UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

On Quaternionic Projective Product Spaces, Bourgin–Yang Theorems and Parametrized Borsuk–Ulam Theorems

Gabriel de Oliveira Lucena

São Carlos-SP
Fevereiro de 2026

UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

On Quaternionic Projective Product Spaces, Bourgin–Yang Theorems and Parametrized Borsuk–Ulam Theorems

Gabriel de Oliveira Lucena

Orientador(a): Prof. Dr. Edivaldo Lopes dos Santos

Tese apresentada ao Programa de Pós-Graduação em Matemática da Universidade Federal de São Carlos como parte dos requisitos para a obtenção do Título de Doutor em Matemática.

São Carlos-SP
Fevereiro de 2026

UNIVERSIDADE FEDERAL DE SÃO CARLOS

Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Matemática

Folha de Aprovação

Defesa de Tese de Doutorado do candidato Gabriel de Oliveira Lucena, realizada em 10/02/2026.

Comissão Julgadora:

Prof. Dr. Edivaldo Lopes dos Santos (UFSCar)

Prof. Dr. Pedro Luiz Queiroz Pergher (UFSCar)

Profa. Dra. Alice Kimie Miwa Libardi (UNESP)

Prof. Dr. Weslem Liberato Silva (UESC)

Prof. Dr. Thiago de Melo (UNESP)

O Relatório de Defesa assinado pelos membros da Comissão Julgadora encontra-se arquivado junto ao Programa de Pós-Graduação em Matemática.

À Teresinha, ao Francisco e ao Guilherme.

Agradecimentos

Agradeço aos meus pais, Teresinha e Francisco, por todo amor que me deram e valores de respeito e amor ao próximo que me ensinaram. Foi o suporte deles, através de muito suor e preocupação, que permitiu que eu tivesse a tranquilidade e conforto necessários para dedicar tantos anos ao estudo. São eles os maiores responsáveis por este trabalho.

Agradeço ao meu irmão, Guilherme, meu melhor amigo. Mesmo sobrecarregado com estudo e trabalho, sempre esteve disposto a revisar o texto dessa tese e dar conselhos de inglês. Agradeço também à Joyce, sua esposa, pelo amor que tem pelo meu irmão e pela grande amizade com a minha mãe.

Agradeço aos meus primos, tios, e todos da família. Em particular, à minha madrinha Conceição, ao meu padrinho Sérgio, e ao meu primo Alan, que concluiu doutorado em 2023, hoje é um importante pesquisador e uma inspiração para eu continuar "calculando curvas imaginarias que não existem".

Agradeço aos meus amigos de escola, Alexia, Douglas, Luis, Murilo, Noelle, Thiago Barata, Thiago Rios, Yoku, Zé e muitos outros. O contato que mantivemos online foi o que me permitiu manter a sanidade tantos anos morando sozinho longe de casa. Agradeço também aos amigos que foram se juntando a esse grupo ao longo do tempo.

Agradeço aos meus amigos da UNESP de Rio Claro, André, Brian, Bruno, Guilherme, Isabela, Lucas, Luana, Luis, Marco, Pedro, Rafael, Rodrigo, Tais, Victor e muitos outros. Agradeço aos meus amigos da licenciatura na USP, Ana Clara, Ana Julia, Anderson, Cainã, Gustavo, Ivan, Jackson, Luisa, Vinícius e muitos outros.

Agradeço a todos os meus amigos da UFSCar. Entre eles, destaco os mais frequentes companheiros de cerveja dos últimos anos, Hindi, Iago, Lejzer, Maurício, Samara e Stefani. Agradeço também ao Thales, que me deu a oportunidade de estudar e trabalhar com ele, e muito valorizou meus comentários, mesmo eu não tendo contribuído tanto quanto eu gostaria.

Agradeço aos meus professores de ensino fundamental e médio. Em especial, à Regina, a primeira a me despertar o interesse por matemática, à Leda, pelo carinho que tinha por mim, à Rosana, pela grande amizade que se estendeu para além da escola, e aos meus professores de matemática, Maria de Lurdes, Alessandro, Ana Maria, Luciano e Auriluci.

Agradeço à Olimpíada Brasileira de Matemática das Escolas Públicas, pela oportunidade de participar do Programa de Iniciação Científica Jr., a todos os amigos que conheci lá, e ao professor Marco. Agradeço também pela oportunidade de participar do Programa de Iniciação Científica e Mestrado, que me deu suporte ao longo de quase toda meu tempo na universidade.

Agradeço a todos os professores do Departamento de Matemática da UNESP de Rio Claro, e também professores de outros departamentos com quem tive contato ao longo do curso. Em especial, à Suzete, pela ajuda em momentos difíceis, à Alice, pela orientação em iniciação científica e ao Thiago pela orientação em iniciação científica e mestrado.

Agradeço aos professores do curso de licenciatura em matemática da USP de São Carlos. Em especial, ao Denner e à Michela.

Agradeço aos professores do Programa de Pós-Graduação em Matemática da UFSCar. Em especial, meu orientador, Edivaldo.

O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Código de Financiamento 001

Resumo

Nós calculamos o anel de cohomologia dos espaços produtos projetivos quaternionicos $\mathbb{H}P_{\overline{n}} = S^{4n_1+3} \times \cdots \times S^{4n_r+3}/S^3$. Usando cálculos similares dos anéis de cohomologia do espaço produto projetivo, devido a Davis, e dos espaços lens produto e espaços produtos projetivos complexos, devidos a González e Velasco, provamos um teorema de Bourgin–Yang para aplicações $f\colon S^{2n_1+1} \times \cdots \times S^{2n_r+1} \to \mathbb{R}^m$ com ação de $\mathbb{Z}_p$ e, para cada grupo $G = \mathbb{Z}_2, \mathbb{Z}_p, S^1$ e S^3, provamos teoremas de Borsuk–Ulam parametrizados para aplicações G-equivariantes de fibrados $f\colon E \to E'$, onde $F \to E \to B$ é um fibrado com ação de G, $E' \to B$ é um fibrado vetorial com ação de G, e F é um produto de esferas. Expandindo as técnicas usadas nas provas, provamos um teorema de Borsuk–Ulam parametrizado geral, para fibrados arbitrários com ação de um grupo G arbitrário. Usamos esse teorema geral para obter diversos teoremas de Borsuk–Ulam parametrizados e de Bourgin–Yang.

Palavras-chave: espaços produto projetivos, aplicação equivariante, dimensão cohomológica, teoremas do tipo Borsuk–Ulam.

Abstract

We compute the cohomology ring for the quaternionic projective product spaces $\mathbb{H}P_{\bar{n}} = S^{4n_1+3} \times \cdots \times S^{4n_r+3}/S^3$. Using similar computations by Davis for the cohomology ring for the projective product spaces, and by González and Velasco for the lens product spaces and complex projective product spaces, we prove a Bourgin–Yang Theorem for maps $f\colon S^{2n_1+1} \times \cdots \times S^{2n_r+1} \to \mathbb{R}^m$ with action of $\mathbb{Z}_p$ and, for each group $G = \mathbb{Z}_2, \mathbb{Z}_p, S^1$ and S^3, we prove a parametrized Borsuk–Ulam theorem for G-equivariant bundle maps $f\colon E \to E'$ where $F \to E \to B$ is fiber bundle with action of G, $E' \to B$ is a vector bundle with action of G, and F is a product of spheres. Expanding on the techniques used in the proofs, we prove a general parametrized Borsuk–Ulam Theorem, for arbitrary fiber bundles with action of an arbitrary group G. We use this general theorem to obtain several parametrized Borsuk–Ulam and Bourgin–Yang Theorems.

Keywords: projective product spaces, equivariant map, cohomological dimension, Borsuk–Ulam type theorems.

Contents

List of Symbols

$X \cup Y$: union of sets;
$X \cap Y$: intersection of sets;
$X \setminus Y$: set difference $\{x \in X \mid x \notin Y\}$;

$\mathbb{Z}$: set of integers;
$\mathbb{R}$: set of real numbers;
$\mathbb{C}$: set of complex numbers;
$\mathbb{H}$: set of quaternion numbers;
$\mathbb{Z}_p$: set of integers mod p;

S^n: n-dimensional sphere;
$\mathbb{R}P^n$: projective space $S^n/\mathbb{Z}_2$;
$L_p(n)$: lens space $S^{2n+1}/\mathbb{Z}_p$;
$\mathbb{C}P^n$: complex projective space S^{2n+1}/S^1;
$\mathbb{H}P^n$: quaternionic projective space S^{4n+3}/S^3;
$\mathbb{R}P_{\overline{n}}$: projective product space $S^{n_1} \times \cdots \times S^{n_r}/\mathbb{Z}_2$;
$L_p(\overline{n})$: lens product space $S^{2n_1+1} \times \cdots \times S^{2n_r+1}/\mathbb{Z}_p$;
$\mathbb{C}P_{\overline{n}}$: complex projective product space $S^{2n_1+1} \times \cdots \times S^{2n_r+1}/S^1$;
$\mathbb{H}P_{\overline{n}}$: quaternionic projective product space $S^{4n_1+3} \times \cdots \times S^{4n_r+3}/S^3$;

$H_n(X;R)$: n-th singular homology group of X with coefficients in R;
$H^n(X;R)$: n-th cohomology group of X with coefficients in R, taken to be Čech cohomology, unless otherwise stated;
$H^*(X;R)$: cohomology ring of X with coefficients in R, taken to be Čech cohomology, unless otherwise stated;
$H_n(X)$: homology with coefficient ring omitted, to be determined by context;
$H^n(X;R)$: cohomology with coefficient ring ommited, to be determined by context;
f^*: induced map in cohomology;
cohom dim(X): For a space X, $\max\{n \in \mathbb{Z} \cup \{\infty\} \mid H^n(X) \neq 0\}$;
$x \smile y = xy$: cup product of $x, y \in H^*(X)$;
$\pi_n(X)$: n-th homotopy group of X;

$X \simeq Y$: isomorphism;
$R[x_1, \ldots, x_r]$: polynomial ring;
$\langle x_1, \ldots, x_r \rangle$: ideal generated by $x_1, \ldots, x_r$;
$\Lambda[x_1, \ldots, x_r]$: exterior algebra;
$R[G]$: group ring for a ring R and group G;
$I_R(G)$: augmentation ideal for a ring R and group G;
$\dim(x)$: integer for which $x \in \bigoplus_{n=1}^{\dim(x)} A^n$ where A is a graded algebra;
$R \oplus S$: direct sum;
$V \otimes W$: tensor product;
$GL_n(\mathbb{R})$: general linear group;

$\overline{X} = X/G$: orbit space of a space X with action of a group G;
$\mathrm{Fix}(X)$: For a space X with action of a group G, $\{x \in X \mid gx = x \text{ for some } g \in G, g \neq 1\}$;
$\mathrm{Free}(X)$: For a space X with action of a group G, $\{x \in X \mid gx = x \implies g = 1\}$;
$X \times_G Y$: Orbit space $X \times Y/G$ where G acts diagonally;
$f|_A$: Restriction of a function $f \colon X \to Y$ to the subset $A \subset X$;

E_G: universal space for the group G;
B_G: classifying space for the group G;
SE: sphere bundle for the bundle $E \to B$;
$\mathbb{C}p$: complexification of the vector bundle $p \colon E \to B$.
$p \oplus q$: whitney sum of the bundles p and q;
$e(p)$: Euler class;

Introduction

When standing on a long straight road, despite the sides of the road being parallel lines, perspective gives us the illusion of these lines meeting at the horizon. This illusion is studied by projective geometry and can be modeled, after some abstraction, by the *projetive plane*, which consists of the sphere with its opposite points “glued together”. It turns out this space has topological properties that make it interesting by itself, which has motivated mathematicians to study and generalize it to settings beyond the original problem it modeled.

Immediately, we can consider higher dimensional spheres to get *projective spaces* of higher dimensions. Also, the gluing of opposite points can be described as the orbit space of a $\mathbb{Z}_2 = \{-1,1\}$ action on the sphere, called antipodal action. This opens the way to consider $\mathbb{Z}_p$, S^1 and S^3 actions on the sphere, which gives us the lens, complex projective and quaternionic projective spaces respectively.

In 2010, Davis [5] considered products of spheres and generalized the projective spaces to the *projective product spaces* $S^{n_1} \times \cdots \times S^{n_r}/\mathbb{Z}_2$, where the action is the diagonal action for the antipodal action, that is, the action given by $-1(x_1,\ldots,x_r) = (-x_1,\ldots,-x_r)$. He studied this spaces and computed their cohomology rings. After Davis, in 2014, González and Velasco [10] computed the cohomology rings for the lens product spaces $S^{2n_1+1} \times \cdots \times S^{2n_r+1}/\mathbb{Z}_p$ and complex projective product spaces $S^{2n_1+1} \times \cdots \times S^{2n_r+1}/S^1$.

A natural continuation would be to consider the quaternionic projective product spaces $S^{4n_1+3} \times \cdots \times S^{4n_r+3}/S^3$. We started the work on this thesis with the initial goal of studying these spaces. We computed their cohoomlogy rings and present it in Chapter 2.

Knowing the cohomology ring for a space opens the way for many applications. An interesting one is that of Borsuk–Ulam theorems. The original Borsuk–Ulam theorem states that for a map $f\colon S^n \to \mathbb{R}^n$ there is a $x \in S^n$ such that $f(x) = f(-x)$. This was extended by C. T. Yang [20] and D. G. Bourgin [3] to a theorem that considers any involution T on the sphere and also estimates the dimension of the space $Z_f = \{x \in S^n \mid f(x) = f(T(x))\}$.

In 2015, de Mattos, Pergher, dos Santos and Singh [13] used Davis’ computation of the cohomology ring for projective product spaces to prove a version of this theorem for maps from a product of spheres with diagonal $\mathbb{Z}_2$ action. In Chapter 3, using the cohomology ring of the lens product spaces computed by González and Velasco, we prove a similar result, for a $\mathbb{Z}_p$ action on a product os spheres. We were not able to continue extending this to S^1 and S^3 actions, as the method we use in the proof relies on the group being finite. There is, however, another type of Borsuk–Ulam theorem for which we did not encounter the same difficulty.

In 1988, Dold proved a parametrized Borsuk–Ulam Theorem, a version of the Borsuk–Ulam Theorem that considered $\mathbb{Z}_2$-equivariant bundle maps, from the total of a sphere bundle to the total space of a vector bundle and, instead of a coincidence point set, considered the zero set of this map. This is very different from the original Borsuk–Ulam Theorem, but can still produce it as a particular case if we consider the base space to be a single point and take the map to be given by $f(x) = g(x) - g(-x)$ for a continuous function g. Many versions of this theorem exist, changing $\mathbb{Z}_2$ to some compact Lie group G and the vector bundle to a fiber bundle $F \to E \to B$. For example, de Mattos, Pergher, dos Santos and Singh [13] considered bundles with fiber $F = S^{n_1} \times \cdots \times S^{n_r}$ and diagonal action of $\mathbb{Z}_2$.

In Chapter 4, we proved a few theorems similar to this. In the first section, we proved versions considering actions of $\mathbb{Z}_p$, S^1 and S^3. We were also able to obtain a better estimate for the $\mathbb{Z}_2$ case. When improving the proofs, we ended up making them less dependent on the cohomology of the spaces involved. This enabled us to prove a more general version of this theorem, which we present in the second section of Chapter 4, that applies to an arbitrary fiber bundle. We then obtain the parametrized Borsuk–Ulam theorems we already mentioned as particular cases of this more general result, together with some new ones.

The results in Chapters 2 and 3, as well as Section 4.2 were submitted as a paper, which we hope will be published soon. We also plan to submit the more general results in the later sections of Chapter 4 as another paper.

We assume the reader is familiar with homology, cohomology and homotopy groups, and is comfortable with all the prerequisites for those, including groups, rings, vector spaces, general topology, etc. In particular, assume that the reader knows Čech cohomology.

We will also use a lot of fiber bundles and group actions. As a quick introduction to them, we have Chapter 1, which acts as a preliminaries chapter for this thesis.

CHAPTER 1

Group Actions and Bundles

The purpose of this chapter is to give a quick presentation on group actions and bundles, which are used in all of the later chapters.

1.1 Group Actions

Definition 1.1. A left *group action*, or simply an action, of a topological group G on a topological space X is a map $\alpha\colon G\times X\to X$, where we denote $\alpha(g,x)=gx$, such that:

- $ex=x$, for all $x\in X$,
- $g_1(g_2x)=(g_1g_2)x$, for all $x\in X$, $g_1,g_2\in G$.

A left *G-space*, or simply *G-space*, is a topological space together with a left group action of G on it.

Similarly, a right group action is a map $\alpha\colon X\times G\to X$ such that $xe=x$ and $(xg_1)g_2=x(g_1g_2)$ for every $x\in X$ and $g_1,g_2\in G$. A right G-space is a topological space together with a right group action of G on it.

The maps $L_g\colon X\to X$, $x\mapsto gx$, are called *left translations*. We will denote them simply by g if convenient. Right translations are defined similarly.

The left translations L_g are homeomorphisms, with inverse $L_{g^{-1}}$. Because of this, actions are sometimes defined to be a group homomorphism $G\to \mathrm{Homeo}(X)$, which in our case would be $g\mapsto L_g$, and the map $G\times X\to X$ can be determined from that.

The morphisms for G-spaces are defined as one would expect.

Definition 1.2. A map $f\colon X\to Y$ between G-spaces X and Y is *G-equivariant*, or a *G-map*, if

$$f(gx)=gf(x)$$

for any $g\in G$ and $x\in X$.

Definition 1.3. Let G be a topological group with neutral element e and X be a G-space.

- We say the action of G on X is *effective*, or that X is an effective G-space if $L_g = \mathrm{id}_X \implies g = e$ or, equivalently, if the kernel of $g \mapsto L_g$ is $\{e\}$. This means that, if $g \neq e$, there is some $x \in X$ such that $gx \neq x$.

- We say the action of G on X is *free*, or that X is a free G-space if, for any $g \in G$ with $g \neq e$ and $x \in X$ we have $gx \neq x$.

- If $A \subset X$, we say that A is *G-invariant* if $ga \in A$ for every $g \in G$ and $a \in A$.

- Denote $\mathrm{Free}(X) = \{x \in X \mid gx \neq x \text{ for all } g \neq e\}$ and $\mathrm{Fix}(X) = \{x \in X \mid gx = x \text{ for some } g \neq e\}$.

The set $\mathrm{Free}(X) = X \setminus \mathrm{Fix}(X)$ is precisely the subset of X where the action of G is free. The set $\mathrm{Fix}(X)$, on the other hand, is the set of points of X that are fixed by some element $g \neq 1$. Such points are sometimes called fixed points of the action, but this term is more commonly used to describe points that are fixed by all elements of G. For this reason, we will avoid calling them fixed points.

Since the relation $x \sim gx$ is an equivalence relation on X, it makes sense to give the correspondent quotient space a name.

Definition 1.4. Let G be a topological group and X a G-space. Then the quotient $X/\sim$ of the equivalence relation $x \sim gx$ is called *orbit space* of X and denoted X/G, or $\overline{X}$ if the group is clear from context. The elements $[x] \in X/G$ are called orbits through x.

Notice that G-equivariant maps $X \to Y$ induce maps $\overline{X} \to \overline{Y}$. The X/G notation is standard in the literature, while $\overline{X}$ isn't, and can mean very different things in different texts. It is, however, a bit shorter and more convenient, and we will use it a lot.

For any space X, denote $\mathrm{cohom\,dim}(X) = \max\{n \in \mathbb{Z} \cup \{\infty\} \mid H^n(X) \neq 0\}$ (we leave the coefficient ring and cohomology theory being used to be determined by context). Since quotients make spaces "smaller", one could expect that $\mathrm{cohom\,dim}(X/G) \leq \mathrm{cohom\,dim}(X)$. This is not true in general. For example, for the action of $\mathbb{Z}$ on $\mathbb{R}$ given by sum, $\mathbb{R}/\mathbb{Z} \simeq S^1$, but $\mathrm{cohom\,dim}(S^1) = 1$ and $\mathrm{cohom\,dim}(\mathbb{R}) = 0$. This can't happen for compact groups. As a consequence of [16, Proposition A.11], we have:

Theorem 1.5. *For any space X with action of a compact group G,*

$$\mathrm{cohom\,dim}(X/G) \leq \mathrm{cohom\,dim}(X).$$

Exemple 1.6. Denote by $\mathbb{R}P^n$ the *projective space* $S^n/\mathbb{Z}_2$, by $L_p(n)$ the *lens space* $L_p(n) = S^{2n+1}/\mathbb{Z}_p$, by $\mathbb{C}P^n$ the *complex projective space* S^{2n+1}/S^1, and by $\mathbb{H}P^n$ the *quaternionic projective space* S^{4n+3}/S^3, $n \leq \infty$ and p an odd prime. The action in each case is given by real, complex or quaternionic multiplication in each coordinate, and regarding $\mathbb{Z}_2 = \{-1, 1\}$, $\mathbb{Z}_p \subset \mathbb{C}$ as roots of the unity, $S^1 \subset \mathbb{C}$, $S^3 \subset \mathbb{H}$,

$S^n \subset \mathbb{R}^{n+1}$, $S^{2n+1} \subset \mathbb{C}^{n+1}$, $S^{4n+3} \subset \mathbb{H}^{n+1}$. We will say these are the canonical actions of the respective groups on spheres.

The cohomology rings are given by

$$\begin{aligned}
H^*(\mathbb{R}P^n;\mathbb{Z}_2) &\simeq \mathbb{Z}_2[s]/\langle s^{n+1}\rangle, && \text{with } \dim(s)=1,\\
H^*(L_p(n);\mathbb{Z}_p) &\simeq \mathbb{Z}_p[\tilde{s},s]/\langle \tilde{s}^2, s^{n+1}\rangle, && \text{with } \dim(s)=2 \text{ and } \dim(\tilde{s})=1,\\
H^*(\mathbb{C}P^n;R) &\simeq R[s]/\langle s^{n+1}\rangle, && \text{with } \dim(s)=2,\\
H^*(\mathbb{H}P^n;R) &\simeq R[s]/\langle s^{n+1}\rangle, && \text{with } \dim(s)=4,
\end{aligned}$$

if $n<\infty$ and

$$\begin{aligned}
H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) &\simeq \mathbb{Z}_2[s], && \text{with } \dim(s)=1,\\
H^*(L_p(\infty);\mathbb{Z}_p) &\simeq \mathbb{Z}_p[\tilde{s},s]/\langle \tilde{s}^2\rangle, && \text{with } \dim(s)=2 \text{ and } \dim(\tilde{s})=1,\\
H^*(\mathbb{C}P^\infty;R) &\simeq R[s], && \text{with } \dim(s)=2,\\
H^*(\mathbb{H}P^\infty;R) &\simeq R[s], && \text{with } \dim(s)=4,
\end{aligned}$$

otherwise.

Definition 1.7. Let $X_1,\dots,X_r$ be spaces with action of a group G. The *diagonal action* of G on $X_1\times\cdots\times X_r$ is the action given by $g(x_1,\dots,x_r)=(gx_1,\dots,gx_r)$. When considering this action, we say that G acts diagonally on $X_1,\dots,X_r$.

For two spaces X and Y with action of G, we denote by $X\times_G Y$ the orbit space $X\times Y/G$, where G acts diagonally.

Proposition 1.8. *Let X be a path connected space with $\mathbb{Z}_p$ action, for some integer $p\geq 2$. Then there is a $\mathbb{Z}_p$-equivariant map $S^1\to X$.*

Proof. Let $\mathbb{Z}_p=\{0,1,\dots,p-1\}$ and $S^1=\mathbb{R}/(x\sim x+kp)$, $x\in\mathbb{R}$, $k\in\mathbb{Z}$. In this setting, the canonical S^1 multiplication becomes $t+s$ and, in particular, the $\mathbb{Z}_p$-action on S^1 is given by $z+t$, $z\in\mathbb{Z}_p$ $t,s\in S^1$.

Choose $x_0\in X$ and let $\alpha\colon[0,1]\to X$ be a path with $\alpha(0)=x_0$, $\alpha(1)=1x_0$. Then $s\mapsto z\alpha(s-z)$, if $s\in[z,z+1]$, $z\in\mathbb{Z}_p$, is the desired $\mathbb{Z}_p$-equivariant map $S^1\to X$. It is continuous by the gluing lemma, and for any $z,w\in\mathbb{Z}_p$, $s\in S^1$, with $s\in[w,w+1]$, we have that $z+s\in[z+w,z+w+1]$ and $z+s\mapsto(z+w)\alpha(z+s-(z+w))=z(w\alpha(s-w))$. □

Proposition 1.9. *Let X be a space with S^1 action and $\pi_1(X)=0$. Then there is a S^1-equivariant map $S^3\to X$.*

Proof. First, we write S^3 in a convenient way. The map

$$\frac{[0,1]\times S^1\times S^1}{\sim}\to S^3 \qquad (h,x,y)\mapsto(hxy,(\sqrt{1-h^2})x)$$

where the equivalence relation $\sim$ is given by $(1,x,y) \sim (1,\lambda x, y/\lambda)$ and $(0,x,y) \sim (0,x,1)$ for every $x,y,\lambda \in S^1$, is an homeomorphism with inverse given by

$$\begin{aligned}(z,w) &\mapsto \left(|z|, \frac{w}{\sqrt{1-|z|^2}}, \frac{\sqrt{1-|z|^2}}{|z|}\frac{z}{w}\right) \text{ if } z \neq 0 \text{ and } w \neq 0\\ (z,0) &\mapsto (1,z,1)\\ (0,w) &\mapsto (0,w,1),\end{aligned}$$

with $z,w \in \mathbb{C}$.

This homeomorphism can be made into a S^1-equivariant homeomorphism by defining the action $\lambda(t,x,y) = (t,\lambda x,y)$.

Let $x_0 \in X$ be any point. Then there is an homotopy $H\colon [0,1] \times S^1 \to X$ from the loop $z \mapsto zx_0$, $z \in S^1$, to the constant loop, and the map $(t,z,w) \mapsto zH(t,w)$ is the desired equivariant map $S^3 \to X$. □

Analogously, we have the following:

Proposition 1.10. *Let X be a space with S^3 action and $\pi_3(X) = 0$. Then there is a S^3-equivariant map $S^7 \to X$.*

Proof. First, we write S^7 in a convenient way. The map

$$\frac{[0,1] \times S^3 \times S^3}{\sim} \to S^3 \qquad (h,x,y) \mapsto (hxy, (\sqrt{1-h^2})x)$$

where the equivalence relation $\sim$ is given by $(1,x,y) \sim (1,\lambda x, y/\lambda)$ and $(0,x,y) \sim (0,x,1)$ for every $x,y,\lambda \in S^3$, is an homeomorphism with inverse given by

$$\begin{aligned}(z,w) &\mapsto \left(|z|, \frac{w}{\sqrt{1-|z|^2}}, \frac{\sqrt{1-|z|^2}}{|z|}\frac{z}{w}\right) \text{ if } z \neq 0 \text{ and } w \neq 0\\ (z,0) &\mapsto (1,z,1)\\ (0,w) &\mapsto (0,w,1).\end{aligned}$$

with $z,w \in \mathbb{H}$.

This homeomorphism can be made into a S^3-equivariant homeomorphism by defining the action $\lambda(t,x,y) = (t,\lambda x,y)$.

Let $x_0 \in X$ be any point. Then there is an homotopy $H\colon [0,1] \times S^1 \to X$ from the map $z \mapsto zx_0$, $z \in S^3$, to the constant map, and the map $(t,z,w) \mapsto zH(t,w)$ is the desired equivariant map $S^7 \to X$. □

1.2 Bundles

In this section, we state the most general and boring definition for a bundle. The bundles we are actually interested in are fibrations and fiber bundles, which we present in the next couple of sections.

Definition 1.11. A *bundle* over B, or simply a bundle, is just another name for a map $p\colon E \to B$, or the triple $\xi = (p, E, B)$. The space E is called *total space*, B is called *base space* and for each $b \in B$, $F_b = p^{-1}(b) \subset E$ is called *fiber* over b.

This definition helps in introducing the standard terminology, but we usually want bundles to have more structure. This will be provided by the bundles presented in the upcoming sections. As usual, accompanying this definition there is a definition for morphisms.

Definition 1.12. Let $p\colon E \to B$ and $q\colon E' \to B'$ be bundles, $f\colon E \to E'$ and $\overline{f}\colon B \to B'$ be maps. If the square

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{\overline{f}} & B'. \end{array}$$

commutes we say that $(f, \overline{f})\colon p \to q$ is a *bundle morphism*, a *bundle map* or a *fiber preserving map* (since the commutativity is equivalent to $f(F_b) \subset F_{\overline{f}(b)}$ for every $b \in B$).

We say that f is a *bundle isomorphism* if there is a bundle map $(g, \overline{g})\colon q \to p$ such that $f \circ g = \mathrm{id}_{E'}$, $g \circ f = \mathrm{id}_E$, $\overline{f} \circ \overline{g} = \mathrm{id}_{B'}$ and $\overline{g} \circ \overline{f} = \mathrm{id}_B$. If a bundle isomorphism $p \to q$ exist, we say that p and q are isomorphic.

We may omit $\overline{f}$ if clear from context, saying simply that f is a bundle morphism, a bundle map or a fiber preserving map. Specifying the map $\overline{f}$ is usually not needed since it is uniquely determined by f when p and q are surjective, which is the case for most bundles we will deal with. In particular, when $\overline{f}$ is the identity and the square can be seem as the triangle

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array} \quad .$$

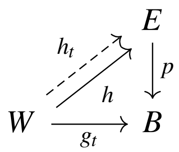
Be careful not to assume that isomorphic bundles can be taken as if being "the same", as we will introduce bundles with some structure that might not be preserved under an generic bundle isomorphism, and may need a more specialized definition of isomorphism.

Definition 1.13. A *section* for a bundle $p\colon E \to B$ is a map $s\colon B \to E$ such that $p \circ s = \mathrm{id}_E$. Notice that $s(B)$ determines the section s. We will also refer to $s(B)$ as a section.

1.3 Fibrations

Fibrations are bundles with the very useful "homotopy lifting property".

Definition 1.14. A *fibration* is a map $p\colon E \to B$ such that, for any homotopy $h_t\colon W \to B$ from any space W and map $g\colon W \to E$ such that $p \circ g = h_0$ there exists a homotopy $g_t\colon W \to E$ such that $p \circ g_t = h_t$ for any $t \in [0,1]$ and $g_0 = g$.



When working only with CW-complexes, manifolds, or some other category, fibrations are often defined to have this property only for maps $p\colon W \to B$ where W is an object in the respective category. We may use these weaker versions of the definition, and leave it to context to determine which version we are using.

Any two fibers F_a and F_b of a fibration over a path connected base space B have the same homotopy type, so we will refer to a space F having the homotopy type of these fibers as *the* fiber of the fibration.

The homotopy equivalence is constructed as follows: take α to be a path from a to b. Then the map $F_a \times I \to B$ given by $(e,t) \mapsto \alpha(t)$ lifts to a map $H\colon F_a \times I \to E$ from which we define the homotopy equivalence $\alpha_*\colon F_a \to F_b$ by $\alpha_*(x) = H(x,1)$. If we consider only loops, the association $\alpha \mapsto \alpha_*$ turns out to be a homomorphism. We state it below as a proposition, see [6, Theorem 6.12] for a complete proof.

Proposition 1.15. *Let $p\colon E \to B$ be a fibration with B path connected. Then for any $a,b \in B$ and path α from a to B, the map α_*, described above, is a homotopy equivalence.*

Futhermore, the homotopy class of α_* depends only on the homotopy class of α relative to a and b, so there is a well defined map

$$\{\textit{paths from } a \textit{ to } b\}/\simeq_{rel\,\{0,1\}} \to \{\textit{homotopy equivalences } F_b \to F_a\}/\simeq,$$
$$[\alpha] \mapsto [(\alpha^{-1})_*].$$

In particular, the map

$$\pi_1(B;b) \to \{\textit{homotopy equivalences } F_b \to F_b\}/\simeq,$$
$$[\alpha] \mapsto [(\alpha^{-1})_*],$$

is a group homomorphism for any $b \in B$.

We also have a definition for fibrations for pairs of spaces.

Definition 1.16. Let $p\colon E \to B$ be a fibration. If $E' \subset E$ is such that the restriction $p|_{E'}\colon E' \to B$ is also a fibration, we say that $p\colon (E,E') \to B$ is a *relative fibration*. In this case, the fibers of elements $b \in B$ are pairs (F_b, F_b') where $F_b' = p|_{E'}^{-1}(b)$.

Again, a pair (F,F') which have the homotopy type of any fiber (F_b,F_b') is called *the* fiber of p.

Fibrations can be very useful to compute cohomology. Knowing the cohomology of the base space and the fiber can give us the cohomology of the total space.

Theorem 1.17 (Leray–Hirsch)**.** *Let $(F,F') \xrightarrow{i} (E,E') \xrightarrow{p} (B)$ be a relative fibration. If $H^n(F,F')$ is an finitely generated free R-module for each $n \in \mathbb{Z}$ and there is a collection $\{c_j\} \subset H^*(E,E')$ for which $\{i^*(c_j)\}$ is a R-basis of $H^*(F,F')$, then*

$$L\colon H^*(B) \otimes H^*(F,F') \to H^*(E,E')$$
$$w \otimes i*(c_j) \mapsto p^*(b)c_j$$

is an isomorphism of R-modules. Therefore, $H^*(E,E')$ is a free graded $H^*(B)$-module with basis $\{c_j\}$.

For a proof, see [7, Theorem 17.8.4]. Notice that this does not give the multiplication structure of $H^*(E,E')$, but in some sense, it almost does. For any two elements $p^*(b)c_j, p^*(b')c_k \in H^*(E,E')$, we have $(p^*(b)c_j)(p^*(b')c_k) = \pm p^*(bb')(c_jc_k)$. Therefore, if the multiplication between elements of the collection $\{c_j\}$ can be computed by some other mean, we will have a complete description of $H^*(E,E')$ as a graded algebra.

1.4 Fiber Bundles

We start with a definition that is a bit weaker then fiber bundles.

Definition 1.18. A *locally trivial bundle* is a bundle $p\colon E \to B$ for which there is covering $\{U_j\}$ of B, a space F and a family of homeomorphisms $\{\varphi_j\colon U_j \times F \to p^{-1}(U_j)\}$ such that the diagram

$$\begin{array}{ccc} U_j \times F & \xrightarrow{\varphi_j} & p^{-1}(U_j) \\ & {\scriptstyle p_{U_j}} \searrow \quad \swarrow {\scriptstyle p} & \\ & U_j & \end{array}$$

commutes for each j. For each $b \in B$, we have $F_b \simeq F$, so F is called the fiber of p. Each homeomorphism $\varphi_j\colon U_j \times F \to p^{-1}(U_j)$ is called a *chart* over U_j.

Adding a few hypothesis, we get the definition we want.

Definition 1.19. Let G be a topological group. A locally trivial bundle $p\colon E \to B$ with fiber F is a *fiber bundle* with structure group G if G acts effectively on F and, for any two charts φ, φ' over U, there is a map $\theta\colon U \to G$ such that

$$\varphi'(u,x) = \varphi(u, \theta(u)x)$$

for every $u \in U$ and $x \in F$. The map θ is called a *transition function* for φ, φ'.

A fiber bundle with fiber F may also be called an F bundle.

Exemple 1.20. For any spaces F and B, the projection $p\colon F\times B\to B$ is a fiber bundle, called *trivial bundle*. The charts are identity maps and the structure group is the trivial group.

Again, we have a definition for pairs. This one requires a bit of care, since we need to make sure the charts still work when restricted.

Proposition 1.21. *Let $p\colon E\to B$ be a fiber bundle with structure group G and $F'\subset F$ be an G-invariant subset of F. Denote $E'=\bigcup_{\varphi \text{ chart over } U}\varphi(U\times F')\subset E$. Then $p|_{E'}\colon E'\to B$ is a fiber bundle with fiber F'.*

Definition 1.22. Let $p\colon E\to B$ be a fiber bundle with structure group G, $F'\subset F$ be an G-invariant subset of F and $E'=\bigcup_{\varphi \text{ chart over } U}\varphi(U\times F')\subset E$. Then we say that $p\colon (E,E')\to B$ is a *relative fiber bundle*, or a *fiber bundle pair* with fiber (F,F') and structure group G.

Later, when dealing with parametrized Borsuk–Ulam theorems, we will study fiber bundles for which there are actions of some group on the total space E. It is desirable that this action respects the fiber bundle structure.

Definition 1.23. By an action of a group H on a fiber bundle $F\to E\xrightarrow{p}B$, we mean actions of H on E and F such that the collection of charts can be taken so that, for any chart $\varphi\colon F\times U\to p^{-1}(U)$ and any $b\in U\subset B$, the map $\varphi_b\colon F\to p^{-1}(\{b\})$, $x\mapsto\varphi(x,b)$ is H-equivariant.

By an action of H on a fiber bundle pair $(F,F_0)\to(E,E_0)\to B$ we mean an action of H on the fiber bundle $F\to E\to B$ such that F_0 is H-invariant. In particular, this induces a action of H on the fiber bundle $F_0\to E_0\to B$.

Notice that $x\in\operatorname{Free}(F)\iff\varphi_b(x)\in\operatorname{Free}(E)$, so the charts can be restricted to homeomorphisms $\operatorname{Free}(F)\times U\to(\pi|_{\operatorname{Free}(E)})^{-1}(U)$, so $(F,\operatorname{Free}(F))\to(E,\operatorname{Free}(E))\to B$ is also an fiber bundle pair with action of G.

A fiber bundle will usually also be a fibration (see [6, Corollary 6.9]).

Proposition 1.24. *If $p\colon E\to B$ is a fiber bundle and B is paracompact, then p is a fibration.*

1.5 G-principal Bundles, Universal Bundles and Classifying Spaces

Here is a particular kind of fiber bundle that will be very important.

Definition 1.25. A *G-principal bundle* is a fiber bundle for which both the structure group and the fiber are G. The action of G on the fiber G is the product of the group G.

Every G-principal bundle is a projection of a G-space on it's orbit space.

Proposition 1.26. *Let $p\colon E \to B$ be a G-principal bundle. Then, there is a free action of G on E for which $B \simeq E/G$.*

The action in the proposition is given by

$$ge = \varphi(\varphi_1^{-1}(e), g\varphi_2^{-1}(e))$$

for every $g \in G$ and $e \in E$, where $\varphi\colon U \times G \to p^{-1}(U)$ is a chart with $e \in p^{-1}(U)$ and φ_1^{-1}, φ_2^{-1} are the coordinate maps of φ^{-1}. This action is independent of the choice of charts φ. For a complete proof, see [6, Proposition 4.4]. We will always assume the total space of a G-principal bundle is a G-space with this action.

With some restrictions, the converse is true (see [4, Theorem II.5.8, pg 88]).

Theorem 1.27. *If X is a completely regular free G-space, and G is a compact Lie group, then $X \to X/G$ is a G-principal bundle.*

Paracompact Hausdorff spaces are completely regular (see [15, Theorem 41.1]), so the theorem applies to them.

Here is a good way of building new fiber bundles from G-principal bundles (see [2, subsection IV.1.3]).

Proposition 1.28. *Let X and Y be spaces with action of G such that $Y \to \overline{Y}$ is a G-principal bundle. Then, the projection $p\colon X \times Y \to Y$ induces a fiber bundle $\overline{p}\colon X \times_G Y \to \overline{Y}$ with fiber X and structural group G.*

This can be proven by defining the charts by $[x,y] \mapsto ((\varphi)_1^{-1}(y)^{-1}x, [y])$, where $\varphi\colon G \times U \to p^{-1}(U)$ is a chart for p and $(\varphi)_1^{-1}$ is the first coordinate map.

For any topological group G, there is a G-principal bundle $E_G \to B_G$ such that, for any other G-principal bundle $E \to B$, there is a G-equivariant map $g\colon E \to E_G$. The spaces E_G and B_G are unique up to G-homotopy equivalence, and the G-equivariant bundle map $g\colon E \to E_G$, together with the map $\overline{g}\colon B \to B_G$ induced in the base spaces, is unique up to G-homotopy (see [7, Section 14.4]).

Definition 1.29. The bundle $E_G \to B_G$ is called *universal G-principal bundle*, B_G is called the *classifying space* for G and $\overline{g}$ is called the *classifying map* for $E \to B$.

For a paracompact Hausdorff space X with action of a compact Lie group G, we denote the classifying map for $X \to X/G$ by $g_{\overline{X}}$.

Here are some important universal bundles.

Theorem 1.30. *For $G = \mathbb{Z}_2$, $\mathbb{Z}_p$, S^1 and S^3, the universal G-principal bundles can be given by $S^\infty \to$*

$\mathbb{R}P^\infty$, $S^\infty \to L_p(\infty)$, $S^\infty \to \mathbb{C}P^\infty$ and $S^\infty \to \mathbb{H}P^\infty$ respectively. Therefore, we have

$$\begin{aligned} H^*(B_{\mathbb{Z}_2};\mathbb{Z}_2) &\simeq \mathbb{Z}_2[s], && \textit{with } \dim(s)=1,\\ H^*(B_{\mathbb{Z}_p};\mathbb{Z}_p) &\simeq \mathbb{Z}_p[\tilde{s},s]/\langle \tilde{s}^2\rangle, && \textit{with } \dim(s)=2 \textit{ and } \dim(\tilde{s})=1,\\ H^*(B_{S^1}) &\simeq R[s], && \textit{with } \dim(s)=2,\\ H^*(B_{S^3}) &\simeq R[s], && \textit{with } \dim(s)=4. \end{aligned}$$

For the products $G = \mathbb{Z}_2^r$, $\mathbb{Z}_p^r$, $(S^1)^r$ and $(S^3)^r$, for some positive integer r, the universal G-principal bundles can be given by $(S^\infty)^r \to (\mathbb{R}P^\infty)^r$, $(S^\infty)^r \to (L_p(\infty))^r$, $(S^\infty)^r \to (\mathbb{C}P^\infty)^r$ and $(S^\infty)^r \to (\mathbb{H}P^\infty)^r$ respectively. Therefore, by the Künneth Formula we have

$$\begin{aligned} H^*(B_{(\mathbb{Z}_2)^r};\mathbb{Z}_2) &\simeq \mathbb{Z}_2[s_1,\dots,s_r], && \textit{with } \dim(s_k)=1,\\ H^*(B_{(\mathbb{Z}_p)^r};\mathbb{Z}_p) &\simeq \mathbb{Z}_p[\tilde{s}_1,s_1,\dots,\tilde{s}_r,s_r]/\langle \tilde{s}_k^2\rangle, && \textit{with } \dim(s_k)=2 \textit{ and } \dim(\tilde{s}_k)=1,\\ H^*(B_{(S^1)^r}) &\simeq R[s_1,\dots,s_r], && \textit{with } \dim(s_k)=2,\\ H^*(B_{(S^3)^r}) &\simeq R[s_1,\dots,s_r], && \textit{with } \dim(s_k)=4, \end{aligned}$$

for $1 \le k \le r$.

1.6 Vector Bundles

Here is another kind of fiber bundle that will be very important.

Definition 1.31. Let K be a field and n a positive integer. A *n-dimensional K-vector bundle* is a fiber bundle $p\colon E \to B$ with fiber K^n and structure group $GL_n(K)$. It's called a complex or quaternionic vector bundle if $K = \mathbb{R}$, $K = \mathbb{C}$ or $K = \mathbb{H}$ respectively and simply a vector bundle if $K = \mathbb{R}$.

Definition 1.32. A *vector bundle map* is a bundle map $p \to q$ between vector bundles which restricts to (vector space) isomorphisms in the fibers.

A vector bundle map that is also a bundle isomorphism is called a *vector bundle isomorphism*. If there is a vector bundle isomorphism $p \to q$, we write $p \simeq q$.

Definition 1.33. The *zero section* of a vector bundle $p\colon E \to B$ is the set consisting of the zeroes of each fiber and will usually be denoted by 0. We denote $E_0 = E \setminus 0$ and $F_0 = F \setminus \{0\}$.

A section $s\colon B \to E$ of p is non-vanishing if $s(b) \neq 0$ for all $b \in B$.

Standard constructions for vector spaces can be applied to the fibers to produce new vector bundles. Here we present direct sum and direct product, but this is possible for other constructions as well (see, for exemple, [7, Section 14.5]).

Definition 1.34. Let $p\colon E \to B$ and $q\colon \tilde{E} \to B$ be vector bundles. Let E' and E'' be the disjoint union of $F_b \oplus \tilde{F}_b$ and $F_b \otimes \tilde{F}_b$ respectively, $b \in B$. Then, the obvious projections $p \oplus q\colon E' \to B$ and $p \otimes q\colon E'' \to B$ are called *whitney sum* and *tensor product* of p and q respectively.

Topologize E' and E'' so that the functions

$$\begin{aligned} (p\oplus q)^{-1}(U) &\to U\times(F\oplus\tilde{F}), \\ \varphi^{-1}(b,x)\oplus\psi^{-1}(b,y) &\mapsto (b,x\oplus y) \end{aligned} \qquad \begin{aligned} (p\otimes q)^{-1}(U) &\to U\times(F\otimes\tilde{F}), \\ \varphi^{-1}(b,x)\otimes\psi^{-1}(b,y) &\mapsto (b,x\otimes y) \end{aligned}$$

are continuous, for φ and ψ charts for p and q respectively, and choose these functions as charts for $p\oplus q$ and $p\otimes q$.

We may write $p\oplus p = 2p$, $p\otimes p = p^2$, $p\oplus p\oplus p = 3p$, $p\otimes p\otimes p = p^3$ and so on.

Definition 1.35. The *complexification* of a real vector bundle $p\colon E\to B$ is the tensor product $\mathbb{C}p = p\otimes q$ where $q\colon B\times\mathbb{C}\to B$ is the trivial bundle. This can be regarded as a complex vector bundle with complex multiplication in each fiber given by $\lambda x\otimes z = x\otimes(\lambda z)$, for any $\lambda\in\mathbb{C}$ and $x\otimes z$ in some fiber $F_b\otimes\mathbb{C}$ of $\mathbb{C}p$.

Isomorphisms for these constructions in vector spaces can also yield vector bundle isomorphisms.

Proposition 1.36. *For K-vector bundles $p\colon E\to B$, $q\colon E'\to B$ and $r\colon E''\to B$, there are vector bundle isomorphisms*

- $r\otimes(p\oplus q)\simeq(r\otimes p)\oplus(r\otimes q)$,
- $(p\oplus q)\otimes r\simeq(p\otimes r)\oplus(q\otimes r)$,
- $p\otimes k\simeq p\simeq k\otimes p$,

where $k\colon B\times K\to B$ is the trivial bundle.

In particular, if p is a real vector bundle, there is a vector bundle isomorphism $\mathbb{C}p\simeq p^2$, where $\mathbb{C}p$ is regarded as a real vector bundle.

CHAPTER 2

Projective Product Spaces

2.1 Introduction and Preliminaries

In this chapter, we will discuss the projective product spaces, introduced by Davis in [5], and its lens, complex and quaternionic versions. The main theorem for this chapter is the computation for the cohomology ring of the quaternionic projective product spaces.

For $\overline{n} = (n_1, \dots, n_r)$, $n_1 \le \cdots \le n_r$, the *projective product space*, denoted $\mathbb{R}P_{\overline{n}}$, is the space $S^{n_1} \times \cdots \times S^{n_r}/\mathbb{Z}_2$, the *lens product space*, denoted $L_p(\overline{n})$, is the space $S^{2n_1+1} \times \cdots \times S^{2n_r+1}/\mathbb{Z}_p$, the *complex projective product space*, denoted $\mathbb{C}P_{\overline{n}}$, is the space $S^{2n_1+1} \times \cdots \times S^{2n_r+1}/S^1$ and the *quaternionic projective product space*, denoted $\mathbb{H}P_{\overline{n}}$, is the space $S^{4n_1+3} \times \cdots \times S^{4n_r+3}/S^3$, where the action in each case is the diagonal action for the cannonical action on each sphere. The cohomology ring for the projective product spaces where computed by Davis:

Theorem 2.1 ([5, Theorem 2.1])**.** *If $n_1 < n_2$ or n_1 is odd,*

$$H^*(\mathbb{R}P_{\overline{n}}; \mathbb{Z}_2) \simeq \frac{\mathbb{Z}_2[u, u_2, \cdots, u_r]}{\langle u^{n_1+1}, u_k^2 \rangle},$$

where $\dim(u) = 1$ *and* $\dim(u_k) = n_k$*,* $2 \le k \le r$*. If n_1 is even, with $n_1 = \cdots = n_{r'}$, we have that*

$$H^*(\mathbb{R}P_{\overline{n}}; \mathbb{Z}_2) \simeq \frac{\mathbb{Z}_2[u, u_2, \cdots, u_r]}{\langle u^{n_1+1}, u_{k'}^2 - u_1^{n_1} u_{k'}, u_k^2 \rangle}.$$

where $\dim(u) = 1$ *and* $\dim(u_k) = n_k$*,* $2 \le k' \le r'$ *and* $r' + 1 \le k \le r$*.*

Later, using similar methods, González and Velasco computed the cohomology ring for the lens and complex projective product spaces

Theorem 2.2 ([10, Theorem 1] and [10, Theorem 2])**.**

$$H^*(L_p(\overline{n}); \mathbb{Z}_p) \simeq \frac{\mathbb{Z}_p[\tilde{u}, u, u_2, \dots, u_r]}{\langle \tilde{u}^2, u^{n_1+1}, u_k^2 \rangle},$$

$$H^*(\mathbb{C}P_{\overline{n}}) \simeq \frac{R[u, u_2, \dots, u_r]}{\langle u^{n_1+1}, u_k^2 \rangle},$$

where $\dim(\tilde{u}) = 1$, $\dim(u) = 2$ *and* $\dim(u_k) = 2n_k + 1$, $2 \leq k \leq r$.

Our goal is to complete the trilogy by computing the cohomology ring for the quaternionic projective product spaces. We use a different method from that on the proofs of Davis and González and Velasco. We will need the following exact sequences(see [1, Corollary 4.1.19] and [1, Theorem 6.3.4]):

Theorem 2.3. *For any fibration $F \to E \to B$, there is an exact sequence*

$$\cdots \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \to \cdots$$

Theorem 2.4. *For any fibration $F \xrightarrow{i} E \xrightarrow{p} B$ such that F is r-connected and B is s-connected, $r \geq 0$, $s \geq 1$, there is an exact sequence*

$$H^0(B) \xrightarrow{p^*} H^0(E) \xrightarrow{i^*} H^0(F) \xrightarrow{\Delta} H^1(B) \to \cdots$$
$$\cdots \to H^{r+s}(F) \xrightarrow{\Delta} H^{r+s+1}(B) \xrightarrow{p^*} H^{r+s+1}(E) \xrightarrow{i^*} H^{r+s+1}(F).$$

2.2 The cohomology ring of the quaternionic projective product space

Theorem 2.5. *There is an R-algebra isomorphism*

$$H^*(\mathbb{H}P_{\overline{n}}) \cong \frac{R[u, u_2, \ldots, u_r]}{\langle u^{n_1+1}, u_k^2 \rangle},$$

where $\dim(u) = 4$ *and* $\dim(u_k) = 4n_k + 3$*, for each* $k \in \{2, \ldots, r\}$.

Proof. We prove this by induction on r. Case $r = 1$ is immediate. Suppose the theorem holds for $r = n - 1$ and denote $\overline{m} = (n_1, \ldots, n_{r-1})$, $S_{\overline{m}} = S^{4n_1+3} \times \cdots \times S^{4n_{r-1}+3}$.

By [4, Theorem 5.8], $S^3 \to S_{\overline{m}} \to \mathbb{H}P_{\overline{m}}$ is a S^3-principal bundle. By [2, Chapter IV, 1.3], the projection $S_{\overline{m}} \times S^{4n_r+3} \to S_{\overline{m}}$ induces a fiber bundle $p\colon S_{\overline{m}} \times_{S^3} S^{4n_r+3} \to \mathbb{H}P_{\overline{m}}$ with fiber S^{4n_r+3} and structure group S^3. Denote the inclusion of the fiber by $i\colon S^{4n_r+3} \to S_{\overline{m}} \times_{S^3} S^{4n_r+3}$. Notice that $S_{\overline{m}} \times_{S^3} S^{4n_r+3} = \mathbb{H}P_{\overline{n}}$.

Since $\mathbb{H}P_{\overline{m}}$ is paracompact (see [4, Theorem II.1.2, pg 88] and [14, Corollary 1]), $S^3 \to S_{\overline{m}} \to \mathbb{H}P_{\overline{m}}$ is also a fibration and there is an exact sequence (see [1, Corollary 4.1.19])

$$\cdots \to \pi_1(S^3) \to \pi_1(S_{\overline{m}}) \to \pi_1(\mathbb{H}P_{\overline{m}}) \to \pi_0(S^3) \to \cdots$$

Since $\pi_0(S^3) = 0$ and $\pi_1(S_{\overline{m}}) = 0$, we have $\pi_1(\mathbb{H}P_{\overline{m}}) = 0$, so $\mathbb{H}P_{\overline{m}}$ is simply connected.

Then, since S^{4n_r+3} is $(4n_r + 2)$-connected, there is a exact sequence (see [1, Theorem 6.3.4])

$$H^0(\mathbb{H}P_{\overline{m}}) \xrightarrow{p^*} H^0(\mathbb{H}P_{\overline{n}}) \to H^0(S^{4n_r+3}) \xrightarrow{\Delta} H^1(\mathbb{H}P_{\overline{m}}) \xrightarrow{p^*} \cdots$$
$$\cdots \to H^{4n_r+2}(S^{4n_r+3}) \xrightarrow{\Delta} H^{4n_r+3}(\mathbb{H}P_{\overline{m}}) \xrightarrow{p^*} H^{4n_r+3}(\mathbb{H}P_{\overline{n}}) \to H^{4n_r+3}(S^{4n_r+3}).$$

There is a right inverse $q\colon \mathbb{H}P_{\overline{m}} \to \mathbb{H}P_{\overline{n}}$ for p, given by $[c_1,\dots,c_{r-1}] \mapsto [c_1,\dots,c_{r-1},c_{r-1}]$, where we consider $c_{r-1} \in S^{4n_r+3}$ by the inclusion. Then q^* is a left inverse for p^* and we have that p^* is injective. The exactness of the sequence gives us $\operatorname{Im}(\Delta) = \ker(p^*) = 0$ so $\ker(\Delta) = H^k(S^{4n_r+3})$, and the restriction $H^k(\mathbb{H}P_{\overline{n}}) \to H^k(S^{4n_r+3})$ is surjective, for each $k \in \{0,\dots,4n_r+3\}$.

Let $u_r \in H^*(\mathbb{H}P_{\overline{n}})$ be such that $i^*(u_r)$ is the generator of $H^{4n_r+3}(S^{4n_r+3})$. Then, by the Leray–Hirsch Theorem, $H^*(\mathbb{H}P_{\overline{n}})$ is a free $H^*(\mathbb{H}P_{\overline{m}})$-module with basis $\{1,u_r\}$, and there is an R-module isomorphism

$$H^*(\mathbb{H}P_{\overline{n}}) \cong H^*(\mathbb{H}P_{\overline{m}}) \otimes H^*(S^{4n_r+3}) = \frac{R[u,u_2,\dots,u_{r-1}]}{\langle u^{n_1+1}, u_k^2\rangle} \otimes \Lambda[u_r] \simeq \frac{R[u,u_2,\dots,u_r]}{\langle u^{n_1+1}, u_k^2\rangle}.$$

Notice cohom $\dim(\mathbb{H}P_{\overline{n}}) = 4n_1 + (4n_2+3) + \cdots + (4n_r+3)$. It remains to prove that $u_r^2 = 0$. The inclusion $j\colon S^{4n_r+3} \hookrightarrow S^{4n_1+3} \times_{S^3} S^{4n_r+3}$ can be factored as $S^{4n_r+3} \xrightarrow{i} S_{\overline{m}} \times_{S^3} S^{4n_r+3} = \mathbb{H}P_{\overline{n}} \xrightarrow{j'} S^{4n_1+3} \times_{s^3} S^{4n_r+3}$. Then $H^*(S^{4n_1+3} \times_{S^3} S^{4n_r+3}) \simeq R[x]/\langle x^{n_1+1}\rangle \otimes \Lambda[x_r]$ as R-modules, where x_r is such that $j(x_r)$ is the generator of $H^{4n_r+3}(S^{4n_r+3})$. Since cohom $\dim(S^{4n_1+3} \times_{S^3} S^{4n_r+3}) = 4n_1 + 4n_r + 3$, we have that $x_r^2 = 0$. We can take $u_r = j'(x_r)$ so that $u_r^2 = j'(x_r^2) = j'(0) = 0$. $\square$

CHAPTER 3

Bourgin–Yang Theorems for a product of spheres

3.1 Introduction and Preliminaries

In 1955, C. T. Yang [20] and D. G. Bourgin [3] proved, independently, the following version of the Borsuk–Ulam Theorem, which gives a minimum "size" for the coincidence point set:

Theorem 3.1. *Let T be a free involution on S^n, $f\colon S^n \to \mathbb{R}^m$ be any continuous function and $A_f = \{x \in S^n \mid f(x) = f(T(x))\}$ be the coincidence point set. If $n \geq m$, then*

$$\text{cohom}\dim(A_f) \geq n - m.$$

Since then, similar theorems have been proven, replacing the sphere and the involution by some other space and a free action of some group. For instance, de Mattos, Pergher, dos Santos and Singh proved the following version:

Theorem 3.2 ([13, Theorem 1.4])**.** *Let $\mathbb{Z}_2$ act by diagonal antipodal action on $M = S^{n_1} \times \cdots \times S^{n_r}$, where $n_1 \leq \cdots \leq n_r$. Let $f\colon M \to \mathbb{R}^m$ be a continuous set and $A_f = \{x \in M \mid f(x) = f(-x)\}$ be the coincidence point set. If $n_1 \geq m$, then*

$$\text{cohom}\dim(A_f) \geq \dim(M) - m.$$

The goal of this chapter is to use Gonzáles and Velasco's computation of the cohomology groups of the lens product spaces to prove Theorem 3.9, which is an original result, similar to the previous theorem, but considers a $\mathbb{Z}_p$ action on a product of spheres.

In the proof of the main theorem of this chapter, we will need Alexander's duality, the Euler class and some lemmas.

Definition 3.3. A *Thom class* for a fiber bundle pair $p\colon (E,E') \to B$, with fiber (F,F') such that $H^*(F,F')$ is an one dimensional free R-module, is an element $u \in H^*(E,E')$ which is mapped to the basis of the $H^*(F,F_0)$ by the restrictions to each fiber.

An *Euler class* for p, denoted $e(p)$, is an element $p^*(u) \in H^*(B)$, where u is a Thom class for p.

For a vector bundle $p\colon E \to B$, we also say that a Thom class and Euler class for the fiber bundle pair $p\colon (E, E_0) \to B$ is a Thom class and Euler class for p (as a vector bundle) respectively.

Euler classes exist for any bundle when taking $\mathbb{Z}_2$ coefficients. For $\mathbb{Z}$ coefficients, it exists for "orientable" bundles. Discussion on what this means won't be needed for our purposes though, as the following properties will help us determine the Euler classes of the bundles we will work with (see [12, Section 17.7 and 17.8]).

Proposition 3.4. *Let $p\colon E \to B$ and $q\colon E' \to B'$ be vector bundles with Euler classes $e(p)$ and $e(q)$ respectively. Then,*

- *for any vector bundle map $f\colon q \to p$, $f^*(e(p))$ is an Euler class for q,*
- *if $B = B'$, the cup product $e(p)e(q)$ is an Euler class for $p \oplus q$,*
- *the Euler class of any vector bundle with a non-vanishing section is zero.*

Definition 3.5. For a ring R and a finite group G, the *group ring* $R[G]$ is the direct sum $\oplus_{g\in G} R$. The elements of the group ring are regarded as formal sums $\sum_{g\in G} r_g g$, and it is made into a ring by defining the distributive multiplication suggested by this notation. The *augmentation ideal* $I_R(G)$ of the group ring $R[G]$ is the kernel of the homomorphism $R[G] \to R$, $\sum r_i g_i \mapsto \sum r_i$.

We will need the following lemmas, which can be found in [19, Lemma 1.1] and [17, Lema 4.1.2] respectively.

Lemma 3.6. *Let M be a topological manifold, G a finite group acting freely on M, $f\colon M \to \mathbb{R}^m$ a continuous map and assume $\mathbb{R}^m$ to be a ring with multiplication given by multiplication of the coordinates. Then $A_f \neq \emptyset$ if, and only if, the vector bundle $\xi_M\colon M \times_G I_{\mathbb{R}^m}(G) \to \overline{M}$ does not have a non-vanishing section.*

Lemma 3.7. *The vector bundle $\eta\colon E\mathbb{Z}_p \times_{\mathbb{Z}_p} I_{\mathbb{C}}(\mathbb{Z}_p) \to B\mathbb{Z}_p$ has Euler class $e_p(\eta) = (-1)s^{p-1} \in H^{2p-2}(B\mathbb{Z}_p; \mathbb{Z}_p)$, where $H^*(B\mathbb{Z}_p; \mathbb{Z}_p) \cong \mathbb{Z}_p[\tilde{s}, s]/\langle \tilde{s}^2 \rangle$ with $\dim(\tilde{s}) = 1$ and $\dim(s) = 2$.*

Finally, the following is Alexander's duality, which can be found, among other references, in [11, Theorem 27.5]).

Theorem 3.8. *Let X be an R-oriented, compact, n-dimensional manifold, $A \subset X$ closed. Then, for all $q \in \mathbb{Z}$, there is an isomorphism*

$$H^q(A) \to H_{n-q}(X, X \setminus A).$$

3.2 A Bourgin–Yang Theorem for diagonal actions of $\mathbb{Z}_p$ on a product of spheres

Theorem 3.9. *Consider cohomology with $\mathbb{Z}_p$ coefficients. Let $\mathbb{Z}_p$ act as complex multiplication on each S^{2n_i+1}, $1 \le i \le r$, and consider the diagonal action on $S_{\overline{n}} = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$, where $n_1 \le \cdots \le n_r$. Let $f\colon S_{\overline{n}} \to \mathbb{R}^m$ be a continuous map and $A_f = \{x \in S_{\overline{n}} \mid f(x) = f(gx), \forall g \in \mathbb{Z}_p\}$ be the coincidence point set. If $2n_1 + 1 > m(p-1)$ then*

$$\operatorname{cohom}\dim(A_f) \ge \dim(S_{\overline{n}}) - m(p-1).$$

Proof. Consider singular cohomology with $\mathbb{Z}_p$ coefficients, and denote Čech cohomology by $\check{H}$. Assume $\mathbb{R}^m$ and $\mathbb{C}^m$ to be rings with multiplication given by multiplication of the coordinates.

There is a commutative diagram

$$\begin{array}{ccc} S_{\overline{n}} \times_{\mathbb{Z}_p} I_{\mathbb{R}^m}(\mathbb{Z}_p) & \longrightarrow & E\mathbb{Z}_p \times_{\mathbb{Z}_p} I_{\mathbb{R}^m}(\mathbb{Z}_p) \\ \Big\downarrow \xi_{S_{\overline{n}}} & & \Big\downarrow \xi \\ L_p(\overline{n}) & \xrightarrow{\quad g_{\overline{S_{\overline{n}}}} \quad} & B\mathbb{Z}_p \end{array}$$

where $S_{\overline{n}} = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$. Notice $\mathbb{C}\xi \cong \eta \oplus \cdots \oplus \eta = m\eta$ by mapping

$$\left[x, \sum (0,\ldots,0,y_z,0,\ldots,0) z \otimes w\right] \mapsto \left[x, \left(0,\ldots,0,\sum (w y_z) z, 0, \ldots, 0\right)\right].$$

Then, by Lemma 3.7,

$$e_p(\xi)^2 = e_p(\mathbb{C}\xi) = e_p(\eta)^m = (-s^{p-1})^m = (-1)^m s^{m(p-1)}$$

and, therefore,

$$e_p(\xi) = a s^{m(p-1)/2}$$

for some $a \in \mathbb{Z}_p$ such that $a^2 = (-1)^m$.

By Theorem 2.2, we have

$$H^*(L_p(\overline{n})) \cong \mathbb{Z}_p[\tilde{u}, u, u_2, \ldots, u_r]/\langle \tilde{u}^2, u^{n_1+1}, u_k^2 \rangle$$

where $\dim(\tilde{u}) = 1$, $\dim(u) = 2$ and $\dim(u_k) = 2n_k + 1$, $2 \le k \le r$. Then, since $m(p-1)/2 \le n_1$, we have

$$e_p(\xi_{S_{\overline{n}}}) = g^*_{\overline{S_{\overline{n}}}}(e_p(\xi)) = g^*_{\overline{S_{\overline{n}}}}(a s^{m(p-1)/2}) = a g_{\overline{S_{\overline{n}}}}(s)^{m(p-1)/2} = a u^{m(p-1)/2} \ne 0.$$

From Proposition 3.4, $\xi_{S_{\overline{n}}}$ does not have a non-vanishing section and from Lemma 3.6, we have that $A_f \ne \emptyset$.

Since A_f is $\mathbb{Z}_p$ invariant, $S_{\overline{n}} \setminus A_f$ is also $\mathbb{Z}_p$ invariant, and there is a exact sequence of the pair $(L_p(\overline{n}), \overline{S_{\overline{n}} \setminus A_f})$:

$$\cdots \longrightarrow H^n(L_p(\overline{n}), \overline{S_{\overline{n}} \setminus A_f}) \xrightarrow{\alpha} H^n(L_p(\overline{n})) \xrightarrow{\beta} H^n(\overline{S_{\overline{n}} \setminus A_f}) \longrightarrow \cdots$$

Let $f' \colon S_{\overline{n}} \setminus A_f \to \mathbb{R}^m$ and $\xi'_{S_{\overline{n}}} \colon S_{\overline{n}} \times_{\mathbb{Z}_p} I_{\mathbb{R}^m}(\mathbb{Z}_p) \to \overline{S_{\overline{n}} \setminus A_f}$ be the restrictions. Since $A_{f'} = \emptyset$, there is an non-vanishing section for $\xi'_{S_{\overline{n}}}$ by Lemma 3.6, and $e_p(\xi'_{S_{\overline{n}}}) = 0$ by Proposition 3.4.

Therefore, $\beta(e_p(\xi_{S_{\overline{n}}})) = e_p(\xi'_{S_{\overline{n}}}) = 0$ and $e_p(\xi_{S_{\overline{n}}}) \in \ker(\beta) = \mathrm{Im}(\alpha)$, so there is an element

$$\mu \in H^{m(p-1)}(L_p(\overline{n}), \overline{S_{\overline{n}} \setminus A_f})$$

such that $\alpha(\mu) = e_p(\xi_{S_{\overline{n}}})$, which is not zero because $e_p(\xi_{S_{\overline{n}}})$ is not zero.

Since $\mathbb{Z}_p$ is a field, the universal coefficient theorem gives us an isomorphism

$$\hom(H_{m(p-1)}(L_p(\overline{n}), \overline{S_{\overline{n}} \setminus A_f}), \mathbb{Z}_p) \cong H^{m(p-1)}(L_p(\overline{n}), \overline{S_{\overline{n}} \setminus A_f}) \neq 0,$$

so $H_{m(p-1)}(L_p(\overline{n}), \overline{S_{\overline{n}} \setminus A_f})$ is not trivial and there is a non zero element

$$\tilde{\mu} \in H_{m(p-1)}(L_p(\overline{n}), \overline{S_{\overline{n}} \setminus A_f}).$$

By Alexander duality, we have $\check{H}^{\dim(S_{\overline{n}})-m(p-1)}(\overline{A_f}) \neq 0$, so cohom $\dim(\overline{A_f}) \geq \dim S_{\overline{n}} - m(p-1)$.

Since $\mathbb{Z}_p$ is a finite group, we have cohom $\dim(A_f) \geq$ cohom $\dim(\overline{A_f}) \geq \dim(S_{\overline{n}}) - m(p-1)$. $\square$

CHAPTER 4

Parametrized Borsuk–Ulam Theorems

4.1 Introduction and Preliminaries

Throughout this chapter, all spaces are considered to be paracompact Hausdorff. In 1988, Dold proved the following version of the Borsuk–Ulam Theorem, which considers sphere bundles, equivariant bundle maps and zero sets instead of coincidence sets. For the sphere bundle, Dold considers the bundle $SE \to B$ where SE denotes the set of unit vectors of the total space E of a vector bundle, assuming it has a norm.

Theorem 4.1 ([9, Corollary 1.5])**.** *Let $E \to B$ and $E' \to B$ be vector bundles of dimension m and n respectively, over a space B, $f\colon SE \to E'$ a fiber preserving map such that $f(-x) = -f(x)$, and $Z_f = \{x \in E \mid f(x) = 0\}$.*

If $m > n$ then

$$\text{cohom dim}(Z_f) \geq \text{cohom dim}(B) + m - n - 1.$$

This is known as a parametrized Borsuk–Ulam Theorem. It produces the Bourgin–Yang Theorem for a continuous map $g\colon S^n \to \mathbb{R}^n$, if we take B to be a single point and define $f(x) = g(x) - g(-x)$. Just like the Bourgin–Yang Theorem, there are many versions of this theorem, replacing the sphere bundle $SE \to B$ by a fiber bundle $F \to E \to B$ with free action of some compact Lie group G.

In Dold's proof, he uses the fact that the restriction $H^*(\overline{SE}) \to H^*(\overline{S^n})$ is surjective to use the Leray–Hirsch Theorem. This is not true for the restriction $H^*(\overline{E}) \to H^*(\overline{F})$ in a general fiber bundle, so this bundle is often also required to admit *cohomlogy extension of the fiber*.

Definition 4.2. A fibration $F \to E \to B$ admits cohomology extension of the fiber if the restriction $H^*(E) \to H^*(F)$ has a left inverse.

In the proofs of the theorems in Section 4.2, as well as in the beginning of Section 4.4, we will see bundles in which cohomology extension of the fiber arise naturally.

In 2015, de Mattos, Pergher, dos Santos and Singh proved the following version of the parametrized Borsuk–Ulam Theorem:

Theorem 4.3 ([13, Theorem 1.3]). *Let $E \to B$ be a $S^{n_1} \times \cdots \times S^{n_r}$ bundle, with $n_1 \leq \cdots \leq n_r$. Suppose that $E \to B$ is equipped with a fiber preserving free $\mathbb{Z}_2$-action such that the induced action on each fiber is equivalent to the diagonal antipodal action, and that the quotient bundle $\overline{E} \to B$ admits a cohomology extension of the fiber with respect to $\mathbb{Z}_2$. Let $E' \to B$ be a m-dimensional vector bundle with fiber preserving $\mathbb{Z}_2$-action which is free outside of the zero section 0. Let $f\colon E \to E'$ be a fiber preserving $\mathbb{Z}_2$-equivariant map, and $Z_f = \{x \in E \mid f(x) = 0\}$. If $n_1 \geq m$, then*

$$\text{cohom dim}(Z_f) \geq \text{cohom dim}(B) + (n_1 - m).$$

We started working on this chapter with the goal to prove parametrized Borsuk–Ulam Theorems similar to this, but with $\mathbb{Z}_p$, S^1 and S^3 actions. We ended up finding an improvement to the proof that allowed us to get a better estimate then the theorem above. This improvement also made the proof less dependent on the cohomology of the orbit spaces of the fibers, which then allowed us to prove a more general theorem, for arbitrary fiber bundles $F \to E \to B$.

In Section 4.2 of this chapter, we prove the theorems for the cases where the group acting on the product of spheres is $\mathbb{Z}_2, \mathbb{Z}_p, S^1$ or S^3, as in our original goal. Then, in Section 4.3, we prove the more general version of the theorem. Finally, in Section 4.4, we obtain the results of the first section again, as a consequence of the more general theorem, as well as some others parametrized Borsuk–Ulam and Bourgin–Yang theorems.

4.2 Parametrized Borsuk–Ulam Theorems for diagonal actions on a product of spheres

To prove the theorems in this section and the next, we will need to use a consequence of the continuity property of Čech cohomology. For more details about Čech cohomology, see [18].

We start with the case $\mathbb{Z}_2$.

Theorem 4.4. *Consider cohomology with $\mathbb{Z}_2$ coefficients. Let $E \to B$ be a $S_{\overline{n}} = S^{n_1} \times \cdots \times S^{n_r}$ bundle, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$, with free, fiber preserving, action of $\mathbb{Z}_2$ on E that induces, on each fiber, the diagonal antipodal action. Let $E' \to B$ be a m-vector bundle with free, fiber preserving, action of $\mathbb{Z}_2$ on E' that induces, on each fiber, the antipodal action. Let $f\colon E \to E'$ be a fiber preserving $\mathbb{Z}_2$-equivariant map, and $Z_f = f^{-1}(0)$, where 0 is the zero section. If the quotient bundle $\overline{E} \to B$ admits cohomology extension of the fiber and $n_1 \geq m$, then*

$$\text{cohom dim}(Z_f) \geq \text{cohom dim}(B) + \left(\sum_{k=1}^{r} n_r\right) - m$$

Proof. Consider cohomology with $\mathbb{Z}_2$ coefficients. Notice that the fiber of the quotient bundle $\overline{E' \setminus 0} \to B$ is the projective space $\overline{\mathbb{R}^m \setminus \{0\}} \simeq \mathbb{R}P^{m-1}$, which has cohomology

$$H^*(\mathbb{R}P^{m-1}) \simeq \mathbb{Z}_2[v]/\langle v^m \rangle.$$

Since the inclusion $\mathbb{R}P^{m-1} \to E' \setminus 0$ is $\mathbb{Z}_2$-equivariant, the classifying map $g_{\overline{\mathbb{R}^m \setminus \{0\}}} \colon \mathbb{R}P^{m-1} \to B_{\mathbb{Z}_2}$ can be factored as $\mathbb{R}P^{m-1} \hookrightarrow \overline{E' \setminus 0} \xrightarrow{g_{\overline{E'\setminus 0}}} B_{\mathbb{Z}_2}$.

Since $B_{\mathbb{Z}_2} = \mathbb{R}P^\infty$,

$$H^*(B_{\mathbb{Z}_2}) = \mathbb{Z}_2[s],$$

where $\dim(s) = 1$, we have that $g_{\overline{\mathbb{R}^m \setminus \{0\}}}(s) = v$. Denote $b = g_{\overline{E' \setminus 0}}(s)$. Then $b \mapsto v$ by the restriction $H^*(\overline{E' \setminus 0}) \to H^*(\mathbb{R}P^{m-1})$.

By the Leray–Hirsch theorem, $H^*(\overline{E' \setminus 0})$ is an $H^*(B)$-module with a basis

$$\{b^\varepsilon \mid 0 \le \varepsilon \le m-1)\}.$$

In particular, there are $w_k \in H^*(B)$, $1 \le k \le m$ so that

$$b^m = w_1 b^{m-1} + w_2 b^{m-2} + \cdots + w_{m-1} b + w_m.$$

Now, lets consider the bundle $E \to B$. There is an $\mathbb{Z}_2$ equivariant map $S^1 \to S_{\overline{n}}$ given by $x \mapsto (x, \ldots, x)$, considering $x \in S^{n_k}$ by the inclusion. This means that the classifying map $g_{\overline{S^1}} \colon \mathbb{R}P^1 \to B_{\mathbb{Z}_2}$ can be factored as $\mathbb{R}P^1 \to \overline{S_{\overline{n}}} \xrightarrow{g_{\overline{S_{\overline{n}}}}} B_{\mathbb{Z}_2}$.

Since

$$H^*(\overline{S_{\overline{n}}}) = \mathbb{Z}_2[u, u_2, \ldots, u_r]/\langle u^{n_1+1}, u_2^2, \ldots, u_r^2 \rangle,$$

where $\dim(u) = 1$, $\dim(u_k) = n_k$ for $2 \le k \le r$, we have that $g_{\overline{S_{\overline{n}}}}(s) = u$. Denote $a = g_{\overline{E}}(s)$. Then $a \mapsto u$ by the restriction $H^*(\overline{E}) \to H^*(\overline{S_{\overline{n}}})$. Since $E \to B$ admits cohomology extension of the fiber, there are also $a_k \in H^{n_k}(E)$ such that $a_k \mapsto u_k$, $2 \le k \le r$.

By the Leray–Hirsch theorem, $H^*(\overline{E})$ is an $H^*(B)$-module with a basis

$$\{a^\varepsilon a_2^{\varepsilon_2} \cdots a_r^{\varepsilon_r} \mid 0 \le \varepsilon \le n_1, \varepsilon_k = 0 \text{ or } 1 \text{ for each } 2 \le k \le r\}.$$

For some nonzero $w \in H^*(B)$ in the top dimension, consider the elements

$$q = w a^{n_1 - m} a_2 \cdots a_r,$$
$$p = a^m - (w_1 a^{m-1} + w_2 a^{m-2} + \cdots + w_{m-1} a + w_m),$$

of $H^*(\overline{E})$. Then,

$$pq = w a^{n_1} a_2 \cdots a_r - \big(w w_1 (a^{n_1-1} a_2 \cdots a_r) + \cdots + w w_{m-1} (a^{n_1-m+1} a_2 \cdots a_r) + w w_m (a^{n_1-m} a_2 \cdots a_r)\big) \neq 0,$$

since it is a linear combination of distinct basis elements, and $w \neq 0$.

Let $i_{\overline{Z_f}}\colon \overline{Z_f} \to \overline{E}$ be the inclusion. We will prove that the $i^*_{\overline{Z_f}}(q)$ is a nonzero element of $H^*(\overline{Z_f})$. For that end, suppose the opposite.

Then, by the continuity of the Čech cohomology, there exists an open subset V of $\overline{E}$ such that $\overline{Z_f} \subset V$ and $i^*_V(q) = 0$, where $i_V : V \to \overline{E}$ is the inclusion. Let

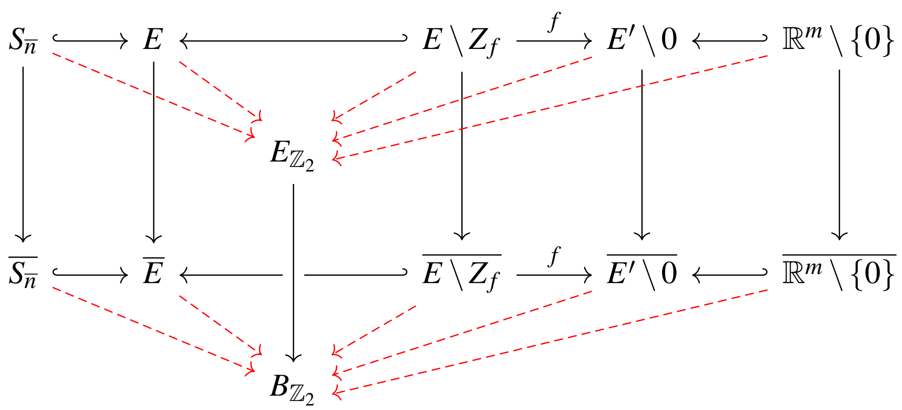
$$j_V : \overline{E} \to (\overline{E}, V)$$

be the natural inclusion and consider the long exact sequence of the pair $(\overline{E}, V)$

$$\cdots \to H^*(\overline{E}, V) \xrightarrow{j_V^*} H^*(\overline{E}) \xrightarrow{i_V^*} H^*(V) \to H^*(\overline{E}, V) \to \cdots$$

Since the sequence is exact, there exists $\mu \in H^*(\overline{E}, V)$ for which $j^*_V(\mu) = q$.

Let $\overline{f}\colon \overline{E} \to \overline{E'}$ be induced by f on the quotient. Denote the restrictions to $E \setminus Z_f \to E' \setminus 0$ and $\overline{E \setminus Z_f} \to \overline{E' \setminus 0}$ by f and $\overline{f}$ also. Then, in the diagram



the vertical arrows are principal $\mathbb{Z}_2$-bundles and, since the vertical squares commute, they are bundle maps and any composition starting on the bottom spaces and ending on $B_{\mathbb{Z}_2}$ is a classifying map. Therefore, the diagram commutes up to $\mathbb{Z}_2$-homotopy and we have,

$$i^*_{\overline{E \setminus Z_f}}(a) = i^*_{\overline{E \setminus Z_f}}(g_{\overline{E}}(s)) = \overline{f}^*(g_{\overline{E' \setminus 0}}(s)) = \overline{f}^*(b).$$

Since the induced maps are all $H^*(B)$-algebra homomorphisms, we have

$$\begin{aligned} i^*_{\overline{E \setminus Z_f}}(p) &= i^*_{\overline{E \setminus Z_f}}\left(a^m - (w_1 a^{m-1} + w_2 a^{m-2} + \cdots + w_{m-1} a + w_m)\right) \\ &= \overline{f}^*\left(b^m - (w_1 b^{m-1} + w_2 b^{m-2} + \cdots + w_{m-1} b + w_m)\right) = \overline{f}^*(0) = 0. \end{aligned}$$

Then, in the long exact sequence of the pair $(\overline{E}, \overline{E \setminus Z_f})$

$$\cdots \to H^*(\overline{E}, \overline{E \setminus Z_f}) \xrightarrow{j^*_{\overline{E \setminus Z_f}}} H^*(\overline{E}) \xrightarrow{i^*_{\overline{E \setminus Z_f}}} H^*(\overline{E \setminus Z_f}) \to H^*(\overline{E}, \overline{E \setminus Z_f}) \to \cdots$$

we have that there exists $\eta \in H^*(\overline{E}, \overline{E \setminus Z_f})$ such that $j^*_{\overline{E \setminus Z_f}}(\eta) = p$.

Naturality of the cup product gives

$$pq = j^*_{\overline{E\setminus Z_f}}(\eta) j^*_V(\mu) = j_{(\overline{E\setminus Z_f})\cup V}(\mu\eta) \in H^*(\overline{E}, (\overline{E\setminus Z_f})\cup V) = H^*(\overline{E},\overline{E}) = 0$$

which is a contradiction. Therefore, we have that $i^*_{Z_f}(q)$ is an nonzero element of dimension

$$\begin{aligned}\dim(i^*_{Z_f}(q)) = \dim(q) &= \dim(wa^{n_1-m}a_2\cdots a_r)\\ &= \dim(w) + (n_1-m)\dim(a) + \dim(a_2) + \cdots + \dim(a_r)\\ &= \text{cohom}\dim(B) + n_1 - m + n_2 + \cdots + n_r,\end{aligned}$$

and the theorem follows from Theorem 1.5. □

The theorem for the case $\mathbb{Z}_p$ is very similar, but there is one more generator in the cohomology rings in the proof.

Theorem 4.5. *Consider cohomology with $\mathbb{Z}_p$ coefficients. Let $E \to B$ be a $S_{\overline{n}} = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$ bundle, $1 \le n_1 \le n_2 \le \cdots \le n_r$, with free, fiber preserving, action of $\mathbb{Z}_p$ on E that induces, on each fiber, the diagonal action for the canonical $\mathbb{Z}_p$ action on each sphere. Let $E' \to B$ be a $2m$-vector bundle with free, fiber preserving, action of $\mathbb{Z}_p$ on E' that induces, on each fiber, the action given by complex multiplication. Let $f\colon E \to E'$ be a fiber preserving $\mathbb{Z}_p$-equivariant map, and $Z_f = f^{-1}(0)$, where 0 is the zero section. If the quotient bundle $\overline{E} \to B$ admits cohomology extension of the fiber and $n_1 \ge m$, then*

$$\text{cohom}\dim(Z_f) \ge \text{cohom}\dim(B) + \left(\sum_{k=1}^{r} 2n_r + 1\right) - 2m$$

Proof. Consider cohomology with $\mathbb{Z}_p$ coefficients. Notice that the fiber of the quotient bundle $\overline{E'\setminus 0} \to B$ is the lens space $\overline{\mathbb{R}^{2m}\setminus\{0\}} \simeq L_p(m-1)$, which has cohomology

$$H^*(L_p(m-1)) \simeq \mathbb{Z}_p[\tilde{v}, v]/\langle \tilde{v}^2, v^m\rangle.$$

Since the inclusion $L_p(m-1) \to E'\setminus 0$ is $\mathbb{Z}_p$-equivariant, the classifying map $g_{\overline{\mathbb{R}^{2m}\setminus\{0\}}}\colon L_p(m-1) \to B_{\mathbb{Z}_p}$ can be factored as $L_p(m-1) \hookrightarrow \overline{E'\setminus 0} \xrightarrow{g_{\overline{E'\setminus 0}}} B_{\mathbb{Z}_p}$.

Since $B_{\mathbb{Z}_p} = L_p(\infty)$,

$$H^*(B_{\mathbb{Z}_p}) = \mathbb{Z}_p[\tilde{s}, s]/\langle \tilde{s}^2\rangle,$$

where $\dim(\tilde{s}) = 1$ and $\dim(s) = 2$, we have that $g_{\overline{\mathbb{R}^{2m}\setminus\{0\}}}(\tilde{s}) = \tilde{v}$ and $g_{\overline{\mathbb{R}^{2m}\setminus\{0\}}}(s) = v$. Denote $\tilde{b} = g_{\overline{E'\setminus 0}}(\tilde{s})$ and $b = g_{\overline{E'\setminus 0}}(s)$. Then $\tilde{b} \mapsto \tilde{v}$, $b \mapsto v$ by the restriction $H^*(\overline{E'\setminus 0}) \to H^*(L_p(m-1))$.

By the Leray–Hirsch theorem, $H^*(\overline{E'\setminus 0})$ is an $H^*(B)$-module with a basis

$$\{\tilde{b}^{\tilde{\varepsilon}} b^{\varepsilon} \mid \tilde{\varepsilon} = 0 \text{ or } 1, 0 \le \varepsilon \le m-1)\}.$$

In particular, there are $w_k \in H^*(B)$, $1 \le k \le 2m$ so that

$$b^m = w_1 b^{m-1}\tilde{b} + w_2 b^{m-1} + \cdots + w_{2m-2} b + w_{2m-1}\tilde{b} + w_{2m}.$$

Now, lets consider the bundle $E \to B$. There is an $\mathbb{Z}_p$ equivariant map $S^3 \to S_{\overline{n}}$ given by $x \mapsto (x,\dots,x)$, considering $x \in S^{2n_k+1}$ by the inclusion. This means that the classifying map $g_{\overline{S^3}}\colon L_p(1) \to B_{\mathbb{Z}_p}$ can be factored as $L_p(1) \to \overline{S_{\overline{n}}} \xrightarrow{g_{\overline{S_{\overline{n}}}}} B_{\mathbb{Z}_p}$.

Since

$$H^*(\overline{S_{\overline{n}}}) = \mathbb{Z}_p[\tilde{u}, u, u_2, \dots, u_r]/\langle \tilde{u}^2, u^{n_1+1}, u_2^2, \dots, u_r^2\rangle,$$

where $\dim(\tilde{u}) = 1$, $\dim(u) = 2$, $\dim(u_k) = 2n_k + 1$ for $2 \le k \le r$, we have that $g_{\overline{S_{\overline{n}}}}(\tilde{s}) = \tilde{z}\tilde{u}$ and $g_{\overline{S_{\overline{n}}}}(s) = zu$ for some nonzero $\tilde{z}, z \in \mathbb{Z}_p$. For simplicity, we may suppose $g_{\overline{S_{\overline{n}}}}(\tilde{s}) = \tilde{u}$ and $g_{\overline{S_{\overline{n}}}}(s) = u$ (this would only take a change of basis). Denote $\tilde{a} = g_{\overline{E}}(\tilde{s})$ and $a = g_{\overline{E}}(s)$. Then $\tilde{a} \mapsto \tilde{u}$, $a \mapsto u$ by the restriction $H^*(\overline{E}) \to H^*(\overline{S_{\overline{n}}})$. Since $E \to B$ admits cohomology extension of the fiber, there are also $a_k \in H^{2n_k+1}(E)$ such that $a_k \mapsto u_k$, $2 \le k \le r$.

By the Leray–Hirsch theorem, $H^*(\overline{E})$ is an $H^*(B)$-module with a basis

$$\{\tilde{a}^{\tilde{\varepsilon}} a^{\varepsilon} a_2^{\varepsilon_2} \cdots a_r^{\varepsilon_r} \mid \tilde{\varepsilon} = 0 \text{ or } 1, 0 \le \varepsilon \le n_1, \varepsilon_k = 0 \text{ or } 1 \text{ for each } 2 \le k \le r\}.$$

For some nonzero $w \in H^*(B)$ in the top dimension, consider the elements

$$\begin{aligned} q &= w\tilde{a}a^{n_1-m}a_2 \cdots a_r, \\ p &= a^m - (w_1 a^{m-1}\tilde{a} + w_2 a^{m-1} + \cdots + w_{2m-2}a + w_{2m-1}\tilde{a} + w_{2m}), \end{aligned}$$

of $H^*(\overline{E})$. Then,

$$\begin{aligned} pq = w\tilde{a}a^{n_1}a_2 \cdots a_r - \big(ww_2(\tilde{a}a^{n_1-1}a_2 \cdots a_r) + \cdots + ww_{2m-2}(\tilde{a}a^{n_1-m+1}a_2 \cdots a_r) \\ + ww_{2m}(\tilde{a}a^{n_1-m}a_2 \cdots a_r)\big) \neq 0, \end{aligned}$$

since it is a linear combination of distinct basis elements, and $w \neq 0$.

Let $i_{\overline{Z_f}}\colon \overline{Z_f} \to \overline{E}$ be the inclusion. We will prove that the $i^*_{\overline{Z_f}}(q)$ is a nonzero element of $H^*(\overline{Z_f})$. For that end, suppose the opposite.

Then, by the continuity of the Čech cohomology, there exists an open subset V of $\overline{E}$ such that $\overline{Z_f} \subset V$ and $i_V^*(q) = 0$, where $i_V\colon V \to \overline{E}$ is the inclusion. Let

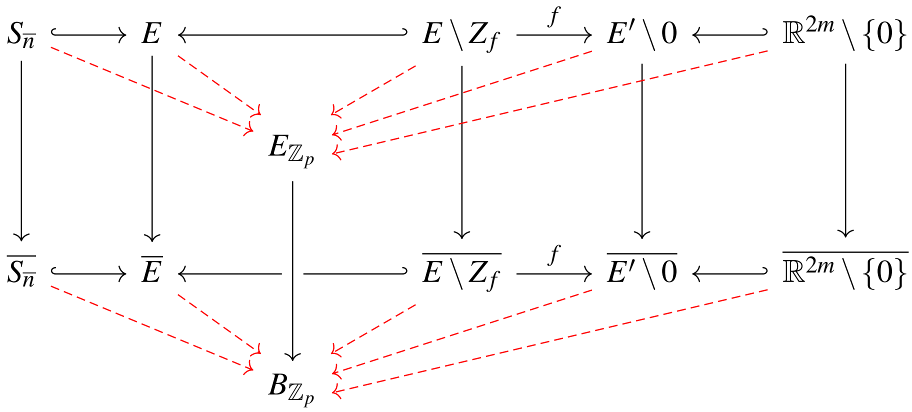
$$j_V\colon \overline{E} \to (\overline{E}, V)$$

be the natural inclusion and consider the long exact sequence of the pair $(\overline{E}, V)$

$$\cdots \to H^*(\overline{E}, V) \xrightarrow{j_V^*} H^*(\overline{E}) \xrightarrow{i_V^*} H^*(V) \to H^*(\overline{E}, V) \to \cdots$$

Since the sequence is exact, there exists $\mu \in H^*(\overline{E}, V)$ for which $j_V^*(\mu) = q$.

Let $\overline{f}\colon \overline{E} \to \overline{E'}$ be induced by f on the quotient. Denote the restrictions to $E \setminus Z_f \to E' \setminus 0$ and $\overline{E \setminus Z_f} \to \overline{E' \setminus 0}$ by f and $\overline{f}$ also. Then, in the diagram



the vertical arrows are principal $\mathbb{Z}_p$-bundles and, since the vertical squares commute, they are bundle maps and any composition starting on the bottom spaces and ending on $B_{\mathbb{Z}_p}$ is a classifying map. Therefore, the diagram commutes up to $\mathbb{Z}_p$-homotopy and we have,

$$i^*_{\overline{E\setminus Z_f}}(a) = i^*_{\overline{E\setminus Z_f}}(g_{\overline{E}}(s)) = \overline{f}^*(g_{\overline{E'\setminus 0}}(s)) = \overline{f}^*(b),$$
$$i^*_{\overline{E\setminus Z_f}}(\tilde{a}) = i^*_{\overline{E\setminus Z_f}}(g_{\overline{E}}(\tilde{s})) = \overline{f}^*(g_{\overline{E'\setminus 0}}(\tilde{s})) = \overline{f}^*(\tilde{b}).$$

Since the induced maps are all $H^*(B)$-algebra homomorphisms, we have

$$\begin{aligned} i^*_{\overline{E\setminus Z_f}}(p) &= i^*_{\overline{E\setminus Z_f}}\left(a^m - (w_1 a^{m-1}\tilde{a} + w_2 a^{m-1} + \cdots + w_{2m-2}a + w_{2m-1}\tilde{a} + w_{2m})\right) \\ &= \overline{f}^*\left(b^m - (w_1 b^{m-1}\tilde{b} + w_2 b^{m-1} + \cdots + w_{2m-2}b + w_{2m-1}\tilde{b} + w_{2m})\right) = \overline{f}^*(0) = 0. \end{aligned}$$

Then, in the long exact sequence of the pair $(\overline{E}, \overline{E\setminus Z_f})$

$$\cdots \to H^*(\overline{E}, \overline{E\setminus Z_f}) \xrightarrow{j^*_{\overline{E\setminus Z_f}}} H^*(\overline{E}) \xrightarrow{i^*_{\overline{E\setminus Z_f}}} H^*(\overline{E\setminus Z_f}) \to H^*(\overline{E}, \overline{E\setminus Z_f}) \to \cdots$$

we have that there exists $\eta \in H^*(\overline{E}, \overline{E\setminus Z_f})$ such that $j^*_{\overline{E\setminus Z_f}}(\eta) = p$.

Naturality of the cup product gives

$$pq = j^*_{\overline{E\setminus Z_f}}(\eta) j^*_V(\mu) = j_{(\overline{E\setminus Z_f})\cup V}(\mu\eta) \in H^*(\overline{E}, (\overline{E\setminus Z_f}) \cup V) = H^*(\overline{E}, \overline{E}) = 0$$

which is a contradiction. Therefore, we have that $i^*_{Z_f}(q)$ is an nonzero element of dimension

$$\begin{aligned} \dim(i^*_{Z_f}(q)) = \dim(q) &= \dim(w\tilde{a}a^{n_1-m}a_2\cdots a_r) \\ &= \dim(w) + \dim(\tilde{a}) + (n_1 - m)\dim(a) + \dim(a_2) + \cdots + \dim(a_r) \\ &= \operatorname{cohom}\dim(B) + 1 + 2n_1 - 2m + (2n_2+1) + \cdots + (2n_r+1), \end{aligned}$$

and the theorem follows from Theorem 1.5. □

The proofs for the other theorems are, again, very similar. We go back to having just one generator in the cohomology rings. Compared to the $\mathbb{Z}_2$ case, the proofs need only to adapt the dimensions of these generators. We will omit most details.

Theorem 4.6. *Let $E \to B$ be a $S_{\overline{n}} = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$ bundle, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$, with free, fiber preserving, action of S^1 on E that induces, on each fiber, the diagonal action for the canonical S^1 action on each sphere. Let $E' \to B$ be a $2m$-vector bundle with free, fiber preserving, action of S^1 on E' that induces, on each fiber, the action given by complex multiplication. Let $f\colon E \to E'$ be a fiber preserving S^1-equivariant map, and $Z_f = f^{-1}(0)$, where 0 is the zero section. If the quotient bundle $\overline{E} \to B$ admits cohomology extension of the fiber and $n_1 \geq m$, then*

$$\text{cohom dim}(Z_f) \geq \text{cohom dim}(B) + \left(\sum_{k=1}^{r} 2n_r + 1\right) - 2m - 1$$

Proof. By the Leray–Hirsch Theorem, $H^*(\overline{E' \setminus 0})$ is an $H^*(B)$-module with a basis

$$\{b^{\varepsilon} \mid 0 \leq \varepsilon \leq m-1)\},$$

where $\dim(b) = 2$. Then, there are $w_k \in H^*(B)$, $\dim(w_k) = 2k$, $1 \leq k \leq m$ so that

$$b^m = w_1 b^{m-1} + \cdots + w_{m-1} b + w_m.$$

On the other hand, $H^*(\overline{E})$ is an $H^*(B)$-module with basis

$$\{a^{\varepsilon} a_2^{\varepsilon_2} \cdots a_r^{\varepsilon_r} \mid 0 \leq \varepsilon \leq n_1, \varepsilon_k = 0 \text{ or } 1 \text{ for each } 2 \leq k \leq r\},$$

where $\dim(a) = 2$ and $\dim(a_k) = 2n_k + 1$ for each $2 \leq k \leq r$.

For some nonzero $w \in H^*(B)$ in the top dimension, consider the elements

$$q = wa^{n_1-m} a_2 \cdots a_r,$$
$$p = a^m - (w_1 a^{m-1} + \cdots + w_{m-1} a + w_m),$$

of $H^*(\overline{E})$. Then,

$$pq = wa^{n_1} a_2 \cdots a_r - \big(ww_1(a^{n_1-1} a_2 \cdots a_r) + \cdots + ww_{m-1}(a^{n_1-m+1} a_2 \cdots a_r) + ww_m(a^{n_1-m} a_2 \cdots a_r)\big) \neq 0,$$

since it is a linear combination of distinct basis elements, and $w \neq 0$.

Analogous to the previous proof, we have that $i_{Z_f}^*(q) \neq 0$ and

$$\begin{aligned}
\dim(i_{Z_f}^*(q)) = \dim(q) &= \dim(wa^{n_1-m} a_2 \cdots a_r) \\
&= \dim(w) + (n_1 - m)\dim(a) + \dim(a_2) + \cdots + \dim(a_r) \\
&= \text{cohom dim}(B) + 2n_1 - 2m + (2n_2+1) + \cdots + (2n_r+1),
\end{aligned}$$

and the theorem follows from Theorem 1.5. □

Theorem 4.7. *Let $E \to B$ be a $S_{\overline{n}} = S^{4n_1+3} \times \cdots \times S^{4n_r+3}$ bundle, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$, with free, fiber preserving, action of S^3 on E that induces, on each fiber, the diagonal action for the canonical S^3 action on each sphere. Let $E' \to B$ be a $4m$-vector bundle with free, fiber preserving, action of S^3 on E' that induces, on each fiber, the action given by quaternionic multiplication. Let $f\colon E \to E'$ be a fiber preserving S^3-equivariant map, and $Z_f = f^{-1}(0)$, where 0 is the zero section. If the quotient bundle $\overline{E} \to B$ admits cohomology extension of the fiber and $n_1 \geq m$, then*

$$\text{cohom dim}(Z_f) \geq \text{cohom dim}(B) + \left(\sum_{k=1}^{r} 4n_r + 3\right) - 4m - 3$$

Proof. By the Leray–Hirsch Theorem, $H^*(\overline{E' \setminus 0})$ is an $H^*(B)$-module with a basis

$$\{b^{\varepsilon} \mid 0 \leq \varepsilon \leq m-1)\},$$

where $\dim(b) = 4$. Then, there are $w_k \in H^*(B)$, $\dim(w_k) = 4k$, $1 \leq k \leq m$ so that

$$b^m = w_1 b^{m-1} + \cdots + w_{m-1} b + w_m.$$

On the other hand, $H^*(\overline{E})$ is an $H^*(B)$-module with basis

$$\{a^{\varepsilon} a_2^{\varepsilon_2} \cdots a_r^{\varepsilon_r} \mid 0 \leq \varepsilon \leq n_1, \varepsilon_k = 0 \text{ or } 1 \text{ for each } 2 \leq k \leq r\},$$

where $\dim(a) = 4$ and $\dim(a_k) = 4n_k + 3$ for each $2 \leq k \leq r$.

For some nonzero $w \in H^*(B)$ in the top dimension, consider the elements

$$\begin{aligned} q &= w a^{n_1 - m} a_2 \cdots a_r, \\ p &= a^m - (w_1 a^{m-1} + \cdots + w_{m-1} a + w_m), \end{aligned}$$

of $H^*(\overline{E})$. Then,

$$\begin{aligned} pq = w a^{n_1} a_2 \cdots a_r - \big(w w_1 (a^{n_1 - 1} a_2 \cdots a_r) + \cdots + w w_{m-1} (a^{n_1 - m + 1} a_2 \cdots a_r) \\ + w w_m (a^{n_1 - m} a_2 \cdots a_r)\big) \neq 0, \end{aligned}$$

since it is a linear combination of distinct basis elements, and $w \neq 0$.

Analogous to the previous proof, we have that $i_{Z_f}^*(q) \neq 0$ and

$$\begin{aligned} \dim(i_{Z_f}^*(q)) = \dim(q) &= \dim(w a^{n_1 - m} a_2 \cdots a_r) \\ &= \dim(w) + (n_1 - m)\dim(a) + \dim(a_2) + \cdots + \dim(a_r) \\ &= \text{cohom dim}(B) + 4n_1 - 4m + (4n_2 + 3) + \cdots + (4n_r + 3), \end{aligned}$$

and the theorem follows from Theorem 1.5. □

4.3 A Parametrized Borsuk-Ulam Theorem for Arbitrary Fiber Bundles

In Dold's parametrized Borsuk–Ulam Theorem, notice that the zero section of the vector bundle $E' \to B$ is the set $\mathrm{Fix}(E') = \{x \in E' \mid gx = x \text{ for some } g \in G, g \neq 1\}$. We will use this interpretation to generalize the theorem by estimating cohom $\dim(Z_f)$ where $Z_f = f^{-1}(\mathrm{Fix}(E'))$ for a G-equivariant bundle map $f\colon E \to E'$. We generalize it a bit further by considering not just $\mathrm{Fix}(E')$, but some subset of E' containing $\mathrm{Fix}(E')$. This is described, more precisely, by considering $(E', E'_0) \to B$ to be a fiber bundle pair with action of G, such that the action of G is free on E'_0, so $\mathrm{Fix}(E') \subset E' \setminus E'_0$, and we estimate cohom $\dim f^{-1}(E' \setminus E'_0)$.

Our generalization needs a very lengthy setup, but we hope to show in the next section that, when considering fibers and actions with known cohomology for the orbit spaces, it is straightforward to check that the hypothesis holds.

Theorem 4.8. *Let G be a compact Lie group, $F \to E \to B$ be a fiber bundle with free action of G, $(F', F'_0) \to (E', E'_0) \to B$ be a fiber bundle pair with action of G that is free on E'_0, E'_0 open, $H^k(\overline{F})$ and $H^k(\overline{F'_0})$ finitely generated free R-modules for each integer k, $f\colon E \to E'$ be a fiber preserving, G-equivariant map, and $Z_f = f^{-1}(E' \setminus E'_0)$.*

Suppose that for some $1 \le r' \le r_0 \le r'' \in \mathbb{Z}$, there are $s_1, \dots, s_{r_0} \in H^*(B_G)$, $u_k = g^*_{\overline{F}}(s_k)$, $u_{r_0+1}, \dots, u_{r''} \in H^*(\overline{F})$, $v_k = g_{\overline{F'_0}}(s_k)$ and positive integers n_k, m_k and $n_{r_0+1}, \dots, n_{r''}$ for each $1 \le k \le r_0$ such that:

- *$\mathscr{U} = \{u_1^{\varepsilon_1} \cdots u_{r''}^{\varepsilon_{r''}} \mid 0 \le \varepsilon_k \le n_k \text{ for each } k\}$ can be completed to a basis of $H^*(\overline{F})$,*
- *$u_1^{\varepsilon_1} \cdots u_{r''}^{\varepsilon_{r''}} = u_1^{\varepsilon'_1} \cdots u_{r''}^{\varepsilon'_{r''}} \implies \varepsilon_k = \varepsilon'_k$ for $\varepsilon_k, \varepsilon'_k \in \{0, \dots, n_k\}$,*
- *$\mathscr{V} = \{v_1^{\varepsilon_1} \cdots v_{r_0}^{\varepsilon_{r_0}} \mid 0 \le \varepsilon_k \le m_k \text{ for each } k\}$ can be completed to a basis of $H^*(\overline{F'_0})$ and contain a basis for $\bigoplus_{k=0}^{(m_1+1)\dim(s_1)} H^k(\overline{F'_0})$,*
- *$n_1 > m_1$ and $n_k \ge m_k$ for each $k \in \{2, \dots, r_0\}$,*
- *$s_k^{n_k+1} = 0$ for all $r' < k \le r_0$.*

We consider $0^0 = 1$, so $u_k = 0$ with $n_k = 0$ or $v_k = 0$ with $m_k = 0$ is allowed in $\mathscr{U}$ and $\mathscr{V}$. If the quotient bundles $\overline{F} \to \overline{E} \to B$ and $(\overline{F'}, \overline{F'_0}) \to (\overline{E'}, \overline{E'_0}) \to B$ admit cohomology extension of the fiber, then

$$\text{cohom}\dim(Z_f) \ge \text{cohom}\dim(B) - \dim(u_1) + \sum_{k=1}^{r'} (n_k - m_k)\dim(u_k) + \sum_{k=r'+1}^{r''} n_k \dim(u_k).$$

Remark 4.9. Notice that the hypothesis that $u_1^{\varepsilon_1} \cdots u_{r''}^{\varepsilon_{r''}} = u_1^{\varepsilon'_1} \cdots u_{r''}^{\varepsilon'_{r''}} \implies \varepsilon_k = \varepsilon'_k$ for $\varepsilon_k, \varepsilon'_k \in \{0, \dots, n_k\}$ is not redundant with $\mathscr{U}$ being linearly independent. For example, consider $S^2 \times S^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$

acts diagonally as the antipodal action in each sphere. Then $H^*(S^2 \times S^2/\mathbb{Z}_2) \simeq \mathbb{Z}_2[u_1,u_2]/\langle u_1^3, u_2^2 - u_1^2u_2\rangle$. The set $\mathscr{U} = \{u_1^{\varepsilon_1}u_2^{\varepsilon_2} \mid 0 \le \varepsilon_1 \le 2, 0 \le \varepsilon_2 \le 2\} = \{1, u_1, u_1^2, u_2, u_2^2, u_1u_2\}$ is a basis for $H^*(S^2 \times S^2/\mathbb{Z}_2)$, but u_2^2 is not written uniquely, since $u_2^2 = u_1^2u_2$. It becomes unique, however, if we take $n_1 = 2$ and $n_2 = 1$ so that, for $\varepsilon_1 \in \{0,1,2\}, \varepsilon_2 \in \{0,1\}$, we have $u_1^2u_2 = u_1^{\varepsilon_1}u_2^{\varepsilon_2 2} \implies \varepsilon_1 = 2, \varepsilon_2 = 1$.

Remark 4.10. Allowing $0^0 = 1$ is needed, for example, for S^1-equivariant maps $f\colon E \to E'$, where $E \to B$ is an fiber bundle with fiber S^3 and $E' \to B$ is an vector bundle of dimension 2. Then, $H^*(B_{S^1}) = R[s]$, where $\dim(s) = 2$ and $g^*_{\overline{\mathbb{R}^2\setminus\{0\}}}(s) = 0$, since $H^2(\overline{\mathbb{R}^2\setminus\{0\}}) = H^2(S^1/S^1) = 0$. In this case, Theorem 4.6 gives us an estimate for $\operatorname{cohom\,dim}(Z_f)$, but Theorem 4.8 would not if we don't allow $0^0 = 1$.

Proof of Theorem 4.8. Let $b_k = g^*_{\overline{E'_0}}(s_k) \in H^*(\overline{E'_0})$ for each $k \in \{1,\dots,r_0\}$. Notice that $b_1^{\varepsilon_1}\cdots b_{r_0}^{\varepsilon_{r_0}} \mapsto v_1^{\varepsilon_1}\cdots v_{r_0}^{\varepsilon_{r_0}}$ by the restriction, $0 \le \varepsilon_k \le m_k$.

For each $v_1^{\varepsilon_1}\cdots v_{r_0}^{\varepsilon_{r_0}}$, we choose one $(\varepsilon'_1,\dots,\varepsilon'_r)$ such that $b_1^{\varepsilon_1}\cdots b_{r_0}^{\varepsilon_{r_0}} \mapsto v_1^{\varepsilon_1}\cdots v_{r_0}^{\varepsilon_{r_0}}$ and let $\mathscr{E}$ be the set of these chosen $(\varepsilon'_1,\dots,\varepsilon'_r)$. We do this to avoid situations where we have $v_1^{\varepsilon_1}\cdots v_{r_0}^{\varepsilon_{r_0}} = v_1^{\varepsilon'_1}\cdots v_{r_0}^{\varepsilon'_{r_0}}$ but $b_1^{\varepsilon_1}\cdots b_{r_0}^{\varepsilon_{r_0}} \neq b_1^{\varepsilon'_1}\cdots b_{r_0}^{\varepsilon'_{r_0}}$, so the mapping $b_1^{\varepsilon_1}\cdots b_{r_0}^{\varepsilon_{r_0}} \mapsto v_1^{\varepsilon_1}\cdots v_{r_0}^{\varepsilon_{r_0}}$ is not injective.

By the Leray–Hirsch theorem, $H^*(\overline{E'_0})$ is a $H^*(B)$-module with a basis that contains $\mathscr{B} = \{b_1^{\varepsilon_1}\cdots b_{r_0}^{\varepsilon_{r_0}} \mid (\varepsilon_1,\dots,\varepsilon_{r_0}) \in \mathscr{E}\}$.

Since $\mathscr{V}$ generates $\bigoplus_{k=0}^{(m_1+1)\dim(s_1)} H^k(\overline{F'_0})$, we have that $\mathscr{B}$ generates $\bigoplus_{k=0}^{(m_1+1)\dim(s_1)} H^k(\overline{E'_0})$, so $b_1^{m_1+1}$ can be written uniquely as

$$b_1^{m_1+1} = \sum_{(\varepsilon_1,\dots,\varepsilon_{r_0})\in\mathscr{E}} w_{(\varepsilon_1,\dots,\varepsilon_{r_0})} b_1^{\varepsilon_1}\cdots b_{r_0}^{\varepsilon_{r_0}}$$

for some $w_{(\varepsilon_1,\dots,\varepsilon_{r_0})} \in H^*(B)$.

Now, let $a_k = g^*_{\overline{E}}(s_k) \in H^*(\overline{E})$ for each $k \in \{1,\dots,r''\}$. Again, notice that $a_1^{\varepsilon_1}\cdots a_{r''}^{\varepsilon_{r''}} \mapsto u_1^{\varepsilon_1}\cdots u_{r''}^{\varepsilon_{r''}}$ by the restriction.

In this case, since we required that $u_1^{\varepsilon_1}\cdots u_{r''}^{\varepsilon_{r''}} = u_1^{\varepsilon'_1}\cdots u_{r''}^{\varepsilon'_{r''}} \implies \varepsilon_k = \varepsilon'_k$, we have that $a_1^{\varepsilon_1}\cdots a_{r''}^{\varepsilon_{r''}} \mapsto u_1^{\varepsilon_1}\cdots u_{r''}^{\varepsilon_{r''}}$ is injective for $0 \le \varepsilon_k \le n_k$, $k \in \{1,\dots,r''\}$.

By the Leray–Hirsch theorem, $H^*(\overline{E})$ is a $H^*(B)$-module with a basis that contains $\{a_1^{\varepsilon_1}\cdots a_{r''}^{\varepsilon_{r''}} \mid 0 \le \varepsilon_k \le n_k \text{ for each } k\}$.

For an arbitrary $w \in H^*(B)$, with $w \neq 0$, consider the elements

$$q = w a_1^{n_1-m_1-1} a_2^{n_2-m_2} \cdots a_{r'}^{n_{r'}-m_{r'}} a_{r'+1}^{n_{r'+1}} \cdots a_{r''}^{n_{r''}},$$
$$p = a_1^{m_1+1} - \sum_{(\varepsilon_1,\dots,\varepsilon_{r_0})\in\mathscr{E}} w_{(\varepsilon_1,\dots,\varepsilon_{r_0})} a_1^{\varepsilon_1}\cdots a_{r_0}^{\varepsilon_{r_0}},$$

of $H^*(\overline{E})$. Then,

$$pq = wa_1^{n_1} a_2^{n_2-m_2} \cdots a_{r'}^{n_{r'}-m_{r'}} a_{r'+1}^{n_{r'+1}} \cdots a_{r''}^{n_{r''}} \\ - \sum w w_{(\varepsilon_1,\dots,\varepsilon_{k_0},0,\dots,0)} a_1^{n_1-m_1-1+\varepsilon_1} a_2^{n_2-m_2+\varepsilon_2} \cdots a_{r'}^{n_{r'}-m_{r'}+\varepsilon_{r'}} a_{r'+1}^{n_{r'+1}} \cdots a_{r''}^{n_{r''}} \neq 0,$$

since each $\varepsilon_k \leq m_k$, so it is a linear combination of distinct basis elements, and $w \neq 0$.

Let $i_{\overline{Z_f}} \colon \overline{Z_f} \to \overline{E}$ be the inclusion. We will prove that the $i_{\overline{Z_f}}^*(q)$ is a nonzero element of $H^*(\overline{Z_f})$. For that end, suppose the opposite.

Then, by the continuity of the Čech cohomology, there exists an open subset V of $\overline{E}$ such that $\overline{Z_f} \subset V$ and $i_V^*(q) = 0$, where $i_V \colon V \to \overline{E}$ is the inclusion. Let

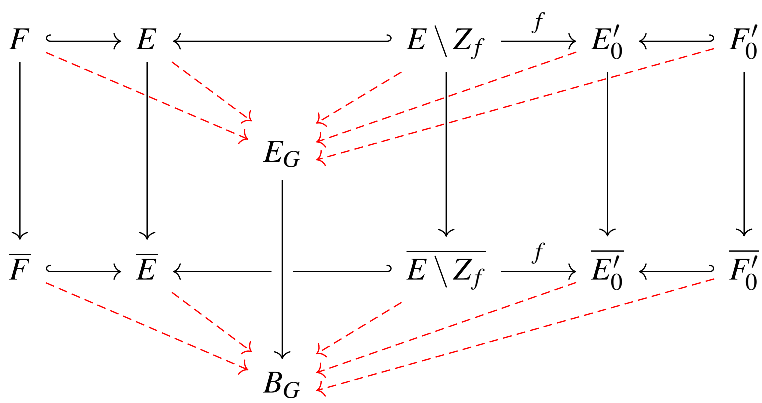
$$j_V \colon \overline{E} \to (\overline{E}, V)$$

be the natural inclusion and consider the long exact sequence of the pair $(\overline{E}, V)$

$$\cdots \to H^*(\overline{E}, V) \xrightarrow{j_V^*} H^*(\overline{E}) \xrightarrow{i_V^*} H^*(V) \to H^*(\overline{E}, V) \to \cdots$$

Since the sequence is exact, there exists $\mu \in H^*(\overline{E}, V)$ for which $j_V^*(\mu) = q$.

Let $\overline{f} \colon \overline{E} \to \overline{E'}$ be induced by f on the quotient. Denote the restrictions to $E \setminus Z_f \to E_0'$ and $\overline{E \setminus Z_f} \to \overline{E_0'}$ by f and $\overline{f}$ also. Then, in the diagram



the vertical arrows are principal G-bundles and, since the vertical squares commute, they are bundle maps and any composition starting on the bottom spaces and ending on B_G is a classifying map. Therefore, the diagram commutes up to G-homotopy and we have, for each $k \in \{1, \dots, r_0\}$,

$$i_{\overline{E \setminus Z_f}}^*(a_k) = i_{\overline{E \setminus Z_f}}^*(g_{\overline{E}}(s_k)) = \overline{f}^*(g_{\overline{E_0'}}(s_k)) = \overline{f}^*(b_k).$$

Since the induced maps are all $H^*(B)$-algebra homomorphisms, we have

$$i_{\overline{E \setminus Z_f}}^*(p) = \overline{f}^* \left(b_1^{m+1} - \sum w_{(\varepsilon_1,\dots,\varepsilon_{r_0})} b_1^{\varepsilon_1} \cdots b_{r_0}^{\varepsilon_{r_0}} \right) = \overline{f}^*(0) = 0.$$

Then, in the long exact sequence of the pair $(\overline{E}, \overline{E \setminus Z_f})$

$$\cdots \to H^*(\overline{E}, \overline{E \setminus Z_f}) \xrightarrow{j_{\overline{E \setminus Z_f}}^*} H^*(\overline{E}) \xrightarrow{i_{\overline{E \setminus Z_f}}^*} H^*(\overline{E \setminus Z_f}) \to H^*(\overline{E}, \overline{E \setminus Z_f}) \to \cdots$$

we have that there exists $\eta \in H^*(\overline{E}, \overline{E\setminus Z_f})$ such that $j^*_{\overline{E\setminus Z_f}}(\eta) = p$.

Naturality of the cup product gives

$$pq = j^*_{\overline{E\setminus Z_f}}(\eta) j^*_V(\mu) = j_{(\overline{E\setminus Z_f})\cup V}(\mu\eta) \in H^*(\overline{E}, (\overline{E\setminus Z_f})\cup V) = H^*(\overline{E},\overline{E}) = 0$$

which is a contradiction. Since $w \in H^*(B)$ was taking arbitrarily, it can be taken in the top dimension of $H^*(B)$, so that q is an nonzero element of dimension

$$\begin{aligned}
\dim(q) =& \dim(w a_1^{n_1-m_1-1} a_2^{n_2-m_2} \cdots a_{r'}^{n_{r'}-m_{r'}} a_{r'+1}^{n_{r'+1}} \cdots a_{r''}^{n_{r''}}) \\
=& \dim(w) + (n_1 - m_1 - 1)\dim(a_1) \\
&+ (n_2 - m_2)\dim(a_2) + \cdots + (n_{r'} - m_{r'})\dim(a_{r'}) \\
&+ n_{r'+1}\dim(a_{r'+1}) + \cdots + n_{r''}\dim(a_{r''}) \\
=& \operatorname{cohom}\dim(B) - \dim(u_1) + \sum_{k=1}^{r'} (n_k - m_k)\dim(u_k) + \sum_{k=r'+1}^{r''} n_k \dim(u_k).
\end{aligned}$$

□

Some of the hypothesis, like requiring cohomology extension of the fiber, are needed just to use the Leray–Hirsh Theorem. By considering B to be a single point we produce a Bourgin–Yang type theorem as a corollary. We can do a bit better then this possible corollary though, as we also don't need to require $H^k(\overline{F})$ and $H^k(\overline{F_0'})$ to be finitely generated and free, and some of the writing can be simplified. We state and prove this Bourgin–Yang version of Theorem 4.8 bellow. The proof is mostly the same, so we omit some details.

Theorem 4.11. *Let G be a compact Lie group, X be a space with free action of G, Y be a space with action of G that is free on an open, G-invariant, subset $Y_0 \subset Y$, $f\colon X \to Y$ be a G-equivariant map, and $Z_f = f^{-1}(Y \setminus Y_0)$.*

Suppose that for some $1 \le r' \le r_0 \le r'' \in \mathbb{Z}$, there are $s_1, \ldots, s_{r_0} \in H^*(B_G)$, $u_k = g^*_{\overline{X}}(s_k)$, $u_{r_0+1}, \ldots, u_{r''} \in H^*(\overline{X})$, $v_k = g_{\overline{Y_0}}(s_k)$ and positive integers n_k, m_k and $n_{r_0+1}, \ldots, n_{r''}$ for each $1 \le k \le r_0$ such that:

- $\mathscr{U} = \{u_1^{\varepsilon_1} \cdots u_{r''}^{\varepsilon_{r''}} \mid 0 \le \varepsilon_k \le n_k$ *for each* $k\}$ *is linearly independent,*
- $u_1^{\varepsilon_1} \cdots u_{r''}^{\varepsilon_{r''}} = u_1^{\varepsilon_1'} \cdots u_{r''}^{\varepsilon_{r''}'} \implies \varepsilon_k = \varepsilon_k'$ *for* $\varepsilon_k, \varepsilon_k' \in \{0, \ldots, n_k\}$,
- $v_1^{m_1+1}$ *is an element of the set generated by* $\mathscr{V} = \{v_1^{\varepsilon_1} \cdots v_{r_0}^{\varepsilon_{r_0}} \mid 0 \le \varepsilon_k \le m_k$ *for each* $k\}$,
- $n_1 > m_1$ *and* $n_k \ge m_k$ *for each* $k \in \{2, \ldots, r_0\}$,
- $s_k^{n_k+1} = 0$ *for all* $r' < k \le r_0$.

We consider $0^0 = 1$, so $u_k = 0$ with $n_k = 0$ or $v_k = 0$ with $m_k = 0$ is allowed in $\mathscr{U}$ and $\mathscr{V}$. Then, we have that

$$\operatorname{cohom\,dim}(Z_f) \geq -\dim(u_1) + \sum_{k=1}^{r'} (n_k - m_k)\dim(u_k) + \sum_{k=r'+1}^{r''} n_k \dim(u_k).$$

Proof. Let $w_{(\varepsilon_1,\dots,\varepsilon_{r_0})} \in R$, for $0 \leq \varepsilon_k \leq m_k$, be such that

$$v_1^{m_1+1} = \sum_{(\varepsilon_1,\dots,\varepsilon_{r_0})\in\mathscr{E}} w_{(\varepsilon_1,\dots,\varepsilon_{r_0})} v_1^{\varepsilon_1}\cdots v_{r_0}^{\varepsilon_{r_0}}.$$

Consider the elements

$$q = u_1^{n_1-m_1-1} u_2^{n_2-m_2}\cdots u_{r'}^{n_{r'}-m_{r'}} u_{r'+1}^{n_{r'+1}}\cdots u_{r''}^{n_{r''}},$$
$$p = u_1^{m_1+1} - \sum_{(\varepsilon_1,\dots,\varepsilon_{r_0})\in\mathscr{E}} w_{(\varepsilon_1,\dots,\varepsilon_{r_0})} u_1^{\varepsilon_1}\cdots u_{r_0}^{\varepsilon_{r_0}},$$

of $H^*(\overline{X})$. Then,

$$pq = u_1^{n_1} u_2^{n_2-m_2}\cdots u_{r'}^{n_{r'}-m_{r'}} u_{r'+1}^{n_{r'+1}}\cdots u_{r''}^{n_{r''}} - \sum w_{(\varepsilon_1,\dots,\varepsilon_{k_0},0,\dots,0)} u_1^{n_1-m_1-1+\varepsilon_1} u_2^{n_2-m_2+\varepsilon_2}\cdots u_{r'}^{n_{r'}-m_{r'}+\varepsilon_{r'}} u_{r'+1}^{n_{r'+1}}\cdots u_{r''}^{n_{r''}} \neq 0,$$

since each $\varepsilon_k \leq m_k$, so it is a linear combination of distinct elements of a basis.

Let $i_{\overline{Z_f}}\colon \overline{Z_f} \to \overline{X}$ be the inclusion. Similar to the previous proof, we can prove that $i^*_{\overline{Z_f}}(q)$ is a nonzero element of $H^*(\overline{Z_f})$, and its dimension is

$$\begin{aligned}\dim(i^*_{\overline{Z_f}}(q)) = \dim(q) &= \dim(u_1^{n_1-m_1-1} u_2^{n_2-m_2}\cdots u_{r'}^{n_{r'}-m_{r'}} u_{r'+1}^{n_{r'+1}}\cdots u_{r''}^{n_{r''}})\\ &= (n_1 - m_1 - 1)\dim(u_1)\\ &\quad + (n_2 - m_2)\dim(u_2) + \cdots + (n_{r'} - m_{r'})\dim(u_{r'})\\ &\quad + n_{r'+1}\dim(u_{r'+1}) + \cdots + n_{r''}\dim(u_{r''})\\ &= -\dim(u_1) + \sum_{k=1}^{r'} (n_k - m_k)\dim(u_k) + \sum_{k=r'+1}^{r''} n_k \dim(u_k).\end{aligned}$$

□

4.4 Applications

In this section, we present applications of Theorem 4.8 and Theorem 4.11. Some known results, as well as the results from Section 4.2, are shown to be obtainable as a consequence of these theorems, and stated as remarks. We also prove some original results, which we state as corollaries.

Let $F \to E \to B$ be a fiber bundle with action of a group G, $(F', F'_0) \to (E', E'_0) \to B$ be a fiber bundle pair with action of G and $f\colon E \to E'$ be a fiber preserving G-equivariant map.

If $F' = \mathbb{R}^m$, $F'_0 = \mathbb{R}^m \setminus \{0\}$ and $G = \mathbb{Z}_2$, which acts on F' as the antipodal action, we have that $\overline{F'_0} = \mathbb{R}P^m$ and $H^*(\mathbb{R}P^m; \mathbb{Z}_2) = \mathbb{Z}_2[v]/\langle v^{m+1}\rangle$, $\dim(v) = 1$.

If $F' = \mathbb{R}^{2m+1}$, $F'_0 = \mathbb{R}^{2m+1} \setminus \{0\}$ and $G = \mathbb{Z}_p$, p an odd prime, acting as complex multiplication on F, $\overline{F'} = L_p(m)$ and $H^*(L_p(m); \mathbb{Z}_p) = \mathbb{Z}_p[\tilde{v}, v]/\langle \tilde{v}^2, v^{m+1}\rangle$, $\dim(\tilde{v}) = 1$, $\dim(v) = 2$.

If $F' = \mathbb{R}^{2m+1}$, $F'_0 = \mathbb{R}^{2m+1} \setminus \{0\}$ and $G = S^1$, acting as complex multiplication on F', $\overline{F'_0} = \mathbb{C}P^m$ and $H^*(\mathbb{C}P^m) = R[v]/\langle v^{m+1}\rangle$, $\dim(v) = 2$.

If $F' = \mathbb{R}^{4m+3}$, $F'_0 = \mathbb{R}^{4m+3} \setminus \{0\}$ and $G = S^3$, acting as quaternionic multiplication on F, $\overline{F'_0} = \mathbb{H}P^m$ and $H^*(\mathbb{H}P^m) = R[v]/\langle v^{m+1}\rangle$, $\dim(v) = 4$.

In all cases above, similar to the bundles $\overline{E' \setminus 0} \to B$ in the beginning of the proofs of Theorems 4.4, 4.5, 4.6 and 4.7, the bundle $\overline{F'_0} \to \overline{E'_0} \to B$ admits cohomology extension of the fiber because the classifying map $\overline{F'_0} \to B_G$ induces a surjective map in cohomology and can be factored as $\overline{F'_0} \to \overline{E'_0} \to B_G$, so the restriction $H^*(\overline{E'_0}) \to H^*(\overline{F'_0})$ is surjective.

To use Theorem 4.8, we will need to determine the image of elements of $H^*(B_G)$ by classifying maps. For spheres, the classifying maps are the inclusions so we have that:

- for S^n with antipodal action of $\mathbb{Z}_2$, $g_{\overline{S^n}}(s) = u$, where $H^*(S^n/\mathbb{Z}_2) = H^*(\mathbb{R}P^n) = \mathbb{Z}_2[u]/\langle u^{n+1}\rangle$,
- for S^{2n+1} with action of $\mathbb{Z}_p$, p an odd prime, given by complex multiplication, $g_{\overline{S^{2n+1}}}(s) = u$, $g_{\overline{S^{2n+1}}}(\tilde{s}) = \tilde{u}$, where $H^*(S^{2n+1}/\mathbb{Z}_p) = H^*(L_p(n)) = \mathbb{Z}_p[\tilde{u}, u]/\langle \tilde{u}^2, u^{n+1}\rangle$,
- for S^{2n+1} with action of S^1 given by complex multiplication, $g_{\overline{S^{2n+1}}}(s) = u$, where $H^*(S^{2n+1}/S^1) = H^*(\mathbb{C}P^n) = R[u]/\langle u^{n+1}\rangle$,
- for S^{4n+3} with action of S^3 given by quaternionic multiplication, $g_{\overline{S^{2n+1}}}(s) = u$, where $H^*(S^{4n+3}/S^3) = H^*(\mathbb{H}P^n) = R[u]/\langle u^{n+1}\rangle$.

Similarly, for products of spheres, we have that:

- for $S_{\overline{n}} = S^{n_1} \times \cdots \times S^{n_r}$ with action of $\mathbb{Z}_2^r$, $g_{\overline{S_{\overline{n}}}}(s_k) = u_k$, where $H^*(S_{\overline{n}}/\mathbb{Z}_2^r) = H^*(\mathbb{R}P^{n_1} \times \cdots \times \mathbb{R}P^{n_r}) = \mathbb{Z}_2[u_1, \ldots, u_r]/\langle u_k^{n_k+1}\rangle$,
- for $S_{\overline{n}} = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$ with action of $\mathbb{Z}_p^r$, p an odd prime, $g_{\overline{S_{\overline{n}}}}(s_k) = u_k$, $g_{\overline{S_{\overline{n}}}}(\tilde{s}_k) = \tilde{u}_k$, where $H^*(S_{\overline{n}}/\mathbb{Z}_p^r) = H^*(L_p(n_1) \times \cdots \times L_p(n)) = \mathbb{Z}_p[\tilde{u}_1, u_1, \ldots, \tilde{u}_r, u_r]/\langle \tilde{u}_k^2, u_k^{n+1}\rangle$,
- for $S_{\overline{n}} = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$ with action of $(S^1)^r$, $g_{\overline{S_{\overline{n}}}}(s_k) = u_k$, where $H^*(S_{\overline{n}}/(S^1)^r) = H^*(\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}) = R[u_1, \ldots, u_r]/\langle u_k^{n+1}\rangle$,
- for $S_{\overline{n}} = S^{4n_1+3} \times \cdots \times S^{4n_r+3}$ with action of $(S^3)^r$, $g_{\overline{S_{\overline{n}}}}(s_k) = u_k$, where $H^*(S_{\overline{n}}/(S^3)^r) = H^*(\mathbb{H}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}) = R[u_1, \ldots, u_r]/\langle u_k^{n+1}\rangle$.

For arbitrary spaces, propositions 1.8, 1.9 and 1.10 will help. If X is any path connected space with action of $G = \mathbb{Z}_2$, then $g^*_{\overline{X}}(s) \neq 0$, since $g^*_{\overline{S^1}}$ can be factored as $g^* \circ g^*_{\overline{X}}$, where g is an $\mathbb{Z}_2$-equivariant map $S^1 \to X$. Similarly, if $G = \mathbb{Z}_p$, p an odd prime, then $g^*_{\overline{X}}(\tilde{s}) \neq 0$. If $G = S^1$ and $\pi_1(X) = 0$, $g^*_{\overline{X}}(s) \neq 0$ and if $G = S^3$ and $\pi_3(X) = 0$, $g^*_{\overline{X}}(s) \neq 0$. If we consider the coefficient ring R to be a field, then $g_{\overline{X}}(s)$ and $g_{\overline{X}}(\tilde{s})$ in each case can be taken to be an element of a basis. For example, if X has action of S^1 and $H^2(X)$ has a single generator, we can suppose, for simplicity, that $g_{\overline{X}}(s)$ is that generator, as this would take only a change of basis.

Remark 4.12. Suppose $F = S^{n-1}$, $F' = \mathbb{R}^m$, $F'_0 = \mathbb{R}^m \setminus \{0\}$ and $G = \mathbb{Z}_2$. Let G act as the antipodal action on the fibers F and F'. Then

$$H^*(\overline{F}) = H^*(\mathbb{R}P^{n-1}) = \mathbb{Z}_2[u]/\langle u^n \rangle,$$
$$H^*(\overline{F'_0}) = H^*(\mathbb{R}P^{m-1}) = \mathbb{Z}_2[v]/\langle v^m \rangle,$$

with $\dim(u) = 1$, $\dim(v) = 1$.

Taking $\mathscr{U} = \{u^\varepsilon \mid 0 \leq \varepsilon \leq n-1\}$ and $\mathscr{V} = \{v^\varepsilon \mid 0 \leq \varepsilon \leq m\}$, we have that $g^*_{\overline{F}}(s) = u$, $g_{\overline{F'_0}}(s) = v$, and Theorem 4.8 gives us Theorem 4.1, due to Dold [9].

Remark 4.13. Suppose $F = S^{n_1} \times \cdots \times S^{n_r}$, $F' = \mathbb{R}^m$, $F'_0 = \mathbb{R}^m \setminus \{0\}$, for positive integers $n_1 \leq \cdots \leq n_r$ and m, $G = \mathbb{Z}_2$. Let G act on the fiber F as the diagonal action for the antipodal action in each sphere, and on F' as the antipodal action. Again, we have

$$H^*(\overline{F'_0}) = H^*(\mathbb{R}P^{m-1}) = \mathbb{Z}_2[v]/\langle v^m \rangle,$$

where $\dim(v) = 1$.

By Theorem 2.1 we have that

$$H^*(\overline{F}; \mathbb{Z}_2) \simeq \frac{\mathbb{Z}_2[u, u_2, \cdots, u_r]}{\langle u^{n_1+1}, u_k^2 \rangle},$$

if $n_1 < n_2$ or n_1 is odd, or

$$H^*(\overline{F}; \mathbb{Z}_2) \simeq \frac{\mathbb{Z}_2[u, u_2, \cdots, u_r]}{\langle u^{n_1+1}, u_{k'}^2 - u_1^{n_1} u_{k'}, u_k^2 \rangle},$$

if n_1 is even, with $n_1 = \cdots = n_{r'}$ for some $r' \leq r$, where $\dim(u) = 1$ and $\dim(u_k) = n_k$, $2 \leq k \leq r$. In either case, taking $\mathscr{U} = \{u^\varepsilon u_2^{\varepsilon_2} \cdots u_r^{\varepsilon_r} \mid 0 \leq \varepsilon \leq n, \varepsilon_k = 0 \text{ or } 1\}$ and $\mathscr{V} = \{v^\varepsilon \mid 0 \leq \varepsilon \leq m\}$, we have that $g^*_{\overline{F}}(s) = u$, $g_{\overline{F' \setminus \{0\}}}(s) = v$, and Theorem 4.8 gives us Theorem 4.4.

We can also obtain a Bourgin–Yang version, either by taking B to be a point or by using Theorem 4.11.

Corollary 4.14. *Consider cohomology with $\mathbb{Z}_2$ coefficients. Let $\mathbb{Z}_2$ act on $S_{\overline{n}} = S^{n_1} \times \cdots \times S^{n_r}$, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$, diagonally as the antipodal action on each sphere, and as the antipodal action on $\mathbb{R}^m$. Let $f\colon S_{\overline{n}} \to \mathbb{R}^m$ be a $\mathbb{Z}_2$-equivariant map, and $Z_f = f^{-1}(0)$. If $n_1 \geq m$, then*

$$\text{cohom}\dim(Z_f) \geq \left(\sum_{k=1}^{r} n_r \right) - m$$

Remark 4.15. Suppose $F = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$, $F' = \mathbb{R}^{2m+1}$, $F'_0 = \mathbb{R}^{2m+1} \setminus \{0\}$, for positive integers $n_1 \le \cdots \le n_r$ and m, $G = \mathbb{Z}_p$, p an odd prime. Let G act on the fiber F diagonally by complex multiplication on each sphere, and on F' by complex multiplication. By Theorem 2.2, we have

$$H^*(\overline{F};\mathbb{Z}_p) \simeq \frac{\mathbb{Z}_p[\tilde{u}, u, u_2, \ldots, u_r]}{\langle \tilde{u}^2, u^{n_1+1}, u_k^2 \rangle},$$

where $\dim(\tilde{u}) = 1$, $\dim(u) = 2$ and $\dim(u_k) = 2n_k + 1$, $2 \le k \le r$. Also, we have that

$$H^*(B_{\mathbb{Z}_p};\mathbb{Z}_p) = H^*(L_p(m);\mathbb{Z}_p) = \mathbb{Z}_p[\tilde{v}, v]/\langle \tilde{v}^2, v^{m+1} \rangle,$$

where $\dim(\tilde{v}) = 1$ and $\dim(v) = 2$.

Taking $\mathscr{U} = \{\tilde{u}^{\tilde{\varepsilon}} u^{\varepsilon} u_2^{\varepsilon_2} \cdots u_r^{\varepsilon_r} \mid \tilde{\varepsilon} = 0 \text{ or } 1, 0 \le \varepsilon \le n, \varepsilon_k = 0 \text{ or } 1\}$ and $\mathscr{V} = \{\tilde{v}^{\tilde{\varepsilon}} v^{\varepsilon} \mid \tilde{\varepsilon} = 0 \text{ or } 1, 0 \le \varepsilon \le m\}$, we have that $g^*_{\overline{F}}(s) = u$, $g^*_{\overline{F}}(\tilde{s}) = \tilde{u}$, $g_{\overline{F' \setminus \{0\}}}(s) = v$, $g_{\overline{F' \setminus \{0\}}}(\tilde{s}) = \tilde{v}$, and Theorem 4.8 gives us Theorem 4.5.

Again, we can obtain a Bourgin–Yang version, either by taking B to be a point or by using Theorem 4.11.

Corollary 4.16. *Consider cohomology with $\mathbb{Z}_p$ coefficients. Let $\mathbb{Z}_p$ act on $S_{\overline{n}} = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$, $1 \le n_1 \le n_2 \le \cdots \le n_r$, diagonally as complex multiplication on each sphere, and as the complex multiplication on $\mathbb{R}^{2m} = \mathbb{C}^m$. Let $f\colon S_{\overline{n}} \to \mathbb{R}^{2m}$ be a $\mathbb{Z}_p$-equivariant map, and $Z_f = f^{-1}(0)$. If $n_1 \ge m$, then*

$$\text{cohom}\dim(Z_f) \ge \left(\sum_{k=1}^{r} 2n_r + 1 \right) - 2m$$

Remark 4.17. Suppose $F = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$, $F' = \mathbb{R}^{2m+1}$, $F'_0 = \mathbb{R}^{2m+1} \setminus \{0\}$, for positive integers $n_1 \le \cdots \le n_r$ and m, $G = S^1$. Let G act on the fiber F diagonally by complex multiplication on each sphere, and on F' by complex multiplication. By Theorem 2.2, we have

$$H^*(\overline{F}) \simeq \frac{R[u, u_2, \ldots, u_r]}{\langle u^{n_1+1}, u_k^2 \rangle}$$

where $\dim(u) = 2$ and $\dim(u_k) = 2n_k + 1$, $2 \le k \le r$. Also, we have that

$$H^*(\overline{F'_0}) = H^*(\mathbb{C}P^m) = R[v]/\langle v^{m+1} \rangle,$$

where $\dim(v) = 2$.

Taking $\mathscr{U} = \{u^{\varepsilon} u_2^{\varepsilon_2} \cdots u_r^{\varepsilon_r} \mid 0 \le \varepsilon \le n, \varepsilon_k = 0 \text{ or } 1\}$ and $\mathscr{V} = \{v^{\varepsilon} \mid 0 \le \varepsilon \le m\}$, Theorem 4.8 gives us Theorem 4.6.

Here is the Bourgin–Yang version.

Corollary 4.18. *Let S^1 act on $S_{\overline{n}} = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$, diagonally as complex multiplication on each sphere, and as the complex multiplication on $\mathbb{R}^{2m} = \mathbb{C}^m$. Let $f\colon S_{\overline{n}} \to \mathbb{R}^{2m}$ be a S^1-equivariant map, and $Z_f = f^{-1}(0)$. If $n_1 \geq m$, then*

$$\text{cohom dim}(Z_f) \geq \left(\sum_{k=1}^{r} 2n_r + 1\right) - 2m - 1$$

Remark 4.19. Suppose $F = S^{4n_1+3} \times \cdots \times S^{4n_r+3}$, $F' = \mathbb{R}^{4m+3}$, $F'_0 = \mathbb{R}^{4m+3} \setminus \{0\}$, for positive integers $n_1 \leq \cdots \leq n_r$ and m, $G = S^3$. Let G act on the fiber F diagonally by quaternionic multiplication on each sphere, and on F' by quaternionic multiplication. By Theorem 2.5, we have

$$H^*(\overline{F}) \simeq \frac{R[u, u_2, \ldots, u_r]}{\langle u^{n_1+1}, u_k^2 \rangle}$$

where $\dim(u) = 4$ and $\dim(u_k) = 4n_k + 3$, $2 \leq k \leq r$. Also, we have that

Taking $\mathscr{U} = \{u^{\varepsilon} u_2^{\varepsilon_2} \cdots u_r^{\varepsilon_r} \mid 0 \leq \varepsilon \leq n, \varepsilon_k = 0 \text{ or } 1\}$ and $\mathscr{V} = \{v^{\varepsilon} \mid 0 \leq \varepsilon \leq m\}$, we have that $g^*_{\overline{F}}(s) = u$, $g_{\overline{F' \setminus \{0\}}}(s) = v$, and Theorem 4.8 gives us Theorem 4.7.

Here is the Bourgin–Yang version.

Corollary 4.20. *Let S^3 act on $S_{\overline{n}} = S^{4n_1+3} \times \cdots \times S^{4n_r+3}$, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$, diagonally as quaternionic multiplication on each sphere, and as the quaternionic multiplication on $\mathbb{R}^{4m} = \mathbb{H}^m$. Let $f\colon S_{\overline{n}} \to \mathbb{R}^{4m}$ be a S^3-equivariant map, and $Z_f = f^{-1}(0)$. If $n_1 \geq m$, then*

$$\text{cohom dim}(Z_f) \geq \left(\sum_{k=1}^{r} 4n_r + 3\right) - 4m - 3$$

The *Dold Manifold* is the orbit space $S^{n_1} \times \mathbb{C}P^{n_2}/\mathbb{Z}_2$, where the action is given by $-1(x, [z]) = (-x, [\overline{z}])$. Next, we consider $\mathbb{C}P^{n_2}/\mathbb{Z}_2$ bundles with this action.

Corollary 4.21. *Consider cohomology with coefficient ring $\mathbb{Z}_2$. Let $\mathbb{Z}_2$ act on $S^{n_1} \times \mathbb{C}P^{n_2}$ by $-1(x, [z]) = (-x, [\overline{z}])$ and as the antipodal action on $\mathbb{R}^m$. Let $f\colon S^{n_1} \times \mathbb{C}P^{n_2} \to \mathbb{R}^m$ be a $\mathbb{Z}_2$-equivariant map, and $Z_f = f^{-1}(0)$. If $n_1 \geq m$, then*

$$\text{cohom dim}(Z_f) \geq n_1 + 2n_2 - m$$

Proof. Notice that $\overline{S^{n_1} \times \mathbb{C}P^{n_2}}$ is the Dold manifold $P(n_1, n_2)$, which has cohomology ring

$$H^*(\overline{S^{n_1} \times \mathbb{C}P^{n_2}}; \mathbb{Z}_2) \simeq \frac{\mathbb{Z}_2[u_1, u_2]}{\langle u_1^{n_1+1}, u_2^{n_2+2} \rangle}$$

where $\dim(u_1) = 1$ and $\dim(u_2) = 2$ (see [8]). Also, we have that

$$H^*(\overline{\mathbb{R}^m \setminus \{0\}}) = H^*(\mathbb{R}P^{m-1}) = \mathbb{Z}_2[v]/\langle v^m \rangle,$$

where $\dim(v) = 1$.

Taking $\mathscr{U} = \{u_1^{\varepsilon} u_2^{\varepsilon_2} \mid 0 \leq \varepsilon \leq n_1, 0 \leq \varepsilon_2 \leq n_2\}$ and $\mathscr{V} = \{v^{\varepsilon} \mid 0 \leq \varepsilon \leq m-1\}$, we have that $g^*_{\overline{F}}(s) = u_1$, $g_{\overline{F' \setminus \{0\}}}(s) = v$, and Theorem 4.11 gives us the result. □

Here is the parametrized version:

Corollary 4.22. *Consider cohomology with coefficient ring $\mathbb{Z}_2$. Let $E \to B$ be a $S^{n_1} \times \mathbb{C}P^{n_2}$ bundle, with free action of $\mathbb{Z}_2$, which acts as $-1(x,[z]) = (-x,[\overline{z}])$ on the fiber. Let $E' \to B$ be a m-vector bundle with action of $\mathbb{Z}_2$, which acts on the fiber as the antipodal action. Let $f\colon E \to E'$ be a fiber preserving $\mathbb{Z}_2$-equivariant map, and $Z_f = f^{-1}(0)$, where 0 is the zero section. If the quotient bundle $\overline{E} \to B$ admit cohomology extension of the fiber and $n_1 \geq m$, then*

$$\text{cohom dim}(Z_f) \geq \text{cohom dim}(B) + n_1 + 2n_2 - m$$

Proof. Let $F = S^{n_1} \times \mathbb{C}P^{n_2}$, $F' = \mathbb{R}^m$, $F'_0 = \mathbb{R}^m \setminus \{0\}$, for positive integers n_1, n_2 and m, $G = \mathbb{Z}_2$. Then $\overline{F}$ is the Dold manifold $P(n_1,n_2)$, which has cohomology ring

$$H^*(\overline{F};\mathbb{Z}_2) \simeq \frac{\mathbb{Z}_2[u_1,u_2]}{\langle u_1^{n_1+1}, u_2^{n_2+2}\rangle}$$

where $\dim(u_1) = 1$ and $\dim(u_2) = 2$ (see [8]). Also, we have that

$$H^*(\overline{F'_0}) = H^*(\mathbb{R}P^{m-1}) = \mathbb{Z}_2[v]/\langle v^m\rangle,$$

where $\dim(v) = 1$.

Taking $\mathscr{U} = \{u_1^{\varepsilon} u_2^{\varepsilon_2} \mid 0 \leq \varepsilon \leq n_1, 0 \leq \varepsilon_2 \leq n_2\}$ and $\mathscr{V} = \{v^{\varepsilon} \mid 0 \leq \varepsilon \leq m-1\}$, we have that $g^*_{\overline{F}}(s) = u_1$, $g_{\overline{F'\setminus\{0\}}}(s) = v$, and Theorem 4.8 gives us the result. □

Now we consider bundles where the fiber is the sphere S^m with two antipodal points N and S glued together. For the sphere S^2, this is the pinched torus. Now we are interested in estimating the size of inverse image of the class $[N] = \{N,S\}$.

Corollary 4.23. *Consider cohomology with coefficient ring $\mathbb{Z}_2$. Let $\mathbb{Z}_2$ act on S^n and on $S^m/\{N,S\}$ as the antipodal action, N and S being the north and south poles respectively. Let $f\colon S^n \to S^m/\{N,S\}$ be a $\mathbb{Z}_2$-equivariant map. Denote $Z_f = f^{-1}([N])$. If $n \geq m$, then*

$$\text{cohom dim}(Z_f) \geq n - m.$$

Proof. Notice that

$$H^*(\overline{S^n};\mathbb{Z}_2) \simeq \mathbb{Z}_2[u]\langle u_1^{n+1}\rangle$$

and

$$H^*(\overline{(S^m/\{N,S\}) \setminus [N]}) = H^*(S^{m-1}/\mathbb{Z}_2) = H^*(\mathbb{R}P^{m-1}) = \mathbb{Z}_2[v]/\langle v^m\rangle$$

where $\dim(u_1) = 1$, $\dim(v) = 1$.

Taking $\mathscr{U} = \{u^{\varepsilon} \mid 0 \leq \varepsilon \leq n\}$ and $\mathscr{V} = \{v^{\varepsilon} \mid 0 \leq \varepsilon \leq m-1\}$, we have that $g^*_{\overline{F}}(s) = u$, $g_{\overline{F'\setminus\{0\}}}(s) = v$, and Theorem 4.11 gives us the result. □

Here is the parametrized version:

Corollary 4.24. *Consider cohomology with coefficient ring $\mathbb{Z}_2$. Let $E \to B$ be a S^{n_1} bundle, with free action of $\mathbb{Z}_2$, which acts as the antipodal action on the fiber. Let $E' \to B$ be a $S^m/\{N,S\}$ bundle, N and S being the north and south poles respectively, with action of $\mathbb{Z}_2$, which acts on the fiber as the action induced by the antipodal action. Let $f\colon E \to E'$ be a fiber preserving $\mathbb{Z}_2$-equivariant map. Denote $Z_f = f^{-1}(\mathrm{Fix}(E'))$. If the quotient bundle $\overline{E} \to B$ admit cohomology extension of the fiber and $n \geq m$, then*

$$\text{cohom}\dim(Z_f) \geq \text{cohom}\dim(B) + n - m.$$

Proof. Take $F = S^n$, $F' = S^m/\{N,S\}$, $F'_0 = F' \setminus [N] = S^m \setminus \{N,S\} \simeq S^{m-1}$, for positive integers and $G = \mathbb{Z}_2$. Then

$$H^*(\overline{F};\mathbb{Z}_2) \simeq \mathbb{Z}_2[u]\langle u_1^{n+1}\rangle$$

and

$$H^*(\overline{F'_0}) = H^*(S^{m-1}/\mathbb{Z}_2) = H^*(\mathbb{R}P^{m-1}) = \mathbb{Z}_2[v]/\langle v^m\rangle$$

where $\dim(u_1) = 1$, $\dim(v) = 1$.

Taking $\mathscr{U} = \{u^\varepsilon \mid 0 \leq \varepsilon \leq n\}$ and $\mathscr{V} = \{v^\varepsilon \mid 0 \leq \varepsilon \leq m-1\}$, we have that $g^*_{\overline{F}}(s) = u$, $g_{\overline{F'\setminus\{0\}}}(s) = v$, and Theorem 4.8 gives us the result. □

So far, we only considered F'_0 to be F' with one point removed. As examples of theorems that need more than that, we will prove parametrized Borsuk–Ulam theorems for bundles with action of the products $\mathbb{Z}_2^r$, $\mathbb{Z}_p^r$, $(S^1)^r$ and $(S^3)^r$.

Corollary 4.25. *Consider cohomology with coefficient ring $\mathbb{Z}_2$. Let $E \to B$ be a $S^{n_1} \times \cdots \times S^{n_r}$ bundle, with free action of $\mathbb{Z}_2^r$, which acts on the fiber coordinatewise as the antipodal action on each sphere S^{n_k}. Let $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_r} \to E' \to B$ be a vector bundle with free action of $\mathbb{Z}_2^r$ which acts on the fiber coordinatewise as the antipodal action on each $\mathbb{R}^{m_k}$.*

Let $f\colon E \to E'$ be a fiber preserving, $\mathbb{Z}_2^r$-equivariant map and $Z_f = f^{-1}(\mathrm{Fix}(E' \setminus E'_0))$. If $n_1 \geq m_1 \geq 1$, $n_k \geq \min\{m_1, m_k\} - 1$ then

$$\text{cohom}\dim(Z_f) \geq \text{cohom}\dim(B) + n_1 - m_1 + \sum_{k=2}^{r} n_k - \min\{m_1, m_k - 1\}.$$

Proof. Notice that $F'_0 = (\mathbb{R}^{m_1} \setminus \{0\}) \times \cdots \times (\mathbb{R}^{m_r} \setminus \{0\})$. Then,

$$H^*(\overline{F};\mathbb{Z}_2) = \mathbb{Z}_2[u_1,\dots,u_r]/\langle u_1^{n_1+1},\dots,u_r^{n_r+1}\rangle,$$

$$H^*(\overline{F'_0};\mathbb{Z}_2) = \mathbb{Z}_2[v_1,\dots,v_r]/\langle v_1^{m_1},\dots,v_r^{m_r}\rangle,$$

where $\dim(u_k) = 1$, $\dim(v_k) = 1$, $k \in \{1,\dots,r\}$.

Since the classifying map $\overline{F'_0} \to B_{\mathbb{Z}_2^r}$ induces a surjective map in cohomology and can be factored as $\overline{F'_0} \to \overline{E'_0} \to B_{\mathbb{Z}_2}$, the restriction $H^*(\overline{E'_0}) \to H^*(\overline{F'_0})$ is surjective and $\overline{F'_0} \to \overline{E'_0} \to B$ admit cohomology extension of the fiber. Similarly, $\overline{F} \to \overline{E} \to B$ also admit cohomology extension of the fiber. Taking $\mathscr{U} = \{u_1^{\varepsilon_1} \cdots u_r^{\varepsilon_r} \mid 0 \le \varepsilon_k \le n_k\}$ and $\mathscr{V} = \{v_1^{\varepsilon_1} \cdots v_r^{\varepsilon_r} \mid 0 \le \varepsilon_1 \le m_1 - 1, 0 \le \varepsilon_k \le \min\{m_1, m_k - 1\}\}$, we have that $g^*_{\overline{F}}(s_k) = u_k$, $g_{\overline{F'_0}}(s_k) = v_k$, and Theorem 4.8 gives us the result. □

For the Bourgin–Yang version, we can choose $\mathscr{V} = \{v_1^{\varepsilon} \mid 0 \le \varepsilon \le m_1 - 1\}$, since $v_1^{m_1} = 0$ is generated by this set. Then, Theorem 4.11 gives us the following.

Corollary 4.26. *Consider cohomology with coefficient ring $\mathbb{Z}_2$. Let $\mathbb{Z}_2^r$ act on $S_{\overline{n}} = S^{n_1} \times \cdots \times S^{n_r}$ coordinatewise as the antipodal action on each sphere S^{n_k} and on $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_r}$ coordinatewise as the antipodal action on each $\mathbb{R}^{m_k}$.*

Let $f\colon S_{\overline{n}} \to \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_r}$ be a $\mathbb{Z}_2^r$-equivariant map and $Z_f = f^{-1}(\mathrm{Fix}(\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_r}))$. If $n_1 \ge m_1 \ge 1$, then

$$\text{cohom dim}(Z_f) \ge \left(\sum_{k=1}^{r} n_k\right) - m_1.$$

For $\mathbb{Z}_p^r$, we have the following:

Corollary 4.27. *Consider cohomology with coefficient ring $\mathbb{Z}_p$, p an odd prime. Let $E \to B$ be a $S^{2n_1+1} \times \cdots \times S^{2n_r+1}$ bundle, with free action of $\mathbb{Z}_p^r$, which acts on the fiber coordinatewise as complex multiplication on each sphere S^{2n_k+1}. Let $\mathbb{R}^{2m_1} \times \cdots \times \mathbb{R}^{2m_r} \to E' \to B$ be a vector bundle with free action of $\mathbb{Z}_p^r$ which acts on the fiber coordinatewise by complex multiplication on each $\mathbb{R}^{2m_k} = \mathbb{C}^{m_k}$.*

Let $f\colon E \to E'$ be a fiber preserving, $\mathbb{Z}_p^r$-equivariant map and $Z_f = f^{-1}(\mathrm{Fix}(E' \setminus E'_0))$. If $n_1 \ge m_1 \ge 2$, $n_k \ge m_k - 1$ then

$$\text{cohom dim}(Z_f) \ge \text{cohom dim}(B) + r + 2(n_1 - m_1) + \sum_{k=2}^{r} 2(n_k - \min\{m_1, m_k - 1\}).$$

Proof. Notice that $F'_0 = (\mathbb{R}^{2m_1} \setminus \{0\}) \times \cdots \times (\mathbb{R}^{2m_r} \setminus \{0\})$. Then,

$$H^*(\overline{F}; \mathbb{Z}_p) = \mathbb{Z}_p[\tilde{u}_1, u_1, \ldots, \tilde{u}_r, u_r]/\langle \tilde{u}_1^2, u_1^{n_1+1}, \ldots, \tilde{u}_r^2, u_r^{n_r+1}\rangle,$$

$$H^*(\overline{F'_0}; \mathbb{Z}_p) = \mathbb{Z}_p[\tilde{v}_1, v_1, \ldots, \tilde{v}_r, v_r]/\langle \tilde{v}_1^2, v_1^{m_1}, \ldots, \tilde{v}_r^2, v_r^{m_r}\rangle,$$

where $\dim(u_k) = 2$, $\dim(\tilde{u}_k) = 1$, $\dim(v_k) = 2$, $\dim(\tilde{(v)}_k) = 1$, $k \in \{1, \ldots, r\}$.

Since the classifying map $\overline{F'_0} \to B_{\mathbb{Z}_p^r}$ induces a surjective map in cohomology and can be factored as $\overline{F'_0} \to \overline{E'_0} \to B_{\mathbb{Z}_p}$, the restriction $H^*(\overline{E'_0}) \to H^*(\overline{F'_0})$ is surjective and $\overline{F'_0} \to \overline{E'_0} \to B$ admit cohomology extension of the fiber. Similarly, $\overline{F} \to \overline{E} \to B$ also admit cohomology extension of the fiber. Taking $\mathscr{U} = \{u_1^{\varepsilon_1} \cdots u_r^{\varepsilon_r} \tilde{u}_1^{\tilde{\varepsilon}_1} \cdots \tilde{u}_r^{\tilde{\varepsilon}_r} \mid 0 \le \varepsilon_k \le n_k, \tilde{\varepsilon}_k = 0 \text{ or } 1\}$ and $\mathscr{V} = \{v_1^{\varepsilon_1} \cdots v_r^{\varepsilon_r} \tilde{v}_1^{\tilde{\varepsilon}_1} \cdots \tilde{v}_r^{\tilde{\varepsilon}_r} \mid 0 \le \varepsilon_1 \le m_1 - 1, 0 \le \varepsilon_k \le \min\{m_1, m_k - 1\}, \tilde{\varepsilon}_k = 0 \text{ or } 1\}$, we have that $g^*_{\overline{F}}(s_k) = u_k$, $g^*_{\overline{F}}(\tilde{s}_k) = u_k$, $g_{\overline{F'_0}}(s_k) = v_k$, $g_{\overline{F'_0}}(\tilde{s}_k) = \tilde{v}_k$, and Theorem 4.8 gives us the result. □

Taking $\mathscr{V} = \{v_1^{\varepsilon} \mid 0 \le \varepsilon \le m_1 - 1\}$ in Theorem 4.11 gives us the following Bourgin–Yang version:

Corollary 4.28. *Consider cohomology with coefficient ring* $\mathbb{Z}_p$. *Let* $\mathbb{Z}_p^r$ *act on* $S_{\overline{n}} = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$ *coordinatewise by complex multiplication on each sphere* S^{n_k} *and on* $\mathbb{R}^{2m_1} \times \cdots \times \mathbb{R}^{2m_r}$ *coordinatewise by complex multiplication on each* $\mathbb{R}^{2m_k} = \mathbb{C}^{m_k}$.

Let $f\colon S_{\overline{n}} \to \mathbb{R}^{2m_1} \times \cdots \times \mathbb{R}^{2m_r}$ *be a* $\mathbb{Z}_p^r$*-equivariant map and* $Z_f = f^{-1}(\mathrm{Fix}(\mathbb{R}^{2m_1} \times \cdots \times \mathbb{R}^{2m_r}))$. *If* $n_1 \geq m_1 \geq 1$, *then*

$$\text{cohom}\dim(Z_f) \geq r - 2m_1 + \sum_{k=1}^{r} 2n_k.$$

For $(S^1)^r$, we have the following:

Corollary 4.29. *Let* $E \to B$ *be a* $S^{2n_1+1} \times \cdots \times S^{2n_r+1}$ *bundle, with free action of* $(S^1)^r$, *which acts on the fiber coordinatewise as complex multiplication on each sphere* S^{2n_k+1}. *Let* $\mathbb{R}^{2m_1} \times \cdots \times \mathbb{R}^{2m_r} \to E' \to B$ *be a vector bundle with free action of* $(S^1)^r$ *which acts on the fiber coordinatewise by complex multiplication on each* $\mathbb{R}^{2m_k} = \mathbb{C}^{m_k}$.

Let $f\colon E \to E'$ *be a fiber preserving,* $(S^1)^r$*-equivariant map and* $Z_f = f^{-1}(\mathrm{Fix}(E' \setminus E'_0))$. *If the quotient bundle* $\overline{E} \to B$ *admit cohomology extension of the fiber and* $n_1 \geq m_1 \geq 1$, $n_k \geq \min\{m_1, m_k - 1\}$ *then*

$$\text{cohom}\dim(Z_f) \geq \text{cohom}\dim(B) + 2(n_1 - m_1) + \sum_{k=2}^{r} 2(n_k - \min\{m_1, m_k - 1\}).$$

Proof. Notice that $F'_0 = (\mathbb{R}^{2m_1+1} \setminus \{0\}) \times \cdots \times (\mathbb{R}^{2m_r+1} \setminus \{0\})$. Then,

$$H^*(\overline{F}) = R[u_1, \ldots, u_r]/\langle u_1^{n_1+1}, \ldots, u_r^{n_r+1}\rangle,$$

$$H^*(\overline{F'_0}) = R[v_1, \ldots, v_r]/\langle v_1^{m_1}, \ldots, v_r^{m_r}\rangle,$$

where $\dim(u_k) = 2$, $\dim(v_k) = 2$, $k \in \{1, \ldots, r\}$.

Since the classifying map $\overline{F'_0} \to B_{(S^1)^r}$ induces a surjective map in cohomology and can be factored as $\overline{F'_0} \to \overline{E'_0} \to B_{S^1}$, the restriction $H^*(\overline{E'_0}) \to H^*(\overline{F'_0})$ is surjective and $\overline{F'_0} \to \overline{E'_0} \to B$ admit cohomology extension of the fiber. Similarly, $\overline{F} \to \overline{E} \to B$ also adimit cohomology extension of the fiber. Taking $\mathscr{U} = \{u_1^{\varepsilon_1} \cdots u_r^{\varepsilon_r} \mid 0 \leq \varepsilon_k \leq n_k\}$ and $\mathscr{V} = \{v_1^{\varepsilon_1} \cdots v_r^{\varepsilon_r} \mid 0 \leq \varepsilon_1 \leq m_1 - 1, 0 \leq \varepsilon_k \leq \min\{m_1, m_k - 1\}\}$, we have that $g^*_{\overline{F}}(s_k) = u_k$, $g_{\overline{F'_0}}(s_k) = v_k$, and Theorem 4.8 gives us the result. □

Taking $\mathscr{V} = \{v_1^{\varepsilon} \mid 0 \leq \varepsilon \leq m_1 - 1\}$ in Theorem 4.11 gives us the following Bourgin–Yang version:

Corollary 4.30. *Let* $(S^1)^r$ *act on* $S_{\overline{n}} = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$ *coordinatewise by complex multiplication on each sphere* S^{n_k} *and on* $\mathbb{R}^{2m_1} \times \cdots \times \mathbb{R}^{2m_r}$ *coordinatewise by complex multiplication on each* $\mathbb{R}^{2m_k} = \mathbb{C}^{m_k}$.

Let $f\colon S_{\overline{n}} \to \mathbb{R}^{2m_1} \times \cdots \times \mathbb{R}^{2m_r}$ *be a* $(S^1)^r$*-equivariant map and* $Z_f = f^{-1}(\mathrm{Fix}(\mathbb{R}^{2m_1} \times \cdots \times \mathbb{R}^{2m_r}))$. *If* $n_1 \geq m_1 \geq 1$, *then*

$$\text{cohom}\dim(Z_f) \geq \left(\sum_{k=1}^{r} 2n_k\right) - 2m_1.$$

For $(S^3)^r$, we have the following:

Corollary 4.31. *Let $E \to B$ be a $S^{4n_1+3} \times \cdots \times S^{4n_r+3}$ bundle, with free action of $(S^3)^r$, which acts on the fiber coordinatewise as quaternionic multiplication on each sphere S^{4n_k+3}. Let $\mathbb{R}^{4m_1} \times \cdots \times \mathbb{R}^{4m_r} \to E' \to B$ be a vector bundle with free action of $(S^3)^r$ which acts on the fiber coordinatewise by quaternionic multiplication on each $\mathbb{R}^{4m_k} = \mathbb{H}^{m_k}$.*

Let $f\colon E \to E'$ be a fiber preserving, $(S^3)^r$-equivariant map and $Z_f = \{x \in E' \mid gf(x) = f(x)$ for some $g \in (S^3)^r, g \neq 1\}$. If the quotient bundle $\overline{E} \to B$ admit cohomology extension of the fiber and $n_1 \geq m_1 \geq 2$, $n_k \geq m_k - 1$ then

$$\text{cohom dim}(Z_f) \geq \text{cohom dim}(B) + 4(n_1 - m_1) + \sum_{k=2}^{r} 4(n_k - \min\{m_1, m_k - 1\}).$$

Proof. Notice that $F'_0 = (\mathbb{R}^{4m_1+3} \setminus \{0\}) \times \cdots \times (\mathbb{R}^{4m_r+3} \setminus \{0\})$. Then,

$$H^*(\overline{F}) = R[u_1, \dots, u_r]/\langle u_1^{n_1+1}, \dots, u_r^{n_r+1} \rangle,$$

$$H^*(\overline{F'_0}) = R[v_1, \dots, v_r]/\langle v_1^{m_1}, \dots, v_r^{m_r} \rangle.$$

where $\dim(u_k) = 4$, $\dim(v_k) = 4$, $k \in \{1, \dots, r\}$.

Since the classifying map $\overline{F'_0} \to B_{(S^3)^r}$ induces a surjective map in cohomology and can be factored as $\overline{F'_0} \to \overline{E'_0} \to B_{S^3}$, the restriction $H^*(\overline{E'_0}) \to H^*(\overline{F'_0})$ is surjective and $\overline{F'_0} \to \overline{E'_0} \to B$ admit cohomology extension of the fiber. Similarly, $\overline{F} \to \overline{E} \to B$ also admit cohomology extension of the fiber. Taking $\mathscr{U} = \{u_1^{\varepsilon_1} \cdots u_r^{\varepsilon_r} \mid 0 \leq \varepsilon_k \leq n_k\}$ and $\mathscr{V} = \{v_1^{\varepsilon_1} \cdots v_r^{\varepsilon_r} \mid 0 \leq \varepsilon_1 \leq m_1 - 1, 0 \leq \varepsilon_k \leq \min\{m_1, m_k - 1\}\}$, we have that $g^*_{\overline{F}}(s_k) = u_k$, $g_{\overline{F'_0}}(s_k) = v_k$, and Theorem 4.8 gives us the result. □

Taking $\mathscr{V} = \{v_1^{\varepsilon} \mid 0 \leq \varepsilon \leq m_1 - 1\}$ in Theorem 4.11 gives us the following Bourgin–Yang version:

Corollary 4.32. *Let $(S^3)^r$ act on $S_{\overline{n}} = S^{4n_1+3} \times \cdots \times S^{4n_r+3}$ coordinatewise by quaternionic multiplication on each sphere S^{4n_k+3} and on $\mathbb{R}^{4m_1} \times \cdots \times \mathbb{R}^{4m_r}$ coordinatewise by quaternionic multiplication on each $\mathbb{R}^{4m_k} = \mathbb{H}^{m_k}$.*

Let $f\colon S_{\overline{n}} \to \mathbb{R}^{4m_1} \times \cdots \times \mathbb{R}^{4m_r}$ be a $(S^3)^r$-equivariant map and $Z_f = f^{-1}(\text{Fix}(\mathbb{R}^{4m_1} \times \cdots \times \mathbb{R}^{4m_r}))$. If $n_1 \geq m_1 \geq 1$, then

$$\text{cohom dim}(Z_f) \geq \left(\sum_{k=1}^{r} 4n_k\right) - 4m_1.$$

Bibliography

[1] Martin Arkowitz. *Introduction to homotopy theory*. Universitext. Springer, New York, 2011, pp. xiv+344.

[2] Armand Borel. *Seminar on transformation groups*. Vol. No. 46. Annals of Mathematics Studies. With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Princeton University Press, Princeton, NJ, 1960, pp. vii+245.

[3] D. G. Bourgin. "On some separation and mapping theorems". In: *Comment. Math. Helv.* 29 (1955), pp. 199–214.

[4] Glen E. Bredon. *Introduction to compact transformation groups*. Vol. Vol. 46. Pure and Applied Mathematics. Academic Press, New York-London, 1972, pp. xiii+459.

[5] Donald M. Davis. "Projective product spaces". In: *J. Topol.* 3.2 (2010), pp. 265–279.

[6] James F. Davis and Paul Kirk. *Lecture notes in algebraic topology*. Vol. 35. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001, pp. xvi+367.

[7] Tammo tom Dieck. *Algebraic topology*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008, pp. xii+567.

[8] Albrecht Dold. "Erzeugende der Thomschen Algebra $\mathfrak{N}$". In: *Math. Z.* 65 (1956), pp. 25–35.

[9] Albrecht Dold. "Parametrized Borsuk-Ulam theorems". In: *Comment. Math. Helv.* 63.2 (1988), pp. 275–285.

[10] Jesús González and Maurilio Velasco. "Complex-projective and lens product spaces". In: *Bol. Soc. Mat. Mex. (3)* 20.2 (2014), pp. 319–333.

[11] Marvin J. Greenberg and John R. Harper. *Algebraic topology*. Vol. 58. Mathematics Lecture Note Series. A first course. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, MA, 1981, xi+311 pp. (loose errata).

[12] Dale Husemoller. *Fibre bundles*. Third. Vol. 20. Graduate Texts in Mathematics. Springer-Verlag, New York, 1994, pp. xx+353.

[13] Denise de Mattos et al. "Zero sets of equivariant maps from products of spheres to Euclidean spaces". In: *Topology Appl.* 202 (2016), pp. 7–20.

[14] E. Michael. "Another note on paracompact spaces". In: *Proc. Amer. Math. Soc.* 8 (1957), pp. 822–828.

[15] James R. Munkres. *Topology*. Second. Prentice Hall, Inc., Upper Saddle River, NJ, 2000, pp. xvi+537.

[16] Daniel Quillen. "The spectrum of an equivariant cohomology ring I". In: *Ann. of Math. (2)* 94 (1971), pp. 549–572.

[17] Marjory Del Vecchio dos SANTOS. “Teorema de Borsuk–Ulam para formas espaciais esféricas”. PhD thesis. São Carlos : Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, 2014.

[18] Edwin H. Spanier. *Algebraic topology*. Corrected reprint of the 1966 original. Springer-Verlag, New York, 1995, pp. xvi+528.

[19] Yuri A. Turygin. “A Borsuk-Ulam theorem for $(\mathbb{Z}_p)^k$-actions on products of (mod p) homology spheres”. In: *Topology Appl.* 154.2 (2007), pp. 455–461.

[20] Chung-Tao Yang. “On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. I”. In: *Ann. of Math. (2)* 60 (1954), pp. 262–282.

Index