UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Nonexistence and Existence of Nontrivial Solutions for a Degenerate Goursat Type Problem

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à Leonila Peñaloza, à Fany Yency
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Resumo

Para um problema do tipo Goursat degenerado, sob diversas condições de contorno e em domínios associados ao problema de Tricomi, examinamos a existência, unicidade e inexistência de soluções, com particular ênfase em fenômenos de expoente crítico no contexto de imersões de Sobolev com peso. Especificamente, para condições de contorno de Dirichlet em um domínio de Tricomi, estabelecemos identidades do tipo Pohožaev e provamos a inexistência de soluções regulares não triviais, bem como identificamos o fenômeno do expoente crítico associado às não linearidades. Nos casos que envolvem condições de contorno de Dirichlet mistas, empregamos o método de Didenko para obter estimativas precisas de energia, demonstrando assim a existência e unicidade de soluções fracas tanto para os casos linear quanto generalizado. Para condições de contorno de Neumann em um domínio limitado, garantimos a compacidade da imersão de Sobolev com peso sob condições adequadas e aplicamos o Teorema do Passo da Montanha para estabelecer rigorosamente a existência de soluções fracas para o problema semilinear correspondente.

Palavras-chave: Equações parciais de tipo misto, problema do tipo Goursat, operador de Gellerstedt, condições de contorno de Dirichlet, condições de contorno de Neumann, identidades do tipo Pohožaev, inexistência de soluções, existência e unicidade de soluções, expoente crítico, imersão de Sobolev com peso.

Abstract

For a degenerate Goursat-type problem under several boundary conditions and in domains associated with the Tricomi problem, we rigorously examine the existence, uniqueness, and nonexistence of solutions, with a particular focus on critical exponent phenomena within the framework of weighted Sobolev embeddings. Specifically, for the Dirichlet boundary conditions in a Tricomi domain, we establish Pohožaev-type identities and prove the nonexistence of nontrivial regular solutions, as well as, identify the critical exponent effect associated with nonlinearities. For cases involving mixed Dirichlet boundary conditions, we employ Didenko's method to derive precise energy estimates, thereby demonstrating the existence and uniqueness of weak solutions for both linear and generalized settings. In the case of Neumann boundary conditions on a bounded domain, we ensure the compactness of the weighted Sobolev embedding under appropriate conditions, and we apply the Mountain Pass Theorem to establish the existence of weak solutions for the corresponding semilinear problem.

Keywords: Mixed-type partial equations, Goursat-type problem, Gellerstedt operator, Dirichlet boundary conditions, Neumann boundary conditions, Pohožaev-type identities, nonexistence of solutions, existence and uniqueness of solutions, critical exponent, weighted Sobolev embedding.

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Introduction

This thesis was prepared as a requirement for the Doctorate degree from the Graduate Program in Mathematics at the Federal University of São Carlos. This work is structured into four chapters, each synthesizing years of in-depth research and commitment to the topics discussed.

The first Chapter introduces the Preliminaries, which provide the foundational basis for the development of this thesis. These preliminaries are organized into three chapters, covering a series of fundamental concepts and known results, including: non-homogeneous dilations, D-star-shaped sets, weighted Sobolev spaces, weighted gradients, dual spaces, admissible domains, weak solutions, interpolation inequalities, Palais-Smale sequences, and the Mountain Pass Theorem, as well as various relationships between the norms of spaces and their respective duals.

Furthermore, we discuss a phenomenon related to the critical exponent of the weighted Sobolev space embedding, as stated in the following proposition.

Proposition 1. *Let $m_1, m_2 \in \mathbb{N}$ be given. Suppose there exists a constant $C > 0$ independent of u such that*

$$||u||_{L^r(\Omega)} \leq C||\nabla_{m_1,m_2} u||_{L^2(\Omega)}, \ \ \forall\, u \in C_0^\infty(\Omega).$$

Then, we have $1 \leq r \leq 2^*(m_1,m_2) := \frac{2(m_1+m_2)+8}{m_1+m_2+m_1m_2}$.

In Chapter 2, we address the mixed-type Dirichlet problem.

$$\begin{cases} \mathscr{O}(u) := -y^{m_1}u_{xx} - x^{m_2}u_{yy} = f(u) & \text{in} \quad \ \ \Omega, \\ \quad u = 0 & \text{on} \quad \Gamma \subset \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded, open, and simply connected subset of $\mathbb{R}^2$ with a boundary $\partial\Omega$ that is C^1-piecewise, $f \in C^0(\mathbb{R})$ and $m_1, m_2 \in \mathbb{N}$.

The importance of studying the problem (1) lies in the fact that several well-known operators can be viewed as particular cases of the operator $\mathscr{O} = -y^{m_1}\partial_x^2 - x^{m_2}\partial_y^2$ addressed in this work and studied in [18], [19], and [20]. For example, the Laplacian operator $-\Delta = -\partial_x^2 - \partial_y^2$ as studied in [3], [2], [12], the Tricomi operator $T = -y\partial_x^2 - \partial_y^2$ in [33], [17], and the Gellerstedt operator $L = -y^m\partial_x^2 - \partial_y^2$ in [15], [11], correspond to particular cases when $m_1 = m_2 = 0$, $m_1 = 1$, $m_2 = 0$ and $m_2 = 0$, respectively. Non-existence results for problems involving these operators are crucial in the theory of PDEs, particularly in the context of semilinear elliptic equations. Regarding the non-existence of sufficiently regular solutions for problems of the form

$$\begin{cases} -\Delta u = div(\nabla u) = f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

and

$$\begin{cases} L(u) = -y^m u_{xx} - u_{yy} = f(u) & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \Gamma \subset \partial\Omega, \end{cases} \tag{3}$$

the critical growth plays an important and crucial role. In [28], it was shown that the problem (2) does not admit any sufficiently regular non-trivial solution in a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ if Ω is star-shaped with respect to some interior point and if $f(u) = |u|^\alpha$ with $\alpha \geq 2^* - 1 = \frac{n+2}{n-2}$, where 2^* is the well-known critical exponent in the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p \leq 2^*$. Similarly, in [15], [17], it was demonstrated that the problem (3) does not admit sufficiently regular non-trivial solutions in a Tricomi domain $\Omega \subset \mathbb{R}^2$ if Ω is star-shaped with respect to the flow of $D = -(m+2)x\partial_x - 2y\partial_y$, and if $f(u) = |u|^\alpha$ with $\alpha > 2^*(m) - 1 = \frac{m+8}{m}$, where $2^*(m)$ is the critical exponent in the weighted Sobolev embedding $W_m^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p < 2^*(m)$. Sabitova, in [30], studied an eigenvalue problem involving the Chaplygin-type operator $sign(y)|y|^m u_{xx} + u_{yy} - \lambda|y|^m u = 0$, proving uniqueness and non-existence results for λ in certain intervals. These results, along with others found in [3], [2], [10], [22], [23], [1], [8], among others, motivated the study of the non-existence of sufficiently regular solutions for the problem (1).

In this chapter, we present the construction of a Pohožajev-type identity for the problem (1), inspired by the classical argument used for the problem (2) in a bounded domain found in [28] and recovered in [15] for the problem (3) in a class of domains related to the operator (Tricomi domains for the operator), as well as in [17] for the problem (3) with $m = 1$. Other recent works for the Grushin operator $G = \Delta_x + (\alpha+1)^2|x|^{2\alpha}\Delta_y$ (see [22], [23], [1], [8]) involving Pohožajev-type identities are [14] and [27]. Furthermore, we explore some variations regarding the domain for the problem (1), as presented in Section 2.1, and construct the respective Pohožajev identities. Finally, we prove the main results of this chapter in Section 2.2, where we conclude, using the Pohožajev-type identity and the Hardy-Sobolev inequality, that the problem (1) in the domains of Section 2.1 does not admit sufficiently regular non-trivial solutions.

During the writing of this chapter, we encountered several difficulties, some of which are practical challenges, such as the knowledge of the solution domains, since m_1 and m_2 are arbitrary but require specific conditions for their existence. Another difficulty, as in all research, involves the calculations; in our case, not only the exponents of the operator, but also the term x^{m_2} influenced the search for a more particular domain to overcome these obstacles, as seen in Theorem 2.9. However, the main challenge was working with weighted Sobolev spaces, as we needed to define a weighted gradient $\nabla_{m_1,m_2} u = (|y|^{\frac{m_1}{2}} u_x, |x|^{\frac{m_2}{2}} u_y)$ and the operator $\mathscr{X} u = (-y^{m_1}u_x, -x^{m_2}u_y)$. We observe that the relation $\Delta u = div(\nabla u)$ and the norm $||u||^2_{H^1(\Omega)} = ||\nabla u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}$ are crucial for the study of non-existence results, as the norm of the solution space and the operator depend on the gradient. Note that, when $m_1 = m_2 = 0$, we obtain $\nabla_{m_1,m_2}u = \nabla u$ and $\mathscr{X}u = (-u_x, -u_y) = -\nabla u$. Observe that $\nabla_{m_1,m_2}u = -\mathscr{X}u$ when $m_1 = m_2 = 0$. However, in general, this relationship does not hold. In conclusion, the norm of the solution space depends on the weighted gradient, which differs from the operator. Specifically, $||u||^2_{H^1{}_{m_1,m_2}} = ||\nabla_{m_1,m_2}u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}$, while the operator $\mathscr{O}u = div(\mathscr{X}u) \neq div(-\nabla_{m_1,m_2}u)$.

Finally, we conclude Chapter 2 by proving the following results, considering Ω_1 as an open, bounded, and simply connected set, with boundary C^1-piecewise, $\partial\Omega_1 = \sigma_1 \cup AC \cup BC$, formed by an arc $\sigma_1 \subset \{(x,y) \in \mathbb{R}^2;\ y > 0\}$, which intersects the $x-$axis at points $A = (2x_0, 0)$ and $B = (0,0)$, where $x_0 < 0$, and by the characteristic curves AC and BC in $\{(x,y) \in \mathbb{R}^2;\ x \leq 0 \text{ and } y \leq 0\}$ of the operator $\mathscr{O}$, passing through points A and B, respectively, and meeting at point C. We say that Ω_1 is a Tricomi domain for the operator $\mathscr{O}$. See Figure 1.

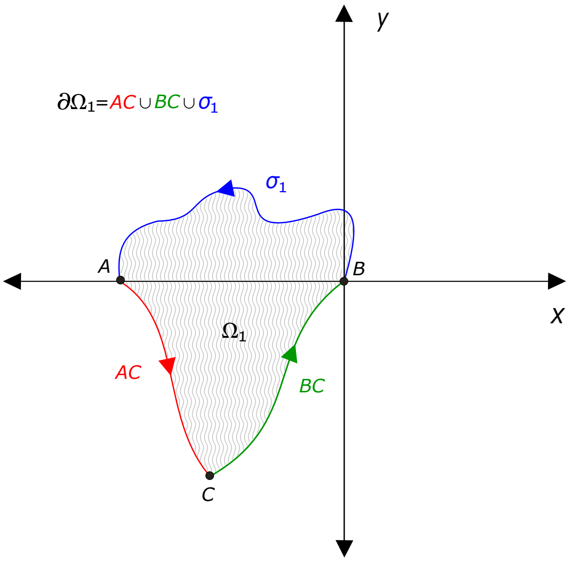

Figure 1: Domain Ω_1 for the problem (1).

Theorem 1. *Let Ω_1 be a Tricomi domain for the operator $\mathscr{O}$ with $m_1 \in \mathbb{N}$ odd and $\frac{m_2}{2} \in \mathbb{N}$ even, whose boundary $\partial\Omega_1$ is C^1-piecewise, with exterior unit normal vector η. If $u \in C^2(\overline{\Omega_1})$ is a solution of the problem* (1)*, then*

$$(m_1+m_2+4)\int_{\Omega_1} F(u) - \frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_1} uf(u) = \frac{1}{2}\left[\int_{BC\cup\sigma_1}\omega_1 + \int_{BC}\omega_2\right],$$

where F is the primitive function of $f \in C^0(\mathbb{R})$ such that $F(0) = 0$, ω_1 and ω_2 are given by

$$\omega_1 = \Big[2Du(-y^{m_1}u_x, -x^{m_2}u_y) + (y^{m_1}u_x^2 + x^{m_2}u_y^2)(-(m_1+2)x, -(m_2+2)y\Big]\cdot\eta\,,$$

$$\omega_2 = \Big[-2F(u)(-(m_1+2)x, -(m_2+2)y) - (m_1+m_2+m_1m_2)u(-y^{m_1}u_x, -x^{m_2}u_y)\Big]\cdot\eta\,,$$

and D is the vector field

$$D = -(m_1+2)x\partial_x - (m_2+2)y\partial_y.$$

Theorem 2. *Let Ω_1 be a Tricomi domain for the operator $\mathscr{O}$, which is D-star-shaped where $D = -(m_1+2)x\partial_x - (m_2+2)y\partial_y$ with $m_1 \in \mathbb{N}$ odd and $\frac{m_2}{2} \in \mathbb{N}$ even. Let $u \in C^2(\overline{\Omega_1})$ be a solution of*

$$\begin{cases} \mathscr{O}(u) := -y^{m_1}u_{xx} - x^{m_2}u_{yy} = u|u|^{\alpha-1} & \text{in} \quad \Omega_1, \\ \quad u = 0 & \text{on} \quad AC \cup \sigma_1 \subseteq \partial\Omega_1, \end{cases}$$

with $\alpha > 2^*(m_1,m_2) - 1 = \frac{m_1+m_2-m_1m_2+8}{m_1+m_2+m_1m_2}$. Then, we have $u \equiv 0$ q.t.p. in Ω_1.

Other similar theorems involving different domains can also be found throughout this chapter; these present challenges, particularly with the necessary estimates arising from the geometry of the domain, which had to be overcome in the proofs through a refinement of the applied technique.

In Chapter 3, we address the linear Tricomi problem.

$$\begin{cases} \mathscr{O}(u) - \lambda u := -y^{m_1}u_{xx} - x^{m_2}u_{yy} - \lambda u = f(x,y) & \text{in} \quad \Omega, \\ \quad u = \gamma & \text{on} \quad AC \cup \sigma \subseteq \partial\Omega, \end{cases} \tag{4}$$

where $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^2$ is a Tricomi domain for the operator $\mathscr{O} = -y^{m_1}\partial_x^2 - x^{m_2}\partial_y^2$. This is, Ω is open, bounded, simply connected set with piecewise C^1 boundary $\partial\Omega_2 = \sigma_2 \cup AC' \cup BC'$ formed by an elliptical arc $\sigma_2 \subset \{(x,y) \in \mathbb{R}^2;\ y > 0\}$ intercepting the axis x at the points $A' = (2x_0, 0)$ and $B' = (0,0)$ with $x_0 > 0$ and the characteristics curves AC' and BC' in $\{(x,y) \in \mathbb{R}^2;\ x \geq 0 \text{ and } y \leq 0\}$ of $\mathscr{O}$ passing through points A' and B' respectively and meet at point C'. See Figure 2.

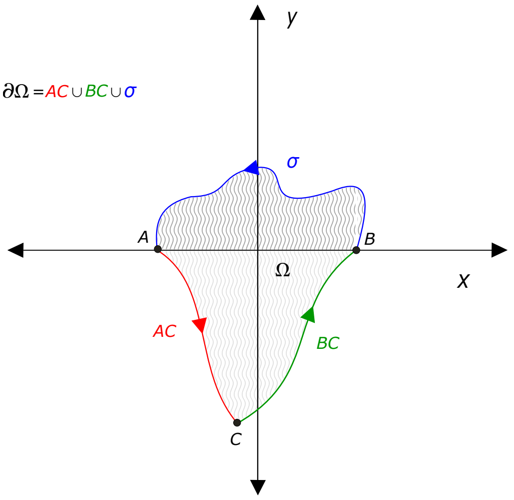

Figure 2: Domain Ω for the problem (4).

As developed later (see also [18]), the characteristic curves AC and BC in $\{(x,y) \in \mathbb{R}^2;\ y \leq 0\}$ of the operator $\mathscr{O}$, for $m_1 \in \mathbb{N}$ odd and $\frac{m_2}{2} \in \mathbb{N}$ even, are given by

$$AC = \left\{(x,y) \in \mathbb{R}^2;\ y_c \leq y \leq 0,\ \frac{2}{m_1+2}(-y)^{\frac{m_1+2}{2}} = \frac{2}{m_2+2}\left[x^{\frac{m_2+2}{2}} + x_1^{\frac{m_2+2}{2}}\right]\right\}, \tag{5}$$

$$BC = \left\{(x,y) \in \mathbb{R}^2;\ y_c \leq y \leq 0,\ \frac{2}{m_1+2}(-y)^{\frac{m_1+2}{2}} = -\frac{2}{m_2+2}\left[x^{\frac{m_2+2}{2}} - x_0^{\frac{m_2+2}{2}}\right]\right\}, \tag{6}$$

and the point $C = \left((\frac{1}{2})^{\frac{2}{m_2+2}}(2x_0), y_c \right)$ where

$$y_c = -\left[-\frac{1}{2}\left(\frac{m_1+2}{m_2+2}\right)(2x_0)^{\frac{m_2+2}{2}} \right]^{\frac{2}{m_1+2}}.$$

Parameterizing (5) and (6) with respect to the variable y, the exterior normal vector will be given by

$$\eta_{AC} = \left(-1, -\left[\frac{m_2+2}{m_1+2}(-y)^{\frac{m_1+2}{2}} + (2x_0)^{\frac{m_2+2}{2}} \right]^{\frac{-m_2}{m_2+2}} (-y)^{\frac{m_1}{2}} \right),$$

and

$$\eta_{BC} = \left(1, -\left[-\frac{m_2+2}{m_1+2}(-y)^{\frac{m_1+2}{2}} \right]^{\frac{-m_2}{m_2+2}} (-y)^{\frac{m_1}{2}} \right), \tag{7}$$

respectively.

The importance of studying problem (4) lies in the fact that the operator $\mathscr{O} = -y^{m_1}\partial_{xx} - x^{m_2}\partial_{yy}$ is a generalization of the Laplacian operator $\Delta = \partial_{xx} + \partial_{yy}$, the Tricomi operator $T = y\partial_{xx} + \partial_{yy}$, as seen in [30, 17, 21] and the Gellerstedt operator $L = y^{m_1}\partial_{xx} + \partial_{yy}$, as discussed in [15, 21, 30]. For this reason, the existence results for weak solutions to problem (4) were motivated by the classical result involving the Laplacian operator in [12] and by the linear problem involving the Tricomi operator in [7]. Furthermore, similar studies for the Keldysh operator $K = k(x)\partial_{xx} + \partial_{yy}$ with $k(x)x > 0, x \neq 0$ and the Grushin operator $G = \Delta_x + (\alpha+1)^2|x|^{2\alpha}\Delta_y$, as mentioned in [25] and [1], contributed to this motivation. The nonexistence results for nontrivial regular solutions to the semilinear problem involving the operator $\mathscr{O} = -y^{m_1}\partial_{xx} - x^{m_2}\partial_{yy}$ developed in the previous chapter (see also [18]), play an important role in our study. Several arguments used in Tricomi domains, as well as the weighted Sobolev spaces associated with the operator, were drawn from this research.

Briefly, we will conduct a historical review of the problem. In [25], Otway studied the Keldysh-type operator, obtaining the existence of distributional solutions as well as certain maximum principles for non-uniformly elliptic operators. Sabitova, in [30], investigated an eigenvalue problem involving a Chaplygin-type operator:

$$sign(y)|y|^m u_{xx} + u_{yy} - \lambda|y|^m u = 0$$

where uniqueness and nonexistence results were obtained for λ within a certain interval. In [21, 9], the problem of eigenfunctions is also addressed. In [26], the existence for the elliptic-hyperbolic Tricomi problem is established, while in [4], estimates for the Dirichlet eigenvalue of the degenerate Laplacian operator are obtained.

In addition to the Laplacian operator, we observe a common feature: the presence of a weight in all of the operators previously mentioned. However, our operator involves two weights. This additional weight introduces further challenges. Indeed, the gradient and the norm are always related through the divergence, which plays a crucial role in the study of PDEs. Alongside this, we introduce a non-zero λ in (4) with the aim of studying the spectral theory of this operator in the future.

Following the ideas of Didenko, we will specifically study the linear Tricomi problem in which $\lambda = \gamma = 0$ in (4) on the same domain Ω

$$\begin{cases} \mathscr{O}(u) := -y^{m_1}u_{xx} - x^{m_2}u_{yy} = f(x,y) & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad AC \cup \sigma \subseteq \partial\Omega. \end{cases} \tag{8}$$

We will also consider, in the same domain Ω, the adjoint problem to the semilinear problem (8)

$$\begin{cases} \mathscr{O}(v) := -y^{m_1}v_{xx} - x^{m_2}v_{yy} = f(x,y) & \text{in} \quad \Omega, \\ \quad v = 0 & \text{on} \quad BC \cup \sigma \subseteq \partial\Omega. \end{cases} \tag{9}$$

During the development of this chapter, as is inevitable, we encountered several difficulties, mainly concerning the domain Ω, to which we imposed some modifications in order to obtain suitable estimates. No less important, and not mentioned in previous works, both for the Gellerstedt operator and for the Tricomi operators, there exists an interaction between the admissibility of the domain for the operator $\mathscr{O}$ (see Definition 1.5) and for the operator $\mathscr{O} - \lambda$ with $\lambda \leq \frac{1}{k_3}$. More specifically, we prove the existence of a non-empty set of Tricomi domains for the operator $\mathscr{O}$ and for an appropriate λ, the set of Tricomi domains for the operator $\mathscr{O} - \lambda$ is also non-empty, justifying the next theorem.

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ be a Tricomi domain with $A = (-x_1, 0)$ and $B = (x_0, 0)$ with $0 < x_0 < x_1$ suppose that Ω and its boundary $\partial\Omega = \sigma \cup AC \cup BC$ satisfies:*

i) $|x| \leq x_1$ *on* $\overline{\Omega}$,

ii) The curve σ is given by a C^2 graph $y = g(x)$ for $-x_1 \leq x \leq x_0$ with $g(-x_1) = 0 = g(x_0)$,

iii) There is a constant

$$h < \left(\frac{x_c^{m_2}}{y_c^{m_1}} \right)^{\frac{1}{2}}$$

such that

$$-h < g'(x) < h \quad \textit{for} \quad -x_1 < x < x_0.$$

Then Ω is admissible for the operator $\mathscr{O} - \lambda$ for all $\lambda \leq \frac{1}{K_3}$, for a suitable $K_3 > 0$.

In addition to this theorem, we establish a result that addresses the existence of weak solutions for the problem (4), as follows.

Theorem 4. *Let Ω be an admissible Tricomi domain for the operator $\mathscr{O} - \lambda$, $f \in L^2(\Omega)$, $\gamma \in \mathbb{R}$ and $\lambda \leq 0$. The problem*

$$\begin{cases} \mathscr{O}(u) - \lambda u = f(x,y) & \textit{in} \quad \Omega, \\ \quad u = \gamma & \textit{on} \quad AC \cup \sigma \subseteq \partial\Omega, \end{cases} \tag{10}$$

admits a unique weak solution in $W^1_{AC\cup\sigma}$ *and the solution operator*

$$S^{\lambda,\gamma}_{AC\cup\sigma} : L^2(\Omega) \to W^1_{AC\cup\sigma}$$

which assigns to $f \in L^2(\Omega)$ *the unique weak solution* $u \in W^1_{AC\cup\sigma}$ *of the problem* (10) *is linear and continuous.*

The Chapter 3 will be divided into two sections. In Section 3.1, we will prove the existence of an admissible domain for the operator $\mathscr{O}$ and, consequently, for the operator $\mathscr{O}-\lambda$. In the section 3.2, we will present the main results of this chapter: we begin by proving the uniqueness and existence of the weak solution for problem (8) and, with the help of this, we prove the uniqueness and existence of the weak solution for problem (4).

In Chapter 4, we will study the existence of non-trivial weak solutions for the mixed-type semilinear Tricomi problem with Neumann boundary conditions.

$$\begin{cases} \mathscr{O}(u) := -|y|^{m_1}u_{xx} - |x|^{m_2}u_{yy} = f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma \subseteq \partial\Omega, \\ \nabla_{m_1,m_2}u\cdot\eta = 0 & \text{on} \quad \partial\Omega\setminus\Gamma, \end{cases} \tag{11}$$

where Ω is an open and bounded subset of $\mathbb{R}^2$, with boundary C^1-piecewise $\partial\Omega$, $f(u) = |u|^{\alpha-2}u$ and $m_1, m_2 \in \mathbb{R}^+$.

Our main focus is on the nonlinear version of problem (11), motivated by the results established in [3], [10], [22], [23], [1], and [16]. Based on the approaches considered by the respective authors, we specify the conditions to ensure the immersion of the weighted Sobolev space, demonstrating both its continuity and compactness. We consider this fundamental tool for variational problems and for the broader theory of partial differential equations. These results are directly related to our problem, as these operators are, in most cases, particular instances of the operator considered here, or, under certain conditions, coincide with the operator presented in this study. More specifically, on the one hand, existence results for weak solutions using variational methods for problems involving the Laplace operator $-\Delta = -\partial_{xx} - \partial_{yy}$, the Tricomi operator $L = -|y|\partial_{xx} - \partial_{yy}$, and the Gellerstedt operator $G = -|y|^{m_1}\partial_{xx} - \partial_{yy}$ have motivated this study. In addition to these, similar results for the Keldysh operator $K = k(x)\partial_{xx} + \partial_{yy}$, the Chaplygin-type operator $sign(y)|y|^m\partial_{xx} + \partial_{yy} - \lambda|y|^m$, and the Grushin operator $G = \Delta_x + (\alpha+1)^2|x|^{2\alpha}\Delta_y$ have also influenced this study and its development. See [2], [12], [5], [16], [25], [30], [26], [22], and [1], among others.

The techniques used to establish the results presented in this chapter adopt an innovative approach for weighted spaces, with a particular focus on the weighted divergence, the weighted gradient, and, consequently the norm. This approach aims to align these new concepts with the operator introduced here. More precisely, we define the weighted divergence $div_{m_1,m_2}(F_1,F_2) := |y|^{\frac{m_1}{2}}(F_1)_x + |x|^{\frac{m_2}{2}}(F_2)_y$ and the weighted gradient $\nabla_{m_1,m_2}u := \left(|y|^{\frac{m_1}{2}}u_x, |x|^{\frac{m_2}{2}}u_y\right)$. By establishing a relationship between these concepts and our operator, which is crucial to solving the variational problems outlined above, specifically $div_{m_1,m_2}(\nabla_{m_1,m_2}u) = \mathscr{O}u$. It is important to note that when $m_1 = m_2 = 0$, we recover the classical gradient $\nabla u = (u_x, u_y)$, the classical divergence operator $div((F_1,F_2)) = (F_1)_x + (F_2)_y$ and the Laplace operator $\mathscr{O}u = \Delta u$. Naturally, the relation $div(\nabla u) = \Delta u$ also holds in this classical case. These aspects introduced numerous challenges in the development of this work.

This chapter is organized into two sections. In Section 4.1, we establish the main results of this chapter: we prove the continuity and, subsequently, the compactness of the weighted Sobolev embedding. Finally, in Section 4.2, we obtain a weak solution for Problem (11) for $f(u) = |u|^{\alpha-2}u$ with $\alpha \in (2, 2^*(m_1,m_2))$. In doing so, we introduce the associated functional and analyze its geometry to apply the Mountain Pass Theorem, thus obtaining a non-trivial weak solution. During the development of these sections, we encountered certain difficulties not addressed in previous references. These difficulties arise from the fact that our operator involves two weights, complicating both the generalization and preservation of the obtained results, as well as the mathematical manipulation of the objects mentioned above. In this context, we refer to the standard estimates and calculations.

Finally, we conclude by stating the main results of Chapter 4.

Theorem 5. *Let Ω be a bounded set of $\mathbb{R}^2$ and $1 \le p < \frac{k+l+2}{1-kl}$ with $kl < 1$. Then, there is a constant C such that*

$$||u||_{L^r(\Omega)} \le C||u||_{W^{1,p}_{k,l}} \; \forall \, r \in [1, 2^*(k,l,p)),$$

this is, the immersion

$$W^{1,p}_{k,l} \hookrightarrow L^r(\Omega)$$

is continuous for $r \in [1, 2^*(k,l,p))$.

Theorem 6. *Let Ω be a bounded set of $\mathbb{R}^2$ and $1 \le p < \frac{k+l+2}{1-kl}$ with $kl < 1$. Then, the immersion*

$$W^{1,p}_{k,l} \hookrightarrow L^r(\Omega)$$

is compact for $r \in [1, 2^*(k,l,p))$.

Theorem 7. *Let Ω be an open and bounded subset of $\mathbb{R}^2$ with $f(u) = u|u|^{\alpha-2}$ and $\alpha \in (2, 2^*(m_1,m_2))$. Then, the problem* (11) *has a weak nontrivial solution.*

CHAPTER 1

Preliminaries

1.1 D-star-shaped domain and weighted Sobolev space

In this section, we will study the invariance of the solutions of the homogeneous equation of the Problem (1) determined by a family of nonhomogeneous dilations that determines an infinitesimal generator D. We will define D-star-shaped domains and the weighted Sobolev space associated to the Problem (1) and, we conclude with a critical exponent phenomenon of Sobolev-Gagliardo-Nirenberg type over D-star-shaped domains.

Define $\phi_\lambda(x,y) = (\lambda^{-\alpha}x, \lambda^{-\beta}y)$, for $\alpha, \beta > 0$ with $\lambda > 0$, a family of nonhomogeneous dilations (i.e. $\alpha \neq \beta$) that generates $\psi_\lambda(u) = u \circ \phi_\lambda = u_\lambda$, a family of nonhomogeneous dilations operators and its infinitesimal generator

$$\left[\frac{d}{d\lambda}u_\lambda\right]_{|\lambda=1} = Du = -\alpha x u_x - \beta y u_y.$$

The flow $\mathscr{F}_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathscr{F}_t(x_0,y_0) = (x(t),y(t))$, for each $t \in \mathbb{R}$, being the unique integral curve of $D = -\alpha x\partial_x - \beta y\partial_y = (-\alpha x, -\beta y)\cdot\nabla$ passing through the point (x_0,y_0) in time $t = 0$. Note that $(x(t),y(t)) = (x_0e^{-\alpha t}, y_0e^{-\beta t})$, because, $\gamma'(t) = (x'(t),y'(t)) = (-\alpha x(t), -\beta y(t)) = D(\gamma(t))$. Hence $x'(t) = -\alpha x(t)$ and $y'(t) = -\beta y(t)$, this is,

$$\frac{d}{dt}x(t) = -\alpha x(t) \ \text{ and } \ \frac{d}{dt}y(t) = -\beta y(t).$$

Therefore, $x(t) = C_1e^{-\alpha t}$ and $y(t) = C_2e^{-\beta t}$ replacing $t = 0$, we found $x(0) = x_0 = C_1$ and $y(0) = y_0 = C_2$, so

$$(x(t),y(t)) = (x_0e^{-\alpha t}, y_0e^{-\beta t}).$$

Furthermore,

$$\begin{aligned}\mathscr{F}_{+\infty}(x_0,y_0) &= \lim_{t\to+\infty}\mathscr{F}_t(x_0,y_0) = \lim_{t\to+\infty}(x(t),y(t))\\ &= \lim_{t\to+\infty}(x_0e^{-\alpha t}, y_0e^{-\beta t}) = (0,0),\end{aligned}$$

for each starting point $(x_0,y_0)\in\mathbb{R}^2$. In conclusion, $\mathbb{R}^2$ is star-shaped with respect to the origin using the field D. We said $\mathbb{R}^2$ is D-star-shaped in the sense of the following definition.

Definition 1.1. Let $\alpha,\beta>0$. A open set $\Omega\subset\mathbb{R}^2$ is said D-star-shaped (or star-shaped with respect to the field $D=-\alpha x\partial_x-\beta y\partial_y$) if for each $(x_0,y_0)\in\overline{\Omega}$ one has to $\mathscr{F}_t(x_0,y_0)\subset\overline{\Omega}$ for each $t\in[0,+\infty]$.

Given D-star-shaped property in Ω, the D-starlike property on $\partial\Omega$ and the relationship between them mentioned in the following lemma can be found in [15].

Lemma 1.2. *Let Ω be an open set with piecewise C^1 boundary $\partial\Omega$. If Ω is D-star-shaped where $D=-\alpha x\partial_x-\beta y\partial_y$ with $\alpha,\beta>0$, then $\partial\Omega$ is starlike, that is,*

$$(\alpha x,\beta y)\cdot\eta(x,y)\geq 0, \tag{1.1}$$

for each regular point $(x,y)\in\partial\Omega$ where $\eta(x,y)$ is the unit exterior normal vector on $\partial\Omega$ at the point (x,y).

Definition 1.3. A subset $\Gamma\subset\partial\Omega$ is said D-starlike if the condition (1.1) holds on Γ, which is equivalent to the condition

$$\alpha x dy-\beta y dx\geq 0 \text{ on } \Gamma,$$

where $\partial\Omega$ is given by the positive orientation leaving the interior of Ω on the left side.

We want to study the behavior of regular solutions of the homogeneous problem associated with the operator

$$\mathscr{O}=-y^{m_1}\partial_x^2-x^{m_2}\partial_y^2 \text{ where } m_1,m_2\in\mathbb{N},$$

with respect to certain nonhomogeneous dilations on bounded domains. Without loss of generality, for bounded domain Ω of $\mathbb{R}^2$ we will consider the family of nonhomogeneous dilations

$$\phi_\lambda(x,y)=(\lambda^{-m_1-2}x,\lambda^{-m_2-2}y), \text{ with } \lambda\in(0,1].$$

If u is a regular solution of homogeneous problem $\mathscr{O}(u)=0$ in Ω, then $u_\lambda=u\circ\phi_\lambda$ is also a regular solution of the homogeneous problem. This is, $\mathscr{O}(u_\lambda)=0$. Indeed, since that $u_\lambda(x,y)=u(\lambda^{-m_1-2}x,\lambda^{-m_2-2}y)$, we obtain,

$$\frac{\partial}{\partial x}u_\lambda(x,y)=\lambda^{-m_1-2}u_x(\lambda^{-m_1-2}x,\lambda^{-m_2-2}y)$$

and

$$\frac{\partial}{\partial y}u_\lambda(x,y)=\lambda^{-m_2-2}u_y(\lambda^{-m_1-2}x,\lambda^{-m_2-2}y).$$

Moreover,

$$\frac{\partial^2}{\partial x^2}u_\lambda(x,y)=\lambda^{-2m_1-4}u_{xx}(\lambda^{-m_1-2}x,\lambda^{-m_2-2}y)$$

and

$$\frac{\partial^2}{\partial y^2}u_\lambda(x,y) = \lambda^{-2m_2-4}u_{yy}(\lambda^{-m_1-2}x, \lambda^{-m_2-2}y).$$

Then,

$$\begin{aligned}\mathscr{O}(u_\lambda(x,y)) =& -y^{m_1}\lambda^{-2m_1-4}u_{xx}(\lambda^{-m_1-2}x, \lambda^{-m_2-2}y)\\ &-x^{m_2}\lambda^{-2m_2-4}u_{yy}(\lambda^{-m_1-2}x, \lambda^{-m_2-2}y)\\ =&\frac{\lambda^{-4}}{\lambda^{-m_1m_2}}[-y^{m_1}\lambda^{-2m_1-m_1m_2}u_{xx}(\lambda^{-m_1-2}x, \lambda^{-m_2-2}y)\\ &-x^{m_2}\lambda^{-2m_2-m_1m_2}u_{yy}(\lambda^{-m_1-2}x, \lambda^{-m_2-2}y)].\end{aligned}$$

The change of variables $\overline{x} = \lambda^{-m_1-2}x$ and $\overline{y} = \lambda^{-m_2-2}y$ give us

$$\overline{x}^{m_2} = \lambda^{-m_1m_2-2m_2}x^{m_2} \text{ and } \overline{y}^{m_1} = \lambda^{-m_1m_2-2m_1}y^{m_1}.$$

By hypotheses, $\mathscr{O}(u(\overline{x},\overline{y})) = -\overline{y}^{m_1}u_{xx}(\overline{x},\overline{y}) - \overline{x}^{m_2}u_{yy}(\overline{x},\overline{y}) = 0$ in Ω. Therefore,

$$\mathscr{O}(u_\lambda(x,y)) = \frac{\lambda^{-4}}{\lambda^{-m_1m_2}}\mathscr{O}(u(\overline{x},\overline{y})) = 0.$$

Consider the following weighted Sobolev space $H^1_{m_1,m_2}(\Omega)$, associated with the problem (1), as the completion of $C^\infty(\Omega)$ with respect to the norm

$$||u||_{H^1_{m_1,m_2}(\Omega)} := \left[\int_\Omega |y|^{m_1}u_x^2 + |x|^{m_2}u_y^2 + u^2\right]^{\frac{1}{2}} \tag{1.2}$$

and the subset $H^1_{m_1,m_2,0}(\Omega) = \overline{C_0^\infty(\Omega)}$ with the norm of $H^1_{m_1,m_2}(\Omega)$ above.

We will prove that actually (1.2) defines a norm for the spaces $H^1_{m_1,m_2}(\Omega)$. First, we define the weighted gradient

$$\nabla_{m_1,m_2}u := \left(|y|^{\frac{m_1}{2}}u_x, |x|^{\frac{m_2}{2}}u_y\right).$$

So

$$||\nabla_{m_1,m_2}u||^2_{L^2(\Omega)} = \int_\Omega \left(|y|^{m_1}u_x^2 + |x|^{m_2}u_y^2\right)$$

and with this we rewrite $||\cdot||_{H^1_{m_1,m_2}(\Omega)}$ by

$$||u||^2_{H^1_{m_1,m_2}(\Omega)} = ||\nabla_{m_1,m_2}u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}.$$

In fact, if $||u||_{H^1_{m_1,m_2}(\Omega)} = 0$ then, $||\nabla_{m_1,m_2}u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)} = 0$ so, $u = 0$ a.e. in Ω. Moreover, if $u = 0$ a.e. in Ω then, $||u||^2_{L^2(\Omega)} = 0$ and $u_x = 0$ and $u_y = 0$ a.e. in Ω. So, $\nabla_{m_1,m_2}u = (0,0)$. Therefore, $||u||_{H^1_{m_1,m_2}(\Omega)} = 0$. Note that,

$$\begin{aligned}|\lambda|\ ||u||_{H^1_{m_1,m_2}(\Omega)} &= |\lambda|\ ||\nabla_{m_1,m_2}u||_{L^2(\Omega)} + |\lambda|\ ||u||_{L^2(\Omega)}\\ &= ||\nabla_{m_1,m_2}(\lambda u)||_{L^2(\Omega)} + ||\lambda u||_{L^2(\Omega)}\\ &= ||\lambda u||_{H^1_{m_1,m_2}(\Omega)},\end{aligned}$$

for all $u \in H^1_{m_1,m_2}(\Omega)$ and for all $\lambda \in \mathbb{R}$.

To finish, just prove

$$||u+v||^2_{H^1_{m_1,m_2}(\Omega)} \leq \Big(||u||_{H^1_{m_1,m_2}(\Omega)} + ||v||_{H^1_{m_1,m_2}(\Omega)}\Big)^2$$

for any $u, v \in H^1_{m_1,m_2}(\Omega)$.

Observe that, $\nabla_{m_1,m_2}(u+v) = \nabla_{m_1,m_2}u + \nabla_{m_1,m_2}v$. Hence,

$$\begin{aligned} ||u+v||^2_{H^1_{m_1,m_2}(\Omega)} \leq& \Big(||\nabla_{m_1,m_2}u||_{L^2(\Omega)} + ||\nabla_{m_1,m_2}v||_{L^2(\Omega)}\Big)^2 + \Big(||u||_{L^2(\Omega)} + ||v||_{L^2(\Omega)}\Big)^2 \\ =& ||u||^2_{H^1_{m_1,m_2}(\Omega)} + ||v||^2_{H^1_{m_1,m_2}(\Omega)} \\ &+ 2||\nabla_{m_1,m_2}u||_{L^2(\Omega)}||\nabla_{m_1,m_2}v||_{L^2(\Omega)} + 2||u||_{L^2(\Omega)}||v||_{L^2(\Omega)}. \end{aligned}$$

On the other hand,

$$\Big(||u||_{H^1_{m_1,m_2}(\Omega)} + ||v||_{H^1_{m_1,m_2}(\Omega)}\Big)^2 = ||u||^2_{H^1_{m_1,m_2}(\Omega)} + ||v||^2_{H^1_{m_1,m_2}(\Omega)} + 2||u||_{H^1_{m_1,m_2}(\Omega)}||v||_{H^1_{m_1,m_2}(\Omega)}.$$

All that's left is to analyze,

$$||\nabla_{m_1,m_2}u||_{L^2(\Omega)}||\nabla_{m_1,m_2}v||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}||v||_{L^2(\Omega)} \leq ||u||_{H^1_{m_1,m_2}(\Omega)}||v||_{H^1_{m_1,m_2}(\Omega)}.$$

In fact,

$$\begin{aligned} &\Big(||\nabla_{m_1,m_2}u||_{L^2(\Omega)}||\nabla_{m_1,m_2}v||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}||v||_{L^2(\Omega)}\Big)^2 \\ &\quad = ||\nabla_{m_1,m_2}u||^2_{L^2(\Omega)}||\nabla_{m_1,m_2}v||^2_{L^2(\Omega)} \\ &\qquad + 2||\nabla_{m_1,m_2}u||_{L^2(\Omega)}||\nabla_{m_1,m_2}v||_{L^2(\Omega)}||u||_{L^2(\Omega)}||v||_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}||v||^2_{L^2(\Omega)}, \end{aligned}$$

and

$$\begin{aligned} &||u||^2_{H^1_{m_1,m_2}(\Omega)}||v||^2_{H^1_{m_1,m_2}(\Omega)} \\ &\quad = \big(||\nabla_{m_1,m_2}u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}\big)\big(||\nabla_{m_1,m_2}v||^2_{L^2(\Omega)} + ||v||^2_{L^2(\Omega)}\big) \\ &\quad = ||\nabla_{m_1,m_2}u||^2_{L^2(\Omega)}||\nabla_{m_1,m_2}v||^2_{L^2(\Omega)} + ||\nabla_{m_1,m_2}u||^2_{L^2(\Omega)}||v||^2_{L^2(\Omega)} \\ &\qquad + ||u||^2_{L^2(\Omega)}||\nabla_{m_1,m_2}v||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}||v||^2_{L^2(\Omega)}. \end{aligned}$$

Therefore, comparing the last two equalities and using that $(||\nabla_{m_1,m_2}u||_{L^2(\Omega)}||v||_{L^2(\Omega)} - ||\nabla_{m_1,m_2}v||_{L^2(\Omega)}||u||_{L^2(\Omega)})^2 \geq 0$, we conclude the desired inequality.

Gathering everything together, it follows that $||\cdot||_{H^1_{m_1,m_2}(\Omega)}$ is a norm in $H^1_{m_1,m_2}(\Omega)$. The next result concerns the behavior of critical exponent which provides an estimate for this, assuming that, there is an immersion of type Sobolev-Gagliardo-Nirenberg on $H^1_{m_1,m_2,0}(\Omega)$. Its proof follows using a similar idea made in [15].

Proposition 1.4. *Let $m_1, m_2 \in \mathbb{N}$ given. Suppose there exists $C > 0$ independent of u such that*

$$||u||_{L^p(\Omega)} \leq C||\nabla_{m_1,m_2}u||_{L^2(\Omega)}, \quad \forall\, u \in C_0^\infty(\Omega). \tag{1.3}$$

Then, $1 \leq p \leq 2^*(m_1,m_2) := \frac{2(m_1+m_2)+8}{m_1+m_2+m_1m_2}$.

Proof. If $u = 0$, (1.3) is trivially held. Take $u \in C_0^\infty(\Omega) \setminus \{0\}$ arbitrary. Using the change of variables theorem gives

$$||u_\lambda||^p_{L^p(\Omega)} = \lambda^{m_1+m_2+4}||u||^p_{L^p(\Omega)}$$

and

$$||\nabla_{m_1,m_2}u_\lambda||^2_{L^2(\Omega)} = \lambda^{m_1+m_2+m_1m_2}||\nabla_{m_1,m_2}u||^2_{L^2(\Omega)}.$$

Replacing u_λ em (1.3), we have

$$\begin{aligned} ||u||_{L^p(\Omega)} &= \lambda^{-\frac{m_1+m_2+4}{p}}||u_\lambda||_{L^p(\Omega)} \\ &\leq C\lambda^{-\frac{m_1+m_2+4}{p}}||\nabla_{m_1,m_2}u_\lambda||_{L^2(\Omega)} \\ &= C\lambda^{\left(-\frac{m_1+m_2+4}{p}+\frac{m_1+m_2+m_1m_2}{2}\right)}||\nabla_{m_1,m_2}u||_{L^2(\Omega)}. \end{aligned}$$

Denote $r = -\frac{m_1+m_2+4}{p} + \frac{m_1+m_2+m_1m_2}{2}$ and suppose that $r > 0$. Then, making $\lambda \to 0^+$, we get $||u||_{L^p(\Omega)} \leq 0$ but $u \neq 0$. In conclusion,

$$r = -\frac{m_1+m_2+4}{p} + \frac{m_1+m_2+m_1m_2}{2} \leq 0,$$

then

$$p \leq \frac{2(m_1+m_2)+8}{m_1+m_2+m_1m_2}.$$

□

1.2 Dual spaces and Admissible Tricomi domain

In this section we will define weighted Sobolev spaces naturally associated with the problems (8) and (9) and their dual spaces and we will establish relationships between them in terms of their norms. Additionally, we will define specific Tricomi domains for our work and prove their existence.

Consider,

$$C^\infty_{AC\cup\sigma}(\overline{\Omega}) = \{u \in C^\infty(\overline{\Omega});\ \ u|_{AC\cup\sigma} = 0\},$$

and

$$C^\infty_{BC\cup\sigma}(\overline{\Omega}) = \{v \in C^\infty(\overline{\Omega});\ \ v|_{BC\cup\sigma} = 0\}.$$

Define $W^1_{AC\cup\sigma}$ and $W^1_{BC\cup\sigma}$ the closures of $C^\infty_{AC\cup\sigma}(\overline{\Omega})$ and $C^\infty_{BC\cup\sigma}(\overline{\Omega})$ respectively, with respect to the $W^{1,2}(\Omega)$ norm. By the theory of spaces with negative norm [13] consider the dual space $W^{-1}_{AC\cup\sigma}$ and $W^{-1}_{BC\cup\sigma}$ of $W^1_{AC\cup\sigma}$ and $W^1_{BC\cup\sigma}$ respectively with the norm

$$||\varphi||_{W^{-1}_{AC\cup\sigma}} = \sup_{0\neq u\in W^1_{AC\cup\sigma}} \frac{|(\varphi,u)_{L^2}|}{||u||_{W^1_{AC\cup\sigma}}},$$

and

$$||\phi||_{W^{-1}_{BC\cup\sigma}} = \sup_{0\neq v\in W^{1}_{BC\cup\sigma}} \frac{|(\phi,v)_{L^2}|}{||v||_{W^{1}_{BC\cup\sigma}}}.$$

We get the following inclusions

$$W^1_{AC\cup\sigma} \subset L^2(\Omega) \subset W^{-1}_{AC\cup\sigma} \quad \text{and} \quad W^1_{BC\cup\sigma} \subset L^2(\Omega) \subset W^{-1}_{BC\cup\sigma}.$$

In [18], we defined $\mathscr{X}u = (-y^{m_1}u_x, -x^{m_2}u_y)$ and we observed that $div(\mathscr{X}u) = \mathscr{O}u$. By the Divergence Theorem [12], we have

$$\int_\Omega div(v(-y^{m_1}u_x, -x^{m_2}u_y)) = \int_{\partial\Omega} v(-y^{m_1}u_x, -x^{m_2}u_y)\cdot\eta \ ,$$

and

$$\int_\Omega div(u(-y^{m_1}v_x, -x^{m_2}v_y)) = \int_{\partial\Omega} u(-y^{m_1}v_x, -x^{m_2}v_y)\cdot\eta \ .$$

By the proprieties of the divergent, we obtain

$$\int_\Omega \nabla v(-y^{m_1}u_x, -x^{m_2}u_y) + \int_\Omega v\, div(-y^{m_1}u_x, -x^{m_2}u_y) = \int_{\partial\Omega} v(-y^{m_1}u_x, -x^{m_2}u_y)\cdot\eta \ ,$$

and

$$\int_\Omega \nabla u(-y^{m_1}v_x, -x^{m_2}v_y) + \int_\Omega u\, div(-y^{m_1}v_x, -x^{m_2}v_y) = \int_{\partial\Omega} u(-y^{m_1}v_x, -x^{m_2}v_y)\cdot\eta \ ,$$

respectively. Replacing, $\nabla v = (v_x, v_y)$, $\nabla u = (u_x, u_y)$ and $div(\mathscr{X}u) = \mathscr{O}u$. We have

$$\int_\Omega (v_x, v_y)(-y^{m_1}u_x, -x^{m_2}u_y) + \int_\Omega v\ \mathscr{O}u = \int_{\partial\Omega} v(-y^{m_1}u_x, -x^{m_2}u_y)\cdot\eta \ ,$$

and

$$\int_\Omega (u_x, u_y)(-y^{m_1}v_x, -x^{m_2}v_y) + \int_\Omega u\ \mathscr{O}v = \int_{\partial\Omega} u(-y^{m_1}v_x, -x^{m_2}v_y)\cdot\eta \ .$$

Therefore, for $u \in W^1_{AC\cup\sigma}$ and $v \in W^1_{BC\cup\sigma}$, we get

$$\int_\Omega (v_x, v_y)(-y^{m_1}u_x, -x^{m_2}u_y) + \int_\Omega v\ \mathscr{O}u = \int_{AC} v(-y^{m_1}u_x, -x^{m_2}u_y)\cdot\eta \ ,$$

and

$$\int_\Omega (u_x, u_y)(-y^{m_1}v_x, -x^{m_2}v_y) + \int_\Omega u\ \mathscr{O}v = \int_{BC} u(-y^{m_1}v_x, -x^{m_2}v_y)\cdot\eta \ .$$

See that $(\mathscr{X}u\cdot\eta)_{|AC} \equiv 0$ and $(\mathscr{X}v\cdot\eta)_{|BC} \equiv 0$. Indeed, we have

$$\begin{aligned}
&(-y^{m_1}u_x, -x^{m_2}u_y)\cdot\eta_{AC} \\
&\quad = (y^{m_1}u_x^2 + x^{m_2}u_y^2)\cdot\left(-1, -\left[\frac{m_2+2}{m_1+2}(-y)^{\frac{m_1+2}{2}} + (2x_0)^{\frac{m_2+2}{2}}\right]^{\frac{-m_2}{m_2+2}}(-y)^{\frac{m_1}{2}}\right) \\
&\quad = y^{m_1}u_x + x^{m_2}(-y)^{\frac{m_1}{2}}\left[x^{\frac{m_2+2}{2}}\right]^{-\frac{m_2}{m_2+2}}u_y \\
&\quad = (-y)^{\frac{m_1}{2}}x^{\frac{m_2}{2}}x^{-\frac{m_2}{2}}\left[x^{\frac{m_2}{2}}u_y - (-y)^{\frac{m_1}{2}}u_x\right] \\
&\quad = (-y)^{\frac{m_1}{2}}x^{\frac{m_2}{2}}\partial_- u.
\end{aligned}$$

Moreover, by hypothesis $u_{|AC} \equiv 0$ which implies $\partial_- u_{|AC} \equiv 0$, follows that $(\mathscr{X} u \cdot \eta)_{|AC} \equiv 0$. Analogously, we obtain $(\mathscr{X} v \cdot \eta)_{|BC} \equiv 0$. Hence

$$\int_\Omega (v_x, v_y)(-y^{m_1} u_x, -x^{m_2} u_y) + \int_\Omega v\, \mathscr{O} u = 0,$$

and

$$\int_\Omega (u_x, u_y)(-y^{m_1} v_x, -x^{m_2} v_y) + \int_\Omega u\, \mathscr{O} v = 0.$$

In conclusion,

$$(\mathscr{O} u, v)_{L^2} = \int_\Omega v\, \mathscr{O} u = \int_\Omega (y^{m_1} u_x v_x + x^{m_2} u_y v_y) = \int_\Omega u\, \mathscr{O} v = (u, \mathscr{O} v)_{L^2}.$$

Note that, for $u \in W^1_{AC\cup\sigma}$

$$\begin{aligned} ||\mathscr{O} u||_{W^{-1}_{BC\cup\sigma}} &= \sup_{0\neq v \in W^1_{BC\cup\sigma}} \frac{|(\mathscr{O} u, v)_{L^2}|}{||v||_{W^1_{BC\cup\sigma}}} \\ &= \sup_{0\neq v \in W^1_{BC\cup\sigma}} \frac{|\int_\Omega (y^{m_1} u_x v_x + x^{m_2} u_y v_y)|}{||v||_{W^1_{BC\cup\sigma}}} \\ &= \sup_{0\neq v \in W^1_{BC\cup\sigma}} \frac{|(u, \mathscr{O} v)_{L^2}|}{||v||_{W^1_{BC\cup\sigma}}} \\ &\leq \sup_{0\neq v \in W^1_{BC\cup\sigma}} \frac{|(u, \mathscr{O} v)_{L^2}|}{K_1 ||v||_{L^2}} \\ &\leq \sup_{0\neq v \in W^1_{BC\cup\sigma}} \frac{||u||_{L^2} ||\mathscr{O} v||_{L^2}}{K_1 ||v||_{L^2}}, \end{aligned}$$

since $\mathscr{O}$ is continuous in L^2. Then,

$$||\mathscr{O}_{AC} u||_{W^{-1}_{BC\cup\sigma}} \leq C_1 ||u||_{W^1_{AC\cup\sigma}}, \quad u \in W^1_{AC\cup\sigma},$$

similarly, for $v \in W^1_{BC\cup\sigma}$

$$||\mathscr{O}_{BC} v||_{W^{-1}_{AC\cup\sigma}} \leq C_2 ||v||_{W^1_{BC\cup\sigma}}, \quad v \in W^1_{BC\cup\sigma},$$

where $\mathscr{O}_{AC} u$ and $\mathscr{O}_{BC} v$ is the unique continuous extensions of $\mathscr{O}$ relative to the dense subspaces $C^\infty_{AC\cup\sigma}(\overline{\Omega})$ and $C^\infty_{BC\cup\sigma}(\overline{\Omega})$ respectively. The following definitions are adaptations of the definitions found in [16] for our purpose.

Definition 1.5. Let $\lambda \in \mathbb{R}$. A Tricomi domain Ω is said admissible for the operator $\mathscr{O} - \lambda$, if there are positive constants C_3 and C_4 such that

$$||u||_{L^2(\Omega)} \leq C_3 ||(\mathscr{O}_{AC} - \lambda) u||_{W^{-1}_{BC\cup\sigma}}, \quad u \in W^1_{AC\cup\sigma}, \tag{1.4}$$

and

$$||v||_{L^2(\Omega)} \leq C_4 ||(\mathscr{O}_{BC} - \lambda) v||_{W^{-1}_{AC\cup\sigma}}, \quad v \in W^1_{BC\cup\sigma}. \tag{1.5}$$

Definition 1.6. Given $f \in L^2(\Omega)$ we say that $u \in W^1_{AC\cup\sigma}$ is a weak solution of (4) if the following relation is hold

$$\mathscr{B}_\lambda(x,y) := \int_\Omega (y^{m_1} u_x v_x + x^{m_2} u_y v_y - \lambda uv)\, dxdy = \int_\Omega fv\, dxdy,$$

for all $v \in C^\infty_{BC\cup\sigma}(\overline{\Omega})$.

1.3 Fundamental tools of the variational method

In this section, we review several key concepts, including the interpolation inequality, Palais-Smale sequences, and the Mountain Pass Theorem, as found in [12], [2], [29], and other sources.

Interpolation inequality

Let $u \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \le p \le q \le \infty$, hence $u \in L^r(\Omega)$ for all $p \le r \le q$ and we have the interpolation inequality

$$||u||_{L^r(\Omega)} \le ||u||^{\alpha}_{L^p(\Omega)} ||u||^{1-\alpha}_{L^q(\Omega)} \quad \text{where} \quad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \; (0 \le \alpha \le 1). \tag{1.6}$$

Definition 1.7. Let $\{u_n\} \subset X$. We say that $\{u_n\}$ is a Palais-Smale (PS) sequence for $\varphi \in C^1(X,\mathbb{R})$ if

$$\varphi(u_n) \to c \text{ and } \varphi'(u_n) \to 0. \tag{1.7}$$

Definition 1.8. Let $\varphi \in C^1(X,\mathbb{R})$. We say φ satisfies the Palais-Smale (PS) condition (1.7), if any PS sequence $\{u_n\}$ at level c admits a convergent subsequence.

Remark 1.9. If φ is a bounded from below operator and satisfies the (PS) condition with $c = \inf_X \varphi$, then c is a critical value of φ.

Theorem 1.10. *Let X be a Banach space and $\varphi \in C^1(X,\mathbb{R})$ be a functional satisfying the PS condition. If $e \in X$ and $0 < r < ||e||$ are such that*

$$a := \max\{\varphi(0), \varphi(e)\} < b := \inf_{||u||=r} \varphi(u),$$

then

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \varphi(\gamma(t))$$

is a critical value of φ with $c \ge b$. Where Γ is the set formed by paths joining the point 0 and e. This is,

$$\Gamma = \{\gamma \in C([0,1],X); \gamma(0) = 0, \gamma(1) = e\}.$$

CHAPTER 2

Nonexistence results for a degenerate Goursat type problem

2.1 Tricomi domains and Pohoẑaev-type identities

We will study the problem (1) on three different domains in this paper, and we will construct Pohoẑaev-type identities for them. We start with the problem

$$\begin{cases} \mathscr{O}(u) := -y^{m_1}u_{xx} - x^{m_2}u_{yy} = f(u) & \text{in} \quad \Omega_1, \\ \quad u = 0 & \text{on} \quad AC \cup \sigma_1 \subseteq \partial\Omega_1, \end{cases} \tag{2.1}$$

where Ω_1 is a Tricomi domain for the operator $\mathscr{O} = -y^{m_1}\partial_x^2 - x^{m_2}\partial_y^2$. This mean that, Ω_1 is open, bounded, simply connected set with piecewise C^1 boundary $\partial\Omega_1 = \sigma_1 \cup AC \cup BC$ formed by an elliptical arc $\sigma_1 \subset \{(x,y) \in \mathbb{R}^2;\ y > 0\}$ intercepting the axis x at the points $A = (2x_0, 0)$ and $B = (0,0)$ with $x_0 < 0$ and by the characteristics curves AC and BC in $\{(x,y) \in \mathbb{R}^2;\ x \leq 0 \text{ and } y \leq 0\}$ of $\mathscr{O}$ passing through points A and B respectively, which meet at point C. See Figure 2.1.

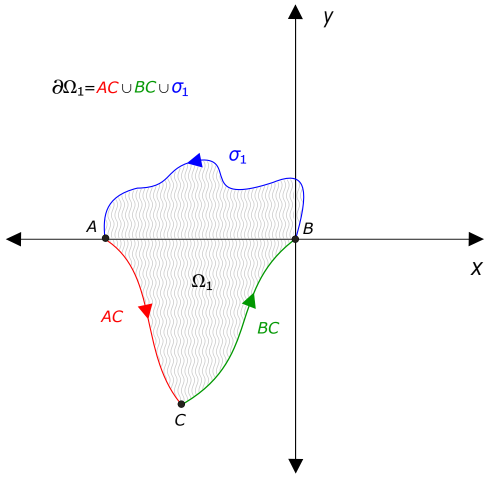

Figure 2.1: Domain Ω_1 for the problem (2.1).

Remark 2.1. The domain Ω_1 depends on m_1, m_2 because the characteristics curves are associated with the operator $\mathscr{O}$, additionally Ω_1 depends on x_0 which determines the parabolic diameter $2|x_0| = |AB|$ and its position in $\mathbb{R}^2$. This is why, we consider the problems (2.3) and (2.6).

Definition 2.2. Let $\Gamma(t) = (\alpha(t), \beta(t))$ be a smooth (C^∞) curve in $\mathbb{R}^2$. We say Γ is a characteristics curve of equation

$$au_{xx} + 2bu_{xy} + cu_{yy} = d,$$

if $a(\beta'(t))^2 - 2b(\alpha'(t))(\beta'(t)) + c(\alpha'(t))^2 = 0$.

For the problem (2.1), $a = -y^{m_1}$, $b = 0$ and $c = -x^{m_2}$. By the Definition 2.2 its characteristic curves satisfies

$$(-y^{m_1})(\beta'(t))^2 + (-x^{m_2})(\alpha'(t))^2 = 0. \tag{2.2}$$

We know Γ can be written locally as the graph of a function because, Γ is a smooth curve. Suppose $\Gamma(x) = (x, y(x))$, in this case, $\alpha(x) = x$ and $\beta(x) = y(x)$ so $\alpha'(x) = 1$ and $\beta'(x) = y'(x)$. By (2.2), we conclude that the characteristics curves of the problem (2.1) are solutions of

$$-y^{m_1}(y'(x))^2 = x^{m_2}.$$

To solve this ordinary differential equation in $\{(x,y) \in \mathbb{R}^2;\ x \leq 0 \text{ and } y \leq 0\}$, without loss of generality, consider $m_1 \in \mathbb{N}$ odd and $\frac{m_2}{2} \in \mathbb{N}$ even . On this conditions the characteristics curves of (2.1) satisfy

$$(-y)^{\frac{m_1}{2}}|y'(x)| = x^{\frac{m_2}{2}}.$$

We will analyze $|y'(x)|$.

Case 1	**Case 2**	**Case 3**
$y'(x) < 0$	$y'(x) = 0$	$y'(x) > 0$
$(-y)^{\frac{m_1}{2}}(-y'(x)) = x^{\frac{m_2}{2}}$	$y(x) = K_3$	$(-y)^{\frac{m_1}{2}}y'(x) = x^{\frac{m_2}{2}}$

In case 1, we get $\frac{2}{m_1+2}(-y)^{\frac{m_1+2}{2}} = \frac{2}{m_2+2}x^{\frac{m_2+2}{2}} + K_1$. In case 2, we get a contradiction with the choice of $\Gamma(x) = (x, y(x))$. In case 3, we get $\frac{2}{m_1+2}(-y)^{\frac{m_1+2}{2}} = -\frac{2}{m_2+2}x^{\frac{m_2+2}{2}} + K_2$.

In order to construct $\partial\Omega_1$ and consequently Ω_1, it is enough to substitute in the characteristic curves found the points $A = (2x_0, 0)$ and $B = (0,0)$. In relation to Definition 1.3 we give the orientation for Ω_1 and particularly for AC when $y'(x) < 0$ and for BC when $y'(x) > 0$. We have

$$k_1 = -\frac{2}{m_2+2}(2x_0)^{\frac{m_2+2}{2}} \text{ and } k_2 = 0.$$

The characteristic curves AC and BC in $\{(x,y) \in \mathbb{R}^2;\ x \leq 0 \text{ and } y \leq 0\}$ of the operator $\mathscr{O}$, for $m_1 \in \mathbb{N}$ odd and $\frac{m_2}{2} \in \mathbb{N}$ even, are given by

$$AC = \left\{(x,y) \in \mathbb{R}^2; y_c \leq y \leq 0,\ \frac{2}{m_1+2}(-y)^{\frac{m_1+2}{2}} = \frac{2}{m_2+2}\left[x^{\frac{m_2+2}{2}} - (2x_0)^{\frac{m_2+2}{2}}\right]\right\},$$

$$BC = \left\{(x,y)\in\mathbb{R}^2;\ y_c \le y \le 0,\ \frac{2}{m_1+2}(-y)^{\frac{m_1+2}{2}} = -\frac{2}{m_2+2}x^{\frac{m_2+2}{2}}\right\},$$

and the point $C = \left((\frac{1}{2})^{\frac{2}{m_2+2}}(2x_0), y_c\right)$ where

$$y_c = -\left[-\frac{1}{2}\left(\frac{m_1+2}{m_2+2}\right)(2x_0)^{\frac{m_2+2}{2}}\right]^{\frac{2}{m_1+2}}.$$

Parameterizing (5) and (6) with respect to the variable y, the exterior normal vector will be given by

$$\eta_{AC} = \left(-1, -\left[\frac{m_2+2}{m_1+2}(-y)^{\frac{m_1+2}{2}} + (2x_0)^{\frac{m_2+2}{2}}\right]^{\frac{-m_2}{m_2+2}}(-y)^{\frac{m_1}{2}}\right),$$

and

$$\eta_{BC} = \left(1, -\left[-\frac{m_2+2}{m_1+2}(-y)^{\frac{m_1+2}{2}}\right]^{\frac{-m_2}{m_2+2}}(-y)^{\frac{m_1}{2}}\right),$$

respectively.

As mentioned in Remark 2.1, we will consider the problem (2.3) and its respective characteristic curves which are found analogously to problem (2.1), namely

$$\begin{cases} \mathscr{O}(u) := -y^{m_1}u_{xx} - x^{m_2}u_{yy} = f(u) & \text{in} \quad \Omega_2, \\ \quad u = 0 & \text{on} \quad AC\cup\sigma_2 \subseteq \partial\Omega_2, \end{cases} \tag{2.3}$$

where Ω_2 is a Tricomi domain for the operator $\mathscr{O} = -y^{m_1}\partial_x^2 - x^{m_2}\partial_y^2$. That is, Ω_2 is open, bounded, simply connected set with piecewise C^1 boundary $\partial\Omega_2 = \sigma_2 \cup AC' \cup BC'$ formed by an elliptical arc $\sigma_2 \subset \{(x,y)\in\mathbb{R}^2;\ y>0\}$ intercepting the axis x at the points $A' = (2x_0, 0)$ and $B' = (0,0)$ with $x_0 > 0$ and the characteristics curves AC' and BC' in $\{(x,y)\in\mathbb{R}^2;\ x \ge 0 \text{ and } y \le 0\}$ of $\mathscr{O}$ passing through points A' and B' respectively and meet at point C'. See Figure 2.2.

The characteristics curves AC' and BC' in $\{(x,y)\in\mathbb{R}^2;\ x\ge 0 \text{ and } y \le 0\}$ of the operator $\mathscr{O}$ for $m_1 \in \mathbb{N}$ odd number and $m_2 \in \mathbb{N}$, are given by

$$AC' = \left\{(x,y) \in\mathbb{R}^2;\ y_c \le y \le 0,\ \frac{2}{m_1+2}(-y)^{\frac{m_1+2}{2}} = -\frac{2}{m_2+2}\left[x^{\frac{m_2+2}{2}} - (2x_0)^{\frac{m_2+2}{2}}\right]\right\}, \tag{2.4}$$

$$BC' = \left\{(x,y)\in\mathbb{R}^2;\ y_{c'} \le y \le 0,\ \frac{2}{m_1+2}(-y)^{\frac{m_1+2}{2}} = \frac{2}{m_2+2}x^{\frac{m_2+2}{2}}\right\}, \tag{2.5}$$

and the point $C' = \left((\frac{1}{2})^{\frac{2}{m_2+2}}(2x_0), y_{c'}\right)$ where

$$y_{c'} = -\left[\frac{1}{2}\left(\frac{m_1+2}{m_2+2}\right)(2x_0)^{\frac{m_2+2}{2}}\right]^{\frac{2}{m_1+2}}.$$

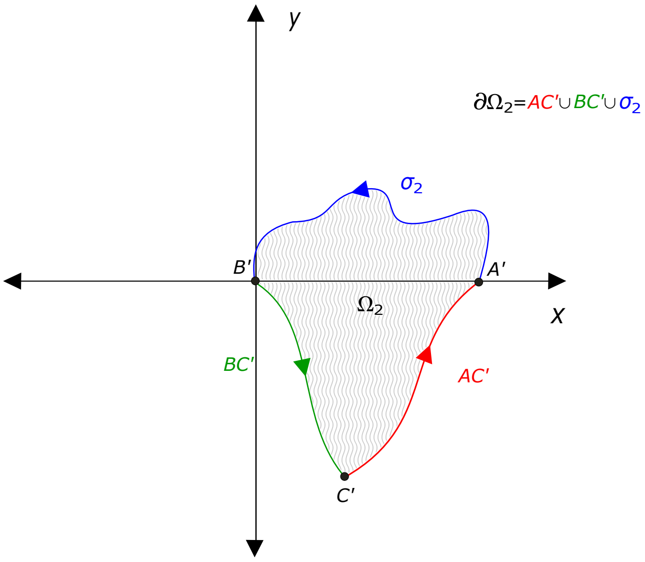

Figure 2.2: Domain Ω_2 for the problem (2.3).

Parameterizing (2.4) and (2.5) in variable y, the exterior normal vector be given by

$$\eta_{AC'} = \left(1, -\left[-\frac{m_2+2}{m_1+2}(-y)^{\frac{m_1+2}{2}} + (2x_0)^{\frac{m_2+2}{2}}\right]^{\frac{-m_2}{m_2+2}} (-y)^{\frac{m_1}{2}}\right),$$

and

$$\eta_{BC'} = \left(-1, -\left[\frac{m_2+2}{m_1+2}(-y)^{\frac{m_1+2}{2}}\right]^{\frac{-m_2}{m_2+2}} (-y)^{\frac{m_1}{2}}\right),$$

respectively.

We also study the problem

$$\begin{cases} \mathscr{O}(u) := -y^{m_1}u_{xx} - x^{m_2}u_{yy} = f(u) & \text{in} \quad \Omega_3, \\ \quad u = 0 & \text{on} \quad AC \cup \sigma_3 \subseteq \partial\Omega_3, \end{cases} \tag{2.6}$$

where Ω_3 is a Tricomi domain for the operator $\mathscr{O} = -y^{m_1}\partial_x^2 - x^{m_2}\partial_y^2$. That is, Ω_3 is open, bounded, simply connected set with piecewise C^1 boundary $\partial\Omega_3 = \sigma_3 \cup AC'' \cup BC''$ formed by an elliptical arc $\sigma_3 \subset \{(x,y) \in \mathbb{R}^2;\ x < 0\}$ intercepting the axis y at the points $A'' = (0, 2y_0)$ and $B'' = (0,0)$ with $y_0 < 0$ and the characteristics curves AC'' and BC'' in $\{(x,y) \in \mathbb{R}^2;\ x \geq 0 \text{ and } y \leq 0\}$ of $\mathscr{O}$ passing through points A'' and B'' respectively and meet at point C''. See Figure 2.3.

To construction of the domain Ω_3, we suppose $\Gamma(y) = (x(y), y)$, locally as the graph of a function, in this case $\alpha(y) = x(y)$ and $\beta(y) = y$, so $\alpha'(y) = x'(y)$ and $\beta'(y) = 1$. Inserting in (2.2), we concluded that the characteristics curves of the problem (2.6) are solutions of

$$-y^{m_1} = x^{m_2}(x'(y))^2.$$

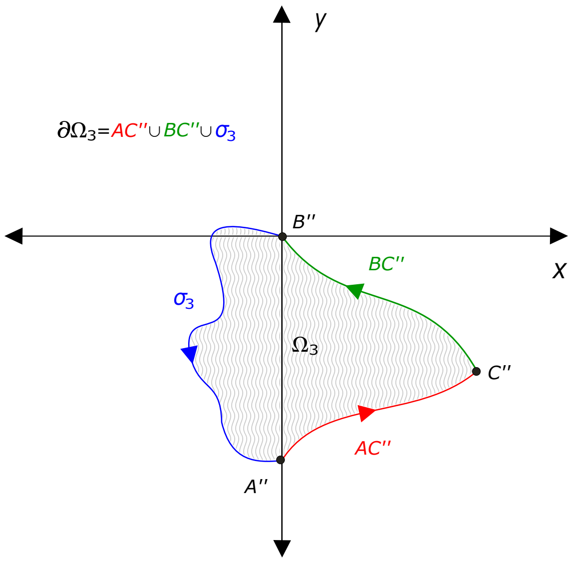

Figure 2.3: Domain Ω_3 for the problem (2.6).

With calculations analogous to those made to find the expressions of the characteristic curves (5) and (6) of problem (2.1), analyzing $|x'(y)|$ instead of $|y'(x)|$, the characteristics curves AC'' and BC'' in $\{(x,y) \in \mathbb{R}^2;\ x \geq 0 \text{ and } y \leq 0\}$ of the operator $\mathscr{O}$, for $m_1 \in \mathbb{N}$ odd number and $m_2 \in \mathbb{N}$, are given by

$$AC'' = \left\{ (x,y) \in \mathbb{R}^2;\ 0 \leq x \leq x_c,\ -\frac{2}{m_1+2}\left[(-y)^{\frac{m_1+2}{2}} - (-2y_0)^{\frac{m_1+2}{2}}\right] = \frac{2}{m_2+2} x^{\frac{m_2+2}{2}} \right\}, \tag{2.7}$$

$$BC'' = \left\{ (x,y) \in \mathbb{R}^2;\ 0 \leq x \leq x_{c''},\ \frac{2}{m_1+2}(-y)^{\frac{m_1+2}{2}} = \frac{2}{m_2+2} x^{\frac{m_2+2}{2}} \right\}, \tag{2.8}$$

and the point $C'' = \left(x_{c''}, \left(\frac{1}{2}\right)^{\frac{2}{m_1+2}} (2y_0) \right)$ where

$$x_{c''} = \left[\frac{1}{2} \left(\frac{m_2+2}{m_1+2} \right) (-2y_0)^{\frac{m_1+2}{2}} \right]^{\frac{2}{m_2+2}}.$$

Parameterizing (2.7) and (2.8) in variable x, the exterior normal vector be given by

$$\eta_{AC''} = \left(\left[-\frac{m_1+2}{m_2+2} x^{\frac{m_2+2}{2}} + (-2y_0)^{\frac{m_1+2}{2}} \right]^{\frac{-m_1}{m_1+2}} x^{\frac{m_2}{2}}, -1 \right),$$

and

$$\eta_{BC''} = \left(\left[\frac{m_1+2}{m_2+2} x^{\frac{m_2+2}{2}} \right]^{\frac{-m_1}{m_1+2}} x^{\frac{m_2}{2}}, 1 \right),$$

respectively.

Now, we will state the theorems with the Pohožaev-type identities for the aforementioned problems (2.1), (2.3), and (2.6), providing a detailed proof of the first one and discussing the key aspects necessary to establish the proofs of the remaining two.

Theorem 2.3. *Let Ω_1 be a Tricomi domain for the operator $\mathscr{O}$ with $m_1 \in \mathbb{N}$ odd and $\frac{m_2}{2} \in \mathbb{N}$ even numbers, whose boundary $\partial\Omega_1$ is piecewise C^1 with the exterior unit normal vector η. If $u \in C^2(\overline{\Omega_1})$ is a solution of the problem* (2.1)*, then*

$$(m_1+m_2+4)\int_{\Omega_1} F(u) - \frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_1} uf(u) = \frac{1}{2}\left[\int_{BC\cup\sigma_1}\omega_1 + \int_{BC}\omega_2\right], \tag{2.9}$$

where F is the primitive function of $f \in C^0(\mathbb{R})$ such that $F(0)=0$, ω_1 and ω_2 are defined by

$$\omega_1 = \Big[2Du(-y^{m_1}u_x, -x^{m_2}u_y) + (y^{m_1}u_x^2 + x^{m_2}u_y^2)(-(m_1+2)x, -(m_2+2)y\Big]\cdot\eta\,, \tag{2.10}$$

$$\omega_2 = \Big[-2F(u)(-(m_1+2)x, -(m_2+2)y) - (m_1+m_2+m_1m_2)u(-y^{m_1}u_x, -x^{m_2}u_y)\Big]\cdot\eta\,, \tag{2.11}$$

and D is the vector field

$$D = -(m_1+2)x\partial_x - (m_2+2)y\partial_y.$$

Proof. The proof will be done in 4 steps. In step 1, we will estimate $\int_{\Omega_1} Du\mathscr{O}u$. While in step 2, we will estimate $\int_{\Omega_1} Duf(u)$ and in step 3, we will estimate $\int_{\Omega_1} uf(u)$. In step 4, we will multiply Problem (2.1) by Du, and we integrate over the domain where the regularity of the solution allows us to use the Divergence Theorem [12] and we relate the previous steps to prove (2.9).

Step 1. Claim,

$$\begin{aligned}\int_{\Omega_1} Du\,\mathscr{O}u =& \frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_1}\big(y^{m_1}u_x^2 + x^{m_2}u_y^2\big)\\ &+ \frac{1}{2}\int_{BC\cup\sigma_1}\Big[2Du\big(-y^{m_1}u_x, -x^{m_2}u_y\big)\\ &+ \big(y^{m_1}u_x^2 + x^{m_2}u_y^2\big)\big(-(m_1+2)x, -(m_2+2)y\big)\Big]\cdot\eta\,.\end{aligned} \tag{2.12}$$

In fact,

$$\begin{aligned}&div\Big((y^{m_1}u_x^2 + x^{m_2}u_y^2)(-(m_1+2)x, -(m_2+2)y)\Big)\\ &\quad=\nabla(y^{m_1}u_x^2 + x^{m_2}u_y^2)(-(m_1+2)x, -(m_2+2)y)\\ &\qquad+ (y^{m_1}u_x^2 + x^{m_2}u_y^2)\,div\Big((-(m_1+2)x, -(m_2+2)y)\Big)\\ &\quad= -2(m_1+2)xy^{m_1}u_xu_{xx} - 2(m_1+2)x^{m_2+1}u_yu_{xy}\\ &\qquad- m_2(m_1+2)x^{m_2}u_y^2 - m_1(m_2+2)y^{m_1}u_x^2 - 2(m_2+2)y^{m_1+1}u_xu_{xy}\\ &\qquad- 2(m_2+2)yx^{m_1}u_yu_{yy} - (m_1+m_2+4)(y^{m_1}u_x^2 + x^{m_2}u_y^2),\end{aligned} \tag{2.13}$$

Define $\mathscr{X}u = (-y^{m_1}u_x, -x^{m_2}u_y)$. Since $Du = -(m_1+2)xu_x - (m_2+2)yu_y$, we found by simple calculations

$$\begin{aligned}-\nabla(Du)\cdot\mathscr{X}u =& -(m_1+2)y^{m_1}u_x^2 - (m_1+2)xy^{m_1}u_xu_{xx}\\ &- (m_2+2)y^{m_1+1}u_xu_{xy} - (m_1+2)x^{m_2+1}u_yu_{xy}\\ &- (m_2+2)x^{m_2}u_y^2 - (m_2+2)yx^{m_2}u_yu_{yy}.\end{aligned} \tag{2.14}$$

So, from (2.13) and (2.14), we get

$$\begin{aligned} -\nabla(Du)\cdot\mathscr{X}u =& \frac{1}{2} div\left((y^{m_1}u_x^2+x^{m_2}u_y^2)(-(m_1+2)x,-(m_2+2)y)\right)\\ &+\frac{(m_1+m_2+m_1m_2)}{2}\left(y^{m_1}u_x^2+x^{m_2}u_y^2\right). \end{aligned} \tag{2.15}$$

Using the Divergence Theorem [12], we gain

$$\int_{\Omega_1} div(Du\,\mathscr{X}u)=\int_{\partial\Omega_1} Du\,\mathscr{X}u\cdot\eta\,,$$

by the proprieties of the divergent

$$\int_{\Omega_1}\nabla(Du)\cdot\mathscr{X}u+\int_{\Omega_1}Du\,div(\mathscr{X}u)=\int_{\partial\Omega_1}Du\,\mathscr{X}u\cdot\eta\,.$$

Note that, $div(\mathscr{X}u)=\mathscr{O}u$. So, we obtain

$$\int_{\Omega_1}Du\,\mathscr{O}u=\int_{\partial\Omega_1}Du\,\mathscr{X}u\cdot\eta+\int_{\Omega_1}-\nabla(Du)\cdot\mathscr{X}u.$$

Inserting (2.15) in the last equation

$$\begin{aligned} \int_{\Omega_1}Du\,\mathscr{O}u=&\int_{\partial\Omega_1}Du\,\mathscr{X}u\cdot\eta\\ &+\frac{1}{2}\int_{\Omega} div\left((y^{m_1}u_x^2+x^{m_2}u_y^2)(-(m_1+2)x,-(m_2+2)y)\right)\\ &+\frac{(m_1+m_2+m_1m_2)}{2}\int_{\Omega}\left(y^{m_1}u_x^2+x^{m_2}u_y^2\right). \end{aligned} \tag{2.16}$$

Using the Divergence Theorem again, we have

$$\begin{aligned} &\int_{\Omega_1} div\left((y^{m_1}u_x^2+x^{m_2}u_y^2)(-(m_1+2)x,-(m_2+2)y)\right)=\\ &\qquad\int_{\partial\Omega_1}\left(y^{m_1}u_x^2+x^{m_2}u_y^2\right)(-(m_1+2)x,-(m_2+2)y)\cdot\eta\,, \end{aligned} \tag{2.17}$$

and inserting (2.17) in (2.16) we obtain

$$\begin{aligned} \int_{\Omega_1}Du\,\mathscr{O}u=&\frac{(m_1+m_2+m_1m_2)}{2}\int_{\Omega_1}(y^{m_1}u_x^2+x^{m_2}u_y^2)\\ &+\frac{1}{2}\int_{\partial\Omega_1}\big[2Du(-y^{m_1}u_x,-x^{m_2}u_y)\\ &+\left(y^{m_1}u_x^2+x^{m_2}u_y^2\right)(-(m_1+2)x,-(m_2+2)y)\big]\cdot\eta\,. \end{aligned}$$

This last equation is the identity (2.12) if the boundary integral vanishes on AC. Just prove that, $(y^{m_1}u_x^2+x^{m_2}u_y^2)_{|AC}\equiv 0$ and $(\mathscr{X}u\cdot\eta)_{|AC}\equiv 0$. Denote by,

$$\partial_+u:=x^{-\frac{m_2}{2}}\left[x^{\frac{m_2}{2}}u_y+(-y)^{\frac{m_1}{2}}u_x\right],$$

and

$$\partial_- u := x^{-\frac{m_2}{2}}\left[x^{\frac{m_2}{2}}u_y - (-y)^{\frac{m_1}{2}}u_x\right].$$

So,

$$\begin{aligned}(\partial_+ u)(\partial_- u) &= x^{-\frac{m_2}{2}}\left[x^{\frac{m_2}{2}}u_y + (-y)^{\frac{m_1}{2}}u_x\right]x^{-\frac{m_2}{2}}\left[x^{\frac{m_2}{2}}u_y - (-y)^{\frac{m_1}{2}}u_x\right]\\ &= x^{-m_2}\left[x^{m_2}u_y^2 - (-y)^{m_1}u_x^2\right]\\ &= x^{-m_2}\left[x^{m_2}u_y^2 + y^{m_1}u_x^2\right].\end{aligned}$$

Moreover, we have

$$\begin{aligned}&(-y^{m_1}u_x, -x^{m_2}u_y)\cdot\eta_{AC}\\ &= (y^{m_1}u_x^2 + x^{m_2}u_y^2)\cdot\left(-1, -\left[\frac{m_2+2}{m_1+2}(-y)^{\frac{m_1+2}{2}} + (2x_0)^{\frac{m_2+2}{2}}\right]^{\frac{-m_2}{m_2+2}}(-y)^{\frac{m_1}{2}}\right)\\ &= y^{m_1}u_x + x^{m_2}(-y)^{\frac{m_1}{2}}\left[x^{\frac{m_2+2}{2}}\right]^{-\frac{m_2}{m_2+2}}u_y\\ &= (-y)^{\frac{m_1}{2}}x^{\frac{m_2}{2}}x^{-\frac{m_2}{2}}\left[x^{\frac{m_2}{2}}u_y - (-y)^{\frac{m_1}{2}}u_x\right]\\ &= (-y)^{\frac{m_1}{2}}x^{\frac{m_2}{2}}\partial_- u.\end{aligned}$$

See that, $\partial_+ u$ and $\partial_- u$ are essentially the directional derivatives along the characteristics BC and AC respectively. By hypothesis $u_{|AC} \equiv 0$ which implies $\partial_- u_{|AC} \equiv 0$, thus $(y^{m_1}u_x^2 + x^{m_2}u_y^2)_{|AC} \equiv 0$ and $(\mathscr{X}u\cdot\eta)_{|AC} \equiv 0$. In conclusion, for m_1, $\frac{m_2}{2}$ odd and even numbers, respectively, (2.12) is true.

Step 2. Claim,

$$\int_{\Omega_1} Du\, f(u) = (m_1+m_2+4)\int_{\Omega_1} F(u) + \int_{BC} F(u)(-(m_1+2)x, -(m_2+2)y)\cdot\eta\,. \tag{2.18}$$

In fact. By the Divergence Theorem, we obtain

$$\int_{\Omega_1} div\big(F(u)(-(m_1+2)x, -(m_2+2)y)\big) = \int_{\partial\Omega_1} F(u)(-(m_1+2)x, -(m_2+2)y)\cdot\eta\,.$$

The divergent´s proprieties gives

$$\begin{aligned}&\int_{\Omega_1}\nabla F(u)\cdot(-(m_1+2)x, -(m_2+2)y)\\ &+\int_{\Omega_1} F(u)\, div\big((-(m_1+2)x, -(m_2+2)y)\big) =\\ &\qquad\int_{\partial\Omega_1} F(u)(-(m_1+2)x, -(m_2+2)y)\cdot\eta\,.\end{aligned}$$

By the hypotheses, F is the primitive function of f so, $\nabla F(u) = f(u)\nabla u$. Also, $div\big((-(m_1+2)x, -(m_2+2)y)\big) = -m_1 - m_2 - 4$. Thus,

$$\begin{aligned}&\int_{\Omega_1} f(u)\nabla u\cdot(-(m_1+2)x, -(m_2+2)y) =\\ &\qquad\int_{\partial\Omega_1} F(u)(-(m_1+2)x, -(m_2+2)y)\cdot\eta + (m_1+m_2+4)\int_{\Omega_1} F(u)\,.\end{aligned}$$

Note that,

$$\nabla u\cdot(-(m_1+2)x,-(m_2+2)y)=-(m_1+2)xu_x-(m_2+2)yu_y=Du.$$

Then,

$$\int_{\Omega_1}Du\,f(u)=(m_1+m_2+4)\int_{\Omega_1}F(u)+\int_{\partial\Omega_1}F(u)(-(m_1+2)x,-(m_2+2)y)\cdot\eta\,.$$

To prove (2.18) just see that $u_{|_{AC\cup\sigma_1}}=0$ by the boundary condition and $F(0)=0$ by hypotheses.

Step 3. Claim,

$$\int_{\Omega_1}u\,f(u)=\int_{\Omega_1}(y^{m_1}u_x^2+x^{m_2}u_y^2)+\int_{BC}u\,(-y^{m_1}u_x,-x^{m_2}u_y)\cdot\eta\,. \tag{2.19}$$

In fact. By the Divergence Theorem, we get

$$\int_{\Omega_1}div\,(u\,(-y^{m_1}u_x,-x^{m_2}u_y))=\int_{\partial\Omega_1}u\,(-y^{m_1}u_x,-x^{m_2}u_y)\cdot\eta\,.$$

The divergent´s proprieties give us

$$\int_{\Omega_1}\nabla u\cdot(-y^{m_1}u_x,-x^{m_2}u_y)+\int_{\Omega_1}u\,div\big((-y^{m_1}u_x,-x^{m_2}u_y)\big)=\int_{\partial\Omega_1}u\,(-y^{m_1}u_x,-x^{m_2}u_y)\cdot\eta\,.$$

Since $\nabla u=(u_x,u_y)$ and $div\big((-y^{m_1}u_x,-x^{m_2}u_y)\big)=\mathscr{O}u$, we get

$$\int_{\Omega_1}u\,\mathscr{O}u=\int_{\Omega_1}(y^{m_1}u_x^2+x^{m_2}u_y^2)+\int_{\partial\Omega_1}u\,(-y^{m_1}u_x,-x^{m_2}u_y)\cdot\eta\,.$$

This last equation is (2.19) just observing that, $\mathscr{O}u=f(u)$ in Ω_1 and $u_{|_{AC\cup\sigma_1}}=0$.

Step 4. Proof the Pohoẑaev identity (2.9).

Multiplying $\mathscr{O}u=f(u)$ by Du in Ω_1, inserting (2.12) from Step 1 and (2.18) from Step 2, gives

$$\begin{aligned}(m_1+m_2+4)\int_{\Omega_1}F(u)=&\frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_1}\big(y^{m_1}u_x^2+x^{m_2}u_y^2\big)\\&+\frac{1}{2}\int_{BC\cup\sigma_1}\Big[2Du\big(-y^{m_1}u_x,-x^{m_2}u_y\big)\\&+\big(y^{m_1}u_x^2+x^{m_2}u_y^2\big)(-(m_1+2)x,-(m_2+2)y)\Big]\cdot\eta\\&-\int_{BC}F(u)(-(m_1+2)x,-(m_2+2)y)\cdot\eta\,.\end{aligned} \tag{2.20}$$

Multiplying (2.19) in Step 3 by $-\frac{m_1+m_2+m_1m_2}{2}$ give us

$$\begin{aligned}-\frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_1}u\,f(u)=&-\frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_1}(y^{m_1}u_x^2+x^{m_2}u_y^2)\\&-\frac{m_1+m_2+m_1m_2}{2}\int_{BC}u\,(-y^{m_1}u_x,-x^{m_2}u_y)\cdot\eta\,.\end{aligned} \tag{2.21}$$

Adding the identities (2.20) and (2.21), grouping the integrals and inserting (2.10) and (2.11), we get the result. □

Theorem 2.4. *Let Ω_2 be a Tricomi domain for the operator $\mathscr{O}$ with $m_1 \in \mathbb{N}$ odd number and $m_2 \in \mathbb{N}$, whose boundary $\partial\Omega_2$ is piecewise C^1 with the exterior unitary normal vector η. If $u \in C^2(\overline{\Omega_2})$ is a solution of the problem* (2.3)*, then*

$$(m_1+m_2+4)\int_{\Omega_2} F(u) - \frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_2} uf(u) = \frac{1}{2}\left[\int_{BC'\cup\sigma_2}\omega_1 + \int_{BC'}\omega_2\right], \tag{2.22}$$

where F is the primitive of $f \in C^0(\mathbb{R})$ such that $F(0)=0$, ω_1 and ω_2 as in (2.10) *and* (2.11) *respectively.*

Theorem 2.5. *Let Ω_3 be a Tricomi domain for the operator $\mathscr{O}$ with $m_1 \in \mathbb{N}$ odd number and $m_2 \in \mathbb{N}$, whose boundary $\partial\Omega_3$ is piecewise C^1 with the exterior unitary normal vector η. If $u \in C^2(\overline{\Omega_3})$ is a solution of the problem* (2.6)*, then*

$$(m_1+m_2+4)\int_{\Omega_3} F(u) - \frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_3} uf(u) = \frac{1}{2}\left[\int_{BC''\cup\sigma_3}\omega_1 + \int_{BC''}\omega_2\right], \tag{2.23}$$

where F is the primitive of $f \in C^0(\mathbb{R})$ such that $F(0)=0$, ω_1 and ω_2 as in (2.10) *and* (2.11) *respectively.*

The Proof of Theorems 2.4 and 2.5 are similar to the proof of the Theorem 2.3. Just observe that, the Pohoẑaev-type identities are similar given in terms of BC, BC' and BC''. Note that, $\mathscr{X}u$, the operator $\mathscr{O}u$ and the vector field Du do not change to the problems (2.1), (2.3) and (2.6), so ω_1 and ω_2 have the same expression independently of the problem. Moreover, the computations done in the Step 2, Step 3 and Step 4 of the Proof of Theorem 2.3 which remain valid to the proof of Theorems 2.4 and 2.5. But, the Step 1 of the proof of Theorem 2.3 depends strongly on the facts $(y^{m_1}u_x^2 + x^{m_2}u_y^2)_{|AC} \equiv 0$ and $(\mathscr{X}u\cdot\eta)_{|AC} \equiv 0$. The proof of Theorems 2.4 and 2.5, we need to define the directional derivatives of u along the characteristic curves AC', BC', AC'' and BC'' and with these, we found $(y^{m_1}u_x^2 + x^{m_2}u_y^2)_{|AC'} \equiv 0$, $(y^{m_1}u_x^2 + x^{m_2}u_y^2)_{|AC''} \equiv 0$ and $(\mathscr{X}u\cdot\eta)_{|AC'} \equiv 0$, $(\mathscr{X}u\cdot\eta)_{|AC''} \equiv 0$ as done in the Step 1 of the proof of Theorem 2.3. Therefore, we can state that the Pohoẑaev-type identities are true to the proposed problems.

2.2 Nonexistence results

Together with the Pohoẑaev-type equations (2.9), (2.22) and (2.23) from Section 2.1, the nonexistence results presented in this Section depend directly on the Hardy-Sobolev inequality because it controls the sign of the boundary integrals on the characteristic curves of Ω_1, Ω_2 and Ω_3. The relationship between the critical growth of power type in nonlinearity and the critical exponent of the weighted Sobolev embedding in the Theorem 1.4 is explicit in each of these nonexistence results. We define a class of weighted functions and a class of absolutely continuous functions where the Hardy-Sobolev inequality will be applied. This inequality can be seen as a weighted Sobolev inequality and the proof can be found in [[24], Theorem 1.14]. Let,

$$\mathscr{W} = \{w : [a,b] \to \mathbb{R}; w \text{ is measurable, positive and finite a.e. in } [a,b]\}$$

and $\mathscr{AC}_L(a,b)$ defined by

$$\left\{\phi : [a,b] \to \mathbb{R};\, \phi_{|J} \in \mathscr{AC}(J),\, \forall J = [c,d] \subset (a,b) \,\text{and}\, \lim_{x\to a^+} \phi(x) = 0\right\}.$$

Lemma 2.6. *(Hardy-Sobolev Inequality). Let $1 < p \leq q < +\infty$ and $v, w \in \mathscr{W}$ be given. Then*

$$\left[\int_a^b |\phi(x)|^q w(x)dx\right]^{\frac{1}{q}} \leq C_L \left[\int_a^b |\phi'(x)|^p v(x)dx\right]^{\frac{1}{p}},$$

for all $\phi \in \mathscr{AC}_L(a,b)$, if and only if,

$$M_L = \sup_{a<x<b} G_L(x) < +\infty,$$

where

$$G_L(x) := \left(\int_x^b w(t)dt\right)^{\frac{1}{q}} \left(\int_a^x v^{1-p'}(t)dt\right)^{\frac{1}{p'}}.$$

Moreover, the best constant C_L holds

$$M_L \leq C_L \leq r(p,q)M_L \text{ where } r(p,q) := \left(1+\frac{q}{p'}\right)^{\frac{1}{q}} \left(1+\frac{p'}{q}\right)^{\frac{1}{q'}}.$$

Theorem 2.7. *Let Ω_1 be a Tricomi domain for the operator $\mathscr{O}$, which is D-star-shaped where $D = -(m_1+2)x\partial_x - (m_2+2)y\partial_y$ with m_1 odd and $\frac{m_2}{2}$ even numbers. Let $u \in C^2(\overline{\Omega_1})$ be a solution of*

$$\begin{cases} \mathscr{O}(u) := -y^{m_1}u_{xx} - x^{m_2}u_{yy} = u|u|^{\alpha-1} & \text{in } \Omega_1, \\ \quad u = 0 & \text{on } AC \cup \sigma_1 \subseteq \partial\Omega_1, \end{cases} \tag{2.24}$$

with $\alpha > 2^*(m_1,m_2) - 1 = \frac{m_1+m_2-m_1m_2+8}{m_1+m_2+m_1m_2}$. Then $u \equiv 0$ a.e. in Ω_1.

Proof. The proof will be done in 5 steps. Step 1, we will analyzed $f(u)$ and $F(u)$. Step 2, we apply the Pohožaev-type equation (Theorem 2.3). Step 3, we will check that $\int_{\sigma_1} \omega_1 \geq 0$. Step 4, we will rewrite $\int_{BC}(\omega_1+\omega_2)$ by a parameterization. Finally, Step 5, we apply Lemma 2.6 to prove that $\int_{BC}(\omega_1+\omega_2) \geq 0$.

Step 1. The problem (2.24) is a particular case of the problem (2.1) where $f(u) = u|u|^{\alpha-1}$. Which implies, $F(u) = \frac{|u|^{\alpha+1}}{\alpha+1}$.

Step 2. Applying the Pohožaev-type equation, Theorem 2.3, we get

$$(m_1+m_2+4)\int_{\Omega_1} \frac{|u|^{\alpha+1}}{\alpha+1} - \frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_1} |u|^{\alpha+1} = \frac{1}{2}\left[\int_{BC\cup\sigma_1} \omega_1 + \int_{BC} \omega_2\right].$$

So, we obtain

$$\left(\frac{(m_1+m_2+4)}{\alpha+1} - \frac{m_1+m_2+m_1m_2}{2}\right)\int_{\Omega_1} |u|^{\alpha+1} = \frac{1}{2}\left[\int_{BC\cup\sigma_1} \omega_1 + \int_{BC} \omega_2\right],$$

and

$$\left(\frac{m_1+m_2-m_1m_2+8-\alpha(m_1+m_2+m_1m_2)}{2(\alpha+1)}\right)\int_{\Omega_1}|u|^{\alpha+1}=\frac{1}{2}\left[\int_{BC\cup\sigma_1}\omega_1+\int_{BC}\omega_2\right].$$

In the next Steps, we will prove that

$$\frac{1}{2}\left[\int_{BC\cup\sigma_1}\omega_1+\int_{BC}\omega_2\right]\geq 0.$$

Since $|u|^{\alpha+1}\geq 0$, then follow the result.

Step 3. By (2.10), we obtain

$$\begin{aligned}\int_{\sigma_1}\omega_1 ds=\int_{\sigma_1}\Big[&2Du(-y^{m_1}u_x,-x^{m_2}u_y)\\&+(y^{m_1}u_x^2+x^{m_2}u_y^2)(-(m_1+2)x,-(m_2+2)y)\Big]\cdot\eta_{\sigma_1}ds,\end{aligned}$$

since $Du=-(m_1+2)xu_x-(m_2+2)yu_y$ and $\eta_{\sigma_1}ds=(-dy,dx)$, then

$$\begin{aligned}\int_{\sigma_1}\omega_1 ds=&\int_{\sigma_1}\Big[(-2(m_1+2)xu_x-2(m_2+2)yu_y)(-y^{m_1}u_x)\\&+(y^{m_1}u_x^2+x^{m_2}u_y^2)(-(m_1+2)x)\Big](-dy)\\&+\int_{\sigma_1}\Big[(-2(m_1+2)xu_x-2(m_2+2)yu_y)(-x^{m_2}u_y)\\&+(y^{m_1}u_x^2+x^{m_2}u_y^2)(-(m_2+2)y)\Big]dx.\end{aligned}$$

Thus, we get

$$\begin{aligned}\int_{\sigma_1}\omega_1 ds=&\int_{\sigma_1}(y^{m_1}u_x^2+x^{m_2}u_y^2)((m_1+2)xdy-(m_2+2)ydx)\\&+\int_{\sigma_1}(2(m_1+2)xx^{m_2}u_y-2(m_2+2)yy^{m_1}u_x)(u_xdx+u_ydy).\end{aligned}$$

By hypotheses $u_{|\sigma_1}\equiv 0$ then $(u_xdx+u_ydy)_{|\sigma_1}=0$. So, we have

$$\int_{\sigma_1}\omega_1 ds=\int_{\sigma_1}(y^{m_1}u_x^2+x^{m_2}u_y^2)((m_1+2)xdy-(m_2+2)ydx).$$

We know that Ω_1 is D-star-shaped and $\partial\Omega_1$ is D-starlike in the sense of Definition 1.3. Therefore, $((m_1+2)xdy-(m_2+2)ydx)\geq 0$. See well, $(y^{m_1}u_x^2+x^{m_2}u_y^2)_{|\sigma_1}\geq 0$ because $\sigma_1\subset\{(x,y)\in\mathbb{R}^2;y>0\}$ and m_2 is even. In conclusion,

$$\int_{\sigma_1}\omega_1 ds\geq 0.$$

Step 4. Denote by

$$I=\int_{BC}(\omega_1+\omega_2).$$

We know that ω_1 and ω_2 are defined by (2.10) and (2.11) respectively. Therefore, the generated flow by $D = -(m_1+2)x\partial_x - (m_2+2)y\partial_y$ is everywhere tangential to BC. Then, we obtain

$$I = \int_{BC} [2Du - (m_1+m_2+m_1m_2)u](-y^{m_1}u_x, -x^{m_2}u_y) \cdot \eta \, ds. \tag{2.25}$$

Remembering, BC is given by $-\frac{m_2+2}{m_1+2}(-y)^{\frac{m_1+2}{2}} = x^{\frac{m_2+2}{2}}$, we get on BC

$$ds = |\eta_{BC}| dy \ \text{ and } \ Du = (m_2+2)(-y)\partial_+ u, \tag{2.26}$$

where $\partial_+ u = u_y + x^{-\frac{m_2}{2}}(-y)^{\frac{m_1}{2}} u_x$ as defined in the Step 1 of the Theorem 2.3.

Inserting (2.26) and (7) in (2.25), we have

$$I = \int_{y_c}^{0} [2(m_2+2)(-y)\partial_+ u - (m_1+m_2+m_1m_2)u] \\ (-y^{m_1}u_x, -x^{m_2}u_y) \cdot \frac{(1, -x^{-\frac{m_2}{2}}(-y)^{\frac{m_1}{2}})}{|\eta_{BC}|} |\eta_{BC}| dy,$$

so,

$$I = \int_{y_c}^{0} [2(m_2+2)(-y)\partial_+ u - (m_1+m_2+m_1m_2)u](-y)^{\frac{m_1}{2}} x^{\frac{m_2}{2}} \partial_+ u \, dy.$$

Then,

$$I = \int_{y_c}^{0} [2(m_2+2)(-y)^{\frac{m_1+2}{2}} x^{\frac{m_2}{2}} (\partial_+ u)^2 - (m_1+m_2+m_1m_2)(-y)^{\frac{m_1}{2}} x^{\frac{m_2}{2}} u \partial_+ u] \, dy. \tag{2.27}$$

Parameterizing the curve BC by

$$\Gamma(t) = \left(\left(-\frac{m_2+2}{m_1+2}(-t)^{\frac{m_1+2}{2}} \right)^{\frac{2}{m_2+2}}, t \right), \qquad t \in [y_c, 0], \tag{2.28}$$

define $\phi(t) = u(\Gamma(t)) \in C^2((y_c, 0)) \cup C^1([y_c, 0])$ and note that

$$\phi'(t) = \nabla u \cdot \Gamma'(t) = \partial_+ u(\Gamma(t)). \tag{2.29}$$

From (2.28) and (2.29), (2.27) becomes

$$I = \int_{y_c}^{0} \Bigg[2(m_2+2)(-t)^{\frac{m_1+2}{2}} \left(-\frac{m_2+2}{m_1+2}(-t)^{\frac{m_1+2}{2}} \right)^{\frac{m_2}{m_2+2}} (\phi'(t))^2 \\ - (m_1+m_2+m_1m_2)(-t)^{\frac{m_1}{2}} \left(-\frac{m_2+2}{m_1+2}(-t)^{\frac{m_1+2}{2}} \right)^{\frac{m_2}{m_2+2}} \phi(t)\phi'(t) \Bigg] \, dt. \tag{2.30}$$

See that,

$$(-t)^{\frac{m_1+2}{2}} \left(-\frac{m_2+2}{m_1+2}(-t)^{\frac{m_1+2}{2}} \right)^{\frac{m_2}{m_2+2}} = \left(-\frac{m_2+2}{m_1+2} \right)^{\frac{m_2}{m_2+2}} (-t)^{\frac{m_1+2m_2+m_1m_2+2}{m_2+2}},$$

and

$$(-t)^{\frac{m_1}{2}}\Big(-\frac{m_2+2}{m_1+2}(-t)^{\frac{m_1+2}{2}}\Big)^{\frac{m_2}{m_2+2}}=\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}(-t)^{\frac{m_1+m_2+m_1m_2}{m_2+2}}.$$

Furthermore,

$$\begin{aligned}&\frac{d}{dt}\left[\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}(-t)^{\frac{m_1+m_2+m_1m_2}{m_2+2}}(\phi(t))^2\right]=\\&\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}\frac{d}{dt}\left[(-t)^{\frac{m_1+m_2+m_1m_2}{m_2+2}}(\phi(t))^2\right]=\\&-\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}\Big(\frac{m_1+m_2+m_1m_2}{m_2+2}\Big)(-t)^{\frac{m_1+m_1m_2-2}{m_2+2}}(\phi(t))^2\\&+2\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}(-t)^{\frac{m_1+m_2+m_1m_2}{m_2+2}}\phi(t)\phi'(t),\end{aligned}$$

then,

$$\begin{aligned}&\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}(-t)^{\frac{m_1+m_2+m_1m_2}{m_2+2}}\phi(t)\phi'(t)=\\&\frac{1}{2}\frac{d}{dt}\left[\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}(-t)^{\frac{m_1+m_2+m_1m_2}{m_2+2}}(\phi(t))^2\right]\\&+\frac{1}{2}\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}\Big(-\frac{m_1+m_2+m_1m_2}{m_2+2}\Big)(-t)^{\frac{m_1+m_1m_2-2}{m_2+2}}(\phi(t))^2.\end{aligned}\tag{2.31}$$

Inserting (2.31) in (2.30) and using the fact that $\phi(y_c)=0=\phi(0)$, we find

$$\begin{aligned}I=&2(m_2+2)\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}\int_{y_c}^{0}(-t)^{\frac{m_1+2m_2+m_1m_2+2}{m_2+2}}(\phi'(t))^2\,dt\\&-\frac{1}{2}\frac{(m_1+m_2+m_1m_2)^2}{m_2+2}\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}\int_{y_c}^{0}(-t)^{\frac{m_1+m_1m_2-2}{m_2+2}}(\phi(t))^2\,dt.\end{aligned}\tag{2.32}$$

Step 5. Claim. $I=\int_{BC}(\omega_1+\omega_2)\geq 0$. This is, the equation (2.32) is nonnegative. Just see,

$$\begin{aligned}&\frac{1}{2}\frac{(m_1+m_2+m_1m_2)^2}{m_2+2}\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}\int_{y_c}^{0}(-t)^{\frac{m_1+m_1m_2-2}{m_2+2}}(\phi(t))^2\,dt\\&\leq 2(m_2+2)\Big(-\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}\int_{y_c}^{0}(-t)^{\frac{m_1+2m_2+m_1m_2+2}{m_2+2}}(\phi'(t))^2\,dt,\end{aligned}$$

simplifying,

$$\frac{1}{2}\frac{(m_1+m_2+m_1m_2)^2}{m_2+2}\int_{y_c}^{0}(-t)^{\frac{m_1+m_1m_2-2}{m_2+2}}(\phi(t))^2\,dt\leq 2(m_2+2)\int_{y_c}^{0}(-t)^{\frac{m_1+2m_2+m_1m_2+2}{m_2+2}}(\phi'(t))^2\,dt,$$

equivalent to say,

$$\left[\int_{y_c}^{0}(-t)^{\frac{m_1+m_1m_2-2}{m_2+2}}(\phi(t))^2\,dt\right]^{\frac{1}{2}}\leq\frac{2(m_2+2)}{m_1+m_2+m_1m_2}\left[\int_{y_c}^{0}(-t)^{\frac{m_1+2m_2+m_1m_2+2}{m_2+2}}(\phi'(t))^2\,dt\right]^{\frac{1}{2}}.$$

For the function $\phi\in\mathscr{AC}_L(y_c,0)$, we use the Lemma 2.6 the Hardy-Sobolev Inequality, with constant $C_L=\frac{2(m_2+2)}{m_1+m_2+m_1m_2}$ in the interval $(a,b)=(y_c,0)$, with exponents $p=q=p'=2$ and with weighted functions $v(t):=(-t)^{\frac{m_1+2m_2+m_1m_2+2}{m_2+2}}$ and $w(t)=(-t)^{\frac{m_1+m_1m_2-2}{m_2+2}}$. Note that,

$$G_L(x)=\left[\int_x^0(-t)^{\frac{m_1+m_1m_2-2}{m_2+2}}dt\right]^{\frac{1}{2}}\left[\int_{y_c}^x(-t)^{-\frac{m_1+2m_2+m_1m_2+2}{m_2+2}}dt\right]^{\frac{1}{2}}.$$

Evaluating the integrals,

$$G_L(x)=\frac{m_2+2}{m_1+m_2+m_1m_2}\left[1-(-y_c)^{-\frac{m_1+m_2+m_1m_2}{m_2+2}}(-x)^{\frac{m_1+m_2+m_1m_2}{m_2+2}}\right]^{\frac{1}{2}}.$$

Thus,

$$M_L=\sup_{a<x<b}G_L(x)=\frac{m_2+2}{m_1+m_2+m_1m_2}<+\infty,$$

and $r(2,2)=2$. Therefore,

$$M_L\leq C_L=\frac{2(m_2+2)}{m_1+m_2+m_1m_2}\leq r(2,2)M_L=\frac{2(m_2+2)}{m_1+m_2+m_1m_2}.$$

In conclusion, $I=\int_{BC}(\omega_1+\omega_2)\geq 0$ and, as mentioned in Step 2 the result follows. □

Theorem 2.8. *Let Ω_2 be a Tricomi domain for the operator $\mathscr{O}$, which is D-star-shaped where $D=-(m_1+2)x\partial_x-(m_2+2)y\partial_y$ with m_1 odd and $\frac{m_2}{2}\in\mathbb{N}$. Let $u\in C^2(\overline{\Omega_2})$ be a regular solution of*

$$\begin{cases}\mathscr{O}(u):=-y^{m_1}u_{xx}-x^{m_2}u_{yy}=u|u|^{\alpha-1}\,in\quad\Omega_2,\\ \qquad u=0\qquad on\quad AC\cup\sigma_2\subseteq\partial\Omega_2,\end{cases}$$

with $\alpha>2^*(m_1,m_2)-1=\frac{m_1+m_2-m_1m_2+8}{m_1+m_2+m_1m_2}$. Then $u\equiv 0$ a.e. in Ω_2.

Proof. The proof will be analogous to the proof of the Theorem 2.7.

Step 1. The same Step 1 of the Theorem 2.7.

Step 2. Applying the Pohožaev-type identity (2.22), we gain

$$(m_1+m_2+4)\int_{\Omega_2}F(u)-\frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_2}uf(u)=\frac{1}{2}\left[\int_{BC'\cup\sigma_2}\omega_1+\int_{BC'}\omega_2\right].$$

Step 3. Analogous to Step 3 of the Theorem 2.7, we have

$$\int_{\sigma_2}\omega_1 ds\geq 0.$$

Step 4. Analogous to Step 4 of the Theorem 2.7, we get

$$\begin{aligned}I=&2(m_2+2)\Big(\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}\int_{y_{c'}}^0(-t)^{\frac{m_1+2m_2+m_1m_2+2}{m_2+2}}(\phi'(t))^2dt\\ &-\frac{1}{2}\frac{(m_1+m_2+m_1m_2)^2}{m_2+2}\Big(\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}}\int_{y_{c'}}^0(-t)^{\frac{m_1+m_1m_2-2}{m_2+2}}(\phi(t))^2dt,\end{aligned}$$

Step 5. The same Step 5. of the Theorem 2.7. □

The following result arises from the difficulties mentioned in the introduction and it can be seen as a weak result of a possible generalization which does not have a final conclusion. We do not have a similar result to the Theorems 2.7 and 2.8 on Ω_3 for the Problem (2.6). But, we can consider the characteristic triangle Ω_4 as being Ω_3 for the Problem (2.6) with $\sigma_3 = \{(x,y) \in \mathbb{R}^2; x = 0 \text{ and } 2y_0 \leq y \leq 0\}$. See Figure 2.4.

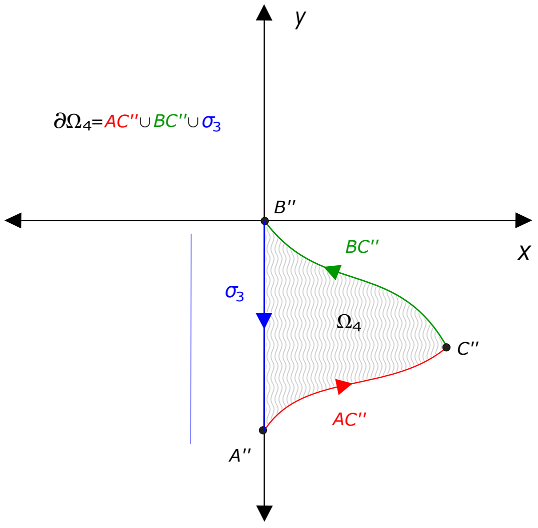

Figure 2.4: Domain Ω_4 for the problem (2.6).

This idea comes from considering $\sigma_1 = \{(x,y) \in \mathbb{R}^2; y = 0 \text{ and } 2x_0 \leq x \leq 0\}$ for the Problem (2.1) and $\sigma_2 = \{(x,y) \in \mathbb{R}^2; y = 0 \text{ and } 0 \leq x \leq 2x_0\}$ for the Problem (2.3) so, we could obtain two corollaries for the Theorems 2.7 and 2.8.

Theorem 2.9. *Let Ω_4 be a Tricomi domain for the operator $\mathscr{O}$, which is D-star-shaped with $D = -(m_1+2)x\partial_x - (m_2+2)y\partial_y$ and m_1 odd and $\frac{m_2}{2} \in \mathbb{N}$. Let $u \in C^2(\overline{\Omega_4})$ be a regular solution of*

$$\begin{cases} \mathscr{O}(u) := -y^{m_1}u_{xx} - x^{m_2}u_{yy} = u|u|^{\alpha-1} & \text{in} \quad \Omega_4, \\ \quad u = 0 & \text{on} \quad AC \cup \sigma_3 \subseteq \partial\Omega_4, \end{cases}$$

with $\alpha > 2^*(m_1,m_2) - 1 = \frac{m_1+m_2-m_1m_2+8}{m_1+m_2+m_1m_2}$. Then $u \equiv 0$ a.e. in Ω_4.

Proof. The proff will be analogous to the proof of the Theorem 2.7.

Step 1. The same Step 1 of the Theorem 2.7.

Step 2. Applying the Pohoẑaev-type identity (2.23), we obtain

$$(m_1+m_2+4)\int_{\Omega_4} F(u) - \frac{m_1+m_2+m_1m_2}{2}\int_{\Omega_4} uf(u) = \frac{1}{2}\left[\int_{BC''\cup\sigma_3}\omega_1 + \int_{BC''}\omega_3\right].$$

Step 3. Analogous to Step 3 of the Theorem 2.7, we have

$$\int_{\sigma_3} \omega_1 ds = 0.$$

Step 4. Analogous to Step 4 of the Theorem 2.7, we get

$$\begin{aligned} I =& 2(m_2+2)\Big(\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}} \int_{y_{c''}}^{0} (-t)^{\frac{m_1+2m_2+m_1m_2+2}{m_2+2}} (\phi'(t))^2\, dt \\ & - \frac{1}{2}\frac{(m_1+m_2+m_1m_2)^2}{m_2+2}\Big(\frac{m_2+2}{m_1+2}\Big)^{\frac{m_2}{m_2+2}} \int_{y_{c''}}^{0} (-t)^{\frac{m_1+m_1m_2-2}{m_2+2}} (\phi(t))^2\, dt\,, \end{aligned}$$

Step 5. The same Step 5. of the Theorem 2.7. ☐

CHAPTER 3

Existence of weak solutions for a class of linear equations

3.1 Admissible Tricomi domain

In this section, we will demonstrate the existence of an admissible Tricomi domain for our operator $\mathscr{O}$. Furthermore, we will prove that this domain is also a admissible Tricomi domain for the operator $\mathscr{O}-\lambda$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a Tricomi domain with $A=(-x_1,0)$ and $B=(x_0,0)$ with $0<x_0<x_1$ suppose that Ω and its boundary $\partial\Omega=\sigma \cup AC \cup BC$ satisfies:*

i) $|x| \leq x_1$ on $\overline{\Omega}$,

ii) The curve σ is given by a C^2 graph $y=g(x)$ for $-x_1 \leq x \leq x_0$ with $g(-x_1)=0=g(x_0)$,

iii) There is a constant

$$h < \left(\frac{x_c^{m_2}}{y_c^{m_1}}\right)^{\frac{1}{2}}$$

such that

$$-h < g'(x) < h \quad \text{for} \quad -x_1 < x < x_0.$$

Then Ω is admissible for the operator $\mathscr{O}-\lambda$ for all $\lambda \leq \frac{1}{K_3}$, for a suitable $K_3>0$.

Proof. Initially, we will demonstrate for case $\lambda=0$. The proof of this case, will be done in 6 steps. In the step 1, we will define an auxiliary Cauchy problem $Dv=u$. While in step 2, we proof existence and uniqueness of the solution for the previous problem and we will define $Iu=v$. In step 3 and step 4, we will estimate $\int_{AC}(y^{m_1}v_x^2+x^{m_2}v_y^2)(a,b)\eta_{AC}$ and $\int_\Omega v\mathscr{O}u$, respectively. We will find $\delta>0$ such that $\int_\Omega v\mathscr{O}u \geq \delta||\nabla v||^2_{L^2(\Omega)}$, in step 5. Finally, we will estimate $(Iu,\mathscr{O}u)_{L^2}$ from above and bellow in the step 6 to prove the inequality (1.4).

Step 1. Consider, $a = -(1+\varepsilon x)$, $b = -h(1+\varepsilon x)$ and $c = 0$ where h is given by the hypotheses $iii)$ and $0 < \varepsilon < \frac{1}{x_0}$. This way, a and b never vanish on $\overline{\Omega}$. Define the auxiliary Cauchy problem as being

$$\begin{cases} Dv = av_x + bv_y = u & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad BC \cup \sigma \subseteq \partial\Omega. \end{cases} \tag{3.1}$$

Step 2. We will use the following change of coordinates

$$(\xi,\eta) = \Phi(x,y) = (y + h(x+x_1), -y + h(x+x_1)).$$

Define, $w(\xi,\eta) = v(x,y)$ and note that,

$$\begin{aligned} Dw &= aw_\xi h + aw_\eta h + bw_\xi - bw_\eta \\ &= (ah+b)w_\xi + (ah-b)w_\eta, \end{aligned}$$

as $ah = b$ then,

$$Dw = 2ahw_\xi.$$

See that, $\Phi(A) = (0,0)$, $\Phi(B) = (h(x_0+x_1), h(x_0+x_1)$ and $\Phi(C) = (y_c + h(x_c+x_1), -y_c + h(x_c + x_1)$.

Since the coordinate lines $\{\eta_0 = Cte\}$ intersect $\Phi(BC \cup \sigma)$ only once, we can conclude that $\Phi(BC \cup \sigma)$ can be written as the graph of a function $\psi(\eta) = \xi$ for $\eta \in [0, -y_c + h(x_c + x_1)]$. See Figure 3.1.

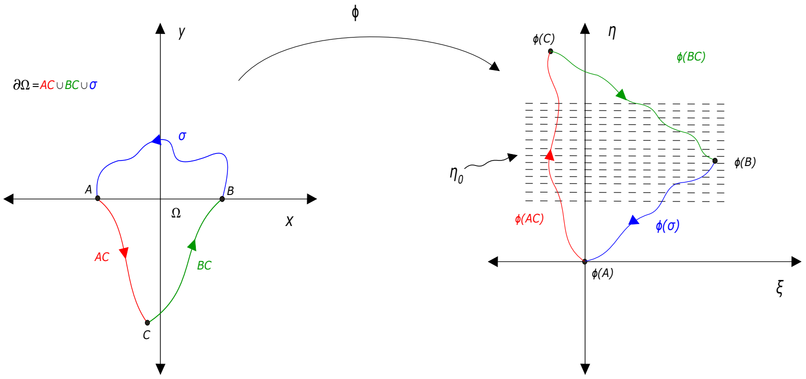

Figure 3.1: Change of coordinates $\Phi : \Omega \to \Phi(\Omega)$.

Integrating on the variable ξ we get,

$$v(x,y) = w(\xi,\eta) = \frac{1}{2h}\int_{\psi(\eta)}^{\xi} (a^{-1}u)(\Phi^{-1}(t,\eta))dt.$$

Define $Iu = v$ as been the operator such that for each $u \in W^1_{AC\cup\sigma}$ associate the unique solution $v \in W^1_{BC\cup\sigma}$ of the auxiliary Cauchy problem (3.1).

Step 3. We will find a suitable expression to

$$\int_{AC}(y^{m_1}v_x^2+x^{m_2}v_y^2)(a,b)\eta_{AC}.$$

By the divergence Theorem in [12], we get

$$\int_{\Omega} div\big((y^{m_1}v_x^2+x^{m_2}v_y^2)(a,b)\big)=\int_{\partial\Omega}(y^{m_1}v_x^2+x^{m_2}v_y^2)(a,b)\eta_{AC},$$

by the proprieties of the divergent

$$\begin{aligned}\int_{\Omega}\nabla(y^{m_1}v_x^2+x^{m_2}v_y^2)(a,b)+\int_{\Omega}(y^{m_1}v_x^2+x^{m_2}v_y^2)\,div(a,b)&\\ =\int_{\partial\Omega}(y^{m_1}v_x^2+x^{m_2}v_y^2)(a,b)\eta_{AC}.&\end{aligned}$$

Developing we have,

$$\begin{aligned}\int_{\Omega}\nabla(y^{m_1}v_x^2+x^{m_2}v_y^2)(a,b)&=\int_{\Omega}(2y^{m_1}v_xv_{xx}+m_2^{m_2-1}v_y^2\\ &\quad+2x^{m_2}v_yv_{xy},m_1y^{m_1-1}v_x^2+2y^{m_1}v_xv_{xy}+2x^{m_2}v_yv_{yy})(a,b)\\ &=\int_{\Omega}(2ay^{m_1}v_xv_{xx}+am_2^{m_2-1}v_y^2\\ &\quad+2ax^{m_2}v_yv_{xy}+bm_1y^{m_1-1}v_x^2+2by^{m_1}v_xv_{xy}+2bx^{m_2}v_yv_{yy}),\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega}(y^{m_1}v_x^2+x^{m_2}v_y^2)\,div(a,b)&=\int_{\Omega}(y^{m_1}v_x^2+x^{m_2}v_y^2)(a_x+b_y)\\ &=\int_{\Omega}(a_xy^{m_1}v_x^2+a_xx^{m_2}v_y^2+b_yy^{m_1}v_x^2+b_yx^{m_2}v_y^2).\end{aligned}$$

We know that $v\equiv 0$ on $BC\cup\sigma$ then $y^{m_1}v_x^2+x^{m_2}v_y^2\equiv 0$ on $BC\cup\sigma$ as done in Step 1. of paper [18]. So,

$$\begin{aligned}\int_{AC}(y^{m_1}v_x^2+x^{m_2}v_y^2)(a,b)\eta_{AC}&=\int_{\Omega}(2ay^{m_1}v_xv_{xx}+am_2^{m_2-1}v_y^2\\ &\quad+2ax^{m_2}v_yv_{xy}+bm_1y^{m_1-1}v_x^2+2by^{m_1}v_xv_{xy}+2bx^{m_2}v_yv_{yy}\\ &\quad+a_xy^{m_1}v_x^2+a_xx^{m_2}v_y^2+b_yy^{m_1}v_x^2+b_yx^{m_2}v_y^2).\end{aligned}\tag{3.2}$$

Step 4. We will find a suitable expression to

$$\int_{\Omega}v\mathscr{O}u.$$

We know that,

$$\int_{\Omega}v\mathscr{O}u=\int_{\Omega}(y^{m_1}u_xv_y+x^{m_2}u_xv_y).\tag{3.3}$$

As $Dv = av_x + bv_y = u$ on Ω then,

$$u_x = a_x v_x + av_{xx} + b_x v_y + bv_{xy} \text{ and } u_y = a_y v_x + av_{xy} + b_y v_y + bv_{yy}. \tag{3.4}$$

Replacing (3.4) in (3.3), we have

$$\begin{aligned}\int_\Omega v\,\mathscr{O}u = \int_\Omega (&a_x y^{m_1} v_x^2 + a y^{m_1} v_x v_{xx} + b_x y^{m_1} v_x v_y + b y^{m_1} v_x v_{xy} \\ &+ a_y x^{m_2} v_x v_y + a x^{m_2} v_y v_{xy} + b_y x^{m_2} v_y^2 + b x^{m_2} v_y v_{yy}).\end{aligned} \tag{3.5}$$

For $\alpha = -\varepsilon y^{m_1} - h(1+\varepsilon x)m_1 y^{m_1-1}$, $\beta = -h\varepsilon y^{m_1}$ and $\gamma = (1+\varepsilon x)m_2 x^{m_2-1} + \varepsilon x^{m_2}$. From (3.2) and (3.5), we obtain

$$\int_\Omega v\,\mathscr{O}u = \int_\Omega \alpha v_x^2 + 2\beta v_x v_y + \gamma v_y^2 + \frac{1}{2}\int_{AC} (y^{m_1} v_x^2 + x^{m_2} v_y^2)(a,b)\eta_{AC}.$$

Step 5. We will find $\delta > 0$ such that

$$\int_\Omega v\,\mathscr{O}u \geq \delta ||\nabla v||^2_{L^2(\Omega)}.$$

We know that $Dv = av_x + bv_y = u$. On AC we have $u \equiv 0$ then $av_x = -bv_y = -ahv_y$. So, $h^2 v_y^2 = v_x^2$. Therefore,

$$(y^{m_1} v_x^2 + x^{m_2} v_y^2)(a,b)\eta_{AC} = (y^{m_1} h^2 + x^{m_2}) v_y^2 (-(1+\varepsilon x), -h(1+\varepsilon x))\eta_{AC}.$$

By the hypothesis $h < (\frac{x_c^{m_2}}{y_c^{m_1}})^{\frac{1}{2}}$. How $-x_1 < x < x_c \leq 0$ and $y_c \leq y \leq 0$ on AC hence $x_c^{m_2} \leq x^{m_2}$ and $y_c^{m_1} \leq y^{m_1}$ because $m_1 \in \mathbb{N}$ and $\frac{m_2}{2} \in \mathbb{N}$ are odd and even numbers, respectively. With all,

$$h < \left(\frac{x_c^{m_2}}{-y_c^{m_1}}\right)^{\frac{1}{2}} \leq \left(\frac{x^{m_2}}{-y^{m_1}}\right)^{\frac{1}{2}}.$$

Then, $y^{m_1} h^2 + x^{m_2} > 0$. In conclusion,

$$\int_{AC} (y^{m_1} v_x^2 + x^{m_2} v_y^2)(a,b)\eta_{AC} \geq 0,$$

and

$$\int_\Omega v\,\mathscr{O}u \geq \int_\Omega \alpha v_x^2 + 2\beta v_x v_y + \gamma v_y^2.$$

where $\alpha = -\varepsilon y^{m_1} - h(1+\varepsilon x)m_1 y^{m_1-1}$, $\beta = -h\varepsilon y^{m_1}$ and $\gamma = (1+\varepsilon x)m_2 x^{m_2-1} + \varepsilon x^{m_2}$. By the hypotheses *ii)*. and *iii)* and the choice of $0 < \varepsilon < \frac{1}{x_0}$ we can proof with the standard calculations that $\alpha\gamma - \beta^2 > 0$.

We can write

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

and see that

$$\begin{pmatrix} v_x & v_y \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \alpha v_x^2 + 2\beta v_x v_y + \gamma v_y^2.$$

On the matrix space $\mathbb{M}_{2x1}$, we can consider the norm $||X||_1 = XAX^T$ para A defined positively, because $\alpha\gamma - \beta^2 > 0$, and we have the stard norm on $\mathbb{M}_{2x1}$ given by $||X||_2 = XX^T$. As $\mathbb{M}_{2x1}$ is a finite dimensional space we know that exist $\delta > 0$ such that $\delta ||X||_2 \leq ||X||_1$ for every matrix $X \in \mathbb{M}_{2x1}$. Consider $X = \begin{pmatrix} v_x & v_y \end{pmatrix}$ we have,

$$\int_\Omega v \mathscr{O} u \geq \int_\Omega \alpha v_x^2 + 2\beta v_x v_y + \gamma v_y^2. \geq \delta \int_\Omega v_x^2 + v_y^2.$$

This is,

$$\int_\Omega v \mathscr{O} u \geq \delta ||\nabla v||^2_{L^2(\Omega)}.$$

Step 6. Note that,

$$(Iu, \mathscr{O}u)_{L^2(\Omega)} = \int_\Omega v \mathscr{O} u.$$

So, $(Iu, \mathscr{O}u)_{L^2(\Omega)} \geq \delta ||\nabla v||^2_{L^2(\Omega)}$. By the generalize Cauchy inequality we get,

$$\delta ||\nabla v||^2_{L^2(\Omega)} \leq (Iu, \mathscr{O}u)_{L^2(\Omega)} \leq ||Iu||_{W^1_{BC\cup\sigma}} ||\mathscr{O}u||_{W^{-1}_{BC\cup\sigma}}$$

On the other hand, by definition for any $v \in W^1_{BC\cup\sigma}$ we have

$$||v||^2_{W^1_{BC\cup\sigma}} = ||\nabla_{m_1,m_2} v||^2_{L^2(\Omega)} + ||v||^2_{L^2(\Omega)}.$$

By the Poincare inequality, exist $k > 0$ such that

$$||v||_{L^2\Omega)} \leq k ||\nabla v||_{L^2(\Omega)}.$$

See that, $||\nabla_{m_1,m_2} v||_{L^2(\Omega)} \leq \bar{k} ||\nabla v||_{L^2(\Omega)}$. In fact, $\nabla_{m_1,m_2} v = (|y|^{\frac{m_1}{2}} v_x, |x|^{\frac{m_2}{2}} v_y)$ as Ω is a bounded set, there is a constant $K \in \mathbb{R}$ such that $|y|^{\frac{m_1}{2}} < K^{m_1}$ and $|x|^{\frac{m_2}{2}} < K^{m_2}$. Therefore,

$$||v||^2_{W^1_{BC\cup\sigma}} \leq \bar{k}^2 ||\nabla v||^2_{L^2(\Omega)} + k^2 ||\nabla v||^2_{L^2(\Omega)} \leq k'^2 ||\nabla v||^2_{L^2(\Omega)}.$$

Since, $Iu = v$ in Ω we get,

$$C||Iu||^2_{W^1_{BC\cup\sigma}} \leq ||Iu||_{W^1_{BC\cup\sigma}} ||\mathscr{O}u||_{W^{-1}_{BC\cup\sigma}},$$

so,

$$C||Iu||_{W^1_{BC\cup\sigma}} \leq ||\mathscr{O}u||_{W^{-1}_{BC\cup\sigma}}.$$

Remembering that D is a first order differential operator with smooth coefficients in Ω we know that there is a constant $0 < C_2 \in \mathbb{R}$ such that

$$||Dv||_{L^2(\Omega)} \leq C_2||v||_{W^1_{BC\cup\sigma}}$$

then,

$$||u||_{L^2(\Omega)} = ||Dv||_{L^2(\Omega)} \leq C_2||v||_{W^1_{BC\cup\sigma}} \leq \frac{C_2}{C}||Iu||_{W^1_{BC\cup\sigma}} \leq \frac{C_2}{C}||\mathscr{O}u||_{W^{-1}_{BC\cup\sigma}}.$$

This proof the estimate (1.4) ($\lambda = 0$). To proof the estimate (1.5) ($\lambda = 0$) is the similar to the previous proof considering $a = (1-\varepsilon x)$, $b = -ah$ and $c = 0$ using the inequality $-h < g'(x)$ of the hypothesis *iii)*.

Now, we will demonstrate for the general case $\lambda \leq \frac{1}{K_3}$. Since Ω is admissible for the operator $\mathscr{O}$ we win the propriety that Ω is admissible for the operator $\mathscr{O} - \lambda$ for a suitable λ. Indeed,

$$||u||_{L^2(\Omega)} \leq C_3||\mathscr{O}_{AC}u - \lambda u + \lambda u||_{W^{-1}_{BC\cup\sigma}}.$$

Then,

$$||u||_{L^2(\Omega)} \leq C_3||\mathscr{O}_{AC}u - \lambda u||_{W^{-1}_{BC\cup\sigma}} + ||\lambda u||_{W^{-1}_{BC\cup\sigma}}.$$

There is a constant $K_3 > 0$ such that $||\lambda u||_{W^{-1}_{BC\cup\sigma}} \leq K_3||\lambda u||_{L^2(\Omega)}$. So,

$$||u||_{L^2(\Omega)} - K_3||\lambda u||_{L^2(\Omega)} \leq C_3||\mathscr{O}_{AC}u - \lambda u||_{W^{-1}_{BC\cup\sigma}}.$$

Therefore, for $\lambda \leq \frac{1}{K_3}$ we consider $\overline{C}_3 = \frac{C_3}{1-\lambda K_3}$ and we get

$$||u||_{L^2(\Omega)} \leq \overline{C}_3||\mathscr{O}_{AC}u - \lambda u||_{W^{-1}_{BC\cup\sigma}}.$$

In conclusion, we proof the estimate (1.4), the proof of estimate (1.5) is similar to the previous one. □

Remark 3.2. The proof of Theorem 3.1 establishes the existence of a nonempty set of admissible Tricomi domains for the Operator $\mathscr{O}$. Consequently, due to the admissibility relation for an appropriate λ, the set of admissible Tricomi domains for the Operator $\mathscr{O} - \lambda$ is also nonempty.

3.2 Existence Results

The existence results presented in this Section depend on the Riesz representation theorem for space with negative norm [13] and the others traditional result of the functional analyse whole proofs can be found in [2].

To prove our main theorem we need to guarantee the existence of a solution for Problem (4) with $\lambda = 0$ and $\gamma = 0$; what we will do next.

Theorem 3.3. *Let Ω be an admissible Tricomi domain for the operator $\mathscr{O}$. For every $f \in L^2(\Omega)$ there exists a unique weak solution $u \in W^1_{AC\cup\sigma}$ of the problem* (8). *Moreover, the solution operator*

$$S_{AC\cup\sigma} : L^2(\Omega) \to W^1_{AC\cup\sigma},$$

which assigns to $f \in L^2(\Omega)$ the unique weak solution $u \in W^1_{AC\cup\sigma}$ of the problem (8) *is linear and continuous.*

Proof. **Uniqueness:** Let $u_1 \in W^1_{AC\cup\sigma}$ and $u_2 \in W^1_{AC\cup\sigma}$ two weak solutions of the problem (8). Suppose that $u_1 \neq u_2$. By the definition, $< \mathscr{O}_{AC}u_1, v >_{AC} = (f,v)_{L^2}$ and $< \mathscr{O}_{AC}u_2, v >_{AC} = (f,v)_{L^2}$. So, $< \mathscr{O}_{AC}u_1 - \mathscr{O}_{AC}u_2, v >_{AC} = 0$. Thus, $\mathscr{O}_{AC}(u_1 - u_2) = 0$. Using (1.4), with $\lambda = 0$, gives $||u_1 - u_2|| = 0$ and this is a contradiction with the suppose.

Existence: Define the linear functional

$$\mathscr{L}_f : \mathscr{O}(C^\infty_{BC\cup\sigma}(\overline{\Omega})) \to \mathbb{R},$$

as being $\mathscr{L}_f(\mathscr{O}v) = (v,f)_{L^2}$. Extending by continuity to $\mathscr{O}(W^1_{BC\cup\sigma})$ gives

$$\mathscr{L}'_f : \mathscr{O}(W^1_{BC\cup\sigma}) \to \mathbb{R},$$

as being $\mathscr{L}'_f(\mathscr{O}v) = (v,f)_{L^2}$.

We know that, $\mathscr{O}(W^1_{BC\cup\sigma}) \subset W^{-1}_{AC\cup\sigma}$ is a vector subspace. By the Hahn-Banach Theorem exist $\overline{\mathscr{L}}_f$ a extension of $\mathscr{L}'_f$. This is, $\overline{\mathscr{L}}_f|_{\mathscr{O}(W^1_{BC\cup\sigma})} = \mathscr{L}'_f$ and particularity $\overline{\mathscr{L}}_f|_{\mathscr{O}(C^\infty_{BC\cup\sigma}(\overline{\Omega}))} = \mathscr{L}_f$. Applying the Riesz Representation Theorem for space with negative norm [13]. Exist $u \in W^1_{AC\cup\sigma}$ such that

$$\overline{\mathscr{L}}_f(b) = < u, b >, \quad \forall\, b \in W^{-1}_{AC\cup\sigma}$$

particularity, for all $b \in \mathscr{O}(W^1_{BC\cup\sigma}) \subset W^{-1}_{AC\cup\sigma}$. Hence for $b = \mathscr{O}v$ gives

$$\mathscr{L}'_f(\mathscr{O}v) = (v,f)_{L^2} = < u, \mathscr{O}v >_{AC}.$$

This is equivalent to the condition *ii)* of the definition 4.3, with $\lambda = 0$, as mentioned early. □

For the purpose of studying spectral theory in the future, we will proof the main theorem.

Theorem 3.4. *Let Ω be an admissible Tricomi domain for the operator $\mathscr{O} - \lambda$, $f \in L^2(\Omega)$, $\gamma \in \mathbb{R}$ and $\lambda \leq 0$. The problem* (4) *admits a unique weak solution in $W^1_{AC\cup\sigma}$ and the solution operator*

$$S^{\lambda,\gamma}_{AC\cup\sigma} : L^2(\Omega) \to W^1_{AC\cup\sigma}$$

which assigns to $f \in L^2(\Omega)$ the unique weak solution $u \in W^1_{AC\cup\sigma}$ of the problem (4) *is linear and continuous.*

Proof. The proof will be done in 3 cases analyzing the similarity with the Theorem 3.3.

Case 1. Consider $\gamma = 0$ and $\lambda = 0$. Here, we have the Theorem 3.3.

Case 2. Consider $\gamma = 0$ and $\lambda < 0$. **Existence:** The proof is analogous to the proof of Theorem (3.3) taking $\mathscr{O} - \lambda I$ in place of $\mathscr{O}$.

Uniqueness: Note that, Ω is an admissible Tricomi domain for the operator $\mathscr{O}$ and we don't know that Ω is an admissible Tricomi domain for the operator $\mathscr{O} - \lambda$. But, this hypotheses is enough. Let $u_1 \in W^1_{AC\cup\sigma}$ and $u_2 \in W^1_{AC\cup\sigma}$ two weak solutions of the problem (4). Suppose that $u_1 \neq u_2$. By definition $(\mathscr{O}_{AC}u_1 - \lambda u_1, v)_{L^2} = (f, v)_{L^2}$ and $(\mathscr{O}_{AC}u_2 - \lambda u_2, v)_{L^2} = (f, v)_{L^2}$ for all $v \in C^\infty_{BC\cup\sigma}(\overline{\Omega})$. So, $(\mathscr{O}_{AC}u_1 - \mathscr{O}_{AC}u_2 - \lambda u_1 + \lambda u_2, v)_{L^2} = 0$ for all $v \in C^\infty_{BC\cup\sigma}(\overline{\Omega})$. Thus, $\mathscr{O}_{AC}(u_1 - u_2) - \lambda(u_1 - u_2) = 0$.

Adding $\lambda||u_1 - u_2||_{W^{-1}_{AC\cup\sigma}}$ to (1.4), we have

$$||u_1 - u_2||_{L^2(\Omega)} + \lambda||u_1 - u_2||_{W^{-1}_{AC\cup\sigma}} \leq C_3||\mathscr{O}_{AC}(u_1 - u_2)||_{W^{-1}_{BC\cup\sigma}} + \lambda||u_1 - u_2||_{W^{-1}_{AC\cup\sigma}}.$$

Moreover, there is a constant $K_1 > 0$ such that $\lambda K_1||u_1 - u_2||_{L^2(\Omega)} \leq \lambda||u_1 - u_2||_{W^{-1}_{AC\cup\sigma}}$, because $L^2 \subset W^{-1}_{AC\cup\sigma}$ and $\lambda < 0$. Then, we obtain

$$||u_1 - u_2||_{L^2(\Omega)} + \lambda K_1||u_1 - u_2||_{L^2} \leq C_3||\mathscr{O}_{AC}(u_1 - u_2)||_{W^{-1}_{BC\cup\sigma}} + \lambda||u_1 - u_2||_{W^{-1}_{AC\cup\sigma}}.$$

Therefore,

$$\begin{aligned}
(1 + \lambda K_1)||u_1 - u_2||_{L^2} &\leq C_3||\mathscr{O}_{AC}(u_1 - u_2)||_{W^{-1}_{BC\cup\sigma}} - (-\lambda)||u_1 - u_2||_{W^{-1}_{AC\cup\sigma}} \\
&= C_3||\mathscr{O}_{AC}(u_1 - u_2)||_{W^{-1}_{BC\cup\sigma}} - |\lambda|\,||u_1 - u_2||_{W^{-1}_{AC\cup\sigma}} \\
&= C_3||\mathscr{O}_{AC}(u_1 - u_2)||_{W^{-1}_{BC\cup\sigma}} - ||\lambda(u_1 - u_2)||_{W^{-1}_{AC\cup\sigma}} \\
&\leq ||\mathscr{O}_{AC}(u_1 - u_2) - \lambda(u_1 - u_2)||_{W^{-1}_{AC\cup\sigma}} = 0.
\end{aligned}$$

Case 3. Consider $\gamma \neq 0$ and $\lambda \leq 0$. Here, take $u = w + \gamma$ where $w \in W^1_{AC\cup\sigma}$ is a weak solution of the next problem

$$\begin{cases} \mathscr{O}(w) - \lambda w = f(x,y) + \lambda\gamma & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad AC \cup \sigma \subseteq \partial\Omega, \end{cases}$$

In fact, $g = f + \lambda\gamma \in L^2$ and $\lambda \leq 0$. By the previous case, there is a unique weak solution $w \in W^1_{AC\cup\sigma}$ for this problem. Replacing, $w = u - \lambda$ in the anterior problem we get the result. □

CHAPTER 4

Weighted Sobolev embedding and nontrivial weak solutions for a class of nonlinear mixed-type equations.

4.1 Weighted Sobolev Embedding

This section aims to present and prove the results, which we consider to be the most significant in this work, owing to the fundamental role of Sobolev embeddings in the research field of partial differential equations. The proof of these results is a generalization of the ideas found in [32]. However, we can find other ideas in [10],[17],[2], [16], [23], [22] and [1].

In the subsequent theorem, we shall consider the general weighted Sobolev space $W^{1,p}_{k,l}$ with $\Omega \subset \mathbb{R}^2$, $k > 0$, $l > 0$ and $1 \leq p < \infty$, as being

$$W^{1,p}_{k,l} = \left\{ u \in L^p(\Omega); |y|^k \frac{\partial u}{\partial x} \in L^p(\Omega), |x|^l \frac{\partial u}{\partial y} \in L^p(\Omega) \right\},$$

with the norm

$$||u||^p_{W^{1,p}_{k,l}} = \int_\Omega |\nabla_{k,l} u|^p + \int_\Omega |u|^p = ||\nabla_{k,l} u||^p_{L^p(\Omega)} + ||u||^p_{L^p(\Omega)}.$$

Consider, the critical exponent $2^*(k,l,p)$ associated with said space as being

$$2^*(k,l,p) = \frac{(k+l+2)p}{k+l+2+klp-p}.$$

As defined in [31], the space $S^p_\alpha(\Omega)$ for $\alpha \geq 0$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ and $X_1, X_2, \ldots, X_n$ are monomials of vector fields, is given by

$$S^p_\alpha(\Omega) = \left\{ u \in L^p(\Omega); (X_{i_0} \cdots X_{i_l}) u \in L^p(\Omega), \, \forall \, 0 \leq l \leq \alpha \right\}, \tag{4.1}$$

with the norm

$$||u||_{S^p_\alpha} = \sum_{0\le l\le \alpha} ||(X_{i_0}\cdots X_{i_l})u||_{L^p(\Omega)}.$$

First, we observe that by taking $n=2$, $\alpha=1$ and the vector fields $X_{i_0}=|y|^k\frac{\partial}{\partial x}$ and $X_{i_1}=|x|^l\frac{\partial}{\partial y}$ em (4.1), we have $S^p_1(\Omega)=W^{1,p}_{k,l}$. Now, by Theorem 13 in [31], we have that the weighted Sobolev space $W^{1,p}_{k,l}$ is continuously embedded in $W^{\frac{1}{2},p}(\Omega)$. Furthermore, by the theory developed in [6] on fractional Sobolev spaces, more specific by the Theorem 7.1, we have that $W^{\frac{1}{2},p}$ is compactly embedded in $L^r(\Omega)$ for all $r\in[1,p]$. Thus, we can conclude that the weighted Sobolev space $W^{1,p}_{k,l}$ is compactly embedded in $L^r(\Omega)$ for all $r\in[1,p]$.

Theorem 4.1. *Let Ω be a bounded set of $\mathbb{R}^2$ and $1\le p<\frac{k+l+2}{1-kl}$ with $kl<1$. Then, there is a constant C such that*

$$||u||_{L^r(\Omega)} \le C||u||_{W^{1,p}_{k,l}} \;\forall\, r\in[1,2^*(k,l,p)),$$

this is, the immersion

$$W^{1,p}_{k,l} \hookrightarrow L^r(\Omega) \tag{4.2}$$

is continuous for $r\in[1,2^*(k,l,p))$.

Proof. To $p=1$. Take M sufficiently large such that $\Omega\subset[-M,M]\times[-M,M]$. We have

$$u(x,y)=\int_{-M}^{x}\frac{\partial u}{\partial x}(t,y)dt \;\;\forall\,(x,y)\in\Omega,$$

and

$$u(x,y)=\int_{-M}^{y}\frac{\partial u}{\partial y}(x,t)dt \;\;\forall\,(x,y)\in\Omega.$$

Therefore, for $\beta>0$ and $\delta>0$ we get

$$|u(x,y)|^\beta \le \left(\int_{-M}^{M}|\frac{\partial u}{\partial x}(t,y)|dt\right)^\beta \;\;\forall\,(x,y)\in\Omega, \tag{4.3}$$

and

$$|u(x,y)|^\delta \le \left(\int_{-M}^{M}|\frac{\partial u}{\partial y}(x,t)|dt\right)^\delta \;\;\forall\,(x,y)\in\Omega. \tag{4.4}$$

Multiplying (4.3) and (4.4) and integrating over $[-M,M]\times[-M,M]$ we obtain

$$\begin{aligned}\int_\Omega |u|^{\beta+\delta}dxdy &\le \int_{-M}^{M}\int_{-M}^{M}\left[\left(\int_{-M}^{M}|\frac{\partial u}{\partial x}(t,y)|dt\right)^\beta\left(\int_{-M}^{M}|\frac{\partial u}{\partial y}(x,t)|dt\right)^\delta\right]dxdy\\ &= \int_{-M}^{M}\left(\int_{-M}^{M}|\frac{\partial u}{\partial x}(t,y)|dt\right)^\beta\left[\int_{-M}^{M}\left(\int_{-M}^{M}|\frac{\partial u}{\partial y}(x,t)|dt\right)^\delta dx\right]dy\\ &= \int_{-M}^{M}\left(\int_{-M}^{M}|\frac{\partial u}{\partial x}(x,y)|dx\right)^\beta dy\int_{-M}^{M}\left(\int_{-M}^{M}|\frac{\partial u}{\partial y}(x,y)|dy\right)^\delta dx. \end{aligned}\tag{4.5}$$

Adding $|y|^k|y|^{-k}$ and $|x|^l|x|^{-l}$ in (4.5) and using the Hölder's inequality on each factor, we conclude

$$\int_\Omega |u|^{\beta+\delta}dxdy \leq \Big(\int_{-M}^{M} |y|^{\frac{-k\beta}{1-\beta}}dy \Big)^{1-\beta} \left[\int_{-M}^{M} |y|^k \Big(\int_{-M}^{M} |\frac{\partial u}{\partial x}(x,y)|dx \Big) dy \right]^{\beta} \cdot$$
$$\Big(\int_{-M}^{M} |x|^{\frac{-l\delta}{1-\delta}}dy \Big)^{1-\delta} \left[\int_{-M}^{M} |x|^l \Big(\int_{-M}^{M} |\frac{\partial u}{\partial y}(x,y)|dy \Big) dx \right]^{\delta}. \qquad (4.6)$$

Taking $0<\beta<\frac{1}{k+1}$ and $0<\delta<\frac{1}{l+1}\cdot$ We have that

$$\Big(\int_{-M}^{M} |y|^{\frac{-k\beta}{1-\beta}}dy \Big)^{1-\beta} \leq C_1$$

and

$$\Big(\int_{-M}^{M} |x|^{\frac{-l\delta}{1-\delta}}dy \Big)^{1-\delta} \leq C_2.$$

So, the inequality (4.6) becomes

$$\int_\Omega |u|^{\beta+\delta}dxdy \leq C_3 \left[\int_{-M}^{M} |y|^k \Big(\int_{-M}^{M} |\frac{\partial u}{\partial x}(x,y)|dx \Big) dy \right]^{\beta} \cdot \left[\int_{-M}^{M} |x|^l \Big(\int_{-M}^{M} |\frac{\partial u}{\partial y}(x,y)|dy \Big) dx \right]^{\delta}.$$
$$= C_3 \left|\left| |y|^k \frac{\partial u}{\partial x}(x,y) \right|\right|_{L^1(\Omega)}^{\beta} \left|\left| |x|^l \frac{\partial u}{\partial y}(x,y) \right|\right|_{L^1(\Omega)}^{\delta}.$$

Therefore,

$$||u||_{L^{\beta+\delta}(\Omega)} \leq C_4 \left|\left| |y|^k \frac{\partial u}{\partial x}(x,y) \right|\right|_{L^1(\Omega)}^{\frac{\beta}{\beta+\delta}} \left|\left| |x|^l \frac{\partial u}{\partial y}(x,y) \right|\right|_{L^1(\Omega)}^{\frac{\delta}{\beta+\delta}},$$

by the Young's inequality, we get

$$||u||_{L^{\beta+\delta}(\Omega)} \leq C_5 \left(\left|\left| |y|^k \frac{\partial u}{\partial x}(x,y) \right|\right|_{L^1(\Omega)} + \left|\left| |x|^l \frac{\partial u}{\partial y}(x,y) \right|\right|_{L^1(\Omega)} \right). \qquad (4.7)$$

For $p>1$ arbitrary, we apply $|u|^\gamma$ with $\gamma>1$ in (4.7). Hence, we obtain

$$\left|\left| |u|^\gamma \right|\right|_{L^{\beta+\delta}(\Omega)} \leq C_5 \left(\left|\left| |y|^k |u|^{\gamma-1} \frac{\partial u}{\partial x}(x,y) \right|\right|_{L^1(\Omega)} + \left|\left| |x|^l |u|^{\gamma-1} \frac{\partial u}{\partial y}(x,y) \right|\right|_{L^1(\Omega)} \right),$$

by the Hölder's inequality, we conclued

$$\left|\left| |u|^\gamma \right|\right|_{L^{\beta+\delta}(\Omega)} \leq C_5 \left(\left|\left| |u|^{\gamma-1} \right|\right|_{L^{p'}(\Omega)} \left|\left| |y|^k \frac{\partial u}{\partial x}(x,y) \right|\right|_{L^p(\Omega)} + \left|\left| |u|^{\gamma-1} \right|\right|_{L^{p'}(\Omega)} \left|\left| |x|^l \frac{\partial u}{\partial y}(x,y) \right|\right|_{L^p(\Omega)} \right),$$
$$= C_5 \left|\left| |u|^{\gamma-1} \right|\right|_{L^{p'}(\Omega)} \left(\left|\left| |y|^k \frac{\partial u}{\partial x}(x,y) \right|\right|_{L^p(\Omega)} + \left|\left| |x|^l \frac{\partial u}{\partial y}(x,y) \right|\right|_{L^p(\Omega)} \right).$$

Taking $\gamma = \frac{p}{\beta+\delta-\beta p-\delta p+p}$, we determine

$$\left|\left||u|^{\gamma}\right|\right|_{L^{\beta+\delta}(\Omega)} = \left[||u||_{L^{(\beta+\delta)\gamma}(\Omega)}\right]^{\gamma}$$

and

$$\left|\left||u|^{\gamma-1}\right|\right|_{L^{p'}(\Omega)} = \left[||u||_{L^{(\beta+\delta)\gamma}(\Omega)}\right]^{\gamma-1}.$$

Thus,

$$||u||_{L^{\frac{(\beta+\delta)p}{\beta+\delta-\beta p-\delta p+p}}(\Omega)} \leq C_5\left(\left|\left||y|^k\frac{\partial u}{\partial x}(x,y)\right|\right|_{L^p(\Omega)} + \left|\left||x|^l\frac{\partial u}{\partial y}(x,y)\right|\right|_{L^p(\Omega)}\right).$$

To finalized, we consider $\beta \to \frac{1}{k+1}$ and $\delta \to \frac{1}{l+1}$. Then, for all $\tau \in \mathbb{R}$ positive small, we have

$$||u||_{L^{\frac{(k+l+2)p}{k+l+2+klp-p}-\tau}(\Omega)} \leq C_5\left(\left|\left||y|^k\frac{\partial u}{\partial x}(x,y)\right|\right|_{L^p(\Omega)} + \left|\left||x|^l\frac{\partial u}{\partial y}(x,y)\right|\right|_{L^p(\Omega)}\right).$$

For $r \in [1, 2^*(k,l,p))$, we can take $\tau \in \mathbb{R}$ small positive value such that $1 \leq r < \frac{(k+l+2)p}{k+l+2+klp-p} - \tau$. Thus, by the interpolation inequality, we obtain that the weighted Sobolev embedding $W^{1,p}_{k,l} \hookrightarrow L^r(\Omega)$ is continuous for $r \in [1, 2^*(k,l,p))$.

□

To conclude this section, we shall prove the compactness of the weighted Sobolev embedding $W^{1,p}_{k,l} \hookrightarrow L^r(\Omega)$ for $r \in [1, 2^*(k,l,p))$, as following in the next theorem.

Theorem 4.2. *Let Ω be a bounded set of $\mathbb{R}^2$ and $1 \leq p < \frac{k+l+2}{1-kl}$ with $kl < 1$. Then, the immersion*

$$W^{1,p}_{k,l} \hookrightarrow L^r(\Omega) \tag{4.8}$$

is compact for $r \in [1, 2^*(k,l,p))$.

Proof. Define $g(\tau) = \frac{(k+l+2)p}{k+l+2+klp-p} - \tau$ for all $\tau > 0$ small. Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence of $W^{1,p}_{k,l}$.

Note that,

$$g(\tau) = \frac{(k+l+2)p}{k+l+2+klp-p} - \frac{\tau}{2} - \frac{\tau}{2} = g(\frac{\tau}{2}) - \frac{\tau}{2} < g(\frac{\tau}{2}).$$

Furthermore, $p < g(\tau)$. Then, by the interpolation inequality (1.6), we obtain

$$||u_n||_{L^{g(\tau)}(\Omega)} \leq ||u_n||^{\alpha}_{L^p(\Omega)}||u_n||^{1-\alpha}_{L^{g(\frac{\tau}{2})}(\Omega)}. \tag{4.9}$$

By the weighted Sobolev embedding (4.2), the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{g(\frac{\tau}{2})}(\Omega)$ because $g(\frac{\tau}{2}) < 2^*(k,l,p)$. Thus (4.9) becomes

$$||u_n||_{L^{g(\tau)}(\Omega)} \leq C||u_n||^{1-\alpha}_{L^p(\Omega)}.$$

Moreover, $W^{1,p}_{k,l} \hookrightarrow L^p(\Omega)$ is a compact immersion. Consequently, $\{u_n\}_{n\in\mathbb{N}}$ is a convergent sequence in $L^p(\Omega)$. Thus, by the preceding inequality, we conclued that $\{u_n\}_{n\in\mathbb{N}}$ is a convergent sequence of $L^{g(\tau)}(\Omega)$. Therefore, the weighted Sobolev immersion $W^{1,p}_{k,l} \hookrightarrow L^{g(\tau)}(\Omega)$ is compact. By the inequality interpolation, we obtain that the weighted Sobolev immersion $W^{1,p}_{k,l} \hookrightarrow L^r(\Omega)$ is compact for $r \in [1, 2^*(k,l,p))$. ☐

4.2 Existence of weak solutions

Building upon the previous sections and incorporating a standard perspective on the geometry of the functional associated with the generalization of the Gellerstedt operator, we establish the existence of nontrivial weak solutions. In this section, we take into account the case $p = 2$, $k = \frac{m_1}{2}$ and $l = \frac{m_2}{2}$ in the Theorems 4.1 and 4.2.

The semilinear mixed-type Tricomi problem with Neumann boundary conditions

$$\begin{cases} \mathscr{O}(u) := -|y|^{m_1}u_{xx} - |x|^{m_2}u_{yy} = u|u|^{\alpha-2} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma \subseteq \partial\Omega, \\ \nabla_{m_1,m_2}u \cdot \eta = 0 & \text{on} \quad \partial\Omega \setminus \Gamma. \end{cases} \tag{4.10}$$

where Ω is a open, bounded subset of $\mathbb{R}^2$ with boundary $\partial\Omega$, $2 < \alpha < 2^*(m_1, m_2)$ and η the exterior normal vector on $\partial\Omega \setminus \Gamma$.

For any vector field $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$, we define the weighted divergence operator

$$div_{m_1,m_2}(\vec{F}) = div_{m_1,m_2}(F_1, F_2) := |y|^{\frac{m_1}{2}}(F_1)_x + |x|^{\frac{m_2}{2}}(F_2)_y.$$

Note that, when $m_1 = m_2 = 0$ we have the standard divergence definition. Now, consider $F_1 = |y|^{\frac{m_1}{2}}u_x$ and $F_2 = |x|^{\frac{m_2}{2}}u_y$, by the definition weighted divergence, we obtain

$$div_{m_1,m_2}(|y|^{\frac{m_1}{2}}u_x, |x|^{\frac{m_2}{2}}u_y) = |y|^{m_1}u_{xx} + |x|^{m_2}u_{yy} = \mathscr{O}(u).$$

This is,

$$div_{m_1,m_2}(\nabla_{m_1,m_2}u) = \mathscr{O}(u)$$

and remembering, if $m_1 = m_2 = 0$ we obtain the standard propriety of divergence for the Laplacian operator $div(\nabla u) = \Delta u$. It is important to observe that $div_{m_1,m_2}(v\nabla_{m_1,m_2}u) = div_{m_1,m_2}(v|y|^{\frac{m_1}{2}}u_x, v|x|^{\frac{m_2}{2}}u_y)$ for all $v \in C^\infty(\Omega)$. So,

$$div_{m_1,m_2}(v\nabla_{m_1,m_2}u) = |y|^{\frac{m_1}{2}}(v_x|y|^{\frac{m_1}{2}}u_x + v|y|^{\frac{m_1}{2}}u_{xx}) + |x|^{\frac{m_2}{2}}(v_y|x|^{\frac{m_2}{2}}u_y + v|x|^{\frac{m_2}{2}}u_{yy}),$$

then,

$$div_{m_1,m_2}(v\nabla_{m_1,m_2}u) = (|y|^{\frac{m_1}{2}}v_x)(|y|^{\frac{m_1}{2}}u_x) + (|x|^{\frac{m_2}{2}}v_y)(|x|^{\frac{m_2}{2}}u_y) + v|y|^{m_1}u_{xx} + v|x|^{m_2}u_{yy}.$$

To conclude, we have

$$\int_\Omega div_{m_1,m_2}(v\nabla_{m_1,m_2}u) = \int_\Omega \nabla_{m_1,m_2}v\nabla_{m_1,m_2}u + \int_\Omega v\, div_{m_1,m_2}(\nabla_{m_1,m_2}u)$$

and applying the divergence theorem, and considering that the weighted divergence div_{m_1,m_2} is equivalent to the classical divergence div in the following sense $div(|y|^{m_1}F_1,|x|^{m_2}F_2) = div_{m_1,m_2}(F_1,F_2)$, we have

$$\int_{\partial\Omega} v\nabla_{m_1,m_2}u\cdot\eta = \int_\Omega \nabla_{m_1,m_2}v\nabla_{m_1,m_2}u - \int_\Omega v\,\mathscr{O}u.$$

We know that, $\int_\Gamma v\nabla_{m_1,m_2}u\cdot\eta = 0$ because, $u_{|\Gamma} = 0$ hence $\frac{\partial u}{\partial T_\Gamma} = 0$ as done in Step 1 of Theorem 2.3. Then,

$$\int_{\Omega\setminus\Gamma} v\nabla_{m_1,m_2}u\cdot\eta = \int_\Omega \nabla_{m_1,m_2}v\nabla_{m_1,m_2}u - \int_\Omega v\,\mathscr{O}u.$$

Moreover,

$$\int_{\Omega\setminus\Gamma} v\nabla_{m_1,m_2}u\cdot\eta = 0$$

by the Newman condition. Then

$$\int_\Omega v\,\mathscr{O}u = \int_\Omega \nabla_{m_1,m_2}v\nabla_{m_1,m_2}u.$$

Then, from (4.10), we get

$$\begin{aligned}\int_\Omega vu|u|^{\alpha-2} &= \int_\Omega (|y|^{\frac{m_1}{2}}v_x, |x|^{\frac{m_2}{2}}v_y)(|y|^{\frac{m_1}{2}}u_x, |x|^{\frac{m_2}{2}}u_y)\\ &= \int_\Omega |y|^{m_1}u_xv_x + |x|^{m_2}u_yv_y.\end{aligned}$$

This leads to the formulation of the following definition.

Definition 4.3. Let $u \in W^{1,2}_{m_1,m_2}$, we say u is a weak solution of (4.10) if the following relation is hold

$$\mathscr{B}(x,y) := \int_\Omega |y|^{m_1}u_xv_x + |x|^{m_2}u_yv_y\, dxdy = \int_\Omega vu|u|^{\alpha-2}\, dxdy,$$

for all $v \in C^\infty(\Omega)$.

Define now, the functional $\varphi : W^{1,2}_{m_1,m_2} \to \mathbb{R}$ associate to the problem (4.10) as being

$$\varphi(u) = \frac{1}{2}\int_\Omega |y|^{m_1}u_x^2 + |x|^{m_2}u_y^2 - \frac{1}{\alpha}\int_\Omega |u|^\alpha.$$

Whose derivative is given by

$$\varphi'(u)\cdot v = \int_\Omega |y|^{m_1}u_xv_x + |x|^{m_2}u_yv_y - \int_\Omega vu|u|^{\alpha-2}.$$

See that,

$$\varphi'(u)\cdot u = \int_\Omega |y|^{m_1}u_x^2 + |x|^{m_2}u_y^2 - \int_\Omega |u|^{\alpha}.$$

This is, the critical points of the φ is the weak solution of the problem (4.10).

Lemma 4.4. *a) $u = 0$ is a strict local minimum point of φ;*

b) give $v \neq 0$ in $W^{1,2}_{m_1,m_2}$ there is a ρ_0 such that $\varphi(\rho_0 v) < 0$.

Proof. a) By the Sobolev Immersion (4.2), we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}||u||^2_{W^{1,2}_{m_1,m_2}} - \frac{1}{\alpha}||u||^{\alpha}_{L^{\alpha}(\Omega)}\\ &\geq \frac{1}{2}||u||^2_{W^{1,2}_{m_1,m_2}} - C||u||^{\alpha}_{W^{1,2}_{m_1,m_2}}.\end{aligned}$$

Therefore, we can observe that $P(x) = \frac{1}{2}x^2 - Cx^{\alpha}$ have roots when $x^2(\frac{1}{2} - Cx^{\alpha-2}) = 0$. This is, $x = 0$ or $x = (\frac{1}{2C})^{\frac{1}{\alpha-2}}$. In conclusion, $\varphi(u) = 0$ when $u = 0$ and consider $r > 0$ small sufficiently. We get that $\varphi(u) > 0 = \varphi(0)$ for all $u \in W^{1,2}_{m_1,m_2}$ with $0 < ||u||_{W^{1,2}_{m_1,m_2}} \leq r < (\frac{1}{2C})^{\frac{1}{\alpha-2}}$. So $u = 0$ is a strict local minimum point.

b) Consider $\delta = \int_\Omega v^{\alpha}$ for some $v \in W^{1,2}_{m_1,m_2}$ given with $||v||_{W^{1,2}_{m_1,m_2}} = 1$, we obtain $\varphi(\rho v) = \frac{1}{2}\rho^2 - \frac{1}{\alpha}\delta\rho^{\alpha}$ doing $\rho \to \infty$ we have

$$\varphi(\rho_0 v) \to -\infty.$$

So, take ρ_0 sufficiently large we get $\varphi(\rho_0 v) < 0$. □

Theorem 4.5. *The problem* (4.10) *has a weak nontrivial solution.*

Proof. Let $\{u_n\}_n \subset W^{1,2}_{m_1,m_2}$ with $n \in \mathbb{N}$ a sequence such that $|\varphi(u_n)| \leq c$ and $\varphi'(u_n) \to 0$. Then, for all n sufficiently large, we have

$$\left|\int_\Omega \left(|\nabla_{m_1,m_2}u_n|^2 - u_n^{\alpha}\right)\right| = |\varphi'(u_n)\cdot u_n| \leq ||u_n||_{W^{1,2}_{m_1,m_2}}$$

because, $||\varphi'(u_n)|| \leq 1$. How $|\varphi(u_n)| \leq c$ we get

$$\begin{aligned}\varphi(u_n) - \frac{1}{\alpha}\varphi'(u_n)\cdot u_n &\leq |\varphi(u_n) - \frac{1}{\alpha}\varphi'(u_n)\cdot u_n|\\ &\leq |\varphi(u_n)| + |\frac{1}{\alpha}\varphi'(u_n)\cdot u_n|\\ &\leq c + \frac{1}{\alpha}||u_n||_{W^{1,2}_{m_1,m_2}}.\end{aligned}$$

In addition,

$$\varphi(u_n) = \frac{1}{2}||u_n||^2 - \frac{1}{\alpha}||u_n||^{\alpha}_{L^{\alpha}}$$

and

$$-\frac{1}{\alpha}\varphi'(u_n)\cdot u_n = -\frac{1}{\alpha}\int_{\Omega}|\nabla_{m_1,m_2}u_n|^2 + \frac{1}{\alpha}\int_{\Omega}|u_n|^{\alpha} = -\frac{1}{\alpha}||u_n||^2_{W^{1,2}_{m_1,m_2}} + \frac{1}{\alpha}||u_n||^{\alpha}_{L^{\alpha}}.$$

Therefore,

$$\frac{1}{2}||u_n||^2_{W^{1,2}_{m_1,m_2}} - \frac{1}{\alpha}||u_n||^{\alpha}_{L^{\alpha}} - \frac{1}{\alpha}||u_n||^2 + \frac{1}{\alpha}||u_n||^{\alpha}_{L^{\alpha}} \leq c + \frac{1}{\alpha}||u_n||_{W^{1,2}_{m_1,m_2}},$$

so,

$$\left(\frac{1}{2} - \frac{1}{\alpha}\right)||u_n||^2_{W^{1,2}_{m_1,m_2}} \leq c + \frac{1}{\alpha}||u_n||_{W^{1,2}_{m_1,m_2}}. \tag{4.11}$$

This implies that $||u_n||_{W^{1,2}_{m_1,m_2}}$ is bounded because $\left(\frac{1}{2} - \frac{1}{\alpha}\right) > 0$. Otherwise, it suffices to divide (4.11) by the norm of u_n in $W^{1,2}_{m_1,m_2}$ and take the limit as n tends to infinity to arrive at a contradiction.

So, $\{u_n\}$ admits a weakly convergent subsequence. Without loss of generality, we have $u_n \rightharpoonup \overline{u}$ in $W^{1,2}_{m_1,m_2}$.

We know that

$$\varphi'(u_n - \overline{u})\cdot(u_n - \overline{u}) = ||u_n - \overline{u}||^2_{W^{1,2}_{m_1,m_2}} - ||u_n - \overline{u}||^{\alpha}_{L^{\alpha}(\Omega)}.$$

Thus,

$$||u_n - \overline{u}||^2_{W^{1,2}_{m_1,m_2}} = \varphi'(u_n - \overline{u})\cdot(u_n - \overline{u}) + ||u_n - \overline{u}||^{\alpha}_{L^{\alpha}(\Omega)},$$

so, by Cauchy-Schwarz inequality,

$$\begin{aligned} ||u_n - \overline{u}||^2_{W^{1,2}_{m_1,m_2}} &\leq o(n)\cdot||u_n - \overline{u})||^2_{W^{1,2}_{m_1,m_2}} + ||u_n - \overline{u}||^{\alpha}_{L^{\alpha}(\Omega)} \\ &\leq o(n)C + ||u_n - \overline{u}||^{\alpha}_{L^{\alpha}(\Omega)} \end{aligned}$$

By the weighted Sobolev compact embedding (4.8), we get $u_n \to \overline{u}$ in L^{α} because $\alpha \in [1, 2^*(m_1,m_2))$. Therefore,

$$||u_n - \overline{u}||^2_{W^{1,2}_{m_1,m_2}} \to 0.$$

Therefore, by Lemma 4.4 and the PS condition, it follows from the Mountain Pass Theorem 1.10 that there exists a critical point u_0 of φ which is the weak solution of the problem (4.10), with $\varphi(u_0) = c > 0$. But, as $\varphi(0) = 0$, it follows that $u_0 \neq 0$, that is, u_0 is a nontrivial solution. □

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