**UNIVERSIDADE FEDERAL DE SÃO CARLOS**
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇãO EM MATEMÁTICA

# Mean curvature flow into an ambient Riemannian manifold evolving by Ricci flow coupled with harmonic map heat flow

Carlos Maurício de Sousa

São Carlos - SP
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# Mean curvature flow into an ambient Riemannian manifold evolving by Ricci flow coupled with harmonic map heat flow

Carlos Maurício de Sousa

Advisor: Prof. Dr. José Nazareno Vieira Gomes

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## Folha de Aprovação

Defesa de Tese de Doutorado do candidato Carlos Maurício de Sousa, realizada em 02/06/2025.

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O Relatório de Defesa assinado pelos membros da Comissão Julgadora encontra-se arquivado junto ao Programa de Pós-Graduação em Matemática.

*Dedico este trabalho*
*à Maria do Amparo,*
*minha mãe,*
*Francisca Isabel*
*e Aderson, meus avós (in memorian),*
*e Francisco Braga (in memorian).*

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*"Se eu vi mais longe, foi por estar sobre ombros de gigantes".*

*Isaac Newton.*

# Resumo

O objetivo principal desta tese é estudar o fluxo da curvatura média em uma variedade diferenciável compacta ambiente M com bordo e com uma métrica Riemanniana que evolui por uma solução autossimilar do fluxo de Ricci acoplado ao fluxo do calor de aplicações harmônicas de uma aplicação de M para uma variedade Riemanniana N. Neste contexto, abordamos um funcional associado ao fluxo de Ricci acoplado ao fluxo do calor de aplicações harmônicas e calculamos a sua variação ao longo de parâmetros que preservam a medida do volume ponderado. Assim, uma extensão da diferencial de Harnack-Hamilton aparece ao considerar o bordo de M evoluindo pelo fluxo da curvatura média que deve se anular no caso de soliton gradiente estável. Em seguida, obtemos uma fórmula do tipo monotonicidade de Huisken para o fluxo da curvatura média no cenário proposto. Como aplicação, consideramos a família normalizada associada ao fluxo da curvatura média para obter resultados de convergência no sentido de Cheeger-Gromov nos casos compacto e não compacto. Além disso, mostramos como construir uma família de sólitons da curvatura média e estabelecemos uma caracterização de tal família.

**Palavras-chave**: Fluxo de Ricci harmônico, fluxo da curvatura média, convergência de Cheeger-Gromov.

# Abstract

The main objective of this thesis is to study the mean curvature flow into an ambient compact smooth manifold M with boundary and with a Riemannian metric that evolves by a self-similar solution of the Ricci flow coupled with the harmonic map heat flow of a map from M to a Riemannian manifold N. In this context, we address a functional associated with the Ricci flow coupled with the harmonic map heat flow and calculate its variation along parameters that preserve the weighted volume measure. So, an extension of the Harnack-Hamilton differential appears by considering the boundary of M evolving by mean curvature flow, which must vanish on the gradient steady soliton case. Next, we obtain a Huisken monotonicity-type formula for the mean curvature flow in the proposed background. As an application, we consider the associated normalized family of the mean curvature flow to obtain results of convergence in the Cheeger-Gromov sense in the compact and noncompact cases. Moreover, we show how to construct a family of mean curvature solitons and we establish a characterization of such a family.

**Keywords**: Ricci harmonic flow, mean curvature flow, Cheeger-Gromov convergence.

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# Introduction

In the last decades, a significant part of the research in Geometric Analysis has focused on the study of intrinsic and extrinsic geometric flows. The most considered intrinsic geometric flow is the Ricci flow, which Hamilton conceived as a tool to prove Thurston's Geometrization Conjecture, settled affirmatively by Perelman, that led to the proof of Poincaré's Conjecture. The mean curvature flow is certainly the most studied in the class of extrinsic geometric flows due to its many geometrical properties. The main feature of the Ricci and mean curvature flows is that their analytical properties are similar to those of the heat flow, since the partial differential equations associated with these geometric flows are parabolic.

This thesis concerns extrinsic flows in ambient Riemannian manifolds under intrinsic flows. More precisely, we study mean curvature flow in Riemannian manifolds evolving by Ricci flow coupled with harmonic map heat flow. Our approach is inspired by many works, as we shall describe.

We begin with the work by Eells and Sampson [ES64], which pioneered the study of harmonic maps that arise from the variation of the energy-type functional (a generalization of Dirichlet's energy functional). There, they aimed to establish existence of harmonic maps which are homotopic to a given map $\phi : (M, g) \to (N, \gamma)$, where $(M, g)$ and $(N, \gamma)$ are closed Riemannian manifolds, i.e., compact and without boundary. For it, they considered the *energy functional* $E$ of $\phi$ as follows

$$E(\phi) := \frac{1}{2}\int_M |\nabla\phi|^2 dM,$$

and they showed that, for a smooth family of maps $\phi_t : (M, g) \to (N, \gamma)$, with $t \in (-\epsilon, \epsilon)$, variational vector field $V$ and $\phi_0 = \phi$, the first variation formula of $E$ is given by

$$\frac{d}{dt}\Big|_{t=0} E(\phi_t) = -\int_M \langle V, \tau_{g,\gamma}\phi\rangle dM,$$

where $\tau_{g,\gamma}\phi$ denotes the tension field of $\phi$, which depend on the Riemannian metrics $g$ and $\gamma$. In particular, a harmonic map $\phi$ (i.e., $\tau_{g,\gamma}\phi = 0$) is a critical point of $E$, see Section 1.1 for more details.

The idea is to deform a given map $\phi \in C^\infty(M, N)$ along the flow given by $\tau_{g,\gamma}\phi_t$ to obtain a harmonic map free-homotopic to $\phi$. When such a deformation is possible, its flow $\phi_t$ becomes a solution of the system of parabolic partial differential equations

$$\frac{\partial \phi_t}{\partial t} = \tau_{g,\gamma}\phi_t, \quad \phi(0) = \phi. \tag{1}$$

System (1) is known as the harmonic map heat flow. For our context, we highlight the following particular result by Eells and Sampson. We observe that they worked in a more general setting by imposing some boundedness on the embedding of $N$ in some Euclidean space $\mathbb{R}^d$; such conditions are automatically fulfilled if $N$ is compact. Eells and Sampson's theorem reads as follows.

**Theorem A** (Eells and Sampson [ES64]). *Let $(M, g)$ and $(N, \gamma)$ be closed Riemannian manifolds, and let $\phi : (M, g) \to (N, \gamma)$ be a smooth map. If $(N, \gamma)$ has non-positive Riemannian curvature, then there exists a unique global smooth solution of* (1) *which converges smoothly to a harmonic map homotopic to $\phi$.*

One of the importance of harmonic maps is that they generalize the concept of harmonic functions. In particular, closed geodesics and minimal surfaces are some examples. More generally, when $\phi$ is an isometric immersion, the tension field has the simplified notation $\tau_g\phi$ and coincides with the mean curvature $H(\phi)$. Hence, $\phi$ is harmonic if and only if it is minimal. Moreover, any isometry of $M$ is harmonic, and any covering map is harmonic.

Hamilton extended Eells and Sampson's theorem for compact Riemannian manifolds with boundary. He showed that the first variation formula of the energy functional $E(\phi)$ of a smooth map $\phi : (M, g) \to (N, \gamma)$, now between Riemannian manifolds with boundary, is given by

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = -\int_M \langle V, \tau_{g,\gamma}\phi\rangle dM + \int_{\partial M} \langle V, \nabla_0\phi\rangle dA.$$

So, a harmonic map $\phi$ with Neumann boundary condition $\nabla_0\phi = 0$ is a critical point of the energy functional.

Hamilton noted that there are three natural boundary value problems to be addressed.

(a) *Dirichlet Problem for a harmonic map from $(M, g)$ to $(N, \gamma)$ with given values on $\partial M$.* Let $\hat{\phi} : \partial M \to N$ be a smooth map, and denote by $\mathscr{M}_{\hat{\phi}}(M, N)$ the closed subspace of maps $\phi : (M, g) \to (N, \gamma)$ such that $\phi|_{\partial M} = \hat{\phi}$, with the $C^\infty$-topology. A relative homotopy class is a connected component of $\mathscr{M}_{\hat{\phi}}(M, N)$. If there is a topological obstruction to extending $\hat{\phi}$, then $\mathscr{M}_{\hat{\phi}}(M, N)$ is empty and nothing more can be said. Otherwise, one has the following result.

**Theorem B** (Hamilton [Ham75]). *Let $(M, g)$ and $(N, \gamma)$ be compact Riemannian manifolds with boundary. Suppose that $N$ has non-positive Riemannian curvature and that $\partial N$ is convex (or empty). Then, the Dirichlet problem for $\phi : (M, g) \to (N, \gamma)$*

$$\begin{cases} \tau_{g,\gamma}\phi = 0 & \text{in } M \\ \phi = \hat{\phi} & \text{on } \partial M \end{cases}$$

*has a solution in every relative homotopy class.*

(b) *Neumann Problem.* If the map $\phi$ is not specified on $\partial M$ at all, he proved that it is enough to impose the auxiliary condition that the normal derivative $\nabla_0\phi = 0$ on $\partial M$.

**Theorem C** (Hamilton [Ham75]). *Let $(M, g)$ and $(N, \gamma)$ be compact Riemannian manifolds with boundary. Suppose that $N$ has non-positive Riemannian curvature and that $\partial N$ is convex (or empty). Then, the Neumann problem*

$$\begin{cases} \tau_{g,\gamma}\phi &= 0 \quad in \quad M \\ \nabla_0\phi &= 0 \quad on \quad \partial M \end{cases}$$

*has a solution in every homotopy class.*

(c) *Mixed Problem.* In contrast with the two above cases, this one considers $\partial N$. It is assumed that $\phi$ maps $\partial M$ into $\partial N$, but in an arbitrary form, and also that the normal derivative $\nabla_0\phi$ taken at a point in $\partial M$ is normal to $\partial N$. This makes sense since $\nabla_0\phi \in TN$. Let $\mathscr{M}_\partial(M, N)$ denote the closed subspace of those $\phi \in \mathscr{M}(M, N)$ such that $\phi(\partial M) \subseteq \partial N$, with the $C^\infty$-topology. A relative homotopy class will now mean a connected component of $\mathscr{M}_\partial(M, N)$. In addition, it is assumed that $\partial N$ is totally geodesic.

**Theorem D** (Hamilton [Ham75]). *Let $(M, g)$ and $(N, \gamma)$ be compact Riemannian manifolds with boundary. Suppose that $N$ has non-positive Riemannian curvature and that $\partial N$ is totally geodesic. Then, the mixed problem*

$$\begin{cases} \tau_{g,\gamma}\phi &= 0 \quad in \quad M \\ \phi(\partial M) \subseteq \partial N & \\ \nabla_0\phi &= 0 \quad on \quad \partial M \end{cases}$$

*has a solution in every relative homotopy class.*

Motivated by previously discussed theory coupled with the promising case of Ricci flow introduced by Hamilton [Ham82], we consider a family of closed hypersurfaces $\Sigma_t$ in $(M, g(t))$ and a family of smooth maps $\phi_t : (M, g(t)) \to (N, \gamma)$ with Riemannian metrics $g(t)$ envolving by some geometric flow and $\Sigma_t$ evolving by mean curvature flow. We now contextualize with results and historical data (see Figure 1) our setting of study.

It is known that the Ricci flow was expected to have a gradient-like structure, as well as the mean curvature flow case from the area functional. Indeed, this was one of Perelman's contributions when he modified the Hilbert-Einstein functional in the context of weighted compact smooth manifolds. He defined the functional $\mathcal{F}(g, f)$ on the space of metrics and smooth functions on a closed smooth manifold, whose variation $\delta\mathcal{F}(g, f)$ provides a gradient-like structure to the Ricci flow with weighted measure-preserving, see [Per02].

Four years later, List [Lis06] presented a connection between Ricci flow on $n-$dimensional closed Riemannian manifold $M$ without boundary and Einstein's static vacuum equations through a coupled system of Ricci flow and heat equation with a coupling constant $\alpha_n = (n-1)/(n-2)$, with $n \geqslant 3$, and then he defined a functional $\mathcal{F}(g, f, w)$ on the space of metrics and cartesian product of smooth functions on a closed smooth manifold without boundary, whose variation $\delta\mathcal{F}(g, f, w)$ provides a gradient-like flow from this coupled system.

In the boundary case, Ecker [Eck07] defined a version of Perelman's $\mathcal{W}$-functional for the Ricci flow on bounded Euclidean domains with smooth boundary. Curiously, Hamilton's differential Harnack expression (see [Ham95b]) on the boundary integrand appears in its time-derivative formula. Based on Ecker's work, Lott [Lot12] defined the functional $I_\infty(g, f)$ on the space of metrics and smooth functions on a compact smooth manifold with boundary to be a weighted version of the Gibbons-Hawking-York action (see [Yor72, GH77]) from which he found an extension of Hamilton's differential Harnack expression on the boundary integrad. Magni, Mantegazza and Tsatis [MMT13] found a Huisken monotonicity-type formula (see [Hui90]) for the mean curvature flow in an ambient smooth manifold with Riemannian metric that evolves by a self-similar solution to the Ricci flow.

More recently, Gomes and Hudson [GH23] considered Lott's program in the context of mean curvature flow in an extended Ricci flow background. They studied variational properties of an appropriate extended version of Lott's functional in the context of List's work, namely, the extended weighted Gibbons-Hawking-York action $I_\infty^{\alpha_n}(g, f, w)$ on a compact smooth manifold with boundary. They obtained evolution equations for the second fundamental form and the mean curvature in an extended Ricci flow background, and then an extension of Hamilton's differential Harnack expression appears as well as a Huisken monotonicity-type formula for the mean curvature solitons in this background.

In a more general context, Müller [Mü09, Mül12] worked on a new geometric flow which consists of a coupled system of the Ricci flow on a closed Riemannian manifold $(M, g)$ with the harmonic map heat flow of a map $\phi : (M, g) \to (N, \gamma)$, where $(N, \gamma)$ is a closed Riemannian manifold. Precisely, he considered a family of Riemannian metrics $g(t)$ on $M$ and a family of smooth maps $\phi(t)$ from $M$ to $N$ to define $(g(t), \phi(t))_{t\in[0,T)}$ as a solution to the coupled system of Ricci flow and harmonic map heat flow ($(RH)_\alpha$ *flow*, for short), namely

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\operatorname{Ric}_{g(t)} + 2\alpha(t)\nabla\phi(t) \otimes \nabla\phi(t), \\ \frac{\partial}{\partial t}\phi(t) = \tau_{g(t),\gamma}\phi(t), \end{cases} \tag{2}$$

where $\alpha(t)$ is a nonnegative coupling constant. For an account of $(RH)_\alpha$ flow, including proof of short-time existence and uniqueness of solutions to (2), see [Mül12, Sect. 4.2].

Müller realized that this coupled system may behave less singularly than the Ricci flow or the standard harmonic map flow alone. In order to interpret system (2) as a gradient flow by means a functional $\mathcal{F}_\alpha(g, f, \phi)$ for a fixed measure, he worked with the heat operator $\square = \frac{\partial}{\partial t} - \Delta_g$

whose formal adjoint $\Box^*$ is given by

$$\Box^* = -\frac{\partial}{\partial t} - \Delta_g + R_g - \alpha|\nabla\phi|^2 \tag{3}$$

along the $(RH)_\alpha$ flow.

Müller's approach motivated the first theorem of this thesis. Next, we continue to establish our study context more precisely.

A gradient soliton to the $(RH)_\alpha$ flow is a self-similar solution $\big(\overline{g}(t), \overline{\phi}(t)\big)$ of (2) given by

$$\begin{cases} \overline{g}(t) = \sigma(t)\psi_t^* g, \\ \overline{\phi}(t) = \psi_t^* \phi, \end{cases}$$

for some initial value $(g, \phi)$, where $\psi_t$ is a smooth one-parameter family of diffeomorphisms of $M$ generated from the flow of $\nabla_g f/\sigma(t)$, $f \in C^\infty(M)$, and $\sigma(t)$ is a positive smooth function on $t$. By setting $\overline{f}(t) = \psi_t^* f$, from (2) we can obtain

$$\begin{cases} \operatorname{Ric}_{\overline{g}} + \nabla^2_{\overline{g}}\overline{f} - \alpha\nabla\overline{\phi}\otimes\nabla\overline{\phi} & = & \dfrac{c}{2(T-t)}\overline{g}, \\ \tau_{\overline{g},\gamma}\overline{\phi} & = & \langle\nabla_{\overline{g}}\overline{\phi}, \nabla_{\overline{g}}\overline{f}\rangle, \end{cases}$$

where $c = 0$ in the steady case (for $t \in \mathbb{R}$ and $\psi_0 = \mathrm{Id}$), $c = 1$ in the shrinking case (for $t \in (-\infty, T)$ and $\psi_{T-1} = \mathrm{Id}$) and $c = 1$ in the expanding case (for $t \in (T, \infty)$ and $\psi_{T+1} = \mathrm{Id}$). Moreover,

$$\frac{\partial}{\partial t}\overline{f} = |\nabla_{\overline{g}}\overline{f}|^2_{\overline{g}}.$$

The function $\overline{f}$ is called the *potential function*.

As in [GH23], we consider mean curvature flow in the following context: let $(g(t), \phi(t))$ be an $(RH)_\alpha$ flow in $M \times I$. Given an $(n-1)$-dimensional compact smooth manifold $\Sigma$ without boundary, and let $\{x(\cdot, t); t \in [0, T)\}$ be a smooth one-parameter family of immersions of $\Sigma$ in $M$. For each $t \in [0, T)$, set $x_t = x(\cdot, t)$ and $\Sigma_t$ for the hypersurface $x_t(\Sigma)$ of $(M, g(t))$, which we can write as $\Sigma_t := \big(\Sigma, x_t^* g(t)\big)$. Suppose that the family $\mathscr{F} := \{\Sigma_t\,;\, t \in [0, T)\}$ evolves under mean curvature flow, MCF for short,

$$\begin{cases} \dfrac{\partial}{\partial t}x(p,t) & = & H(p,t)e(p,t), \\ x(p,0) & = & x_0(p), \end{cases}$$

where $H(p, t)$ and $e(p, t)$ are the mean curvature and the unit normal of $\Sigma_t$ at $p \in \Sigma$, respectively. In this setting, we say that $\mathscr{F}$ is a *MCF in the* $(g(t), \phi(t)) - (RH)_\alpha$ *flow background*. In the particular case $\big(g(t), \phi(t)\big) = \big(\overline{g}(t), \overline{\phi}(t)\big)$ is a self-similar solution to the $(RH)_\alpha$ flow on $M$ with potential function $\overline{f}$, a hypersurface $\Sigma_t \in \mathscr{F}$ is a *mean curvature soliton*, if

$$H(p,t) + e(p,t)\overline{f} = 0 \quad \text{on} \quad \Sigma.$$

Here, $e(\,\cdot\,,t)$ must be the inward unit normal vector field on $\Sigma_t$.

Now, we consider an $n$-dimensional compact smooth manifold $M$ with boundary $\partial M$. Let $\mathrm{met}(M)$ be the set of all Riemannian metrics $g$ on $M$. We define the functional $\mathcal{F}_\infty^\alpha$ on the product $\mathscr{P}(M,N) := \mathrm{met}(M) \times C^\infty(M) \times C^\infty(M,N)$ as

$$\mathcal{F}_\infty^\alpha(g,f,\phi) := \int_M \Big(R_\infty - \alpha|\nabla\phi|^2\Big)e^{-f}dM + 2\int_{\partial M} H_\infty e^{-f}dA,$$

where $R_\infty := R_g + 2\Delta_g f - |\nabla f|_g^2$ is the *weighted scalar curvature* of $g$, the function $H_\infty := H_g + e_0 f$ is the *weighted mean curvature* with respect to the inward unit normal vector field $e_0$ on $\partial M$, the forms $dM$ and $dA$ are the $n$-dimensional Riemannian measure of $(M,g)$, and the $(n-1)$-dimensional Riemannian measure of $(\partial M, g)$, respectively.

We observe that $\mathcal{F}_\infty^\alpha$ is the proper extension for our context of the energy functionals $E$, $\mathcal{F}(g,f)$, $\mathcal{F}(g,f,w)$, $I_\infty$, $I_\infty^{\alpha_n}$ and $\mathcal{F}_\alpha$ previously mentioned. Furthermore, it is already clear that $R_\infty$ arises quite naturally, as observed by Perelman [Per02, Sect. 1.3], and $H_\infty$ is in fact the appropriate geometric object when we are using a weighted measure (see, e.g., [Gro03, Sect. 9.4.E]).

Our first main result is a variational formula for $\mathcal{F}_\infty^\alpha$ from which we obtain a gradient-like structure for $(RH)_\alpha$ flow and an extension of Hamilton's differential Harnack expression of the mean curvature flow in Euclidean space. It reads as follows (see Sections 1.1 and 2.1 for definitions and notations).

**Theorem 1.** *Let $M$ be an $n$-dimensional compact smooth manifold with boundary $\partial M$, and let $\mathscr{F}$ be the MCF of $\partial M$ in the $\big(g(t),\phi(t)\big) - (RH)_\alpha$ flow background with Neumann boundary condition $\nabla_0\phi = 0$. If $u := e^{-f}$ is a solution to the conjugate heat equation*

$$\Box^* u = 0 \quad in \quad M \times [0,T)$$

*with $e_0 u = Hu$ on $\partial M$, then*

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_\infty^\alpha = {} & 2\int_M \Big(|\mathrm{Ric} + \nabla^2 f - \alpha\nabla\phi\otimes\nabla\phi|^2 + \alpha|\tau_{g,\gamma}\phi - \langle\nabla\phi,\nabla f\rangle|^2\Big)e^{-f}dM \\ & + 2\int_{\partial M}\Big(\frac{\partial H}{\partial t} - 2\langle\widehat{\nabla} f,\widehat{\nabla} H\rangle + \mathcal{A}(\widehat{\nabla} f,\widehat{\nabla} f) + 2R^{0i}\widehat{\nabla}_i f - \frac{1}{2}\nabla_0 R - HR_{00} \\ & + \alpha\mathcal{A}(\widehat{\nabla}\phi,\widehat{\nabla}\phi)\Big)e^{-f}dA,\end{aligned}$$

*where $\mathcal{A}$ is the second fundamental form of $\partial M$, and $\widehat{\nabla}$ denotes the gradient on $\partial M$.*

For the proof of Theorem 1, we first study the MCF in an extended Ricci flow background , and then "translate" the results for the context of the $(RH)_\alpha$ flow. We also obtain an extension of the Hamilton's differential Harnack expression from the mean curvature flow in Euclidean space, but now, to the more general context of MCF in the $(RH)_\alpha$ flow which must vanishes on the gradient steady soliton to this flow, see Corollary 2.24. In particular, when $\phi$ is a real-valued smooth function on $M$, we recover [GH23, Cor. 4].

Our second main result is a Huisken monotonicity-type formula for the MCF in an $(RH)_\alpha$ flow background.

**Theorem 2.** *Let $(M, g)$ be an $n$-dimensional Riemannian manifold, and let $\Sigma$ be an $(n-1)$-dimensional compact smooth manifold without boundary. Consider $\mathscr{F}$ the MCF of $\Sigma$ in the $(\overline{g}, \overline{\phi})-(RH)_\alpha$ flow background with potential function $\overline{f}$. Denote by $dA_{\overline{g}}$ the $(n-1)$-dimensional Riemannian measure on $\Sigma$ and set $\mathrm{Area}_{\overline{f}}(\Sigma_t) := \int_\Sigma e^{-\overline{f}} dA_{\overline{g}}$. Under these conditions, the function $\Phi(t)$ given by:*

*(i)* $\mathbb{R} \ni t \mapsto \mathrm{Area}_{\overline{f}}(\Sigma_t)$ *in the steady case,*

*(ii)* $(-\infty, T) \ni t \mapsto [4\pi(T-t)]^{-(n-1)/2}\,\mathrm{Area}_{\overline{f}}(\Sigma_t)$ *in the shrinking case, and*

*(iii)* $(T, \infty) \ni t \mapsto [4\pi(t-T)]^{-(n-1)/2}\,\mathrm{Area}_{\overline{f}}(\Sigma_t)$ *in the expanding case,*

*is non-increasing. Moreover, $\Phi(t)$ is constant if and only if $\mathscr{F}$ is a family of mean curvature solitons.*

In Section 2.3 we address the construction of a family $\mathscr{G}$ of mean curvature solitons in a $(RH)_\alpha$ flow background, and we establish a characterization of such a family, as follows.

**Theorem 3.** *If $\Sigma$ is an $f$-minimal hypersurface of $(M, g)$, then $\mathscr{G}$ is a family of mean curvature solitons in the $(\overline{g}, \overline{\phi})-(RH)_\alpha$ flow background on $M$. Moreover, any family $\mathscr{F}$ of mean curvature solitons in the $(\overline{g}, \overline{\phi})-(RH)_\alpha$ flow background on $M$ is given by $\mathscr{G}$ up to reparametrization.*

Coming back to talk about Theorem 2, we observe that it is a useful tool for studying an analogous result to Huisken's work on mean curvature solitons in the $(\overline{g}, \overline{\phi})-(RH)_\alpha$ flow background. It is important to mention that such a result provides a fundamental insight into the behavior of the MCF in Euclidean space, showing that shrinking self-similar solutions are intrinsically linked to singularities of type-I. These solutions emerge as critical points of the Gaussian area-type functional, see [Hui90] for details.

The MCF in a shrinking $\big(\overline{g}(t), \overline{\phi}(t)\big)-(RH)_\alpha$ flow background develops a singularity of type-I when there exists a constant $C > 1$ such that

$$\max_{p\in\Sigma} |\mathcal{A}(p,t)| \leqslant \frac{C}{\sqrt{T-t}},$$

where $\mathcal{A}(\cdot, t)$ stands for the second fundamental form of $\Sigma_t$.

The most appropriate setting to give an application of Theorem 2 is to consider the family $\mathscr{F}$ and its associated normalized family $\widetilde{\mathscr{F}}$ as follows.

Let $\mathscr{F} = \{\Sigma_t; t \in [0,T)\}$ be the MCF in the shrinking $\left(\overline{g}, \overline{\phi}\right) - (RH)_\alpha$ flow background with potential function $\overline{f}(t)$ and $\overline{g}(t) = (T-t)\psi_t^* g$, for $t \in (-\infty, T)$. Setting $s = -\log(T-t)$ and $\widetilde{\Sigma}_s = (\Sigma, \widetilde{x}_s^* g)$, where $\widetilde{x}_s = \psi_t \circ x_t$, we have

$$\frac{\partial \widetilde{x}_s}{\partial s} = \left(\nabla_g f + H_g\right)(\widetilde{x}_s) \quad \text{on} \quad \Sigma \times [-\log T, \infty).$$

We call the family $\widetilde{\mathscr{F}} := \{\widetilde{\Sigma}_s; s \in [-\log T, \infty)\}$ the *normalized MCF* in $(M,g)$.

Our next main result is a convergence theorem for $\widetilde{\mathscr{F}}$ in the $C^\infty$ Cheeger-Gromov sense.

**Theorem 4.** *Assume that $(M,g)$ is an $n$-dimensional compact Riemannian manifold, and let $(\overline{g}(t), \overline{\phi}(t))$ be a shrinking self-similar solution to the $(RH)_\alpha$ flow on $M$ with potential function $\overline{f}$ and initial value $(g,\phi)$. Given an $(n-1)-$dimensional compact smooth manifold $\Sigma$ without boundary, and let $\mathscr{F}$ be the MCF of $\Sigma$ in the shrinking $(\overline{g}, \overline{\phi}) - (RH)_\alpha$ flow background which develops a singularity of type-I. Consider the associate normalized MCF $\widetilde{\mathscr{F}}$ in $(M,g)$. Then, for any increasing sequence $\{s_j\}_{j=1}^\infty$ and points $\{p_j\}_{j=1}^\infty$ in $\Sigma$, there exist subsequences $s_{j_k}$ and $p_{j_k}$ in $\Sigma$, such that the family of immersion maps $\widetilde{x}_{s_{j_k}} : \Sigma \to (M,g)$ from pointed manifolds $(\Sigma, p_{j_k})$ converges to an immersion map $x_\infty : \Sigma_\infty \to (M,g)$ from an $(n-1)-$dimensional complete pointed Riemannian manifold $(\Sigma_\infty, x_\infty^* g, p_\infty)$ in the $C^\infty$ Cheeger–Gromov sense. Furthermore, $(\Sigma_\infty, x_\infty^* g)$ is a $f_\infty$-minimal hypersurface of $(M,g)$, where $f_\infty = f \circ x_\infty$.*

The proof of Theorem 4 is based on the Arzelà–Ascoli theorem in the context of Riemannian manifolds which provides a sequence of isometric immersions on an exhaustion of $\Sigma_\infty$ converging to a limiting global solution which is $f_\infty$-minimal hypersurface of $(M,g)$. We prove global supremum estimates depending only on the initial bounds on Riemann tensor $\mathrm{Rm}_g$ and Hessian operator $\nabla_g^2\phi$, whereas the interior estimates depend on the full $C^\infty$ norm of $\overline{g}$ (see Chapter 3). This is possible since the estimates for the derivatives of $g$ and $\phi$ can be combined in the right way.

This thesis is structured as follows. We begin in Chapter 1 with some definitions and basic concepts about Riemannian geometry and maps between Riemannian manifolds. In Chapter 2 we obtain the variational formula for the functional $\mathcal{F}_\infty^\alpha$, and an extension of Hamilton's differential Harnack expression for our context. Moreover, we provide the proof of Theorems 1, 2 and 3. In Chapter 3 we proceed with proofs of preliminary results which are formulated in the more general context of complete Riemannian manifolds with bounded geometry, and then we prove Theorem 4. This chapter ends with the proof of this latter theorem for the case of a complete noncompact Riemannian manifold with some additional uniformity conditions.

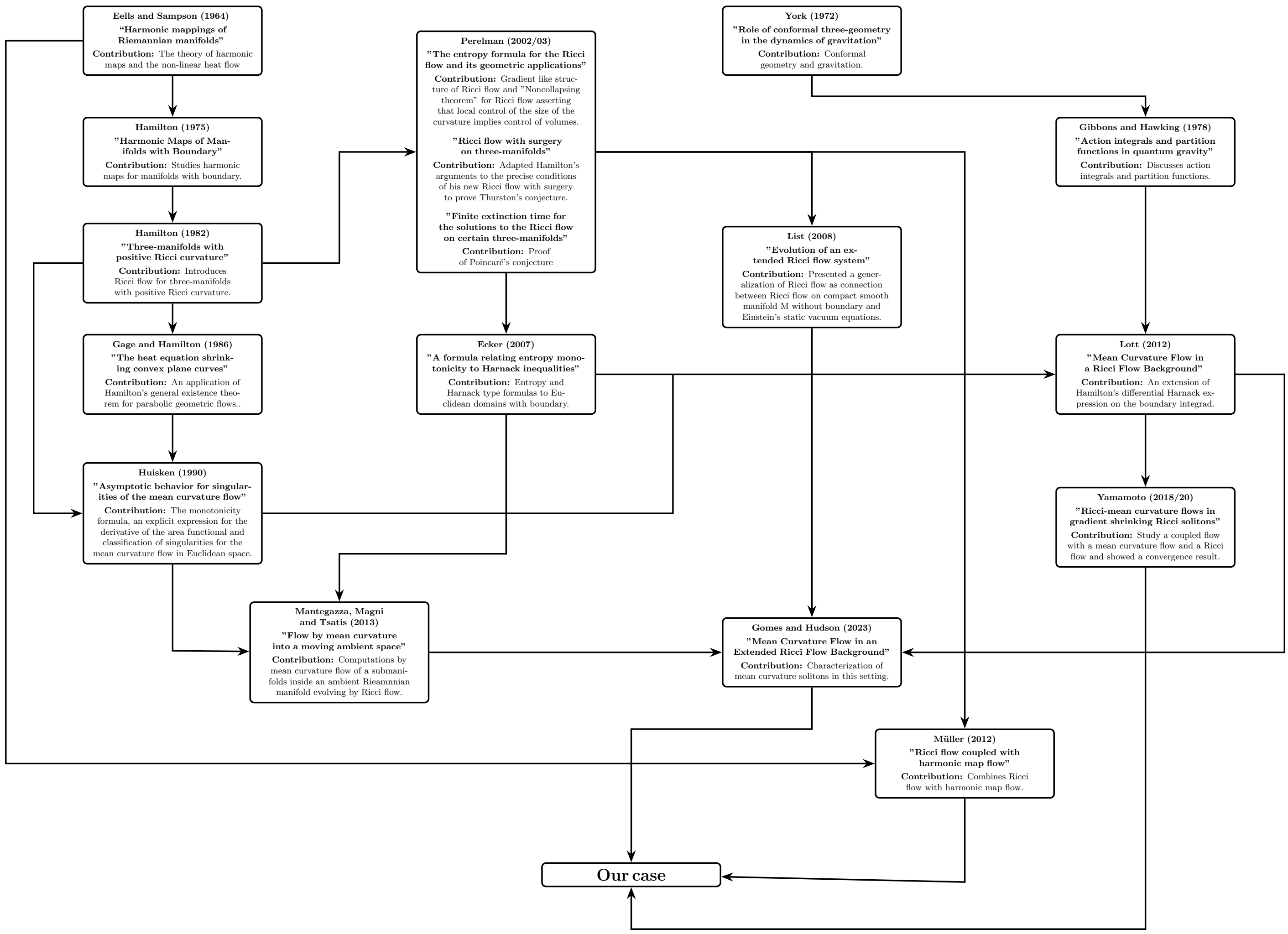



Figure 1: Flowchart of the chronological development of referenced works.

Each arrow connects earlier works (tail) to later publications (head), showing how research evolved over time until reaching this thesis (our case).

# CHAPTER 1

# Preliminaries

We begin with some definitions and basic concepts about Riemannian geometry and maps between Riemannian manifolds.

## 1.1 Basic Riemannian geometry

Throughout this text, all manifolds are assumed to be orientable and connected. Consider a smooth map $\phi : (M^n, g) \to (N^m, \gamma)$ between Riemannian manifolds $(M^n, g)$ and $(N^m, \gamma)$ with boundaries $\partial M$ and $\partial N$, respectively. We shall denote the local coordinates at $p \in M$ by $\{x^i\}$, the local coordinate basis by $\{\partial_i\}$ and the local dual coordinate basis by $\{dx^i\}$. Near $\partial M$, we take $x^0$ to be a local defining function for $\partial M$. We denote the local coordinates for $\partial M$ by $\{x^{\hat{i}}\}$. We choose these coordinates near a point at $\partial M$ so that $\partial_0\big|_{\partial M}$ coincides with the inward-pointing unit normal field $e_0$ along the boundary. For $N$ we shall denote by $\{y^\alpha\}$ the local coordinate at $\phi(p)$, the local coordinate basis by $\{\partial_\alpha\}$ and $\phi^\alpha := y^\alpha \circ \phi$. We shall use the convention that repeated Latin indices are summed over from 0 to $n-1$ and repeated Greek indices are summed over from 0 to $m-1$. In general, we are using the Einstein convention of summing over repeated indices. In dealing with flows, we shall usually simplify the notation by suppressing the parameter $t$.

The metric on $M$ is denoted by $g = \langle , \rangle$ and $\langle \partial_i, \partial_j \rangle = g_{ij}$, and its inverse is denoted by $g^{ij}$ so that $g_{ij}g^{jk} = \delta_i^k$. The forms $dM$ and $dA$ are the $n$-dimensional Riemannian measure of $(M, g)$, and the $(n-1)$-dimensional Riemannian measure of $(\partial M, g|_{\partial M})$, respectively. We also use the classical notation $h^{ij} = g^{ik}g^{jl}h_{kl}$, for any 2-tensor field $h$ on $M$.

We denote the Levi-Civita connection on $TM$ by $\nabla$ and on $T\partial M$ by $\widehat{\nabla}$. By simplicity, we also denote $\nabla_i := \nabla_{\partial_i}$, $\nabla^i := g^{ij}\nabla_j$ and $X^i = g^{ij}X_j$, where $X_j = \langle X, \partial_j \rangle$.

In what concerns $\partial M$, we write $\mathcal{A}_{\hat{i}\hat{j}} := \langle \nabla_{\partial_{\hat{i}}} \partial_{\hat{j}}, e_0 \rangle$ for its second fundamental form, and

$H := g^{\hat{i}\hat{j}}\mathcal{A}_{\hat{i}\hat{j}}$ for its mean curvature. Hence,

$$\mathcal{A}^{\hat{i}\hat{j}} = g^{\hat{i}\hat{k}}g^{\hat{j}\hat{l}}\mathcal{A}_{\hat{k}\hat{l}} \quad \text{and} \quad \mathcal{A}^{\hat{k}}_{\hat{i}} = g^{\hat{k}\hat{l}}\mathcal{A}_{\hat{l}\hat{i}}.$$

For all $X, Y \in \Gamma(TM)$ and $\omega \in \Gamma(T^*M)$, we have

$$\nabla_X^{T^*M}\omega(Y) = X\big(\omega(Y)\big) - \omega\big(\nabla_X Y\big),$$

The smooth map $\phi$ induces the fiber bundle $\phi^*TN$ over $M$ as follows

$$\phi^*TN = \big\{(p,u); p \in M, u \in T_{\phi(p)}N\big\} = \bigcup_{p\in M}\{p\} \times T_{\phi(p)}N.$$

The Levi-Civita covariant derivative $\nabla^{TN}$ of the metric $\gamma$ on $N$ induces the following covariant derivative on $\phi^*TN$,

$$\nabla_X^{\phi^*TN}U := \nabla_{\phi_* X}^{TN}U,$$

for all $X \in \Gamma(TM)$ and $U \in \Gamma(TN)$.

The Riemannian curvature tensor is defined as

$$\mathrm{Rm}(X,Y)Z := \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z = R^l_{ijk}Y^jX^iZ^k\partial_l.$$

where

$$\begin{aligned} R^l_{ijk}\partial_l &= R(\partial_i,\partial_j)\partial_k = \nabla_j\nabla_i\partial_k - \nabla_i\nabla_j\partial_k, \\ R^l_{ijk} &= \partial_j\Gamma^l_{ik} - \partial_i\Gamma^l_{jk} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}, \\ \Gamma^k_{ij} &= \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \end{aligned}$$

When lowering the index to the fourth position, we obtain

$$R_{ijkl} = g_{ml}R^m_{ijk},$$

so that $R^s{}_{ijk} = g^{ls}R_{ijkl}$, where $R_{ijkl} = \langle R(\partial_i,\partial_j)\partial_k,\partial_l\rangle$ is the Riemann curvature. The Ricci tensor $R_{ij}$ is defined as $R_{ij} = g^{kl}R_{ikjl}$, and the scalar curvature is its trace $R_g = g^{ik}R_{ik} = g^{ik}g^{jl}R_{ijkl}$. Thus, for a vector field $X$, one has

$$[\nabla_i,\nabla_j]X^k = \nabla_i\nabla_j X^k - \nabla_j\nabla_i X^k = -R^k_{ijm}X^m = g^{kl}R_{ijlm}X^m.$$

Taking the trace in the second Bianchi identity

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0$$

we obtain

$$g^{im}\nabla_i R_{jklm} = -g^{im}\nabla_j R_{kilm} - g^{im}\nabla_k R_{ijlm} = -\nabla_j R_{kl} + \nabla_k R_{jl}.$$

We now trace with $g^{jl}$ to get

$$g^{im}\nabla_i R_{km} = -g^{jl}\nabla_j R_{kl} + \nabla_k R$$

that immediately implies

$$\nabla^l R_{kl} := g^{jl}\nabla_j R_{kl} = \frac{1}{2}\nabla_k R.$$

Now, we compute

$$\nabla\partial_i(\partial_s) := \nabla_{\partial_s}\partial_i := \Gamma^k_{si}\partial_k = \Gamma^k_{ji}dx^j(\partial_s)\partial_k =: \Gamma^k_{ij}dx^j \otimes \partial_k(\partial_s)$$

and

$$\begin{aligned}\nabla dx^i(\partial_r, \partial_s) &:= \nabla^{T^*M}_{\partial_r}\big(dx^i(\partial_s)\big) - dx^i\big(\nabla_{\partial_r}\partial_s\big)\\ &= \nabla_{\partial_r}(\delta_{is}) - dx^i(\Gamma^l_{rs}\partial_l)\\ &= -\Gamma^i_{jk}dx^j(\partial_r)dx^k(\partial_s)\\ &=: -\Gamma^i_{jk}dx^j \otimes dx^k(\partial_r, \partial_s).\end{aligned}$$

In short

$$\nabla\partial_i = \Gamma^k_{ij}dx^j \otimes \partial_k \quad \text{and} \quad \nabla dx^i = -\Gamma^i_{jk}dx^j \otimes dx^k. \tag{1.1}$$

Moreover

$$\begin{aligned}\left(\nabla\partial_\lambda|_\phi\right)(\partial_i) := \nabla^{\phi^*TN}_{\partial_i}\partial_\lambda|_\phi := \nabla_{\phi_*\partial_i}\partial_\lambda|_\phi = \nabla_{\nabla_i\phi^\alpha\partial_\alpha|_\phi}\partial_\lambda|_\phi &= \nabla_i\phi^\alpha\nabla_{\partial_\alpha|_\phi}\partial_\lambda|_\phi\\ &= \nabla_i\phi^\alpha\phi^*(\nabla_{\partial_\alpha}\partial_\lambda)\\ &= \nabla_i\phi^\alpha\phi^*(\Gamma^\beta_{\alpha\lambda}\partial_\beta)\\ &= (\Gamma^\beta_{\alpha\lambda}\circ\phi)\nabla_i\phi^\alpha\partial_\beta|_\phi\end{aligned}$$

and

$$\begin{aligned}\nabla dy^\lambda(\partial_i, \partial_\alpha|_\phi) := \left(\nabla^{T^*M}_{\partial_i}dy^\lambda\right)(\partial_\alpha|_\phi) &:= \nabla_{\partial_i}dy^\lambda(\partial_\alpha|_\phi) - dy^\lambda(\nabla_{\partial_i}\partial_\alpha|_\phi)\\ &= -dy^\lambda\left((\Gamma^\theta_{\alpha\beta}\circ\phi)\nabla_i\phi^\beta\partial_\theta|_\phi\right)\\ &= -(\Gamma^\lambda_{\alpha\beta}\circ\phi)\nabla_i\phi^\beta.\end{aligned}$$

In short

$$\nabla\partial_\lambda|_\phi = (\Gamma^\beta_{\alpha\lambda}\circ\phi)\nabla_i\phi^\alpha dx^i \otimes \partial_\beta|_\phi \quad \text{and} \quad \nabla dy^\lambda = -(\Gamma^\lambda_{\alpha\beta}\circ\phi)\nabla_i\phi^\beta dx^i \otimes dy^\alpha. \tag{1.2}$$

For a smooth function $f : M \to \mathbb{R}$, we write its gradient as $\nabla f = \nabla^i f\partial_i$ so that $\nabla^i f = g^{ij}\nabla_j f$ and $|\nabla f|^2_g = g^{ij}\nabla_i f\nabla_j f$, where $\nabla_j f = \langle\nabla f, \partial_j\rangle$. In addition, we set $\nabla^i\phi^\lambda = g^{ij}\nabla_j\phi^\lambda = g^{ij}\langle\nabla\phi^\lambda, \partial_j\rangle$, which gives the following expression for the Hessian tensor:

$$\nabla^i\nabla^j f := g^{ik}g^{jl}\nabla^2 f(\partial_k, \partial_l).$$

It is important to mention some basic results of the conformal theory. Let $\overline{g} = cg$, for some positive constant $c$, and let $\overline{\nabla}$ be the covariant derivative of $\overline{g}$. By Koszul's formula $\overline{\nabla} = \nabla$, moreover,

(i) $\overline{g}^{ij} = \dfrac{1}{c} g^{ij}$;

(ii) $\nabla_{\overline{g}} f = \dfrac{1}{c} \nabla_g f$;

(iii) $\overline{\mathrm{Rm}} = \mathrm{Rm}$;

(iv) $\overline{\mathrm{Ric}}(X,Y) = \mathrm{Ric}(X,Y)$;

(v) $\overline{\mathrm{Ric}}(X) = \dfrac{1}{c} \mathrm{Ric}(X)$;

(vi) $\overline{\nabla}^2 f(X,Y) = \nabla^2 f(X,Y)$;

(vii) $\overline{\nabla}_X \overline{\nabla} f = \dfrac{1}{c} \nabla_X \nabla f$.

We recall that the derivative $\nabla\phi$ maps linearly sections of $TM$ to sections of $TN$ along $\phi$, i.e., in terms of the bundle $\phi^*TN$, we can interpret $\nabla\phi$ as a section of the vector bundle of homomorphisms $\mathrm{Hom}(TM;\phi^*TN)$. Furthermore, since this latter bundle is isomorphic to the induced bundle $T^*M \otimes \phi^*TN$, we can introduce a connection $\nabla$ on $\Gamma(T^*M \otimes \phi^*TN)$ to obtain the second derivative $\nabla\nabla\phi$ as the derivative of $\nabla\phi$ with respect to the connection on $\Gamma(T^*M \otimes \phi^*TN)$, thus, it is a section of the bundle $T^*M \otimes T^*M \otimes \phi^*TN$. The *tension field* $\tau_{g,\gamma}\phi$ (or Laplacian $\Delta\phi$) is the trace of $\nabla\nabla\phi$ with respect to the inner product on $TM$. This defines $\tau_{g,\gamma}\phi$ as a section of the bundle $\phi^*TN$. Precisely,

$$\begin{aligned} \nabla\phi : TM &\longrightarrow \phi^*TN \\ X &\longmapsto d\phi(X) \end{aligned}$$

where

$$d\phi(\partial_j) = d(y^\lambda \circ \phi)(\partial_j)\partial_\lambda|_\phi = d\phi^\lambda(\partial_j)\partial_\lambda|_\phi = \langle \nabla\phi^\lambda, \partial_j \rangle_M \partial_\lambda|_\phi = \nabla_j \phi^\lambda \partial_\lambda|_\phi$$

and

$$d\phi(X) = \nabla_j \phi^\lambda dx^j(X)\partial_\lambda|_\phi.$$

Therefore,

$$\nabla\phi = \partial_j \phi^\lambda dx^j \otimes \partial_\lambda|_\phi = \nabla_j \phi^\lambda dx^j \otimes \partial_\lambda|_\phi.$$

Taking $X = \nabla f$ and writing $\nabla f = g^{ik}\nabla_i f \partial_k$, we get

$$\langle \nabla\phi, \nabla f \rangle := \nabla\phi(\nabla f) = g^{ik}\nabla_j \phi^\lambda \nabla_i f dx^j(\partial_k)\partial_\lambda|_\phi = \langle \nabla f, \nabla\phi^\lambda \rangle \partial_\lambda|_\phi.$$

Taking $X = e_0$, we have

$$\nabla_0\phi := \nabla\phi(e_0) = \nabla_j \phi^\lambda dx^j(e_0)\partial_\lambda|_\phi = e_0\phi^\lambda \partial_\lambda|_\phi.$$

By using (1.1) and (1.2), one has

$$\begin{aligned}(\nabla_{\partial_i}\nabla\phi)(\partial_j, dy^\lambda) &= \nabla_{\partial_i}\nabla\phi(\partial_j, dy^\lambda) - \nabla\phi(\nabla_{\partial_i}\partial_j, dy^\lambda) - \nabla\phi(\partial_j, \nabla_{\partial_i} dy^\lambda)\\ &= \partial_i\partial_l\phi^\theta dx^l(\partial_j)\partial_\theta(dy^\lambda) - \Gamma^k_{ij}\nabla_l\phi^\theta dx^l(\partial_k)\partial_\theta(dy^\lambda)\\ &\quad + \left(\Gamma^\lambda_{\alpha\beta}\circ\phi\right)\nabla_i\phi^\beta\nabla_l\phi^\theta dx^l(\partial_j)\partial_\theta(dy^\alpha)\\ &= \partial_i\partial_j\phi^\lambda - \Gamma^k_{ij}\nabla_k\phi^\lambda + (\Gamma^\lambda_{\alpha\beta}\circ\phi)\nabla_i\phi^\alpha\nabla_j\phi^\beta.\end{aligned}$$

By setting

$$(\nabla_i\nabla_j\phi)^\lambda := \partial_i\partial_j\phi^\lambda - \Gamma^k_{ij}\nabla_k\phi^\lambda + (\Gamma^\lambda_{\alpha\beta}\circ\phi)\nabla_i\phi^\alpha\nabla_j\phi^\beta, \tag{1.3}$$

we obtain

$$\nabla\nabla\phi = (\nabla_i\nabla_j\phi)^\lambda dx^i\otimes dx^j\otimes\partial_\lambda|_\phi.$$

The tension field of $\phi$ with respect to the metrics $g$ and $\gamma$ is given by

$$\begin{aligned}\tau_{g,\gamma}\phi &= \mathrm{tr}_g(\nabla\nabla\phi)\\ &= g^{ij}\Big(\partial_i\partial_j\phi^\lambda - \Gamma^k_{ij}\nabla_k\phi^\lambda + (\Gamma^\lambda_{\alpha\beta}\circ\phi)\nabla_i\phi^\alpha\nabla_j\phi^\beta\Big)\partial_\lambda|_\phi\\ &= \Big(\Delta_g\phi^\lambda + (\Gamma^\lambda_{\alpha\beta}\circ\phi)g(\nabla\phi^\alpha,\nabla\phi^\beta)\Big)\partial_\lambda|_\phi.\end{aligned}$$

Notice that $\tau_{g,\gamma}\phi$ is a generalization of the Laplacian on $C^\infty(M)$. By definition, the map $\phi$ is harmonic if $\tau_{g,\gamma}\phi = 0$.

We shall use the inner product on the bundle $T^*M\otimes\phi^*TN$ induced by $g$ and $\gamma$ as follows

$$\langle\nabla\phi,\nabla\phi\rangle_{T^*M\otimes\phi^*TN} := g^{ij}\phi^*\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta. \tag{1.4}$$

Since there is no danger of confusion, we shall write

$$T(\nabla\phi,\nabla\phi) := T^{ij}\phi^*\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta$$

for any 2-tensor $T$ on $M$, and the same notation $\langle\cdot,\cdot\rangle$ for the inner products on $M$, $N$ and $T^*M\otimes\phi^*TN$. Besides, for the sake of simplicity, we write

$$\nabla\phi\otimes\nabla\phi(\partial_i,\partial_j) := \phi^*\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta.$$

As in (1.4), we have

$$\langle S,T\rangle = g^{ik}g^{jl}\phi^*\gamma_{\alpha\beta}S^\alpha_{ij}T^\beta_{kl}$$

for any $S,T\in T^*M\otimes T^*M\otimes\phi^*TN$.

As in (1.3), we obtain the higher-order derivatives

$$(\nabla^{r+1}\phi)^\lambda = \partial_{i_1\cdots i_{r+1}}\phi^\lambda - \sum_{j=1}^r\Gamma^k_{i_j\,i_{r+1}}\partial_{i_1\cdots i_{j-1}ki_{j+1}\cdots i_r}\phi^\lambda + (\Gamma^\lambda_{\alpha\beta}\circ\phi)\nabla_{i_{r+1}}\phi^\alpha\nabla^r\phi^\beta$$

where $\partial_{i_1\cdots i_\ell}\phi^\lambda = \frac{\partial \phi^\lambda}{\partial x^{i_1}\cdots\partial x^{i_\ell}}$, $\nabla^r = \nabla_{i_1}\cdots\nabla_{i_r}$ and

$$\langle S, T\rangle = g^{i_1 j_1}\cdots g^{i_r j_r}\phi^*\gamma_{\alpha\beta}S^\alpha_{i_1\cdots i_r}T^\beta_{j_1\cdots j_r}$$

for any $S, T \in (T^*M)^{\otimes r}\otimes \phi^*TN$.

We finalize this section recalling the Dirichlet's energy functional $E : C^\infty(M,N) \to \mathbb{R}$ on maps between Riemannian manifolds with boundary, which is given by

$$E(\phi) := \int_M \frac{1}{2}|\nabla\phi|^2 dM,$$

where $|\nabla\phi|^2 := \langle\nabla\phi, \nabla\phi\rangle_{T^*M\otimes\phi^*TN}$ and $E(\phi)$ is called the *energy* of the map $\phi$.

Let $\phi_t : (M,g) \to (N,\gamma)$ be a smooth family of maps, with $t \in (-\epsilon,\epsilon)$, variational vector field $V$ and $\phi_0 = \phi$. Note that $V \in \Gamma(\phi^*TN)$, since the variation $\{\phi_t\}_{t\in(-\epsilon,\epsilon)}$ determines a curve in $C^\infty(M,N)$, and $V$ represents the tangent vector of this curve at $t=0$. Hamilton [Ham75] showed that the first variation formula of $E$ is given by

$$\frac{d}{dt}\Big|_{t=0}E(\phi_t) = -\int_M \langle V, \tau_{g,\gamma}\phi\rangle dM + \int_{\partial M}\langle V, \nabla_0\phi\rangle dA.$$

So, a harmonic map $\phi$ with Neumann boundary condition $\nabla_0\phi = 0$ is a critical point of Dirichlet's energy functional $E$.

## 1.2 Mean curvature flow

Let $(M,g)$ be an $n$-dimensional Riemannian manifold, and consider an $(n-1)$-dimensional compact smooth manifold $\Sigma$ without boundary (also known as a closed hypersurface). Let $\{x(\cdot,t); t\in[0,T)\}$ be a smooth one-parameter family of immersions of $\Sigma$ in $(M,g)$. Define $\Sigma_t := (\Sigma, x_t^*g)$, where $x_t = x(\cdot,t)$, and assume that the family $\mathscr{F} := \{\Sigma_t; t\in[0,T)\}$ evolves under mean curvature flow, MCF for short, that is,

$$\begin{cases} \frac{\partial}{\partial t}x(p,t) = H(p,t)e(p,t), \\ x(p,0) = x_0(p), \end{cases} \tag{1.5}$$

where $H(p,t)$ and $e(p,t)$ are the mean curvature and the unit normal of $\Sigma_t$ at the point $p\in\Sigma$, respectively. Besides, whenever there is no danger of confusion, we are writing $g(t)$ on $\Sigma$ instead of $x_t^*g(t)$, and $e_0$ to denote the inward unit normal vector field on $x_0(\Sigma)$, which we can identify with $\Sigma$. Also, we shall usually simplify the notation by suppressing the parameter $t$.

For short-time existence and uniqueness of a solution for (1.5) we refer to Gage and Hamilton [GH86] from which we know that a solution for this problem can be obtained as an application of Hamilton's general existence theorem for parabolic geometric flows in [Ham82].

In the case of smooth closed curves, Gage and Hamilton [GH86] proved that all smooth closed convex curves shrink to a point without forming any other singularities in finite time, where as Grayson [Gra87] proved that every simple smooth closed non-convex curve will become convex.

In the case of closed hypersurfaces in Euclidean space, Huisken [Hui90] showed that the shrinking self-similar solutions to the MCF appear as stationary points for the Gaussian area-type functional playing the role of the energy-type functional, which is non-increasing along the flow. He also proved, through his monotonicity formula, that the MCF converges to a self-shrinker in Euclidean space after scaling when it develops a *type-I singularity*, which is defined as follows.

**Definition 1.1.** If there exists a independent constant $C > 1$ so that

$$\sup_{p\in\Sigma} |\mathcal{A}(p,t)| \leq \frac{C}{\sqrt{2(T-t)}}, \tag{1.6}$$

we say that the MCF develops at time $T$ a type-I singularity.

System (1.5) is a nonlinear parabolic system; for this reason, the MCF may develop singularities, which happen in finite time.

The well-known examples of MCF are given by families of concentric round hyperspheres in $\mathbb{R}^n$ and cylinders in $\mathbb{R}^{n+1}$.

**Example 1.2.** Let $\mathbb{S}^{n-1}(R)$ be the $(n-1)$-sphere of radius $R$, and let $x(p,t) = r(t)x_0(p)$ be a family of immersions of $\mathbb{S}^{n-1}(R)$ into $\mathbb{R}^n$, where $r(t) = \sqrt{R^2 - 2(n-1)t} = \sqrt{2(n-1)(T-t)}$ and $x_0$ is the standard inclusion map. Note that at time $T = \frac{R^2}{2(n-1)}$ the sphere shrinks to a point, so the flow becomes singular. Moreover, the norm of the second fundamental form evolves as $|\mathcal{A}(\,\cdot\,,t)| = \frac{\sqrt{n-1}}{r(t)} = \frac{1}{\sqrt{2(T-t)}}$.

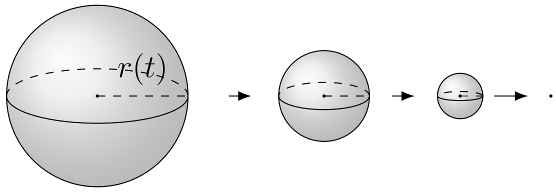



Figure 1.1: Sphere collapsed in finite time.

Another example is given by cylinders $\mathbb{S}^n(R) \times \mathbb{R}$.

**Example 1.3.** Let $\mathbb{S}^n(R) \times \mathbb{R}$ be the cylinders, and let $\widetilde{x}(p,s,t) = (x(p,t),s) = (r(t)x_0(p),s)$ be the family of immersions of $\mathbb{S}^n(R) \times \mathbb{R}$ into $\mathbb{R}^{n+2}$, where $r(t) = \sqrt{R^2 - 2nt} = \sqrt{2n(T-t)}$ and collapse to $\mathbb{R}$ at time $T = \frac{R^2}{2n}$. Moreover, $|\mathcal{A}(\,\cdot\,,t)| = \frac{\sqrt{n}}{r(t)} = \frac{1}{\sqrt{2(T-t)}}$.

Spheres and cylinders are special examples of *homothetically shrinking flows*, that is, hypersurfaces that simply move by contraction during the evolution by mean curvature. An

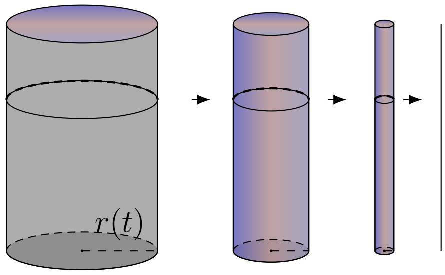



Figure 1.2: The Cylinder also collapsed in finite time.

interesting question is what happens to these solutions in the long run? How does the geometry of these hypersurfaces change? The first thing that you can say about the change of the geometry is that in all these flows, the area decreases. One way to carry out this analysis is to understand the behavior of the time derivative of the metric, the second fundamental form, the mean curvature flow, and the area element. This is the content of the following Huisken result by

**Proposition 1.4.** *Let $\mathscr{F}$ be a family moving by MCF in Euclidean space $\mathbb{R}^n$, $n \geq 3$. Then*

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2H\mathcal{A}_{ij}, \\
\frac{\partial}{\partial t} \mathcal{A}_{ij} &= (\widehat{\Delta}\mathcal{A})_{ij} - 2H\mathcal{A}_{ik}\mathcal{A}^k_j + \mathcal{A}^{kl}\mathcal{A}_{kl}\mathcal{A}_{ij}, \\
\frac{\partial}{\partial t} H &= \widehat{\Delta} H + \mathcal{A}^{ij}\mathcal{A}_{ij} H, \\
\frac{\partial}{\partial t} dA &= -H^2 dA.
\end{aligned}$$

For a proof, see [Hui84, Lem. 3.2, Thm. 3.4 and Cor. 3.5] or [Man11, Sect. 2.3].

From the point of view of variational theory, it is known that minimal hypersurfaces can be interpreted as critical points of the area functional. A hypersurface of the area functional is called *critical* if $\frac{d}{dt}\big|_{t=0} \operatorname{Area}(X) = 0$ for all variational vector field $X = \frac{\partial}{\partial t}\big|_{t=0} x_t$, where $\{x_t\}$ is a smooth one-parameter family of immersions of the hypersurface in $M$. Huisken [Hui90] considers a *weighted area functional* with the backward heat kernel $\rho(t,x)$ on $\mathbb{R}^n$, $n \geq 3$, centered at some $x_0 \in \mathbb{R}^n$ and with maximal time $T > 0$, i.e.,

$$\rho(t,x) = \frac{e^{\frac{-|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{\frac{n}{2}}}, \quad \text{for} \quad t < T,$$

and obtained the following monotonicity formula.

**Theorem 1.5** (Huisken [Hui90]). *Let $\mathscr{F}$ be a family moving by MCF in Euclidean space $\mathbb{R}^n$, $n \geq 3$, for $t < T$. Then*

$$\frac{d}{dt}\Big([4\pi(T-t)]^{\frac{1}{2}} \int_{\Sigma_t} \rho(t,x)dA_t\Big) = -[4\pi(T-t)]^{\frac{1}{2}} \int_{\Sigma_t} \Big|H + \frac{1}{2(T-t)}(x-x_0)^{\perp}\Big|^2 \rho(t,x)dA_t$$

*where $(x-x_0)^{\perp}$ is the normal component of $x - x_0$.*

Now, we come back to the Riemannian manifold ambient case. First, we recall a well known property for $f$-minimal hypersurfaces, which are closely related to the purpose of this thesis.

Let $\{x_t\}$ be a smooth one-parameter family of immersions of $\Sigma$ in $M$, and let $X = \frac{\partial}{\partial t}\big|_{t=0} x_t$ be the variational vector field along $\Sigma$. Let us consider the *$f$-weighted area functional* given by

$$\text{Area}_f(\Sigma_t) = \int_\Sigma e^{-x_t^* f} dA_t = \int_{\Sigma_t} e^{-f} dA,$$

where $dA_t$ stands for the area element on $\Sigma_t$. Recall that

$$\frac{d}{dt}\Big|_{t=0} \text{Area}_f(\Sigma_t) = -\int_\Sigma H_\infty \langle X, e_0 \rangle e^{-f} dA,$$

where $H_\infty := H + e_0 f$. So the critical points of the $f$-weighted area functional on $\Sigma$ are *$f$-minimal hypersurfaces*, i.e., $H_\infty = 0$. Some results concerning $f$-minimal hypersurfaces can be found, e.g., in [CZ15, CMZ15, SY15, Wei17, ALR20] and [CVZ21].

## 1.3 Basic analysis on smooth manifolds

A compactness theorem by Hamilton [Ham95a] for the Ricci flow asserts that any sequence of complete solutions to the Ricci flow with their curvature uniformly bounded above and their injectivity radii bounded from below will contain a convergent subsequence. This result stems from the convergence theory established by Cheeger and Gromov, see [Gro81], [CG86] and [CG90]. While regularity is a critical aspect in many applications of this theory, the proof of the compactness theorem for the Ricci flow benefits from the fact that sequences of Ricci flow solutions exhibit excellent regularity properties. Specifically, because bounds on the curvature of a Ricci flow solution imply bounds on all its curvature derivatives, the compactness theorem ensures $C^\infty$-convergence on compact sets. The approach involves examining progressively shorter time intervals leading up to a singularity of the Ricci flow and rescaling the solution within each interval to obtain solutions over longer time periods with uniformly bounded curvature. The limiting solution derived from this process provides insights into the structure of the singularity.

One of the notions of convergence for subsets in the Euclidean space is given by Hausdorff convergence. In the context of Riemannian manifolds, this concept of convergence is extended through the Gromov-Hausdorff topology, which is a generalization of the Hausdorff one presented by Gromov in [Gro81] that makes it possible to study convergence in these more general spaces, see [ACGL20] for more details.

### 1.3.1 *The compact-open $C^k$ topology*

Given smooth manifolds $M$ and $N$, and an integer $k \geq 0$, let $C^k(M, N)$ be the space of all functions from $M$ to $N$ that are $k$-times continuously differentiable. We equip $C^k(M, N)$

with the compact-open $C^k$ topology, which is a topology on $C^k(M,N)$ generated by a subbasis consisting of sets given by:

$$\begin{aligned} &S^k(\phi, x, U, y, V, K, \varepsilon) \\ &= \left\{ \psi \in C^k(M,N); \phi(K) \subseteq V, \max_{|\alpha|\leq k} \sup_{p\in x(K)} \left| \partial^\alpha (y\circ\psi\circ x^{-1})(p) - \partial^\alpha(y\circ\phi\circ x^{-1})(p) \right| < \varepsilon \right\}, \end{aligned}$$

where $\phi \in C^k(M,N)$, $(x,U)$ and $(y,V)$ are charts for $M$ and $N$, respectively, $K$ is a compact subset of $U$ with $\phi(K) \subseteq V$, and $\varepsilon > 0$.

In summary, the compact-open $C^k$ topology is generated by sets that impose conditions on the behavior of functions on compact subsets of $M$ and the images of these subsets being contained within compact subsets of $N$.

In the case where $M$ is compact, given any $\phi \in C^k(M,N)$, we can find finitely many charts $\{(x_i,U_i)\}_{i=1}^r$ for $M$ and $\{(y_i,V_i)\}_{i=1}^r$ for $N$, and compact subsets $\{K_i\}_{i=1}^r$ of $M$ such that $\bigcup_{i=1}^r K_i = M$ and $\phi(K_i) \subseteq V_i$. The sets

$$\mathscr{B}(\phi,\varepsilon) = \bigcap_{i=1}^r S^k(\phi, x_i, U_i, y_i, V_i, K_i, \varepsilon)$$

form a neighborhood basis for $C^k(M,N)$. Moreover, we say that a sequence $\{\phi_j\}_{j\in\mathbb{N}}$ of maps $\phi_j \in C^k(M,N)$ converges to a function $\phi \in C^k(M,N)$ in this topology (or in $C^k$ for short) if and only if given any $\varepsilon > 0$ there exists $J \in \mathbb{N}$ such that $\phi_j \in \mathscr{B}(\phi,\varepsilon)$ for all $j \geq J$.

This is the smallest topology on $C^\infty(M,N)$ under which the inclusions $C^\infty(M,N) \to C^k(M,N)$ are embeddings. Once these definitions are well established, we will omit the charts system to simplify the notation.

The following theorem provides a consistent precompactness for sequences of MCF, it extends the classical Arzelà–Ascoli theorem to apply in our setting of uniform convergence. For a proof, see [ACGL20].

**Theorem 1.6.** (**Arzelà–Ascoli**) *Let $(M,g)$ and $(N,\gamma)$ be Riemannian manifolds with connections ${}^M\nabla$ and ${}^N\nabla$, respectively, and $M$ compact. Given a sequence $\{\phi_j\}$ in $C^{k+1}(M,N)$ such that $\phi_j(M) \subset L$ for some compact $L \subset N$ and*

$$\sup_{p\in M} \sum_{s=1}^{k+1} \|\phi_j^* \nabla^s \phi_j(p)\| < \infty$$

*for every $j \in \mathbb{N}$. Then there exists a subsequence of $\{\phi_j\}_{j\in\mathbb{N}}$ that converges in the compact-open $C^k$ topology to some $\phi \in C^k(M,N)$.*

### 1.3.2 Sequence convergence of sections on a bundle

In addition to map convergence, we are also interested in the $C^p$-convergence and $C^\infty$-convergence of metrics of sequences, which are sections of a vector bundle.

**Definition 1.7.** ($C^p$**-convergence)** Let $E$ be a vector bundle over a smooth manifold $M$, and let $g$ and $\nabla$ be metrics and connections on $E$ and $TM$, respectively. For an open set $\Omega \subset M$ with compact closure $\overline{\Omega}$ and a sequence $\{\xi_k\}$ of sections of $E$, we say $\{\xi_k\}$ converges in $C^p$ to $\xi_\infty \in \Gamma(E\mid_{\overline{\Omega}})$ if for every $\varepsilon > 0$, there exists $k_0 = k_0(\varepsilon)$ such that

$$\sup_{0\leq\alpha\leq p}\sup_{x\in\overline{\Omega}}|\nabla^\alpha(\xi_k - \xi_\infty)|_g < \varepsilon, \text{ for } \quad k > k_0.$$

We say $\{\xi_k\}$ converges in $C^\infty$ to $\xi_\infty$ on $\overline{\Omega}$ if it converges in $C^p$ for every $p \in \mathbb{N}$.

If we are dealing with a compact set, the choice of metric and connection on $E$ and $TM$ does not affect convergence.

To define smooth convergence on compact subsets, we use an exhaustion of $M$, i.e., a sequence of open sets $\{U_k\}_{k=1}^\infty$ in $M$ such that $\overline{U_k}$ is compact, $\overline{U_k} \subset U_{k+1}$ for all $k$, and $\bigcup_{k\geq 1} U_k = M$. For a compact subset $K \subset M$, there exists $k_0$ such that $K \subset U_k$ for all $k \geq k_0$. If $M$ is compact, then $U_k = M$ for sufficiently large $k$.

**Definition 1.8.** ($C^\infty$**-convergence on compact sets)** Let $\{U_k\}_{k=1}^\infty$ be an exhaustion of a smooth manifold $M$, and let $E$ be a vector bundle over $M$. Given metrics $g$ and connections $\nabla$ on $E$ and $TM$, and a sequence $\{\xi_i\}$ of sections of $E$ defined on open sets $K_i \subset M$, with $\xi_\infty \in \Gamma(E)$, we say $\{\xi_i\}$ converges smoothly on compact sets to $\xi_\infty$ if, for every $k \in \mathbb{N}$, there exists $i_0$ such that $\overline{U_k} \subset K_i$ for all $i \geq i_0$, and $(\xi_i\mid_{\overline{U_k}})_{i\geq i_0}$ converges in $C^\infty$ to $\xi_\infty$ on $\overline{U_k}$.

**Remark 1.9.** Note that this notion of convergence is independent of the choice of metrics and connections on $E$ and $TM$. We consider $\{\xi_k\} = \{g_k\}$ a sequence of Riemannian metric on $M$ in Definitions 1.7 and 1.8.

We now introduce the notion of bounded geometry. Recall that the injectivity radius $\operatorname{inj}(p, g)$ at a point $p$ in a Riemannian manifold $(M, g)$ is the maximum radius for which the exponential map on $p$ is a diffeomorphism. The injectivity radius of $M$ is defined as $\operatorname{inj}(M, g) := \inf\{\operatorname{inj}(p, g) : p \in M\}$. It is important to mention that if $M$ is not complete, $\operatorname{inj}(M, g) = 0$.

**Definition 1.10 (Bounded geometry).** We say that a complete Riemannian manifold $(M, g)$ has bounded geometry if for every integer $j \geqslant 0$ there exist positive constants $C_j$ and $\eta$ such that the Riemann curvature tensor $\operatorname{Rm}_g$ and the injectivity radius $\operatorname{inj}(M, g)$ satisfy

$$|\nabla^j \operatorname{Rm}_g| < C_j \quad \text{and} \quad \operatorname{inj}(M, g) \geqslant \eta > 0.$$

Notice that $\mathbb{R}^n$ with the standard metric and compact Riemannian manifolds have bounded geometry. Actually, bounded geometry provides a natural generalization to the known settings of compact and Euclidean spaces. In particular, the class of manifolds of bounded geometry allows us to uniformly apply constructions that are well-known for compact manifolds. For more details and examples we refer to Eldering [Eld13].

The notion of convergence of immersions from pointed manifolds is defined as follows. It is the immersion map version of the Cheeger-Gromov convergence.

**Definition 1.11.** (**Cheeger-Gromov convergence**) Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold with bounded geometry. Assume that for each $k \geqslant 1$ we have an $m$-dimensional pointed manifold $(\Sigma_k, p_k)$ and an immersion $x_k : \Sigma_k \to M$. Then, we say that a sequence of immersions $\{x_k : \Sigma_k \to M\}_{k=1}^{\infty}$ converges to an immersion $x_\infty : \Sigma_\infty \to M$ from an $m$-dimensional pointed manifold $(\Sigma_\infty, p_\infty)$ if there exist

1. An exhaustion $\{U_k\}_{k=1}^{\infty}$ of $\Sigma_\infty$ with $p_\infty \in U_k$ and
2. A sequence of diffeomorphisms $\Psi_k : U_k \to V_k \subset \Sigma_k$ with $\Psi_k(p_\infty) = p_k$ such that the sequence of maps $x_k \circ \Psi_k : U_k \to M$ converges in $C^\infty$ to $x_\infty : \Sigma_\infty \to M$ uniformly on compact sets in $\Sigma_\infty$.

Observe that a sequence of pointed Riemannian manifolds $(\Sigma_k, g_k, p_k)$ converges geometrically to a pointed complete Riemannian manifold $(\Sigma_\infty, g_\infty, p_\infty)$ in the sense of Definition 1.11, if there are embeddings $\Psi_k : U_k \to V_k \subset \Sigma_k$ with $\Psi_k(p_\infty) = p_k$ such that the pullback metrics $\Psi_k^* g_k$ converge uniformly on compact subsets of $\Sigma_\infty$ to $g_\infty$ in the $C^\infty$-topology.

### 1.3.3 Convergence of submanifolds

**Theorem 1.12** (Hamilton [Ham95a]). *Let $\{(M_k, g_k, p_k)\}_{k=1}^{\infty}$ be a sequence of $n$-dimensional complete pointed Riemannian manifolds. Suppose that*

*(1) for each integer $p \geq 0$, there exists a constant $0 < C_p < \infty$ such that*

$$|\nabla^p \mathrm{Rm}_{g_k})|_{g_k} \leq C_p, \quad \text{for all } k \geq 1$$

*(2) there exists a constant $0 < \eta < \infty$ such that*

$$\mathrm{inj}(p_k, g_k) \geq \eta, \quad \text{for all } k \geq 1.$$

*Then, there exist a complete pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ and a subsequence $\{k_\ell\}_{\ell=1}^{\infty}$ such that the subsequence $\{(M_{k_\ell}, g_{k_\ell}, p_{k_\ell})\}_{\ell=1}^{\infty}$ converges to $(M_\infty, g_\infty, p_\infty)$.*

To prove the convergence of submanifolds in a Riemannian manifold, we need the following estimate for the injectivity radius of a submanifold.

**Theorem 1.13** (Chen and Lin [CY07]). *Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold with*

$$|\mathrm{Rm}_g|(p) \leqslant C \quad \text{and} \quad \mathrm{inj}(p, g) \geqslant \eta > 0, \ \textit{for all } p \in M$$

*and two constants $C$ and $\eta$. Let $x : \Sigma \to (M, g)$ be a complete isometrically immersion manifold with*

$$|\mathcal{A}(x)| \leqslant D$$

*for some constant $D > 0$. Then there exists a constant $\delta = \delta(C, \eta, D, n) > 0$ such that*

$$\mathrm{inj}(p, x^*g) \geqslant \delta, \ \textit{for all } p \in \Sigma.$$

**Remark 1.14.** Notice that if $\{x_k : (\Sigma_k, p_k) \to (M, g)\}_{k=1}^{\infty}$ converges to $x_\infty : (\Sigma_\infty, p_\infty) \to M$, then $\{(\Sigma_k, x_k^*g, p_k)\}_{k=1}^{\infty}$ converges to $(\Sigma_\infty, x_\infty^*g, p_\infty)$ in the sense of Cheeger–Gromov convergence.

### 1.3.4 The maximum principle

The following result is proved in [CK14, Thm. 4.4], which was also used by [Mül12, Prop. 5.1].

**Proposition 1.15.** *Let $u : M \times [0, T) \to \mathbb{R}$ be a smooth function satisfying*

$$\frac{\partial}{\partial t} u \geqslant \Delta_{g(t)} u + \langle X(t), \nabla u \rangle_{g(t)} + F(u), \tag{1.7}$$

*where $g(t)$ is a smooth one-parameter family of metrics on $M$, $X(t)$ is a smooth one-parameter family of vector fields on $M$, and $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. Suppose $u(\cdot, 0)$ is bounded below by a constant $C_0 \in \mathbb{R}$ and let $\phi(t)$ be a solution to*

$$\frac{\partial}{\partial t} \phi = F(\phi), \quad \phi(0) = C_0.$$

*Then, $u(x, t) \geqslant \phi(t)$ for all $x \in M$ and all $t \in [0, T)$ for which $\phi(t)$ is defined.*

**Remark 1.16.** Similarly, if inequality (1.7) is reversed to

$$\frac{\partial}{\partial t} u \leqslant \Delta_{g(t)} u + \langle X(t), \nabla u \rangle_{g(t)} + F(u),$$

and $u(\cdot, 0)$ is bounded above by $C_0$, then $u(x, t) \leqslant \phi(t)$ for all $x \in M$ and all $t \in [0, T)$ for which the solution $\phi(t)$ of the corresponding ODE exists.

## 1.4 Approximate isometries

In this section, we introduce some essential concepts for constructing the limit manifold. The following definitions and propositions can be found in [CCG$^+$07].

**Definition 1.17** (**Approximate isometry**). For any $0 < \epsilon < 1$ and $k \geq 0$, a smooth map $\Phi : (\Sigma, h_\infty) \to (M, g)$ is an $(\epsilon, k)$-**pre-approximate isometry** if

$$\sup_{p\in\Sigma} |\Phi^* g - h_\infty|_{h_\infty} \leqslant \epsilon, \qquad \sup_{1\leqslant\alpha\leqslant k} \sup_{p\in\Sigma} |\nabla^\alpha_{h_\infty}(\Phi^* g)|_{h_\infty} \leqslant \epsilon.$$

An $(\epsilon, k)$-pre-approximate isometry is an $(\epsilon, k)$-**approximate isometry** if it is a diffeomorphism and

$$\sup_{p\in M} |(\Phi^{-1})^* h_\infty - g|_g \leqslant \epsilon, \qquad \sup_{1\leqslant\alpha\leqslant k} \sup_{p\in M} |\nabla^\alpha_g [(\Phi^*)^{-1} h_\infty]|_g \leqslant \epsilon,$$

i.e., $\Phi^{-1} : (M, g) \to (\Sigma, h_\infty)$ is also an $(\epsilon, k)$-pre-approximate isometry.

Note that the condition $|(\Phi^{-1})^* h_\infty - g|_g \leqslant \epsilon$ is equivalent to $|h_\infty - \Phi^* g|_{\Phi^* g} \leqslant \epsilon$, and $|\nabla^\alpha_g [(\Phi^{-1})^* h_\infty]|_g \leqslant \epsilon$ is equivalent to $|\nabla^\alpha_{\Phi^* g} h_\infty|_{\Phi^* g} \leqslant \epsilon$. Another way to express the condition $\sup_{p\in M} |\Phi^* g - h_\infty|_{h_\infty} \leqslant \epsilon$ is

$$\begin{aligned} \left| g_{ab} \frac{\partial \Phi^a}{\partial x^i} \frac{\partial \Phi^b}{\partial x^j} - (h_\infty)_{ij} \right|_g &= \left( \left[ g_{ab} (h_\infty)^{ik} \frac{\partial \Phi^a}{\partial x^i} \right] \frac{\partial \Phi^b}{\partial x^j} - I^k_j \right) \left( \frac{\partial \Phi^c}{\partial x^k} \left[ g_{cd} (h_\infty)^{jl} \frac{\partial \Phi^d}{\partial x^l} \right] - I^j_k \right) \\ &= \left| (d\Phi)^T (d\Phi) - \mathrm{id} \right|^2, \end{aligned}$$

where id is the identity map on $TM$, the transpose arises from the two metrics $g$ and $h_\infty$, and $\|T\|^2 = \mathrm{trace}(T^2)$.

Approximate isometries allow pointwise bounding of metric tensors as follows.

**Proposition 1.18** (**Approximate isometries and norms**). *Let $\epsilon \in (0, 1)$.*

*(i) If $\Phi : (\Sigma, h_\infty) \to (M, g)$ is an $(\epsilon, 0)$-pre-approximate isometry and $X$ is a vector field on $\Sigma$, then*

$$|X|^2_{\Phi^* g} \leqslant (1 + \epsilon) |X|^2_{h_\infty}.$$

*(ii) If $\Phi : (\Sigma, h_\infty) \to (M, g)$ is an $(\epsilon, 0)$-approximate isometry and $X$ is a vector field on $\Sigma$, then*

$$\frac{1}{1+\epsilon} |X|^2_{h_\infty} \leqslant |X|^2_{\Phi^* g} \leq (1 + \epsilon) |X|^2_{h_\infty}.$$

The following proposition shows how $(\epsilon, 0)$-approximate isometries deform distances by small amounts.

**Proposition 1.19** (**Distance**). *If $\Phi : (\Sigma, h_\infty) \to (M, g)$ is an $(\epsilon, 0)$-pre-approximate isometry, then*

$$\Phi\left(B_{h_\infty}(x_0, r)\right) \subset B_g\left(\Phi(x_0), \sqrt{1+\epsilon} \cdot r\right),$$

*where $r$ is the distance function from $x_0$, and $B_{h_\infty}$ and $B_g$ stand for geodesic balls.*

# CHAPTER 2

# The $(RH)_\alpha$ flow, evolution equations and some applications

In this chapter, we obtain the variational formula for the functional $\mathcal{F}_\infty^\alpha$, and an extension of Hamilton's differential Harnack expression for our context. Moreover, we provide the proof of Theorems 1, 2 and 3.

## 2.1 Evolution of $\mathcal{F}_\infty^\alpha$ under the coupled system $(RH)_\alpha$ flow

In this section $g(t)$ stands for a smooth one-parameter family of Riemannian metrics on an $n$-dimensional smooth manifold $M$, and $\phi(t)$ a smooth one-parameter family of smooth maps from $M$ to an $m$-dimensional Riemannian manifold $(N,\gamma)$, with $g(0)=g$ and $\phi(0)=\phi$. Moreover, consider the product

$$\mathscr{P}(M,N) := \mathrm{met}(M) \times C^\infty(M) \times C^\infty(M,N),$$

where $\mathrm{met}(M)$ denotes the set of all Riemannian metrics on $M$.

**Definition 2.1** ([Mül12])**.** The pair $(g(t),\phi(t))_{t\in[0,T)}$ is a solution to the coupled system of Ricci flow and harmonic map heat flow ($(RH)_\alpha$ *flow*, for short) with a nonnegative coupling constant $\alpha(t)$, if it satisfies

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\operatorname{Ric}_{g(t)} + 2\alpha(t)\nabla\phi(t)\otimes\nabla\phi(t), \\ \frac{\partial}{\partial t}\phi(t) = \tau_{g(t),\gamma}\phi(t), \end{cases} \tag{2.1}$$

where $\tau_{g,\gamma}\phi$ is the tension field.

For an account of $(RH)_\alpha$ flow, including proof of short-time existence and uniqueness of solutions to (2.1), we refer to Müller [Mül12, Sect. 4.2].

Müller realized that this coupled system may behave less singular than the Ricci flow or the standard harmonic map flow alone. In order to interpret system (2.1) as a gradient flow by means a functional $\mathcal{F}_\alpha$ for a fixed measure, he considered the heat operator $\Box = \frac{\partial}{\partial t} - \Delta_g$ whose formal adjoint is

$$\Box^* = -\frac{\partial}{\partial t} - \Delta_g + R_g - \alpha|\nabla\phi|^2 \tag{2.2}$$

along the $(RH)_\alpha$ flow.

A *gradient soliton* to the $(RH)_\alpha$ flow is a self-similar solution $\big(\overline{g}(t), \overline{\phi}(t)\big)$ of (2.1), given by

$$\begin{cases} \overline{g}(t) = \sigma(t)\psi_t^* g, \\ \overline{\phi}(t) = \psi_t^*\phi, \end{cases}$$

for some initial value $(g, \phi)$, where $\psi_t$ is a smooth one-parameter family of diffeomorphisms of $M$ generated from the flow of $\nabla_g f/\sigma(t)$, $f \in C^\infty(M)$, and $\sigma(t)$ is a positive smooth function of $t$.

A gradient soliton to the $(RH)_\alpha$ flow is characterized by next proposition.

**Proposition 2.2** ([Mül12]). *Suppose there exists a triple $(g, f, \phi)$ satisfying*

$$\begin{cases} \mathrm{Ric}_g + \nabla_g^2 f - \alpha\nabla\phi\otimes\nabla\phi = \lambda g, & (2.3.a) \\ \tau_{g,\gamma}\phi = \langle\nabla_g f, \nabla_g\phi\rangle_g & (2.3.b) \end{cases}$$

*on $M$, for some constants $\lambda \in \mathbb{R}$ and $\alpha \geqslant 0$. Take $\psi_t$ the one-parameter family of diffeomorphisms generated by $Y_t = \nabla_g f/\sigma(t)$ to define $\overline{g}(t) = \sigma(t)\psi_t^* g$ and $\overline{\phi}(t) = \psi_t^*\phi$, with $\psi_0 = \mathrm{Id}$ and $\sigma(t) = 1 - 2\lambda t > 0$, where $t \in (-\infty, 1/2\lambda)$, for $\lambda > 0$; $t \in \mathbb{R}$, for $\lambda = 0$; and $t \in (1/2\lambda, +\infty)$, for $\lambda < 0$. The pair $(\overline{g}(t), \overline{\phi}(t))$ is a self-similar solution of $(RH)_\alpha$ flow on $M$. Conversely, if $\big(\sigma(t)\psi_t^* g, \psi_t^*\phi\big)$ is a self-similar solution of $(RH)_\alpha$ flow on $M$, then the triple $(g, f, \phi)$ satisfies* (2.3.a) *(as well* (2.3.b)*).*

*Proof.* We start by computing

$$\begin{aligned}\frac{\partial}{\partial t}\overline{g}(t) &= \sigma'(t)\psi_t^* g + \sigma(t)\psi_t^*(\mathcal{L}_{Y_t} g) = \psi_t^*(-2\lambda g + 2\nabla_g^2 f) = -2\psi_t^*(\lambda g - \nabla_g^2 f) \\ &= -2\psi_t^*(\mathrm{Ric}_g - \alpha\nabla\phi\otimes\nabla\phi) = -2\mathrm{Ric}_{\overline{g}(t)} + 2\alpha\nabla\overline{\phi}(t)\otimes\nabla\overline{\phi}(t)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\phi}(t) &= \psi_t^*(\mathcal{L}_{Y_t}\phi) = \frac{1}{\sigma(t)}\psi_t^*\mathcal{L}_{\nabla_g f}\phi = \frac{1}{\sigma(t)}\psi_t^*\langle\nabla_g f, \nabla_g\phi\rangle_g \\ &= \frac{1}{\sigma(t)}\psi_t^*\tau_{g,\gamma}\phi = \tau_{\overline{g}(t),\gamma}\overline{\phi}(t).\end{aligned}$$

So, the pair $\big(\overline{g}(t), \overline{\phi}(t)\big)$ is a solution of $(RH)_\alpha$ flow on $M$. Reciprocally, if $\big(\sigma(t)\psi_t^* g, \psi_t^*\phi\big)$ is a self-similar solution of $(RH)_\alpha$ flow on $M$, a straightforward computation shows that $(g, f, \phi)$ satisfies (2.3.a) (as well (2.3.b)). □

From Proposition 2.2 we can interpret a gradient soliton to the $(RH)_\alpha$ flow on $M$ as a triple $(g, f, \phi)$ satisfying (2.3.a) (as well (2.3.b)). It is *steady* if $\lambda = 0$, *shrinking* if $\lambda > 0$ and *expanding* if $\lambda < 0$. The function $f$ is called the *potential function*.

Now, we assume that $M$ is compact with boundary $\partial M$ to define the functional $\mathcal{F}^\alpha_\infty$ on $\mathscr{P}(M, N)$ as follows

$$\mathcal{F}^\alpha_\infty(g, f, \phi) := \int_M \Big(R_\infty - \alpha|\nabla\phi|^2\Big) e^{-f} dM + 2\int_{\partial M} H_\infty e^{-f} dA, \tag{2.4}$$

where $R_\infty := R_g + 2\Delta_g f - |\nabla f|^2$ is the *weighted scalar curvature* of $g$, the function $H_\infty := H + e_0 f$ is the *weighted mean curvature* with respect to the inward unit normal vector field $e_0$ on $\partial M$.

We observe that the functional $\mathcal{F}^\alpha_\infty$ is the proper extension to our context of the energy functional $\mathcal{F}_\alpha$ introduced by Müller [Mül12] for the case of $M$ and $N$ both without boundary, of the action $I_\infty$ introduced by Lott [Lot12] when studying a mean curvature flow in a Ricci flow background, and of the functional $I^{\alpha_n}_\infty$ introduced by Gomes and Hudson [GH23] when studying a mean curvature flow in an extended Ricci flow background. These latter two cases are on a compact smooth manifold with boundary. Furthermore, it is already clear that the function $R_\infty$ arises quite naturally, as observed by Perelman [Per02], and $H_\infty$ is the appropriate geometric object when we are using a weighted measure (see, e.g., [Gro03], [Lot12] or [GH23]).

We shall adopt the following notation. Given $(g, f, \phi) \in \mathscr{P}(M, N)$, take variations $(g_{ij} + th_{ij}, f + t\ell, \phi + t\vartheta)$, with $h_{ij} \in \Gamma(\mathrm{Sym}^2(T^*M)), \ell \in C^\infty(M)$ and $\vartheta \in C^\infty(M, N)$ with $\vartheta(x) \in T_{\phi(x)}N$. We denote by $\delta$ the derivative $\frac{d}{dt}|_{t=0}$, and then $\delta g = h$, $\delta f = \ell$, and $\delta\phi = \vartheta$. Moreover, we are using the weighted volume element $d\mu = e^{-f}dM$, which is *weighted measure-preserving* if and only if $\frac{\mathrm{tr}_g h}{2} - \ell = 0$ on $M$, since $\delta(e^{-f}dM) = (\frac{\mathrm{tr}_g h}{2} - \ell)e^{-f}dM$. Also, for the sake of simplicity, we are writing $\gamma_{\alpha\beta}$ on $M$ instead of $\phi^*\gamma_{\alpha\beta}$.

With these in mind, we compute the variation of $\mathcal{F}^\alpha_\infty$ as follows.

**Proposition 2.3.** *Under weighted measure-preserving, we have*

$$\begin{aligned}\delta\mathcal{F}^\alpha_\infty =& \int_M \Big(-h^{ij}(R_{ij} + \nabla^2 f(\partial_i, \partial_j) - \alpha\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta) + 2\alpha\langle\tau_{g,\gamma}\phi - \langle\nabla f, \nabla\phi\rangle\,, \vartheta\rangle\Big)e^{-f}dM\\ &- \int_{\partial M}\big(h^{\hat{i}\hat{j}}\mathcal{A}_{\hat{i}\hat{j}} + h^{00}(H + e_0 f)\big)e^{-f}dA + 2\alpha\int_{\partial M}\langle\nabla_0\phi, \vartheta\rangle e^{-f}dA.\end{aligned}$$

*Proof.* First, note that by (2.4) we can write

$$\mathcal{F}^\alpha_\infty(g, f, \phi) = I_\infty(g, f) - \alpha E(g, f, \phi),$$

where $E(g, f, \phi) := \int_M |\nabla\phi|^2 e^{-f}dM$. Thus, we can use Proposition 2 in [Lot12], which guarantees that

$$\delta I_\infty = -\int_M h^{ij}\left(R_{ij} + \nabla^2 f(\partial_i, \partial_j)\right)e^{-f}dM - \int_{\partial M}\Big(h^{\hat{i}\hat{j}}\mathcal{A}_{\hat{i}\hat{j}} + h^{00}\left(H + e_0 f\right)\Big)e^{-f}dA.$$

Hence, it is enough to prove that

$$\delta E=\int_M\Big(-h^{ij}\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta-2\langle\tau_{g,\gamma}\phi-\langle\nabla f,\nabla\phi\rangle\,,\vartheta\rangle\Big)e^{-f}dM-2\int_{\partial M}\langle\nabla_0\phi,\vartheta\rangle e^{-f}dA.$$

Indeed, notice that

$$\delta E(h,\ell,\vartheta)=\int_M\Big(\delta\big(|\nabla\phi|^2\big)+|\nabla\phi|^2\Big(\frac{\operatorname{tr}_g h}{2}-\ell\Big)\Big)e^{-f}dM$$

and

$$\delta\big(|\nabla\phi|^2\big)=-h^{ij}\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta+2g^{ij}\gamma_{\alpha\beta}\nabla_i\vartheta^\alpha\nabla_j\phi^\beta,$$

where $\vartheta=\vartheta^\alpha\partial_\alpha$. So, under weighted measure-preserving, we have

$$\delta E=\int_M\Big(-h^{ij}\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta e^{-f}+2g^{ij}\gamma_{\alpha\beta}\nabla_i\vartheta^\alpha\nabla_j\phi^\beta e^{-f}\Big)dM.$$

The result follows from integration by parts. □

**Remark 2.4.** By considering $M$ compact without boundary in Proposition 2.3, we recover the results by Müller [Mül12, (3.4)], for $\phi\in C^\infty(M,N)$; and by List [Lis08], for $\phi\in C^\infty(M)$. In the compact case with boundary, we also recover the results by Gomes and Hudson [GH23, Prop. 1], for $\phi\in C^\infty(M)$; and by Lott [Lot12], for $\phi$ constant.

The next two corollaries provide the critical points of $\mathcal{F}^\alpha_\infty$ under weighted measure-preserving.

**Corollary 2.5.** *If the induced metric on $\partial M$ is fixed, then the critical points of $\mathcal{F}^\alpha_\infty$ under weighted measure-preserving are gradient steady solitons of $(RH)_\alpha$ flow that satisfy $H+e_0f=0$ and $\nabla_0\phi=0$ on $\partial M$.*

*Proof.* By hypotheses $\frac{\operatorname{tr}_g h}{2}-\ell=0$ on $M$ and $h^{\hat{i}\hat{j}}=0$ on $\partial M$. Hence, by Proposition 2.3 we obtain

$$\begin{aligned}\int_M\Big(\langle h,\alpha\nabla\phi\otimes\nabla\phi-\operatorname{Ric}_g-\nabla^2_g f\rangle+2\alpha\big\langle\vartheta,\tau_{g,\gamma}\phi-\langle\nabla\phi,\nabla f\rangle_g\big\rangle\Big)e^{-f}dM&\\+\int_{\partial M}\Big(2\alpha\langle\vartheta,\nabla_0\phi\rangle-\langle h,(H+e_0f)e_0^\flat\otimes e_0^\flat\rangle\Big)e^{-f}dA=0,&\end{aligned}\tag{2.5}$$

for all $(h,\vartheta)\in\Gamma(\operatorname{Sym}^2(T^*M))\times C^\infty(M,N)$, where "$\flat$" stands for musical isomorphism. Now, we assume $h$ and $\vartheta$ are compactly supported, so that

$$\int_M\Big(\langle h,\alpha\nabla\phi\otimes\nabla\phi-\operatorname{Ric}_g-\nabla^2_g f\rangle+2\alpha\big\langle\vartheta,(\tau_{g,\gamma}\phi-\langle\nabla\phi,\nabla f\rangle_g)\big\rangle\Big)e^{-f}dM=0.$$

So, $(g,f,\phi)$ must be a gradient steady soliton of $(RH)_\alpha$ flow, and then, from (2.5) we get

$$\int_{\partial M}\Big(2\alpha\langle\vartheta,\nabla_0\phi\rangle-\langle h,(H+e_0f)e_0^\flat\otimes e_0^\flat\rangle\Big)e^{-f}dA=0,$$

for all $(h,\vartheta)\in\Gamma(\operatorname{Sym}^2(T^*M))\times C^\infty(M,N)$, that implies $H+e_0f=0$ and $\nabla_0\phi=0$ on $\partial M$. □

If we relax the fixed induced metric assumption on the boundary, then we obtain the next result.

**Corollary 2.6.** *The critical points of $\mathcal{F}_\infty^\alpha$ under weighted measure-preserving are gradient steady solitons of $(RH)_\alpha$ flow with totally geodesic boundary satisfying the conditions $e_0 f = 0$ and $\nabla_0\phi = 0$ on $\partial M$.*

*Proof.* As in the first part of the proof of Corollary 2.5, we obtain that the critical points of $\mathcal{F}_\infty^\alpha$ under weighted measure-preserving are gradient steady solitons to the $(RH)_\alpha$ flow. Then

$$\int_{\partial M} \Big(2\alpha\langle\vartheta, \nabla_0\phi\rangle - \langle h, \mathcal{A} + (H + e_0 f)e_0^\flat \otimes e_0^\flat\rangle\Big)e^{-f}dA = 0,$$

for all $(h, \vartheta) \in \Gamma(\mathrm{Sym}^2(T^*M)) \times C^\infty(M, N)$, that implies $\mathcal{A} = 0$, $e_0 f = 0$ and $\nabla_0\phi = 0$ on the boundary. □

**Remark 2.7.** We recover the results by Gomes and Hudson [GH23], for the case $\phi \in C^\infty(M)$; and by Lott [Lot12, Cor. 4], for $\phi$ constant.

### 2.1.1 The modified $(RH)_\alpha$ flow setting

Our main results works in the following setting. We say that a family $(g(t), \phi(t))_{t\in[0,T)} \in \mathrm{met}(M) \times C^\infty(M, N)$ evolves by *modified $(RH)_\alpha$ flow* if it satisfies the system

$$\begin{cases} \dfrac{\partial}{\partial t}g = -2(\mathrm{Ric}_g + \nabla^2 f - \alpha\nabla\phi \otimes \nabla\phi), & (2.6.a) \\ \dfrac{\partial}{\partial t}\phi = \tau_{g,\gamma}\phi - \langle\nabla\phi, \nabla f\rangle_g. & (2.6.b) \end{cases}$$

and

$$\frac{\partial}{\partial t}f = -R_g - \Delta_g f + \alpha|\nabla\phi|_g^2 \tag{2.7}$$

in $M \times [0, T)$, with $H + e_0 f = 0$ and $\nabla_0\phi = 0$ on $\partial M$.

We can find motivations for considering the modified $(RH)_\alpha$ flow in Proposition 2.3 and its corollaries. For instance, we can see that the $(RH)_\alpha$ flow can be interpreted as the gradient flow of $\mathcal{F}_\infty^\alpha$ for any weighted measure-preserving on $M$. More importantly, this approach will be useful in the study of mean curvature flow in the $(g(t), \phi(t)) - (RH)_\alpha$ flow background, which is the main research object of this thesis.

**Remark 2.8.** We observe that along the modified $(RH)_\alpha$ flow, the measure $e^{-f}dM$ remains fixed. In fact, from (2.6.a) we have $h_{ij} = 2(-R_{ij} - \nabla^2 f(\partial_i, \partial_j) + \alpha\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta)$. Tracing equation (2.6.a) and using (2.7), we obtain $\frac{\mathrm{tr}_g h}{2} - \ell = 0$ on $M$.

In what follows, we establish the tools to work on the modified $(RH)_\alpha$ flow setting. The first is the time-derivative of $\mathcal{F}_\infty^\alpha$ under this flow.

**Proposition 2.9.** *If $(g(t),\phi(t))_{t\in[0,T)}$ evolves by modified $(RH)_\alpha$ flow, then*

$$\begin{aligned}\frac{d}{dt}\mathcal{F}^\alpha_\infty =&2\int_M \Big(|\mathrm{Ric}+\nabla^2 f-\alpha\nabla\phi\otimes\nabla\phi|^2+\alpha|\tau_{g,\gamma}\phi-\langle\nabla\phi,\nabla f\rangle|^2\Big)e^{-f}dM\\ &+2\int_{\partial M}\Big(\widehat{\Delta}H-2\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)+\mathcal{A}^{\hat{i}\hat{j}}\mathcal{A}_{\hat{i}\hat{j}}H+\mathcal{A}^{\hat{i}\hat{j}}R_{\hat{i}\hat{j}}\\ &+2R^{0\hat{i}}\widehat{\nabla}_{\hat{i}}f-\widehat{\nabla}_{\hat{i}}R^{0\hat{i}}-\alpha\mathcal{A}(\widehat{\nabla}\phi,\widehat{\nabla}\phi)\Big)e^{-f}dA.\end{aligned}$$

*In particular, if both $\big(R_{\hat{i}\hat{j}}+\nabla^2 f(\partial_i,\partial_j)-\alpha\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_{\hat{j}}\phi^\beta\big)|_{\partial M}$ and $\big(R_{\hat{i}0}+\nabla_{\hat{i}}\nabla_0 f\big)|_{\partial M}$ vanish, then the boundary integrand vanishes.*

*Proof.* By (2.6.a) and (2.6.b) we have $h_{ij}=2(-R_{ij}-\nabla^2 f(\partial_i,\partial_j)+\alpha\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta)$ and $\vartheta=\tau_{g,\gamma}\phi-\langle\nabla\phi,\nabla f\rangle$, respectively. Tracing equation (2.6.a) and using (2.7), we obtain $\frac{\mathrm{tr}_g h}{2}-\ell=0$ on $M$, which allows us to use Proposition 2.3 to get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}^\alpha_\infty =&2\int_M \Big(|\mathrm{Ric}+\nabla^2 f-\alpha\nabla\phi\otimes\nabla\phi|^2+\alpha|\tau_{g,\gamma}\phi-\langle\nabla\phi,\nabla f\rangle|^2\Big)e^{-f}dM\\ &+2\int_{\partial M}\big(\mathcal{A}^{\hat{i}\hat{j}}(R_{\hat{i}\hat{j}}+\nabla^2 f(\partial_{\hat{i}},\partial_{\hat{j}})-\alpha\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_{\hat{j}}\phi^\beta)\big)e^{-f}dA,\end{aligned}$$

where we used that $H+e_0 f=0$ and $\nabla_0\phi=0$ on $\partial M$. On the other hand, Lemma 1 in Lott [Lot12] guarantees that

$$\begin{aligned}&\mathcal{A}^{\hat{i}\hat{j}}\left(R_{\hat{i}\hat{j}}+\nabla^2 f(\partial_{\hat{i}},\partial_{\hat{j}})\right)e^{-f}-\widehat{\nabla}_{\hat{i}}\Big(\Big(R^{\hat{i}0}+\nabla^{\hat{i}}\nabla^0 f\Big)e^{-f}\Big)\\ =&\Big(\widehat{\Delta}H-2\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)+\mathcal{A}^{\hat{i}\hat{j}}\mathcal{A}_{\hat{i}\hat{j}}H+\mathcal{A}^{\hat{i}\hat{j}}R_{\hat{i}\hat{j}}+2R^{0\hat{i}}\widehat{\nabla}_{\hat{i}}f-\widehat{\nabla}_{\hat{i}}R^{0\hat{i}}\Big)e^{-f},\end{aligned}$$

where $\nabla^{\hat{i}}\nabla^0 f=g^{\hat{i}\hat{k}}g^{0i}\nabla^2 f(\partial_{\hat{k}},\partial_i)$. Then

$$\begin{aligned}&\mathcal{A}^{\hat{i}\hat{j}}(R_{\hat{i}\hat{j}}+\nabla^2 f(\partial_{\hat{i}},\partial_{\hat{j}})-\alpha\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_{\hat{j}}\phi^\beta)e^{-f}-\widehat{\nabla}_{\hat{i}}\Big(\Big(R^{\hat{i}0}+\nabla^{\hat{i}}\nabla^0 f\Big)e^{-f}\Big)\\ =&\Big(\widehat{\Delta}H-2\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)+\mathcal{A}^{\hat{i}\hat{j}}\mathcal{A}_{\hat{i}\hat{j}}H+\mathcal{A}^{\hat{i}\hat{j}}R_{\hat{i}\hat{j}}+2R^{0\hat{i}}\widehat{\nabla}_{\hat{i}}f-\widehat{\nabla}_{\hat{i}}R^{0\hat{i}}\\ &-\alpha\mathcal{A}(\widehat{\nabla}\phi,\widehat{\nabla}\phi)\Big)e^{-f},\end{aligned}\tag{2.8}$$

and from Stokes' theorem

$$\int_{\partial M}\widehat{\nabla}_{\hat{i}}\Big(\Big(R^{\hat{i}0}+\nabla^{\hat{i}}\nabla^0 f\Big)e^{-f}\Big)dA=0,$$

which is enough to obtain the first part of the theorem. In particular, if both $\big(R_{\hat{i}\hat{j}}+\nabla^2 f(\partial_{\hat{i}},\partial_{\hat{j}})-\alpha\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_{\hat{j}}\phi^\beta\big)|_{\partial M}$ and $\big(R_{\hat{i}0}+\nabla^2 f(\partial_{\hat{i}},e_0)\big)|_{\partial M}$ vanish, then from equation (2.8) the boundary integrand vanishes. ☐

In the next result, we establish the evolution equations of the geometric quantities of $\partial M$ under modified $(RH)_\alpha$ flow. For its proof, we shall need the following identity

$$\begin{aligned}\widehat{\nabla}^2 H(\partial_{\hat{i}},\partial_{\hat{j}})=&(\widehat{\Delta}\mathcal{A})_{\hat{i}\hat{j}}+\widehat{\nabla}_{\hat{i}}R_{\hat{j}0}+\widehat{\nabla}_{\hat{j}}R_{\hat{i}0}-\nabla_0 R_{\hat{i}\hat{j}}+\mathcal{A}^{\hat{k}}_{\hat{i}}R_{0\hat{k}0\hat{j}}+\mathcal{A}^{\hat{k}}_{\hat{j}}R_{0\hat{k}0\hat{i}}-\mathcal{A}_{\hat{i}\hat{j}}R_{00}\\ &+2\mathcal{A}^{\hat{k}\hat{l}}R_{\hat{k}\hat{i}\hat{l}\hat{j}}-HR_{0\hat{i}0\hat{j}}-H\mathcal{A}^{\hat{k}}_{\hat{i}}\mathcal{A}_{\hat{j}\hat{k}}+\mathcal{A}^{\hat{k}\hat{l}}\mathcal{A}_{\hat{k}\hat{l}}\mathcal{A}_{\hat{i}\hat{j}}+\nabla_0 R_{0\hat{i}0\hat{j}}.\end{aligned}\tag{2.9}$$

Identity (2.9) has already been observed by Lott [Lot12]. Its proof can be obtained from Simons [Sim68] or, alternatively, from Huisken [Hui86]. Indeed, in our notations, Lemma 2.1 in [Hui86] becomes

$$\begin{aligned}\widehat{\nabla}^2 H(\partial_{\hat{i}},\partial_{\hat{j}}) = (\widehat{\Delta}\mathcal{A})_{\hat{i}\hat{j}} - H\mathcal{A}_{\hat{i}\hat{k}}\mathcal{A}^{\hat{k}}_{\hat{j}} + \mathcal{A}^{\hat{k}\hat{l}}\mathcal{A}_{\hat{k}\hat{l}}\mathcal{A}_{\hat{i}\hat{j}} - HR_{0\hat{i}0\hat{j}} + \mathcal{A}_{\hat{i}\hat{j}}R^{\hat{k}}_{0\hat{k}0} - \mathcal{A}^{\hat{k}}_{\hat{j}}R^{\hat{l}}_{\hat{k}\hat{l}\hat{i}}\\ - \mathcal{A}^{\hat{k}}_{\hat{i}}R^{\hat{l}}_{\hat{k}\hat{l}\hat{j}} + 2\mathcal{A}^{\hat{k}\hat{l}}R_{\hat{k}\hat{i}\hat{l}\hat{j}} + \nabla_{\hat{j}}R^{\hat{k}}_{0\hat{k}\hat{i}} - \nabla_0 R^{\hat{k}}_{\hat{i}\hat{k}\hat{j}} + \nabla_{\hat{i}}R^{\hat{k}}_{0\hat{k}\hat{j}}.\end{aligned}$$

Hence, (2.9) follows from the equality $\nabla_{\hat{i}}R_{\hat{j}0} = \widehat{\nabla}_{\hat{i}}R_{\hat{j}0} - \mathcal{A}_{\hat{i}\hat{j}}R_{00} + \mathcal{A}^{\hat{k}}_{\hat{i}}R_{\hat{j}\hat{k}}$.

**Proposition 2.10.** *If $(g(t),\phi(t))_{t\in[0,T)}$ evolves by modified $(RH)_\alpha$ flow, then the following evolution equations hold on $\partial M$*

$$\frac{\partial}{\partial t}g_{\hat{i}\hat{j}} = -(\mathcal{L}_{\widehat{\nabla} f}g)_{\hat{i}\hat{j}} - 2(R_{\hat{i}\hat{j}} - \alpha\gamma_{\alpha\beta}\widehat{\nabla}_{\hat{i}}\phi^\alpha\widehat{\nabla}_{\hat{j}}\phi^\beta) - 2H\mathcal{A}_{\hat{i}\hat{j}}, \tag{2.10}$$

$$\frac{\partial}{\partial t}\phi = \tau_{\hat{g},\gamma}\phi + \nabla_0\nabla_0\phi - \mathcal{L}_{\widehat{\nabla} f}\phi, \tag{2.11}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{\hat{i}\hat{j}} = (\widehat{\Delta}\mathcal{A})_{\hat{i}\hat{j}} - (\mathcal{L}_{\widehat{\nabla} f}\mathcal{A})_{\hat{i}\hat{j}} - \mathcal{A}^{\hat{k}}_{\hat{i}}R^{\hat{l}}_{\hat{k}\hat{l}\hat{j}} - \mathcal{A}^{\hat{k}}_{\hat{j}}R^{\hat{l}}_{\hat{k}\hat{l}\hat{i}} + 2\mathcal{A}^{\hat{k}\hat{l}}R_{\hat{k}\hat{i}\hat{l}\hat{j}} - 2H\mathcal{A}_{\hat{i}\hat{k}}\mathcal{A}^{\hat{k}}_{\hat{j}}\\ + \mathcal{A}^{\hat{k}\hat{l}}\mathcal{A}_{\hat{k}\hat{l}}\mathcal{A}_{\hat{i}\hat{j}} + \nabla_0 R_{0\hat{i}0\hat{j}}\end{aligned} \tag{2.12}$$

*and*

$$\frac{\partial}{\partial t}H = \widehat{\Delta}H - \langle\widehat{\nabla} f,\widehat{\nabla} H\rangle + 2\mathcal{A}^{\hat{i}\hat{j}}R_{\hat{i}\hat{j}} + \mathcal{A}^{\hat{i}\hat{j}}\mathcal{A}_{\hat{i}\hat{j}}H + \nabla_0 R_{00} - 2\alpha\mathcal{A}(\widehat{\nabla}\phi,\widehat{\nabla}\phi). \tag{2.13}$$

*Proof.* We start by substituting $\nabla_{\hat{i}}\nabla_{\hat{j}}f = \widehat{\nabla}_{\hat{i}}\widehat{\nabla}_{\hat{j}}f + H\mathcal{A}_{\hat{i}\hat{j}}$ (as $H + e_0 f = 0$) into equation (2.6.a) to get

$$\frac{\partial}{\partial t}g_{\hat{i}\hat{j}} = -2(R_{\hat{i}\hat{j}} + \widehat{\nabla}_{\hat{i}}\widehat{\nabla}_{\hat{j}}f + H\mathcal{A}_{\hat{i}\hat{j}} - \alpha\gamma_{\alpha\beta}\widehat{\nabla}_{\hat{i}}\phi^\alpha\widehat{\nabla}_{\hat{j}}\phi^\beta),$$

which is equation (2.10). Likewise, equation (2.11) is only a restriction on boundary of equation (2.6.b), since $\nabla_0\phi = 0$.

$$\begin{aligned}\frac{\partial}{\partial t}\phi &= \tau_{g,\gamma}\phi - \langle\nabla\phi,\nabla f\rangle\\ &= g^{ij}\left(\partial_i\partial_j\phi^\lambda - \Gamma^k_{ij}\partial_k\phi^\lambda + (\Gamma^\lambda_{\alpha\beta}\circ\phi)\partial_i\phi^\alpha\partial_j\phi^\beta\right)\partial_\lambda|_\phi - \langle\nabla\phi^\lambda,\widehat{\nabla} f + e_0 f e_0\rangle\partial_\lambda|_\phi\\ &= \Delta\phi^\lambda\partial_\lambda|_\phi + g^{\hat{i}\hat{j}}(\Gamma^\lambda_{\alpha\beta}\circ\phi)\partial_{\hat{i}}\phi^\alpha\partial_{\hat{j}}\phi^\beta\partial_\lambda|_\phi + g^{00}(\Gamma^\lambda_{\alpha\beta}\circ\phi)\partial_0\phi^\alpha\partial_0\phi^\beta\partial_\lambda|_\phi - \langle\nabla\phi^\lambda,\widehat{\nabla} f\rangle\partial_\lambda|_\phi\\ &= \widehat{\Delta}\phi^\lambda\partial_\lambda|_\phi - g\big(\nabla\phi^\lambda,\overrightarrow{H}\big)\partial_\lambda|_\phi + g^{00}\nabla^2\phi^\lambda(e_0,e_0)\partial_\lambda|_\phi + g^{\hat{i}\hat{j}}(\Gamma^\lambda_{\alpha\beta}\circ\phi)\partial_{\hat{i}}\phi^\alpha\partial_{\hat{j}}\phi^\beta\partial_\lambda|_\phi\\ &\quad + g^{00}(\Gamma^\lambda_{\alpha\beta}\circ\phi)\partial_0\phi^\alpha\partial_0\phi^\beta\partial_\lambda|_\phi - \langle\nabla\phi^\lambda,\widehat{\nabla} f\rangle\partial_\lambda|_\phi\\ &= \tau_{\hat{g},\gamma}\phi + \nabla_0\nabla_0\phi - \mathcal{L}_{\widehat{\nabla} f}\phi.\end{aligned}$$

To prove (2.12) we first observe that by (2.6.a)

$$\frac{1}{2}h_{ij} = -(R_{ij} + \nabla_i\nabla_j f - \alpha\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta)$$

and

$$\delta\mathcal{A}_{\hat{i}\hat{j}} = \frac{1}{2}(\nabla_{\hat{i}} h_{\hat{j}0} + \nabla_{\hat{j}} h_{\hat{i}0} - \nabla_0 h_{\hat{i}\hat{j}}) + \frac{1}{2}h_{00}\mathcal{A}_{\hat{i}\hat{j}}.$$

Since $\nabla_0\phi = 0$ implies $\frac{1}{2}h_{00} = R_{00} + \nabla_0\nabla_0 f$, the previous equation is rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{\hat{i}\hat{j}} &= -\nabla_{\hat{i}}(R_{\hat{j}0} + \nabla_{\hat{j}}\nabla_0 f - \alpha\gamma_{\alpha\beta}\nabla_{\hat{j}}\phi^\alpha\nabla_0\phi^\beta) - \nabla_{\hat{j}}(R_{\hat{i}0} + \nabla_{\hat{i}}\nabla_0 f - \alpha\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_0\phi^\beta) \\ &\quad + \nabla_0(R_{\hat{i}\hat{j}} + \nabla^2 f(\partial_{\hat{i}}, \partial_{\hat{j}}) - \alpha\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_{\hat{j}}\phi^\beta) - (R_{00} + \nabla_0\nabla_0 f)\mathcal{A}_{\hat{i}\hat{j}}.\end{aligned}$$

Now we will compute some terms of this equation. The first one of them is

$$\nabla_{\hat{i}}\nabla_{\hat{j}}\nabla_0 f = \widehat{\nabla}_{\hat{i}}\nabla_{\hat{j}}\nabla_0 f - \mathcal{A}_{\hat{i}\hat{j}}\nabla_0\nabla_0 f + \mathcal{A}^{\hat{k}}_{\hat{i}}\nabla_{\hat{j}}\nabla_{\hat{k}} f.$$

Replacing $\nabla_{\hat{j}}\nabla_0 f = -\widehat{\nabla}_{\hat{j}} H_g + \mathcal{A}^{\hat{k}}_{\hat{j}}\widehat{\nabla}_{\hat{k}} f$ (since $H + e_0 f = 0$), we obtain

$$\begin{aligned}\nabla_{\hat{i}}\nabla_{\hat{j}}\nabla_0 f &= -\widehat{\nabla}_{\hat{i}}\widehat{\nabla}_{\hat{j}} H + \widehat{\nabla}_{\hat{i}}\left(\mathcal{A}^{\hat{k}}_{\hat{j}}\widehat{\nabla}_{\hat{k}} f\right) - \mathcal{A}_{\hat{i}\hat{j}}\nabla_0\nabla_0 f + \mathcal{A}^{\hat{k}}_{\hat{i}}\widehat{\nabla}_{\hat{j}}\widehat{\nabla}_{\hat{k}} f + H\mathcal{A}^{\hat{k}}_{\hat{i}}\mathcal{A}_{\hat{j}\hat{k}} \\ &= -\widehat{\nabla}_{\hat{i}}\widehat{\nabla}_{\hat{j}} H + \left(\widehat{\nabla}_{\hat{i}}\mathcal{A}^{\hat{k}}_{\hat{j}}\right)\widehat{\nabla}_{\hat{k}} f + \mathcal{A}^{\hat{k}}_{\hat{j}}\widehat{\nabla}_{\hat{i}}\widehat{\nabla}_{\hat{k}} f - \mathcal{A}_{\hat{i}\hat{j}}\nabla_0\nabla_0 f + \mathcal{A}^{\hat{k}}_{\hat{i}}\widehat{\nabla}_{\hat{j}}\widehat{\nabla}_{\hat{k}} f \\ &\quad + H\mathcal{A}^{\hat{k}}_{\hat{i}}\mathcal{A}_{\hat{j}\hat{k}}.\end{aligned}$$

The second one of them is

$$\nabla_0\nabla^2 f(\partial_{\hat{i}}, \partial_{\hat{j}}) - \nabla_{\hat{j}}\nabla_{\hat{i}}\nabla_0 f = \nabla_0\nabla_{\hat{j}}\nabla_{\hat{i}} f - \nabla_{\hat{j}}\nabla_0\nabla_{\hat{i}} f = -R_{0\hat{j}\hat{k}\hat{i}}\widehat{\nabla}^{\hat{k}} f - R_{0\hat{j}0\hat{i}}\nabla_0 f.$$

The third one of them is

$$\begin{aligned}\nabla_0\big(\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_{\hat{j}}\phi^\beta\big) &:= \partial_0\big(\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_{\hat{j}}\phi^\beta\big)\big) - \nabla\phi\otimes\nabla\phi(\nabla_{\partial_0}\partial_{\hat{i}}, \partial_{\hat{j}}) - \nabla\phi\otimes\nabla\phi(\partial_{\hat{i}}, \nabla_{\partial_0}\partial_{\hat{j}}) \\ &= \partial_0(\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_{\hat{j}}\phi^\beta) - \gamma_{\alpha\beta}\langle\nabla\phi^\alpha, \nabla_{\partial_0}\partial_{\hat{i}}\rangle\nabla_{\hat{j}}\phi^\beta - \nabla_{\hat{i}}\phi^\alpha\gamma_{\alpha\beta}\langle\nabla\phi^\beta, \nabla_{\partial_0}\partial_{\hat{j}}\rangle \\ &= \gamma_{\alpha\beta}\nabla_{\hat{i}}\nabla_0\phi^\alpha\nabla_{\hat{j}}\phi^\beta + \gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_{\hat{j}}\nabla_0\phi^\beta.\end{aligned}$$

By interchanging 0 and $j$ we also obtain

$$\nabla_{\hat{j}}\left(\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_0\phi^\beta\right) = \gamma_{\alpha\beta}\nabla_{\hat{i}}\nabla_{\hat{j}}\phi^\alpha\nabla_0\phi^\beta + \gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^\alpha\nabla_0\nabla_{\hat{j}}\phi^\beta.$$

All this implies that

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{\hat{i}\hat{j}} &= \widehat{\nabla}_{\hat{i}}\widehat{\nabla}_{\hat{j}} H - (\widehat{\nabla}_{\hat{i}}\mathcal{A}_{\hat{k}\hat{j}} - R_{0\hat{j}\hat{i}k})\widehat{\nabla}^{\hat{k}} f - \mathcal{A}^{k}_{\hat{i}}\widehat{\nabla}_{\hat{j}}\widehat{\nabla}_{\hat{k}} f - \mathcal{A}^{k}_{\hat{j}}\widehat{\nabla}_{\hat{i}}\widehat{\nabla}_{\hat{k}} f + R_{0\hat{i}0\hat{j}} H \\ &\quad - \nabla_{\hat{i}} R_{\hat{j}0} - \nabla_{\hat{j}} R_{\hat{i}0} + \nabla_0 R_{\hat{i}\hat{j}} - \mathcal{A}_{\hat{i}\hat{j}} R_{00} - H\mathcal{A}^{\hat{k}}_{\hat{i}}\mathcal{A}_{\hat{j}\hat{k}}.\end{aligned}$$

Using Codazzi-Mainardi equation $R_{0\hat{j}\hat{i}\hat{k}} = \widehat{\nabla}_{\hat{i}}\mathcal{A}_{\hat{j}\hat{k}} - \widehat{\nabla}_{\hat{k}}\mathcal{A}_{\hat{i}\hat{j}}$ one has

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{\hat{i}\hat{j}} &= \widehat{\nabla}_{\hat{i}}\widehat{\nabla}_{\hat{j}} H - \left(\widehat{\nabla}_{\hat{k}}\mathcal{A}_{\hat{i}\hat{j}}\right)\widehat{\nabla}^{\hat{k}} f - \mathcal{A}^{\hat{k}}_{\hat{i}}\widehat{\nabla}_{\hat{j}}\widehat{\nabla}_{\hat{k}} f - \mathcal{A}^{\hat{k}}_{\hat{j}}\widehat{\nabla}_{\hat{i}}\widehat{\nabla}_{\hat{k}} f + R_{0\hat{j}0\hat{i}} H \\ &\quad - \nabla_{\hat{i}} R_{\hat{j}0} - \nabla_{\hat{j}} R_{\hat{i}0} + \nabla_0 R_{\hat{i}\hat{j}} - \mathcal{A}_{\hat{i}\hat{j}} R_{00} - H\mathcal{A}^{\hat{k}}_{\hat{i}}\mathcal{A}_{\hat{j}\hat{k}} \\ &= \widehat{\nabla}_{\hat{i}}\widehat{\nabla}_{\hat{j}} H - \left(\mathcal{L}_{\widehat{\nabla} f}\mathcal{A}\right)_{\hat{i}\hat{j}} - \nabla_{\hat{i}} R_{\hat{j}0} - \nabla_{\hat{j}} R_{\hat{i}0} + \nabla_0 R_{\hat{i}\hat{j}} - \mathcal{A}_{\hat{i}\hat{j}} R_{00} + R_{0\hat{i}0\hat{j}} H \\ &\quad - H\mathcal{A}^{\hat{k}}_{\hat{i}}\mathcal{A}_{\hat{j}\hat{k}}.\end{aligned}$$

From Simons' identity (2.9) we get

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{\hat{\imath}\hat{\jmath}} &= (\widehat{\Delta}\mathcal{A})_{\hat{\imath}\hat{\jmath}} - \Big(\mathcal{L}_{\widehat{\nabla} f}\mathcal{A}\Big)_{\hat{\imath}\hat{\jmath}} - (\nabla_{\hat{\imath}} R_{\hat{\jmath}0} - \widehat{\nabla}_{\hat{\imath}} R_{\hat{\jmath}0}) - (\nabla_{\hat{\jmath}} R_{\hat{\imath}0} - \widehat{\nabla}_{\hat{\jmath}} R_{\hat{\imath}0}) - 2\mathcal{A}_{\hat{\imath}\hat{\jmath}} R_{00} \\ &\quad + \mathcal{A}^{\hat{k}}_{\hat{\imath}} R_{0\hat{k}0\hat{\jmath}} + \mathcal{A}^{\hat{k}}_{\hat{\jmath}} R_{0\hat{k}0\hat{\imath}} + 2\mathcal{A}^{\hat{k}l} R_{\hat{k}\hat{\imath}\hat{l}\hat{\jmath}} - 2H\mathcal{A}^{\hat{k}}_{\hat{\imath}}\mathcal{A}_{\hat{\jmath}\hat{k}} + \mathcal{A}^{\hat{k}\hat{l}}\mathcal{A}_{\hat{k}\hat{l}}\mathcal{A}_{\hat{\imath}\hat{\jmath}} \\ &\quad + \nabla_0 R_{0\hat{\imath}0\hat{\jmath}}.\end{aligned}$$

As $\nabla_{\hat{\imath}} R_{\hat{\jmath}0} = \widehat{\nabla}_{\hat{\imath}} R_{\hat{\jmath}0} - \mathcal{A}_{\hat{\imath}\hat{\jmath}} R_{00} + \mathcal{A}^{\hat{k}}_{\hat{\imath}} R_{\hat{\jmath}\hat{k}}$ we conclude that

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{\hat{\imath}\hat{\jmath}} &= (\widehat{\Delta}\mathcal{A})_{\hat{\imath}\hat{\jmath}} - \Big(\mathcal{L}_{\widehat{\nabla} f}\mathcal{A}\Big)_{\hat{\imath}\hat{\jmath}} - \mathcal{A}^{\hat{k}}_{\hat{\imath}} R^{\hat{l}}_{\hat{k}\hat{l}\hat{\jmath}} - \mathcal{A}^{\hat{k}}_{\hat{\jmath}} R^{\hat{l}}_{\hat{k}\hat{l}\hat{\imath}} + 2\mathcal{A}^{\hat{k}\hat{l}} R_{\hat{k}\hat{\imath}\hat{l}\hat{\jmath}} - 2H\mathcal{A}^{\hat{k}}_{\hat{\imath}}\mathcal{A}_{\hat{\jmath}\hat{k}} \\ &\quad + \mathcal{A}^{\hat{k}\hat{l}}\mathcal{A}_{\hat{k}\hat{l}}\mathcal{A}_{\hat{\imath}\hat{\jmath}} + \nabla_0 R_{0\hat{\imath}0\hat{\jmath}}.\end{aligned}$$

Finally, we show equation (2.13). For it, note that

$$\delta H = -h_{\hat{\imath}\hat{\jmath}}\mathcal{A}^{\hat{\imath}\hat{\jmath}} + g^{\hat{\imath}\hat{\jmath}}\delta\mathcal{A}_{\hat{\imath}\hat{\jmath}}$$

and

$$g^{\hat{\imath}\hat{\jmath}}(\mathcal{L}_{\widehat{\nabla} f}\mathcal{A})_{\hat{\imath}\hat{\jmath}} - 2\mathcal{A}^{\hat{\imath}\hat{\jmath}}\widehat{\nabla}_{\hat{\imath}}\widehat{\nabla}_{\hat{\jmath}} f = \widehat{\nabla}_{\widehat{\nabla} f}(g^{\hat{\imath}\hat{\jmath}}\mathcal{A}_{\hat{\imath}\hat{\jmath}}) = \langle\widehat{\nabla} f, \widehat{\nabla} H\rangle.$$

So,

$$\begin{aligned}\frac{\partial}{\partial t}H &= 2(R_{\hat{\imath}\hat{\jmath}} + \widehat{\nabla}_{\hat{\imath}}\widehat{\nabla}_{\hat{\jmath}} f + H\mathcal{A}_{\hat{\imath}\hat{\jmath}})\mathcal{A}^{\hat{\imath}\hat{\jmath}} + g^{\hat{\imath}\hat{\jmath}}\Big((\widehat{\Delta}\mathcal{A})_{\hat{\imath}\hat{\jmath}} - \big(\mathcal{L}_{\widehat{\nabla} f}\mathcal{A}\big)_{\hat{\imath}\hat{\jmath}} - \mathcal{A}^{\hat{k}}_{\hat{\imath}} R^{\hat{l}}_{\hat{k}\hat{l}\hat{\jmath}} \\ &\quad - \mathcal{A}^{\hat{k}}_{\hat{\jmath}} R^{\hat{l}}_{\hat{k}\hat{l}\hat{\imath}} + 2\mathcal{A}^{\hat{k}\hat{l}} R_{\hat{k}\hat{\imath}\hat{l}\hat{\jmath}} - 2H\mathcal{A}^{\hat{k}}_{\hat{\imath}}\mathcal{A}_{\hat{\jmath}\hat{k}} + \mathcal{A}^{\hat{k}\hat{l}}\mathcal{A}_{\hat{k}\hat{l}}\mathcal{A}_{\hat{\imath}\hat{\jmath}} + \nabla_0 R_{0\hat{\imath}0\hat{\jmath}}\Big) \\ &\quad - 2\alpha\mathcal{A}(\widehat{\nabla}\phi, \widehat{\nabla}\phi) \\ &= 2\mathcal{A}^{\hat{\imath}\hat{\jmath}} R_{\hat{\imath}\hat{\jmath}} + 2H\mathcal{A}^{\hat{\imath}\hat{\jmath}}\mathcal{A}_{\hat{\imath}\hat{\jmath}} + \widehat{\Delta} H - \Big(g^{\hat{\imath}\hat{\jmath}}\big(\mathcal{L}_{\widehat{\nabla} f}\mathcal{A}\big)_{\hat{\imath}\hat{\jmath}} - 2\mathcal{A}^{\hat{\imath}\hat{\jmath}}\widehat{\nabla}_{\hat{\imath}}\widehat{\nabla}_{\hat{\jmath}} f\Big) - 2\mathcal{A}^{\hat{k}\hat{\jmath}}\mathcal{A}_{\hat{\jmath}\hat{k}} H \\ &\quad + \mathcal{A}^{\hat{k}\hat{l}}\mathcal{A}_{\hat{k}\hat{l}} H + \nabla_0 R_{00} - 2\alpha\mathcal{A}(\widehat{\nabla}\phi, \widehat{\nabla}\phi) \\ &= \widehat{\Delta} H - \langle\widehat{\nabla} f, \widehat{\nabla} H\rangle + 2\mathcal{A}^{\hat{\imath}\hat{\jmath}} R_{\hat{\imath}\hat{\jmath}} + \mathcal{A}^{\hat{\imath}\hat{\jmath}}\mathcal{A}_{\hat{\imath}\hat{\jmath}} H + \nabla_0 R_{00} - 2\alpha\mathcal{A}(\widehat{\nabla}\phi, \widehat{\nabla}\phi).\end{aligned}$$

This finishes the proof. □

As a consequence of Proposition 2.10, we have the following refinement of the formula obtained in Proposition 2.9.

**Corollary 2.11.** *If $(g(t), \phi(t))_{t\in[0,T)}$ evolves by modified $(RH)_\alpha$ flow, then*

$$\begin{aligned}\frac{d}{dt}\mathcal{F}^\alpha_\infty &= 2\int_M \Big(\big|\mathrm{Ric} + \nabla^2 f - \alpha\nabla\phi\otimes\nabla\phi\big|^2 + \alpha|(\tau_{g,\gamma}\phi - \nabla\phi(\nabla f)|^2\Big)e^{-f} dM \\ &\quad + 2\int_{\partial M}\Big(\frac{\partial H}{\partial t} - \langle\widehat{\nabla} f, \widehat{\nabla} H\rangle + \mathcal{A}(\widehat{\nabla} f, \widehat{\nabla} f) + 2R^{0\hat{\imath}}\widehat{\nabla}_{\hat{\imath}} f - \frac{1}{2}\nabla_0 R - HR_{00} \\ &\quad + \alpha\mathcal{A}(\widehat{\nabla}\phi, \widehat{\nabla}\phi)\Big)e^{-f} dA.\end{aligned}$$

*In particular, if both $\big(R_{\hat{\imath}\hat{\jmath}} + \nabla^2 f(\partial_{\hat{\imath}}, \partial_{\hat{\jmath}}) - \alpha\gamma_{\alpha\beta}\nabla_{\hat{\imath}}\phi^\alpha\nabla_{\hat{\jmath}}\phi^\beta\big)|_{\partial M}$ and $\big(R_{\hat{\imath}0} + \nabla_{\hat{\imath}}\nabla_0 f\big)|_{\partial M}$ vanish, then the boundary integrand vanishes.*

*Proof.* From equation (2.13) of Proposition 2.10, the boundary integrand term of Proposition 2.9 can be rewritten as

$$\begin{aligned}
&\widehat{\Delta} H - 2\langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + \mathcal{A}(\widehat{\nabla} f, \widehat{\nabla} f) + \mathcal{A}^{\hat{i}\hat{j}}\mathcal{A}_{\hat{i}\hat{j}} H + \mathcal{A}^{\hat{i}\hat{j}} R_{\hat{i}\hat{j}} + 2R^{0\hat{i}}\widehat{\nabla}_{\hat{i}} f - \widehat{\nabla}_{\hat{i}} R^{0\hat{i}} \\
&\quad - \alpha\mathcal{A}(\widehat{\nabla}\phi, \widehat{\nabla}\phi) \\
&= \frac{\partial H}{\partial t} - \langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + \mathcal{A}(\widehat{\nabla} f, \widehat{\nabla} f) - \mathcal{A}^{\hat{i}\hat{j}} R_{\hat{i}\hat{j}} + 2R^{0\hat{i}}\widehat{\nabla}_{\hat{i}} f - \widehat{\nabla}_{\hat{i}} R^{0\hat{i}} - \nabla_0 R_{00} \\
&\quad + \alpha\mathcal{A}(\widehat{\nabla}\phi, \widehat{\nabla}\phi).
\end{aligned}$$

Contracted Bianchi Identity and the fact that $\nabla_{\hat{i}} R_{\hat{j}0} = \widehat{\nabla}_{\hat{i}} R_{\hat{j}0} - \mathcal{A}_{\hat{i}\hat{j}} R_{00} + \mathcal{A}^{k}_{\hat{i}} R_{\hat{j}k}$ imply

$$\frac{1}{2}\nabla_0 R = \nabla_{\hat{i}} R^{\hat{i}0} + \nabla_0 R_{00} = \widehat{\nabla}_{\hat{i}} R^{\hat{i}0} - H R_{00} + \mathcal{A}^{\hat{i}\hat{j}} R_{\hat{i}\hat{j}} + \nabla_0 R_{00}.$$

The main result of the corollary follows from these two latter equations. If, in addition, both $R_{\hat{i}\hat{j}} + \nabla^2 f(\partial_{\hat{i}}, \partial_{\hat{j}}) - \alpha\gamma_{\alpha\beta}\nabla_{\hat{i}}\phi^{\alpha}\nabla_{\hat{j}}\phi^{\beta}$ and $R_{\hat{i}0} + \nabla_{\hat{i}}\nabla_0 f$ vanish on $\partial M$, then by Proposition 2.9 the integrand of $\partial M$, namely

$$\frac{\partial H}{\partial t} - \langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + \mathcal{A}(\widehat{\nabla} f, \widehat{\nabla} f) + 2R^{0\hat{i}}\widehat{\nabla}_{\hat{i}} f - \frac{1}{2}\nabla_0 R - H R_{00} + \alpha\mathcal{A}(\widehat{\nabla}\phi, \widehat{\nabla}\phi)$$

vanishes. □

## 2.2 Hypersurfaces in the $(RH)_\alpha$ flow background

Before embarking on the proof of our first main theorem, let us put it in context. Let $M$ be an $n$-dimensional smooth manifold, and consider an $(RH)_\alpha$ flow $(g(t), \phi(t))$ in $M \times I$. Given an $(n-1)$-dimensional compact smooth manifold $\Sigma$ without boundary, and let $\{x(\cdot, t); t \in [0, T)\}$ be a smooth one-parameter family of immersions of $\Sigma$ in $M$. For each $t \in [0, T)$, set $x_t = x(\cdot, t)$ and $\Sigma_t$ for the hypersurface $x_t(\Sigma)$ of $(M, g(t))$, which we can write as $\Sigma_t := (\Sigma, x_t^* g(t))$. Suppose that the family $\mathscr{F} := \{\Sigma_t \,;\, t \in [0, T)\}$ evolves under mean curvature flow, MCF for short, namely

$$\begin{cases} \dfrac{\partial}{\partial t} x(p,t) &= H(p,t) e(p,t), \\ x(p,0) &= x_0(p), \end{cases}$$

where $H(p, t)$ and $e(p, t)$ are the mean curvature and the unit normal of $\Sigma_t$ at the point $p \in \Sigma$, respectively. In this setting, we say that $\mathscr{F}$ is a *MCF in the* $(g(t), \phi(t))-(RH)_\alpha$ *flow background.* In the particular case $\big(g(t), \phi(t)\big) = \big(\overline{g}(t), \overline{\phi}(t)\big)$ is a self-similar solution to the $(RH)_\alpha$ flow on $M$ with potential function $\overline{f}$, a hypersurface $\Sigma_t \in \mathscr{F}$ is a *mean curvature soliton*, if

$$H(p,t) + e(p,t)\overline{f} = 0 \quad \text{on} \quad \Sigma.$$

Here, $e(\,\cdot\,, t)$ must be the inward unit normal vector field on $\Sigma_t$.

**Proposition 2.12.** *Let $(M,g)$ be an $n$-dimensional Riemannian manifold, and let $\Sigma$ be an $(n-1)$-dimensional compact smooth manifold without boundary. Consider $\mathscr{F}$ the MCF of $\Sigma$ in the $(g(t),\phi(t))-(RH)_\alpha$ flow background with the Neumann boundary condition $\nabla_0\phi = 0$ on $\Sigma$. Then, the following evolution equations hold on $\Sigma_t$*

$$\frac{\partial}{\partial t} g_{\hat{i}\hat{j}} = -2(R_{\hat{i}\hat{j}} - \alpha\gamma_{\alpha\beta}\widehat{\nabla}_{\hat{i}}\phi^\alpha\widehat{\nabla}_{\hat{j}}\phi^\beta) - 2H\mathcal{A}_{\hat{i}\hat{j}}, \tag{2.14}$$

$$\frac{\partial}{\partial t}\phi = \tau_{\widehat{g},\gamma}\phi + \nabla_0\nabla_0\phi, \tag{2.15}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{\hat{i}\hat{j}} = {}& (\widehat{\Delta}\mathcal{A})_{\hat{i}\hat{j}} - \mathcal{A}^{\hat{k}}_{\ \hat{i}}R^{\hat{l}}_{\hat{k}\hat{l}\hat{j}} - \mathcal{A}^{\hat{k}}_{\ \hat{j}}R^{\hat{l}}_{\hat{k}\hat{l}\hat{i}} + 2\mathcal{A}^{\hat{k}\hat{l}}R_{\hat{k}\hat{i}\hat{l}\hat{j}} - 2H\mathcal{A}_{\hat{i}\hat{k}}\mathcal{A}^{\hat{k}}_{\ \hat{j}} \\ & + \mathcal{A}^{\hat{k}\hat{l}}\mathcal{A}_{\hat{k}\hat{l}}\mathcal{A}_{\hat{i}\hat{j}} + \nabla_0 R_{0\hat{i}0\hat{j}}\end{aligned} \tag{2.16}$$

*and*

$$\frac{\partial}{\partial t}H = \widehat{\Delta}H + 2\mathcal{A}^{\hat{i}\hat{j}}R_{\hat{i}\hat{j}} + \mathcal{A}^{\hat{i}\hat{j}}\mathcal{A}_{\hat{i}\hat{j}}H + \nabla_0 R_{00} - 2\alpha\mathcal{A}(\widehat{\nabla}\phi,\widehat{\nabla}\phi). \tag{2.17}$$

*Proof.* First, assume $\Sigma_t = \partial X_t$ with each $X_t$ compact. Given an interval $[a,b]\subset[0,T)$ and the MCF of $\Sigma$ in the $(g(t),\phi(t))_{t\in[a,b]} - (RH)_\alpha$ flow background with $\nabla_a\phi = 0$ on $\Sigma = \partial X_a$. We can find a positive solution $u(t) = e^{-f(t)}$ for

$$\begin{cases} \Box^*_{g(t)}u = 0 & \text{in } \bigcup_{t\in[a,b]}(X_t\times\{t\}), \\ e_t u = H_{g(t)}u & \text{on } \bigcup_{t\in[a,b]}(\partial X_t\times\{t\}), \end{cases} \tag{2.18}$$

by solving it backwards in time from $t=b$, where $\Box^*_{g(t)}$ is defined as in (2.2). Indeed, choosing diffeomorphisms $r_t : X_a \to X_t$, we reduce the problem of solving (2.18) to a parabolic equation on a fixed domain. For it, take $\widetilde{g}(t) = r_t^* g(t)$, $\widetilde{\phi}(t) = r_t^*\phi(t)$, $\widetilde{f}(t) = r_t^* f(t)$ and $\widetilde{u}(t) = r_t^* u(t)$, it is straightforward to compute that

$$\begin{cases} \Box^*_{\widetilde{g}(t)}\widetilde{u} + \langle \nabla_{\widetilde{g}(t)}\widetilde{u}, \dfrac{\partial r_t}{\partial t}\rangle = 0 & \text{in } X_a\times[a,b], \\ \widetilde{e}_t\widetilde{u} = H_{\widetilde{g}(t)}\widetilde{u} & \text{on } \partial X_a\times[a,b] \end{cases} \tag{2.19}$$

which is equivalent to (2.18). Now, by using $s = b-t$, we have that (2.19) is equivalent to the following parabolic equation

$$,\begin{cases} \frac{\partial}{\partial s}\widetilde{u}(s) = \Delta_{\widetilde{g}}\widetilde{u} - R_{\widetilde{g}}\widetilde{u} + \alpha|\nabla_{\widetilde{g}}\widetilde{\phi}|^2\widetilde{u} + \langle\nabla_{\widetilde{g}}\widetilde{u}, \dfrac{\partial r_t}{\partial s}\rangle & \text{in } X_a\times[a,b], \\ \widetilde{e}_s\widetilde{u} = H_{\widetilde{g}}\widetilde{u} & \text{on } \partial X_a\times[a,b]. \end{cases} \tag{2.20}$$

It guarantees the existence of a solution $u(t) = e^{-f(t)}$ for (2.18). Thus, we can take a one-parameter family $\{\psi_t\}_{t\in[a,b]}$ of diffeomorphisms generated by $\{-\nabla_{g(t)}f(t)\}_{t\in[a,b]}$, with $\psi_a = \mathrm{Id}$ and $\psi_t(X_a) = X_t$ for all $t$. By setting $\widetilde{g}(t) = \psi_t^* g(t)$, $\widetilde{\phi}(t) = \psi_t^*\phi(t)$, $\widetilde{f}(t) = \psi_t^* f(t)$ and $\widetilde{\gamma}(t) = \psi_t^*\gamma(t)$, we have that $\widetilde{g}(t)$, $\widetilde{\phi}(t)$, $\widetilde{f}(t)$ and $\widetilde{\gamma}(t)$ are defined on $X_a$. We claim that

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{g}_{ij} = -2(\widetilde{R}_{ij} + \widetilde{\nabla}^2\widetilde{f}(\partial_i,\partial_j) - \alpha\widetilde{\gamma}_{\alpha\beta}\widetilde{\nabla}_i\widetilde{\phi}^\alpha\widetilde{\nabla}_j\widetilde{\phi}^\beta), \\ \frac{\partial}{\partial t}\widetilde{\phi} = \tau_{\widetilde{g},\widetilde{\gamma}}\widetilde{\phi} - \langle\widetilde{\nabla}\widetilde{\phi},\widetilde{\nabla}\widetilde{f}\rangle_{\widetilde{g}} \end{cases} \tag{2.21}$$

and

$$\frac{\partial}{\partial t}\widetilde{f} = -\Delta_{\widetilde{g}}\widetilde{f} - R_{\widetilde{g}} + \alpha|\widetilde{\nabla}\widetilde{\phi}|^2_{\widetilde{g}} \tag{2.22}$$

in $X_a \times [a,b]$ with $H_{\widetilde{g}} + e_a\widetilde{f} = 0$ and $\nabla_a\phi = 0$ on $\partial X_a = \Sigma$. Indeed, to prove (2.21), we compute

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{g}_{ij} &= \psi_t^*\Big(\frac{\partial}{\partial t}g_{ij}\Big) + \psi_t^*\Big(\mathcal{L}_{\frac{d}{dt}\psi_t}g\Big)_{ij}\\ &= \psi_t^*\Big(-2(R_{ij} - \alpha\gamma_{\alpha\beta}\nabla_i\phi^\alpha\nabla_j\phi^\beta)\Big) - \psi_t^*\Big(\mathcal{L}_{\left(\nabla_{g(t)}f(t)\right)}g\Big)_{ij}\\ &= -2(\widetilde{R}_{ij} + \widetilde{\nabla}^2\widetilde{f}(\partial_i,\partial_j) - \alpha\widetilde{\gamma}_{\alpha\beta}\widetilde{\nabla}_i\widetilde{\phi}^\alpha\widetilde{\nabla}_j\widetilde{\phi}^\beta).\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{\phi} &= \psi_t^*\Big(\frac{\partial}{\partial t}\phi\Big) + \psi_t^*\mathcal{L}_{\frac{d}{dt}\psi_t}\phi\\ &= \psi_t^*\Big(\tau_{g,\gamma}\phi\Big) - \psi_t^*\mathcal{L}_{\left(\nabla_{g(t)}f(t)\right)}\phi\\ &= \tau_{\widetilde{g},\widetilde{\gamma}}\widetilde{\phi} - \langle\widetilde{\nabla}\widetilde{\phi},\widetilde{\nabla}\widetilde{f}\rangle_{\widetilde{g}}.\end{aligned}$$

To prove (2.22), we use that $\Delta e^{-f} = (|\nabla f|^2 - \Delta f)e^{-f}$ and (2.18) to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{f} &= \psi_t^*\Big(\frac{\partial}{\partial t}f\Big) + \psi_t^*\mathcal{L}_{\frac{d}{dt}\psi_t}f\\ &= \psi_t^*\Big(|\nabla f|^2 - \Delta f - R + \alpha|\nabla\phi|^2\Big) - \psi_t^*\mathcal{L}_{\left(\nabla_{g(t)}f(t)\right)}f\\ &= -\Delta_{\widetilde{g}}\widetilde{f} - R_{\widetilde{g}} + \alpha|\widetilde{\nabla}\widetilde{\phi}|^2_{\widetilde{g}}.\end{aligned}$$

For the boundary condition, it is enough to note that $e_t u = H_{g(t)}u$ implies $e_t f(t) + H_{g(t)} = 0$, and then $0 = \psi_t^* e_t f(t) + \psi_t^* H_{g(t)} = e_a\widetilde{f}(t) + H_{\widetilde{g}(t)}$. Hence, $(\widetilde{g}(t),\widetilde{\phi}(t))$ evolves by modified $(RH)_\alpha$ flow in $X_a \times [a,b]$, thus, we can apply Proposition 2.10 for the compact manifold $X_a$ with boundary $\partial X_a$, from which we obtain

$$\begin{aligned}\frac{\partial}{\partial t}g_{\hat{i}\hat{j}} &= \frac{\partial}{\partial t}\Big((\psi_t^*)^{-1}\psi_t^* g_{\hat{i}\hat{j}}\Big) = \frac{\partial}{\partial t}\Big((\psi_t^*)^{-1}\widetilde{g}_{\hat{i}\hat{j}}\Big) = (\psi_t^*)^{-1}\Big(\frac{\partial}{\partial t}\widetilde{g}_{\hat{i}\hat{j}} + \Big(\mathcal{L}_{\frac{d}{dt}\psi_t^{-1}}\widetilde{g}\Big)_{\hat{i}\hat{j}}\Big)\\ &= -2(R_{\hat{i}\hat{j}} - \alpha\gamma_{\alpha\beta}\widehat{\nabla}_{\hat{i}}\phi^\alpha\widehat{\nabla}_{\hat{j}}\phi^\beta) - 2H\mathcal{A}_{\hat{i}\hat{j}},\end{aligned}$$

on $\Sigma_t$ that is (2.14). Likewise, from (2.11) one has

$$\frac{\partial}{\partial t}\phi = (\psi_t^*)^{-1}\Big(\frac{\partial}{\partial t}\widetilde{\phi} + \mathcal{L}_{\frac{d}{dt}\psi_t^{-1}}\widetilde{\phi}\Big) = \tau_{\widehat{g},\gamma}\phi + \nabla_0\nabla_0\phi,$$

which is (2.15). Next, equation (2.12) implies

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{\hat{i}\hat{j}} &= (\psi_t^*)^{-1}\Big(\frac{\partial}{\partial t}\widetilde{\mathcal{A}}_{\hat{i}\hat{j}} + \Big(\mathcal{L}_{\frac{d}{dt}\psi_t^{-1}}\widetilde{\mathcal{A}}\Big)_{\hat{i}\hat{j}}\Big)\\ &= (\widehat{\Delta}\mathcal{A})_{\hat{i}\hat{j}} - \mathcal{A}^{\hat{k}}_{\hat{i}}R^{\hat{l}}_{\hat{k}\hat{l}\hat{j}} - \mathcal{A}^{\hat{k}}_{\hat{j}}R^{\hat{l}}_{\hat{k}\hat{l}\hat{i}} + 2\mathcal{A}^{\hat{k}\hat{l}}R_{\hat{k}\hat{i}\hat{l}\hat{j}} - 2H\mathcal{A}_{\hat{i}\hat{k}}\mathcal{A}^{\hat{k}}_{\hat{j}} + \mathcal{A}^{\hat{k}l}\mathcal{A}_{\hat{k}\hat{l}}\mathcal{A}_{\hat{i}\hat{j}}\\ &\quad + \nabla_0 R_{0\hat{i}0\hat{j}},\end{aligned}$$

and from (2.13) we get

$$\begin{aligned}\frac{\partial}{\partial t}H &= (\psi_t^*)^{-1}\Big(\frac{\partial}{\partial t}H_{\widetilde{g}} + \mathcal{L}_{\frac{d}{dt}\psi_t^{-1}}H_{\widetilde{g}}\Big)\\ &= \widehat{\Delta}H + 2\mathcal{A}^{\hat{i}\hat{j}}R_{\hat{i}\hat{j}} + \mathcal{A}^{\hat{i}\hat{j}}\mathcal{A}_{\hat{i}\hat{j}}H + \nabla_0 R_{00} - 2\alpha\mathcal{A}(\widehat{\nabla}\phi,\widehat{\nabla}\phi).\end{aligned}$$

For finishing, we observe that the result could be derived from a local calculation on $\Sigma_t$, hence, it is also valid without the assumption that $\Sigma_t$ bounds a compact domain. □

**Remark 2.13.** We point out that (2.14) holds regardless of the assumption $\nabla_0\phi = 0$ on $\Sigma$.

**Remark 2.14.** If $M$ is the Euclidean space with its standard metric $g_{st}$, $g(t) = g_{st}$ and $\phi(t) = \phi$ is a constant, then Eqs. (2.14), (2.16) and (2.17) are the same as in [Hui84, Lem. 3.2, Thm. 3.4 and Cor. 3.5], see also Mantegazza [Man11, Sect. 2.3]. Moreover, we recover Prop. 4 in Gomes and Hudson [GH23], for $\phi \in C^\infty(M)$; and Prop. 4 in Lott [Lot12], for $\phi$ constant.

**Theorem 2.15.** *Let $M$ be an $n$-dimensional compact smooth manifold with boundary $\partial M$, and let $\mathscr{F}$ be the MCF of $\partial M$ in the $(g(t),\phi(t)) - (RH)_\alpha$ flow background with the Neumann boundary condition $\nabla_0\phi = 0$. If $u := e^{-f}$ is a solution to the conjugate heat equation*

$$\Box^* u = 0 \quad \text{in} \quad M \times [0,T) \tag{2.23}$$

*with $e_0 u = Hu$ on $\partial M$, then*

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_\infty^\alpha &= 2\int_M \Big(|\mathrm{Ric} + \nabla^2 f - \alpha\nabla\phi\otimes\nabla\phi|^2 + \alpha|\tau_{g,\gamma}\phi - \langle\nabla\phi,\nabla f\rangle|^2\Big)e^{-f}dM\\ &\quad + 2\int_{\partial M}\Big(\frac{\partial H}{\partial t} - 2\langle\widehat{\nabla} f,\widehat{\nabla} H\rangle + \mathcal{A}(\widehat{\nabla} f,\widehat{\nabla} f) + 2R^{0i}\widehat{\nabla}_i f - \frac{1}{2}\nabla_0 R - HR_{00}\\ &\quad + \alpha\mathcal{A}(\widehat{\nabla}\phi,\widehat{\nabla}\phi)\Big)e^{-f}dA,\end{aligned}$$

*where $R$ is the scalar curvature of $(M,g)$, $\mathcal{A}$ is the second fundamental form of $\partial M$, and $\widehat{\nabla}$ denotes the gradient on $\partial M$.*

*Proof.* The hypotheses on $\{\partial M_t\,;\, t\in[0,T)\}$ and $u$ allow us to use $\widetilde{g}(t)$, $\widetilde{\phi}(t)$ and $\widetilde{f}(t)$ on $M$ as in the proof of Proposition 2.12. Thus, the result follows immediately from Corollary 2.11 and the fact that the identity

$$\frac{\partial}{\partial t}H_{\widetilde{g}} = \frac{\partial}{\partial t}H_g - \langle\widehat{\nabla} f,\widehat{\nabla} H\rangle$$

holds on $\partial M_t$ for all $t \in [0,T)$. □

## 2.3 Characterization of mean curvature solitons

Let $(\overline{g}(t),\overline{\phi}(t))$ be a self-similar solution to the $(RH)_\alpha$ flow on an $n$-dimensional smooth manifold $M$, given by

$$\begin{cases}\overline{g}(t) = \sigma(t)\psi_t^* g,\\ \overline{\phi}(t) = \psi_t^*\phi,\end{cases} \tag{2.24}$$

for some initial value $(g, \phi)$, where $\psi_t$ is a smooth one-parameter family of diffeomorphisms of $M$ generated from the flow of $\nabla_g f/\sigma(t)$, for some potential function $f \in C^\infty(M)$, and $\sigma(t)$ is a positive smooth function on $t$. By taking derivative of $\overline{g}(t)$ in (2.24) and as in the proof of Proposition 2.2, one has

$$\mathrm{Ric}_{\overline{g}(t)} + \nabla^2_{\overline{g}(t)} \overline{f}(t) - \alpha \nabla\overline{\phi}(t) \otimes \nabla\overline{\phi}(t) = -\frac{\sigma'(t)}{2\sigma(t)} \overline{g}(t).$$

Solving $\sigma'(t)/\sigma(t) = -c/(T-t)$ and using conformal theory, we immediately obtain that the triple $(g(t), f(t), \phi(t))$ satisfies (2.3.a) (as well (2.3.b)) along the $(RH)_\alpha$ flow. More precisely, we have the following result.

**Proposition 2.16.** *Let $M$ be an $n$-dimensional smooth manifold, and let $(\overline{g}(t), \overline{\phi}(t))$ be a self-similar solution to the $(RH)_\alpha$ flow on $M$. Then, the following identities hold*

$$\begin{cases} \mathrm{Ric}_{\overline{g}} + \nabla^2_{\overline{g}} \overline{f} - \alpha \nabla\overline{\phi} \otimes \nabla\overline{\phi} & = & \dfrac{c}{2(T-t)} \overline{g}, \\ \tau_{\overline{g},\gamma} \overline{\phi} & = & \langle \nabla_{\overline{g}} \overline{f}, \nabla_{\overline{g}} \overline{\phi} \rangle_{\overline{g}}, \end{cases} \tag{2.25}$$

*for all $t$, where $c = 0$ in the steady case (for $t \in \mathbb{R}$ and $\psi_0 = \mathrm{Id}$), $c = 1$ in the shrinking case (for $t \in (-\infty, T)$ and $\psi_{T-1} = \mathrm{Id}$), $c = 1$ in the expanding case (for $t \in (T, +\infty)$ and $\psi_{T+1} = \mathrm{Id}$), besides*

$$\frac{\partial}{\partial t} \overline{f} = |\nabla_{\overline{g}} \overline{f}|^2_{\overline{g}}. \tag{2.26}$$

*The function $\overline{f}$ is still called the potential function.*

Now, we will show how to construct a family of mean curvature solitons and establish a characterization of such a family. For it, let $M$ be an $n$-dimensional smooth manifold, and let $(\overline{g}(t), \overline{\phi}(t))$ be a self-similar solution to the $(RH)_\alpha$ flow on $M$ for some initial value $(g, \phi)$ and with potential function $\overline{f} = \psi_t^* f$, where $\{\psi_t\}$ is the smooth one-parameter family of diffeomorphisms of $M$ generated by $Y_t = \nabla_g f/\sigma(t)$, with $\sigma(t) = \kappa(T-t)$ and $\psi_{T-\kappa} = \mathrm{Id}$, where $\kappa = 1$ in the shrinking case (for $t \in (-\infty, T)$), $\kappa = -1$ in the expanding case (for $t \in (T, +\infty)$) and $\sigma(t) = 1$ in the steady case (for $t \in \mathbb{R}$) with $\psi_0 = \mathrm{Id}$.

Given an $(n-1)$-dimensional compact smooth manifold $\Sigma$ without boundary, let $\{x(\,\cdot\,, t)\}$ be a smooth one-parameter family of immersions of $\Sigma$ in $M$, where $x(\cdot, t) := \psi(\cdot, -t + 2(T-\kappa))$, for $\kappa = 1$, with $t \in \big(2(T-1), T\big)$ and $\psi_{T-1} = \mathrm{Id}$ in the shrinking case; and for $\kappa = -1$, with $t \in (T, 2(T+1))$ and $\psi_{T+1} = \mathrm{Id}$ in the expanding case and $x(\,\cdot\,, t) := \psi(\,\cdot\,, -t)$ in the steady case, where $\psi_t$ is a smooth one-parameter family of diffeomorphisms of $M$ generated from the flow of $\nabla_g f/\kappa(T-t)$. Note that $x(\,\cdot\,, T-\kappa) = \psi(\,\cdot\,, T-\kappa) = \mathrm{Id}$ in the shrinking and expanding cases and $x(\,\cdot\,, 0) = \psi(\,\cdot\,, 0) = \mathrm{Id}$ in steady case. If $\mathscr{G} := \{\Sigma_t\}$ evolves by MCF in the $(\overline{g}, \overline{\phi}) - (RH)_\alpha$ flow background on $M$, then it is a family of mean curvature solitons. Indeed, since $\overline{g}(t) = \sigma(t)\psi_t^* g$, we have $\nabla_g f = \sigma(t)\nabla_{\overline{g}(t)}\overline{f}$, and then

$$H(p,t) = \overline{g}(t)\Big(\frac{\partial}{\partial t}x(p,t), e(p,t)\Big) = -\overline{g}(t)\Big(\frac{\nabla_g f(p)}{\sigma(t)}, e(p,t)\Big)$$
$$= -\overline{g}(t)\Big(\nabla_{\overline{g}(t)}\overline{f}(p), e(p,t)\Big) = -e(p,t)\overline{f}(p),$$

it proves our claim. A sufficient condition for ensuring that $\mathscr{G}$ is a family of mean curvature solitons is that the hypersurface $\Sigma$ must be $f$-minimal. Besides, we will see that any family $\mathscr{F}$ of mean curvature solitons is given by family $\mathscr{G}$ up to reparametrization, as stated below.

**Theorem 2.17.** *If $\Sigma$ is an $f$-minimal hypersurface of $(M,g)$, then $\mathscr{G}$, as defined above, is a family of mean curvature solitons in the $(\overline{g},\overline{\phi}) - (RH)_\alpha$ flow background on $M$. Moreover, any family $\mathscr{F}$ of mean curvature solitons in the $(\overline{g},\overline{\phi}) - (RH)_\alpha$ flow background on $M$ is given by $\mathscr{G}$ up to reparametrization.*

*Proof.* Let $\Sigma$ be a hypersurface of $(M,g)$ satisfying $H + e_0 f = 0$ on $\Sigma$, where $e_0$ is the unit normal vector field on $\Sigma$. Take $\mathscr{G} = \{\Sigma_t\}$ the smooth one-parameter family of isometric immersions of $\Sigma$ in $M$ as above, so that $e_0 = \sqrt{\sigma(t)}e(\,\cdot\,,t)$, and then $\mathcal{A}_{e_0} = \sqrt{\sigma(t)}\mathcal{A}_{e(\cdot,t)}$ that implies $H = \sqrt{\sigma(t)}H(\cdot,t)$. So, $H(\,\cdot\,,t) + e(\,\cdot\,,t)\overline{f} = 0$. Thus,

$$\Big(\frac{\partial}{\partial t}x(\,\cdot\,,t)\Big)^{\perp} = \overline{g}(t)\Big(\frac{\partial}{\partial t}x(\,\cdot\,,t), e(\,\cdot\,,t)\Big)e(\,\cdot\,,t) = -\overline{g}(t)\Big(\frac{\nabla_g f}{\sigma(t)}, e(\,\cdot\,,t)\Big)e(\,\cdot\,,t)$$
$$= -\overline{g}(t)\Big(\nabla_{\overline{g}(t)}\overline{f}, e(\,\cdot\,,t)\Big)e(\,\cdot\,,t) = -e(\,\cdot\,,t)(\overline{f})e(\,\cdot\,,t) = H(\,\cdot\,,t)e(\,\cdot\,,t).$$

Now, we affirm that if a smooth family of hypersurfaces $\Sigma_t$ satisfies $\langle \frac{\partial}{\partial t}x(p,t), e(p,t)\rangle = H(p,t)$, then it can be everywhere locally reparametrized to a mean curvature flow. Indeed, if $\frac{\partial}{\partial t}x(p,t) = H(p,t)e(p,t) + X(p,t)$, where $X(p,t) \in dx_t(Tp\Sigma)\ \ \forall p \in \Sigma$, take $\{\varphi_t\}$ the smooth one-parameter family of diffeomorphisms of $\Sigma$ generated by $Y(p,t) = -[dx_t]^{-1}(X(p,t))$ and then consider the reparametrization $\widetilde{x}(p,t) = x(\varphi_t(p),t)$. By a straightforward computation $\widetilde{\Sigma}_t$ evolves by MCF in the $(\overline{g},\overline{\phi}) - (RH)_\alpha$ flow background on $M$. Finally, by a simple analysis of this proof, we also show that any family $\mathscr{F}$ of mean curvature solitons is given by $\mathscr{G}$ up to reparametrization. □

**Remark 2.18.** The previous theorem recovers Thm. 3 in Gomes and Hudson [GH23], for $\phi \in C^\infty(M)$; and Prop. 4.3 in Yamamoto [Yam20] in the case of gradient shrinking Ricci soliton and $\phi$ constant.

We conclude this section with the proof of Theorem 2.20. We begin by determining how the area evolves under MCF in an $(RH)_\alpha$ flow background.

**Lemma 2.19.** *Let $(\overline{g}(t),\overline{\phi}(t))$, $\overline{f}$ and $\mathscr{F}$ be as in the statement of Theorem 2.20. Then, the following equation holds on $\Sigma_t$*

$$\frac{d}{dt}(dA_{\overline{g}}) = -(\overline{R}_i^i + H_{\overline{g}}^2 - \alpha|\widehat{\nabla}_{\overline{g}}\overline{\phi}|_{\overline{g}}^2)dA_{\overline{g}}.$$

*Proof.* The lemma follows from the well-known formula

$$\frac{d}{dt}(dA_{\overline{g}}) = \frac{1}{2}\mathrm{tr}_{(\overline{g}_{ij}(t))}\Big(\frac{\partial}{\partial t}\overline{g}_{ij}\Big)dA_{\overline{g}}$$

and equation (2.14) in Proposition 2.12 (see also Remark 2.13). □

**Theorem 2.20.** *Let $(M, g)$ be an $n$-dimensional Riemannian manifold, and let $\Sigma$ be an $(n-1)$-dimensional compact smooth manifold without boundary. Consider $\mathscr{F}$ the MCF of $\Sigma$ in the $(\overline{g}, \overline{\phi})-(RH)_\alpha$ flow background with potential function $\overline{f}$. Denote by $dA_{\overline{g}}$ the $(n-1)$-dimensional Riemannian measure on $\Sigma$ and set $\mathrm{Area}_{\overline{f}}(\Sigma_t) := \int_\Sigma e^{-\overline{f}}dA_{\overline{g}}$. Under these conditions, the function $\Phi(t)$, given by:*

*(i)* $\mathbb{R} \ni t \mapsto \mathrm{Area}_{\overline{f}}(\Sigma_t)$ *in the steady case,*

*(ii)* $(-\infty, T) \ni t \mapsto [4\pi(T-t)]^{-(n-1)/2}\,\mathrm{Area}_{\overline{f}}(\Sigma_t)$ *in the shrinking case, and*

*(iii)* $(T, \infty) \ni t \mapsto [4\pi(t-T)]^{-(n-1)/2}\,\mathrm{Area}_{\overline{f}}(\Sigma_t)$ *in the expanding case,*

*is non-increasing. Moreover, $\Phi(t)$ is constant if and only if $\mathscr{F}$ is a family of mean curvature solitons.*

*Proof.* Lemma 2.19 and a straightforward computation yield

$$\frac{d}{dt}\int_{\Sigma_t} e^{-\overline{f}}dA_{\overline{g}} = -\int_{\Sigma_t}\Big(\frac{d}{dt}\overline{f} + \overline{R}^i_i + H^2_{\overline{g}} - \alpha|\widehat{\nabla}_{\overline{g}}\overline{\phi}|^2_{\overline{g}}\Big)e^{-\overline{f}}dA_{\overline{g}}.$$

By chain rule $\frac{d}{dt}\overline{f} = \frac{\partial}{\partial t}\overline{f}\frac{dt}{dt} + \overline{g}(t)(\nabla_{\overline{g}(t)}\overline{f}, \frac{\partial x}{\partial t})$ that implies

$$\frac{d}{dt}\int_{\Sigma_t} e^{-\overline{f}}dA_{\overline{g}} = -\int_{\Sigma_t}\Big(\frac{\partial}{\partial t}\overline{f} + H_{\overline{g}}e_t\overline{f} + \overline{R}^i_i + H^2_{\overline{g}} - \alpha|\widehat{\nabla}_{\overline{g}}\overline{\phi}|^2_{\overline{g}}\Big)e^{-\overline{f}}dA_{\overline{g}}.$$

First, consider a steady $(\overline{g}(t), \overline{\phi}(t)) - (RH)_\alpha$ flow background. In this case, we can take traces in the first equation of (2.25) on $\Sigma_t$ to get

$$0 = \overline{R}^i_i + \overline{\nabla}_i\overline{\nabla}^i\overline{f} - \alpha|\widehat{\nabla}_{\overline{g}}\overline{\phi}|^2_{\overline{g}} = \overline{R}^i_i + \widehat{\nabla}^i\widehat{\nabla}_i\overline{f} - H_{\overline{g}}e_t\overline{f} - \alpha|\widehat{\nabla}_{\overline{g}}\overline{\phi}|^2_{\overline{g}}.$$

Then, using (2.26), we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\Sigma_t} e^{-\overline{f}}dA_{\overline{g}} &= -\int_{\Sigma_t}\Big(|\nabla_{\overline{g}}\overline{f}|^2_{\overline{g}} - \widehat{\Delta}_{\overline{g}}\overline{f} + 2H_{\overline{g}}e_t\overline{f} + H^2_{\overline{g}}\Big)e^{-\overline{f}}dA_{\overline{g}}\\ &= -\int_{\Sigma_t}\Big(|\widehat{\nabla}_{\overline{g}}\overline{f}|^2_{\overline{g}} + (e_t\overline{f})^2 - \widehat{\Delta}_{\overline{g}}\overline{f} + 2H_{\overline{g}}e_t\overline{f} + H^2_{\overline{g}}\Big)e^{-\overline{f}}dA_{\overline{g}}\\ &= -\int_{\Sigma_t}\Big(H_{\overline{g}} + e_t\overline{f}\Big)^2 e^{-\overline{f}}dA_{\overline{g}},\end{aligned}$$

where in the second line we have used the equality

$$\widehat{\Delta}_{\overline{g}}e^{-\overline{f}} = (|\widehat{\nabla}_{\overline{g}}\overline{f}|^2_{\overline{g}} - \widehat{\Delta}_{\overline{g}}\overline{f})e^{-\overline{f}}$$

and Stokes' theorem. Since the boundary integrand in the right-hand side is nonnegative, we have immediately the result of the theorem for the steady case.

For the shrinking case, we claim that the function

$$(-\infty, T) \ni t \mapsto [4\pi(T-t)]^{-(n-1)/2} \int_{\Sigma_t} e^{-\overline{f}} dA_{\overline{g}}$$

is non-increasing during the flow. Indeed, as above, we take traces in the first equation of (2.25) on $\Sigma_t$ to obtain

$$\frac{n-1}{2(T-t)} = \overline{R}_i^i + \overline{\nabla}^i \overline{\nabla}_i \overline{f} - \alpha |\widehat{\nabla}_{\overline{g}} \overline{\phi}|^2_{\overline{g}} = \overline{R}_i^i + \widehat{\nabla}^i \widehat{\nabla}_i \overline{f} - H_{\overline{g}} e_t \overline{f} - \alpha |\widehat{\nabla}_{\overline{g}} \overline{\phi}|^2_{\overline{g}}.$$

Then,

$$\begin{aligned}
&\frac{d}{dt}\Big([4\pi(T-t)]^{-(n-1)/2} \int_{\Sigma_t} e^{-\overline{f}} dA_{\overline{g}}\Big) \\
&= -[4\pi(T-t)]^{-(n-1)/2} \int_{\Sigma_t} \Big(|\widehat{\nabla}_{\overline{g}} \overline{f}|^2_{\overline{g}} + (e_t \overline{f})^2 - \widehat{\Delta}_{\overline{g}} \overline{f} + 2H_{\overline{g}} e_t \overline{f} + H^2_{\overline{g}} \\
&\quad + \frac{n-1}{2(T-t)}\Big) e^{-\overline{f}} dA_{\overline{g}} + \frac{n-1}{2}[4\pi(T-t)]^{-\frac{(n-1)}{2}-1} \int_{\Sigma_t} e^{-\overline{f}} dA_{\overline{g}} \\
&= -[4\pi(T-t)]^{-(n-1)/2} \int_{\Sigma_t} \Big(H_{\overline{g}} + e_t \overline{f}\Big)^2 e^{-\overline{f}} dA_{\overline{g}}.
\end{aligned} \tag{2.27}$$

This proves the claim, and so the theorem for the shrinking case. Finally, in a similar way, one proves the expanding case. □

**Remark 2.21.** For the shrinking case in Theorem 2.20, we recover Huisken's monotonicity formula [Hui90, Thm. 3.1], by taking $M = \mathbb{R}^n$, $g_{ij}(\tau) = \delta_{ij}$, $\overline{f}(x, \tau) = |x|^2/(4\tau)$ and $\overline{\phi}(\tau) = \phi$ constant.

**Remark 2.22.** We recover Huisken-type monotonicity formulas [MMT13, Prop. 3.1] for hypersurface case and [Lot12, Prop. 8 and Rmk. 5] by taking $\overline{\phi}(\tau) = \phi$ constant and by taking $\overline{\phi} \in C^\infty(M, \mathbb{R})$ recover [GH23, Thm. 3]

## 2.4 Extension of Hamilton's differential Harnack expression

Here, we will see as the boundary integrand term of the time-derivative of the functional $\mathcal{F}^\alpha_\infty$ provides an extension of Hamilton's differential Harnack expression for mean curvature flow in Euclidean space to the more general context of mean curvature flow in a $(RH)_\alpha$ flow.

Let $\mathscr{F}$ be a family of mean curvature solitons in the steady $(\overline{g}, \overline{\phi}) - (RH)_\alpha$ flow background. Then, the equations

$$\overline{R}_{\hat{i}\hat{j}} + \overline{\nabla}^2 \overline{f}(\partial_{\hat{i}}, \partial_{\hat{j}}) - \alpha \gamma_{\alpha\beta} \overline{\nabla}_{\hat{i}} \overline{\phi}^\alpha \overline{\nabla}_{\hat{j}} \overline{\phi}^\beta = 0 \quad \text{and} \quad \overline{R}_{\hat{i}0} + \overline{\nabla}^2 \overline{f}(\partial_{\hat{i}}, e_0) - \alpha \gamma_{\alpha\beta} \overline{\nabla}_{\hat{i}} \overline{\phi}^\alpha \overline{\nabla}_0 \overline{\phi}^\beta = 0$$

on $\Sigma_t$ become

$$\overline{R}_{\hat{i}\hat{j}} + \widehat{\nabla}^2\overline{f}(\partial_{\hat{i}}, \partial_{\hat{j}}) + H_{\overline{g}}\mathcal{A}_{\hat{i}\hat{j}} - \alpha\gamma_{\alpha\beta}\widehat{\nabla}_{\hat{i}}\overline{\phi}^{\alpha}\widehat{\nabla}_{\hat{j}}\overline{\phi}^{\beta} = 0, \tag{2.28}$$

and

$$\overline{R}_{\hat{i}0} - \widehat{\nabla}_{\hat{i}}H_{\overline{g}} + \mathcal{A}^{\hat{k}}_{\hat{i}}\widehat{\nabla}_{\hat{k}}\overline{f} - \alpha\gamma_{\alpha\beta}\nabla_0\overline{\phi}^{\beta}\widehat{\nabla}_{\hat{i}}\overline{\phi}^{\alpha} = 0. \tag{2.29}$$

**Example 2.23.** For instance, consider $M = \mathbb{R}^n$, $\overline{g}(t) = g_{st}$ and $\overline{\phi}(t) = \phi$ constant, and let $L$ be a linear function on $\mathbb{R}^n$. Defining $\overline{f} = L + t|\nabla L|^2$, we have that $\overline{f}$ satisfies (2.26). Changing $\overline{f}$ to $-f$, equations (2.28) and (2.29) then become

$$\widehat{\nabla}^2\overline{f}(\partial_i, \partial_j) - H\mathcal{A}_{ij} = 0 \quad \text{and} \quad \widehat{\nabla}_i H + \mathcal{A}^k_i\widehat{\nabla}_k f = 0,$$

respectively, which appear in [Ham95b, p. 219] as equations for a translating soliton.

Consider a bounded domain $\Omega$ with smooth boundary $\partial\Omega := \Sigma$ in Euclidean space $\mathbb{R}^n$, and take a solution $u = e^{-f}$ to the conjugate heat equation (2.23) in $\Omega \times [0, T)$ with $e_0 u = Hu$ on $\Sigma$. If $\mathscr{F}$ is a MCF in a $(g(t), \phi(t)) - (RH)_\alpha$ flow background with $g(t)$ Ricci flat and $\nabla_0\phi = 0$ on $\Sigma$, then the boundary integrand in Theorem 2.15 becomes

$$\mathcal{Z}(V) + \alpha\mathcal{A}(\widehat{\nabla}\phi, \widehat{\nabla}\phi),$$

where $V = -\widehat{\nabla}f$ and $\mathcal{Z}(V) := \frac{\partial H}{\partial t} + 2\langle V, \widehat{\nabla}H\rangle + \mathcal{A}(V, V)$ is Hamilton's differential Harnack expression for the case of MCF in Euclidean space, which vanishes in the particular case $\mathscr{F}$ is a translating soliton (see [Ham95b, Def. 4.1 and Lem. 3.2]).

The next result suggests an extension $\mathcal{Z}^{\alpha}_{\overline{g},\overline{\phi}}$ of $\mathcal{Z}$ for the more general case of MCF in a $(RH)_\alpha$ flow background, whose characterization of nullity should be on the steady case. For this, we observe that, if we consider a steady $(\overline{g}(t), \overline{\phi}(t)) - (RH)_\alpha$ flow background on $M$ with potential function $\overline{f}$, and $\Sigma$ is a mean curvature soliton at $t = 0$, then its ensuing mean curvature flow $\{\Sigma_t\}$ consists of mean curvature solitons, and $\{\Sigma_t\}$ differs from $\{\psi_t(\Sigma)\}$ by diffeomorphisms. In Section 2.3, we give a more general description that includes the shrinking and expanding soliton cases.

**Corollary 2.24.** *Let $M$ be an $n$-dimensional smooth manifold, and let $(\overline{g}(t), \overline{\phi}(t))$ be a steady self-similar solution to the $(RH)_\alpha$ flow on $M$ with potential function $\overline{f}$ and initial value $(g, \phi)$. Given an $(n-1)$-dimensional compact smooth manifold $\Sigma$ without boundary, and let $\mathscr{F}$ be a MCF of $\Sigma$ in the steady $(\overline{g}, \overline{\phi}) - (RH)_\alpha$ flow background which satisfies $H + e_0 f = 0$ and $\nabla_0\phi = 0$ on $\Sigma$. Under these conditions, the identity*

$$\mathcal{Z}(-\widehat{\nabla}_{\overline{g}}\overline{f}) + 2\overline{R}^{0\hat{i}}\widehat{\nabla}_{\hat{i}}\overline{f} - \frac{1}{2}\overline{\nabla}_0\overline{R} - H_{\overline{g}}\overline{R}_{00} + \alpha\mathcal{A}(\widehat{\nabla}_{\overline{g}}\overline{\phi}, \widehat{\nabla}_{\overline{g}}\overline{\phi}) = 0$$

*holds for all $t \in [0, T)$, where $\mathcal{A}$ is the second fundamental form of $\Sigma$, and $\widehat{\nabla}$ denotes the gradient on $\Sigma$.*

*Proof.* If $(\overline{g}(t), \overline{\phi}(t))$ is a gradient steady soliton on $M \times [0,T)$, then the positive function $u = e^{-\overline{f}(t)}$ on $\bigcup_{t\in[0,T)}(X_t \times \{t\}) \subset M \times [0,T)$ satisfies the conjugated heat equation (2.18) with $e_0 u = Hu$ and $\nabla_0 \phi = 0$ on $\partial X_0 = \Sigma$, where the boundary conditions follows from the assumptions on $\Sigma$. To see this, first observe that $\Delta_{\overline{g}} u = (|\nabla_{\overline{g}}\overline{f}|^2_{\overline{g}} - \Delta_{\overline{g}}\overline{f})u$. Now taking traces in the first equation of (2.25) and using (2.26), we obtain

$$\frac{\partial}{\partial t}u = -u|\nabla_{\overline{g}}\overline{f}|^2_{\overline{g}} = -\Delta_{\overline{g}}u + R_{\overline{g}}u - \alpha|\nabla_{\overline{g}}\overline{\phi}|^2_{\overline{g}}u.$$

Thus, we can define $\widetilde{g}(t)$, $\widetilde{\phi}(t)$ and $\widetilde{f}(t)$ on $X_0$ as in the proof of Proposition 2.12, so that $(\widetilde{g}(t), \widetilde{\phi}(t))$ evolves by modified $(RH)_\alpha$ flow on $X_0 \times [0,T)$. Besides, again we use that $(\overline{g}(t), \overline{\phi}(t))$ is a gradient steady soliton and that $\nabla_0 \phi = 0$ on $\Sigma$, to get

$$\big(\widetilde{R}_{\hat{i}\hat{j}} + \widetilde{\nabla}^2\widetilde{f}(\partial_{\hat{i}}, \partial_{\hat{j}}) - \alpha\gamma_{\alpha\beta}\widetilde{\nabla}_{\hat{i}}\widetilde{\phi}^\alpha\widetilde{\nabla}_{\hat{j}}\widetilde{\phi}^\beta\big)|_\Sigma = 0 \quad \text{and} \quad \big(\widetilde{R}_{\hat{i}0} + \widetilde{\nabla}^2\widetilde{f}(\partial_{\hat{i}}, e_0)\big)|_\Sigma = 0.$$

As in the proof of Theorem 2.15, the result of the corollary follows from Corollary 2.11 and the identity

$$\frac{\partial}{\partial t}H_{\widetilde{g}} = \frac{\partial}{\partial t}H_{\overline{g}} - \langle \widehat{\nabla}_{\overline{g}}\overline{f}, \widehat{\nabla}_{\overline{g}}H_{\overline{g}}\rangle_{\overline{g}}.$$

This completes the proof. □

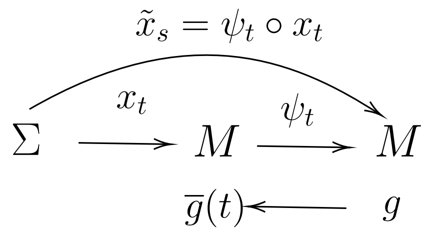
**Remark 2.25.** Suppose $M = \mathbb{R}^n$, $\overline{g}(t) = g_{st}$ and $\overline{\phi}(t) = \phi$ constant. Let $L$ be a linear function on $\mathbb{R}^n$ and define $\overline{f} = L + t|\nabla L|^2$. Letting $V(t) = -\widehat{\nabla}\overline{f}$, Corollary 2.24 coincides with [Ham95b, Lem. 3.2].

# CHAPTER 3

# Applications of a Huisken monotonicity-type formula for MCF in the $(RH)_\alpha$ flow background

In this chapter, we prove our fourth main theorem, which is an application of Huisken's monotonicity-type formula in the $(g(t), \phi(t)) - (RH)_\alpha$ flow background. The most appropriate setting to do this is to consider the normalized family $\widetilde{\mathscr{F}}$ of MCF as follows.

Assume that $(M, g)$ is an $n$-dimensional Riemannian manifold, and let $(\overline{g}(t), \overline{\phi}(t))$ be a shrinking self-similar solution to the $(RH)_\alpha$ flow on $M$ with potential function $\overline{f}$ and initial value $(g, \phi)$. Given an $(n-1)$-dimensional compact smooth manifold $\Sigma$ without boundary, and let $\mathscr{F}$ be a MCF of $\Sigma$ in the shrinking $(\overline{g}, \overline{\phi}) - (RH)_\alpha$ flow background.




Write $H_{\overline{g}} = H(p, t)$, $e_t = e(p, t)$ and $\widetilde{x}_s = \psi_t \circ x_t$, for simplicity. Setting $s = -\log(T - t)$ and $\widetilde{\Sigma}_s = \big(\Sigma, \widetilde{x}_s^* g\big)$, we have $s \in [-\log T, \infty)$, $\frac{ds}{dt} = \frac{1}{T-t}$ and $\frac{dt}{ds} = T - t$. Since $H_{\overline{g}} = \frac{1}{\sqrt{T-t}} H_{\psi_t^* g}$ and $e_t = \frac{1}{\sqrt{T-t}} e_{\psi_t^* g}$, we obtain, on $\widetilde{\Sigma}_s$,

$$\begin{aligned}\frac{\partial \tilde{x}_s}{\partial s} &= \left(\frac{d\psi_t}{dt} \circ x_t + (\psi_t)_* \Big(\frac{\partial x_t}{\partial t}\Big)\right) \frac{dt}{ds} \\ &= \left(\frac{\nabla_g f \circ \psi_t}{T - t} \circ x_t + (\psi_t)_* H_{\overline{g}} e_t\right) \frac{dt}{ds} \\ &= (\nabla_g f + H_g)(\widetilde{x}_s).\end{aligned}$$

We call the family $\widetilde{\mathscr{F}} := \{\widetilde{\Sigma}_s; s \in [-\log T, \infty)\}$ the *normalized MCF* in $(M, g)$.

Now, we proceed with the proofs of preliminary results, which are formulated in the more general context of complete Riemannian manifolds with bounded geometry, so that we can work on compact and complete noncompact Riemannian manifolds by using convergence techniques that have been presented in Chapter 1.

## 3.1 Bounded geometry in the $(RH)_\alpha$ flow background

In what follows, when $(g, \phi)$ is the initial value of a self-similar solution $(\overline{g}(t), \overline{\phi}(t))$ to the $(RH)_\alpha$ flow on a noncompact smooth manifold $M$, we are assuming that there exist positive constants $C'_j$ such that

$$|\nabla_g^j(\nabla\phi \otimes \nabla\phi)| \leqslant C'_j \quad \text{on} \quad M, \tag{3.1}$$

for every integer $j \geqslant 0$. Such a condition is natural since it is just necessary on initial condition of the flow (see estimate (3.4)). The motivation for this additional assumption can be seen in Müller's work [Mül12]. Of course, in the compact case, (3.1) is trivially satisfied.

In order to apply the Arzelá-Ascoli theorem, we need a type of interior estimate for $|\widehat{\nabla}_g^k \mathcal{A}(\widetilde{x}_s)|$ on $\widetilde{\Sigma}_s \in \widetilde{\mathscr{F}}$ as done by Ecker and Huisken for MCF in Euclidean space, by Yamamoto for MCF in a Ricci flow background, and by Gomes, Hudson and Yamamoto for MCF in an extended Ricci soliton background.

**Proposition 3.1.** *Assume that $(M, g)$ is an $n$-dimensional Riemannian manifold with bounded geometry, and let $(\overline{g}(t), \overline{\phi}(t))$ be a shrinking self-similar solution to the $(RH)_\alpha$ flow on $M$ with potential function $\overline{f}$ and initial value $(g, \phi)$ satisfying (3.1). Given an $(n-1)$-dimensional compact smooth manifold $\Sigma$ without boundary, and let $\mathscr{F}$ be the MCF of $\Sigma$ in the shrinking $(\overline{g}, \overline{\phi}) - (RH)_\alpha$ flow background which develops a singularity of type-I. Consider the associate normalized MCF $\widetilde{\mathscr{F}}$ in $(M, g)$. Then, for every integer $k \geqslant 0$ there exist positive constants $C_k$ such that*

$$|\widehat{\nabla}_g^k \mathcal{A}(\widetilde{x}_s)|_g \leqslant C_k \text{ on } \Sigma \times [-\log T, \infty),$$

*where $\widehat{\nabla}_g$ is defined from the Levi–Civita connection on $\widetilde{\Sigma}_s$.*

*Proof.* The proof follows a standard approach as in [GHY24, Hui90, Mül12, Yam20], which is by induction on $k$. First of all, since $\overline{g} = (T-t)\psi_t^* g$ and $\widetilde{x}_s = \psi_t \circ x_t$, one has

$$|\widehat{\nabla}_g^k \mathcal{A}_g(\widetilde{x}_s)|_g = (T-t)^{\frac{1}{2}+\frac{1}{2}k} |\widehat{\nabla}_{\overline{g}}^k \overline{\mathcal{A}}_{\overline{g}}(x_t)|_{\overline{g}}, \tag{3.2}$$

$$|\nabla_g^k \mathrm{Rm}_g|_g = (T-t)^{1+\frac{1}{2}k} |\nabla_{\overline{g}}^k \mathrm{Rm}_{\overline{g}}|_{\overline{g}}, \tag{3.3}$$

$$|\nabla_g^k \big(\nabla\phi \otimes \nabla\phi\big)|_g = (T-t)^{1+\frac{1}{2}k} |\nabla_{\overline{g}}^k \big(\nabla\overline{\phi} \otimes \nabla\overline{\phi}\big)|_{\overline{g}}, \tag{3.4}$$

where $\nabla_g$, $\widehat{\nabla}_g$, $\nabla_{\overline{g}}$ and $\widehat{\nabla}_{\overline{g}}$ are defined from the Levi–Civita connection on $(M, g)$, $(\Sigma, \widetilde{x}_s^* g)$, $(M, \overline{g})$ and $(\Sigma, x_t^* \overline{g})$, respectively. Thus, the degree of $\widehat{\nabla}_{\overline{g}}^k \mathcal{A}_{\overline{g}}$ is $\frac{1}{2} + \frac{1}{2}k$ and of $\nabla_{\overline{g}}^k \mathrm{Rm}_{\overline{g}}$ and

$\nabla^k_{\overline{g}}\big(\nabla\overline{\phi}\otimes\nabla\overline{\phi}\big)$ are $1+\frac{1}{2}k$. Also, we write $\mathcal{A}_{\overline{g}}(x_t)$ and $\mathcal{A}_g(\widetilde{x}_s)$ by $\overline{\mathcal{A}}$ and $\mathcal{A}$, respectively, while $\mathrm{Rm}_{\overline{g}}$ and $\mathrm{Rm}_g$ by $\overline{\mathrm{Rm}}$ and $\mathrm{Rm}$, respectively.

For tensors $\mathcal{T}_1$ and $\mathcal{T}_2$, we write $\mathcal{T}_1 * \mathcal{T}_2$ to mean a tensor formed by a sum of terms each one of them obtained by contracting some indices of the pair $\mathcal{T}_1$ and $\mathcal{T}_2$ by using $g, x^*g$, in particular,

$$|\mathcal{T}_1 * \mathcal{T}_2| \leqslant C|\mathcal{T}_1||\mathcal{T}_2|$$

where $C>0$ is a constant which depends only on the algebraic structure of $\mathcal{T}_1 * \mathcal{T}_2$. For $a,b\in\mathbb{Q}$, we consider $V_{a,b}$ the set of all (time-dependent) tensors $\mathcal{T}$ on $M$ which can be expressed as

$$\mathcal{T} = (\nabla^{k_1}_{\overline{g}}\overline{\mathrm{Rm}} * \cdots * \nabla^{k_I}_{\overline{g}}\overline{\mathrm{Rm}}) * \big(\nabla^{m_1}_{\overline{g}}(\alpha\nabla\overline{\phi}\otimes\nabla\overline{\phi}) * \cdots * \nabla^{m_K}_{\overline{g}}(\alpha\nabla\overline{\phi}\otimes\nabla\overline{\phi})\big) * (\widehat{\nabla}^{\ell_1}_{\overline{g}}\overline{\mathcal{A}} * \\ \cdots * \widehat{\nabla}^{\ell_J}_{\overline{g}}\overline{\mathcal{A}}) * (\overset{p}{*} Dx)$$

with $I, J, K, p, k_1,\dots,k_I, m_1,\dots,m_K, \ell_1,\dots,\ell_J \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{I}\left(1+\frac{1}{2}k_i\right) + \sum_{k=1}^{K}\left(1+\frac{1}{2}m_k\right) + \sum_{j=1}^{J}\left(\frac{1}{2}+\frac{1}{2}\ell_j\right) = a \text{ and } \sum_{j=1}^{J}\ell_j \leqslant b,$$

and we define a vector space $\mathcal{V}_{a,b}$ as the set of all tensors $\mathcal{T}$ on $M$ which can be expressed as $\mathcal{T} = a_1\mathcal{T}_1 + \cdots + a_r\mathcal{T}_r$ for some $r\in\mathbb{N}$, $a_1,\dots,a_r\in\mathbb{R}$ and $\mathcal{T}_1,\dots,\mathcal{T}_r\in V_{a,b}$.

The first step for the induction follows from the singularity of type-I assumption (see (1.6)). For a fixed $k\geqslant 1$, assume that there exist positive constants $C_0, C_1,\dots,C_{k-1}$ such that

$$|\widehat{\nabla}^i_g\mathcal{A}| \leqslant C_i \text{ on } \Sigma\times[-\log T,\infty)$$

for $i=0,1,\dots,k-1$. We consider the evolution equation of $|\widehat{\nabla}^k_g\mathcal{A}|^2$, and finally we will prove the bound of $|\widehat{\nabla}^k_g\mathcal{A}|^2$ by parabolic maximum principle. $|\widehat{\nabla}^k_g\mathcal{A}|^2 = (T-t)^{k+1}|\widehat{\nabla}^k_{\overline{g}}\overline{\mathcal{A}}|^2$ from (3.2) and $\frac{\partial}{\partial s} = (T-t)\frac{\partial}{\partial t}$,

$$\frac{\partial}{\partial s}|\widehat{\nabla}^k_g\mathcal{A}|^2 = -(k+1)|\widehat{\nabla}^k_g\mathcal{A}|^2 + (T-t)^{k+2}\frac{\partial}{\partial t}|\widehat{\nabla}^k_{\overline{g}}\overline{\mathcal{A}}|^2 \leqslant (T-t)^{k+2}\frac{\partial}{\partial t}|\widehat{\nabla}^k_{\overline{g}}\overline{\mathcal{A}}|^2.$$

As used in the proof of [Yam20, Prop. 4.9], there exist tensors $\mathcal{E}[k]\in\mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}$, $\mathcal{C}[k]\in\mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k+1}$ and $\mathcal{G}[k]\in\mathcal{V}_{\frac{1}{2}+\frac{1}{2}k,k-1}$ such that

$$\frac{\partial}{\partial t}|\widehat{\nabla}^k_{\overline{g}}\overline{\mathcal{A}}|^2 = \widehat{\Delta}_{\overline{g}}|\widehat{\nabla}^k_{\overline{g}}\overline{\mathcal{A}}|^2 - 2|\widehat{\nabla}^{k+1}_{\overline{g}}\overline{\mathcal{A}}|^2 + \mathcal{E}[k]*\widehat{\nabla}^k_{\overline{g}}\overline{\mathcal{A}} + \mathcal{C}[k]*\mathcal{G}[k],$$

where $\widehat{\Delta}_{\overline{g}}$ is the Laplacian on $\big(\Sigma, x_t^*\overline{g}(t)\big)$. Setting $\widehat{\Delta}_g$ for the Laplacian on $\big(\Sigma,\widetilde{x}_s^*g\big)$, one has $(T-t)\widehat{\Delta}_{\overline{g}} = \widehat{\Delta}_g$. Hence

$$(T-t)^{k+2}(\widehat{\Delta}_{\overline{g}}|\widehat{\nabla}^k_{\overline{g}}\overline{\mathcal{A}}|^2 - 2|\widehat{\nabla}^{k+1}_{\overline{g}}\overline{\mathcal{A}}|^2) = \widehat{\Delta}_g|\widehat{\nabla}^k_g\mathcal{A}|^2 - 2|\widehat{\nabla}^{k+1}_g\mathcal{A}|^2.$$

Since $\mathcal{G}[k] \in \mathcal{V}_{\frac{1}{2}+\frac{1}{2}k,k-1}$, there exist $r \in \mathbb{N}$, $a_1, \dots, a_r \in \mathbb{R}$ and $\mathcal{G}[k]_1, \dots, \mathcal{G}[k]_r \in V_{\frac{1}{2}+\frac{1}{2}k,k-1}$ such that $\mathcal{G}[k] = \sum_{i=1}^r a_i \mathcal{G}[k]_i$. Hence, $|\mathcal{G}[k]| \leqslant \sum_{i=1}^r |a_i||\mathcal{G}[k]_i|$. By definition of $V_{\frac{1}{2}+\frac{1}{2}k,k-1}$, each $\mathcal{G}[k]_i$ can be expressed as

$$(\nabla_{\overline{g}}^{k_1} \overline{\mathrm{Rm}} * \cdots * \nabla_{\overline{g}}^{k_I} \overline{\mathrm{Rm}}) * \big(\nabla_{\overline{g}}^{m_1}(\alpha \nabla \overline{\phi} \otimes \nabla \overline{\phi}) * \cdots * \nabla_{\overline{g}}^{m_K}(\alpha \nabla \overline{\phi} \otimes \nabla \overline{\phi})\big) * (\widehat{\nabla}_{\overline{g}}^{\ell_1} \overline{\mathcal{A}} * \\ \cdots * \widehat{\nabla}_{\overline{g}}^{\ell_J} \overline{\mathcal{A}}) * (\overset{p}{*} Dx)$$

with $I, J, K, p, k_1, \dots, k_I, m_1, \dots, m_K, \ell_1, \dots, \ell_J \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{I} \left(1 + \frac{1}{2} k_i\right) + \sum_{k=1}^{K} \left(1 + \frac{1}{2} m_k\right) + \sum_{j=1}^{J} \left(\frac{1}{2} + \frac{1}{2} \ell_j\right) = \frac{1}{2} + \frac{1}{2} k \text{ and } \sum_{j=1}^{J} \ell_j \leqslant k - 1.$$

Hence, by using (3.2), (3.3) and (3.4),

$$\begin{aligned}(T-t)^{\frac{1}{2}+\frac{1}{2}k} |\mathcal{G}[k]_i| \leqslant & C (T-t)^{\frac{1}{2}+\frac{1}{2}k} |\nabla_{\overline{g}}^{k_1} \overline{\mathrm{Rm}}| \cdots |\nabla_{\overline{g}}^{k_I} \overline{\mathrm{Rm}}||\nabla_{\overline{g}}^{m_1}(\nabla \overline{\phi} \otimes \nabla \overline{\phi})| \cdots \\ & |\nabla_{\overline{g}}^{m_K}(\nabla \overline{\phi} \otimes \nabla \overline{\phi})||\widehat{\nabla}_{\overline{g}}^{\ell_1} \overline{\mathcal{A}}| \cdots |\widehat{\nabla}_{\overline{g}}^{\ell_J} \overline{\mathcal{A}}||Dx|^p \\ = & C(\sqrt{n-1})^p |\nabla_g^{k_1} \mathrm{Rm}| \cdots |\nabla_g^{k_I} \mathrm{Rm}||\nabla^{m_1}(\nabla \phi \otimes \nabla \phi)| \cdots \\ & |\nabla^{m_K}(\nabla \phi \otimes \nabla \phi)||\nabla_g^{\ell_1} \mathcal{A}| \cdots |\nabla_g^{\ell_J} \mathcal{A}|\end{aligned}$$

for some constant $C > 0$. Here note that $|Dx| = \sqrt{n-1}$. Since $(M, g)$ has bounded geometry and we are assuming (3.1), the derivatives $|\nabla_g^{k_i} \mathrm{Rm}|$ and $|\nabla_g^{m_k}(\nabla \phi \otimes \nabla \phi)|$ are bounded. Furthermore, since $\ell_j \leqslant k-1$, each $|\nabla_g^{\ell_j} \mathcal{A}|$ is bounded by assumption of induction. So, there exists a constant $C' > 0$ such that

$$(T-t)^{\frac{1}{2}+\frac{1}{2}k} |\mathcal{G}[k]| \leqslant C'.$$

In the same way, since $\mathcal{E}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}$, there exist $r' \in \mathbb{N}$, $b_1, \dots, b_{r'} \in \mathbb{R}$ and $\mathcal{E}[k]_1, \dots, \mathcal{E}[k]_{r'} \in V_{\frac{3}{2}+\frac{1}{2}k,k}$ such that $\mathcal{E}[k] = \sum_{i=1}^{r'} b_i \mathcal{E}[k]_i$.

Hence, $|\mathcal{E}[k]| \leqslant \sum_{i=1}^{r'} |b_i| \, |\mathcal{E}[k]_i|$ . By definition of $V_{\frac{3}{2}+\frac{1}{2}k,k}$, each $\mathcal{E}[k]_i$ can be expressed as

$$(\nabla_{\overline{g}}^{k_1} \overline{\mathrm{Rm}} * \cdots * \nabla_{\overline{g}}^{k_I} \overline{\mathrm{Rm}}) * \big(\nabla_{\overline{g}}^{m_1}(\alpha \nabla \overline{\phi} \otimes \nabla \overline{\phi}) * \cdots * \nabla_{\overline{g}}^{m_K}(\alpha \nabla \overline{\phi} \otimes \nabla \overline{\phi})\big) * (\widehat{\nabla}_{\overline{g}}^{\ell_1} \overline{\mathcal{A}} * \\ \cdots * \widehat{\nabla}_{\overline{g}}^{\ell_J} \overline{\mathcal{A}}) * (\overset{p}{*} Dx)$$

with $I, J, K, p, k_1, \dots, k_I, m_1, \dots, m_K, \ell_1, \dots, \ell_J \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{I} \left(1 + \frac{1}{2} k_i\right) + \sum_{k=1}^{K} \left(1 + \frac{1}{2} m_k\right) + \sum_{j=1}^{J} \left(\frac{1}{2} + \frac{1}{2} \ell_j\right) = \frac{3}{2} + \frac{1}{2} k \text{ and } \sum_{j=1}^{J} \ell_j \leqslant k - 1.$$

If $\max\{\ell_1, \dots, \ell_J\} \leqslant k-1$, we can prove that $(T-t)^{\frac{3}{2}+\frac{1}{2}k} |\mathcal{E}[k]_i|$ is bounded by same argument as the case of $\mathcal{G}[k]_i$. If $\max\{\ell_1, \dots, \ell_J\} = k$, one can see that the possible forms of $\mathcal{E}[k]_i$ are

$$\begin{aligned} \overline{\mathcal{A}} * \overline{\mathcal{A}} * \widehat{\nabla}_{\overline{g}}^{k} \overline{\mathcal{A}} * (\overset{p}{*} Dx), \\ \overline{\mathrm{Rm}} * \widehat{\nabla}_{\overline{g}}^{k} \overline{\mathcal{A}} * (\overset{p}{*} Dx), \\ \nabla \overline{\phi} \otimes \nabla \overline{\phi} * \widehat{\nabla}_{\overline{g}}^{k} \overline{\mathcal{A}} * (\overset{p}{*} Dx). \end{aligned}$$

For each term, we can see by same argument as the case of $\mathcal{G}[k]_i$ that there exists a constant $\widetilde{C} > 0$ such that $(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{E}[k]_i| \leqslant \widetilde{C}|\widehat{\nabla}^k\mathcal{A}|$. Thus, there exists a constant $C'' > 0$ such that

$$(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{E}[k]| \leqslant C''(1+|\widehat{\nabla}^k\mathcal{A}|).$$

Since $\mathcal{C}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k+1}$, there exist $r'' \in \mathbb{N}, c_1, \dots, c''_r \in \mathbb{R}$ and $\mathcal{C}[k]_1, \dots, \mathcal{C}[k]_{r''} \in V_{\frac{3}{2}+\frac{1}{2}k,k+1}$ such that $\mathcal{C}[k] = \sum_{i=1}^{r'} c_i\mathcal{C}[k]_i$. Hence, one has $|\mathcal{C}[k]| \leqslant \sum_{i=1}^{r'}|c_i||\mathcal{C}[k]_i|$. By definition of $V_{\frac{3}{2}+\frac{1}{2}k,k+1}$, each $\mathcal{C}[k]_i$ can be expressed as

$$(\nabla_{\overline{g}}^{k_1}\overline{\mathrm{Rm}} * \cdots * \nabla_{\overline{g}}^{k_I}\overline{\mathrm{Rm}}) * \big(\nabla_{\overline{g}}^{m_1}(\alpha\nabla\overline{\phi}\otimes\nabla\overline{\phi}) * \cdots * \nabla_{\overline{g}}^{m_K}(\alpha\nabla\overline{\phi}\otimes\nabla\overline{\phi})\big) * (\widehat{\nabla}_{\overline{g}}^{\ell_1}\overline{\mathcal{A}} * \\ \cdots * \widehat{\nabla}_{\overline{g}}^{\ell_J}\overline{\mathcal{A}}) * (\overset{p}{*}Dx)$$

with $I, J, K, p, k_1, \dots, k_I, m_1, \dots, m_K, \ell_1, \dots, \ell_J \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{I}\left(1+\frac{1}{2}k_i\right) + \sum_{k=1}^{K}\left(1+\frac{1}{2}m_k\right) + \sum_{j=1}^{J}\left(\frac{1}{2}+\frac{1}{2}\ell_j\right) = \frac{3}{2}+\frac{1}{2}k \text{ and } \sum_{j=1}^{J}\ell_j \leqslant k+1.$$

If $\max\{\ell_1, ..., \ell_J\} \leqslant k-1$, we can prove that $(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{C}[k]_i|$ is bounded by same argument as the case of $\mathcal{G}[k]_i$ If $\max\{\ell_1, \dots, \ell_J\} = k$, one can see that the possible forms of $\mathcal{C}[k]_i$ are

$$\begin{aligned} \overline{\mathcal{A}} * \overline{\mathcal{A}} * \widehat{\nabla}_{\overline{g}}^k\overline{\mathcal{A}} * (\overset{p}{*}Dx), \\ \overline{\mathrm{Rm}} * \widehat{\nabla}_{\overline{g}}^k\overline{\mathcal{A}} * (\overset{p}{*}Dx), \\ \nabla\overline{\phi}\otimes\nabla\overline{\phi} * \widehat{\nabla}_{\overline{g}}^k\overline{\mathcal{A}} * (\overset{p}{*}Dx) \end{aligned}$$

and $(T-t)^{\frac{3}{2}+\frac{1}{2}k}\ |\mathcal{C}[k]_i| \leqslant \widetilde{C}|\widehat{\nabla}^k\mathcal{A}|$ as the case of $\mathcal{E}[k]_i$. If $\max\{\ell_1, \dots, \ell_J\} = k+1$, one can see that the possible form of $\mathcal{C}[k]_i$ is

$$\widehat{\nabla}_{\overline{g}}^{k+1}\overline{\mathcal{A}} * (\overset{p}{*}Dx),$$

and $(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{C}[k]_i| \leqslant \widetilde{C}'|\widehat{\nabla}^{k+1}\mathcal{A}|$ for some constant $\widetilde{C}' > 0$. Hence we can see that there exists a constant $C''' > 0$ such that

$$(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{C}[k]| \leqslant C'''(1+|\widehat{\nabla}^k\mathcal{A}|+|\widehat{\nabla}^{k+1}\mathcal{A}|).$$

Now, we compute

$$\begin{aligned} \frac{\partial}{\partial s}|\widehat{\nabla}^k\mathcal{A}|^2 \leqslant& (T-t)^{k+2}\frac{\partial}{\partial t}|\widehat{\nabla}_{\overline{g}}^k\overline{\mathcal{A}}|^2 \\ \leqslant& \widehat{\Delta}_g|\widehat{\nabla}^k\mathcal{A}|^2 - 2|\widehat{\nabla}^{k+1}\mathcal{A}|^2 + C''(1+|\widehat{\nabla}^k\mathcal{A}|)|\widehat{\nabla}^k\mathcal{A}| \\ &+ C'C'''(1+|\widehat{\nabla}^k\mathcal{A}|+|\widehat{\nabla}^{k+1}\mathcal{A}|). \end{aligned}$$

Since $-|\widehat{\nabla}^{k+1}\mathcal{A}|^2 + C'C'''|\widehat{\nabla}^{k+1}\mathcal{A}| \leqslant (C'C''')^2/4$, and then

$$\begin{aligned} \frac{\partial}{\partial s}|\widehat{\nabla}^k\mathcal{A}|^2 \leqslant& \widehat{\Delta}_g|\widehat{\nabla}^k\mathcal{A}|^2 - |\widehat{\nabla}^{k+1}\mathcal{A}|^2 \\ &+ C''|\widehat{\nabla}^k\mathcal{A}|^2 + (C''+C'C''')|\widehat{\nabla}^k\mathcal{A}| + C'C''' + (C'C''')^2/4. \end{aligned}$$

By putting $\overline{C}_k := C'' + (C'' + C'C''') + C'C''' + (C'C''')^2/4$, it result

$$\frac{\partial}{\partial s}|\widehat{\nabla}^k\mathcal{A}|^2 \leqslant \widehat{\Delta}_g|\widehat{\nabla}^k\mathcal{A}|^2 - |\widehat{\nabla}^{k+1}\mathcal{A}|^2 + \overline{C}_k(1 + |\widehat{\nabla}^k\mathcal{A}|^2). \tag{3.5}$$

Hence, immediately

$$\frac{\partial}{\partial s}|\widehat{\nabla}^k\mathcal{A}|^2 \leqslant \widehat{\Delta}_g|\widehat{\nabla}^k\mathcal{A}|^2 + \overline{C}_k(1 + |\widehat{\nabla}^k\mathcal{A}|^2). \tag{3.6}$$

Note that inequality (3.5) also holds for $k-1$, that is,

$$\frac{\partial}{\partial s}|\widehat{\nabla}^{k-1}\mathcal{A}|^2 \leqslant \widehat{\Delta}_g|\widehat{\nabla}^{k-1}\mathcal{A}|^2 - |\widehat{\nabla}^k\mathcal{A}|^2 + \overline{C}_{k-1}(1 + |\widehat{\nabla}^{k-1}\mathcal{A}|^2), \tag{3.7}$$

for some constant $\overline{C}_{k-1} > 0$. Hence by combining inequality (3.6) and (3.7),

$$\begin{aligned}\frac{\partial}{\partial s}(|\widehat{\nabla}^k\mathcal{A}|^2 + 2\overline{C}_k|\widehat{\nabla}^{k-1}\mathcal{A}|^2) &\leqslant \widehat{\Delta}_g(|\widehat{\nabla}^k\mathcal{A}|^2 + 2\overline{C}_k|\widehat{\nabla}^{k-1}\mathcal{A}|^2)\\ &\quad + \overline{C}_k - \overline{C}_k|\widehat{\nabla}^k\mathcal{A}|^2 + 2\overline{C}_k\overline{C}_{k-1}(1 + |\widehat{\nabla}^{k-1}\mathcal{A}|^2).(33)\end{aligned}$$

Since,

$$\begin{aligned}\overline{C}_k - \overline{C}_k|\widehat{\nabla}^k\mathcal{A}|^2 + 2\overline{C}_k\overline{C}_{k-1}(1 + |\widehat{\nabla}^{k-1}\mathcal{A}|^2) &= -\overline{C}_k(|\widehat{\nabla}^k\mathcal{A}|^2 + 2\overline{C}_k|\widehat{\nabla}^{k-1}\mathcal{A}|^2)\\ &\quad + \overline{C}_k(1 + 2\overline{C}_{k-1} + 2(\overline{C}_k + \overline{C}_{k-1})|\widehat{\nabla}^{k-1}\mathcal{A}|^2)\end{aligned}$$

and $|\widehat{\nabla}^{k-1}\mathcal{A}|^2$ is bounded by assumption of induction, one can see that there exists a constant $\overline{\overline{C}}_k > 0$ such that

$$\begin{aligned}\frac{\partial}{\partial s}(|\widehat{\nabla}^k\mathcal{A}|^2 + 2\overline{C}_k|\widehat{\nabla}^{k-1}\mathcal{A}|^2 - \overline{\overline{C}}_k) &\leqslant \widehat{\Delta}_g(|\widehat{\nabla}^k\mathcal{A}|^2 + 2\overline{C}_k|\widehat{\nabla}^{k-1}\mathcal{A}|^2 - \overline{\overline{C}}_k)\\ &\quad - \overline{C}_k(|\widehat{\nabla}^k\mathcal{A}|^2 + 2\overline{C}_k|\widehat{\nabla}^{k-1}\mathcal{A}|^2 - \overline{\overline{C}}_k).\end{aligned}$$

So, doing $\mu := e^{\overline{C}_k s}(|\widehat{\nabla}^k\mathcal{A}|^2 + 2\overline{C}_k|\widehat{\nabla}^{k-1}\mathcal{A}|^2 - \overline{\overline{C}}_k)$,

$$\frac{\partial}{\partial s}\mu \leqslant \widehat{\Delta}_g\mu.$$

Since $\Sigma$ is compact, $\mu$ is bounded at initial time $s = -\log T$, for the shrinking case. Then, by parabolic maximum principle, it follows that $\mu$ is also bounded on $\Sigma \times [-\log T, \infty)$, that is, there exists a constant $\widetilde{C}_k > 0$ such that $\mu \leqslant \widetilde{C}_k$ on $\Sigma \times [-\log T, \infty)$. Finally,

$$|\widehat{\nabla}^k\mathcal{A}|^2 \leqslant e^{-\overline{C}_k s}\widetilde{C}_k - 2\overline{C}_k|\widehat{\nabla}^{k-1}\mathcal{A}|^2 + \overline{\overline{C}}_k \leqslant C_k,$$

where $C_k := T^{\overline{C}_k}\widetilde{C}_k + \overline{\overline{C}}_k$. So, we have completed the proof. □

From now on consider $\mathcal{S} := \mathrm{Ric} - \alpha\nabla\phi \otimes \nabla\phi$ and its trace $S := R - \alpha|\nabla\phi|^2$. Keep in mind the following lemma.

**Lemma 3.2** (Müller [Mül12]). *Let $(g(t), \phi(t))$ be a solution to the $(RH)_\alpha$ flow with $\alpha(t) = \alpha$ for some positive constant $\alpha$. Then, $S$ satisfies the following evolution equation*

$$\frac{\partial S}{\partial t} = \Delta S + 2|\mathcal{S}|^2 + 2\alpha|\tau_g \phi|^2.$$

*In particular, $S$ is nonnegative along the gradient shrinking soliton to the $(RH)_\alpha$ flow on $M$.*

Müller also proved a Hamilton type equation for the $(RH)_\alpha$ flow as follows

$$S_g + |\nabla f|^2 - 2\lambda f = C,$$

for some constant $C$. For the shrinking case, we can assume (by rescaling of the metric $g$ and by adding $C$ to $f$, if necessary)

$$S_g + |\nabla f|^2 - f = 0.$$

Thus, if $\overline{g}(t) = \sigma(t)\psi_t^* g$, where $\sigma(t) = T - t$, then $S_{\psi_t^* g} + |\nabla_{\psi_t^* g}\overline{f}|^2 - \overline{f} = 0$, where $\bar{f} = \psi_t^* f$, and by conformal theory, $S_{\overline{g}} = S_{\psi_t^* g}/\sigma(t)$ and $|\nabla_{\overline{g}}\overline{f}| = |\nabla_{\psi_t^* g}\overline{f}|/\sigma(t)$. So, we obtain

$$S_{\overline{g}} + |\nabla_{\overline{g}}\overline{f}|^2 - \frac{\overline{f}}{T-t} = 0. \tag{3.8}$$

The following lemma provides the variation of the weighted Area-type functional with respect to the family $\widetilde{\mathscr{F}}$.

**Lemma 3.3.** *Let $(M, g)$ be an $n$-dimensional Riemannian manifold, and let $(\overline{g}(t), \overline{\phi}(t))$ be a shrinking self-similar solution to the $(RH)_\alpha$ flow on $M$ with potential function $\overline{f}$ and initial value $(g, \phi)$. Given an $(n-1)-$dimensional compact smooth manifold $\Sigma$ without boundary, and let $\mathscr{F}$ be the MCF of $\Sigma$ in the shrinking $(\overline{g}, \overline{\phi}) - (RH)_\alpha$ flow background. Consider the associate normalized MCF $\widetilde{\mathscr{F}}$ in $(M, g)$. Then*

$$\frac{d}{ds}\int_\Sigma e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g} = -\int_\Sigma \big(H_g(\widetilde{x}_s) + e_s(f\circ\widetilde{x}_s)\big)^2 e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g}, \;\; s \in [-\log T, +\infty).$$

*Proof.* As $\overline{f}\circ x_t = f\circ\widetilde{x}_s$ and $x_t^*\overline{g} = (T-t)\widetilde{x}_s^* g$ both on $\Sigma$, we have

$$(4\pi(T-t))^{-\frac{n-1}{2}} e^{-\overline{f}} dA_{\overline{g}} = (4\pi)^{-\frac{n-1}{2}} e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g}. \tag{3.9}$$

$$(T-t)\left(H_{\overline{g}} + e_t\overline{f}\right)^2 = \left(H_g(\widetilde{x}_s) + e_s(f\circ\widetilde{x}_s)\right)^2. \tag{3.10}$$

The lemma follows from equalities (2.27),(3.9), (3.10) and the chain rule. □

**Lemma 3.4.** *Assume that $(M, g)$ is an $n-$dimensional Riemannian manifold with bounded geometry, and let $(\overline{g}(t), \overline{\phi}(t))$ be a shrinking self-similar solution to the $(RH)_\alpha$ flow on $M$ with potential function $\overline{f}$ and initial value $(g, \phi)$ satisfying* (3.1). *Given an $(n-1)-$dimensional compact smooth manifold $\Sigma$ without boundary, and let $\mathscr{F}$ be the MCF of $\Sigma$ in the shrinking*

$(\overline{g},\overline{\phi}) - (RH)_\alpha$ *flow background which develops a singularity of type-I. Consider the associate normalized MCF* $\widetilde{\mathscr{F}}$ *in* $(M,g)$*. Then, there exists a constant* $C>0$ *such that*

$$\int_\Sigma e^{-\frac{f}{2}\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g} \leqslant C$$

*uniformly on* $[-\log T,\infty)$.

*Proof.* We begin by substituting $\bar{\rho}_t = \frac{1}{[4\pi(T-t)]^{\frac{n}{2}}} e^{-\frac{\overline{f}}{2}}$, $u_t = [4\pi(T-t)]^{1/2}$, $h = -2\operatorname{Ric}_{\overline{g}} + 2\alpha\nabla\overline{\phi}\otimes\nabla\overline{\phi}$ and $V = H_{\overline{g}} e_t$ into Prop. 3.2 of [Yam20] to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Sigma u_t\, x_t^*\bar{\rho}_t\, dA_{\overline{g}} \\
&= -\int_\Sigma u_t\Big(H_{\overline{g}} + \frac{1}{2}e_t\overline{f}\Big)^2 x_t^*\bar{\rho}_t\, dA_{\overline{g}} \\
&\quad + \int_\Sigma u_t x_t^* \left(\Delta_{\overline{g}}\bar{\rho}_t + \frac{\partial\bar{\rho}_t}{\partial t} - \bar{\rho}_t S_{\overline{g}}\right) dA_{\overline{g}} \\
&\quad + \int_\Sigma \left(\frac{\partial u_t}{\partial t} - \widehat{\Delta}_{\overline{g}} u_t + u_t\Big(\frac{1}{2}\operatorname{Hess}_{\overline{g}}\overline{f} - \frac{h}{2}\Big)(e_t,e_t)\right) x_t^*\overline{\rho}_t\, dA_{\overline{g}}.
\end{aligned}$$

By using $\Delta_{\overline{g}}\overline{f} = -S_{\overline{g}} + \frac{n}{2(T-t)}$ (take traces in (2.25)), $|\nabla\overline{f}|^2 = \frac{\overline{f}}{T-t} - S_{\overline{g}}$ (see (3.8)), $S_{\overline{g}} \geqslant 0$ and (2.26), we obtain

$$\Delta_{\overline{g}}\overline{\rho}_t + \frac{\partial\overline{\rho}_t}{\partial t} - \bar{\rho}_t S_{\overline{g}} = \overline{\rho}_t\Big(-\frac{\overline{f}}{4(T-t)} - \frac{S_{\overline{g}}}{4} + \frac{n}{4(T-t)}\Big) \leqslant \frac{\overline{\rho}_t}{4(T-t)}(n-f).$$

Furthermore, since $u$ satisfies

$$\frac{\partial u_t}{\partial t} - \widehat{\Delta}_{\overline{g}} u_t + u_t\left(\operatorname{Hess}_{\overline{g}}\overline{f} - \frac{h}{2}\right)(e_t,e_t) = 0,$$

we get

$$\frac{\partial u_t}{\partial t} - \widehat{\Delta}_{\overline{g}} u_t + u_t\left(\frac{1}{2}\operatorname{Hess}_{\overline{g}}\overline{f} - \frac{h}{2}\right)(e_t,e_t) = -\frac{1}{2}u_t\operatorname{Hess}_{\overline{g}}\overline{f}(e_t,e_t).$$

Since $\operatorname{Hess}_{\overline{g}}\overline{f} = \frac{1}{2(T-t)}\overline{g} - \operatorname{Ric}_{\overline{g}} + \alpha\nabla\overline{\phi}\otimes\nabla\overline{\phi}$, we obtain

$$-\frac{1}{2}u_t\operatorname{Hess}_{\overline{g}}\overline{f}(e_t,e_t) = u_t\left(-\frac{1}{4(T-t)} + \frac{1}{2}(\operatorname{Ric}_{\overline{g}} - \alpha\nabla\overline{\phi}\otimes\nabla\overline{\phi})(e_t,e_t)\right).$$

It is clear that

$$\begin{aligned}
(\operatorname{Ric}_{\overline{g}} - \alpha\nabla\overline{\phi}\otimes\nabla\overline{\phi})(e_t,e_t) &\leqslant |\operatorname{Ric}_{\overline{g}} - \alpha\nabla\overline{\phi}\otimes\nabla\overline{\phi}|_{\overline{g}} \\
&\leqslant \frac{|\operatorname{Ric}_g|_g + \alpha|\nabla\phi\otimes\nabla\phi|_g}{T-t} \leqslant \frac{C''}{T-t},
\end{aligned}$$

where $C'' := \max_M\left\{|\operatorname{Ric}_g|_g + \alpha|\nabla\phi\otimes\nabla\phi|_g\right\}$ is a constant since $(M,g)$ has bounded geometry and we are assuming (3.1). Hence,

$$\frac{d}{dt}\int_\Sigma u_t\, x_t^*\overline{\rho}_t\, dA_{\overline{g}} < \frac{1}{4(T-t)}\int_\Sigma\Big(C_0 - \overline{f}\circ x_t\Big)u_t\, x_t^*\overline{\rho}_t\, dA_{\overline{g}},$$

where $C_0 := n + 2C''$. Since $s = -\log(T-t)$. Consequently,

$$\frac{d}{ds}\int_\Sigma e^{-\frac{f}{2}\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g} = (4\pi)^{\frac{n-1}{2}}(T-t)\frac{d}{dt}\int_\Sigma u_t\, x_t^*\overline{\rho}_t\, dA_{\overline{g}}$$
$$< \frac{1}{4}\int_\Sigma \Big(C_0 - f\circ\widetilde{x}_s\Big) e^{-\frac{f}{2}\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g}.$$

The lemma follows from the analysis of the sign on the previous inequality. □

**Lemma 3.5.** *Assume that $(M,g)$ is an $n$-dimensional Riemannian manifold with bounded geometry, and let $(\overline{g}(t),\overline{\phi}(t))$ be a shrinking self-similar solution to the $(RH)_\alpha$ flow on $M$ with the potential function $\overline{f}$ and initial value $(g,\phi)$ satisfying* (3.1). *Given an $(n-1)-$dimensional compact smooth manifold $\Sigma$ without boundary, and let $\mathscr{F}$ be the MCF of $\Sigma$ in the shrinking $(\overline{g},\overline{\phi})-(RH)_\alpha$ flow background which develops a singularity of type-I. Consider the associate normalized MCF $\widetilde{\mathscr{F}}$ in $(M,g)$. Then, there exists a constant $C'>0$ such that*

$$\left|\frac{d^2}{d^2 s}\int_\Sigma e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g}\right| = \left|\frac{d}{ds}\int_\Sigma \big(H_g(\widetilde{x}_s)+e_s(f\circ\widetilde{x}_s)\big)^2 e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g}\right| \leqslant C'$$

*uniformly on $[-\log T,\infty)$.*

*Proof.* By Lemma 3.2 we have $S+|\nabla f|^2 - f = 0$ and $S\geqslant 0$ along the shrinking self-similar solution to the $(RH)_\alpha$ flow on $M$. So, $0\leqslant|\nabla f|^2\leqslant f$ and $0\leqslant S\leqslant f$. The result of the lemma follows from Lemma 3.4 and the same steps as done by Yamamoto in [Yam20]. □

Now, we show that given a sequence of isometric immersions in $\widetilde{\mathscr{F}}$ of an exhaustion on $\Sigma_\infty$, there exists a limiting global solution that converges to $x_\infty:\Sigma_\infty\to(M,g)$ using Arzelà–Ascoli theorem in the context of pointed Riemannian manifolds. For it, we work in a way that allows a generalization to the more general context of bounded geometry and satisfying (3.1). Note that these hypotheses for the compact case are automatically satisfied.

**Proposition 3.6.** *Let $(M,g)$ be an $n$-dimensional Riemannian manifold with bounded geometry, and let $(\overline{g}(t),\overline{\phi}(t))$ be a shrinking self-similar solution to the $(RH)_\alpha$ flow on $M$ with potential function $\overline{f}$ and initial value $(g,\phi)$ satisfying* (3.1). *Given an $(n-1)-$dimensional compact smooth manifold $\Sigma$ without boundary, and let $\mathscr{F}$ be the MCF of $\Sigma$ in the shrinking $(\overline{g},\overline{\phi})-(RH)_\alpha$ flow background which develops a singularity of type-I. Then, for any increasing sequence $\{s_j\}_{j=1}^\infty$ and points $\{p_j\}_{j=1}^\infty$ in $\Sigma$ (assume Remark 3.7, if $M$ is noncompact), there exist subsequences $s_{j_k}$ and $p_{j_k}$ so that the family of immersions map $\widetilde{x}_{s_{j_k}}:\Sigma\to(M,g)$ from pointed manifolds $(\Sigma,p_{j_k})$ converges to an immersion map $x_\infty:\Sigma_\infty\to(M,g)$ from an $(n-1)-$dimensional complete pointed Riemannian manifold $(\Sigma_\infty, x_\infty^* g, p_\infty)$ in the $C^\infty$ Cheeger–Gromov sense.*

*Proof.* Take an increasing sequence $\{s_j\}_{j=1}^\infty$ and points $\{p_j\}_{j=1}^\infty$ in $\Sigma$, and denote the Riemann curvature tensor of each $\widetilde{\Sigma}_{s_j}$ by $\widehat{\mathrm{Rm}}(\widetilde{x}_{s_j}^* g)$.

Firstly, we will show that there exists a subsequence of pointed Riemannian manifolds $\{(\Sigma, \widetilde{x}^*_{s_{j_k}} g, p_{j_k})\}$ which converges to some complete pointed Riemannian manifold $(\Sigma_\infty, h_\infty, p_\infty)$ in the $C^\infty$ Cheeger-Gromov sense. In fact, since $(M, g)$ has bounded geometry, there are positive constants $D_p$ and $\eta$ such that

$$|\nabla^p \mathrm{Rm}_g| \leqslant D_p < \infty \quad \text{and} \quad \mathrm{inj}(M, g) \geqslant \eta > 0 \tag{3.11}$$

for every integer $p \geqslant 0$. Besides, since assumption (3.1) holds, then by Proposition 3.1 there are positive constants $C_p$ (which does not depend on $s_j$), such that

$$|\widehat{\nabla}^p_g \mathcal{A}(\widetilde{x}_{s_j})|_g \leqslant C_p. \tag{3.12}$$

Thus, we are able to apply Theorem 1.13 which guarantees the existence of a positive constant $\delta = \delta(C_0, D_0, \eta, n)$ such that the injectivity radius of each $(\Sigma, \widetilde{x}^*_{s_j} g)$ satisfies

$$\mathrm{inj}(\Sigma, \widetilde{x}^*_{s_j} g) \geqslant \delta > 0.$$

From (3.11), (3.12), the Gauss equation and its iterated derivatives, we can see that there exist positive constants $\widetilde{C}_p$ (which also do not depend on $s_j$), such that

$$|\widehat{\nabla}^p \widehat{\mathrm{Rm}}(\widetilde{x}^*_{s_j} g)| \leqslant \widetilde{C}_p < \infty,$$

for every integer $p \geqslant 0$. Then, by Theorem 1.12, there exists a subsequence $\{(\Sigma, \widetilde{x}^*_{s_{j_k}} g, p_{j_k})\}$ which converges to some complete pointed Riemannian manifold $(\Sigma_\infty, h_\infty, p_\infty)$, i.e, there exist an exhaustion $\{U_{j_k}\}_{k=1}^\infty$ of $\Sigma_\infty$ with $x_\infty \in U_{j_k}$ and diffeomorphisms $\Psi_{j_k} : U_{j_k} \to \Psi_{j_k}(U_{j_k}) \subset \Sigma$ with $\Psi_{j_k}(p_\infty) = p_{j_k}$ such that $\Psi^*_{j_k}(\widetilde{x}^*_{s_{j_k}} g)$ converges in $C^\infty$ to $h_\infty$ uniformly on compact sets in $\Sigma_\infty$.

Secondly, we will use the standard diagonal argument to construct an immersion map $x_\infty : \Sigma_\infty \to (M, g)$. For this, we assume that there is a Nash isometric embedding $\Theta : (M, g) \to (\mathbb{R}^d, g_{\mathrm{st}})$ in some higher dimensional Euclidean space such that, for each integer $j \geqslant 0$, the norm $|\nabla^j_g \mathcal{A}(\Theta)| \leqslant \overline{D}_j$, for some constants $\overline{D}_j > 0$, where $\mathcal{A}(\Theta)$ is the second fundamental form of $\Theta$.

By setting,

$$\bar{x}_{s_{j_k}} := \Theta \circ \widetilde{x}_{s_{j_k}} \circ \Psi_{j_k} : U_{j_k} \to (\mathbb{R}^d, g_{\mathrm{st}})$$

Take a sequence of radii $R_1 < R_2 < \cdots \to \infty$, and consider balls $B_i := B_{h_\infty}(p_\infty, R_i) \subset \Sigma_\infty$.

First of all, we work on $B_1$. Since $U_{j_k}$ is an exhaustion, there exists $k_1$ such that $\overline{B_1} \subset U_{j_k}$ for all $k \geqslant k_1$. Hence we have a sequence of $C^\infty$-maps $\bar{x}_{s_{j_k}} = \Theta \circ \widetilde{x}_{s_{j_k}} \circ \Psi_{j_k} : (U_{j_k} \supset) B_1 \to (\mathbb{R}^d, g_{\mathrm{st}})$ restricted on $B_1$ for all $k \geqslant k_1$.

(0): $C^0$-estimate. First, we derive a $C^0$-bound for $\bar{x}_{s_{j_k}}$. If $M$ is compact, then the image $\Theta(M)$ is a compact set in $\mathbb{R}^d$ and contained in some ball

$$B_{g_{\mathrm{st}}}(0, \hat{C}_0) = \{\, y \in \mathbb{R}^d \mid |y|_{g_{\mathrm{st}}} < \hat{C}_0 \,\}$$

with radius $\hat{C}_0$. Since each image $\bar{x}_{s_{j_k}}(B_1)$ is contained in $\Theta(M)$, we have

$$|\bar{x}_{s_{j_k}}|_{g_{\mathrm{st}}} \leqslant \hat{C}_0 \quad \text{on} \quad \overline{B_1}.$$

It is clear that the constant $\hat{C}_0$ does not depend on $j_k$.

**Remark 3.7.** At this point of the proof we are assuming that the sequence $\{\widetilde{x}_{s_{j_k}}(p_{j_k})\}$ and the norms $|\nabla_g^j \mathcal{A}(\Theta)|$ are uniformly bounded in $M$, which are true in the compact case. Note that we will need to prove or assume these facts for the noncompact case.

If $M$ is noncompact, we need some additional argument to get a $C^0$-bound. Since

$$|\bar{x}^*_{s_{j_k}} g_{\mathrm{st}} - h_\infty|_{h_\infty} = |\Psi^*_{j_k}(\widetilde{x}^*_{s_{j_k}} g) - h_\infty|_{h_\infty} \to 0$$

uniformly on $\overline{B_1}$, for a given $\epsilon > 0$ there exists $k_1'(\geqslant k_1)$ such that on $\overline{B_1}$

$$|\bar{x}^*_{s_{j_k}} g_{\mathrm{st}} - h_\infty|_{h_\infty} < \epsilon \quad \text{for} \quad k \geqslant k_1',$$

and from Proposition 1.19 this implies that

$$|\bar{x}_{s_{j_k}}(p_\infty) - \bar{x}_{s_{j_k}}(p)|_{g_{\mathrm{st}}} \leqslant \sqrt{1+\epsilon}\, d_{h_\infty}(p_\infty, p) \leq \sqrt{1+\epsilon} R_1$$

for all $p \in B_1$ and $k \geqslant k_1'$. Furthermore, by assumption for the noncompact case, $\{\widetilde{x}_{s_{j_k}}(p_{j_k})\}_{k=1}^\infty$ is a bounded sequence in $M$. Hence $\bar{x}_{s_{j_k}}(p_\infty) = (\Theta \circ \widetilde{x}_{s_{j_k}})(p_{j_k})$ is also a bounded sequence in $\mathbb{R}^d$, that is, there exists a constant $\hat{C}_0'$ such that $|\bar{x}_{s_k}(p_\infty)|_{g_{\mathrm{st}}} \leqslant \hat{C}_0'$. Hence we have

$$|\bar{x}_{s_k}|_{g_{\mathrm{st}}} \leqslant \hat{C}_0' + \sqrt{1+\epsilon} R_1 =: \hat{C}_0$$

for $k \geqslant k_1'$. It is clear that $\hat{C}_0$ does not depend on $k$. Hence, we get a $C^0$-bound.

(1): $C^1$-estimate. Next, we consider a $C^1$-bound for $\bar{x}_{s_{j_k}}$. One can see that $\nabla_{g_{\mathrm{st}}} \bar{x}_{s_{j_k}} = D\bar{x}_{s_{j_k}}$. Since $\bar{x}_{s_{j_k}} : (B_1, \bar{x}^*_{s_{j_k}} g_{\mathrm{st}}) \to (\mathbb{R}^d, g_{\mathrm{st}})$ is an isometric immersion, we have a $C^1$-bound

$$|\nabla_{g_{\mathrm{st}}} \bar{x}_{s_{j_k}}|_{\bar{x}^*_{s_{j_k}} g_{\mathrm{st}} \otimes g_{\mathrm{st}}} = |D\bar{x}_{s_{j_k}}|_{\bar{x}^*_{s_{j_k}} g_{\mathrm{st}} \otimes g_{\mathrm{st}}} = \sqrt{n-1} =: \hat{C}_1.$$

(2): $C^2$-estimate. Next, we derive a $C^2$-bound for $\bar{x}_{s_{j_k}}$. Let $\hat{\nabla}$ be the connection on $(\otimes^p T^*\Sigma) \otimes \mathbb{R}^d$ $(p \geqslant 0)$ over $B_1$ induced by metric $\bar{x}^*_{s_{j_k}} g_{\mathrm{st}}$ and $g_{\mathrm{st}}$. Note that $\hat{\nabla} = \nabla_{g_{\mathrm{st}}}$ for $p = 0$. Since $\hat{\nabla} \bar{x}_{s_{j_k}} = D\bar{x}_{s_{j_k}}$, we have

$$\hat{\nabla}^2 \bar{x}_{s_{j_k}} = \mathcal{A}(\bar{x}_{s_{j_k}}),$$

where $\mathcal{A}(\bar{x}_{s_{j_k}})$ is the second fundamental form of the isometric immersion $\bar{x}_{s_{j_k}} = \Theta \circ \widetilde{x}_{s_{j_k}} \circ \Psi_{j_k} : (B_1, \bar{x}^*_{s_{j_k}} g_{\mathrm{st}}) \to (\mathbb{R}^d, g_{\mathrm{st}})$. Ideed,

$$\begin{aligned}
\hat{\nabla}^2 \bar{x}_{s_{j_k}}((\bar{x}_{s_{j_k}})_* X, (\bar{x}_{s_{j_k}})_* Y) &= \hat{\nabla}_{(\bar{x}_{s_{j_k}})_* X}[\hat{\nabla} \bar{x}_{s_{j_k}}]((\bar{x}_{s_{j_k}})_* Y) \\
&= \big((\bar{x}_{s_{j_k}})_* X\big)\big(\hat{\nabla} \bar{x}_{s_{j_k}}((\bar{x}_{s_{j_k}})_* Y)\big) - \hat{\nabla} \bar{x}_{s_{j_k}}\big(\hat{\nabla}_{(\bar{x}_{s_{j_k}})_* X}(\bar{x}_{s_{j_k}})_* Y\big) \\
&= \mathcal{A}(\bar{x}_{s_{j_k}})((\bar{x}_{s_{j_k}})_* X, (\bar{x}_{s_{j_k}})_* Y), \quad \forall X, Y \in T\overline{B}_1.
\end{aligned}$$

The claim follows from the Gauss equation. Hence, by using the composition rule for the second fundamental forms of immersions, we have

$$\begin{aligned}\hat{\nabla}^2\bar{x}_{s_{j_k}}(X,Y) =& \mathcal{A}(\bar{x}_{s_{j_k}})(X,Y)\\ =& \mathcal{A}(\Theta)((\widetilde{x}_{s_{j_k}}\circ\Psi_{j_k})_*X,(\widetilde{x}_{s_{j_k}}\circ\Psi_{j_k})_*Y)+\Theta_*(\mathcal{A}(\widetilde{x}_{s_{j_k}})(\Psi_{j_k*}X,\Psi_{j_k*}Y))\end{aligned}$$

for any tangent vectors $X$ and $Y$ on $M$. By using the notion of $*$-product, this identity is written as

$$\hat{\nabla}^2\bar{x}_{s_{j_k}} = \mathcal{A}(\Theta)*(\overset{2}{*}D(\widetilde{x}_{s_{j_k}}\circ\Psi_{j_k}))+\mathcal{A}(\widetilde{x}_{s_{j_k}})*D\Theta*(\overset{2}{*}D\Psi_{j_k}). \tag{3.13}$$

Since $|D(\widetilde{x}_{s_{j_k}}\circ\Psi_{j_k})|_{\bar{x}^*_{s_{j_k}}g_{\mathrm{st}}\otimes g} = |D\Psi_{j_k}|_{\bar{x}^*_{s_{j_k}}g_{\mathrm{st}}\otimes\widetilde{x}^*_{s_{j_k}}g} = \sqrt{n-1}$ and $|D\Theta|_{g\otimes g_{\mathrm{st}}} = \sqrt{n}$, we have

$$|\hat{\nabla}^2\bar{x}_{s_{j_k}}|_{\bar{x}^*_{s_{j_k}}g_{\mathrm{st}}\otimes g_{\mathrm{st}}} \leqslant \hat{C}_2'|\mathcal{A}(\Theta)|_{g\otimes g_{\mathrm{st}}}+\hat{C}_2''|\mathcal{A}(\widetilde{x}_{s_{j_k}})|_{x^*_{s_{j_k}}g\otimes g}$$

for some constants $\hat{C}_2'$ and $\hat{C}_2''$ which do not depend on $j_k$. Furthermore, by assumptions, we have $|\mathcal{A}(\Theta)|_{g\otimes g_{\mathrm{st}}} \leqslant \tilde{D}_0$ and $|\mathcal{A}(\widetilde{x}_{s_{j_k}})|_{x^*_{s_{j_k}}g\otimes g} \leqslant D_0$. Hence we have a $C^2$-bound

$$|\hat{\nabla}^2\bar{x}_{s_{j_k}}|_{\bar{x}^*_{s_{j_k}}g_{\mathrm{st}}\otimes g_{\mathrm{st}}} \leqslant \hat{C}_2'\tilde{D}_0+\hat{C}_2''D_0 =: \hat{C}_2.$$

It is clear that $\hat{C}_2$ does not depend on $j_k$.

<u>$(p)$: $C^p$-estimate.</u> By differentiating (3.13), we can get a $C^p$-bound. We only observe a $C^3$-bound. Note that for any tangent vectors $X$ and $Y$ on $M$ we have

$$\big(\nabla_{\bar{x}^*_{s_{j_k}}g_{\mathrm{st}}\otimes g}D(\widetilde{x}_{s_{j_k}}\circ\Psi_{j_k})\big)(X,Y) = \mathcal{A}(\widetilde{x}_{s_{j_k}}\circ\Psi_{j_k})(X,Y) = \mathcal{A}(\widetilde{x}_{s_{j_k}})(\Psi_{j_k*}X,\Psi_{j_k*}Y).$$

By using the notion of $*$-product, this identity is written as

$$\nabla_{\bar{x}^*_{s_{j_k}}g_{\mathrm{st}}\otimes g}D(\widetilde{x}_{s_{j_k}}\circ\Psi_{j_k}) = \mathcal{A}(\widetilde{x}_{s_{j_k}})*(\overset{2}{*}D\Psi_{j_k}).$$

Furthermore, note that $\nabla_{g\otimes g_{\mathrm{st}}}D\Theta = \mathcal{A}(\Theta)$ and $\nabla_{\bar{x}^*_k g_{\mathrm{st}}\otimes\widetilde{x}^*_k g}D\Psi_k = 0$. Hence we have

$$\begin{aligned}\hat{\nabla}^3\bar{x}_{s_{j_k}} =& \nabla_{g\otimes g_{\mathrm{st}}}\mathcal{A}(\Theta)*(\overset{2}{*}D(\widetilde{x}_{s_{j_k}}\circ\Psi_{j_k}))\\ &+2A(\Theta)*D(\widetilde{x}_{s_{j_k}}\circ\Psi_k)*\mathcal{A}(\widetilde{x}_{s_{j_k}})*(\overset{2}{*}D\Psi_{j_k})\\ &+\nabla_{\widetilde{x}^*_{s_{j_k}}g\otimes g}\mathcal{A}(\widetilde{x}_{s_{j_k}})*D\Theta*(\overset{2}{*}D\Psi_{j_k})\\ &+\mathcal{A}(\widetilde{x}_{s_{j_k}})*\mathcal{A}(\Theta)*(\overset{2}{*}D\Psi_{j_k}).\end{aligned}$$

By assumptions, the norms of all tensors that appear in the above inequality are bounded. Hence we have a $C^3$-bound

$$|\hat{\nabla}^3\bar{x}_{s_{j_k}}|_{\bar{x}^*_{s_{j_k}}g_{\mathrm{st}}\otimes g_{\mathrm{st}}} \leq \hat{C}_3$$

for some constant $\hat{C}_3$ which does not depend on $k$. For higher derivatives, one can prove that there exists a constant $\hat{C}_p > 0$ which does not depend on $k$ such that

$$|\hat{\nabla}^p \bar{x}_{s_{j_k}}|_{\bar{x}^*_{s_{j_k}} g_{\mathrm{st}} \otimes g_{\mathrm{st}}} \leq \hat{C}_p,$$

by induction.

In the above argument, we have proved that there exist constants $\hat{C}_p$ ($p \geqslant 0$) which do not depend on $k$ such that $|\hat{\nabla}^p \bar{x}_{s_{j_k}}|_{\bar{x}^*_{s_{j_k}} g_{\mathrm{st}} \otimes g_{\mathrm{st}}} \leqslant \hat{C}_p$. Thus, one can prove that there exist constants $C_p$ ($p \geqslant 0$) which do not depend on $k$ such that

$$|\nabla^p \bar{x}_{s_{j_k}}|_{h_\infty \otimes g_{\mathrm{st}}} \leqslant C_p.$$

Now, by a standard argument as the Arzelà–Ascoli theorem (see Theorem 1.6), there exists a smooth map $\bar{x}_{1,\infty} : \overline{B}_1 \to (\mathbb{R}^d, g_{\mathrm{st}})$ such that the sequence of immersions $\bar{x}_{s_{j_k}} : \overline{B}_1 \to (\mathbb{R}^d, g_{\mathrm{st}})$ converges to $\bar{x}_{1,\infty} : \overline{B}_1 \to (\mathbb{R}^d, g_{\mathrm{st}})$ up to subsequence. Since all images $\bar{x}_{s_{j_k}}(\overline{B}_1)$ are contained in $\Theta(M)$, the image $\bar{x}_{1,\infty}(\overline{B}_1)$ is also contained in $\Theta(M)$. Furthermore $\bar{x}_{1,\infty} : \overline{B}_1 \to (\mathbb{R}^d, g_{\mathrm{st}})$ has the property that

$$\bar{x}^*_{1,\infty} g_{\mathrm{st}} = h_\infty,$$

since $|\bar{x}^*_{1,\infty} g_{\mathrm{st}} - h_\infty|_{h_\infty} \leqslant |\bar{x}^*_{1,\infty} g_{\mathrm{st}} - \bar{x}^*_{s_{j_k}} g_{\mathrm{st}}|_{h_\infty} + |\bar{x}^*_{s_{j_k}} g_{\mathrm{st}} - h_\infty|_{h_\infty}$ and the right hand side converges to 0 as $k \to \infty$ on $\overline{B}_1$. Thus, especially, $\bar{x}_{1,\infty} : \overline{B}_1 \to (\mathbb{R}^d, g_{\mathrm{st}})$ is an immersion map.

Next, for the subsequence of $\bar{x}_{s_{j_k}}$ which converges to $\bar{x}_{1,\infty}$, we work on $B_2$. Then all the above arguments also work on $B_2$ and we can show that there exists a smooth immersion map $\bar{x}_{2,\infty} : \overline{B}_2 \to \Theta(M) \subset \mathbb{R}^d$ with $\bar{x}^*_{2,\infty} g_{\mathrm{st}} = h_\infty$ and $\bar{x}_{2,\infty} = \bar{x}_{1,\infty}$ on $\overline{B}_1$ and the sequence of immersions $\bar{x}_{s_{j_k}} : \overline{B}_2 \to (\mathbb{R}^d, g_{\mathrm{st}})$ converges to $\bar{x}_{2,\infty} : \overline{B}_2 \to (\mathbb{R}^d, g_{\mathrm{st}})$ up to subsequence. By iterating this construction and the diagonal argument, finally we get a smooth immersion map $\bar{x}_\infty : \Sigma_\infty \to \Theta(M) \subset (\mathbb{R}^d, g_{\mathrm{st}})$ with $\bar{x}^*_\infty g_{\mathrm{st}} = h_\infty$ and the sequence of immersions $\bar{x}_{s_{j_k}} : \Sigma_\infty \to (\mathbb{R}^d, g_{\mathrm{st}})$ converges to $\bar{x}_\infty : \Sigma_\infty \to (\mathbb{R}^d, g_{\mathrm{st}})$ up to subsequence, and the map defined by $x_\infty := \Theta^{-1} \circ \bar{x}_\infty : \Sigma_\infty \to (M, g)$ is the required one satisfying the properties in the statement. □

## 3.2 The compact case

Now, we are in a position to prove the fourth main theorem of this thesis.

**Theorem 3.8.** *Assume that $(M, g)$ is an $n$-dimensional compact Riemannian manifold, and let $(\overline{g}(t), \overline{\phi}(t))$ be a shrinking self-similar solution to the $(RH)_\alpha$ flow on $M$ with potential function $\overline{f}$ and initial value $(g, \phi)$. Given an $(n-1)-$dimensional compact smooth manifold $\Sigma$ without boundary, and let $\mathscr{F}$ be the MCF of $\Sigma$ in the shrinking $(\overline{g}, \overline{\phi}) - (RH)_\alpha$ flow background which develops a singularity of type-I. Consider the associate normalized MCF $\widetilde{\mathscr{F}}$ in $(M, g)$. Then, for any increasing sequence $\{s_j\}_{j=1}^\infty$ and points $\{p_j\}_{j=1}^\infty$ in $\Sigma$, there exist subsequences $s_{j_k}$ and $p_{j_k}$ in $\Sigma$, such that the family of immersion maps $\widetilde{x}_{s_{j_k}} : \Sigma \to (M, g)$ from pointed manifolds $(\Sigma, p_{j_k})$*

*converges to an immersion map $x_\infty : \Sigma_\infty \to (M, g)$ from an $(n-1)-$dimensional complete pointed Riemannian manifold $(\Sigma_\infty, x_\infty^* g, p_\infty)$ in the $C^\infty$ Cheeger–Gromov sense. Furthermore, $(\Sigma_\infty, x_\infty^* g)$ is a $f_\infty$-minimal hypersurface of $(M, g)$, where $f_\infty = f \circ x_\infty$.*

*Proof.* Take an increasing sequence $\{s_j\}_{j=1}^\infty$ and a sequence of points $\{p_j\}_{j=1}^\infty$ in $\Sigma$. From Proposition 3.6, there exist subsequences $s_{j_k}$ and $p_{j_k}$ so that the family of immersions map $\widetilde{x}_{s_{j_k}} : \Sigma \to (M, g)$ from pointed manifolds $(\Sigma, p_{j_k})$ converges to an immersion map $x_\infty : \Sigma_\infty \to (M, g)$ from an $(n-1)-$dimensional complete pointed Riemannian manifold $(\Sigma_\infty, x_\infty^* g, p_\infty)$ in the $C^\infty$ Cheeger–Gromov sense.

Next, we will prove that $\Sigma_\infty$ is a $f_\infty$-hypersurface of $(M, g)$, where $f_\infty = f \circ x_\infty$. We denote $\widetilde{x}_{s_{j_k}}$ by $\widetilde{x}_k$ for short. Then, there exist an exhaustion $\{U_k\}_{k=1}^\infty$ of $\Sigma_\infty$ with $p_\infty \in U_k$ and a sequence of diffeomorphisms $\Psi_k : U_k \to V_k := \Psi_k(U_k) \subset \Sigma$ with $\Psi_k(p_\infty) = p_{j_k}$ such that $\Psi_k^*(\widetilde{x}_k^* g)$ converges in $C^\infty$ to $\widetilde{x}_\infty^* g$ uniformly on compact sets in $\Sigma_\infty$, and furthermore the sequence of maps $\widetilde{x}_k \circ \Psi_k : U_k \to (M, g)$ converges in $C_\infty$ to $x_\infty : \Sigma_\infty \to (M, g)$ uniformly on compact sets in $\Sigma_\infty$. Let $K \subset \Sigma_\infty$ be any compact set. So, for any compact set $K \subset \Sigma_\infty$ there exists $k_0$ such that $K \subset U_k$, for all $k \geqslant k_0$. Since $\widetilde{x}_k \circ \Psi_k : U_k \to (M, g)$ converges to $x_\infty : \Sigma_\infty \to (M, g)$ in $C^\infty$ uniformly on $K$. One has,

$$\begin{aligned}
&\int_K \left[H(\widetilde{x}_{s_k} \circ \Psi_k) + e_k \left(f \circ (\widetilde{x}_{s_k} \circ \Psi_k)\right)\right]^2 e^{-f\circ(\widetilde{x}_{s_k}\circ\Psi_k)} dA_{(\widetilde{x}_{s_k}\circ\Psi_k)^* g} \\
&\to \int_K \left(H(x_\infty) + e_\infty f_\infty\right)^2 e^{-f_\infty} dA_{x_\infty^* g}
\end{aligned}$$

as $k \to \infty$, and

$$\begin{aligned}
&\int_K \left[H(\widetilde{x}_{s_k} \circ \Psi_k) + e_k f(\widetilde{x}_{s_k} \circ \Psi_k)\right]^2 e^{-f\circ(\widetilde{x}_{s_k}\circ\Psi_k)} dA_{(\widetilde{x}_{s_k}\circ\Psi_k)^* g} \\
= &\int_{\Psi_k(K)} \left[H(\widetilde{x}_{s_k}) + e_k(f \circ \widetilde{x}_{s_k})\right]^2 e^{-f\circ\widetilde{x}_{s_k}} dA_{\widetilde{x}_{s_k}^* g} \\
\leqslant &\int_\Sigma \left[H(\widetilde{x}_{s_k}) + e_k(f \circ \widetilde{x}_{s_k})\right]^2 e^{-f\circ\widetilde{x}_{s_k}} dA_{\widetilde{x}_{s_k}^* g}.
\end{aligned}$$

Hence, it is enough to prove the following:

$$\int_\Sigma [H(\widetilde{x}_{s_k}) + e_k(f \circ \widetilde{x}_{s_k})]^2 e^{-f\circ\widetilde{x}_{s_k}} dA_{\widetilde{x}_{s_k}^* g} \to 0 \tag{3.14}$$

as $k \to \infty$. We will argue by contradiction. Assume that there exist a constant $\delta > 0$ and a subsequence $\{\ell\} \subset \{k\}$ with $\ell \to \infty$ such that

$$\int_\Sigma [H(\widetilde{x}_{s_\ell}) + e_\ell(f \circ \widetilde{x}_{s_\ell})]^2 e^{-f\circ\widetilde{x}_{s_\ell}} dA_{\widetilde{x}_{s_\ell}^* g} \geqslant \delta.$$

Then

$$\int_\Sigma [H(\widetilde{x}_s) + e_s(f \circ \widetilde{x}_s)]^2 e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g} \geqslant \frac{\delta}{2},$$

for $s \in [s_\ell, s_\ell + \frac{\delta}{2C'}]$, where we used Lemma 3.5 and $C'$ is the constant appeared in that lemma. Hence,

$$\int_{-\log T}^{\infty} \int_\Sigma [H(\widetilde{x}_s) + e_s(f \circ \widetilde{x}_s)]^2 e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g} ds = \infty.$$

On the other hand, by monotonicity formula in Lemma 3.3

$$\frac{d}{ds} \int_\Sigma e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g} = -\int_\Sigma [H(\widetilde{x}_s) + e_s(f \circ \widetilde{x}_s)]^2 e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g} \leqslant 0.$$

Thus, the weighted volume

$$\int_\Sigma e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g}$$

is monotone decreasing and nonnegative. Therefore, it converges to some value

$$\alpha := \lim_{s\to\infty} \int_\Sigma e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g} < \infty,$$

and then we obtain the following contradiction:

$$\int_{-\log T}^{\infty} \int_\Sigma [H(\widetilde{x}_s) + e_s(f \circ \widetilde{x}_s)]^2 e^{-f\circ\widetilde{x}_s} dA_{\widetilde{x}_s^* g} = -\alpha + \int_\Sigma e^{-f\circ\widetilde{x}_a} dA_{\widetilde{x}_a^* g} < \infty,$$

where $a := -\log T$, which proves (3.14). So, the proof of the theorem is complete. □

## 3.3 The noncompact case

In this section, we address the case of complete noncompact Riemannian manifolds $(M, g)$ with some additional uniformity conditions. We begin with a brief discussion on the reduced distance along the $(RH)_\alpha$ flow (see [Mül12, Sect. 8]), initially defined by Perelman into the Ricci flow setting (see [Per02, Sect. 7]).

Let $(\overline{g}(t), \overline{\phi}(t))$ be a shrinking self-similar solution of the $(RH)_\alpha$ flow in $M \times [0, T)$. For any smooth curve $\gamma : [t_1, t_2] \to M$ with $0 \leqslant t_1 < t_2 < T$, consider the $\mathcal{L}$-length of $\gamma$ by

$$\mathcal{L}(\gamma) := \int_{t_1}^{t_2} \sqrt{t_2 - t}\big(S_{\overline{g}} + |\dot{\gamma}|^2\big) dt,$$

where $|\dot{\gamma}|$ is the norm of $\dot{\gamma}(t)$ measured by $\overline{g}$ and $S_{\overline{g}} = R_{\overline{g}} - \alpha_n |\nabla\overline{\phi}|^2_{\overline{g}}$. For a fixed point $(q_2, t_2)$ in the space-time $M \times [0, T)$, Müller defined the reduced distance

$$\ell_{q_2,t_2} : M \times [0, t_2) \to \mathbb{R}$$

based at $(q_2, t_2)$ by

$$\ell_{q_2,t_2}(q_1, t_1) := \frac{1}{2\sqrt{t_2 - t_1}} \inf_\gamma \mathcal{L}(\gamma),$$

where the infimum is taken over all smooth curve $\gamma : [t_1, t_2] \to M$ with $\gamma(t_1) = q_1$ and $\gamma(t_2) = q_2$.

In what follows, we assume that there exists a Nash isometric embedding $\Theta : (M, g) \to (\mathbb{R}^d, g_{\text{st}})$ in some higher-dimensional Euclidean space, such that, for every integer $j \geqslant 0$, the second fundamental form $\mathcal{A}(\Theta)$ of $\Theta$ satisfies

$$|\nabla_g^j \mathcal{A}(\Theta)| \leqslant \overline{D}_j \tag{3.15}$$

for some constants $\overline{D}_j > 0$. We observe that, under this assumption, $(M, g)$ must have bounded geometry by Gauss equation (and its iterated derivatives) and Thm. 2.1 of [CY07].

**Theorem 3.9.** *Assume that $(M, g)$ is an $n$-dimensional complete noncompact Riemannian manifold, and let $(\overline{g}(t), \overline{\phi}(t))$ be a shrinking self-similar solution to the $RH_\alpha$ flow on $M$ with potential function $\overline{f}$ and initial value $(g, \phi)$ satisfying* (3.1) *and* (3.15). *Given an $(n-1)$-dimensional compact smooth manifold $\Sigma$ without boundary, and let $\mathscr{F}$ be the MCF of $\Sigma$ in the shrinking $(\overline{g}, \overline{\phi}) - (RH)_\alpha$ flow background which develops a singularity of type-I. Consider the normalized MCF $\widetilde{\mathscr{F}}$ in $(M, g)$. In addition, assume that there exists a point $q_0 \in \Sigma$ such that the reduced distance $\ell_{x_t(q_0),t}$ converges pointwise to $f$ (as $t \to T$) on $M \times [0, T)$. Then, for any increasing sequence $\{s_j\}_{j=1}^{\infty}$ there exists a subsequence $s_{j_k}$ such that the family of immersion maps $\widetilde{x}_{s_{j_k}} : \Sigma \to (M, g)$ from pointed manifold $(\Sigma, q_0)$ converges to an immersion map $x_\infty : \Sigma_\infty \to (M, g)$ from an $(n-1)-$dimensional complete pointed Riemannian manifold $(\Sigma_\infty, x_\infty^* g, q_\infty)$ in the $C^\infty$ Cheeger–Gromov sense. Furthermore, $(\Sigma_\infty, x_\infty^* g)$ is an $f_\infty-$minimal hypersurface of $(M, g)$, where $f_\infty = f \circ x_\infty$.*

*Proof.* As we had already mentioned in the proof of Proposition 3.6, it is enough to show that $\{\widetilde{x}_{s_j}(q_0)\}_{j=1}^{\infty}$ is a bounded sequence in $M$, since the remainder of the proof is the same as in Theorem 3.8.

We start by taking $t_1$, $t_2$ with $0 \leqslant t_1 < t_2 < T$, and $\{x_t(q_0)\}_{t \in [t_1, t_2]}$ as a curve joining $x_{t_1}(q_0)$ and $x_{t_2}(q_0)$. Thus,

$$\begin{aligned} \ell_{x_{t_2}(q_0),t_2}(x_{t_1}(q_0), t_1) &\leqslant \frac{1}{2\sqrt{t_2 - t_1}} \int_{t_1}^{t_2} \sqrt{t_2 - t} \left( S_{\overline{g}} + \left| \frac{\partial x_t}{\partial t} \right|^2 \right) dt \\ &= \frac{1}{2\sqrt{t_2 - t_1}} \int_{t_1}^{t_2} \sqrt{t_2 - t} \big( S_{\overline{g}} + H_{\overline{g}}^2 \big) dt. \end{aligned}$$

Since $\mathscr{F}$ develops a singularity of type-I, we have $(T - t) H_{\overline{g}}^2$ is bounded. Moreover, bounded geometry assumption and the fact that $(T - t) S_{\overline{g}} = S_g$ imply $S_{\overline{g}} + H_{\overline{g}}^2 \leqslant \frac{C}{T-t}$ for some positive constant $C$ which does not depend on $t$. Hence,

$$\begin{aligned} &\ell_{x_{t_2}(q_0),t_2}(x_{t_1}(q_0), t_1) \\ \leqslant\ & \frac{C}{2\sqrt{t_2 - t_1}} \int_{t_1}^{t_2} \frac{\sqrt{t_2 - t}}{T - t} dt \leqslant \frac{C}{2\sqrt{t_2 - t_1}} \int_{t_1}^{t_2} \frac{1}{\sqrt{T - t}} dt \leqslant C \frac{\sqrt{T - t_1}}{\sqrt{t_2 - t_1}}. \end{aligned}$$

By assumption $\ell_{x_t(q_0),t}$ converges pointwise to $f$ (as $t \to T$) on $M \times [0, T)$ and by taking the limit as $t_2 \to T$, we have $f(x_{t_1}(q_0), t_1) \leqslant C$. As $f(x_t(q_0), t) = \overline{f}(x_t(q_0)) = f(\widetilde{x}_s(q_0))$, one has

$f(\widetilde{x}_s(q_0)) \leqslant C$ for all $s \in [-\log T, \infty)$. Thm. 5.1 in Wang [Wan16] ensures that there exist positive constants $C_1$ and $C_2$ such that

$$\frac{1}{4}(r - C_1)^2 \leqslant f \leqslant \frac{1}{4}(r + C_2)^2$$

on $M$, where $r(q) = d_g(q_0, q)$ is the distance function from any fixed point $q_0 \in M$. Then

$$d_g(q_0, \widetilde{x}_s(q_0)) \leqslant 2\sqrt{C} + C_1,$$

which means that $\{\widetilde{x}_{s_j}(q_0)\}_{j=1}^{\infty}$ is bounded in $(M, g)$ and the proof is complete. □

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