right) + \left[\left(\frac{b^+}{b^-} \right) \int_{\widehat{A_1A}} \left(\sum_{i+j=0}^{n} b_{ij}^{-} x^i y^j \mathrm{d}x - \sum_{i+j=0}^{n} a_{ij}^{-} x^i y^j \mathrm{d}y \right) \right] \\
=& \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \Bigg[\sum_{i+j=0}^{n} b_{ij}^{+} \left(\sqrt{\frac{2h}{b^+}} \cos\theta \right)^i \left(\sqrt{\frac{2h}{b^+}} \sin\theta \right)^j \left(-\sqrt{\frac{2h}{b^+}} \sin\theta \right) \\
& - \sum_{i+j=0}^{n} a_{ij}^{+} \left(\sqrt{\frac{2h}{b^+}} \cos\theta \right)^i \left(\sqrt{\frac{2h}{b^+}} \sin\theta \right)^j \left(\sqrt{\frac{2h}{b^+}} \cos\theta \right) \Bigg] \mathrm{d}\theta \\
& + \left(\frac{b^+}{b^-} \right) \int_{\frac{-\pi}{2}}^{-\frac{3\pi}{2}} \Bigg[\sum_{i+j=0}^{n} b_{ij}^{-} \left(\sqrt{\frac{2h}{b^+}} \cos\theta \right)^i \left(\sqrt{\frac{2h}{b^+}} \sin\theta \right)^j \left(-\sqrt{\frac{2h}{b^+}} \sin\theta \right) \\
& - \sum_{i+j=0}^{n} a_{ij}^{-} \left(\sqrt{\frac{2h}{b^+}} \cos\theta \right)^i \left(\sqrt{\frac{2h}{b^+}} \sin\theta \right)^j \left(\sqrt{\frac{2h}{b^+}} \cos\theta \right) \Bigg] \mathrm{d}\theta \\
=& \sum_{i+j=0}^{n} \left(\frac{2h}{b^+} \right)^{\frac{i+j+1}{2}} \left[\int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} -b_{ij}^{+} \cos^i\theta \sin^{j+1}\theta - a_{ij}^{+} \cos^{i+1}\theta \sin^j\theta \right] \mathrm{d}\theta \\
& + \sum_{i+j=0}^{n} \left(\frac{2h}{b^+} \right)^{\frac{i+j+1}{2}} \left[\frac{b^+}{b^-} \int_{\frac{-\pi}{2}}^{-\frac{3\pi}{2}} -b_{ij}^{-} \cos^i\theta \sin^{j+1}\theta - a_{ij}^{-} \cos^{i+1}\theta \sin^j\theta \right] \mathrm{d}\theta \\
\overset{\theta'=\theta+\pi}{=}& \sum_{i+j=0}^{n} \left(\frac{2h}{b^+} \right)^{\frac{i+j+1}{2}} \left[\int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} -b_{ij}^{+} \cos^i\theta \sin^{j+1}\theta - a_{ij}^{+} \cos^{i+1}\theta \sin^j\theta \right] \mathrm{d}\theta \\
& + \sum_{i+j=0}^{n} \left(\frac{2h}{b^+} \right)^{\frac{i+j+1}{2}} \left[\frac{b^+}{b^-} \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} -b_{ij}^{-} \cos^i(\theta'-\pi) \sin^{j+1}(\theta'-\pi) - a_{ij}^{-} \cos^{i+1}(\theta'-\pi) \sin^j(\theta'-\pi) \right] \mathrm{d}\theta' \\
=& \sum_{i+j=0}^{n} \left(\frac{2h}{b^+} \right)^{\frac{i+j+1}{2}} \left[\int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} -b_{ij}^{+} \cos^i\theta \sin^{j+1}\theta - a_{ij}^{+} \cos^{i+1}\theta \sin^j\theta \right] \mathrm{d}\theta \\
& + \sum_{i+j=0}^{n} \left(\frac{2h}{b^+} \right)^{\frac{i+j+1}{2}} \left[\frac{b^+}{b^-} \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} -b_{ij}^{-} (-\cos\theta')^i (-\sin\theta')^{j+1} - a_{ij}^{-} (-\cos\theta')^{i+1} (-\sin\theta')^j \right] \mathrm{d}\theta' \\
=& \sum_{i+j=0}^{n} \left(\frac{2h}{b^+} \right)^{\frac{i+j+1}{2}} \left[\int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} -b_{ij}^{+} \cos^i\theta \sin^{j+1}\theta - a_{ij}^{+} \cos^{i+1}\theta \sin^j\theta \right] \mathrm{d}\theta \\
& + \sum_{i+j=0}^{n} \left(\frac{2h}{b^+} \right)^{\frac{i+j+1}{2}} \left[\left(\frac{b^+}{b^-} \right) \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} -(-b_{ij}^{-})(-1)^{i+j} \cos^i\theta' \sin^{j+1}\theta' + a_{ij}^{-}(-1)^{i+j} \cos^{i+1}\theta' \sin^j\theta' \right] \mathrm{d}\theta' \\
=& \sum_{i+j=0}^{n} \frac{\left(\sqrt{2}\right)^{i+j+1} \left(\sqrt{h}\right)^{i+j+1}}{\left(\sqrt{b^+}\right)^{i+j+1}} \Bigg[\left(-b_{ij}^{+} + (-1)^{i+j} \frac{b^+}{b^-} b_{ij}^{-} \right) \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos^i\theta \sin^{j+1}\theta \ \mathrm{d}\theta \\
& - \left(a_{ij}^{+} + (-1)^{i+j+1} \frac{b^+}{b^-} a_{ij}^{-} \right) \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos^{i+1}\theta \sin^j\theta \ \mathrm{d}\theta \Bigg] \\
=& \sqrt{h} \Bigg[\sum_{i+j=0}^{n} \frac{\left(\sqrt{2}\right)^{i+j+1} \left(\sqrt{h}\right)^{i+j}}{\left(\sqrt{b^+}\right)^{i+j+1}} \Bigg(\left(-b_{ij}^{+} + (-1)^{i+j} \frac{b^+}{b^-} b_{ij}^{-} \right) \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos^i\theta \sin^{j+1}\theta \ \mathrm{d}\theta \\
& - \left(a_{ij}^{+} + (-1)^{i+j+1} \frac{b^+}{b^-} a_{ij}^{-} \right) \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos^{i+1}\theta \sin^j\theta \ \mathrm{d}\theta \Bigg) \Bigg] \\
=& \sqrt{h} \left(B_0 + B_1\sqrt{h} + B_2 \left(\sqrt{h}\right)^2 + \cdots + B_n \left(\sqrt{h}\right)^n \right),
\end{aligned} \tag{3.24}
$$

where

$$B_l = \sum_{i+j=l} \frac{\left(\sqrt{2}\right)^{l+1}}{\left(\sqrt{b^+}\right)^{l+1}} \left[\left(-b_{ij}^+ + (-1)^{i+j} \frac{b^+}{b^-} b_{ij}^- \right) \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos^i \theta \sin^{j+1} \theta \mathrm{d}\theta \right.$$
$$\left. - \left(a_{ij}^+ + (-1)^{i+j+1} \frac{b^+}{b_-} a_{ij}^- \right) \cdot \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos^{i+1} \theta \sin^j \theta \mathrm{d}\theta \right], \quad 0 \leq l \leq n.$$

Since $\sqrt{h} \neq 0$ and $B_0 + B_1\sqrt{h} + \cdots + B_n(\sqrt{h})^n$ is a polynomial in $\sqrt{h}$ of degree n, it follows that (3.24) has at most n zeros for $h > 0$, provided that $M(h)$ is not identically zero by Theorem 3.4.

To show that n zeros can indeed appear, we consider a particular case by letting $a_{ij}^- = b_{ij}^\pm = 0$ and $b^+ = 1$. Since the sine function is odd and the cosine function is even, we have

$$\cos^{i+1}(-\theta)\sin^j(-\theta) = (\cos(-\theta))^{i+1} (\sin(-\theta))^j = (-1)^j \cos^{i+1} \theta \sin^j \theta.$$

If j is even, the function $f(\theta) = \cos^{i+1} \theta \sin^j \theta$ is even, whereas if j is odd, $f(\theta)$ is odd. Since the integral of an odd function over an interval symmetric about the origin vanishes, it follows that

$$\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos^{i+1} \theta \sin^j \theta \, \mathrm{d}\theta = 0 \quad \text{if } j \text{ is odd.}$$

Conversely, if j is even, the symmetry of the integrand implies that

$$\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos^{i+1} \theta \sin^j \theta \, \mathrm{d}\theta = -2 \int_0^{\frac{\pi}{2}} \cos^{i+1} \theta \sin^j \theta \, \mathrm{d}\theta.$$

Hence, we obtain

$$\begin{aligned} M(h) &= 2 \sum_{i+2k}^{n} a_{i,2k}^+ r^{i+2k+1} \int_0^{\frac{\pi}{2}} \cos^{i+1} \theta \sin^{2k} \theta \mathrm{d}\theta \\ &= 2r(C_0 + C_1 r + \cdots + C_n r^n), \end{aligned}$$

where $r = \sqrt{2h}$ and the coefficients are given by

$$C_l = \sum_{i+2k=l} a_{i,2k}^+ \int_0^{\frac{\pi}{2}} \cos^{i+1} \theta \sin^{2k} \theta \, \mathrm{d}\theta, \quad \text{for } 0 \leq l \leq n.$$

By fixing $a_{i,2k}^+$ for $k > 0$ and treating the remaining $a_{i,0}^+$, for $i = 0, \ldots, n$, as free parameters, we can ensure that the coefficients C_l are independent. Consequently, these parameters can be chosen to produce n simple zeros of $M(h)$ in a neighborhood of $r = 0$. □

Proposition 3.6. *(Liu and Han [18], page 1384). Consider a concrete piecewise system with piecewise-linear perturbation:*

$$Z = \begin{cases} Y^+ = \begin{pmatrix} y + \varepsilon p_1(x) \\ -x \end{pmatrix}, & \text{if } x > 0, \\ Y^- = \begin{pmatrix} y + \varepsilon p_2(x) \\ -x \end{pmatrix}, & \text{if } x \leq 0, \end{cases} \tag{3.25}$$

where

$$p_1(x) = \begin{cases} 5x, & x \in (0,1], \\ 5x-4, & x \in (1,+\infty), \end{cases} \qquad p_2(x) = 2-x.$$

Then, system (3.6) *admits three limit cycles for* $\varepsilon > 0$ *small.*

Proof. For $\varepsilon = 0$, observe that the Hamiltonian function of system (3.6) is $H(x,y) = (x^2+y^2)/2$, for all x. So, we obtain

$$\begin{aligned} H^+(0,a(h)) = h &\implies \frac{0^2+(a(h))^2}{2} = h \implies a(h) = \sqrt{2h}, \\ H^-(0,a_1(h)) = H^-(0,a(h)) &\implies \frac{0^2+(a_1(h))^2}{2} = \frac{0^2+2h}{2} \implies a_1(h) = -\sqrt{2h}. \end{aligned}$$

Hence, $A = (0,\sqrt{2h})$ and $A_1 = (0,-\sqrt{2h})$. For $\sqrt{2h} \leq 1$, using

$$x = \sqrt{2h}\cos\theta, \qquad y = \sqrt{2h}\sin\theta,$$

since $\sqrt{2h}\cos\theta \leq \sqrt{2h} \leq 1$, we have that

$$\begin{aligned} M(h) &= \frac{\sqrt{2h}}{\sqrt{2h}}\left[\frac{-\sqrt{2h}}{-\sqrt{2h}}\int_{\widehat{AA_1}} 0\,\mathrm{d}x - p_1(x)\,\mathrm{d}y + \int_{\widehat{A_1A}} 0\,\mathrm{d}x - p_2(x)\,\mathrm{d}y\right] \\ &= -\int_{\widehat{AA_1}} p_1(x)\,\mathrm{d}y - \int_{\widehat{A_1A}} p_2(x)\,\mathrm{d}y \\ &= -\int_{\sqrt{2h}}^{-\sqrt{2h}} 5x\,\mathrm{d}y - \int_{-\sqrt{2h}}^{\sqrt{2h}}(2-x)\,\mathrm{d}y \\ &= \int_{-\sqrt{2h}}^{\sqrt{2h}} 5\sqrt{2h-y^2}\,\mathrm{d}y - \int_{-\sqrt{2h}}^{\sqrt{2h}}\left(2-\left(-\sqrt{2h-y^2}\right)\right)\,\mathrm{d}y \\ &= \int_{-\sqrt{2h}}^{\sqrt{2h}} 5\sqrt{2h-y^2}\,\mathrm{d}y - \int_{-\sqrt{2h}}^{\sqrt{2h}}\left(2+\sqrt{2h-y^2}\right)\,\mathrm{d}y \\ &= 5\pi h - (4\sqrt{2h}+\pi h) \\ &= 4(\pi h - \sqrt{2h}). \end{aligned} \tag{3.26}$$

For $\sqrt{2h} > 1$, let $\theta_1 \in (0,\pi/2)$ satisfying $\cos\theta_1 = 1/\sqrt{2h}$. Since the cosine function is decreasing in $(0,\pi/2)$, then

$$\begin{aligned} \sqrt{2h}\cos\theta > 1,\ &\text{if } \theta \in (0,\theta_1), \\ \sqrt{2h}\cos\theta < \sqrt{2h}\cdot\frac{1}{\sqrt{2h}} = 1,\ &\text{if } \theta \in \left(\theta_1,\frac{\pi}{2}\right). \end{aligned}$$

Hence,

$$
\begin{aligned}
M(h) =&\frac{\sqrt{2h}}{\sqrt{2h}}\left[\frac{-\sqrt{2h}}{-\sqrt{2h}}\int_{\widehat{AA_1}} 0\ \mathrm{d}x - p_1(x)\ \mathrm{d}y + \int_{\widehat{A_1A}} 0\ \mathrm{d}x - p_2(x)\ \mathrm{d}y\right]\\
=&-\int_{\widehat{AA_1}} p_1(x)\ \mathrm{d}y - \int_{\widehat{A_1A}} p_2(x)\ \mathrm{d}y\\
=&-\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} p_1(\sqrt{2h}\cos\theta)(\sqrt{2h}\cos\theta)\ \mathrm{d}\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p_2(\sqrt{2h}\cos\theta)(\sqrt{2h}\cos\theta)\ \mathrm{d}\theta\\
=&2\int_0^{\frac{\pi}{2}} \sqrt{2h}\cos\theta\, p_1(\sqrt{2h}\cos\theta)\ \mathrm{d}\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{2h}\cos\theta - 2h\cos^2\theta\ \mathrm{d}\theta\\
=&2\int_0^{\frac{\pi}{2}} \sqrt{2h}\cos\theta\, p_1(\sqrt{2h}\cos\theta)\ \mathrm{d}\theta - \left[2\sqrt{2h}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right)\right.\\
&\left. -2h\left(\frac{\frac{\pi}{2}}{2} - \frac{-\frac{\pi}{2}}{2} + \frac{\sin(\pi)}{4} - \frac{\sin(-\pi)}{4}\right)\right]\\
=&2\left[\int_0^{\frac{\pi}{2}}\left(\sqrt{2h}\cos\theta\, p_1(\sqrt{2h}\cos\theta)\right)\ \mathrm{d}\theta\right] - 4\sqrt{2h} - \pi h\\
=&2\left[\int_0^{\theta_1}\sqrt{2h}\cos\theta\left(\left(5\sqrt{2}\cos\theta\right) - 4\right)\ \mathrm{d}\theta + 5\int_{\theta_1}^{\frac{\pi}{2}}\left(2h\cos^2\theta\right)\ \mathrm{d}\theta\right] - \pi h - 4\sqrt{2h}\\
=&2\left[5\int_0^{\frac{\pi}{2}} 2h\cos^2\theta\ \mathrm{d}\theta - 4\int_0^{\theta_1}\sqrt{2h}\cos\theta\ \mathrm{d}\theta\right] - \pi h - 4\sqrt{2h}\\
=&20h\left[\frac{\frac{\pi}{2}}{2} + \frac{\sin\left(2\frac{\pi}{2}\right)}{4} - \frac{0}{2} + \frac{\sin(0)}{4}\right] - 8\sqrt{2h}[\sin(\theta_1) - \sin(0)] - \pi h - 4\sqrt{2h}\\
=&5\pi h - 8\sqrt{2h}\sin\left(\arccos\left(\frac{1}{\sqrt{2h}}\right)\right) - \pi h - 4\sqrt{2h}\\
=&4\pi h - 8\sqrt{2h}\sqrt{1 - \left(\frac{1}{\sqrt{2h}}\right)^2} - 4\sqrt{2h}\\
=&4(\pi h - 2\sqrt{2h-1} - \sqrt{2h}).
\end{aligned}
\tag{3.27}
$$

For the geometrical aspect on the deduction of $M(h)$, see Figure 3.5.

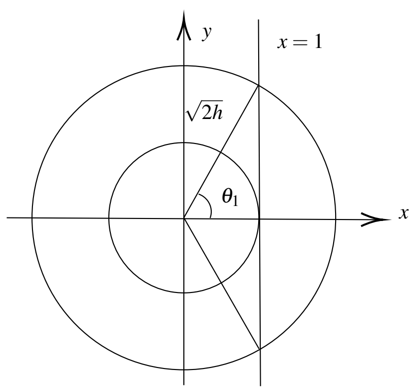


Figure 3.5: The geometric aspect on the deduction of $M(h)$.

From (3.26) and (3.27), we have

$$M(h) = \begin{cases} 4(\pi h - \sqrt{2h}), & \text{if } 0 < h \le \frac{1}{2}, \\ 4(\pi h - 2\sqrt{2h-1} - \sqrt{2h}), & \text{if } h > \frac{1}{2}. \end{cases} \tag{3.28}$$

With the help of Maple software, we find that the function $M(h)$ in (3.28) has three positive zeros $h_1 \in (0, 1/2)$ and $h_2, h_3 \in (1/2, +\infty)$ with

$$h_1 = \frac{2}{\pi^2} \approx 0.2026423672, \quad h_2 \approx 0.5623206104, \quad h_3 \approx 1.359655126.$$

See Figure 3.6 for the graphic of the piecewise function $M(h)$.

Furthermore, observe that

$$M'(h_1) = \frac{2\sqrt{2}}{\sqrt{h_1}} + 8\pi \neq 0, \quad M'(h_2) = \frac{8}{\sqrt{2h_2 - 1}} - \frac{2\sqrt{2}}{\sqrt{h_2}} + 8\pi \neq 0,$$
$$M'(h_3) = \frac{8}{\sqrt{2h_3 - 1}} - \frac{2\sqrt{2}}{\sqrt{h_3}} + 8\pi \neq 0.$$

Hence, system (3.25) admits three limit cycles.

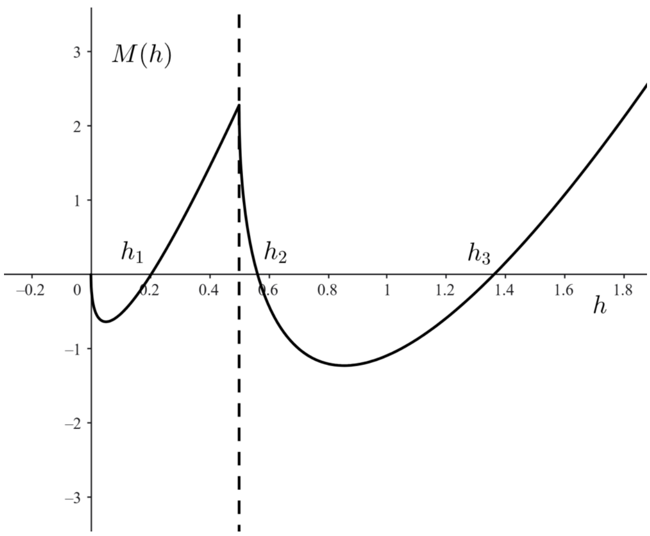


Figure 3.6: The graph of the piecewise function $M(h)$ in (3.28).

□

In the following, we consider the quadratic and cubic polynomials perturbation of a nonlinear center.

Proposition 3.7. *(Liu and Han [18], page 1386). Consider the piecewise system*

$$Z = \begin{cases} Y^+ = \begin{pmatrix} y + \varepsilon \sum\limits_{i+j=0}^{n} a_{ij} x^i y^j \\ -x + x^2 \end{pmatrix}, & \text{if } x > 0, \\ Y^- = \begin{pmatrix} y + \varepsilon \sum\limits_{i+j=0}^{n} b_{ij} x^i y^j \\ -x - x^2 - x^3 \end{pmatrix}, & \text{if } x \le 0, \end{cases} \tag{3.29}$$

for $\varepsilon > 0$ small. Then, for $n = 2$ or 3, system (3.29) *can have respectively at most six or nine limit cycles near the origin.*

Proof. For $\varepsilon = 0$, the Hamiltonian function associated with system (3.29) is defined piecewise as

$$H(x,y) = \begin{cases} H^+(x,y) = \dfrac{1}{2}(x^2+y^2) - \dfrac{1}{3}x^3, & \text{for } x > 0, \\ H^-(x,y) = \dfrac{1}{2}(x^2+y^2) + \dfrac{1}{3}x^3 + \dfrac{1}{4}x^4, & \text{for } x \le 0. \end{cases}$$

According to assumption (II), let $A(h) = (0, a(h))$ and $A_1(h) = (0, a_1(h))$ be the points where the orbit with energy h intersects the switching line $\Sigma = \{x = 0\}$. It follows that

$$H^+(A(h)) = h \implies \frac{(a(h))^2}{2} = h \implies a(h) = \sqrt{2h},$$

$$H^-(A_1(h)) = H^-(A(h)) \implies \frac{(a_1(h))^2}{2} = \frac{(a(h))^2}{2} \implies a_1(h) = -\sqrt{2h}.$$

So, $A = (0, \sqrt{2h})$ and $A_1 = (0, -\sqrt{2h})$. According to formula (3.14), we have

$$\begin{aligned} M(h,\delta) &= \left(\iint_{\mathrm{int}(\widehat{AA_1} \cup \overrightarrow{A_1A})} f_x^+ \, \mathrm{d}x\,\mathrm{d}y + \int_{\overrightarrow{A_1A}} f^+(0,y)\,\mathrm{d}y \right) \\ &\quad + \left(\iint_{\mathrm{int}(\widehat{A_1A} \cup \overrightarrow{AA_1})} f_x^- \, \mathrm{d}x\,\mathrm{d}y + \int_{\overrightarrow{AA_1}} f^-(0,y)\,\mathrm{d}y \right) \\ &= \left(\iint_{\mathrm{int}(\widehat{AA_1} \cup \overrightarrow{A_1A})} f_x^+ \, \mathrm{d}x\,\mathrm{d}y \right) + \left(\iint_{\mathrm{int}(\widehat{A_1A} \cup \overrightarrow{AA_1})} f_x^- \, \mathrm{d}x\,\mathrm{d}y \right) \\ &\quad + \left(\int_{\overrightarrow{A_1A}} f^+(0,y) - f^-(0,y)\,\mathrm{d}y \right) \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{3.30}$$

where

$$I_1 = \iint_{\mathrm{int}(\widehat{AA_1} \cup \overrightarrow{A_1A})} f_x^+ \, \mathrm{d}x\,\mathrm{d}y, \quad I_2 = \iint_{\mathrm{int}(\widehat{A_1A} \cup \overrightarrow{AA_1})} f_x^- \, \mathrm{d}x\,\mathrm{d}y, \quad I_3 = \int_{\overrightarrow{A_1A}} f^+(0,y) - f^-(0,y)\,\mathrm{d}y.$$

To describe the integration regions, we introduce the polar coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$. Thus, the level curves $H^{\pm}(x,y) = h$ can be expressed as

$$\frac{r^2}{2} - \frac{\cos^3\theta}{3} r^3 = h, \quad \cos\theta > 0, \tag{3.31}$$

and

$$\frac{r^2}{2} - \frac{\cos^3\theta}{3} r^3 + \frac{\cos^4\theta}{4} r^4 = h, \quad \cos\theta \le 0. \tag{3.32}$$

Statement 1. For $h > 0$ sufficiently small, equation (3.31) has a unique positive solution $r = r(\theta, h)$ for each $\theta \in (-\pi/2, \pi/2)$.

In fact, let $p(r) = -r^3\cos^3\theta/3 + r^2/2 - h$, where $h > 0$ and $\cos\theta > 0$. By Descartes' Rule of Signs (see Theorem A.13), the sequence of coefficients of $p(r)$, which is $(-,+,-)$, has two sign changes. Thus, $p(r)$ has either two or zero positive roots. For the negative roots, we examine $p(-r) = r^3\cos^3\theta/3 + r^2/2 - h$; the sequence $(+,+,-)$ has exactly one sign change, implying $p(r)$ has exactly one negative root. Consequently, the following cases for the roots of $p(r)$ are possible:

(i) One negative real root and two complex conjugate roots;

(ii) One negative real root and two positive real roots.

If case (i) were to occur, the discriminant $\Delta_1 = h\left(1-6h\cos^6\theta\right)/2$ of the cubic equation $p(r) = -r^3\cos^3\theta/3 + r^2/2 - h = 0$ would be negative. If $\Delta_1 < 0$, then since $h > 0$, we must have $1 - 6h\cos^6\theta < 0$, which implies $h > 1/(6\cos^6\theta)$. This contradicts the assumption that h is a small positive parameter. Thus, $\Delta_1 > 0$ for h sufficiently small, ensuring the existence of two distinct positive roots as described in case (ii).

Furthermore, the critical points of $p(r)$ are determined by

$$p'(r) = 0 \iff r - r^2\cos^3\theta = 0 \iff r(1 - r\cos^3\theta) = 0$$
$$\iff r = 0 \quad \text{or} \quad r = \frac{1}{\cos^3\theta}.$$

Evaluating the second derivative, we find $p''(r) = 1 - 2r\cos^3\theta$, which implies

$$p''(0) = 1 > 0 \quad \text{and} \quad p''\left(\frac{1}{\cos^3\theta}\right) = -1 < 0.$$

Hence, $r = 0$ is a local minimum and $r = 1/\cos^3\theta$ is a local maximum of p. Since $p(r)$ has two positive roots for h sufficiently small, at least one of them must lie in the interval $(0, 1/\cos^3\theta]$. This interval is closed on the right to account for the case where the two positive roots coincide (which occurs exactly when the discriminant Δ_1 vanishes).

From this, and applying the algorithm described in Appendix C, the positive solution r can be expressed as a power series in $m = \sqrt{2h}$:

$$r = m + e_2(\theta)m^2 + e_3(\theta)m^3 + \cdots + e_k(\theta)m^k + \cdots \triangleq \Phi_1(\theta, m), \tag{3.33}$$

where the first coefficients are given by

$$e_2(\theta) = \frac{1}{3}\cos^3\theta, \qquad e_3(\theta) = \frac{5}{18}\cos^6\theta, \qquad e_4(\theta) = \frac{8}{27}\cos^9\theta,$$
$$e_5(\theta) = \frac{77}{216}\cos^{12}\theta, \qquad e_6(\theta) = \frac{112}{243}\cos^{15}\theta, \qquad e_7(\theta) = \frac{2431}{3888}\cos^{18}\theta,$$
$$e_8(\theta) = \frac{935}{1024}\cos^{21}\theta, \qquad e_9(\theta) = \frac{1062347}{839808}\cos^{24}\theta.$$

Statement 2. For $h > 0$ sufficiently small, equation (3.32) has a unique positive solution $r = r(\theta, h)$ for each $\theta \in [\pi/2, 3\pi/2]$.

In fact, let $p(r) = r^4\cos^4\theta/4 + r^3\cos^3\theta/3 + r^2/2 - h$, where $h > 0$ and $\cos\theta \leq 0$. Since $\cos\theta \leq 0$, let $c = \cos\theta \leq 0$. The sequence of coefficients for $p(r)$ is $(c^4/4, c^3/3, 1/2, -h)$. Note that:

$$c^4 \geq 0, \quad c^3 \leq 0, \quad 1/2 > 0, \quad -h < 0.$$

The sequence of signs is $(+,-,+,-)$, which presents exactly three sign changes. By Descartes' Rule of Signs (see Theorem A.13), $p(r)$ has either three or one positive real roots. For negative roots, $p(-r)$ has signs $(+,+,+,-)$, yielding exactly one sign change and thus one negative real root. Consequently, the following cases for the roots of $p(r)$ are possible:

(i) One negative root and three positive roots;

(ii) One negative root, one positive root and two complex roots.

Furthermore, observe that the derivative of $p(r)$ is given by

$$p'(r) = r^3 \cos^4 \theta + r^2 \cos^3 \theta + r = r\left(r^2 \cos^4 \theta + r \cos^3 \theta + 1\right).$$

Let $q(r) = r^2 \cos^4 \theta + r \cos^3 \theta + 1$ and let Δ_2 denote its discriminant. We find that

$$\Delta_2 = (\cos^3 \theta)^2 - 4(\cos^4 \theta)(1) = \cos^4 \theta \left(\cos^2 \theta - 4\right).$$

Since $\cos^2 \theta \leq 1$ and $\cos^4 \theta > 0$ for $\theta \neq \pi/2, 3\pi/2$, it follows that $\Delta_2 < 0$. Consequently, the quadratic function $q(r)$ has no real roots, implying that $r = 0$ is the unique real critical point of $p(r)$. Since $p(0) = -h < 0$ and $p(r) \to \infty$ as $r \to \pm\infty$, it follows from the monotonicity of p on $(0,\infty)$ and $(-\infty,0)$ that p has exactly one positive real root and one negative real root. Thus, case (ii) is established.

From this, and applying the algorithm described in Appendix C, the positive solution r for the region $x \leq 0$ can be expressed as a power series in $m = \sqrt{2h}$:

$$r = m + s_2(\theta)m^2 + s_3(\theta)m^3 + \cdots + s_k(\theta)m^k + \cdots \triangleq \Phi_2(\theta, m), \tag{3.34}$$

where the first coefficients are given by:

$$\begin{aligned}
s_2(\theta) &= \frac{1}{3}\cos^3 \theta, \\
s_3(\theta) &= \frac{5}{18}\cos^6 \theta - \frac{1}{4}\cos^4 \theta, \\
s_4(\theta) &= \frac{8}{27}\cos^9 \theta - \frac{1}{2}\cos^7 \theta, \\
s_5(\theta) &= \frac{77}{216}\cos^{12} \theta - \frac{7}{8}\cos^{10} \theta + \frac{7}{32}\cos^8 \theta, \\
s_6(\theta) &= \frac{112}{243}\cos^{15} \theta - \frac{40}{27}\cos^{13} \theta + \frac{5}{6}\cos^{11} \theta, \\
s_7(\theta) &= \frac{2431}{3888}\cos^{18} \theta - \frac{715}{288}\cos^{16} \theta + \frac{143}{64}\cos^{14} \theta - \frac{33}{128}\cos^{12} \theta, \\
s_8(\theta) &= \frac{640}{729}\cos^{21} \theta - \frac{112}{27}\cos^{19} \theta + \frac{140}{27}\cos^{17} \theta - \frac{35}{24}\cos^{15} \theta, \\
s_9(\theta) &= \frac{1062347}{839808}\cos^{24} \theta - \frac{323323}{46656}\cos^{22} \theta + \frac{230945}{20736}\cos^{20} \theta - \frac{12155}{2304}\cos^{18} \theta + \frac{715}{2048}\cos^{16} \theta.
\end{aligned}$$

When $n = 3$, according to (3.30) and (3.31), the first integral is given by

$$
\begin{aligned}
I_1 &= \sum_{i+j=1, i\geq 1}^{3} i a_{ij} \iint_{\mathrm{int}\left(\widehat{AA_1}\cup\overrightarrow{A_1A}\right)} x^{i-1} y^j \,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{i+j=1, i\geq 1}^{3} i a_{ij} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^{\Phi_1(\theta,m)} |r|\cdot r^{i-1}\cos^{i-1}\theta \cdot r^j \sin^j\theta \,\mathrm{d}r\right] \mathrm{d}\theta \\
&= \sum_{i+j=1, i\geq 1}^{3} i a_{ij} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{i-1}\theta \sin^j\theta \left[\int_0^{\Phi_1(\theta,m)} r^{i+j} \,\mathrm{d}r\right] \mathrm{d}\theta \\
&= \frac{a_{10}}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Phi_1^2(\theta,m)\,\mathrm{d}\theta + \frac{2a_{20}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\Phi_1^3(m,\theta)\,\mathrm{d}\theta + \frac{a_{11}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\Phi_1^3(\theta,m)\,\mathrm{d}\theta \\
&\quad + \frac{a_{12}}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\Phi_1^4(\theta,m)\,\mathrm{d}\theta + \frac{2a_{21}}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\sin\theta\Phi_1^4(\theta,m)\,\mathrm{d}\theta \\
&\quad + \frac{3a_{30}}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\Phi_1^4(\theta,m)\,\mathrm{d}\theta.
\end{aligned}
$$

Since $\Phi_1(\theta,m)$ is an even function with respect to θ, it follows that $\Phi_1^3(\theta,m)$ and $\cos\theta\Phi_1^4(\theta,m)$ are also even. Consequently, the functions $\sin\theta\Phi_1^3(\theta,m)$ and $\cos\theta\sin\theta\Phi_1^4(\theta,m)$ are odd. Their integrals over the symmetric interval $[-\pi/2,\pi/2]$ vanish, leading to

$$
\begin{aligned}
I_1 = &\frac{a_{10}}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Phi_1^2(\theta,m)\,\mathrm{d}\theta + \frac{2a_{20}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\Phi_1^3(\theta,m)\,\mathrm{d}\theta \\
&+ \frac{a_{12}}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\Phi_1^4(\theta,m)\,\mathrm{d}\theta + \frac{3a_{30}}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\Phi_1^4(\theta,m)\,\mathrm{d}\theta.
\end{aligned}
$$

Similarly, according to (3.30) and (3.32), we obtain

$$
\begin{aligned}
I_2 &= \sum_{i+j=1, i\geq 1}^{3} \frac{i b_{ij}}{i+j+1}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^{i-1}\theta\sin^j\theta\Phi_2^{i+j+1}(\theta,m)\,\mathrm{d}\theta \\
&= \frac{b_{10}}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \Phi_2^2(\theta,m)\,\mathrm{d}\theta + \frac{2b_{20}}{3}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta\Phi_2^3(\theta,m)\,\mathrm{d}\theta \\
&\quad + \frac{b_{12}}{4}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin^2\theta\Phi_2^4(\theta,m)\,\mathrm{d}\theta + \frac{3b_{30}}{4}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2\theta\Phi_2^4(\theta,m)\,\mathrm{d}\theta,
\end{aligned}
$$

where we have again applied the parity properties of $\Phi_2(\theta,m)$ and the trigonometric functions over the symmetric interval $[\pi/2, 3\pi/2]$ with respect to the horizontal axis.

Furthermore, the line integral I_3 along the switching line segment $\overline{A_1A}$ is given by

$$
I_3 = \sum_{j=0}^{3}(a_{0j}-b_{0j})\int_{-m}^{m} y^j\,\mathrm{d}y = \sum_{j=0}^{3}\frac{c_j[1-(-1)^{j+1}]}{j+1}m^{j+1} = 2c_0 m + \frac{2c_2}{3}m^3,
$$

where $c_j = a_{0j} - b_{0j}$ and $m = \sqrt{2h}$ represents the intersection points $a(h)$ and $a_1(h)$.

Hence, by applying the algorithm described in Appendix C and taking $m=\sqrt{2h}$, the Melnikov function $M(h,\delta)$ is expanded as:

$$M(h,\delta)=\sqrt{h}\left[B_0(\delta)+B_1(\delta)\sqrt{h}+\cdots+B_9(\delta)\left(\sqrt{h}\right)^9+O((\sqrt{h})^{10})\right],\quad 0<h\ll 1, \tag{3.35}$$

where the coefficients $B_k(\delta)$ are given by:

$$
\begin{aligned}
B_0(\delta)=&2\sqrt{2}c_0,\\
B_1(\delta)=&(a_{10}+b_{10})\pi,\\
B_2(\delta)=&2\sqrt{2}\left[\frac{4}{9}(a_{10}-b_{10})+\frac{4}{3}(a_{20}-b_{20})+\frac{2}{3}c_2\right],\\
B_3(\delta)=&4\left[\frac{5}{48}a_{10}\pi+\frac{1}{4}(a_{20}+b_{20})\pi+\frac{1}{96}b_{10}\pi+\frac{1}{8}(a_{12}+b_{12})\pi+\frac{3}{8}(a_{30}+b_{30})\pi\right],\\
B_4(\delta)=&4\sqrt{2}\left[\frac{128}{405}a_{10}+\frac{32}{45}a_{20}+\frac{4}{45}(a_{12}-b_{12})+\frac{16}{15}(a_{30}-b_{30})+\frac{88}{405}b_{10}-\frac{8}{45}b_{20}\right],\\
B_5(\delta)=&8\Bigg[\frac{385}{3456}a_{10}\pi+\frac{35}{144}a_{20}\pi-\frac{1295}{13824}b_{10}\pi-\frac{35}{288}b_{20}\pi+\frac{5}{288}a_{12}\pi+\frac{35}{96}a_{30}\pi+\frac{1}{576}b_{12}\pi\\
&+\frac{25}{192}b_{30}\pi\Bigg],\\
B_6(\delta)=&8\sqrt{2}\left[\frac{512}{1215}a_{10}+\frac{512}{567}a_{20}+\frac{208}{1215}b_{10}+\frac{272}{405}b_{20}+\frac{128}{2835}a_{12}+\frac{256}{189}a_{30}+\frac{88}{2835}b_{12}+\frac{64}{135}b_{30}\right],\\
B_7(\delta)=&16\Bigg[\frac{85085}{497664}a_{10}\pi+\frac{5005}{13824}a_{20}\pi+\frac{129745}{3981312}b_{10}\pi-\frac{8855}{55296}b_{20}\pi+\frac{385}{27648}a_{12}\pi\\
&+\frac{5005}{9216}a_{30}\pi-\frac{1295}{110592}b_{12}\pi-\frac{13055}{36864}b_{30}\pi\Bigg],\\
B_8(\delta)=&16\sqrt{2}\Bigg[\frac{32768}{45927}a_{10}+\frac{16384}{10935}a_{20}+\frac{512}{10935}a_{12}+\frac{8192}{3645}a_{30}-\frac{85216}{229635}b_{10}\\
&-\frac{8608}{76545}b_{20}+\frac{208}{10935}b_{12}+\frac{26176}{25515}b_{30}\Bigg],\\
B_9(\delta)=&32\Bigg[\frac{7436429}{23887872}a_{10}\pi+\frac{323323}{497664}a_{20}\pi+\frac{17017}{995328}a_{12}\pi+\frac{323323}{331776}a_{30}\pi\\
&+\frac{51352301}{382205952}b_{10}\pi+\frac{1082081}{3981312}b_{20}\pi+\frac{25949}{7981312}b_{20}\pi+\frac{25949}{7962624}b_{12}\pi-\frac{35035}{2654208}b_{30\pi}\Bigg],
\end{aligned}
$$

For the parameter vector $\delta=(a_{10},b_{10},a_{20},b_{20},a_{12},b_{12},a_{30},b_{30},c_0,c_2)\in\mathbb{R}^{10}$, we verify that:

$$\operatorname{rank}\left(\frac{\partial(B_0,B_1,B_2,B_3,B_4,B_5,B_6,B_7,B_8)}{\partial(a_{10},b_{10},a_{20},b_{20},a_{12},b_{12},a_{30},b_{30},c_0,c_2)}\right)=9.$$

Thus, by Theorem 3.4, the Melnikov function $M(h,\delta)$ can have at most nine simple zeros for $0<h\ll 1$ given an appropriate choice of δ near the origin.

Specifically, in the case where $n=2$, we consider the parameter vector $\delta=(a_{10},b_{10},a_{20},b_{20},c_0,c_2)$. In this setting, direct computation yields:

$$\operatorname{rank}\left(\frac{\partial(B_0,B_1,B_2,B_3,B_4,B_5)}{\partial(a_{10},b_{10},a_{20},b_{20},c_0,c_2)}\right)=6.$$

Therefore, by Theorem 3.4, system (3.29) can undergo a bifurcation of at most six limit cycles from the period annulus in a neighborhood of the origin. □

CHAPTER 4

Conclusion

Throughout this dissertation, we have seen that some concepts and results from piecewise systems are similar to those of continuous dynamical systems. Likewise, the two methods studied for computing limit cycles, the Lyapunov constants and the Melnikov functions, also have similarities when applied to continuous systems.

Lyapunov constants are applied to perturbed systems that have a weak focus. Since we want to verify the existence of limit cycles, first we applied the blow-up method to the singularity. After that, we analyze the coefficients associated with the series expansion of the first return map, or Poincaré map. Moreover, we saw that Lyapunov constants provide an upper bound for the number of limit cycles that can bifurcate from a weak focus, making it necessary to verify the fixed points of the first return map.

Similarly, Melnikov functions are used to obtain an upper bound on the number of limit cycles. One way to verify the exact number of limit cycles is to check whether a zero of the Melnikov function is simple, that is, whether its first derivative does not vanish at that zero. In addition, there is an interesting difference compared to the previous method: this method is applied, in particular, to perturbations of periodic orbits in dynamical systems.

Therefore, the two methods have both similarities and differences and should be applied depending on the system under consideration. Although we were unable to find the first integral of one of the cases, we believe that the results developed and the algorithm implemented in Maple were of great value to this dissertation.

APPENDIX A

Results of Analysis, Calculus and Algebra

In this appendix, we compile the essential mathematical definitions and theorems presented throughout this work. As these are standard results in the literature, we shall omit their proofs, limiting ourselves to precise statements and providing the corresponding references for the interested reader.

A.1 Basic Analysis and Calculus

Theorem A.1. *(Lima [17], page 185) (Rolle's Theorem). Let $f : [a,b] \to \mathbb{R}$ continuous, such that $f(a) = f(b)$. If f is differentiable at (a,b), then there exists a point $c \in (a,b)$ where $f'(c) = 0$.*

Theorem A.2. *(Rudin [20], page 108) (Mean Value Theorem). If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which*

$$f(b) - f(a) = (b-a)f'(x).$$

Theorem A.3. *(Rudin [20], page 134) (Fundamental Theorem of Calculus). If a function $f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

A.2 Multivariable Calculus

Theorem A.4. *(Spivak [21], page 35) (Inverse Function Theorem). Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing x_0, and $\det f'(x_0) \neq 0$. Then, there are an open set U containing x_0 and an open set V containing $f(x_0)$ such that $f : U \to V$ has a continuous inverse $f^{-1} : V \to U$ which is differentiable, and for all $y \in V$ satisfies*

$$\left(f^{-1}\right)'(y) = \left[f'\left(f^{-1}(y)\right)\right]^{-1}.$$

Theorem A.5. *(Spivak [21], page 42) (Implicit Function Theorem). Let* $f : \mathbb{R}^k \oplus \mathbb{R}^m \to \mathbb{R}^m$ *a function of class k. If* $x_0 \in \mathbb{R}^k$ *and* $y_0 \in \mathbb{R}^m$*, suppose that* $f(x_0, y_0) = 0$ *and* $\det[f'(x_0, y_0)] \neq 0$*. Then, there exist an open set* $U \in \mathbb{R}^k$ *containing* x_0*, an open set* $V \in \mathbb{R}^m$ *containing* y_0 *and a function* $g : U \to V$ *of class* C^k *such that*

(i) For all $x_0 \in W$*, there exists a unique* $y = g(x)$ *such that* $f(x, g(x)) = 0$*, for all* $x \in W$*;*

(ii) The function g is differentiable.

Theorem A.6. *(Spivak [21], page 59) (Fubini's Theorem). Let* $U \subset \mathbb{R}^n$ *and* $V \subset \mathbb{R}^m$ *be closed rectangles, and let* $f : U \times V \to \mathbb{R}$ *be integrable. For* $x \in U$*, let* $g_x : V \to \mathbb{R}$ *be defined by* $g_x(y) = f(x, y)$ *and let*

$$L(x) = \underline{\int_V} g_x = \underline{\int_V} f(x,y)\ \mathrm{d}y,$$
$$R(x) = \overline{\int_V} g_x = \overline{\int_V} f(x,y)\ \mathrm{d}y.$$

Then L and R are integrable on U and

$$\int_{U\times V} f = \int_U L = \int_U \left(\underline{\int_V} f(x,y)\ \mathrm{d}y \right)\ \mathrm{d}x,$$
$$\int_{U\times V} f = \int_U R = \int_U \left(\overline{\int_V} f(x,y)\ \mathrm{d}y \right)\ \mathrm{d}x,$$

Theorem A.7. *(Spivak [21], page 67) (Change Variable Theorem). Let* $U \subset \mathbb{R}^n$ *be an open set and* $g : U \to \mathbb{R}^n$ *a bijective and continuously differentiable function such that* $\det g'(x) \neq 0$*, for all* $x \in U$*. If* $f : g(U) \to \mathbb{R}$ *is integrable, then*

$$\int_{g(U)} f(y)\ \mathrm{d}y = \int_U (f \circ g(x))\, |\det g'(x)|\ \mathrm{d}x.$$

A.3 Differential forms and integration on manifolds

Proposition A.8. *(Properties of the Wedge Product on* $\mathbb{R}^n$*). Let* $\omega, \omega' \in \Lambda^k(\mathbb{R}^n)$*,* $\eta, \eta' \in \Lambda^l(\mathbb{R}^n)$ *and* $a, a' \in \mathbb{R}$*.*

(i) $(a\omega + \omega a') \wedge \eta = a\,(\omega \wedge \eta) + a'\,(\omega' \wedge \eta)$*;*
$\eta\,(a\omega + a'\omega') = a\,(\eta \wedge \omega) + a'\,(\eta \wedge \omega')$*;*

(ii) $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$*.*

Proof. See Lee [16, page 357] and apply the Theorem to the case $V = (\mathbb{R}^n, +, \cdot)$, where V is a finite-dimension vector field.

Definition A.9. *Let $\omega \in \Lambda^k(\mathbb{R}^n)$, then*

$$\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1,\ldots,i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

The ***differential*** *of ω is a $k+1$-form $\mathrm{d}\omega$ given by*

$$\begin{aligned}\mathrm{d}\omega &= \sum_{i_1<\cdots<i_k} \mathrm{d}\omega_{i_1,\ldots,i_k} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\ &= \sum_{i_1<\cdots<i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1,\ldots,i_k}) \cdot \mathrm{d}x^\alpha \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.\end{aligned}$$

Proposition A.10. *(Spivak [21], page 92) (Properties of the Exterior Derivative on $\mathbb{R}^n$). Let $\omega \in \Lambda^k(\mathbb{R}^n)$ and let $\eta \in \Lambda^l(\mathbb{R}^n)$.*

(i) d *is linear over* $\mathbb{R}$*;*

(ii) $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$*;*

(iii) $\mathrm{d} \circ \mathrm{d} \equiv 0$.

Theorem A.11. *(Lee [16], page 412) (Stokes's Theorem). Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M. Then*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Theorem A.12. *(Lee [16], page 415) (Green's Theorem). Suppose that D is a compact regular domain in $\mathbb{R}^2$, and P, Q are smooth real-valued functions on D. Then*

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y = \int_{\partial D} P\, \mathrm{d}x + Q\, \mathrm{d}y.$$

A.4 Miscellaneous results

Theorem A.13. *(Wang [22], page 526) (Descarte's Rule of Signs). Let $p(x) = a_0 x^{b_0} + a_1 x^{b_1} + \cdots + a_n x^{b_n}$ denote a polynomial with nonzero real coefficients a_i, where b_i are integers satisfying $0 \leq b_0 < b_1 < b_2 < \cdots < b_n$. Then the number of positive real zeros of $p(x)$ (counted with multiplicities) is either equal to the number of variations in sing in the sequence $a_0, \ldots, a_n$ of the coefficients or less than that by an even whole number. The number of negative zeros of $p(x)$ (counted with multiplicities) is either equal to the number of variations in sign in the sequence of the coefficients of $p(-x)$ or less than that by an even whole number.*

Definition A.14. *(Conway [7], page 103). A function f has an* ***isolated singularity*** *at $z = x_0$ if there is an $r > 0$ such that f is defined and analytic in $B(x_0, r) - \{x_0\}$ but not in $B(x_0, r)$. The point x_0 is called a* ***removable singularity*** *if there is an analytic function $g : B(x_0, r) \to \mathbb{R}$ such that $g(z) = f(z)$, for $0 < |z - x_0| < r$.*

APPENDIX B

Method of computation of the Lyapunov constants in Maple

```
    restart

    with(Groebner): #Package for calculating Gröbner bases
    with(combinat): #Package contains combinatorial functions

    #Non-linear part of system
    N := 7;
    Ap := p02*y^2 + p11*x*y + p20*x^2;
    Bp := q02*y^2 + q11*x*y + q20*x^2;
    Am := 0;
    Bm := 0;

    #Variables
    xp := -y + Ap;
    yp := x + Bp;
    xm := -y + Am;
    ym := x + Bm;

    #Polar coordinate change
      #The positive system in polar coordinates
      Rp := simplify(subs([x = r*cos(theta), y = r*sin(theta)], (x*xp + y*yp)/
 r)):
      Tp := simplify(subs([x = r*cos(theta), y = r*sin(theta)], (x*yp - xp*y)/
 r^2)):

      #The negative system in polar coordinates
      Rm := simplify(subs([x = r*cos(theta), y = r*sin(theta)], (x*xm + y*ym)/
 r)):
      Tm := simplify(subs([x = r*cos(theta), y = r*sin(theta)], (x*ym - xm*y)/
 r^2)):

      #Removing the dependence on t
      dRpdTp := convert(simplify(series(Rp/Tp, r, N + 1)), polynom):
      dRmdTm := convert(simplify(series(Rm/Tm, r, N + 1)), polynom):

    #Series of r(theta) with coefficient p[1]..p[N+1]

```

```
     #Positive serie
     #In this case, we will denote p^+[i] by pp[i]
     fp := rho + sum(pp[i](theta)*rho^i, i = 2 .. N + 1):
     dfp := diff(fp, theta) - mtaylor(subs(r = fp, dRpdTp), rho, N + 1):

     for j to N do
       print(j);
       simplify(coeff(dfp, rho, j));
       subs(diff(pp[j](theta), theta) = dpp, %);
       pp[j] := unapply(subs(s = theta, int(solve(%, dpp), theta = 0 .. s)), theta);
       Lp[j] := pp[j](Pi);
     end do:

     #Negative serie
     #In this case, we will denote p^-[i] by pm[i]
     fm := rho + sum(pm[i](theta)*rho^i, i = 2 .. N + 1):
     dfm := diff(fm, theta) - mtaylor(subs(r = fm, dRmdTm), rho, N + 1):

     for j to N do
       print(j);
       simplify(coeff(dfm, rho, j));
       subs(diff(pm[j](theta), theta) = dpm, %);
       pm[j] := unapply(subs(s = theta, int(solve(%, dpm), theta = 0 .. s)), theta);
       Lm[j] := pm[j](Pi);
     end do:

   #Calculation of simplified Lp[j] and Lm[j] coefficients
     #Positive case
     Vp1:=Lp[1]:
     Vp2:=simplify(Lp[2],[Vp1]):
     for l from 3 to N do
       Vp[l] := simplify(Lp[l], [seq(Vp[k], k = 2..l-1)]):  # Vp[l] depends of Vp[2], Vp[3], ..., Vp[l-1]
     end do:

     #Negative case
     Vm1:=Lm[1]:
     Vm2:=simplify(Lm[2],[Vm1]):
     for l from 3 to N do
       Vm[l] := simplify(Lm[l], [seq(Vm[k], k = 2..l-1)]):  # Vm[l] depends of Vm[2], Vm[3], ..., Vm[l-1]
     end do:

   #Calculation of Lyapunov coefficients
     V1 := 0;
     V2 := 0;
     for l from 3 to N do
     V[l] := Vp[l] + Vm[l];
     end do;
```

APPENDIX C

Larger expressions of Proposition (3.7) in Maple

```
#To calculate the coefficients e_i(theta)
restart
with (Linear Algebra):

n := 10:
H := r^2/2 - cos(theta)^3/3*r^3 - m^2/2:
phi := m:
for i from 2 to n do
phi := phi + e[i]*m^i:
od:
temp := subs(r = phi, H):
for i from 3 to n do
temp1 := subs(m = 0, diff(temp, m $ i)/i!):
e[i - 1] := solve(temp1, e[i - 1]):
print(i - 1, e[i - 1]);
od:

```

```
#To calculate the coefficients s_i(theta)
restart
with (Linear Algebra):

n := 10:
H := r^2/2 - cos(theta)^3/3*r^3 + cos(theta)^4/4*r^4 - m^2/2:
phi := m:
for i from 2 to n do
phi := phi + s[i]*m^i:
od:
temp := subs(r = phi, H):
for i from 3 to n do
temp1 := subs(m = 0, diff(temp, m $ i)/i!):
s[i - 1] := solve(temp1, s[i - 1]):
print(i - 1, s[i - 1]);
od:

```

```
#Values of sin and cos
```

```
   sin((3*pi)/2) := -1;
   sin(pi/2) := 1;
   cos(pi/2) := 0;
   cos((3*pi)/2) := 0;

   #To calculate the constants c
   f := m^9*e[9] + m^8*e[8] + m^7*e[7] + m^6*e[6] + m^5*e[5] + m^4*e[4] + m^3*e[3] + m^2*e[2] + m:
   g1 := simplify(a10/2*int(f^2, theta = -pi/2 .. pi/2)):
   g3 := simplify((2*a20)/3*int(cos(theta)*f^3, theta = -pi/2 .. pi/2)):
   g4 := simplify(a12/4*int(sin(theta)^2*f^4, theta = -pi/2 .. pi/2)):
   g6 := simplify((3*a30)/4*int(cos(theta)^2*f^4, theta = -pi/2 .. pi/2)):
   c1 := coeff(g1, m^2) + coeff(g3, m^2) + coeff(g4, m^2) + coeff(g6, m^2):
   c2 := coeff(g1, m^3) + coeff(g3, m^3) + coeff(g4, m^3) + coeff(g6, m^3):
   c3 := coeff(g1, m^4) + coeff(g3, m^4) + coeff(g4, m^4) + coeff(g6, m^4):
   c4 := coeff(g1, m^5) + coeff(g3, m^5) + coeff(g4, m^5) + coeff(g6, m^5):
   c5 := coeff(g1, m^6) + coeff(g3, m^6) + coeff(g4, m^6) + coeff(g6, m^6):
   c6 := coeff(g1, m^7) + coeff(g3, m^7) + coeff(g4, m^7) + coeff(g6, m^7):
   c7 := coeff(g1, m^8) + coeff(g3, m^8) + coeff(g4, m^8) + coeff(g6, m^8):
   c8 := coeff(g1, m^9) + coeff(g3, m^9) + coeff(g4, m^9) + coeff(g6, m^9):
   c9 := coeff(g1, m^10) + coeff(g3, m^10) + coeff(g4, m^10) + coeff(g6, m^10):

   #To calculate the constants d
   f1 := m^9*s[9] + m^8*s[8] + m^7*s[7] + m^6*s[6] + m^5*s[5] + m^4*s[4] + m^3*s[3] + m^2*s[2] + m:
   h1 := simplify(b10/2*int(f1^2, theta = pi/2 .. (3*pi)/2)):
   h3 := simplify((2*b20)/3*int(cos(theta)*f1^3, theta = pi/2 .. (3*pi)/2)):
   h4 := simplify(b12/4*int(sin(theta)^2*f1^4, theta = pi/2 .. (3*pi)/2)):
   h6 := simplify((3*b30)/4*int(cos(theta)^2*f1^4, theta = pi/2 .. (3*pi)/2)):
   d1 := coeff(h1, m^2) + coeff(h3, m^2) + coeff(h4, m^2) + coeff(h6, m^2):
   d2 := coeff(h1, m^3) + coeff(h3, m^3) + coeff(h4, m^3) + coeff(h6, m^3):
   d3 := coeff(h1, m^4) + coeff(h3, m^4) + coeff(h4, m^4) + coeff(h6, m^4):
   d4 := coeff(h1, m^5) + coeff(h3, m^5) + coeff(h4, m^5) + coeff(h6, m^5):
   d5 := coeff(h1, m^6) + coeff(h3, m^6) + coeff(h4, m^6) + coeff(h6, m^6):
   d6 := coeff(h1, m^7) + coeff(h3, m^7) + coeff(h4, m^7) + coeff(h6, m^7):
   d7 := coeff(h1, m^8) + coeff(h3, m^8) + coeff(h4, m^8) + coeff(h6, m^8):
   d8 := coeff(h1, m^9) + coeff(h3, m^9) + coeff(h4, m^9) + coeff(h6, m^9):
   d9 := coeff(h1, m^10) + coeff(h3, m^10) + coeff(h4, m^10) + coeff(h6, m^10):

   factor(c1+d1);
   factor(c2+d2);
   factor(c3+d3);
   factor(c4+d4);
   factor(c5+d5);
   factor(c6+d6);
   factor(c7+d7);
   factor(c8+d8);
   factor(c9+d9);
```

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