UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

An obstruction to topological embeddings between generalized manifolds

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São Carlos-SP
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"Can’t change what you’ve done
Start fresh next semester"
Next Semester - TØP

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Resumo

Este trabalho apresenta um estudo sobre uma certa classe de obstruções à existência de uma homotopia própria entre uma aplicação contínua própria e um mergulho topológico próprio entre variedades generalizadas, sem assumir qualquer estrutura diferenciável. Essa classe foi estudada por Carlos Biasi, Janey Daccach e Osamu Saeki em [1], utilizando variedades generalizadas, e em [2], assumindo uma estrutura suave. Considerando tal classe, obtemos algumas aplicações interessantes envolvendo classes de Euler, obstrução de Haefliger, conjuntos de auto-interseção e espaços projetivos reais.

Palavras-chave: dualidade de Poincaré; variedades generalizadas; obstruções; operações quadrado de Steenrod; classes de Stiefel–Whitney, espaços projetivos reais

Abstract

This work presents a study of a certain class of obstructions to the existence of a proper homotopy between a proper continuous map and a proper topological embedding between generalized manifolds, without assuming any smooth structure. Such a class was studied by Carlos Biasi, Janey Daccach, and Osamu Saeki in [1] using generalized manifolds, and in [2] assuming a smooth structure. Using this class, we derive some interesting applications involving Euler classes, Haefliger's obstruction, self-intersection sets, and real projective spaces.

Keywords: Poincaré duality; generalized manifolds; obstructions, Steenrod squaring operations; Stiefel-Whitney classes, real projective spaces.

Contents

Introduction

The existence and classification of embeddings between topological manifolds is an important topic in many areas of mathematics. A natural question arises: given a continuous map between two topological manifolds, under what conditions can it be homotopic to an embedding? Classical techniques that address this problem often rely on differential topology and the use of smooth structures. However, many topological spaces of interest do not admit a differentiable structure, which motivates the development of tools (here algebraic tools) that can be use in a more general topological setting.

This work presents a study of a certain class of obstruction to the existence of a proper homotopy between a proper continuous map and a proper topological embedding between generalized manifolds, without assuming any smooth structure. Such a class of obstruction was first introduced by Carlos Biasi, Janney Daccach, and Osamu Saeki in [2], under the assumption of smoothness. Later, in [1], the authors generalized the construction to a more general class of spaces; known as generalized manifolds; which have local homological properties similar to those of Euclidean spaces.

In Chapter 1 we present the basic definitions and algebraic tools needed throughout this work. We recall the notions of direct and inverse limits in the category of $\mathbb{Z}_2$-modules and introduce the definitions of singular homology with closed support and singular cohomology with compact support. We also discuss the Steenrod squaring operations and some of their properties.

In Chapter 2 we provide a precise definition of generalized manifolds and explore their properties. Using the Poincaré duality, we define the Wu classes and, subsequently, the Stiefel–Whitney classes.

In Chapter 3 we define an obstruction class $\theta_f \in \check{H}^c_{m-k}(M;\mathbb{Z}_2)$ associated to a proper map $f : M \to N$ between generalized manifolds with dimension m and $m+k$, respectively, $k > 0$. We prove that if this class does not vanish, then f cannot be properly homotopic to a topological embedding.

The Chapter 4 is dedicated to presenting some applications of the obstruction class defined in Chapter 3. First, we study the relationship between the Euler normal class of a proper topological embedding and its Stiefel–Whitney classes. We extend a classical result of Haefliger on the vanishing of Stiefel–Whitney classes for proper embeddings to ENR $\mathbb{Z}_2$-homology manifolds. We also introduce the notion of self-intersection set of a continuous map and investigate its connection with the obstruction class θ_f for compact generalized manifolds. Finally, we generalize the results given in Section 4 of [1] to continuous maps between real projective spaces $\mathbb{R}P^n$ and certain generalized manifolds, under the assumption that n is a power of 2. More specifically, the main result in the Section 4.4 (Lemma 4.19) is original and constitutes a generalization of the results given in [1] where the authors used $n = 2$ and X is a 3-dimensional generalized manifold.

CHAPTER 1

Foundational Concepts

This chapter presents some basics definitions and results that will be used throughout this work. It is important to observe that we will assume some previous knowledge of singular homology and singular cohomology, including some important properties of these modules.

1.1 Algebraic Limits for homology and cohomology

The main algebraic objects in this work are the singular homology with closed support and singular cohomology with compact support. In order to define these objects, we first introduce the notion of algebraic limits considering systems of modules and homomorphisms.

1.1.1 Algebraic limits for modules

We will introduce the algebraic limits for direct systems and inverse systems, together with the proof of their existence. Here **I** will always denote a quasi-ordered set. The following discussion can be found in [12].

Definition 1.1. Let $\{X_i\}_{i\in\mathbf{I}}$ be a collection of modules and let $\{\varphi_{i',i} : X_i \longrightarrow X_{i'}\}_{i,i'\in\mathbf{I}}$ be a family of homomorphisms. We say that the family $\{X_i, \varphi_{i',i}\}_{i\in\mathbf{I}}$ is a *direct system of modules* if, for all $i \leq i' \leq i''$ in **I**, the following conditions are satisfied:

- $\varphi_{i'',i'} \circ \varphi_{i',i} = \varphi_{i'',i}$;
- $\varphi_{i,i} = id_{X_i}$,

where $id_{X_i} : X_i \longrightarrow X_i$ is the identity map.

Definition 1.2. Given a direct system of modules $\{X_i, \varphi_{i',i}\}_{i,i'\in\mathbf{I}}$, we say that a module X together with a family of homomorphism $\{\varphi_i : X_i \longrightarrow X\}_{i\in\mathbf{I}}$ are a *direct limit of modules* if satisfies the following conditions:

(i) for every $i \leq i'$ in $\mathbf{I}$, $\varphi_{i'} \circ \varphi_{i',i} = \varphi_i$;

(ii) (Universal property) for any module Y and any family of homomorphisms $\{\psi_i : X_i \longrightarrow Y\}_{i \in \mathbf{I}}$, satisfying the condition that, for all $i \leq i'$ in $\mathbf{I}$, $\psi_{i'} \circ \varphi_{i',i} = \psi_i$, there exists a unique homomorphism $\psi : X \longrightarrow Y$ such that the following diagram commutes

$$\begin{tikzcd}
X_i \arrow[rr, "\varphi_{i',i}"] \arrow[dr, "\varphi_i"] \arrow[ddr, bend right, "\psi_i"'] & & X_{i'} \arrow[dl, "\varphi_{i'}"'] \arrow[ddl, bend left, "\psi_{i'}"] \\
& X \arrow[d, "\psi"] & \\
& Y &
\end{tikzcd}$$

and we denote

$$\psi := \varinjlim_{i \in \mathbf{I}} \psi_i.$$

Some natural questions arise from the definition of the direct limit of modules: Does every direct system of modules admit a direct limit? If such a direct limit always exists, can we prove that it is unique?

Proposition 1.3. *Let $\{X_i, \varphi_{i',i}\}_{i,i' \in \mathbf{I}}$ be a direct system of modules. If its limit exists, it has to be unique up to isomorphism.*

Proof. Suppose that $\{M, \varphi_i : X_i \longrightarrow M\}_{i \in \mathbf{I}}$ and $\{N, \psi_i : X_i \longrightarrow N\}_{i \in \mathbf{I}}$ are two direct limits for the same direct system $\{X_i, \varphi_{i',i}\}_{i,i' \in \mathbf{I}}$. By Definition 1.2, we have that $\psi_{i'} \circ \varphi_{i',i} = \psi_i$. Using the universal property, there exists a unique homomorphism $\psi : M \longrightarrow N$ such that $\psi \circ \varphi_i = \psi_i$. In an analogous way, there exists a unique homomorphism $\psi' : N \longrightarrow M$ such that $\psi' \circ \psi_i = \varphi_i$.

Observe that ψ is a isomorphism. In fact, for all $i \in \mathbf{I}$,

$$\psi_i = \psi \circ \varphi_i = \psi \circ \psi' \circ \psi_i$$

and

$$\varphi_i = \psi' \circ \psi_i = \psi' \circ \psi \circ \varphi_i.$$

Using the universal property, id_N and id_M are the unique homomorphism such that $id_N \circ \psi_i = \psi_i$ and $id_M \circ \varphi_i = \varphi_i$. Then ψ and ψ' are isomorphism.

□

Definition 1.4. Let $\{X_i, \varphi_{i',i}\}_{i,i' \in \mathbf{I}}$ be a direct system of modules and let $\{X, \varphi_i : X_i \longrightarrow X\}_{i \in \mathbf{I}}$ be its direct limit. We denote

$$X := \varinjlim_{i \in \mathbf{I}} X_i.$$

Proposition 1.5. *The direct limit of a direct system $\{M_i, \varphi_{i',i}\}_{i,i' \in I}$ always exists.*

Proof. First, define

$$M' = \bigoplus_{i \in \mathbf{I}} M_i$$

where $\bigoplus_{i \in \mathbf{I}} M_i$ is defined as the set of all functions

$$\varphi : \mathbf{I} \longrightarrow \bigcup_{i \in \mathbf{I}} M_i$$

where $\varphi(i) \neq 0$ for only a finite number of indices $i \in \mathbf{I}$. For each $i \in \mathbf{I}$, consider the homomorphism $\phi'_i : M_i \longrightarrow M'$ defined by

$$\begin{array}{rcl} \phi'_i : M_i & \longrightarrow & M' \\ x & \longmapsto & \phi'_i(x) \end{array}$$

where

$$\begin{array}{rcl} \phi'_i(x) : \mathbf{I} & \longrightarrow & \bigcup_{i \in \mathbf{I}} M_i \\ j & \longmapsto & \left\{ \begin{array}{ll} x & \text{se } j = i \\ 0 & \text{se } j \neq i. \end{array} \right. \end{array}$$

In this way, for each $i \in \mathbf{I}$, we identify $\phi'_i(x) \in M'$ as

$$(\phi'_i(x)(j))_{j \in \mathbf{I}} = (x_j)_{j \in \mathbf{I}}$$

where x_j is non zero just for $j = i$ and $x_i = x$. Observe that ϕ'_i is a monomorphism for all $i \in \mathbf{I}$.

For each $i \leq i'$ in $\mathbf{I}$ consider the composition:

$$\phi'_{i'} \circ \varphi_{i',i} : M_i \longrightarrow M',$$

and defined the set

$$S = \{(\phi'_{i'} \circ \varphi_{i',i}(x) - \phi'_i(x), \forall\, i \leq i', \forall\, x \in M_i\} \subset \bigoplus_{i \in \mathbf{I}} M_i = M'.$$

Denote by $[S]$ the submodule of M' generated by S.

We want to prove that $\{M, \varphi_i : M_i \longrightarrow M\}$ is the direct limit of $\{M_i, \varphi_{i',i}\}_{i,i' \in \mathbf{I}}$, where $\varphi_i := \pi \circ \phi'_i$, $M = M'/[S]$ and

$$\begin{array}{rcl} \pi : M' & \longrightarrow & M \\ x & \longmapsto & x + [S] \end{array}$$

is the natural projection.

Observe that, for each $x \in M_i$,

$$\begin{aligned}
\varphi_{i'} \circ \varphi_{i',i}(x) &= \pi \circ \phi'_{i'} \circ \varphi_{i'',i}(x) \\
&= \phi'_{i'} \circ \varphi_{i'',i}(x) + [S] \\
&= \phi'_i(x) + [S] \\
&= \pi \circ \phi'_i(x) \\
&= \varphi_i(x).
\end{aligned}$$

For a module N and a family of homomorphisms $\{\psi_i : M_i \longrightarrow N\}_{i\in\mathbf{I}}$ where $\psi_{i'} \circ \varphi_{i',i} = \psi_i$, we want to prove that there exists a unique homomorphism $\psi : M \longrightarrow N$ such that $\psi \circ \varphi_{i'} = \psi_{i'}$.

For each $x_i \in M_i$, $\psi_i(x_i) \in N$ and $(x_i)_{i\in\mathbf{I}} \in M'$, consider

$$\sum_{i\in\mathbf{I}} \psi_i(x_i) \in N.$$

In this way, define $\psi : M' \longrightarrow N$ as

$$\psi((x_i)_{i\in\mathbf{I}}) = \sum_{i\in\mathbf{I}} \psi_i(x_i).$$

It can be easily been proved that $[S] \subset Ker(\psi)$ and there exists a unique homomorphism $\bar{\psi} : M \longrightarrow N$ such that $\bar{\psi}(x + [S]) = \psi(x)$ (see [15]).

Note that, for $x \in M_i$,

$$\begin{aligned}
\bar{\psi} \circ \varphi_i(x) &= \bar{\psi}(\pi \circ \phi'_i)(x) \\
&= \bar{\psi}(\phi'_i(x) + [S]) \\
&= \psi(\phi'_i(x)) \\
&= \psi_i(x).
\end{aligned}$$

This prove that $\{M, \varphi_i\}_{i\in\mathbf{I}}$ is the direct limit for the directed system $\{M_i, \varphi_{i',i}\}_{i,i'\in\mathbf{I}}$.

□

The definition of the inverse system of a module and its inverse limit is similar to the definition of direct system of module and its direct limit; all we need to do is reverse the arrows of the homomorphisms.

Definition 1.6. Let $\{X_i\}_{i\in\mathbf{I}}$ be a collection of modules and let $\{\varphi_{i,i'} : X_{i'} \longrightarrow X_i\}_{i,i'\in\mathbf{I}}$ be a family of homomorphisms. We say that $\{X_i, \varphi_{i,i'}\}_{i\in\mathbf{I}}$ is an *inverse system of modules* if, for all $i \leq i' \leq i''$ in $\mathbf{I}$, the following conditions are satisfied:

- $\varphi_{i,i'} \circ \varphi_{i',i''} = \varphi_{i,i''}$;

- $\varphi_{i,i} = id_{X_i}$.

Definition 1.7. Given an inverse system of modules $\{X_i, \varphi_{i,i'}\}_{i,i' \in \mathbf{I}}$, we say that a module X together with a family of homomorphisms $\{\varphi_{i,i'} : X_{i'} \longrightarrow X_i\}_{i \in \mathbf{I}}$ are an *inverse limit of modules* if satisfies the following properties:

(i) for every $i \leq i'$ in $\mathbf{I}$, $\varphi_{i,i'} \circ \varphi_{i'} = \varphi_i$;

(ii) (Universal property) for any module Y and any family of homomorphisms $\{\psi_i : Y \longrightarrow X_i\}_{i \in \mathbf{I}}$ satisfying the condition that, for all $i \leq i'$ in $\mathbf{I}$, $\varphi_{i,i'} \circ \psi_{i'} = \psi_i$, there exists a unique homomorphism $\psi : Y \longrightarrow X$ such that following diagram commutes

$$\begin{tikzcd}
X_{i'} \arrow[rr, "\varphi_{i,i'}"] & & X_i \\
& X \arrow[ul, "\varphi'_i"'] \arrow[ur, "\varphi_i"] & \\
& Y \arrow[u, "\psi"] \arrow[uul, bend left, "\psi_{i'}"] \arrow[uur, bend right, "\psi_i"'] &
\end{tikzcd}$$

and we denote

$$\psi := \varprojlim_{i \in \mathbf{I}} \psi_i.$$

Using the same argumentation of Proposition 1.3 one can show that if the inverse limit of an inverse system of modules exists, then it is unique up to isomorphism. Furthermore, the inverse limit of an inverse system always exists.

Definition 1.8. Let $\{X_i, \varphi_{i',i}\}_{i,i' \in \mathbf{I}}$ be an inverse system of modules and let $\{X, \varphi_i : X \longrightarrow X_i\}_{i \in \mathbf{I}}$ be its direct limit. We denote

$$X := \varprojlim_{i \in \mathbf{I}} X_i.$$

Proposition 1.9. *The inverse limit of an inverse system $\{M_i, \varphi_{i,i'}\}_{i,i' \in I}$ always exists.*

Proof. Let us show that $\{M, \varphi_i\}_{i \in \mathbf{I}}$ is the inverse limit for $\{M_i, \varphi_{i,i'}\}_{i,i' \in \mathbf{I}}$, where

$$M = \left\{ (m_i)_{i \in I} \in \prod_{i \in I} M_i \;\middle|\; \phi_{i,i'}(m_{i'}) = m_i \text{ for all } i \leq i' \in \mathbf{I} \right\}$$

and

$$\varphi_k : M \longrightarrow M_k$$

is defined by $\varphi_k((m_i)_{i\in\mathbf{I}}) = m_k$.

In fact, for every $i \leq i'$ in $\mathbf{I}$ and each $(m_i)_{i\in\mathbf{I}} \in M$,

$$\varphi_{i,i'} \circ \varphi_{i'}((m_i)_{i\in\mathbf{I}}) = \varphi_{i,i'}(m_i') = m_i = \varphi_i\big((m_i)_{i\in\mathbf{I}}\big).$$

Now, for the universal property, suppose that there exists a family $\{N, \psi_i : Y \longrightarrow X_i\}_{i\in\mathbf{I}}$ such that $\varphi_{i,i'} \circ \psi_{i'} = \psi_i$. Define the homomorphism

$$\begin{aligned} \psi : N &\longrightarrow M \\ y &\longmapsto \big(\psi_i(y)\big)_{i\in\mathbf{I}} \end{aligned}$$

as $\psi(y) = \big(\psi_i(y)\big)_{i\in\mathbf{I}}$. Note that $\varphi_i \circ \psi = \psi_i$ for all $i \in \mathbf{I}$. Moreover, if there exists another homomorphism $\tilde{\psi} : N \longrightarrow M$ such that $\varphi_i \circ \tilde{\psi} = \psi_i$, its easy to see that $\tilde{\psi} = \psi$.

□

1.1.2 Homology with closed support and cohomology with compact support

Consider M a topological space and let $\mathbf{I}_M$ be the quasi-ordered set of all compact subsets of M where, for $K, K' \in \mathbf{I}_M$, we have that $K \leq K'$ if and only if $K \subset K'$.

For every $K \leq K'$ in $\mathbf{I}_M$, the inclusion of pairs

$$i : (M, M - K') \hookrightarrow (M, M - K)$$

induces a homomorphism in singular homology

$$\phi_{K,K'} := i_* : H_q(M, M - K'; \mathbb{Z}_2) \longrightarrow H_q(M, M - K; \mathbb{Z}_2)$$

for all $q \in \mathbb{N}$. Moreover, for every $K \leq K' \leq K''$ in $\mathbf{I}_M$ we have that $\phi_{K,K} = id_{H_q(M,M-K;\mathbb{Z}_2)}$ and $\phi_{K,K'} \circ \phi_{K',K''} = \phi_{K,K''}$. This provides that

$$\{H_q(M, M - K; \mathbb{Z}_2), \phi_{K,K'}\}_{K,K'\in\mathbf{I}_M}$$

is an inverse system of modules.

Definition 1.10. The algebraic limit of the inverse system $\{H_q(M, M - K; \mathbb{Z}_2), \phi_{K,K'}\}_{K,K'\in\mathbf{I}_M}$ is called the *homology with closed support of* M and it is denoted by

$$H_q^c(M) = \varprojlim_{K\in\mathbf{I}_M} H_q(M, M - K; \mathbb{Z}_2).$$

In [22], a characterization for the homology with closed support is given by: Let M be a topological space and consider the collection $\mathscr{K} = \{K\}_{K\in\mathbf{I}_M}$. A compatible family of $\mathscr{K}$ is a collection of classes $\{z_K \in H_q(M, M - K; \mathbb{Z}_2)\}_{K\in\mathscr{K}}$, for a natural number q, such that, if $K, K' \in \mathscr{K}$ then z_K and $z_{K'}$ are identified under homomorphism induced by inclusion

$$H_q(M,M-K,\mathbb{Z}_2) \longrightarrow H_q(M,M-(K\cap K'),\mathbb{Z}_2) \longleftarrow H_q(M,M-K',\mathbb{Z}_2).$$

All the $\mathscr{K}$ compatible family form a module with the componentwise operations. Observe that this characterization is equivalent to the one given in Definition1.10.

To define the cohomology with compact support we must do a similar construction. For every $K \leq K'$ in $\mathbf{I}_M$, consider the pair inclusion

$$i:(M,M-K') \hookrightarrow (M,M-K).$$

This induces a homomorphism in cohomology

$$\varphi_{K',K} := i^* : H^q(M,M-K;\mathbb{Z}_2) \longrightarrow H^q(M,M-K';\mathbb{Z}_2)$$

for all $q \in \mathbb{N}$. For $K \leq K'$ in $\mathbf{I}_M$, it follows that $\varphi_{K,K} = id_{H^q(M,M-K;\mathbb{Z}_2)}$ and $\varphi_{K'',K'} \circ \varphi_{K',K} = \varphi_{K'',K}$. This asserts that

$$\{H^q(M,M-K;\mathbb{Z}_2), \varphi_{K',K}\}_{K,K'\in\mathbf{I}_M}$$

forms is a direct system of modules.

Definition 1.11. The algebraic limit of the direct system $\{H^q(M,M-K;\mathbb{Z}_2), \varphi_{K',K}\}_{K,K'\in\mathbf{I}_M}$ is called the *cohomology with compact support of* M and denoted by

$$H^q_c(M;\mathbb{Z}_2) = \varinjlim_{K\in\mathbf{I}_M} H^q(M,M-K;\mathbb{Z}_2).$$

Remark 1.12. Let M be a compact topological space. Then

$$H^c_q(M;\mathbb{Z}_2) \cong H_q(M;\mathbb{Z}_2)$$

and

$$H^q_c(M;\mathbb{Z}_2) \cong H^q(M;\mathbb{Z}_2).$$

Suppose that $f: M \longrightarrow N$ is a continuous map between topological spaces. Can we define an induced homomorphism

$$f_*: H^c_q(M;\mathbb{Z}_2) \longrightarrow H^c_q(N;\mathbb{Z}_2)$$

for all $q \in \mathbb{N}$? If f is a proper continuous map, we guarantee that f induces a homomorphism using the universal property as follows: for every $K \subset N$ compact, the properness of f asserts that $f^{-1}(K) \subset M$ is compact and $f(M - f^{-1}(K)) \subset N-K$. For $f^{-1}(K) \subset M$, consider the homomorphisms

$$\phi_{f^{-1}(K)}: H^c_q(M;\mathbb{Z}_2) \longrightarrow H_q(M,M-f^{-1}(K);\mathbb{Z}_2)$$

given by the inverse limit and the induced homomorphism

$$f_*: H_q(M,M-f^{-1}(K);\mathbb{Z}_2) \longrightarrow H_q(N,N-K;\mathbb{Z}_2)$$

Define the composition

$$\psi_K := f_* \circ \phi_{f^{-1}(K)} : H_q^c(M;\mathbb{Z}_2) \longrightarrow H_q(N,N-K;\mathbb{Z}_2)$$

For all $K \leq K'$ compact subsets of N, the following diagram commutes

$$\begin{tikzcd}
(M,M-f^{-1}(K')) \arrow[r, "f"] \arrow[d, "i"] & (N,M-K') \arrow[d, "i"] \\
(M,M-f^{-1}(K)) \arrow[r, "f"] & (N,M-K)
\end{tikzcd}$$

where i denote the inclusions. We induce a commutative diagram

$$\begin{tikzcd}
H_q(M,M-f^{-1}(K');\mathbb{Z}_2) \arrow[r, "f_*"] \arrow[d, "\phi_{f^{-1}(K),f^{-1}(K')}"] & H_q(N,N-K';\mathbb{Z}_2) \arrow[d, "\phi'_{K,K'}"] \\
H_q(M,M-f^{-1}(K);\mathbb{Z}_2) \arrow[r, "f_*"] & H_q(N,N-K;\mathbb{Z}_2).
\end{tikzcd}$$

Then,

$$\begin{aligned}
\phi'_{K,K'} \circ \psi_{K'} &= \phi'_{K,K'} \circ f_* \circ \phi_{f^{-1}(K')} \\
&= f_* \circ \phi_{f^{-1}(K),f^{-1}(K')} \circ \phi_{f^{-1}(K')} \\
&= f_* \circ \phi_{f^{-1}(K)} \\
&= \psi_K.
\end{aligned}$$

Using the universal property, exists a unique homomorphism

$$f_* : H_q^c(M;\mathbb{Z}_2) \longrightarrow H_q^c(N;\mathbb{Z}_2)$$

such that $\phi'_K \circ f_* = \psi_K$.

A similar argument can be used to show that every proper map $f : M \to N$ between topological spaces induces a homomorphism

$$f^* : H_c^q(N;\mathbb{Z}_2) \to H_c^q(M;\mathbb{Z}_2).$$

Proposition 1.13. *Let $f : M \longrightarrow N$ be a homeomorphism between topological spaces. Then*

$$f_* : H_q^c(M;\mathbb{Z}_2) \longrightarrow H_q^c(N;\mathbb{Z}_2)$$

and

$$f^* : H_c^q(N;\mathbb{Z}_2) \longrightarrow H_c^q(M;\mathbb{Z}_2)$$

are isomorphisms.

Proof. We will prove this proposition for homology, for cohomology you must turn around the arrows.

Note that $f : M \longrightarrow N$ and $f^{-1} : N \longrightarrow M$ are proper maps. We induce the homomorphisms

$$f_* : H^c_q(M;\mathbb{Z}_2) \longrightarrow H^c_q(N;\mathbb{Z}_2)$$

and

$$f^{-1}_* : H^c_q(N;\mathbb{Z}_2) \longrightarrow H^c_q(M;\mathbb{Z}_2).$$

We will prove that $(f^{-1})_* = (f_*)^{-1}$.

Note that $id_{H^c_q(N;\mathbb{Z}_2)}$ is the unique homomorphism that makes the following diagram commutes

$$\begin{tikzcd}
H_q(N,N-K';\mathbb{Z}_2) \arrow[rr, "\phi'_{K,K'}"] & & H_q(N,N-K;\mathbb{Z}_2) \\
& H^c_q(N;\mathbb{Z}_2) \arrow[ul, "\phi'_{K'}"'] \arrow[ur, "\phi'_K"] \arrow[d, "id_{H^c_q(N;\mathbb{Z}_2)}"] & \\
& H^c_q(N;\mathbb{Z}_2), \arrow[uul, bend left, "\phi'_{K'}"] \arrow[uur, bend right, "\phi'_K"'] &
\end{tikzcd}$$

where $\{H^c_q(N;\mathbb{Z}_2), \phi'_K\}_{K\in\mathbf{I}_N}$ is the inverse limit for the inverse system $\{H_q(N,N-K;\mathbb{Z}_2), \phi'_{K,K'}\}_{K,K'\in\mathbf{I}_N}$.

Denote the inverse limit of $\{H_q(M,M-f^{-1}(K);\mathbb{Z}_2), \phi_{f^{-1}(K),f^{-1}(K')}\}_{K,K'\in\mathbf{I}_N}$ as

$$\{H^c_q(M;\mathbb{Z}_2), \phi_{f^{-1}(K)}\}_{K\in\mathbf{I}_N}.$$

For each compact $K \subset N$, consider

$$\psi_K := f_* \circ \phi_{f^{-1}(K)} : H^c_q(M;\mathbb{Z}_2) \longrightarrow H_q(N,N-K;\mathbb{Z}_2)$$

and

$$\psi'_{f^{-1}(K)} := (f^{-1})_* \circ \phi'_K : H^c_q(M;\mathbb{Z}_2) \longrightarrow H_q(N,N-K;\mathbb{Z}_2).$$

Using the discussion above, we have that $\psi_K = \phi'_K \circ f_*$ and $\psi'_{f^{-1}(K)} = \phi_{f^{-1}(K)} \circ (f^{-1})_*$. Then

$$\begin{aligned}
\phi'_K \circ f_* \circ (f^{-1})_* &= \psi_K \circ (f^{-1})_* \\
&= f_* \circ \phi_{f^{-1}(K)} \circ (f^{-1})_* \\
&= f_* \circ \psi'_{f^{-1}(K)} \\
&= f_* \circ (f^{-1})_* \circ \phi'_K \\
&= \phi'_K.
\end{aligned}$$

A similar argument works to show that $\phi_{f^{-1}(K)} \circ (f^{-1})_* \circ f_* = \phi_{f^{-1}(K)}$. This prove that f_* is an isomorphism. To prove that f^* is an isomorphism turn around the arrows.

□

Now, consider U an open subset of M and the inclusion $i : U \hookrightarrow M$. Since i is a proper map, for every $q \in \mathbb{N}$, there exists an induced homomorphism in cohomology

$$i^* : H_c^q(M;\mathbb{Z}_2) \longrightarrow H_c^q(U;\mathbb{Z}_2).$$

Furthermore, one can verify that there exists a natural homomorphism

$$i_\# : H_c^q(U;\mathbb{Z}) \to H_c^q(M;\mathbb{Z}).$$

To define the homomorphism $i_\#$ consider $K \in \mathbf{I}_U$. By the excision axiom in cohomology, the inclusion of pairs $i : (U, U-K) \longrightarrow (M, M-K)$ induces an isomorphism

$$i^* : H^q(M, M-K;\mathbb{Z}_2) \longrightarrow H^q(U, U-K;\mathbb{Z}_2).$$

Denote $\{H_c^q(U;\mathbb{Z}_2), \varphi_K\}_{K\in\mathbf{I_U}}$ the direct limit of the direct system $\{H^q(U, U-K;\mathbb{Z}_2), \varphi_{K',K}\}_{K,K'\in\mathbf{I}_U}$ and $\{H_c^q(M;\mathbb{Z}_2), \phi_K\}_{K\in\mathbf{I}_M}$ the direct limit of the direct system $\{H^q(M, M-K;\mathbb{Z}_2), \phi_{K',K}\}_{K,K'\in\mathbf{I}_M}$.

Define the composition

$$\psi_K := \phi_K \circ (i^*)^{-1} : H^q(U, U-K;\mathbb{Z}_2) \longrightarrow H_c^q(M;\mathbb{Z}_2).$$

Note that

$$\begin{aligned} \psi_{K'} \circ \varphi_{K',K} &= \phi_{K'} \circ i^{*-1} \circ \varphi_{K',K} \\ &= \phi_{K'} \circ \phi_{K',K} \circ i^{*-1} \\ &= \phi_K \circ i^{*-1} \\ &= \phi_K \end{aligned}$$

Using the universal property, there exists a unique homomorphism

$$i_\# : H_c^q(U;\mathbb{Z}_2) \longrightarrow H_c^q(M;\mathbb{Z}_2)$$

such that $i_\# \circ \varphi_K = \psi_K$.

Using the definitions of homology with closed support and cohomology with compact support, by [1], we can consider the cup product

$$\begin{gathered} \smile: H^i(M;\mathbb{Z}_2) \times H^j(M;\mathbb{Z}_2) \longrightarrow H^{i+j}(M;\mathbb{Z}_2) \text{ or} \\ \smile: H_c^i(M;\mathbb{Z}_2) \times H^j(M;\mathbb{Z}_2) \longrightarrow H_c^{i+j}(M;\mathbb{Z}_2) \text{ or} \\ \smile: H_c^i(M;\mathbb{Z}_2) \times H_c^j(M;\mathbb{Z}_2) \longrightarrow H_c^{i+j}(M;\mathbb{Z}_2), \end{gathered}$$

the cap product

$$\begin{gathered} \frown: H^i(M;\mathbb{Z}_2) \times H_j(M;\mathbb{Z}_2) \longrightarrow H_{j-i}(M;\mathbb{Z}_2) \text{ or} \\ \frown: H^i(M;\mathbb{Z}_2) \times H_j^c(M;\mathbb{Z}_2) \longrightarrow H_{j-i}^c(M;\mathbb{Z}_2) \text{ or} \\ \frown: H_c^i(M;\mathbb{Z}_2) \times H_c^j(M;\mathbb{Z}_2) \longrightarrow H_{j-i}(M;\mathbb{Z}_2) \end{gathered}$$

and the Kronecker index

$$\langle\cdot,\cdot\rangle : H_c^i(M;\mathbb{Z}_2)\times H_i^c(M;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \text{ or}$$
$$\langle\cdot,\cdot\rangle : H^i(M;\mathbb{Z}_2)\times H_i(M;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2.$$

Since $\mathbb{Z}_2$ is a field, we can consider the simplified universal coefficient theorem:

Theorem 1.14. *Let M be a topological space. The homomorphism*

$$\begin{array}{rcl} \psi_X : H^k(X;\mathbb{Z}_2) & \longrightarrow & \mathrm{Hom}(H_k(X;\mathbb{Z}_2),\mathbb{Z}_2) \\ x & \longmapsto & \psi_X(x) : H_k(X;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \end{array}$$

is an isomorphism where, for all $y \in H_k(X;\mathbb{Z}_2)$

$$\psi_X(x)(y) = \langle x,y\rangle.$$

Moreover, we have a version of the universal coefficient theorem for compact and closed supports:

Theorem 1.15 (Universal coefficient theorem)**.** *Let M be a topological space. The homomorphism*

$$\begin{array}{rcl} \psi_X : H_c^k(X;\mathbb{Z}_2) & \longrightarrow & \mathrm{Hom}(H_k^c(X;\mathbb{Z}_2),\mathbb{Z}_2) \\ x & \longmapsto & \psi_X(x) : H_k^c(X;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \end{array}$$

is an isomorphism where, for all $y \in H_k^c(X;\mathbb{Z}_2)$

$$\psi_X(x)(y) = \langle x,y\rangle.$$

There are some important properties of the cup product, the cap product and the kronecker index:

Proposition 1.16. *Let* $f : M \longrightarrow N$ *be a proper map between topological spaces and consider the classes* $\alpha \in H^i(M;\mathbb{Z}_2), \beta \in H^i(M;\mathbb{Z}_2), \gamma \in H_{j-i}(M;\mathbb{Z}_2), \gamma' \in H_k(M;\mathbb{Z}_2)$ *Then,*

- $\langle\alpha, f_*(\beta)\rangle = \langle f^*(\alpha),\beta\rangle,$
- $\langle\alpha,\beta \frown \gamma\rangle = \langle\beta \smile \alpha,\gamma\rangle,$
- $\alpha \frown (\beta \frown \gamma') = (\alpha \smile \beta) \frown \gamma',$
- $f_*(f^*(\alpha) \frown \beta) = \alpha \frown f_*(\beta).$

Note that, we can consider homology with closed support and cohomology with compact support in Proposition 1.16.

1.2 Čech homology and Čech cohomology

In this section we will consider the definitions given in [18].

Let X a topological space and $A \subset X$. Consider $\alpha_1 = \{X_i\}_i$ an open cover of X, and $\alpha_2 = \{A_j\}_j$ an open cover of A. If $\alpha_2 \subset \alpha_1$, we say that $\alpha = (\alpha_1, \alpha_2)$ is an *open cover for the pair* (X,A).

We want to define a directed set of all open covers of the pair (X,A). For this, we say that an open cover $\beta = (\beta_1, \beta_2)$ of (X,A) is finer than, or equal to, $\alpha = (\alpha_1, \alpha_2)$, denoted by $\alpha \leq \beta$, if and only if for each open $V \in \beta_1$ exists $U \in \alpha_1$ such that $V \subset U$; and if $V \in \beta_2$, then $U \in \alpha_2$. We denote by $\Omega(X,A)$ the directed set of all open covers of the pair (X,A).

For each $\alpha = (\alpha_1, \alpha_2) \in \Omega(X,A)$, let $K_{\alpha_1}(X)$ and $K_{\alpha_2}(A)$ be the simplicial complexes whose vertices are elements of X and A, respectively. The quotient $K_\alpha(X,A) := K_{\alpha_1}(X)/K_{\alpha_2}(A)$ is a chain complex and for each $q \in \mathbb{N}$. We can consider $H_q(K_\alpha(X,A);\mathbb{Z}_2)$ and $H^q(K_\alpha(X,A);\mathbb{Z}_2)$. Furthermore, for $\alpha \leq \beta \in \Omega(X,A)$, the identity map on the vertices induces a homomorphism

$$\pi_{\beta,\alpha} : K_\beta(X,A) \longrightarrow K_\alpha(X,A).$$

Hence $\{H_q(K_\alpha(X,A);\mathbb{Z}_2), \pi_{\beta,\alpha*}\}_{\Omega(X,A)}$ is an inverse system of modules and we have the direct system of modules $\{H^q(K_\alpha(X,A);\mathbb{Z}_2), \pi^*_{\beta,\alpha}\}_{\Omega(X,A)}$.

Definition 1.17. Given the direct system $\{H_q(K_\alpha(X,A);\mathbb{Z}_2), \pi^*_{\beta,\alpha}\}_{\Omega(X,A)}$, we define the *Čech homology of the pair* (X,A) by the inverse limit

$$\check{H}_q(X,A;\mathbb{Z}_2) = \varprojlim_{\Omega(X,A)} H_q(K_\alpha(X,A);\mathbb{Z}_2).$$

Definition 1.18. Given the direct system $\{H^q(K_\alpha(X,A);\mathbb{Z}_2), \pi^*_{\beta,\alpha}\}_{\Omega(X,A)}$, we define the *Čech cohomology of the pair* (X,A) by the direct limit

$$\check{H}^q(X,A;\mathbb{Z}_2) = \varinjlim_{\Omega(X,A)} H^q(K_\alpha(X,A);\mathbb{Z}_2).$$

Proposition 1.19. *Let $f : (X,A) \longrightarrow (Y,B)$ be a proper continuous map. There exist the induced maps*

$$f_* : \check{H}^c_q(X,A;\mathbb{Z}_2) \longrightarrow \check{H}^c_q(Y,B;\mathbb{Z}_2)$$

and

$$f^* : \check{H}^q_c(Y,B;\mathbb{Z}_2) \longrightarrow \check{H}^q_c(X,A;\mathbb{Z}_2).$$

Proposition 1.20. *There exist natural homomorphisms*

$$\eta_{(X,A)*} : H_q(X,A;\mathbb{Z}_2) \longrightarrow \check{H}_q(X,A;\mathbb{Z}_2)$$

and

$$\eta^*_{(X,A)} : \check{H}^q(X,A;\mathbb{Z}_2) \longrightarrow H^q(X,A;\mathbb{Z}_2)$$

which commute with the homomorphisms induced by any proper continuous map $f : (X,A) \longrightarrow (Y,B)$.

Using [4], we can consider the following definitions:

Definition 1.21. The *Čech homology with compact support* is defined by the inverse limit

$$\check{H}_q^c(X;\mathbb{Z}_2) = \varprojlim_{K\subset X} \check{H}_q(X, X-K;\mathbb{Z}_2).$$

Definition 1.22. The *Čech cohomology with compact support* is defined by the direct limit

$$\check{H}_c^q(X;\mathbb{Z}_2) = \varinjlim_{K\subset X} \check{H}^q(X, X-K;\mathbb{Z}_2).$$

Consider the composition

$$\begin{tikzcd}
H_q^c(X;\mathbb{Z}_2) \arrow[r, "\phi_K"] \arrow[rr, bend right, "\psi_K"'] & H_q(X,X-K;\mathbb{Z}_2) \arrow[r, "\eta_{(X,X-K)}"] & \check{H}_q(X,X-K;\mathbb{Z}_2)
\end{tikzcd}$$

Note that the following diagram commutes

$$\begin{tikzcd}
H_q^c(X;\mathbb{Z}_2) \arrow[r, "\phi_{K'}"] \arrow[dr, "\phi_K"'] & H_q(X,X-K';\mathbb{Z}_2) \arrow[r, "\eta_{(X,X-K')}"] \arrow[d, "\phi_{K,K'}=i_*"] & \check{H}_q(X,X-K';\mathbb{Z}_2) \arrow[d, "\phi_{K,K'}=i_*"] \\
 & H_q(X,X-K;\mathbb{Z}_2) \arrow[r, "\eta_{(X,X-K)}"] & \check{H}_q(X,X-K;\mathbb{Z}_2).
\end{tikzcd}$$

Then $\phi_{K,K'} \circ \psi_{K'} = \psi_K$.

By the universal property, there exists a unique homomorphism

$$\eta_X : H_q^c(X;\mathbb{Z}_2) \longrightarrow \check{H}_q^c(X;\mathbb{Z}_2).$$

Proposition 1.23. *For a proper continuous map $f : M \longrightarrow N$, the following diagram is commutative*

$$\begin{tikzcd}
H_q^c(X;\mathbb{Z}_2) \arrow[r, "\eta_X"] \arrow[d, "f_*"] & \check{H}_q^c(X;\mathbb{Z}_2) \arrow[d, "f_*"] \\
H_q^c(Y;\mathbb{Z}_2) \arrow[r, "\eta_Y"] & \check{H}_q^c(Y;\mathbb{Z}_2)
\end{tikzcd}$$

1.3 Steenrod squaring operations

This section aims to introduce an important family of additive cohomology operations: the Steenrod squaring operations in cohomology. Later, this family leads naturally to the definitions of Wu classes and Stiefel–Whitney classes for generalized manifolds. Assuming the existence of Steenrod squaring operations in cohomology, we will define the Steenrod squaring operation in homology.

1.3.1 Steenrod squaring operation in cohomology

Definition 1.24. For every every pair of topological spaces (X,A) and $m \geq 0$, the *Steenrod squaring operation in cohomology* is an addictive family of operations

$$\{Sq^k : H^m(X,A;\mathbb{Z}_2) \longrightarrow H^{m+k}(X,A;\mathbb{Z}_2)\}_{k\geq 0}$$

which satisfies the following properties:

1. for every continuous map $f:(X,A) \longrightarrow (Y,B)$ between pairs of topological spaces the following diagram commutes
$$\begin{tikzcd} H^m(X,A;\mathbb{Z}_2) \arrow[d, "Sq^k"] & H^m(Y,B;\mathbb{Z}_2) \arrow[l, "f^*"'] \arrow[d, "Sq^k"] \\ H^{m+k}(X,A;\mathbb{Z}_2) & H^{m+k}(Y,B;\mathbb{Z}_2); \arrow[l, "f^*"] \end{tikzcd}$$
2. for every $x \in H^m(X,A;\mathbb{Z}_2)$, $Sq^0(x) = x$, $Sq^m(x) = x \smile x$ and $Sq^k(x) = 0$ for all $k > m$;
3. for every $x \in H^{m-q}(X,A;\mathbb{Z}_2)$ and $y \in H^q(X,A;\mathbb{Z}_2)$ we have that
$$Sq^k(x \smile y) = \sum_{i+j=k} Sq^i(x) \smile Sq^j(y).$$

It was proved in [11] that this family of operations in cohomology always exists and they are unique. Using the existence of this family, for every $m \geq 0$, we consider the operation

$$\begin{array}{rcl} Sq^* : H^m(X,A;,\mathbb{Z}_2) & \longrightarrow & H^*(X,A;,\mathbb{Z}_2) \\ x & \longmapsto & \bigoplus_{i=0}^{m} Sq^i(x) \end{array}$$

Definition 1.25. The natural extension of Sq^* to $H^*(X,A;\mathbb{Z}_2)$

$$Sq : H^*(X,A;\mathbb{Z}_2) \longrightarrow H^*(X,A;\mathbb{Z}_2)$$

is called the *the total Steenrod squaring operation in cohomology.*

Remark 1.26. For $x = x_0 \oplus x_1 \cdots \oplus x_\alpha \in H^*(X,A;\mathbb{Z}_2)$, it follows that

$$\begin{aligned} Sq(x) &= Sq(x_0 \oplus x_1 \oplus \cdots \oplus x_\alpha) \\ &= Sq^*(x_0) \oplus Sq^*(x_1) \oplus \cdots \oplus Sq^*(x_\alpha) \\ &= \sum_{i=0}^{0} Sq^i(x_0) \oplus \sum_{i=0}^{1} Sq^i(x_1) \oplus \cdots \oplus \sum_{i=0}^{\alpha} Sq^i(x_\alpha) \\ &= \sum_{i+j=0} Sq^i(x_j) \oplus \sum_{i+j=1} Sq^i(x_j) \oplus \cdots \oplus \sum_{i+j=2\alpha} Sq^i(x_j). \end{aligned}$$

By the characterization given in 1.26, the naturality of each Sq^k implies the naturality of Sq.

Considering the properties of the Steenrod squaring operation for cohomology, we have this important result (see [19]):

Proposition 1.27. *For every $x, y \in H^*(X, A; \mathbb{Z}_2)$, the total Steenrod squaring operation in cohomology satisfies the following*

- $Sq(x \smile y) = Sq(x) \smile Sq(y)$*;*
- $Sq(x \times y) = Sq(x) \times Sq(y)$

Remark 1.28. The total Steenrod squaring operation in cohomology is an automorphism (see [19]). Denote $\overline{Sq}$ as the inverse of the automorphism Sq.

There are some interesting results about $\overline{Sq}$.

Proposition 1.29. *For all $x, y \in H^*(X, A; \mathbb{Z}_2)$,*

$$\overline{Sq}(y) \smile \overline{Sq}(x) = \overline{Sq}(x \smile y).$$

Proof. Considering Proposition1.27, for every $x, y \in H^*(X, Y; \mathbb{Z}_2)$, it follows that

$$\begin{aligned} Sq(\overline{Sq}(x) \smile \overline{Sq}(y)) &= Sq(\overline{Sq}(x)) \smile Sq(\overline{Sq}(y)) \\ &= x \smile y \\ &= Sq(\overline{Sq}(x \smile y)). \end{aligned}$$

Since Sq is an automorphism we have that

$$\overline{Sq}(y) \smile \overline{Sq}(x) = \overline{Sq}(x \smile y).$$

□

Proposition 1.30. *The naturality of Sq provides the naturality of $\overline{Sq}$.*

Proof. Let $f : (X, A) \longrightarrow (Y, B)$ be a continuous proper map between topological spaces. Using the naturality of Sq,

$$\begin{aligned} f^* \circ \overline{Sq} &= \overline{Sq} \circ Sq \circ f^* \circ \overline{Sq} \\ &= \overline{Sq} \circ f^* \circ Sq \circ \overline{Sq} \\ &= \overline{Sq} \circ f^*. \end{aligned}$$

□

1.3.2 Steenrod squaring operation in homology

We define the Steenrod squaring operation in homology considering a transpose operation for Sq (see [19]).

Definition 1.31. For every every pair (X,A) of topological spaces, the *total Steenrod squaring operation in homology* is an operation

$$\widetilde{Sq} : H_*(X,A;\mathbb{Z}_2) \longrightarrow H_*(X,A;\mathbb{Z}_2)$$

which is characterized by the following property

$$\langle x, \widetilde{Sq}(\beta)\rangle = \langle \overline{Sq}(x), \beta\rangle$$

for $x \in H^*(X,A;\mathbb{Z}_2)$ and $\beta \in H_*(X,A;\mathbb{Z}_2)$

Proposition 1.32. *For each $a \in H^*(X,A;\mathbb{Z}_2)$ and $\beta \in H_*(X,A;\mathbb{Z}_2)$,*

$$\widetilde{Sq}(a \frown \beta) = Sq(a) \frown \widetilde{Sq}(\beta).$$

Proof. For all $x \in H^*(X,A;\mathbb{Z}_2)$:

$$\begin{aligned}
\langle x, \widetilde{Sq}(a \frown \beta)\rangle &= \langle \overline{Sq}(x), a \frown \beta\rangle \\
&= \langle \overline{Sq}(x) \smile a, \beta\rangle \\
&= \langle \overline{Sq}(x \smile Sq(a)), \beta\rangle \\
&= \langle x \smile Sq(a), \widetilde{Sq}(\beta)\rangle \\
&= \langle x, Sq(a) \frown \widetilde{Sq}(\beta)\rangle.
\end{aligned}$$

Using the universal coefficient Theorem, this proves that

$$\widetilde{Sq}(a \frown \beta) = Sq(a) \frown \widetilde{Sq}(\beta).$$

□

Proposition 1.33. *The total steenrod squaring operation in homology is natural.*

Proof. For each proper continuous map $f : (X,A) \longrightarrow (Y,B)$ between topological spaces, $x \in H^*(Y,B;\mathbb{Z}_2)$ and $\beta \in H_*(X,A;\mathbb{Z}_2)$, we have that

$$\begin{aligned}
\langle x, f_*(\widetilde{Sq}(\beta))\rangle &= \langle f^*(x), \widetilde{Sq}(\beta)\rangle \\
&= \langle \overline{Sq}(f^*(x)), \beta\rangle \\
&= \langle f^*(\overline{Sq}(x)), \beta\rangle \\
&= \langle \overline{Sq}(x), f_*(\beta)\rangle \\
&= \langle x, \widetilde{Sq}(f_*(\beta))\rangle.
\end{aligned}$$

Using the universal coefficient Theorem, this proves that

$$f_* \circ \widetilde{Sq} = \widetilde{Sq} \circ f_*.$$

□

The Steenrod squaring operations can be defined in homology with closed support and cohomology with compact support. Therefore, all results extend to the compact and closed support setting.

CHAPTER 2

Generalized manifolds and Stiefel-Whitney Classes

In the paper "A Primary Obstruction to Topological Embeddings and Its Applications", Carlos Biasi, Janney Daccach and Osamu Saeki defined a primary obstruction using the smooth structure on topological manifolds (see [2]). Later, they extended their results to a more general class of topological spaces, not necessarily with a differentiable structure (see [1]). These spaces are required to have local homology at each point isomorphic to the local homology at each point of a euclidean space.

In this chapter, we aim to define such spaces, known as generalized manifolds, and explore some of their properties. Moreover, in order to define an analogous obstruction, we will need to study the notion of Stiefel–Whitney classes for generalized manifolds.

2.1 Generalized manifolds

This section provides a rigorous definition of generalized manifolds, along with some illustrative examples that will be important in this work. Throughout, topological manifolds are assumed to be second-countable, Hausdorff, and locally Euclidean. The definitions and some of the properties presented in this section can be found in [8], [14], [17], [20] and [22].

Definition 2.1. A topological space X is said to be a *locally compact space* if, for every $x \in X$, there exists a neighborhood $V_x \subset X$ and a compact $K \subset X$ such that $x \in V_x \subset K$.

Note that a locally compact Hausdorff space can be characterized as a topological space that admits a basis of precompact open subsets.

Definition 2.2. Let M be a topological space, $U \subset M$ an open subset, and let $\mathscr{U} = \{U_\alpha \in M\}_{\alpha \in A}$ be an open cover of U:

- $\mathscr{U}$ is a *locally finite cover* of U if, for every $x \in U$, there exists an open neighborhood $A_x \subset U$ of x such that A_x intersects only finitely many elements of $\mathscr{U}$;
- an open cover $\mathscr{V} = \{V_\beta \in M\}_{\beta \in B}$ of U is a *refinement* of $\mathscr{U}$ if, for every $V_\beta \in \mathscr{V}$, there exists $\alpha \in A$ such that $V_\beta \subset U_\alpha$;
- U is a *paracompact space* if every open cover of U admits a locally finite refinement;
- M is *hereditarily paracompact* if every open subset of M is paracompact.

Remark 2.3. Every hereditarily paracompact space is paracompact.

Definition 2.4. A topological space M is said to be *homologically locally connected* ($HLC^\infty_{\mathbb{Z}_2}$) if, for every $x \in M$ and every open neighborhood $U_x \in M$ of x, there exists another neighborhood $V_x \subset U_x$ of x such that the inclusion $i : V_x \hookrightarrow U_x$ induces a trivial homomorphism in the reduced homology

$$i_* : \tilde{H}_q(V_x;\mathbb{Z}_2) \longrightarrow \tilde{H}_q(U_x;\mathbb{Z}_2)$$

for every $q \in \mathbb{N}$.

At this point, we can introduce the principal topological object that will play a central role in this work.

Definition 2.5. A locally compact Hausdorff space M is said to be a *generalized manifold of dimension m* if:

(i) M is hereditarily paracompact;

(ii) for every open subset $U \subset M$, there exists $l \in \mathbb{N}$ such that $H^{l+1}_c(U,\mathbb{Z}_2) \cong 0$, that is, $dim_{\mathbb{Z}_2} M < \infty$;

(iii) M is $HLC^\infty_{\mathbb{Z}_2}$;

(iv) for every $x \in M$ and $q \in \mathbb{N}$, we have that

$$H_q(M, M-\{x\};\mathbb{Z}_2) \cong H_q(\mathbb{R}^m, \mathbb{R}^m - \{0\};\mathbb{Z}_2).$$

Remark 2.6. The item (i) is important to establish that every open subset of a generalized manifold is itself a generalized manifold. Observe that every open subset of a locally compact Hausdorff space is also a locally compact Hausdorff space.

The items (i) and (iii) are important to show that the sheaf theoretic homologies and cohomologies are equivalent to the singular homologies and cohomologies (see [1] and [8]). Using this, we may consider the definitions of homology with closed support and cohomology with compact support given in Section 1.1.2.

The item (iv) provides that the dimension of a generalized manifold is a topological invariant.

Proposition 2.7. *Let M be an m- dimensional generalized manifold. Then every open subset of M is also an m-dimensional generalized manifold.*

Proof. Let M be a generalized manifold and $U \subset M$ an open subset. Observe that every open subset of U is an open subset of M. In this way, to prove that U is also a generalized manifold we proceed as follows:

- since M is hereditarily paracompact, every open subset of U is also paracompact;
- for every open subset $\tilde{U} \subset U$, there exists $l \in \mathbb{N}$ such that $H_c^{l+1}(\tilde{U};\mathbb{Z}_2) \cong 0$. This prove that $dim_{\mathbb{Z}_2}M < \infty$;
- let $x \in U$ and $\tilde{U}_x$ be a neighborhood for x. Since $\tilde{U}_x$ is also a neighborhood of x in M, there exists another neighborhood $V_x \subset \tilde{U}_x$ of x such that the inclusion $i : V_x \hookrightarrow \tilde{U}_x$ induces the zero homomorphism in the reduced homology. Since V_x is also an open subset of U, it follows that U is $HLC^{\infty}_{\mathbb{Z}_2}$;
- for every open subset $U \subset M$, by the excision theorem, the inclusion $i : U \hookrightarrow M$ induces an isomorphism
$$i_* : H_q(U,U-\{x\};\mathbb{Z}_2) \xrightarrow{\cong} H_q(M,M-\{x\};\mathbb{Z}_2)$$
for all $x \in U$ and
$$H_q(U,U-\{x\};\mathbb{Z}_2) \cong H_q(\mathbb{R}^m,\mathbb{R}^m-\{0\};\mathbb{Z}_2)$$

□

In Chapter 4, we shall study a specific class of topological spaces, the ENR $\mathbb{Z}_2$-homology spaces. Note that, for a topological space X and a subset $A \subset X$, we say that A is a *retract* of X if there exists a continuous map $r : X \longrightarrow A$ such that $r \circ i = id_A$, where $i : A \hookrightarrow X$.

Definition 2.8. A topological space M is said to be a *Euclidean neighborhood retract* (ENR) if there exists a positive integer n, an embedding $\phi : M \longrightarrow \mathbb{R}^n$ and a neighborhood $V \subset \mathbb{R}^n$ of the image $\phi(M)$ such that $\phi(M)$ is a retract of V.

Definition 2.9. We define an ENR $\mathbb{Z}_2$-homology manifold as a topological space M that is an ENR space and satisfies the (iv) property in Definition 2.5

Note that, an ENR $\mathbb{Z}_2$-homology manifold is an important example of generalized manifold (see [1]). Using this fact, since every topological manifold is an ENR $\mathbb{Z}_2$-homology manifold, we assert that every topological manifold is a generalized manifold.

Another important class of topological spaces are the absolute neighborhood retracts (ANR).

Definition 2.10. A topological space M is called an *absolute neighborhood retract* (ANR) if, for every normal space X, every closed subset $A \subset X$, and every continuous map $f : A \longrightarrow M$, we can extended this map over some neighborhood of A in X.

For normal spaces, there is an important characterization of ANR spaces (see [17]):

Proposition 2.11. *A normal space X is an absolute neighborhood retract (ANR) if and only if, whenever X is embedded as a closed subset of a normal space Y, there exists a neighborhood of X in Y that retracts onto X.*

Note that, every paracompact Hausdorff space is a normal space and every generalized manifold is paracompact. Therefore, we conclude that every *ENR* $\mathbb{Z}_2$- homology manifold is a normal space. By Proposition 2.11, if a topological space M is an *ENR* $\mathbb{Z}_2$- homology manifold, then M is an *ANR* space. Using this fact, the homomorphism given in Section 1.2

$$\eta_M : H_q^c(M;\mathbb{Z}_2) \longrightarrow \check{H}_q^c(M;\mathbb{Z}_2)$$

is an isomorphism.

2.2 Poincaré Duality

The work of Borel (see [5], [6]) and Bredon (see [7]), establishes that the Poincaré duality holds for all generalized manifolds when we consider the coefficients in $\mathbb{Z}_2$. Based on this, for an m-dimensional generalized manifold M, there exists a class $[M] \in H_m^c(M;\mathbb{Z})$, called the *fundamental class of M*, such that, by the Poincaré duality, the homomorphism

$$\begin{aligned} D_M : H^i(M;\mathbb{Z}_2) &\longrightarrow H_{m-i}^c(M;\mathbb{Z}_2) \\ \alpha &\longmapsto \alpha \frown [M] \end{aligned}$$

or

$$\begin{aligned} D_M : H_c^i(M;\mathbb{Z}_2) &\longrightarrow H_{m-i}(M;\mathbb{Z}_2) \\ \beta &\longmapsto \beta \frown [M] \end{aligned}$$

is an isomorphism for each $\alpha \in H^i(M;\mathbb{Z}_2)$ and $\beta \in H_c^i(M;\mathbb{Z}_2)$.

Remark 2.12. If we consider M a connected generalized manifolds with dimension m, we have that $[M]$ is the generator of $H_m^c(M;\mathbb{Z}_2)$.

Using the Poincaré duality, for each generalized manifold and its open subsets, we can show that:

Proposition 2.13. *Let M be an m-dimensional generalized manifold, $U \subset M$ an open subset and $i : U \hookrightarrow M$ the inclusion map. The following diagram commutes:*

$$\begin{tikzcd}
H_c^q(M;\mathbb{Z}_2) \arrow[r, "D_M"] & H_{m-q}(M;\mathbb{Z}_2) \\
H_c^q(U;\mathbb{Z}_2) \arrow[r, "D_U"] \arrow[u, "i_\#"] & H_{m-q}(U;\mathbb{Z}_2) \arrow[u, "i_*"]
\end{tikzcd}$$

where $i_\# : H_c^q(M;\mathbb{Z}_2) \longrightarrow H_c^q(U;\mathbb{Z}_2)$ is given in Section 1.1.

Proof. See [12].

□

Corollary 2.14. *Let N be an $m+k$-dimensional generalized manifold, $V \subset N$ an open subset and $i : V \hookrightarrow N$ the inclusion. The dual diagram is commutative*

$$\begin{tikzcd}
\mathrm{Hom}(H_c^q(N;\mathbb{Z}_2),\mathbb{Z}_2) \arrow[d, "i_\#^*"] & \mathrm{Hom}(H_{m+k-q}(N;\mathbb{Z}_2),\mathbb{Z}_2) \arrow[l, "(D_N)^*"'] \arrow[d, "(i_*)^*"] \\
\mathrm{Hom}(H_c^q(V;\mathbb{Z}_2),\mathbb{Z}_2) & \mathrm{Hom}(H^c_{m+k-q}(V;\mathbb{Z}_2),\mathbb{Z}_2) \arrow[l, "(D_V)^*"']
\end{tikzcd}$$

where $i_\#^*, (D_N)^*, (D_V)^*$ and $(i_*)^*$ are the dual homomorphism for $i_\#, D_N, D_V$ and i_*, respectively .

Proof. For all $\alpha \in \mathrm{Hom}(H_{m+k-q}(N;\mathbb{Z}_2),\mathbb{Z}_2)$,

$$\begin{aligned}
(i_\#^* \circ (D_N)^*)(\alpha) &= i_\#^*((D_N)^*(\alpha)) \\
&= i_\#^*(\alpha \circ D_N) \\
&= \alpha \circ D_N \circ i_\#
\end{aligned}$$

and

$$\begin{aligned}
((D_V)^* \circ (i_*)^*)(\alpha) &= (D_V)^*((i_*)^*(\alpha)) \\
&= (D_V)^*(\alpha \circ i_*) \\
&= \alpha \circ i_* \circ D_V.
\end{aligned}$$

By the Proposition 2.13,

$$\alpha \circ D_N \circ i_\# = \alpha \circ i_* \circ D_V.$$

This prove that the diagram commutes.

□

2.3 Stiefel-Whitney class

In this section, we introduce the Wu classes for generalized manifolds, and use them to define the Stiefel–Whitney classes of a generalized manifold.

2.3.1 The Wu class of a generalized manifold

Let M be an m-generalized manifold. For all $\alpha \in H^k(M;\mathbb{Z}_2)$ and $\beta \in H_c^{m-k}(M;\mathbb{Z}_2)$, we define the homomorphism

$$\phi : H^k(M;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_c^{m-k}(M;\mathbb{Z}_2),\mathbb{Z}_2),$$

where $\phi(\alpha)$ is the homomorphism given by

$$\begin{array}{rcl}\phi(\alpha) : H_c^{m-k}(M;\mathbb{Z}_2) & \longrightarrow & \mathbb{Z}_2\\ \beta & \longmapsto & \langle \alpha, \beta \frown [M]\rangle = \langle \beta \smile \alpha, [M]\rangle.\end{array}$$

Let $D_M^* : \mathrm{Hom}(H_k(M;\mathbb{Z}_2),\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_c^{m-k}(M;\mathbb{Z}_2),\mathbb{Z}_2)$ be the dual homomorphism of the Poincaré duality in M and $\psi_M(\alpha) : H^k(M;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_k(M;\mathbb{Z}_2),\mathbb{Z}_2)$ the isomorphism given by the universal coefficient Theorem. Observe that

$$\begin{array}{rcl}D_M^* \circ \psi_M(\alpha)(\beta) & = & \psi_M(\alpha)(D_M(\beta))\\ & = & \langle \alpha, \beta \frown [M]\rangle\\ & = & \langle \beta \smile \alpha, [M\rangle]\\ & = & \phi(\alpha)(\beta).\end{array}$$

This prove that ϕ is an isomorphism.

Consider the homomorphism

$$\begin{array}{rcl}\langle Sq^k(\cdot), [M]\rangle : H_c^{m-k}(M;\mathbb{Z}_2) & \longrightarrow & \mathbb{Z}_2\\ x & \longmapsto & \langle Sq^k(x), [M]\rangle.\end{array}$$

Using the fact that ϕ is an isomorphism, there exists a unique class $v_k \in H^k(M,\mathbb{Z}_2)$ such that

$$\langle x \smile v_k, [M]\rangle = \langle Sq^k(x), [M]\rangle,$$

for all $x \in H_c^{m-k}(M,\mathbb{Z}_2)$.

By definition 1.24, for each $x \in H_c^{m-k}(M;\mathbb{Z}_2)$, we have that $Sq^k(x) = 0$ for $k > m-k$. Then, $v_k = 0$ for all $\frac{m}{2} < k$. Takes $k = 0$, then

$$\begin{array}{rcl}\langle x \smile v_0, [M]\rangle & = & \langle Sq^0(x), [M]\rangle\\ & = & \langle x, [M]\rangle.\end{array}$$

This ensures that $v_0 = 1$.

Definition 2.15. Under the same hypotheses as in the discussion above, the element $v_k \in H^k(M;\mathbb{Z})$ is called the *k-th Wu class* of M . The *total Wu class of M* is then given by

$$v(M) = 1 \oplus v_1 \oplus \cdots \oplus v_p,$$

where p is the smallest integer greater than $\frac{m}{2}$.

It should be noted that we can consider $[M]$ as an element of $H^*(M;\mathbb{Z}_2)$. Given an element $x = x_1 \oplus \cdots \oplus x_n \in H^*(M;\mathbb{Z}_2)$ consider $[M] = 0 \oplus \cdots \oplus [M] \oplus \cdots \oplus 0 \in H^*(M;\mathbb{Z}_2)$, then

$$\begin{aligned}
\langle x \smile v(M), [M]\rangle &= \langle \sum_{i+j=m} x_i \smile v_j, [M]\rangle \\
&= \sum_{i+j=m} \langle x_i \smile v_j, [M]\rangle \\
&= \sum_{i+j=m} \langle Sq^j(x_i), [M]\rangle \\
&= \langle \sum_{i+j=m} Sq^j(x_i), [M]\rangle \\
&= \langle Sq(x), [M]\rangle.
\end{aligned}$$

2.3.2 The Stiefel-Whitney class of a generalized manifolds

Using the total Steenrod squaring operation in cohomology and the Wu classes of a generalized manifolds, we define the Stiefel-Whitney class of a generalized manifold as follows:

Definition 2.16. Let M be an m-generalized manifold. The *Stiefel-Whitney class* of M is defined by

$$w(M) = Sq(v(M)).$$

By Remark 1.26 and the Definition 2.15, the k-th term of $w(M)$ is characterized by

$$w_k(M) = \sum_{i+j=k} Sq^i(v_j).$$

We call the class $w_k(M)$ as the *k-th Stiefel-Whitney class* of M.

Remark 2.17. The 0-th Stiefel-Whitney Class of a generalized manifolds M is $w_0(M) = 1$.

We can see that $w(M) \in H^*(M,\mathbb{Z}_2)$ admits an inverse element in the graded ring

$$(H^*(M;\mathbb{Z}_2), +, \smile)$$

and we denote

$$\overline{w}(M) := w(M)^{-1}.$$

The inverse element of $w(M) = 1 \oplus w_1 \oplus \cdots \oplus w_\beta \in H^*(M;\mathbb{Z}_2)$ is given by $\overline{w}(M) = \overline{w}_0 \oplus \overline{w}_1 \oplus \cdots \oplus \overline{w}_\beta$, where

$$\begin{aligned}
\overline{w}_0 &= 1, \\
\overline{w}_1 &= w_1 \\
&\cdots \\
\overline{w}_p &= \sum_{i=1}^{p-1} w_i \smile \overline{w}_{p-i} + w_p \; for \; p \in \{2, 3, \cdots, \beta\}.
\end{aligned}$$

In fact, since we are considering coefficients in $\mathbb{Z}_2$, it follows that

- the term of degree 0 of $w(M) \smile \overline{w}(M)$ is

$$\begin{aligned}[w(M) \smile \overline{w}(M)]_0 &= 1 \smile \overline{w}_0 \\ &= 1 \smile 1 \\ &= 1;\end{aligned}$$

- the term of degree 1 of $w(M) \smile \overline{w}(M)$ is

$$\begin{aligned}[w(M) \smile \overline{w}(M)]_1 &= w_0 \smile \overline{w}_1 + w_1 \smile \overline{w}_0 \\ &= w_1 + w_1 \\ &= 0;\end{aligned}$$

- the term of degree 2 of $w(M) \smile \overline{w}(M)$ is

$$\begin{aligned}[w(M) \smile \overline{w}(M)]_2 &= 1 \smile \overline{w}_2 + w_1 \smile \overline{w}_1 + w_2 \smile \overline{w}_0 \\ &= (w_1 \smile w_1 + w_2) + (w_1 \smile w_1 + w_2 \smile 1) \\ &= 0;\end{aligned}$$

- for $p > 1$, the term $p+1$ of $w(M) \smile \overline{w}(M)$ is

$$\begin{aligned}[w(M) \smile \overline{w}(M)]_{p+1} &= \overline{w}_{p+1} + \sum_{j=1}^{p+1} w_j \smile \overline{w}_{p+1-j} \\ &= \overline{w}_{p+1} + (\sum_{j=1}^{p} w_j \smile \overline{w}_{p+1-j}) + (w_{p+1} \smile \overline{w}_0) \\ &= \overline{w}_{p+1} + \overline{w}_{p+1} \\ &= 0.\end{aligned}$$

Since we are working with coefficients in $\mathbb{Z}_2$ it is not necessary to verify that $\overline{w}(M) \smile w(M) = 1$.

Now, we present some examples of Stiefel–Whitney classes that will be relevant for this work (see [19]).

Exemple 2.18. Consider $M = \mathbb{R}^m$ with $m > 0$. For each $k > 0$, $H^k(M;\mathbb{Z}_2) \cong \{0\}$ then $v(M) = v_0 = 1$ and

$$\begin{aligned}w(M) &= Sq(1) \\ &= Sq^0(1) \oplus Sq^1(1) \oplus \cdots \oplus Sq^m(1).\end{aligned}$$

Observe that, for each $j > 0$, we have that $Sq^j(1) = 0$. This shows that $w(\mathbb{R}^m) = \overline{w}(\mathbb{R}^m) = 1$.

Exemple 2.19. Consider $M = \mathbb{R}P^n$ with $n = 2^k$ and $k \geq 0$. Then

$$w(\mathbb{R}P^n) = 1 \oplus a^1 \oplus a^n$$

and

$$\overline{w}(\mathbb{R}P^n) = 1 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^{n-1},$$

where a^k is the generator of $H^k(\mathbb{R}P^n; \mathbb{Z}_2)$.

Exemple 2.20. For $M = \mathbb{R}P^n$, we have that $w(\mathbb{R}P^n) = 1$ if and only if $(n+1) = 2^k$, for some $k \geq 0$.

Let M be a generalized manifold. We can characterized the image of the fundamental class $[M]$ by the total Steenrod squaring operation in cohomology using the following proposition:

Proposition 2.21. *For each generalized manifold M,*

$$\widetilde{Sq}([M]) = \overline{w}(M) \frown [M].$$

Proof. By the characterization of $w(M)$, for all $x \in H^*(M; \mathbb{Z}_2)$,

$$\begin{aligned} \langle x \smile \overline{w}(M), [M] \rangle &= \langle Sq \circ \overline{Sq}(x \smile \overline{w}(M)), [M] \rangle \\ &= \langle \overline{Sq}(x \smile \overline{w}(M)) \smile v(M), [M] \rangle. \end{aligned}$$

Since $w(M) = Sq(v(M))$, it follows that

$$v(M) = \overline{Sq}(w(M))$$

and then

$$\begin{aligned} \langle x, \overline{w}(M) \frown [M] \rangle &= \langle x \smile \overline{w}(M), [M] \rangle \\ &= \langle \overline{Sq}(x \smile \overline{w}(M)) \smile v(M), [M] \rangle \\ &= \langle \overline{Sq}(x \smile \overline{w}(M)) \smile \overline{Sq}(w(M)), [M] \rangle \\ &= \langle \overline{Sq}(x \smile \overline{w}(M) \smile w(M)), [M] \rangle \\ &= \langle \overline{Sq}(x), [M] \rangle \\ &= \langle x, \widetilde{Sq}[M] \rangle. \end{aligned}$$

Using the universal coefficient Theorem,

$$\widetilde{Sq}([M]) = \overline{w}(M) \frown [M].$$

□

CHAPTER 3

The classes $\theta(f)$ and θ_f

In this chapter, we define the main algebraic object of this work. Given a proper continuous map between generalized manifolds, we associate to it a Čech homology class with closed support that become an obstruction to the existence of a homotopy between this map and a proper embedding. In other words, if this class does not vanishes, the map cannot be proper homotopic to a proper embedding. To consider this class associated with a proper continuous map between generalized manifolds, we first introduce the following definitions:

Definition 3.1. Let $f : M \longrightarrow N$ be a proper continuous map between generalized manifolds of dimensions m and $m+k$, respectively, with $k > 0$. The *total Stiefel-Whitney class of the stable normal bundle* of f is defined by the expression

$$w(f) = f^*(w(N)) \smile \overline{w}(M) \in H^*(M;\mathbb{Z}_2).$$

The element of degree q of $w(f)$ can be expressed as

$$w_q(f) = \sum_{i+j=q} f^*(w_i(N)) \smile \overline{w}_j(M) \in H^q(M;\mathbb{Z}_2).$$

Definition 3.2. Consider $f : M \longrightarrow N$ a proper continuous map between an m-generalized manifold into an $m+k$-generalized manifold, $k > 0$. Using the Poincaré duality, there exists a unique class $U_f \in H^k(N)$ such that

$$U_f \frown [N] = f_*([M]) \in H^c_m(N).$$

The class U_f is called the *Poincaré dual* class of $f_*([M])$.

At this point, we have all the definitions required to define the obstruction, which was studied by Carlos Biasi, Janney Daccach and Osamu Saeki in [1], and using the smooth structure in [2].

Definition 3.3. Let M be an m-generalized manifold and N an $m+k$-generalized manifold, $k > 0$. For every proper continuous map $f : M \longrightarrow N$, we define the class

$$\theta(f) = \big(f^*(U_f) - w_k(f)\big) \frown [M] \in H^c_{m-k}(M,\mathbb{Z}_2).$$

The homology class given in Definition 3.3 is invariant under proper homotopies.

Remark 3.4. We say that two proper continuous maps $f : M \longrightarrow N$ and $g : M \longrightarrow N$ are *properly homotopic* if there exists a homotopy $F : M \times [0,1] \longrightarrow N$ between f and g, such that F a proper continuous map.

Proposition 3.5. *Let $f : M \longrightarrow N$ be a proper continuous map between an m-generalized manifold into an $m+k$-generalized manifold, $k > 0$. The class $\theta(f) \in H^c_{m-k}(M;\mathbb{Z}_2)$ is invariant under proper homotopies.*

Proof. Suppose that $g : M \longrightarrow N$ is a proper continuous map in the same way of f. If g is properly homotopic to f we have that $f^* = g^*$. Then

$$\begin{aligned} w_k(g) &= \sum_{i+j=q} g^*(w_i(N)) \smile \overline{w}_j(M) \\ &= \sum_{i+j=q} f^*(w_i(N)) \smile \overline{w}_j(M) \\ &= w_k(f). \end{aligned}$$

Furthermore, $U_f, U_g \in H^k(N;\mathbb{Z}_2)$ are the only classes such that

$$\begin{aligned} U_f \frown [N] &= f_*([M]) \\ &= g_*([M]) \\ &= U_g \frown [N] \end{aligned}$$

By the Poincaré duality, $U_f = U_g$. It follows that

$$\begin{aligned} \theta(f) &= \big(f^*(U_f) - w_k(f)\big) \frown [M] \\ &= \big(g^*(U_g) - w_k(g)\big) \frown [M] \\ &= \theta(g). \end{aligned}$$

□

For every proper continuous map between generalized manifolds, we can discuss some interesting results using the class introduced in Definition 3.3.

Theorem 3.6. *Let $f : M \longrightarrow N$ be a proper continuous map between an m-dimensional generalized manifold into a $m+k$-dimensional generalized manifold, with $k > 0$. Then $f_*(\theta(f)) \in H^c_{m-k}(N;\mathbb{Z}_2)$ always vanishes.*

Proof. By the definition of $\theta(f)$ given in 3.3, we note that

$$\begin{aligned} f_*(\theta(f)) &= f_*\big((f^*(U_f) - w_k(f)) \frown [M]\big) \\ &= f_*\big((f^*(U_f) \frown [M]) - (w_k(f) \frown [M])\big) \\ &= f_*(f^*(U_f) \frown [M]) - f_*(w_k(f) \frown [M]) \\ &= \underbrace{U_f \frown f_*([M])}_{①} - \underbrace{f_*(w_k(f) \frown [M])}_{②}. \end{aligned}$$

If we prove that ① = ②, then $f_*(\theta(f)) = 0$.

Analyzing ①, we observe that

$$U_f \frown f_*([M]) = U_f \frown (U_f \frown [N]) = (U_f \smile U_f) \frown [N].$$

The term of degree $m-k$ in $f_*(w(f) \frown [M])$ is precisely the element represented by ②. Therefore, we have to show that $(U_f \smile U_f) \frown [N]$ is equal to the $m-k$-term of $f_*(w(f) \frown [M])$.

Since $w(f) = f^*(w(N)) \smile \overline{w}(M)$, it follows that

$$\begin{aligned} f_*(w(f) \frown [M]) &= f_*\big((f^*(w(N)) \smile \overline{w}(M)) \frown [M]\big) \\ &= f_*\big(f^*(w(N)) \frown (\overline{w}(M) \frown [M])\big) \\ &= w(N) \frown f_*(\overline{w}(M) \frown [M]) \\ &= w(N) \frown f_* \circ \widetilde{Sq}([M]) \\ &= Sq(v(N)) \frown \widetilde{Sq} \circ f_*([M]) \\ &= \widetilde{Sq}(v(N) \frown f_*([M])) \\ &= \widetilde{Sq}(v(N) \frown (U_f \frown [N])) \\ &= \widetilde{Sq}\big((v(N) \smile U_f) \frown [N]\big). \end{aligned}$$

Using the universal coefficient theorem, for all $\varepsilon \in H_c^{m-k}(N;\mathbb{Z}_2)$, we have to show that

$$\langle \varepsilon, \Delta \rangle = \langle \varepsilon, (U_f \smile U_f) \frown [N] \rangle = \langle \varepsilon \smile (U_f \smile U_f), [N] \rangle,$$

where Δ denotes the $m-k$-term of

$$f_*(w(f) \frown [M]) = \widetilde{Sq}\big((v(N) \smile U_f) \frown [N]\big).$$

Observe that,

$$\begin{aligned} \langle \varepsilon, \widetilde{Sq}((v(N) \smile U_f) \frown [N]) \rangle &= \langle \overline{Sq}(\varepsilon), (v(N) \smile U_f) \frown [N] \rangle \\ &= \langle \overline{Sq}(\varepsilon) \smile (v(N) \smile U_f), [N]) \rangle \\ &= \langle (\overline{Sq}(\varepsilon) \smile U_f) \smile v(N), [N \rangle] \\ &= \langle Sq(\overline{Sq}(\varepsilon) \smile U_f), [N] \rangle \\ &= \langle \varepsilon \smile Sq(U_f), [N] \rangle \end{aligned}$$

Recall that, in order to compute the Kronecker index $\langle \varepsilon \smile Sq(U_f), [N]\rangle$, we need the $2k$-term of $Sq(U_f)$. The $2k$-term of $Sq(U_f)$ is precisely

$$[Sq(U_f)]_{2k} = U_f \smile U_f.$$

It follows that

$$\langle \varepsilon, \Delta\rangle = \langle \varepsilon \smile (U_f \smile U_f), [N]\rangle$$

and this prove the equality $f_*(\theta(f)) = 0$.

□

For each proper continuous map $f : M \longrightarrow N$ between generalized manifolds, we aim to study the relation between $\theta(f)$ and $\theta(f_V)$, where $f_V : M \longrightarrow V$ denotes the restriction of the map f to an open $V \subset N$ which contains $f(M)$.

Lemma 3.7. *Let $f : M \longrightarrow N$ be a proper continuous map between an m-dimensional generalized manifold into a $(m+k)$-dimensional manifold, $k > 0$, and let $V \subset N$ be an open subset such that $f(M) \subset V$. Then*

$$\theta(f) = \theta(f_V) \in H^c_{m-k}(M, \mathbb{Z}_2),$$

where $f_V : M \xrightarrow{f} V$.

Proof. Under the hypothesis of Lemma 3.7, since V is an open subset of a generalized manifold with dimension $m+k$, then V is itself a generalized manifold with dimension $m+k$.

Consider the natural homomorphism induced in cohomology by the inclusion $i : V \hookrightarrow N$ (see Section 1.1.2),

$$i_\# : H^m_c(V;\mathbb{Z}_2) \longrightarrow H^m_c(N;\mathbb{Z}_2).$$

Let

$$i^*_\# : \mathrm{Hom}(H^m_c(N;\mathbb{Z}_2), \mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H^m_c(V;\mathbb{Z}_2), \mathbb{Z}_2)$$

be the dual homomorphism of $i_\#$.

Define the isomorphisms

$$\varphi_N := (D_N)^* \circ \psi_N \circ D_N^{-1} : H^c_{m+k-q}(N;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H^{m+k-q}_c(N;\mathbb{Z}_2), \mathbb{Z}_2)$$

and

$$\varphi_V := (D_V)^* \circ \psi_V \circ D_V^{-1} : H^c_{m+k-q}(V;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H^{m+k-q}_c(V;\mathbb{Z}_2), \mathbb{Z}_2).$$

The following diagram is commutative (see Corollary 2.14 and Remark 3.8)

$$\begin{tikzcd}
 & H^c_{m+k-q}(N;\mathbb{Z}_2) \arrow[r, "\varphi_N"] & \mathrm{Hom}(H^{m+k-q}_c(N;\mathbb{Z}_2),\mathbb{Z}_2) \arrow[r, "(D_N^{-1})^*"] \arrow[dd, "i^*_\#"] & \mathrm{Hom}(H_q(N;\mathbb{Z}_2)\mathbb{Z}_2) \arrow[r, "\psi_N^{-1}"] \arrow[dd, "(i_*)^*"] & H^q(N;\mathbb{Z}_2) \arrow[dd, "i^*"] \\
H^c_{m+k-q}(M;\mathbb{Z}_2) \arrow[ur, bend left, "f_*"] \arrow[dr, bend right, "f_{V*}"'] & & & & \\
 & H^c_{m+k-q}(V;\mathbb{Z}_2) \arrow[r, "\varphi_V"] \arrow[uu, "i_*"] & \mathrm{Hom}(H^{m+k-q}_c(V;\mathbb{Z}_2),\mathbb{Z}_2) \arrow[r, "(D_V^{-1})^*"] & \mathrm{Hom}(H_q(V;\mathbb{Z}_2),\mathbb{Z}_2) \arrow[r, "\psi_V^{-1}"] & H^q(N;\mathbb{Z}_2)
\end{tikzcd}$$

where

$$\psi_N : H^q(N;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_q(N;\mathbb{Z}_2),\mathbb{Z}_2)$$

and

$$\psi_V : H^q(V;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_q(V;\mathbb{Z}_2),\mathbb{Z}_2)$$

are the isomorphism defined by universal coefficient Theorem in 1.14.

From the commutative diagram, we obtain that

$$\begin{aligned}
\varphi_V \circ f_{V*}([M]) &= i_{\#}^* \circ \varphi_N \circ f_*([M]) \\
&= i_{\#}^* \circ \varphi_N \circ D_N(U_f) \\
&= i_{\#}^* \circ (D_N)^* \circ \psi_N(U_f) \\
&= (D_V)^* \circ \psi_V \circ i^*(U_f) \\
&= \varphi_V \circ D_V \circ i^*(U_f) \\
&= \varphi_V(i^*(U_f) \frown [V]).
\end{aligned}$$

Since φ_V is an isomorphism, we conclude that

$$f_{V*}([M]) = i^*(U_f) \frown [N]$$

and

$$U_{f_V} = i^*(U_f).$$

Moreover, we will prove that, for each $q \leq m+k$, the term of degree q of $v(V)$, denoted by $[v(V)]_q = v'_q$, is equal to the term of degree q of $i^*(v(N))$ denoted by $[i^*(v(N))]_q = i^*(v_q)$.

For each $\alpha \in H^q(N;\mathbb{Z}_2)$ we have that $\psi_N(\alpha)$ is the homomorphism

$$\alpha^* : H_q(N;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

such that $\alpha^*(\gamma) = \langle \alpha, \gamma \rangle$ for all $\gamma \in H_q(N;\mathbb{Z}_2)$.

By the Corollary 2.14, we have that

$$\alpha^* \circ D_N \circ i_{\#} = \alpha^* \circ i_* \circ D_V.$$

Using this identity, for all $x \in H_c^{m+k-q}(N;\mathbb{Z}_2)$ and $\alpha \in H^q(N;\mathbb{Z}_2)$, we obtain

$$\begin{aligned}
\langle i_{\#}(x) \smile \alpha, [N] \rangle &= \langle \alpha, i_{\#}(x) \frown [N] \rangle \\
&= \langle \alpha, D_N \circ i_{\#}(x) \rangle \\
&= \langle \alpha, i_* \circ D_V(x) \rangle \\
&= \langle i^*(\alpha), D_V(x) \rangle \\
&= \langle i^*(\alpha), x \frown [V] \rangle \\
&= \langle x \smile i^*(\alpha), [V] \rangle.
\end{aligned}$$

This prove that

$$\underbrace{\langle i_{\#}(x) \smile \alpha, [N]\rangle}_{\text{Ⓘ}} = \underbrace{\langle x \smile i^*(\alpha), [V]\rangle}_{\text{Ⓘⓘ}}$$

Now, by the definition of $v_q \in H^q(N;\mathbb{Z}_2)$, we have that

$$\underbrace{\langle i_{\#}(y) \smile v_q, [N]\rangle}_{\text{(III)}} = \underbrace{\langle Sq^q(i_{\#}(y)), [N]\rangle}_{\text{(VI)}}.$$

Considering equation Ⓘ with $x = y \in H_c^{m+k-q}(N;\mathbb{Z}_2)$ and $\alpha = v_q$, we have

$$\langle i_{\#}(y) \smile v_q, [N]\rangle = \langle y \smile i^*(v_q), [V]\rangle.$$

Considering equation Ⓘ with $x = Sq^q(y) \in H_c^{m+k}(N;\mathbb{Z}_2)$ and $\alpha = 1 \in H^0(N;\mathbb{Z}_2)$, we obtain

$$\begin{aligned}
\langle Sq^q(i_{\#}(y)), [N]\rangle &= \langle i_{\#}(Sq^q(y)) \smile 1, [N]\rangle \\
&= \langle Sq^q(y) \smile i^*(1), [V]\rangle \\
&= \langle Sq^q(y) \smile 1, [V]\rangle \\
&= \langle Sq^q(y), [V]\rangle.
\end{aligned}$$

Since (III) = (VI), this prove that $v'_q = i^*(v_q)$.

Therefore

$$\begin{aligned}
w(V) &= Sq(v(V)) \\
&= Sq \circ i^*(v(N)) \\
&= i^* \circ Sq(v(N)) \\
&= i^*(w(N)).
\end{aligned}$$

Finally, observe that $f_V^* \circ i^* = f^*$. By Definition 3.3, it follows that

$$\begin{aligned}
\theta(f) &= \left[(f^*(U_f) - w_k(f))\right] \frown [M] \\
&= \Big[f^*(U_f) - \big(\sum_{i+j=k} f^*(w_i(N)) \smile w_j(M)^{-1}\big)\Big] \frown [M] \\
&= \Big[f_V^*(U_{f_V}) - \big(\sum_{i+j=k} f_V^* \circ i^*(w_i(N)) \smile w_j(M)^{-1}\big)\Big] \frown [M] \\
&= \Big[f_V^*(U_{f_V}) - \big(\sum_{i+j=k} f_V^*(w_i(V)) \smile w_j(M)^{-1}\big)\Big] \frown [M] \\
&= \theta(f_V).
\end{aligned}$$

□

Remark 3.8. Prove that the following diagram commutes

$$\begin{tikzcd}
H^c_{m+k-q}(N;\mathbb{Z}_2) \arrow[r, "\varphi_N"] & \mathrm{Hom}(H_c^{m+k-q}(N;\mathbb{Z}_2),\mathbb{Z}_2) \arrow[d, "i^*_\#"] \\
H^c_{m+k-q}(V;\mathbb{Z}_2) \arrow[u, "i_*"] \arrow[r, "\varphi_V"] & \mathrm{Hom}(H_c^{m+k-q}(V;\mathbb{Z}_2),\mathbb{Z}_2)
\end{tikzcd}$$

is equivalent to prove that this diagram commutes

$$\begin{tikzcd}
H^c_{m+k-q}(N;\mathbb{Z}_2) \arrow[r, "D_N^{-1}"] & H^q(N;\mathbb{Z}_2) \arrow[r, "\psi_N"] \arrow[d, "i^*"] & \mathrm{Hom}(H_q(N;\mathbb{Z}_2),\mathbb{Z}_2) \arrow[r, "(D_N)^*"] \arrow[d, "(i_*)^*"] & \mathrm{Hom}(H_c^{m+k-q}(N;\mathbb{Z}_2),\mathbb{Z}_2) \arrow[d, "i^*_\#"] \\
H^c_{m+k-q}(V;\mathbb{Z}_2) \arrow[u, "i_*"] \arrow[r, "D_V^{-1}"] & H^q(V;\mathbb{Z}_2) \arrow[r, "\psi_V"] & \mathrm{Hom}(H_q(V;\mathbb{Z}_2)\mathbb{Z}_2) \arrow[r, "(D_V)^*"] & \mathrm{Hom}(H_c^{m+k-q}(V;\mathbb{Z}_2),\mathbb{Z}_2)
\end{tikzcd}$$

Note that, we just have to prove that the first and the second rectangles commute.

For each $x \in H^q(N;\mathbb{Z}_2)$

$$\begin{aligned}
i_* \circ D_V \circ i^*(x) &= i_*(i^*(x) \frown [V]) \\
&= x \frown i_*([V]) \\
&= x \frown [N] \\
&= D_N(x).
\end{aligned}$$

For each $\alpha \in H^q(N;\mathbb{Z}_2)$ and $\gamma \in H_q(V;\mathbb{Z}_2)$

$$\begin{aligned}
((i_*)^* \circ \psi_N)(\alpha)(\gamma) &= (i_*)^*(\psi_N(\alpha))(\gamma) \\
&= \psi_N(\alpha)(i_*(\gamma)) \\
&= \langle \alpha, i_*(\gamma) \rangle \\
&= \langle i^*(\alpha), \gamma \rangle \\
&= \psi_V(i^*(\alpha))(\gamma) \\
&= (\psi_V \circ i^*)(\alpha)(\gamma).
\end{aligned}$$

In Section 1.2, for every generalized manifold M, we introduced a natural homomorphism

$$\eta_M : H^c_{m-k}(M;\mathbb{Z}_2) \longrightarrow \check{H}^c_{m-k}(M;\mathbb{Z}_2).$$

In this way, for every proper continuous map $f : M \longrightarrow N$ between generalized manifolds with dimension m and $m+k$, respectively, $k > 0$, we can consider

$$\eta_M(\theta(f)) = \theta_f \in \check{H}^c_{m-k}(M;\mathbb{Z}_2).$$

If M is a compact generalized manifold or if it is an *ENR* $\mathbb{Z}_2$-homology manifold, it can be shown that η_M is an isomorphism (see [8], [18] and [22]. Using this fact, for those spaces, every time θ_f vanishes as an element of $\check{H}^c_{m-k}(M;\mathbb{Z}_2)$, then $\theta(f)$ vanishes as an element of $H^c_{m-k}(M;\mathbb{Z}_2)$.

Exists an equivalent definition to the Čech homology with compact support every time we consider a closed subset of a generalized manifold.

Lemma 3.9. *Let $f : M \longrightarrow N$ be a proper map between generalized manifolds. Then $f(M)$ is closed.*

Proof. Suppose that $f : M \longrightarrow N$ is a proper continuous map between generalized manifolds. Since every generalized manifold is a locally compact Hausdorff space we may admits that N has a basis of precompact open subsets (see [17]).

We want to prove that $N - f(M)$ is an open subset of N. For every $y \in N - f(M)$, exists an open neighborhood V_y of y where $\overline{V_y}$ is compact. Since $\overline{V_y}$ is compact in a Hausdorff space, we assert that $\overline{V_y}$ is a closed subset in N. By the properness of f, we have that $f^{-1}(\overline{V_y})$ is a compact subset of M. Note that

$$\begin{aligned} f(f^{-1}(\overline{V_y})) &= f(M \cap f^{-1}(\overline{V_y})) \\ &= f(M) \cap \overline{V_y} \end{aligned}$$

is a compact subset. This show that $f(M) \cap \overline{V_y}$ is a closed subset of N. Observe that

$$\begin{aligned} U_y &:= V_y \cap \left[N - (f(M) \cap \overline{V_y})\right] \\ &= V_y \cap \left[(N - f(M)) \cup (N - \overline{V_y})\right] \\ &= V_y \cap (N - f(M)) \end{aligned}$$

is an open neighborhood of y where $U_y \subset N - f(M)$. This shows that $f(M)$ is a closed subset of N.

□

Remark 3.10. Suppose that $f : M \longrightarrow N$ is a topological embedding between generalized manifolds and $f(M)$ is closed. Let V be a compact subset of N. Since $M \approx f(M)$,

$$\begin{aligned} V \cap f(M) &\approx f^{-1}(V) \cap M \\ &\approx f^{-1}(V). \end{aligned}$$

Observe that $V \cap f(M)$ is compact because it is closed in a compact. This implies that $f^{-1}(V)$ is compact in M, and then f is a proper topological embedding.

Corollary 3.11. *Let $f : M \longrightarrow N$ be a topological embedding between generalized manifolds. Then f is a proper continuous map if and only if $f(M)$ is closed is a subset of N.*

Let $f : M \longrightarrow N$ a proper continuous map between generalized manifolds. By Lemma 3.7, $f(M)$ is a closed subspace of N. Consider the collection $\mathbf{I} = \{V \subset N\}$ of all open neighborhoods of $f(M)$, with the following partial relation: $V \leq V'$ if and only if $V' \subset V$.

For every $V \leq V' \leq V''$ in $\mathbf{I}$, the inclusion $i : V' \hookrightarrow V$ induces a homomorphism

$$\phi_{V,V'} := i_* : H_q^c(V';\mathbb{Z}_2) \longrightarrow H_q^c(V;\mathbb{Z}_2),$$

where $\phi_{V,V} = id_{H_q^c(V;\mathbb{Z}_2)}$ and $\phi_{V,V'} \circ \phi_{V',V''} = \phi_{V,V''}$. This ensures that the family

$$\{H_q^c(V;\mathbb{Z}_2), \phi_{V,V'}\}_{V,V' \in \mathbf{I}}$$

is an inverse system.

Definition 3.12. For the inverse system $\{H_q^c(V;\mathbb{Z}_2), \phi_{V,V'}\}_{V,V' \in \mathbf{I}}$ let $\{\check{H}_{m-k}^c(f(M);\mathbb{Z}_2), \phi_V\}_{V \in \mathbf{I}}$ be its inverse limit. We denote

$$\check{H}_q^c(f(M);\mathbb{Z}_2) = \varprojlim_{V \in \mathbf{I}} H_q^c(V;\mathbb{Z}_2).$$

Remark 3.13. Using the Proposition 1.23, we can consider the commutative diagram

$$\begin{tikzcd}
H_{m-k}^c(M;\mathbb{Z}_2) \arrow[r, "\bar{f}_*"] \arrow[d, "\eta_M"] & H_{m-k}^c(f(M);\mathbb{Z}_2) \arrow[d, "\eta_{f(M)}"] \\
\check{H}_{m-k}^c(M;\mathbb{Z}_2) \arrow[r, "\bar{f}_*"] & \check{H}_{m-k}^c(f(M);\mathbb{Z}_2)
\end{tikzcd}$$

where $\eta_f(M)$ is the only homomorphism such that, for every $V \in \mathbf{I}$, $\phi_V \circ \eta_{f(M)} = i_{V*}$, that means

$$\eta_{f(M)} = \varprojlim_{V \in \mathbf{I}} i_{V*}.$$

There are some immediate corollaries of Theorem 3.6 and Lemma 3.7.

Corollary 3.14. *Let $f : M \longrightarrow N$ be a proper continuous map between an m- dimensional generalized manifold into an $(m+k)$- dimensional generalized manifold, $k > 0$. Considering the restriction map $\bar{f} := M \xrightarrow{f} f(M)$, then $\bar{f}_*(\theta_f) \in \check{H}_{m-k}^c(f(M);\mathbb{Z}_2)$ vanishes.*

Proof. Let $\mathbf{I} = \{V \subset N\}$ the collection of open neighborhoods of $f(M)$ in N. For each $V \in \mathbf{I}$ consider the proper continuous map $f_{V*} := M \xrightarrow{f} V$. Since V is open, we can apply the Theorem 3.6 and conclude that $f_{V*}(\theta(f_V)) = 0$. Using Lemma 3.7 we have that $f_{V*}(\theta(f)) = f_{V*}(\theta(f_v)) = 0$. By the inclusion $i_V : f(M) \hookrightarrow V$, we obtain

$$i_{V*} \circ \bar{f}_*(\theta(f)) = f_{V*}(\theta(f)) = 0.$$

Using the definition of $H_{m-k}^c(f(M);\mathbb{Z}_2)$ we have that

$$\varprojlim_{V \in \mathbf{I}} i_{V*} \circ \bar{f}_* \quad = \quad \eta_{f(M)} \circ \bar{f}_*.$$

Then

$$\begin{aligned}
0 &= \varprojlim_{V \in \mathbf{I}} (i_{V*} \circ \bar{f}_*)(\theta(f)) \\
&= \varprojlim_{V \in \mathbf{I}} i_{V*} \circ \bar{f}_*(\theta(f)) \\
&= \eta_{f(M)} \circ \bar{f}_*(\theta(f)) \\
&= \bar{f}_* \circ \eta_M(\theta(f)) \\
&= \bar{f}_*(\theta_f).
\end{aligned}$$

□

Corollary 3.15. *Let $f : M \longrightarrow N$ be a proper continuous map between an m- dimensional generalized manifold into an $(m+k)$-dimensional generalized manifold, $k > 0$. If*

$$\bar{f}_* : \check{H}^c_{m-k}(M;\mathbb{Z}_2) \longrightarrow \check{H}^c_{m-k}(f(M);\mathbb{Z}_2)$$

is a monomorphism , then $\theta_f \in \check{H}^c_{m-k}(M;\mathbb{Z}_2)$ vanishes.

Proof. If

$$\bar{f}_* : \check{H}^c_{m-k}(M;\mathbb{Z}_2) \longrightarrow \check{H}^c_{m-k}(f(M);\mathbb{Z}_2)$$

is a monomorphism, using Corollary 3.14, we obtain that $\theta_f = 0$.

□

Corollary 3.16. *Let $f : M \longrightarrow N$ be a proper topological embedding between an m-dimensional generalized manifold into an $(m+k)$-dimensional generalized manifold, $k > 0$. Then $\theta_f \in \check{H}^c_{m-k}(M;\mathbb{Z}_2)$ vanishes.*

Proof. Suppose that $f : M \longrightarrow N$ is a proper topological embedding. Then $f_v : M \xrightarrow{f} f(M)$ is an homeomorphism. This implies that

$$\bar{f}_* : \check{H}^c_{m-k}(M;\mathbb{Z}_2) \longrightarrow \check{H}^c_{m-k}(f(M);\mathbb{Z}_2)$$

is a monomorphism. Using Corolarry 3.15, we conclude that $\theta_f = 0$.

□

For every proper continuous map $f : M \to N$ between ENR-$\mathbb{Z}_2$-homology manifolds, if θ_f does not vanish, then $\theta(f)$ does not vanish either. Therefore, by Corollary 3.16, f cannot be properly homotopic to a proper topological embedding.

CHAPTER 4

Applications

In order to conclude our work, we present some applications of the results established in Chapter 3, using the properties of the classes $\theta(f) \in H^c_{m-k}(M;\mathbb{Z}_2)$ and $\theta_f \in \check{H}^c_{m-k}(M;\mathbb{Z}_2)$ considering a continuous proper map $f : M \longrightarrow N$ between generalized manifolds with dimension n and $n+k$, respectively.

4.1 The normal Euler class of a proper topological embedding

The goal of this section is to establish a relation between the Stiefel-Whitney class of the stable normal bundle of a proper topological embedding between *ENR* $\mathbb{Z}_2$-homology manifolds and its normal Euler class. For this, we need the definition of the *Thom Class* for a proper topological embedding. To define this class, we will consider an analogous construction used in [9] and [12].

Let M and N be ENR $\mathbb{Z}_2$-homology manifolds with dimension m and $m+k$, respectively, $k > 0$. Let $f : M \longrightarrow N$ be a proper topological embedding. Define the class $\tau(f) \in H^k(N, N - f(M);\mathbb{Z}_2)$ as follows:

Since f is a proper topological embedding, $f(M)$ is a closed subset of N and we may consider the closed inclusion $e : f(M) \hookrightarrow N$. Observe that $f(M)$ is also an ENR $\mathbb{Z}_2$-homology manifold and, for natural numbers $0 \leq q \leq m-k$, the natural homomorphism given in 1.2.

$$\eta_{f(M)} : \check{H}_c^{m+k-q}(f(M);\mathbb{Z}_2) \longrightarrow H_c^{m+k-q}(f(M);\mathbb{Z}_2)$$

is an isomorphism. Also, consider the isomorphisms of the Alexander Duality given in [9]

$$\widetilde{D}_N : \check{H}_c^{m+k-q}(f(M);\mathbb{Z}_2) \longrightarrow H_q(N, N - f(M);\mathbb{Z}_2)$$

and

$$\widetilde{D}_{f(M)} : \check{H}_c^{m+k-q}(f(M);\mathbb{Z}_2) \longrightarrow H_{q-k}(f(M);\mathbb{Z}_2).$$

Define the isomorphism

$$e! := \widetilde{D}_{f(M)} \circ \widetilde{D}_N^{-1} : H_q(N, N - f(M);\mathbb{Z}_2) \longrightarrow H_{q-k}(f(M);\mathbb{Z}_2).$$

For $q = k$,

$$\begin{aligned} H_k(N, N - f(M); \mathbb{Z}_2) &\cong H_0(f(M); \mathbb{Z}_2) \\ &\cong \bigoplus_{\lambda} H_0(N_\lambda; \mathbb{Z}_2) \\ &\cong \bigoplus_{\lambda} \mathbb{Z}_2 \end{aligned}$$

where each N_λ is a connected component of $f(M)$. Denote v_λ the generator of $H_k(N, N - f(M); \mathbb{Z}_2)$ which corresponds with the generator 1_λ of $H_0(f(M); \mathbb{Z}_2)$ related with the λ-component. By the universal coefficient Theorem, it follows that

$$\begin{aligned} H^k(N, N - f(M); \mathbb{Z}_2) &\cong \mathrm{Hom}(H_k(N, N - f(M); \mathbb{Z}_2), \mathbb{Z}_2) \\ &\cong \mathrm{Hom}(H_0(f(M); \mathbb{Z}_2), \mathbb{Z}_2). \end{aligned}$$

We assert that there exists a unique $\tau \in H^k(N, N - f(M); \mathbb{Z}_2)$ such that $\langle \tau, v_\lambda \rangle = 1_\lambda$. This means that we can define the class τ in terms of the generators of $H_k(N, N - f(M); \mathbb{Z}_2)$. The class τ is called the *Thom class*.

Now, considering the restriction $\bar{f} : M \xrightarrow{f} f(M)$ and the inclusion $i : N \hookrightarrow (N, N - f(M))$, we have the composition

$$\begin{tikzcd} M \arrow[r, "\bar{f}"] \arrow[rr, bend right, "f"'] & f(M) \arrow[r, "e"] & N \arrow[r, "i"] & (N, N - f(M)) \end{tikzcd}$$

i.e. $i \circ f = i \circ e \circ \bar{f}$.

The image of τ by the homomorphism

$$(i \circ e)^* := e^* \circ i^* : H^k(N, N - f(M); \mathbb{Z}_2) \longrightarrow H^k(f(M), \mathbb{Z}_2)$$

is called *normal Euler class* and denoted by χ.

For the proper embedding $f : M \longrightarrow N$, the *Thom Class of f* is defined by $\tau_f := \tau$. The *normal Euler class of f* is define by $\chi_f = f^* \circ i^*(\tau)$.

Note that

$$\begin{aligned} \chi_f &= f^* \circ i^*(\tau) \\ &= \bar{f}^* \circ e^* \circ i^*(\tau) \\ &= \bar{f}^*(\chi). \end{aligned}$$

Using [9], we consider the following proposition:

Proposition 4.1. *Let $e : f(M) \hookrightarrow N$ be a closed inclusion and consider the composition*

$$H_q(f(M); \mathbb{Z}_2) \xrightarrow{(i_* \circ e_*)} H_q(N, N - f(M); \mathbb{Z}_2) \xrightarrow{e_!} H_{q-k}(f(M); \mathbb{Z}_2).$$

Then,

$$e_! \circ (i_* \circ e_*)(\varepsilon) = \chi \frown \varepsilon$$

for all $\varepsilon \in H_q(f(M);\mathbb{Z}_2)$.

The cap product considered in Proposition 4.1 is

$$\frown : H^k(M;\mathbb{Z}_2) \times H_q(f(M);\mathbb{Z}_2) \longrightarrow H_{q-k}(f(M);\mathbb{Z}_2).$$

If we consider the homomorphism $e_!$ as the composition

$$H_q^c(N,N-f(M);\mathbb{Z}_2) \xrightarrow{\widetilde{D}_N^{-1}} \check{H}^{m+k-q}(f(M);\mathbb{Z}_2) \xrightarrow{\widetilde{D}_{f(M)}} H_{q-k}^c(f(M);\mathbb{Z}_2),$$

we have a similar result of Proposition 4.1 for the cap product

$$\frown : H^k(M;\mathbb{Z}_2) \times H_q^c(f(M);\mathbb{Z}_2) \longrightarrow H_{q-k}^c(f(M);\mathbb{Z}_2).$$

Proposition 4.2. *Let* $e : f(M) \hookrightarrow N$ *be a closed inclusion as in the previous discussion and consider the composition*

$$H_q^c(f(M);\mathbb{Z}_2) \xrightarrow{(i_* \circ e_*)} H_q^c(N,N-f(M);\mathbb{Z}_2) \xrightarrow{e_!} H_{q-k}^c(f(M);\mathbb{Z}_2).$$

Then,

$$e_! \circ (i_* \circ e_*)(\varepsilon) = \chi \frown \varepsilon$$

for all $\varepsilon \in H_q^c(f(M);\mathbb{Z}_2)$.

The following lemma can be found in [3]:

Lemma 4.3. *Let* $f : M \longrightarrow N$ *be a proper embedding between ENR* $\mathbb{Z}_2$*-homology manifolds with dimension* m *and* $m+k$*, respectively,* $k > 0$*. Then, the diagram commutes*

$$\begin{tikzcd}
H^k(N;\mathbb{Z}_2) \arrow[r, "e^*"] \arrow[dd, "D_N"] & H^k(f(M),\mathbb{Z}_2) \arrow[d, "\eta_{f(M)}^{-1}"] \\
 & \check{H}^k(f(M);\mathbb{Z}_2) \arrow[d, "\widetilde{D}_N"] \\
H_m^c(N;\mathbb{Z}_2) \arrow[r, "i_*"] & H_m^c(N,N-f(M),\mathbb{Z}_2)
\end{tikzcd}$$

where $i : N \longrightarrow (N,N-f(M))$ *and* $e : f(M) \hookrightarrow N$ *are the inclusion.*

Remark 4.4. The Poincaré duality is a particular case of Alexander duality. When we consider ENR $\mathbb{Z}_2$-homology manifolds the following diagram is commutative

$$\begin{tikzcd}
\check{H}^k(f(M);\mathbb{Z}_2) \arrow[r, "\widetilde{D}_{f(M)}"] & H^c_{m-k}(f(M),\mathbb{Z}_2) \\
\check{H}^k(f(M);\mathbb{Z}_2) \arrow[u, "\eta^{-1}_{f(M)}"] \arrow[ur, "D_{f(M)}"] &
\end{tikzcd}$$

for $\eta_{f(M)}$ given in Section 1.2 and the Poincaré Duality $D_{f(M)}$ for $f(M)$.

Finally, we have the result that relates the Stiefel-Whitney of the stable normal bundle of a proper topological embedding between *ENR* $\mathbb{Z}_2$-homology manifolds and its normal Euler class.

Proposition 4.5. *Consider M an m-dimensional ENR $\mathbb{Z}_2$-homology manifold and N an $m+k$-dimensional ENR $\mathbb{Z}_2$-homology manifold, with $k>0$. For a proper topological embedding $f: M \longrightarrow N$ it follows that*

$$w_k(f) = f^*(U_f) = \chi_f.$$

Proof. Using the previous discussion,

$$\begin{aligned}
\bar{f}_*(\chi_f \frown [M]) &= \bar{f}_*(\bar{f}^*(\chi) \frown [M]) \\
&= \chi \frown \bar{f}_*([M]) \\
&= e_! \circ (i_* \circ e_*) \circ \bar{f}_*([M]) \\
&= e_! \circ i_* \circ f_*([M]) \\
&= e_! \circ i_*(U_f \frown [N]) \\
&= e_! \circ i_* \circ D_N(U_f) \\
&= e_! \circ \widetilde{D}_N \circ \eta^{-1}_{f(M)} \circ e^*(U_f) \\
&= \widetilde{D}_{f(M)} \circ \widetilde{D}_N^{-1} \circ \widetilde{D}_N \circ \eta^{-1}_{f(M)} \circ e^*(U_f) \\
&= \widetilde{D}_{f(M)} \circ \eta^{-1}_{f(M)} \circ e^*(U_f) \\
&= D_{f(M)} \circ e^*(U_f) \\
&= e^*(U_f) \frown [f(M)] \\
&= e^*(U_f) \frown \bar{f}_*[M] \\
&= \bar{f}_*(\bar{f}^* \circ e^*(U_f) \frown [M]) \\
&= \bar{f}_*(f^*(U_f) \frown [M]).
\end{aligned}$$

Since $\bar{f}_*$ is an isomorphism, it follows that $\chi_f \frown [M] = f^*(U_f) \frown [M]$. The Poincaré duality provides that $\chi_f = f^*(U_f)$. By the Corollary 3.16, we have that $\theta_f = 0$ and then $\theta(f) = 0$. This proves that

$$0 = \big(f^*(U_f) - w_k(f)\big) \frown [M];$$

Finally $\chi_f = f^*(U_f) = w_k(f)$.

$\square$

4.2 The Haefliger obstruction

In [13], Haefliger proved that, for all compact topological manifolds M and N of dimensions m and $m+k$, respectively, with $k>0$, if $f: M \to N$ is a topological embedding, then $w_i(f)=0$, for all $i>k$. In this section, we will prove a similar result considering *ENR* $\mathbb{Z}_2$-homology manifolds.

Lemma 4.6. *Let $f: M \longrightarrow N$ a proper continuous map between generalized manifolds with dimensions m and $m+k$, respectively, $k>0$. For each $i>k$, define*

$$\begin{array}{rcl} \tilde{f}_i : M & \longrightarrow & N \times \mathbb{R}^{i-k} \\ x & \longmapsto & \tilde{f}_i(x) = (f(x), 0). \end{array}$$

Then

$$w(f) = w(\tilde{f}_i).$$

Proof. Observe that, the composition of inclusions

$$N \overset{id}{\hookrightarrow} N \times \{0\} \overset{i}{\hookrightarrow} N \times \mathbb{R}^{i-k}$$

induces an isomorphism in cohomology

$$\varphi := id^* \circ i^* : H^q(N \times \mathbb{R}^{i-k}; \mathbb{Z}_2) \longrightarrow H^q(N; \mathbb{Z}_2).$$

Note that

$$\tilde{f}_i := i \circ id \circ f : M \longrightarrow N \times \mathbb{R}^{i-k}$$

induced a homomorphism in cohomology

$$\tilde{f}_i^* := f^* \circ id^* \circ i^* : H^q(N \times \mathbb{R}^{i-k}; \mathbb{Z}_2) \longrightarrow H^q(M; \mathbb{Z}_2)$$

and

$$\begin{array}{rcl} \tilde{f}_i \circ \varphi^{-1} & = & f^* \circ id^* \circ i^* \circ (i^*)^{-1} \circ (id^*)^{-1} \\ & = & f^*. \end{array}$$

If we prove that $\varphi^{-1}(v(N)) = v(N \times \mathbb{R}^{i-k})$, then

$$\begin{array}{rcl} \varphi^{-1}(w(N)) & = & \varphi^{-1}(Sq(v(N))) \\ & = & Sq(\varphi^{-1}(v(N))) \\ & = & Sq(v(N \times \mathbb{R}^{i-k})) \\ & = & w(N \times \mathbb{R}^{i-k}) \end{array}$$

and

$$\begin{aligned} w(\tilde{f}_i) &= \tilde{f}_i^*(w(N\times\mathbb{R}^{i-k}))\smile\overline{w}(M) \\ &= \tilde{f}_i^*\circ\varphi^{-1}(w(N))\smile\overline{w}(M) \\ &= f^*(w(N))\smile\overline{w}(M) \\ &= w(f). \end{aligned}$$

For each $j\in\mathbb{N}$, consider $x\in H_c^{m+i-j}(N\times\mathbb{R}^{i-k};\mathbb{Z}_2)$. Then

$$\langle x\smile\widetilde{v}_j,[N\times\mathbb{R}^{i-k}]\rangle=\langle Sq^j(x),[N\times\mathbb{R}^{i-k}]\rangle,$$

where $\widetilde{v}_j$ is the j-th Wu classe of $N\times\mathbb{R}^{i-k}$. Let v_j be the j-th Wu class of N and consider

$$v'_j=\varphi^{-1}(v_j),$$

if we prove that

$$\langle x\smile v'_j,[N\times\mathbb{R}^{i-k}]\rangle=\langle Sq^j(x),[N\times\mathbb{R}^{i-k}]\rangle,$$

then $\widetilde{v}_j=v'_j$. This implies that $v(N)=\varphi(v(N\times\mathbb{R}^{i-k})$.

Using the isomorphism $i_*\circ id_*:H_j(N;\mathbb{Z}_2)\longrightarrow H_j(N\times\mathbb{R}^{i-k};\mathbb{Z}_2)$, it follows that

$$\begin{aligned} H_c^{m+i-j}(N\times\mathbb{R}^{i-k};\mathbb{Z}_2) &\cong H_j(N\times\mathbb{R}^{i-k};\mathbb{Z}_2) \\ &\cong H_j(N;\mathbb{Z}_2) \\ &\cong H_j(N;\mathbb{Z}_2)\otimes\mathbb{Z}_2 \\ &\cong H_j(N;\mathbb{Z}_2)\otimes H_0(\mathbb{R}^{i-k};\mathbb{Z}_2) \\ &\cong H_c^{m+k-j}(N;\mathbb{Z}_2)\otimes H_c^{m+k-j}(N;\mathbb{Z}_2). \end{aligned}$$

For each $x\in H_c^{m+i-j}(N\times\mathbb{R}^{i-k};\mathbb{Z}_2)$, assume that $x=x'\times\varepsilon$ for $x'\in H_c^{m+k-j}(N;\mathbb{Z}_2)$ and ε the generator of $H_c^{m+k-j}(N;\mathbb{Z}_2)$. Furthermore, we can consider $v'_j=v_j\times 1$, where 1 is the generator of $H^0(\mathbb{R}^{i-k};\mathbb{Z}_2)$. Then

$$\begin{aligned} \langle(x'\times\varepsilon)\smile(v_j\times 1),[N]\times[\mathbb{R}^{i-k}]\rangle &= \langle(x'\smile v_j)\times(\varepsilon\smile 1),[N]\times[\mathbb{R}^{i-k}]\rangle \\ &= \langle x'\smile v_j,[N]\rangle\langle\varepsilon,[\mathbb{R}^{i-k}]\rangle \\ &= \langle Sq^j(x'),[N]\rangle\langle\varepsilon,[\mathbb{R}^{i-k}]\rangle \\ &= \langle Sq^j(x')\times\varepsilon,[N]\times[\mathbb{R}^{i-k}]\rangle \\ &= \langle Sq^j(x'\times\varepsilon),[N]\times[\mathbb{R}^{i-k}]\rangle. \end{aligned}$$

This implies that

$$\langle x\smile v'_j,[N\times\mathbb{R}^{i-k}]\rangle=\langle Sq^j(x),[N\times\mathbb{R}^{i-k}]\rangle$$

and $v_j=\varphi(\widetilde{v}_j)$.

□

Remark 4.7. For each $x \in H^k(X;\mathbb{Z}_2)$, we denote $|x| = k$. We have the following properties:

- $\langle x \times y, a \times b\rangle = (-1)^{|x||y|}\langle x,a\rangle\langle y,b\rangle$
- $(x_1 \times y_1) \smile (x_2 \times y_2) = (-1)^{|y_1||x_2|}(x_1 \smile x_2) \times (y_1 \smile y_2)$

For each generalized manifold N with dimension $m+k$, using the homomorphism φ given in Lemma 4.6 and the Poincaré duality, we obtain that

$$\begin{aligned} H^c_m(N \times \mathbb{R}^{i-k};\mathbb{Z}_2) &\cong H^i(N \times \mathbb{R}^{i-k};\mathbb{Z}_2) \\ &\cong H^i(N;\mathbb{Z}_2) \\ &\cong H^c_{m-(i-k)}(N;\mathbb{Z}_2) \\ &\cong H^c_{m-(i-k)}(N;\mathbb{Z}_2) \otimes \mathbb{Z}_2. \end{aligned}$$

Proposition 4.8. *Let M and N be ENR $\mathbb{Z}_2$-homology manifolds, with dimension m and $m+k$, respectively, $k > 0$. For each proper topological embedding $f : M \longrightarrow N$, if $i > k$, then $w_i(f) \in H^i(M;\mathbb{Z}_2)$ vanishes.*

Proof. For $i > k$, using the proper continuous map $f : M \longrightarrow N$, we have the following proper continuous map

$$\begin{aligned} \tilde{f}_i : M &\longrightarrow N \times \mathbb{R}^{i-k} \\ x &\longmapsto \tilde{f}_i(x) = (f(x), 0). \end{aligned}$$

Note that $\tilde{f}_i([M]) \in H^c_m(N \times \mathbb{R}^{i-k};\mathbb{Z}_2)$ can be seem as an element $\mu \times 1 \in H^c_{m-(i-k)}(N;\mathbb{Z}_2) \otimes \mathbb{Z}_2$. Then $\mu = 0$ and the Poincaré dual class of $\tilde{f}_i([M])$ must be $U_{\tilde{f}_i} = 0$. Since $\tilde{f}_i$ is a proper topological embedding, we can apply Corollary 3.16 and $\eta_M(\theta(\tilde{f}_i)) = 0$, where $\eta_M : H^c_m(M;\mathbb{Z}_2) \longrightarrow \check{H}^c_m(M;\mathbb{Z}_2)$ is an isomorphism. Then

$$\begin{aligned} \theta(\tilde{f}_i) &= \left(\tilde{f}_i(U_{\tilde{f}_i}) - w_i(\tilde{f}_i)\right) \frown [M] \\ &= -w_i(\tilde{f}_i) \frown [M] \\ &= 0. \end{aligned}$$

Using the Poincaré duality, we conclude that $w_i(\tilde{f}_i) = 0$. By the Lemma 4.6,

$$w_i(f) = w_i(\tilde{f}_i) = 0.$$

□

A direct application of Proposition 4.8 is exhibit a sufficient condition under which Euclidean spaces cannot be embedded as closed subsets of an *ENR* $\mathbb{Z}_2$-homology manifold.

Corollary 4.9. *Let M be an ENR $\mathbb{Z}_2$-homology manifold with dimension m and $\overline{w}_i(M) \neq 0$, for some $i > 0$. Then M cannot be embedded as a closed subset of $\mathbb{R}^{m+j}$, for $j \leq i$.*

Proof. Suppose that M can be embedded as a closed subset of $\mathbb{R}^{m+j}$, for $j \leq i$. Then, for each $j \leq i$, there exists a proper embedding $f : M \longrightarrow \mathbb{R}^{m+j}$.

If $j < i$, using the definition of the i-th Stiefel-Whitney Class of the stable normal bundle of f, it follows that

$$\begin{aligned} w_i(f) &= \sum_{\alpha+\beta=i} f^*(w_\alpha(\mathbb{R}^{m+j})) \smile \overline{w}_\beta(M) \\ &= f^*(w_0(\mathbb{R}^{m+j}) \smile \overline{w}_i(M)) \\ &= 1 \smile \overline{w}_i(M) \\ &= \overline{w}_i(M) \\ &\neq 0. \end{aligned}$$

Note that Proposition 4.8 provides that $w_i(f) = 0$.

If $j = i$, using the Poincaré duality and $U_f \in H^j(\mathbb{R}^{m+j};\mathbb{Z}_2) \cong \{0\}$, it follows that

$$\begin{aligned} \theta(f) &= [f^*(U_f) - \overline{w}_j(M)] \frown [M] \\ &= -\overline{w}_j(M) \frown [M] \\ &\neq 0. \end{aligned}$$

Corollary 3.16 guarantee that $\theta(f) = 0$.

Then M cannot be embedded as a closed subset of $\mathbb{R}^{m+j}$ for $j \leq i$.

□

4.3 The self-intersection set of a proper continuous map

In this section, we define the self-intersection set of a continuous map. Using this, we will discuss the relation of the class θ_f, defined in Chapter 3, and the image of the induced homomorphism given by inclusion considering the closure of the self-intersection set as a subset of a compact subset.

Definition 4.10. Let $f : M \longrightarrow N$ a continuous map between topological spaces. The set

$$M(f) = \{x \in M | f^{-1}(f(x)) \neq \{x\}\}$$

is called the *self-intersection set of f*.

Note that the self-intersection set of a continuous map is set of the elements in the source in which the map is not injective.

Lemma 4.11. *Consider the diagram of modules and homomorphisms*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\gamma_{i+1}"] & A_i \arrow[r, "g_i"] \arrow[d, "\alpha_i"] & B_i \arrow[r, "h_i"] \arrow[d, "\beta_i"] & C_i \arrow[r] \arrow[d, "\gamma_i"] & \cdots \\
\cdots \arrow[r] & C'_{i+1} \arrow[r, "f'_{i+1}"] & A'_i \arrow[r, "g'_i"] & B'_i \arrow[r, "h'_i"] & C'_i \arrow[r] & \cdots
\end{tikzcd}$$

where each row is a exact sequence and each rectangle commutes. Suppose that, for $k \in \mathbb{N}$, *each* $\gamma_k : C_k \longrightarrow C'_k$ *is an isomorphism. Then, for all* $i \geq 0$, *there exist homomorphisms*

$$\begin{array}{rcl} \varphi_i : A_i & \longrightarrow & A'_i \oplus B_i \\ a & \longmapsto & \varphi(a) = (\alpha_i(a), g_i(a)); \end{array}$$

$$\begin{array}{rcl} \psi_i : A'_i \oplus B_i & \longrightarrow & B'_i \\ (b,c) & \longmapsto & \psi(b,c) = \beta_i(c) - g'_i(b) \end{array}$$

and

$$\begin{array}{rcl} \eta_i : B'_i & \longrightarrow & A_{i-1} \\ d & \longmapsto & \eta_i(d) = f_i \circ \gamma_i^{-1} \circ h'_i(d), \end{array}$$

such that the following sequence is an exact sequence

$$\cdots \longrightarrow A_i \xrightarrow{\varphi_i} A'_i \bigoplus B_i \xrightarrow{\psi_i} B'_i \xrightarrow{\eta_i} A_{i-1} \longrightarrow \cdots.$$

Proof. See [12]

□

Considering the closure of the self-intersection of a map, we have the following:

Proposition 4.12. *Let* M *be a compact* m*-dimensional generalized manifold and* N *an* $m+k$*-dimensional generalized manifold,* $k > 0$. *For each proper continuous map* $f : M \longrightarrow N$, *consider* $A = \overline{M(f)}$. *Then, there exists an element* $\mu \in \check{H}_{m-k}(A;\mathbb{Z}_2)$ *such that*

$$j_*(\mu) = \theta_f \in \check{H}_{m-k}(M;\mathbb{Z}_2)$$

and

$$(f_{|A})_*(\mu) = 0 \in \check{H}_{m-k}(f(A);\mathbb{Z}_2),$$

where $j : A \hookrightarrow M$ *and* $f_{|A} : A \xrightarrow{f} N$.

Proof. If $A = \emptyset$, we have that $\bar{f} : M \xrightarrow{f} f(M)$ is an embedding and $\bar{f}_*$ is a monomorphism. Takes $\mu = 0$.

Observes that (M,A) and $(f(M), f(A))$ are compact pairs. Then, we have commutative a diagram

$$\begin{tikzcd}
\check{H}_{i+1}(M;\mathbb{Z}_2) \arrow[r] \arrow[d, "\bar{f}_*"] & \check{H}_{i+1}(M,A;\mathbb{Z}_2) \arrow[r] \arrow[d, "\bar{f}_*"] & \check{H}_i(A;\mathbb{Z}_2) \arrow[r, "j_*"] \arrow[d, "(\bar{f}_{|A})_*"] & \check{H}_i(M;\mathbb{Z}_2) \arrow[d, "\bar{f}_*"] \\
\check{H}_{i+1}(f(M);\mathbb{Z}_2) \arrow[r] & \check{H}_{i+1}(f(M),f(A);\mathbb{Z}_2) \arrow[r] & \check{H}_i(f(A);\mathbb{Z}_2) \arrow[r, "j'_*"] & \check{H}_i(f(M);\mathbb{Z}_2)
\end{tikzcd}$$

where each row an exact sequence (see [10]).

Since $f_{|M-A} : M - A \longrightarrow f(M) - f(A)$ is a homeomorphism, we say that $f : (M,A) \longrightarrow (f(M), f(A))$ is a *relative homeomorphism.* In [10], Eilenberg and Steenrod guarantee that the Čech homology is invariant under relative homeomorphism for compact pairs, this mean that

$$\bar{f}_* : \check{H}_i(M,A;\mathbb{Z}_2) \longrightarrow \check{H}_i(f(M), f(A);\mathbb{Z}_2)$$

is an isomorphism for each $i \geq 0$.

Using Lemma 4.11, there exists an exact sequence

$$\cdots \longrightarrow \check{H}_{m-k}(A;\mathbb{Z}_2) \xrightarrow{\varphi} \check{H}_{m-k}(f(A);\mathbb{Z}_2) \bigoplus \check{H}_{m-h}(M;\mathbb{Z}_2) \xrightarrow{\psi} \check{H}_{m-k}(f(M);\mathbb{Z}_2) \longrightarrow \cdots$$

where $j : A \hookrightarrow M$, $j' : f(A) \hookrightarrow f(M)$, $\varphi = ((f_{|A})_*, j_*)$ and $\psi(b,c) = \bar{f}_*(c) + j'_*(b)$. Observe that

$$\psi(0, \theta(f)) \;\; = \;\; \bar{f}_*(\theta_f) + j'_*(0).$$

Corollary 3.14 provides that $\bar{f}_*(\theta_f) = 0$.

By the exactness of the above sequence, there exists $\mu \in \check{H}_{m-k}(A;\mathbb{Z}_2)$ such that

$$\begin{aligned} \varphi(\mu) &= ((\bar{f}_{|A})_*(\mu), j_*(\mu)) \\ &= (0, \theta_f) \end{aligned}$$

Then $j_*(\mu) = \theta_f$ and $(\bar{f}_{|A})_*(\mu) = 0$.

□

Using the same idea of Proposition 4.12, if $v \in \check{H}_{m-k}(f(A);\mathbb{Z}_2)$ satisfies $j'_*(v) = 0$, we can apply the exactness property and there exists μ_v such that $j_*(\mu_v) = \theta_f$ and $(\bar{f}_{|A})_*(\mu_v) = v$.

Using Proposition 4.12 we cannot guarantee that $\mu \neq 0$. In [21], Ronga gives some additional hypothesis in which $\mu \neq 0$ when $A \neq \emptyset$.

To the next corollary, we need the definition of topological dimension for a topological space (see [16]).

Definition 4.13. The *topological dimension* of a topological space X at a point $x \in X$ is defined inductively, for $n \geq 0$:

- we say that X has dimension 0 at $x \in X$ if, for each neighborhood U of x, there exists a neighborhood V of x with $V \subset U$ and $\partial V = \varnothing$;

- we say that X has dimension less than or equal to n at $x \in X$ if, for each neighborhood U of x, there exists a neighborhood V of x such that $V \subset U$ and ∂V has dimension less than or equal to $n-1$.

By convention, the empty set has dimension -1.

Definition 4.14. A topological space X has dimension less than or equal to n if has dimension less than or equal to n at each of its points.

Lemma 4.15. *Let X be a compact topological space with topological dimension less than or equal to n. Then, given any closed $C \subset X$, $H_{n+1}(X,C;\mathbb{Z}_2) \cong \{0\}$.*

Proof. See [16].

□

Corollary 4.16. *Let $f : M \longrightarrow N$ be a continuous map between a compact m-dimensional generalized manifold into an $m+k$- generalized manifold, $k > 0$. If the topological dimension of $\overline{M(f)}$ is strictly less than $m-k$, then $\theta(f) \in H_{m-k}(M;\mathbb{Z}_2)$ vanishes.*

Proof. Since the topological dimension of $A = \overline{M(f)}$ is less than $m-k$ and A is compact, by Lemma 4.15, we have that

$$\check{H}_{m-k}(A;\mathbb{Z}_2) \cong \{0\}.$$

Then $\check{H}_{m-k}(A;\mathbb{Z}_2) \cong \{0\}$ and using Proposition 4.12 it follows that $\mu = 0$. Hence $\theta(f) = 0$.

□

4.4 Applications on Real Projective Spaces

To conclude our work, we apply the results from Chapter 3 to continuous maps between real projective spaces $\mathbb{R}P^n$ and certain generalized manifolds. Here n is always a power of 2. The main result of this section is original and constitutes a generalization of the results given in [1] where the authors used $n = 2$ and X is a 3-dimensional generalized manifold.

Note that, $\mathbb{R}P^n$ is an n-dimensional compact connected generalized manifold. Then, for $0 \le q \le n$,

$$\begin{aligned} \check{H}^c_q(\mathbb{R}P^n;\mathbb{Z}_2) &\cong H^c_q(\mathbb{R}P^n;\mathbb{Z}_2) \\ &\cong H_q(\mathbb{R}P^n;\mathbb{Z}_2) \\ &\cong \mathbb{Z}_2. \end{aligned}$$

Moreover, a continuous map $f : \mathbb{R}P^n \to X$ into a generalized manifold is always a proper map ($\mathbb{R}P^n$ is compact and X is Hausdorff).

The following lemmas only require the structure of $\mathbb{R}P^n$ as a topological space. Assume that X is an arbitrary Hausdorff topological space.

Lemma 4.17. *Consider* $f : \mathbb{R}P^n \longrightarrow X$ *a continuous map. Denote by*

$$(f_*)^* : \mathrm{Hom}(H_k(X;\mathbb{Z}_2),\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_k(\mathbb{R}P^n;\mathbb{Z}_2),\mathbb{Z}_2)$$

the dual of the induced homomorphism

$$f_* : H_k(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H_k(X;\mathbb{Z}_2).$$

Let

$$\psi : H^k(X;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_k(X;\mathbb{Z}_2),\mathbb{Z}_2)$$

and

$$\phi : H^k(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_k(\mathbb{R}P^n;\mathbb{Z}_2),\mathbb{Z}_2)$$

be the isomorphisms given by the universal coefficient Theorem. Then, the following diagram is commutative

$$\begin{tikzcd}
H^k(X;\mathbb{Z}_2) \arrow[r, "\psi", "\cong"'] \arrow[d, "f^*"] & \mathrm{Hom}(H_k(X),\mathbb{Z}_2) \arrow[d, "(f_*)^*"] \\
H^k(\mathbb{R}P^n;\mathbb{Z}_2) \arrow[r, "\phi", "\cong"'] & \mathrm{Hom}(H_k(\mathbb{R}P^n),\mathbb{Z}_2)
\end{tikzcd}$$

Proof. For every $x \in H_k(X;\mathbb{Z}_2)$ and $y \in H_k(\mathbb{R}P^n;\mathbb{Z}_2)$, it follows that

$$\begin{aligned}
(f_*)^* \circ \psi(x)(y) &= (f_*)^*(\psi(x))(y) \\
&= \psi(x)(f_*(y)) \\
&= \langle x, f_*(y)\rangle \\
&= \langle f^*(x), y\rangle \\
&= \phi(f^*(x))(y) \\
&= \phi \circ f^*(x)(y).
\end{aligned}$$

This prove that the diagram commutes.

□

Lemma 4.18. *Let* $f : \mathbb{R}P^n \longrightarrow X$ *be a continuous map. Then, for* $0 \leq k \leq n$, *the induced homomorphism*

$$f_* : H_k(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H_k(X;\mathbb{Z}_2)$$

is injective if and only if the induced homomorphism

$$f^* : H^k(X;\mathbb{Z}_2) \longrightarrow H^k(\mathbb{R}P^n;\mathbb{Z}_2)$$

is surjective.

Proof. First, observe that, for $0 \leq k \leq n$, $H_k(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Using the universal coefficient Theorem, it follows that

$$\begin{aligned} H^k(\mathbb{R}P^n;\mathbb{Z}_2) &\cong \mathrm{Hom}(H_k(\mathbb{R}P^n;\mathbb{Z}_2),\mathbb{Z}_2) \\ &\cong \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}_2) \\ &\cong \mathbb{Z}_2. \end{aligned}$$

Suppose that f_* is an injective homomorphism and f^* is not surjective. Since f^* is not surjective we assert that f^* is the trivial homomorphism. Using the Lemma 4.17, for all $x \in \mathrm{Hom}(H_k(X;\mathbb{Z}_2),\mathbb{Z}_2)$,

$$(f_*)^*(x) = \phi \circ f^* \circ \psi^{-1}(x) = 0$$

and $(f_*)^*$ is the trivial homomorphism.

Consider ε the generator of $H_k(\mathbb{R}P^n;\mathbb{Z})$. Observe that $f_*(\varepsilon) \neq 0$. Since $H_k(X;\mathbb{Z})$ is a vector space and $f_*(\varepsilon) \neq 0$, we can choose a basis $(f_*(\varepsilon),x_1,\ldots,x_j)$ for $H_k(X;\mathbb{Z})$, where each $x_i \in H_k(X;\mathbb{Z})$, $i = 1,\ldots,j$. Define the linear functional

$$\alpha : H_k(X;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

such that $\alpha(x_i) = 0$ and $\alpha(f_*(\varepsilon)) \neq 0$, for $i = 1,\cdots,j$. Then

$$0 = (f_*)^*(\alpha)(\varepsilon) = \alpha \circ f_*(\varepsilon) \neq 0.$$

This is a contradiction. This prove that f^* has to be a surjective homomorphism.

Now, suppose that f^* is a surjective homomorphism and f_* is not injective. Since f_* is not injective and $H_k(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$, we conclude that f_* is the trivial homomorphism. There exists a class $x \in H^k(X;\mathbb{Z}_2)$ such that $f^*(x)$ is the generator of $H^k(\mathbb{R}P^n)$, this means that

$$\phi(f^*(x)) = id_{H_k(\mathbb{R}P^n;\mathbb{Z}_2)} : H_k(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H_k(\mathbb{R}P^n;\mathbb{Z}_2).$$

Using the commutative diagram

$$\begin{aligned} (f_*)^* \circ \psi(x) &= \phi \circ f^*(x) \\ &= id_{H_k(\mathbb{R}P^n;\mathbb{Z}_2)} \end{aligned}$$

and

$$\begin{aligned} (f_*)^* \circ \psi(x) &= \psi(x) \circ f_* \\ &= 0. \end{aligned}$$

This is a contradiction and it proves that f_* has to be an injective homomorphism.

□

Now, considering the fundamental class $[\mathbb{R}P^n] \in H_n(\mathbb{R}P^n;\mathbb{Z}_2)$, we have the following:

Lemma 4.19. *Let $f : \mathbb{R}P^n \longrightarrow X$ be a continuous map into a Hausdorff space X. Suppose that $f_*([\mathbb{R}P^n]) \in H_n(X;\mathbb{Z}_2)$ is the trivial class. Then $f_* : H_1(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H_1(X;\mathbb{Z}_2)$ is the trivial homomorphism.*

Proof. Suppose that $f_* : H_1(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H_1(X;\mathbb{Z}_2)$ is not the trivial homomorphism, then f_* is injective. Using Lemma 4.18, we have that f^* is surjective. There exists a class $\varepsilon \in H^1(X;\mathbb{Z}_2)$ such that $f^*(\varepsilon)$ generates $H^1(\mathbb{R}P^n;\mathbb{Z}_2)$. Using the properties of the cup product considering $\mathbb{R}P^n$

$$\underbrace{f^*(\varepsilon) \smile f^*(\varepsilon) \smile \ldots \smile f^*(\varepsilon)}_{n-times} = f^*(\varepsilon^n) \neq 0$$

This prove that $f^*(\varepsilon^n)$ generates $H^n(\mathbb{R}P^n;\mathbb{Z}_2)$ and $f^* : H^n(X;\mathbb{Z}_2) \longrightarrow H^n(\mathbb{R}P^n;\mathbb{Z}_2)$ is surjective.

Since f^* is surjective, f_* is injective. But $f_*([\mathbb{R}P^n]) = 0$ and we have a contradiction. This proves that $f_* : H_1(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H_1(X;\mathbb{Z}_2)$ is the trivial homomorphism.

□

A direct consequence of the Lemma 4.19 is a generalization of the results given in [1] where the authors used $n = 2$ and X a 3-dimensional generalized manifold.

Proposition 4.20. *Let $f : \mathbb{R}P^n \longrightarrow X$ be a continuous map into an $n+1$-dimensional generalized manifold. Suppose that $f^*(1) \in H^0(\mathbb{R}P^n;\mathbb{Z}_2)$ is nonzero and $f_*([\mathbb{R}P^n]) = 0$. Then, the induced homomorphism*

$$\bar{f}_* : H_{n-1}(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow \check{H}_{n-1}(f(\mathbb{R}P^n);\mathbb{Z}_2)$$

is trivial for $\bar{f} : \mathbb{R}P^n \xrightarrow{f} f(\mathbb{R}P^n)$.

Proof. By Lemma 4.19, $f_* : H_1(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H_1(X;\mathbb{Z}_2)$ is the trivial homomorphism. In this case, $f^* : H^1(X;\mathbb{Z}_2) \longrightarrow H^1(\mathbb{R}P^n;\mathbb{Z}_2)$ is also the trivial homomorphism and

$$\begin{aligned}
\theta(f) &= \left[f^*(U_f) - \sum_{i+j=1} f^*(w_i(X) \smile \bar{w}_j(\mathbb{R}P^n)\right] \frown [\mathbb{R}P^n] \\
&= -\left(f^*(1) \smile \overline{w}_1(\mathbb{R}P^n)\right) \frown [\mathbb{R}P^n] \\
&= -\overline{w}_1(\mathbb{R}P^n) \frown [\mathbb{R}P^n].
\end{aligned}$$

By Example 2.19, we have that $\overline{w}_k(\mathbb{R}P^n) \neq 0$ and then $\theta(f) \neq 0$.

Since $H_{n-1}(\mathbb{R}P^n;\mathbb{Z}_2) \overset{\eta_{\mathbb{R}P^n}}{\cong} \check{H}_{n-1}(\mathbb{R}P^n;\mathbb{Z}_2)$ and $\theta(f) \neq 0$, we have that $\theta_f \neq 0$. This shows that θ_f is the generator of $H_{n-1}(\mathbb{R}P^n;\mathbb{Z}_2)$. By Corollary 3.2, $\bar{f}_*\theta_f = 0$ and then $\bar{f}_*$ is the trivial homomorphism.

□

We can generalize the results in Proposition 4.20 to continuous maps between $\mathbb{R}P^n$ and an $n+2$-dimensional generalized manifolds, for $n > 2$.

Lemma 4.21. *Let $f : \mathbb{R}P^n \longrightarrow X$ be a continuous map into a Hausdorff space. Suppose that the induced homomorphism $f_* : H^n(X;\mathbb{Z}_2) \longrightarrow H^n(\mathbb{R}P^n;\mathbb{Z}_2)$ is trivial, $n > 2$. Then, the homomorphism $f_* : H^2(X;\mathbb{Z}_2) \longrightarrow H^2(\mathbb{R}P^n;\mathbb{Z}_2)$ is trivial.*

Proof. If $f_* : H^2(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H^2(X;\mathbb{Z}_2)$ is non trivial, there exists $\varepsilon \in H^2(X;\mathbb{Z}_2)$ such that $f^*(\varepsilon)$ is the generator of $H^2(\mathbb{R}P^n;\mathbb{Z}_2)$. Since n is even, exists $\alpha \in \mathbb{N}$ such that $n = 2\alpha$. Then

$$\underbrace{f^*(\varepsilon) \smile \cdots \smile f^*(\varepsilon)}_{\alpha-times} = f^*(\varepsilon^\alpha) \neq 0.$$

Observe that $f_* : H^n(X;\mathbb{Z}_2) \longrightarrow H^n(\mathbb{R}P^n;\mathbb{Z}_2)$ is the zero homomorphism and we have a contradiction. This proves that $f^* : H^2(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H^2(X;\mathbb{Z}_2)$ is the zero homomorphism.

□

Proposition 4.22. *Let $f : \mathbb{R}P^n \longrightarrow X$ be a continuous map into an $n+2$-dimensional generalized manifold, $n > 2$. Suppose that $f^*(1) \in H^0(\mathbb{R}P^n;\mathbb{Z}_2)$ is nonzero and $f_*([\mathbb{R}P^n]) = 0$. Then, the induced homomorphism*

$$\bar{f}_* : H_{n-2}(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow \check{H}_{n-2}(f(\mathbb{R}P^n);\mathbb{Z}_2)$$

is trivial, for $\bar{f} : \mathbb{R}P^n \xrightarrow{f} f(\mathbb{R}P^n)$.

Proof. By Lemma 4.19 and Lemma 4.21, $f_* : H_k(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H_k(X;\mathbb{Z}_2)$ is the trivial homomorphism for $k = 1,2$. In this case, $f^* : H^k(X;\mathbb{Z}_2) \longrightarrow H^k(\mathbb{R}P^n;\mathbb{Z}_2)$ is also the zero homomorphism and

$$\begin{aligned} \theta(f) &= \Big[f^*(U_f) - \sum_{i+j=2} f^*(w_i(X)) \smile \overline{w_j}(\mathbb{R}P^n)\Big] \frown [\mathbb{R}P^n] \\ &= -\big(f^*(1) \smile \overline{w}_2(\mathbb{R}P^n)\big) \frown [\mathbb{R}P^n] \\ &= -\overline{w}_2(\mathbb{R}P^n) \frown [\mathbb{R}P^n] \end{aligned}$$

By Example 2.19, we have that $\overline{w}_k(\mathbb{R}P^n) \neq 0$ and then $\theta(f) \neq 0$.

Since $H_{n-2}(\mathbb{R}P^n;\mathbb{Z}_2) \overset{\eta_{\mathbb{R}P^n}}{\cong} \check{H}_{n-2}(\mathbb{R}P^n;\mathbb{Z}_2)$ and $\theta(f) \neq 0$, we have that $\theta_f \neq 0$. This shows that θ_f is the generator of $H_{n-2}(\mathbb{R}P^n;\mathbb{Z}_2)$. By Corollary 3.2, $\bar{f}_*\theta_f = 0$ and then $\bar{f}_*$ is the trivial homomorphism.

□

Remark 4.23. Observe that the hypothesis that n is a power of 2 is essential to guarantee that $\overline{w}_k(\mathbb{R}P^n) \neq 0$.

Corollary 4.24. *Let $f : \mathbb{R}P^n \longrightarrow X$ be a continuous map into an $n+k$-dimensional generalized manifold X, with $k = 1,2$ and $n > 2$. If $f_*([\mathbb{R}P^n]) = 0$, then f cannot be a topological embedding.*

Proof. Suppose that $f : \mathbb{R}P^n \longrightarrow X$ is a topological embedding. Then $\bar{f} : \mathbb{R}P^n \longrightarrow f(RP^n)$ is a homeomorphism and

$$\bar{f}_* : H_{n-k}(\mathbb{R}P^n; \mathbb{Z}_2) \longrightarrow \check{H}_{n-k}(f(\mathbb{R}P^n); \mathbb{Z}_2)$$

is an isomorphism. Observe that, by Proposition 4.20 and Proposition 4.22, $\bar{f}_* = 0$. We have a contradiction and $f : \mathbb{R}P^n \longrightarrow X$ cannot be a proper topological embedding.

□

Remark 4.25. Note that Corollary 4.24 is also true for $n = 2$ when $k = 1$.

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