UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Torsion invariant on cellular complexes

Gustavo de Oliveira Cardoso dos Santos

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Dissertation submitted to the Graduate Program in Mathematics of Universidade Federal de São Carlos in partial fulfillment of the requirements for the degree of Master in Mathematics.

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UNIVERSIDADE FEDERAL DE SÃO CARLOS

Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Matemática

Folha de Aprovação

Defesa de Dissertação de Mestrado do candidato Gustavo de Oliveira Cardoso dos Santos, realizada em 20/08/2024.

Comissão Julgadora:

Prof. Dr. Luiz Roberto Hartmann Junior (UFSCar)

Prof. Dr. Mauro Flavio Spreafico (UniTrento)

Prof. Dr. Thiago de Melo (UNESP)

O Relatório de Defesa assinado pelos membros da Comissão Julgadora encontra-se arquivado junto ao Programa de Pós-Graduação em Matemática.

Dedico este trabalho à minha mãe, Marilene,
e à minha dinda, Roseli (in memoriam).

Acknowledgement

Pode até parecer clichê, mas não tenho como iniciar esses agradecimentos sem antes de mais nada, agradecer a Deus. Tenho sentido a presença do nosso Senhor ao meu lado, cada vez mais forte, ao longo da minha trajetória, me guiando a cada conquista e a cada frustração, me mantendo firme em meus objetivos.

Agradeço principalmente à minha mãe, Marilene, pelo apoio incondicional ao longo da minha vida. Enquanto muitas pessoas falam sobre se fazer todo o possível, eu posso dizer com tranquilidade que a senhora fez muitas vezes o impossível se tornar possível para mim. É a responsável por eu encarar os meus estudos como uma prioridade, desde que me entendo por gente. Muito obrigado por, além de ser mãe, ser também pai, amiga e em alguns momentos quase uma psicóloga. Espero estar sendo um filho e um ser humano à altura do que a senhora me criou para ser.

À minha dinda, Roseli, que Deus a tenha, por todo o incentivo e apoio dado a mim e à minha mãe por tantos anos. Só posso esperar que, onde quer que ela esteja, sinta orgulho do caminho que tenho trilhado. À minha tia-avó Josefa, minha avó do coração, uma pessoa que com certeza terá sempre meu profundo respeito, consideração e afeto. Que todos os meus familiares se sintam contemplados nesses agradecimentos, por todo o apoio, seja direto ou indireto, que resultaram na conclusão de mais esta etapa.

Ao professor Oscar Ocampo, me faltam palavras para dimensionar a sua importância ao longo da minha jornada acadêmica. Me recordo do dia, pouco depois de começarmos a trabalhar, em que me perguntou se eu tinha a intenção de fazer um mestrado e minha resposta naquele momento era de incerteza. É interessante perceber hoje que o seu encaminhamento fez o mestrado se tornar, para mim, um caminho nada mais do que natural. Além de ter sido meu orientador de iniciações científicas, é um amigo e um mentor que levo para toda a vida. Aos professores da minha graduação, agradeço em especial a Cristina Lizana, Eliane Santos, Graça Luzia e Vinicius Laass.

Aos meus amigos que levo da graduação para a vida, João Mota, Rosiele Barbosa e Ruan Vieira, só tenho a agradecer por terem me apoiado em meus momentos difíceis, por terem comemorado junto comigo as minhas vitórias, pelas conversas, por muitas vezes acreditarem em mim, mais do que eu mesmo, e pelos momentos memoráveis vividos nos últimos anos. Não poderia deixar de agradecer também a Caio Lima e Ênio Leite pelo apoio e amizade.

Ao professor Luiz Hartmann, eu devo dizer que desde a primeira conversa que tivemos a respeito dessa orientação, de alguma forma tive a certeza que eu havia encontrado o caminho que estava procurando. Muito obrigado por todo o apoio e incentivo, pela confiança depositada e pelo alto nível de disponibilidade ao longo desse mestrado. Ao professor Alex Carlucci, agradeço por me mostrar que a pós-graduação pode ser feita com empatia e acolhimento.

Além disso, não há como falar do meu mestrado sem falar dos amigos que fiz ao longo dessa caminhada. Agradeço a Carolina Santana, minha irmã de mestrado, que juntamente com Danilo Santos, Lucas Maekawa e Rodrigo Thomaz, me trouxeram um sentimento de pertencimento que foi tão importante na minha chegada na UFSCar. Agradeço também a Olavo de Melo, que aceitava ir em dia anterior de prova de topologia estudar no DM. Ainda tive a sorte de seguir conhecendo Amanda Araújo, Diana Bello, Lara Budtinger, Marina de Miguel, Mynor Melara e Rafael Belli. Muito obrigado a todos vocês pelas risadas que tornaram meus dias mais leves, pelo incentivo nos momentos mais cansativos e pelas belas memórias que levo do meu mestrado. Pelo acolhimento e presteza em São Carlos, eu agradeço a D. Lídia e Sr. Mozair.

Que fique registrado aqui também o meu agradecimento às recomendações dos professores Marcio Fenille e Oscar Ocampo, que participaram da banca de qualificação deste trabalho, e dos professores Mauro Spreafico e Thiago de Melo, que participaram da banca de defesa desta dissertação.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, and by the Grupo Carrefour - TAC 11.6.21.

Resumo

Classificar espaços a menos de homeomorfismo, de equivalência de homotopia, ou de equivalência combinatorial, é um dos problemas centrais da Topologia. Para isso, dispomos de diversos invariantes na Topologia Algébrica, como a característica de Euler, os grupos de homotopia, os grupos de homologia e os grupos de cohomologia. Em 1935, K. Reidemeister publicou um trabalho sobre a classificação de uma certa classe de 3-variedades, que possuem grupos de homologia e grupos de homotopia isomorfos, mas não são homeomorfas. Algumas delas não possuem nem o mesmo tipo de homotopia. Para essa classificação, Reidemeister utilizou um invariante combinatorial/topológico chamado invariante de torção. Baseados em [20], [18], [19] e [13], abordaremos aqui algumas versões desse invariante para CW-complexos, a saber a torção de Whitehead, a torção de Reidemeister e a torção de Reidemeister de interseção para pseudovariedades com singularidades isoladas.

Palavras-chave: CW-complexos; dualidade de Poincaré; grupos de Whitehead; pseudovariedades; torção de Reidemeister; torção de Reidemeister de interseção; torção de Whitehead.

Abstract

Classifying spaces up to homeomorphism, homotopy equivalence, or combinatorial equivalence is one of the main problems in Topology. To achieve this, we have several invariants in Algebraic Topology, such as Euler characteristic, homotopy groups, homology groups and cohomology groups. In 1935, K. Reidemeister published a work on the classification of a certain class of 3-manifolds that have isomorphic homology groups and homotopy groups but are not homeomorphic. Some of them do not even have the same type of homotopy. For this classification, Reidemeister used a combinatorial/topological invariant called torsion invariant. Based in [20], [18], [19] and [13], we will discuss here some versions of this invariant for CW-complexes, namely Whitehead torsion, Reidemeister torsion, and Reidemeister intersection torsion for pseudomanifolds with isolated singularities.

Keywords: CW-complexes; Poincaré duality; Whitehead groups; pseudomanifolds; Reidemeister torsion; intersection Reidemeister torsion; Whitehead torsion.

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Introduction

The development of torsion theory began in 1935 with the work [24] of K. Reidemeister, on the classification of the of 3-dimensional lens spaces that have isomorphic homology groups and homotopy groups but are not homeomorphic. Some of them are not even homotopically equivalent. Since it is not possible to use homology groups or homotopy groups to distinguish them, Reidemeister used the torsion invariant. This can be considered a milestone for Algebraic Topology. In subsequent works, W. Franz [8] and G. de Rham [5] also made important contributions to the construction of the invariant that became known as the Reidemeister torsion, R-torsion or Franz–Reidemeister torsion.

J. H. C. Whitehead [30] defined the torsion of a homotopy equivalence between finite CW-complexes. This is a generalization of the invariant developed by Reidemeister, Franz and de Rham but with a more "sensitive" feature, being a simple homotopy invariant. T. A. Chapman proved in [1] that the Whitehead torsion is a topological invariant for compact connected CW-complexes. Furthermore, a proof of the topological invariance of the torsion for polyhedra is given by R. D. Edwards in [6].

The applications of torsion theory considerably exceed its initial motivations. Extremely relevant connections can be made between the Reidemeister torsion and knot theory. In 1962, J. Milnor observed in [19] that the classical Alexander polynomial of knots can be interpreted as the Reidemeister torsion of its knot complement in the 3-sphere S^3. The work of Milnor was extended by V. Turaev in [29], obtaining an equivalence between the Alexander polynomial and the torsion for closed 3-manifolds.

Furthermore, the torsion theory has interesting applications in dynamical systems theory. In 1983, a connection between the Lefschetz type dynamical zeta functions and the Reidemeister torsion was established by D. Fried in [9]. In general terms, Fried used the Reidemeister torsion to count the fixed point classes of all iterates of a map f, i.e., the periodic points of f. In [7], A. Fel'shtyn addresses how the dynamical zeta functions in the Nilsen theory are closely related to Reidemeister torsion.

In this text, throughout four chapters, we will make algebraic and topological constructions of the torsion invariants, assuming some definitions and the main results involving fundamental groups, covering spaces and homology theory. For more details on the topics mentioned see, for example, [14] and [22]. Moreover, a certain level of familiarity with the concept of modules is also recommended, details of which can be found in [16]. This work is a compilation of significant results from torsion theory, with a historical arc spanning almost 60 years, starting with three papers from the 1960s and ending with recent works from 2020.

In Chapter 1, based on [14], [22], [25] and [27], we will cover some preliminaries that form the foundations of our constructions. We begin with some definitions and properties of CW-complexes, where, in Theorem 1.1.8, we show that a CW-complex induces a CW-structure on its universal covering space. In particular, pseudomanifolds with isolated singularities are CW-complexes that deserve some special attention. We then proceed with the construction of cellular homology and conclude the chapter with a discussion on duality in manifolds. In Remark 1.3.10 we provide the motivation for our invariant in the case of singular spaces.

In Chapter 2, based on [2] and [20], we will focus on a purely algebraic construction of the torsion invariant. From an algebraic perspective, torsion is a generalization of the notion of determinant, where we now consider the class of a matrix in the reduced Whitehead group of a ring. An important result is the "gluing" formula, given by Theorem 2.3.3 and Theorem 2.3.4, where the latter considers a correction factor given by the homology of the chain complexes involved. Another notable result is Theorem 2.5.2, where we prove that the torsion of a chain complex is invariant under algebraic subdivision.

In Chapter 3, based on [2], [20], [18] and [19], we begin a topological approach of our invariant, building on the algebraic constructions previously made. In Theorem 3.1.5, we prove that the torsion of a CW-complex is invariant under combinatorial subdivision, which is one of the main results of the whole theory. Furthermore, in Section 3.3, we work with the lens spaces, which were the motivators for Reidemeister's work in 1935. We also show in Theorem 3.4.5 that the Reidemeister torsion satisfies a type of duality induced by Poincaré duality, as constructed in Section 1.3. We conclude the chapter with an interesting application of the torsion invariant in Theorem 3.5.3, where we prove that homeomorphic spaces are not necessarily combinatorially equivalent, thus refuting the Hauptvermutung conjecture.

In Chapter 4, based on [13], we finish our work by defining the invariant torsion for pseudomanifolds with isolated singularities. In general, singular spaces do not satisfy the duality theorem, and to recover this property, M. Goresky and R. MacPherson define intersection homology in [11]. Intersection homology recovers duality but loses homotopy invariance, failing to satisfy one of the Eilenberg–Steenrod axioms and is therefore a generalized homology theory. The first results for intersection torsion in compact PL singular spaces were given by A. Dar in [4]. Following the work of L. Hartmann and M. Spreafico, we aim to define the intersection torsion invariant for pseudomanifolds with a CW-decomposition given by a mapping cone. The cellular construction of the intersection chain complexes in Section 4.2 deserves special attention. Results such as Theorem 4.3.11 and Theorem 4.3.12, characterize the intersection torsion as an honest real number. Moreover, in Theorem 4.4.2 we show the duality to intersection Reidemeister torsion.

All figures in this text are due to the author.

CHAPTER 1

Preliminaries

In this initial chapter, we intend to familiarize the reader with some of the objects and results on which our text is based. The topological spaces that interest us are CW-complexes, and consequently, cellular homology theory will play an important role in our constructions. Another topic that will be important for torsion theory is the Poincaré duality.

1.1 CW-complexes

Remembering that a space is called a *cell* of dimension m if it is homeomorphic to the unit n-ball B^m. It is called an *open cell* of dimension m if it is homeomorphic to $\operatorname{Int} B^m$. In each case the integer m is uniquely determined by the space in question.

Now we define the type of topological space with which we will work in our results.

Definition 1.1.1. *A cellular complex, or CW-complex, is a space X and a collection of disjoint open cells e_α whose union is X such that:*

1. *X is Hausdorff;*
2. *for each open m-cell e_α of the collection, there exists a continuous map $\varphi_\alpha \colon B^m \to X$ that maps $\operatorname{Int} B^m$ homeomorphically onto e_α and carries the boundary of B^m, denoted by ∂B^m, into finite union of open cells, each of dimension less than m;*
3. *a set A is closed in X if $A \cap \overline{e}_\alpha$ is closed i $\overline{e}_\alpha$, for each α.*

The finiteness part of Condition 2 was called “closure-finiteness" by J. H. C. Whitehead. The Condition 3 expresses the fact that X has what he called the “weak topology" relative to the colletion $\{\overline{e}_\alpha\}$. These terms are the origin of the letters C and W in the term CW-complex.

The map φ_α is called *characteristic map* for the open cell e_α. Note that the maps φ_α are not uniquely specified in the definition of a CW-complex. Only the space X and the collection $\{e_\alpha\}$ are specified.

Example 1.1.2. *A 1-dimensional CW-complex is what we call* graph*, i.e., a set of vertices (the 0-cells) and edges (the 1-cells).*

Example 1.1.3. *The sphere n-dimensional S^n has the structure of a CW-complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to considering S^n as the quotient space $B^n/\partial B^n$. See the case $n = 2$ in Figure 1.1.*

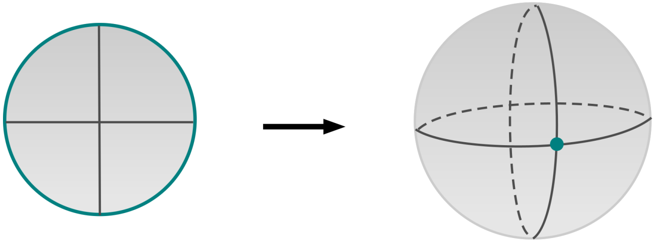

Figure 1.1: A CW-structure for S^2.

We have that the dimension of a CW-complex X is the largest dimension of a cell of X, if such exists, otherwise it is said to be infinite.

Definition 1.1.4. *Let X be a CW-complex. Let Y be a subspace that equals an union of open cells of X. Suppose that for each open cell e_α of X contained in Y, its closure is also contained in Y. Then we shall show that Y is a closed set in X, and that Y is a CW-complex in its own right. The subspace Y is called* CW-subcomplex of X*.*

In particular, denoting by X^n the union of the open cells of X of dimension at most n, we have that X^n satisfies the conditions of previous definition. The subspace X^n is thus a subcomplex of X, which is called the *p-skeleton* of X.

If X and K are CW-complexes, a map $f\colon X \to K$ is *cellular* if $f(X^n) \subset K^n$, for all n. More generally, if (X,Y) and (K,L) are pairs of CW-complexes, a map $f\colon (X,Y) \to (K,L)$ is *cellular* if $f(X^n \cup Y) \subset (K^n \cup L)$, for all n.

Definition 1.1.5. *A CW-complex X for which the maps φ_α can be taken to be homeomorphisms, and for which each set $\overline{e}_\alpha - e_\alpha$ equals the union of finitely many open cells of X, is called a* regular CW-complex*.*

The two results below are very important for our constructions, since they characterizes a CW-structure for several of the spaces which we work throughout this text.

Proposition 1.1.6. *If X and Y are finite CW-complexes, then $X \times Y$ is a CW-complex.*

Proof. We have that a cell for $X \times Y$ is given by $e_\alpha \times e'_\beta$, where e_α is a cell of X and e'_β is a cell of Y. Moreover the respective characteristic map is defined as $\varphi_\alpha \times \varphi'_\beta$. ☐

Theorem 1.1.7 ([25, Theorem 8.27]). *If X is a CW-complex and Y is a CW-subcomplex, then X/Y is a CW-complex.*

Now we have two more results that will be assumed constantly. One states that a covering space of a CW-complex has an induced cellular structure, and the other defines a chain complex of free modules associated with a universal covering space of a CW-complex. More importantly, the latter also defines a basis for each chain module of this complex.

Theorem 1.1.8 ([26, Theorem 2, Section 6.9, Part III]). *Suppose that X is a CW-complex and $p\colon \widetilde{X} \to X$ is a covering of X. Then*

$$\{\widetilde{e}_\alpha;\ e_\alpha \in X \text{ and } \widetilde{e}_\alpha \text{ is a lift of } e_\alpha \text{ to } \widetilde{X}\}$$

is a cell structure on $\widetilde{X}$ with respect to which X becomes a CW-complex. If $\varphi_\alpha\colon I^n \to X$ is a characteristic map for the cell e_α, if $\widetilde{e}_\alpha$ is a lift of e_α and if $\widetilde{\varphi}_\alpha\colon I^n \to \widetilde{X}$ is a lift of φ_α such that $\widetilde{\varphi}_\alpha(x) \in \widetilde{e}_\alpha$ for some $x \in \operatorname{Int} I^n$, then $\widetilde{\varphi}_\alpha$ is a characteristic map for $\widetilde{e}_\alpha$.

Proposition 1.1.9 ([2, (3.15)]). *Suppose that $p\colon \widetilde{X} \to X$ is the universal covering and that Π is the group of covering transformations of $\widetilde{X}$. Assume that Y is a subcomplex of X and $\widetilde{Y} = p^{-1}(Y)$. For each cell e_α of $X - Y$, consider a specific characteristic map $\varphi_\alpha\colon I^n \to X$ and a specific lift $\widetilde{\varphi}_\alpha\colon I^n \to \widetilde{X}$ of φ_α be chosen. Then*

$$\{\langle\widetilde{\varphi}_\alpha\rangle;\ e_\alpha \in X - Y\}$$

is a basis for $\mathscr{C}(\widetilde{X},\widetilde{Y})$ as a complex of $\mathbb{Z}(\Pi)$-modules.

Definition 1.1.10. *If X is a topological space, define an equivalence relation on $X \times I$ by $(x,t) \sim (x',t')$ if $t = t' = 1$. The quotient space $X \times I/\sim$ is called the* cone over X*, and it is denoted by $C(X)$.*

We may also regard $C(X)$ as the quotient space $X \times I/X \times \{1\}$. The identified point as the equivalence class of $(x,1)$ is called the *vertex*. We have essentially introduced a new point, namely the vertex v, that does not belong to X and joined each point in X to v by a line segment. The Figure 1.2 gives an idea of the structure of the cone over a space X with vertex v.

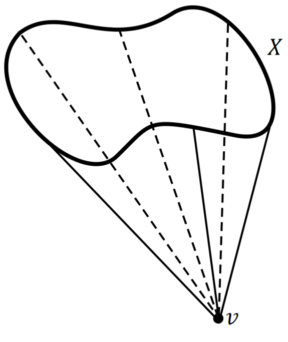

Figure 1.2: The cone over a space X.

By Theorem 1.1.7, we have that $C(X)$ has a CW-structure induced by CW-structure of X. Moreover, note that the cone $C(X)$ always come with a natural inclusion $i\colon X \to C(X)$, where $i(x) = [1,x]$, that is cellular and defines the cellular pair $(C(X),X)$.

Definition 1.1.11. *A topological pair (X,A) is called a* relative homology n-manifold *if for each point x of X not in A, the local homology group*

$$H_i(X, X-\{x\}) \cong \left\{ \begin{array}{ll} \mathbb{Z}, & i=n \\ 0, & i \neq n \end{array} \right. .$$

In the case where A is empty, we said simply that X is a homology n-manifold.

If M is an n-manifold with boundary, then the pair $(M,\partial M)$ is a relative homology n-manifold. More generally, if (X,A) is any pair such that $X-A$ is an n-manifold, then (X,A) is a relative homology n-manifold.

Let N be a regular oriented CW-complex that is a homology m-manifold. A regular oriented CW-structure on $C(N)$ is given as follows. Let v be the vertex of $C(N)$. For each cell e in N denote by $[v,e]$ the cone over $\overline{e}$ with vertex v. Since N is regular, the set of the cells $[v,e]$ and e, with e a cell of N, coincides with the standard CW-structure of $C(N)$ induced by $I \times N$, and gives a regular oriented CW-structure on $C(N)$.

The cone of a CW-complex has a very interesting structure, where it is a manifold except by its vertex. Let us see some spaces with similar behavior.

Definition 1.1.12. *An n-dimensional pseudomanifold is an n-dimensional finite, regular CW complex which satisfies the following three conditions:*

1. *every cell is a face of some n-cell;*
2. *every $(n-1)$-dimensional cell is a face of at most two n-cells;*
3. *given any two n-cells, e^n and $\hat{e}^n$, there exists a sequence of n-cells*

   $$\{e_0^n, e_1^n, \dots, e_k^n\},$$

   such that $e_0^n = e^n$, $e_k^n = \hat{e}^n$ and e_{i-1} and e_i have a common $(n-1)$-dimensional face, for all $j = 1,2,\dots,k$.

The boundary of an n-dimensional pseudomanifold X, denoted by ∂X, is defined to be the subcomplex of X generated by the $(n-1)$-cells which are faces of exactly one n-cell of X.

Example 1.1.13. *The wedge sum $S^2 \vee S^2$, i.e., two 2-spheres joined at a point is a pseudomanifold. See Figure 1.3.*

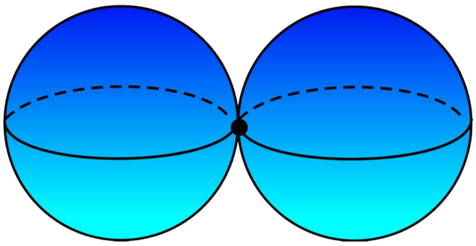

Figure 1.3: Wedge sum of two S^2.

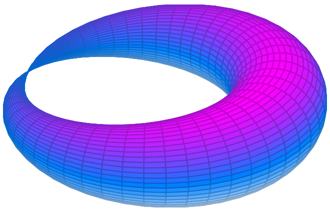

Figure 1.4: Pinched torus.

Example 1.1.14. *The pinched torus, i.e., the torus having collapsed one of the generators of its homology group H_1 to a point is a pseudomanifold. See Figure 1.4.*

Remember that a *filtration* of a topological space X is a sequence of subspaces $\{X^n; n \in \mathbb{N}\}$ with $X^n \subset X^{n+1}$, for all n.

Definition 1.1.15. *A stratification of a pseudomanifold m-dimensional X with isolated singularities is a filtration by closed subspaces*

$$X_0 \subset X_1 \subset \cdots \subset X_{m-1} \subset X_m = X,$$

such that for each point $p \in X_i - X_{i-1}$ there is a filtered space

$$p = V_i \subset \cdots \subset V_{m-1} \subset V_m = V$$

and a mapping $V \times B^i \to X$ which, for each j, takes $V_j \times B^i$ homeomorphically to a neighborhood of p in X_j.

Thus, if $X_i - X_{i-1}$ is not empty, it is a manifold of dimension i, and is called the i-dimensional *stratum* of the stratification. Every pseudomanifold admits a stratification.

Definition 1.1.16. *An n-dimensional pseudomanifold with isolated singularities and boundary disjoint from the singular set is an n-dimensional pseudomanifold X with boundary such that there exists a finite subset of* 0*-cells $\Sigma = \{x_0, \ldots, x_k\}$ in the interior of X such that $X - \Sigma$ is an n-dimensional homology manifold with boundary. The subspace Σ is called the singular locus of X.*

A subcomplex Y disjoint from the singular set of X is called *proper subcomplex*. We call the pair (X,Y) a *proper relative pseudomanifold*.

1.2 Cellular homology

We will then construct the homology groups of a CW-complex.

Lemma 1.2.1 ([14, Lemma 2.34]). *If X is a CW-complex , then:*

1. *$H_k(X^n,X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n-cells of X;*
2. *$H_k(X^n)$ is zero for $k > n$. In particular, if X is finite-dimensional then $H_k(X)$ is zero for $k >$ dim$X$;*
3. *The inclusion $i\colon X^n \hookrightarrow X$ induces an isomorphism $i_*\colon H_k(X^n) \longrightarrow H_k(X)$ if $k < n$.*

Let X be a CW-complex. Using the Lemma 1.2.1, from the long exact sequences for the pairs (X^{n+1},X^n), (X^n,X^{n-1}) and (X^{n+1},X^{n-2}) we have the following diagram

$$\begin{tikzcd}
 & & & & 0 \\
 & & 0 \arrow[d] & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & \\
 & & H_n(X^n) \arrow[ur] \arrow[d, "j_n"] & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1},X^n) \arrow[r] \arrow[ur, "\partial_{n+1}"] & H_n(X^n,X^{n-1}) \arrow[r] \arrow[d, "\partial_n"'] & H_{n-1}(X^{n-1},X^{n-2}) \arrow[r] & \cdots \\
 & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & 0 \arrow[ur] & & &
\end{tikzcd}$$

where d_{n+1} and d_n are defined as the compositions $j_n \circ \partial_{n+1}$ and $j_{n-1} \circ \partial_n$, which are just “relativizations" of the boundary maps ∂_{n+1} and ∂_n. The composition $d_n \circ d_{n+1}$ includes two successive maps in one of the exact sequences, hence is zero. Thus the horizontal row in the diagram is a chain complex, called the *cellular chain complex* of X since $H_n(X^n,X^{n-1})$ is free with basis in one-to-one correspondence with the n-cells of X, so one can think of elements of $H_n(X^n,X^{n-1})$ as linear combinations of n-cells of X.

Definition 1.2.2. *The homology groups of this cellular chain complex are called the cellular homology groups of X. Temporarily we denote these groups by $H_n^{CW}(X)$.*

Theorem 1.2.3. *We have that $H_n^{CW}(X) \cong H_n(X)$.*

Proof. From the diagram above, $H_n(X)$ can be identified with $H_n(X^n)/\operatorname{Im}\partial_{n+1}$. Since j_n is injective, it maps $\operatorname{Im}\partial_{n+1}$ isomorphically onto

$$\operatorname{Im} j_n \circ \partial_{n+1} = \operatorname{Im} d_{n+1}$$

and $H_n(X^n)$ isomorphically onto $\operatorname{Im} j_n = \ker\partial_n$. Since j_{n-1} is injective, $\ker\partial_n = \ker d_n$. Thus j_n induces an isomorphism of the quotient $H_n(X^n)/\operatorname{Im}\partial_{n+1}$ onto $\ker d_n/\operatorname{Im} d_{n+1}$. ☐

We have some immediate applications of the previous result.

Corollary 1.2.4. *If X is a CW-complex with no n-cells then $H_n(X) = 0$.*

Corollary 1.2.5. *More generally, if X is a CW-complex with k n-cells, then $H_n(X)$ is generated by at most k elements. Since $H_n(X^n, X^{n-1})$ is free abelian on k generators, the subgroup $\ker d_n$ must be generated by at most k elements, hence also the quotient $\ker d_n/\operatorname{Im} d_{n+1}$.*

Corollary 1.2.6. *If X is a CW-complex having no two of this cells in adjacent dimensions, then $H_n(X)$ is free abelian with basis in one-to-one correspondence with n-cells of X.*

Proof. This is because the cellular boundary maps d_n are automatically zero in this case. ☐

Example 1.2.7. *Since $\mathbb{C}P^n$ has a structure of CW-complex with one n-cell of each even dimension $2k \leq 2n$, by Corollary 1.2.6 we have*

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & \text{for } i = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Example 1.2.8. *Since S^n has a structure of CW-complex with one 0-cell and one n-cell, by Corollary 1.2.6, for $n \geq 2$ we have*

$$H_i(S^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

1.3 Poincaré duality

Let K and L be nonempty complexes in some euclidean space. Let $s = v_0 \ldots v_m$ be the general simplex of K and let $t = w_0 \ldots w_n$ be the general simplex of L. Suppose that whenever $s \in K$ and $t \in L$, the points $v_0, \ldots, v_m, w_0, \ldots, w_n$ are independent. Then we denote by $s * t$ the simplex $v_0 \ldots v_m w_0 \ldots w_n$ they span. If the collection consisting of all simplices $s * t$ and their faces is a simplicial complex, then this complex is called the *join* of K and L, and is denoted by $K * L$.

Example 1.3.1. *If K consists of a single vertex v, then $v * L$ is the cone over L, as already defined. See Figure 1.2.*

Example 1.3.2. *If K consists of two points, then $K * L$ is the suspension of L and is denoted by $S(L)$.*

Definition 1.3.3. *Let s be a simplex of the complex K. The* star *of s in K is the union of the interiors of all simplices of K having s as a face, and is denoted by* $\mathrm{St}\, s$. *The closure of* $\mathrm{St}\, s$ *is the union of all simplices of K having s as a face, and is denoted by* $\overline{\mathrm{St}}\, s$. *The link of s in K is the union of all simplices of K lying in* $\overline{\mathrm{St}\, s}$ *that are disjoint from s, and is denoted by* $\mathrm{Link}(s)$.

Example 1.3.4. *In the 2-dimensional complex in Figure 1.5, the link of the vertex g consist of the hexagon $abcdefa$ and the vertex h. The link of the vertex h is the vertex g.*

Example 1.3.5. *In the 3-dimensional complex in Figure 1.6, the link of the 1-simplex fg is the pentagon $P = abcdea$, and the link of the vertex f is the cone $P * g$. The link of the 1-simplex ab is the 1-simplex fg, and the link of the vertex a is the union of the 2-simplicies bfg and efg.*

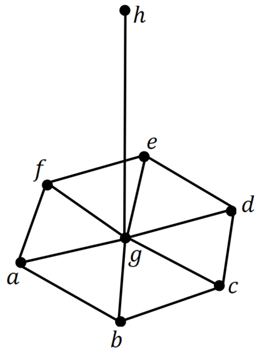

Figure 1.5: Links of vertices in a 2-complex.

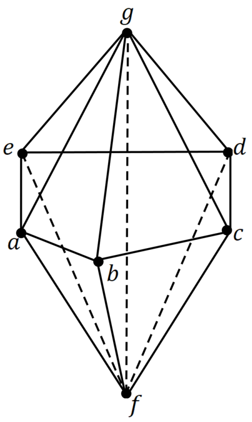

Figure 1.6: Links of vertices in a 3-complex.

Let X be a locally finite simplical complex, and we denote by $\mathrm{sd} X$ its first barycentric subdivision. The simplices of $\mathrm{sd} X$ are of the form

$$\widehat{\sigma}_{i_1}\widehat{\sigma}_{i_2}\ldots\widehat{\sigma}_{i_k},$$

where $\widehat{\sigma}_{i_k}$ is the barycenter of σ_{i_k} and each σ_{i_k} is face of $\sigma_{i_{k+1}}$. We partially order the vertices of $\mathrm{sd} X$ by decreasing dimension of the simplices of X of which they are the barycenters. This ordering induces a linear ordering on the vertices of each simplex of $\mathrm{sd} X$. Given a simplex σ of X, the union of all open simplices of $\mathrm{sd} X$ of which $\widehat{\sigma}$ is the *initial* vertex is just $\mathrm{Int}\, \sigma$.

We define $D(\sigma)$ to be the union of all open simplices of $\mathrm{sd} X$ of which $\widehat{\sigma}$ is the *final* vertex. We called this set the *dual block to* σ.

The blocks $D(\sigma)$ will play similar to that of the open cells of a CW-complex. We call the closure $\overline{D}(\sigma)$ of $D(\sigma)$ the *closed block dual to* σ. The polytope of a subcomplex of $\mathrm{sd} X$ is equals the union of all simplices of $\mathrm{sd} X$ of which $\widehat{\sigma}$ is the final vertex.

Example 1.3.6. *Let X be the 2-dimensional complex in Figure 1.7. The complex* $\mathrm{sd} X$ *is indicated by black and blue lines. The block dual to any 2-simplex σ consists of its barycenter $\widehat{\sigma}$ alone. The block*

$D(ab)$ dual to the 1-simplex $s = ab$ consists of its barycenter $\widehat{s}$ and the two open line segments joining $\widehat{s}$ to the barycenters of the two triangles having ab as a face. The dual block to the vertex e is the region of dotted lines, and the corresponding closed block consists of this region plus its boundary in blue. Note the interesting fact that $\overline{D}(e) - D(e)$ is the union of the lower-dimensional blocks $D(s_i)$ as s_i ranges over all 1-simplices and 2-simplices having e as a face.

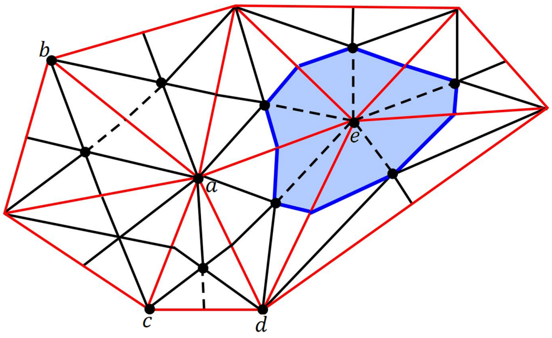

Figure 1.7: Some blocks dual in a 2-dimensional complex.

Example 1.3.7. *Let X be the complex in Figure 1.8. It is the join of the polygon $cdefc$ with the line segment ab. The block dual to each 3-simplex of X is its barycenter. The closed block dual to the 2-simplex abc consists of the two line segments joining the barycenter of abc with the barycenters of the two 3-simplices having abc as a face. The closed block $\overline{D}(ab)$ dual to the 1-simplex ab is the closed octogonal region in Figure 1.9. The set $\partial D(ab)$ is the union of the blocks dual to the 2-simplices and 3-simplices of X having ab as a face.*

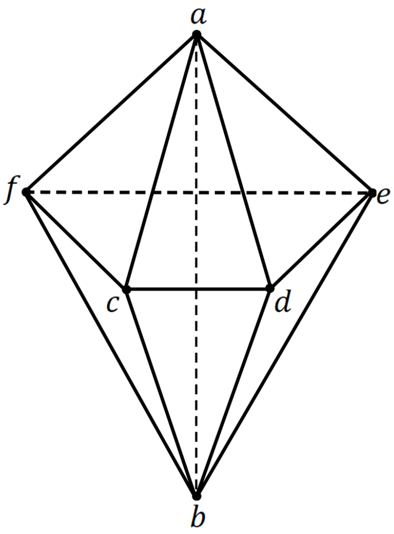

Figure 1.8: Join of $cdefc$ with ab.

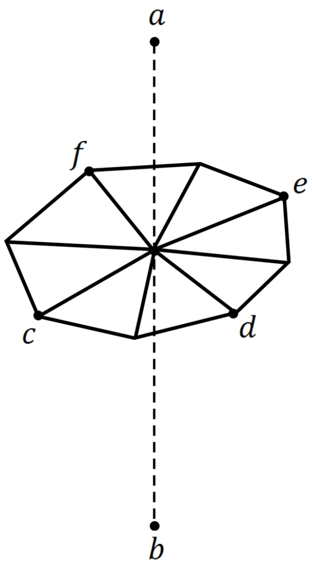

Figure 1.9: Blocks dual.

Definition 1.3.8. *Let X be a locally finite complex that is a homology n-manifold. The collection of the dual blocks $D(\sigma)$ will be called the* dual block decomposition *of X. The union of the blocks of dimension at most p will be denoted by X_p, and called the* dual p-skeleton *of X. The* dual chain complex $\mathscr{D}(X)$ *of X is defined with chain groups given by*

$$D_p(X) = H_p(X_p, X_{p-1}),$$

and the boundary homomorphism is defined by the exact sequence of the triple (X_p, X_{p-1}, X_{p-2}) (see Lemma 1.2.1).

Therefore, now we can state the result that this section proposes.

Theorem 1.3.9 (Poincaré duality[22, Theorem 65.1])**.** *Let X be a compact triangulated homology n-manifold. If X is orientable, then for all p, there is an isomorphism*

$$H^p(X) \cong H_{n-p}(X),$$

with coefficients in any group G. Moreover, if X is non-orientable, there is for all p an isomorphism

$$H^p(X) \cong H_{n-p}(X),$$

with coefficients in $\mathbb{Z}_2$.

A crucial point of the proof consists of showing that there is a biunivocal correspondence between the p-simplices of X and the dual $n-p$ blocks of X that maps each simplex to its dual block. Hence the free abelian groups $C^p(X)$ and $D_{n-p}(X)$ are isomorphic, by an isomorphism ϕ that carries the basis element σ^* for $C^p(X)$, where σ is an oriented p-simplex, to a generator of $H_{n-p}(\overline{D}(\sigma), \partial\overline{D}(\sigma))$. If X is orientable, it is showed that the sign of $\phi(\sigma^*)$ may be chosen that the following diagram commutes

$$\begin{tikzcd}
C^{p-1}(X) \arrow[r, "\phi"] \arrow[d, "\partial^{p-1}"] & D_{n-p+1}(X) \arrow[d, "\partial_{n-p+1}"] \\
C^p(X) \arrow[r, "\phi"] & D_{n-p}(X).
\end{tikzcd}$$

This is basically what will prove the existence of the Poincaré duality isomorphism in the case of integer coefficients.

Remark 1.3.10. *The homology groups isomorphic to the singular homology groups do not satisfy the duality for singular spaces, in general. For this reason M. Goresky and R. MacPherson define the intersection homology in [11], in order to recover the duality property in pseudomanifolds. In Chapter 4 we will use a cellular construction of intersection homology given by L. Hartmann and M. Spreafico in [13].*

CHAPTER 2

Algebraic theory of torsions

In this chapter we will introduce the torsion invariant with the construction of the purely algebraic part of the theory. This makes sense for at least two reasons. First, to give the reader an idea of how strong are the algebraic structures that define this invariant. Second, to give us a sense of the importance of the torsions theory regardless of having a topological space in mind.

2.1 The groups $K_G(R)$

Let $GL(n,R)$ be the group of non-singular $n \times n$ matrices, i.e., matrices which have a two-sided inverse, over the ring with unity R. We denote the unity of R by 1_R. There is a natural injection of $GL(n,R)$ into $GL(n+1,R)$ given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1_R \end{pmatrix}.$$

The *infinite general linear group of R* is defined as the direct limit

$$GL(R) = \lim_{\rightarrow} GL(n,R) = \frac{\sqcup_n GL(n,R)}{\sim},$$

where

$$A \sim \begin{pmatrix} A & 0 \\ 0 & 1_R \end{pmatrix}.$$

Alternatively, $GL(R)$ may be thought of as the group consisting of all infinite non-singular matrices which are eventually the identity. For convenience of notation we shall identify each $A \in GL(n,R)$ with its image in $GL(R)$.

Let $E^n_{i,j}$, with $i \neq j$, be the $n \times n$ matrix with all entries zero except for a 1_R, which denote an unity element in R, in the (i,j)-spot. An *elementary matrix* is a matrix of the form $(I_n + aE^n_{i,j})$, for some $a \in R$. We denote by $E(R)$ the subgroup of $GL(R)$ generated by the elementary matrices. Elements of $E(R)$ will be denoted by E, E_1, E_2 etc.

In order to study $GL(R)/E(R)$, define an equivalence relation in $GL(R)$ by: $A \sim B$ if and only if there are elements $E_1, E_2 \in E(R)$ such that $A = E_1 B E_2$.

We will prove in Proposition 2.1.2 that $E(R)$ is normal showing that $E(R)$ is the commutator subgroup of $GL(R)$. For this, note that $A \sim B$ if and only if A can be gotten from B by a finite sequence of operations which consist of adding a left multiple of one row to another, or a right multiple of the column to another. More generally, instead of using rows, if P_1 and P_2 are disjoint $p \times n$ and $q \times n$ submatrices of the non-singular $n \times n$ matrix A and if X is a $p \times q$ matrix, the following hold

$$I_R\colon A = \begin{pmatrix} \cdots\cdots \\ P_1 \\ \cdots\cdots \\ P_2 \\ \cdots\cdots \end{pmatrix} \sim B = \begin{pmatrix} \cdots\cdots\cdots \\ P_1 + XP_2 \\ \cdots\cdots\cdots \\ P_2 \\ \cdots\cdots\cdots \end{pmatrix}$$

$$II_R\colon \text{If } p = q,\ A = \begin{pmatrix} \cdots\cdots \\ P_1 \\ \cdots\cdots \\ P_2 \\ \cdots\cdots \end{pmatrix} \sim B = \begin{pmatrix} \cdots\cdots \\ P_2 \\ \cdots\cdots \\ -P_1 \\ \cdots\cdots \end{pmatrix}.$$

I_R is immediate from the definition of matrix multiplication. II_R follows from I_R by the sequence

$$\begin{pmatrix} \cdots\cdots \\ P_1 \\ \cdots\cdots \\ P_2 \\ \cdots\cdots \end{pmatrix} \sim \begin{pmatrix} \cdots\cdots\cdots \\ P_1 + P_2 \\ \cdots\cdots\cdots \\ P_2 \\ \cdots\cdots\cdots \end{pmatrix} \sim \begin{pmatrix} \cdots\cdots\cdots\cdots\cdots \\ P_1 + P_2 \\ \cdots\cdots\cdots\cdots\cdots \\ P_2 - (P_1 + P_2) \\ \cdots\cdots\cdots\cdots\cdots \end{pmatrix} = \begin{pmatrix} \cdots\cdots\cdots \\ P_1 + P_2 \\ \cdots\cdots\cdots \\ -P_1 \\ \cdots\cdots\cdots \end{pmatrix} \sim \begin{pmatrix} \cdots\cdots\cdots\cdots\cdots \\ (P_1 + P_2) - P_1 \\ \cdots\cdots\cdots\cdots\cdots \\ -P_1 \\ \cdots\cdots\cdots\cdots\cdots \end{pmatrix} = \begin{pmatrix} \cdots\cdots \\ P_2 \\ \cdots\cdots \\ -P_1 \\ \cdots\cdots \end{pmatrix}.$$

The corresponding operations on columns give rise to analogous equivalences which we call I_C and II_C:

$$I_C\colon A = \begin{pmatrix} \vdots & P_1 & \vdots & P_2 & \vdots \end{pmatrix} \sim B = \begin{pmatrix} \vdots & P_1 + XP_2 & \vdots & P_2 & \vdots \end{pmatrix},$$

$$II_C\colon \text{If } p = q,\ A = \begin{pmatrix} \vdots & P_1 & \vdots & P_2 & \vdots \end{pmatrix} \sim B = \begin{pmatrix} \vdots & P_2 & \vdots & -P_1 & \vdots \end{pmatrix}.$$

Proposition 2.1.1. *If A and B are elements of $GL(R)$ then $AB \sim BA$.*

Proof. For sufficiently large n we may assume that A and B are both $n \times n$ matrices. Then

$$AB = \begin{pmatrix} AB & 0 \\ 0 & I_n \end{pmatrix} \sim \begin{pmatrix} AB & A \\ 0 & I_n \end{pmatrix} \sim \begin{pmatrix} 0 & A \\ -B & I_n \end{pmatrix} \sim \begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix}$$

Similarly $BA = \begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix}$.

Finally, using II_C and II_R, respectively, we have

$$\begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \sim \begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix}.$$

$\square$

Proposition 2.1.2. *The subgroup $E(R)$ is the commutator subgroup of $GL(R)$.*

Proof. If $E \in E(R)$ and $X \in GL(R)$, then $(XE)X^{-1} \sim X^{-1}(XE)$ by the Proposition 2.1.1, and so $XEX^{-1} = E_1EE_2$. Given a commutator $ABA^{-1}B^{-1}$ we apply this with $X = BA$ to get

$$(AB)(A^{-1}B^{-1}) = [E_1(BA)E_2](BA)^{-1} = E_1[(BA)E_2(BA)^{-1}] \in E(R).$$

Hence the commutator subgroup is contained in $E(R)$.

Conversely, a typical generator of $E(R)$ is of the form $(I_n + aE^n_{i,k})$. Noticing that $(I_n + aE^n_{i,j})^{-1} = (I_n - aE^n_{i,j})$, we can see that this generator is a commutator because

$$(I_n + aE^n_{i,k}) = (I_n + aE^n_{i,j})(I_n + aE^n_{j,k})(I_n - aE^n_{i,j})(I_n - aE^n_{j,k}).$$

□

The following result is immediate .

Corollary 2.1.3. *If H is a subgroup of $GL(R)$ containing $E(R)$ then H is a normal subgroup and $GL(R)/H$ is abelian.*

Suppose that G is a subgroup of the group of the units of R. Let E_G be the group generated by $E(R)$ and all matrices of the form

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & I_m \end{pmatrix},$$

where $g \in G$ and, possibly, $m \to \infty$. Then we define

$$K_G(R) := \frac{GL(R)}{E_G}.$$

By Corollary 2.1.3, $K_G(R)$ is an abelian group. We denote the quotient map by $\tau\colon GL(R) \to K_G(R)$ and we call $\tau(A)$ the *torsion of the matrix A*. Since $K_G(R)$ is abelian and, as by convenience, we will use the additive notation, we have $\tau(AB) = \tau(A) + \tau(B)$.

Example 2.1.4. *For $G = \{1_R\}$, we have that $K_1(R) = GL(R)/E(R)$, which is called Whitehead group of R.*

Example 2.1.5. *If $G = \{\pm 1_R\}$, we denote $K_G(R)$ by $\overline{K}_1(R)$, and call reduced Whitehead group of R.*

Example 2.1.6. *If G is a given group, $R = \mathbb{Z}(G)$ and $T = G \cup (-G)$ is the group of the trivial units of $\mathbb{Z}(G)$, then $K_T(\mathbb{Z}(G))$ is called Whitehead group of G and is denoted by $Wh(G)$.*

If G and G' are subgroups of the units of R and R', respectively, then any ring homomorphism $f\colon R \to R'$ such that $f(G) \subset G'$ induces a group homomorphism $f_*\colon K_G(R) \to K_{G'}(R')$ given by

$$f_*\tau((a_{i,j})) = \tau((f(a_{i,j}))).$$

The map f_* is well-defined, since if $(a_{i,j}) \in E_G$, then $(f(a_{i,j})) \in E_{G'}$. Thus we have a covariant functor $\mathscr{C} \to \mathscr{D}$, where $\mathscr{C}$ is the category whose objects are the pairs (R,G) and the morphisms are the ring homomorphisms $f\colon R \to R'$ with $f(G) \subset G'$ and $\mathscr{D}$ is the category whose objects are the abelian groups $K_G(R)$ and the morphisms are the group homomorphisms $f_*\colon K_G(R) \to K_{G'}(R')$.

Hence, we have a covariant functor from the category of groups and group homomorphisms to the category of abelian groups and group homomorphisms given by

$$G \mapsto Wh(G),$$

$$\{f\colon G \to G'\} \mapsto \{f_*\colon Wh(G) \to Wh(G')\},$$

where f first induces the ring homomorphism $\mathbb{Z}(G) \to \mathbb{Z}(G')$ given by $\sum_i n_i g_i \mapsto \sum_i n_i f(g_i)$, which induces the map f_* as in the previous paragraph.

Lemma 2.1.7. *If $g \in G$ and $f\colon G \to G$ is a group homomorphism such that $f(x) = gxg^{-1}$ for all x then $f_*\colon Wh(G) \to Wh(G)$ is the identity map.*

Proof. Consider the induced homomorphism $\mathbb{Z}(G) \to \mathbb{Z}(G)$ given by

$$\sum_i n_i g_i \mapsto \sum_i n_i f(g_i) = \sum_i n_i (g g_i g^{-1}) = g(\sum_i n_i g_i) g^{-1}.$$

Hence we have $f_*\colon Wh(G) \to Wh(G)$ and then

$$f_*(\tau(A)) = \tau(f(A)) = \tau(xIAx^{-1}I) = \tau(xI) + \tau(A) + \tau(x^{-1}I) = \tau(A).$$

Therefore f_* is the identity map. □

Theorem 2.1.8. *Suppose that R is a commutative ring and G is a subgroup of the group $U(R)$ of all units of R. Let $SK_1(R)$ be the image $\tau_G(SL(R))$, where $\tau_G\colon GL(R) \to K_G(R)$ and $SL(R)$ is the subgroup of $GL(R)$ of matrices of determinant 1. Then there is a split short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & SK_1(R) \arrow[r, "i"] & K_G(R) \arrow[r, shift left, "{[\det]}"] & U(R)/G \arrow[l, shift left, "s"] \arrow[r] & 0, \end{tikzcd}$$

where i is the inclusion, $[\det](\tau A)$ denote the coset of $(det A)$ in $U(R)/G$, and $s(u \cdot G) = \tau_G(u)$. In particular, if R is a field, $[\det]$ is an isomorphism.

Proof. The sequence above is exact, since if $\tau(A) \in \operatorname{Im} i$ then $i(\tau(A)) = \tau(A)$ and $[\det](\tau(A)) = (\det A) \cdot G = 1 \cdot G = G$. Thus $\tau(A) \in \ker[\det]$. Now, given $\tau(B) \in \ker[\det]$, we have $\det B = g \in G$. For $A = I + (g^{-1} - 1)E_{ii}$, $B \sim AB$, where $\det A = g^{-1}$, we obtain $AB \in SL(R)$ and $\tau(AB) = \tau(B) \in SK_1(R)$. Thus $i(AB) = \tau(AB) = \tau(B)$ and hence $\ker[\det] = \operatorname{Im} i$. Moreover

$$[\det] \circ s(u \cdot G) = [\det(\tau(u))] = (\det u) \cdot G \ = u \cdot G$$

and so $[\det] \circ s = 1_{U(R)/G}$. We conclude that the sequence is split.

Now we claim that the group $SK_1(R) = \tau_G(SL(R))$ is independent of G. In fact, we have

$$\begin{tikzcd} GL(R) \arrow[r, "\tau_1"] \arrow[rr, bend right, "\tau_G"'] & K_1(R) \arrow[r, "\pi"] & K_G(R), \end{tikzcd}$$

where π is the natural projection, and then $\pi|_{\tau_1(SL(R))}\colon \tau_1(SL(R)) \to \tau_G(SL(R))$ is an isomorphism, since given $A \in SL(R)$ such that $\pi(A) = 0$, we have that $A \in E(R)$ and hence $\ker(\pi) = \{0\}$. If $\tau(B) \in \tau_G(SL(R))$ and $\det B = g$ then, as before, there is A with $\det A = g^{-1}$, $\pi(A) = 0$ and $AB \sim B$ in E_G so that $AB \in \tau_1(SL(R))$ and $\pi(AB) = B$. Hence $\pi_{\tau_1(SL(R))}$ is surjective and the isomorphism follows. In particular, if R is a field, so $U(R) = R - \{0\} = R^\times$ and G is a subgroup of $U(R)$. Since $SK_1(R)$ does not depend of G, for $G = R^\times$ we have $SK_1(R) = 0$. Thus $SK_1(R) = 0$ for any subgroup G of $U(R)$. By the exactness of the sequence, [det] is an isomorphism. ☐

The group $SK_1(R)$ is called *special Whitehead group*. As a consequence of the Theorem 2.1.8, we will have a particular interest in the direct sum decomposition

$$K_1(R) \cong U(R) \oplus SK_1(R).$$

Example 2.1.9. *For the ring $\mathbb{Z}$, it is well known that $SL(n,\mathbb{Z})$ is generated by its transvections, i.e., by the elementary matrices $T_{i,j} = (I_n + 1E^n_{i,j})$. Thus we have $SK_1(\mathbb{Z}) = 0$. Then*

$$K_1(\mathbb{Z}) \cong U(\mathbb{Z}) = \{\pm 1\}.$$

Moreover, it follows that $\overline{K}_1(\mathbb{Z}) = 0$.

2.2 Elementary constructions with free modules

Let F be a free R-module and let $b = \{b_1, \ldots, b_k\}$ and $c = \{c_1, \ldots, c_k\}$ be two different bases for F. Setting $c_i = \sum a_{ij} b_j$ we obtain a non-singular matrix (a_{ij}) with entries in R. The corresponding element of the reduced Whitehead group $\overline{K}_1(R)$ will be denoted by $\left[\frac{c}{b}\right]$. If $\left[\frac{c}{b}\right] = 0$ then we will say that b is *equivalent* to c, denoting by $b \sim c$.

The identities

$$\left[\frac{d}{b}\right] = \left[\frac{d}{c}\right] + \left[\frac{c}{b}\right] \text{ and } \left[\frac{b}{b}\right] = 0$$

show that this is an equivalence relation.

The above identities follow directly from the definition of change of basis matrix. It is immediate that $b \sim b$ since $\left[\frac{b}{b}\right] = 0$. Moreover if $b \sim c$, we know that

$$\left[\frac{b}{c}\right] + \left[\frac{c}{b}\right] = \left[\frac{b}{b}\right].$$

Hence $\left[\frac{b}{c}\right] = 0$ since $\left[\frac{c}{b}\right] = 0 = \left[\frac{b}{b}\right]$, so that $c \sim b$. Finally if $b \sim c$ and $c \sim d$, we have

$$\left[\frac{d}{b}\right] = \left[\frac{d}{c}\right] + \left[\frac{c}{b}\right] = 0,$$

so that $b \sim d$. Thus $\sim$ is a reflexive, symmetric and transitive relation. It follows that $\sim$ is an equivalence relation.

Example 2.2.1. *For a free R-module of rank 2, the basis* $\beta = \{b_1 + ab_2, b_2\}$ *is equivalent to the basis* $\alpha = \{b_1, b_2\}$.

In fact, we know that the matrix representing the change of basis from α *to* β *is given by*

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \in E(R).$$

Thus $\left[\frac{\beta}{\alpha}\right] = 0$ *and hence* $\alpha \sim \beta$.

Next consider a short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow S \longrightarrow 0,$$

of free modules. Given bases $m = \{m_1, \dots, m_k\}$ for M and $s = \{s_1, \dots, s_l\}$ for S, we construct a basis ms for N as follows. Lift each $s_i \in S$ to an element $\widetilde{s_i} \in N$. Then

$$m\widetilde{s} = \{m_1, \dots, m_k, \widetilde{s}_1, \dots, \widetilde{s}_l\}$$

is the required basis.

Remark 2.2.2. *If* $\bar{m}$ *and* $\bar{s}$ *are alternative bases for the modules M and S respectively, then*

$$\left[\frac{\widetilde{\bar{m}\bar{s}}}{m\widetilde{s}}\right] = \left[\frac{\bar{m}}{m}\right] + \left[\frac{\bar{s}}{s}\right], \tag{2.1}$$

since the matrix representing the change of base from $m\widetilde{s}$ *to* $\widetilde{\bar{m}\bar{s}}$ *is given by*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A denote the matrix representing the change of base from m to $\bar{m}$ *and B denote the matrix representing the change of base from s to* $\bar{s}$.

Of course this basis ms depends on the choice of the $\widetilde{s_i}$. However the equivalence class of ms depends only on m and s. Indeed, if

$$m\widehat{s} = \{m_1, \dots, m_k, \widehat{s}_1, \dots, \widehat{s}_l\}$$

is obtained with other lifting choice, then

$$\left[\frac{m\widehat{s}}{m\widetilde{s}}\right] = \left[\frac{m}{m}\right] + \left[\frac{s}{s}\right] = 0.$$

The following reformulation will often be convenient. Suppose that we are given free modules

$$F_0 \subset F_1 \subset F_2$$

together with bases b_1 for the quotient module F_1/F_0 and b_2 for the quotient module F_2/F_1. Then obtain a basis $b_1\widetilde{b_2}$ for F_2/F_0, using the exact sequence

$$0 \longrightarrow F_1/F_0 \longrightarrow F_2/F_0 \longrightarrow F_2/F_1 \longrightarrow 0,$$

Remark 2.2.3. *This construction is associative, in the sense that given modules*

$$F_0 \subset F_1 \subset F_2 \subset F_3$$

and bases b_i *for the free modules* F_i/F_{i-1}*, the basis* $(b_1\widetilde{b_2})\widetilde{b_3}$ *for* F_3/F_0 *is equivalent to the basis* $b_1\widetilde{(b_2\widetilde{b_3})} = b_1(\widetilde{b_2}\widetilde{\widetilde{b_3}})$*, since this equivalence class depend only on* b_1*,* b_2 *and* b_3*.*

More generally, given modules

$$F_0 \subset F_1 \subset \cdots \subset F_k$$

and given bases b_i for the quotients F_i/F_{i-1}, we obtain a basis $b_1\widetilde{b_2}\ldots\widetilde{b_k}$ for F_k/F_0 which is well defined up to equivalence.

It is be convenient to represent situations such as this by diagrams such as the following.

$$\begin{tikzcd}
F_0 \arrow[r, "b_1"'] \arrow[rrr, bend left, "b_1\widetilde{b_2}\widetilde{b_3}"] & F_1 \arrow[r, "b_2"'] \arrow[rr, bend right, "b_2\widetilde{b_3}"'] & F_2 \arrow[r, "b_3"'] & F_3
\end{tikzcd}$$

Each arrow represents an inclusion map. The letter associated with each arrow denotes a basis (or equivalence class of bases) for the corresponding quotient module.

Our construction is also commutative, in the following sense: Let M and N be submodules of S, and let $M+N$ be the smallest submodule containing both. Thus we obtain a diagram:

$$\begin{tikzcd}
& S & \\
& M+N \arrow[u] & \\
M \arrow[ur, "c"] & & N \arrow[ul, "b"'] \\
& M\cap N \arrow[ul, "b"] \arrow[ur, "c"'] & \\
& 0 \arrow[u] &
\end{tikzcd}$$

Suppose that $M/M\cap N$ is free with basis b and that $N/(M\cap N)$ is free with basis c. By Isomorphism theorem, we have $M/(M\cap N) \cong (M+N)/N$ and $N/(M\cap N) \cong (M+N)/M$. Using the left hand inclusions we form the basis $b\widetilde{c}$ for $(M+N)/(M\cap N)$, and using the right hand inclusions we form the basis $c\widetilde{b}$. Then clearly

$$b\widetilde{c} \sim c\widetilde{b}. \tag{2.2}$$

It is essential at this point that we are working in $\overline{K}_1(R)$ rather than $K_1(R)$.

2.3 The torsion of a chain complex

Let $\mathscr{C}$ be a finite chain complex

$$\begin{tikzcd} C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_1 \arrow[r] & C_0 \end{tikzcd},$$

of modules over the ring R such that each C_i is free with preferred basis c_i and each homology group $H_i = H_i(\mathscr{C})$ is free with a preferred basis h_i. If $H_i = 0$, by definition we have an unique basis. Let B_i be the image of the boundary homomorphism $\partial_{i+1}\colon C_{i+1} \to C_i$ and let Z_{i+1} be its kernel. For the time being, in order to simplify the discussion we will assume that each B_i is also a free module, but we will see in Section 2.4 that this hypothesis can be omitted.

Choose a basis b_i, for each B_i. Using the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & B_i \arrow[r] & Z_i \arrow[r] & H_i \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & Z_i \arrow[r] & C_i \arrow[r] & B_{i-1} \arrow[r] & 0, \end{tikzcd}$$

we see that the bases b_i, h_i, b_{i-1} combine to yield a new basis $b_i\widetilde{h}_i\widetilde{b}_{i-1}$ for C_i.

Definition 2.3.1. *We define the torsion of $\mathscr{C}$ as the element of $\overline{K}_1(R)$ given by*

$$\tau_{c_i,h_i}(\mathscr{C}) = \sum_{i=0}^{n}(-1)^i\left[\frac{b_i\widetilde{h}_i\widetilde{b}_{i-1}}{c_i}\right].$$

Remark 2.3.2. *This does not depend on the choice of the b_i. In fact, choosing the different bases $\overline{b}_i$, we have*

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i\left[\frac{\overline{b}_i\widetilde{h}_i\widetilde{\overline{b}}_{i-1}}{c_i}\right] &= \sum_{i=0}^{n}(-1)^i\left(\left[\frac{\overline{b}_i\widetilde{h}_i\widetilde{\overline{b}}_{i-1}}{b_i\widetilde{h}_i\widetilde{b}_{i-1}}\right] + \left[\frac{b_i\widetilde{h}_i\widetilde{b}_{i-1}}{c_i}\right]\right)\\
&= \sum_{i=0}^{n}(-1)^i\left(\left[\frac{\overline{b}_i}{b_i}\right] + \left[\frac{h_i}{h_i}\right] + \left[\frac{\overline{b}_{i-1}}{b_{i-1}}\right] + \left[\frac{b_i\widetilde{h}_i\widetilde{b}_{i-1}}{c_i}\right]\right)\\
&= \sum_{i=0}^{n}(-1)^i\left[\frac{b_i\widetilde{h}_i\widetilde{b}_{i-1}}{c_i}\right].
\end{aligned}$$

Of course, the torsion of $\mathscr{C}$ depend on the c_i and h_i. Naturally, when the homology groups of a chain complex are zero, we can omit the base h_i from the notation of the torsion of this complex. It is important to remember that the sum which we are using in the definition of torsion is simply a choice to denote the operation of the quotient group $\overline{K}_1(R)$.

Consider a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{C}' \arrow[r, "\alpha"] & \mathscr{C} \arrow[r, "\beta"] & \mathscr{C}'' \arrow[r] & 0, \end{tikzcd}$$

in the category of chain complexes and chain mappings over R. We will assume that the modules C'_i, C_i and C''_i are free with preferred bases c'_i, c_i and c''_i, respectively, which are compatible, in the sense that $c_i \sim c'_i\widetilde{c}''_i$. By definition, we have the following commutative diagram of chain modules

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C'_{i+1} \arrow[r, "\alpha_{i+1}"] \arrow[d, "\partial'_{i+1}"] & C_{i+1} \arrow[r, "\beta_{i+1}"] \arrow[d, "\partial_{i+1}"] & C''_{i+1} \arrow[r] \arrow[d, "\partial''_{i+1}"] & 0 \\
0 \arrow[r] & C'_i \arrow[r, "\alpha_i"] \arrow[d] & C_i \arrow[r, "\beta_i"] \arrow[d] & C''_i \arrow[r] \arrow[d] & 0. \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Assuming such conditions, first consider the special case where all three complexes are acyclic.

Theorem 2.3.3. *If the homology groups $H_*(\mathscr{C}')$, $H_*(\mathscr{C})$ and $H_*(\mathscr{C}'')$ are all zero, then*

$$\tau_{c_i}(\mathscr{C}) = \tau_{c'_i}(\mathscr{C}') + \tau_{c''_i}(\mathscr{C}'').$$

Proof. Consider the diagram of submodules of C_i.

$$\begin{tikzcd}
 & C_i & \\
 & C'_i + B_i \arrow[u, "b''_{i-1}"'] & \\
C'_i \arrow[uur, bend left, "c''_i"] \arrow[ur, "b''_i"'] & & B_i \arrow[uul, bend right, "b_{i-1}"'] \arrow[ul, "b'_{i-1}"] \\
 & B'_i \arrow[ul, "b'_{i-1}"] \arrow[ur, "b''_i"'] & \\
 & 0 \arrow[u, "b'_i"'] \arrow[uul, bend left, "c'_i"] \arrow[uur, bend right, "b_i"'] &
\end{tikzcd}$$

Each capital letter in the diagram represents a submodule of C_i. Notice that we are committing an abuse of notation when we identified a set with your image under an injective function. We claim that $\alpha_i(C'_i) \cap B_i = \alpha_i(B'_i)$. Indeed, it is clear that $\alpha_i(B'_i) \subset \alpha_i(C'_i) \cap B_i$. For other hand, if $\gamma \in \alpha_i(C'_i) \cap B_i$, we have that $\alpha_i(c'_i) = \gamma = \partial_{i+1}(c_{i+1})$, for some $c'_i \in C'_i$ and some $c_{i+1} \in C_{i+1}$. We know that

$$\alpha_{i-1}(\partial'_i(c'_i)) = \partial_i(\alpha_i(c'_i)) = \partial_i(\partial_{i+1}(c_{i+1})) = 0.$$

Since α_{i-1} is injective, $\partial'_i(c'_i) = 0$. By aciclycity of $\mathscr{C}'$, we have that $c'_i \in B'_i$. Thus $\alpha_i(c'_i) \in \alpha_i(B'_i)$ and hence $\alpha_i(C'_i) \cap B_i \subset \alpha_i(B'_i)$.

The isomorphisms

$$C_i/(C'_i+B_i)\cong B''_{i-1},$$

$$(C'_i+B_i)/B_i\cong C'_i/(C'_i\cap B_i)=C'_i/B'_i\cong C'_i/Z'_i\cong B'_{i-1} \text{ and}$$

$$(C'_i+B_i)/C'_i\cong B_i/(C'_i\cap B_i)=B_i/B'_i\cong B''_i$$

follows from the Isomorphism theorem and the exactness of the sequences

$$0\longrightarrow \alpha_i(C'_i)+B_i\xrightarrow{i}\mathscr{C}\xrightarrow{\partial''_i\circ\beta_i}B''_{i-1}\longrightarrow 0,$$

$$0\longrightarrow B'_i\xrightarrow{\alpha_i}B_i\xrightarrow{\beta_i}B''_i\longrightarrow 0.$$

Each lower case latter in the diagram represents a basis for the corresponding quotient module. We are assuming that the modules B'_i, B_i and B''_i are free with bases b'_i, b_i and b''_i, respectively. Since we can choose any bases we like, we may as well choose b_i to be equal to $b'_i\widetilde{b}''_i$. Hence, pushing to the left across the diagram, we have

$$b_i\widetilde{b}_{i-1}=(b'_i\widetilde{b}''_i)(\widetilde{b'_{i-1}\widetilde{b}''_{i-1}})=b'_i(\widetilde{b}''_i\widetilde{b}'_{i-1})\widetilde{b''_{i-1}}\sim(b'_i\widetilde{b}'_{i-1})(\widetilde{b}''_i\widetilde{b''_{i-1}})=(b'_i\widetilde{b}'_{i-1})(\widetilde{b''_i\widetilde{b}''_{i-1}}),$$

using the Equation (2.2). Therefore

$$\begin{aligned}
\tau_{c_i}(\mathscr{C})&=\sum_{i=0}^{n}(-1)^i\left[\frac{b_i\widetilde{b}_{i-1}}{c_i}\right]\\
&=\sum_{i=0}^{n}(-1)^i\left[\frac{(b'_i\widetilde{b}'_{i-1})(\widetilde{b''_i\widetilde{b}''_{i-1}})}{c'_i\widetilde{c}''_i}\right]\\
&=\sum_{i=0}^{n}(-1)^i\left(\left[\frac{b'_i\widetilde{b}'_{i-1}}{c'_i}\right]+\left[\frac{b''_i\widetilde{b}''_{i-1}}{c''_i}\right]\right)\\
&=\tau_{c'_i}(\mathscr{C}')+\tau_{c''_i}(\mathscr{C}'')
\end{aligned}$$

using the Equation (2.1). This completes the proof. □

More generally suppose that the homology groups H'_i, H_i and H''_i of $\mathscr{C}'$, $\mathscr{C}$ and $\mathscr{C}''$, respectively, are not zero, but are free with preferred bases h'_i, h_i and h''_i, respectively. Then the exact homology sequence

$$H'_n\longrightarrow H_n\longrightarrow H''_n\longrightarrow H'_{n-1}\longrightarrow\cdots\longrightarrow H'_0\longrightarrow H_0\longrightarrow H''_0,$$

can be seen as a free acyclic chain complex $\mathscr{H}$ of dimension $3n+2$. To be more precise, we define $\mathscr{H}$ the chain complex, with chain module $\overline{H}_i$, by setting $\overline{H}_{3i}=H''_i$, $\overline{H}_{3i+1}=H_i$ and $\overline{H}_{3i+2}=H'_i$. Hence, we can consider the basis $\overline{h}_i$ of $\overline{H}_i$, where $\overline{h}_{3i}=h''_i$, $\overline{h}_{3i+1}=h_i$ and $\overline{h}_{3i+2}=h'_i$. Thus the torsion of $\mathscr{H}$ is defined.

Theorem 2.3.4. *Assuming the hypotheses above, we have*

$$\tau_{c_i,h_i}(\mathscr{C}) = \tau_{c'_i,h'_i}(\mathscr{C}') + \tau_{c''_i,h''_i}(\mathscr{C}'') + \tau_{\overline{h}_i}(\mathscr{H}).$$

Proof. Let X'_i, X_i and X''_i be the kernels of the homomorphisms $H'_i \to H_i$, $H_i \to H''_i$ and $H''_i \to H_{i-1}$, respectively. We have the following short exact sequences:

$$0 \longrightarrow X'_i \longrightarrow H'_i \longrightarrow X_i \longrightarrow 0,$$

$$0 \longrightarrow X_i \longrightarrow H_i \longrightarrow X''_i \longrightarrow 0,$$

$$0 \longrightarrow X''_i \longrightarrow H''_i \longrightarrow X'_{i-1} \longrightarrow 0.$$

We are assuming that the modules X'_i, X_i and X''_i are free with bases x'_i, x_i and x''_i, respectively.

Consider the following diagram of submodules of C_i.

$$\begin{tikzcd}
 & & C_i & \\
 & & \partial_i^{-1}(C'_{i-1}) \arrow[u, "b''_{i-1}"] & \\
 & & C'_i+Z_i \arrow[u, "x'_{i-1}"] & \\
 & C'_i+B_i \arrow[ur, "x''_i"'] \arrow[uur, bend left, "h''_i"] & & Z_i \arrow[ul, "b'_{i-1}"'] \arrow[uuul, bend right, "b_{i-1}"'] \\
C'_i \arrow[ur, "b''_i"'] \arrow[uuuurr, bend left, "c''_i"] & & Z'_i+B_i \arrow[ul, "b'_{i-1}"'] \arrow[ur, "x''_i"'] & \\
 & Z'_i \arrow[ul, "b'_{i-1}"] \arrow[ur, "b''_i"'] & & B_i \arrow[ul, "x_i"] \arrow[uu, bend right, "h_i"'] \\
 & & Z'_i\cap B_i \arrow[ul, "x_i"'] \arrow[ur, "b''_i"'] & \\
 & & B'_i \arrow[u, "x'_i"] \arrow[uul, bend left, "h'_i"] & \\
 & & 0 \arrow[u, "b'_i"] \arrow[uuur, bend right, "b_i"'] \arrow[uuuull, bend left, "c'_i"] \arrow[uuuuuuuu, bend left=80, "c_i"] &
\end{tikzcd}$$

Again we are committing an abuse of notation in the diagram, when we identified a set with your image under an injective function. Furthermore, we formally define the set

$$\partial_i^{-1}(C'_{i-1}) = \{c_i \in C_i;\ \exists\, c'_{i-1} \in C'_{i-1} \text{ with } \alpha_{i-1}(c'_{i-1}) = \partial_i(c_i)\}.$$

We claim that the sequence

$$0 \longrightarrow \partial_i^{-1}(C'_{i-1}) \xrightarrow{i} C_i \xrightarrow{\partial''_i\circ\beta_i} B''_{i-1} \longrightarrow 0,$$

is a short exact sequence, where i denotes the inclusion. In fact, we have that $\partial_i'' \circ \beta_i$ is onto since β_i is onto. We only need to show that $\ker \partial_i'' \circ \beta_i = \operatorname{Im} i$. If $c_i \in \operatorname{Im} i$, then exist $k_i \in \partial_i^{-1}(C_{i-1}')$ such that $k_i = i(k_i) = c_i$. By definition, there is $c_{i-1}' \in C_{i-1}'$ with $\alpha_{i-1}(c_{i-1}') = \partial_i(c_i)$. Thus

$$\partial_i'' \circ \beta_i(c_i) = \beta_{i-1} \circ \partial_i(c_i) = \beta_{i-1} \circ \alpha_{i-1}(k_i) = 0.$$

So $\operatorname{Im} i \subset \ker \partial_i'' \circ \beta_i$. Furthermore, if $x_i \in \ker \partial_i'' \circ \beta_i$ then $\beta_{i-1} \circ \partial_i(x_i) = \partial_i'' \circ \beta_i(x_i) = 0$. Therefore, $\partial_i(x_i) \in \ker \beta_{i-1} = \operatorname{Im} \alpha_{i-1}$ and hence exist $x_{i-1}' \in C_{i-1}'$ such that $\alpha_{i-1}(x_{i-1}') = \partial_i(x_i)$. It follows that $x_i \in \partial_i^{-1}(C_{i-1}')$ and so $\operatorname{Im} i \subset \ker(\partial_i'' \circ \beta_i)$. We conclude that $\ker \partial_i'' \circ \beta_i = \operatorname{Im} i$, which guarantees the exactness of the sequence and hence

$$C_i/\partial_i^{-1}(C_{i-1}') \cong B_{i-1}''.$$

Now we define the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \alpha_i(C_i') + B_i \arrow[r, "i"] & \partial_i^{-1}(C_{i-1}') \arrow[r, "\overline{\beta}_i"] & H_i'' \arrow[r] & 0, \end{tikzcd}$$

where $\overline{\beta_i}$ maps γ on the coset of $\beta_i(\gamma)$ in H_i'', which we denote by $\overline{\beta_i(\gamma)}$. Suppose that $\omega \in \alpha_i(C_i') + B_i$. In this case, we have that $\omega = \alpha_i(c_i') + \partial_{i+1}(c_{i+1})$, for some $c_i' \in C_i'$ and $c_{i+1} \in C_{i+1}$. Hence, $\partial_i(\omega) = \partial_i(\alpha_i(c_i') + \partial_{i+1}(c_{i+1})) = \partial_i(\alpha_i(c_i')) = \alpha_{i-1}(\partial_i'(c_i'))$. Then, $\omega \in \partial_i^{-1}(C_{i-1}')$ and the inclusion is well defined. Moreover, $\overline{\beta_i}$ is well defined since for $\gamma_1, \gamma_2 \in \partial_i^{-1}(C_{i-1}')$, if $\gamma_1 = \gamma_2$ then the coset of $\beta_i(\gamma_1)$ and the coset of $\beta_i(\gamma_2)$ are equal. The surjectivity of $\overline{\beta_i}$ follows from the surjectivity of β_i. In this way, we only need to show that $\ker \overline{\beta}_i = \operatorname{Im} i$. Well, if $\omega \in \operatorname{Im} i$, so exist $c_i' \in C_i'$ and $c_{i+1} \in C_{i+1}$ such that $\omega = \alpha_i(c_i') + \partial_{i+1}(c_{i+1})$. Thus,

$$\begin{aligned} \overline{\beta}_i(\omega) &= \overline{\beta_i(\alpha_i(c_i') + \partial_{i+1}(c_{i+1}))} \\ &= \overline{\beta_i(\partial_{i+1}(c_{i+1}))} \\ &= \overline{\partial_{i+1}''(\beta_{i+1}(c_{i+1}))} \\ &= \overline{0}, \end{aligned}$$

i.e, $\operatorname{Im} i \subset \ker(\beta_i)$. On the other hand, if $\gamma \in \ker(\overline{\beta_i})$, so $\beta_i(\gamma) = \partial_{i+1}''(c_{i+1}'')$, for some $c_{i+1}'' \in C_{i+1}''$. Since β_i is onto, for all i, exists $c_{i+1} \in C_{i+1}$ such that $\beta_{i+1}(c_{i+1}) = c_{i+1}''$. Then $\beta_i(\gamma) = \partial_{i+1}''(c_{i+1}'') = \partial_{i+1}''(\beta_{i+1}(c_{i+1})) = \beta_i(\partial_{i+1}(c_{i+1}))$ so that $\gamma - \partial_{i+1}(c_{i+1}) \in \ker \beta_i = \operatorname{Im} \alpha_i$. Hence there exists $c_i' \in C_i'$ such that $\alpha_i(c_i') = \gamma - \partial_{i+1}(c_{i+1})$. In this way, $\gamma \in \operatorname{Im} i$ and we conclude that $\ker(\beta_i) \subset \operatorname{Im} i$. It follows that

$$\partial_i^{-1}(C_{i-1}')/(\alpha_i(C_i') + B_i) \cong H_i''.$$

Similarly, using the maps inclusion and β_i, we can prove that

$$B_i/(\alpha_i(Z_i') \cap B_i) \cong B_i'' \cong (\alpha_i(C_i') + B_i)/\alpha_i(C_i').$$

Moreover, we also have the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \alpha_i(C'_i)\cap Z_i \arrow[r, "i"] & \alpha_i(C'_i) \arrow[r, "p"] & \alpha_i(C'_i)/\alpha_i(Z'_i) \arrow[r] & 0. \end{tikzcd}$$

Indeed, suppose that $\omega \in \operatorname{Im} i$. We have $\omega = \alpha_i(c'_i)$, for some $c'_i \in C'_i$ and $\partial_i\omega = 0$. Hence $\alpha_{i-1}\circ\partial'_i(c'_i) = \partial_i\circ\alpha_i(c'_i) = \partial_i(\omega) = 0$. Thus $\partial'_i(c'_i) = 0$ and $c'_i \in Z'_i$. Therefore $\alpha_i(c'_i) \in \ker p$, i.e., $\omega \in \ker p$. Thereby $\operatorname{Im} i \subset \ker p$. It remains for us to show that $\ker p \subset \operatorname{Im} i$. In fact, if $\gamma \in \ker p$ then $\gamma \in \alpha_i(Z'_i) \subset \alpha_i(C'_i)$. Thus $\gamma = \alpha_i(c'_i)$, for some $c'_i \in C'_i$. We have that $\partial_i(\gamma) = \partial_i(\alpha_i(z'_i)) = \alpha_{i-1}(\partial'_i(z'_i)) = 0$. We conclude that $\gamma \in (\alpha_i(C'_i)\cap Z_i) = \operatorname{Im} i$. It follows that

$$(\alpha_i(C'_i)+Z_i)/Z_i \cong \alpha_i(C'_i)/(\alpha_i(C'_i)\cap Z_i) \cong \alpha_i(C'_i)/\alpha_i(Z'_i).$$

Similarly, using the inclusion and projection, we can prove that

$$(C'_i+B_i)/(Z'_i+B_i) \cong C'_i/Z'_i \cong B'_{i-1} \text{ and}$$

$$(C'_i+Z_i)/(C'_i+B_i) \cong Z_i/(Z'_i+B_i) \cong X''_i.$$

The other isomorphisms can be verified without difficulty using definitions, hypotheses and some results, as Isomorphism theorem.

Again we have $c_i \sim c'_i\widetilde{c}''_i$ by hypothesis. We are free to choose the bases b'_i, b_i and b''_i, so we may choose b_i equal to $b'_i\widetilde{x}'_i\widetilde{b}''_i$ (see the diagram above). By definition

$$\begin{aligned}
\tau_{\overline{h}_i}(\mathscr{H}) &= \sum_{i=0}^{n}(-1)^{3i+2}\left[\frac{x'_i\widetilde{x}_i}{h'_i}\right] + \sum_{i=0}^{n}(-1)^{3i+1}\left[\frac{x_i\widetilde{x}''_i}{h_i}\right] + \sum_{i=0}^{n}(-1)^{3i}\left[\frac{x''_i\widetilde{x}'_{i-1}}{h''_i}\right] \\
&= \sum_{i=0}^{n}(-1)^{3i}\left(\left[\frac{x'_i\widetilde{x}_i}{h'_i}\right] - \left[\frac{x_i\widetilde{x}''_i}{h_i}\right] + \left[\frac{x''_i\widetilde{x}'_{i-1}}{h''_i}\right]\right) \\
&= \sum_{i=0}^{n}(-1)^{i}\left(\left[\frac{x'_i\widetilde{x}_i}{h'_i}\right] - \left[\frac{x_i\widetilde{x}''_i}{h_i}\right] + \left[\frac{x''_i\widetilde{x}'_{i-1}}{h''_i}\right]\right)
\end{aligned}$$

Hence, we have that $\tau_{c'_i,h'_i}(\mathscr{C}') + \tau_{c''_i,h''_i}(\mathscr{C}'') + \tau_{\overline{h}_i}(\mathscr{H})$ is equal to

$$\sum_{i=0}^{n}(-1)^{i}\left(\left[\frac{b'_i\widetilde{h}'_i\widetilde{b}'_{i-1}}{c'_i}\right] + \left[\frac{b''_i\widetilde{h}''_i\widetilde{b}''_{i-1}}{c''_i}\right] + \left[\frac{x'_i\widetilde{x}_i}{h'_i}\right] - \left[\frac{x_i\widetilde{x}''_i}{h_i}\right] + \left[\frac{x''_i\widetilde{x}'_{i-1}}{h''_i}\right]\right)$$

Thus, using the Equation (2.1), we have

$$\begin{aligned}
\tau_{c'_i,h'_i}(\mathscr{C}')+\tau_{c''_i,h''_i}(\mathscr{C}'')+\tau_{\overline{h}_i}(\mathscr{H}) &= \sum_{i=0}^{n}(-1)^i\Bigg(\left[\frac{b'_i\widetilde{h}'_i\widetilde{b}'_{i-1}}{b'_i\widetilde{x}'_i\widetilde{x}_i\widetilde{b}'_{i-1}}\right]+\left[\frac{b'_i\widetilde{x}'_i\widetilde{x}_i\widetilde{b}'_{i-1}}{c'_i}\right]\\
&\quad+\left[\frac{b''_i\widetilde{h}''_i\widetilde{b}''_{i-1}}{b''_i\widetilde{x}''_i\widetilde{x}'_{i-1}\widetilde{b}''_{i-1}}\right]+\left[\frac{b''_i\widetilde{x}''_i\widetilde{x}'_{i-1}\widetilde{b}''_{i-1}}{c''_i}\right]\\
&\quad+\left[\frac{x'_i\widetilde{x}_i}{h'_i}\right]+\left[\frac{h_i}{x_i\widetilde{x}''_i}\right]+\left[\frac{x''_i\widetilde{x}'_{i-1}}{h''_i}\right]\Bigg)\\
&= \sum_{i=0}^{n}(-1)^i\Bigg(\left[\frac{b'_i}{b'_i}\right]+\left[\frac{h'_i}{x'_i\widetilde{x}_i}\right]+\left[\frac{b'_{i-1}}{b'_{i-1}}\right]+\left[\frac{b'_i\widetilde{x}'_i\widetilde{x}_i\widetilde{b}'_{i-1}}{c'_i}\right]\\
&\quad+\left[\frac{b''_i}{b''_i}\right]+\left[\frac{h''_i}{x''_i\widetilde{x}'_{i-1}}\right]+\left[\frac{b''_{i-1}}{b''_{i-1}}\right]+\left[\frac{b''_i\widetilde{x}''_i\widetilde{x}'_{i-1}\widetilde{b}''_{i-1}}{c''_i}\right]\\
&\quad+\left[\frac{x'_i\widetilde{x}_i}{h'_i}\right]+\left[\frac{h_i}{x_i\widetilde{x}''_i}\right]+\left[\frac{x''_i\widetilde{x}'_{i-1}}{h''_i}\right]\Bigg)\\
&= \sum_{i=0}^{n}(-1)^i\Bigg(\left[\frac{b'_i\widetilde{x'_i\widetilde{x}_i}\widetilde{b}'_{i-1}}{c'_i}\right]+\left[\frac{b''_i\widetilde{\widetilde{x}''_i\widetilde{x}'_{i-1}}\widetilde{b}''_{i-1}}{c''_i}\right]+\left[\frac{h_i}{x_i\widetilde{x}''_i}\right]\Bigg)
\end{aligned}$$

Therefore, using the Equation (2.2) and the fact that we take $b_i=b'_i\widetilde{x}'_i\widetilde{b}''_i$ for each i, we have

$$\begin{aligned}
\tau_{c'_i,h'_i}(\mathscr{C}')+\tau_{c''_i,h''_i}(\mathscr{C}'')+\tau_{\overline{h}_i}(\mathscr{H}) &= \sum_{i=0}^{n}(-1)^i\left[\frac{(b'_i\widetilde{x}'_i\widetilde{x}_i\widetilde{b}'_{i-1})(b''_i\widetilde{\widetilde{x}''_i\widetilde{x}'_{i-1}}\widetilde{b}''_{i-1})\widetilde{h}_i}{c'_i\widetilde{c}''_i\widetilde{(x_i\widetilde{x}''_i)}}\right]\\
&= \sum_{i=0}^{n}(-1)^i\left[\frac{(b'_i\widetilde{x}'_i\widetilde{b}''_i)\widetilde{h}_i(b'_{i-1}\widetilde{\widetilde{x}'_{i-1}\widetilde{b}''_{i-1}})\widetilde{(x_i\widetilde{x}''_i)}}{(c'_i\widetilde{c}''_i)\widetilde{(x_i\widetilde{x}''_i)}}\right]\\
&= \sum_{i=0}^{n}(-1)^i\left[\frac{b_i\widetilde{h}_i\widetilde{b}_{i-1}}{c_i}\right]\\
&= \tau_{c_i,h_i}(\mathscr{C}).
\end{aligned}$$

□

2.4 Some constructions with stably free modules

A R-module M is called *stably free* if there exists a free R-module F such that $M\oplus F$ is free. Since we consider just finitely generated modules, the following definition makes sense. If $M\oplus F\cong F'$ then the difference between the rank of F' and the rank of F will be called the *rank* of M.

Example 2.4.1. *Any free module F is a stably free since $F\cong F\oplus 0$.*

Example 2.4.2. *Suppose that R denote the ring $\mathbb{R}[x,y,z]/\langle x^2+y^2+z^2-1\rangle$. Let T be the set*

$$\{(f,g,h)\in R^3;\ xf+yg+zh=0\}.$$

We have that T is stably free, but it is not a free module. This example is well known in the study of stably free modules. For more details, see [3] and [15], or more generally [28].

Lemma 2.4.3. *Consider a short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0, \end{tikzcd}$$

of R-modules. If Y and Z are stably free, then X is also stably free.

Proof. Since Z is stably free, we have equivalently that the short exact sequence above splits. Hence, we have that $Y \cong X \oplus Z$. Thus if $Z \oplus F \cong F'$ and $Y \oplus F \cong F''$, where F, F' and F'' are free, it follows that

$$X \oplus F' \cong X \oplus Z \oplus F \cong Y \oplus F \cong F'',$$

i.e., X is stably free. ☐

Now consider a free chain complex $\mathscr{C}$ over R given by

$$\begin{tikzcd} C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_1 \arrow[r] & C_0 \end{tikzcd},$$

i.e., where each C_i is free, and assume that the homology modules $H_i = H_i(\mathscr{C})$ are also free. Using the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & B_i \arrow[r] & Z_i \arrow[r] & H_i \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & Z_i \arrow[r] & C_i \arrow[r] & B_{i-1} \arrow[r] & 0, \end{tikzcd}$$

we have that Z_0 is free and as H_0 is free we know that $Z_0 \cong B_0 \oplus H_0$, and hence B_0 is stably free. By Lemma 2.4.3 since B_0 and C_1 are stably free, we have that Z_1 is stably free. Similarly, since Z_1 and H_1 are stably free, then B_1 is stably free. Assuming that B_{i-1} is stably free, we conclude that Z_i is stably free and hence B_i is stably free. It follows by induction that all the modules Z_i and B_i are stably free.

We will show that the constructions of the Section 2.2 can be carried out using stably free modules in place of free modules. It will then follow easily that all of the constructions and proofs of the Section 2.2 can be carried out without the extra hypothesis that the B_i and X_i are free.

Let F_i be a standard free module of rank i, with standard basis $\{f_1, \ldots, f_i\}$. We may think of F_i as the submodule of F_{i+1} generated by the first i basis elements.

Definition 2.4.4. *A stable basis, or simply s-basis, b for a stably free module M is a basis $\{b_1, \ldots, b_{r+t}\}$ for some free module $M \oplus F_t$, for any non-negative integer t.*

Given two s-bases $b = \{b_1, \ldots, b_{r+t}\}$ and $c = \{c_1, \ldots, c_{r+u}\}$ for M, the symbol

$$\left[\frac{c}{b}\right] \in \overline{K}_1(R)$$

is defined as follows. Choose an integer $v \geq \max\{t,u\}$. Extend $\{b_1,\ldots,b_{r+t}\}$ to a basis for $M \oplus F_v$ by setting

$$b_{r+i} = 0 \oplus f_i,\ i \geq t+1.$$

Similarly extend $\{c_1,\ldots,c_{r+u}\}$ to a basis for $M \oplus F_v$ by setting

$$c_{r+i} = 0 \oplus f_i,\ i \geq u+1.$$

Now $c_i = \sum a_{ij} b_j$ where the matrix $(a_{ij}) \in GL(r+v,R)$ represents the required element $\left[\frac{c}{b}\right] \in \overline{K}_1(R)$. Clearly this construction can be done for any integer $v \geq \max\{t,u\}$. As in Section 2.2, we write $b \sim c$ if $\left[\frac{c}{b}\right] = 0$. The identities

$$\left[\frac{d}{b}\right] = \left[\frac{d}{c}\right] + \left[\frac{c}{b}\right] \text{ and } \left[\frac{b}{b}\right] = 0$$

are again satisfied.

Next consider a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0. \end{tikzcd}$$

Given s-bases x for X and z for Z we construct an s-basis $x\tilde{z}$ for Y as follows. Suppose that x is a s-basis for $X \oplus F_t$ and that z is a basis for $Z \oplus F_u$. Consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & F_t \arrow[r, "\alpha"] & F_{t+u} \arrow[r, "\beta"] & F_u \arrow[r] & 0, \end{tikzcd}$$

where α denotes the inclusion homomorphism, and β is defined by

$$\beta(f_i) = \begin{cases} 0, & \text{if } i \leq t \\ f_{i-t}, & \text{if } i > t \end{cases}.$$

Forming the direct sum

$$\begin{tikzcd} 0 \oplus 0 \arrow[r] & X \oplus F_t \arrow[r] & Y \oplus F_{t+u} \arrow[r] & Z \oplus F_u \arrow[r] & 0 \oplus 0, \end{tikzcd}$$

the bases x and z combine to yield a basis $x\tilde{z}$ for $Y \oplus F_{t+u}$, which is well defined up to equivalence. This is the required construction.

2.5 An algebraic subdivision theorem

In this section, we will prove an algebraic theorem that will later allow us to manage the geometric operations involved in subdividing a complex.

Let $\mathscr{C}$ be a chain complex and let

$$\mathscr{C}^{(0)} \subset \mathscr{C}^{(1)} \subset \cdots \subset \mathscr{C}^{(n)} = \mathscr{C}$$

be a filtration of $\mathscr{C}$ by subcomplexes such that the homology group $H_i(\mathscr{C}^{(\lambda)}/\mathscr{C}^{(\lambda-1)})$ is zero for $i \neq \lambda$. We fix the set $\mathscr{C}^{(-1)} = 0$.

Then we can define a new chain complex $\overline{\mathscr{C}}$ by setting

$$\overline{C}_\lambda = H_\lambda(\mathscr{C}^{(\lambda)}/\mathscr{C}^{(\lambda-1)}).$$

The boundary homomorphism $\overline{\partial}_\lambda : \overline{C}_\lambda \to \overline{C}_{\lambda-1}$ is obtained from the exact sequence of the triple $\mathscr{C}^{(\lambda)}$, $\mathscr{C}^{(\lambda-1)}$, $\mathscr{C}^{(\lambda-2)}$. The following is well known.

Lemma 2.5.1. *The homology groups $H_i(\overline{\mathscr{C}})$ are canonically isomorphic to the groups $H_i(\mathscr{C})$.*

Proof. We have that

$$H_i(\mathscr{C}^{(\lambda)}/\mathscr{C}^{(\mu)}) = 0, \text{ for } i > \lambda \text{ or } i \leq \mu$$

by induction on $k = \lambda - \mu$, using the homology exact sequence of the triple $\mathscr{C}^{(\lambda)}$, $\mathscr{C}^{(\lambda-k)}$, $\mathscr{C}^{(\lambda-k-1)}$. Hence

$$H_i(\mathscr{C}) \cong H_i(\mathscr{C}^{(i+1)}) \cong H_i(\mathscr{C}^{(i+1)}/\mathscr{C}^{(i-2)}).$$

Now consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
\overline{C}_{\lambda+1} \arrow[r, "\varphi"] \arrow[dr, "\overline{\partial}_{\lambda+1}"'] & H_\lambda(\mathscr{C}^{(\lambda)}/\mathscr{C}^{(\lambda-2)}) \arrow[r, "\psi"] \arrow[d, "\alpha"] & H_\lambda(\mathscr{C}^{(\lambda+1)}/\mathscr{C}^{(\lambda-2)}) \arrow[r] & 0 \\
 & \overline{C}_\lambda \arrow[d, "\overline{\partial}_\lambda"] & & \\
 & \overline{C}_{\lambda-1} & &
\end{tikzcd}$$

where the vertical line comes from the homology exact sequence of the triple $\mathscr{C}^{(\lambda)}$, $\mathscr{C}^{(\lambda-1)}$, $\mathscr{C}^{(\lambda-2)}$, and the horizontal line from the triple $\mathscr{C}^{(\lambda+1)}$, $\mathscr{C}^{(\lambda)}$, $\mathscr{C}^{(\lambda-2)}$. It is clear the injectivity of α and the surjectivity of ψ. Therefore, the cycle group $\overline{Z}_\lambda$ can be identified with $H_\lambda(\mathscr{C}^{(\lambda)}/\mathscr{C}^{(\lambda-2)})$, and $\overline{B}_\lambda$ can be identified with $\operatorname{Im}\varphi$, since $\overline{\partial}_{\lambda+1}(\overline{C}_{\lambda+1}) = \alpha(\varphi(\overline{C}_{\lambda+1}))$. Hence

$$H_\lambda(\overline{\mathscr{C}}) = \overline{Z}_\lambda/\overline{B}_\lambda \cong H_\lambda(\mathscr{C}^{(\lambda+1)}/\mathscr{C}^{(\lambda-2)}) \cong H_\lambda(\mathscr{C}).$$

□

We wish to examine the behavior of the torsion in a filtered chain complex like the one constructed above. For this, we assume further hypotheses.

Each $C_i^{(\lambda)}/C_i^{(\lambda-1)}$ should be free with preferred basis $c_i^{(\lambda)}$, so that C_i is free with basis

$$c_i = c_i^0 \tilde{c}_i^1 \tilde{c}_i^2 \dots \tilde{c}_i^n.$$

Also each $\overline{C}_\lambda = H_\lambda(\mathscr{C}^{(\lambda)}/\mathscr{C}^{(\lambda-1)})$ should be free with preferred basis $\bar{c}_\lambda$, and the modules

$$H_i(\mathscr{C}) \cong H_i(\overline{\mathscr{C}})$$

should be free with preferred basis h_i. Finally $\mathscr{C}$ should be finitely generated so that the torsions of $\mathscr{C}$ and $\overline{\mathscr{C}}$ are defined. With these hypotheses we have the following result.

Theorem 2.5.2 (Algebraic subdivision)**.** *If each quotient complex $\mathscr{C}^{(k)}/\mathscr{C}^{(k-1)}$ has torsion equal to zero, then $\tau_{c_i,h_i}(\mathscr{C}) = \tau_{\bar{c}_i,h_i}(\overline{\mathscr{C}})$.*

Proof. Let $\overline{\mathscr{C}}^{(k)}$ be the truncated chain complex

$$\cdots \longrightarrow 0 \longrightarrow \overline{C}_k \longrightarrow \overline{C}_{k-1} \longrightarrow \cdots \longrightarrow \overline{C}_0 \longrightarrow 0,$$

which is obtained from $\overline{\mathscr{C}}$ by chopping off the chain modules of degree greater than k. We wish to look at the torsion of $\overline{\mathscr{C}}^{(k)}$. It follows from Lemma 2.5.1 that $H_i(\overline{\mathscr{C}}^{(k)}) \cong H_i(\mathscr{C}^{(k)})$. Note that $H_i(\overline{\mathscr{C}}^{(k)}) \cong H_i(\mathscr{C}^{(k)})$ is zero for $i > k$, and is isomorphic to $H_i(\mathscr{C})$, with preferred basis h_i, for $i < k$. In order for the torsion to be defined we must also choose a preferred basis (or at least s-basis) for $H_i(\overline{\mathscr{C}}^{(k)}) \cong H_i(\mathscr{C}^{(k)})$. But clearly $H_k(\overline{\mathscr{C}}^{(k)})$ can be identified with the cycle group $\overline{Z}_k$ of $\overline{\mathscr{C}}$, which is known to be stably free. Hence some preferred s-basis $\bar{z}_k$ can be chosen. It then follows that $\tau_{\bar{c}_i,h'_i}(\overline{\mathscr{C}}^{(k)})$ and $\tau_{c_i,h'_i}(\mathscr{C}^{(k)})$ are defined, where

$$h'_i = \begin{cases} 0, & \text{for} \quad i > k \\ h_i, & \text{for} \quad i < k \\ \bar{z}_k, & \text{for} \quad i = k \end{cases}.$$

We will prove by induction on k that

$$\tau_{\bar{c}_i,h'_i}(\overline{\mathscr{C}}^{(k)}) = \tau_{c_i,h'_i}(\mathscr{C}^{(k)}).$$

Taking $k = n$, this will complete the proof.

Let $\mathscr{H}$ be the homology exact sequence of the pair $(\overline{\mathscr{C}}^{(k)}, \overline{\mathscr{C}}^{(k-1)})$ or, equivalently, of the pair $(\mathscr{C}^{(k)}, \mathscr{C}^{(k-1)})$. Then according to the Theorem 2.3.4 we have

$$\tau_{\bar{c}_i,h'_i}(\overline{\mathscr{C}}^{(k)}) = \tau_{\bar{c}_i,h'_i}(\overline{\mathscr{C}}^{(k-1)}) + \tau_{\bar{c}_k,\bar{z}_k}(\overline{\mathscr{C}}^{(k)}/\overline{\mathscr{C}}^{(k-1)}) + \tau_{\overline{h}_i}(\mathscr{H}),$$

$$\tau_{c_i,h'_i}(\mathscr{C}^{(k)}) = \tau_{c_i,h'_i}(\mathscr{C}^{(k-1)}) + \tau_{c_i^{(k)},\bar{c}_k}(\mathscr{C}^{(k)}/\mathscr{C}^{(k-1)}) + \tau_{\overline{h}_i}(\mathscr{H}).$$

We know that $\tau_{\bar{c}_i,h'_i}(\overline{\mathscr{C}}^{(k-1)}) = \tau_{c_i,h'_i}(\mathscr{C}^{(k-1)})$, by induction, and $\tau_{c_i^{(k)},\bar{c}_k}(\mathscr{C}^{(k)}/\mathscr{C}^{(k-1)}) = 0$, by hypothesis.

Moreover $\tau_{\bar{c}_k,\bar{z}_k}(\overline{\mathscr{C}}^{(k)}/\overline{\mathscr{C}}^{(k-1)})$ is zero, by the definition of the truncated chain complex $\overline{\mathscr{C}}^{(k)}$. It follows that $\tau_{\bar{c}_i,h'_i}(\overline{\mathscr{C}}^{(k)}) = \tau_{c_i,h'_i}(\mathscr{C}^{(k)})$. This completes the proof. □

CHAPTER 3

Algebraic-topological approach of torsions

Since we concluded the previous chapter with an interesting result about the invariance under algebraic subdivision of the torsion, this is an appropriate moment to start a discussion about combinatorial and topological invariance of the torsion. In this chapter, we will define the torsion of a CW-complex and we will see that although this torsion is intrinsically related to the fundamental group and the homology groups of a space, at times it can see a little more than these other two invariants.

3.1 Torsion for CW-complexes

For any pair (X,Y) of CW-complexes, the associated chain complex $\mathscr{C}(X,Y)$ is defined by setting

$$C_q(X,Y) = H_q(|X^q \cup Y|, |X^{q-1} \cup Y|),$$

where H_q denotes the singular homology with integer coefficients, and where $|X^q|$ denotes the underlying topological space of the p-skeleton of X. Each chain group $C_q(X,Y)$ is free abelian with one generator for each q-cell of $X-Y$. Note that the homology group $H_q(\mathscr{C}(X,Y))$ of this chain complex is canonically isomorphic to the singular group $H_q(|X|,|Y|)$. The idea is similar to the Lemma 2.5.1. These isomorphic groups will be denoted by $H_q(X,Y)$.

Now we consider (K,L) a pair consisting of a finite, connected CW-complex K, and a subcomplex L. The fundamental group $\pi_1(K)$ will be denoted by Π. The *Whitehead torsion* of the pair (K,L) will be defined as follows.

Now consider $\widetilde{L}$ and $\widetilde{K}$ the universal covering of L and K, respectively. We know that $\widetilde{L}$ and $\widetilde{K}$ are CW-complexes, where $\widetilde{L}$ is a subcomplex of $\widetilde{K}$. The fundamental group Π will be identified with the group of covering transformations, so that each $\sigma \in \Pi$ determines a mapping

$$\sigma\colon (\widetilde{K},\widetilde{L}) \to (\widetilde{K},\widetilde{L}).$$

It is immediate that σ is a cellular map. Hence each $\sigma \in \Pi$ determines a induced chain map

$$\sigma_{\#} \colon \mathscr{C}(\widetilde{K},\widetilde{L}) \to \mathscr{C}(\widetilde{K},\widetilde{L}).$$

This action makes each chain group $C_q(\widetilde{K},\widetilde{L})$ into a module over the integral group ring $\mathbb{Z}(\Pi)$. By Proposition 1.1.9, we obtain a free chain complex

$$C_n(\widetilde{K},\widetilde{L}) \longrightarrow C_{n-1}(\widetilde{K},\widetilde{L}) \longrightarrow \cdots \longrightarrow C_0(\widetilde{K},\widetilde{L})$$

over $\mathbb{Z}(\Pi)$ with a preferred basis. We assume that the homology groups $H_q(\widetilde{K},\widetilde{L})$ of this complex are free with a preferred basis h_q. Hence the torsion

$$\tau_{h_q}(\mathscr{C}(\widetilde{K},\widetilde{L})) \in \overline{K}_1(\mathbb{Z}(\Pi))$$

would be defined, as in the Section 2.3. The cellular structure of (K,L) determines a class of preferred bases, as follows. Let $e_1,\dots,e_\alpha$ be the q-cells of $K-L$. For each e_i choose a representative cell $\widetilde{e}_i$ of $\widetilde{K}$ lying over e_i. Furthermore choose an orientation, so that $\widetilde{e}_i$ determines a basis element of $C_q(\widetilde{K},\widetilde{L})$, which we may also denote by $\widetilde{e}_i$. Then $c_q = \{\widetilde{e}_1,\dots,\widetilde{e}_\alpha\}$ is the required basis for $C_q(\widetilde{K},\widetilde{L})$.

Using these bases, the torsion $\tau_{h_q}(\mathscr{C}(\widetilde{K},\widetilde{L}))$ is apparently defined as an element of $\overline{K}_1(\mathbb{Z}(\Pi))$. However we have made an arbitrary choice of the representative cells $\widetilde{e}_i$, which leads to a certain arbitrariness in the resulting torsion. To eliminate the indeterminacy it is necessary to pass to the quotient group

$$Wh(\Pi) = \overline{K}_1(\mathbb{Z}(\Pi))/\Pi.$$

Note that the quotient $\overline{K}_1(\mathbb{Z}(\Pi))/\Pi$ is equivalent to the quotient $GL(\mathbb{Z}(\Pi))/\pm\Pi$, as we did in Example 2.1.6.

Definition 3.1.1. *The image of $\tau_{h_q}(\mathscr{C}(\widetilde{K},\widetilde{L}))$ in the quotient group $Wh(\Pi)$ is called the* Whitehead torsion *$\tau_{h_q}(K,L)$ of the pair (K,L).*

Notice that this torsion is well defined, since $\tau_{h_q}(\mathscr{C}(\widetilde{K},\widetilde{L}))$ does not depend on the choice of c_p, where $0 \le p \le n$. Of course, $\tau_{h_q}(\mathscr{C}(\widetilde{K},\widetilde{L}))$ depends on the choice of the q-cells $e_1,\dots,e_\alpha$, but we will show that it does not depend on the choice of the lifts. In fact, suppose that $\widetilde{e}_i$ is replaced by a different representative cell $\pm\sigma_{\#}(\widetilde{e}_i)$, i.e., the basis $c_p = \{\widetilde{e}_{p_1},\dots,\widetilde{e}_{p_i},\dots,\widetilde{e}_{p_\alpha}\}$ is replaced by the basis $c'_q = \{\widetilde{e}_{p_1},\dots,\pm\sigma_{\#}(\widetilde{e}_{p_i}),\dots,\widetilde{e}_{p_\alpha}\}$. Then, in dimension q, we have

$$\left[\frac{b_p\widetilde{h}_p\widetilde{b}_{p-1}}{c'_p}\right] = \left[\frac{b_p\widetilde{h}_p\widetilde{b}_{p-1}}{c_p}\right] - \left[\frac{c'_p}{c_p}\right].$$

The matrix representing the change of basis from c_p to c'_p is given by

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & I_m \end{pmatrix},$$

where $\sigma \in \Pi$. Thus $\left[\frac{c'_p}{c_p}\right]$ is annihilated when we pass the quotient group $Wh(\Pi)$, and hence we have the desired. Since this is hold for each q, we conclude that $\tau_{h_q}(\mathscr{C}(\widetilde{K},\widetilde{L}))$ does not depend from the choice of the bases c_q of the chain modules $C_q(\widetilde{K},\widetilde{L})$.

Remark 3.1.2. *In making use of the group $Wh(\Pi)$ we never need to worry about base points, since, by Lemma 2.1.7, any inner automorphism of Π induce the identity automorphism of $Wh(\Pi)$.*

Definition 3.1.3. *A CW-complex X is a subdivision of a CW-complex K if the underlying space $|X|$ is equal to $|K|$, and if each open cell of X is contained in an open cell of K, so that the identity map $K \to X$ is cellular. See the Figure 3.1. Similarly the pair (X,Y) is a subdivision of the pair (K,L) if X is a subdivision of K and Y is a subdivision of L.*

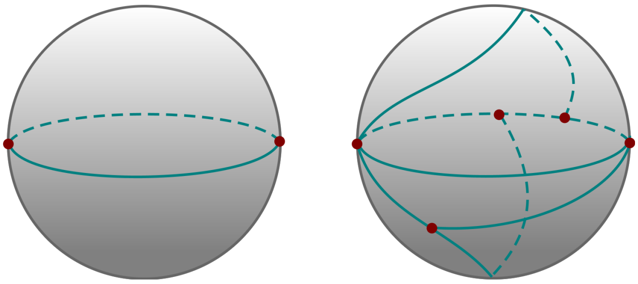

Figure 3.1: A subdivision of a CW-complex.

Lemma 3.1.4. *Suppose that each $H_q(\widetilde{K},\widetilde{L})$ is $\mathbb{Z}(\Pi)$-free with preferred basis h_p, where each basis element can be represented by a cycle contained in a single component of $\widetilde{K}-\widetilde{L}$. Assume that each component of $K-L$ is simply connected. Then $\tau_{h_q}(K,L)=0$.*

Proof. First suppose that $\mathscr{C}(\widetilde{K},\widetilde{L})$ is acyclic and $K-L$ has a single component Γ. Choose a representative component $\widetilde{\Gamma}$ of $\widetilde{K}-\widetilde{L}$. Clearly $\widetilde{\Gamma}$ projects homeomorphically onto Γ. For each cell e of $K-L$ choose the representative cell $\widetilde{e}$ as the unique cell in $\widetilde{\Gamma}$ which lies over e, as illustrated in the Figure 3.2.

Now notice that no representative cell $\widetilde{e}^k$ of $\widetilde{K}-\widetilde{L}$ can be incident to a proper translate $\sigma\widetilde{e}^{k-1}$, $\sigma \neq 1$, of a representative cell, since ∂e^k is contained in the component Γ, so that $\sigma\widetilde{e}^{k-1}$ must be contained in a component $\sigma\widetilde{\Gamma}$ which is disjoint from $\widetilde{\Gamma}$. This means that the boundary $\partial\widetilde{e}^k$ can be expressed as a linear combination of representative $k-1$ cells with coefficients which are integers, rather than elements of the group ring.

For the case that $\mathscr{C}(\widetilde{K},\widetilde{L})$ is not acyclic, we replace each preferred basis element $b \in H_i(\widetilde{K},\widetilde{L})$ by the translate $\sigma(b)$, chosen so that $\sigma(b)$ is represented by a cycle contained in the representative components $\widetilde{\Gamma}$. Since $K-L$ is simply connected, we see that this change of the basis will not alter the torsion.

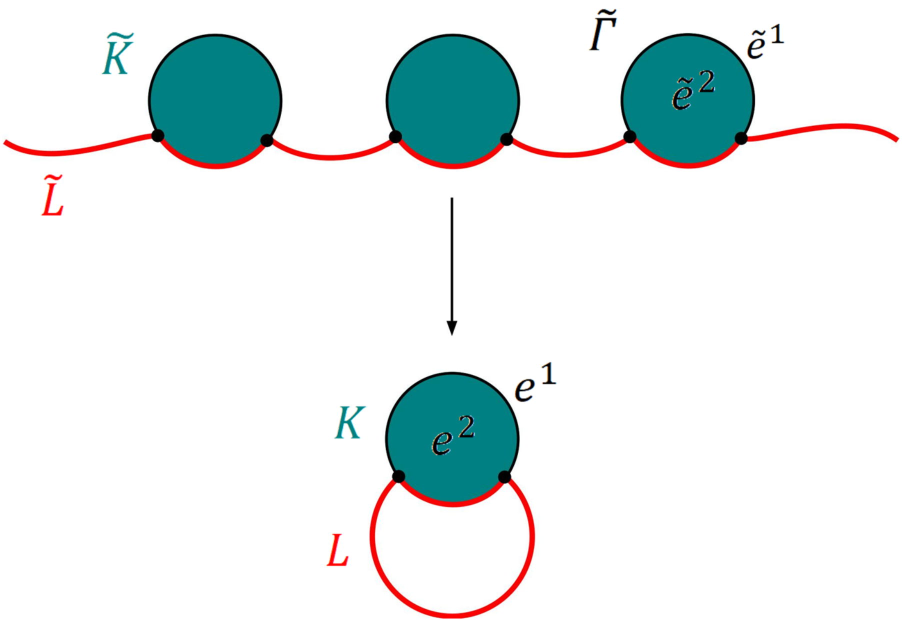

Figure 3.2: Choice of representative cells in $\widetilde{K}$.

Thus in computing the torsion of the complex $\mathscr{C}(\widetilde{K},\widetilde{L})$ we need only work with the subring $\mathbb{Z} \subset \mathbb{Z}(\Pi)$. It follows that the torsion of $\mathscr{C}(\widetilde{K},\widetilde{L})$ belongs to the subgroup

$$\overline{K}_1(\mathbb{Z}) = 0 \subset \overline{K}_1(\mathbb{Z}(\Pi)).$$

Therefore $\tau_{h_q}(K,L) = 0$.

If $K-L$ has several components the proof is essentially the same. It is only necessary to choose a representative component $\widetilde{\Gamma}_i$ lying over each component Γ_i of $K-L$, and choose representative cells $\widetilde{e}$ in $\widetilde{\Gamma}_i$ as before. □

Theorem 3.1.5 (Combinatorial invariance)**.** *The torsion $\tau_{h_q}(K,L)$ is invariant under subdivision of the pair (K,L).*

Proof. Let (X,Y) be a subdivision of the pair (K,L). We consider $\mathscr{C} = \mathscr{C}(\widetilde{X},\widetilde{Y})$ the chain complex associated with the subdivided pair, so that

$$\tau_{h_q}(\mathscr{C}) \mapsto \tau_{h_q}(X,Y) \in Wh(\Pi).$$

We will define a sequence of subcomplexes

$$\mathscr{C}^{(0)} \subset \mathscr{C}^{(1)} \subset \cdots \subset \mathscr{C}^{(n)} = \mathscr{C}$$

so that the associated chain group

$$\overline{C}_\lambda = H_\lambda(\mathscr{C}^{(\lambda)}/\mathscr{C}^{(\lambda-1)})$$

can be identified with $C_\lambda(\widetilde{K},\widetilde{L})$. In particular the torsion $\tau_{h_q}(\overline{\mathscr{C}}) \in \overline{K}_1(\mathbb{Z}(\Pi))$ will correspond to $\tau_{h_q}(K,L) \in Wh(\Pi)$. We will then show that

$$\tau_{h_q}(\mathscr{C}) = \tau_{h_q}(\overline{\mathscr{C}}),$$

and therefore

$$\tau_{h_q}(X,Y) = \tau_{h_q}(K,L).$$

The construction is as follows. Let $X(q)$ be the subcomplex of X consisting of all cells of X which are contained in the subspace $|K^q \cup L| \subset |X|$. Let $\widetilde{X}(q)$ be the corresponding subcomplex of $\widetilde{X}$. Finally, we consider $\mathscr{C}^{(q)} = \mathscr{C}(\widetilde{X}(q),\widetilde{Y})$.

Then

$$H_i(\mathscr{C}^{(q)}/\mathscr{C}^{(q-1)}) = H_i(\widetilde{X}(q),\widetilde{X}(q-1)) \cong H_i(|\widetilde{K}^q \cup \widetilde{L}|,|\widetilde{K}^{q-1} \cup \widetilde{L}|).$$

This group is trivial for $i \neq q$, and is equal to $C_q(\widetilde{K},\widetilde{L})$ for $i = q$. Hence the associated complex $\overline{\mathscr{C}}$ can be identified with $\mathscr{C}(\widetilde{K},\widetilde{L})$.

In order to apply the Theorem 2.5.2, we must verify that the torsion

$$\tau_{c_q}(\mathscr{C}^{(q)}/\mathscr{C}^{(q-1)}) = \tau_{c_q}(\mathscr{C}(\widetilde{X}(q),\widetilde{X}(q-1)))$$

is zero.

In fact, we have that each component of $|X(q)| - |X(q-1)|$ is a q-cell of the complex K, and therefore is simply connected. Moreover, each preferred generator for $H_q(\widetilde{X}(q),\widetilde{X}(q-1)) \cong C_q(\widetilde{K},\widetilde{L})$ is represented by a cycle which lies in a single component of $|\widetilde{X}(q)| - |\widetilde{X}(q-1)|$. Thus, by Lemma 3.1.4 $\tau_{c_q}(\mathscr{C}(\widetilde{X}(q),\widetilde{X}(q-1))) = 0$.

Hence, by Theorem 2.5.2, we conclude that $\tau_{h_q}(X,Y) = \tau_{h_q}(K,L)$. This completes the proof.

□

Remark 3.1.6. *The invariance under subdivision will be sufficient for our purposes. Note, however, a much stronger result due to Chapman: The torsion is invariant under any homeomorphisms between compact connected CW-complexes. For more details see [1].*

Example 3.1.7. *For the n-sphere S^n, we have that $\tau_{h_q}(S^n,\emptyset) \in \overline{K}_1(\mathbb{Z}(\Pi)) = \overline{K}_1(\mathbb{Z}) = 0$, for all $n \geq 2$.*

Note that, in this case the torsion invariant cannot distinguish the spheres, influenced by the action of $\pi_1(S^n)$, which is trivial for $n \geq 2$. An alternative to trying to see something different is to use a representation of the fundamental group, changing the Whitehead group considered. In the next section we will do this construction.

3.2 Change of rings: Reidemeister torsion

First we will define a tensor product, a structure that will be very important to carry out the change of rings, and hence of reduced Whitehead groups.

Definition 3.2.1. *Let A be a right R-module, and let B be a left R-module. Let F be the free abelian group on the set $A \times B$. Let K be the subgroup of F generated by all elements of the forms:*

1. $(a+a',b)-(a,b)-(a',b)$;
2. $(a,b+b')-(a,b)-(a,b')$;
3. $(ar,b)-(a,rb)$,

for all $a,a' \in A$, $b,b' \in B$, and $r \in R$. The quotient group F/K is called the tensor product *of A and B, and it is denoted by $A \otimes_R B$. The coset $(a,b)+K$ is denoted by $a \otimes b$, and the coset of $(0,0)$ is denoted by 0.*

Now we consider the general problem of changing rings. Let $\mathscr{C}$ be a free complex of left R-modules, where each C_q has preferred basis c_q. Let

$$\varphi\colon R \to R'$$

be a ring homomorphism. Then a free complex $\mathscr{C}^\varphi$ over R' is obtained by setting

$$C_q^\varphi = R' \otimes_\varphi C_q = R' \otimes_R C_q,$$

where $r' \otimes rx = r'\varphi(r) \otimes x$, for all $r' \in R'$, $r \in R$ and $x \in C_q$. In other words, we use the homomorphism φ to make R' into a right R-module. Furthermore the basis $c_q = \{c_{q_1}, \ldots, c_{q_\alpha}\}$ for C_q determines a basis $c_q' = \{1 \otimes c_{q_1}, \ldots, 1 \otimes c_{q_\alpha}\}$ for C_q^φ. Assuming that the homology modules $H_q(\mathscr{C}^\varphi)$ are free with preferred basis h_q, then the torsion of the complex $\mathscr{C}^\varphi$ is well defined. This torsion is called the *torsion of $\mathscr{C}$ with respect to the representation φ and depending of the bases c_q and h_q*, and denoted by $\tau^\varphi_{c_q,h_q}(\mathscr{C})$. Thus we have that

$$\tau^\varphi_{c_q,h_q}(\mathscr{C}) = \tau_{c_q',h_q}(\mathscr{C}^\varphi) = \sum_{q=0}^{n}(-1)^q \left[\varphi\left(\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right)\right] = \varphi_*(\tau_{c_q,h_q}(\mathscr{C})).$$

If $\varphi\colon G \to \mathrm{Aut}_{R'}(N)$ is a representation of a group G in the group of the automorphisms of some free right module N over R', then we have the induced ring homomorphism

$$\mathbb{Z}(G) \to \mathbb{Z}(\mathrm{Aut}_{R'}(N)) \subset M_n(R'),$$

where n is the rank of the module N. Hence we may form the complex

$$C_q^\varphi = N \otimes_\varphi C_q,$$

that however is only a complex of $\mathbb{Z}$-modules, unless N is also a left R'-module, in which case we obtain a complex of free finitely generated R'-modules. Assuming that the homology modules $H_q(\mathscr{C}^\varphi)$ are free with preferred bases h_q, then the torsion of complex $\mathscr{C}^\varphi$ with respect to bases c_q and h_q is a class in $\overline{K}_1(M_n(R')) = \overline{K}_1(R')/\{\pm 1\}\varphi(G)$.

Now, as in Section 3.1, let (K,L) be a pair of finite CW-complexes. We again assume that K is connected, with fundamental group Π, and that the chain complex $\mathscr{C}(\widetilde{K},\widetilde{L})$ is free over $\mathbb{Z}(\Pi)$ with preferred bases. We will discuss two different procedures for changing this ring.

In the first procedure, let $\mathbb{F}$ be a field, we consider a homomorphism

$$\varphi\colon \Pi \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}) \cong \mathbb{F}^{\times} = \mathbb{F} - \{0\}.$$

Then φ gives rise to a ring homomorphism

$$\mathbb{Z}(\Pi) \to \mathbb{F}$$

which will also be denoted by φ. Using φ we can form the chain complex

$$\mathscr{C}^{\varphi}(K,L) = \mathbb{F} \otimes_{\varphi} \mathscr{C}(\widetilde{K},\widetilde{L})$$

over $\mathbb{F}$. If each $H_q(\mathscr{C}^{\varphi}(K,L))$ is $\mathbb{Z}(\Pi)$-free with preferred basis h_q, then the torsion

$$\tau_{h_i}(\mathscr{C}^{\varphi}(K,L)) \in \overline{K}_1(\mathbb{F}) = \mathbb{F}^{\times}/\{\pm 1\}\varphi(\Pi)$$

is well defined. This torsion is called the *torsion of (K,L) with respect to the representation φ and depending of the basis h_q*, and denoted by $\tau^{\varphi}_{h_q}(K,L)$.

In the second procedure, we consider $\varphi\colon \Pi \to O(n)$ a real orthogonal representation of the group Π. Then φ extends to a unique ring homomorphism

$$\mathbb{Z}(\Pi) \to \mathbb{Z}(O(n)) \subset M_n(\mathbb{R}).$$

Using φ we can form the complex

$$\mathscr{C}^{\varphi}(K,L) = M_n(\mathbb{R}) \otimes_{\varphi} \mathscr{C}(\widetilde{K},\widetilde{L}).$$

Assuming that each $H_q(\mathscr{C}^{\varphi}(K,L))$ is $\mathbb{Z}(\Pi)$-free with preferred basis h_q, then the torsion

$$\tau^{\varphi}_{h_q}(K,L) \in \overline{K}_1(M_n(\mathbb{R})) = \overline{K}_1(\mathbb{R})$$

is defined.

The group $\overline{K}_1(\mathbb{R})$ can be identified either with the multiplicative group $\mathbb{R}^+$ or with the additive group $\mathbb{R}$, using the correspondence

$$(a_{ij}) \mapsto |\det(a_{ij})| \ \text{ or } \ (a_{ij}) \mapsto \log|\det(a_{ij})|,$$

respectively. Is doesn't really matter which identification is used, but the multiplicative notation is more usual. In this case, we have

$$\begin{aligned}\tau_{h_q}(\mathscr{C}^{\varphi}(K,L)) &= \sum_{q=0}^{n} (-1)^q \left[\varphi\left(\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right)\right] \\ &= \prod_{q=0}^{n} \left|\det\left(\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right)\right|^{(-1)^q}.\end{aligned}$$

Definition 3.2.2. *The positive real number corresponding to* $\tau^{\varphi}_{h_q}(K,L) = \tau_{h_q}(\mathscr{C}^{\varphi}(K,L)) \in \overline{K}_1(\mathbb{R})$ *is called the* R-torsion *of the pair* (K,L) *with respect to real orthogonal representation* φ*, depending of the bases* h_q*.*

For the R-torsion, there is no ambiguity coming from the lack of uniqueness of representative cells in the universal covering space. If the p-cell $\widetilde{e}$ of $\widetilde{K}$ is replaced by a translate $\sigma(\widetilde{e})$, then $\tau^{\varphi}_{h_q}(K,L)$ will be multiplied or divided by $|\det\varphi(\sigma)|$ according as p is odd or even. But this determinant is ± 1, since $\varphi(\sigma)$ is orthogonal.

Remark 3.2.3. *Instead of considering orthogonal representations, W. Müller defines R-torsions using unimodular representations, i.e., a representation* $\varphi\colon\ \pi_1(K) \to GL(E)$ *on a finite dimensional real or complex vector space* E*, where* $|\det\varphi(\gamma)| = 1$*, for all* $\gamma \in \pi_1(K)$*. For more details, see [21].*

Remark 3.2.4. *As in Section 3.1, the torsion with respect to a representation also are invariant under subdivision.*

Example 3.2.5. *Let* S^n *be the n-sphere. We know that* $\pi_1(S^n)$ *is trivial, for* $n \geq 2$*. Moreover, we have*

$$H_i(S^n) = \begin{cases} \mathbb{Z}, & \textit{if } i = 0, n, \\ 0, & \textit{otherwise.} \end{cases}$$

Choosing $\lambda \neq \pm 1$*, we have bases* $h_0 = \lambda c_0$ *for* $H_0(S^n)$*, and* $h_n = \lambda^{-1} c_n$ *for* $H_n(S^n)$*. Thus, denoting by* ρ *the trivial real orthogonal representation of* $\pi_1(S^n)$*, we have*

$$\tau^{\rho}_{h_q}(S^n) = \left|\det\left(\frac{b_0\widetilde{h}_0}{c_0}\right)\right| \left|\det\left(\frac{b_n\widetilde{h}_n\widetilde{b}_{n-1}}{c_n}\right)\right|^{(-1)^n} = \lambda(\lambda^{-1})^{(-1)^n} = \lambda\lambda^{(-1)^{n+1}}.$$

We see that with an appropriate representation of the fundamental group, we can differentiate combinatorially (topologically) the spheres in even and odd dimensions, since for n even, we have $\tau^{\rho}_{h_q}(S^n) = 1$, and for n odd, we have $\tau^{\rho}_{h_q}(S^n) = \lambda^2$. It is interesting to note that the Euler characteristic of S^n is 2 if n is even, and 0 if n is odd. Thus the behavior of the R-torsion with relation to the space, in this case, is the opposite to Euler characteristic.

D. Fried comments in [10] that *"the Euler characteristic counts points while the torsion counts circles"*, in reference to the alternating nature of these two invariants. We can also interpret that the Euler characteristic of the boundary of an n-cell e^n of S^n is comparable with the Reidemeister torsion of the n-sphere, with a suitable choice of basis for the homology modules and representation of the fundamental group, based on Poincaré duality.

The Reidemeister torsion allows us search distinctions at the boundaries of the cells that make up the cellular structure of the space.

3.3 The 3-dimensional lens spaces

Let B^3 be the closed unit ball in $\mathbb{R}^3$ and let D^2_+ and D^2_- be the closed upper and lower hemispheres of ∂B^3. Suppose that integers p, q are given with $p \geq 2$ and $\gcd(p,q) = 1$. Let R be the rotation of

D^2_- through $2\pi q/p$ radians, and define $h\colon D^2_- \to D^2_+$ by $h(x,y,z) = (R(x,y),-z)$. In this setting the 3-dimensional *lens space* $L(p,q)$ is often defined as the quotient space under the equivalence relation generated by h, i.e,

$$L(p,q) = \frac{B^3}{\left[x \sim y \text{ if } x \in D^2_- \text{ and } y = h(x)\right]},$$

as illustrated in Figure 3.3.

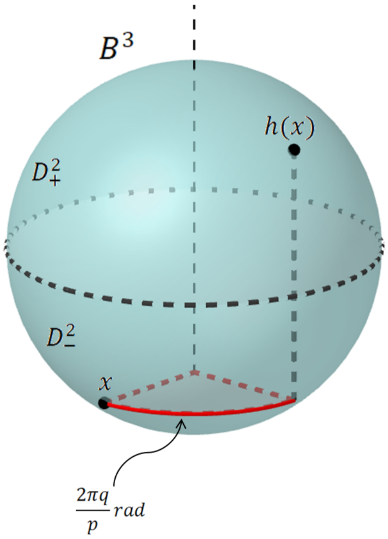

Figure 3.3: The quotient space $L(p,q)$.

Now let S_1 be the unit circle in $\mathbb{R}^2 \times \{0\}$, we define $R_1\colon S_1 \to S_1$ be rotation through $2\pi q/p$ radians. Moreover, we consider $S_2 = \{z-\text{axis}\} \cup \{\infty\}$. Dividing S_2 into p line segments, where one of which is an infinite line segment which has ∞ as an interior point, and an other line segment is from $(0,0,-1)$ to $(0,0,1)$, we define $R_2\colon S_2 \to S_2$ the simplicial isomorphism which shifts each vertex to the next higher one, except that the highest vertex now becomes the lowest.

Since every point of $S^3 - (S_1 \cup S_2)$ belong to an unique arc from S_1 to S_2 we may define $g = R_1 * R_2\colon S^3 \to S^3$ by $g[z_1,z_2,t] = [R_1(z_1),R_2(z_2),t]$, where $[a,b,t]$ denote the point in the arc from a to b, situated in the arc-length t.

For example, as shown in the Figure 3.4, we have the following sets in Table 3.1.

v	$v * S_1$
$(0,0,0)$	$B^2 \times \{0\}$
$(0,0,1)$	D^2_+
$(0,0,-1)$	D^2_-
∞	$\{(x,y,z); x^2+y^2 \geq 1\}$

Table 3.1: Some examples of $v * S_1$.

If we take $\Pi = \{1,g,g^2,\dots,g^{p-1}\}$ and $\psi\colon S^3 \to S^3/\Pi$ the quotient map, we have that $\psi g^k = \psi$, since Π acts freely on S^3. Moreover, by definition of the application g, if $(x,y,z) \in g^k(B^3)$, then

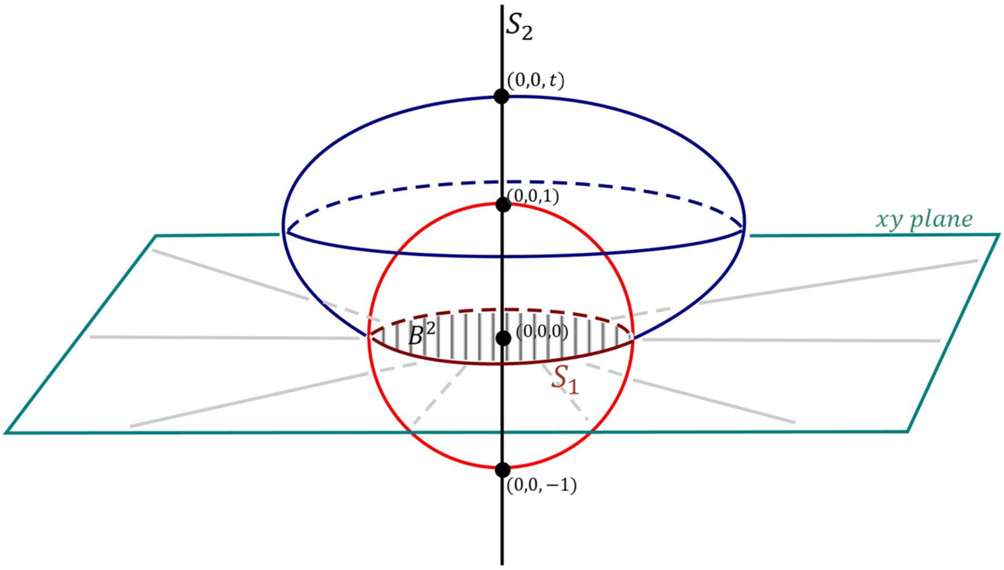

Figure 3.4: The S^3 seen as $S_1 * S_2$.

$(x,y,z) \in S^3$. Given $(x,y,z) \in S^3$, we know that (x,y,z) belongs to some plane parallel to xy-plane, where the point $(0,0,z)$ is at some of the p line segments of S_2, so that we can take (x,y,z) on $g^k(B^3)$, for some k, in a similar way to stereographic projection. It follows that $S^3 = \bigcup_k g^k(B^3)$. Hence

$$\psi(B^3) = \bigcup_k \psi(g^k(B^3)) = \psi\left(\bigcup_k g^k(B^3)\right) = \psi(S^3) = S^3/\Pi.$$

Then S^3/Π is homeomorphic to the quotient space of B^3 under the identifications induced by $\psi|_{B^3}$. By definition of g and h, we have $g|_{D^2_-} = h|_{D^2_-}$, $g(B^3) \cap B^3 = D^2_+$ and $(g^k(B^3) \cap B^3) - S_1 = \emptyset$, if $k \not\equiv \pm 1(\text{mod} p)$. Thus

$$S^3/\Pi \cong B^3/h = L(p,q).$$

Also consider the following subcomplexes of S^3:

$$\begin{aligned}\tilde{e}_0 &= v_0,\\ \tilde{e}_1 &= I_0 = [v_0, e^{2\pi iq/p}],\\ \tilde{e}_2 &= S_1 * v_0 \cong D^2_-,\\ \tilde{e}_3 &= S_1 * I_0 \cong B^3.\end{aligned}$$

The $\tilde{e}_k$ $(0 \leq k \leq 3)$ are closed cells, which give a CW-decomposition of S^3. The maps g^t takes oriented cells isomorphically in an orientation preserving manner to oriented cells and the basic cellular chains satisfy

$\partial\tilde{e}_1 = e^{2\pi iq/p} - v_0 = (g-1)\tilde{e}_0$ (see Figure 3.5),
$\partial\tilde{e}_2 = v_0 + e^{2\pi iq/p} + e^{4\pi iq/p} + \ldots + e^{2(p-1)\pi iq/p} = (1+g+g^2+\ldots+g^{p-1})\tilde{e}_1$ (see Figure 3.6),
$\partial\tilde{e}_3 = S^1 * e^{2\pi iq/p} - S^1 * v_0 = (g^r-1)\tilde{e}_2$, where $rq \equiv 1(\text{mod} p)$ (see Figure 3.7).

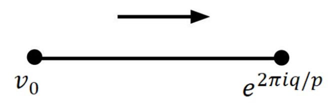

Figure 3.5: Boundary of $\tilde{e}_1$.

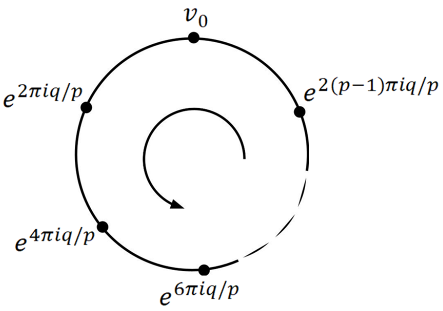

Figure 3.6: Boundary of $\tilde{e}_2$.

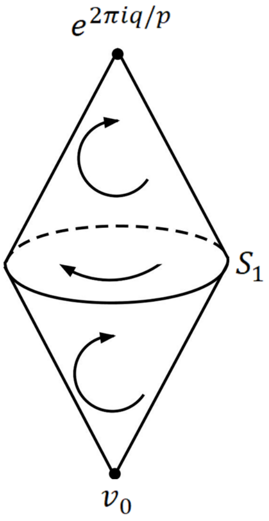

Figure 3.7: Boundary of $\tilde{e}_3$.

$L(p,q)$ obtains a natural CW-structure, with exactly one cell in each dimension from the cell structure on S^3 via projection $\psi\colon S^3 \to L(p,q)$. The cells are the sets $e_k = \psi(\tilde{e}_k)$, where $0 \leq k \leq 3$, with characteristic maps $\psi|_{\tilde{e}_k}\colon \tilde{e}_k \to L(p,q)$. The orientation we have chosen for $\tilde{e}_k$ induces an orientation for e_k.

To compute $H_*(L(p,q))$, we note that

$$\begin{aligned}\partial e_{2i} &= \partial\psi(\tilde{e}_{2i}) = \psi\partial(\tilde{e}_{2i})\\ &= \psi(\tilde{e}_{2i-1} + g\tilde{e}_{2i-1} + \ldots + g^{p-1}\tilde{e}_{2i}) = pe_{2i-1}\\ \partial e_{2i+1} &= \psi\partial(\tilde{e}_{2i+1}) = \psi(g^r\tilde{e}_{2i} - \tilde{e}_{2i}) = 0.\end{aligned}$$

Thus the cellular chain complex is

$$\begin{tikzcd} 0 \arrow[r] & C_3 \arrow[r, "0"] & C_2 \arrow[r, "\times p"] & C_1 \arrow[r, "0"] & C_0 \arrow[r] & 0. \end{tikzcd}$$

Hence the homology groups of $L(p,q)$ with integral coefficients are

$$H_n(L(p,q)) = \begin{cases} \mathbb{Z}, & \text{if } n = 0,3,\\ \mathbb{Z}_p, & \text{if } n = 1,\\ 0, & \text{otherwise.}\end{cases}$$

Since the 3-sphere is the universal covering space of the lens space and Π is the group of covering transformations, $\pi_1(L(p,q)) = \mathbb{Z}_p$ and $\pi_i(L(p,q)) = \pi_i(S^3)$ for $i \neq 1$. Note that p is an invariant for these spaces.

Now we will compute the Reidemeister torsion of the spaces $L(7,1)$ and $L(7,2)$, which will be important for the next sections. Note that the operation used in the following computations is product, since in this case the torsions belong to the group $\mathbb{C}^\times$.

A homomorphism ϕ from Π to the complex numbers $\mathbb{C}$ takes the generator g into some p^{th} root of unity λ. If $\lambda \neq 1$ then

$$1+\lambda+\lambda^2+\ldots+\lambda^{p-1}=0,$$

so that the boundary relations in

$$C_*^{\phi}(L(p,q)) = \mathbb{C}\otimes_{\phi} C_*(S^3)$$

become

$$\partial\tilde{e}_1 = (\lambda-1)\tilde{e}_0, \quad \partial\tilde{e}_2 = 0, \quad \partial\tilde{e}_3 = (\lambda^r-1)\tilde{e}_2.$$

Since $\lambda-1$ and λ^r-1 are non-zero, then the chain complex $C_*^{\phi}(L(p,q))$ is acyclic. Moreover, it is clear that

$$\widetilde{\partial\tilde{e}_1} = \tilde{e}_1, \quad \widetilde{\partial\tilde{e}_2} = 0, \quad \widetilde{\partial\tilde{e}_3} = \tilde{e}_3.$$

Thus the torsion of $L(p,q)$ with relation to the representation ϕ is

$$\begin{aligned}
\tau^{\phi}(L(p,q)) =& \tau(\mathscr{C}^{\phi}(L(p,q))) \\
=& \left[\frac{\partial\tilde{e}_1}{\tilde{e}_0}\right]\left[\frac{\partial\tilde{e}_2\widetilde{\partial\tilde{e}_1}}{\tilde{e}_1}\right]^{-1}\left[\frac{\partial\tilde{e}_3\widetilde{\partial\tilde{e}_2}}{\tilde{e}_2}\right]\left[\frac{\partial\tilde{e}_4\widetilde{\partial\tilde{e}_3}}{\tilde{e}_3}\right]^{-1} \\
=& \det\left(\frac{(\lambda-1)\tilde{e}_0}{\tilde{e}_0}\right)\det\left(\frac{(\lambda^r-1)\tilde{e}_2}{\tilde{e}_2}\right) \\
=& (\lambda-1)(\lambda^r-1).
\end{aligned}$$

This complex number is well defined up to multiplication by numbers of form $\pm\lambda^k$. Taking the absolute value of $\tau^{\phi}(L(p,q))$ we obtain a well defined real number which for simplicity we will denote by $|\tau|$.

Applying this construction to $L(7,1)$, to an approximation of two decimal places, we obtain $|\tau|$ = 0,75 or 2,44 or 3,80, depending on the choice of ϕ. In particular, for $\lambda = e^{2\pi i/7}$, since $r=1$ we have

$$\begin{aligned}
(\lambda-1)(\lambda^r-1) &= (e^{2\pi i/7}-1)^2 \\
&= -0,47 - i0,59 \\
&\Rightarrow |\tau| = 0,75
\end{aligned}$$

On the other hand for $L(7,2)$ we obtain $|\tau|$ = 1,69 or 1,36 or 3,05. In particular, for $\lambda = e^{2\pi i/7}$, since $r=4$ we have

$$\begin{aligned}
(\lambda-1)(\lambda^r-1) &= (e^{2\pi i/7}-1)(e^{8\pi i/7}-1) \\
&= 1,05 - i1,32 \\
&\Rightarrow |\tau| = 1,69
\end{aligned}$$

Thus the spaces $L(7,1)$ and $L(7,2)$ are not combinatorially equivalent.

Remark 3.3.1. *Note that the values of the torsions obtained above are the inverses of the values obtained by J. Milnor in [18].*

3.4 A duality theorem for Reidemeister torsion

Definition 3.4.1. *Let V be a vector space over $\mathbb{F}$, and consider $r \in \mathbb{N}$. We consider the vector space over $\mathbb{F}$*

$$V^{\otimes r} = \underbrace{V \otimes \ldots \otimes V}_{r \text{ times}}$$

and the subspace

$$A = \langle v_1 \otimes \ldots \otimes v_i \otimes \ldots \otimes v_j \otimes \ldots \otimes v_r;\ v_1, \ldots, v_r \in V, 1 \leq i < j \leq r, v_i = v_j \rangle.$$

The vector space $V^{\otimes r}/A$ over $\mathbb{F}$ is called the r^{th} exterior power of V and is denoted by $\Lambda^r V$. We denote an element $(v_1 \otimes \ldots \otimes v_r) + A$ by $v_1 \wedge \ldots \wedge v_r$.

More details with respect to exterior algebra also can be found in [17] and [12].

Now we consider a vector space V of dimension r over $\mathbb{F}$. By simply we denote by ΛV the r^{th} exterior power $\Lambda^r V$. This is a one-dimensional vector space since $\dim \Lambda V = \binom{r}{r} = 1$.

Given a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

and given generators $a \in \Lambda A$ and $b \in \Lambda B$, construct a generator $b/a \in \Lambda C$ as follows. Choose a basis $\{x_1, \ldots, x_n\}$ for B so that the last $n-m$ vectors $\{x_{m+1}, \ldots, x_n\}$ form a basis for the subspace $A \subset B$. Thus a and b can be written in the form

$$a = \lambda x_{m+1} \wedge \ldots \wedge x_n,\ b = \mu x_1 \wedge \ldots \wedge x_n$$

for appropriate $\lambda, \mu \in F$ with $\lambda, \mu \neq 0$.

Define b/a to be the image in ΛC of

$$\mu \lambda^{-1} x_1 \wedge \ldots \wedge x_m.$$

Compare the construction of these generators b/a with the construction of the bases $m\widetilde{s}$ in Section 2.2.

Now consider a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_0 \arrow[r] & 0 \end{tikzcd},$$

and a preferred generator $vc_q \in \Lambda C_q$, for each q. Using the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & B_{n-2} \arrow[r] & 0, \end{tikzcd}$$

one construct a generator

$$vc_{n-1}/vc_n \in \Lambda B_{n-2}.$$

Proceeding inductively, using the sequences

$$\begin{tikzcd} 0 \arrow[r] & B_q \arrow[r] & C_q \arrow[r] & B_{q-1} \arrow[r] & 0, \end{tikzcd}$$

we constructs generators

$$vc_q/(vc_{q+1}/(\ldots/(vc_{n-1}/vc_n)\ldots)) \in \Lambda B_{q-1}$$

for each q. We will denote these generators by

$$(vc_q/vc_{q+1}/\ldots/vc_{n-1}/vc_n).$$

In particular, we obtain a generator

$$(vc_1/vc_2/\ldots/vc_{n-1}/vc_n) \in \Lambda B_0 = \Lambda C_0.$$

Let $c_0 = \{c_{0_1}, \ldots, c_{0_n}\}$ be a basis of C_0. We can write

$$(vc_1/vc_2/\ldots/vc_{n-1}/vc_n) = \lambda c_{0_1} \wedge \ldots \wedge c_{0_n}.$$

Given the generator $vc_0 = \mu c_{0_1} \wedge \ldots \wedge c_{0_n} \in \Lambda C_0$, we obtain a element

$$\mu\lambda^{-1} = \tau_{c_q}(\mathscr{C})^{-1} \in \mathbb{F}^{\times},$$

which is the torsion of $\mathscr{C}$ depending of the bases $c_q = \{c_{q_1}, \ldots, c_{q_k}\}$, where we can write each generator $vc_q = c_{q_1} \wedge \ldots \wedge c_{q_k}$. Note that we can also write

$$(vc_1/vc_2/\ldots/vc_{n-1}/vc_n) = \tau_{c_q}(\mathscr{C}) vc_0. \tag{3.1}$$

Now we will apply the duality for this situation. For any vector space E, note that $\Lambda(E^*)$ is isomorphic to the dual $(\Lambda E)^*$ of ΛE. In fact, given the elements $x_1, \ldots, x_n \in E$ and $y_1^*, \ldots, y_n^* \in E^*$, define

$$(y_1^* \wedge \ldots \wedge y_n^*)(x_1 \wedge \ldots \wedge x_n) = \det(y_i^*(x_j)),$$

where $(y_i^*(x_j))$ is the $n \times n$ matrix which the i^{th} line vector is given by $(y_i^*(x_1), \ldots, y_i^*(x_n))$.

Example 3.4.2. *If $x_1, \ldots, x_n$ is a basis for E and $x_1^*, \ldots, x_n^*$ is the dual basis for E^* then*

$$(x_1^* \wedge \ldots \wedge x_n^*)(x_1 \wedge \ldots \wedge x_n) = \det(x_i^*(x_j)) = \det Id = 1.$$

Example 3.4.3. *Given the basis $v = \{(4,2),(6,2)\}$ of $\mathbb{R}^2$, we have the dual basis $v^* = \{(-1,3),(1,-2)\}$. Choosing the standard basis $c = \{(1,0),(0,1)\}$, we have that the matrix of change of the basis c for v is given by*

$$A = \begin{pmatrix} 4 & 6 \\ 2 & 2 \end{pmatrix}.$$

And the matrix of change of the basis c^* for v^* is given by

$$(A^{-1})^T = \begin{pmatrix} -1/2 & 1/2 \\ 3/2 & -1 \end{pmatrix}.$$

We conclude that the determinant of the matrix of change of the basis c^* for v^* id the inverse of the determinant of the matrix of change of the basis c for v.

Consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

together with the dual sequence

$$0 \longrightarrow C^* \longrightarrow B^* \longrightarrow A^* \longrightarrow 0.$$

Let $a \in \Lambda A$, $b \in \Lambda B$, $b^* \in \Lambda B^*$, and $c^* \in \Lambda C^*$ be generators with $b^*(b) = 1$.

Lemma 3.4.4. *Then* $c^*(b/a)$ *is equal to* $\pm((b^*/c^*)(a))^{-1}$.

Proof. Choose a basis $\{x_1, \dots, x_n\}$ for B so that $\{x_{m+1}, \dots, x_n\}$ form a basis for A. Let $\{x_1^*, \dots, x_n^*\}$ be the dual basis for B^*, so that $\{x_1^*, \dots, x_m^*\}$ form a basis for $C^* \subset B^*$. Define the elements $\lambda, \mu, \eta \in \mathbb{F}^\times$ by

$$\begin{aligned} a &= \lambda x_{m+1} \wedge \dots \wedge x_n, \\ b &= \mu x_1 \wedge \dots \wedge x_n, \\ b^* &= \overline{\mu}^{-1} x_1^* \wedge \dots \wedge x_n^*, \\ c^* &= \overline{\eta} x_1^* \wedge \dots \wedge x_m^*, \end{aligned}$$

where $\overline{\mu}$ and $\overline{\eta}$ denote the conjugates of μ and η, respectively. Then $c^*(b/a) = \mu\lambda^{-1}\eta$ and $(b^*/c^*)(a) = \pm\lambda\mu^{-1}\eta^{-1}$. Thus we conclude that

$$c^*(b/a) = \pm((b^*/c^*)(a))^{-1}.$$

□

Theorem 3.4.5. *If the torsion* $\tau^\varphi(M)$ *is defined, then* $\tau^\varphi(M, \partial M)$ *is defined, and conversely. Furthermore*

$$\tau^\varphi(M, \partial M) = \overline{\tau^\varphi(M)}^{(-1)^n}.$$

Proof. Given an acyclic chain complex

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$$

with generators $vc_q \in \Lambda C_q$, form the dual complex

$$0 \longrightarrow C_0^* \longrightarrow \cdots \longrightarrow C_{n-1}^* \longrightarrow C_n^* \longrightarrow 0$$

and the dual generators $vc_q^* \in \Lambda C_q^*$. Just as before we construct generators

$$(vc_q^*/vc_{q-1}^*/\dots/vc_1^*/vc_0^*) \in \Lambda B_{q+1}^*$$

by induction on q. Writing $(vc^*_{n-1}/\ldots/vc^*_1/vc^*_0) = \gamma c^*_{n_1}\wedge\ldots\wedge c^*_{n_k}$ and $\eta c^*_{n_1}\wedge\ldots\wedge c^*_{n_k}$, the torsion of the dual complex $\mathscr{C}^*$ depending of the bases $c^*_q = \{c^*_{q_1},\ldots,c^*_{q_k}\}$, is the defined as the element

$$\eta\gamma^{-1} = \tau_{c^*_q}(\mathscr{C}^*)^{-1} \in \mathbb{F}^\times.$$

Note that we can also write

$$(vc^*_{n-1}/vc^*_{n-2}/\ldots/vc^*_0) = \tau_{c^*_q}(\mathscr{C}^*)vc^*_n. \tag{3.2}$$

Note that using the Equation (3.1), we have

$$\lambda = vc^*_0((v_1/\ldots/vc_{n-1}/vc_n)) = vc^*_0(\tau_{c_q}(\mathscr{C})vc_0) = \tau_{c_q}(\mathscr{C}),$$

where $\lambda\in\mathbb{F}^\times$. Inductively, applying the Lemma 3.4.4 to the dual exact sequences

$$\begin{tikzcd} 0 \arrow[r] & B_q \arrow[r] & C_q \arrow[r] & B_{q-1} \arrow[r] & 0, \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & B^*_q \arrow[r] & C^*_q \arrow[r] & B^*_{q+1} \arrow[r] & 0, \end{tikzcd}$$

we obtain the identity

$$\pm\lambda = \pm\vartheta^*_{q-1,\ldots,0}(vc_q/\vartheta_{q+1,\ldots,n}) = \left(\left(vc^*_q/\vartheta^*_{q-1,\ldots,0}\right)(\vartheta_{q+1,\ldots,n})\right)^{(-1)^q},$$

where $\vartheta_{q+1,\ldots,n} = (vc_{q+1}/\ldots/vc_{n-1}/vc_n)$ and $\vartheta^*_{q-1,\ldots,0} = (vc^*_{q-1}/\ldots/vc^*_1/vc^*_0)$.

In particular, taking $q = n-1$ and using the Equation (3.2) we have

$$\pm\lambda = \left((vc^*_{n-1}/vc^*_{n-2}/\ldots/vc^*_0)\right)(vc_n) = \Big((\tau_{c^*_q}(\mathscr{C}^*)vc^*_n)(vc_n)\Big)^{(-1)^{n-1}} = \overline{\tau_{c^*_q}(\mathscr{C}^*)}^{(-1)^{n-1}}.$$

Therefore the torsion $\tau_{c^*_q}(\mathscr{C}^*)$ of the dual complex satisfies the identity

$$\tau_{c_q}(\mathscr{C})\overline{\tau_{c^*_q}(\mathscr{C}^*)}^{(-1)^n} = \lambda(\pm\lambda^{-1}) = \pm 1. \tag{3.3}$$

Now we know that the q-cells of M determine a preferred basis for $C_q(M)$ and hence a preferred generator

$$vc_q \in \Lambda C_q = \Lambda(\mathbb{F}\otimes_\varphi C_q(M)).$$

This generator is well defined up to multiplication by elements in $\pm\varphi(\Pi)$. Given $\tau_{c_q}(\mathscr{C})\in\mathbb{F}^\times$ then $\tau^\varphi(M)$ is defined as the image in the quotient $\mathbb{F}^\times/\pm\varphi(\Pi)$.

The $(n-q)$-cells of $M'-\partial M'$ determine a dual basis for $C_{n-q}(M',\partial M')$ and hence determine the dual generator

$$vc^*_q \in \Lambda C^*_q = \Lambda(\mathbb{F}\otimes_\varphi C_{n-q}(M',\partial M')).$$

Thus given $\tau_{c^*_q}(\mathscr{C}^*)$ is defined then

$$\tau^\varphi(M',\partial M') = \pm\varphi(\Pi)\tau_{c^*_q}(\mathscr{C}^*) \in \mathbb{F}^\times/\varphi(\Pi).$$

By Equation (3.3), we have that

$$\tau^{\varphi}(M)\overline{\tau^{\varphi}}(M',\partial M')^{(-1)^n} = \pm\varphi(\Pi).$$

Since M and M' have a common subdivision, we have

$$\tau^{\varphi}(M,\partial M') = \tau^{\varphi}(M',\partial M')$$

□

Corollary 3.4.6. *The torsion invariant* $\tau = \tau^{\varphi}(M)$ *satisfies the identity*

$$\tau\overline{\tau}^{(-1)^n} = \pm\varphi(\Pi).$$

Now suppose that $\mathscr{C}$ is a finite chain complex

$$\begin{tikzcd} C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_1 \arrow[r] & C_0 \end{tikzcd},$$

where each homology group $H_i = H_i(\mathscr{C})$ is free with a preferred basis h_i.

Note that using the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & B_{n-1} \arrow[r] & Z_{n-1} \arrow[r] & H_{n-1} \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0, \end{tikzcd}$$

we construct the generators

$$vc_n/vz_n \in \Lambda B_{n-1} \text{ and } (vc_n/vz_n)vh_{n-1} \in \Lambda Z_{n-1}.$$

Proceeding inductively, we constructs generators

$$vc_q/(vc_{q+1}/(\ldots/(vc_n/vz_n)vh_{n-1}\ldots)vh_q) \in \Lambda B_{q-1}$$

for each q. We will denote these generators by

$$(vc_q/vc_{q+1}vh_q/\ldots/vc_nvh_{n-1}/vz_n).$$

In particular, we obtain the generators

$$(vc_1/vc_2vh_1/\ldots/vc_nvh_{n-1}/vh_n) \in \Lambda B_0, \text{ and}$$

$$(vc_1vh_0/vc_2vh_1/\ldots/vc_nvh_{n-1}/vh_n) \in \Lambda Z_0 = \Lambda C_0.$$

Here it is more convenient to replaced vz_n by vh_n, since $\Lambda Z_n = \Lambda H_n$. Let $c_0 = \{c_{0_1},\ldots,c_{0_n}\}$ be a basis of C_0. We can write

$$(vc_1vh_0/vc_2vh_1/\ldots/vc_nvh_{n-1}/vh_n) = \lambda c_{0_1}\wedge\ldots\wedge c_{0_n}.$$

Given the generator $vc_0 = \mu c_{0_1}\wedge\ldots\wedge c_{0_n} \in \Lambda C_0$, we obtain a element

$$\mu\lambda^{-1} = \tau_{c_q,h_q}(\mathscr{C})^{-1} \in \mathbb{F}^{\times},$$

which is the torsion of $\mathscr{C}$ depending of the bases c_q and h_q.

Therefore, we have the following result.

Theorem 3.4.7. *If the torsion* $\tau^{\varphi}_{h_q}(M)$ *is defined, then* $\tau^{\varphi}_{h_q}(M,\partial M)$ *is defined, and conversely. Furthermore*

$$\tau^{\varphi}_{h_q}(M,\partial M) = \overline{\tau^{\varphi}_{h_q}(M)}^{(-1)^n}.$$

Proof. Using the above construction, the proof is analogous to the Theorem 3.4.5. □

3.5 An application of the torsion to the Hauptvermutung

The Hauptvermutung is now a refuted conjecture stating that any two triangulations of homeomorphic spaces are combinatorially equivalent. This conjecture was formulated in 1908, by E. Steinitz and H. F. F. Tietze.

The propose of this section is present a beautiful application of the invariant that we have constructed so far, in addition to making clearer the way in which the torsion invariant was used by J. Milnor in [18] to prove that the Hauptvermutung is false for the case of pseudomanifolds of dimension greater than or equal to 6. For this reason, some results will be assumed since the their proofs use tools that are beyond the objective of this text.

Lemma 3.5.1 ([18, Lemma 1])**.** *If* $\pm qq'$ *is a quadratic residue modulo p, and if* $n > 3$*, then* $L(p,q)\times\mathbb{R}^n$ *is diffeomorphic to* $L(p,q')\times\mathbb{R}^n$*.*

As an example, for $p = 7$, we obtain that $L(7,1)\times\mathbb{R}^n$ is diffeomorphic to $L(7,2)\times\mathbb{R}^n$, for $n > 3$, since $1\cdot 2 \equiv 3^2$ is a quadratic residue modulo 7.

Now, we notice that it is possible to define torsion of even more general way, as follows. Let G be a discrete group which acts freely on a CW-complex K, and let $\varphi\colon G\to\mathbb{P}$ be a multiplicative homomorphism from G to a commutative ring $\mathbb{P}$. If

1. the quotient complex K/G has only finitely many cells, and
2. the homology group $H_i(\mathbb{P}\otimes_{\varphi} C_*(K;\mathbb{Z}))$ is trivial, for each i,

then the torsion $\Delta_{\varphi}(K)$, will be defined. The torsion is a unit of $\mathbb{P}$ which is well defined up to multiplication by elements of the form $\pm\varphi(g)$, where $g\in G$. We will use the notation

$$\Delta_{\varphi}(K)\in U(\mathbb{P})/\pm\varphi(G),$$

where $U(\mathbb{P})\subset\mathbb{P}$ denotes the group of units. This element $\Delta_{\varphi}(K)$ is also invariant under subdivision, as previously.

Note that if $\widetilde{K}$ is taken to be the universal covering space of a finite CW-complex K, and G is the group of covering transformations, then we have that $\Delta_{\varphi}(\widetilde{K})$ is exactly $\tau^{\varphi}(K)$, as we defined in Section 3.2.

Theorem 3.5.2. *Let L_q be the 3-dimensional lens manifold $L(7,q)$, suitably triangulated, and denote σ^n an n-simplex. Consider the finite simplicial complex X_q is obtained from the product $L_q \times \sigma^n$ by adjoining a cone over the boundary $L_q \times \partial\sigma^n$, i.e.,*

$$X_q = L_q \times \sigma^n \sqcup_{L_q \times \partial\sigma^n} C(L_q \times \partial\sigma^n).$$

Then, for $n \geq 3$, the complexes X_1 and X_2 are homeomorphic.

Proof. Let x_0 be the vertex of the cone. For $n > 3$, the complement $X_q - \{x_0\}$ is homoeomorphic to $L_q \times \mathbb{R}^n$. In fact, let $h\colon \sigma^n \to D^n$ be a homeomorphism, and define $f\colon X_q - \{x_0\} \to L_q \times \mathbb{R}^n$ by

$$f(y,z) = (y, h(z)),$$

$$f(t(y,z') + (1-t)x_0) = \left(y, \frac{h(z')}{t}\right),$$

for $y \in L_q$, $z \in \sigma^n$, $z' \in \partial\sigma^n$, and $0 < t \leq 1$. It is clear that f is continuous, with inverse defined directly by h, for the pairs $(x,s) \in L_q \times D^n$, and for the pairs $(x,w) \in L_q \times (\mathbb{R}^n - \mathrm{Int} D^n)$, we use h in addition the homeomorphism between $L_q \times (\mathbb{R}^n - \mathrm{Int} D^n)$ and $C(L_q \times \partial D^n) - \{x_0\}$, in a similar way to stereographic projection. Therefore, f is a homeomorphism between $X_q - \{x_0\}$ and $L_q \times \mathbb{R}^n$. Hence X_q is homeomorphic to the single ponit compactification of $L_q \times \mathbb{R}^n$. We conclude that X_1 is homeomorphic to X_2, since by Lemma 3.5.1, $L_1 \times \mathbb{R}^n$ is homeomorphic to $L_2 \times \mathbb{R}^n$. The case $n = 3$ is a little more delicate, where it is necessary to use elements that are beyond the objective of this text. For more details see [18]. □

Now we will make a construction that will give us a negative answer to Hauptvermutung when $n \geq 3$.

Theorem 3.5.3. *No finite cell subdivision of the simplicial complex X_1 is isomorphic to a cell subdivision of X_2.*

Proof. Each X_q is a manifold except at one exceptional point x_0. Removing this point we obtain a space $X_q - \{x_0\}$ which is homeomorphic to $L(7,q) \times \mathbb{R}^n$. We have that $\Pi = \pi_1(X_q - \{x_0\})$ is a cyclic group of order 7, since $\Pi \cong \pi_1(L(7,q) \times \mathbb{R}^n) \cong \pi_1(L(7,q)) \cong \mathbb{Z}_7$.

Suppose that K_q denote the single point compactification of the universal covering space of $X_q - \{x_0\}$. Thus the group Π operates on K_q with a single fixed point. The quotient space K_q/Π is equal to X_q. It is clear that any cell structure on the pair (X_q, x_0) induces a cell structure on the pair (K_q, k_0) and hence on K_q.

The simplest cell structure on X_q has five cells: namely the four cells $e_i \times \mathrm{Int} D^n$ of $L(7,q) \times \mathrm{Int} D^n \cong L(7,q) \times \mathbb{R}^n \cong X_q - \{x_0\}$, together with the vertex x_0. The corresponding cell structure on K_q has twenty-eight cells of the form $g^r(e_i) \times \mathrm{Int} D^n$, for $0 \leq r \leq 6$, together with one vertex which will be denoted by k_0. Remember from Section 3.3 that g is a generator of Π.

Consider the chain complex $C_*(K_q, k_0)$. This complex is free over the group ring $\mathbb{Z}(\Pi)$ with four preferred generators $e_i \times \operatorname{Int} D^n$. It is equal to the chain complex $C_*^{\phi}(\tilde{L}(7,q))$ except for a shift of n in the dimension. Hence the torsion $\Delta_{\phi}(K_q, k_0)$ is defined and is equal to $\tau^{\phi}(L(7,q))^{(-1)^n}$, where ϕ is the representation which we use to calculate the torsions of $L(7,1)$ and $L(7,2)$ in Section 3.3.

Therefore the torsion invariant Δ_{ϕ} distinguishes (K_1, k_0) from (K_2, k_0). It follows that the pairs (X_1, x_0) and (X_2, x_0) are not combinatorially equivalent and hence no cell subdivision of X_1 is isomorphic to a cell subdivision of X_2.

□

A natural question that may arise is if the Theorem 3.5.3 produce a counterexample for the Chapman's result in Remark 3.1.6. The answer for this speculation is negative. We know that the complexes $X_1 - \{x_0\}$ and $X_2 - \{x_0\}$ are homeomorphic, with universal covering $K_1 - \{k_0\}$ and $K_2 - \{k_0\}$, respectively. It is possible conclude that the torsion of the complexes $\Delta_{\phi}(K_1 - \{k_0\})$ and $\Delta_{\phi}(K_2 - \{k_0\})$ are different and hence $\tau^{\phi}(X_1 - \{x_0\})$ and $\tau^{\phi}(X_2 - \{x_0\})$ are different. But the spaces $X_1 - \{x_0\}$ and $X_2 - \{x_0\}$ clearly are not compact, because they are open. Therefore the spaces $X_1 - \{x_0\}$ and $X_2 - \{x_0\}$ not satisfies the hypothesis of Chapman's result.

On the other hand, does not make sense consider the spaces X_1 and X_2 for this purpose, although these two complexes are compact and connected, satisfying the hypotheses in Remark 3.1.6. Note that at no time we calculate the torsions of K_1 and K_2, and even if that were the case, not necessarily K_q is the universal covering of X_q, as well as Π is not the group of covering transformations of K_q, the group Π doesn't even acts freely on K_q.

CHAPTER 4

Intersection Reidemeister torsion for spaces with isolated singularities

In this final chapter, we present a study of the torsion invariant for some singular spaces. So far, we have seen that the torsion invariant has several properties, particularly a type of duality induced by Poincaré duality. Since this invariant depends on the homology of the space and we aim to define it for singular spaces, ensuring that the torsion continues to satisfy the duality property, we will need to resort to intersection homology theory. Hence, this "new" torsion will be called intersection Reidemeister torsion.

4.1 The cone of a chain complex and algebraic mapping cone

Let $\mathscr{C}$ be a finite chain complex of free finitely generated left R-modules

$$\begin{tikzcd} C_m \arrow[r] & C_{m-1} \arrow[r] & \cdots \arrow[r] & C_1 \arrow[r] & C_0. \end{tikzcd}$$

We assume that each C_q has a preferred basis denoted by c_q. The *cone* of $\mathscr{C}$ is the algebraic mapping cone of the chain identity of the augmented complex $\mathscr{C}$, i.e., the chain complex $C(\mathscr{C})$ of length $m+1$ with chain modules given by

$$\dot{C}_q = \begin{cases} C_{q-1} \oplus C_q, & \text{if} \quad q > 0, \\ \langle v \rangle \oplus C_0, & \text{if} \quad q = 0, \end{cases}$$

where $\langle v \rangle$ denote the module generated by $v \in R$, and boundary operator is given by

$$\dot{\partial}_q = \begin{cases} \begin{pmatrix} \partial_{q-1} & 0 \\ 1 & -\partial_q \end{pmatrix}, & q > 1, \\ \begin{pmatrix} \varepsilon & 0 \\ 1 & -\partial_1 \end{pmatrix}, & q = 1, \\ 0, & q = 0, \end{cases}$$

where $\varepsilon \colon C_0 \to \langle v \rangle$ is the augmentation, i.e., $\sum_i n_i \sigma_i \mapsto \sum_i n_i$. In particular

$$\begin{tikzcd} \cdots \arrow[r] & \dot{C}_1 = C_0 \oplus C_1 \arrow[r] & \dot{C}_0 = \langle v \oplus 0, 0 \oplus c_0 \rangle \arrow[r] & 0. \end{tikzcd}$$

Let $\dot{B}_q$ be the image of the boundary homomorphism $\dot{\partial}_{q+1} \colon \dot{C}_{q+1} \to \dot{C}_q$ and let $\dot{Z}_{q+1}$ be its kernel. For $q > 0$,

$$\dot{H}_q = \frac{\dot{Z}_q}{\dot{B}_q} = \frac{\ker \dot{\partial}_q}{\operatorname{Im} \dot{\partial}_{q+1}} = \frac{\{\dot{c} = \partial_q(y) \oplus y;\ y \in C_q\}}{\dot{Z}_q - \{0 \oplus B_q\}} \cong 0,$$

while

$$\dot{H}_0 = \frac{\langle v \oplus 0, 0 \oplus c_0 \rangle}{\langle v \oplus c_0 \rangle} \cong \langle v \oplus 0 \rangle.$$

The chain inclusion

$$\begin{aligned} j_q \colon C_q &\to \dot{C}_q \\ c_q &\longmapsto 0 \oplus c_q \end{aligned}$$

induces the exact sequence

$$0 \longrightarrow \mathscr{C} \xrightarrow{\ j\ } C(\mathscr{C}) \xrightarrow{\ p\ } (C(\mathscr{C}), \mathscr{C}) \longrightarrow 0,$$

where j denote the chain inclusion induced by j_q, for each q, and p is the projection. The chain module of the relative complex is $C''_q = \dot{C}_q / C_q = (C_{q-1} \oplus C_q)/C_q \cong C_{q-1}$, for $q > 0$, and $C''_0 \cong \langle v \rangle$. Moreover the boundary $\partial''_q = \partial_{q-1}$, for $q > 1$, and $\partial''_1 = \varepsilon$. Bases for the chain modules are $c''_q = c_{q-1}$ for $q > 0$, and $c''_0 = \{v\}$.

The short exact sequence of chain complex above induces a homology long exact sequence

$$\cdots \longrightarrow H_q(C(\mathscr{C})) = 0 \longrightarrow H_q(C(\mathscr{C}), \mathscr{C}) \xrightarrow{p_{*,q}} H_{q-1}(\mathscr{C}) \longrightarrow H_{q-1}(C(\mathscr{C})) = 0,$$

for $q > 1$, where the boundary is in fact the identity. Therefore, $H_q(C(\mathscr{C}), \mathscr{C}) \cong H_{q-1}(\mathscr{C})$, for $q > 1$, and it is trivial otherwise. If h_q is a basis for $H_q(\mathscr{C})$, then a basis h''_q for $H_q(C(\mathscr{C})$ is given by h_{q-1}, for $q > 1$.

To define the torsion of $C(\mathscr{C})$, we will need to determine the bases of the chain modules, of the images of the boundaries, and of the homology.

Since each chain module C_q of $\mathscr{C}$ has a preferred basis c_q, then each chain module $\dot{C}_q$ of $C(\mathscr{C})$ has a preferred basis $\dot{c}_q = \{c_{q-1} \oplus 0, 0 \oplus c_q\}$, for $q > 0$, and $\dot{c}_0 = \{0 \oplus c_0\}$. The unique non trivial homology group of the cone is in dimension zero, and it is free, therefore the torsion of chain complex $C(\mathscr{C})$ is defined, and is related to the torsion of $\mathscr{C}$. We denote by $\dot{h}_0$ a fixed basis for $\dot{H}_0$.

Assume that each $\dot{B}_q$ has a preferred basis $\dot{b}_q$. Following the construction in Section 2.3, since $H_q(C(\mathscr{C}))$ is trivial for $q > 0$, we have that $\dot{b}_q \widetilde{\dot{b}}_{q-1}$ is other basis for $\dot{C}_q$, for $q > 0$. A basis for $\dot{C}_0$ is given by $\dot{b}_0 \widetilde{\dot{h}}_0$. Applying the definition

$$\tau_{\dot{c}_q, \dot{h}_q}(C(\mathscr{C})) = \left[\frac{\dot{b}_0 \widetilde{\dot{h}}_0}{\dot{c}_0} \right] + \sum_{q=1}^{m+1} (-1)^q \left[\frac{\dot{b}_q \widetilde{\dot{b}}_{q-1}}{\dot{c}_q} \right].$$

We want to write the torsion of the cone as a function of the torsion of $\mathscr{C}$. For $q > 1$, if $c = x \oplus y \in \dot{C}_q = C_{q-1} \oplus C_q$,

$$\dot{\partial}_q(c) = \begin{pmatrix} \partial_{q-1} & 0 \\ 1 & -\partial_q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \partial_{q-1}(x) \\ x - \partial_q(y) \end{pmatrix}.$$

Thus, when $q > 1$,

$$\dot{\partial}_q(c_{q-1}\oplus 0) = \partial_{q-1}(c_{q-1})\oplus c_{q-1},$$

$$\dot{\partial}_q(0\oplus \widetilde{b}_{q-1}) = 0\oplus -b_{q-1}.$$

Thereby we can not consider the elements c_{q-1} coming from of b_{q-1}. Therefore, we have that

$$\widetilde{\dot{b}}_{q-1} = \{\widetilde{h}_{q-1}\widetilde{b}_{q-2}\oplus 0, 0\oplus \widetilde{b}_{q-1}\},$$

$$\dot{b}_q = \{0\oplus \widetilde{h}_q, b_{q-1}\oplus \widetilde{b}_{q-1}, 0\oplus -b_q\},$$

and hence the new basis for $\dot{C}_q$. with $q > 1$ is

$$\begin{aligned}\dot{b}_q\widetilde{\dot{b}}_{q-1} &= \{0\oplus \widetilde{h}_q, b_{q-1}\oplus \widetilde{b}_{q-1}, 0\oplus -b_q, \widetilde{h}_{q-1}\widetilde{b}_{q-2}\oplus 0, 0\oplus \widetilde{b}_{q-1}\}\\ &\cong \{0\oplus b_q\widetilde{h}_q\widetilde{b}_{q-1}, b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}\oplus 0\}.\end{aligned}$$

At $q = 0, 1$, recall the right end of the chain complex of $C(\mathscr{C})$, where

$$\dot{\partial}_1(x\oplus y) = \begin{pmatrix}\varepsilon & 0\\ 1 & -\partial_1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\varepsilon(x)\\ x-\partial_1(y)\end{pmatrix} = \varepsilon(x)\oplus x-\partial_1(y)$$

we see that $\dot{b}_0 = \{\varepsilon(\widetilde{h}_0)\oplus \widetilde{h}_0, 0\oplus -b_0\}$. Then the new basis at $q = 1$ is

$$\dot{b}_1\widetilde{\dot{b}}_0 = \{0\oplus b_1\widetilde{h}_1\widetilde{b}_0, b_0\widetilde{h}_0\oplus 0\}.$$

Since $\widetilde{\dot{h}}_0 = \{\alpha v\oplus 0, 0\oplus \widetilde{h}_0\}$, the new basis at $q = 0$ is

$$\begin{aligned}\dot{b}_0\widetilde{\dot{h}}_0 &= \{\varepsilon(\widetilde{h}_0)\oplus \widetilde{h}_0, 0\oplus -b_0, \alpha v\oplus 0, 0\oplus \widetilde{h}_0\}\\ &\cong \{0\oplus b_0\widetilde{h}_0, \alpha v\oplus 0\}.\end{aligned}$$

Therefore, for $q > 1$

$$\begin{aligned}\left[\frac{\dot{b}_q\widetilde{\dot{b}}_{q-1}}{\dot{c}_q}\right] &= \left[\frac{\{0\oplus b_q\widetilde{h}_q\widetilde{b}_{q-1}, b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}\oplus 0\}}{\{c_{q-1}\oplus 0, 0\oplus c_q\}}\right]\\ &= \left[\frac{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}}{c_{q-1}}\right] + \left[\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right].\end{aligned}$$

For $q = 1$, we have

$$\left[\frac{\dot{b}_1\widetilde{\dot{b}}_0}{\dot{c}_1}\right] = \left[\frac{b_0\widetilde{h}_0}{c_0}\right] + \left[\frac{b_1\widetilde{h}_1\widetilde{b}_0}{c_1}\right],$$

and for $q = 0$

$$\left[\frac{\dot{b}_0\widetilde{\dot{h}}_0}{\dot{c}_0}\right] = \left[\frac{\alpha v}{v}\right] + \left[\frac{b_0\widetilde{h}_0}{c_0}\right].$$

It follows that

$$\begin{aligned}\tau_{\dot{c}_q,\dot{h}_q}(C(\mathscr{C})) &= \left[\frac{\alpha v}{v}\right] + \left[\frac{b_0\widetilde{h}_0}{c_0}\right] + \sum_{q=1}^{m+1}(-1)^q\left(\left[\frac{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}}{c_{q-1}}\right] + \left[\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right]\right)\\ &= \left[\frac{\alpha v}{v}\right]\\ &= \pm\alpha,\end{aligned}$$

since as the complex $\mathscr{C}$ has dimension m, then

$$\left[\frac{b_0\widetilde{h}_0}{c_0}\right] + \sum_{q=1}^{m+1}(-1)^q\left(\left[\frac{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}}{c_{q-1}}\right] + \left[\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right]\right) = 0.$$

Note that if $R = \mathbb{Z}$, then $\alpha = 1$.

Now we can discuss the second algebraic structure proposed for this section and characterize its torsion, as we was done for the cone of a chain complex. Let $i\colon \mathscr{C} \to \mathscr{D}$ be an inclusion of chain complexes, we consider its algebraic mapping cone

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "i"] \arrow[d, "j"] & \mathscr{D} \arrow[d, "\bar{j}"] \\
C(\mathscr{C}) \arrow[r, "\bar{i}"] & \ddot{\mathscr{C}} = C(\mathscr{C}) \sqcup_i \mathscr{D},
\end{tikzcd}$$

where $j, \bar{i}$ and $\bar{j}$ are also inclusions of chain complexes.

By definition, $\ddot{\mathscr{C}}$ is the following complex

$$\begin{aligned}\ddot{C}_q &= C_{q-1} \oplus D_q,\\ \ddot{\partial}_q &= \partial^{\mathscr{C}}_{q-1} \oplus (i_{q-1} - \partial^{\mathscr{D}}_q) = \begin{pmatrix}\partial^{\mathscr{C}}_{q-1} & 0\\ i_{q-1} & -\partial^{\mathscr{D}}_q\end{pmatrix}.\end{aligned}$$

At several moments throughout the text, we will denote $\partial^{\mathscr{C}}_q$ simply by ∂_q, and $i_{q-1}(c_{q-1})$ simply by c_{q-1}, when there is no risk of confusion.

Note that considering the inclusion $\bar{i}\colon C(\mathscr{C}) \to \ddot{\mathscr{C}}$, we see that the modules of the quotient complex $\ddot{\mathscr{C}}/C(\mathscr{C})$ are

$$\ddot{C}_q/\dot{C}_q = (C_{q-1} \oplus D_q)/(C_{q-1} \oplus C_q) \cong D_q/C_q,$$

i.e., the inclusion $\bar{j}\colon \mathscr{D} \to \ddot{\mathscr{C}}$ induces a chain isomorphism on classes

$$\widetilde{j}\colon (\mathscr{D},\mathscr{C}) \to (\ddot{\mathscr{C}}, C(\mathscr{C})).$$

We have the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C} \arrow[r, "i"] & \mathscr{D} \arrow[r, "p"] & \mathscr{D}/\mathscr{C} \arrow[r] & 0,
\end{tikzcd}$$

e

$$\begin{tikzcd}
0 \arrow[r] & C(\mathscr{C}) \arrow[r, "\bar{i}"] & \ddot{\mathscr{C}} \arrow[r, "\bar{p}"] & \ddot{\mathscr{C}}/C(\mathscr{C}) \arrow[r] & 0,
\end{tikzcd}$$

that induces the following commutative diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{C} \arrow[r, "i"] \arrow[d, "j"] & \mathscr{D} \arrow[r, "p"] \arrow[d, "\overline{j}"] & \mathscr{D}/\mathscr{C} \arrow[r] \arrow[d, "\widetilde{j}"] & 0 \\
0 \arrow[r] & C(\mathscr{C}) \arrow[r, "\overline{i}"] \arrow[d, "q"] & \ddot{\mathscr{C}} \arrow[r, "\overline{p}"] \arrow[d, "\overline{q}"] & \ddot{\mathscr{C}}/C(\mathscr{C}) \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & C(\mathscr{C})/\mathscr{C} \arrow[r, "\widetilde{i}"] \arrow[d] & \ddot{\mathscr{C}}/\mathscr{D} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Hence, we have the following commutative diagram of homology exact sequences

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & & \\
0 \arrow[r] & H_{q+1}(C(\mathscr{C})/\mathscr{C}) \arrow[r, "\widetilde{i}_{*,q+1}"] \arrow[d] & H_{q+1}(\ddot{\mathscr{C}}/\mathscr{D}) \arrow[r] \arrow[d, "\ddot{\delta}_{q+1}"] & 0 \arrow[d] & \\
\cdots \arrow[r] & H_q(\mathscr{C}) \arrow[r, "i_{*,q}"] \arrow[d, "j_{*,q}"] & H_q(\mathscr{D}) \arrow[r, "p_{*,q}"] \arrow[d, "\overline{j}_{*,q}"] & H_q(\mathscr{D},\mathscr{C}) \arrow[r] \arrow[d, "\widetilde{j}_{*,q}"] & \cdots \\
\cdots \arrow[r] & H_q(C(\mathscr{C})) \arrow[r, "\overline{i}_{*,q}"] \arrow[d, "q_{*,q}"] & H_q(\ddot{\mathscr{C}}) \arrow[r, "\overline{p}_{*,q}"] \arrow[d, "\overline{q}_{*,q}"] & H_q(\ddot{\mathscr{C}},C(\mathscr{C})) \arrow[r] \arrow[d] & \cdots, \\
0 \arrow[r] & H_q(C(\mathscr{C})/\mathscr{C}) \arrow[r, "\widetilde{i}_{*,q}"] \arrow[d] & H_q(\ddot{\mathscr{C}}/\mathscr{D}) \arrow[r] \arrow[d] & 0 & \\
 & \vdots & \vdots & &
\end{tikzcd}$$

where $\ddot{\delta}_q$ is the connecting homomorphism, and $\overline{p}_{*,q}$ is an isomorphism, for all $q > 1$, since the homology of the cone is trivial.

Moreover, using the diagram above, we have

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(\mathscr{C}) \arrow[r, "i_{*,q}"] \arrow[d, "\varphi_q"] & H_q(\mathscr{D}) \arrow[r, "p_{*,q}"] \arrow[d, "id"] & H_q(\mathscr{D}/\mathscr{C}) \arrow[r, "\delta_q"] \arrow[d, "\psi_q"] & H_{q-1}(\mathscr{C}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{q+1}(\ddot{\mathscr{C}}/\mathscr{D}) \arrow[r, "\ddot{\delta}_{q+1}"] & H_q(\mathscr{D}) \arrow[r, "\overline{j}_{*,q}"] & H_q(\ddot{\mathscr{C}}) \arrow[r, "\overline{q}_{*,q}"] & H_q(\ddot{\mathscr{C}}/\mathscr{D}) \arrow[r] & \cdots,
\end{tikzcd} \tag{4.1}$$

where $\varphi_q = \widetilde{i}_{*,q}\ddot{\delta}_{q+1}^{-1}$ and $\psi_q = \overline{p}_{*,q}^{-1}\widetilde{j}_{*,q}$ are isomorphisms. Let h_q, $h_q^{\mathscr{D}}$ and h_q'' be bases of $H_q(\mathscr{C})$, $H_q(\mathscr{D})$ and $H_q(\mathscr{D}/\mathscr{C})$, respectively. Then, we fix the bases $\ddot{h}_q = \psi(h_q'')$ and $\ddot{h}_{q+1}'' = \varphi_q(h_q)$ for $H_q(\ddot{\mathscr{C}})$ and $H_{q+1}(\ddot{\mathscr{C}}/\mathscr{D})$.

Remark 4.1.1. *We denote by y'_q a set on $H_q(\mathscr{C})$ such that its image is a basis for the image of $i_{*,q}$, and by $\ddot{y}''_{q+1}$ the corresponding set under φ. By y_q a set in $H_q(\mathscr{D})$ such that its image is a basis for $p_{*,q}$. By y''_q a set in $H_q(\mathscr{D}/\mathscr{C})$ such that its image is a basis for the image of δ_q and by $\ddot{y}_q$ its image under ψ_q.*

Let $\mathscr{H}$ be the upper sequence of the Diagram 4.1, we have that

$$\tau_{\overline{h}_q}(\mathscr{H}) = \sum_{q=0}^{m+1} (-1)^q \left(\left[\frac{\delta_{q+1}(y''_{q+1})y'_q}{h_q} \right] - \left[\frac{i_{*,q}(y'_q)y_q}{h^{\mathscr{D}}_q} \right] + \left[\frac{p_{*,q}(y_q)y''_q}{h''_q} \right] \right),$$

where $\overline{h}_{3q} = h''_q$, $\overline{h}_{3q+1} = h^{\mathscr{D}}_q$ and $\overline{h}_{3q+2} = h_q$.

Using the chain isomorphism $\widetilde{j}$, we immediately obtain the induced isomorphism

$$\widetilde{j}_{*,q} \colon H_q(\mathscr{D},\mathscr{C}) \to H_q(\ddot{\mathscr{C}}, C(\mathscr{C})).$$

For $q > 0$, we have the isomorphism

$$\overline{p}^{-1}_{*,q} \widetilde{j}_{*,q} \colon H_q(\mathscr{D},\mathscr{C}) \to H_q(\ddot{\mathscr{C}}).$$

For further use we also calculate the homology. For $q > 0$, if $c = x \oplus y \in \ddot{C}_q = C_{q-1} \oplus D_q$,

$$\ddot{\partial}_q(c) = \begin{pmatrix} \partial^{\mathscr{C}}_{q-1}(x) \\ x - \partial^{\mathscr{D}}_q(y) \end{pmatrix}.$$

Thus the kernel of $\ddot{\partial}_q$ is generated by the elements of the form $\partial^{\mathscr{D}}_q(y_q) \oplus y_q$, with $y_q \in C_q \subset D_q$, and $0 \oplus z^{\mathscr{D}}_q$, where $z^{\mathscr{D}}_q \in \ker \partial^{\mathscr{D}}_q$. The image of $\ddot{\partial}_{q+1}$ is generated by the elements of the form $b^{\mathscr{C}}_{q-1} \oplus 0$, $0 \oplus c_q$, and $0 \oplus b^{\mathscr{D}}_q$, where $c_q \in C_q$, $b^{\mathscr{C}}_{q-1}$ is basis of $B^{\mathscr{C}}_{q-1}$, and $b^{\mathscr{D}}_q$ is basis of $B^{\mathscr{D}}_q$.

Thus

$$\begin{aligned} H_q(\ddot{\mathscr{C}}) = \ker \ddot{\partial}_q / \operatorname{Im} \ddot{\partial}_{q+1} &= \frac{\langle \{\partial^{\mathscr{D}}_q(y) \oplus y; y \in C_q\}, 0 \oplus z^{\mathscr{D}}_q \rangle}{\langle b^{\mathscr{C}}_{q-1} \oplus 0, 0 \oplus c_q, 0 \oplus b^{\mathscr{D}}_q \rangle} \\ &\cong H_q(\mathscr{D},\mathscr{C}). \end{aligned}$$

If d_q is a basis for each D_q of $\mathscr{D}$ and c_q is a basis for each C_q of $\mathscr{C}$. We define a basis d''_q for the pair $(\mathscr{D},\mathscr{C})$ as the elements of d_q that lie in $\mathscr{D} - \mathscr{C}$. We have a short exact sequence of chain complex of free modules with preferred basis c_q, d_q, d''_q

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{C} \arrow[r, "i"] & \mathscr{D} \arrow[r, "p"] & \mathscr{D}/\mathscr{C} \arrow[r] & 0, \end{tikzcd}$$

where $d_q = c_q \widetilde{d''_q}$.

Using the chain isomorphism $\widetilde{j} \colon (\mathscr{D},\mathscr{C}) \to (\ddot{\mathscr{C}}, C(\mathscr{C}))$, we have that $\widetilde{j}_q(d''_q)$ is a chain basis for $(\ddot{\mathscr{C}}, C(\mathscr{C}))$. Considering the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C(\mathscr{C}) \arrow[r, "\overline{i}"] & \ddot{\mathscr{C}} \arrow[r, "\overline{p}"] & (\ddot{\mathscr{C}}, C(\mathscr{C})) \arrow[r] & 0, \end{tikzcd}$$

and $\overline{s} = \overline{p}^{-1}|_{j(\mathscr{D}-\mathscr{C})} \colon (\ddot{\mathscr{C}}, C(\mathscr{C})) \to \ddot{\mathscr{C}}$. Then, a coherent basis for $\ddot{C}_q$ is given by

$$\bar{i}_q(\dot{c}_q)\overline{s}_q(\widetilde{j}_q(d''_q)).$$

We have

$$\ddot{c}_q = \{c_{q-1} \oplus 0, 0 \oplus d_q\} = \{c_{q-1} \oplus 0, 0 \oplus c_q, 0 \oplus d''_q\}.$$

For homology, note that the induced map $\overline{p}_{*,q}$ is an isomorphism, for $q > 0$, while in dimension $q = 0$ the map $\bar{i}_{*,q}$. We denote by h''_q a preferred basis for $H_q(\ddot{\mathscr{C}}, \mathscr{C})$ and by $\ddot{h}_q$ a preferred basis for $H_q(\ddot{\mathscr{C}})$. Since $\widetilde{j}_{*,q}$ is an isomorphism, a preferred basis for $H_q(\ddot{\mathscr{C}}, \mathscr{C})$ is $\widetilde{j}_{*,q}(h''_q)$. Since $\overline{p}_{*,q}$ is an isomorphism for $q > 0$, we have $\ddot{h}_q = \overline{p}_{*,q}^{-1}\widetilde{j}_{*,q}(h''_q)$. Thus, we have

$$\ddot{h}_q = \{0 \oplus h''_q\}.$$

Hence, in order to define the torsion of $\ddot{\mathscr{C}}$, it is sufficient to assume that $H_q(\mathscr{D}, \mathscr{C})$ is free with the basis above.

A preferred basis for the image of $\ddot{\partial}_{q+1} \colon \ddot{C}_{q+1} \to \ddot{C}_q$, denoted by $\ddot{B}_q$, is

$$\ddot{b}_q = \{0 \oplus \widetilde{h}_q, b_{q-1} \oplus \widetilde{b}_{q-1}, 0 \oplus b_q, 0 \oplus b''_q\},$$

where the lift of the basis of $\ddot{B}_{q-1}$ is

$$\begin{aligned}
\widetilde{\ddot{b}}_{q-1} &= \{\widetilde{h}_{q-1}\widetilde{b}_{q-2} \oplus 0, 0 \oplus \widetilde{b}^{\mathscr{D}}_{q-1}\} \\
&= \{\widetilde{h}_{q-1}\widetilde{b}_{q-2} \oplus 0, 0 \oplus \widetilde{b}_{q-1}, 0 \oplus \widetilde{b}''_{q-1}\}.
\end{aligned}$$

Hence a new basis for $\ddot{C}_q$, with $q > 1$, is given by

$$\begin{aligned}
\ddot{b}_q\widetilde{\ddot{h}}_q\widetilde{\ddot{b}}_{q-1} &= \{0 \oplus \widetilde{h}_q, b_{q-1} \oplus \widetilde{b}_{q-1}, 0 \oplus b_q, 0 \oplus b''_q, 0 \oplus \widetilde{h}''_q, \widetilde{h}_{q-1}\widetilde{b}_{q-2} \oplus 0, 0 \oplus \widetilde{b}_{q-1}, 0 \oplus \widetilde{b}''_{q-1}\} \\
&\cong \{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2} \oplus 0, 0 \oplus b_q\widetilde{h}_q\widetilde{b}_{q-1}, 0 \oplus b''_q\widetilde{h}''_q\widetilde{b}''{}_{q-1}\}.
\end{aligned}$$

We may now compute the torsion:

$$\begin{aligned}
\tau_{x_q, y_q}(\ddot{\mathscr{C}}) &= \sum_{q=0}^{m+1} (-1)^q \left[\frac{\{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2} \oplus 0, 0 \oplus b_q\widetilde{h}_q\widetilde{b}_{q-1}, 0 \oplus b''_q\widetilde{h}''_q\widetilde{b}''{}_{q-1}\}}{\{c_{q-1} \oplus 0, 0 \oplus c_q, 0 \oplus d''_q\}} \right] \\
&= \sum_{q=0}^{m+1} (-1)^q \left(\left[\frac{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}}{c_{q-1}} \right] + \left[\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q} \right] \right) + \sum_{q=0}^{m+1} (-1)^q \left[\frac{b''_q\widetilde{h}''_q\widetilde{b}''{}_{q-1}}{d''_q} \right] \\
&= \sum_{q=0}^{m+1} (-1)^q \left[\frac{b''_q\widetilde{h}''_q\widetilde{b}''{}_{q-1}}{d''_q} \right] \\
&= \tau_{d''_q, h''_q}(\mathscr{D}/\mathscr{C}),
\end{aligned}$$

where $x_q = \bar{i}_q(\dot{c}_q)\overline{s}_q(\widetilde{j}_q(d''_q))$ and $y_q = \bar{i}_{*,q}(\dot{h}_q)\overline{p}_{*,q}^{-1}(\widetilde{j}_{*,q}(h''_q))$.

Remark 4.1.2. *Making the suitable identifications of the preferred homology bases, we have*

$$\tau_{\ddot{c}_q, \ddot{h}_q}(\ddot{\mathscr{C}}) = \tau_{d''_q, h''_q}(\mathscr{D}/\mathscr{C}).$$

4.2 Torsion for intersection chain complexes

Definition 4.2.1. *A perversity is a function* $\mathfrak{p}\colon \{2,\dots,m\} \to \mathbb{Z}$ *such that* $\mathfrak{p}(2)=0$ *and* $\mathfrak{p}(i+1)=\mathfrak{p}(i)$ *or* $\mathfrak{p}(i)+1$.

Let us look at some examples.

Example 4.2.2. *We have that:*

1. *the function* $\mathfrak{z}\colon \{2,\dots,m\} \to \mathbb{Z}$*, where* $\mathfrak{z}(i)=0$*, is called the zero perversity;*
2. *the function* $\mathfrak{t}\colon \{2,\dots,m\} \to \mathbb{Z}$*, given by* $\mathfrak{t}(i)=i-2$ *is called the top perversity;*
3. *the function* $\mathfrak{m}\colon \{2,\dots,m\} \to \mathbb{Z}$*, defined by* $\mathfrak{m}(i)=\lfloor i/2 \rfloor -1$*, where* $\lfloor \cdot \rfloor$ *denote the floor function, is called the lower middle perversity;*
4. *given a perversity* $\mathfrak{p}$*, the complementary perversity* $\mathfrak{p}^c$ *is defined as* $\mathfrak{p}^c(i)=\mathfrak{t}(i)-\mathfrak{p}(i)=i-\mathfrak{p}(i)-2$.

Let $\mathscr{C}$ be a chain complex of free modules

$$C_m \longrightarrow C_{m-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0,$$

and let $\mathscr{C}^{(k)}$ be the truncated chain complex

$$C_k \longrightarrow C_{k-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0,$$

where $0 \leq k \leq m$.

Let $\mathfrak{p}$ be a perversity, and define the constant $\mathfrak{a}=\mathfrak{a}(m+1)=m+1-\mathfrak{p}(m+1)$. Suppose that the chain modules of $\mathscr{D}$ and $\mathscr{C}$ are defined as $H_q(\mathscr{D}^{(q)},\mathscr{D}^{(q-1)})$ and $H_q(\mathscr{C}^{(q)},\mathscr{C}^{(q-1)})$, respectively. Considering the mapping cone $\ddot{\mathscr{C}}=C(\mathscr{C})\sqcup_i \mathscr{D}$, for each q, we have

$$\begin{aligned}
\ddot{C}_q &= C_{q-1}\oplus D_q \\
&= H_{q-1}(\mathscr{C}^{(q-1)},\mathscr{C}^{(q-2)})\oplus H_q(\mathscr{D}^{(q)},\mathscr{D}^{(q-1)}) \\
&= H_q(\mathscr{C}^{(q-1)}\sqcup_i \mathscr{D}^{(q)},\mathscr{C}^{(q-2)}\sqcup_i \mathscr{D}^{(q-1)}) \\
&= H_q(\ddot{\mathscr{C}}^{(q)},\ddot{\mathscr{C}}^{(q-1)}).
\end{aligned}$$

We define the chain complex $\mathscr{E}_q$ so that

$$\mathscr{E}_q=\begin{cases} \mathscr{D}^{(q)}, & \text{for } q<\mathfrak{a}, \\ \ddot{\mathscr{C}}^{(q)}, & \text{for } q\geq \mathfrak{a}, \end{cases}$$

with boundary homomorphism $\partial_q^{\mathscr{E}_q}$ defined by the compositions of the homology exact sequences associated to the chain pairs $(\mathscr{E}_q,\mathscr{E}_{q-1})$:

$$\begin{tikzcd}
 & & & \vdots \arrow[d] & \\
 & & & H_{q-1}(\mathscr{E}_{q-2}) \arrow[d] & \\
\cdots \arrow[r] & H_q(\mathscr{E}_{q-1}) \arrow[r] & H_q(\mathscr{E}_q,\mathscr{E}_{q-1}) \arrow[r] \arrow[dr, "\partial_q^{\mathscr{E}_q}"'] & H_{q-1}(\mathscr{E}_{q-1}) \arrow[r] \arrow[d] & \cdots . \\
 & & & H_{q-1}(\mathscr{E}_{q-1},\mathscr{E}_{q-2}) \arrow[d] & \\
 & & & \vdots &
\end{tikzcd}$$

We define the *intersection chain complex* $I^{\mathfrak{p}}\ddot{\mathscr{C}}$ of perversity $\mathfrak{p}$ of $\ddot{\mathscr{C}}$ with modules given by

$$I^{\mathfrak{p}}\ddot{C}_q = H_q(\mathscr{E}_q,\mathscr{E}_{q-1}).$$

We have that

$$I^{\mathfrak{p}}\ddot{C}_q = \begin{cases} D_q, & q<\mathfrak{a}, \\ H_{\mathfrak{a}}(\ddot{\mathscr{C}}^{(\mathfrak{a})}),\mathscr{D}^{(\mathfrak{a}-1)}), & q=\mathfrak{a}, \\ \ddot{C}_q, & q>\mathfrak{a}, \end{cases}$$

with boundary

$$I^{\mathfrak{p}}\ddot{\partial}_q = \begin{cases} j'_{q-1}\circ\partial'_q = \partial_q^{\mathscr{D}}, & q\leq\mathfrak{a}-1, \\ j'_{\mathfrak{a}-1}\circ\partial''_{\mathfrak{a}}, & q=\mathfrak{a}, \\ j''_{\mathfrak{a}}\circ\partial'''_{\mathfrak{a}+1}, & q=\mathfrak{a}+1, \\ j'''_{q-1}\circ\partial'''_q = \ddot{\partial}_q, & q\geq\mathfrak{a}+2, \end{cases}$$

where the homomorphisms j'''_q,∂'''_q, j''_q,∂''_q and j'_q,∂'_q comes from the exact sequences

$$\cdots \longrightarrow H_q(\mathscr{D}^{(q)}) \xrightarrow{j'_q} H_q(\mathscr{D}^{(q)},\mathscr{D}^{(q-1)}) \xrightarrow{\partial'_q} H_{q-1}(\mathscr{D}^{(q-1)}) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H_q(\ddot{\mathscr{C}}^{(q)}) \xrightarrow{j''_q} H_q(\ddot{\mathscr{C}}^{(q)},\mathscr{D}^{(q-1)}) \xrightarrow{\partial''_q} H_{q-1}(\mathscr{D}^{(q-1)}) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H_q(\ddot{\mathscr{C}}^{(q)}) \xrightarrow{j'''_q} H_q(\ddot{\mathscr{C}}^{(q)},\ddot{\mathscr{C}}^{(q-1)}) \xrightarrow{\partial'''_q} H_{q-1}(\ddot{\mathscr{C}}^{(q-1)}) \longrightarrow \cdots.$$

Note that for the dimensions $\mathfrak{a}+1$ and $\mathfrak{a}$, we have the following sequence

$$\begin{tikzcd}
H_{\mathfrak{a}+1}(\ddot{\mathscr{C}}^{(\mathfrak{a}+1)},\ddot{\mathscr{C}}^{(\mathfrak{a})}) \arrow[d, "\partial'''_{\mathfrak{a}+1}"] & & \\
H_{\mathfrak{a}}(\ddot{\mathscr{C}}^{(\mathfrak{a})}) \arrow[r, "j''_{\mathfrak{a}}"] & H_{\mathfrak{a}}(\ddot{\mathscr{C}}^{(\mathfrak{a})},\mathscr{D}^{(\mathfrak{a}-1)}) \arrow[r, "\partial''_{\mathfrak{a}}"] & H_{\mathfrak{a}-1}(\mathscr{D}^{(\mathfrak{a}-1)}) \arrow[d, "j'_{\mathfrak{a}-1}"] \\
 & & H_{\mathfrak{a}-1}(\mathscr{D}^{(\mathfrak{a}-1)},\mathscr{D}^{(\mathfrak{a}-2)}).
\end{tikzcd}$$

We know that $(j'_{\mathfrak{a}-1} \circ \partial''_{\mathfrak{a}}) \circ (j''_{\mathfrak{a}} \circ \partial'''_{\mathfrak{a}+1})$ is zero, since by the exactness of the sequences we have $\operatorname{Im} j''_{\mathfrak{a}} = \ker \partial''_{\mathfrak{a}}$.

The following result characterizes the complex in dimension a.

Lemma 4.2.3. *We have that*

$$I^{\mathfrak{p}}\ddot{C}_q = \begin{cases} D_q, & q < \mathfrak{a}, \\ Z_{\mathfrak{a}-1} \oplus D_{\mathfrak{a}}, & q = \mathfrak{a}, \\ \ddot{C}_q, & q > \mathfrak{a}, \end{cases}$$

with boundary

$$I^{\mathfrak{p}}\ddot{\partial}_q = \begin{cases} \partial^{\mathscr{D}}_q, & q \leq \mathfrak{a}-1, \\ \ddot{\partial}_q, & q \geq \mathfrak{a}. \end{cases}$$

Proof. We need only to consider the case $q = \mathfrak{a}$. Since

$$I^{\mathfrak{p}}\ddot{C}_{\mathfrak{a}} = H_{\mathfrak{a}}(\ddot{\mathscr{C}}^{(\mathfrak{a})}, \mathscr{D}^{(\mathfrak{a}-1)}),$$

the relevant diagram is

$$\begin{tikzcd}
 & D_{\mathfrak{a}-1} \arrow[r] \arrow[d] & \cdots \\
\ddot{C}_{\mathfrak{a}} \arrow[r] & \ddot{C}_{\mathfrak{a}-1} \arrow[r] \arrow[d] & \cdots, \\
 & \ddot{C}_{\mathfrak{a}-1}/\bar{j}_{\mathfrak{a}-1}(D_{\mathfrak{a}-1}) &
\end{tikzcd}$$

where the vertical sequence is the exact sequence associated to the inclusion $\bar{j}$.

Since $\ddot{C}_{\mathfrak{a}-1} = C_{\mathfrak{a}-2} \oplus D_{\mathfrak{a}-1}$, and the inclusion $\bar{j}_{\mathfrak{a}-1} \colon D_{\mathfrak{a}-1} \to \ddot{C}_{\mathfrak{a}}$ is $\bar{j}_{\mathfrak{a}-1}(x) = 0 \oplus x$, the associated exact sequence reads

$$\begin{tikzcd}
0 \arrow[r] & D_{\mathfrak{a}-1} \arrow[r, "\bar{j}_{\mathfrak{a}-1}"] & \ddot{C}_{\mathfrak{a}-1} = C_{\mathfrak{a}-2} \oplus D_{\mathfrak{a}-1} \arrow[r] & C_{\mathfrak{a}-2} \arrow[r] & 0 .
\end{tikzcd}$$

Whence, we need to compute the kernel of the composition $\psi \circ \ddot{\partial}_{\mathfrak{a}}$, where ψ is the projection:

$$\ker(\psi \circ \ddot{\partial}_{\mathfrak{a}}) = \{(x,y) \in \ddot{C}_{\mathfrak{a}} = C_{\mathfrak{a}} \oplus D_{\mathfrak{a}}; \psi(\ddot{\partial}_{\mathfrak{a}}(x,y)) = \partial_{\mathfrak{a}-1}(x) = 0\} = Z_{\mathfrak{a}-1} \oplus D_{\mathfrak{a}},$$

where $Z_{\mathfrak{a}-1}$ is the kernel of the boundary $\partial_{\mathfrak{a}-1} \colon C_{\mathfrak{a}-1} \to C_{\mathfrak{a}-2}$. □

Lemma 4.2.4. *We have that*

$$I^{\mathfrak{p}}H_q(\ddot{\mathscr{C}}) := H_q(I^{\mathfrak{p}}\ddot{\mathscr{C}}) = \begin{cases} H_q(\mathscr{D}), & 0 \leq q \leq \mathfrak{a}-2, \\ p_{*,\mathfrak{a}-1}(H_{\mathfrak{a}-1}(\mathscr{D})) \subset H_{\mathfrak{a}-1}(\mathscr{D}/\mathscr{C}), & q = \mathfrak{a}-1, \\ H_q(\mathscr{D}, \mathscr{C}), & \mathfrak{a} \leq q \leq m+1. \end{cases}$$

Proof. By Lemma 4.2.3, if $q \leq \mathfrak{a}-2$ then $I^{\mathfrak{p}}\ddot{C}_q = D_q$ and $I^{\mathfrak{p}}\ddot{\partial}_q = \partial^{\mathscr{D}}_q$, thus $H_q(I^{\mathfrak{p}}\ddot{\mathscr{C}}) = H_q(\mathscr{D})$. If $q \geq \mathfrak{a}+1$, then $I^{\mathfrak{p}}\ddot{C}_q = \ddot{C}_q$ and $I^{\mathfrak{p}}\ddot{\partial}_q = \ddot{\partial}_q$, hence $H_q(I^{\mathfrak{p}}\ddot{\mathscr{C}}) = H_q(\ddot{\mathscr{C}}) \cong H_q(\mathscr{D}, \mathscr{C})$, since $q > 0$.

For other cases, the relevant part of the complex is

$$\begin{tikzcd}
\cdots \arrow[r, "\bar{j}_{\mathfrak{a}-1}"] & I^{\mathfrak{p}}\ddot{C}_{\mathfrak{a}} = Z_{\mathfrak{a}-1} \oplus D_{\mathfrak{a}} \arrow[r] & I^{\mathfrak{p}}\ddot{C}_{\mathfrak{a}-1} = D_{\mathfrak{a}-1} \arrow[r] & \cdots .
\end{tikzcd}$$

When $q = \mathfrak{a}-1$, we have that $\ker I^{\mathfrak{p}}\ddot{\partial}_{\mathfrak{a}-1} = \ker \partial^{\mathscr{D}}_{\mathfrak{a}-1}$, while $\operatorname{Im} I^{\mathfrak{p}}\ddot{\partial}_{\mathfrak{a}} = Z_{\mathfrak{a}-1} + \operatorname{Im}\partial^{\mathscr{D}}_{\mathfrak{a}}$, and hence $H_{\mathfrak{a}-1}(I^{\mathfrak{p}}\mathscr{C}) = \operatorname{Im} p_{*,\mathfrak{a}-1} \subset H_{\mathfrak{a}-1}(\mathscr{D},\mathscr{C})$.

If $q = \mathfrak{a}$, then for $x \oplus y \in Z_{\mathfrak{a}-1} \oplus D_{\mathfrak{a}}$, we have

$$I^{\mathfrak{p}}\ddot{\partial}_{\mathfrak{a}}(x \oplus y) = \partial^{\mathscr{C}}_{\mathfrak{a}-1}(x) \oplus x - \partial^{\mathscr{D}}_{\mathfrak{a}}(y) = 0 \oplus x - \partial^{\mathscr{D}}_{\mathfrak{a}}(y).$$

and this is zero, if $\partial^{\mathscr{D}}_{\mathfrak{a}}(y) = x$. Since for $u \oplus v \in \ddot{C}_q$, we have

$$I^{\mathfrak{p}}\ddot{\partial}_{\mathfrak{a}+1}(u \oplus v) = \partial_{\mathfrak{a}}(u) \oplus u - \partial^{\mathscr{D}}_{\mathfrak{a}+1}(v).$$

Since $\partial_{\mathfrak{a}}(u) \in \ker \partial_{\mathfrak{a}-1}$, then $I^{\mathfrak{p}}\partial_{\mathfrak{a}+1} = Z_{\mathfrak{a}-1} + \partial^{\mathscr{D}}_{\mathfrak{a}+1}(D_{\mathfrak{a}+1})$. It follows that $I^{\mathfrak{p}}H_{\mathfrak{a}}(\ddot{\mathscr{C}}) = H_{\mathfrak{a}-1}(\mathscr{D},\mathscr{C})$.

□

Remark 4.2.5. *Suppose that each D_q of $\mathscr{D}$ has a preferred basis d_q and that C_q of $\mathscr{C}$ has a preferred basis c_q. Let d''_q be a consistent preferred basis for $(\mathscr{D},\mathscr{C})$, in the sense that $d_q = c_q\widetilde{d''_q}$. Then $I^{\mathfrak{p}}\ddot{C}_q$ of $I^{\mathfrak{p}}\ddot{\mathscr{C}}$ is free and a preferred basis is given by*

$$I^{\mathfrak{p}}\ddot{c}_q = \begin{cases} d_q, & q \leq \mathfrak{a}-1, \\ \{z_{\mathfrak{a}-1} \oplus 0, 0 \oplus c_{\mathfrak{a}}\}, & q = \mathfrak{a}, \\ \{c_{q-1} \oplus 0, 0 \oplus d_q\}, & q \geq \mathfrak{a}+1. \end{cases}$$

Note that a submodule of a free module over a principal ideal damain, is also free. Thus the submodule $Z_{\mathfrak{a}-1}$ of $C_{\mathfrak{a}-1}$ is free. Moreover, assuming that $H_q(\mathscr{D})$ is free with preferred basis $h^{\mathscr{D}}_q$, and $H_q(\mathscr{D},\mathscr{C})$ is free with preferred basis h''_q. Then, we have that $H_q(I^{\mathfrak{p}}\ddot{\mathscr{C}})$ is also free and has preferred basis $I^{\mathfrak{p}}\ddot{h}_q$ defined by

$$I^{\mathfrak{p}}\ddot{h}_q = \begin{cases} h^{\mathscr{D}}_q, & q \leq \mathfrak{a}-2, \\ p_{*,\mathfrak{a}-1}(h^{\mathscr{D}}_{\mathfrak{a}-1}), & q = \mathfrak{a}-1, \\ h''_q, & q \geq \mathfrak{a}. \end{cases}$$

Remark 4.2.6. *In particular, if $\mathscr{D} = \mathscr{C}$, we have defined the intersection chain complex $I^{\mathfrak{p}}C(\mathscr{C})$ with perversity $\mathfrak{p}$ of $C(\mathscr{C})$. Thus, it follows from Lemma 4.2.3 that*

$$I^{\mathfrak{p}}\dot{C}_q = \begin{cases} C_q, & q < \mathfrak{a}, \\ Z_{\mathfrak{a}-1} \oplus C_{\mathfrak{a}}, & q = \mathfrak{a}, \\ \dot{C}_q, & q > \mathfrak{a}, \end{cases}$$

with boundary

$$I^{\mathfrak{p}}\dot{\partial}_q = \begin{cases} \partial_q, & q \leq \mathfrak{a}-1, \\ \dot{\partial}_q, & q \geq \mathfrak{a}+2. \end{cases}$$

Furthermore, by Lemma 4.2.4, we have

$$I^{\mathfrak{p}}H_q(C(\mathscr{C})) := H_q(I^{\mathfrak{p}}C(\mathscr{C})) = \begin{cases} H_q(\mathscr{C}), & 0 \leq q \leq \mathfrak{a}-2, \\ 0, & \mathfrak{a}-1 \leq q \leq m+1. \end{cases}$$

Similarly to Remark 4.2.5, each $I^{\mathfrak{p}}\dot{C}_q$ is free with basis $I^{\mathfrak{p}}\dot{c}_q$ and the intersection homology $I^{\mathfrak{p}}H_q(C(\mathscr{C}))$ is free and we denote by $I^{\mathfrak{p}}\dot{h}_q$ a preferred basis. We use the notation $I^{\mathfrak{p}}\dot{b}_q$ to denote a preferred basis of the image of $I^{\mathfrak{p}}\partial_{q+1} \colon I^{\mathfrak{p}}\dot{C}_{q+1} \to I^{\mathfrak{p}}\dot{C}_q$.

For $q < \mathfrak{a}-1$, a new basis for $I^{\mathfrak{p}}\dot{C}_q$ is given by

$$I^{\mathfrak{p}}\dot{b}_q\widetilde{I^{\mathfrak{p}}\dot{h}_q}\widetilde{I^{\mathfrak{p}}\dot{b}}_{q-1} = b_q\widetilde{h}_q\widetilde{b}_{q-1}.$$

For $q > \mathfrak{a}+1$, a new basis for $I^{\mathfrak{p}}\dot{C}_q$ is given by

$$I^{\mathfrak{p}}\dot{b}_q\widetilde{I^{\mathfrak{p}}\dot{h}_q}\widetilde{I^{\mathfrak{p}}\dot{b}}_{q-1} = \dot{b}_q\widetilde{\dot{b}}_{q-1} = \{0\oplus b_q\widetilde{h}_q\widetilde{b}_{q-1}, b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}\oplus 0\}.$$

To deal with the remaining cases the relevant part of the complex is

$$\cdots \longrightarrow I^{\mathfrak{p}}\dot{C}_{\mathfrak{a}} = Z_{\mathfrak{a}-1}\oplus C_{\mathfrak{a}} \longrightarrow I^{\mathfrak{p}}\dot{C}_{\mathfrak{a}-1} = C_{\mathfrak{a}-1} \longrightarrow \cdots.$$

For $q = \mathfrak{a}+1$, it is clear that we have the same as the cases when $q > \mathfrak{a}+1$, since $\mathrm{Im}\, I^{\mathfrak{p}}\dot{\partial}_{\mathfrak{a}+1} = \mathrm{Im}\,\dot{\partial}_{\mathfrak{a}+1} \subset \ker\dot{\partial}_{\mathfrak{a}} = \ker I^{\mathfrak{p}}\dot{\partial}_{\mathfrak{a}}$.

For $q = \mathfrak{a}$, by Lemma 4.2.4 and Remark 4.2.6, the intersection homology is trivial. Since for each $z\oplus x \in Z_{\mathfrak{a}-1}\oplus C_{\mathfrak{a}}$, we have $\dot{\partial}_{\mathfrak{a}}(z\oplus x) = 0\oplus z - \partial_{\mathfrak{a}}(x)$. Thus we can choose $\widetilde{I^{\mathfrak{p}}\dot{b}}_{\mathfrak{a}-1} = \{\widetilde{h}_{\mathfrak{a}-1}\oplus 0, 0\oplus \widetilde{b}_{\mathfrak{a}-1}\}$ and $I^{\mathfrak{p}}\dot{b}_{\mathfrak{a}} = b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}$. Hence, the new basis of $I^{\mathfrak{p}}\dot{C}_{\mathfrak{a}}$ is

$$I^{\mathfrak{p}}\dot{b}_{\mathfrak{a}}I^{\mathfrak{p}}\dot{b}_{\mathfrak{a}-1} = \{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}\oplus 0, 0\oplus b_{\mathfrak{a}}\widetilde{h}_{\mathfrak{a}}\widetilde{b}_{\mathfrak{a}-1}\}.$$

For $q = \mathfrak{a}-1$, we have that $I^{\mathfrak{p}}\dot{b}_{\mathfrak{a}-2} = b_{\mathfrak{a}-2}$ and $\widetilde{I^{\mathfrak{p}}\dot{b}}_{\mathfrak{a}-2} = \widetilde{b}_{\mathfrak{a}-2}$. Since $\mathrm{Im}\, I^{\mathfrak{p}}\dot{\partial}_{\mathfrak{a}} = Z_{\mathfrak{a}-1}$ then $I^{\mathfrak{p}}H_{\mathfrak{a}-1}(C(\mathscr{C})) = 0$. Thus the new basis is

$$I^{\mathfrak{p}}\dot{b}_{\mathfrak{a}-1}\widetilde{I^{\mathfrak{p}}\dot{b}}_{\mathfrak{a}-2} = b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}\widetilde{b}_{\mathfrak{a}-2}.$$

Therefore, the torsion of the intersection chain complex of the cone with perversity $\mathfrak{p}$ is

$$\begin{aligned}
\tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}h_q}(I^{\mathfrak{p}}C(\mathscr{C})) &= \sum_{q=0}^{m+1}(-1)^q\left[\frac{I^{\mathfrak{p}}\dot{b}_q\widetilde{I^{\mathfrak{p}}\dot{h}_q}\widetilde{I^{\mathfrak{p}}\dot{b}}_{q-1}}{I^{\mathfrak{p}}\dot{c}_q}\right] \\
&= \sum_{q=0}^{\mathfrak{a}-2}\left[\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right] + (-1)^{\mathfrak{a}-1}\left[\frac{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}\widetilde{b}_{\mathfrak{a}-2}}{c_{\mathfrak{a}-1}}\right] \\
&\quad + (-1)^{\mathfrak{a}}\left(\left[\frac{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}}{z_{\mathfrak{a}-1}}\right] + \left[\frac{b_{\mathfrak{a}}\widetilde{h}_{\mathfrak{a}}\widetilde{b}_{\mathfrak{a}-1}}{c_{\mathfrak{a}}}\right]\right) \\
&\quad + \sum_{q=\mathfrak{a}+1}^{m+1}(-1)^q\left(\left[\frac{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}}{c_{q-1}}\right] + \left[\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right]\right).
\end{aligned}$$

Notice that

$$(-1)^{\mathfrak{a}}\left[\frac{b_{\mathfrak{a}}\widetilde{h}_{\mathfrak{a}}\widetilde{b}_{\mathfrak{a}-1}}{c_{\mathfrak{a}}}\right] + \sum_{q=\mathfrak{a}+1}^{m+1}(-1)^q\left(\left[\frac{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}}{c_{q-1}}\right] + \left[\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right]\right) = 0,$$

and

$$
\begin{aligned}
\left[\frac{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}\widetilde{b}_{\mathfrak{a}-2}}{c_{\mathfrak{a}-1}}\right] &= \left[\frac{z_{\mathfrak{a}-1}\widetilde{b}_{\mathfrak{a}-2}}{c_{\mathfrak{a}-1}}\right] + \left[\frac{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}\widetilde{b}_{\mathfrak{a}-2}}{z_{\mathfrak{a}-1}\widetilde{b}_{\mathfrak{a}-2}}\right] \\
&= \left[\frac{z_{\mathfrak{a}-1}\widetilde{b}_{\mathfrak{a}-2}}{c_{\mathfrak{a}-1}}\right] + \left[\frac{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}}{z_{\mathfrak{a}-1}}\right] \\
\Rightarrow \left[\frac{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}}{z_{\mathfrak{a}-1}}\right] &= \left[\frac{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}\widetilde{b}_{\mathfrak{a}-2}}{c_{\mathfrak{a}-1}}\right] - \left[\frac{z_{\mathfrak{a}-1}\widetilde{b}_{\mathfrak{a}-2}}{c_{\mathfrak{a}-1}}\right].
\end{aligned}
$$

Thus, we have proved the following result.

Proposition 4.2.7. *Let $I^{\mathfrak{p}}\dot{h}_q$ be a preferred homology basis for $H_q(I^{\mathfrak{p}}C(\mathscr{C}))$, then the torsion of the intersection chain complex of the cone of $\mathscr{C}$ with perversity $\mathfrak{p}$ is*

$$
\begin{aligned}
\tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}q}(I^{\mathfrak{p}}C(\mathscr{C})) &= \sum_{q=0}^{\mathfrak{a}-1}\left[\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right] + (-1)^{\mathfrak{a}}\left[\frac{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}}{z_{\mathfrak{a}-1}}\right] \\
&= \sum_{q=0}^{\mathfrak{a}-2}\left[\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right] + (-1)^{\mathfrak{a}-1}\left[\frac{z_{\mathfrak{a}-1}\widetilde{b}_{\mathfrak{a}-2}}{c_{\mathfrak{a}-1}}\right].
\end{aligned}
$$

Now we will calculate the intersection torsion for the mapping cone $\ddot{\mathscr{C}}$. Remember that we define the bases $I^{\mathfrak{p}}\ddot{c}_q$ of $I^{\mathfrak{p}}\ddot{C}_q$ and $I^{\mathfrak{p}}\ddot{h}_q$ of $I^{\mathfrak{p}}H_q(\ddot{\mathscr{C}})$ in Remark 4.2.5. We use the notation $I^{\mathfrak{p}}\ddot{b}_q$ to denote a preferred basis of the image of $I^{\mathfrak{p}}\ddot{\partial}_{q+1}\colon I^{\mathfrak{p}}\ddot{C}_{q+1} \to I^{\mathfrak{p}}\ddot{C}_q$.

For $q < \mathfrak{a}-1$, a new basis for $I^{\mathfrak{p}}\ddot{C}_q$ is given by

$$
I^{\mathfrak{p}}\ddot{b}_q\widetilde{I^{\mathfrak{p}}\ddot{h}}_q\widetilde{I^{\mathfrak{p}}\ddot{b}}_{q-1} = b_q^{\mathscr{D}}\widetilde{h}_q^{\mathscr{D}}\widetilde{b}_{q-1}^{\mathscr{D}}.
$$

For $q > \mathfrak{a}+1$, we have that

$$
\begin{aligned}
I^{\mathfrak{p}}\ddot{b}_q\widetilde{I^{\mathfrak{p}}\ddot{h}}_q\widetilde{I^{\mathfrak{p}}\ddot{b}}_{q-1} &= \ddot{b}_q\widetilde{\ddot{h}}_q\widetilde{\ddot{b}}_{q-1} \\
&= \{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}\oplus 0, 0\oplus b_q\widetilde{h}_q\widetilde{b}_{q-1}, 0\oplus b''_q\widetilde{h}''_q\widetilde{b}''{}_{q-1}\}.
\end{aligned}
$$

To deal with the remaining cases the relevant part of the complex is

$$
\cdots \longrightarrow I^{\mathfrak{p}}\ddot{C}_{\mathfrak{a}} = Z_{\mathfrak{a}-1}\oplus D_{\mathfrak{a}} \longrightarrow I^{\mathfrak{p}}\ddot{C}_{\mathfrak{a}-1} = D_{\mathfrak{a}-1} \longrightarrow \cdots.
$$

For $q = \mathfrak{a}+1$, it is clear that we have exactly the same as the cases when $q > \mathfrak{a}+1$, since $\operatorname{Im} I^{\mathfrak{p}}\ddot{\partial}_{\mathfrak{a}+1} = \operatorname{Im}\ddot{\partial}_{\mathfrak{a}+1} \subset \ker\ddot{\partial}_{\mathfrak{a}} = \ker I^{\mathfrak{p}}\ddot{\partial}_{\mathfrak{a}}$.

For $q = \mathfrak{a}$, the intersection homology is the homology of the quotient complex. We can choose

$$
\begin{aligned}
I^{\mathfrak{p}}\ddot{b}_{\mathfrak{a}} &= \{0\oplus\widetilde{h}_{\mathfrak{a}}, b_{\mathfrak{a}-1}\oplus\widetilde{b}_{\mathfrak{a}-1}, 0\oplus b_{\mathfrak{a}}^{\mathscr{D}}\} \\
&= \{0\oplus\widetilde{h}_{\mathfrak{a}}, b_{\mathfrak{a}-1}\oplus\widetilde{b}_{\mathfrak{a}-1}, 0\oplus b_{\mathfrak{a}}, 0\oplus b''_{\mathfrak{a}}\},
\end{aligned}
$$

and

$$\begin{aligned}\widetilde{I^{\mathfrak{p}}\ddot{b}}_{\mathfrak{a}-1} &= \{\widetilde{h}_{\mathfrak{a}-1}\oplus 0, 0\oplus \widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-1}\}\\ &= \{\widetilde{h}_{\mathfrak{a}-1}\oplus 0, 0\oplus \widetilde{b}_{\mathfrak{a}}, 0\oplus \widetilde{b}''_{\mathfrak{a}}\}.\end{aligned}$$

Thus we have a new basis for $I^{\mathfrak{p}}\ddot{C}_{\mathfrak{a}}$ given by

$$\begin{aligned}I^{\mathfrak{p}}\ddot{b}_{\mathfrak{a}}\widetilde{I^{\mathfrak{p}}\ddot{h}}_{\mathfrak{a}}\widetilde{I^{\mathfrak{p}}\ddot{b}}_{\mathfrak{a}-1} &= \{0\oplus \widetilde{h}_{\mathfrak{a}}, b_{\mathfrak{a}-1}\oplus \widetilde{b}_{\mathfrak{a}-1}, 0\oplus b_{\mathfrak{a}}, 0\oplus b''_{\mathfrak{a}}, 0\oplus \widetilde{h}''_{\mathfrak{a}}, 0\oplus \widetilde{b}_{\mathfrak{a}-1}, \widetilde{h}_{\mathfrak{a}-1}\oplus 0, 0\oplus \widetilde{b}''_{\mathfrak{a}-1}\}\\ &= \{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}\oplus 0, 0\oplus b_{\mathfrak{a}}\widetilde{h}_{\mathfrak{a}}\widetilde{b}_{\mathfrak{a}-1}, 0\oplus b''_{\mathfrak{a}}\widetilde{h}''_{\mathfrak{a}}\widetilde{b}''_{\mathfrak{a}-1}\}.\end{aligned}$$

For $q=\mathfrak{a}-1$, considering $h_{\mathfrak{a}-1}=\delta_{\mathfrak{a}}(y''_{\mathfrak{a}})y'_{\mathfrak{a}-1}$, by definition of the connecting homomorphism $\delta_{\mathfrak{a}}$, we have that $\widetilde{\delta_{\mathfrak{a}}(y''_{\mathfrak{a}})}\in\langle b''_{\mathfrak{a}}\rangle$. Thus, we take $\widetilde{I^{\mathfrak{p}}\ddot{b}}_{\mathfrak{a}-1}=\{y'_{\mathfrak{a}-1}, \widetilde{b}_{\mathfrak{a}}, \widetilde{b}''_{\mathfrak{a}}\}$, and hence

$$I^{\mathfrak{p}}\ddot{b}_{\mathfrak{a}-1}=\{i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1}), b_{\mathfrak{a}}, b''_{\mathfrak{a}}\}.$$

Here, we are careful to denote $i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1})$, so that there is no confusion.

The intersection homology in dimension $\mathfrak{a}-1$ is $p_{*,\mathfrak{a}-1}(H_{\mathfrak{a}-1}(\mathscr{D}))\subset H_{\mathfrak{a}-1}(\mathscr{D}/\mathscr{C})$. We choose the subset $y_{\mathfrak{a}-1}$ of $h^{\mathscr{D}}_{\mathfrak{a}-1}$ as the lift of $p_{*,\mathfrak{a}-1}(h^{\mathscr{D}}_{\mathfrak{a}-1})$. Moreover we have that $I^{\mathfrak{p}}\ddot{b}_{\mathfrak{a}-2}=b^{\mathscr{D}}_{\mathfrak{a}-2}$. Then the new basis of $I^{\mathfrak{p}}\ddot{C}_{\mathfrak{a}-1}$ is

$$\begin{aligned}I^{\mathfrak{p}}\ddot{b}_{\mathfrak{a}-1}\widetilde{I^{\mathfrak{p}}\ddot{h}}_{\mathfrak{a}-1}\widetilde{I^{\mathfrak{p}}\ddot{b}}_{\mathfrak{a}-2} &= \{b_{\mathfrak{a}-1}, i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1}), b''_{\mathfrak{a}-1}, y_{\mathfrak{a}-1}, \widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-2}\}\\ &= \{b_{\mathfrak{a}-1}b''_{\mathfrak{a}-1}, i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1}), y_{\mathfrak{a}-1}, \widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-2}\}\\ &= b^{\mathscr{D}}_{\mathfrak{a}-1}i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1})y_{\mathfrak{a}-1}\widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-2}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\tau_{I^{\mathfrak{p}}\ddot{c}_q, I^{\mathfrak{p}}\ddot{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{C}}) &= \sum_{q=0}^{m+1}(-1)^q\left[\frac{I^{\mathfrak{p}}\ddot{b}_q\widetilde{I^{\mathfrak{p}}\ddot{h}}_q\widetilde{I^{\mathfrak{p}}\ddot{b}}_{q-1}}{I^{\mathfrak{p}}\ddot{c}_q}\right]\\ &= \sum_{q=0}^{\mathfrak{a}-2}(-1)^q\left[\frac{b^{\mathscr{D}}_q\widetilde{h}^{\mathscr{D}}_q\widetilde{b}^{\mathscr{D}}_{q-1}}{d_q}\right]+(-1)^{\mathfrak{a}-1}\left[\frac{b^{\mathscr{D}}_{\mathfrak{a}-1}i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1})y_{\mathfrak{a}-1}\widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-2}}{d_{\mathfrak{a}-1}}\right]\\ &\quad +(-1)^{\mathfrak{a}}\left[\frac{\{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}\oplus 0, 0\oplus b_{\mathfrak{a}}\widetilde{h}_{\mathfrak{a}}\widetilde{b}_{\mathfrak{a}-1}, 0\oplus b''_{\mathfrak{a}}\widetilde{h}''_{\mathfrak{a}}\widetilde{b}''_{\mathfrak{a}-1}\}}{\{z_{\mathfrak{a}-1}\oplus 0, 0\oplus c_{\mathfrak{a}}, 0\oplus d''_{\mathfrak{a}}\}}\right]\\ &\quad +\sum_{q=\mathfrak{a}+1}^{m+1}(-1)^q\left[\frac{\{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}\oplus 0, 0\oplus b_q\widetilde{h}_q\widetilde{b}_{q-1}, 0\oplus b''_q\widetilde{h}''_q\widetilde{b''}_{q-1}\}}{\{c_{q-1}\oplus 0, 0\oplus c_q, 0\oplus d''_q\}}\right]\\ &= \sum_{q=0}^{\mathfrak{a}-2}(-1)^q\left[\frac{b^{\mathscr{D}}_q\widetilde{h}^{\mathscr{D}}_q\widetilde{b}^{\mathscr{D}}_{q-1}}{d_q}\right]+(-1)^{\mathfrak{a}-1}\left[\frac{b^{\mathscr{D}}_{\mathfrak{a}-1}i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1})y_{\mathfrak{a}-1}\widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-2}}{d_{\mathfrak{a}-1}}\right]\\ &\quad +(-1)^{\mathfrak{a}}\left(\left[\frac{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}}{z_{\mathfrak{a}-1}}\right]+\left[\frac{b_{\mathfrak{a}}\widetilde{h}_{\mathfrak{a}}\widetilde{b}_{\mathfrak{a}-1}}{c_{\mathfrak{a}}}\right]+\left[\frac{b''_{\mathfrak{a}}\widetilde{h}''_{\mathfrak{a}}\widetilde{b}''_{\mathfrak{a}-1}}{d''_{\mathfrak{a}}}\right]\right)\\ &\quad +\sum_{q=\mathfrak{a}+1}^{m+1}(-1)^q\left(\left[\frac{b_{q-1}\widetilde{h}_{q-1}\widetilde{b}_{q-2}}{c_{q-1}}\right]+\left[\frac{b_q\widetilde{h}_q\widetilde{b}_{q-1}}{c_q}\right]+\left[\frac{b''_q\widetilde{h}''_q\widetilde{b''}_{q-1}}{d''_q}\right]\right).\end{aligned}$$

Note that

$$\begin{aligned}\left[\frac{b^{\mathscr{D}}_{\mathfrak{a}-1}i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1})y_{\mathfrak{a}-1}\widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-2}}{d_{\mathfrak{a}-1}}\right] &= \left[\frac{b^{\mathscr{D}}_{\mathfrak{a}-1}i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1})y_{\mathfrak{a}-1}\widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-2}}{b^{\mathscr{D}}_{\mathfrak{a}-1}h^{\mathscr{D}}_{\mathfrak{a}-1}\widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-2}}\right] + \left[\frac{b^{\mathscr{D}}_{\mathfrak{a}-1}h^{\mathscr{D}}_{\mathfrak{a}-1}\widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-2}}{d_{\mathfrak{a}-1}}\right] \\ &= \left[\frac{i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1})y_{\mathfrak{a}-1}}{h^{\mathscr{D}}_{\mathfrak{a}-1}}\right] + \left[\frac{b^{\mathscr{D}}_{\mathfrak{a}-1}h^{\mathscr{D}}_{\mathfrak{a}-1}\widetilde{b}^{\mathscr{D}}_{\mathfrak{a}-2}}{d_{\mathfrak{a}-1}}\right].\end{aligned}$$

Thus, we have proved the following result.

Proposition 4.2.8. *Let $\mathfrak{p}$ be a perversity and consider $I^{\mathfrak{p}}\ddot{h}_q$ a preferred basis of $I^{\mathfrak{p}}H_q(\ddot{\mathscr{C}})$, then the intersection torsion of chain complex of the mapping cone of the pair $(\mathscr{D},\mathscr{C})$ with perversity $\mathfrak{p}$ is*

$$\begin{aligned}\tau_{I^{\mathfrak{p}}\ddot{c}_q,I^{\mathfrak{p}}\ddot{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{C}}) = \sum_{q=0}^{\mathfrak{a}-1}(-1)^q\left[\frac{b^{\mathscr{D}}_q\widetilde{h}^{\mathscr{D}}_q\widetilde{b}^{\mathscr{D}}_{q-1}}{d_q}\right] + (-1)^{\mathfrak{a}-1}\left[\frac{i_{*,\mathfrak{a}-1}(y'_{\mathfrak{a}-1})y_{\mathfrak{a}-1}}{h^{\mathscr{D}}_{\mathfrak{a}-1}}\right] + (-1)^{\mathfrak{a}}\left[\frac{b_{\mathfrak{a}-1}\widetilde{h}_{\mathfrak{a}-1}}{z_{\mathfrak{a}-1}}\right] \\ + \sum_{q=\mathfrak{a}}^{m+1}(-1)^q\left[\frac{b''_q\widetilde{h}''_q\widetilde{b}''_{q-1}}{d''_q}\right].\end{aligned}$$

Proposition 4.2.9. *The torsion of the intersection chain complex of the mapping cone of the pair $(\mathscr{D},\mathscr{C})$ with perversity $\mathfrak{p}$ satisfies*

$$\tau_{I^{\mathfrak{p}}\ddot{c}_q,I^{\mathfrak{p}}\ddot{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{C}}) = \tau_{I^{\mathfrak{p}}\dot{c}_q,I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}C(\mathscr{C})) + \tau_{d''_q,h''_q}(\mathscr{D}/\mathscr{C}) + \tau_{I^{\mathfrak{p}}\overline{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}}),$$

where

$$\tau_{I^{\mathfrak{p}}\overline{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}}) = (-1)^{\mathfrak{a}-1}\left[\frac{p_{*,\mathfrak{a}-1}(h^{\mathscr{D}}_{\mathfrak{a}-1})y''_{\mathfrak{a}-1}}{h''_{\mathfrak{a}-1}}\right] + \sum_{q=0}^{\mathfrak{a}-2}(-1)^q\left(\left[\frac{y''_{q+1}y'_q}{h_q}\right] - \left[\frac{i_{*,q}(y'_q)y_q}{h^{\mathscr{D}}_q}\right] + \left[\frac{p_{*,q}(y_q)y''_q}{h''_q}\right]\right).$$

Proof. Consider the short exact sequence of chain complexes of free modules with preferred bases:

$$0 \longrightarrow I^{\mathfrak{p}}C(\mathscr{C}) \xrightarrow{\bar{i}} I^{\mathfrak{p}}\ddot{\mathscr{C}} \xrightarrow{\overline{p}} I^{\mathfrak{p}}\ddot{\mathscr{C}}/I^{\mathfrak{p}}C(\mathscr{C}) \longrightarrow 0.$$

We assume that some basis for $Z_{\mathfrak{a}-1}$ has been fixed.

By definition, we have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \dot{C}_{\mathfrak{a}+1} \arrow[r] \arrow[d, "I^{\mathfrak{p}}\bar{i}_{\mathfrak{a}+1}=\bar{i}_{\mathfrak{a}+1}"] & Z_{\mathfrak{a}-1}\oplus C_{\mathfrak{a}} \arrow[r] \arrow[d, "I^{\mathfrak{p}}\bar{i}_{\mathfrak{a}}=id\oplus i_{\mathfrak{a}}"] & C_{\mathfrak{a}-1} \arrow[r] \arrow[d, "I^{\mathfrak{p}}\bar{i}_{\mathfrak{a}-1}=i_{\mathfrak{a}-1}"] & \cdots \\
\cdots \arrow[r] & \ddot{C}_{\mathfrak{a}-1} \arrow[r] \arrow[d, "I^{\mathfrak{p}}\overline{p}_{\mathfrak{a}+1}=\overline{p}_{\mathfrak{a}+1}"] & Z_{\mathfrak{a}-1}\oplus D_{\mathfrak{a}} \arrow[r] \arrow[d, "I^{\mathfrak{p}}\overline{p}_{\mathfrak{a}}=0\oplus p_{\mathfrak{a}}"] & D_{\mathfrak{a}-1} \arrow[r] \arrow[d, "I^{\mathfrak{p}}\overline{p}_{\mathfrak{a}-1}=p_{\mathfrak{a}-1}"] & \cdots \\
\cdots \arrow[r] & \ddot{C}_{\mathfrak{a}-1}/\dot{C}_{\mathfrak{a}+1} \arrow[r] & D_{\mathfrak{a}}/C_{\mathfrak{a}} \arrow[r] & D_{\mathfrak{a}-1}/C_{\mathfrak{a}-1} \arrow[r] & \cdots .
\end{tikzcd}$$

Using the chain isomorphism $\widetilde{j}\colon \mathscr{D}/\mathscr{C} \to \ddot{\mathscr{C}}/C(\mathscr{C})$ induced by the inclusion $\overline{j}\colon \mathscr{D} \to \ddot{\mathscr{C}}$, and making the suitable identifications of the bases of the chain modules and of the homology modules, we have the short exact sequence of chain complexes

$$0 \longrightarrow I^{\mathfrak{p}}C(\mathscr{C}) \xrightarrow{\bar{i}} I^{\mathfrak{p}}\ddot{\mathscr{C}} \xrightarrow{\psi} \mathscr{D}/\mathscr{C} \longrightarrow 0,$$

where $\psi = \widetilde{j}^{-1} \circ I^{\mathfrak{p}}\overline{p}$, and each chain module of $\mathscr{D}/\mathscr{C}$ is free with preferred basis d''_q.

By Theorem 2.3.4, we have that

$$\tau_{I^{\mathfrak{p}}\ddot{c}_q, I^{\mathfrak{p}}\ddot{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{C}}) = \tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}C(\mathscr{C})) + \tau_{d''_q, h''_q}(\mathscr{D}/\mathscr{C}) + \tau_{I^{\mathfrak{p}}\overline{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}}),$$

where $I^{\mathfrak{p}}\ddot{\mathscr{H}}$ is the associated long homology sequence, with $I^{\mathfrak{p}}\overline{h}_{3i} = I^{\mathfrak{p}}\dot{h}_i$, $I^{\mathfrak{p}}\overline{h}_{3i+1} = I^{\mathfrak{p}}\ddot{h}_i$, and $I^{\mathfrak{p}}\overline{h}_{3i+2} = h''_i$.

Note that in degrees $q > \mathfrak{a} - 1$, the homology of $I^{\mathfrak{p}}C(\mathscr{C})$ is trivial, and therefore the sequence $I^{\mathfrak{p}}\ddot{\mathscr{H}}$ is a sequence of isomorphisms and zero maps, with trivial torsion. The relevant part of the homology sequence associated to $I^{\mathfrak{p}}\ddot{\mathscr{H}}$ is

$$\cdots \longrightarrow H_{\mathfrak{a}-1}(\mathscr{D}) \xrightarrow{p_{*,\mathfrak{a}-1}} p_{*,\mathfrak{a}-1}(H_{\mathfrak{a}-1}(\mathscr{D})) \longrightarrow H_{\mathfrak{a}-1}(\mathscr{D}/\mathscr{C}) \longrightarrow \cdots.$$

With the choices of the homology bases as in Remark 4.1.1, we have $I^{\mathfrak{p}}\ddot{h}_{\mathfrak{a}-1} = p_{*,\mathfrak{a}-1}(h^{\mathscr{D}}_{\mathfrak{a}-1})$ in $H_{\mathfrak{a}-1}(I^{\mathfrak{p}}\ddot{\mathscr{C}})$ and $I^{\mathfrak{p}}\ddot{h}_{\mathfrak{a}-1} = h''_{\mathfrak{a}-1}$ in $H_{\mathfrak{a}-1}(\mathscr{D}/\mathscr{C})$. Therefore $p_{*,\mathfrak{a}-1}$ is the inclusion, $I^{\mathfrak{p}}y_{\mathfrak{a}-1} = p_{*,\mathfrak{a}-1}(h^{\mathscr{D}}_{\mathfrak{a}-1})$ and the matrix representing the change of basis is the identity matrix in $H_{\mathfrak{a}-1}(I^{\mathfrak{p}}\ddot{\mathscr{C}})$.

In $H_{\mathfrak{a}-1}(\mathscr{D}/\mathscr{C})$, identifying $p_{*,\mathfrak{a}-1}(h^{\mathscr{D}}_{\mathfrak{a}-1}) = p_{*,\mathfrak{a}-1}(y_{\mathfrak{a}-1})$, the matrix of the change of basis is

$$\left(\frac{p_{*,\mathfrak{a}-1}(y_{\mathfrak{a}-1})y''_{\mathfrak{a}-1}}{h''_{\mathfrak{a}-1}}\right).$$

The remaining part of the sequence associated to $I^{\mathfrak{p}}\ddot{\mathscr{H}}$ is

$$\cdots \longrightarrow H_{\mathfrak{a}-1}(\mathscr{D}/\mathscr{C}) \xrightarrow{\delta_{\mathfrak{a}-1}} H_{\mathfrak{a}-2}(\mathscr{C}) \xrightarrow{i_{*,\mathfrak{a}-2}} H_{\mathfrak{a}-2}(\mathscr{D}) \longrightarrow H_{\mathfrak{a}-2}(\mathscr{D}/\mathscr{C}) \longrightarrow \cdots$$

and coincides with the homology sequence associated to the pair $(\mathscr{D}, \mathscr{C})$. Using the bases described in Remark 4.1.1, we have that the torsion of $I^{\mathfrak{p}}\ddot{\mathscr{H}}$ is given by

$$(-1)^{\mathfrak{a}-1}\left[\frac{p_{*,\mathfrak{a}-1}(h^{\mathscr{D}}_{\mathfrak{a}-1})y''_{\mathfrak{a}-1}}{h''_{\mathfrak{a}-1}}\right] + \sum_{q=0}^{\mathfrak{a}-2}(-1)^q\left(\left[\frac{y''_{q+1}y'_q}{h_q}\right] - \left[\frac{i_{*,q}(y'_q)y_q}{h^{\mathscr{D}}_q}\right] + \left[\frac{p_{*,q}(y_q)y''_q}{h''_q}\right]\right).$$

□

4.3 Cellular intersection torsion for pseudomanifolds with isolated singularities

We will make the cellular constructions, which we will need to define the intersection torsion for pseudomanifolds with isolated singularities.

Definition 4.3.1. *The space $Z = Y \sqcup_W C(W)$ is called a space with an isolated singularity. The space $X = Z - \{x_0\}$, where x_0 is the vertex of the cone, is a manifold with boundary.*

From the topological point of view, the relevant space is Z, that is an n-dimensional pseudomanifold with singular locus Σ of dimension zero. By definition, Z has a CW-decomposition K, where K is a n-dimensional pseudomanifold such that $Z-\Sigma$ is an homology n-dimensional manifold with proper boundary. This is a particular type of regular CW-complex.

In particular, we assume $\Sigma = x_0$, and we call K a *pseudomanifold with an isolated singularity and proper boundary*. In this section we work with the CW-complex K, it is clear that all the results will hold for the corresponding space Z. Note that Z is the single point compactification of $Z-\{x_0\}$.

Lemma 4.3.2. *An n-dimensional pseudomanilfold K with an isolated singularity is the push out of a regular CW-complex M that is an homology n-manifold with boundary $\partial M = N$ by attaching the cone $C(N)$.*

Proof. Setting, for the 0-cell $e_0 = x_0$ of K, $N = \mathrm{Link}(e_0)$, and $M = \overline{K - C(N)}$, we have the result. □

Definition 4.3.3. *Let (K,L) be a proper relative pseudomanifold with one isolated singularity x_0. We call the decomposition $K = M \sqcup_N C(N)$, where $N = \mathrm{Link}(x_0)$ and $M = \overline{K - C(N)}$, the standard decomposition of K.*

Lemma 4.3.4. *Let K be a pseudomanifold with one isolated singularity x_0, and proper boundary, i.e., $\partial K \cap \{x_0\} = \emptyset$. Let $K = M \sqcup_N C(N)$ be the standard decomposition of K. Then, the inclusion $i\colon N \to K$ induces a trivial map in homotopy, and M/N, $K/C(N)$ and K have the same homotopy type.*

Proof. If γ is any cycle in a non trivial class of $\pi_1(N)$, the image $i_*([\gamma])$ is a class in $\pi_1(C(N))$. Since $\pi_1(C(N))$ is trivial, we have that γ is a trivial homotopy in K. It is clear that M/N and $K/C(N)$ have a same homotopy type. Moreover, by [25, Theorem 8.33], since M a CW-complex and N is a CW-subcomplex, then the inclusion $i\colon N \to M$ is a cofibration, i.e., for any two maps $f\colon M\times 0 \to K$ and $G\colon N \times I \to K$ which coincide when restricted to $N \times 0$, there is a map $F\colon M \times I \to K$, such that F restricted to $M\times 0$ is f and F restricted to $N \times I$ is G. See the diagram below,

$$\begin{tikzcd}
N\times 0 \arrow[r] \arrow[d, "i\times id"'] & N\times I \arrow[d, "i\times id"'] \arrow[ddr, "G"] & \\
M\times 0 \arrow[r] \arrow[drr, "f"'] & M\times I \arrow[dr, "F"] & \\
& & K.
\end{tikzcd}$$

By [23, Lemma 3.2.2], since $i\colon N \to M$ is a cofibration, then we have that M/N has the same homotopy type of K. It follows that M/N, $K/C(N)$ and K have the same homotopy type. □

Let $K = M \sqcup_N C(N)$ be the standard decomposition of K. We have that the inclusion $i\colon N \to K$ induces a group homomorphism $i_*\colon \pi_1(N) \to \pi_1(K)$ that factors through $\pi_1(C(N))$ in the sense of

Seifert–Van Kampen Theorem, i.e., $\pi_1(K)$ is the quotient $\pi_1(M)/\mathscr{N}(i_*(\pi_1(N)))$, where $\mathscr{N}(i_*(\pi_1(N)))$ is the normalized of $i_*(\pi_1(N))$. See the commutative diagram below, induced by inclusions,

$$\begin{tikzcd}
 & \pi_1(M) \arrow[dr] & \\
\pi_1(N) \arrow[ur] \arrow[rr, "i_*"] \arrow[dr] & & \pi_1(K). \\
 & \pi_1(C(N)) \arrow[ur] &
\end{tikzcd}$$

Thus, the universal covering space $\widetilde{K}$ has an induced decomposition

$$\widetilde{K} = \left(\widetilde{M}/i_*(\pi_1(N))\right) \sqcup_{\widetilde{N}} \bigsqcup_{g\in\Pi} C(\widetilde{N}_g),$$

where $\widetilde{N}_g$ denotes the boundary of the g-sheet of $\widetilde{M}/i_*(\pi_1(N))$, and $\Pi = \pi_1(K)$. We know that the cellular chain complex $\mathscr{C}(\widetilde{K})$ is a complex of free modules $C_q(\widetilde{K})$ over $\mathbb{Z}(\Pi)$ and finitely generated by any lift of the cells of K, since K is finite (see Proposition 1.1.9). However, note that Π acts trivially onto the cells of $C(N)$, and therefore Π acts globally on its lifts without mixing the sheets. In other words, we may identify the Π-action on the following chain complexes

$$C_q\left(\bigsqcup_{g\in\Pi} C(\widetilde{N}_g)\right) = \Pi C_q(C(N)),$$

where $\Pi C_q(C(N))$ means the $\mathbb{Z}(\Pi)$-module with basis the q-cells of $C(N)$. Using the decomposition of the chain modules as defined in Section 4.1 for the mapping cone, we may write

$$C_q(\widetilde{K}) = \Pi C_{q-1}(N) \oplus C_q(\widetilde{M}),$$

where $C_q(\widetilde{K})$ is the a $\mathbb{Z}(\Pi)$-module with basis given by the lift of cells of K, we make Π acting on the universal covering of M as subspace of $\widetilde{K}$.

The key point here to observe that the boundary operator $\ddot{\partial}_q$ of $C_q(\widetilde{K})$ when restricted to the chain of the $\bigsqcup_{g\in\Pi} C(\widetilde{N}_g)$ does not mix the sheets. Using the standard decomposition of the boundary operator

$$\ddot{\partial}_q = \partial^{\mathscr{C}}_{q-1} \oplus (i_{q-1} - \partial^{\mathscr{D}}_q) = \begin{pmatrix} \partial^{\mathscr{C}}_{q-1} & 0 \\ i_{q-1} & -\partial^{\mathscr{D}}_q \end{pmatrix},$$

where $\mathscr{C}$ is the complex of the $\mathbb{Z}$-modules $C_q = \Pi C_q(N)$, and $\mathscr{D}$ is the complex of the $\mathbb{Z}(\Pi)$-modules $D_q = C_q(\widetilde{M})$, what happens is that if $c = x \oplus y \in C_q(\widetilde{K})$, then we may write $c = gt \oplus y$, for some $g \in \Pi$ and $t \in C_q(C(N))$, and hence

$$\ddot{\partial}_q(x \oplus y) = \partial^{\mathscr{C}}_{q-1}(gt) \oplus (i_{q-1} - \partial^{\mathscr{D}}_q)(y) = g\partial^{\mathscr{C}}_{q-1}(t) \oplus (i_{q-1} - \partial^{\mathscr{D}}_q)(y).$$

Therefore, we have proved the following result.

Lemma 4.3.5. *Let $\mathscr{C}(\widetilde{K})$ be the cellular chain complex of $\mathbb{Z}(\Pi)$-modules of the of the universal covering*

$$\widetilde{K} = \left(\widetilde{M}/i_*(\pi_1(N))\right) \sqcup_{\widetilde{N}} \bigsqcup_{g\in\Pi} C(\widetilde{N}_g)$$

of the mapping cone $K = M \sqcup_N C(N)$ of the inclusion $i\colon N \to M$. Then $\mathscr{C}(\widetilde{K})$ coincides with the mapping cone

$$\Pi C(\mathscr{C}(N)) \sqcup_i \mathscr{C}(\widetilde{M}),$$

of the chain inclusion $i\colon \Pi\mathscr{C}(N) \to \mathscr{C}(\widetilde{M})$ of chain complexes of $\mathbb{Z}(\Pi)$-modules.

We now proceed to define the cellular intersection chain complexes. Let (K,L) be a proper relative n-pseudomanifold with an isolated singularity, and let $\mathfrak{p}$ be a perversity. As constructed in Section 1.1, there exists a stratification of K, given by

$$K_{(0)} \subset K_{(1)} \subset \cdots \subset K_{(n-1)} \subset K_{(n)} = K.$$

Definition 4.3.6. *If q is an integer, we have that a cell e of $K_{(q)}$ is said $(\mathfrak{p},q)$-allowable if* $\dim(\bar{e}\cap\Sigma) \leq q-n+\mathfrak{p}(n)$. *The intersection cellular family $I^{\mathfrak{p}}\mathscr{F}(K)$ with perversity $\mathfrak{p}$ associated to K is the subfamily of $(\mathfrak{p},q)$-allowable cells of K, namely $I^{\mathfrak{p}}\mathscr{F}(K) = \{I^{\mathfrak{p}}K_{(q)}\}_{q=0}^{m}$, where*

$$I^{\mathfrak{p}}K_{(q)} = \{e \in K_{(q)}; e \text{ is } (\mathfrak{p},q)-allowable\}.$$

Note that for each q there are cellular inclusions $I^{\mathfrak{p}}K_{(q)} \to I^{\mathfrak{p}}K_{(q+1)}$, and $I^{\mathfrak{p}}K_{(q)} \to K_{(q)}$.

For a regular orientable CW-complex N that is an homology m-manifold, we find that

$$I^{\mathfrak{p}}C(N)_{(q)} = \begin{cases} N_{(q)}, & q < m+1-\mathfrak{p}(m+1), \\ C(N)_{(q)}, & q \geq m+1-\mathfrak{p}(m+1), \end{cases}$$

$$I^{\mathfrak{p}}K_{(q)} = \begin{cases} M_{(q)}, & q < m+1-\mathfrak{p}(m+1), \\ K_{(q)}, & q \geq m+1-\mathfrak{p}(m+1). \end{cases}$$

For each perversity $\mathfrak{p}$, the family $I^{\mathfrak{p}}\mathscr{F}(K)$ defines a filtration of K given by

$$0 \subset I^{\mathfrak{p}}K_{(0)} \subset I^{\mathfrak{p}}K_{(1)} \subset \cdots \subset I^{\mathfrak{p}}K_{(n-1)} \subset I^{\mathfrak{p}}K_{(n)} = K,$$

which is not cellular, but it has an associated chain complex with chain modules defined as $I^{\mathfrak{p}}C_q(K) = H_q(I^{\mathfrak{p}}K_{(q)}, I^{\mathfrak{p}}K_{(q-1)})$ and where the boundary operators are defined using the exact sequence of the pairs.

Note that $I^{\mathfrak{p}}C_q(K)$ and $I^{\mathfrak{p}}C_q(K,L)$ are free finitely generated $\mathbb{Z}$-modules, and that the intersection cellular chain module of perversity $\mathfrak{p}$ is the submodule of the cellular chain module $C_q(K)$ generated by the $(\mathfrak{p},q)$-allowable cells with $(\mathfrak{p},q-1)$-allowable boundary.

Using the description of the universal covering in Lemma 4.3.5, we see that the action of the fundamental group commutes with the requirement of allowability.

Definition 4.3.7. *Considering the intersection cellular family of $\widetilde{K}$, we obtain a chain complex of $\mathbb{Z}(\Pi)$-modules $I^{\mathfrak{p}}C_q(\widetilde{K}) = H_q(I^{\mathfrak{p}}\widetilde{K}_{(q)}, I^{\mathfrak{p}}\widetilde{K}_{(q-1)})$, which we call the* intersection cellular chain complex *of K with perversity $\mathfrak{p}$ associated to $\widetilde{K}$ with coefficients in $\mathbb{Z}(\Pi)$.*

Analogously, we define the cellular intersection complex $I^{\mathfrak{p}}\mathscr{C}(\widetilde{K},\widetilde{L})$ of $\mathbb{Z}(\Pi)$-modules

$$I^{\mathfrak{p}}C_q(\widetilde{K},\widetilde{L}) = H_q(I^{\mathfrak{p}}\widetilde{K}_{(q)} \cup \widetilde{L}, I^{\mathfrak{p}}\widetilde{K}_{(q-1)} \cup \widetilde{L}).$$

Lemma 4.3.8. *Let K be an n-dimensional pseudomanifold without boundary and with an isolated singularity x_0, and $\mathfrak{p}$ a perversity. Let $K = M \sqcup_N C(N)$ be the standard decomposition of K. Then, the cellular intersection chain complex $I^{\mathfrak{p}}\mathscr{C}(\widetilde{K})$ coincides with the intersection chain complex*

$$\mathscr{C}(\widetilde{M}) \sqcup_i I^{\mathfrak{p}}(\Pi C(\mathscr{C}(N)))$$

of the mapping cone of the chain inclusion induced by inclusion $i\colon N \to M$. This chain complex is a chain complex of free finitely generated $\mathbb{Z}(\Pi)$-modules with preferred class of equivalent bases, with representatives:

$$I^{\mathfrak{p}}\ddot{c}_q = \begin{cases} d_q, & q \leq \mathfrak{a}-1, \\ \{z_{\mathfrak{a}-1} \oplus 0, 0 \oplus c_{\mathfrak{a}}\}, & q = \mathfrak{a}, \\ \{c_{q-1} \oplus 0, 0 \oplus d_q\}, & q \geq \mathfrak{a}+1. \end{cases}$$

Remembering that we can rewrite $\{c_{q-1} \oplus 0, 0 \oplus d_q\}$ as $\{c_{q-1} \oplus 0, 0 \oplus c_q, 0 \oplus d''_q\}$.

Proof. The non critical degree are similar to the Remark 4.2.5. For the the critical degree $\mathfrak{a}$, since $Z_{\mathfrak{a}-1}$ is a submodule of $\Pi C_{\mathfrak{a}-1}(N)$, and we are working with the principal ideal domain $\mathbb{Z}$, then $Z_{\mathfrak{a}-1}$ is free. Moreover, any two bases differs by a matrix with integer coefficients, and are therefore in the same class in the reduced Whitehead group. □

In particular, the intersection cellular chain complex $I^{\mathfrak{p}}\mathscr{C}(C(N))$ coincides with the intersection chain complex $I^{\mathfrak{p}}C(\mathscr{C}(N))$, and the relative case is similar.

We consider now

$$I^{\mathfrak{p}}\mathscr{C}^{\varphi}(K) = V \otimes_{\varphi} I^{\mathfrak{p}}\mathscr{C}^{\varphi}(K) = I^{\mathfrak{p}}(\mathscr{C}^{\varphi_0}(N) \sqcup_i \mathscr{C}^{\varphi}(M)),$$

where $\varphi\colon \Pi \to O(V)$ is the real orthogonal representation of Π. Moreover, by Lemma 4.3.4 we have that $H_q(\mathscr{C}^{\varphi}(K))$, $H_q(\mathscr{C}^{\varphi}(M,N))$ are naturally isomorphic, with bases $\ddot{h}_q$ and h''_q, respectively. We denote by h_q and $h_q^{\mathscr{D}}$ bases for the homology vector spaces $H_q(\mathscr{C}^{\varphi}(N))$ and $H_q(\mathscr{C}^{\varphi}(N))$, respectively. Considering the long exact sequence

$$\cdots \longrightarrow H_q(\mathscr{C}^{\varphi}(N)) \xrightarrow{i_{*,q}} H_q(\mathscr{C}^{\varphi}(M)) \xrightarrow{p_{*,q}} H_q(\mathscr{C}^{\varphi}(M,N)) \longrightarrow \cdots.$$

Then, by Remark 4.2.5 we have a preferred basis for $I^{\mathfrak{p}}\ddot{h}_q$ for the intersection homology of K given by

$$I^{\mathfrak{p}}\ddot{h}_q = \begin{cases} h_q^{\mathscr{D}}, & q \leq \mathfrak{a}-2, \\ p_{*,\mathfrak{a}-1}(h_{\mathfrak{a}-1}^{\mathscr{D}}), & q = \mathfrak{a}-1, \\ h''_q, & q \geq \mathfrak{a}. \end{cases}$$

Remark 4.3.9. *The cellular intersection chain complex* $I^{\mathfrak{p}}\mathscr{C}(C(N))$ *has a natural class of equivalent of preferred bases, and this induces a natural class of preferred bases for the chain complex* $I^{\mathfrak{p}}C(\mathscr{C}(N))$. *Representatives of this class are:*

$$I^{\mathfrak{p}}\dot{c}_q = \begin{cases} c_q, & q \leq \mathfrak{a}-1, \\ z_{\mathfrak{a}-1} \oplus c_{\mathfrak{a}}, & q = \mathfrak{a}, \\ \dot{c}_q, & q \geq \mathfrak{a}+1. \end{cases}$$

where c_q *is the preferred cell basis of* $C_q(N)$, $\dot{c}_q$ *the preferred basis* $\dot{C}_q(N)$, *and* $z_{\mathfrak{a}-1}$ *i any integral basis for* $Z_{\mathfrak{a}-1} \subset C_{\mathfrak{a}-1}(N)$. *Furthermore, assuming that* $H_q(N)$ *is free with preferred basis* h_q. *Then* $H_q(I^{\mathfrak{p}}C(\mathscr{C}))$ *is free (stably free) and has the preferred basis*

$$I^{\mathfrak{p}}\dot{h}_q = \begin{cases} h_q, & q \leq \mathfrak{a}-2, \\ 0, & q \geq \mathfrak{a}-1. \end{cases}$$

Hereafter, unless otherwise stated, the bases $I^{\mathfrak{p}}\dot{c}_q$ and $I^{\mathfrak{p}}\dot{h}_q$ will be the described in Remark 4.3.9.

Proposition 4.3.10. *Let N be a connected finite regular CW-complex, and let* $\mathfrak{p}$ *be a perversity. Assume that* $H_q(N)$ *is free with preferred basis* h_q. *Then, the torsion of the cellular intersection chain complex* $I^{\mathfrak{p}}\mathscr{C}(C(N))$ *depending of the bases* $I^{\mathfrak{p}}\dot{c}_q$ *and* $I^{\mathfrak{p}}\dot{h}_q$ *is*

$$\tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}\mathscr{C}(C(N))) = \tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}C(\mathscr{C}(N))) = \sum_{q=0}^{\mathfrak{a}-1} (-1)^q \left[\frac{b_q h_q b_{q-1}}{c_q}\right] + (-1)^{\mathfrak{a}} \left[\frac{b_{\mathfrak{a}-1} h_{\mathfrak{a}-1}}{z_{\mathfrak{a}-1}}\right].$$

Proof. It follows by Proposition 4.2.7. ☐

Now we consider the complex of real vector spaces

$$I^{\mathfrak{p}}\mathscr{C}^{\varphi}(C(N)) = V \otimes_{\varphi} I^{\mathfrak{p}}\mathscr{C}(C(N)),$$

where $\varphi\colon \pi_1(C(N)) \to O(V)$ is a representation of $\pi_1(C(N))$ on a real vector space V. Since $\pi_1(C(N))$ is trivial, we have that φ is a trivial representation and the complex $V \otimes_{\varphi} I^{\mathfrak{p}}\mathscr{C}(C(N)) = V \otimes I^{\mathfrak{p}}\mathscr{C}(C(N))$, that will only depend on the representation through its rank.

Theorem 4.3.11. *Let N be a connected finite regular CW-complex, and let* $\mathfrak{p}$ *be a perversity. Let* n_q *be the standard basis, and* h_q *any other preferred basis for* $H_q(\mathscr{C}^{\varphi_0}(N))$. *Then the torsion of the cellular intersection chain complex* $I^{\mathfrak{p}}\mathscr{C}^{\varphi_0}(C(N)) = V \otimes I^{\mathfrak{p}}\mathscr{C}(C(N))$ *depending of the bases* $I^{\mathfrak{p}}\dot{c}_q$ *and* $I^{\mathfrak{p}}\dot{h}_q$ *is well defined and given by*

$$\tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}\mathscr{C}^{\varphi_0}(C(N))) = \prod_{q=0}^{\mathfrak{a}-2} \left|\det\left(\frac{h_q}{n_q}\right)\right|^{(-1)^q} \prod_{q=0}^{\mathfrak{a}-2} \left(\#T(H_q(N))\right)^{(-1)^q},$$

where $\#T(H_q(N))$ *denote the cardinality of the torsion part of* $H_q(N)$.

Proof. Consider the chain complex $\mathscr{C}(N)$. By [22, Theorem 11.4], we have that

$$C_q := C_q(N) \cong U_q \oplus \widetilde{F}(H_q(\mathscr{C}(N))) \oplus \widetilde{T}(H_q(\mathscr{C}(N)),$$

where $\widetilde{F}(H_q(\mathscr{C}(N)))$ denote the lift of the free part of $H_q(\mathscr{C}(N))$, $\widetilde{T}(H_q(\mathscr{C}(N))$ denote the lift of the torsion part of $H_q(\mathscr{C}(N))$, and $\partial_q(U_q) = \widetilde{T}(H_{q-1}(\mathscr{C}(N))$. Suppose that $u_q = \dim U_q$ and $m_q = \dim(C_q)$. Then we have that the standard basis of C_q is the set e_q, where $\{e_{q,1},\dots,e_{q,u_q}\}$ is a basis of U_q, $\{e_{q-1,m_{q-1}-u_q},\dots,e_{q-1,m_{q-1}}\}$ is a basis of $\widetilde{T}(H_q(\mathscr{C}(N))$, $\{e_{q,u_{q+1}},\dots,e_{q,m_q}\}$ is a basis for Z_q, which denote the kernel of $\partial_q\colon C_q \to C_{q-1}$. Moreover we have that

$$\partial_q(e_{q,j}) = k_{q-1,j}e_{q-1,m_{q-1}-u_q+j},$$

for $1 \leq q \leq u_q$, with $k_{q-1,j} \in \mathbb{Z}$, $k_{q-1,j} \neq 0$, and $k_{q-1,m_{q-1}-u_q+1}|k_{q-1,m_{q-1}-u_q+2}|\dots|k_{q-1,m_{q-1}}$. The homology $H_q(\mathbb{R}\otimes\mathscr{C}(N))$ is free and has standard basis n_q, com $\tilde{n}_q = \{e_{q,u_q+1},\dots,e_{q,m_q-u_q+1}\}$.

Taking $\tilde{b}_{q-1} = \{e_{q,1},\dots,e_{q,u_q}\}$, we have that

$$b_q\tilde{n}_q\tilde{b}_{q-1} = \{k_{q,m_q-u_{q+1}}e_{q,m_q-u_{q+1}}\} \cup \{e_{q,u_q+1},\dots,e_{q,m_q-u_q+1}\} \cup \{e_{q,1},\dots,e_{q,u_q}\}$$

is a basis of $\mathbb{R}\otimes C_q$, and hence

$$\left|\det\left(\frac{b_q\tilde{n}_q\tilde{b}_{q-1}}{e_q}\right)\right| = \left|\prod_{j=1}^{t_q} k_{q,j}\right| = \#T(H_q(\mathscr{C}(N))).$$

The same for $\mathscr{C}^{\varphi_0}(N)$, and for $I^{\mathfrak{p}}\mathscr{C}^{\varphi_0}(C(N))$. Thus, denoting by n_q the standard basis for the free part of the homology with integral coefficients $H_q(N)$, by v_q the standard basis of the cycles in the chain module with integral coefficients $C_q(N)$ and by h_q any fixed basis for the homology $H_q(N)$ with coefficients in $\mathbb{Z}$. We have that the class in $\mathbb{R}$ is

$$\left[\frac{b_q\tilde{n}_q\tilde{b}_{q-1}}{e_q}\right] = \left|\det\left(\frac{b_q\tilde{n}_q\tilde{b}_{q-1}}{e_q}\right)\right| = \#T(H_q(\mathscr{C}(N))).$$

Thus, for $q < \mathfrak{a}-1$,

$$\left|\det\left(\frac{I^{\mathfrak{p}}\dot{b}_q\widetilde{I^{\mathfrak{p}}\dot{n}_q}\widetilde{I^{\mathfrak{p}}\dot{b}_{q-1}}}{I^{\mathfrak{p}}\dot{e}_q}\right)\right| = \#T(H_q(N)).$$

If $q = \mathfrak{a}-1$, we have

$$\left|\det\left(\frac{I^{\mathfrak{p}}\dot{b}_{\mathfrak{a}-1}\widetilde{I^{\mathfrak{p}}\dot{n}_{\mathfrak{a}-1}}\widetilde{I^{\mathfrak{p}}\dot{b}_{\mathfrak{a}-2}}}{I^{\mathfrak{p}}\dot{e}_{\mathfrak{a}-1}}\right)\right| = \left|\det\left(\frac{b_{\mathfrak{a}-1}\tilde{h}_{\mathfrak{a}-1}}{v_{\mathfrak{a}-1}}\right)\right|.$$

Since Z_q is uniquely determined, and its basis is given by the first s elements of the basis e_q, with $s = \dim Z_q$, we have that

$$\left|\det\left(\frac{I^{\mathfrak{p}}\dot{b}_{\mathfrak{a}-1}\widetilde{I^{\mathfrak{p}}\dot{n}_{\mathfrak{a}-1}}\widetilde{I^{\mathfrak{p}}\dot{b}_{\mathfrak{a}-2}}}{I^{\mathfrak{p}}\dot{e}_{\mathfrak{a}-1}}\right)\right| = \#T(H_{\mathfrak{a}-1}(N)).$$

□

Therefore, we obtain the main result of this section.

Theorem 4.3.12. *Let K be an n-dimensional pseudomanifold without boundary and with an isolated singularity x_0, and a perversity $\mathfrak{p}$. Let $K = M \sqcup_N C(N)$ be the standard decomposition of K. Let $\varphi\colon \pi_1(K) \to O(V)$ be some real orthogonal representation. Assume that the bases of $I^{\mathfrak{p}}C^{\varphi}_q(K)$ and $H_q(I^{\mathfrak{p}}\mathscr{C}^{\varphi}(K))$ are defined, for all q. Then, the torsion of the cellular intersection chain complex $I^{\mathfrak{p}}\mathscr{C}^{\varphi}(K)$ depending of the bases $I^{\mathfrak{p}}\ddot{c}_q$ and $I^{\mathfrak{p}}\ddot{h}_q$ is well defined and given by*

$$\tau_{I^{\mathfrak{p}}\ddot{c}_q, I^{\mathfrak{p}}\ddot{h}_q}(I^{\mathfrak{p}}\mathscr{C}^{\varphi}(K)) = \tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}\mathscr{C}^{\varphi}(C(N)))\,\tau_{d''_q, h''_q}(\mathscr{C}^{\varphi}(M,N))\,\tau_{I^{\mathfrak{p}}\bar{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}}),$$

where

$$\tau_{I^{\mathfrak{p}}\bar{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}}) = \left|\det\left(\frac{p_{*,\mathfrak{a}-1}(h^{\mathscr{D}}_{\mathfrak{a}-1})y''_{\mathfrak{a}-1}}{h''_{\mathfrak{a}-1}}\right)\right|^{(-1)^{\mathfrak{a}-1}} \prod_{q=0}^{\mathfrak{a}-2}\left|\frac{\det\left(\dfrac{\delta_{q+1}(y''_{q+1})y'_q}{h_q}\right)\det\left(\dfrac{p_{*,q}(y_q)y''_q}{h''_q}\right)}{\det\left(\dfrac{i_{*,q}(y'_q)y_q}{h^{\mathscr{D}}_q}\right)}\right|^{(-1)^q}.$$

Proof. It follows by Proposition 4.2.9. □

Example 4.3.13. *Consider the pseudomanifold X_q defined in Section 3.5, and take in particular $n = 4$, i.e.,*

$$X_q = L_q \times \sigma^4 \sqcup_{L_q \times \partial\sigma^4} C(L_q \times \partial\sigma^4),$$

where $L_q = L(7,q)$ and σ^4 is homeomorphic to closed disk D^4. By Seifert–Van Kampen Theorem, we have that

$$\pi_1(X_q) \cong \frac{\pi_1(L_q \times \sigma^4)}{\mathscr{N}(i_*(\pi_1(L_q \times \partial\sigma^4)))} \cong \frac{\mathbb{Z}_7}{\mathbb{Z}_7} \cong 0.$$

Thus the representation $\varphi\colon \pi_1(X_q) \to \mathbb{R}$ is the trivial representation. Moreover, let $\mathfrak{m}$ be the lower middle perversity. In this case, since the complex has dimension 7, we have that the constant $\mathfrak{a} = 7 - \mathfrak{m}(7) = 7 - 2 = 5$. By Künneth Theorem, we know that

$$H_k(L_q \times \partial\sigma^4) \cong H_k(L_q \times S^3) \cong \bigoplus_{i+j=k} H_i(L_q) \otimes H_j(S^3).$$

Thus, we obtain that

$$H_i(L_q \times \partial\sigma^4) \cong \begin{cases} \mathbb{Z}, & i = 0,6, \\ \mathbb{Z}_7, & i = 1,4, \\ \mathbb{Z}\oplus\mathbb{Z}, & i = 3, \\ 0, & \textit{otherwise}. \end{cases}$$

We fix the basis h_i for $I^{\mathfrak{m}}H_i(C(L_q \times \partial\sigma^4))$. Note that, we have

$$h_0 = \lambda^C_0 e^{L_q}_0 \times e^{\partial\sigma^4}_0 \quad \textit{and} \quad h_3 = \{\lambda^C_{3,1} e^{L_q}_0 \times e^{\partial\sigma^4}_3, \lambda^C_{3,2} e^{L_q}_3 \times e^{\partial\sigma^4}_0\},$$

where $\lambda^C_0, \lambda^C_{3,1}, \lambda^C_{3,2} \in \mathbb{Z}$. Hence

$$\det\left(\frac{h_0}{n_0}\right) = \lambda^C_0 \quad \textit{and} \quad \det\left(\frac{h_3}{n_3}\right) = \det\begin{pmatrix} \lambda^C_{3,1} & 0 \\ 0 & \lambda^C_{3,2} \end{pmatrix} = \lambda^C_{3,1}\lambda^C_{3,2}.$$

Thus, by Theorem 4.3.11

$$\begin{aligned}\tau_{I^{\mathrm{m}}\dot{c}_i, I^{\mathrm{m}}h_i}(I^{\mathrm{m}}\mathscr{C}^{\varphi}(C(L_q\times\partial\sigma^4))) &= \left|\det\left(\frac{h_0}{n_0}\right)\right|\left|\det\left(\frac{h_3}{n_3}\right)\right|^{-1}\left(\#T(H_1(L_q\times\partial\sigma^4))\right)^{-1}\\ &= \left|\lambda_0^C\right|\left|\frac{1}{\lambda_{3,1}^C\lambda_{3,2}^C}\right|\cdot\frac{1}{7}.\end{aligned}$$

Furthermore, we have that the pair $(L_q\times\sigma^4, L_q\times\partial\sigma^4)$ *has the same cellular structure of the pair* (X_q, x_0) *defined in the proof of Theorem 3.5.3, and then* $\mathscr{C}^{\varphi}(L_q\times\sigma^4, L_q\times\partial\sigma^4)$ *is free over* $\mathbb{Z}(\Pi)$ *with four cells* $e''_{i+4} = e_i\times\mathrm{Int}D^4$, *where* $0\le i\le 3$. *Thus, the torsion of* $\mathscr{C}^{\varphi}(L_q\times\sigma^4, L_q\times\partial\sigma^4)$ *coincides with the torsion of* $\mathscr{C}^{\varphi}(L_q)$. *We fix the basis* h''_i *for*

$$H_i(L_q\times\sigma^4, L_q\times\partial\sigma^4)\cong\begin{cases}\mathbb{Z}, & i=4,7,\\ \mathbb{Z}_7, & i=5,\\ 0, & \textit{otherwise}.\end{cases}$$

Hence, for $i=4$, *we have* $\varphi(\partial\tilde{e}''_5) = \varphi((g-1)\tilde{e}''_4) = 0$ *and* $\widetilde{h''_4} = \lambda''_4\tilde{e}''_4$. *Thus the matrix of change of basis is*

$$\left(\frac{\lambda''_4\tilde{e}''_4}{\tilde{e}''_4}\right),$$

where $\lambda''_4\in\mathbb{Z}$. *For* $i=5$, *we have* $\varphi(\partial\tilde{e}''_6) = \varphi((1+g+\ldots+g^6)\tilde{e}''_5) = 7(\tilde{e}''_5)$ *and the homology is trivial. Thus the matrix of change of basis is*

$$\left(\frac{7\tilde{e}''_5}{\tilde{e}''_5}\right).$$

For $i=6$, *we have that the homology is trivial and* $\varphi(\partial\tilde{e}''_7) = \varphi((g^r-1)\tilde{e}''_6) = 0$, *thus the matrix of change of basis is the identity. For* $i=7$, *we have that* $\widetilde{h''_7} = \lambda''_7\tilde{e}''_7$. *Thus the matrix of change of basis is*

$$\left(\frac{\lambda''_7\tilde{e}''_7}{\tilde{e}''_7}\right),$$

where $\lambda''_7\in\mathbb{Z}$. *Therefore,*

$$\begin{aligned}\tau_{d''_i, h''_i}(\mathscr{C}^{\varphi}(L_q\times\sigma^4, L_q\times\partial\sigma^4)) &= \left|\det\left(\frac{\lambda''_4\tilde{e}_4}{\tilde{e}_4}\right)\right|\left|\det\left(\frac{7\tilde{e}_5}{\tilde{e}_5}\right)\right|^{-1}\left|\det\left(\frac{\lambda''_7\tilde{e}_7}{\tilde{e}_7}\right)\right|^{-1}\\ &= \frac{1}{7}\left|\frac{\lambda''_4}{\lambda''_7}\right|.\end{aligned}$$

Again by Künneth Theorem, we know that

$$H_i(L_q\times\sigma^4)\cong\begin{cases}\mathbb{Z}, & i=0,3,\\ \mathbb{Z}_7, & i=1,\\ 0, & \textit{otherwise}.\end{cases}$$

Moreover, the exact homology sequence is given by

$$\begin{tikzcd}
\cdots \arrow[r] & I^{\mathfrak{m}}H_i(C(L_q\times\partial\sigma^4)) \arrow[r, "i_{*,i}"] & I^{\mathfrak{m}}H_i(X_q) \arrow[r, "p_{*,i}"] & H_i(L_q\times\sigma^4, L_q\times\partial\sigma^4) \arrow[d, "\delta_i"] \\
& & & I^{\mathfrak{m}}H_{i-1}(C(L_q\times\partial\sigma^4)). \arrow[d] \\
& & & \vdots
\end{tikzcd}$$

The three non-trivial cases will be developed below. In the first case, the relevant part of the sequence is

$$0 \longrightarrow I^{\mathfrak{m}}H_7(X_q) \xrightarrow{p_{*,7}} H_7(L_q\times\sigma^4, L_q\times\partial\sigma^4) \longrightarrow 0.$$

Note that

$$I^{\mathfrak{m}}H_i(X_q) \cong \begin{cases} H_i(L_q\times\sigma^4), & i\leq 3,\\ 0, & i=4,\\ H_i(L_q\times\sigma^4, L_q\times\partial\sigma^4), & i>4. \end{cases}$$

Consider the set y_7 in $I^{\mathfrak{m}}H_7(X_q)\cong H_7(L_q\times\sigma^4, L_q\times\partial\sigma^4)\cong\mathbb{Z}$, such that $p_{*,7}(y_7)$ is a basis for $p_{*,7}(I^{\mathfrak{m}}H_i(X_q))$. Taking $y_7=\lambda_7^{X_q}h_7^{X_q}$, with $\lambda_7^{X_q}\in\mathbb{Z}$, we have

$$\det\left(\frac{y_7}{h_7^{X_q}}\right)=\det\left(\frac{\lambda_7^{X_q}h_7^{X_q}}{h_7^{X_q}}\right)=\lambda_7^{X_q}.$$

Moreover,

$$\det\left(\frac{p_{*,7}(y_7)}{h_7''}\right)=\det\left(\frac{\lambda_7^{X_q}\lambda_7''h_7^{X_q}}{h_7''}\right)=\lambda_7^{X_q}\lambda_7''.$$

In the second case, the relevant part of the sequence is

$$0 \longrightarrow H_4(L_q\times\sigma^4, L_q\times\partial\sigma^4) \xrightarrow{\delta_4} I^{\mathfrak{m}}H_3(C(L_q\times\partial\sigma^4)) \xrightarrow{i_{*,3}} I^{\mathfrak{m}}H_3(X_q) \longrightarrow 0.$$

Consider the set y_4'' in $H_4(L_q\times\sigma^4, L_q\times\partial\sigma^4)$, such that $\delta_4(y_4'')$ is a basis for $\delta_4(H_4(L_q\times\sigma^4, L_q\times\partial\sigma^4))$. Taking $y_4''=\lambda_4''h_4''$, we have

$$\det\left(\frac{y_4''}{h_4''}\right)=\det\left(\frac{\lambda_4''h_4''}{h_4''}\right)=\lambda_4''.$$

Moreover, we have that $p_4^{-1}(h_4'')\in C_4(L_q\times\sigma^4)$, and hence $\partial_4(p_4^{-1}(h_4''))$ in $C_3(L_q\times\sigma^4)$ is written as

$$\{\lambda_{3,1}^{C}e_0^{L_q}\times e_3^{\sigma^4}, \lambda_{3,2}^{C}e_3^{L_q}\times e_0^{\sigma^4}\}.$$

Thus, $i_3^{-1}(\partial_4(p_4^{-1}(h_4'')))=\lambda_{3,1}^{C}e_0^{L_q}\times e_3^{\partial\sigma^4}$. Therefore,

$$\delta_4(y_4'')=\lambda_4''\lambda_{3,1}^{C}e_0^{L_q}\times e_3^{\partial\sigma^4}.$$

Now we can choose the set $y_3' = \lambda_{3,2}^C e_3^{L_q} \times e_0^{\partial\sigma^4}$ *in* $I^{\mathrm{m}}H_3(C(L_q \times \partial\sigma^4)) \cong H_3(L_q \times \partial\sigma^4) \cong \mathbb{Z} \oplus \mathbb{Z}$*, such that* $i_{*,3}(y_3')$ *is a basis for* $i_{*,3}(I^{\mathrm{m}}H_3(C(L_q \times \partial\sigma^4)))$*. Thus,* $i_{*,3}(y_3') = \lambda_{3,2}^C e_3^{L_q} \times e_0^{\sigma^4}$*. Considering*

$$h_3 = \{\lambda_{3,1}^C e_0^{L_q} \times e_3^{\partial\sigma^4}, \lambda_{3,2}^C e_3^{L_q} \times e_0^{\partial\sigma^4}\}$$

and

$$h_3^{X_q} = \{\lambda_3^{X_q} e_3^{L_q} \times e_0^{\sigma^4}\},$$

where $\lambda_3^{X_q} \in \mathbb{Z}$*, we obtain that*

$$\det\left(\frac{\delta_4(y_4'')y_3'}{h_3}\right) = \det\begin{pmatrix}\lambda_4'' & 0\\ 0 & 1\end{pmatrix} = \lambda_4'',$$

and

$$\det\left(\frac{i_{*,3}(y_3')}{h_3^{X_q}}\right) = \frac{\lambda_{3,2}^C}{\lambda_3^{X_q}}.$$

In the third case, the relevant part of the sequence is

$$0 \longrightarrow I^{\mathrm{m}}H_0(C(L_q \times \partial\sigma^4)) \xrightarrow{i_{*,0}} I^{\mathrm{m}}H_0(X_q) \longrightarrow 0$$

Consider the set y_0' *in* $I^{\mathrm{m}}H_0(C(L_q \times \partial\sigma^4)) \cong H_0(L_q \times \partial\sigma^4) \cong \mathbb{Z}$*, such that* $i_{*,0}(y_0')$ *is a basis for* $i_{*,0}(I^{\mathrm{m}}H_0(C(L_q \times \partial\sigma^4)))$*. Taking* $y_0' = \lambda_0^C h_0$*, with* $\lambda_0^C \in \mathbb{Z}$*, we have*

$$\det\left(\frac{y_0'}{h_0}\right) = \det\left(\frac{\lambda_0^C h_0}{h_0}\right) = \lambda_0^C.$$

Moreover,

$$\det\left(\frac{i_{*,0}(y_0')}{h_0^{X_q}}\right) = \det\left(\frac{\lambda_0^C \lambda_0^{X_q} h_0^{X_q}}{h_0^{X_q}}\right) = \lambda_0^C \lambda_0^{X_q}.$$

By Theorem 4.3.12, we conclude that

$$\begin{aligned}\tau^{\varphi}_{I^p\ddot{h}_q}(X_q) &= \left|\frac{1}{7}\lambda_0^C \frac{1}{\lambda_{3,1}^C \lambda_{3,2}^C}\right| \left|\frac{1}{7}\frac{\lambda_4''}{\lambda_7''}\right| \left|\lambda_4'' \frac{\lambda_{3,2}^C}{\lambda_4'' \lambda_3^{X_q}} \frac{\lambda_0^C}{\lambda_0^C \lambda_0^{X_q}}\right| \\ &= \frac{1}{7^2}\left|\frac{\lambda_0^C \lambda_4''}{\lambda_{3,1}^C \lambda_7'' \lambda_3^{X_q} \lambda_0^{X_q}}\right|.\end{aligned}$$

4.4 Duality in intersection torsion

Considering the Section 1.3, a chain complex $\mathscr{C}$ of dimension m of vector spaces C_q over a field $\mathbb{F}$ is dualizable (or a Poincaré complex) if there exists a second chain complex $\mathscr{C}'$ of dimension m of vector spaces C^q over the field $\mathbb{F}$, and a chain isomorphism $\mathscr{P}_q\colon C^q \to C^*_{m-q}$. This is equivalent to the existence of a chain isomorphism $\mathscr{Q}_q = \mathscr{P}^*_{m-1}\colon C_q = (C_q^*)^* \to (C^{m-q})^*$. These maps induce the following isomorphism in homology:

$$\mathscr{P}_{*,q}\colon H_q(\mathscr{C}') \to H_{m-q}(\mathscr{C}^*), \text{ and } \mathscr{Q}_{*,q}\colon H_q(\mathscr{C}) \to H_{m-q}(\mathscr{C}')^*.$$

If the spaces C_q of $\mathscr{C}$ have bases c_q, then the spaces C^q naturally have bases $c^q = \mathscr{P}_{*,q}^{-1}(c_{m-q}^*)$. If $H_q(\mathscr{C})$ is free and has preferred basis h_q, then $H_q(\mathscr{C}')$ is also free with basis $h^q = \mathscr{P}_{*,q}^{-1}(h_{m-q}^*)$. Then, by Theorem 3.4.7 the torsion satisfy

$$\tau_{c^q,h^q}(\mathscr{C}') = \tau_{c_q,h_q}(\mathscr{C})^{(-1)^{m+1}}.$$

Consider $X = Y \sqcup_W C(W)$, where Y is a compact connected orientable smooth manifold of dimension $n = m+1$ with boundary W. Let (M,N) be a triangulation of (Y,W). We have the cell decomposition $K = M \sqcup_N C(N)$ of X. We construct a new cell complex X^* that makes the role of the dual decomposition as follows.

Let e be a cell of $M-N$, then denote by e^* the dual cell of e in the dual block decomposition M' of M. The family of these block cells is F'. Let c be a cell of N. Then, we have three block cells associated to c. The first is $\overline{x}'$, the dual block cell of c in $\overline{N}'$, the second is c', the dual block cell of c in M', and the third is the $C(\overline{c}')$, the cone over $\overline{c}'$. The cells $\overline{c}'$ and $C(\overline{c}')$ have part of the boundary in common, and this part of boundary is precisely the cell $\overline{c}'$. We define the dual block cell c^* of c to be the mapping cone $c' \sqcup_j C(\overline{c}')$, where $j\colon \overline{c}' \to c'$ is the inclusion.

The family K^* of the dual block cells c^* for $c \in K$ is a cell decomposition of K, and we have the bijection

$$\mathscr{Q}\colon M \to K^*.$$

This induces a chain isomorphism

$$\mathscr{Q}_q\colon C_q(M) \to C_{m+1-q}(K^*)^*,$$

and therefore an isomorphism in homology

$$\mathscr{Q}_{*,q}\colon H_q(M) = H_q(\mathscr{C}(M)) \to H_{m+1-q}(\mathscr{C}(K^*))^* = H_{m+1-q}(K^*)^*.$$

On other hand, the restriction of $\mathscr{Q}$ on the subcomplex $M-N$ induces a bijection onto the subcomplex F' given by

$$\overline{\mathscr{Q}} = \mathscr{Q}|_{M-N}\colon M-N \to F'.$$

We have the chain isomorphism

$$\overline{\mathscr{Q}}_q\colon C_q(M,N) = C_q(K,C(N)) \to C_{m+1-q}(F')^* = C_{m+1-q}(M')^*,$$

that induces an isomorphism in homology

$$\overline{\mathscr{Q}}_{*,q}\colon H_q(K) = H_q(\mathscr{C}(K,C(N))) \to H_{m+1-q}(M)^* = H_{m+1-q}(\mathscr{C}(M')).$$

In particular, since by Lemma 4.3.4, X has the same homotopy type of Y/W, taking the real coefficients by a real orthogonal representation $\varphi\colon \pi_1(X) \to O(V)$, we have the chain isomorphisms

$$\mathscr{Q}_q\colon C_q^{\varphi}(Y) \to C_{m-q}^{\varphi}(X)^*,$$

$$\overline{\mathscr{Q}}_q \colon C^{\varphi}_q(Y,W) \to C^{\varphi}_{m-q}(Y)^*,$$

inducing isomorphisms in homology

$$\mathscr{Q}_{*,q} \colon H_q(Y) = H_q(\mathscr{C}^{\varphi}(Y)) \to H_{m-q}(X)^* = H_{m-q}(\mathscr{C}^{\varphi}(X))^*,$$

$$\widehat{\mathscr{Q}}_{*,q} \colon H_q(X) = H_q(\mathscr{C}^{\varphi}(X)) \to H_q(Y)^* = H_q(\mathscr{C}^{\varphi}(Y))^*.$$

The maps introduced above permit to define the following chain map between the intersection chain complexes. Let $K = M \sqcup_N C(N)$ be an n-dimensional pseudomanifold without boundary and with an isolated singularity, and let $\mathfrak{p}$ be a perversity. Then, we have a chain map $I^{\mathfrak{p}}\mathscr{Q}_q \colon I^{\mathfrak{p}}\mathscr{C}^{\varphi}(K) \to I^{\mathfrak{p}^c}\mathscr{C}^{\varphi}(K^*)$. The construction of this map is based on the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & I^{\mathfrak{p}}C^{\varphi}_{\mathfrak{a}+1}(K) \arrow[r] \arrow[d, "="] & I^{\mathfrak{p}}C^{\varphi}_{\mathfrak{a}}(K) \arrow[r] \arrow[d, "="] & I^{\mathfrak{p}}C^{\varphi}_{\mathfrak{a}-1}(K) \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & C^{\varphi}_{\mathfrak{a}+1}(M) \arrow[r] \arrow[d, "\mathscr{Q}_{\mathfrak{a}+1}"] & C^{\varphi}_{\mathfrak{a}}(M) \arrow[r] \arrow[d, "\mathscr{Q}_{\mathfrak{a}}"] & C^{\varphi}_{\mathfrak{a}-1}(M,N) \arrow[r] \arrow[d, "\overline{\mathscr{Q}}_{\mathfrak{a}-1}"] & \cdots \\
\cdots \arrow[r] & C^{\varphi}_{m+1-\mathfrak{a}-1}(K^*)^* \arrow[r] \arrow[d, "="] & C^{\varphi}_{m+1-\mathfrak{a}}(K^*)^* \arrow[r] \arrow[d, "="] & C^{\varphi}_{m+1-\mathfrak{a}+1}(M')^* \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & I^{\mathfrak{p}^c}C^{\varphi}_{\mathfrak{a}^c-3}(K^*)^* \arrow[r] & I^{\mathfrak{p}^c}C^{\varphi}_{\mathfrak{a}^c-2}(K^*)^* \arrow[r] & I^{\mathfrak{p}^c}C^{\varphi}_{\mathfrak{a}^c-1}(K^*)^* \arrow[r] & \cdots .
\end{tikzcd}$$

This maps $\mathscr{Q}_q$ induces the isomorphisms in homology

$$I^{\mathfrak{p}}\mathscr{Q}_{*,q} \colon H_q(I^{\mathfrak{p}}\mathscr{C}^{\varphi}(K)) \to H_{n-q}(I^{\mathfrak{p}^c}(K^*))^*.$$

Proposition 4.4.1. *Let N be an m-dimensional connected finite regular CW-complex, and let $\mathfrak{p}$ be a perversity. Then*

$$\tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}\mathscr{C}^{\varphi_0}(C(N))) = \left(\tau_{I^{\mathfrak{p}^c}\dot{c}^{rel,q}, I^{\mathfrak{p}^c}\dot{h}^{rel,q}}(I^{\mathfrak{p}^c}\mathscr{C}^{\varphi_0}(C(N'),N'))\right)^{(-1)^m},$$

where $I^{\mathfrak{p}}\dot{c}^{rel,q}, I^{\mathfrak{p}}\dot{h}^{rel,q}$ are the bases of chain modules and homology modules, respectively, of the relative intersection chain complex $I^{\mathfrak{p}}\mathscr{C}^{\varphi_0}(C(N'),N')$.

Proof. By the construction in Theorem 4.3.11, we have

$$\tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}\mathscr{C}^{\varphi_0}(C(N))) = \tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}C(\mathscr{C}^{\varphi_0}(N))) = \prod_{q=0}^{\mathfrak{a}-2}\left[\frac{h_q}{n_q}\right]^{(-1)^q}\prod_{q=0}^{\mathfrak{a}-2}\left[\frac{b_q\widetilde{n}_q\widetilde{b}_{q-1}}{c_q}\right]^{(-1)^q}.$$

Using the Poincaré isomorphism $I^{\mathfrak{p}}\mathscr{Q}_q$, for $q \leq \mathfrak{a}-2$ we have

$$\begin{aligned}
\tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}\mathscr{C}^{\varphi_0}(C(N))) &= \prod_{q=0}^{\mathfrak{a}-2}\left[\frac{h_q}{n_q}\right]^{(-1)^q}\prod_{q=0}^{\mathfrak{a}-2}\left[\frac{b_q\widetilde{n}_q\widetilde{b}_{q-1}}{c_q}\right]^{(-1)^q} \\
&= \prod_{q=0}^{\mathfrak{a}-2}\left[\frac{h^{m-q}}{n^{m-q}}\right]^{(-1)^{q+1}}\prod_{q=0}^{\mathfrak{a}-2}\left[\frac{b^{m-q}\widetilde{n}^{m-q}\widetilde{b}^{m-q-1}}{c^{m-q}}\right]^{(-1)^{q+1}} \\
&= \prod_{q=m}^{m-\mathfrak{a}+2}\left[\frac{h^q}{n^q}\right]^{(-1)^{m+q+1}}\prod_{q=m}^{m-\mathfrak{a}+2}\left[\frac{b^q\widetilde{n}^q\widetilde{b}^{q-1}}{c^q}\right]^{(-1)^{m+q+1}}.
\end{aligned}$$

Considering $\mathfrak{a}^c = m - \mathfrak{a} + 3$, we obtain

$$\begin{aligned}\tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}\mathscr{C}^{\varphi_0}(C(N))) &= \left(\prod_{q=\mathfrak{a}^c-1}^{m}\left[\frac{h^q}{n^q}\right]^{(-1)^{q+1}}\right)^{(-1)^m}\left(\prod_{q=\mathfrak{a}^c-1}^{m}\left[\frac{b^q\widetilde{n}^q\widetilde{b}^{q-1}}{c^q}\right]^{(-1)^{q+1}}\right)^{(-1)^m}\\ &= \left(\tau_{I^{\mathfrak{p}^c}\dot{c}^{rel,q}, I^{\mathfrak{p}^c}\dot{h}^{rel,q}}(I^{\mathfrak{p}^c}\mathscr{C}^{\varphi_0}(C(N'),N'))\right)^{(-1)^m}.\end{aligned}$$

□

Theorem 4.4.2. *We have that*

$$\tau_{I^{\mathfrak{p}}\ddot{c}_q, I^{\mathfrak{p}}\ddot{h}_q}(I^{\mathfrak{p}}\mathscr{C}^{\varphi}(K)) = \left(\tau_{I^{\mathfrak{p}}\ddot{c}^*_q, I^{\mathfrak{p}}\ddot{h}^*_q}(I^{\mathfrak{p}^c}\mathscr{C}^{\varphi}(K^*))\right)^{(-1)^m}.$$

Proof. By duality of the torsion, we have

$$\tau_{d''_q, h''_q}(\mathscr{C}^{\varphi}(M,N)) = \left(\tau^{\varphi}_{d^q, h^q_{\mathscr{D}}}(\mathscr{C}^{\varphi}(M'))\right)^{(-1)^m}.$$

By Proposition 4.4.1,

$$\tau_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{\mathfrak{p}}\mathscr{C}^{\varphi}(C(N))) = \left(\tau_{I^{\mathfrak{p}^c}\dot{c}^{rel,q}, I^{\mathfrak{p}^c}\dot{h}^{rel,q}}(I^{\mathfrak{p}^c}(\mathscr{C}^{\varphi}(N'),N'))\right)^{(-1)^m}.$$

Then

$$\begin{aligned}\tau^{\varphi}_{I^{\mathfrak{p}}\ddot{c}_q, I^{\mathfrak{p}}\ddot{h}_q}(I^{\mathfrak{p}}\mathscr{C}(K)) &= \tau^{\varphi_0}_{I^{\mathfrak{p}}\dot{c}_q, I^{\mathfrak{p}}\dot{h}_q}(I^{p}\mathscr{C}(C(N)))\tau^{\varphi}_{d''_q, h''_q}(\mathscr{C}(M,N))\tau_{I^{\mathfrak{p}}\overline{h}q}(I^{\mathfrak{p}}\ddot{\mathscr{H}})\\ &= \left(\tau^{\varphi}_{I^{\mathfrak{p}^c}\dot{c}^{rel,q}, I^{\mathfrak{p}^c}\dot{h}^{rel,q}}(I^{\mathfrak{p}^c}\mathscr{C}(C(N'),N'))\right)^{(-1)^m}\left(\tau^{\varphi}_{d^q, h^q_{\mathscr{D}}}(\mathscr{C}(M'))\right)^{(-1)^m}\tau_{I^{\mathfrak{p}}\overline{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}}).\end{aligned}$$

Using the sequence

$$0 \longrightarrow \mathscr{C}(N') \longrightarrow I^{\mathfrak{p}^c}\mathscr{C}(C(N')) \longrightarrow I^{\mathfrak{p}^c}\mathscr{C}(C(N'),N') \longrightarrow 0,$$

by Theorem 2.3.3

$$\tau_{I^{\mathfrak{p}^c}\dot{c}^{rel,q}, I^{\mathfrak{p}^c}\dot{h}^{rel,q}}(I^{\mathfrak{p}^c}\mathscr{C}(C(N'),N')) = \tau^{\varphi}_{I^{\mathfrak{p}^c}\dot{c}^q, I^{\mathfrak{p}^c}\dot{h}^q}(I^{\mathfrak{p}^c}\mathscr{C}(C(N')))\left(\tau^{\varphi}_{c^q, h^q}(\mathscr{C}(N'))\right)^{-1},$$

and using the sequence

$$0 \longrightarrow \mathscr{C}(N') \longrightarrow \mathscr{C}(M') \longrightarrow \mathscr{C}(M',N') \longrightarrow 0,$$

by Theorem 2.3.4

$$\tau^{\varphi}_{d^q, h^q_{\mathscr{D}}}(\mathscr{C}(M')) = \tau^{\varphi}_{d''^q, h''^q}(\mathscr{C}(M',N'))\tau^{\varphi}_{c^q, h^q}(\mathscr{C}(N'))\tau_{\overline{h}_q}(\mathscr{H}).$$

Thus, we have that

$$\begin{aligned}\tau^{\varphi}_{I^{\mathfrak{p}}\ddot{c}_q, I^{\mathfrak{p}}\ddot{h}_q}(I^{\mathfrak{p}}\mathscr{C}(K)) = &\left(\tau^{\varphi}_{I^{\mathfrak{p}^c}\dot{c}^q, I^{\mathfrak{p}^c}\dot{h}^q}(I^{\mathfrak{p}^c}\mathscr{C}(C(N'))) + \tau_{\overline{h}_q}(\mathscr{H})\right)^{(-1)^m}\left(\tau_{d''^q, h''^q}(\mathscr{C}(M',N'))\right)^{(-1)^m}\\ &\tau_{I^{\mathfrak{p}}\ddot{\overline{h}}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}})\end{aligned}$$

Consider the diagram of the homology modules with coefficients in a field $\mathbb{F}$

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(L) \arrow[r, "i_{*,q}"] \arrow[d, "\mathscr{Q}_q"] & H_q(M) \arrow[r, "p_{*,q}"] \arrow[d, "\widehat{\mathscr{Q}}_q"] & H_q(M,L) \arrow[r, "\delta_q"] \arrow[d, "\overline{\mathscr{Q}}_q"] & H_{q-1}(N) \arrow[r] \arrow[d, "\mathscr{Q}_{q-1}"] & \cdots \\
\cdots \arrow[r] & H^{m-q}(L')^* \arrow[r, "(\delta^{m+1-q})^*"'] & H^{m+1-q}(M',N')^* \arrow[r, "(p^{*,m+1-q})^*"'] & H^{m+1-q}(M')^* \arrow[r, "(i^{*,m+1-q})^*"'] & H^{m+1-q}(N')^* \arrow[r] & \cdots,
\end{tikzcd}$$

where $y'_q \in H_q(L)$, $y_q \in H_q(M)$, and $y''_q \in H_q(M,L)$ denote the elements such that its image is a basis for the image of $i_{*,q}$, $p_{*,q}$, and δ_q, respectively. Moreover, $x'_{m-q} \in H^{m-q}(L')^*$, $x''_{m+1-q} \in H^{m+1-q}(M',N')^*$, and $x_{m+1-q} \in H^{m+1-q}(M')^*$ denote the elements such that its images is a basis for the image of $(\delta^{m+1-q})^*$, $(p^{*,m+1-q})^*$, and $(i^{*,m+1-q})^*$, respectively. In addition, consider the bases h_q, $h_q^{\mathscr{Q}}$, and h''_q for $H_q(L)$, $H_q(M)$, and $H_q(M,L)$, respectively, and also their duals corresponding.

By commutativity of the diagram, we have $\overline{\mathscr{Q}}_q \circ p_{*,q}(y_q) = (p^{*,m+1-q})^* \circ \widehat{\mathscr{Q}}_q(y_q)$. This allow us to fix the set of elements

$$(y^{m+1-q})^* = \overline{\mathscr{Q}}_q \circ p_{*,q}(y_q)$$

in $H^{m+1-q}(M')^*$. The notation means that we can choose the elements y^{m+1-q} in $H^{m+1-q}(M')$ in such a way that their duals satisfy the above equation. Then, $((p^{*,m+1-q})^* \circ \widehat{\mathscr{Q}}_q(y_q))(y^{m+1-q}) = 1$, we have that $\widehat{\mathscr{Q}}_q(y_q)$ are the duals of the $p^{*,m+1-q}(y^{m+1-q})$, i.e.,

$$\widehat{\mathscr{Q}}_q(y_q) = (p^{*,m+1-q}(y^{m+1-q}))^*.$$

In a similar way, by commutativity of the diagram we have $\widehat{\mathscr{Q}}_q \circ i_{*,q}(y_q) = (\delta^{m+1-q})^* \circ \mathscr{Q}_q(y'_q)$, then we fix $y''_{m+1-q} = \widehat{\mathscr{Q}}_q \circ i_{*,q}(y'_q)$ in $H^{m+1-q}(M',N')^*$. Since $((\delta^{m+1-q})^* \circ Q_q(y'_q))(y''^{m+1-q}) = 1$, we have $\mathscr{Q}_q(y'_q) = (\delta^{m+1-q}(y''^{m+1-q}))^*$. And eventually, $\mathscr{Q}_{q-1} \circ \delta(y''_q) = (i^{*,m+1-q})^* \circ \overline{\mathscr{Q}}_q(y''_q)$, so that fixing $(y'^{m+1-q})^* = \mathscr{Q}_{q-1} \circ \delta_q(y''_q)$ in $H^{m+1-q}(N')^*$, since $((i^{*,m+1-q})^* \circ \overline{\mathscr{Q}}_q(y''_q))(y'^{m+1-q}) = 1$, it follows that $\overline{\mathscr{Q}}_q(y''_q) = ((i^{*,m+1-q})^*(y'^{m+1-q}))^*$.

Now we will analyse the torsion. We may choose

$$\begin{aligned}
x_{m+1-q} &= \overline{\mathscr{Q}}_q(y''_q), \\
x'_{m+1-q} &= \mathscr{Q}_q(y'_q), \\
x''_{m+1-q} &= \widehat{\mathscr{Q}}_q(y_q).
\end{aligned}$$

Thus the matrix representing the change of bases in $H^{m+1-q}(N')^*$, on the one hand is

$$\begin{aligned}
\left(\frac{((i^{*,m+1-q})^*(x_{m+1-q}))x'_{m+1-q}}{(h^{m+1-q})^*}\right) &= \left(\frac{(i^{*,m+1-q} \circ \overline{\mathscr{Q}}_q(y''_q))\mathscr{Q}_{q-1}(y'_{q-1})}{\mathscr{Q}_{q-1}(h_{q-1})}\right) \\
&= \left(\frac{(\overline{\mathscr{Q}}_{q-1} \circ \delta_q(y''_q))\mathscr{Q}_{q-1}(y'_{q-1})}{\mathscr{Q}_{q-1}(h_{q-1})}\right) \\
&= \left(\frac{\delta_q(y''_q)y'_{q-1})}{h_{q-1}}\right),
\end{aligned}$$

and on the other

$$\begin{aligned}
\left(\frac{((i^{*,m+1-q})^{*}(x_{m+1-q}))x'_{m+1-q}}{(h^{m+1-q})^{*}}\right) &= \left(\frac{(y'^{m+1-q})^{*}(\delta^{m+2-q}(y''^{m+2-q}))^{*}}{(h^{m+q-1})^{*}}\right)\\
&= \left(\left(\frac{y'^{m+1-q}\delta^{m+2-q}(y''^{m+2-q})}{h^{m+q-1}}\right)^{T}\right)^{-1}.
\end{aligned}$$

For the matrix representing the change of bases in $H^{m+1-q}(M')^{*}$, we have

$$\begin{aligned}
\left(\frac{((p^{*,m+1-q})^{*}(x''_{m+1-q}))x_{m+1-q}}{(h_{\mathscr{D}}^{m+1-q})^{*}}\right) &= \left(\frac{((p^{*,m+1-q})^{*}\circ\widehat{\mathscr{Q}}_q(y_q))\overline{\mathscr{Q}}_q(y''_q)}{\overline{\mathscr{Q}}_q(h''_q)}\right)\\
&= \left(\frac{(\overline{\mathscr{Q}}_q\circ p_{*,q}(y_q))\overline{\mathscr{Q}}_q(y''_q)}{\overline{\mathscr{Q}}_q(h''_q)}\right)\\
&= \left(\frac{p_{*,q}(y_q)y''_q}{h''_q}\right)\\
&= \left(\frac{(y^{m+1-q})^{*}(i^{*,m+1-q}(y'^{m+1-q}))^{*}}{(h_{\mathscr{D}}^{m+1-q})^{*}}\right)\\
&= \left(\left(\frac{y^{m+1-q}i^{*,m+1-q}(y'^{m+1-q})}{h_{\mathscr{D}}^{m+1-q}}\right)^{T}\right)^{-1}.
\end{aligned}$$

For the matrix representing the change of bases in $H^{m+1-q}(M',N')^{*}$, we have

$$\begin{aligned}
\left(\frac{((\delta^{*,m+1-q})^{*}(x'_{m+1-q}))x''_{m+1-q}}{(h''^{m+1-q})^{*}}\right) &= \left(\frac{((\delta^{*,m+1-q})^{*}\circ\mathscr{Q}_q(y'_q))\widehat{\mathscr{Q}}_q(y_q)}{\widehat{\mathscr{Q}}_q(h_q^{\mathscr{D}})}\right)\\
&= \left(\frac{(\widehat{\mathscr{Q}}_q\circ i_{*,q}(y'_q))\widehat{\mathscr{Q}}_q(y_q)}{\widehat{\mathscr{Q}}_q(h_q^{\mathscr{D}})}\right)\\
&= \left(\frac{i_{*,q}(y'_q)y_q}{h_q^{\mathscr{D}}}\right)\\
&= \left(\frac{(y''^{m+1-q})^{*}(p^{*,m+1-q}(y^{m+1-q}))^{*}}{(h''^{m+1-q})^{*}}\right)\\
&= \left(\left(\frac{y''^{m+1-q}p^{*,m+1-q}(y^{m+1-q})}{h''^{m+1-q}}\right)^{T}\right)^{-1}.
\end{aligned}$$

Then we can calculate the torsion of the dual homology complex.

$$
\begin{aligned}
\tau_{\bar{h}_q}(\mathscr{H}) &= \prod_{q=0}^{m+1} \left(\left[\frac{\delta^{q+1}(y''^{q+1})y'^{q}}{h^q} \right] \left[\frac{i^{*,q}(y'^q)y^q}{h^q_{\mathscr{D}}} \right]^{-1} \left[\frac{p^{*,q}(y^q)y''^q}{h''^q} \right] \right)^{(-1)^q} \\
&= \prod_{q=0}^{m+1} \left(\left[\frac{\delta_{m+1-q}(y''_{m+1-q})y'_{m-q}}{h_{m-q}} \right] \left[\frac{i_{*,m+1-q}(y'_{m+1-q})y_{m+1-q}}{h^{\mathscr{D}}_{m+1-q}} \right]^{-1} \right. \\
&\qquad \left. \left[\frac{p_{*,m+1-q}(y_{m+1-q})y''_{m+1-q}}{h''_{m+1-q}} \right] \right)^{(-1)^{q+1}} \\
&= \left(\prod_{q=0}^{m+1} \left(\left[\frac{\delta_q(y''_q)y'_{q-1})}{h_{q-1}} \right] \left[\frac{i_{*,q}(y'_q)y_q}{h^{\mathscr{D}}_q} \right]^{-1} \left[\frac{p_{*,q}(y_q)y''_q}{h''_q} \right] \right)^{(-1)^q} \right)^{(-1)^m} .
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\tau_{I^{\mathfrak{p}}\bar{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}}) &= \prod_{q=0}^{\mathfrak{a}-2} \left(\left[\frac{\delta_{q+1}(y''_{q+1})y'_q}{h_q} \right] \left[\frac{i_{*,q}(y'_q)y_q}{h^{\mathscr{D}}_q} \right]^{-1} \left[\frac{p_{*,q}(y_q)y''_q}{h''_q} \right] \right)^{(-1)^q} \\
&\qquad \left[\frac{p_{*,\mathfrak{a}-1}(y_{\mathfrak{a}-1})y''_{\mathfrak{a}-1}}{h''_{\mathfrak{a}-1}} \right]^{(-1)^{\mathfrak{a}-1}} \\
&= \prod_{q=0}^{\mathfrak{a}-2} \left(\left[\frac{y'^{m-q}\delta^{m+1-q}(y''^{m+1-q})}{h^{m-q}} \right] \left[\frac{y''^{m+1-q}p^{*,m+1-q}(y^{m+1-q})}{h''^{m+1-q}} \right]^{-1} \right. \\
&\qquad \left. \left[\frac{y^{m+1-q}i^{*,m+1-q}(y'^{m+1-q})}{h_{\mathscr{D}}^{m+1-q}} \right] \right)^{(-1)^{q+1}} \left[\frac{y^{m+2-a}i^{*,m+2-\mathfrak{a}}(y'^{m+2-\mathfrak{a}})}{h_{\mathscr{D}}^{m+2-\mathfrak{a}}} \right]^{(-1)^{\mathfrak{a}}} \\
&= \left[\frac{y^{m+2-\mathfrak{a}}i^{*,m+2-\mathfrak{a}}(y'^{m+2-\mathfrak{a}})}{h_{\mathscr{D}}^{m+2-\mathfrak{a}}} \right]^{(-1)^{\mathfrak{a}}} \left(\prod_{q=m+3-\mathfrak{a}}^{m+1} \left(\left[\frac{y'^{q-1}\delta^q(y''^q)}{h^{q-1}} \right] \right. \right. \\
&\qquad \left. \left. \left[\frac{y''^q p^{*,q}(y^q)}{h''^q} \right]^{-1} \left[\frac{y^q i^{*,q}(y'^q)}{h^q_{\mathscr{D}}} \right] \right)^{(-1)^q} \right)^{(-1)^m} .
\end{aligned}
$$

Thus, we have that

$$
\begin{aligned}
\tau_{\bar{h}_q}(\mathscr{H})^{(-1)^m}\tau_{I^{\mathfrak{p}}\bar{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}}) = {} & \left(\prod_{q=0}^{m+1}\left(\left[\frac{\delta^{q+1}(y''^{q+1})y'^q}{h^q}\right]\left[\frac{i^{*,q}(y'^q)y_q}{h^q_{\mathscr{D}}}\right]^{-1}\left[\frac{p^{*,q}(y^q)y''^q}{h''^q}\right]\right)^{(-1)^q}\right)^{(-1)^m}\\
& \left[\frac{y^{m+2-\mathfrak{a}}i^{*,m+2-\mathfrak{a}}(y'^{m+2-\mathfrak{a}})}{h_{\mathscr{D}}^{m+2-\mathfrak{a}}}\right]^{(-1)^{\mathfrak{a}}}\\
& \left(\prod_{q=m+3-\mathfrak{a}}^{m+1}\left(\left[\frac{y'^{q-1}\delta^q(y''^q)}{h^{q-1}}\right]\left[\frac{y''^q p^{*,q}(y^q)}{h''^q}\right]^{-1}\right.\right.\\
& \left.\left.\left[\frac{y^q i^{*,q}(y'^q)}{h^q_{\mathscr{D}}}\right]\right)^{(-1)^q}\right)^{(-1)^m}\\
= {} & \left(\prod_{q=0}^{m+1}\left(\left[\frac{\delta^{q+1}(y''^{q+1})y'^q)}{h^q}\right]\left[\frac{i^{*,q}(y'^q)y^q}{h^q_{\mathscr{D}}}\right]^{-1}\left[\frac{p^{*,q}(y^q)y''^q}{h''^q}\right]\right)^{(-1)^q}\right)^{(-1)^m}\\
& \left[\frac{y^{m+2-\mathfrak{a}}i^{*,m+2-\mathfrak{a}}(y'^{m+2-\mathfrak{a}})}{h_{\mathscr{D}}^{m+2-\mathfrak{a}}}\right]^{(-1)^{\mathfrak{a}}}\left[\frac{y'^{m+2-\mathfrak{a}}\delta^{m+3-\mathfrak{a}}(y''^{m+3-\mathfrak{a}})}{h^{m+2-\mathfrak{a}}}\right]^{(-1)^{1-\mathfrak{a}}}\\
& \left(\prod_{q=m+3-\mathfrak{a}}^{m+1}\left(\left[\frac{y'^q\delta^{q+1}(y''^{q+1})}{h^q}\right]\left[\frac{y''^q p^{*,q}(y^q)}{h''^q}\right]^{-1}\right.\right.\\
& \left.\left.\left[\frac{y^q i^{*,q}(y'^q)}{h^q_{\mathscr{D}}}\right]\right)^{(-1)^q}\right)^{(-1)^m}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\tau_{\bar{h}_q}(\mathscr{H})^{(-1)^m}\tau_{I^{\mathfrak{p}}\bar{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}}) = {} & \left(\prod_{q=0}^{m+2-\mathfrak{a}}\left(\left[\frac{\delta^{q+1}(y''^{q+1})y'^q}{h^q}\right]\left[\frac{i^{*,q}(y'^q)y^q}{h^q_{\mathscr{D}}}\right]^{-1}\right.\right.\\
& \left.\left.\left[\frac{p^{*,q}(y^q)y''^q}{h''^q}\right]\right)^{(-1)^q}\right)^{(-1)^m}\left[\frac{y^{m+2-\mathfrak{a}}i^{*,m+2-\mathfrak{a}}(y'^{m+2-\mathfrak{a}})}{h_{\mathscr{D}}^{m+2-\mathfrak{a}}}\right]^{(-1)^{\mathfrak{a}}}\\
& \left[\frac{y'^{m+2-\mathfrak{a}}\delta^{m+3-\mathfrak{a}}(y''^{m+3-\mathfrak{a}})}{h^{m+2-\mathfrak{a}}}\right]^{(-1)^{1-\mathfrak{a}}}\\
= {} & \left(\prod_{q=0}^{m+1-\mathfrak{a}}\left(\left[\frac{\delta^{q+1}(y''^{q+1})y'^q}{h^q}\right]\left[\frac{i^{*,q}(y'^q)y^q}{h^q_{\mathscr{D}}}\right]^{-1}\right.\right.\\
& \left.\left.\left[\frac{p^{*,q}(y^q)y''^q}{h''^q}\right]\right)^{(-1)^q}\right)^{(-1)^m}\left[\frac{p^{*,m+2-\mathfrak{a}}(y^{m+2-\mathfrak{a}})y''^{m+2-\mathfrak{a}}}{h''^{m+2-\mathfrak{a}}}\right]^{(-1)^{m+2-\mathfrak{a}}}.
\end{aligned}
$$

Therefore, we conclude that

$$\tau_{\overline{h}_q}(\mathscr{H})^{(-1)^m}\tau_{I^{\mathfrak{p}}\overline{h}_q}(I^{\mathfrak{p}}\ddot{\mathscr{H}}) = \left(\prod_{q=0}^{\mathfrak{a}^c-2}\left(\left[\frac{\delta^{q+1}(y''^{q+1})y'^{q}}{h^q}\right]\left[\frac{i^{*,q}(y'^{q})y^{q}}{h^q_{\mathscr{D}}}\right]^{-1}\right.\right.$$
$$\left.\left.\left[\frac{p^{*,q}(y^{q})y''^{q}}{h''^{q}}\right]\right)^{(-1)^q}\right)^{(-1)^m}\left[\frac{p^{*,\mathfrak{a}^c-1}(y^{\mathfrak{a}^c-1})y''^{\mathfrak{a}^c-1}}{h''^{\mathfrak{a}^c-1}}\right]^{(-1)^{\mathfrak{a}^c-1}},$$

where $\mathfrak{a}^c = m+1-\mathfrak{p}^c(m+1) = \mathfrak{p}(m+1)+2$. It follows the desired result. □

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