UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

HARDY AND BERGMAN SPACES WITH APPLICATIONS

Leonardo de Almeida Carvalho

São Carlos – SP
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Leonardo de Almeida Carvalho

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I dedicate this work to my family, whose effort made it possible.

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Resumo

O presente trabalho aborda questões de dualidade e aproximação em espaços de Bergman, com base em resultados recentes de Chakrabarti, Edholm e McNeal (2019), e inclui um estudo complementar sobre espaços de Hardy no disco unitário. São explorados três problemas centrais: a caracterização do espaço dual, a aproximação por funções holomorfas bem comportadas e a construção das funções analíticas mais próximas em L^p. O trabalho concentra-se em domínios limitados em $\mathbb{C}^n$, com foco especial em domínios de Reinhardt, em particular os triângulos generalizados de Hartogs, onde surgem projeções do tipo sub-Bergman que ampliam a teoria clássica.

Palavras-chave: Espaços de Bergman, dualidade, aproximação, domínios de Reinhardt, projeções do tipo sub-Bergman.

Abstract

The present work addresses issues of duality and approximation in Bergman spaces, based on recent results by Chakrabarti, Edholm, and McNeal (2019), and includes a complementary study on Hardy spaces in the unit disk. Three central problems are explored: the characterization of the dual space, the approximation by well-behaved holomorphic functions, and the construction of the closest analytic functions in L^p. The study focuses on bounded domains in $\mathbb{C}^n$ with special emphasis on Reinhardt domains, particularly the generalized Hartogs triangles where sub-Bergman projections arise, extending the classical theory.

Keywords: Bergman spaces, duality, approximation, Reinhardt domains, sub-Bergman projections.

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Let $\Omega \subset \mathbb{C}$ be a domain.

(1) $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

(2) $\mathbb{C}^n_* = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : z_j \neq 0 \text{ for all j}\}$.

(3) $dV(z)$ denotes Lebesgue measure in $\mathbb{C}^n$.

(4) $\mathbb{R}_{>0} = \{r \in \mathbb{R} : r > 0\}$ and $\mathbb{R}^n_{>0} = \{(r_1, \ldots, r_n) \in \mathbb{R}^n : r_j > 0 \text{ for all } 1 \leq j \leq n\}$. Also, $\mathbb{R}_{\geq 0} = \{r \in \mathbb{R} : r \geq 0\}$.

(5) $\mathbb{D}_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$ with $a \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$. For the unit disk centered at the origin we use $\mathbb{D}$.

(6) $\overline{\mathbb{D}_r(a)} = \{z \in \mathbb{C} : |z - a| \leq r\}$ with $a \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$.

(7) $L^p(\Omega) = \{f : \Omega \to \mathbb{C} : \int_\Omega |f(z)|^p dV(z) < \infty\}$, $0 \leq p < \infty$. If $p = \infty$, we define $L^\infty(\Omega) = \{f : \Omega \to \mathbb{C} : \operatorname{ess\,sup}_{z \in \Omega} |f(z)| < \infty\}$.

(8) $\mathcal{O}(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is holomorphic}\}$.

(9) $A^p(\Omega) = \mathcal{O}(\Omega) \cap L^p(\Omega)$.

(10) $\mathbb{D}^n_r(a) = \mathbb{D}_{r_1}(a_1) \times \cdots \times \mathbb{D}_{r_n}(a_n)$ with $a \in \mathbb{C}^n$ and $r \in \mathbb{R}^n_{>0}$. For the unit polydisk centered at the origin we use $\mathbb{D}^n$.

(11) $\mathbb{T}(a, r) = \partial\mathbb{D}_{r_1}(a_1) \times \cdots \times \partial\mathbb{D}_{r_n}(a_n)$ with $a \in \mathbb{C}^n$ and $r \in \mathbb{R}^n_{>0}$. If $a = (0, 0, \ldots, 0) \in \mathbb{C}^n$, we use $\mathbb{T}(r)$. For the unit torus we use $\mathbb{T}^n$.

(12) $V_j = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_j = 0\}$ for $j = 1, \ldots, n$.

(13) $\mathbb{H}_\gamma = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\}$. If $\gamma = 1$, we simply write $\mathbb{H}$.

(14) $B(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\}$ with $a \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$. For the unit ball in $\mathbb{R}^n$ centered at the origin we use B^n.

(15) $\overline{B(a, r)} = \{x \in \mathbb{R}^n : |x - a| \leq r\}$ with $a \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$.

(16) $\partial B(a,r) = \{x \in \mathbb{R}^n : |x-a| = r\}$ with $a \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$. For the unit sphere in $\mathbb{R}^n$ centered at the origin we use Σ^{n-1}.

Multi-index notation:

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we write:

(1) $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

(2) $\alpha! = \alpha_1! \cdots \alpha_n!$.

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we write:

(1) $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

(2) $\mathcal{S}(\Omega, L^p) = \{\alpha \in \mathbb{Z}^n : e_\alpha \in A^p(\Omega)\}$, where $e_\alpha(z) = z^\alpha$.

CHAPTER 1

Introduction

The study of Bergman spaces has evolved from several sources, with part of its development inspired by the theory of Hardy spaces. For any $0 < p < \infty$, a function f analytic in the unit disk $\mathbb{D}$ is said to belong to the Hardy space H^p if the integrals $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ remain bounded as $r \to 1$. It belongs to the Bergman space A^p if the area integral $\int_{\mathbb{D}} |f(z)|^p d\sigma$ is finite. It is well known that $H^p \subset A^p$, and that H^∞ consists of all bounded analytic functions.

The structural properties of functions in H^p were actively studied between 1915 and 1930, beginning with a classical paper by G. H. Hardy. With the rise of functional analysis in the 1930s, H^p spaces came to be regarded as examples of Banach spaces for $1 \leq p \leq \infty$. This fresh perspective not only introduced new challenges but also provided powerful techniques to tackle classical problems. In Chapter 2, we explore the foundational concepts that lead to the definition of Hardy spaces on the unit disk, including harmonic and subharmonic functions as well as the factorization of holomorphic functions.

At the same time, Stefan Bergman [1] developed an elegant theory of Hilbert spaces of analytic functions in planar domains and higher-dimensional complex spaces, relying heavily on a reproducing kernel now known as the Bergman kernel. Bergman's work focused on spaces of analytic functions that are square-integrable over the given domain with respect to the Lebesgue area or volume measure. This kernel, along with its properties and other key results that underpin the theory of Bergman spaces, is studied in Chapter 3. As counterparts to Hardy spaces, Bergman spaces raised analogous questions but proved, in many ways, more challenging. In brief, although Hardy space problems were largely settled by the 1970s, their counterparts for Bergman spaces were generally viewed as intractable.

The landscape shifted significantly in the 1990s, when problems previously considered intractable began to be solved. Major breakthroughs attracted other workers to the field and inspired a period of intense research on Bergman spaces and related topics – a line of investigation that remains active to this day. In this spirit, Chapters 4, 5, 6 and 7 of this work focus on a study of the paper '*duality and approximation of Bergman Spaces*' by Chakrabarti, Edholm, and McNeal [2].

If $\Omega \subset \mathbb{C}^n$ is a domain and $p > 0$, three basic questions about function theory on A^p motivate this paper:

($Q1$) What is the dual space of $A^p(\Omega)$?

($Q2$) Can an element in $A^p(\Omega)$ be norm approximated by holomorphic functions with better global behavior?

($Q3$) For $g \in L^p(\Omega)$, how does one construct $G \in A^p(\Omega)$ that is nearest to g?

The questions are stated broadly at this point. In Chapter 6 we shall see precise formulations, accompanied by the results presented throughout the previous chapters.

At first glance ($Q1$-3) appear independent. One of the interesting points of this paper is to show the questions are highly interconnected. On planar domains some connections were shown in [6] and [14]. However, with support on the irregularity of the Bergman projection described in [10], there are bounded pseudoconvex domains $D \subset \mathbb{C}^2$ such that

(a) the dual space of $A^p(D)$ cannot be identified with $A^q(D)$ where $\frac{1}{p} = \frac{1}{q} = 1$,

(b) there are functions in $A^p(D)$, $p < 2$, that cannot be L^p-approximated by functions in A^2, and

(c) the L^2-nearest holomorphic function to a general $g \in L^p(D)$ is not in $A^p(D)$.

The negative examples highlight the contrast with the positive answers to ($Q1$-3) and are demonstrated in Chapter 7. These results are called *breakdowns* of the function theory, to indicate a break with expectations coming from previously studied special cases.

In Chapter 4, we state and demonstrate general results in the sense that they can be formulated for arbitrary bounded domains $\Omega \subset \mathbb{C}^n$. In particular, in Chapter 5, we narrow the focus to Reinhardt domains, where the theory of holomorphic functions in several complex variables plays a central role, supported by a rich framework that includes series representations. This chapter also lays the groundwork for possible representations of the dual of Bergman spaces.

Finally, in Chapter 6, we turn to the setting where ($Q1$-3) begin to unfold – focusing especially on the generalized Hartogs triangles. The extra symmetries of this family of Reinhardt domains allow precise descriptions of L^p allowable monomials, orthogonality relations, and integrability in general. The main results are Theorem 6.2 and 6.15, which construct sub-Bergman projections that are L^p bounded where the usual Bergman projection $\mathcal{B}$ is not. In fact, it is on these domains that the breakdowns of Chapter 7 occur, highlighting the limitations of the general theory.

CHAPTER 2

An Introduction to Hardy Spaces in One Complex Variable

The main references for this chapter are [12] and [15].

2.1 Harmonic functions, Poisson representation

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a domain. A function $u \in C^2(\Omega)$ is said to be *harmonic* if it satisfies Laplace's equation

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0 \quad \text{in } \Omega.$$

Remark 2.2. We will be concerned with harmonic functions defined on a domain in the complex plane. For the complex plane we shall use the complex coordinate $z = x + iy$, with x and y real.

There is a natural relation between harmonic and holomorphic functions in the complex plane. The Laplacian operator can, in fact, be factored as

$$\Delta = \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Note that the equation $\frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} = 0$ for a complex valued function $F = u + iv$, with u and v real, is equivalent to

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}, \end{cases}$$

which is called the Cauchy-Riemman system, whose solutions $F = u + iv$ are precisely the holomorphic functions. Any holomorphic function is, therefore, harmonic. In addition, if

$F = u + iv$ is holomorphic, taking complex conjugates in the identity $\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = 0$, we get

$$\begin{aligned}\overline{\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}} = 0 &\Leftrightarrow \overline{\frac{\partial F}{\partial x}} - i\overline{\frac{\partial F}{\partial y}} = 0\\ &\Leftrightarrow \left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}\right) = 0\\ &\Leftrightarrow \frac{\partial \overline{F}}{\partial x} - i\frac{\partial \overline{F}}{\partial y} = 0.\end{aligned}$$

It shows $\overline{F} = u - iv$ is also harmonic. Consequently, both $u = (F + \overline{F})/2$ and $v = (F - \overline{F})/2i$ are harmonic functions. In other words, the real and imaginary parts of a holomorphic function are harmonic. On the other hand, consider the following results:

Lemma 2.3. *Let $u : \Omega \subset \mathbb{C} \to \mathbb{R}$ be a function of class $C^2(\Omega)$. Then u is harmonic in Ω if, and only if, $u_x - iu_y$ is holomorphic on Ω.*

Proof. Suppose u is harmonic in Ω. Then $u_x, u_y \in C^1(\Omega)$ and

- $(u_x)_x = (-u_y)_y$,
- $(u_x)_y = -(-u_y)_x$.

The functions u_x and $-u_y$ satisfy the Cauchy-Riemann system. Then $u_x - iu_y$ is holomorphic in Ω. On the other hand, if $u_x - iu_y$ is holomorphic then

- $(u_x)_x = (-u_y)_y$,
- $(u_x)_y = -(-u_y)_x$.

Consequently

$$u_{xx} + u_{yy} = -u_{yy} + u_{yy} = 0.$$

Thus, u is harmonic. □

Theorem 2.4. *Let u be a real harmonic function on a simply connected domain $\Omega \subset \mathbb{C}$.*[1] *Then there is a harmonic conjugate*[2] *of u in Ω.*

Proof. Let $f = u_x - iu_y$ in Ω. By Lemma 2.3, f is holomorphic on Ω. Since Ω is simply connected, f has a primitive in Ω [4, Chapter IV, Corollary 6.16], say F. Write $F = \tilde{u} + i\tilde{v}$. Then

$$u_x - iu_y = f = F' = \tilde{u}_x + i\tilde{v}_x = \tilde{u}_x - i\tilde{u}_y.$$

[1] A domain $\Omega \subset \mathbb{C}$ is said to be simply connected if it is open, connected, and every closed curve in Ω can be shrunk continuously to a point within Ω. Intuitively, this means that Ω has no holes.

[2] Given a real harmonic function u defined on a simply connected domain $\Omega \subset \mathbb{C}$, a function $v\colon \Omega \to \mathbb{R}$ is called a *harmonic conjugate* of u if the function $f = u + iv$ is holomorphic on Ω. In this case, v is also harmonic and satisfies the Cauchy–Riemann equations with u.

Consequently, $u_x = \tilde{u}_x$ and $u_y = \tilde{u}_y$ in Ω. Since Ω is connected, we see that $\tilde{u} - u$ is a real constant. Let $C = \tilde{u} - u \in \mathbb{R}$. Thus, $F - C = u + i\tilde{v}$ is holomorphic on Ω and $\tilde{v}$ is a harmonic conjugate of u. □

So, we have seen that a real function u defined on a simply connected domain of $\mathbb{C}$ is harmonic if, and only if, it is the real part of some holomorphic function.

Suppose u is a real harmonic function on the disk

$$\mathbb{D}_R(0) = \{z \in \mathbb{C} : |z| < R\}.$$

Then we know $u(z) = \Re(F(z))$ for some holomorphic function F. Let

$$F(z) = \sum_{k=0}^{\infty} c_k z^k$$

be the power series representation of F. Note that we can write $u(z) = (F(z) + \overline{F}(z))/2$ and get a series representation for u. Let us do that using the polar form of $z = re^{i\theta}$ with $r = |z|$ and $-\pi \leq \theta \leq \pi$. We obtain:

$$u(re^{i\theta}) = \frac{1}{2}\left(\sum_{k=0}^{\infty} c_k r^k e^{ik\theta} + \sum_{k=0}^{\infty} \overline{c_k} r^k e^{-ik\theta}\right) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta},$$

with $a_k = c_k/2$ for $k > 0$, $a_0 = \Re(c_0)$ and $a_k = \overline{c_{-k}}/2$ for $k < 0$. We conclude that any u harmonic in $\mathbb{D}_R(0)$ has a series representation

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta} \tag{2.1}$$

converging uniformly on compact subsets of $\mathbb{D}_R(0)$.

Suppose $R > 1$. Since (2.1) converges uniformly for $r = 1$, we see that a_k is the Fourier coefficient, corresponding to the frequency k, of the function $t \mapsto u(e^{it})$, that is,

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) e^{-ikt} dt.$$

Substituting this integral for a_k in (2.1), we get:

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik(\theta - t)} dt.$$

For $0 \le r < 1$ the series converges uniformly, its sum being

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} r^{|k|} e^{ikt} &= 1 + 2\Re\left(\sum_{k=1}^{\infty} (re^{it})^k\right) \\
&= 1 + 2\Re\left(\frac{1}{1-re^{it}} - 1\right) \\
&= 1 + 2\Re\left(\frac{1}{1 - r\cos(t) - ir\sin(t)} - 1\right) \\
&= 1 + 2\Re\left(\frac{1 - r\cos(t) + ir\sin(t)}{1 - 2r\cos(t) + r^2\cos^2(t) + r^2\sin^2(t)} - 1\right) \\
&= 1 + 2\left(\frac{1 - r\cos(t)}{1 + r^2 - 2r\cos(t)} - 1\right) \\
&= 1 + 2\left(\frac{1 - r\cos(t)}{1 + r^2 - 2r\cos(t)} - \frac{1 + r^2 - 2r\cos(t)}{1 + r^2 - 2r\cos(t)}\right) \\
&= 1 + 2\left(\frac{-r^2 + r\cos(t)}{1 + r^2 - 2r\cos(t)}\right) \\
&= \frac{1 + r^2 - 2r\cos(t)}{1 + r^2 - 2r\cos(t)} + \frac{-2r^2 + 2r\cos(t)}{1 + r^2 - 2r\cos(t)} \\
&= \frac{1 - r^2}{1 + r^2 - 2r\cos(t)}.
\end{aligned}$$

This function is the Poisson kernel for the unit disk and will be denoted by $P_r(t)$.

For a function u harmonic in $\mathbb{D}_R(0)$, $R > 1$, we have obtained the Poisson representation:

$$\begin{aligned}
u(re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} u(e^{it})dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)u(e^{it})dt,
\end{aligned} \tag{2.2}$$

where $0 \le r < 1$ and $-\pi \le \theta \le \pi$. Note that (2.2) exhibits the function $u_r(e^{it}) = u(re^{it})$ as the convolution (on the torus group $\mathbb{T} = \{e^{it} : t \in \mathbb{R}\}$ which we identify with the interval $[-\pi, \pi]$) of the functions $u(e^{it})$ and P_r.

Formula (2.2) provides the key to the solution of the Dirichlet problem for the disk. This basic problem consists in finding a function u continuous on $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$ and harmonic in $\mathbb{D}$, whose restriction to the boundary of $\mathbb{D}$, $u(e^{it})$, coincides with a previously given continuous function $f(t)$ on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$. The natural candidate for the solution will be the integral in (2.2) with $f(t)$ in place of $u(e^{it})$, that is, the function $t \mapsto u(re^{it})$ for $0 \le r < 1$, is the convolution of f and the Poisson kernel P_r. We write $u(re^{i\theta}) = P_r * f(\theta)$ or $u = P(f)$ and say that u is the Poisson integral of f. That this function u is indeed a solution will be seen shortly. First, we shall show that the Poisson representation (2.2) remains valid for a much wider class of harmonic functions in the unit disk. One consequence will be the uniqueness of the solution to the Dirichlet problem. We start with the following:

Theorem 2.5. *Let u be a harmonic function in $\mathbb{D}$ such that*

$$\sup_{0\leq r<1}\int_{-\pi}^{\pi}|u(re^{it})|^p dt<\infty \tag{2.3}$$

for some $p>1$. Then there is a function $f\in L^p([-\pi,\pi])$ such that

$$u(re^{i\theta})=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)f(t)dt. \tag{2.4}$$

That is: u is the Poisson integral of some L^p function f.

Proof. Let $r_n\uparrow 1$ (that is, r_n is an increasing sequence converging to 1). Consider the functions

$$f_n(t)=u(r_ne^{it}).$$

From condition (2.3) $\{f_n\}$ is a bounded sequence in $L^p([-\pi,\pi])=L^q([-\pi,\pi])'$, where $\frac{1}{p}+\frac{1}{q}=1$. Thus, the sequence $\{f_n\}$ is in a closed ball of the normed dual to $L^q([-\pi,\pi])$. The Banach-Alaoglu theorem asserts that such ball is weak*-compact and therefore, since $L^q([-\pi,\pi])$ is separable, also metrizable [19, Theorems 3.15 and 3.16]. It follows that $\{f_n\}$ has a subsequence converging in the weak*-topology to a certain $f\in L^p([-\pi,\pi])$, that is: for every $g\in L^q([-\pi,\pi])$,

$$\int_{-\pi}^{\pi}g(t)f_n(t)dt\to\int_{-\pi}^{\pi}g(t)f(t)dt \quad \text{as } n\to\infty. \tag{2.5}$$

For each n, the function $z\mapsto u(r_nz)$ is harmonic in $\mathbb{D}_{r_n^{-1}}(0)$, a disk of radius bigger than one. Therefore, we have the Poisson representation:

$$u(r_nre^{i\theta})=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}u(r_ne^{it})dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)f_n(t)dt.$$

Letting $n\to\infty$, the left hand side tends to $u(re^{i\theta})$ while the right hand side tends to $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)f(t)dt$, according to (2.5). This yields the Poisson representation (2.4). □

Let us observe that the theorem remains valid for $p=\infty$ substituting for (2.3) the condition:

$$\sup_{0\leq r<1}\|u_r\|_{L^\infty}<\infty, \tag{2.6}$$

where u_r is the function $t\mapsto u(re^{it})$. All one needs to realize is that still $L^\infty([-\pi,\pi])=L^1([-\pi,\pi])'$.

A relevant question at this point in whether condition (2.3) with $p=1$ will imply a Poisson representation. The proof of Theorem 2.5 does not extend to this case because $L^1([-\pi,\pi])$ is not a dual space. However, $L^1([-\pi,\pi])$ can be isometrically imbedded into the space $M([-\pi,\pi])$ of Borel measures on $[-\pi,\pi]$ by assigning to each $f\in L^1([-\pi,\pi])$, the measure $d\mu(t)=f(t)dt$. The space $M([-\pi,\pi])$ is the dual of the space $C([-\pi,\pi])$ of continuous functions on $[-\pi,\pi]$ with the supremum norm (see [11, Corollary 7.18]). Then we can repeat the argument used in the proof of Theorem 2.5 and obtain:

Theorem 2.6. *Let u be a harmonic function in $\mathbb{D}$, such that:*

$$\sup_{0\leq r<1}\int_{-\pi}^{\pi}|u(re^{it})|dt<\infty. \tag{2.7}$$

Then, there is a Borel measure μ on $[-\pi,\pi]$, such that:

$$u(re^{i\theta})=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)d\mu(t).$$

In other words, u is the Poisson integral of the measure μ (we shall write $u=P(\mu)$). The functions u are often called Poisson-Stieltjes integrals.

Let us observe that condition (2.7) is automatically satisfied if $u\geq 0$. Indeed, in that case:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|u(re^{it})|dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(re^{it})dt=u(0).$$

The last identity is an instance of the mean value property of harmonic functions. We obtain, therefore, the following:

Corollary 2.7. *Any positive harmonic function in $\mathbb{D}$ is the Poisson integral of some positive measure on $\mathbb{T}$.*

The measure is positive because it is obtained as a weak*-limit of positive measures.

For $p=\infty$, Theorem 2.5 and its proof imply that the solution u of the Dirichlet problem on $\mathbb{D}$ with boundary function f is, if any, $P(f)$, the Poisson integral of f. We shall presently see that $P(f)$ is indeed a solution. This will be based upon the fact that the Poisson kernel gives rise to an approximate identity. To see what this means, we examine closely the Poisson kernel

$$P_r(t)=\frac{1-r^2}{1+r^2-2r\cos(t)}=\sum_{k=-\infty}^{\infty}r^{|k|}e^{ikt},$$

where $0\leq r<1$ and $t\in\mathbb{R}$. It is obviously a 2π-periodic continuous function of t. It is also positive, since

$$P_r(t)=\frac{1-r^2}{1+r^2-2r\cos(t)}=\frac{1-r^2}{|1-re^{it}|^2}.$$

Besides

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(t)dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty}r^{|k|}e^{ikt}dt=1. \tag{2.8}$$

In fact, since $\int_{-\pi}^{\pi}e^{ikt}dt=0$ for all $k\neq 0$, it follows that

$$\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}r^{|k|}\int_{-\pi}^{\pi}e^{ikt}dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}1dt=\frac{1}{2\pi}(\pi-(-\pi))=1.$$

And finally, for any $\delta>0$ is $\sup\limits_{\delta<|t|\leq\pi}P_r(t)\to 0$ as $r\to 1$. Indeed, for $\delta\leq|t|\leq\pi$, it holds that

$$P_r(t)\leq\frac{1-r^2}{1+r^2-2r\cos(\delta)}\leq\frac{1-r^2}{1-\cos^2(\delta)}\to 0 \quad \text{as } r\to 1.$$

The last inequality is due to the fact that the denominator is minimal for $r = \cos(\delta)$.

In general, an approximate identity on the torus $\mathbb{T}$ will be a family ϕ_α of 2π-periodic functions in $L^1([-\pi,\pi])$, with indices α ranging over a directed set and satisfying the following three conditions:

i) $\sup_\alpha \frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi_\alpha(t)|dt = k < \infty$;

ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_\alpha(t)dt = 1$ for every α;

iii) $\int_{\delta<|t|\leq\pi} |\phi_\alpha(t)|dt \to 0$ for every $\delta > 0$.

Clearly, the Poisson kernel gives an approximate identity P_r. In this case $k = 1$ in *i)* and an even stronger version of *iii)* holds as we have seen. The fact that the Poisson kernel is positive and satisfies *ii)* gives converses to Theorems 2.5 and 2.6.

Theorem 2.8. *Let $f \in L^p([-\pi,\pi])$, $1 \leq p \leq \infty$, and let $u = P(f)$ be its Poisson integral, that is:*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)f(t)dt,$$

where $0 \leq r < 1$ and $-\pi \leq \theta \leq \pi$. Then $u(z)$ is harmonic in $\mathbb{D}$. Besides, if $p < \infty$, we have:

$$\int_{-\pi}^{\pi} |u(re^{it})|^p dt \leq \int_{-\pi}^{\pi} |f(t)|^p dt \tag{2.9}$$

for every $r < 1$, and if $p = \infty$

$$|u(z)| \leq \|f\|_{L^\infty} \tag{2.10}$$

for every $z \in \mathbb{D}$.

Proof. If the Fourier series of f is $\sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$, then

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}.$$

In fact,

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)f(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} r^{|k|}e^{ik(\theta-t)}f(t)dt \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} r^{|k|}e^{ik(\theta-t)}f(t)dt \\ &= \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}, \end{aligned}$$

where a_k is the Fourier coefficient, corresponding to the frequency k, of the function $t \mapsto f(t)$, that is,

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

If f is real-valued, so is u, and, clearly,

$$u(z) = \Re \left(a_0 + 2 \sum_{k=1}^{\infty} a_k z^k \right),$$

that is, u is the real part of a holomorphic function. Consequently, u is harmonic in $\mathbb{D}$. (2.9) can be obtained very easily by writing

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) f(\theta - t) dt,$$

and applying Minkowski's inequality for integrals:

$$\begin{aligned}
\|u(re^{i\cdot})\|_{L^p} &= \left(\int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta \right)^{1/p} \\
&= \left(\int_{-\pi}^{\pi} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) f(\theta - t) dt \right|^p d\theta \right)^{1/p} \\
&\leq \left(\int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) |f(\theta - t)| dt \right)^p d\theta \right)^{1/p} \\
&\leq \int_{-\pi}^{\pi} \left(\left(\frac{1}{2\pi} \right)^p \int_{-\pi}^{\pi} [P_r(t)]^p |f(\theta - t)|^p d\theta \right)^{1/p} dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) \left(\int_{-\pi}^{\pi} |f(\theta - t)|^p d\theta \right)^{1/p} dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) \|f(\cdot - t)\|_{L^p} dt \\
&= \|f\|_{L^p}.
\end{aligned}$$

The dot $\cdot$ stands for the variable with respect to which norms are taken. Finally, if $p = \infty$:

$$|u(z)| = |u(re^{i\theta})| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) |f(t)| dt \leq \|f\|_{L^\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) dt \right) = \|f\|_{L^\infty}.$$

□

Theorem 2.9. *Let μ be a complex Borel measure on $[-\pi, \pi]$ and $u = P(\mu)$ its Poisson integral. Then $u(z)$ is harmonic in $\mathbb{D}$ and*

$$\int_{-\pi}^{\pi} |u(re^{it})| dt \leq \int_{-\pi}^{\pi} d|\mu|(t) \tag{2.11}$$

(the last integral denotes the total variation of μ).

Proof. If the Fourier series of μ is $\sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$, that is, if

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt} d\mu(t),$$

then

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}.$$

As before, u is clearly harmonic. Besides

$$\int_{-\pi}^{\pi} |u(re^{i\theta})| d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_r(\theta - t) d|\mu|(t) d\theta = \int_{-\pi}^{\pi} d|\mu|(t)$$

by Fubini's theorem. □

Thus, we have seen that, in the class of harmonic functions in $\mathbb{D}$, conditions (2.3) for $p > 1$ characterizes those which are Poisson integrals of L^p functions, and condition (2.3) for $p = 1$ characterizes those which are Poisson integrals of Borel measures of finite total variation.

We shall study next the boundary behavior of Poisson integrals. This will allow us to solve the Dirichlet problem and several variants of it by means of Poisson integrals and will also give us a better understanding of how u determines f in Theorem 2.5 or μ in Theorem 2.6. First we study the convergence in the L^p norm. We can state a general result valid for every approximate identity ϕ_α.

Theorem 2.10. *a) If $f \in L^p([-\pi, \pi])$ with $1 \leq p < \infty$ and f_α stands for the convolution*

$$f_\alpha(\theta) = (f * \phi_\alpha)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)\phi_\alpha(t) dt,$$

it follows that $f_\alpha \to f$ in L^p, that is:

$$\int_{-\pi}^{\pi} |f_\alpha(t) - f(t)|^p dt \to 0.$$

b) If f is a continuous 2π-periodic function, we have $f_\alpha \to f$ uniformly on $\mathbb{T}$.

Proof. Note

$$f_\alpha(\theta) - f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(\theta - t) - f(\theta))\phi_\alpha(t) dt$$

because of property *ii)* of the approximate identity. Then, Minkowski's inequality for integrals implies that:

$$\begin{aligned}
\|f_\alpha - f\|_{L^p} &= \left(\int_{-\pi}^{\pi} |f_\alpha(\theta) - f(\theta)|^p d\theta\right)^{1/p} \\
&\le \left(\int_{-\pi}^{\pi} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta-t) - f(\theta)||\phi_\alpha(t)|dt\right)^p d\theta\right)^{1/p} \\
&\le \int_{-\pi}^{\pi} \left(\left(\frac{1}{2\pi}\right)^p \int_{-\pi}^{\pi} |f(\theta-t) - f(\theta)|^p |\phi_\alpha(t)|^p d\theta\right)^{1/p} dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi_\alpha(t)|\|f(\cdot - t) - f\|_{L^p} dt \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} |\phi_\alpha(t)|\|f(\cdot - t) - f\|_{L^p} dt + \frac{1}{2\pi}\int_{\delta<|t|\le\pi} |\phi_\alpha(t)|\|f(\cdot - t) - f\|_{L^p} dt
\end{aligned}$$

for an arbitrary $\delta > 0$. The first term in the sum is bounded by

$$\left(\sup_{|t|<\delta} \|f(\cdot - t) - f\|_{L^p}\right)\left(\sup_\alpha \frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi_\alpha(t)|dt\right) = k \sup_{|t|<\delta} \|f(\cdot - t) - f\|_{L^p}$$

with $k < \infty$ (property *i)* of the approximate identity). But

$$\sup_{|t|<\delta} \|f(\cdot - t) - f\|_{L^p}$$

can be made small by taking δ small. Indeed, it is clear that $\|f(\cdot - t) - f\|_{L^p} \to 0$ as $t \to 0$ (note that we are taking f continuous when $p = \infty$. For $p < \infty$, we just need to approximate f in the L^p norm by continuous functions in order to justify the claim). The second term in this sum is bounded by

$$\frac{1}{\pi}\|f\|_{L^p}\int_{\delta<|t|\le\pi} |\phi_\alpha(t)|dt,$$

which, according to property *iii)* of the approximate identity, tends to zero as α moves in the directed set of indices, no matter how small δ is. Finally, given $\varepsilon > 0$, we first choose $\delta > 0$ small enough to have

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} \|f(\cdot - t) - f\|_{L^p}|\phi_\alpha(t)|dt < \frac{\varepsilon}{2}$$

independently of α. Then, with this δ fixed, all we have to do is to take α far enough in the order of the directed set of indices to render

$$\frac{1}{2\pi}\int_{\delta<|t|\le\pi} \|f(\cdot - t) - f\|_{L^p}|\phi_\alpha(t)|dt < \frac{\varepsilon}{2}.$$

This will be sufficient to have $\|f_\alpha - f\|_{L^p} < \varepsilon$.

□

Taking as approximate identity the Poisson kernel P_r, we obtain:

Corollary 2.11. *Let f be a 2π-periodic function on $\mathbb{R}$, and let $u = P(f)$. Then:*

a) *If $f \in L^p([-\pi,\pi])$ with $1 \le p < \infty$, we have:*

$$\int_{-\pi}^{\pi} |u(re^{it}) - f(t)|^p dt \to 0 \text{ as } r \to 1.$$

b) *If f is continuous, then $u(re^{it}) \to f(t)$ uniformly in t as $r \to 1$.*

Remark 2.12. Thus, in Theorem 2.5, the function f is the limit in $L^p(\mathbb{T})$ of the functions $u_r(t) = u(re^{it})$, $1 < p < \infty$.

Remark 2.13. Part b) of the Corollary 2.11 implies that, for f continuous on $\mathbb{T}$, $u = P(f)$ is, indeed, the solution of the classical Dirichlet problem. Of course, part a) implies that for $1 \le p < \infty$, $u = P(f)$ is the solution of an L^p version of the Dirichlet problem.

Theorem 2.14. *Let ϕ_α be an approximate identity on the torus $\mathbb{T}$. Then:*

a) *If $f \in L^\infty([-\pi,\pi])$ and $f_\alpha = f * \phi_\alpha$, it follows that $f_\alpha \to f$ in the weak*-topology of L^∞.*

b) *If $\mu \in M(\mathbb{T})$ and $f_\alpha = \mu * \phi_\alpha$, it follows that $f_\alpha \to \mu$ in the weak*-topology of $M(\mathbb{T})$.*

Proof. a) We have to see that $\int f_\alpha(\theta)\psi(\theta)d\theta$ converges to $\int f(\theta)\psi(\theta)d\theta$ for every $\psi \in L^1([-\pi,\pi])$. But

$$\begin{aligned}\int_{-\pi}^{\pi} f_\alpha(\theta)\psi(\theta)d\theta &= \int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\phi_\alpha(\theta - t)dt\right)\psi(\theta)d\theta \\ &= \int_{-\pi}^{\pi} (\psi * \phi_\alpha(-\cdot))(t)f(t)dt \to \int_{-\pi}^{\pi} \psi(t)f(t)dt\end{aligned}$$

since $\psi * \phi_\alpha(-\cdot) \to \psi$ in L^1 and $f \in L^\infty$.

The proof of b) is entirely similar, only this time we take ψ continuous and apply part b) of the theorem. □

Corollary 2.15. a) *If $f \in L^\infty([-\pi,\pi])$ and $u = P(f)$, we have $u_r(t) = u(re^{it}) \to f(t)$ in the weak*-topology of L^∞.*

b) *If $\mu \in M(\mathbb{T})$ and $u = P(\mu)$, we have $u_r(t)dt \to d\mu(t)$ in the weak*-topology of M.*

It should be noted that if an integrable function f or a complex Borel measure μ has Fourier series $\sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$, then, the Poisson integral of f or μ is the function $u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}$ which, for each fixed r, can be viewed as an average of the partial sums

$$S_n(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta}.$$

In fact, note that

$$\begin{aligned} S_n(\theta) &= \sum_{k=-n}^{n} a_k e^{ik\theta} \\ &= a_0 + \sum_{k=1}^{n} a_k e^{ik\theta} + \sum_{k=-1}^{-n} a_k e^{ik\theta} \\ &= a_0 + \underbrace{\sum_{k=1}^{n} a_k e^{ik\theta}}_{A_n(\theta)} + \underbrace{\sum_{k=1}^{n} a_{-k} e^{-ik\theta}}_{B_n(\theta)} . \end{aligned}$$

Set $S_0(\theta) = A_0(\theta) + B_0(\theta) = a_0$, where $A_0(\theta) = a_0$ and $B_0(\theta) = 0$. Then

$$\begin{aligned} u(re^{i\theta}) &= \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \\ &= a_0 + \sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=-1}^{-\infty} a_n r^{|n|} e^{in\theta} \\ &= a_0 + \sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} a_{-n} r^n e^{-in\theta} \\ &= a_0 + \sum_{n=1}^{\infty} r^n (A_n(\theta) - A_{n-1}(\theta)) + \sum_{n=1}^{\infty} r^n (B_n(\theta) - B_{n-1}(\theta)) \\ &= a_0 + \left(\sum_{n=1}^{\infty} r^n A_n(\theta) - \sum_{n=1}^{\infty} r^n A_{n-1}(\theta) \right) + \left(\sum_{n=1}^{\infty} r^n B_n(\theta) - \sum_{n=1}^{\infty} r^n B_{n-1}(\theta) \right) \\ &= a_0 - rA_0(\theta) + \left(\sum_{n=1}^{\infty} r^n A_n(\theta) - \sum_{n=1}^{\infty} r^{n+1} A_n(\theta) \right) + \left(\sum_{n=1}^{\infty} r^n B_n(\theta) - \sum_{n=1}^{\infty} r^{n+1} B_n(\theta) \right) \\ &= (1-r)A_0(\theta) + (1-r)\sum_{n=1}^{\infty} r^n A_n(\theta) + (1-r)\sum_{n=1}^{\infty} r^n B_n(\theta) \\ &= (1-r)A_0(\theta) + (1-r)\left(\sum_{n=1}^{\infty} r^n (A_n(\theta) + B_n(\theta)) \right) \\ &= (1-r)\sum_{n=0}^{\infty} r^n (A_n(\theta) + B_n(\theta)) \\ &= (1-r)\sum_{n=0}^{\infty} r^n S_n(\theta). \end{aligned}$$

The functions $u_r(\theta) = u(re^{i\theta})$ are called the Abel means of (the Fourier series of) f or μ. Thus, every theorem about the boundary behaviour of the function u can be read as a theorem on the Abel summability of the Fourier series of f or μ. So far, we have established the Abel summability in the L^p norm. Now we shall analize the problem of pointwise summability or, in other words, we shall study the pointwise behaviour of a Poisson-Stieltjes integral. We shall no longer obtain results for a general approximate identity. Now, the particular structure of the Poisson kernel (more specifically the fact that $P_r(t)$ is decreasing as a function of $|t|$), will be decisive.

Theorem 2.16 (Fatou's Theorem). *Let μ be a Borel measure on $\mathbb{T}$, and call*

$$F(\theta) = \int_0^\theta d\mu(t).$$

We know that F is a function of bounded variation and, hence, it has a (finite) derivative at almost every point θ (see [11, Section 3.5]). Let θ_1 be one of those points at which $F'(\theta_1)$ exists and is finite. Let $u = P(\mu)$. Then $u(z)$ converges to $F'(\theta_1)$ as z tends to $e^{i\theta_1}$ non-tangentially. By this we mean that, for every $c > 0$, $u(z) \to F'(\theta_1)$ as $z = re^{i\theta}$ tends to $e^{i\theta_1}$ remaining in the region $\{re^{i\theta} : |\theta - \theta_1| < c(1-r)\}$. We shall indicate this type of convergence by writing $u(z) \to F'(\theta_1)$ as $z \xrightarrow{N.T.} e^{i\theta_1}$.

Proof. First of all, just to simplify the writing, we may clearly assume that $\theta_1 = 0$, and also that $F'(0) = 0$. Otherwise we consider the measure $d\lambda(t) = d\mu(t) - F'(0)dt$, then

$$\tilde{F}(\theta) = \int_0^\theta d\lambda(t) = \int_0^\theta d\mu(t) - \int_0^\theta F'(0)dt = F(\theta) - F'(0)\theta$$

satisfies $\tilde{F}'(0) = 0$ and the result follows. Take $c > 0$. We shall show that $u(re^{i\theta})$ can be made small uniformly in θ for $|\theta| < c(1-r)$, by just taking r close enough to 1. Let $\varepsilon > 0$. Take $\delta > 0$ such that $|F(t)| < \varepsilon|t|$ every time that $|t| < \delta$.[3] Look only at $r's$ so close to 1 that if $re^{i\theta}$ is in our region of approach, then $|\theta| < \delta/4$, in other words, let $c(1-r) < \delta/4$. Then, for $re^{i\theta}$ in our region,

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} d\mu(t) = \underbrace{\frac{1}{2\pi}\int_{\delta<|t|\leq\pi} P_r(\theta-t)d\mu(t)}_{I} + \underbrace{\frac{1}{2\pi}\int_{-\delta}^{\delta} P_r(\theta-t)d\mu(t)}_{J}.$$

For the first term in this sum, since $|t| > \delta$ and $|\theta| < \delta/4$, we have

$$|\theta - t| \geq |t| - |\theta| > \delta - \frac{\delta}{4} = \frac{3\delta}{4} > \frac{\delta}{2}$$

and therefore

$$|I| \leq \sup_{\delta<|t|\leq\pi} P_r(\theta-t) \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} d|\mu|(t) = \sup_{|t|>\delta/2} P_r(t) \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} d|\mu|(t). \tag{2.12}$$

[3]Since $F'(0) = 0$, it means that

$$\lim_{t\to 0} = \frac{F(t)}{t} = 0.$$

Clearly, this tends to 0 as $r \to 1$, and we just need to worry about the other term. Writing $d\mu(t) = F'(t)dt$ and integrating the second term, J, by parts we get

$$\begin{aligned} J &= \frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} d\mu(t) \\ &= \frac{1}{2\pi}\left[\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}F(t)\right]_{-\delta}^{\delta} - \frac{1}{\pi}\int_{-\delta}^{\delta} \frac{(1-r^2)r\sin(\theta-t)}{(1+r^2-2r\cos(\theta-t))^2}F(t)dt \\ &\leq K\left(\sup_{|t|>\delta/2} P_r(t)\right) + \frac{1}{\pi}\int_{-\delta}^{\delta} \frac{(1-r^2)r\sin(t-\theta)}{(1+r^2-2r\cos(\theta-t))^2}F(t)dt. \end{aligned}$$

The constant K appears because the function F has bounded variation and is bounded on the interval $[-\delta, \delta]$. This guarantees that the boundary terms $F(\delta)$ and $F(-\delta)$ are finite and can be controlled by some constant depending on the total variation and the supremum norm of F on $[-\delta, \delta]$. Moreover, since $|\theta| < \delta/4$,

$$|\theta \pm \delta| \geq \delta - |\theta| > \frac{3\delta}{4} > \frac{\delta}{2}.$$

It implies that $P_r(\theta-\delta)$ and $P_r(\theta+\delta)$ can be uniformly bounded above by $\sup_{|t|>\delta/2} P_r(t)$. Again we just need to study this last term. Suppose, just for definiteness, that $\theta > 0$. So we can decompose the integral as

$$\frac{1}{\pi}\int_{-\delta}^{\delta} = \frac{1}{\pi}\int_{-\delta}^{0} + \frac{1}{\pi}\int_{0}^{2\theta} + \frac{1}{\pi}\int_{2\theta}^{\delta}. \tag{2.13}$$

Now we look at each of the terms in this sum. On the first one we use that

$$|F(t)| < \varepsilon|t| = \varepsilon(-t) \leq \varepsilon(\theta - t)$$

in such a way that, after changing $\theta - t$ to t, we obtain

$$\begin{aligned} \left|\frac{1}{\pi}\int_{-\delta}^{0}\right| &\leq \frac{1}{\pi}\int_{-\delta}^{0} \frac{(1-r^2)r|\sin(t-\theta)|}{(1+r^2-2r\cos(\theta-t))^2}|F(t)|dt \\ &\leq -\frac{1}{\pi}\int_{-\delta}^{0} \frac{(1-r^2)r\sin(t-\theta)}{(1+r^2-2r\cos(\theta-t))^2}\varepsilon(\theta-t)dt \\ &\overset{s=\theta-t}{=} \frac{\varepsilon}{\pi}\int_{\theta+\delta}^{\theta} \frac{(1-r^2)r\sin(-s)}{(1+r^2-2r\cos(s))^2} s ds \\ &= \frac{\varepsilon}{\pi}\int_{\theta}^{\theta+\delta} \frac{(1-r^2)r\sin(s)}{(1+r^2-2r\cos(s))^2} s ds \\ &\leq \frac{\varepsilon}{\pi}\int_{0}^{\pi} \frac{(1-r^2)r\sin(s)}{(1+r^2-2r\cos(s))^2} s ds \\ &= -\frac{\varepsilon}{2\pi}\left[\frac{(1-r^2)s}{1+r^2-2r\cos(s)}\right]_0^{\pi} + \frac{\varepsilon}{2\pi}\int_0^{\pi} P_r(s)ds \\ &= -\frac{\varepsilon}{2}\left(\frac{1-r}{1+r}\right) + \frac{\varepsilon}{2}. \end{aligned}$$

For the second term in (2.13) we use that $|F(t)| < \varepsilon t$ and $|\theta - t| < \theta$ (since $0 < t < 2\theta$). In this way,

$$\begin{aligned}\left|\frac{1}{\pi}\int_0^{2\theta}\right| &\le \frac{1}{\pi}\int_0^{2\theta}\frac{(1-r)^2 r|\sin(t-\theta)|}{(1+r^2-2r\cos(\theta-t))^2}|F(t)|dt\\ &\le \frac{1}{\pi}\int_0^{2\theta}\frac{(1-r)^2 r\theta}{(1+r^2-2r\cos(\theta-t))^2}\varepsilon t dt\\ &\le \frac{\varepsilon}{\pi}\int_0^{2\theta}\frac{(1+r)(1-r)r\theta}{(1-r)^4}tdt\\ &\le \frac{2\varepsilon\theta}{\pi(1-r)^3}\left[\frac{t^2}{2}\right]_0^{2\theta}\\ &= \frac{4\theta^3\varepsilon}{\pi(1-r)^3}\\ &\le \frac{4c^3}{\pi}\varepsilon.\end{aligned}$$

As for the third and last term in (2.13), we use

$$|F(t)| < \varepsilon t = \varepsilon(2t - t) \le \varepsilon(2t - 2\theta) = 2\varepsilon(t-\theta)$$

and proceed exactly as with the first term.

Finally, we see that the integral in (2.13) can be made small arbitrarily by ε . It shows $J < \varepsilon$ since r is close enough to 1. Because of this and by (2.12), we can conclude the theorem. □

When $d\mu(t) = f(t)dt$ with $f \in L^1([-\pi, \pi])$, we know that for almost every θ is

$$\frac{1}{t}\int_0^t |f(\theta+s) - f(\theta)|ds \to 0 \quad \text{as } t \to 0. \tag{2.14}$$

Those θ for which this holds are called Lebesgue points for f (see [20, p. 138]) . Therefore we get

Corollary 2.17. *Let $f \in L^1([-\pi, \pi])$, and let $u = P(f)$. Then, for every Lebesgue point θ, $u(z) \to f(\theta)$ as $z \xrightarrow{N.T.} e^{i\theta}$. In particular, this is true almost everywhere.*

Proof. Write $d\mu(t) = f(t)dt$ and consider F as in Theorem 2.16. If θ is a Lebesgue point for f, hence by definition (2.14) holds. Note that

$$\begin{aligned}F'(\theta) - f(\theta) &= \lim_{t\to 0}\left[\frac{1}{t}(F(\theta+t) - F(\theta))\right] - f(\theta)\\ &= \lim_{t\to 0}\frac{1}{t}\left[\int_\theta^{\theta+t} f(s)ds - \int_0^t f(\theta)ds\right]\\ &= \lim_{t\to 0}\frac{1}{t}\left[\int_0^t f(\theta+s) - f(\theta)ds\right].\end{aligned}$$

Using (2.14) we see that $F'(\theta) = f(\theta)$ at every Lebesgue point. By Theorem 2.16,

$$u(z) \to F'(\theta) = f(\theta) \quad \text{as } z \xrightarrow{\text{N.T.}} e^{i\theta}$$

and whenever θ is a Lebesgue point of f. □

Remark 2.18. When $d\mu(t) = f(t)dt + d\sigma(t)$, where $f \in L^1([-\pi, \pi])$ and σ is singular, it is known that $F'(\theta) = f(\theta)$ for almost everywhere θ (see [20]), so that every Lebesgue-Stieltjes integral has non-tangential boundary values almost everywhere. This applies in particular to the harmonic functions in $\mathbb{D}$ satisfying any of the conditions (2.3), (2.6) or (2.7). This result contains the classical theorem of Fatou stating that "any function holomorphic and bounded in $\mathbb{D}$ has non-tangential boundary values almost everywhere.".

The next result aims to provide a contrast with the previous theorem, particularly regarding the notion of non-tangential limits. In this result, we will use the Blaschke product, which will be defined later (see Theorems 2.54 and 2.55). It is recommended that the reader first consult these theorems (only the statements) and then return here for a clearer understanding of the result below.

Theorem 2.19. *Let C_0 be any simple closed curve passing through the point $z = 1$ situated, except for that point, totally inside the circle $|z| = 1$, and tangent to the circle at that point. Let C_θ be the curve C_0 rotated around $z = 0$ by an angle θ. Then there is a Blaschke product $B(z)$ which, for almost all θ_0, does not tend to any limit as $z \to e^{i\theta_0}$ inside C_{θ_0}.*

Proof. We may suppose that for r close to 1 the circle $|z| = r < 1$ meets C_0 at exactly two points (otherwise we replace the region bounded by C_0 by a smaller having the required property). Let l_n denote the lenght of the arc of $|z| = 1 - 1/n$ situated inside C_0, and let

$$m_n = \left\lfloor \frac{2\pi}{l_n} \right\rfloor + 1.$$

Let S_n be a set of m_n equally spaced points situated on $|z| = 1 - 1/n$. The circular distance between any two consecutive points is less than l_n, so that every C_θ contains in its interior a point of S_n. The sum σ_n of the distances of the points of S_n from the circumference $|z| = 1$ is

$$\frac{m_n}{n} \leq \frac{1 + 2\pi/l_n}{n} = \frac{l_n + 2\pi}{nl_n} = o(1),$$

since the tangency of C_0 to $|z| = 1$ implies that $nl_n \to \infty$. Let us take n_k increasing so rapidly that

$$\sum_{k=1}^{\infty} \sigma_{n_k} < \infty,$$

and let $B(z)$ be the Blaschke product with zeros at the points of $S_{n_1} \cup S_{n_2} \cdots$. Note that we are taking

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{a_n - z}{1 - z\overline{a_n}} \frac{|a_n|}{a_n},$$

where $k = 0$ and $\{a_n\}$ is composed of all the points of $S_{n_1} \cup S_{n_2} \cdots$. In fact,

$$\sum_{n=1}^{\infty} 1 - |a_n| = \sum_{k=1}^{\infty} \sigma_{n_k} < \infty.$$

Finally, since $B(z)$ has infinitely many zeros inside every C_θ, the limit of $B(z)$ as $z \to e^{i\theta}$ in the interior of C_θ must be zero if it exists at all. However, by the Theorem 2.55 such a limit exists for almost no θ. □

Remark 2.20. Consider l_n defined as in the proof of the previous theorem. In addition, define $\overline{l_n}$ to be the lenght of the equivalent arc of $|z| = 1 - 1/n$, but now inside the region $\{re^{i\theta} : |\theta - \theta_1| < c(1-r)\}$. Exactly as in the proof of Fatou's Theorem (Theorem 2.16), we may assume $\theta_1 = 0$. It implies that

$$\overline{l_n} = 2c\left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right).$$

Clearly there exists N_1 large enough such that $l_{N_1} > \overline{l_{N_1}}$, and therefore $l_n > \overline{l_n} = 2c\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)$ for all $n > N_1$. Moreover, choose N_2 such that $1 - \frac{1}{n} > \frac{1}{2}$ for all $n > N_2$ (for instance, $N_2 = 2$ works). Given any $M > 0$, we can just take $c = M$ and define $N = \max\{N_1, N_2\}$. Then, for all $n > N$, we have $l_n > \overline{l_n}$ and $nl_n > 2M(1 - \frac{1}{n}) > M$ for all $n > N$. In other words, $nl_n \to \infty$ as $n \to \infty$. This is essentially the justification for why the previous theorem fails if we work with non-tangential convergence. In fact, we need that $nl_n \to \infty$ as $n \to \infty$ in order to obtain $\sigma_n \to 0$, which does not happen in the non-tangential case (since $n\overline{l_n} \to 2c$).

When a harmonic function u satisfies condition (2.3) for some $p > 1$, then u can be recovered from its boundary function f. Indeed, we know that $u = P(f)$. However, if u just satisfies (2.7) (that is, if $p = 1$), then it is no longer true that u is the Poisson integral of its boundary function. For example, let

$$u(re^{it}) = P_r(t) = \sum_{k=-\infty}^{\infty} r^{|k|}e^{ikt}.$$

This is, of course, a harmonic function, and it clearly satisfies (2.7). Its boundary function is 0. Indeed, $P_r(t) \to 0$ as $r \to 1$ for every $t \neq 0$ in $[-\pi, \pi]$. However, $u > 0$, and it cannot be the Poisson integral of 0, which is identically 0. Actually, in this case $u = P(\delta)$ where δ is the Diract delta or, in other words, the unit mass concentrated at 0 in $[-\pi, \pi]$. This difference in the behaviour of an L^p-bounded harmonic function for $p = 1$ or $p > 1$ is a basic fact and, as we shall see, is the natural starting point for the theory of Hardy spaces.

Theorem 2.21. *Let u be a continuous function on a domain $\Omega \subset \mathbb{R}^n$. Then u is harmonic in Ω if, and only if, u satisfies the following property (known as mean value property): For every $x_0 \in \Omega$ and for every $r > 0$ such that $\overline{B(x_0, r)} = \{x \in \mathbb{R}^n : |x - x_0| \leq r\} \subset \Omega$,*

$$u(x_0) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u(x_0 + r\sigma)d\sigma, \tag{2.15}$$

where $\Sigma_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere in $\mathbb{R}^n$, $d\sigma$ is the Lebesgue measure on Σ_{n-1} and

$$|\Sigma_{n-1}| = \int_{\Sigma_{n-1}} d\sigma.$$

Proof. Suppose that u is harmonic in Ω, so that $\triangle u = 0$ in Ω. Let $x_0 \in \Omega$ and $r > 0$ be such that $\overline{B(x_0, r)} \subset \Omega$. For $0 < s \leq r$, let $f(s)$ stand for the average of u over the sphere of center x_0 and radius s, that is,

$$f(s) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} u(x_0 + s\sigma) d\sigma.$$

Since u is continuous and continuously differentiable ($u \in C^2(\Omega)$), then

$$f'(s) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} \sum_{j=1}^{n} u_{x_j}(x_0 + s\sigma)\sigma_j d\sigma.$$

Note that $\sum_{j=1}^{n} u_{x_j}(x_0 + s\sigma)\sigma_j = D_\sigma u(x_0 + s\sigma)$, which is the derivative of u in the direction of the outer normal in the point $x_0 + s\sigma$. Then

$$f'(s) = \frac{1}{s^{n-1}|\Sigma_{n-1}|} \int_{\partial B(x_0,s)} D_\sigma u(x) d\sigma_s(x),$$

where $\sigma = \frac{x - x_0}{s}$, $\partial B(x_0, s)$ is the boundary of the ball, that is, $\partial B(x_0, s) = \sum(x_0, s)$, the sphere of center x_0 and radius s; and $d\sigma_s$ is the natural Lebesgue measure on $\partial B(x_0, s)$. By applying Green's theorem we get

$$f'(s) = \frac{1}{s^{n-1}|\Sigma_{n-1}|} \int_{B(x_0,s)} \triangle u = 0.$$

Thus $f(s)$ is constant for $0 < s \leq r$. But clearly $f(s) \to u(x_0)$ for $s \to 0$ and $f(s) \to f(r)$ for $s \to r$. In fact,

$$\begin{aligned} |f(s) - u(x_0)| &= \left| \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} (u(x_0 + s\sigma) - u(x_0)) d\sigma \right| \\ &\leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} |u(x_0 + s\sigma) - u(x_0)| d\sigma \to 0 \quad \text{as } s \to 0, \end{aligned}$$

since u is harmonic and then continuous. Moreover,

$$\begin{aligned} |f(s) - f(r)| &= \left| \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} (u(x_0 + s\sigma) - u(x_0 + rs\sigma)) d\sigma \right| \\ &\leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} |u(x_0 + s\sigma) - u(x_0 + r\sigma)| d\sigma \to 0 \quad \text{as } s \to r, \end{aligned}$$

since u is uniformly continuous (its derivative is bounded in the convex open set $B(x_0, r)$). Therefore $f(r) = u(x_0)$ and (2.15) is proved.

The converse is equally easy if we assume, to start with, that u is twice differentiable. If this is the case and we assume that the mean value property holds, then, with the same notation used above we have $f(s) = u(x_0)$ (remember that x_0 was taken arbitrarily), constant on $[0, r]$. Suppose that $\triangle u \not\equiv 0$, say $\triangle u(\overline{x}) > 0$ for some $\overline{x} \in \Omega$. From the fact that $\triangle u$ is continuous, so we must have that $\triangle u > 0$ in some $\overline{B(\overline{x}, r')} \subset \Omega$, $r' > 0$. Then taking $x_0 = \overline{x}$ and $r = r'$, we see that

$$0 = f'(s) = \frac{1}{s^{n-1}|\Sigma_{n-1}|} \int_{B(x_0,s)} \triangle u > 0,$$

which is a absurd!

In case u is just a continuous function in Ω satisfying the mean value property, we shall show u is harmonic by reducing the problem to the case of a smooth function considered previously. Since the problem is local we can assume that Ω is bounded and u is bounded. We shall use a C^∞ function ϕ with support in $B(0,1)$ and having $\int_{\mathbb{R}^n} \phi = 1$ and the approximate identity in $\mathbb{R}^n$, ϕ_ε, to which it gives rise:

$$\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\varepsilon^{-1}x) \quad \text{for } \varepsilon > 0.$$

It can be checked quite easily that ϕ_ε is indeed an approximate identity in $\mathbb{R}^n$. The definition is entirely similar to the one given for the torus $\mathbb{T}$ (right before Theorem 2.8). We shall also require ϕ to be radial, that is: $\phi(x) = \psi(|x|)$. We can now approximate u by the functions

$$u_\varepsilon(x) = (u * \phi_\varepsilon)(x) = \int_{\mathbb{R}^n} u(x-y)\phi_\varepsilon(y)dy = \int_{\mathbb{R}^n} u(y)\phi_\varepsilon(x-y)dy$$

(it is understood that u is extended to $\mathbb{R}^n$ by making it equal to 0 outside of Ω). We see in the last integral that the smoothness of ϕ implies that u_ε is smooth too. Now observe that u_ε satisfies the mean value property in

$$\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

Indeed if $x_0 \in \Omega_\varepsilon$ and $\overline{B(x_0, r)} \subset \Omega_\varepsilon$, we have

$$\begin{aligned}
\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u_\varepsilon(x_0 + r\sigma)d\sigma &= \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}}\int_{\mathbb{R}^n} u(x_0 + r\sigma - y)\phi_\varepsilon(y)dyd\sigma \\
&= \int_{\mathbb{R}^n} \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u(x_0 - y + r\sigma)d\sigma\phi_\varepsilon(y)dy \\
&= \int_{\mathbb{R}^n} u(x_0 - y)\phi_\varepsilon(y)dy \\
&= u_\varepsilon(x_0),
\end{aligned}$$

since u satisfies the mean value property in Ω and $\overline{B(x_0, r+\varepsilon)} \subset \Omega$. Consequently, u_ε is harmonic in Ω_ε. But on the other hand for $x \in \Omega_\varepsilon$ is:

$$\begin{aligned}
u_\varepsilon(x) &= \int_{\mathbb{R}^n} u(x-y)\phi_\varepsilon(y)dy \\
&= \int_0^\infty r^{n-1}\int_{\Sigma_{n-1}} u(x - r\sigma)\phi_\varepsilon(r\sigma)d\sigma dr \\
&= \varepsilon^{-n}\int_0^\infty r^{n-1}\int_{\Sigma_{n-1}} u(x - r\sigma)d\sigma\psi(\varepsilon^{-1}r)dr \\
&= \varepsilon^{-n}\int_0^\infty r^{n-1}|\Sigma_{n-1}|u(x)\psi(\varepsilon^{-1}r)dr \\
&= u(x)\int_{\mathbb{R}^n} \phi_\varepsilon(x)dx \\
&= u(x).
\end{aligned}$$

This ends the proof that u is harmonic because in a neighbourhood of a given $x \in \Omega$, u coincides with u_ε for ε small enough and we already know that u_ε is harmonic. □

Remark 2.22. It has to be observed that the mean value property in Theorem 2.21 is equivalent to the fact that for every $x_0 \in \Omega$ and every $r > 0$ such that $\overline{B(x_0, r)} \subset \Omega$,

$$u(x_0) = \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} u(x)dx. \tag{2.16}$$

Indeed if the first mean value holds and $\overline{B(x_0, r)} \subset \Omega$, we have:

$$u(x_0) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} u(x_0 + s\sigma)d\sigma$$

for all $0 < s \le r$. Integrating both sides against s^{n-1} from 0 to r, we get:

$$\begin{aligned}
\frac{r^n}{n} u(x_0) &= \frac{1}{|\Sigma_{n-1}|} \int_0^r s^{n-1} \int_{\Sigma_{n-1}} u(x_0 + s\sigma)d\sigma ds \\
&= \frac{1}{|\Sigma_{n-1}|} \int_0^r s^{n-1} \frac{1}{s^{n-1}} \int_{\partial B(x_0,s)} u(y)d\sigma_s(y)ds \\
&= \frac{1}{|\Sigma_{n-1}|} \int_0^r \int_{\partial B(x_0,s)} u(y)d\sigma_s(y)ds \\
&= \frac{1}{|\Sigma_{n-1}|} \int_{B(x_0,r)} u(x)dx.
\end{aligned}$$

Thus, since $|B(x_0, r)| = \frac{|\Sigma_{n-1}|}{n} r^n$,

$$u(x_0) = \frac{n}{r^n |\Sigma_{n-1}|} \int_{B(x_0,r)} u(x)dx = \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} u(x)dx,$$

which is exaclty (2.16). Conversely, if we assume the second mean value property and $x_0 \in \Omega$, we shall have, for all r in an interval to the right of 0:

$$u(x_0) = \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} u(x)dx = \frac{n}{r^n |\Sigma_{n-1}|} \int_0^r s^{n-1} \int_{\Sigma_{n-1}} u(x_0 + s\sigma)d\sigma ds.$$

The right hand side, as a function of r, will have derivative 0:

$$\begin{aligned}
0 &= -\frac{n^2}{r^{n+1}|\Sigma_{n-1}|} \int_{B(x_0,r)} u(x)dx + \frac{n}{r^n|\Sigma_{n-1}|} r^{n-1} \int_{\Sigma_{n-1}} u(x_0 + r\sigma)d\sigma \\
&= -\frac{n}{r} \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} u(x)dx + \frac{n}{r^n|\Sigma_{n-1}|} r^{n-1} \int_{\Sigma_{n-1}} u(x_0 + r\sigma)d\sigma \\
&= -\frac{n}{r} u(x_0) + \frac{n}{r^n|\Sigma_{n-1}|} r^{n-1} \int_{\Sigma_{n-1}} u(x_0 + r\sigma)d\sigma.
\end{aligned}$$

Thus

$$u(x_0) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} u(x_0 + r\sigma)d\sigma,$$

which is (2.15).

A consequence of the mean value property is the so called maximum principle for the harmonic functions, which can be stated as follows:

Corollary 2.23. *Let u be a real-valued harmonic function in a domain $\Omega \subset \mathbb{R}^n$. Then u cannot attain a maximum value unless it is constant.*

Proof. Suppose that u does attain a maximum value, that is, there exists $x_0 \in \Omega$ such that $u(x) \leq u(x_0) = m$ for every $x \in \Omega$. Take $r > 0$ such that $\overline{B(x_0, r)} \subset \Omega$. Then

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} u(x)dx = u(x_0) = m.$$

Since $u(x) \leq m$ for every x and u is continuous, if we had $u(\overline{x}) < m$ for some $\overline{x} \in B(x_0, r)$, then we would have $u(x) < m$ for some $B(\overline{x}, r') \subset B(x_0, r)$ and the average of u over $B(x_0, r)$ would have to be $< m$. In fact,

$$\begin{aligned}
m &= u(x_0) \\
&= \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} u(x)dx \\
&= \frac{1}{|B(x_0,r)|}\left(\int_{B(x_0,r)\setminus B(\overline{x},r')} u(x)dx - \int_{B(\overline{x},r')} u(x)dx\right) \\
&= \frac{1}{|B(x_0,r)|}\left(\int_{B(x_0,r)\setminus B(\overline{x},r')} u(x)dx - \frac{|B(\overline{x},r')|}{|B(\overline{x},r')|}\int_{B(\overline{x},r')} u(x)dx\right) \\
&\leq \frac{1}{|B(x_0,r)|}(|B(x_0,r)\setminus B(\overline{x},r')|m + |B(\overline{x},r')|u(\overline{x})) \\
&< \frac{1}{|B(x_0,r)|}(|B(x_0,r)\setminus B(\overline{x},r')|m + |B(\overline{x},r')|m) \\
&< m\frac{(|B(x_0,r)\setminus B(\overline{x},r')| + |B(\overline{x},r')|)}{|B(x_0,r)|} \\
&= m\frac{|B(x_0,r)|}{|B(x_0,r)|} \\
&= m,
\end{aligned}$$

which is clearly a contradiction. Thus $u(x) = m$ for every $x \in B(x_0, r)$. This shows that the set A of points of Ω where $u(x) = m$ is an open set. But $B = \Omega - A = \{x \in \Omega : u(x) < m\}$ is also open because u is continuous. Since A is not empty and Ω is connected, B has to be necessarily empty. Consequently $u(x) = m$ for every $x \in \Omega$. □

Here is an equivalent formulation of the maximum and minimum principles:

Corollary 2.24. *Let u be a real-valued function, continuous on the closure $\overline{\Omega}$ of a bounded domain $\Omega \subset \mathbb{R}^n$, and harmonic in Ω. Then u attains its maximum and its minimum at the boundary of Ω (only at the boundary if u is not a constant).*

From this we can derive a uniqueness result for the solution of the Dirichlet problem for a bounded domain. Namely:

Corollary 2.25. *Let u_1 and u_2 be two functions continuous on the closure $\overline{\Omega}$ of a bounded domain Ω, harmonic in Ω and such that $u_1(x) = u_2(x)$ for every $x \in \partial\Omega$, the boundary of Ω. Then $u_1(x) = u_2(x)$ for every $x \in \overline{\Omega}$.*

Proof. We may assume that u_1 and u_2 are real-valued. Consider the function $u = u_1 - u_2$, which is harmonic in Ω and continuous on $\overline{\Omega}$, with $u(x) = 0$ for every $x \in \partial\Omega$. By the maximum principle, u attains its maximum and minimum on the boundary, where it vanishes. Hence, $u \equiv 0$ in $\overline{\Omega}$, and thus $u_1 = u_2$ in $\overline{\Omega}$. □

Now we shall briefly discuss the Dirichlet problem for a ball, say the unit ball B^n of $\mathbb{R}^n$, that is:

$$B^n = B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}.$$

We have already solved this problem for $n = 2$. The solution u was the Poisson integral of the boundary function f:

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta - t)} f(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|re^{i\theta}|^2}{|re^{i\theta}-e^{it}|^2} f(t)dt. \end{aligned}$$

We shall see that for general n, the solution is given by:

$$u(x) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} \frac{1-|x|^2}{|x-s|^n} f(s)ds. \tag{2.17}$$

In addition, we shall write

$$\frac{1-|x|^2}{|x-s|^n} = P(x,s),$$

the Poisson kernel for the ball. The fact that (2.17) is indeed the solution of the Dirichlet problem for B^n depends upon the following properties of the Poisson kernel:

a) $P(x,s)$ is harmonic in $x \in B^n$ for each fixed $s \in \Sigma_{n-1}$.

Proof. Let $N = 1 - |x|^2 = |s|^2 - |x|^2$ and $S = |x-s|^2$. Then

$$P(x,s) = \frac{1-|x|^2}{|x-s|^n} = NS^{-n/2}.$$

We have

$$\partial_i P(x,s) = (\partial_i N)S^{-n/2} - \frac{n}{2}N(\partial_i S)S^{-((n/2)+1)}$$

and

$$\begin{aligned}
\partial_i^2 P(x,s) &= (\partial_i^2 N)S^{-n/2} - n(\partial_i N)(\partial_i S)S^{-((n/2)+1)} \\
&\quad - \frac{n}{2}N(\partial_i^2 S)S^{-((n/2)+1)} - \frac{n}{2}\left(-\frac{n}{2}+1\right)N(\partial_i S)^2 S^{-((n/2)+2)} \\
&= (\partial_i^2 N)S^{-n/2} - n(\partial_i N)(\partial_i S)S^{-((n/2)+1)} \\
&\quad - \frac{n}{2}N(\partial_i^2 S)S^{-((n/2)+1)} + \frac{n(n+2)}{4}N(\partial_i S)^2 S^{-((n/2)+2)}.
\end{aligned}$$

Then

$$S^{((n/2)+1)}\partial_i^2 P(x,s) = (\partial_i^2 N)S - n(\partial_i N)(\partial_i S) - \frac{n}{2}N(\partial_i^2 S) + \frac{n(n+2)}{4}N(\partial_i S)^2 S^{-1}.$$

Furthermore,

- $\partial_i N = -2x_i$;
- $\partial_i^2 N = -2$;
- $\partial_i S = 2(x_i - s_i)$;
- $\partial_i^2 S = 2$.

Then

- $\sum_{i=1}^n \partial_i^2 N = \sum_{i=1}^n -2 = -2n$;
- $\sum_{i=1}^n (\partial_i N)(\partial_i S) = \sum_{i=1}^n (-2x_i)(2(x_i - s_i)) = 4\sum_{i=1}^n -x_i^2 + x_i s_i = -4|x|^2 + 4\langle x, s\rangle$;
- $\sum_{i=1}^n \partial_i^2 S = \sum_{i=1}^n 2 = 2n$;
- $\sum_{i=1}^n (\partial_i S)^2 = \sum_{i=1}^n (2(x_i - s_i))^2 = 4\sum_{i=1}^n (x_i - s_i)^2 = 4|x-s|^2 = 4S$.

Consequently,

$$\begin{aligned}
S^{((n/2)+1)}\triangle P(x,s) &= \sum_{i=1}^n (\partial_i^2 N)S - n\sum_{i=1}^n (\partial_i N)(\partial_i S) - \frac{n}{2}\sum_{i=1}^n N(\partial_i^2 S) + \frac{n(n+2)}{4}\sum_{i=1}^n N(\partial_i S)^2 S^{-1} \\
&= -2nS + 4n|x|^2 - 4n\langle x,s\rangle - n^2 N + n(n+2)N \\
&= 2nN + 4n|x|^2 - 4n\langle x,s\rangle - 2nS.
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{S^{((n/2)+1)}}{2n}\triangle P(x,s) &= N + 2|x|^2 - 2\langle x,s\rangle - S \\
&= |s|^2 - |x|^2 + 2|x|^2 - 2\langle x,s\rangle + |x-s|^2 \\
&= (|s|^2 + |x|^2 - 2\langle x,s\rangle) + |x-s|^2 \\
&= |x-s|^2 - |x-s|^2 \\
&= 0,
\end{aligned}$$

which implies $\triangle P(x,s) = 0$, since $\dfrac{S^{((n/2)+1)}}{2n} \neq 0$. □

b) $P(x,s) \geq 0$ and $\dfrac{1}{|\Sigma_{n-1}|}\displaystyle\int_{\Sigma_{n-1}} P(x,s)ds = 1$ for every $x \in B^n$.

Proof. Clearly $P(x,s) \geq 0$ and, by *a)*,

$$1 = \frac{1-|0|^2}{|0-s|^n} = P(0,s) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(rx',s)dx'$$

for every r with $0 < r < 1$. Note that $|rx'-s| = |rs-x'|$, since

$$|rx'-s|^2 = r^2\underbrace{|x'|^2}_{=1} - 2r\langle x',s\rangle + \underbrace{|s|^2}_{=1} = r^2|s|^2 - 2r\langle x',s\rangle + |x'|^2 = |rs-x'|^2.$$

Consequently,

$$P(rx',s) = P(rs,x').$$

Thus

$$1 = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(rs,x')dx',$$

which is the equality in *b)*. □

c) For every $\delta > 0$,

$$\lim_{r\to 1}\int_{|s-x'|>\delta} P(rx',s)ds = 0$$

uniformly in $x' \in \Sigma_{n-1}$.

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be chosen arbitrarily. Since $|1-r^2|$ is continuous, it must converge to zero as $r \to 1$, then there exists $\delta_2 > 0$ such that

$$|r-1| < \delta_2 \Rightarrow |1-r^2| < \frac{\varepsilon\delta^n}{|\Sigma_{n-1}|2^n}.$$

Note that if $|x'-s| > \delta$ and $|x'-rx'| = |1-r| < \frac{\delta}{2}$, then

$$|rx'-s| \geq |x'-s| - |x'-rx'| = |x'-s| - |1-r| > \delta - \frac{\delta}{2} = \frac{\delta}{2} \quad \text{for every } x' \in \Sigma_{n-1}.$$

Setting $\delta_0 = \min\{\delta_2, \delta/2\}$, for $|r-1| < \delta_0$, we have

$$\begin{aligned}
\left|\int_{|s-x'|>\delta} \frac{1-|rx'|^2}{|rx'-s|^n}ds - 0\right| &\leq (1-r^2)\int_{|s-x'|>\delta} \frac{1}{|rx'-s|^n}ds \\
&< (1-r^2)\int_{|s-x'|>\delta} \frac{1}{(\delta/2)^n}ds \\
&\leq \frac{2^n(1-r^2)}{\delta^n}\int_{|\Sigma_{n-1}|} ds \\
&= \frac{2^n|\Sigma_{n-1}|}{\delta^n}(1-r^2) \\
&< \frac{2^n}{\delta^n}|\Sigma_{n-1}|\frac{\varepsilon\delta^n}{|\Sigma_{n-1}|2^n} \\
&= \varepsilon.
\end{aligned}$$

We then prove that

$$\lim_{r\to 1}\int_{|s-x'|>\delta} P(rx',s)ds = 0.$$

□

With these three properties we can prove the following:

Theorem 2.26. *Let f be a continuous function on Σ_{n-1}. Then the function u, defined in $\overline{B(0,1)}$ as*

$$u(x) = \begin{cases} \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x,s)f(s)ds, & \text{if } |x|<1,\\ f(x), & \text{if } |x|=1, \end{cases}$$

is continuous in $\overline{B(0,1)}$ and harmonic in $B(0,1)$. It is, therefore, the solution of the Dirichlet problem in B^n with boundary function f.

Proof. The harmonicity of u in $B(0,1)$ follows from the harmonicity of $P(x,s)$ as a function of x. In fact, first of all, note that

$$u(x) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x,s)f(s)ds$$

is continuous, since the right side is continuous at variable x. Let $x_0 \in B(0,1)$. Taking $r>0$ such that $\overline{B(x_0,r)} \subset B(0,1)$, since $P(x,s)$ is harmonic in $x\in B(0,1)$, it must satisfy the mean value property. Then

$$\begin{aligned} u(x_0) &= \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x_0,s)f(s)ds\\ &= \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}}\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} P(x,s)dx\right)f(s)ds\\ &= \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}\left(\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x,s)f(s)ds\right)dx\\ &= \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} u(x)dx. \end{aligned}$$

We then prove that u satisfies the mean value property, which implies that u is harmonic in $B(0,1)$ by Theorem 2.21.

For the continuity, we just need to show that $u(rx')\to f(x')$ as $r\to 1$ uniformly in $x'\in\Sigma_{n-1}$. We write

$$u(rx') - f(x') = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(rx',s)(f(s)-f(x'))ds$$

by using property *b)* of the Poisson kernel. Then

$$\begin{aligned} |u(rx')-f(x')| &\le \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(rx',s)|f(s)-f(x')|ds\\ &= \frac{1}{|\Sigma_{n-1}|}\left(\int_{|s-x'|<\delta} + \int_{|s-x'|\ge\delta}\right). \end{aligned}$$

Since f is continuous, given $\varepsilon > 0$, δ can be chosen in such a way that the first term in the sum is $\frac{\varepsilon}{2}$ independently of x'. Then, with this choice of $\delta > 0$, r can be chosen so close to 1 that the second term in the sum is also smaller than $\frac{\varepsilon}{2}$ independently of x' (by property c)). □

If we want to solve the Dirichlet problem for a ball centered at $x_0 \in \mathbb{R}^n$ and having radius R, all we have to do is to reduce the problem to the unit ball by translation and dilation. Let f be the boundary function. Then consider for $x' \in \Sigma_{n-1}$ the function

$$g(x') = f(x_0 + Rx')$$

and solve the Dirichlet problem in the unit ball with boundary function g. The solution will be

$$v(x) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} P(x,s)g(s)ds.$$

Then $u(x) = v\left(\dfrac{x - x_0}{R}\right)$ will be the solution of the original Dirichlet problem for $B(x_0, R)$. We can write

$$u(x) = v\left(\frac{x - x_0}{R}\right) = \frac{R^{n-2}}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} \frac{R^2 - |x - x_0|^2}{|x - x_0 - Rs|^n} f(x_0 + Rs)ds.$$

The existence of solution for the Dirichlet problem in a ball allows us to make the following observation:

Theorem 2.27. *Suppose that u is continuous in an open set Ω and satisfies the following seemingly weaker form of the mean value property: for each $x_0 \in \Omega$, there is a sequence of positive numbers $r_j \downarrow 0$ (the r_j's depending possibly on the particular x_0) such that for each r_j:*

$$u(x_0) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} u(x_0 + r_j x')dx'.$$

Then u is harmonic in Ω.

Proof. Let $x_0 \in \Omega$ and suppose that $\overline{B(x_0, R)} \subset \Omega$. Let v be the solution of the Dirichlet problem for $B(x_0, R)$ with boundary function coinciding with u. We shall show that $u = v$ in $B(x_0, R)$. Since x_0 is arbitrary, this will end the proof of the theorem. Suppose that $u - v > 0$ for some points of $B(x_0, R)$, and let $m = \max\{u(x) - v(x) : x \in \overline{B(x_0, R)}\} > 0$. Since $u - v = 0$ on the boundary of $B(x_0, R)$, the set of points of $\overline{B(x_0, R)}$ where $u - v$ attains the value m, will be a compact subset K of $B(x_0, R)$. Let x_1 be a point of this compact K having maximal distance to x_0. For each $r > 0$ small enough, at least half of the sphere $\partial B(x_1, r)$ is not in K. But then $u(x_1) - v(x_1)$, which is the average of $u - v$ over each of the spheres of center x_1 and radius r_j (each of the radius corresponding to x_1) will have to be $< m$, which is a contradiction.

In fact, since $u(\overline{x}) - v(\overline{x}) < m$ for some $\overline{x} \in B(x_1, r_j) - K$, then we must have $u(x) - v(x) < m$ for some $B(\overline{x}, r_j') \subset B(x_1, r_j)$ and, consequently,

$$
\begin{aligned}
u(x_1) - v(x_1) &= \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} u(x_1 + r_j x')dx' - \frac{1}{|B(x_1, r_j)|} \int_{B(x_1,r_j)} v(x)dx \\
&= \frac{1}{|B(x_1, r_j)|} \int_{B(x_1,r_j)} u(x)dx - \frac{1}{|B(x_1, r_j)|} \int_{B(x_1,r_j)} v(x)dx \\
&= \frac{1}{|B(x_1, r_j)|} \int_{B(x_1,r_j)} (u(x) - v(x))dx \\
&= \frac{1}{|B(x_1, r_j)|} \left(\int_{B(x_1,r_j)\setminus B(\overline{x},r_j')} (u(x) - v(x))dx + \int_{B(\overline{x},r_j')} (u(x) - v(x))dx \right) \\
&= \frac{1}{|B(x_1, r_j)|} \left(\int_{B(x_1,r_j)\setminus B(\overline{x},r_j')} (u(x) - v(x))dx + \frac{|B(\overline{x}, r_j')|}{|B(\overline{x}, r_j')|} \int_{B(\overline{x},r_j')} (u(x) - v(x))dx \right) \\
&\leq \frac{1}{|B(x_1, r_j)|} (|B(x_1, r_j)\setminus B(\overline{x}, r_j')|m + |B(\overline{x}, r_j')|(u(\overline{x}) - v(\overline{x}))) \\
&< \frac{1}{|B(x_1, r_j)|} (|B(x_1, r_j)\setminus B(\overline{x}, r_j')|m + |B(\overline{x}, r_j')|m) \\
&= m \frac{(|B(x_1, r_j)\setminus B(\overline{x}, r_j')| + |B(\overline{x}, r_j')|)}{|B(x_1, r_j)|} \\
&= m \frac{|B(x_1, r_j)|}{|B(x_1, r_j)|} \\
&= m.
\end{aligned}
$$

If we had $u - v < 0$ for some point in $B(x_0, R)$, we would proceed exactly in the same way but using the minimum instead of the maximum. □

The previous theorem allows us to give a very simple proof of the following reflection principle:

Theorem 2.28. *Let Ω be a domain in $\mathbb{R}^n$, symmetric with respect to the hyperplane $x_{n_+} = 0$. Suppose that u is a function continuous in Ω, harmonic in $\Omega^+ = \{x \in \Omega : x_n > 0\}$ and odd in the variable x_n, that is:*

$$u(x_1, \ldots, x_{n-1}, -x_n) = -u(x_1, \ldots, x_{n-1}, x_n)$$

for every $x = (x_1, \ldots, x_n) \in \Omega$. Then, u is harmonic in Ω.

Proof. We just need to show that u satisfies the mean value property in the form appearing in the Theorem 2.27. But this is obvious, because:

i) If $x_0 \in \Omega^+$ we know that u is harmonic in Ω^+ and, consequently, for those $r's$ such that $\overline{B(x_0, r)} \subset \Omega^+$, $u(x_0)$ coincides with the average of u over the sphere $\partial B(x_0, r)$.

ii) If $x_0 \in \Omega^- = \{x \in \Omega : x_n < 0\}$, we have the same situation because for $x = (x_1, \dots, x_n) \in \Omega^-$,

$$u(x_1, \dots, x_{n-1}, -x_n) = -u(x_1, \dots, x_{n-1}, x_n)$$

and this implies that $\Delta u(x) = 0$ also in Ω^-.

iii) If $x_0 = (x_1, \dots, x_{n-1}, 0)$, we have $u(x_0) = 0$ and also, the average of u over each sphere centered at x_0 is necessarily 0 because u takes opposite values at a point and its symmetric in the other halfspace.

□

As a last application of the mean value property we shall prove the following extension of a classical theorem of Liouville:

Theorem 2.29. *The only bounded harmonic functions in $\mathbb{R}^n$ are the constants.*

Proof. Suppose u is harmonic in $\mathbb{R}^n$ and also $|u(x)| \le M < \infty$ for every $x \in \mathbb{R}^n$. Let x_1 and x_2 be two arbitrary points chosen in $\mathbb{R}^n$. Then, using the mean value property we have

$$u(x_1) - u(x_2) = \frac{1}{|B(x_1, r)|} \int_{B(x_1,r)} u(x)dx - \frac{1}{|B(x_2, r)|} \int_{B(x_2,r)} u(x)dx.$$

Let r be much larger than $|x_1 - x_2| = d$. Then

$$\begin{aligned}
|u(x_1) - u(x_2)| &= \left| \frac{1}{|B(x_1, r)|} \int_{B(x_1,r)} u(x)dx - \frac{1}{|B(x_2, r)|} \int_{B(x_2,r)} u(x)dx \right| \\
&= \frac{1}{|B(0, r)|} \left| \int_{B(x_1,r)} u(x)dx - \int_{B(x_2,r)} u(x)dx \right| \\
&\le \frac{1}{r^n |B(0, 1)|} \left(\left| \int_{B(x_1,r)\setminus B(x_2,r)} u(x)dx \right| + \left| \int_{B(x_2,r)\setminus B(x_1,r)} u(x)dx \right| \right) \\
&\le \frac{n}{r^n |\Sigma_{n-1}|} \left(\int_{B(x_1,r)\setminus B(x_2,r)} |u(x)|dx + \int_{B(x_2,r)\setminus B(x_1,r)} |u(x)|dx \right) \\
&\le \frac{n}{r^n |\Sigma_{n-1}|} \left(M \int_{B(x_1,r)\setminus B(x_2,r)} dx + M \int_{B(x_2,r)\setminus B(x_1,r)} dx \right) \\
&= \frac{Mn}{r^n |\Sigma_{n-1}|} \int_{B(x_1,r)\triangle B(x_2,r)} dx \\
&= \frac{Mn}{r^n |\Sigma_{n-1}|} |B(x_1, r) \triangle B(x_2, r)|,
\end{aligned}$$

where $B(x_1, r) \triangle B(x_2, r)$ is the symmetric difference of the two balls. Then, since

$$(B(x_1, r) \setminus B(x_2, r)) \subset B(x_1, r) \setminus B(x_2, r - d),$$

it follows that

$$|B(x_1, r) \setminus B(x_2, r)| \le |B(x_1, r) \setminus B(x_2, r - d)|$$

and, consequently,

$$\begin{aligned}
|B(x_1,r)\setminus B(x_2,r)| &\le |B(x_1,r)\setminus B(x_2,r-d)|\\
&= |B(x_1,r)| - |B(x_2,r-d)|\\
&= |B(0,1)|r^n - |B(0,1)|(r-d)^n\\
&= |B(0,1)|(r^n-(r-d)^n)\\
&= |B(0,1)|[(r-(r-d))(r^{n-1}+r^{n-2}(r-d)+\cdots+r(r-d)^{n-2}+(r-d)^{n-1})]\\
&\le |B(0,1)|[d(r^{n-1}+r^{n-2}r+\cdots+rr^{n-2}+r^{n-1})]\\
&= |B(0,1)|nr^{n-1}d\\
&= \frac{|\Sigma_{n-1}|}{n}nr^{n-1}d\\
&= |\Sigma_{n-1}|r^{n-1}d,
\end{aligned}$$

from which we conclude that

$$|u(x_1)-u(x_2)| \le \frac{2M_n d}{r}.$$

Now just let $r\to\infty$. □

2.2 Subharmonic functions

Definition 2.30. A subharmonic function on an open set $\Omega\subset\mathbb{R}^n$ is a function v defined on Ω, with values $-\infty\le v(x)<\infty$ and satisfying the following two conditions:

i) v is upper semicontinuous in Ω.

ii) For every $x_0\in\Omega$, there is a ball $B(x_0,r(x_0))\subset\Omega$, $r(x_0)>0$, such that for every r with $0<r<r(x_0)$,

$$v(x_0)\le\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0+r\sigma)d\sigma. \tag{2.18}$$

Remark 2.31. To say that v is upper semicontinuous in Ω means that for every $t\in\mathbb{R}$, the set $\{x\in\Omega : v(x)<t\}$ is open. In fact, given $x_0\in\{x\in\Omega : v(x)<t\}$ and choosing $\varepsilon=t-v(x_0)>0$, it follows from definition that there is $\delta>0$ such that

$$x\in B(x_0,\delta)\cap\Omega\Rightarrow v(x)<v(x_0)+\varepsilon=t.$$

This is, in turn, clearly equivalent to the following:

$$\text{For every } x_0\in\Omega : \limsup_{\Omega\ni x\to x_0} v(x)\le v(x_0). \tag{2.19}$$

In fact, given $x_0\in\Omega$, suppose that for all $\varepsilon>0$, there exists $\delta>0$ such that

$$x\in B(x_0,\delta)\cap\Omega\Rightarrow v(x)<v(x_0)+\frac{\varepsilon}{2}.$$

We just have to observe that

$$\begin{aligned}\limsup_{\Omega\ni x\to x_0} v(x) &= \inf_{\bar{\delta}>0}\left(\sup_{x\in B(x_0,\bar{\delta})-\{x_0\}} v(x)\right)\\ &\le \sup_{x\in B(x_0,\delta)-\{x_0\}} v(x)\\ &\le v(x_0)+\frac{\varepsilon}{2}\\ &< v(x_0)+\varepsilon.\end{aligned}$$

And since $\varepsilon > 0$ is arbitrary, we can just take $\varepsilon \to 0$. On the other hand, suppose that for every $x_0 \in \Omega$, it holds

$$\limsup_{\Omega\ni x\to x_0} v(x) \le v(x_0).$$

Let

$$L = \limsup_{\Omega\ni x\to x_0} v(x) = \inf_{\delta>0}\left(\sup_{x\in B(x_0,\delta)-\{x_0\}} v(x)\right).$$

By definition of infimum, given $\varepsilon > 0$, there is $\delta_0 > 0$ such that

$$\sup_{x\in B(x_0,\delta_0)-\{x_0\}} v(x) < L+\varepsilon.$$

Then, for this δ_0, we have

$$x \in B(x_0,\delta_0)\cap\Omega \Rightarrow v(x) < L+\varepsilon \le v(x_0)+\varepsilon.$$

Like continuity, upper semicontinuity is a pointwise property. When the inequality in (2.19) holds for a given $x_0 \in \Omega$, it is said that v is upper semicontinuous at x_0.

Lemma 2.32. *Let $K \subset \mathbb{R}^n$ be a compact subset. Let (K_n) be a sequence of non-empty closed subsets of K with $K_{i+1} \subset K_i$ for each i. Then*

$$A = \bigcap_{n=1}^{\infty} K_n \ne \varnothing.$$

Proof. Since K_n is closed in the compact set K, each K_n is compact. Pick $a_n \in K_n$ for all n. The sequence $\{a_n\}$ lies in K_1, so by compactness, it has a convergent subsequence $\{a_{n_k}\}$ with limit a in K_1. Now, observe that, except possibly for the first term, this subsequence is also contained in K_2. Since K_2 is closed, the limit a must lie in K_2. Continuing in this fashion, for any m, the tail $\{a_{n_k}\}_{k\ge m}$ lies in K_m, and since K_m is closed, so $a \in K_m$. Therefore, $a \in A = \bigcap_{n=1}^{\infty} K_n$, and hence $A \ne \varnothing$. □

Remark 2.33. Observe that the upper semicontinuity of v, together with the fact that $v(x) < \infty$ for every $x \in \Omega$, imply that v is bounded above on every compact $K \subset \Omega$. Indeed, let

$K_j = \{x \in K : v(x) \geq j\}$ for $j = 1, 2, \ldots$. These $K_j's$ are compact sets and $K \supset K_1 \supset K_2 \supset \cdots$. Since $\bigcap_{j=1}^{\infty} K_j = \{x \in K : v(x) = \infty\} = \varnothing$ by assumption, we must have $K_j = \varnothing$ for some j, by Lemma 2.32. That is: $v(x) < j$ for every $x \in K$.

This is what allows us to assign a meaning to the integrals appearing in (2.18). Each of these integrals is defined as the integral of the positive part of the function $v(x_0 + r\sigma)$ minus the integral of its negative part. This difference makes sense because the integral of the positive part is finite, being the integral of a bounded function. A priori, the integrals in (2.18) can be either a real number or $-\infty$. Actually, we shall eventually show that, unless v is identically equal to $-\infty$, none of these integrals can be $-\infty$, in such a way that v is integrable over the corresponding spheres.

It has to be noted that, for v subharmonic, (2.18) implies that $v(x_0) \leq \lim_{x \to x_0 \text{ in } \Omega} \sup v(x)$ and, consequently, we actually have equality in (2.19). In fact, let $x_0 \in \Omega$ and fix $r(x_0) > 0$ as in part *ii*) of the Definition 2.30. If $0 < r < r(x_0)$, then

$$\begin{aligned}
\int_{B(x_0,r)} v(y)dy &= \int_0^r \int_{\partial B(x_0,\delta)} v(y) d\sigma_s(y) d\delta \\
&= \int_0^r \delta^{n-1} \left(\int_{\Sigma_{n-1}} v(x_0 + \delta\sigma) d\sigma \right) d\delta \\
&\geq \int_0^r \delta^{n-1} |\Sigma_{n-1}| v(x_0) d\delta \\
&= |\Sigma_{n-1}| v(x_0) \frac{r^n}{n},
\end{aligned}$$

which means that

$$v(x_0) \leq \frac{n}{|\Sigma_{n-1}| r^n} \int_{B(x_0,r)} v(y)dy = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} v(y)dy.$$

Then we get:

$$\begin{aligned}
v(x_0) &\leq \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} v(y)dy \\
&= \frac{1}{|B(x_0,r)|} \left(\int_{B(x_0,r)-\{x_0\}} v(y)dy + v(x_0) \int_{\{x_0\}} dy \right) \\
&= \frac{1}{|B(x_0,r)|} \left(\left(\sup_{y \in B(x_0,r)-\{x_0\}} v(y) \right) \int_{B(x_0,r)-\{x_0\}} dy \right) \\
&= \frac{1}{|B(x_0,r)|} \left(\left(\sup_{y \in B(x_0,r)-\{x_0\}} v(y) \right) \int_{B(x_0,r)} dy \right) \\
&= \left(\sup_{y \in B(x_0,r)-\{x_0\}} v(y) \right) \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} dy \\
&= \sup_{y \in B(x_0,r)-\{x_0\}} v(y).
\end{aligned}$$

Since it holds for every $0 < r < r(x_0)$, we can take r arbitrarily small and, finally,

$$v(x_0) \le \lim_{r\to 0^+} \left(\sup_{y\in B(x_0,r)-\{x_0\}} v(y) \right) = \inf_{r>0} \left(\sup_{y\in B(x_0,r)-\{x_0\}} v(y) \right) = \limsup_{x\to x_0 \text{ in } \Omega} v(x).$$

Proposition 2.34. *v is upper semicontinuous in Ω if, and only if, for every compact $K \subset \Omega$, v is the limit over K of a decreasing sequence of continuous functions.*

Proof. First of all, let Λ be a directed set. We see that the infimum v of a family $\{v_\alpha\}_{\alpha\in\Lambda}$ of upper semicontinuous functions is itself upper semicontinuous. Indeed,

$$\{x : v(x) < t\} = \bigcup_\alpha \{x : v_\alpha(x) < t\}$$

is an open set, since it is union of open sets. In particular, if v is the limit of a decreasing sequence of continuous functions in K, v will be upper semicontinuous in K.

For the converse, suppose that v is upper semicontinuous in Ω and let K be a compact subset of Ω. Given

$$0 < \varepsilon < \operatorname{dist}(K, \Omega^{\complement}),$$

we know that

$$U_\varepsilon = \bigcup_{x\in K} B(x, \varepsilon)$$

defines an open covering of K, from which we can extract a finite subcovering, say

$$B(x_1, \varepsilon), \dots, B(x_j, \varepsilon)$$

(with $x_k \in K$ and $B(x_k, \varepsilon) \subset \Omega$, $k \in \{1, \dots, j\}$). Now let $\phi_k \ge 0$ be continuous, with support contained in $\overline{B(x_k, \varepsilon)}$ and such that $\sum_{k=1}^{j} \phi_k(x) = 1$ for every $x \in K$. The $\phi_k' s$ form what is known as a partition of the unity in K subordinated to the covering U_ε (see [19, Theorem 6.20]).

Let $m_j = \sup_{B(x_j,\varepsilon)} v$ and consider the continuous function $\psi(x) = \sum_j m_j \phi_j(x)$. We can do this for a decreasing sequence $\varepsilon_1 > \varepsilon_2 > \dots \to 0$, obtaining corresponding functions $\psi_1, \psi_2, \dots$. Then inductively let $u_1 = \psi_1, u_2 = \min(\psi_2, u_1), \dots, u_j = \min(\psi_j, u_{j-1}), \dots$. In this way, we obtain continuous functions $u_1 \ge u_2 \ge \cdots$. Note that $\psi_n(x) \ge v(x)$ for all $n \in \mathbb{N}$, which implies that $u_n(x) \ge v(x)$ for all $n \in \mathbb{N}$. In fact, the above construction provides, for each ε_l chosen, a finite subcovering of open balls

$$B_l^1, \dots, B_l^{j_l}, \quad l \in \mathbb{N}.$$

Now let

$$m_{j_l} = \sup_{B_l^{j_l}} v.$$

Since $x \in K$, then x must be in at least one of these balls B_l^ν, $\nu \in \{1, \dots, j_l\}$. We get the sup of v over these balls and pick the minimum, say m_p. All other balls, which do not contain x, will make their respective ϕ_ν vanishing. Then

$$\psi_l(x) = \sum_{k=1}^{j_l} m_k \phi_k(x) \geq m_p \sum_{k=1}^{j_l} \phi_k(x) \geq v(x).$$

We conclude that $\psi_l(x) \geq v(x)$ for all $l \in \mathbb{N}$ and all $x \in K$, which implies that $u_1 \geq u_2 \geq \cdots \geq v$ by construction.

Finally, it remains to show that $u_l \to v$ as $l \to \infty$. We need to show that for each $x_0 \in K$ and $\delta > 0$, there is $l_0 \in \mathbb{N}$ such that

$$l > l_0 \Rightarrow u_l(x_0) - v(x_0) < \delta \Leftrightarrow u_l(x_0) < v_n(x_0) + \delta.$$

Since v is upper semicontinuous in Ω, by definition we have

$$\limsup_{\Omega \ni x \to x_0} v(x) \leq v(x_0) \text{ for all } x_0 \in \Omega,$$

or equivalently,

$$L := \inf_{r>0} \left(\sup_{x \in B(x_0,r) - \{x_0\}} v(x) \right) \leq v(x_0).$$

For any $\delta > 0$, using the definition of infimum, it follows that there is $r' > 0$ such that

$$\sup_{x \in B(x_0,r')} v(x) < L + \frac{\delta}{2} \leq v(x_0) + \frac{\delta}{2} < v(x_0) + \delta.$$

So we just have to take l_0 such that

$$l > l_0 \Rightarrow \varepsilon_l < \frac{1}{2} r'.$$

Thus, if $l > l_0$, for every ball B_l^ν, $\nu \in \{1, \dots, j_l\}$, such that $x_0 \in B_l^\nu$ we have that $m_\nu = \sup_{B_l^\nu} v \leq \sup_{B(x_0,r')} v$, from which we conclude that $\psi_l(x_0) < v(x_0) + \delta$ and then $u_l(x_0) < v(x_0) + \delta$. □

Of course, any real-valued harmonic function is subharmonic, since it is continuous and has the mean value property, which is stronger than (2.18). However, the subharmonicity is all that is needed for the maximum principle. We can state:

Theorem 2.35. *Let v be a subharmonic function in a domain $\Omega \subset \mathbb{R}^n$. Then v cannot attain a maximum value unless it is constant.*

Proof. Suppose that v does attain a maximum value, that is, there exists $x_0 \in \Omega$ such that $v(x) \leq v(x_0) = m$ for every $x \in \Omega$. Take $0 < r < r(x_0)$, where $r(x_0)$ is the same that the one in Definition 2.30. Since $v(x) \leq m$ for every x and v is upper semicontinuous, if we had

$v(\overline{x}) < m$ for some $\overline{x} \in B(x_0, r)$, then we would have $v(x) < m$ for some $B(\overline{x}, r') \subset B(x_0, r)$ and the average of u over $B(x_0, r)$ would have to be $< m$. In fact,

$$\begin{aligned}
m &= v(x_0) \\
&\leq \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} v(x)dx \\
&= \frac{1}{|B(x_0,r)|} \left(\int_{B(x_0,r)\backslash B(\overline{x},r')} v(x)dx - \int_{B(\overline{x},r')} v(x)dx \right) \\
&= \frac{1}{|B(x_0,r)|} \left(\int_{B(x_0,r)\backslash B(\overline{x},r')} v(x)dx - \frac{|B(\overline{x},r')|}{|B(\overline{x},r')|} \int_{B(\overline{x},r')} v(x)dx \right) \\
&\leq \frac{1}{|B(x_0,r)|} (|B(x_0,r)\backslash B(\overline{x},r')|m + |B(\overline{x},r')|v(\overline{x})) \\
&< \frac{1}{|B(x_0,r)|} (|B(x_0,r)\backslash B(\overline{x},r')|m + |B(\overline{x},r')|m) \\
&< m \frac{(|B(x_0,r)\backslash B(\overline{x},r')| + |B(\overline{x},r')|)}{|B(x_0,r)|} \\
&< m \frac{|B(x_0,r)|}{|B(x_0,r)|} \\
&= m,
\end{aligned}$$

which is clearly a contradiction. Thus $v(x) = m$ for every $x \in B(x_0, r)$. This shows that the set A of points of Ω where $v(x) = m$ is an open set. But $B = \Omega - A = \{x \in \Omega : v(x) < m\}$ is also open because v is upper semicontinuous. Since A is not empty and Ω is connected, B has to be necessarily empty. Consequently $v(x) = m$ for every $x \in \Omega$. □

The maximum principle can also be given in this form:

Corollary 2.36. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $v : \overline{\Omega} \to [-\infty, \infty)$ be upper semicontinuous in $\overline{\Omega}$ and subharmonic in Ω. Then v has a maximum in Ω and attains it at the boundary (only at the boundary if v is not a constant).*

Proof. The fact that v has a maximum in $\overline{\Omega}$ follows simply from the upper semicontinuity (exactly in the same way that the fact that it is bounded above). In fact, define

$$m = \sup_{x \in \overline{\Omega}} v(x).$$

We already know that $m < \infty$, by Remark 2.33. For all $n \in \mathbb{N}$, take $x_n \in \overline{\Omega}$ such that

$$m - \frac{1}{n} < v(x_n) \leq m.$$

This gives rise to a sequence $\{x_n\} \subset \overline{\Omega}$ with

$$\lim_{n \to \infty} v(x_n) = m.$$

Since $\overline{\Omega}$ is compact, $\{x_n\}$ admits a convergent subsequence, say $x_{n_k} \to x_0 \in \overline{\Omega}$. Then, by the upper semicontinuity of v,

$$\begin{aligned} m \geq v(x_0) &\geq \limsup_{k\to\infty} v(x_{n_k}) \\ &\geq \lim_{k\to\infty} v(x_{n_k}) \\ &= m. \end{aligned}$$

Thus $v(x_0) = m$ and, by definition, $m < \infty$. Finally, we just need to apply the previous theorem. Provided v is not constant, v does not attain its maximum value in Ω. □

The maximum principle can be used to establish the following characterization of subharmonic functions, which is the best justification for the name "subharmonic".

Theorem 2.37. *Let $v : \Omega \to [-\infty, \infty)$ be upper semicontinuous in the open set Ω. Then, the following conditions are equivalent:*

a) v is subharmonic in Ω.

b) Whenever u is a real-valued continuous function in $\overline{G}$, harmonic in G, G being an open and bounded set with $\overline{G} \subset \Omega$, and u satisfies $v(x) \leq u(x)$ for every $x \in \partial G$, then $v(x) \leq u(x)$ for every $x \in G$.

Proof. Suppose *a*) holds and u satisfies the assumption made in *b*). We can assume G is connected. Then the function $v - u$ is upper semicontinuous in $\overline{G}$, subharmonic in G and $v - u \leq 0$ in ∂G. It follows then from Corollary 2.36 that $v - u \leq 0$ also in G.

Conversely, assuming that *b*) holds, let us prove that v is subharmonic. Let $x_0 \in \Omega$ and $\overline{B(x_0, r)} \subset \Omega$. We shall prove that

$$v(x_0) \leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(x_0 + r\sigma) d\sigma$$

holds. Since v is upper semicontinuous, we know by Proposition 2.34 that there is a decreasing sequence of functions $u_1 \geq u_2 \geq \cdots \geq u_j \geq \cdots$ continuous in $\partial B(x_0, r)$, converging to v in $\partial B(x_0, r)$. Abusing the notation a little, let us denote also by u_j the continuous function in $\overline{B(x_0, r)}$ and harmonic in $B(x_0, r)$ which coincides with u_j in $\partial B(x_0, r)$, that is, the solution of the Dirichlet problem in $B(x_0, r)$ with boundary function u_j. It follows from *b*) that $v(x) \leq u_j(x)$ for every j and every $x \in B(x_0, r)$. Moreover, the subharmonicity of u_{j+1}, together with the fact already proved that *a*) implies *b*) yield $u_j(x) \geq u_{j+1}(x)$ for every j and every $x \in B(x_0, r)$. We can write:

$$v(x_0) \leq \lim_{j\to\infty} u_j(x_0) = \lim_{j\to\infty} \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} u_j(x_0 + r\sigma) d\sigma = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(x_0 + r\sigma) d\sigma.$$

It proves what we wanted. The interchanging of the limit and the integral sign is justified by the monotone convergence theorem (see [21, Corollary 1.9]). □

Remark 2.38. Observe that in the course of proving Theorem 2.37 we have also proved that if v is subharmonic in Ω, then

$$v(x_0) \leq \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0+r\sigma)d\sigma$$

holds whenever $\overline{B(x_0,r)} \subset \Omega$. This is stronger than what we assumed in the definition of subharmonicity.

Proposition 2.39. *Let v be a subharmonic function in a domain $\Omega \subset \mathbb{R}^n$, and suppose that v is not identically equal to $-\infty$. Then, whenever $\overline{B(x_0,r)} \subset \Omega$, we have*

$$\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0+r\sigma)d\sigma > -\infty.$$

Proof. Let $\overline{B(x_0,r)} \subset \Omega$, and let $u_1 \geq u_2 \geq \cdots u_j \geq \cdots$ be a sequence of continuous functions in $\partial B(x_0,r)$ converging to v in $\partial B(x_0,r)$. As before, consider u_j extended as a harmonic function in $B(x_0,r)$, continuous in $\overline{B(x_0,r)}$. Remember from Theorem 2.26 that

$$u_j(x) = \frac{r^{n-2}}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} \frac{r^2-|x-x_0|^2}{|x-x_0-rs|^n} u_j(x_0+rs)ds,$$

where $x \in B(x_0,r)$ and u_j is the boundary function, considering the same abuse of notation made previously. Then, for every $x \in \mathbb{R}^n$ with $|x|<1$ is:

$$\begin{aligned}
v(x_0+rx) &\leq \lim_{j\to\infty} u_j(x_0+rx)\\
&= \lim_{j\to\infty}\frac{r^{n-2}}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} \frac{r^2-|x_0+rx-x_0|^2}{|x_0+rx-x_0-rs|^n}u_j(x_0+rs)ds\\
&= \lim_{j\to\infty}\frac{r^{n-2}}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} \frac{r^2(1-|x|^2)}{r^n|x-s|^n}u_j(x_0+rs)ds\\
&= \lim_{j\to\infty}\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} \frac{1-|x|^2}{|x-s|^n}u_j(x_0+rs)ds \qquad (2.20)\\
&= \lim_{j\to\infty}\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x,s)u_j(x_0+rs)ds\\
&= \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x,s)\lim_{j\to\infty}u_j(x_0+rs)ds\\
&= \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x,s)v(x_0+rs)ds,
\end{aligned}$$

where we have used the monotone convergence theorem [21] (note that $P(x,s)\geq 0$). We claim that

$$A = \{x\in\Omega : \int_{\Sigma_{n-1}} v(x+r\sigma)d\sigma = -\infty \text{ for some } r>0 \text{ with } \overline{B(x,r)}\subset\Omega\}$$

is an open set. In fact, given $x_0 \in A$, it follows that

$$\int_{\Sigma_{n-1}} v(x_0+r\sigma)d\sigma = -\infty \text{ for some } r>0 \text{ with } \overline{B(x_0,r)}\subset\Omega,$$

then also

$$\int_{\Sigma_{n-1}} P(x,s)v(x_0+rs)ds = -\infty,$$

because $P(x,s)$ is positive and bounded as a function of s. It follows then from (2.20) that $v(x) = -\infty$ for every $x \in B(x_0,r)$, which implies that A is open. But its complement $B = \Omega - A$ is also open. Indeed, if $x_0 \in B$, so for all $r > 0$ with $\overline{B(x_0,r)} \subset \Omega$,

$$\int_{\Sigma_{n-1}} v(x_0+r\sigma)d\sigma > -\infty.$$

Note that no point of A can belong to $B(x_0,r)$ because this would imply that

$$\int_{\Sigma_{n-1}} v(x_0+r'\sigma)d\sigma = -\infty$$

for some $r' < r$, since v would be equal to $-\infty$ in a whole open subset of the sphere $\partial B(x_0,r')$. Therefore, either $A = \varnothing$ or $A = \Omega$, and in this latter case v is identically $-\infty$. Thus we get $A = \varnothing$. □

Proposition 2.40. *Let v be a subharmonic function in $B(0,R) \subset \mathbb{R}^n$. Then,*

$$m(r) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(r\sigma)d\sigma$$

is an increasing function in the interval $[0,R)$.

Proof. Let $r_1 < r_2 < R$. Let $u_1 \geq u_2 \geq \cdots \geq u_j \geq \cdots$ be a sequence of functions in $\partial B(0,r_2)$ converging to v in $\partial B(0,r_2)$. For each j, we also denote by u_j the function continuous in $\overline{B(0,r_2)}$ and harmonic in $B(0,r_2)$ coinciding with our original u_j in $\partial B(0,r_2)$. Then

$$\begin{aligned}
m(r_1) &= \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(r_1\sigma)d\sigma \\
&\leq \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u_j(r_1\sigma)d\sigma \\
&= u_j(0) \\
&= \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u_j(r_2\sigma)d\sigma \xrightarrow{j\to\infty} \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(r_2\sigma)d\sigma = m(r_2).
\end{aligned}$$

Thus $m(r_1) \leq m(r_2)$ and our statement is proved. □

Remark 2.41. It has to be observed that in the Definition 2.30 of a subharmonic function, condition *ii*) can be replace by: For every $x_0 \in \Omega$ and every $r > 0$ such that $\overline{B(x_0,r)} \subset \Omega$,

$$v(x_0) \leq \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} v(x)dx.$$

In fact, let v be a subharmonic function on an open set $\Omega \subset \mathbb{R}^n$, with values $-\infty \leq v(x) < \infty$. It follows from Remark 2.38 that

$$v(x_0) \leq \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0+r\sigma)d\sigma$$

holds whenever $\overline{B(x_0,r)} \subset \Omega$. To conclude, we just observe that

$$\begin{aligned}\int_{B(x_0,r)} v(y)dy &= \int_0^r \int_{\partial B(x_0,\delta)} v(y)d\sigma_s(y)d\delta \\ &= \int_0^r \delta^{n-1}\left(\int_{\Sigma_{n-1}} v(x_0+\delta\sigma)d\sigma\right)d\delta \\ &\geq \int_0^r \delta^{n-1}|\Sigma_{n-1}|v(x_0)d\delta \\ &= |\Sigma_{n-1}|v(x_0)\frac{r^n}{n},\end{aligned}$$

which means that

$$v(x_0) \leq \frac{n}{|\Sigma_{n-1}|r^n}\int_{B(x_0,r)} v(y)dy = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} v(y)dy.$$

On the other hand, suppose that for every $x_0 \in \Omega$ and every $r > 0$ such that $\overline{B(x_0,r)} \subset \Omega$,

$$v(x_0) \leq \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} v(x)dx.$$

Suppose that v does attain a maximum value, that is, there exists $x_0 \in \Omega$ such that $v(x) \leq v(x_0) = m$ for every $x \in \Omega$. Take $r > 0$ such that $\overline{B(x_0,r)} \subset \Omega$. Since $v(x) \leq m$ for every x and v is upper semicontinuous, if we had $v(\overline{x}) < m$ for some $\overline{x} \in B(x_0,r)$, then we would have $v(x) < m$ for some $B(\overline{x},r') \subset B(x_0,r)$ and the average of v over $B(x_0,r)$ would have to be $< m$ (see the proof of Theorem 2.35). Thus $v(x) = m$ for every $x \in B(x_0,r)$. This allows us to conclude that the set A of points of Ω where $v(x) = m$ is an open set. But $B = \Omega - A = \{x \in \Omega : v(x) < m\}$ is also open because v is upper semicontinuous. Since A is not empty and Ω is connected, B has to be necessarily empty. Consequently $v(x) = m$ for every $x \in \Omega$. In other words, v satisfies the maximum principle. From this we can infer that v satisfies the property $b)$ in Theorem 2.37, which implies that v is subharmonic in Ω.

Remark 2.42. For $v \in C^2(\Omega)$, the method of proof of Theorem 2.21 gives a necessary and sufficient condition for v to be subharmonic in terms of $\triangle v$, namely, we must have $\triangle v(x) \geq 0$ for every $x \in \Omega$. The proof of necessity uses also Proposition 2.40.

In fact, let $v \in C^2(\Omega)$ and suppose that $\triangle v \geq 0$ in Ω. Let $x_0 \in \Omega$ and $r > 0$ be such that $\overline{B(x_0,r)} \subset \Omega$. For $0 < s \leq r$, let $f(s)$ stand for the average of v over the sphere of center x_0 and radius s, that is,

$$f(s) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0+s\sigma)d\sigma.$$

Since v is continuous and continuously differentiable ($v \in C^2(\Omega)$), then

$$f'(s) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} \sum_{j=1}^{n} v_{x_j}(x_0+s\sigma)\sigma_j d\sigma.$$

Note that $\sum_{j=1}^{n} v_{x_j}(x_0+s\sigma)\sigma_j = D_\sigma v(x_0+s\sigma)$, which is the derivative of u in the direction of the outer normal in the point $x_0+s\sigma$. And therefore

$$f'(s) = \frac{1}{s^{n-1}|\Sigma_{n-1}|}\int_{\partial B(x_0,s)} D_\sigma v(x) d\sigma_s(x),$$

where $\sigma = \frac{x-x_0}{s}$, $\partial B(x_0,s)$ is the boundary of the ball; and $d\sigma_s$ is the natural Lebesgue measure on $\partial B(x_0,s)$. By applying Green's theorem we get

$$f'(s) = \frac{1}{s^{n-1}|\Sigma_{n-1}|}\int_{B(x_0,s)} \triangle v \geq 0.$$

Then $f(s)$ is non-decreasing for $0 < s \leq r$. Now, it is clear that

$$\lim_{s\to 0} f(s) = v(x_0).$$

Thus

$$\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0+r\sigma)d\sigma = f(r) \geq \lim_{s\to 0} f(s) = v(x_0).$$

On the other hand, suppose that v is subharmonic. Let us suppose that $\triangle v(x_0) < 0$ for some $x_0 \in \Omega$. Consider $r > 0$ such that $\overline{B(x_0,r)} \subset \Omega$. If $v \in C^2(\Omega)$, so $\triangle v$ is continuous in Ω and then there is a radius $r_0 \in (0, \mathrm{dist}(x_0,r))$ such that $\triangle v < 0$ in $B(x_0,R)$ for all $R \in (0,r_0)$. But then, we compute as above that the function f is decreasing and hence

$$v(x_0) > \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0+r\sigma)d\sigma,$$

which shows that v is not subharmonic in Ω (contradiction).

Proposition 2.43. *Suppose that v is a nonnegative continuous function on an open set Ω such that v is of class C^2 on the open set $\Omega_0 = \{x \in \Omega : v(x) > 0\}$ and satisfies $\triangle v \geq 0$ on Ω_0. Then v is subharmonic in the whole set Ω.*

Proof. We already know that v is subharmonic in Ω_0, so that we only need to consider points $x_0 \in \Omega$ such that $v(x_0) = 0$. Let x_0 be one of such points, and suppose $\overline{B(x_0,r)} \subset \Omega$. Let us prove that v satisfies the condition $b)$ of Theorem 2.37 in order to show that v is subharmonic.

Let u be harmonic in $B(x_0,r)$ coinciding with v in the boundary of $B(x_0,r)$. Note that $u \geq 0$, since it is harmonic and then satisfies the maximum principle. It is enough to show that $v(x) \leq u(x)$ for every $x \in B(x_0,r)$. Suppose that $v(x) - u(x) > 0$ for some $x \in B(x_0,r)$, and define then

$$\max_{x\in\overline{B(x_0,r)}} (v(x)-u(x)) = \delta > 0.$$

Let $A = \{x \in B(x_0,r) : v(x) - u(x) = \delta\}$. If $x \in A$, we have

$$v(x) = u(x) + \delta \geq \delta > 0,$$

so that $A \subset \Omega_0$. On the other hand, A is non-empty closed subset of $B(x_0, r)$. It turns out that A is also open. Indeed, let $x \in A$ and consider $B(x, r') \subset B(x_0, r) \cap \Omega_0$. Then, since $v - u$ is subharmonic in Ω_0, we have

$$\delta = v(x) - u(x) \leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} (v(x + r'\sigma) - u(x + r'\sigma))d\sigma \leq \delta.$$

It follows that $v - u = \delta$ all over $B(x, r')$. Since A is both open and closed, it must be $A = B(x_0, r)$, but this is impossible because $v(x_0) - u(x_0) = -u(x_0) \leq 0$. Then $v(x) \leq u(x)$ for every $x \in B(x_0, r)$ and v is subharmonic by Theorem 2.37. □

Theorem 2.44. *Let v be a subharmonic function in Ω, and suppose ϕ is a function increasing and convex*[4] *in $\mathbb{R}$. Then the composite function $\phi \circ v$ is also subharmonic (define $\phi(-\infty)$ so that ϕ becomes continuous at $-\infty$. That way, the composition will always make sense).*

Proof. First, we have to see that $\phi \circ v$ is upper semicontinuous, that is, that $(\phi \circ v)^{-1}([-\infty, t))$ is always open. But

$$(\phi \circ v)^{-1}([-\infty, t)) = v^{-1}(\phi^{-1}([-\infty, t))$$

and, since ϕ is increasing and continuous,

$$\phi^{-1}([-\infty, t)) = [-\infty, s)$$

unless it is empty ($\phi(s) = t$). Since v is upper semicontinuous then $v^{-1}([-\infty, s))$ is open. Thus, $(\phi \circ v)^{-1}([-\infty, t))$ is always open. Now, let $\overline{B(x_0, r)} \subset \Omega$. Then

$$\phi(v(x_0)) \leq \phi\left(\frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(x_0 + r\sigma)d\sigma\right) \leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} \phi(v(x_0 + r\sigma))d\sigma,$$

the last inequality being a consequence of convexity (Jensen's inequality). □

We shall give now our main example of a subharmonic function, namely, $\ln(|F(z)|)$ for F holomorphic not identically 0 in a plane domain. The subharmonicity of this function will be one of our main tools. If F is never 0, then $\ln(|F(z)|) = \Re(\ln(F(z)))$, where $\ln(F(z))$ can be defined locally as a holomorphic function. It follows that for F holomorphic without zeros $\ln(|F(z)|)$ is actually a harmonic function. In the general case, some work will be needed to get rid of the zeros.

Lemma 2.45.

$$\int_{-\pi}^{\pi} \ln(|1 - e^{it}|)dt = 0.$$

[4] A function $f : \mathbb{R} \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}$ and $t \in [0, 1]$, we have $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$.

Proof. First of all, exploring the 2π-periodicity of the integrand, we can write:

$$\int_{-\pi}^{\pi} \ln(|1-e^{it}|)dt = \int_{0}^{2\pi} \ln(|1-e^{it}|)dt.$$

Now, since $\sin^2(t) = (1-\cos(2t))/2$, it follows that

$$\begin{aligned} |1-e^{it}| &= |(1-\cos(t)) + i\sin(t)| \\ &= \sqrt{1^2 - 2\cos(t) + \cos^2(t) + \sin^2(t)} \\ &= \sqrt{2(1-\cos(t))} \\ &= \sqrt{2(2\sin^2(t/2))} \\ &= 2|\sin(t/2)|. \end{aligned}$$

Then

$$\begin{aligned} \int_{-\pi}^{\pi} \ln(|1-e^{it}|)dt &= \int_{0}^{2\pi} \ln(2|\sin(t/2)|)dt \\ &\overset{t/2=\theta}{=} 2\int_{0}^{\pi} \ln(2\sin(\theta))d\theta \\ &= 2\int_{0}^{\pi} \ln(2) + 2\int_{0}^{\pi} \ln(\sin(\theta))d\theta \\ &= 2\pi\ln(2) + 2\underbrace{\int_{0}^{\pi} \ln(\sin(\theta))d\theta}_{I}. \end{aligned}$$

Let us calculate I:

$$\int_{0}^{\pi} \ln(\sin(\theta))d\theta = \int_{0}^{\pi/2} \ln(\sin(\theta))d\theta + \int_{\pi/2}^{\pi} \ln(\sin(\theta))d\theta.$$

But

$$\begin{aligned} &\int_{\pi/2}^{\pi} \ln(\sin(\theta))d\theta \\ &\qquad \overset{t=\theta-\pi/2}{=} \int_{0}^{\pi/2} \ln(\sin(t+\pi/2))dt \\ &\qquad = \int_{0}^{\pi/2} \ln(\cos(t))dt. \end{aligned}$$

Therefore, using that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we get:

$$\begin{aligned}
\int_0^{\pi} \ln(\sin(\theta))d\theta &= \int_0^{\pi/2} \ln(\sin(\theta))d\theta + \int_0^{\pi/2} \ln(\cos(\theta))d\theta \\
&= \int_0^{\pi/2} \ln(\sin(\theta)\cos(\theta))d\theta \\
&= \int_0^{\pi/2} \ln\left(\frac{2\sin(\theta)\cos(\theta)}{2}\right) d\theta \\
&= \int_0^{\pi/2} [\ln(2\sin(\theta)\cos(\theta)) - \ln(2)]d\theta \\
&= \int_0^{\pi/2} \ln(\sin(2\theta))d\theta - \int_0^{\pi/2} \ln(2)d\theta \\
&\overset{u=2\theta}{=} \frac{1}{2}\left(\int_0^{\pi} \ln(\sin(u))du - \pi\ln(2)\right),
\end{aligned}$$

which implies that

$$I = \int_0^{\pi} \ln(\sin(\theta))d\theta = -\pi\ln(2).$$

Thus

$$\int_{-\pi}^{\pi} \ln(|1 - e^{it}|)dt = 2\pi\ln(2) - 2\pi\ln(2) = 0.$$

□

Lemma 2.46. *Let F be holomorphic, not identically 0 on an open set $\Omega \subset \mathbb{C}$. Then the set*

$$Z(F) = \{z \in \Omega : F(z) = 0\} = F^{-1}(\{0\})$$

is a discrete and closed subset of Ω.

Proof. It is clear that $Z(F)$ is closed in Ω, since F is continuous. If $Z(F) = \varnothing$, the result is trivial. Let us suppose that $Z(F) \neq \varnothing$. Take $z_0 \in Z(F)$. Since F is not identically 0, there exists (see [4, Chapter IV, Corollary 3.9]) $m \in \mathbb{N}$ and $g : \Omega \to \mathbb{C}$ holomorphic satisfying $g(z_0) \neq 0$ and

$$F(z) = F(z_0) + (z - z_0)^m g(z) = (z - z_0)^m g(z).$$

By continuity, there exists $\delta > 0$ such that

$$g(z) \neq 0 \text{ for all } z \in B(z_0, \delta).$$

Consequently,

$$F(z) = (z - z_0)^m g(z) \neq 0 \text{ for all } z \in B^*(z_0, \delta).$$

Therefore, z_0 is isolated in $Z(F)$, and since z_0 was arbitrary, $Z(F)$ is discrete. □

Theorem 2.47 (Jensen's formula). *Let F be holomorphic in $\mathbb{D}_R(0)$ and suppose that $F(0) \neq 0$. Let $0 < r < R$ and call $z_1, z_2, \ldots, z_n$ the zeros of F in $\overline{\mathbb{D}_r(0)}$ listed according to their multiplicities. Then:*

$$\ln|F(0)| + \sum_{j=1}^{n} \ln\left(\frac{r}{|z_j|}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|F(re^{it})|dt.$$

Proof. Let us enumerate the zeros of F such that $z_1, \ldots, z_m \in \mathbb{D}_r(0)$ and $|z_{m+1}| = \cdots = |z_n| = r$. Define

$$G(z) = F(z)\prod_{j=1}^{m}\frac{r^2 - z\,\overline{z_j}}{r(z - z_j)}\prod_{j=m+1}^{n}\frac{z_j}{z - z_j}.$$

Note that G is holomorphic and nowhere zero in $\mathbb{D}_{r+\varepsilon}(0)$ for some $\varepsilon > 0$. Otherwise for every $n \in \mathbb{N}$ sufficiently large and defining $\varepsilon_n = r + 1/n < R$, we construct a sequence $\{z_n\}$ of zeros in $\overline{\mathbb{D}_{\varepsilon_1}(0)} \subset \mathbb{D}_R(0)$, which would imply the existence of a convergent subsequence. But we know from Lemma 2.46 that the set of zeros of a holomorphic function not identically 0 must be a discrete and closed subset. It shows $\ln|G(z)|$ is harmonic in $\mathbb{D}_{r+\varepsilon}(0)$. We shall have:

$$\ln|G(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|G(re^{it})|dt.$$

But

$$\ln|G(0)| = \ln|F(0)| + \sum_{j=1}^{m}\ln\left(\frac{r}{|z_j|}\right) = \ln|F(0)| + \sum_{j=1}^{n}\ln\left(\frac{r}{|z_j|}\right).$$

Since

$$\begin{aligned}\left|\frac{r^2 - re^{it}\overline{z_j}}{r(re^{it} - z_j)}\right| &= \left|\frac{r - e^{it}\overline{z_j}}{re^{it} - z_j}\right| \\ &= \left|\frac{e^{it}(re^{-it} - \overline{z_j})}{re^{it} - z_j}\right| \\ &= \frac{|re^{-it} - \overline{z_j}|}{|re^{it} - z_j|} \\ &= 1\end{aligned}$$

and

$$\left|\frac{z_j}{re^{it} - z_j}\right| = \left|\frac{re^{it_j}}{r(e^{it} - e^{it_j})}\right| = \frac{1}{|1 - e^{i(t-t_j)}|},$$

we finally obtain

$$\ln|F(0)| + \sum_{j=1}^{n}\ln\left(\frac{r}{|z_j|}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|F(re^{it})|dt + \sum_{j=m+1}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(\frac{1}{|1 - e^{i(t-t_j)}|}\right)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|F(re^{it})|dt$$

by Lemma 2.45 □

Corollary 2.48. *Let F be holomorphic, not identically 0 on an open set $\Omega \subset \mathbb{C}$. Then, the functions $\ln|F(z)|$, $\ln^+|F(z)| = \max(\ln|F(z)|, 0)$ and $|F(z)|^\alpha$ for every $0 < \alpha < \infty$, are all subharmonic in Ω.*

Proof. First of all, let us see that the function $\ln(|F(z)|)$ is subharmonic. It is a continuous function with values in $[-\infty, \infty)$. Besides, if $\overline{\mathbb{D}_r(z_0)} \subset \Omega$, we have:

$$\ln(|F(z_0)|) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln(|F(z_0 + re^{it})|)dt.$$

This is clear if $F(z_0) = 0$. If this is not the case, it follows from Theorem 2.47 applied to the function $z \mapsto F(z_0 + z)$ which is holomorphic in $\mathbb{D}_{r+\varepsilon}(0)$ for some $\varepsilon > 0$ and does not vanish at 0.

As for the functions $\ln^+(|F(z)|)$ and $|F(z)|^\alpha$, $\alpha > 0$, they result from composition $\ln(|F(z)|)$ with the functions $\phi(t) = \max(t, 0)$ and $\phi(t) = e^{\alpha t}$ respectively, which are increasing and convex. The subharmonicity follows from Theorem 2.44. □

Theorem 2.49. *For $F \in \mathcal{O}(\mathbb{D})$ and $0 \leq r < 1$, we define:*

- $m_0(F, r) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^+(|F(re^{it})|)dt\right)$.
- $m_p(F, r) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{it})|^p dt\right)^{1/p}$, $0 < p < \infty$.
- $m_\infty(F, r) = \sup_t |F(re^{it})|$.

Then, for each $F \in \mathcal{O}(\mathbb{D})$ and each $0 \leq p \leq \infty$, $m_p(F, r)$ is an increasing function of r in $[0, 1)$.

Proof. This is just a consequence of Colorollary 2.48 and Proposition 2.40. □

Definition 2.50. For $0 < p \leq \infty$, we shall define the Hardy space $H^p(\mathbb{D})$ (also denoted simply by H^p when the context is clear) to be the following class of functions:

$$H^p(\mathbb{D}) = \{F \in \mathcal{O}(\mathbb{D}) : \|F\|_{H^p} \equiv \sup_{0 \leq r < 1} m_p(F, r) < \infty\}.$$

For $p = 0$, we have the Nevanlinna class N, defined by:

$$N = \{F \in \mathcal{O}(\mathbb{D}) : \sup_{0 \leq r < 1} m_0(F, r) < \infty\}.$$

Remark 2.51. If $0 < p < q < \infty$, we clearly have $H^\infty \subset H^q \subset H^p \subset N$. In fact, the first inclusion is trivial, since $F \in H^\infty$ implies that

$$\sup_{0 \leq r < 1} \left(\sup_t |F(re^{it})|\right) < \infty.$$

Then $|F(re^{it})| \leq M$ for some positive constant M independent of r and t and, consequently, $|F(re^{it})|^q \leq M^q < \infty$, from which we conclude that

$$\sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{it})|^q dt\right)^{1/q} < \infty,$$

that is, $F \in H^q(\mathbb{D})$.

For the second inclusion, we just apply Hölder's inequality for the exponent conjugates $\left(\frac{q}{p}, \frac{q}{q-p}\right)$:

$$\begin{aligned}\int_{-\pi}^{\pi} |F(re^{it})|^p dt &= \int_{-\pi}^{\pi} (|F(re^{it})|^p \cdot 1) dt \\ &\leq \left(\int_{-\pi}^{\pi} (|F(re^{it})|^p)^{q/p} dt\right)^{p/q} \left(\int_{-\pi}^{\pi} 1 dt\right)^{1-p/q} \\ &= \left(\int_{-\pi}^{\pi} |F(re^{it})|^q dt\right)^{p/q} (2\pi)^{1-p/q} \\ &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{it})|^q dt\right)^{p/q} (2\pi),\end{aligned}$$

which implies that

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{it})|^p dt\right)^{1/p} \leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{it})|^q dt\right)^{1/q}.$$

Then

$$\sup_{0\leq r<1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{it})|^p dt\right)^{1/p} \leq \sup_{0\leq r<1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{it})|^q dt\right)^{1/q} < \infty,$$

since $F \in H^q(\mathbb{D})$.

Finally, we observe that $\ln^+ t \leq C_p t^p$, $t \geq 0$. Then,

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+(|F(re^{it})|) dt\right) \leq \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} C_p|F(re^{it})|^p dt\right) < \infty,$$

since $F \in H^p(\mathbb{D})$. Thus $F \in N$.

Proposition 2.52. *Let $F \in N$ be such that $F(0) \neq 0$. Then*

$$\sup_{0\leq r<1}\frac{1}{2\pi}\int_{\pi}^{-\pi} |\ln(|F(re^{it})|)| dt < \infty.$$

Proof. Note that

$$-\infty < \ln(|F(0)|) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(|F(re^{it})|) dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+(|F(re^{it})|) dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^-(|F(re^{it})|) dt.$$

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^-(|F(re^{it})|) dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+(|F(re^{it})|) dt - \ln(|F(0)|)$$

and, consequently,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} |\ln(|F(re^{it})|)| dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+(|F(re^{it})|) dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^-(|F(re^{it})|) dt \\ &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} \ln^+(|F(re^{it})|) dt - \ln(|F(0)|).\end{aligned}$$

From this inequality, our claim follows immediately since

$$\sup_{0\le r<1} m_0(F,r) = \sup_{0\le r<1} \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+(|F(re^{it})|)dt\right) < \infty.$$

□

Next, we shall use Jensen's formula to derive a basic fact, namely: that the zeros of $F \in N$ can not be too far from the boundary.

Theorem 2.53. *Suppose $F \in N$ and F is not identically 0. Let z_j be the zeros of F listed according to their multiplicities. Then:*

$$\sum_j (1-|z_j|) < \infty. \tag{2.21}$$

Proof. We may assume $|z_1| \le |z_2| \le \dots$ and $F(0) \neq 0$. Applying Jensen's formula, we get, for every $0<r<1$,

$$\begin{aligned}\ln(|F(0)|) + \sum_{|z_j|\le r} \ln\left(\frac{r}{|z_j|}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(|F(re^{it})|)dt \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+(|F(re^{it})|)dt \\ &\le M < \infty\end{aligned}$$

with M independent of r. Given n, we just need to take $r<1$ close enough to 1, to have $|z_n| \le r$ and, consequently $|z_j| \le r$ for $j = 1,2,\dots,n$. Then, for n fixed,

$$\sum_1^n \ln\left(\frac{1}{|z_j|}\right) \le M - n\ln(r) - \ln(|F(0)|)$$

is true for all r bigger than certain $r(n)$ which depends on n. Letting r tend to 1 we obtain

$$\sum_1^n \ln\left(\frac{1}{|z_j|}\right) \le M - \ln(|F(0)|).$$

Since this is true for every n, we have

$$\sum_1^\infty \ln\left(\frac{1}{|z_j|}\right) \le M - \ln(|F(0)|) < \infty.$$

Now, observing that $1-|z_j| \le \ln\left(\frac{1}{|z_j|}\right)$, we get (2.21).[5] □

[5]In fact, if $f(t) = 1-t+\ln(t)$, with $0<t<1$, we get $f'(t)>0$, which implies that f is increasing and, consequently, $f(t) \le f(1) = 0$.

Theorem 2.54. *Let (z_j) be a sequence of complex numbers $0 < |z_j| < 1$, such that (2.21) holds. Let k be a non-negative integer. Then the "Blaschke product"*

$$B(z) = z^k \prod_{j=1}^{\infty} \frac{z_j - z}{1 - z\overline{z_j}} \frac{|z_j|}{z_j}$$

converges uniformly on each compact subset of the unit disk, to a function $B \in H^\infty$ whose zeros are precisely the z_j's plus a zero of order k at 0 if $k > 0$.

Proof. All we need to prove is that the series

$$\sum_{j=1}^{\infty} \left| 1 - \frac{z_j - z}{1 - z\overline{z_j}} \frac{|z_j|}{z_j} \right|$$

converges uniformly on compact subsets of $\mathbb{D}$ (see [20, Theorem 15.4]). But for $|z| \leq r < 1$ we have the estimate:

$$\begin{aligned} \left| 1 - \frac{z_j - z}{1 - z\overline{z_j}} \frac{|z_j|}{z_j} \right| &= \left| \frac{z_j - z|z_j|^2 - (z_j - z)|z_j|}{z_j - z|z_j|^2} \right| \\ &= (1 - |z_j|) \left| \frac{z_j + z|z_j|}{z_j - z|z_j|^2} \right| \\ &\leq (1 - |z_j|) \frac{1 + r}{1 - r}, \end{aligned}$$

so that (2.21) is sufficient for our purposes. In fact, (2.21) implies the convergence of the above series in $|z| \leq r$. Since $r < 1$ is arbitrary, the result follows. □

Since $B \in H^\infty$, Fatou's Theorem (Theorem 2.16) implies that B has non-tangential limits at almost every boundary point. We shall write

$$B(e^{it}) = \lim B(z) \text{ as } z \xrightarrow{\text{N.T.}} e^{it}.$$

In general, if for some function F in $\mathbb{D}$, the non-tangential boundary value of F is known to exist at e^{it}, we shall denote it by $F(e^{it})$. If $F \in H^p$ with $p \geq 1$, we know that $F(e^{it})$ exists almost everywhere, since F is a Poisson (or Poisson-Stieltjes) integral according to Theorems 2.5 and 2.6. In the next section we shall extend this result to any $p > 0$.

Theorem 2.55. *Let B the Blaschke product appearing in Theorem 2.54. Then $|B(e^{it})| = 1$ for almost everywhere t and*

$$\lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln(|B(re^{it})|) dt = 0.$$

Proof. We know that the limit exists because the integral is an increasing (and bounded, since $B \in H^\infty$) function of r. Also $|B(z)| \leq 1$. In fact, note that

$$\left| \frac{z_j - z}{1 - z\overline{z_j}} \right| < 1,$$

since

$$\begin{aligned}\left|\frac{z_j - z}{1 - z\overline{z_j}}\right| < 1 &\Leftrightarrow |z_j - z|^2 < |1 - z\overline{z_j}|^2 \\ &\Leftrightarrow (z_j - z)(\overline{z_j} - \overline{z}) < (1 - z\overline{z_j})(1 - \overline{z}z_j) \\ &\Leftrightarrow |z_j|^2 + |z|^2 < 1 + |z|^2|z_j|^2 \\ &\Leftrightarrow |z_j|^2(1 - |z|^2) < 1 - |z|^2,\end{aligned}$$

and the last inequality is true for all $|z_j| < 1$ and $|z| < 1$. Then for all $n \in \mathbb{N}$:

$$\begin{aligned}|B_n(z)| &= |z|^k \prod_{j=1}^{n} \left|\frac{z_j - z}{1 - z\overline{z_j}} \frac{|z_j|}{z_j}\right| \\ &\leq \prod_{j=1}^{n} \left|\frac{z_j - z}{1 - z\overline{z_j}}\right| \\ &< \prod_{j=1}^{n} 1 = 1,\end{aligned}$$

which implies that $|B(z)| = \lim_{n\to\infty} |B_n(z)| \leq 1$. We conclude that

$$\ln\left(\frac{1}{|B(re^{it})|}\right) \geq 0$$

and Fatou's Lemma [20] can be used to obtain:

$$\int_{-\pi}^{\pi} \ln\left(\frac{1}{|B(e^{it})|}\right) dt \leq \lim_{r\to 1} \int_{-\pi}^{\pi} \ln\left(\frac{1}{|B(re^{it})|}\right) dt,$$

or equivalently,

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \ln(|B(re^{it})|)dt \leq \int_{-\pi}^{\pi} \ln(|B(e^{it})|)dt \leq 0.$$

If we prove that this limit is also ≥ 0, we shall be done. Note that

$$|B_n(e^{it})| = |e^{it}|^k \prod_{j=1}^{n} \left|\frac{z_j - e^{it}}{1 - e^{it}\overline{z_j}} \frac{|z_j|}{z_j}\right| = \prod_{j=1}^{n} \frac{1}{|e^{it}|} \frac{|z_j - e^{it}|}{|e^{-it} - \overline{z_j}|} = 1 \text{ a.e.}$$

and also $|B_n(re^{it})| \to 1$ uniformly as $r \to 1$, since B_n is holomorphic in a neighbourhood of $\overline{\mathbb{D}}$. Then:

$$\lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln(|B(re^{it})|)dt = \lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln\left(\left|\frac{B(re^{it})}{B_n(re^{it})}\right|\right) dt.$$

But

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln\left(\left|\frac{B(re^{it})}{B_n(re^{it})}\right|\right) dt \geq \ln\left(\left|\frac{B(0)}{B_n(0)}\right|\right) = \sum_{n+1}^{\infty} \ln(|z_j|).$$

Thus, for every n:

$$\sum_{n+1}^{\infty} \ln(|z_j|) \leq \lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln(|B(re^{it})|)dt.$$

Now we realize that

$$\sum_{1}^{\infty} \ln\left(\frac{1}{|z_j|}\right) \leq \sum_{1}^{\infty} C(1-|z_j|) < \infty$$

for a C depending on how small is the smallest z_j. It follows that

$$\sum_{n+1}^{\infty} \ln(|z_j|) \to 0 \text{ as } n \to \infty$$

and, consequently,

$$0 \leq \lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln(|B(re^{it})|)dt.$$

□

2.3 Factorization of holomorphic functions

Theorem 2.56. *Let $F \in H^p$, $1 < p \leq \infty$. Then:*

(*a*) *For almost t, the limit*

$$F(e^{it}) = \lim F(z) \text{ when } z \xrightarrow{N.T.} e^{it}$$

exists. The function $f(t) = F(e^{it})$ is in L^p and $F = P(f)$.

(*b*)

$$\int_0^{2\pi} |F(re^{it}) - f(t)|^p dt \xrightarrow{r\to 1} 0, \text{ if } p < \infty.$$

If $p = \infty$, $F(re^{it}) \to f(t)$, in the weak-topology of L^∞, when r tends to 1. For each $1 < p \leq \infty$, we have $\|F\|_{H^p} = \|f\|_{L^p}$.*

(*c*) *F is the Cauchy's integral of its boundary function, that is,*

$$F(z) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{F(\xi)}{\xi - z} d\xi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F(e^{it})e^{it}}{e^{it} - z} dt.$$

Proof. (*a*) Theorem 2.5 implies that $F = P(f)$ for some $f \in L^p$. Applying Corollary 2.17, we have $f(t) = F(e^{it})$ for almost t.

(*b*) The convergence follows immediately from Corollary 2.11 for $p < \infty$ and from Corollary 2.15 for $p = \infty$. Fatou's Lemma [20] implies that $\|f\|_{L^p} \leq \|F\|_{H^p}$. The opposite inequality follows from (2.9).

(*c*) For $r < 1$, we can use Cauchy's formula [4]:

$$F(rz) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{F(r\xi)}{\xi - z} d\xi.$$

So we just take $r \to 1$ and use (*b*).

□

Theorem 2.57. *Let $F \in N$ not identically 0 and let B be the Blaschke product formed by the zeros of F. Then $F(z) = B(z)G(z)$, with $G \in N$ and $G(z) \neq 0$ in $\mathbb{D}$. Moreover, $\|G\|_N = \|F\|_N$ and if $F \in H^p$, we also have $G \in H^p$ and $\|G\|_{H^p} = \|F\|_{H^p}$.*

Proof. Note that G is well-defined, by Theorems 2.53 and 2.54. From the inequality

$$ln^+(a \cdot b) \leq ln^+ a + \ln^+ b,$$

we have

$$\int_{\pi}^{\pi} \ln^+ |G(re^{it})| dt \leq \int_{-\pi}^{\pi} \ln^+ |F(re^{it})| dt + \int_{-\pi}^{\pi} \ln^+ \frac{1}{|B(re^{it})|} dt.$$

Remember that $|B(re^{it})| \leq 1$, which implies that

$$\ln^+ \frac{1}{|B(re^{it})|} = \ln \frac{1}{|B(re^{it})|}$$

and then the last integral on RHS tends to zero, by Theorem 2.55. So $\|G\|_N \leq \|F\|_N$. But also $|F(z)| \leq |G(z)|$, since $|B(z)| \leq 1$, which implies that $\|F\|_N \leq \|G\|_N$. Hence $\|G\|_N = \|F\|_N$.

Suppose now that $F \in H^p$. Let $B_n(z)$ the partial finite product of $B(z)$ and

$$G_n(z) = \frac{F(z)}{B_n(z)}.$$

For each n, $|B_n(re^{it})| \to 1$, uniformly in t, when $r \to 1$, since B_n is holomorphic in a neighbourhood of $\overline{\mathbb{D}}$. If $p < \infty$,

$$\begin{aligned}
\|G_n\|_{H^p}^p &= \lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{F(re^{it})}{B_n(re^{it})} \right|^p dt \\
&= \lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\left| \frac{F(re^{it})}{B_n(re^{it})} \right|^p - |F(re^{it})|^p \right) dt + \|F\|_{H^p}^p \\
&= \|F\|_{H^p}^p.
\end{aligned}$$

If $p = \infty$,

$$\begin{aligned}
\|G_n\|_{H^\infty} &= \lim_{r\to 1} \sup_t \left(\left| \frac{F(re^{it})}{B_n(re^{it})} \right| \right) \\
&= \lim_{r\to 1} \sup_t \left(\left| \frac{F(re^{it})}{B_n(re^{it})} \right| - F(re^{it}) \right) + \|F\|_{H^\infty} \\
&= \|F\|_{H^\infty}.
\end{aligned}$$

But $|F(z)| \leq |G_n(z)|$ and then $\|G_n\|_{H^p} = \|F\|_{H^p}$ for all p and for all n. Fixed r and taking $n \to \infty$, we have $|G_n(re^{it})|^p \nearrow |G(re^{it})|^p$ for all p. If $p < \infty$, we apply the monotone convergence theorem [21, Corollary 1.9] to get:

$$\begin{aligned} m_p^p(G,r) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(re^{it})|^p dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{n\to\infty} |G_n(re^{it})|^p dt \\ &= \lim_{n\to\infty} m_p^p(G_n,r) \\ &\le \lim_{n\to\infty} \left(\lim_{r\to 1} m_p^p(G_n,r)\right) \\ &= \lim_{n\to\infty} \|G_n\|_{H^p}^p \\ &= \|F\|_{H^p}^p. \end{aligned}$$

Then $\|G\|_{H^p} \le \|F\|_{H^p}$ for all p. The other inequality follows from $|F(z)| \le |G(z)|$ and hence

$$\|G\|_{H^p} = \|F\|_{H^p}.$$

For $p = \infty$, we just note that

$$\begin{aligned} m_\infty(G,r) &= \sup_t |G(re^{it})| \\ &= \sup_t \left(\lim_{n\to\infty} |G_n(re^{it})|\right) \\ &= \sup_t \left(\sup_n |G_n(re^{it})|\right) \\ &= \sup_n \left(\sup_t |G_n(re^{it})|\right) \\ &\le \sup_n \left(\lim_{r\to 1} \sup_t |G_n(re^{it})|\right) \\ &= \lim_{n\to\infty} \left(\lim_{r\to 1} m_\infty(G_n,r)\right) \\ &= \lim_{n\to\infty} \|G_n\|_{H^\infty} \\ &= \|F\|_{H^\infty}. \end{aligned}$$

□

We will study now the boundary properties of a function $F \in H^p$, $0 < p \le 1$, with $F(z) \ne 0$ in $|z| < 1$. Suppose $F \in H^p$, $0 < p \le 1$, and also that F has no zeros in $\mathbb{D}$. Since $\mathbb{D}$ is simply connected, $F(z) = [G(z)]^n$ for some $G \in \mathcal{O}(\mathbb{D})$, with $1 < np$. Since $|G(z)|^{np} = |F(z)|^p$, we have that $G \in H^{np}$, with $\|G(z)\|_{H^{np}}^n = \|F(z)\|_{H^p}$. We know from Theorem 2.56 that

$$G(e^{it}) = \lim G(z) \text{ when } z \xrightarrow{N.T.} e^{it}$$

exists almost everywhere and the function $G(e^{it}) \in L^{np}$. So

$$F(e^{it}) = \lim F(z) \text{ when } z \xrightarrow{N.T.} e^{it}$$

also exists almost everywhere and the function $F(e^{it}) \in L^p$.

Conclusion: Every $F \in H^p$, $0 < p \leq 1$ that does not vanish has non-tangential boundary function $F(e^{it})$ belonging to L^p.

We also have the following result for H^p, $0 < p \leq 1$:

Statement: Every $F \in H^p$, $0 < p \leq 1$, with no zeros in $\mathbb{D}$, satisfies

$$\int_{-\pi}^{\pi} |F(re^{it}) - F(e^{it})|^p dt \xrightarrow{r \to 1} 0.$$

The proof of this fact is by induction, that is, we show that if it is is true for $2p$, it is also true for p. Indeed, let $F \in H^p$ with no zeros in $\mathbb{D}$ and write $F(z) = [G(z)]^2$ (here we are using the fact that $\mathbb{D}$ is simply connected), with $\|G(z)\|_{H^{2p}}^2 = \|F(z)\|_{H^p}$. Then

$$\begin{aligned}
\int_{-\pi}^{\pi} |F(re^{it}) - F(e^{it})|^p dt &= \int_{-\pi}^{\pi} |G^2(re^{it}) - G^2(e^{it})|^p dt \\
&= \int_{-\pi}^{\pi} |G(re^{it}) + G(e^{it})|^p |G(re^{it}) - G(e^{it})|^p dt \\
&\leq \left(\frac{2\pi}{2\pi} \int_{-\pi}^{\pi} |G(re^{it}) + G(e^{it})|^{2p} dt \right)^{p/2p} \times \left(\frac{2\pi}{2\pi} \int_{-\pi}^{\pi} |G(re^{it}) - G(e^{it})|^{2p} dt \right)^{p/2p} \\
&\leq C_p \|G\|_{H^{2p}}^p \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(re^{it}) - G(e^{it})|^{2p} \right)^{1/2},
\end{aligned}$$

which tends to zero when r tends to 1. We use this to prove the following theorem:

Theorem 2.58. *Let $F \in H^p$, $0 < p < \infty$. Then:*

(a) *The non-tangential limit*

$$F(e^{it}) = \lim F(z) \text{ when } z \xrightarrow{N.T.} e^{it}$$

exists for almost every t, and the function

$$f : t \mapsto F(e^{it}) \in L^p([-\pi, \pi]).$$

(b)

$$\int_{-\pi}^{\pi} |F(re^{it}) - f(t)|^p dt \to 0 \text{ when } r \to 1.$$

(c) $\|F\|_{H^p} = \|f\|_{L^p}$.

Proof. (a) By Theorem 2.56, we just consider the case $0 < p \leq 1$. Suppose f is not identically 0. Let B be the Blaschke product formed by the zeros of F. Write $F(z) = B(z)G(z)$. By Theorem 2.57, $\|F\|_{H^p} = \|G\|_{H^p}$ and G has no zeros. Since B and G have non-tangential limit almost everywhere, the same is true for F, that is, for almost every t,

$$F(e^{it}) = B(e^{it})G(e^{it}) = \lim F(z) \text{ when } z \xrightarrow{N.T.} e^{it}.$$

Moreover, since $|B(e^{it})| = 1$ almost everywhere, we have that $|F(e^{it})| = |G(e^{it})|$ almost everywhere and, consequently, $F(e^{it}) \in L^p$, once $G(e^{it})$ is in L^p by the above conclusion. It shows (a).

(b) We observe that

$$\begin{aligned}\int_{-\pi}^{\pi} |F(re^{it}) - f(t)|^p dt &= \int_{-\pi}^{\pi} |B(re^{it})G(re^{it}) - B(e^{it})G(e^{it})|^p dt \\ &\leq \left(\int_{-\pi}^{\pi} |B(re^{it})G(re^{it}) - B(re^{it})G(e^{it})|^p dt + \int_{-\pi}^{\pi} |B(re^{it})G(e^{it}) - B(e^{it})G(e^{it})|^p dt\right) \\ &\leq \int_{-\pi}^{\pi} |G(re^{it}) - G(e^{it})|^p + \int_{-\pi}^{\pi} |B(re^{it}) - B(e^{it})|^p |G(e^{it})|^p dt,\end{aligned}$$

and the last expression tends to zero as r tends to 1. Indeed, by the above statement, the first integral tends to zero, since $G \in H^p$ and has no zeros, and the second goes to zero by the dominated convergence theorem (see [20]).

(c) If $p \leq 1$, we note that

$$|F(re^{it})|^p = |F(re^{it}) - f(t) + f(t)|^p \leq (|F(re^{it}) - f(t)| + |f(t)|)^p \leq |F(re^{it}) - f(t)|^p + |f(t)|^p,$$

which implies that

$$|F(re^{it})|^p - |f(t)|^p \leq |F(re^{it}) - f(t)|^p.$$

But we also have that

$$|f(t)|^p - |F(re^{it})|^p \leq |F(re^{it}) - f(t)|^p$$

just interchanging $F(re^{it})$ and $f(t)$. Therefore

$$||F(re^{it})|^p - |f(t)|^p| \leq |F(re^{it}) - f(t)|^p$$

and the result follows from (b).

□

Corollary 2.59. *Let $F \in H^p$, $0 < p < \infty$, and suppose that its boundary function $F(e^{it}) \in L^q$. Then $F \in H^q$.*

Proof. For $q \leq p$, we already know that $F \in H^q$, since $H^p \subset H^q$. Suppose $q > p$. If $p > 1$, the result follows from Theorem 2.56, since F is the Poisson's integral of its boundary function which are in L^q. For the case $p \leq 1$, we use that $F(z) = B(z)G(z)$, where B is the Blaschke product formed by the zeros of F and G is a function in H^p which has no zeros, with $\|G\|_{H^p} = \|F\|_{H^p}$. Let n be an integer such that $1 < np$ and write $G(z) = [H(z)]^n$. So $H \in H^{np}$ and $\|H\|_{H^{np}}^n = \|G\|_{H^p} = \|F\|_{H^p}$. The boundary function satisfies

$$|H(e^{it})|^n = |G(e^{it})| = |F(e^{it})|.$$

Using that $F(e^{it}) \in L^q$, we have that $H(e^{it}) \in L^{nq}$, from which we conclude that $H \in H^{nq}$ (by the first part of the proof) and then $F \in H^q$. □

Corollary 2.60. *Every $F \in H^1$ is the Poisson's integral and the Cauchy's integral of its boundary function.*

Proof. Let $F \in H^1$ and take $0 < s < 1$. We have for $z = re^{it}$ in $\mathbb{D}$,

$$F(sre^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} F(se^{it})dt.$$

Taking $s \to 1$, we see the LHS of the above equation tends to $F(re^{it})$, while, by the Theorem 2.58, we have

$$\left|\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}[F(se^{it})-F(e^{it})]dt\right| \le C_r \int_{-\pi}^{\pi} |F(se^{it})-F(e^{it})|dt \xrightarrow{s\to 1} 0.$$

Thus

$$F(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} F(e^{it})dt.$$

The representation of Cauchy is obtained in a similar way, considering

$$F(sz) = \frac{1}{2\pi i}\int_{|\xi|=1} \frac{F(s\xi)}{\xi - z} d\xi.$$

□

Remark 2.61. A good question is if the previous result remains true for $p < 1$. The answer is no and we support this by an example: let

$$F(re^{i\theta}) = P_r(\theta) + iQ_r(\theta) = \frac{1+z}{1-z},$$

then $F \in H^p(\mathbb{D})$ for all $p < 1$ and

$$\lim_{z \xrightarrow{\text{N.T.}} e^{i\theta}} F(z) = iQ_1(\theta) \quad \text{a.e.},$$

but $F(re^{i\theta}) \neq \frac{i}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)Q_1(t)dt = 0$. In fact, note that

$$P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos(\theta)} \quad \text{and} \quad Q_r(\theta) = \frac{2r\sin(\theta)}{1+r^2-2r\cos(\theta)}.$$

So

$$\int_{-\pi}^{\pi} |F(re^{it})|^p dt = \int_{-\pi}^{\pi} |P_r(t)+iQ_r(t)|^p dt \le \int_{-\pi}^{\pi} |P_r(t)|^p dt + \int_{-\pi}^{\pi} |Q_r(t)|^p dt. \tag{2.22}$$

For the first integral, we just apply Hölder's inequality for the exponent conjugates $\left(\frac{1}{p}, \frac{1}{1-p}\right)$:

$$\begin{aligned}
\int_{-\pi}^{\pi} |P_r(t)|^p dt &= \int_{-\pi}^{\pi} (|P_r(t)|^p \cdot 1) dt \\
&\le \left(\int_{-\pi}^{\pi} (|P_r(t)|^p)^{1/p} dt\right)^p \left(\int_{-\pi}^{\pi} 1 dt\right)^{1-p} \\
&= \left(\int_{-\pi}^{\pi} |P_r(t)| dt\right)^p (2\pi)^{1-p} \\
&= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(t)| dt\right)^p (2\pi),
\end{aligned}$$

which implies that

$$\lim_{r\to 1^-} |P_r(t)|^p dt \leq \lim_{r\to 1^-} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(t)|dt\right)^p (2\pi) < \infty,$$

since we already know, by (2.8), that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(t)|dt = 1.$$

It remains to investigate the convergence of the second integral. First, remember that

$$\cos(2x) = 1 - 2\sin^2(x),$$

which implies that $1 + r^2 - 2r\cos(t) = (1-r)^2 + 4r\sin^2(t/2)$. Then

$$\begin{aligned}
\int_{-\pi}^{\pi} |Q_r(t)|^p dt &= \int_{-\pi}^{\pi} \left|\frac{2r\sin(t)}{1+r^2-2r\cos(t)}\right|^p dt \\
&= \int_{-\pi}^{\pi} \left|\frac{4r\sin(t/2)\cos(t/2)}{(1-r)^2+4r\sin^2(t/2)}\right|^p dt \\
&\leq \int_{-\pi}^{\pi} \left|\frac{4r\sin(t/2)\cos(t/2)}{4r\sin^2(t/2)}\right|^p dt \\
&= \int_{-\pi}^{\pi} \left|\frac{\cos(t/2)}{\sin(t/2)}\right|^p dt \\
&= 2\int_{0}^{\pi} \left|\frac{\cos(t/2)}{\sin(t/2)}\right|^p dt \\
&= 4\int_{0}^{\pi/2} \left|\frac{\cos(u)}{\sin(u)}\right|^p du \\
&= 4\int_{0}^{\pi/2} \left(\frac{\cos(u)}{\sin(u)}\right)^p du.
\end{aligned}$$

Now, using that $\sin(x) \geq (2/\pi)x$ for all $0 \leq x \leq \pi/2$, we obtain:

$$\begin{aligned}
\int_{-\pi}^{\pi} |Q_r(t)|^p dt &\leq 4\int_{0}^{\pi/2} \left(\frac{\cos(u)}{\sin(u)}\right)^p du \\
&\leq 4\int_{0}^{\pi/2} \left(\frac{1}{(2/\pi)u}\right)^p du \\
&= 2^{2-p}\pi^p \int_{0}^{\pi/2} u^{-p} du \\
&= C_p \int_{0}^{\pi/2} u^{-p} du,
\end{aligned}$$

which converges for all $p < 1$. So

$$\lim_{r\to 1^-} \int_{-\pi}^{\pi} |Q_r(t)|^p dt < \infty \text{ for all } p < 1.$$

Finally, we conclude by (2.22) that $F \in H^p(\mathbb{D})$ for all $p < 1$.

Corollary 2.62. *If $F \in H^1$, so for every $z \in \mathbb{D}$,*

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\Re(F(e^{it}))dt + i\Im(F(0)).$$

Proof. If $z = re^{i\theta}$, then

$$\Re\left(\frac{e^{it}+z}{e^{it}-z}\right) = \frac{1-|z|^2}{|e^{it}-z|^2} = P_r(\theta - t).$$

Moreover,

$$G(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\Re(F(e^{it}))dt$$

is a holomorphic function[6], whose real part coincides with the real part of F. So $F(z) - G(z)$ is constant. Since

$$G(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re(F(e^{it}))dt = \Re(F(0)),$$

we have $F(z) = G(z) + i\Im(F(0))$. □

2.3.1 An application

Theorem 2.63. *Let $F(z) = \sum_{n=0}^{\infty} a_n z^n \in H^1$, and let $\{c_n\}$ be the Fourier coefficients of its boundary function $F(e^{it})$. Then $c_n = a_n$ for all $n \geq 0$ and $c_n = 0$ for all $n < 0$.*

[6] To prove that

$$G(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\Re(F(e^{it}))\,dt$$

is holomorphic in the unit disk $\mathbb{D}$, we use Morera's theorem (see [4]). Let γ be any closed triangular path in $\mathbb{D}$. Then,

$$\int_\gamma G(z)\,dz = \frac{1}{2\pi}\int_\gamma \left(\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\Re(F(e^{it}))\,dt\right)dz.$$

By the integrability of $\Re(F(e^{it}))$ (since $F \in H^1$, its boundary values satisfy $\Re(F(e^{it})) \in L^1([-\pi,\pi])$), and the continuity and boundedness of the kernel

$$K(z,t) := \frac{e^{it}+z}{e^{it}-z}$$

for z in the compact path γ (as the denominator never vanishes for $|z| < 1$), the function $(z,t) \mapsto K(z,t)\Re(F(e^{it}))$ is integrable on $\gamma \times [-\pi,\pi]$. Therefore, by Fubini's theorem, the order of integration can be interchanged:

$$\int_\gamma G(z)\,dz = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_\gamma K(z,t)\,dz\right)\Re(F(e^{it}))\,dt.$$

Since for each fixed t, $z \mapsto K(z,t)$ is holomorphic in $\mathbb{D}$, Cauchy's theorem implies

$$\int_\gamma K(z,t)\,dz = 0.$$

Hence,

$$\int_\gamma G(z)\,dz = 0,$$

for every closed triangular path γ in $\mathbb{D}$. By Morera's theorem, this shows that G is holomorphic in $\mathbb{D}$.

Proof. Let $F_r(e^{it}) := F(re^{it})$. For $n \geq 0$, Cauchy's integral formula yields

$$a_n = \frac{1}{2\pi i}\int_{|\xi|=r}\frac{F(\xi)}{\xi^{n+1}}\,d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_r(e^{it})e^{-int}\,dt.$$

But

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})e^{-int}\,dt.$$

Therefore, for $n \geq 0$,

$$\begin{aligned} |r^n a_n - c_n| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}(F_r(e^{it}) - F(e^{it}))\,dt\right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |F_r(e^{it}) - F(e^{it})|\,dt \\ &\leq \|F_r - F\|_{L^1} \longrightarrow 0 \quad \text{ as } r \to 1, \end{aligned}$$

by Theorem 2.58. Therefore $r^n a_n \to c_n$, and since $r \to 1$, we conclude $a_n = c_n$ for all $n \geq 0$.

Now fix $n < 0$. Note that each function

$$F_r(e^{it}) = \sum_{k=0}^{\infty} a_k r^k e^{ikt}$$

has a Fourier series supported only on nonnegative integers, so its n-th Fourier coefficient is zero:

$$c_n(F_r) = 0 \quad \text{for all } r \in (0,1).$$

As $F_r \to F$ in $L^1([-\pi,\pi])$, and the Fourier coefficient map

$$f \mapsto c_n(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$$

is a continuous linear functional on L^1, we obtain:

$$c_n = c_n(F) = \lim_{r\to 1} c_n(F_r) = 0.$$

Therefore, $c_n = 0$ for all $n < 0$.

□

In fact, the previous result can be extended to a larger class of holomorphic functions. But first consider the following characterization theorem:

Theorem 2.64 (Continuity Criterion)**.** *Let u be a linear functional over $C^\infty(\mathbb{T})$ then u is continuous if and only if there are $C > 0$ and $N \in \mathbb{N}$ such that*

$$|\langle u, \phi\rangle| \leq C \max_{0\leq j\leq N} \|D^j\phi\|_{L^\infty} \text{ for all } \phi \in C^\infty(\mathbb{T}). \tag{2.23}$$

Remark 2.65. We may rewrite the above inequality as

$$|\langle u, \phi\rangle| \leq C \cdot p_N(\phi),$$

in which $p_N(\phi) = \max_{0\leq j\leq N} \|D^j\phi\|_{L^\infty}$. In fact, this is precisely the seminorm estimate characterizing continuity of the linear functional $u : C^\infty(\mathbb{T}) \to \mathbb{C}$ (see [19, Section 1.46] for a detailed discussion of the space $C^\infty(\mathbb{T})$).

Proof. Let u be a continuous linear functional and suppose that the inequality (2.23) does not hold for any choice of C and N. Taking $C = N = n$, $n \in \mathbb{N}$, we see that there is a function $\phi_n \in C^\infty(\mathbb{T})$ such that

$$\delta_n = |\langle u, \phi_n\rangle| > n \max_{0\leq j\leq n} \|D^j\phi_n\|_{L^\infty}. \tag{2.24}$$

In particular, $\delta_n > 0$ and ϕ_n is not identically zero. Define $\psi_n = \frac{\phi_n}{\delta_n}$ and observe that $\psi_n \to 0$ in $C^\infty(\mathbb{T})$.[7] In fact, let m be a nonnegative integer and $n \in \mathbb{N}$ such that $n > m$, then, it follows from (2.24) that

$$\|D^m\psi_n\|_{L^\infty} \leq \max_{0\leq j\leq n} \|D^j\psi_n\|_{L^\infty} < \frac{1}{n}.$$

But

$$|\langle u, \psi_n\rangle| = |\delta_n^{-1}\langle u, \phi_n\rangle| = 1 \text{ for all } n \in \mathbb{N},$$

which contradicts the continuity of u. For the converse, suppose that $\phi_l \to 0$ in $C^\infty(\mathbb{T})$, that is,

$$\max_{0\leq j\leq N} \|D^j\phi_l\|_{L^\infty} \to 0,$$

since the derivatives of all orders of ϕ_l tend uniformly to zero on $\mathbb{T}$. Then, using (2.23), it follows that

$$\langle u, \phi_l\rangle \to 0,$$

which shows the continuity of u.

□

Lemma 2.66. *Let* $P_r(t) = \dfrac{1-r}{1+r^2-2r\cos(t)}$ *be the Poisson kernel, with* $0 \leq r < 1$. *Then*

$$\|D_t^j P_r(\theta - t)\|_{L^\infty} \leq C_j(1-r)^{-j-1}, \qquad 0 \leq r < 1.$$

Proof. In fact, let g be defined on a neighborhood of x^0 and have derivatives up to order n at x^0; let f be defined on a neighborhood of $y^0 = g(x^0)$ and have derivatives up to order n at y^0. Then the j-th derivative of the composition $h(x) = f(g(x))$ at x^0 is given by the formula of Faà di Bruno [3]:

$$h_j = \sum_{k=1}^{j} f_k \sum_{p(j,k)} j! \prod_{i=1}^{j} \frac{g_i^{\lambda_i}}{(\lambda_i!)(i!)^{\lambda_i}},$$

in which

[7] Remember that a sequence $\{\phi_j\}$ of functions in $C^\infty(\mathbb{T})$ converges to zero in $C^\infty(\mathbb{T})$ if for all positive integer m, the derivatives of order m of functions ϕ_j converges uniformly to zero in $\mathbb{T}$ when $j \to \infty$.

- $h_j = \dfrac{d^j h}{dx^j}(x^0)$;
- $f_k = \dfrac{d^k f}{dy^k}(y^0)$;
- $g_i = \dfrac{d^i g}{dx^i}(x^0)$;
- $p(j,k) = \{(\lambda_1, \ldots, \lambda_j) : \lambda_i \in \mathbb{N}_0, \sum_{i=1}^{j} \lambda_i = k, \sum_{i=1}^{j} i\lambda_i = j\}$.

Taking $f(t) = \dfrac{1}{t}$ and $g(t) = \dfrac{1 + r^2 - 2r\cos(\theta - t)}{1 - r^2}$, one follows that

$$|f^{(j)}(t)| = \frac{j!}{t^{j+1}} \text{ and } |g^{(j)}(t)| \leq \frac{2r}{1 - r^2}.$$

Note that

$$g(t) = \frac{1 + r^2 - 2r\cos(\theta - t)}{1 - r^2} \geq \frac{(1 - r)^2}{1 - r^2} = \frac{1 - r}{1 + r}.$$

Then

$$\begin{aligned}
|D_t^j P_r(\theta - t)| &= |h_j| \\
&\leq \sum_{k=1}^{j} \frac{k!}{[g(t)]^{k+1}} \sum_{p(j,k)} j! \prod_{i=1}^{j} \frac{(2r)^{\lambda_i}}{(\lambda_i!)(i!(1 - r^2))^{\lambda_i}} \\
&\leq \sum_{k=1}^{j} \frac{k!(1 + r)^{k+1}}{(1 - r)^{k+1}} \sum_{p(j,k)} j! \prod_{i=1}^{j} \frac{2^{\lambda_i}}{(\lambda_i!)(i!(1 - r^2))^{\lambda_i}} \\
&\leq \frac{1}{(1 - r)^{j+1}} \sum_{k=1}^{j} k! 2^{k+1} \sum_{p(j,k)} j! \prod_{i=1}^{j} \frac{2^{\lambda_i}}{(\lambda_i!)(i!(1 - r^2))^{\lambda_i}} \\
&= C_j (1 - r)^{-j-1}.
\end{aligned}$$

Taking the sup on both sides we get the desired estimate. □

Theorem 2.67. *Let $F(z)$ be holomorphic in $\mathbb{D}$. The following conditions are equivalent:*

(i) *For every $\phi \in C^\infty(\mathbb{T})$ there exists the limit*

$$\langle f, \phi \rangle := \lim_{r \nearrow 1} \int_{-\pi}^{\pi} F(re^{i\theta}) \phi(\theta) d\theta. \tag{2.25}$$

(ii) *There is a distribution $f \in \mathcal{D}'(\mathbb{T})$ so that F is the Poisson integral of f*

$$F(re^{i\theta}) = \frac{1}{2\pi} \langle f(t), P_r(\theta - t) \rangle. \tag{2.26}$$

(iii) *There are constants $C > 0$, $\alpha \geq 0$, such that*

$$|F(re^{i\theta})| \leq \frac{C}{(1 - r)^\alpha}, \; 0 \leq r < 1. \tag{2.27}$$

Proof. Recall that

$$\mathcal{D}(\mathbb{T}) = \{\phi : \mathbb{T} \to \mathbb{C} : \phi \in C^\infty(\mathbb{T})\}.$$

We define a distribution to be a continuous linear functional $u : \mathcal{D}(\mathbb{T}) \to \mathbb{C}$. The space of all distributions on $\mathbb{T}$ is denoted by $\mathcal{D}'(\mathbb{T})$.
$(i) \Rightarrow (ii)$. Consider a sequence $r_k \nearrow 1$ and write

$$f_k(\theta) = F(r_k e^{i\theta}).$$

Then $f_k \in \mathcal{D}'(\mathbb{T})$ and $\langle f_k, \phi\rangle \to \langle f, \phi\rangle$ for every $\phi \in C^\infty(\mathbb{T})$, by item (i). By the Continuity Criterion (Theorem 2.64), for each f_k, there are $C_k > 0$ and $N_k \in \mathbb{N}$ such that

$$|\langle f_k, \phi\rangle| \le C_k \max_{0\le j\le N_k} \|D^j\phi\|_{L^\infty} \text{ for all } \phi \in C^\infty(\mathbb{T}),$$

which means that $\{f_k : k \in \mathbb{N}\}$ is pointwise bounded on $C^\infty(\mathbb{T})$. By the uniform boundedness principle [19] in $C^\infty(\mathbb{T})$, we obtain that the same set is also strongly bounded, which implies that the constants C_k and the order N_k are uniformly bounded (with respect to k), say by C and N, respectively. Hence we obtain

$$|\langle f_k, \phi\rangle| \le C \max_{0\le j\le N} \|D^j\phi\|_{L^\infty}$$

and the same estimate holds for f in the place of f_k, that is,

$$|\langle f, \phi\rangle| \le C \max_{0\le j\le N} \|D^j\phi\|_{L^\infty}, \tag{2.28}$$

proving that f is a distribution of order N. The Poisson formula applied to $z \mapsto F(r_k z)$ which is in $H^\infty(\mathbb{D})$ can be written as

$$F(rr_k e^{i\theta}) = \frac{1}{2\pi}\langle F(r_k e^{it}), P_r(\theta - t)\rangle.$$

Therefore, taking $k \to \infty$ and using (2.25), we obtain (2.26).

$(ii) \Rightarrow (iii)$. Suppose that (2.26) holds with $f \in \mathcal{D}'(\mathbb{T})$. Of course f satisfies (2.28) for convenient C and N, by the Continuity Criterion. Applying this estimate to $\phi(t) = P_r(\theta - t)$ and using the Lemma 2.66, we easily obtain (2.27).

$(iii) \Rightarrow (i)$. Suppose first that $0 \le \alpha < 1$ in (2.27). Adding a constant to F the same estimate (with another C) still holds and there is no loss of generality in assuming that $F(0) = 0$. In fact, if we take $G(z) = F(z) - F(0)$ and the estimate holds for G, then

$$\begin{aligned}
|F(z)| &\le |G(z)| + |F(0)| \\
&\le \frac{C}{(1-r)^\alpha} + |F(0)| \\
&= \frac{1}{(1-r)^\alpha}(C + (1-r)^\alpha |F(0)|) \\
&\le \frac{1}{(1-r)^\alpha}(C + F(0)) = \frac{C'}{(1-r)^\alpha},
\end{aligned}$$

in which $C' = C + F(0)$. Write

$$F(re^{i\theta}) = \int_0^r \frac{\partial}{\partial s} F(se^{i\theta}) ds$$

using the Cauchy-Riemman equations[8] for F and integration by parts we have, for each $\phi \in C^\infty(\mathbb{T})$,

$$\begin{aligned}
|\langle F(re^{it}) - F(r'e^{it}), \phi(t)\rangle| &= \left| \int_{-\pi}^{\pi} \left(F(re^{i\theta}) - F(r'e^{i\theta})\right) \phi(\theta)\, d\theta \right| \\
&= \left| \int_{-\pi}^{\pi} \left(\int_{r'}^{r} \frac{\partial}{\partial s} F(se^{i\theta})\, ds \right) \phi(\theta)\, d\theta \right| \\
&= \left| \int_{r'}^{r} \int_{-\pi}^{\pi} \frac{\partial}{\partial s} F(se^{i\theta}) \phi(\theta)\, d\theta\, ds \right| \\
&= \left| \int_{r'}^{r} \int_{-\pi}^{\pi} \left(-\frac{i}{s} \frac{\partial}{\partial \theta} F(se^{i\theta}) \right) \phi(\theta)\, d\theta\, ds \right| \\
&= \left| \int_{r'}^{r} \frac{1}{s} \left(\int_{-\pi}^{\pi} \frac{\partial}{\partial \theta} F(se^{i\theta}) \phi(\theta)\, d\theta \right) ds \right| \\
&= \left| \int_{r'}^{r} \frac{1}{s} \left(F(se^{i\theta})\phi(\theta) \Big|_{\theta=-\pi}^{\theta=\pi} - \int_{-\pi}^{\pi} F(se^{i\theta}) \frac{\partial \phi}{\partial \theta}(\theta)\, d\theta \right) ds \right| \\
&= \left| \int_{r'}^{r} \frac{1}{s} \left(0 - \int_{-\pi}^{\pi} F(se^{i\theta}) \frac{\partial \phi}{\partial \theta}(\theta)\, d\theta \right) ds \right| \\
&= \left| \int_{r'}^{r} \frac{1}{s} \int_{-\pi}^{\pi} F(se^{i\theta}) \frac{\partial \phi}{\partial \theta}(\theta)\, d\theta\, ds \right|.
\end{aligned}$$

[8]Here we are using the Cauchy-Riemman equations in polar coordinates:

$$\begin{cases} \dfrac{\partial u}{\partial s} = \dfrac{1}{s}\dfrac{\partial v}{\partial \theta}, \\ \dfrac{\partial v}{\partial s} = -\dfrac{1}{s}\dfrac{\partial u}{\partial \theta}. \end{cases}$$

Calculating the derivative of F with respect to s:

$$\frac{\partial F}{\partial s} = \frac{\partial u}{\partial s} + i\frac{\partial v}{\partial s} = \frac{1}{s}\frac{\partial v}{\partial \theta} - i\frac{1}{s}\frac{\partial u}{\partial \theta}.$$

On the other hand,

$$\frac{\partial F}{\partial \theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta},$$

and multiplying by $\dfrac{i}{s}$, we get

$$\frac{i}{s}\frac{\partial F}{\partial \theta} = \frac{i}{s}\frac{\partial u}{\partial \theta} - \frac{1}{s}\frac{\partial v}{\partial \theta}.$$

Therefore,

$$\frac{\partial F}{\partial s} = -\frac{i}{s}\frac{\partial F}{\partial \theta}.$$

Assuming that $r, r' > \frac{1}{2}$, then $s > \frac{1}{2}$ and $\frac{1}{s} < 2$. So using (2.27):

$$\begin{aligned}
|\langle F(re^{it}) - F(r'e^{it}), \phi(t)\rangle| &= \left| \int_{-\pi}^{\pi} \int_{r'}^{r} F(se^{i\theta}) \frac{\partial \phi}{s\partial\theta}(\theta) ds d\theta \right| \\
&< 2 \left| \int_{-\pi}^{\pi} \int_{r'}^{r} F(se^{i\theta}) \frac{\partial \phi}{\partial\theta}(\theta) ds d\theta \right| \\
&\leq C(\phi) \int_{r'}^{r} \frac{1}{(1-s)^{\alpha}} ds \qquad (2.29) \\
&= C(\phi) \left(\frac{-(1-s)^{1-\alpha}}{1-\alpha} \Big|_{r}^{r'} \right) \\
&= C(\phi) \left(\frac{(1-r)^{1-\alpha}}{1-\alpha} - \frac{(1-r')^{1-\alpha}}{1-\alpha} \right).
\end{aligned}$$

The right hand side converges to 0 when $r, r' \to 1$ since $\alpha < 1$, showing that the limit

$$\lim_{r \nearrow 1} \int_{-\pi}^{\pi} F(re^{i\theta}) \phi(\theta) d\theta$$

exists in this case. In fact, let $r_n \nearrow 1$ and take $r = r_n, r' = r_m$, we just verified that the sequence of complex numbers is Cauchy, and therefore convergent. The estimate (2.29) also shows that the limit is independent of the sequence $r_n \nearrow 1$.

If $1 \leq \alpha < 2$, let us assume that $F(0) = F'(0) = 0$. In fact, if we take $G(z) = F(z) - F(0) - zF'(0)$ and the estimate holds for G, then

$$\begin{aligned}
|F(z)| &\leq |G(z) + F(0)| + |z||F'(0)| \\
&\leq |G(z)| + |F(0)| + |F'(0)| \\
&\leq \frac{C}{(1-r)^{\alpha}} + |F(0)| + |F'(0)| \\
&= \frac{1}{(1-r)^{\alpha}} (C + (1-r)^{\alpha}|F(0)| + (1-r)^{\alpha}|F'(0)|) \\
&\leq \frac{1}{(1-r)^{\alpha}} (C + |F(0)| + |F'(0)|) \\
&\leq \frac{C'}{(1-r)^{\alpha}},
\end{aligned}$$

in which $C' = C + |F(0)| + |F'(0)|$. Writing

$$F(re^{i\theta}) = \int_0^r ds \int_0^s \frac{\partial^2 F}{\partial \sigma^2}(\sigma e^{i\theta}) d\sigma,$$

we can proceed as before:

$$F(re^{i\theta}) - F(r'e^{i\theta}) = \int_{r'}^{r} \int_0^s \frac{\partial^2 F}{\partial \sigma^2}(\sigma e^{i\theta}) d\sigma ds.$$

So for all $\phi \in C^{\infty}(\mathbb{T})$,

$$\begin{aligned}
\left|\langle F(re^{it}) - F(r'e^{it}), \phi(t)\rangle\right| &= \left| \int_{-\pi}^{\pi} \left(\int_{r'}^{r} \int_0^s \frac{\partial^2 F}{\partial \sigma^2}(\sigma e^{i\theta})\, d\sigma\, ds \right) \phi(\theta)\, d\theta \right| \\
&= \left| \int_{r'}^{r} \int_0^s \int_{-\pi}^{\pi} \frac{\partial^2 F}{\partial \sigma^2}(\sigma e^{i\theta}) \phi(\theta)\, d\theta\, d\sigma\, ds \right|.
\end{aligned}$$

But

$$\frac{\partial^2 F}{\partial \sigma^2}(\sigma e^{i\theta}) = \frac{1}{\sigma^2}\left(\frac{\partial^2 F}{\partial \theta^2} + i\frac{\partial F}{\partial \theta}\right)(\sigma e^{i\theta}),$$

so we obtain

$$\int_{-\pi}^{\pi} \frac{\partial^2 F}{\partial \sigma^2}(\sigma e^{i\theta})\phi(\theta)\, d\theta = \frac{1}{\sigma^2}\int_{-\pi}^{\pi} \left(\frac{\partial^2 F}{\partial \theta^2} + i\frac{\partial F}{\partial \theta}\right)(\sigma e^{i\theta})\phi(\theta)\, d\theta.$$

We now integrate each term by parts on the circle. First, for the second-order term:

$$\int_{-\pi}^{\pi} \frac{\partial^2 F}{\partial \theta^2}(\sigma e^{i\theta})\phi(\theta)\, d\theta = \int_{-\pi}^{\pi} F(\sigma e^{i\theta})\frac{\partial^2 \phi}{\partial \theta^2}(\theta)\, d\theta,$$

where we used the smoothness and 2π-periodicity of F and ϕ. Likewise, for the first-order term,

$$\int_{-\pi}^{\pi} \frac{\partial F}{\partial \theta}(\sigma e^{i\theta})\phi(\theta)\, d\theta = -\int_{-\pi}^{\pi} F(\sigma e^{i\theta})\frac{\partial \phi}{\partial \theta}(\theta)\, d\theta.$$

Combining both,

$$\int_{-\pi}^{\pi} \frac{\partial^2 F}{\partial \sigma^2}(\sigma e^{i\theta})\phi(\theta)\, d\theta = \frac{1}{\sigma^2}\int_{-\pi}^{\pi} F(\sigma e^{i\theta})\left(\frac{\partial^2 \phi}{\partial \theta^2} - i\frac{\partial \phi}{\partial \theta}\right)(\theta)\, d\theta.$$

Substituting this into the previous expression, we find

$$\begin{aligned} |\langle F(re^{it}) - F(r'e^{it}), \phi(t)\rangle| &= \left|\int_{r'}^{r}\int_0^s \frac{1}{\sigma^2}\int_{-\pi}^{\pi} F(\sigma e^{i\theta})\left(\frac{\partial^2 \phi}{\partial \theta^2} - i\frac{\partial \phi}{\partial \theta}\right)(\theta)\, d\theta\, d\sigma\, ds\right| \\ &\leq \int_{r'}^{r}\int_0^s \frac{1}{\sigma^2}\int_{-\pi}^{\pi} |F(\sigma e^{i\theta})| \cdot \left|\frac{\partial^2 \phi}{\partial \theta^2}(\theta) - i\frac{\partial \phi}{\partial \theta}(\theta)\right| d\theta\, d\sigma\, ds. \end{aligned}$$

Since $\phi \in C^\infty(\mathbb{T})$, the derivatives of ϕ are bounded, and we obtain

$$|\langle F(re^{it}) - F(r'e^{it}), \phi(t)\rangle| \leq C(\phi)\int_{r'}^{r}\int_0^s \frac{|F(\sigma re^{i\theta})|}{\sigma^2} d\sigma\, ds.$$

To estimate this integral, we split it at a fixed $\delta \in (0,1)$ (say $\delta = \frac{1}{2}$):

$$\int_0^s \frac{|F(\sigma re^{i\theta})|}{\sigma^2}\, d\sigma = \int_0^\delta \frac{|F(\sigma re^{i\theta})|}{\sigma^2}\, d\sigma + \int_\delta^s \frac{|F(\sigma re^{i\theta})|}{\sigma^2}\, d\sigma.$$

Since F vanishes to second order at $z = 0$, there exists $M > 0$ such that $\dfrac{|F(\sigma e^{i\theta})|}{\sigma^2} \leq M$ for all $\sigma \in (0, \delta)$. Therefore, the first integral is

$$\int_0^\delta \frac{|F(\sigma re^{i\theta})|}{\sigma^2}\, d\sigma \leq \int_0^\delta M d\sigma = M\delta < \infty,$$

and so

$$\int_{r'}^{r}\int_0^\delta \frac{|F(\sigma re^{i\theta})|}{\sigma^2}\, d\sigma \leq \int_{r'}^{r'} M\delta = (r - r')M\delta \to 0 \quad \text{as } r, r' \to 1.$$

For the second, we use the information on item (iii) and the fact that $\frac{1}{\sigma^2} < \frac{1}{\delta^2}$:

$$\begin{aligned}\int_\delta^s \frac{|F(\sigma e^{i\theta})|}{\sigma^2} d\sigma &\le \frac{C}{\delta^2}\int_\delta^s \frac{1}{(1-\sigma)^\alpha}\, d\sigma \\ &\overset{1-\sigma=u}{=} \frac{C}{\delta^2}\int_{1-s}^{1-\delta} u^{-\alpha} du \\ &= \begin{cases} \frac{C}{\delta^2}\frac{(1-\delta)^{1-\alpha}-(1-s)^{1-\alpha}}{1-\alpha}, & \text{if } 1<\alpha<2 \\ \frac{C}{\delta^2}\ln\left(\frac{1-\delta}{1-s}\right), & \text{if } \alpha = 1\end{cases} .\end{aligned}$$

Therefore, for $1<\alpha<2$,

$$\begin{aligned}\int_{r'}^r \int_\delta^s \frac{|F(\sigma e^{i\theta})|}{\sigma^2} d\sigma ds &\le \frac{C}{\delta^2(1-\alpha)}\int_r^{r'} (1-\delta)^{1-\alpha} - (1-s)^{1-\alpha} \\ &= \frac{C(1-\delta)^{1-\alpha}}{\delta^2(1-\alpha)}(r-r') - \frac{C}{\delta^2(1-\alpha)}\int_{r'}^r (1-s)^{1-\alpha} ds \\ &= \frac{C(1-\delta)^{1-\alpha}}{\delta^2(1-\alpha)}(r-r') - \frac{C}{\delta^2(1-\alpha)}\frac{(1-r)^{2-\alpha}-(1-r')^{2-\alpha}}{2-\alpha},\end{aligned}$$

which converges to zero as $r, r' \to 1$. Similarly, if $\alpha = 1$,

$$\begin{aligned}\int_{r'}^r \int_\delta^s \frac{|F(\sigma e^{i\theta})|}{\sigma^2} d\sigma ds &\le \frac{C}{\delta^2}\int_{r'}^r \ln\left(\frac{1-\delta}{1-s}\right) ds \\ &= \frac{C}{\delta^2}\ln(1-\delta)(r-r') - \frac{C}{\delta^2}\int_r^{r'} \ln(1-s) ds.\end{aligned}$$

and

$$\int_r^{r'} \ln(1-s)ds = -(1-r)\ln(1-r) + (1-r) + (1-r')\ln(1-r') - (1-r'),$$

which also tends to zero as $r, r' \to 1$. Therefore, the full double integral satisfies

$$\int_{r'}^r \int_0^s \frac{|F(\sigma e^{i\theta})|}{\sigma^2}\, d\sigma\, ds \longrightarrow 0 \quad \text{as } r, r' \nearrow 1.$$

Taking a convenient number of derivatives, we can adapt the argument for any $\alpha > 0$. □

Remark 2.68. If a function $F \in \mathcal{O}(\mathbb{D})$ satisfies the equivalent properties in Theorem 2.67 we say that F is of tempered growth at the boundary.

Corollary 2.69. *The functions in $H^p(\mathbb{D})$, $0 < p \le \infty$ are of tempered growth at the boundary. In particular, they admit the Poisson representation in the sense of distributions.*

Proof. Let $F \in H^p$, we will prove that F satisfies (2.27). It is enough to prove the case $p < \infty$. Fix $z \in \mathbb{D}$ and consider the mean value inequality for the subharmonic function $|F|^p$ (see Corollary 2.48):

$$|F(z)|^p \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(z+se^{it})|^p dt,\ 0 < s < \rho := \frac{1-|z|}{2}.$$

Integrating this inequality with respect to sds we obtain

$$\int_0^\rho s|F(z)|^p ds \leq \frac{1}{2\pi}\int_0^\rho \int_{-\pi}^{\pi} |F(z+se^{it})|^p sdtds \Leftrightarrow |F(z)|^p \leq \frac{1}{\rho^2\pi}\int\int_{D_\rho(z)} |F(\zeta)|^p dA(\zeta),$$

in which $D_\rho(z)$ is the disc centered in z with radius ρ and dA is the area element. We can use polar coordinates $re^{i\theta}$ centered at the origin and include $D_\rho(z)$ in the annulus $|z|-\rho < r < |z|+\rho$. Then

$$\begin{aligned} |F(z)|^p &\leq \frac{1}{\rho^2\pi}\int\int_{D_\rho(z)} |F(\zeta)|^p dA(\zeta) \\ &\leq \frac{1}{\rho^2\pi}\int_{|z|-\rho}^{|z|+\rho}\int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta dr \\ &\leq \frac{2\pi\|F\|_{H^p}^p}{\rho^2\pi}(2\rho) \\ &= \frac{4\|F\|_{H^p}^p}{\rho} \\ &= \frac{8\|F\|_{H^p}^p}{(1-|z|)} \\ &= \frac{C}{(1-|z|)}, \end{aligned}$$

in which $C = 8\|F\|_{H^p}^p$. Therefore we proved (2.27) with $\alpha = 1/p$. □

Remark 2.70. The space of holomorphic functions with tempered growth is strictly bigger than $\bigcup_{p>0} H^p(\mathbb{D})$, actually for any $\alpha > 0$ there is a holomorphic function on $\mathbb{D}$ that satisfies (2.27) while its radial limit $\lim\limits_{r\to 1} F(re^{i\theta})$ does not exist for almost every θ, in particular, $F \notin H^p(\mathbb{D})$ for any $p > 0$. The construction of such a function can be found in [6, Chapter 5].

Definition 2.71. The space of functions which are boundary values of functions in $H^p(\mathbb{D})$ in the sense of Fatou's Theorem (pointwise radial limits) will be denoted by $\mathcal{H}^p(\mathbb{T})$. The space of distributions which are boundary values of function in $H^p(\mathbb{D})$ in the weak sense will be denoted by $\mathcal{H}^p_b(\mathbb{T})$.

CHAPTER 3

Preliminaries

In this chapter, we will present some basic results that we will use throughout the work. The main references for this chapter are [16] and [17].

3.1 Representation of holomorphic functions in several complex variables

Definition 3.1. Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $f : \Omega \to \mathbb{C}$ is *holomorphic* if for each $j = 1, \ldots, n$ and each fixed $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$ the function

$$\zeta \mapsto f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)$$

is holomorphic, in the classical one-variable sense (see [4]), on the set

$$\Omega(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \equiv \{\zeta \in \mathbb{C} : (z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n) \in \Omega\}.$$

In other words, we require that f be holomorphic in each variable separately. Define $\mathcal{O}(\Omega)$ to be the set of all holomorphic functions on Ω. Functions holomorphic on $\mathbb{C}^n$ are called *entire holomorphic functions*.

Definition 3.2. A domain $\mathcal{R} \subset \mathbb{C}^n$ is called *Reinhardt* if $(z_1, \ldots, z_n) \in \mathcal{R}$ implies $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in \mathcal{R}$ for all $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$. Additionally, denote by $|\mathcal{R}|$ the subset of $(\mathbb{R}_{>0} \cup \{0\})^n$ defined

$$|\mathcal{R}| = \{(|z_1|, \ldots, |z_n|) : z = (z_1, \ldots, z_n) \in \mathcal{R}\},$$

and call this set the *Reinhardt shadow* of $\mathcal{R}$.

Remark 3.3. The unit disk, polydisks, and balls in $\mathbb{C}^n$ are classical examples of Reinhardt domains. Later on, we will introduce a nontrivial example of such a domain, which will play an important role in addressing (Q1-3).

Proposition 3.4. *Let $\mathcal{R} \subset \mathbb{C}^n$ be a Reinhardt domain and let $f \in \mathcal{O}(\mathcal{R})$. Define*

$$a_\alpha(f, r) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}(r)} \frac{f(\zeta)}{\zeta^{\alpha+1}} d\zeta, \qquad \alpha \in \mathbb{Z}^n, \ r \in \mathcal{R} \cap \mathbb{R}^n_{>0}.$$

Then:

(1) *For any $\alpha \in \mathbb{Z}^n$, the number $a_\alpha(f, r)$ is independent of $r \in \mathcal{R} \cap \mathbb{R}^n_{>0}$. In particular, we define $a_\alpha(f) := a_\alpha(f, r)$.*

(2) *Consequently, $\mathcal{R} \subset \mathcal{D}_f$, where $\mathcal{D}_f$ denotes the domain of convergence of the Laurent series $\sum_{\alpha \in \mathbb{Z}^n} a_\alpha(f) z^\alpha$.*

(3)
$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha(f) z^\alpha, \qquad z \in \mathcal{R}. \tag{3.1}$$

(4) *If $\mathcal{R} \cap V_j \neq \varnothing$, $j = 1, \ldots, n$ (in particular, if $0 \in \mathcal{R}$), then $a_\alpha(f) = 0$ for all $\alpha \in \mathbb{Z}^n \backslash \mathbb{Z}^n_+$.*

Proof. The proof can be found in [16, p. 51]. □

Remark 3.5. The Laurent series (3.1) converges uniformly on compact subsets of $\mathcal{R}$. For more details, see [16, p. 43].

3.2 Introduction to the Bergman kernel

Definition 3.6. Let $\Omega \subset \mathbb{C}^n$ be a domain and $p > 0$. The *Bergman space* $A^p(\Omega)$ is defined by

$$A^p(\Omega) := \mathcal{O}(\Omega) \cap L^p(\Omega).$$

Whenever it is clear from the context, we may simply write A^p.

Remark 3.7. For $p \geq 1$, $A^p(\Omega)$ is a normed space, with the norm inherited from $L^p(\Omega)$. If $0 < p < 1$, the space $A^p(\Omega)$ is no longer a normed space in the traditional sense. In this case, the triangle inequality is replaced by the inequality $\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$. In particular, $A^p(\Omega)$ is always a linear metric space.

To begin the discussion, we first consider the special case $p = 2$.

Lemma 3.8. *Let $K \subset \Omega$ be compact. There is a constant $C_K > 0$, depending on K and on n, such that*

$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_{A^2(\Omega)} \quad \forall f \in A^2(\Omega). \tag{3.2}$$

Proof. Since K is compact, there is an $r(K) = r > 0$ so that for any $z \in K$, $B(z,r) \subset \Omega$.[1] Therefore, for each $z \in K$ and $f \in A^2(\Omega)$, it follows from the mean value property for holomorphic functions and Hölder's inequality that

$$\begin{aligned}
|f(z)| &= \left| \frac{1}{\mathrm{vol}(B(z,r))} \int_{B(z,r)} f(w)dV(w) \right| \\
&\leq \frac{1}{\mathrm{vol}(B(z,r))} \int_{B(z,r)} |f(w)|dV(w) \\
&\leq \frac{1}{\mathrm{vol}(B(z,r))} \left(\int_{B(z,r)} 1dV(w) \right)^{1/2} \left(\int_{B(z,r)} |f(w)|^2 dV(w) \right)^{1/2} \\
&= \mathrm{vol}(B(z,r))^{-1/2} \|f\|_{L^2(B(z,r))} \\
&\leq C(n) r^{-n} \|f\|_{A^2(\Omega)} \coloneqq C_K \|f\|_{A^2(\Omega)}.
\end{aligned}$$

□

Lemma 3.9. *The space $A^2(\Omega)$ is a Hilbert space with the inner product*

$$\langle f, g \rangle \coloneqq \int_\Omega f(z)\overline{g(z)}dV(z).$$

Proof. We only need to establish completeness, as all other properties are automatically inherited from $L^2(\Omega)$. Let $\{f_j\} \subset A^2(\Omega)$ be a sequence that is Cauchy in the L^2-norm. Since $L^2(\Omega)$ is complete, there exists a limit function $f \in L^2(\Omega)$ such that $f_j \to f$ in the L^2-norm. We need to see that f is holomorphic. Fix a compact set $K \subset \Omega$. From Lemma 3.8, we see that

$$\sup_{z \in K} |f_j(z) - f_k(z)| \leq C_K \|f_j - f_k\|_{L_2}.$$

This implies that $\{f_j\}$ is a Cauchy sequence with respect to the sup norm on K. Since the space of continuous functions $C(K)$, endowed with the sup norm, is complete, there exists a continuous function g_K such that

$$\sup_{z \in K} |f_j(z) - g_K(z)| \to 0.$$

Moreover, if $K_1 \subset K_2$, the restrictions of g_{K_1} and g_{K_2} to K_1 coincide, since both are uniform limits of $\{f_j\}$ on K_1. This allows us to define a function $g : \Omega \to \mathbb{C}$ by

$$g(z) \coloneqq g_K(z) \quad \text{for any compact } K \ni z.$$

[1] For each $z \in K \subset \Omega$, since Ω is open, there exists $r_z > 0$ such that $B(z, r_z) \subset \Omega$. Then $\{B(z, r_z/2)\}_{z \in K}$ defines an open covering of K. By compactness of K, we can extract a finite subcovering $\{B(z_j, r_j)\}_{j=1}^N$, where $r_j = r_{z_j}/2$ for each j. Now set

$$r \;\coloneqq\; \min_{1 \leq j \leq N} r_j \;>\; 0.$$

If $z \in K$, then $z \in B(z_j, r_j)$ for some j, hence $B(z,r) \subset B(z_j, 2r_j) = B(z_j, r_{z_j}) \subset \Omega$. Thus, $B(z,r) \subset \Omega$ for all $z \in K$.

The definition is well-defined since the functions g_K agree on the intersection of any two compact sets. By a classical result (see [18, p. 10]), holomorphic functions are closed under locally uniform convergence, and thus g is holomorphic. It remains to show $f = g$ almost everywhere. Indeed, since $f_j \to f$ in L^2, there exists a subsequence $\{f_{j_k}\}$ that converges pointwise almost everywhere to f. On the other hand, since $f_j \to g$ uniformly on every compact subset of Ω, in particular over the singleton $\{z\}$ for any point $z \in \Omega$, it follows that

$$f_{j_k}(z) \to g(z) \qquad \text{for every } z \in \Omega.$$

Therefore, at every point where the pointwise convergence limit of the subsequence exists – which holds almost everywhere – we must have $f(z) = g(z)$. Hence, $f = g$ almost everywhere. Since g is holomorphic and f coincides with g almost everywhere in the L^2-sense, it follows that f is holomorphic as well, and therefore $f \in A^2(\Omega)$.

□

Remark 3.10. The proof of Lemma 3.9 shows that L^2-convergence of holomorphic functions implies uniform convergence on compact subsets (and hence pointwise convergence).

Remark 3.11. Lemma 3.9 says that $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, since it is a Hilbert space. For $p \neq 2$, $p \geq 1$, $A^p(\Omega)$ is a Banach space, since it is closed[2], but is no longer Hilbert.

Lemma 3.12. *For each fixed $z \in \Omega$, the functional*

$$\Phi_z : f \mapsto f(z), \qquad f \in A^2(\Omega)$$

is a continuous linear functional on $A^2(\Omega)$.

Proof. This is immediate from Lemma 3.8 if we take K to be the singleton $\{z\}$. □

We can now invoke the Riesz representation theorem to guarantee the existence of a unique function $b_z \in A^2(\Omega)$ such that

$$\Phi_z(f) = f(z) = \langle f, b_z \rangle.$$

Definition 3.13. The *Bergman kernel* is the function $\mathbb{B}_\Omega(z, w) = \overline{b_z(w)}$, $z, w \in \Omega$. It has the reproducing property

$$f(z) = \int_\Omega \mathbb{B}_\Omega(z, w) f(w) dV(w), \qquad \forall f \in A^2(\Omega).$$

Whenever it is clear from the context, we may simply write $\mathbb{B}(z, w)$.

Proposition 3.14. *The Bergman kernel $\mathbb{B}_\Omega(z, w)$ is conjugate symmetric:*

$$\mathbb{B}_\Omega(z, w) = \overline{\mathbb{B}_\Omega(w, z)}.$$

[2] This follows from the Cauchy estimates in several complex variables, which provide bounds on holomorphic functions in terms of their values on compact subsets of the domain. These estimates imply that $A^p(\Omega)$ is a closed subspace of the Banach space $L^p(\Omega)$. See [18, p. 11] for a more detailed exposition.

Proof. Taking $f(z) = b_w(z) = \overline{\mathbb{B}_\Omega(w,z)}$ for some $w \in \Omega$, we conclude that

$$\begin{aligned}
\overline{\mathbb{B}_\Omega(w,z)} &= b_w(z) \\
&= \int_\Omega \mathbb{B}_\Omega(z,\zeta) b_w(\zeta) dV(\zeta) \\
&= \int_\Omega \mathbb{B}_\Omega(z,\zeta)\overline{\mathbb{B}_\Omega(w,\zeta)} dV(\zeta) \\
&= \overline{\int_\Omega \mathbb{B}_\Omega(w,\zeta)\overline{\mathbb{B}_\Omega(z,\zeta)} dV(\zeta)} \\
&= \overline{\int_\Omega \mathbb{B}_\Omega(w,\zeta) b_z(\zeta) dV(\zeta)} \\
&= \overline{b_z(w)} \\
&= \mathbb{B}_\Omega(z,w),
\end{aligned}$$

as we wanted to prove. □

Proposition 3.15. *The Bergman kernel is uniquely determined by the properties that it is an element of $A^2(\Omega)$ in z, is conjugate symmetric, and reproduces $A^2(\Omega)$.*

Proof. Let $K(z,w)$ be another such kernel. Then

$$\begin{aligned}
\mathbb{B}_\Omega(z,w) &= \overline{\mathbb{B}_\Omega(w,z)} \\
&= \int_\Omega K(z,\zeta)\overline{\mathbb{B}_\Omega(w,\zeta)} dV(\zeta) \\
&= \overline{\int_\Omega \mathbb{B}_\Omega(w,\zeta)\overline{K(z,\zeta)} dV(\zeta)} \\
&= \overline{\overline{K(z,w)}} \\
&= K(z,w).
\end{aligned}$$

□

Theorem 3.16. *Suppose that $\{f_j\}$ is a sequence of holomorphic functions on Ω and that $\|f_j\|_K \coloneqq \sup_{z\in K} |f_j(z)|$ is bounded (independently of j) for each compact subset K of Ω. Then there is a subsequence $\{f_{j_k}\}$ that converges uniformly on each compact subset of Ω.*

Proof. The proof can be found in [5, p. 7] □

Lemma 3.17. *Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions converging pointwise to a function $f : \Omega \to \mathbb{C}$. Suppose that $\{f_n\}$ is uniformly bounded on every compact subset $K \subset \Omega$. Then $f_n \to f$ uniformly on compact subsets of Ω.*

Proof. Let $K \subset \Omega$ be a compact subset. Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$. Since $\{f_n\}$ is uniformly bounded on K, so must be $\{f_{n_k}\}$. By Theorem 3.16, every subsequence of $\{f_n\}$

admits a subsequence converging uniformly on each compact subset of Ω. Therefore, there exists a subsequence $\{f_{n_{k_l}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_l}} \to g$ uniformly on K for some g. However, since $f_n \to f$ pointwise, we must have $g = f$ by uniqueness. Thus $f_{n_{k_l}} \to f$ uniformly on K.

Now, to prove that $f_n \to f$ uniformly on K, we argue by contradiction and assume the opposite. Then there exists $\varepsilon_0 > 0$ such that for every $M \in \mathbb{N}$, there exists an index $n_M \geq M$ and a point $z_M \in K$ satisfying $|f_{n_M}(z_M) - f(z_M)| \geq \varepsilon_0$. By construction, $\{f_{n_M}\}$ defines a subsequence of $\{f_n\}$ which does not converge uniformly to f on K. This would mean that $\{f_{n_M}\}$ could not have a subsequence converging uniformly to f, which leads us to a contradiction. Hence $f_n \to f$ uniformly on K. And since $K \subset \Omega$ was taken arbitrarily, $f_n \to f$ uniformly on compact subsets of Ω. □

Since $L^2(\Omega)$ is a separable Hilbert space, its closed subspace $A^2(\Omega)$ is also separable, as any closed subspace of a separable Hilbert space is separable. Therefore, there exists a complete orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$ for $A^2(\Omega)$.

Proposition 3.18. *Let K be a compact subset of Ω. Then the series*

$$\sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)} \tag{3.3}$$

converges uniformly on $K \times K$ to the Bergman kernel $\mathbb{B}_\Omega(z, w)$.

Proof. Since $\mathbb{B}_\Omega(z, w) \in A^2(\Omega)$ as a function of z, it admits an expansion in terms of the orthonormal basis, that is,

$$\mathbb{B}_\Omega(\cdot, w) = \sum_{j=1}^{\infty} \langle \mathbb{B}_\Omega(\cdot, w), \phi_j \rangle \phi_j(\cdot).$$

By Lemma 3.15, we have

$$\langle \mathbb{B}_\Omega(\cdot, w), \phi_j \rangle = \overline{\langle \phi_j, \mathbb{B}_\Omega(\cdot, w) \rangle} = \overline{\phi_j(w)}.$$

Thus

$$\mathbb{B}_\Omega(\cdot, w) = \sum_{j=1}^{\infty} \phi_j(\cdot)\overline{\phi_j(w)}$$

for fixed $w \in \Omega$ and so $\mathbb{B}_\Omega(\cdot, w)$ converges in the norm. Recall that pointwise convergence is dominated by $L^2(\Omega)$ convergence in $A^2(\Omega)$ (see Remark 3.10). Therefore $\mathbb{B}_\Omega(\cdot, w)$ converges pointwise to $\sum_{j=1}^{\infty} \phi_j(\cdot)\overline{\phi_j(w)}$.

Now, let $K \subset \Omega$ be a compact subset. By the Riesz-Fischer and Riesz representation theorems, we obtain

$$\begin{aligned}\sup_{z\in K}\left(\sum_{j=1}^{\infty}|\phi_j(z)|^2\right)^{1/2} &= \sup_{z\in K}\|\{\phi_j(z)\}_{j=1}^{\infty}\|_{\ell^2}\\ &= \sup_{z\in K}\sup_{\|\{a_j\}\|_{\ell^2}=1}\left|\sum_{j=1}^{\infty}a_j\phi_j(z)\right|\\ &= \sup_{z\in K}\sup_{\|f\|_{A^2}=1}|f(z)|\\ &\le C_K.\end{aligned} \tag{3.4}$$

In the last inequality we have used Lemma 3.8. Therefore,

$$\left|\sum_{j=1}^{\infty}\phi_j(z)\overline{\phi_j(w)}\right| \le \sum_{j=1}^{\infty}|\phi_j(z)\overline{\phi_j(w)}| \le \left(\sum_{j=1}^{\infty}|\phi_j(z)|^2\right)^{1/2}\left(\sum_{j=1}^{\infty}|\phi_j(w)|^2\right)^{1/2}$$

and the convergence is uniform over $z,w\in K$. In fact, set $\overline{\Omega}:=\{\overline{z}:z\in\Omega\}$ and for each $m\in\mathbb{N}$ define $f_m:\Omega\times\overline{\Omega}\to\mathbb{C}$ by

$$f_m(z,w)=\sum_{j=1}^{m}\phi_j(z)\overline{\phi_j(\overline{w})}.$$

Then $\{f_m\}$ is a sequence of holomorphic functions that is uniformly bounded on compact subsets of $\Omega\times\overline{\Omega}$ and converges pointwise to $\mathbb{B}_\Omega(z,\overline{w})$ by the arguments presented above. Therefore, by Lemma 3.17, $\{f_m\}$ must converge uniformly on compact subsets of $\Omega\times\overline{\Omega}$. Finally, let K be a compact subset of Ω. For all $(z,w)\in K\times K$ and for each $m\in\mathbb{N}$,

$$\sum_{j=1}^{m}\phi_j(z)\overline{\phi_j(w)}=f_m(z,\overline{w}),$$

which converges uniformly to $\mathbb{B}_\Omega(z,\overline{\overline{w}})=\mathbb{B}_\Omega(z,w)$. Hence $\sum_{j=1}^{\infty}\phi_j(z)\overline{\phi_j(w)}$ converges uniformly to $\mathbb{B}_\Omega(z,w)$ on $K\times K$. □

Remark 3.19. It is important to highlight that the proof of Proposition 3.18 shows that

$$\sum_{n=1}^{\infty}\phi_j(z)\overline{\phi_j(w)}$$

equals the Bergman kernel $\mathbb{B}_\Omega(z,w)$ no matter what the choice of complete orthonormal basis $\{\phi_j\}$ for $A^2(\Omega)$.

Now, since $A^2(\Omega)$ is a closed subspace, we have $L^2(\Omega)=A^2(\Omega)\oplus A^2(\Omega)^{\perp}$ and there exists an orthogonal projection onto $A^2(\Omega)$ (see [21, pp. 177-178]). This is called the *Bergman projection*, which we will denote by $\mathcal{B}$. In the following, we will show that the integral operator defined by the Bergman kernel is equal to the Bergman projection.

Proposition 3.20. *Let $\Omega \subset \mathbb{C}^n$ be a domain, then the mapping*

$$P : f \mapsto \int_\Omega \mathbb{B}_\Omega(\cdot, w) f(w) dV(w)$$

is the Hilbert space orthogonal projection of $L^2(\Omega, dV)$ onto $A^2(\Omega)$.

Proof. Given $f \in L^2(\Omega)$, we apply the reproducing kernel to $\mathcal{B}f \in A^2(\Omega)$, giving

$$\mathcal{B}f(z) = \langle \mathcal{B}f, \mathbb{B}_\Omega(\cdot, z)\rangle.$$

Since $\mathcal{B}$ is self-adjoint and $\mathbb{B}_\Omega(\cdot, z) \in A^2(\Omega)$, we obtain

$$\mathcal{B}f(z) = \langle \mathcal{B}f, \mathbb{B}_\Omega(\cdot, z)\rangle = \langle f, \mathbb{B}_\Omega(\cdot, z)\rangle.$$

□

Remark 3.21. The Bergman projection is, therefore, given by

$$\mathcal{B}f(z) = \int_\Omega \mathbb{B}_\Omega(z, w) f(w) dV(w), \qquad f \in L^2(\Omega). \tag{3.5}$$

Proposition 3.22. *For $z \in \Omega \subset\subset \mathbb{C}^n$, it holds that $\mathbb{B}_\Omega(z, z) > 0$.*

Proof. We have

$$\mathbb{B}_\Omega(z, z) = \sum_{j=1}^{\infty} |\phi_j(z)|^2 \geq 0.$$

If, in fact, $\mathbb{B}_\Omega(z, z) = 0$ for some z, then $\phi_j(z) = 0$ for all j. This would imply that $f(z) = 0$ for every $f \in A^2(\Omega)$, which is absurd, since there exist functions in $A^2(\Omega)$ that do not vanish at z; for example $f \equiv 1$. □

3.2.1 Bergman Kernel on the unit disk

Let us now calculate the Bergman kernel of the unit disk. For this, let $\Omega = \mathbb{D}$, where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

The following theorem gives us equivalent ways to recognize an orthonormal basis in a Hilbert space.

Theorem 3.23. *Let $\mathcal{H}$ be a Hilbert space. The following properties of an orthonormal set $\{\phi_n\}$ are equivalent:*

(*i*) *Finite linear combinations of elements in $\{\phi_n\}$ are dense in $\mathcal{H}$.*

(*ii*) *If $f \in \mathcal{H}$ and $\langle f, \phi_n\rangle = 0$ for all $n \geq 0$, then $f = 0$.*

(*iii*) *If* $f \in \mathcal{H}$ *and* $S_N(f) = \sum_{n=0}^{N} a_n \phi_n$, *where* $a_n = \langle f, \phi_n \rangle$, *then* $S_N(f) \to f$ *as* $N \to \infty$ *in the norm.*

(*iv*) *If* $a_n = \langle f, \phi_n \rangle$, *then* $\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2$.

Proof. The proof can be found in [21, p. 165] □

In this section, let us write $A^2(\mathbb{D}) = A^2$.

Lemma 3.24. *Let* $\phi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n$ *for all* $n \geq 0$. *Then* $\{\phi_n\}_{n=0}^{\infty}$ *is an orthonormal basis for* A^2.

Proof. First of all, we observe that the monomials $1, z, z^2, \dots$ form an orthogonal set in A^2. In fact, an easy computation gives

$$\int_{\mathbb{D}} z^n \overline{z^m} dV(z) = \int_0^{2\pi} \int_0^1 r^n e^{in\theta} r^m e^{-im\theta} r dr d\theta = \int_0^{2\pi} e^{i(n-m)\theta} d\theta \int_0^1 r^{n+m+1} dr = \frac{2\pi}{n+m+2} \delta_{nm},$$

where δ_{nm} is the Kronecker delta. This shows that the functions

$$\phi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n, \quad n = 0, 1, 2, \dots,$$

form an orthonormal set in A^2. It remains to show that they form a basis. Let $f \in A^2$ and suppose $\langle f, \phi_n \rangle = 0$ for all $n \geq 0$. Since each $f \in A^2$ is holomorphic in $\mathbb{D}$, then $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for $|z| < 1$. Fix $n \in \mathbb{N}_0$. Then, by converting to polar coordinates and using Fubini's theorem, we obtain

$$\begin{aligned}
0 &= \langle f, \phi_n \rangle \\
&= \int_{\mathbb{D}} f(z) \overline{\phi_n(z)} dV(z) \\
&= \int_0^1 \int_0^{2\pi} f(re^{i\theta}) \overline{\phi_n(re^{i\theta})} r d\theta dr \\
&= \int_0^1 \int_0^{2\pi} \left(\sum_{k=0}^{\infty} a_k r^k e^{ik\theta} \right) \left(\sqrt{\frac{n+1}{\pi}} r^n e^{-in\theta} \right) r d\theta dr \\
&= \sqrt{\frac{n+1}{\pi}} \int_0^1 \sum_{k=0}^{\infty} a_k r^{n+k+1} \left(\int_0^{2\pi} e^{i(k-n)} d\theta \right) dr \qquad (3.6) \\
&= \sqrt{\frac{n+1}{\pi}} \int_0^1 \sum_{k=0}^{\infty} a_k r^{n+k+1} (2\pi \delta_{kn}) dr \\
&= \sqrt{\frac{n+1}{\pi}} \int_0^1 2\pi a_n r^{2n+1} dr \\
&= a_n \sqrt{\frac{n+1}{\pi}} \frac{\pi}{n+1}.
\end{aligned}$$

It implies $a_n = 0$. And since $n \in \mathbb{N}_0$ was arbitrary, it follows from (3.6) that $a_n = 0$ for all $n \geq 0$. That is, $f = 0$. Thus by Theorem 3.23 *(ii)*, $\{\phi_n\}_{n=0}^{\infty}$ forms an orthonormal basis for A^2. □

We recall that for $|z| < 1$, the function $f(z) = \dfrac{1}{1-z}$ is holomorphic and has power series representation

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}. \tag{3.7}$$

Lemma 3.25. *If* $|z| < 1$, *then* $\displaystyle\sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2}$.

Proof. First of all, note that

$$\sum_{n=0}^{\infty}(n+1)z^n = \sum_{n=1}^{\infty} nz^n + \sum_{n=0}^{\infty} z^n. \tag{3.8}$$

Writing $S(z) = \sum_{n=1}^{\infty} nz^n = z + 2z^2 + 3z^3 + \cdots$, it follows that

$$\begin{aligned} S(z) - zS(z) = S(z)(1-z) &= z + z^2 + z^3 + \cdots \\ &= -1 + (1 + z + z^2 + z^3 + \cdots) \\ &= -1 + \frac{1}{1-z} \\ &= \frac{z}{1-z}. \end{aligned}$$

Then $S(z) = \dfrac{z}{(1-z)^2}$. By (3.8) we conclude that

$$\sum_{n=0}^{\infty}(n+1)z^n = \frac{z}{(1-z)^2} + \frac{1}{1-z} = \frac{1}{(1-z)^2},$$

as we wanted to show. □

We are now ready to calculate the Bergman kernel on $\mathbb{D}$ by means of Theorem 3.18. Using the orthonormal basis found in Lemma 3.24 we see that

$$\begin{aligned} \mathbb{B}_{\mathbb{D}}(z,w) &= \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(w)} \\ &= \sum_{n=0}^{\infty} \left(\sqrt{\frac{n+1}{\pi}} z^n\right)\left(\sqrt{\frac{n+1}{\pi}}\overline{w}^n\right) \\ &= \frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)(z\overline{w})^n. \end{aligned} \tag{3.9}$$

Therefore, by Lemma 3.25, we have precisely that

$$\mathbb{B}_{\mathbb{D}}(z,w) = \frac{1}{\pi}\frac{1}{(1-z\overline{w})^2}. \tag{3.10}$$

3.2.2 Bergman Kernel on the polydisk

Theorem 3.26. *Suppose $D_j \subset\subset \mathbb{C}^n$, $j = 1, 2$, are bounded domains with Bergman kernels $\mathbb{B}_{D_1}$ and $\mathbb{B}_{D_2}$. Then the Bergman kernel $\mathbb{B}_{D_1\times D_2}$ for the product domain $D = D_1 \times D_2$ is given by*

$$\mathbb{B}_{D_1\times D_2}((z_1, z_2), (w_1, w_2)) = \mathbb{B}_{D_1}(z_1, w_1)\mathbb{B}_{D_2}(z_2, w_2)$$

for all $(z_1, z_2), (w_1, w_2) \in D_1 \times D_2$.

Proof. Define

$$G((z_1, z_2), (a_1, a_2)) := \mathbb{B}_{D_1}(z_1, a_1)\mathbb{B}_{D_2}(z_2, a_2).$$

We want to show that G is the Bergman kernel of $D = D_1 \times D_2$. First, fix $(a_1, a_2) \in D$. Since $\mathbb{B}_{D_1}(\cdot, a_1) \in A^2(D_1)$ and $\mathbb{B}_{D_2}(\cdot, a_2) \in A^2(D_2)$, it follows that $G(\cdot, (a_1, a_2)) \in A^2(D)$. Now, for any $f \in A^2(D)$, using Fubini's theorem and the reproducing properties of $\mathbb{B}_{D_1}$ and $\mathbb{B}_{D_2}$, we compute:

$$\begin{aligned}
\int_D f(z_1, z_2)\overline{G((z_1, z_2), (a_1, a_2))}dV(z_1, z_2) &= \int_{D_1}\int_{D_2} f(z_1, z_2)\overline{\mathbb{B}_{D_1}(z_1, a_1)\mathbb{B}_{D_2}(z_2, a_2)}dV(z_2)dV(z_1) \\
&= \int_{D_1} \mathbb{B}_{D_1}(a_1, z_1)\left(\int_{D_2} f(z_1, z_2)\mathbb{B}_{D_2}(a_2, z_2)dV(z_2)\right)dV(z_1) \\
&= \int_{D_1} \mathbb{B}_{D_1}(a_1, z_1)f(z_1, a_2)dV(z_1) \\
&= f(a_1, a_2).
\end{aligned}$$

Thus, by the uniqueness property of the Bergman kernel, we conclude that $G \equiv \mathbb{B}_D$. □

Corollary 3.27. *The Bergman kernel $\mathbb{B}_{\mathbb{D}^n}$ for the unit polydisk $\mathbb{D}^n$ in $\mathbb{C}^n$ is given by*

$$\mathbb{B}_{\mathbb{D}^n}(z, w) = \frac{1}{\pi^n}\prod_{j=1}^{n}\frac{1}{(1 - z_j\overline{w_j})^2}.$$

3.2.3 Bergman Kernel on the unit ball

Theorem 3.28. *The Bergman kernel for the unit ball*

$$B^n = \{z \in \mathbb{C}^n : |z| < 1\}$$

is given by

$$\mathbb{B}_{B^n}(z, w) = \frac{n!}{\pi^n(1 - \langle z, \overline{w}\rangle)^{n+1}},$$

where $\langle z, \overline{w}\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$.

Proof. See [17, pp. 60–63, Chapter 1] or [18, Chapter 4, Section 4.4]. □

CHAPTER 4

General Domains

Throughout this chapter, unless otherwise stated, we assume that $\Omega \subset \mathbb{C}^n$ is a domain.

Definition 4.1. The *Hartogs triangle* is

$$\mathbb{H} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$

Moreover, for $\gamma > 0$, the power-generalized Hartogs triangle of expoent γ is

$$\mathbb{H}_\gamma := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\}. \tag{4.1}$$

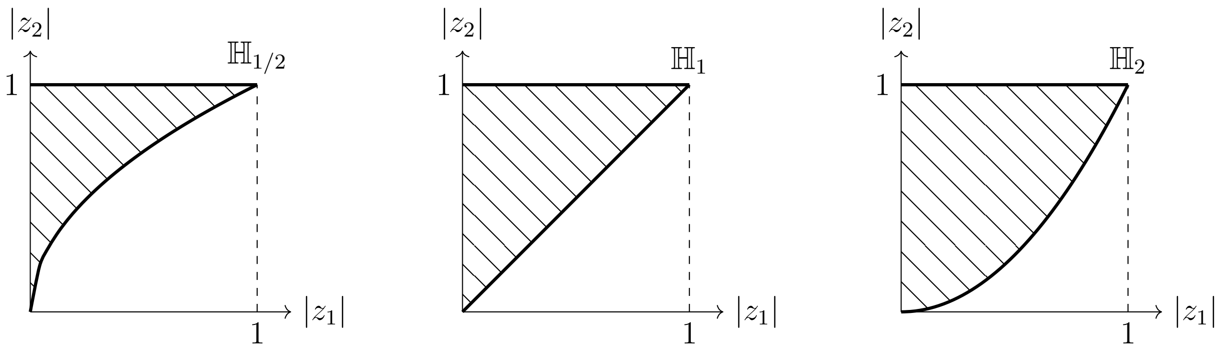

Figure 4.1: Hartogs triangle $\mathbb{H}_\gamma$ for $\gamma = \frac{1}{2}, 1$ and 2.

4.1 Two auxiliary operators

We recall now that the Bergman projection is given by

$$\mathcal{B}f(z) = \int_\Omega \mathbb{B}(z, w) f(w) dV(w), \qquad f \in L^2(\Omega). \tag{4.2}$$

Let $P : L^2(\Omega) \to A^2(\Omega)$ be a bounded operator given by an integral formula

$$Pf(z) = \int_\Omega \mathbb{P}(z, w) f(w) dV(w). \tag{4.3}$$

For $p > 0$ fixed, consider the conditions

(H1) $\exists C > 0$ such that $\|Pf\|_p \leq C\|f\|_p \quad \forall f \in L^p(\Omega)$. ($P$ is bounded on L^p)

(H2) $Ph = h \quad \forall h \in A^p(\Omega)$. ($P$ reproduces A^p)

Let us define two auxiliary operators related to P. The operator $|P|$ is defined by

$$|P|f(z) = \int_\Omega |\mathbb{P}(z,w)|f(w)dV(w), \tag{4.4}$$

where $|\mathbb{P}(z,w)|$ denotes absolute value. The triangle inequality shows that if $|P|$ satisfies (H1), then P does as well. In fact,

$$0 < |Pf(z)| = \left| \int_\Omega \mathbb{P}(z,w)f(w)dV(w) \right| \leq \int_\Omega |\mathbb{P}(z,w)||f(w)|dV(w) = |P|(|f|)(z).$$

Then

$$\begin{aligned}
\left(\int_\Omega |Pf(z)|^p dV(z) \right)^{1/p} &\leq \left(\int_\Omega [|P|(|f|)(z)]^p dV(z) \right)^{1/p} \\
&\leq C \left(\int_\Omega |(|f|)(z)|^p dV(z) \right)^{1/p} \\
&= C\|f\|_p,
\end{aligned}$$

since $f \in L^p(\Omega) \Rightarrow |f| \in L^p(\Omega)$ and $\|f\|_p = \|(|f|)(\cdot)\|_p$.

The operator $P^\dagger$ is defined by

$$P^\dagger f(w) = \int_\Omega \overline{\mathbb{P}(z,w)} f(z)dV(z). \tag{4.5}$$

Note that $\langle Pf, g \rangle = \langle f, P^\dagger g \rangle$ holds when Fubini's theorem can be applied, so $P^\dagger$ is the formal adjoint of P.

4.2 Extending the Bergman projection

If $\Omega \subset \mathbb{C}^n$ is bounded, $L^t(\Omega) \subset L^s(\Omega)$ for any $1 \leq s < t$. Thus for $p \geq 2$, $f \in L^p(\Omega)$ implies that $\mathcal{B}f \in A^2(\Omega)$ and is given by the integral (4.2). Moreover, $\int_\Omega \mathbb{B}(z,w)f(w)dV(w)$ is taken as the definition of $\mathcal{B}f$, whenever the integral converges (even if $f \notin L^2(\Omega)$). For $p < 2$ and $f \in L^p(\Omega)$, this integral does not necessarily converges. Even when it converges, directly determining the size of the integral is difficult – it is therefore desirable to evaluate $\mathcal{B}f$ as a limit.

4.2.1 Boundedness of the Bergman Kernel

Various hypotheses on Ω guarantee convergence of (4.2) for $f \in L^p(\Omega)$, $p < 2$. For example, let $\mathbb{D}$ be the unit disk in the complex plane and fix $z \in \mathbb{D}$. Then for $f \in L^1(\mathbb{D})$,

$$\begin{aligned}\left|\int_{\mathbb{D}} \mathbb{B}_{\mathbb{D}}(z,w)f(w)dV(w)\right| &= \left|\frac{1}{\pi}\int_{\mathbb{D}} \frac{1}{(1-z\overline{w})^2} f(w)dV(w)\right| \\ &\le C_z \int_{\mathbb{D}} |f(w)|dV(w) \\ &< \infty.\end{aligned}$$

Here $C_z = \sup_{w\in\mathbb{D}} |\mathbb{B}_{\mathbb{D}}(z,w)| < \infty$, since $z \in \mathbb{D}$ is fixed. In fact,

$$|1 - z\overline{w}| \ge |1 - |z\overline{w}|| = 1 - |z||\overline{w}| > 1 - |z| \Rightarrow |1 - z\overline{w}|^{-2} \le (1 - |z|)^{-2}.$$

But this argument fails for the domains $\mathbb{H}_\gamma$ (see Definition 4.1). Consider $\mathbb{H}_k$ for $k \in \mathbb{Z}^+$ with $k > 1$ to illustrate. Let $\mathbb{B}(z,w) = \mathbb{B}_{\mathbb{H}_k}(z_1, z_2, w_1, w_2)$ denote the Bergman kernel. Theorem 1.2 of [8] says

$$\mathbb{B}(z,w) = \frac{p_k(z_1\overline{w_1}) \cdot [(z_2\overline{w_2})^2 + (z_1\overline{w_1})^k] + z_2\overline{w_2} \cdot q_k(z_1\overline{w_1})}{k\pi^2(1 - z_2\overline{w_2})^2(z_2\overline{w_2} - (z_1\overline{w_1})^k)^2},$$

for explicit polynomials $p_k(s)$, $q_k(s)$ of the complex variable s. Two crucial facts are that $p_k(0) = k - 1$ and $q_k(0) = 1$. Let $z = (z_1, z_2) \in \mathbb{H}_k$ be a fixed point (note $z_2 \neq 0$) and $w_\delta = (0, \delta)$, $\delta > 0$, be a point in $\mathbb{H}_k$ on the z_2 axis. Then

$$\mathbb{B}(z, w_\delta) = \frac{(k-1)(z_2\delta)^2 + z_2\delta}{k\pi^2(1 - z_2\delta)^2(z_2\delta)^2} \approx \frac{1}{\delta}.$$

Letting $\delta \to 0$ shows that $\mathbb{B}(z, \cdot) \notin L^\infty(\mathbb{H}_k)$.

Other arguments are required to show $\mathcal{B}$ is defined on L^p for $p < 2$ on domains like $\mathbb{H}_{m/n}$. In [10], estimates on $|\mathbb{B}_{m/n}(z,w)|$ and a variant of Schur's test show $|\mathcal{B}|$ is defined (and bounded) on $L^p(\mathbb{H}_{m/n})$ for an interval of $p < 2$; see Theorem 6.1.

4.2.2 Limits of exhaustions

If $|\mathcal{B}|$ is bounded on $L^p(\Omega)$, the integral (4.2) is finite. Computing $\mathcal{B}f$ can be done as a principal value, a consequence of the folowing fact:

Proposition 4.2. *Let Ω be a domain in $\mathbb{C}^n$. Suppose P is an operator of the form* (4.3) *such that $|P|$ is bounded on $L^p(\Omega)$ for a given $1 < p < \infty$. For $t \in (0,1)$, let $\Omega_t \subset \Omega$ such that if $t < t'$, then $\Omega_{t'} \subset \Omega_t$, and $\bigcup_{t\in(0,1)} \Omega_t = \Omega$.*[1] *Then if $f \in L^p(\Omega)$, for almost every $z \in \Omega$ we have*

$$Pf(z) = \lim_{t\to 0} \int_{\Omega_t} \mathbb{P}(z,w)f(w)dV(w).$$

[1] If $\mathbb{C}^n \backslash \Omega \neq \varnothing$, take $\Omega_t = \{z \in \Omega : \text{dist}(z, \mathbb{C}^n\backslash\Omega) > t\}$. Otherwise, if $\Omega = \mathbb{C}^n$, take $\Omega_t = B(0, \log(1/t))$.

Proof. Let $f \in L^p(\Omega)$. Because $|P|$ is bounded, we obtain

$$\underbrace{\int_\Omega \left\{ \left| \int_\Omega |\mathbb{P}(z,w)||f(w)|dV(w) \right|^p \right\} dV(z)}_{=\||P|(|f|)\|_p^p} \leq C\|f\|_p^p.$$

In particular, for a.e. $z \in \Omega$, the quantity $\{\cdot\}$ above is $< \infty$. Thus $|\mathbb{P}(z,\cdot)||f(\cdot)| \in L^1(\Omega)$ for a.e. $z \in \Omega$.

Let χ_t be the indicator function on Ω_t. Note that

$$|\chi_t(w)\mathbb{P}(z,w)||f(w)| \leq |\mathbb{P}(z,w)||f(w)|$$

for any $z \in \Omega$. Fix z such that $|\mathbb{P}(z,\cdot)||f(\cdot)| \in L^1(\Omega)$. The dominated convergence theorem implies

$$\begin{aligned} \lim_{t\to 0} \int_{\Omega_t} \mathbb{P}(z,w)f(w)dV(w) &= \lim_{t\to 0} \int_\Omega \chi_t(w)\mathbb{P}(w,z)f(w)dV(w) \\ &= \int_\Omega \lim_{t\to 0} \chi_t(w)\mathbb{P}P(w,z)f(w)dV(w) \\ &= \int_\Omega \mathbb{P}(w,z)f(w)dV(w) \\ &= Pf(z), \end{aligned}$$

as claimed. ☐

4.3 Consequences of $(H1)$

Two functional analysis results are derived from assumptions about L^p boundedness of the Bergman projection. Conditions $(H1)$ and $(H2)$ enter the hypotheses and conclusions respectively.

4.3.1 $(H2)$ and density

Lemma 4.3. *Let Ω be a domain in $\mathbb{C}^n$. Assume $\mathcal{B}$ is bounded on $L^p(\Omega)$ for a given $1 < p < \infty$. The following statements are equivalent:*

(i) $A^2(\Omega) \cap A^p(\Omega)$ *is dense in* $A^p(\Omega)$.

(ii) $\mathcal{B}h = h \quad \forall h \in A^p(\Omega)$.

Proof. Assume (i). Then for each $h \in A^p(\Omega)$, there is a sequence $\{h_\nu\} \subset A^2(\Omega) \cap A^p(\Omega)$ such that $h_\nu \to h$ in $A^p(\Omega)$. Since $\mathcal{B}$ is assumed continuous on $L^p(\Omega)$, $\mathcal{B}h_\nu \to \mathcal{B}h$. However, $\mathcal{B}h_\nu = h_\nu$, since $h_\nu \in A^2(\Omega)$. Then $\mathcal{B}h = h$.

Assume (ii). Let $h \in A^p(\Omega)$. Since $L^2(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$ [21], there exist a sequence $\{g_\nu\} \subset L^2(\Omega)\cap L^p(\Omega)$ such that $g_\nu \to h$ in L^p. Set $h_\nu = \mathcal{B}g_\nu$. Then $h_\nu \in A^2(\Omega)\cap A^p(\Omega)$ and

$$h_\nu \to \mathcal{B}h,$$

since $\mathcal{B}$ is L^p bounded. As $\mathcal{B}h = h$ by assumption, (i) holds. □

4.3.2 Generalized self-adjointness

The Bergman projection $\mathcal{B}$ is self-adjoint on $A^2(\Omega)$: $\langle \mathcal{B}f, g\rangle = \langle f, \mathcal{B}g\rangle$ if $f, g \in L^2(\Omega)$. This does not automatically imply that $\langle \mathcal{B}f, g\rangle = \langle f, \mathcal{B}g\rangle$ if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ for general conjugate exponents p and q.

However this relation holds when $|\mathcal{B}|$ satisfies (H1), a consequence of the following general result.

Proposition 4.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Assume there exists an operator P of the form (4.3) and that $|P|$ is bounded on $L^p(\Omega)$ for a given $1 < p < \infty$. Let q be conjugate to p. Then*

(i) *$|P^\dagger|$ is bounded on $L^q(\Omega)$.*

(ii) $\langle Pf, g\rangle = \langle f, P^\dagger g\rangle \quad \forall f \in L^p(\Omega), \forall g \in L^q(\Omega).$

Proof. Let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Tonelli's theorem implies

$$\begin{aligned}\langle |P|(|f|), |g|\rangle &= \int_\Omega \left(\int_\Omega |\mathbb{P}(z,w)||f(w)|dV(w)\right)|g(z)|dV(z)\\ &= \int_\Omega |f(w)|\left(\int_\Omega |\mathbb{P}(z,w)||g(z)|dV(z)\right)dV(w)\\ &= \langle |f|, |P^\dagger|(|g|)\rangle.\end{aligned}$$

Hölder's inequality and boundedness of $|P|$ on L^p yield

$$\langle |f|, |P^\dagger|(|g|)\rangle = \langle |P|(|f|), |g|\rangle \le \||P|(|f|)\|_p\|g\|_q \le C\|f\|_p\|g\|_q.$$

Taking the supremum over $\|f\|_p = 1$ shows $\||P^\dagger|(|g|)\|_q \le C\|g\|_q$ as claimed.

Fubini's theorem[2] now applies to give (ii):

$$\begin{aligned}\langle Pf, g\rangle &= \int_\Omega \left(\int_\Omega \mathbb{P}(z,w)f(w)dV(w)\right)\overline{g(z)}dV(z)\\ &= \int_\Omega f(w)\left(\int_\Omega \mathbb{P}(z,w)\overline{g(z)}dV(z)\right)dV(w)\\ &= \int_\Omega f(w)\overline{\left(\int_\Omega \overline{\mathbb{P}(z,w)}g(z)dV(z)\right)}dV(w)\\ &= \langle f, P^\dagger g\rangle.\end{aligned}$$

□

Remark 4.5. The Bergman kernel is conjugate symmetric, that is, $\overline{\mathbb{B}(z,w)} = \mathbb{B}(w,z)$. Thus if $|\mathcal{B}|$ is L^p bounded, (ii) says $\langle \mathcal{B}f, g\rangle = \langle f, \mathcal{B}g\rangle$ for $f \in L^p(\Omega)$, $g \in L^q(\Omega)$.

4.4 Representing $A^p(\Omega)'$ by $A^q(\Omega)$.

The sought for representation is through L^2 pairing. For $1 < p < \infty$ define the conjugate-linear map

$$\Phi_p(g)(f) = \int_\Omega f\overline{g}dV, \qquad g \in A^q,\ f \in A^p. \tag{4.6}$$

Hölder's inequality implies Φ_p maps $A^q(\Omega)$ continuously into $A^p(\Omega)'$.

Remark 4.6. Φ_p is a bounded antilinear operator.

The goal is to understand when Φ_p is surjective. The preliminary results hold generally.

4.4.1 General behavior

Proposition 4.7. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and $1 < p < \infty$.*

(i) If $p \leq 2$, then Φ_p is injective.

(ii) If $p \geq 2$, then Φ_p has dense image in $A^p(\Omega)'$.

---

[2]

$$\begin{aligned}\int_\Omega\int_\Omega |\mathbb{P}(z,w)||f(w)||g(z)|dV(w)dV(z) &= \int_\Omega |g(z)|\left(\int_\Omega |\mathbb{P}(z,w)||f(w)|dV(w)\right)dV(z)\\ &= \int_\Omega |g(z)||P|(|f|)(z)dV(z)\\ &\leq \|g\|_q\||P|(|f|)\|_p\\ &\leq C\|g\|_q\|f\|_p < \infty.\end{aligned}$$

Proof. Let q be the conjugate exponent of p.

For part (i), suppose that $g \in \ker(\Phi_p)$. Note in particular that $g \in A^q(\Omega)$. Since $p \le 2$, it follows that $p \le 2 \le q$, which implies $A^q(\Omega) \subset A^2(\Omega) \subset A^p(\Omega)$. Therefore $g \in A^2(\Omega) \subset A^p(\Omega)$ and $\Phi_p(g)$ can act on g:

$$0 = \Phi_p(g)(g) = \|g\|^2_{L^2(\Omega)} \Rightarrow g \equiv 0.$$

Consider part (ii). Since $p \ge 2$, so necessarily $q \le 2$. By part (i), the map $\Phi_q : A^p(\Omega) \to A^q(\Omega)'$ is injective. Define $A^p(\Omega)^\# := \{\overline{f} : f \in A^p(\Omega)'\}$, the space of all bounded antilinear functionals on $A^p(\Omega)$. Define the transpose $\Phi_q' : (A^q(\Omega)')' \to A^p(\Omega)^\#$ of Φ_q by

$$\Phi_q'(\lambda)(f) = \lambda(\Phi_q f), \quad \lambda \in (A^q(\Omega)')', \ f \in A^p(\Omega).$$

Since Φ_q is injective, the transposed map Φ_q' has dense image.[3]

Now recall $L^q(\Omega)$ is reflexive. Since $A^q(\Omega) \subset L^q(\Omega)$ is closed, then $A^q(\Omega)$ is also reflexive. Thus the evaluation map $\varepsilon : A^q(\Omega) \to (A^q(\Omega)')'$ defined by

$$\varepsilon(g)(\phi) = \phi(g), \quad \phi \in A^q(\Omega)', \ g \in A^q(\Omega),$$

is an isometric isomorphism. Let $\mathcal{C} : A^p(\Omega)^\# \to A^p(\Omega)'$ be the conjugation map defined by $(\mathcal{C} \circ \lambda)(g) = \overline{\lambda(g)}$. $\mathcal{C}$ is an antilinear isometric isomorphism between $A^p(\Omega)^\#$ and $A^p(\Omega)'$. To complete the proof of part (ii) it suffices to show

$$\Phi_p = \mathcal{C} \circ \Phi_q' \circ \varepsilon, \tag{4.7}$$

since ε and $\mathcal{C}$ are isometric isomorphism and Φ_q' has dense image. In fact, assume (4.7). Let $f \in A^p(\Omega)'$. Of course $\overline{f} \in A^p(\Omega)^\#$. Since Φ_q' has dense image, there exists a sequence $\{g_n\} \subset (A^q(\Omega)')'$ such that $\Phi_q'(g_n) \to \overline{f}$. But ε is also an isometric isomorphism, which means that for each g_n there exists $h_n \in A^q(\Omega)$ such that $\varepsilon(h_n) = g_n$. Thus, we have that $\Phi_q'(\varepsilon(h_n)) \to \overline{f}$, which implies that

$$\Phi_p(h_n) = \mathcal{C}(\Phi_q'(\varepsilon(h_n)) \to \mathcal{C}(\overline{f}) = f.$$

[3] Note that $\ker(\Phi_q) = \operatorname{Ran}(\Phi_q')^\perp$, where

$$\operatorname{Ran}(\Phi_q')^\perp = \{f \in A^p(\Omega) : \psi(f) = 0 \text{ for all } \psi \in \operatorname{Ran}(\Phi_q')\} = \{f \in A^p(\Omega) : \lambda(\Phi_q f) = 0 \text{ for all } \lambda \in (A^q(\Omega)')'\}.$$

Since Φ_q is injective, we have $\{0\} = \ker(\Phi_q) = \operatorname{Ran}(\Phi_q')^\perp$, which implies that $\operatorname{Ran}(\Phi_q')$ is dense in $A^p(\Omega)^\#$. In fact, if it is not dense, then there exists some $\phi \in A^p(\Omega)^\# \backslash \overline{\operatorname{Ran}(\Phi_q')}$. By the Hahn-Banach theorem, we can find a bounded linear functional F on $A^p(\Omega)^\#$ such that $F(\phi) \neq 0$ and $F \equiv 0$ on $\overline{\operatorname{Ran}(\Phi_q')}$. Consequently, $F(\psi) = 0$ for all $\psi \in \operatorname{Ran}(\Phi_q')$, which implies by reflexivity ($A^p(\Omega) \cong (A^p(\Omega)^\#)'$) that there is $f \in A^p(\Omega)$ such that $\overline{\psi(f)} = 0$ for all $\psi \in \operatorname{Ran}(\Phi_q')$. In particular, $\psi(f) = 0$ for all $\psi \in \operatorname{Ran}(\Phi_q')$, or equivalently, $\lambda(\Phi_q f) = 0$ for all $\lambda \in (A^q(\Omega)')'$. This shows that $f \in \operatorname{Ran}(\Phi_q')^\perp$. However, we know that $F(\phi) = \overline{\phi(f)} \neq 0$, which leads to a contradiction, since it implies that $f \neq 0$ and $\operatorname{Ran}(\Phi_q')^\perp = \{0\}$.

Let us prove (4.7). For $f \in A^p(\Omega)$, $g \in A^q(\Omega)$,

$$\begin{aligned}(\mathcal{C} \circ \Phi_q' \circ \varepsilon)(g)(f) &= \overline{\Phi_q'(\varepsilon(g))(f)} \\ &= \overline{\varepsilon(g)(\Phi_q f)} \\ &= \overline{(\Phi_q f)(g)} \\ &= \overline{\int_\Omega g\overline{f} dV} \\ &= \int_\Omega f\overline{g} dV \\ &= \Phi_p(g)(f),\end{aligned}$$

which establishes (4.7). □

Remark 4.8. Proposition 4.7 shows Φ_p is generally almost surjective if $p \geq 2$. To show it is actually surjective would require establishing closed range. This is equivalent to an estimate of the form

$$\|\Phi_p g\|_{A^p(\Omega)'} \gtrsim \operatorname{dist}(g, \ker(\Phi_p)),$$

for all $g \in A^p(\Omega)$, where $\ker(\Phi_p)$ denotes the null space of Φ_p. In fact, this estimate implies that the induced map

$$\Psi : A^q(\Omega)/\ker(\Phi_p) \to \operatorname{Ran}(\Phi_p),$$

given by $\Psi([g]) = \Phi_p(g)$, is a topological isomorphism. Consequently, $\operatorname{Ran}(\Phi_p)$ is Banach, hence closed. On the other hand, if $\operatorname{Ran}(\Phi_p) \subset A^p(\Omega)'$ is closed, then it is a Banach space. The induced map

$$\Psi : A^q(\Omega)/\ker(\Phi_p) \to \operatorname{Ran}(\Phi_p)$$

is a topological isomorphism, in view of the open mapping theorem [19], which implies the estimate above.

Corollary 4.9. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Suppose the map $\Phi_p : A^q \to A^p(\Omega)'$ is surjective for a given $1 < p < \infty$. Let q be conjugate to p. Then there is a natural identification*

$$A^p(\Omega)' \cong \frac{A^q(\Omega)}{\ker(\Phi_p)}.$$

Furthermore, the map

$$\Phi_q : A^p(\Omega) \to A^q(\Omega)'$$

is injective and has closed range.

Proof. Let $\Psi : \dfrac{A^q(\Omega)}{\ker(\Phi_p)} \to A^p(\Omega)'$ be defined by

$$\Psi([g]) := \Phi_p(g).$$

It is well-defined. In fact, if $g_1, g_2 \in A^q(\Omega)$ and $g_1 - g_2 \in \ker(\Phi_p)$, then $g_1 = g_2 + m$ for some $m \in \ker(\Phi_p)$ and

$$\Psi([g_1]) = \Phi_p(g_1) = \Phi_p(g_2 + m) = \Phi_p(g_2) = \Psi([g_2]).$$

It is straightforward to verify that it is antilinear. Let $g \in \ker(\Psi)$. Then $0 = \Psi([g]) = \Phi_p(g)$, that is, $g \in \ker(\Phi_p)$. Thus, $[g] = 0$ and it shows that Ψ is injective. Surjectivity of Ψ follows easily from the assumption that Φ_p is surjective. We now show that Ψ is continuous. For all $f \in A^p(\Omega)$ and $\tilde{g} \in [g] = g + \ker(\Phi_p)$, we have that

$$|\Psi([g])(f)| = |\Phi_p(g)(f)| = |\Phi_p(\tilde{g})(f)| \leq \|f\|_{A^p(\Omega)} \|\tilde{g}\|_{A^q(\Omega)}.$$

Taking the supremum over all $f \in A^p(\Omega)$ with $\|f\|_{A^p(\Omega)} = 1$, we get

$$\|\Psi([g])\|_{A^p(\Omega)'} \leq \|\tilde{g}\|_{A^q(\Omega)}.$$

And since it holds for all $\tilde{g} \in [g]$, it follows that

$$\|\Psi([g])\|_{A^p(\Omega)'} \leq \inf_{\tilde{g} \in [g]} \|\tilde{g}\|_{A^q(\Omega)} = \|[g]\|_{A^q(\Omega)/\ker(\Phi_p)}.$$

To conclude, the continuity of the inverse Ψ^{-1} follows from the open mapping theorem [19], since $\dfrac{A^q(\Omega)}{\ker(\Phi_p)}$ and $A^p(\Omega)'$ are Banach spaces.

Now let us check that Φ_q is injective and has closed image. Let $f \in \ker(\Phi_q)$. Then $f \in A^p(\Omega)$ and

$$(\Phi_q f)(g) = \int_\Omega g\overline{f}\,dV = 0 \text{ for all } g \in A^q(\Omega).$$

If $f \neq 0$, we can find a bounded linear functional $F \in A^p(\Omega)'$ such that $F(f) \neq 0$. Since Φ_p is surjective, it must exist $g \in A^q(\Omega)$ such that $\Phi_q(g) = F$, that is,

$$0 = \Phi_q(g)(f) = F(f) \neq 0,$$

which is a contradiction. Thus $f = 0$ and Φ_q is injective.

Finally, since Φ_p is surjective and both $\mathcal{C}$ and ε are isomorphisms, it follows from (4.7) that Φ_q' must also be surjective. It implies that Φ_q has closed range (see [19, p. 101]).

□

4.4.2 Surjectivity of Φ_p

Surjectivity of Φ_p follows from existence of an operator satisfying $(H1)$ and $(H2)$ whose formal adjoint maps into $\mathcal{O}(\Omega)$.

Theorem 4.10. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Let $1 < p < \infty$ be given and q be the conjugate exponent of p. Suppose there exists P of the form (4.3) and $\mathcal{G} \subset A^p(\Omega)$ such that*

(i) $|P|$ is bounded on $L^p(\Omega)$,

(*ii*) $PF = F \quad \forall F \in \mathcal{G}$,

(*iii*) $\mathrm{Ran}(P^\dagger) \subset A^q(\Omega)$.

Then $\Phi_p : A^q(\Omega) \to \mathcal{G}'$ *is surjective.*

Remark 4.11. (*a*) The case $\mathcal{G} = A^p(\Omega)$ is included in Theorem 4.10.

(*b*) If $P = \mathcal{B}$, hypothesis (*iii*) is a consequence of (*i*) by Proposition 4.4.

Proof. Let $\lambda \in \mathcal{G}'$. We want to find a $h \in A^q(\Omega)$, such that $\lambda = \Phi_p(h)$. Extend λ by the Hahn-Banach theorem to a functional on $L^p(\Omega)$, still denoted λ, with the same norm. Then there is a $g \in L^q(\Omega)$, with $\|g\|_{L^q} = \|\lambda\|_{(L^p)'}$, such that

$$\lambda(f) = \int_\Omega f\bar{g}dV = \langle f, g\rangle \text{ for all } f \in L^p(\Omega).$$

Let $h = P^\dagger g$. By (*iii*), $h \in A^q(\Omega)$. Then for $F \in \mathcal{G}$ we have

$$\begin{aligned}\Phi_p(h)(F) &= \langle F, h\rangle \\ &= \langle F, P^\dagger g\rangle \\ &= \langle PF, g\rangle \\ &= \langle F, g\rangle \\ &= \lambda(F).\end{aligned}$$

The third equality follows from Proposition 4.4, the fourth follows from (*ii*). □

An elementary necessary condition for surjectivity of Φ_p is worth recording.

Proposition 4.12. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded domain. Suppose that for some* p, $1 < p < 2$, $A^2(\Omega) \cap A^p(\Omega)$ *is not dense in* $A^p(\Omega)$. *Then* Φ_p *is not surjective.*

Proof. Since Ω is bounded, $A^2(\Omega) \subset A^p(\Omega)$. The hypothesis thus says that $A^2(\Omega)$ is not dense in $A^p(\Omega)$. By the Hahn-Banach theorem, there exists a non-trivial $\psi \in A^p(\Omega)'$ which vanishes on $A^2(\Omega)$. Let q be the conjugate exponent of p. Suppose there were a non-trivial function $g \in A^q(\Omega)$ such that $\psi(h) = \int_\Omega h\bar{g}dV$ for all $h \in A^p(\Omega)$. Since $q > 2$, $g \in A^2(\Omega)$ and ψ acts on g. But then $0 = \psi(g) = \int_\Omega |g|^2 dV$, contradicting the fact g is not identically zero. □

4.5 Approximation on $A^p(\Omega)$

Functions in $A^p(\Omega)$, $1 < p < 2$, can be approximated by functions in $A^2(\Omega)$ if ($H1$) and ($H2$) hold.

Theorem 4.13. *Let* $\Omega \subset \mathbb{C}^n$ *be a domain. For a given* $1 < p < 2$, *suppose there exists an operator* P *of the form* (4.3) *and* $\mathcal{G} \subset A^p(\Omega)$ *such that* *(i)* *P is bounded on $L^p(\Omega)$.*

(ii) $Ph = h \quad \forall h \in \mathcal{G}$.

Then every $f \in \mathcal{G}$ can be approximated in the L^p norm by a sequence $\{f_n\} \subset A^2(\Omega)$.

Proof. Since $f \in \mathcal{G} \subset L^p(\Omega)$, there exists a sequence $\phi_n \in C_c^\infty(\Omega)$ such that $\|\phi_n - f\|_p \to 0$ as $n \to \infty$. Letting $f_n \coloneqq P\phi_n$, hypotheses (i) and (ii) give

$$\|f_n - f\|_p = \|P(\phi_n - f)\|_p \lesssim \|\phi_n - f\|_p.$$

Since $P : L^2(\Omega) \to A^2(\Omega)$, the claimed result holds. □

Remark 4.14. Ω is not assumed to be bounded in Theorem 4.13.

CHAPTER 5

Reinhardt Domains

Throughout this section, let $\mathcal{R} \subset \mathbb{C}^n$ be a bounded Reinhardt domain (see Definition 3.2). But why focus on such a special class of domains? Reinhardt domains, characterized by their invariance under rotations in each complex coordinate separately, provide a particularly convenient setting for the study of holomorphic functions in several complex variables. A key property is that any holomorphic function defined on a Reinhardt domain admits a power series expansion in terms of monomials, which converge normally in the interior of the domain. This allows a natural and effective use of multi-index notation and enables explicit computations in terms of monomial coefficients.

This structure greatly facilitates the analysis of function spaces such as A^p, especially when studying objects like the Bergman kernel and the Bergman projection. In L^2-type settings, monomials form an orthogonal and complete system, enabling explicit series representations. In such settings, monomials are orthogonal to each other, meaning that their pairwise inner products vanish. This independence simplifies computations and makes it possible to express functions as sums of monomials. Consequently, both the Bergman kernel and the Bergman projection become more transparent and easier to handle.

Moreover, many classical and instructive domains – such as the polydisc and the Hartogs triangle – are Reinhardt. These domains balance analytical tractability with geometric richness, providing a fertile ground for both explicit computation and theoretical development in several complex variables. For further details, see Chapter 1 of [16].

5.1 Integration on Reinhardt domains

For $r \in |\mathcal{R}|$ and f a continuous function on $\mathcal{R}$, let f_r be the function on the unit torus $\mathbb{T}^n = \{|z_j| = 1, \text{ for } j = 1, \ldots, n\} \subset \mathbb{C}^n$ defined by

$$f_r(e^{i\theta_1}, \ldots, e^{i\theta_n}) = f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}).$$

Abbreviate this relation by

$$f_r(e^{i\theta}) = f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}), \tag{5.1}$$

using vector notation on r and θ. Fubini's theorem implies

$$\begin{aligned} \|f\|^p_{L^p(\mathcal{R})} &= \int_{\mathcal{R}} |f(z)|^p dV(z) \\ &= \int_{\mathbb{T}^n} \int_{|\mathcal{R}|} |f(re^{i\theta})|^p r_1 r_2 \ldots r_n dr d\theta \\ &= \int_{|\mathcal{R}|} \left(\int_{\mathbb{T}^n} |f(re^{i\theta})|^p d\theta \right) r_1 r_2 \ldots r_n dr \\ &= \int_{|\mathcal{R}|} \|f_r\|^p_{L^p(\mathbb{T}^n)} r_1 r_2 \ldots r_n dr, \end{aligned} \tag{5.2}$$

a form of polar coordinate integration on $\mathcal{R}$.

5.2 Holomorphic monomials

For a multi-index $\alpha \in \mathbb{Z}^n$, let e_α denote the monomial function of exponent α:

$$e_\alpha(z) = z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \qquad z \in \mathbb{C}^\cdot$$

If $f \in \mathcal{O}(\mathcal{R})$, then f has a unique Laurent series expansion

$$f = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha(f) e_\alpha \tag{5.3}$$

converging uniformly on compact subsets of $\mathcal{R}$. The map

$$a_\alpha : \mathcal{O}(\mathcal{R}) \to \mathbb{C} \tag{5.4}$$

will be called the α-th coefficient functional. The uniqueness of the Laurent expansion shows the map a_α is well-defined (see Proposition 3.4).

5.3 The coefficient functionals

In this section, expansion (5.3) of an $f \in A^p(\mathcal{R})$ is shown to consist only of monomials in $A^p(\mathcal{R})$. For $1 \le p \le \infty$, define the set $\mathcal{S}(\mathcal{R}, L^p)$ of *L^p-allowable multi-indices for $\mathcal{R}$* by

$$\mathcal{S}(\mathcal{R}, L^p) := \{\alpha \in \mathbb{Z}^n : e_\alpha \in A^p(\mathcal{R})\}. \tag{5.5}$$

Since $\mathcal{R}$ is bounded, for $p_1 < p_2$ it holds that $\mathcal{S}(\mathcal{R}, L^{p_2}) \subset \mathcal{S}(\mathcal{R}, L^{p_1})$.

Proposition 5.1. *For each $\alpha \in \mathcal{S}(\mathcal{R}, L^p)$ and $1 \le p \le \infty$, the coefficient functional*

$$a_\alpha : A^p(\mathcal{R}) \to \mathbb{C}$$

is bounded. Moreover $\|a_\alpha\|_{A^p(\mathcal{R})'} = \dfrac{1}{\|e_\alpha\|_{L^p(\mathcal{R})}}$.

Proof. Let $\mathbb{T}(r) = \{|z_j| = r_j : j = 1, \dots, n\} \subset \mathcal{R}$ be a torus. For $f \in A^p(\mathcal{R})$, it follows from Proposition 3.4 that

$$\begin{aligned} a_\alpha(f) &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}(r)} \frac{f(\zeta)}{\zeta^\alpha} \cdot \frac{d\zeta}{\zeta_1} \cdots \frac{d\zeta_n}{\zeta_n} \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \frac{f(re^{i\theta})}{(re^{i\theta})^\alpha} \cdot \frac{ir_1 e^{i\theta_1} d\theta_1}{r_1 e^{i\theta_1}} \cdots \frac{ir_n e^{i\theta_n} d\theta_n}{r_n e^{i\theta_n}} \\ &= \frac{1}{(2\pi)^n} \cdot \frac{1}{r^\alpha} \int_{\mathbb{T}^n} f_r(e^{i\theta}) e^{-i\langle \alpha, \theta \rangle} d\theta, \end{aligned}$$

where $d\theta = d\theta_1 d\theta_2 \dots d\theta_n$ is the volume element of the unit torus. Hölder's inequality implies

$$\begin{aligned} |a_\alpha(f)| &\leq \frac{1}{(2\pi)^n} \cdot \frac{1}{r^\alpha} \|f_r\|_{L^p(\mathbb{T}^n)} \|1\|_{L^q(\mathbb{T}^n)} \\ &= \frac{1}{(2\pi)^n} \cdot \frac{1}{r^\alpha} \|f_r\|_{L^p(\mathbb{T}^n)} (2\pi)^{\frac{n}{q}} \\ &= \frac{(2\pi)^{-\frac{n}{p}}}{r^\alpha} \|f_r\|_{L^p(\mathbb{T}^n)}, \end{aligned} \tag{5.6}$$

using that $\frac{1}{q} = 1 - \frac{1}{p}$. When $p = \infty$, interpret $(2\pi)^{-\frac{n}{p}}$ as 1.

For $1 \leq p < \infty$, it follows from (5.6) that

$$|a_\alpha(f)|^p \cdot (2\pi)^n (r^\alpha)^p \leq \|f_r\|^p_{L^p(\mathbb{T}^n)}.$$

So if $\alpha \in \mathcal{S}(\mathcal{R}, L^p)$,

$$\begin{aligned} |a_\alpha(f)|^p \cdot \|e_\alpha\|^p_{L^p(\mathcal{R})} &= |a_\alpha(f)|^p \cdot \int_{\mathcal{R}} |z^\alpha|^p dV(z) \\ &= |a_\alpha(f)|^p \cdot \int_{\mathbb{T}^n} \int_{\mathcal{R}} (r^\alpha)^p r_1, \dots, r_n dr d\theta \\ &= |a_\alpha(f)|^p \cdot (2\pi)^n \int_{|\mathcal{R}|} (r^\alpha)^p r_1 \dots r_n dr \\ &\leq \int_{|\mathcal{R}|} \|f_r\|^p_{L^p(\mathbb{T}^n)} r_1 \dots r_n dr \\ &= \|f\|^p_{L^p(\mathcal{R})}. \end{aligned} \tag{5.7}$$

If $\alpha \in \mathcal{S}(\mathcal{R}, L^\infty)$, the trivial estimate $|a_\alpha(f)| \leq \inf\limits_{r \in |\mathcal{R}|} \frac{\|f\|_\infty}{r^\alpha} = \frac{\|f\|_\infty}{\sup_{z \in \mathcal{R}} |z^\alpha|} = \frac{\|f\|_\infty}{\|e_\alpha\|_\infty}$ holds. This estimate and (5.7) imply that for all $1 \leq p \leq \infty$,

$$\|a_\alpha\|_{A^p(\mathcal{R})'} \leq \frac{1}{\|e_\alpha\|_{L^p(\mathcal{R})}}.$$

Since $e_\alpha \in A^p(\mathcal{R})$ and $a_\alpha(e_\alpha) = 1 = \frac{\|e_\alpha\|_{L^p(\mathcal{R})}}{\|e_\alpha\|_{L^p(\mathcal{R})}}$, in fact $\|a_\alpha\|_{A^p(\mathcal{R})'} = \frac{1}{\|e_\alpha\|_{L^p(\mathcal{R})}}$. □

Proposition 5.1 implies that Laurent expansions of functions in A^p only have monomials that belong to L^p:

Corollary 5.2. *Let $\mathcal{R}$ be a bounded Reinhardt domain and $1 \le p \le \infty$. Let $f \in A^p(\mathcal{R})$, with Laurent expansion given by (5.3). Then if $\alpha \notin \mathcal{S}(\mathcal{R}, L^p)$, $a_\alpha(f) = 0$. Thus*

$$f(z) = \sum_{\alpha \in \mathcal{S}(\mathcal{R}, L^p)} a_\alpha(f) e_\alpha(z).$$

Proof. Assume $a_\alpha(f) \neq 0$. Choose a decreasing family of relatively compact Reinhardt domains $R_\varepsilon \subset \mathcal{R}$ such that $\mathcal{R}_\varepsilon \to \mathcal{R}$ as $\varepsilon \to 0$.[1] It follows from Proposition 5.1 that

$$|a_\alpha(f)|^p \|e_\alpha\|^p_{L^p(\mathcal{R}_\varepsilon)} \le \|f\|^p_{L^p(\mathcal{R}_\varepsilon)}.$$

As $\varepsilon \to 0$, the right hand side tends to $\|f\|_{L^p(\mathcal{R})} < \infty$, but the left hand side tends to ∞, since $\|e_\alpha\|_{L^p(\mathcal{R}_\varepsilon)} \to \infty$. This contradiction proves the result. □

Remark 5.3. Take $n = 1$, let $U^* = \{0 < |z| < 1\}$ be the punctured disc and $p = \infty$. Clear $\mathcal{S}(U^*, L^\infty) = \mathbb{N}_0$. Corollary 5.2 thus says every $f \in A^\infty(U^*)$ is of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and consequently f extends holomorphically to the unit disc. This recaptures Riemman's removable singularity theorem. A similar argument holds on $A^p(U^*)$ for any $p \ge 2$.

5.4 Norm convergence of Laurent series

If $\mathcal{R}$ is a bounded Reinhardt domain, $f \in A^p(\mathcal{R})$ and $p \in [1, \infty]$, Corollary 5.2 says

$$f(z) = \sum_{\alpha \in \mathcal{S}(\mathcal{R}, L^p)} a_\alpha(f) e_\alpha(z), \tag{5.8}$$

with uniform convergence on compact subsets of $\mathcal{R}$. The goal of this section is to show the series also converges in the A^p norm if $p \in (1, \infty)$.

[1] Let $\mathcal{R}_\varepsilon = \{z \in \mathcal{R} : \operatorname{dist}(z, \mathbb{C}^n \backslash \mathcal{R}) > \varepsilon\}$. For each $\varepsilon > 0$, $\mathcal{R}_\varepsilon$ is Reinhardt. In fact, first observe that $\mathbb{C}^n \backslash \mathcal{R}$ is Reinhardt: if $z \in \mathbb{C}^n \backslash \mathcal{R}$ and $w = e^{i\theta} z \in \mathcal{R}$, then by the Reinhardt property of $\mathcal{R}$, we must have $z = e^{-i\theta} w \in \mathcal{R}$, a contradiction. Now let $z \in \mathcal{R}_\varepsilon$. To prove that $e^{i\theta} z \in \mathcal{R}_\varepsilon$, take any $w \in \mathbb{C}^n \backslash \mathcal{R}$. Since $\mathbb{C}^n \backslash \mathcal{R}$ is Reinhardt, it follows that $e^{-i\theta} w \in \mathbb{C}^n \backslash \mathcal{R}$. Then

$$\|e^{i\theta} z - w\| = |e^{i\theta}| \|z - e^{-i\theta} w\| = \|z - e^{-i\theta} w\| \ge \operatorname{dist}(z, \mathbb{C}^n \backslash \mathcal{R}) > \varepsilon.$$

Therefore, $\operatorname{dist}(e^{i\theta} z, \mathbb{C}^n \backslash \mathcal{R}) > \varepsilon$, that is, $e^{i\theta} z \in \mathcal{R}_\varepsilon$. Note also that $\mathcal{R}_\varepsilon$ is open: let $z \in \mathcal{R}_\varepsilon$. Then $\operatorname{dist}(z, \mathbb{C}^n \backslash \mathcal{R}) > \varepsilon$. Set $\delta := \operatorname{dist}(z, \mathbb{C}^n \backslash \mathcal{R}) - \varepsilon > 0$. If $z' \in B(z, \delta)$, then for all $w \in \mathbb{C}^n \backslash \mathcal{R}$, the triangle inequality gives

$$\|z' - w\| \ge \|z - w\| - \|z' - z\|.$$

In particular, since $\|z' - z\| < \delta$, we obtain

$$\|z' - w\| > \|z - w\| - \delta.$$

Taking the infimum over $w \in \mathbb{C}^n \backslash \mathcal{R}$, we get

$$\operatorname{dist}(z', \mathbb{C}^n \backslash \mathcal{R}) > \varepsilon,$$

which shows $z' \in \mathcal{R}_\varepsilon$. Thus $B(z, \delta) \subset \mathcal{R}_\varepsilon$.

Since the index set of the series is a subset of an n-dimensional lattice, a choice of truncation is required. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ is a multi-index, let $|\alpha|_\infty = \max\{|\alpha_j|, j = 1, \dots, n\}$. For a formal series $g(z) = \sum_{\alpha\in\mathbb{Z}^n} b_\alpha e_\alpha(z)$ and a positive integer N, let

$$S_N g = \sum_{|\alpha|_\infty \le N} b_\alpha e_\alpha.$$

Call this the "square partial sum" of the series defining g.

Theorem 5.4. *Let $\mathcal{R}$ be a bounded Reinhardt domain in $\mathbb{C}^n$, $1 < p < \infty$ and $f \in A^p(\mathcal{R})$. Then*

$$\|S_N f - f\|_p \to 0 \text{ as } N \to \infty.$$

The proof of Theorem 5.4 is broken into parts.

5.4.1 Reduction and estimate

The following fact reduces matters to an estimate plus a simpler density result.

Lemma 5.5. *Let T_k, $k = 1, 2, \dots,$ be a sequence of bounded linear operators from a Banach space X to a Banach space Y. Suppose that there is a dense subset D of X, so that for each $x \in D$, $T_k x \to 0$ in the norm of Y as $k \to \infty$. Then the following are equivalent:*

(1) $\lim_{k\to\infty} \|T_k x\| = 0$ *for each* $x \in X$.

(2) *There is a $C > 0$ such that for each k, we have $\|T_k\|_{\text{op}} \le C$.*

Proof. Assume (1). Then (2) holds by the uniform boundedness principle [19].

Assume (2). Fix $x \in X$ and $\varepsilon > 0$. Since D is dense in X, there exists $p \in D$ such that $\|x - p\|_X < \frac{\varepsilon}{2C}$. Therefore

$$\|T_k x\|_Y \le \|T_k x - T_k p\|_Y + \|T_k p\|_Y < \frac{\varepsilon}{2} + \|T_k p\|_Y.$$

Choosing k so large that $\|T_k p\|_Y < \frac{\varepsilon}{2}$ yields (1). □

The estimate for Theorem 5.4 is

Lemma 5.6. *Let $\mathcal{R}$ be a bounded Reinhardt domain. For each $1 < p < \infty$, there exists a constant C_p such that*

$$\|S_N f\|_p \le C_p \|f\|_p \qquad \textit{for all } N \in \mathbb{Z}^+,\ f \in A^p(\mathcal{R}).$$

Proof. Denote the unit torus by $\mathbb{T}^n = \{z \in \mathbb{C}^n : |z_j| = 1, \text{ for } j = 1, \dots, n\}$. If $g \in L^p(\mathbb{T}^n)$, let $\sigma_N g$ denote the square partial sum of its Fourier series,

$$\sigma_N g = \sum_{|\nu|_\infty \le N} \widehat{g}(\nu) e^{i\nu\cdot\theta},$$

where $\widehat{g}(\nu) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} g(\theta)e^{-i\nu\cdot\theta}d\theta$. According to [13][2], for each $1 < p < \infty$, there exists a constant C_p such that

$$\|\sigma_N g\|_p \le C_p \|g\|_p \qquad \text{for all } N \in \mathbb{Z}^+,\ g \in L^p(\mathbb{T}^n).$$

For $r \in |\mathcal{R}|$ and $f \in A^p(\mathcal{R})$, set $f_r(e^{i\theta})$ as in (5.1). Since

$$f(z) = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} a_\alpha(f)e_\alpha(z),$$

with uniform convergence on compact subsets of $\mathcal{R}$, we may write

$$f_r(e^{i\theta}) = f(re^{i\theta}) = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} a_\alpha(f)(re^{i\theta})^\alpha = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} a_\alpha(f)r^\alpha e^{i\alpha\cdot\theta}.$$

Additionally,

$$S_N f(re^{i\theta}) = \sum_{|\alpha|_\infty\le N} a_\alpha(f)(re^{i\theta})^\alpha = \sum_{|\alpha|_\infty\le N} a_\alpha(f)r^\alpha e^{i\alpha\cdot\theta}.$$

The Fourier coefficients of f_r are given by:

$$\begin{aligned}
\widehat{f_r}(\nu) &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f_r(e^{i\theta})e^{-i\nu\cdot\theta}d\theta \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\left(\sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} a_\alpha(f)r^\alpha e^{i\alpha\cdot\theta}\right)e^{-i\nu\cdot\theta}d\theta \\
&= \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} a_\alpha(f)r^\alpha\left(\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} e^{i(\alpha-\nu)\cdot\theta}d\theta\right) \\
&= \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} a_\alpha(f)r^\alpha\delta_{\alpha\nu} \\
&= a_\nu(f)r^\nu,
\end{aligned}$$

where $\delta_{\alpha\nu}$ is the Kronecker delta. The interchange between the sum and the integral is justified by the uniform convergence of the Laurent series on the compact torus

$$\mathbb{T}(r) = \left\{z \in \mathbb{C}^n : z_j = r_j e^{i\theta_j},\ \theta_j \in [0, 2\pi),\ 1 \le j \le n\right\} \subset \mathcal{R},$$

for fixed $r \in |\mathcal{R}|$. Therefore,

$$\sigma_N f_r(e^{i\theta}) = \sum_{|\nu|_\infty\le N} \widehat{f_r}(\nu)e^{i\nu\cdot\theta} = \sum_{|\nu|_\infty\le N} a_\nu(f)r^\nu e^{i\nu\cdot\theta} = S_N f(re^{i\theta}).$$

[2] See Chapter 4 of [13], specifically Corollary 4.1.3 and Theorem 4.1.8

From this and from (5.2), we conclude that

$$\begin{aligned}
\|S_N f\|_p^p &= \int_{|\mathcal{R}|}\int_{\mathbb{T}^n} |S_N f(re^{i\theta})|^p d\theta r_1 r_2 \dots r_n dr \\
&= \int_{|\mathcal{R}|}\int_{\mathbb{T}^n} |\sigma_N f_r(e^{i\theta})|^p d\theta r_1 r_2 \dots r_n dr \\
&= \int_{|\mathcal{R}|} \|\sigma_N f_r\|^p_{L^p(\mathbb{T}^n)} r_1 r_2 \dots r_n dr \\
&\le C_p \int_{|\mathcal{R}|} \|f_r\|^p_{L^p(\mathbb{T}^n)} r_1 r_2 \dots r_n dr \\
&= C_p \|f\|_p^p,
\end{aligned}$$

as we wanted to prove. □

5.4.2 Series expansion of functionals

The dense set D needed in Lemma 5.5 is found by duality. Given a functional $\lambda \in A^p(\mathcal{R})'$, consider the finite sum

$$S'_N \lambda = \sum_{|\alpha|_\infty \le N} \lambda(e_\alpha) a_\alpha, \tag{5.9}$$

where a_α are the coefficient functionals in Proposition 5.1.

Proposition 5.7. *For each $\lambda \in A^p(\mathcal{R})'$,*

$$\|S'_N \lambda - \lambda\|_{(A^p)'} \to 0 \text{ as } N \to \infty. \tag{5.10}$$

Proof. For $f \in A^p(\mathcal{R})$

$$S'_N \lambda(f) = \sum_{|\alpha|_\infty \le N} \lambda(e_\alpha) a_\alpha(f) = \lambda\left(\sum_{|\alpha|_\infty \le N} a_\alpha(f) e_\alpha\right) = \lambda(S_N f).$$

It follows from Lemma 5.6 that

$$|S'_N \lambda(f)| = |\lambda(S_N f)| \le \|\lambda\|_{(A^p)'} \|S_N f\|_p \le C \|\lambda\|_{(A^p)'} \|f\|_p.$$

Thus $\|S'_N\|_{op} \le C$ where S'_N is viewed as an operator on the Banach space $A^p(\mathcal{R})'$.

Claim: The span of $\{a_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}$ is dense in $A^p(\mathcal{R})'$.

To prove the claim, let $\mu \in (A^p(\mathcal{R})')'$ be an element of the double dual of $A^p(\mathcal{R})$ such that $\mu(f) = 0$ for each f in the span of $\{a_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}$. By the Hahn-Banach theorem, it suffices to show that $\mu = 0$ on $A^p(\mathcal{R})'$.

Since $A^p(\mathcal{R})$ is closed in $L^p(\mathcal{R})$, $A^p(\mathcal{R})$ is reflexive. Therefore there exists a $g \in A^p(\mathcal{R})$ such that $\mu(f) = f(g)$ for all $f \in A^p(\mathcal{R})'$. Taking $f = a_\alpha$, it follows that $a_\alpha(g) = 0$, that is, the α-th coefficient of the Laurent expansion of the holomorphic function g vanishes for each α. This implies $g = 0$, which shows $\mu = 0$ and establishes the claim.

To complete the proof, in Lemma 5.5 let $X = Y = A^p(\mathcal{R})'$, $T_N = S_N' - \text{id}$ and D be the linear span of $\{a_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}$. Note that for each element $\lambda \in D$, there is an N_0 such that $T_N\lambda = 0$ for $N \geq N_0$. In fact, let $\lambda \in D \subset A^p(\mathcal{R})'$. Then

$$\lambda = \sum_{|\alpha|_\infty \leq M} c_\alpha a_\alpha.$$

For $f \in A^p(\mathcal{R})$, we have $\lambda(f) = \sum_{|\alpha|_\infty \leq M} c_\alpha a_\alpha(f)$ and

$$S_N'\lambda(f) = \lambda(S_N f) = \sum_{|\alpha|_\infty \leq M} c_\alpha a_\alpha(S_N f).$$

Now remember that $S_N f = \sum_{|\alpha|_\infty \leq N} a_\alpha(f) e_\alpha$ and $a_\alpha(S_N f)$ represents the coefficient of e_α in the expansion of $S_N f$, that is,

$$a_\alpha(S_N f) = \begin{cases} a_\alpha(f), & \text{if } |\alpha|_\infty \leq N, \\ 0, & \text{if } |\alpha|_\infty > N. \end{cases}$$

Then

$$\begin{aligned} T_N\lambda(f) &= S_N'\lambda(f) - \lambda(f) \\ &= \sum_{|\alpha|_\infty \leq M} c_\alpha a_\alpha(S_N f) - \sum_{|\alpha|_\infty \leq M} c_\alpha a_\alpha(f) \\ &= \sum_{|\alpha|_\infty \leq M} c_\alpha (a_\alpha(S_N f) - a_\alpha(f)). \end{aligned}$$

Taking $N \geq M$, we have $T_N\lambda(f) = 0$ for all f, which implies that $T_N\lambda = 0$. The hypotheses of Lemma 5.4 are thus satisfied, which implies (5.10) □

5.4.3 Proof of Theorem 5.4

In Lemma 5.5, take $X = Y = A^p(\mathcal{R})$, and $T_N = S_N - \text{id}$. For each Laurent polynomial p, note that $T_N p = 0$ for large enough N. The result will follow from Lemma 5.5 provided it is shown that $D \coloneqq \{\text{Laurent polynomials} \in A^p(\mathcal{R})\}$ is a dense subspace of $A^p(\mathcal{R})$.

By Corollary 5.2, D is the linear span of $\{e_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}$. To show this last set is dense, suppose $\lambda \in A^p(\mathcal{R})'$ satisfies $\lambda(e_\alpha) = 0$ for all $\alpha \in \mathcal{S}(\mathcal{R}, L^p)$. Definition (5.9) shows $S_N'\lambda = 0$ for each N. However Proposition 5.7 implies $\lambda = \lim S_N'\lambda = 0$. Thus, the Hahn-Banach theorem implies $\text{span}\{e_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}$ is dense in $A^p(\mathcal{R})$.

5.5 Computing the projection term-by-term

If $\Omega \subset \mathbb{C}^n$ is a bounded domain and $p \geq 2$, $\mathcal{B}h = h$ for all $h \in A^p(\Omega)$ since $A^p(\Omega) \subset A^2(\Omega)$. For $1 < p < 2$, this generally fails, even if $\mathcal{B}$ is L^p bounded.

However on a bounded Reinhardt domain, if $|\mathcal{B}|$ satisfies $(H1)$ and h is in the form (5.8), $\mathcal{B}h$ can be computed merely by discarding monomials.

Proposition 5.8. *Let $\mathcal{R}$ be a bounded Reinhardt domain. For given $1 < p < 2$, suppose $|\mathcal{B}|$ is bounded on $L^p(\mathcal{R})$.*

(i) If $\gamma \in \mathcal{S}(\mathcal{R}, L^p)\backslash\mathcal{S}(\mathcal{R}, L^2)$, then $e_\gamma \in \ker(\mathcal{B})$.

(ii) If $f \in A^p(\mathcal{R})$ has expansion (5.8), then

$$\mathcal{B}f = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^2)} a_\alpha(f)e_\alpha.$$

The square partial sums of the series in (ii) converge in $L^p(\mathcal{R})$.

Proof. To see (i), choose a decreasing family $\{\mathcal{R}_t : 0 < t < 1\}$ of relatively compact Reinhardt subdomains of $\mathcal{R}$ whose union is $\mathcal{R}$.[3] Since $e_\gamma \in A^p(\mathcal{R})$, then $e_\gamma \in \mathcal{O}(\mathcal{R})$ and e_γ is continuous on $\mathcal{R}$. Of course e_γ is bounded on the compact subset $\overline{\mathcal{R}_t} \subset \mathcal{R}$, which allows us to conclude that $e_\gamma \in L^2(\mathcal{R}_t)$. For each $\beta \in \mathcal{S}(\mathcal{R}, L^2)$, orthogonality implies $\langle e_\gamma, e_\beta\rangle_{\mathcal{R}_t} = 0$ since $\gamma \notin \mathcal{S}(\mathcal{R}, L^2)$. In fact, by Fubini's theorem,

$$\begin{aligned}
\langle e_\beta, e_\gamma\rangle_{\mathcal{R}_t} &= \int_{\mathcal{R}_t} e_\beta(z)\overline{e_\gamma(z)}dV(z)\\
&= \int_{\mathcal{R}_t} z^\beta\overline{z}^\gamma dV(z)\\
&= \int_{\mathcal{R}_t}\prod_{j=1}^n z_j^{\beta_j}\overline{z_j}^{\gamma_j}dV(z)\\
&= \int_{\mathbb{T}^n}\int_{|\mathcal{R}_t|}\left(\prod_{j=1}^n (r_je^{i\theta_j})^{\beta_j}(r_je^{-i\theta_j})^{\gamma_j}\right)\cdot\prod_{j=1}^n r_jdr_jd\theta_j\\
&= \left(\int_{\mathbb{T}^n}\prod_{j=1}^n e^{i(\beta_j-\gamma_j)\theta_j}d\theta_j\right)\cdot\left(\int_{|\mathcal{R}_t|}\prod_{j=1}^n r_j^{\beta_j+\gamma_j+1}dr_j\right)\\
&= \prod_{j=1}^n\left(\int_0^{2\pi} e^{i(\beta_j-\gamma_j)\theta_j}d\theta_j\right)\cdot\left(\int_{|\mathcal{R}_t|}\prod_{j=1}^n r_j^{\beta_j+\gamma_j+1}dr_j\right)\\
&= \begin{cases} 0, & \text{if } \beta\neq\gamma,\\ (2\pi)^n\int_{|\mathcal{R}_t|}\prod_{j=1}^n r_j^{2\beta_j+1}dr_j, & \text{if } \beta=\gamma.\end{cases}
\end{aligned}$$

Let $\mathbb{B}(z, w)$ denote the Bergman kernel of $\mathcal{R}$ (see Definition 3.13 and Proposition 3.18). Since $\mathbb{B}(z, w) = \displaystyle\sum_{\beta\in\mathcal{S}(\mathcal{R},L^2)}\frac{e_\beta(z)\overline{e_\beta(w)}}{\|e_\beta\|_2^2}$, it follows that

$$\int_{\mathcal{R}_t}\mathbb{B}(z, w)e_\gamma(w)dV(w) = \int_{\mathcal{R}_t}\sum_{\beta\in\mathcal{S}(\mathcal{R},L^2)}\frac{e_\beta(z)\overline{e_\beta(w)}}{\|e_\beta\|_2^2}e_\gamma(w)dV(w).$$

[3]Let $\mathcal{R}_t = \{z \in \mathcal{R} : \mathrm{dist}(z, \mathbb{C}^n\backslash\mathcal{R}) > t\}$.

Since the sum converges uniformly on compact subsets of $\mathcal{R}$ and $\mathcal{R}_t \subset\subset \mathcal{R}$, we can interchange the sum and the integral:

$$\int_{\mathcal{R}_t} \mathbb{B}(z,w)e_\gamma(w)dV(w) = \sum_{\beta\in\mathcal{S}(\mathcal{R},L^2)} \frac{e_\beta(z)}{\|e_\beta\|_2^2}\langle e_\gamma, e_\beta\rangle_{\mathcal{R}_t} = 0.$$

Proposition 4.2 thus yields

$$\mathcal{B}e_\gamma = 0. \tag{5.11}$$

To see (ii), let $f \in A^p(\mathcal{R})$. From Theorem 5.4, $f = \lim S_N f$ with convergence in $L^p(\mathcal{R})$. Since $\mathcal{B}$ is continuous on $L^p(\mathcal{R})$ ($|\mathcal{B}|$ is $L^p(\mathcal{R})$ bounded by assumption),

$$\mathcal{B}f = \lim_{N\to\infty} \mathcal{B}(S_N f) = \lim_{N\to\infty} \mathcal{B}\left(\sum_{|\alpha|_\infty\le N} a_\alpha(f)e_\alpha\right) = \lim_{N\to\infty} \sum_{|\alpha|_\infty\le N} a_\alpha(f)\mathcal{B}(e_\alpha),$$

all limits taken in L^p. Thus (5.11) and the fact that $\mathcal{B}(e_\alpha) = e_\alpha$ if $e_\alpha \in L^2(\mathcal{R})$ yields (ii). □

Remark 5.9. Proposition 5.8 does not assert that $\mathcal{B}f \in A^2(\mathcal{R})$ for general $f \in A^p(\mathcal{R})$ when $1 < p < 2$. Note that when $1 < p < 2$,

$$\sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} a_\alpha e_\alpha \in A^p(\mathcal{R}) \not\Rightarrow \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^2)} a_\alpha e_\alpha \in A^2(\mathcal{R}),$$

though each of the monomials in the right sum is in $A^2(\mathcal{R})$.

5.6 Sub-Bergman projections

Throughout the section, assume $p \ge 2$. If $\Omega \subset \mathbb{C}^n$ is a bounded domain, let

$$G^{2,p}(\Omega) := \overline{\text{span}}_{A^2(\Omega)} A^p(\Omega). \tag{5.12}$$

$G^{2,p}(\Omega) \subset L^2(\Omega)$ is a closed subspace. The L^p sub-Bergman projection is defined as the orthogonal projection

$$\widetilde{\mathcal{B}^p_\Omega} : L^2(\Omega) \to G^{2,p}(\Omega).$$

The representing kernel

$$\widetilde{\mathcal{B}^p_\Omega} f = \int_\Omega \widetilde{\mathbb{B}^p_\Omega}(z,w)f(w)dV(w) \tag{5.13}$$

is the L^p sub-Bergman kernel. The same arguments used in Proposition 3.20 yield formula (5.13). Subscripts are dropped when the domain is unambigous. Since $A^p(\Omega) \subset G^{2,p}(\Omega)$, it follows that $\widetilde{\mathcal{B}^p_\Omega} f = f$ for all $f \in A^p(\Omega)$.

On a Reinhardt domain, the sub-Bergman projection assumes a concrete form.

Proposition 5.10. *Let $\mathcal{R}$ be a bounded Reinhardt domain in $\mathbb{C}^n$ and $p \ge 2$. Then* (*i*) $G^{2,p}(\mathcal{R}) = \overline{\text{span}}_{A^2(\mathcal{R})}\{e_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}$.

(*ii*) $\widetilde{\mathbb{B}^p}(z,w) = \displaystyle\sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} \frac{e_\alpha(z)\overline{e_\alpha(w)}}{\|e_\alpha\|_2^2}$.

Proof. This follows from Corollary 5.2 and Theorem 5.4. Since two norms are involved, details are given for clarity. Note that $\overline{\text{span}}_{A^p(\mathcal{R})}(\mathcal{F}) \subset \overline{\text{span}}_{A^2(\mathcal{R})}(\mathcal{F})$ for any $\mathcal{F} \subset A^2(\mathcal{R})$, since $p \geq 2$ and $\mathcal{R}$ is bounded. Let $g \in G^{2,p}(\mathcal{R})$ and $\varepsilon > 0$. Definition (5.12) says there exists $g' \in A^p(\mathcal{R})$ such that $\|g - g'\|_2 < \varepsilon$. Corollary 5.2 and Theorem 5.4 imply there exists $g'' \in \text{span}\{e_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}$ such that $\|g' - g''\|_2 \leq C\|g' - g''\|_p < \varepsilon$, C depending on the diameter of $\mathcal{R}$. Conversely, each monomial e_α with $\alpha \in \mathcal{S}(\mathcal{R}, L^p)$ belongs to $A^p(\Omega) \subset G^{2,p}(\mathcal{R})$, hence $\overline{\text{span}}_{A^2(\mathcal{R})}\{e_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\} \subset G^{2,p}(\mathcal{R})$. Thus (*i*) holds.

For (*ii*), since $\widetilde{\mathcal{B}^p}$ orthogonally projects onto $G^{2,p}(\mathcal{R})$ and $\{e_\alpha/\|e_\alpha\|_2\}_{\alpha\in\mathcal{S}(\mathcal{R},L^p)}$ is a complete orthonormal basis for $G^{2,p}(\mathcal{R})$ (by item (*i*)), it follows that for any $f \in G^{2,p}(\mathcal{R})$,

$$f = \widetilde{\mathcal{B}^p} f = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} \frac{\langle f, e_\alpha\rangle}{\|e_\alpha\|_2^2} e_\alpha. \tag{5.14}$$

The series converges in $A^2(\mathcal{R})$. The kernel representation (*ii*) now follows as in ordinary Bergman theory. In fact, for fixed $w \in \mathcal{R}$, the function $\widetilde{\mathbb{B}^p}(z,w) \in G^{2,p}(\mathcal{R})$ as a function of z, and applying (5.14) to $f(z) = \widetilde{\mathbb{B}^p}(z,w)$ we obtain

$$\widetilde{\mathbb{B}^p}(\cdot,w) = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} \frac{\langle \widetilde{\mathbb{B}^p}(\cdot,w), e_\alpha\rangle}{\|e_\alpha\|_2^2} e_\alpha(\cdot) = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} \frac{\overline{\langle e_\alpha, \widetilde{\mathbb{B}^p}(\cdot,w)\rangle}}{\|e_\alpha\|_2^2} e_\alpha(\cdot) = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} \frac{e_\alpha(\cdot)\overline{e_\alpha(w)}}{\|e_\alpha\|_2^2}$$

with convergence in $A^2(\mathcal{R})$, and hence also pointwise, which completes the proof of (*ii*). □

Let q be conjugate to p. Note that $q \leq 2$. Subspaces of $A^p(\mathcal{R})$ and $A^q(\mathcal{R})$ enter the next result, and also appear in the description of dual spaces in the next section. Generalizing (5.12), define the subspace of $A^q(\mathcal{R})$

$$G^{q,p}(\mathcal{R}) := \overline{\text{span}}_{A^q(\mathcal{R})}\{e_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}. \tag{5.15}$$

Extending Proposition 5.8 (*i*), define the subspace of $A^q(\mathcal{R})$

$$N^{q,p}(\mathcal{R}) := \overline{\text{span}}_{A^q(\mathcal{R})}\{e_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^q)\backslash\mathcal{S}(\mathcal{R}, L^p)\}. \tag{5.16}$$

$\widetilde{\mathcal{B}^p}$ is not necessarily bounded on $L^p(\mathcal{R})$. When $|\widetilde{\mathcal{B}^p}|$ is L^p bounded, the following holds:

Proposition 5.11. *Let $\mathcal{R}$ be a bounded Reinhardt domain in $\mathbb{C}^n$. Let $p \geq 2$ and q be the conjugate to p. Suppose $|\widetilde{\mathcal{B}^p}|$ is bounded on $L^p(\mathcal{R})$. Then*

(*i*) *$\widetilde{\mathcal{B}^p}$ is a projection from $L^p(\mathcal{R})$ onto $A^p(\mathcal{R})$.*

(*ii*) *Let* $\widetilde{\mathcal{B}^p}^\dagger$ *be the formal adjoint defined by* (4.5). *Then* $\widetilde{\mathcal{B}^p}^\dagger$ *is bounded on* $L^q(\mathcal{R})$. *For* $f \in L^q(\mathcal{R})$,

$$\widetilde{\mathcal{B}^p}^\dagger f = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^p)} \frac{\langle f, e_\alpha \rangle}{\|e_\alpha\|_2^2} e_\alpha. \tag{5.17}$$

The square partial sums of the series converge in $A^q(\mathcal{R})$.

(*iii*) *Consider* $\widetilde{\mathcal{B}^p}^\dagger$ *restricted to* $A^q(\mathcal{R})$. *Then* $\ker(\widetilde{\mathcal{B}^p}^\dagger) = N^{q,p}(\mathcal{R})$, $\operatorname{Ran}(\widetilde{\mathcal{B}^p}^\dagger) = G^{q,p}(\mathcal{R})$, *and* $\widetilde{\mathcal{B}^p}^\dagger h = h$ *for all* $h \in G^{q,p}(\mathcal{R})$.

Proof. The proof of (*i*) follows directly from definition of $\widetilde{\mathcal{B}^p_\mathcal{R}}$ ($|\widetilde{\mathcal{B}^p}|$ bounded on $L^p(\mathcal{R})$ implies $\widetilde{\mathcal{B}^p}$ bounded on $L^p(\mathcal{R})$) and the fact that the intersection $G^{2,p}(\mathcal{R}) \cap L^p(\mathcal{R}) = A^p(\mathcal{R})$.

The first statement in (*ii*) follows from Proposition 4.4 (*i*). Representation (5.17) follows from Proposition 5.10 (*ii*). Convergence of the series in $A^q(\mathcal{R})$ follows from Theorem 5.4. In fact, since $\mathbb{B}^p$ is conjugate symmetric and $\widetilde{\mathcal{B}^p}$ is bounded on $L^p(\mathcal{R})$, Proposition 4.4 yields

$$\langle \widetilde{\mathcal{B}^p} f, g \rangle = \langle f, \widetilde{\mathcal{B}^p} g \rangle \quad \text{for all } f \in L^p(\mathcal{R}),\ g \in L^q(\mathcal{R}).$$

Since $L^p(\mathcal{R}) \subset L^2(\mathcal{R}) \subset L^q(\mathcal{R})$, it follows that for every $f \in L^2(\mathcal{R}) \cap L^q(\mathcal{R})$, we have

$$\widetilde{\mathcal{B}^p}^\dagger f = \widetilde{\mathcal{B}^p} f = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^p)} \frac{\langle f, e_\alpha \rangle}{\|e_\alpha\|_2^2} e_\alpha,$$

with the square partial sums converging in $A^q(\mathcal{R})$, since $\widetilde{\mathcal{B}^p}^\dagger f \in G^{2,p}(\mathcal{R}) \subset A^2(\mathcal{R}) \subset A^q(\mathcal{R})$.

Now recall that $L^2(\mathcal{R}) \cap L^q(\mathcal{R})$ is dense in $L^q(\mathcal{R})$. Given $f \in L^q(\mathcal{R})$, let $\{f_n\} \subset L^2(\mathcal{R}) \cap L^q(\mathcal{R})$ be such that $f_n \to f$ in the L^q-norm. Since $\widetilde{\mathcal{B}^p}^\dagger$ is bounded on L^q, we have $\widetilde{\mathcal{B}^p}^\dagger f_n \to \widetilde{\mathcal{B}^p}^\dagger f$ in $L^q(\mathcal{R})$. Moreover, since $A^q(\mathcal{R})$ is closed and each $\widetilde{\mathcal{B}^p}^\dagger f_n$ lies in $A^q(\mathcal{R})$, it follows that $\widetilde{\mathcal{B}^p}^\dagger f \in A^q(\mathcal{R})$. Consequently, $\widetilde{\mathcal{B}^p}^\dagger f$ admits a unique expansion

$$\widetilde{\mathcal{B}^p}^\dagger f = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^p)} a_\alpha(f) e_\alpha.$$

For each n, recall that

$$\widetilde{\mathcal{B}^p}^\dagger f_n = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^p)} \frac{\langle f_n, e_\alpha \rangle}{\|e_\alpha\|_2^2} e_\alpha.$$

If $\alpha \in \mathcal{S}(\mathcal{R}, L^p) \subset \mathcal{S}(\mathcal{R}, L^q)$, Proposition 5.1 implies that the coefficient functional $a_\alpha : A^q(\mathcal{R}) \to \mathbb{C}$ is continuous. Therefore,

$$\lim_{n\to\infty} a_\alpha(\widetilde{\mathcal{B}^p}^\dagger f_n) = a_\alpha(\widetilde{\mathcal{B}^p}^\dagger f).$$

But

$$\lim_{n\to\infty} a_\alpha(\widetilde{\mathcal{B}^p}^\dagger f_n) = \lim_{n\to\infty} \frac{\langle f_n, e_\alpha \rangle}{\|e_\alpha\|_2^2} = \frac{\langle f, e_\alpha \rangle}{\|e_\alpha\|_2^2}.$$

By the uniqueness of the series expansion, we conclude that

$$a_\alpha(\widetilde{\mathcal{B}^p}^\dagger f) = \frac{\langle f, e_\alpha \rangle}{\|e_\alpha\|_2^2}.$$

Moreover, if $\alpha \in \mathcal{S}(\mathcal{R}, L^q) \setminus \mathcal{S}(\mathcal{R}, L^p)$, then clearly $a_\alpha(\widetilde{\mathcal{B}^p}^\dagger f_n) = 0$ for all n. By continuity of a_α, we have

$$a_\alpha(\widetilde{\mathcal{B}^p}^\dagger f) = \lim_{n\to\infty} a_\alpha(\widetilde{\mathcal{B}^p}^\dagger f_n) = 0,$$

which establishes (5.17).

For (iii), let $\alpha \in \mathcal{S}(\mathcal{R}, L^q)\setminus\mathcal{S}(\mathcal{R}, L^p)$. Then (5.17) shows $\widetilde{\mathcal{B}^p}^\dagger(e_\alpha) = 0$. If $f \in N^{q,p}(\mathcal{R})$, then there exists a sequence $\{f_n\}$ in the span$\{e_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^q)\setminus\mathcal{S}(\mathcal{R}, L^p)\}$ such that $f_n \to f$ in $A^q(\mathcal{R})$ and $\widetilde{\mathcal{B}^p}^\dagger f_n = 0$ for all n. Consequently, $\widetilde{\mathcal{B}^p}^\dagger f_n \to \widetilde{\mathcal{B}^p}^\dagger f$ in $A^q(\mathcal{R})$, which implies $\widetilde{\mathcal{B}^p}^\dagger f = 0$. On the other hand, if $f \in A^q(\mathcal{R})\setminus N^{q,p}(\mathcal{R})$, the Laurent series expansion of f must contain a nonzero coefficient of a monomial e_β with $\beta \in \mathcal{S}(\mathcal{R}, L^p)$. Formula (5.17) shows $\widetilde{\mathcal{B}^p}^\dagger(f) \neq 0$. Thus $\ker \widetilde{\mathcal{B}^p}^\dagger = N^{q,p}(\mathcal{R})$.

Finally, (5.17) shows that the range of $\widetilde{\mathcal{B}^p}^\dagger$ is the closure of the linear span of the family $\{e_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}$, that is, the subspace $G^{q,p}(\mathcal{R})$. Clearly $\mathrm{Ran}(\widetilde{\mathcal{B}^p}^\dagger) \subset G^{q,p}(\mathcal{R})$. To prove the reverse inclusion, let e_α with $\alpha \in \mathcal{S}(\mathcal{R}, L^p)$. Taking $f = e_\alpha \in L^p(\mathcal{R}) \subset L^2(\mathcal{R}) \subset L^q(\mathcal{R})$, we have

$$\widetilde{\mathcal{B}^p}^\dagger f = \sum_{\beta\in\mathcal{S}(\mathcal{R},L^p)} \frac{\langle e_\alpha, e_\beta\rangle}{\|e_\beta\|_2^2} e_\beta = \sum_{\beta\in\mathcal{S}(\mathcal{R},L^p)} \delta_{\alpha\beta} e_\beta = e_\alpha.$$

It shows that $e_\alpha \in \mathrm{Ran}(\widetilde{\mathcal{B}^p}^\dagger)$. If $f \in G^{q,p}(\mathcal{R})$, then there exists a sequence $\{g_n\}$ in span$\{e_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}$ such that $g_n \to f$ in $A^q(\mathcal{R})$. For each n, there exists $f_n \in L^q(\mathcal{R})$ such that $\widetilde{\mathcal{B}^p}^\dagger f_n = g_n$, since $\mathrm{Ran}(\widetilde{\mathcal{B}^p}^\dagger)$ must contain all finite linear combinations of such e_α. Since $\widetilde{\mathcal{B}^p}^\dagger$ is bounded on L^q and idempotent[4], we have

$$\widetilde{\mathcal{B}^p}^\dagger f_n = \widetilde{\mathcal{B}^p}^\dagger(\widetilde{\mathcal{B}^p}^\dagger f_n) \to \widetilde{\mathcal{B}^p}^\dagger f \text{ in } A^q(\mathcal{R}).$$

But also $\widetilde{\mathcal{B}^p}^\dagger f_n \to f$ in $A^q(\mathcal{R})$, which implies $f = \widetilde{\mathcal{B}^p}^\dagger f \in \mathrm{Ran}(\widetilde{\mathcal{B}^p}^\dagger)$. It also shows that $\widetilde{\mathcal{B}^p}^\dagger$ restricts to the identity on $G^{q,p}(\mathcal{R})$. □

5.7 Representation of $A^p(\mathcal{R})'$

Proposition 5.12. *Let $\mathcal{R}$ be a bounded Reinhardt domain in $\mathbb{C}^n$. Let $p \geq 2$ and q be conjugate to p. Suppose $|\widetilde{\mathcal{B}^p}|$ is bounded on $L^p(\mathcal{R})$.*

[4]The idempotency of the operator $\widetilde{\mathcal{B}^p}^\dagger$ follows directly from its formal definition: for $f \in L^q(\mathcal{R})$, we have

$$\widetilde{\mathcal{B}^p}^\dagger f = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} \frac{\langle f, e_\alpha\rangle}{\|e_\alpha\|_2^2} e_\alpha,$$

where the series converges in $A^q(\mathcal{R})$. Applying the operator again, we obtain

$$\widetilde{\mathcal{B}^p}^\dagger(\widetilde{\mathcal{B}^p}^\dagger f) = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} \frac{\langle \widetilde{\mathcal{B}^p}^\dagger f, e_\alpha\rangle}{\|e_\alpha\|_2^2} e_\alpha = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} \frac{\langle f, \widetilde{\mathcal{B}^p} e_\alpha\rangle}{\|e_\alpha\|_2^2} e_\alpha = \sum_{\alpha\in\mathcal{S}(\mathcal{R},L^p)} \frac{\langle f, e_\alpha\rangle}{\|e_\alpha\|_2^2} e_\alpha = \widetilde{\mathcal{B}^p}^\dagger f,$$

since each $e_\alpha \in A^p(\mathcal{R}) \subset G^{2,p}(\mathcal{R})$ and $\widetilde{\mathcal{B}^p} e_\alpha = e_\alpha$.

(*i*) *The map* $\Phi_p : A^q(\mathcal{R}) \to A^p(\mathcal{R})'$ *is surjective and* $\ker(\Phi_p) = N^{q,p}(\mathcal{R})$.

(*ii*) *There is an explicit linear homeomorphism of Banach spaces*

$$A^p(\mathcal{R})' \cong G^{q,p}(\mathcal{R}). \tag{5.18}$$

(*iii*) *There is a topological direct sum representation*

$$A^q(\mathcal{R})' = \Phi_q(A^p(\mathcal{R})) \oplus \overline{\operatorname{span}}_{A^q(\mathcal{R})'}\{a_\alpha : \mathcal{S}(\mathcal{R}, L^q)\backslash\mathcal{S}(\mathcal{R}, L^p)\}. \tag{5.19}$$

Proof. Let $P = \widetilde{\mathcal{B}^p_\mathcal{R}}$ for notational economy.

To see (*i*), check the hypotheses of Theorem 4.10. Hypothesis (*i*) of Theorem 4.10 is satisfied by assumption. Hypothesis (*ii*) of the same theorem holds since $A^p(\mathcal{R}) \subset G^{2,p}(\mathcal{R})$. Proposition 4.4 implies $P^\dagger$ is L^q bounded. Using the same argument as in the proof of Proposition 5.11, we see hypothesis (*iii*) is satisfied. Theorem 4.10 thus says Φ_p is surjective. To determine $\ker \Phi_p$, direct computation gives

$$\Phi_p(e_\alpha)(e_\beta) = \int_\mathcal{R} e_\beta \overline{e_\alpha} dV = \|e_\beta\|_2^2 \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol. Thus $N^{q,p}(\mathcal{R}) \subset \ker \Phi_p$. On the other hand, if $f \in A^q(\mathcal{R})\backslash N^{q,p}(\mathcal{R})$, there exists $\beta \in \mathcal{S}(\mathcal{R}, L^p)$ such that in the expansion (5.8) $a_\beta \neq 0$. Then $\Phi_p(f)(e_\beta) = a_\beta \|e_\beta\|_2^2 \neq 0$, showing $\ker \Phi_p = N^{q,p}(\mathcal{R})$.

For (*ii*), first note the direct sum representation

$$A^q(\mathcal{R}) = N^{q,p}(\mathcal{R}) \oplus G^{q,p}(\mathcal{R}). \tag{5.20}$$

In fact, $N^{q,p}(\mathcal{R}) \cap G^{q,p}(\mathcal{R}) = \{0\}$ holds since the sets are spanned by independent sets of monomials. If $f \in A^q(\mathcal{R})$, write

$$f = (f - P^\dagger f) + P^\dagger f.$$

Proposition 5.11 (*iii*) implies $\ker P^\dagger = N^{q,p}(\mathcal{R})$ and $\operatorname{Ran} P^\dagger = G^{q,p}(\mathcal{R})$. Therefore (5.20) holds. By (*i*), $\Phi_p : A^q(\mathcal{R}) \to A^p(\mathcal{R})'$ is surjective and $\ker \Phi_p = N^{q,p}(\mathcal{R})$. Thus (5.20) and Corollary 4.9 give

$$A^p(\mathcal{R})' \cong \frac{A^q(\mathcal{R})}{\ker(\Phi_p)} = \frac{A^q(\mathcal{R})}{N^{q,p}(\mathcal{R})} \cong G^{q,p}(\mathcal{R}),$$

as claimed.

For (*iii*), let $N^{q,p}(\mathcal{R})^\circ$ be the annihilator of $N^{q,p}(\mathcal{R})$:

$$N^{q,p}(\mathcal{R})^\circ = \{\lambda \in A^q(\mathcal{R})' : \lambda(f) = 0 \quad \forall f \in N^{q,p}(\mathcal{R})\}.$$

The decomposition (5.20) implies a natural isomorphism $N^{q,p}(\mathcal{R})^\circ = G^{q,p}(\mathcal{R})'$.[5] Proposition 5.7 implies that $N^{q,p}(\mathcal{R})^\circ$ can be identified with $\lambda \in A^q(\mathcal{R})'$ of the form

$$\lambda = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^p)} \lambda(e_\alpha) a_\alpha, \tag{5.21}$$

the square partial sums of the series converging in $A^q(\mathcal{R})'$. Then

$$N^{q,p}(\mathcal{R})^\circ \subset \overline{\operatorname{span}}_{A^q(\mathcal{R})'}\{a_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}.$$

On the other hand, if $\lambda \in \overline{\operatorname{span}}_{A^q(\mathcal{R})'}\{a_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}$, we have

$$\lambda = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^p)} c_\alpha a_\alpha$$

for complex c_α. Of course $\lambda \in A^q(\mathcal{R})'$. Since for $f \in N^{q,p}(\mathcal{R})$ we have

$$f = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^q)\setminus\mathcal{S}(\mathcal{R},L^p)} k_\alpha e_\alpha$$

for complex k_α, it follows from linearity and continuity of the coefficient functional a_α (see Proposition 5.1) that

$$\lambda(f) = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^p)} c_\alpha a_\alpha(f) = 0,$$

which implies $\lambda \in N^{q,p}(\mathcal{R})^\circ$. Thus,

$$N^{q,p}(\mathcal{R})^\circ = \overline{\operatorname{span}}_{A^q(\mathcal{R})'}\{a_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^p)\}. \tag{5.22}$$

The same analysis shows

$$G^{q,p}(\mathcal{R})^\circ = \overline{\operatorname{span}}_{A^q(\mathcal{R})'}\{a_\alpha : \alpha \in \mathcal{S}(\mathcal{R}, L^q)\setminus S(\mathcal{R}, L^p)\}. \tag{5.23}$$

(5.20) yields a direct sum decomposition of the dual spaces

$$A^q(\mathcal{R})' = N^{q,p}(\mathcal{R})^\circ \oplus G^{q,p}(\mathcal{R})^\circ.$$

Of course $N^{q,p}(\mathcal{R})^\circ \cap G^{q,p}(\mathcal{R})^\circ = \{0\}$ by (5.22) and (5.23). Now let $\lambda \in A^q(\mathcal{R})'$. Proposition 5.7 implies that

$$\lambda = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^q)} \lambda(e_\alpha) a_\alpha = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^q)\setminus\mathcal{S}(\mathcal{R},L^p)} \lambda(e_\alpha) a_\alpha + \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^p)} \lambda(e_\alpha) a_\alpha, \tag{5.24}$$

[5]The isomorphism between $N^{q,p}(\mathcal{R})^\circ$ and $G^{q,p}(\mathcal{R})'$ is induced by the restriction map $\Psi(\lambda) = \lambda|_{G^{q,p}(\mathcal{R})}$, where $\lambda \in N^{q,p}(\mathcal{R})^\circ$. This map is well-defined, linear, continuous, injective and surjective (since every functional on $G^{q,p}(\mathcal{R})$ can be extended to zero on $N^{q,p}(\mathcal{R})$). Hence, Ψ defines a topological isomorphism between $N^{q,p}(\mathcal{R})^\circ$ and $G^{q,p}(\mathcal{R})'$, based on the open mapping theorem [19].

the square partial sums of the series converging in $A^q(\mathcal{R})'$. The final step is to show the identity $\Phi_q(A^p(\mathcal{R})) = N^{q,p}(\mathcal{R})^\circ$. However if $\alpha \in \mathcal{S}(\mathcal{R}, L^p)$, a direct computation yields

$$a_\alpha = \frac{1}{\|e_\alpha\|_2^2} \cdot \Phi_q(e_\alpha). \tag{5.25}$$

In fact, since $\Phi_q(e_\alpha) \in A^q(\mathcal{R})'$, Proposition 5.7 implies

$$\Phi_q(e_\alpha) = \sum_{\beta \in \mathcal{S}(\mathcal{R},L^q)} \Phi_q(e_\alpha)(e_\beta)a_\beta = \sum_{\beta \in \mathcal{S}(\mathcal{R},L^q)} \left(\int_{\mathcal{R}} e_\beta(z)\overline{e_\alpha(z)}dV(z) \right) a_\beta = \|e_\alpha\|_2^2 a_\alpha.$$

If $f \in A^p(\mathcal{R})$, then by (5.8) we have the expansion

$$f = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^p)} a_\alpha(f)e_\alpha.$$

Similarly, for all $g \in N^{q,p}(\mathcal{R}) \subset A^q(\mathcal{R})$, we have the expansion

$$g = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^q) \setminus \mathcal{S}(\mathcal{R},L^p)} a_\alpha(g)e_\alpha.$$

Therefore $\Phi_q(f)(g) = 0$, which shows $\Phi_q(A^p(\mathcal{R})) \subset N^{q,p}(\mathcal{R})^\circ$. On the other hand, by (5.22) and (5.25) we can rewrite

$$N^{q,p}(\mathcal{R})^\circ = \overline{\operatorname{span}}_{A^q(\mathcal{R})'} \left\{ \frac{1}{\|e_\alpha\|_2^2} \Phi_q(e_\alpha) : \alpha \in \mathcal{S}(\mathcal{R}, L^p) \right\}.$$

Let $\lambda \in N^{q,p}(\mathcal{R})^\circ$. Then

$$\lambda = \sum_{\alpha \in \mathcal{S}(\mathcal{R},L^p)} \frac{c_\alpha}{\|e_\alpha\|_2^2} \Phi_q(e_\alpha),$$

with convergence in $A^q(\mathcal{R})'$. In other words, setting

$$S_N\lambda := \sum_{|\alpha|_\infty \le N} \frac{c_\alpha}{\|e_\alpha\|_2^2} \Phi_q(e_\alpha),$$

we have $S_N\lambda \to \lambda$ in $A^q(\mathcal{R})'$ as $N \to \infty$. Since each $S_N\lambda$ lies in $\Phi_q(A^p(\mathcal{R}))$ and Φ_q has closed range (see Corollary 4.9), we conclude that $\lambda \in \Phi_q(A^p(\mathcal{R}))$. It shows $N^{q,p}(\mathcal{R})^\circ \subset \Phi_q(A^p(\mathcal{R}))$. Thus, $\Phi_q(A^p(\mathcal{R})) = N^{q,p}(\mathcal{R})^\circ$ and this implies (5.19). $\square$

CHAPTER 6

Generalized Hartogs Triangles

The main result in [10] is that the Bergman projection $\mathcal{B}_{\mathbb{H}_\gamma} = \mathcal{B}_\gamma$ is "defective" as an L^p operator and, moreover, whether $\gamma \in \mathbb{Q}$ or not determines the extent of its deficiency. The precise result is

Theorem 6.1 ([10])**.** *Let $\mathbb{H}_\gamma$ be given by (4.1).*

(i) Let $\gamma = \dfrac{m}{n}$, where $m, n \in \mathbb{Z}^+$ with $\gcd(m,n) = 1$.

Then $\mathcal{B}_\gamma : L^p(\mathbb{H}_\gamma) \to A^p(\mathbb{H}_\gamma)$ boundedly if and only if $p \in \left(\dfrac{2m+2n}{m+n+1}, \dfrac{2m+2n}{m+n-1}\right)$.

(ii) Let γ be irrational.

Then $\mathcal{B}_\gamma : L^p(\mathbb{H}_\gamma) \to A^p(\mathbb{H}_\gamma)$ boundedly if and only if $p = 2$.

Focus on $\mathbb{H}_{m/n}$, $\frac{m}{n} \in \mathbb{Q}$, and integrability exponents $p \geq 2$. The proof of (i) in Theorem 6.1 actually shows more: the Bergman projection on $\mathbb{H}_{m/n}$ fails to generate A^p functions from L^p data for certain p. To apply Theorems 4.10 and 4.13, operators are needed that create A^p functions for p outside the range in Theorem 6.1 (i).

The sub-Bergman projections defined in Section 5.6 are such operators. Verification of this is done over several sections, leading to

Theorem 6.2. *Let $\mathbb{H}_{m/n}$, where $m, n \in \mathbb{Z}^+$ with $\gcd(m,n) = 1$, be given by (4.1). For each $p \geq 2$, the sub-Bergman projection $\widetilde{\mathcal{B}^p} : L^p(\mathbb{H}_{m/n}) \to A^p(\mathbb{H}_{m/n})$ satisfies*

(i) $|\widetilde{\mathcal{B}^p}|$ is bounded on $L^p(\mathbb{H}_{m/n})$.

(ii) $\widetilde{\mathcal{B}^p} h = h \quad \forall h \in A^p(\mathbb{H}_{m/n})$.

Theorem 6.2 is proved in Section 6.3. If q is conjugate to p, $|\widetilde{\mathcal{B}^p}|$ also maps $L^q(\mathbb{H}_{m/n})$ into $A^q(\mathbb{H}_{m/n})$ boundedly, but the map is no longer surjective, see Remark 6.17. An explicit description of the set of L^p-allowable multi-indices plays a crucial role in the proof of Theorem 6.2.

6.1 Integrability and orthogonality

6.1.1 Holomorphic monomials in $L^p(\mathbb{H}_{m/n})$

Let $\mathbb{H}_{m/n}$, $m,n \in \mathbb{Z}^+$ with $\gcd(m,n) = 1$, be a fixed power-generalized Hartogs triangle throughout the section.

Lemma 6.3. *Let $p \in [1,\infty)$. The set of L^p-allowable multi-indices is*

$$\mathcal{S}(\mathbb{H}_{m/n}, L^p) = \left\{\alpha = (\alpha_1,\alpha_2) : \alpha_1 \geq 0,\ n\alpha_1 + m\alpha_2 \geq \left\lfloor -\frac{2}{p}(m+n)+1 \right\rfloor \right\}. \tag{6.1}$$

For $\alpha \in \mathcal{S}(\mathbb{H}_{m/n}, L^p)$,

$$\|e_\alpha\|^p_{L^p(\mathbb{H}_{m/n})} = \frac{4m\pi^2}{n(p\alpha_1+2)^2 + m(p\alpha_1+2)(p\alpha_2+2)}. \tag{6.2}$$

Proof. Recall from Definition 4.1 that

$$\mathbb{H}_{m/n} = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^{m/n} < |z_2| < 1\} = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| < |z_2|^{n/m} < 1\}.$$

Note there are points in $\mathbb{H}_{m/n}$ where $z_1 = 0$, which forces $\alpha_1 \geq 0$. Computing in polar coordinates

$$\begin{aligned}\|e_\alpha\|^p_{L^p(\mathbb{H}_{m/n})} &= \int_{\mathbb{H}_{m/n}} |z^\alpha|^p dV(z) \\ &= \int_{|z_2|<1} \int_{|z_1|<|z_2|^{n/m}<1} |z_1|^{p\alpha_1}|z_2|^{p\alpha_2} dA(z_1)dA(z_2).\end{aligned}$$

For $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, we have $|z_1|^{p\alpha_1} = r_1^{p\alpha_1}$, $|z_2|^{p\alpha_2} = r_2^{p\alpha_2}$, $dA(z_1) = r_1dr_1d\theta_1$ and $dA(z_2) = r_2dr_2d\theta_2$. Then

$$\begin{aligned}\|e_\alpha\|^p_{L^p(\mathbb{H}_{m/n})} &= \int_0^{2\pi}\int_0^1\int_0^{2\pi}\int_0^{r_2^{n/m}} r_1^{p\alpha_1}r_2^{p\alpha_2} r_1dr_1d\theta_2r_2dr_2d\theta_2 \\ &= 4\pi^2\int_0^1 r_2^{p\alpha_2+1}\left(\int_0^{r_2^{n/m}} r_1^{p\alpha_1+1}dr_1\right)dr_2 \\ &= 4\pi^2\int_0^1 r_2^{p\alpha_2+1}\left(\frac{r_2^{\frac{n}{m}(p\alpha_1+2)}}{p\alpha_1+2}\right)dr_2 \\ &= \frac{4\pi^2}{p\alpha_1+2}\int_0^1 r_2^{p\alpha_2+1+\frac{n}{m}(p\alpha_1+2)}dr_2.\end{aligned}$$

This integral converges if and only if the exponent $p\alpha_2 + 1 + \frac{n}{m}(p\alpha_1+2) > -1$. From here, (6.2) easily follows:

$$\begin{aligned}\|e_\alpha\|^p_{L^p(\mathbb{H}_{m/n})} &= \frac{4\pi^2}{p\alpha_1+2}\int_0^1 r_2^{p\alpha_2+1+\frac{n}{m}(p\alpha_1+2)}dr_2 \\ &= \frac{4\pi^2}{p\alpha_1+2}\cdot\frac{1}{p\alpha_2+2+\frac{n}{m}(p\alpha_1+2)} \\ &= \frac{4m\pi^2}{n(p\alpha_1+2)^2+m(p\alpha_1+1)(p\alpha_2+2)}.\end{aligned}$$

To see (6.1), notice that since $\alpha_1, \alpha_2, m, n \in \mathbb{Z}$,

$$\begin{aligned} p\alpha_2 + 2 + \frac{n}{m}(p\alpha_1 + 2) > 0 &\Leftrightarrow m(p\alpha_2 + 2) + n(p\alpha_1 + 2) > 0 \\ &\Leftrightarrow mp\alpha_2 + 2m + np\alpha_1 + 2n > 0 \\ &\Leftrightarrow p(n\alpha_1 + m\alpha_2) + 2(m+n) > 0 \\ &\Leftrightarrow p(n\alpha_1 + m\alpha_2) > -2(m+n) \\ &\Leftrightarrow n\alpha_1 + m\alpha_2 > -\frac{2}{p}(m+n) \\ &\Leftrightarrow n\alpha_1 + m\alpha_2 \geq \left\lfloor -\frac{2}{p}(m+n) + 1 \right\rfloor . \end{aligned}$$

□

Let us examine the sets $\mathcal{S}(\mathbb{H}_{m/n}, L^p)$ as functions of $p \in [1, \infty)$. The floor function in (6.1) shows that

$$\mathcal{S}(\mathbb{H}_{m/n}, L^p) = \mathcal{S}(\mathbb{H}_{m/n}, L^{p \pm \varepsilon})$$

if $\varepsilon > 0$ is small, unless $-\frac{2}{p}(m+n) + 1 \in \mathbb{Z}$. The lattice points in $\mathcal{S}(\mathbb{H}_{m/n}, L^p)$ are therefore stable except for certain exceptional p. Call these exceptional values *thresholds*. Note that $\mathcal{S}(\mathbb{H}_{m/n}, L^t) \subset \mathcal{S}(\mathbb{H}_{m/n}, L^s)$ if $s < t$, so $\mathcal{S}(\mathbb{H}_{m/n}, L^p)$ jumps to a smaller set of lattice points as p increases past a threshold value.

Example 6.4. For $m = n = 1$, consider the function

$$C(p) = \left\lfloor -\frac{4}{p} + 1 \right\rfloor .$$

The following table lists some values of $C(p)$:

p	$-\frac{4}{p} + 1$	$C(p)$
1	-3	-3
5/4	-2.2	-3
4/3	-2	-2
2	-1	-1
3	≈ -0.333	-1
4	0	0
5	0.2	0

Table 6.1: Values of $C(p)$ for selected $p \in [1, 5]$.

Note that the value of $C(p)$ changes *discretely* whenever $-\frac{4}{p} + 1$ crosses an integer. As p increases, $C(p)$ approaches zero from below, making the condition $\alpha_1 + \alpha_2 \geq C(p)$ *more restrictive.* The thresholds here occur at $p = 1$, $p = 4/3$, $p = 2$ and $p = 4$.[1]

[1]Intuitively, a threshold can be thought of as a doorway between two rooms. Crossing it signifies the transition from one room (set) to another, much like how a threshold marks the point at which a set changes from one condition to another.

The next result makes this stabilization precise and shows there are only a finite number of thresholds for a given $\mathbb{H}_{m/n}$.

Proposition 6.5. *There are exactly $2m+2n$ thresholds associated to $\mathbb{H}_{m/n}$. They occur when $p_k = \dfrac{2m+2n}{1-k}$ for $k \in \{1-2m-2n, 2-2m-2n, \dots, -1, 0\}$.*

Consider the corresponding partition of $[1,\infty)$

$$[1,\infty) = \bigcup_{k=1-2m-2n}^{0} [p_k, p_{k+1}), \qquad p_k = \frac{2m+2n}{1-k}. \tag{6.3}$$

Then for any $p \in [p_k, p_{k+1})$,

$$S(\mathbb{H}_{m/n}, L^p) = \{(\alpha_1,\alpha_2) : \alpha_1 \ge 0,\ n\alpha_1 + m\alpha_2 \ge k\} = \mathcal{S}(\mathbb{H}_{m/n}, L^{p_k}), \tag{6.4}$$

and

$$S(\mathbb{H}_{m/n}, L^\infty) = \{(\alpha_1,\alpha_2) : \alpha_1 \ge 0,\ n\alpha_1 + m\alpha_2 \ge 0\} = \mathcal{S}(\mathbb{H}_{m/n}, L^{2m+2n}). \tag{6.5}$$

Proof. Define $\ell_{m,n}(p) := -\frac{2}{p}(m+n)+1$, $p \in [1,\infty)$. The function $\ell_{m,n}(p)$ is increasing and takes values in the interval $[1-2m-2n, 1)$. Note $\ell_{m,n}(p) = k \in \mathbb{Z}$ if and only if $p = \frac{2m+2n}{1-k}$.

Rewrite the partition in (6.3):

$$[1,\infty) = \bigcup_k \left[\frac{2m+2n}{1-k}, \frac{2m+2n}{-k}\right) := \bigcup_k J_k,$$

where the union is taken over $k \in \{1-2m-2n, 2-2m-2n, \dots, -1, 0\}$. Suppose $p, p' \in J_k$ for some J_k. Since

$$J_k = \left[\frac{2m+2n}{1-k}, \frac{2m+2n}{-k}\right) = [p_k, p_{k+1})$$

and $\ell_{m,n}$ is (strictly) increasing, we get

$$\ell_{m,n}(p), \ell_{m,n}(p') \in [\ell_{m,n}(p_k), \ell_{m,n}(p_{k+1})) = [k, k+1),$$

which in turn implies $\lfloor \ell_{m,n}(p) \rfloor = k = \lfloor \ell_{m,n}(p') \rfloor$, and shows (6.4) holds (just take $p' = p_k$).

To see (6.5), let $\alpha = (\alpha_1, \alpha_2) \in \mathcal{S}(\mathbb{H}_{m/n}, L^{2m+2n})$. Equation (6.1) says that $\alpha_1 \ge 0$ and $n\alpha_1 + m\alpha_2 \ge 0$. Since $|z_1|^m < |z_2|^n < 1$ if $z \in \mathbb{H}_{m/n}$, it follows that

$$|z_1^{\alpha_1} z_2^{\alpha_2}|^m = \left|\frac{z_1^m}{z_2^n}\right|^{\alpha_1} |z_2|^{n\alpha_1 + m\alpha_2} < 1,$$

which says $\alpha \in \mathcal{S}(\mathbb{H}_{m/n}, L^\infty)$. □

Example 6.6. Consider the domain $\mathbb{H}_2$. Proposition 6.5 says there are 6 thresholds associated to $\mathbb{H}_2$:

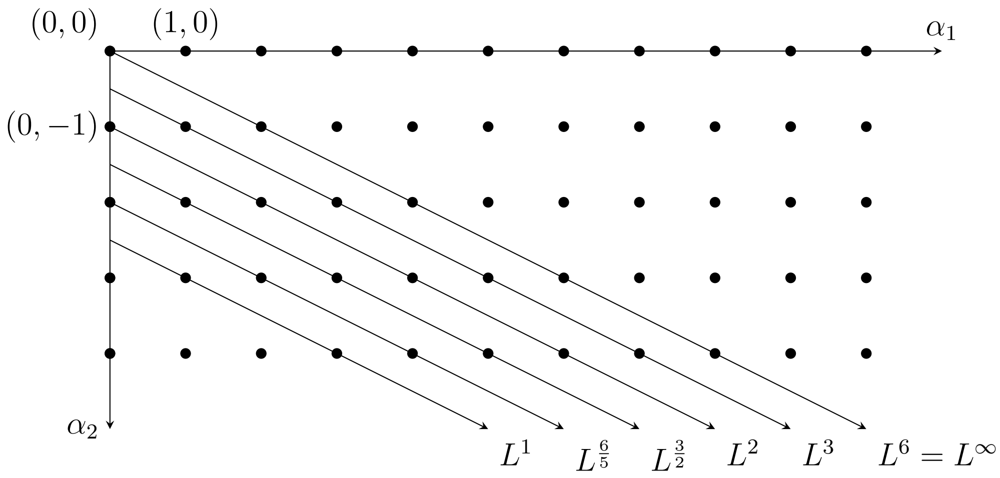

Figure 6.1: Thresholds associated to $\mathbb{H}_2$.

The lines come from (6.4). The lattice points on the first five lines represent L^p-integrable monomials for all p up to but not including the p value of the next line, while the lattice points on and above the $p = 6$ line correspond to bounded monomials on $\mathbb{H}_2$.

Choose $\beta \in \mathcal{S}\left(\mathbb{H}_2, L^{\frac{3}{2}}\right)$ and $\delta \in \mathcal{S}(\mathbb{H}_2, L^2)$ with $\beta \neq \delta$. The first observation is that the L^2 pairing

$$\langle e_\beta, e_\delta \rangle_{\mathbb{H}_2} \tag{6.6}$$

is defined. Note that $\frac{3}{2}$ and 2 are not conjugate. If β also belonged to $\mathcal{S}(\mathbb{H}_2, L^2)$, Hölder's inequality would imply (6.6) is finite. Thus assume β lies on the line $L^{\frac{3}{2}}$ in the diagram. Proposition (6.5) says $e_\beta \in L^t(\mathbb{H}_2)$ for all $\frac{3}{2} \le t < 2$ and that $e_\delta \in L^s(\mathbb{H}_2)$ for all $2 \le s < 3$. There are infinitely many pairs of conjugate exponents in these two intervals[2], so once again Hölder's inequality shows (6.6) is defined. The second observation is that (6.6) equals 0. This follows since $\beta \neq \delta$ and the monomials $\{e_\alpha\}$ are orthogonal on $\mathbb{H}_2$.

The same conclusion holds for any multi-indices $\beta \neq \delta$ chosen with $\beta \in \mathcal{S}\left(\mathbb{H}_2, L^{\frac{6}{5}}\right)$ (respectively $\beta \in \mathcal{S}(\mathbb{H}_2, L^1)$) and $\delta \in \mathcal{S}(\mathbb{H}_2, L^3)$ (respectively $\delta \in \mathcal{S}(\mathbb{H}_2, L^6)$). The following corollary of Proposition 6.5 gives the general version:

Corollary 6.7. *Let $\gamma = \frac{m}{n}$, $k \in \{1 - 2m - 2n, 2 - 2m - 2n, \ldots, -1, 0\}$, and define $j(k) := 1 - k - 2m - 2n$. Set*

$$p_k = \frac{2m + 2n}{1 - k}, \qquad p_{j(k)} = \frac{2m + 2n}{1 - j(k)} = \frac{2m + 2n}{2m + 2n + k}.$$

Then for any choice of multi-indices $\beta \in \mathcal{S}(\mathbb{H}_\gamma, L^{p_k})$ and $\delta \in \mathcal{S}(\mathbb{H}_\gamma, L^{p_{j(k)}})$ with $\beta \neq \delta$, the inner product

$$\langle e_\beta, e_\delta \rangle_{\mathbb{H}_\gamma} = 0.$$

[2] Examples of conjugate exponent pairs (p, q) with $p < 2$ and $q < 3$: $\left(\frac{5}{3}, \frac{5}{2}\right)$, $\left(\frac{8}{5}, \frac{8}{3}\right)$, and $\left(\frac{7}{4}, \frac{7}{3}\right)$. In each case, the relation $\frac{1}{p} + \frac{1}{q} = 1$ holds.

Proof. First, observe that

$$\begin{aligned}\frac{1}{p_k}+\frac{1}{p_{j(k)}} &= \frac{1-k}{2m+2n}+\frac{2m+2n+k}{2m+2n}\\ &= \frac{2m+2n+1}{2m+2n}\\ &= 1+\frac{1}{2m+2n}>1.\end{aligned}$$

However, Proposition (6.5) says $\beta \in \mathcal{S}(\mathbb{H}_\gamma, L^t)$ for all $t < p_{k+1}$ and that $\delta \in \mathcal{S}(\mathbb{H}_\gamma, L^s)$ for all $s < p_{j(k-1)}$. Since

$$\begin{aligned}\frac{1}{p_{k+1}}+\frac{1}{p_{j(k-1)}} &= \frac{-k}{2m+2n}+\frac{2m+2n+k-1}{2m+2n}\\ &= \frac{2m+2n-1}{2m+2n}\\ &= 1-\frac{1}{2m+2n}<1,\end{aligned}$$

there are t, s with $t \in [p_k, p_{k+1})$ and $s \in [p_{j(k)}, p_{j(k-1)})$ such that $\frac{1}{t}+\frac{1}{s}=1$. The remainder of the proof follows from the same argument given above. □

Remark 6.8. Corollary 6.7 is nontrivial only because p_k and $p_{j(k)}$ are not conjugate: indeed, $\frac{1}{p_k}+\frac{1}{p_{j(k)}}>1$. No analogue of Corollary 6.7 exists for $\mathbb{H}_\gamma$, $\gamma \in \mathbb{Q}$.

6.2 Constructing A^p fuctions

Construction of the sub-Bergman kernels and projection operators is based on the decomposition of monomials in Proposition 6.5.

6.2.1 Type-A operators on $\mathbb{H}_{m/n}$

A lemma from [10] is recalled that relates estimates on a class of kernels defined on $\mathbb{H}_{m/n} \times \mathbb{H}_{m/n}$ to mapping properties of the associated integral operators. If $\Omega \subset \mathbb{C}^n$ is a domain and K is an almost everywhere positive, measurable function on $\Omega \times \Omega$, let $\mathcal{K}$ denote the integral operator associated to K:

$$\mathcal{K}f(z)=\int_\Omega K(z,w)f(w)dV(w).$$

Definition 6.9. For $A \in \mathbb{R}_{>0}$, call $\mathcal{K}$ an operator of type-A on $\mathbb{H}_{m/n}$ if its kernel satisfies

$$K(z_1,z_2,w_1,w_2) \lesssim \frac{|z_2 w_2|^A}{|1-z_2\overline{w_2}|^2|z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2},$$

for a constant independent of $(z,w) \in \mathbb{H}_{m/n} \times \mathbb{H}_{m/n}$.

The basic L^p mapping result is

Proposition 6.10 ([10]). *If $\mathcal{K}$ is an operator of type-A on $\mathbb{H}_{m/n}$, then $\mathcal{K} : L^p(\mathbb{H}_{m/n}) \to L^p(\mathbb{H}_{m/n})$ boundedly if*

$$\frac{2m+2n}{Am+2n+2m-2nm} < p < \frac{2n+2m}{2nm-Am}, \tag{6.7}$$

whenever

$$n(2-m^{-1})-1 < A < 2n. \tag{6.8}$$

Remark 6.11. The range of L^p boundedness as A tends to the upper and lower bounds in (6.8) is significant. As $A \to$ 2n, the interval in (6.7) increases to $(1,\infty)$. Thus, an operator of type-$2n$ on $\mathbb{H}_{m/n}$ is L^p bounded for all $1 < p < \infty$. In the other direction, note the left endpoint $n(2-m^{-1})-1 \geq 0$ for all choices of $m, n \in \mathbb{Z}^+$. As A decreases to this endpoint, the interval in (6.7) collapses towards the point $\{2\}$. However an operator of type $n(2-m^{-1})-1$ is not necessarily bounded on any L^p space, including L^2.

6.2.2 Splitting monomials by integrability class

Abbreviate the L^p-allowable multi-indices given by Proposition 6.5:

$$\mathcal{S}(\mathbb{H}_{m/n}, L^{p_k}) = \{(\alpha_1,\alpha_2) : \alpha_1 \geq 0,\ n\alpha_1 + m\alpha_2 \geq k\} := S_k,$$

where $p_k = \dfrac{2m+2n}{1-k}$ and $k \in \{1-2m-2n, 2-2m-2n, \dots, -1, 0\}$.

The L^p sub-Bergman kernels for $p \geq 2$ are defined

$$\widetilde{\mathbb{B}^p}(z,w) := \sum_{\alpha \in S_k} \frac{e_\alpha(z)\overline{e_\alpha(w)}}{\|e_\alpha\|_2^2}, \qquad p \in [p_k, p_{k+1}). \tag{6.9}$$

The stabilization in Proposition 6.5 accounts for the identical definition of $\widetilde{\mathbb{B}^p}(z,w)$ for all $p \in [p_k, p_{k+1})$. Note that only S_k for $k \in [1-m-n, 0]$ occurs in any of the kernels (6.9), since $p \geq 2$. Proposition 6.5 also says $S_0 = \mathcal{S}(\mathbb{H}_{m/n}, L^\infty)$. Consequently, denote the sum

$$\sum_{\alpha \in \mathcal{S}_0} \frac{e_\alpha(z)\overline{e_\alpha(w)}}{\|e_\alpha\|_2^2} := \widetilde{\mathbb{B}^\infty}(z,w) \tag{6.10}$$

and call $\widetilde{\mathbb{B}^\infty}(z,w)$ the L^∞ sub-Bergman kernel on $\mathbb{H}_{m/n}$. The sum definig $\widetilde{\mathbb{B}^\infty}(z,w)$ consists only of L^∞ monomials.

As an aid to calculating the sums (6.9) and (6.10), define

$$s_k = \{\alpha : \alpha_1 \geq 0,\ n\alpha_1 + m\alpha_2 = k\}, \tag{6.11}$$

and consider the functions

$$b^{p_k}(z,w) = \sum_{\alpha \in s_k} \frac{e_\alpha(z)\overline{e_\alpha(w)}}{\|e_\alpha\|_2^2}. \tag{6.12}$$

Orthogonality of $\{e_\alpha\}$ yields the decomposition

$$\sum_{j=k}^{-1} b^{p_j}(z,w) + \widetilde{\mathbb{B}^{\infty}}(z,w) = \widetilde{\mathbb{B}^{p_k}}(z,w) \tag{6.13}$$

for negative integers $k \geq 1-m-n$.[3]

6.2.3 Analyzing the sub-Bergman kernels

The first step is to obtain an upper bound on b^{p_k} connected to Definition 6.9.

Proposition 6.12. *The following estimate holds for all $z, w \in \mathbb{H}_{m/n}$:*

$$|b^{p_k}(z,w)| \lesssim \frac{|z_2\overline{w_2}|^{2n+\frac{k}{m}}}{|z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2}. \tag{6.14}$$

Recall $k<0$ in (6.14), $k \in \{1-m-n, 2-m-n, \dots, -1\}$.

Proof. Since $\gcd(m,n)=1$, there is a unique pair (β_1,β_2) with $0 \leq \beta_1 \leq m-1$ and $n\beta_1 + m\beta_2 = k$.[4] Notice that the subsequent lattice points on this line are of the form $(\beta_1 +$

[3]Note that $\widetilde{\mathbb{B}^{p_k}}$ is the sub-Bergman kernel associated to the sub-Bergman projection $\widetilde{\mathcal{B}^{p_k}}$ from $L^2(\mathbb{H}_{m/n})$ onto

$$\overline{\text{span}}_{A^2(\mathbb{H}_{m/n})}\{e_\alpha : \alpha \in S_k\}.$$

Set $\mathcal{G}_j := \overline{\text{span}}_{A^2(\mathbb{H}_{m/n})}\{e_\alpha : \alpha \in s_j\}$ for $j \in \{k,\dots,-1\}$ and $\mathcal{G}_0 := \overline{\text{span}}_{A^2(\mathbb{H}_{m/n})}\{e_\alpha : \alpha \in S_0\}$. Since the index set S_k decomposes as a disjoint union

$$S_k = s_k \sqcup \cdots \sqcup s_{-1} \sqcup S_0,$$

we can write

$$\overline{\text{span}}_{A^2(\mathbb{H}_{m/n})}\{e_\alpha : \alpha \in S_k\} = \mathcal{G}_k \oplus \cdots \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0.$$

Each $\mathcal{G}_j$ is a closed subspace of $A^2(\mathbb{H}_{m/n})$, hence a Hilbert space. Therefore, the orthogonal projection $L^2(\mathbb{H}_{m/n}) \to \mathcal{G}_j$ is well-defined and represented by integration against the kernel b^{p_j}. In particular, $b^{p_0} = \widetilde{\mathbb{B}^{\infty}}$. Orthogonality of $\{e_\alpha\}$ implies the subspaces $\mathcal{G}_j$ are pairwise orthogonal, from which it follows that

$$\widetilde{\mathbb{B}^{p_k}}(z,w) = b^{p_k}(z,w) + \cdots + b^{p_0}(z,w).$$

This establishes the decomposition (6.13).

[4]Since $\gcd(m,n)=1$ and $k \in \mathbb{Z}$, the equation

$$n\beta_1 + m\beta_2 = k$$

admits integer solutions. Considering the equation modulo m, we have

$$n\beta_1 \equiv k \pmod{m}.$$

Furthermore, n admits a multiplicative inverse modulo m, denoted n^{-1}. Multiplying both sides by n^{-1} yields

$$\beta_1 \equiv n^{-1}k \pmod{m}.$$

This means β_1 differs from $n^{-1}k$ by a multiple of m, which can be written as

$$\beta_1 = n^{-1}k + tm, \quad t \in \mathbb{Z}.$$

$jm, \beta_2 - jn)$. Equation (6.2) says for all $\alpha \in \mathcal{S}(\mathbb{H}_{m/n}, L^2)$,

$$\begin{aligned}\|e_\alpha\|_2^2 &= \frac{4m\pi^2}{n(2\alpha_1+2)^2 + m(2\alpha_1+2)(2\alpha_2+2)} \\ &= \frac{m\pi^2}{n(\alpha_1+1)^2 + m(\alpha_1+1)(\alpha_2+1)} \\ &= \frac{m\pi^2}{(\alpha_1+1)(n(\alpha_1+1)+m(\alpha_2+1))} \\ &= \frac{m\pi^2}{(\alpha_1+1)(n\alpha_1+m\alpha_2+m+n)}.\end{aligned} \tag{6.15}$$

In what follows, let $s := z_1\overline{w_1}$ and $t := z_2\overline{w_2}$. Definition (6.12) and (6.15) imply

$$\begin{aligned}b^{p_k}(z,w) &= \sum_{\alpha\in s_k} \frac{e_\alpha(z)\overline{e_\alpha(w)}}{\|e_\alpha\|_2^2} \\ &= \frac{m+n+k}{m\pi^2} \sum_{\alpha\in s_k} (\alpha_1+1)s^{\alpha_1}t^{\alpha_2} \\ &= \frac{m+n+k}{m\pi^2} \sum_{j=0}^{\infty} (\beta_1+jm+1)s^{\beta_1+jm}t^{\beta_2-jn} \\ &= \frac{m+n+k}{m\pi^2} \cdot t^{k/m} \sum_{j=0}^{\infty} (\beta_1+jm+1)s^{\beta_1+jm}(t^{-n/m})^{\beta_1+jm} \\ &= \frac{m+n+k}{m\pi^2} \cdot t^{k/m} \sum_{j=0}^{\infty} (\beta_1+jm+1)u^{\beta_1+jm},\end{aligned} \tag{6.16}$$

By the division algorithm,

$$n^{-1}k = qm + r, \quad 0 \le r < m.$$

Choosing $t = -q$ gives

$$\beta_1 = r,$$

which is the unique integer in $[0, m-1]$ satisfying the congruence. This choice ensures

$$\beta_2 = \frac{k - n\beta_1}{m}$$

is an integer, completing the unique integer solution (β_1, β_2) of the original equation.

where $u := st^{-n/m}$. Writing this series in closed form yields

$$\begin{aligned}
\frac{m+n+k}{m\pi^2}\cdot t^{k/m}\sum_{j=0}^{\infty}(\beta_1+jm+1)u^{\beta_1+jm} &= \frac{m+n+k}{m\pi^2}\cdot t^{k/m}u^{\beta_1}\sum_{j=0}^{\infty}(\beta_1+jm+1)u^{jm}\\
&= \frac{m+n+k}{m\pi^2}\cdot t^{k/m}u^{\beta_1}\left((\beta_1+1)\sum_{j=0}^{\infty}u^{jm}\right)+\left(m\sum_{j=0}^{\infty}ju^{jm}\right)\\
&= \frac{m+n+k}{m\pi^2}\cdot t^{k/m}u^{\beta_1}\left((\beta_1+1)\cdot\frac{1}{1-u^m}\right)+\left(m\cdot\frac{u^m}{(1-u^m)^2}\right)\\
&= \frac{m+n+k}{m\pi^2}\cdot t^{k/m}u^{\beta_1}\cdot\frac{(\beta_1+1)+(m-\beta_1-1)u^m}{(1-u^m)^2}\\
&= \frac{m+n+k}{m\pi^2}\cdot t^{k/m}(st^{-n/m})^{\beta_1}\cdot\frac{(\beta_1+1)+(m-\beta_1-1)(st^{-n/m})^m}{(1-(st^{-n/m})^m)^2}\\
&= \frac{m+n+k}{m\pi^2}\cdot s^{\beta_1}t^{\beta_2}\cdot\frac{(\beta_1+1)t^{2n}+(m-\beta_1-1)s^mt^n}{(t^n-s^m)^2}.
\end{aligned}$$

Noting that $|s|^m < |t|^n$, the bound (6.14) follows:

$$\begin{aligned}
|b^{p_k}(z,w)| &\le \left|\frac{m+n+k}{m\pi^2}\right|\cdot|s|^{\beta_1}|t|^{\beta_2}\cdot\frac{|\beta_1+1||t|^{2n}+|m-\beta_1-1||s|^m|t|^n}{|t^n-s^m|^2}\\
&< \left|\frac{m+n+k}{m\pi^2}\right|\cdot|t|^{(\beta_1 n)/m}|t|^{(k-n\beta_1)/m}\cdot\frac{|\beta_1+1||t|^{2n}+|m-\beta_1-1||t|^{2n}}{|t^n-s^m|^2}\\
&< \left|\frac{m+n+k}{m\pi^2}\right|\cdot|t|^{k/m}\cdot\frac{(|\beta_1+1|+|m-\beta_1-1|)|t|^{2n}}{|t^n-s^m|^2}\\
&< C_{m,n,k}\frac{|t|^{2n+k/m}}{|t^n-s^m|^2}\\
&= C_{m,n,k}\frac{|z_2\overline{w_2}|^{2n+\frac{k}{m}}}{|z_2^n\overline{w_2}^n-z_1^m\overline{w_1}^m|^2}.
\end{aligned}$$

□

Let B^{p_k} be the integral operator

$$\mathrm{B}^{p_k}(f)(z) := \int_{\mathbb{H}_{m/n}} b^{p_k}(z,w)f(w)dV(w). \tag{6.17}$$

The operator B^{p_k} is the orthogonal projection from $L^2(\mathbb{H}_{m/n}) \to \overline{\text{span}}_{L^2}\{e_\alpha : \alpha \in s_k\}$. Note that each s_k is a set of points in the lattice point diagram lying on a single line.

Corollary 6.13. *Let $p_k = \frac{2m+2n}{1-k}$ for each integer $1-m-n \le k \le -1$ and q_k be conjugate to*

p_k.[5] *The projection* B^{p_k} *is an operator of type-A for* $A = 2n + \frac{k}{m}$. *Thus,* B^{p_k} *is* L^p*-bounded for*

$$p \in \left(\frac{2n+2m}{2n+2m+k}, \frac{2n+2m}{-k} \right) = (q_{k+1}, p_{k+1}). \tag{6.18}$$

Proof. Set $A = 2n + \frac{k}{m}$ in Proposition 6.10. □

The second step is to show the kernel $\widetilde{\mathbb{B}^{\infty}}(z,w)$ satisfies bounds related to Definition 6.9 and is more involved.

Proposition 6.14. *The* L^∞ *sub-Bergman kernel on* $\mathbb{H}_{m/n}$ *satisfies*

$$|\widetilde{\mathbb{B}^{\infty}}(z,w)| \lesssim \frac{|z_2\overline{w_2}|^{2n}}{|1 - z_2\overline{w_2}|^2 |z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2}. \tag{6.19}$$

Proof. Recall the description of $\mathcal{S}(\mathbb{H}_{m/n}, L^\infty)$ given by (6.5) and let $r \in \{0, 1, \ldots, m-1\}$. Since $\gcd(m,n) = 1$, there is a unique (α_1, α_2) with both $n\alpha_1 + m\alpha_2 = r$ and $0 \le \alpha_1 \le m-1$. Set this $\alpha_1 = \sigma(r)$. The function σ is a permutation of the set $\{0, 1, \ldots, m-1\}$ with $\sigma(0) = 0$.

Each $\alpha \in \mathcal{S}(\mathbb{H}_{m/n}, L^\infty) = \{(\alpha_1, \alpha_2) : \alpha_1 \ge 0,\ n\alpha_1 + m\alpha_2 \ge 0\}$ can be uniquely described by a line of the form $n\alpha_1 + m\alpha_2 = k$ and an α_1 value. Again letting $r \in \{0, 1, \ldots, m-1\}$, parametrize k and α_1 by

$$n\alpha_1 + m\alpha_2 = md + r, \qquad d = 0, 1, \ldots$$

$$\alpha_1 = mj + \sigma(r), \qquad j = 0, 1, \ldots$$

[5] Since $p_k = \frac{2m+2n}{1-k}$, we want to find q_k such that $\frac{1}{p_k} = \frac{1}{q_k} = 1$. We have:

$$\begin{aligned} \frac{1}{p_k} + \frac{1}{q_k} = 1 &\Leftrightarrow \frac{1-k}{2m+2n} + \frac{1}{q_k} = 1 \\ &\Leftrightarrow \frac{1}{q_k} = 1 - \frac{1-k}{2m+2n} \\ &\Leftrightarrow \frac{1}{q_k} = \frac{2m+2n-1+k}{2m+2n} \\ &\Leftrightarrow q_k = \frac{2m+2n}{2m+2n-1+k}. \end{aligned}$$

For ease of notation set $s = z_1\overline{w_1}$, $t = z_2\overline{w_2}$. From equations (6.10) and (6.15),

$$\begin{aligned}
\widetilde{\mathbb{B}^{\infty}}(z,w) &= \sum_{\alpha\in\mathcal{S}(\mathbb{H}_{m/n},L^{\infty})} \frac{e_\alpha(z)\overline{e_\alpha(w)}}{\|e_\alpha\|_2^2} \\
&= \frac{1}{m\pi^2}\sum_{\alpha\in\mathcal{S}(\mathbb{H}_{m/n},L^{\infty})} (\alpha_1+1)(n\alpha_1+m\alpha_2+m+n)s^{\alpha_1}t^{\alpha_2} \\
&= \frac{1}{m\pi^2}\sum_{r=0}^{m-1}\sum_{d,j=0}^{\infty}(mj+\sigma(r)+1)(md+r+m+n)s^{mj+\sigma(r)}t^{d+\frac{r}{m}-nj-\frac{n}{m}\sigma(r)} \\
&= \frac{1}{m\pi^2}\sum_{r=0}^{m-1}u^{\sigma(r)}t^{\frac{r}{m}}\left(\sum_{j=0}^{\infty}(mj+\sigma(r)+1)u^{mj}\right)\left(\sum_{d=0}^{\infty}(md+r+m+n)t^d\right) \\
&:= \frac{1}{m\pi^2}\sum_{r=0}^{m-1}u^{\sigma(r)}t^{\frac{r}{m}}I_r(u)J_r(t),
\end{aligned} \tag{6.20}$$

where we have introduced the new variable $u = st^{-n/m}$. Note both $|t| < 1$ and $|u| < 1$ on $\mathbb{H}_{m/n}$. For fixed r, estimate the sums $I_r(u)$ and $J_r(t)$ given in (6.20):

$$\begin{aligned}
|I_r(u)| &= \left|\sum_{j=0}^{\infty}(mj+1)u^{mj}+\sigma(r)\sum_{j=0}^{\infty}u^{mj}\right| \\
&\le \left|m\sum_{j=0}^{\infty}ju^{mj}\right| + \left|\sum_{j=0}^{\infty}u^{mj}\right| + \left|\sigma(r)\sum_{j=0}^{\infty}u^{mj}\right| \\
&= \frac{m|u|^m}{|1-u^m|^2}+\frac{1}{|1-u^m|}+\frac{\sigma(r)}{|1-u^m|} \\
&< \frac{m}{|1-u^m|^2}+\frac{|1-u^m|}{|1-u^m|^2}+\frac{(m-1)|1-u^m|}{|1-u^m|^2} \\
&\le \frac{m+2+2(m-1)}{|1-u^m|^2} \\
&\lesssim \frac{1}{|1-u^m|^2} \\
&= \frac{|t|^{2n}}{|t^n-s^m|^2},
\end{aligned} \tag{6.21}$$

and

$$\begin{aligned} |J_r(t)| &= \left| m \sum_{d=0}^{\infty} (d+1)t^d + (r+n) \sum_{d=0}^{\infty} t^d \right| \\ &\leq \left| m \sum_{d=0}^{\infty} dt^d \right| + \left| m \sum_{d=0}^{\infty} t^d \right| + \left| (r+n) \sum_{d=0}^{\infty} t^d \right| \\ &= \frac{m|t|}{|1-t|^2} + \frac{m}{|1-t|} + \frac{(r+n)}{|1-t|} \\ &< \frac{m}{|1-t|^2} + \frac{m|1-t|}{|1-t|^2} + \frac{(r+n)|1-t|}{|1-t|^2} \\ &\leq \frac{m+2m+2(m-1+n)}{|1-t|^2} \\ &\lesssim \frac{1}{|1-t|^2}. \end{aligned} \tag{6.22}$$

Note both bounds hold for all $r \in \{0, 1, \ldots, m-1\}$. Combining (6.21) and (6.22) with (6.20) gives the result:

$$\begin{aligned} |\widetilde{\mathbb{B}^{\infty}}(z,w)| &\leq \frac{1}{m\pi^2} \sum_{r=0}^{m-1} |u|^{\sigma(r)} |t|^{\frac{r}{m}} |I_r(u)||J_r(t)| \\ &< \frac{1}{m\pi^2} \sum_{r=0}^{m-1} |I_r(u)||J_r(t)| \\ &\lesssim \frac{|t|^{2n}}{|1-t|^2|t^n - s^m|^2} \\ &= \frac{|z_2\overline{w_2}|^{2n}}{|1-z_2\overline{w_2}|^2|z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2}. \end{aligned}$$

□

6.3 Proof of Theorem 6.2

For $p \in [2, \infty)$, the L^p sub-Bergman projection is

$$\widetilde{\mathcal{B}^p} f(z) := \int_{\mathbb{H}_{m/n}} \widetilde{\mathbb{B}^p}(z,w) f(w) dV(w),$$

with kernel given by (6.9). Notice the identical kernels in definition (6.9) imply $\widetilde{\mathcal{B}^p} = \widetilde{\mathcal{B}^{p'}}$ for all $p, p' \in [p_k, p_{k+1})$. Similarly, $\widetilde{\mathcal{B}^{\infty}}$ denotes the L^{∞} sub-Bergman projection on $\mathbb{H}_{m/n}$, the operator whose kernel is defined by (6.10).

Proposition 6.15. *Let $p_k = \frac{2m+2n}{1-k}$, for $k \in \{1-m-n, 2-m-n, \ldots, -1\}$, and let q_k denote the conjugate exponent of p_k. Interpret $p_1 = \infty$ and $q_1 = 1$.*

Let $p \in [p_k, p_{k+1})$. The following hold:

(*i*) $|\widetilde{\mathcal{B}^p}|$ *is* $L^{p'}$ *bounded for all* $p' \in (q_{k+1}, p_{k+1})$.

(*ii*) $|\widetilde{\mathcal{B}^\infty}|$ *is a bounded operator on* L^p *for all* $p \in (1, \infty)$.

Proof. Estimate (6.19) shows that $\widetilde{\mathcal{B}^\infty}$ is a type-A operator with $A = 2n$. Proposition 6.10 then implies (*ii*). For $1 - m - n \le k \le -1$, apply the triangle inequality to equation (6.13) together with estimates (6.14) and (6.19) to see that $|\widetilde{\mathcal{B}^{p_k}}|$ is a type-A operator with $A = 2n + \frac{k}{m}$. In fact,

$$
\begin{aligned}
|\widetilde{\mathbb{B}^{p_k}}(z,w)| &\le \left(\sum_{j=k}^{-1} |b^{p_j}(z,w)| \right) + |\widetilde{\mathbb{B}^\infty}(z,w)| \\
&\lesssim \left(\sum_{j=k}^{-1} \frac{|z_2\overline{w_2}|^{2n+\frac{j}{m}}}{|z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} \right) + \frac{|z_2\overline{w_2}|^{2n}}{|1 - z_2\overline{w_2}|^2 |z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} \\
&= \frac{|z_2\overline{w_2}|^{2n+\frac{k}{m}}}{|z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} \left(\sum_{j=k}^{-1} |z_2\overline{w_2}|^{\frac{j-k}{m}} \right) + \frac{|z_2\overline{w_2}|^{2n}}{|1 - z_2\overline{w_2}|^2 |z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} \\
&= \frac{|z_2\overline{w_2}|^{2n+\frac{k}{m}}}{|z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} \left(\sum_{w=0}^{-1-k} \underbrace{|z_2\overline{w_2}|^{w}}_{\le 1} \right) + \frac{|z_2\overline{w_2}|^{2n}}{|1 - z_2\overline{w_2}|^2 |z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} \\
&\le \frac{C_k|z_2\overline{w_2}|^{2n+\frac{k}{m}}}{|z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} + \frac{|z_2\overline{w_2}|^{2n}}{|1 - z_2\overline{w_2}|^2 |z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} \\
&= \frac{C_k|1 - z_2\overline{w_2}|^2 |z_2\overline{w_2}|^{2n+\frac{k}{m}} + |z_2\overline{w_2}|^{2n+\frac{k}{m}} |z_2\overline{w_2}|^{-\frac{k}{m}}}{|1 - z_2\overline{w_2}|^2 |z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} \\
&= \frac{\left[C_k|1 - z_2\overline{w_2}|^2 + |z_2\overline{w_2}|^{-\frac{k}{m}} \right] |z_2\overline{w_2}|^{2n+\frac{k}{m}}}{|1 - z_2\overline{w_2}|^2 |z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} \\
&< \frac{[4C_k + 1] |z_2\overline{w_2}|^{2n+\frac{k}{m}}}{|1 - z_2\overline{w_2}|^2 |z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2} \\
&\lesssim \frac{|z_2\overline{w_2}|^{2n+\frac{k}{m}}}{|1 - z_2\overline{w_2}|^2 |z_2^n\overline{w_2}^n - z_1^m\overline{w_1}^m|^2}.
\end{aligned}
$$

Proposition 6.10 then implies (*i*). □

Remark 6.16. Note that if $p \in [p_k, p_{k+1})$ with $k \in \{1 - m - n, \dots, -1\}$, then necessarily $p \in (q_{k+1}, p_{k+1})$. Since the function $p \mapsto q$, where q is the conjugate exponent of p, is (strictly) decreasing, it follows that $q_k \ge q > q_{k+1}$, which implies $q_{k+1} < q \le p < p_{k+1}$.

To complete the proof of Theorem 6.2, recall that $\widetilde{\mathcal{B}^p}$ is defined as the orthogonal projection from $L^2(\mathbb{H}_{m/n})$ onto $G^{2,p}(\mathbb{H}_{m/n})$, the target space given by equation (5.12). Since $A^p(\mathbb{H}_{m/n}) \subset G^{2,p}(\mathbb{H}_{m/n})$, reproduction property (*ii*) of Theorem 4.10 holds.

Remark 6.17. Again let $p \geq 2$ with $p \in [p_k, p_{k+1})$. If $p' \in (q_{k+1}, p_{k+1})$, then its conjugate $q' \in (q_{k+1}, p_{k+1})$.[6] Proposition 6.15 shows $|\widetilde{\mathcal{B}^p}|$ is both $L^{p'}$ and $L^{q'}$ bounded. In particular, $|\widetilde{\mathcal{B}^p}|$ is bounded on $L^q(\mathbb{H}_{m/n})$, where q is conjugate to p.

On the other hand, reproduction of the space $A^{q'}$ fails for all $q' < 2$. Indeed, a slight modification of the proof Proposition 5.8 shows: if $f \in A^{q'}(\mathbb{H}_{m/n})$, then

$$\widetilde{\mathcal{B}^p}(f)(z) = \sum_{\alpha \in \mathcal{S}(\mathbb{H}_{m/n}, L^p)} a_\alpha(f) e_\alpha(z).$$

Lemma 6.3 implies $\mathcal{S}(\mathbb{H}_{m/n}, L^{q'})$ is a *strict* superset of $\mathcal{S}(\mathbb{H}_{m/n}, L^2)$ which in turn contains $\mathcal{S}\mathbb{H}_{m/n}, L^p)$. Thus non-trivial elements in $A^{q'}$ are mapped to 0. Ramifications of this are seen in the next subsection.

6.4 Duality, approximation and minimization

The sub-Bergman projection give precise answers to version of (Q1-3) on the domains $\mathbb{H}_{m/n}$.

6.4.1 Duality

The dual space of $A^p(\mathbb{H}_{m/n})$ for all $1 < p < \infty$ can be concretely described. The representation is particularly cogent when $p > 2$.

Proposition 6.18. *Let $p > 2$ with conjugate q. The dual space $A^p(\mathbb{H}_{m/n})'$ can be identified with a proper subset of $A^q(\mathbb{H}_{m/n})$. Namely,*

$$A^p(\mathbb{H}_{m/n})' \cong \left\{ f \in A^q(\mathbb{H}_{m/n}) : f = \sum_{\alpha \in \mathcal{S}(\mathbb{H}_{m/n}, L^p)} a_\alpha(f) e_\alpha \right\}. \tag{6.23}$$

Additionally,

$$A^q(\mathbb{H}_{m/n})' \cong \Phi_q(A^p(\mathbb{H}_{m/n})) \oplus \overline{\text{span}}_{A^q(\mathbb{H}_{m/n})'} \{a_\alpha : \alpha \in \mathcal{S}(\mathbb{H}_{m/n}, L^q) \backslash \mathcal{S}(\mathbb{H}_{m/n}, L^p)\}. \tag{6.24}$$

Proof. Since $|\widetilde{\mathcal{B}^p}|$ is bounded on L^p, Proposition 5.12 applies. Equation (6.23) follows from part (*ii*) of Proposition 5.12, noting the right hand side of (6.23) is $G^{q,p}(\mathbb{H}_{m/n})$. Equation (6.24) follows from part (*iii*) of the same proposition. □

6.4.2 Approximation of A^p functions

The form of *(Q2)* addressed is the following: given $p \in (1, \infty)$ and $r > p$, when can $f \in A^p(\mathbb{H}_{m/n})$ be approximated by $A^r(\mathbb{H}_{m/n})$ functions in the L^p norm? As in Proposition 6.18, the answer is most appealing when $p > 2$.

[6] Let $p' \in (q_{k+1}, p_{k+1})$. Its conjugate exponent is given by $q' = \frac{p'}{p'-1}$. Since the map $p' \mapsto q' = \frac{p'}{p'-1}$ is (strictly) decreasing, it follows that $q' \in \left(\frac{p_{k+1}}{p_{k+1}-1}, \frac{q_{k+1}}{q_{k+1}-1} \right) = (q_{k+1}, p_{k+1})$.

Proposition 6.19. *Let $p \geq 2$ be given and $r > p$. Then $f \in A^p(\mathbb{H}_{m/n})$ can be approximated by $A^r(\mathbb{H}_{m/n})$ functions in the L^p norm if and only if $\widetilde{\mathcal{B}^r} f = f$.*[7]

Proof. Suppose $f \in A^p(\mathbb{H}_{m/n})$ and $\widetilde{\mathcal{B}^r} f = f$. Since $f \in L^p(\mathbb{H}_{m/n})$, there is a sequence $\{\phi_n\} \subset C_c^\infty(\mathbb{H}_{m/n})$ satisfying $\|\phi_n - f\|_p \to 0$ as $n \to \infty$. Set $f_n := \widetilde{\mathcal{B}^r}\phi_n$. Note $f_n \in A^r(\mathbb{H}_{m/n})$ by Proposition 6.15. Since $r > p \geq 2$, then $r \in [p_k, p_{k+1})$ for some k. Of course $p < p_{k+1}$. Let q be the conjugate exponent of p. It follows that $q \leq 2 \leq p < p_{k+1}$ and therefore $q > q_{k+1}$, which implies

$$q_{k+1} < p < p_{k+1}.$$

Thus, by Proposition 6.15, $|\widetilde{\mathcal{B}^r}|$ is bounded on L^p, and hence

$$\|f_n - f\|_p = \left\|\widetilde{\mathcal{B}^r}(\phi_n - f)\right\|_p \lesssim \|\phi_n - f\|_p,$$

so f is approximable as claimed.

For the converse, let $f \in A^p(\mathbb{H}_{m/n})$, and suppose there exists a sequence $\{g_n\} \subset A^r(\mathbb{H}_{m/n})$ such that $g_n \to f$ in $A^p(\mathbb{H}_{m/n})$. Assume, for contradiction, $\widetilde{\mathcal{B}^r} f \neq f$. Then, by Proposition 5.11, there exists

$$\beta \in \mathcal{S}(\mathbb{H}_{m/n}, L^p) \backslash \mathcal{S}(\mathbb{H}_{m/n}, L^r)$$

such that $a_\beta(f) \neq 0$. Since $g_n \in A^r(\mathbb{H}_{m/n})$, we have $a_\beta(g_n) = 0$ for all n. By Proposition 5.1, the coefficient functional a_β is continuous on $A^p(\mathbb{H}_{m/n})$, which implies

$$|a_\beta(f)| = |a_\beta(g_n - f)| \lesssim \|g_n - f\|_{A^p} \to 0 \quad \text{as } n \to \infty,$$

a contradiction. □

For $1 < p < 2$, the results are more complicated. In the first case, the sub-Bergman projections $\widetilde{\mathcal{B}^r}$ are only defined if $r \geq 2$; consequently no approximation theorem for the range $1 < p < r < 2$ follows from results in this paper. Additionally, the approximation result that does follows – for the range $1 < p < 2 \leq r$ – requires consideration of the partition (6.3) in Proposition 6.5.

Proposition 6.20. *Let $1 < p < 2$ and p' be conjugate to p. In the partition (6.3), choose k so that $p' < p_{k+1} = \frac{2m+2n}{-k}$.*

Fix $r \in [p_k, p_{k+1})$. Then $f \in A^p(\mathbb{H}_{m/n})$ can be approximated by $A^r(\mathbb{H}_{m/n})$ functions in the L^p norm if and only if $\widetilde{\mathcal{B}^r} f = f$.

Proof. Since $p' < p_{k+1}$, simple algebra shows that $q_{k+1} < p$, where q_{k+1} is the conjugate exponent to p_{k+1}. And since $p \in (q_{k+1}, p_{k+1})$, Proposition 6.15 implies $\widetilde{\mathcal{B}^r}$ is bounded on L^p.

The rest of the proof is the same as for Proposition 6.19. □

[7]Note that although f may belong to $f \in A^p(\mathbb{H}_{m/n}) \backslash A^r(\mathbb{H}_{m/n})$, the boundedness of $\widetilde{\mathcal{B}^r}$ on L^p ensures that we can apply $\widetilde{\mathcal{B}^r} f$ for any $f \in A^p(\mathbb{H}_{m/n})$.

6.5 L^2-nearest approximant in A^p

Question (Q3) can be cast as a broad minimization problem. Suppose $\|\cdot\|_X$ is an auxiliary norm on the space $L^p(\Omega)$, $\Omega \subset \mathbb{C}^n$ fixed.
Problem: Given $g \in L^p(\Omega)$, find $G \in A^p(\Omega)$ so

$$\|g - G\|_X \leq \|g - h\|_X \tag{6.25}$$

for all $h \in A^p(\Omega)$.

For general $\|\cdot\|_X$, techniques needed for this problem mostly await development. But when $X = L^2(\Omega)$ the sub-Begman operators give results. Recall that for $p \geq 2$, $\widetilde{\mathcal{B}^p}$ is the orthogonal projection from L^2 onto $G^{2,p}$, the latter space is given in Proposition 5.10. If Ω is bounded, the diagram

$$\begin{array}{ccc} L^p(\Omega) & \hookrightarrow & L^2(\Omega) \\ \downarrow ? & & \downarrow \widetilde{\mathcal{B}^p} \\ A^p(\Omega) & \hookrightarrow & G^{2,p}(\Omega) \end{array}$$

summarizes relations between the function spaces, with $\hookrightarrow$ denoting injection. Consider "closest" to mean closest measured by the L^2 norm in the following. If $g \in L^2(\Omega)$, the unique closest element in $G^{2,p}(\Omega)$ is $\widetilde{\mathcal{B}^p}g$. However when $\Omega = \mathbb{H}_{m/n}$, Theorem 6.2 says that $\widetilde{\mathcal{B}^p}$ restricts to a bounded operator on $L^p(\mathbb{H}_{m/n})$. It follows that $\widetilde{\mathcal{B}^p}g$ is also the closest element in $A^p(\Omega)$ to g. Thus,

Proposition 6.21. *Let $p \geq 2$ and $g \in L^p(\mathbb{H}_{m/n})$. The function $\widetilde{\mathcal{B}^p}g$ satisfies*

$$\|g - \widetilde{\mathcal{B}^p}g\|_{L^2} \leq \|g - h\|_{L^2}$$

for all $h \in A^p(\mathbb{H}_{m/n})$, with equality if and only if $h = \widetilde{\mathcal{B}^p}g$.

CHAPTER 7

Breakdown on the Hartogs Triangle

The breakdowns of function theory can be observed on the Hartogs triangle, using results established in the previous sections and in [10].

Since $\mathbb{H}$ is Reinhardt, every $f \in \mathcal{O}(\mathbb{H})$ has a unique Laurent expansion, written

$$f(z) = \sum_{\alpha \in \mathbb{Z}^2} a_\alpha z^\alpha, \qquad z = (z_1, z_2) \in \mathbb{H}.$$

Since $z_2 \neq 0$ on $\mathbb{H}$ but there are points in $\mathbb{H}$ where $z_1 = 0$, the summation is taken over the set $\{\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_1 \geq 0\}$. If $f \in A^p(\mathbb{H})$, results in Chapter 5 show the Laurent expansion of f need only be summed over the smaller set of L^p-allowable multi-indices, see (5.5). Corollay 5.2 implies

$$f(z) = \sum_{\alpha \in \mathcal{S}(\mathbb{H}, L^p)} a_\alpha z^\alpha \quad \text{if } f \in A^p(\mathbb{H}). \tag{7.1}$$

A special case of [10, Theorem 1.1 and Remark 4.9] is

Theorem 7.1. *The absolute value of the Bergman projection $|\mathcal{B}|$ on $\mathbb{H}$ is bounded from $L^p(\mathbb{H})$ to $A^p(\mathbb{H})$ if and only if $p \in (\frac{4}{3}, 4)$.*

7.1 Failure of representation

The dual space $A^p(\mathbb{H})'$ is not isomorphic to $A^q(\mathbb{H})$ for $p \in \left(\frac{4}{3}, 2\right)$ and q conjugate to p. This is illustrated with the pair $p = \frac{5}{3}$ and $q = \frac{5}{2}$. The argument works with minor changes for any $p \in \left(\frac{4}{3}, 2\right)$.

Before defining a functional on $A^{5/3}(\mathbb{H})$, a computation is useful:

Example 7.2. The holomorphic function $h(z_1, z_2) = z_2^{-2} (= z_1^0 z_2^{-2})$ satisfies

(*i*) $h \in A^{5/3}(\mathbb{H})$ and $h \notin A^2(\mathbb{H})$.

(*ii*) $\mathcal{B}h$ is well-defined and $\mathcal{B}h \equiv 0$.

Proof. Lemma 6.3 shows that $(0,-2) \in \mathcal{S}(\mathbb{H}, L^{5/3})$ and $(0,-2) \notin \mathcal{S}(\mathbb{H}, L^2)$.[1] Thus (i) holds.

Since $\frac{5}{3} \in \left(\frac{4}{3}, 4\right)$, Theorem 7.1 says $|\mathcal{B}|$ is bounded on $L^{5/3}(\mathbb{H})$. It follows from Proposition 5.8 that $\mathcal{B}h$ is well-defined and $\mathcal{B}h \equiv 0$. ☐

A non-representable functional is now given using the coefficients in (7.1).

Example 7.3. The coefficient functional

$$a_{(0,-2)} : A^{5/3}(\mathbb{H}) \to \mathbb{C}$$

assigning to $f \in A^{5/3}(\mathbb{H})$ the coefficient of z_2^{-2} in its Laurent expansion is bounded on $A^{5/3}(\mathbb{H})$. However, there does not exist $\phi \in A^{5/2}(\mathbb{H})$ such that

$$a_{(0,-2)}(f) = \langle f, \phi \rangle_{\mathbb{H}}.$$

Proof. Uniqueness of the Laurent expansion shows the functional $a_{(0,-2)}$ is well-defined. Boundedness of $a_{(0,-2)}$ follows from Proposition 5.1.

To prove non-representability, let $h(z) = z_2^{-2} \in \mathcal{O}(\mathbb{H})$ as above. Example 7.2 says $h \in A^{5/3}(\mathbb{H})$ but $h \notin A^2(\mathbb{H})$. Since $(0,-2) \notin \mathcal{S}(\mathbb{H}, L^2)$, Corollary 6.7 shows that for all $g \in A^2(\mathbb{H})$

$$\langle h, g \rangle_{\mathbb{H}} = 0. \tag{7.2}$$

The fact that $a_{(0,-2)}$ cannot be represented by $\langle \cdot, \phi \rangle_{\mathbb{H}}$ for some $\phi \in A^{5/2}(\mathbb{H})$ is now straightforward. Suppose such a representation held. Note $a_{(0,-2)}(h) = 1$ by definition. Since $A^{5/2}(\mathbb{H}) \subset A^2(\mathbb{H})$, (7.2) implies $\langle h, \phi \rangle_{\mathbb{H}} = 0$ for all $\phi \in A^{5/2}(\mathbb{H})$, a contradiction. ☐

7.2 Failure of approximation on A^p

There are functions $f \in A^{5/3}(\mathbb{H})$ for which no sequence of functions $f_n \in A^2(\mathbb{H})$ converges to f in the $L^{5/3}$ norm. As in the previous subsection, minor changes in the argument give an analogous result for any $p \in (\frac{4}{3}, 2)$.

Proposition 7.4. *$A^2(\mathbb{H})$ is not dense in $A^{5/3}(\mathbb{H})$.*

Proof. Let $a_{(0,-2)} \in A^{5/3}(\mathbb{H})'$ and $h \in A^{5/3}(\mathbb{H}) \backslash A^2(\mathbb{H})$ be as in the previous section. By Corollary 5.2, since $(0,-2) \notin \mathcal{S}(\mathbb{H}, L^2)$, $a_{(0,-2)}$ vanishes on the linear subspace $A^2(\mathbb{H})$ of $A^{5/3}(\mathbb{H})$. If $A^2(\mathbb{H})$ were dense in $A^{5/3}(\mathbb{H})$, continuity would imply $a_{(0,-2)} \equiv 0$ on $A^{5/3}(\mathbb{H})$. However, $a_{(0,-2)}(h) = 1$, which contradicts this vanishing. ☐

[1] Let $C(p) := \left\lfloor -\frac{2}{p}(m+n) + 1 \right\rfloor$. Consider $m = n = 1$. For $p = \frac{5}{3}$, we get $C(p) = -2$; for $p = 2$, $C(p) = -1$. Now compare this with Lemma 6.3.

In fact a stronger statement is true: there are functions in $A^{5/3}(\mathbb{H})$ that cannont be approximated uniformly on compact subsets of $\mathbb{H}$ by functions in $A^2(\mathbb{H})$. To see this, suppose that $\{f_n\}$ is a sequence in $A^2(\mathbb{H})$ such that $f_n \to h$ uniformly on compact subsets of $\mathbb{H}$. Recall the Cauchy representation of a coefficient of a Laurent series:

$$a_{(0,-2)}(f) = \frac{1}{(2\pi i)^2} \int_T \frac{f(\zeta)}{\zeta_2^{-2}} \cdot \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2},$$

where T is a torus contained in $\mathbb{H}$, for example $\{(z_1, z_2) : |z_1| = \frac{1}{4}, |z_2| = \frac{1}{2}\} \subset \mathbb{H}$. Since $f_n \to h$ uniformly on T as $n \to \infty$, it follows that $a_{(0,-2)}(f_n) \to 1 = a_{(0,-2)}(h)$ as $n \to \infty$. This is a contradiction, since Corollary 5.2 implies $a_{(0,-2)}(f_n) = 0$ for each n.

7.3 Failure of approximation on L^p

For $p \geq 4$, there are explicit functions $g \in L^p(\mathbb{H})$ such that $\mathcal{B}g \notin A^p(\mathbb{H})$. Note that $L^p(\mathbb{H}) \subset L^2(\mathbb{H})$ for this range of p, so $\mathcal{B}g$ is well-defined. As $g \mapsto \mathcal{B}g$ associates the L^2-nearest holomorphic function to a general g, this is a different failure of approximation than in the previous section.

Since Theorem 7.1 says there does not exist C such that $\|\mathcal{B}f\|_p \leq C\|f\|_p$ for all $f \in L^p$, the uniform boundedness principle [19] implies the existence of such g. But the explicit form of such "extremal functions" (though non-unique) is useful for other purposes. The proofs in [9, 10] actually show

Example 7.5. On $\mathbb{H}$, let $\psi(z_1, z_2) = \overline{z_2}$. Then $\mathcal{B}\psi \notin L^p(\mathbb{H})$ for any $p \geq 4$.

Proof. The proof of Proposition 5.1 in [10] shows that $\mathcal{B}\psi = C z_2^{-1}$, for a constant $C \neq 0$. An elementary computation in polar coordinates (see Lemma 6.3) shows $z_2^{-1} \notin L^p(\mathbb{H})$ if $p \geq 4$. □

Since $\psi \in L^\infty(\mathbb{H})$, thus in $L^p(\mathbb{H})$ for all $p > 0$, Example 7.5 demonstrates the breakdown mentioned above.

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