



UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Persistent Leray's spectral sequence and separation property for generalized manifold

Telmo Irineo Acosta Vellozo

São Carlos-SP
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Resumo

Nesta tese trabalhamos em dois temas diferentes. Por um lado, conseguimos construir uma sequência espectral que calcula a cohomologia persistente de um espaço a partir da cohomologia persistente em cada aberto e suas interseções de uma cobertura que é a pré-imagem por uma aplicação, de uma cobertura, de um espaço conhecido.

Por outro lado, obtivemos resultados de separação por aplicações de codimensão-1 para variedades generalizadas. Mais especificamente, provamos resultados que nos permitem estimar o número de componentes do complemento da imagem de uma aplicação.

Palavras-chave: Cohomologia persistente, aplicação de codimensão-1, separação, sequência espectral, variedades generalizadas.

Abstract

In this thesis we work on two different topics. On the one hand, we manage to construct a spectral sequence that calculates the persistent cohomology of a space from the persistent cohomology in each open and its interceptions of a covering that is the pre-image by a function of a covering of a known space.

On the other hand, we achieved separation results by codimension-1 maps to generalized manifold. More specifically, we proved results that allow us to estimate the number of related components of the image complement of the map.

Keywords: Codimension-1 maps, generalized manifold, persistent cohomology, separation, spectral sequence.

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Introduction

In this thesis we focus on two different topics, one on the application of algebraic topology, which is persistent cohomology, and the other on the separation property for generalized manifold.

Next we are going to tell a little about the evolution of the relationship between persistent homology and spectral sequences, to motivate our approach. Given a filtration of a space, on the one hand we have an spectral sequence associated and on the other hand we have the persistent homology associated, in [33], Zomorodian and Gunnar noted that these are related. In [17], Eldelsbrunner and Harer developed an algorithm using this relation to calculate persistent homology called Spectral Sequence Algorithm. In [1], Bauer, Kerber and Reinnghaus implemented algorithm for computing persistent homology in a distributed memory environment using the relation between spectral sequence and persistent homology, that was faster of those already existing. After, in [31], Torras implemented an algorithm using the Mayer-Vietoris spectral sequences to calculate persistent homology, this type of spectral sequental is constructed from a partition of space and the homology in each subset of this partition.

On the other hand, persistent cohomology has been developed, with faster algorithms and obtaining more information than persistent homology (see in [15]).

In the first part of this thesis we achieved to show that the persistent cohomology of a filtered finite dimensional simplicial complex is a graded module over a polynomial ring, like Zomorodian and Gunnar had done to persistent homology in [33], then applying the Leray's spectral sequence to the sheaf of this graded module over a cover of a known space, we get Leray's Theorem for Persistence Module (Theorem2.2.2).

The second part of this thesis is focused on obtaining results for generalized manifold through extension of the results obtained to smooth manifold and topological manifold. Given a continuous map $f : M \rightarrow N$ between generalized manifolds, is often an important problem to study the topology of the complement $N - f(M)$ of $f(M)$ in N . In this thesis, we consider the codimension 1 case; i.e. the case where $\dim N - \dim M = 1$, and study the number of connect components or path connect components of $N - f(M)$. Such a problem was first considered by Vaccaro in [32], who found a PL-immersed S^2 in \mathbb{R}^3 whose complement is connected. On the other hand, it is known by Jordan-Brouwer Theorem that the number of connected components of the complement of an embedded- $n - 1$ -sphere in \mathbb{R}^n is equal 2. An other important result in this way was obtained by Feighn in [19],

taking that if $H_1(N; \mathbb{Z}_2) = 0$, then every proper C^2 -immersion $f : M \rightarrow N$ disconnect N where M and N are smooth manifold.

For topological manifold there are many results with different approach. In [26], Nuño Ballesteros and Romero Fuster showed that, if $H_1(N; \mathbb{Z}_2) = 0$, then every $f : M \rightarrow N$ proper continuous map whose $A(f) = \{x \in M : f^{-1}f(x) \neq x\}$ the self-intersection set is not dense in any component connect of M , the complement $N - f(M)$ is disconnected. In [25], Nuño Ballesteros with the same hypothesis as in [26] and $N - f(A)$ connected, gave a formula for the number of connect components of $N - f(M)$ in terms of the Cech cohomology. An other result in to estimate the number of connected components in $N - f(M)$ was given in [29] by Saeki, he showed that, if $H_1(N; \mathbb{Z})$ is a torsion group or $f_*[M] \in H_{n-1}(N; \mathbb{Z})$ is a torsion element, then the connect component of $N - f(M)$ is estimate for below with the multiplicity of the normal crossing point. Other results using normal crossing point for immersions was obtaining in [7][8][9]. In [5], Biasi and Saeki got a result for smooth manifold using normal crossing point and primary obstruction to topological embeddings.

In this second part of this thesis we present three results of separation by codimension-1 map in generalized manifold. The first one, Theorem 3.2.1 is an extension of the result of Ballesteros in [25] and as a result of this we got a version of Jordan-Brouwer Theorem to generalized manifold, Corollary 3.2.1. The second one, Theorem 3.3.2, in this theorem we adapt the arguments in the proof of Saeki in [29], obtaining a result similar to generalized manifold. And the third one, Theorem 3.4.2 uses primary obstruction to topological embeddings, in this theorem we adapt the arguments in the proof of Biasi and Saeki in [5], obtaining a result similar to generalized manifold.

This thesis has three chapters. The first is the background, where you will find the necessary requirements to understand the other chapters. The other two chapters are completely independent. The second is about persistent cohomology and persistent Leray's spectral sequence. The third is about separation properties for codimension-1 maps. In all this work map means continuous map.

Background

In this first section we adapt what has been developed in [10] for differential forms to our needs.

1.1 Mayer-Vietoris Principle

1.1.1 Mayer-Vietoris Principle to countably many open sets

Instead of a cover with two open sets as in the usual Mayer-Vietoris sequence, consider the open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of the space M , where the index set J is a countable ordered set. Denote the intersections $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ by $U_{\alpha_0 \alpha_1 \dots \alpha_n}$. There is a sequence of inclusions of open sets

$$M \longleftarrow \bigsqcup_{\alpha_i} U_{\alpha_i} \xleftarrow{(\partial_i, \partial_j)} \bigsqcup_{\alpha_i < \alpha_j} U_{\alpha_i \alpha_j} \xleftarrow{(\partial_i, \partial_j, \partial_k)} \bigsqcup_{\alpha_i < \alpha_j < \alpha_k} U_{\alpha_i \alpha_j \alpha_k} \longleftarrow \cdots$$

where \bigsqcup is disjoint union and ∂_i is the inclusion with "ignores" the i th open set; for example,

$$\partial_0 : U_{\alpha_0 \alpha_1 \alpha_2} \hookrightarrow U_{\alpha_1 \alpha_2}.$$

This sequence of inclusions of open sets induces a sequence of restrictions of cochain complexes

$$C^*(M) \xrightarrow{r} \prod C^*(U_{\alpha_i}) \xrightarrow{(\delta_i, \delta_j)} \prod_{\alpha_i < \alpha_j} C^*(U_{\alpha_i \alpha_j}) \xrightarrow{(\delta_{\alpha_i}, \delta_{\alpha_j}, \delta_{\alpha_k})} \prod_{\alpha_i < \alpha_j < \alpha_k} C^*(U_{\alpha_i \alpha_j \alpha_k}) \longrightarrow \cdots$$

where δ_0 , for instance, is induced from the inclusion

$$\partial_0 : \bigsqcup_{\alpha_0} U_{\alpha_0 \alpha_1 \dots \alpha_n} \longrightarrow U_{\alpha_1 \dots \alpha_n}$$

and therefore is the restriction

$$\delta_0 : C^*(U_{\alpha_1 \dots \alpha_n}) \longrightarrow \prod_{\alpha_0} C^*(U_{\alpha_0 \alpha_1 \dots \alpha_n}).$$

We define the difference operator $\delta : \prod C^*(U_{\alpha_0 \alpha_1 \dots \alpha_{n-1}}) \longrightarrow \prod C^*(U_{\alpha_0 \alpha_1 \dots \alpha_n})$ to be the alternating difference $\delta_0 - \delta_1 + \cdots + (-1)^n \delta_n$.

Definition 1.1.1. If $w \in \prod C^q(U_{\alpha_0 \dots \alpha_p})$, then w has "components" $w_{\alpha_0 \dots \alpha_p} \in C^q(U_{\alpha_0 \dots \alpha_p})$ and

$$(\delta w)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i w_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}},$$

where on the right-hand side the restriction operation to $U_{\alpha_0 \dots \alpha_{p+1}}$ has been suppressed and caret denotes omission.

Proposition 1.1.1. $\delta^2 = 0$

Proof. Basically this is true because in $(\delta^2 w)_{\alpha_0 \dots \alpha_{p+1}}$ we omit two indices α_i, α_j twice with the opposite signs. To be precise,

$$\begin{aligned} (\delta^2 w)_{\alpha_0 \dots \alpha_{p+2}} &= \sum (-1)^i (\delta w)_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+2}} = \\ &= \sum_{j < i} (-1)^i (-1)^j w_{\alpha_0 \dots \hat{\alpha}_j \dots \hat{\alpha}_i \dots \alpha_{p+2}} + \sum_{j > i} (-1)^i (-1)^{j-1} w_{\alpha_0 \dots \hat{\alpha}_i \dots \hat{\alpha}_j \dots \alpha_{p+2}} = 0. \end{aligned}$$

□

Convention: Up until now the indices in $w_{\alpha_0 \dots \alpha_p}$ are all in increasing order $\alpha_0 < \dots < \alpha_p$. More generally we will allow indices in any order, even with repetitions, subject to the convention that when two indices are interchanged, the form becomes its negative:

$$w_{\dots \alpha \dots \beta \dots} = -w_{\dots \beta \dots \alpha \dots}$$

We need to check that this convention is consistent with the definition of the difference operator δ above, that is to say: $(\delta w)_{\dots \alpha_r \dots \alpha_s \dots} = -(\delta w)_{\dots \alpha_s \dots \alpha_r \dots}$. Basically this is true because: suppose $r < s$, then $\sum_{i < r} (-1)^i w_{\dots \hat{\alpha}_i \dots \alpha_r \dots \alpha_s \dots} = -\sum_{i < r} (-1)^i w_{\dots \hat{\alpha}_i \dots \alpha_s \dots \alpha_r \dots}$ because α_r and α_s are interchanged, the same happens when $r < i < s$ and $i > s$; $w_{\dots \hat{\alpha}_r \dots \alpha_s \dots} = -w_{\dots \alpha_s \dots \hat{\alpha}_r \dots}$ because α_s and α_{r+1} are interchanged, the same happens when we take out the indice α_s .

Theorem 1.1.1 (The Generalized Mayer-Vietoris Sequence). *The sequence*

$$0 \longrightarrow C^*(M) \xrightarrow{r} \prod C^*(U_{\alpha_i}) \xrightarrow{\delta} \prod C^*(U_{\alpha_i \alpha_j}) \xrightarrow{\delta} \prod C^*(U_{\alpha_i \alpha_j \alpha_k}) \xrightarrow{\delta} \dots$$

is exact; in other words, the δ -cohomology of this complex vanishes identically.

Proof. Clearly $C^*(M)$ is the kernel of the first δ since an element of $\prod C^*(U_{\alpha_i})$ is a global cochain on M if and only if its components agree on the overlaps.

Now let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open cover $\mathcal{U} = \{U_\alpha\}$. Suppose $w \in \prod C^*(U_{\alpha_0 \dots \alpha_p})$ is a p -cocycle. Define a $(p-1)$ -cochain τ by

$$\tau_{\alpha_0 \dots \alpha_{p-1}} = \sum_{\alpha} \rho_\alpha w_{\alpha \alpha_0 \dots \alpha_{p-1}}.$$

Then

$$(\delta\tau)_{\alpha_0\dots\alpha_p} = \sum_i (-1)^i \tau_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} = \sum_{i,\alpha} (-1)^i \rho_\alpha w_{\alpha\alpha_0\dots\hat{\alpha}_i\dots\alpha_p}.$$

Because w is a cocycle,

$$(\delta w)_{\alpha\alpha_0\dots\alpha_p} = w_{\alpha_0\dots\alpha_p} + \sum_i (-1)^{i+1} w_{\alpha\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} = 0,$$

then

$$w_{\alpha_0\dots\alpha_p} = \sum_i (-1)^i w_{\alpha\alpha_0\dots\hat{\alpha}_i\dots\alpha_p}.$$

So

$$(\delta\tau)_{\alpha_0\dots\alpha_p} = \sum_\alpha \rho_\alpha \sum_i (-1)^i w_{\alpha\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} = \sum_\alpha \rho_\alpha w_{\alpha_0\dots\alpha_p} = w_{\alpha_0\dots\alpha_p}.$$

This shows that every cocycle is a coboundary. \square

In fact, the definition of τ in this proof gives a homotopy operator on the complex. Write Kw for τ :

$$(Kw)_{\alpha_0\dots\alpha_{p-1}} = \sum_\alpha \rho_\alpha w_{\alpha\alpha_0\dots\alpha_{p-1}}.$$

Then

$$\begin{aligned} (\delta Kw)_{\alpha_0\dots\alpha_p} &= \sum_i (-1)^i (Kw)_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} = \sum_{i,\alpha} (-1)^i \rho_\alpha w_{\alpha\alpha_0\dots\hat{\alpha}_i\dots\alpha_p}, \\ (K\delta w)_{\alpha_0\dots\alpha_p} &= \sum_\alpha \rho_\alpha (\delta w)_{\alpha\alpha_0\dots\alpha_p} = \\ &= \left(\sum_\alpha \rho_\alpha\right) w_{\alpha_0\dots\alpha_p} + \sum_{i,\alpha} (-1)^{i+1} \rho_\alpha w_{\alpha\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} = w_{\alpha_0\dots\alpha_p} - (\delta Kw)_{\alpha_0\dots\alpha_p}. \end{aligned}$$

Therefore, K is an operator from $\prod C^*(U_{\alpha_0\dots\alpha_p})$ to $\prod C^*(U_{\alpha_0\dots\alpha_{p-1}})$ such that

$$\delta K + K\delta = 1.$$

Note that if ϕ is a cocycle, then $\delta K\phi = \phi$. So on cocycles K is a right inverse to δ . Given such ϕ , the set of all solution η of $\delta\eta = \phi$ consists of $K\phi + \delta$ -coboundaries.

Now let $K^{p,q} = C_\delta^p(\mathcal{U}, C_d^q) = \prod C_d^q(U_{\alpha_0\dots\alpha_p})$ be the double complex consists of the " p -cochains of the cover \mathcal{U} with values in the q -cochains". The horizontal maps of the double complex are the difference operators δ and the vertical ones the coboundary morphisms d . The double complex may be made into a single complex $\mathbf{C}^n(\mathcal{U}) = \bigoplus_{p+q=n} C_\delta^p(\mathcal{U}, C_d^q)$ with the differential operator given by $D = D' + D'' = \delta + (-1)^p d$.

A D -cocycle is a string such as $\phi = a_0 + \dots + a_n$ with $da_0 = 0$, $\delta a_0 = -D''a_1, \dots, \delta a_{n-1} = -D''a_n$ and $\delta a_n = 0$.

A D -coboundary is a string such as $\phi = a_0 + \dots + a_n = D(b_0 + \dots + b_{n+1})$, where $D''b_0 = 0$, $a_1 = \delta b_0 + D''b_1, \dots, a_n = \delta b_n + D''b_{n+1}$ and $\delta b_{n+1} = 0$.

Theorem 1.1.2 (Generalized Mayer-Vietoris Principle). *The double complex $C_{\delta}^*(\mathcal{U}, C_d^*) = \bigoplus_{p,q \geq 0} C_{\delta}^p(\mathcal{U}, C_d^q)$ computes the cohomology of M ; more precisely, the restriction map $r : C_d^*(M) \rightarrow C_{\delta}^*(\mathcal{U}, C_d^*)$ induces an isomorphism in cohomology:*

$$r^* : H_d^*(M) \rightarrow H_D\{C_{\delta}^*(\mathcal{U}, C_d^*)\}.$$

Proof. Since $Dr = (\delta + d)r = dr = rd$, the following diagram is commutative

$$\begin{array}{ccc} C_d^*(M) & \xrightarrow{r} & C_{\delta}^*(\mathcal{U}), \\ d \uparrow & & \uparrow D \\ C_d^*(M) & \xrightarrow{r} & C_{\delta}^*(\mathcal{U}) \end{array}$$

r is a chain map, and so it induces a map r^* in cohomology.

Step 1. r^ is surjective.*

Let ϕ be a cocycle relative to D . By δ -exactness the lowest component of ϕ is δ of something. By subtracting $D(\text{something})$ from ϕ , we can remove the lowest component of ϕ and still stay in the same cohomology class as ϕ . After iterating this procedure enough times we can move ϕ in its cohomology class to a cocycle ϕ' with only the top component. ϕ' is a global d -cocycle because $d\phi' = D\phi = 0$ and $\delta\phi' = 0$.

Step 2. r^ is injective.*

If $r(w) = D\phi$, we can choose ϕ' in the same cohomology class of ϕ , such that ϕ' consists of only top component, we can do it for the δ -exactness. Then because $\delta\phi' = 0$, it is actually a global chain of M . So w is a d -boundary. \square

1.2 Čech Cohomology

In this section we follow [10] and [16].

1.2.1 Presheaves

Definition 1.2.1. *A presheaf \mathcal{F} on a topological space X is a function that assigns to every open set U in X an abelian group $\mathcal{F}(U)$ and to every inclusion of open sets $i_U^V : V \rightarrow U$ a group homomorphism, called the restriction, $\mathcal{F}(i_U^V) : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ satisfying the following properties:*

1. $\mathcal{F}(i_V^V) = \text{identity map}$
2. *transitivity:* $\mathcal{F}(i_V^W)\mathcal{F}(i_U^V) = \mathcal{F}(i_U^W)$.

The restriction $\mathcal{F}(i_U^V) : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is often denoted ρ_V^U . A homomorphism of two presheaves

$f : \mathcal{F} \longrightarrow \mathcal{G}$, is a collection of maps $f_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ which commute with the restrictions:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \rho_V^U \downarrow & & \downarrow \rho_V^U \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

Let $\text{Open}(X)$ be the category whose objects are the open sets in X and whose morphisms are inclusions of open sets. In functorial language, a presheaf is simply a contravariant functor from the category $\text{Open}(X)$ to the category of Abelian groups, and a homomorphism of two presheaves, $f : \mathcal{F} \longrightarrow \mathcal{G}$ is a *natural transformation* from the functor \mathcal{F} to the functor \mathcal{G} .

Example 1.2.1. Let $\pi : E \longrightarrow M$ be a map between topological spaces. Define a presheaf \mathcal{H}^q on M by $\mathcal{H}^q(U) = H^q(\pi^{-1}U)$, and if $V \subset U$ is an inclusion, let $\rho_V^U : H^q(\pi^{-1}U) \longrightarrow H^q(\pi^{-1}V)$ be the natural restriction map.

1.2.2 Čech Cohomology

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ be an open cover of the topological space X . The 0-cochains on \mathcal{U} with values in the presheaf \mathcal{F} are functions which assign to each open set U_α an element of $\mathcal{F}(U_\alpha)$, i.e., $C^0(\mathcal{U}, \mathcal{F}) = \prod_{\alpha \in J} \mathcal{F}(U_\alpha)$. Similarly the 1-cochains are the elements of $C^1(\mathcal{U}, \mathcal{F}) = \prod_{\alpha < \beta} \mathcal{F}(U_\alpha \cap U_\beta)$ and so on.

The sequence of inclusions

$$U_{\alpha_i} \xleftarrow{(\partial_i, \partial_j)} U_{\alpha_i \alpha_j} \xleftarrow{(\partial_i, \partial_j, \partial_k)} U_{\alpha_i \alpha_j \alpha_k} \longleftarrow \dots$$

gives rise to a sequence of group homomorphisms

$$\prod_{\alpha_i} \mathcal{F}(U_{\alpha_i}) \longrightarrow \prod_{\alpha_i < \alpha_j} \mathcal{F}(U_{\alpha_i \alpha_j}) \longrightarrow \dots$$

We define $\delta : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$ to be the alternating difference of the $\mathcal{F}(\partial_i)$'s

$$\delta = \mathcal{F}(\delta_0) - \mathcal{F}(\delta_1) + \dots + (-1)^{p+1} \mathcal{F}(\delta_{p+1}).$$

Explicitly, if $w \in C^p(\mathcal{U}, \mathcal{F})$, then

$$(\delta w)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i w_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}},$$

where on the right-hand side the restriction of $w_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}$ to $U_{\alpha_0 \dots \alpha_{p+1}}$ is suppressed. It follows from the transitivity of the restriction homomorphism that $\delta^2 = 0$ (proof as in Proposition 1.1.1). Thus $C^*(\mathcal{U}, \mathcal{F})$ is a differential complex with differential operator δ . The cohomology of this complex,

denoted by $H_\delta C^*(\mathcal{U}, \mathcal{F})$ or $H^*(\mathcal{U}, \mathcal{F})$, is called the Čech cohomology of the cover \mathcal{U} with values in \mathcal{F} .

The cover $\mathcal{B} = \{V_\beta\}_{\beta \in J}$ is a *refinement* of the cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$, written $\mathcal{U} < \mathcal{B}$, if there is a map $\phi : J \rightarrow I$ such that $V_\beta \subset U_{\phi(\beta)}$. The map ϕ induces a map

$$\phi^\# : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^q(\mathcal{B}, \mathcal{F})$$

in the obvious manner: $(\phi^\# w)(V_{\beta_0 \dots \beta_q}) = w(U_{\phi(\beta_0) \dots \phi(\beta_q)})$.

Lemma 1.2.1. $\phi^\#$ is a chain map, i.e., it commutes with δ .

Proof. $(\delta(\phi^\# w))(V_{\beta_0 \dots \beta_{q+1}}) = \sum (-1)^i (\phi^\# w)(V_{\beta_0 \dots \hat{\beta}_i \dots \beta_{q+1}}) = \sum (-1)^i w(U_{\phi(\beta_0) \dots \phi(\hat{\beta}_i) \dots \phi(\beta_{q+1})})$

$$(\phi^\# \delta w)(V_{\beta_0 \dots \beta_{q+1}}) = (\delta w)(U_{\phi(\beta_0) \dots \phi(\beta_{q+1})}) = \sum (-1)^i w(U_{\phi(\beta_0) \dots \phi(\hat{\beta}_i) \dots \phi(\beta_{q+1})})$$

□

Lemma 1.2.2. Given $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open cover and $\mathcal{B} = \{V_\beta\}_{\beta \in J}$ a refinement, if ϕ and ψ are two refinement maps $: J \rightarrow I$, then there is a homotopy operator between $\phi^\#$ and $\psi^\#$.

Proof. Define $K : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^{q-1}(\mathcal{B}, \mathcal{F})$ by

$$(Kw)(V_{\beta_0 \dots \beta_{q-1}}) = \sum (-1)^i w(U_{\phi(\beta_0) \dots \phi(\beta_i) \psi(\beta_i) \dots \psi(\beta_{q-1})})$$

and show that $\psi^\# - \phi^\# = \delta K + K \delta$. □

A *direct system of groups* is a collection of groups $\{G_i\}_{i \in I}$ indexed by a directed set I such that for any pair $a < b$ there is a group homomorphism $f_b^a : G_a \rightarrow G_b$ satisfying

1. $f_a^a = \text{identity}$,
2. $f_c^a = f_c^b \circ f_b^a$, if $a < b < c$.

On the disjoint union $\bigsqcup G_i$ we introduce an equivalence relation \sim by decreeing two elements $g_a \in G_a$ and $g_b \in G_b$ to be equivalent if for some upper bound c of a and b , we have $f_c^a(g_a) = f_c^b(g_b)$ in G_c . The *direct limit* of the direct system, denoted by $\varinjlim G_i$ is the quotient of $\bigsqcup G_i$ by the equivalent relation \sim . We make direct limit into a group by defining $[g_a] + [g_b] = [f_c^a(g_a) + f_c^b(g_b)]$ where $[g_a]$ is the equivalence class of g_a .

It follows from the two lemmas above that if $\mathcal{U} < \mathcal{B}$, then there is a well-defined map in cohomology $H^*(\mathcal{U}, \mathcal{F}) \rightarrow H^*(\mathcal{B}, \mathcal{F})$ making $\{H^*(\mathcal{U}, \mathcal{F})\}_{\mathcal{U}}$ into a direct system of groups. The direct limit of this direct

$$\check{H}^*(X, \mathcal{F}) = \varinjlim H^*(\mathcal{U}, \mathcal{F})$$

is the Čech cohomology of X with values in presheaf \mathcal{F} .

For more details you can look at [10] page 113.

If the space X is ENR (euclidean neighborhood retract) there is another way to define Čech Cohomology when the preseaf is constant.

Definition 1.2.2. Let $Y \subset X \subset E$, where E is ENR and Y, X locally compact. Let $\Lambda = \Lambda(X, Y)$ be the set of all pairs (V, W) of neighborhood of X, Y such that $W \supset V$. Under reversed inclusion Λ is directed $((V, W) \leq (\tilde{V}, \tilde{W}), \text{ if and only if, } \tilde{V} \subset V \text{ and } \tilde{W} \subset W)$, and $\{H^*(V, W; R)\}$ together with the restriction homomorphism is a direct system of graded abelian groups. We define $\check{H}^q(X, Y; R) = \varinjlim \{H^q(V, W; R)\}$ the q -th Čech Cohomology group of X mod Y with coefficients in R .

With this last definition is easier to verify the Eilenberg-Steenrod axioms. For more details you can look at [16] page 281. A result important for us is the next proposition that you can find the proof in [16] Proposition 6.12.

Proposition 1.2.1. If $Y \subset X$ is a pair of ENR's then $\check{H}^*(X, Y) \simeq H^*(X, Y)$, i.e. for ENR's Čech cohomology coincides with ordinary cohomology.

Note 1.2.1. Oddly enough, the corresponding Čech homology groups defined using inverse limits are not so well-behaved. This is because the exactness axiom fails due to the algebraic fact that an inverse limit of exact sequences need not be exact, as a direct limit would be. However, there is a way around this problem using a more refined definition. This is Steenrod homology theory, which you can learn about in [24].

1.3 Wu class and Stiefel-Whitney class

In this section we follow [23].

To define *Wu class*, we first need to define the sequence of operations Sq^k , that operations are called *Steenrod squares*. The *Steenrod squares*, $\{Sq^k\}_{k \geq 0}$, are additive cohomology operations

$$Sq^k : H^m(X; \mathbb{Z}_2) \longrightarrow H^{m+k}(X; \mathbb{Z}_2),$$

that are defined for all m and satisfies the following properties

1. If $f : X \longrightarrow Y$ is a continuous function, then $Sq^k \circ f^* = f^* \circ Sq^k$.
2. If $x \in H^m(X; \mathbb{Z}_2)$, then $Sq^0(x) = x$, $Sq^m(x) = x \smile x$ and $Sq^k(x) = 0$, for $k > m$.
3. If $x, y \in H^*(X; \mathbb{Z}_2)$, the *Cartan formula* is true:

$$Sq^k(x \smile y) = \sum_{i+j=k} Sq^i(x) \smile Sq^j(y),$$

where \smile is the cup product. The operation *total square* is given by

$$Sq(x) = x + Sq^1(x) + Sq^2(x) + \cdots + Sq^m(x).$$

Note that using the operation *total square*, we have the Cartan formula, that is gives by the equation

$$Sq(x \smile y) = Sq(x) \smile Sq(y).$$

Let M be a closed, oriented and connected m -smooth manifold. Then the application

$$H^k(M; \mathbb{Z}_2) \rightarrow \text{Hom}(H^{m-k}(M; \mathbb{Z}_2); \mathbb{Z}_2),$$

given by $v \mapsto \langle v \smile, [M] \rangle$ is an isomorphism, where $[M]$ is the fundamental class of M and \langle, \rangle is the Kronecker index. Consider the homomorphism $h : H^{m-k}(M; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$, given by $x \mapsto \langle Sq^k(x), [M] \rangle$. Since the application above is an isomorphism, there is only one $v_k \in H^k(M; \mathbb{Z}_2)$ such that $\langle v_k \smile x, [M] \rangle = h(x)$. The class v_k is called *kth Wu class* of M and $v = 1 + v_1 + v_2 + \cdots + v_m$ is called the *total Wu class* of M .

The *Stiefel-Whitney classes* are characterized by the next Theorem (Theorem 17 in [23]).

Theorem 1.3.1 (Wu). *The total Stiefel-Whitney class of M satisfies $w(M) = Sq(v)$, i.e., $w_k = \sum_{i+j=k} Sq^i(v_j)$, where v is the total Wu class of M .*

Since w is thus defined entirely in terms of cohomology and homology operations, we have the next result.

Corollary 1.3.1. *The Stiefel-Whitney classes are invariants of homotopy type.*

1.4 Spectral sequences

In this section we follow [10].

1.4.1 Exact Couples

An *exact couple* is an exact sequence of Abelian groups of the form

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & \swarrow k & \searrow j \\ & B & \end{array}$$

where i, j and k are group homomorphisms. Define $d : B \rightarrow B$ by $d = j \circ k$. Then $d^2 = j(kj)k = 0$, so the homology group $H(B) = (\ker d) / (\text{im } d)$ is defined. Here A and B are assumed to be Abelian so that the quotient $H(B)$ is a group.

Out of a given exact couple we can construct a new exact couple, called the *derived couple*,

$$\begin{array}{ccc} A' & \xrightarrow{i'} & A' \\ & \swarrow k' & \searrow j' \\ & B' & \end{array}$$

by making the following definitions:

1. $A' = i(A); B' = H(B)$.
2. i' is induced from i ; to be precise, $i'(ia) = i(ia)$.
3. If $a' = ia$ is in A' , with a in A , then $j'a' = [ja]$, where $[]$ denotes the homology class in $H(B)$. To show that j' is well-defined we have to check two things:
 - (a) ja is a cycle. This follows from $d(ja) = j(kj)a = 0$.
 - (b) The homology class $[ja]$ is independent of the choice of a . Suppose $a' = i\bar{a}$ for some other \bar{a} in A . Then because $0 = i(a - \bar{a})$, we have $a - \bar{a} = kb$ for some b in B . Thus $ja - j\bar{a} = jkb = db$, so $[ja] = [j\bar{a}]$.
4. k' is induced from k . Let $[b]$ be a homology class in $H(B)$. Then $jk b = 0$ so that $kb = ia$ for some a in A . Define $k'[b] = kb \in A'$.

Now we are going to show that with these definitions, the derived couple is an exact couple.

1. Exactness at B' .

- (a) $im j' \subset ker k'$: $k'j'(a') = k'j'(ia) = k'j(a) = kj(a) = 0$.
- (b) $ker k' \subset im j'$: since $k'(b) = k(b) = 0$, it follows that $b = j(a) = j'(ia) \in im j'$.

2. $im k' = ker i'$.

- (a) $im k' \subset ker i'$: $i'k'[b] = i'k(b) = ik(b) = 0$.
- (b) $ker i' \subset im k'$: $0 = i'(a') = i(i(a))$, then exist $b \in B$ such that $k(b) = i(a)$ and $d(b) = 0$ because $d(b) = jk(b) = ji(a) = 0$. Then $a' = k'[b]$.

3. $im i' = ker j'$.

- (a) $im i' \subset ker j'$: $j'i'(a') = j'i'(ia) = j'(iia) = [ji(a)] = [0]$.
- (b) $ker j' \subset im i'$: $j'(a') = [0]$, then exist $a \in A$ and $b \in B$ such that $a' = i(a)$ and $j(a) = d(b) = jk(b)$. For this exist $\bar{a} \in A$ such that $i(\bar{a}) = a - k(b)$. Then $a' = i(a) = i(i(\bar{a}) + k(b)) = i(i\bar{a})$.

1.4.2 The Spectral Sequence of a Filtered Complex

Let K be a differential complex with differential operator D . Usually K comes with a grading $K = \bigoplus_{k \in \mathbb{Z}} C^k$ and $D : C^k \rightarrow C^{k+1}$ increases the degree by 1. A *subcomplex* K' of K is a subgroup such that $DK' \subset K'$. A sequence of subcomplexes

$$K = K_0 \supset K_1 \supset K_2 \supset \dots$$

is called a *filtration* on K . This makes K into a *filtered complex*, with *associated graded complex*

$$GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}.$$

For notational reasons we usually extend the filtration to negative indices by defining $K_p = K$ for $p < 0$. Let A be the group

$$A = \bigoplus_{p \in \mathbb{Z}} K_p$$

A is again a differential complex with operator D . Define $i : A \rightarrow A$ to be the inclusion $K_{p+1} \hookrightarrow K_p$ and define B the quotient

$$B = \bigoplus_{p \in \mathbb{Z}} K_p/K_{p+1}.$$

Then we have the next short exact sequence

$$0 \longrightarrow A \xrightarrow{i} A \xrightarrow{j} B \longrightarrow 0.$$

Then B is the associated graded complex GK of K . In the short exact sequence above each group is a complex with the operator induced from D . In the graded case we get from this short exact sequence a long exact sequence of cohomology groups

$$\cdots \longrightarrow H^k(A) \xrightarrow{i_1} H^k(A) \xrightarrow{j_1} H^k(B) \xrightarrow{k_1} H^{k+1}(A) \longrightarrow \cdots,$$

which we may write as

$$\begin{array}{ccc} H(A) & \xrightarrow{i_1} & H(A) \\ & \swarrow k_1 & \searrow j_1 \\ & H(B) & \end{array}$$

Define $A_1 = H(A)$ and $B_1 = H(B)$, we have the next exact couple

$$\begin{array}{ccc} A_1 & \xrightarrow{i} & A_1 \\ & \swarrow k_1 & \searrow j_1 \\ & B_1 & \end{array}$$

where the map i need no longer be an inclusion. We suppress the subscript of i_1 to avoid cumbersome notation later. As we saw in the last section, the exact couple above gives rise to a sequence of exact couples:

$$\begin{array}{ccc} A_r & \xrightarrow{i} & A_r \\ & \swarrow k_r & \searrow j_r \\ & B_r & \end{array}$$

each being the derived couple of its predecessor. For the sake of the exposition consider now the case where the filtered complex terminates after K_3 :

$$\cdots = K_{-1} = K_0 \supset K_1 \supset K_2 \supset K_3 \supset 0.$$

Then A_1 is the direct sum of all the terms in the following sequence

$$\dots \leftarrow \widetilde{H(K)} \leftarrow \widetilde{H(K)} \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow 0.$$

This is of course not an exact sequence. Next, A_2 by definition is the image of A_1 under i in A_1 and so is the direct sum of the groups in the sequence

$$\dots \leftarrow \widetilde{H(K)} \leftarrow \widetilde{H(K)} \leftarrow \supset iH(K_1) \leftarrow iH(K_2) \leftarrow iH(K_3) \leftarrow 0.$$

Note that here the map $iH(K_1) \hookrightarrow H(K)$ is an inclusion. Similarly A_3 is the sum of

$$\dots \leftarrow \widetilde{H(K)} \leftarrow \widetilde{H(K)} \leftarrow \supset iH(K_1) \leftarrow \supset iiH(K_2) \leftarrow iiiH(K_3) \leftarrow 0$$

and A_4 is the sum of

$$H(K) \supset iH(K_1) \supset iiH(K_2) \supset iiiH(K_3) \supset 0.$$

Since all the maps became inclusions in A_4 , all the A 's are stationary after the four derived couple and we define A_∞ to be the stationary value:

$$A_4 = A_5 = A_6 = \dots = A_\infty.$$

Furthermore, since

$$\begin{array}{ccc} A_4 & \xrightarrow{i} & A_4 \\ & \swarrow k_4 & \searrow j_4 \\ & B_4 & \end{array}$$

is exact and $i : A_4 \rightarrow A_4$ is the inclusion, the map $k_4 : B_4 \rightarrow A_4$ must be the zero map. Therefore, after the fourth stage all the differentials of the exact couples are zero and the B 's also became stationary,

$$B_4 = B_5 = B_6 = \dots = B_\infty.$$

The exact couple

$$\begin{array}{ccc} A_\infty & \xrightarrow{i_\infty} & A_\infty \\ & \swarrow k_\infty & \searrow j_\infty \\ & B_\infty & \end{array}$$

A_∞ is the direct sum of the next filtration of $H(K)$

$$\dots = H(K) = H(K) \supset iH(K_1) \supset iiH(K_2) \supset iiiH(K_3) \supset 0$$

and the i_∞ is the inclusion. Since B_∞ is isomorphic to the quotient of i_∞ (using the Five Lemma at the short sequence $0 \rightarrow A \xrightarrow{i_\infty} A \rightarrow B \rightarrow 0$ is exact), it is the direct sum of the successive quotients in i_∞ , then B_∞ is the associated graded complex of the filtered complex $H(K)$.

We now return to the general case. The sequence of subcomplexes

$$\dots = K = K \supset K_1 \supset K_2 \supset K_3 \supset \dots$$

induces a sequence in cohomology

$$\cdots \xleftarrow{\sim} H(K) \xleftarrow{\sim} H(K) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \xleftarrow{\cdots},$$

where the maps i are of course no longer inclusions. Let F_p be the image of $H(K_p)$ in $H(K)$. Then there is a sequence of inclusions

$$H(K) = F_0 \supset F_1 \supset F_2 \supset F_3 \supset \cdots,$$

making $H(K)$ into a filtered complex; this filtration is called the *induced filtration* on $H(K)$. A filtration K_p on the filtered complex K is said to have *length* l if $K_p \neq 0$ and $K_p = 0$ for $p > l$. By the same argument as the special case above, we see that whenever the filtration on K has finite length, then A_r and B_r are eventually stationary and the stationary value B_∞ is the associated graded complex $\bigoplus F_p/F_{p+1}$ of the filtered complex $H(K)$.

It is customary to write E_r for B_r . Hence,

$$E_1 = H(B) \text{ with differential } d_1 = j_1 \circ k_1,$$

$$E_2 = H(E_1) \text{ with differential } d_2 = j_2 \circ k_2,$$

$$E_3 = H(E_2) \text{ with differential } d_3 = j_3 \circ k_3,$$

$$\vdots$$

A sequence of differential groups $\{E_r, d_r\}$ with E_r is the homology of its predecessor E_{r-1} is called a *spectral sequence*. If E_r eventually became stationary, we denote the stationary value by E_∞ , and if E_∞ is equal to the associated graded group of some filtered group H , then we say that the spectral sequence *converges* to H .

Now suppose the filtered complex K comes with a grading: $K = \bigoplus_{n \in \mathbb{Z}} K^n$. To distinguish the grading degree n from the filtration degree p , we will often call n the *dimension*. The filtration $\{K_p\}$ on K induces a filtration on K^n .

Theorem 1.4.1. *Let $K = \bigoplus_{n \in \mathbb{Z}} K^n$ be a graded filtered complex with filtration $\{K_p\}$ and let $H_D^*(K)$ be the cohomology of K with the filtration $\{F_p\}$ where F_p is the image of $H(K_p)$ in $H(K)$. Suppose for each dimension n the filtration $\{K_p^n\}$ has a finite length. Then the short exact sequence*

$$0 \longrightarrow \bigoplus K_{p+1} \longrightarrow \bigoplus K_p \longrightarrow \bigoplus K_p/K_{p+1} \longrightarrow 0$$

induces a spectral sequence which converges to $H_D^(K)$.*

Proof. By treating the convergence question one dimension at a time, this proof reduces to the ungraded situation. To be absolutely sure, we will write out the details. As before,

$$A_r = \bigoplus_{p \in \mathbb{Z}} i^{r-1} H(K_p);$$

if $r \geq p + 1$, then $i^r H(K_{p+1}) = F_p$ and

$$i : i^r H(K_{p+1}) \longrightarrow i^r H(K_p)$$

is an inclusion. With a grading on each derived couple, i and j preserve the dimension, but k increases the dimension by 1. Given n , let $l(n)$ be the length of $\{K_p^n\}_{p \in \mathbb{Z}}$ and let $r \geq l(n+1) + 1$. Then for any integer p ,

$$i^r H^{n+1}(K_{p+1}) = F_{p+1}^{n+1}$$

and

$$i : i^r H^{n+1}(K_{p+1}) \longrightarrow i^r H^{n+1}(K_p)$$

is an inclusion. It follows that

$$i_r : A_r^{n+1} \longrightarrow A_r^{n+1}$$

is an inclusion and

$$k_r : B_r^n \longrightarrow A_r^{n+1}$$

is the zero map. Therefore, as $r \longrightarrow \infty$, the group B_r^n becomes stationary and we can define B_∞^n to be this stationary value. Note that

$$A_\infty^n = \bigoplus F_p^n$$

and that i_∞ sends F_{p+1}^n into F_p^n for every n . Because

$$i_\infty : \bigoplus F_{p+1} \longrightarrow \bigoplus F_p$$

is an inclusion, B_∞ is the associated graded complex $\bigoplus F_p / F_{p+1}$ of $H_D^*(K)$. \square

1.4.3 The Spectral Sequence of a Double Complex

Let $K = \bigoplus K^{p,q}$ be a double complex, that is, a family of modules doubly indexed by the integers with horizontal differential operator $\delta : K^{p,q} \longrightarrow K^{p+1,q}$ and vertical differential operator $d : K^{p,q} \longrightarrow K^{p,q+1}$ such that $\delta^2 = d^2 = 0$ and $d\delta = \delta d$. We can form a single complex out of it in the usual way, by letting $K = \bigoplus C^k$, where $C^k = \bigoplus_{p+q=k} K^{p,q}$, and defining the differential operator $D : C^k \longrightarrow C^{k+1}$ to be $D = \delta + (-1)^p d$. Then the sequence of subcomplexes indicated below is a filtration on K :

$$K_p = \bigoplus_{i \geq p} \bigoplus_{q \geq 0} K^{i,q}$$

To show that D and K_p are well-defined we have to check:

1. $D^2 = 0$. This follows from $D^2 = d^2 + \delta d - d\delta + \delta^2 = 0$.
2. $DK_p \subset K_p$. This follows from $DK^{i,q} = \delta K^{i,q} + (-1)^i dK^{i,q} \subset K^{i+1,q} \oplus K^{i,q+1} \subset K_{p+1} \subset K_p$ for $i \geq p$.

The direct sum $A = \bigoplus K_p$ is also a double complex. Here, as always, we form a single complex $A = \bigoplus A^k$ out of this double complex by summing de bidegrees: $A^k = C^k \cap A$ consists of all elements in A whose total degree is k . There is an inclusion $i : A^k \longrightarrow A^k$ given by $i : A^k \cap K_{p+1} \longrightarrow A^k \cap K_p$. The single complex A inherits the differential operator D from K .

Similarly, $B = \bigoplus K_p/K_{p+1}$ can be made into a single complex with operator D . Note that the differential operator D on B is $(-1)^p d$; therefore,

$$E_1 = H_D(B) = H_d(K).$$

Recall that the coboundary operator $k_1 : H(B) \longrightarrow H(A)$ is the defined by de following diagram:

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & A^{k+1} \cap K_{p+1} & \xrightarrow{(***)} & A^{k+1} \cap K_p & \longrightarrow & B^{k+1} \cap K_p/K_{p+1} \longrightarrow 0 \\
 & & \uparrow D & & \uparrow (** & & \uparrow \\
 0 & \longrightarrow & A^k \cap K_{p+1} & \longrightarrow & A^k \cap K_p & \xrightarrow{(*)} & B^k \cap K_p/K_{p+1} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow
 \end{array}$$

Let b in $A^k \cap K_p$ represent a cocycle $[b]$ in $B^k \cap K_p/K_{p+1}$, this corresponds to Step $(*)$ in the diagram. To get $k_1([b])$, we compute Db (in Step $(**)$) and take its inverse under i (in the Step $(***)$). Since b represents an element of $E_1 = H_D(B) = H_d(K)$, $db = 0$ and $Db = \delta b + (-1)^p db = \delta b$. Thus $Kk_1[b] = [\delta b]$; so the differential $d_1 = j_1 k_1$ on E_1 is given by δ . Consequently

$$E_2 = H_\delta H_d(K).$$

Theorem 1.4.2. *Given a double complex $K = \bigoplus_{p,q \geq 0} K^{p,q}$ there is a spectral sequence $\{E_r, d_r\}$ converging to the total cohomology $H_D(K)$ such that each E_r has a bigrading with*

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1}$$

and

$$E_1^{p,q} = H_d^{p,q}(K),$$

$$E_2^{p,q} = H_\delta^p H_d^q(K);$$

furthermore, the associated graded complex of the total cohomology is given by

$$GH_D^n(K) = \bigoplus_{p+q=n} E_\infty^{p,q}(K).$$

Note 1.4.1. *Given two spectral sequences, $\{(E_r^{*,*}, d_r)\}$ and $\{(\hat{E}_r^{*,*}, \hat{d}_r)\}$, a morphism of spectral sequences is a sequence of homomorphisms of bigraded modules, $f_r : (E_r^{*,*}, d_r) \longrightarrow (\hat{E}_r^{*,*}, \hat{d}_r)$, for all*

r , of bidegree $(0,0)$, such that f_r commutes with the differentials, that is, $f_r \circ d_r = \hat{d}_r \circ f_r$, and each f_{r+1} is induced by f_r on homology, that is,

$$f_{r+1} : E_{r+1}^{*,*} \cong H(E_r^{*,*}, d_r) \xrightarrow{H(f)} H(\hat{E}_r^{*,*}, \hat{d}_r) \cong \hat{E}_{r+1}^{*,*} .$$

One way to build a morphism of spectral sequences is to have a morphism in the initial page that commute with the initial differential. Furthermore, such a morphism induces a mapping $f_\infty : E_\infty \rightarrow \hat{E}_\infty$. If we have graded filtered complexes $K = K_0 \supseteq K_1 \supseteq \dots \supseteq K_n$ and $\hat{K} = \hat{K}_0 \supseteq \hat{K}_1 \supseteq \dots \supseteq \hat{K}_n$ and a morphism $f : K \rightarrow \hat{K}$ respect the filtration, that is, $f(K_i) \subset \hat{K}_i$ and commute with the differentials, then this morphism induce a morphism of spectral sequences between the induced spectral sequences.

Persistence and Leray's theorem

2.1 Persistence

The concept of persistence emerged independently in the work of Frosini, Ferri, and collaborators in Bologna, Italy, in the doctoral work of Robins at Boulder, Colorado, and within the biogeometry project of Edelsbrunner at Duke, North Carolina.

Persistent homology/cohomology is an algebraic method for measuring topological features of shapes and of functions. In the last 20 years the research and application of persistent homology/cohomology has been intense, for an introduction and survey you can read [20].

In this section we achieved to show that the persistent cohomology of a filtered finite dimensional simplicial complex is a graded module over a polynomial ring, like Zomorodian and Gunnar had done to persistent homology in [33]

2.1.1 Simplicial Complexes

A *simplicial complex* is a set K , together with a collection \mathcal{S} of subsets of K called *simplices* such that for all $v \in K$, $\{v\} \in \mathcal{S}$, and if $\alpha \subset \sigma \in \mathcal{S}$, then $\alpha \in \mathcal{S}$. We call the sets $\{v\}$ the *vertices* of K . When it is clear from the context what \mathcal{S} is, we refer to the set K as a complex. We say $\sigma \in \mathcal{S}$ is a *k-simplex of dimension k* if $|\sigma| = k + 1$. If $\alpha \subset \sigma$, α is a *face* of σ and σ is a *coface* of α . An *orientation* of a simplex $\sigma = \{v_0, \dots, v_k\}$ is an equivalence class of orderings of the vertices of σ , where $(v_0, \dots, v_k) \sim (v_{\tau(0)}, \dots, v_{\tau(k)})$ if the sign of the permutation τ is 1. We denote an oriented simplex by $[\sigma]$. A *k-simplex* may be realized geometrically as the convex hull of $k + 1$ affinely independent points in \mathbb{R}^d , $d \geq k$. A realization gives us the familiar low-dimensional simplices: *vertices, edges, triangles* and *tetrahedra*, for $0 \leq k \leq 3$. Within a realized complex, the simplices must meet along common faces. A *subcomplex* of K is a subset $L \subset K$ that also is a simplicial complex.

2.1.2 Chain Complex

The k -th chain group C_k of K is the free Abelian group on its set of oriented k -simplices, where $[\sigma] = -[\tau]$ if $\sigma = \tau$ and σ and τ are differently oriented. An element $c \in C_k$ is a k -chain, $c = \sum_i n_i [\sigma_i]$, $\sigma_i \in K$ with coefficients n_i in a field \mathbb{F} . The boundary operator $\partial_k : C_k \rightarrow C_{k-1}$ is a homomorphism defined linearly on chain c by its value on any simplex $\sigma = [v_0, \dots, v_k] \in C_k$,

$$\partial_k \sigma = \sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k],$$

where \hat{v}_i means that v_i is deleted from the sequence. The boundary operator connects the chain groups into a chain complex

$$C_* : \dots \rightarrow C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \rightarrow \dots$$

We may also define subgroups of C_k using the boundary operator: the cycle group $Z_k = \ker \partial_k$ and the boundary group $B_k = \text{im } \partial_{k+1}$.

Proposition 2.1.1. $\partial_k \partial_{k+1} = 0$.

Proof. The same idea that in 1.1.1 □

This fact, along with the definitions, implies that the defined subgroups are nested, $B_k \subseteq Z_k \subseteq C_k$. For generality, we often define null boundary operators in dimensions where C_k is empty.

Definition 2.1.1. The k -th homology group is $H_k = Z_k/B_k$. Its elements are classes of homologous cycles. The rank β_k of the free submodule of H_k , is Betti number of the module H_k .

2.1.3 Cochain complex

The k -th cochain group C^k of the complex K is $\{\varphi \in \text{Hom}(C_k, \mathbb{F}) : \varphi(v_{\tau(0)}, \dots, v_{\tau(k)}) = \text{sign}(\tau) \varphi(v_0, \dots, v_k)\}$ where \mathbb{F} is a field and τ is a permutation. Thus a k -cochain is an \mathbb{F} -homomorphism $\gamma : C_k \rightarrow \mathbb{F}$. The boundary operator $\partial_k : C_k \rightarrow C_{k-1}$ induces a unique homomorphism in the cochains $\delta^k : C^{k-1} \rightarrow C^k$ called coboundary operator that satisfies $\gamma(\partial_k([z])) = \delta^k(\gamma)([z])$ for all k -chains z and $(k-1)$ -cochains γ , such that

$$\delta^k(\gamma)([x_0, \dots, x_k]) = \sum_{i=0}^{k+1} (-1)^i \gamma([x_0, \dots, \hat{x}_i, \dots, x_k]).$$

The coboundary operator connects the cochain groups into a cochain complex

$$C^* : \dots \leftarrow C^{k+1} \xleftarrow{\delta^{k+1}} C^k \xleftarrow{\delta^k} C^{k-1} \leftarrow \dots$$

We may also define subgroups of C^k using the coboundary operator: the cocycle group $Z^k = \ker \delta^k$ and the coboundary group $B^k = \text{im } \delta^{k-1}$.

Proposition 2.1.2. $\delta^k \delta^{k-1} = 0$

Proof. Because $\partial_k \partial_{k+1} = 0$. □

This fact implies: $B^k \subseteq Z^k \subseteq C^k$.

Definition 2.1.2. The k -th cohomology group is $H^k = Z^k / B^k$. Its elements are classes of homologous cocycles.

2.1.4 Persistent Cohomology

Given a filtered complex

$$\emptyset \subseteq K_n \subseteq K_{n-1} \subseteq \cdots \subseteq K_1 \subseteq K_0 = K,$$

the i -th complex K_i has associated the coboundary operator δ_i^k and groups C_i^k , Z_i^k , B_i^k and H_i^k for $i, k \geq 0$, and we have the homomorphism restrictions $v_i^k : H_i^k \rightarrow H_{i+1}^k$. Then we have the sequence

$$0 \longrightarrow H_0^* \xrightarrow{v_0^*} H_1^* \xrightarrow{v_1^*} \cdots \xrightarrow{v_{n-2}^*} H_{n-1}^* \xrightarrow{v_{n-1}^*} H_n^* \longrightarrow 0.$$

Definition 2.1.3. The p -persistent k -th cohomology group is

$$H_{i,p}^k = Z_i^k / (B_{i+p}^k \cap Z_i^k).$$

Note 2.1.1. If we define $v_{i,p}^*$ the homomorphism restriction from H_i^* to H_{i+p}^* , then $H_{i,p}^*$ is the image of $v_{i,p}^*$.

Definition 2.1.4. A persistence module \mathcal{M} is a family of R -modules M_i , together with homomorphisms $\varphi_i : M_i \rightarrow M_{i+1}$.

For example, the sequence of cohomology of the filtration is a persistence module, where φ_i are the restrictions.

Definition 2.1.5. A persistence module $\{M_i, \varphi_i\}$ is of finite type if each component module is a finitely generated R -module, and if the maps φ_i are isomorphisms for $i \geq m$ and $i \leq n$ for some integers m and n .

If we have a filtration of a finite K , then the cohomology of the filtration is a persistence module \mathcal{M} of finite type.

2.1.5 Correspondence

Suppose we have a persistence module $\mathcal{M} = \{M_i, \varphi_i\}_{i \in \mathbb{Z}}$ over a ring R . We now equip $R[t]$ with the standard grading and define a graded module over $R[t]$ by

$$\alpha(\mathcal{M}) = \bigoplus_{i \in \mathbb{Z}} M_i,$$

where the R -module structure is simply the sum of the structures on the individual components, and where the action of t is given by

$$t \cdot (\cdots, m_0, m_1, m_2, \cdots) = (\cdots, \varphi_0(m_0), \varphi_1(m_1), \varphi_2(m_2), \cdots).$$

That is, t simply shifts elements of the module up in the graduation.

In the case of the persistence module coming from a cohomology of the filtration of a complex K , we use the follow notation $PH(K)$ to $\alpha(K)$ and we can write

$$PH^*(K) = \bigoplus_{i=0}^n H_i^*$$

where n is the size of the filtration of K .

Theorem 2.1.1. *The correspondence α defines an equivalence of categories between the category of persistence modules of finite type over R and the category of finitely generated graded modules over $R[t]$.*

The proof is the Artin-Rees theory in commutative algebra [18].

2.2 Leray's theorem

Álvaro Torras Casas in his doctoral thesis [31] explored Mayer-Vietoris spectral sequence associated to filtered covers on filtered complexes. From this study, he managed to create an algorithm to calculate persistent homology.

In this section we will explore Leray's spectral sequence associated to filtered covers on filtered complexes. The difference between Mayer-Vietoris spectral sequences and Leray's spectral sequence is while Mayer-Vietoris spectral sequence works on the homology of the open cover of the space, Leray's spectral sequence works on the cohomology of preimage of a map of an open cover of a known space. There are at least two advantages in working with Leray's spectral sequence instead of Mayer-Vietoris spectral sequence, the first is that the algorithms to calculate persistent cohomology are faster and the second is that it works with open cover of known spaces.

Consider $f : X \rightarrow Y$ a map from one space to another, we want to study how the cohomology groups of X relate to those of Y . Let \mathcal{U} be any cover of Y such that $f^{-1}\mathcal{U}$ is a locally finite cover of X . By Generalized Mayer-Vietoris principle

$$H_d^*(X) \simeq H_D\{C_\delta^*(f^{-1}\mathcal{U}, C_d^*)\}.$$

By spectral sequence of double complex, if K is the double complex $C_\delta^*(f^{-1}\mathcal{U}, C_d^*)$ on X , then by Theorem 1.4.2 the spectral sequence of K converges to

$$H_D\{C_\delta^*(f^{-1}\mathcal{U}, C_d^*)\}$$

and

$$E_2^{p,q} = H_\delta^p H_d^q \{C_\delta^*(f^{-1}\mathcal{U}, C_d^*)\}.$$

Thus we have the next version of Leray's theorem.

Theorem 2.2.1 (Leray's Theorem). *Let X and Y be spaces, let \mathcal{U} be a cover of Y such that $f^{-1}\mathcal{U}$ is a cover for X locally finite and let $f : X \rightarrow Y$ be a map. Then there is a spectral sequence converging to $H_d^*(X)$ and $E_2^{p,q} = H_\delta^p H_d^q \{C_\delta^*(f^{-1}\mathcal{U}, C_d^*)\}$.*

Another way to prove the Theorem 2.2.1 is to take the vertical filtration, since the construction was done using horizontal filtration and the page two of spectral sequence of double complex to vertical filtration stabilizes.

Example 2.2.1. *Consider the vertical projection of a circle S^1 onto a segment I . Cover I with three open sets U_0, U_1, U_2 as shown in figure 2.1*

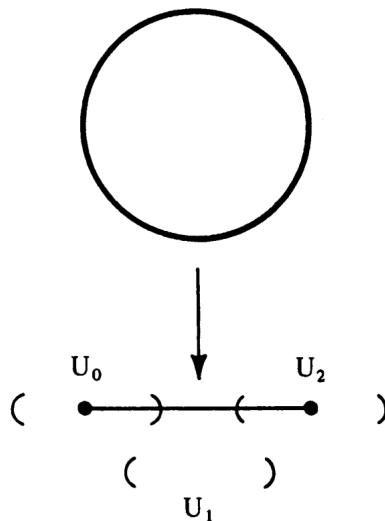


Figure 2.1:

$H_d^0(f^1(U_0)) \simeq H_d^0(f^{-1}(U_2)) \simeq \mathbb{K}$, $H_d^0(f^{-1}(U_1)) \simeq H_d^0(f^{-1}(U_{01})) \simeq H_d^0(f^{-1}(U_{12})) \simeq \mathbb{K}^2$ and $H_d^i = 0$ for all $i > 0$. Then the page E_1 is

$$\mathbb{K} \oplus \mathbb{K}^2 \oplus \mathbb{K} \xrightarrow{\delta} \mathbb{K}^2 \oplus \mathbb{K}^2$$

with δ given by $(b, (c_1, c_2), d) \rightarrow ((c_1 - b, c_2 - b), (d - c_1, d - c_2))$. Thus $\ker \delta = \{(b, (b, b), b)\}$ and $H_\delta^0 H_d^0 \simeq \mathbb{K}$. Since $\text{im } \delta$ is 3-dimensional, $H_\delta^1 H_d^0 \simeq \mathbb{K}$. So the page E_2 is

$$\mathbb{K} \quad \mathbb{K}.$$

In this case, then, $E_2 = E_\infty$.

Note that $H_d^q(f^{-1}\mathcal{U})$ is a presheaf on Y , we will use the notation \mathcal{H}^q for this presheaf. Then in the Leray's theorem, $E_2^{p,q} = H_\delta^p(\mathcal{U}, \mathcal{H}^q)$ is the Čech cohomology of the cover \mathcal{U} with values in \mathcal{H} .

Now we are in a condition to state the most important theorem of this section.

Theorem 2.2.2 (Leray's Theorem for Persistence Module). *Let X and Y be topological spaces, let $f : X \rightarrow Y$ be a map, let \mathcal{U} be a cover of Y such that $f^{-1}\mathcal{U}$ is a cover for X locally finite and a filtration $X = K_0 \supseteq K_1 \supseteq \dots \supseteq K_N = \emptyset$. Then there is a spectral sequence converging to $PH^*(X)$ and $E_2^{p,q} = H_\delta^p(\mathcal{U}, PH^q(f^{-1}\mathcal{U}))$.*

Proof. Let $X = K_0 \supseteq K_1 \supseteq \dots \supseteq K_N = \emptyset$ be a filtration of X , then for the presheaf $H_d^q(K_i \cap f^{-1}(U))$, we will use the notation \mathcal{H}_i^q and the notation $E_{2,i}^{p,q}$ for $H_\delta^p(\mathcal{U}, \mathcal{H}_i^q)$. As the Leray's theorem is true for the restriction of the map f to each subspace K_i , we have a spectral sequence converging to $H_d^*(K_i)$ whose page E_2 is $E_{2,i}^{p,q}$. If we consider $\bigoplus_i E_{2,i}^{p,q} = \bigoplus_i H_\delta^p(\mathcal{U}, \mathcal{H}_i^q) = H_\delta^p(\mathcal{U}, \bigoplus_i \mathcal{H}_i^q)$. For the other side the direct limit commute with the direct sum, then we have a spectral sequence converging to $\bigoplus_i H_d^*(K_i) = PH(X)$ whose page E_2 is $H_\delta^p(\mathcal{U}, \bigoplus_i \mathcal{H}_i^q) = H_\delta^p(\mathcal{U}, PH^q(f^{-1}\mathcal{U}))$.

Now we will to show that the restriction homomorphism $\eta_i^* : C_i^* \rightarrow C_{i+1}^*$ induces a homomorphism on the direct sum of Leray's sequence of filtration, for this we need to show:

1. $d \circ \eta_i^q = \eta_i^{q+1} \circ d$. That is the same that next diagram is commutative

$$\begin{array}{ccc} C_{i+1}^q(f^{-1}(U_{\alpha_0 \dots \alpha_p})) & \xrightarrow{d} & C_{i+1}^{q+1}(f^{-1}(U_{\alpha_0 \dots \alpha_p})) \\ \eta_i^q \uparrow & & \eta_i^{q+1} \uparrow \\ C_i^q(f^{-1}(U_{\alpha_0 \dots \alpha_p})) & \xrightarrow{d} & C_i^{q+1}(f^{-1}(U_{\alpha_0 \dots \alpha_p})). \end{array}$$

Let $\tau \in C_i^q(f^{-1}(U_{\alpha_0 \dots \alpha_p}))$ and $z = (z_0, \dots, z_{q+1}) \in C_{q+1}^{i+1}(f^{-1}(U_{\alpha_0 \dots \alpha_p}))$, then

$$\eta_i^{q+1} \circ d(\tau)(z) = d(\tau)(\iota_{q+1}^{i+1}(z)),$$

where $\iota_{q+1}^{i+1} : C_{q+1}^{i+1}(f^{-1}(U_{\alpha_0 \dots \alpha_p})) \rightarrow C_{q+1}^i(f^{-1}(U_{\alpha_0 \dots \alpha_p}))$ is the inclusion, then

$$\eta_i^{q+1} \circ d(\tau)(z) = \sum_{i=0}^p (-1)^i \tau \iota_{q+1}^{i+1}(z_0, \dots, \hat{z}_i, \dots, z_{q+1}),$$

and for the other side

$$d \circ \eta_i^q(\tau)(z) = \tau \iota_q^{i+1} \left(\sum_{i=0}^p (-1)^i (z_0, \dots, \hat{z}_i, \dots, z_{q+1}) \right) = \sum_{i=0}^p (-1)^i \tau \iota_q^{i+1}(z_0, \dots, \hat{z}_i, \dots, z_{q+1}).$$

Then $d \circ \eta_i^q = \eta_i^{q+1} \circ d$.

2. $\delta \circ \eta_i^q = \eta_i^q \circ \delta$. That is the same that next diagram is commutative

$$\begin{array}{ccc} \prod_{\alpha_0 < \dots < \alpha_p} C_{i+1}^q(f^{-1}(U_{\alpha_0 \dots \alpha_p})) & \xrightarrow{\delta} & \prod_{\alpha_0 < \dots < \alpha_{p+1}} C_{i+1}^q(f^{-1}(U_{\alpha_0 \dots \alpha_{p+1}})) \\ \eta_i^q \uparrow & & \eta_i^q \uparrow \\ \prod_{\alpha_0 < \dots < \alpha_p} C_i^q(f^{-1}(U_{\alpha_0 \dots \alpha_p})) & \xrightarrow{\delta} & \prod_{\alpha_0 < \dots < \alpha_{p+1}} C_i^q(f^{-1}(U_{\alpha_0 \dots \alpha_{p+1}})). \end{array}$$

Let $\tau \in \prod_{\alpha_0 < \dots < \alpha_p} C_i^q(f^{-1}(U_{\alpha_0 \dots \alpha_p}))$ and $z \in C_i^q(f^{-1}(U_{\alpha_0 \dots \alpha_{p+1}}))$, then

$$\delta \circ \eta_i^q(\tau)(z) = \sum_{i=0}^{p+1} (-1)^i (\eta_i^q(\tau))_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}(z) = \sum_{i=0}^{p+1} (-1)^i (\tau \iota_q^{i+1})_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}(z),$$

and for the other side

$$\eta_i^q \circ \delta(\tau)(z) = \delta(\tau)(\iota_q^{i+1}(z)) = \sum_{i=0}^{p+1} (-1)^i \tau_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}(\iota_q^{i+1}(z)),$$

as $(\tau \iota_q^{i+1})_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}(z) = \tau_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}(\iota_q^{i+1}(z))$, we have $\delta \circ \eta_i^q = \eta_i^q \circ \delta$.

3. η respects the filtration (horizontal filtration of the double complex). This is true because the restriction acting in the index of the summation, not in index p or q .

We will call $\eta_{r,i}^{p,q} : E_{r,i}^{p,q} \rightarrow E_{r,i+1}^{p,q}$ the homomorphism induced by η_i^* and we define

$$\eta_{r,i}^n : \bigoplus_{p+q=n} E_{r,i}^{p,q} \rightarrow \bigoplus_{p+q=n} E_{r,i+1}^{p,q}$$

as $\eta_{r,i}^n = \bigoplus_{p+q=n} \eta_{r,i}^{p,q}$. Now we can define an action on the spectral sequence $\bigoplus_{p+q=n} \bigoplus_i E_{2,i}^{p,q} = \bigoplus_{p+q=n} H_{\delta}^p(\mathcal{U}, \bigoplus_i \mathcal{H}_i^q)$ given by

$$t(\dots, m_{r,0}^n, m_{r,1}^n, m_{r,2}^n, \dots) = (\dots, \eta_{r,0}^n(m_{r,0}^n), \eta_{r,1}^n(m_{r,1}^n), \eta_{r,2}^n(m_{r,2}^n), \dots).$$

where $m_{r,i}^n = \bigoplus_{p+q=n} m_{r,i}^{p,q}$.

Then this action define an action on the limit of the spectral sequence, the limit is $\bigoplus_i H^*(K_i)$, we can show that it homomorphism is isomorphic to the restriction $\eta_i^* : H_d^*(K_i) \rightarrow H_d^*(K_{i+1})$ taking the vertical filtration on the double complex. Then we conclude that the spectral sequence $\bigoplus_i E_{2,i}^{p,q}$ converge to $\alpha(\{H_d^*(K_i), \eta_i^*\})$. Then we have the theorem. \square

Example 2.2.2. Consider the projection of the torus T onto the circle S^1 as shown in figure 2.2 and take the filtration $T = K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 = \emptyset$ as shown in figure 2.3.

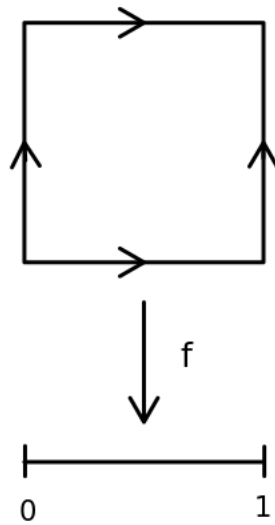


Figure 2.2:

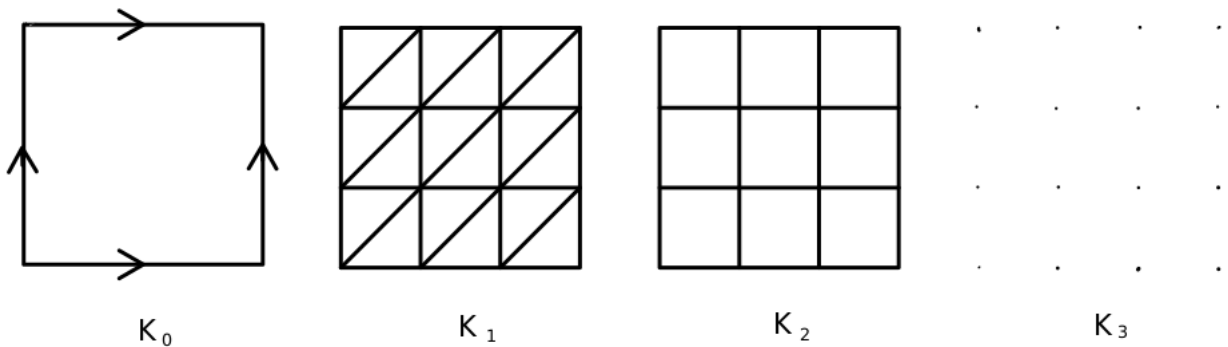


Figure 2.3: Filtration of torus.

For an analytical illustration take the torus as $[0, 1]^2 / \sim$ and the circle $[0, 1] / \sim$ and consider the cover of S^1 : $U_0 = (\frac{9}{10}, \frac{2}{5}), U_1 = (\frac{2}{5}, \frac{4}{5})$ and $U_2 = (\frac{1}{2}, \frac{1}{10})$.

In the filtration $K_3 = \{(0, 0), (0, \frac{1}{3}), (0, \frac{2}{3}), (\frac{1}{3}, 0), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, 0), (\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3})\}$ (the vertices), K_2 is the complex formed by the horizontal and vertical edges between the vertices of K_3 , K_1 is the complex formed by K_2 union the anti-diagonal edges and K_0 is the torus.

Then $H^*(K_3) = \{\mathbb{K}^9, 0, 0\}$, $H^*(K_2) = \{\mathbb{K}, \mathbb{K}^{10}, 0\}$, $H^*(K_1) = \{\mathbb{K}, \mathbb{K}^{19}, 0\}$ and $H^*(K_0) = \{\mathbb{K}, \mathbb{K}^2, \mathbb{K}\}$,
 $H^*(K_3 \cap f^{-1}(U_i)) = \{\mathbb{K}^6, 0, 0\}$, $H^*(K_2 \cap f^{-1}(U_i)) = \{\mathbb{K}, \mathbb{K}^2, 0\}$, $H^*(K_1 \cap f^{-1}(U_i)) = \{\mathbb{K}, \mathbb{K}^7, 0\}$,
 $H^*(K_0 \cap f^{-1}(U_i)) = \{\mathbb{K}, \mathbb{K}, 0\}$, $H^*(K_3 \cap f^{-1}(U_{ij})) = \{\mathbb{K}^3, 0, 0\}$, $H^*(K_2 \cap f^{-1}(U_{ij})) = \{\mathbb{K}, \mathbb{K}, 0\}$,
 $H^*(K_1 \cap f^{-1}(U_{ij})) = \{\mathbb{K}, \mathbb{K}, 0\}$, $H^*(K_0 \cap f^{-1}(U_{ij})) = \{\mathbb{K}, \mathbb{K}, 0\}$.

$$E_{1,3} =$$

$$0 \xrightarrow{\delta} 0$$

$$\mathbb{K}^6 \oplus \mathbb{K}^6 \oplus \mathbb{K}^6 \xrightarrow{\delta} \mathbb{K}^3 \oplus \mathbb{K}^3 \oplus \mathbb{K}^3$$

$$E_{2,3} =$$

$$0 \quad 0$$

$$\mathbb{K}^9 \quad 0$$

$$E_{1,2} =$$

$$\mathbb{K}^4 \oplus \mathbb{K}^4 \oplus \mathbb{K}^4 \xrightarrow{\delta} \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$$

$$\mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \xrightarrow{\delta} \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$$

$$E_{2,2} =$$

$$\mathbb{K}^9 \quad 0$$

$$\mathbb{K} \quad \mathbb{K}$$

$$E_{1,1} =$$

$$\mathbb{K}^7 \oplus \mathbb{K}^7 \oplus \mathbb{K}^7 \xrightarrow{\delta} \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$$

$$\mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \xrightarrow{\delta} \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$$

$$E_{2,1} =$$

$$\mathbb{K}^{18} \quad 0$$

$$\mathbb{K} \quad \mathbb{K}$$

$$E_{1,0} =$$

$$\mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \xrightarrow{\delta} \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$$

$$\mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \xrightarrow{\delta} \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$$

$$E_{2,0} =$$

$$\mathbb{K} \quad \mathbb{K}$$

$$\mathbb{K} \quad \mathbb{K}$$

Separation Property for Generalized Manifold

3.1 Generalized Manifold

Definition 3.1.1. *A locally compact space X is a generalized m -manifold if the following conditions are satisfied:*

1. *X is an ENR (Euclidean Neighborhood Retract), if there exist a subspace Y of some \mathbb{R}^n homeomorphic to X , a neighborhood V of Y and a retraction $r : V \rightarrow Y$;*
2. *$H_*(X, X - \{x\}; R) \simeq H_*(\mathbb{R}^m, \mathbb{R}^m - \{0\}; R)$ for every $x \in X$, where R is \mathbb{Z} or \mathbb{Z}_2 .*

The space X is a generalized m -manifold with boundary if the condition 2 is replaced by $H_m(X, X - \{x\}; R) \simeq R$ or 0 , and if $bd(X) = \{x \in X : H_m(X, X - \{x\}; R) \simeq 0\}$ is a generalized $(m - 1)$ -manifold embedded in X

Such manifolds have been studied throughout the second half of the twentieth century, for example, it is a known fact that there are generalized manifold that are not homeomorphic to topological manifold(see [12][13]). Another known fact is that the following dualities apply (see [11] [21]).

Theorem 3.1.1 (Poincaré’s Duality Theorem for Generalized Manifold). *The duality map $D_M : H_c^k(M; R) \rightarrow H_{n-k}(M; R)$ given by $D_M(\alpha) = \alpha \frown [M]$, is an isomorphism for all k whenever M is an R -oriented generalized n -manifold.*

Theorem 3.1.2 (Alexander’s Duality Theorem for Generalized Manifold). *Let X be a oriented generalized n -manifold. If B is a closed subset of X , then $\check{H}_c^{n-i}(B; R) \simeq H_i(X, X - B; R)$ for each integer i , where \check{H}_c^* represent Čech cohomology with support compact.*

Now we will present the most important results concerning the topology of generalized manifold. These results are obtained assuming basically two conditions that we will define below.

Definition 3.1.2. A generalized n -manifold, $n \geq 5$, has the disjoint disks property (DDP) if given $\varepsilon > 0$ and a pair of maps $f, g : D^2 \rightarrow X$, there are a pair of maps $f', g' : D^2 \rightarrow X$ such that $d(f, f') < \varepsilon$, $d(g, g') < \varepsilon$ and $f'(D^2) \cap g'(D^2) = \emptyset$.

Definition 3.1.3. A resolution for a generalized n -manifold X is a map $\phi : M \rightarrow X$ such that $\phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \rightarrow U$ is an homotopic equivalence for all open $U \supset X$, where M is a topological manifold. If X admits a resolution, we say that X is resolvable.

Now we are in a condition to state one of the most important theorems for generalized manifold.

Theorem 3.1.3 (Edwards's approximation theorem). *Let X be a generalized n -manifold with DDP. If $\phi : M \rightarrow X$ is a resolution for X , then ϕ is limit of a sequence of homeomorphisms $h_i : M \rightarrow X$.*

Corollary 3.1.1. *Let X be a generalized n -manifold resolvable, $n \geq 5$. X is a topological manifold if, and only if, X have the DDP.*

In this way, the corollary 3.1.1 provides a criterion to characterize topological manifolds based on the generalized manifolds. In [27][28], F. Quinn associates to any connect generalized n -manifold, $n \geq 4$, a local index $\iota(X) \in 1 + 8\mathbb{Z}$ and he shows that $\iota(X) = 1$ if, and only if, X is resolvable. Combining it with Edwards's approximation theorem we have the next theorem of characterization to topological manifold.

Theorem 3.1.4 (Edwards-Quinn). *If $n \geq 5$, a generalized n -manifold X with DDP is a topological manifold if, and only if, $\iota(X) = 1$.*

This characterization open the question: Are there generalized manifold X with $\iota(X) \neq 1$? The next theorem that appeared in [12][13] answer this question.

Theorem 3.1.5 (Bryant, Ferry, Mio, Weinberger). *Let M be a closed simply-connected topological n -manifold and $\sigma \in 8\mathbb{Z} + 1$. There is an generalized n -manifold X homotopy equivalent to M with $\iota(X) = \sigma$.*

3.2 Separation by codimension-1 map depending on the self-intersection set

In this section we will see the first result of separation by codimension-1 map to generalized manifold, Theorem 3.2.1, which is an extension of Theorem 2.2 in [25], from topological manifold to generalized manifold.

Lemma 3.2.1. *Consider the following commutative diagram of R -modules, where the rows are exact and g is an isomorphism*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ D & \xrightarrow{\lambda} & A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

Then, $\ker(h) \simeq \operatorname{coker}(f + \lambda)$, where $f + \lambda : A \oplus D \rightarrow A'$ is the induced map.

Proof. We apply the ker-coker lemma to the following diagram, which is obtained in a natural way from the above

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow \bar{f} & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & \operatorname{coker}(\lambda) & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

getting the exact sequence:

$$\ker(\bar{f}) \longrightarrow \ker(g) \longrightarrow \ker(h) \longrightarrow \operatorname{coker}(\bar{f}) \longrightarrow \operatorname{coker}(g) \longrightarrow \operatorname{coker}(h).$$

Since $\ker(g) = \operatorname{coker}(g) = 0$, we have that $\ker(h) \simeq \operatorname{coker}(\bar{f})$ and the lemma follows from the isomorphism

$$\operatorname{coker}(\bar{f}) = \frac{A'/\operatorname{Im}(\lambda)}{(\operatorname{Im}(f) + \operatorname{Im}(\lambda))/\operatorname{Im}(\lambda)} \simeq \frac{A'}{\operatorname{Im}(f) + \operatorname{Im}(\lambda)} = \operatorname{coker}(f + \lambda).$$

□

Theorem 3.2.1. *Let $f : X \rightarrow Y$ be a proper map from a connected generalized n -manifold to a connected generalized $(n + 1)$ -manifold with $H_1(Y; \mathbb{Z}_2) = 0$ and let A be the closure of the self-intersection set $A(f) = \{x \in X : f^{-1}f(x) \neq x\}$. Suppose that $A \neq X$ and $Y - f(A)$ is connected. Then $\beta_0(Y - f(X)) = 2 + \dim_{\mathbb{Z}_2} \operatorname{coker}(i^* + f|_A^*)$, where $i^* + f|_A^* : \check{H}_c^{n-1}(X; \mathbb{Z}_2) \oplus \check{H}_c^{n-1}(f(A); \mathbb{Z}_2) \rightarrow \check{H}_c^{n-1}(A; \mathbb{Z}_2)$ is the induced map.*

Proof. To simplify the notation, we shall omit the coefficient group \mathbb{Z}_2 in all homology and cohomology groups.

Since f is proper (and hence closed), $f(A)$, $f(X)$ are closed and we can consider the Čech cohomology of the pairs (X, A) , $(f(X), f(A))$ and get the following commutative diagram, where the rows are exact:

$$\begin{array}{ccccccc} \longrightarrow & \check{H}_c^{n-1}(f(X)) & \longrightarrow & \check{H}_c^{n-1}(f(A)) & \longrightarrow & \check{H}_c^n(f(X), f(A)) & \longrightarrow \\ & \downarrow (1) & & \downarrow (2) & & \downarrow (3) & \\ \longrightarrow & \check{H}_c^{n-1}(X) & \longrightarrow & \check{H}_c^{n-1}(A) & \longrightarrow & \check{H}_c^n(X, A) & \longrightarrow \\ & & & & & & \\ \longrightarrow & \check{H}_c^n(f(X)) & \longrightarrow & \check{H}_c^n(f(A)) & \longrightarrow & \check{H}_c^{n+1}(f(X), f(A)) & \longrightarrow \\ & \downarrow (4) & & \downarrow (5) & & \downarrow (6) & \\ \longrightarrow & \check{H}_c^n(X) & \longrightarrow & \check{H}_c^n(A) & \longrightarrow & \check{H}_c^{n+1}(X, A) & \longrightarrow \end{array}$$

But some of these cohomology groups are computed using the Alexander's duality:

$$\check{H}_c^n(X) \simeq H_0(X) \simeq \mathbb{Z}_2,$$

$\check{H}_c^n(f(X)) \simeq H_1(Y, Y - f(X)) \simeq \tilde{H}_0(Y - f(X))$, where the last isomorphism comes from the exact

sequence of the pair $(Y, Y - f(X))$:

$$0 = H_1(Y) \longrightarrow H_1(Y, Y - f(X)) \longrightarrow \tilde{H}_0(Y - f(X)) \longrightarrow \tilde{H}_0(Y) = 0 .$$

This gives a formula for the number of connected components of $Y - f(X)$:

$$\beta_0(Y - f(X)) = 1 + \dim_{\mathbb{Z}_2} \check{H}_c^n(f(X)). \quad (3.1)$$

We apply also the Alexander's duality to A and $f(A)$:

$$\check{H}_c^n(A) \simeq H_0(X, X - A) = 0,$$

$\check{H}_c^n(f(A)) \simeq H_1(Y, Y - f(A)) = 0$, where the last equality comes from the exact sequence of the pair $(Y, Y - f(A))$:

$$0 = H_1(Y) \longrightarrow H_1(Y, Y - f(A)) \longrightarrow \tilde{H}_0(Y - f(A)) = 0 .$$

On the other hand, for the Theorem 5 in [30] pag. 318, the maps (3) and (6) in the above diagram are isomorphisms. Then we can apply the Five Lemma to the maps (2), ..., (6) and deduce that $f^* : \check{H}_c^n(f(X)) \longrightarrow \check{H}_c^n(X)$ is an epimorphism. Therefore $\dim_{\mathbb{Z}_2} \check{H}_c^n(f(X)) = 1 + \dim_{\mathbb{Z}_2} \ker(f^*)$.

But the above lemma 3.2.1 implies that $\ker(f^*) = \text{coker}(i^* + f|_A^*)$, then $\beta_0(Y - f(X)) = 2 + \dim_{\mathbb{Z}_2} \text{coker}(i^* + f|_A^*)$.

□

Now we are going to see some particular results according to the closure of the self-intersection set A . If $A = \emptyset$ we have the next version of Jordan-Brouwer Theorem.

Corollary 3.2.1 (Jordan-Brouwer Theorem to generalized manifold). *Let $f : X \longrightarrow Y$ be a proper embedding from a connected generalized n -manifold to a connected generalized $(n + 1)$ -manifold with $H_1(Y; \mathbb{Z}_2) = 0$. Then the number of connect components of $Y - f(X)$ is 2.*

Proof. The self-intersection set A is empty, then $X \neq A$, $Y - f(A)$ is connect, $\check{H}_c^{n-1}(A; \mathbb{Z}_2) = 0$ and $\dim_{\mathbb{Z}_2} \text{coker}(i^* + f|_A^*) = 0$.

□

An other result is in the case where $\dim A < n$ and A compact.

Proposition 3.2.1. *Let $f : X \longrightarrow Y$ be a proper map from a connected generalized n -manifold to a connected generalized $(n + 1)$ -manifold with $H_1(Y; \mathbb{Z}_2) = 0$ and let A be the closure of the selfintersection set $A(f) = \{x \in X : f^{-1}f(x) \neq x\}$. Suppose that $\dim A < n$ and A compact, then $Y - f(X)$ is disconnected.*

Proof. As $\dim A < n$ and A compact we have that $\check{H}_c^n(A) = \check{H}^n(A) = H^n(A) = 0$, and hence (5) in the diagram

$$\begin{array}{ccccccc}
\longrightarrow & \check{H}_c^{n-1}(f(X)) & \longrightarrow & \check{H}_c^{n-1}(f(A)) & \longrightarrow & \check{H}_c^n(f(X), f(A)) & \longrightarrow \\
& \downarrow (1) & & \downarrow (2) & & \downarrow (3) & \\
\longrightarrow & \check{H}_c^{n-1}(X) & \longrightarrow & \check{H}_c^{n-1}(A) & \longrightarrow & \check{H}_c^n(X, A) & \longrightarrow \\
& & & & & & \\
\longrightarrow & \check{H}_c^n(f(X)) & \longrightarrow & \check{H}_c^n(f(A)) & \longrightarrow & \check{H}_c^{n+1}(f(X), f(A)) & \longrightarrow \\
& \downarrow (4) & & \downarrow (5) & & \downarrow (6) & \\
\longrightarrow & \check{H}_c^n(X) & \longrightarrow & \check{H}_c^n(A) & \longrightarrow & \check{H}_c^{n+1}(X, A) & \longrightarrow
\end{array}$$

is an epimorphism. Then the Five Lemma again to the maps (2), ..., (6) implies that (4) is an epimorphism. Then we have that $\check{H}_c^n(f(X)) \neq 0$ and by equation 3.1, $\beta_0(Y - f(X)) \geq 2$. \square

3.3 Separation by codimension-1 map depending on the normal crossing point

In this section we will show the second result on separation by codimension-1 map to generalized manifold, Theorem 3.3.2, in this theorem we adapt the arguments of the Saeki's proof of the Theorem 2.2 in [29], obtaining a similar result to generalized manifold. In the following of this section $R = \mathbb{Z}$.

Definition 3.3.1. A subset A of X is locally n -co-connect (n -LCC) in X if for each $x \in A$ and neighborhood U of x , there is a neighborhood V of x in X lying in U such that the inclusion induced homomorphism $\pi_i(V - A) \rightarrow \pi_i(U - A)$ is trivial for all $i \leq n$.

Definition 3.3.2. Let $f : M \rightarrow N$ be a map from a generalized $(n - 1)$ -manifold into a generalized n -manifold. A point $q \in N$ is called a normal crossing point if there exist a neighborhood U of q and an integer m ($0 \leq m \leq n$) such that $f^{-1}(q)$ consists of m distinct points q_i ($1 \leq i \leq m$) and that there are pairwise disjoint neighborhood W_i in M of q_i such that $f|_{W_i} : W_i \rightarrow N$ are proper embedding and $U \cap f(M) = U \cap (\cup_{i=1}^m f(W_i))$ corresponds to union of m distinct subspaces in general position. We call m the multiplicity of q . Here, m distinct subspaces H_i ($1 \leq i \leq m$) are in general position if for every sequence of integers $i(1), \dots, i(s)$ with $1 \leq i(1) < \dots < i(s) \leq m$ we have $\text{codim}(H_{i(1)} \cap \dots \cap H_{i(s)}) = \text{codim}(H_{i(1)}) + \dots + \text{codim}(H_{i(s)})$ and H_i separate any other H_j in two connected components.

Definition 3.3.3. By a pseudo-isotopy of a space M , we mean a homotopy $H_t : M \rightarrow M$, $t \in [0, 1]$, such that H_t is a homeomorphism for $0 \leq t < 1$.

The following result was proven in [22] as Theorem 2.4.9.

Theorem 3.3.1. Let X be a generalized n -manifold, $n \geq 3$, and F a closed subset of X with dimension $n - 1$. Then for all map $f : I \rightarrow X$ can be ε -approximated by a 0-LCC embedding $f' : I \rightarrow X$ such that $f'(I) \cap F$ have dimension 0.

Proposition 3.3.1. *Suppose that X is a simply-connected generalized n -manifold, $n \geq 3$. Let M be a compact arc and let Q be a closed submanifold of X with dimension $n - 1$. Then there exist an pseudo-isotopy H_t of X such that $H_1(M) \cap H_1(Q)$ is a point.*

Proof. For Theorem 3.3.1 we can assume that $F = M \cap Q$ have dimension 0 and is 0-LCC. Let A be a compact arc in M such that $F \subset A$ and let B be a compact arc in Q such that $F \subset B$. Then $A \cup B$ is union of arcs and circles, how X is simply-connected, there exist a pseudo-isotopy H_t such that $H_1(A \cup B) = \{x\}$ a point in F . Then $H_1(M) \cap H_1(Q) = \{x\}$. \square

Definition 3.3.4. *Let $f : X \rightarrow Y$ be a map. We said that f is an immersion, if for each $x \in X$ there is a neighborhood B of x that $f|_B : B \rightarrow Y$ is an embedding.*

The next idea we take from [14].

Note 3.3.1. *Suppose that X is an oriented generalized n -manifold, $n \geq 5$, and that M is a closed oriented topological m -dimensional submanifold of X . Let $V \subset M$ be an open m -ball neighborhood of a point $x \in M$ and let U be an open set in X such that $U \cap M = V$. Then for Alexander's Duality, we have*

$$H_{n-p}(U, U - V) \simeq \check{H}_c^p(V) \simeq \begin{cases} \mathbb{Z} & \text{if } p = m, \\ 0 & \text{otherwise.} \end{cases}$$

*Suppose $q = n - m$ and that Q is a closed oriented q -dimensional submanifold of X such that $Q \cap V = \{x\}$. Let $W = Q \cap U$, then $H_q(W, W - \{x\}) \simeq \mathbb{Z}$. Thus, the inclusion $H_q(W, W - \{x\}) \rightarrow H_q(U, U - V)$ is multiplication by an integer k , with we define **intersection number of M and Q at x** ($I_x(M, Q)$). If $M \cap Q$ is a finite set F , define the **intersection number of M and Q** ($I(M, Q)$) to be the sum over F of the intersection numbers at $x \in F$ or zero if $F = \emptyset$. If $f : Y \rightarrow X$ is a immersion map from a closed generalized manifold, for all $p \in F(Y)$, $f^{-1}(p) = \{p_1, \dots, p_s\}$ and there are U, U_1, \dots, U_s disjoint neighborhood of p, p_1, \dots, p_s respectively such that $f|_{U_i} : U_i \rightarrow X$ is an embedding and $f^{-1}U = \bigcup_i U_i$. If $V \cap f(Y) = \{x\}$ we can define **intersection number of M and $f(Y)$ at x** as $I_x(M, f(Y)) = \sum_{i=1}^s I_x(M, f(U_i))$. If $M \cap f(Y)$ is a finite set F , define the **intersection number of M and $f(Y)$** ($I(M, f(Y))$) to be the sum over f of the intersection number at $x \in F$ or zero if $F = \emptyset$.*

Note 3.3.2. *Since intersection number is thus defined in terms of homology, it is homotopy type invariant.*

Definition 3.3.5. *A polyhedron P in a generalized manifold M is tame in M , if, for every point $x \in P$ there is a triangulation of P containing x as a vertex and a triangulation of a neighborhood of x in M containing the star of x in P as a subcomplex.*

The proof of the next proposition is part of the Theorem 5.3 in [14]

Proposition 3.3.2. *Let X be an oriented generalized n -manifold, $n \geq 5$ with DDP. Let M be a closed oriented topological 1-dimensional submanifold of X and let Q be an tame closed oriented $n - 1$ -dimensional PL submanifold of X such that $M \cap Q = \{x\}$. Then the intersection number of M and Q at x is $+1$ or -1 .*

Theorem 3.3.2. *Let $f : M \rightarrow N$ be an PL immersion map from a closed oriented connected PL $(n - 1)$ -manifold M into an oriented simply-connected generalized n -manifold N , with DDP and $n \geq 5$, such that $f(M)$ is tame in N . If f has a normal crossing point of multiplicity 2, then the number of path connected components of $N - f(M)$ is greater than or equal to 3.*

Proof. Let $q \in N$ be a normal crossing point of multiplicity m , U an open neighborhood of q in N and W_i neighborhood of q_i for each i , as in the definition of normal crossing point.

For x and y in $U - f(M)$ we define $x \sim y$ if x and y are connected by a path in $N - f(M)$.

We want to show that the number l of elements of $(U - f(M))/\sim$ is greater than or equal to $m + 1$.

First suppose $m = 1$. We can apply the Corollary 3.2.1 for $f|_{W_i}^U : W_i \rightarrow U$, this map is a proper embedding, then the number of connected components of $U - f(M)$ is 2. Suppose $l = 1$, then any two points x and y of different components are connected by an arc α in $N - f(M)$ and for Proposition 3.3.1 there exist an arc β in U that join x and y and intercept $f(M)$ in one point z . Then $\alpha \cup \beta$ is a cycle that $(\alpha \cup \beta) \cap f(M) = \{z\}$, then the intersection number of $\alpha \cup \beta$ and $f(M)$ is 1 or -1 , but $\alpha \cup \beta$ is contractible because N is simply-connected, then the intersection number of $\alpha \cup \beta$ and $f(M)$ is zero. That is absurd, then $l \geq 2$.

Now suppose $m = 2$, for definition $f(W_1)$ separate $f(W_2)$ in two connect components and for Corollary 3.2.1, $f(W_1)$ separate U in two connect components that we call U^+ and U^- . Take $q_1 \in f(W_2) \cap U^+$ and $q_2 \in f(W_2) \cap U^-$. Note that q_1 and q_2 are normal crossing points with multiplicity one, then the number of elements of $(U^+ - f(M))/\sim$ is greater or equal to 2, similarly we see that the number of elements in $(U^- - f(M))/\sim$ is also greater or equal 2. Suppose $l \leq 2$, then every point in $U^+ - f(M)$ is connected by a path in $N - f(M)$ to a point in $U^- - f(M)$. Take a point $z_1 \in U^+ - f(M)$, then take a path α_1 in $N - f(M)$ which connect z_1 with $z'_1 \in U^- - f(M)$. There exist an arc α'_1 in U with intercepts $f(M)$ in one point x_1 and with connect z'_1 with a point $z_2 \in U^+ - f(M)$. If we repeat this procedure inductively to obtain z_i, α_i, z'_i, x_i and α'_i such that $I_{x_i}(\alpha'_i(I), f(M)) = I_{x_1}(\alpha'_1(I), f(M))$ (we can do this because it always crosses $f(W_1)$ on the same side) for all i . Then z_s and z_t belong to the same connected component of $U^+ - f(M)$ for some $s < t$, since the number of connected components of $U^+ - f(M)$ is finite. Connect z_s and z_t by a path α'' in $U^+ - f(M)$ and let $\beta = \alpha_s \cup \alpha'_s \cup \dots \cup \alpha'_{t-1} \cup \alpha_{t-1} \cup \alpha''$ be the resulting closed path in N with is a cycle and $I(\beta(I), f(M)) = \pm(t - s) \neq 0$. But β is contractible because N is simply-connected, then $I(\beta(I), f(M)) = 0$. That is absurd, then $l \geq 3$

Since the number of path connect components of $N - f(M)$ is greater or equal to the number of elements of $(U - f(M))/\sim$, we have the result. \square

Note 3.3.3. *It should be noted that the generalized manifold constructed in [12][13] for any index,*

are simply-connected, then we can apply the Theorem 3.3.2 for this generalized manifolds.

3.4 Separation by codimension-1 map depending on the primary obstruction to topological embedding

In this section we will show the third result on separation by codimension-1 map to generalized manifold, Theorem 3.4.2, in which we use the concept of primary obstruction to topological embedding that has been defined and studied in [2][3][4][5][6]. In this theorem we adapt the arguments used in [5] to prove the Proposition 6.4 for smooth manifold that is about separation by codimension-1 map.

In the following of this section $R = \mathbb{Z}_2$, let M and N be generalized manifolds of dimensions m and n , respectively, such that $k = n - m > 0$ and $f : M \rightarrow N$ is a proper map. Let A be the closure of the self-intersection set $A(f) = \{x \in M : f^{-1}f(x) \neq x\}$. Let denote by $U_f \in H^k(N)$ the Poincaré dual of $f_*[M] \in H_m^c(N)$; in other words $f_*[M] = U_f \frown [N]$. Note that $f_*[M]$ is well-defined, since f is a proper map.

Let the total Stiefel-Whitney class of M and N denoted by $w(M) \in H^*(M)$ and $w(N) \in H^*(N)$ respectively and let $\bar{w}(M) \in H^*(M)$; i.e. $\bar{w}(M) = w(M)^{-1}$.

Definition 3.4.1. $w(f) = (f^*(w(N)) \smile \bar{w}(M))$ is called total Stiefel-Whitney class of the stable normal bundle of f . We denote by $w_k(f) \in H^k(M)$ the degree k term of $w(f)$, with is the k -th Stiefel-Witney class of stable normal bundle of f .

Definition 3.4.2. $\theta(f) = (f^*U_f - w_k(f)) \frown [M] \in H_{m-k}^c(M)$ is called primary obstruction to topological embedding.

Note 3.4.1. This homology class is a proper homotopy invariant of f and has the property that, when M and N are generalized manifolds, if f is properly homotopic to a embedding, then $\theta(f)$ vanishes, this was showed in [2].

Note that when M is compact, $\theta(f) \in H_{m-k}(M)$, since we have $H_{m-k}(M) \simeq H_{m-k}^c(M)$.

Next three results are Theorem 3.1, Corollary 3.15 and Corollary 3.17 in [2].

Theorem 3.4.1. Let $f : M \rightarrow N$ be a proper map of an generalized m -manifold M into an generalized $m + k$ -manifold N with $k > 0$. Then $f_*\theta(f) \in H_{m-k}^c(N)$ always vanishes.

Corollary 3.4.1. Let $f : M \rightarrow N$ be a map from an compact generalized m -manifold M into an generalized $m + k$ -manifold N with $k > 0$. Set $B = f(A)$. Then there exists an element $\mu \in \check{H}_{m-k}(A)$ such that $j_*\mu = \theta(f) \in \check{H}_{m-k}(M) = H_{m-k}(M)$ and $(f|_A)_*(\mu) = 0 \in \check{H}_{m-k}(B)$, where $j : A \rightarrow M$ is the inclusion map. (When $A = \emptyset$, we regard $\check{H}_{m-k}(A) = 0 = \check{H}_{m-k}(B)$)

Corollary 3.4.2. *Let $f : M \rightarrow N$ be a map from an compact generalized m -manifold M into an generalized $(m+k)$ -manifold with $k \geq 0$. If the topological dimension of A is strictly less than $m-k$, then $\theta(f) \in H_{m-k}(M)$ vanishes.*

Lemma 3.4.1. *Let $f : M \rightarrow N$ be a map from an compact generalized m -manifold M into an generalized $(m+k)$ -manifold with $k \geq 0$. Suppose that $\mu \in \check{H}_{m-k}(A)$ is not zero. Then if $\theta(f)$ vanishes, $\bar{f}_* : \check{H}_{m-k+1}(M) \rightarrow \check{H}_{m-k+1}(f(M))$ is not surjective.*

Proof. Suppose that f is not an embedding; i.e., $A \neq \emptyset$. Consider the following diagram of Čech homologies with exact rows:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \check{H}_i(A) & \longrightarrow & \check{H}_i(M) & \longrightarrow & \check{H}_i(M,A) \longrightarrow \check{H}_{i-1}(A) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \check{H}_i(B) & \longrightarrow & \check{H}_i(f(M)) & \longrightarrow & \check{H}_i(f(M),B) \longrightarrow \check{H}_{i-1}(B) \longrightarrow \cdots \end{array}$$

where the vertical homomorphisms are induced by f . Note that the homomorphism $f_* : \check{H}_i(M,A) \rightarrow \check{H}_i(f(M),B)$ is an isomorphism by excision. Then it is not difficult to extract the following exact sequence:

$$\begin{aligned} \check{H}_{m-k+1}(A) &\xrightarrow{\alpha} \check{H}_{m-k+1}(B) \oplus \check{H}_{m-k+1}(M) \xrightarrow{((j'')_* \oplus \bar{f}_*)} \check{H}_{m-k+1}(f(M)) \\ &\rightarrow \check{H}_{m-k}(A) \xrightarrow{\alpha} \check{H}_{m-k}(B) \oplus \check{H}_{m-k}(M) \end{aligned}$$

where $j'' : B \rightarrow f(M)$ is the inclusion map, $\alpha = ((f|_A)_*, i_*)$ where $i : A \rightarrow M$ is the inclusion map. By corollary 3.4.1 and $\theta(f) = 0$, then $\ker(\alpha) \neq 0$. Therefore $\bar{f}_* : \check{H}_{m-k+1}(M) \rightarrow \check{H}_{m-k+1}(f(M))$ can not be surjective. \square

Theorem 3.4.2. *Let $f : M \rightarrow N$ be a map from a compact generalized m -manifold M into a generalized $(m+1)$ -manifold N with $H_1(N) = 0$. If $A \neq M$, $\mu \neq 0$ and $w_1(f) = 0$, then the number of connected components of $N - f(M)$ is greater than or equal to three.*

Proof. By hypothesis $A \neq M$, then $0 = H_0(M, M - A)$, by Alexander's duality $\check{H}_c^m(A) \simeq 0$, how M is compact hence A compact, $H^m(A) = \check{H}^m(A) = \check{H}_c^m(A) = 0$ and by universal coefficient theorem $0 = \check{H}_m(A) = H_m(A)$.

Consider the following exact sequence of the pair $(N, N - f(M))$

$$H_1(N) \rightarrow H_1(N, N - f(M)) \rightarrow \check{H}_0(N - f(M)) \rightarrow \check{H}_0(N)$$

by hypothesis $H_1(N) = 0 = \check{H}_0(N)$, then $H_1(N, N - f(M))$ is isomorphic to $\check{H}_0(N - f(M))$. On the other hand $H_1(N, N - f(M))$ is isomorphic to $\check{H}^m(f(M))$ by Alexander's Duality. Thus we see that $\check{H}^m(f(M)) \simeq H^m(f(M))$ (because $f(M)$ is compact) which is isomorphic to $H_m(f(M))$ by universal coefficient theorem, then $\beta_0(N - f(M)) = \dim H_m(f(M)) + 1$.

On the other hand we have the following exact sequence

$$\check{H}_m(A) \xrightarrow{\alpha} \check{H}_m(B) \oplus \check{H}_m(M) \xrightarrow{((j'')_* \oplus \tilde{f}_*)} \check{H}_m(f(M))$$

where $j'' : B \rightarrow f(M)$ is the inclusion map, $\alpha = ((f|_A)_*, i_*)$ where $i : A \rightarrow M$ is the inclusion map.

As $\check{H}_m(A) = 0$, then $\tilde{f}_* : \check{H}_m(M) \rightarrow \check{H}_m(f(M))$ is a monomorphism. By our hypothesis $w_1(f) = 0$ and $f^*U_f = 0$ because $H_1(N) = 0$ then $\theta(f) = 0$ and by hypothesis $\mu \neq 0$. Therefore for lemma 3.4.1 $f_* : \check{H}_m(M) \rightarrow \check{H}_m(f(M))$ is not surjective, then $\dim \check{H}_m(f(M)) \geq \dim \check{H}_m(M) + 1 = 2$, consequently $\beta_0(N - f(M)) \geq 3$ □

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